Finite diffeomorphism invariant observables in quantum gravity

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ABSTRACT

Two sets of spatially diffeomorphism invariant operators are constructed in the loop representation formulation of quantum gravity. This is done by coupling general relativity to an anti-symmetric tensor gauge field and using that field to pick out sets of surfaces, with boundaries, in the spatial three manifold. The two sets of observables then measure the areas of these surfaces and the Wilson loops for the self-dual connection around their boundaries. The operators that represent these observables are finite and background independent when constructed through a proper regularization procedure. Furthermore, the spectra of the area operators are discrete so that the possible values that one can obtain by a measurement of the area of a physical surface in quantum gravity are valued in a discrete set that includes integral multiples of half the Planck area.

These results make possible the construction of a correspondence between any three geometry whose curvature is small in Planck units and a diffeomorphism invariant state of the gravitational and matter fields. This correspondence relies on the approximation of the classical geometry by a piecewise flat Regge manifold, which is then put in correspondence with a diffeomorphism invariant state of the gravity-matter system in which the matter fields specify the faces of the triangulation and the gravitational field is in an eigenstate of the operators that measure their areas.

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1 Introduction

The problem of observables is probably the key problem presently confronting non-perturbative approaches to the quantization of the gravitational field. The problem is difficult because it reflects, in two different ways, the problem of making sense of a diffeomorphism invariant quantum field theory. Already at the classical level, it is non-trivial to construct and interpret functions of the dynamical variables that are invariant under spacetime diffeomorphisms. While, it is true, one can describe in words a certain limited number of them, what is needed is much more than this. First of all, we must have an infinite number of observables, corresponding to the number of physical degrees of freedom of the theory. Second, to make the transition to the quantum theory one must know their Poisson algebra.

When we come to the quantum theory, new problems emerge because essentially all diffeomorphism invariant observables involve products of local field observables, and so are not directly defined in the representation space of any quantum field theory. They must be defined through a limiting procedure which is analogous to the regularization procedures of conventional quantum field theories. Furthermore, it is necessary to confront the fact that none of the conventional regularization or renormalization procedures can be applied to this case. This is because they all depend on the presence of a background metric. It is then necessary to define new regularization procedures which may be applied to field theories constructed without a background metric. Additional background structure does come in the definitions of the regulated operators; what must be shown is that in the limits that the regulators are removed the resulting action is finite and background independent.

In the last year and a half, some progress\cite{1, 2} has been made on the problem of observables in the context of a nonperturbative approach to quantum gravity based on the Ashtekar variables\cite{3} and the loop representation\cite{6, 7}. This approach is based on taking as a starting point a quantum kinematical framework which is based on non-Fock representations\cite{8, 9, 2} of certain non-local observable algebras. It seems necessary to pick non-Fock representations as the starting point for the construction of diffeomorphism invariant quantum field theories to avoid the dependence of the Fock structure on a fixed background metric.

What has been learned using this approach may be summarized as

\footnote{Reviews of previous work in this direction are found in \cite{4, 5, 2}.}
follows\textsuperscript{1, 2}:

a) It seems impossible to construct background independent renormalization procedures for products of local fields. This is because any local renormalization procedure is ambiguous up to a local density. As a consequence, because we are working in a formalism in which the basic local observable is a frame field, this theory has no operator which can represent the measurement of the metric at a point. This further means that diffeomorphism invariant operators cannot normally be constructed from integrals over products of local fields.

b) Despite this, there are several non-local observables that can be constructed as operators acting on kinematical states by a regularization procedure appropriate to the non-perturbative theory. In all of the cases in which a well defined operator exists in the limit that the regularization is removed, that operator is finite and background independent. Among these operators are those that measure the area of any given surface, the volume of any given region and the spatial integral of the norm of any given one form. By measurements of these observables the metric can be determined, in spite of the fact that there is no local operator which can represent the metric.

c) The connection between background independence and finiteness is most likely general, as there are arguments that for any operator that can be constructed through a point splitting regularization procedure of the type used in \textsuperscript{1, 2}, background independence implies finiteness\textsuperscript{10}.

d) The spectra of the operators that measure areas and volumes are discrete series of rational numbers times the appropriate Planck units.

e) Using these results, the semiclassical limit of the theory can be understood. Given any classical three metric, slowly varying at the Planck scale, it is possible to construct a kinematical quantum state which has the property that it is an eigenstate of the above mentioned operators and the eigenvalues agree with the corresponding classical values in terms of that metric, up to terms of order of the inverse of the measured quantity in Planck units.

These results are very encouraging, but they are subject to an important limitation. They concern the kinematical level of the theory, which is the original state space on which the unconstrained quantum theory is defined. The physical states, which are those states that satisfy the Hamiltonian and diffeomorphism constraints, live in a subspace of this space.

It would then be very desirable to find results analogous to these holding for physical operators. In this paper I report results which bring us significantly closer to that goal. These include the construction of a number of
operators which are invariant under spatial diffeomorphisms. For example, as I will show below, it is possible to construct a diffeomorphism invariant operator that measures the area of surfaces which are picked out by the values of some dynamical fields. Just as in the kinematical case, the spectrum of this operator is discrete and includes the integral multiples of half the Planck area.

The basic idea on which these results are based is to use matter fields to define physical reference frames and then use these to construct diffeomorphism invariant observables. Of course, the idea of using matter fields to specify dynamically a coordinate system is very old. It goes back to Einstein[12], who pointed out that in order to realise the operational definition of lengths and times in terms of rulers and clocks in general relativity, it was necessary to consider the whole dynamical system of the rulers and clocks, together with the gravitational field. The application of this idea to the quantum theory was first discussed by DeWitt[13], and has been recently revived by Rovelli[14, 16] and by Kuchar and Torre[15].

In a paper closely related to this one, Rovelli has used a scalar field to pick out a set of surfaces whose areas are then measured. In this paper I take for the matter field an antisymmetric tensor gauge field, with dynamics as first written down by Kalb and Ramond[17]. There are two reasons for this. First, as we will see below, the coupling of the antisymmetric tensor gauge field to gravity is particularly simple in the Ashtekar formalism, which allows us to hope that it will be possible to get results about physical observables, which must commute also with the Hamiltonian constraint. The second reason is that, as I will describe, the configurations of the Kalb-Ramond field can be associated with open surfaces, which has certain advantages.

Now, it is clear on the kinematical level that if one measures the area of every two dimensional surface one determines the spatial metric completely. One can then imagine that if one has a finite, but arbitrarily large, set of surfaces, one can use measurements of their areas to make a partial measurement of the metric. Such an arrangement can serve as a model of an apparatus that might be used to measure the gravitational field, because indeed, any real physical measuring device returns a finite amount of information and thus makes only a partial measurement of a quantum field. As I discuss below, there are a number of results and lessons that can be learned about measurement theory for quantum gravity by using a finite collection of surfaces as a model of a measuring apparatus.

