HERMITIAN MAASS LIFT FOR GENERAL LEVEL

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Abstract. For an imaginary quadratic field $K$ of discriminant $-D$, let $\chi = \chi_K$ be the associated quadratic character. We will show that the space of special hermitian Jacobi forms of level $N$ is isomorphic to the space of plus forms of level $DN$ and nebentypus $\chi$ (the hermitian analogue of Kohnen’s plus space) for any integer $N$ prime to $D$. This generalizes the results of Krieg from $N = 1$ to arbitrary level. Combining this isomorphism with the recent work of Berger and Klosin and a modification of Ikeda’s construction we prove the existence of a lift from the space of elliptic modular forms to the space of hermitian modular forms of level $N$ which can be viewed as a generalization of the classical hermitian Maass lift to arbitrary level.

1. Introduction

The Saito-Kurokawa lift, established by the work of many authors (classically by Maass [19, 20, 21], Andrianov [3], Eichler-Zagier [9] for the full level and reinterpreted in representation theoretic language by Piatetski-Shapiro [26]), has been of interest and importance in number theory, for instance, in proving part of the Bloch-Kato conjecture by Skinner-Urban [30], providing evidence for the Bloch-Kato conjecture by Brown [7] and Agarwal-Brown [1]. For these applications, one needs a generalization of the Saito-Kurokawa lift to higher level, which was established by the work of Manickam-Ramakrishnan-Vasudevan [22, 23, 24], Ibukiyama [11], Agarwal-Brown [2] and Schmidt [28] for square-free level. (Saito-Kurokawa lift is known to exist for all level.)

The Hermitian analogue of Saito-Kurokawa lift is a lifting of elliptic modular forms to hermitian modular forms of degree 2, typically referred to as the Maass lift in the literature, has been studied since 1980 with the paper of Kojima [17]. Such a lift is of arithmetic interest, for example, in providing evidence for the Bloch-Kato conjecture by Klosin [14], [15] and for the $p$-adic theory. For these arithmetic applications, there is a need for the generalization of the Hermitian Maass lift.

As with the classical Saito-Kurokawa lift, the first construction of the Hermitian Maass lift by Kojima was a composition of the three lifts: Let $K = \mathbb{Q}(i)$ and assume that $k$ is divisible by 4. Then we have

$$\mathcal{G}_{k-1}(4, \chi) \longrightarrow \mathcal{G}_{k-1}^+(4, \chi) \longrightarrow \mathcal{J}_{k,1}^*(1) \longrightarrow \mathcal{M}_{k,2}(1)$$

where

(i) the lift from $\mathcal{G}_{k-1}(4, \chi)$ to $\mathcal{G}_{k-1}^+(4, \chi)$ is an analogue of Shimura–Shitani isomorphism (but it is not an isomorphism here though and unlike Shitani’s map, this map is not obtained by certain cycle integrals) and $\mathcal{G}_{k-1}^+(4, \chi)$ is an analogue of Kohnen’s plus space [16];

(ii) the lift from $\mathcal{G}_{k-1}^+(4, \chi)$ to the space of special Hermitian Jacobi forms $\mathcal{J}_{k,1}^*(1)$; and finally,

(iii) the Hermitian analogue of the original Maass lift from $\mathcal{J}_{k,1}^*(1)$ to $\mathcal{M}_{k,2}(1)$.
See section 2 for our definition and notation of the spaces and groups mentioned.

In [18], Krieg generalized Kojima’s result to general imaginary quadratic fields $K$. He established the maps

$$(1.2) \quad \mathcal{M}_{k-1}^+(D, \chi) \longrightarrow \mathcal{J}_{k,1}^*(1) \longrightarrow \mathcal{M}_{k,2}(1)$$

Moreover, Krieg defined the Hermitian Maaß space $\mathcal{M}_{k,2}(1) \subset \mathcal{M}_{k,2}(1)$ and he showed that the image of the second map is $\mathcal{M}_{k,1}^+(1)$. Also, $\mathcal{J}_{k,1}^*(1) \rightarrow \mathcal{M}_{k,1}^+(D, \chi)$ is an isomorphism. As for the map $\mathcal{G}_{k-1}(D, \chi)$ to $\mathcal{G}_{k-1}(D, \chi)$, Krieg remarked that if the discriminant $D$ is prime, the space of cusp forms $\mathcal{G}_{k-1}(D, \chi)$ has a basis consisting of primitive form $f_1, \ldots, f_a, f_{a+1}, \ldots, f_{a+b}$ with $f_i^* = f_i$ for all $i = a + 1, \ldots, a + b$ and the forms $f_i - f_i^*$ is a basis for cusp forms in $\mathcal{M}_{k-1}^+(D, \chi)$. Here, $f^*(\tau) = \overline{f(-\overline{\tau})}$ is the form obtained by complex conjugating all Fourier coefficients of $f$. Ikeda [12] gives a generalization of this construction to arbitrary discriminant $D$, which we modify to accommodate forms of higher level (c.f. section 5.3). We remark that for level $N = 1$, there are alternative approaches to the Hermitian Saito-Kurokawa lift via the theory of compatible Eisenstein series by Ikeda [12], via Imai’s Converse Theorem [13] recently by Matthes [27] (but only for $K = \mathbb{Q}(i)$), and theta lifting by Atobe [5].

In this paper, we generalize this result to higher level: Let $K$ be an imaginary quadratic field of discriminant $-D$, we describe the (first two) maps

$$\mathcal{M}_{k-1}(DN, \chi) \longrightarrow \mathcal{M}_{k-1}^+(DN, \chi) \longrightarrow \mathcal{J}_{k,1}^*(N) \longrightarrow \mathcal{M}_{k,2}(N) \tag{1.3}$$

For general level $N$ such that $(D, N) = 1$, Berger and Klosin, in their recent work [6], generalized the definition of Krieg’s Maaß space $\mathcal{M}_{k,2}(1)$ to that of higher level $\mathcal{M}_{k,2}(N)$. Using a modification of Kojima and Krieg’s result and Ibukiyama’s construction of the classical Maaß lift for general level, they then constructed the Hermitian analogue of the Maaß lift $\mathcal{J}_{k,1}^*(N) \rightarrow \mathcal{M}_{k,2}(N)$ i.e. the last map in the above diagram (1.3) and showed that its image is the Maaß space $\mathcal{M}_{k,2}(N)$. When $D$ is prime, they defined $\mathcal{M}_{k-1}^+(DN, \chi)$ and constructed a map $\mathcal{J}_{k,1}^*(N) \rightarrow \mathcal{M}_{k-1}^+(DN, \chi)$ which they showed is injective and that it is surjective onto the space of $p$-old cusps forms when $N = p$ is a split prime in $K$.

In this article, we generalize Krieg’s method to higher level to provide the surjectivity for the map in the second main result of Berger and Klosin [6] i.e. to show that in fact, $\mathcal{J}_{k,1}^*(N) \rightarrow \mathcal{M}_{k-1}^+(DN, \chi)$ is always an isomorphism, under appropriate modification of the space $\mathcal{M}_{k-1}^+(DN, \chi)$ defined in [6].

The main technical challenge to overcome is the lack of a simple set of generators for the congruence subgroup $\Gamma_0(N)$ (even though there are known generators given in [8]), typically used to reduce the verification of the functional equation satisfied by a Jacobi form to a finite set of functional equations that can be verified by analytical mean. A minor challenge is to avoid the computation of the inverse of a certain matrix $M_{u,v}(\sigma)$ which Krieg [18] and Berger-Klosin [6] used. (In Krieg’s case, the matrix involved $M_{u,v}(\sigma)$ is not too complicated since he only needs the one for $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the reflection matrix. In Berger-Klosin’s case, they need to exploit a relationship between the matrix $M_{\pi u, \pi v}(\sigma)$ and $M_{u,v}(\sigma)$; c.f. Section 3.2 of [6] for details.)

For the outline of the article, we recall the definition of Hermitian modular forms, the Hermitian Maaß space and Hermitian Jacobi forms in section 2 as well as review the relevant
results of Berger and Klosin [6] and define our lift. Then we proceed to generalize Krieg’s
work, namely giving a characterization of the space of plus forms in section 3 and obtain
an arithmetic criterion for the lift we define to be a Hermitian Jacobi form in section 4. In
section 5, we verify that the arithmetic criterion is true; thus, establishing the main
result of the article (Theorem 5.1). We also explain our modification of Ikeda’s construction to
get a surjective map \(\mathfrak{M}_{k-1}(DN, \chi) \to \mathfrak{M}_{k-1}^+(DN, \chi)\), thus completing the Hermitian Saito-
Kurokawa lift for general level depicted in (1.3). In the last section 6, we show that the
Maaß space is stable under Hecke operator \(T_p\) for inert prime \(p\) in \(K\).

2. HERMITIAN MODULAR FORMS, MAASS SPACE AND JACOBI FORMS

Recall that we fix an imaginary quadratic field \(K = \mathbb{Q}(\sqrt{-D})\) of discriminant \(-D = -D_K\).
We fix an embedding \(K \to \mathbb{C}\) to treat \(K\) as a subfield of \(\mathbb{C}\) and identify the non-trivial
automorphism of \(\text{Gal}(K/\mathbb{Q})\) as the restriction of complex conjugation of \(\mathbb{C}\) to \(K\). Thus, if
\(x \in K\) then its norm \(|x|^2 = x\bar{x} \in \mathbb{Q}\) makes sense. We denote by \(\mathfrak{o}_K\) the ring of integers of \(K\)
and by \(\mathfrak{o}_K := \frac{\mathfrak{o}_K}{\sqrt{\Delta}}\) the inverse different of \(K\).

Throughout the article, we shall assume the weight \(k\) is divisible by \(|\mathfrak{o}_K^\times|\), the number of
units in \(K\). In particular, \(k\) is even. (When \(k\) is not divisible by \(|\mathfrak{o}_K^\times|\), Berger and Klosin
showed that the Maaß space is zero.) In this section, we recall the notion of Hermitian
modular form, Hermitian Jacobi form and the Hermitian Maaß space as well as the main
results of [6] relating to these forms.

2.1. Hermitian modular forms. Let \(R\) be a ring and \(S\) an \(R\)-algebra equipped with an
involution \(\iota : S \to S\). If \(A\) is any \(R\)-algebra then \(S \otimes_R A\) has an obvious induced involution
\(\iota \otimes \text{Id}\). For each matrix \(g \in M_n(S \otimes_R A)\), we denote \(g^t\) the matrix \((\iota \otimes \text{Id})(g_{ij})\) obtained by
applying the induced involution to every entry of \(g\). We also denote \(g^t\) the transpose of \(g\);
and when \(\iota\) could be inferred without ambiguity, \(g^*\) is used to denote \((g^t)^t\).

If the triple \([R, S, \iota]\) is reasonable (namely, so that the subsequently mentioned Weil
restriction of scalars \(\text{Res}_{S/R} \text{GL}_{2n/S}\) exists, for instance, \(S\) is a finitely generated projective
\(R\)-module), we have the unitary similitude \textit{algebraic} groups defined over \(R\) and its sub-
groups:

\[
\begin{align*}
\text{GU}(n, n)_{[R, S, \iota]} & = \{ g \in \text{Res}_{S/R} \text{GL}_{2n/S} \mid g^*J_{2n}g = \mu(g)J_{2n}, \mu(g) \in \text{G}_m/R \} \\
\text{U}(n, n)_{[R, S, \iota]} & = \{ g \in \text{GU}(n, n) \mid \mu(g) = 1 \} \\
\text{SU}(n, n)_{[R, S, \iota]} & = \{ g \in \text{U}(n, n) \mid \det g = 1 \}
\end{align*}
\]

where \(J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\), \(I_n\) is the \(n \times n\) identity matrix and \(\text{GL}_{2n/S}, \text{G}_m/R\) are the general
linear group (over \(S\)) and multiplicative group (over \(R\)) respectively. Here, \(\text{Res}_{S/R}\) denotes
Weil restriction of scalars.

The case we are interested in is \(R = \mathbb{Z}, A = \mathfrak{o}_K\) and \(\iota\) being complex conjugation i.e.
the restriction of the non-trivial element of \(\text{Gal}(K/\mathbb{Q})\) to \(\mathfrak{o}_K\). Thus, we omit the subscript
\([R, S, \iota]\) if \(R = \mathbb{Z}, S = \mathfrak{o}_K\) and \(\iota\) is complex conjugation. Also, due to frequent usage, we also
denote \(G_n = \text{GU}(n, n), U_n = \text{U}(n, n)\).

For \(n > 1\), let \(i_n := iI_n\) and define the hermitian upper half-plane of degree \(n\) as the
complex manifold

\[
\mathbb{H}_n := \{ Z \in M_n(\mathbb{C}) \mid -i_n(Z - Z^t) > 0 \}
\]
where for a matrix $A \in M_n(\mathbb{C})$, the notation $A > 0$ means that $A \in M_n(\mathbb{R})$ and $A$ is positive-definite. The subgroup
\[ G_+^n(\mathbb{R}) := \{ g \in G_n(\mathbb{R}) \mid \mu(g) > 0 \} \]
acts transitively on $\mathbb{H}_n$ via the familiar formula
\[ g Z := (A Z + B)(C Z + D)^{-1} \]
if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in M_n(\mathbb{C})$.

A subgroup $\Gamma \subseteq G_+^n(\mathbb{R})$ is called a congruence subgroup if
(i) $\Gamma$ is commensurable with $U_n(\mathbb{Z})$, and
(ii) $\Gamma \supseteq \Gamma_n(N) := \{ g \in U(n, n)(\mathbb{Z}) \mid g \equiv I_{2n} \mod N \}$ for some positive $N \in \mathbb{Z}$.

For $g \in G_+^n(\mathbb{R})$ and $Z \in \mathbb{H}_n$, we define the factor of automorphy
\[ j(g; Z) := \det(CZ + D) \]
and for integer $k \geq 0$ and function $F : \mathbb{H}_n \to \mathbb{C}$, we define the action $|_k g$ by
\[ F|_k g (Z) := j(g; Z)^{-k} F(gZ). \]

**Definition 2.1.** A function $F : \mathbb{H}_n \to \mathbb{C}$ is called a Hermitian semi-modular form of weight $k$ and level $\Gamma$ if
\[ F|_k g = F \quad \text{for all } g \in \Gamma. \]

If $F$ is further holomorphic then it is called a Hermitian modular form. We denote $\mathcal{M}'_k(\Gamma)$ for the space of all such Hermitian semi-modular forms and by $\mathcal{M}_k(\Gamma) \subseteq \mathcal{M}'_k(\Gamma)$ for the subspace of Hermitian modular forms.

Subsequently, we are only interested in the case where congruence subgroup $\Gamma = \Gamma_{0,n}(N)$ with
\[ \Gamma_{0,n}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n, n)(\mathbb{Z}) \mid C \equiv 0 \mod N v_K \right\} \]
so we denote
\[ \mathcal{M}_{k,n}(N) := \mathcal{M}_k(\Gamma_{0,n}(N)) \]

The standard notations $\mathcal{M}_k(N, \psi)$ and $\mathcal{G}_k(N, \psi)$ are used to denote the space of elliptic modular (resp. cusp) forms on the standard Hecke congruence subgroup
\[ \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\} \]
and nybentypus $\psi$ where $\psi$ is a Dirichlet character modulo $N$.

