A CLASSIFICATION OF IRREDUCIBLE WAKIMOTO MODULES FOR THE AFFINE LIE ALGEBRA $A_1^{(1)}$

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ABSTRACT. By using methods developed in Adamović (Comm. Math. Phys. 270 (2007) 141-161) we study the irreducibility of certain Wakimoto modules for $\mathfrak{sl}_2$ at the critical level. We classify all $\chi \in \mathbb{C}((z))$ such that the corresponding Wakimoto module $W_{\chi}$ is irreducible. It turns out that zeros of Schur polynomials play important role in the classification result.

1. INTRODUCTION

In the representation theory of affine Kac-Moody Lie algebras, representations at the critical level belong to one of the most important cases. The Kac-Kazhdan conjecture for characters motivates explicit realizations of irreducible highest weight modules at the critical level. These representations can be realized by using Wakimoto modules (cf. [F], [FF1], [FF2], [FP], [S], [W]). In [A2] we introduced an infinite-dimensional Lie superalgebra $\mathcal{A}$ which is a certain limit of N=2 superconformal algebras obtained by using Kazama-Suzuki mappings (cf. [A1], [FST], [KS]). We also constructed a family of functors which send irreducible $\mathcal{A}$–modules to irreducible modules for the affine Lie algebra $A_1^{(1)}$ at the critical level. By using this construction we proved irreducibility of a large family of Wakimoto modules $W_{\chi}$ parameterized by $\chi \in \mathbb{C}((z))$. In this paper we shall completely solve the irreducibility problem for Wakimoto modules $W_{\chi}$. We shall describe all $\chi \in \mathbb{C}((z))$ such that $W_{\chi}$ is irreducible.

We first consider $\mathcal{A}$–modules $\widetilde{F}_\chi$ constructed by using representations of the infinite-dimensional Clifford algebra and also parameterized by $\chi \in \mathbb{C}((z))$. The functor $\mathcal{L}_0$ sends $\widetilde{F}_\chi$ to the Wakimoto module $W_{-\chi}$ (cf. [A2]). Then $W_{-\chi}$ is irreducible $A_1^{(1)}$–module if and only if $\widetilde{F}_\chi$ is irreducible $\mathcal{A}$–module (cf. Theorems 5.2 and 5.3). So we only need to classify $\chi \in \mathbb{C}((z))$ such that $\widetilde{F}_\chi$ is irreducible. By combining results from [A2] and results from the present paper, we obtain the following classification result.

Theorem 1.1. Assume that $\chi \in \mathbb{C}((z))$. Then the Wakimoto module $W_{-\chi}$ is an irreducible $A_1^{(1)}$–module (resp. $\widetilde{F}_\chi$ is irreducible $\mathcal{A}$–module) if and only if $\chi$ satisfies one of the following conditions:

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There is \( p \in \mathbb{Z}_{\geq 0}, p \geq 1 \) such that
\[
\chi(z) = \sum_{n=-p}^{\infty} \chi_n z^n \in \mathbb{C}(z) \quad \text{and} \quad \chi_p \neq 0.
\]

(ii) \[
\chi(z) = \sum_{n=0}^{\infty} \chi_n z^n \in \mathbb{C}(z) \quad \text{and} \quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}).
\]

(iii) There is \( \ell \in \mathbb{Z}_{>0} \) such that
\[
\chi(z) = \ell + 1 + \sum_{n=1}^{\infty} \chi_n z^n \in \mathbb{C}(z)
\]
and \( S_{\ell}(-\chi) \neq 0 \), where \( S_{\ell}(-\chi) = S_{\ell}(-\chi, -\chi, -\chi, \ldots) \) is a Schur polynomial.

We also prove that when the Wakimoto module \( W_{-\chi} \) is reducible, then it contains an irreducible submodule.

Although the methods used in this paper can be mainly applied for the affine Lie algebra \( A_1^{(1)} \), we believe that the main classification result can be extended for higher rank case. We hope to study this problem in our future publications.

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2. Clifford vertex superalgebras

The Clifford algebra \( CL \) is a complex associative algebra generated by \( \Psi^\pm(r), r \in \frac{1}{2} + \mathbb{Z} \), and relations
\[
\{\Psi^\pm(r), \Psi^\mp(s)\} = \delta_{r+s,0}; \quad \{\Psi^\pm(r), \Psi^\pm(s)\} = 0
\]
where \( r, s \in \frac{1}{2} + \mathbb{Z} \).

Let \( F \) be the irreducible \( CL \)–module generated by the cyclic vector \( 1 \) such that
\[
\Psi^\pm(r)1 = 0 \quad \text{for} \quad r > 0.
\]

As a vector space,
\[
F = \bigwedge (\Psi^-(n-\frac{1}{2}), n \in \mathbb{Z}_{\geq 0}) \otimes \bigwedge (\Psi^+(n-\frac{1}{2}), n \in \mathbb{Z}_{\geq 0})
\]
where \( \bigwedge (x_i, i \in I) \) denotes the exterior algebra with generators \( x_i, i \in I \).

