The influence of atomic collisions on the spectrum of light scattered from an $f$-deformed Bose–Einstein condensate

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Received 18 March 2009, in final form 20 August 2009
Published 22 September 2009
Online at stacks.iop.org/JPhysB/42/195501

Abstract
In this paper, we investigate the spectrum of light scattered from a Bose–Einstein condensate (BEC) in the framework of an $f$-deformed boson model. We use an $f$-deformed quantum model in which Gardiner’s phonon operators for the BEC are deformed by an operator-valued function, $f(\hat{n})$, of the particle-number operator $\hat{n}$. We also consider the collisions between the atoms as a special kind of $f$-deformation where the collision rate $\kappa$ is regarded as the corresponding deformation parameter. By applying the small fluctuation approximation, we obtain the spectrum of light scattered from the $f$-deformed BEC. By analysing the scattering spectrum we find that by increasing the values of the deformation parameters $\kappa$ and $\eta = \frac{1}{N}$ ($N$ is the total number of condensate atoms) the spectrum shows deviation from the spectrum associated with the non-deformed Bose–Einstein condensate.

1. Introduction
The experimental realization of a Bose–Einstein condensate (BEC) [1–5] in a gas of atoms has generated much interest in studying the properties of BECs and their manipulation. Apart from the fundamental studies to achieve low enough temperatures and high enough atomic densities for generating BECs, there is another effort concerning detection and observation on the condensate or, more generally, of a system of quantum statistically degenerate atoms. This goal will be dominated by scattering light from the system of cooled atoms.

Analysing the properties of light scattered from a sample of cold bosonic atoms can provide a means of detection effects associated with the formation of a BEC [6, 7]. In search of signatures of the BEC, Javanainen [6] has calculated the spectrum of light scattered from a low-density degenerate Bose gas, in the limit of large detuning. He pointed out that the spectrum contains distinct qualitative features associated with Bose–Einstein statistics. In [7], the far-off-resonant light scattering from a BEC has been used to probe density correlation. It has been shown that the spectrum of light scattered from an interacting BEC has a single sharp line. The frequency of the sharp line which signals the presence of a Bose condensate in light scattering [6] contains essential information about the condensate and the interaction. Quantum optical studies of light scattered from degenerate atomic gases were initiated by Svistunov and Shlyapnikov [8] and Politzer [9]. They considered scattering of weak light from a BEC in the limit of a very large trap. However, several authors [10, 11] have considered detailed studies of the scattered light from a BEC confined in traps of a more realistic size and shape. On the experimental side, Andrews et al [12] have reported a non-destructive optical detection of a Bose condensate. They have used dispersive light scattering to spatially image a trapped condensate. This detection technique can be used to measure the spectrum of the scattered light from the condensate when the driving light is detuned far from the resonance of the optical transitions.

More recently, Inouye et al [13] studied the scattering of a single beam of light from the condensate. This phenomenon led to a new process—self-stimulated Rayleigh scattering—above low threshold intensity. However, later experiments with shorter and stronger laser pulses led to a more profound understanding of BEC experiments. Schneble
et al [14] reported the results of an experiment where the incident cw laser was replaced by an optical pulse, giving rise to the remarkable result that for short pulses, backward scattering atoms were observable in addition to the forward peaks. Hilliard et al [15] studied experimentally superradiant scattering of light by the Bose condensed atomic sample in a regime where pump depletion is important. Bar-Gill et al [16] studied experimentally superradiance in a BEC by using a two-frequency pump beam. By controlling the frequency difference between the beam components, they measured the spectrum of the backward superradiant atomic modes. Saba et al [17] demonstrated an experimental technique based on stimulated light scattering to determine the relative phase of two BECs. Experimental observation of the superradiant Rayleigh scattering with ultracold and Bose condensed atoms in an optical ring cavity was reported by Slama et al [18, 19]. The experiments of superradiance in a BEC were followed by several theoretical works [20–22]. In [20], a detailed theoretical analysis of superradiant Rayleigh scattering from atomic BECs in the framework of the spatially dependent semiclassical Maxwell–Schrödinger equations has been presented. Pu et al [21] have investigated theoretically four-wave mixing of optical and matter waves resulting from the scattering of the short light pulse off an atomic BEC. The mechanisms of pairwise scattering [14] of photons with cold atoms have recently been considered by Guo et al [22]. They have shown that considerable motional correlation can be established between atoms during the scattering process which can effectively squeeze the scattering spectrum. All of the above-mentioned studies reveal the significance of further investigation of light scattering from a BEC.

Most of the properties of atomic BECs are dominated by two-body collisions, which can be characterized by the s-wave scattering length $a$. The sign and the absolute value of the scattering length determine stability, internal energy, formation rate, size and collective excitation of a condensate. Therefore, the spectrum of light scattered from the BEC is crucial in understanding the effect of two-body collisions in the BEC. In order to study the dynamics of a BEC gas in a realistic experimental situation, it is necessary to consider the two-body collision effect within the condensate. Inclusion of two-body correlations only and disregard of all higher-body collisions are ideally suited for the BEC [23]. BEC is possible only at extremely low temperatures and extremely low densities. The number of trapped atoms is typically of the order of a few hundred to a few million. This is extremely small compared to the Avogadro number. Under these conditions, in an experimental situation [24], only two-body collisions are relevant and there are practically no three- and higher-body collisions. We have recently shown [25] that the collision effect transforms the standard harmonic oscillator model into an $f$-deformed one and the atomic collisions within the condensate may be described in the framework of $f$-deformation.

