I will discuss recently solved Quantum Mechanical—Quantum Electrodynamical problem in this lectures. It was solved numerically in papers [1, 2], and then analytical solution was found in papers [3, 4].

We will use convenient in atomic physics Gauss units: $e^2 = \alpha = 1/137$. Magnetic fields $B > m_e^2 e^3$ we will call strong while $B > m_e^3 / e^2$ we will call superstrong.

Important quantity in the problem under consideration is Landau radius $a_H = 1/\sqrt{eB}$ called magnetic length in condensed matter physics.

Let us consider hydrogen atom in external homogeneous magnetic field $B$. At strong $B$ Bohr radius $a_B$ is larger than $a_H$, so there are two time scales in the problem: fast motion in the perpendicular to magnetic field plane and slow motion along the magnetic field. That is why adiabatic approximation is applicable: averaging over fast motion we get one-dimensional motion of electron along the magnetic field in effective potential

$$U(z) \approx \frac{-e^2} {\sqrt{z^2 + a_H^2}}.$$  

The energy of a ground state can be estimated as

$$E_0 = -2m \left( \int U(z)dz \right)^2 \approx -me^4 \ln \left( B/m^2 e^3 \right)$$

and it goes to minus infinity when $B$ goes to infinity.

We will see that radiative corrections qualitatively change this result: ground state energy goes to finite value when $B$ goes to infinity. This happens due to screening of the Coulomb potential.

Since at strong $B$ reduction of the number of space dimensions occurs and motion takes place in one space and one time dimensions it is natural to begin

1 The article is published in the original.
let us take as an interpolating formula the following expression:

\[ \bar{P}(t) = \frac{2t}{3 + 2t}. \]  

(9)

The accuracy of this approximation is better than 10%. Substituting (9) into (7) we get:

\[
\Phi = 4\pi g \int_{-\infty}^{\infty} \frac{e^{ikz}d\ell}{k^2 + 4g^2(k^2/2m^2)/(3 + k^2/2m^2)}
= \frac{4\pi g}{1 + 2g^2/3m^2} \left[ \frac{1}{k^2} + \frac{2g^2/3m^2}{k^2 + 6m^2 + 4g^2} \right] e^{ikz}d\ell
\]

(10)

In the case of heavy fermions \( m \gg g \) the potential is given by the tree level expression; the corrections are suppressed as \( g^2/m^2 \). In the case of light fermions \( m \ll g \):

\[
\Phi(z)\big|_{m \ll g} = \begin{cases} 
\pi e^{-2|z|}, & z \ll \frac{1}{g} \ln \left( \frac{g}{m} \right) \\
-2\pi g \left( \frac{3m^2}{3g^2} \right) |z|, & z \gg \frac{1}{g} \ln \left( \frac{g}{m} \right) 
\end{cases} 
\]  

(11)

For \( m = 0 \) we have Schwinger model—the first gauge invariant theory with a massive vector boson. Light fermions make a continuous transition from \( m > g \) to \( m = 0 \) case. The next two figures correspond to \( g = 0.5, m = 0.1 \). The expression for \( \mathcal{V} \) contains \( \mathcal{P} \).

To find the modification of the Coulomb potential in \( D = 4 \) we need an expression for \( \Pi \) in strong \( B \).

One starts from electron propagator \( G \) in strong \( B \). Solutions of the Dirac equation in homogenous constant in time \( B \) are known, so one can write spectral representation of electron Green function. Denominators contain \( k^2 - m^2 - 2eB \), and for \( B \gg m/e \) and \( k^2 \ll eB \) in sum over levels lowest Landau level (LLL, \( n = 0 \)) dominates. In coordinate representation transverse part of LLL wave function is:

\[ \Psi \sim \exp((-x^2 - y^2)eB) \]

which in momentum representation gives \( \Psi \sim \exp((k_x^2 - k_y^2)/eB) \) (gauge in which \( \vec{A} \equiv 1/2[\vec{B} \times \vec{r}] \) is used).