Define the following fields on \( F \)
\[
\Psi^+(z) = \sum_{n \in \mathbb{Z}} \Psi^+(n + \frac{1}{2}) z^{-n-1}, \quad \Psi^-(z) = \sum_{n \in \mathbb{Z}} \Psi^-(n + \frac{1}{2}) z^{-n-1}.
\]
The fields $\Psi^+(z)$ and $\Psi^-(z)$ generate on $F$ the unique structure of a simple vertex superalgebra (cf. [K2], [FB]).

Define the following Virasoro vector in $F$:

$$\omega^{(f)} = \frac{1}{2} (\Psi^+ \Psi^- z^{-1}) + \Psi^+ \Psi^- z^{-1} 1.$$  

Then the components of the field $L^{(f)}(z) = Y(\omega^{(f)}, z) = \sum_{n \in \mathbb{Z}} L_n(z) z^{-n-1}$ defines on $F$ a representation of the Virasoro algebra with central charge 1.

Set

$$J^{(f)}(z) = Y(\Psi^+ \Psi^- z^{-1} 1, z) = \sum_{n \in \mathbb{Z}} J_n(z) z^{-n-1}.$$  

Then we have

$$[J^{(f)}, \Psi^+ (m + \frac{1}{2})] = \pm \Psi^+ (m + n + \frac{1}{2}).$$

Let $\tilde{F} = \text{Ker}_F \Psi^+ (\frac{1}{2})$ be the subalgebra of the vertex superalgebra $F$ generated by the fields

$$\tilde{\partial} \Psi^+ (z) = \sum_{n \in \mathbb{Z}} -n \Psi^+ (n + \frac{1}{2}) z^{-n-1} \text{ and } \Psi^- (z) = \sum_{n \in \mathbb{Z}} \Psi^- (n + \frac{1}{2}) z^{-n-1}.$$  

Then $\tilde{F}$ is a simple vertex superalgebra and it is $\frac{1}{2} \mathbb{Z}_{\geq 0}$-graded with respect to the operator $L^{(0)}$. Let us describe the basis of $\tilde{F}$. A superpartition is a sequence $\lambda = (\lambda_n)_{n \in \mathbb{Z}_{\geq 0}}$ in $S \cup \{0\}$, $S \subset \mathbb{Q}_+$, such that

$$\lambda_1 > \lambda_2 > \cdots \text{ and } \lambda_n = 0 \text{ for } n \text{ sufficiently large.}$$  

Define the length of partition by $\ell(\lambda) = \max \{ n \mid \lambda_n \neq 0 \}$. If $\ell(\lambda) = \ell$ we write $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Let $\phi$ denotes the superpartition with all the entries being zero. Then we define $\ell(\phi) = 0$.

Let $\mathcal{P}$ be the set of all superpartitions in $\left( \frac{1}{2} + \mathbb{Z}_{\geq 0} \right) \cup \{0\}$ and $\overline{\mathcal{P}}$ be the set of all superpartitions in $\left( \frac{1}{2} + \mathbb{Z}_{\geq 0} \right) \cup \{0\}$. Then we have

$$\mathcal{P} = \bigcup_{r=0}^{\infty} \mathcal{P}_r, \quad \overline{\mathcal{P}} = \bigcup_{r=0}^{\infty} \overline{\mathcal{P}}_r$$

where $\mathcal{P}_0 = \overline{\mathcal{P}}_0 = \{ \phi \}$, and

$$\mathcal{P}_r = \{ \lambda = (\lambda_1, \ldots, \lambda_r) \in \left( \frac{1}{2} + \mathbb{Z} \right)^r \mid \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 1/2 \}$$

$$\overline{\mathcal{P}}_r = \{ \lambda = (\lambda_1, \ldots, \lambda_r) \in \left( \frac{1}{2} + \mathbb{Z} \right)^r \mid \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 3/2 \}.$$  

For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_r, \mu = (\mu_1, \ldots, \mu_s) \in \overline{\mathcal{P}}_s$ we set

$$v_{\lambda,\mu} := \Psi^- (\lambda_1) \cdots \Psi^- (\lambda_r) \Psi^+ (\mu_1) \cdots \Psi^+ (\mu_s) 1$$

$$v_{\phi,\mu} := \Psi^- (\lambda_1) \cdots \Psi^- (\lambda_r) 1, \quad v_{\phi,\mu} := \Psi^+ (\mu_1) \cdots \Psi^+ (\mu_s) 1,$$

$$v_{\phi,\phi} = 1.$$  

Then the set

$$\{ v_{\lambda,\mu} \mid (\lambda, \mu) \in \mathcal{P} \times \overline{\mathcal{P}} \}$$  

(2.1)
3. The vertex superalgebra $V$ and its modules

In this section we shall recall definition of the vertex superalgebra $V$ and certain results from [A2]. Let $M(0) = \mathbb{C}[\gamma^+(n), \gamma^-(n) \mid n < 0]$ be the commutative vertex algebra generated by the fields

$$\gamma^\pm(z) = \sum_{n < 0} \gamma^\pm(n) z^{-n-1}.$$ (cf. [F]). Let $\chi^\pm(z) = \sum_{n \in \mathbb{Z}} \chi^\pm_n z^{-n-1} \in \mathbb{C}(z)$. Let $M(0, \chi^+, \chi^-)$ denotes the 1–dimensional irreducible $M(0)$–module with the property that every element $\gamma^\pm(n)$ acts on $M(0, \chi^+, \chi^-)$ as multiplication by $\chi^\pm_n \in \mathbb{C}$.

