Large data global well-posedness and scattering for the focusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}$

Yongming Luo

Abstract

We consider the focusing cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t U + \Delta U = -|U|^2 U \quad \text{on } \mathbb{R}^2 \times \mathbb{T}. \quad (3\text{NLS})$$

Different from the 3D Euclidean case, the (3NLS) is mass-critical and non-scale-invariant on the waveguide manifold $\mathbb{R}^2 \times \mathbb{T}$, hence the underlying analysis becomes more subtle and challenging. We formulate thresholds using the 2D Euclidean ground state of the focusing cubic NLS and show that solutions of (3NLS) lying below the thresholds are global and scattering in time. The proof relies on several new established Gagliardo-Nirenberg inequalities, whose best constants are formulated in term of the 2D Euclidean ground state. It is also worth noting the interesting fact that the thresholds for global well-posedness and scattering do not coincide. To the author’s knowledge, this paper also gives the first large data scattering result for focusing NLS on product spaces.

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on the waveguide manifold $\mathbb{R}_x^2 \times T_y$. The equation (1.1) arises from various domains in applied sciences such as nonlinear optics and Bose-Einstein condensation. We refer to \cite{12,13,14} for a detailed introduction on the physical background of (1.1). It is worth noting at this point that although large data problems for the defocusing analogues of (1.1) have been extensively studied \cite{22,19,13,53,52}, results concerning the focusing models are relatively less well-known. The purpose of this paper is to give a first step towards the large data scattering\(^1\) for NLS of focusing type on product spaces.

A first systematic study on NLS on compact manifolds might at least date back to \cite{7}, where the authors studied the cubic NLS on a bounded domain or an exterior domain in $\mathbb{R}^2$. Concerning NLS on tori, Bourgain studied in his seminal papers \cite{1,5} the NLS and KdV equations on rational tori. Particularly, using number theoretical methods Bourgain proved (endpoint and non-endpoint) Strichartz estimates for NLS and KdV on rational tori, which were also utilized to establish different local and global well-posedness results. As a byproduct of the proof of the $\ell^2$-decoupling conjecture \cite{6}, the Strichartz estimates on rational tori were later extended to irrational ones, including both endpoint and non-endpoint cases.

For NLS on more general compact manifolds, a systematic study was initiated in a series of works \cite{8,9,10} by Burq, Gérard and Tzvetkov, where the authors proved Strichartz and multilinear estimates, local and global well-posedness results for NLS on compact manifolds. Using the theory of atomic spaces initiated by Koch and Tataru \cite{59}, Herr, Tataru, and Tzvetkov were able to prove local and global well-posedness results for NLS on tori, product spaces and Zoll-manifolds in the energy-critical case \cite{26,27,28}. However, at the energy-critical level the well-posedness results also depend on the profile of the initial data, and a large data result can in general not be obtained using only the \textit{a priori} bounds deduced from the conservation laws. Following the nowadays well-known concentration compactness arguments initiated by Kenig and Merle \cite{55} and the so-called Black-Box-Theory, Ionescu, Pausader and Staffilani \cite{30,31,32} showed that defocusing energy-critical NLS on $T^d$, $\mathbb{R} \times T^d$ and on the hyperbolic space $\mathbb{H}^3$ are always globally well-posed. By appealing to suitable variational arguments, Yu, Yue and Zhao \cite{51,50} utilized the Black-Box-Theory to prove that solutions of the focusing energy-critical NLS on $T^d$ and $\mathbb{R} \times T^d$ lying below ground states are always globally well-posed.

The above mentioned models can be generalized to the NLS

$$i \partial_t U + \Delta_{x,y} U = \pm |U|^\alpha U$$

(1.2)

on the manifold $\mathbb{R}^d \times M$, where $M$ is an $n$-dimensional compact manifold. Loosely speaking, the dispersion of an NLS-wave on a compact manifold is much weaker than on $\mathbb{R}^d$, hence it is not expected that scattering takes place for large $n$. Indeed, even in the defocusing case, a global but not scattering solution of NLS on $T^d$ does exist, see for instance \cite{14}. Nonetheless, in view of the classical long time dynamics results for NLS on Euclidean spaces\(^2\), the rather weak dispersion effect corresponding to the manifold $M$ can be compensated by the stronger dispersion on $\mathbb{R}^d$, and scattering solutions\(^3\) are expected when

(i) The nonlinearity is at most energy-critical w.r.t. the space dimension $d+n$, and

(ii) The nonlinearity is at least mass-critical w.r.t. the space dimension $d$.

In other words, we expect that a general scattering theory as the one formulated in the Euclidean case should be available for $\alpha$ lying in the range $\left[\frac{4}{d}, \frac{4}{d+n-2}\right]$. Particularly, it is necessary that $n \leq 2$. In this direction, the first contribution was made by Tzvetkov and Visciglia \cite{16}, where the authors studied well-posedness and scattering of solutions of (1.2) with small initial data in non-isotropic Sobolev spaces. The same authors studied later in \cite{57} the special case where $M = T^d$ and (1.2) is defocusing. Particularly, they proved that (1.2) is always globally well-posed for $\alpha \in (0, \frac{4}{d+1})$ and additionally scattering for $\alpha \in \left(\frac{4}{d+1}, \frac{4}{d+n-2}\right)$ in $H^1(\mathbb{R}^d \times T^d)$. In the case where $M = T^2$ and the nonlinearity is mass-critical ($\alpha = \frac{4}{d+1}$) or energy-critical ($\alpha = \frac{4}{d+n-2}$), the first breakthrough was made by Hani and Pausader \cite{22}, where they studied defocusing (1.2) with $\alpha = 4$, $d = 1$ and $n = 2$, which is the well-known defocusing quintic NLS on $\mathbb{R} \times T^2$. Particularly, (1.2) in this case is both mass- and energy-critical. Based on a conjecture on large data scattering of the large scale resonant system of (1.2), which was later solved by Cheng, Guo and Zhao \cite{13}, Hani and Pausader proved that a solution of (1.2) is always global and scattering. Utilizing

\(^{1}\)We mainly focus on the (much harder) scattering problem. In fact, due to the energy-subcritical nature of (1.2) a large data global well-posedness result follows immediately from Lemma 3.14 given below. See Theorem 1.7 for details.

\(^{2}\)We refer an NLS on Euclidean space to an NLS on $\mathbb{R}^d$.

\(^{3}\)We are referring here to scattering in the $H^1$-energy space, which is the minimal space that admits all useful conservation laws. Such heuristics do not hold when the scattering is considered in spaces of higher differentiation order, see \cite{55}.
the methodologies of [22] for (1.2), the ones of [19] for the corresponding large scale resonant system of (1.2) and the Black-Box-Theory, the large data scattering problem for defocusing (1.2) with critically algebraic nonlinearities on $\mathbb{R}^d \times \mathbb{T}^n$ has been completely resolved [22, 19, 13, 53, 52]. We also refer to [23, 8, 44, 21, 1] for further interesting topics in this direction.

Let us now focus on the focusing cubic NLS (1.1) and explain briefly the mass-criticality of (1.1). Indeed, we may simply assume that (1.1) is independent of the $y$-variable, and in this case (1.1) reduces to the 2D Euclidean focusing cubic NLS, which is known to be mass-critical. In order to incorporate the full impact of $\mathbb{T}$ into the problem, we should instead consider the following scaling transformation inspired by Hani and Pausader [22]: it is easy to verify that (1.1) remains invariant under the scaling transformation $\lambda \to \lambda^4$

\[
U(t, x, y) = \sum_{k \in \mathbb{Z}} (\mathcal{F}_y U)(t, x, k)e^{iky}.
\]

Hence we may represent the nonlinear potential $|U|^2U$ by

\[
(|U|^2U)(y) = \sum_{k} \sum_{(k_1, k_2, k_3) \in \mathcal{I}_k} \mathcal{F}_y U(k_1) \mathcal{F}_y^* U(k_2) \mathcal{F}_y U(k_3)e^{iky},
\]

where

\[
\mathcal{I}_k := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1 - k_2 + k_3 = k\}.
\]

We may further decompose $\mathcal{I}_k$ into the resonant part (RS) and non-resonant part (NRS):

\[
\mathcal{I}_k = \{(k_1, k_2, k_3) \in \mathcal{I}_k : k_1^2 - k_2^2 + k_3^2 = k^2\} \cup \{(k_1, k_2, k_3) \in \mathcal{I}_k : k_1^2 - k_2^2 + k_3^2 \neq k^2\} =: \mathcal{R}S_k \cup \mathcal{N}RS_k.
\]

The idea is as follows: the RS part can be seen as a non-perturbative component that should be dealt in a more complex and serious way. Nevertheless, by applying a normal form transformation, the NRS part can be relaxed in the large scale limitation. We refer to [22, Lem. 5.7] or [12, Lem. 3.11] for details of a rigorous verification of such intuitive heuristics. This suggests us to study the large scale resonant system

\[
i\partial_t u_j + \Delta_x u_j = -\left( \sum_{(j_1, j_2, j_3) \in \mathcal{R}S_j} u_{j_1} \bar{u}_{j_2} u_{j_3} \right) u_j, \quad j \in \mathbb{Z}
\]

on $\mathbb{R}^2_x$. By fundamental counting combinatorics, the large scale resonant system can be reformulated to

\[
i\partial_t u_j + \Delta_x u_j = -\left( \sum_i |u_i|^2 + \sum_{i \neq j} |u_i|^2 \right) u_j, \quad j \in \mathbb{Z},
\]

which will be the main model under consideration in the remaining part of the present paper.

Before we turn to the main results, we recall several conservation laws and symmetry invariance of the NLS which will be useful for the upcoming proofs. For the NLS (1.1), we have following classical conservation laws:

- **Mass**: $\mathcal{M}(U) = \|U\|_{L^2(\mathbb{R}^2 \times \mathbb{T})}^2$.
- **Energy**: $\mathcal{E}(U) = \frac{1}{2} \|\nabla_x U\|_{L^2(\mathbb{R}^2 \times \mathbb{T})}^2 - \frac{1}{4} \|U\|_{L^4(\mathbb{R}^2 \times \mathbb{T})}^4$.
- **Momentum**: $\mathcal{P}(U) = \text{Im} \int_{\mathbb{R}^2 \times \mathbb{T}} \nabla_x \bar{U} dx dy.$

\[\text{Since (1.1) is energy-subcritical, the small scale limit } \lambda \to 0 \text{ is irrelevant.}\]
Moreover, by direct calculation it is also immediate that \((1.1)\) and \((1.3)\) are invariant under the Galilean transformation

\[
U(t, x, y) \mapsto e^{i\xi \cdot x - i|\xi|^2} U(t, x - 2\xi t, y),
\]

\[
u(t, x) \mapsto e^{i\xi \cdot x - i|\xi|^2} \nu(t, x - 2\xi t)
\]

for arbitrary \(\xi \in \mathbb{R}^2\), where the Galilean transformation in \((1.4)\) is understood componentwise.

### 1.1 Main results

We begin with formulating the large data scattering result for the large scale resonant system \((1.3)\). Following the idea in [18], we define the Weinstein problem by

\[
C_{\text{GW,rs}} := \inf_{u \in H^1(\mathbb{R}^2)} \frac{\|u\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^2)}^2}{\|u\|_{L^4(\mathbb{R}^2)}^4}
\]

As revealed in [18], the Weinstein problem \((1.3)\) is closely related to the Gagliardo-Nirenberg inequality and provides a sharp threshold for well-posedness problems of NLS of focusing type. Our first result gives a precise description of the constant \(C_{\text{GW,rs}}\) in terms of the 2D Euclidean ground state of the focusing cubic NLS.

**Proposition 1.1** (Large scale Gagliardo-Nirenberg inequality). Define

\[
C_{\text{GW,2d}} := \inf_{u \in H^1(\mathbb{R}^2)} \frac{\|u\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^2)}^2}{\|u\|_{L^4(\mathbb{R}^2)}^4}
\]

Then \(C_{\text{GW,rs}} = \frac{1}{4} C_{\text{GW,2d}}\).

**Remark 1.2.** By Pohozaev’s identity (see for instance [3]) it is immediate that

\[
C_{\text{GW,rs}} = \frac{1}{2} C_{\text{GW,2d}} = \frac{1}{4} \mathcal{M}(Q_{2d}),
\]

where \(Q_{2d}\) is the unique positive, radial solution of

\[
-\Delta Q_{2d} + Q_{2d} = Q_{2d}^3 \quad \text{on } \mathbb{R}^2.
\]

Proposition 1.1 motivates the following large data scattering result for \((1.3)\):

**Theorem 1.3** (Large data scattering for the large scale resonant system). Let \(u_0 \in h^1 L^2\) satisfy \(M_0(u) < \frac{1}{4} \mathcal{M}(Q_{2d})\). Then a solution \(u\) of \((1.3)\) with \(u(0) = u_0\) is global and scattering in time, i.e. there exist \(\phi^\pm \in h^1 L^2\) such that

\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} \phi^\pm\|_{h^1 L^2} = 0.
\]

The proof follows the standard concentration compactness arguments from [25]. In order to exclude the minimal-blowup solution, we invoke the so-called long time Strichartz estimate to rule out the rapid cascade and quasi-soliton scenarios. Such long time Strichartz estimates were initiated by Dodson [17, 18, 15, 16, 19] for the study of Euclidean mass-critical NLS. The one we use in this paper is the vector-valued variant deduced in [19] for the defocusing analogue of \((1.3)\). We point out that ruling out the
rapid cascade scenario is just a straightforward modification of the same arguments given in [18, 49], by combining also Proposition 1.1. However, the Planchon-Vega-type interaction Morawetz inequality [11] applied in [49] cannot be used for the focusing model to rule out the quasi-soliton scenario. Alternatively, we utilize the potentials constructed in [16] to achieve this goal.

We now turn our attention to the main model (1.1). As usual, the focusing nature of (1.1) generally does not admit scattering for arbitrary initial data, and a suitable variational analysis for formulating scattering thresholds will be necessary. The starting point of our variational analysis is the following scale-invariant (w.r.t. \( x \)-variable) Gagliardo-Nirenberg inequality of additive type.

**Proposition 1.4** (Gagliardo-Nirenberg inequality on \( \mathbb{R}^2 \times \mathbb{T} \)). Let \( C_\Gamma \) be the best constant of the inequality

\[
\| u \|_{L^4(\mathbb{T})} \leq C_\Gamma \| u \|_{L^2(\mathbb{T})}^{\frac{2}{3}} \| \nabla_y u \|_{L^2(\mathbb{T})}
\]

for functions \( u \in H^1(\mathbb{T}) \) with \( \int_\mathbb{T} u \, dy = 0 \). Let also

\[
\mathcal{G}_{GN,2d} := \inf_{u \in H^1(\mathbb{R}^2)} \frac{\| u \|_{L^2(\mathbb{R}^2)}^{\frac{2}{3}} \| \nabla_x u \|_{L^2(\mathbb{R}^2)}^{\frac{4}{3}}}{\| u \|_{L^6(\mathbb{R}^2)}}.
\]

Then there exists some \( c \in (0, C_\Gamma^\frac{3}{4}) \mathcal{G}_{GN,2d}(1 + (2\pi)^{-\frac{5}{4}}) \) such that for all \( u \in H^1(\mathbb{R}^2 \times \mathbb{T}) \) we have

\[
\| u \|_{L^4(\mathbb{R}^2 \times \mathbb{T})} \leq \| \nabla_x u \|_{L^2(\mathbb{R}^2 \times \mathbb{T})} \left( \left( \pi \mathcal{M}(Q_{2d}) \right)^{\frac{1}{4}} \| u \|_{L^4(\mathbb{R}^2 \times \mathbb{T})} + c \| u \|_{L^4(\mathbb{R}^2 \times \mathbb{T})} \| \nabla_y u \|_{L^2(\mathbb{R}^2 \times \mathbb{T})} \right)^{\frac{1}{2}}.
\]  

**Remark 1.5.** Here follow several comments on Proposition 1.4.

(i) The existence of \( \mathcal{G}_{GN,2d} \) follows from the classical Euclidean Gagliardo-Nirenberg inequality. The existence of \( C_\Gamma \) will be proven in the proof of Proposition 1.4.

(ii) The constant \( (\pi \mathcal{M}(Q_{2d}))^{-\frac{1}{4}} \) is sharp in the sense that there exists no non-negative number \( \tilde{c} \geq 0 \) such that \( (1.6) \) holds when \( (\pi \mathcal{M}(Q_{2d}))^{-\frac{1}{4}} \) is replaced by a smaller number and \( c \) is replaced by \( \tilde{c} \). Indeed, we can simply take \( u \) independent of \( y \in \mathbb{T} \) and the second term in \( (1.6) \) is equal to zero. Then replacing \( (\pi \mathcal{M}(Q_{2d}))^{-\frac{1}{4}} \) by any smaller number would lead to a contradiction to \( (1.6) \). On the other hand, \( (1.7) \) can not hold for \( c = 0 \). To see this, we can simply insert \( u(x, y) = Q_{2d}(x) \phi(y) \) into \( (1.7) \) to obtain the contradiction \( \| \phi \|_{L^4(\mathbb{T})} \leq (2\pi)^{-\frac{5}{4}} \| \phi \|_{L^4(\mathbb{T})} \) for all \( \phi \in H^1(\mathbb{T}) \). However, we do not know if the number \( C_\Gamma^\frac{3}{4} \mathcal{G}_{GN,2d}(1 + (2\pi)^{-\frac{5}{4}}) \) is optimal.