Any Hermitian modular form of level $N$ possesses Fourier expansion
\[ F(Z) = \sum_{T \in S_n(\mathbb{Q})} c_F(T) e[\text{Tr}(TZ)] \]
where $c_F(T) \in \mathbb{C}$,
\[ S_n = L_n^\vee := \{ x \in S_n(\mathbb{Q}) \mid \text{Tr}(x L) \subset \mathbb{Z} \} \]
is the dual lattice of the lattice $L_n = S_n(\mathbb{Z}) \cap M_n(\mathfrak{o}_K)$ of integral Hermitian matrices (with respect to the trace pairing $\text{Tr}$) and
\[ S_n := \{ h \in \text{Res}_{\mathfrak{o}_K/\mathbb{Z}} M_n(\mathfrak{o}_K) \mid h^* = h \} \]
is the $\mathbb{Z}$-group scheme of Hermitian matrices. Also, we adopt the notation $e[x] := e^{2\pi i x}$.

Explicitly, one has, for example

$$S_2 = \left\{ \left( \frac{\ell}{T}, \frac{t}{m} \right) \in M_2(K) \bigg| \ell, m \in \mathbb{Z}; t \in \mathfrak{a}_K \right\}.$$  

**Definition 2.2.** The Maass space $\mathcal{M}_k^\epsilon(N) \subseteq \mathcal{M}_{k,2}(N)$ consists of forms $F \in \mathcal{M}_{k,2}(N)$ such that there exists a function $\alpha_F : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ so that the Fourier coefficients of $F$ can be described using $\alpha_F$ in the following way

$$c_F(T) = \sum_{d \in \mathbb{Z}_+, d \mid \epsilon(T)} d^{k-1} \alpha_F \left( \frac{D_K \det T}{d^2} \right)$$

for all $T \in S_2(\mathbb{Z}), T \geq 0, T \neq 0$ where $c_F(T)$ is the $T$-th Fourier coefficient of $F$ and

$$\epsilon(T) := \max \left\{ q \in \mathbb{Z}_+ \mid \frac{1}{q} T \in S_2(\mathbb{Z}) \right\}.$$

2.2. **Hermitian Jacobi forms.** We recall the notion of Jacobi forms from [6]. First, any $M \in U(1,1)(\mathbb{Z})$ can be written as $\epsilon A$ for some $A \in SL_2(\mathbb{Z})$ and $\epsilon \in \mathfrak{o}_K^\times$. Let $\mathfrak{H}$ denote the complex upper half plane. We define the action of the Jacobi group $U(1,1)(\mathbb{Z}) \ltimes \mathfrak{o}_K^2$ on the functions $\varphi : \mathfrak{H} \times \mathbb{C}^2 \to \mathbb{C}$ by

$$\varphi|_{k,m}[\epsilon A] := (c\tau + d)^{-k} e \left[ -m \frac{czw}{c\tau + d} \right] \varphi \left( \frac{a\tau + b}{c\tau + d}, \frac{\epsilon z}{c\tau + d} \right)$$

$$\varphi|_{k,m}[\lambda, \mu] := e \left[ m(\lambda^2 \tau + \lambda z + \lambda w) \right] \varphi(\tau, z + \lambda \tau + \mu, w + \lambda \tau + \mu)$$

As noted in [6], the $\epsilon$ in the expression $M = \epsilon A$ is not unique but the action described here is well-defined as long as $k$ is divisible by $|\mathfrak{o}_K^\times|$. 

**Definition 2.3.** A Hermitian Jacobi forms of weight $k$, index $m$ and level $N$ is a holomorphic function $\varphi : \mathfrak{H} \times \mathbb{C}^2 \to \mathbb{C}$ such that

(i) $\varphi|_{k,m}[\epsilon A] = \varphi$ for all $A \in \Gamma_0(N)$;

(ii) $\varphi|_{k,m}[\lambda, \mu] = \varphi$ for all $\lambda, \mu \in \mathfrak{o}_K$; and

(iii) for each $M \in SL_2(\mathbb{Z})$, the function $\varphi|_{k,m}[M]$ has a Fourier expansion of the form

$$\varphi|_{k,m}[M](\tau, z, w) = \sum_{\ell \in \mathbb{Z}_{\geq 0}, \nu \in \mathfrak{o}_K^\times} c^M_{\varphi}(\ell, t) e \left[ \frac{\ell}{\nu} \tau + tz + tw \right]$$

where $\nu = \nu(M) \in \mathbb{Z}_+$ depends on $M$ and $\nu(I_2) = 1$. The $c^M_{\varphi}(\ell, t) \in \mathbb{C}$ are the Fourier coefficients. We drop the superscript for $M = I_2$ i.e. $c^M_{\varphi}(\ell, t) = c^I_\varphi(\ell, t)$.

We denote by $\mathfrak{J}_{k,m}(N)$ for the space of Hermitian Jacobi forms of weight $k$, index $m$ and level $N$.

Writing a matrix $Z \in \mathbb{H}_2$ as $Z = \begin{pmatrix} \tau & z \\ w & \tau^* \end{pmatrix}$, we can think of a Hermitian modular form as a function of four complex variables $(\tau, z, w, \tau^*)$ where we can collect its Fourier expansion to express

$$F(Z) = \sum_{m=0}^\infty \varphi^F_m(\tau, z, w) e \left[ m\tau^* \right].$$
It is well-known (c.f. [18] Section 3 for level $N = 1$ and [10] Satz 7.1 for general level) that the functions $\varphi^F_m$'s are Jacobi forms of index $m$, normally referred to as Fourier–Jacobi coefficients of $F$. Proposition 2.4 of [6] says that just like the Siegel case, a form $F$ in the Hermitian Maass space $\mathcal{M}^+_k(N)$ is completely determined by its first Jacobi form $\varphi^F_1$ and one of the major results of the same paper (Theorem 2.8) shows that in fact $\mathcal{M}^+_k(N) \cong \mathcal{J}^*_k,1(N)$ where $\mathcal{J}^*_k,1(N) \subset \mathcal{J}^*_k(N)$ is the subspace of special Jacobi forms.

**Definition 2.4.** A form $\varphi \in \mathcal{J}^*_k,1(N)$ is called a special Jacobi form if its Fourier coefficients $c_\varphi(\ell, t)$ only depend on $\ell - |t|^2$.

Thus, we have the analogue of the classical Maass lift for Hermitian modular forms.

### 2.3. Theta decomposition

Recall that $\mathfrak{d}_K = i \sqrt{D}/\mathfrak{o}_K$ is the different ideal of $\mathfrak{o}_K$. Let $[\mathfrak{d}_K] := i \sqrt{D}/\mathfrak{o}_K$. For each $u \in [\mathfrak{d}_K]$, we define the theta function

$$\vartheta_u(\tau, z, w) := \sum_{\alpha \in u + \mathfrak{o}_K} e\left([\alpha]^2 \tau + \bar{\alpha} z + aw\right).$$

For fixed $\tau$, the theta functions are linearly independent (c.f. Haverkamp [10] Proposition 5.1) and they transformed in the following way:

$$\vartheta_u|_{1,1} \sigma = \sum_{\nu \in [\mathfrak{d}_K]} M_{u,v}(\sigma) \vartheta_v$$

for each $\sigma \in \text{SL}_2(\mathbb{Z})$ where

)$i \sqrt{D} \sum_{\gamma \in u + \mathfrak{o}_K} e\left[a\gamma] = \frac{\gamma u - \gamma v + d|v|^2}{c}\right]$ if $c \neq 0$;

$$M_{u,v} \left(\frac{a}{c} \frac{b}{d}\right) = \begin{cases} \frac{1}{c \sqrt{D}} \sum_{\gamma \in u + \mathfrak{o}_K} e\left[a\gamma] = \frac{\gamma u - \gamma v + d|v|^2}{c}\right] & \text{if } c \neq 0; \\
\text{sign}[a] \delta_{u,u} e[ab|u|^2] & \text{otherwise}
\end{cases}$$

according to [6] Lemma 3.1 or the Lemma in Section 4 of [18]. Since the transformation $|_{k,m} \sigma$ is associative, one could think of $M$ as a group homomorphism $\text{SL}_2(\mathbb{Z}) \rightarrow \text{GL}_d(\mathbb{C})$ where $d$ is the number of classes in $[\mathfrak{d}_K]$ sending each $\sigma$ to the matrix $(M_{u,v}(\sigma))_{u,v \in [\mathfrak{d}_K]}$ whose rows and columns are indexed by $[\mathfrak{d}_K]$ should we fix an ordering of $[\mathfrak{d}_K]$.

By section 3 of [6], any special Jacobi form $\varphi \in \mathcal{J}^*_k(N)$ can be expressed uniquely as a combination

$$\varphi = \sum_{u \in [\mathfrak{d}_K]} f_u \vartheta_u \quad \text{where} \quad f_u(\tau) = \sum_{\ell \geq 0} \alpha^*_\varphi(\ell) e\left[\frac{\ell \tau}{D}\right].$$

Here, $\alpha^*_\varphi(D(m - |u|^2)) = c^J_\varphi(m, u)$ are the Fourier coefficients of $\varphi$ as in definition 2.3. We call (2.2) the theta decomposition of $\varphi$.

### 2.4. The injection $\mathcal{J}^*_k(N) \rightarrow \mathcal{M}^+_k,1(DN, \chi)$

Following Krieg, we decompose the character $\chi_K = \prod_{p \mid D} \chi_p$. More concretely, $\chi_p(n) = (n \mid p)$ is just the Legendre symbol mod $p$ for odd prime $p \mid D$ whereas

$$\chi_2(n) = \begin{cases} (-4 \mid n) = (-1)^{(n-1)/2} & \text{if } 4 \mid D \text{ and } 8 \nmid D, \\
(-8 \mid n) = (-1)^{(n^2-4n+3)/8} & \text{if } 8 \mid D \text{ and } 8 \nmid D, \\
(8 \mid n) = (-1)^{(n^2-1)/8} & \text{if } 8 \mid D \text{ and } 8 \equiv 1 \mod 4,
\end{cases}$$

8 mod 4.
Then we define
\[
a_D(\ell) := |\{u \in [0_K] \mid D|u|^2 \equiv -\ell \mod D\}|
\]
which could be computed as
\[
a_D(\ell) = \prod_{p \mid D} (1 + \chi_p(-\ell))
\]
according to Krieg [18], Section 4, equation (5). Now, we define the space
\[
\mathcal{M}_{k-1}^+(DN, \chi) := \{g \in \mathcal{M}_{k-1}(DN, \chi) \mid \alpha_\ell(g) = 0 \text{ if } a_D(-\ell) = 0\}
\]
similar to that of [18] but for arbitrary \(N\) as opposed to \(N = 1\) in [18]. When \(D\) is prime, this is the same as the space defined in [6]. We state the following generalization of proposition 3.5 of [6]:

**Proposition 2.5.** Suppose that \((D, N) = 1\). Then the map \(\mathfrak{J}_k(N) \to \mathcal{M}_{k-1}^+(DN, \chi)\) given by \(\varphi \mapsto f := f_{0|k-1}W_D\) is an injection where \(W_D = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} D & y \\ 0 & 1 \end{pmatrix}\) and \(y = N^{-1} \mod D\).

The Fourier coefficients of \(f\) satisfy
\[
a_\ell(f) = i \frac{a_D(\ell)}{\sqrt{D}} \chi(N) \alpha_{\varphi}^*(\ell).
\]

Note that we are basically dropping the assumptions \(D\) prime and \(N\) odd in [6]. The proof is the same as the proof given in [6]. One only needs to check that the simple lemma used in the proof also works for non-prime \(D\) and \(N\) being even number:

**Lemma 2.6** (Lemma 3.3 in [6]). Suppose \((D, N) = 1\) and \(t \in \mathbb{Z}\) such that \((t, N) = 1\). Then
\[
\sum_{\gamma \in \mathcal{O}_K/N\mathcal{O}_K} e\left[\frac{t|\gamma|^2}{N}\right] = \chi(N).N.
\]

**Proof.** By Chinese Remainder Theorem, the statement can be reduced to the case where \(N\) is a prime power. When \(N\) is an odd prime power, the left hand side is a product of two Gauss sum as in Haverkamp [10], Lemma 0.4. When \(N = 2^r\) is a power of 2, one can use induction on \(r\). \(\square\)

Notice that Proposition 2.5 already tells us what the lifting should be, if it exists. Suppose we have \(g \in \mathcal{M}_{k-1}^+(DN, \chi)\) and \(\varphi_g \in \mathfrak{J}_k^1(N)\) is its pre-image. Write \(\varphi_g = \sum f_{u,g} \vartheta_u\) as in (2.2). Then by Proposition 2.5, \(g = f_{0,g}|k-1W_D\) and the Fourier coefficients of \(g\) satisfies \(a_\ell(g) = i \frac{2D(\ell)}{\sqrt{D}} \chi(N) \alpha_{\varphi_g}^*(\ell)\) which leads to
\[
\alpha_{\varphi_g}^*(\ell) = \frac{-i \sqrt{D}}{a_D(\ell) \chi(N)} a_\ell(g)
\]
and the decomposition (2.2) tells us that
\[
f_{u,g}(\tau) = \sum_{\ell \geq 0, \ell \equiv -D|u|^2 \mod D} \alpha_{\varphi_g}^*(\ell) e\left[\frac{\ell \tau}{D}\right] = \sum_{\ell \geq 0, \ell \equiv -D|u|^2 \mod D} \frac{-i \sqrt{D}}{a_D(\ell) \chi(N)} a_\ell(g) e\left[\frac{\ell \tau}{D}\right]
\]
Note that this is the lift given by Krieg in Section 6 of [18] for level \( N = 1 \), twisted by the factor of \( \chi(N) \) to account for the level.

In section 5, we will show that for any \( g \in \mathfrak{M}^+_{k-1}(D N, \chi) \), the function \( \varphi_g = \sum f_{a,g} \vartheta_u \) with the \( f_{a,g} \) as derived above is indeed a Hermitian Jacobi form; and so the map in Proposition 2.5 is an isomorphism. To do that, we generalize Krieg’s characterization of plus forms and obtain the arithmetical condition for \( \varphi_g \) to be a Hermitian Jacobi form.

