Scaling cosmologies of $\mathcal{N} = 8$ gauged supergravity

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Abstract
We construct exact cosmological scaling solutions in $\mathcal{N} = 8$ gauged supergravity. We restrict to solutions for which the scalar fields trace out geodesic curves on the scalar manifold. Under these restrictions it is shown that the axionic scalars are necessarily constant. The potential is then a sum of exponentials and has a very specific form that allows for scaling solutions. The scaling solutions describe eternal accelerating and decelerating power-law universes, which are unstable. An uplift of the solutions to 11-dimensional supergravity is carried out and the resulting time-dependent geometries are discussed. In the discussion we briefly comment on the fact that $\mathcal{N} = 2$ gauged supergravity allows accelerating stable scaling solutions.

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1. Introduction
In order to understand the possible (late-time) cosmological scenarios in string theory it is natural to study this in a supergravity context with a higher dimensional origin [1–5]. That way one can learn how supersymmetry and a higher dimensional origin constrain the possibilities. In light of this some investigations on the vacuum structure of gauged extended supergravities have been carried out; for instance de Sitter vacua in such theories can be found generically [1, 3, 6–11]. However, it is only for $\mathcal{N} = 2$ supergravity that stable de Sitter vacua have been constructed [10, 11], unfortunately for those examples the higher dimensional origin is unclear.

The possibilities for dark energy go beyond a positive cosmological constant, see [12] for a recent overview. An interesting possibility is the existence of cosmological scaling solutions. There are many definitions used but a common feature is that scaling cosmologies correspond to attractors, repellers and saddle points of the cosmological dynamical system. The definition used here is that the ratio of the energy densities of different constituents
remains constant during evolution. For a matter-scaling solution the energy density of the background barotropic fluid evolves in a constant ratio with respect to the scalar field energy density. Since such solutions can correspond to an attractor they are typically used in attempts to alleviate the cosmic coincidence problem [13]. Scaling cosmologies are characterized by a scale factor that is power law, \( a(\tau) \sim \tau^P \). We refer to [12–14] for more phenomenological issues concerning scaling solutions.

The scaling solutions studied in this paper are of two kinds, the matter scaling explained above and a scalar-dominated scaling solution in which the energy density of the barotropic fluid vanishes and the potential energy of the scalar fields scales as the kinetic energy\(^3\).

In contrast to de Sitter solutions scaling cosmologies have not been given that much attention in supergravity. In [4] an unstable accelerating \( (P > 1) \) scaling solution with \( P = 3 \) was found in \( \mathcal{N} = 8 \) supergravity as an alternative to acceleration from de Sitter solutions. In [15] an example of a stable scaling solution, with \( P = 1 \) was found in \( \mathcal{N} = 4 \) gauged supergravity. Finally, [16] considered scaling solutions of six-dimensional gauged chiral supergravity compactified to four dimensions.

It is the aim of this paper to systematically find the scaling cosmologies in \( \mathcal{N} = 8, D = 4 \) supergravity and to check their stability. Although \( \mathcal{N} = 8 \) supergravity is not realistic from a particle physics point of view it has attractive features; there is only the supergravity multiplet and the different theories only differ in the gauge group. This simplicity makes it easier to oversee all the possibilities for finding interesting solutions. It is believed that many (if not all) \( \mathcal{N} = 8 \) theories have a higher dimensional origin. For the gaugings we consider in this paper the higher dimensional origin is known explicitly [17].

To perform an exhaustive study of cosmological solutions in supergravity theories is notoriously difficult because of the many scalar fields and the corresponding complicated potentials. For instance in \( \mathcal{N} = 8 \) supergravity there are 70 scalars that parametrize the \( E_7(\mathbb{C})/SU(8) \)-coset space and the complexity of the potential depends on the gauge group of the theory. In this respect, ungauged supergravity is easier since there is no potential. If we restrict to FLRW-universes and redefine cosmic time \( \tau \) to a new time coordinate \( s \) via \( \mathrm{d}\tau = a(\tau)^3 \mathrm{d}s \), the scalar field action in ungauged supergravities reads

\[
S_{\text{scalar}} = \int G_{ij}(\phi)\phi^i\phi^j \, \mathrm{d}s,
\]

where \( \cdot \) indicates derivation with respect to \( s \). This is the action for geodesic curves parametrized by the affine parameter \( s \). Therefore cosmologies driven by massless fields are geodesics on the scalar manifold. For the moduli spaces that appear in maximal supergravity there have been investigations on the geodesic curves and their higher dimensional interpretation [18, 19].

In the case that there is a scalar potential, [20] stated that the scalars of all scaling solutions describe geodesic curves on the scalar manifold. However the proof of [20] shows that a geodesic can indeed give rise to a scaling solution but the converse statement—a scaling solution must be a geodesic—is not proven. An example of a scaling solution that does not describe a geodesic can be found for the axion–dilaton system of [21]. We consider those scaling solutions that are geodesic as an interesting subclass. Since geodesics on symmetric spaces are well understood we can perform a systematic search for geodesic scaling cosmologies of gauged supergravities. Ideally we would like to study all possible scaling cosmologies and not just the class that describes geodesics.

