Analysis of a full discretization of stochastic Cahn–Hilliard equation with unbounded noise diffusion

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Abstract: In this article, we develop and analyze a full discretization, based on the spatial spectral Galerkin method and the temporal drift implicit Euler scheme, for the stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise. By introducing an appropriate decomposition of the numerical approximation, we first use the factorization method to deduce the a priori estimate and regularity estimate of the proposed full discretization. With the help of the variation approach, we then obtain the sharp spatial and temporal convergence rate in negative Sobolev space in mean square sense. Furthermore, the sharp mean square convergence rates in both time and space are derived via the Sobolev interpolation inequality and semigroup theory. To the best of our knowledge, this is the first result on the convergence rate of temporally and fully discrete numerical methods for the stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise.

MSC 2010 subject classifications: Primary 60H35; Secondary 35R60; 60H15; 65M75.
Keywords and phrases: stochastic Cahn–Hilliard equation, multiplicative space-time white noise, full discretization, strong convergence rate.

*This work was funded by National Natural Science Foundation of China (No. 91630312, No. 91530118, No.11021101 and No. 11290142).
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1. Introduction

Stochastic Cahn–Hilliard equation is a fundamental phase field model and can be used to describe the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably (see e.g. [4, 18]). There exists a lot of works focusing on the well-posedness of the stochastic Cahn–Hilliard equation (see e.g. [13, 5, 1]). Recently, the existence and uniqueness of the solution, as well as its regularity estimates, for the Stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise with diffusion coefficient of sublinear growth is proven in [7].

In this article, we consider the numerical approximation of the following stochastic Cahn–Hilliard equation with multiplicative space-time white noise

\[
\begin{align*}
    dX(t) + A(AX(t) + F(X(t)))dt &= G(X(t))dW(t), \quad t \in (0, T) \\
    X(0) &= X_0.
\end{align*}
\]

Here \(0 < T < \infty\), \(\mathbb{H} := L^2(\mathcal{O})\) with \(\mathcal{O} = [0, L]\), \(-A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}\) is the Laplacian operator under homogenous Dirichlet boundary condition, and \(\{W(t)\}_{t \geq 0}\) is a cylindrical Wiener process on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The nonlinearity \(F\) is assumed to be the Nemytskii operator of \(f'\), where \(f\) is a polynomial of degree 4, i.e., \(c_4 \xi^4 + c_3 \xi^3 + c_2 \xi^2 + c_1 \xi + c_0\) with \(c_i \in \mathbb{R}, i = 0, \cdots, 4, c_4 > 0\). A typical example is the double well potential \(f = \frac{1}{4}(\xi^2 - 1)^2\). When \(G = I\), Eq. (1.1) corresponds to the stochastic Cahn–Hilliard–Cook equation. The diffusion coefficient \(G\) is assumed to be the Nemytskii operator of \(g\), where \(g\) is a global Lipschitz function with the sublinear growth condition \(|g(\xi)| \leq C(1 + |\xi|^\alpha)\), \(\alpha < 1\).

Due to the lack of the analytical expression of the exact solution, the numerical approximation of the stochastic Cahn–Hilliard equation has attracted a lot of attentions and is far from well understood (see e.g. [6, 15, 17, 16, 12, 19]). The authors in [15, 17] show the strong convergence of the finite element method and its implicitly full discretization for Eq. (1.1) driven by additive spatial regular noise. The authors in [16] study the exponential integrability property of the spectral Galerkin method and deduce the strong convergence rate of the spectral Galerkin method for Eq. (1.1) with \(d = 1\) driven by additive trace class noise. Recently, the convergence rates of the finite element method and its implicitly full discretization are obtained in [19] for Eq. (1.1) driven by additive spatial regular noise. For Eq. (1.1) driven by additive spatial regular noise.
space-time white noise, we are only aware that the authors in [12] derive the optimal strong convergence rate in space and super-convergence rate in time for a full discretization, based on the spatial spectral Galerkin method and the temporal accelerated implicit Euler method. Recently, the authors in [7] show the strong convergence and the optimal strong convergence rate of the spatial spectral Galerkin method for Eq. (1.1) driven by multiplicative space-time white noise.

To the best of our knowledge, there exists no result about the strong convergence of temporally and fully discrete numerical approximations for stochastic Cahn–Hilliard equation driven by multiplicative space-time white noise. The present work considers the full discretization with the spatial spectral Galerkin method and the temporal drift implicit Euler method, and makes further contributions on the strong convergence of numerical schemes for SPDEs with non-globally monotone continuous nonlinearity, especially for Eq. (1.1). The presence of the unbounded elliptic operator in front of the cubic nonlinearity and the roughness of the driving noise, make the convergence analysis of numerical approximation much more challenging and demanding.

To overcome such difficulties, several steps and techniques are introduced. Instead of studying the strong convergence problem in $H$ directly, we use the similar idea in [12] to present the convergence analysis of numerical approximation in a negative Sobolev space at the first step. To this end, the optimal spatial and temporal regularity estimates of this numerical approximation are presented by using the equivalence between Eq. (1.1) and a random PDE, as well as the monotonicity of $-AF$ in $H^{-1}$ and the factorization method. Then the optimal strong convergence error estimate in $H^{-1}$ is derived with the help of the variation approach and an appropriate error decomposition. Based on the Sobolev interpolation inequality and the smoothy effect of the semigroup generated by the bi-Laplacian operator, we recover the optimal convergence rate in mean square sense of the numerical approximation for Eq. (1.1) driven by multiplicative space-time white noise. Let $\delta t$ be the time stepsize such that $T = K\delta t$, $K \in \mathbb{N}^+$, $X_0 \in H^\gamma$, $\gamma \in (0, \frac{3}{2})$, $N \in \mathbb{N}^+$ and $p \geq 1$, then the numerical solution $X_k^N$, $k \leq K$, is strongly convergent to $X$ and satisfies

$$\|X_k^N - X(t_k)\|_{L^2(\Omega; H)} \leq C(X_0, T, p, \gamma)(\delta t^{\gamma} + \lambda_N^{-(\frac{3}{2})})$$

for a positive constant $C(X_0, T, p, \gamma)$. We also remark that this approach is also available for deducing the strong convergence rates of numerical schemes.
for Eq. (1.1) with $d \leq 3$ driven by multiplicative regular noise and for proving the strong convergence of numerical schemes for Eq. (1.1) under other boundary conditions.

The outline of this paper is as follows. In the next section, some preliminaries and assumptions are listed. Section 3 is devoted to giving the regularity and a priori estimates of the spectral Galerkin method and the implicit full discretization. In Section 4, we derive the optimal mean square convergence rate of the implicit full discretization via the variation approach and the Sobolev interpolation equality.

## 2. Preliminaries

In this section, we give some preliminaries and notations. Given two separable Hilbert spaces $(\mathcal{H}, \| \cdot \|_H)$ and $(\mathcal{H}, \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}}))$, $\mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})$ and $\mathcal{L}_1(\mathcal{H}, \tilde{\mathcal{H}})$ are the Banach spaces of all linear bounded operators and the nuclear operators from $\mathcal{H}$ to $\tilde{\mathcal{H}}$, respectively. The trace of an operator $T \in \mathcal{L}_1(\mathcal{H})$ is $tr[T] = \sum_{k \in \mathbb{N}} \langle T f_k, f_k \rangle_\mathcal{H}$, where $\{f_k\}_{k \in \mathbb{N}}$ ($\mathbb{N} = \{0, 1, 2, \cdots \}$) is any orthonormal basis of $\mathcal{H}$. In particular, if $T \geq 0$, $tr[T] = \|T\|_{\mathcal{L}_1}$. Denote by $\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})$ the space of Hilbert–Schmidt operators from $\mathcal{H}$ into $\tilde{\mathcal{H}}$, equipped with the usual norm given by $\| \cdot \|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})} = (\sum_{k \in \mathbb{N}} \|f_k\|_{\tilde{\mathcal{H}}}^2)^{1/2}$. The following useful property and inequality hold

$$\|S T\|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})} \leq \|S\|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})} \|T\|_{\mathcal{L}(\mathcal{H})}, \quad T \in \mathcal{L}(\mathcal{H}), \quad S \in \mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}}),$$

$$(2.1)$$

$$tr[Q] = \|Q^\frac{1}{2}\|_{\mathcal{L}_2(\mathcal{H})}^2 = \|T\|_{\mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}})}^2, \quad Q = TT^*, \quad T \in \mathcal{L}_2(\mathcal{H}, \tilde{\mathcal{H}}),$$

where $T^*$ is the adjoint operator of $T$.