3. Characterization of plus forms

For each \( m \mid D \), set \( \psi_m := \prod_{p \mid m \text{ prime}} \chi_p \) as in section 5 of [18]. For a character \( \psi \) of modulus \( M \), the (twisted) Gauss sum associated to \( \psi \) is defined as

\[
G(\psi; b) := \sum_{a \in \mathbb{Z}/M} \psi(a) e\left[ b \frac{a}{M} \right].
\]

and the standard Gauss sum \( G(\psi) := G(\psi; 1) \). We remark that \( \chi_K \) is of conductor \( D \) and that if \( (m, \frac{D}{m}) = 1 \) then \( \psi_m \) is of conductor \( m \). It is well–known from one of Gauss’ proofs of quadratic reciprocity (c.f. [25], Lemma 4.8.1) that

\[
G(\psi_m) := \varepsilon(\psi_m) \sqrt{m}
\]

where

\[
\varepsilon(\psi_m) := \begin{cases} 
1 & \text{if } \psi_m(-1) = 1, \\
i & \text{if } \psi_m(-1) = -1.
\end{cases}
\]

Fix natural numbers \( m, n \) such that \( D = mn \) and \( (m, n) = 1 \). We consider the Hecke operator (without normalization factor)

\[
g|_{k-1} U_m := \sum_{j=0}^{m-1} g|_{k-1}  \begin{pmatrix} 1 & j \\ 0 & m \end{pmatrix}
\]

and choose a matrix \( P_m \in \text{SL}_2(\mathbb{Z}) \) such that

\[
P_m \equiv \begin{cases} 
J \mod m^2 \\
I \mod (nN)^2
\end{cases}
\]

where \( J := J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) are the usual generators for \( \text{SL}_2(\mathbb{Z}) \). Set

\[
Q_m := P_m \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}
\]

and define the operator \( V_m \) by

\[
g|_{k-1} V_m := g|_{k-1} U_m |_{k-1} Q_m.
\]

Similar to [18] Section 5, one has

**Lemma 3.1.** \( Q_m \Gamma_0(N D) Q_m^{-1} = \Gamma_0(N D) \) and so \( g \mapsto g|_{k-1} Q_m \) is an involution of \( \mathfrak{M}_{k-1}(D N, \chi) \).
Proposition 3.2. If $k$ is even and $g \in \mathcal{M}_{k-1}(DN, \chi)$ then we have
\[ g|_{k-1}V_m = G(\psi_m)m^{1-k}g \]
if and only if
\[ \alpha_\ell(g) = 0 \text{ for all } \ell \text{ such that } \psi_m(-\ell) = -1. \]

Proof. The proof is verbatim as in [18] with appropriate modification to take care of the level.

\[ \square \]

Corollary 3.3. Let $k$ be even and $D = 2^e d$ with $2 \nmid d$. Then
\[ \mathcal{M}_{k-1}^+(DN, \chi) = \left\{ g \in \mathcal{M}_{k-1}(DN, \chi) \left| g|_{k-1}V_p = G(\psi_p)p^{1-k}g \text{ for } p = 2^e \text{ and } p | d \text{ odd prime} \right. \right\}. \]
Furthermore, if $g \in \mathcal{M}_{k-1}^+(DN, \chi)$ and $(m, \frac{D}{m}) = 1$ then
\[ g|_{k-1}V_m = G(\psi_m)m^{1-k}g. \]

Note that Proposition 3.2 and Corollary 3.3 are the generalization of Proposition and Corollary A (resp.) in Section 5 of [18].

For any $j \in \mathbb{Z}/D\mathbb{Z}$, set $\sigma_j := \begin{pmatrix} 1 & j \\ 0 & D \end{pmatrix}$. Our goal is to compute $g|_{k-1}\sigma_j\sigma$ for $g \in \mathcal{M}_{k-1}^+(DN, \chi)$ and any $\sigma \in \Gamma_0(N)$. Fix such a $\sigma = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. For any $j \in \mathbb{Z}$, let $\mu = \mu_\sigma(j) := (x + zj, D)$ and $m = m_\sigma(j) := D\mu$ be the $\mu$-component of $D$ i.e.
\[ m = \prod_{p | \mu \text{ prime}} p^{\text{val}_p(D)} \]
and let $n := \frac{D}{m}$. Then we have

Lemma 3.4. (i) $(x + zj, n) = 1$;
(ii) $(r(x + zj) - nz, m) = 1$ for any $r \in \mathbb{Z}$; in particular, $(m, z) = 1$;
(iii) $(x + zj, y + tj) = 1$;
(iv) $(y + tj, m) = 1$.

Proposition 3.5. For any $g \in \mathcal{M}_{k-1}^+(DN, \chi)$, we have
\[ g|_{k-1}\sigma_j\sigma = \sum_{s=0}^{m-1} \frac{G(\psi_m; (x + zj)s - \lambda)}{m \psi_n(x + zj)} g|_{k-1} \begin{pmatrix} 1 & ns + \kappa \\ 0 & D \end{pmatrix} \]
where $\kappa = \kappa_\sigma(j) \in \mathbb{Z}$ is any integer such that
\[ \kappa \equiv \frac{y + tj}{x + zj} \mod n \]
and
\[ \lambda := \frac{y + jt - \kappa(x + zj)}{n} \in \mathbb{Z}. \]
In case \( D \) is square-free, the formula can be simplified to
\[
g|_{k-1}\sigma_j\sigma = \frac{G(\psi_m; nz)}{m \psi_n(x + zj)} \sum_{s=0}^{m-1} g|_{k-1} \begin{pmatrix} ns + \kappa \\ 0 \\ D \end{pmatrix}.
\]

Proof. Let \( P_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). We have
\[
g|_{k-1}\sigma_j\sigma = \frac{m^{k-1}}{G(\psi_m)} g|_{k-1} V_m |_{k-1} \begin{pmatrix} x + zj & y + jt \\ Dz & Dt \end{pmatrix}
\]
by Corollary 3.3
\[
= \frac{m^{k-1}}{G(\psi_m)} \sum_{r=0}^{m-1} g|_{k-1} \begin{pmatrix} a + rc & b + rd \\ mc & md \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x + zj & y + jt \\ Dz & Dt \end{pmatrix}
\]
\[
= \frac{1}{G(\psi_m)} \sum_{r=0}^{m-1} g|_{k-1} \begin{pmatrix} (a + rc)(x + zj) + (b + rd)nz & (a + rc)(y + jt) + (b + rd)nt \\ mc(x + zj) + dDz & mc(y + jt) + (b + rd)mdnt \end{pmatrix}
\]
Let us temporarily denote the above matrix by \( \gamma = \gamma_r \) and denote by \( A, B \) the entries of the first column of \( \gamma \) i.e.
\[
A = (a + rc)(x + zj) + (b + rd)nz
\]
\[
B = mc(x + zj) + dDz
\]
Observe that \( \det \gamma = D \) and that the lower row of \( \gamma \) is divisible by \( D \). Thus, the matrix obtained by dividing both entries on the lower row of \( \gamma \) by \( D \), namely the matrix
\[
\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} (a + rc)(y + jt) + (b + rd)nt \\ mc(y + jt) + (b + rd)mdnt \end{pmatrix}
\]
is in \( \text{SL}_2(\mathbb{Z}) \) whence \( (A, B) = 1 \) and so \( (A, B) = (A, mnB) = (A, m)(A, n) = 1 \) by Lemma 3.4 for \( A \equiv \begin{pmatrix} r(x + zj) - nz & 1 \\ x + zj & 0 \end{pmatrix} \mod m \) and \( \begin{pmatrix} A \\ B \end{pmatrix} \in \Gamma_0(ND) \) and so we have
\[
g|_{k-1}\gamma = \frac{\psi_n(x + zj)}{\psi_m(r(x + zj) - nz)} g|_{k-1} \begin{pmatrix} 1 \\ 0 \\ D \end{pmatrix}
\]
\[
F_r = \begin{pmatrix} (a + rc)(y + jt) + (b + rd)nt \\ (a + rc)(x + zj) + (b + rd)nz \end{pmatrix} \mod D \equiv \begin{pmatrix} r(y + jt) - nt & 1 \\ r(x + zj) - nz & 0 \end{pmatrix} \mod m
\]
\[
y + jt \\ x + zj \mod n
\]
so that we get
\[
g|_{k-1}\sigma_j\sigma = \sum_{r=0}^{m-1} \frac{\psi_n(x + zj)}{G(\psi_m; r(x + zj) - nz)} g|_{k-1} \begin{pmatrix} 1 \\ 0 \\ F_r \end{pmatrix}
\]
As \( r \) runs through the residues \( \mod m \), the \( F_r \) also goes through the subset of residues
\[
\{ns + \kappa \mid 0 \leq s \leq m - 1\} \subseteq \mathbb{Z}/D\mathbb{Z}.
\]
A simple change-of-variable from $r$ to $s$ and use observation that $\frac{1}{G(m,b)} = G(m,-b) / m$ to move the Gauss sum to the numerator, we derive the formula claimed in the proposition.

With the assumption that $D$ is square-free, we always have $\mu = m$ so $m \mid x + zj$ and the factor $G(m; r(x + zj) - nz) = G(m; -nz)$ is independent of $s$. □

4. Arithmetic Criterion

To simplify a lot of subsequent formulas, we shall also denote $a_u := a_D(-D|u|^2)$ for $u \in [D_K]$. Given a plus form

$$ g = \sum_{\ell=0}^{\infty} a_\ell(g)e[\ell \tau] \in \mathcal{M}^+_k(DN, \chi), $$

we define

$$ g_u := -\frac{i\sqrt{D}}{a_u} \sum_{\ell=0}^{\infty} a_\ell(g)e[\ell \tau] \mod D $$

for all $u \in [D_K]$ and

$$ \varphi_g := \sum g_u \vartheta_u. $$

If $\varphi_g$ is a Jacobi form of level $N$ then it will be the lifting of $\chi(N)g = \pm g$, by the argument at the end of section 2.

In this section, we shall derive transformation formula for $g_u$ using result of the previous section and then deduce an arithmetic condition for $\varphi_g$ to be a Hermitian Jacobi form (Proposition 4.3).

**Proposition 4.1.** Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. For each $j \in \mathbb{Z}$, we choose $\kappa = \kappa_\sigma(j) \in \mathbb{Z}$ such that

$$ \kappa \equiv \frac{b + dj}{a + cj} \mod n $$

as before; in addition that if $m \neq \mu$ (i.e. $m = 2\mu$ or $m = 4\mu$), then

$$ \kappa \equiv \frac{(b + dj + c) / 2^f}{(a + cj) / 2^f} \mod \frac{m}{\mu} $$

where $f = \text{val}_2(a + cj) = \text{val}_2(\mu)$ is the maximum power of two dividing $\mu$. Then

$$ g_u|_{k-\sigma} = \frac{1}{Da_u} \sum_{v \in [D_K]} \sum_{j=0}^{D-1} R_\sigma(v,j) G(\psi_m; nc) \psi_n(a + cj) e[|u|^2j - |v|^2\kappa] g_v $$

where

$$ R_\sigma(v,j) = \begin{cases} \frac{1}{2} \left( 1 + e \left[ -\frac{(a+\sigma c)(|v|^2)}{2m} \right] \right) & \text{if } m = 4\mu, \\ 1 & \text{if } m \neq 4\mu. \end{cases} $$

It is not difficult to see that if $m \neq \mu$ then $2^f|b+dj+c$ from $ad-bc = 1$ and $2^f|a+cj$ and so $(b+ dj + c) / 2^f \mod \frac{m}{\mu}$ is well-defined. Also, note that such $\kappa$ always exists for $(n, \frac{m}{\mu}) | (n,m) = 1$ so we can solve for $\kappa$ by Chinese Remainder Theorem.
Proof. For \( g \in \mathcal{M}_k^+(DN, \chi) \), observe the following

(i) \( g_u = \frac{-iD^{k-3/2}}{a_u} \sum_{j=0}^{D-1} e[|u|^2j] g_{k-1} \sigma_j \)

(ii) \( g = \frac{i}{\sqrt{D}} \sum_{v \in [0,1]} g_v |g_{k-1} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \)

(iii) \( g_v|_{k-1} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \sigma_j = D^{1-k} e[-|v|^2j] g_v \)

and then use Proposition 3.5, we arrive at

\[
g_u|_{k-1} \sigma = \frac{-iD^{k-3/2}}{a_u} \sum_{j=0}^{D-1} \sum_{s=0}^{m-1} \frac{e[|u|^2j] G(\psi_m; s(a + cj) - \lambda)}{m \psi_n(a + cj)} g_{k-1} \sigma_{n + \kappa} \text{ by (i) and Proposition 3.5}
\]

\[
= \frac{-iD^{k-3/2}}{a_u} \sum_{j, s} \frac{e[|u|^2j] G(\psi_m; s(a + cj) - \lambda)}{m \psi_n(a + cj)} \sum_{v \in [0,1]} g_v|_{k-1} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \sigma_{n + \kappa} \text{ by (ii)}
\]

\[
= \frac{1}{Da_u} \sum_{v \in [0,1]} \sum_{j, s} G(\psi_m; s(a + cj) - \lambda) e[|u|^2j - |v|^2(n s + \kappa)] g_v
\]

\[
= \frac{1}{Da_u} \sum_v \sum_{j=1}^{D-1} \frac{G(\psi_m) e[|u|^2j - |v|^2\kappa]}{m \psi_n(a + cj)} \left( \sum_{s=0}^{m-1} \psi_m(s(a + cj) - \lambda) e[-D|v|^2s \frac{\kappa}{m}] \right) g_v
\]

With the appropriate requirement for \( \kappa \) as in the statement of the proposition, a careful case-by-case analysis yields the inner most sum

\[
\sum_{s=0}^{m-1} \psi_m(s(a + cj) - \lambda) e[-D|v|^2s \frac{\kappa}{m}]
\]

\[
= \begin{cases} 
\frac{m}{2} \psi_m(nc) \left( 1 + e \left[ -\frac{(a+ cj)D|v|^2}{2m} \right] \chi_2(5 - 2nc) \right) \delta_{\mu,(D|v|^2,m)} & \text{if } m = 4\mu, \\
\psi_m(nc) \delta_{\mu,(D|v|^2,m)} & \text{if } m \neq 4\mu.
\end{cases}
\]

where

\[
\delta_{X,Y} := \begin{cases} 
1 & \text{if } X = Y, \\
0 & \text{otherwise}
\end{cases}
\]

is the usual Kronecker delta.

\[ \square \]

Remark 4.2. We remark that in case \( m = 4\mu \) (which can only happen if \( 8 \mid D \)) and \( (D|v|^2, m) = \mu \), we have \( e \left[ -\frac{(a+ cj)D|v|^2}{2m} \right] = e \left[ -\frac{(a+ cj)D|v|^2}{4\mu} \right] \in \{ \pm i \} \) only depends on \( j \mod 8 \).

Also, since \( \chi_2(5) = -1 \) for both possibilities of \( \chi_2 = (\pm 8 \mid *) \), we have

\[
\chi_2(5 - 2nc) = (-1)^{nc+1} \chi_2(-1) = \begin{cases} 
-\chi_2(-1) & \text{if } nc \equiv 1 \mod 4, \\
\chi_2(-1) & \text{if } nc \equiv -1 \mod 4.
\end{cases}
\]
Proposition 4.3. Let \( g \in \mathcal{M}^+_{k-1}(DN, \chi) \). Then \( \varphi_g = \sum g_u \vartheta_u \) is a Hermitian Jacobi form of level \( N \) if and only if the equation

\[
(4.1) \sum_u \frac{M_{u,v}(\sigma)g_u}{Da_u} \sum_{j=0}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a + cj) e \left[ |u|^2 j - |w|^2 \kappa \right] = \delta^{\text{mod} D}_{D|w|^2, D|v|^2}
\]

holds for any \( \sigma \in \Gamma_0(N) \) and any \( v, w \in [\mathfrak{d}_K] \) such that \( g_w \neq 0 \).

