Direct extraction of QCD $\Lambda_{\overline{MS}}$ from moments of structure functions in neutrino-nucleon scattering, using the CORGI approach

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Abstract

We use recently calculated next-to-next-to-leading (NNLO) anomalous dimension coefficients for the $n = 1, 3, 5, \ldots, 13$ moments of the $xF_3$ structure function in $\nu N$ scattering, together with the corresponding three-loop Wilson coefficients, to obtain improved QCD predictions for the moments. The Complete Renormalization Group Improvement (CORGI) approach is used, in which all dependence on renormalization or factorization scales is avoided by a complete resummation of ultraviolet logarithms. The Bernstein Polynomial method is used to compare these QCD predictions to the $xF_3$ data.
of the CCFR collaboration, and direct fits for $\Lambda^{(5)}_{\overline{MS}}$, with $N_f = 5$ effective quark flavours, over the range $20 < Q^2 < 125.9 \text{ Gev}^2$, were performed. We obtain $\Lambda^{(5)}_{\overline{MS}} = 202^{+54}_{-45} \text{ MeV}$, corresponding to the three-loop running coupling $\alpha_s(M_Z) = 0.1174^{+0.0043}_{-0.0043}$. Including target mass corrections as well we obtain $\Lambda^{(5)}_{\overline{MS}} = 228^{+35}_{-36} \text{ MeV}$, corresponding to $\alpha_s(M_Z) = 0.1196^{+0.0027}_{-0.0031}$.
1 Introduction

The recent measurements of the CCFR collaboration provide the most precise determination of the deep inelastic scattering (DIS) structure functions of neutrinos and anti-neutrinos on nucleons [1]. In this paper we wish to compare CCFR measurements of the structure function $x F_3(x, Q^2)$, with QCD predictions for its moments in order to determine $\Lambda_{\overline{MS}}$ [2, 3, 4, 5]. We intend to follow essentially the same method of analysis employed in Refs. [2, 3], in which one constructs averages of the measured structure function with respect to suitably chosen Bernstein polynomials. The polynomials are chosen so that the range of $x$ for which the experimental $x F_3$ is not determined makes only a small contribution to the averages. These experimental Bernstein averages are then fitted to the QCD predictions for the corresponding linear combinations of moments. The analysis uses the available next-next-to-leading order (NNLO) anomalous dimension and coefficient function results for the odd moments with $n = 1, 3, 5, \ldots, 13$ [6]. Refs. [2, 3] also consider the $F_2(x, Q^2)$ structure function in $ep$ DIS at NNLO. The key difference in our analysis is that the QCD predictions for the moments of $F_3$ are obtained in the “Complete Renormalization Group Improvement” (CORGI) approach [7, 8], in which all dependence on the renormalization scale, $\mu$, and factorization scale, $M$, are eliminated. Customarily in the standard RG-improvement of QCD predictions for leptoproduction moments one chooses $\mu = x Q$ and $M = y Q$, with $x$ and $y$ undetermined dimensionless constants, so the renormalization scale and factorization scale are proportional to the physical DIS energy scale $Q$. Refs. [2, 3] make the standard choice $\mu = M = Q$. In the CORGI improvement one instead keeps $\mu$ and $M$ independent of $Q$. One is then forced to resum to all-orders the RG-predictable ultraviolet (UV) logarithms of $Q$, the logarithms of $M$ and $\mu$ contained in the renormalized coupling constant, and the perturbative coefficients, then cancel, and one is left with predictions which are independent of $\mu$ and $M$. Crucially in this way one also generates the correct physical $Q$-dependence of the moments, whereas in standard improvement one omits an infinite subset of UV logarithms, so that the $Q$-dependence involves the unphysical parameters $x$ and $y$. The approach is closely related to the Effective Charge formalism of Grunberg [9]. We find that our CORGI fits result in somewhat larger values of $\alpha_s(M_Z)$ than those reported in [3].

The plan of the paper is to give a brief review of the CORGI approach
for lepton production moments in Section 2. Section 3 will contain a short description of the Bernstein Polynomial averages to be employed in the fits, and Section 4 details the results of the fits to the CORGI predictions for the moments. Section 5 contains a discussion and Conclusions.