Given a Banach space $(\mathcal{E}, \| \cdot \|_\mathcal{E})$, we denote by $\gamma(\mathcal{H}, \mathcal{E})$ the space of $\gamma$-radonifying operators endowed with the norm $\|T\|_{\gamma(\mathcal{H}, \mathcal{E})} = (\mathbb{E} \| \sum_{k \in \mathbb{N}} \gamma_k T f_k \|_{\mathcal{E}}^2)^{1/2}$, where $(\gamma_k)_{k \in \mathbb{N}}$ is a Rademacher sequence on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. For convenience, let $L^q = L^q(\mathcal{O})$, $1 \leq q < \infty$ and $E = \mathcal{C}(\mathcal{O})$ equipped with the usual inner product and norm. We also need the following Burkholder inequality in $E$,

$$\left\| \sup_{t \in [0,T]} \left\| \int_0^t \phi(r) dW(r) \right\|_E \right\|_{L^p(\Omega)} \leq C_p \|\phi\|_{L^p(\Omega; L^2([0,T]; \gamma(\mathcal{H}, E)))}$$

$$\leq C_p \left( \mathbb{E} \left( \int_0^T \left\| \sum_{k \in \mathbb{N}} (\phi(t) e_k)^2 \right\|^2_E dt \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} .$$

$$(2.2)$$
where \( \{e_k\}_{k \in \mathbb{N}} \) is any orthonormal basis of \( \mathbb{H} \).

Next, we introduce some assumptions and spaces associated with \( A \). We denote by \( H^k := H^k(\mathcal{O}) \) the standard Sobolev space. Denote \( A = -\Delta \) the Dirichlet Laplacian operator with \( D(A) = \{v \in H^2(\mathcal{O}) : v = 0 \text{ on } \partial \mathcal{O} \} \).

It is known that \( A \) is a positive definite, self-adjoint and unbounded linear operator on \( H^k \). Thus there exists an orthonormal eigensystem \( \{(\lambda_j, e_j)\}_{j \in \mathbb{N}} \) such that \( 0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots \) with \( \lambda_j \sim j^{\frac{2}{d}} \). We define \( \mathbb{H}^{\alpha}, \alpha \in \mathbb{R} \) as the space of the series \( v := \sum_{j=1}^{\infty} v_j e_j, \ v_j \in \mathbb{R} \), such that \( \|v\|_{\mathbb{H}^{\alpha}} := \left(\sum_{j=1}^{\infty} \lambda_j^{\alpha} v_j^2 \right)^{\frac{1}{2}} < \infty \). Equipped with the norm \( \| \cdot \|_{\mathbb{H}^{\alpha}} \) and corresponding inner product, the Hilbert space \( \mathbb{H}^{\alpha} \) equals \( D(A^{\frac{\alpha}{2}}) \). Throughout this article, we denote by \( C \) a generic constant which may depend on several parameters but never on the projection parameter \( N \) and may change from occurrence to occurrence. We remark that the convergence analysis of the numerical approximation still holds for the general \( Q \)-Wiener process case.

In the following, we present the well-posedness result of the considered equation whose proofs can be found in [7].

**Theorem 2.1.** Let \( T > 0, \ X_0 \in \mathbb{H}^{\gamma}, \gamma \in (0, \frac{2}{d}), \ p \geq 1 \). Then Eq. (1.1) possesses a unique mild solution \( X \) in \( L^p(\Omega; C([0,T]; \mathbb{H})) \). Moreover, there exist \( C(X_0, T, p, \gamma) > 0 \) such that

\[
\|X\|_{L^p(\Omega; C([0,T]; \mathbb{H}))} \leq C(X_0, T, p, \gamma)
\]

and

\[
\|X(t) - X(s)\|_{L^p(\Omega; \mathbb{H})} \leq C(X_0, T, p, \gamma)|t - s|^\frac{\gamma}{2},
\]

where \( s, t \in [0, T] \).

Denote \( P^N \) the spectral Galerkin projection into the linear space spanned by the first \( N \) eigenvectors \( \{e_1, \cdots, e_N\} \). Then the spectral Galerkin approximation satisfies the following SPDE

\[
dX_N(t) + A(AX_N(t) + P^N F(X(t)))dt = P^N G(X_N(t))dW(t), \quad t \in (0, T],
\]

\[
X_N(0) = P^N X_0.
\]

To analyze the convergence error of the full discretization, we give a useful lemma on the properties of the spectral Galerkin approximation.
Lemma 2.1. Let $X_0 \in \mathbb{H}^\gamma$, $\gamma \in (0, \frac{3}{2})$, $\sup_{N \in \mathbb{N}^+} \|X_0^N\|_E \leq C(X_0)$ and $p \geq 1$. There exists a unique solution $X^N$ of Eq. (2.5) satisfying
\[ \|X^N\|_{L^p(\Omega; C([0,T];\mathbb{H}^\gamma))} \leq C(X_0, T, p, \gamma) \] (2.6)
and
\[ \|X^N(t) - X^N(s)\|_{L^p(\Omega; \mathbb{H}^\gamma)} \leq C(X_0, T, p, \gamma)(t-s)^{\frac{\gamma}{2}}, \]
where $C(X_0, T, p, \gamma) > 0$ and $s, t \in [0, T]$.
If in addition assume that $\sup_{N \in \mathbb{N}^+} \|X_0^N\|_E \leq C(X_0)$, then for $\beta < \gamma$, there exist $C(X_0, T, p, \beta) > 0$ such that
\[ \|X^N - X\|_{L^p(\Omega; C([0,T];\mathbb{H}^\beta))} \leq C(X_0, T, p, \beta)\lambda_N^{-\frac{\beta}{2}}. \] (2.7)

To end this section, we give some useful lemmas from [7].

Lemma 2.2. Let $g : L^4 \to H$ be the Nemytskii operator of a polynomial of second degree. Then it holds that for any $\beta \in (0, 1)$ that
\[ \|g(x)y\|_{H^{\beta-1}} \leq C\left(1 + \|x\|^2_E + \|x\|^2_{H^{\beta}}\right)\|y\|_{H^{-\beta}}, \]
where $x \in E, x \in H^\beta$ and $y \in H$.

3. A priori and regularity estimates of the full discretization

After giving the regularity estimates of $X$ and $X^N$, we propose the full discretization of Eq. (1.1) and give its a priori estimate and regularity estimate in this section.

Let $\delta t$ be the time stepsize such that $T = K\delta t$, $K \in \mathbb{N}^+$. The full discrete numerical scheme starting from $X_0^N := P^N X_0$, based on the temporal drift implicit Euler scheme and the spatial spectral Galerkin method, is defined as
\[ X_{k+1}^N = X_k^N - A^2 X_{k+1}^N \delta t - AP^N F(X_{k+1}^N) + G(X_k^N)\delta_k W, \ t \leq K - 1 \] (3.1)
where $\delta_k W = W(t_{k+1}) - W(t_k)$, $k \leq K - 1$. Then we also have the mild form of $X_k^N$,
\[ X_{k+1}^N = T_{\delta t} X_k^N - T_{\delta t} \delta t AP^N(F(X_{k+1}^N)) + T_{\delta t} G(X_k^N)\delta_k W, \ t \leq K - 1, \] (3.2)
where $T_{\delta t} = (I + A^2 \delta t)^{-1}$.

In order to study the moment estimate of the numerical scheme, we decompose $X_N^k$ into $Y_N^k$ and $Z_N^k$, which satisfies for $k \leq K - 1$ and $N \in \mathbb{N}^+$,

\begin{align*}
Y_{k+1}^N &= Y_k^N - A^2 Y_{k+1}^N \delta t - AF(Y_k^N + Z_k^N) \delta t, \\
Z_{k+1}^N &= Z_k^N - A^2 Z_{k+1}^N \delta t + G(Y_k^N + Z_k^N) \delta t W,
\end{align*}

with $Y_0^N = X_0^N$, $Z_0^N = 0$. We would like to mention that the similar decomposition has been used to derive the strong convergence rates of numerical approximations for non-global Lipschitz SPDEs driven by additive noise (see e.g. [2, 3, 8]). The following lemmas and propositions are devoted to deducing the a priori estimates of $Y_N^k$, $Z_N^k$ and $X_N^k$, $k \leq K$, $N \in \mathbb{N}^+$.

**Lemma 3.1.** For $2 \leq p \leq \infty$ and $m > 0$, it holds that

$$\|T_{\delta t}^m f\|_{L^p} \leq C(m \delta t)^{-\frac{1}{2}\left(1 - \frac{1}{p}\right)} \|f\|, \quad f \in \mathbb{H}, \ t > 0.$$ 

**Proof.** For $f \in \mathbb{H}$, we have $T_{\delta t}^k f = \sum_{i \in \mathbb{N}^+} \frac{1}{1 + \lambda_i^2 \delta t}^k \langle f, e_i \rangle e_i$. From the uniform boundedness of $e_i$, $i \in \mathbb{N}^+$ and the fact that $c\delta^2 \leq \lambda_i^h \leq C\delta^2$, it follows that

$$\|T_{\delta t}^m P^h f\|_E = \left\| \sum_{i=1}^{N^h} \frac{1}{1 + \lambda_i^2 \delta t}^m \langle f, e_i \rangle e_i \right\|_E \leq \left( \sum_{i=1}^{N^h} \frac{1}{1 + \lambda_i^2 \delta t}^m \right)^{\frac{1}{2}} \|f\| \leq C(m \delta t)^{-\frac{1}{2}} \|f\|,$$

and

$$\|T_{\delta t}^m P^h f\| \leq \left\| \sum_{i=1}^{N^h} \frac{1}{1 + \lambda_i^2 \delta t}^m \langle f, e_i \rangle e_i \right\| \leq C \|f\|.$$

The Riesz–Thorin interpolation theorem leads to the desired result.

