Time Evolution and Probability in Quantum Theory

The Central Role of Born’s Rule

Stephen Bruce Sontz
CIMAT
Guanajuato, Mexico

Abstract

In this treatise I introduce the time dependent Generalized Born’s Rule for the probabilities of quantum events, including conditional and consecutive probabilities, as the unique fundamental time evolution equation of quantum theory. Then these probabilities, computed from states and events, are to be compared with relative frequencies of observations. Schrödinger’s equation still is valid in one model of the axioms of quantum theory, which I call the Schrödinger model. However, the role of Schrödinger’s equation is auxiliary, since it serves to help compute the continuous temporal evolution of the probabilities given by the Generalized Born’s Rule. In other models, such as the Heisenberg model, the auxiliary equations are quite different, but the Generalized Born’s Rule is the same formula (covariance) and gives the same results (invariance). Also some aspects of the Schrödinger model are not found in the isomorphic Heisenberg model, and they therefore do not have any physical significance. One example of this is the infamous collapse of the quantum state. Other quantum phenomena, such as entanglement, are easy to analyze in terms of the Generalized Born’s Rule without any reference to the unnecessary concept of collapse. Finally, this leads to the possibility of quantum theory with other sorts of auxiliary equations instead of Schrödinger’s equation, and examples of this are given. Throughout this treatise the leit motif is the central importance of quantum probability and most especially of the simplifying role of the time dependent Generalized Born’s Rule in quantum theory.
A Lilia por tu amor, por tu apoyo, por todo
Back Cover Blurb

This treatise presents a careful re-organization at an advanced level in terms of the probability theory of basic quantum theory based on quantum states and quantum events. This leads to the recognition that the fundamental time evolution equation of quantum theory is the *Generalized Born’s Rule*, not Schrödinger’s equation, which here plays a secondary role in one model of quantum theory. References to classical physics are kept to a minimum, since this is meant to be a stand-alone, axiomatic presentation of standard quantum theory. This is not a generalization of quantum theory nor an interpretation of it. The intended audience consists of those with a detailed previous knowledge of quantum theory, classical probability and functional analysis. Advanced undergraduate students may find something of interest, but this is not a continuation of an introductory course in quantum theory.
Preface

What I can not create
I do not understand

Richard Feynman

After finishing my recently published introductory book [31] on quantum theory, I began thinking of writing a sequel on the usual advanced topics used in applications. I am thinking of Hartree-Fock, variational methods, atomic and molecular physics, solid state physics, scattering theory, perturbation theory and so on. But these topics are covered in many fine texts which get the physics (mostly) right as well as getting the mathematics (mostly) right. Some even get everything right! There seemed to be no point in writing yet another big square book on all that well known material. And I am tired of the retellings of the story of the historical path that led to the quantum theory. Besides history hides the internal logical structure of quantum theory. So references to history will be few and far between.

I decided to resolve clearly at an advanced level, but at the same time in the simplest possible way, the unaddressed basic problem of quantum theory: its logical structure without any reference to classical physics. And it seemed to me that the central role of quantum probability in quantum theory was not sufficiently appreciated in the scientific community, especially the roles of conditional and consecutive probability. That seemed to be a good starting point, and so I aimed at writing an expository article directed at a specialized public to explain that. I soon realized that I had it wrong. Quantum probability is not a central part of quantum theory. Rather the basics of quantum theory are exactly the same as quantum probability. What I mean is that the Generalized Born’s Rule, and not Schrödinger’s equation as I previously thought, is the fundamental time evolution equation of quantum theory. That is the essence of this treatise.
My approach is so different from the current trend to give ‘interpretations’ of quantum theory that I felt obliged to give an apology separately.

How I happened to arrive at this understanding may be of interest. I was confused by the equivalence of the Schrödinger picture and the Heisenberg picture. Most particularly, how could the Schrödinger picture have collapse of the state as a basic aspect of the theory while the equivalent Heisenberg picture has states which are time independent? I then realized that the equivalence of these pictures is a well known statement about time dependent probabilities. And therefore, probabilities and how to calculate them are the central issue of quantum theory. That is exactly what the Generalized Born’s Rule is about.

I am telling a new story, but all the characters in it are familiar. So, this is a new organization of standard material with Born’s rule placed front and center. This requires introducing a Generalized Born’s Rule that might be new to you, though it is implicit in standard textbook quantum mechanics and in many cases is related to Lüders rule. However most importantly, the really new idea in this treatise is that the Generalized Born’s Rule is the one and only fundamental time evolution equation in quantum theory. Quantum probability is not simply an important part of quantum theory, as I have previously thought. Rather it is the central part. Schrödinger’s equation is still present in one model of quantum theory and still plays a crucial role in understanding quantum systems, but it is not the fundamental time evolution equation. It turns out that the model based on Schrödinger’s equation, which I call Schrödinger’s model, has attributes that are not present in other isomorphic models. The collapse of the wave function is an example of this.

Another new aspect of this story is that there are many non-isomorphic models of the basic axioms of quantum theory. This seems to fly in the face of trying to find the unique and universal laws of nature. But the point is that quantum theory is a work in progress and in this treatise only the basics are being considered. In particular, this is not a complete exposition of known quantum theory, since that would require many volumes. So, many topics are deliberately not included. This is not a theory of everything, but a theory of something. However, I am considering quantum theory consistent with Galilean relativity and at the same time quantum theory consistent with special relativity. If things work out that way, I am even considering quantum theory consistent with general relativity. So I hope the reader realizes the importance of non-uniqueness.
This is a story about events (as well as states and probabilities) and could be confused with the consistent histories interpretation of quantum theory. (See [13].) I deal with this confusion in the chapter Interpretations. For now let me note that I am not giving any ‘interpretation’ of quantum theory. Nor do I care to, since ‘interpretations’ are neither falsifiable nor verifiable. Rather this is a scientific treatise. However, I will touch on some issues that are often considered in an ‘interpretation’ of quantum theory. This is not an attempt to solve philosophical problems, but instead a way of throwing some light on quantum situations by using just the basics: events, states and probabilities.

This story is addressed to a reader with a rather substantial background in quantum physics as well as in the mathematics used in describing that physics. This is not an elementary treatise, nor even the immediate sequel of such an introduction as for example found in my book [31]. Advanced undergraduate students in physics or mathematics might find this material to be quite challenging, though they should not be discouraged from trying. But my intended audience consists of professional scientists and some graduate students with lots of math and physics knowledge, much more than can be found in [31]. The prerequisites that the reader should have include, but are not limited to, an extensive prior knowledge of quantum physics and Hilbert space theory. I assume knowledge of functional analysis up to at least the spectral theory for self-adjoint operators. And this will be a stand-alone story. References to classical physics will be incidental, not part of the main story. It is not clear whether those with some knowledge of classical mechanics have an advantage or an impediment. Some easy pedagogical examples, which are some how illustrative of the general theory, will also be given. But mostly, the presentation will be quite general. The experts will note, however, that I limit myself to examples that are type I von Neumann algebras.

This treatise is organized as follows. The first chapter states and discusses the Axioms. Chapter 2 is a long introduction to Quantum Probability. Here the important topics of conditional and consecutive probability are discussed. This is not new, neither in theory nor experiment, since one has to assign theoretical probabilities to sequences of events and one measures relative frequencies of such sequences. In one section this theory is compared and contrasted with its well known intellectual competitor: Classical Probability as a topic in measure theory. But I do not consider comparisons with classical physics, Bell’s inequalities, hidden variables and such side issues, since my focus is on quantum theory as a new type of probability theory and nothing
Chapter 3 then applies this theory to Entanglement. In Chapters 4 to 7 I discuss various issues that have filled volumes for decades. These are Schrödinger’s Cat, the Measurement Problem, the EPR paper, Determinism and Probability. I do not resolve any of these issues, but try to shed some light on what they are (or should be) about.

In the rest of the treatise I consider topics that strictly speaking have little or nothing to do with quantum theory. I include them since it is an appropriate moment for commenting on them. In Chapter 8 I attempt to dismiss Interpretation, in the contemporary sense of that word, from scientific consideration. Chapter 9 is devoted to the Wave Function about which I expect the reader to know a lot already. I include it in order to rid quantum theory of residual classical language that slips into discussions of it. Chapter 10 concerns various extensions and alternatives to quantum theory. Some of these are currently well accepted, while others are wild speculation at best. In the final Chapter 11 it is my turn to tilt at windmills with an unlikely proposal.

Strangely enough, I do not have to give any applications of the basic time evolution equation of quantum theory. This is because that is done quite well in the standard textbooks, usually in the Schrödinger model. Readers with a lot of background can get the basic gist of this treatise by reading the sections on Axiom 5 and Quantum Conditional Probability and then skipping forward to the Chapter on Entanglement. I would like to point out that I originally thought that Entanglement could not be explained without using the collapse of the state associated with a measurement. So I examined this case carefully in order to understand the fundamental role of the collapse condition. What I found out surprised me; an analysis in terms of the conditional probability of sequences of quantum events suffices.

There is a multitude of topics that I could have included, but I decided to make my point in a slim book. I am reminded of the lawyer who was summing up in court the defense of his client on and on. His partner seated at his side took a legal notepad and wrote on it: “Sit down! You have made your point!” So, I hope that I have said enough to make my point.

This treatise was much easier to write than [31], since here I can rely on the previous knowledge of my intended reader. One day I hope to write a book on basic quantum theory in a manner accessible to a general public, since I feel that I owe it to my friends who are not scientists. That will be a much more difficult task. In many ways this treatise is prologue.

Storytelling is a part of human culture and, by inference, a part of human
nature. It is an art practiced in all societies throughout history. It pre-exists written language. But it is a mistake to think that a universal part of the human condition is necessarily a talent shared by all people. Quite simply, there are many persons who can not narrate a story in a fashion comprehensible to others. I am not merely speaking of those born with cerebral damage. There are also those who do not understand that a story is for an audience that does not yet know completely the topic at hand and, consequently, the storyteller must go into the details, step by step, to communicate with that audience. Young children seem to always be like that; when they tell a story it is as if they expect the listener to already know it and they are merely repeating something known by everyone. One has to learn that the ‘other’ is different from the ‘ego’ and so must be accommodated accordingly. But some people never learn this, or learn it incompletely. Such people we can deem as ‘pre-literate’ though this is meant to be purely descriptive and not judgmental. At best they are like children who can repeat fragments without realizing what are the processes, both mental and observational, by which this all comes about. Such people are not able to organize nor communicate scientific knowledge; They only listen to ‘authority’ and repeat it. A fancy way of naming storytelling is simply ‘narrative’.

Curiosity is also a part of human culture and, by inference, a part of human nature. And similarly, not everyone is curious or, better said, not everyone is curious in the same way. Moreover, those who are curious may not have much talent or experience for dealing with that curiosity in a systematic way. Scientific activity is a combination of curiosity and narrative. While some sort of scientific activity in this vague sense is found in all cultures, the contemporary version of science is not a necessary aspect of human culture. Many people are ‘pre-scientific’ very much in the same way as many people are pre-literate.

The contemporary concept of science is not an invention of the modern world, even though it is a human invention of an intellectual sort. One can learn how science, basically as understood nowadays, was practiced in Hellenistic society beginning around 300 B.C. from the fascinating book [28] by L. Russo. As Russo points out, some ‘modern’ scientists do not grasp completely what Archimedes and Co. were doing in Ancient Antiquity. One necessary scientific activity, then as now, is the construction of mathematical models which relate to observations. It is a two-way street, but logically the flow starts on the observational side. In this approach the historical process is not of any interest. However, I will directly jump over to the theoretical side.
by starting with an axiomatization of quantum theory. This methodology is what Maxwell’s Equations are all about, though they are not typically called Maxwell’s Axioms. Maybe they should be renamed.

I thank Micho Đurđevich for graciously commenting on a preliminary version of this treatise.

Many people have helped me in understanding the matters discussed here. Sometimes that help was a detailed explanation, but often it was a casual question or comment that focused my attention on an important matter. A very partial list of those that I have known must include Luigi Accardi, David Brydges, Micho Đurđevich, Leonard Gross, George Hagedorn, Brian Hall, Ira Herbst, Jim Howland, Jaime Cruz Sampedro, Larry Thomas and Carlos Villegas. During my undergraduate days I was advised to read the masters. So I am indebted also to those whom I have only known through their published work. Most notable among those are Valentine Bargmann, Paul Dirac, Richard Feynman, David Hilbert, John von Neumann, Erwin Schrödinger, and Eugene Wigner. I thank all those mentioned as well as many more, too numerous to recall and list their names. Of course, any errors or shortcomings are due to my imperfect understanding of non-intuitive matters.

Stephen Bruce Sontz
Centro de Investigación en Matemáticas, A.C. (CIMAT)
Guanajuato, Mexico
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Apology

Avant tout il faut savoir poser des problèmes. 
Et quoi qu’on dise, dans la vie scientifique, 
les problèmes ne se posent pas d’eux-mêmes.

Gaston Bachelard

If this treatise receives any reception beyond total negligence, it most likely will consist mainly of negative criticisms. In anticipation of the worst of these, I present a defense. I only ask those imagined critics to try to take what I now say into consideration.

I will be criticized for not treating some favorite topic. But this treatise is meant to lay the groundwork for building a better understanding of quantum theory. No claim is made to completeness. Nor would this be possible in anything less than an encyclopedic set of volumes. If some think that it is important that other topics be studied from this viewpoint, then it behooves them to undertake that study. I intend to stand not only on the shoulders of the giants who preceded me, but also on the shoulders of those who follow.

I will be criticized for not discussing classical physics, Bell’s inequalities, quantization schemes, semi-classical limits or any of a number of topics that are not part of quantum theory itself. But my subject matter is quantum theory, nothing else.

I will be criticized for not seeking a deterministic, classical explanation for quantum situations that have probability 1, despite the fact that probability theory always includes the limiting case of probability 1 with no need for explanations outside of that theory itself.

I will be criticized for being ‘over-mathematical’ when, in fact, I am just trying to isolate the purely mathematical aspects of quantum theory in order to avoid their intrusion into the difficult process of creating physical intuition for a probability theory.
I will be criticized for relying on axioms. However, this is a time tested methodology for organizing an explanatory system of thought in order to provide objective criteria for valid advances and to enhance understanding. It also serves the important pedagogical goal of giving a student a way to learn and dominate the subject.

I will be criticized for giving yet another interpretation of quantum theory, no better, and quite possibly, worse than others. But interpretation is not my concern. Interpretation is felt to be required when obscurity, confusion and contradiction dominate. But by isolating and logically organizing the basics, I think that the need for interpretation recedes.

I will be criticized because this is mostly well known material that has already appeared in the scientific literature long ago. Well, point partially granted. However, I have firstly re-organized that well known material and secondly put the Generalized Born’s Rule at its head as the fundamental time evolution equation of quantum theory. I have not tried, as many have, to derive Born’s Rule from more basic principles. Rather it is an axiom. Moreover, I have emphasized that this is actually a time evolution equation, even though it is not a differential equation.

I will be criticized because this is new material which rejects standard quantum theory. But nothing is rejected, not even the collapse condition, although it is language that I would prefer to avoid. I am only trying to put things in their proper perspective given the new viewpoint provided by the Generalized Born’s Rule.

I will be criticized for not solving some problem or other, whether it be a problem of physics, mathematics or epistemology. I have posed my own problem and outlined my resolution of it. Explicitly, the initial problem was how to understand the collapse condition as a necessary part of the Heisenberg picture of the quantum theory of entanglement. This is a part of the scientific project, not all of it. I encourage others to work on their problems.

I will be criticized because I do not have the requisite standing in the scientific community to opine publicly on these matters. This is but a way to avoid confronting the treatise on its intrinsic merits and should be recognized as such. And rejected as such.
Contents

Preface iv

Apology x

1 Axioms 1
   1.1 Axiom 1 - Kinematics . . . . . . . . . . . . . . . . . 3
   1.2 Axiom 2 - States and Events . . . . . . . . . . . . 4
   1.3 Axiom 3 - Spin and Statistics . . . . . . . . . . . . 7
   1.4 Axiom 4 - Time Independent Born’s Rule . . . . . . 8
   1.5 Axiom 5 - Dynamics . . . . . . . . . . . . . . . . . . 10
   1.6 Models of the Axioms . . . . . . . . . . . . . . . . . 11
   1.7 How Many Time Evolutions? . . . . . . . . . . . . . 17
   1.8 Relation to Observation . . . . . . . . . . . . . . . . 17

2 Quantum Probability 19
   2.1 Introduction . . . . . . . . . . . . . . . . . . . . 19
   2.2 The Physical Assumptions . . . . . . . . . . . . . 20
   2.3 The Mathematical Model . . . . . . . . . . . . . . 21
   2.4 The Finite Dimensional Case . . . . . . . . . . . . 25
   2.5 Quantum Probability: The First Steps . . . . . . . 28
   2.6 Quantum Condiitional Probability . . . . . . . . . 32
   2.7 Quantum Consecutive Probability . . . . . . . . . 35
   2.8 Quantum Probability of Two Events . . . . . . . . 38
   2.9 Generalized Born’s Rule with a State . . . . . . . 45
   2.10 Generalized Born’s Rule with no State . . . . . . 49
   2.11 Probability Amplitudes . . . . . . . . . . . . . . . 51
   2.12 Quantum Integrals . . . . . . . . . . . . . . . . . . 52
   2.13 Born’s rule redux . . . . . . . . . . . . . . . . . . . 53
# List of Notations and Abbreviations

| Symbol   | Meaning                                                                 |
|----------|-------------------------------------------------------------------------|
| $A$      | Self-adjoint operator                                                  |
| $B$      | Borel subset of $\mathbb{R}$                                           |
| $\mathcal{B}$ | The $\sigma$-algebra of Borel subsets of $\mathbb{R}$              |
| $\mathcal{C}\infty$ | Having all derivatives                  |
| $\mathbb{C}$ | The set of complex numbers                                             |
| $\mathbb{C}\mathbb{P}(\mathcal{H})$ | The complex projective space of $\mathcal{H}$ |
| $D$      | Density matrix                                                         |
| $\dim$   | Dimension                                                              |
| $e$      | Base of the natural exponentials and natural logarithms                 |
| $E$      | A quantum event or, equivalently, a projection                          |
| $E^c$    | $I - E$, the complementary event of $E$                                |
| $\mathcal{E}$ | The set of all quantum events                                         |
| $E_t$    | One-parameter group of bijections of $\mathcal{E}$ where $t \in \mathbb{R}$ |
| $\mathcal{E}(X)$ | Expected value of the random variable $X                             |
| $\mathcal{E}_\psi(A)$ | Expected value of the operator $A$ given the pure state $\psi         |
| $f$      | A function                                                             |
| $\mathcal{H}$ | Hamiltonian operator                                                  |
| $\mathcal{H}_{\text{free}}$ | Free Hamiltonian operator                                           |
| $\mathcal{H}_{\text{int}}$ | Interacting Hamiltonian operator                                     |
| $\mathcal{H}$ | Hilbert space                                                         |
| $(\mathcal{H}, E, S)$ | Model of quantum theory                                               |
| $\hbar$  | Planck’s normalized constant                                           |
| $i$      | The complex unit $\sqrt{-1}$                                           |
| $I$      | Identity operator                                                      |
| $I_\mathcal{H}$ | Identity operator of $\mathcal{H}$                                    |
| $id$     | Identity map                                                           |
| $l$      | Linear functional                                                      |
| $\mathcal{L}(\mathcal{H})$ | The set $\{T : \mathcal{H} \to \mathcal{H} | T$ is linear and bounded$}$ |
| $L^2(\mathbb{R}^3)$ | Hilbert space of equivalence classes of $\mathbb{R}^3 \to \mathbb{C}$ |
| $m_k$    | $k$th moment of the measure $\mu$                                      |
| $\mathcal{M}(\Omega)$ | vector space of all measurable functions $f : \Omega \to \mathbb{C}|$ |
Notations

\(\mathbb{N}\) The set of non-negative integer numbers

povm Positive operator valued measure

pvms Projection valued measure

\(P_A\) The projection valued measure of the self-adjoint operator \(A\)

\(P(E | \rho)\) The quantum probability of the event \(E\) given the density matrix \(\rho\)

\(P(E | \psi)\) The quantum probability of the event \(E\) given the pure state \(\psi\)

\(P(T \in B | \rho)\) The quantum probability that \(T\) lies in the set \(B\) given the density matrix \(\rho\)

\(P(T \in B | \psi)\) The quantum probability that \(T\) lies in the set \(B\) given the pure state \(\psi\)

\(P(E_1 | E_2)\) Conditional probability of event \(E_1\), given event \(E_2\)

\(P(E_1, E_2 | \psi)\) Quantum consecutive probability of the event \(E_1\) and then later the event \(E_2\), given the state \(\psi\)

\(\mathbb{R}\) The set of real numbers

Res(\(A\)) Resolvent set of the operator \(A\)

\(S\) The set of all quantum states

\(S_t\) One-parameter group of bijections of \(S\) where \(t \in \mathbb{R}\)

\(S_1, S_2, S_3\) 2 \times 2 spin matrices

\(S^2\) 4 \times 4 total spin matrix

Spec(\(A\)) The spectrum of the operator \(A\)

\(SU(2)\) The special unitary Lie group of 2 \times 2 matrices

Supp \(P\) Support of a povm \(P\)

t time

\(T\) Linear operator

\(Tr\) The trace of an trace class operator

\(U\) Unitary operator; open set

\(U(s,t)\) Time evolution operator

\(\mathcal{V}\) A von Neumann algebra

\(V_j\) Spectral subspace

\(X\) Classical random variable

\(\delta_{ij}\) Kronecker delta

\(\varepsilon_1, \varepsilon_2\) Standard orthonormal basis of \(\mathbb{C}^2\)

\(\lambda\) Real or complex number

\(\{\lambda\}\) The set whose only element is \(\lambda\)

\(\Lambda\) The empty sequence of events
| Symbol | Notation |
|--------|----------|
| $\mu$  | Measure  |
| $\mu_X$ | Distribution of the random variable $X$ |
| $\pi$  | Quantum probability |
| $\rho$ | Density matrix |
| $\sigma$ | Standard deviation |
| $\sigma_k$ | $k$th central moment of the measure $\mu$ |
| $\phi$ | Unit vector in $\mathcal{H}$; pure state |
| $\phi_t$ | Pure state that depends on time $t \in \mathbb{R}$ |
| $|\phi\rangle \langle \phi|$ | Dirac notation for the density matrix associated with a pure state $\phi$ |
| $\langle \phi |$ | Dirac bra notation of the pure state $\phi$ |
| $|\phi\rangle$ | Dirac ket notation of the pure state $\phi$ |
| $\chi_S$ | The characteristic function of the set $S$ |
| $\psi$ | Unit vector in $\mathcal{H}$; pure state |
| $\psi_t$ | Pure state that depends on time $t \in \mathbb{R}$ |
| $\Omega$ | Classical probability space; region in Minkowski space-time |
| $(\Omega, \mathcal{F}, P)$ | Classical probability space, its $\sigma$-algebra and its probability function |
| $\emptyset$ | The empty set |
| $\langle \cdot, \cdot \rangle$ | Inner product |
| $|| \cdot ||$ | Norm |
| $|| \cdot ||_{op}$ | Operator norm |
| $\bigvee$ | Intersection of sets |
| $\cap$ | Union of sets |
| $\wedge$ | Infimum of lattice elements |
| $\bigvee$ | Supremum of lattice elements |
| $\otimes$ | Tensor product |
I had been told that Euclid proved things and was much disappointed that he started with axioms.

Bertrand Russell

In [31] I put the axioms in the penultimate chapter of the book. This was a pedagogical choice. The intended audience for that book consists of people with no prior background in physics, and so I wanted to present first the particulars and get to the logic behind it all later. In fact, since many novices are allergic to axioms and logic, I even made that chapter optional. But this treatise addresses a much more advanced audience. I not only expect your interest in these details, but also your understanding of their import. One point is that there is a way to associate certain aspects of this mathematical theory with physical phenomena. A theory, any theory, is interesting and important if there are a sufficient number of physical phenomena that are adequately described by it. There is no need to claim that this theory, or any theory, is adequate for describing all possible physical phenomena. It is in this sense that I use the expressions ‘physical characteristic’ or ‘observable quantity’, for example. These are like ‘point’ and ‘line’ in geometry; they are undefined expressions subject to possible correspondence with certain physical phenomena.

These axioms are intended to provide a logical starting point for understanding standard ‘textbook’ quantum mechanics as is used on a daily basis by scientists and engineers. If you prefer to start with other axioms that
Axioms

have these as logical consequences, then you are implicitly accepting the rest of this treatise, provided that I have made no mistakes. If you prefer to start with other axioms that contradict these, then the points of discrepancy should be subjected to experimental tests. These axioms are not intended to be complete, but are meant to give an explicit, logical basis for the rest of the treatise. Nor are they intended to be the final, most efficient way to do this.

I will use the spectral theorem for self-adjoint operators acting in a Hilbert space. I try to use standard notations and conventions. Throughout this treatise self-adjoint operators are understood to be densely defined. These axioms are not the same as those given in [31]. Nor is this intended to be a complete list of axioms. The statement of each axiom terminates with the symbol ■.

We let \( \mathbb{C} \) denote the field of complex numbers and \( i = \sqrt{-1} \in \mathbb{C} \). We also define \( \mathcal{L}(\mathcal{H}) := \{ T : \mathcal{H} \to \mathcal{H} \mid T \text{ is linear and bounded} \} \), where \( \mathcal{H} \) always denotes a complex Hilbert space. We note that \( \mathcal{L}(\mathcal{H}) \) when equipped with the operator norm, denoted by \( || \cdot ||_{op} \), and the adjoint operation, denoted by \( T \mapsto T^* \), is both a \( C^* \)-algebra as well as a von Neumann algebra. The inner product on \( \mathcal{H} \), denoted as \( \langle \cdot, \cdot \rangle \), is anti-linear in the first entry and linear in the second. The norm on \( \mathcal{H} \) is denoted as \( || \cdot || \).

Here is some Dirac notation we will use. Every element in the dual Hilbert space \( \mathcal{H}' := \{ l : \mathcal{H} \to \mathbb{C} \mid l \text{ is linear and bounded} \} \), according to the Riesz representation theorem, can be written for a unique \( \phi \in \mathcal{H} \) as \( l = \langle \phi \rangle \), where the bra \( \langle \phi \rangle \) is defined by \( \langle \phi | \psi := \langle \phi, \psi \rangle \) for all \( \psi \in \mathcal{H} \). Every vector \( \psi \in \mathcal{H} \), a Hilbert space, can also be denoted as a ket \( | \psi \rangle \). Then the definition of bra becomes \( \langle \phi | \psi := \langle \phi, \psi \rangle \). For a pair of vectors \( \phi, \psi \in \mathcal{H} \) we define

\[
| \psi \rangle \langle \phi | := | \psi \rangle \otimes \langle \phi | \in \mathcal{H} \otimes \mathcal{H}'.
\]

This is then identified with the element in \( \mathcal{L}(\mathcal{H}) \) defined for all \( \alpha \in \mathcal{H} \) by

\[
| \psi \rangle \langle \phi | \alpha := \langle \phi, \alpha \rangle \psi.
\]

Its operator norm satisfies \( || | \psi \rangle \langle \phi | ||_{op} = || \psi || \cdot || \phi || \), while its operator adjoint satisfies \( (| \psi \rangle \langle \phi |)^* = | \phi \rangle \langle \psi | \). If both \( \psi \) and \( \phi \) are also non-zero, then \( | \psi \rangle \langle \phi | \) is a rank 1 operator. If \( \phi \) is a unit vector, then \( | \phi \rangle \langle \phi | \) is both a projection and a self-adjoint, positive, trace 1 operator, that is, a density matrix.

We let \( \hbar \) denote the (normalized) Planck's constant. After the Table of Contents a list of other standard notations and abbreviations from physics and mathematics can be found.
1.1 Axiom 1 - Kinematics

**Axiom 1: (Kinematics)** For every quantum system there is an associated non-commutative von Neumann algebra \( \mathcal{V} \) in \( L(\mathcal{H}) \). The projection operators in \( \mathcal{V} \) are called *(quantum) events* and are associated with the physical events of the system. There is a non-empty set of self-adjoint operators, each of which is associated with a physically measurable observable of the system. In particular, every event is such an observable. Also, each such self-adjoint operator \( A \) (also called an *observable*) is associated to \( \mathcal{V} \), which means that its projection valued measure (pvm) \( P_A \) satisfies \( P_A(B) \in \mathcal{V} \) for all Borel subsets \( B \) of \( \mathbb{R} \). ■

The first condition in the axiom implies that \( \dim_{\mathbb{C}} \mathcal{H} \geq 2 \). This axiom does not assert that every observable quantity of a physical entity has a corresponding self-adjoint operator in quantum theory. The most important example of such an observable is the time of a physical event. Since the events in \( \mathcal{V} \) are assumed to be observables and the events (≡ projections) generate \( \mathcal{V} \), it follows that the observables generate \( \mathcal{V} \). The criterion of non-commutativity reflects Dirac’s often expressed opinion that the essential characteristic of quantum theory is that the observables do not commute.

Since this is a presentation based on events and, in particular events of the form \( P_A(B) \) (using the notation of the axiom), it is more convenient, almost obligatory, to use von Neumann algebras instead of the more general structure of \( C^* \)-algebras. The point is that if one starts with a \( C^* \)-algebra \( \mathcal{C} \) and one has a self-adjoint \( A \in \mathcal{C} \), then the associated events \( P_A(B) \) do not necessarily lie in \( \mathcal{C} \), though they do lie in the smallest von Neumann algebra containing \( \mathcal{C} \).

The mathematical definition of a quantum event as a projection operator must be considered with care, since it does not correspond to the word ‘event’ in common English usage. A projection operator \( E \notin \{0, I\} \) has spectrum \( \{0, 1\} \). Sometimes it is said that a quantum event is a YES-NO phenomenon. In other words, a quantum event has exactly two possible eigenvalues. But a physical event colloquially means that only one thing has occurred. For example, suppose there is one beta unstable, radioactive nucleus in an atomic trap. Suppose that it decays in a certain time period. Common usage has it that beta decay occurred. But if it does not decay in that time period, common usage has it that nothing happened, that there was no physical event. But the second alternative is just the NO or 0 eigenvalue of the event.
of beta decay in that time interval. In either case we have the same quantum event, but with two distinct values. For example, we say that $P_A(B)$ is the quantum event that the observable $A$ takes a value in $B$. But in general this quantum event has two eigenvalues. As with any non-trivial self adjoint operator a quantum event can assume more than one value. We can also think of the eigenvalue 1 of the event $P_A(B)$ as meaning that a measurement associated with $A$ produced a value in $B$, while the eigenvalue 0 says that the same measurement produced a value in $\mathbb{R} \setminus B$, the set complement of $B$ in the real line. But measurements do not play a distinguished role in this treatise; they are just a particular type of event. And events do play a central role here.

