Cosmological Einstein-Skyrme solutions with non-vanishing topological charge

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Time-dependent analytic solutions of the Einstein-Skyrme system –gravitating Skyrmions–, with topological charge one are analyzed in detail. In particular, the question of whether these Skyrmions reach a spherically symmetric configuration for $t \to +\infty$ is discussed. It is shown that there is a static, spherically symmetric solution described by the Ermakov-Pinney system, which is fully integrable by algebraic methods. For $\Lambda > 0$ this spherically symmetric solution is found to be in a “neutral equilibrium” under small deformations, in the sense that under a small squashing it would neither blow up nor disappear after a long time, but it would remain finite forever (plastic deformation). Thus, in a sense, the coupling with Einstein gravity spontaneously breaks the spherical symmetry of the solution. However, in spite of the lack of isotropy, for $t \to \infty$ (and $\Lambda > 0$) the space time is locally flat and the anisotropy of the Skyrmion only reflects the squashing of spacetime.

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2. THE ACTION INTEGRAL

We are interested in self-gravitating Skyrmions for the SU(2) group described by the action

\[ I[g, U] = \int d^4x \sqrt{-g} \left( \frac{R - 2\Lambda}{2\kappa} + \frac{K}{4} \text{Tr}[A^\mu A_\mu + \frac{\lambda}{8} F_{\mu\nu} F^{\mu\nu}] \right). \]  

Here \( A_\mu \) is a shorthand for the Maurer-Cartan form \( U^{-1} \nabla_\mu U \), with \( U \in SU(2) \) and \( F_{\mu\nu} = [A_\mu, A_\nu] \); \( A_\mu = A^j_\mu t_j \) where \( t_j = -i\sigma_j \) are the SU(2) generators, and \( \sigma_j \) are the Pauli matrices. In our conventions \( c = \hbar = 1 \), the spacetime signature is \((-\ldots, +, +, +)\) and Greek indices run over spacetime. Moreover, \( R \) is the Ricci scalar, \( \Lambda \) is the cosmological constant and \( \kappa \) is the gravitational constant. Here \( K \) and \( \lambda \) are (positive) coupling constants, related to the experimentally determined phenomenological parameters \( F_\pi \) and \( e \) through[11]

\[ K = \frac{1}{4} F_\pi^2, \quad K\lambda = \frac{1}{e^2}, \]
\[ F_\pi = 186 \text{ MeV}, \quad e = 5.45. \]

The Skyrme equation, obtained by varying (1) with respect to \( U \), together with Einstein’s equations are

\[ \nabla^\mu A_\mu + \frac{\lambda}{4} \nabla^\mu [A^\nu, F_{\mu\nu}] = 0, \]
\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \]

where \( G_{\mu\nu} \) is the Einstein tensor and the energy-momentum tensor for the Skyrme field is

\[ T_{\mu\nu} = -\frac{K}{2} \text{Tr} \left[ A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} A^\alpha A_\alpha + \frac{\lambda}{4} \left( g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right]. \]

2.1. Static self-gravitating Skyrmion

The spacetime geometry for the static solutions of the coupled system (2) is the product \( R \times S^3 \),

\[ ds^2 = -dt^2 + \frac{\rho_0^2}{4} \left[(d\gamma + \cos \theta d\varphi)^2 + d\theta^2 + \sin^2 \theta d\varphi^2\right], \]

where \( 0 \leq \gamma \leq 2\pi, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \) are the coordinates on the 3-sphere of constant radius \( \rho_0 \). Following[27, 30, 32, 34], we adopt the standard parametrization of the SU(2)-valued scalar \( U(x^\mu) \) as

\[ U^\pm(x^\mu) = Y^0(x^\mu) I \pm Y^i(x^\mu) t_i, \quad (Y^0)^2 + Y^i Y_i = 1, \]
where $I$ is the $2 \times 2$ identity matrix. The unit vector $Y^A = (Y^0, Y^i)$ defines the embedded three sphere, which is naturally given by

$$
\begin{align*}
Y^0 &= \cos \alpha, \\
n^1 &= \sin \Theta \cos \Phi, \\
n^2 &= \sin \Theta \sin \Phi, \\
n^3 &= \cos \Theta.
\end{align*}
$$

