ON ALGEBRAIC EQUIVALENCES AMONG THE 27 ABEL-PRYM CURVES ON A GENERIC ABELIAN 5-FOLD

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Abstract. This article shows that on a generic principally polarized abelian variety of dimension five the $\mathbb{Q}$-vector space of algebraic equivalences among the 27 Abel-Prym curves has dimension 20.

1. Introduction

Let $X$ be a principally polarized abelian variety (ppav) over $\mathbb{C}$. Let $A^*(X)$ denote the Chow ring of $X$ modulo algebraic equivalence with $\mathbb{Q}$-coefficients (this notation differs from the one in [Fu98, 10.3, p.185]). Besides the intersection product, the ring $A^*(X)$ is endowed with Pontryagin product defined by

$$x_1 \ast x_2 = m_*(p_1^*x_1 \cdot p_2^*x_2),$$

where $m: X \times X \to X$ is the addition morphism, and $p_j: X \times X \to X$ is the projection onto the $j$th factor, cf. [BL, p.530]. Moreover, $A^*(X)$ carries a bi-grading,

$$A(X) = \bigoplus_{l,s} A^l(X)_{(s)}.$$ 

The $l$-grading is by codimension. The Beauville grading $(s)$ is defined by the condition $x \in A^l(X)_{(s)}$ if and only if $k^*x = k^{2l-s}x$ for all $k \in \mathbb{Z}$, where $k$ also denotes the endomorphism of $X$ given by $x \mapsto kx$ (the second grading was originally defined in [B86] for rational equivalence but it is well-defined for algebraic equivalence as well). The $(s)$-component of a cycle $Z$ is denoted by $Z_{(s)}$. Also, let us recall the Fourier transform $\mathcal{F}_X: A^*(X) \to A^*(X)$ that is given by $z \mapsto p_{2,*}(p_1^*z \cdot e^\ell)$, where $\ell$ is the class of the Poincaré bundle on $X \times X$ (here we identify $X$ and its dual abelian variety using the principal polarization).

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Let $P := \text{Prym}(\tilde{C}/C)$ be the Prym variety of a connected étale double cover $\tilde{C} \to C$ of smooth curves, see [Mu74]. In the sequel we assume that $\tilde{C}$ is not hyperelliptic and we let $p := \dim P$. After fixing a base-point $\tilde{o} \in \tilde{C}$ we have the Abel-Prym map $\psi: \tilde{C} \to P$ whose image will be denoted by $\tilde{C}$ as well and is called an Abel-Prym curve ($\psi$ is a closed embedding, cf. [BL04, Cor.12.5.6, p.380]).

Given an Abel-Prym curve $\tilde{C} \subset P$, by [A12, Def.1, p.708] there is an associated tautological (sub)ring $\mathcal{T}(P, \tilde{C}) \subset A^*(P)$, defined to be the smallest subring of $A^*(P)$ for the intersection product that contains $[\tilde{C}]$ and is stable under $\ast, \mathcal{F}_P$ and $k^\ast$ for all $k \in \mathbb{Z}$. By [A12, Thm.4, p.710], $\mathcal{T}(P, \tilde{C})$ is generated by the cycles $\zeta_n := \mathcal{F}_P([\tilde{C}](n-1))$ for $1 \leq n \leq p-1$ odd (the components of $[\tilde{C}]$ of even Beauville degree vanish because $\tilde{C}$ has a symmetric translate). Also, by [A12, Rem.3, p.711] the ring $\mathcal{T}(P, \tilde{C})$ does not depend on the choice of an Abel-Prym curve if $P$ is a generic ppav of dimension $p \neq 5$. As a corollary of the main result of this article (Theorem 2.10), we obtain that for a generic ppav $P$ of dimension 5 the tautological ring $\mathcal{T}(P, \tilde{C})$ does depend on the choice of an Abel-Prym curve $\tilde{C} \subset P$. This gives 27 tautological rings, one for each choice of an Abel-Prym curve (see the next paragraph for the explanation of the number 27). However, by [A11, p.35] each of these 27 rings is isomorphic to the quotient $\mathbb{Q}[x_1, x_3]/(x_1^6, x_2^2, x_1^2x_3)$ of the polynomial ring in two variables via the homomorphism $\zeta_i \mapsto x_i$ (the cycles $\zeta_n$ vanish for $n > 3$ by [A11, Cor.3.3.8, p.34]).