In the last few years, there has been a great deal of interest in quantum groups and their associated algebras, which are specific deformations of Lie algebras. Algebraic models have been used very successfully in several research areas of physics and mathematics such as exactly solvable statistical models [26], non-commutative geometry [27], nuclear quantum many body problems [28] and rational conformal field theories [29]. Recent interest in quantum groups has led us to introduce the concept of $q$-deformed boson oscillators [30–32], which are deformations of the standard bosonic harmonic oscillator algebra. The $q$-deformed boson oscillator has been interpreted as a nonlinear oscillator with a very specific type of nonlinearity which classically corresponds to an intensity dependence of the oscillator frequency [33]. In addition, a general type of nonlinearity for which the intensity dependence of frequency of oscillations is described by a generic function $f(\tilde{n})$, the so-called $f$-deformation, has been introduced [34]. One of the most important properties of $f$-deformed bosons is their relation to nonlinearity of a special type [35]. The relation between deformed radiation field and nonlinear quantum optical processes has been studied [36].

Recently, new insight into the description of BEC has been obtained by understanding and applying the $f$-deformed bosons. The case of a finite number of atoms, $N$, has been investigated in [37]. It provides a physical and natural realization of the $f$-deformed boson by using Gardiner’s phonon operators [38] for the description of BEC. Gardiner’s phonon operators satisfy an $f$-deformed commutation relation. As $N \to \infty$ the standard bosonic commutation relation is regained. The type of quantum nonlinearity introduced by $f$-deformation provides a compact description of physical effects in BEC. For example, it has been revealed how an $f$-deformed BEC produces a correction to the Planck distribution formula [33, 35]. In [35] it has been shown that the quantum nonlinearity introduced by $f$-deformation changes the specific heat behaviour. It is reasonable to expect that the spectrum of light scattered from a BEC is modified in the presence of deformations. In [39] it has been pointed out that intrinsic deformation due to the number of condensate atoms extensively changes the spectrum of light scattered from a BEC. It is useful to ask how the results of [39] are modified by considering the effect of collisions between the atoms within the condensate as a special kind of $f$-deformation.

In this paper, we intend to study the spectrum of light scattered from atoms within an $f$-deformed BEC. The system under consideration is an $f$-deformed BEC of a trapped atomic gas composed of two-level atoms, in which Gardiner’s phonon operators for BEC are deformed by an operator-valued function $f(\tilde{n})$. By considering the effect of collisions between the atoms within the condensate as a special kind of $f$-deformation, in which the collision rate $\kappa$ is regarded as the deformation parameter, and by using the small fluctuation approximation, we study the spectrum of light scattered from an $f$-deformed BEC. Such a system offers extra degrees of flexibility ($\kappa, N$) for the response of an $f$-deformed BEC to the laser light. We show that the $f$-deformed BEC exhibits nonlinear characteristics, such that the nonlinearity increases by adjusting the deformation parameters ($\kappa, N$) and the nonlinearity may lead to a deviation from the typical predicted spectrum’s shape.

The present paper is organized as follows. In section 2 we present our model and we give an analytical expression
for the light scattered from the \( f \)-deformed BEC. We use the deformation algebra to study the condensate with large but finite number of atoms. Here the deformation parameter is no longer phenomenological and is defined by the total number of atoms. We show that the atomic collisions within the condensate can be regarded as an extra deformation on the intrinsically defomed Gardiner’s phonon operators for the BEC. In the presence of atomic collisions within the condensate, we analyse the light scattering from the \( f \)-deformed BEC. Summary and conclusions are given in section 3.

2. The model and analytical solution for the spectrum of light scattered from an \( f \)-deformed BEC

We consider a system consisting of a weakly interacting BEC of two-level atoms in a trap interacting resonantly with a classical radiation field. We introduce the creation and annihilation operators \( \hat{b}^\dagger (\hat{a}^\dagger) \) and \( \hat{b} (\hat{a}) \), respectively, for the atoms in the excited (ground) state. The frequency of the radiation field and the atomic transition frequency of the atoms in the excited (ground) state. The frequency of transforming to the slowly varying annihilation operator \( \hat{a}^\dagger \).

The fast frequency dependence of \( \hat{b}(t) \) can be eliminated by transforming to the slowly varying annihilation operator \( \hat{a}^\dagger = \hat{a}^\dagger e^{it\omega_r} \).

