Dimer models and exceptional collections

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Abstract

We construct a full strong exceptional collection consisting of line bundles on any two-dimensional smooth toric weak Fano stack. The total endomorphism algebra of the resulting collection is isomorphic to the path algebra of a quiver with relations associated with a dimer model and a perfect matching on it.

1 Introduction

A dimer model is a bicolored graph on a real 2-torus which encodes the information of a quiver with relations. The main result of [IU] states that for any smooth quasi-projective toric Calabi-Yau 3-fold $M$, there is a dimer model $G$ such that

- the moduli space $\mathcal{M}_\theta$ of $\theta$-stable representations of the quiver $\Gamma$ with relations associated with $G$ of dimension vector $(1, \ldots, 1)$ is isomorphic to $M$ if we choose a suitable stability parameter $\theta$, and

- the direct sum $\mathcal{V} = \bigoplus_v \mathcal{L}_v$ of the tautological bundles is a tilting object whose endomorphism algebra is isomorphic to the path algebra $C\Gamma$ of the quiver $\Gamma$ with relations.

This gives a description of the derived category of coherent sheaves on any smooth toric Calabi-Yau 3-fold in terms of a quiver with relations:

$$D^b \text{coh} M \cong D^b \text{mod} C\Gamma.$$ 

The same result can also be obtained by combining the existence of an isoradial dimer model by Gulotta [Gul08], the Calabi-Yau property of an isoradial dimer model by Broomhead [Bro] (cf. also [MR10, Dav, Bocb, IU10]) and the Calabi-Yau trick by Bridgeland, King and Reid [BKR01] (cf. also [vdB04]).

The aim of this paper is to give a similar description for the derived category of coherent sheaves on a two-dimensional smooth toric weak Fano stack. Here, a smooth toric Deligne-Mumford stack with the trivial generic stabilizer is said to be a weak Fano stack if the anti-canonical bundle is nef and big. Let $X$ be such a stack and $K_X$ be the total space of its canonical bundle. Then a projective crepant resolution of the coarse moduli space of $K_X$ is a smooth toric Calabi-Yau 3-fold $M$, and one has an equivalence

$$\Phi : D^b \text{coh} M \to D^b \text{coh} K_X.$$
of derived categories. The image $\Phi(\mathcal{V})$ of the tilting object on $\mathcal{M}$ is a tilting object on $K_X$, which gives a generator on $X$ by the derived restriction to the zero-section.

The basic strategy is to find a suitable equivalence $\Phi$ so that the resulting generator on $X$ will be not only an object of the derived category but a direct sum of line bundles. Then its restriction to the zero section will be a tilting object. To be more precise, we first construct line bundles before choosing a derived equivalence as follows:

1. We start with the moduli space $\mathcal{M}_\theta$ with an arbitrary generic stability parameter $\theta$, which may not be lying over the coarse moduli space of $K_X$. This will enable us to use an arbitrary central perfect matching to describe the quiver with relations that is derived equivalent to $X$ in Theorem 7.2.

2. Let $\mathcal{L}_v$ be the line bundle on $K_X$ obtained as the proper transform of $L_v$ on $\mathcal{M}_\theta$ and put $\mathcal{V} = \bigoplus_v L_v$. Then one has an isomorphism

$$\text{End}(\mathcal{V}) \cong \text{End}(\mathcal{V}) \cong C \Gamma.$$

3. The acyclicity

$$\text{Ext}^k(\mathcal{V}, \mathcal{V}) = 0, \quad k \neq 0$$

of $\mathcal{V}$ implies that of $\mathcal{V}$.

4. Now one can use the Calabi-Yau trick [BKR01, vdB04, BK04] to show that $\mathcal{V}$ is a tilting object.

5. The restriction $\mathcal{E}$ of $\mathcal{V}$ to the zero-section is a tilting object on $X$, which is a direct sum of line bundles.

6. The endomorphism algebra of $\mathcal{E}$ is the quotient of the endomorphism algebra of $\mathcal{V}$ by the ideal consisting of elements vanishing on the zero section. The isomorphism $\text{End}(\mathcal{V}) \cong \text{End}(\mathcal{V})$ gives a description of this ideal in terms of a perfect matching on the dimer model.

This gives a proof of a particular case of a conjecture of King, together with a description of the total morphism algebra:

**Theorem 1.1.** Any two-dimensional smooth toric weak Fano stack has a full strong exceptional collection consisting of line bundles, such that the total morphism algebra of the collection is isomorphic to the path algebra of a quiver with relations associated with a consistent dimer model and a perfect matching on it.

The original conjecture of King [Kin97, Conjecture 9.3] states that a smooth complete toric variety has a full strong exceptional collection consisting of line bundles. This is shown to be false by Hille and Perling [HP06], who subsequently gave a necessary and sufficient condition for a smooth complete toric surface to have such a collection [HP]. Kawamata [Kaw06] shows that a smooth projective toric stack has a full exceptional collection consisting of sheaves. Borisov and Hua [BH09] suggested to extend the conjecture to stacks, with an additional assumption that the toric stack be weak Fano, and proved it for toric Fano stacks of Picard number or dimension at most two. This modified conjecture turned out to be false by Efimov [Efi10]. A fine moduli interpretation of any smooth
projective toric varieties, which was one of the original motivations of King, is obtained by Craw and Smith [CS08]. The concept of dimer models and the idea to use them to construct full strong exceptional collections on toric surfaces came from string theorists; see e.g. [FHM+06, FHV+06, FV06, HHV06, HK05] and references therein.

The organization of this paper is as follows: We collect basic facts on line bundles on toric stacks in Section 2, and basic definitions on dimer models in Section 3. The relation between exceptional collections, tilting objects and derived equivalences is summarized in Section 4. We recall the main result of [IU] in Section 5. A tilting object on the total space of the canonical bundle of a toric weak Fano stack will be constructed in Section 6, which will be restricted to the image of the zero-section to produce a full strong exceptional collection in Section 7. In Section 8, we use the same idea as in Section 7 to give a description of the derived category of coherent sheaves on the union of toric divisors in \( \mathcal{M}_\theta \) in terms of a dimer model.