**Lemma 3.2.** Let $X_0 \in \mathbb{H}$, $T > 0$ and $q \geq 1$. There exists a unique solution $\{X_N^k\}_{k \leq K}$ of Eq. (3.1) satisfying

$$\sup_{k \leq K} \mathbb{E} \left[ \|X_N^k\|_{\mathbb{H}^1}^p \right] \leq C(X_0, T, q)$$

for a positive constant $C(X_0, T, q)$.
Proof. Taking inner product with \( Y_{k+1}^N \) in \( H^{-1} \) on both sides of Eq. (3.3) leads to
\[
\|Y_{k+1}^N\|_{H^{-1}}^2 \leq \|Y_k^N\|_{H^{-1}}^2 - 2\|\nabla Y_{k+1}^N\|\|\nabla Y_{k+1}^N\| \delta t - 2\langle F(Y_{k+1}^N + Z_{k+1}^N), Y_{k+1}^N \rangle \delta t.
\]
From the monotonicity of \(-F\), the equivalence of norms in \( H^1 \) and \( H^1 \) and the Young inequality, it follows that for some small \( \epsilon > 0 \),
\[
\|Y_{k+1}^N\|_{H^{-1}}^2 + 8(c_4 - \epsilon)\|Y_k^N\|_{L^4}^4 \delta t + (2 - \epsilon)\|Y_k^N\|_{H^1}^2 \delta t \leq \|Y_k^N\|_{H^{-1}}^2 + C(\epsilon) \left( 1 + \|Z_{k+1}^N\|_{L^4}^4 + \|A^{-\frac{1}{2}} Z_{k+1}^N\|^2 \right) \delta t
\]
\[
\leq \|Y_k^N\|_{H^{-1}}^2 + C(\epsilon) \left( 1 + \|Z_{k+1}^N\|_{L^4}^4 \right) \delta t.
\]
From the mild form of \( Z_k^N \), the Hölder inequality and the Burkholder inequality, it follows that for \( p \geq 2 \) and \( q \geq 4 \),
\[
\mathbb{E}[\|Z_{k+1}^N\|_{L^p}^q] = \mathbb{E}\left[ \left\| \sum_{i=0}^k T_{\delta t}^{k+1-i} G(Y_i^N + Z_i^N) \delta W_i \right\|_{L^p}^q \right]
\leq \mathbb{E}\left[ \left( \sum_{i=0}^k \sum_{j \in \mathbb{N}^+} \|T_{\delta t}^{k+1-i} G(Y_i^N + Z_i^N) c_j\|_{L^p}^2 \delta t \right)^{\frac{q}{2}} \right]
\leq \mathbb{E}\left[ \left( \sum_{i=0}^k (\delta t)^{-\frac{1}{2} + \frac{1}{p} + \frac{1}{q}} (1 + \|Y_i^N\|^{2\alpha} + \|Z_i^N\|^{2\alpha} \delta t)^{\frac{q}{2}} \right) \right]
\leq C \left( (k + 1 - i) \delta t \right)^{-\frac{1}{2} + \frac{1}{p} + \frac{1}{q}} \mathbb{E}\left[ \left( \sum_{i=0}^k (1 + \|Y_i^N\|^{4\alpha} + \|Z_i^N\|^{4\alpha} \delta t)^{\frac{q}{2}} \right) \right].
\]
Using the Young inequality, we obtain for \( 0 \leq s \leq t \),
\[
\mathbb{E}[\|Z_{k+1}^N\|_{L^p}^q] \leq CT^\frac{q}{2} \left( 1 + \mathbb{E}\left[ \left( \sum_{i=0}^k \|Y_i^N\|^{4\alpha} \delta t \right)^{\frac{q}{2}} \right] \right) \leq CT^\frac{q}{2} \left( 1 + \mathbb{E}\left[ \left( \sum_{i=0}^k \|Y_i^N\|^{4\alpha} \delta t \right)^{\frac{q}{2}} \right] \right).
\]
Gronwall’s inequality yields that for \( 0 \leq s \leq t \leq T \),
\[
\mathbb{E}[\|Z_{k+1}^N\|_{L^p}^q] \leq C(T) \left( 1 + \mathbb{E}\left[ \sum_{i=0}^k \|Y_i^N\|^{4\alpha} \delta t \right]^{\frac{q}{2}} \right).
\] (3.7)
Now taking mth moment, \( m \in \mathbb{N}^+ \) on (3.6) and letting \( p = 4, q = 4m \), we have

\[
\mathbb{E}\left[ \left( \sum_{i=0}^{k} \| Y_N^k \|_{L^4}^4 \delta t \right)^m \right] 
\leq C\| Y^N(0) \|_{H^m-1}^{2m} + C(\epsilon, T) \sum_{i=0}^{k} \left( 1 + \mathbb{E}[\| Z_N^i \|_{L^4}^4] \right) \delta t 
\leq C\| Y^N(0) \|_{H^m-1}^{2m} + C(\epsilon, T) \sum_{i=0}^{k} \left( C(\epsilon_1) + \epsilon_1 \mathbb{E}\left[ \left( \sum_{j=0}^{i} \| Y_N^k \|_{L^p}^4 \delta t \right)^m \right] \right) \delta t,
\]

where \( \epsilon_1 > 0 \) is a small number such that \( C(\epsilon, T) \epsilon_1 T < \frac{1}{2} \). The above estimation leads to

\[
\mathbb{E}\left[ \sum_{i=0}^{k} \| Y_N^k \|_{L^4}^4 \delta s \right]^m \leq C\| Y^N(0) \|_{H^m-1}^{2m} + C(m, \epsilon, \epsilon_1, T),
\]

which in turns yields that for \( m \in \mathbb{N}^+ \),

\[
\mathbb{E}\left[ \| Y_{k+1}^N \|_{H^m-1}^{2m} \right] + \mathbb{E}\left[ \left( \sum_{i=0}^{k} \| \nabla Y_N^k \|_{L^4}^4 \delta t \right)^m \right] + \mathbb{E}\left[ \left( \sum_{i=0}^{k} \| Y_N^k \|_{L^4}^4 \delta t \right)^m \right] \leq C(X_0, T, m),
\]

and for \( p \geq 2, q \geq 4, \)

\[
\mathbb{E}\left[ \| Z_{k+1}^N \|_{L^p}^q \right] \leq C(X_0, T, m). \tag{3.8}
\]

Combining the above estimates, we completes the proof.

Lemma 3.3. Let \( X_0 \in \mathbb{H}, T > 0 \) and \( q \geq 1 \). There exists a positive constant \( C(T, q, X_0) \) such that

\[
\sup_{k \leq K} \mathbb{E}\left[ \| Z_k^N \|_E^q \right] \leq C(T, q, X_0).
\]

Proof. By the Burkholder inequality and the smoothy effect of \( T_{\delta t} \), we have for \( q \geq 2, \)

\[
\mathbb{E}\left[ \| Z_{k+1}^N \|_E^q \right] = \mathbb{E}\left[ \| \sum_{i=0}^{k} T_{\delta t}^{k+1-i} G(Y_N^k + Z_N^k) \delta W_i \|_E^q \right]
\]
\[ \leq \mathbb{E} \left[ \left( \sum_{i=0}^{k} \sum_{j \in \mathbb{N}^+} \left\| T_{\delta t}^{k+1-i} G(Y_i^N + Z_i^N) e_j \right\|_E^2 \delta t \right)^{\frac{q}{2}} \right] \]

\[ \leq \mathbb{E} \left[ \left( \sum_{i=0}^{k} \left( (k + 1 - i) \delta t \right)^{-\frac{1}{2}} (1 + \left\| Y_i^N \right\|^{2\alpha} + \left\| Z_i^N \right\|^{2\alpha} \delta t) \right)^{\frac{q}{2}} \right]. \]

The Hölder inequality yields that for any \( l > 2, \)
\[ \mathbb{E}[\| Z_{k+1}^N \|_E^q] \]
\[ \leq C \mathbb{E} \left[ \left( \sum_{i=0}^{k} ((k + 1 - i) \delta t)^{-\frac{l}{2(\alpha l - 1)}} (\sum_{i=0}^{k} (1 + \left\| Y_i^N \right\|^{2\alpha l} + \left\| Z_i^N \right\|^{2\alpha l} \delta t)^{\frac{l}{2}} \right) \right]. \]

Since \( \alpha < 1, \) it always exists \( l > 2 \) such that \( 2\alpha l < 4. \) Then taking such \( l \) and combining the estimates (3.8)-(3.2), we obtain
\[ \mathbb{E}[\| Z_{k+1}^N \|_E^q] \leq C(\alpha, T) \mathbb{E} \left[ \left( \sum_{i=0}^{k} (1 + \left\| Y_i^N \right\|^{4} + \left\| Z_i^N \right\|^{2\alpha l} \delta t) \right)^{\frac{q}{2}} \right] \leq C(X_0, T, q, \alpha). \]

Lemma 3.4. Let \( X_0 \in \mathbb{H}, T > 0 \) and \( q \geq 1. \) There exists a positive constant \( C(X_0, T, q) \) such that
\[ \mathbb{E} \left[ \sup_{k \leq K} \| Z_k^N \|_E^q \right] \leq C(X_0, T, q). \]

Proof. Define the continuous extension of \( Z_k^N \) as
\[ \hat{Z}^N(t) = \int_0^T T_{\delta t}^{-\frac{1}{2}([t])} P^N G(X_{\delta t}^N) dW(s), \hat{Z}^N(0) = 0, \]
where \( \hat{Z}^N(t_k) = Z^N_k \), \( t_k = k\delta t \). Thus it suffices to estimate \( \mathbb{E} \left[ \sup_{t \in [0,T]} \| \hat{Z}^N(t) \|_E^q \right] \).