The creation and annihilation operators \( \hat{b}^\dagger (t) \) and \( \hat{b}(t) \) obey the standard bosonic commutation relation \( [\hat{b}(t), \hat{b}^\dagger (t)] = 1 \). We can therefore write the Heisenberg equation of motion for the operator \( \hat{b}(t) \) in the following form:

\[
\dot{\hat{b}}(t) = -i\Delta \hat{b}(t) - ig\sqrt{N_c} + \sqrt{2}\Gamma \hat{b}(t),
\]

where \( \Delta = \omega_r - \Omega \), the damping rate \( \Gamma \) is given by \( \Gamma = \gamma N_c \), [41, with \( \gamma \) the one-atom linewidth, and \( b_m(t) \) is the vacuum noise operator, having the following correlation properties:

\[
\langle \hat{b}_m(t) \hat{b}_m(t') \rangle = \langle \hat{b}_m(\delta(t - t')) \rangle = 0.
\]

Now, we shall proceed by making the small fluctuation approximation [42]. According to this approximation, when the system reaches the steady state, \( \hat{b}(t) (\hat{b}^\dagger (t)) \) can be divided into two parts: \( \hat{b}(t) = \hat{b}_0 + \delta \hat{b}(t) (\hat{b}^\dagger (t) = \hat{b}^\dagger_0 + \delta \hat{b}^\dagger (t)) \), where \( \hat{b}_0 (\hat{b}^\dagger_0) \) denotes the steady-state value of \( \hat{b}(t) (\hat{b}^\dagger (t)) \) and \( \delta \hat{b}(t) (\delta \hat{b}^\dagger (t)) \) is the small fluctuation with a zero mean value.

Under this approximation, equation (5) can be solved and we get

\[
\dot{\delta} \hat{b}(t) = \frac{-i\hat{g} \sqrt{N_c}}{ \Gamma + i\Delta } 
\]

Therefore, we obtain the Fourier component of the operator \( \delta \hat{b}(t) \) in the following form:

\[
\delta \hat{b}(\omega) = \frac{\sqrt{2\Gamma}}{ \Gamma + i\Delta } \hat{b}_m(\omega).
\]

Now, we give an analytical expression for the spectrum of light scattered from the BEC. According to [6], the spectrum is obtained by calculating correlation functions for the matter field operators. Hence, in the steady-state regime, the standard definition of the spectrum is

\[
S(\omega) = \int d\omega' \langle \delta \hat{b}^\dagger(\omega) \delta \hat{b}(\omega') \rangle.
\]

By using equations (6) and (10), we obtain

\[
S(\omega) = \int d\omega' \langle \delta \hat{b}^\dagger(\omega) \delta \hat{b}(\omega') \rangle = 0.
\]

This means that, in the long time limit, only the equal time correlations survive.

As pointed out before, to study the dynamics of the BEC gas, the Bogoliubov approximation [40] in quantum many-body theory is usually applied, in which the creation and annihilation operators for condensate atoms are replaced by a c-number. However, this approximation destroys the conservation of the total particle number. To overcome this problem, Gardiner [38] suggested a modified Bogoliubov approximation by introducing phonon operators which conserve the total atomic particle number \( N \) and obey the \( f \)-deformed commutation relation of the Heisenberg--Weyl algebra such that, as \( N \rightarrow \infty \), the usual commutation relation of the Heisenberg--Weyl algebra is regained. Then Gardiner’s phonon approach gives an elegant infinite atomic particle-number approximation theory for BEC taking into account
the conservation of the total atomic number [37]. We consider Gardner’s phonon operators defined by [38]

$$b_q = \frac{q}{\sqrt{N}} \hat{a}^* \hat{b}, \quad b_q^* = \frac{q}{\sqrt{N}} \hat{a} \hat{b}^*$$.  

These operators obey the deformed commutation relation

$$[b_q, b_q^*] = 1 - \frac{2}{N} b_q^* b_q = 1 - 2\eta b_q^* b_q$$,  

with $\eta = \frac{1}{N}$. When $\eta \to 0$ or $N \to \infty$, the standard (non-deformed) bosonic commutation relation is regained. In general, the deformed operator $b_q$ is related to the non-deformed operator $\hat{b}$ through an operator-valued function $f_1$ as

$$\hat{b}_q \approx \hat{b}_1(f_b \hat{b}; \eta)$$.

In our particular case, we have

$$f_1(b^* \hat{b}; \eta) = \sqrt{1 - \eta(b^* \hat{b} - 1)}$$.

Here the deformation parameter is no longer phenomenological and is defined by the total number of atoms. For small deformation, the deformed boson operators $b_q^*$ and $b_q$ could be expressed in terms of the standard bosonic operators $\hat{b}$ and $\hat{b}$ ($[\hat{b}, \hat{b}^*] = 1$) as

$$\hat{b}_q \approx \hat{b}_1\left[1 - \frac{\eta}{2}(b^* \hat{b} - 1)\right] = \hat{b} - \frac{1}{2N} \hat{b}^* \hat{b} \hat{b}$$.

Using the algebra (20) and deformation (21), we get

$$f(\hat{N})^2 = \left\{\begin{array}{ll}
q^\alpha q^\gamma \hat{N} - q^{-\gamma} \hat{N}, & \alpha \neq \gamma \\
q^{\beta+\gamma}(N-1), & \alpha = \gamma.
\end{array}\right.$$  

By introducing new deformation parameters $q = e^\gamma$, $\alpha = \upsilon + \mu$, $\gamma = \upsilon - \mu$, we obtain

$$f(\hat{N})^2 = \frac{\sinh(\tau \mu \hat{N})}{\sinh(\tau \mu)} \exp[\tau(\beta + \upsilon(N - 1))]$$.