With this information one can solve (2a) for $\alpha$, $\Theta$ and $\Phi$ as functions of $\gamma$, $\theta$ and $\varphi$. It can be directly checked that the configuration

$$
\Phi = \frac{\gamma + \varphi}{2}, \quad \tan \Theta = \frac{\cot \left(\frac{\theta}{2}\right)}{\cos \left(\frac{\varphi - \gamma}{2}\right)}, \quad \tan \alpha = \frac{\sqrt{1 + \tan^2 \Theta}}{\tan \left(\frac{\varphi - \gamma}{2}\right)},
$$

identically satisfies the Skyrme equations (2a) in the background metric (1). This was already noted long ago by Manton and Ruback [37] (see also [38]). Those authors, however, did not produce a consistent solution taking into account the back-reaction of the Skyrmion on the geometry. In other words, they did not attempt to solve the Einstein equations (2b) with the stress-energy tensor (3) generated by a Skyrmion $U$ of the form (5), (6), (7). Plugging (7) into (6) and (5), the only nonvanishing components of $T_{\mu}^\nu$ are found to be

$$
T^t_t = -\frac{3K(\lambda + \rho_0^2)}{2\rho_0^4}, \quad T^\gamma_\gamma = T^\theta_\theta = T^\varphi_\varphi = \frac{K(\lambda - \rho_0^2)}{2\rho_0^4}.
$$

It can be observed that although the solution $U$ explicitly depends on the angles $\gamma$, $\theta$ and $\varphi$, the energy-momentum tensor does not, which means that the back reaction should not upset the isometries of the background geometry [41]. Solving Einstein’s equations with the energy-momentum tensor (8) algebraically fixes the radius of the three-dimensional sphere and the cosmological constant in terms of the remaining parameters in the action,

$$
\rho_0^2 = \frac{2\lambda \kappa K}{2 - \kappa K}, \quad \Lambda = \frac{3(2 - \kappa K)^2}{8\lambda \kappa K}.
$$

Hence, the metric (1) together with the static Skyrmion (5), (6) and (7) define a self-consistent solution of the full Einstein-Skyrme system (2) provided the conditions (9) are satisfied. Note that this requires $\lambda$, $(2 - \kappa K)$ and $\lambda$ to have the same sign, which we take tentatively positive. This solution is the self-gravitating generalization of the Skyrmions in [37]. It is useful to stress here that the above constraint is only needed if one wants a static solution with $a(t) = 1$. On the other hand, all rest of the analysis of the present paper will hold for generic values of the coupling constants and cosmological constant.

Our result can also be seen as a generalization of the hedgehog ansatz discussed in [27], that allows for the construction of exact multi-Skyrmion configurations composed by elementary spherically symmetric Skyrmions with non-trivial winding number in four-dimensions [29, 31].

On any three-dimensional constant time hypersurface, the winding number for the configuration is

$$
w = \frac{-1}{24\pi^2} \int \text{Tr}[\epsilon^{ijk} A_i A_j A_k] = +1,
$$

which implies that this Skyrmion cannot be continuously deformed to the trivial $SU(2)$ vacuum, $U = 1$ [3].

### 2.2. Bianchi-IX Self-gravitating Skyrmions

Remarkably, the above static Skyrmion can be promoted to a time-dependent solution in which the space-time metric is of the Bianchi type-IX described by the metric

$$
ds^2 = -dt^2 + \frac{\rho(t)^2}{4} \left[ a^2(t) (d\gamma + \cos \theta d\varphi)^2 + d\theta^2 + \sin^2 \theta d\varphi^2 \right],
$$

where $\rho(t)$ is a global scaling factor and $a(t)$ is a squashing coefficient. As can be directly verified a Skyrmion of the same form as before (5), with $Y^0$ and $Y^i$ still given by (6) still identically satisfies the Skyrme field equations in a time-dependent background geometry of the form (11). The technical reason why this happens is that the scale factor $\rho$ and the squashing parameter $a$ depend only on time, while the Skyrme ansatz depends only on the spatial coordinates. This is actually consistent with an ansatz for the Skyrmion in which the full Skyrme system is consistently
reduced to a single scalar equation for the profile [27, 28]. The Skyrmion in this case still has baryon charge +1.