2 Lepton production moments in the CORGI approach

Let us define the Mellin moments for the $\nu N$ structure function $xF_3(x, Q^2)$,

$$\mathcal{M}_n(Q^2) = \int_0^1 x^{n-1} F_3(x, Q^2) \, dx .$$

Adopting the notation of [7] we have the factorized form for the $n$th moment,

$$\mathcal{M}_n(Q^2) = A(n) \left( \frac{c a}{1 + c a} \right)^{d(n)/b} \exp(\mathcal{I}(a)) \left( 1 + r_1(n)\tilde{a} + r_2(n)\tilde{a}^2 + r_3(n)\tilde{a}^3 + \ldots \right) ,$$

where the first three terms correspond to the operator matrix element $<O_n(M)>$, with $M$ the factorization scale, and $a \equiv \alpha_s(M)/\pi$ in terms of the RG-improved coupling. $A(n)$ is an undetermined scheme-independent overall constant, and will be one of the parameters varied in the fits. $\mathcal{I}(a)$ is a function of the anomalous dimension coefficients $d_i(n)$,

$$\frac{M}{<O_n>} \frac{\partial <O_n>}{\partial M} = \gamma_{O_n} = -d(n)a - d_1(n)a^2 - d_2(n)a^3 - \ldots .$$

The first anomalous dimension coefficient, $d(n)$, [10] is independent of the factorization scheme (FS), the higher coefficients $d_i(n) \ (i \geq 1)$ define the FS. The coupling $a$ satisfies the beta-function equation,

$$M \frac{\partial a}{\partial M} = \beta(a) = -ba^2(1 + ca + c_2a^2 + c_3a^3 + \ldots) .$$

Here $b = (33 - 2N_f)/6$ and $c = (153 - 19N_f)/12b$ are the first two universal coefficients of the beta-function. The remaining coefficients are scheme-dependent and determine the renormalization scheme (RS). The final factor in Eq.(2) is the coefficient function. We use the notation $\tilde{a} \equiv \alpha_s(\mu)/\pi$ , where
\( \mu \) is the renormalization scale.

The self-consistency of perturbation theory means that there is a dependence \( r_k(n)(\mu, M, c_2, \ldots, c_k; d_1, d_2, \ldots, d_k) \), on the parameters specifying the FS and RS [7]. The coefficient \( r_1 \) depends on the factorization scale \( M \), with,

\[
r_1(n)(M) = d(n) \left( \frac{\ln M}{\Lambda} - \ln \frac{Q}{\Lambda_{M_n}} \right) - \frac{d_1(n)}{b} .
\]

Here \( \Lambda_{M_n} \) is an FS and RS-invariant dimensionful constant associated with the moment, and it is the second UV logarithm in Eq.(5) which determines the physical \( Q \)-dependence of \( M_n(Q^2) \). \( \Lambda_{M_n} \) is directly related to \( \Lambda_{\overline{MS}} \) of modified minimal subtraction by,

\[
\Lambda_{M_n} = \Lambda_{\overline{MS}} \left( \frac{2c}{b} \right)^{\frac{1}{b}} exp \left( \frac{d_1(n)}{bd(n)} + \frac{r_1(n)}{d(n)} \right) ,
\]

where \( d_1(n) \) is the \( \overline{MS} \) NLO anomalous dimension coefficient, and \( r_1(n) \) is computed in the \( \overline{MS} \) scheme with \( M = Q \). The \( (2c/b)^{-c/b} \) factor corresponds to the standard convention for defining \( \Lambda_{\overline{MS}} \) [1]. Using Eq.(5) one can trade \( M \) for \( r_1 \) as a parameter on which \( r_k \) depends, similarly \( \mu \) can be traded for \( \tilde{r}_1 \equiv r_1(M = \mu) \). The self-consistency of perturbation theory allows one to obtain the partial derivatives of the \( r_k \) coefficients with respect to the FS and RS parameters \( \{r_1, \tilde{r}_1, c_2, \ldots, c_k; d_1, d_2, \ldots, d_k\} \). On integrating these we obtain expressions for the \( r_k \) as multinomials in the FS and RS parameters, with scheme-independent constants of integration, \( X_k \). Thus at NNLO we have,

\[
r_2(n) = \left( \frac{1}{2} - \frac{b}{2d(n)} \right) r_1^2(n) + \frac{b}{d} r_1(n) \tilde{r}_1(n) + \frac{cd_1(n)}{2b} - \frac{d_2(n)}{2b} - \frac{d(n)c_2}{2b} + X_2(n) .
\]

