Distance-regular graphs of $q$-Racah type and the $q$-tetrahedron algebra

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In Memory of Donald Higman

Abstract

In this paper we discuss a relationship between the following two algebras: (i) the subconstituent algebra $T$ of a distance-regular graph that has $q$-Racah type; (ii) the $q$-tetrahedron algebra $\mathbb{X}_q$ which is a $q$-deformation of the three-point $\mathfrak{sl}_2$ loop algebra. Assuming that every irreducible $T$-module is thin, we display an algebra homomorphism from $\mathbb{X}_q$ into $T$ and show that $T$ is generated by the image together with the center $Z(T)$.

Keywords. Tetrahedron algebra, quantum affine algebra, distance-regular graph, $Q$-polynomial.

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1 Introduction

In [20] B. Hartwig and the second author gave a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra $\mathfrak{X}$ by generators and relations, and displayed an isomorphism from $\mathfrak{X}$ to the three-point $\mathfrak{sl}_2$ loop algebra. The algebra $\mathfrak{X}$ is called the tetrahedron algebra [20, Definition 1.1]. In [24] we introduced a $q$-deformation $\mathfrak{X}_q$ of $\mathfrak{X}$ called the $q$-tetrahedron algebra. In [24] and [25] we described the finite-dimensional irreducible $\mathfrak{X}_q$-modules. In [26, Section 4] we displayed four homomorphisms into $\mathfrak{X}_q$ from the quantum affine algebra $U_q(\mathfrak{sl}_2)$. In [26, Section 12] we found a homomorphism from $\mathfrak{X}_q$ into the subconstituent algebra of a distance-regular graph that is self-dual with classical parameters. In the present paper we do something similar for a distance-regular graph said to have $q$-Racah type. This type is described as follows. Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$ (See Section 4 for formal definitions). We say that $\Gamma$ has $q$-Racah type whenever $\Gamma$ has a $Q$-polynomial structure with eigenvalue...
sequence \( \{\theta_i\}_{i=0}^D \) and dual eigenvalue sequence \( \{\theta^*_i\}_{i=0}^D \) that satisfy

\[
\theta_i = \eta + uq^{2i-D} + vq^{D-2i} \quad (0 \leq i \leq D),
\]

\[
\theta^*_i = \eta^* + u^*q^{2i-D} + v^*q^{D-2i} \quad (0 \leq i \leq D),
\]

where \( q, u, v, u^*, v^* \) are nonzero and \( q^2 \neq 1 \) for \( 1 \leq i \leq D \). Assume \( \Gamma \) has \( q \)-Racah type.

Fix a vertex \( x \) of \( \Gamma \) and let \( T = T(x) \) denote the corresponding subconstituent algebra [32, Definition 3.3]. Recall that \( T \) is generated by the adjacency matrix \( A \) and the dual adjacency matrix \( A^* = A^*(x) \) [32, Definition 3.10]. An irreducible \( T \)-module \( W \) is called thin whenever the intersection of \( W \) with each eigenspace of \( A \) and each eigenspace of \( A^* \) has dimension at most 1 [32, Definition 3.5]. Assuming each irreducible \( T \)-module is thin, we display invertible central elements \( \Phi, \Psi \) of \( T \) and a homomorphism \( \vartheta : \mathfrak{B}_q \to T \) such that

\[
A = \eta I + u\Phi \Psi^{-1} \vartheta(x_{01}) + v\Psi \Phi^{-1} \vartheta(x_{12}),
\]

\[
A^* = \eta^* I + u^*\Phi \Psi \vartheta(x_{23}) + v^*\Psi \Phi^{-1} \vartheta(x_{30}),
\]

where the \( x_{ij} \) are the standard generators of \( \mathfrak{B}_q \). It follows that \( T \) is generated by the image \( \vartheta(\mathfrak{B}_q) \) together with \( \Phi, \Psi \). In particular \( T \) is generated by \( \vartheta(\mathfrak{B}_q) \) together with the center \( Z(T) \).

This paper is organized as follows. In Section 2 we recall the definition of \( \mathfrak{B}_q \). In Section 3 we describe how \( \mathfrak{B}_q \) is related to \( U_q(\hat{\mathfrak{sl}}_2) \). In Section 4 we recall the basic theory of a distance-regular graph \( \Gamma \), focussing on the \( Q \)-polynomial property and the subconstituent algebra. In Section 5 we discuss the split decomposition of \( \Gamma \). In Section 6 we give our main results.

Throughout the paper \( \mathbb{C} \) denotes the field of complex numbers.

## 2 The \( q \)-tetrahedron algebra \( \mathfrak{B}_q \)

In this section we recall the \( q \)-tetrahedron algebra. We fix a nonzero scalar \( q \in \mathbb{C} \) such that \( q^2 \neq 1 \) and define

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, 2, \ldots
\]

We let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \) denote the cyclic group of order 4.

