Boundedness of derivatives and anti-derivatives of holomorphic functions as a rare phenomenon

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Abstract
In this article we prove a general result which in particular suggests that, on a simply connected domain $\Omega$ in $\mathbb{C}$, all the derivatives and anti-derivatives of the generic holomorphic function are unbounded. A similar result holds for the operator $\tilde{T}_N$ of partial sums of the Taylor expansion with center $\zeta \in \Omega$ at $z = 0$, seen as functions of the center $\zeta$. We also discuss a universality result of these operators $\tilde{T}_N$.

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1 Introduction
Let $\Omega$ be a domain in the complex plane and consider the space $\mathcal{H}ol(\Omega)$ of all the functions that are holomorphic on $\Omega$ with the topology of uniform convergence on compacta. In the first section of this article we show that, for a function $f \in \mathcal{H}ol(\Omega)$, the phenomenon of its $k$-derivative or $k$-anti-derivative being bounded on $\Omega$ is a rare phenomenon in the topological sense, provided that $\Omega$ is simply connected. We do this by using Baire’s Theorem and we prove that the set $\mathcal{D}$ of all the functions $f \in \mathcal{H}ol(\Omega)$ with the property that the derivatives and the anti-derivatives of $f$ of all orders are unbounded on $\Omega$ is a dense $G_\delta$ set in $\mathcal{H}ol(\Omega)$.

If a function $f$ is holomorphic in an open set containing $\zeta$, then $S_N(f, \zeta)(z)$ denotes the $N$-th partial sum of the Taylor expansion of $f$ with center $\zeta$ at $z$. If $\Omega$ is a simply connected domain and $\zeta \in \Omega$, we define the class $\mathcal{U}(\Omega, \zeta)$ as follows:

Definition 1.1. The set $\mathcal{U}(\Omega, \zeta)$ is the set of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, with $K^c$ connected, and for every function $h$ which is continuous on $K$ and holomorphic in the interior of $K$, there exists a sequence $\{\lambda_n\} \in \{0, 1, 2, \ldots\}$ such that

$$\sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \longrightarrow 0, \quad n \to \infty$$

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Denote $D = \{ z \in \mathbb{C} : |z| < 1 \}$. It is shown in [4] that $U(D, 0)$ is a dense $G_\delta$ set in $\mathcal{H}ol(D)$. More generally, in [3] it is shown that $U(\Omega, \zeta)$ is a dense $G_\delta$ set in $\mathcal{H}ol(\Omega)$, where $\Omega$ is any simply connected domain and $\zeta \in \Omega$. Next, for $\Omega$ as above, we define the set $U(\Omega)$:

**Definition 1.2.** The set $U(\Omega)$ is the set of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, with $K^c$ connected, and every function $h$ which is continuous on $K$ and holomorphic in the interior of $K$, there exists a sequence $\{\lambda_n\} \in \{0, 1, 2, \ldots\}$ such that, for every compact set $L \subset \Omega$, $\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \to 0$, $n \to \infty$.

Again in [3] it is shown that $U(\Omega)$ is a dense $G_\delta$ set in $\mathcal{H}ol(\Omega)$. Furthermore, in [1] it is shown that $U(\Omega, \zeta) = U(\Omega)$, provided that $\Omega$ is contained in a half-plane. This result is generalized in [2], where it is shown that $U(\Omega, \zeta) = U(\Omega)$ for any simply connected domain $\Omega$ and $\zeta \in \Omega$.

In the second section of this article, we fix a $\zeta_0 \in \Omega$ and, for $N \geq 1$, we consider the function

$$S_N(f, \zeta_0) : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto \sum_{n=0}^{N} \frac{f^{(n)}(\zeta_0)}{n!}(z - \zeta_0)^n = S_N(f, \zeta_0)(z)$$

V. Nestoridis suggested that, contrary to the functions in $U(\Omega, \zeta)$, whose Taylor partial sums are considered as functions of $z$ with the center $\zeta$ fixed, we fix $z = 0$ and let the center $\zeta$ vary in $\Omega$. Thus, for $N \geq 0$, we obtain an operator

$$\tilde{T}_N : \mathcal{H}ol(\Omega) \to \mathcal{H}ol(\Omega)$$

$$f \mapsto \tilde{T}_N(f)$$

where

$$\tilde{T}_N(f) : \Omega \to \mathbb{C}$$

$$\zeta \mapsto \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(-\zeta)^n = \tilde{T}_N(f)(\zeta)$$

for any $f \in \mathcal{H}ol(\Omega)$ and $N \geq 0$. The set of functions $f \in \mathcal{H}ol(\Omega)$ such that $\tilde{T}_N(f)$ is unbounded on $\Omega$ for all $N \geq 0$ is residual in $\mathcal{H}ol(\Omega)$. This led V. Nestoridis to conjecture that, if $0 \notin \Omega$, then the class $S(\Omega)$ of all functions $f \in \mathcal{H}ol(\Omega)$ with the property that the set $\{\tilde{T}_N(f) : N = 0, 1, 2, \ldots\}$ is dense in $\mathcal{H}ol(\Omega)$ is a dense $G_\delta$ set in $\mathcal{H}ol(\Omega)$. In this article we show that either $S(\Omega) = \emptyset$ or $S(\Omega)$ is a dense $G_\delta$ set in $\mathcal{H}ol(\Omega)$. The question of whether $S(\Omega) \neq \emptyset$ will be examined in a future article. However, we do show that, if $0 \notin \Omega$, then the set $S_\delta(\Omega)$ of the functions $f \in \mathcal{H}ol(\Omega)$ with he property that the closure
of the set \( \{ \tilde{T}_{\lambda}(f) \} \) contains the constant functions on \( \Omega \) is residual in \( \mathcal{H}ol(\Omega) \). We do this by proving that \( S_{\lambda}(\Omega) \) contains the set \( U(\Omega) \), which is already proven to be a dense \( G_\delta \) set in \( \mathcal{H}ol(\Omega) \) ([3]).

In the last part of the article, answering a question by T. Hatziafratis, we prove that, for a countable set \( E \subset T = \{ z \in \mathbb{C} : |z| = 1 \} \), the generic holomorphic function on \( \mathbb{D} \) has unbounded derivatives and anti-derivatives on each ray \( [0, z) \), \( z \in E \). We also obtain a more general result, where in fact we do not use Baire’s Theorem and, therefore, the topological vector space used need not be a Fréchet space.

2 Preliminaries

Regarding the terminology used, a set \( \Omega \subset \mathbb{C} \) is called a domain if it is open and connected in \( \mathbb{C} \). A \( G_\delta \) set in \( \mathcal{H}ol(\Omega) \) is a countable intersection of open sets in \( \mathcal{H}ol(\Omega) \) and an \( F_\sigma \) set is a countable union of closed sets in \( \mathcal{H}ol(\Omega) \). Furthermore, a subset \( E \) of \( \mathcal{H}ol(\Omega) \) is called dense if there exists no non-empty open subset \( U \) of \( \mathcal{H}ol(\Omega) \) such \( U \) and \( E \) are disjoint. The set \( E \) is nowhere dense in \( \mathcal{H}ol(\Omega) \) if every non-empty open set \( U \) has an open non-empty subset \( V \) such that \( E \) and \( V \) are disjoint. This is equivalent to the closure of \( E \) having an empty interior in \( \mathcal{H}ol(\Omega) \). A \( G_\delta \) dense subset of \( \mathcal{H}ol(\Omega) \) is a \( G_\delta \) subset which is also dense. Because the space \( \mathcal{H}ol(\Omega) \) is metrizable complete, Baire’s theorem implies that a subset of \( \mathcal{H}ol(\Omega) \) is \( G_\delta \) dense iff it is the countable intersection of open and dense subsets of \( \mathcal{H}ol(\Omega) \). A subset of \( \mathcal{H}ol(\Omega) \) is called residual if it contains a \( G_\delta \) dense set. Equivalently, if its complement is contained in an \( F_\sigma \) set of the first category.