It is traditional to speak in quantum theory in terms of the self-adjoint operators acting in a given Hilbert space $\mathcal{H}$. This is not logically necessary, since in functional analysis one proves that these mathematical structures are in bijective correspondence with two other structures. These structures are on the one hand projection valued measures (pvm’s) with values in $\mathcal{L}(\mathcal{H})$ defined on the Borel $\sigma$-algebra of $\mathbb{R}$ and on the other hand strongly continuous unitary groups acting on $\mathcal{H}$. For example, one can define the commutativity of two self-adjoint operators $S$ and $T$ by one of these equivalent definitions:

- Let $P_S$ and $P_T$, denote the pvm’s of $S$ and $T$, respectively. Then we say that $S$ and $T$ commute if $P_S(B)$ commutes with $P_T(C)$ for all Borel subsets $B, C$ of $\mathbb{R}$. (Notice that each of the families $\{P_S(B) \mid B \in \mathcal{B}(\mathbb{R})\}$ and $\{P_T(C) \mid C \in \mathcal{B}(\mathbb{R})\}$ is commutative. This definition requires that their union is also a commutative family.)

- Let $e^{irS}$ and $e^{isT}$ for $r, s \in \mathbb{R}$ denote the unitary groups of $S$ and $T$, respectively. Then we say that $S$ and $T$ commute if $e^{irS}$ commutes with $e^{isT}$ for all $r, s \in \mathbb{R}$. (Notice that here as well each of the families of unitary operators $\{e^{irS} \mid r \in \mathbb{R}\}$ and $\{e^{isT} \mid s \in \mathbb{R}\}$ is commutative. This definition requires that their union is also a commutative family.)

The next axiom is really just a continuation of Axiom 1. It is included as a separate axiom because of tradition.

### 1.2 Axiom 2 - States and Events

**Axiom 2: (States and Events)** Every quantum system is described by
Axiom 2

probabilities that are computed by using its set of events (see Axiom 1) and the set of states of its von Neumann algebra $\mathcal{V}$. ■

Unfortunately, the terminology ‘state’ is so widely used that there is no hope of ever replacing it with a neutral term. It is a term that drips with meanings from classical physics as well as from everyday life. But in quantum theory as presented here it is a mathematical term with a mathematical definition.

In the following we will mainly discuss the case when the von Neumann algebra $\mathcal{V} = \mathcal{L}(\mathcal{H})$, since many quantum systems are covered by this case. In introductory texts it is the only case considered, since that already puts a lot of mathematical burden on beginners. I did this myself in [31]. While this is the only case familiar to many physicists, there are other von Neumann algebras used in quantum theory. When a physicist speaks of a Hilbert space as being the setting for the discussion of a quantum system, the underlying, usually implicit assumption is that the appropriate von Neumann algebra is $\mathcal{L}(\mathcal{H})$. But that could be an error. The problem is to find the correct von Neumann algebra, and not the correct Hilbert space as is often thought. This is just a part of the problem of quantization. An important point here is that the lattice of events is central to quantum theory, and its structure depends on the von Neumann algebra being used. Note that the quantization problem is not a problem within quantum theory itself, but rather a problem of how to arrive at the quantum theory of a particular physical system or maybe of a class of physical systems. This non-trivial and important problem will not be addressed in this treatise.

Definition 1.2.1 A state is as a linear functional $l : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ satisfying

- \textit{(Positivity Preservation)}
  \[ l(T) \geq 0 \text{ for all } T \in \mathcal{L}(\mathcal{H}) \text{ that are positive (meaning that } T = T^* \geq 0). \]

- \textit{(Normalization)}
  \[ l(I) = 1, \text{ where } I \in \mathcal{L}(\mathcal{H}) \text{ denotes the identity operator.} \]

It turns out that every state $l$ is a bounded linear functional with norm $\|l\| = 1$. So $l \in (\mathcal{L}(\mathcal{H}))'$, the dual Banach space of the Banach space $\mathcal{L}(\mathcal{H})$. We let $\mathcal{S}$ denote the set of all states. Then $\mathcal{S}$ is a convex subset of $\mathcal{L}(\mathcal{H})$.

An immediate consequence of this axiom is that every \textit{unit vector} $\psi \in \mathcal{H}$, that is $\|\psi\| = 1$, determines a state $l_\psi$ defined by $l_\psi(T) := \langle \psi, T \psi \rangle$ for all $T \in \mathcal{L}(\mathcal{H})$. Moreover, the state $l_\psi$ does not uniquely determine a unit vector.
that defines it, since \( l_\psi = l_\varphi \) for all unit vectors \( \varphi \) that satisfy \( \varphi = \lambda \psi \) for some \( \lambda \in \mathbb{C} \). (Necessarily we have \( |\lambda| = 1 \).) The easy proof of these facts is left to the reader. Such a state \( l_\psi \) is called a pure state.

There are more states. Every trace class operator \( D = D^* \geq 0 \) satisfying \( \text{Tr}(D) = 1 \) is called a density matrix. Any such density matrix \( D \) defines a state \( l_D \) by \( l_D(T) := \text{Tr}(DT) \) for all \( T \in \mathcal{L}(\mathcal{H}) \). We say that \( l_D \) is a mixed state. It is an exercise in functional analysis to show that \( l_D \) is a state and that this state uniquely determines the density matrix \( D \). Also, any state \( l_\psi \), where \( \psi \) is a unit vector, defines a unique density matrix \( D = |\psi\rangle\langle\psi| \) (in Dirac notation) such that \( l_D = l_\psi \). However, if \( \dim_{\mathbb{C}} \mathcal{H} \geq 2 \), it is easy to construct mixed states which are not pure states.

I have carefully distinguished the terminology here. Often the expressions ‘density matrix’ and ‘mixed state’ are conflated as if they were synonyms. But that is not quite so. Gleason’s Theorem (see [12]) speaks to this question.

In common parlance, especially among physicists, the expression mixed state is often reserved for those states \( l_D \) which are not pure states. This is due in part to the emphasis given to pure states in many applications. Indeed, often one only considers the pure states. It is important to note that the set of pure states is in bijective correspondence with the set of all one-dimensional subspaces of the Hilbert space \( \mathcal{H} \). The latter set is called the (complex) projective space associated to \( \mathcal{H} \) and is denoted as \( \mathbb{C}\text{P}(\mathcal{H}) \). This projective space has many interesting mathematical properties.

The set of quantum events should be in bijection with the physical events of a given quantum system. If not, then one has not correctly chosen the von Neumann algebra for the system. The choice of the self-adjoint operators and the states, which are physically relevant, is not so obvious. It is generally accepted that the self-adjoint operators should be associated with the von Neumann algebra, as specified in Axiom 1, though it may be the case that not all such self-adjoint operators will have a physical significance. As for the states, my preference is to include first only those states \( |\psi\rangle\langle\psi| \) (with \( \psi \) a unit vector in \( \mathcal{H} \)) that are also events in the von Neumann algebra and then the mixed states formed from these, that is, their closed convex hull. To my way of thinking, this is not an ad hoc super-selection rule, but rather a way of carefully choosing a self-consistent model for a quantum system. Such a choice of model is called a quantization. A rule of thumb is that a quantization should be based on a von Neumann algebra that is generated by the smallest set of events needed to understand the quantum system. However, quantization remains to this day as much an art as a science.
### 1.3 Axiom 3 - Spin and Statistics

**Axiom 3: (Spin and Statistics)** The most basic quantum entities are either *bosons*, all of which have integer spin, or *fermions*, all of which have half-integer spin. All other quantum entities are composites of these.

The Hilbert space $\mathcal{H}$ for a single boson or a single fermion carries a *unitary representation* of the Lie group $SU(2)$. That representation defines the value of the spin. It seems that all ‘matter’ is composed of fermions, while all ‘interactions’ are mediated by bosons. But *dark matter* could be something else; we simply do not know. Also, gravitation is thought by many physicists to be mediated by *gravitons*, a spin 2 boson. However, we simply do not know if that is correct. So, this axiom may be changed some day.

The Hilbert space of composite quantum entities is given in terms of the Hilbert spaces of the constituent bosons and fermions by a non-intuitive mathematical construction. This axiom is included because of its importance in quantum theory. However, it is not presented in total detail since it is not going to play a role in this treatise, which focuses on probability and leaves spin and statistics to a side.

This axiom is a complicated condition that seems to have nothing to do with the other axioms, and it introduces spin into quantum theory in a seemingly *ad hoc* manner. If you like to think in terms of analogies, then the Fifth Axiom of Euclid comes to mind. While generations have fretted over deeper explanations of the origin of probability in quantum theory, nary anybody is much concerned with an underlying explanation of spin. But spin seems to be as much an essential, non-classical ingredient of quantum theory as is probability. The reader can refer to the literature on this topic, most notably in quantum field theory.

Another mathematical structure introduced in this axiom is that of a representation. There is no denying that this is an important concept in quantum physics. For example, see [32]. However, it will not play any role in this treatise. Perhaps some comments on this omission are in order. For example, the group symmetries of space and time, respectively, lead to the conservation of linear momentum and energy, respectively. Even though they are part of relativistic quantum theory, these symmetries are not a part of basic quantum theory as presented in this treatise. This is common in most introductory textbooks on quantum theory, which is a bit of an anomaly given that quantum Hamiltonians are often based on classical Hamiltonians.
representing the total energy of a system. Actually, the basics presented here are not explicitly invariant under representations of the Galilean group nor the Poincaré group, but rather consistent with either of these groups. Thinking that quantum theory automatically carries a representation of the Poincaré group led to the mistaken conviction that parity is conserved in all interactions in quantum theory. We also will not be considering quantum interactions explicitly, except in so far as they are a type of quantum event.

It may seem strange that the non-intuitive concept of spin enters an axiom, while the intuitive concept of position does not. Note that position, together with linear momentum, enter quantum theory via a representation of the Weyl-Heisenberg group. This could be taken as a shortcoming of the approach taken in this thesis. However, none of these particular observables will play a role here, though all of them could be present in a more complete axiomatization of quantum theory.

1.4 Axiom 4 - Time Independent Born’s Rule

**Axiom 4: (Time Independent Born’s Rule)** Let $T$ be a self-adjoint operator (perhaps associated with an observable quantity of a physical entity), $\rho$ be a density matrix and $B$ be a Borel subset of $\mathbb{R}$. Then the quantum probability that $T$ lies in the set $B$ given $\rho$ is defined as $P(T \in B \mid \rho) := Tr(P_T(B)\rho)$. Here $Tr$ is the trace of a trace class operator, and $P_T$ is the pvm associated to the self-adjoint operator $T$.

This axiom, or a simple consequence of it about expected values, is part of standard quantum theory as found in the textbooks and as practiced in the scientific community. Only later on will we put time dependence into this and, most importantly, elevate the resulting formula to be the basic time evolution equation of quantum theory.

For fixed $T$ and $\rho$ the assignment $B \mapsto P(T \in B \mid \rho) \in [0, 1]$ for $B$ a Borel subset of $\mathbb{R}$ is a probability measure on $\mathbb{R}$ in the sense of measure theory. The physical significance of quantum probability is that it corresponds to the relative frequency of the empirically observed quantities associated to the self-adjoint operator $T$.

To show the normalizations of this probability measure, we note that

$$P(T \in \emptyset \mid \rho) = Tr(P_T(\emptyset)\rho) = Tr(0\rho) = Tr(0) = 0$$
and

\[
P(T \in \mathbb{R} \mid \rho) = Tr(P_T(\mathbb{R}) \rho) = Tr(I \rho) = Tr(\rho) = 1.
\]

To show \(\sigma\)-additivity let \(\{B_j \mid j \in \mathbb{N}\}\) be a countable family of disjoint Borel subsets of \(\mathbb{R}\). We can calculate the trace of a trace class operator using any orthonormal basis of \(\mathcal{H}\). We choose an orthonormal basis \(\{\phi_k\}\) which diagonalizes the trace class operator \(\rho\). Specifically, \(\rho \phi_k = \lambda_k \phi_k\) where \(\lambda_k \geq 0\) for each index \(k\). (Also \(\sum_k \lambda_k = 1\).)

Then we have

\[
P(T \in \bigcup_j B_j \mid \rho) = Tr(P_T(\bigcup_j B_j) \rho)
= \sum_k \langle \phi_k, P_T(\bigcup_j B_j) \rho \phi_k \rangle
= \sum_k \langle \phi_k, \sum_j P_T(B_j) \rho \phi_k \rangle \quad \text{using strong operator topology}
= \sum_k \langle \phi_k, \sum_j P_T(B_j)(\lambda_k \phi_k) \rangle
= \sum_k \sum_j \lambda_k \langle \phi_k, P_T(B_j) \phi_k \rangle \quad \text{using Fubini’s theorem}
= \sum_j \sum_k \langle \phi_k, P_T(B_j)(\lambda_k \phi_k) \rangle
= \sum_j \sum_k \langle \phi_k, P_T(B_j) \rho \phi_k \rangle
= \sum_j Tr(P_T(B_j) \rho)
= \sum_j Tr(P_T(B_j) \phi) = \sum_j P(T \in B_j \mid \rho).
\]

We can apply Fubini’s theorem since all of the terms in the double sum are non-negative.

We discuss mainly the case of pure states \(\phi \in \mathcal{H}\) where \(||\phi|| = 1\), in which case \(\rho = |\phi\rangle\langle \phi|\). We then introduce the notation

\[
P(T \in B \mid \phi) := Tr(P_T(B) |\phi\rangle\langle \phi|) = \langle \phi, P_T(B) \phi \rangle,
\]
which we read as: The probability that $T$ has a value in $B$ given $\phi$. The notation as well as the way of reading it suggests that this is related to the concept of conditional probability in classical probability. We shall come back to this point.

There are alternative formulas for Born’s rule for the case of pure states. Here are two useful ones:

$$P(T \in B \mid \phi) = ||P_T(B)\phi||^2 = Tr(P_T(B)E_{\phi}).$$

Here $E_{\phi} := |\phi\rangle\langle\phi|$ in Dirac notation is a rank 1 projection, and $Tr$ denotes the trace of a trace class operator. Note that $E_{\phi}$ is a both a quantum event and a density matrix.

Axiom 4 is the simplest form of Born’s rule. It will be generalized and as such its central importance in quantum theory will become apparent.

## 1.5 Axiom 5 - Dynamics

**Axiom 5: (Dynamics)** Every physical system has two associated actions of one-parameter groups. These are denoted as $S_t$ and $E_t$, where $t \in \mathbb{R}$ is considered as a parameter in the theory which corresponds to time. (By one-parameter group we mean that $S_aS_b = S_{a+b}$ for all $a, b \in \mathbb{R}$ and that $S_0 = I$, the appropriate identity map. Similar formulas hold for $E_t$.)

The one-parameter group $S_t$ maps the convex set $S$ of all states (including mixed states) to itself, while the one-parameter group $E_t$ maps the set $E$ of all quantum events to itself.

The dynamics (or time evolution) of the physical system itself for an initial observable represented by a self-adjoint operator $T = T^*$ and an initial density matrix $\rho$ is given by the time dependent Born’s rule

$$t \mapsto Tr(E_t(P_T(B))S_t\rho) \quad \text{for } t \in \mathbb{R}, \quad (1.5.1)$$

provided that conservation of probability holds, which by definition means that for fixed $\rho$ and $T = T^*$ the mapping $B \mapsto Tr(E_t(P_T(B))S_t\rho)$ is a probability measure for every $t \in \mathbb{R}$. Here initial means at time $t = 0$. ■

The reason for having two one-parameter groups in (1.5.1) will become apparent in the next section.

Following tradition I have stated these as five separate axioms. Nonetheless, logically speaking Axioms 1 and 2 form one statement on kinematics,
while Axioms 4 and 5 are one statement on dynamics. Axiom 3 as noted above stands out as being quiet different, although in a technical sense it is also a statement on kinematics. A long, convoluted statement.

A special case of Axiom 5 is when $S_t$ maps the set of pure states to itself. Then we have that

$$S_t|\phi\rangle\langle\phi| = |\phi_t\rangle\langle\phi_t|$$

(1.5.2)

for some unit vector $\phi_t \in \mathcal{H}$ which is not uniquely determined, but is unique modulo a phase factor. In this case the dynamics is given by another version of Born’s rule:

$$t \mapsto Tr(E_t(P_T(B)) |\phi_t\rangle\langle\phi_t|) = \langle\phi_t, E_t(P_T(B))\phi_t\rangle = ||E_t(P_T(B))\phi_t||^2$$

(1.5.3)

provided again that conservation of probability holds.

1.6 Models of the Axioms

The time evolution of neither the events (given by $E_t$) nor of the states (given by $S_t$) is fundamental. What is fundamental in the sense that it corresponds to observations is the combination of these two one-parameter groups in the above expressions for the time dependent probability. Neither of these two one-parameter groups is uniquely determined by the axioms. There are various models of these axioms for which $S_t$ and $E_t$ are quite different. Some of these models are typically called pictures in the literature. Axiom 5 was written to include the three most commonly used models, which we present next. But other models are possible.

I am not the first to say that quantum theory is a way for calculating probabilities, and nothing else. I am saying here that different models have different ways for doing those calculations. Moreover, the fundamental, final formula for arriving at those calculations is the same in all models (namely, the Generalized Born’s Rule), which is covariance, and that the resulting number is the same in all models, which is invariance.

The most commonly used model is called the Schrödinger picture, but I prefer to call it the Schrödinger model. This model has the property that $E_t = I$ for all $t \in \mathbb{R}$, that is, the events have trivial time evolution. One also says that the observables, which are events since they are pvm’s, are time independent in this model. In this model the time evolution maps pure states to pure states and is given by Schrödinger’s equation. The solution of
this equation with an initial condition $\phi$ at time $t = 0$ is given by functional analysis as $\phi_t = e^{-itH/h} \phi$, where $H = H^*$ is the Hamiltonian in Schrödinger’s equation. (The minus sign in the exponent of $e^{-itH/h}$ is a purely conventional. It has no physical significance; it does not need to be ‘explained’.) Taking the initial condition $\phi$ to be a unit vector, it follows by the unitarity of $e^{-itH/h}$ that $\phi_t$ is a unit vector for all $t \in \mathbb{R}$. Then $S_t$ in the Schrödinger model is defined on pure states by the formula (1.5.2), and the time dependent Born’s rule gives the probability that the observable $T = T^*$ is in the Borel set $B$ as a function of time $t \in \mathbb{R}$ as

$$t \mapsto \langle \phi_t, P_T(B) \phi_t \rangle.$$ 

Since $\phi_t$ is a unit vector, for each fixed time $t$ this is a probability measure as we have seen earlier. That is to say, conservation of probability holds in the Schrödinger model.

Notice that Schrödinger’s equation is pushed into the background even in the Schrödinger model. All that is important is the Hamiltonian $H$ which in and of itself specifies the flow $t \mapsto \phi_t = e^{-itH/h} \phi$ in the Hilbert space $\mathcal{H}$. However, there is a preference for thinking that differential equations are basic and that their solutions, in this case the flow, are secondary. Of course, to understand this flow in specific cases it often is a good idea to solve Schrödinger’s equation. In introductory texts it is usually considered to be pedagogically advantageous to give Schrödinger’s equation a central role. And I do this in [31] for example. But this should be taken by those with more knowledge with a ton, not a grain, of salt.

Next, we extend $S_t$ to act on density matrices $\rho$ as follows. First, by the spectral theorem there exists an orthonormal basis $\{\phi_k\}$ of $\mathcal{H}$ and real numbers $\lambda_k \geq 0$ such that $\sum_k \lambda_k = 1$ and $\rho = \sum_k \lambda_k \langle \phi_k | \phi_k \rangle$. We also define $U_t := e^{-itH/h}$, the unitary time evolution operator, for all $t \in \mathbb{R}$. The time evolution of the ket is $|\phi_k \rangle \mapsto U_t |\phi_k \rangle$, while that of its dual bra is $\langle \phi_k | \mapsto \langle \phi_k | U_t^*$. So, $|\phi_k \rangle \langle \phi_k | \mapsto U_t |\phi_k \rangle \langle \phi_k | U_t^*$. Putting this together, the time evolution of $\rho$ is defined by

$$\rho \mapsto \sum_k \lambda_k (U_t |\phi_k \rangle \langle \phi_k |) U_t^* = U_t \left( \sum_k |\lambda_k \phi_k \rangle \langle \phi_k | \right) U_t^* = U_t \rho U_t^*.$$

This model is so widely used that properties specific to it are often thought to be general properties of quantum theory. I will discuss this misunderstanding in more detail later on.
Another model is called the Heisenberg picture, which I prefer to call the Heisenberg model. In this model one has $S_t = I$ for all $t \in \mathbb{R}$. One says that the states are time independent in this model. In particular, $S_t$ maps pure states to pure states. The time evolution of a quantum event $P$ is given in this model by the Heisenberg equation

$$P \mapsto E_t P := U_t^* P U_t,$$

where $U_t$ for $t \in \mathbb{R}$ is a strongly continuous unitary group acting on $\mathcal{H}$. It is important to note that this flow is basic; it is not arrived at as the solution of a differential equation. Next it follows that $U_t := e^{-itH/\hbar}$ with $H = H^*$ by Stone’s theorem. (The minus sign in the exponent of $e^{-itH/\hbar}$ is again purely conventional. It has no physical significance; it does not need to be ‘explained’.) If $P_A$ is the pvm of a self-adjoint operator $A$, then the same time evolution applies to it:

$$P_A \mapsto E_t P_A := U_t^* P_A U_t,$$

where $U_t^* P_A U_t(B) := U_t^* P_A(B) U_t$ for all Borel subsets $B$ of $\mathbb{R}$. It turns out from functional analysis that this time evolution maps a pvm, which is a family of projections, to another family of projections, which turns out itself to be a pvm. It follows that the pvm $P_A$ of a self-adjoint operator maps to the pvm of another self-adjoint operator. Explicitly, one can show that

$$E_t P_A = U_t^* P_A U_t = P_{U_t^* A U_t}.$$

Again, the Heisenberg model has certain specific properties which are not general properties of quantum theory. Such a particular property of the Heisenberg model is that $E_t$ extends naturally to a time evolution of $\mathcal{L}(\mathcal{H})$ defined by $E_t T := U_t^* T U_t$ for all $T \in \mathcal{L}(\mathcal{H})$. The group $E_t$ in any of these manifestations is difficult to accommodate with ordinary intuition. This says that observables are time dependent, which might seem sensible enough if one is speaking of position or angular momentum. However, events are self-adjoint operators; so they are observables too. So, in the Heisenberg model events are time dependent! On the other hand states, which intuitively tell us everything about a system at any moment, are time independent. This is backwards from the common intuition of what ‘events’ are and what ‘states’ are. In part this is due to a poor choice in terminology. A quantum event $E \notin \{0, I\}$ is an observable that has spectrum $\{0, 1\}$. This corresponds to two
Models of Axioms

possible observed values. Sometimes an event is called a Yes-No experiment. We tend to think that the value 1 (Yes) means that the event occurred, while the value 0 (No) means that the event did not occur. But this is misleading; an event is an observable that can give either of these two values. In either case the event has been observed. In any event (in another sense yet of that word) quantum theory is non-intuitive.

Also notice that the time evolution of events in the Schrödinger model extends naturally to $\mathcal{L}(\mathcal{H})$ by setting $E_t := I_{\mathcal{L}(\mathcal{H})}$ for all $t \in \mathbb{R}$. This particular extension property is not a necessary aspect of quantum theory.

To show that conservation of probability holds in the Heisenberg model let $\rho$ be a density matrix and $T = T^*$ be a self-adjoint operator. Then

$$Tr(E_t(P_T(B)) \rho) = Tr(U_t^* P_T(B) U_t \rho) = Tr(P_T(B) U_t \rho U_t^*) = Tr(P_T(B) \rho_t), \quad (1.6.2)$$

where $\rho_t := U_t \rho U_t^*$, the time evolved density matrix for every $t \in \mathbb{R}$, is also a density matrix. But we have already proved in our discussion of the Schrödinger model that the last expression gives a probability measure on $\mathbb{R}$.

The same Born’s rule (1.5.3) is used in the Schrödinger model and the Heisenberg model. What that means is that the same formula is used in both. This is covariance. But more is true. The two models give the same numerical probability using (1.5.3) provided that the same self-adjoint operator $H$ is used in both models. This is invariance. (In other words, compatible sign conventions are being used in the two models. This does have a significance, since it makes the following proof work.)

Here is the proof of this well-known, yet important result. We assume that in both models at time $t = 0$ the state is $\phi$ and that the pvm is $P_T$. Of course, in the Schrödinger model only the state can be time dependent, while in the Heisenberg model only the pvm can be time dependent. Be aware that all of the following expressions are time dependent. Then for all $t \in \mathbb{R}$ we have in an obvious notation that

$$P^{Sch}(T \in B \mid \phi_t) = ||P_T(B) \phi_t||^2 = ||P_T(B) e^{-itH/\hbar} \phi||^2 = ||e^{itH/\hbar} P_T(B) e^{-itH/\hbar} \phi||^2 = ||U_t^* P_T(B) U_t \phi||^2 = ||E_t(P_T(B)) \phi||^2 = P^{Heis}(E_t T \in B \mid \phi).$$
This is, of course, a special case of the following calculation. Let $\rho$ be a density matrix. Then we see that
\[
P_{\text{Sch}}(T \in B \mid \rho_t) = \text{Tr}(P_T(B) \rho_t) = \text{Tr}(P_T(B) U_t \rho U_t^*) \\
= \text{Tr}(U_t^* P_T(B) U_t \rho) = \text{Tr}(E_t P_T(B) \rho) \\
= P_{\text{Heis}}(E_t T \in B \mid \rho).
\]
(1.6.3)
This calculation should be compared with (1.6.2).

But still other models are used for which both $S_t$ and $E_t$ are non-trivial. The interaction pictures are such models. There are a multitude of such models. One starts by taking any one-parameter family of unitary operators $U_t$ for $t \in \mathbb{R}$ acting on the Hilbert space $\mathcal{H}$. Next one changes the Schrödinger model to become the new interaction model by transforming the pure states by $\psi \mapsto U_t \psi =: \psi'$ and mapping the pvm's by $P_S \mapsto U_t P_S U_t^* =: P_{S'}$, where $S$ is a self-adjoint operator and $U_t S U_t^* =: S'$. One then sees that
\[
P(S \in B \mid \psi) = \langle \psi, P_S(B) \psi \rangle = \langle U_t \psi, U_t P_S(B) U_t^* U_t \psi \rangle = P(S' \in B \mid \psi').
\]
This shows that the time dependent probability as calculated in each model is exactly the same even though the time dependence of both the states and the pvm’s has been changed. The transformation from the Schrödinger model to the Heisenberg model is a special case of this. Another very special case is to take $U_t := U$ for all $t \in \mathbb{R}$ where $U$ is a fixed unitary transformation.

Note that in the interaction model $U_t$ need not be a unitary group nor does $t \mapsto U_t$ need to have any sort of continuity. The lack of continuity, for example, may seem to you to be a mathematical trick with no physical intuition behind it. If so, you are right. The interaction model is just used as a convenient mathematical technique to help one calculate probabilities, and its intermediary steps have no physical significance.

All of these models are isomorphic (defined below) to the Schrödinger model, but there are other models which are not. The point of the axioms is to capture certain features, which are to be considered as basic to quantum theory. They are not meant to be categorical, in the same sense as, for example, the axioms of Euclidean plane geometry are meant to describe completely their topic. Rather the axioms for quantum theory are meant to be like the axioms in mathematics of a vector space, of which there are many non-isomorphic objects. Similarly, as we shall discuss in detail later, there are many non-isomorphic quantum theories. Here is that important definition.
**Definition 1.6.1** A model of quantum theory is a triple \((H, E, S)\) where \(H\) is a Hilbert space, \(E : \mathbb{R} \rightarrow E\) is a one-parameter group of bijections of the set \(E\) of events in \(L(H)\) to itself, and \(S : \mathbb{R} \rightarrow S\) is a one-parameter group of bijections of the set \(S\) of states in \(L(H)'\) to itself such that for every \(t \in \mathbb{R}\) the time dependent Born’s rule

\[
t \mapsto \operatorname{Tr}(E_t(P_T(B))S_t\rho)
\]

is a probability measure on \(\mathbb{R}\) for every density matrix \(\rho\), for all self-adjoint operators \(T\) and all Borel subsets \(B\) of \(\mathbb{R}\). As noted earlier this last condition is called conservation of probability.

An isomorphism of the quantum theories \((H, E, S)\) and \((H', E', S')\) is a one-parameter family of unitary transformations \(U_t : H \rightarrow H'\) which are onto and satisfy condition (1.6.4) below. Here the parameter is \(t \in \mathbb{R}\). In the model \((H, E, S)\) we let \(T = T^*\) be a self-adjoint operator and \(\rho\) be a density matrix. These correspond in the usual way in the model \((H', E', S')\) to the self-adjoint operators \(T' := U_tTU_t^*\) and to the density matrices \(\rho'(A) := \rho(U_t^*AU_t)\) for all \(A \in L(H')\). Then for all times \(t \in \mathbb{R}\) and for all Borel subsets \(B\) of \(\mathbb{R}\) we require that

\[
\operatorname{Tr}(E_t(P_T(B))S_t\rho) = \operatorname{Tr}(E'_t(P_{T'}(B))S'_t\rho')
\]

which is called preservation of probability. (This condition is based on the equality of probabilities (1.6.3) for the Schrödinger and Heisenberg models.)

Finally, we say that two models of quantum theory are isomorphic if there exists an isomorphism between them.

Please be careful to note the difference between conservation of probability and preservation of probability.

There will surely be those who wish to maintain that the time changing state and the Schrödinger model are correct while the corresponding theory in the Heisenberg model is not correct. This just amounts to rejecting the importance of the previous definition. In that case the test is to design an experiment whose results would be different in the two models. And then do the experiment to see which model is wrong. (Maybe both will be wrong!) But the Schrödinger model as presented here would have to be augmented with another measurable property besides the Generalized Born’s Rule, which is the same in the two models. In other words one would have to propose a property that only holds in the Schrödinger model.
1.7 How Many Time Evolutions?

A standard criticism of standard quantum theory is that it has two time evolutions: one from Schrödinger’s equation and the other from the collapse of the state. It is widely held that neither of these can be the consequence of the other. After all, Schrödinger’s equation gives a continuous time evolving state while the collapse is discontinuous.

Of course, one could try to explain collapse as a continuous change that is so quick as to appear discontinuous. Or, on the other hand, one might try to explain continuous time evolution as a rapid succession of discontinuous jumps, much as a motion picture is not a picture of continuous motion but rather many static pictures in succession giving the impression of motion. While these remain as logical possibilities, neither is readily implementable.

So, leaving these possibilities to a side, these two time evolutions are not only incompatible, but also neither can be the basic description of time evolution in quantum theory. There are logically two alternatives here. The first is to accept that nature is described by two independent time evolutions, and that’s the end of the matter. The second is that there is one time evolution which is somehow more basic than these two. I am advocating for the second option in a rather specific way.

1.8 Relation to Observation

It is crucial to understand that the relation of quantum theory to physical observations is based only on those statements in the theory that are model independent within an isomorphism class of models. Unitary transformations are the isomorphisms of Hilbert spaces. Physically, this means that the formulation of a theory in the context of one specific Hilbert space can always be translated into the context of a different isomorphic Hilbert space via the application of a unitary transformation. Such a unitary transformation changes the model being used to understand the physics. Only properties that hold in all isomorphic models can possibly be relevant to understanding physical phenomena. A property that holds in one model, but not in all other isomorphic models, must absolutely never be considered as having any physical significance. Such properties can be useful as intermediate steps in the analysis of a physical system, but nothing more than that. Such model dependent properties do not need to be ‘interpreted’ nor ‘explained’ vis a vis
their physical significance. They simply have no physical significance.

