Boundary $G/G$ theory
and
topological Poisson-Lie sigma model

Fernando Falceto
Depto. Física Teórica, Univ. Zaragoza,
E-50009 Zaragoza, Spain

Krzysztof Gawędzki
C.N.R.S., I.H.E.S., F-91440 Bures-sur-Yvette, France
and Laboratoire de Physique, ENS-Lyon,
46, Allée d’Italie, F-69364 Lyon, France

Abstract

We study a boundary version of the gauged WZW model with a Poisson-Lie group $G$ as the target. The Poisson-Lie structure of $G$ is used to define the Wess-Zumino term of the action on surfaces with boundary. We clarify the relation of the model to the topological Poisson sigma model with the dual Poisson-Lie group $G^*$ as the target and show that the phase space of the theory on a strip is essentially the Heisenberg double of $G$ introduced by Semenov-Tian-Shansky.
1 Introduction

The topological Poisson sigma models introduced in [20] have become an important laboratory for field-theoretical methods applied to mathematical problems, see [4][5][13]. In those applications, it is important to consider the models on two-dimensional surfaces with boundary, for example on a disc. In particular, the perturbative treatment of the quantum theory in such a geometry leads to a formal deformation quantization of the algebra of functions on the target, with the leading order of the $\ast$-product of functions determined by the Poisson bracket. In the specific case when the target manifold is a Poisson-Lie group, this approach should provide a royal route to quantum groups. Although this expectation has not yet materialized, it remains that the case of topological Poisson-Lie sigma models deserves a special attention.

It has been observed quite early [3] that for the target space which is the Poisson-Lie group $G^\ast$ dual to a simple complex Lie group $G$ equipped with an $r$-matrix Poisson structure, the Poisson sigma model in the bulk is essentially equivalent on the classical level to the gauged Wess-Zumino-Witten model, the so called $G/G$ coset theory [12]. The latter is a topological field theory whose quantum partition and correlation functions on closed surfaces compute the Verlinde dimensions of the spaces of non-abelian theta functions [21].

The purpose of this note is four-fold. First, we present a covariant description of the relation between the classical Poisson-Lie sigma model with $G^\ast$ target and the $G/G$ coset theory, obtained in [3] in the Hamiltonian approach. Second, we extend the discussion to space-times with boundary. In the latter case, the requirement of equivalence to the Poisson-Lie sigma model fixes the boundary contributions in the gauged WZW model. More exactly, the precise equivalence of two theories requires a replacement of the target $G^\ast$ in the Poisson sigma model by its quotient by a discrete subgroup and a restriction of the target in the gauged WZW model to an open subset of $G$. As a result, the action of the $G/G$ theory becomes well defined for any value of the coupling constant $k$.

Third, we describe the canonical structure of the resulting classical field theory on a cylinder and on a strip. In the first case, it is essentially the space of conjugacy classes of commuting pairs of elements in $G$. In the second case, the phase space appears to be a version of the Heisenberg double of Poisson-Lie group $G$ introduced by Semenov Tian-Shansky in [18], a Poisson-Lie replacement for the cotangent bundle $T^*G$. This provides an example
of a general construction of a Poisson Lie groupoid of a Poisson manifold described in [9].

Finally, we discuss briefly the quantization of the theory for the Poisson structures induced by the standard classical $r$ matrices. The restriction of field values in the gauged WZW theory imposed by the equivalence to the topological Poisson-Lie theory appears to be not quite innocuous, at least for generic values of $k$. It leads, for example, to an infinite-dimensional space of boundary states, despite a topological character of the theory. Nevertheless, as is generally the case for two-dimensional topological field theories [13], this space has a structure of an associative non-commutative algebra. It may be viewed as the quantum deformation $\mathcal{U}_q(\mathfrak{g})$ of the enveloping algebra of Lie algebra $\mathfrak{g}$ of $G$. Another problem is the absence of unitarity that has a classical source: the action of the gauged WZW model equivalent to the Poisson-Lie sigma model is not real in the Minkowski signature with respect to the conjugation of fields $g$ and $A$ that fix the compact real forms of $G$ and $\mathfrak{g}$. For $q$ a root of unity (i.e. for integer $k$), one may reduce the algebra of the boundary states of the $G/G$ theory considered here to a finite-dimensional one to recover a genuine model of quantum topological field theory. This is a different version of the boundary $G/G$ theory than the one constructed recently in [11] within an approach respecting unitary at all stages. The gain from the present approach is its direct relation to the topological Poisson sigma models and Poisson-Lie theory.

The paper is organized as follows. In Sect. 2 we recall various facts from the theory of Poisson-Lie groups needed later. Sect. 3 defines a version of the gauged WZW model on surfaces with boundary by restricting appropriately the target. Sect. 4 establishes the relations between the restricted $G/G$ theory and the Poisson-Lie sigma models with a discrete quotient of $G^*$ as the target. The phase-space structure of the theory in closed and open geometries is discussed in Sect. 5 and the remarks about quantization of the theory are contained in Sect. 6.

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2 Basic Poisson-Lie theory

Let us start by recalling few basic facts from the theory of Poisson-Lie groups, see [8] [10] [12] [16]. A Poisson-Lie group is a Lie group equipped with a Poisson structure which makes the product $m : G \times G \to G$ a Poisson map.
if \( G \times G \) is considered with the product Poisson structure. Linearization of the Poisson structure at the unit 1 of \( G \) provides a Lie algebra structure on \( g^* = T^*_1(G) \) by the formula

\[
[d\xi(1), d\zeta(1)]^* = d\{\xi, \zeta\}(1).
\]

The fact that the product in \( G \) is Poisson is reflected in the compatibility condition

\[
\langle [X, Y], [v, w]^* \rangle + \langle \text{ad}^*_w X, \text{ad}^*_Y v \rangle - \langle \text{ad}^*_v Y, \text{ad}^*_w X \rangle - \langle \text{ad}^*_v Y, \text{ad}^*_X v \rangle = 0.
\]

On the other hand if \( G \) is connected and simply connected, this is the only condition for integrating \([, ,]^*\) to a Poisson structure on \( G \) that makes it Poisson-Lie and the Poisson structure is unique. Since \( g \) and \( g^* \) enter symmetrically in (2), one has also a connected and simply connected Poisson-Lie group \( G^* \) with Lie algebra \((g^*, [ , ,]^*)\) and a Poisson structure whose linearization at 1 gives the bracket \([, ,] \). \( G^* \) is the dual Poisson-Lie group of \( G \).

Condition (2) allows to define a Lie bracket in \( g \oplus g^* \) by the formula

\[
[X + v, Y + w] = [X, Y] + [v, w]^* - \text{ad}^*_w X + \text{ad}^*_Y v + \text{ad}^*_v X - \text{ad}^*_w Y.
\]

The corresponding simply connected group \( D \), together with its local subgroups \( G \) and \( G^* \) form a so called local double Lie group [16]. \( D \) can be equipped canonically with a Poisson structure, non-degenerate around the unit, such that the local embeddings of \( G \) and \( G^* \) in \( D \) are Poisson maps (see [18], [2] for details). \( D \) with this Poisson structure is called the Heisenberg double. It can be considered a generalization of \( T^*G \) for Poisson-Lie groups.

The above construction may be described in more concrete terms that provide an explicit realization for the group \( G^* \) and for the Poisson structures of \( G \) and \( G^* \). We shall take \( G \) to be a complex, simple, connected, simply connected Lie group and we shall work in the complex category. The non-degenerate, invariant, bilinear form \( \text{tr} \) on \( g \) permits to establish an isomorphism between \( g \) and \( g^* \). The Poisson structures will be represented by a bivector field \( \Gamma \) so that the Poisson bracket is given by the contraction of \( \Gamma \), i.e. \( \{\xi, \zeta\}(g) = \iota(\Gamma_g) d\xi \wedge d\zeta \). For the Poisson structure on \( G \), the components of \( \Gamma \) contracted with the right-invariant forms \( \Lambda(X) = \text{tr}(dgg^{-1}X) \), with \( X \in g \), will be denoted by \( \gamma_g(X, Y) = \iota(\Gamma_g) \Lambda(X) \wedge \Lambda(Y) \).

