On subgroups in division rings of type $2$

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Abstract

Let $D$ be a division ring with the center $F$. We say that $D$ is a division ring of type 2 if for every two elements $x, y \in D$, the division subring $F(x, y)$ is a finite dimensional vector space over $F$. In this paper we investigate multiplicative subgroups in such a ring.

Key words: Division ring, type 2, finitely generated subgroups.

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1 Introduction

In the theory of division rings, one of the interesting problems is the question of what groups can not occur as multiplicative groups of non-commutative division rings. There are some very interesting results answering to this question. Among them we note the famous discovery of Wedderburn in 1905, which states that if $D^*$ is a finite group, then $D$ is commutative, where $D^*$ denotes the multiplicative group of $D$. Later, L. K. Hua (see, for example, in [8, p. 223]) proved that the multiplicative group of a non-commutative division ring cannot be solvable. Recently, in [7] it was shown that the group $D^*$ can not be even locally nilpotent. Note also Kaplansky’s Theorem (see [8,(15.15), p. 259]) which states that if the group $D^*/F^*$ is torsion, then $D$ is commutative. There are another results of such a kind that can be found for example, in [1]-[3], [5]-[7],...

In this paper we consider this question for division rings of type 2. Recall that a division ring $D$ with the center $F$ is said to be division ring of type 2 if for every two elements $x, y \in D$, the division subring $F(x, y)$ is a finite dimensional vector space over $F$.

Throughout this paper the following notations will be used consistently: $D$ denotes a division ring with the center $F$ and $D^*$ is the multiplicative group of $D$. If $S$ is a nonempty subset of $D$, then we say that $S$ is algebraic over $F$ if every element of $S$ is algebraic over $F$. We denote also by $F[S]$ and $F(S)$ the subring and the division subring of $D$ generated by $S$ over $F$, respectively. The symbol $D'$ is used to denote the derived group $[D^*, D^*]$. We say that a division ring $D$ is centrally finite if it is a finite dimensional vector space over $F$. An element $x \in D$ is said to be radical over a subring $K$ of $D$ if there exists some positive integer $n(x)$ depending on $x$ such that $x^{n(x)} \in K$. A nonempty subset $S$ of $D$ is radical over $K$ if every element of $S$ is radical over $K$. We denote by $N_{D/F}$ and $RN_{D/F}$ the norm and the reduced norm respectively. Finally, if $G$ is any group then we always use the symbol $Z(G)$ to denote the center of $G$.

2 Finitely generated subgroups

The main purpose in this section is to prove that in a non-commutative division ring $D$ of type 2 with the center $F$ there are no finitely generated subgroups containing $F^*$.

**Lemma 2.1** Let $D$ be a division ring with the center $F$, $D_1$ be a division subring of $D$ containing $F$. Suppose that $D_1$ is a finite dimensional vector space over $F$ and $a \in D_1$. Then, $N_{D_1/F}(a)$ is periodic if and only if $N_{F(a)/F}(a)$ is periodic.
Proof. Let \( F_1 = Z(D_1) \supset F, m^2 = [D_1 : F_1] \) and \( n = [F_1(a) : F_1] \). By [4, Lem. 3, p.145 and Cor. 4, p. 150], we have

\[
N_{D_1/F_1}(a) = [RN_{D_1/F_1}(a)]^m = [N_{F_1(a)/F_1}(a)]^{m^2/n}.
\]

Now, using the Tower formulae for the norm, from the equality above we get

\[
N_{D_1/F}(a) = [N_{F_1(a)/F}(a)]^{m^2/n}.
\]

Since \( a \in F(a), N_{F_1(a)/F(a)} = a^k \), where \( k = [F_1(a) : F(a)] \). Therefore

\[
N_{F(a)/F}(a^k) = N_{F(a)/F}(N_{F_1(a)/F(a)}(a)) = N_{F_1(a)/F}(a).
\]

It follows that \( N_{D_1/F}(a) = [N_{F(a)/F}(a)]^{km^2/n} \), and the conclusion is now obvious. \( \square \)

The following proposition is useful. In particular, it is needed to prove the next theorem.

**Proposition 2.1** Let \( D \) be a division ring with the center \( F \). If \( N \) is a subnormal subgroup of \( D^* \) then \( Z(N) = N \cap F^* \).