As an example, a Schrödinger operator acting in $L^2(\mathbb{R}^3)$ could have an eigenstate which is represented by a $C^\infty$ function. That function corresponds to a unit vector in any isomorphic Hilbert space, whose elements need not be represented by functions on a differential manifold. So the concept of $C^\infty$ does not apply in that isomorphic model. Nonetheless, the eigenvalue of the corresponding self-adjoint operator for the corresponding eigenstate is the same real number for all isomorphic models and has a physical significance, provided that the original Schrödinger operator represents a physical system. For example, the Segal-Bargmann space on $\mathbb{C}^3$ (see [2]) can be used instead of $L^2(\mathbb{R}^3)$. (A unitary transformation effecting this change of model is the Segal-Bargmann transform.) In the Segal-Bargmann model all pure states are $C^\infty$ functions. So in that model the property of a pure state being $C^\infty$ is unremarkable. As we shall see later, the failure to recognize this elementary aspect of quantum theory leads to pointless discussions about the ‘meaning’ or ‘interpretation’ of model dependent properties.
Chapter 2
Quantum Probability

2.1 Introduction

This chapter is meant to motivate, define and study the fundamentals of quantum probability. I start by explaining how certain classical probability measures (in the sense of the formulation of Kolmogorov in his book [20]) arise as a consequence of various hypotheses, which are accepted generally by physicists who work in quantum theory. The idea that probability is an added-on ‘interpretation’ of quantum theory is a misconception, that leads many to think that quantum probability can be replaced by using some alternative ‘interpretation’. While this has been known since at least since the publication of von Neumann’s book [25] in 1932, it seems not to be well known, neither in the mathematics nor the physics communities. Actually, just about everything in this chapter can be found in the literature, with the possible exception of the Generalized Born’s Rule as the fundamental time evolution equation of quantum theory. I have tried to re-organize this known, but not well known, material which has too often been overlooked by people who were, I am afraid, more interested in ‘interpreting’ quantum theory than in understanding it.

Before going into details it is necessary to understand what is meant by a
probability theory in the most general terms. But this is rather clear. It is any theory which contains formulas whose values are real numbers lying in the closed interval [0, 1]. The application of probability theory to experimental science is that these real numbers, called probabilities, have the significance that they describe relative frequencies of certain observations. There are many more details on both the theoretical and experimental sides, but this is the basic idea. One of the more important achievements of contemporary mathematics is the introduction of many new probability theories, such as free probability.

I understand Quantum Probability to mean that part of Quantum Theory which gives rise to probabilities and the rules that apply to them. As such it is not an independent theory, since it is essentially linked to the rest of Quantum Theory. In other words, Quantum Probability is a part of physics. It also is an example, (the first historically, I guess), of a Non-commutative Probability Theory, which is an ongoing research area in basic mathematics. But I shall not deal with this more general mathematical topic, except to note that it (typically?) does not come as part and parcel of a theory with time evolution nor is it intended to be a physical theory. For a glimpse into some beautiful advanced topics in quantum probability see [1], [27], [30] and references therein.

2.2 The Physical Assumptions

A given quantum system is described by a formulation based on a specific complex, non-zero Hilbert space $\mathcal{H}$ which is often, but not necessarily, infinite dimensional. This is implicit in Dirac’s notation, though it is usually accepted explicitly by those in the quantum physics community. Note that the case when $\mathcal{H}$ has dimension 1 is included, even though the resulting quantum physics is trivial. But we do exclude the case $\mathcal{H} = 0$, since that leads to no physics whatsoever, since there are no states and no events. So, $\mathcal{H} \neq 0$ in this treatise.

Within such a mathematical model quantum physics is assumed by most in the physics community to satisfy these properties among others:

- The state of a quantum system is described by vectors $\psi \in \mathcal{H}$ with $|\psi| = 1$ with two such unit vectors $\psi_1, \psi_2$ describing the same state provided that there exists $\lambda \in \mathbb{C}$ such that $\psi_1 = \lambda \psi_2$. (Necessarily $|\lambda| = 1$.)
• Most physical measurements (with the remarkable exception of time measurements) are represented by self-adjoint, densely defined linear operators acting on a linear subspace of \( \mathcal{H} \).

• The same physical measurement performed on an ensemble of quantum systems, which are all described by the same state, does not in general yield the same measured value, but rather values with some relative frequencies of numbers in some subset of \( \mathbb{R} \) with *more* than one number.

Various comments are in order. The first property only describes *pure states* and not the more general *mixed states*, which also enter into quantum theory. However, what we will be saying is easily generalized to the setting with all states, including the mixed states, under consideration.

The second property may have a converse, namely that every self-adjoint, densely defined linear operator corresponds to a physical measurement. This converse is denied by theories with *superselection rules*. But this possibility does not concern us, since we will not be using the converse statement.

The third property concerns what one observes in experiment. The only way to deny it is to re-define the meaning of the word ‘same’. But we will not use this property is the following argument, but rather offer a explanation of it using classical probability theory.

### 2.3 The Mathematical Model

The central mathematical fact is that there is a complete description of densely defined, self-adjoint operators which act in a Hilbert space. This is the content of the *Spectral Theorem*. This description is given in terms of *projection-valued measures*, which we will define momentarily. However, this section is not a complete presentation of the appropriate functional analysis, which I expect the reader already to know. Mainly it serves to establish notation and as a quick review.

First, we recall some basics from Hilbert space theory. We let \( \langle \cdot, \cdot \rangle \) denote the inner product in \( \mathcal{H} \), a complex Hilbert space, and \( \| \cdot \| \) its associated norm. One has \( \| \psi \| := \langle \psi, \psi \rangle^{1/2} \) for all \( \psi \in \mathcal{H} \). A function \( T : \mathcal{H} \to \mathcal{H} \) is said to be *linear* if, just as in linear algebra, we have

\[
T(\alpha_1 \psi_1 + \alpha_2 \psi_2) = \alpha_1 T(\psi_1) + \alpha_2 T(\psi_2)
\]
for all $\psi_1, \psi_2 \in \mathcal{H}$ and all $\alpha_1, \alpha_2 \in \mathbb{C}$. Furthermore, we say that such a function is a \textit{(linear) operator}. If such an operator $T$ has the property that there exists some real number $M \geq 0$ such that $||T\psi|| \leq M||\psi||$ holds for all $\psi \in \mathcal{H}$, then we say that the operator $T$ is \textit{bounded}.

A \textit{projection} $E : \mathcal{H} \to \mathcal{H}$ is by definition a linear operator such that $E$ is idempotent (meaning that $E^2 = E$) and $E$ is self-adjoint (meaning that $E^* = E$). Projections are bounded operators. It is important to note that 0, the zero operator (i.e., the operator which maps every $\psi \in \mathcal{H}$ to $0 \in \mathcal{H}$) is a projection. Also, $I_\mathcal{H}$, the identity operator acting on $\mathcal{H}$ is a projection, where by definition $I_\mathcal{H} \psi := \psi$ for all $\psi \in \mathcal{H}$. Also, we define

$$L(\mathcal{H}) := \{ T : \mathcal{H} \to \mathcal{H} \mid T \text{ is linear and bounded} \}.$$

For each $T \in L(\mathcal{H})$ we define its \textit{operator norm} as

$$||T||_{op} := \sup \{ ||T\psi|| \mid \psi \in \mathcal{H} \text{ and } ||\psi|| \leq 1 \}.$$

With this norm $L(\mathcal{H})$ becomes a complete, normed space, which is to say that it is a \textit{Banach space}. (Complete means that every Cauchy sequence with respect to the metric associated to the norm is convergent to an element in $L(\mathcal{H})$. The \textit{associated metric} is defined by $d_{op}(T_1, T_2) := ||T_1 - T_2||_{op}$ for all $T_1, T_2 \in L(\mathcal{H})$.)

As examples we mention that for any projection $E \neq 0$ we have that $||E||_{op} = 1$. Also, $||I_\mathcal{H}||_{op} = 1$ since $\mathcal{H} \neq 0$.

We recall from standard measure theory that the smallest $\sigma$-algebra, denoted as $\mathcal{B}(\mathbb{R})$, of subsets of $\mathbb{R}$ that contains all finite open interval $(a, b)$ (with $a, b \in \mathbb{R}$ and $a < b$) is called the \textit{Borel $\sigma$-algebra} of $\mathbb{R}$. Also, a set in $\mathcal{B}(\mathbb{R})$ is called a \textit{Borel set}. Notice that the empty set $\emptyset$ and the whole real line $\mathbb{R}$ are Borel sets.

We now have enough to give this definition.

\textbf{Definition 2.3.1} A projection-valued measure (or simply \textit{pvm}) associated to a Hilbert space $\mathcal{H}$ is a function $P : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ satisfying these properties:

- $P(B) \in L(\mathcal{H})$ is a projection for every Borel set $B \in \mathcal{B}(\mathbb{R})$.
- $P(\emptyset) = 0$, the zero operator; and $P(\mathbb{R}) = I_\mathcal{H}$, the identity operator acting on $\mathcal{H}$.
• Let \( \{B_j \mid j \in J\} \) be a finite or countably infinite family of Borel sets in \( \mathcal{B}(\mathbb{R}) \), which is disjoint (meaning that \( B_j \cap B_k = \emptyset \) whenever \( j \neq k \)). Then we have the \( \sigma \)-additivity condition

\[
P\left( \bigcup_{j \in J} B_j \right) = \sum_{j \in J} P(B_j),
\]

where the possibly infinite sum on the right side has the meaning that

\[
P\left( \bigcup_{j \in J} B_j \right) \psi = \sum_{j \in J} P(B_j) \psi
\]

holds for every \( \psi \in \mathcal{H} \). Here the possibly infinite sum on the right side is understood to mean the convergence with respect to the metric topology on \( \mathcal{H} \) induced by its norm. That metric, denoted as \( d \), is defined by \( d(\psi_1, \psi_2) := ||\psi_1 - \psi_2|| \) for all \( \psi_1, \psi_2 \in \mathcal{H} \). Also notice that \( \bigcup_{j \in J} B_j \) is a Borel set, by definition of \( \sigma \)-algebra, and so the left side is also well defined.

A curious consequence of this definition is that for every pvm we have \( P(B \cap C) = P(B)P(C) \) for all \( B, C \in \mathcal{B}(\mathbb{R}) \). And this in turn implies that the family of operators \( \{P(B) \mid B \in \mathcal{B}(\mathbb{R})\} \) is commutative.

The point of measure theory is to use measures in order to develop a well behaved theory of integrals. In courses this is typically done with measures with non-negative real values (such as Lebesgue measure), though also the generalization to measures with all real values or with complex values is sometimes presented. These generalizations are essentially the same as the theory with non-negative valued measures. Actually, this also works out for a measure \( \mu : \mathcal{B}(\mathbb{R}) \to X \), where \( X \) is any Hausdorff topological vector space. The topology gives a meaning to convergent infinite sums in \( X \). In particular, for any Borel measurable, function \( f : \mathbb{R} \to \mathbb{C} \) we can consider whether the integral \( \int_\mathbb{R} f(\lambda) d\mu(\lambda) \) exists as a uniquely defined element in \( X \). The usual technical, measure-theoretic details apply. In the case of interest here the vector space is the complex vector space \( \mathcal{L}(\mathcal{H}) \), but not with its norm topology. Instead, we use the strong operator topology on \( \mathcal{L}(\mathcal{H}) \). Without going into all the technical details, let’s simply note that a sequence of operators \( A_n \in \mathcal{L}(\mathcal{H}) \) converges to an operator \( A \in \mathcal{L}(\mathcal{H}) \) in the strong operator topology if and only if for every \( \phi \in \mathcal{H} \) the sequence \( A_n \phi \in \mathcal{H} \) converges to \( A\phi \in \mathcal{H} \) in the norm topology of \( \mathcal{H} \). The strong
operator topology is the same as the operator norm topology on \( \mathcal{L}(\mathcal{H}) \) if and only if the dimension of \( \mathcal{H} \) is finite. Using the norm topology in the infinite dimensional case gives an integral inadequate for spectral theory and hence inadequate for quantum theory.

A pvm \( P \) is a measure taking values in \( \mathcal{L}(\mathcal{H}) \), a Hausdorff topological vector space when endowed with the strong operator topology. And so we have a well defined (and well behaved in the sense of measure theory) integral \( \int_{\mathbb{R}} dP(\lambda) f(\lambda) \in \mathcal{L}(\mathcal{H}) \) for any essentially bounded, Borel function \( f : \mathbb{R} \to \mathbb{C} \). Also, an integral can even be defined for some unbounded Borel functions, but that leads to ever more technical details, which the reader can find in any good functional analysis text. Unfortunately, in the statement of the Spectral Theorem, which we state next, the integrand is an unbounded Borel function in the important case when the operator is not bounded.

**Theorem 2.3.1 (Spectral Theorem)** Suppose \( A \) that is a densely defined self-adjoint operator acting in a Hilbert space \( \mathcal{H} \). Then there exists a unique pvm \( P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \) such that

\[
A = \int_{\mathbb{R}} \lambda dP(\lambda). \tag{2.3.1}
\]

Many comments are in order, including why this theorem is named so. As a start we have the annoying fact that the function that we are integrating in (2.3.1) is the function \( f(\lambda) = \lambda \) for all \( \lambda \in \mathbb{R} \), and this is not a bounded function, though it is a Borel function. Here measure theory helps, if the pvm has bounded support. The definition of the support of a pvm \( P \) is much the same as in regular measure theory:

\[
\text{Supp } P := \mathbb{R} \setminus \cup \{ U \subset \mathbb{R} \mid P(U) = 0 \text{ and } U \text{ is open} \}.
\]

The only, quite minor difference is that 0 in the condition \( P(U) = 0 \) refers to the projection operator 0. By definition, \( \text{Supp } P \) is a closed subset of \( \mathbb{R} \), being the complement of an open subset. It also cannot be empty, since if it were we would have that \( I_{\mathcal{H}} = P(\mathbb{R}) = 0 \), which is impossible since \( \mathcal{H} \neq 0 \).

By measure theory we always have

\[
A = \int_{\text{Supp } P} \lambda dP(\lambda).
\]

So, if \( \text{Supp } P \) is a bounded subset of \( \mathbb{R} \), then this integral exists, since the integrand on this subset is a bounded Borel function.
Here is where we see spectral theory entering the theory. First, we review some material from the functional analysis of Hilbert spaces.

**Definition 2.3.2** Suppose that \( A : \mathcal{D} \to \mathcal{H} \) is a linear operator defined on the dense subspace \( \mathcal{D} \) of a Hilbert space \( \mathcal{H} \). Then we define the resolvent set of \( A \) to be the following subset of the complex numbers:

\[
\text{Res}(A) := \{ \lambda \in \mathbb{C} \mid \exists T_\lambda \in \mathcal{L}(\mathcal{H}) \text{ s.t. } (\lambda I - A)T_\lambda = I_\mathcal{H} \text{ and } T_\lambda(\lambda I - A) = \text{id}_\mathcal{D} \}.
\]

One says \( \lambda \) is in the resolvent set provided that the (densely defined) operator \( \lambda I - A \) has a globally defined, bounded inverse. Then the spectrum is defined as the complementary subset of \( \mathbb{C} \), that is

\[
\text{Spec}(A) := \mathbb{C} \setminus \text{Res}(A).
\]

In functional analysis one proves that for self-adjoint operators \( A \) we have that \( \text{Spec}(A) \) is a closed, non-empty subset of the real line \( \mathbb{R} \) and, moreover, that \( \text{Spec}(A) \) is a bounded subset if and only if \( A \) is a bounded operator. (The diligent reader will have noticed that we have not defined what it means for a densely defined operator to be bounded. Even worse, we have not defined what it means for a densely defined operator to be self-adjoint. See your favorite functional analysis text for these details.)

Now we come to a relation between the pvm and spectral theory.

**Theorem 2.3.2** Let \( A \) be a densely defined self-adjoint operator, and let \( P_A \) be the unique pvm associated to \( A \) by the spectral theorem. Then

\[
\text{Supp} P_A = \text{Spec}(A).
\]

Putting these results together we see that for bounded self-adjoint operators the integral in (2.3.1) is well-defined. Of course, it remains to prove the equation (2.3.1). As is typical in functional analysis there are a lot of technical details to deal with in order to give meaning to the integral in (2.3.1) for an unbounded self-adjoint operator \( A \) and, having done that, to prove the equation (2.3.1) holds.

### 2.4 The Finite Dimensional Case

Leaving those technical details to the texts, let us understand in detail what the Spectral Theorem says in the case when \( \mathcal{H} \) is finite dimensional, since
Finite Dimensional Case

this is often not presented in a course of functional analysis. In that case we have a situation in linear algebra. The self-adjoint operator is then identified with a Hermitian $n \times n$ matrix $A$ with $n \geq 1$. Since only the whole space $\mathcal{H}$ is a dense subspace (because $\dim \mathcal{H}$ is finite), the matrix $A$ maps all of $\mathcal{H}$ to itself. In linear algebra one proves that the set of eigenvalues is a finite, non-empty subset of $\mathbb{R}$. Let $\{\lambda_1, \cdots, \lambda_k\}$ where $1 \leq k \leq n$, denote that subset of eigenvalues, that is these are the $k$ distinct eigenvalues of $A$. The diagonalization theorem for $A$ says that there is a basis of $A$ with respect to which $A$ has diagonal form with the eigenvalues appearing along the diagonal, each with a number of times equal to its multiplicity. Let’s put this statement into another equivalent notation. For each $1 \leq j \leq k$ we define the spectral subspace associated with the eigenvalue $\lambda_j$ by

$$V_j := \{\psi \in \mathcal{H} \mid A\psi = \lambda_j \psi\}.$$ 

Since $\lambda_j$ is an eigenvalue, there is an eigenvector (non-zero, by definition) in $V_j$. In short, $V_j \neq 0$. One proves readily that $V_i$ and $V_j$ are orthogonal subspaces, since they correspond to eigenvalues $\lambda_i \neq \lambda_j$. Then the diagonalization of $A$ is realized by choosing a basis $B_j$ of $V_j$ for each $1 \leq j \leq k$ and proving that the union $\mathcal{B} := \cup_j B_j$ is a basis of $\mathcal{H}$. Since $A$ restricted to $V_j$ is simply multiplication by $\lambda_j$, the matrix of $A$ in the basis $\mathcal{B}$ is the sought for diagonal matrix representation of $A$. What this means is that we have an orthogonal decomposition

$$\mathcal{H} = V_1 \oplus \cdots \oplus V_k \quad (2.4.1)$$

such that $A$ acts as multiplication by a scalar on each summand.

We now put this into the language of projections. For each spectral subspace $V_j$ there is a unique projection such that $\text{Ran} P_j = V_j$. The very fact that these are projections says that $P_j^* P_j = P_j$ and $P_j^2 = P_j$. Moreover, the fact that $V_i$ is orthogonal to $V_j$ for all $i \neq j$ implies that $P_i P_j = 0$ for $i \neq j$. Note that this property is not shared by numbers (say, real or complex numbers as you wish) since in that case the product $ab = 0$ implies either $a = 0$ or $b = 0$. But these non-zero projections satisfy $P_i P_j = 0$ for $i \neq j$.

In any event we use this convenient condensed notation: $P_i P_j = \delta_{ij} P_i$ for all $1 \leq i, j \leq k$, where $\delta_{ij}$ is the Kronecker delta. And we can write (2.4.1) in this notation as:

$$I_{\mathcal{H}} = P_1 + \cdots + P_k,$$
which is called a resolution of the identity. This is not too exciting, since there are many, many resolutions of the identity. What makes this resolution of the identity interesting is that it comes from the diagonalization of \( A \). In fact, since \( A \) acts as multiplication by the eigenvalue \( \lambda_j \) on the subspace \( V_j \), it follows quickly that

\[
A = \lambda_1 P_1 + \cdots + \lambda_k P_k,
\]

which is called the spectral representation of the Hermitian matrix \( A \). This is completely equivalent to the diagonalization of \( A \). We next re-write the finite sum on the right side of (2.4.2) as an integral with respect to a pvm \( P \). We clearly want \( P \) to satisfy \( P(\{\lambda_j\}) = P_j \) for each \( 1 \leq j \leq k \). We also want \( P(\mathbb{R} \setminus \text{Spec} A) = 0 \), the zero operator. If we can do that, then by standard identities of measure theory (only now applied to a pvm), we have

\[
\int_{\mathbb{R}} \lambda dP(\lambda) = \int_{\mathbb{R} \setminus \text{Spec} A} \lambda dP(\lambda) + \int_{\text{Spec} A} \lambda dP(\lambda)
\]

\[
= 0 + \sum_{j=1}^{k} \int_{\{\lambda_j\}} \lambda dP(\lambda)
\]

\[
= \sum_{j=1}^{k} \lambda_j \int_{\{\lambda_j\}} \lambda dP(\lambda)
\]

\[
= \lambda_1 P_1 + \cdots + \lambda_k P_k
\]

as desired. So it only remains to define the pvm with the required properties. But for every Borel subset \( B \subset \mathbb{R} \) we simply define

\[
P(B) := \sum_{j=1}^{k} \chi_B(\lambda_j^j) P_j,
\]

where the characteristic function \( \chi_S \) of any set \( S \) is defined as

\[
\chi_S(\lambda) := \begin{cases} 
1 & \text{if } \lambda \in S, \\
0 & \text{if } \lambda \notin S.
\end{cases}
\]

An alternative way to write this is as

\[
P(B) = \sum_{\{j \mid \lambda_j \in B\}} P_j,
\]
that is, add up all the projections associated to the eigenvalues that lie in $B$.

One readily checks that $P$ so defined is a pvm and that it satisfies the desired properties. The moral of this story is that the diagonalization of a Hermitian matrix is exactly the finite-dimensional special case of the Spectral Theorem. Another way to phrase this is to say that the Spectral Theorem generalizes the diagonalization of a Hermitian matrix to the infinite dimensional case. Notice that in this generalization a finite sum has been replaced by an integral, which in certain cases will be a finite sum and in other cases will be an infinite sum. However, the complete generalization to the infinite dimensional case requires integrals and the corresponding measure theory with pvm’s.

2.5 Quantum Probability: The First Steps

With all this mathematical background we are ready to see probability theory in the context of quantum theory. This way of introducing probability into quantum physics is what is called Quantum Probability.

The first thing to note is that a pvm is quite similar, though not identical, to a probability measure. This is seen for example in the normalization conditions. But even more impressive is that there is a partial order on self-adjoint operators, which we can restrict to the special case of projection operators (which are, as you will recall, self-adjoint). It turns out that for any projection $P$ we have $0 \leq P \leq I_H$, which are inequalities of self-adjoint operators. So projections lie between the two extreme projections possible. This is analogous to classical probability theory, where any probability $p$ lies between the two extreme probabilities, namely $0 \leq p \leq 1$, which are inequalities of numbers.

To make the relation tighter with classical probability, we note that there is a converse of the Spectral Theorem.

**Theorem 2.5.1 (Converse to Spectral Theorem)** Suppose that $\mathcal{H}$ is a Hilbert space and $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be a pvm. Then

$$A := \int_{\mathbb{R}} \lambda dP(\lambda)$$

(2.5.1)

defines a self-adjoint operator, densely defined in $\mathcal{H}$. Moreover, the pvm $P_A$ associated to $A$ satisfies $P_A = A$. 
Moreover, and more importantly, this result together with the Spectral
Theorem give a bijection between the set of all self-adjoint operators acting in
$\mathcal{H}$ and the set of all pvm’s taking values in $L(\mathcal{H})$. One way the bijection sends
the self-adjoint operator $A$ to its pvm $P_A$. The other way a pvm $P$ is sent to
the self-adjoint operator given in (2.5.1). This is an amazing result, since it
sets up a dictionary between objects of analysis (self-adjoint operators) and
objects of measure theory (pvm’s).

Therefore the second basic assumption of quantum theory which says that
physical measurements are represented by self-adjoint operators translates to
saying that physical measurements are represented by pvm’s. We are getting
very close to classical probability theory using just a basic assumption of
quantum theory. To take the next step we consider a self-adjoint operator $A$
(or equivalently, its pvm $P_A$) and a state $\psi \in \mathcal{H}$. (Recall that $||\psi|| = 1$.)
Here we are using the first basic assumption of quantum theory. From these
mathematical objects, given to us by quantum physics, we define for every
Borel subset $B \subset \mathbb{R}$ and unit vector $\psi \in \mathcal{H}$ the expression on the left side
here:

$$P(A \in B \mid \psi) := \langle \psi, P_A(B) \psi \rangle.$$  \hfill (2.5.2)

We read the left side in full detail as follows: The probability that the
measurement of $A$ gives a value in $B$ given that the quantum system is
‘described’ by the state $\psi$. A more colloquial reading is: The probability
that $A$ is in $B$ given $\psi$. Of course, this terminology should be justified. The
first result to prove is that the right side satisfies $0 \leq \langle \psi, P_A(B) \psi \rangle \leq 1$.
The next thing to prove is that for the state $\psi$ fixed and for the self-adjoint
operator $A$ fixed, the function defined for every Borel subset $B$ of $\mathbb{R}$ by
$B \mapsto \langle \psi, P_A(B) \psi \rangle$ is a classical probability measure on the real line $\mathbb{R}$, that
is, it is $\sigma$-additive and satisfies the normalization of a classical probability
measure:

$$\langle \psi, P_\emptyset(B) \psi \rangle = 0 \quad \text{and} \quad \langle \psi, P_A(\mathbb{R}) \psi \rangle = 1.$$

All of this follows trivially from the definitions and standard properties of
Hilbert space. I hope the reader appreciates that the tricky bit in this theory
is getting the definitions right. In functional analysis one calls the probability
measures in (2.5.2) the spectral measures. But in the physics community,
(2.5.2) is known as Born’s rule, and although Max Born stated it differently
it was worthy of his receiving the Nobel prize mainly for this achievement in
quantum physics. It is no exaggeration to say that Born’s rule is Quantum
Probability.
Let us pause for an interlude to remark that there are some equivalent formulas for Born’s rule \((2.5.2)\). Continuing with the notation established above and using standard results in Hilbert space theory we have

\[
P(A \in B \mid \psi) = \langle \psi, P_A(B) \psi \rangle = ||P_A(B) \psi||^2 = Tr(P_A(B) E_\psi) = Tr(E_\psi P_A(B))
\]

where \(E_\psi := |\psi\rangle\langle \psi|\) is a rank 1 quantum event (using Dirac notation) and \(Tr\) denotes the trace of a trace class operator. The last two formulas have the advantage of expressing the probability of Born’s rule in terms of the trace.

This is also a good time to note that there is something more general going on here. The event \(P_A(B)\) is the formulas above can be replaced by any event \(E\) to give this definition of the probability of a quantum event, given a pure state \(\psi\), as

\[
P(E \mid \psi) := \langle \psi, E \psi \rangle = ||E\psi||^2 = Tr(E_\psi E).
\]

(2.5.3)

The reader should note that this is a new meaning for the word “probability”. Quantum events do not form a \(\sigma\)-algebra if \(\dim \mathcal{H} \geq 2\), However, there are enough properties to justify this terminology. First, \(0 \leq P(E \mid \psi) \leq 1\) by basic Hilbert space theory. So we get a real number that can be compared with a relative frequency. Next, it satisfies the normalizations

\[
P(I \mid \psi) = 1 \quad \text{and} \quad P(0 \mid \psi) = 0,
\]

where the 0 on the left side of the last equation is the zero operator. This topic will be presented in more detail in Section 2.8.

And definition (2.5.3) in turn is a particular case of the expected value of an observable \(A = A^*\) with respect to \(\psi\), which is defined as

\[
\mathcal{E}_\psi(A) := \langle \psi, A \psi \rangle.
\]

(2.5.4)

And one more generalization is obtained by using the definition (2.5.4) for all \(A \in \mathcal{L}(\mathcal{H})\). The linear map \(\mathcal{E}_\psi : \mathcal{L}(\mathcal{H}) \to \mathbb{C}\) so obtained gives an example of a non-commutative integral. The elegant theory of non-commutative integration is found in many places in the literature.

Let’s return to the main argument. So by applying two basic properties of quantum theory (which really are widely accepted) we have arrived at an infinite number of classical probability measures for a given observable. As I
like to say: This is a feature, not a bug. It only remains to understand what, if any, role these classical probability measures play in quantum physics. It is difficult to argue that they have no relevance or meaning in quantum physics, since they arise from two basic properties of quantum theory plus some (well, admittedly a lot of) mathematics. It is difficult to argue that there they are, but who cares? The standard way of dealing with this situation is to use these probability measures to save (‘explain’ if you wish) the phenomena of the relative frequencies that occur in the third basic property of quantum theory. However, do notice that our argument for arriving at these classical probability measures did not use this third basic property as an assumption.

It might appear that I have proved Axiom 4 in Chapter 1 from other axioms of quantum theory. That is not correct, since Axiom 4 is only a mathematical definition within the theory. However, it is a definition with a lot of physical significance since it is related to observations. Using the probability measures in (2.5.2) to understand measured relative frequencies is the physical significance of Axiom 4 as well as of the time dependent Born’s rule (1.5.1) in Axiom 5.

However, two generally accepted first principles of quantum theory have led us to the doorstep of Axiom 4. We just need an extra shove to cross the threshold to arrive at Axiom 4. This is quite different from Schrödinger’s equation, though some authors put it on an equal footing with Born’s Rule. But mathematical considerations alone do not get one close to Schrödinger’s equation. One could just as well arrive at the Klein-Gordon equation or the Dirac equation. I consider Born’s rule as very nearly inevitable, but not Schrödinger’s equation. Later I shall argue that it is reasonable to replace Schrödinger’s equation, although I prefer to retain Born’s Rule.

But we have yet to see the generalized Born’s rule. The next section is a step in that direction.
2.6 Quantum Conditional Probability

The probability of an event occurring, given that another event has already occurred, is a standard topic in classical probability. This corresponds to the relative frequencies measured for such a pair of events. This is known as conditional probability. So we now consider the quantum analogue of conditional probability. Recall that if \((\Omega, \mathcal{F}, P)\) is a classical probability space we define the conditional probability of an event \(E_1 \in \mathcal{F}\) given the event \(E_2 \in \mathcal{F}\) as

\[
P(E_1|E_2) := \begin{cases} 
\frac{P(E_1 \cap E_2)}{P(E_2)} & \text{if } P(E_2) \neq 0 \\
0 & \text{otherwise.} 
\end{cases} 
\] (2.6.1)

We immediately have \(P(E_1|E_2)P(E_2) = P(E_1 \cap E_2)\) for all \(E_1, E_2 \in \mathcal{F}\). From this we get Bayes Theorem

\[
P(E_1|E_2)P(E_2) = P(E_1 \cap E_2) = P(E_2 \cap E_1) = P(E_2|E_1)P(E_1). 
\]

We shall see that these simple formulas do not carry over to quantum probability. So this classical probability theory serves as an analogy, nothing more.