As it is shown in ref. [16], for a general Poisson-Lie structure on \( G \),

\[
\gamma_g(X, Y) = \frac{1}{2} \text{tr}(XrY - XAd_g rAd_g^{-1}Y). \tag{4}
\]
for an antisymmetric linear operator \( r : \mathfrak{g} \to \mathfrak{g} \) such that
\[
    r[rX,Y] + r[X,rY] - [rX,rY] = \alpha[X,Y], \quad \text{with} \quad \alpha \in \mathbb{C}.
\] (5)

The linearization of the Poisson-Lie structure at the unit \( 1 \in G \) defines a new Lie bracket in \( \mathfrak{g} \), namely
\[
    [X,Y]_r = \frac{1}{2}([X,rY] + [rX,Y]).
\] (6)

The bracket \([ , ]\) in \( \mathfrak{g} \) corresponds to \([ , ]^*\) in \( \mathfrak{g}^* \), via the linear isomorphism induced by the bilinear form \( \text{tr} \).

In the following we shall take \( \alpha = 1 \) in (5) which (provided \( \alpha \neq 0 \)) can be achieved by a trivial rescaling of the bilinear structure in \( \mathfrak{g} \) or of the Poisson bracket in \( G \). In this case
\[
    [r_X, r_Y] = \frac{1}{2}(r_X + r_Y) \quad \text{and the embedding}
\]
\[
    \sigma : \mathfrak{g} \to \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}
\]
\[
    X \mapsto (r_+X, r_-X)
\] (8)
defines an homomorphism from \((\mathfrak{g}, [ , ])\) to \((\mathfrak{d}, [ , ]_{\mathfrak{d}})\) with \([ , ]_{\mathfrak{d}} = ([ , ]_{\mathfrak{d} \oplus \mathfrak{d}})\).

This way one may realize the algebra \((\mathfrak{g}^*, [ , ]^*)\) as the subalgebra \( \mathfrak{g}_r = \sigma(\mathfrak{g}) \subset \mathfrak{d} \) and the dual group \( G^*_r \) as the subgroup \( G_r \) in \( G \times G \) corresponding to the Lie subalgebra \( \mathfrak{g}_r \). We shall denote by \((g_+, g_-)\) the elements of \( G_r \). We may identify \( \mathfrak{g} \) with \( \mathfrak{g}_r^* \) using \( \sigma \) and \( \text{tr} \). Explicitly, the pairing between \((r_+X, r_-X) \in \mathfrak{g}_r \) and \( Y \in \mathfrak{g} \) is given by
\[
    \text{tr}XY = \text{tr}(r_+X - r_-X)Y.
\] (9)

The original bracket \([ , ]\) on \( \mathfrak{g} \) integrates then to a Poisson-Lie structure \( \Gamma^r \) on \( G_r \).

One may use the right-invariant forms on \( G_r \), \( \Lambda^r(X) = \text{tr}[(dg_+g_+^{-1} - dg_-g_-^{-1})X] \) for \( X \in \mathfrak{g} \), in order to compute the components of \( \Gamma^r \). They take the form
\[
    \gamma^r_{g_+, g_-}(X,Y) = \text{tr}[X(Ad_{g_+} - Ad_{g_-})(r_+Ad_{g_-}^{-1} - r_-Ad_{g_+}^{-1})Y].
\] (10)

Another point from the Poisson-Lie group theory that we shall need is the description of the Heisenberg double. In our case we shall take
\[ D = G \times G \] with the subgroups \( G_r \) and \( G_d = \{(g, g) \mid g \in G\} \) corresponding to \( G^* \) and \( G \). Since the intersection of the Lie algebras \( g_r \cap g_d \) is trivial, the intersection \( G_r \cap G_d \) is composed of elements \((g_0, g_0)\) where \( g_0 \) belongs to a discrete subgroup \( G_0 \subset G \). For a later use, let us note that if \((r_+, X, r_-X) \in g_r\) then \((Ad_{g_0} r_+ X, Ad_{g_0} r_- X)\) also belongs to \( g_r \) and hence is equal to \((r_+ Ad_{g_0} X, r_- Ad_{g_0} X)\). This implies that

\[ r_\pm Ad_{g_0} = Ad_{g_0} r_\pm. \] (11)

We shall be interested in the description of the Poisson structure of \( D \) on its main symplectic leaf \( D_0 = G_d G_r \cap G_r G_d \) which contains a neighborhood of the unit of \( D \). In \( D_0 \) the Poisson bracket is non degenerate and therefore it may be inverted to produce the corresponding symplectic structure \( \Omega \). If we take \((hg_+, hg_-) = (\tilde{g}_+, \tilde{g}_-) \in D_0\), then

\[
\Omega((hg_+, hg_-)) = \text{tr}[dh^{-1}(d\tilde{g}_+ \tilde{g}_-^{-1} - d\tilde{g}_- \tilde{g}_+^{-1}) + \tilde{h}^{-1} dh(g_+^{-1} dg_+ - g_-^{-1} dg_-)]
\] (12)

Note that the above form is indeed well defined on \( D_0 \) and that it is invariant under the left and right diagonal actions of \( G_0 \).

Consider the map

\[ \pi : G_r \ni (g_+, g_-) \mapsto g_- g_+^{-1} \in G. \] (13)

\(\pi(g_+, g_-) = \pi(\tilde{g}_+, \tilde{g}_-)\) if and only if \( \tilde{g}_\pm = g_\pm g_0 \) for \( g_0 \in G_0 \). It is easy to see that \( \pi \) is a submersion so that \( \pi(G_r) \) is an open connected subset of \( G_r \) containing the unit. In fact, \( \pi : G_r \rightarrow \pi(G_r) \) is a \( G_0 \) principal bundle, hence a discrete covering of \( \pi(G_r) \cong G_r / G_0 \). The symplectic leaves of the Poisson structure \( \Gamma^r \) on \( G_r \) are connected components of the pre-images by \( \pi \) of conjugacy classes in \( G \). The Poisson structure \( \Gamma^r \) projects by \( \pi \) to a Poisson structure on \( \pi(G_r) \) whose contraction with \( \Lambda(X) \wedge \Lambda(Y) \) is

\[ \gamma^r_g(X,Y) = \text{tr}[X(r_- - Ad_g r_+)(1 - Ad_g^{-1})Y]. \] (14)

The symplectic leaves of the projected structure are the connected components of intersection of \( \pi(G_r) \) with the conjugacy classes in \( G \). In fact, formula (14) defines a Poisson structure on the whole group manifold \( G \). This is a different Poisson structure on \( G \) than that given by (4). In particular, it does not make \( G \) with its standard multiplication a Poisson-Lie group.
In the connection between topological Poisson-sigma model with the dual group $G^*$ as the target and the gauged group $G$ WZW model, the operator $r$ plays the crucial role. It determines, as we have seen, the Poisson structure of the target in the former and in the latter it may be used to write the closed form $\chi = \frac{1}{3} \text{tr}[(g^{-1}dg)^{\wedge 3}]$ on $G$ entering the Wess-Zumino term of the action as a differential of a 2-form on the open subset $\pi(G_r)$ of $G$. More exactly, the pullback of $\chi$ by $\pi$ has the form

$$
\pi^* \chi = \frac{1}{3} \text{tr}[(g^{-1}_- dg_- - g^{-1}_+ dg_+)^{\wedge 3}]
= d \text{tr}[g^{-1}_- dg_- \wedge g^{-1}_+ dg_+]
+ \frac{1}{3} \text{tr}[(g^{-1}_- dg_-)^{\wedge 3}] - \frac{1}{3} \text{tr}[(g^{-1}_+ dg_+)^{\wedge 3}].
$$