3. THE TIME-DEPENDENT SYSTEM

The full Einstein-Skyrme field equations [2] with the metric [11], reduce to

\[ 2a\rho^2(2\rho \dot{a} + 3a\dot{\rho} \dot{\rho} - 2a^2 \rho^2 (\Lambda \rho^2 + a^2 - 4) - \kappa K[(2\rho^2 + \lambda)a^2 + \rho^2 + 2\lambda] = 0 , \]

\[ 2a^2 \rho^2(2\dot{\rho} \dot{\rho} + \dot{\rho}^2) - 2a^2 \rho^2 (\Lambda \rho^2 + a^2 - 4) - \kappa K[(2\rho^2 + \lambda)a^2 - \rho^2 - 2\lambda] = 0 , \]

\[ a\rho^3(\dot{\rho} \dot{a} + 3\dot{\rho} \dot{\rho}) + (a^2 - 1)[\kappa K(\lambda + \rho^2) + 4a^2 \rho^2] = 0 . \]

The function \( a(t) \) describes the deviations from spherical symmetry. For \( a(t) = \pm 1 \) the spatial sections are spheres and so the solution has full spherical symmetry (which is expected for a gravitating soliton of charge 1 which, on a flat background, has spherical symmetry). Thus, an interesting question would be whether or not the solutions approach the “most symmetric configuration”. Alternatively, when this condition is violated spherical symmetry is “spontaneously” broken. The flat Skyrmion of charge \( \pm 1 \) means that the solutions approach the “most symmetric configuration”. Alternatively, when this condition (see, for instance, [5]), whereas if Eq. (13) does not hold, the gravitating Skyrmion is not spherically symmetric.

As seen in [34], assuming \( a(t) = \pm 1 \) turns (12a), (12b) and (12c) into a consistent one-dimensional dynamical system for \( \rho(t) \), which can be solved explicitly, as discussed in the following sections. A preliminary analysis of the interesting properties of this system for generic \( a(t) \) was presented in [34]. In the present paper, we will generalize the analysis of [34] clarifying the issue of the final state of the dynamical system. In particular, we address the question of whether (13) holds and in which sense this is a stable condition. The integrability properties of the reduced dynamical system for \( a(t) = \pm 1 \) will also be analyzed.

3.1. Minisuperspace Lagrangian and Hamiltonian

It is convenient to write the dynamical system made of Eqs. (12a), (12b) and (12c) using Hamiltonian formalism. The first step is to observe that Eqs. (12a), (12b) follow from the variational principle of the following Lagrange function,

\[ L (x^k, \dot{x}^k) = L_{GR} + V_A + V_{Sk}, \]

where \( L_{GR} \) is the Lagrangian of general relativity (GR) in the mini-superspace geometries of the form [11], i.e.

\[ L_{GR}(a, \dot{a}, \rho, \dot{\rho}) = (2\rho^2 \dot{a} \dot{\rho} + 3a \rho \dot{\rho}^2) + (a^2 - 4) a \rho. \]

It can be checked that varying \( L \) with respect to \( a \) and \( \rho \) yields (12b) (12c), where \( V_A \) and \( V_{Sk} \) are the potential terms which correspond to the cosmological constant and to the Skyrmion field,

\[ V_A(a, \rho) = \Lambda \rho^3, \quad V_{Sk}(a, \rho) = \frac{\kappa K((2\rho^2 + \lambda)a^2 + \rho^2 + 2\lambda)}{2a\rho}. \]

Since Lagrangian (14) describes an autonomous system invariant under time translations generated by \( \partial_t \), Noether’s theorem implies energy conservation, which turns out to be the left hand side of (12a). The fact that the energy vanishes reflects the fact that in General Relativity it is constrained to be zero by invariance under time reparametrizations, \( t \to \tau(t) \). In a generic time choice the metric (11) is

\[ ds^2 = -N^2(\tau) d\tau^2 + \frac{\rho^2(\tau)}{4} \left[ a^2(\tau)(d\gamma + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right], \]

where \( N(\tau) d\tau = dt \). In this parametrization the Lagrangian is

\[ \tilde{L}(N, a, \dot{a}, \rho, \dot{\rho}) = \frac{1}{N} \left( 2\rho^2 \dot{a} \dot{\rho} + 3a \rho \dot{\rho}^2 \right) - N(a^2 - 4) a \rho + NV_A + NV_{Sk}. \]
Here it is manifest that the only dynamical degrees of freedom of the system are metric coefficients \( \rho \) and \( a \) and the Skyrminon does not bring in new dynamical variables. Then, varying with respect to the variables \( N, a \) and \( \rho \) yields equations \[12a, 12b\] and \[12c\], respectively. The corresponding Hamiltonian for this system is
\[
\mathcal{H} \equiv N \left[ \frac{p_\rho p_\rho}{2\rho^2} - \frac{3a}{4\rho^2} p_\rho^2 - (a^2 - 4) a \rho - V_\Lambda - V_{Sk} \right],
\]
and the Legendre transformation from \( a, \rho, N \) to \( p_a, p_\rho, \pi_N \) reads
\[
p_a = \frac{2p_\rho^2}{N} \dot{\rho} , \quad p_\rho = \frac{2\rho^2}{N} \dot{a} + \frac{3a}{\rho} \rho , \quad \pi_N = 0.
\]