Let $\mathcal{P}: \mathcal{R}_6 \to \mathcal{A}_5$ be the Prym map from the moduli space $\mathcal{R}_6$ of connected étale double covers $\tilde{C} \to C$ of smooth genus 6 curves to the moduli space $\mathcal{A}_5$ of 5-dimensional ppav’s. By [DS81], the morphism $\mathcal{P}$ is generically finite of degree 27. By the tetragonal construction described in [Do92, 2.5, p.76], given $(\tilde{C} \to C) \in \mathcal{R}_6$ and a degree 4 linear pencil $g_1^4$ on $C$ there exist étale double covers $\tilde{C}_0/C_0$ and $\tilde{C}_1/C_1$ of tetragonal curves whose associated Prym is isomorphic (as a ppav) to $P = \text{Prym}(\tilde{C}/C)$. Furthermore, by [Do92, Thm.4.2, p.90], the correspondence on the generic fiber of $\mathcal{P}$ induced by the tetragonal construction is isomorphic to the incidence correspondence of the 27 lines on a smooth cubic surface. Also, the monodromy group of the cover $\mathcal{R}_6 \to \mathcal{A}_5$ is the Weyl group $W$ of the $E_6$ lattice.

It follows from [Fa96] and [A12] that the 27 Abel-Prym curves on a generic ppav of dimension 5 are not pairwise algebraically equivalent, see Lemma 2.4. In this article we determine all the
algebraic equivalences (with \( \mathbb{Q} \)-coefficients) among these 27 effective 1-cycles. The main result is the following.

**Theorem** (=Theorem 2.10). The \( \mathbb{Q} \)-vector space of algebraic equivalences of the 27 Abel-Prym curves on a generic principally polarized abelian 5-fold has dimension 20.

The key ingredient for proving the above theorem is Lemma 2.9. This lemma implies that the action of the monodromy group of the cover \( \mathcal{R}_6 \to \mathcal{A}_5 \) on the 27 Abel-Prym curves on a generic ppav of dimension 5 preserves algebraic equivalences among the Abel-Prym curves (see [Har79] for a discussion of monodromy in algebraic geometry and its applications).

Throughout the paper we work over the field of complex numbers.

2. Algebraic equivalences

**Proposition 2.1** (Connectedness principle). Let \( Y \) be a smooth connected but not necessarily complete curve and let \( f: X \to Y \) be a proper morphism. If \( X \) is connected then the locus

\[
\text{CF}(f) := \{ y \in Y \mid X_y \text{ is connected} \}
\]

parametrizing connected fibers is Zariski closed in \( Y \).

**Proof.** Let \( f = h \circ g \) be the Stein factorization, where \( g: X \to Z \) has connected fibers and \( h: Z \to Y \) is a finite morphism. Since \( Z = g(X) \) and \( X \) is connected then \( Z \) is connected. If \( Z \) is a point, the conclusion holds trivially. Assume that \( Z \) is not a point, let \( Z_1, \ldots, Z_k \) be the irreducible components of \( Z \) and let \( h_i \) be the restriction of \( h \) to \( Z_i \). If \( k \geq 2 \) then there are at most finitely many points with connected fibers since each \( Z_i \) is mapping onto \( Y \). Thus it remains to consider the case \( k = 1 \), i.e., the case when \( Z \) is an irreducible curve and \( h: Z \to Y \) is a finite surjective morphism. We may and shall assume that \( Z \) is reduced. Consider the normalization \( \tilde{Z} \to Z \), which is a finite birational morphism onto \( Z \), and the composed morphism \( \tilde{h}: \tilde{Z} \to Z \to Y \). It suffices to show that the locus \( \text{CF}(\tilde{h}) \) is Zariski closed in \( Y \). If \( \tilde{h} \) has degree 1 then \( \tilde{h} \) is an isomorphism and \( \text{CF}(\tilde{h}) = Y \). If \( \tilde{h} \) has degree \( \geq 2 \), then \( \text{CF}(\tilde{h}) \), which must be contained in the branch locus, is at most a finite set of points and hence is Zariski closed in \( Y \). \( \square \)
Remark 2.2. The conclusion of the above proposition becomes false in general if $Y$ is a reducible curve, $\dim Y > 1$ or if $X$ is not connected.

Lemma 2.3. If $\tilde{C}_1/C_1, \tilde{C}_2/C_2, \tilde{C}_3/C_3$ is a tetragonally related triple on a Prym variety $P$ then $[\tilde{C}_1](2) + [\tilde{C}_2](2) + [\tilde{C}_3](2) = 0$ in $A^*(P)$.

