Abstract. A cosmological model with a gravitational Lagrangian $L_g(R) \propto R + AR^n$ is set up to account for the presently observed re-acceleration of the universe. The evolution equation for the scale factor $a$ of the universe is analyzed in detail for the two parameters $n = 2$ and $n = 4/3$, which were preferred by previous studies of the early universe. The initial conditions are specified at a red-shift parameter $z \approx 0$. The fit to the observable data fixes the free parameter $A$. The analysis shows that the model with $n = 2$ agrees better with present data. Then, if we set $w(q) = -1$ at $z = 0$, corresponding to the deceleration parameter $q \approx -1/2$, we find that at $z \approx 0.5$, $w(q)$ has evolved to $w \approx -0.72$, corresponding to $q \approx 0$. At $z \approx 1$ we find $w \approx 0$ corresponding to $q \approx 1/2$. These results are compared with the flat Friedmann model with cold matter and Lambda-term (LCDM model) for the same initial conditions at $z \approx 0$. The other choice of the model with $n = 4/3$ allows for big crunch. However this possibility is eliminated by the fit of $A$ to the present data.

1. Introduction

The discovery of a re-acceleration in the present expansion of the universe is interpreted as evidence for the existence of a cosmological Λ-term in the present equation of motion. At present, the most popular explanation is based on quintessence models which extend Einstein’s action by scalar fields with different potentials. The additional parameters are chosen to comply with observable data. For a review see [1, 2].

When studying such models it is important to match them with presently popular models of the early universe. Also here scalar fields have been used to modify Einstein’s equations. But another important set of models makes use of curvature terms generated by fluctuating quantum fields as a Casimir effect, a mechanism discovered by Sakharov [3]. This gives rise to additional terms $\Delta L(R)$ which behave quite differently from the Einstein-Hilbert Lagrangian [4, 5].

Starting from a scale invariant renormalizable action proportional to $R^2$ it is possible to generate for small $R$ an Einstein action [6] plus terms of the form

$$\Delta L(R) = AR^2 + CR^2 \ln |R^2/R_s^2|$$

(1)

where $A, C$ are parameters. The effect of such terms is studied with general methods developed an arbitrary function $\Delta L(R)$ in the Friedmann models [7].
Such additional terms can, of course, be generated alternatively by an Einstein action coupled to a scalar field $\sigma$ with a suitably chosen potential \[8, 9\]. For instance, $\Delta L(R) = AR^2$ is obtained from $\Delta L(R, \sigma) = \sigma R - \sigma^2/4A$ by extremization in $\sigma$.

There have been ample discussion of models in which $\Delta L(R)$ is chosen to fit observable data \[10\]. Our choice (1) has the advantage having a physical origin and possessing only three parameters which can be fixed completely by present data at $z = 0$, thus allowing us to calculate uniquely the future and past evolution of the universe.

2. Basic Equations

As suggested by observations, we consider the flat cosmological Friedmann model with the metric

$$ds^2 = d\tau^2 - a(\tau)^2(dx^2 + dy^2 + dz^2).$$

(2)

If $H_0$ denotes the presently observable Hubble constant (the subscript 0 will always indicate the present epoch), the reduced curvature tensor $\rho_k \equiv H_0^{-2} R_k^l$ has the following matrix elements as a function of the reduced time $\theta \equiv H_0^\tau$, with the notation $\dot{a} \equiv da/d\theta$:

$$\rho_0^0 = -3\ddot{a}/a,$$

(3)

$$\rho_i^i = \rho = -6 \left( \dot{a}/a + \ddot{a}/a^2 \right).$$

(4)

The variation of Einstein’s Lagrangian with an additional term $\Delta L(R) \equiv f(R)$ gives

$$G_{i}^{k} = \frac{8\pi G}{H_0^2} T_{i}^{k} + \tilde{T}_{i}^{k}; \quad G_{i}^{k} = \rho_{i}^{k} - \frac{1}{2} \delta_{i}^{k} \rho.$$  

(5)

Here $T_{i}^{k}$ corresponds only to cold matter in the present universe and

$$\tilde{T}_{i}^{k} = \left\{ \frac{\partial f}{\partial \rho} \rho_{i}^{k} - \frac{1}{2} \delta_{i}^{k} f + \left( \delta_{i}^{k} g^{lm} - \delta_{l}^{i} g^{km} \right) \left( \frac{\partial f}{\partial \rho} \right)_{;m} \right\}$$

(6)

specifies the effective quintessence with the nontrivial dependence on curvature.