Let \( \Omega_1, \Omega_2 \) be two domains in \( \mathbb{C} \) and \( T : \mathcal{H}ol(\Omega_1) \to \mathcal{H}ol(\Omega_2) \) be a linear operator with the property that for every \( z \in \Omega_2 \), the function \( f \mapsto T(f)(z) \) is continuous in \( \mathcal{H}ol(\Omega_1) \). Observe that this latter property is weaker than \( T \) being continuous. Define

\[
\mathcal{U}_T = \{ f \in \mathcal{H}ol(\Omega_1) : T(f) \text{ is unbounded on } \Omega_2 \}
\]

**Proposition 2.1.** If \( \Omega_1, \Omega_2 \) are two domains in \( \mathbb{C} \) and \( T \) is as above, then either \( \mathcal{U}_T = \emptyset \) or \( \mathcal{U}_T \) is a dense \( G_\delta \) set in \( \mathcal{H}ol(\Omega_1) \).

**Proof.** If \( \mathcal{U}_T \neq \emptyset \), for \( m \geq 1 \) define

\[
U_m = \{ f \in \mathcal{H}ol(\Omega_1) : |T(f)(z)| \leq m \text{ for all } z \in \Omega_2 \}
\]

Then

\[
\mathcal{U}_T = \left( \bigcup_{m=1}^{\infty} U_m \right)^c = \bigcap_{m=1}^{\infty} U_m^c
\]

We will show that \( U_m \) is closed and nowhere dense in \( \mathcal{H}ol(\Omega_1) \) for each \( m \geq 1 \).
To see that it is closed, take a sequence \( \{f_n\} \) in \( U_m \) such that \( f_n \to f \) uniformly on compact subsets of \( \Omega_1 \) for some function \( f \). Then \( f \in Hol(\Omega_1) \) and, for \( z \in \Omega_2 \) we have

\[
|T(f)(z)| \leq |T(f)(z) - T(f_n)(z)| + |T(f_n)(z)|
\]

Taking \( n \to \infty \) we get that \( |T(f)(z)| \leq m \) because of the continuity of \( f \to T(f)(z) \), i.e. \( f \in U_m \). Thus, \( U_m \) is closed.

To see that \( U_m \) is nowhere dense, it suffices to show that \( U_m^c = \emptyset \). Suppose \( f \in U_m^c \).

Since \( \mathcal{U}_T \neq \emptyset \), there exists a function \( g \in \mathcal{H}ol(\Omega_1) \) such that \( T(g) \) is unbounded on \( \Omega_2 \). Then \( \{f + \frac{1}{n}g\}_n \) is a sequence in \( \mathcal{H}ol(\Omega_1) \) and, if \( K \) is a compact subset of \( \Omega_1 \), we have

\[
\|(f + \frac{1}{n}g) - f\|_K = \sup_{z \in K} |f(z) + \frac{1}{n}g(z) - f(z)|
\]

\[
= \sup_{z \in K} \left| \frac{1}{n}g(z) \right| = \frac{1}{n} \|g\|_K
\]

By taking \( n \to \infty \) and observing that \( \|g\|_K < \infty \), \( g \) being holomorphic on \( \Omega_1 \supset K \), we obtain that \( f + \frac{1}{n}g \to f \) uniformly on \( K \). But \( K \) was an arbitrary compact subset of \( \Omega_1 \), so \( f + \frac{1}{n}g \to f \) uniformly on compact subsets of \( \Omega_1 \).

Since \( f \in U_m^c \), there exists an \( n_0 \) such that \( f + \frac{1}{n_0}g \in U_m \). By the linearity of \( f \to T(f) \) this means that

\[
\frac{1}{n_0} |T(g)(z)| \leq |T(f)(z) + \frac{1}{n_0} T(g)(z)| + |T(f)(z)|
\]

\[
\leq m + m
\]

or \( |T(g)(z)| \leq 2mn_0 \), for all \( z \in \Omega_2 \), which is contradictory to the fact that \( T(g) \) is unbounded on \( \Omega_2 \). Thus, \( U_m^c = \emptyset \) and the proof is complete. \( \square \)

**Proposition 2.2.** For \( n \in \mathbb{Z} \), let \( T_n : \mathcal{H}ol(\Omega_1) \to \mathcal{H}ol(\Omega_2) \) be linear and such that for every \( z \in \Omega_2 \), the function \( f \to T(f)(z) \) is continuous in \( \mathcal{H}ol(\Omega_1) \). If \( \mathcal{U}_{T_n} \neq \emptyset \) for all \( n \in \mathbb{Z} \) then the set \( \bigcap_n \mathcal{U}_{T_n} \) is dense \( G_\delta \) in \( \mathcal{H}ol(\Omega_1) \).

**Proof.** The space \( \mathcal{H}ol(\Omega_1) \) with the metric of uniform convergence on compacta is a complete metric space, so by Baire’s Theorem any countable intersection of dense \( G_\delta \) sets in \( \mathcal{H}ol(\Omega_1) \) is again a dense \( G_\delta \) set in \( \mathcal{H}ol(\Omega_1) \). Since \( \mathcal{U}_{T_n} \neq \emptyset \), it is a dense \( G_\delta \) set in \( \mathcal{H}ol(\Omega) \) by Proposition (2.1), \( n \in \mathbb{Z} \), and the desired result follows immediately. \( \square \)

Observe that Propositions (2.1) and (2.2) still hold if we replace \( \mathcal{H}ol(\Omega_2) \) by \( \mathbb{C}^X \), where \( X \) is any non-empty set and \( \mathbb{C}^X \) is the set of all functions from \( X \) to \( \mathbb{C} \).

## 3 Boundedness of derivatives and anti-derivatives as a rare phenomenon

**Proposition 3.1.** Let \( \Omega \subset \mathbb{C} \) be open and non-empty. The set \( A_0 \) of all functions \( f \in \mathcal{H}ol(\Omega) \) that are bounded on \( \Omega \) is a set of the first category in \( \mathcal{H}ol(\Omega) \).
Proof. For $m \in \mathbb{N}$ define

$$A_m = \left\{ f \in \mathcal{H}(\Omega) : |f(z)| \leq m, \text{ for all } z \in \Omega \right\}$$

It is obvious that

$$A_0 = \bigcup_{m=1}^{+\infty} A_m$$

We will show that every $A_m$ is closed and has an empty interior in $\mathcal{H}(\Omega)$.

For $m \in \mathbb{N}$, the set $A_m$ is closed in $\mathcal{H}(\Omega)$: Let $\{f_n\}$ be a sequence in $A_m$ and $f$ a function on $\Omega$ such that $f_n \to f$ uniformly on compact subsets of $\Omega$. By the Weierstrass theorem, $f \in \mathcal{H}(\Omega)$ and, for $z \in \Omega$

$$|f(z)| = \lim_{n \to \infty} |f_n(z)| \leq m$$

Therefore, $f \in A_m$ and $A_m$ is closed in $\mathcal{H}(\Omega)$ for each $m = 1, 2, ...$.

Next we show that $A_m^0 = \emptyset$ for all $m = 1, 2, ...$: First observe that there exists a function $g \in \mathcal{H}(\Omega)$ that is unbounded on $\Omega$. Indeed, if $\Omega$ is unbounded take $g(z) = z$, $z \in \Omega$, and if $\Omega$ is bounded, take $\zeta_0 \in \partial \Omega$ and $g(z) = \frac{1}{z - \zeta_0}$.