Proof. Taking $\tilde{C}_1/C_1$ as a reference double cover, the curves $\tilde{C}_2, \tilde{C}_3 \subset P$ are the special subvarieties associated to some $g^1_4$ on $C_1$, see [BS2, 3(c), p.366]. By [A12, Ex.1, p.724] we have the relation $[\tilde{C}_2](2) + [\tilde{C}_3](2) = -[\tilde{C}_1](2)$. □

Lemma 2.4. On a generic ppav of dimension 5 there is a pair of Abel-Prym curves that are not algebraically equivalent.

Proof. By [Fa96], for a generic ppav $P \in A_5$ with Prym realization $P = \text{Prym}(\tilde{C}/C)$ we have $[\tilde{C}](2) \neq 0$. It is well known that $C$ has five $g^1_4$’s. Choosing a $g^1_4$ on the base curve $C$, the special subvarieties $V_0, V_1 \subset P$ associated to the $g^1_4$ are also Abel-Prym curves, [BS2, 3(c), p.366]. By Lemma 2.3 we have the relation $[V_0](2) + [V_1](2) + [\tilde{C}](2) = 0$. If the Abel-Prym curves $V_0, V_1$ and $\tilde{C}$ are pairwise algebraically equivalent then we have $3[\tilde{C}](2) = 0$. Since we are working with $\mathbb{Q}$-coefficients this implies $[\tilde{C}](2) = 0$, which is a contradiction. □

Lemma 2.5. If $\tilde{C}_1, \ldots, \tilde{C}_{27}$ are the 27 Abel-Prym curves on a generic ppav $P \in A_5$ then the vector space of algebraic equivalences among $\tilde{C}_1, \ldots, \tilde{C}_{27}$ induced by the tetragonal construction has dimension 20.

Proof. Let $\mathcal{V}$ be the $\mathbb{Q}$-vector space with basis $\tilde{C}_1, \ldots, \tilde{C}_{27}$ and consider the linear map

$$cl: \mathcal{V} \rightarrow A^4(P)$$

that takes an element of $\mathcal{V}$ to its class modulo algebraic equivalence. Let $\mathcal{V}_{\text{tet}} \subset \mathcal{V}$ be the subspace spanned by the vectors $\tilde{C}_i + \tilde{C}_j + \tilde{C}_k$, where $\tilde{C}_i, \tilde{C}_j, \tilde{C}_k$ are tetragonally related. The vector space $R_{\text{tet}} := \ker(cl: \mathcal{V}_{\text{tet}} \rightarrow A^4(P))$ is the space of algebraic equivalences induced by the tetragonal construction.

It follows immediately from [Do92, Thm.4.2, p.90] that tetragonally related triples are in bijection with the 45 triples of lines forming triangles on a smooth cubic surface in $\mathbb{P}^3$. Let us fix an ordering $L_1, \ldots, L_{27}$ of the 27 lines on the cubic surface. Each of the triangles determines
a vector in a 27 dimensional \( \mathbb{Q} \)-vector space with basis \( L_1, \ldots, L_{27} \) with 1’s in the coordinates corresponding to the lines forming the triangle, see [Dol10] and [Hum96, Ch.4] for details on the geometry of cubic surfaces. An explicit calculation shows that the dimension of the span of these 45 vectors is 21. Therefore, \( \dim V_{\text{tet}} = 21 \).

For each \( i \in \{1, \ldots, 27\} \) we have the formula \( [\tilde{C}_i] = [\tilde{C}_i]_{(0)} + [\tilde{C}_i]_{(2)} \) in \( A^4(P) \). The vanishing of the odd degree Beauville components of \( [\tilde{C}_i] \) is a consequence of the fact that \( \tilde{C}_i \) has a symmetric translate. The vanishing of components of degree \( \geq 4 \) was proven in [A11, Cor.3.3.8]. Furthermore, using the Fourier transform on \( A^*(P) \) it is easy to see that \( [\tilde{C}_i]_{(0)} \) for all \( i \in \{1, \ldots, 27\} \). Using Lemma 2.3, this implies that for every tetragonally related triple \( \tilde{C}_i, \tilde{C}_j, \tilde{C}_k \) the image vector \( cl(\tilde{C}_i + \tilde{C}_j + \tilde{C}_k) \) lies in the 1-dimensional subspace of \( A^4(P) \) spanned by \( [C_1]_{(0)} \). Therefore, \( \dim R_{\text{tet}} = 20 \). □

Let \( L_1, \ldots, L_{27} \) be the lines on a smooth cubic surface. It is well known that the group of symmetries of these lines is the Weyl group \( W \) of the \( E_6 \) lattice.