Here, for any integers \( X, Y, M \),

\[
\delta^{\text{mod} M}_{X,Y} := \begin{cases} 
1 & \text{if } X \equiv Y \text{ mod } M, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. From transformation formula for \( \vartheta_u \), we find that \( \varphi_g \) is Jacobi form of level \( N \) if and only if

\[
\sum_{u \in [\mathfrak{d}_K]} M_{u,v}(\sigma)g_u|_{k-1}\sigma = g_v \quad \text{for all } v \in [\mathfrak{d}_K] \text{ and } \sigma \in \Gamma_0(N)
\]

By Proposition 4.1, the above equation is the same as

\[
\sum_{u \in [\mathfrak{d}_K]} \frac{M_{u,v}(\sigma)}{Da_u} \sum_{w \in [\mathfrak{d}_K]} \sum_{j=0}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a + cj) e \left[ |u|^2 j - |w|^2 \kappa \right] g_w = g_v.
\]

Now observe that \( g_w = g_{w'} \) if \( D|w|^2 \equiv D|w'|^2 \) mod \( D \) and conversely, if \( D|w|^2 \not\equiv D|w'|^2 \) mod \( D \) (or equivalently, \( |w|^2 \not\equiv |w'|^2 \) in \( \mathbb{Q}/\mathbb{Z} \)) then \( g_w \) and \( g_{w'} \) have disjoint Fourier expansion so if they are non-zero, they must be linearly independent. So grouping the summation over \( w \) by \( D|w|^2 \) mod \( D \), we find that the above equation holds if and only if

\[
(4.2) \sum_{u \in [\mathfrak{d}_K]} \frac{M_{u,v}(\sigma)}{Da_u} \sum_{w' \in [\mathfrak{d}_K]} \sum_{j=0}^{D-1} R_\sigma(w', j) \mathbf{G}(\psi_m; nc) \psi_n(a + cj) e \left[ |u|^2 j - |w'|^2 \kappa \right] = \delta^{\text{mod} D}_{D|w'|^2, D|v|^2}
\]

for all \( w \in [\mathfrak{d}_K] \) such that \( g_w \neq 0 \). Now if \( D|w'|^2 \equiv D|w|^2 \) mod \( D \) then

- \( e^{-|w'|^2 \kappa} = e^{-|w|^2 \kappa} \) as \( \kappa \in \mathbb{Z} \);
- \( (D|w'|^2, m) = (D|w|^2, m) \) so the sum over \( j \) such that \( (D|w'|^2, m) = \mu \) goes through the same set of \( j \)'s independent of \( w' \in [\mathfrak{d}_K] \); and also
- \( R_\sigma(w', j) = R_\sigma(w, j) \): obviously if \( m \neq 4\mu \) and in case \( m = 4\mu \), the only term that involve the \( w \) in the definition of \( R_\sigma \) is

\[
e \left[ \frac{(a + cj)D|w'|^2}{2m} \right]
\]

which equals

\[
e \left[ \frac{(a + cj)D|w|^2}{2m} \right]
\]

as long as \( D|w'|^2 \equiv D|w|^2 \) mod \( D \).
Thus, we can replace all occurrences of \( w' \) in the summand of (4.2) with \( w \) and the (4.2) reduces to the equivalent

\[
\sum_{u \in \mathfrak{O}_K} \frac{M_{u,v}(\sigma)}{Da_u} \sum_{w' \in \mathfrak{O}_K} R_{\sigma}(w, j) \mathbf{G}(\psi_{m}; nc) \psi_{n}(a + cj) e \left[ |u|^2 j - |w|^2_k \right] = \delta_{D|w|^2, D|v|^2}^{\text{mod } D} S
\]

One can now see that the summand \( S \) is free of the summation variable \( w' \) and so the middle sum over \( w' \) is just \( a_w S \) where \( a_w \) is the number of \( w' \in \mathfrak{O}_K \) such that \( D|w'|^2 \equiv D|w|^2 \mod D \). Hence, we have (4.1).

**Remark 4.4.** Observe that the equation (4.1) does not have modular forms in it: It purely depends on the arithmetic of the field \( K \). Thus, if it is true, one could expect that it is true in general for all \( \sigma \in \text{SL}_2(\mathbb{Z}) \) and \( v, w \in \mathfrak{O}_K \). (Note that even if the equation holds for all \( \text{SL}_2(\mathbb{Z}) \), this does not imply that \( \varphi_g \) is Jacobi form of level 1; since the computation of \( g|_{k-1} \sigma \sigma \) we used along the way is only valid for \( \sigma \in \Gamma_0(N) \).

**Remark 4.5.** We remark here that one technique to show (4.1) is true for higher level is to use existence of lifting for level 1 given by Krieg. In section 5 of [18], Krieg defined the Eisenstein series

\[
E^*_{k-1}(\tau) = \sum_{\substack{D = mn \in \mathbb{N}^2 \backslash \{0\} \atop (m,n) = 1}} \psi^*_{m}(-1) f_{k-1}(\tau; \psi^*_m, \psi^*_n)
\]

and he showed that

\[
\mathfrak{M}^+_k(D, \chi) = \mathcal{C}E^*_{k-1} \oplus \mathcal{S}^*_k(D, \chi)
\]

for all even \( k > 2 \). The Fourier coefficient of \( E^*_{k-1} \) is given by

\[
\alpha_{\ell}(E^*_{k-1}) = a_D(\ell) \sum_{d \mid \ell} \chi(d) d^{k-2}
\]

for \( ( \ell, D ) = 1 \). Thus, \( (E^*_{k-1})_{w} \neq 0 \) for all \( w \in \mathfrak{O}_K \) such that \( (D|w|^2, D) = 1 \). Thus, by Krieg’s isomorphism for the case \( N = 1 \), we know that (4.1) is true for all \( \sigma \in \Gamma \) and all \( w, v \) such that \( (D|w|^2, D) = 1 \). When \( D \) is prime, the remaining class \( w \in \mathfrak{O}_K \) such that \( (D|w|^2, D) \neq 1 \) is \( w = 0 \) and one easily works out the equation to show surjectivity in the case of interest in [6]. We will not pursue this direction.

Our next result reduces the verification of the equation in Proposition 4.3 to that of representatives of each equivalence class of \( \Gamma_0(N) \) under the equivalence relation induced by \( \Gamma_0(D) \), namely \( \alpha \sim \beta \) if \( \alpha = \beta \gamma \) for some \( \gamma \in \Gamma_0(D) \).

**Proposition 4.6.** If the equation (4.1) in Proposition 4.3 holds for \( \sigma \in \text{SL}_2(\mathbb{Z}) \) and every \( w, v \in \mathfrak{O}_K \) then it also holds for \( \sigma \gamma \) and every \( w, v \in \mathfrak{O}_K \) for any \( \gamma \in \Gamma_0(D) \). In particular, (4.1) holds for all \( \sigma \in \Gamma_0(D) \).

**Proof.** Fix a matrix \( \gamma = \begin{pmatrix} x & y \\ Dz & t \end{pmatrix} \in \Gamma_0(D) \). From

\[
\sigma \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ Dz & t \end{pmatrix} = \begin{pmatrix} ax + bDz & ay + bt \\ cx + dDz & cy + dt \end{pmatrix}
\]
we find that $\mu_{\sigma\gamma}(j) = \mu_{\sigma}(j)$ which implies $m_{\sigma\gamma}(j) = m_{\sigma}(j)$ and $n_{\sigma\gamma}(j) = n_{\sigma}(j)$. For any $w \in [0, \infty]$ such that $(D|w|^2, m) = \mu$, it is easy to check that

$$ R_{\sigma\gamma}(w, j) = R_{\sigma}(w, j) $$

and finally, we can take

$$ \kappa_{\sigma\gamma}(j) = ty + t^2\kappa_{\sigma}(j). $$

First, assume $z > 0$. In Lemma 3.1 of [6], it was mentioned that $M_{u,v}(\gamma) = \delta_{u,tv}e [xy|u|^2] \chi(t)$ and so from the fact that $M$ is a group homomorphism $M(\sigma\gamma) = M(\sigma)M(\gamma)$, one has $M_{u,v}(\sigma\gamma) = M_{u,tv}(\sigma)e [xy|tv|^2] \chi(t)$. So the left hand side of (4.1) for $\sigma\gamma$ is

$$ \sum_u \frac{M_{u,tv}(\sigma)\sigma}{Da_u} \sum_{j=0}^{D-1} R_{\sigma}(w, j)G(\psi_m; n(cx + Dz)) $$

$$ \times \psi_n((ax + bDz) + (cx + Dz)j)e [|u|^2j - |w|^2\kappa_{\sigma\gamma}] $$

$$ = \sum_u \frac{M_{u,tv}(\sigma)e [xy|tv|^2] \chi(t)a_w}{Da_u} \sum_{j=0}^{D-1} R_{\sigma}(w, j)G(\psi_m; ncx)\psi_n(ax + cxj) $$

$$ \times \psi_n((ax + bDz) + (cx + Dz)j)e [|u|^2j - |w|^2(ty + t^2\kappa_{\gamma})] $$

$$ = \sum_u \frac{M_{u,tv}(\sigma)e [xy|tv|^2 - ty|w|^2] \chi(t)\sigma}{Da_u} \sum_{j=0}^{D-1} G(\psi_m; nc)\psi_n(a + cj)e [|u|^2j - |tw|^2\kappa_{\gamma}] $$

$$ = e [ty(|v|^2 - |w|^2)] \sum_u \frac{M_{u,tv}(\sigma)a_{tw}}{Da_u} \sum_{j=0}^{D-1} G(\psi_m; nc)\psi_n(a + cj)e [|u|^2j - |tw|^2\kappa_{\gamma}] $$

by observing that $xt \equiv 1 \mod D$ whence $e [xyt^2|v|^2 - ty|w|^2] = e [ty(|v|^2 - |w|^2)]$ as $(xt - 1)|v|^2 \in \mathbb{Z}$. Since $(t, D) = 1$, multiplication by $t$ induces automorphism of $[0, \infty]$ so that we have $a_{tw} = a_w$, $(D|tw|^2, m) = (D|w|^2, m)$.

As indicated, the last sum is the left hand side of (4.1) for $\sigma$ and $tv, tw$ in place of $v, w$. So we get $\delta_{wtw^2, D|tw|^2}^D$ for the answer by the hypothesis that the equation holds for $\sigma$. Note that the same as $\delta_{D|tw|^2, D|tv|^2}^D = \delta_{D|w|^2, D|v|^2}^D$ again by $(t, D) = 1$. Evidently, if $D|w|^2 \equiv D|v|^2 \mod D$ then $e [ty(|v|^2 - |w|^2)] = 1$. So (4.1) holds for $\sigma\gamma$ and $v, w$.

The case $z = 0$ can be handled similarly. Note that this case includes the matrix $-I_2$ which accounts for the case where $z < 0$. Recall that $M_{w', v}(\gamma) = \text{sign}[x] \delta_{w', xv}e [xy|w'|^2]$ from the definition so that

$$ M_{u,v}(\sigma\gamma) = \sum_{w'} M_{u,w'}(\sigma)M_{w', v}(\gamma) $$

$$ = \sum_{w'} M_{u,w'}(\sigma)e [xy|w'|^2] $$

$$ = \text{sign}[x] M_{u,xv}(\sigma)e [xy|xv|^2] $$
and we get
\[
\sum_u \frac{\text{sign}[x]}{D a_u} M_{u,xv}(\sigma) e[xy|xv|^2] a_w \sum_{j=0}^{D-1} R_\sigma(w,j) e[\psi_m; n cx] \psi_n(ax + cj) \times \\
\times e[|u|^2 j - |w|^2 ty + t^2 \kappa] \\
= \text{sign}[x] \chi(x) e[xy|xv|^2 - |w|^2 ty] \sum_u \frac{M_{u,xv}(\sigma) a_w}{D a_u} \sum_{j=0}^{D-1} R_\sigma(w,j) e[\psi_m; n c] \psi_n(a + cj) \times \\
\times e[|u|^2 j - |tw|^2 \kappa]
\]

The inner sum is again the left hand side of (4.1) with \(xv\) and \(tw\) in place of \(v, w\) respectively so we get \(\delta_{D[xv]^2, D[tw]^2}\) as a result by assumption. Note that \(z = 0\) implies \(x = t = \pm 1\) so \(|xy|^2 = |x|^2\) and \(|ty|^2 = |w|^2\). Thus, if \(D|xy|^2 = D|ty|^2\) mod \(D\) then if \(x = t = 1\) then \(\text{sign}[x] \chi(x)e[xy|xv|^2 - |w|^2 ty] = \text{sign}[1] \chi(1)e|y|v^2 - |w|^2 y| = 1\) obviously. Otherwise, if \(x = t = -1\) then \(\text{sign}[x] \chi(x)e[xy|xv|^2 - |w|^2 ty] = \text{sign}[-1] \chi(-1)e[-y|v^2 + |w|^2 y] = 1\).

For the remaining statement, one easily check that the equation holds for \(\sigma = I_2\) and all \(v, w \in [\mathcal{D}_K]\).

\[\square\]

5. Hermitian Maaß lift

This section is dedicated to proving the main theorem of the article

**Theorem 5.1.** Suppose that \((D, N) = 1\). Let \(g \in \mathcal{M}^{+}_{k-1}(DN, \chi)\). For any \(u \in [\mathcal{D}_K]\), define
\[
g_u(\tau) := \chi(N) \frac{-i \sqrt{D}}{a_D(-D|u|^2)} \sum_{\ell = -D|u|^2 \text{ mod } D} a_{\ell}(g) e\left[\frac{\ell \tau}{D}\right]
\]
and
\[
\varphi_g := \sum_{u \in [\mathcal{D}_K]} g_u \vartheta_u \quad \text{where} \quad \vartheta_u(\tau, z, w) := \sum_{a \in w + \vartheta_1} e\left[|a|^2 \tau + \overline{a}z + aw\right].
\]

Then \(\varphi_g \in \mathfrak{M}_k^+(N)\) and \(\varphi_g \mapsto g\) under the injective map \(\mathfrak{M}_k^+(N) \to \mathcal{M}_k^+(DN, \chi)\) in Proposition 2.5.

In particular, we have an isomorphism \(\mathfrak{M}_k^+(N) \cong \mathcal{M}_k^{+}(DN, \chi)\).

