Thermomechanical generation of fissure patterns on the surface of heated circular wood samples

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Abstract

We discuss the observation of primary crack patterns on the surface of heated medium density fiberboard (MDF) round samples in inert atmosphere. A constant heat flux irradiates the wood surface, and the primary cracks seem to appear instantaneously at a temperature below the pyrolysis point, before any actual charring. Such fissures were originally believed to form mainly by the action of physicochemical processes; on the contrary, we show here that below the pyrolysis temperatures this occurs by means of thermomechanical surface instability.

The crack patterns can indeed be explained qualitatively by the simultaneous thermal expansion and softening of the hot surface layer, which is restrained by the colder wood beneath. This generates membrane compressive stresses leading to surface instability. Physically, this is a consequence of the thermomechanical properties of wood, which is a natural thermoplastic.

In this paper, the macro-crack topology is reproduced by a full 3D thermomechanical instability model. We obtain the patterns by solving the according eigenvalue problem numerically, by Finite Element Method (FEM). We also formulate the model in 2D, assuming a circular soft thin plate bonded to an elastic foundation, and solve it both analytically and numerically.

Finally, we compare our results with analogous crack patterns appearing on the surface of square samples, which we discussed in a previous study. We conclude that very different pattern symmetries (orthotropic, isotropic and circular) might be explained by the same model of thermomechanical surface instability.

Keywords: Wood cracking, Thermomechanical buckling, Analytical models

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1. Introduction

When wood is heated, a well-known first effect that can be observed with rising temperature is the instantaneous appearance of regular crack patterns on its surface. These constitute preferred paths for the generation of new flames, and enhance the overall combustion and charring processes once the pyrolysis temperature ($\approx 300^\circ$C) is reached. For these reasons, engineers preoccupied with fire safety have studied the crack patterns formation extensively [1].

Explanation was searched for a long time in physicochemical processes such as charring, drying and shrinkage, which occur at temperatures above the pyrolysis point [2, 3]. However, such previous investigations have not been able to physically explain the cracks pattern topology [4, 5, 6].

On the contrary, by investigating the physics of processes at temperatures below the pyrolysis point, i.e. before any actual charring, the topology of patterns observed on heated oven-dry square wood samples in nitrogen atmosphere was explained by a thermomechanical surface instability that induces wrinkling [7]. This means that, under these conditions, it is not necessary to consider chemical processes in order to explain the principal crack pattern formation.

Physical explanation is to be found indeed in the thermomechanical properties of wood, which is a natural thermoplastic [8, 9]: when it reaches the glass transition temperature $T_g \approx 200^\circ$C, dry wood simultaneously softens and elongates extensively. This induces in the hot layer restrain thermal stresses, which under certain conditions eventually lead to wrinkling [7, 10, 11]. As detailed in [7], the cracks appear along the node lines of the buckling modes because there always exists a major principal extensional stress, which is perpendicular to these nodes. The cracks initiate at such locations where the mechanical resistance of the material decreases with increasing temperature.

The above mechanism has proven to be effective in reproducing the fissure patterns for square wood and Medium Density Fiberboard (MDF) samples [7]. In this paper we wonder whether this is not related to that specific sample shape, namely if our explanation is general enough to apply to the topology observed on circular specimens as well.

To this aim, we consider the formation of crack patterns in experiments performed on round oven-dry MDF samples, which are heated from above in nitrogen atmosphere. We try to match the observed crack topology by formulating a buckling model of a circular plate bonded to an elastic foundation. We successfully reproduce the observed patterns with a full 3D model of thermomechanical buckling, solved numerically via Finite Element Method (FEM). Such model is formulated also in 2D, both analytically and numerically, then successfully validated against the literature [12].

In conclusion, we argue that the same model of thermomechanical surface buckling can explain the very different crack patterns emerging on both square (Figures 1 and 2) and circular specimens, Figs 5 and 6. As a side result, our numerical 2D analysis finds additional modes ($n = 2, 3$) compared to the analytical solutions given in [12] for a steel disk.

Let us remark that we are focusing on soft matter, i.e. on the rubbery state
of the heated layer, when thermal decomposition is still negligible as it is prior to charring or burning. Moreover, the full mechanism of cracking formation might require further analysis: for instance, we do not investigate the crack initiation and propagation. However, we know for sure that cracks appear at locations with maximal extensional stress; in this paper we limit ourselves to explaining the physical origin and topology of the observed crack patterns.