Once we have a set of finite spatially diffeomorphism invariant operators, defined by using matter fields as a quantum reference frame, it is interest-
ing to try to employ the same strategy to construct physical observables\(^2\). One can add to the theory additional matter degrees of freedom which can represent physical clocks and use these to construct operators which commute with the Hamiltonian constraint but describe measurements made at particular times as measured by the physical clock. This idea is developed in \([24]\). Furthermore, once one has physical operators that correspond to measurements localized in space and time by the use of matter fields to form a spacetime reference system, it is possible to give a formulation of a measurement theory which may be applied to quantum cosmology. A sketch of such a measurement theory is also developed there.

This paper is organized as follows. In the next section I show how to couple an antisymmetric tensor gauge field to gravity. This is followed by section 3 in which I show how to quantize the tensor gauge field in terms of a surface representation that is closely analogous to the abelian loop representation for Maxwell theories\(^3\). I then show how to combine these results with the loop representation of quantum gravity and how to construct diffeomorphism invariant states of the coupled system. Here it is also shown how to construct the diffeomorphism invariant operator that measures the area of the surface picked out by the quantum state of the antisymmetric tensor gauge field. In section 4 I consider a straightforward extension of these results in which certain degrees of freedom are added which result in quantum states which are labeled by open rather than closed surfaces. In the next section, which is section 5, I show how the matter field may also be used to construct a diffeomorphism invariant loop operator. This has the effect of adding Wilson loops of the left-handed spacetime connection around the edges of the surfaces picked out by the quantum states of the matter. These results are then used in section 6, where I show how to construct quantum reference systems by combining the surfaces from a number of independent matter degrees of freedom to construct simplicial complexes.

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\(^2\)Note that, as pointed out by Rovelli\([16]\), there is a model diffeomorphism invariant theory in 3 + 1 dimensions whose Hamiltonian constraints is proportional to a linear combination of the gauge and diffeomorphism constraints. This is the Husain-Kuchar model\([18]\), which corresponds, at the classical level, to a limit of general relativity in which the speed of light has been taken to zero. The physical state space of this model is just the space of diffeomorphism invariant states of quantum gravity and the physical inner product is known and takes a simple form in the loop representation\([18]\). The observables I construct here are then examples of physical observables if we adjoin to the Husain-Kuchar model the antisymmetric gauge field.

\(^3\)Rodolfo Gambini has kindly informed me that many of the results of this section were found previously in \([19]\).
We find here a very interesting correspondence between certain piecewise-flat Regge manifolds and the elements of a basis of diffeomorphism invariant states of the coupled matter-gravity system. I also sketch an approach to a measurement theory for quantum gravity which is based on the results described here\cite{24}. Finally, the implications of these results are the subject of the conclusion.

2 Coupling an antisymmetric tensor gauge theory to gravity

An antisymmetric tensor gauge field\cite{17} is a two form, $C_{ab} = -C_{ba}$ subject to a gauge transformation generated by a one form $\Lambda_a$ by,

$$\delta C_{ab} = d\Lambda_{ab}. \quad (1)$$

It’s field strength is a three form which will be denoted $W_{abc} = dC_{abc}$. The contribution to the action for these fields coupled to gravity is, in analogy with electromagnetism,

$$S_C = \frac{k}{4} \int d^4x \sqrt{g} g^{ad} g^{be} g^{cf} W_{abc} W_{def} \quad (2)$$

where $k$ is a coupling constant with dimensions of inverse action and all the quantities are, just for moment, four dimensional.

In the Hamiltonian theory\cite{3} its conjugate momenta is given by $\tilde{\pi}_{ab} = -\tilde{\pi}_{ba}$ so that

$$\{C_{ab}(x), \tilde{\pi}_{cd}(y)\} = \delta^{[c}_{a} \delta^{d]}_{b} \delta^3(y,x) \quad (3)$$

where the delta function is understood to be a density with the weight on the first entry and $\tilde{\pi}_{cd}$ is also a density. We will also find it convenient to work with the dualized fields

$$\tilde{W}^* = \frac{1}{3!} \varepsilon^{abc} W_{abc} \quad (4)$$

and

$$\pi^*_a = \frac{1}{2} \varepsilon_{abc} \tilde{\pi}_{bc}. \quad (5)$$

\footnote{From now on we consider that the indices $a, b, c, ...$ are spatial indices, while indices $i, j, k, ...$ will be internal $SO(3)$ indices. Densities, as usual, are sometimes, but not always, indicated by a tilde.}
The gauge transform (1) is then generated by the constraint

\[ G = \partial_c \tilde{\pi}^{cd} = d\pi^*_cd = 0 \]  

This field can be coupled to gravity in the Ashtekar formalism by adding to the Hamiltonian constraint the term

\[ \mathcal{C}_{\text{matter}} = \frac{k}{2} (\tilde{W}^*)^2 + \frac{1}{2k} \pi^*_a \pi^*_b \tilde{q}^{ab} \]  

and adding to the diffeomorphism constraint the term

\[ \mathcal{D}^\text{matter}_a = \pi^*_a \tilde{W}^* \].

We may note that the term added to the Hamiltonian constraint is naturally a density of weight two, so that it is polynomial without the necessity of changing the weight of the constraint by multiplying by a power of the determinant of the metric, as is necessary in Maxwell or Yang-Mills theory [20].

The antisymmetric tensor gauge field can be understood to be a theory of surfaces in three dimensions in the same sense that Maxwell theory is a theory of the Faraday flux lines [21]. By (6) there is a scalar field \( \phi \) such that locally

\[ \pi^*_a = d\phi_a \]  

The equipotential surfaces of \( \phi \) define a set of surfaces which are the analogues of the Faraday lines of electromagnetism. Further, any two dimensional surface \( \mathcal{S} \) defines a distributional configuration of the \( \pi^{ab} \) by,

\[ \pi^{ab}_\mathcal{S}(x) = \int d^3 \mathcal{S}^{ab}(\sigma) \delta^3(x, \mathcal{S}(\sigma)) \]  

here \( \sigma \) are coordinates on the surface. Note that \( \pi^{ab}_\mathcal{S} \) is automatically divergence free. These are completely analogous to the distributional configurations of the electric field [2, 22] that may be associated to a curve \( \gamma \) by

\[ E^a_\gamma(x) = \int d\gamma^a(s) \delta^3(x, \gamma(s)) \]  

We can define a diffeomorphism invariant observable which depends on both the metric and \( \tilde{\pi}^{cd} \) which has the property that when \( \tilde{\pi}^{cd} \) has such a distributional configuration it measures the area of that surface. This can be...
defined either by generalizing the definition of the area observable\(^1, 2\) or more directly by

\[
A(\pi, \tilde{E}) \equiv Q(\pi, \tilde{E}) = \int \sqrt{q^{ab} \pi^a \pi^b} \quad (12)
\]

An equivalent expression for this is given by

\[
A(\pi, \tilde{E}) = \lim_{N \to \infty} \sum_{N=1}^{N} \sqrt{A_{\text{approx}}^2[R_i]} \quad (13)
\]

where space has been partitioned into \(N\) regions \(R_i\) such that in the limit \(N \to \infty\) the regions all shrink to points. Here, the observable that is measured on each region is defined by\(^3\):

\[
A_{\text{approx}}^2[R] \equiv \int_R d^3x \int_R d^3y \ T^{ab}(x, y)\pi^a(x)\pi^b(y) \quad (14)
\]