\( \triangle \)

In view of (1.7) and Remark 1.5 we define

\[
c_\Gamma := \inf \{ c > 0 : (1.7) \) holds for \( c > 0 \}.
\]

Having all the preliminaries we are able to formulate the large data scattering result for (1.1):
Theorem 1.7. Let the conditions in Theorem 1.6 up to (1.10), be retained, and let (1.10) be replaced by the weaker condition
\[ M(U_0) \in (0, 2\pi M(Q_{2d})) \] (1.13)
Then a solution \( U \) of (1.1) with \( U(0) = U_0 \) is global.

Proof. The proof is short, thus we already record it at the beginning of the paper. Since (1.1) is energy-subcritical, it is well-known (see for instance [11]) that global well-posedness is equivalent to the statement that for all \( n \in \mathbb{N} \) we have
\[ \sup_{t \in [-n, n]} \| \nabla_{x,y} U(t) \|_{L^2(\mathbb{R}^2 \times \mathbb{T})} < \infty, \]
which follows immediately from Lemma 3.14 below. \( \square \)

1.2 Notation and definitions

We will use the notation \( A \lesssim B \) whenever there exists some positive constant \( C \) such that \( A \leq CB \).
Similarly we define \( A \gtrsim B \) and we will use \( A \sim B \) when \( A \lesssim B \lesssim A \). For simplicity, we will in most cases ignore the dependence of the function spaces on their underlying domains and hide this dependence in their indices. For example \( L^2_\ell L^2_\ell = L^2(\mathbb{Z}, L^2(\mathbb{R}^2)) \), \( H^1_{r,y} = H^1(\mathbb{R}^2 \times \mathbb{T}) \) and so on. However, when the space is involved with time, we will still display the underlying domain such as \( L^2_{r,x}(t) \), \( L^\infty_t L^2_x(\mathbb{R}) \) etc. The space \( h^1\dot{H}^1_x \) is defined through the norm
\[ \| f \|_{H^1_x} := \sum_j \| f \|_{H^1_x}^2 \]
for \( f : \mathbb{Z} \to \dot{H}^1_x \). We denote by \( g_{\xi_0, x_0, \lambda_0} \) the \( L^2_x \)-symmetry transformation defined by
\[ g_{\xi_0, x_0, \lambda_0} f(x) := \lambda_0^{-1} e^{i\xi_0 \cdot x} f(\lambda_0^{-1}(x-x_0)) \]
for \((\xi_0, x_0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \). We define the Fourier transformation of a function \( f \) w.r.t. \( x \in \mathbb{R}^2 \) or \( y \in \mathbb{T} \) by
\[ \hat{f}_y(x, k) = F_y f(x, k) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{T}} f(x, y) e^{-iky} dy, \]
\[ \hat{f}_x(\xi, y) = F_x f(\xi, y) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^2} f(x, y) e^{-i\xi \cdot x} dx, \]
\[ \hat{f}_{\xi,k}(x) = F_{x,y} f(\xi, k) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^2 \times \mathbb{T}} f(x, y) e^{-i(\xi \cdot x + ky)} dx dy. \]
Let \( \phi \in C^\infty_c(\mathbb{R}^2) \) be a fixed radial, non-negative and radially decreasing function such that \( \psi(x) = 1 \) if \( |x| \leq 1 \) and \( \psi(x) = 0 \) for \( |x| \geq \frac{11}{10} \). Then for \( N > 0 \), we define the Littlewood-Paley projectors w.r.t \( x \)-variable by
\[ P_{\leq N} f(x) = F^{-1}_x \left( \phi \left( \frac{x}{N} \right) \hat{f}(\xi) \right)(x), \]
\[ P_N f(x) = F^{-1}_x \left( \left( \phi \left( \frac{x}{N} \right) - \phi \left( \frac{2x}{N} \right) \right) \hat{f}(\xi) \right)(x), \]
\[ P_{> N} f(x) = F^{-1}_x \left( \left( 1 - \phi \left( \frac{x}{N} \right) \right) \hat{f}(\xi) \right)(x). \]
We also record the following well-known Bernstein inequalities which will be frequently used throughout the paper: For all \( s \geq 0 \) and \( 1 \leq p \leq \infty \) we have
\[ \| P_{> N} f \|_{L^p} \lesssim N^{-s} \| \nabla_x^s P_{> N} f \|_{L^p}, \]
\[ \| \nabla_x^s P_{\leq N} f \|_{L^p} \lesssim N^s \| P_{\leq N} f \|_{L^p}, \]
\[ \| \nabla_x^s \| P_N f \|_{L^p} \sim N^{\pm s} \| P_N f \|_{L^p}, \]
\[ \| P_{\leq N} f \|_{L^p} \lesssim N^{\frac{s}{p} - \frac{1}{p}} \| P_{\leq N} f \|_{L^p}, \]
\[ \| P_{> N} f \|_{L^p} \lesssim N^{-\frac{s}{p} - \frac{1}{p}} \| P_{> N} f \|_{L^p}, \]
\[ \|P_N f\|_{L^s} \lesssim N^{\frac{2}{s} - \frac{2}{r}} \|P_N f\|_{L^p}. \]

Next we introduce the concept of an \textit{admissible} pair. A pair \((q, r)\) is said to be \(H^s\)-admissible if \(q, r \in [2, \infty], s \in \left[0, \frac{2}{d}\right], q + r = \frac{2}{s} - s\) and \((q, d) \neq (2, 2)\). For any \(L^2\)-admissible pairs \((q_1, r_1)\) and \((q_2, r_2)\) we have the following Strichartz estimate: if \(u\) is a solution of
\[
i \partial_t u + \Delta_x u = F
\]
in \(I \subset \mathbb{R}\) with \(t_0 \in I\) and \(u(t_0) = u_0\), then
\[
\|u\|_{L^q_t L^r_x(I)} \lesssim \|u_0\|_{L^q_x} + \|F\|_{L^{q'_r}_t L^{r'_x}(I)},
\]
where \((q'_r, r'_x)\) is the H"older conjugate of \((q_2, r_2)\). For a proof, we refer to [33, 11]. Combining with Minkowski’s inequality, for a vector \(u = (u_j)_{j \in \mathbb{Z}}\) satisfying
\[
i \partial_t u_j + \Delta_x u_j = F_j \quad \forall j \in \mathbb{Z}
\]
we also have the Strichartz estimate
\[
\|u\|_{L^q_t L^r_x(I)} \lesssim \|u_0\|_{L^q_x} + \|F\|_{L^{q'_r}_t L^{r'_x}(I)}.
\]

For \(d = 2\), we define the spaces \(S_0, S_1\) by
\[
S_0 := L^\infty_t L^2_x \cap L^2_t L^\infty_x, \quad S_1 := L^\infty_t \dot{H}^1 \cap L^2_t W^{1, \infty}_x,
\]
where \((2^+, \infty^-)\) is an \(L^2\)-admissible pair with some sufficiently small \(2^+ \in (2, \infty)\). In the following, an admissible pair is always referred to an \(L^2\)-admissible pair if not otherwise specified.

Finally, we denote by \(F(u_m) = -(\sum_i |u_i|^2 + \sum_{i \neq j} |u_i|^2) u_m\) the nonlinear potential of [13] for the component \(u_m\). When the vector \(w\) is given by \(w = (w_m)_m = (P u_m)_m\), where \(P\) is some frequency projector, then we similarly define \(F(w_m) = -(\sum_i |w_i|^2 + \sum_{i \neq j} |w_i|^2) w_m\).

\section{Scattering for the large scale resonant system}

\subsection{Proof of Proposition 1.1}

We begin with the proof of Proposition 1.1.

\textit{Proof of Proposition 1.1}. Recall that \(Q_{2d}\) is the unique radially symmetric and positive solution of
\[-\Delta Q_{2d} + Q_{2d} = Q_{2d}^3 \quad \text{on} \ \mathbb{R}^2.
\]
It is well-known by Pohozaev’s identity (see [3]) that
\[
C_{GN, 2d} = \frac{\|Q_{2d}\|_{L^\infty_x}^2 \|\nabla Q_{2d}\|_{L^2_x}^2}{\|Q_{2d}\|_{L^4_x}^4}.
\]

We now set \(u^n = Q_{2d} \sum_{|i| \leq n} e_i\). Then direct calculation yields
\[
\int_{\mathbb{R}^2} \left( \sum_i |u_i^n|^2 \right)^2 + \sum_j \left( \sum_{i \neq j} |u_i^n|^2 \right) |u_j|^2 \, dx = (4n^2 + 4n + 1) \|Q_{2d}\|_{L^2_x}^2 \|\nabla Q_{2d}\|_{L^2_x}^2.
\]
Hence \(C_{GN, rs} \leq \frac{4n^2 + 4n + 1}{8n^2 + 6n + 1} C_{GN, 2d}\). Sending \(n \to \infty\) we obtain \(C_{GN, rs} \leq \frac{1}{2} C_{GN, 2d}\). Using Minkowski, (1.5) and Hölder we obtain
\[
\int_{\mathbb{R}^2} \left( \sum_j |u_j|^2 \right)^2 \, dx \leq \left( \sum_j \left( \int_{\mathbb{R}^2} |u_j|^4 \, dx \right)^{\frac{1}{2}} \right)^2 \leq \left( \sum_j C_{GN, 2d}^2 \|u_j\|_{L^2} \|\nabla u_j\|_{L^2} \right)^2 \leq C_{GN, 2d} \|u\|_{L^2_t L^4_x} \|\nabla u\|_{L^2_t L^4_x}.
\]

The desired inequality then follows from the rough estimate
\[
\int_{\mathbb{R}^2} \left( \sum_i |u_i|^2 \right)^2 + \sum_j \left( \sum_{i \neq j} |u_i|^2 \right) |u_j|^2 \, dx \leq 2 \int_{\mathbb{R}^2} \left( \sum_j |u_j|^2 \right)^2 \, dx.
\]
\[
\square
\]

7
2.2 Existence of a minimal blow-up solution for the large scale resonant system

Next, we establish a result concerning the existence of a minimal blow-up solution of \((1.3)\) when assuming that Theorem \([18, 59] \) does not hold. The proof is almost identical to [19, Thm. 3.3], where we only need to add the additional mass constraint to the inductive hypothesis, thus we omit the details here.

**Theorem 2.1** (Existence of a minimal blow-up solution). Suppose that Theorem \([18, 59] \) does not hold. Then there exists a solution \(u_c\) of \((1.3)\) such that \(M_0(u_c) < \frac{1}{4} M(Q_{2d})\) and

\[
\|u_c\|_{L^4_t L^4_x((\inf I_{\max}, 0])} = \|u_c\|_{L^4_t L^4_x([0, \sup I_{\max})]} = \infty.
\]

Moreover, the set \(\{u_c(t) : t \in I_{\max}\}\) is precompact in \(h^1 L^2_t\) modulo \(L^2_x\)-symmetries.

2.3 Properties of the almost periodic solution

In this subsection we collect some useful properties of the minimal blow-up solution \(u_c\).

**Lemma 2.2** (Arzela-Ascoli characterization of \(h^1 L^2\)-compactness, [49]). Let \(u\) be an almost periodic solution of \((1.3)\). Then there exist functions \(x : I \to \mathbb{R}^d\), \(\xi : I \to \mathbb{R}^d\), \(C : (0, \infty) \to (0, \infty)\) and \(N : I \to (0, \infty)\) such that for any \(\eta > 0\) and any int we have

\[
\sum_j |u_j(t, x)|^2 dx + \sum_j |u_j(t, \xi)|^2 \lesssim \int_{[0, \eta]} u(t, \xi(t))^2 d\xi < \eta^2. \tag{2.1}
\]

**Lemma 2.3** (Normalisation of the symmetry functions, [38]). We may additionally assume that the minimal blow-up solution \(u_c\) deduced from Theorem 2.1 satisfies the following:

(i) The maximal interval \(I\) contains at least \([0, \infty)\).

(ii) We have \(\|u_c\|_{L^4_t L^4_x([0, \infty))} = \infty\).

(iii) The functions \(x, \xi, N\) can be chosen such that \(x(0) = \xi(0) = 0, N(0) = 1\) and \(N(t) \leq 1\) for all \(t \in [0, \infty)\).

**Lemma 2.4** (Local constancy of \(N(t)\), [38]). Let \(u : I \times \mathbb{R}^d \to \mathbb{C}\) be a non-zero maximal-lifespan solution of \((1.3)\) that is almost periodic modulo symmetries and has the frequency scale function \(N\). Then there exists a small \(\delta = \delta(u) > 0\) such that for every \(t_0 \in I\) we have

\[
[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I.
\]

Moreover, \(N(t) \sim_u N(t_0)\) whenever \(|t - t_0| \leq \delta N(t_0)^{-2}\).

**Lemma 2.5** (Spacetime bound, [38]). Let \(u : I \times \mathbb{R}^d \to \mathbb{C}\) be a non-zero maximal-lifespan solution of \((1.3)\) that is almost periodic modulo symmetries and has the frequency scale function \(N\). Let \(J\) be any subinterval of \(I\). Then

\[
\int_J N^2(t) dt \lesssim \|u\|_{L^4_t L^4_x(J)}^4 \lesssim 1 + \int_J N^2(t) dt.
\]

The following result is an immediate consequence of Lemma 2.4 and Lemma 2.5.

**Corollary 2.6.** Suppose that for some interval \(J\) we have \(\|u\|_{L^4_t L^4_x(J)} = 1\). Then \(\sup_{t \in J} N(J) \leq u_{\text{inf}} J N(t)\). Moreover, if a time interval \(J\) can be partitioned into consecutive intervals \(J = J_1 \cup J_2\) with \(\|u\|_{L^4_t L^4_x(J)} = 1\), then \(\sum_{J_1} \sup_{t \in J_1} N(t) \sim_u \int_J N(t)^3 dt\).

2.4 Impossibility of solutions of rapid cascade type

In this section we rule out the rapid cascade scenario, i.e. the case \(\int_0^\infty N(t)^3 dt < \infty\). We firstly state the following lemma proved in [18, 49], which confirms the higher regularity of the minimal blow-up solution in the rapid cascade case.
Lemma 2.7 ([18, 49]). Let $u$ be the almost periodic solution of (1.3) given by Theorem 2.4. If $\int_0^\infty N(t)^3 \, dt = K < \infty$, then $\|u\|_{L^\infty_t \ell^2 H^2_x(0, \infty)} < \infty$.

Lemma 2.8 (Impossibility of almost periodic solution of rapid cascade type). Let $u$ be the almost periodic solution of (1.3) given by Theorem 2.4. If $\int_0^\infty N(t)^3 \, dt = K < \infty$, then $u \equiv 0$.

Proof. From Lemma 2.4 it follows that $u$ is almost periodic modulo $L^\infty_t \ell^2 H^2_x$. By Proposition 1.1, conservation of mass and energy follows from Proposition 1.1 again, we infer that $u \in L^\infty_t \ell^2 H^2_x(0, \infty)$ deduced from Lemma 2.7.