Substituting electron Green functions into polarization operator we get:

\[
\Pi_{\mu\nu} \sim e^2 eB \int dq_x dq_y \exp \left( \frac{q_x^2 + q_y^2}{eB} \right) \times \exp \left( \frac{(q + k)^2}{eB} \right) dq_z d\gamma_\nu d\gamma_\mu
\]

(12)
\[
\Phi = \frac{4\pi e}{(k_1^2 + k_2^2)} \left(1 - \frac{\alpha}{3\pi} \ln \left(\frac{eB}{m^2}\right) + \frac{2e^2B}{\pi} \exp\left(-\frac{k_1^2}{2eB}\right) \psi^2 \left(\frac{k_1^2}{4m^2}\right)\right).
\]

(13)

\[
\Phi(z) = 4\pi e \int \frac{e^{ikz}dk_{1}dk_{2}}{(k_1^2 + k_2^2)} \left(1 - \frac{2e^2B}{\pi} \exp\left(-\frac{k_1^2}{2eB}\right)\right) \frac{1}{(2\pi)^2}.
\]

(14)

\[
\Phi(z) = \frac{e}{|z|} \left[1 - e^{-\sqrt{6m_e}|z|} + e^{-\sqrt{2/\pi}e^3B|z|}\right].
\]

(15)

\[
\Phi(z) \bigg|_{e^{-3B/m_e^2}} = \frac{e}{|z|} \left[1 + O\left(\frac{e^3B}{m_e^2}\right)\right]
\]

(16)

For magnetic fields \( B \ll 3m_e^2/e^3 \) the potential is the Coulomb up to small power suppressed terms: in full accordance with the \( D = 2 \) case, where \( g^2 \) plays the role of \( e^2B \). In the opposite case of superstrong magnetic fields \( B \gg 3m_e^2/e^3 \) we get:

\[
\Phi(z) = \begin{cases}
\frac{e}{|z|} & |z| > \frac{1}{m}, \\
\frac{e^{-\sqrt{2/\pi}e^3B|z|} - 1}{m} & 1/m > |z| > \frac{1}{\sqrt{2/\pi}e^3B}, \\
\frac{1}{\sqrt{2/\pi}e^3B} \ln \left(\frac{e^3B}{\sqrt{2/\pi}e^3B}\right) & |z| < 1/m.
\end{cases}
\]

(17)

\[
V(z) = -e\Phi(z).
\]

(18)

Spectrum of the Dirac equation in constant in space and time magnetic field is well known:

\[
e^2 = m^2 + p_x^2 + (2n + 1 + \sigma_z)eB,
\]

(19)

\( n = 0, 1, 2, 3, \ldots; \sigma_z = \pm 1 \). For \( B > B_{cr} = m_e/e \) the electrons are relativistic with only one exception: electrons from lowest Landau level \( (n = 0, z = -1) \) can be non-relativistic.

In what follows we will study the spectrum of electrons from LLL in the Coulomb field of the proton modified by the superstrong \( B \).

Spectrum of Schrödinger equation in cylindrical coordinates \( (\rho, z) \) in the gauge, where \( \vec{A} = 1/2[\vec{B}r] \) is:

\[
E_{p,n,\sigma_z} = \left(n + \frac{|m| + m + 1 + \sigma_z}{2}\right)eB + \frac{p_\rho^2}{m_e} + \frac{p_z^2}{2m_e}.
\]

(20)

LLL corresponds to \( n = 0, \sigma_z = -1, m = 0, -1, -2, \ldots \).

A wave function factorizes on those describing free motion along a magnetic field with momentum \( p_\rho \) and those describing motion in perpendicular to magnetic field plane:

\[
R_{0m}(\vec{p}) = [\pi(2a_H^2)^{1/2}m!\left|m!\right|^{-1/2}]
\times \rho^{m} \exp \left[-im\rho - \rho^2/(4a_H^2)\right]
\]

(21)

\[
\Psi_{n0m-1} = R_{0m}(\vec{p}) \chi_n(z),
\]

(22)

where \( \chi_n(z) \) is the solution of the Schrödinger equation for electron motion along a magnetic field:

\[
\left[-\frac{1}{2m} \frac{d^2}{dz^2} + U_{eff}(z)\right] \chi_n(z) = E_n \chi_n(z).
\]

(23)

Without screening the effective potential is given by the following formula:

\[
U_{eff}(z) = -e^2 \int \frac{R_{0m}(\vec{p})^2}{\sqrt{\rho^2 + z^2}} d\rho,
\]

(24)
For $|z| \gg a_H$, the effective potential equals to the Coulomb potential:
\[
U_{\text{eff}}(z) \big|_{z \gg a_H} = \frac{e^2}{|z|}
\]
(25)
and effective potential is regular at $z = 0$:
\[
U_{\text{eff}}(0) \sim -\frac{e^2}{|a_H|}.
\]
(26)

Since $U_{\text{eff}}(z) = U_{\text{eff}}(-z)$, the wave functions are odd or even under reflection $z \to -z$; the ground states (for $m = 0, -1, -2, ...$) are described by even wave functions.