(15)

Let us show that the last two terms cancel each other. Define $\mu = g^{-1}_+ dg_+ - g^{-1}_- dg_- \in \Lambda^1(G_r) \otimes \mathfrak{g}$. One has $r \pm \mu = g^{-1}_\pm dg_\pm$ and

$$
\text{tr}[(g^{-1}_+ dg_+)^{\wedge 3}] = \text{tr}[(r_+ \mu)^{\wedge 3}]
= \frac{1}{2} \text{tr}[r_+ (\mu \wedge r \mu + r \mu \wedge \mu) \wedge r_+ \mu]
= \frac{1}{2} \text{tr}[(\mu \wedge r \mu + r \mu \wedge \mu) \wedge r_- r_+ \mu]
$$

where we have used (7) and the antisymmetry of $r$ with respect to the bilinear form $\text{tr}$. The same result is obtained for $\text{tr}[(g^{-1}_- dg_-)^{\wedge 3}]$ (of course $r_+ r_- = r_- r_+$). Coming back to (15) we infer that $\chi = d\rho$ on $\pi(G_r)$ where $\rho$ is a 2-form on $\pi(G_r)$ such that

$$
\pi^* \rho = \text{tr}[g^{-1}_- dg_- \wedge g^{-1}_+ dg_+].
$$

(17)

A straightforward computation leads to an equivalent expression for the symplectic form $\Omega$ of (12) on the leaf $D_0$ in the Heisenberg double:

$$
\Omega = \text{tr}[(g^{-1}dg + dgg^{-1} + g^{-1}h^{-1}dhg)h^{-1}dh]
+ \rho(g) - \rho(hgh^{-1}).
$$

(18)

The right hand side may be viewed as the 2-form on the space of pairs $(h, g)$ such that both $g = g_- g_+^{-1}$ and $hgh^{-1} = \tilde{g}_- \tilde{g}_+^{-1}$ are in $\pi(G_r)$. We shall denote it then as $\Omega(h, g)$. It is in this form that the symplectic form of the Heisenberg double will appear below.
3 Restricted $G/G$ coset theory

The $G/G$ coset theory is the gauged version of the WZW model \[22\] with the group $G$ as the target. The fields of this model on a two-dimensional oriented surface $\Sigma$ equipped with a conformal or a pseudo-conformal structure are $g : \Sigma \to G$ and a $g$-valued 1-form on $\Sigma$. The action functional of the model has the form

\[
S_{WZW}(g, A) = \frac{1}{4\pi} \int_{\Sigma} \text{tr} [(g^{-1} \partial_l g)(g^{-1} \partial_r g)] dx^l \wedge dx^r + S_W(g)
\]

\[+ \frac{1}{2\pi} \int_{\Sigma} \text{tr} [\partial_l gg^{-1} A_r - A_l g^{-1} g^r \partial_r g] dx^l \wedge dx^r \]

\[+ \frac{1}{2\pi} \int_{\Sigma} \text{tr} [A_l A_r - gA_l g^{-1} A_r] dx^l \wedge dx^r \tag{19}\]

where $x^l = z$, $x^r = \bar{z}$ are the complex variables in the Euclidean signature and the light-cone ones $x^l = x + t$, $x^r = x - t$ in the Minkowski signature. The derivatives $\partial_l = \frac{\partial}{\partial x^l}$, $\partial_r = \frac{\partial}{\partial x^r}$ and $A = A_l dx^l + A_r dx^r$. The Wess-Zumino term $S_W(g)$ is often written as $\frac{1}{4\pi} \int_{\Sigma} g^* d^{-1} \chi$ where $d^{-1} \chi$ stands for a 2-form whose differential is equal to the 3-form $\chi$ on $G$. Since such 2-forms do not exist globally, some choices are needed. On closed surfaces, one may replace $\int_{\Sigma} g^* d^{-1} \chi$ by $\int_{B} g^* \chi$ where $B$ is an oriented 3-manifold with $\partial B = \Sigma$ and $\tilde{g} : B \to G$ extends $g$. This gives a well defined amplitude $e^{ikS_{WZW}}$ for integer $k$. When $\Sigma$ has a boundary, the definition of the amplitude is more problematic and, in general, it makes sense only as an element of a product of line bundles over the loop group $LG$ \[3\]. To extract a numerical value of such an amplitude one has to use sections of the line bundle that are not globally defined.

What we shall do here, is to restrict the values of the field $g$ of the gauged WZW model to the subset $\pi(G_r) \subset G$ and to define

\[
S_{WZ}(g) = \frac{1}{4\pi} \int_{\Sigma} g^* \rho \tag{20}\]

where $\rho$ is given by \[17\]. In the geometric language, this corresponds to a particular choice of a section in the (trivial) restriction of the line bundle over $LG$ to the loop space $L\pi(G_r)$. We shall call the resulting field theory the restricted $G/G$ coset model.
The model defined above has simple transformation properties under the gauge transformations of fields
\[ h^g = hgh^{-1}, \quad h^A = hAh^{-1} + hdh^{-1} \] (21)
for \( h : \Sigma \to G \) such that \( h^g \) takes also values in \( \pi(G_r) \) (this is always accomplished for \( h \) sufficiently close to unity). The action transforms according to
\[ S_{WZW}(h^g, h^A) = S_{WZW}(g, A) + \int_{\Sigma} \Omega(h, g) \] (22)
where \( \Omega(h, g) \) is the closed 2-form given by (18). In particular, it follows that the action is invariant under infinitesimal gauge transformations that vanish on the boundary.

The equations of motion of the model \( \delta S_{WZW} = 0 \) for field variations vanishing at the boundary are
\[ D_l(g^{-1}D_rg)dx^l \wedge dx^r + F(A) = 0 \] (23,a)
\[ g^{-1}D rg = 0, \quad gD_lg^{-1} = 0 \] (23,b)
where \( D \) stands for the covariant derivative \( D_lr = \partial_lr + [A_{l,r}, \cdot] \) and \( F(A) = dA + A \wedge A \) is the field strength of \( A \).

Although (23,a) is a second order differential equation we can write a system of first order equations equivalent to (23), by simply taking
\[ F(A) = 0, \] (24,a)
\[ g^{-1}Dg = 0. \] (24,b)

The situation we find is analogous to the following simple mechanical system.
Starting from Lagrangian
\[ L = \frac{1}{2} \dot{x}^2 + y \dot{x} \]
the equations of motion are
\[ \ddot{x} + y = 0 \quad \dot{x} = 0 \]
or equivalently
\[ y = 0 \quad \dot{x} = 0. \]
As we have first order equations of motion it should be possible to get the system from a first order Lagrangian. We can achieve this goal by adding
to the Lagrangian terms quadratic in the equations of motion which do not
change the dynamics. The phase space structure of the theory remains then
unchanged. In the Hamiltonian language, all we do is to trade secondary
constraints for primary ones, which does not make any difference in the
canonical analysis of the theory. In our simple example, the new Lagrangian

\[ L' = y\dot{x} \]

is equivalent to the previous one. Note, on the other hand, that the new
action is topological in the sense that it is invariant under diffeomorphism
of time.

In the following we will proceed in a similar way for the restricted \( G/G \)

coset model, using its first order equations of motion to eliminate from the
action the piece quadratic in derivatives. We shall identify the result with a
topological Poisson sigma model with group \( G_r \) equipped with the Poisson
structure \( \Gamma^r \) as the target. We shall call the latter theory the topological
Poisson-Lie sigma model.