We shall apply these results in the following subsection to illustrate the atomic collisions within the condensate as a specific kind of $f$-deformation.

2.2. An example: the atomic collision effect

As a particular physical example, we consider the atomic collisions within the condensate. The effective interaction Hamiltonian contains a nonlinear term proportional to $(\hat{a}^* \hat{a})^2$ (appendix A):  

$$\hat{H}_I = \frac{\hbar}{\kappa} (\hat{a}^* \hat{a})^2$$,  

where the collision rate is denoted by $\kappa$. According to the kinetic theory, the collision rate is $\kappa \approx \rho \pi a^2 v_{rms}$, where $\rho$ is the density of the atoms, $a$ is the scattering length and $v_{rms}$ is the root-mean-square speed of the atoms. By expanding the Hamiltonian (25) and considering small values of $\upsilon$ and $\mu^2$, we obtain

$$\hat{H}(\hat{N}) = \frac{\hbar \omega_0}{2} \left[(2\hat{N} + 1) + \frac{1}{6} \mu^2 \hat{N} + \left(\frac{1}{2} \mu^2 + 2\upsilon\right) \hat{N} \right] + O(\upsilon^2, \mu^2, \mu^4)$$,  

where the interaction Hamiltonian reads

$$\hat{H}_I(\hat{N}) \approx \frac{\hbar \omega_0}{2} \left[\frac{1}{6} \mu^2 \hat{N} + \left(\frac{1}{2} \mu^2 + 2\upsilon\right) \hat{N} \right]$$.
The Hamiltonian (28) reproduces the Hamiltonian (26) by setting \( \mu^2 = 0 \) and \( \nu = \frac{\kappa}{2\omega_0} \). Thus, we see that the atomic collision effect transforms the standard (nonlinear) harmonic oscillator model into an \( f \)-deformed one. Alternatively, we could set, \( ab \ ini\to \) in equation (22), \( f(\mathcal{N}) = \sqrt{\kappa^*N + (1 - \kappa')} \) with \( \kappa' = \frac{\kappa}{2\omega_0} \) to obtain the Hamiltonian (26). Therefore, the parameters of the generalized deformed algebra are related to the rate of atomic collisions \( \kappa \). As an interesting point we note that when \( \mu^2 = -3\nu \) and \( \nu = \frac{2\kappa}{2\omega_0} \), the Hamiltonian (27) reproduces the Kerr-like Hamiltonian, where the nonmonochromatic field Hamiltonian contains, to the lowest order, a nonlinear term proportional to \( \mathcal{N}(\mathcal{N} - 1) \):

\[
\hat{H}_{\text{Kerr}}(\mathcal{N}) = \frac{\hbar\omega_0}{2}(2\mathcal{N} + 1) + \frac{k}{2}\mathcal{N}(\mathcal{N} - 1). \tag{29}
\]

Therefore, one can infer that up to the first-order approximation, the nonlinearity of the model under consideration due to the atomic collisions within the condensate may be described as a Kerr-type nonlinearity.

Subsequently, by considering the effect of collisions between the atoms within the condensate, we can apply the extra deformation on the intrinsically deformed Gardiner’s phonon operators for BEC by an operator-valued function \( f_2(\hat{n}) = \sqrt{\kappa^*\hat{n} + (1 - \kappa')} \) of the particle number operator \( \hat{n} = \hat{N}_q = \hat{b}^\dagger\hat{b} q \). Here the nonlinearity is related to the collisions between the atoms within the condensate. The operator-valued function \( f_2(\hat{n}) \) reduces to 1 as soon as \( \kappa \to 0 \). It means that the deformation increases with the collision rate \( \kappa \).

The deformed Gardiner’s phonon operators \( \hat{B}_q \) and \( \hat{B}^+_q \) are related to the operators \( \hat{b}_q \) and \( \hat{b}^+_q \) through the operator-valued function \( f_2(\hat{n}) \) as

\[
\hat{B}_q = f_2(\hat{n}) \hat{b}_q, \quad \hat{B}^+_q = f_2(\hat{n}) \hat{b}^+_q. \tag{30}
\]

Therefore, the deformed version of the Hamiltonian (2) can be written as

\[
\hat{H} = \hbar\omega \hat{B}^+_q \hat{B}_q + \hbar\sqrt{N}\sum_k [g(t) \hat{B}^+_q \hat{c}_k + g^*(t) \hat{B}_q \hat{c}_k] + \hbar\sqrt{N} \sum_k \xi(k) [\hat{b}^*_k \hat{c}_k + \hat{c}^*_k \hat{b}_k]. \tag{31}
\]

Note that, in the last term of the rhs of equation (31), the nonlinear character of \( \hat{B}_q \) has been neglected due to the weak-coupling assumption with the reservoir. For small deformation, we obtain

\[
\hat{B}_q = \hat{b}
\]

\[
\hat{B}^+_q = \left[ 1 - \frac{\kappa'}{2}(1 - \hat{b}^*_q \hat{b}_q) \right] \hat{b}^+_q. \tag{32}
\]