\(^3\) This is also true for matter-scaling solutions so the only difference is that the fluid vanishes.
We make some important restrictions in this paper. We consider only flat FLRW-universes and ignore scaling solutions that exist on the boundary of the scalar manifold\(^4\). We refer to [23] for a general treatment of scaling solutions in the presence of spatial curvature, that also includes solutions on the boundary of the scalar manifold.

2. Scaling solutions and geodesics

The action we consider contains \(N\) Klein–Gordon fields \(\phi^i\) coupled to gravity:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} G_{ij}(\phi) \partial_i \phi^i \partial^j \phi^j - V(\phi) \right] + S_{\text{matter}},
\]

where \(\kappa^2 = 8\pi G\) with \(G\) being Newton’s constant and \(S_{\text{matter}}\) the action that describes a barotropic fluid with constant equation-of-state-parameter \(\gamma - 1\). In a flat FLRW-background, \(ds^2 = -d\tau^2 + a(\tau)^2 d\vec{x}^2\), the equations of motion read\(^5\):

\[
\ddot{\phi}^i + \Gamma^i_{jk} \dot{\phi}^j \dot{\phi}^k + 3H \dot{\phi}^i = -G^i_{\phi} \partial_j V,
\]

\[
\dot{\rho} + 3\gamma H \rho = 0,
\]

\[
p = (\gamma - 1) \rho,
\]

\[
H^2 = \frac{\kappa^2}{3} (T + V + \rho),
\]

\[
H = -\kappa^2 (T + \frac{3}{2} \gamma \rho),
\]

with \(T\) being the kinetic energy of the scalars, \(T = \frac{1}{2} G_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j\) and \(H = \dot{a}/a\), the Hubble parameter. We choose units in which \(\kappa^2 = \frac{1}{2}\).

We define a scaling cosmology as a solution of the above equations for which \(V(\tau) \sim T(\tau) \sim \rho(\tau)\).

From the Friedmann equation (6) and the acceleration equation (7) we find that the scale factor is power law \(a \sim \tau^P\) and vice versa. Hence for a scaling solution we have

\[
V(\tau) \sim T(\tau) \sim \rho(\tau) \sim H^2(\tau) \sim \dot{H}(\tau) \sim \frac{1}{\tau^2}.
\]

Scaling solutions for multiple fields are well studied for flat scalar manifolds \(G_{ij} = \delta_{ij}\) with (multiple) exponential potentials [22–27], where the scaling solutions all have the following form:

\[
\phi^i = a^i \ln(\tau) + b^i.
\]

We note that the scalars trace out straight lines, that is geodesics on a flat space. If we take \(s = \ln \tau\) as a parameter the straight lines have a constant velocity \(\left[\sum a^2_i\right]^{1/2}\) and the parameter \(s\) is an affine parameter. The reason that geodesics can describe scaling cosmologies comes from the constraint \(V \sim T\), which implies

\[
L_{\text{scalar}} \sim T(\tau) - V(\tau) \sim T(\tau).
\]

Since \(T = \frac{1}{2} G_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j \sim \tau^{-2}\) for scaling it is clear that \(s = \ln \tau\) is an affine parameter since \(G_{ij} \partial_i \phi \partial_j \phi = \text{const}\). The geodesic equation is

\[
\ddot{\phi}^i + \Gamma^i_{jk} \dot{\phi}^j \dot{\phi}^k = 0,
\]

\(^4\) They are called non-proper solutions in [22, 23].

\(^5\) We use the mostly plus convention for the metric and the Riemann tensor reads \(R_{\mu\nu}^{\alpha\beta} = \partial_\rho \Gamma^\rho_{\mu\nu} + \cdots\).
where a prime denotes differentiation with respect to \( s \). Consistency with the Klein–Gordon equation gives rise to a first-order equation [20]:

\[
(3P - 1)\phi^i = -e^2 G^{ij} \partial_j \ln V.
\]  

(13)

### 3. The \( \mathcal{N} = 8 \) scalar potential

The action of ungauged \( \mathcal{N} = 8 \) supergravity in four dimensions exhibits a rigid \( SL(8, \mathbb{R}) \)-symmetry [28]. The equations of motion allow for a larger, non-compact \( E_7(7) \)-symmetry. The theory contains 70 scalars that parametrize the coset \( E_7(7)/SU(8) \). De Wit and Nicolai gauged the \( SO(8) \)-subgroup of \( SL(8, \mathbb{R}) \) by introducing minimal couplings that break supersymmetry [29]. In order to restore supersymmetry, further terms have to be added to the action. In this way, one generates a potential that is proportional to the square of the gauge coupling constant. Later on, starting from this example, the so-called CSO\( (p, q, r) \)-gaugings were found [30–32].