We proceed as in Ref. \[11\] by introducing the new variable

$$y \equiv (\dot{a}a)^2$$

(7)

which allows to reduce the order of the equations. Then the $i = k = 0$ component of Eq. (5) leads to

$$y + \left[ f,_{\rho} \left( y - a \frac{dy}{da} \right) - \frac{a^4}{6} f(\rho) + a y \frac{df,_{\rho}}{da} \right] = \frac{\rho_c}{\rho_*} a^4 \equiv \Omega_c a,$$

(8)

where $\rho_c$ is the $a$-dependent cold dark matter (CDM) energy density, $\rho_* = 3H_0^2/8\pi G$ is the critical density. Choosing the value of the scale factor $a(\theta_0)$ equal to 1 at present, one has $\rho_c = \rho_0/a^3$ and $(\rho_c/\rho_*) a^4 \equiv \Omega_c a$.

In order to investigate the evolution of the cosmological model it is enough to obtain the solution of Eq. (8) with appropriate initial conditions. But for interpretation of the solution and for its comparison with observations it is necessary track for changes of
CDM energy density $\rho_c$ and quintessence energy density $\rho_v$ separately. For this purpose we use $i = k = 0$ component of Einstein’s equations (3) and (7),

$$G_0^0 = \rho_0^0 - \frac{1}{2} \rho = \frac{8\pi G}{H_0^2}(\rho_c + \rho_v).$$

(9)

Multiplying this equation by $a^4$ and using Eqs. (3) and (7), we have

$$y = (\rho_c + \rho_v) a^4 / \rho_*$$

(10)

accounting for the evolution of cold matter energy density. From this we find for the quintessence energy density:

$$\frac{\rho_v}{\rho_*} = (y - \Omega_c a) / a^4.$$  

(11)

By solving Eq. (8), we can find the evolution of $\rho_v$. At known $y$, the Hubble parameter and the curvature components are obtained from

$$h(a) = \sqrt{y / a^4}; \quad \rho = -\frac{3}{a^3} \frac{dy}{da}; \quad \rho_0 = 3 \left(y - \frac{a dy}{2 da}\right).$$

(12)

We now use Eqs. (3) and (4), rewritten as

$$\frac{\rho}{6} + H^2 = -\frac{\ddot{a}}{a},$$

(13)

to derive an expression for the deceleration parameter of the universe

$$q \equiv -\frac{\ddot{a}a}{a^2}$$

(14)

in terms of the curvature components as follows:

$$q = \frac{a^2 \rho}{a^2 6} + 1.$$  

(15)

For a comparison with observable data we relate the parameter $q$ to quintessence and CDM energy densities. Let us suppose that the quintessence pressure and energy density are proportional to each: $p_v = \omega \rho_v$. Then the $i = k = 0$ component of Einstein’s equations becomes

$$3 \left(\frac{\dot{a}}{a}\right)^2 = \mathcal{K}(\rho_c + \rho_v)$$

(16)

where $\mathcal{K} = 8\pi G$ and the equation for the scalar curvature gives

$$6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = \mathcal{K}(\rho_c + \rho_v - 3p_v) = \mathcal{K}[\rho_c + \rho_v(1 - 3\omega)].$$

(17)

From these equations we derive

$$q = -\frac{\ddot{a}a}{a^2} = \frac{1}{2} \left(1 + \frac{3\delta_v \omega}{1 + \delta_v}\right); \quad \delta_v = \rho_v / \rho_c.$$  

(18)

The proportionality constant $\omega$ is related to the deceleration parameter $q$ by

$$\omega = \left(\frac{2q - 1}{3}\right) \left(\frac{\delta_v + 1}{\delta_v}\right).$$

(19)

