Structure of the Bogoliubov-Valatin canonical basis set

Aurel Bulgac\(^1\)* and Matthew Kafker\(^1\)†

\(^1\)Department of Physics, University of Washington, Seattle, Washington 98195–1560, USA

(Dated: March 10, 2022)

We discuss the mathematical properties of the Bogoliubov-Valatin basis set of quasiparticle wave functions for a fermion system, with particular emphasis on the properties of the canonical basis set. The properties of the canonical basis set, apart from their definition, are largely unknown. In particular, what is the physically required size of the canonical wave function set in order to correctly describe superfluid systems. While the cardinality of the set of quasiparticle wave functions for an isolated system in vacuum is \(c = |\mathbb{R}^3|\), the basis set for a finite system in a finite volume is countable, with cardinality \(K_0 = |\mathbb{Z}| = |\mathbb{N}|\). However, the size of the canonical basis set for an isolated system in a finite volume or for a periodic system is typically much smaller than the size of the entire basis set, and it is determined by the level of the spatial resolution. We show how one can get insight into the character of the canonical wave functions and we justify the minimum number of canonical wave functions needed for a given system.

The mathematical framework for describing superfluid fermionic systems was formulated in terms of quasiparticles by Bogoliubov [1] and Valatin [2]. Zumino [3] and Bloch and Messiah [4] have shown that one can introduce a particular set of quasiparticles, with similar properties to the set used by Bardeen et al. [5] (BCS), the canonical set of states. The ground state of a superfluid or superconducting system has a particularly simple form expressed in terms of this set of single-particle wave functions, well suited for calculations. The microscopic calculations however suffered for a long time from a very annoying divergence of the order parameter, which was handled typically in an \textit{ad hoc} manner, using physical arguments, devoid however of accurate estimates of what might have been missing. The presence of this divergence prevented Oliveira et al. [6] to extend the Density Functional Theory (DFT) \[7\] from normal to superconductor systems, in a manner similar to the Kohn-Sham DFT formulation [8], with local pairing fields. The nature of the divergence was clarified in 1980 [9], which later lead to an extension of the DFT framework in the Kohn-Sham spirit to both finite and infinite superfluid systems [10–13]. In particular, for decades in nuclear physics the pairing correlations were treated with arbitrary cutoffs on the number and the character of the single-particle orbitals in modeling pairing phenomena. The typical argument used in nuclear calculations of superfluid nuclei was that the energy of the ground state converged quite rapidly as a function of the chosen cutoff. Anderson [14], in discussing the treatment of electronic systems, characterized this kind of situation as the “Quantum Chemists’ Fallacy No. 1 and 2,” of which even Wigner was partially guilty, as “you may get pretty good energetics out of a qualitatively wrong state.” The perfect example is the case of a superconductor, in which in spite of the fact that the contribution from the condensation energy is negligible, the wave function with pairing correlations leads to qualitative changes, which otherwise in a typical chemical physicist approach would have been completely overlooked. We show here how using the canonical basis set one can get an insight into how many single-particle states and why one should introduce them in an accurate treatment of pairing correlations.

The creation \(\alpha^\dagger_k\) and annihilation \(\alpha_k\) quasiparticle operators are represented with a unitary transformation from the field operators as follows \[15\]

\[
\alpha^\dagger_k = \int d\xi \left[ u_k(\xi)\psi^\dagger(\xi) + v_k(\xi)\psi(\xi) \right],
\]

\[
\alpha_k = \int d\xi \left[ v^*_k(\xi)\psi^\dagger(\xi) + u^*_k(\xi)\psi(\xi) \right],
\]

and with the reverse relations

\[
\psi^\dagger(\xi) = \sum_k \left[ v^*_k(\xi)\alpha^\dagger_k + u^*_k(\xi)\alpha_k \right],
\]

\[
\psi(\xi) = \sum_k \left[ v^*_k(\xi)\alpha^\dagger_k + u^*_k(\xi)\alpha_k \right].
\]

Here \(\psi^\dagger(\xi)\) and \(\psi(\xi)\) are the field operators for the creation and annihilation of a particle with coordinate \(\xi = (r, \sigma, \tau)\) (spatial, spin, and isospin coordinates), and the integral implies also a summation over discrete variables when appropriate. In a finite volume, with periodic boundary conditions for example, the index \(k\) is always discrete. For a finite isolated system in vacuum [9, 16] the sum over \(k\) stands for a summation over the discrete indices and an integral over the continuous ones respectively.