### 3.2. Isotropic space-time and the Ermakov-Pinney equation

For the spherically symmetric space-time \( a^2 = 1 \), \[12a\] is identically satisfied, while \[12b\] and \[12c\] reduce to the following system \[34\]:
\[
\ddot{\rho} = \frac{\Lambda}{3} \rho^2 + \frac{\lambda \kappa K}{2\rho^2} + \frac{\kappa K - 2}{2}, \quad \ddot{a} = \frac{\Lambda}{3} \rho - \frac{\lambda \kappa K}{2\rho^2}.
\]

As noted before, \[21\] is the vanishing energy constraint, while \[22\] is a particular case of the well-known Ermakov-Pinney (EP) equation\[39, 40\], which is also found in various physical systems (see for instance \[41, 42\]). One of its features is that it is invariant under a larger than expected symmetry, \( SL(2, R) \) in this case. The representation of the symmetry algebra depends on whether \( \Lambda \lesssim 0 \). Specifically, the generators of the \( SL(3, R) \) Lie algebra are: the autonomous symmetry \( \Gamma^1 = \partial_t \), and the two generators \( \Gamma^2 \) and \( \Gamma^3 \) with representations
\[
\Gamma^2_{(\lambda > 0)} = \frac{2}{\omega} \sinh(\omega t) \partial_t + \cosh(\omega t) \rho \partial_\rho, \quad \Gamma^3_{(\lambda > 0)} = \frac{2}{\omega} \cosh(\omega t) \partial_t + \sinh(\omega t) \rho \partial_\rho,
\]
for positive cosmological constant, where \( \omega^2 := 4|\Lambda|/3 \), or
\[
\Gamma^2_{(\lambda < 0)} = \frac{2}{\omega} \sin(\omega t) \partial_t + \cos(\omega t) \rho \partial_\rho, \quad \Gamma^3_{(\lambda < 0)} = \frac{2}{\omega} \cos(\omega t) \partial_t - \sin(\omega t) \rho \partial_\rho,
\]
for negative cosmological constant, while when \( \Lambda = 0 \) the generators take the simple form
\[
\Gamma^2_{(\Lambda = 0)} = 2t \partial_t + \rho \partial_\rho, \quad \Gamma^3_{(\Lambda = 0)} = t^2 \partial_t + t \rho \partial_\rho.
\]

The solution of the EP equation \[22\] can be expressed using a generic solution of the associated linear equation \( \ddot{\rho} - \frac{4}{3} \rho \) \[10, 41\], as
\[
\omega^2 \rho^2 = -(K - 2) + (K - 2 + \rho_0^2 \omega^2) \cosh(\omega t) \pm \omega \sqrt{2\rho_0^2(K - 2) + 2\kappa K \lambda + \rho_0^4 \omega^2 \sinh \omega t} \quad (\Lambda > 0),
\]
for \( \Lambda > 0 \), and
\[
\omega^2 \rho^2 = K - 2 + (-(K - 2) + \rho_0^2 \omega^2) \cos(\omega t) \pm \omega \sqrt{2\rho_0^2(K - 2) + 2\kappa K \lambda - \rho_0^4 \omega^2 \sin \omega t} \quad (\Lambda < 0),
\]
for \( \Lambda < 0 \), where \( \rho_0 = \rho(0) \) and the second integration constant has been eliminated by the constraint equation \[21\]. Furthermore, for \( \Lambda = 0 \) the solution is a power law,
\[
\rho^2 = \rho_1 (t - t_0)^2 + \rho_0
\]

\[1\] The EP equation has the form \( \ddot{u} + \omega^2 u + bu^{-3} = 0 \) and admits exact solutions \( u = F(y_1, y_2) \) where \( y_1, y_2 \) are the independent solutions of the associated problem \( \ddot{y} + \omega^2 y = 0 \) \[43\].
where \( \rho_0 = \frac{\lambda \kappa K}{2} \) and \( \rho_1 = \kappa K \cdot \frac{2}{a^2} \).