Now suppose that \(S, T\) are self-adjoint operators acting in \(\mathcal{H}\) and that \(B, C\) are Borel subsets of \(\mathbb{R}\). The probability that \(S\) has a value in \(B\) for a state \(\psi\) is given, as we have seen, by Born’s rule:

\[
P(S \in B \mid \psi) = \langle \psi, P_S(B)\psi \rangle = ||P_S(B)\psi||^2. 
\] (2.6.2)

We wish to define the conditional probability that \(T \in C\) given that \(S \in B\) for a state \(\psi\). This will be denoted as \(P(T \in C \mid S \in B, \psi)\). We assume that this has the form \(P(T \in C \mid \psi_1)\) for some state \(\psi_1\), that is to say, it will be a quantum probability of a certain form. The choice of \(\psi_1\) depends on the first event \(S \in B\) and the state \(\psi\). The usual definition is

\[
\psi_1 := \frac{P_S(B)\psi}{||P_S(B)\psi||} 
\] (2.6.3)

provided \(P_S(B)\psi \neq 0\). This definition has been given several unfortunate names. I will comment on those names and the significance of this definition after we have seen some of this material developed further. But first the important definition that this leads up to.
Definition 2.6.1 Let $S, T$ be self-adjoint operators, $B, C$ Borel subsets of $\mathbb{R}$ and $\psi$ a unit vector. Then the (quantum) conditional probability that $T$ has a value in $C$, given both that a state $\psi$ is given and that $S$ has a given value in $B$, is defined by

$$P(T \in C \mid S \in B, \psi) := P(T \in C \mid \psi_1) = P\left( T \in C \mid \frac{P_S(B)\psi}{||P_S(B)\psi||} \right)$$

if $P_S(B)\psi \neq 0$. Also, if $P_S(B)\psi = 0$ we define $P(T \in C \mid S \in B, \psi) := 0$.

Some might prefer to leave the expression $P(T \in C \mid S \in B, \psi)$ undefined in case $P_S(B)\psi = 0$. Taking $S, T, B, \psi$ fixed and if also $P_S(B)\psi \neq 0$, then the formula $C \mapsto P(T \in C \mid S \in B, \psi) = P(T \in C \mid \psi_1)$ shows that this expression defines a probability measure on $\mathbb{R}$. By using the formula in the next proposition one can see that the expression $P(T \in C \mid S \in B, \psi)$ is not even finitely additive in $B$.

Proposition 2.6.1 With the notation of this definition we have that

$$P(T \in C \mid S \in B, \psi) = \frac{||P_T(C)P_S(B)\psi||^2}{P(S \in B \mid \psi)} \quad (2.6.4)$$

for $P(S \in B \mid \psi) \neq 0$.

Proof: Expanding this definition out we see that the quantum conditional probability satisfies

$$P(T \in C \mid S \in B, \psi) = P\left( T \in C \mid \frac{P_S(B)\psi}{||P_S(B)\psi||} \right)$$

$$= \left\langle \frac{P_S(B)\psi}{||P_S(B)\psi||}, P_T(C) \frac{P_S(B)\psi}{||P_S(B)\psi||} \right\rangle = \frac{1}{||P_S(B)\psi||^2} \langle P_S(B)\psi, P_T(C)P_S(B)\psi \rangle$$

$$= \frac{||P_T(C)P_S(B)\psi||^2}{P(S \in B \mid \psi)} . \quad \blacksquare$$

For computations the expression (2.6.4) is much more useful than the definition. Also it immediately follows for any value of $P_S(B)\psi$ that

$$P(T \in C \mid S \in B, \psi) P(S \in B \mid \psi) = ||P_T(C)P_S(B)\psi||^2.$$
This definition of conditional probability immediately allows us to define independence. This topic is discussed in the next section in the more general context of quantum events.

Another immediate consequence of the definition is that for any $T = T^*$ and Borel subset $C$ of $\mathbb{R}$ we have $P(T \in C \mid T \in C, \psi) = 1$ for all $\psi$ satisfying $P(T \in C \mid \psi) \neq 0$. This should be compared with the equally trivial identity that holds in classical probability theory: $P(E \mid E) = 1$ for every event $E$, provided that $P(E) \neq 0$.

After seeing all this the reader should realize that the definition (2.6.3) plays an intermediate, minor role and could have been incorporated directly into the definition (2.6.4) without further ado. But much ado has been spent on this so-called collapse condition (2.6.3). It has also been dubbed the quantum jump and the projection postulate. So much ink has been spilled to ‘explain’ or ‘justify’ this formula (often under the term ‘interpretation’) that I am obliged to make some comments about it. The points of contact with observational data are the relevant, essential features of this, or any, theory. And for quantum theory those are its probability measures on $\mathbb{R}$. This is the central importance of Born’s rule (2.5.2), which on the theoretical side is simply a definition. Similarly, the Generalized Born’s Rule (2.9.2) below is simply a definition in the theory. Its importance comes from its relevance for understanding experimentally measured relative frequencies. When looking at different models of quantum theory, it is perfectly acceptable to have discrepancies over matters that do not have anything to do with observations. What is relevant are the probabilities, not the specific manners for calculating them in a particular model.

For example, in the Schrödinger model the states vary in time while the pvm’s do not. On the contrary, in the Heisenberg model the opposite obtains: The pvm’s vary in time while the states do not. What is the same in both are the probabilities for the same events. This is what makes these two models isomorphic. It is important to note that Born’s rule and its generalization are exactly the same formulas in both of these models. This is covariance. When the time dependence is introduced into these equations (but in different manners depending on the model chosen), the calculated time dependent probabilities are identically equal in these two models. This is invariance. In the Schrödinger model the collapse formula (2.6.3) does have a mathematical sense to it, since states are allowed to change in that model. However, it is devoid of any physical significance since it is not part of other equivalent models, such as the Heisenberg model in which states do
not change. Do be careful! The physical significance of the theory comes only later in the formula for the quantum conditional probability and the application of it to quantum events.

2.7 Quantum Consecutive Probability

Classical probability is a guiding analogy in the next definition. But both experiment and theory indicate that a time ordered sequence of two specific observations yielding certain values should have an associated probability. After all the relative frequencies of such sequences are often measured.

**Definition 2.7.1** Let $S, T$ be self-adjoint operators, $B, C$ Borel subsets of $\mathbb{R}$ and $\psi$ a unit vector. Then the (quantum) consecutive probability is defined as

$$P(S \in B, T \in C \mid \psi) := ||P_T(C)P_S(B)\psi||^2.$$  \hfill (2.7.1)

The left side of this should be read as follows: The probability that $S$ takes a value in $B$ and subsequently $T$ takes a value in $C$, given the state $\psi$.

Of course, one motivation of this definition is that it allows us to write (2.6.4) in this form reminiscent of classical conditional probability:

$$P(T \in C \mid S \in B, \psi) = \frac{P(S \in B, T \in C \mid \psi)}{P(S \in B \mid \psi)}$$ \hfill (2.7.2)

provided that $P(S \in B \mid \psi) \neq 0$.

Some authors have the previous definition only for the case of commuting operators. Some say it is not meaningful for non-commutating operators, while others react adversely the fact that $P_T(C)P_S(B)$ is not necessarily an event and so should not have an associated probability. However, I have found this definition in the general non-commutative case in the literature though I think it has been unduly neglected. But that literature is vast, and my reading of it is necessarily limited.

The expressions (2.6.4) and (2.7.1) are generalization of Born’s rule for calculating quantum probabilities. They are special cases of the generalized Born’s rule, which will be presented later.

We previously showed that Born’s rule (2.6.2) is invariant if we transform the states by $\psi \mapsto U_t\psi =: \psi'$ and the pvm’s by $P_S \mapsto U_tP_SU_t^* =: P_{S'}$, where $U_t$ is unitary for all $t \in \mathbb{R}$. We will now show that the conditional probability
and the generalized Born’s rule \((2.7.1)\) are also invariant under this transformation. Of course, we also transform \(P_T \mapsto U_t P_T U_t^* =: P_{T'}\). Given this notation we now have for \(P(S' \in B) \mid \psi') = P(S \in B) \mid \psi) \neq 0\) that the conditional probability \((2.6.4)\) satisfies

\[
P(T' \in C \mid S' \in B, \psi') = \frac{|P_{T'}(C) P_{S'}(B) \psi'|^2}{P(S' \in B) \mid \psi')} = \frac{|[U_t P_T(C) U_t^* U_t P_S(B) U_t^* U_t \psi]|^2}{P(S \in B) \mid \psi) = \frac{|P_T(C) P_S(B) \psi|^2}{P(S \in B) \mid \psi)} = P(T \in C \mid S \in B, \psi).
\]

On the other hand, if \(P(S' \in B) \mid \psi') = P(S \in B) \mid \psi) = 0\), then

\[
P(T \in C \mid S \in B, \psi) = 0 = P(T' \in C \mid S' \in B, \psi').
\]

This proves that the quantum conditional probability is invariant under this transformation. The proof that the generalized Born’s rule is also invariant is much the same. The importance of these invariances is that these quantum probabilities give the same results in all isomorphic quantum theories.

Note that the self-adjoint operators \(S\) and \(T\) commute (by definition!) if and only if every projection \(P_S(B)\) commutes with every projection \(P_T(C)\). So for commuting \(S\) and \(T\) we have by \((2.7.1)\) that

\[
P(S \in B, T \in C \mid \psi) = P(T \in C, S \in B \mid \psi).
\]

This says that in this case changing the temporal order of the two events does not change the combined probability. For such events simultaneity of the two events makes sense and, again, gives the same combined probability \((2.7.3)\).

However, it is not too difficult to find examples (thinking of spin \(1/2\), say) of non-commuting self-adjoint operators \(S\) and \(T\) for which there exist Borel sets \(B, C\) and a state \(\psi\) satisfying

\[
P(S \in B, T \in C \mid \psi) \neq P(T \in C, S \in B \mid \psi).
\]

This implies that for such \(S\) and \(T\) the time order of the two events must not include simultaneity. This contrasts with the case of commuting \(S\) and \(T\) where simultaneity is allowed.
Perhaps a word or two is in order about the notion of simultaneity. I have not assumed Galilean invariance of this theory. Nor have I assumed Lorentz invariance. Either one of these invariances can be incorporated into quantum theory. Of course, these lead to non-isomorphic models of quantum theory. But in either case there is a well defined notion of simultaneity of events with respect to an inertial frame of reference. That is the notion of simultaneity that should be applied here. I suppose that this is how simultaneity should be understood in a quantum theory of gravitation, even though such a theory does not exist at the time of writing this. This also relates to the question of the time ordering of events in general. In a Galilean theory this order is absolute, but in a Lorentzian theory it is only absolute for events that are light-like or time-like with respect to each other. (I am excluding time reversal as a symmetry, of course.) This motivates the extra hypothesis for Lorentz invariant quantum theories that two events that are space-like should commute.

This discussion has relevance to the well known—**and false**—assertion that “the electron can be at two places at the same time”. Let $Q$ represent the position operator for the electron. It is clear that $Q$ commutes with itself. So for any state $\psi$

$$P(Q \in B, Q \in C | \psi) = P(Q \in C, Q \in B | \psi) \quad (2.7.5)$$

for any pair of Borel subset $B, C$ of $\mathbb{R}$. This is important, since it includes simultaneous events, which is the case we want to consider. It also says that the probability of these simultaneous events makes sense in quantum theory. It is now just a matter of computing the probability in $(2.7.5)$. So we start that calculation as follows:

$$P(Q \in C, Q \in B | \psi) = ||P_Q(B)P_Q(C)\psi||^2 = ||P_Q(B \cap C)\psi||^2.$$ 

The last equality is a property of pvm’s that is proved in functional analysis. The expression ‘two places’ I interpret as meaning that the sets $B$ and $C$ are disjoint, that is, $B \cap C = \emptyset$, the empty set. But, by the definition of pvm we have that $P_Q(\emptyset) = 0$. We conclude that the probability in $(2.7.5)$ is 0 for disjoint Borel sets $B, C$ and for any state $\psi$. So much for the theory. What about observations? Well, I claim that there has never been an observation of an electron in two places at the same time. Of course, if such an observation were ever made, this aspect of quantum theory would have to be modified. But the principle of conservation of electric charge would also have to be
modified. So, if there is anything ‘strange’ or ‘mysterious’ to understand here, I do not know what that might be. I also will not discuss the relation of this result with classical physics, because I do not wish to discuss classical physics at all in this treatise. As a final point let me note that this argument is quite abstract; it applies to any self-adjoint operator and any pair of disjoint Borel subsets of \( \mathbb{R} \).

The inequality (2.7.4) is equivalent to

\[
P(T \in C \mid S \in B, \psi) P(S \in B \mid \psi) \neq P(S \in B \mid T \in C, \psi) P(T \in C \mid \psi)
\]

which says that Bayes Theorem does not generalize to quantum probability. Notice that the inequality (2.7.4) also contrasts with the classical case where \( P(E_1 \cap E_2) = P(E_2 \cap E_1) \). Moreover, (2.7.4) precludes defining a classical probability measure \( \pi \) on \( \mathbb{R}^2 \) such that \( S \) takes a value in \( B \) and \( T \) takes a value in \( C \) in any order is equal to \( \pi(B \times C) \). Nonetheless, (2.6.4) has reasonable ‘marginals’, namely

\[
P(S \in \mathbb{R}, T \in C \mid \psi) = \|P_T(C)P_S(\mathbb{R})\psi\|^2 = \|P_T(C)\psi\|^2 = P(T \in C \mid \psi),
\]

since \( P_S(\mathbb{R}) = I \), the identity map. Similarly,

\[
P(S \in B, T \in \mathbb{R} \mid \psi) = \|P_T(\mathbb{R})P_S(B)\psi\|^2 = \|P_S(B)\psi\|^2 = P(S \in B \mid \psi).
\]

On the other hand, taking \( S, T, B \) and \( \psi \) fixed, the map that sends the Borel subset \( C \) of \( \mathbb{R} \) to \( P(T \in C \mid S \in B, \psi) \) is a classical probability measure on \( \mathbb{R} \) as already remarked before.

### 2.8 Quantum Probability of Two Events

The results of the previous section have been presented in terms of events associated to pvm’s. So we have considered events such as \( S \in B = P_S(B) \) and so forth. But it is useful to express these results more abstractly in terms of arbitrary events. Here are the definitions. The first part of the next definition is Born’s rule for one event, given a pure state. The second part is Born’s rule for one event, given a density matrix; it is a generalization of the first part.

**Definition 2.8.1** Let \( E \) be an event, \( \psi \) be a unit vector and \( \rho \) be a density matrix. Then the probability of \( E \) given \( \psi \) is defined by Born’s rule

\[
P(E \mid \psi) := \langle \psi, E\psi \rangle = \|E\psi\|^2.
\] (2.8.1)
Moreover, the probability of $E$ given $\rho$ is defined by
\[
P(E \mid \rho) := \text{Tr}(E \rho).
\] (2.8.2)

In the sequel (2.8.2) will be regarded as a special case of the Generalized Born’s Rule.

Let’s recall the standard justification of (2.8.2) in terms of (2.8.1). So we take an orthonormal basis $\phi_k$ which diagonalizes the density matrix $\rho$, say $\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$ where $0 \leq \lambda_k \leq 1$ and $\sum_k \lambda_k = 1$. Then we compute
\[
P(E \mid \rho) = \text{Tr}(E \rho)
= \sum_k \langle \phi_k, E \rho \phi_k \rangle
= \sum_k \langle \phi_k, E \lambda_k \phi_k \rangle
= \sum_k \lambda_k \langle \phi_k, E \phi_k \rangle
= \sum_k \lambda_k P(E \mid \phi_k).
\]

In short, $P(E \mid \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|) = \sum_k \lambda_k P(E \mid \phi_k)$. This exhibits $P(E \mid \rho)$ as an infinite combination of the component probabilities $P(E \mid \phi_k)$, each with its corresponding weight factor $\lambda_k$. An immediate consequence of the previous formula is $0 \leq P(E \mid \rho) \leq 1$. Also the following normalizations are easily shown and provide more justification for (2.8.2):
\[
P(I \mid \rho) = 1 \quad \text{and} \quad P(0 \mid \rho) = 0.
\]

Finally, there is a form of $\sigma$-additivity which says that for any countable family $\{E_j \mid j \in \mathbb{N}\}$ of orthogonal events, meaning that $E_j \wedge E_k = 0$ for all $j \neq k$, we have by a standard argument that
\[
P \left( \bigvee_j E_j \mid \rho \right) = \sum_j P(E_j \mid \rho).
\] (2.8.3)

While this looks like the $\sigma$-countable condition for classical measures, it is quite different. Notice that a necessary, but not sufficient, condition for the family $\{E_j\}$ to be orthogonal is that $\{E_j\}$ is a commutative family, a very restrictive condition. These facts motivate the next definition.
Definition 2.8.2 A (quantum) probability on the set \( \mathcal{E} \) of quantum events is a function \( \pi : \mathcal{E} \to [0, 1] \) such that

- Normalization: \( \pi(0) = 0 \) and \( \pi(I) = 1 \).
- \( \sigma \)-additivity: For any countable family \( \{E_j \mid j \in \mathbb{N}\} \) of orthogonal events in \( \mathcal{E} \) we have
  \[
  \pi \left( \bigvee_j E_j \right) = \sum_j \pi(E_j).
  \]

It follows from the discussion above that for any density matrix \( \rho \) the function \( E \mapsto P(E \rho) \) is a probability on \( \mathcal{E} \). The question of the converse arises, that is, whether all probabilities on \( \mathcal{E} \subset \mathcal{L}(\mathcal{H}) \) have this form. The answer is yes, if \( \dim \mathcal{H} \geq 3 \) by Gleason’s theorem in [12].

The next definition gives two more versions of Born’s rule, but now for two time ordered events. It is based on (2.6.4) and (2.7.1) of the previous section.

Definition 2.8.3 Let \( E_1, E_2 \) be quantum events and \( \psi \) be a unit vector. Then the (quantum) consecutive probability of the event \( E_1 \) and then later the event \( E_2 \), given the state \( \psi \), is

\[
P(E_1, E_2 \mid \psi) := ||E_2E_1\psi||^2.
\]

(Notice that on the right side the earlier event \( E_1 \) goes on the right.)

The (quantum) condition probability of \( E_2 \), given that \( E_1 \) has occurred and given a state \( \psi \), is defined provided that \( E_1\psi \neq 0 \) by

\[
P(E_2 \mid E_1, \psi) := \frac{P(E_1, E_2 \mid \psi)}{P(E_1 \mid \psi)} = \frac{||E_2E_1\psi||^2}{||E_1\psi||^2}.
\]

If \( E_1\psi = 0 \), then we define \( P(E_2 \mid E_1, \psi) := 0 \).

We say that the event \( E_2 \) is (quantum) independent of the event \( E_1 \) with respect to a state \( \psi \) if \( E_1\psi \neq 0 \) and

\[
P(E_2 \mid E_1, \psi) = P(E_2 \mid \psi).
\]

(2.8.4)

If \( S \) and \( T \) are self-adjoint operators, then the ordered pair \( (S, T) \) is said to be independent with respect to \( \psi \) if for all Borel subsets \( B, C \) of \( \mathbb{R} \) the event \( S \in B \) is independent of the event \( T \in C \) with respect to \( \psi \), that is

\[
P(S \in B, T \in C \mid \psi) = P(S \in B \mid \psi) P(T \in C \mid \psi).
\]

(The motivation for this equation is given later.)
Quantum independence is not necessarily a symmetric relation due to the non-commutativity of quantum theory. Specifically, $E_2$ is independent of $E_1$ given $\psi$ if and only if $E_1 \psi \neq 0$ and

$$||E_2 E_1\psi||^2 = ||E_1\psi||^2 ||E_2\psi||^2.$$ 

This equation trivially holds if $E_1 \psi = 0$. So the condition $E_1 \psi \neq 0$ need not be imposed here and will be dropped when we use this formula. Therefore the last equation is equivalent to

$$P(E_1, E_2 \mid \psi) = P(E_1 \mid \psi) P(E_2 \mid \psi),$$

which gives the promised motivation of the last part of the previous definition. Using (2.8.4) we see that $E_1$ is independent of $E_2$ given $\psi$ if and only if

$$||E_1 E_2\psi||^2 = ||E_1\psi||^2 ||E_2\psi||^2$$

or equivalently

$$P(E_2, E_1 \mid \psi) = P(E_1 \mid \psi) P(E_2 \mid \psi).$$

Clearly if $E_1$ and $E_2$ commute, then this is a symmetric relation, that is, $E_1$ is independent from $E_2$, given $\psi$, if and only if $E_2$ is independent from $E_1$, given $\psi$. Also $P(E_1, E_1 \mid \psi) = P(E_1 \mid \psi)$ trivially holds.

This definition is taking us ever deeper into non-commutative territory. We assume $\dim \mathcal{H} \geq 2$, since that guarantees that $\mathcal{L}(\mathcal{H})$ is non-commutative. For each state we already have a probability (cp. Definition 2.8.2) defined on the set of quantum events $\mathcal{E}$ in $\mathcal{L}(\mathcal{H})$, and $\mathcal{E}$ is not a $\sigma$-algebra. But now we have, given a state, a probability defined on ordered pairs of events. (Neither $E_1$ nor $E_2$ is associated to a specific time. We only require the time order that $E_1$ occurs first and then later $E_2$.) This is yet another step beyond the probability theory of Kolmogorov based on $\sigma$-algebras. The question arises as to what are the properties of quantum consecutive probability.

Throughout the following we let $\psi$ be any unit vector. First notice that $0 \leq P(E_1, E_2 \mid \psi) \leq 1$, since $||E_2 E_1|| \leq 1$. Also, we have for any event $E$ the intuitively transparent formulas for the marginals

$$P(E, I \mid \psi) = P(E \mid \psi) \quad \text{and} \quad P(I, E \mid \psi) = P(E \mid \psi).$$

So, any event $E$ and $I$ are independent, and in both orders, since $P(I \mid \psi) = 1$. Moreover, at the other extreme we have the normalizations

$$P(E, 0 \mid \psi) = 0 \quad \text{and} \quad P(0, E \mid \psi) = 0.$$
And so any event $E$ and the ‘never-YES’ event 0 are independent (and again in both orders), since $P(0 \mid \psi) = 0$.

Also, we have a form of $\sigma$-additivity: Let the event $F = \bigvee_j F_j$, where $\{F_j\}$ is a countable family of pairwise orthogonal events. Then for any event $E$ by a standard argument we have for the consecutive probability that

$$P(E, F \mid \psi) = \sum_j P(E, F_j \mid \psi).$$

This is $\sigma$-additivity in the second event of the ordered pair of events.

However, $\sigma$-additivity fails in general in the first event for consecutive probability. We continue with the above notation and compute

$$P(F, E \mid \psi) = \|EF\psi\|^2 = \|E(\bigvee_j F_j \psi)\|^2 = \langle E(\bigvee_j F_j \psi), E(\bigvee_k F_k \psi) \rangle = \sum_{j,k} \langle EF_j \psi, EF_k \psi \rangle = \sum_j \langle EF_j \psi, EF_j \psi \rangle + \sum_{j \neq k} \langle EF_j \psi, EF_k \psi \rangle = \sum_j P(F_j, E \mid \psi) + \sum_{j \neq k} \langle \psi, F_j EF_k \psi \rangle.$$

The first summation is what one expects from $\sigma$-additivity. But the second summation is a typical quantum term, which already in the case $F = F_1 \lor F_2$ is producing interference with the first term. That is, the second sum, which is called an interference term, can either increase or decrease the first term. However, if $E$ commutes with all of the events $F_j$ (which already commute among themselves by orthogonality), then we have for each pair $j \neq k$ that

$$\langle \psi, F_j EF_k \psi \rangle = \langle \psi, EF_j F_k \psi \rangle = 0$$

by the orthogonality condition $F_j F_k = 0$. Therefore, in the commutative case this characteristic quantum interference term vanishes identically.

Also it is important to remark that this interference term arises from the rules for computing quantum probabilities and from nothing else. Of course, those rules are based on Hilbert space properties, particularly on the fact that the set of events is not a $\sigma$-algebra. Note, for example, that there is no particle/wave duality being invoked here. In fact, there is no wave
equation. There is no Superposition Principle for solutions or for states. There is no mention of the Uncertainty Principle, of Complementarity or of the Measurement Problem. There is no so-called ‘self-interference’ of a particle with itself. Even Schrödinger’s equation is absent from the derivation of this result. If this interference term is not intuitive for you, it means that quantum probability is not intuitive for you.

The phrase ‘consecutive probability’ is never even defined in classical probability theory for two good reasons. First, the order of events is not important. Second, the conjunction of two events $A_1, A_2$ in a $\sigma$-algebra is their intersection $A_1 \cap A_2$, which is again in the $\sigma$-algebra, that is, it is also an event. So, in classical probability the probability of two (and by induction any finite sequence of) events is itself the probability of just one event. This is not so in quantum probability. It is easy to construct events $E_1$ and $E_2$ in $\mathcal{L}(\mathbb{C}^2)$ such that $0 \neq E_1 E_2 \neq E_2 E_1 \neq 0$ and yet $E_1 \wedge E_2 = 0$, the ‘never-YES’ event. In this case neither $E_1 E_2$ nor $E_2 E_1$ is an event. Nonetheless, the probabilities of sequences $P(E_1, E_2 | \psi)$ and $P(E_2, E_1 | \psi)$ make good sense and are not equal in general.

**Example 2.8.1** Here is a general example. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then define $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$, the Hilbert space tensor product. Let $F_1$ (resp., $F_2$) be an event acting on $\mathcal{H}_1$ (resp., $\mathcal{H}_2$). Put $E_1 := F_1 \otimes I$ and $E_2 := I \otimes F_2$. Then $E_1$ and $E_2$ commute, but that is not enough to have independence. We also have to choose appropriately a unit vector $\psi \in \mathcal{H}$. We choose $\psi = \psi_1 \otimes \psi_2$, where $\psi_1$ (resp., $\psi_2$) is a unit vector in $\mathcal{H}_1$ (resp., $\mathcal{H}_2$) such that $E_1 \psi_1 \neq 0$ and $E_2 \psi_2 \neq 0$. Then it is an easy exercise to show that $E_1$ is independent of $E_2$ with respect to $\psi$ and vice versa. Of course, this example gets very interesting if we use a unit vector that does not factorize, and it turns out that the same two events are then not independent.

To generalize the last definition for the case of a density matrix, consider this expression:

$$||E_2 E_1 \psi||^2 = \langle E_2 E_1 \psi, E_2 E_1 \psi \rangle = \langle \psi, (E_2 E_1)^* E_2 E_1 \psi \rangle$$

$$= \langle \psi, (E_2 E_1)^* E_2 E_1 E_\psi \psi \rangle$$

$$= Tr((E_2 E_1)^* E_2 E_1 E_\psi \psi).$$

where $E_\psi = |\psi \rangle \langle \psi|$ is a rank 1 projection operator. So, using the same notation, the corresponding definitions for a density matrix $\rho$ are

$$P(E_1, E_2 | \rho) := Tr((E_2 E_1)^* E_2 E_1 \rho),$$

(2.8.5)
the consecutive conjunctive probability of first \( E_1 \) and then later \( E_2 \), given \( \rho \). Notice that \((E_2E_1)^*E_2E_1\rho\) is a trace class operator, since \( \rho \) is. So its trace in (2.8.5) is well defined. However, its trace norm need not be equal to its trace.

Also the conditional probability of \( E_2 \), given \( E_1 \) and \( \rho \), is defined by

\[
P(E_2 \mid E_1, \rho) := \frac{P(E_1, E_2 \mid \rho)}{P(E_1 \mid \rho)} = \frac{Tr((E_2E_1)^*E_2E_1\rho)}{Tr(E_1\rho)},
\]

provided that \( P(E_1 \mid \rho) \neq 0 \). The formula (2.8.6) is usually derived from Lüders rule. (See [22].) Here I have by-passed Lüders rule and given (2.8.6) directly as a definition. One can easily manipulate (2.8.6) to arrive at the formula usually found in the literature for this conditional probability. For example see [5], where a uniqueness result for this formula is also found. Since I arrived at (2.8.6) without being aware of the literature (such as [5]), my notation is not standard.

Since these formulas may be unfamiliar in the non-commutative context, let’s see a justification of (2.8.5). As before, we let \( \phi_k \) be an orthonormal basis that diagonalizes \( \rho \), that is, \( \rho = \sum_k \lambda_k \langle \phi_k | \phi_k \rangle \) where \( 0 \leq \lambda_k \leq 1 \) and \( \sum_k \lambda_k = 1 \). Then we see that

\[
P(E_1, E_2 \mid \rho) = Tr((E_2E_1)^*E_2E_1\rho) = \sum_k \langle \phi_k, (E_2E_1)^*E_2E_1\rho \phi_k \rangle
\]

\[
= \sum_k \langle E_2E_1\phi_k, E_2E_1\lambda_k \phi_k \rangle = \sum_k \lambda_k |E_2E_1\phi_k|^2
\]

\[
= \sum_k \lambda_k P(E_1, E_2 \mid \phi_k).
\]

So \( P(E_1, E_2 \mid \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|) = \sum_k \lambda_k P(E_1, E_2 \mid \phi_k) \), showing \( P(E_1, E_2 \mid \rho) \) as an infinite combination of the component probabilities \( P(E_1, E_2 \mid \phi_k) \), each with its corresponding weight factor \( \lambda_k \). As before, these probabilities are easily shown to be real numbers in the interval \([0, 1]\).

As in the case of pure states, we define the event \( E_2 \) to be independent of the event \( E_1 \) given a density matrix \( \rho \), if \( P(E_1 \mid \rho) \neq 0 \) and

\[
P(E_2 \mid E_1, \rho) = P(E_2 \mid \rho).
\]

This is equivalent to

\[
P(E_1, E_2 \mid \rho) = P(E_1 \mid \rho)P(E_2 \mid \rho).
\]

In general this is not a symmetric relation, though it is if \( E_1 \) and \( E_2 \) commute.
Example 2.8.2 The events $E_1 = F_1 \otimes I$ and $E_2 = I \otimes F_2$ of Example 2.8.1 are independent in either order with respect to any density matrix $\rho = \rho_1 \otimes \rho_2$, where $\rho_1$ (resp., $\rho_2$) is a density matrix acting on $\mathcal{H}_1$ (resp., $\mathcal{H}_2$), provided that $P(F_1 \mid \rho_1) \neq 0$ and $P(F_2 \mid \rho_2) \neq 0$. The details are left to the reader.

2.9 Generalized Born’s Rule with a State

Rather than spell out more details of the case of two events, we continue with the generalization to a finite sequence of time ordered events, which is now readily at hand. This is the central definition of this treatise.

Definition 2.9.1 Generalized Born’s Rule with a State. Suppose that $E_1, E_2, \ldots, E_n$ for an integer $n \geq 1$ is an ordered sequence of events (possibly with repetitions) and let $\psi$ be a unit vector. Then the consecutive (conjunctive) probability that first $E_1$ occurs and then $E_2$ occurs and so on continuing until $E_n$ occurs, given $\psi$, is defined as

$$P(E_1, E_2, \ldots, E_n \mid \psi) := ||E_n \cdots E_2 E_1 \psi||^2.$$  \hfill (2.9.1)

Let $\Lambda$ denote the empty sequence of events, not to be confused with $\emptyset$, the empty set. Note that $\Lambda$ is vacuously ordered. Then we define $P(\Lambda \mid \psi) := 1$.