To show the equivalence between these two expressions, we may start with (12) and regulate it the way it is done in the quantum theory by introducing a background euclidean coordinate system and a set of test fields \(f_\epsilon(x, y)\) by

\[
f_\epsilon(x, y) \equiv \frac{\sqrt{q(x)}\theta[\frac{\epsilon}{2} - |x^1 - y^1|][\theta[\frac{\epsilon}{2} - |x^2 - y^2|][\frac{\epsilon}{2} - |x^3 - y^3|]} \quad (15)
\]

In these coordinates

\[
\lim_{\epsilon \to 0} f_\epsilon(x, y) = \delta^3(x, y) \quad (16)
\]

We can then write

\[
A(\pi, \tilde{E}) = Q(\pi, \tilde{E}) = \lim_{\epsilon \to 0} \int d^3x \int d^3y \int d^3z T^{ab}(y, z)\pi^a(y)\pi^b(z)f_\epsilon(x, y)f_\epsilon(x, z) \quad (17)
\]

When the expression inside the square root is slowly varying in \(x\) we can reexpress it in the following way. We divide space into regions \(R_i\) which

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\(^6\)Here \(T^{ab}(x, y)\) is defined as follows\(^2\). Let there be a procedure, based on a background flat metric, to associate a circle \(\gamma_{x,y}\) to every two points in the three manifold \(\Sigma\). Then define \(T^{ab}(x, y) \equiv Tr_{\gamma_{x,y}}U_{\gamma_{x,y}}E^a(x)U_{\gamma_{x,y}}E^b(y)\), where \(U_{\gamma}(x, y) \equiv PexpG \int_{\gamma} A\) is parallel transport along the curve \(\gamma\) from \(x\) to \(y\).
are cubes of volume $\varepsilon^3$ centered on the points $x_i = (n\varepsilon, m\varepsilon, p\varepsilon)$ for $n, m, p$ integers. We then write,

\[
A(\pi, \tilde{E}) = \lim_{\varepsilon \to 0} \sum_i \varepsilon^3 \sqrt{\int d^3y \int d^3z T^{ab}(y, z) \pi^a(y) \pi^b(z) f_\varepsilon(y, x_i) f_\varepsilon(z, x_i)}
\]

\[
= \lim_{N \to \infty} \sum_{N=1}^{N} \sqrt{A^2_{\text{approx}}[R_i]} \tag{18}
\]

If we now plug into these expressions the distributional form (10) it is straightforward to show that

\[
A(\pi_\mathcal{S}, \tilde{E}) = \int_{\mathcal{S}} \sqrt{h}
\]

where $h$ is the determinant of the metric of the two surface, which is given by $h = \tilde{q}^{ab} n_a n_b$ where $n^a$ is the unit normal of the surface.

### 3 Quantization

It is straightforward to construct an algebra of loops and closed surfaces to coordinatize the gauge constraint surface, corresponding to the imposition of both $G = 0$ and the $SU(2)$ Gauss’s law of the gravitational fields. We may associate to every closed surface $\mathcal{S}$ a gauge invariant observable,

\[
T[\mathcal{S}] \equiv e^{ik \int_{\mathcal{S}} C} \tag{20}
\]

Conjugate to $T[\mathcal{S}]$ we have the observables $\tilde{\pi}^{ab}(x)$ which satisfy the algebra,

\[
\{T[\mathcal{S}], \tilde{\pi}^{ab}(x)\} = ik \int d^2S^{ab}(\sigma) \delta^3(x, \mathcal{S}(\sigma))T[\mathcal{S}] \tag{21}
\]

We would like now to construct a representation of this algebra as an algebra of operators. We can construct a surface representation in which the states are functions of a set of closed surfaces $\Psi[\{\mathcal{S}\}]$. In order to implement the abelian gauge invariance we require that these states are invariant under reparametrization invariance and satisfy two relations. First, we require that

\[
\Psi[\mathcal{S} \circ \mathcal{S}'] = \Psi[\mathcal{S} \cup \mathcal{S}'] \tag{22}
\]

\footnote{Such a representation was first constructed in [19].}
where $\mathcal{S}$ and $\mathcal{S}'$ are any two surfaces that touch at one point and $\mathcal{S} \circ \mathcal{S}'$ is the surface made by combining them. Second, we require that $\Psi[\mathcal{S}] = \Psi[\mathcal{S}']$ whenever $e^{\int_{\mathcal{S}} F} = e^{\int_{\mathcal{S}'} F}$ for every two form $F$. We then define the representation by,

$$\hat{T}[\mathcal{S}'] \Psi[\mathcal{S}] = \Psi[\mathcal{S} \cup \mathcal{S}']$$  \hspace{1cm} (23)

and

$$\hat{\pi}^{ab}(x) \Psi[\mathcal{S}] = \hbar k \int d^2 S^{ab}(\sigma) \delta^3(x, \mathcal{S}(\sigma)) \Psi[\mathcal{S}].$$  \hspace{1cm} (24)

It then follows that the operators satisfy

$$[\hat{T}[\mathcal{S}], \hat{\pi}^{ab}(x)] = -\hbar k \int d^2 S^{ab}(\sigma) \delta^3(x, \mathcal{S}(\sigma)) \hat{T}[\mathcal{S}]$$  \hspace{1cm} (25)

We should now say a word about dimensions. In order that the interpretation of $A[\pi, \tilde{E}]$ as an area work, it is necessary that $\tilde{\pi}^{bc}$ have dimensions of inverse length, from which it follows from (7) that $k$ have dimensions inverse to $\hbar$ and that the dimensions of $C^{ab}$ are mass/length. This choice is consistent with both the Poisson bracket and the requirement that the exponenten in (20) be dimensionless.

We now bring gravity in via the standard loop representation\[6\]. The states are then functions, $\Psi[\alpha, \mathcal{S}]$, of loops and surfaces. We may introduce a set of bra’s $<\alpha, \mathcal{S}|$ labeled by loops and surfaces so that,

$$\Psi[\alpha, \mathcal{S}] = <\alpha, \mathcal{S}| \Psi >$$  \hspace{1cm} (26)

We then want to express the area observable (13) as a diffeomorphism invariant operator and show that it does indeed measure areas. It is straightforward to show that the bra’s $<\alpha, \mathcal{S}|$ are, for nonintersecting loops $\alpha$, eigenstates of the operator $\hat{A}$. This operator may be constructed by using the expression (13) as a regularization, in the way described in detail in [3]. A straightforward calculation shows that for the case of nonintersecting loops\[8\]

$$<\alpha, \mathcal{S}| \hat{A}_{\text{approx}}^2 [\mathcal{R}] = (\frac{\hbar k l^2_{\text{Planck}}}{2})^2 I[\alpha, \mathcal{S} \cap \mathcal{R}]^2 <\alpha, \mathcal{S}|$$  \hspace{1cm} (27)

\[8\]The appearance of the Planck area is due to the presence of $G$ in the definition of the parallel propogators for the Ashtekar connection, $A$, as in the kinematical case \[1\] 3. This is due to the fact that it is $G A_{n}$ that has dimensions of inverse length. The dimensionality of the gravitational constant thus manifests itself in the appearance of the Planck area in the operator algebra for quantum gravity.
where $I[\gamma, S]$ is the intersection number given by,

$$I[\gamma, S] \equiv \int d\gamma^a(s) \int d^2 S^{bc}(\sigma) \delta^3(S(\sigma), \gamma(s)) \epsilon_{abc}$$  \hspace{1cm} (28)$$

and where $S \cap R$ means the part of the surface that lies inside the region.