4 Topological Poisson-Lie sigma model

In general, the Poisson sigma model is a two-dimensional sigma model

with the target manifold \( \mathcal{X} \) equipped with a Poisson structure \( \Gamma \). The fields
of the model are \( X : \Sigma \to \mathcal{X} \) and a 1-form \( \psi \) on \( \Sigma \) with values in the pullback
by \( X \) of the cotangent bundle of \( \mathcal{X} \). The action functional has the form

\[ S_{Ps}(X, \psi) = -\frac{1}{2\pi} \int_{\Sigma} \left[ (dX, \psi) + \frac{1}{2} \langle \Gamma \circ X, \psi \wedge \psi \rangle \right] \tag{25} \]

where \( \langle , \rangle \) denotes the pairing between the tangent and the cotangent
vectors to \( \mathcal{X} \).

We shall be interested in the Poisson-Lie case where \( \mathcal{X} = G_r \) and \( \Gamma = \Gamma^r \).

In this case, \( X = (g_+, g_-) \). The tangent bundle to \( G_r \) may be identified with
the trivial bundle \( G_r \times g_r \) by the right translations. With this identification,
\( dX = (dg_+ g_+^{-1}, dg_- g_-^{-1}) \). Similarly, the cotangent bundle to \( G_r \) may be
identified by the right translations with the trivial bundle \( G_r \times g \) and, using
this identification, field \( \psi \) becomes a \( g \)-valued 1-form that we shall denote
by \( A \). Recalling the pairing (10) between \( g_r \) and \( g \) and the formula (10) for
the components of \( \Gamma^r \), we infer that

\[ S_{Ps}(g_+, g_-, A) = -\frac{1}{2\pi} \int_{\Sigma} \text{tr}[(dg_+ g_+^{-1} - dg_- g_-^{-1}) \wedge A] \]
\[ + \frac{1}{2} A \wedge (\text{Ad} g_+ - \text{Ad} g_-)(r_+ \text{Ad}^{-1} g_+ - r_- \text{Ad}^{-1} g_-) A. \]  

(26)

The Poisson-Lie sigma model with \( G_r \) target and fields \((g_+, g_-)\) and \( A \) is closely related to the gauged WZW model with the target \( \pi(G_r) \) and fields \( g = g_- g_+^{-1} \) and \( A \) that we considered in the previous section. The action of the latter is given by equation (19) with the Wess-Zumino term defined by (20). More explicitly,

\[
S_{WZW}(g_- g_-, A) = \frac{1}{4\pi} \int \sum \text{tr}((g^{-1}_+ \partial_t g_+)(g^{-1}_- \partial_r g_-) + (g^{-1}_- \partial_t g_-)(g^{-1}_+ \partial_r g_+))
\]

\[-2(g^{-1}_+ \partial_t g_+)(g^{-1}_- \partial_r g_-)]dx^l \wedge dx^r
\]

\[-\frac{1}{2\pi} \int \sum \text{tr}(A_l g^{-1}_- \partial_t g + (g \partial_t g^{-1}_+))A_r
\]

\[+ g A_t g^{-1}_+ A_r - A_t A_r]dx^l \wedge dx^r. \]  

(27)

The equations of motion are (24,a) and (24,b). The latter reads in terms of \((g_+, g_-)\) as

\[ P_{l, \pm} \equiv g^{-1}_\pm \partial_l g_\pm + r_\pm (\text{Ad}^{-1} g_\pm - \text{Ad}^{-1} g_-) A_l = 0, \]

\[ P_{r, \pm} \equiv g^{-1}_\pm \partial_r g_\pm + r_\pm (\text{Ad}^{-1} g_\pm - \text{Ad}^{-1} g_+) A_r = 0. \]  

(28)

After a straightforward calculation one obtains the identity:

\[
S_{WZW}(g_- g_+, A) = \frac{1}{4\pi} \int \sum \text{tr}(P_{l,+} P_{r,+} + P_{l,-} P_{r,-} - 2P_{l,+} P_{r,-})dx^l \wedge dx^r
\]

\[+ S_{P\sigma}(g_+, g_- A) \]  

(29)

where \( S_{P\sigma} \) is the action (26) of the Poisson-Lie sigma model.

Note that the difference between \( S_{WZW} \) and \( S_{P\sigma} \) is quadratic in the equations of motion for \( S_{WZW} \). This implies that neither the classical solutions nor the phase space structure of both models (based on first functional derivatives of the action evaluated on-shell) differ, as long as the boundary conditions coincide. The latter will be chosen to require in both models that \( A \) vanishes on vectors tangent to boundary, with no conditions on \( g \) nor \((g_+, g_-)\), see below. The relation between the gauged WZW model and the \( G_r \) Poisson sigma model was first established in [3] in the Hamiltonian formalism. The above discussion provides a covariant version of that result valid in an arbitrary space-time topology.
A warning is in order here: the equivalence of the models is restricted for the moment to fields $g$ of the $G/G$ theory which are of the form $g_- g_+^{-1}$ for $(g_+, g_-) : \Sigma \to G_r$. The general $\pi(G_r)$-valued fields $g$ have such form only locally, with local representatives possibly differing by the right action of elements $g_0 \in G_0$. To extend the relation to the general case, let us replace the gauge field $A$ by another $g$-valued 1-form

$$
\eta = \text{Ad}_{g_-}(r_- \text{Ad}_{g_+}^{-1} - r_+ \text{Ad}_{g_-}^{-1}) A,
$$

(30)

Given the $\pi(G_r)$-valued field $g$, 1-form $\eta$ is unambiguously defined, as follows from (11). Conversely, the gauge field $A$ may be recovered from $\eta$:

$$
A = (r_- \text{Ad}_{g_-}^{-1} - r_+) \eta.
$$

(31)

Actually (30) and (31) solve the equation

$$
\text{tr}(dgg^{-1} \wedge \eta) = \text{tr}[(dg_+ g_+^{-1} - dg_- g_-^{-1}) \wedge A].
$$

(32)

Now, in terms of $g$ and $\eta$,

$$
S_{P \sigma} = -\frac{1}{2\pi} \int_{\Sigma} \text{tr}[(dgg^{-1} \wedge \eta) + \frac{1}{2} \eta \wedge (r_- - \text{Ad}_{g_+})(1 - \text{Ad}_{g_-}^{-1})].
$$

(33)

We recognize the action of the Poisson sigma model with the target $\pi(G_r) \cong G_r/G_0$ equipped with the Poisson structure (14) projected from $G_r$. The relations (30) and (31) define now a one-to-one correspondence between the classical solution $(g, A)$ and of the gauged WZW theory and the solution $(g, \eta)$ of the Poisson sigma model with the action (33), both theories with $\pi(G_r)$ target.

5 Canonical structure

Having established the connection between the restricted $G/G$ coset theory and the orbifold version of the Poisson-Lie sigma model with $G_r$ target, we shall study the phase space structure of the theory on space-times $\Sigma = \mathbb{R} \times (\mathbb{R}/2\pi \mathbb{Z})$ (closed geometry) or $\Sigma = \mathbb{R} \times [0, \pi]$ (open geometry), with the coordinates $(t, x)$ and Minkowski signature. We shall use the first order formalism, specially convenient for extracting the symplectic structure of the phase space, see [10].