Let now $F$ be the vertex superalgebra generated by the fields $\Psi^\pm(z)$ and $\gamma^\pm(z)$. Therefore $F = F \otimes M(0)$. As in [A2], denote by $V$ the vertex subalgebra of the vertex superalgebra $F$ generated by the following vectors

\begin{align*}
\tau^\pm &= (\Psi^\pm(-\frac{3}{2}) + \gamma^\pm(-1)\Psi^\pm(-\frac{1}{2}))1, \\
j &= \frac{\gamma^+(1) - \gamma^-(1)}{2}, \\
\nu &= \frac{2\gamma^+(1)\gamma^-(1) + \gamma^+(2) + \gamma^-(-2)}{4}1.
\end{align*}

The vertex superalgebra structure on $V$ is generated by the following fields

\begin{align*}
G^\pm(z) &= Y(\tau^\pm, z) = \sum_{n \in \mathbb{Z}} G^\pm(n + \frac{1}{2})z^{-n-2}, \\
S(z) &= Y(\nu, z) = \sum_{n \in \mathbb{Z}} S(n)z^{-n-2}, \\
T(z) &= Y(j, z) = \sum_{n \in \mathbb{Z}} T(n)z^{-n-1}.
\end{align*}

Denote by $\mathcal{A}$ the Lie superalgebra with basis $S(n), T(n), G^\pm(r), C$, $n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}$ and (anti)commutation relations given by

\begin{align*}
[S(m), S(n)] &= [S(m), T(n)] = [S(m), G^\pm(r)] = 0, \\
[T(m), T(n)] &= [T(m), G^\pm(r)] = 0, \\
[C, S(m)] &= [C, T(n)] = [C, G^\pm(r)] = 0, \\
\{G^+(r), G^-(s)\} &= 2S(r + s) + (r - s)T(r + s) + \frac{C}{3}(r^2 - \frac{1}{2})\delta_{r+s,0}, \\
\{G^+(r), G^+(s)\} &= \{G^-(r), G^-(s)\} = 0
\end{align*}

for all $n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z}$. 

is a basis of $\bar{F}$.
By using the commutator formulae for vertex superalgebras, we have that the components of fields (3.5)-(3.7) satisfy the (anti)commutation relation for the Lie superalgebra \( A \) such that the central element \( C \) acts as multiplication by \( c = -3 \). Let \( V^{\text{com}} \) be the commutative vertex subalgebra of \( V \) generated by the fields \( T(z) \) and \( S(z) \). Clearly, \( V^{\text{com}} \cong M_T(0) \otimes M_S(0) \), where \( M_T(0) \) (resp. \( M_S(0) \)) is the subalgebra of \( V^{\text{com}} \) generated by the field \( T(z) \) (resp. \( S(z) \)).

Recall that \( V \) admits the following \( \mathbb{Z} \)-gradation:

\[
V = \bigoplus_{m \in \mathbb{Z}} V^m
\]

\[
V^m = \text{span}_\mathbb{C} \{ G^+(-n_1 - \frac{3}{2}) \cdots G^+(-n_r - \frac{3}{2}) G^-(-k_1 - \frac{3}{2}) \cdots G^-(-k_s - \frac{3}{2}) w | w \in V^{\text{com}}, n_i, k_j \in \mathbb{Z}_{\geq 0}, r - s = m \}.
\]

Now we shall consider a family of irreducible \( V \)-modules.

For \( \chi^+, \chi^- \in \mathbb{C}((z)) \) we set \( F(\chi^+, \chi^-) := F \otimes M(0, \chi^+, \chi^-) \).

Then \( F(\chi^+, \chi^-) \) is a module for the vertex superalgebra \( V \), and therefore for the Lie superalgebra \( A \).