It then follows from (13) that

$$<\alpha, S| \hat{A} = \frac{hkl^2_{\text{Planck}}}{2} I^+[\alpha, S] <\alpha, S|$$  \hspace{1cm} (29)$$

where $I^+[\alpha, S]$ represents the positive definite unoriented intersection number which simply counts all intersections positively. Thus, we see that the operator assigns to the surface an area which is given by $hkl^2_{\text{Planck}}/2$ times the number of intersections of the loop with the surface.

The action (29) of the area operator is diffeomorphism invariant, because the surface is picked out by the configuration of the field. (One may check that this is also the case when the loop has an intersection at the surface.)

The operator is then well defined acting on states of the form

$$\Phi[\{\alpha, S\}] = <\{\alpha, S\}|\Phi>$$  \hspace{1cm} (30)$$

where $\{\ldots\}$ denotes equivalence classes under diffeomorphisms. On the space of diffeomorphism invariant states we can impose the natural inner product. Again, restricted to the case of nonintersecting loops this must have the form,

$$<\{\alpha, S\}|\{\beta, S'\} = \delta_{\{\alpha, S\}\{\beta, S'\}}$$  \hspace{1cm} (31)$$

where the delta function is a krono(k)cker delta of knot classes. The definition of the inner product on intersecting loops may be obtained by imposing reality conditions. The complete set of reality conditions at the diffeomorphism invariant level is not known, but it is known that an inner product that satisfies (31) is consistent with the requirement that $\hat{A}$ be a hermitian operator.

We may then conclude that the spectrum of $\hat{A}$ is discrete. It consists first of the series integer multiples of $hkl^2_{\text{Planck}}/2$, together with a discrete series of other eigenvalues that come from eigenstates similar to those discussed in [2] in which the loops have intersections at the surfaces.

Finally, may then note that if we require that the diffeomorphism invariant operator yield, when acting on kinematical states of the kind described in

9When the loop $\alpha$ has intersections at the surface $S$ there are additional terms in the action of the area operator[2].
the same areas as the kinematical area operator, we get the condition that,

$$k = \frac{1}{\bar{\hbar}}$$  \hspace{1cm} (32)

With its coupling thus set by \(\hbar\), the antisymmetric tensor gauge field is then in a sense purely a quantum phenomena.

## 4 Adding a boundary

In the next section I am going to make use of the quantum antisymmetric tensor gauge field to construct a quantum reference system for measuring the diffeomorphism invariant states of the gravitational field. For this and other purposes, it is convenient to have states which are labeled by open surfaces in addition to those described in the previous section in which gauge invariance restricts the surfaces to be closed. As I will now describe, there is a very simple way to do this, which is analogous to the Abelian Higgs model and was described first by Kalb and Ramond\[17\]. We will see that by coupling the \(C_{ab}\) field to a vector field in a way that preserves the gauge invariance \(1\) we open up the possibility for our surfaces to have boundaries.

Let us consider then adding to the system described by \(2\) an ordinary abelian gauge field, \(b_a\), with an Abelian gauge group given by

$$\delta b_a = \partial_a \phi$$ \hspace{1cm} (33)

where \(\phi\) is a scalar field. We may couple this field to the Abelian tensor gauge field by supplementing the gauge transformations \(1\) by

$$\delta b_a = \Lambda_a$$ \hspace{1cm} (34)

Thus, we see that this vector field can be set to zero by a gauge transform. A field strength for \(b_a\) that is invariant under both abelian gauge invariances may be defined by

$$F_{ab} = db_{ab} - C_{ab}$$ \hspace{1cm} (35)

To define the dynamics of this coupled system we add to the action the term

$$S_b = \frac{k}{4} \int d^4x \sqrt{g} g^{ab} g^{cd} F_{ac} F_{bd}$$ \hspace{1cm} (36)

We can define a constrained Hamiltonian system by adding \(2\) and \(36\) to the gravitational action. If the conjugate momenta to the \(b_a\) are labeled as
\( \tilde{p}^a \) the diffeomorphism and gauge constraints (8) and (6) are now

\[
D_a = \tilde{W}^* \pi_a + \tilde{p}^c F_{ac}
\]

and

\[
G^a = \partial_b \tilde{\pi}^{ab} + \tilde{p}^a
\]

The Hamiltonian constraint has additional terms, which are given by

\[
\frac{1}{2k} \tilde{p}^a \tilde{p}^b q_{ab} + \frac{k}{2} \det(q) F_{ac} F_{bd} q^{ab} q^{cd}
\]

Note that the new terms are non-polynomial, when expressed in terms of the canonical variables \( \tilde{E}_i^a \), as in the case of the Maxwell and Yang-Mills theories[20]. (As in that case this can be remedied by multiplying through by \( \det(q_{ab}) \).) Finally, there is a new constraint,

\[
g = \partial_c \tilde{p}^c
\]

which generates (33). This, however, is not independent of (38) as

\[
\partial_a G^a = g
\]

As a result, there are now three independent gauge constraints and six each of canonical coordinates and momenta. Thus, the theory now has three degrees of freedom per point. The two additional degrees of freedom are reflected in the fact that in addition to the one gauge invariant field \( \tilde{W}^* \), we now have the gauge invariant two form \( F_{ab} \). Three of these four gauge invariant degrees of freedom are independent, because we have

\[
dF_{abc} = -W_{abc}
\]

As a result, we can associate gauge invariant observables to each open surface \( \mathcal{S} \). This is given by

\[
T[\mathcal{S}] = e^{\frac{i}{\hbar} \int_\mathcal{S} F}
\]

The poisson brackets of this with the canonical momenta \( \tilde{\pi}^{ab} \) and \( p^a \) are given by

\[
\{ \tilde{\pi}^{ab}(x), T[\mathcal{S}] \} = -\frac{1}{k} \int d^2 \delta^{ab}(\sigma) \delta^2(x, \mathcal{S}(\sigma)) T[\mathcal{S}]
\]

\[
\{ p^a(x), T[\mathcal{S}] \} = \frac{1}{k} \int ds \delta^2(x, \partial \mathcal{S}(s)) \partial S(s) T[\mathcal{S}]
\]
The surface representation defined by (23) and (24) can be extended in the obvious way. The arguments of the states are now open surfaces and the obvious combination laws of surfaces hold. In addition to (24), which still holds, there is the operator

\[ \hat{p}^a(x) \Psi[S] = \frac{\hbar}{k} \int ds \delta^x(x, \partial S(s)) \partial S(s) \Psi[S]. \quad (46) \]

Finally, one may check that the gravitational degrees of freedom may be added and the area operator defined, so that all the results of the previous section extend naturally to the surface representation with boundaries.