**Definition 2.1** [24, Definition 10.1] Let \( \mathfrak{B}_q \) denote the unital associative \( \mathbb{C} \)-algebra that has generators

\[
\{x_{ij} \mid i, j \in \mathbb{Z}_4, \ j - i = 1 \text{ or } j - i = 2 \}
\]

and the following relations:

(i) For \( i, j \in \mathbb{Z}_4 \) such that \( j - i = 2 \),

\[
x_{ij}x_{ji} = 1.
\]
(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1, 1), (1, 2), (2, 1)$,

$$\frac{qx_{hi}x_{ij} - q^{-1}x_{ij}x_{hi}}{q - q^{-1}} = 1.$$  \hspace{1cm} (ii)

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$x_{hi}^3x_{jk} - [3]_q x_{hi}^2x_{jk}x_{hi} + [3]_q x_{hi}x_{jk}x_{hi}^2 - x_{jk}x_{hi}^3 = 0. \hspace{1cm} (1)$$

We call $\mathcal{Q}$ the $q$-tetrahedron algebra or "q-tet" for short. We refer to the $x_{ij}$ as the standard generators for $\mathcal{Q}$.

**Note 2.2** The equations (1) are the cubic $q$-Serre relations [29, p. 10].

We make some observations.

**Lemma 2.3** [24, Lemma 6.3] There exists a $\mathbb{C}$-algebra automorphism $\varrho$ of $\mathcal{Q}$ that sends each generator $x_{ij}$ to $x_{i+1,j+1}$. Moreover $\varrho^4 = 1$.

**Lemma 2.4** [24, Lemma 6.5] There exists a $\mathbb{C}$-algebra automorphism of $\mathcal{Q}$ that sends each generator $x_{ij}$ to $-x_{ij}$.

**3 The quantum affine algebra $U_q(\widehat{sl}_2)$**

In this section we consider how $\mathcal{Q}$ is related to the quantum affine algebra $U_q(\widehat{sl}_2)$. We start with a definition.

**Definition 3.1** [7, p. 266] The quantum affine algebra $U_q(\widehat{sl}_2)$ is the unital associative $\mathbb{C}$-algebra with generators $K_i^{\pm 1}, e_i^{\pm}, i \in \{0, 1\}$ and the following relations:

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_0K_1 = K_1K_0,$$

$$K_i e_i^{\pm} K_i^{-1} = q^{\pm 1} e_i^{\pm},$$

$$K_i e_j^{\pm} K_i^{-1} = q^{\mp 2} e_j^{\pm}, \quad i \neq j,$$

$$[e_i^{\pm}, e_j^{\mp}] = \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$[e_0^{\pm}, e_1^{\mp}] = 0,$$

$$e_i^{\pm} e_j^{\mp} - [3]_q (e_i^{\pm})^2 e_j^{\mp} + [3]_q e_i^{\pm} e_j^{\mp} (e_i^{\pm})^2 - e_j^{\mp} (e_i^{\pm})^3 = 0, \quad i \neq j.$$  \hspace{1cm} (3)

The following presentation of $U_q(\widehat{sl}_2)$ will be useful.
Proposition 3.2 ([23, Theorem 2.1], [38]) The quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ is isomorphic to the unital associative $\mathbb{C}$-algebra with generators $x_i^{\pm 1}$, $y_i$, $z_i$, $i \in \{0, 1\}$ and the following relations:

$$x_ix_i^{-1} = x_i^{-1}x_i = 1,$$

$\forall x_0x_1$ is central,

$$\frac{qy_iy_i - q^{-1}y_ix_i}{q - q^{-1}} = 1,$$

$$\frac{qy_iy_i - q^{-1}z_iy_i}{q - q^{-1}} = 1,$$

$$\frac{qz_ix_i - q^{-1}x_iz_i}{q - q^{-1}} = 1,$$

$$\frac{qz_iz_j - q^{-1}y_jz_i}{q - q^{-1}} = x_0^{-1}x_1^{-1}, \quad i \neq j,$$

$$y_i^2y_j - [3]_qy_i^2y_jy_i + [3]_qy_iy_jy_i^2 - y_jy_i^2 = 0, \quad i \neq j,$$

$$z_i^2z_j - [3]_qz_i^2z_jz_i + [3]_qz_iz_j^2z_i - z_jz_i^2 = 0, \quad i \neq j.$$

An isomorphism with the presentation in Definition 3.1 is given by:

$$x_i^{\pm 1} \mapsto K_i^{\pm 1},$$

$$y_i \mapsto K_i^{-1} + e_i^-, \quad z_i \mapsto K_i^{-1} - K_i^{-1}e_i^+q(q^{-1})^2.$$

The inverse of this isomorphism is given by:

$$K_i^{\pm 1} \mapsto x_i^{\pm 1},$$

$$e_i^- \mapsto y_i - x_i^{-1}, \quad e_i^+ \mapsto (1 - x_iy_i)q^{-1}(q^{-1})^{-2}.$$

Theorem 3.3 [24, Proposition 7.4] For $i \in \mathbb{Z}_4$ there exists a $\mathbb{C}$-algebra homomorphism from $U_q(\widehat{\mathfrak{sl}_2})$ to $\mathbb{K}_q$ that sends

$$x_1 \mapsto x_{i,i+2}, \quad x_i^{-1} \mapsto x_{i+2,i}, \quad y_1 \mapsto x_{i+2,i+3}, \quad z_1 \mapsto x_{i+3,i},$$

$$x_0 \mapsto x_{i+2,i}, \quad x_0^{-1} \mapsto x_{i,i+2}, \quad y_0 \mapsto x_{i,i+1}, \quad z_0 \mapsto x_{i+1,i+2}.$$

Proof: Compare the defining relations for $U_q(\widehat{\mathfrak{sl}_2})$ given in Proposition 3.2 with the relations in Definition 2.1. □