By keeping only the lowest order of \( \eta = \frac{1}{N} \) for very large total number of atoms \( N \) and by keeping only the first-order term of the collision rate \( \kappa \) for very low temperature, we get

\[
\hat{B}_q \approx \left( \hat{b} - \frac{1}{2N} \hat{b}^* \hat{b} \hat{b} \hat{b}^* \right) \left[ 1 - \frac{\kappa'}{2} \left( 1 - \frac{1}{N} \hat{b}^* \hat{b} \hat{b} \hat{b}^* \right) \right] \hat{B}_q. \tag{33}
\]

By assuming that all the atoms are initially in the condensate phase, we take \( N = N_c \) and thus by using equation (33) the deformed Hamiltonian (32) can be expressed in terms of the non-deformed operators \( \hat{b} \) and \( \hat{b}^* \) as:

\[
\hat{H}_{\text{eff}} = \hbar\omega \hat{b}^* \hat{b} + \hbar\omega(\kappa' - \eta) \hat{b}^* \hat{b} + \hbar\sqrt{N}[g(t) \hat{b}^* + \text{H.c.}] + \hbar\sqrt{N} \sum_k [\xi(k) \hat{b}^*_k \hat{c}_k + \xi(k) \hat{b}^*_k \hat{c}^*_k + \text{H.c.}.] \tag{34}
\]

Thus, we see that the atomic collisions increase the nonlinearity of the Hamiltonian. In figures 1 and 2, we plot the quantity \( \|b\| - |\beta_0| \) as a function of the total number of atoms \( N \) for \( \kappa = 0 \) (in the absence of atomic collisions), \( \Delta = 0 \), \( g = 2.5\gamma \), \( \delta/\gamma = 50 \) and \( \arg(b) = \arg(\beta_0) = \pi/2 \).

By solving the above equation, we get the steady-state value of the field. The solution of equation (36) reduces to expression (7) as soon as \( \kappa, \eta \to 0 \), as depicted in figures 1 and 2. In figures 1 and 2, we plot the quantity \( \|\beta\| - |\beta_0| \) as a function of the total number of atoms \( N \) and the deformation parameter \( \kappa' \), respectively. As is seen, by decreasing the deformation parameter \( \eta = 1/N \) and \( \kappa' \), the quantity \( \|\beta\| - |\beta_0| \) tends to zero and the solution of equation (36) reduces to expression (7).
The quantity $||\beta||$ as a function of the deformation parameter $\kappa$, when the total number of atoms $N = 100$ and the values of other parameters are the same as those in figure 1.

where

$$A = -i\Delta - \Gamma + i\sqrt{N}g(\eta - \kappa') (\beta + \beta^*)$$

$$- 4i\hat{\omega}(\eta - \kappa')\beta^2,$$

$$B = i\sqrt{N}g(\eta - \kappa')\beta + 2i\hat{\omega}(\eta - \kappa')\beta^2.$$  \hspace{1cm} (38)

Therefore, the Fourier component of the operator $\hat{\delta}\hat{b}(t)$ reads

$$\hat{\delta}\hat{b}(\omega) = \frac{1}{E(\omega)} \{ i\omega - A^{*}\} \hat{b}_{\omega}(\omega) + B\hat{b}_{\omega}^*(\omega),$$

where

$$E(\omega) = |A|^2 - |B|^2 - \omega^2 - i\omega(A + A^*).$$  \hspace{1cm} (40)

Finally, by using equations (6) and (39), the spectrum of light scattered from the $f$-deformed BEC under consideration is obtained as

$$S(\omega) = \int d\omega' \langle \hat{\delta}\hat{b}^*(\omega')\hat{\delta}\hat{b}(\omega) \rangle = \frac{|B|^2}{|E(\omega)|^2}. $$  \hspace{1cm} (41)

When the limit $\kappa, \eta \to 0$ is taken, the result of equation (11), $S(\omega) = 0$, is recovered.