The Hermitian number density and the skew-
symmetric anomalous density matrices are defined as

\[ n(\xi, \zeta) = \langle \Phi | \psi^\dagger(\zeta) \psi(\xi) | \Phi \rangle = \sum_k v_k^*(\xi) v_k(\zeta), \quad (5) \]

\[ \pi(\xi, \zeta) = \langle \Phi | \psi(\xi) \psi^\dagger(\zeta) | \Phi \rangle = \sum_k u_k(\xi) u_k^*(\zeta), \quad (6) \]

\[ \kappa(\xi, \zeta) = \langle \Phi | \psi(\xi) \psi(\xi) | \Phi \rangle = \sum_k v_k^*(\xi) u_k(\zeta), \quad (7) \]

\[ n(\xi, \zeta) + \pi(\xi, \zeta) = \delta(\xi - \zeta), \quad (8) \]

where the quasiparticle vacuum is defined as

\[ \alpha_k | \Phi \rangle = 0, \quad | \Phi \rangle = N \prod_k \alpha_k | 0 \rangle, \quad \langle \Phi | \alpha_k \alpha_k^\dagger | \Phi \rangle = \delta_{kl}, \quad (9) \]

and where \( N \) is a normalization factor (determined up to an arbitrary phase), \( \alpha_k | 0 \rangle \neq 0 \) for all \( k \), and \( | 0 \rangle \) is the vacuum state. For any \( k \), if the norm \( \int d\xi |v_k(\xi)|^2 = 0 \) or \( \alpha_k | 0 \rangle = 0 \) the corresponding factor \( \alpha_k \) should be skipped in the definition of \( | \Phi \rangle \). The new density matrix \( \pi(\xi, \zeta) \) is used in the discussion of the canonical basis set below.

The anti-commutation relations for the field operators \( \psi^\dagger(\xi), \psi(\xi) \) and for the quasiparticle operators \( \alpha_k^\dagger, \alpha_k \) imply that [15]

\[ \int d\xi \left[ u_k^*(\xi) u_l(\xi) + v_k^*(\xi) v_l(\xi) \right] = \delta_{kl}, \quad (10) \]

\[ \int d\xi \left[ u_k(\xi) v_l(\xi) + v_k(\xi) u_l(\xi) \right] = 0, \quad (11) \]

\[ \sum_k \left[ u_k(\xi) u_k^*(\zeta) + v_k^*(\xi) v_k(\zeta) \right] = \delta(\xi - \zeta), \quad (12) \]

\[ \sum_k \left[ u_k(\xi) v_k^*(\zeta) + v_k^*(\xi) u_k(\zeta) \right] = 0. \quad (13) \]

Eq. (12) means that the quasiparticle wave functions \( u_k(\xi), v_k(\xi) \) form (in general) an over complete set, as for an arbitrary function \( g(\xi) \) one has the decomposition

\[ g(\xi) = \sum_k u_k^*(\xi) \int d\zeta u_k(\zeta) g(\zeta) + \sum_k v_k(\xi) \int d\zeta v_k^*(\zeta) g(\zeta). \quad (14) \]

Additionally one can show that [15]

\[ \int d\zeta \left[ n(\xi, \zeta) n(\zeta, \eta) + \kappa(\xi, \zeta) \kappa^*(\zeta, \eta) \right] = n(\xi, \eta), \quad (15) \]

\[ \int d\zeta n(\xi, \zeta) \kappa(\zeta, \eta) = \int dy \kappa(\xi, \zeta) n^*(\zeta, \eta). \quad (16) \]

For a finite system the quasiparticle components \( v_k(\xi) \) always have a finite norm [9]

\[ \int d\xi | v_k(\xi) |^2 < \infty, \quad (17) \]

\[ \langle \psi | v_k | v_l \rangle \neq \delta_{kl}. \]

\[ \int d\xi | v_k(\xi) |^2 < \infty, \quad (17) \]

\[ \langle \psi | v_k | v_l \rangle \neq \delta_{kl}. \]

\[ \int d\xi | v_k(\xi) |^2 < \infty, \quad (17) \]

\[ \langle \psi | v_k | v_l \rangle \neq \delta_{kl}. \]

\[ \int d\xi | v_k(\xi) |^2 < \infty, \quad (17) \]

\[ \langle \psi | v_k | v_l \rangle \neq \delta_{kl}. \]