The observed value of $q$ at red-shift parameter $z = a_0/a - 1 \ll 1$ defines the values of $y_0$ and $y_0'$ in Eq. (9) in the present point $a_0 = 1$. The data [1] seem to imply that $w < 0$ corresponding to the inflation stage.
3. Quintessence Model with \( f = A \rho^n \)

Instead of our fluctuation-generated additional terms in (1), a model with an additional term \( \Delta L(R) = f(R) = A \rho^n \propto R^n \) has been considered in Ref. [10]. Let us review this case briefly to see the difference with respect to our theory. The basic equation for dark energy from (8) reads

\[
A \left( \frac{\gamma}{a^2} \right) \left( \frac{y'}{a^3} \right)^{n-2} \left[ n(n-1)yy'' + \frac{1-n}{2} y'^2 + (4-3n)n\frac{yy'}{a} \right] = \Omega_c a - y, \tag{20}
\]

where \( \gamma = (-3)^{n-1}, y' = dy/da \), where \( A \) is still arbitrary. Variation of the Lagrangian \( L(R) = f(R) \) gives, instead of Eq. (6),

\[
\hat{T}^k_i = \mathbb{N} T^k_i, \tag{21}
\]

where \( T^k_i \) is energy-momentum tensor of usual matter. The authors in Ref. [10] analyze these equation without taking into account of usual matter. In this approximation, the solution of our theory would be obtained by setting the tensor in Eq. (6) equal to zero: \( \hat{T}^k_i = 0 \). For the choice \( f(R) = AR^n \), the corresponding equation is obtained by setting the brackets in Eq. (20) equal to zero. The solution is

\[
y = a^k; \quad k = \frac{(4n-5)2n}{(n-1)(2n-1)}. \tag{22}
\]

By Eq. (7), this is corresponds to the following expression for the scale factor

\[
a = \theta^\alpha; \quad \alpha = \frac{(n-1)(2n-1)}{2-n}. \tag{23}
\]

This solutions has a constant deceleration parameter \( q \). Further, to comply with the observable \( H_0 \), Ref. [10] determined certain intervals of \( n \) with different ages of the universe. This allowed him to find the deceleration parameter \( q \) and parameter \( w \) (see Section 2) as:

\[
q = \left( 1/\alpha - 1 \right); \quad w = \left( 2 - 3\alpha \right)/3\alpha. \tag{24}
\]

Present data force us to reject such a pure power dependence of the scale factor in (23). The latest data show that at present \( (z = 0) \), the universe is accelerating its expansion \( (w \approx -1) \) but at \( z = 1 \) there was no inflation \( (w \approx 0) \). Apart from that, the influence of cold matter cannot be neglected [12].

Thus we need another ansatz for \( f(\rho) \) to account for present observations. From (20) we see that a simple power in the asymptotic solutions is absent for \( n = 2 \) and \( n = 4/3 \). These happen to be the same parameters which were of special importance in a previous theory of the early universe. At \( n = 4/3 \), one of us [13] has obtained for the first time a cosmological model without a singularity. That model passes through a regular minimum, had inflationary stage and tends asymptotically to the classical Friedmann solutions. In addition, the usefulness of an additional term \( R^2 \) in the models of the early universe has been pointed out before. Therefore we analyze the possibility of using of such powers for construction of models with variable parameters \( q \) and \( w \).
4. Models with \( n = 2 \) and \( n = 4/3 \)

4.1. \( f(\rho) = -A\rho^2 \)

This model was often used in the theory of the early universe. It describes the stage of fast oscillations of \( a(\theta) \) (the so-called scalaron stage, which was introduced by A. Starobinsky in [14]). The damping of such oscillations was connected with creation of unstable particles and filling of the early universe by a hot plasma.

Here we want to analyze this model in the opposite regime when the period of oscillations is commensurable with age of the universe \( 1/H_0 \). One can easily see the oscillations of the model by inserting the specified form of quintessence in (5) and (6)

\[
\frac{d^2 \rho}{d\theta^2} + 3\frac{\dot{a}d\rho}{a d\theta} + (\rho + \Omega_c a) = 0. \tag{25}
\]

The scalar curvature performs oscillations near the value of \( \rho \) which corresponds to the model of cold dust matter in the Friedmann universe.