In this formalism, the basic object is a 2-form $\alpha$ on a space $\mathcal{P}$ which one may take as a bundle over the space-time $\Sigma$. (In general space-time
dimension $d$, $\alpha$ is a $d$-form). Fields $\Phi$ are sections of $\mathcal{P}$ and the first order action is given by

$$S(\Phi) = \int_\Sigma \Phi^* \alpha$$

(34)

In the open geometry, fields $\Phi$ may be required to satisfy boundary conditions restricting their values on $\partial \Sigma$ to a subbundle $\mathcal{P}_\partial$ of $\mathcal{P}|_{\partial \Sigma}$. A term $\int_{\partial \Sigma} \Phi^* \beta$ where $\beta$ is a 1-form on $\mathcal{P}_\partial$ ($(d-1)$-form in general) may be added to the action. The variational equations $\delta S(\Phi) = 0$ take the geometric form

$$\Phi^* (\iota(\delta \Phi) d \alpha) = 0 \quad \text{on} \quad \Sigma$$

(35,a)

$$\Phi^* (\iota(\delta \Phi)(\alpha + d \beta)) = 0 \quad \text{along} \quad \partial \Sigma$$

(35,b)

for any vector fields $\delta \Phi$ giving the infinitesimal variation of $\Phi$ tangent on $\partial \Sigma$ to the subbundle $\mathcal{P}_\partial$. Let us denote by $\tilde{P}$ the space of the solutions of these equations defined on $\Sigma$ (and subject to the boundary condition).

Let now $\Sigma_{12}$ be a a chunk of $\Sigma$ between two (maximal) curves ($(d-1)$-dimensional surfaces, in general) $C_1$ and $C_2$ transverse to the time axis. Let $S_{12}(\Phi)$ be defined by restricting the integration in (34) to $\Sigma_{12}$ (and to $\partial \Sigma \cap \Sigma_{12}$ in the $\beta$-term). If $\Phi$ belongs to the space $\tilde{P}$ of solutions and the vector field $\delta \Phi$ gives a variation tangent to $\tilde{P}$ then, as a straightforward calculation shows,

$$\delta S_{12}(\Phi) = \Xi_{C_2}(\delta \Phi) - \Xi_{C_1}(\delta \Phi)$$

(36)

where $\Xi_C$ is a 1-form on the space of solutions defined by

$$\Xi_C(\delta \Phi) = \int_C \Phi^* (\iota(\delta \Phi) \alpha) - \int_{\partial C} \Phi^* (\iota(\delta \Phi) \beta).$$

(37)

Clearly $\tilde{\Omega} = d \Xi$ is a closed 2-form on $\tilde{P}$ that does not depend on the choice of $C$. The explicit formula for $\tilde{\Omega}$ reads

$$\tilde{\Omega}(\delta_1 \Phi, \delta_2 \Phi) = \int_C \Phi^* (\iota(\delta_2 \Phi) \iota(\delta_1 \Phi) d \alpha))$$

$$- \int_{\partial C} \Phi^* (\iota(\delta_2 \Phi) \iota(\delta_1 \Phi)(\alpha + d \beta)).$$

(38)

The 2-form $\tilde{\Omega}$ does not have to be non-degenerate, but, since it is closed, its degeneration distribution is involutive. By definition, the phase space $P$ of
the field theory is the space of leaves of this distribution. \( \tilde{\Omega} \) descends to a symplectic form on \( P \) (more exactly, on its non-singular part).

In our case, we could use either the Poisson-Lie formulation of the theory whose action is already first order in field derivatives or the coset one. The latter requires introduction of variables \( \xi_{l,r} \) representing the light-cone derivatives of field \( g \) to obtain its first order formulation. As stressed above, both approaches lead to the same phase space.

### 5.1 Poisson sigma models

Let us start by few comments about the canonical structure of the topological Poisson sigma models with a general target \( X \), an interesting issue in its own.

For a Poisson sigma model we take \( \mathcal{P} = \Sigma \times T^* X \oplus T^* X \) with local coordinates \( (x^l, x^r, X^a, \psi_{at}, \psi_{ar}) \) and

\[
\alpha = -\frac{1}{2\pi} \left[ dX^a \wedge (\psi_{at} dx^l + \psi_{ar} dx^r) + \Gamma^{ab}(X)\psi_{at} \psi_{bt} dx^l \wedge dx^r \right]
\]

so that the action \( (14) \) with \( \beta = 0 \) reproduces the action \( (25) \) with \( \psi = \psi_l dx^l + \psi_r dx^r = \psi_t dt + \psi_x dx \). In the bulk, the equations of motion are:[20]

\[
\partial_\mu X^a = \Gamma^{ab}(X)\psi_{b\mu},
\]

\[
\partial_\mu \psi_{ar} - \partial_\nu \psi_{a\mu} = \partial_\alpha \Gamma^{bc}(X)\psi_{b\mu} \psi_{c\nu}.
\]

In the open geometry, one has, additionally, the boundary equation

\[
\delta X^a \psi_{at}(t, 0) = 0 = \delta X^a \psi_{at}(t, \pi).
\]

It will be solved by imposing the condition \( \psi_t = 0 \) and keeping \( X \) free on the boundary. On more general surfaces \( \Sigma \) with boundary, we shall require that \( \psi \) vanishes on vectors tangent to \( \partial \Sigma \). Such boundary conditions were used in [4] in connection with the Kontsevich deformation quantization [14].

Note that \( X \) evolves within fixed symplectic leaves and that it is constant on the boundary. The 2-form \( \tilde{\Omega} \) on the space of solutions is

\[
\tilde{\Omega} = \frac{1}{2\pi} \int_C dX^a \wedge d\psi_{ax} \, dx
\]
with the integral over any maximal curve of constant $t$. The form $\tilde{\Omega}$ has a large degeneration kernel given by the vector fields

$$
\delta X^a = \Gamma^{ab}(X)\epsilon_b, \quad \delta \psi_{a\mu} = \partial_{\mu}\epsilon_a - \partial_a \Gamma^{bc}\psi_{b\mu}\epsilon_c
$$

(43)

with $\epsilon_a(t, x)$ vanishing on the boundary in the open geometry. The latter should be viewed as corresponding to the infinitesimal gauge transformations in the Poisson sigma model context \cite{20}. The phase space of the theory $P$ is the space of the classical solutions modulo the degeneration induced by the vector fields (43). At non-singular points, the tangent space to $P$ may be identified with the space of vector fields $(\delta X, \delta \psi_x)$ at fixed $t$ such that

$$
D_1(\delta X, \delta \psi_x) \equiv \partial_x \delta X^a - \Gamma^{ab}(X)\delta \psi_{bx} - \partial_t \Gamma^{ab}(X)\psi_{bx}\delta X^c = 0
$$

(44)

modulo

$$
(\delta X, \delta \psi_x) = (\Gamma^{ab}(X)\epsilon_b, \partial_x\epsilon_a - \partial_a \Gamma^{bc}(X)\psi_{bx}\epsilon_c) \equiv D_0 \epsilon
$$

(45)

with $\epsilon(t, 0) = \epsilon(t, \pi) = 0$ in the open geometry. Operators $(D_0, D_1)$ form an elliptic complex. We infer that the phase space $P$ is finite-dimensional in non-singular points and has dimension equal to the index of $(D_0, D_1)$.