Now assume that there exists $f \in A_m^0$ for some fixed $m = 1, 2, ...$. Then $\{f + \frac{1}{n}g\}_n$ is a sequence in $\mathcal{H}(\Omega)$ and $f + \frac{1}{n}g \to f$ uniformly on compact subsets of $\Omega$, $n \to \infty$. But $f \in A_m^0$, hence there exists an $n_0 \in \mathbb{N}$ such that $f + \frac{1}{n_0}g \in A_m^0$. This means that

$$|f(z) + \frac{1}{n_0}g(z)| \leq m, \text{ for all } z \in \Omega$$

But then, for any $z \in \Omega$ we would have

$$\left| \frac{1}{n_0}g(z) \right| = \left| f(z) + \frac{1}{n_0}g(z) - f(z) \right|$$

$$\leq |f(z) + \frac{1}{n_0}g(z)| + |f(z)|$$

$$\leq m + m,$$

Therefore, $|g(z)| \leq 2mn_0$ for all $z \in \Omega$, which is contradictory to the fact that $g$ is unbounded on $\Omega$. Thus, $A_m^0 = \emptyset$ and the proof is complete.

For $f \in \mathcal{H}(\Omega)$, we denote by $f^{(k)}$ the $k$-derivative of $f$, $k \geq 1$. By $f^{(0)}$ we denote $f$ itself.

**Proposition 3.2.** Let $\Omega \subset \mathbb{C}$ be open and non-empty and $k \in \mathbb{N}$. The set $A_k$ of all functions $f \in \mathcal{H}(\Omega)$ such that $f^{(k)}$ is bounded on $\Omega$ is a set of the first category in $\mathcal{H}(\Omega)$.  


Proof. For $m \in \mathbb{N}$, define

$$A_m = \left\{ f \in \mathcal{Hol}(\Omega) : |f^{(k)}(z)| \leq m, \text{ for all } z \in \Omega \right\}$$

It is obvious that

$$A_k = \bigcup_{m=1}^{+\infty} A_m$$

We will show that each $A_m$ is closed and has empty interior in $\mathcal{Hol}(\Omega)$.

To see that it is closed, take a sequence \( \{f_n\} \) in $A_m$ and a function $f$ on $\Omega$ such that $f_n \rightarrow f$ uniformly on compact subsets of $\Omega$. By the Weierstrass theorem we have that $f \in \mathcal{Hol}(\Omega)$ and $f^{(k)}_n \rightarrow f^{(k)}$ uniformly on compact subsets of $\Omega$. Therefore, for any $z \in \Omega$ we have that

$$|f^{(k)}(z)| = \lim_{n \to \infty} |f^{(k)}_n(z)| \leq m$$

i.e. $f \in A_m$. Thus, $A_m$ is closed.

To see that $A_m^c = \emptyset$, first observe that there exists a function $g \in \mathcal{Hol}(\Omega)$ such that $g^{(k)}$ is unbounded on $\Omega$. Indeed, if $\Omega$ is unbounded take $g(z) = z^{k+1}$ and if $\Omega$ is bounded take $\zeta_0 \in \partial \Omega$ and $g(z) = \frac{1}{z - \zeta_0}$.

Now assume that there exists $f \in A_m^c$. Then \( \{f + \frac{1}{n_0}g\}_n \) is a sequence in $\mathcal{Hol}(\Omega)$ and $f + \frac{1}{n_0}g \rightarrow f$ uniformly on compact subsets of $\Omega$, $n \to \infty$. But $f \in A_m^c$, hence there exists an $n_0 \in \mathbb{N}$ such that $f + \frac{1}{n_0}g \in A_m^c$. This means that

$$|f^{(k)}(z) + \frac{1}{n_0}g^{(k)}(z)| \leq m, \text{ for all } z \in \Omega$$

where the linearity of the derivative operator is used. But then, for any $z \in \Omega$ we would have

$$\left| \frac{1}{n_0}g^{(k)}(z) \right| = \left| f^{(k)}(z) + \frac{1}{n_0}g^{(k)}(z) - f^{(k)}(z) \right|$$

$$\leq \left| f^{(k)}(z) + \frac{1}{n_0}g^{(k)}(z) \right| + \left| f^{(k)}(z) \right|$$

$$\leq m + m,$$

Thus $|g^{(k)}(z)| \leq 2mn_0$ for all $z \in \Omega$, which is contradictory to the fact that $g^{(k)}$ is unbounded on $\Omega$. Thus, $A_m^c = \emptyset$ and the proof is complete.

Proposition 3.3. Let $\Omega \subset \mathbb{C}$ be open and non-empty. The set $\mathcal{E}$ of all functions $f \in \mathcal{Hol}(\Omega)$ with the property that $f^{(k)}$ is unbounded on $\Omega$, for all $k \in \mathbb{N}$, is a dense $G_\delta$ set in $\mathcal{Hol}(\Omega)$.

Proof. Using the notation previously established it is obvious that

$$\mathcal{E} = \bigcap_{k=0}^{\infty} A_k^c$$

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By Propositions (3.1) and (3.2) we have that for each $k \geq 0$, the set $A_k$ is the countable union of closed, nowhere dense sets in $\text{Hol}(\Omega)$, so its complement $A_k^c$ must be a dense $G_\delta$ set in $\text{Hol}(\Omega)$. By Baire’s Theorem, the set $E$ is a dense $G_\delta$ set in $\text{Hol}(\Omega)$ as a countable intersection of dense $G_\delta$ sets in a complete metric space.

From now on, and throughout the remainder of this section, consider an $\Omega \subset \mathbb{C}$ which is non-empty, open and simply connected. Fix $\zeta_0 \in \Omega$ and, for $f \in \text{Hol}(\Omega)$ define

$$T(f)(z) = \int_{\gamma_z} f(\xi)d\xi,$$

for all $z \in \Omega$

$$T^{(k)}(f)(z) = \int_{\gamma_z} T^{(k-1)}(f)(\xi)d\xi,$$

for all $z \in \Omega, k \geq 2$

where $\gamma_z$ is any polygonal line in $\Omega$ that starts at $\zeta_0$ and ends at $z$. Since $\Omega$ is assumed to be simply connected, each $T^{(k)}$ is well-defined and holomorphic in $\Omega$ and its $k$-derivative is $f$.

**Proposition 3.4.** The operator

$$T : \text{Hol}(\Omega) \rightarrow \text{Hol}(\Omega)$$

$$f \mapsto T(f)$$

is linear and continuous on $\text{Hol}(\Omega)$.

**Proof.** The linearity of $T$ is obvious from the linearity of the integral. For the continuity, take a sequence $\{f_n\}$ in $\text{Hol}(\Omega)$ and a function $f$ on $\Omega$ such that $f_n \rightarrow f$ uniformly on compact subsets of $\Omega$. By the Weierstrass theorem we have that $f \in \text{Hol}(\Omega)$. We must show that $T(f_n) \rightarrow T(f)$ on compact subsets of $\Omega$.

Let $K$ be a compact subset of $\Omega$. Either $\Omega = \mathbb{C}$ or $\Omega \neq \mathbb{C}$.