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2 Line bundles on toric stacks

We recall the definition of toric stacks from [BCS05]. Let \( N \) be a free abelian group of rank \( n \). Note in [BCS05] \( N \) is allowed to have torsions so that the associated stack may have generic stabilizers but we consider the torsion free case in this paper. A stacky fan \( \Sigma = (\Sigma, \{v_i\}_{i=1}^r) \) consists of a fan \( \Sigma \) in \( N_{\mathbb{R}} = N \otimes \mathbb{R} \) and the set \( \{v_i\}_{i=1}^r \) of generators of one-dimensional cones in \( \Sigma \). The toric stack \( X_\Sigma \) associated with \( \Sigma \) is defined as the quotient stack

\[
X_\Sigma = [(\mathbb{C}^r \setminus SR(\Sigma))/K],
\]

where the Stanley-Reisner locus \( SR(\Sigma) \) consists of points \((z_1, \ldots, z_r)\) such that there is no cone in \( \Sigma \) which contains all \( v_i \) for which \( z_i = 0 \), and

\[
K = \text{Ker}(\phi \otimes \mathbb{C}^\times)
\]

is the kernel of the tensor product with \( \mathbb{C}^\times \) of the map \( \phi : \tilde{N} = \mathbb{Z}^r \to N \) sending the \( i \)-th coordinate vector \( e_i \) to \( v_i \) for \( i = 1, \ldots, r \). We sometimes write

\[
U_\Sigma = \mathbb{C}^r \setminus SR(\Sigma)
\]

so that

\[
X_\Sigma = [U_\Sigma/K].
\]

It follows from the definition that the category of coherent sheaves on \( X_\Sigma \) is equivalent to the category of \( K \)-equivariant coherent sheaves on \( U_\Sigma \);

\[
\text{coh} X_\Sigma \cong \text{coh}^K U_\Sigma.
\]
Let $\tilde{M} = \text{Hom}(N, \mathbb{Z})$ and $\tilde{\tilde{M}} = \text{Hom}(\tilde{N}, \mathbb{Z})$ be the abelian group dual to $N$ and $\tilde{N}$ respectively. The tori $T = \text{Spec} \mathbb{C}[M]$ and $\tilde{T} = \text{Spec} \mathbb{C}[\tilde{M}]$ act naturally on $X_\Sigma$ and $U_\Sigma$, and the category of $T$-equivariant coherent sheaves on $X_\Sigma$ is equivalent to the category of $\tilde{T}$-equivariant coherent sheaves on $U_\Sigma$:

$$\text{coh}^T X_\Sigma \cong \text{coh}^{\tilde{T}} U_\Sigma.$$ 

As a smooth toric variety, $\text{Pic} U_\Sigma$ is generated by invariant divisors $D_i = \{z_i = 0\}$, which are clearly trivial. Hence one has

$$\text{Pic}^T X_\Sigma \cong \text{Hom}(\tilde{T}, \mathbb{C}^\times) = \tilde{M}.$$ 

The Picard group of $X_\Sigma$ can be calculated using the exact sequence

$$1 \to \text{Hom}(T, \mathbb{C}^\times) \to \text{Pic}^T X_\Sigma \to \text{Pic} X_\Sigma \to 1.$$ 

The divisor $D_i = \{z_i = 0\}$ naturally corresponds to the $i$-th coordinate vector in $\tilde{M}$, which defines a line bundle $O(D_i) \in \text{Pic}^T X_\Sigma$. Its image in $\text{Pic} X_\Sigma$ will again be denoted by $O(D)$. Any line bundle $L \in \text{Pic}^T X_\Sigma$ can be represented as $O(D)$, where $D = (f)$ is the divisor of any $T$-invariant rational section

$$f \in (\mathbb{C}[z_1^{\pm 1}, \ldots, z_r^{\pm 1}] \otimes L)^T,$$

which is unique up to scalar multiples.

The cohomology of $T$-equivariant line bundle is given as follows:

**Proposition 2.1.** Let $\Sigma$ be a simplicial stacky fan and $O(D)$ be the $T$-equivariant line bundle associated with a divisor $D = \sum_{i=1}^r a_i D_i$. Then the $T$-invariant part of the cohomology group of $O(D)$ is given by

$$H^p_T(X_\Sigma, O(D)) \cong H^p_Z(|\Sigma|),$$

where $|\Sigma|$ is the support of the fan $\Sigma$ underlying the stacky fan $\Sigma$,

$$\psi_D : |\Sigma| \to \mathbb{R}$$

is the piecewise-linear function which is linear on each cone of $\Sigma$ satisfying

$$\psi_D(v_i) = a_i,$$

and

$$Z = \{x \in |\Sigma| \mid \psi_D(x) \geq 0\}.$$ 

See e.g. [Ful93, Section 3.5] for a proof of Proposition 2.1. The cohomology group of a line bundle $L$ on $X_\Sigma$ is the direct sum

$$H^p(X_\Sigma, L) = \bigoplus_{D : L \cong O(D)} H^p_T(X_\Sigma, O(D))$$

over the set of toric divisors $D$ such that $L \cong O(D)$, which is a torsor over the lattice $M$. 


3 Dimer models

A dimer model is a bicolored graph on a torus $T = \mathbb{R}^2/\mathbb{Z}^2$ consisting of a set $B \subset T$ of black nodes, another set $W \subset T$ of white nodes, and a set $E$ of edges consisting of embedded line segments connecting vertices of different colors.

A connected component of the complement $T \setminus E$ is called a face of the graph. A bicolored graph on $T$ is said to be a dimer model if any face is simply-connected. We only deal with dimer models satisfying a consistency condition described in [IU10, Definition 3.5]. See also [MR10, Dav, Bro, Bocb] for more about consistency conditions of dimer models. Mathematical literature on dimer models also includes [Sze08, NN, Naga, Nagb, Shi06, Shi08, Shi09, Shi10, Moz09, BM].

A quiver consists of a set $V$ of vertices, a set $A$ of arrows, and two maps $s,t : A \to V$ from $A$ to $V$. For an arrow $a \in A$, $s(a)$ and $t(a)$ are said to be the source and the target of $a$ respectively. A path on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \ldots, a_1)$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \ldots, n - 1$. We also allow for a path of length zero, starting and ending at the same vertex. The path algebra $CQ$ of a quiver $Q = (V, A, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths:

$$(b_m, \ldots, b_1) \cdot (a_n, \ldots, a_1) = \begin{cases} (b_m, \ldots, b_1, a_n, \ldots, a_1) & s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A quiver with relations is a pair of a quiver and a two-sided ideal $I$ of its path algebra. For a quiver $\Gamma = (Q, I)$ with relations, its path algebra $C\Gamma$ is defined as the quotient algebra $CQ/I$.

A dimer model $G = (B, W, E)$ encodes the information of a quiver $\Gamma = (Q, I)$ with relations in the following way: The set $V$ of vertices of $Q$ is the set of faces of the graph, and the set $A$ of arrows of $Q$ is the set $E$ of edges of the graph. The directions of the arrows are determined by the colors of the vertices of the graph, so that the white vertex $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by minus 90 degrees.

For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white vertex connected to $a \in E = A$ clockwise and the latter going around the black vertex connected to $a$ counterclockwise. Then the ideal $I$ of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

A perfect matching (or a dimer configuration) on a dimer model $G = (B, W, E)$ is a subset $D$ of $E$ such that for any vertex $v \in B \cup W$, there is a unique edge $e \in D$ connected to $v$. The two-sided ideal of $C\Gamma$ generated by arrows $a$ in $D \subset E = A$ will be denoted by $I_D$.