The constant of integration \( X_k(n) \) can only be determined given a N\(^k\)LO perturbative calculation, whereas the remaining terms are “RG-predictable” and can be obtained from lower orders. The so-called Complete Renormalization Group Improvement (CORGI) result at N\(^k\)LO corresponds to resumming to all-orders the RG-predictable terms, i.e. those not involving \( X_i(n), (i > k) \) [7]. It is easy to show that this is equivalent to working with standard RG-improvement in an FS and RS in which all the parameters are zero. In this
scheme one has \[ a = \tilde{a} = a_0, \] where the CORGI coupling \[ a_0(n) \] can be expressed in terms of the Lambert W function \cite{12, 13, 14}, defined implicitly by \[ W(z) \exp(W(z)) = z, \]

\[ a_0(n) = -\frac{1}{c[1 + W_{-1}(z_n(Q))]} \]

\[ z_n(Q) \equiv -\frac{1}{e} \left( \frac{Q}{\Lambda_{M_n}} \right)^{b/c}. \] (8)

Here the “−1” subscript denotes the branch of the Lambert W function required for asymptotic freedom, the nomenclature being that of Ref.\cite{13}. The \[ \Lambda_{M_n} \] is the scheme-invariant, related to \[ \Lambda_{\overline{MS}} \] by Eq.(6). One obtains the \[ N^3\text{LO CORGI result}, \]

\[ M_n(Q^2) = A(n) \left( \frac{ca_0(n)}{1 + ca_0(n)} \right)^{d(n)/b} \left( 1 + X_2(n)a_0^2(n) + X_3(n)a_0^3(n) + \ldots + X_k(n)a_0^k(n) \right). \] (9)

Substituting the explicit expression for \[ a_0 \] in terms of \[ W_{-1} \] in Eq.(8) we obtain the NNLO CORGI result,

\[ M_n(Q^2) = A(n)[-W_{-1}(z_n(Q))]^{-d(n)/b} \left( 1 + X_2(n)a_0^2(n) \right), \]

\[ z_n(Q) \equiv -\frac{1}{e} \left( \frac{Q}{\Lambda_{M_n}} \right)^{b/c}. \] (10)

So we see that the moment is directly proportional to a power of the Lambert function \[ W_{-1}. \] The NNLO CORGI invariants \[ X_2(n) \] can be computed from the \[ \overline{MS} \] results for \[ r_1(n), r_2(n), d_1(n), d_2(n) \] \cite{6}. The NNLO anomalous dimension coefficient \[ d_2(n) \] is only known for odd moments, \[ n = 1, 3, 5, \ldots, 13 \] \cite{6}. On rearranging Eq.(7) one has,

\[ X_2(n) = \left( \frac{-1}{2} - \frac{b}{2d(n)} \right)r_1^2(n) - \frac{cd_1(n)}{2b} + \frac{d_2(n)}{2b} + \frac{d(n)c_2}{2b} + r_2(n) \right). \] (11)

We tabulate the resulting \[ X_2(n) \] CORGI invariants for \[ n = 3, 5, \ldots, 13, \] and the ratio \[ \Lambda_{M_n}/\Lambda_{\overline{MS}} \] determined from Eq.(6), in Table 1. We assume \[ N_f = 5 \] active quark flavours.
Table 1: The numerical values of the ratio $\frac{\Lambda_{Mn}}{\Lambda_{MS}}$ and the CORGI invariants $X_2(n)$, for the odd moments $n = 3, 5, \ldots, 13$ of $xF_3$.

| $n$ | $\frac{\Lambda_{Mn}}{\Lambda_{MS}}$ | $X_2(n)$  |
|-----|----------------------------------|-----------|
| 3   | 2.268568660                     | -0.9283306650 |
| 5   | 2.999798808                     | 1.750455480 |
| 7   | 3.489368710                     | 3.858776378 |
| 9   | 3.870927376                     | 5.663749685 |
| 11  | 4.188431945                     | 7.264781157 |
| 13  | 4.462684796                     | 8.714133383 |