**Proof.** If \( N \) is contained in \( F^* \) then there is nothing to prove. Thus, suppose that \( N \) is non-central. By [10, 14.4.2, p. 439], \( C_D(N) = F \). Hence \( Z(N) \subseteq N \cap F^* \). Since the inclusion \( N \cap F^* \subseteq Z(N) \) is obvious, \( Z(N) = N \cap F^* \). \( \square \)

**Theorem 2.1** Let \( D \) be a division ring of type 2. Then \( Z(D') \) is a torsion group.

**Proof.** By Proposition 2.1 \( Z(D') = D' \cap F^* \). Any element \( a \in Z(D') \) can be written in the form \( a = c_1 c_2 \ldots c_r \), where \( c_i = [x_i, y_i] \) with \( x_i, y_i \in D^* \) for \( i \in \{1, \ldots, r\} \). Put \( D_1 = D_2 := F(c_1, c_2), D_3 := F(c_1 c_2, c_3), \ldots, D_r := F(c_1 \ldots c_{r-1}, c_r) \) and \( F_i = Z(D_i) \) for \( i \in \{1, \ldots, r\} \). Since \( D \) is of type 2, \([D_i : F] < \infty \).

Since \( N_{F(x_i, y_i)/F}(c_i) = 1 \), by Lemma 2.1 \( N_{F(c_1)/F}(c_i) \) is periodic. Again by Lemma 2.1 \( N_{D_i/F}(c_i) \) is periodic. Therefore, there exists some positive integer \( n_i \) such that \( N_{D_i/F}(c_i^{n_i}) = 1 \). Recall that \( D_2 = D_1 \). Hence we get

\[
N_{D_2/F}(c_1 c_2)^m = N_{D_1/F}(c_1)^m N_{D_2/F}(c_2)^m = 1,
\]

where \( m = n_1 n_2 \). Again by Lemma 2.1 \( N_{F(c_1 c_2)/F}(c_1 c_2) \) is periodic; hence \( N_{D_1/F}(c_1 c_2) \) is periodic. By induction, \( N_{D_r/F}(c_1 \ldots c_{r-1}) \) is periodic. Suppose that \( N_{D_r/F}(c_1 \ldots c_{r-1})^n = 1 \). Then

\[
N_{D_r/F}(a^n) = N_{D_r/F}(c_1 \ldots c_{r-1})^n N_{D_r/F}(c_r)^n = 1.
\]
Hence, $a^{n[D_r:F]} = 1$. Therefore, $a$ is periodic. Thus $Z(D')$ is torsion.

From the discussion before Corollary 8 in [9], we can obtain the following result as a corollary of the theorem above.

**Corollary 2.1** Let $D$ be a non-commutative ring of type 2 with the center $F$. Then $D' \setminus Z(D')$ contains no elements purely inseparable over $F$.

In [2, Theorem 1], it was proved that if $D$ is a centrally finite division ring and $D^*$ is finitely generated, then $D$ is commutative. Here, in the first, we note that if $D^*$ is finitely generated then $D$ is even a finite field. Further, we shall prove that in a division ring $D$ of type 2 with the center $F$, there are no finitely generated subgroups containing $F^*$. Consequently, if $D$ is of type 2 and $D^*$ is finitely generated then $D$ is a finite field.

**Lemma 2.2** Let $K$ be a field. If the multiplicative group $K^*$ of $K$ is finitely generated, then $K$ is finite.

**Proof.** If $\text{char}(K) = 0$, then $K$ contains the subfield $\mathbb{Q}$ of rational numbers. Since $K^*$ is finitely generated, in view of [10, 5.5.8, p. 113], $\mathbb{Q}^*$ is finitely generated, that contradicts to the well-known property of the group $\mathbb{Q}^*$. Thus, we have $\text{char}(K) = p > 0$. Suppose that $K^* = \langle a_1, a_2, \ldots, a_r \rangle$. Then, $K = \mathbb{F}_p(a_1, a_2, \ldots, a_r)$, where $\mathbb{F}_p$ is the prime subfield of $K$. We shall prove that $a_i$ is algebraic over $\mathbb{F}_p$ for every $i \in \{1, 2, \ldots, r\}$. Clearly, if this will be done then $K$ will be finite. Suppose that $a = a_i$ is transcendental over $\mathbb{F}_p$ for some $i$. Since the subgroup $\mathbb{F}_p(a)^*$ is finitely generated, it can be written in the form

$$\mathbb{F}_p(a)^* = \left\langle \frac{f_1(a)}{g_1(a)}, \frac{f_2(a)}{g_2(a)}, \ldots, \frac{f_n(a)}{g_n(a)} \right\rangle,$$