In the first case, i.e. $\Omega = \mathbb{C}$, for $z \in K$ we take $\gamma_z$ to be the line segment $[\zeta_0, z]$. Set $M = \max\{|\zeta_0|, \max_{z \in K}|z|\}$ and observe that $M$ is well defined and finite because $K$ is compact in $\mathbb{C}$. Define $L = \overline{D(0, M)} = \{z \in \mathbb{C} : |z| \leq M\}$. Then $L$ is compact in $\mathbb{C}$, $K \subset L$ and $\gamma_z \subset L$, for all $z \in K$. Therefore, for $z \in K$ we have

$$|T(f_n)(z) - T(f)(z)| = \left| \int_{\gamma_z} f_n(\xi)d\xi - \int_{\gamma_z} f(\xi)d\xi \right|$$

$$= \left| \int_{\gamma_z} (f_n(\xi) - f(\xi))d\xi \right|$$

$$\leq \|f_n - f\|_L |z - \zeta_0|$$

$$\leq 2M\|f_n - f\|_L$$

Thus $\|T(f_n) - T(f)\|_K \leq 2M\|f_n - f\|_L \rightarrow 0$, $n \rightarrow \infty$.

In the second case, i.e. $\Omega \neq \mathbb{C}$, since $\Omega$ is a simply connected domain, by the Riemann Mapping Theorem there exists an analytic function $\phi : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{C}$ such that $\phi$ is univalent and $\phi(\mathbb{D}) = \Omega$. Obviously $\phi$ is a homeomorphism between $\mathbb{D}$ and
Since the set \( \{z_0\} \cup K \subset \Omega \) is compact, the set \( \phi^{-1}(\{z_0\} \cup K) \subset \mathbb{D} \) is also compact. Therefore, there exists an \( r \), with \( 0 < r < 1 \), such that \( \phi^{-1}(\{z_0\} \cup K) \subset \overline{D(0,r)} = \{z \in \mathbb{C} : |z| \leq r\} \). Define \( L = \phi(\overline{D(0,r)}) \subset \phi(\mathbb{D}) = \Omega \). Then \( L \) is compact and \( K \subset L \). For \( z \in K \) we have that \( \phi^{-1}(z) \in \overline{D(0,r)} \), hence the line segment \( [\phi^{-1}(z), \phi^{-1}(z)] \subset \overline{D(0,r)} \). Therefore, if \( \sigma : [0,1] \to \mathbb{C} \) is a parametrization of \( [\phi^{-1}(z), \phi^{-1}(z)] \), then \( \text{Length}(\sigma) \leq 2r \). Take \( \gamma_z = \phi([\phi^{-1}(z), \phi^{-1}(z)]) \subset \phi(\overline{D(0,r)}) = L \) and observe that \( \gamma_z \) is rectifiable: \( \phi \circ \sigma : [0,1] \to \Omega \) is a parametrization of \( \gamma_z \) and

\[
\text{Length}(\gamma_z) = \int_0^1 |\gamma_z'(t)| \, dt
\]

\[
= \int_0^1 |(\phi \circ \sigma)'(t)| \, dt
\]

\[
= \int_0^1 |(\phi'(\sigma(t)))| |\sigma'(t)| \, dt
\]

\[
\leq \max \{ |\phi'(z)| : z \in \overline{D(0,r)} \} \text{Length}(\sigma)
\]

\[
\leq \max \{ |\phi'(z)| : z \in \overline{D(0,r)} \} \cdot 2r
\]

which is of course finite because \( \phi' \) is continuous on the compact set \( \overline{D(0,r)} \).

We then have

\[
|T(f_n)(z) - T(f)(z)| = | \int_{\gamma_z} f_n(\xi) \, d\xi - \int_{\gamma_z} f(\xi) \, d\xi |
\]

\[
= | \int_{\gamma_z} (f_n(\xi) - f(\xi)) \, d\xi |
\]

\[
\leq \| f_n - f \|_L \cdot \text{Length}(\gamma_z)
\]

\[
\leq \| f_n - f \|_L \cdot \max \{ |\phi'(z)| : z \in \overline{D(0,1)} \} \cdot 2r
\]

Thus \( \|T(f_n) - T(f)\|_K \leq \| f_n - f \|_L \cdot \max \{ |\phi'(z)| : z \in \overline{D(0,1)} \} \cdot 2r \to 0 \), \( n \to \infty \).

In any case we have shown that \( T(f_n) \to T(f) \) uniformly on \( K \). Since \( K \) was an arbitrary compact subset of \( \Omega \), the continuity of \( T \) follows.

**Corollary 3.5.** Let \( k \geq 1 \). The operator

\[
T^{(k)} : \mathcal{Hol}(\Omega) \to \mathcal{Hol}(\Omega)
\]

\[
f \mapsto T^{(k)}(f)
\]

is linear and continuous on \( \mathcal{Hol}(\Omega) \).

**Proof.** We have that \( T^{(k)} = T \circ T \circ ... \circ T \), the composition of \( T \) \( k \) times. Therefore linearity and continuity both follow by Proposition 3.4.

**Corollary 3.6.** If \( f_n \to f \) uniformly on compact subsets of \( \Omega \) and \( k \geq 1 \), then \( T^{(k)}(f_n) \to T^{(k)}(f) \) pointwise in \( \Omega \).
Proof. By the Weierstrass Theorem, \( f \in \mathcal{H}ol(\Omega) \). By Corollary (3.5) we have that \( T^{(k)}(f_n) \rightarrow T^{(k)}(f) \) uniformly on compact subsets of \( \Omega \) and therefore \( T^{(k)}(f_n) \rightarrow T^{(k)}(f) \) pointwise in \( \Omega \).

Proposition 3.7. Let \( \Omega \subset \mathbb{C} \) be a simply connected domain and \( k \geq 1 \). The set \( B_k \) of all \( f \in \mathcal{H}ol(\Omega) \) such that \( T^{(k)}(f) \) is bounded on \( \Omega \) is a set of the first category in \( \mathcal{H}ol(\Omega) \).

Proof. For \( m \in \mathbb{N} \), define

\[
B_m = \{ f \in \mathcal{H}ol(\Omega) : |T^{(k)}(f)(z)| \leq m \text{ for all } z \in \Omega \}
\]

Then \( B_k = \bigcup_{m=1}^{\infty} B_m \). We will show that each \( B_m \) is closed and nowhere dense in \( \mathcal{H}ol(\Omega) \).

To see that it is closed, take a sequence \( \{f_n\} \) in \( B_m \) such that \( f_n \rightarrow f \) uniformly on compact subsets of \( \Omega \). By Corollary (3.7), \( T^{(k)}(f_n) \rightarrow T^{(k)}(f) \) pointwise in \( \Omega \). Therefore, for \( z \in \Omega \) we have that

\[
|T^{(k)}(f)(z)| \leq |T^{(k)}(f)(z) - T^{(k)}(f_n)(z)| + |T^{(k)}(f_n)(z)| + m
\]

Taking \( n \rightarrow \infty \) we obtain \( |T^{(k)}(f)(z)| \leq m \) and therefore \( f \in B_m \). Thus, \( B_m \) is closed.

To see that \( B_k^0 = \emptyset \), first observe that there exists a function \( g \in \mathcal{H}ol(\Omega) \) such that \( T^{(k)}(g) \) is unbounded on \( \Omega \); indeed, if \( \Omega \) is unbounded take \( g(z) = 1, z \in \Omega \), and if \( \Omega \) is bounded take \( \zeta_0 \in \partial \Omega \) and \( g(z) = \frac{1}{(z - \zeta_0)^{k+1}} \). Now assume that \( f \in B_m^0 \). Then \( f + \frac{1}{m} g \rightarrow f \) uniformly on compact subsets of \( \Omega \), \( n \rightarrow \infty \). Therefore, there exists an \( n_0 \) such that \( f + \frac{1}{n_0} g \in B_m \). By the linearity of \( f \mapsto T^{(k)}(f) \) this means that

\[
|T^{(k)}(f)(z) + \frac{1}{n_0} T^{(k)}(g)(z)| = |T^{(k)}(f + \frac{1}{n_0} g)(z)| \leq m
\]

for all \( z \in \Omega \). But then

\[
\frac{1}{n_0} |T^{(k)}(g)(z)| \leq |T^{(k)}(f)(z) + \frac{1}{n_0} T^{(k)}(g)(z)| + |T^{(k)}(f)(z)|
\]

\[
\leq m + m
\]

or \( |T^{(k)}(g)(z)| \leq 2mn_0 \), for all \( z \in \Omega \), which is contradictory to the fact that \( T^{(k)}(g) \) is unbounded on \( \Omega \). Thus, \( B_m^0 = \emptyset \) and the proof is complete.