The generators of this subgroup are denoted by

\[
\Lambda_{ab} = -\Lambda_{ba}, \quad a, b = 1, \ldots, 8
\]

and they obey the following algebra:

\[
[\Lambda_{ab}, \Lambda_{cd}] = \Lambda_{ad} \eta_{bc} - \Lambda_{ac} \eta_{bd} - \Lambda_{bd} \eta_{ac} + \Lambda_{bc} \eta_{ad},
\]  

(14)

where

\[
\eta_{ab} = \begin{pmatrix}
1_{p \times p} & 0 & 0 \\
0 & -1_{q \times q} & 0 \\
0 & 0 & 0_{r \times r}
\end{pmatrix}.
\]  

(15)

Other gaugings have subsequently been found, see for instance [33] but for the rest of the paper we only consider the CSO-gaugings.

Obtaining an explicit expression for the potential that can be used to search for vacua is a difficult task. Often one makes truncations to get manageable expressions. Therefore we focus on the \( SL(8, \mathbb{R})/SO(8) \)-submanifold of \( E_7(7)/SU(8) \) that contains the 35 scalar fields while the other 35 pseudoscalar fields are consistently truncated. The action for the metric and the 35 scalars is given by [34]:

\[
\mathcal{L} = \sqrt{-g} \left\{ R + \frac{1}{4} \text{Tr}[\partial M \partial M^{-1}] - V \right\},
\]  

(16)

where \( M = LL^T \) with \( L \) being the coset representative of the \( SL(8, \mathbb{R})/SO(8) \)-coset. The potential is given by

\[
V = \text{Tr}\left[ (\eta M)^2 \right] - \frac{1}{2} \left( \text{Tr}[\eta M] \right)^2.
\]  

(17)

The scalar field equations of motion derived from the Langrangian are

\[
\partial [M^{-1} \partial M] = 4(\eta M)^2 - 2 \text{Tr}[\eta M] \eta M - \frac{4}{n} V \mathbb{1},
\]  

(18)

with \( n \) being the dimension of the matrices (for now \( n = 8 \)).

The coset \( SL(8, \mathbb{R})/SO(8) \) contains seven dilatons \( \phi^i \) and 28 axions \( \chi^a \). In the solvable gauge the coset representative \( L \) is written as

\[
L = e^{\alpha E} e^{-\frac{1}{2} \psi H},
\]  

(19)

where the sum over the indices \( \alpha \) is a sum over the positive root generators and the sum over the indices \( i \) is a sum over the Cartan generators.

For what follows we need some properties of the weights \( \vec{\beta}_a \) of the \( SL(n, \mathbb{R}) \)-algebra in the fundamental representation:

\[
\sum_a \beta_{ai} = 0, \quad \sum_a \beta_{ai} \beta_{aj} = 2 \delta_{ij}, \quad \vec{\beta}_a \cdot \vec{\beta}_b = 2 \delta_{ab} - \frac{2}{n}.
\]  

(20)

The last two identities hold in a convenient basis that is given in appendix A.
Because the axions appear at least squared in the potential it is consistent to put them to zero. Then the matrix $M$ simplifies to $M = \text{diag}(e^{-\vec{\beta}_a \cdot \vec{\phi}})$, such that the kinetic term becomes canonical and the potential is a sum of exponentials:

$$L_{\text{scalar}} = -\frac{1}{2} \delta_{ij} \partial \phi^i \partial \phi^j - \frac{1}{2} \sum_{a=1}^{pq} e^{-2\vec{\beta}_a \cdot \vec{\phi}} + \sum_{a<b} \eta_{ab} \eta_{bb} e^{-(\vec{\beta}_a + \vec{\beta}_b) \cdot \vec{\phi}}. \tag{21}$$

4. Dilatonic scaling cosmologies

The potential in (21) is an example of a general exponential potential, which is a sum of $M$ exponential terms that depend on $N$ scalar fields:

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[\vec{\alpha}_a \cdot \vec{\phi}], \tag{22}$$

where $\vec{\alpha}_a \cdot \vec{\phi} = \sum_{i=1}^{N} \alpha_{ai} \phi^i$. Scaling solutions of such potentials have been studied in great detail [22–27, 35] and for convenience we reformulate some essential properties.