The conditional probability that the sequence of events $E_1, \ldots, E_n$ occurs in that time order given that the sequence of events $F_1, \ldots, F_k$ has already occurred in that time order, given $\psi$, is defined as

$$P(E_1, \ldots, E_n \mid F_1, \ldots, F_k, \psi) := \frac{P(F_1, \ldots, F_k, E_1, \ldots, E_n \mid \psi)}{P(F_1, \ldots, F_k \mid \psi)} = \frac{||E_n \cdots E_1 F_k \cdots F_1 \psi||^2}{||F_k \cdots F_1 \psi||^2}.$$

provided the denominator is not zero. (The definition on the first line is for all integers $k \geq 0$, while the second line only holds for $k \geq 1$.) Otherwise we define

$$P(E_1, \ldots, E_n \mid F_1, \ldots, F_k, \psi) := 0.$$

These definitions are special cases of the following corresponding definitions given the same sequences of events and a density matrix $\rho$.

The consecutive probability of the ordered sequence of events $E_1, \ldots, E_n$ for $n \geq 1$ is defined as

$$P(E_1, \ldots, E_n \mid \rho) := Tr((E_n \cdots E_1)^* E_n \cdots E_1 \rho).$$
Also, we define \( P(\Lambda | \rho) := 1 \).

We say that a family of events \( \{ E_\alpha | \alpha \in A \} \) which has an order induced from a linear order on \( A \) is independent with respect to \( \rho \) if for every finite ordered subset \( E_{\alpha_1}, E_{\alpha_2}, \ldots, E_{\alpha_n} \) for \( n \geq 1 \) with \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \) we have

\[
P(E_{\alpha_1}, E_{\alpha_2}, \ldots, E_{\alpha_n} | \rho) = P(E_{\alpha_1} | \rho) P(E_{\alpha_2} | \rho) \cdots P(E_{\alpha_n} | \rho).
\]

The ordered sequence \( T_1, \ldots, T_n \) of self-adjoint operators is independent with respect to \( \rho \) if for every sequence of Borel subsets \( B_1, \ldots, B_n \) of \( \mathbb{R} \) the ordered sequence of events \( T_1 \in B_1, \ldots, T_n \in B_n \) is independent with respect to \( \rho \).

This can be defined for arbitrary families as well.

With the same notation as above the conditional probability is defined as

\[
P(E_1, \ldots, E_n | F_1, \ldots, F_k, \rho) := \frac{P(F_1, \ldots, F_k, E_1, \ldots, E_n | \rho)}{P(F_1, \ldots, F_k | \rho)} \quad (2.9.2)
\]

provided \( P(F_1, \ldots, F_k | \rho) \neq 0 \); otherwise it is defined to be 0.

All of the previous definitions of quantum probability are special cases of (2.9.2), which is called the Generalized Born’s Rule.

All of these probabilities are real numbers in the interval \([0, 1]\), and they have other obvious properties. However, the language of \( \sigma \)-algebras has been left far behind.

The ordering here of events reflects the time order of their occurrences, but does not associate them to specific times. However, since events are self-adjoint operators, they can have a time dependence in some models (such as the Heisenberg model). But this dependence does not impact the time order of these events.

Consider the union \( \{ F_1, \ldots, F_k, E_1, \ldots, E_n \} \) of the ordered sequences as a new ordered sequence of events. If this new ordered sequence is independent, then necessarily we have that

\[
P(E_1, \ldots, E_n | F_1, \ldots, F_k, \rho) = P(E_1, \ldots, E_n | \rho) P(F_1, \ldots, F_k | \rho).
\]

But this is not a sufficient condition for independence of the new ordered sequence, except in the case when \( n = k = 1 \).

Having defined independence of an ordered sequence of observables (which are quantum random variables), it is desirable to define identically distributed observables as well. Then we will be able to speak of an ordered sequence of independent, identically distributed (iid) observables.
Definition 2.9.2 Let $S$ and $T$ be self-adjoint operators. Then we say that $S$ and $T$ are identically distributed with respect to a density matrix $\rho$ if $P(S \in B \mid \rho) = P(T \in B \mid \rho)$ for all Borel subsets $B$ of $\mathbb{R}$.

Notice that this definition does not depend on the order of $S$ and $T$. Clearly, this is a symmetric relation. In fact, it is an equivalence relation on the set of self-adjoint operators. The construction of a finite sequence of iid observables can be done easily using tensor products.

Everything in the formulas of these definitions is well known to anyone working in classical probability where each sequence of events becomes just one event. What is really new is not so much generalizing to sequences of non-commuting events, but rather identifying all of this as the Generalized Born’s Rule (2.9.2) of quantum theory. Moreover, (2.9.2) is the fundamental time evolution equation of quantum theory, provided that the secondary, model dependent time evolutions are given for the set of events and for the set of states.

The normalization conditions $P(\Lambda \mid \psi) = 1$ and $P(\Lambda \mid \rho) := 1$ could seem non-intuitive. In their defense they make the conditional probabilities work out when $k = 0$. But they seem to say that the probability that nothing happens is 1. I suspect that this is a trap of language. Think of ever longer sequences of events. Easily such very long sequences can have probability 0 or very near 0. But then thinking of starting with such a long sequence of events with small probability and then considering shorter and shorter subsequences of it. The intuition is that the probability increases. And when one arrives at the empty sequence of events, one then has the most probable situation. In more mundane terms we can say that events impose restrictions on probability and that, by removing all such restrictions, one gets the most probable outcome, i.e., probability 1.

Continuing with the intuition in the previous paragraph, we can define the probability of an infinite sequence of events.

Definition 2.9.3 Let $\{E_j \mid j \in \mathbb{N}\}$ be an infinite sequence of linearly ordered events with $E_i$ preceding $E_j$ if and only if $i < j$. Let $\rho$ be a density matrix. Then we define the probability of the ordered sequence, given $\rho$, as

$$P(\{E_j\} \mid \rho) := \lim_{n \to \infty} P(E_1, \ldots, E_n \mid \rho)$$

$$= \lim_{n \to \infty} Tr((E_n \cdots E_1)^* E_n \cdots E_1 \rho)$$

$$= \inf_n Tr((E_n \cdots E_1)^* E_n \cdots E_1 \rho).$$
If this limit exists, then clearly $0 \leq P(\{E_j\} \mid \rho) \leq 1$. It also satisfies obvious normalization condition if all $E_j = I$ or if any $E_j = 0$.

The limit in this definition exists since the sequence is both bounded below by 0 and non-increasing. To prove the latter statement we compare the $n + 1$-st term with the $n$-th term. Write the density matrix as $\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$ where $0 \leq \lambda_k \leq 1$, $\sum_k \lambda_k = 1$ and $\{\phi_k\}$ is an orthonormal basis of $\mathcal{H}$ which diagonalizes $\rho$. The result is trivially true if $E_{n+1} = 0$. So we assume that $E_{n+1} \neq 0$ which implies that $||E_{n+1}|| = 1$. Then we have

$$
Tr( (E_{n+1}E_n \cdots E_1)^* E_{n+1} E_n \cdots E_1 \rho) = 
$$

$$
= \sum_k \langle \phi_k, (E_{n+1}E_n \cdots E_1)^* E_{n+1} E_n \cdots E_1 \rho \phi_k \rangle
$$

$$
= \sum_k \langle E_{n+1}E_n \cdots E_1 \phi_k, E_{n+1} E_n \cdots E_1 \rho \phi_k \rangle
$$

$$
= \sum_k \lambda_k \langle E_{n+1}E_n \cdots E_1 \phi_k, E_{n+1} E_n \cdots E_1 \phi_k \rangle
$$

$$
= \sum_k \lambda_k ||E_{n+1} E_n \cdots E_1 \phi_k||^2
$$

$$
\leq \sum_k \lambda_k ||E_{n+1}||^2 ||E_n \cdots E_1 \phi_k||^2
$$

$$
= \sum_k \lambda_k ||E_n \cdots E_1 \phi_k||^2
$$

$$
= \sum_k \lambda_k \langle E_n \cdots E_1 \phi_k, E_n \cdots E_1 \phi_k \rangle
$$

$$
= \sum_k \langle E_n \cdots E_1 \phi_k, E_n \cdots E_1 \lambda_k \phi_k \rangle
$$

$$
= \sum_k \langle \phi_k, (E_n \cdots E_1)^* E_n \cdots E_1 \rho \phi_k \rangle
$$

$$
= Tr( (E_n \cdots E_1)^* E_n \cdots E_1 \rho).
$$

This proves that the sequence is non-increasing. And this fact is behind the assertion that the conditional probabilities are $\leq 1$. However, this result does not mean that more events lowers the probability. It only says that adding more events after a given sequence of events lowers the probability. It is well known one can find events $E_1$ and $E_2$ acting on $\mathbb{C}^2$ such that $E_2 E_1 = 0$ and so, in particular, $P(E_2 E_1 \mid \psi) = 0$, but that there exists an event $F$ satisfying $P(E_2 F E_1 \mid \psi) > 0$. (Think about light polarizing filters.)
Since $E_1$ is the first event and $E_n$ is the last event, the expression $E_n \cdots E_1$ is well-known in quantum field theory. It is called a time-ordered product. Clearly, the actual calculation of these probabilities can be rather challenging in practice. Such probabilities could be difficult to check in the laboratory as well. One typically prefers experimental situations with few events in play. However, nature does not always smile favorably on the experimental scientist. Even if one wished to study just two consecutive events, there may be other uncontrolled intermediate events so that one is studying a situation with many events instead of just two. Such undesired intermediate events are pejoratively dubbed as noise (as if they were not physical phenomena which one could study) and the experimenter then works hard to eliminate them or, at least, to minimize their collective impact. Neither at an experimental level nor at a theoretical level is there any ‘mystery’ about such noise that needs special explanation. It is simply very annoying. But to their merit some physicists do try to study this noise which they rename as decoherence.

It is important to remark, and trivial to verify, that the probabilities in Definition 2.9.1 are invariant under isomorphisms between models. So it makes sense to suppose these probabilities could have physical significance. The Generalized Born’s Rule clearly applies to the special case when all of the events have the form $P(T \in B)$, where $T$ is a self-adjoint operator and $B$ is a Borel subset of $\mathbb{R}$.

Also, these quantum probabilities can be defined for events and states associated to any von Neumann algebra.

### 2.10 Generalized Born’s Rule with no State

It makes mathematical sense to drop the state from the formulas and arrive at a definition of probability for sequences of events with no mention of a state. This also has some physical intuition behind it, though as we shall see it clashes with ideas from classical probability. So we present these concepts only as definitions without ever using them later in this treatise.

**Definition 2.10.1** Suppose that $E_1, E_2, \ldots, E_n$ for an integer $n \geq 1$ is an ordered sequence of events. Then the consecutive (conjunctive) probability that first $E_1$ occurs and then $E_2$ occurs and so on continuing until $E_n$ occurs is defined as

$$P(E_1, E_2, \ldots, E_n) := ||E_n \cdots E_2 E_1||^2.$$  

(2.10.1)
Let $\Lambda$ denote the empty sequence of events. Then we define $P(\Lambda) := 1$.

The conditional probability that the sequence of events $E_1, \ldots, E_n$ occurs in that time order given that the sequence of events $F_1, \ldots, F_k$ has already occurred in that time order is defined as

$$P(E_1, \ldots, E_n \mid F_1, \ldots, F_k) := \frac{P(F_1, \ldots, F_k, E_1, \ldots, E_n)}{P(F_1, \ldots, F_k)} = \frac{||E_n \cdots E_1 F_k \cdots F_1||^2}{||F_k \cdots F_1||^2}$$

provided the denominator is not zero. (The definition on the first line is for all integers $k \geq 0$, while the second line only holds for $k \geq 1$.) Otherwise we define

$$P(E_1, \ldots, E_n \mid F_1, \ldots, F_k) := 0.$$ 

This definition will be used in the Chapter on Entanglement. So it does have relevance to physics. But it is non-intuitive, at least for me. Here is a puzzling particular case. If $E \neq 0$ is an event, then

$$P(E) = ||E||^2 = 1.$$  

This contradicts the intuition that non-zero events should have non-trivial probabilities. In particular, the map $P : \mathcal{E} \to [0, 1]$ is not $\sigma$-additive, where $\mathcal{E}$ is the set of all events in $\mathcal{H}$. Therefore, $P$ is not a pvm. Also, when restricted to a subset of $\mathcal{E}$ which is a $\sigma$-algebra (the commutative classical case), the range of $P$ is just $\{0, 1\}$. So in Chapter 3 on Entanglement only the conditional probability with a state will be used although a similar analysis with the less intuitive conditional probability with no state can be made and gives the same results.

This probability with no state contrasts sharply with the probability of an event, given a state $\psi$, for which we have that

$$P(E \mid \psi) = ||E \psi||^2$$

can be any number in the interval $[0, 1]$ for events $0 \neq E \neq I$. This accords with the classical idea that the probability of an event reflects to some degree the state in which the system 'finds' itself. But $P(E) = 1$ for all non-zero events seems to mean something decidedly weaker. Since $P(0) = 0$ is the only exceptional case and 0 is the event that has only the value NO, one is
led to think that $P(E) = 1$ says that YES is a possible value of $E$. Again, this is a very weak condition.

Let’s see how this works in a familiar example. Let $Q$ denote the position operator defined in the Hilbert space $L^2(\mathbb{R})$. Let $a, b \in \mathbb{R}$ satisfy $a < b$. Then the quantum event

$$Q \in [a, b] = P_Q([a, b]) \neq 0$$

and so $P(Q \in [a, b]) = 1$. On the other hand

$$P(Q \in [a, b] \mid \psi) = ||(Q \in [a, b])\psi||^2$$

can assume any value in $[0, 1]$, depending on the value of the unit vector $\psi$. Since there exist some states such that the event $Q \in [a, b]$ gives the value YES, the probability of the event itself is 1. However, the quantum event

$$Q \in [a] = P_Q([a]) = 0$$

and so $P(Q \in [a]) = 0$. This is a way of saying in the language of quantum probability that the position observable can not give sharp numerical values, but can give values in non-trivial intervals.

### 2.11 Probability Amplitudes

In many formulations of quantum theory it is emphasized that the quantum probabilities are calculated as the absolute value squared of a probability amplitude. That is implicit in this approach in the appropriate case when we have two unit vectors $\psi, \phi \in \mathcal{H}$, in which case we define their probability amplitude to be $A(\phi, \psi) := \langle \phi, \psi \rangle$. We use the notation $E_{\phi} = |\phi\rangle\langle\phi|$, the event that ‘$\phi$ occurs’. Then the probability of $E_{\phi}$, given $\psi$, is

$$P(E_{\phi} \mid \psi) = ||E_{\phi}\psi||^2 = ||\langle\phi, \psi\rangle\phi||^2 = |A(\phi, \psi)|^2.$$ 

For fixed $\phi$ the map $\psi \mapsto A(\phi, \psi)$ is linear provided that $\psi$ is allowed to be any vector in $\mathcal{H}$, but that $\psi \mapsto |A(\phi, \psi)|^2$ is not linear. Of course, the more relevant concept here is convexity. So, suppose that $\psi = \sum_k \lambda_k \phi_k$, where $\{\phi_k\}$ is an orthonormal basis and the complex numbers $\lambda_k$ satisfy $\sum_k |\lambda_k|^2 = 1$. Then we have a form of $\sigma$-additivity of amplitudes:

$$A(\phi, \psi) = \sum_k \lambda_k \langle \phi, \phi_k \rangle.$$
But in general
\[ |A(\phi, \psi)|^2 \quad \text{and} \quad \sum_k |\lambda_k \langle \phi, \phi_k \rangle|^2 = \sum_k |\lambda_k|^2 |A(\phi, \phi_k)|^2 \]
are not equal due to the well known interference terms.

Probability amplitudes can also be defined for all the other examples of quantum probability, but they are elements in the Hilbert space instead of being complex numbers. For example, for a quantum event \( E \) and a unit vector \( \psi \) we have \( P(E | \psi) = |A(E, \psi)|^2 \), where we define the probability amplitude as \( A(E, \psi) := E \psi \in \mathcal{H} \).

Essentially, probability amplitudes add more notation to the theory but without shedding much more light on it. However, that language is available if you wish to use it.

### 2.12 Quantum Integrals

Having defined and studied quantum probability, it is now straightforward to define and study the quantum theory of integration as was mentioned earlier as \( \mathcal{E}(A | \rho) = \text{Tr}(A \rho) \) for any \( A \in \mathcal{L}(\mathcal{H}) \) and density matrix \( \rho \). We can say that this is the (non-commutative) integral of \( A \) with respect to the state \( \rho \). Using terminology from classical probability theory we can call this the expectation of \( A \) with respect to \( \rho \). In quantum physics one says that this is the expected value of \( A \) in the state \( \rho \); this is well known since the early days of quantum theory, although only after Born’s seminal paper appeared. So the next definition seems to be natural, though its importance is not clear.

**Definition 2.12.1** Let \( A_1, A_2, \ldots, A_n \) be an ordered sequence in \( \mathcal{L}(\mathcal{H}) \) and let \( \rho \) be a density matrix. We define the time-ordered integral of this sequence of operators with respect to \( \rho \) to be

\[
\mathcal{E}(A_1, A_2, \ldots, A_n | \rho) := \text{Tr}(A_n \cdots A_2 A_1 \rho). \tag{2.12.1}
\]

This is also called the expectation of the ordered sequence \( A_1, A_2, \ldots, A_n \) with respect to \( \rho \).

Of course, the time ordered integral of an ordered sequence of operators is equal to the integral of a single operator, since

\[
\mathcal{E}(A_1, A_2, \ldots, A_n | \rho) = \mathcal{E}(A_n \cdots A_2 A_1 | \rho).
\]
But the point is that non-commutativity of $\mathcal{L}(\mathcal{H})$ makes the order of the operators important. And that is what is underlining this definition. The corresponding definition in usual measure theory, which is a commutative integration theory, would not have such importance.

There is a temptation to say that (2.12.1) is a state that is associated to the probability for ordered sequences of events. However, it has some properties that argue against being so named. For example, for an ordered sequence $E_1, E_2, \ldots, E_n$ of events with $n \geq 2$ we have in general that

$$
\mathcal{E}(E_1, E_2, \ldots, E_n | \rho) \neq P(E_1, E_2, \ldots, E_n | \rho).
$$

So with this definition the expectation does not extend the probability. Also, there is no apparent positivity property. One ‘nice’ property that it does have is the normalization $\mathcal{E}(I, I, \ldots, I | \rho) = Tr \rho = 1$. It also has reasonable marginals such as $\mathcal{E}(I, E_2, \ldots, E_n | \rho) = \mathcal{E}(E_2, \ldots, E_n | \rho)$ and so forth.

But now we also have quantum conditional probability at our disposal, and so we should have quantum conditional expectation as well.

**Definition 2.12.2** Let $E_1, \ldots, E_k$ be an ordered sequence of events in $\mathcal{L}(\mathcal{H})$ and $A_1, \ldots, A_n$ be an ordered sequence of operators in $\mathcal{L}(\mathcal{H})$. Also let $\rho$ be a density matrix. Then the (quantum) conditional expectation is defined as

$$
\mathcal{E}(A_1, \ldots, A_n | E_1, \ldots, E_k, \rho) := \frac{\mathcal{E}(A_1, \ldots, A_n, E_1, \ldots, E_k | \rho)}{\mathcal{E}(E_1, \ldots, E_k | \rho)}
$$

provided $\mathcal{E}(E_1, \ldots, E_k | \rho) \neq 0$. (Note that the denominator here is not $P(E_1, \ldots, E_k | \rho)$.)

Note in the Schrödinger and Heisenberg models that the time evolution extends naturally to $\mathcal{L}(\mathcal{H})$. Consequently, given a Hamiltonian these integrals are also time dependent.

### 2.13 Born’s rule redux

Until the end of the last section, I had been rather cavalier in using the term *Born’s rule*. What I mean by it is any formula in quantum theory that is a special case of (2.9.2). Since M. Born was the first to give such a formula in quantum theory, I have decided to credit him by calling (2.9.2) and its immediate consequences the *Generalized Born’s Rule*. Actually, I have not
yet presented Born’s rule in its original form. It might be instructive for the
reader to see this in detail. To do this I shall dive ever so shallowly into
historical waters.

The time independent version of Schrödinger’s equation $H\psi = E\psi$ for
a self-adjoint partial differential operator $H = H^*$ is an eigenvalue problem
with two unknowns for which it must be solved: the eigenvalue $E \in \mathbb{R}$
and its corresponding non-zero eigenvector $\psi$. Already in Schrödinger’s first
paper \[29\] on the subject it was realized that $E$ represents an energy, but
the physical significance of $\psi$ was left unresolved in that paper. However,
it seemed reasonable that the solution $\psi$ should also have some physical
significance. And a similar concern arises with the solution $\psi_t$ of the time
dependent Schrödinger equation.

In modern terminology M. Born addressed this in the specific case that
$\psi \in L^2(\mathbb{R}^3)$. However, to avoid a lot of sub-indices let’s consider the case
$\psi \in L^2(\mathbb{R})$, since the same same ideas apply. So, $\mathcal{H} = L^2(\mathbb{R})$ is the Hilbert
space for this situation. The basic assumption is that the position of the
system is a relevant observable, that is, the values of its pvm lie in the
von Neumann algebra of the system. Here the self-adjoint position operator
$Q : D(Q) \to L^2(\mathbb{R})$ is defined on the dense subspace

$$D(Q) := \{ \psi \in L^2(\mathbb{R}) \mid x\psi(x) \in L^2(\mathbb{R}) \}$$

by the formula $Q\psi(x) := x\psi(x)$. But more importantly, the pvm of $Q$ is
$P_Q(B) \phi = \chi_B \phi$ for all Borel subsets $B$ of $\mathbb{R}$ and all $\phi \in L^2(\mathbb{R})$. Here $\chi_B$ is
the characteristic function of the Borel set $B$, defined for all $x \in \mathbb{R}$ as

$$\chi_B(x) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

All of these results about $Q$ come from functional analysis. We continue
by using Born’s rule as given in Axiom 4 to calculate the probability that
the position of the system is in a Borel subset $B$ of $\mathbb{R}$ given a unit vector
$\psi \in L^2(\mathbb{R})$ as follows:

$$P(Q \in B \mid \psi) = \langle \psi, P_Q(B)\psi \rangle = \int_{\mathbb{R}} dx \, \psi(x)^* P_Q(B)\psi(x) =$$

$$= \int_{\mathbb{R}} dx \, \psi(x)^* \chi_B(x)\psi(x) = \int_B dx \, \psi(x)^* \psi(x)$$

$$= \int_B dx \, |\psi(x)|^2.$$
The last expression on the right here is the formula given by Born for the probability that the position of the system is in $B$ given the unit vector $\psi$. Of course, Born came to this conclusion without using all the tools of quantum probability, which came later. Actually, Born initiated the field of quantum probability by indicating the physical significance of this expression. This formula for the probability is one way of viewing the physical significance of the solution $\psi \in L^2(\mathbb{R})$ of the eigenvalue problem $H\psi = E\psi$. It we consider $P(T \in B | \psi)$ for some other self-adjoint operator $T$, we can give $\psi$ some other physical significance. I do not wish to elevate this last comment to the level of a general principle of complementarity; it is merely another application of Born’s rule as given in Axiom 4.

Concerning a solution $\psi_t \in L^2(\mathbb{R})$ of the time dependent Schrödinger’s equation, Born’s rule asserts that

$$P(Q \in B | \psi_t) = \int_B dx |\psi_t(x)|^2$$

(2.13.1)

is the probability at every time $t \in \mathbb{R}$ that the position of the system is in the Borel set $B$. So we see time dependent probability in quantum theory in this simple example. However, in this treatise we take equation (2.13.1) to be the fundamental time evolution equation of the position observable $Q$. Moreover, we see in this example that the time dependent Schrödinger’s equation plays a secondary role in understanding the physical significance of $\psi_t$.

### 2.14 Comparison with Classical Probability

This section, just as the rest of this treatise, does not address the issues of comparing quantum theory with classical physical theory. Rather, it is a comparison of Kolmogorov’s formulation in [20] of classical probability in terms of measure theory with the quantum theory of probability that has been presented here.

A key difference is the mathematical structure of the set of events. In classical probability the events are elements of a $\sigma$-algebra $\mathcal{F}$ whose elements are subsets of a non-empty sample space $\Omega$. In particular this is a Boolean algebra. This means the various rules of Boolean algebra hold including the de Morgan identities. Another way of saying this is that the events obey the rules of classical logic that go back at least to the works of Aristotle.
Thinking that events tell us that nature has certain properties, this means that those properties also satisfy the rules classical logic.

In quantum theory the events are the closed subspaces (≡ projections) of a complex Hilbert space $\mathcal{H}$. (Note that the role of the complex numbers seems to be essential.) These form a complete orthomodular lattice for which the de Morgan identities fail if $\dim \mathcal{H} \geq 2$. Consequently, if one assigns 'properties' to these events, then classical logic will not apply to them. So we must think differently about quantum events. Of course, we can always say that the event itself is a property, but this is quite distinct from how classical probability is structured.

Quantum events when viewed as projections lie inside a larger structure, namely the complex vector space $L(\mathcal{H})$. To draw comparisons it is convenient to embed classical events in a larger structure, namely the complex vector space $M(\Omega) := \{ X : \Omega \to \mathbb{C} \mid X \text{ is measurable} \}$. Then a classical event $A \in \mathcal{F}$ is associated with its characteristic function $\chi_A \in M(\Omega)$. One has $\chi_A^2 = \chi_A = \chi_A^*$. Conversely, for every $X \in M(\Omega)$ satisfying $X^2 = X = X^*$, there exists a unique set $A \in \mathcal{F}$ such that $X = \chi_A$. So, we can define classical events equivalently as those $X \in M(\Omega)$ satisfying $X^2 = X = X^*$. This compares favorably with the definition of a quantum event as those $E \in L(\mathcal{H})$ satisfying $E^2 = E = E^*$.

In classical probability the observables are called random variables and are defined as those elements of $M(\Omega)$ that are real valued, that is, those $X \in M(\Omega)$ satisfying $X = X^*$. The essential range of any $X \in M(\Omega)$ is the spectrum of $X$, denoted $\text{Spec}(X)$. (See a text on measure theory for definitions.) If $X$ is a random variable, then the elements of $\text{Spec}(X)$ form a non-empty subset of $\mathbb{R}$, and its elements correspond to the values seen in the observations associated to $X$. For each Borel subset $B$ of $\mathbb{R}$ there is an event in $\mathcal{F}$ that is denoted as $X \in B$ and is called the event that $X$ is observed to have a value in $B$. It is defined as $X \in B := X^{-1}(B)$. If a probability measure $P$ on $\mathcal{F}$ is given, then the probability that $X$ has a value in the Borel set $B$ is defined to be $P(X \in B) = P(X^{-1}(B))$. For $X$ and $P$ given, the map $B \mapsto P(X \in B) =: \mu_X(B)$ is a probability measure on $\mathbb{R}$ which is called the distribution of $X$ (with respect to $P$). The phrase in parentheses is often omitted since $P$ is implicit in a many contexts. The probability measure $P$, being a measure, has a theory of integration that comes with it for free. So, integrals $\mathcal{E}(X) := \int_{\Omega} dP(\omega) X(\omega)$ are defined for a wide class of $X \in M(\Omega)$, including all bounded, Borel measurable functions. We say that $\mathcal{E}(X)$ is the expected value of $X$ (with respect to $P$).
An important identity for the expected value is

$$E(X) = \int_{\Omega} dP(\omega) X(\omega) = \int_{\mathbb{R}} d\mu_X(\lambda) \lambda$$

in the sense that if one of these two integrals exists, then so does the other and the equality holds. This expresses the expected value as the first moment of a probability measure on \(\mathbb{R}\). More generally, for any bounded, Borel function \(f : \mathbb{R} \to \mathbb{C}\) and all \(\omega \in \Omega\) we define \(f(X)(\omega) := f(X(\omega))\), which is itself bounded and Borel measurable. Then we have that the probability measure \(\mu_X\) satisfies

$$E(f(X)) = \int_{\Omega} dP(\omega) f(X)(\omega) = \int_{\mathbb{R}} d\mu_X(\lambda) f(\lambda).$$

The observables in quantum theory are self-adjoint operators \(T = T^*\), but the condition \(T \in \mathcal{L}(\mathcal{H})\) is not required though it may hold. The spectrum of any self-adjoint operator (bounded or unbounded) is a non-empty, closed subset of \(\mathbb{R}\), and its elements correspond to the values seen in the observations associated to \(T\). For each Borel subset \(B\) of \(\mathbb{R}\) there is an event in \(\mathcal{L}(\mathcal{H})\) that is denoted as \(T \in B := P_T(B)\), where \(P_T\) is the pvm associated to \(T\) by spectral theory. If a density matrix \(\rho\) is given, then the probability that \(T\) has a value in the Borel subset \(B\) of \(\mathbb{R}\) is defined by Born’s rule to be

$$P(T \in B \mid \rho) = Tr(P_T(B) \rho).$$

In quantum theory, only rarely would \(\rho\) be omitted from the notation on the left side. Probability theory in quantum theory exists even prior to choosing a state \(\rho\), since a self-adjoint operator \(T\) has its unique associated pvm \(P_T\). This has properties similar to those of a probability measure, except that it takes values that are quantum events. So a pvm has a codomain consisting of quantum events. On the other hand a classical probability measure has classical events in its domain. The integral \(\int_{\mathbb{R}} dP_T(\lambda) f(\lambda)\) exists in \(\mathcal{L}(\mathcal{H})\) for a wide class of Borel measurable functions \(f : \mathbb{R} \to \mathbb{C}\), including all bounded functions. The ‘expected value’ of this pvm gives

$$T = \int_{\mathbb{R}} dP_T(\lambda) \lambda.$$
Comparison with Classical Probability

by the spectral theorem. Maybe you did not expect this result. Actually,

$$ f(T) := \int_{\mathbb{R}} dP_T(\lambda) f(\lambda) $$

defines a functional calculus for all bounded, Borel functions $f : \mathbb{R} \rightarrow \mathbb{C}$.

Another curious point is that a classical probability measure $P$ satisfies $0 \leq P(A) \leq 1$ for every event $A$. This is an inequality of real numbers. On the other hand, a pvm $P$ on $\mathbb{R}$ satisfies $0 \leq P(B) \leq I$ for every Borel subset of $\mathbb{R}$. This is an inequality of self-adjoint operators. So, the linearly ordered interval of real numbers $[0, 1]$ for probabilities in the classical case is replaced by the lattice of projections in the ‘interval’ $[0, I]$ of self-adjoint operators. An even more curious point is that the interval $[0, 1]$ only contains real numbers, while the ‘interval’ $[0, I]$ contains self-adjoint operators that are not projections. This opens the door to considering positive operator valued measures, which will be discussed later.

Yet another way of relating quantum probability to classical probability is by restricting a pvm to a sub-lattice $\mathcal{E}'$ of the lattice events of $\mathcal{E}$ such that $\mathcal{E}'$ is a $\sigma$-algebra. Then one can put any classical probability measure whatsoever on $\mathcal{E}'$. This probability measure need not be the restriction of a spectral measure associated to a pvm defined $\mathcal{E}$, in which case one is considering a structure unrelated to quantum theory. However, if one starts with a spectral measure on $\mathcal{E}$, one can restrict it to many such $\sigma$-algebra sub-lattices in order to ‘view’ the pvm in a variety of classical ways. This could be what some would call complementarity, though such a specific description is not usually given. Conversely, one could have a classical probability measure on $\mathcal{E}'$ and ask whether this is the restriction of a spectral measure on $\mathcal{E}$. And if it is, then whether that spectral measure is unique. This is close to the setting of the Kadison-Singer conjecture (see [18]), which is now a proved theorem (see [23]). In that context one has a maximal commutative sub-$*$-algebra $\mathcal{A}$ of a $C^*$-algebra $\mathcal{C}$ and a state $\phi : \mathcal{A} \rightarrow \mathbb{C}$. Then the theorem says that there exists a unique extension $\tilde{\phi} : \mathcal{C} \rightarrow \mathbb{C}$ of $\phi$ which is a state. Colloquially, under these hypotheses one commutative ‘snapshot’ of a state suffices to characterize it.