4 Exceptional collections and tilting objects

Let $X$ be a smooth stack and $\mathcal{T} = D^b \text{coh } X$ be the derived category of coherent sheaves on $X$.

Definition 4.1.
1. An object $E$ of $\mathcal{T}$ is acyclic if $\text{Ext}^k(E, E) = 0$ for $k \neq 0$.

2. An acyclic object $E$ is exceptional if $\text{End}(E)$ is spanned by the identity morphism.

3. A sequence $(E_1, \ldots, E_n)$ of exceptional objects is an exceptional collection if $\text{Ext}^k(E_i, E_j) = 0$ for $1 \leq j < i \leq n$.

4. An exceptional collection $(E_1, \ldots, E_n)$ is strong if $\text{Ext}^k(E_i, E_j) \neq 0$ implies $k = 0$.

5. An exceptional collection is full if it generates $\mathcal{T}$ as a triangulated category.

6. An object $V$ is a generator if $\mathbb{R}\text{Hom}(V, X) = 0$ implies $X \cong 0$.

7. An acyclic generator is called a tilting object.

Note that a sequence $(E_1, \ldots, E_n)$ of line bundles on a smooth proper stack is a full strong exceptional collection if and only if $V = \bigoplus_{i=1}^n E_i$ is a tilting object. The algebra $\text{End}(V) = \bigoplus_{i,j=1}^n \text{Hom}(E_i, E_j)$ is called the total endomorphism algebra of the collection. It is a finite-dimensional algebra which can be described as the path algebra of a quiver with relations.

A tilting object induces a derived equivalence;

**Theorem 4.2** ([Ric89, Bon89]). If a smooth stack $X$ has a tilting object $V$, then the functor

$$\mathbb{R}\text{Hom}(V, \bullet) : D^b\text{coh} X \to D^b\text{mod End}(V)$$

induces an equivalence of triangulated categories.

## 5 A tilting bundle on a smooth toric Calabi-Yau 3-fold

Let $\{v_i\}_{i=1}^r$ be a set of points on a lattice $N$ of rank two, and $\Sigma = (\overline{\Sigma}, \{v_i\}_{i=1}^r)$ be a two-dimensional complete stacky fan whose two-dimensional cones are

$$\overline{\Sigma}_i = \mathbb{R}_+ v_i + \mathbb{R}_+ v_{i+1}, \quad i = 1, \ldots, r,$$

with $v_{r+1} = v_1$. The toric stack $X_N$ associated with $\Sigma$ is a weak Fano stack if and only if all the $v_i$ are on the boundary of the lattice polygon

$$\Delta = \text{Conv}\{v_i\}_{i=1}^r$$

developed as the convex hull of $\{v_i\}_{i=1}^r$. The torus $\text{Spec} \mathbb{C}[\overline{M}]$ acting on $X_N$ will be denoted by $\overline{T}$, where $\overline{M} = \text{Hom}(N, \mathbb{Z})$. Let

$$p : X_\Sigma \to X_N$$

be the total space of the canonical bundle of $X_N$. The stacky fan $\Sigma$ corresponding to the total space of the canonical bundle is given by $(\Sigma, \{\tilde{v}_i\}_{i=0}^r)$, where the generators of one-dimensional cones are given by

$$\tilde{v}_0 = (0, 0, 1), \quad \tilde{v}_i = (v_i, 1), \quad i = 1, \ldots, r$$
and three-dimensional cones of $\Sigma$ consists of
\[\sigma_i = \mathbb{R}_+ \tilde{v}_i + \mathbb{R}_+ \tilde{v}_{i+1} + \mathbb{R}_+ \tilde{v}_0, \quad i = 1, \ldots, r.\]

We write the lattice containing $\Sigma$ and its dual as $N$ and $M$ respectively. Let $T = \text{Spec} \mathbb{C}[M]$ be the torus acting on $X_\Sigma$. The toric divisor associated with the one-dimensional cone generated by $\tilde{v}_i$ will be denoted by $D_i$.

Let $R = H^0(\mathcal{O}_{X_\Sigma})$ be the coordinate ring of the three-dimensional affine toric variety associated with the cone over $\Delta$. The following are shown in [IU08, IU]:

- There is a consistent dimer model $G = (B, W, E)$ on $T = (\mathcal{M} \otimes \mathbb{R})/\mathcal{M}$ such that the moduli space $\mathcal{M}_\theta$ of $\theta$-semi-stable representations of $\Gamma$ with dimension vector $(1, \ldots, 1)$ is a crepant resolution of $\text{Spec} R$ where $\theta$ is a generic stability parameter for the quiver $\Gamma$ with relations associated with the dimer model $G$.

- For a prime toric divisor $D$ in $\mathcal{M}_\theta$, there is a perfect matching, which we write $D$ again by abuse of notation, such that the divisor is the zero locus of the arrows contained in the perfect matching.

- For any perfect matching $D$ on $G$, there is a stability parameter $\theta$ such that $D$ corresponds to a prime toric divisor on $\mathcal{M}_\theta$.

- The tautological bundle
  \[\mathcal{V} = \bigoplus_{v \in \mathcal{V}} \mathcal{L}_v\]
  on $\mathcal{M}_\theta$ is a tilting object in $D^b \text{coh} \mathcal{M}_\theta$ such that
  \[\text{End} \mathcal{V} \cong \mathbb{C}\Gamma.\]

The fan describing the toric variety $\mathcal{M}_\theta$ is a refinement of the fan consisting of the cone over $\Delta$ and its faces, and will be denoted by $\tilde{\Sigma}$. The perfect matchings corresponding to $\tilde{v}_i$ for $i = 0, \ldots, r$ will be denoted by $D_i$. The generators of the one-dimensional cones of $\tilde{\Sigma}$ which does not belong to $\Sigma$ will be denoted by $v_{r+1}, \ldots, v_r$, and the corresponding toric divisors will be written as $D_{r+1}, \ldots, D_r$. A perfect matching which is not on a vertex of $\Delta$ depends on the choice of $\theta$ [IU, Proposition 6.5]. A perfect matching which corresponds to $\tilde{v}_0$ under some stability parameter $\theta$ is called a central perfect matching.