Notice that for \( \alpha_1 > \frac{1}{p} + \gamma \), \( p > 1 \), \( \gamma = \frac{1}{8} \),

\[
\hat{Z}^N(t) = \int_0^t (t - s)^{\alpha_1 - 1} T_{\delta t}^{-(\frac{1}{\alpha_1} - \frac{\gamma}{2})} \hat{Y}_{\alpha_1,N}(s) ds,
\]

where \( \hat{Y}_{\alpha_1,N}(s) = \int_0^s (s - r)^{-\alpha_1} T_{\delta t}^{-(\frac{1}{\alpha_1} - \frac{\gamma}{2})} P^N(G(Y^N_r) + \hat{Z}^N([r]_{\delta t})) dW(r) \). By the H"older inequality, we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| \hat{Z}^N(t) \|_E^q \right] \leq C \mathbb{E} \left[ \| \hat{Y}_{\alpha_1,N} \|_{L^q([0,T];\mathbb{H})}^q \right].
\]

From the H"older and Burkholder inequalities, it follows that for \( q \geq \max(p,2) \),

\[
\mathbb{E} \left[ \| \hat{Y}_{\alpha_1,N} \|_{L^q([0,T];\mathbb{H})}^q \right] \leq C(T,q) \int_0^T \mathbb{E} \left[ \left( \int_0^s (s - r)^{-\alpha_1} T_{\delta t}^{-(\frac{1}{\alpha_1} - \frac{\gamma}{2})} P^N G(Y^N_r) + \hat{Z}^N([r]_{\delta t})) dW(r) \right)^q ds \right] ds
\]

\[
\leq C(T,q) \int_0^T \mathbb{E} \left[ \left( \int_0^s (s - r)^{-2\alpha_1} \sum_{i \in \mathbb{N}^+} \| T_{\delta t}^{-(\frac{1}{\alpha_1} - \frac{\gamma}{2})} P^N G(Y^N_r) + \hat{Z}^N([r]_{\delta t}))e_i \|_d^2 dr \right)^q \right] ds.
\]

To give a upper bound of the above term, the following smoothe effect of \( T_{\delta t} \) is required, that is, for any \( \kappa > 0 \) and \( \beta > 0 \),

\[
\| A^\beta T_{\delta t}^\kappa x \| \leq C(\kappa \delta t)^{-\frac{\beta}{2}} \| x \|, \quad x \in \mathbb{H}.
\]

(3.11)

Indeed, we show this property in the case that \( \lambda_i^2 \delta t < 1 \) and \( \lambda_i^2 \delta t \geq 1 \), respectively. Due to the fact \( |\frac{1}{1+z}| \leq e^{-Cz} \) for \( z \in [0,1] \), we have that \( \| T_{\delta t}^\kappa e_i \| \leq e^{-C\kappa \delta t} \) when \( \lambda_i^2 \delta t < 1 \). Hence,

\[
\sup_{\lambda_i^2 \delta t < 1} \| A^\beta T_{\delta t}^\kappa e_i \| \leq \sup_{\lambda_i^2 \delta t < 1} \lambda_i^\beta e^{-C\lambda_i^2 \kappa \delta t} \leq C(\kappa \delta t)^{-\frac{\beta}{2}}.
\]

In the case that \( \lambda_i^2 \delta t \geq 1 \), it holds that for \( 0 < \kappa < 1 \),

\[
\sup_{\lambda_i^2 \delta t \geq 1} \| A^\beta T_{\delta t}^\kappa e_i \| \leq C \sup_{\lambda_i^2 \delta t \geq 1} \lambda_i^\beta \left( \frac{1}{1 + \lambda_i^2 \delta t} \right)^\kappa \leq C \delta t^{-\frac{\beta}{2}}
\]
and for $\kappa \geq 1$,
\[
\sup_{\lambda_i^2 \delta t \geq 1} ||A_\delta T_{\delta t}^\kappa e_i|| \leq C \sup_{\lambda_i^2 \delta t \geq 1} \lambda_i^\beta \frac{1}{1 + \lambda_i^\kappa \delta t} \leq C(\kappa \delta t)^{-\frac{\beta}{2}}.
\]

Therefore, the estimate (3.11) holds. By the Parseval equality and the smoothy effect of $T_{\delta t}$, we get for any sufficient small $\epsilon > 0$,
\[
\mathbb{E}\left[\|\hat{Y}_{\alpha,N}\|_L^g_{(0,T)}\right] \leq C(T, q) \int_0^T \mathbb{E}\left[\left(\int_0^t (s-r)^{-2\alpha_1} \sum_{i \in \mathbb{N}^+} \sum_{j \in \mathbb{N}^+} \langle P^N G(Y_{[r]}^N) + \tilde{Z}^N([r]_{\delta t}), T_{\delta t}^{-\frac{\lambda_i^\kappa}{1 + \lambda_i^\kappa \delta t}} e_j \rangle^2 dr \right)^{\frac{q}{2}}\right] ds
\]
\[
\leq C(T, q) \int_0^T \mathbb{E}\left[\left(\int_0^t (s-r)^{-2\alpha_1} \sum_{j \in \mathbb{N}^+} ||T_{\delta t}^{-\frac{\lambda_i^\kappa}{1 + \lambda_i^\kappa \delta t}} e_j||^2 ||G(Y_{[r]}^N) + \tilde{Z}^N([r]_{\delta t})||^2 dr \right)^{\frac{q}{2}}\right] ds
\]
\[
\leq C(T, q) \int_0^T \mathbb{E}\left[\left(\int_0^t (s-r)^{-2\alpha_1} (s-[r]_{\delta t})^{-\frac{1}{2} - \epsilon} (1 + ||Y_{[r]}^N||^{2\alpha} + ||\tilde{Z}^N([r]_{\delta t})||^{2\alpha}) dr \right)^{\frac{q}{2}}\right] ds.
\]

Since $\alpha < 1$, one can choose a positive number $l > 2$ and a large enough number $p$ such that $2\alpha l < 4$ and $(2\alpha_1 + \frac{1}{2} + \epsilon)(l-1) < 1$. Then by using a priori estimates (3.7) and (3.8), we obtain
\[
\mathbb{E}\left[\|\hat{Y}_{\alpha,N}\|_{L^p(0,t)}^g\right] \leq C(T, q, \alpha) \int_0^t \left(\int_0^s (s-r)^{-2\alpha_1 (2\alpha_1 + \frac{1}{2} + \epsilon) \frac{l-1}{2l}} dr \right)^{\frac{q(l-1)}{2l}} ds
\]
\[
\times \mathbb{E}\left[\left(\int_0^s (1 + ||Y_{[r]}^N||^{2\alpha} + ||\tilde{Z}^N([r]_{\delta t})||^{2\alpha}) dr \right)^{\frac{q}{2}}\right] ds
\]
\[
\leq C(T, q, \alpha, Y_{0}^N),
\]
which completes the proof.

\begin{proof}
By Lemma 3.4, it suffices to bound $Y_{k}^N$ in $\mathbb{H}$. Taking inner product on both sides of Eq. (3.3) with $Y_{k+1}^N$ in $\mathbb{H}$, we obtain
\[
\|Y_{k+1}^N\|^2 \leq ||Y_{k}^N||^2 - (2 - \epsilon) \delta t ||AY_{k+1}^N||^2
\]

**Proposition 3.1.** Let $X_0 \in \mathbb{H}$, $T > 0$, $q \geq 1$. There exists a positive constant $C(X_0, T, q)$ such that
\[
\mathbb{E}\left[\sup_{k \leq K} ||X_k^N||^q\right] \leq C(X_0 T, q).
\]

**Proof.** By Lemma 3.4, it suffices to bound $Y_{k}^N$ in $\mathbb{H}$. Taking inner product on both sides of Eq. (3.3) with $Y_{k+1}^N$ in $\mathbb{H}$, we obtain
\[
\|Y_{k+1}^N\|^2 \leq ||Y_{k}^N||^2 - (2 - \epsilon) \delta t ||AY_{k+1}^N||^2
\]
\[ -8(c_4 - \epsilon)\|Y_{k+1}^N + Z_{k+1}^N\|^2 + C(\epsilon)(1 + \|\nabla Y_{k+1}^N\|^2 + \|Z_{k+1}^N\|^2 + \|Z_{k+1}^N\|_4^4 + \|Z_{k+1}^N\|_8^8)\delta t. \]

Thus it is concluded from (3.8)-(3.9) and Lemma 3.4 that for \( p \geq 1 \),

\[
\mathbb{E}\left[ \sup_{k \leq K} \|Y_{k+1}^N\|^{2p} \right] + \mathbb{E}\left[ \left( \sum_{k=0}^{K-1} \|AY_{k+1}^N\|^2 \delta t \right)^p \right] \\
\leq C(\epsilon, T) \mathbb{E}\left[ \left( \sum_{k=0}^{K-1} \|\nabla Y_{k+1}^N\|^2 \delta t \right)^p \right] \\
+ C(\epsilon, T) \mathbb{E}\left[ \left( \sum_{k=0}^{K-1} \|Y_{k+1}^N\|_4^4 \delta t \right)^p \sup_{k \leq K} \|Z_k^N\|_E^2 \right] \\
+ C(\epsilon, T) \sum_{k=0}^{K-1} \mathbb{E}\left[ 1 + \|Z_{k+1}^N\|_8^8 \delta t \right] \leq C(X_0, p, T),
\]

which, together with the Hölder inequality, completes the proof. \( \square \)

In the following, we present the discrete continuity of the discrete convolution \( \{Z_k^N\}_{k \leq K} \) and the numerical approximation \( \{X_k^N\}_{k \leq K} \).

