Cosmology in time asymmetric extensions of general relativity

Genly Leon\textsuperscript{a} and Emmanuel N. Saridakis\textsuperscript{b,a}

\textsuperscript{a}Instituto de Física, Pontificia Universidad de Católica de Valparaíso, Casilla 4950, Valparaíso, Chile
\textsuperscript{b}Physics Division, National Technical University of Athens, Zografou Campus 15780, Athens, Greece

E-mail: genly.leon@ucv.cl, Emmanuel_Saridakis@baylor.edu

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Abstract. We investigate the cosmological behavior in a universe governed by time asymmetric extensions of general relativity, which is a novel modified gravity based on the addition of new, time-asymmetric, terms on the Hamiltonian framework, in a way that the algebra of constraints and local physics remain unchanged. Nevertheless, at cosmological scales these new terms can have significant effects that can alter the universe evolution, both at early and late times, and the freedom in the choice of the involved modification function makes the scenario able to produce a huge class of cosmological behaviors. For basic ansatzes of modification, we perform a detailed dynamical analysis, extracting the stable late-time solutions. Amongst others, we find that the universe can result in dark-energy dominated, accelerating solutions, even in the absence of an explicit cosmological constant, in which the dark energy can be quintessence-like, phantom-like, or behave as an effective cosmological constant. Moreover, it can result to matter-domination, or to a Big Rip, or experience the sequence from matter to dark energy domination. Additionally, in the case of closed curvature, the universe may experience a cosmological bounce or turnaround, or even cyclic behavior. Finally, these scenarios can easily satisfy the observational and phenomenological requirements. Hence, time asymmetric cosmology can be a good candidate for the description of the universe.

Keywords: dark energy theory, modified gravity

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1 Introduction

The standard model of cosmology includes two accelerated phases of expansion, at early and late times respectively. Such a behavior cannot be obtained within the standard paradigm of physics, namely in the framework of general relativity and Standard Model of particles. Hence, additional degrees of freedom should be included in the picture. If these extra degrees of freedom are attributed to new, exotic ingredients of the universe content, then concerning late times one has the concept of dark energy (for reviews see [1, 2]) and concerning early times the concept of inflaton field(s) (for reviews see [3, 4]). On the other hand, if the extra degrees of freedom are of gravitational origin, then one obtains the paradigm of modified gravity (see [5, 6] and references therein). The latter approach has the additional motivation of improving the UltraViolet behavior of gravity and alleviating the difficulties towards its quantization [7, 8]. Note that there are not strict boundaries between the above approaches, since one can partially or completely transform from one to the other, or construct theories where both extensions are imposed.

In the usual approach to gravitational modification one adds higher-order corrections to the Einstein-Hilbert action, like in $F(R)$ gravity [9–13], in Gauss-Bonnet and $f(G)$ gravity [14, 15], in Lovelock gravity [16, 17], in Weyl gravity [18, 19], in Hořava-Lifshitz gravity [20–22], in Galileon modifications [23–26], in nonlinear massive gravity [27–30] etc.
different class of gravitational modifications arise when one starts from the equivalent torsional formulation of gravity and add higher-order correction, like in $f(T)$ gravity [31–34], in $f(T, T_G)$ gravity [35–37], etc.

Recently, a new class of modified gravity was proposed [38]. In particular, working in the Hamiltonian framework the authors constructed a theory that breaks the time reversal invariance of general relativity. Although the algebra of constraints and local physics are unchanged, new terms appear at cosmological scales, that can alter the universe evolution, both at early and late times.

In the present work we are interesting in investigating in detail the cosmological implications of the above time asymmetric extensions of general relativity. In order to achieve this independently of the initial conditions and the specific universe evolution, we apply the dynamical systems method [39, 40] which allows us to extract the global behavior of the scenario, bypassing the complexity of the involved equations. Indeed, due to the freedom in choosing the relevant extra modification function, the capabilities of the scenario are found to be huge. The plan of the work is the following: in section 2 we present the time asymmetric extension of general relativity and we apply it in a cosmological framework. In section 3 we perform a detailed dynamical analysis, extracting the stable late time solutions and the corresponding observables, and in section 4 we discuss their physical implications. Lastly, section 5 is devoted to the conclusions.