3 Bernstein Averages of moments

In phenomenological investigations of structure functions, for a given value of $Q^2$, only a limited number of experimental points, covering a partial range of values of $x$, are available. Therefore, one cannot directly determine the moments. A method devised to deal with this situation is to take averages of the structure function weighted by suitable polynomials. We can compare theoretical predictions with experimental results for the Bernstein averages, which are defined by [2]

$$F_{nk}(Q^2) \equiv \int_0^1 dx p_{nk}(x) F_3(x, Q^2)$$  \hspace{1cm} (12)

where $p_{nk}(x)$ are modified Bernstein polynomials,

$$p_{nk}(x) = \frac{\Gamma(n + \frac{3}{2})}{\Gamma(k + \frac{1}{2})\Gamma(n - k + 1)} x^{2k}(1 - x^2)^{(n-k)} ,  \hspace{1cm} (13)$$

and are normalized to unity, $\int_0^1 dx p_{nk}(x) = 1$. Therefore the integral (12) represents an average of the function $F_3(x)$ in the region $\bar{x}_{nk} - \frac{1}{2} \Delta x_{nk} \leq x \leq \bar{x}_{nk} - \frac{1}{2} \Delta x_{nk}$ where $\bar{x}_{nk}$ is the average of $x$ which is very near to the maximum of $p_{nk}(x)$, and $\Delta x_{nk}$ is the spread of $\bar{x}_{nk}$. The key point is that values of $F_3$ outside this interval contribute little to the integral (12), as $p_{nk}(x)$ decreases to zero very quickly. So, by suitably choosing $n, k$, we manage to adjust the region where the average is peaked to that in which we have experimental...
Using the binomial expansion in Eq.(13), it follows that the averages of $F_3$ with $p_{nk}(x)$ as weight functions, can be obtained in terms of odd moments,

$$F_{nk} = 2 \frac{(n-k)! \Gamma(n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})\Gamma(n-k+1)} \sum_{l=0}^{n-k} \frac{(-1)^l}{l!(n-k-l)!} \int_0^1 dx x^{(2(k+l)+1)-1} F_3,$$  \tag{14}

using Eq.(1) then,

$$F_{nk} = 2 \frac{(n-k)! \Gamma(n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})\Gamma(n-k+1)} \sum_{l=0}^{n-k} \frac{(-1)^l}{l!(n-k-l)!} M_{2(k+l)+1}. \tag{15}$$

For the NNLO QCD fits to be performed we are restricted to considering odd moments of $xF_3$ for which the NNLO anomalous dimension coefficient $d_2(n)$ has been computed, $n = 3, 5, \ldots, 13$. We can only include a Bernstein average, $F_{nk}$, if we have experimental points covering the whole range $[\bar{x}_{nk} - \frac{1}{2}\Delta x_{nk}, \bar{x}_{nk} - \frac{1}{2}\Delta x_{nk}]$, this means that we can use only the 10 averages $F_{21}^{(exp)}(Q^2), F_{31}^{(exp)}(Q^2), F_{32}^{(exp)}(Q^2), F_{41}^{(exp)}(Q^2), F_{42}^{(exp)}(Q^2), F_{51}^{(exp)}(Q^2), F_{52}^{(exp)}(Q^2), F_{61}^{(exp)}(Q^2), F_{62}^{(exp)}(Q^2), F_{63}^{(exp)}(Q^2)$. To obtain these experimental averages from the CCFR data for $xF_3$, we fit $xF_3(x, Q^2)$ for each bin in $Q^2$ separately, to the convenient phenomenological expression,

$$xF_3^{(phen)} = Ax^B(1-x)^C,$$  \tag{16}

this form ensures zero values for $xF_3$ at $x = 0$, and $x = 1$. A theoretical justification of Eq.(16) may be found in Ref.[15]. Using Eq.(16) with the fitted values of $A, B, C$, one can then compute $F_{nk}^{(exp)}(Q^2)$ using Eq.(12), in terms of Gamma functions. The resulting experimental Bernstein averages are plotted in Figure 1. The errors in the $F_{nk}^{(exp)}(Q^2)$ correspond to allowing the CCFR data for $xF_3$ to vary within the experimental error bars, including experimental systematic errors. We have only included data for $Q^2 \geq 20\text{GeV}^2$, and our QCD fits will assume $N_f = 5$ active quark flavours. This has the merit of simplifying the analysis by avoiding evolution through flavour thresholds, whilst only reducing the number of fitted $F_{nk}^{(exp)}$ points by eight.
Figure 1: Fit to $xF_3$ using Bernstein averages
4 NNLO QCD fits to Bernstein averages for $F_3$