4 Distance-regular graphs; preliminaries

We now turn our attention to distance-regular graphs. After a brief review of the basic definitions we recall the $Q$-polynomial property and the subconstituent algebra. For more information we refer the reader to [1, 3, 19, 32].
Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We call $V$ the standard module. We endow $V$ with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \bar{v}$ for $u, v \in V$, where $t$ denotes transpose and $\bar{\cdot}$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of $V$ with a 1 in the $y$ coordinate and 0 in all other coordinates. We observe $\{ \hat{y} \mid y \in X \}$ is an orthonormal basis for $V$.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $R$. Let $\partial$ denote the path-length distance function for $\Gamma$, and set $D := \max\{ \partial(x, y) \mid x, y \in X \}$. We call $D$ the diameter of $\Gamma$. For an integer $k \geq 0$ we say that $\Gamma$ is regular with valency $k$ whenever each vertex of $\Gamma$ is adjacent to exactly $k$ distinct vertices of $\Gamma$. We say that $\Gamma$ is distance-regular whenever for all integers $h, i, j \ (0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{ z \in X \mid \partial(x, z) = i, \partial(z, y) = j \}|$$

is independent of $x$ and $y$. The $p^h_{ij}$ are called the intersection numbers of $\Gamma$. We abbreviate $c_i = p^1_{1i-1}$ ($1 \leq i \leq D$), $b_i = p^1_{i+11}$ ($0 \leq i \leq D - 1$), $a_i = p^1_{i1}$ ($0 \leq i \leq D$).

For the rest of this paper we assume $\Gamma$ is distance-regular; to avoid trivialities we always assume $D \geq 3$. Note that $\Gamma$ is regular with valency $k = b_0$. Moreover $k = c_i + a_i + b_i$ for $0 \leq i \leq D$, where $c_0 = 0$ and $b_D = 0$.

We mention a fact for later use. By the triangle inequality, for $0 \leq h, i, j \leq D$ we have $p^h_{ij} = 0$ (resp. $p^h_{ij} \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two.

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $(x, y)$-entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call $A_i$ the $i$th distance matrix of $\Gamma$. We abbreviate $A = A_1$ and call this the adjacency matrix of $\Gamma$. We observe (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i \ (0 \leq i \leq D)$; (iv) $A_i^t = A_i \ (0 \leq i \leq D)$; (v) $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \ (0 \leq i, j \leq D)$, where $I$ (resp. $J$) denotes the identity matrix (resp. all 1’s matrix) in $\text{Mat}_X(\mathbb{C})$. Using these facts we find $\{ A_i \}_{i=0}^{D}$ is a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{C})$, called the Bose-Mesner algebra of $\Gamma$. It turns out that $A$ generates $M$ [1, p. 190]. By [3, p. 45], $M$ has a second basis $\{ E_i \}_{i=0}^{D}$ such that (i) $E_0 = |X|^{-1} J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $\overline{E_i} = E_i \ (0 \leq i \leq D)$; (iv) $E_i^t = E_i \ (0 \leq i \leq D)$; (v) $E_i E_j = \delta_{ij} E_i \ (0 \leq i, j \leq D)$. We call $\{ E_i \}_{i=0}^{D}$ the primitive idempotents of $\Gamma$.

We recall the eigenvalues of $\Gamma$. Since $\{ E_i \}_{i=0}^{D}$ form a basis for $M$ there exist complex scalars $\{ \theta_i \}_{i=0}^{D}$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Observe $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. By [1, p. 197] the scalars $\{ \theta_i \}_{i=0}^{D}$ are in $\mathbb{R}$. Observe $\{ \theta_i \}_{i=0}^{D}$ are mutually distinct since $A$ generates $M$. We call $\theta_i$ the eigenvalue of $\Gamma$ associated with $E_i \ (0 \leq i \leq D)$. Observe

$$V = E_0 V + E_1 V + \cdots + E_D V$$

(orthogonal direct sum).
For $0 \leq i \leq D$ the space $E_i V$ is the eigenspace of $A$ associated with $\theta_i$.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\text{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, so $M$ is closed under $\circ$. Thus there exist complex scalars $q_{ij}^h$ ($0 \leq h, i, j \leq D$) such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D).$$

By [2, p. 170], $q_{ij}^h$ is real and nonnegative for $0 \leq h, i, j \leq D$. The $q_{ij}^h$ are called the Krein parameters of $\Gamma$. The graph $\Gamma$ is said to be $Q$-polynomial (with respect to the given ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two [3, p. 235]. See [4, 5, 6, 10, 11, 14, 15, 30] for background information on the $Q$-polynomial property.

From now on we assume $\Gamma$ is $Q$-polynomial with respect to $\{E_i\}_{i=0}^D$. We call the sequence $\{\theta_i\}_{i=0}^D$ the eigenvalue sequence for this $Q$-polynomial structure.