Lemma 2.6. The representation of \( W \) on \( \bigoplus_{i=1}^{27} \mathbb{Q}L_i \) decomposes into a direct sum of irreducible representations of dimensions 1, 6 and 20.

Proof. The Weyl group \( W \) has order 51840 and 25 conjugacy classes. Computing the character table for \( W \) and using the orthogonality relations between irreducible characters, it can be seen that \( \bigoplus_{i=1}^{27} \mathbb{Q}L_i \) decomposes into irreducible representations of dimensions 1, 6 and 20. Our calculations were carried out on Magma. □

Remark 2.7. The vectors determined by the 45 triangles in the cubic surface span a 21-dimensional subrepresentation, whose quotient is the representation of \( W \) on the \( \mathbb{Q} \)-span of the \( E_6 \) lattice, which is well-known to be absolutely irreducible, [Hum73, Lem.B, p.53]. Under the map of the free abelian group on \( L_1, \ldots, L_{27} \) to the Picard group of the cubic surface, each triangle is mapped to \( -K \) (the anti-canonical divisor class) and the perp of \( K \) is precisely the \( E_6 \) lattice.
Let $\mathcal{A}_5^0$ be the complement in $\mathcal{A}_5$ of the union of the branch divisor of $\mathcal{R}_6 \to \mathcal{A}_5$ and the singular locus of $\mathcal{A}_5$. Let $V$ be a non-empty connected Zariski open subset of $\mathcal{A}_5^0$ over which the Prym map $\mathcal{R}_6 \to \mathcal{A}_5$ is finite. We have the following key lemma.

**Lemma 2.8.** A general (smooth, connected but not complete) curve section $S$ of $V$ has the property that the map $\pi_1(S) \to \pi_1(V)$ induced by the inclusion is surjective.

**Proof.** If $V$ were projective, the result of the lemma would follow immediately by the Lefschetz hyperplane theorem for homotopy groups. Since $V$ is not projective, we have to compactify $V$ and resolve the resulting space before applying Lefschetz theorem. Although the argument is standard, we include it here for completeness.

Let $\bar{V}$ be a smooth projective variety containing $V$ and such that the complement $\bar{V} - V$ is a simple normal crossings (snc) divisor. To obtain $\bar{V}$, we may take the closure of $V$ in some projective space and apply Hironaka’s theorem to resolve the singularities and make the complement of $V$ an snc divisor (we identify $V$ with its proper transform in $\bar{V}$).

Let $D_1, \ldots, D_n$ be the irreducible components of $\bar{V} - V$. For a general curve section $\bar{S}$ of $\bar{V}$ we have $\bar{S} \cap D_i \neq \emptyset$ for $1 \leq i \leq n$ and $S := \bar{S} \cap V$ is a non-empty Zariski open set in $\bar{S}$. Also, by the theorem of Lefschetz (cf. [Mi63 Thm.7.4, p.41]) and induction we have a surjection $\pi_1(S) \to \pi_1(V)$.

Given a smooth connected complex manifold $M$ and a smooth connected divisor $\Gamma \subset M$, using van Kampen’s theorem we may check that there is an exact sequence

$$\mathbb{Z}\ell_\Gamma \to \pi_1(M - \Gamma) \to \pi_1(M) \to 1,$$

where $\ell_\Gamma$ is the class of a loop in $M$ that goes once around $\Gamma$.

Since $\bar{S} \cap D_1 \neq \emptyset$ then $\ell_1$ is in the image of $\pi_1(S) \to \pi_1(\bar{V} - D_1)$. Therefore, using the above exact sequence we have a natural diagram with all maps being surjective

$$
\begin{array}{ccc}
\pi_1(S) & \to & \pi_1(\bar{S}) \\
\downarrow & & \downarrow \\
\pi_1(\bar{V} - D_1) & \to & \pi_1(\bar{V}).
\end{array}
$$

Now, removing $D_i$’s one at a time and applying the above argument, we conclude by induction that there is a surjection $\pi_1(S) \to \pi_1(V)$. □
Recall that a connected cover \( f: X \rightarrow Y \) of topological spaces is said to be Galois (or regular, or normal, see \([\text{Hat02}, \text{p.70}]\)) if for every \( y \in Y \) and every pair of lifts \( x_1, x_2 \in X \) of \( y \) there is a deck transformation of \( X \) taking \( x_1 \) to \( x_2 \). Given a connected cover \( f: X \rightarrow Y \) of topological spaces we shall denote its Galois closure by \( \tilde{X} \rightarrow Y \). More precisely, \( \tilde{X} \) is the connected cover of \( Y \) corresponding to the normal subgroup of \( \pi_1(Y) \) obtained as the intersection of all the conjugates of \( f_*(\pi_1(X)) \) in \( \pi_1(Y) \). Also, the group of deck transformations of \( \tilde{X} \) over \( Y \) will be referred to as the monodromy group of \( X \) over \( Y \) and will be denoted by \( M(X/Y) \) in the sequel.