Another notable difference is that the probability of a sequence of events in quantum probability does not reduce in general to the probability of a single event, as happens in classical probability theory. This entails a separate definition of quantum probability for sequences of events. The properties of there multi-event probabilities include new features absent in classical
probability theory, such as interference terms and dependence on the order of events.

Also notice that classical probability measures on $\mathbb{R}$ arise naturally in quantum probability, but classical probability does not involve quantum probability. And finally in classical probability theory, there is no basic time evolution equation, although time dependent stochastic processes are a part of that theory. However, quantum probability is intrinsically a part of quantum theory, which has time evolution as a major facet of the theory. In fact, the generalized Born’s rule (2.9.2) is the fundamental time evolution equation of quantum theory.

### 2.15 Expected Value

Expected value is a mostly unremarkable, quite secondary aspect of quantum probability. However, many times I have heard colleagues speak about it incorrectly. Unfortunately, this misunderstanding can be found in print, too. So I think that it is necessary to give a clarification of this topic.

First, let’s repeat what quantum theory says about probability, namely Born’s rule for $\psi$ a unit vector, $A = A^*$ and $B$ a Borel subset of $\mathbb{R}$, which with a bit of new notation is

$$\mu(B) := P(A \in B \mid \psi) = \langle \psi, P_A(B) \psi \rangle.$$

As a function of the Borel subset $B$ of $\mathbb{R}$ with both $A$ and $\psi$ fixed, $\mu$ is a probability measure on the real line $\mathbb{R}$. We have seen this basic point already many times. One can now apply concepts from standard probability theory to this probability measure $\mu$. And this is indeed done. For example, one can consider the moments of any probability measure. So, for every integer $k \geq 1$ we define the $k$th moment of $\mu$ as

$$m_k := \int_{\mathbb{R}} d\mu(\lambda) \lambda^k,$$

provided that this integral converges absolutely. The expected value of $\mu$ is defined to be $m_1$, the first moment, provided again that the integral converges absolutely. The idea is that $m_1$ is a statistical estimator given by probability theory of the empirically observed sample average

$$\bar{m} := \frac{\lambda_1 + \cdots + \lambda_n}{n}.$$
of \( n \geq 1 \) measured values (or sample) \( \lambda_1, \ldots, \lambda_n \). The values measured are typically not all the same and, indeed, this is what motivates one to turn to probability theory in order to understand them. It is quite common that the expected value is not going to be equal to any of the measured values, since this is a common property of the sample average. For example, the sample average of the number of children per family (in a sample of families in a city, say) could be 2.3, which is not an integer.

As a brief aside, let me note that one can continue by defining the central moments for every integer \( k \geq 2 \) by

\[
\sigma_k := \int_{\mathbb{R}} d\mu(\lambda) (\lambda - m_1)^k
\]

provided that the expected value \( m_1 \) exists and that the integral in (2.15.1) also is absolutely convergent. A frequently used central moment is \( \sigma_2 \), which is also called the variance. And the standard deviation of \( \mu \) is defined by \( \sigma := \sqrt{\sigma_2} \). Standard deviations (or equivalently variances) enter into the Roberts inequality that expresses mathematically the Heisenberg Uncertainty Principle.

It is well known that the moments (or equivalently the central moments) do not uniquely determine \( \mu \) in all cases. Even in the most favorable case when these moments exist for all \( k \) this moment problem might not have a solution \( \mu \) and, if it does, that solution might not be unique. Colloquially, one can say that the moments carry some information about the probability measure \( \mu \), but not in general all the possible information.

We have enough context now for discussing a common misunderstanding. The false assertion is often made that the only information that quantum theory provides about a system is the expected value of observables in a given state. Actually, Born’s rule provides the probability measure for any observable in a given state, which is a lot more than just the first moment of that probability measure. This aspect of quantum theory dates back at least to 1932 in the seminal book [25] of von Neumann and so should be well known in the physics community. My experience is that it is known by some, but not by others. An example of this confusion is the ‘proof’ that Schrödinger’s model is equivalent to the Heisenberg model by showing that the expected value is the same in both models. While this is a necessary condition, by itself it is not sufficient.

To complete this section here is the derivation of the usual formula for calculating the expected value of \( A = A^* \) given a pure state \( \psi \) in the dense
domain of $A$. 

$$m_1 = \int_{\mathbb{R}} \langle \psi, dP_A(\lambda) \psi \rangle \lambda = \langle \psi, \left( \int_{\mathbb{R}} dP_A(\lambda) \right) \psi \rangle = \langle \psi, A\psi \rangle.$$ 

### 2.16 Dynamics: The Generalized Born’s Rule

The **dynamics**, also known as the **time evolution**, of a physical system is given in quantum theory by a further generalization of the time dependent generalized Born’s rule (2.9.2). Contrary to the confused opinion of many authors (including me in [31]), there is only one fundamental time evolution equation in quantum theory. It is the same equation in all models. It is the only equation which we subject to experimental verification. But (2.9.2) is still not adequate for all purposes, since it assumes that there is no time evolution of the quantum system between events. So now we suppose that a system is given with state $\psi$ at time $t_0$ and that we are interested in the events $E_1, \ldots, E_n$ at the times $t_1, t_2, \ldots, t_n$, where $t_0 < t_1 < \cdots < t_n$. The probability of this sequence of events at these times is

$$P(E_1, \ldots, E_n, t_0, t_1, \ldots, t_n | \psi) = ||E_n U(t_n, t_{n-1}) \cdots E_2 U(t_2, t_1) E_1 U(t_1, t_0) \psi||^2,$$

where $U(s, t)$ is the time evolution operator of the system for the times $s < t$. In the Schrödinger model $U(s, t) = \exp(-i(t-s)H/\hbar)$, where $H$ is the Hamiltonian of the quantum system. I imagine that this formula is obvious to those who think in the Schrödinger model. But whether this formula is obvious or not is an independent consideration. It is an axiom. It is the final version of the **Generalized Born’s Rule**.

**Axiom 5 Updated**: Let $\rho$ be a density matrix. Suppose that $E_1, \ldots, E_n$ is a sequence of events and that $t_0 < t_1 < \cdots < t_n$ is a sequence of times. Suppose that we have **time evolution operators** $U(s, t) \in \mathcal{L}(\mathcal{H})$ for all $s < t$. We define the **consecutive probability with time evolution** $P(E_1, \ldots, E_n, t_0, t_1, \ldots, t_n | \rho)$ to be

$$Tr(E_n U(t_n, t_{n-1}) \cdots E_2 U(t_2, t_1) E_1 U(t_1, t_0) \rho).$$

(2.16.1)

The **conditional probability with time evolution** reads much like (2.9.2) with the time evolution operators appropriately interspersed between the event operators. The exact, general formula becomes unwieldy to write down. ■
This axiom holds in all models if we extend the group $E_t$ to act on all bounded operators and so, in particular, on the operators $U(s,t)$. In the standard models (Schrödinger, Heisenberg, interaction) the action of $E_t$ is via conjugation by operators in a unitary group. This conjugation acts on all of $\mathcal{L}(\mathcal{H})$ as well. As we have come to expect, the value of the probability (2.16.1) is invariant under isomorphisms of models. It might appear strange that the time evolution operators depend on the model, rather than being the same in all models as in the familiar Schrödinger model. But this is exactly one of the things that happens in the interaction model, which also should be familiar for the reader. Recall that in the interaction model one writes the Schrödinger Hamiltonian as $H = H_{\text{free}} + H_{\text{int}}$, the sum of a ‘free’ term and of an ‘interacting’ term. Then the operators in the Schrödinger model, including the time evolution operators, are transformed to operators in the interaction model by conjugating them with $\exp(-itH_{\text{free}}/\hbar)$. The splitting of $H$ into two terms is chosen to facilitate subsequent calculations, not to change the results. And that is exactly the point of having different isomorphic models of quantum theory.

Again, the manner for computing this time dependent probability does depend on the model, which leads to much confusion. In the Schrödinger model, which is the most familiar and most widely used model, all the events are time independent and only the state could possibly change with time. As is well known the time evolution of the state in this model is given by a family of equations known collectively as Schrödinger’s equation, which all have the same form $i\hbar \psi''(t) = H\psi(t)$, where $H$ is a self-adjoint operator, known as the Hamiltonian, acting in the Hilbert space. The multitude of physical systems covered by this approach is due to the fact that physicists are very adept at finding the appropriate Hamiltonian for many systems. However, the time dependent state which solves Schrödinger’s equation has no physical significance. It is an artifact of the model and nothing else. But it is one of the ingredients that for the case $\rho = |\psi\rangle\langle\psi|$ in (2.16.1) goes into computing the time dependent probability, which indeed does have physical significance. In other models one must calculate the time dependence of the elements in (2.9.2) using other auxiliary equations. But again these auxiliary time dependent elements do not have any physical significance.

It will surely be taken as heresy on my part to say that Schrödinger’s equation is without physical significance. However, if that is the fate of its solutions, then that must be the fate of Schrödinger’s equation itself. It is a stepping stone, a useful tool. This idea flies in the face of long-standing
Dynamics

traditions in physics, especially those that favor differential equations as the most fundamental elements of a physical theory. There is an expectation that the time evolution of a physical system should be expressed as a differential equation involving time as one of the variables. But (2.16.1) does not have that form. Taking the derivative of (2.16.1) in order to find a differential equation that it must solve leads to a relation of the time derivative of the probability with the time evolutions of the events and of the state. But both of those time evolutions are model dependent.

This preference for differential equations manifests itself in the way the equation of motion is written in the Heisenberg model. Typically this is presented as an ill-defined differential equation whose so-called ‘solution’ is then given. It is that ‘solution’ (1.6.1) which is the actual time evolution equation, in spite of the fact that it is not a differential equation.

Another apparent problem in taking the generalized Born’s rule (2.16.1) as the fundamental time evolution equation of quantum theory is that this ignores the history of its discovery. The physics community is fascinated with the story of how these ideas emerged and who gets the credit for each of them. And I am complicit in this tradition to the extent that I do use the names of scientists when discussing their discoveries. At one point while writing this treatise I thought of removing ‘Lie’ from Lie group to give one example of how one might eliminate history from this narrative. Born’s rule in the traditional narrative is seen as a later add-on, an embellishment of ideas that already ‘worked’ but somehow seemed incomplete. And besides that there already was a known equation, namely Schrödinger’s equation, that was recognized by one and all as the time evolution differential equation of quantum theory. And this history then continues with Born’s rule being severely criticized by some and rejected by others. This paragraph is one of my few excursions into the history of quantum theory, and the point behind it is that the historical sequence of discoveries is not the logical structure of quantum theory. The generalized Born’s rule (2.16.1) is the fundamental time evolution equation of quantum theory.

If you think that my purpose here is to remove Schrödinger’s equation from its central position in quantum theory, then you are reading correctly. For example, on Wikipedia the various topics on quantum theory (which there is called quantum mechanics) are presented in a box with Schrödinger’s equation at its head. This should be replaced with the generalized Born’s rule according to my thesis.

It is not well understood that one is actually only using the conditional
Quantum Information probability (2.9.2) when analyzing many physical phenomena. Let’s recall how that works in the case of two events. Given an event \( E_1 \) and a unit vector \( \psi \), the conditional probability of a subsequent event \( E_2 \) is

\[
P(E_2 \mid E_1, \psi) = \frac{||E_2 E_1 \psi||^2}{||E_1 \psi||^2} = ||E_2 \tilde{\psi}||^2 = P(E_2 \mid \tilde{\psi}),
\]

where \( \tilde{\psi} := E_1 \psi / ||E_1 \psi|| \) is the ‘collapse’ of the original state \( \psi \). Of course, this is just reading backwards the motivating argument for the definition of conditional probability. The point now is that the right side of this equation can always be translated into the conditional probability on the left side.

Another common way of speaking of this situation is that the first event \( E_1 \) ‘prepares’ a state \( \tilde{\psi} \), which the event \( E_2 \) then ‘measures’. However, it all reduces to quantum conditional probability of quantum events.

Another curious aspect of quantum theory is that the one-event Born’s rule in Axiom 4 is rarely ever used to understand observations. A careful analysis almost always reveals that one is actually considering a conditional probability of two events.

### 2.17 Quantum Information

As with any other aspect of basic quantum theory, information must be defined in a way that is invariant under isomorphisms of models. Since the only structure left invariant is the Generalized Born’s Rule, information must be defined in terms of it. In a more elaborated quantum theory with more structures, and a correspondingly more restricted notion of isomorphism, it could be possible to introduce other definitions of information. Following the usual convention, I will formulate this topic in terms of entropy, which is the negative of information.

Suppose that \( T = T^* \) is a self-adjoint operator and that \( \rho \) is a density matrix. Then, as we have seen, for \( B \) a Borel subset of \( \mathbb{R} \) the map

\[
B \mapsto P(T \in B \mid \rho) = Tr(P_T(B) \rho)
\]

is a probability measure on \( \mathbb{R} \). We suppose that this measure is absolutely continuous with respect to Lebesgue measure. Let \( f_{T,\rho} : \mathbb{R} \to [0, \infty) \) be a density function for \( P(T \in B \mid \rho) \), that is

\[
P(T \in B \mid \rho) = \int_B d\lambda f_{T,\rho}(\lambda).
\]
This condition uniquely determines $f_{T,\rho}$ except on a subset of $\mathbb{R}$ of Lebesgue measure zero. One can make $f_{T,\rho}$ unique by requiring that it satisfy the cadlag condition.

This is a model independent property in the sense that if $T'$ and $\rho'$ correspond respectively in an isomorphic model to $T$ and $\rho$, then $f_{T',\rho'} = f_{T,\rho}$ almost everywhere with respect to Lebesgue measure. Then we define the entropy of $T$ given $\rho$ as

$$H(T \mid \rho) := -\int_{\mathbb{R}} d\lambda f_{T,\rho}(\lambda) \log(f_{T,\rho}(\lambda))$$

where the usual definition $0 \log 0 := 0$ is being used. The symbol log can mean the logarithm to the base $e$ or any other base $b > 1$. Once a base is chosen, that fixes a normalization condition on the entropy. Continuing to use the notation established above, we have $H(T \mid \rho) = H(T' \mid \rho')$.

Similarly, the conditional entropy of $T$ given an event $S \in B$ and a density matrix $\rho$ can be defined if $P(T \in C \mid S \in B, \rho)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with density function $f_{S,B,\rho} : \mathbb{R} \to [0, \infty)$ by

$$H(T \mid S \in B, \rho) := -\int_{\mathbb{R}} d\lambda f_{S,B,\rho}(\lambda) \log(f_{S,B,\rho}(\lambda)).$$

Since there is no joint probability measure in general for a pair of self-adjoint operators $S$ and $T$, it seems to not be possible to define in general a conditional probability of $T$, given $S$ and $\rho$. Of course, if $S$ and $T$ commute, then this can be done. In general the probabilities $P(S \in B, T \in C)$ and $P(S \in B, T \in C \mid \rho)$, where $B, C$ are Borel subsets of $\mathbb{R}$ and $\rho$ is a density matrix, give numbers that do carry information about the ordered pair $S, T$. 
Chapter 3

Entanglement

I would not call [entanglement] one but rather the characteristic trait of quantum mechanics . . .

(italics in original)

Erwin Schrödinger

Entanglement has been one of the most puzzling topics in quantum theory. This is largely due to a detailed discussion about two events that presumably ‘know’ nothing of each other, but nonetheless show an uncanny deterministic relation between them. One wonders how a probabilistic theory could ever explain this determinism. Worse yet, these two events can be space-like with respect to each other and so relativity theory forbids information from passing from one to the other. Yet in some strange form this seems to be what is happening. There are many experiments that agree with entanglement, including some carefully constructed experiments where the two events are indeed space-like. And no experiment, as far as I am aware, contradicts entanglement. How can we deal with this conundrum? Must quantum theory be modified or abandoned? Is there some spooky action at a distance that accounts for entanglement? Is physics non-local? How can the collapse of a state by one detector possibly affect the other detector?

The point here is that in a probabilistic theory some of the calculated probabilities are 1. Experimental confirmation of entanglement should not produce existential angst, much less to appeals for new scientific principles to ‘explain’ the ‘mystery’ of this agreement. So what is going on here in terms of quantum theory? Quite simply, entanglement always involves three events, never just two. (Or possibly more than three!) There always is an event
that precedes in time the two other events. Of course, those two later events ‘know’ about that earlier event. The appropriate probability to be computed is that of the conditional probability of these three (or more) events.

Here is some notation for a simple example of entanglement. We have an event $E_0$, which we will call the initial event, and then two events $E_1$ and $E_2$, each of which occurs after $E_0$. The time relation between $E_1$ and $E_2$ is irrelevant to this analysis. This example captures the essence of the EPR paper and is discussed further in Chapter 6. We could assume that $E_1$ and $E_2$ commute, but that assumption is not used until the end of this chapter.

Notice that space-like separated events must commute according to special relativity, while some time-like (or light-like) separated events can commute.

We start by taking $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We let $S_1$, $S_2$ and $S_3$ denote the standard $2 \times 2$ spin matrices. In the sequel we will only use

$$S_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for convenience in doing calculations. Similar results hold for the other spin matrices. Then we next consider these self-adjoint operators acting on $\mathcal{H}$:

$$S^2, \quad S_r := S_3 \otimes I, \quad S_l := I \otimes S_3,$$

where

$$S^2 = (S_1 \otimes I + I \otimes S_1)^2 + (S_2 \otimes I + I \otimes S_2)^2 + (S_3 \otimes I + I \otimes S_3)^2$$

is the standard total spin operator. The three events to be considered are defined as

$$E_0 := P_{S^2}(\{0\}), \quad E_1 := P_{S_r}(\{1/2\}), \quad E_2 := P_{S_l}(\{-1/2\}).$$

The way one verbalizes this can be misleading, but I will venture to do it anyway. The event $E_0$ says that initially the system has total spin 0. The event $E_1$ says that a detector to the right of the place where $E_0$ occurred measures the $z$ component of the spin to be $1/2$. The event $E_2$ says that a detector to the left of the place where $E_0$ occurred measures the $z$ component of the spin to be $-1/2$. The words ‘right’ and ‘left’ are not really important. They merely give a visual rendition to the formalism.

Notice that in this example the self-adjoint operators $S_r$ and $S_l$ commute. Consequently, the events $E_1$ and $E_2$ also commute.
Given these definitions it is an exercise, which we now do, to compute the appropriate conditional probability for these 3 events. The reader should be aware that the intermediate steps in this calculation are without physical significance. The only physically significant part of this calculation is the last step where the value of the conditional probability is given.

Let $\varepsilon_1, \varepsilon_2$ be the standard orthonormal basis of $\mathbb{C}^2$, namely

$$\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

We first identify the pertinent quantum events in terms of this basis using Dirac notation:

$$E_1 = P_{S_z}(1/2) = |\varepsilon_1\rangle\langle\varepsilon_1| \otimes I_2 \quad \text{and} \quad E_2 = P_{S_l}(1/2) = I_2 \otimes |\varepsilon_2\rangle\langle\varepsilon_2|,$$

where $I_2$ is the identity operator acting on $\mathbb{C}^2$. Notice that these are rank 2 projections. The operator $S^2$ is not central in $\mathcal{L}(\mathcal{H})$. It is well known that its spectrum is $\{0, 2\}$. The eigenspace for the eigenvalue 2 has dimension 3 (a triplet for spin 1), while the eigenspace for the eigenvalue 0 has dimension 1 (a singlet for spin 0), and this latter eigenspace has an orthonormal basis consisting of the one vector

$$\psi_0 := \frac{1}{\sqrt{2}}(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1).$$

So we have the quantum event

$$E_0 = P_{S^z}(\{0\}) = |\psi_0\rangle\langle\psi_0|,$$

which is a projection with rank 1. We now compute the quantum conditional probability

$$P(E_1 \mid E_2, E_0 \mid \psi) = \frac{||E_1E_2E_0\psi||^2}{||E_2E_0\psi||^2} \quad (3.0.1)$$

provided that $||E_2E_0\psi|| \neq 0$, (When not stated otherwise, $\psi \in \mathcal{H}$ is a unit vector.) Intuitively, the expression (3.0.1) gives the conditional probability that the detector on the right measures the $z$-component of spin to be $1/2$ given that the detector on the left measures the $z$-component of spin to be $-1/2$ and that (previously) the initial state has been prepared to have spin 0. These words in everyday language only serve to reassure the dubious that this formalism has its motivation. We first have for all $\psi \in \mathcal{H}$ that

$$E_0 \psi = |\psi_0\rangle\langle\psi_0|\psi = \langle\psi_0, \psi\rangle\psi_0.$$
Entanglement

Continuing we have for all $\psi \in \mathcal{H}$ that
\[
E_2 E_0 \psi = (I_2 \otimes |\varepsilon_2\rangle \langle \varepsilon_2|)(\psi_0, \psi) \psi_0
\]
\[
= 2^{-1/2}\langle \psi_0, \psi \rangle (I_2 \otimes |\varepsilon_2\rangle \langle \varepsilon_2|)(\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)
\]
\[
= 2^{-1/2}\langle \psi_0, \psi \rangle (\varepsilon_1 \otimes |\varepsilon_2\rangle \langle \varepsilon_2|\varepsilon_2 \otimes |\varepsilon_2\rangle \langle \varepsilon_2|\varepsilon_1)
\]
\[
= 2^{-1/2}\langle \psi_0, \psi \rangle (\varepsilon_1 \otimes \varepsilon_2)
\]
\[
= 2^{-1/2}|\varepsilon_1 \otimes \varepsilon_2\rangle \langle \psi_0|\psi.
\]

This is non-zero if $\langle \psi_0, \psi \rangle \neq 0$, which we assume from now on. We conclude
\[
E_2 E_0 = 2^{-1/2}|\varepsilon_1 \otimes \varepsilon_2\rangle \langle \psi_0|.
\]

(3.0.2)

Clearly, $E_2 E_0$ is a non-zero operator that is not a projection. Finally, for all $\psi \in \mathcal{H}$ we evaluate again that
\[
E_1 E_2 E_0 \psi = (|\varepsilon_1\rangle \langle \varepsilon_1| \otimes I_2)2^{-1/2}\langle \psi_0, \psi \rangle (\varepsilon_1 \otimes \varepsilon_2)
\]
\[
= 2^{-1/2}\langle \psi_0, \psi \rangle (|\varepsilon_1\rangle \langle \varepsilon_1| \otimes I_2)(\varepsilon_1 \otimes \varepsilon_2)
\]
\[
= 2^{-1/2}\langle \psi_0, \psi \rangle (\varepsilon_1 \otimes \varepsilon_2)
\]
\[
= 2^{-1/2}|\varepsilon_1 \otimes \varepsilon_2\rangle \langle \psi_0|\psi
\]
\[
= E_2 E_0 \psi.
\]

This implies that we have two equal operators (not projections)
\[
E_1 E_2 E_0 = E_2 E_0.
\]

(3.0.3)

This is more than we need to conclude that the conditional probability (3.0.1) is equal to 1, namely
\[
P(E_1 | E_2, E_0 | \psi) = \frac{||E_1 E_2 E_0 \psi||^2}{||E_2 E_0 \psi||^2} = 1.
\]

The equality (3.0.3) of these two operators is an intermediate step which has no physical significance. For example, the operator $E_2 E_0$ is not an event since $E_0$ and $E_2$ do not commute. Note that $E_0$ and $E_1$ also do not commute. This lack of commutativity is good news. It means that we are dealing with a truly quantum situation.

Similarly, one can calculate that
\[
P(E_2 | E_1, E_0 | \psi) = \frac{||E_2 E_1 E_0 \psi||^2}{||E_1 E_0 \psi||^2} = 1
\]
provided that $||E_1E_0 \psi|| \neq 0$. The details are left to the reader. Using
this straightforward method for 3 events with the same time order, one can
evaluate the conditional probability for other combinations of components
of spin. Typically those probabilities will lie strictly between 0 and 1, but
in the specific example given here a value of 1 is the result. The fact that
a certain probability is 1 does not invalidate the use of probability theory,
but is merely a special case of it. In particular, probability 1 does not imply
that this is a deterministic situation. In other specific cases probability 0 will
occur. For example, we have
\[
P(E^c_2 \mid E_1, E_0 \mid \psi) = \frac{|| (I - E_2) E_1 E_0 \psi ||^2}{|| E_1 E_0 \psi ||^2} = 0,
\]
where $E^c_2 = I - E_2$ is the event complementary to $E_2$. One of the intermediate
steps used to show this (using a formula above) is $(I - E_2) E_1 E_0 = 0$.

In any probabilistic theory, some of the calculated probabilities will turn
out to be 1. I do not see anything ‘spooky’ about that. Nor do I see anything
‘non-local’ in this analysis. Nor do I see that an element of ‘reality’ is required
to ‘explain’ a probability one situation. If you are seeking ‘reality’ here, why
not say that all probabilities are real? See Chapter 6 for more on this.

Of course, we can discard the information about the event $E_2$ and ask
only for the conditional probability of $E_1$ given $E_0$. Intuitively, this is the
point of view of the experimenter on the right. So we want to compute
\[
P(E_1 \mid E_0, \psi) = \frac{|| E_1 E_0 \psi ||^2}{|| E_0 \psi ||^2}.
\]

We first note that we have $E_0 \psi = \langle \psi_0, \psi \rangle \psi_0$ for all $\psi \in \mathcal{H}$ as before. Next, we calculate for all $\psi \in \mathcal{H}$ that
\[
E_1 E_0 \psi = (|\varepsilon_1 \rangle \langle \varepsilon_1 | \otimes I_2) (\langle \psi_0, \psi \rangle \psi_0)
= 2^{-1/2} \langle \psi_0, \psi \rangle (|\varepsilon_1 \rangle \langle \varepsilon_1 | \otimes I_2) (\varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1)
= 2^{-1/2} \langle \psi_0, \psi \rangle (\varepsilon_1 \otimes \varepsilon_2)
= 2^{-1/2} |\varepsilon_1 \otimes \varepsilon_2 \rangle \langle \psi_0 | \psi \rangle.
\]
Consequently, $E_1 E_0 = 2^{-1/2} |\varepsilon_1 \otimes \varepsilon_2 \rangle \langle \psi_0 | \psi \rangle$ and therefore
\[
P(E_1 \mid E_0, \psi) = \frac{|| E_1 E_0 \psi ||^2}{|| E_0 \psi ||^2} = \frac{|| 2^{-1/2} |\varepsilon_1 \otimes \varepsilon_2 \rangle \langle \psi_0 | \psi \rangle ||^2}{|| \langle \psi_0, \psi \rangle \psi_0 ||^2} = 1/2,
\]
Entanglement

since \( \| \varepsilon_1 \otimes \varepsilon_2 \rangle \langle \psi_0 | \psi \| = | \langle \psi_0 , \psi \rangle | \). In other words, the axiom for calculating the conditional probability gives the expected result. Similarly, one can show that \( P(E_2 \mid E_0, \psi) = 1/2 \).

We can also compute the conditional probability that \( E_1 \) and \( E_2 \) occur given \( E_0 \). This makes sense if we assume that the events \( E_1 \) and \( E_2 \) commute. The conditional probability for this situation is

\[
P(E_1, E_2 \mid E_0, \psi) = \frac{\| E_1 E_2 E_0 \psi \|^2}{\| E_0 \psi \|^2}
\]

provided \( \| E_0 \psi \| \neq 0 \). Recall (3.0.2) and (3.0.3): \( E_1 E_2 E_0 = 2^{-1/2} | \langle \varepsilon_1 \otimes \varepsilon_2 \rangle \psi_0 \rangle \). This equality of operators, not of events, has no physical significance, but does allow us to proceed to the next part of the calculation. So here it is:

\[
P(E_1, E_2 \mid E_0, \psi) = \frac{\| E_1 E_2 E_0 \psi \|^2}{\| E_0 \psi \|^2} = \frac{\| 2^{-1/2} | \langle \varepsilon_1 \otimes \varepsilon_2 \rangle \psi_0 \rangle \|^2}{\| | \langle \psi_0 \rangle \| \psi_0 \|^2} = 1/2.
\]

So, we have calculated a different conditional probability, and we get a probability that is neither 0 nor 1. Notice that this result agrees with what little intuition we can muster concerning quantum theory. In possibly misleading ordinary language it says that after measuring the system to have spin 0 the probability is 1/2 that the detector on the right measures the \( z \)-component of spin to be \( +1/2 \) and the detector on the left measures the \( z \)-component of spin to be \( -1/2 \).

What about the rest of the probability? Simple, similar calculations which are left to the reader show that \( P(E_1^c, E_2^c \mid E_0, \psi) = 1/2 \), where the complementary event of an event \( E \) is defined as \( E^c := I - E \). One also easily verify that \( P(E_1^c, E_2 \mid E_0, \psi) = 0 \) and \( P(E_1, E_2^c \mid E_0, \psi) = 0 \). All of these probabilistic statements can be expressed in ordinary language, and they all agree with commonly held ‘intuition’.

One mathematical fact behind this example is that both events \( E_1 \) and \( E_2 \) are products of the form \( F_1 \otimes F_2 \), where each factor is an event in \( L(C^2) \), but the event \( E_0 \) is not of that form. Here the factorizations are with respect to the given definition \( H = C^2 \otimes C^2 \). This fact is not invariant in general under a unitary transformation to another isomorphic Hilbert space with its own given factorization as a product of two Hilbert spaces of dimension 2. Nonetheless, it does make sense to speak of entangled events. It does not make sense to speak of entangled particles. A good rule of thumb is that when someone speaks of \textit{two} entangled particles, it is best to rephrase that as a
clearer statement about three entangled events. Entanglement is a property of events. Of course, you may object that I never defined entanglement.

**Definition 3.0.1** A sequence of time ordered events is entangled if some appropriate conditional probability involving them is equal to 1. The noun describing such events is entanglement.

Notice that in this definition the Hilbert space need not be represented as a tensor product. Typically, I expect that three entangled events are such that in some unitarily equivalent tensor product Hilbert space two of them are factorizable as a tensor product of events, while one of them is not.

Another mathematical fact of this example is that the first event $E_0$ has rank 1. Some might object that this event only serves to ‘prepare’ the ‘real’ initial state $\psi_0 = E_0\psi / |E_0\psi|$, where $\psi \in \mathcal{H}$ is some ‘pre-initial’ state satisfying $E_0\psi \neq 0$. You could think of it that way, but by doing so you are missing the point. States do not change in this analysis. Everything follows from only events and their conditional probabilities.

The analysis of entanglement given in this chapter does not depend in any way on Schrödinger’s equation. Both the states and the events have been taken to be time independent. So this example is in both the Schrödinger model and in the Heisenberg model. Typically in other presentations, the time dependence is imposed on the spatial structure rather than on the spin structure in a way consistent with the presentation here. Most importantly we learn that Schrödinger’s equation is not a basic structure in quantum theory, while quantum probability theory is basic.