For \( f \in \mathcal{H}ol(\Omega) \), where \( \Omega \subset \mathbb{C} \) is a simply connected domain, we denote

\[
f^{(k)} = \begin{cases} 
\text{the } k^{th} \text{ derivative of } f, & \text{if } k > 0 \\
f, & \text{if } k = 0 \\
T^{(-k)}(f), & \text{if } k < 0
\end{cases}
\]

where \( T^{(k)}(f) \) as defined above. Collecting all the above results together we get
Theorem 3.8. Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then the set $\mathcal{D}$ of all functions $f \in \text{Hol}(\Omega)$ with the property that $f^{(k)}$ is unbounded on $\Omega$ for all $k \in \mathbb{Z}$ is a dense $G_\delta$ subset of $\text{Hol}(\Omega)$.

Proof. For $k \in \mathbb{Z}$ define

$$D_k = \{ f \in \text{Hol}(\Omega) : f^{(k)} \text{ unbounded on } \Omega \}$$

Then $\mathcal{D} = \bigcap_{k \in \mathbb{Z}} D_k$. By Propositions (3.1), (3.2) and (3.7) we have that each $D_k$ is a dense $G_\delta$ set in $\text{Hol}(\Omega)$, because its complement is a countable union of closed, nowhere dense sets in $\text{Hol}(\Omega)$. Since $\text{Hol}(\Omega)$ is a complete metric space, Baire’s Theorem gives that any countable intersection of dense $G_\delta$ sets is again a dense $G_\delta$ set.

At this point observe that Proposition (3.3) and Theorem (3.8) are immediate corollaries to Proposition (2.2):

The operator

$$\Lambda : \text{Hol}(\Omega) \to \text{Hol}(\Omega)$$

$$f \mapsto f'$$

is linear and continuous by the Weierstrass Theorem. If additionally $\Omega$ is simply connected, the same holds for the operator

$$\tilde{\Lambda} : \text{Hol}(\Omega) \to \text{Hol}(\Omega)$$

$$f \mapsto \int_{\gamma_0} f(\xi)d\xi$$

by Proposition (3.4), the primitive of $f$ being defined as in the discussion preceding that same Proposition.

Now define $\Lambda_k$ to be $k$ compositions of $\Lambda$ with itself, $k \geq 1$, $\Lambda_0$ to be the identity function on $\text{Hol}(\Omega)$ and $\Lambda_k$ to be $(-k)$ compositions of $\tilde{\Lambda}$ with itself, $k \leq -1$. Then each $\Lambda_k$ is linear and continuous in $\text{Hol}(\Omega)$ and, furthermore, $\mathcal{U}_{\Lambda_k} \neq \emptyset$, for all $k \in \mathbb{Z}$. Therefore, the set $\bigcap_{k \in \mathbb{Z}} \mathcal{U}_{\Lambda_k}$ is a dense $G_\delta$ subset of $\text{Hol}(\Omega)$. But this is exactly the set $\mathcal{D}$ of Theorem (3.8).

4 Universality of operators related to the partial sums

Now assume that $\Omega$ is a domain in $\mathbb{C}$. For $N \geq 0$ we define:

$$S_N : \text{Hol}(\Omega) \to \text{Hol}(\Omega \times \mathbb{C})$$

$$f \mapsto S_N(f, \cdot)(\cdot) = S_N(f)$$

where

$$S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!} (z - \zeta)^n, \ z \in \Omega, \ z \in \mathbb{C}$$
Then $S_N$ is obviously linear. By the Weierstrass Theorem it is also continuous; indeed suppose $K = K_1 \times K_2$ is a compact subset of $\Omega \times \mathbb{C}$, where $K_1, K_2$ are compact subsets of $\Omega$ and $\mathbb{C}$ respectively, and $f_k \rightarrow f$ uniformly on compact subsets of $\Omega$. Set $M = \max_{(\zeta, z) \in K} |z - \zeta|$. Then, for $(\zeta, z) \in K$ we have that

$$|S_N(f_k, \zeta)(z) - S_N(f, \zeta)(z)| = \left| \sum_{n=0}^{N} \frac{f_k^{(n)}(\zeta) - f^{(n)}(\zeta)}{n!} (z - \zeta)^n \right|$$

$$\leq \sum_{n=0}^{N} \frac{|f_k^{(n)}(\zeta) - f^{(n)}(\zeta)|}{n!} |z - \zeta|^n$$

$$\leq \sum_{n=0}^{N} \ould{f_k^{(n)} - f^{(n)}}_{K_1} M^n$$

which means that

$$\|S_N(f_k) - S_N(f)\|_K \leq \sum_{n=0}^{N} \ould{f_k^{(n)} - f^{(n)}}_{K_1} M^n$$

and therefore $S_N(f_k) \rightarrow S_N(f)$ uniformly on $K$, for each $N = 0, 1, 2, \ldots$

Now fix $\zeta_0 \in \Omega$ and, for $N \geq 0$, define

$$T_N : \mathcal{H}ol(\Omega) \rightarrow \mathcal{H}ol(\mathbb{C})$$

$$f \mapsto S_N(f, \zeta_0)(\cdot)$$

Then each $T_N$ is linear and continuous in $\mathcal{H}ol(\Omega)$ and

$$\mathcal{U}_{T_N} = \{ f \in \mathcal{H}ol(\Omega) : S_N(f, \zeta_0) \text{ is unbounded in } \mathbb{C} \}$$

But $S_N(f, \zeta_0)$ is a polynomial, so it is bounded in $\mathbb{C}$ if and only if it is constant in $\mathbb{C}$. Therefore

$$\mathcal{U}_{T_N} = \{ f \in \mathcal{H}ol(\Omega) : S_N(f, \zeta_0) \text{ is non-constant in } \mathbb{C} \}$$

For $N = 0$ we have that $S_N(f, \zeta_0)(z) = f(\zeta_0)$, $z \in \mathbb{C}$, so $\mathcal{U}_{T_N} = \emptyset$.

for $N \geq 1$, we have that

$$S_N(f, \zeta_0)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta_0)}{n!} (z - \zeta_0)^n$$

is constant if and only if $f'(\zeta_0) = f''(\zeta_0) = \ldots = f^{(N)}(\zeta_0) = 0$. But there always exists a function $f \in \mathcal{H}ol(\Omega)$ such that $f^{(k)}(\zeta_0) \neq 0$, for all $k \in \mathbb{N}$, for example $f(z) = e^z$.

Therefore, $\mathcal{U}_{T_N} \neq \emptyset$, for all $N \geq 1$. By Proposition (2.2) we have that the set $\bigcap_{N=1}^{\infty} \mathcal{U}_{T_N}$ of all the functions $f \in \mathcal{H}ol(\Omega)$ with the property that the function $S_N(f, \zeta_0)$ is unbounded
in \( \mathbb{C} \) for all \( N \geq 1 \), is a dense \( G_\delta \) set in \( \text{Hol}(\Omega) \).