**Proposition 3.2.** Let \( X_0 \in \mathbb{H} \), \( T > 0 \), \( q \geq 1 \). Then for \( \gamma \in (0, \frac{3}{2}) \), there exists \( C(X_0, T, p, \gamma) > 0 \) such that

\[
\mathbb{E}\left[ \sup_{k \leq K} \|Z_k^N\|_{\mathbb{H}^\gamma}^q \right] \leq C(X_0, T, p, \gamma)
\]

and

\[
\mathbb{E}\left[ \|Z_k^N - Z_{k_1}^N\|^q \right] \leq C(X_0, T, p, \gamma)(k - k_1)\delta t^{\frac{2q}{4}}.
\]

where \( k, k_1 \leq K \).

**Proof.** Similar arguments in Lemma 3.4 yield that for \( \alpha_1 > \frac{1}{p} + \frac{2}{4}, p > 1 \),

\[
\mathbb{E}\left[ \sup_{k \leq K} \|Z_k^N\|_{\mathbb{H}^\gamma}^q \right] \leq \mathbb{E}\left[ \sup_{t \in [0,T]} \|\tilde{Z}^N(t)\|_{\mathbb{H}^\gamma}^q \right] \leq C\mathbb{E}\left[ \|\tilde{Y}_{\alpha_1,N}\|_{L^p([0,T];\mathbb{H})}^p \right].
\]

Further, the Hölder and Burkholder inequalities imply that for any sufficient small \( \epsilon > 0 \) and \( q \geq \max(p, 2) \),

\[
\mathbb{E}\left[ \|\tilde{Y}_{\alpha_1,N}\|_{L^p(0,T;\mathbb{H})}^q \right]
\]
\[
\leq C(T, q) \int_0^T \mathbb{E} \left[ \left( \int_0^s (s - r)^{-2\alpha_1} \sum_{j \in \mathbb{N}^+} \|T_{[\delta t]}^{-\left(\frac{q}{2}\right)} e_j\|^2 (G(Y_N^r) + \hat{Z}_N([r]_{\delta t}))^2 dr \right)^{\frac{q}{2}} \right] ds
\]
\[
\leq C(T, q) \int_0^T \mathbb{E} \left[ \left( \int_0^s (s - r)^{-2\alpha_1} (s - [r]_{\delta t})^{-\frac{1}{4} - \epsilon} (1 + \|Y_N^r\|^2 + \|\hat{Z}_N([r]_{\delta t})\|^{2\alpha}) dr \right)^{\frac{q}{2}} \right] ds.
\]

Then choosing a large enough \( p \) such that \( 2\alpha_1 + \frac{1}{4} + \epsilon < 1 \), and using Proposition 3.1 and Lemma 3.3, we complete the proof for the regularity estimate in space.

The mild form of \( \hat{Z}_N(t) \) and the Burkholder inequality yield that for \( 0 \leq s \leq t \leq T, q \geq 1 \) and sufficient small \( \epsilon_1 \),

\[
\mathbb{E} \left[ \|\hat{Z}_N(t) - \hat{Z}_N(s)\|^{q} \right]
\leq \mathbb{E} \left[ \left\| \int_0^s \left( T_{[\delta t]}^{-\left(\frac{q}{2}\right)} - T_{[\delta t]}^{-\left(\frac{q}{2}\right)} \right) P_N(G(Y_N^r) + \hat{Z}_N([r]_{\delta t})) dW(r) \right\|^{q} \right]
\leq C \mathbb{E} \left[ \left( \int_0^s \| T_{[\delta t]}^{-\left(\frac{q}{2}\right)} - T_{[\delta t]}^{-\left(\frac{q}{2}\right)} \| P_N(G(Y_N^r) + \hat{Z}_N([r]_{\delta t})) e_i \| ^2 dr \right)^{\frac{q}{2}} \right]
\leq C \mathbb{E} \left[ \left( \int_0^s \sum_{i \in \mathbb{N}^+} \| T_{[\delta t]}^{-\left(\frac{q}{2}\right)} - T_{[\delta t]}^{-\left(\frac{q}{2}\right)} \| P_N(G(Y_N^r) + \hat{Z}_N([r]_{\delta t})) e_i \| ^2 dr \right)^{\frac{q}{2}} \right]
\]

Then combining the above estimate, Proposition 3.1 and Lemma 3.3, we complete the proof. \( \square \)

Next, we focus on the regularity estimates of the numerical solution \( \{X_N^k\}_{k \leq K} \).

**Proposition 3.3.** Let \( X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2}) \), \( T > 0 \) and \( q \geq 1 \). Then there exists \( C(X_0, T, q, \gamma) > 0 \) such that

\[
\mathbb{E} \left[ \sup_{k \leq K} \|X_N^k\|_{\mathbb{H}^\gamma}^q \right] \leq C(X_0, T, q, \gamma),
\]

and

\[
\mathbb{E} \left[ \|X_N^k - X_N^\delta\|_{\mathbb{H}^\gamma}^q \right] \leq C(X_0, T, q, \gamma)((k - k_1)\delta t)^{\frac{\gamma}{2}}.
\]
where $k_1, k \leq K$.

**Proof.** Due to the above regularity estimate in Lemma 3.2, it only suffices to give the regularity estimate of $\{Y_k^N\}_{k \leq K}$. To this end, we first give the upper bound of $\|Y_k^N\|_{L^6}$. For convenience, we only give the proof for the case that $\gamma \in [1, \frac{3}{2})$. From the Sobolev embedding theorem, the smoothy effect of $e^{-At}$, the Gagliardo–Nirenberg and Young inequalities, it follows that

$$
\|Y_{k+1}^N\|_{L^6}^6 \leq \|T_{\delta t}^{k+1}Y_0^N\|_{L^6}^6 + \left\| \sum_{j=0}^k T_{\delta t}^{k+1-j} AF(Y_{j+1}^N + Z_{j+1}^N) \right\|_{L^6}^6 \delta t 
$$

$$
\leq C\|Y_0^N\|_{H^1} + C\sum_{j=0}^k (k + 1 - j)^{-\frac{3}{2}} \delta t^{-\frac{3}{2}} \left( 1 + \|Z_{j+1}^N\|_{L^6}^3 + \|Y_{j+1}^N\| \right) \delta t 
$$

$$
+ C\sum_{j=0}^k (k + 1 - j)^{-\frac{3}{2}} \delta t^{-\frac{3}{2}} \sup_{j \leq K} \|Y_j^N\|_{H^1}^6 + C\sum_{j=0}^k \|(-A)Y_{j+1}^N\|_{L^6}^2 \delta t. 
$$

The a priori estimates of $Y_k^N$ and $Z_k^N$ yield that for $q \geq 1$,

$$
\mathbb{E}\left[ \sup_{k \leq K} \|Y_k^N\|_{L^6}^q \right] \leq C(X_0, T, q). 
$$

Now, we are in the position to give the desired regularity estimate. From the mild form of $Y_k^N$ and the above a priori estimate in $L^6$, it follows that

$$
\mathbb{E}\left[ \sup_{k \leq K} \|Y_k^N\|_{H^{3/2}}^q \right] 
$$

$$
\leq C\mathbb{E}\left[ \|Y_0^N\|_{H^{3/2}}^q \right] + C\delta t \mathbb{E}\left[ \left( \sum_{j=0}^{K-1} \|T_{\delta t}^{K-j} AF(Y_{j+1}^N + Z_{j+1}^N)\|_{H^{3/2}}^q \right) \delta t \right] 
$$

$$
\leq C(q)\|X^N(0)\|_{H^{3/2}}^q + C(q)\sum_{j=0}^{K-1} (T - t_j)^{-\frac{3}{2}} \delta t \mathbb{E}\left[ \sup_{j \leq K} \left( 1 + \|Y_j^N\|_{L^6}^3 + \|Z_j^N\|_{L^6}^3 \right)^q \right] 
$$

$$
\leq C(q)\|X^N(0)\|_{H^{3/2}}^q + C(T, p)\mathbb{E}\left[ \sup_{j \leq K} \left( 1 + \|Y_j^N\|_{L^6}^3 + \|Z_j^N\|_{L^6}^3 \right)^q \right] 
$$

$$
\leq C(X_0, T, q). 
$$

For convenience, we assume that $k > k_1$. The mild form of $Y_k^N$ and $Y_{k_1}^N$, together with the smoothy effect of $T_{\delta t}$, yields that

$$
\mathbb{E}\left[ \|Y_k^N - Y_{k_1}^N\|_q^q \right] \leq \mathbb{E}\left[ \|T_{\delta t}^{k-k_1} (T_{\delta t}^{k_1} - I) Y_0^N\|_q^q \right] 
$$
\[ + \mathbb{E} \left[ \left( \sum_{j=0}^{k_1-1} \left\| (T_{\Delta t}^{k_1-j} (T_{\Delta t}^{k-k_1} - I)AF(Y_{j+1}^N + Z_{j+1}^N) \right\| \delta t \right)^q \right] \]

\[ + \mathbb{E} \left[ \left( \sum_{j=k_1}^{k} \left\| T_{\Delta t}^{k-j} AF(Y_{j+1}^N + Z_{j+1}^N) \right\| \delta t \right)^q \right] \]

\[ \leq C(X_0, T, q, \gamma) |(k - k_1) \delta t|^{4/3}. \]