It is easy to compute the latter at special solution $X = X_0 = \text{const.}$, $\psi = 0$ of the classical equations. Let $(\delta X, \delta \psi_x)$ be a corresponding solution of (44) in the open geometry. By subtracting the pair (45) for a unique $\epsilon$ vanishing on the boundary, $(\delta X, \delta \psi_x)$ may be brought to a linear form $(\delta' X, \delta' \psi_x)$ with

$$
\delta' X^a(x) = x\Gamma^{ab}(X_0)\delta' \psi_{bx} + \delta X^a(0), \quad \delta' \psi_x = \frac{1}{\pi} \int_0^\pi \delta \psi_x \, dx
$$

(46)

otherwise unconstrained. It follows that the virtual dimension of $P$ is equal to $2\dim(\mathcal{X})$ in this case. As explained in \cite{5}, see also \cite{6}, symplectic space $P$ may be identified for the open geometry with the Poisson groupoid of $\mathcal{X}$ (again modulo singularities). In particular, a point in $P$ contains the information about the homotopy class $[X]$ of paths in a fixed symplectic leaf of $\mathcal{X}$ obtained by restricting the solution $X$ to any spatial hypercurve. Such paths run between the $(t$-independent) points $X(t, 0)$ and $X(t, \pi)$. The homotopy classes $[X]$ may be composed if the end of one agrees with the beginning of the other and this composition is consistent with the groupoid structure in $P$. We refer for the details to \cite{5}. 15
Similarly, in the closed geometry, one may achieve that
\[
\delta' X^a(x) = \delta X^a(0) - \Gamma^{ab}(X_0) \epsilon_b(0), \quad \delta' \psi_x = \frac{1}{2\pi} \int_0^{2\pi} \delta \psi_x \, dx
\] (47)
but now with the constraint \(\Gamma^{ab}(X_0) \delta' \psi_{bx} = 0\) that follows from (44) and the remaining gauge transformations by constant \(\epsilon\). The virtual dimension of \(P\) is then equal to \(2(\text{dim}(X) - \text{dim}(\mathcal{L}))\), where \(\mathcal{L}\) is the symplectic leaf of \(X\) containing \(X_0\). The non-singular part of \(P\) is a symplectic manifold of dimension equal to twice the transverse dimension of the foliation of \(X\) by the symplectic leaves. A point in \(P\) contains the information about the homotopy class \([X]\) of loops in a symplectic leaf. In particular, there is a natural projection from \(P\) to the space of leaves and a natural injection of the latter into \(P\) obtained by assigning the constant solutions \(X = X_0, \psi = 0\) to points \(X_0 \in \mathcal{X}\). Away from the singularities, this injection is a local embedding onto a Lagrangian submanifold of \(P\). Altogether, one obtains an interesting structure that certainly deserves further study and a name in the general context of Poisson manifolds. We propose to call the closed-geometry phase space \(P\) the transversal of the Poisson manifold \(\mathcal{X}\).

The general discussion applies, of course, to the case of Poisson-Lie model (26) with target \(G_r\) or to its orbifold (33) with target \(G_r/G_0 \cong \pi(G_r)\) In particular, the latter has the same phase space with the same symplectic structure as the restricted \(G/G\) coset model. In the \(G/G\) model, the degeneracy directions in the space of classical solutions are given by the standard gauge transformations and the quotient is somewhat easier to describe explicitly, as we shall do now.

5.2 \(G/G\) coset theory

In the first order formulation of the \(G/G\) theory we consider the space \(\mathcal{P} = \Sigma \times \pi(G_r) \times g^1\) of points \((x, g, \xi_l, \xi_r, A_l, A_r)\), where \(x = (x^l, x^r)\), the 2-form
\[
\alpha = \frac{1}{4\pi} \text{tr}[-\xi_l \xi_r dx^l \wedge dx^r - \xi_l g^{-1} dg \wedge dx^l + \xi_r g^{-1} dg \wedge dx^r] + \frac{1}{4\pi} \rho(g) + \frac{1}{2\pi} \text{tr}[A_l dg g^{-1} \wedge dx^l + A_r g^{-1} dg \wedge dx^r + A_l A_r dx^l \wedge dx^r - g A_l g^{-1} A_r dx^l \wedge dx^r] \tag{48}
\]
with vanishing 1-form \(\beta\). For a field configuration
\[
\Phi(x) = (g(x), \xi_l(x), \xi_r(x), A_l(x), A_r(x)),
\]

16
the first order action (34) coincides with the second-order one of (19) if
\[ \xi_l = g^{-1} \partial_l g \quad \text{and} \quad \xi_r = g^{-1} \partial_r g. \] (49)

In the first order approach we do not impose those equations but extract them from the extremality condition \( \delta S(\Phi) = 0 \) which, in our case, imply relations (49) together with the old equations (24) and, in the open geometry, impose also the boundary equation
\[ \text{tr}[(\delta g_+ g_+^{-1} - \delta g_- g_-^{-1}) A_t] = 0 \quad \text{on} \quad \partial \Sigma \] (50)
for any \( \delta g \) compatible with the boundary conditions. We solve (50) by taking \( A_t = A_l - A_r = 0 \) on the boundary of \( \Sigma \) and \( g \) free. These boundary conditions correspond to the ones in the the Poisson sigma models considered above.

Closed geometry

In order to solve the classical equations (24) on the cylinder, we introduce the space \( \mathcal{G} \) of maps \( \hat{h} : \mathbb{R}^2 \to G \) such that \( \hat{h}(t, x + 2\pi) = \hat{h}(t, x) h_1^{-1} \) for some \( h_1 \in G \) and all \((t, x)\). A general solution of (24,a) has then the form
\[ A = \hat{h} d\hat{h}^{-1}, \quad \hat{h} \in \mathcal{G}. \] (51)

The solutions of (24,b) are in turn given by
\[ g = \hat{g} \hat{h}^{-1}, \quad \hat{g} \in \mathcal{G} \] (52)
with constant \( \hat{g} \), provided that \( g(t, x) \) takes values in \( \pi(G_r) \). In particular, it follows that \( \hat{g} \) commutes with \( h_1 \). All pairs \((\hat{h}\gamma^{-1}, \gamma \hat{g}\gamma^{-1})\) for constant \( \gamma \in G \) correspond to the same solution of equations (24) and this is the only ambiguity. We may identify then the space of solutions with a subspace of \( (\mathcal{G} \times G) / G \).

The closed 2-form (42) is easy to compute by plugging in the general solution (51), (52) in terms of \( \hat{h}(t, x) \) and \( \hat{g} \) to the formula (38). One obtains:
\[ \tilde{\Omega}(\hat{h}, \hat{g}) = \frac{1}{4\pi} \text{tr}[h_1^{-1} dh_1 \wedge (\hat{g}^{-1} d\hat{g} + d\hat{g}\hat{g}^{-1} + \hat{g}^{-1} h_1^{-1} dh_1 \hat{g})]. \] (53)

It is easy to check that the form (53) projects to the orbit space \( (\mathcal{G} \times G) / G \).
In fact, it depends only on the orbits under simultaneous conjugations of the commuting pairs \((h_1, \hat{g})\) of elements in \( G \). It follows that \( \tilde{\Omega} \) has a huge
kernel composed of the vectors $(\delta \hat{h}, \delta \hat{g})$ such that $(\delta \hat{h}^{-1})_t(x)$ is periodic in $x$ and otherwise arbitrary and that $\hat{g}^{-1} \delta \hat{g} = 0$. The corresponding flows generate the group of gauge transformations $h : \Sigma \rightarrow G$ that act by left multiplication on $\hat{h}$. It is the same local gauge symmetry that was already identified in previous Section in the Lagrangian formalism.

We may describe now precisely the reduced phase space $P$, i.e. the transversal of the Poisson manifold $\pi(G_r) \cong G_r/G_0$. It is composed of the $G$-orbits of pairs $([\hat{h}], \hat{g})$ where $\hat{g} \in G$ and $[\hat{h}]$ is the homotopy class of maps $\hat{h} : \mathbb{R} \rightarrow G$ such $\hat{h} \hat{g} \hat{h}^{-1}$ takes values in $\pi(G_r)$ and that $\hat{h}(x + 2\pi) = \hat{h}(x)h_1^{-1}$ for $h_1 \in G$ commuting with $\hat{g}$ (the homotopies leave $h_1$ unchanged). The $G$-orbits are composed of pairs $([\hat{h}^{-1}], \gamma \hat{g} \gamma^{-1})$, $\gamma \in G$. The phase space $P$ covers the space of conjugacy classes of commuting pairs $(h_1, \hat{g})$.