The present article is organized as follows: in Section 2 we discuss the experimental setup and measurements, while in Section 3 the 3D model is described into detail. This is then compared to the observations in Section 4 and conclusions are given in Section 5.

The Appendix contains an analytical formulation of a 2D model for the buckling of a thin layer bonded to an elastic substrate in Appendix A. Finally, the corresponding numerical solution is obtained via variational formulation in Appendix B.

2. Experimental setup and observations

2.1. Experimental setup

The experimental rig consisted of the gas supply system (which provided nitrogen stored in a bottle and air from ambient environment through two pipes), a low pressure compartment and the control system.

The samples were weighted via an electric balance, and two thermocouples measured the surface and internal temperatures of the sample. A digital camera in front of the observation window was recording the experiment, more details are given in [7].

The tested samples were produced by cutting a large piece of MDF board into several circular pieces with a diameter of 100 mm, Fig.3, while a sample holder made of Kaowool with high heat insulation was used to carry the samples, as shown in Fig.4. To identify the effect of heat flux and to compare it to the previous patterns of rectangular samples [7], 20 kW/m² representing low heat flux and 50 kW/m² denoting high heat flux were used. The experiments were all performed under 95 kPa, which is close to the regular atmospheric
pressure. Finally, we used nitrogen atmosphere in order to prevent surface oxidation reactions that, after charring, would transform it into ash.

For calculating the temperature and density profiles inside the samples we used the pyrolysis model in Fire Dynamics Simulator (FDS) version 6.3.2 [13]. The model solves the coupled heat conduction and pyrolysis reaction equations with a one-dimensional finite difference method, summarized in [7].

2.2. Observations

It was found from the experiment observations that under low heat flux (20 kW/m²) there are two major cracks of which one is in circular shape close to the sample edge, while the other is a straight line right across the sample center, as in Fig.5. Nevertheless, the case with high heat flux (50 kW/m²) is more complicated since the crack pattern can be divided into two zones. The cracks close to the sample edge presented a radial pattern, while those near the center distributed more randomly (Fig.6).

Interestingly, this shows indeed that for this circular symmetry two different heat fluxes provide two distinct crack patterns on MDF. We observed something analogous in the case of square samples, where two distinct geometries are determined by two different materials (orthotropic fir and isotropic MDF) [7]. However, Figs.5 and 6 show that the topology induced by circular geometry is very sensitive to the penetration depth, which is related to the heat flux magnitude as we explain in Section 4.

3. A full 3D model of thermomechanical buckling and its validation

In this section we describe the full 3D model used to investigate the physical problem of wood surface wrinkling. Consider a three dimensional hot layer, subjected to in-plane thermal elongations that are restrained by an elastic, colder...
substrate to which it is perfectly bonded. The according thermoelectricity formulation is based on the equilibrium equations

$$\text{div}\sigma + \rho \mathbf{f} = 0,$$

(1)

with $\sigma$ the stress tensor, $\rho$ its density and $\mathbf{f}$ the resultant of external forces. We label by $\mathbf{D}$ the symmetric elasticity tensor and by $\epsilon^{(\text{th})}$ the thermal strain tensor. The general Hook’s law then gives for the stress tensor of the material the following expression,

$$\sigma = \mathbf{D} : \left( \epsilon - \epsilon^{(\text{th})} \right).$$

(2)

The total deformation is thus written as

$$\epsilon = \frac{1}{2} \left[ (\nabla \mathbf{u})^T + \nabla \mathbf{u} + (\nabla \mathbf{u})^T \nabla \mathbf{u} \right],$$

(3)

where $\mathbf{u}$ is the displacement. The above deformation (or strain tensor) can be also rewritten as the direct sum of thermal and elastic strains $\epsilon^{(e)},$

$$\epsilon = \epsilon^{(\text{th})} + \epsilon^{(e)} = \alpha \Delta T + \epsilon^{(e)}.$$  

(4)

$\alpha$ is the thermal expansion tensor and $\Delta T$ is the temperature change. The MDF properties are derived from [14].