\section{A tilting bundle on the canonical bundle}

Let
\[\varphi : X_\Sigma \to X_\Sigma\]
be the natural morphism from a stack to its coarse moduli space, and
\[\varphi : \widetilde{X}_\Sigma \to X_\Sigma\]
be a crepant resolution. Let further $X_\Sigma$ be an arbitrary crepant resolution of $\text{Spec} R$, so that $\widetilde{X}_\Sigma$ and $X_\Sigma$ are isomorphic in codimension one.
For a line bundle $L$ on $X_{\tilde{\Sigma}}$, let $L'$ be its proper transform on $\tilde{X}_\Sigma,
L = (\varphi_* L')^{\vee \vee}
be the reflexive sheaf of rank one on $X_{\tilde{\Sigma}}$ obtained as the double dual of the direct image of $L'$, and
$L = (\varphi^* L)^{\vee \vee}
be the line bundle on $X_{\Sigma}$ obtained as the double dual of the pull-back of $L$ to $X_{\Sigma}$. If $L$ is isomorphic to $O_{X_{\tilde{\Sigma}}} (\tilde{D})$ for a divisor $\tilde{D} = \sum_{i=0}^{r} a_i D_i$, then $L$ is isomorphic to $O_{X_{\Sigma}} (D)$ where $D = \sum_{i=0}^{r} a_i D_i$ is obtained from $\tilde{D}$ by forgetting the toric divisors contracted by $\varphi$, and $L$ is isomorphic to $O_{X_{\Sigma}} (D)$ where $D = \sum_{i=0}^{r} a_i D_i$ is the pull-back of $D$ by $\varphi$.

**Lemma 6.1.** The spaces of global sections of $L$, $L'$, $L$ and $L$ are related as follows:
$H^0 (X_{\tilde{\Sigma}}, L) = H^0 (\tilde{X}_\Sigma, L') \subset H^0 (X_{\Sigma}, L) = H^0 (X_{\Sigma}, L)$.
Moreover, if $H^0 (X_{\tilde{\Sigma}}, L)$ is a reflexive $R$-module, then the inclusion in the middle is an isomorphism.

**Proof.** The first equality follows from the fact that $X_{\tilde{\Sigma}}$ and $\tilde{X}_\Sigma$ are isomorphic in codimension one. The inclusion $\varphi_* L' \subset L$ implies $H^0 (\tilde{X}_\Sigma, L') \subset H^0 (X_{\Sigma}, L)$. Since both $L$ and $\varphi_* L$ are reflexive, the inclusion $L \hookrightarrow \varphi_* L$ is an isomorphism and the second equality follows. Finally, the four spaces have structures of torsion free $R$-modules which coincides on the smooth locus of Spec $R$. Therefore, the reflexivity assumption implies the last assertion. $\square$

The description of the cohomology of a $T$-equivariant line bundle on $X_{\tilde{\Sigma}}$ admits the following simplification in the present situation: Let $D = \sum_{i=0}^{r} a_i D_i$ be a toric divisor on $X_{\tilde{\Sigma}}$ and $\psi_D : |\Sigma| \to \mathbb{R}$ be the piecewise-linear function associated with $D$. Put
$\Delta = |\tilde{\Sigma}| \cap (\mathbb{R}^2 \times \{1\})$
and
$\mathcal{Z} = Z \cap \Delta$,
where
$Z = \{x \in |\tilde{\Sigma}| \mid \psi_D (x) \geq 0\}$.
Then one has
$H^p (\tilde{X}_\Sigma, O (D)) \cong H^p (|\Sigma|) \cong H^p (\mathcal{Z})$
since everything is linear on the third coordinate.

The cohomology of a $T$-equivariant line bundle on $X_{\Sigma}$ has an analogous description, which can be simplified further as follows: For a toric divisor $D = \sum_{i=0}^{r} a_i D_i$ on $X_{\Sigma}$, let $sgn (D) = (\text{sgn } a_0; \text{sgn } \overrightarrow{D}) = (\text{sgn } a_0; \text{sgn } a_1, \ldots, \text{sgn } a_r)$ be the sequence of the signatures of the coefficients of $D$, where
$sgn a = \begin{cases} + & a \geq 0, \\
- & a < 0. \end{cases}$
Consider \(\text{sgn}(\overline{D})\) as a cyclic sequence. A \(-\)-interval in \(\text{sgn}(\overline{D})\) is a succession of \(-\) bounded by one + and the next +. Then it follows from Proposition 2.1 that

\[
\begin{align*}
\text{rank } H^0_T(O(D)) &= \begin{cases} 1 & \text{sgn}(D) = (+; + \cdots +), \\ 0 & \text{otherwise}, \end{cases} \\
\text{rank } H^1_T(O(D)) &= \begin{cases} 0 & \text{sgn } a_0 = - \text{ or } \text{sgn}(\overline{D}) = (+; + \cdots +), \\ \#\{-\text{intervals}\} - 1 & \text{otherwise}, \end{cases} \\
\text{rank } H^2_T(O(D)) &= \begin{cases} 1 & \text{sgn}(D) = (+; - \cdots -), \\ 0 & \text{otherwise}. \end{cases}
\end{align*}
\]

**Proposition 6.2.** Assume that a line bundle \(\mathcal{L}\) and its dual \(\mathcal{L}'\) on \(X_{\Sigma}\) are acyclic.

1. \(H^0(\mathcal{L})\) is Cohen-Macaulay and hence reflexive as an \(R\)-module.

2. Assume further that \(\mathcal{L}\) satisfies the following condition:

\[(\ast)\] if one has \(\text{sgn } a_i = \text{sgn } a_j = -\) where the line segment \([\tilde{v}_i, \tilde{v}_j]\) between \(\tilde{v}_i\) and \(\tilde{v}_j\) lies on the boundary of \(\Delta\), then \(\text{sgn } a_k = -\) for any \(k\) such that \(\tilde{v}_k \in [\tilde{v}_i, \tilde{v}_j]\).

Then the corresponding line bundle \(\mathcal{L}\) on \(X_{\Sigma}\) is acyclic.

**Proof.** The first statement follows from arguments in [TU10 Proposition A.2] as follows: Since \(R\) is Gorenstein and \(f\) is crepant, we have \(f^\ast R \cong O_{X_{\tilde{\Sigma}}}\). Then the Grothendieck duality for \(f\) and the acyclicity of \(\mathcal{L}'\) imply

\[
\text{Hom}_R^i(H^0(\mathcal{L}), \mathcal{R}) \cong \text{Hom}_{X_{\tilde{\Sigma}}}^i(\mathcal{L}, f^\ast \mathcal{R}) \cong \text{Hom}_{X_{\tilde{\Sigma}}}^i(\mathcal{L}, O_{X_{\tilde{\Sigma}}}),
\]

which is 0 for \(i \neq 0\) by the acyclicity of \(\mathcal{L}'\). This implies that \(H^0(\mathcal{L})\) is a Cohen-Macaulay \(R\)-module, and therefore is reflexive since \(\dim R \geq 2\).