By the results of previous section, we only need to show the equation (4.1) holds for all representatives \(\sigma \in \Gamma_0(1) = \text{SL}_2(\mathbb{Z})\) of \(\Gamma_0(1)/\Gamma_0(D)\) and all \(v, w \in [\mathcal{D}_K]\). (This is stronger than the criterion in Proposition 4.3.) From the well–known representatives for \(\Gamma_0(1)/\Gamma_0(D)\) and the fact that the class of the identity matrix is already settled in Proposition 4.6, we reduce the verification to that of \(\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) with \(c \mid D\) and \(c > 0\). Fix such an arbitrary \(\sigma\) and two arbitrary classes \(v, w \in [\mathcal{D}_K]\). Let
\[
A_u := \sum_{j \text{ mod } D} \frac{a_w}{a_u} R_\sigma(w,j) G(\psi_m; nc) \psi_n(a + cj) e\left[|u|^2 j - |w|^2 \kappa\right]
\]
and

\[ A := \sum_{u \in [d_K]} \frac{M_{u,v}(\sigma)}{D} A_u. \]

be a factor of the inner summand and the left hand side (respectively) of (4.1). Under our assumption on the lower left entry \( c \) of \( \sigma \), we shall evaluate \( A_u \) and then \( A \) explicitly to establish (4.1), namely to show that

\[ A = \delta_{a \equiv a \mod D} \delta_{D|w/|2} D_{|v/2}. \]

For clarity and due to the fact that the complete set of representatives of \([d_K]\) are of different forms, we consider the two cases of odd and even discriminant \( D \) separately. The proof strategy is pretty much the same; except for the fact that the even discriminant case is technically much more complicated.

In this section, to simplify our formula, we recycle the previous notation \( e[\bullet] \) as follow:

For any integer \( M \neq 0 \) and any rational number \( r \in \mathbb{Z}(M) \) where

\[ \mathbb{Z}(M) := \left\{ \frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, (a, b) = 1 \text{ and } (b, M) = 1 \right\}, \]

the notation \( e\left[\frac{r}{M}\right] \) will always mean the root of unity \( e^\frac{s\pi i}{M} \) where \( s \) is any integer such that \( r \equiv s \mod M \). For example, if \( r \) is odd, then \( e\left[\frac{r}{M}\right] = e\left[\frac{s}{M}\right] \) where \( s \) is a multiplicative inverse of \( r \mod 2^e \). This way, we do not have to write the longer notation \( e\left[\frac{r-1 \mod 2^e}{2^e}\right] \) or resort to defining a lot of new symbol for multiplicative inverse.

5.1. The odd discriminant case. Let us consider the case where the discriminant \(-D\) is odd. In this case, \( D \equiv 3 \mod 4 \) and we also know that \( D \) is square-free. The representatives for \([d_K]\) can be taken from

\[ \left\{ \frac{ix}{\sqrt{D}} \bigg| x \in \mathbb{Z}/D\mathbb{Z} \right\}. \]

Here, \( x \in \mathbb{Z}/D\mathbb{Z} \) refers to the choice of \( x \in \mathbb{Z} \) from any complete sets of representatives of \( \mathbb{Z}/D\mathbb{Z} \). Note that \( D|u|^2 \mod D \) is independent of the representative for \( u \).

We record the following lemma which is a slight generalization of Lemma A in [18] Section 6.

**Lemma 5.2.** Given odd prime \( p \) and \( x, y, z \in \mathbb{Z}, p \nmid z \), let \( \psi = (*|p) \) be the Legendre symbol for the prime \( p \). Then one has

\[ \sum_{j=1}^{p-1} \psi(j)e\left[\frac{jx^2 + j^{-1}y^2}{p}\right] = G(\psi; z) \frac{\psi(x^2) + \psi(y^2)}{1 + \psi(y^2)} \sum_{\gamma \mod p} e\left[\frac{2xyz\gamma}{p}\right]. \]

Set \( D^* = \frac{D}{c} \). We proceed to evaluate the left hand side of (4.1). Under our assumption that \( c|D \) and \( c > 0 \), it is easy to evaluate the matrix explicitly

\[ M_{u,v}(\sigma) = -i \sqrt{D} e\left[\frac{ax^2 - 2xy + dy^2}{Dc}\right] G(\psi; a) \delta_{x,dy} \mod c \]

if \( u, v \in [d_K] \) are represented by \( u = \left[\frac{ix}{\sqrt{D}}\right], v = \left[\frac{iy}{\sqrt{D}}\right] \) for some \( x, y \in \mathbb{Z} \).
Choose any \( h \in \mathbb{Z} \) such that \( h \equiv \begin{cases} b & \text{mod } c \\ c^{-1} & \text{mod } D^* \end{cases} \) and for any \( u \in [\mathfrak{d}_K] \), we set

\[
F_u := \sum_{\gamma \equiv D|\mathfrak{d}|^{2} \mod D^*} e \left[ \frac{2xh^2 \gamma}{D^*} \right] \quad \text{if } u = \left[ \frac{ix}{\sqrt{D}} \right].
\]

(The right hand side is the same for any \( x \in \mathbb{Z} \) as long as \( x^2 \equiv D|u|^2 \mod D \). Also, recall that we fix \( w \in [\mathfrak{d}_K] \).)

Then using Lemma 5.2, we find that for any fixed \( u \in [\mathfrak{d}_K] \), the inner sum over \( j \) on the left hand side of (4.1) equals

\[
A_u = c \delta^{\mod c} D|u|^2 D|dw|^2 F_u, w G(\psi_D^*; \psi)(a) e \left[ -|w|^2 dh - \frac{D|u|^2 ah^2}{D^*} \right].
\]

Take representative \( v = \left[ \frac{yw}{\sqrt{D}} \right] \in [\mathfrak{d}_K] \). By mean of (5.1) and (5.2) and change of variable from \( u = \left[ \frac{ix}{\sqrt{D}} \right] \) to \( x \mod D \), one has

\[
A = \sum_{x \mod D} \frac{1}{D^*} e \left[ \frac{ax^2 - 2xy + dy^2}{Dc} \right] G(\psi; a) \delta^{\mod c} \frac{D}{c} F \left[ \frac{ix}{\sqrt{D}} \right] G(\psi_D^*; \psi)(a) e \left[ -|w|^2 dh - \frac{x^2 ah^2}{D^*} \right].
\]

Here, note that \( G(\psi; a)G(\psi_D^*; \psi) = i \sqrt{D} \) since \( \psi(-1) \psi_D^*(-1) = \chi(-1) = -1 \).

Evidently, if \( D|v|^2 \equiv D|w|^2 \neq D|dw|^2 \mod c \) then the last sum is an empty sum (there is no \( x \) satisfying the two conditions) and thus is zero which is the same as \( \delta^{\mod c} D|v|^2 D|w|^2 \).

Let us consider the case \( D|v|^2 \equiv D|dw|^2 \mod c \) and the sum reduces to just sum over \( x \mod D \) such that \( x \equiv dy \mod c \). Making change of summation variable to \( x \mod D^* \) by identifying \( x \) with \( cx + dy \) and we continue with

\[
A = \sum_{x \mod D^*} \frac{1}{D^*} e \left[ \frac{a(cx + dy)^2 - 2(cx + dy)y + dy^2}{Dc} \right] - \frac{|w|^2 dh \left( \frac{D|u|^2 ah^2}{D^*} \right)}{\sum_{\gamma \mod D^*} e \left[ \frac{2(cx + dy)h^2 \gamma}{D^*} \right] e \left[ \frac{2(cx + dy)^2 h^2}{D^*} \right]} \]

The inner sum over \( x \mod D^* \) is either \( D^* \) or 0 depending on whether \( D^* \mid 2by + 2h \gamma - 2dyah \) or not. Note that \( D^* \mid 2by + 2h \gamma - 2dyah \) is equivalent to \( \gamma \equiv y \mod D^* \). And so the outer
sum over $\gamma$ is empty i.e. $A = 0$ if $y^2 = D|v|^2 \not\equiv D|w|^2 \mod D$. We have thus shown that
$A = 0$ if $D|v|^2 \not\equiv D|w|^2 \mod D$ i.e. if either $D|v|^2 \not\equiv D|w|^2 \mod D^*$ or $D|v|^2 \not\equiv D|w|^2 \mod c$; which matches the right hand side $\delta^D_{D|v|^2,D|w|^2}$ of (4.1) by Chinese Remainder Theorem.

In case $D|v|^2 \equiv D|w|^2 \mod D$, the sum over $\gamma$ is just the summand at $\gamma = y$ i.e.

$$A = e \left[ -\frac{D|w|^2dh}{D} + \frac{bdy^2}{D} + \frac{2dhy^2y - ad^2y^2h^2}{D^*} \right]$$

$$= e \left[ -hd + bd + 2cdh^2 - acd^2h^2 \right]$$

$$= 1$$

because $-hd + bd + 2cdh^2 - acd^2h^2 \equiv 0 \mod D$ by definition of $h$. It is obvious mod $c$ since $h \equiv b \mod c$. We have $c(-hd + bd + 2cdh^2 - acd^2h^2) \equiv d + cbd + 2d - ad^2 \equiv d - (ad-bc)d \equiv 0 \mod D^*$ and so $-hd + bd + 2cdh^2 - acd^2h^2 \equiv 0 \mod D^*$ as well.

This concludes our verification of (4.1); thus establishing the lifting when $D$ is odd.

5.2. The even discriminant case. Now we move on to the even discriminant case. In this case the representatives for $[D_K]$ can be taken from

$$\left\{ \frac{x_1}{2} + \frac{i x_2}{\sqrt{D}} \right\} \mod \frac{D}{2} \text{ and } x_2 \in \mathbb{Z}/(\frac{D}{2}) \mathbb{Z}$$

Decompose $D = 2^e D'$, $c = 2^f c'$ where $e = \text{val}_2(D), f = \text{val}_2(c)$ are 2-adic valuation of $D$ and $c$ respectively; in other words, $2 \nmid c'$ and $2 \nmid D'$. Let $c^*$ be the $c$ component of $D$ i.e. $c^* = 2^e c'$ if $f \geq 1$ and $c^* = c = c'$ if $f = 0$ i.e. $2 \nmid c$. (Note that $D'$ is square-free so $c'$ is also square-free.) Let $D^* := \frac{D}{c^*}, f_1 := \text{val}_2((D^*, D|w|^2))$,

$$f_2 := e - \log_2 \left( \frac{c^*}{c'} \right) = \begin{cases} 0 & \text{if } 2 \mid c, \\ e & \text{if } 2 \nmid c. \end{cases}$$

Note that $f_2$ is the power of 2 dividing $D^*$ for

$$D^* = \frac{D}{c^*} = \frac{2^e D'}{2^e - f_2 c'} = 2f_2 \frac{D'}{c'}$$

so

$$f_1 = \min(f_2, \text{val}_2(D|w|^2)).$$

For any $u \in [D_K]$, set

$$E_u := \begin{cases} G \chi_2(D|w|^2 + D|w|^2) \chi_2(D|w|^2) \chi_2(D|w|^2 + D|w|^2) & \text{if } 2 \nmid c \\ \delta_{D|w|^2}^{2e-1} \chi_2(D|w|^2) e \left[ \frac{D(D|w|^2 - D|w|^2) \mod \frac{D}{c'}}{2^e} \right] \chi_2(1 + ac) & \text{if } 2 \mid c \end{cases}$$

$$F_u := \sum_{\gamma \mod \frac{D}{c'}} \chi_2 \left( \frac{2x \frac{1}{cc} - f_2 \gamma}{\frac{D'}{c'}} \right) \text{ for any } x \text{ such that } x^2 \equiv D|w|^2 \mod \frac{D'}{c'}$$
\[ E'_u := \begin{cases} 
1 & \text{if } e - f_1 = 0, \\
(-1)^{D|u|^2} & \text{if } e - f_1 = 1, \\
(-1)^{D|u|^2/2} & \text{if } e - f_1 = 2 \text{ and } D|u|^2 \equiv 0 \text{ mod } 2, \\
(-1)^{\frac{D|u|^2 + D|w|^2}{2}} & \chi_2(-1) \text{ if } e - f_1 = 2 \text{ and } D|u|^2 \equiv 1 \text{ mod } 2 
\end{cases} \]

\[ K_u := e \left[ -\frac{D|u|^2}{c'} \frac{ab}{D/\psi} - \frac{D|u|^2 a_{cc'}}{D^*} - \frac{D|w|^2 d_{cc'}}{D^*} \right] \]

\[ = e \left[ -\frac{D|u|^2}{c'} \frac{ab}{D/\psi} - \frac{\frac{1}{D/\psi} D|u|^2 a_{cc'}}{D'/c'} - \frac{\frac{1}{D/\psi} D|w|^2 d_{cc'}}{D'/c'} - \frac{\frac{1}{D/\psi} D|w|^2 d_{cc'}}{2f_2} - \frac{1}{D/\psi} D|w|^2 d_{cc'} \right] \]

Then with Lemma 5.2 and a lot of computations, we find that the inner sum over \( j \) on the left hand side of (4.1) is likewise given by

\[ A_u = B_u + (1 - \delta_{f_1,\theta}) C_u \]

where

\[ B_u = c' \delta_{D|u|^2, D|du|^2} F_u K_u E_u \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \text{G}(\psi_{D^*}; 2f_2 c^* c) \psi_{c^*}(a) \]

\[ C_u = c \delta_{D|u|^2, D|du|^2} F_u K_u E_u' \frac{1}{1 + \chi_2(D|u|^2)} \text{G}(\psi_{D^*}); c^* c) \psi_{c^*}(a) \]

In the formula above, the \( B_u \) and \( (1 - \delta_{f_1,\theta}) C_u \) are the partition of the sum over \( j \) in the definition of \( A_u \) into those such that \( 2 \mid (D|w|^2, m) \) and \( 2 \mid (D|w|^2, D^*) \) respectively. Certainly, there exists \( j \) such that \( 2 \mid (D|w|^2, m) \) only when \( 2 \mid (D|w|^2, D^*) \) i.e. \( f_1 \neq 0 \); the fact we indicated by the factor \( (1 - \delta_{f_1,\theta}) \) in the formula.

Note that \( 1 - \delta_{0, f_1} \neq 0 \) only when \( c \) is odd and \( 2 \mid D|w|^2 \); in which case, \( f_2 = e, \text{G}(\psi_{D^*}; 2f_2 c^* c) = \text{G}(\psi_{D^*}; 2e), \text{G}(\psi_{D^*}) = \text{G}(\psi_{D^*}; 2e) \text{G}(\chi_2; \frac{D^*}{2e}) \) and notice that \( \frac{D^*}{2e} = \frac{D'}{c'} \) so if \( 1 - \delta_{0, f_1} \neq 0 \), we have a simpler formula

\[ A_u = c' \delta_{D|u|^2, D|du|^2} F_u K_u E_u \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \text{G}(\psi_{D^*}; 2f_2 c^* c) \psi_{c^*}(a) [E_u + (1 - \delta_{f_1,\theta}) E_u' \text{G}(\chi_2; D'/c')] \]

As for the matrix, let \( u = \left[ \frac{a_1}{2} + \frac{ia_2}{\sqrt{D}} \right] \) and \( v = \left[ \frac{b_1}{2} + \frac{ib_2}{\sqrt{D}} \right] \) be representatives then we have

\[ M_{u,v}(\sigma) = e \left[ \frac{D/\psi}{c'} \frac{bdy_2^2}{c} \right] e \left[ \frac{D/\psi f}{i\sqrt{D}} \right] e \left[ \frac{1}{D'|c'} (ax_2^2 - 2x_2y_2 + dy_2^2) \right] M^*_{u,v}(\sigma) \]

where the factor

\[ M^*_{u,v}(\sigma) := 2f^2 \delta_{x_2, dy_2} \left[ \frac{1}{c'} (ax_1^2 - 2x_1y_1 + dy_1^2) \right] + \frac{1}{D'|c'} (ax_2^2 - 2x_2y_2 + dy_2^2) \]

Here, we take a convention that \( \delta_{X,Y}^{mod 1/2} = 1 \) for all \( X, Y \in \mathbb{Z} \).