We recall the dual Bose-Mesner algebra of $\Gamma$. For the rest of this paper we fix a vertex $x \in X$. We view $x$ as a “base vertex.” For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y,y)$-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X). \quad (2)$$

We call $E_i^*$ the $i$th dual idempotent of $\Gamma$ with respect to $x$ [32, p. 378]. We observe (i) $\sum_{i=0}^D E_i^* = I$; (ii) $E_i^* E_j^* = E_i^* (0 \leq i \leq D)$; (iii) $E_i^* E_i^* (0 \leq i \leq D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^* (0 \leq i, j \leq D)$. By these facts $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$. We call $M^*$ the dual Bose-Mesner algebra of $\Gamma$ with respect to $x$ [32, p. 378]. For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y,y)$-entry $(A_i^*)_{yy} = |X| (E_i)_{xy}$ for $y \in X$. Then $\{A_i^*\}_{i=0}^D$ is a basis for $M^*$ [32, p. 379]. Moreover (i) $A_0^* = I$; (ii) $A_i^* = A_i^* (0 \leq i \leq D)$; (iii) $A_i^* A_i^* = A_i^* (0 \leq i \leq D)$; (iv) $A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$ ($0 \leq i, j \leq D$) [32, p. 379]. We call $\{A_i^*\}_{i=0}^D$ the dual distance matrices of $\Gamma$ with respect to $x$. We abbreviate $A^* = A_1^*$ and call this the dual adjacency matrix of $\Gamma$ with respect to $x$. The matrix $A^*$ generates $M^*$ [32, Lemma 3.11].

We recall the dual eigenvalues of $\Gamma$. Since $\{E_i^*\}_{i=0}^D$ form a basis for $M^*$ there exist complex scalars $\{\theta_i^*\}_{i=0}^D$ such that $A^* = \sum_{i=0}^D \theta_i^* E_i^*$. Observe $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$ for $0 \leq i \leq D$. By [32, Lemma 3.11] the scalars $\{\theta_i^*\}_{i=0}^D$ are in $\mathbb{R}$. The scalars $\{\theta_i^*\}_{i=0}^D$ are mutually distinct since $A^*$ generates $M^*$. We call $\theta_i^*$ the dual eigenvalue of $\Gamma$ associated with $E_i^*$ ($0 \leq i \leq D$). We call the sequence $\{\theta_i^*\}_{i=0}^D$ the dual eigenvalue sequence for the given $Q$-polynomial structure.

We recall the subconstituents of $\Gamma$. From (2) we find

$$E_i^* V = \text{span} \{ \hat{y} \mid y \in X, \partial(x, y) = i \} \quad (0 \leq i \leq D). \quad (3)$$

By (3) and since $\{ \hat{y} \mid y \in X \}$ is an orthonormal basis for $V$ we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}.$$
For $0 \leq i \leq D$ the space $E^*_i V$ is the eigenspace of $A^*$ associated with $\theta^*_i$. We call $E^*_i V$ the $i$th subconstituent of $\Gamma$ with respect to $x$.

We recall the subconstituent algebra of $\Gamma$. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the subconstituent algebra (or Terwilliger algebra) of $\Gamma$ with respect to $x$ [32, Definition 3.3]. Observe that $T$ has finite dimension. Moreover $T$ is semisimple since it is closed under the conjugate transpose map [13, p. 157]. We note that $A, A^*$ together generate $T$. By [32, Lemma 3.2] the following are relations in $T$:

\[
E^*_i A E^*_j = 0 \quad \text{iff} \quad p^h_{ij} = 0, \quad (0 \leq b, i, j \leq D), \quad (4)
\]

\[
E^*_h A^* E^*_j = 0 \quad \text{iff} \quad q^h_{ij} = 0, \quad (0 \leq h, i, j \leq D). \quad (5)
\]

See [8, 9, 12, 16, 17, 21, 31, 32, 33, 34] for more information on the subconstituent algebra.

We recall the $T$-modules. By a $T$-module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let $W$ denote a $T$-module and let $W'$ denote a $T$-module contained in $W$. Then the orthogonal complement of $W'$ in $W$ is a $T$-module [18, p. 802]. It follows that each $T$-module is an orthogonal direct sum of irreducible $T$-modules. In particular $V$ is an orthogonal direct sum of irreducible $T$-modules.

Let $W$ denote an irreducible $T$-module. Observe that $W$ is the direct sum of the nonzero spaces among $E_0^* W, \ldots, E_D^* W$. Similarly $W$ is the direct sum of the nonzero spaces among $E_0 W, \ldots, E_D W$. By the endpoint of $W$ we mean $\min\{i | 0 \leq i \leq D, E_i^* W \neq 0\}$. By the diameter of $W$ we mean $|\{i | 0 \leq i \leq D, E_i^* W \neq 0\}| - 1$. By the dual endpoint of $W$ we mean $\min\{i | 0 \leq i \leq D, E_i W \neq 0\}$. By the dual diameter of $W$ we mean $|\{i | 0 \leq i \leq D, E_i W \neq 0\}| - 1$. It turns out that the diameter of $W$ is equal to the dual diameter of $W$ [30, Corollary 3.3]. By [32, Lemma 3.4] $\dim E^*_i W \leq 1$ for $0 \leq i \leq D$ if and only if $\dim E_i W \leq 1$ for $0 \leq i \leq D$; in this case $W$ is called thin.

We finish this section with a few comments.

**Lemma 4.1** [32, Lemma 3.4, Lemma 3.9, Lemma 3.12] Let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then $\rho, \tau, d$ are nonnegative integers such that $\rho + d \leq D$ and $\tau + d \leq D$. Moreover the following (i)-(iv) hold.

(i) $E^*_i W \neq 0$ if and only if $\rho \leq i \leq \rho + d, \quad (0 \leq i \leq D)$.

(ii) $W = \sum_{h=0}^d E^*_{\rho+h} W \quad \text{(orthogonal direct sum)}$.