4.1. Scaling for multiple exponential potentials

It follows from the autonomous system approach in [22] that it is convenient to separate exponential potentials in two classes I and II. Class I is characterized by the fact that the $\vec{\alpha}_a$-vectors are linearly independent whereas for class II they are linearly dependent. Models that belong to the first class are known in the literature under the name generalised assisted inflation [27]. Class I generically allows exact scaling solutions, whereas class II can have exact scaling solutions only when the $\vec{\alpha}_a$-vectors are affinely related [22, 23]. Affinely related means that there exists a set of $R$ independent $\vec{\alpha}_a$ such that after relabelling $a = 1, \ldots, R$, the remaining $\vec{\alpha}_b$ are expressed as $\vec{\alpha}_b = \sum_{a=1}^{R} c_{ba} \vec{\alpha}_a$, where the coefficients $c_{ba}$ fulfil the constraint:

$$\sum_{a=1}^{R} c_{ba} = 1, \quad \text{for all } b = R, \ldots, M. \tag{23}$$

Both types of potentials that allow for scaling solutions have the unique property that after an orthogonal field redefinition $\vec{\phi} \rightarrow \vec{\phi}'$ the potential can always be written as the following product [23, 25]:

$$V(\phi') = e^{\psi N} U(\phi_2, \ldots, \phi_N). \tag{24}$$

Let us prove (24) for class I and then for class II with affinely related $\vec{\alpha}_a$-vectors. For the proof we always assume that a field rotation is performed such that the minimal number, $R$, of scalars appears in the potential and that consequently $N - R$ scalar fields are free. This number $R$ equals the number of linearly independent $\vec{\alpha}_a$-vectors [22]. So class I has $R = M$ and class II $R < M$.

If the $\vec{\alpha}_a$ are linearly independent there exists a (unit) vector $\vec{E}$ such that

$$\vec{\alpha}_a \cdot \vec{E} = c, \tag{25}$$

6 The kinetic term allows a truncation of all the axions since the dilatons parametrize a geodesic complete submanifold.
where \( c \) is a number which is independent of the index \( a \). Since, if we multiply (25) with \( \alpha_{aj} \)
and sum over \( a \) we obtain

\[
\sum_{ai} \alpha_{aj} \alpha_{ai} E' = c \sum_{a} \alpha_{aj}.
\]

(26)

The matrix \( B_{ij} = \sum_{a} \alpha_{aj} \alpha_{ai} \) has an inverse (because \( R = M \)) and the equation can be solved to find \( E' \). If we now write the scalar fields in a different basis:

\[
\vec{\phi} = \vec{\phi}_{1} + \vec{\phi}_{\perp}.
\]

(27)

then we have in the new basis that \( \alpha_{aj} = c \) for all \( a \) and consequently the potential takes the form (24).

Now assume the \( \vec{\alpha}_{a} \) are linearly dependent in an affine way. Consider the \( R \) independent vectors \( \vec{\alpha}_{a} \) with \( a = 1, \ldots, R \). For this subset we can repeat the same procedure as above to find a unit vector \( \vec{E} \) that obeys (25). Then we have in the new basis that \( \alpha_{aj} = c \) for \( a = 1, \ldots, R \). Consider \( \alpha_{a_{1}} \) for \( b > R \):

\[
\alpha_{b1} = \sum_{a=1}^{R} c_{ba} \alpha_{a1} = c \sum_{a=1}^{R} c_{ba} = c.
\]

(28)

Again the potential can be factorized as in (24).

It is easy to prove the inverse, if the potential can be written as (24) then either the \( \vec{\alpha}_{a} \) are linearly independent or they are dependent in an affine way.

As proven in [23, 25] the exact scaling solution is such that it is the overall scalar \( \phi_{1} \) that is non-constant and the other scalars are constant. Therefore the exact scaling solutions of multiple exponential potentials are such that the potential is truncated to a single exponential potential.

The requirement for such a truncation is twofold. First, it must be possible to rewrite the potential like in (24) and second the function \( U \) must have stationary points (\( \partial U = 0 \)) in order to have a truncation consistent with the equations of motion. If this is satisfied the truncated action is given by

\[
S = \int \sqrt{-g} \left[ R - \frac{1}{2}(\partial \phi)^{2} - \Lambda e^{\phi} \right] + (S_{\text{Matter}}),
\]

(29)

where \( \Lambda \) is the function \( U \) at the stationary point. If the scaling solution exists it is given by

\[
a \sim \tau^{P}, \quad \phi = -c \ln \tau + \frac{\ln \left( \frac{6P - 2}{c^{2} \Lambda} \right)}{c}, \quad \rho = 6 \left( 1 - \frac{1}{c^{2} P} \right) \frac{P^{2}}{\tau^{2}}.
\]

(30)

- Let us first assume that the barotropic fluid vanishes, then the scaling solution is the scalar-dominated solution with \( P = 1/c^{2} \). The scaling solution exists when \( \Lambda > 0 \) and \( P > 1/3 \) or \( \Lambda < 0 \) and \( P < 1/3 \). An inflationary solution (\( P > 1 \)) requires \( c^{2} < 1 \). The scaling solution with \( \Lambda < 0 \) is never stable and the scaling solution with \( \Lambda > 0 \) is stable if the extremum of \( U \) is a minimum and the fluid perturbations imply an extra stability condition \( \frac{1}{P} > \frac{c^{2}}{2 \gamma} \).