We mention that \( \mathcal{U}_{T_1} \) is an open dense set in \( \text{Hol}(\Omega) \) because \( \mathcal{U}_{T_1} = \{ f \in \text{Hol}(\Omega) : f'(\zeta_0) \neq 0 \} \). Similarly, \( \mathcal{U}_{T_N} \) is also an open dense set in \( \text{Hol}(\Omega) \), so \( \bigcap_{N=1}^{\infty} \mathcal{U}_{T_N} \) is \( G_\delta \) dense in \( \text{Hol}(\Omega) \). So this corollary of Proposition (2.2) is well known and obvious. A similar result holds if we replace \( \mathbb{C} \) by any unbounded domain \( \Omega \); in particular this holds for \( \Omega_2 = \Omega \) if \( \Omega \) is unbounded.

Now fix \( z = 0 \) and, for \( N \geq 0 \), define

\[
\tilde{T}_N : \text{Hol}(\Omega) \to \text{Hol}(\Omega)
\]

\[
f \mapsto S_N(f, \cdot)(0)
\]

Each \( \tilde{T}_N \) is linear and continuous in \( \text{Hol}(\Omega) \).

For \( N = 0 \), we have that \( S_0(f, \zeta)(0) = f(\zeta), \zeta \in \Omega \), and therefore

\[
\mathcal{U}_{\tilde{T}_N} = \{ f \in \text{Hol}(\Omega) : f \text{ is unbounded in } \Omega \}
\]

which is a dense \( G_\delta \) set in \( \text{Hol}(\Omega) \) by Proposition (3.1).

For \( N \geq 1 \), if \( \Omega = \mathbb{C} \), take \( f(z) = e^z, z \in \mathbb{C} \). Since \( z \mapsto e^z \) dominates the polynomials in \( \mathbb{C} \), we have that \( S_N(f, \zeta)(0) \) is unbounded in \( \mathbb{C} \). If \( \Omega \neq \mathbb{C} \), take \( \zeta_0 \in \partial \Omega \) and \( f(z) = \frac{1}{z - \zeta_0}, z \in \Omega \). Then \( f \in \text{Hol}(\Omega) \) and

\[
S_N(f, \zeta)(0) = \sum_{n=0}^{N} \frac{\zeta^n}{(\zeta - \zeta_0)^{n+1}}, \zeta \in \Omega
\]

which is a rational function with poles only at \( z = \zeta_0 \). Hence \( \lim_{\zeta \to \zeta_0} |S_N(f, \zeta)(0)| = \infty \) and \( S_N(f, \cdot)(0) \) is unbounded in \( \Omega \).

Therefore, \( \mathcal{U}_{\tilde{T}_N} \neq \emptyset \) for all \( N \geq 0 \), so by Corollary (2.2) we have that the set \( \bigcap_{N=0}^{\infty} \mathcal{U}_{\tilde{T}_N} \) of all functions \( f \in \text{Hol}(\Omega) \) with the property that \( S_N(f, \cdot)(0) \) is unbounded in \( \Omega \) for all \( N \geq 0 \), is a dense \( G_\delta \) set in \( \text{Hol}(\Omega) \).

Next we consider the following class \( \mathcal{S}(\Omega) \) of functions on \( \Omega \):

**Definition 4.1.** Let \( \Omega \) be an open, non-empty subset of \( \mathbb{C} \). We define \( \mathcal{S}(\Omega) \) to be the set of all functions \( f \in \text{Hol}(\Omega) \) such that \( \{ \tilde{T}_N(f) \}_{N \geq 0} \) is dense in \( \text{Hol}(\Omega) \).

From now on and unless otherwise stated we assume that \( \Omega \) is a simply connected domain in \( \mathbb{C} \). Our goal is to show that either \( \mathcal{S}(\Omega) = \emptyset \) or \( \mathcal{S}(\Omega) \) is a dense \( G_\delta \) set in \( \text{Hol}(\Omega) \). To this end, first observe that, \( \text{Hol}(\Omega) \) is separable: the set \( \{ p_j \} \) of all polynomials with coefficients having rational coordinates is dense in \( \text{Hol}(\Omega) \) by the Runge Theorem. Now consider an exhaustive sequence \( \{ K_m \} \) of compact subsets of \( \Omega \), i.e. a sequence \( \{ K_m \} \) of compact subsets of \( \Omega \) such that

1. \( \Omega = \bigcup_{m=1}^{\infty} K_m \)
2. $K_m$ lies in the interior of $K_{m+1}$, for $m = 1, 2, ...$

3. Every compact subset of $\Omega$ lies in some $K_m$

4. Every component of $K_m^c$ contains a component of $\Omega^c$, $m = 1, 2, ...$

(See [5]) Now we can show that $S(\Omega)$ can be expressed as a set which will be shown to be a $G_\delta$ one in $Hol(\Omega)$:

**Proposition 4.2.** $S(\Omega) = \bigcap_{s, j, m = 1}^{\infty} \bigcup_{N = 0}^{\infty} \{ f \in Hol(\Omega) : \sup_{\zeta \in K_m} |\tilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{s} \}$

**Proof.** That $S(\Omega)$ is a subset of the set on the right is an immediate consequence of the definition of $S(\Omega)$.

Consider now a function $f$ in the set on the right, a function $g \in Hol(\Omega)$, a compact subset $K$ of $\Omega$ and an $\epsilon > 0$. There exists an $m \geq 1$ such that $K \subset K_m$ and an $s \geq 1$ such that $\frac{1}{s} < \epsilon$. For these $g$, $K_m$ and $s$, there exists a $j \geq 1$ such that

$$\sup_{\zeta \in K} |p_j(\zeta) - g(\zeta)| \leq \sup_{\zeta \in K_m} |p_j(\zeta) - g(\zeta)| < \frac{1}{2s}$$

For these $K_m$, $s$ and $j$, there exists an $N \geq 0$ such that

$$\sup_{\zeta \in K} |\tilde{T}_N(f)(\zeta) - p_j(\zeta)| \leq \sup_{\zeta \in K_m} |\tilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{2s}$$

By the triangle inequality, for $z \in K$, we have

$$|\tilde{T}_N(f)(z) - g(z)| \leq |\tilde{T}_N(f)(z) - p_j(z)| + |p_j(z) - g(z)|$$

$$\leq \sup_{\zeta \in K} |\tilde{T}_N(f)(\zeta) - p_j(\zeta)| + \sup_{\zeta \in K} |p_j(\zeta) - g(\zeta)|$$

$$< \frac{1}{2s} + \frac{1}{2s}$$

Therefore, $\sup_{\zeta \in K} |\tilde{T}_N(f)(\zeta) - g(\zeta)| \leq \frac{1}{s} < \epsilon$, so $\{\tilde{T}_N(f)\}$ is dense in $Hol(\Omega)$. 

**Proposition 4.3.** $S(\Omega)$ is a $G_\delta$ set in $Hol(\Omega)$.

**Proof.** By Proposition 4.2, it suffices to show that, for $j, s, m \geq 1$ and $N \geq 0$, the set

$$E_{j,s,m,N} := \{ f \in Hol(\Omega) : \sup_{\zeta \in K_m} |\tilde{T}_N(f)(\zeta) - p_j(\zeta)| < \frac{1}{s} \}$$

is open in $Hol(\Omega)$.