(iii) $E_i W \neq 0$ if and only if $\tau \leq i \leq \tau + d, \quad (0 \leq i \leq D)$.

(iv) $W = \sum_{h=0}^d E_{\tau+h} W \quad \text{(orthogonal direct sum)}$.

**Lemma 4.2** [26, Lemma 12.1] For $Y \in \text{Mat}_X(\mathbb{C})$ the following are equivalent:

(i) $Y \in T$;

(ii) $YW \subseteq W$ for all irreducible $T$-modules $W$. 

7
5 The split decomposition

We are going to make use of a certain decomposition of $V$ called the \emph{split decomposition}. The split decomposition was defined in [37] and discussed further in [26, 28]. In this section we recall some results on this topic.

**Definition 5.1** [37, Definition 5.1] For $-1 \leq i, j \leq D$ we define

\[
V_{i,j}^{\downarrow \downarrow} = (E_0^i V + \cdots + E_i^i V) \cap (E_0 V + \cdots + E_j V),
\]
\[
V_{i,j}^{\downarrow \uparrow} = (E_0^i V + \cdots + E_i^i V) \cap (E_D V + \cdots + E_{D-j}^j V).
\]

In the above two equations we interpret the right-hand side to be 0 if $i = -1$ or $j = -1$.

**Definition 5.2** [37, Definition 5.5] With reference to Definition 5.1, for $(\mu, \nu) = (\downarrow, \downarrow)$ or $(\mu, \nu) = (\downarrow, \uparrow)$ we have $V_{i-1,j}^{\mu \nu} \subseteq V_{i,j}^{\mu \nu}$ and $V_{i,j-1}^{\mu \nu} \subseteq V_{i,j}^{\mu \nu}$. Therefore

\[V_{i-1,j}^{\mu \nu} + V_{i,j-1}^{\mu \nu} \subseteq V_{i,j}^{\mu \nu}.
\]

Referring to the above inclusion, we define $\tilde{V}_{i,j}^{\mu \nu}$ to be the orthogonal complement of the left-hand side in the right-hand side; that is

\[\tilde{V}_{i,j}^{\mu \nu} = (V_{i-1,j}^{\mu \nu} + V_{i,j-1}^{\mu \nu})^\perp \cap V_{i,j}^{\mu \nu}.
\]

The following is a mild generalization of [37, Corollary 5.8].

**Lemma 5.3** With reference to Definition 5.2 the following holds for $(\mu, \nu) = (\downarrow, \downarrow)$ and $(\mu, \nu) = (\downarrow, \uparrow)$:

\[V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu \nu} \quad \text{(direct sum).}
\]

**Proof:** For $(\mu, \nu) = (\downarrow, \downarrow)$ this is just [37, Corollary 5.8]. For $(\mu, \nu) = (\downarrow, \uparrow)$, in the proof of [37, Corollary 5.8] replace the sequence $\{E_i\}_{i=0}^{D}$ by $\{E_{D-i}\}_{i=0}^{D}$. \qed

**Note 5.4** Following [28, Definition 6.4] we call the sum (6) the $(\mu, \nu)$-split decomposition of $V$.

We now recall how the split decompositions are related to the irreducible $T$-modules. We start with a definition.

**Definition 5.5** [37, Definition 4.1] Let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. By the \emph{displacement of $W$ of the first kind} we mean the scalar $\rho + \tau + d - D$. By the \emph{displacement of $W$ of the second kind} we mean the scalar $\rho - \tau$. By the inequalities in Lemma 4.1, each kind of displacement is at least $-D$ and at most $D$.

**Lemma 5.6** [37, Theorem 6.2] For $-D \leq \delta \leq D$ the following coincide:
(i) The subspace of $V$ spanned by the irreducible $T$-modules for which $\delta$ is the displacement of the first kind;

(ii) $\sum \tilde{V}_{ij}^\uparrow$, where the sum is over all ordered pairs $i, j$ ($0 \leq i, j \leq D$) such that $i + j = \delta + D$.

**Lemma 5.7** For $-D \leq \delta \leq D$ the following coincide:

(i) The subspace of $V$ spanned by the irreducible $T$-modules for which $\delta$ is the displacement of the second kind;

(ii) $\sum \tilde{V}_{ij}^\downarrow$, where the sum is over all ordered pairs $i, j$ ($0 \leq i, j \leq D$) such that $i + j = \delta + D$.

**Proof:** In the proof of [37, Theorem 6.2], replace the sequence $\{E_i\}_{i=0}^D$ by the sequence $\{E_{D-i}\}_{i=0}^D$. \hfill $\square$

6 A homomorphism $\vartheta : \boxtimes_q \rightarrow T$

We now impose an assumption on $\Gamma$.

**Assumption 6.1** We fix complex scalars $q, \eta, \eta^*, u, u^*, v, v^*$ with $q, u, u^*, v, v^*$ nonzero and $q^{2i} \neq 1$ for $1 \leq i \leq D$. We assume that $\Gamma$ has a $Q$-polynomial structure with eigenvalue sequence

$$\theta_i = \eta + uq^{2i-D} + vq^{D-2i} \quad (0 \leq i \leq D)$$

and dual eigenvalue sequence

$$\theta^*_i = \eta^* + u^*q^{2i-D} + v^*q^{D-2i} \quad (0 \leq i \leq D).$$

Moreover we assume that each irreducible $T$-module is thin.