The thermally induced displacement field $\mathbf{u}$ in principle has components $(u, v, w)$ along the axes $(x, y, z)$, but one can exploit the $U(1)$ symmetry of the circular sample and perform the substitution $(x, y, z) \rightarrow (r, \theta, z)$, where $\theta \in [0, 2\pi]$. Eq.(1) can be solved using the appropriate boundary conditions for a plate with free edge laying on an elastic foundation, namely null Kirchhoff effective shear force $V_r$ and zero radial bending moment $M_r$ (see also Appendix A for the 2D counterpart). Our formulation considers only the action of thermal stresses, therefore in the equilibrium equation (1) we set the resultant of external forces to zero, namely $\mathbf{f} = 0$. We thus obtain an eigenvalue problem, which provides the most stressed locations of the hot surface, these correspond indeed

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1At such locations the major extensional stress is maximal, as demonstrated in [7].
to the nodes of the resulting eigenmodes. As soon as the mechanical resistance of the material decreases with increasing temperature, the cracks will initiate along these nodes.

The model validation is done by comparing its predictions with a well-known 2D analytical solution for the buckling of a steel disk [12]. Using the Poisson ratio for steel, namely $\nu \sim 0.3$, we solve the equilibrium equations (1) numerically by Finite Element Method (FEM) with the program COMSOL Multiphysics [15], and obtain the light purple dots in Fig. 7. The agreement is very good; the only difference is the larger eigenvalues of the 2D solution [12], which is a dimensional reduction of the 3D problem. This is expected, since it is well known that the smallest eigenvalue $\lambda_{cr}$ is enhanced at lower dimensions [10].

As discussed in the Appendix, we formulate our thermomechanical model also in 2D, which is solved both analytically and numerically. Our 2D curve is successfully validated, as it overlaps with the one found in the literature [12] for any value of the relative stiffness parameter $\gamma \equiv R(k/D)^{1/4}$, with $D \approx Eh_c^3/[12(1 - \nu^2)]$ the flexural rigidity, $E$ the Young modulus and $h_c$ the plate thickness. It is easily seen that for low values of $\gamma$, our 2D and 3D solutions coincide, while when $\gamma \gtrsim 5$ the 3D solution returns slightly lower values for $\lambda_{cr}$. Moreover, compared to the analytical solution given for a steel disk, it seems that the numerical calculation finds additional modes for $n = 2, 3$. This is shown in Fig. B.10.
Figure 7: Validation of the model (for $\nu = 0.3$) and both 3D and 2D prediction for $\nu = 0.02$.

Figure 8: Model predictions and comparison with measurements.
4. Interpretation of experiments by the thermomechanical model

In this section we apply our thermomechanical model to the specific case of a round MDF sample, in order to verify that it is indeed able to reproduce the crack patterns observed in the experiments, Figs.5 and 6. To this aim, we substitute the Poisson ratio for wood $\nu = 0.02$ in our 3D and 2D solutions to compute the corresponding critical eigenmodes. The result is plotted in Fig.8. First of all, we notice the upper bound on $\lambda_{cr}$ given by the 2D solution, as discussed. For several ranges of $\gamma$ we verify the accumulation of modes, reflecting the well-known fact that the system is very sensitive to perturbations [17]. This pattern is found also in the observations.

Furthermore, one can see that when the ratio of foundation stiffness vs bending rigidity $\gamma$ is low, the buckling mode tends to be more global (zeroth and first mode on the right side of Fig.8). On the contrary, when the spring constant $k$ is large compared to the bending stiffness, we are in the presence of surface wrinkling (for instance 4th and higher modes).

A thicker heated layer therefore tends to exhibit global buckling modes (long wave lengths), while for a thinner one the surface wrinkles (short wave lengths). Phenomenologically, this corresponds respectively to Fig.5, i.e. low heat flux 20 W/m2, and to Fig.6, high heat flux 50 W/m2. The penetration depth is indeed inversely proportional to the incident heat flux: when this is low, it takes longer to the surface to reach the same temperature, and the heat can travel deeper inside the material. This leads to a longer penetration depth [18].