Since \( M(0, \chi^+, \chi^-) \) is one-dimensional, we have that as a vector space

\[
F(\chi^+, \chi^-) \cong F \cong \bigwedge (\Psi^\pm (-i - \frac{1}{2}) | i \geq 0).
\]

Now let \( \chi(z) \in \mathbb{C}((z)) \). Define:

\[
\tilde{F}_\chi := \tilde{F} \otimes M(0, 0, \chi) \subset F(0, \chi).
\]

The operator \( J_f(0) \) acts semisimply on \( \tilde{F}_\chi \) and it defines the following \( \mathbb{Z} \)-gradation

\[
\tilde{F}_\chi = \bigoplus_{j \in \mathbb{Z}} \tilde{F}^j_\chi, \quad \tilde{F}^j_\chi = \{ v \in \tilde{F}_\chi | J_f(0)v = jv \}.
\]

The \( A \)-module structure on \( \tilde{F}_\chi \) is uniquely determined by the following action of the Lie superalgebra \( A \) on \( \tilde{F} \):

\[
G^+(i - \frac{1}{2}) = -i \Psi^+(i - \frac{1}{2}),
\]

\[
G^-(i - \frac{1}{2}) = -i \Psi^-(i - \frac{1}{2}) + \sum_{k=-p}^{\infty} \chi_{-k} \Psi^-(k + i - \frac{1}{2}).
\]

Now we shall first recall the following irreducibility result:

**Proposition 3.1** ([A2], Proposition 5.2). Assume that \( p \in \mathbb{Z}_{\geq 0} \) and that

\[
\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{-1} \in \mathbb{C}((z))
\]
satisfies the following conditions

\[
\chi_p \neq 0, \quad (3.11)
\]

\[
\chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}) \quad \text{if} \ p = 0. \quad (3.12)
\]

Then \( \widetilde{F}_\chi \) is an irreducible \( \mathcal{V} \)-module.

### 4. Schur Polynomials and Irreducibility of \( \widetilde{F}_\chi \)

In this section we shall extend the irreducibility result from Proposition 3.1. We shall always assume that \( \chi(z) \) has the form

\[
\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}(z), \quad (4.13)
\]

where \( \ell \in \mathbb{Z} \). Then the \( \mathcal{A} \)-module structure on \( \widetilde{F}_\chi \) is uniquely determined by the following action of the Lie superalgebra \( \mathcal{A} \) on \( \widetilde{F}\):

\[
G^+(i - \frac{1}{2}) = -i \Psi^+(i - \frac{1}{2}), \quad (4.14)
\]

\[
G^-(i - \frac{1}{2}) = (\ell + 1 - i) \Psi^-(i - \frac{1}{2}) + \sum_{n=1}^{\infty} \chi_{-n} \Psi^-(n + i - \frac{1}{2}). \quad (4.15)
\]

By Proposition 3.1 we know that if \( \ell \) is generic or \( \ell = 0 \), then \( \widetilde{F}_\chi \) is an irreducible module. We shall consider the case when \( \ell \in \mathbb{Z}_{>0} \), and find a sufficient condition on \( \chi(z) \) so that \( \widetilde{F}_\chi \) is irreducible.

For every \( s \in \mathbb{Z}_{>0} \), we define

\[
\Omega_s = \Psi^+(-s - \frac{1}{2}) \Psi^+(-s + \frac{1}{2}) \cdots \Psi^+(-\frac{3}{2}) 1 \in \widetilde{F}_\chi.
\]

We shall need the following lemma. The proof will use only the action of the operators \( G^+(i - \frac{1}{2}), i \in \mathbb{Z} \).

**Lemma 4.1.** Assume that \( U \subset \widetilde{F}_\chi \) is any submodule, \( U \neq \{0\} \). Then there is \( s \in \mathbb{Z}_{>0} \) such that

\[
\Omega_s \in U.
\]

**Proof.** For \( \lambda \in \mathcal{P} \) and \( t \in \overline{\mathcal{P}} \), we set

\[
G^+_{\lambda} = \begin{cases} 
1 & \text{if } \lambda = \phi \\
G^+(\lambda_1) \cdots G^+(\lambda_r) & \text{if } \lambda = (\lambda_1, \ldots, \lambda_r)
\end{cases}
\]

\[
G^+_t = \begin{cases} 
1 & \text{if } t = \phi \\
G^+(-t_1) \cdots G^+(-t_r) & \text{if } t = (t_1, \ldots, t_r)
\end{cases}
\]

Let \( v \in U, v \neq 0 \). Then \( v \) has unique decomposition

\[
v = \sum_{(\lambda,\mu) \in \mathcal{P} \times \overline{\mathcal{P}}} C_{\lambda,\mu} v_{\lambda,\mu} \quad (C_{\lambda,\mu} \in \mathbb{C})
\]
in the basis (2.1). Let \( \ell = \max \{ \ell(\lambda) \mid C_{\lambda,\mu} \neq 0 \} \). We can choose \((\bar{\lambda}, \bar{\mu}) \in \mathcal{P} \times \overline{\mathcal{P}}\) such that

1. \( C_{\bar{\lambda},\bar{\mu}} \neq 0, \ell(\bar{\lambda}) = \ell \),
2. \( \ell(\overline{\mathcal{P}}) = \ell_1 = \min \{ \ell(\mu) \mid \mu \in T_1 \} \) where \( T_1 = \{ \mu \in \overline{\mathcal{P}} \mid C_{\bar{\lambda},\mu} \neq 0 \} \).