We note that the functional form of the exact solutions are related with the representation of the corresponding admitted \( \mathfrak{sl}(2, \mathbb{R}) \) Lie algebra. From the exact solutions in which \( a^2(t) = 1 \) we observe that for positive cosmological constant the space-time \((11)\) has a de Sitter evolution, while for negative cosmological constant the scale factor \( \rho(t) \) is periodic with frequency \( \omega \). Finally for zero cosmological constant and for \( t \to \infty \) the space-time \((11)\) describes the Milne universe.

3.3. Einstein static universe

In order to examine the stability properties of the static Einstein universe around the isotropic solutions \((26)\) and \((27)\), let us consider the critical points for the field equations \((12a)-(12c)\). The critical points of the Hamiltonian \((19)\) are given by the conditions

\[
\frac{\partial V_{\text{eff}}}{\partial a} = 0 \quad \text{and} \quad \frac{\partial V_{\text{eff}}}{\partial \rho} = 0, \tag{29}
\]

where \( V_{\text{eff}} = -\left(a^2 - 4\right)a\rho - V_\Lambda - V_{Sk} \). Taking into account the additional the constraint \((12a)\) –which reduces to \( V_{\text{eff}} = 0 \), the critical points in the \((\rho, a)\)-plane are identified as

\[
P_\pm : \rho_c = \left[\frac{3(2 - \kappa K)}{4\Lambda}\right]^{1/2} = \left[\frac{3\lambda \kappa K \Lambda}{2}\right]^{1/4}, \quad a_c = \pm 1. \tag{30}
\]

Observe that for \( \kappa K > 2 \) the critical points \( P_\pm \) exist provided both \( \Lambda \) and \( \lambda \) are negative, while the opposite happens if \( \kappa K < 2 \) \((\lambda > 0, \Lambda > 0)\). Last but not least, for zero cosmological constant \( P_\pm \) exist if and only if \( \kappa K = 2 \) and \( \lambda = 0 = \Lambda \).

Finally, we note that these critical points in momentum space are located at \((p_a, p_\rho) = (0, 0)\) and therefore they correspond to static configurations. It should be noted that the critical points \( P_\pm \) are exact solutions of the field equations and describe isotropic Einstein static spacetimes \([44, 45]\) and therefore perturbing around them is a meaningful test for the stability of the solutions. In the next section we examine the stability of the critical points \( P_\pm \) in the linearized approximation of the time-dependent field equations.

4. Stability of the spherically symmetric Skyrmion

Let us now study the evolution of an infinitesimal perturbation around the classical solution near the critical point for \( a = 1 \),

\[
a := 1 + u(t), \quad \rho := \rho_E + v(t), \tag{31}
\]

where \( \rho_E \) stands for the exact solution of the EP equation \((22)\), and \( u \) and \( v \) are the small perturbations. Substituting this into \((12)\) and keeping up to first order in \( u \) and \( v \), one finds (from now on we drop the label \( E \) from the exact

\[\text{For } \kappa K < 0 \text{ and } \lambda < 0 \text{ there would be an additional possible critical point with } a_0 \neq 0 \text{ at, } \hat{P}_0 : \hat{\rho}_c = \left[\frac{8 - \kappa K}{2\Lambda}\right]^{1/2} = \left[-\lambda a_0^2 + 4\right]^{1/2} a_0^{1/2} \] with \( a_c = a_0 \neq 0 \) and \( \kappa K = -2a_0^2(a_0^2 + 4) \). The critical point \( \hat{P}_0 \) can be neglected in the standard situations where \( \kappa K \geq 0 \).
specific initial conditions. In that case, the solution is straightforward to see that in general $u \rightarrow \infty$ in the vicinity of 1.

Near a sphere. The main reason is that, when $\Lambda < 0$, the value depending on the initial conditions. Although the solution is not strictly stable around $a = 1$, for later times $a^2(t)$ increases as we will see in the next Section. The peculiar neutral equilibrium feature of the present system means that if the initial data are close to $a^2 = 1$, for later times $a^2(t)$ approaches $a_0^2$ in the vicinity of 1.