2 Time asymmetric extensions of general relativity and cosmology

Let us briefly review the time asymmetric extension of general relativity [38]. In a first subsection we present the gravitational model itself, while in a second subsection we apply it in a cosmological framework.

2.1 Time asymmetric extension of general relativity

In this formulation one starts with the Hamiltonian form of general relativity with a cosmological constant [41]

$$S^{\text{GR}} = \int dt \int_\Sigma \left\{ \pi^{ab} g_{ab} - N \mathcal{H}^{\text{ADM}} - N^a D_a \right\},$$

(2.1)

where

$$\mathcal{H}^{\text{ADM}} = -\frac{1}{G} \sqrt{g} (\mathcal{R} - 2\Lambda) + \frac{G}{\sqrt{g}} \left( \frac{1}{2} \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) + \mathcal{H}^\Psi = 0$$

(2.2)

is the usual Hamiltonian constraint. In the above expressions $g_{ab}$ is the spatial metric, with $\pi^{ab}$ its canonical momenta and $\pi = g_{ab} \pi^{ab}$ the corresponding trace, while $N$ and $N^a$ are the usual lapse and shift functions. In this formalism, the Hamiltonian constraint (2.2), along with the diffeomorphism constraint

$$D_a = D_b \pi^b_a + D^\Psi_a = 0,$$

(2.3)

form a first class algebra, where the terms $\mathcal{H}^\Psi$ and $D^\Psi_a$ correspond to the matter content and $D_a$ is the covariant derivative. Obviously, the above expressions respect the time reversal symmetry

$$t \rightarrow -t$$

(2.4a)

$$g_{ab} \rightarrow g_{ab}$$

(2.4b)

$$\pi^{ab} \rightarrow -\pi^{ab}.$$
In order to acquire well defined cosmological evolution equations one must use a gauge fixing, and it proves convenient to use the “constant mean curvature gauge condition” (CMC) \[38\]

\[
\pi - \sqrt{\bar{g}} < \pi >= 0,
\]  
where \(< \cdot \cdot \cdot >\) denotes the spatial average of a density \(\rho\) defined through \(< \rho >= \frac{\int_{\Sigma} \rho}{\int_{\Sigma} \sqrt{\bar{g}}}\), with \(V = \int_{\Sigma} \sqrt{\bar{g}}\) the spatial volume. The CMC condition (2.5) is a gauge fixing of the Hamiltonian constraint (2.2), and thus they form a second class system. However, note that the CMC condition (2.5) and the diffeomorphism constraint (2.3) form a system of four first class constraints \[42–44\], as it is the case for the Hamiltonian constraint along with the diffeomorphism constraint. One can show that, restricting to constraints that are local in \(g_{ab}\) and \(\pi^{ab}\), there are no other pairs of systems of four first class constraints that one is the gauge fixing of the other, however one has the freedom to add a term linear in \(\pi\) to the Hamiltonian constraint [44]. This new term \(\pi/L\), with \(L\) the length-scale where this term becomes significant, breaks the time reversal symmetry (2.4a)–(2.4c), and this feature gave to the obtained gravitational modification the name “time asymmetric extension of general relativity”. One can extend the above extra, time-asymmetric, term of the Hamiltonian constraint, by assuming that the length-scale in which it becomes important is driven by a function of spatially averaged quantities, such as the spatial volume \(V\). Hence, in summary, one can extend (2.2) to a modified Hamiltonian constraint of the form \[38\]

\[
H^{\text{new}} = -\frac{1}{G} \sqrt{\bar{g}} (R - 2\Lambda) + G \left( \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) + f(V) \pi + H^\Psi = 0,
\]  
where \(f(V)\) is an arbitrary function of \(V\).