**Remark 6.2** In the notation of Bannai and Ito [1, p. 263] the $Q$-polynomial structure from Assumption 6.1 is type I with $s \neq 0, s^* \neq 0$. We caution the reader that the scalar denoted $q$ in [1, p. 263] is the same as our scalar $q^2$.

**Example 6.3** The ordinary cycles are the only known distance-regular graphs that satisfy Assumption 6.1 [3].

Under Assumption 6.1 we will display a $\mathbb{C}$-algebra homomorphism $\vartheta : \boxtimes_q \rightarrow T$. To describe this homomorphism we define two matrices in $\text{Mat}_X(\mathbb{C})$, called $\Phi$ and $\Psi$.

**Definition 6.4** With reference to Lemma 5.3 and Assumption 6.1, let $\Phi$ (resp. $\Psi$) denote the unique matrix in $\text{Mat}_X(\mathbb{C})$ that acts on $\tilde{V}_{ij}^\uparrow$ (resp. $\tilde{V}_{ij}^\downarrow$) as $q^{i+j-D}I$ for $0 \leq i, j \leq D$. Observe that each of $\Phi, \Psi$ is invertible.
Lemma 6.5 Under Assumption 6.1 let \( W \) denote an irreducible \( T \)-module with endpoint \( \rho \), dual endpoint \( \tau \), and diameter \( d \). Then \( \Phi \) and \( \Psi \) act on \( W \) as \( q^{\rho+\tau+d-D}I \) and \( q^{\rho-\tau}I \) respectively.

Proof: Concerning \( \Phi \), abbreviate \( \delta = \rho + \tau + d - D \) and recall that this is the displacement of \( W \) of the first kind. We show that \( \Phi \) acts on \( W \) as \( q^{\delta}I \). Let \( V_0 \) denote the common subspace from parts (i), (ii) of Lemma 5.6. By Lemma 5.6(i) we have \( W \subseteq V_0 \). In Lemma 5.6(ii) \( V_0 \) is expressed as a sum. The matrix \( \Phi \) acts on each term of this sum as \( q^{\delta}I \) by Definition 6.4, so \( \Phi \) acts on \( V_0 \) as \( q^{\delta}I \). By these comments \( \Phi \) acts on \( W \) as \( q^{\delta}I \) and this proves our assertion concerning \( \Phi \). Our assertion concerning \( \Psi \) is similarly proved using the displacement of the second kind and Lemma 5.7. \( \square \)

Lemma 6.6 Under Assumption 6.1 the matrices \( \Phi \) and \( \Psi \) are central elements of \( T \).

Proof: The matrices \( \Phi \) and \( \Psi \) are contained in \( T \) by Lemma 4.2 and Lemma 6.5. These matrices are central in \( T \) since by Lemma 6.5 they act as a scalar multiple of the identity on every irreducible \( T \)-module. \( \square \)

The following is our main result.

Theorem 6.7 Under Assumption 6.1 there exists a \( \mathbb{C} \)-algebra homomorphism \( \vartheta : \mathbb{R}_q \rightarrow T \) such that both

\[
\begin{align*}
A &= \eta I + u\Phi\Psi^{-1}\vartheta(x_{01}) + v\Psi\Phi^{-1}\vartheta(x_{12}), \\
A^* &= \eta^* I + u^*\Phi\Psi\vartheta(x_{23}) + v^*\Psi\Phi^{-1}\vartheta(x_{30}).
\end{align*}
\]

(7) (8)

We will prove the above theorem after a few lemmas.

Lemma 6.8 Under Assumption 6.1 let \( W \) denote an irreducible \( T \)-module with endpoint \( \rho \), dual endpoint \( \tau \), and diameter \( d \). Then there exists a \( \mathbb{R}_q \)-module structure on \( W \) such that the adjacency matrix \( A \) acts as \( \eta I + uq^{2\tau+d-D}x_{01} + vq^{D-d-2\tau}x_{12} \) and the dual adjacency matrix \( A^* \) acts as \( \eta^* I + u^*q^{2\rho+d-D}x_{23} + v^*q^{D-d-2\rho}x_{30} \). This \( \mathbb{R}_q \)-module structure is irreducible.

Proof: By [22, Example 1.4] and since the \( T \)-module \( W \) is thin the pair \( A, A^* \) acts on \( W \) as a Leonard pair in the sense of [35, Definition 1.1]. In the notation of [35, Definition 5.1] this Leonard pair has an eigenvalue sequence \( \{\theta_{\tau+i}\}_{i=0}^d \) and a dual eigenvalue sequence \( \{\theta^*_{\rho+i}\}_{i=0}^d \) in view of Lemma 4.1. To motivate what follows we note that

\[
\begin{align*}
\theta_{\tau+i} &= \eta + uq^{2\tau+d-D}q^{2i-d} + vq^{D-d-2\tau}q^{d-2i}, \\
\theta^*_{\rho+i} &= \eta^* + u^*q^{2\rho+d-D}q^{2i-d} + v^*q^{D-d-2\rho}q^{d-2i}
\end{align*}
\]