For the second statement, first assume that \(H^2_T(O(D)) \neq 0\). Then one has \(\text{sgn}(D) = (+; - \cdots -)\) so that

\[
\psi_D|_{\Delta(\tilde{v})} < 0
\]

for any \(\tilde{v} \in \partial \Delta\). It follows that \(\partial \Delta \subset \Delta \setminus \overline{Z}\) and \(\tilde{v}_0 \in \overline{Z}\), which implies

\[
H^2_T(X_{\tilde{\Sigma}}, \mathcal{L}) \cong H^2_Z(\Delta) \cong H^1(\Delta \setminus \overline{Z}) \neq 0.
\]

Next assume that \(H^1_T(O(D)) \neq 0\), which implies that \(\text{sgn } a_0 = +\) and there are more than two \(-\)-intervals in \(\text{sgn}(\overline{D})\). If \(O(\overline{D})\) is acyclic, then \(\Delta \setminus \overline{Z}\) is connected (and simply-connected). Now consider the toric divisor

\[
\overline{D}' = \sum_{i=0}^{\tilde{r}} (-\tilde{a}_i - 1) \overline{D}_i
\]

which satisfies

\[
O(\overline{D}') \cong O(-\overline{D})
\]
as a non-equivariant line bundle (the subtraction of one from all the coefficients of $\tilde{D}^\vee$ corresponds to a change of a $\mathbb{T}$-linearization). Put

$$Z^\vee := \{ x \in \Delta \mid \psi_{\tilde{D}^\vee}(x) \geq 0 \} = \{ x \in \Delta \mid \psi_{\tilde{D}}(x) \leq -1 \}.$$ 

Then $Z$ is a deformation retract of $\Delta \setminus Z^\vee$. Now the condition that $L^\vee$ is acyclic implies that $\Delta \setminus Z^\vee$ and therefore $Z$ are connected, which contradicts the connectedness of $\Delta \setminus Z$ and the existence of multiple connected components of $\partial \Delta \cap (\Delta \setminus Z)$.

**Lemma 6.3.** Let $L_v$ and $L_w$ be the tautological bundles on the moduli space $M_\theta$ of quiver representations associated with a consistent dimer model with generic stability parameter $\theta$ and put $L = L'_v \otimes L'_w$. Then $L$ satisfies the condition $(\ast)$ in Proposition 6.2 where $X_\Sigma = M_\theta$.

**Proof.** It follows from [IU] that the union $\bigcup_{D \in [\tilde{v}_i, \tilde{v}_j]} D$ of perfect matchings on the line segment $[\tilde{v}_i, \tilde{v}_j] \subset \partial \Delta$ is the union of isolated edges with the zig-zag paths corresponding to these line segments dividing the torus $M_R/M$ into strips, and the open subscheme $U$ of the moduli space $M_\theta \cong X_\Sigma$ consisting of quiver representations such that any arrow not in $\bigcup_{D \in [\tilde{v}_i, \tilde{v}_j]} D$ is non-zero is isomorphic to the product of the moduli space $N$ of representations of the McKay quiver of type $A_n$ in dimension two and a one-dimensional torus;

$$U \cong N \times \mathbb{C}^\times.$$ 

A tautological bundle $L_v$ on $M_\theta$ restricts to the outer tensor product $L'_v \otimes O_{\mathbb{C}}^\times$ of a tautological bundle $L'_v$ on $N$ and a trivial bundle on $\mathbb{C}^\times$. Since the line bundle $(L'_v)^\vee \otimes L'_w$ on $N$ is acyclic, the proof of Lemma 6.3 reduces to Lemma 6.4 below.

**Lemma 6.4.** Let $\Sigma$ be a two-dimensional fan whose two-dimensional cones are given by $\mathbb{R}_+ v_i + \mathbb{R}_+ v_{i+1}, \quad i = 0, 1, \ldots, n$

where

$$v_i = (i, 1) \in N \cong \mathbb{Z}^2.$$ 

Let

$$D = \sum_{i=0}^{n} a_i D_i$$

be a divisor on $X_\Sigma$, where $D_i$ is the toric divisor associated with the one-dimensional cone of $\Sigma$ generated by $v_i$. If a line bundle $O(D)$ is acyclic and the signatures of $a_0$ and $a_n$ are negative, $\text{sgn}(a_0) = \text{sgn}(a_n) = -1$, then the signatures of all the $a_i$ are negative.

**Proof.** This is a corollary of the following fact, which in turn follows immediately from Proposition 2.1: $H^2_c(O(D)) = 0$ for any divisor $D$, and $H^1_c(O(D))$ is the number of $-$-intervals minus one if there is any, and zero otherwise.

Now by applying Lemma 6.4 and Proposition 6.2 to $L'_v \otimes L'_w$ for tautological bundles $L_v$ and $L_w$, one shows that $V = \bigoplus_v L'_v$ is an acyclic bundle satisfying

$$\text{End}(V) \cong \mathbb{C}\Gamma.$$ 

The following definition is due to Bezrukavnikov and Kaledin.
Definition 6.5 ([BK04, Definition 2.1]). A non-zero object of an abelian category is almost exceptional if $\text{Ext}^i(M,M) = 0$ for $i > 0$ and the algebra $\text{End}(M)$ has finite homological dimension.

The equivalence

$$D^b \text{coh } \mathcal{M}_0 \cong D^b \text{ mod } \mathcal{C}\Gamma,$$

implies that $\mathcal{C}\Gamma$ has finite homological dimension. Then the acyclicity of $\mathcal{V}$ and the isomorphism

$$\text{End}(\mathcal{V}) \cong \mathcal{C}\Gamma$$

shows the following:

Lemma 6.6. The vector bundle $\mathcal{V}$ on $X_\Sigma$ is almost exceptional.

The proof of [BKR01, Lemma 4.2] actually shows the following slightly stronger statement:

Lemma 6.7. The derived category of coherent sheaves on a smooth Deligne-Mumford stack without a generic stabilizer is indecomposable.

There is a morphism

$$\pi : X_\Sigma \to \text{Spec } R.$$

Since $R$ is Gorenstein, $X_\Sigma$ is smooth and the morphism $\pi$ is crepant, the Grothendieck duality implies that the identity functor is a Serre functor of $D^b \text{ coh } X_\Sigma$ with respect to $R$, in the sense that there is a functorial isomorphism

$$\mathbb{R} \text{Hom}_R(\mathbb{R}\pi_* \mathbb{R} \text{Hom}_{X_\Sigma}(\mathcal{F}, \mathcal{G}), R) \cong \mathbb{R}\pi_* \mathbb{R} \text{Hom}_{X_\Sigma}(\mathcal{G}, \mathcal{F})$$

satisfying the compatibility conditions in [BK89]. This suffices to show the following:

Theorem 6.8. The functor

$$\Phi = \mathbb{R} \text{Hom}(\mathcal{V}, \bullet) : D^b \text{ coh } X_\Sigma \to D^b \text{ mod } \mathcal{C}\Gamma$$

is an equivalence of triangulated categories.