The critical reader may object that I have done nothing other than just repeat the standard analysis for an entanglement experiment in a disguised form. But that ignores certain key aspects of this presentation. Firstly, it is an analysis of three events, not of two particles. And secondly, there is no reference to collapse, which is only a way of describing some intermediate steps in other, more common ways of dealing with entanglement.

And collapse does not require an ‘explanation’ any more than any other mathematical algorithm that gives the right results. The terminology for this particular step is unfortunate, since it leads one to think that some sort of ‘understanding’ of a physical process is needed. Calculations are meant to help us understand physical phenomena, or as it has been said since Antiquity: to save the phenomena. (See [28] for the original meaning of this phrase.)
But for those who cling to the collapse language, let me note that collapse also occurs in the context of classical physics, despite the often made claim that entanglement is a purely quantum effect. One can easily produce two space-like events which are highly correlated and are described classically. As in the quantum case, one simply allows an event in the common past of these two events to have an impact on them. As in the quantum case, the two later events would be binary: one of two related possibilities. For example, a dollar bill is torn half and each half is taken to a distant place where they are examined at space-like separated events by, as is conventional, Alice and Bob. One has the left half and the other has the right half. So we can say that when Alice sees that she has the left half that this collapses the event of what Bob sees. This is not very useful language, but it could be said. The mythical determinism of classical mechanics has nothing to do with this, since it is after all a chimera. I will come back to this controversial point in a later chapter.

Also, I have done an analysis here in terms of the conditional probability of events, which is never done when analyzing entanglement as far as I am aware. Rather the standard discussion uses these words: particle, state, measurement and collapse. It turns out that the conditional probability in the particular case (3.0.1) just happens to be 1. Again I want to emphasize strongly that the intermediate steps in the calculation have absolutely no physical significance. Only the calculated probability matters. The rules of quantum probability say that this particular combination of events occurs in 100% of repetitions of the experiment. This deserves to be checked by experiment. And it has been. The conclusion is that quantum theory is verified. This seems to be widely accepted in the physics community.

Let me make clear that entanglement is a property of events and not of anything else. It makes no sense to entangle particles because particles are not events. It makes no sense to entangle a system with its environment, since neither a system nor its environment is an event. It makes no sense to entangle an observer of a quantum system with that system, and so on.

For those accustomed to thinking in terms of the Schrödinger model it might be difficult to analyze phenomena using a sequence of quantum events instead of a sequence of quantum states. Of course, I am claiming that all entanglement phenomena can be analyzed as is done here using a sequence of quantum events and their conditional probability. This is a sweeping claim, although it depends on the rigorous definition I have given of entanglement. More specifically, I am claiming that situations that are traditionally called
entanglement in the literature are described by the definition 3.0.1. Clearly, this claim will and should be challenged. Let me note that it should be kept in mind when evaluating this claim that quantum states are themselves rank 1 quantum events. But do note that in the example of this chapter there are also quantum events of rank 2.

Now if you insist on using the word ‘collapse’, that could be acceptable as long as you do not assign a physical significance to it, as long as you do not treat it as a physical process. However, in practice that does not happen. Many physicists go down a rabbit hole by saying that collapse is a profound topic which requires some deeper ‘explanation’ or ‘interpretation’. Then it becomes a source of thousands of pages of no significance at all. Many are so used to working only with the Schrödinger model that they are unaware that some aspects of that model, such as collapse, are not present in other equivalent models, such as the Heisenberg model. And it is only the model independent probabilistic aspects of quantum theory that have a physical significance. Therefore I advocate for discarding “collapse” from the quantum vocabulary much as “equant” has been discarded from astronomical vocabulary. It is not needed. It is best to realize that there is no there there.

Analogies are difficult to find, since other branches of science do not rely on two distinct ways of speaking about time evolution. However, biologists sometimes speak of the evolution of species in teleological terms, which they take to be incorrect, yet useful, shortcuts for describing Darwinian evolution. They might say that a certain species of animals evolved to have a thicker subcutaneous layer of fat and white fur in order to survive to a colder, snowy climate. This can be easily misinterpreted by those who have not studied biology. However, biologists realize that this is just a manner of speech. My point is that the two ways of speaking about time evolution in quantum theory (Schrödinger’s equation and collapse of the state) are at best a short cut, a manner of speech, that takes the place of the Generalized Born’s Rule.

There is a sort of critique of formal rules and axioms that objects to the lack of prior justification of the rules. In short, where do the rules come from? That they work well for all observations is not accepted as a satisfactory answer, especially if the rule clashes with an ‘intuition’ of some sort or other. Of course, any proposed explanation of ‘why’ the rule is correct is itself subject to the same criticism. And so on ad infinitum. Such criticism could be applied to the Generalized Born’s Rule for computing quantum probabilities, to which I respond: Hypotheses non fingo.
Chapter 4

Schrödinger’s Cat

Nothing divided people more deeply than how they felt about cats.

Kingsley Amis

But there is another problem with entanglement. It has to do with how it is verified. The two events $E_1$ and $E_2$ can occur with space-like separation. In fact, this has been done precisely to see if quantum theory holds in that case. But how is the comparison of results from a series of measurements made? After all, at the time of the measurement neither experimenter can possibly know what the other has measured. So they record their measurements and get together at a later time (in the intersection of their respective future light cones) and compare data. Of course, such persistent data are not exactly events. This will lead us to consider the ‘paradox’ of Schrödinger’s cat.

Think of special relativity theory where events in space-time are described as points in a Minkowski space. Or perhaps, since points are an idealization, events are small regions in space-time, all of whose points are very close to each other. One of the conceptual difficulties with relativity theory is that one discusses objects as if they were events, but somehow persisting in time. These are called *world-lines*. For example, the physics professor holds up his piece of chalk for all the class to see. He asks how long it is. Everyone agrees that it is about 2cm long. But no! That is only one inertial frame of reference. The endpoints of the chalk are actually world-lines of events in Minkowski space and the spatial distance between these endpoints at the same time (relative to another frame of reference) is the length of the chalk (in that other frame of reference). The length in another appropriate inertial
frame is $10^{-9}$ m, a nanometer. Students are not so easily convinced and rightly so. After all, the concept of event has been changed behind their backs. And what color is the chalk? Well, the white light of the classroom reflects mostly the blue end of the spectrum. All agree that it is a piece of blue chalk. But in another inertial frame the frequency is shifted, and the reflected light from the chalk is red! Even classical physics can be very non-intuitive.

Returning to quantum theory, what is persistent data? I doubt that it is an event which lingers for an indefinitely long time. This seems to be changing concepts, much as in the case of relativity theory. Actually, I have not given any localization information in speaking about quantum events; they are just (time) ordered projections in a Hilbert space. Let me remedy that by using ideas of Haag from his version of local quantum field theory in [15]. As usual everything is within the context of one given Hilbert space $\mathcal{H}$. One thinks of open regions in Minkowski space. For any such region $\Omega$ one has an associated von Neumann algebra $\mathcal{V}(\Omega)$ in $\mathcal{L}(\mathcal{H})$. It is well known that a von Neumann algebra has `lots’ (in a sense that can be made precise\footnote{For example, the smallest von Neumann algebra in $\mathcal{L}(\mathcal{H})$ containing the events of a given von Neumann algebra $\mathcal{V} \subset \mathcal{L}(\mathcal{H})$ is $\mathcal{V}$ itself.}) of projections, that is, of quantum events. One says that any projection in $\mathcal{V}(\Omega)$ is an event \textit{localized} to the set $\Omega$. Intuitively, one is thinking that it is something that happened within that region of space-time. These quantum events are the only ones which correspond to something happening in $\Omega$; all other quantum events in $\mathcal{L}(\mathcal{H})$ are not part of the quantum theory in $\Omega$.

Suppose that $\Omega_1$ and $\Omega_2$ are space-like separated regions, meaning that for all $\omega_1 \in \Omega_1$ and all $\omega_2 \in \Omega_2$ we have that $\omega_1$ and $\omega_2$ are space-like events. Under these hypotheses we assume, following Haag, that for all quantum events $E_1 \in \mathcal{V}(\Omega_1)$ and $E_2 \in \mathcal{V}(\Omega_2)$, we have that $E_1$ and $E_2$ commute. Of course, the ‘small’ open sets are the ones that matter, especially those that are small in the temporal direction. I do not know how to incorporate a tubular neighborhood around a very long, especially infinitely long, world-line, into quantum theory in an intuitive way. But events associated with these sorts of regions are what one would use in quantum theory to describe the data recorded from scientific experiments. Or to describe any sort of information that is held indefinitely and can be shared. Of course, the entire scientific enterprise is based on sharing data. And this is what happens also with every entanglement experiment. But in that context it is not considered
to be a problem. It seems to be ‘natural’ that the experimenters share their data at some future time. And everyone is in agreement that entanglement has been experimentally verified this way. This is the way one verifies all quantum experiments, actually all experiments in all the sciences! And this is the basis of the often heard statement that quantum theory has been remarkably successful.

But wait! Enter Schrödinger’s cat, and the persistence of data from a quantum event becomes a problem. Let’s take the innocent cat out of harm’s way by changing the set-up. As in the original version, there is a radioactive source which has a probability of $1/2$ of decaying within a certain period of time. A diabolical machine is in place that can detect this decay always and is screened from any possible background noise. (We will see later why it is just as diabolical as Schrödinger’s machine.) If the decay is detected, then the machine prints YES on a sheet of paper that is blank except for the experiment run number. Otherwise, it does nothing during the time of the experiment. When the time period ends, the machine ejects the sheet of paper. To check that everything is working well, the experimenter repeats this many times, counts the number of times that YES appears and then divides by the total number of experimental runs. Within statistical precision the relative frequency of YES agrees with $1/2$. But are these sheets of paper quantum events? Is there a self-adjoint operator $P$ whose spectrum is $\{0, 1\}$ with the quantum event $P = 1$ corresponding to YES appearing and $P = 0$ otherwise? If so, then $P$ is a quantum event. And even if this is true, how do the results $1$ and $0$ persist in time? After all, the data from these events has to be published, lest the experimenter perish. And the sheets of paper can be preserved for future reference an indeterminable number of times for a more or less indefinite future.

Here is my best attempt at dealing with these questions. I think it makes sense to use the projection $P$, which in the cat context tells us whether the cat is dead or alive. This operator generates a commutative von Neumann algebra of dimension 2, namely $\mathcal{V} = \{\alpha I + \beta P \mid \alpha, \beta \in \mathbb{C}\}$ in $\mathcal{L}(\mathcal{H})$, where the dimension of $\mathcal{H}$ could be quite large, even infinite. Now following Haag the only quantum events available in this quantum theory are $0, P, I - P$, and $I$. Notice that these are 4 distinct events, since $P \neq 0$ and $P \neq I$ by the construction of the experiment. Let me note that the event 0 corresponds to starting the experiment with a dead cat, while the event $I$ corresponds to using an immortal cat.

So what happens now if we take normalized eigenvectors $\psi_1, \psi_2$ of $P$,
Schrödinger’s Cat

one for the eigenvalue 1 and the other for the eigenvalue 0? We can form
the normalized eigenvector (pure state) $\phi := 2^{-1/2}(\psi_1 + \psi_2)$ in the Hilbert
space $\mathcal{H}$. But this state is not available in this quantum theory, because
the corresponding quantum event $|\phi\rangle\langle\phi|$ is not in $\mathcal{V}$. In other words, the
‘superposition’ state of a live cat and a dead cat is not in this quantum theory.
It is important to note that this argument does not use a superselection rule.
Rather it is a mistake to take $\mathcal{L}(\mathcal{H})$ as the von Neumann algebra for this
situation. The error lies in thinking that the only von Neumann algebras
suitable for doing quantum theory are isomorphic to $\mathcal{L}(\mathcal{K})$ for some Hilbert
space $\mathcal{K}$. The von Neumann algebra $\mathcal{V}$ is not of this type, since dim $\mathcal{V} = 2$.
Rather we have $\mathcal{V} \cong L^\infty(\{0, 1\})$. Worse yet from my point of view, every pair
of self-adjoint operators in $\mathcal{V}$ commutes. This does not a bone fide
quantum theory make, methinks. I require at least a speck of non-commutativity. And
any added self-adjoint operator would have to correspond to some physical
aspect of the ‘quantum cat’. But if we only observe the results ‘live cat’ or
‘dead cat’, then $\mathcal{V}$ suffices to save the phenomena. We do not need a theory
that includes events for results that are not observed. More importantly,
we do not–actually can not–give meaning to quantum events that do not
correspond to physical results, since it is just such a correspondence which
is the meaning of a quantum event ($\equiv$ projection operator).

However, it seems to be folklore that “one can not write something down
on paper in quantum theory”. This is related to the no-clone theorem of
quantum theory. (See [26].) But my diabolical machine does write something
down on paper. And this is why it is diabolical. It defies quantum theory.
What are we to make of that? Some physicists hold to the view that all
physical phenomena are described by quantum theory. So they face the
challenge of describing in the context of quantum theory how data can be
recorded and shared. It is not permitted to appeal to classical physics in
such an argument. It has to be a demonstration totally within quantum
theory itself. This is essentially what is known as the measurement problem
of quantum theory. Other physicists assert that the diabolical machine as
well as all measuring devices are classical, not quantum, systems. But it
is still not easy to see how classical mechanics with conservative forces can
explain how records can be made and preserved. A more plausible way would
be a classical system with dissipative forces that has at least two stable sinks
in phase space. (If there are exactly two stable sinks, this classical system is
called a bit. And bits do seem to exist.) That would not be a Hamiltonian
system, and so the usual methods of quantization do not apply. In other
words. It would be difficult, though maybe possible, to describe the recording device using quantum theory. But some how or another a classical system has to be coupled to a quantum system in such a way that information flows from the quantum system to the classical system, where that information is preserved. It seems to be folklore that this is not possible, but I will not express an opinion. I do not propose any solution.

Nonetheless, I wish to note that there do seem to be recording devices in nature that are usually considered to be quantum systems. Let us think of a nucleus that is unstable via beta decay. The nucleus changes from one element in the periodic table to another when it decays. This is recorded in many ways; for example the electric charge of the nucleus changes. So the electric charge serves as a way of recording whether the nucleus has decayed or not. And if the daughter nucleus is stable, then this data is recorded for all posterity. Nuclei are certainly not classical systems. And most physicists would agree that they are quantum systems. So this is an example of how some data (though possibly not all data) can be recorded in a quantum system. The wheel has come full circle. It was a radioactive decay that initiated the events of the diabolical machine. In short, forget about the diabolical machine, the box, and the cat inside the box. None of that needs any more explaining than beta decay itself does. So we find the same puzzle at the end of the story as at the beginning, but now in a quantum system.

The same problem is present in classical physics, though it seems to never be discussed. Again, it is not simply a question of a physical system and some observation of it. Rather, the information from that situation is recorded and shared. The question is: “Can one write something down on paper in classical theory”. The answer may be yes, but that is not so obvious. Classical mechanics describes motion as a trajectory in phase space. What does that theory have to do with preserving information? A piece of paper with something written on it is a system with many degrees of freedom, that is to say, its phase space has an enormously high dimension. And what does a trajectory in that phase space have to do with the transmission of information in the scientific community? These are not rhetorical questions. My point is that the physics community is not agitated over the fact that these serious questions are not addressed when discussing classical mechanics, say in a university course or in a popular exposition for a general public. Typically, one hears that there is no measurement problem in classical physics, but that there is such a problem in quantum physics. What’s sauce for the goose is sauce for the gander.
Chapter 5

The Measurement Problem

Truth is truth
To the end of reckoning
William Shakespeare

The measurement problem is not a problem that one can address, let alone solve, in the context of an axiomatic theory. Let me illustrate this with an example which I hope is not very controversial: Euclidean geometry. In the full axiomatization of this theory (which eluded Euclid’s efforts) certain statements are proved about the size of angles and the areas of geometric figures, such as squares in the Pythagorean theorem. These statements are in contradiction with statements in non-Euclidean geometries about the ‘same things’. It is generally accepted that a way to test if Euclidean geometry holds is to measure the three physical angles of physical triangles and see whether their sum is indeed $\pi$ radians (in modern units). It does not pertain to Euclidean geometry nor to any of its competitors to explain how to measure angles or areas. That is not understood to be part of the task of the theory itself. This question is not considered to be one which the theory has to answer: How does one measure an angle? In fact, one assumes that physical triangles with three physical angles exist and, moreover, that all of these incompatible geometries are speaking of these same physical objects. There are no Euclidean angles, no projective geometry angles, and so on. But there are measured angles. Similarly, there is no classical energy, no quantum energy and so on. For an experimental scientist measurement is an activity, not a problem.

And I see the axiomatization of quantum theory in the same way. As
presented in this treatise quantum theory is an axiomatic theory of states, events and probabilities. How these are measured is not addressed in the mathematical theory. (But I do come out in favor of relative frequency as the way to measure probability.) One should think of ‘state, event, probability’ as three basic concepts much like ‘point, line, angle’ in geometry. Of course, the latter geometric words do get a significance in terms of observations. And so do the former three concepts of quantum theory. But those are scientific questions outside the scope of the axiomatic theory.

In this treatise the question is how to deal with measurement in the context of quantum theory based on states, events and probability. Appeals to classical physics, or any other theory, are beside the point. One is looking for a consistent, but not circular, exposition of the topic. This will not solve the measurement problem, but rather explain what are the issues that a scientific approach must address.

To start off I would like to introduce the idea that measurement is done by devices (or physical systems, if you wish) that implement sequences of two events in such a way that the conditional probability of the second event, given the first event (and maybe a state) is 1. It is definitely not assumed that such a device always has the same sequence of two events. Such a device is essentially a function that has a certain set of events as the possible first event (input) of the sequence and then has the second event (output) in that sequence. It should further be required that this function is one-to-one, so that the second event in the sequence uniquely specifies the corresponding first event. Colloquially, the second event is characterized by the first event, and it in turn uniquely characterizes the first event. (As an aside, let me note that the same idea also applies in classical physics. But this is a treatise on quantum theory.)

Let me emphasize this one more time. Sometimes the claim is heard that a measuring device must be classical since its results are ‘determined’. The assertion is made that this is a deterministic process, and hence the device can not be a quantum system but must be classical system. But this logic is fallacious. Probability theories include the special case of conditional probability 1, as we have seen in Chapter 3 on Entanglement.

Whether such devices actually exist physically is another question, though it is generally assumed that they do. Anyway there is nothing in quantum theory which precludes such devices from existing. So, measuring devices as defined here are consistent with quantum theory. Sequences of two events that have conditional probability 1 are consistent with a probabilistic theory;
they are just a special case. How to explain particular physical devices using quantum theory is something that depends on the specific details of those devices. Though I doubt there is a general answer for all devices, I could be proved wrong. In short, I suspect that giving a general quantum theory of measuring devices is an intractable problem, at least with our current level of understanding. This seems to be born out by the overwhelming lack of progress in addressing this issue. But for those who think otherwise and want to research this problem, please remember the important role of approximation. The point is that devices which implement sequences of events with probability nearly equal to 1 can be quite accurate, though not perfect, measuring instruments.

Quantum computing can be viewed in the same way. Though speaking of a sequence of states being changed by quantum operations is valid, that language does not survive in the equivalent Heisenberg model where the state is time independent. I would rather say a hard-wired quantum computer is a physical device that has a final (output) quantum event that is determined with high conditional probability by a given first (input) quantum event and any state. Only in this case, the function being implemented this way need not be one-to-one; distinct input could result with high probability in the same output. A programmable quantum computer would do the same, again with high probability, but the function would depend on a program. To my understanding this is what ‘classical’ computers do, but with a set of events that is a Boolean algebra. A fully quantum computer would use all quantum events.

Let’s look at Schrödinger’s cat using these ideas. A radioactive nucleus with charge $Z$ is under consideration. Let $E_1$ be the (quantum) event that has value 1 if and only if beta decay occurs in the time interval. Let $E_2$ be the (quantum!) event that has value 1 if and only if the cat dies in the same time interval. The device is constructed so that for any state $\psi$ we have that the following conditional probabilities hold:

$$P(E_2 \mid E_1, \psi) = 1 \quad \text{decay occurs; cat dies},$$
$$P(E_2^c \mid E_1, \psi) = 0 \quad \text{decay occurs; cat does not die},$$
$$P(E_2 \mid E_1^c, \psi) = 0 \quad \text{decay does not occur; cat dies},$$
$$P(E_2^c \mid E_1^c, \psi) = 1 \quad \text{decay does not occur; cat does not die}.$$ 

Here $E_1^c = I - E$ is the complementary event to the event $E$. For example, the event $E_1^c$ means that the beta decay did not occur in the time interval.
It is not to be confused with the expression ‘nothing happened’ in ordinary language, which misses the point. No one, starting with Schrödinger himself, ever seems to doubt the possibility of constructing a device which has these conditional probabilities.

Seeing a live cat is a way of measuring the charge of the nucleus and getting the result that it is $Z$. And seeing a dead cat is a way of measuring the charge of the nucleus and getting the result that it is $Z + 1$. Of course, there are other ways of measuring the charge of the nucleus. One might say they are more ‘direct’ than using the poor cat as the measuring device. But I claim that they will also be based on sequences of quantum events which have conditional probabilities equal only to either 0 or 1. Or maybe just with conditional probabilities only very near 0 or very near 1.
Chapter 6

The EPR paper

Face au réel, ce qu’on croire savoir
offusque ce qu’on devrait savoir.
Gaston Bachelard

Ever since its publication in 1935 the famous EPR paper [10] (named for its authors Einstein, Podolsky and Rosen), has generated much discussion and controversy, despite the fact that its conclusion is correct. It has even been referred to as the EPR paradox! The authors come in the last paragraph of [10] to the following conclusion:

While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question whether or not such a description exists. We believe, however, that such a theory is possible.

The argument of this paper is an example of entanglement, though that word is not used, presented in the Schrödinger picture. To achieve their conclusion the authors have to give some idea of their concept of ‘physical reality’. This they do by saying the following:

*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*

( Italics in the original)

They then add how to regard this condition:
Regarded not as a necessary, but merely
as a sufficient condition of reality . . .

One can quibble with their condition, but for the moment let’s accept it and try to understand the paper a little better. At the beginning of the paper the authors state:

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) “Is the theory correct?” and (2) “Is the description given by the theory complete?” . . .

A little further on they continue:

It is the second question that we wish to consider here, as applied to quantum mechanics.

This is also the question posed in the title of the paper. However a careful consideration of both their argument and of their very words shows that they did not answer this second question. Just to be clear here again is the essential part of their conclusion:

. . . the wave function does not provide a complete description of the physical reality . . .

So the implicit assumption is that the wave function is the only theoretical element that is involved in the description of quantum theory. That is how they can jump from showing the incompleteness of the wave function to the incompleteness of quantum theory as a whole.

This is an error of logic, of course. But it is easy to see how the authors arrived at it. Apparently, their tacit assumption is that the Schrödinger equation is the fundamental time evolution equation of quantum theory and that equation has a unique solution (the wave function), given an adequate initial condition. This is second nature for those accustomed to thinking in terms of physical theories based on differential equations; this is intuitive. Unfortunately, as we have seen, it is wrong. The details in the course of their argument about how the state changes due to measurement are not viewed as contradicting this general idea of how physical theories work. Of course, these details are nonetheless just how one calculates conditional probability in the Schrödinger picture, though that wording is not used.

However, we can suppress this tacit assumption and see that the actual conclusion of the paper is correct. Indeed, the wave function alone does
not suffice to provide a complete description. As presented in this treatise, events and probabilities (and their possible time evolution) are also part of basic quantum theory. It might well be that the concept and importance of quantum event were not known to the authors of the EPR paper in 1935. But probability was a contemporaneously available concept. Indeed, probability one is singled out for special consideration in their sufficient condition for an element of physical reality. However, later commentators, and perhaps these authors as well, have turned this into a necessary condition if they assert that something with probability less that one is not an element of physical reality. While one may wish to take their condition as both necessary as well as sufficient, that is a giant step beyond what the authors of EPR actually assert.

Some confusion arises since rank 1 events are also states (‘wave functions’ for these authors). But we saw that entanglement in the simplest example of Chapter 3 involves events that are rank 2 operators and therefore not wave functions. Similarly, the EPR thought experiment involves events which are not wave functions. This may seem to be just a minor technical detail, but ignoring it leads to a fatal flaw in the EPR argument. Using the notation of Chapter 3, the occurrence of \( E_1 \), a rank 2 event, does not tell us the state of the particle on the left in the Hilbert space \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \), a 4-dimensional space. As shown in Chapter 3 entanglement can easily be understood by calculating the conditional probability of events without assigning ‘meaning’ to the intermediate steps. But if one prefers the language of states, then the correct statement is that the occurrence of the event \( E_1 \) tells us that the state is in the 2-dimensional subspace \( \text{Ran} \ E_1 \). Similarly, \( E_2 \) does not tell us the state of the particle on the right. The implicit error here is the assumption that the measurements of the detectors are being modeled by events in two different 2-dimensional Hilbert spaces. So one is discussing this quantum system with three distinct Hilbert spaces, and therefore three different von Neumann algebras. This is a clear violation of Axiom 1. In short the EPR thought experiment does not assign, by using their sufficient condition, an ‘element of reality’ to the state of either particle.

Let’s examine this in a bit more detail. To describe the detector on the left we only need the von Neumann algebra \( \mathcal{L}(\mathbb{C}^2) \). A different copy of the same von Neumann algebra suffices for the other detector. But if the space-like separated events of the detectors are compared and found to have some correlations among them, then a scientific puzzle has been noted that requires further explanation. Of course, we already know what that explanation is in
this case. And a part of that is including the two copies of the von Neumann algebra of the individual detectors as sub-algebras of $\mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, the new von Neumann algebra for the combined system.

Let’s take this one step further and suppose that the detector on the left not only says that $E_1$ occurred, but the experimenter on the left is told that the earlier event $E_0$ had already occurred. To accommodate the event $E_0$ would require using a new model, the simplest one of which would be the von Neumann algebra $\mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Using this model the experimenter on the left then has sufficient information for calculating the probability of the event $E_4$ of any spin $1/2$ measurement by the detector on the right. As I will argue in a moment, this is sufficient for asserting that the value ($\equiv$ eigenvalue) of the event $E_4$ has an element of physical reality.

Now I wish to suggest that their sufficient condition for element of physical reality is unnecessarily linked to probability one events. Leaving aside the vague restriction about not disturbing the system, I propose another more general, sufficient condition for an element of physical reality:

If we can predict with non-zero probability the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

I might even be disposed to deleting the word ‘non-zero’ from the above condition. After all, violation of a conservation law has a physical reality of a certain sort—the reality of never occurring (to date). This condition seems to be more in the spirit of basic quantum theory viewed as being a new type of probability theory. Of course, in some sense I am proposing a broader definition of ‘element of physical reality’. However, this condition is intuitive in the sense that people act as if it holds. For example, a house burning down and resulting in (financial) value zero has an element of physical reality due to its non-zero probability, even though it has never happened and most likely never will. Yet this element of physical reality is sufficient to motivate buying fire insurance for the house.

Anyway, with this enhanced sufficient condition physical quantities such as position or spin component are elements of physical reality. Moreover, such physical quantities do find a counterpart in quantum theory. Actually, their pvm’s and probabilities are counterparts. This would make quantum theory nearer to being complete than the authors might have thought according to their own criterion given in the EPR paper:
Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in physical theory.*

(Italics in original)

Clearly, quantum theory remains incomplete but for other reasons such as its inability to predict the properties of the elementary particles, the values of fundamental constants, the nature of gravitation and the origin of the universe to name some examples.
Chapter 7

Determinism and Probability

The epistemological value of probability theory is based on the fact that chance phenomena, considered collectively and on a grand scale, create non-random regularity.

Andrei Kolmogorov

There is a probabilistic aspect in all the standard formulations of quantum theory. There is rarely a probabilistic aspect in standard formulations of classical physics and, when there is, it is chalked up to being due to a lack of complete information rather than a fundamental aspect of that theory. But it is also said that classical physics is deterministic. And this is not a logical consequence of what I just said. Rather it is known as jumping to a conclusion. The determinism ascribed to classical physics is a myth repeated over and over by so many experts that it has become heresy to question it. Note that a mathematical formulation of determinism is not found in the standard texts on classical physics. The best one can find in the scientific literature are various theorems in mathematics which state that some classes of time dependent differential equations have unique solutions provided they are accompanied by appropriate initial conditions. But it turns out that there are other time dependent differential equations that do not have this property and, according to theorem, such equations are non-linear. Also there is nothing in classical Newtonian mechanics which tells us which sort of differential equations will arise in that theory, that is, whether they have unique solutions with appropriate initial conditions or not. But it is known that the second law of motion of Newtonian mechanics is non-linear in almost all examples.
Since this is the relevant moment, let me pause from continuing this discussion in order to call out a confusion that is rampant in the physics community. The point is that differential equations do not automatically imply determinism. To suppose otherwise is wrong-headed, to say the least. Some differential equations are consistent with determinism, but others are not. We often work with a linear approximation. Linear differential equations with appropriate initial conditions do have unique solutions, which are also global in time. But this property of a particular approximation does not always hold in other non-linear contexts. So it is a non sequitur to say that a theory based on differential equations is automatically deterministic.

Let me be clear about this difference between classical mechanics and quantum theory. The theory of classical mechanics does not answer the question of whether determinism or probability applies to its subject matter. On the other hand quantum theory at its most basic level makes statements about probability and, according to this treatise, about nothing else. One should not say that quantum theory eliminated determinism from physics; rather it put probability into physics. And probability is very non-intuitive.

However, we must pause, since it appears that Schrödinger’s equation is deterministic. After all, for any self-adjoint Hamiltonian $H$ there is a unique solution, global in time $t$, given any initial state $\varphi$ in the Hilbert space, namely, $\psi(t) = e^{-iHt/\hbar} \varphi$ in the Schrödinger model. At a purely mathematical level, this is a sort of determinism. But this does not give us determinism of a quantum system, since the states $\psi(t)$ by themselves only determine probabilities when combined with the events associated to a system. And this role of probability is not the lockstep relation of cause and effect as envisioned by the usual idea of determinism. The Heisenberg model also has a mathematical determinism, but now of the events as given by Heisenberg’s equation (1.6.1). And again this is not a physical determinism, since in general events are not sufficient for computing quantum probabilities.

And moving on now to experiment the situation is even shakier. The idea is that by knowing all about a physical system completely at one instant of time, the future of that system is known or, as one says, determined. How could one possibly know that? How could one possibly falsify that? If some systems behave in what seems to be a deterministic manner (at least for some experimentally finite period of time), that does not mean they will continue to behave in that manner forever. And if they do not behave in a deterministic manner, it is unacceptable to say that is due to lack of knowledge about the initial situation. How could one possibly know in all cases that there is more
to know? It is logically possibly that there is nothing more to know. There is a lot of muddled thinking about determinism. Of course, there is a lot of muddled thinking about probability theory, too.

Something that is truly confusing for me is that people, who can not look at a probability 1 event without running to the refuge of deterministic language, can accept without any qualms any number of probability 0 events in quantum systems. Yet such events are literally duals of each other. One says that a certain quantum transition is forbidden (i.e., has probability 0), since it violates some conservation law. However, that same conservation law holds with probability 1. And this all should be explained in terms of quantum theory, not with determinism.