A numerical simulation for the case $\Lambda > 0$ is shown in figure 1. In this case $\rho(t)$ is periodic and may vanish for specific initial conditions. In that case, the solution $u(t)$ from Eq. (32a) reaches a singularity for which $\dot{u}(t) \rightarrow \infty$. It is straightforward to see that in general $u(t)$ is not a decreasing function which means that the EP solution is unstable.

$$0 = \ddot{u} + 3 \frac{\dot{\rho}}{\rho} \dot{u} + 2 \left[ K \kappa + 4 + K \kappa \rho^{-2} \right] \rho^{-2} u, \quad (32a)$$

$$0 = \ddot{v} + \rho^{-2} \left[ 4 + 4(\dot{\rho})^2 - 4K + 12 \dot{\rho} - 8\Lambda \rho^2 \right] v$$

$$\quad + \frac{1}{2} \rho^{-3} \left[ \kappa K - 2\Lambda \rho^4 - \kappa K \lambda - 4 \rho + 2(\dot{\rho})^2 - 2K \kappa \rho^2 + 4^3 \rho \right] u$$

$$\quad + \frac{1}{2} \left[ 1 + (\dot{\rho})^2 - \kappa K + 2\dot{\rho} - \Lambda \rho^2 \right] \rho^{-1} + \frac{\kappa K}{4} (-\lambda + 2\Lambda + 1) \rho^{-3}, \quad (32b)$$

$$0 = \frac{\dot{\rho}}{\rho} u + \left[ \frac{2}{\rho^2} + 3 \left( \frac{\dot{\rho}}{\rho} \right) - \frac{\kappa K \lambda}{2} - \frac{\kappa K}{\rho^5} - \Lambda \right] u$$

$$\quad + 3 \frac{\dot{\rho}}{\rho} v + \left[ 3 \frac{\dot{\rho}}{\rho^2} + 3 \left( \frac{\dot{\rho}}{\rho} \right) - \frac{3 \kappa K}{\rho^2} - 2\Lambda \right] v +$$

$$\quad + \left[ \frac{3}{2 \rho^2} + 3 \left( \frac{\dot{\rho}}{\rho} \right) - \frac{3 \kappa K}{4 \rho^2} - \frac{\Lambda}{2} \right] - \kappa K \lambda \rho \] \rho^{-1} (2\Lambda + \lambda). \quad (32c)$$

Since the solution $\rho(t)$ for $\Lambda < 0$ is explicitly known, Eq. (32a) is an ODE for $u(t)$ that can be directly solved. If $\Lambda > 0$, $\Lambda > 0$ implies $\rho \sim \rho_0 \rho^{(\omega/2)t}$ for $t \rightarrow \infty$. In this limit, Eq. (32a) reduces to $\ddot{u} + (3\omega/2) \dot{u} = 0$, whose solution is

$$u(t) = u_0 e^{-3\omega t} + c, \quad (33)$$

where $u_0$ and $c$ are arbitrary constants fixed by the initial conditions of the perturbations. This means that for $t \rightarrow \infty$, $a$ can approach any constant value $1 + c$ and there is nothing special about $a = 1$ or $a \neq 1$. In fact, Eq. (32a) has the form of a damped oscillator driven by an effective harmonic potential $u^2 [K \kappa + 4 + K \kappa \rho^{-2}] \rho^{-2}$, which vanishes exponentially for $t \rightarrow \infty$, as well as all of its derivatives. This is a case of the so-called “neutral equilibrium” [40].

Having found $u$, Eq. (32b) can now be solved for $v$. Substituting the asymptotic expression for $\rho$, (32b) takes the form

$$0 = \ddot{v} + \frac{\omega}{2} \dot{v} - 2\omega^2 v, \quad (34)$$

whose solution is

$$v(t) = v_0 e^{\omega t} \quad (35)$$

with $m = (-1 \pm \sqrt{3}) \omega/4$. This means that $v(t)$ either vanishes or blows up for large $t$. Which of the two branches actually occurs is decided by the constraint equation (32c). This last equations is identically satisfied by the exponentially decaying perturbation and is grossly violated by the unstable branch. It is therefore verified that under a small perturbation around the critical point $\{ \rho = \rho_0; a = 1 \}$ the solution settles to $\{ \rho = \rho_E; a = 1 + c \}$.
4.1. Asymptotically isotropic space-time