Let \( U := P^{-1}(V) \) be the inverse image of \( V \) under the Prym map \( P: \mathcal{R}_6 \rightarrow \mathcal{A}_5 \). By our construction, the cover \( U \rightarrow V \) is finite étale of degree 27. Unfortunately \( V \) does not carry a universal family of ppav’s, which we shall need in the sequel. In what follows, we shall show that there is a finite cover \( V' \rightarrow V \) such that \( V' \) carries a universal family of ppav’s and the monodromy group \( M(U'/V') \) of the pull-back \( U' \rightarrow V' \) of \( U \rightarrow V \) surjects onto \( W \). Let us argue that taking ppav’s with level 3 structure suffices for this purpose, i.e., we may take the connected étale cover \( \lambda: V' \rightarrow V \) with Galois group \( G := \text{Sp}(10, \mathbb{Z}/3)/\langle \pm I \rangle \). First, it is well known that the moduli space of 5-dimensional ppav’s with level 3 structure carries a universal family (see [BL04, Prop.8.8.2, p.233]), and in particular, so does \( V' \). Therefore, it suffices to prove that \( M(U'/V') \) surjects onto \( W \). By [Do92], there is a surjection \( \pi_1(V) \rightarrow W \) whose kernel \( N := \ker(\pi_1(V) \rightarrow W) \) is the image of \( \pi_1(\tilde{V}) \) under the natural homomorphism. The group \( M := \lambda_*(\pi_1(V')) = \ker(\pi_1(V) \rightarrow G) \) determines the Galois cover \( V' \rightarrow V \). The situation is summarized in the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(U') & \longrightarrow & \pi_1(U) \\
\downarrow & & \downarrow \\
\pi_1(V') & \longrightarrow & \pi_1(V) \\
\downarrow & & \downarrow \\
\lambda & & \\
\pi_1(\tilde{V}) & \hookrightarrow & \pi_1(\tilde{V}) \\
\downarrow & & \downarrow \\
W & & \\
\end{array}
\]

where the cokernel of the vertical map on the left is the monodromy group \( M(U'/V') \). To prove surjectivity of \( M(U'/V') \rightarrow W \) induced by the homomorphisms in the above diagram, it suffices to show that \( M \) surjects onto \( W \). By considering the indices of \( M \) and \( N \) in \( \pi_1(V) \) we
see immediately that \( N \) is not contained in \( M \). Indeed, the index of \( M \) is the order of \( G \) and is given by a standard formula. However, it suffices to note that \( H = \text{GL}(5, \mathbb{Z}/3)/(\pm I) \) injects as a subgroup of \( G \) and \( |H| = \frac{1}{2}(3^5 - 1)(3^5 - 3)(3^5 - 3^2)(3^5 - 3^3)(3^5 - 3^4) \), which is already much larger than 51840. Therefore, the image \( \bar{N} \) of \( N \) in the quotient \( \pi_1(V)/M \cong G \) is non-trivial.

Since \( N \) is a normal subgroup of \( \pi_1(V) \), \( \bar{N} \) is normal in the quotient group \( G \). Since \( G \) is a simple group (cf. [Di48, Thm.1, p.12]) and \( \bar{N} \) is non-trivial, we must have \( \bar{N} = G \). Thus, we have \( MN = \pi_1(V) \), which shows that the quotient map \( M \rightarrow \pi_1(V)/N \cong W \) is surjective.

By Lemma 2.8, there exists a smooth connected curve section \( S \) of \( V \) whose topological fundamental group surjects onto that of \( V \). Let \( S' := \lambda^{-1}(S) \) and let \( R' \) be the pull-back of \( U' \) to \( S' \). The covering \( \mu: S' \rightarrow S \) is determined by the inverse image \( Q \) of \( \lambda_\ast(\pi_1(V')) \) in \( \pi_1(S) \) under the homomorphism \( \pi_1(S) \rightarrow \pi_1(V) \) induced by the inclusion (in particular, \( Q = \mu_\ast\pi_1(S') \)). Therefore, since \( \lambda_\ast \) is injective, from the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(V') & \xrightarrow{\lambda_\ast} & \pi_1(V) \\
\downarrow & & \downarrow \\
\pi_1(S') & \xrightarrow{\mu_\ast} & \pi_1(S)
\end{array}
\]

we see that the vertical map on the left is surjective. Using the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(\tilde{U}') & \longrightarrow & \pi_1(V') \\
\downarrow & & \downarrow \\
\pi_1(\tilde{R}') & \longrightarrow & \pi_1(S')
\end{array}
\]

and surjectivity of \( \pi_1(S') \rightarrow \pi_1(V') \) we conclude that the monodromy group of \( R' \rightarrow S' \) still surjects onto \( W \).