From this perspective it is not so strange to say that quantum theory is probabilistic and, in fact, that is thought to be one of the basic aspects of the theory. I have done just that in this treatise. What is strange is that a deterministic time dependent differential equation is claimed to be another basic aspect of quantum theory. Of course, I refer to the time dependent Schrödinger’s equation

\[ \frac{i\hbar}{\hbar} \frac{d\psi}{dt} = H\psi. \]  

(7.0.1)

Let me remind the reader what the mathematical theory actually says about this equation. We assume that \( H \) is a self-adjoint densely defined linear operator acting in a Hilbert space \( \mathcal{H} \). Then a theorem says that for any \( \phi \) in the dense domain of \( H \) there exists a unique solution \( \psi_t \) of (7.0.1) for all \( t \in \mathbb{R} \) such that \( \psi_0 = \phi \). Moreover, that theorem asserts that \( \psi_t = e^{-itH/\hbar}\phi \), where the globally defined unitary operators \( e^{-itH/\hbar} \) are defined for all \( t \in \mathbb{R} \) by the functional calculus of spectral theory applied to the operator \( H \).

In the Schrödinger model one has Schrödinger’s equation, a deterministic equation, and a probabilistic state collapse condition. It seems that the time evolution chugs along on its merry way changing the state in a continuous deterministic way, when—poof!—somehow a probabilistic event occurs that changes the state in a discontinuous way. How can there be two distinct types of time evolution? This perennial puzzle about quantum theory arises as we have seen by focusing on details in the Schrödinger model which are model dependent. Of course, in the equivalent Heisenberg model there is a deterministic time evolution of the pvm’s while the state remains constant in time. In both models the (identical!) probability measures are continuous functions of time. The continuity of these probability measures follows from the continuity of \( t \mapsto e^{-itH/\hbar} \) in the strong operator topology.
For now I would like to discuss another sort of probability in one aspect of quantum theory. But this aspect is quite different. Again, consider a nucleus that can undergo beta decay. Before the decay occurs the electron and anti-neutrino of the final state do not exist. They are created by the decay process. And no matter where the nucleus is in the universe, the electron has exactly the same characteristic properties as any other electron in the universe. Its mass, electric charge, spin and magnetic moment are always the same. In other words these are probability one properties of any electron. The same holds for the anti-neutrino; it is the same as any other anti-neutrino produced in beta decay. To make this sound more ‘paradoxical’ imagine two beta decays of the same isotope nucleus, but with a space-like separation between these two events. How can one event possibly ‘know’ about the other? Yet they produce the same decay products, although the momenta of the decay products are not always the same. Also, these decays violate parity conservation, but they do that in the same way always. On the other hand an alpha decay of a nucleus always preserves parity. These are all rightly called particle properties, though one could also describe them as spooky action at a distance.
Chapter 8

Interpretation

Interpretation is the revenge of intellect upon art.

Susan Sontag

The astute reader will have noticed that I have only used the expression *physical significance* instead of ‘interpretation’ as was done in days of yore for describing the relation between theory and observation. This is due to the unfortunate situation that ‘interpretation’ has come to denote something quite different from ‘save the phenomena’, a phrase also from days of yore. (See 28.) Nowadays ‘interpretation’ refers to extra-scientific statements that are intended to ‘save the theory’. They are neither verifiable nor falsifiable and consequently have no role whatsoever in science. Some how they hang in there as some sort of assurance that everything is ‘intuitive’ after all. The theses of this treatise are scientific and are subject to the standard critiques of scientific methodology. Any assertion that I make may be right or it may be wrong. But in the contemporary sense of the word I do not give any ‘interpretation’ of quantum theory. Still, some comments are called for about some of the more common ‘interpretations’ of quantum theory.

The collapse of the wave function is quite often the central concern of an interpretation. This results in tilting against windmills. There is nothing about the collapse condition that merits much attention once one realizes that it is a mathematical step used in the Schrödinger model of quantum theory, but that it has no correlate in other isomorphic models. Typically, discussions that deal exclusively with the Schrödinger model fall into the trap of treating a characteristic specific to that model as universally applicable to quantum theory, when they clearly are not due to the lack of that characteristic in
another isomorphic model of quantum theory. The more-or-less standard Copenhagen Interpretation simply asserts that collapse is something that happens. But collapse is not a physical process, but rather a mathematical formula. Of course, the Copenhagen Interpretation is not just one assertion. And it morphs. So it is difficult to totally debunk it based on a detailed analysis of what it says. But it is clearly barking up the wrong tree.

Another recurrent goal of ‘interpretations’ is to explain what probability ‘means’ or how it arises in quantum theory. This can be seen in the Many Worlds Interpretation which deals with this issue by positing a hypothesis that is neither verifiable nor falsifiable. (See [3] and references therein.) If such a hypothesis makes some people content, what can I possibly say? While in this treatise no attempt is made to explain where probability comes from, explicit hypotheses are given as to what its properties are. I refer here to the generalized Born’s rule together with its underlying mathematics. This can be checked experimentally. This theory is not designed to make anyone happy, though that may happen too. It is meant to save the phenomena of the relative frequencies measured in experiments. Also The Many Worlds Interpretation introduces a wave function of the Universe; this is supposed to describe its state. This would make some sense if it were included in a Hilbert space theory with an appropriate von Neumann algebra, whose events and pvm’s had some physical significance at a cosmological level.

The consistent history approach (see [13]) is based on histories of quantum events and their probabilities and therefore bears a superficial resemblance to the thesis of this treatise. In all fairness this is not referred to as an interpretation by Griffiths in [13]. But for better or worse, it is often taken to be such. Leaving that point to a side, let me deal with what it says about quantum theory. The idea is to introduce classical probability spaces (and lots of them) in order to define probability in the context of quantum theory. This leads to new constructs of which the most important is that of a history of (usually finitely many, though possibly very many) time ordered quantum events. Each history then defines a (usually finite, though possibly very large) classical sample space on which probabilities are defined and then related to Born’s rule. The formalism depends on forming an inner product from the trace as $Tr(A^*B)$ for an unspecified class of bounded operators $A$ and $B$. Of course, for infinite dimensional Hilbert spaces this is the rather small space of Hilbert-Schmidt operators. As noted in [13] this only works well for all operators in the finite dimensional case. This places an enormous restriction on the scope of this approach. But then the first attempt to
assign probabilities to the events in a history runs into an obstacle, namely, finite additivity fails. To avoid this a sufficient orthogonality condition (See Eq. (10.20) in [13]) is given. While this condition is not necessary, without some other condition being imposed finite additivity does fail in general. The author does not seem to consider $\sigma$-additivity, which becomes relevant when histories with infinitely many events are considered. But even so, then the restriction is imposed to consider only consistent (or weakly consistent) histories as having a physical meaning. This is a rather strong restriction with nothing corresponding in standard quantum mechanics. In short this approach adds new features to standard quantum theory and produces a theory which says less.

Another novelty in [13] is to re-define quantum logic by declaring that, for example, $E \land F$ does not have a meaning if the events $E$ and $F$ do not commute. The justification is that in this case neither $EF$ nor $FE$ is an event, a well known fact that does not prevent one from defining $E \land F$ as their unique infimum as in done in standard quantum logic. Of course, throwing quantum logic to the wolves may not be a great loss, but I wonder what exactly is gained by this. This gives a new logic that not only says less than standard quantum logic but also diminishes the role of non-commutativity in quantum theory. The author of [13] also seems to be unaware of the existence of spectral measures, which are classical probability measures on $\mathbb{R}$, nor of their role in Born’s rule, something which was published in [25] in 1932.

By now the contrast of the current histories approach with the theses of this treatise should be clear enough. I do not construct new sample spaces, but use the spectral measures on $\mathbb{R}$ available in [25]. I do not require extra special conditions to guarantee $\sigma$-additivity, since this is a consequence of spectral theory. I use absolutely standard spectral theory (and the measure theory behind much of it) plus a straightforward generalization of Born’s rule to define conditional and other multi-event probabilities on $\mathbb{R}$. I consider this treatise to be a new organization of quantum theory and in no way an interpretation of it.

I think that the entire enterprise of finding the ‘correct interpretation’ of quantum theory as far as scientific activity is concerned is quite besides the point, and so is not interesting for me at all. Simply put, I am interested in scientific theories of nature rather than who-knows-what theories of theories. I have not even mentioned other popular ‘interpretations’ of quantum theory. Rather I include this chapter only to distance the contents of this treatise from all that. Interpretation is the revenge of intellect upon science.
Chapter 9

The Wave Function

When we want to understand something strange, previously unknown to anyone, we have to begin with an entirely different set of questions. What is it? How does it work?

Margaret Mead

I do not want to discuss classical physics at all. But the wave function has the dubious role of being an aspect of quantum theory that is often considered in language that is classical. Here is some of that language. In classical physics the time evolution is described in terms of the points in a particular space, which is called the phase space. These points are the classical pure states. The phase space typically has high dimension. Then the classical dynamics specifies the possible one-dimensional curves in phase space. These curves are parameterized by time. Each point on such a curve represents a complete description of the physical system at the corresponding time.

All of this is temptingly analogous to the Schrödinger model of quantum theory, where the time dependent solution of Schrödinger’s equation, which is inappropriately called a wave function, gives a curve in a space whose points are called pure states. But analogies are not explanations. Unfortunately, this one is completely misleading, since in the isomorphic Heisenberg model the state is always constant in time. In the classical case the dynamical curve wending its way through a high dimensional phase space is used to deduce

\[\text{Schrödinger’s equation is not a wave equation.}\]
how physical objects move in three dimensional physical space. Analogously, the time dependent wave function is used to deduce properties that one can visualize in terms of the geometry of three dimensional physical space.

However, the analogy fails in part because Schrödinger’s equation is not the fundamental time evolution equation of quantum theory. It also fails because, although it is not widely recognized, classical mechanics also has a Heisenberg-type model in the case when the classical Hamiltonian flow \( \phi_t : \mathcal{P} \rightarrow \mathcal{P} \) is a global diffeomorphism of the phase space \( \mathcal{P} \). Classical observables are functions \( f : \mathcal{P} \rightarrow \mathbb{R} \), and the classical pure states are the points in \( \mathcal{P} \). What one observes is the time dependent quantity \( f(\phi_t(p_0)) \), where \( p_0 \) is the initial state. This is usually considered in a Schrödinger-type model as \( f(p_t) \) with the observable \( f \) being constant in time and the initial state \( p_0 \) evolving to \( p_t := \phi_t(p_0) \) at time \( t \). This easily generalizes to the time evolution of mixed states (\( \equiv \) probability measures on \( \mathcal{P} \)) in which case this is the Liouville formulation of classical mechanics as used mostly in statistical mechanics. But \( f(p_t) = f_t(p_0) \) where we take the state \( p_0 \) to be constant in time and let the observable evolve from \( f \) to \( f_t := f \circ \phi_t \) at time \( t \). This Heisenberg-type model is called the Hamiltonian picture in [11].

There are also interaction-like models in classical mechanics, in which one takes any two families of bijections \( \alpha_t, \beta_t : \mathcal{P} \rightarrow \mathcal{P} \) for \( t \in \mathbb{R} \) that satisfy \( \alpha_t \circ \beta_t = \phi_t \). Then one lets the states flow according to \( p \mapsto \beta_t(p) \) while the observables flow according to \( f \mapsto f \circ \alpha_t \). Despite \( \phi_t \) being a \( C^\infty \) function, its factors \( \alpha_t \) and \( \beta_t \) need not even be continuous. But nonetheless, such a non-intuitive factorization could help one do calculations. I am not aware of such interaction-type models being used in classical mechanics. One could also generalize classical mechanics by dropping the condition \( \alpha_t \circ \beta_t = \phi_t \).

One way classical thinking can creep into quantum theory is when the Hilbert space for the quantum theory is \( L^2(\mathbb{R}^3) \), since a normalized wave function \( \psi \) in that space has an associated probability density \( |\psi|^2 \) on \( \mathbb{R}^3 \). So, computer displays of these densities are made and can be seen in textbooks, in popular expositions and on the Internet. The graph of \( |\psi|^2 : \mathbb{R}^3 \rightarrow [0, \infty) \) is a subset of \( \mathbb{R}^4 \), making direct visualization a bit tricky for most of us. But level sets in \( \mathbb{R}^3 \) can be encoded in computer memory and then their 2-dimensional projections can be displayed on a screen or printed on paper. If one uses instead a curve of normalized solutions of the time dependent Schrödinger’s equation, then one can make a computer video. Of course, it is easy to misinterpret what the moving blob in such a video means. But a more profound problem is that, except for toy models, \( L^2(\mathbb{R}^3) \) is not the
Hilbert space used to describe quantum systems. In the following examples spin and statistics are omitted, since even without that the basic idea is clear.

Consider the ever popular example of the hydrogen atom. In the texts the stationary states \( \psi \in L^2(\mathbb{R}^3) \) are found, together with their corresponding energy levels. Many images have been made for \(|\psi|^2\). The formulas for the stationary states as well as the images made from them are all quite pretty. But all of that is misleading, since the (electrically neutral) hydrogen atom is a two body problem, whose Hilbert space in the Schrödinger model is \( L^2(\mathbb{R}^6) \) with Euclidean coordinates \( x_1, x_2, x_3 \) for the electron and \( y_1, y_2, y_3 \) for the nucleus. A mathematical technique (change to center-of-mass coordinates) shows that the Hamiltonian of this system can be transformed in such a way as to give an equivalent problem with two Schrödinger operators, each acting in \( L^2(\mathbb{R}^3) \). But neither one of these operators is the Hamiltonian of the hydrogen atom nor of any other physical entity. The wave functions for each of these two Schrödinger operators can be combined and then with an inverse change of coordinates can be written as \( \psi(x_1, x_2, x_3, y_1, y_2, y_3) \) in \( L^2(\mathbb{R}^6) \). But this last step is rarely done in the elementary texts. Rather only one of these two Schrödinger operators acting in \( L^2(\mathbb{R}^3) \) is analyzed. For example this is what I did in [31].

Sometimes, instead of presenting the change of coordinates, a model is used with the nucleus fixed and immovable at some point in space. Then the electron is analyzed with a one-body Schrödinger operator. Among other things the mass parameter in such an approach should not be the electron mass. In fact, the mass parameter to be used depends on the isotope of the nucleus. But justifying that is difficult, to say the least. So the electron mass is used. Worse yet, this ‘approximation’ contradicts the supposedly sacred Uncertainty Principle by fixing the values of both the position and the momentum of the nucleus.

The neutral hydrogen molecule is a four body problem and so has its wave functions in \( L^2(\mathbb{R}^{12}) \). The water molecule is a 21 body problem and so has its wave function in \( L^2(\mathbb{R}^{63}) \). And so it goes. But you can find images of the wave function of the ground state of the water molecule \( \text{H}_2\text{O} \) in the literature. This is done by changing to center-of-mass coordinates, fixing the values all of the variables for the 3 nuclei as well as integrating out all the variables for 17 of the 18 electrons. The modulus squared of the resulting ‘wave function’ in \( L^2(\mathbb{R}^3) \) can then be published in a chemistry textbook or on the cover of a popular science magazine. This sort of visualization loses a lot of information that is encoded in the correct wave function of the
molecule. Of course, it is approximately correct information that remains. (It is only approximate since there are only approximations of the original wave function itself.) But one is easily mislead into thinking that one has visualized the spatial structure of the molecule.

I will certainly be criticized for downplaying the role of Schrödinger’s equation in quantum theory. But that equation stands firm on the basis of its many successes, which are found in the scientific literature. I emphasize in [31] the central role of Schrödinger’s equation in the scientific activity of quantum physics. The two-slit experiment is but one grain of sand on the vast beach of successes of the Schrödinger model. They all speak for themselves.

I need not repeat those details. I only wish to place things in the correct perspective with respect to the Generalized Born’s Rule. Those who wish to continue working with the Schrödinger model do not need permission to do so. Their results, if obtained correctly, will have scientific value. However, if one gives a physical significance to the model dependent aspects of the Schrödinger model, then I claim one is overinterpreting the mathematics.

I most likely expect to be criticized for taking a viewpoint that is ‘too mathematical’ in this treatise. But ironically it is those persons ascribing deep meanings to mathematical structures with no physical significance who are the ones that are ‘too mathematical’.

This topic is related to quantization, even though that is not a part of quantum theory but rather a way of arriving at a quantum theory from some other starting point. Typically, one starts from classical physics, but that is not essential. Actually, second quantization starts with one quantum theory in order to produce another. I do not wish to belittle research on quantization. It has its importance, but only as a way to quantum theory. In this treatise I want to describe the basics of the quantum world, and not go into the details of the journey for getting there. I have even done research on quantization, but this is not the place to go into that. My only goal now is to comment on its lack of relevance to doing and understanding quantum theory itself. Even worse, quantization is used to leak ideas of classical physics into quantum theory. Of course, some classical ideas do transfer to quantum theory, but others not. So that issue must be faced and understood on a case by case basis.

\[2^\text{In poker the saying is that the cards speak for themselves. But if they do not speak clearly to you, see for example [4] for an explanation of the two-slit experiment in terms of events, states and probabilities}\]
Chapter 10

Beyond Conventional Quantum Theory

The most interesting ideas are heresies.

Susan Sontag

The main thesis of this treatise is that there is one basic time evolution equation in quantum theory. That equation is the generalized Born’s rule. But there is also a secondary equation in the Schrödinger model, namely Schrödinger’s equation. In the isomorphic Heisenberg model there is a quite different secondary time evolution equation. These are equations that have a unique solution for a given initial condition. This looks suspiciously like determinism, but it is only a mathematical property. We hear it said about Schrödinger’s equation that if we know the state of the system at some initial time, then we know it at all future (and even past!) times. That is a big ‘if’ and then some. In the first place, some states are not known and are even unknowable. In the second place, collapse in the Schrödinger model precludes knowing both future and past.

We use Schrödinger’s equation for the simple reason that it has been so successful. But its success has not been complete as is evident from the number of confusions, among other things, that it gives rise to. Now the central role of the generalized Born’s rule clarifies matters, as we have seen. Because of this I have been lead to wonder if the Schrödinger model is the only possible quantum theory (up to isomorphism). And I have come to think not. Still I must explain why the Schrödinger model has been so successful and what could be an alternative to it. Before getting into that let me note that
Accardi and his collaborators have a well developed theory which is not an alternative to standard quantum theory, but rather approximates standard Hamiltonian unitary flows for systems with many degrees of freedom with singular Hamiltonian unitary flows in terms of fewer variables. See [1] and references therein for this important achievement.

I shall use classical mechanics as an analogy. Now, classical Newtonian mechanics is a theory of masses, forces and motion. The basic axioms \( \text{(called } \text{Newton's Laws of Motion) } \) of that theory give the relations among these. But that theory is divided into two parts. This division is based on the type of forces of a particular situation. These are called \textit{conservative forces} and (rather unimaginatively) \textit{non-conservative forces}. (I dislike the alternative phrase \textit{dissipative force} since it suggests that heat should be introduced into classical mechanics.) The theory for conservative forces has a lot of quite wonderful properties. This is related to a new form of energy, the \textit{potential energy}, which then casts a new light on \textit{kinetic energy} (which was originally confused with the expression \( mv^2 \)) to give the principle of \textit{conservation of mechanical energy}. And the potential energy opens the door to an elegant theory. One can then define the Hamiltonian function, Poisson brackets and Hamilton’s equations of classical motion. Under certain hypotheses these equations of motion have a unique \textit{local} (in time) solution given appropriate initial conditions. This seems to be determinism, but \textit{caveat emptor}. In the first place, local solutions can and often do have singularities, which preclude extrapolating into the indefinite future. In the second place, the Hamiltonian theory does not apply to situations with non-conservative forces. In the context of conservative forces one arrives eventually at symplectic geometry. I have even heard some colleagues maintain that classical mechanics is a topic in symplectic geometry. But that is not so.

Classical mechanics includes the analysis of situations with friction and other non-conservative forces, in which case there is no potential energy and no conservation of mechanical energy. The motion of a boat in water is a simple example of classical mechanics with non-conservative forces. Other examples abound. Actually, there is such a sparsity of examples with only conservative forces that the everyday ‘common sense’ of most people conceals their importance. Ask any high school physics teacher how difficult it is to teach classical mechanics with only conservative forces. (Angular momen-

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I believe there are at least four axioms. An overlooked axiom states that force is a vector quantity.
tum is essentially impossible.) Yet it is an error to think that conservative forces are basic while non-conservative forces are not. Within the context of classical mechanics neither is more basic than the other.

I suppose that something analogous occurs in quantum theory, namely that the Schrödinger model is similar to conservative classical mechanics. That model also has a lot of quite wonderful properties, but that does not imply that they are necessarily properties of all quantum systems. I find it quite plausible that a different equation for the time evolution of the states could be posited and that (together with time independent pvm’s as just one example) would give an acceptable quantum theory that is not isomorphic to the Schrödinger model. Such an alternative to the Schrödinger model might not be deducible from the Schrödinger model. Rather it could be an independent model, also consistent with the basic axioms of quantum theory. Let me be completely clear here. I am saying that Schrödinger’s equation is not a necessary part of quantum theory, just as conservative forces are not a necessary part of classical mechanics. Of course, Schrödinger’s equation is an important, interesting part of quantum theory much as conservative classical mechanics is an important, interesting part of classical mechanics.

Another well studied extension of quantum theory is to include positive operator valued measures (povm), a generalization of projection (quantum event) valued measures. These arise so naturally that many researchers consider this to still be quantum theory. It is fine to think of them that way. My only point is that they are not included in the axiomatization of Chapter 1. If they are included in new axioms, then the definition of isomorphism of quantum theories should also be modified. And the Generalized Born Rule would also have to be changed. But I would expect that the new Generalized Born Rule would continue to be the fundamental time evolution equation of quantum theory.

One of the quite wonderful properties of the Schrödinger model is called unitarity. It says that the time evolution of a state of a quantum system is realized using the unitary group $e^{-iHt/\hbar}$ for all $t \in \mathbb{R}$, where $H$ is a special self-adjoint operator associated to the system. It is well known that this unitarity group leads to a unique global (in time) solution of the appropriate initial value problem. This leads to time reversal invariance of the theory, but not necessarily of the physical quantum system. I find it perfectly plausible to replace the unitary group with a semi-group that violates unitarity as well as time reversal invariance. And that semi-group would describe the model dependent time evolution of the states. As always, the only time evolution
relevant to observations, to saving the phenomena, is that of the generalized Born’s rule. And such a quantum theory would not be less basic than the ‘friction-less’ Schrödinger model. Nor would it nor should it be derivable from the Schrödinger model. Again, let me note that a different approach is taken in [1].

Of course, such semi-groups exist and have been studied extensively. My point is that they should not be considered as approximations in any sense of that word to a more ‘basic’ Schrödinger model. Neither should they be considered as being derived nor even capable of being derived from the Schrödinger model. As with Schrödinger’s equation itself, they are secondary mathematical structures with no physical significance in themselves, except that they provide a key ingredient for calculating the time evolution of the generalized Born’s rule. That is, of course, a legitimate and important role, which can be recognized as such by including an axiom on semi-groups into quantum theory as a replacement of Schrödinger’s equation. The idea of using semi-groups in quantum theory is found in the literature going back to the 1970’s. See [17], [19] and [24].

My point is that semi-groups should be taken as seriously as Schrödinger’s equation is. Any supposedly ‘deeper explanation’ of the semi-group as a merely phenomenological and convenient approximation to a more ‘general’ formulation based on the Schrödinger model may be beside the point. Any theory should be viewed as phenomenological if it saves the phenomena, which is one goal of any theory. Further, any theory should be viewed as convenient if it aids our understanding of nature, which is another goal of any theory.

A quite nice and successful family of semi-groups is what I wish to call the Lindbladian model, which is based on quantum dynamical semi-groups. This choice of name honors the author of [21]. However, as noted in [21] there are earlier articles on this topic, some of which I cited above. Also see [6], [7] and [8]. Unfortunately, many authors including Lindblad himself say that they are using a formulation in either the Schrödinger picture or the Heisenberg picture. I prefer to view their work as dealing with a new model not previously considered in the quantum literature. One of the motivations for introducing this model is to facilitate the description of irreversible quantum processes. This is achieved by allowing non-Hamiltonian systems that do not satisfy the principle of unitarity. However, one very nice feature of this model (and models isomorphic to it) is that it includes standard quantum theory as a special case. So it is also compatible with reversible quantum processes and
unitarity. Another nice feature is that it gives a context in which *decoherence* can be studied. The main difficulty with this model is that it can be less than obvious how to introduce a specific semi-group for studying a specific system. This difficulty makes intuition hard to come by, at least for me. In standard quantum theory this is the problem of *quantization*, which is usually resolved on a case-by-case basis by appealing to conservative classical mechanics.

To understand better the role of unitarity in this model and other models of quantum theory we now turn to Wigner’s theorem. Wigner’s theorem is one of those wonderful results related to standard quantum theory. It is based on the idea that the Hilbert space expressions with physical significance for quantum theory have the form

\[ |\langle \psi, \varphi \rangle|^2 \]  

(10.0.1)

for unit vectors \( \psi, \varphi \in H \).

We have already noted that these other Hilbert space structures do not have any physical significance: vector sum, scalar product, inner product.

First, let’s note that (10.0.1) is a probability that comes from Born’s rule. Let \( E = |\varphi\rangle\langle \varphi | \) be the rank 1 event associated to \( \varphi \). Then we calculate

\[
P(E | \psi) = \langle \psi, E\psi \rangle \\
= \langle \psi, |\varphi\rangle\langle \varphi | \psi \rangle \\
= \langle \psi, \langle \varphi, \psi \rangle \varphi \rangle \\
= \langle \varphi, \psi \rangle \langle \psi, \varphi \rangle \\
= |\langle \psi, \varphi \rangle|^2.
\]

Consequently, the expression in (10.0.1) is the probability that the event \( E \) associated to \( \varphi \) occurs given the state \( \psi \). Since (10.0.1) is invariant if we interchange of \( \psi \) and \( \varphi \), (10.0.1) also is the probability that the event associated to \( \psi \) occurs given the state \( \varphi \). So our axiomatization of Born’s rule gives the usual physical significance to the expression (10.0.1).

Moreover, as we noted before, the expression (10.0.1) is well defined for pure states, not just for unit vectors in the Hilbert space \( H \). Specifically, this means that if we replace \( \psi \) with \( \lambda \psi \), where \( \lambda \in \mathbb{C} \) satisfies \( |\lambda| = 1 \) and also \( \varphi \) with \( \mu \varphi \) with \( |\mu| = 1 \), then the expression (10.0.1) does not change.

Next, Wigner introduces the idea that a transformation of the set \( S \) of pure states to itself is an isomorphism from the point of view of quantum theory if it is a bijection \( T : S \rightarrow S \) such that the expression (10.0.1) is preserved; this is a special case of *conservation of probability*. Such a bijection
is called a Wigner transformation. Then, Wigner proves that every Wigner transformation is the quotient of a bijection \( U : \mathcal{H} \to \mathcal{H} \) that is linear and unitary or else of a bijection \( V : \mathcal{H} \to \mathcal{H} \) that is anti-linear and anti-unitary. This leads to the powerful corollary that every group of Wigner transformations \( T_t : \mathcal{S} \to \mathcal{S} \) for \( t \in \mathbb{R} \) is the quotient of a group of unitary transformations \( U_t : \mathcal{H} \to \mathcal{H} \). With a little continuity in the group \( T_t \) and hence also in the group \( U_t \), we can apply Stone’s theorem to write \( U_t = e^{-itH/\hbar} \) for a unique densely defined self-adjoint operator \( H \) acting in the Hilbert space \( \mathcal{H} \). We have arrived at the unitary group of the Schrödinger model (also of the Heisenberg model) from general principles.

But if the Schrödinger model as well as models isomorphic to it are not the only possible models of quantum theory, then one is obliged to conclude that some hypothesis used in Wigner’s proof is not a fundamental principle of quantum theory. I propose that the conservation of probability may not be a fundamental principle of quantum theory. In my opinion it could be analogous to the principle of conservation of mechanical energy in classical mechanics. Conservation of mechanical energy holds in many interesting, important cases but is not a universal principle of classical mechanics. Analogously, I am proposing that conservation of probability—and the consequent unitarity—are not necessarily universal principles of quantum theory. They are fine in the ‘fiction-less’ case of quantum theory as seen in the Schrödinger model, but not in general. In this regard one has to view Wigner’s theorem as a No-Go theorem. It tells us neither conservation of probability nor unitarity go as an aspect of a more general quantum theory that is not isomorphic to the Schrödinger model. We now see that in the Lindbladian model the semi-group in general does not come from a Wigner transformation. And it is precisely this that makes the Lindbladian model so interesting and meriting further research. I realize that this idea is controversial, but experiment will be the judge.

Finally, I conclude with a rather wild speculation. Since events and states are all that enter the generalized Born’s rule, together with rules for their time evolution, it seems natural to consider a theory with only these elements. So, no Hilbert space. Maybe even no \( \hbar \). There would be a set \( \mathcal{E} \) of events and a set \( \mathcal{S} \) of states. There would be a function

\[
P : \mathcal{E} \times \mathcal{S} \to [0, 1].
\]

For \( E \in \mathcal{E} \) and \( S \in \mathcal{S} \) we would say that \( P(E, S) \) is the probability of the event \( E \), given the state \( S \). Adding in two one-parameter groups of bijections
of $\mathcal{E}$ and $\mathcal{S}$, respectively, we would have a time dependent probability theory. Models of such a theory would be in abundance. Isomorphism would be defined in terms of preservation of probability. Some other mathematical structures would eventually have to be imposed on these sets in order to get a viable theory. For example, consecutive and conditional probabilities could be introduced. My main point here is that the lattice of projections in a von Neumann algebra might not be the best place to look for a model of physical events.
Chapter 11

A Modest Proposal

Bidding good-bye is such sweet sorrow
that I could bid adieu til it be morrow
Romeo in Romeo and Juliette
William Shakespeare

It seems to me that the expression ‘quantum mechanics’ does not adequately describe the physical theory. It is based on two misleading words. The first is ‘mechanics’ which originally meant the study of machines but changed to mean the study of motion. Unfortunately, the concept of motion (as understood as an object moving along a curve) is not central to the physics we wish to discuss at the atomic and smaller length scales.

The word ‘quantum’ is even more unfortunate. It was used to contrast the new theory with classical theory where all measured values could assume a continuum of possible values. All of a sudden there were physical quantities whose measured values were in a discrete set. These measured values were said to be ‘quantized’. While this is an important part of the story, it is not the whole story. Even in this new theory there are still some physical quantities with values in a continuum, such as the energy of a free particle. Other quantities are always ‘quantized’, such as the energy of a harmonic oscillator (realized as a diatomic molecule, say). But there are mixed cases, such as the hydrogen atom which has ‘quantized’ energy levels (associated with the bound states) as well as a continuum of energy levels (associated with scattering states).

What is important here are the probabilities that arise from the pvm of a self-adjoint operator and more generally its spectral theory. So it seems more
appropriate to say *Spectral Probability Physics* instead of saying Quantum Mechanics for the physical theory based on self-adjoint operators.

With this modest proposal I bid my gentle reader good-bye.
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