If \( T_1 = \{ \phi \} \), we set \( f = G^+_{\bar{\lambda}} \). Otherwise, let \( s \in \mathbb{Z}_{>0} \) be such that

\[
\max \{ \mu_1 \mid \mu = (\mu_1, \ldots, \mu_l) \in T_1 \} = s + \frac{1}{2}.
\]

If \( \overline{\mathcal{P}} = (\overline{\mathcal{P}}_1, \ldots, \overline{\mathcal{P}}_{\ell_1}) \) and \( 0 < \ell_1 = \ell(\overline{\mathcal{P}}) < s \), there are unique \( t_1 > \cdots > t_p, p = s - \ell_1, \) such that

\[
\{ t_1, \ldots, t_p \} = \{ \frac{1}{2}, \ldots, s + \frac{1}{2} \} \setminus \{ \overline{\mathcal{P}}_1, \ldots, \overline{\mathcal{P}}_{\ell_1} \}.
\]

Define now \( t \in \overline{\mathcal{P}} \) in the following way:

\[
t = \begin{cases} 
\phi & \text{if } \ell_1 = s \\
(s + 1/2, \ldots, 3/2) & \text{if } \ell_1 = 0 \\
(t_1, \ldots, t_p) & \text{if } 0 < \ell_1 < s
\end{cases}
\]

Then we set

\[
f = G^+_{\bar{\lambda}} G^+_{\bar{\mu}}.
\]

By construction we have that \( G^+_{\bar{\lambda}} \) annihilates basis vectors \( v_{\lambda,\mu} \) such that \( \ell(\lambda) \leq \ell, \lambda \neq \bar{\lambda} \), and \( G^+_{\bar{\mu}} \) annihilates all \( v_{\bar{\lambda},\mu} \), where \( \mu \in T_1 \setminus \{ \overline{\mathcal{P}} \} \). Therefore,

\[
f v_{\lambda,\mu} = 0 \quad \text{if} \quad C_{\lambda,\mu} \neq 0 \quad \text{and} \quad (\lambda, \mu) \neq (\bar{\lambda}, \overline{\mathcal{P}}),
\]

\[
f v_{\bar{\lambda},\overline{\mathcal{P}}} = \nu \Omega_s \quad (\nu \neq 0) \quad \text{if} \quad T_1 \neq \{ \phi \},
\]

\[
f v_{\overline{\mathcal{P}},\overline{\mathcal{P}}} = \nu_1 \mathbf{1} \quad (\nu_1 \neq 0) \quad \text{if} \quad T_1 = \{ \phi \}.
\]

The proof follows.

In order to present new irreducibility criterion, we shall first recall the definition of Schur polynomials.

Define the Schur polynomials \( S_r(x_1, x_2, \ldots) \) in variables \( x_1, x_2, \ldots \) by the following equation:

\[
\exp \left( \sum_{n=1}^{\infty} \frac{x_n}{n} y^n \right) = \sum_{r=0}^{\infty} S_r(x_1, x_2, \ldots) y^r.
\]

We shall also use the following formula for Schur polynomials:

\[
S_r(x_1, x_2, \ldots) = \frac{1}{r!} \det \begin{pmatrix}
x_1 & x_2 & \cdots & x_r \\
-x+1 & x_1 & x_2 & \cdots & x_{r-1} \\
0 & -x+2 & x_1 & \cdots & x_{r-2} \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & x_1
\end{pmatrix}
\]
Lemma 4.2. We have
\[ G^{-\frac{1}{2}} \cdots G^{-\left(\ell - \frac{1}{2}\right)} \Omega_\ell = (-1)^\ell \ell! S_\ell(-\chi) \mathbf{1} \]
where \( S_\ell(-\chi) = S_\ell(-\chi_1, \ldots, -\chi_\ell, \ldots) \).

Proof. By using action (4.15) we get:
\[
G^{-\frac{1}{2}} \cdots G^{-\left(\ell - \frac{1}{2}\right)} \Omega_\ell = (-1)^\ell \ell \prod_{i=1}^{\ell} \chi_i - 1 - 1 \mathbf{1}.
\]
(Here we use elementary properties of determinants and formula (4.17) for Schur polynomials).

Proposition 4.1. Assume that \( \ell \in \mathbb{Z}_{>0} \),
\[
\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_n z^n \in \mathbb{C}((z))
\]
such that
\[ S_\ell(-\chi) \neq 0. \]
Then \( \tilde{F}_\chi \) is an irreducible \( \mathcal{V} \)-module.