To summarize some of the constructions so far, we now have over the smooth curve \( S' \), a degree 27 finite étale map \( R' \rightarrow S' \) and a family \( P' \rightarrow S' \) of ppav's (with level 3 structure), such that for each point \( s' \in S' \), each one of the 27 points in the fiber of \( R' \rightarrow S' \) over \( s' \) (equivalently, in the fiber in \( \mathcal{R}_6 \) over the image point of \( V \subset \mathcal{A}_5 \)) corresponds to exactly one of 27 Abel-Prym curves in the abelian variety \( P'_{s'} \). Each Abel-Prym curve is uniquely determined up to translations (since the automorphism group of the ppav \( P'_{s'} \) is \( \{\pm 1\} \), and an Abel-Prym curve always has some translates that are invariant under \(-1\)). Since translation in the abelian
variety defines an obvious algebraic equivalence between a 1-cycle and any of its translates, maintaining a careful distinction between an Abel-Prym curve $C_i$ and the collection of all its translates, is not the essential point in our arguments; we are going to suppress the distinction, for simplicity. What is critical is that the mapping $R' \to S'$ still has monodromy group $W$.

Additionally, for simplicity of notation, we shall now write $R$ and $S$ for $R'$ and $S'$, respectively, in the sequel. Thus, with this updated notation, let $\mathcal{P}$ be the universal family of 5-dimensional ppav’s (with level 3 structure) over $S$. The fiber over $s \in S$ will be denoted by $\mathcal{P}_s$.

Let $\mathcal{V}$ be the local system on $S$ whose fiber over a point $s \in S$ is the 27-dimensional vector space spanned by the Abel-Prym curves in $\mathcal{P}_s$. Given a contractible open set $U \subset S$ in the analytic topology we fix a trivialization

$$\mathcal{V}(U) \simeq \bigoplus_{i=1}^{27} \mathbb{Q}C_i,$$

and let $C_{i,s}$ denote the fiber of $C_i$ over $s \in U$. Thus, the fiber of $\mathcal{V}$ over $s \in U$ is $\mathcal{V}_s = \bigoplus_{i=1}^{27} \mathbb{Q}C_{i,s}$.

To simplify the notation, for $q = (q_1, \ldots, q_{27}) \in \mathbb{Q}^{27}$ define

$$Z_s(q) := \sum_{i=1}^{27} q_i C_{i,s}.$$

Since $S$ has a base for the analytic topology consisting of contractible sets, the assignment

$$U \mapsto \mathcal{T}(U) := \{ Z_s(q) \mid Z_s(q) \sim_{\text{alg}} 0 \text{ for all } s \in U \}$$

defines a presheaf on $S$ in the analytic topology. In the sequel, we let $\mathcal{T}$ denote the associated sheaf on $S$.

**Lemma 2.9.** The subsheaf $\mathcal{T}$ of $\mathcal{V}$ is a local subsystem of $\mathcal{V}$.

**Proof.** Given a contractible open set $U \subset S$ in the analytic topology, a trivialization $\mathcal{V}(U) \simeq \bigoplus_{i=1}^{27} \mathbb{Q}C_i$ and $q = (q_1, \ldots, q_{27}) \in \mathbb{Q}^{27}$ as above, we claim that the locus

$$\{ s \in U \mid Z_s(q) \sim_{\text{alg}} 0 \}$$

is either all of $U$ or a countable set of points in $U$. Assuming this claim, let us prove the lemma. Fix $s \in S$ for the remainder of the proof and consider the monodromy representation

$$\rho: \pi_1(S, s) \to \text{Aut}(\mathcal{V}_s).$$
It is well-known that there is a natural bijection between isomorphism classes of local systems and monodromy representations up to conjugation, see [V07, Cor. 3.10, p. 71]. Under this bijection subrepresentations correspond to local subsystems. Therefore, to show that $T$ is a local subsystem of $V$, it suffices to prove that for each $[\gamma] \in \pi_1(S, s)$ the automorphism $\rho([\gamma])$ of $V$ maps $T$ to itself. The image of $\gamma$: $[0, 1] \to S$ can be covered by finitely many contractible open sets $U_1, \ldots, U_n$ for the analytic topology. Assume $s \in U_1$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n - 1$.