Generic commuting pairs may be brought by conjugation to the form $(e^{2\pi \mu}, e^{2\pi \nu})$ with $\mu$ and $\nu$ in the Cartan subalgebra of $g$. In this parametrization, the expression for the symplectic form simplifies to

$$\tilde{\Omega}(\hat{h}, \hat{g}) = 2\pi \text{tr}[d\mu d\nu].$$

Note that at non-singular points $P$ has dimension equal to twice the rank of $G$. It is equal to twice the transverse dimension of the foliation of $\pi(G_r)$ by the symplectic leaves (the conjugacy classes) at generic points, in accordance with the general discussion.

**Open geometry**

On the strip, the classical equation (24) together with the boundary condition $A_t = 0$ may be solved similarly. We take as $\mathcal{G}$ the space of maps $\hat{h} : \Sigma \rightarrow G$ constant on every connected component of $\partial \Sigma$, i.e. $\hat{h}(t, 0) = h_0$ and $\hat{h}(t, \pi) = h_\pi$. Then (51) and (52) still gives a general solution of (24), provided that $g$ takes values in $\pi(G_r)$. All pairs $(\hat{h}^{-1}, \gamma \hat{g} \gamma^{-1})$, $\gamma \in G$, correspond to the same solution. The space of solutions may be again identified with a subspace of $(\mathcal{G} \times G)/G$. Of course we could fix now the $\gamma$-ambiguity by restricting $\hat{h}$ to be 1 on one of the component of the boundary but, for a moment, we prefer not to make any choice.

The 2-form (38) is for the open geometry given by

$$\tilde{\Omega}(\hat{h}, \hat{g}) = \frac{1}{4\pi} \text{tr}[(\hat{g}^{-1} d\hat{g} - d\hat{g} \hat{g}^{-1} + \hat{g}^{-1} h_\pi^{-1} d\pi \hat{g}) \wedge h_\pi^{-1}dh_\pi] - \frac{1}{4\pi} \rho(h_\pi \hat{g} h_\pi^{-1})$$

$$- \frac{1}{4\pi} \text{tr}[(\hat{g}^{-1} d\hat{g} - d\hat{g} \hat{g}^{-1} + \hat{g}^{-1} h_0^{-1} d\pi \hat{g}) \wedge h_0^{-1}dh_0] + \frac{1}{4\pi} \rho(h_0 \hat{g} h_0^{-1})$$

(55)
which, again, is unambiguously defined on \((G \times G)/G\). Indeed, it descends to the quotient space \((G \times G \times G)/G\) of the \(G\)-orbits of triples \((h_0, h_\pi, \hat{g})\). The kernel of \(\tilde{\Omega}\) is given by the vectors \((\delta\hat{h}, \delta\hat{g})\) such that \((\delta\hat{h}\hat{h}^{-1})(t, x)\) vanishes at \(x = 0\) and \(x = \pi\) and that \(g^{-1}\delta\hat{g} = 0\). They generate the gauge transformations \(h : \Sigma \to G\) equal to 1 on the boundary and acting by left multiplication on \(\tilde{\hat{h}}\).

The reduced phase space \(P\) of the model is composed of the \(G\)-orbits of pairs \(([\hat{h}], \hat{g})\) where \([\hat{h}]\) is a homotopy class of paths \(\hat{h} : [0, \pi] \to G\) such that \(\hat{h}\hat{g}\hat{h}^{-1}\) takes values in \(\pi(G_r)\). The homotopies are supposed to be fixed on the boundary. Space \(P\) covers the space of \(G\)-orbits of triples \((h_0, h_\pi, \hat{g})\) such that \(h_0\hat{g}h_0^{-1}\) and \(h_\pi\hat{g}h_\pi^{-1}\) lie in the same connected component of \(\pi(G_r) \cap C_{\hat{g}}\) where \(C_{\hat{g}}\) is the conjugacy class of \(\hat{g}\). Recall that such components form the symplectic leaves of the Poisson structure (14) on \(\pi(G_r)\). There are two natural maps from \(P\) to \(\pi(G_r)\) induced by \(([\hat{h}], \hat{g}) \mapsto h_0\hat{g}h_0^{-1}\) and \(([\hat{h}], \hat{g}) \mapsto h_\pi\hat{g}h_\pi^{-1}\) that map into the same symplectic leaf of \(\pi(G_r)\) and an embedding of \(\pi(G_r)\) into a Lagrangian submanifold of \(P\) defined by \(\hat{g} \mapsto ([1], \hat{g})\).

Besides, \(P\) is a groupoid with the multiplication composing the \(G\)-orbits of the pairs \(([\hat{h}], \hat{g})\) and \(([\hat{h}'], \hat{g}')\) such that \(h_\pi\hat{g}h_\pi^{-1} = h_0\hat{g}'h_0^{-1}\) to the \(G\)-orbit of \(([\hat{h}''], \hat{g})\), where

\[
\hat{h}''(x) = \begin{cases} 
\hat{h}(2x) & \text{for } 0 \leq x \leq \frac{\pi}{2} \\
\hat{h}'(2x - \pi)h_0^{-1}h_\pi & \text{for } \frac{\pi}{2} \leq x \leq \pi.
\end{cases}
\]

The symplectic space \(P\) together with the above projections, the embedding and the partial multiplication satisfies the axioms of the Poisson groupoid of \(\pi(G_r) \cong G_r/G_0\) and provides an example of the general construction discussed in [5].

Coming back to the expression (55) for the symplectic form \(\tilde{\Omega}\), we may simplify it using the freedom to choose a representative in the orbits of pairs \(([\hat{h}], \hat{g})\). Setting, for example, \(h_0 = 1\), we obtain the formula

\[
\tilde{\Omega}(\hat{h}, \hat{g}) = \frac{1}{4\pi} \Omega(h_\pi, \hat{g})
\]

where the last 2-form is given by the right hand side of (18). As we see, the Poisson groupoid of \(\pi(G_r) \cong G_r/G_0\) (as well as that of \(G_r\) itself) is closely
related to the Heisenberg double of $G$. Its dimension is twice the dimension of $G$, as was indicated by the general discussion.

An alternative description of the symplectic structure on $P$ is obtained by taking representatives of (generic) orbits in the form $(h_0, h_\pi, e^{2\pi \tau})$ with $\tau$ in the Cartan subalgebra of $\mathfrak{g}$ (this does not fix the ambiguity completely). In the above parametrization

$$\tilde{\Omega} = \Omega^{PL}(\tau, h_0) - \Omega^{PL}(\tau, h_\pi)$$

where

$$\Omega^{PL}(\tau, h) = \text{tr}[d\tau \wedge (h^{-1} dh) + \frac{1}{4\pi}(h^{-1} dh \wedge e^{2\pi \tau} h^{-1} dh \ e^{-2\pi \tau})]$$

$$- \frac{1}{4\pi} \rho(h e^{2\pi \tau} h^{-1}).$$

The form $\Omega^{PL}$ appeared for the first time in [7], see also [2] where its relation to the Heisenberg double has been stressed.