Since \( D = 2^e c' (\frac{D'}{c'}) \) and the three factors are relatively prime,

\[ \delta_{D|u|^2, D|u|^2}^{mod D} = \delta_{D|u|^2, D|u|^2}^{mod 2^e} \times \delta_{D|u|^2, D|u|^2}^{mod c'} \times \delta_{D|u|^2, D|u|^2}^{mod D'/c'} \]
Similar to the odd discriminant case, we show that \( A = \delta^{\text{mod } D}_{D|v|^2, D|w|^2} \) by showing that \( A = 1 \) only when \( D|v|^2 \equiv D|w|^2 \mod c' \), \( D|v|^2 \equiv D|w|^2 \mod \frac{D'}{c'} \) and \( D|v|^2 \equiv D|w|^2 \mod 2c' \); and \( A = 0 \) otherwise.

In all cases, we see that in order that \( \frac{M_{\omega,s}(\sigma)}{D}A_u \neq 0 \), \( u \) must satisfies the two congruences \( D|u|^2 \equiv D|dw|^2 \mod c' \) and \( x_2 \equiv dy_2 \mod c' \). The second condition implies \( D|v|^2 \equiv D|dv|^2 \mod c' \). Thus, if \( D|v|^2 \neq D|w|^2 \mod c' \), \( A = 0 \). So let us now assume that \( D|v|^2 \equiv D|w|^2 \mod c' \) and the sum reduces to over \( u \) such that \( x_2 \equiv dy_2 \mod c' \) (for then \( D|u|^2 \equiv D|dw|^2 \mod c' \) holds automatically). Identifying \( u \) with a pair \((x_1, x_2) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(\frac{D}{2})\mathbb{Z}\), we can rewrite the sum

\[
A = \frac{G(\psi_c; 2f)G(\psi_{D'}; 2e+f)}{i2^e \sqrt{D}} A^+ A^-
\]

where

\[
A^+ = \frac{1}{D'/c'} \sum_{x_2 \equiv dy_2 \mod c'} F_{x_2} \left[ \frac{1}{D'/c'} \left( ax_2^2 - 2x_2y_2 + dy_2^2 \right) \right]
\]

\[
A^- = \sum_{x_1 \mod 2, x_2 \mod 2^{e-1}} \left[ -\frac{1}{D'/c'} D|u|^2 \frac{a}{c'} - \frac{1}{D'/c'} D|w|^2 \frac{d}{c'} \right] \frac{1 + \chi_2(D|w|^2)}{2f} \times
\]

\[
\left[ E_u + (1 - \delta f_1, 0) E'_u \right] \frac{G(\psi_c; 2f)G(\psi_{D'}; 2e+f)}{i2^e \sqrt{D}} A^+ A^-
\]

We shall show that

\[
(5.3) \quad A^+ = \delta^{\text{mod } D'}/c'
\]

and

\[
(5.4) \quad A^- = \delta^{\text{mod } 2e}_{D|v|^2, D|w|^2}
\]

which completes the proof that \( A = \delta^{D}_{D|v|^2, D|w|^2} \). But first, using property of Gauss sum and quadratic reciprocity, one can simplify (5.4) to

\[
(5.5) \quad A^- = \delta^{\text{mod } 2e}_{D|v|^2, D|w|^2} (-1)^{(e-1)(D'/c'-1)} \psi_{D'}(2f) \psi_c(2e) 2^e \psi(\psi_2; D')
\]

We compute \( A^+ \) to establish (5.3):

\[
A^+ = \frac{1}{D'/c'} \sum_{x_2 \equiv dy_2 \mod c'} \sum_{x_2 \equiv dy_2 \mod c'} e \left[ \frac{2x_2^2 - 2f^2 \gamma_2}{D'/c'} \right] \times
\]

\[
\times e \left[ \frac{1}{D'/c'} \left( ax_2^2 - 2x_2y_2 + dy_2^2 \right) - \frac{1}{D'/c'} D|w|^2 \frac{a}{c'} \right]
\]

since \( 2f^2 \equiv 2^e c'^2 - 2f \gamma_2 = 2^e c'^2 \).
Thus, we showed that $A = 0$ if $D|v|^2 \neq D|w|^2 \mod D'/c'$.

To prove (5.5), observe that it is in principle a finite check since we have finitely many (but a lot of) possibilities for $e, f$ and $a, b, c, d, D', D|v|^2, D|v|^2$ mod $2^e$. Note that $D|u|^2 \equiv 2e^{-2}D'x_2^2 + x_2^2 \mod 2^e$ only depend on $x_1$ mod 2 and $x_2$ mod $2^{e-1}$. Also, we must have $\chi_2(D|u|^2) \neq -1$ so $\chi_2(D|u|^2) = 0$ if $2 \mid D|u|^2$ and is 1 otherwise. In other words, $\chi_2(D|u|^2) = \delta_{D|u|^2,1}$. In particular, if $e = 2$ then $D|u|^2 \equiv 1$ mod 4 if $2 \nmid D|u|^2$. For illustration, we establish (5.5) in the case $e = 2$. The case $e = 3$ is similarly solved by case-by-case analysis.

From now on, we assume $e = 2$ and “left hand side” (LHS) and “right hand side” (RHS) refers to that of equation (5.5). As the matrix $\sigma$ is fixed, we drop it in the notation $M^*_{u,v}(\sigma)$.

Note that when $e = 2$, we must have $D' \equiv 1$ mod 4 so that $D = 2eD' = 4D'$ is a fundamental discriminant.

### 5.2.1. The case $f = 2$ and $e = 2$.

Note that in this case, $c$ is even so $a, d$ are both odd. Then

$$A^- = \sum_{x_1 \mod 2 \atop x_2 \mod 2^{e-1}} \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} E_u M^*_{u,v}(\sigma).$$

since $f_2 = 0$ and $1 - \delta_{f_1,0} = 0$. (Note that the above holds as long as $2 \mid c$ i.e. $f \geq 1$.)

By definition, $M^*_{u,v}(\sigma) \neq 0$ only when $x_1 \equiv dy_1 \mod 2$ and $x_2 \equiv dy_2 \mod 2$ in which case $\chi_2(D|u|^2) = \chi_2(D^2D|v|^2) = \chi_2(D|v|^2)$ and (5.6) reduces to

$$A^- = \sum_{x_2 \mod 2^{e-1} \atop x_1 = dy_1, x_2 \equiv dy_2 \mod 2} \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|v|^2)} E_u M^*_{u,v}(\sigma)$$

$$= \sum_{x_2 \mod 2^{e-1} \atop x_1 = dy_1, x_2 \equiv dy_2 \mod 2} E_u M^*_{u,v}(\sigma)$$

Likewise, observe that $E_u \neq 0$ only when $D|v|^2 - D|w|^2 \equiv c \mod 2^{e-1}$ so we must have $D|v|^2 \equiv D|w|^2 \mod 2^{e-1}$ in order that $A^- \neq 0$. It also follows from the definition of

$$E_u = \delta_{D|u|^2 - D|w|^2, c} \chi_2(a) e \left[ \frac{D'D|w|^2ab}{2^e} \right] \times \left( 1 + e \left[ \frac{D'(D|u|^2 - D|w|^2)(2bc + 1) - D|w|^2cd}{2^e} \right] \chi_2(1 + ac) \right)$$
that we further need \( D|v|^2 \equiv D|w|^2 \text{ mod } 2^e \) in order that the last factor \( 1 + \epsilon [\ldots] \chi_2(1 + ac) \neq 0 \). Hence, \( A^- = 0 \) unless \( D|v|^2 \equiv D|w|^2 \text{ mod } 2^e \).

Now let us assume \( D|v|^2 \equiv D|w|^2 \text{ mod } 2^e \). We find that
\[
E_u = 2^e \chi_2(a) e \left[ -\frac{D'D|w|^2ab}{2^e} \right]
\]
is independent of \( u \) (if its parameter satisfies the summation condition \( x_1 \equiv dy_1 \text{ mod } 2 \) and \( x_2 \equiv dy_2 \text{ mod } 2 \)) and so in case \( e = f = 2 \), one has
\[
A^- = 2^e \chi_2(a) e \left[ -\frac{D'D|w|^2ab}{2^e} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|v|^2)} \sum_{x_2 \text{ mod } 2^e-1} M_{u,v}^*(\sigma)
\]
\[
= 2^e \chi_2(a) e \left[ -\frac{D'D|w|^2ab}{2^e} \right] e \left[ \frac{1}{4} (a(dy_1)^2 - 2(dy_1)y_1 + (dy_1)^2) + \frac{1}{16} (d'(dy_2)^2 - 2(dy_2)y_2 + (dy_2)^2) \right]
\]
\[
= 2^e \chi_2(a) e \left[ -\frac{D'D|w|^2ab}{2^e} \right] e \left[ \frac{bdy_1^2}{4} + \frac{bd'y_2^2}{4} \right] \left( 1 + e \left[ \frac{ac'}{4} \right] \right)^2
\]
\[
= 2^e \chi_2(a) \left( 1 + e \left[ \frac{ac'}{4} \right] \right)^2
\]
\[
= 2^e \chi_2(a) \left( 1 + 2e \left[ \frac{ac'}{4} \right] + e \left[ \frac{ac'}{2} \right] \right)
\]
\[
= 2^e \chi_2(a) e \left[ \frac{ac'}{4} \right] = 2i\chi_2(a)\chi_2(ac') = 2^{e+1} i\chi_2(c').
\]

In this case, the right hand side of (5.5) is \( 2^e (-1)^{(e'-1)/2} = 2^e i\chi_2(c') \). That is because \( G(\psi_2, D') = 2i \) and since \( D' \equiv 1 \text{ mod } 4 \), we find that \( D' \equiv c' \text{ mod } 4 \) and so
\[
(-1)^{(e'-1)/(D'/c'-1)/4} = (-1)^{(e'-1)/2} = (-1)^{(e'-1)/2}.
\]

So we are done.

5.2.2. The case \( f = 1 \) and \( e = 2 \). For the case \( f = 1 \), we have
\[
M_{u,v}^* = 2^{2e-2} D'^{x_1 \equiv x_1 - dy_1 \text{ mod } 2} e \left[ \frac{1}{8} (ax_1^2 - 2x_1y_1 + (dy_1)^2) + \frac{1}{2e} (ax_2^2 - 2x_2y_2 + (dy_2)^2) \right]
\]
So \( M_{u,v}^* \neq 0 \) only for \( x_2 = dy_2 + D' \equiv dy_2 + 1 \text{ mod } 2 \) and \( x_1 = dy_1 + 1 \text{ mod } 2 \) (then \( D|v|^2 \equiv (dy_1 + 1)^2 + (dy_2 + 1)^2 \equiv D|v|^2 + 2d(y_1 + y_2) + 2 \text{ mod } 4 \) so \( D|v|^2 \equiv D|w|^2 \text{ mod } 2 \) and we still have \( \chi_2(D|v|^2) = \chi_2(D|w|^2) \). We likewise find that with such conditions on \( x_1, x_2 \), the factor \( E_\ell \neq 0 \) only when \( D|v|^2 \equiv D|w|^2 \text{ mod } 4 \): first, we need \( D|v|^2 \equiv D|w|^2 \text{ mod } 2 \) so
\[ D|v|^2 \equiv D|w|^2 \mod 2 \] and given that, the factor
\[
1 + e \left[ \frac{D'(D|v|^2 - D|w|^2(2bc + 1) - D|w|^2cd)}{2e} \right] \chi_2(1 + ac)
\]
\[
= 1 + e \left[ \frac{D|v|^2 + 2d(y_1 + y_2) + 2 - D|w|^2 - 2D|w|^2c'd'}{4} \right] (-1)^{ac'} \text{ since } D' \equiv 1 \mod 4 \text{ and } c = 2c'
\]
\[
= 1 - e \left[ \frac{D|v|^2 - D|w|^2}{4} + \frac{d(y_1 + y_2) + 1 - D|w|^2c'd'}{2} \right]
\]
in the definition of \( E_u \) is zero unless \( D|v|^2 \equiv D|w|^2 \mod 4 \). When \( D|v|^2 \equiv D|w|^2 \mod 4 \), the value of the above factor is 2. So if \( D|v|^2 \equiv D|w|^2 \mod 4 \) then the summation in (5.7) is just a singleton where \( x_1 = dy_1 + 1 \) and \( x_2 = dy_2 + 1 \):
\[
A^- = 2^e \chi_2(a) e \left[ -\frac{D'D|w|^2ab}{2e} \right] 2^e \left[ \frac{\frac{1}{4}(a(dy_1 + 1)^2 - 2(dy_1 + 1)y_1 + dy_1^2)}{2^e+1} \right] \times \\
\times e \left[ \frac{\frac{1}{4}(a(dy_2 + 1)^2 - 2(dy_2 + 1)y_2 + dy_2^2)}{8} \right] \\
= 2^{e+1} \chi_2(a) e \left[ -\frac{D|v|^2ab}{4} + \frac{\frac{1}{4}(2(ad - 1)y_1 + (ad - 1)dy_1^2 + a)}{8} \right] \times \\
\times e \left[ \frac{D'(2(ad - 1)y_2 + (ad - 1)dy_2^2 + a)}{8} \right] \\
= 2^{e+1} \chi_2(a) e \left[ -\frac{D|v|^2ab}{4} + \frac{\frac{1}{4}(2 \cdot 2bc' \cdot y_1 + 2bc'dy_1^2 + a) + D'(2 \cdot 2bc' \cdot y_2 + 2bc'dy_2^2 + a)}{8} \right] \\
= 2^{e+1} \chi_2(a) e \left[ -\frac{D|v|^2ab}{4} + \frac{4by_1 + 2bdy_1^2 + D'(4by_2 + 2bdy_2^2) + ac' + aD'}{8} \right] \\
= 2^{e+1} \chi_2(a) e \left[ -\frac{D|v|^2ab}{4} + \frac{by_1 + D'by_2}{2} + \frac{bdy_1^2 + D'bdy_2^2}{4} + \frac{ac'(D' + 1)}{8} \right] \\
= 2^{e+1} \chi_2(a) e \left[ \frac{b((d - a)D|v|^2 + 2(y_1 + y_2))}{4} + \frac{ac'(D' + 1)}{8} \right] \\
= 2^{e+1} \chi_2(a) e \left[ \frac{c'a(D' + 1)}{8} \right]
\]
Note here to derive the last equation above, we reason that \( a \equiv d \equiv 1 \mod 2 \) so we either have \( a - d \equiv 2 \mod 4 \) whence \( \frac{d-a}{2} D|v|^2 + y_1 + y_2 \equiv D|v|^2 + y_1 + y_2 \equiv y_1^2 + y_2^2 + y_1 + y_2 \equiv 0 \mod 2 \) and so \( b((d - a)D|v|^2 + 2(y_1 + y_2)) \equiv 0 \mod 4 \); or \( a - d \equiv 0 \mod 4 \) whence \( ad \equiv 1 \mod 4 \) and so we must have \( 2 \mid b \) which also leads to \( b((d - a)D|v|^2 + 2(y_1 + y_2)) \equiv 0 \mod 4 \) since the other factor is clearly divisible by 2 and so
\[
\frac{b((d - a)D|v|^2 + 2(y_1 + y_2))}{4} \in \mathbb{Z}
\]
always.
Once again, we match the RHS perfectly as in the previous case for $e = f = 2$. The difference is the involvement of the $\psi_{D'}(2)$ since the exponent $f$ is odd:

$$(-1)^{(e'-1)(D'/c'-1)}\psi_{D'}(2)^f \psi_{c'}(2^e)2^e \mathbf{G}(\chi_2; D') = (-1)^{(e'-1)(D'/c'-1)}\psi_{D'}(2)2^{e+1}i$$

and we observe that

$$e \left[ \frac{c'a(D' + 1)}{8} \right] = e \left[ \frac{c'aD' + 1}{4} \right]$$

$$\begin{align*}
&= \begin{cases} 
  e \left[ \frac{c'a}{4} \right] & \text{if } \frac{D' + 1}{2} \equiv 1 \mod 4 \iff D' \equiv 1 \mod 8 \\
  e \left[ -\frac{c'a}{4} \right] & \text{if } \frac{D' + 1}{2} \equiv -1 \mod 4 \iff D' \equiv 5 \mod 8 
\end{cases} \\
&= \psi_{D'}(2)e \left[ \frac{c'a}{4} \right] \text{ since } e \left[ \frac{c'a}{4} \right] \in \{\pm i\}
\end{align*}$$

and so (5.5) reduces to

$$\chi_2(a)e \left[ \frac{c'a}{4} \right] = (-1)^{(e'-1)(D'/c'-1)}i = \chi_2(c')i$$

which was established in the previous case.