Now using interpolation and (2.1) we obtain

$$\lim_{t \to \infty} \|P_{\xi(t), \leq C(\eta)} N(t) u(t)\|_{\ell^2 H^1_x} = 0,$$

which combining with interpolation and the fact that $u \in L^\infty_t \ell^2 H^1_x(0, \infty)$ deduced from Lemma 2.7.

for all $t \in (0, \infty)$. Since $\eta$ is chosen arbitrarily, we conclude that

$$\lim_{t \to \infty} \|u(t)\|_{\ell^2 H^1_x} = 0.$$

By Proposition 1.1, conservation of mass and energy

$$E(u(0)) = \lim_{t \to \infty} E(u(t)) \lesssim \lim_{t \to \infty} \|u(t)\|_{\ell^2 H^1_x}^2 = 0.$$

But using Proposition 1.1 again, we infer that

$$\|\nabla u(0, x)\|_{\ell^2 L^2_x} \leq 2 \left(1 - \frac{2M_0(u)}{M(Q_{2d})}\right)^{-1} E(u(0)) = 0,$$

which in turn implies $u \equiv 0$. This completes the proof. \qed

2.5 Impossibility of solutions of quasi-soliton type

In this section we rule out the quasi-soliton scenario, i.e. the case $\int_0^\infty N(t)^3 \, dt = \infty$. First, we denote by $\varepsilon_3$ the small constant related to the long time Strichartz estimate, which is the same constant defined in [49, Sec. 5]. The construction of the long time Strichartz estimate is however very cumbersome and will not be directly applied for the upcoming proofs, thus we omit the details. Let $T > 0$ and define $K = \int_0^T N(t)^3 \, dt$. Define $w = (w_m)_m = P_{\langle \varepsilon_3^{-1} K \rangle} u$. Then

$$i\partial_t w_m + \Delta w_m = F(w_m) + \left(P_{\langle \varepsilon_3^{-1} K \rangle} F(u_m) - F(w_m)\right) = F(w_m) + N_m.$$

Let $a(t, x) = (a_j(t, x))_{j=1,2} : I \times \mathbb{R}^2 \to \mathbb{R}^2$ be some to be determined potentials. We define the frequency localized interaction Morawetz action $M(t)$ by

$$M(t) = 2 \sum_{n,m} \int \int |w_n(t, y)|^2 a(t, x - y) \text{Im} \{\bar{w}_m \nabla w_m\}(t, x) \, dx \, dy.$$

Integration by parts yields

$$\frac{d}{dt} M(t) = 4 \int \int \sum_{n} |w_n|^2 (t, y) \partial_t a_j(t, x - y) \Re \left( \sum_{m} \partial_j \bar{w}_m \partial_k w_m \right)(t, x) \, dx \, dy$$

$$- 4 \int \int \Im \left( \sum_{m} \bar{w}_m \partial_k w_n (t, y) \partial_j a_j(t, x - y) \right) \Im \left( \sum_{m} \bar{w}_m \partial_j w_m (t, x) \right) \, dx \, dy$$

(2.2)
By Duhamel’s formula, Strichartz estimate and conservation of mass we know that
\[ \text{Proof.} \]
Particularly, \( \phi \) and decreasing function such that
\[ \text{Lemma 2.11 (Impossibility of almost periodic solution of quasi-soliton type)} \]
\[ \text{Proof.} \]
(1.3) Let \( J \) be a time interval such that
\[ \text{Lemma 2.10. Let } J \text{ be a time interval such that } \|u\|_{L^4_t L^6_x(J)} \lesssim 1. \text{ Then} \]
\[ \text{as } R \to \infty. \]
\[ \text{Proof. By Duhamel’s formula, Strichartz estimate and conservation of mass we know that } \|u\|_{L^4_t L^6_x(J)} \lesssim 1 \text{ for arbitrary admissible (p, q). Then the desired claim follows from (2.2) and interpolation between } L^\infty_t L^2_x L^2_t \text{ and } L^4_t L^4_x L^2 t \text{ for some admissible } (p, q) \text{ with } p \in (2, 4). \]
Having all the preliminaries we are in the position to rule out the quasi-soliton scenario.
\[ \text{Lemma 2.11 (Impossibility of almost periodic solution of quasi-soliton type). Let } u \text{ be the almost periodic solution of } (1.3) \text{ given by Theorem 2.7. If } \int_0^\infty N(t)^3 \, dt = \infty, \text{ then } u \equiv 0. \]
\[ \text{Proof. First, we construct the potentials } a_j(t, x) \text{ as follows: For } R > 0, \text{ let } \varphi \in C^\infty_c(\mathbb{R}^2; [0, 1]) \text{ be a radial and decreasing function such that } \varphi \leq 1, \varphi(z) \equiv 1 \text{ on } B_{R-\sqrt{3}}(0) \text{ and supp } \varphi \subset B_R(0). \text{ Next, define} \]
\[ \phi(x) = \frac{1}{|B_R(0)|} \int \varphi(x-z) \varphi(z) \, dz. \]
\[ \text{Particularly, } \phi \text{ is non-negative and radial, supported on } B_{2R}(0) \text{ and supp}_{x \in \mathbb{R}^2} \phi(x) \leq 4. \text{ Moreover, we have} \]
\[ \phi(x-y) = \frac{1}{|B_R(0)|} \int \varphi(x-z) \varphi(y-z) \, dz. \]
\[ \text{With slight abuse of notation we identify } \varphi : \mathbb{R}^2 \to [0, \infty) \text{ with the same function } \varphi : [0, \infty) \to [0, \infty). \text{ The same convention is made for other radial functions.} \]
By \cite[Lem. 6.6]{29}, the function $\phi$ is also decreasing. Finally, for $M > 0$ define

$$
\psi_M(x) = \psi_M(|x|) = \frac{1}{|x|} \int_0^{|x|} \phi\left(\frac{s}{M}\right) ds.
$$

As immediate consequences, we have

$$
r\psi'_M(r) = \phi\left(\frac{r}{M}\right) - \psi_M(r) \leq 0, \\
x_j\psi_M(|x|) \leq 8RM, \\
\nabla(x_j\psi_M(|x|)) \leq \frac{24RM}{|x|}.
$$

We now define

$$a_j(t, x) = N(t)x_j\psi_{RN(t)-1}(x).$$

Then the assumptions of $a_j$ in Lemma \ref{Lemma:2.9} are satisfied (with $R$ replaced by $24R^2$) and in view of Lemma \ref{Lemma:2.9}, the proof of Lemma \ref{Lemma:2.11} follows as long as we can prove

$$
\int_0^T (2.3) + (2.4) + (2.6) + (2.7) dt \gtrsim K
$$

for sufficiently large $K$. Let us first take (2.6). Straightforward calculation results in

$$
(2.6)
= -\sum_{m,n} \int N(t)\Delta(\phi(R^{-1}N(t)|x-y|) + \psi_{RN(t)-1}(x-y))|w_m(t,x)||w_n(t,y)|^2 dxdy.
$$

Moreover, by product rule and chain rule

$$
\Delta(\phi(x) + \psi_1(x)) = (\phi''(|x|) + \psi'_1(|x|)) + |x|^{-1}(\phi'(|x|) + \psi'_1(|x|)).
$$

First recall that $\sup_{s \geq 0} |\varphi(s)| \lesssim 1$, $\sup_{s \geq 0} |\varphi'(s)| \lesssim R^{-\frac{1}{2}}$ and $\varphi(s)$ is supported on $R - \sqrt{R} \leq |s| \leq R$. By definition of $\phi$, we obtain

$$
\phi'(r) = \frac{d}{dr}(\phi(re_1)) = |B_R(0)|^{-1} \int \varphi'(|re_1 - z|)\frac{r - z_1}{|re_1 - z|}\varphi(z) dt \lesssim R^{-\frac{3}{2}}.
$$

In the same manner, we deduce $\phi''(r) \lesssim R^{-2}$. Now we observe that

$$
\phi'(0) = -|B_R(0)|^{-1} \int \frac{z_1}{|z|}\varphi'(|z|)\varphi(z) dt = 0,
$$

thus $\phi'(r) = \int_0^r \phi''(r) dr \lesssim R^{-2}r$. Next, using (2.11)

$$
r^{-1}\psi_1'(r) = r^{-3}\int_0^r \phi'(t) dt ds \lesssim R^{-2}. \quad (2.12)
$$

Similarly we infer that $\psi''_1(r) \lesssim R^{-2}$, which in turn implies

$$
\Delta(\phi(x) + \psi_1(x)) \lesssim R^{-2}.
$$

Therefore by conservation of mass

$$
\int_0^T (2.3) dt \gtrsim -\int_0^T \frac{N(t)^3}{R^4} ||w(t)||_{L^2} dt = o_R(1)K.
$$

Let us now consider (2.3), (2.4) and (2.5). Define the radial and angular derivatives $\nabla_{r,y}$ and $\nabla_y$ centered at a point $y \in \mathbb{R}^2$ by

$$
\nabla_{r,y,j} = \frac{(x_j - y_j)}{|x - y|} \partial_j, \quad \nabla_y,j = \partial_j - \nabla_{r,y,j}.
$$
Then it is straightforward by direct calculation to verify that for a function $f$ we have the decomposition
\[
|\nabla f|^2 = |\nabla_{x,y} f|^2 + |\nabla_y f|^2.
\]
Combining with (2.11) and Cauchy-Schwarz we infer that
\[
2.13 \quad + \quad 2.14
= 4 \sum_{m,n} \int \int N(t) \phi(R^{-1} N(t)|x-y|) |\nabla w_m(t,x)|^2 |w_n(t,y)|^2 \, dx \, dy
- 4 \sum_{m,n} \int \int N(t) \phi(R^{-1} N(t)|x-y|) \Im(\bar{w}_n \partial_k w_n)(t,y) \Im(\bar{w}_m \partial_k w_m)(t,x) \, dx \, dy
+ 4 \sum_{m,n} \int \int N(t) \phi(R^{-1} N(t)|x-y|) - \psi_{R/N(t)}(x-y)) |w_n(t,y)|^2 |\nabla_y w_m(t,x)|^2 \, dx \, dy
- 4 \sum_{m,n} \int \int N(t) \phi(R^{-1} N(t)|x-y|) - \psi_{R/N(t)}(x-y)) \Im(\bar{w}_n \nabla_x w_n)(t,y) \Im(\bar{w}_m \nabla_y w_m)(t,x) \, dx \, dy
\geq 4 \sum_{m,n} \int \int N(t) \phi(R^{-1} N(t)|x-y|) |\nabla w_m(t,x)|^2 |w_n(t,y)|^2 \, dx \, dy
- 4 \sum_{m,n} \int \int N(t) \phi(R^{-1} N(t)|x-y|) \Im(\bar{w}_n \partial_k w_n)(t,y) \Im(\bar{w}_m \partial_k w_m)(t,x) \, dx \, dy
= 4\beta R(0)^{-1} \sum_{m,n} N(t) \int \int \left( \phi(R^{-1} N(t) x - z) |\nabla w_m(t,x)|^2 \right) \left( \phi(R^{-1} N(t) y - z) |w_n(t,y)|^2 \right) \, dz \, dx \, dy
- 4\beta R(0)^{-1} \sum_{m,n} N(t) \int \int \left( \phi(R^{-1} N(t) x - z) \Im(\bar{w}_m \partial_k w_m)(t,x) \right) \times \left( \phi(R^{-1} N(t) y - z) \Im(\bar{w}_n \partial_k w_n)(t,y) \right) \, dz \, dx \, dy. \tag{2.13}
\]
The sum of (2.13) and (2.14) is invariant under the Galilean transformation $w \mapsto e^{-ix \cdot \beta_0} w$ for arbitrary $\beta_0 \in \mathbb{R}^2$ (which can be easily checked by carefully expanding the terms in (2.14) and using product and chain rules, we omit the straightforward but tedious details here). Thus we choose $t \mapsto \beta(t)$ such that
\[
\sum_m \phi(R^{-1} N(t)x - z) \Im(e^{-ix \cdot \beta(t)} w_m \nabla(e^{-ix \cdot \beta(t)} w_m))(t,x) = 0
\]
and we are left with the term
\[
4\beta R(0)^{-1} \sum_{m,n} N(t) \int \int \left( \phi(R^{-1} N(t)x - z) |\nabla (e^{-ix \cdot \beta(t)} w_m)(t,x)|^2 \right) \times \left( \phi(R^{-1} N(t)y - z) |w_n(t,y)|^2 \right) \, dz \, dx \, dy. \tag{2.15}
\]
The existence of such a function $t \mapsto \beta(t)$ is guaranteed by (2.20) and the fact that $u \neq 0$. On the other hand, (2.11), (2.12) and the fact that $F(w_m) \bar{w}_m \leq 0$ result in
\[
2.20
= \sum_{m,n} \int \int N(t) (2\psi_{R(t)} \cdot 1)(x-y) + |x| \psi_{R(t)} \cdot 1(x-y))(F(w_m) \bar{w}_m)(t,x) |w_n(t,y)|^2 \, dx \, dy
\geq \sum_{m,n} \int \int 2N(t) \psi_{R(t)} \cdot 1(x-y)(F(w_m) \bar{w}_m)(t,x) |w_n(t,y)|^2 \, dx \, dy
= |\beta R(0)|^{-1} \sum_{m,n} \int \int 2N(t) \phi(R^{-1} N(t)x - z)(F(w_m) \bar{w}_m)(t,x) \times \left( \phi(R^{-1} N(t)y - z) |w_n(t,y)|^2 \right) \, dz \, dx \, dy
\]
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By Proposition 1.1 we know that there exists some \( \varepsilon > 0 \) such that

\[
\int 2N(t) (\psi_{RN(t)-1} (x-y) - \phi(R^{-1}N(t) |x-y|)) (F(w_m) \bar{w}_m)(t,x) |w_n(t,y)|^2 \, dx \, dy < 0
\]

\[\geq |B_R(0)|^{-1} \sum_{m,n} \int 2N(t) (\varphi(R^{-1}N(t)x-z))(F(w_m) \bar{w}_m)(t,x) \times (\varphi(R^{-1}N(t)y-z)|w_n(t,y)|^2) \, dz \, dx \, dy \]  \hspace{1cm} (2.16)

\[+ o_R(1) \sum_m \int N(t)(F(w_m) \bar{w}_m)(t,x) \, dx. \]  \hspace{1cm} (2.17)

We also note that (2.16) and (2.17) are Galilean invariant. Now we consider the sum of (2.15) and (2.16). Let \( \chi \in C^\infty_c(\mathbb{R}^2; [0,1]) \) be radial, decreasing, supp \( \chi \subset B_{R^{-\sqrt{R}(0)}} \) and \( \chi \equiv 1 \) on \( B_{R^{-2\sqrt{R}(0)}} \). Also denote \( \tilde{\chi}(x) = \chi(R^{-1}N(t)x-z) \). Then

\[
4 \sum_m \int \varphi(R^{-1}N(t)x-z)|\nabla(e^{-ix \cdot \hat{\beta}}w_m(t,x))|^2 \, dx \\
+ 2 \sum_m \int \varphi(R^{-1}N(t)x-z)(F(e^{-ix \cdot \hat{\beta}}w_m)e^{-ix \cdot \hat{\beta}}w_m)(t,x) \, dx \hspace{1cm} (2.18)
\]

\[
= 4 \sum_m \int \tilde{\chi}(x) |\nabla(e^{-ix \cdot \hat{\beta}}w_m(t,x))|^2 \, dx + 2 \sum_m \int \tilde{\chi}(x) (F(e^{-ix \cdot \hat{\beta}}w_m)e^{-ix \cdot \hat{\beta}}w_m)(t,x) \, dx \\
+ 4 \sum_m \int (\varphi - \chi)(R^{-1}N(t)x-z)|\nabla(e^{-ix \cdot \hat{\beta}}w_m(t,x))|^2 \, dx \\
+ 2 \sum_m \int (\varphi - \chi)(R^{-1}N(t)x-z)(F(e^{-ix \cdot \hat{\beta}}w_m)e^{-ix \cdot \hat{\beta}}w_m)(t,x) \, dx \hspace{1cm} (2.19)
\]

\[
= 8 \left( \frac{1}{2} \sum_m \int |\nabla(\tilde{\chi}(x)e^{-ix \cdot \hat{\beta}}w_m(t,x))|^2 \, dx \right) + 2 \sum_m \int \frac{1}{4} F(\tilde{\chi}e^{-ix \cdot \hat{\beta}}w_m)\tilde{\chi}e^{-ix \cdot \hat{\beta}}w_m(t,x) \, dx \\
+ 4 \sum_m \int (\varphi - \chi)(R^{-1}N(t)x-z)|\nabla(e^{-ix \cdot \hat{\beta}}w_m(t,x))|^2 \, dx \\
+ 2 \sum_m \int (\varphi - \chi)(R^{-1}N(t)x-z)(F(e^{-ix \cdot \hat{\beta}}w_m)e^{-ix \cdot \hat{\beta}}w_m)(t,x) \, dx \hspace{1cm} (2.20)
\]

\[
+ 4 \sum_m \int \tilde{\chi}(x) \operatorname{div} \nabla(\tilde{\chi}(x)w_m(t,x)) \, dx. \hspace{1cm} (2.21)
\]

By definition we have \( \varphi - \chi \geq 0 \), thus \( 2.20 \geq 0 \). For \( 2.22 \), using the definition of \( \chi \) we have

\[
2.22 \geq -\frac{C N(t)^2}{R^4} \sum_m \int_{R^{-2\sqrt{R}(0)}} \tilde{\chi}(x) |w_m(t,x)|^2 \, dx.
\]

This in turn implies

\[
\int_0^T \sum_{m,n} \int \int (2.22) \times (\varphi(R^{-1}N(t)y-z)|w_n(t,y)|^2) \, dz \, dx \, dy \, dt \\
\geq -CR^{-3} \sum_{m,n} \int_0^T N(t)^3 |w_m(t,x)|^2 |w_n(t,y)|^2 \\
\times (|B_R(0)|^{-1} \int \chi(R^{-1}N(t)x-z)\varphi(R^{-1}N(t)y-z) \, dz) \, dx \, dy \, dt = \alpha_R(1) K.
\]

By Proposition 1.1, we know that there exists some \( \varepsilon > 0 \) such that

\[
2.19 \geq -\varepsilon \sum_m \int F(\tilde{\chi}e^{-ix \cdot \hat{\beta}}w_m)\tilde{\chi}e^{-ix \cdot \hat{\beta}}w_m(t,x) \, dx. \hspace{1cm} (2.23)
\]