To this end, consider functions $g_k \in Hol(\Omega)$, $k \geq 1$, and $g \in E_{j,s,m,N}$ such that $g_k \rightarrow g$ uniformly on compact subsets of $\Omega$. It suffices to find a $k_0$ such that $g_k \in E_{j,s,m,N}$, for all $k \geq k_0$. Since $g \in E_{j,s,m,N}$, there exists a $\delta > 0$ such that

$$\sup_{\zeta \in K_m} |\tilde{T}_N(g)(\zeta) - p_j(\zeta)| < \frac{1}{s} - 2\delta$$
Set $M = \max \{ e^{\left| z \right|} : \zeta \in K_m \}$. By the Weierstrass Theorem we have that $g_k^{(i)} \rightarrow g^{(i)}$ uniformly on compact subsets of $\Omega$, $i = 0, 1, ..., N$, so there exists a $k_0 \in \mathbb{N}$ such that

$$\|g_k^{(i)} - g^{(i)}\|_{K_m} < \frac{\delta}{M}$$

for all $i = 0, 1, \ldots, N$. Therefore, for $z \in K_m$ and $k \geq k_0$ we have

$$|\tilde{T}_N(g_k)(z) - p_j(z)| \leq |\tilde{T}_N(g_k)(z) - \tilde{T}_N(g)(z)| + |\tilde{T}_N(g)(z) - p_j(z)|$$

$$= \left| \sum_{n=0}^{N} \frac{g^{(n)}_k(z) - g^{(n)}(z)}{n!} (-z^n) \right| + |\tilde{T}_N(g)(z) - p_j(z)|$$

$$\leq \sum_{n=0}^{N} \frac{|g^{(n)}_k(z) - g^{(n)}(z)|}{n!} |z|^n + \sup_{\zeta \in K_m} |\tilde{T}_N(g)(\zeta) - p_j(\zeta)|$$

$$< \sum_{n=0}^{N} \frac{|g^{(n)}_k(z) - g^{(n)}(z)|}{n!} |K_m| |z|^n + \frac{1}{s} - 2\delta$$

$$< \frac{\delta}{M} \sum_{n=0}^{N} \frac{|z|^n}{n!} + \frac{1}{s} - 2\delta$$

$$\leq \frac{\delta}{M} \sum_{n=0}^{\infty} \frac{|z|^n}{n!} + \frac{1}{s} - 2\delta$$

$$= \frac{\delta}{M} (|z| + \frac{1}{s}) - 2\delta$$

$$\leq \frac{\delta}{M} M + \frac{1}{s} - 2\delta$$

$$= \frac{1}{s} - \delta$$

Since the $z \in K_m$ was arbitrary, we have that

$$\sup_{\zeta \in K_m} |\tilde{T}_N(g_k)(\zeta) - p_j(\zeta)| \leq \frac{1}{s} - \delta < \frac{1}{s}$$

for all $k \geq k_0$. Hence $g_k \in E_{j,s,m,N}$, $k \geq k_0$. This completes the proof. \qed

**Proposition 4.4.** Let $\Omega$ be a simply connected domain in $\mathbb{C}$. Either $S(\Omega) = \emptyset$ or $S(\Omega)$ is a dense $G_\delta$ set in $\text{Hol}(\Omega)$.

**Proof.** If $S(\Omega) \neq \emptyset$, by Proposition [4.3] it suffices to show that $S(\Omega)$ is dense in $\text{Hol}(\Omega)$.

Let $f \in S(\Omega)$. Observe that, if $p$ is a polynomial, then $f + p \in S(\Omega)$. Indeed, $f + p \in \text{Hol}(\Omega)$ and, for all $N > \deg p$, we have that $\tilde{T}_N(f + p) = \tilde{T}_N(f) + q_p$, where

$$q_p(\zeta) = \sum_{n=0}^{N} \frac{(-1)^n p^{(n)}(\zeta)}{n!} \zeta^n, \quad \zeta \in \Omega$$

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is again a polynomial. For a function $g \in \text{Hol}(\Omega)$, we have that $g - q_p \in \text{Hol}(\Omega)$, and therefore there exists a sequence $\{\lambda_n\}$ in $\mathbb{N}$ such that $\tilde{T}_{\lambda_n}(f) \to g - q_p$ uniformly on compact subsets of $\Omega$. But then $\tilde{T}_{\lambda_n}(f + p) = \tilde{T}_{\lambda_n}(f) + q_p \to g$ uniformly on compact subsets of $\Omega$, i.e. $\{\tilde{T}_N(f + p)\}$ is dense in $\text{Hol}(\Omega)$ and $f + p \in S(\Omega)$.

Now the density of $S(\Omega)$ in $\text{Hol}(\Omega)$ follows easily because by Runge’s Theorem the polynomials are dense in $\text{Hol}(\Omega)$.

At this point observe that, if $0 \in \Omega$, then $S(\Omega) = \emptyset$. Indeed, for $f, g \in \text{Hol}(\Omega)$ such that $f(0) \neq g(0)$, we have that, for any $N \in \mathbb{N}$ and any compact subset $L$ of $\Omega$ such that $0 \in L$,

$$\sup_{\zeta \in L} |\tilde{T}_N(f)(\zeta) - g(\zeta)| \geq |\tilde{T}_N(f)(0) - g(0)| = |f(0) - g(0)| > 0$$

so there is no subsequence of $\{\tilde{T}_N(f)\}$ that converges to $g$ uniformly on compact subsets of $\Omega$.

Definition 4.5. Let $\Omega$ be open in $\mathbb{C}$. The set $S_t(\Omega)$ is the set of all $f \in \text{Hol}(\Omega)$ with the property that, for every $c \in \mathbb{C}$ there exists a sequence $\{\lambda_n\}$ in $\mathbb{N}$ such that, for every $L \subset \Omega$ compact,

$$\sup_{\zeta \in L} |\tilde{T}_{\lambda_n}(f)(\zeta) - c| \to 0, \quad n \to \infty$$

Proposition 4.6. The set $S_t(\Omega)$ is a $G_\delta$ set in $\text{Hol}(\Omega)$.

Proof. Let $\{z_j\}_{j \in \mathbb{N}}$ be an enumeration of the points in the complex plane with rational coordinates. Following the proof of Propositions (4.2) and (4.3), we get that

$$S_t(\Omega) = \bigcap_{s,j,m=1}^{\infty} \bigcup_{N=0}^{\infty} \{f \in \text{Hol}(\Omega) : \sup_{\zeta \in K_m} |\tilde{T}_N(f)(\zeta) - z_j| < \frac{1}{s}\}$$

and that the set

$$\{f \in \text{Hol}(\Omega) : \sup_{\zeta \in K_m} |\tilde{T}_N(f)(\zeta) - z_j| < \frac{1}{s}\}$$

is open in $\text{Hol}(\Omega)$, $m, j, s \geq 1$, $N \geq 0$. \hfill $\Box$

Observe again that, if $0 \in \Omega$, then $S_t(\Omega) = \emptyset$. Indeed, for $f \in \text{Hol}(\Omega)$, $c \in \mathbb{C}$ with $f(0) \neq c$ and $L \subset \Omega$ compact, we have that

$$\sup_{\zeta \in L} |\tilde{T}_N(f)(\zeta) - c| \geq |\tilde{T}_N(f)(0) - c| = |f(0) - c| > 0$$

for all $N \in \mathbb{N}$. However, we can show that $S_t(\Omega)$ is dense in $\text{Hol}(\Omega)$ if $\Omega$ is a simply connected domain and $0 \notin \Omega$:

Theorem 4.7. Let $\Omega$ be a simply connected domain with $0 \notin \Omega$. Then $S_t(\Omega)$ contains a dense $G_\delta$ set in $\text{Hol}(\Omega)$.  

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Proof. Since Ω is a simply connected domain, the class \( U(Ω) \) is a dense \( G_δ \) set in \( \mathcal{H}o\mathcal{l}(Ω) \). We will show that \( U(Ω) \subset S_t(Ω) \).