Given an element $Z_s(q) \in T$, by the above claim we know that $Z_t(q) \in T$ for all $t \in U_1$. Applying the same argument to $U_2, \ldots, U_n$, we conclude that the parallel transport of $Z_s(q)$ stays in the fibers of $T$. Therefore, $\rho([\gamma])$ maps $T$ to itself.

In the remainder of the proof we show the above claim. Let $\tilde{R} \to S$ be the Galois closure of the cover $R \to S$. Let $\tilde{V}$ be the pull-back of $V$ to $\tilde{R}$ under the composition $\tilde{R} \to R \to S$. Since the claim is about algebraic equivalences in the fibers of $V$, it suffices to show the claim for $\tilde{V}$ globally over $\tilde{R}$. The local system $\tilde{V}$ trivializes over $\tilde{R}$ and we may write

$$\tilde{V} \simeq \bigoplus_{i=1}^{27} QC_i.$$ 

Let $\tilde{P} := P \times_S \tilde{R} \to \tilde{R}$ be the pull-back of the universal family. By clearing the denominators and rearranging the terms we may rewrite $Z_s(q) \sim_{\text{alg}} 0$ as

$$\sum_{i \in I} a_i C_{i,s} \sim_{\text{alg}} \sum_{j \in J} b_j C_{j,s}, \quad (2.1)$$

where $a_i, b_j$ are non-negative integers, $I \cup J = \{1, \ldots, 27\}$ and $I \cap J = \emptyset$. Furthermore, (2.1) holds if and only if there exists an effective cycle $E_s$ on $\tilde{P}$ such that the cycles $E_s + \sum_{i \in I} a_i C_{i,s}$ and $E_s + \sum_{j \in J} b_j C_{j,s}$ lie in the same connected component of the Chow variety of $\tilde{P}$, see [K96, 4.1.3, p. 122]. In what follows we may and shall assume that the effective cycle $E_s$ is effectively algebraically equivalent to a sufficiently high multiple of the cycle $\Xi^4_s$, where $\Xi_s$ is the theta divisor on $\tilde{P}$ (see [K96, 4.1.2, p. 121] for the definition of effective algebraic equivalence). More precisely, we may arrange, by taking another finite cover if necessary, that there is a family over $\tilde{R}$ of theta divisors $\Xi_s \subset \tilde{P}$, and we take $E_s$ to be the intersection of four general elements of the linear system $|n\Xi_s|$ for $n \gg 0$. To emphasize the role of $n$ we write $E_{n,s}$ for such $E_s$. This assumption is needed for the following two reasons. First, to ensure that $E_{n,s}$ extends to a cycle over all of $\tilde{R}$, i.e., that there is an effective cycle $E_n$ flat over $\tilde{R}$ whose fiber over $s \in \tilde{R}$ is $E_{n,s}$.
Second, to ensure that is suffices to consider at most countably many effective relative cycles \( \mathcal{E}_n \) (indexed by \( n \)) in order to exhibit algebraic equivalence (provided it holds) of \( \sum_{i \in I} a_i C_{i,t} \) and \( \sum_{j \in J} b_j C_{j,t} \) on \( \tilde{\mathcal{P}}_t \) by adding on the fiber of \( \mathcal{E}_n \) over \( t \).

Consider the relative Chow variety of \( \tilde{\mathcal{P}} \) over \( \tilde{\mathcal{R}} \) that parametrizes effective cycles on \( \tilde{\mathcal{P}} \) which lie over 0-dimensional subschemes of \( \tilde{\mathcal{R}} \), see [K96, Def.3.1.1, p.41 and Def.3.20, p.51]. Let \( E_{n,t} \) denote the fiber of \( E_n \) over \( t \in \tilde{\mathcal{R}} \). It is easily seen that the cycles \( E_{n,t} + \sum_{i \in I} a_i C_{i,t} \) and \( E_{n,t} + \sum_{j \in J} b_j C_{j,t} \) are algebraically equivalent on \( \tilde{\mathcal{P}} \) (but not necessarily on \( \tilde{\mathcal{P}}_t \)), and therefore, lie in the same connected component \( \mathcal{X} \) of the relative Chow variety of \( \tilde{\mathcal{P}} \) over \( \tilde{\mathcal{R}} \). Applying Proposition 2.1 to the natural morphism \( \mathcal{X} \to \tilde{\mathcal{R}} \) we conclude that the set of \( t \in \tilde{\mathcal{R}} \) such that \( E_{n,t} + \sum_{i \in I} a_i C_{i,t} \) and \( E_{n,t} + \sum_{j \in J} b_j C_{j,t} \) lie in the same connected component of the Chow variety of \( \tilde{\mathcal{P}}_t \) is Zariski closed in \( \tilde{\mathcal{R}} \).