\[ \int d\xi | v_k(\xi) |^2 < \infty, \quad (17) \]

\[ \langle \psi | v_k | v_l \rangle \neq \delta_{kl}. \]

\[ \int d\xi | v_k(\xi) |^2 < \infty, \quad (17) \]

\[ \langle \psi | v_k | v_l \rangle \neq \delta_{kl}. \]
canonical occupation probabilities \( n_k \) from Eq. (20) and define the single particle energies as \( e_k = \langle \phi_k | H | \phi_k \rangle \), where \( H \) is the normal mean field single-particle Hamiltonian within the Hartree-Fock-Bogoliubov (HFB) and Superfluid Local Density Approximation (SLDA) frameworks, and in which case \( \langle \phi_k | H | \phi_l \rangle \neq 0 \) if \( k \neq l \). The simple relationship between the HF occupation probabilities and the single-particle energies becomes thus more difficult to interpret physically and justify within HFB and SLDA frameworks.

Since the total particle number is not well defined within HFB and SLDA, as the gauge symmetry is broken, one has to restore this symmetry. In the canonical representation the gauge symmetry is significantly easier to within HFB and SLDA, as the gauge symmetry is broken, and anomalous densities can be determined without the knowledge of the quasiparticle wavefunctions (qpwfs) and of the corresponding quasiparticle energies.

The quasiparticle representation in which the number density matrix and the generalized number density matrix and the general
ded that a typical nuclear mean field can sustain a set with cardinality \( \mathbb{A} \), not necessarily identical to those define in Eq. (25),

\[
\phi_T(\xi) = i\sigma_y \phi_k^*(\xi),
\]

where \( \sigma_y \) is the Pauli matrix.

We will illustrate the properties of the canonical wave functions with some generic numerical results obtained for a 1-dimensional example, which, however, retains all the qualitative features of a 3-dimensional system. For the sake of simplicity we have chosen a 1-dimensional system with potential and pairing fields

\[
V(x) = \frac{V_0}{1 + \cosh(x/a)},
\]

\[
\Delta(x) = \frac{\Delta_0}{1 + \cosh(x/a)},
\]
where we will use the notation for the spatial coordinate $-\infty < x < \infty$, $V_0 = -50$ MeV, $\Delta_0 = 3$ MeV, $R = r_0 A^{1/3} = 14.9$ fm, $a = 0.5$ fm, and $\mu = -5$ MeV.
(We avoid using a Woods-Saxon potential well in order not to generate singularities of the derivatives of the wave functions.) We solved the non-self consistent SLDA or HFB equations for the qpwfs [12, 21], using the Discrete Variable Representation method [22]

\[
\begin{pmatrix}
H - \mu & \Delta \\
\Delta & -H + \mu
\end{pmatrix}
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix} = E_k
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
\]  \quad (34)

in a box of size $L = 80$ fm and with four different lattice constants $dx = 1, 0.5, 0.25, 0.125$ fm and where

\[
H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)
\]  \quad (35)

and $m$ is the nucleon mass, in the absence of spin-orbit interaction. The Eq. (34) are for the components $u_k(x)$ with spin-up and $v_k(x)$ with spin-down. The equations for the components $u_k(x)$ with spin-down and $v_k(x)$ with spin-up are obtained from these equations by changing the sign of the pairing field $\Delta(x)$ only [12, 21].

The SLDA equations for cold fermionic gases and nuclei have the same structure in this case. It is straightforward to extend this type of analysis to more complicated geometries, for example the pasta phase in neutron star crusts, or the superconductor-normal metal-superconductor (SNS) junctions in condensed matter physics. The case discussed here is equivalent to a NSN junction. This analysis equally applies to infinite periodic systems.