Let us now examine the isotropization of spacetime for large $t$. According to [47], if a solution of the field equations \([12a]-[12c]\), in the limit $t \to +\infty$, satisfies the conditions: (a) the global scale factor $\rho(t)$ is going to infinity, i.e. $\rho(t) \to +\infty$, (b) the anisotropic parameter $a(t)$ becomes constant, $a(t) \to a_0$, (c) the weak energy condition is not violated $T_{00} > 0$, while it holds $T_{0i}/T_{00} \to 0$ and (d) the ratio of the shear $\sigma$ with the expansion rate $\theta$
FIG. 3: Qualitative behaviour of the general solution perturbed around the stable solution $a = 1$ which is given by the Ermakov-Pinney equation. The plots are for the function $\sigma(t)/\theta(t)$ which follow from the solution of the field equations with various values of the free parameters where $\Lambda > 0$. The initial conditions are that of fig 1.

vanishes, i.e. $\sigma/\theta \to 0$, then the space-time will be asymptotically isotropic. $T^{\mu\nu}$ is the energy momentum tensor, the kinematic quantities $\sigma$ and $\theta$ are defined by the observer $w^\mu = \delta^\mu_0$ ($w^\nu w_\nu = -1$), such as $\sigma^2 = \sigma_{\mu\nu} \sigma^{\mu\nu}$, where $\sigma_{\mu\nu} = u(\kappa; \lambda) (h^\Lambda_{\mu\nu} - \frac{1}{3} \theta h_{\mu\nu})$ and $\theta = u(\mu; \nu) h^{\mu\nu}$ in which $h_{\mu\nu}$ is the projective tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$.

In figure 3 the evolution of the anisotropy parameter $\sigma/\theta$ is presented from where we can see that the ration vanishes.

For $\Lambda > 0$, conditions (a), (b) and (c) are satisfied. Figure 5 shows the evolution of $a(t)$, $\rho(t)$ for the system $(12a)-(12c)$ with $\Lambda > 0$ and initial conditions far from the point $a_0 = 1$. We observe that as $t \to +\infty$, conditions (a) and (b) are satisfied, while figure 6 shows that condition (d) is also satisfied, because $\sigma = \sqrt{\frac{\dot{a}}{a}}$, which implies that space-time is asymptotically isotropic. On the other hand, for $\Lambda < 0$, condition (c), $T^{00} > 0$, can be violated which means that the “isotropization” is not guaranteed.

The present analysis shows that in general, the exact solution with $a(t) = \pm 1$ is unstable. However the spacetime is asymptotically isotropic for large values of $t$. That means that in the late-time the only fluid-term which survives is that of the cosmological constant. That result revises the previous analysis of [36].

5. CONCLUSIONS

We have analyzed the gravitating, time-dependent analytic solutions of the Einstein-Skyrme system with topological charge one introduced in [34]. In particular, we have shown that these solutions –whose analogues in flat space-times would be spherically symmetric–, reach an isotropic asymptotic state for $t \to +\infty$. This question was also analyzed numerically in [36]. In addition, we have shown that the isotropic solution, given by the Ermakov-Pinney equation, itself is not stable configuration, but a state of neutral equilibrium, like a spontaneously broken vacuum. Thus, the isotropy of the charge 1 Skyrmion on flat spaces may be broken by the coupling with Einstein gravity. However, despite this fact, the asymptotic solutions for $\Lambda > 0$ of the dynamical system describing the time-dependent gravitating
Skyrmion are asymptotically isotropic in large scales. The main reason is that, when $\Lambda > 0$, the “destabilizing” terms in the dynamical system (leading to the instability of the isotropic solution) are suppressed for $t \to +\infty$. Consequently, such terms only act for a finite amount of time after which the value of $a(t)$ freezes. To the best of the authors’ knowledge, this is the first explicit example of a symmetry breaking induced by the coupling with Einstein gravity of a topological soliton (which on flat spaces would be isotropic) in a realistic theory such as the Skyrme model. Moreover, we have discussed in detail the integrability of the isotropic solution in terms of the Ermakov-Pinney system.
FIG. 5: Qualitative behaviour of the solution of the field equations (12a)-(12c) for initial conditions far for $a(0) = 1$. The left plot is for the scale factor $a(t)$, while the right plot is for the scale factor $\rho(t)$. The lines are for various values of the free parameters, where $\Lambda > 0$.

FIG. 6: Qualitative behaviour of the anisotropic parameter $\sigma(t)/\theta(t)$, for the solutions of fig. 5.

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