Combining the above regularity estimates together, we finish the proof for \( \gamma \in [1, \frac{3}{2}). \) Similar arguments, together with

\[ \| e^{-A^2 t} X_0^N \|_{L^6} \leq Ct^{-\frac{1}{12}} \| X_0 \|, \]

lead to the desired result for the case that \( \gamma \in (0, 1). \)

**Remark 3.1.** Similarly, one can obtain the a priori estimate in \( E. \) More precisely, it holds that for \( q \geq 1, \)

\[ \mathbb{E} \left[ \| X^N(t) \|_E^q \right] \leq C(X_0, T, q, 1 + t^{-\frac{q}{2}}). \]

4. Mean square convergence rate of full discretization

Based on the a priori and regularity estimates of the numerical approximation, we focus on the strong convergence and strong convergence rate of the proposed scheme for Eq. (1.1).

For convenience, we only present the convergence analysis in mean square sense. Notice that

\[ \| X_k^N - X(t_k) \| \leq \| X_k^N - X^N(t_k) \| + \| X^N(t_k) - X(t_k) \|. \]

By Lemma 2.1 and [7, Remark 3.2], we have that if \( X_0 \in \mathbb{H}^\gamma, \gamma \in (0, \frac{3}{2}), \) then for \( \alpha \in (0, \gamma) \) it holds that

\[ \| X^N - X \|_{L^2(\Omega; C([0,T];\mathbb{H}))} \leq C(T, X_0, \gamma) \lambda_N^{-\frac{\alpha}{2}}. \] (4.1)

for a positive constant \( C(T, X_0, \gamma). \) Thus we need to estimate the error of \( \| X_k^N - X^N(t_k) \|. \) To this end, we introduce an auxiliary process in the following part.

4.1. Strong error estimate between the auxiliary process and the spectral Galerkin approximation

To deduce the mean square convergence rate in time, we introduce an auxiliary process \( \tilde{X}_k^N, k \leq K \) with \( \tilde{X}_0^N = X^N(0), \) defined by

\[ \tilde{X}_{k+1}^N = \tilde{X}_k^N - A^2 \delta t \tilde{X}_{k+1}^N - P^N AF(X^N(t_{k+1})) \delta t + P^N G(X^N(t_{k+1})) \delta W_k. \]
Then we split the error of $X^N_k - X^N(t_k)$ as
\[ \|X^N_k - X^N(t_k)\| \leq \|X^N(t_k) - \tilde{X}_k^N\| + \|\tilde{X}_k^N - X^N_k\|. \]

The first error is bounded by the following lemma. The second error will be dealt with the interpolation arguments. We would like to mention that compared with the existing literatures (see e.g. \cite{9, 10, 11, 14, 16}), this is also a new approach to deducing strong convergence rates of temporal and full discretizations for SPDEs with non-monotone coefficients.

**Lemma 4.1.** Let $X_0 \in \mathbb{H}^b$, $\gamma \in (0, \frac{3}{2})$, $N \in \mathbb{N}^+$ and $p \geq 1$. There exists a positive constant $C(X_0, T, p, \gamma)$ such that
\[ \|X^N(t_k) - \tilde{X}_k^N\|_{L^p(\Omega; \mathbb{H})} \leq C(X_0, T, p, \gamma)\delta t^{\frac{\gamma}{2}}. \] (4.2)

**Proof.** Denote $[s]_{\delta t} := \max\{0, \delta t, \ldots, k\delta t, \ldots\} \cap [0, s]$ and $[s] = \frac{[s]_{\delta t}}{\delta t}$. The mild forms of $X^N(t_k)$ and $\tilde{X}_k^N$ yield that
\[ \|X^N(t_k) - \tilde{X}_k^N\|_{L^p(\Omega; \mathbb{H})} \leq \left\| \int_0^{t_k} (e^{-A^2(s-t)} - T^{k-[s]}_{\delta t}) A P^N F(X^N(s)) ds \right\|_{L^p(\Omega; \mathbb{H})} \]
\[ + \left\| \int_0^{t_k} T^{k-[s]}_{\delta t} A P^N (F(X^N(s)) - F(X^N([s]_{\delta t} + \delta t))) ds \right\|_{L^p(\Omega; \mathbb{H})} \]
\[ + \left\| \int_0^{t_k} (e^{-A^2(s-t)} - T^{k-[s]}_{\delta t}) P^N G(X^N(s)) dW(s) \right\|_{L^p(\Omega; \mathbb{H})} \]
\[ + \left\| \int_0^{t_k} T^{k-[s]}_{\delta t} P^N (G(X^N(s)) - G(X^N([s]_{\delta t}))) dW(s) \right\|_{L^p(\Omega; \mathbb{H})} \]
\[ =: I_1 + I_2 + I_3 + I_4. \] (4.3)

By the properties of $e^{-A^2t}$ and $T^k_{\delta t}$, the priori estimates of $Y^N$ and $Z^N$, and $\|e^{-A^2t}X_0^N\|_{L^6} \leq C t^{-\frac{\gamma}{12}}\|X_0\|$, the first term is estimated as for small $\epsilon > 0$,
\[ \gamma < \frac{3}{2}, \]
\[ I_1 \leq C \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - [s]_{\delta t})^{-1+\epsilon} \delta t^{\frac{2}{3}+\frac{\gamma}{2}-\epsilon} \left(1 + \left\|X^N(s)\right\|_{L^p(\Omega; \mathbb{H}^b)}^3\right) ds \leq C(T, X_0, p)\delta t^{\frac{\gamma}{2}-\epsilon}. \]

The similar arguments in the proof of \cite[Proposition 3.2]{7} yield that
\[ \mathbb{E} \left[ \|X^N(t)\|_{\mathbb{H}^b}^2 \right] \leq C(X_0, T, q) (1 + t^{-\frac{\gamma}{2}}), \]
which implies that

\[
I_2 \leq \left\| \int_0^{t_k} T_{\delta t}^{k-s} A P^N \left( F'(X^N(s)) (X^N(s) - X^N([s]_{\delta t} + \delta t)) \right) ds \right\|_{L^p(\Omega; H)}
\]

\[
\leq \left\| \int_0^{t_k} (t_k - [s]_{\delta t})^{-\frac{1}{2}} \left\| F'(X^N(s)) \right\|_E \left\| X^N(s) - X^N([s]_{\delta t} + \delta t) \right\|_{L^p(\Omega)} ds \right\|
\]

\[
\leq C(T, X_0) \delta t^\gamma.
\]

The Burkholder inequality and Sobolev embedding theorem lead that for small \( \epsilon > 0 \),

\[
I_3 \leq C(T, p) \left( \left\| \int_0^{t_k} \left( e^{-A^2(t_k-s)} - T_{\delta t}^{k-s} \right) P^N G(X^N(s)) \right\|_{L^2_{t,s}} \right)^2
\]

\[
\leq C(T, p) \left( \left\| \int_0^{t_k} A^{1+\epsilon + \gamma} (e^{-A^2(t_k-s)} - T_{\delta t}^{k-s}) A^{-\frac{1}{2}} \right\|_{L^2_{t,s}} \right)^2
\]

\[
\leq C(T, p) \left( \left\| \int_0^{t_k} A^{1+\epsilon + \gamma} (e^{-A^2(t_k-s)} - T_{\delta t}^{k-s}) A^{-\frac{1}{2}} \right\|_{L^2_{t,s}} \right)^2
\]

\[
\leq C(T, X_0, T, p) \delta t^\gamma.
\]

Similar arguments yield that

\[
I_4 \leq C(T, p) \left( \left\| \int_0^{t_k} \left( T_{\delta t}^{k-s} P^N (G(X^N(s)) - G(X^N([s]_{\delta t}))) \right) \right\|_{L^2_{t,s}} \right)^2
\]

\[
\leq C(T, p) \left( \left\| \int_0^{t_k} A^{1+\epsilon} T_{\delta t}^{k-s} \right\|_{L^2_{t,s}} \right)^2 \left\| A^{-\frac{1}{2}} (G(X^N(s)) - G(X^N([s]_{\delta t}))) \right\|_{L^2_{t,s}} \right)^2
\]

\[
\leq C(T, p) \left( \left\| \int_0^{t_k} (t_k - [s]_{\delta t})^{-\frac{1}{2}} \left\| X^N(s) - X^N([s]_{\delta t}) \right\|_{L^2_{t,s}} \right\|_{L^2_{t,s}} \right)^2
\]

\[
\leq C(T, X_0, T, p) \delta t^\gamma.
\]

Combining (4.3) and the above regularity estimates, we complete the proof.