where $f_i(X), g_i(X) \in \mathbb{F}_p[X], g_i(a) \neq 0$ and $(f_i(X), g_i(X)) = 1$. Take some positive integer $m$ such that

$$m > \max \{ \deg(f_i), \deg(g_i) | i \in \{1, 2, \ldots, n\} \}$$

and an irreducible polynomial $f(X) \in \mathbb{F}_p[X]$ of degree $m$ (such a polynomial always exists). Then, we have

$$f(a) = \left( \frac{f_1(a)}{g_1(a)} \right)^{m_1} \left( \frac{f_2(a)}{g_2(a)} \right)^{m_2} \cdots \left( \frac{f_n(a)}{g_n(a)} \right)^{m_n},$$

with $m_1, m_2, \ldots, m_n \in \mathbb{Z}$. Since $a$ is transcendental, $\mathbb{F}_p[a] \simeq \mathbb{F}_p[X]$, so from the last equality it follows that there exists some $i \in \{1, 2, \ldots, n\}$ such that $f(X)$ divides either $f_i(X)$ or $g_i(X)$. But this is impossible by the choice of degree $m$ of $f(X)$. Thus, we have
proved that \( a_i \) is algebraic over \( \mathbb{F}_p \) for any \( i \in \{1, 2, \ldots, n\} \). Therefore, \( K \) is a finite field.

Now we can prove the following theorem, which shows that in a non-commutative division ring \( D \) of type 2 there are no finitely generated subgroups of \( D^* \), containing \( F^* \).

**Theorem 2.2** Let \( D \) be a non-commutative division ring of type 2 with center \( F \) and suppose that \( N \) is a subgroup of \( D^* \) containing \( F^* \). Then \( N \) is not finitely generated.

**Proof.** Suppose that there is a finitely generated subgroup \( N = \langle x_1, \ldots, x_n \rangle \) of \( D^* \) containing \( F^* \). Then, in virtue of [11], 5.5.8, p. 113], \( F^* N'/N' \) is a finitely generated abelian group, where \( N' \) denotes the derived subgroup of \( N \).

**Case 1:** \( \text{char}(D) = 0 \).

Then, \( F \) contains the field \( \mathbb{Q} \) of rational numbers and it follows that \( \mathbb{Q}^*/(\mathbb{Q}^* \cap N') \simeq \mathbb{Q}^* N'/N' \). Since \( F^* N'/N' \) is finitely generated, \( \mathbb{Q}^* N'/N' \) is finitely generated and consequently \( \mathbb{Q}^*/(\mathbb{Q}^* \cap N') \) is finitely generated. Consider an arbitrary element \( a \in \mathbb{Q}^* \cap N' \). Then \( a \in F^* \cap D' = \mathbb{Z}(D') \). By Theorem 2.1, \( a \) is periodic. Since \( a \in \mathbb{Q} \), we get \( a = \pm 1 \). Thus, \( \mathbb{Q}^* \cap N' \) is finite. Since \( \mathbb{Q}^*/(\mathbb{Q}^* \cap N') \) is finitely generated, \( \mathbb{Q}^* \) is finitely generated, that is impossible.

**Case 2:** \( \text{char}(D) = p > 0 \).

Denote by \( \mathbb{F}_p \) the prime subfield of \( F \), we shall prove that \( F \) is algebraic over \( \mathbb{F}_p \). In fact, suppose that \( u \in F \) and \( u \) is transcendental over \( \mathbb{F}_p \). Then, the group \( \mathbb{F}_p (u)^*/(\mathbb{F}_p (u)^* \cap N') \) considered as a subgroup of \( F^* N'/N' \) is finitely generated. Consider an arbitrary element \( f(u)/g(u) \in \mathbb{F}_p (u)^* \cap N' \), where \( f(X), g(X) \in \mathbb{F}_p [X], ((f(X), g(X)) = 1 \) and \( g(u) \neq 0 \). As above, we have \( f(u)^*/g(u)^* = 1 \) for some positive integer \( s \). Since \( u \) is transcendental over \( \mathbb{F}_p \), it follows that \( f(u)/g(u) \in \mathbb{F}_p \). Therefore, \( \mathbb{F}_p (u)^* \cap N' \) is finite and consequently, \( \mathbb{F}_p (u)^* \) is finitely generated. But, in view of Lemma 2.2, \( \mathbb{F}_p (u) \) is finite, that is a contradiction. Hence \( F \) is algebraic over \( \mathbb{F}_p \) and it follows that \( D \) is algebraic over \( \mathbb{F}_p \). Now, in virtue of Jacobson’s Theorem [8, (13.11), p. 219], \( D \) is commutative, that is a contradiction.