If \( \sum_{i \in I} a_i C_{i,t} \) and \( \sum_{j \in J} b_j C_{j,t} \) are algebraically equivalent on \( \tilde{\mathcal{P}}_t \), there exists \( m \geq n \) and a relative cycle \( \mathcal{E}_m \) (constructed as above) such that \( E_{m,t} + \sum_{i \in I} a_i C_{i,t} \) and \( E_{m,t} + \sum_{j \in J} b_j C_{j,t} \) lie in the same component of the Chow variety of \( \tilde{\mathcal{P}}_t \). Furthermore, since the choice of the four effective divisors in \( \mid m \Xi_t \mid \) that define \( E_{m,t} \) is unrestricted (as long as they intersect properly), it suffices to consider at most countably many such cycles \( \mathcal{E}_m \) in order to exhibit algebraic equivalence of \( \sum_{i \in I} a_i C_{i,t} \) and \( \sum_{j \in J} b_j C_{j,t} \) on \( \tilde{\mathcal{P}}_t \) (for any \( t \)). Therefore, we conclude that the locus of \( t \in \tilde{\mathcal{R}} \) such \( \sum_{i \in I} a_i C_{i,t} \) and \( \sum_{j \in J} b_j C_{j,t} \) are algebraically equivalent on \( \tilde{\mathcal{P}}_t \) is either all of \( \tilde{\mathcal{R}} \) or a countable union of points in \( \tilde{\mathcal{R}} \).

**Theorem 2.10.** The \( \mathbb{Q} \)-vector space of algebraic equivalences of the 27 Abel-Prym curves on a generic principally polarized abelian 5-fold has dimension 20.

**Proof.** By Lemma 2.5 the fibers of \( \mathcal{T} \) have dimension \( \geq 20 \). By definition of \( \mathcal{T} \), for each \( s \in S \) the 1-dimensional sub-representation \( \mathcal{L}_s \) of \( \mathcal{V}_s \) spanned by the Beauville degree 0 graded piece of an Abel-Prym curve intersects \( \mathcal{T}_s \) trivially. By Lemma 2.4, the composition \( \mathcal{T}_s \hookrightarrow \mathcal{V}_s \to \mathcal{V}_s / \mathcal{L}_s \) is not surjective for generic \( s \in S \). Therefore, for generic \( s \in S \) we have \( \dim \mathcal{T}_s \leq 25 \). By Lemma 2.9, \( \mathcal{T} \) is invariant under the monodromy of the Galois closure of \( \mathcal{R} \to S \), which by our construction surjects onto the Weyl group \( W \). By Lemma 2.6 we conclude that the fibers of \( \mathcal{T} \) have dimension 20. 

\[ \square \]
Corollary 2.11. On a generic ppav $P$ of dimension 5 with Abel-Prym curves $	ilde{C}_1, \ldots, \tilde{C}_{27}$ we have: (1) $\mathcal{T}(P, \tilde{C}_i) \neq \mathcal{T}(P, \tilde{C}_j)$ if $i \neq j$; (2) $[\tilde{C}_i]_{(2)} \neq 0$ in $A^*(P)$ for $1 \leq i \leq 27$.

Proof. (1) For each $i \in \{1, \ldots, 27\}$, the bi-graded piece of $\mathcal{T}(P, \tilde{C}_i)$ of codimension 4 and Beauville degree 2 is spanned by $[\tilde{C}_i]_{(2)}$. If $\mathcal{T}(P, \tilde{C}_i) = \mathcal{T}(P, \tilde{C}_j)$ then $[\tilde{C}_i]_{(2)}$ and $[\tilde{C}_j]_{(2)}$ are proportional, which is easily seen to contradict the structure of the vector space of relations among Abel-Prym curves from the proof of Theorem 2.10.

(2) We may check that the class $[\tilde{C}_i]_{(2)}$ does not belong to the 20-dimensional $\mathbb{Q}$-vector space of algebraic equivalences among $\tilde{C}_1, \ldots, \tilde{C}_{27}$ from Theorem 2.10. □

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