Using Eq.(15) the ten Bernstein averages $F_{nk}(Q^2)$ can be written in terms of odd moments $\mathcal{M}_n(Q^2)$,

$$F_{21}(Q^2) = 7.5 \left( \mathcal{M}_3(Q^2) - \mathcal{M}_5(Q^2) \right)$$
$$F_{31}(Q^2) = 13.125 \left( \mathcal{M}_3(Q^2) - 2\mathcal{M}_5(Q^2) + \mathcal{M}_7(Q^2) \right)$$
$$F_{32}(Q^2) = 17.5 \left( \mathcal{M}_5(Q^2) - \mathcal{M}_7(Q^2) \right)$$
$$F_{41}(Q^2) = 19.687 \left( \mathcal{M}_3(Q^2) - 3\mathcal{M}_5(Q^2) + 3\mathcal{M}_7(Q^2) - \mathcal{M}_9(Q^2) \right)$$
$$F_{42}(Q^2) = 39.375 \left( \mathcal{M}_5(Q^2) - 2\mathcal{M}_7(Q^2) + \mathcal{M}_9(Q^2) \right)$$
$$F_{51}(Q^2) = 27.070 \left( \mathcal{M}_3(Q^2) - 4\mathcal{M}_5(Q^2) + 6\mathcal{M}_7(Q^2) - \mathcal{M}_{11}(Q^2) \right)$$
$$F_{52}(Q^2) = 72.187 \left( \mathcal{M}_5(Q^2) - 3\mathcal{M}_7(Q^2) + 3\mathcal{M}_9(Q^2) - 4\mathcal{M}_{11}(Q^2) \right)$$
$$F_{61}(Q^2) = 35.191 \left( \mathcal{M}_3(Q^2) - 5\mathcal{M}_5(Q^2) + 10\mathcal{M}_7(Q^2) - \mathcal{M}_9(Q^2) \right.\left. + 5\mathcal{M}_{11}(Q^2) - \mathcal{M}_{13}(Q^2) \right)$$
$$F_{62}(Q^2) = 117.30 \left( \mathcal{M}_5(Q^2) - 4\mathcal{M}_7(Q^2) + 6\mathcal{M}_9(Q^2) - 4\mathcal{M}_{11}(Q^2) \right.\left. + \mathcal{M}_{13}(Q^2) \right)$$
$$F_{63}(Q^2) = 187.69 \left( \mathcal{M}_7(Q^2) - 3\mathcal{M}_9(Q^2) + 3\mathcal{M}_{11}(Q^2) - \mathcal{M}_{13}(Q^2) \right)$$

We shall use the NNLO CORGI result of Eq.(10) for the QCD prediction of $\mathcal{M}_n(Q^2)$. The basic fit parameters will be the unknown normalization constants $A(n)$, $n = 3, 5, 7, \ldots, 13$, and $\Lambda_{\overline{MS}}$, related to the CORGI $\Lambda_{\mathcal{M}_n}$ by Eq.(6), see Table 1. Thus there are 7 parameters to be simultaneously fitted to the experimental $F_{nk}(Q^2)$ averages. Defining a global $\chi^2$ for all the experimental data points of Figure 1, we found an acceptable fit with minimum $\chi^2$/d.o.f. = 1.2905/43. The best fit is indicated by the curves in Figure 1. Allowing $\chi^2$ within 1 of the minimum to estimate an error gives,

$$\Lambda^{(5)}_{\overline{MS}} = 202^{+54}_{-45} \text{ MeV} , \quad (18)$$

which corresponds to the three-loop $\overline{MS}$ coupling at the $Z$-mass,

$$\alpha_s(M_Z) = 0.1174^{+0.0043}_{-0.0043} . \quad (19)$$
The minimum $\chi^2$ values for all 7 fitting parameters are tabulated in Table 2.