Now we insert (2.23) into the original integral and integrate over \( z \), with the following observations: First, if \( |x-y| \geq R^2 N(t)^{-1} \), then the supports of \( \chi^4(R^{-1}N(t)x-z) \) and \( \varphi(R^{-1}N(t)y-z) \) will be disjoint.
Second, if \(|x - y| \leq 4^{-1} R^2 N(t)^{-1}\), then
\[
\inf_{|x - y| \leq 4^{-1} R^2 N(t)^{-1}} |B_R(0)|^{-1} \int \chi^4(R^{-1} N(t)x - z) \varphi(R^{-1} N(t)y - z) dz \geq 1.
\]
Therefore,
\[
\int_0^T \sum_{m,n} \int \int (2.23) \times (\varphi(R^{-1} N(t)y - z) |w_n(t,y)|^2) \, dz \, dx \, dy \, dt \\
\geq - C \varepsilon \sum_{m,n} \int_0^T N(t) \int \int_{|x - y| \leq \frac{R^2}{2}} (F(w_m)w_m)(t,x)|w_n(t,y)|^2 \, dx \, dy \, dt.
\]
Let us finally take (2.21). Notice that since \(\varphi - \chi^4\) is supported on \(|x| \leq [R - 2\sqrt{R}, R]\), we obtain
\[
|B_R(0)|^{-1} \int (\varphi - \chi^4)(R^{-1} N(t)x - z) \varphi(R^{-1} N(t)y - z) dz \lesssim R^{-1},
\]
which implies
\[
\int_0^T \sum_{m,n} \int \int (2.23) \times (\varphi(R^{-1} N(t)y - z) |w_n(t,y)|^2) \, dz \, dx \, dy \, dt \\
= o_R(1) \sum_{m,n} \int_0^T N(t) \int (F(w_m)\bar{w}_m)(t,x) \, dx \, dt.
\]
Summing up at this point, we have thus so far proved
\[
\int_0^T (2.23) + (2.4) + (2.5) + (2.6) + (2.7) \, dt \\
\geq - C \varepsilon \sum_{m,n} \int_0^T N(t) \int \int_{|x - y| \leq \frac{R^2}{2}} (F(w_m)w_m)(t,x)|w_n(t,y)|^2 \, dx \, dy \, dt \\
+ o_R(1) \sum_{m,n} \int_0^T N(t) \int (F(w_m)\bar{w}_m)(t,x) \, dx \, dt + o_R(1) K + \int_0^T (2.7) \, dt.
\]
Now we recall
\[
F(w_m) = -(\sum_i |w_i|^2 + \sum_{i \neq m} |w_i|)w_m.
\]
Hence for any time interval \(J\) we have \(\| \sum_m F(w_m)\bar{w}_m \|_{L^1_{\ell^2(J)}} \sim \|w\|^4_{L^1_{\ell^2(L)}L^2(J)}\). By rewriting \(w\) to \(w = u - \rho_{\ell^2} u\) and using Lemma 2.2 Lemma 2.5 Lemma 2.10 and conservation of mass, we conclude that if \(\|w\|^4_{L^1_{\ell^2(L)}L^2(J)} = 1\) for some interval \(J\), then
\[
- \int \sum_{m,n} \int_{|x - y| \leq \frac{R^2}{2}} (F(w_m)\bar{w}_m)(t,x)|w_n(t,y)|^2 \, dx \, dy \, dt \\
\geq - \int \sum_{m} \int (F(u_m)\bar{w}_m)(t,x) \, dx \, dt + o_R(1) + o_K(1)
\]
\[
\geq \|u\|^4_{L^1_{\ell^2(L)}L^2(J)} + o_R(1) + o_K(1) = 1 + o_R(1) + o_K(1)
\]
as \(R,K \to \infty\). We now partition \([0,T]\) into \([0,T] = \cup J_\ell\) such that \(\|u\|^4_{L^1_{\ell^2(L)}L^2(J_\ell)} = 1\) for all \(J_\ell\). Then for sufficiently large \(R\) and \(K\)
\[
\int_0^T N(t) \sum_{m,n} \int_{|x - y| \leq \frac{R^2}{2}} (F(w_m)\bar{w}_m)(t,x)|w_n(t,y)|^2 \, dx \, dy \, dt \\
\sim \sum_{J_\ell} N(J_\ell) \int_{J_\ell} \sum_{m,n} \int_{|x - y| \leq \frac{R^2}{2}} (F(w_m)\bar{w}_m)(t,x)|w_n(t,y)|^2 \, dx \, dy \, dt.
\]
\[
\sum_{\ell} N(J_\ell) \sim \int_0^T N(t)^3 \, dt = K.
\]

In the same manner,
\[
\alpha_R(1) \sum_m \int_0^T N(t) \int (F(w_m)w_m)(t, x) \, dx \, dt = \alpha_R(1)K.
\]

Let us finally take (2.7). Direct calculation shows
\[
= 2 \sum_{m,n} \int \int N'(t)\phi(R^{-1}N(t)|x - y|)(x_j - y_j) \Im(\bar{w}_m \partial_j w_m)(t, x)|w_n(t, y)|^2 \, dxdy
\]
\[
= 2|B_R(0)|^{-1} \sum_{m,n} \int \int \varphi(R^{-1}N(t)x - z)\varphi(R^{-1}N(t)y - z)
\]
\[
\times N'(t)(x_j - y_j) \Im(\bar{w}_m \partial_j w_m)(t, x)|w_n(t, y)|^2 \, dxdy.
\]

Again, in order to keep the supports of \( \varphi(R^{-1}N(t)x - z) \) and \( \varphi(R^{-1}N(t)y - z) \) to be not disjoint, it is necessary that \( |x - y| \leq R^2 N(t)^{-1} \). Moreover, one easily verifies that (2.26) is Galilean invariant. Combining with Young’s inequality we infer that for arbitrary \( \gamma > 0 \) there exists some \( C(\gamma) > 0 \) such that
\[
\lesssim |B_R(0)|^{-1}\gamma N(t) \sum_{m,n} \int \int \varphi(R^{-1}N(t)x - z)\varphi(R^{-1}N(t)y - z)
\]
\[
\times |\nabla (e^{-ix\beta(t)}w_m)(t, x)|^2 |w_n(t, y)|^2 \, dxdy
\]
\[+ R^4|B_R(0)|^{-1}C(\gamma)N(t)^{-3}N'(t)^2 \sum_{m,n} \int \int \varphi(R^{-1}N(t)x - z)\varphi(R^{-1}N(t)y - z)
\]
\[\times |w_m(t, x)|^2 |w_n(t, y)|^2 \, dxdy.
\]
(2.25) can be absorbed to the first term in (2.18) by choosing \( \gamma \) small. For (2.26), using conservation of mass we have
\[
\int_0^T (2.25) \, dt \lesssim C(\gamma)R^4 \int_0^T |N'(t)| \, dt.
\]

In general the best we can hope for estimating (2.27) would be \( \int_0^T |N'(t)| \, dt \lesssim \int_0^T N(t)^3 \, dt = K \), which is obviously insufficient for our purpose. The idea is to modify \( N(t) \) in a way such that \( N(t) \) is less oscillatory so that \( |N'(t)| \) is small. This can be done using the so-called smoothing algorithm initiated by Dodson [18] Sec. 6.1], where the level of the peaks of the function \( N(t) \) are inductively reduced. The adaptation of the smoothing method to our model is however verbatim, we thus omit the details here. In a nutshell, after applying the smoothing algorithm we may replace the function \( N(t) \) by a new function \( \tilde{N}(t) \) such that \( \tilde{N}(t) \leq N(t) \) and
\[
\int_0^T |\tilde{N}'(t)| \, dt \ll K
\]
for sufficiently large \( K \). Now we fix \( R = R(\varepsilon) \), then applying the smoothing algorithm to get a new function \( \tilde{N}(t) \) that is determined by the number \( R \). Since \( \tilde{N}(t) \leq N(t) \), the number \( R \) is not affected. Possibly we also need to shrink the value of \( \varepsilon_3 \), but this does not effect the results from Lemma (2.9) since nevertheless we need to take \( K \) very large. Summing up, we conclude
\[
\int_0^T (2.28) + (2.29) + (2.30) + (2.31) \, dt
\]
\[\geq 2^{-1}C\varepsilon K + o_R(1)K + o_K(1)K + R^4o_K(1)K.
\]
as \( R, K \to \infty \). The desired claim follows by firstly taking \( R = R(\varepsilon) \) sufficiently large, then modifying the frequency scale function \( \tilde{N}(t) \), shrinking the number \( \varepsilon_3 \), and finally taking \( K = K(R) \) sufficiently large to apply Lemma (2.9).

\[\Box\]

**Proof of Theorem 1.5.** This follows immediately from Lemma (2.8) and Lemma (2.11)

\[\Box\]
3 Scattering for the focusing cubic NLS on $\mathbb{R}^2 \times T$

In this final section we give the proof of Theorem 1.6. Up to the variational and virial analysis, many of the arguments given in this section are similar to the ones from [12], where the authors studied the defocusing analogue of (1.1). Nonetheless, the linear profile decomposition established in [12] is insufficient for our purpose: the linear profile decomposition in [12] is given at the $L^2_t H^1_y$-level, while in our case we need a linear profile decomposition at the $H^1_{x,y}$-level in order to apply the variational arguments. We follow the same lines in [34][36][40] to construct such a linear profile decomposition.

3.1 Small data well-posedness and stability theories

We collect in this section the small data and stability theories for (1.1) and some useful inequalities.

**Lemma 3.1** (Strichartz estimates on $\mathbb{R}^2 \times T$, [47]). Let $\gamma \in \mathbb{R}$, $s \in [0, 1)$ and $p, q, p, q$ satisfy $p, q \in (2, \infty)$ and $2p^{-1} + 2q^{-1} = 2\gamma^{-1} + 2\overline{\gamma}^{-1} = 1 - s$. Then for any time interval $I \ni t_0$ we have

$$\| e^{i(t-t_0)\Delta_x} f \|_{L^2_t L^2_y H^\gamma_y(I)} \lesssim \| f \|_{H^\gamma_y}.$$

Moreover, the Strichartz estimate for the Duhamel term

$$\| \int_{t_0}^t e^{i(t-s)\Delta_x} F(s) \, ds \|_{L^2_t L^2_y H^\gamma_y(I)} \lesssim \| F \|_{L^p_t L^q_y H^\gamma_y(I)}$$

holds in the case $s = 0$.

**Lemma 3.2** (Fractional calculus on $T$, [47]). For $s \in \left(\frac{1}{2}, 1\right]$ we have

$$\| u_1 u_2 \|_{H^s_y} \lesssim \| u_1 \|_{H^s_y} \| u_2 \|_{L^\infty_t L^\infty_y} + \| u_2 \|_{H^s_y} \| u_1 \|_{L^\infty_t L^\infty_y},$$

$$\| u_1 u_2 u_3 \|_{H^s_y} \lesssim \sum_{i=1}^3 \| u_i \|_{H^s_y} \prod_{j=1, j \neq i}^3 \| u_j \|_{L^\infty_y}.$$

**Lemma 3.3** (Small data well-posedness, [47]). Let $I$ be an open interval containing 0. Define

$$X(I) := (L^\infty_t L^2_x H^1_y(I) \cap L^2_t L^\infty_y H^1_y(I)) \cap (L^\infty_t \dot{H}^1_x L^2_y(I)) \cap L^2_t \dot{W}^{1, \infty}_x L^2_y(I)) =: S_0 H^1_y(I) \cap S_1 L^2_y(I).$$

Let also $s \in \left(\frac{1}{2}, 1\right]$. Assume that

$$\| U_0 \|_{H^s_y} \leq A$$

for some $A > 0$. Then there exists $\delta = \delta(A)$ such that if

$$\| e^{it\Delta} U_0 \|_{L^2_t L^2_y H^s_y(I)} \leq \delta,$$

then there exists a unique solution $U \in X(I)$ of (1.1) with $U(0) = U_0$ such that

$$\| U \|_{X(I)} \lesssim A,$$

$$\| U \|_{L^2_t L^2_y H^s_y(I)} \leq 2 \| e^{it\Delta} U_0 \|_{L^2_t L^2_y H^s_y(I)}.$$

**Lemma 3.4** (Scattering criterion). If $U$ is a global solution of (1.1) and there exists some $s \in \left(\frac{1}{2}, 1\right]$ such that

$$\| U \|_{L^4_t L^4_y H^s_y(\mathbb{R})} + \| U \|_{L^\infty_t H^s_y(\mathbb{R})} < \infty,$$

then $U$ is scattering in $H^1_{x,y}$. Moreover, we have

$$\| U \|_{L^4_t L^4_y H^s_y(\mathbb{R})} \leq C(\| U \|_{L^4_t L^4_y H^s_y(\mathbb{R})}, \| U \|_{L^\infty_t H^s_y(\mathbb{R})}),$$

(3.2)
Proof. That (3.1) implies scattering was proved by [12, Thm. 2.9]. Now using Duhamel’s formula and Strichartz estimate we also infer that (3.1) implies
\[\|U\|_{L^8_t L^6_x L^8_y(R^3)} \leq C(\|U\|_{L^1_t L^2_x H^s_y(R^3)}, \|U\|_{L^\infty_t H^1_{x,y}(R^3)}) < \infty.\]

Then by Duhamel’s formula, Strichartz and Hölder
\[\|U\|_{L^1_t L^2_x L^2_y(R^3)} \lesssim \|U\|_{L^\infty_t H_{x,y}^1(R^3)} + \|U\|^3_{L^1_t L^2_x L^2_y(R^3)} + \|U\|^2_{L^1_t L^2_x L^2_y(R^3)}\]
\[\lesssim \|U\|_{L^\infty_t H_{x,y}^1(R^3)}(1 + \|U\|^2_{L^1_t L^2_x L^2_y(R^3)}),\]
which implies (3.2). \qed

Lemma 3.5 (Long time stability, [12]). Let \( U \) be a solution of \((1.1)\) on the time interval \( I \ni 0 \) and let \( Z \) be a solution of
\[i\partial_t Z + \Delta Z = -|Z|^2 Z + \varepsilon\]
on \( I \). Let also \( s \in (\frac{1}{2}, 1) \) be given. Assume that
\[\|U\|_{L^\infty_t L^2_x H^s_y(I)} \leq M, \tag{3.3}\]
\[\|Z\|_{L^1_t L^2_x H^s_y(I)} \leq L,
\[\|Z(0) - U(0)\|_{L^2_x H^s_y} \leq M'.\]

Assume also the smallness conditions
\[\|e^{it\Delta}(Z(0) - U(0))\|_{L^1_t L^2_x H^s_y(I)} \leq \varepsilon,
\[\|e\|_{L^4_t L^4_x H^s_y(I)} \leq \varepsilon \tag{3.4}\]
for some \( 0 < \varepsilon \leq \varepsilon_1 \) where \( \varepsilon_0 = \varepsilon_0(M, M', L) > 0 \) is a small constant. Then
\[\|Z - U\|_{L^1_t L^2_x H^s_y(I)} \leq C(M, M', L)\varepsilon,
\[\|Z - U\|_{S_0 H^s_y(I)} \leq C(M, M', L)M',
\[\|U\|_{S_0 H^s_y(I)} \leq C(M, M', L).\]

3.2 Linear profile decomposition

In this section we establish a linear profile decomposition for a bounded sequence in \( H^1_{x,y} \). Firstly we fix some notation. For each \( j \in \mathbb{Z} \), define \( C_j \) by
\[C_j := \left\{ \Pi_{i=1}^2 \{2^j k_1, 2^j k_i + 1 \} \right\} \subset \mathbb{R} : k \in \mathbb{Z}^2 \}
and \( \mathcal{C} := \cup_{j \in \mathbb{Z}} C_j \). Given \( Q \in \mathcal{C} \) we define \( f_Q \) by \( f_Q(x) := \chi_Q \chi_Q \Delta f \), where \( \chi_Q \) is the characteristic function of the cube \( Q \).

Lemma 3.6 (Improved Strichartz estimate, [12]). For \( f \in H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T}) \), we have the following refined Strichartz estimate
\[\|e^{it\Delta_x} f\|_{L^4_t L^2_x H^1_y} \lesssim \||f||_{L^2_x H^1_y} \left( \sup_{Q \in \mathcal{C}} |Q|^{-\frac{1}{2}} \left\| |e^{it\Delta_x} f| \right\|_{L^4_{t,x,y} \mathbb{R}^3} \right) \frac{1}{4}.\]

Lemma 3.7 (Inverse Strichartz inequality). Let \( (f_n) \subset H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T}) \). Suppose that
\[\lim_{n \to \infty} \|f_n\|_{H^1_{x,y}} = A < \infty \quad \text{and} \quad \lim_{n \to \infty} \|e^{it\Delta_x} f_n\|_{L^1_{t,x,y}}(\mathbb{R}^3) = \varepsilon > 0.\]
Then up to a subsequence, there exist \( \phi \in L^2_{t,x} H^1_y(\mathbb{R}^2 \times \mathbb{T}) \) and \( (t_n, x_n, \xi_n, \lambda_n) \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \) such that \( \limsup_{n \to \infty} |\xi_n| < \infty \) and \( \lim_{n \to \infty} \lambda_n =: \lambda_\infty \in (0, \infty) \). Moreover,
\[\lambda_n e^{-i\xi_n (\lambda_n x + x_n)} (e^{it_n\Delta_y} f_n)(\lambda_n x + x_n, y)\]
\[ \lim_{n \to \infty} \| f_n \|_{L^2_{t,x} H^4_y} = \sup_{n \in \mathbb{N}} \| f_n - \phi_n \|_{L^2_{t,x} H^4_y} \leq \| \phi \|_{L^2_{t,x} H^4_y} + A^2 \left( \frac{\varepsilon}{A} \right)^4, \]

\[ \lim_{n \to \infty} \| f_n \|_{L^2_{t,x} L^2_y} - \| f_n - \phi_n \|_{L^1_{t,x} L^2_y} = 0, \]

\[ \lim_{n \to \infty} \| f_n \|_{L^2_{t,x} L^2_y} = 0, \]

\[ \lim_{n \to \infty} \| f_n \|_{L^2_{t,x} H^4_y} = 0. \]