Notice also how the 3D model can fully explain the vertical crack on the specimen side, along the thickness (Figure 8). Such a crack usually appears along the direction of the principal plane. Finally, the additional patterns that appear in the central area of the specimens (see e.g. Fig.5) can be probably explained by secondary bifurcations. This is evident from the Y-shaped cracks (or sulci, see [19]) within the "bubbles" shown in Fig.2. However, a rigorous study of these patterns requires a full non-linear analysis that goes beyond the scope of this article.

5. Conclusions

In this paper we have considered the formation of crack patterns on the surface of circular wood samples subjected to radiation heat flux in inert atmosphere. Our analysis shows that a model of thermomechanical surface buckling is able to reproduce the observed cracks pattern topology, and provides a possible physical explanation for their formation.

Interestingly, the crack patterns for both circular and square wood specimens [7] are recreated by the same model of surface instability for a plate over an elastic foundation: the macroscopic physical mechanism is indeed identical.

We should however remark that our explanation for the crack patterns is valid for temperatures which are lower than those which induce chemical de-
composition (pyrolysis point $T_p \approx 300^\circ C$).

If one aims to explore what happens for higher temperatures, the emerging chemico-physical phenomena concur in creating a much more complex phenomenology. For instance, it is well known that once the solid materials like cellulose, rubber, and plastics are burned, they release combustible gases via pyrolysis, under the effects of external and flame heat fluxes. The materials would lose their structural integrity by charring, deforming and developing defects such as cracks, bubbles and voids. For wooden materials, the char shrinkage and cracking are typical “charring behaviors” which reduce the heat barrier effect of the char layer during flaming combustion and pyrolysis. These defects enhance the combustion process by allowing oxygen and external heat flux to travel further into the material; they also allow pyrolysis gases to escape to the surface for subsequent combustion.

Finally, to refine and fully validate our model, further experiments on wooden samples will be needed. These should include at least measurements of temperature profile and of deformations (displacements and strains), to capture the transition to surface instability. Also, in order to identify precisely the relevant temperature range, one needs the experimental determination of the glass-transition point. Measurements of thermal expansion coefficient and elasticity modulus in function of temperature at the macroscopic scale are also required.

Nevertheless, even taking into account these limitations, the topology of cracks observed on MDF samples seem to be well explained by a thermomechanically driven surface instability that occurs before pyrolysis. Specifically, in this paper we have confirmed that such phenomenon is totally general, independently of the particular shape of the specimens and of the different crack patterns.

Aknowledgements

This work was supported by the National Natural Science Foundation of China (NSFC) under Grant No. 51406192. SK has been supported by the Academy of Finland through the project Adaptive isogeometric methods for thin-walled structures (decision numbers 270007, 273609, 304122). Access and licenses for the commercial software Abaqus FEA have been provided by CSC – IT Center for Science (www.csc.fi).

Appendix A. Approximate 2D analytical solution for a thin plate

The governing equation for a thin plate bounded by an elastic Winkler foundation is generally written as (see \cite{ref7} and references quoted therein)

$$\nabla^4 w + \lambda^2 \nabla^2 w + \gamma^4 w = 0,$$

(A.1)

We anyway expect that, under our conditions, the eventual presence of oxygen or moisture would not change these results appreciably, if not at all.
for the displacement field \( w \), with lengths normalized by the plate radius \( R \). We treated the case of rectangular plate in [7], where \( \nabla \) is the gradient in Cartesian coordinates. For a circular plate instead, the load parameter is \( \lambda^2 \equiv R^2 N_r / D \), and \( N_r \) is the uniform radial load at the edge. \( D \approx Eh^3 / [12(1 - \nu^2)] \) is the flexural rigidity, with the Young modulus \( E \) and the hot plate thickness \( h \). 

\[ \gamma^4 \equiv k R^4 / D \] is a stiffness parameter, where \( k \) is the spring constant of the foundation (modeled as described into detail in [7], where \( \gamma \equiv \beta \)). For a radial symmetry, assuming \( n \in \mathbb{N} \) nodal diameters, the solution of the buckling equation \( \text{A.1} \) can be written in polar coordinates \((r, \theta)\) as

\[ w = u(r) \cos(n\theta), \quad \theta \in [0, 2\pi] \tag{A.2} \]

so that Eq.(A.1) can be recast as follows,

\[ L^2 u + \lambda^2 Lu + \gamma^4 u = 0 \tag{A.3} \]

with the Laplacian operator in polar coordinates

\[ L \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \tag{A.4} \]

The eigenvalue problem for our case is given by Eq.(A.3) and the following boundary conditions for a plate with free edge on an elastic foundation [12],

\[ u''(1) + \nu[u'(1) - n^2 u(1)] = 0 \tag{A.5} \]

\[ u'''(1) + u''(1) - [1 + n^2(2 - \nu) - \lambda^2]u'(1) + n^2(3 - \nu)u(1) = 0 \tag{A.6} \]

namely zero moment and resultant shear, respectively.