### 6 Quantization

Formula (60) should be compared to the one for the symplectic form of the cotangent bundle $T^* G \cong G \times \mathfrak{g}^* \cong G \times \mathfrak{g} \ni (g, p)$. Upon parametrization $h = h_\pi h_0^{-1}, \ p = h_0 \tau h_0^{-1}$, the canonical symplectic form on the cotangent bundle $\Omega_{T^* G} = d \text{tr}[p(h^{-1} dh)]$ becomes

$$\Omega_{T^* G} = \omega(\tau, h_\pi) - \omega(\tau, h_0)$$

where

$$\omega(\tau, h) = \text{tr}[d\tau \wedge (h^{-1} dh) - \tau(h^{-1} dh)^\wedge 2] = \lim_{k \to \infty} k \Omega^{PL}(\tau/k, h).$$

Let us recall the basic elements of the standard quantization of $T^* G$ adapted to the setup of complex manifolds. The space of quantum states $\mathcal{H}$ and its dual $\mathcal{H}^\ast$ are taken as

$$\mathcal{H} = A(G), \quad \mathcal{H}^\ast = \prod_\lambda \text{End}(V_\lambda)$$

where $A(G)$ is the space of analytic functions on $G$ that are finite linear combinations of matrix elements of irreducible analytic representation of $G$.
of highest weight $\lambda$ acting in spaces $V_\lambda$ of dimension $d_\lambda < \infty$. The duality between $\mathcal{H}$ and $\mathcal{H}^*$ is given by
\[
\langle f, (u_\lambda) \rangle = \sum_\lambda d_\lambda \int f(g) \text{tr}[g^{-1}_\lambda u_\lambda] dg
\]
where the integral is over the compact form of $G$ and $g_\lambda \in \text{End}(V_\lambda)$ represents $g$. Note that $\mathcal{H}^*$, as a direct product of matrix algebras, carries a natural algebra structure. The product in this algebra corresponds by the duality to the coproduct $f(g) \mapsto f(g_1 g_2)$ in $\mathcal{A}(G)$. Conversely, the commutative pointwise product of functions in $\mathcal{A}(G)$ induces by the duality the coproduct in $\mathcal{H}^*$. With the antipode defined as the transpose of the antipode $f(g) \mapsto f(g^{-1})$ of $\mathcal{A}(G)$, $\mathcal{H}^*$ becomes the Hopf algebra, the dual of $\mathcal{A}(G)$. As such, it may be identified with a completion of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ that embeds into it by $\mathcal{U}(\mathfrak{g}) \ni u \mapsto (u_\lambda) \in \mathcal{H}^*$ where $u_\lambda$ represents $u$ in $V_\lambda$. The algebra $\mathcal{U}(\mathfrak{g})$ may be thought of as obtained by the quantization of the polynomial functions on $T^*G$ depending on the momentum $p = h_0 e^{2\pi \tau h_0^{-1}}$. On the other hand, the quantization of the analytic functions on $T^*G$ that depend on $h = h_\pi h_0^{-1}$ (and Poisson-commute) gives rise to the commutative algebra $\mathcal{A}(G)$ that we identified with $\mathcal{H}$.

In the deformed case, one may take as the dual space of boundary states
\[
\mathcal{H}^*_q = \prod_\lambda \text{End}(V^q_\lambda)
\]
where $V^q_\lambda$ are the highest weight representations of the quantum deformation $\mathcal{U}_q(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ corresponding to the classical $r$-matrix $r$, whenever those are available. Again $\mathcal{H}^*$ has a natural structure of algebra and may be thought of for generic $q$ (related to the coupling constant $k$) as a completion of $\mathcal{U}_q(\mathfrak{g})$ and, also, as a quantization of the space of functions on $P$ that depend on the variable $h_0 e^{2\pi \tau h_0^{-1}}$. Similarly, for generic $q$, the classical function on $P$ that depend on $h = h_\pi h_0^{-1}$ give rise upon quantization to the (non-commutative) algebra $\mathcal{F}_q(G)$ of function on the quantum group that may be identified with the space of states $\mathcal{H}_q$ \cite{19}. The duality interchanges products and coproducts. The algebraic aspects of the quantization of the Heisenberg double were also studied in \cite{1}.

When $q$ is a root of unity then the algebra $\mathcal{U}_q(\mathfrak{g})$ has a finite-dimensional quotient $\mathcal{U}_q(\mathfrak{g})$ \cite{17} which has a finite series of irreducible representations labeled by the so called integrable weights $\lambda$ which may be used in \cite{66}. The finite-dimensional boundary algebra $\mathcal{H}^*_q$ obtained this way may be equipped
with the non-degenerate bilinear form given by $\pm S_0^\lambda \text{tr}$ on each component $\text{End}(V_\lambda^\vee)$, where $S_0^\lambda$ is the modular matrix. Together with the fusion algebra of bulk states, one obtains then a genuine example of a topological two-dimensional field theory in the sense of [15], see also [11]. It may be thought of as a “vertex version” of the boundary $G/G$ theory studied in [11]. In this case, however, the algebra $\mathcal{H}_q^*$ does not carry a natural coproduct (the map $\bar{U}_q(g) \ni u \mapsto (u^\lambda) \in \mathcal{H}_q^*$ is not an isomorphism and it may not be used to transport the coproduct from $\bar{U}_q(g)$).

References

[1] A. Yu. Alekseev, L. D. Faddeev, $(T^*G)_t$: a toy model for conformal field theory, Commun. Math. Phys. 141 (1991), 413-422.

[2] A. Yu. Alekseev, A. Z. Malkin, Symplectic structures associated to Lie-Poisson groups, Commun. Math. Phys. 162 (1994), 147-174.

[3] A. Yu. Alekseev, P. Schaller, T. Strobl, The Topological $G/G$ WZW Model in the Generalized Momentum Representation, Phys. Rev. D52 (1995), 7146-7160.

[4] A. S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000), 591-611.

[5] A. S. Cattaneo, G. Felder, Poisson sigma models and symplectic groupoids, arXiv:math.SG/0003023.

[6] A. S. Cattaneo, G. Felder, Poisson sigma models and deformation quantization, Mod. Phys. Lett. A 16 (2001), 179-190.

[7] F. Falceto, K. Gawędzki, On quantum group symmetries of conformal field theories, in: XXth International Conference on Differential Geometric Methods in Theoretical Physics, eds. Cato, S., Rocha, A., Singapore: World Scientific 1992, pp. 972-985.

[8] F. Falceto, K. Gawędzki, Lattice Wess-Zumino-Witten model and quantum groups, J. Geom. Phys. 11 (1993), 251-279.

[9] K. Gawędzki, Topological actions in two-dimensional quantum field theories, in: Non-perturbative quantum field theory, eds. G. ’t Hooft, A. Jaffe, G. Mack, P. K. Mitter, R. Stora, Plenum Press, New York 1988, pp. 101-142.
[10] K. Gawędzki, *Classical origin of quantum group symmetries in Wess-Zumino-Witten conformal field theory*, Commun. Math. Phys. **139** (1991), 201-213.

[11] K. Gawędzki, *Boundary WZW, G/H, G/G and CS theories*, arXiv: hep-th/0108044

[12] K. Gawędzki, A. Kupiainen, *Coset construction from functional integral*, Nucl. Phys. **B 320** (1989), 625-668.

[13] K. Gawędzki, I. Todorov, P. Tran-Ngoc-Bich, *Canonical quantization of the boundary Wess-Zumino-Witten model*, arXiv:hep-th/0101170.

[14] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, IHES M97/72 preprint, arXiv:q-alg/9709040.

[15] Lazaroiu, C. I.: *On the structure of open-closed topological field theory in two-dimensions*, Nucl.Phys. **B 603** (2001), 497-530.

[16] J. Lu, A. Weinstein, *Poisson-Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom **31** (1990), 501-526.

[17] G. Lusztig, *Finite dimensional Hopf algebras arising from quantum groups*, J. Amer. Math. Soc. **3** (1990), 257-296.

[18] M. A. Semenov-Tian-Shansky, *Dressing transformations and Poisson-Lie group actions*, Publ. RIMS **21** (1985), 1237-1260.

[19] M. A. Semenov-Tian-Shansky, *Poisson Lie groups, quantum duality principle and the quantum double*, Theor.Math.Phys. **93** (1992), 1292-1307.

[20] P. Schaller, T. Strobl, *Poisson structure induced (topological) field theories*, Mod.Phys.Lett. **A 9** (1994), 3129-3136.

[21] C. Sorger, *La formule de Verlinde*, Astérisque **237** (1996), 87-114.

[22] E. Witten, *Non-abelian bosonization in two dimensions*, Commun. Math. Phys. **92** (1984), 455-472.