From Theorem 2.1 and Lemma 2.2 we get the following result, which generalizes Theorem 1 in [2]:

**Corollary 2.2** Let \( D \) be a division ring of type 2. If the multiplicative group \( D^* \) is finitely generated, then \( D \) is a finite field.

If \( M \) is a maximal finitely generated subgroup of \( D^* \), then \( D^* \) is finitely generated. So, the next result follows immediately from Corollary 2.2.
Corollary 2.3 Assume that $D$ is a division ring of type 2. If the multiplicative group $D^*$ has a maximal finitely generated subgroup, then $D$ is a finite field.

By the same way as in the proof of Theorem 2.1, we obtain the following corollary.

Corollary 2.4 Assume that $D$ is a division ring of type 2 with the center $F$ and $S$ is a subgroup of $D^*$. If $N = SF^*$, then $N/N'$ is not finitely generated.

Proof. Suppose that $N/N'$ is finitely generated. Since $N' = S'$ and $F^*/(F^* \cap S') \simeq S'F^*/S'$, it follows that $F^*/(F^* \cap S')$ is a finitely generated abelian group. Now, by the same way as in the proof of Theorem 2.1, we can conclude that $D$ is commutative.

The following result follows immediately from Corollary 2.4.

Corollary 2.5 Assume that $D$ is a division ring of type 2. Then, $D^*/D'$ is not finitely generated.

3 The radicality of subgroups

In this section we study subgroups of $D^*$, that are radical over some subring of $D$. To prove the next theorem we need the following useful property of division rings of type 2.

Lemma 3.1 Let $D$ be a division ring of type 2 with the center $F$ and $N$ be a subnormal subgroup of $D^*$. If for every elements $x, y \in N$, there exists some positive integer $n_{xy}$ such that $x^{n_{xy}} y = y x^{n_{xy}}$, then $N \subseteq F$.

Proof. Since $N$ is subnormal in $D^*$, there exists the following series of subgroups

$$N = N_1 \triangleleft N_2 \triangleleft \ldots \triangleleft N_r = D^*.$$  

Suppose that $x, y \in N$ and $K := F(x, y)$. By putting $M_i = K \cap N_i, \forall i \in \{1, \ldots, r\}$ we obtain the following series of subgroups

$$M_1 \triangleleft M_2 \triangleleft \ldots \triangleleft M_r = K^*.$$  

For any $a \in M_1 \leq N_1 = N$, suppose that $n_{ax}$ and $n_{ay}$ are positive integers such that

$$a^{n_{ax}} x = x a^{n_{ax}} \quad \text{and} \quad a^{n_{ay}} y = y a^{n_{ay}}.$$  

Then, for $n := n_{ax} n_{ay}$ we have

$$a^n = (a^{n_{ax}})^{n_{ay}} = (x a^{n_{ax}} x^{-1})^{n_{ay}} = x a^{n_{ax} n_{ay}} x^{-1} = x a^n x^{-1},$$
and
\[ a^n = (a^{n_1})^{n_2} = (ya^{n_3}y^{-1})^{n_4} = y^a n_1 n_2 = y^n. \]

Therefore \( a^n \in Z(K) \). Hence \( M_1 \) is radical over \( Z(K) \). By [5, Theorem 1], \( M_1 \subseteq Z(K) \). In particular, \( x \) and \( y \) commute with each other. Consequently, \( N \) is abelian group. By [10, 14.4.4, p. 440], \( N \subseteq F \).

**Theorem 3.1** Let \( D \) be a division ring of type 2 with the center \( F \), \( K \) be a proper division subring of \( D \) and suppose that \( N \) is a normal subgroup of \( D^* \). If \( N \) is radical over \( K \), then \( N \subseteq F \).

**Proof.** Suppose that \( N \) is not contained in the center \( F \). If \( N \setminus K = \emptyset \), then \( N \subseteq K \). By [10, p. 433], either \( K \subseteq F \) or \( K = D \). Since \( K \neq D \) by the supposition, it follows that \( K \subseteq F \). Hence \( N \subseteq F \), that contradicts to the supposition. Thus, we have \( N \setminus K \neq \emptyset \).