Proof. For \( R > 0 \), denote by \( f^R \) the function such that \( \mathcal{F}_x(f^R) = \chi_R \mathcal{F}_x f \), where \( \chi_R \) is the characteristic function of the ball \( B_R(0) \subset \mathbb{R}^2 \). First we obtain that

\[ \sup_{n \in \mathbb{N}} \| f_n - f^R \|_{L^2_{t,x} H^4_y} \leq \sup_{n \in \mathbb{N}} \sum_k \| \mathcal{F}_x f_n(\xi, k) \|^2 \right\} \]

\[ \leq R^{-2} \sup_{n \in \mathbb{N}} \sum_k \| \mathcal{F}_x f_n(\xi, k) \|^2 \]
Since $T$ is compact, we simply assume that $y_n \equiv 0$. Define
\[ h_n(x,y) := \lambda_n e^{-i\xi_n(x_n x + n \Delta y)} (\lambda_n x + x_n, y), \]
\[ h_n^R(x,y) := \lambda_n e^{-i\xi_n(x_n x + n \Delta y)} (\lambda_n x + x_n, y). \]
It is easy to verify that $\|h_n\|_{L^2}^2 \geq \|f_n\|_{L^2}^2$. By the $L^2_y$-boundedness of $(f_n)$ we know that there exists some $\phi \in L^2_y$ such that $h_n \rightharpoonup \phi$ weakly in $L^2_y$. Arguing similarly, we infer that $(h_n^R)$ converges weakly to some $\phi^R \in L^2_y$. By definition of $\phi$ and $\phi^R$ we see that
\[ \|\phi - \phi^R\|_{L^2_y}^2 = \lim_{n \to \infty} (h_n - h_n^R, \phi - \phi^R)_{L^2_y} \leq (\limsup_{n \to \infty} \|h_n - h_n^R\|_{L^2_y}) \|\phi - \phi^R\|_{L^2_y}. \]
Using (3.10) we then obtain that
\[ \phi^R \rightharpoonup \phi \quad \text{in} \quad L^2_y \quad \text{as} \quad R \to \infty. \] (3.13)

Now define the function $\chi_1$ such that $\mathcal{F}_x^y \chi_1$ is the characteristic function of the cube $[-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$. Also let $\chi_2 = \delta_0 \in H^{-1}_y$, where $\delta_0$ is the Dirac function at zero. Since $H^1_y \hookrightarrow H^{-1}_y$, we infer that $\|\chi_2\|_{H^{-1}_y} \lesssim 1$. From (3.12), the weak convergence of $h_n^R$ to $\phi^R$ in $L^2_y$ and change of variables it follows
\[ (\phi^R, \chi_1 \chi_2)_{L^2_y} = \lim_{n \to \infty} \lambda_n^{\frac{d}{2}} \|h_n^R\|_{L^2_y} \|\chi_1 \chi_2\|_{L^2_y} \gtrsim \varepsilon^{12} A^{-11}. \]
On the other hand, using duality we infer that
\[ |(\phi^R, \chi_1 \chi_2)_{L^2_y}| \leq \|\phi^R\|_{L^2_y} \|\chi_1\|_{L^2_y} \|\chi_2\|_{H^{-1}_y} \lesssim \|\phi^R\|_{L^2_y}. \]
Thus
\[ \|\phi^R\|_{L^2_y} \gtrsim C \varepsilon A^{-11} \] (3.14)
for some $C = C(d) > 0$ which is uniform for all $R \geq K_1$. Now using (3.13) and (3.14) we finally deduce that
\[ \|\phi\|_{L^2_y}^2 \geq \|\phi^R\|_{L^2_y}^2 - C \varepsilon^{12} A^{-11} \geq C \varepsilon^{12} A^{-11} \]
for sufficiently large $R$, which gives the lower bound of (3.6). From now on we fix $R$ such that the lower bound of (3.6) is valid for this chosen $R$ and let $(t_n, x_n, \xi_n, \lambda_n)$ be the corresponding symmetry parameters. Since $L^2_y$ is a Hilbert space, from the weak convergence of $h_n$ to $\phi$ in $L^2_y$ we obtain that
\[ \lim_{n \to \infty} (h_n, \phi)_{L^2_y} - \|h_n\|_{L^2_y}^2 - \|\phi\|_{L^2_y}^2 = 2 \lim_{n \to \infty} \text{Re} \langle \phi, h_n - \phi \rangle_{L^2_y} = 0. \]
Combining with the fact that
\[ \|P_{\Lambda \xi} \phi - \phi\|_{L^2_y} \to 0 \quad \text{as} \quad n \to \infty \]
for $\lambda_n \to \infty$ we deduce the equalities in (3.6) and (3.7). Since $L^2_y \subset L^2_y$ is also a Hilbert space, follows verbatim. In the case $\lambda_n \xi_n \to \infty$, using the boundedness of $(\lambda_n \xi_n)$ and chain rule, we also infer that $\|h_n\|_{L^2_y} \lesssim \|f_n\|_{L^2_y}$. By the $H^1_y$-boundedness of $(f_n)$ and uniqueness of weak limit we deduce additionally that $\phi \in H^1_{x,y}$ and (3.5) follows.

Next we show that we may assume $\xi_n \equiv 0$ under the additional condition $\limsup_{n \to \infty} |\lambda_n \xi_n| < \infty$. Define
\[ \mathcal{T}_{a,b} u(x) := be^{ia \cdot x} u(x) \]
for $a \in \mathbb{R}^d$ and $b \in \mathbb{C}$ with $|b| = 1$. Let also
\[ (\lambda \xi)_\infty := \lim_{n \to \infty} \lambda_n \xi_n, \]
\[ e^{i(\xi \cdot x)_\infty} := \lim_{n \to \infty} e^{i\xi_n \cdot x_n}. \]
By the boundedness of \((\lambda_n, \zeta_n)\) we infer that \(T_{\lambda_n, e^{i\zeta_n \cdot x}}\) is an isometry on \(L^2_{x,y}\) and converges strongly to \(T_{\lambda, e^{i\zeta \cdot x}}\) as operators on \(H^1_{x,y}\). We may replace \(h_n\) by \(\lambda_n(e^{it_n \Delta}f_n)(\lambda_n x + x_n, y)\) and \(\phi\) by \(T_{\lambda, e^{i\zeta \cdot x}}\) and \(\phi\) carry over.

Finally, we prove (3.7). In the case \(\lambda_\infty < \infty\) we additionally know that \(\phi \in H^1_{x,y}\) and \(\zeta \equiv 0\). Using the fact that \(H^1_{x,y}\) is a Hilbert space, and change of variables we obtain

\[
o_n(1) = \|f_h\|_{H^1_{x,y}} - \|f_h - \phi\|_{H^1_{x,y}} = \lambda_n^2 \|f_h\|_{H^1_{x,y}} - \|f_h - \phi\|_{H^1_{x,y}}.
\]

Combining with the lower boundedness of \((\lambda_n)\), this implies that

\[
\|f_h\|_{H^1_{x,y}} - \|f_h - \phi\|_{H^1_{x,y}} = \lambda_n^2 o_n(1) = o_n(1),
\]

which gives (3.7) in the case \(\lambda_\infty < \infty\). Assume now \(\lambda_\infty = \infty\). Using change of variables and chain rule we obtain that

\[
\|f_h\|_{H^1_{x,y}} - \|f_h - \phi\|_{H^1_{x,y}} = \|f_h\|_{H^1_{x,y}} - \|f_h - \phi\|_{H^1_{x,y}} = \lambda_n^2 o_n(1) = o_n(1).
\]

Using the boundedness of \((\xi_n)\) and (3.8) we infer that \(I_1 \to 0\). For \(I_2\), using Bernstein and the boundedness of \((\xi_n)\) in \(\mathbb{R}^d\) and of \((h - P_{\leq \lambda_\infty} \phi)\) in \(L^2_{x,y}\) we see that

\[
I_2 \lesssim \lambda_n^{-1} \|h - P_{\leq \lambda_\infty} \phi\|_{L^2_{x,y}} \|\nabla_x P_{\leq \lambda_\infty} \phi\|_{L^2_{x,y}} \lesssim \lambda_n^{-2 - \theta} \to 0.
\]

Finally, \(I_3\) can be similarly estimated using Bernstein inequality, we omit the details here. Summing up we conclude (3.8).

\[\square\]

**Remark 3.8.** By redefining the symmetry parameters we may w.l.o.g. assume that

\[
(i) \lambda_n \equiv 1 \quad \text{or} \quad \lambda_n \to \infty,
\]

\[
(ii) \quad t_n \equiv 0 \quad \text{or} \quad \frac{t_n}{\lambda_n} \to \pm \infty
\]

and the linear profiles \(\phi_n\) take the form

\[
\phi_n = \begin{cases} e^{i t_n \Delta_x} \phi(x - x_n, y), & \text{if } \lambda_\infty = 1, \\ \lambda_n^{-1} e^{i x \cdot \zeta_n} [e^{i t_n \Delta_x} P_{L^2_{x,y}} \phi](\lambda_n^{-1} (x - x_n), y), & \text{if } \lambda_\infty = \infty. \end{cases}
\]

\[\square\]

**Lemma 3.9.** We have

\[
\|f_h\|_{L^2_{x,y}}^4 = \|\phi_h\|_{L^2_{x,y}}^4 + \|f_h - \phi_h\|_{L^2_{x,y}}^4 + o_n(1). \tag{3.15}
\]

**Proof.** Assume first that \(\lambda_\infty = 1\) and \(t_n \to \pm \infty\). In this case, we have \(\phi \in H^1_{x,y}\). For \(\beta > 0\) let \(\psi \in C^\infty_c(\mathbb{R}^2) \cap C^0_{\text{per}}(\mathbb{T})\) such that \(\|\phi - \psi\|_{H^1_{x,y}} \leq \beta\). Define also \(\psi_n := e^{it_n \Delta_x} \psi(x - x_n, y)\). Then by dispersive estimate we deduce that

\[
\|\psi_n\|_{L^2_{x,y}} \lesssim |t_n|^{-\frac{1}{2}} \|\psi\|_{L^4_{x,y}} \to 0.
\]

Now let \(\zeta \in C^\infty(\mathbb{R}^3; [0,1])\) be a cut-off function such that \(\text{supp} \zeta \subset \mathbb{R}^2 \times [-2\pi, 2\pi]\) and \(\zeta \equiv 1\) on \(\mathbb{R}^2 \times [-\pi, \pi]\). Then by Gagliardo-Nirenberg inequality, product rule and periodicity along the \(y\)-direction we have

\[
\|\psi_n - \phi_n\|_{H^1_{x,y}((\mathbb{R}^2 \times \mathbb{T}))} \lesssim \|\zeta(\psi_n - \phi_n)\|_{L^2_{x,y}(\mathbb{R}^3)} \lesssim \|\zeta(\psi_n - \phi_n)\|_{H^1_{x,y}(\mathbb{R}^3)} \lesssim \|\psi_n - \phi_n\|_{H^1_{x,y}((\mathbb{R}^2 \times \mathbb{T}))} \leq \beta,
\]

\[20\]
which in turn implies \( \| \phi_n\|_{L^4_{x,y}} = o_n(1) \). Therefore by triangular inequality
\[
\| f_n\|_{L^4_{x,y}} - \| f_n - \phi_n\|_{L^4_{x,y}} \leq \| \phi_n\|_{L^4_{x,y}} = o_n(1)
\]
and follows. Now we assume \( \lambda_\infty = 1 \) and \( t_n \equiv 0 \). Then we use the Brezis-Lieb lemma to deduce
\[
\| h_n\|_{L^4_{x,y}} = \| \phi\|_{L^4_{x,y}} + \| h_n - \phi\|_{L^4_{x,y}} + o_n(1).
\]
follows then by undoing the transformation. Finally, we take the case \( \lambda_\infty = \infty \). Using Gagliardo-
Nirenberg, chain rule, Bernstein, Minkowski and the embedding \( H^1_y \hookrightarrow L^4_y \)
\[
\| \phi_n\|_{L^4_{x,y}} \leq \| \phi\|_{L^4_{x,y}(\mathbb{R}^2)} \| \nabla (P_{\leq \lambda_n} \omega)\|_{L^4_y} \leq \lambda_n^{\frac{1}{8}} \| \phi\|_{L^4_y} \leq \lambda_n^{\frac{1}{8}} \| \phi\|_{L^4_y} \rightarrow 0
\]
as \( n \rightarrow \infty \). The desired claim then follows again by triangular inequality.

\[ \square \]

**Lemma 3.10** (Linear profile decomposition for bounded \( H^1_{x,y} \)-sequence). Let \((\psi_n)\) be a bounded sequence in \( H^1_{x,y} \). Then up to a subsequence, there exist nonzero linear profiles \((\phi^j)\) \( \subset L^2_y H^1_x \), remainders \((w^j_n)_{j,n} \subset L^2_y H^1_x \), parameters \((t_n^j, x_n^j, \xi_n^j, \lambda_n^j)_{j,n} \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \) and \( K^* \equiv \mathbb{N} \cup \{ \infty \} \), such that
(i) For any finite \( 1 \leq j \leq K^* \) the parameters satisfy
\[
1 > j \lim_{n \to \infty} \frac{\xi_n^j}{|x_n^j|}, \quad \lim_{n \to \infty} t_n^j =: t_\infty^j \in \{0, \pm \infty\}, \quad \lim_{n \to \infty} \lambda_n^j =: \lambda_\infty^j \in \{1, \infty\},
\]
\[
t_n^j \equiv 0 \quad \text{if} \quad t_\infty^j = 0, \quad \lambda_n^j \equiv 1 \quad \text{if} \quad \lambda_\infty^j = 1,
\]
\[
\xi_n^j \equiv 0 \quad \text{if} \quad \lambda_\infty^j = 1.
\]
(ii) For any finite \( 1 \leq k \leq K^* \) we have the decomposition
\[
\psi_n = \sum_{j=1}^{k} T^j_n P^j_n \phi^j + w^k_n.
\]
Here, the operators \( T^j_n \) and \( P^j_n \) are defined by
\[
T^j_n u(x) := \begin{cases} 
|e^{it_n^j \Delta_x} u(x - x_n^j, y)|, & \text{if} \ \lambda_n^j = 1, \\
g_{\xi_n^j, x_n^j, \lambda_n^j} |e^{it_n^j \Delta_x} u(x, y)|, & \text{if} \ \lambda_n^j = \infty
\end{cases}
\]
and
\[
P^j_n u := \begin{cases} 
u, & \text{if} \ \lambda_n^j = 1, \\
P_{\leq \lambda_n^j} u, & \text{if} \ \lambda_n^j = \infty
\end{cases}
\]
for some \( \theta \in (0, 1) \). Moreover,
\[
\phi^j \in \begin{cases} H^1_{x,y}, & \text{if} \ \lambda_n^j = 1, \\
L^2_y H^1_x, & \text{if} \ \lambda_n^j = \infty.
\end{cases}
\]
(iii) The remainders \((w^k_n)_{k,n}\) satisfy
\[
\lim_{k \to K^*} \lim_{n \to \infty} \| e^{it_n \Delta_x} w^k_n \|_{L^4_{x,y}(\mathbb{R})} = 0.
\]
(iv) The parameters are orthogonal in the sense that
\[ \frac{\lambda_n^j}{\lambda_n^k} + \lambda_n^j |\xi_n - \xi_n^k| + \|f_k \left( \frac{\lambda_n^j}{\lambda_n^k} \right)^2 - t_n^j \| + \left| x_n^j - x_n^k - 2t_n^k (\lambda_n^j)^2 (\xi_n - \xi_n^k) \right| \to \infty \] (3.16)
for any \( j \neq k \).

(v) For any finite \( 1 \leq k \leq K^* \) we have the energy decompositions
\[ \|\psi_n\|_L^2 \leq \sum_{j=1}^{k} \|T_n^j P_n^j \phi_j\|_L^2 + \|u_n^k\|_L^2 + o_n(1), \] (3.17)
\[ \|\nabla \psi_n\|_L^2 = \sum_{j=1}^{k} \|\nabla T_n^j P_n^j \phi_j\|_L^2 + \|\nabla u_n^k\|_L^2 + o_n(1), \] (3.18)
\[ \|\nabla g \psi_n\|_L^2 = \sum_{j=1}^{k} \|\nabla g T_n^j P_n^j \phi_j\|_L^2 + \|\nabla g u_n^k\|_L^2 + o_n(1), \] (3.19)
\[ \|\psi_n\|_L^4 = \sum_{j=1}^{k} \|T_n^j P_n^j \phi_j\|_L^4 + \|u_n^k\|_L^4 + o_n(1). \] (3.20)

\textbf{Proof.} We construct the linear profiles iteratively and start with \( k = 0 \) and \( u_n^0 := \psi_n \). We assume initially that the linear profile decomposition is given and its claimed properties are satisfied for some \( k \). Define \( \varepsilon_k := \lim_{n \to \infty} \|e^{it_\Delta} u_n^k\|_{L^2_{t,x,y}(\mathbb{R})} \).

If \( \varepsilon_k = 0 \), then we stop and set \( K^* = k \). Otherwise we apply Lemma 3.7 to \( u_n^k \) to obtain the sequence \( (\phi^{k+1}, u_n^{k+1}, \xi_n^{k+1}, \lambda_n^{k+1}) \). We should still need to check that the items (iii) and (iv) are satisfied for \( k + 1 \). That the other items are also satisfied for \( k + 1 \) follows directly from the construction of the linear profile decomposition. If \( \varepsilon_k = 0 \), then item (iii) is automatic; otherwise we have \( K^* = \infty \).