5 A diffeomorphism invariant loop operator

Given that the matter fields specify a set of surfaces with boundaries, we may imagine constructing a diffeomorphism invariant holonomy operator, analogous to the \( T^0[\alpha] \) operators of the kinematical theory, in which the loop \( \alpha \) is given by the boundary \( \partial S_I \) of the surface determined by the \( I \)’th matter field.

To do this we first need to construct an appropriate diffeomorphism invariant classical observable that will measure the holonomy of \( A^i_a \) on such loops. This can be done by using the fact that \( \hat{p}^a \), by virtue of its being a divergence free vector density, defines a congruence of flows. These flows may be labeled by a two dimensional coordinate \( \sigma^\alpha \), with \( \alpha = 1, 2 \), which may be considered to be scalar fields on \( \Sigma \) that are constant along the flows. The idea is to define a generalization of the trace of the holonomy from a curve to a congruence by taking the infinite product of the traces of the holonomies over each curve in the congruence. This may be done in the following way.

Each divergence free vector density may be written as a two form in terms of the two functions \( \sigma^\alpha \) as \([21]\),

\[ p_{ab}^* = (d\sigma^1 \wedge d\sigma^2)_{ab} \quad (47) \]

where the \( \sigma^\alpha \) are two scalar functions that are constant along the curves of the congruences and so may be taken to label them. The curves of the congruences may be written as \( \gamma_p^a(\sigma, s) \) and satisfy,

\[ \dot{\gamma}_p^a(\sigma, s) = \frac{d\gamma_p^a(\sigma, s)}{ds} = \hat{p}^a \quad (48) \]
We may note that because each $\tilde{p}^a$ is divergence free it is the case that through every point $x$ of $\Sigma$ there passes at most one curve of the congruence. We will denote this curve by $\gamma_p(x)$. We may take as a convention that if no curve of the congruence passes through $x$ we have $\gamma_p(x) = x$, which is just the degenerate curve whose image is just the point $x$. Further, note that we assume that either appropriate boundary conditions have been imposed which fix the gauge at the boundary or we are working in the context of a closed manifold $\Sigma$, for which the curves $\gamma_p(x)$ are closed.

We may then define a classical observable which is the trace of the holonomy of the connection around the curve $\gamma_p(x)$.

$$W[p, A](x) = \text{Tr}U_{\gamma_p(x)}$$ (49)

where $U_{\gamma}$ is the usual path ordered holonomy of $A$ on the curve $\gamma$. We may note that the observable $W[p, A](x)$ transforms as a scalar field.

We may now write a diffeomorphism and gauge invariant observable which is

$$T[p, A] \equiv e^{\int d\sigma_1 d\sigma_2 L N T r U_{\gamma_p, \sigma}}$$ (50)

To show that this is indeed diffeomorphism invariant, as well as to facilitate expressing it as a quantum operator, it is useful to rewrite it in the following way. Let $S_\tilde{p}$ be an arbitrary two surface subject only to the condition that it intersects each curve in the congruence determined by $\tilde{p}^a$ exactly once so that $I[\gamma_p, \sigma, S_\tilde{p}] = 1$. Then we may write

$$T[p, A] = e^{\int d^2 S_\tilde{p}^a \tilde{p}^*_a L N W[p, A]}$$ (51)

The diffeomorphism invariance of this observable is now manifest. To see why this form may be translated to a diffeomorphism invariant quantum operator, we may note that it reduces to a simple form if we plug in for $\tilde{p}^a$ the distributional divergence free vector density

$$\tilde{p}_\alpha^a \equiv \int ds \delta^3(x, \alpha(s)) \dot{\alpha}^a(s).$$ (52)

It is then not hard to show that

$$T[p_\alpha, A] = T r P e^{\int A} = T[\alpha]$$ (53)

We may now define a quantum operator $\hat{T}$ corresponding to (51) by replacing $\tilde{p}^a$ with the corresponding operator (46),

$$\hat{T} = T[\hat{p}, \hat{A}].$$ (54)
As all the operators in its definition commute, there is no ordering issue. It is then straightforward to show that

\[ < \{S, \gamma\} | \hat{T} = < \{S, \gamma \cup \partial S\} \]  \hspace{1cm} (55)

That is, the action of the \( \hat{p} \) operators in (51) is by (46) to turn the operator into a loop operator for the holonomy around the surface. The result is that what the operator \( \hat{T} \) does is to add a loop to the diffeomorphism equivalence class \( \{S, \gamma\} \) which is exactly the boundary of the surface. Thus, we have succeeding in constructing a diffeomorphism invariant loop operator.

I close this section by noting two extensions of this result. First, if one considers the case of Maxwell-Einstein theory, where both fields are treated in the loop representation\(^{10}\), one has an analogous operator, where \( \tilde{p}^a \) should be taken to be just the electric field. In this case, if a diffeomorphism invariant quantum state is given by \( \Psi[\{\alpha, \gamma\}] \) where \( \alpha \) are the abelian loops that represent the electromagnetic field and \( \gamma \) are the loops that represent the gravitational field and \( \hat{T} \) is the operator just described we have

\[ \hat{T} \Psi[\{\alpha, \gamma\}] = \Psi[\{\alpha, \gamma \cup \alpha\}] \]  \hspace{1cm} (56)

That is, the operator puts a loop of the self-dual gravitational connection over each loop of the electromagnetic potential.

Second, all of the considerations of this paper apply to the system which is gotten by taking the \( G \to 0 \) limit of general relativity in the Ashtekar formalism\(^{30}\). This limit yields a chirally asymmetric theory whose phase space consists of all self-dual configurations together with their linearized anti-self-dual perturbations. In this case there are operators \( \tilde{e}_i^a \) and \( A_i^a \) which are cannonically conjugate, but the internal gauge symmetry is the abelian \( U(1)^3 \) reduction of the internal \( SU(2) \) gauge symmetry. One then has an operator analogous to (51), which is just

\[ T[e, A]|_{G\to 0} \equiv e \int_\sigma \tilde{e}_i^a A_i^a \]  \hspace{1cm} (57)

The corresponding quantum operator has the effect of increasing the winding numbers of loops that are already present. It is also interesting to note that in this case, \( T[e, A]|_{G\to 0} \) commutes with the Hamiltonian constraint, so that it is actually a constant of the motion \(^{30}\).