Now, to complete the proof of our theorem we shall show that the elements of \( N \) satisfy the requirements of Lemma 3.1. Thus, suppose that \( a, b \in N \). We examine the following cases:

**Case 1:** \( a \in K \).

a) \( b \not\in K \).

We shall prove that there exists some positive integer \( n \) such that \( a^n b = ba^n \). Thus, suppose that \( a^n b \neq ba^n \), \( \forall n \in \mathbb{N} \). Then, \( a + b \neq 0 \), \( a \neq \pm 1 \) and \( b \neq \pm 1 \). So we have

\[ x = (a + b)a(a + b)^{-1}, y = (b + 1)a(b + 1)^{-1} \in N. \]

Since \( N \) is radical over \( K \), we can find some positive integers \( m_x \) and \( m_y \) such that

\[ x^{m_x} = (a + b)a^{m_x}(a + b)^{-1}, y^{m_y} = (b + 1)a^{m_y}(b + 1)^{-1} \in K. \]

Putting \( m = m_x m_y \), we have

\[ x^m = (a + b)a^m(a + b)^{-1}, y^m = (b + 1)a^m(b + 1)^{-1} \in K. \]

Direct calculations give the equalities

\[ x^m b - y^m b + x^m a - y^m = x^m(a + b) - y^m(b + 1) = (a + b)a^m - (b + 1)a^m = a^m(a - 1), \]

from that we get the following equality

\[ (x^m - y^m)b = a^m(a - 1) + y^m - x^m a. \]
If \((x^m - y^m) \neq 0\), then \(b = (x^m - y^m)^{-1}[a(a^m - 1) + y^m - x^m a] \in K\), that is a contradiction to the choice of \(b\). Therefore \((x^m - y^m) = 0\) and consequently, \(a^m(a - 1) = y^m(a - 1)\). Since \(a \neq 1\), \(a^m = y^m = (b + 1)a^m(b + 1)^{-1}\) and it follows that \(a^m b = ba^m\), that is a contradiction.

b) \(b \in K\).

Consider an element \(x \in N \setminus K\). Since \(xb \notin K\), by Case 1, there exist some positive integers \(r, s\) such that

\[
a^r x b = x b a^r \quad \text{and} \quad a^s x = x a^s.
\]

From these equalities it follows that

\[
a^{rs} = (xb)^{-1}a^{rs}(xb) = b^{-1}(x^{-1}a^{rs}x)b = b^{-1}a^{rs}b,
\]

and consequently, \(a^{rs} b = ba^{rs}\).

Case 2: \(a \notin K\).

Since \(N\) is radical over \(K\), there exists some positive integer such that \(a^m \in K\). By Case 1, there exists some positive integer \(m\) such that \(a^m b = ba^m\).

In [1, Theorem 5] it was shown that if \(D\) is a centrally finite division ring with the center \(F\) whose characteristic is different from the index of \(D\) over \(F\) then \(D^*\) contains no maximal subgroups that are radical over \(F\). Now, in the case of division ring of type 2, we can prove the following theorem.

**Theorem 3.2** Let \(D\) be a division ring of type 2 with the center \(F\) such that \([D : F] = \infty\) and \(\text{char} F = p > 0\). Then the group \(D^*\) contains no maximal subgroups that are radical over \(F\).

**Proof.** Suppose that \(M\) is a maximal subgroup of \(D^*\) that is radical over \(F\). Put \(G = D' \cap M\). For each \(x \in G\), there exists a positive integer \(n(x)\) such that \(x^{n(x)} \in F\). It follows that \(x^{n(x)} \in D' \cap F = Z(D')\). By Theorem 2.1, \(Z(D')\) is periodic, so \(x\) is periodic. Thus, \(G\) is a periodic group. Since \(M' \leq G, M'\) is a periodic too. For any \(x, y \in M'\), put \(H = \langle x, y \rangle\) and \(D_1 = F(x, y)\). Then \(n := [D_1 : F] < \infty\) and \(H\) is a periodic subgroup of \(D_1^* \leq GL_n(F)\). By [8, (9.9'), p. 154], \(H\) is finite. Since \(\text{char} F = p > 0\), by [8, (13.3), p.215], \(H\) is cyclic . In particular, \(x\) and \(y\) commute with each other, and consequently, \(M'\) is abelian. It follows that \(M\) is a solvable group. Thus \(M\) is a solvable maximal subgroup of \(D^*\). By [1, Cor. 10] and [3, Th. 6], \([D : F] < \infty\), that is a contradiction. \(\blacksquare\)
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