The above modification of the Hamiltonian constraint gives rise to a novel class of gravitational modifications. The new term leaves the constraint algebra and the local physical degrees of freedom unchanged [38]. The only complexity comes from the fact that it affects the propagation of chiral fermions, since the left-handed spacetime connection \(D_a \Psi_A\) does depend on \(\pi^{ab}\). In order to handle this issue, one introduces the Ashtekar geometry \[45\], alongside the usual spacetime geometry characterized by the spacetime metric \(g_{\mu\nu}\). Thus, although the gravitational effects and the propagation of photons are governed by the conventional spacetime geometry, the propagation of chiral fermions is determined by the Ashtekar geometry which contains all the information of time irreversible behavior. Nevertheless, since in this work we are interested in the late-time background cosmological evolution, in which the matter sector is effectively described by a perfect fluid, and where radiation (a part of which is composed by chiral fermions) is negligible, in the following we do not discuss the above issue in more details. Hence, the time asymmetric modified gravity that we focus in this work is characterized by the action

\[
S = \int dt \int_{\Sigma} \left\{ \pi^{ab} g_{ab} - N H^{\text{new}} - N^a D_a \right\},
\]  
where \(H^{\text{new}}\) is given by (2.6).

### 2.2 Cosmological application of time asymmetric gravity

Let us now apply the time asymmetric extension of general relativity in a cosmological framework. In particular, we focus on a Friedmann-Robertson-Walker (FRW) spacetime metric of the form

\[
ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),
\]  
(2.8)
where $a(t)$ is the scale factor, $k = -1, 0, 1$ for spatially open, flat or close geometry respectively, and with $d\Omega^2$ the two-dimensional sphere line element. Note that time-asymmetric extension of general relativity singles out a specific $3 + 1$ decomposition, selected by the constant mean curvature gauge condition, and moreover it introduces a dependence on the spatial slices volume, and thus the spacetime must be spatially compact. This is indeed the case in the above cosmological metric, where $k$ refers to positive, negative or zero constant spatial curvature. In particular, all of these cases are consistent with a non-trivial spatially compact topology, with $k = +1$ corresponding to spheres, $k = 0$ to tori, while for $k = -1$ the infinite number of compact manifolds with constant negative curvature are classified by Thurston [46] (see also [47]). Inserting the above metric in the total action $S + S_m$, with $S$ given by (2.7) and $S_m$ the matter action, and performing the variation in the ADM formalism, we easily obtain the Friedmann equations as [38]

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_m + f(V(a))^2$$

(2.9)

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho_m + p_m) + a f(V(a)) \frac{\partial f(V(a))}{\partial a},$$

(2.10)

where $H = \dot{a}/a$ is the Hubble parameter, $V(a) \propto a^3$ is the spatial volume, and $G$ is the gravitational constant. Additionally, we have considered the matter action $S_m$ to correspond to a perfect fluid with energy density $\rho_m$ and pressure $p_m$ respectively. We stress here that in action (2.7) we do not include an explicit cosmological constant, since our goal is exactly to investigate whether the universe acceleration can arise solely from a general modification term $f(V)$ (which definitely in the specific case $f(V) = \text{const.}$ gives rise to an effective cosmological constant).

Defining for convenience $g(a) = \frac{a}{G} f(V(a))$, the above modified Friedmann equations become

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_m + \frac{G^2 g(a)^2}{a^2}$$

(2.11)

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho_m + p_m) + \frac{G^2 g(a) g'(a)}{a} - \frac{G^2 g(a)^2}{a^2},$$

(2.12)

and thus the modification is included in the arbitrary function $g(a)$. Furthermore, we can rewrite the Friedmann equations (2.11), (2.12) in the usual form

$$H^2 = \frac{\kappa^2}{3} (\rho_m + \rho_{DE})$$

(2.13a)

$$\dot{H} = -\frac{\kappa^2}{2} (\rho_m + p_m + \rho_{DE} + p_{DE}),$$

(2.13b)

if we define the energy density and pressure of the effective dark energy sector as

$$\rho_{DE} = \frac{3G}{8\pi} \frac{g(a)^2}{a^2}$$

(2.14)

$$p_{DE} = -\frac{G}{8\pi} \left[ \frac{g(a)^2}{a^2} + \frac{2g(a) g'(a)}{a} \right],$$