Depending on the relative magnitude of \( \gamma \) and \( \lambda \), we obtain three general different solutions which are bounded at the origin. If \( J_n \) is the Bessel function of the first kind of order \( n \), for \( \lambda > \sqrt{2} \gamma \) we obtain [12]

\[ u(r) = C_1 J_n(\alpha r) + C_2 J_n(\beta r) \tag{A.7} \]

where

\[ \alpha = \sqrt{\frac{\lambda^2 + \sqrt{\lambda^4 - 4\gamma^4}}{2}}, \quad \beta = \sqrt{\frac{\lambda^2 - \sqrt{\lambda^4 - 4\gamma^4}}{2}} \tag{A.8} \]

Substituting in the boundary conditions \( \text{A.5} \) and \( \text{A.6} \), one finds the following criticality condition,

\[ f(\lambda) = [\alpha^2 J''_n(\alpha r) + \nu(\alpha J'_n(\alpha r) - n^2 J_n(\alpha r))] \]

\[ \times [\beta^2 J''_n(\beta r) + \beta^2 J'_n(\beta r)] - [1 + n^2(2 - \nu) - \lambda^2]J_n(\beta r) + n^2(3 - \nu)J_n(\beta r)] \]

\[ - [\beta^2 J''_n(\beta r) + \nu(\beta J'_n(\beta r) - n^2 J_n(\beta r))] \]

\[ \times [\alpha^2 J''_n(\alpha r) + \alpha^2 J'_n(\alpha r)] - [1 + n^2(2 - \nu) - \lambda^2]J_n(\alpha r) + n^2(3 - \nu)J_n(\alpha r)] \]

\[ = 0 \tag{A.9} \]

The lowest value of \( \lambda \) satisfying the above condition is the critical \( \lambda_{cr} \). 

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For $\lambda = \sqrt{2}\gamma$, the solution is instead
\[ u = C_1 J_n(\gamma r) + C_2 r J_{n+1}(\gamma r), \] (A.10)
which gives the criticality condition
\[ f(\lambda) = [\gamma^2 J''_n(\gamma r) + \nu(\gamma J'_n(\gamma r) - n^2 J_n(\gamma r))] \]
\[ \times \left\{ 3\gamma^2 J''_{n+1}(\gamma r) + \gamma^3 r J'''_{n+1}(\gamma r) + 2\gamma J'_{n+1}(\gamma r) + \gamma^2 r J''_{n+1}(\gamma r) \right\} \]
\[ - [1 + n^2(2 - \nu) - \lambda^2](J_{n+1}(\gamma r) + \gamma r J'_{n+1}(\gamma r)) + n^2(3 - \nu)rJ_{n+1}(\gamma r) \]
\[ - \left\{ \gamma^2 J''_n(\gamma r) + \gamma^3 r J'''_n(\gamma r) - [1 + n^2(2 - \nu) - \lambda^2]J'_n(\gamma r) + n^2(3 - \nu)J_n(\gamma r) \right\} \]
\[ \times [2\gamma J'_{n+1}(\gamma r) + \gamma^2 r J''_{n+1}(\gamma r) + \nu J_{n+1}(\gamma r) + \gamma r J'_{n+1}(\gamma r) - n^2 r J_{n+1}(\gamma r)] = 0. \] (A.11)

Finally, for $\lambda < \sqrt{2}\gamma$, the solution is written as
\[ u = C_1 \text{Re}[J_n(i\delta r)] + C_2 \text{Im}[J_n(i\delta r)], \] (A.12)
where $\delta = -1$ and
\[ \delta = \sqrt{-\lambda^2 + i\sqrt{4\gamma^4 - \lambda^4}}. \] (A.13)