\(^{10}\)The loop representation for Maxwell fields is described in\(^{28, 29}\) and the coupling of Maxwell to gravity in the Ashtekar formalism is described in\(^{20}\).
6 A quantum reference system

I would now like to describe how the preceding results can be used to construct a physical interpretation of a very large class of diffeomorphism invariant states. As I mentioned in the introduction, the idea of using matter fields to provide a dynamically defined coordinate system with respect to which a diffeomorphism invariant interpretation of the gravitational fields can be defined was introduced into quantum gravity by De Witt’s paper in which he applied the Bohr-Rosenfeld analysis to the problem of the measurability of the quantum gravitational field. It is interesting to note that in this paper DeWitt concluded that it was impossible to make measurements in quantum gravity that resolved distances shorter than the Planck scale. The results of the present paper reinforce this result and add to it two important dimensions: first that, at least in one approach, it is impossible to measure things smaller than Planck scales because the fundamental geometrical quantities are quantized in Planck units and second, that it is areas and volumes, and not lengths, whose measurements are so quantized.

Let us consider, for simplicity, that there are many species of antisymmetric-tensor gauge fields, $(C^I_{ab}, b^I_a)$, labeled by the index $I = 1, \ldots, N$, where $N$ can be taken arbitrarily large. This is a harmless assumption as long as we are concerned only with spatially diffeomorphism invariant states. I will come back to this point in the conclusion.

By the straightforward extension of all the results of the previous section, quantum states are now functions of $N$ surfaces, $S_I$, so that

$$\Psi[\{\gamma, S_I\}] = \langle \{\gamma, S_I\} | \Psi \rangle$$

(58)

We may note that the space of diffeomorphism equivalence classes, $\{\gamma, S_I\}$ of loops and $N$ labeled open surfaces is countable. The diffeomorphism invariant state space of quantum gravity coupled to the $N$ antisymmetric tensor gauge fields then has a countable basis given by

$$\Psi_{\{\alpha, S'_I\}}[\{\gamma, S_I\}] = \delta_{\{\alpha, S'_I\}}[\{\gamma, S_I\}$$

(59)

in the case that the loop $\gamma$ is not self-intersecting. In the intersecting case, the form of the basis elements is more complicated because of the presence of the non-trivial relations among intersecting loops which result from the identities satisfied by $SU(2)$ holonomies. For the kinematical case, these

\footnote{For the volume operator see \cite{2}.}

\footnote{Note that each surface may be disconnected.}
relations, and the effect on the characteristic inner product are described in [2]. For the present, diffeomorphism invariant, case they have not yet been completely worked out. However, for the results I will describe below it is sufficient to restrict attention to diffeomorphism equivalence classes involving only non-intersecting loops.

Let us now consider a particular subspace of states of this form which are defined in the following way. Let us consider a particular triangulation of the three manifold, $\Sigma$, labeled $\mathcal{T}$. It consists of some number, $M$, of tetrahedra, labeled $T_\alpha$, where $\alpha = 1, \ldots, M$, that have been joined by identifying faces. Let us call the faces $\mathcal{F}_I$ and let us consider only $\mathcal{T}$ that contain exactly $N$ faces so that $I = 1, \ldots, N$. The idea is then to use this triangulation to construct a quantum coordinate system by identifying each face $\mathcal{F}_I$ with the surface $S_I$ which is an excitation of the $I$'th matter field.

We do this in the following way. For each such triangulation of $\Sigma$ we can consider a subspace of states, which I will denote $S_T$, which consists of all states that have the form

$$\Psi[\{\gamma, S_I\}] = \delta_{\{\mathcal{F}_I\}}\{S_I\}\psi[\{\gamma, S_I\}],$$

where the $\delta_{\{\mathcal{F}_I\}}\{S_I\}$ is, again, a topological Kronocker delta that is equal to one if and only if each surface $S_I$ can be put in correspondence with the face $\mathcal{F}_I$ such that all the topological relations among the surfaces are preserved. Such an arrangement of surfaces can be taken to constitute a quantum reference frame. The states in $S_T$ can then take any value as we vary over the countable set of diffeomorphism equivalence classes in which the loops are knotted and linked with the surfaces in $\mathcal{T}$ and with each other in all possible diffeomorphically inequivalent ways.

If we impose an additional restriction, we can make a correspondence between a basis for $S_T$ and a countable set of peicewise flat three dimensional manifolds based on the simplicial complex $\mathcal{T}$. This restriction is the following: in any three dimensional simplicial complex the number of faces, $F(\mathcal{T})$ is greater than or equal to the number of links, $L(\mathcal{T})$ [2]. For a reason that will be clear in a moment, let us restrict attention to $\mathcal{T}$ such that $F(\mathcal{T}) = L(\mathcal{T})$.

Let us then consider the characteristic basis for $S_T$ given by (59) with $\{S_I\} = \mathcal{T}$. In any such state we may then associate a definite value for the area of each face in $\mathcal{T}$, which is given by the eigenvalue of $\hat{A}^I$.

We may then associate to each set of areas $\mathcal{A}^I$ a piecewise flat manifold, which I will call $\mathcal{M}_{\{\mathcal{A}^I, S_I\}}$, which is composed of flat tetrahedra glued together with the topology of $\mathcal{T}$ such that the areas of the faces are given by
the $\mathcal{A}^I$. We know that generically this can be done, because such piecewise geometries are determined by the edge lengths of the triangulation, and we have assumed that the number of edges in $\mathcal{T}$ is equal to the number of faces. Thus, we may in general invert the $N$ relations between the edge lengths and the areas of the faces to find the edge lengths. However, when doing this, we need to be careful of one point, which the following.

Note that we have chosen the signs while taking the square root in (13) so that all areas are positive. However, if we consider a tetrahedron in $\mathcal{T}$, there is no reason for the areas of the four sides to satisfy the tetrahedral identities, which imply that the sum of the areas of any three sides is greater than the area of the fourth side. This means that we cannot associate to each tetrahedron of $\mathcal{T}$ a metrically flat tetrahedra, if we require that the signature of its metric be positive definite. Instead, we must associate a flat metric of either positive or negative signature, depending on whether or not the classical tetrahedral identities are satisfied. Thus, whether a particular surface of a particular tetrahedra is spacelike, timelike or null depends on how the identities are satisfied in that tetrahedra.

However, each surface bounds two tetrahedra and there is no reason that the signature of the metric may not change as the surface is crossed. Thus, a surface may be, for example, timelike with respect to its imbedding in one of the tetrahedra it bounds, and spacelike in another, as long as the absolute values of the areas are the same. Similarly, when the edge lengths are determined from the areas it is necessary to use the appropriate formula for each tetrahedra, which depends on the signature of the metric in that tetrahedra.

Thus, the result is that the piecewise flat manifold $\mathcal{M}_{\{A^I, S_I\}}$ that is determined from the $N$ areas $A^I$ in general contains flat tetrahedra with different signatures, patched together so that the absolute values of the areas match. Additional conditions, which are precisely the tetrahedral identities, must be satisfied if the geometry of $\mathcal{M}_{\{A^I, S_I\}}$ is to correspond to a positive definite metric on $\Sigma$.