- If on the other hand a barotropic fluid is nonzero there exists a matter-scaling solution [35]. The matter-scaling solution is such that the energy density of the barotropic fluid and the scalar fields are non-vanishing and have a fixed ratio. The scale factor of a matter-scaling solution is that of a universe containing only the barotropic fluid, that is \( a \sim \tau^{P} \) with \( P_{\text{matter}} = \frac{2}{27} \). The solution exists when \( \Lambda > 0 \) and \( \frac{1}{P} < \frac{2}{27} \). When \( U \) is in a minimum the solution is stable.

7 In [36] the same was proven for purely positive exponential terms and a special class of dilaton couplings.
4.2. The CSO-dilaton potentials

For the CSO\((p, q, r)\)-gaugings \((p + q + r = 8)\) with \(r > 0\) the potential (17) can be written in such a way that a smaller coset matrix \(\tilde{\mathcal{M}}\) appears. The result is [34]

\[ V = e^{\phi} U(\mathcal{M}) = e^{c\phi} \left[ \text{Tr}\left( \tilde{\eta} \tilde{\mathcal{M}} \right)^2 - \frac{1}{2} \left( \text{Tr}[\tilde{\eta} \tilde{\mathcal{M}}] \right)^2 \right], \]  

(31)

where

\[ c^2 = \frac{8}{p + q} - 1, \quad \tilde{\eta} = \text{diag}(\tilde{1}_p, -\tilde{1}_q), \]  

(32)

and the scalars appearing in \(\tilde{\mathcal{M}}\) are those coming from the \(SL(p + q, \mathbb{R})/SO(p + q)\)-coset and together with \(\phi\) they span the manifold \(GL(p + q, \mathbb{R})/SO(p + q) = \mathbb{R} \times SL(p + q, \mathbb{R})/SO(p + q)\).

If we restrict to the dilatons, (21) becomes

\[ V = e^{\phi} U(\phi) = e^{\phi} \left[ \frac{1}{2} \sum_a \tilde{\rho}_a^2 \phi - \sum_{a < b} \tilde{\eta}_{ab} \tilde{\eta}_{bb} e^{-\tilde{\sigma}_a \tilde{\sigma}_b} \phi \right], \]  

(33)

where the vectors \(\tilde{\rho}_a\) are the weights of \(SL(p + q, \mathbb{R})\). For \(c \neq 0\) this potential clearly belongs to class II with affinely related \(\tilde{\alpha}_a\)-vectors. Therefore the potential is of the appropriate form for scaling solutions!

To find a scaling solution it is sufficient to find a stationary point of \(U\) that has the correct sign to allow for a scaling solution. The stationary points of \(U\) are most easily found using Lagrange multipliers as was shown in [9]. The outcome of this calculation is summarized in table 1.

We remark that there is no scalar-dominated scaling solution between \(1/3 < P_{\text{scalar}} < 1\). This implies that in these models a matter-scaling solution can never coexist with a scalar-dominated scaling cosmology.

The accelerating scaling solutions \((P > 1)\) are found for the CSO\((3, 3, 2)\)-gauging and the CSO\((4, 3, 1)\)-gauging. The first was found by Townsend in [5] where it was constructed by a reduction of a de Sitter vacuum in five-dimensional \(SO(3, 3)\)-gauged supergravity. The second possibility with \(P = 7\) has as far as we know not been found before. The cosmologies of the CSO\((1, 1, 6)\)-gauging were considered before [37] where the solutions were obtained from a reduction of seven-dimensional pure gravity on a group manifold.

Since the matrix \(\delta \partial_i U\) evaluated at an extremum is not positive definite, the solutions are unstable.

5. Axion–dilaton scaling cosmologies

Once the axions are turned on the system is much more complex and the construction of solutions becomes a difficult task. But if we restrict to scaling cosmologies that describe
geodesics the problem boils down to parametrizing the geodesics and to check when they are solutions. In what follows we describe a way to parametrize the geodesics in terms of isometry transformations of straight lines.

Since the submanifold spanned by the dilaton fields is flat, the geodesics on that part are straight lines
\[ \phi'(s) = v^i s + \phi'(0), \quad (34) \]
in terms of the affine parameter introduced in section 2. Since \( SL(n, \mathbb{R}) \) is a symmetry of the geodesic equations it maps geodesics to geodesics. Transforming the above straight line generates a general geodesic, which is not a straight line. The following lemma shows that in this way all geodesics can be obtained. The proof is left for the appendix.

**Lemma 5.1.** Every geodesic on the symmetric space \( SL(n; \mathbb{R})/SO(n) \) can be obtained by acting with isometries on a straight geodesic through the origin.