Proof. First we shall prove that the vacuum vector is a cyclic vector for the \( U(A) \)-action, i.e.,
(4.18) \[ U(A) \mathbf{1} = \tilde{F}. \]
Take an arbitrary basis element
(4.19) \[ v = \Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2}) \Psi^-(k_1 + \frac{1}{2}) \cdots \Psi^-(k_s + \frac{1}{2}) \mathbf{1} \in \tilde{F}, \]
where \( n_i, k_i \in \mathbb{Z}_{>0}, n_1 > n_2 > \cdots > n_r, k_1 > k_2 > \cdots > k_s \geq 0. \)
Let \( N \in \mathbb{Z}_{>0} \) such that \( N \geq k_1. \) By using (4.15) we get that
\[
G^-(N - \frac{1}{2}) \cdots G^\left(-\frac{3}{2}\right) G^\left(-\frac{1}{2}\right) \mathbf{1} = C \Psi^-(N - \frac{1}{2}) \cdots \Psi^\left(-\frac{3}{2}\right) \Psi^\left(-\frac{1}{2}\right) \mathbf{1},
\]
where
\[ C = (\ell + 1)(\ell + 2) \cdots (\ell + N + 1) \]
So $C \neq 0$, and we have that
\[ \Psi^0 \left( -N - \frac{1}{2} \right) \cdots \psi^0 \left( -\frac{3}{2} \right) \psi^0 \left( -\frac{1}{2} \right) \mathbf{1} \in U(\mathcal{A}). \]

By using this fact and the action of elements $G^+(i - \frac{1}{2})$, $i \in \mathbb{Z}$, we obtain that $v \in U(\mathcal{A}). \mathbf{1}$. In this way we proved (4.18).

It is enough to prove that every vector $u \in \tilde{F}_\chi$ is cyclic. So let $U = U(\mathcal{A}).u$. By using Lemma 4.1 we have that there is $s \in \mathbb{Z}_{>0}$ such that $\Omega_s \in U$. Assume that $s > \ell$. Then clearly
\[ G^{-}(\ell + \frac{3}{2}) \cdots G^{-}(s + \frac{1}{2}) \Omega_s = C_1 \Omega_\ell \]
for certain non-zero constant $C_1$. Similarly, if $s < \ell$ we see that
\[ G^+(\ell + \frac{1}{2}) \cdots G^+(s + \frac{3}{2}) \Omega_s = C_2 \Omega_\ell, \quad (C_2 \neq 0). \]

Therefore we conclude that $\Omega_\ell \in U$.

Applying Lemma 4.2 we get
\[ G^{-}(\ell + \frac{1}{2}) \cdots G^{-}(s - \frac{1}{2}) \Omega_\ell = \nu \mathbf{1}, \quad (\nu \neq 0). \]

Thus $\mathbf{1} \in U = U(\mathcal{A}).u$. Now relation (4.18) gives that $u$ is a cyclic vector in $\tilde{F}_\chi$. The proof follows.

**Proposition 4.2.** Assume that $\ell \in \mathbb{Z}_{>0}$ and $S_\ell(-\chi) = 0$.

(i) Then $U_\chi = U(\mathcal{A}).\Omega_\ell$ is a proper submodule of $\tilde{F}_\chi$. In particular, $\tilde{F}_\chi$ is reducible.

(ii) $U_\chi$ is an irreducible $\mathcal{V}$–module.

**Proof.** Assume that $S_\ell(-\chi) = 0$. Define
\[
\begin{align*}
w &= G^{-}(\ell + \frac{1}{2}) \cdots G^{-}(\ell - \frac{1}{2}) \Omega_\ell \\
 &= \left\{ \left( (-1)^{\ell-1}(\ell - 1)! \Psi^+(-\ell - \frac{1}{2}) + a_1 \Psi^+(-\ell + \frac{1}{2}) + \cdots + a_{\ell-1} \Psi^+(-\frac{3}{2}) \right) \mathbf{1} \right\}
\end{align*}
\]
where $a_1, \ldots, a_{\ell-1}$ are certain complex numbers.

Therefore $w \neq 0$. By using Lemma 4.2 the assumption $S_\ell(-\chi) = 0$ and the definition of $w$ we get
\[ G^+(n - \frac{1}{2})w = 0 \quad \text{for} \quad n \in \mathbb{Z}_{>0}. \]

One can easily show that
\[ G^+(\ell + \frac{1}{2}) \cdots G^+(\ell - \frac{3}{2})w = C \Omega_\ell \quad (C \neq 0), \]
which implies that $U_\chi = U(\mathcal{A}).w$. Every element of $U_\chi$ is a linear combination of vectors
\[ G^+(n_1 - \frac{1}{2}) \cdots G^+(n_r - \frac{1}{2}) G^+(m_1 - \frac{1}{2}) \cdots G^+(m_s - \frac{1}{2})w, \]
for $n_1, m_1 \in \mathbb{Z}_{>0}$, $n_1 > n_2 > \cdots > n_r$, $m_1 > m_2 > \cdots > m_s$. But a vector (4.22) is either zero (if $G^+(n_1 - \frac{1}{2}) \cdots G^+(m_s - \frac{1}{2})w = 0$) or has the following non-trivial summand of lowest degree in $\tilde{F}$ (with respect to $L^f(0)$)
\[ C \Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2}) \Psi^+(-m_1 - \frac{1}{2}) \cdots \Psi^+(-m_s - \frac{1}{2})w \]
where $C \neq 0$. From this one gets that $1 \not\in U_\chi$. Therefore $\tilde{F}_\chi$ is a reducible module with the proper submodule $U_\chi$. This proves assertion (i).