In this case the criticality condition holds as follows,
\[ f(\lambda) = \{\text{Re}''[J_n] + \nu\text{Re}'[J_n] - n^2\text{Re}[J_n]\} \]
\[ \times \left\{ \text{Im}''[J_n] + \text{Im}'[J_n] - [1 + n^2(2 - \nu) - \lambda^2]\text{Im}'[J_n] + n^2(3 - \nu)\text{Im}[J_n] \right\} \]
\[ - (\text{Re} \leftrightarrow \text{Im}) = 0. \] (A.14)

Solving Eqs. (A.9), (A.11) and (A.14) returns the values of $\lambda_{cr}$ that are plotted in Figs.[7] and [8].

**Appendix B. Variational formulation and isogeometric analysis of a 2D stability problem**

The weak (or variational) formulation of the stability problem for a plate on elastic foundation corresponding to (A.3) with (A.5) and (A.6) reads as follows: Find $u \in U \subset H^2(\Omega)$ such that
\[ a(u, v) = \lambda^2 m(u, v) \quad \forall v \in V \subset H^2(\Omega), \quad \Omega = (0, 1) \] (B.1)
where the bilinear forms $a$ and $m$: $U \times V \rightarrow \mathbb{R}$, respectively, are defined as
\[ a(u, v) = (\nu - 1) \int_0^1 \left[ u'' \left( \frac{u'}{r} - n^2 \frac{v}{r^2} \right) + v'' \left( \frac{u'}{r} - n^2 \frac{u}{r^2} \right) - 2n^2 \left( \frac{u'}{r} - \frac{u}{r^2} \right) \left( \frac{v'}{r} - \frac{v}{r^2} \right) \right] r dr \]
\[ + \int_0^1 \left( u'' + \frac{u'}{r} - n^2 \frac{u}{r^2} \right) \left( v'' + \frac{v'}{r} - n^2 \frac{v}{r^2} \right) r dr + \gamma^4 \int_0^1 u v r dr, \] (B.2)
\[ m(u, v) = \int_0^1 (u'v' + n^2 \frac{u}{r} v) r dr, \] (B.3)

with \( u(r) \) and \( v(r) \) standing, respectively, for trial and test functions, where the prime denotes differentiation with respect to the radial coordinate \( r \). The homogeneous Neumann boundary conditions at free edge \( r = 1 \) are fulfilled automatically. In the corresponding conforming Galerkin formulation, one finds \( u_h \in U_h \subset U \) such that

\[ a(u_h, v_h) = \lambda^2 m(u_h, v_h) \quad \forall v_h \in V_h \subset V. \] (B.4)

An isogeometric NURBS-based discretization of the solution domain naturally provides \( C^{p-1} \) global regularity [20], where \( p \) is a B-spline order. For \( p \geq 2 \), the corresponding isoparametric discrete function space is a subset of an \( H^2 \) Sobolev space, which provides a conforming Galerkin version of the method.

The numerical implementation utilizing user-defined finite elements of the commercial software Abaqus FEA is described in [21]. For steel plates with \( \nu = 0.3 \), we consider three types of boundary conditions corresponding to clamped, simply supported and free edges. The critical load \( \lambda_{cr} \) against the foundation stiffness \( \gamma \) is presented for different cases of radial symmetries \( (n = 0, \ldots, 4) \) in Figs.B.9, B.10 and B.11, respectively. Solid curves represent the values calculated via Abaqus user elements. Circle marks, with values analytically defined in [12], are used for the verification of the numerical implementation. The lowest curves build the border line of critical buckling loads. For the simply supported case, it should be mentioned that the border line of \( \lambda_{cr} \) is composed of several curves corresponding to \( n = 0, 1, \ldots \), while in [12] only two curves \((n = 0 \text{ and } n = 1)\) define the border line.

The case when \( \nu = 0.02 \), which concerns the MDF samples in our experiments, is shown in Fig.B.12. Diamond marks correspond to analytical values, while circle marks stand for those calculated with the FEM software COMSOL.
Figure B.9: $\nu = 0.3$, Clamped edge

Figure B.10: $\nu = 0.3$, Simply supported edge

Figure B.11: $\nu = 0.3$, Free edge

Figure B.12: $\nu = 0.02$, Free edge

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