This 1-dimensional model is equivalent to solving the SLDA equations for a spherical system, but only for s-wave orbitals, therefore for even orbitals $\phi_k(x) = \phi_k(-x)$ and $x \geq 0$ in the present formulation. The 1-dimensional normal number density $n(x)$ here is only for the fermions with spin-down, which in the case of even fermion particle number is identical to the normal number density of the spin-up particles. As shown in Ref. [9] the anomalous density $\kappa(x)$ has longer exponential tails than the number density $n(x)$. This longer tail of the pairing field becomes particularly important as one approaches the nucleon drip-line. This behavior should be also apparent in the profiles of $V(x)$ and $\Delta(x)$, an aspect which we however neglected here and which does not change the qualitative behavior of these densities, see Fig. 1. Fig. 1 also shows that with increasing spatial resolu-
tion \((dx \to 0)\) the normal density is more accurately reproduced at larger distances. We have also checked that Eqs. (23) and (24) correctly reproduce the normal and anomalous densities as well using the canonical wave functions. In Ref. [9] it was argued that the pairing field \(\Delta(\xi)\) has longer tails than the mean field potential, as the asymptotic behavior of the normal density is controlled by the behavior \(\left|v_k(\xi)\right|^2\) closest to the Fermi level, thus with with smaller eigenvalues \(E_k > 0\), while that of the anomalous density is controlled by only one such qpwf. It appears however that the different weights with which the canonical basis wave functions enter in Eqs. (23) and (24) can lead to the same expected asymptotic behavior of these densities as well.

The canonical occupation probabilities \(n_k\) shown in Fig. 2 have a conspicuous behavior not discussed previously in literature. For smaller lattice constant \(dx\) the maximum momentum cutoff \(p_{\text{cutoff}} = \frac{\hbar}{dx}\) is larger and then the spectrum of \(n_k\) extends to higher energies. The profile of \(n_k\) has two obvious “knees,” one close to the Fermi level, the infrared (IR) knee, and a second one at a high energy, the ultraviolet (UV) knee. The canonical wave functions \(\phi_k(x)\) have the expected spatial behavior as long as their support is commensurate with the support of the number density matrix \(n(x, y)\) as discussed above, see Eq. (20), the text below, and Figs. 3 and 4 in the case of \(dx = 0.125\ \text{fm}\). However, as soon as the support of the canonical wave functions \(\phi_k(x)\) is essentially outside the support of the density matrix \(n(x, y)\), see Fig. 5, for which the index \(k\) is on the right of the UV-knee in Fig. 2, the corresponding \(n_k\) decay significantly faster with \(k\). Both the profiles and the numerical values of \(n_k\) for these canonical states can be obtained with greater accuracy from Eq. (30). These canonical occupation probabilities do not identically vanish simply due to obvious quantum localization effects, but they are increasingly smaller with increasing resolution and decreasing lattice constant \(dx\). In the limit \(dx \to 0\) the UV-knee \(\to \infty\) and at the same time the number of canonical states localized outside the system also tends to infinity. These non-localized canonical states however are irrelevant in describing the physical properties of the system.

Around the IR-knee in Fig. 2 the canonical occupation probabilities have the expected BCS behavior [5]. It is clear however that in between the IR-knee and UV-knee there is a region where the canonical occupation probabilities have a power law behavior. Such a behavior, due to the short-range character of the nuclear forces, has been predicted in 1980 by Sartor and Mahaux [23] and recently put clearly in evidence experimentally by O. Hen et al. [24]. Shima Tan [25] has proven analytically the emergence of this behavior for fermions interacting with a zero-range interaction in 3D. The nuclear pairing is typically simulated in theory with a \(\delta\)-potential, which naturally leads to a local pairing field \(\Delta(\xi)\) [9], as the case we discuss here, and thus this power law behavior of the canonical occupation probabilities is expected. Tan [25] has shown that asymptotically \(n_k \propto 1/k^\alpha\). This power law behavior of the number density is directly related to the divergence of the anomalous density matrix Eq. (7). In case of a 3-dimensional system it was shown in Ref. [9] that the anomalous density matrix \(\kappa(\xi, \zeta) \propto \frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}\) when \(\left|\mathbf{r} - \mathbf{r}'\right| \to 0\), where \(\mathbf{r}\) and \(\mathbf{r}'\) are the spatial components of \(\xi\) and \(\zeta\) respectively.

Within SLDA or any treatment of pairing with a local pairing field \(\Delta(\xi)\) the theory requires regularization and renormalization [10, 26]. In particular, without such a procedure the kinetic energy of a 3-dimensional superfluid system diverges as \(\int d^3k \frac{\hbar^2 k^2}{2m} n_k \to \infty\), where \(k\) stands here for the wave vector. We have checked that \((\epsilon_k)^2 n_k \approx \text{const.}\) in our example in the region beyond the IR-knee and up to the UV-knee, see Fig. 2, confirming the theoretical prediction of Refs. [23, 25]. In the case of pure finite-range nucleon interactions, with no zero-range components, there is an upper momentum cutoff controlled by the interaction range. The expectation value of the average kinetic energy \(\epsilon_k\) of each canonical state shown in Fig. 6, for all states localized inside the system, thus up to the UV-knee in Fig. 2, increases as expected theoretically for standing waves in a cavity. Beyond the UV-knee, for the canonical states localized mostly outside the system, see Fig. 5, \(\epsilon_k\) drops in value, see Fig. 6, and their contribution to the total kinetic energy is commensurate with what one expects from numerical discretization errors \((dx \neq 0)\) of the continuum.