for \( 0 \leq i \leq d \). In both equations above the coefficients of \( q^{2i-d} \) and \( q^{d-2i} \) are nonzero; therefore the action of \( A, A^* \) on \( W \) is a Leonard pair of \( q \)-Racah type in the sense of [36, Example 5.3]. Referring to this Leonard pair, let \( \{\varphi_i\}_{i=1}^d \) (resp. \( \{\phi_i\}_{i=1}^d \) denote the first (resp. second) split sequence [35, Section 7] associated with the eigenvalue sequence \( \{\theta_{\tau+i}\}_{i=0}^d \)
and the dual eigenvalue sequence $\{\theta^*_r\}_{i=0}^d$. By [35, Section 7] each of $\varphi_i$, $\phi_i$ is nonzero for $1 \leq i \leq d$. By [36, Example 5.3] there exists a nonzero $r \in \mathbb{C}$ such that

$$\varphi_i = (q^i - q^{-i})(q^{d+i+1} - q^{-d-i-1})$$

$$\times (q^{d-i} - q^{-d-i})(urq^{rq^2+2\rho+d+i-2D} - v^* q^{2D-2d-2\rho+1-i}),$$

$$\phi_i = (q^i - q^{-i})(q^{d+i+1} - q^{-d-i-1})$$

$$\times (urq^{rq^2+d+D+1-i} - vq^{D-2d-2r+i})(u^* q^{2\rho+d-i-1} - v^* r^{-1} q^{D-2\rho-i})$$

for $1 \leq i \leq d$. Observe that $r$ is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$ since each of $\varphi_1, \varphi_2, \ldots, \varphi_d$ is nonzero. By [35, Section 7] there exists a basis $\{v_i\}_{i=0}^d$ of $W$ such that

$$A_{vi} = \theta_{r+d-i}v_i + v_{i+1} \quad (0 \leq i \leq d-1), \quad A_{vd} = \theta_r v_d,$$

$$A^*v_i = \theta_{* \rho+i} v_i + \phi_i v_{i-1} \quad (1 \leq i \leq d), \quad A^*v_0 = \theta_{* \rho} v_0.$$

For convenience we adjust this basis slightly. For $1 \leq i \leq d$ define

$$\gamma_i = (q^i - q^{-i}) (urq^{rq^2+d+D+1-i} - vq^{D-2d-2r+i}).$$

In the above equation the right-hand side is nonzero since it is a factor of $\phi_i$, so $\gamma_i \neq 0$. Define $u_i = \gamma_1 \gamma_2 \cdots \gamma_i^{-1} v_i$ for $0 \leq i \leq d$ and note that $\{u_i\}_{i=0}^d$ is a basis for $W$. By the construction

$$A_{ui} = \theta_{r+d-i}u_i + \gamma_{i+1} u_{i+1} \quad (0 \leq i \leq d-1), \quad A_{ud} = \theta_r u_d,$$

$$A^*u_i = \theta_{* \rho+i} u_i + \phi_i \gamma_i^{-1} u_{i-1} \quad (1 \leq i \leq d), \quad A^*u_0 = \theta_{* \rho} u_0.$$

We let each standard generator of $\mathfrak{gl}_q$ act linearly on $W$; to define this action we specify what it does to the basis $\{u_i\}_{i=0}^d$. Here are the details:

$$x_{01}.u_i = q^{d-2i}u_i + (q^d - q^{2d-i-2}) q^{d-r}u_{i+1} \quad (0 \leq i \leq d-1), \quad x_{01}.u_d = q^{-d} u_d,$$

$$x_{12}.u_i = q^{2i-d}u_i + (q^d - q^{2i+2-d}) u_{i+1} \quad (0 \leq i \leq d-1), \quad x_{12}.u_d = q^d u_d,$$

$$x_{23}.u_i = q^{2i-d}u_i + (q^d - q^{2i-2-d}) u_{i-1} \quad (1 \leq i \leq d), \quad x_{23}.u_0 = q^{-d} u_0,$$

$$x_{30}.u_i = q^{d-2}u_i + (q^d - q^{d+2}) q^{d-1} u_{i-1} \quad (1 \leq i \leq d), \quad x_{30}.u_0 = q^d u_0,$$

$$x_{13}.u_i = q^{2i-d} u_i \quad (0 \leq i \leq d),$$

$$x_{31}.u_i = q^{d-2} u_i \quad (0 \leq i \leq d),$$

$$x_{02}.u_i = (1 - qr^{-d-1})(1 - q^{2d-2i+2})(1 - q^{2d-2i+4}) \cdots (1 - q^{2d-2i}) q^{d-2i}$$

$$\times (1 - q^{d-2i-2}) u_0$$

$$+ (1 - qr^{d+1})(1 - qr^{-d}) \sum_{h=1}^{i} \frac{(q^{2i+2} - 1)r}{q^{2i+1}(1 - qr^{d-2i-2})} u_{i+1} \quad (0 \leq i \leq d-1),$$

$$x_{02}.u_d = \frac{(1 - q^2)(1 - q^4) \cdots (1 - q^d) q^{-d}}{(1 - qr^{1-d})(1 - qr^{3-d}) \cdots (1 - qr^{d-1})} u_0$$