An isometry transformation is nonlinear on the level at the coordinates \( \phi^i, \chi^a \) but for the scalar matrix \( M \) it works as
\[ M(s) \rightarrow \Omega M(s) \Omega^T, \quad \det \Omega = 1. \quad (35) \]

The consequence of the above lemma is that the problem of finding geodesic scaling solutions reduces to an algebraic problem. If a geodesic scaling solution exists the scalars take the form:
\[ \varphi = v_0 s + d_0, \quad \hat{M} = \Omega D \Omega^T, \quad (36) \]
where \( D = \text{diag}(e^{-\beta_i \cdot \phi}) \) with \( \phi = \hat{\phi} s \). For simplicity we work in the truncated system defined by \( \hat{M} \) and \( \varphi \), as explained in the previous section.

For the scaling solutions we have
\[ V(s) = \alpha e^{-2s}, \quad H = P e^{-s}. \quad (37) \]
With \( \alpha \) being some constant. If we substitute this into the equations of motion for \( \hat{M} \) and \( \varphi \), we find the following matrix equation:
\[ e^{-[(2+cv_0)s-cd_0]} A^{-1} \left\{ (1 - 3 P) D^{-1} D' + \frac{4 \alpha}{n} I \right\} = 4 D AD - 2 \text{Tr}(AD) D, \quad (38) \]
where \( A = \Omega^T \eta \Omega \). For a given matrix \( D \) that corresponds to a straight line, we can always make an orthogonal field redefinition on the dilatons such that the matrix \( D \) simplifies to
\[ D = \text{diag}(e^{-\|v\| s}, e^{\|v\| s}, 1, \ldots, 1). \quad (39) \]
It is then not too difficult to check that the solutions of equation (38) necessarily have \( \|v\| = 0 \). This implies that \( \hat{M} \) is constant and the only running field is the overall dilaton \( \varphi \). In particular, if we act with the rigid CSO-symmetry on the dilaton solutions we are guaranteed to find new solutions but only with constant axions.

### 6. Higher dimensional origin

In [17] it was shown that the non-compact gaugings are associated with 11-dimensional supergravity solutions that have a non-compact internal space. For the CSO(\( p, q, r \))-gaugings the internal space \( \mathcal{N}^{p,q,r} \) is a hypersurface in \( \mathbb{R}^8 \), defined by the following equation:
\[ T_{AB} e^A e^B = R^2 \quad \text{and} \quad T = L^T \eta L, \quad (40) \]

\( \dagger \) This is not an isometry transformation on the coset, but corresponds to choosing a new coordinate system on the coset. This new coordinate system does not affect equation (38).
where $z^A$ are Cartesian coordinates of $\mathbb{R}^8$ and $R$ is determined by the flux of the 4-form field strength in 11 dimensions. In the previous sections we fixed the flux parameter $R = 1$. For arbitrary flux parameter the potential has an extra factor $R^{-2}$ in front of the expression (17).

The metric on the hypersurface is then induced from the Euclidean metric on $\mathbb{R}^8$. Given a solution in four dimensions with a metric $g_4$ and some scalars as only non-vanishing fields, the 11-dimensional metric $g_{11}$ is then determined as

$$g_{11} = \Delta^4 g_4(x) + \Delta^{-2} g_H(x, y),$$

where $g_H(x, y)$ is the metric on $\mathcal{H}^{p,q,r}$ and $\Delta(x, y)$ is a warp factor

$$\Delta = \frac{T^2_{AB} z^A z^B}{R^2}.$$ (42)

From the explicit solutions for the scalar matrix $M$ we notice that our scaling solutions correspond to $SO(p) \times SO(q)$-invariant directions in the scalar coset

$$T = e^{4\phi} \begin{pmatrix} X_{\mathbb{R}^p \times \mathbb{R}} & 0 & 0 \\ 0 & -\tilde{X}_{\mathbb{R}^q \times \mathbb{R}} & 0 \\ 0 & 0 & 0_{p \times q} \end{pmatrix}.$$ (43)

The constants $X$ and $\tilde{X}$ are the constant diagonal components of the $SL(p+q, \mathbb{R})/SO(p+q)$-scalar matrix $\tilde{M} = \text{diag}(X_{\mathbb{R}^p}, \tilde{X}_{\mathbb{R}^q})$.

We follow the same spirit of [8] and take the Euclidean metric on $\mathbb{R}^8$ as

$$ds^2 = d\sigma^2 + \sigma^2 d\Omega^2_{p-1} + d\tilde{\sigma}^2 + \tilde{\sigma}^2 d\Omega^2_{q-1} + \sum_{A=p+q+1}^8 dz^A dz^A,$$ (44)

with $d\Omega^2_n$ being the round metric on the unit $n$-sphere. In terms of these non-Cartesian coordinates, the hypersurface (40) is explicitly given by

$$\sigma^2 - \left(\frac{\tilde{X}}{X}\right) \tilde{\sigma}^2 = \frac{R^2}{X} e^{c\phi} \sim \tau$$ (45)