Next, we show the optimal regularity of \( \tilde{X}_k^N \) and give the convergence analysis of \( \tilde{X}_k^N - X_k^N \) in \( H^{-1} \).

**Lemma 4.2.** Let \( X_0 \in H^\gamma \), \( \gamma \in (0, \frac{3}{2}) \), \( T > 0 \) and \( p \geq 1 \). Then \( \tilde{X}_k^N \) satisfies

\[
\mathbb{E} \left[ \sup_{k \leq K} \left\| \tilde{X}_k^N \right\|_{H^\gamma}^p \right] \leq C(X_0, T, p).
\]
Proof. Since the auxiliary process involves the informations of $X^N$, the proof is similar to that of [7, Proposition 3.1].

**Lemma 4.3.** Under the condition of Lemma 4.1, there exist $\delta t_0 \leq 1$ and $C(X_0, T, \gamma) > 0$ such that for any $\delta t \leq \delta t_0$, $N \in \mathbb{N}^+$, we have

$$
\| \tilde{X}_k^N - X_k^N \|_{L^2(\Omega; \mathbb{H}^{-1})} \leq C(X_0, T)\delta t \tilde{\tau}.
$$

(4.4)

**Proof.** From the definitions of $X_{k+1}^N$ and $\tilde{X}_{k+1}^N$, it follows that for small $\epsilon > 0$,

$$
\| X_{k+1}^N - \tilde{X}_{k+1}^N \|_{\mathbb{H}^{-1}}^2
\leq \| X_k^N - \tilde{X}_k^N \|_{\mathbb{H}^{-1}}^2 - 2\| A^\frac{1}{2}(X_{k+1}^N - \tilde{X}_{k+1}^N) \|^2 \delta t
- 2\langle F(X_{k+1}^N) - F(X_k^N), X_{k+1}^N - \tilde{X}_{k+1}^N \rangle \delta t
+ \langle X_{k+1}^N - \tilde{X}_{k+1}^N, (G(X_k^N) - G(X_k^N)) \delta W_k \rangle_{\mathbb{H}^{-1}}
\leq \| X_k^N - \tilde{X}_k^N \|_{\mathbb{H}^{-1}}^2 - (2 - \epsilon)\| A^\frac{1}{2}(X_{k+1}^N - \tilde{X}_{k+1}^N) \|^2 \delta t
+ 2C(\epsilon)\| X_{k+1}^N - \tilde{X}_{k+1}^N \|_{\mathbb{H}^{-1}}^2 \delta t - 2C(\epsilon)\| F(X_{k+1}^N) - F(X_k^N) \|_{\mathbb{H}^{-1}}^2 \delta t
+ \langle X_{k+1}^N - X_k^N, (G(X_k^N) - G(X_k^N)) \delta W_k \rangle_{\mathbb{H}^{-1}}
+ \langle X_k^N - \tilde{X}_k^N, (G(X_k^N) - G(X_k^N)) \delta W_k \rangle_{\mathbb{H}^{-1}}
+ \langle \tilde{X}_k^N - \tilde{X}_{k+1}^N, (G(X_k^N) - G(X_k^N)) \delta W_k \rangle_{\mathbb{H}^{-1}}
=: \| X_k^N - \tilde{X}_k^N \|_{\mathbb{H}^{-1}}^2 - (2 - \epsilon)\| A^\frac{1}{2}(X_{k+1}^N - \tilde{X}_{k+1}^N) \|^2 \delta t
+ 2C(\epsilon)\| X_{k+1}^N - \tilde{X}_{k+1}^N \|_{\mathbb{H}^{-1}}^2 \delta t + II_1^k + II_2^k + II_3^k + II_4^k.
$$

Taking $2C(\epsilon)\delta t$ small enough, for $l \leq K - 1$, we get

$$
\| X_{l+1}^N - \tilde{X}_{l+1}^N \|_{\mathbb{H}^{-1}}^2 + \sum_{k=0}^l \| A^\frac{1}{2}(X_{k+1}^N - \tilde{X}_{k+1}^N) \|^2 \delta t
\leq 2C(\epsilon)\sum_{k=0}^{l-1} \| X_{k+1}^N - \tilde{X}_{k+1}^N \|_{\mathbb{H}^{-1}}^2 \delta t + C\sum_{k=0}^l (II_1^k + II_2^k + II_3^k + II_4^k).
$$

(4.5)

Taking expectation on both sides of (4.5) and using the Hölder inequality, we have

$$
\mathbb{E} \left[ \| X_{l+1}^N - \tilde{X}_{l+1}^N \|_{\mathbb{H}^{-1}}^2 \right] + \mathbb{E} \left[ \sum_{k=0}^l \| A^\frac{1}{2}(X_{k+1}^N - \tilde{X}_{k+1}^N) \|^2 \delta t \right]
$$
\[
\leq C(\varepsilon) \sum_{k=0}^{l-1} \mathbb{E}\left[\|X_{k+1}^N - \tilde{X}_{k+1}^N\|_{\mathbb{H}^{-1}}^2\right] \delta t + C\mathbb{E}\left[\sum_{k=0}^{l} II_1^k\right] \\
+ C\mathbb{E}\left[\sum_{k=0}^{l} II_2^k\right] + C\mathbb{E}\left[\sum_{k=0}^{l} II_3^k\right] + C\mathbb{E}\left[\sum_{k=0}^{l} II_4^k\right].
\]

By using the a priori estimates of \(X_{k+1}^N\), \(X^N\), Lemma 4.1 and Lemma 2.2, we have

\[
\mathbb{E}\left[\sum_{k=0}^{l} II_1^k\right] \\
\leq C\mathbb{E}\left[\sum_{k=0}^{l} \|F(\tilde{X}_{k+1}^N) - F(X^N(t_{k+1}))\|_{\mathbb{H}^{-1}}^2\delta t\right] \\
\leq C(X_0, p)\mathbb{E}\left[\sum_{j=0}^{l} (1 + \|X^N(t_{j+1})\|_E^4 + \|\tilde{X}_{j+1}^N\|_E^4) \|\tilde{X}_{j+1}^N - X^N(t_{j+1})\|_{\mathbb{H}^{-1}}^2\delta t\right] \\
\leq C(X_0, T, p)\left(\sum_{j=0}^{l} ((j+1)\delta t)^{-\frac{3}{2}}\delta t\right)\delta t^{\frac{3}{2}}.
\]

The mild form of \(X_k^N\) yields that

\[
\mathbb{E}\left[\sum_{k=0}^{l} \left(II_2^k + II_4^k\right)\right] \\
\leq C\mathbb{E}\left[\sum_{k=0}^{l} \langle (T_{\delta t} - I)(X_k^N - \tilde{X}_k^N), (G(X_k^N) - G(X^N(t_k)))\delta W_k\rangle_{\mathbb{H}^{-1}}\right] \\
+ C\mathbb{E}\left[\sum_{k=0}^{l} \langle T_{\delta t} A(F(X_{k+1}^N) - F(X^N(t_{k+1})))\delta t, (G(X_k^N) - G(X^N(t_k)))\delta W_k\rangle_{\mathbb{H}^{-1}}\right] \\
+ C\mathbb{E}\left[\sum_{k=0}^{l} \langle T_{\delta t} (G(X_k^N) - G(X^N(t_k)))\delta W_k, (G(X_k^N) - G(X^N(t_k)))\delta W_k\rangle_{\mathbb{H}^{-1}}\right] \\
=: III_1 + III_2 + III_3.
\]