To attempt to include target mass corrections (TMC) in the fits we amended the expression for $\mathcal{M}_n(Q^2)$ to [16],

$$\mathcal{M}_n^{TMC}(Q^2) = \mathcal{M}_n(Q^2) + \frac{n(n+1)}{n+2} \frac{m_p^2}{Q^2} \mathcal{M}_{n+2}(Q^2) + \frac{(n+2)(n+1)m_p^4}{2(n+2)} \frac{1}{Q^4} \mathcal{M}_{n+4}(Q^2) + O\left(\frac{m_p^6}{Q^6}\right). \quad (20)$$

This results from the series expansion in powers of $\frac{m_p^2}{Q^2}$ of the Nachtmann moments [17]. Here $m_p$ is the proton mass. The influence of the $O\left(\frac{m_p^4}{Q^4}\right)$ terms is very small, and we shall neglect them. Minimizing $\chi^2$ with the amended expression for the moments then gives a best fit of comparable quality, with

$$\Lambda_{MS}^{(TMC)} = \Lambda_{MS} + 228^{+35}_{-36} \text{ MeV}. \quad (21)$$

Corresponding to the coupling at the $Z$-mass,

$$\alpha_s(M_Z) = 0.1196^{+0.0027}_{-0.0031}. \quad (22)$$

The best fit values of the 7 fitting parameters including TMC, are tabulated in Table 3. We can compare this value of $\alpha_s(M_Z)$ with the corresponding values $\alpha_s(M_Z) = 0.1153\pm0.0041$ obtained in Ref.[3], and $\alpha_s(M_Z) = 0.1187\pm0.0026$ obtained in Ref.[4]. Finally we can confirm the expectation that at the energy scales $Q^2 \geq 20 \text{ GeV}^2$ included in our fit, higher-twist (HT) effects should be small. We modified the expression for $\mathcal{M}_n(Q^2)$ by an additional term [2],

$$\mathcal{M}_n^{(HT)}(Q^2) = n \left(\frac{\rho \Lambda_{MS}^2}{Q^2}\right) \mathcal{M}_n(Q^2), \quad (23)$$

where $\rho$ is an additional phenomenological parameter which will be fitted to the data. The best 8 fit parameters are tabulated in Table 4. The inclusion of HT terms shifts the central value of $\Lambda_{MS}^{(5)}$ by only 2 Mev (cf. Table 2), confirming as expected that HT effects are negligible. Our fitted value of the HT parameter $\rho = -0.77 \pm 0.23$ is to be compared with the value $-0.14 \pm 0.6$ obtained in Ref.[3], and $-0.31 \pm 0.80$ obtained in Ref.[4].

We should stress that the errors in the values of $\alpha_s(M_Z)$ quoted in Eqs.(19),(22), reflect the errors in the $F_{nk}^{(exp)}$ values in Fig. 1 , to which
the NNLO CORGI predictions for the moments of Eq.(10) have been fitted. In the CORGI approach all dependence on the unphysical factorization and renormalization scales is eliminated by the complete resummation of RG-predictable UV logarithms, the remaining uncertainty in the QCD prediction then resides in the unknown N^3LO CORGI invariant X_3(n). It would in principle be straightforward to use Padé approximants to estimate the unknown d_3(n) anomalous dimension coefficients, as in Ref.[4], and perform fits with an estimated X_3(n), but we have not done so in this work.

Table 2: Numerical values of fitting parameters, for the best fit of Figure 1.

|          |          |          |          |
|----------|----------|----------|----------|
| Λ_{MS}   | 202^{+44}_{-45} MeV |
| A(3)     | 0.497^{−0.012}_{+0.022} |
| A(5)     | 0.159^{−0.008}_{+0.008} |
| A(7)     | 0.07^{−0.0013}_{+0.0012} |
| A(9)     | 0.04^{−0.0013}_{+0.0014} |
| A(11)    | 0.03^{−0.0069}_{+0.0069} |
| A(13)    | 0.029^{−0.0020}_{+0.0019} |

Table 3: Numerical values of fitting parameters, including TMC.

|          |          |          |          |
|----------|----------|----------|----------|
| Λ_{MS}   | 228^{+35}_{−36} MeV |
| A(3)     | 0.481^{−0.011}_{+0.012} |
| A(5)     | 0.16^{−0.0044}_{+0.0044} |
| A(7)     | 0.08^{−0.0024}_{+0.0021} |
| A(9)     | 0.05^{−0.0001}_{+0.0027} |
| A(11)    | 0.03^{−0.0038}_{+0.0028} |
| A(13)    | 0.009^{−0.0025}_{+0.0029} |
Table 4: Numerical values of fitting parameters, including HT.