We may note also that the correspondence between the piecewise flat three geometry, $\mathcal{M}_{\{A^I, S_I\}}$, and the diffeomorphism equivalence classes $\{\gamma, S_I\}$ is not one to one. Given $\mathcal{M}_{\{A^I, S_I\}}$ we have fixed only the topology of the surfaces and their intersection numbers with the loops. There remain a countable set of diffeomorphism equivalence classes with these specifications; they are distinguished by the knotting of the loops and their linking with each other.
Of this remaining information, a certain amount may be said to correspond to information about the spatial geometry that cannot be resolved by measurements made using the quantum coordinate system $\mathcal{T}$. We may imagine further refining the quantum reference system by introducing new surfaces by subdividing the tetrahedra in $\mathcal{T}$. If we consider how this may be done while keeping the topological relations of the loops with themselves and with the original set of surfaces fixed, we may see that there is a sense in which we can obtain a more precise measurement of the spatial quantum geometry associated with the topology of the loops, $\gamma$. Of course, there is also a danger that by subdividing too much we may reach a point where additional surfaces tell us nothing more about the quantum geometry; the information about the matter state and the quantum geometry is entangled and cannot be easily separated. In a further work, I hope to return to the problem of how to disentangle the geometrical from the matter information in such measurements.

At the same time, it is clear that there is information in the topology of the loops that is not about the spatial geometry and so cannot be resolved by further refinement of the simplex based on the matter state. This includes information about the routings through intersections. It is clear from this and earlier considerations that the routings through the intersections carry information about the degrees of freedom conjugate to the three geometry.

One can obtain this conjugate information by measuring the operators $T^I \equiv T[\partial S_I]$ defined in the previous section. If one measures all $N$ of these operators, rather than the $N$ areas, one determines the parallel transport of the Ashtekar connection around the edges of each of the faces of the simplex $\mathcal{T}$. Essentially, this means that one determines, instead of the areas of the faces, the left handed curvatures evaluated at the faces. There is also a classical description that can be associated with this measurement, it is described in [24].

I would like to close this section by describing the sense in which the results just described suggest an approach to a measurement theory for quantum gravity\textsuperscript{13}. The idea is to extend the principle enunciated by Bohr that what is observed in quantum mechanics must be described in terms of the whole system which includes a specification of both the atomic system and the measuring apparatus. In the case of quantum gravity, the quantum system is no longer an atom, it is the whole spacetime geometry. As the quantum system is no longer microscopic, but in fact encompasses the whole

\textsuperscript{13}These remarks are enlarged in [24].
universe, we can no longer treat the measuring instrument classically while we treat the spacetime geometry quantum mechanically. Thus, it is necessary that a measuring system that is to be used to determine something about the spacetime geometry must be prepared for the measurement by putting it in some definite quantum state.

In this paper I have described two conjugate sets of measurements, which determine either the areas of or the left handed parallel transport around areas a set of $N$ surfaces. However, the basic features of the measurement process and how we describe it should extend to more general measurements.

Any measurement theory must have two components: preparation and measurement. If we are to use this measuring instrument to probe the quantum geometry, we must prepare the quantum state of the measuring instrument appropriately. As we are interested in describing the theory at a diffeomorphism invariant level, we must give a diffeomorphism invariant specification of the quantum state of the measuring instrument such that, when we act on the combined gravity matter state with the area operators we measure a set of areas which are meaningful.

Now, the requirement of diffeomorphism invariance forbids us from preparing the measuring system in some state and then taking the direct product of that "apparatus state" with a state of the system. Instead, the preparation of the measuring system must be described by restricting the system to an appropriate diffeomorphism invariant subspace of the combined apparatus-gravity system. Thus, what I have done above is to prepare the quantum state of the whole system in a way appropriate to the specification of the measuring instrument by restricting the topological relations among the $N$ surfaces so that they are faces of a given simplex $\mathcal{T}$. This is done by restricting the quantum state of the system, prior to the measurement, to be of the form (60). After we have made this restriction, we can be sure that the results of the $N$ measurements will be a set of $N$ areas that can be ascribed to the faces of the simplex $\mathcal{T}$. Thus, the result of the measurement of the area operators on prepared states of the form (60) is to produce a partial description of the spatial geometry which is given by the piecewise flat manifold $\mathcal{M}_{\gamma,S_T}$.

Now, the $N$ area operators $\hat{A}^I$ commute with each other, but they do not make a complete set of commuting observables. This is because to each such piecewise flat manifold, which encodes the results of the $N$ observables, there are a countably infinite number of diffeomorphism invariant states in the subspace $\mathcal{S}_\mathcal{T}$ which are degenerate as far as the values of the $\hat{A}^I$ are
concerned.

We would then like to ask whether we can add operators to the $\hat{A}^I$ to make a complete set of commuting operators. We certainly can extend the set, by subdividing some or all of the tetrahedra in $T$ to produce a simplicial complex with more surfaces. This would correspond to introducing more matter fields, which would make it possible to specify more surfaces whose area is to be measured and by so doing make a more refined measurement of the quantum geometry. But, notice that there is a natural limit to how much one can refine one’s observations of the quantum geometry because one can never measure the area of any surface to be less than one-half Planck area.

Further, note that no matter how large $N$ is, and no matter how the $N$ surfaces are arranged topologically, there are always a countably infinite set of states associated to each measurement of the $\hat{A}^I$'s. In this sense, it seems that one can never construct a physical measuring system that suffices to extract all the information out of the quantum gravitational field. This is, of course, just a reflection of the fact that the quantum gravitational field has an infinite number of degrees of freedom, while any physical measuring instrument can only record a finite amount of information. However, note that we have come to an expression of this fact in a way that is completely diffeomorphism invariant. In particular, we have a characterization of a field with an infinite number of degrees of freedom in which we do not say how many degrees of freedom are associated to each "point." This is very good, as we know that no diffeomorphism invariant meaning can be given to a point of space or spacetime.

Of course, there remains one difficulty with carrying out this type of interpretation, which is that the problem of time in quantum gravity must be resolved so that we know how to speak of the time of the observation. In 

I show that the resolution of the problem of time may be carried out using the ideas of spacetime diffeomorphism invariant observables of Rovelli

To do this one constructs the physical time dependent operators that correspond to the $\hat{A}^I$ and $T^I$. These depend on a time parameter $\tau$ which is the reading of a physical clock built into the measuring instrument. The $N$ operators $\hat{A}^I(\tau)$ will then commute with the Hamiltonian constraint, and so act on physical states and their eigenvalues return the values of the areas of the $N$ surfaces when the physical clock reads $\tau$. Given this, there seems to

\footnote{Note that such observables have been described in the 2 + 1 case by Carlip, in the Gowdy model by Husain and in the Bianchi-I model by Tate.}
be no obstacle to the observer employing the projection postulate and saying that the quantum state of the matter plus gravity system is projected into a subspace of the physical Hilbert space spanned by the appropriate eigenstates of the $A^I(\tau)$ and that this is something that occurs just after the measurement is made, in spite of the fact that she and her apparatus are living inside the quantum system under study.