Proof. The proof is completely parallel to [BK04, Proposition 2.2]: The functor $\Phi$ has a left adjoint

$$\Psi = \bullet \otimes_{\mathcal{C}\Gamma} \mathcal{V} : D^b \text{ mod } \mathcal{C}\Gamma \to D^b \text{ coh } X_\Sigma,$$

which produces a semiorthogonal decomposition

$$D^b \text{ coh } X_\Sigma = (C, C^\perp), \quad (6.1)$$

where $C$ is the essential image of $\Psi$ and $C^\perp$ is the right orthogonal of $\mathcal{V}$. The $R$-Calabi-Yau property of $D^b \text{ coh } X_\Sigma$ implies that (6.1) is an orthogonal decomposition, and the indecomposability of $D^b \text{ coh } X_\Sigma$ shows that $C^\perp$ is empty. \qed


7 A tilting bundle on a toric weak Fano surface

We use the same notation as in the previous sections.

Lemma 7.1. If $V$ be a tilting object on $X_{\Sigma}$ which is a direct sum of line bundles, then the restriction $E = \iota^*V$ of $V$ by the zero-section $\iota : X_{\Sigma} \to X_{\Sigma}$ is again a tilting object.

Proof. $E$ is a generator since it is the restriction of a generator to a closed subscheme. Vanishing of higher Ext-groups $\text{Hom}^{-q}(E, E)$ follows from that of $\text{Hom}^{-q}(V, V)$ by

$$\text{Hom}^{*}(V, V) = \text{Hom}^{*}(p^*E, p^*E) = \text{Hom}^{*}(E, E \otimes p_* \mathcal{O}_{X_{\Sigma}}) = \text{Hom}^{*}(E, E \otimes \bigoplus_{n=0}^{\infty} \mathcal{K}_{X_{\Sigma}}(-n)),$$

where $\mathcal{K}_{X_{\Sigma}}$ is the canonical sheaf of $X_{\Sigma}$ and the isomorphism $V \cong p^*E$ comes from the assumption that $V$ is a direct sum of line bundles.

Note that the existence of a tilting object in $D^b \text{coh} X_{\Sigma}$ which is a direct sum of line bundles is equivalent to the existence of a full strong exceptional collection consisting of line bundles.

Theorem 7.2. Let $D_0$ be an arbitrary central perfect matching. Then there is a full strong exceptional collection on $X_{\Sigma}$ consisting of line bundles such that the total morphism algebra satisfies

$$\text{End}(E) \cong \mathbb{C} \Gamma / \mathcal{I}_{D_0}.$$ 

Proof. Choose a generic stability parameter $\theta$ such that $D_0$ is $\theta$-stable as in [IU08, Lemma 6.2]. Then we can apply Theorem 6.8 and Lemma 7.1 to see that the restrictions of $L_v$ form an full strong exceptional collection.

To obtain the description of the endomorphism algebra $\text{End}(E)$ of $E$, first note that $\text{End}(E)$ is the quotient of $\text{End}(V)$ by the ideal generated by sections vanishing at the toric divisor $D_0$, which is the image of the zero-section $\iota : X_{\Sigma} \to X_{\Sigma}$. Now the theorem follows from the following facts, which are obvious:

1. An element of $\text{End}(V)$ vanishes on the divisor $D_0$ in $X_{\Sigma}$ if and only if it vanishes at the generic point of $D_0$.

2. This is equivalent to the vanishing of the corresponding element of $\text{End}(V)$ at the generic point of $D_0 \subset \mathcal{M}_{\theta}$.

3. A path of the quiver gives an element of $\text{End}(V)$ vanishing on $D_0$ if and only if it is contained in $\mathcal{I}_{D_0}$.
Remark 7.3. It follows from Theorem 7.2 that the derived category $D^b \text{mod} C/\mathcal{I}_{D_0}$ of modules over $C/\mathcal{I}_{D_0}$ does not depend on the choice of a central perfect matching $D_0$. Osamu Iyama pointed out that this result also follows from the theory of 2-APR tilting [IO].

Remark 7.4. The collection of line bundles obtained in Theorem 7.2 is not only a full strong exceptional collection but satisfies the condition that the rolled-up helix algebra

$$A = \bigoplus_{i,j=1}^{n} \bigoplus_{k,n=0}^{\infty} \text{Hom}^k(E_i, E_j \otimes K^{\otimes(-n)})$$

is concentrated in $k = 0$. The converse statement that any collection of line bundles satisfying this condition comes from a dimer model is an immediate consequence of the main result of Bocklandt [Boca] (cf. also [Bocc, Theorem 3.7]). Indeed, given such a collection $(E_1, \ldots, E_n)$, one has $A \cong \text{End}_{X_\Sigma}(\oplus p^* E_i)$ where $p : X_\Sigma \to X_\Sigma$ is the canonical bundle of $X_\Sigma$. Then it is a toric non-commutative crepant resolution of the coordinate ring $R = H^0(O_{X_\Sigma})$ of the affine toric 3-fold obtained by contracting the zero-section by [TU10, Proposition A.2].

8 A tilting bundle on the union of toric divisors

Let $Y = \mathcal{M}_\theta$ be the smooth toric Calabi-Yau 3-fold obtained as the moduli space of representations of the quiver with relations associated with a consisteint dimer model $G = (B,W,E)$ and $V$ be the tilting object on $Y$ obtained as the direct sum of tautological line bundles as in Section 1. For a vertex $v \in V$ of the quiver $(V, A, s, t)$ associated with $G$, the small cycle $\omega_v \in \Gamma$ is defined as $\omega_v = p_+(a) \cdot a$, where $a$ is any arrow such that $s(a) = v$. Let further $W = \sum_{v \in V} \omega_v$ be the central element of $A \cong \Gamma$ obtained as the sum of the small cycles $\omega_v$ starting from each vertex $v \in Q_0$, and $A_0 = A/(W)$ be the quotient ring by the two-sided ideal generated by $W$. Since the center of $A \cong \text{End}(V)$ is isomorphic to $R \cong H^0(O_Y)$, the element $W$ defines a regular function on $Y$. Let $i : Y_0 \to Y$ be the inclusion of the zero locus of $W$, which is the union of toric divisors.

Lemma 8.1. The restriction $V_0 = i^*V$ is a tilting object in $D^b \text{coh} Y_0$, whose endomorphism algebra is isomorphic to $A_0$.