We are now in a position to investigate the influence of atomic collisions on the spectrum of light scattered from the $f$-deformed BEC under consideration. In fact, due to the nonlinearity associated with $f$-deformation, it is reasonable to expect that the scattering spectrum $S$ is modified. In figure 3, we show the three-dimensional plot of the spectrum $S$ as a function of the normalized frequency $\omega/\gamma$ and total number of atoms $N$ in the absence of atomic collisions ($\kappa = 0$). We see that the spectrum $S$ vanishes for $\eta \to 0$ ($N \to \infty$) and the usual predicted spectrum’s shape is regained. In other words, there are no quantum fluctuations in the absence of deformation. The spectrum $S$ shows a peak with the decrease of the number of atoms $N$, since the increase of the deformation parameter $\eta = 1/N$ results in large nonlinearity, which may lead to observable effects on the spectrum of the scattered light. Indeed, by applying the deformation on the atomic operators of BEC, we get nonzero values in the scattering spectrum $S$ which is a signature of quantum fluctuations. To observe the influence of atomic collisions on light scattered from an $f$-deformed BEC, for a given number of atoms $N$, we plot the scattering spectrum $S$ as a function of the normalized frequency $\omega/\gamma$ and the deformation parameter $\kappa'$. This is shown in figure 4. We see that the spectrum $S$ increases with the deformation parameter $\kappa'$. The nonzero values in the spectrum $S$ confirm that the quantum fluctuations arise only when we apply the deformation on the atomic operators of BEC. The reason is due to the fact that considering the effect of two-body collisions between the atoms within the BEC requires a deformation of the bosonic field [39]. The effect of $f$-deformation leads naturally to the occurrence of nonlinear interactions in the model under consideration. Hence, the presence of nonlinearity leads to a deviation from the usual predicted spectrum’s shape. In the absence of atomic collisions the results of [39] are recovered.
In the end, it is necessary to make a brief discussion on the self-consistency of the small fluctuation approximation. Our treatment indicates that under the small fluctuation approximation around the steady state, only the coherent part of the light scattering survives, while the incoherent part becomes zero, for both the cases of non-deformed and \( f \)-deformed condensates. In order to show that this result is true, it is sufficient to check the self-consistency of the small fluctuation approximation. For this purpose, we calculate the incoherent part of the total exciton number in the steady state, \( \bar{n} = \langle \hat{b}^\dagger \hat{b} \rangle \). In the following, we first treat the case of an \( f \)-deformed condensate (in the presence of atomic collisions) and then consider the case of a non-deformed condensate as a limiting case. The linearized equations for \( \hat{b}(t) \) and \( \hat{b}^\dagger(t) \) read (cf equation (37))

\[
\frac{\partial}{\partial t} \left( \hat{b}(t) \right) = -\mathbf{A} \left( \hat{b}(t) \right) + \mathbf{D}^{1/2} \left( \hat{b}^\dagger(t) \right),
\]

with

\[
\mathbf{A} = - \left( \begin{array}{cc} A & B \\ B^* & A^* \end{array} \right), \quad \mathbf{D} = \left( \begin{array}{cc} 2\Gamma & 0 \\ 0 & 2\Gamma \end{array} \right),
\]

where \( A \) and \( B \) are given by (38). The matrix of the spectrum function \( S(\omega) \) is given in terms of \( \mathbf{A} \) and \( \mathbf{D} \) by [45, 46]

\[
S(\omega) = (\mathbf{A} + \i \omega \mathbf{I})^{-1} \mathbf{D}(\mathbf{A}^T - \i \omega \mathbf{I})^{-1},
\]

where \( \mathbf{I} \) is the identity matrix and \( T \) means transpose. We get accordingly

\[
S(\omega) = \frac{2\Gamma}{(|A|^2 - |B|^2 - \alpha^2)^2 + \omega^2(A + A^*)^2} \\
\times \left( \begin{array}{cc} |A|^2 + |B|^2 + \i \omega(A - A^*) & -2\alpha^2B \\ -2\alpha^2B^* & -2\alpha^2|A|^2 \end{array} \right) \\
\times \left( \begin{array}{cc} 1 & \i \omega \alpha^2 \\ -\i \omega \alpha^2 & 1 \end{array} \right).
\]

Now, we can calculate \( \bar{n} = \langle \hat{b}^\dagger \hat{b} \rangle \) from matrix \( C \) [47]:

\[
C = \left( \begin{array}{c} \langle \hat{b}^\dagger \hat{b} \rangle \\ \langle \hat{b}^\dagger \hat{b}^\dagger \hat{b} \rangle \\ \langle \hat{b}^\dagger \hat{b} \rangle \end{array} \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) \, d\omega.
\]

Hence, by using equations (45) and (46), we obtain

\[
\bar{n} = \langle \hat{b}^\dagger \hat{b} \rangle = C_{21} = -\frac{2\Gamma}{\pi} \i AB^* \times \int_{-\infty}^{+\infty} \left( \begin{array}{c} 1 \\ \i \omega \alpha^2 \\ -\i \omega \alpha^2 \end{array} \right) \, d\omega.
\]

The above integration can be carried out by contour integration easily, and the result is \( \bar{n} = 0 \). Therefore, the self-consistency of the small fluctuation approximation confirms our result indicating that in the steady state the incoherent part of the light scattering from the \( f \)-deformed condensate vanishes. For the case of a non-deformed condensate, one can take the limit \( \kappa, \eta \to 0 \). In this limit, according to equation (38), we get \( A = -i\Delta - \Gamma \) and \( B = 0 \) and accordingly, equation (47) results in again \( \bar{n} = 0 \).

### 3. Summary and conclusions

In summary, we have studied the spectrum of light scattered from an \( f \)-deformed BEC of a gas of two-level atoms in which Gardiner’s phonon operators are deformed by an operator-valued function \( f(\hat{h}) \), of the particle-number operator \( \hat{h} \). By considering the effect of collisions between the atoms within the condensate, we have applied the extra deformation on the intrinsically deformed Gardiner’s phonon operators for BEC. We have found that the presence of deformation parameters \( \eta \) and \( \kappa \) introduces nonlinearity, which may lead to observable effects on the spectrum of light scattered from an \( f \)-deformed BEC. Also, we have found that the deformation parameters \( \eta \) and \( \kappa \) play an important role in determining the spectrum of light scattered from the BEC. The scattering spectrum vanishes for \( \kappa, \eta \to 0 \), i.e. in the absence of deformations the usual predicted spectrum’s shape is recovered. As pointed out before, the collision rate is \( \kappa \approx \rho \pi a^2 v_{\text{rms}} \). Therefore, we can adjust the value of the collision rate \( \kappa \) by changing the density of the BEC. For example, lowering the temperature of the BEC increases the condensate density \( \rho \), hence increases the collision rate \( \kappa \), leading to the increase of deformation. By applying the deformation on atomic operators of BEC, it is possible to obtain large nonlinearity that leads to nonzero values in the scattering spectrum which is a signature of quantum fluctuations.