Thus, despite various assertions to the contrary, there seems to be no difficulty in applying a Copenhagen-like description of the measuring process to the case of quantum cosmology in spite of the fact that the measuring instrument is inside the universe. As long as we can prepare the measuring instrument in such a way that the quantum state of the whole matter-gravity state space is inside a subspace of the state space associated with a particular specification of the measuring instrument one can assign meaning to a set of commuting observations. The implications of this are discussed further in [24].

Finally, we may note that one finds that the $A^I(\tau = 0)$ are equal to the area operators constructed in this paper [24], so that the quantization of areas becomes a physical prediction based on the spectra of a set of physical operators of the theory.

7 Conclusions

I would like to close by making a number of comments about the implications of the results obtained here.

1) We see that in each case when we have succeeded in constructing the definition of an operator in such a way that when it is diffeomorphism invariant it is automatically finite. This is in accord with the general arguments that all spatially invariant diffeomorphism invariant operators must be finite that were given previously in [2, 10]. This suggests strongly that the problem of constructing a finite theory of quantum gravity can be to a great extent resolved at the diffeomorphism invariant level. The reason is that once one imposes spatial diffeomorphism invariance there is no longer any physical meaning that can be given to a point in space. As a result, although the theory still has an infinite number of degrees of freedom, in the sense discussed in section 6, it is no longer meaningful to speak of the field as having a certain number of degrees of freedom per point. Instead, there seems to be a natural limitation to how many degrees of freedom there can be inside of a Planck volume due to the discreteness of the spectra of the geometrical
operators that measure area and volume\footnote{4}. This in turn suggests that the problem of finiteness has little to do with the dynamics of the theory or the choice of matter couplings, which are coded in the Hamiltonian constraint.

2) We also see that the conclusions of previous analyses of the measurement problem in quantum gravity by DeWitt and others are confirmed. The key conclusion of these works was that it should be impossible to meaningfully resolve distance scales shorter than the Planck scale. We see that this is the case here, because the possible values of physical areas that can be gotten from a diffeomorphism invariant measurement procedure are quantized in units of the Planck area.

We also see that any particular configuration of the matter fields that are used to define the reference system can only be used to resolve a certain finite amount of information about the space time geometry. This is a consequence of using the quantum theory to describe the reference system as well as the gravitational field. This is certainly consistent with the general observation that diffeomorphism invariant measurements are about relations between the gravitational field and the measuring instruments. If we want a measurement system which is able to resolve \(N\) different spatial distances, it had better come equipped with \(N\) distinguishable components.

3) We see that a large class of doubts about the physical applicability of the description of quantum states of geometry by means of the loop representation can now be put to rest. Note that any spacial geometry in which the components of the curvatures are small in Planck units can be approximated by a Regge manifold in which the areas of the faces are integral multiples of half the Planck area. As a result we see from the correspondence between Regge manifolds and quantum states arrived at in section 6 that any such spatial geometry can be associated with a diffeomorphism invariant quantum state in the loop representation. This allows us to extend the discussion of the classical limit of quantum gravity developed in \cite{1} to the diffeomorphism invariant level.

4) It would be very interesting to be able to characterize the quantum geometry associated with diffeomorphism invariant states of the pure gravitational field. The results obtained with matter fields as reference systems suggest that there should be a basis of states which are diagonal in some set of diffeomorphism invariant operators which measure the three geometry and that this basis contains the characteristic states of non-intersecting knots and links. The problem is to construct an appropriate set of diffeomorphism invariant classical observables which are functions only of the gravitational field and translate them into quantum operators while preserv-
ing the diffeomorphism invariance. We already know how to construct a few such operators, which measure the areas of extremal surfaces and, in the case that it is spatially compact, the volume of the universe\footnote{2}.

One approach to the construction of such observables could be by mimicking the results of this paper by constructing observables that measure the areas of surfaces on the faces of a given simplex, and asking that all the areas are extremized as the whole simplex is moved around in the geometry. Constructions along this line are presently under study.

5) Given the present results, a new approach to the construction of the full dynamical theory becomes possible. This is to impose an inner product consistent with the reality conditions at the diffeomorphism invariant level and then project the Hamiltonian constraint into the resulting Hilbert space of diffeomorphism invariant states. The physical state space would then be found as a subspace of the space of diffeomorphism invariant states. constraint The main difficulties facing such an approach are the problem of expressing both the reality conditions and the Hamiltonian constraint in diffeomorphism invariant forms.

6) As I commented above, the form of the Hamiltonian constraint (7) for gravity coupled to the simple, massless antisymmetric tensor gauge field is particularly simple. It would be very interesting to see if solutions to the Hamiltonian constraint for the coupled gravity matter system could be obtained, if not exactly, in the context of some perturbative expansion. It is interesting to note that exact solutions can be obtained in the strong coupling limit in which $k$ is taken to zero (because $k$ is inverse to what would usually be written as the coupling constant). In this limit, only the second term of (7) survives. It is easy to show, using a regularization of the type introduced in \footnote{3} that one then has a class of solutions of the form $\Psi([S,\gamma])$ in which the loops never intersect the surfaces or in which the loops always lie in the surfaces. It would be very interesting to then develop a strong coupling expansion to construct approximate expressions for solutions for finite $k$. It would also be very interesting to see if one could recover from the semiclassical limit of the gravity matter system the solutions to the Schroedinger equation described in \footnote{19}.

7) It is interesting to note that surfaces play an interesting role in two mathematical developments connected to the Ashtekar variables and the loop representation. In \footnote{33} Baez extends the loop representation to the case in which the spatial manifold has a boundary, and shows that in this case there is an interesting algebra of operators that acts on the diffeomorphism invariant states. In \footnote{34}, Crane proposes a new interpretative scheme for
quantum gravity in which Hilbert spaces of states coming from conformal field theories are defined on surfaces which are identified with observers and measuring instruments. Both proposals need to be completed by the construction of explicit diffeomorphism invariant observables associated to surfaces, and it would be interesting to see if the operators described here can thus play a role in these proposals.

8) Finally, I would like to address the issue of the use of $N$ separate matter fields to label the operators that measure the areas of the $N$ surfaces. This is clearly necessitated by the idealization in which I use the values of a field to specify a set of physical surfaces in a very simple way. The point is that there must be a physical way to distinguish the $N$ different surfaces in terms of the configurations of the matter fields. In real life, in which measuring instruments of arbitrary complexity are constructed from a small number of fields there is no difficulty with specifying quantum states associated to some degree of precision with an arbitrary number and configuration of surfaces. In a realistic situation the configuration is complex enough to allow an intrinsic labeling of the different surfaces. Of course, another issue also arises when we construct the surfaces out of realistic physical fields, which is that there will be restrictions to the accuracy of the measurements of the areas due to the fact that matter is made out of atoms. It is not, however, impossible that such limitations can be overcome by a clever use of matter and other fields to specify very small surfaces. What the present results suggest, however, is that no matter how clever we are with the design of our measuring instruments, it will be impossible to measure the area of any physical surface to be less than half the Planck area.

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15 This accords with the observation stressed by Barbour that real physical observables are well defined because the world is sufficiently complex to allow the events of spacetime to be distinguished by the values of the physical fields. [35]
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