Let \( f \in U(Ω) \) and \( c \in \mathbb{C} \). Take \( K = \{0\} \), which is disjoint from \( Ω \) because \( 0 \notin Ω \). Then \( K \) is a compact set in \( \mathbb{C} \), \( K \cap Ω = \emptyset \), \( K^c \) is connected, and the function \( h(z) = c \), \( z \in K \), is continuous on \( K \) and (trivially) analytic in the interior of \( K \). By definition of the class \( U(Ω) \), there exists a sequence \( \{λ_n\} \in \mathbb{N} \) such that, for every compact set \( L \subset Ω \),

\[
\sup_{ζ \in L} \sup_{z \in K} |S_{λ_n}(f, ζ)(z) - h(z)| \to 0, \quad n \to \infty
\]

or

\[
\sup_{ζ \in L} |S_{λ_n}(f, ζ)(0) - c| \to 0, \quad n \to \infty
\]

But this is exactly

\[
\sup_{ζ \in L} |\tilde{T}_{λ_n}(f)(ζ) - c| \to 0, \quad n \to \infty
\]

Therefore, \( f \in S_t(Ω) \). This completes the proof. \( \Box \)

5 A more general statement

During a seminar on these topics, T. Hatziafratis posed the following question: Let \( E \) be a countable dense subset of \( T = \{z \in \mathbb{C} : |z| = 1\} \). Is it true that, for the generic function \( f \in \mathcal{H}o\mathcal{l}(D) \), all the derivatives and anti-derivatives of \( f \) are unbounded on every radius joining 0 to a point of \( E \)?

The answer to this question is affirmative. To see this, we examine a more general case:

Proposition 5.1. Let \( Ω \subset \mathbb{C} \) be an open set, \( X \) a non-empty subset of \( Ω \).
If \( T: \mathcal{H}o\mathcal{l}(Ω) \to \mathcal{H}o\mathcal{l}(Ω) \) is a linear operator with the property that, for every \( z \in Ω \), the mapping \( \mathcal{H}o\mathcal{l}(Ω) \ni f \mapsto T(f)(z) \in \mathbb{C} \) is continuous, and

\[
S = S(T, Ω, X) = \{f \in \mathcal{H}o\mathcal{l}(Ω) : T(f) \text{ is unbounded on } X\},
\]

then either \( S = \emptyset \) or \( S \) is a dense \( G_δ \) set in \( \mathcal{H}o\mathcal{l}(Ω) \).

Proof. To show that \( S \) is a \( G_δ \) set, for \( m \geq 1 \), define

\[
S_m = \{f \in \mathcal{H}o\mathcal{l}(Ω) : \exists z \in X \text{ such that } |T(f)(z)| > m\}
\]

Then \( S = \bigcap_{m=1}^{∞} S_m \). Since the mapping \( f \mapsto T(f)(z) \) is continuous, the set \( S_m \) is open in \( \mathcal{H}o\mathcal{l}(Ω) \), for each \( m \geq 1 \). Hence, \( S \) is a \( G_δ \) set in \( \mathcal{H}o\mathcal{l}(Ω) \).

To show that \( S \) is dense in \( \mathcal{H}o\mathcal{l}(Ω) \) if it is not empty, let \( g \in S \), i.e. \( g \in \mathcal{H}o\mathcal{l}(Ω) \) and \( T(g) \) is unbounded on \( X \), and let \( f \in \mathcal{H}o\mathcal{l}(Ω) \). If \( T(f) \) is unbounded on \( X \), then \( f \in S \) and \( f \) is (trivially) the limit in \( \mathcal{H}o\mathcal{l}(Ω) \) of a sequence of functions in \( S \). If \( T(f) \) is bounded on \( X \) by, say, \( M_1 \), then, for a fixed \( n \geq 1 \), the function \( T(f + \frac{1}{n} g) \) is unbounded.
on $X$. Indeed, suppose it is bounded on $X$ by a positive number $M_2$. Then, if $z \in X$, by the linearity of $T$ we would have

$$|T(g)(z)| = n|T\left(\frac{1}{n}g\right)(z)|$$

$$= n|T(f + \frac{1}{n}g)(z) - T(f)(z)|$$

$$\leq n|T(f + \frac{1}{n}g)(z)| + n|T(f)(z)|$$

$$\leq n M_2 + n M_1$$

But this means that $T(g)$ is bounded on $X$ by $n(M_1 + M_2)$, which is contradictory to the fact that $T(g)$ is unbounded on $X$. Therefore, $T(f + \frac{1}{n}g)$ is unbounded on $X$ for every $n \geq 1$; in other words $f + \frac{k}{n}g \in S$, for every $n \geq 1$. But $f + \frac{k}{n}g \to f, n \to \infty$, uniformly on compact subsets of $\Omega$, so $f$ is again the limit in $\mathcal{H}ol(\Omega)$ of a sequence of functions in $S$. Since $f$ was an arbitrary function in $\mathcal{H}ol(\Omega)$, $S$ is dense in $\mathcal{H}ol(\Omega)$ and the proof is complete. \hfill \Box

Consider now countable $T^{(k)}$ and $X_m$ such that $S(T^{(k)}, \Omega, X_m) \neq \emptyset$, for all $k, m$. Then Baire’s Theorem gives that $\bigcap_{k,m} S(T^{(k)}, \Omega, X_m)$ is a dense $G_{\delta}$ set in $\mathcal{H}ol(\Omega)$. This answers the aforementioned question in the affirmative, because if $\zeta_m \in E$ and $X_m$ is the radius joining 0 to $\zeta_m$, then the function $g(z) = \frac{1}{z - \zeta_m}$, $z \in \mathbb{D}$, belongs to $S(T^{(k)}, \mathbb{D}, X_m)$ for all $k \geq 0$, where $T$ is the differentiation operator.

More generally, we can replace $\mathbb{D}$ with any open non-empty set $\Omega$ in $\mathbb{C}$, $T$ being the differentiation operator and $X_m \subset \Omega$ having at least one accumulation point in $\partial \Omega$. If $\Omega$ is simply connected, then we obtain the analogous result for both the integration operator and the operator related to Taylor partial sums $\tilde{T}_N$ that was defined before.

Observing that in the proof of Proposition (5.1) no properties of $\mathcal{H}ol(\Omega)$ were used other than those of a topological vector space, we can obtain the best generalization of our result, where completeness is not assumed and the proof does not use Baire’s Theorem:

**Proposition 5.2.** Let $\mathcal{V}$ be a topological vector space over the field $\mathbb{R}$ or $\mathbb{C}$ and $X$ a non-empty set. Denote by $F(X)$ the set of all complex-valued functions on $X$ and consider a linear operator $T : \mathcal{V} \to F(X)$ with the property that, for all $x \in X$, the mapping $\mathcal{V} \ni \alpha \mapsto T(\alpha)(x) \in \mathbb{C}$ is continuous. Let $S = \{\alpha \in \mathcal{V} : T(\alpha) \text{ is unbounded on } X\}$. Then either $S = \emptyset$ or $S$ is a dense $G_{\delta}$ set in $\mathcal{V}$.

**Proof.** That $S$ is a $G_{\delta}$ set follows from the fact that $S = \bigcap_{m=1}^{\infty} \bigcup_{x \in X} \{\alpha \in \mathcal{V} : |T(\alpha)(x)| > m\}$ and the continuity of $\alpha \mapsto T(\alpha)(x)$. The proof that $S$ is dense if it is non-empty is identical to the proof of Proposition (5.1). \hfill \Box

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