| Parameter | Value         |
|-----------|---------------|
| $\Lambda_{\text{MS}}$ | $204_{-46}^{+53}$ MeV |
| $A(3)$   | $0.497_{-0.014}^{+0.011}$ |
| $A(5)$   | $0.159_{-0.0013}^{+0.0008}$ |
| $A(7)$   | $0.07_{-0.0012}^{+0.0015}$ |
| $A(9)$   | $0.04_{-0.0015}^{+0.0014}$ |
| $A(11)$  | $0.03_{-0.0009}^{+0.0010}$ |
| $A(13)$  | $0.029_{-0.0029}^{+0.0020}$ |
| $\rho$   | $-0.77_{-0.23}^{+0.23}$ |

5 Discussion and Conclusions

In this paper we have used a similar method of analysis to that of [3] to fit QCD predictions for the moments of the $\nu N$ DIS structure function $x F_3$, to suitably constructed Bernstein polynomial averages of the CCFR experimental data for $xF_3$ [1]. The key difference in our approach has been the use of Complete Renormalization Group Improved (CORGI) [7, 8] NNLO QCD predictions, hence avoiding the need to make particular ad hoc choices of the dimensionful renormalization scale, $\mu$, and factorization scale, $M$. The most important motivation for the CORGI approach is that by completely resumming all the UV logarithms one correctly generates the physical dependence of the moments $M_n(Q^2)$ on the DIS energy scale $Q$. From Eq.(10) this is seen to be,

$$M_n(Q^2) \approx A(n) \left[ -W_{-1} \left( -\frac{1}{e} \left( \frac{Q}{\Lambda_{M_n}} \right)^{-b/c} \right) \right]^{-d(n)/b} \left( 1 + O \left( \frac{1}{b \ln(Q/\Lambda_{M_n})} \right)^2 \right),$$

so that the large-$Q$ behaviour is controlled by the Lambert $W_{-1}$ function, and the ratio $Q/\Lambda_{M_n}$, with $\Lambda_{M_n}$ the FS and RS-invariant defined in Eq.(5). In contrast with standard RG-improvement and choosing $\mu = M = xQ$, with
x a dimensionless constant, one has instead,

\[ \mathcal{M}_n(Q^2) \approx A(n) \left[ -W_{-1} \left( -\frac{1}{e} \left( \frac{xQ}{\Lambda_{\overline{MS}}} \right)^{-b/c} \right) \right]^{-d(n)/b} \left( 1 + O \left( \frac{1}{b \ln(xQ/\Lambda_{\overline{MS}})} \right) \right), \]

which manifestly depends on the unphysical parameter x. Here \( \tilde{\Lambda}_{\overline{MS}} = (2c/b)^{-c/b} \times \Lambda_{\overline{MS}} \). In Refs. [2, 3, 4] the standard choice \( x = 1 \) is made.

It seems clear to us that CORGI QCD predictions should be used in the fits. However, the value of \( \alpha_s(M_Z) = 0.1196 \pm 0.003 \) that we obtain, including TMC effects, is consistent with the result \( \alpha_s(M_Z) = 0.1153 \pm 0.0063 \) obtained in Ref. [3] using a similar method of analysis, but with the standard \( x = 1 \) scale choices. We are in even closer agreement with the value \( \alpha_s(M_Z) = 0.1195 \pm 0.004 \) reported in Ref. [4], but these authors use a very different method of analysis involving the Jacobi Polynomial technique, together with the standard \( x = 1 \) scale choice. This indicates that the incomplete resummation of UV logarithms implicit in the standard scale choice \( \mu = M = Q \), does not greatly modify the fit. This is underwritten by the reasonably small NNLO CORGI invariants \( X_2(n) \), and ratios \( \Lambda_{\overline{MS}}/\Lambda_{\overline{MS}} \), appearing in Table 1. From Eq.(6) and Table 1 one can see that the CORGI result corresponds to using an \( n \)-dependent scale choice, ranging from \( x = 0.44 \) for the \( n = 3 \) moment to \( x = 0.22 \) for the \( n = 13 \) moment. The CORGI approach also differs in that \( c_2 = 0 \) and \( d_1(n) = d_2(n) = 0 \), rather than the \( \overline{MS} \) values. It should be noted, however, that use of the standard physical choice of renormalization scale is in general likely to result in misleading determinations of \( \Lambda_{\overline{MS}} \), as has been discussed in Ref. [3] in the case of \( e^+e^- \) jet observables.

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