More conveniently, we may use the \( i = k = 0 \) component of the generalized gravitational equation (20)

\[
6A \left[ y''y - y'^2/4 - 2yy'/a \right] = -a^2(y - \Omega_c a). \tag{26}
\]

Introducing the quadratic scale variable \( \xi = a^2 \), this becomes

\[
24A \left[ \frac{d^2 y}{d\xi^2} - \frac{1}{4} \left( \frac{dy}{d\xi} \right)^2 - \frac{y}{2\xi} \frac{dy}{d\xi} \right] = \Omega_c \sqrt{\xi} - y. \tag{27}
\]

The initial conditions are specified at \( a_0 = 1 \), where the reduced Hubble parameter \( a_0/\dot{a}_0 \) is unity. This condition defines \( y_0 = 1 \) at given \( \xi_0 = 1 \) [see Eq. (7)].

At \( y_0 = 1 \), the value of derivative \( (dy/da)_0 \) is determined by observations in the following way. Using the observable estimation for \( \Omega_v = 0.7 \) and \( \Omega_c = 0.3 \), the value of \( \delta_{v0} \) in Eq. (18) is approximately equal to 2. The observations for \( z << 1 \) show that \( w \approx -1 \) and, using (18), we find

\[
q_0 \approx \frac{1}{2}(1 - 2) = -1/2. \tag{28}
\]

At \( \xi_0 = 1 \), we obtain from (18)

\[
(dy/d\xi)_0 \approx 3/2. \tag{29}
\]

The only open parameter left is \( A \) in Eq. (27). It specifies the amount of dark energy in the universe.

The solution \( y(\xi) \) in the interval \( 0.01 < \xi < 1 \) is presented in Fig. 1. The age of the universe \( T \) is defined by the value of the Hubble parameter

\[
\frac{\dot{a}}{a} = h(\xi) = \sqrt{y(\xi)/\xi}, \quad \theta = TH_0 = \int_\epsilon^1 \frac{d\ln \xi}{2h(\xi)} = \int_\epsilon^1 \frac{d\xi}{2\sqrt{y(\xi)}}, \quad \epsilon \to 0. \tag{30}
\]

Note that Eq. (27) yields for \( \xi \to 0 \) the regular behavior \( y \to \text{const} \). Thus the integral at \( \epsilon \to 0 \) has no singularity.
4.2. \( f(\rho) = -A\rho^{4/3} \)

The special feature of the model with \( n = 4/3 \) consists in absence of the scale factor \( a \) in explicit form in the Einstein’s equations if matter is neglected \[13\]. This allows one to find the general solution of Eq. \[20\] with given \( n \). One can show that de Sitter’s solution arises in the limit \( a \gg 1 \). The curvature of this limiting solution is determined by the parameter \( A \).

The evolution equation for this case is from \[20\]:

\[
y \frac{d^2 y}{d\xi^2} - \frac{3}{8} \left( \frac{dy}{d\xi} \right)^2 + \frac{1}{2} \frac{dy}{d\xi} = \frac{9}{16} \frac{1}{3^{1/3} A} \left( \frac{2}{\xi} \frac{dy}{d\xi} \right)^{2/3} \left[ \Omega_c \sqrt{\xi} - y \right]. \tag{31}
\]

By analogy with previous subsection, we obtain for this model (see Fig. 1) an age of the universe \( T \approx 0.79/H_0 \approx 10.8 \) Gyr with a parameter \( A = 0.33 \).

Leaving the analysis of the solution to Section \[5\] let us consider here only the following issue. For \( \xi > 1 \), the influence of cold matter becomes negligible. Then Eq. \[31\] has a first integral. It can be easily seen from \[20\] for \( n = 4/3 \):

\[
y \frac{d^2 y}{da^2} - \frac{3}{8} \left( \frac{dy}{da} \right)^2 = D \left( \frac{dy}{da} \right)^{2/3} y, \quad D = \frac{4}{9} (3)^{-1/3}/A. \tag{32}
\]