Assume now that $U \subset U_\chi$ is a non-zero submodule. Then Lemma 4.1 implies that there is $s \in \mathbb{Z}_{\geq 0}$ such that $\Omega_s \in U$. By using relations (4.20) and (4.21) from the proof of Proposition 4.1 we see that $\Omega_\ell \in U$. Therefore $U = U(A)\Omega_\ell = U_\chi$ and $U_\chi$ is an irreducible $A$–module. This proves assertion (ii).

\textbf{Proposition 4.3.} Assume that $\ell \in \mathbb{Z}, \ell < 0$.

(i) $\tilde{F}_\chi$ is reducible and $J_\chi = U(A)\cdot 1$ is its proper submodule.

(ii) $J_\chi$ is an irreducible $\mathcal{V}$–module.

\textbf{Proof.} Let $q = -\ell - 1$. Then

$$G(n - \frac{1}{2}) = -(q + n)\Psi^-(n - \frac{1}{2}) + \sum_{n=1}^{\infty} \chi_n\Psi^-(n + i - \frac{1}{2}).$$

By using similar arguments as in the proof of Proposition 4.2 one can see that $\Psi^-(\frac{q}{2}) \not\in J_\chi$ which gives reducibility of $\tilde{F}_\chi$. The proof that submodule $J_\chi$ is irreducible is completely analogous to that of Proposition 4.2 (ii).

Note that $U_\chi$ and $J_\chi$ are $\mathbb{Z}$–graded $\mathcal{V}$–modules with respect to $J^f(0)$:

\begin{align*}
U_\chi &= \bigoplus_{i \in \mathbb{Z}} U^i_\chi, \quad U^i_\chi = \{v \in U_\chi | J^f(0)v = iv\}, \\
J_\chi &= \bigoplus_{i \in \mathbb{Z}} J^i_\chi, \quad J^i_\chi = \{v \in J_\chi | J^f(0)v = iv\}.
\end{align*}

Now we are able to classify $\chi \in \mathbb{C}((z))$ such that $\tilde{F}_\chi$ is irreducible. We have proved the following classification result.

\textbf{Theorem 4.1.} Assume that $\chi \in \mathbb{C}((z))$. Then the $\mathcal{V}$–module $\tilde{F}_\chi$ is irreducible if and only if $\chi$ satisfies one of the following conditions:

(i) There is $p \in \mathbb{Z}_{\geq 0}, p \geq 1$ such that

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n}z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_p \neq 0.$$ 

(ii)

$$\chi(z) = \sum_{n=0}^{\infty} \chi_{-n}z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}).$$
There is $\ell \in \mathbb{Z}_{\geq 0}$ such that 

$$\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}(z)$$

and $S_{\ell}(-\chi) \neq 0$.

5. WAKIMOTO MODULES

We shall first recall the definition of the Wakimoto modules at the critical level (cf. [F], [W]). The Weyl vertex algebra $W$ is generated by the fields 

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n},$$

whose components satisfy the commutation relations for the infinite-dimensional Weyl algebra 

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m,0}.$$

Assume that $\chi(z) \in \mathbb{C}(z)$.

On the vertex algebra $W$ exists the structure of the $A_1^{(1)}$–module at the critical level defined by 

$$e(z) = a(z), \quad h(z) = -2 : a^*(z) a(z) : -\chi(z), \quad f(z) = - : a^*(z)^2 a(z) : -2 \partial_z a^*(z) - a^*(z) \chi(z).$$

This module is called the Wakimoto module and it is denoted by $W_{-\chi}$.

Let $F_{-1}$ be the lattice vertex superalgebra $V_L$ associated to the lattice $L = \mathbb{Z} \beta$, where $\langle \beta, \beta \rangle = -1$ (cf. [A1], [K2], [LL]). Then $F_{-1}$ has the following $\mathbb{Z}$–gradation (cf. [A2]): 

$$F_{-1} = \bigoplus_{j \in \mathbb{Z}} F^j_{-1}, \quad F^j_{-1} = \{ v \in F_{-1} | \beta(0)v = -jv \}.$$

In [A2], we constructed mappings $L_s, s \in \mathbb{Z}$, from the category of $\mathcal{V}$–modules to the category of $A_1^{(1)}$–modules at the critical level. Let $V_{-2}(sl_2)$ denotes the universal affine vertex algebra for $A_1^{(1)}$ at the critical level, and $M_T(0)$ be the commutative subalgebra of $\mathcal{V}$ generated by the field $T(z)$.