Proof. $V_0$ is a generator just as in Lemma 7.1. One has

$$\mathbb{R}\text{Hom}_{Y_0}(V_0, V_0) = \mathbb{R}\Gamma(V_0^\vee \otimes V_0)$$

$$= \mathbb{R}\Gamma(i^*(V_0^\vee \otimes V))$$

$$= \mathbb{R}\Gamma(i_*i^*(V_0^\vee \otimes V))$$

$$= \mathbb{R}\Gamma(\{V_0^\vee \otimes V \xrightarrow{W} V_0^\vee \otimes V\})$$

$$= \{ \text{End}(V) \xrightarrow{W} \text{End}(V) \}$$

which shows that $\text{End}(V_0) \cong \text{End}(V)/W \text{End}(V)$ and $\text{Ext}_{Y_0}^i(V_0, V_0) = 0$ for $i > 0$. \qed
Since a tilting object induces an equivalence of bounded derived categories (see e.g. [TU10, Lemma 3.3] for a proof without smoothness assumption), we have the following:

**Corollary 8.2.** The functor

\[ \Phi_0 = \mathbb{R}\text{Hom}(\mathcal{Y}_0, \bullet) : D^b\text{coh} \mathcal{Y}_0 \to D^b\text{mod} \mathcal{A}_0 \]

is an equivalence of triangulated categories.

The bounded stable derived category of \( Y_0 \) is the quotient category

\[ D^b_{\text{sing}}(Y_0) = D^b\text{coh} \mathcal{Y}_0 / \text{perf} \mathcal{Y}_0. \]

of the bounded derived category of coherent sheaves on \( Y_0 \) by the full subcategory consisting of perfect complexes (i.e. bounded complexes of locally-free sheaves). The bounded stable derived category of \( \mathcal{A}_0 \) is defined similarly as the quotient category

\[ D^b_{\text{sing}}(\mathcal{A}_0) = D^b\text{mod} \mathcal{A}_0 / \text{perf} \mathcal{A}_0 \]

of the bounded derived category \( D^b\text{mod} \mathcal{A}_0 \) of finitely-generated right \( \mathcal{A}_0 \)-modules by the full triangulated subcategory \( \text{perf} \mathcal{A}_0 \) consisting of perfect complexes (i.e. bounded complexes of projective modules). Since perfect complexes are characterized in a purely categorical way as homologically finite objects [Orl06] (i.e. objects \( A \) such that for any object \( B \), the group \( \text{Hom}(A, B[i]) \) is trivial for all but a finite number of \( i \in \mathbb{Z} \)), an equivalence of bounded derived categories induces an equivalence of bounded stable derived categories:

**Corollary 8.3.** One has an equivalence

\[ D^b_{\text{sing}}(Y_0) \cong D^b_{\text{sing}}(\mathcal{A}_0) \]

of triangulated categories.

Stable derived categories are introduced by Buchweitz [Buc87] motivated by the theory of matrix factorizations by Eisenbud [Eis80]. They are rediscovered by Orlov [Orl04] under the name triangulated categories of singularities following an idea of Kontsevich, and plays essential role in homological mirror symmetry. In particular, the stable derived category \( D^b_{\text{sing}}(Y_0) \) is expected to be equivalent to the derived category of the wrapped Fukaya category of an affine curve (see [AAE+] and references therein). On the other hand, the stable derived category can be regarded as the derived category of a curved algebra (i.e. a pair of an algebra and its central element, cf. e.g. [Pos]), and the curved algebra \((C \Gamma, W)\) is the colimit of the following covariant functor \( F \) from the category \( \mathcal{C} \) to the category of curved algebras: A dimer model gives a one-dimensional CW complex with nodes as 0-cells and edges as 1-cells, and the category \( \mathcal{C} \) is the category whose objects are open stars of 0-cells and 1-cells and whose morphisms are inclusions. In other words, the category \( \mathcal{C} \) has nodes and edges as objects, and there is a unique non-identity morphism \( e \to n \) for each adjacency of an edge \( e \in E \) and a node \( n \in B \cup W \). The functor \( F \) sends an edge \( e \in E \) to the curved algebra \( F(e) = (\mathcal{A}_e, W_e) \) consisting of the path algebra \( \mathcal{A}_e \) of the cyclic quiver \( s_e \xrightarrow{a_e} t_e \xrightarrow{p_e} s_e \) with two vertices \( s_e, t_e \) and two arrows \( a_e, p_e \) and the central
element $W_e = a_e \cdot p_e + p_e \cdot a_e$. For a node $n$, let $(e_0, \ldots, e_r)$ be the set of edges adjacent to $n$, ordered clockwise if $n \in B$ and counter-clockwise if $n \in W$. Then the value $\mathcal{F}(n)$ for a node $n$ is the curved algebra $(\mathcal{A}_n, W_n)$ consisting of the path algebra $\mathcal{A}_n$ of the cyclic quiver

$$s_{e_0} \xrightarrow{a_{e_0}} t_{e_0} = s_{e_1} \xrightarrow{a_{e_1}} t_{e_1} = s_{e_2} \xrightarrow{a_{e_2}} \cdots \xrightarrow{a_{e_r}} t_{e_r} = s_{e_0}$$

with $r + 1$ vertices and $r + 1$ arrows and the central element

$$W_n = \sum_{i=0}^{r} a_{e_{i-1}} \cdots a_{e_0} a_{e_r} \cdots a_{e_{i+1}} a_{e_i}.$$

For each adjacency $e_i \to n$, the map $\mathcal{F}(e_i \to n) : \mathcal{F}(e_i) \to \mathcal{F}(n)$ sends $a_{e_i}$ and $p_{e_i}$ in $\mathcal{A}_{e_i}$ to $a_{e_i}$ and $a_{e_{i-1}} \cdots a_{e_0} a_{e_r} \cdots a_{e_{i+1}} a_{e_i}$ in $\mathcal{A}_n$ respectively. It is an interesting problem to relate this to an idea of Kontsevich [Kon] to describe the Fukaya category of a Stein manifold in terms of a constructible sheaf of dg categories on its Lagrangian skeleton.

**References**

[AAC+] Mohammed Abouzaid, Denis Auroux, Alexander I. Efimov, Ludmil Katzarkov, and Dmitri Orlov, *Homological mirror symmetry for punctured spheres*, arXiv:1103.4322.

[BCS05] Lev A. Borisov, Linda Chen, and Gregory G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. 18 (2005), no. 1, 193–215 (electronic). MR MR2114820 (2006a:14091)

[BH09] Lev Borisov and Zheng Hua, *On the conjecture of King for smooth toric Deligne-Mumford stacks*, Adv. Math. 221 (2009), no. 1, 277–301. MR MR2509327

[BK89] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337. MR MR1039961 (91b:14013)

[BK04] R. V. Bezrukavnikov and D. B. Kaledin, *McKay equivalence for symplectic resolutions of quotient singularities*, Tr. Mat. Inst. Steklova 246 (2004), no. Algebr. Geom. Metody, Svyazi i Prilozhi., 20–42. MR MR2101282 (2006e:14006)

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. 14 (2001), no. 3, 535–554 (electronic). MR MR1824990 (2002f:14023)

[BM] Martin Bender and Sergey Mozgovoy, *Crepant resolutions and brane tilings II: Tilting bundles*, arXiv:0909.2013.

[Boca] Raf Bocklandt, *Calabi Yau algebras and weighted quiver polyhedra*, arXiv:0905.0232.