Our approach is based on the Heisenberg equations of motion for the exciton operators, and the small fluctuation approximation around the steady-state solution has been used. A discussion on the self-consistency of this approximation has also been presented for both the cases of non-deformed and \( f \)-deformed condensates.

### Acknowledgments

The authors would like to express their gratitude to the referees, whose valuable comments have improved the paper. They are also grateful to the Office of Graduate Studies of the University of Isfahan for their support.

### Appendix A. The effective Hamiltonian in the presence of atomic collisions

In order to derive the effective interaction Hamiltonian (26) in the presence of atomic collisions, we use the same mathematical procedure established in [44]. We introduce the second quantized Hamiltonian for a BEC confined in a trap potential \( V(\vec{r}) \), in the following form:

\[
\hat{H}(t) = \int d^3r \left[ \frac{\hbar^2}{2m} \vec{\nabla} \hat{\psi} \cdot \vec{\nabla} \hat{\psi} + V(\vec{r}) \hat{\psi}^\dagger \hat{\psi} + \frac{U_0}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right],
\]

(A.1)

where \( \hat{\psi}(\vec{r}, t) \) and \( \hat{\psi}^\dagger(\vec{r}, t) \) are the Heisenberg picture field operators which, respectively, annihilate and create an atom at position \( \vec{r} \) and obey the bosonic commutation relation \( [\hat{\psi}(\vec{r}, t), \hat{\psi}^\dagger(\vec{r}, t)] = \delta(\vec{r} - \vec{r}^\prime) \). \( U_0 = 4\pi \hbar^2 a/m \) measures the strength of the two-body interaction, \( a \) being the s-wave
scattering length. The Schrödinger equation for the state vector of the system is

$$\hat{i} \hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(0)|\Psi(t)\rangle,$$

(A.2)

and in the time-dependent Hartree approximation, the state vector for a system of $N$ particles is written as

$$|\Psi(t)\rangle = (N!)^{-1/2} \left[ \int d^3r \hat{\psi}_N(\vec{r}, t) \hat{\psi}_N(\vec{r}, 0) \right]^N |0\rangle,$$

(A.3)

where $\hat{\psi}_N(\vec{r}, t)$ is the normalized single-particle wavefunction and $|0\rangle$ is the vacuum state. By setting $\hat{\psi}_N(\vec{r}, t) = \exp(-i\mu_0 t/\hbar)\hat{\phi}_N(\vec{r})$ and by using equation (A.2), we obtain the time-independent Gross–Pitaevskii equation

$$\mu_0 \hat{\phi}_N(\vec{r}) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + NU_0 |\hat{\phi}_N|^2 \right] \hat{\phi}_N(\vec{r}),$$

(A.4)

where the macroscopic wavefunction $\hat{\phi}_N(\vec{r})$ for the $N$-particle system is normalized to unity. In particular, we write the Heisenberg field annihilation operator as a mode expansion over single-particle states as

$$\hat{\psi}(\vec{r}, t) = \sum_\alpha \hat{a}_\alpha(t) |\phi_\alpha(\vec{r})\rangle e^{-i\mu_\alpha t/\hbar},$$

(A.5)

where $\{|\phi_\alpha(\vec{r})\rangle\}$ are a complete orthonormal basis set and $\{\mu_\alpha\}$ are the corresponding eigenvalues. Here, we have chosen $\phi_0(\vec{r}) = \hat{\phi}_N(\vec{r})$ ($N$ is the mean particle number of atoms) as one member of the complete orthonormal set and have identified $\mu_0 = \mu_N$ and $\hat{a} = \hat{a}_0$. The second term $\hat{\psi}(\vec{r}, t)$ accounts for the non-condensate terms. Then substituting the mode expansion in the second-quantized Hamiltonian (A.1), retaining only the first term representing the condensate, and using the Gross–Pitaevskii equation (A.4), we obtain the following single-mode Hamiltonian for the condensate in the Schrödinger picture:

$$\hat{H}_s = \hat{H}(0) = \hat{a}^\dagger \hat{a} e_N + \frac{\hbar k}{2} \hat{a}^\dagger \hat{a},$$

(A.6)

where

$$\hbar k = U_0 \int d^3r |\phi_N|^4,$$

(A.7)

with $k$ being the collision rate between condensate atoms and

$$e_N = \int d^3r \left[ \frac{\hbar^2}{2m} |\nabla \phi_N|^2 + V(\vec{r})|\phi_N|^2 \right].$$

(A.8)