To calculate the ground state of hydrogen atom in the textbook “Quantum Mechanics” by L.D. Landau and E.M. Lifshitz the shallow-well approximation is used:
\[
E_{\text{sw}} = -2m_e\left[\int_{a_H}^{a_H} U(z)dz\right]^2
\]
(27)
\[
= -(m_e e^2/2)\ln\left(B/(m_e^2 e^4)\right).
\]

Let us derive this formula. The starting point is one-dimensional Schrödinger equation:
\[
-\frac{1}{2\mu} \frac{d^2}{dz^2} \chi(z) + U(z)\chi(z) = E_0 \chi(z).
\]
(28)

Neglecting $E_0$ in comparison with $U$ and integrating we get:
\[
\chi'(a) = 2\mu \int_0^a U(x)\chi(x)dx,
\]
(29)
where we assume $U(x) = U(-x)$, that is why $x$ is even.

The next assumptions are: 1. the finite range of the potential energy: $U(x) \neq 0$ for $a > x > -a$; 2. $\chi$ under-
(condition for the potential to form a shallow well) we get that, indeed, \( |E_{\text{eff}}| \ll |E| \) and that the variation of \( \chi \) inside the well is small, \( \Delta z/\chi \sim \mu |U| a^2 \ll 1 \). Concerning the one-dimensional Coulomb potential, it satisfies this condition only for \( a \ll 1/(m e^2) \equiv a_b \).

This explains why the accuracy of log\(^2\) formula is very poor.

Much more accurate equation for atomic energies in strong magnetic field was derived by B. M. Karnakov and V. S. Popov [5]. It provides a several percent accuracy for effective potential transforms into series are negligible.

The energies of the ODD states are:

\[
E_{\text{odd}} = -\frac{m e^4}{2\pi^2} + O\left(\frac{m e^2}{B}\right), \quad n = 1, 2, \ldots \tag{34}
\]

So, for superstrong magnetic fields \( B \approx m_e^2/e^3 \) the deviations of odd states energies from the Balmer series are negligible.

When screening is taken into account an expression for effective potential transforms into

\[
\tilde{U}_{\text{ef}}(z) = -e^2 \int \frac{R_{am}(\tilde{\rho}) \tilde{\rho}^2 d^2 \rho}{\sqrt{\tilde{\rho}^2 + z^2}} \times \left[ 1 - e^{-\left(6m_e^2/2\pi\right)B + 6m_e^2z} \right] \tag{35}
\]

The original KP equation for LLL splitting by the Coulomb potential is:

\[
\ln (H) = \lambda + 2\ln \lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1 + |m|), \tag{36}
\]

where \( \psi(x) \) is the logarithmic derivative of the gamma function; it has simple poles at \( x = 0, -1, -2, \ldots \)

The modified KP equation, which takes screening into account looks like:

\[
\ln \left(\frac{H}{1 + e^2 H/3\pi}\right) = \lambda + 2\ln \lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) \tag{37}
\]

\[+ \ln 2 + 4\gamma + \psi(1 + |m|),
\]

\[E = -\left(m e^4/2\right)\lambda^2. \]

In particular, for a ground state \( \lambda = 11.2, E_0 = -1.7 \text{ keV}. \)

In conclusion,

1. (1) analytical expression for charged particle electric potential in \( d = 1 \) is given; for \( m < g \) screening take place at all distances;

2. (2) analytical expression for charged particle electric potential \( \Phi(z, \rho = 0) \) at superstrong \( B \) at \( d = 3 \) is found; screening take place at distances \( |z| \ll 1/m_e \);

3. (3) an algebraic formula for the energy levels of a hydrogen atom originating from the lowest Landau level in superstrong \( B \) has been obtained.

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