[Bochb] , *Consistency conditions for dimer models*, arXiv:1104.1592.
[Bocc] ______, Generating toric noncommutative crepant resolutions, arXiv:1104.1592.

[Bon89] A. I. Bondal, Representations of associative algebras and coherent sheaves, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 1, 25–44. MR MR992977 (90i:14017)

[Bro] Nathan Broomhead, Dimer models and Calabi-Yau algebras, arXiv:0901.4662.

[Buc87] Ragnar-Olaf Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, Available from https://tspace.library.utoronto.ca/handle/1807/16682, 1987.

[CS08] Alastair Craw and Gregory G. Smith, Projective toric varieties as fine moduli spaces of quiver representations, Amer. J. Math. 130 (2008), no. 6, 1509–1534. MR MR2464026

[Dav] Ben Davison, Consistency conditions for brane tilings, arXiv:0812.4185.

[Efi10] Alexander I. Efimov, Maximal lengths of exceptional collections of line bundles, arXiv:1010.3755, 2010.

[Eis80] David Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Trans. Amer. Math. Soc. 260 (1980), no. 1, 35–64. MR MR570778 (82d:13013)

[FHM+06] Sebastián Franco, Amihay Hanany, Dario Martelli, James Sparks, David Vegh, and Brian Wecht, Gauge theories from toric geometry and brane tilings, J. High Energy Phys. (2006), no. 1, 128, 40 pp. (electronic). MR MR2201204

[FHV+06] Sebastián Franco, Amihay Hanany, David Vegh, Brian Wecht, and Kristian D. Kennaway, Brane dimers and quiver gauge theories, J. High Energy Phys. (2006), no. 1, 096, 48 pp. (electronic). MR MR2201227

[Ful93] William Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR MR1234037 (94g:14028)

[FV06] Sebastián Franco and David Vegh, Moduli spaces of gauge theories from dimer models: proof of the correspondence, J. High Energy Phys. (2006), no. 11, 054, 26 pp. (electronic). MR MR2270405 (2007j:81161)

[Gul08] Daniel R. Gulotta, Properly ordered dimers, R-charges, and an efficient inverse algorithm, J. High Energy Phys. (2008), no. 10, 014, 31. MR MR2453031 (2010b:81116)

[HHV06] Amihay Hanany, Christopher P. Herzog, and David Vegh, Brane tilings and exceptional collections, J. High Energy Phys. (2006), no. 7, 001, 44 pp. (electronic). MR MR2240899 (2008b:81224)
[HK05] Amihay Hanany and Kristian D. Kennaway, *Dimer models and toric diagrams*, hep-th/0503149, 2005.

[HP] Lutz Hille and Markus Perling, *Exceptional sequences of invertible sheaves on rational surfaces*, arXiv:0810.1936.

[HP06] ______, *A counterexample to King’s conjecture*, Compos. Math. 142 (2006), no. 6, 1507–1521. MR MR2278758 (2007h:14074)

[IO] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, arXiv:0905.0059.

[IU] Akira Ishii and Kazushi Ueda, *Dimer models and the special McKay correspondence*, arXiv:0905.0059.

[IU08] ______, *On moduli spaces of quiver representations associated with dimer models*, Higher dimensional algebraic varieties and vector bundles, RIMS Kôkyûroku Bessatsu, B9, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008, pp. 127–141. MR MR2509696

[IU10] ______, *A note on consistency conditions on dimer models*, arXiv:1012.5449, to appear in RIMS Kôkyûroku Bessatsu, 2010.

[Kaw06] Yujiro Kawamata, *Derived categories of toric varieties*, Michigan Math. J. 54 (2006), no. 3, 517–535. MR MR2280493 (2008d:14079)

[Kin97] Alastair King, * Tilting bundles on some rational surfaces*, preprint available at http://www.maths.bath.ac.uk/~masadk/papers/, 1997.

[Kon] Maxim Kontsevich, *Symplectic geometry of homological algebra*, Mathematische Arbeitstagung 2009.

[Moz09] Sergey Mozgovoy, *Creant resolutions and brane tilings I: Toric realization*, arXiv:0908.3475, 2009.

[MR10] Sergey Mozgovoy and Markus Reineke, *On the noncommutative Donaldson-Thomas invariants arising from brane tilings*, Adv. Math. 223 (2010), no. 5, 1521–1544. MR 2592501

[Naga] Kentaro Nagao, *Derived categories of small toric Calabi-Yau 3-folds and counting invariants*, arXiv:0809.2994.

[Nagb] ______, *Refined open noncommutative Donaldson-Thomas invariants for small crepant resolutions*, arXiv:0907.3784.

[NN] Kentaro Nagao and Hiraku Nakajima, *Counting invariant of perverse coherent sheaves and its wall-crossing*, arXiv:0809.2992.

[Orl04] D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova 246 (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240–262. MR MR2101296
[Ori06] Triangulated categories of singularities, and equivalences between Landau-Ginzburg models, Mat. Sb. 197 (2006), no. 12, 117–132. MR 2437083 (2009g:14013)

[Pos] Leonid Positselski, Two kinds of derived categories, koszul duality, and comodule-contramodule correspondence, arXiv:0905.2621.

[Ric89] Jeremy Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (1989), no. 3, 436–456. MR MR1002456 (91b:18012)

[Sti06] Jan Stienstra, Mahler measure, Eisenstein series and dimers, Mirror symmetry. V, AMS/IP Stud. Adv. Math., vol. 38, Amer. Math. Soc., Providence, RI, 2006, pp. 151–158. MR MR2282959 (2008c:11120)

[Sti08] Hypergeometric systems in two variables, quivers, dimers and dessins d’enfants, Modular forms and string duality, Fields Inst. Commun., vol. 54, Amer. Math. Soc., Providence, RI, 2008, pp. 125–161. MR MR2454323

[Sti09] Chow forms, Chow quotients and quivers with superpotential, Motives and algebraic cycles, Fields Inst. Commun., vol. 56, Amer. Math. Soc., Providence, RI, 2009, pp. 327–336. MR 2572726 (2010k:14088)

[Sti10] Computation of principal \(A\)-determinants through dimer dynamics, New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008), Adv. Stud. Pure Math., vol. 59, Math. Soc. Japan, Tokyo, 2010, pp. 349–369. MR 2683214

[Sze08] Balázs Szendrői, Non-commutative Donaldson-Thomas invariants and the conifold, Geom. Topol. 12 (2008), no. 2, 1171–1202. MR MR2403807 (2009e:14100)

[TU10] Yukinobu Toda and Hokuto Uehara, Tilting generators via ample line bundles, Adv. Math. 223 (2010), no. 1, 1–29. MR 2563209

[vdB04] Michel van den Bergh, Non-commutative crepant resolutions, The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 749–770. MR MR2077594 (2005e:14002)

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