Using (3.6), (3.17), (3.18) and (3.19) we obtain that
\[ \sum_{j \in N} A_j^{24} \left( \frac{\varepsilon_j}{A_j} \right)^{24} \leq \sum_{j \in N} ||\phi_j||_{L^2 H^1_y}^2 = \sum_{j \in N} \lim_{n \to \infty} ||T_n^j P_n^j \phi_j||_{L^2 H^1_y}^2 \leq \lim_{n \to \infty} ||\psi_n||_{L^2 H^1_y}^2 = A_0 \]
where \( A_j := \lim_{n \to \infty} ||u_n^j||_{L^2 H^1_y} \). By (3.17) and (3.19) we know that \( (A_j)_j \) is monotone decreasing, thus also bounded. Hence
\[ A_j^2 \left( \frac{\varepsilon_j}{A_j} \right)^{24} \to 0 \quad \text{as} \ j \to \infty. \]

Combining with the boundedness of \( (A_j)_j \) we immediately conclude that \( \varepsilon_j \to 0 \) and the proof of item (iii) is complete. Finally we show item (iv). Assume that item (iv) does not hold for some \( j < k \). By the construction of the profile decomposition we have
\[ u_n^{k-1} = u_n^j - \sum_{l=j+1}^{k-1} g_n^l e^{-it_n^l \Delta} P_n^l \phi_l. \]

Then by definition of \( \phi^k \) we know that
\[ \phi^k = \text{w-lim} \lim_{n \to \infty} e^{-it_n^k \Delta} ([g_n^k]^{-1} u_n^{k-1}) \]
\[ = \text{w-lim} \lim_{n \to \infty} e^{-it_n^k \Delta} ([g_n^k]^{-1} u_n^j) - \sum_{l=j+1}^{k-1} \text{w-lim} \lim_{n \to \infty} e^{-it_n^k \Delta} ([g_n^k]^{-1} P_n^l \phi_l), \]
where the weak limits are taken in the \( L^2 H^1_y \)-topology. We aim to show \( \phi^k \) is zero, which leads to a contradiction and proves item (iv). For the first summand, we obtain that
\[ e^{-it_n^k \Delta} ([g_n^k]^{-1} u_n^j) = (e^{-it_n^k \Delta} (g_n^k)^{-1} g_n^l e^{it_n^l \Delta}) [e^{-it_n^k \Delta} (g_n^k)^{-1} u_n^j]. \]
Direct calculation yields

\[ e^{-\mu_n \Delta} (g_n^k)^{-1} g_n^j e^{it_n \Delta_x} = \beta_n^{j,k} g_n^k \lambda_n^j (\xi - \xi_n^j) \frac{t_n - x_n^j - 2it_n (\lambda_n^j)^2 (\xi - \xi_n^j)}{\lambda_n^j} e^{-it_n \left( \frac{\lambda_n^j}{\lambda_n^k} \right)^2 - t_n^2} \Delta_x. \]  

(3.21)

with \( \beta_n^{j,k} = e^{it_n \lambda_n^k} x_n^j (\lambda_n^k)^2 (\xi - \xi_n^j)^2 \). Therefore, the failure of item (iv) will lead to the strong convergence of the adjoint of \( e^{-\mu_n \Delta} (g_n^k)^{-1} g_n^j e^{it_n \Delta_x} \) on \( L^2_y H^1_y \). By construction of the profile decomposition we have

\[ e^{-\mu_n \Delta} (g_n^k)^{-1} w_n^j \to 0 \quad \text{in} \ L^2_y H^1_y \]

and we conclude that the first summand weakly converges to zero in \( L^2_y H^1_y \). Now we consider the single terms in the second summand. We can rewrite each single summand to

\[ e^{-\mu_n \Delta} (g_n^k)^{-1} P_n^j \phi^l \to 0 \quad \text{in} \ L^2_y H^1_y. \]

Due to the construction of the profile decomposition and the inductive hypothesis we know that \( \phi^l \in L^2_y H^1_y \) and item (iv) is satisfied for the pair \((j,l)\). Using the fact that

\[ \| P_{\leq (\lambda_n^j)^2} \phi^l - \phi^l \|_{L^2_y H^1_y} \to 0 \quad \text{when} \ \lambda_n^j \to \infty \]

and density arguments, it suffices to show that

\[ I_n := e^{-\mu_n \Delta} (g_n^j)^{-1} g_n^l e^{it_n \Delta_x} \phi \to 0 \quad \text{in} \ L^2_y H^1_y \]

for arbitrary \( \phi \in C_c^{\infty}(\mathbb{R}^2) \cap C_{\mathrm{per}}^{\infty}(T) \). Using (3.21) we obtain that

\[ I_n = \beta_n^{j,l} g_n^j \lambda_n^j (\xi - \xi_n^j) \frac{t_n - x_n^j - 2it_n (\lambda_n^j)^2 (\xi - \xi_n^j)}{\lambda_n^j} e^{-it_n \left( \frac{\lambda_n^j}{\lambda_n^l} \right)^2 - t_n^2} \Delta_x \phi. \]

Assume first that \( \lim_{n \to \infty} \frac{\lambda_n^j}{\lambda_n^l} = \infty \). Then for any \( \psi \in C_c^{\infty}(\mathbb{R}^2) \cap C_{\mathrm{per}}^{\infty}(T) \) we have

\[ \| (I_n, \psi)_{L^2_y H^1_y} \| \leq \min \left\{ \left( \frac{\lambda_n^j}{\lambda_n^l} \right)^{-1} \| F_x \psi \|_{L^2_y H^1_y}, \left( \frac{\lambda_n^l}{\lambda_n^j} \right)^{-1} \| F_x \psi \|_{L^2_y H^1_y} \right\} \to 0. \]

So we may assume that \( \lim_{n \to \infty} \frac{\lambda_n^j}{\lambda_n^l} \in (0, \infty) \). Suppose now \( t_n^j (\lambda_n^j)^2 - t_n^l \to \pm \infty \). Then the weak convergence of \( I_n \) to zero in \( L^2_y H^1_y \) follows immediately from the dispersive estimate. Hence we may also assume that \( \lim_{n \to \infty} t_n^j (\lambda_n^j)^2 - t_n^l \in \mathbb{R} \). Finally, it is left with the options

\[ \left| \frac{x_n^j - x_n^l - 2t_n^l (\lambda_n^j)^2 (\xi - \xi_n^j)}{\lambda_n^l} \right| \to \infty. \]

In the latter case, we utilize the fact that the symmetry group composing by unbounded translations in \( L^2_y \) weakly converges to zero as operators in \( L^2_y H^1_y \) to deduce the claim: In the former case, we can use the same arguments as the ones for the translation symmetry by considering the Fourier transformation of \( I_n \) (w.r.t. \( x \)) in the frequency space. This completes the desired proof of item (iv).

\[ \square \]

**Remark 3.11.** By interpolation and Strichartz we have for \( s \in (\frac{1}{2}, 1) \)

\[ \lim_{k \to K^*} \lim_{n \to \infty} \| e^{it \Delta_x} w_n \|_{L^s_x H^s_y(\mathbb{R})} \approx \lim_{k \to K^*} \lim_{n \to \infty} \| e^{it \Delta_x} w_n \|_{L^s_x H^s_y(\mathbb{R})} \]

\[ \lesssim \lim_{k \to K^*} \lim_{n \to \infty} \| e^{it \Delta_x} w_n \|_{L^1_x H^{1-s}_y(\mathbb{R})} \| w_n \|_{H^s_y} \]

\[ \lesssim \lim_{k \to K^*} \lim_{n \to \infty} \| e^{it \Delta_x} w_n \|_{L^1_x L^{1-s}_y(\mathbb{R})} = 0. \]  

(3.22)
3.3 Large scale approximation

The following lemma shows that large scale nonlinear profiles can be well approximated by the large scale resonant system \((1.3)\).

**Lemma 3.12 (Large scale approximation).** Let \((\lambda_n) \subset (0, \infty)\) such that \(\lambda_n \to \infty\), \((t_n) \subset \mathbb{R}\) such that either \(t_n \equiv 0\) or \(t_n \to \pm \infty\) and \((\xi_n) \subset \mathbb{R}^d\) such that \((\xi_n)\) is bounded. Let \(\phi \in L^2_\theta H^1_y\) and define

\[
\phi_n := g_{\xi_n, x_n, \lambda_n} e^{i t_n \Delta} P_{\leq \lambda_n} \phi
\]

for some \(\theta \in (0, 1)\). Assume also that \(\mathcal{M}(\phi) < \pi \mathcal{M}(Q_{2d})\). Then for all sufficiently large \(n\) the solution \(u_n\) of \((1.1)\) with \(U_n(0) = \phi_n\) is global and scattering in time with

\[
\lim_{n \to \infty} \|U_n\|_{L^4_t, H^1_y(\mathbb{R})} \leq C(\|\phi\|_{L^2_\theta H^1_y}).
\]

Furthermore, for every \(\beta > 0\) there exists \(N_\beta \in \mathbb{N}\) and \(\psi_\beta \in C^\infty_0(\mathbb{R} \times \mathbb{R}^2) \otimes C^\infty_{per}(\mathbb{T})\) such that

\[
\left\| U_n - \lambda_n^{-1} e^{-it|\xi_n|^2} e^{i \xi_n \cdot x} \psi_\beta \left( \frac{t}{\lambda_n} + t_n, \frac{x - x_n - 2it \xi_n}{\lambda_n}, y \right) \right\|_{L^4_t, H^1_y(\mathbb{R})} \leq \beta
\]

(3.23)

for all \(n \geq N_\beta\).

**Proof.** The proof is almost identical to the one of \([12, \text{Lem. 3.11}]\), we only need to replace the large data scattering result \([49, \text{Thm. 1.1}]\) therein for the defocusing analogue of \((1.3)\) to Theorem \([13]\) (hence we also impose the mass restriction \(\mathcal{M}(\phi) < \pi \mathcal{M}(Q_{2d})\)). We omit therefore the repeating arguments. \(\Box\)

**Remark 3.13.** We explain where the prefactor \(\pi\) comes from. By our definition of the Fourier series, for \(\phi \in H^1_{x,y}\) we have the Fourier inverse formula

\[
\phi(x, y) = \sum_{k \in \mathbb{Z}} (2\pi)^{-1} e^{iky} \hat{F}_y \phi(x, k)
\]

and the Plancherel's isometry formula

\[
\|\phi\|^2_{L^2_{x,y}} = \|F_y \phi\|^2_{L^2_y}.
\]

The initial data \(V_\phi\) for the large scale proxy is defined by

\[
V_\phi := \left( e^{iky} (2\pi)^{-\frac{1}{2}} \hat{F}_y \phi(x, k) \right)_{k \in \mathbb{Z}}.
\]

In order to apply Theorem \([13]\) we then demand

\[
(2\pi)^{-1} \|\phi\|^2_{L^2_{x,y}} = (2\pi)^{-1} \|F_y \phi\|^2_{L^2_y} < 2^{-1} \mathcal{M}(Q_{2d}).
\]

\(\triangle\)

3.4 Variational analysis

We begin with the proof of Proposition \([13]\).

**Proof of Proposition \([13]\).** For a function \(u\) we define 

\[
m(u) := (2\pi)^{-1} \int u(y) dy.
\]

Then

\[
\|u\|_{L^4_{x,y}} \leq \|m(u)\|_{L^4_{x,y}} + \|u - m(u)\|_{L^4_{x,y}}.
\]

We will show that \(\|m(u)\|_{L^4_{x,y}}\) and \(\|u - m(u)\|_{L^4_{x,y}}\) are bounded by the first term and second term of \((1.7)\) respectively, which will complete the proof. For \(\|m(u)\|_{L^4_{x,y}}\) we use \((16)\) and Jensen to infer

\[
\|m(u)\|_{L^4_{x,y}}^4 = 2\pi \|m(u)\|_{L^4_{x,y}}^4 \\
\leq 2\pi C_{GN, 2d}^{-1} \|m(u)\|_{L^2_y}^2 \|\nabla_x m(u)\|_{L^2_y}^2 \\
\leq 2\pi C_{GN, 2d}^{-1} (2\pi)^{-1} \|u\|_{L^2_{x,y}}^2 \|\nabla_x u\|_{L^2_{x,y}}^2 = (\pi \mathcal{M}(Q_{2d}))^{-1} \|u\|_{L^2_{x,y}}^2 \|\nabla_x u\|_{L^2_{x,y}}^2.
\]

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To estimate the second term, we recall the Sobolev inequality on torus for functions with zero mean (see for instance \cite{2}): for $s > 0$ and $1 < p < q < \infty$ with $s/d = 1/p - 1/q$ we have

$$\|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|_{L^p(\mathbb{T}^d)}.$$ 

Therefore, setting $s = \frac{1}{2}, p = 2$ and $q = 4$ we obtain

$$\|u\|^4_{L^4_x} \lesssim \|u\|^4_{H^\frac{1}{2}} \lesssim \|u\|^3_{L^3_x} \|\nabla u\|_{L^2_y},$$

and the existence of the number $C_T$ from Proposition 1.4 follows. Using Hölder we conclude

$$\|u - m(u)\|^4_{L^4_{x,v}} \leq C_T\|u - m(u)\|^3_{L^3_x L^2_v} \|\nabla u\|_{L^2_{x,v}}.$$

Followed by Minkowski, Gagliardo-Nirenberg (bounding $L^6_\mathbb{T}$ by $L^2_\mathbb{T} - H^1_\mathbb{T}$ in 2D), Hölder and Jensen, we see that

$$\|u - m(u)\|^3_{L^3_x L^2_v} \leq \|u - m(u)\|^3_{L^3_x L^2_v},$$

which completes the desired proof. \hfill \Box

We next prove a crucial energy trapping result based on Proposition 1.4.

**Lemma 3.14 (Energy trapping).** Let $c_\ast$ and $\Gamma$ be defined through (1.8) and (1.9) respectively. Suppose that $U_0 \in H^1_{x,v}$ satisfies (1.13), (1.11) and (1.12). Let $U$ be the solution of (1.1) with $U(0) = U_0$. Then for all $t \in I_{\text{max}}$, where $I_{\text{max}}$ is the maximal lifespan of $U$, we have

$$\|\nabla_y U(t)\|^2_{L^2_{x,v}} \leq \Gamma(U_0).$$

Moreover, if there exists some $\beta \in (0, 1)$ such that

$$\mathcal{M}(U_0) \leq (1 - \beta)2\pi \mathcal{M}(Q_{2d}),$$

$$\mathcal{E}(U_0) \leq (1 - \beta)2^{-1}\Gamma(U_0),$$

then

$$\|\nabla_{x,y} U(t)\|^2_{L^2_{x,v}} \lesssim_{\beta} \mathcal{E}(U_0),$$

$$\|\nabla_x U(t)\|^2_{L^2_{x,v}} \lesssim \frac{1}{2}\|\nabla_x U(t)\|^2_{L^2_{x,v}} - \frac{1}{4}\|U(t)\|^4_{L^4_{x,v}} =: \mathcal{E}_s(U(t)).$$

**Proof.** Using (1.7) we have

$$\mathcal{E}(U_0) \geq \frac{1}{2}\|\nabla_y U(t)\|^2_{L^2_{x,v}} + \frac{1}{4}\|\nabla_x U(t)\|^2_{L^2_{x,v}} \left(2 - \left((\pi \mathcal{M}(Q_{2d}))^{-\frac{1}{2}} \mathcal{M}(U) \right)^\frac{1}{2} + c_\ast \mathcal{M}(U) \right)^\frac{1}{2} \|\nabla_y U(t)\|^2_{L^4_{x,v}}.$$  

$$= \frac{1}{2}\|\nabla_y U(t)\|^2_{L^2_{x,v}} + \frac{1}{4}\|\nabla_x U(t)\|^2_{L^2_{x,v}} \Xi(\|\nabla_y U(t)\|^2_{L^2_{x,v}}).$$

Similarly,

$$\mathcal{E}_s(U(t)) \geq \frac{1}{4}\|\nabla_x U(t)\|^2_{L^2_{x,v}} \Xi(\|\nabla_y U(t)\|^2_{L^2_{x,v}}).$$

One easily verifies that $\Gamma(U)$ is a root of $\Xi$. Since $\|\nabla_y U_0\|^2_{L^2_{x,v}} < \Gamma(U)$, if there exists some $t \in I_{\text{max}}$ such that $\|\nabla_y U(t)\|^2_{L^2_{x,v}} \geq \Gamma(U)$, then by continuity there exists some $s \in (0, t]$ such that $\|\nabla_y U(s)\|^2_{L^2_{x,v}} = \Gamma(U)$. But then we obtain the contradiction

$$2^{-1}\Gamma(U) > \mathcal{E}(U_0) = \mathcal{E}(U(s)) = 2^{-1}\Gamma(U),$$

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which implies \((3.24)\). Next, we take \((3.27)\) and \((3.28)\). \((3.29)\) implies \(\|\nabla_y U(t)\|_{L^2_y}^2 \leq (1 - \beta)\Gamma(U)\), which in turn implies

\[
2\frac{d}{dt} - \left((-\beta \tau(Q_{2d})) + c \mathcal{M}(Q_{2d}) + c \mathcal{M}(U) \|\nabla_y U(t)\|_{L^2_y}^2\right) \geq (1 - (1 - \beta)\frac{d}{dt})(1 - (1 - \beta)\frac{d}{dt})2\frac{d}{dt} > 0,
\]

which combining with \((3.29)\) and \((3.30)\) implies \((3.27)\) and \((3.28)\).