Because the ratio $\tilde{X}/X$ appears often we call it $\lambda$. If we introduce new coordinates $r, \rho$ in the following way

$$\tilde{\sigma} = \rho r, \quad \sigma = \rho (1 + \lambda r^2)^{1/2},$$ (46)

then hypersurface (45) is defined by $\rho^2 = \frac{R^2}{X} e^{-c\phi}$ and the metric on $\mathcal{H}^{p,q,r}$ is found to be

$$ds^2_{H} = \frac{R^2}{X} e^{-c\phi} \left[ 1 + \frac{(\lambda + \lambda^2) r^2}{1 + \lambda r^2} \right] dr^2 + (1 + \lambda r^2) d\Omega^2_{p-1} + r^2 d\Omega^2_{q-1} + \sum_{A=p+q+1}^8 (dz^A)^2.$$ (47)

The warp factor is

$$\Delta = X(1 + (\lambda + \lambda^2) r^2) e^{c\phi}$$ (48)

and the 11-dimensional metric is then given by (41).

The internal hyperbolic spaces have no non-compact isometries as opposed to the maximally symmetric hyperboloid [39] and we cannot create a compact orbifold from the internal hyperbolic spaces. The scaling solutions describe internal hyperboloids with compact $SO(p) \times SO(q)$-symmetry and an overall time-dependent breathing mode playing the role of a four-dimensional quintessence field.

9 The flux parameter $R$ is also the inverse of the gauge coupling constant.

10 In terms of the $SO(p) \times SO(q)$ invariant scalars $s$ and $t$ defined in [1, 8, 38], our solutions have constant $s$ and running $t$. 
7. Discussion

In this paper we investigated scaling solutions in $\mathcal{N} = 8$ gauged supergravity. When restricted to the dilatons, the potential becomes a sum of exponentials. We showed that, when the gauge group is a contraction of $SO(p,q)$, the exponentials exhibit a special form, they have so-called affine couplings. This special form is necessary for the existence of exact scaling solutions. We find eternal accelerating solutions for which the barotropic fluid vanishes. From the point of view of 11-dimensional supergravity, the solutions correspond to time-dependent geometries with non-compact internal spaces. If we assume the presence of a barotropic fluid we also find matter-scaling solutions. The solutions we obtained have one running scalar and all other scalars are trapped in a saddle point or maximum of the potential, and are therefore unstable. As explained in [3] unstable vacua are not necessarily a bad thing for cosmology. If a de Sitter vacuum is at a saddle or a maximum of the potential, the universe will stop accelerating at some point and collapses to a singularity since the potential becomes negative. However the typical time before the collapse is comparable to the age of our universe. We expect that a similar statement can be made for the accelerating scaling vacua.

These dilatonic solutions describe geodesics in the scalar manifold. We showed that geodesic scaling solutions with non-constant axions do not exist.

Scaling solutions correspond to critical points of the cosmological dynamical system and therefore describe the early- or late-time behaviour of general cosmological solutions. From this point of view there is a similarity with cosmological billiards in supergravity where the asymptotic behaviour of cosmological solutions correspond to Kasner-type metrics and the axionic fields are constant [19]. A more complete analysis would involve solutions that interpolate between scaling vacua in order to understand how the cosmic billiard behaviour is realized in gauged extended supergravity.

In light of our findings one could wonder whether stable scaling solutions with eternal acceleration are possible at all in supergravity? If one lowers the amount of supersymmetry then stable solutions are possible. In $\mathcal{N} = 4$ gauged supergravity, a (non-accelerating) stable scaling solution was found in [15] and, as we outline below, stable eternal accelerating scaling solutions are present in $\mathcal{N} = 2$ theories. These stability properties are similar for de Sitter vacua in supergravity, where stable vacua are only found for $\mathcal{N} \leq 2$. So until now the only stable solutions that violate the strong energy condition are found in $\mathcal{N} \leq 2$ supergravity.

The existence of stable scaling solutions in $\mathcal{N} = 2$ gauged supergravity follows from the fact that there exist stable de Sitter vacua in $D = 5$ [11]. If a certain supergravity has a de Sitter solution in $4+n$ dimensions, that implies that the system can be truncated to just gravity and a positive cosmological constant, $\Lambda$. If we reduce this theory on an $n$-torus and consider only one breathing mode $\psi$ for the overall volume of the $n$-torus,

$$ds^2 = e^{\sqrt{\frac{n}{n+2}} \psi} dx_4^2 + e^{-2 \sqrt{\frac{n}{n+2}} \psi} d\xi^2,$$

we find

$$S_4 = \int dx^4 \sqrt{-g_4} \left[ R - \frac{1}{2} (\partial \psi)^2 - \Lambda e^{\sqrt{\frac{n}{n+2}} \psi} \right].$$

This theory has an accelerating scaling solution

$$dx_4^2 = -d\tau^2 + r^{2 \frac{n+2}{n}} dx_3^2, \quad \sqrt{\frac{n}{n+2}} \psi = -2 \ln \tau + c.$$  

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11 We would like to thank P Fré for pointing out this similarity.
Plugging this in the metric ansatz (49) and redefining time via $\tilde{t} = \sqrt{c} \ln t$ we find that the uplift of the four-dimensional scaling solution is

$$d s^2_{4+n} = -d\tilde{t}^2 + e^{\frac{4n}{\sqrt{c}} \tilde{t}} \, dx^2_{3n}.$$  \hfill (52)

This is a $4 + n$-dimensional de Sitter universe in flat FLRW-coordinates. When the de Sitter solution is stable, so is the scaling solution obtained via reduction.