The second term on the right-hand side of equation (A.6) accounts for many-body interactions. Finally, we introduce an interaction picture defined by the transformation

$$\hat{H}_I(t) = \hat{U}^\dagger(t) \hat{H}_s \hat{U}(t) - \frac{i}{\hbar} \frac{\partial}{\partial t} \hat{U}(t) \hat{U}^\dagger(t),$$

(A.9)

$$\hat{U}(t) = \exp\left(-i(e_N - \hbar k/2)\hat{a}^\dagger \hat{a} t/\hbar \right),$$

which yields the single-mode Hamiltonian

$$\hat{H}_I = \frac{\hbar k}{2} (\hat{a}^\dagger \hat{a})^2.$$

(A.10)

### Appendix B. The Heisenberg equation of motion for the operator $\hat{b}(t)$

We consider the Heisenberg equation of motion for the operators $\hat{b}(t)$ and $\hat{c}_k(t)$ with the effective Hamiltonian (34):

$$\dot{\hat{b}} = -\frac{i}{\hbar} \left[ \hat{b}, \hat{H}_{eff} \right] = -i\omega_b(\kappa' - \eta)\hat{b}^\dagger(t)\hat{b}(t)$$

$$- i\gamma(t)\sqrt{N} - i\gamma(t)\sqrt{\frac{\kappa' - \eta}{2}} \left[ 2\hat{b}^\dagger(t)\hat{b}(t) + \hat{b}^2(t) \right]$$

$$- i\sqrt{N} \sum_k \xi(k)\hat{c}_k^\dagger(t)$$

(B.1)

$$\dot{\hat{c}}_k = -\frac{i}{\hbar} \left[ \hat{c}_k, \hat{H}_{eff} \right] = -i\Omega_k \hat{c}_k(t) - i\sqrt{N}\xi(k)\hat{b}(t).$$

(B.2)

The equation of motion for the reservoir operator $\hat{c}_k(t)$ can be formally integrated to yield

$$\hat{c}_k(t) = \hat{c}_k(0) e^{-i\Omega_k t} - i\sqrt{N}\xi(k) \int_0^t dt' \hat{b}(t') e^{-i\Omega_k(t-t')}.$$

(B.3)

where the first term describes the free evolution of the reservoir modes and the second term arises from their interaction with the harmonic oscillator. The reservoir operator $\hat{c}_k(t)$ can be removed by substituting the formal solution of $\hat{c}_k(t)$ into equation (34). By using definition (3), we obtain

$$\dot{\hat{b}}(t) = -i\Delta\hat{b}(t) - 2i\omega_b(\kappa' - \eta)\hat{b}^\dagger(t)\hat{b}(t)$$

$$- i\gamma(t)\sqrt{N} - i\gamma(t)\sqrt{\frac{\kappa' - \eta}{2}} \left[ 2\hat{b}^\dagger(t)\hat{b}(t) + \hat{b}^2(t) \right]$$

$$- i\sqrt{N} \sum_k \xi(k)\hat{c}_k^\dagger(t)$$

(B.4)

where

$$\hat{f}_k(t) = -i \sum_k \xi(k)\hat{c}_k(0) e^{-i\Omega_k(t-t')}.$$

(B.5)

Here, $\hat{f}_k(t)$ is a noise operator because it depends on the reservoir operator $\hat{c}_k(0)$. The fluctuations in the expectation values involving the harmonic oscillator will therefore depend on the evolution of the reservoir operators. The noise operator varies rapidly due to the presence of all the reservoir frequencies.

In view of the Weisskopf–Wigner approximation, the summation in equation (B.4) leads to a delta function $\delta(t-t')$ and the integration can then be carried out. Therefore, we obtain

$$\int_0^t dt' \hat{b}(t') e^{-i\Omega_k(t-t')} \equiv \frac{1}{2} \xi N\hat{b}(t),$$

(B.6)

where $\zeta$ is the damping constant and defined as

$$\zeta = 2\pi \left[ \xi(\Omega)^2 D(\Omega) \right].$$

(B.7)

Here, the density of states $D(\Omega) = \frac{\Omega^2}{2\pi^2}$, with $V$ the quantization volume, and $\xi(\Omega) = \xi(\Omega)^2$ is the coupling constant evaluated at $k = \frac{\Omega}{2\pi}$. It is possible to make the substitutions $\frac{1}{2} \xi N\hat{b}(t) = \Gamma \hat{b}(t)$ and $\hat{f}_k(t)\sqrt{\mathcal{N}}_c = \sqrt{2\Gamma} \hat{b}_m(t)$, so we can rewrite equation (B.4) in the following form:

$$\dot{\hat{b}}(t) = -i\Delta\hat{b}(t) - 2i\omega_b(\kappa' - \eta)\hat{b}^\dagger(t)\hat{b}(t)$$

$$- i\gamma(t)\sqrt{N} - i\gamma(t)\sqrt{\frac{\kappa' - \eta}{2}} \left[ 2\hat{b}^\dagger(t)\hat{b}(t) + \hat{b}^2(t) \right]$$

$$+ \sqrt{2\Gamma} \hat{b}_m(t) - \Gamma \hat{b}(t).$$

(B.8)
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