$$+ (1 - qr^{d+1}) \sum_{h=1}^{d} \frac{(1 - q^2)(1 - q^4) \cdots (1 - q^{2d-2h}) q^{-d}}{(1 - qr^{1-d})(1 - qr^{3-d}) \cdots (1 - qr^{d+1-2h})} u_h,$$
\[ x_{20}u_0 = (1 - rq^{d+1}) \sum_{h=0}^{d-1} \frac{(1 - q^2)(1 - q^4) \cdots (1 - q^{2h})rq^{h-d-h-d}}{(1 - rq^{1-d})(1 - rq^{3-d}) \cdots (1 - rq^{2h-d+1})} u_h \]
\[ + \frac{(1 - q^2)(1 - q^4) \cdots (1 - q^{2d})rq^{d-d^2}}{(1 - rq^{1-d})(1 - rq^{3-d}) \cdots (1 - rq^{d-1})} u_d. \]
\[ x_{20}u_i = \frac{q^d - q^{2i-2-d}}{1 - rq^{2i-d-1}} u_{i-1} \]
\[ + (1 - rq^{d+1})(1 - rq^{d-1}) \sum_{h=1}^{d-1} \frac{(1 - q^{2i+2})(1 - q^{2i+4}) \cdots (1 - q^{2h})rq^{h-i-q^{(d+1)i-(d-1)h-d}}}{(1 - rq^{2i-d-1})(1 - rq^{2i-d+1}) \cdots (1 - rq^{d-1})} u_h \]
\[ + (1 - rq^{d-1})(1 - rq^{d-1}) \sum_{h=1}^{d-1} \frac{(1 - q^{2i+2})(1 - q^{2i+4}) \cdots (1 - q^{2d})rq^{d+i-d^2}}{(1 - rq^{2i-d-1})(1 - rq^{2i-d+1}) \cdots (1 - rq^{d-1})} u_d \quad (1 \leq i \leq d). \]

In the above formulæ the denominators are nonzero since \( r \) is not among \( q^{d-1}, q^{d-3}, \ldots, q^{1-d} \).

One checks (or see [27]) that the above actions satisfy the defining relations for \( \mathbb{E}_q \) from Definition 2.1, so these actions induce a \( \mathbb{E}_q \)-module structure on \( W \). Combining the action of \( A \) (resp. \( A^* \)) on \( \{u_i\}_{i=0}^d \) with the actions of \( x_{01}, x_{12} \) (resp. \( x_{23}, x_{30} \)) on \( \{u_i\}_{i=0}^d \) we find that both
\[ A = \eta I + uq^{2r+d-D}x_{01} + vq^{D-d-2r}x_{12}, \]
\[ A^* = \eta^* I + u^* q^{2r+d-D}x_{23} + v^* q^{D-d-2r}x_{30} \]
on \( W \). By these equations and since the \( T \)-module \( W \) is irreducible we find the \( \mathbb{E}_q \)-module \( W \) is irreducible. The result follows. \( \square \)

**Lemma 6.9** Under Assumption 6.1 let \( W \) denote an irreducible \( T \)-module and consider the \( \mathbb{E}_q \)-action on \( W \) from Lemma 6.8. Then the following equations hold on \( W \):
\[ A = \eta I + u\Phi^\varepsilon x_{01} + v\Psi^\varepsilon x_{12}, \]
\[ A^* = \eta^* I + u^* \Phi^\varepsilon x_{23} + v^* \Psi^\varepsilon x_{30}. \]

**Proof:** Combine Lemma 6.5 and Lemma 6.8. \( \square \)

It is now a simple matter to prove Theorem 6.7.

**Proof of Theorem 6.7:** We start with a comment. Let \( W \) and \( W' \) denote irreducible \( T \)-modules, and consider the \( \mathbb{E}_q \)-module structure on \( W \) and \( W' \) from Lemma 6.8. From the construction we may assume that if the \( T \)-modules \( W \) and \( W' \) are isomorphic then the \( \mathbb{E}_q \)-modules \( W \) and \( W' \) are isomorphic. With our comment out of the way we proceed to the main argument. The standard module \( V \) decomposes into a direct sum of irreducible \( T \)-modules. Each irreducible \( T \)-module in this decomposition supports a \( \mathbb{E}_q \)-module structure from Lemma 6.8. Combining these \( \mathbb{E}_q \)-modules we get a \( \mathbb{E}_q \)-module structure on \( V \). This module structure induces a \( \mathbb{C} \)-algebra homomorphism \( \vartheta : \mathbb{E}_q \to \text{Mat}_X(\mathbb{C}) \). The map \( \vartheta \) satisfies (7), (8) in view of Lemma 6.9. To finish the proof it suffices to show that \( \vartheta(\mathbb{E}_q) \subseteq T \).
To this end we pick $\zeta \in \mathbb{X}_q$ and show $\vartheta(\zeta) \in T$. Since $T$ is semisimple, and by our preliminary comment, there exists $B \in T$ that acts on each irreducible $T$-module in the above decomposition as $\vartheta(\zeta)$. The $T$-modules in this decomposition span $V$ so $\vartheta(\zeta)$ coincides with $B$ on $V$; therefore $\vartheta(\zeta) = B$ and in particular $\vartheta(\zeta) \in T$ as desired. We have now shown that $\vartheta(\mathbb{X}_q) \subseteq T$ and the result follows.

**Remark 6.10** In Theorem 6.7 we obtained a $\mathbb{C}$-algebra homomorphism $\vartheta : \mathbb{X}_q \to T$. In Theorem 3.3 we displayed four $\mathbb{C}$-algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ into $\mathbb{X}_q$. Composing these homomorphisms with $\vartheta$ we obtain four $\mathbb{C}$-algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ into $T$.

We conjecture that the conclusion of Theorem 6.7 still holds if we weaken Assumption 6.1 by no longer requiring that each irreducible $T$-module is thin.

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