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Appendix A. Weights of $SL(8, \mathbb{R})$

A convenient basis for the weights of $SL(8, \mathbb{R})$ in the fundamental representation is given by

$$\vec{\beta}_1 = (1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.1)

$$\vec{\beta}_2 = (-1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.2)

$$\vec{\beta}_3 = (0, 0, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.3)

$$\vec{\beta}_4 = (0, 0, 0, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.4)

$$\vec{\beta}_5 = (0, 0, 0, 0, \frac{4}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.5)

$$\vec{\beta}_6 = (0, 0, 0, 0, 0, \frac{5}{\sqrt{15}}, \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.6)

$$\vec{\beta}_7 = (0, 0, 0, 0, 0, 0, \frac{6}{\sqrt{21}}, \frac{1}{\sqrt{28}})$$  \hfill (A.7)

$$\vec{\beta}_8 = (0, 0, 0, 0, 0, 0, 0, \frac{7}{\sqrt{28}}).$$  \hfill (A.8)

Appendix B. Proof of lemma 5.1

**Lemma B.1.** Every geodesic on the symmetric space $SL(n; \mathbb{R})/SO(n)$ can be obtained by acting with isometries on a straight geodesic; by a straight geodesic we mean a geodesic that has the velocity vector in the Cartan subalgebra.

**Proof.** Let us write $\gamma : I \subset \mathbb{R} \rightarrow \mathcal{M} = SL(n; \mathbb{R})/SO(n)$ for a geodesic. Without loss of generality we may assume that the identity point $e \cong SO(n)$ lies in $\gamma(I)$ since we can always multiply the geodesic by a representative of $\gamma(t_0)^{-1}$ with $t_0 \in I$ and this geodesic goes through the origin $e$. If we can prove that by a rotation at the origin using the compact subgroup $SO(n)$
we can direct any tangent vector completely in the Cartan directions, the theorem is proved. Hence we need to know how the isotropy group acts on the tangent space at the origin; the isotropy group keeps $e$ fixed and induces a transformation on $T M_e$. Let $\Omega \in SO(n)$ generate the left-translation $x \mapsto \Omega \cdot x$. Suppose $X$ is a vector at the origin; $X \in \mathfrak{p}$ with $\mathfrak{p}$ being the orthogonal complement of $so(n)$ in $\mathfrak{sl}(n, \mathbb{R})$. Then $X$ generates the curve $t \mapsto e^{X \cdot t}$, that is, $X = \frac{d}{dt} |_{t=0} e^{X \cdot t}$. The action $t_{\Omega}$ of $\Omega$ on $X$ is found by

$$t_{\Omega}(X) = \frac{d}{dt} |_{t=0} (\Omega e^{X \cdot t} \cdot e) = \frac{d}{dt} |_{t=0} (e^{Ad\Omega X \cdot t} \cdot e) = Ad\Omega(X).$$

(B.1)

Hence we deduce that $t_{\Omega}(X) = Ad\Omega(X)$. The adjoint representation of $\mathfrak{sl}(n; \mathbb{R})$ decomposes under the subgroup $so(n)$ as follows:

$$n^2 - 1 \to \frac{1}{2} n(n - 1) \oplus \frac{1}{2} n(n + 1) - 1. \quad (B.2)$$

The above branching rule can be easily derived by investigating the fundamental representation of $\mathfrak{sl}(n; \mathbb{R})$; the subalgebra $so(n)$ is formed by the antisymmetric $n \times n$ matrices and the orthogonal complement to $so(n)$ is formed by the symmetric traceless matrices. A symmetric traceless matrix $(m_{ij})$ transforms under the adjoint action of $\omega_{ij} = -\omega_{ji} \in so(n)$ as $m_{ij} \mapsto \omega_{ji} m_{ij} + \omega_{ij} m_{ji}$, that is according to $\frac{1}{2} n(n + 1) - 1$. Hence $\mathfrak{p}$ forms an irreducible representation of $so(n)$ and since every symmetric matrix can be diagonalized using an orthogonal transformation, any tangent vector $X \in \mathfrak{p}$ can be rotated completely into the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{p}$, which we identify with the diagonal traceless matrices in the vector representation. Hence the lemma is proved. \(\square\)

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