**Theorem 5.1** ([A2], Theorem 6.2). Assume that $U$ is a $\mathcal{V}$–module which admits the following graduation:

$$U = \bigoplus_{j \in \mathbb{Z}} U^j, \quad \mathcal{V}^i \cdot U^j \subset U^{i+j}.$$

Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} L_s(U) \quad L_s(U) = \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}.$$
and each $\mathcal{L}_s(U)$ is an $V_{-2}(sl_2) \otimes M_T(0)$–module. If $U$ is irreducible, then $\mathcal{L}_s(U)$ is an irreducible $A_1^{(1)}$–module at the critical level.

In particular, the map $\mathcal{L}_0$ sends $\mathcal{V}$–module $\tilde{F}_\chi$ to the Wakimoto module $W_{-\chi}$ and

$$W_{-\chi} \cong \mathcal{L}_0(\tilde{F}_\chi) = \bigoplus_{j \in \mathbb{Z}} \tilde{F}_\chi^j \otimes F_{-1}^j.$$

Recall first:

**Theorem 5.2.** ([A2]) Assume that $\tilde{F}_\chi$ is an irreducible $\mathcal{V}$–module. Then $W_{-\chi}$ is irreducible $A_1^{(1)}$–module at the critical level.

In the case of Wakimoto modules the converse is also true.

**Theorem 5.3.** Assume that $\tilde{F}_\chi$ is reducible. Then the Wakimoto module $W_{-\chi}$ is also reducible.

**Proof.** Assume that $N \subset \tilde{F}_\chi$ is any proper submodule. Take $s \in \mathbb{Z}_{>0}$ such that $\Omega_s \in N$ (cf. Lemma 4.1) and define $U = U(A).\Omega_s \subseteq \tilde{N}$. Then $U$ admits the $\mathbb{Z}$–gradation

$$U = \bigoplus_{j \in \mathbb{Z}} U^j$$

where

$$U^j = \{v \in U \mid J_f(0)v = jv\} \subset \tilde{F}_\chi^j.$$

Then by using Theorem 5.1 we conclude that

$$\mathcal{L}_0(U) = \bigoplus_{j \in \mathbb{Z}} U^j \otimes F_{-1}^j$$

is an $A_1^{(1)}$–module and it is a proper submodule of the Wakimoto module $W_{-\chi}$. The proof follows. \square

**Corollary 5.1.** The Wakimoto module $W_{-\chi}$ is irreducible if and only if $\chi(z) \in \mathbb{C}((z))$ satisfies one of the conditions (i)-(iii) of Theorem 4.1.

In the case when the module $W_{-\chi}$ is reducible, it contains irreducible submodule.

**Corollary 5.2.** Let $\chi(z) = \frac{\ell+1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1}$ and $\ell \in \mathbb{Z}$.

(i) Assume that $\ell \in \mathbb{Z}_{>0}$ and $S_\ell(-\chi) = 0$. Then

$$\mathcal{L}_0(U_\chi) = \bigoplus_{i \in \mathbb{Z}} U_\chi^i \otimes F_{-1}^i$$

is an irreducible submodule of $W_{-\chi}$.

(ii) Assume that $\ell < 0$. Then

$$\mathcal{L}_0(J_\chi) = \bigoplus_{i \in \mathbb{Z}} J_\chi^i \otimes F_{-1}^i$$

is an irreducible submodule of $W_{-\chi}$. 
Proof. Propositions 4.2 and 4.3 imply that $U_\chi$ and $J_\chi$ are irreducible $\mathcal{V}$-modules which are $\mathbb{Z}$ graded with gradations (4.23) and (4.24). Then Theorem 5.1 implies that $\mathcal{L}_0(U_\chi)$ and $\mathcal{L}_0(J_\chi)$ are irreducible $A_1^{(1)}$-modules. The proof follows. \hfill \square

Remark 5.1. In the case of reducible Wakimoto modules from Corollary 5.2 one can consider the action of $sl_2$ on $W_{-\chi}$ and the maximal $sl_2$-integrable submodule $W_{-\chi}^{\text{int}}$ of $W_{-\chi}$. It is clear that $W_{-\chi}^{\text{int}}$ is a proper $A_1^{(1)}$-module of $W_{-\chi}$. By combining our results and the results from [FG] and [ACM] one can easily show that $\mathcal{L}_0(U_\chi) = W_{-\chi}^{\text{int}}$ when $\ell > 0$ (resp. $\mathcal{L}_0(J_\chi) = W_{-\chi}^{\text{int}}$ when $\ell < 0$). So our method shows that the maximal integrable submodule of the Wakimoto module $W_{-\chi}$ is irreducible at the critical level.

Remark 5.2. It is interesting to look at the case $\ell = 1$ and $\chi(z) = \frac{2}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1}$. Then the Wakimoto module $W_{-\chi}$ is irreducible if $\chi_{-1} \neq 0$ and reducible if $\chi_{-1} = 0$. The reducible Wakimoto modules $W_{-\chi}$ such that

$$\chi(z) = \frac{2}{z} + \sum_{n=2}^{\infty} \chi_{-n} z^{n-1}$$

(i.e., $\chi_{-1} = 0$) were studied in [FFR].

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