At the end of this section, we introduce the MEI-functional \(D\) which plays a fundamental role for setting up the inductive hypothesis. Such functional was firstly introduced in [36] and is quite useful for building up a multi-directional inductive hypothesis scheme. Define the domain \(\Omega \subset \mathbb{R}^2\) by

\[
\Omega := \left((-\infty, 0] \times \mathbb{R}\right) \cup \left\{(c, h) \in \mathbb{R}^2 : c \in (0, \pi\mathcal{M}(Q_{2d})), h \in (-\infty, 2^{-1}\Gamma(c))\right\}.
\]

Then we define the MEI-functional \(D : \mathbb{R}^2 \to [0, \infty]\) by

\[
D(c, h) = \begin{cases} 
\pi(c\mathcal{M}(Q_{2d}) - c)^{-1} + h(2^{-1}\Gamma(c) - h)^{-1}, & \text{if } (c, h) \in \Omega, \\
\infty, & \text{otherwise.}
\end{cases}
\]

For \(U \in H^1_{x,y}\), define \(D(U) := D(\mathcal{M}(U), \mathcal{E}(U))\). Also define the quantity \(K(U) := \Gamma(U) - \|\nabla_y U\|_{L^2_y}^2\) and the set \(A\) by

\[
A := \{U \in H^1_{x,y} : 0 < \mathcal{M}(U) < \pi\mathcal{M}(Q_{2d}), \mathcal{E}(U) < 2^{-1}\Gamma(U), K(U) > 0\}.
\]

By conservation of mass and energy we know that if \(U\) is a solution of \((1.1)\), then \(D(U(t))\) is a conserved quantity, thus in the following we simply write \(D(U) = D(U(t))\). Moreover, by Lemma 3.14 we know that if \(U(t) \in A\) for some \(t\) in the lifespan of \(U\), then \(U(t) \in A\) for all \(t\) in the maximal lifespan of \(U\). In this case, we simply write \(U \in A\).

We end this section by giving some useful properties of the MEI-functional.

**Lemma 3.15.** Let \(U, U_1, U_2\) be solutions of \((1.1)\). The following statements hold:

(i) Let \(K(U(t)) > 0\) for some \(t\) in the lifespan of \(U\). Then \(0 < D(U) < \infty\) if and only if \(U \in A\).

(ii) Let \(U_1, U_2 \in A\) satisfy \(\mathcal{M}(U_1) \leq \mathcal{M}(U_2)\) and \(\mathcal{E}(U_1) \leq \mathcal{E}(U_2)\), then \(D(U_1) \leq D(U_2)\). If in addition either \(\mathcal{M}(U_1) < \mathcal{M}(U_2)\) or \(\mathcal{E}(U_1) < \mathcal{E}(U_2)\), then \(D(U_1) < D(U_2)\).

(iii) Let \(D_0 \in (0, \infty)\). Then

\[
\|\nabla_{x,y} U\|_{L^2_y}^2 \lesssim_{D_0} \mathcal{E}(U), \\
\|U\|_{H^1_{x,y}}^2 \lesssim_{D_0} \mathcal{E}(U) + \mathcal{M}(U) \lesssim_{D_0} D(U)
\]

uniformly for all \(U \in A\) with \(D(U) \leq D_0\).

**Proof.** (i) follows immediately from Lemma 3.14 (ii) follows directly from the definition of \(D\) and \(\Gamma(U)\). Now we take (iii). Since \(U \in A\), we know that \(\mathcal{M}(U) \in (0, \pi\mathcal{M}(Q_{2d}))\) and using Lemma 3.14 also \(\mathcal{E}(U) \in [0, 2^{-1}\Gamma(U)]\). By definition of \(D_0\) we infer that

\[
D_0 \geq D(U) \geq \frac{\mathcal{M}(U)}{\pi\mathcal{M}(Q_{2d}) - \mathcal{M}(U)}.
\]

which in turn implies

\[
\mathcal{M}(U) \leq \frac{D_0}{1 + D_0}\pi\mathcal{M}(Q_{2d}).
\]

Similarly, we deduce

\[
\mathcal{E}(U) \leq \frac{D_0}{1 + D_0}2^{-1}\Gamma(U)
\]

and \((3.31)\) follows from Lemma 3.14 The first inequality of \((3.32)\) follows already from \((3.31)\). Next, we obtain that \((3.33)\) also implies

\[
\mathcal{M}(U) \leq \frac{D(U)\pi\mathcal{M}(Q_{2d})}{1 + D(U)} \leq D(U)\pi\mathcal{M}(Q_{2d}).
\]

Together with \(D(U) \geq \mathcal{E}(U)\), which is deduced directly from the definition of \(D\), the desired claim follows.
3.5 Existence of a minimal blow-up solution

Having all the preliminaries we are ready to construct a minimal blow-up solution of (1.1). For convenience, we simply fix the number $s$ in Lemma 3.3 to $s = \frac{3}{2}$. This number can be replaced by any number from the interval $\left(\frac{1}{2}, 1\right)$, but we need to restrict the number to be smaller than one in order to apply (3.22). Define

$$\tau(D_0) := \sup \left\{ \|U\|_{L^1_{t,x}H^1_y(t_{\text{max}})} : U \text{ is solution of (1.1)}, U(0) \in \mathcal{A}, D(U) \leq D_0 \right\}$$

and

$$D^* := \sup \{ D_0 > 0 : \tau(D_0) < \infty \}.$$ 

By Lemma 3.3 and Lemma 3.15 we know that $D^* > 0$ and $\tau(D_0) < \infty$ for sufficiently small $D_0$. We will therefore assume $D^* < \infty$ and aim to derive a contradiction, which will imply $D^* = \infty$ and the proof of Theorem 3.6 will be complete in view of Lemma 3.15. By the inductive hypothesis we can find a sequence $(U_n)_n$ which are solutions of (1.1) with $(U_n(0))_n \subset \mathcal{A}$ and maximal lifespan $(I_n)_n$ such that

$$\lim_{n \to \infty} \|U_n\|_{L^1_{t,x}H^1_y(\inf I_n)} = \lim_{n \to \infty} \|U_n\|_{L^1_{t,x}H^1_y(0, \sup I_n)} = \infty,$$

$$\lim_{n \to \infty} D(U_n) = D^*.$$

Up to a subsequence we may also assume that

$$(\mathcal{M}(U_n), \mathcal{E}(U_n)) \to (\mathcal{M}_0, \mathcal{E}_0) \quad \text{as} \quad n \to \infty.$$

By continuity of $D$ and finiteness of $D^*$ we know that

$$D^* = D(\mathcal{M}_0, \mathcal{E}_0), \quad \mathcal{M}_0 \in (0, \pi \mathcal{M}(Q_{2d})), \quad \mathcal{E}_0 \in [0, 2^{-1} \Gamma(U)).$$

From Lemma 3.15 it follows that $(U_n(0))_n$ is a bounded sequence in $H^1_{x,y}$, hence Lemma 3.10 is applicable for $(U_n(0))_n$. We define the nonlinear profiles as follows: For $\lambda_\infty^k = \infty$, we define $U^k_n$ as the solution of (1.1) with $U^k_n(0) = T^k_n \phi^k$. For $\lambda_\infty^k = 1$ and $\tau_\infty^k = 0$, we define $U^k_n$ as the solution of (1.1) with $U^k_n(0) = \phi^k$. For $\lambda_\infty^k = 1$ and $\tau_\infty^k \to \pm \infty$, we define $U^k$ as the solution of (1.1) that scatters forward (backward) to $e^{it\Delta} \phi^k$ in $H^1_{x,y}$. In both cases for $\lambda_\infty^k = 1$ we define

$$U^k_n := U^k(t + t_n, x - x_n^k, y).$$

Then $U^k_n$ is also a solution of (1.1). In all cases we have for each finite $1 \leq k \leq K^*$

$$\lim_{n \to \infty} \|U^k_n(0) - T^k_n \phi^k\|_{H^1_{x,y}} = 0.$$

(3.35)

In the following, we establish a Palais-Smale type lemma which is essential for the construction of the minimal blow-up solution.

**Lemma 3.16** (Palais-Smale-condition). Let $(U_n)_n$ be a sequence of solutions of (1.1) with maximal lifespan $I_n$, $U_n \in \mathcal{A}$ and $\lim_{n \to \infty} D(U_n) = D^*$. Assume also that there exists a sequence $(t_n)_n \subset \prod I_n$ such that

$$\lim_{n \to \infty} \|U_n\|_{L^1_{t,x}H^1_y(\inf I_n, t_n)} = \lim_{n \to \infty} \|U_n\|_{L^1_{t,x}H^1_y(t_n, \sup I_n)} = \infty.$$

(3.36)

Then up to a subsequence, there exists a sequence $(x_n)_n \subset \mathbb{R}^2$ such that $(U_n(t_n, \cdot + x_n, y))_n$ strongly converges in $H^1_{x,y}$.

**Proof.** By time translation invariance we may assume that $t_n \equiv 0$. Let $(U^j_n)_{j,n}$ be the nonlinear profiles corresponding to the linear profile decomposition of $(U_n(0))_n$. Define

$$\psi^k_n := \sum_{j=1}^k U^j_n + e^{it\Delta} \phi^k.$$

We will show that there exists exactly one non-trivial bad linear profile, relying on which the desired claim follows. We divide the remaining proof into three steps.
Step 1: Positive energies of the linear profiles

Since the nonlinearity is focusing, it is a priori unclear whether the linear profiles have non-negative energies. We show that this is indeed the case for sufficiently large $n$. Using (3.17) to (3.20) we conclude that for any finite $1 \leq k \leq K^*$

$$M_0 = \sum_{j=1}^{k} \mathcal{M}(T_n^j P_n^j \phi^j) + \mathcal{M}(w_n^k) + o_n(1),$$

(3.37)

$$\mathcal{E}_0 = \sum_{j=1}^{k} \mathcal{E}(T_n^j P_n^j \phi^j) + \mathcal{E}(w_n^k) + o_n(1)$$

(3.38)

$$\|\nabla_y U_n(0)\|_{L^2_y}^2 = \sum_{j=1}^{k} \|\nabla_y T_n^j P_n^j \phi^j\|_{L^2_y}^2 + \|\nabla_y w_n^k\|_{L^2_y} + o_n(1).$$

(3.39)

By (3.39) and the fact that $U_n(0) \in \mathcal{A}$ we know that for given $1 \leq k \leq K^*$ we have $\mathcal{K}(T_n^k P_n^k \phi^k) > 0$ and $\mathcal{K}(w_n^k) > 0$ for sufficiently large $n$. If in this case $\mathcal{E}(T_n^k P_n^k \phi^k)$ were negative, then

$$\mathcal{E}(T_n^k P_n^k \phi^k) < 0 \leq 2^{-1} \mathcal{K}(T_n^k P_n^k \phi^k),$$

which contradicts Lemma 3.14 and we conclude that $\mathcal{E}(T_n^k P_n^k \phi^k) \geq 0$ for given $1 \leq k \leq K^*$ and all $n \geq N_1$ for some large $N_1 = N_1(k)$. The same holds for $w_n^k$ and the proof of Step 1 is complete.

Step 2: Decoupling of nonlinear profiles

In this step, we show that the nonlinear profiles are asymptotically decoupled in the sense that

$$\lim_{n \to \infty} \|U_n^i U_n^j\|_{L^2_{t,x} \dot{H}^1_y(R)} = \lim_{n \to \infty} \sum_{\lambda_1, \lambda_2 = 0}^1 \|\partial_t^{\lambda_1} \partial_x^{\lambda_2} U_n^i \partial_t^{\lambda_1} \partial_x^{\lambda_2} U_n^j\|_{L^2_{t,x} \dot{H}^1_y(R)} = 0$$

(3.40)

for any fixed $1 \leq i, j \leq K^*$ with $i \neq j$, provided that

$$\lim_{n \to \infty} (\|U_n^i\|_{L^1_{t,x} \dot{H}^1_y(R)} + \|U_n^j\|_{L^1_{t,x} \dot{H}^1_y(R)}) < \infty.$$ 

We claim that for any $\beta > 0$ there exists some $\psi^i_{\beta}, \psi^j_{\beta} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2) \cap C^\infty_{per}(T)$ such that

$$\|U_n^i - (\lambda_n^i)^{-1} e^{-it|\xi_n^i|^2} e^{i\xi_n^i \cdot x} \psi^i_{\beta}\left(\frac{t}{(\lambda_n^i)^2} + t_n, \frac{x - x_n}{\lambda_n^i}, y\right)\|_{L^1_{t,x} \dot{H}^1_y(R)} \leq \beta,$$

$$\|U_n^j - (\lambda_n^j)^{-1} e^{-it|\xi_n^j|^2} e^{i\xi_n^j \cdot x} \psi^j_{\beta}\left(\frac{t}{(\lambda_n^j)^2} + t_n, \frac{x - x_n}{\lambda_n^j}, y\right)\|_{L^1_{t,x} \dot{H}^1_y(R)} \leq \beta.$$ 

Indeed, for $\lambda_{\infty}^i, \lambda_{\infty}^j = \infty$, this follows already from (5.22), while for $\lambda^i_{\infty}, \lambda^j_{\infty} = 1$ we choose some $\psi^i_{\beta}, \psi^j_{\beta} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ such that

$$\|U^i - \psi^i_{\beta}\|_{L^1_{t,x} \dot{H}^1_y(R)} \leq \beta, \quad \|U^j - \psi^j_{\beta}\|_{L^1_{t,x} \dot{H}^1_y(R)} \leq \beta.$$ 

Define

$$\Lambda_n(\psi_{\beta}) := (\lambda_n^i)^{-1} \psi^i_{\beta}\left(\frac{t}{(\lambda_n^i)^2} + t_n, \frac{x - x_n}{\lambda_n^i}, y\right).$$

Using Hölder we infer that

$$\|\partial_t^{\lambda_1} \partial_x^{\lambda_2} U_n^i \partial_t^{\lambda_1} \partial_x^{\lambda_2} U_n^j\|_{L^2_{t,x} \dot{H}^1_y(R)} \lesssim \beta + \|\partial_t^{\lambda_1} \Lambda_n(\psi^i_{\beta}) \partial_t^{\lambda_1} \Lambda_n(\psi^j_{\beta})\|_{L^2_{t,x} \dot{H}^1_y(R)}.$$ 

Since $\beta$ can be chosen arbitrarily small, it suffices to show

$$\lim_{n \to \infty} \|\partial_t^{\lambda_1} \Lambda_n(\psi^i_{\beta}) \partial_t^{\lambda_1} \Lambda_n(\psi^j_{\beta})\|_{L^2_{t,x} \dot{H}^1_y(R)} = 0.$$  

(3.41)
Assume that $\frac{\lambda_i}{n} + \frac{\lambda_j}{n} \to \infty$. By symmetry we may w.l.o.g. assume that $\frac{\lambda_i}{n} \to 0$. Using change of variables we obtain that

$$
\| \partial_{\theta}^2 \Lambda_n(\psi_i^j(\theta)) \partial_{\theta}^2 \Lambda_n(\psi_i^j(\theta)) \|_{L^2_{t,x}L^2_{\theta}(\mathbb{R})} = \frac{\lambda_i}{\lambda_n} \| \partial_{\theta}^2 \psi_i^j(t) \partial_{\theta}^2 \psi_i^j \left( \left( \frac{\lambda_i}{\lambda_n} \right)^2 t - \left( \frac{\lambda_i}{\lambda_n} \right)^2 t_n - t_n, y \right),
$$

(3.42)

$$
\lesssim \lambda_n (\frac{\lambda_i}{\lambda_n})^{-1} \| \psi_i^j \|_{L^2_{t,x}H^1_{\theta}(\mathbb{R})} \| \psi_i^j \|_{L^2_{t,x}W^{1,\infty}(\mathbb{R})} \to 0
$$

and the claim follows. Suppose therefore $\frac{\lambda_i}{\lambda_n} + \frac{\lambda_j}{\lambda_n} \to \lambda_0 \in (0, \infty)$. If $\left( \frac{\lambda_i}{\lambda_n} \right)^2 t_n - t_n \to \pm \infty$, then by (3.42) the supports of the integrands become disjoint in the temporal direction. We may therefore further assume that $\left( \frac{\lambda_i}{\lambda_n} \right)^2 t_n - t_n \to t_0 \in \mathbb{R}$. If $\left| \frac{\lambda_i}{\lambda_n} t_n - 2t_n \left( \frac{\lambda_i}{\lambda_n} \right)^2 (\xi_i^j - \xi_j^i) \right| \to \infty$ and $\xi_i^j = \xi_j^i$ for infinitely many $n$, then the supports of the integrands become disjoint in the $x$-spatial direction; If $\left| \frac{\lambda_i}{\lambda_n} t_n - 2t_n \left( \frac{\lambda_i}{\lambda_n} \right)^2 (\xi_i^j - \xi_j^i) \right| \to \infty$ and $\xi_i^j \neq \xi_j^i$ for infinitely many $n$, then we apply the change of temporal variable $t \to \frac{t - t_n}{\lambda_n (\xi_i^j - \xi_j^i)}$ to see the decoupling of the supports of the integrands in the $x$-spatial direction. Finally, if $\frac{\lambda_i}{\lambda_n} t_n - 2t_n \left( \frac{\lambda_i}{\lambda_n} \right)^2 (\xi_j^i - \xi_i^j) \to x_0 \in \mathbb{R}^d$, then by (3.10) we must have $\lambda_i \left| \xi_j^i - \xi_i^j \right| \to \infty$. Hence for all $t \neq 0$ the integrand converges pointwise to zero. Using the dominated convergence theorem (setting $\| \psi_i^j \|_{L^2_{t,x}W^{1,\infty}(\mathbb{R})} \psi_i^j$ as the majorant) we finally conclude (3.34).