**Figure 1.** For the models with \( n = 2 \) (left graph) and with \( n = 4/3 \) (right graph) the following parameters subject to \( \xi \) are shown.
The first integral is

\[
\left( \frac{dy}{da} \right)^{4/3} = \frac{8}{3} D(y + C\sqrt{y}),
\]

where \( C \) denotes an integration constant. For \( C > 0 \), this equation has the asymptotical solution \( y \approx \gamma a^4 \), \( \gamma = 2(D/3)^{3/2} \), which coincides with the de Sitter solution

\[
y = (a\dot{a})^2 = \gamma a^4; \quad a = a_0 \exp(\sqrt{\gamma} \theta).
\]

For \( C < 0 \), the solution possess a scale parameter \( a \) where \( dy/da = 0 \). Beyond that, \( y \) starts decreasing. The solution reaches a point \( y \to 0 \). By definition, \( y \) can never be negative. Analytic continuation of the solution beyond the specified point leads to an increase of \( y \) at decreasing of \( a \). As \( a \) approaches zero we obtain a new singularity. The zero of \( y \) corresponds to a maximum of the scale factor \( a \) with subsequent contraction of the universe ending in a singularity.

This is an example for a big crunch (see [1] and references therein). It is important to note that regime of a crunch is realized here for a flat cosmological model.

Note that a contraction of cosmological model to a point cannot really be described by a homogenous model. In the limiting regime, the universe becomes certainly inhomogeneous and anisotropic because of fast growth of perturbations. Using of the solution of Eq. (33) with parameters \( D \) and \( C \) corresponding to observable data for \( z \ll 1 \) guarantees an absence of a crunch in future (\( C > 0 \)).

5. Discussion

After fitting the model’s parameters to the present observable data - acceleration of the universe, the Hubble parameter at red-shift parameter \( z = 0 \), and the age of the universe - there are no free parameters in the model. Its predictions for large \( z \) can be compared with observations.

As can be seen in Fig. 1 (in case of \( n = 2 \)), the acceleration of the expansion changes at \( z \approx 0.46 \) to a deceleration (\( w \approx -0.72 \), \( q \approx 0 \)). Near \( z = 1 \) the variant of dust-like dark energy (\( w = 0 \)) is realized. It corresponds to latest observational data [1].

A special feature of the model is that for \( a \to 0 \), the variable \( y = (\dot{a}a)^2 \) tends to a constant, which is equivalent to an evolution of the universe filled by hot matter (\( p = \epsilon/3 \)) in the Friedmann cosmology. When the solution is oscillating, the inflation at \( z \approx 0 \) cannot be eternal although the period of oscillations is comparable with the age of universe.

In case of \( n = 4/3 \), the crossover from a decelerated expansion to an accelerated one takes place at \( z \approx 0.30 \) when \( w \approx -0.60 \).

It is interesting to compare our model with the simplest LCDM model (for review see [1]). Using the notation of Section (2), we obtain

\[
y = \Omega_c \xi^{1/2} + \Omega_v \xi^2
\]

with the same values \( \Omega_c \approx 0.3 \) and \( \Omega_v \approx 0.7 \).
Geometric model of dark energy

Our results show the following: the age of the universe is near $1/H_0 \approx 13.7$ Gyr. This age is larger than the age from the pure $\rho^2$-model ($\approx 11.4$ Gyr) and from the $\rho^{4/3}$-model ($\approx 10.8$ Gyr). The parameter $w$ of the LCDM model calculating with use of Eq. (19) has the value $-1$. We may use Eq. (18) to determine the deceleration parameter $q$ and compare it with the $\rho^2$ and $\rho^{4/3}$ models. At $z = 1$, the deceleration parameter is $q \approx 0.2$, whereas the $\rho^2$ and $\rho^{4/3}$ models have $q \approx 0.5$. This value is closer to observable data. At $z \approx 0.5$, the parameter $q$ in the $\rho^2$-model is close to zero and in the LCDM model $q \approx -0.1$. Apparently, the $\rho^2$-model corresponds better to observations. The LCDM model is mathematically very simple. But it leaves the value of the $\Lambda$-term an unexplained fundamental constant. For dynamical models of the $\Lambda$-term (for example, the $\rho^2$-model) this value evolves from a large Planck value in the early universe to small value at present.

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