A closer analysis of Fig. 4 clearly show that some canonical wave functions \(\phi_k(x)\) oscillate much faster than the density \(n(x, x)\) and that our estimate of the maximum expected number of relevant canonical wave functions, see Eq. (29), is an understatement when the maximum momentum cutoff \(\frac{\hbar}{dx} > \sqrt{2m|U|}\). The assumption made, when deriving the estimate for \(N_{\text{max}}\), that the canonical wave functions \(\phi_k(x)\) cannot oscillate with

---

**FIG. 5.** The same as Fig. 3 for \(\phi_{300}(x)\) and \(n_{300} = 2.6E - 13\).
a wave vector greater than \( \approx \sqrt{2m|V|/\hbar^2} \), was not accurate. Coupling of the qpwfs components \( v_\xi(\xi) \) to the continuum states, facilitated by \( \Delta(\xi) \), leads to spatial oscillations with any wave vector. In the limit \( dx \to 0 \) the cardinality of the set of canonical wave functions \( \phi_k(\xi) \) is either \( N_0 \) for a finite system in a finite volume or \( \epsilon \) for an isolated finite system in vacuum. Therefore, one should use the best estimate of the number \( N_{\text{max}} \) for the cutoff momentum \( p_{\text{max}} = \frac{2\pi}{dx} \) and from the condition of accommodating a standing wave in our “square well” potential with \( 2R \approx 14.9 \) fm in our numerical example one obtains the approximate position of the UV-knee at \( k_{\text{max}} = \frac{2\pi}{dx} + \mathcal{O}(1) \approx 240 \) for \( dx = 0.125 \) fm (as \( k \) counts the number of half-wave lengths inside the potential well), in perfect agreement with our numerical identification of the UV-knee in Fig. 2. When coupling a bound state through the pairing field \( \Delta \) with the continuum, the strength of the bound state is spread over a large energy range with very long tails, with a Lorentzian shape of the spectral distribution [9]. Moreover, in time-dependent phenomena, even in the absence of a true pairing condensate (when the long range order is lost) and at high excitation energies (with corresponding temperatures well above the pairing phase transition \( T_c \)) the remnant pairing field leads to many single-particle transitions and the quantum Boltzmann one-body entropy increases considerably [27]. With this in mind one can now provide a better estimate of the size of the canonical basis set for a 3-dimensional system in a finite simulation box with sides of length \( L_x = N_x dx, L_y = N_y dy, L_z = N_z dz, (dx = dy = dz) \), ignoring spin and isospin degrees of freedom,

\[
N_{\text{max}} = \frac{4\pi}{3} \left( \frac{\hbar}{dx} \right)^3 \frac{4\pi}{3} \frac{r_0^3}{(2\pi\hbar)^3} \frac{1}{2A} \left( \frac{r_0}{dx} \right)^3. \tag{36}
\]

At the same time the total number of single-particle quantum states in such a box is

\[
N_{\text{spwfs}} = L_x L_y L_z \left( \frac{2\pi\hbar}{dx} \right)^3 = N_x N_y N_z, \tag{37}
\]

which is typically significantly larger. For example for a typical simulation box for a heavy nucleus with volume \( 30^3 \) fm\(^3\) and \( dx = 1 \) fm the total number of qpwfs is \( N_{\text{spwfs}} = 27,000 \gg N_{\text{max}} \approx 2.2A < 1,000 \). This would correspond to a momentum cutoff \( p_{\text{cutoff}} \approx 600\text{MeV}/c \), in agreement with current values used for the nuclear forces within the chiral effective theory. When accounting for both \( u_\xi(\xi) \) and \( v_\xi(\xi) \) components a factor of 2 should be included. Additionally one has to account for the spin and isospin degrees of freedom. This estimate is likely very optimistic, and in practice one would have to consider larger values for the parameter \( r_0 \) in Eq. (36), in order to correctly describe the surface diffuseness of a nucleus and its deformation in a fixed basis set for static problems.