**Step 3: Existence of at least one bad profile**

First we claim that there exists some $1 \leq J \leq K^*$ such that for all $j \geq J + 1$ and all sufficiently large $n$, $U_{J,n}^j$ is global and

$$
\sup_{J + 1 \leq j \leq K^*} \lim_{n \to \infty} \| U_{J,n}^j \|_{L^1_{t,x}H^1_{\theta}(\mathbb{R})} \lesssim 1.
$$

(3.43)

Indeed, using (3.17) to (3.19) we infer that

$$
\lim_{k \to K^*} \lim_{n \to \infty} \sum_{j=1}^k \| T_{J,n}^j P_{J,n}^j \phi_{J,n}^j \|_{H^1_{t,x}} < \infty.
$$

Then (3.43) follows from Lemma 3.3. In the same manner, by Lemma 3.3 we infer that

$$
\sup_{J + 1 \leq j \leq K^*} \lim_{n \to \infty} \sum_{j=1}^k \| U_{J,n}^j \|_{L^1_{t,x}H^1_{\theta}(\mathbb{R})} \lesssim 1.
$$

(3.44)

We now claim that there exists some $1 \leq J_0 \leq J$ such that

$$
\lim_{n \to \infty} \| U_{J_0,n}^j \|_{L^1_{t,x}H^1_{\theta}(\mathbb{R})} = \infty.
$$

We argue by contradiction and assume that

$$
\lim_{n \to \infty} \| U_{J,n}^j \|_{L^1_{t,x}H^1_{\theta}(\mathbb{R})} < \infty \quad \forall 1 \leq j \leq J.
$$

(3.45)

To proceed, we first show that

$$
\sup_{J + 1 \leq j \leq K^*} \lim_{n \to \infty} \left\| \sum_{j=J+1}^k U_{J,n}^j \right\|_{L^1_{t,x}H^1_{\theta}(\mathbb{R})} \lesssim 1.
$$

(3.46)

Indeed, using triangular inequality, (3.40) and (3.44) we immediately obtain

$$
\sup_{J + 1 \leq j \leq K^*} \lim_{n \to \infty} \left\| \sum_{j=J+1}^k \partial_{\theta}^2 U_{J,n}^j \right\|_{L^2_{t,x}L^2_{\theta}(\mathbb{R})}^{4}
$$

29
\[ \leq \sup_{j+1 \leq k \leq K} \lim_{n \to \infty} \left( \sum_{j=1}^{k} \| U_n^j \|_{L_t^2 x H_y^2 (\mathbb{R})}^2 + \sum_{i,j=1, i \neq j}^{k} \| \partial_x^{i} U_n^j \partial_x^{j} U_n^i \|_{L_t^2 x L_y^2 (\mathbb{R})}^2 \right)^2 \leq 1 \]

for \( s = 0, 1 \). Combining (3.46) with (3.45) we deduce that

\[ \sup_{1 \leq k \leq K^*} \lim_{n \to \infty} \left\| \sum_{j=1}^{k} U_j^2 \right\|_{L_t^1 x H_y^1 (\mathbb{R})} \leq 1. \]  \quad (3.47)

Therefore, using (3.41) to (3.43), (3.35) and Strichartz we confirm that the conditions (3.3) to (3.4) are satisfied for sufficiently large \( k \) and \( n \), where we set \( U = U_n \) and \( Z = \Psi^k_n \) therein. As long as we can show that (3.4) is satisfied for \( s = \frac{3}{4} \), we are able to apply Lemma 3.3 and Lemma 3.4 to obtain the contradiction

\[ \limsup_{n \to \infty} \| U_n \|_{L_t^1 x H_y^1 (\mathbb{R})} < \infty. \]  \quad (3.48)

Direct calculation shows that

\[ e = i \partial_t \Psi^k_n + \Delta_{x,y} \Psi^k_n + |\Psi^k_n|^2 \Psi^k_n \]

\[ = \left( \sum_{j=1}^{k} (i \partial_t U_n^j + \Delta_{x,y} U_n^j) \right) + |\Psi^k_n|^2 \Psi^k_n \]

\[ = (I_1 + I_2). \]

In the following we show the asymptotic smallness of \( I_1 \) and \( I_2 \). Since \( U_n^j \) solves (1.1), we can rewrite \( I_1 \) to

\[ I_1 = -\left( \sum_{j=1}^{k} |U_n^j|^2 U_n^j - \left| \sum_{j=1}^{k} U_n^j \right|^2 \right) = O \left( \sum_{i,j=1, i \neq j}^{k} |U_n^i|^2 |U_n^j|^2 + \sum_{p,i,j=1, i \neq j}^{k} |U_n^p U_n^i| \right) =: I_{11} + I_{12}. \]

We only consider the summand \( I_{11} \), the summand \( I_{12} \) can be dealt similarly. By Hölder we have

\[ \| |U_n^i|^2 |U_n^j| \|_{L_t^1 \times L_y^{\frac{4}{3}}} \leq \| U_n^i U_n^j \|_{L_t^{\frac{4}{3}} \times L_y^2} \| U_n^i \|_{L_t^2 \times L_y^\infty} \| U_n^j \|_{L_t^2 \times L_y^\infty} \]

\[ \leq \| U_n^i \|_{L_t^2 \times L_y^\infty} \| U_n^j \|_{L_t^2 \times L_y^{\frac{8}{3}}} \| U_n^j \|_{L_t^2 \times L_y^\infty}. \]

Then (3.43), (3.45) and (3.46) imply

\[ \lim_{k \to K^*} \lim_{n \to \infty} \| I_{11} \|_{L_t^1 \times L_y^{\frac{4}{3}}} = 0. \]

On the other hand,

\[ \| |U_n^i|^2 |U_n^j| \|_{L_t^1 \times L_y^{\frac{4}{3}}} \leq \| U_n^i \|_{L_t^2 \times L_y^\infty} \| U_n^j \|_{L_t^2 \times L_y^\infty} \leq 1. \]

Combining with the inequality \( \| f \|_{L_y^{\frac{8}{3}}} \leq \| f \|_{L_y^2} \| f \|_{L_y^\infty} \), we infer that

\[ \lim_{k \to K^*} \lim_{n \to \infty} \| I_{11} \|_{L_t^1 \times L_y^{\frac{4}{3}}} = 0 \]

Next, we prove the asymptotic smallness of \( I_2 \). Direct calculation shows

\[ I_2 = O \left( \Psi^k_n (e^{it \Delta_{x,y} U_n^j})^2 + (\Psi^k_n)^2 (e^{it \Delta_{x,y} U_n^k})^2 + (e^{it \Delta_{x,y} U_n^k})^3 \right). \]

But then (3.47), (3.22) and Lemma 3.3 immediately yield

\[ \lim_{k \to K^*} \lim_{n \to \infty} \| I_2 \|_{L_t^1 \times L_y^{\frac{8}{3}}} = 0 \]

and Step 2 is complete.
Step 3: Reduction to one bad profile and conclusion

From Step 2 we conclude that there exists some $1 \leq J_1 \leq K^*$ such that

$$
\limsup_{n \to \infty} \|U_n^j\|_{L_t^4_x H_y^1(\mathbb{R})} = \infty \quad \forall 1 \leq j \leq J_1,
$$

$$
\limsup_{n \to \infty} \|U_n^j\|_{L_t^4_x H_y^1(\mathbb{R})} < \infty \quad \forall J_1 + 1 \leq j \leq K^*.
$$

By Lemma 3.12 (which is applicable due to (3.37)) we deduce that $\lambda_\infty^c = 1$ for all $1 \leq j \leq J_1$. If $J_1 > 1$, then using (3.37), (3.38), the asymptotic positivity of energies deduced from Step 1 and Lemma 3.15 we know that $\limsup_{n \to \infty} D(U_n^1) < D^*$, which violates (3.49) due to the inductive hypothesis. Thus $J_1 = 1$ and

$$
U_n(0, x) = e^{it_n^1 \Delta_{x,y}} \phi^1(x - x_n^1) + w_n^1(x).
$$

In particular, $\phi^1 \in H_x^1, y$. Similarly, we must have $\mathcal{M}(w_n^1) = o_n(1)$ and $\mathcal{E}(w_n^1) = o_n(1)$, otherwise we could deduce again the contradiction (3.48) using Lemma 3.15. Combining with Lemma 3.17 we conclude that $w_n^1 \in H_y^1, x$. Finally, we exclude the cases $t_n^1 \rightarrow \pm \infty$. We only consider the case $t_n^1 \rightarrow \infty$, the case $t_n^1 \rightarrow -\infty$ can be similarly dealt. Indeed, using Strichartz we obtain that

$$
\|e^{it \Delta_{x,y}} U_n(0)\|_{L_t^4_x H_y^1(\mathbb{R})} \lesssim \|e^{it \Delta_{x,y}} \phi^1\|_{L_t^4_x H_y^1(\mathbb{R})} + \|w_n^1\|_{H^1_y} \rightarrow 0
$$

and using Lemma 3.3 we infer the contradiction (3.48) again. This completes the desired proof. \[ \square \]

**Lemma 3.17** (Existence of a minimal blow-up solution). Suppose that $D^* < \infty$. Then there exists a global solution $U_c$ of (1.1) such that $D(u_c) = D^*$ and

$$
\|U_c\|_{L_t^4_x H_y^1(\mathbb{R})} = \|u_c\|_{L_t^4_x H_y^1(\mathbb{R})} = \infty.
$$

Moreover, $U_c$ is almost periodic in $H_x^1, y$ modulo $\mathbb{R}^2_x$-translations, i.e. the set $\{U(t) : t \in \mathbb{R}\}$ is precompact in $H_x^1, y$ modulo translations w.r.t. the $x$-variable.

**Proof.** As discussed at the beginning of this section, under the assumption $D^* < \infty$ one can find a sequence $(U_n)_n$ of solutions of (1.1) that satisfies the preconditions of Lemma 3.10. We apply Lemma 3.10 to infer that $(U_n(0))_n$ (up to modifying time and space translation) is precompact in $H_x^1, y$. We denote its strong $H_x^1, y$-limit by $\psi$. Let $U_c$ be the solution of (1.1) with $U_c(0) = \psi$. Then $D(U_c(t)) = D(\psi) = D^*$ for all $t$ in the maximal lifespan $I_{\text{max}}$ of $U_c$ (recall that $D$ is a conserved quantity).

We first show that $U_c$ is a global solution. We only show that $s_0 := \sup I_{\text{max}} = \infty$, the negative direction can be similarly proved. If this does not hold, then by Lemma 3.3 there exists a sequence $(s_n)_n \subset \mathbb{R}$ with $s_0 < s_n$ such that

$$
\lim_{n \to \infty} \|U_c\|_{L_t^4_x H_y^1([-\infty, s_n])} = \|u_c\|_{L_t^4_x H_y^1([-\infty, s_n])} = \infty.
$$

Define $V_n(t) := u_c(t + s_n)$. Then (3.50) is satisfied with $t_n \equiv 0$. We then apply Lemma 3.10 to the sequence $(V_n(0))_n$ to conclude that there exists some $\varphi \in H_x^1, y$ such that, up to modifying the space translation, $U_c(s_n)$ strongly converges to $\varphi$ in $H_x^1, y$. But then using Strichartz we obtain

$$
\|e^{it \Delta_{x,y}} U_c(s_n)\|_{L_t^4_x H_y^1(\mathbb{R})} = \|e^{it \Delta_{x,y}} \varphi\|_{L_t^4_x H_y^1(\mathbb{R})} + o_n(1) = o_n(1).
$$

By Lemma 3.3 we can extend $U_c$ beyond $s_0$, which contradicts the maximality of $s_0$. Now by (3.34) and Lemma 3.3 it is necessary that

$$
\|U_c\|_{L_t^4_x H_y^1([-\infty, 0])} = \|U_c\|_{L_t^4_x H_y^1(\mathbb{R})} = \infty.
$$

We finally show that the orbit $\{U_c(t) : t \in \mathbb{R}\}$ is precompact in $H_x^1, y$ modulo $\mathbb{R}^2_x$-translations. Let $(\tau_n)_n \subset \mathbb{R}$ be an arbitrary time sequence. Then (3.50) implies

$$
\|U_c\|_{L_t^4_x H_y^1([-\infty, \tau_n])} = \|U_c\|_{L_t^4_x H_y^1(\mathbb{R})} = \infty.
$$

The claim follows by applying Lemma 3.10 to $(U(\tau_n))_n$. \[ \square \]
3.6 Extinction of the minimal blow-up solution

We exclude in this final section the minimal blow-up solution that we deduced from the last section. The following lemma is an immediate consequence of the fact that $U_c$ is almost periodic in $H_{x,y}^1$ and conservation of momentum. The proof is standard, we refer to [20] for details of the proof.

**Lemma 3.18.** Let $U_c$ be the minimal blow-up solution given by Lemma 3.17. Then there exists some function $x : \mathbb{R} \to \mathbb{R}^2$ such that

(i) For each $\varepsilon > 0$ there exists $R > 0$ so that

$$
\int_{|x + x(t)| \geq R} |\nabla_{x,y} U_c(t)|^2 + |U_c(t)|^2 + |U_c(t)|^4 \, dx dy \leq \varepsilon \quad \forall t \in \mathbb{R}.
$$

(ii) The center function $x(t)$ obeys the decay condition $x(t) = o(t)$ as $|t| \to \infty$.

**Proof of Theorem 1.6.** We will show the contradiction that the minimal blow-up solution $U_c$ given by Lemma 3.17 is equal to zero, which will finally imply Theorem 1.6. First we notice that since $U$ is a non-zero almost-periodic solution in $H_{x,y}^1$, we have

$$
\inf_{t \in \mathbb{R}} \|\nabla_{x,y} U_c(t)\|_{L^2_{x,y}}^2 =: \rho > 0.
$$

Next, let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth radial cut-off function satisfying

$$
\chi = \begin{cases}
|x|^2, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| \geq 2.
\end{cases}
$$

Then for $R > 0$, we define the local virial action $z_R(t)$ by

$$
z_R(t) := \int R^2 \chi\left(\frac{x}{R}\right) |U_c(t, x)|^2 \, dx dy.
$$

Direct calculation yields

$$
\partial_t z_R(t) = 2 \text{Im} \int R \nabla_x \chi\left(\frac{x}{R}\right) \cdot \nabla_x U_c(t) \overline{U_c(t)} \, dx dy,
$$

$$
\partial_{tt} z_R(t) = 4 \int \partial_{x,y}^2 \chi\left(\frac{x}{R}\right) \partial_{x,y} U_c \partial_{x,y} \overline{U_c} \, dx dy - \frac{1}{R^2} \int \Delta_x^2 \chi\left(\frac{x}{R}\right) |U_c|^2 \, dx dy - \int \Delta_x \chi\left(\frac{x}{R}\right) |U_c|^4 \, dx dy.
$$

We then obtain that

$$
\partial_{tt} z_R(t) = 16 \mathcal{E}_c(U(t)) + A_R(U_c(t)),
$$

where $\mathcal{E}_c$ is defined by (3.28) and $A_R(U_c(t))$ by (3.29) and (3.30) hold for this $\beta$. By (3.29) we deduce that there exists some $c_\beta > 0$ such that

$$
16 \mathcal{E}_c(U(t)) \geq c_\beta \|\nabla_{x,y} U_c(t)\|_{L^2_{x,y}}^2 \geq c_\beta \rho =: 2\eta > 0.
$$

From Lemma 3.18 it follows that there exists some $R_0 \geq 1$ such that

$$
\int_{|x + x(t)| \geq R_0} |\nabla_{x,y} U_c(t)|^2 + |U_c(t)|^2 + |U_c(t)|^4 \, dx dy \leq \frac{\eta}{C_1}.
$$
Thus for any $R \geq R_0 + \sup_{t \in [t_0, t_1]} |x(t)|$ with some to be determined $t_0, t_1 \in [0, \infty)$, we have

$$\partial_{tt} z_R(t) \geq \eta_1$$

(3.52)

for all $t \in [t_0, t_1]$. By Lemma 3.18 we know that for any $\eta_2 > 0$ there exists some $t_0 \gg 1$ such that $|x(t)| \leq \eta_2 t$ for all $t \geq t_0$. Now set $R = R_0 + \eta_2 t_1$. Integrating (3.52) over $[t_0, t_1]$ yields

$$\partial_t z_R(t_1) - \partial_t z_R(t_0) \geq \eta_1 (t_1 - t_0).$$

(3.53)

Using (3.51), Cauchy-Schwarz and Lemma 3.15 we have

$$|\partial_t z_R(t)| \leq C_2 D^* R = C_2 D^* (R_0 + \eta_2 t_1)$$

(3.54)

for some $C_2 = C_2(D^*) > 0$. (3.53) and (3.54) give us

$$2C_2 D^* (R_0 + \eta_2 t_1) \geq \eta_1 (t_1 - t_0).$$

Setting $\eta_2 = \frac{\eta_1}{4C_2 D^*}$, dividing both sides by $t_1$ and then sending $t_1$ to infinity we obtain $\frac{1}{2} \eta_1 \geq \eta_1$, which implies $\eta_1 = 0$, a contradiction. This completes the proof.

Acknowledgments

The author acknowledges the funding by Deutsche Forschungsgemeinschaft (DFG) through the Priority Programme SPP-1886.

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