5.2.3. The case $f = 0$ and $e = 2$. If $f = 0$ then the equation (5.5) reads

$$\sum_{x_1 \mod 2, x_2 \mod 2^{e-1}} e \left[ -\frac{1}{D'/c'}D|u|^2 \frac{a}{w^c} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \times$$

$$\times [\mathbf{G}(\chi_2; D') \chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f,0})E'_u \mathbf{G}(\chi_2; D'/c')]M_{u,v}^*$$

$$= \delta_{D|u|^2,D|w|^2}(1)\psi_{c'}(2^e)2^e \mathbf{G}(\chi_2; D') \times$$

$$\begin{align*}
&= \sum_{x_1 \mod 2, x_2 \mod 2^{e-1}} e \left[ -\frac{1}{D'/c'} \left( 2^{e-2}D'x_1^2 + \frac{a^2}{d} \right) + \frac{1}{D'/c'}D|w|^2 \frac{d}{c} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \times \\
&\times [\chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f,0})E'_u] \times \\
&\times e \left[ \frac{1}{4} (\frac{ax_2^2 - 2x_1y_1 + dy_1^2}{d} + \frac{1}{D'/c'}(ax_2^2 - 2x_2y_2 + dy_2^2)) \right]
\end{align*}$$

(5.8)

whose left hand side can be further simplified to

$$(1 + \chi_2(D|w|^2))e \left[ -\frac{d}{D'/c'}(D|u|^2 - D|w|^2) \right] \times$$

$$\times \sum_{x_1 \mod 2, x_2 \mod 2^{e-1}} \chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f,0})E'_u \times$$

$$\begin{align*}
&\times e \left[ -\frac{1}{e} x_1y_1 \frac{1}{2} - \frac{1}{D'/c'}x_2y_2 \right]
\end{align*}$$
\[
\begin{align*}
&\frac{1}{4} \left( \frac{1}{d/d'c} \left( \frac{e}{2} \right) \right) + \frac{1}{d/d'c} \left( \frac{e}{2} \right) e = \frac{1}{4} \left( \frac{1}{d/d'c} \left( \frac{e}{2} \right) \right) e = \frac{d}{d/d'c} \left( \frac{e}{2} \right) e = \frac{d}{d/d'c} D|v|^2.
\end{align*}
\]

If \( 2 \mid D|w|^2 \) then \( \chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{q,0}) E_u = \chi_2(D|u|^2 + D|w|^2) \) is non-zero only when \( D|u|^2 \equiv 0 \, \text{mod} \, 2 \); in other words, we need \( x_1 \equiv x_2 \, \text{mod} \, 2 \). In this case, we always have \( \chi_2(D|w|^2) = 1 \) and \( \chi_2(D|u|^2) = 0 \). So the left hand side of (5.8) is just
\[
2e \left[ \frac{1}{d/d'c}(D|u|^2 - D|w|^2) \right] \sum_{x_1 \, \text{mod} \, 2 \atop x_2 \, \text{mod} \, 2 \, \text{and} \, x_2 = x_1} \chi_2(D|u|^2 + D|w|^2) e \left[ \frac{1}{d/d'c}(-2x_1 y_1) + \frac{1}{d/d'c}(-2x_2 y_2) \right]
\]

If \( e = 2 \) then note that \( D' \equiv 1 \, \text{mod} \, 4 \) and \( D|w|^2 \equiv 1 \, \text{mod} \, 4 \) and so
\[
\begin{align*}
&\sum_{x_1 \, \text{mod} \, 2 \atop x_2 = x_1} \chi_2(D|u|^2 + D|w|^2) e \left[ \frac{1}{d/d'c}(-2x_1 y_1) + \frac{1}{d/d'c}(-2x_2 y_2) \right] \\
= &\sum_{x_1 \, \text{mod} \, 2 \atop x_2 = x_1} \chi_2(2x_1^2 + D|w|^2) e \left[ \frac{1}{d/d'c}(-2x_1 y_1) + \frac{1}{d/d'c}(-2x_2 y_2) \right] \\
= &\sum_{x_1 \, \text{mod} \, 2 \atop x_2 = x_1} (-1)^{2x_1^2 + D|w|^2} e \left[ \frac{-1}{d/d'c}(y_1 + y_2)x_1 \right] \\
= &\sum_{x_1 \, \text{mod} \, 2 \atop x_2 = x_1} (-1)^{2x_1^2} (-1)(y_1 + y_2)x_1 \\
= &1 - (-1)^{y_1 + y_2}
\end{align*}
\]

so \( A^- = 0 \) unless \( y_1 + y_2 \equiv 1 \, \text{mod} \, 2 \), in which case \( D|w|^2 \equiv 1 \, \text{mod} \, 2 \) and so \( D|u|^2 \equiv D|w|^2 \equiv 1 \, \text{mod} \, 4 \). When that happen the left hand side is just 4. And the remaining factor on the right hand side \((-1)^{(c'-1)(D'/c'-1)/4} \psi_c(2^e) \chi_2(c) = 1\). That is because we must have \( c' \equiv D'/c' \, \text{mod} \, 4 \) here and so \((-1)^{(c'-1)(D'/c'-1)/4} = (-1)^{(c'-1)/2} = \chi_2(c)\).

Now let us consider \( 2 \mid D|w|^2 \). We either have \( e - f_1 = 0 \) i.e. \( D|w|^2 \equiv 0 \, \text{mod} \, 4 \) whence \( E_u = 1 \) or \( e - f_1 = 1 \) i.e. \( D|w|^2 \equiv 2 \, \text{mod} \, 4 \) whence \( E_u = (-1)^{D|w|^2} \). And so
\[
\begin{align*}
\text{LHS}_{(5.8)} &= e \left[ \frac{-d}{d/d'c} \left( \frac{e}{2} \right)(D|u|^2 - D|w|^2) \right] \sum_{x_1, x_2 \, \text{mod} \, 2} \frac{\chi_2(D|u|^2 + D|w|^2) + E_u}{1 + \chi_2(D|u|^2)} e \left[ -\frac{1}{d/d'c}x_1 y_1 \right] - \frac{1}{d/d'c}x_2 y_2 \right] \\
= &e \left[ \sum_{x_1, x_2 \, \text{mod} \, 2} \frac{\chi_2(D|u|^2 + D|w|^2) + E_u}{1 + \chi_2(D|u|^2)} (-1)^{x_1 y_1 + x_2 y_2} \right. \\
= &e \left[ \sum_{x_1 \, \text{mod} \, 2 \atop x_2 = x_1} \frac{\chi_2(D|u|^2 + D|w|^2) + E_u}{1 + \chi_2(D|u|^2)} (-1)^{x_1 y_1 + x_2 y_2} \right. \\
\end{align*}
\]
is non-zero only when $y$ mod 2. Together with the second condition, we get $|D|w|^2 = 0$ in case (n,q) = 0 mod 4, $|D|w|^2 \equiv 0$ mod 4,

and $|D|w|^2 \equiv 0$ mod 4. The first implies $y_1 \equiv y_2$ mod 2; together with the second condition, we get $|D|w|^2 = 0$ mod 4. So we find that LHS of (5.8) is 0 if $|D|v|^2 \equiv D|w|^2$ mod 4 and is 4 if $|D|v|^2 \equiv D|w|^2$ mod 4 in case 2 | $D|w|^2$ as well.

5.3. The map from elliptic forms to plus forms. With Berger-Klosin’s result, we have completed most of the Saito-Kurokawa lift. All that remains is to construct a surjective linear map

$$\mathcal{G}_{k-1}(DN, \chi) \to \mathcal{G}_{k-1}(DN, \chi).$$

For this linear map, we modify Ikeda’s construction in [12]. We recall the notation of [12]. For any positive integer $M$ and prime $q$, we denote

$$M_q = q^{val_q(M)}$$

where val$_q$ is the $q$-adic valuation and for a Dirichlet character $\psi$ modulo $M$, we denote $\psi_q$ to be the character mod $M_q$ defined by $\psi_q(n) = \psi_q(n')$ for $(n, q) = 1$ where $n'$ is any integer such that

$$n' \equiv \begin{cases} n \quad \text{mod } M_q \\ 1 \quad \text{mod } M/M_q \end{cases}$$

and $\psi_q(n) = 0$ in case $(n, q) > 1$. Let $\psi_0$ be the character $\psi/\psi_q$. (When $\psi = \chi_K$, this notation agrees with what we used previously in section 2.) We remark that our weight $k-1$ corresponds to the weight $2k + 1$ in Ikeda’s paper.
As in [12], we observe that for a primitive form \( f \in \mathfrak{S}_{k-1}(Dm, \chi) \) and every subset \( Q \subset Q_D := \{ \text{prime divisors of } D \} \), there exists a primitive form \( f_Q \) whose Fourier coefficients at prime \( p \) is given by

\[
a_{f_Q}(p) = \begin{cases} \chi_Q(p)a_f(p) & \text{if } p \not\in Q, \\ \chi'_Q(p)a_f(p) & \text{otherwise.} \end{cases}
\]

where \( \chi_Q := \prod_{q \in Q} \chi_q \) and \( \chi'_Q := \prod_{q \in Q_D \setminus Q} \chi_q = \frac{\chi}{\chi_Q} \).

For any primitive form \( f \in \mathfrak{S}_{k-1}(Dm, \chi) \) where \( m \mid N \) and for any \( \ell \) such that \( \ell m \mid N \), we define

\[
f^*[\ell] := \sum_{Q \subset Q_D} \chi_Q(-\ell)f_Q
\]

Let us recycle the notation \( \sigma_{\ell} := \left( \begin{array}{c} \ell \\ 1 \end{array} \right) \). Then any general form \( f \in \mathfrak{S}_{k-1}(DN, \chi) \) can be expressed uniquely as

\[
f = \sum_{i \in I} \alpha_i \left( f_i |_{k-1} \sigma_{\ell_i} \right)
\]

where \( I \) is some finite indexing set, \( \alpha_i \in \mathbb{C}^\times \), \( f_i \in \mathfrak{S}_{k-1}(Dm_i, \chi) \) are primitive forms (under appropriate independence assumption) and \( m_i, \ell_i \) are natural numbers such that \( m_i \ell_i \mid N \) and we set

\[
f^* := \sum_{i \in I} \alpha_i \left( f_i^*[\ell_i] \mid \sigma_{\ell_i} \right).
\]

It is easy to verify that \( f^* \in \mathfrak{S}_{k-1}^+(DN, \chi) \). This is because Lemma 15.4 in [12] (which we restate below) holds verbatim and Corollary 15.5 follows with a minor twist:

**Lemma 5.3** (Lemma 15.4 for higher level). Let \( f \in \mathfrak{S}_{k-1}(Dm, \chi) \) be a primitive form with \( (m, D) = 1 \). For any \( M \in \mathbb{Z} \), we split the prime factorization of \( M \) into primes not dividing \( D \), primes dividing \( D \) and primes dividing \( D \) in \( Q_D \):

\[
M' = \prod_{p \not\in D} p^{\text{val}_p(M)} \quad M'_Q = \prod_{p \mid D, p \not\in Q} p^{\text{val}_p(M)} \quad M_Q = \prod_{p \mid D, p \in Q} p^{\text{val}_p(M)}
\]

as in the notation of Ikeda. Then the \( M \)-th Fourier coefficient of \( f_Q \) are related to that of \( f \) by

\[
a_{f_Q}(M) = a_f(M'M'_Q)a_f(M_Q)\prod_{q \in Q} \chi_q(M)
\]

where \( \chi = \bigotimes' \chi_q \) is the idele character corresponding to the quadratic character \( \chi \).

**Corollary 5.4.** Let \( f \) be as in previous lemma. Then

\[
a_{f^*[\ell]}(M) = a_f(M') \prod_{q \mid D} (a_f(M_q) + \chi_q(\ell)\overline{\chi_q(-M)a_f(M_q)})
\]

\[
= a_f(M')a_D(\ell M) \prod_{q \mid (D, M)} (a_f(M_q) + \chi_q(-1)\chi_q(\ell)a_f(M_q)\overline{\chi_q(M)})
\]

With this computation, it is evident that the form \( f^*[\ell]|_{\sigma_{\ell}} \in \mathfrak{S}_{k-1}^+(Dm\ell, \chi) \) as \( a_D(\ell^2 M) = a_D(M) \) as long as \( (D, \ell) = 1 \).
Proposition 5.5 (Ikeda [12] Proposition 15.17 for higher level). The map $\mathcal{G}_{k-1}(DN, \chi) \to \mathcal{G}_{k-1}^+(DN, \chi)$ where $f \mapsto f^*$ is surjective.