The Burkholder inequality and Sobolev interpolation inequality lead to

\[
III_1 \leq \mathbb{E}\left[\left(\sum_{k=0}^{l} \sum_{j \in \mathbb{N}^+} \langle (T_{\delta t} - I)X_k^N, (G(X_k^N) - G(X^N(t_k)))e_j\rangle_{\mathbb{H}^{-1}}^2\delta t\right)^{\frac{3}{2}}\right]
\]
Thus $III_1 = 0$. For the second term,

$$III_2 \leq C \mathbb{E} \left[ \sum_{k=0}^{l} \langle T_{\delta t} A(F(X_k^N) - F(X_k^N(t_k))) \rangle \delta t, (G(X_k^N) - G(X_k^N(t_k))) \delta W_k \rangle_{H^{-1}} \right]$$

$$+ C \mathbb{E} \left[ \sum_{k=0}^{l} \langle T_{\delta t} A(F(X_{k+1}^N) - F(X_k^N)) \rangle \delta t, (G(X_k^N) - G(X_k^N(t_k))) \delta W_k \rangle_{H^{-1}} \right]$$

$$+ C \mathbb{E} \left[ \sum_{k=0}^{l} \langle T_{\delta t} A(F(X_k^N(t_k+1)) - F(X_k^N(t_k))) \rangle \delta t, (G(X_k^N) - G(X_k^N(t_k))) \delta W_k \rangle_{H^{-1}} \right]$$

$$=: III_{21} + III_{22} + III_{23}.$$ 

Similar to $III_1$, we have $III_{21} = 0$. The estimations of $III_{22}$ and $III_{23}$ are similar, we only give the estimate of $III_{22}$. The continuity of $X_k^N$ and Lemma 4.1 lead to

$$III_{22} \leq C \mathbb{E} \left[ \sum_{k=0}^{l} \langle T_{\delta t} A(F(X_k^N) - F(X_k^N)) \rangle \delta t, (G(X_k^N) - G(X_k^N(t_k))) \delta W_k \rangle_{H^{-1}} \right]$$

$$\leq C \mathbb{E} \left[ \sum_{k=0}^{l} \| T_{\delta t} A(F(X_k^N) - F(X_k^N)) \delta t \|_{H^{-1}}^{2} \right] + C \sum_{k=0}^{l} \mathbb{E} \left[ \| (G(X_k^N) - G(X_k^N(t_k))) \delta W_k \|_{H^{-1}}^{2} \right]$$

$$\leq C(T, X_0) \delta t^{\frac{2}{2} + \frac{1}{2}} + C \sum_{k=0}^{l} \mathbb{E} \left[ \| X_k^N - \tilde{X}_k^N \|_{H_{1}}^{2} \right] \delta t + C \sum_{k=0}^{l} \mathbb{E} \left[ \| \tilde{X}_k^N - X_k^N(t_k) \|_{H_{1}}^{2} \right] \delta t$$

$$\leq C(T, X_0) \delta t^{\frac{2}{2} + \frac{1}{2}} + \epsilon \sum_{k=0}^{l} \mathbb{E} \left[ \| X_k^N - \tilde{X}_k^N \|_{H_{1}}^{2} \right] \delta t$$

$$+ C(\epsilon) \sum_{k=0}^{l} \mathbb{E} \left[ \| X_k^N - \tilde{X}_k^N \|_{H_{1}}^{2} \right] \delta t + C \sum_{k=0}^{l} \mathbb{E} \left[ \| \tilde{X}_k^N - X_k^N(t_k) \|_{H_{1}}^{2} \right] \delta t.$$ 

The Burkholder inequality yields that

$$III_3 \leq C \sum_{k=0}^{l} \sum_{j \in N^+} \mathbb{E} \left[ \| (G(X_k^N) - G(X_k^N(t_k))) e_j \|_{H_{1}}^{2} \right] \delta t$$
\[
\leq C(\epsilon) \sum_{k=0}^{l} \mathbb{E} \left[ \|X_k^{N} - X^{N}(t_k)\|_{L^2(\Omega; H)}^2 \right] \delta t
\]
\[
+ \epsilon \sum_{k=0}^{l} \mathbb{E} \left[ \|(X_k^{N} - X^{N}(t_k))\|_{L^2(\Omega; H)}^2 \right] \delta t.
\]

Notice that \( \mathbb{E} \left[ \sum_{k=0}^{l} I_{3}^{k} \right] = 0 \). Combining all the estimates of \( II_{1}^{k}-II_{4}^{k} \), we complete the proof. \( \square \)

**Theorem 4.1.** Let \( X_0 \in H^\gamma, \gamma \in (0, \frac{3}{2}) \), \( T < 0 \). Then there exists \( \delta t_0 \leq 1 \) such that for \( N \in \mathbb{N}^+ \) and \( \delta t \in (0, \delta t_0) \),

\[
\|X_k^{N} - X^{N}(t_k)\|_{L^2(\Omega; H)} \leq C(T, X_0, \gamma)\delta t^{4},
\]

where \( C(T, X_0, p, \gamma) > 0 \) and \( \alpha \in (0, \gamma) \).

**Proof.** The triangle inequality yields that

\[
\|X_k^{N} - X^{N}(t_k)\|_{L^2(\Omega; H)} \leq \|\tilde{X}_k^{N} - X^{N}(t_k)\|_{L^2(\Omega; H)} + \|X_k^{N} - \tilde{X}_k^{N}\|_{L^2(\Omega; H)}
\]

\[
\leq C(T, X_0)\delta t^{4} + \|X_k^{N} - \tilde{X}_k^{N}\|_{L^2(\Omega; H)}.
\]

The mild form of \( X_I^{N} \) and \( \tilde{X}_I^{N} \) lead to

\[
\|X_I^{N} - \tilde{X}_I^{N}\|_{L^2(\Omega; H)} \leq C\left( \sum_{k=0}^{l-1} T_{\delta t}^{l-k} A(F(X_{k+1}^{N}) - F(X^{N}(t_{k+1})))\delta t \right)_{L^2(\Omega; H)}
\]

\[
+ C\left( \sum_{k=0}^{l-1} T_{\delta t}^{l-k} (G(X_{k}^{N}) - G(X^{N}(t_{k})))\delta W_{k} \right)_{L^2(\Omega; H)}.
\]

The smoothness of \( T_{\delta t} \), Sobolev interpolation inequality, Remark 3.1 and Lemma 2.2 imply that for \( \beta \in (0, 1) \),

\[
\left\| \sum_{k=0}^{l-1} T_{\delta t}^{l-k} A(F(X_{k+1}^{N}) - F(X^{N}(t_{k+1})))\delta t \right\|_{L^2(\Omega; H)}
\]

\[
\leq \sum_{k=0}^{l-1} (t_l - t_k)^{-\frac{\beta}{4}} \left( 1 + \|X_{k+1}^{N}\|_{H}^2 + \|X_{k+1}^{N}\|_{H^\beta}^2 + \|X^{N}(t_{k+1})\|_{E}^2 + \|X^{N}(t_{k+1})\|_{H^\beta}^2 \right) ^{1-\beta}
\]

\[
\|X_{k+1}^{N} - X^{N}(t_{k+1})\|_{H^\beta - 1}^{\beta} \|X_{k+1}^{N} - X^{N}(t_{k+1})\|_{L^2(\Omega)}^{1-\beta} \delta t.
\]
\[
\leq \sum_{k=0}^{l-1} (t_l - t_k)^{\frac{3}{4}} \left( 1 + \|X_{k+1}^N\|_E^2 + \|X_{k+1}^N\|_{H^\beta}^2 + \|X^N(t_{k+1})\|_E^2 + \|X^N(t_{k+1})\|_{H^\beta}^2 \right)^{\frac{1}{4}} \|X^N_{k+1} - X^N(t_{k+1})\|_{H^{-1}}^{\frac{\beta}{2}} \|X^N_{k+1} - X^N(t_{k+1})\|_{H^\beta}^{\frac{1}{2}} \delta t^{\frac{\alpha}{2}} + \|X^N_{k+1} - X^N(t_{k+1})\|_{H^{-1}}^{\frac{\beta}{2}} \|X^N_{k+1} - X^N(t_{k+1})\|_{H^\beta}^{\frac{1}{2}} \delta t^{\frac{\alpha}{2}} + C(T, X_0, \gamma) \delta t^{\frac{\alpha}{2}}.
\]

The Burkholder inequality yields that
\[
\|\sum_{k=0}^{l-1} T^{l-k}_\delta (G(X^N_k) - G(X^N(t_k)))\delta W_k\|_{L^2(\Omega; H)}^2 \\
\leq \sum_{k=0}^{l-1} \sum_{j \in \mathbb{N}^+} \mathbb{E} \|T^{l-k}_\delta (G(X^N_k) - G(X^N(t_k))) e_j\|_{H}^2 |\delta t| \\
\leq \sum_{k=0}^{l-1} (t_l - t_k)^{-\frac{1}{4}} \mathbb{E} \|X^N_k - \tilde{X}^N_k\|_{H^{-1}}^2 \delta t + C(T, X_0, \gamma) \delta t^{\frac{\alpha}{2}}.
\]

The discrete Gronwall inequality leads to the desired result.

As a consequence of (4.1) and Theorem 4.1, we present the sharp mean square convergence rate result of the proposed scheme.

**Corollary 4.1.** Let \(X_0 \in H^\gamma\), \(\gamma \in (0, \frac{3}{2})\), \(T < 0\). Then there exists \(\delta t_0 \leq 1\) such that for \(N \in \mathbb{N}^+\) and \(\delta t \in (0, \delta t_0)\),
\[
\|X^N_k - X(t_k)\|_{L^2(\Omega; H)} \leq C(T, X_0, \gamma) (\delta t^{\frac{\alpha}{2}} + \lambda_N^{\frac{\alpha}{2}})
\]
where \(C(T, X_0, p, \gamma) > 0\) and \(\alpha \in (0, \gamma)\).

From the above arguments in the proof of Theorem 4.1, it is not hard to see that under the same condition of Corollary 4.1, \(\sup_{k \leq K} \|X^N_k - X(t_k)\|_{L^2(\Omega; \mathbb{R})} \leq C(T, X_0, \gamma) (\delta t^{\frac{\alpha}{2}} + \lambda_N^{\frac{\alpha}{2}})\), \(\alpha \in (0, \gamma)\) also holds.

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