**Acknowledgements**

The funding from the Office of Science, Grant No. DE-FG02-97ER41014 and also the partial support provided in part by NNSA cooperative Agreement DE-NA0003841 to AB is greatly appreciated.

[1] N. N. Bogoliubov, “On a new method in the theory of superconductivity,” Il Nuovo Cimento 7, 853 (1958).
[2] J. G. Valatin, “Comments on the theory of superconductivity,” Il Nuovo Cimento 7, 843 (1958).
[3] B. Zumino, “Normal Forms of Complex Matrices,” J. Math. Phys. 3, 1055 (1962).
[4] C. Bloch and A. Messiah, “The canonical form of an antisymmetric tensor and its application to the theory of superconductivity,” Nucl. Phys. 39, 95 (1962).
[5] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, “Theory of superconductivity,” Phys. Rev. 108, 1175 (1957).
[6] L. N. Oliveira, E. K. U. Gross, and W. Kohn, “Density-functional theory for superconductors,” Phys. Rev. Lett. 60, 2430 (1988).
[7] P. Hohenberg and W. Kohn, “Inhomogeneous Electron Gas,” Phys. Rev. 136, B864 (1964).
[8] W. Kohn and L. J. Sham, “Self-Consistent Equations Including Exchange and Correlation Effects,” Phys. Rev. 140, A1133 (1965).
[9] A. Bulgac, “Hartree-Fock-Bogoliubov approximation for finite systems (1980),” (), arXiv:nucl-th/9907088.
[10] A. Bulgac and Y. Yu, “Renormalization of the Hartree-Fock-Bogoliubov Equations in the Case of a Zero Range Pairing Interaction,” Phys. Rev. Lett. 88, 042504 (2002).
[11] Y. Yu and A. Bulgac, “Energy Density Functional Approach to Superfluid Nuclei,” Phys. Rev. Lett. 90, 222501 (2003).
[12] A. Bulgac, “Local-density-functional theory for superfluid fermionic systems: The unitary Fermi gas,” Phys. Rev. A 76, 040502 (2007).
[13] A. Bulgac, “Time-Dependent Density Functional Theory for Fermionic Superfluids: from Cold Atomic gases, to Nuclei and Neutron Star Crust,” Physica Status Solidi B 256, 2000592 (2019).
[14] P. A. Anderson, Basic Notions of Condensed Matter Physics (Benjamin/Cummins Publishing Company Inc. London, 1984).
[15] P. Ring and P. Schuck, The Nuclear Many-Body Problem, 1st ed. (Springer-Verlag, Berlin Heidelberg New York, 2004).
[16] S. T. Belyaev, A. V. Smirnov, S. V. Tolokonnikov, and S. A. Fayans, “Pairing in nuclei in the coordinate representation,” Sov. J. Nucl. Phys. 45, 783 (45).
[17] A. Bulgac, “Projection of good quantum numbers for reaction fragments,” Phys. Rev. C 100, 034612 (2019).
[18] A. Bulgac, “Restoring broken symmetries for nuclei and reaction fragments,” Phys. Rev. C 104, 054601 (2021).
[19] S. Jin, A. Bulgac, K. Roche, and G. Wiazlowski, “Coordinate-space solver for superfluid many-fermion systems with the shifted conjugate-orthogonal conjugate-gradient method,” Phys. Rev. C 95, 044302 (2017).
[20] A. Bohr and B. R. Mottelson, Nuclear Structure, Vol. I (Benjamin Inc., New York, 1969).
[21] A. Bulgac, “Framework for Polarized Superfluid Fermion Systems (2020),” (), arXiv:2005.06763.
[22] A. Bulgac and M. M. Forbes, “Use of the discrete variable representation basis in nuclear physics,” Phys. Rev. C 87, 051301(R) (2013).
[23] R. Sartor and C. Mahaux, “Self-energy, momentum distribution, and effective masses of a dilute Fermi gas,” Phys. Rev. C 21, 1546 (1980).
[24] O. Hen et al., “Momentum sharing in imbalanced Fermi systems,” Science 346, 614 (2014).
[25] S. Tan, “Large momentum part of a strongly correlated Fermi gas,” Ann. Phys. 323, 2971 (2008).
[26] A. Bulgac, “Local density approximation for systems with pairing correlations,” Phys. Rev. C 65, 051305 (2002).
[27] A. Bulgac, “Pure quantum extension of the semiclassical Boltzmann-Uehling-Uhlenbeck equation,” Phys. Rev. C 105, L021601 (2022).