Proof. Suppose that $g \in \mathcal{G}_{k-1}^+(DN, \chi)$ which we can express as

$$g = \sum_{i \in \mathcal{I}} \alpha_i g_i |_{k-1} \sigma_{\ell_i}$$

with similar meaning for $I, \alpha_i$; for example, $g_i$’s are primitive forms in $\mathcal{G}_{k-1}(Dm_i, \chi)$ and $m_i \ell_i | N$. Following Ikeda, for each subset $Q \subset Q_D$, we consider the form

$$g' = \sum_{i \in \mathcal{I}} \alpha_i \chi_Q(-\ell_i)(g_i)_Q | \sigma_{\ell_i}$$

and observe that $g - g'$ has its $n$-th Fourier coefficient vanishing for all $(n, D_Q) = 1$ (i.e. $q \nmid n$ for all $q \in Q$; equivalently, $n_Q = 1$). To see that, one has

$$a_{g-g'}(n) = \sum_{i \in \mathcal{I}} \alpha_i a_{g_i}(n/\ell_i) - \sum_{i \in \mathcal{I}} \alpha_i \chi_Q(-\ell_i) a_{(g_i)_Q}(n/\ell_i)$$

$$= \sum_{i \in \mathcal{I}} \alpha_i [a_{g_i}(n/\ell_i) - \chi_Q(-\ell_i) a_{g_i}((n/\ell_i)'(n/\ell_i)'_Q) a_{g_i}((n/\ell_i)_Q) \prod_{q \in Q} \chi_q(n/\ell_i)]$$

by Lemma 5.3

$$= \sum_{i \in \mathcal{I}} \alpha_i [a_{g_i}(n/\ell_i) - \chi_Q(-n) a_{g_i}(n/\ell_i)]$$

$$= \sum_{i \in \mathcal{I}} \alpha_i [1 - \chi_Q(-n)] a_{g_i}(n/\ell_i)$$

$$= [1 - \chi_Q(-n)] \left( \sum_{i \in \mathcal{I}} \alpha_i a_{g_i}(n/\ell_i) \right)$$

$$= [1 - \chi_Q(-n)] a_g(n)$$

Note the fact that $(n/\ell_i)_Q = 1$ due to $(n, Q) = n_Q = 1$ whence $(n/\ell_i)'(n/\ell_i)'_Q = n/\ell_i$. Now by assumption $a_g(n) = 0$ whenever $a_D(n) = 0$, we find that $a_{g-g'}(n) = 0$ if $a_D(n) = 0$. If $(n, Q) = 1$ and $a_D(n) \neq 0$ then we must have $\chi_q(-n) \neq -1$ for all $q | D$; in other words, either $\chi_q(-n) = 0$ or $\chi_q(-n) = 1$ for all $q | D$. We can’t conclude that $\chi_q(-n) = 1$ for all $q | D$ but this should be the case for all $q \in Q$ by assumption $(n, Q) = 1$. And so the factor $1 - \chi_Q(-n) = 0$ and we still have $a_{g-g'}(n) = 0$.

We have shown that $g - g' \in \mathcal{G}_{k-1}(DN, \chi)$ has vanishing $n$-th Fourier coefficient for all $(n, D_Q) = 1$. By Miyake [25] Theorem 4.6.8, we find that $g - g' = 0$. Thus, we proved that

$$g = \sum_{Q \subset Q_D} \sum_Q \alpha_i \chi_Q(-\ell_i)(g_i)_Q | \sigma_{\ell_i}$$

for every $Q \subset Q_D$. Summing both sides over all $Q$, we get

$$2^{Q_D} g = \sum_{Q} \sum \alpha_i \chi_Q(-\ell_i)(g_i)_Q | \sigma_{\ell_i}$$

\(^1\)Here, we take a convention that $a_g(r) = 0$ if $r \notin \mathbb{Z}$. 
\[ g = \sum_{i=1}^{\alpha_i g_i^* [\ell_i]} \sigma_{\ell_i} \]

so \( g \) is in the image of the map. \( \square \)

**Remark 5.6.** We remark that when \( D \) is prime and \( N = 1 \), we get Krieg's remark mentioned in the introduction. For then, \( Q = \{ D \} \), \( \mathcal{S}_{k-1}(DN, \chi) \) has a basis of primitive form and we have \( f_0 = f \) and \( f_Q = f^p \) for any primitive form \( f \). And we find that \( f^* = f_0 - f_Q = f - f^p \).

6. Hecke equivariant

Now we establish Hecke equivariance of our Maass lift for certain Hecke operators.

First, we characterize the Maass space, similar to Andrianov’s characterization of Maass space in [3]. The following lemma is generalization of the Lemma in [18], Section 7 for higher level Maass space. Intuitively, it expresses the fact that the Fourier coefficients of the hermitian modular forms in the Maass space only depends on \( \epsilon(T) \) and \( \det(T) \), which is evident in Definition 22.

**Lemma 6.1.** Suppose that \( F \in \mathcal{M}_{k,2}(N) \) with Fourier expansion
\[
F(Z) = \sum_{T \in S_2(Q)} c_F(T) e^{\text{Tr}(TZ)}.
\]

Then \( F \in \mathcal{M}_{k}^*(N) \) if and only if there exists a function \( \beta : \mathbb{Z}^+ \times \mathbb{Z}_{\geq 0} \to \mathbb{C} \) such that

(i) for all \( T \in S_2(Q), T \geq 0, T \neq 0 \):
\[
c_F(T) = \beta(\epsilon(T), D^{\det(T)}{\epsilon(T)^2})
\]

(ii) for all \( d \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}^+ \) and all primes \( p \nmid Nq \):
\[
(1 - p^{k-1}W) \sum_{v=0}^{\infty} \beta(p^v q, d)W^v = \sum_{v=0}^{\infty} \beta(q, p^{2v}d)W^v
\]

holds as a formal power series in \( W \); in other words, \( \beta(p^v q, d) - p^{k-1} \beta(p^{v-1} q, d) = \beta(q, dp^{2v}) \) for all \( v \geq 0 \).

(iii) \( \beta(u, v) = \beta(1, vu^2) \) for all \( u \mid N^\infty \), i.e. if every prime divisor of \( u \) divides \( N \).

The main utility of the lemma is the fact that the function \( \beta \) relates directly to the Fourier coefficients of the form.

Now we recall the Hecke operator for Hermitian modular forms. The argument in Lemma 3.1 and Lemma 3.2 of [4] can be used to show that the commensurator of \( \Gamma_{0,n}(N) \) in \( \text{GU}(n,n)(\mathbb{Q})^+ \) is the whole group \( \text{GU}(n,n)(\mathbb{Q})^+ \). (This was also claimed in [18], Section 1.) Modifying Andrianov’s definition for Siegel modular forms, we define the following subsemigroup of \( \text{GU}(n,n)(\mathbb{Q})^+ \)
\[
\Delta_{0,n}(N) := \Gamma_{0,n}(N) \left\{ g \in \text{GU}(n,n)(\mathbb{Q})^+ \cap \text{GL}_{2n}(K(N)) \mid g \equiv \begin{pmatrix} I_n & 0 \\ 0 & \mu(g)I_n \end{pmatrix} \mod N \right\} \Gamma_{0,n}(N)
\]
where \( K(N) = \bigcap_{p \mid N} (\mathcal{O}_K)_p \subset K \) denotes the subring of our imaginary quadratic field \( K \) consisting of \( N \)-integral elements.
Then \(\Delta_{0,n}(N), \Gamma_{0,n}(N)\) is a Shimura pair and we have the associated \(\mathbb{Z}\)-ring of Hecke operators as in [25], Chapter 2 or [29], Chapter 3. For a rational prime \(p\), let us consider the Hecke operator

\[ T_p = T_p(N) := \Gamma_{0,2}(N) \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \Gamma_{0,2}(N). \]

In the proof of Theorem in Section 7 of [18], Krieg showed that when \(N = 1\) and \(p\) is inert in \(K\), one has the decomposition

\begin{equation}
\Gamma_{0,2}(1) \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \Gamma_{0,2}(1) = \bigsqcup_{(D,B) \in R} \Gamma_{0,2}(1) \begin{pmatrix} p\hat{D} & B \\ 0 & D \end{pmatrix}
\end{equation}

where \(\hat{D} := (D^*)^{-1} = (D^*)^{-1}\) and \(R\) is the finite set of following pair of matrices \((D,B)\) where

- \(D = I_2, B = 0\);
- \(D = pI_2, B = \begin{pmatrix} \gamma & b \\ 0 & \delta \end{pmatrix}\) where \(b \in \mathfrak{o}_K/p\mathfrak{O}_K\) and \(\gamma, \delta = 1,2,\ldots, p;\)
- \(D = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}\) where \(\gamma = 1,2,\ldots, p;\) and
- \(D = \begin{pmatrix} 1 & d \\ 0 & p \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}\) where \(d \in \mathfrak{o}_K/p\mathfrak{O}_K\) and \(\gamma = 1,2,\ldots, p.\)

We extend his result to higher level.

**Lemma 6.2.** Suppose that \(p \nmid N\) is inert in \(K\) and let \(\xi, \lambda \in \mathbb{Z}\) be such that \(p\xi - N\lambda = 1.\) Then the set

\[ R_N := \left\{ \begin{pmatrix} \xi pI_2 & \lambda I_2 \\ NI_2 & I_2 \end{pmatrix}, \begin{pmatrix} 1 & \gamma & b \\ 0 & 1 & b \gamma \\ N & 0 & 1 + N\gamma \end{pmatrix}, \begin{pmatrix} 1 & 0 & \gamma \\ 0 & \xi p & 0 \lambda \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \xi p & 0 & \lambda & \lambda d \\ -\hat{d} & 1 & 0 & \gamma \\ N & 0 & 1 & d \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}
\]

where \(\gamma, \delta = 1,2,\ldots, p\) and \(b, d \in \mathfrak{o}_K/p\mathfrak{O}_K\)

is a complete set of representatives for

\[ \underbrace{\Gamma_{0,2}(N) \cap \alpha^{-1}\Gamma_{0,2}(N)\alpha \setminus \Gamma_{0,2}(N)}_{\alpha^{-1}\Gamma_{0,2}(Np)\alpha} \text{ where } \alpha = \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix}. \]

**Proof.** It follows from (6.1) and Shimura [29], Proposition 3.1 that \(R_N\) is a complete set of representatives for \(\alpha^{-1}\Gamma_{0,2}(p)\alpha \setminus \Gamma_{0,2}(1)\).

These representatives for \(\alpha^{-1}\Gamma_{0,2}(p)\alpha \setminus \Gamma_{0,2}(1)\) must be thus distinct classes in \(\alpha^{-1}\Gamma_{0,2}(Np)\alpha \setminus \Gamma_{0,2}(N)\) as well since the condition for two matrices to be in the same class is independent of the level.

It remains to see that there are no more classes. In other words, any matrix \(\begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in \Gamma_{0,2}(N)\) lies in one of the right \(\alpha^{-1}\Gamma_{0,2}(Np)\alpha\)-coset of some matrix in \(R_N\): Let \(\rho \in \Gamma_{0,2}(N)\) be arbitrary and let \(\rho' \in R_N\) be the element such that \(\rho\) lies in the right coset \(\alpha^{-1}\Gamma_{0,2}(p)\alpha\) \(\rho'\).

(Such an element exists because \(R_N\) is a complete set of representatives for \((\alpha^{-1}\Gamma_{0,2}(p)\alpha) \setminus \Gamma_{0,2}(1)\) that we extracted from Krieg.) Then there exists a matrix \(\rho'' \in \Gamma_{0,2}(p)\) such that \(\rho = \alpha^{-1}\rho'\alpha\).
\(\alpha^{-1}\rho''\alpha \rho'\). We then have \(\alpha^{-1}\rho''\alpha = \rho(\rho')^{-1} \in \Gamma_{0,2}(N)\) so that \(\rho'' \in \alpha\Gamma_{0,2}(N)\alpha^{-1}\). From a simple observation that
\[
\alpha \left( \begin{array}{cc}
A & B \\
NC & D
\end{array} \right) \alpha^{-1} = \left( \begin{array}{cc}
I_n & 0 \\
0 & pI_n
\end{array} \right) \left( \begin{array}{cc}
A & B \\
NC & D
\end{array} \right) \left( \begin{array}{cc}
I_n & 0 \\
0 & \frac{1}{p}I_n
\end{array} \right) = \left( \begin{array}{cc}
A & \frac{1}{p}B \\
pNC & D
\end{array} \right)
\]
is in \(\Gamma_{0,2}(p)\) if and only if \(B \equiv 0 \mod p\) and we find \(\alpha\Gamma_{0,2}(N)\alpha^{-1} \cap \Gamma_{0,2}(p) = \Gamma_{0,2}(Np)\) under the assumption that \(p \nmid N\). This proves \(\rho'' \in \Gamma_{0,2}(Np)\); in other words, \(\rho\) is in the right \(\alpha^{-1}\Gamma_{0,2}(Np)\alpha\)-coset of \(\rho'\).

Thus, our set \(R_N\) is the complete set of representatives for \(\alpha^{-1}\Gamma_{0,2}(Np)\alpha \backslash \Gamma_{0,2}(N)\).

\[\square\]

**Corollary 6.3.** Suppose that \(p \nmid N\) is inert in \(K\). We have the following decomposition
\[
\Gamma_{0,2}(N) \alpha \Gamma_{0,2}(N) = \bigsqcup_{(D,B) \in R} \Gamma_{0,2}(N) \left( \begin{array}{cc}
pD & B \\
0 & D
\end{array} \right).
\]

**Theorem 6.4.** Suppose that \(p\) is an inert prime in \(K\) and \(p \nmid N\). Then the Maaß space \(\mathfrak{M}_k^\alpha(N)\) is stable under \(T_p\). In other words, if \(F \in \mathfrak{M}_k^\alpha(N)\) then \(G = F|_k T_p\) is also in the Maaß space.

**Proof.** Let \(\beta_F\) be the function associated to \(F\) by Lemma 6.1. Following [18], Section 7, we define
\[
\beta_G(p^vq,r) := \beta_F(p^{v-1}q,r) + p^{4-2k}\beta_F(p^{v+1}q,r)
\]
\[
+ \begin{cases}
  p^{1-k}\beta_F(p^{v+1}q,p^{-2}r) + p^{3-k}\beta_F(p^{v-1}q,p^2r) & \text{if } p^2 \mid r \\
  p^{1-k}\beta_F(p^vq,r) + p^{3-k}\beta_F(p^{v-1}q,p^2r) & \text{if } p \mid r, p^2 \nmid r \\
  p^{1-k}(p+1)\beta_F(p^vq,r) + p^{1-k}(p^2-p)\beta_F(p^{v-1}q,p^2r) & \text{if } p \nmid r 
\end{cases}
\]
where \(p \nmid q\). Then Krieg [18] already showed that \(\beta_G\) satisfies (i) and (ii) in Lemma 6.1 for \(G\). The last property is evident from the definition of \(\beta_G\) and the corresponding property (ii) and (iii) of \(\beta_F\).

\[\square\]

**Remark 6.5.** The Maaß space is not stable under all Hecke operators but its adelic analogue should be stable under all Hecke operators; as illustrated by Klosin [15], Section 5.

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\[\footnote{There appears to be a mistake in Krieg’s paper [18], it should be \(p^{4-2k}\), not \(p^{6-2k}\).}\]
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