(2.15)

i.e. attributing the dark energy sector to the new terms that time asymmetric gravity brings to the Friedmann equations. In this case, the dark energy equation-of-state parameter becomes:

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = -\frac{1}{3} \left[ 1 + \frac{2ag'(a)}{g(a)} \right].$$

(2.16)
In summary, the modified gravity at hand is determined by the arbitrary function \( f(V(a)) \). Hence, according to the choice of \( f(V(a)) \) one obtains distinct classes of cosmological models.

3 Late-time cosmology

In this section we are interested in investigating in detail the late-time cosmology of the time asymmetric extension of general relativity. Since the gravitational modification is determined by the function \( f(V) \), we will choose two basic ansatzes, namely the power law and the exponential one. In particular, we will consider

- Model I: \( f(V) = g_1 V^m \), which implies that the auxiliary function \( g(a) \) becomes \( g(a) = \frac{g_1}{G} \left( \frac{a}{a_0} \right)^p \), with \( p = 3m + 1 \), with \( g_1 \) a constant and \( p \) a parameter, and where \( a_0 \) is a constant which can be set to 1 for convenience.

- Model II: \( f(V) = g_2 e^{\lambda V} \), which implies that \( g(a) = g_2 \frac{a}{G} e^{\lambda a/a_0^3} \), with \( g_2 \) a constant, \( \lambda \) a parameter, and with \( a_0 \) a constant which can be set to 1.

In order to study the cosmological behavior in a general way, independently of the initial conditions and the specific universe evolution, we will apply the dynamical systems method, which allows to extract the global features of a cosmological scenario [48–55]. In this procedure, one first transforms the involved cosmological equations into an autonomous system and then he extracts its critical points. Hence, perturbing linearly around these critical points, and expressing the perturbations in terms of a perturbation matrix, allows to determine the type and stability of each critical point by examining the eigenvalues of this matrix.

3.1 Model I: \( f(V) = g_1 V^m \)

In the case where \( f(V) = g_1 V^m \), i.e. when \( g(a) = \frac{g_1}{G} a^p \) (with \( p = 3m + 1 \)), with \( g_1 \) a constant and \( p \) a parameter, the Friedmann equations \((2.11), (2.12)\) become

\[
H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_m + g_1^2 a^{2p-2},
\]

\[
\dot{H} - \frac{k}{a^2} = -4\pi G (\rho_m + p_m) + (p - 1)g_1^2 a^{2p-2},
\]

and thus the effective dark energy \((2.14)\) and pressure \((2.15)\) respectively become

\[
\rho_{DE} = \frac{3g_1^2}{8\pi G} a^{2p-2}
\]

\[
p_{DE} = -\frac{g_1^2}{8\pi G} a^{2p-2}(1 + 2p),
\]

and hence \((2.16)\) leads to

\[
w_{DE} = -\frac{1}{3}(1 + 2p).
\]

Additionally, we can define the “total” equation-of-state parameter

\[
w_{\text{tot}} \equiv -1 - \frac{2\dot{H}}{3H^2} = \frac{8\pi G a^2 w_m \rho_m - g_1^2 (2p + 1) a^{2p} + k}{8\pi G a^2 \rho_m + 3g_1^2 a^{2p} - 3k},
\]

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and the deceleration parameter as
\[ q = -1 - \frac{\dot{H}}{H^2} = \frac{1 + 3w_{\text{tot}}}{2}, \]
with \( w_m \equiv p_m/\rho_m \) the matter equation of state. In the following we assume the usual energy conditions, which lead to \( 0 \leq w_m \leq 1 \). Finally, note that for \( g_1 = 0 \) we re-obtain standard general relativity.

### 3.1.1 Zero or negative curvature

In the case \( k = 0, -1 \) as auxiliary variables it proves convenient to use the various density parameters, namely
\[ \Omega_k = -\frac{k}{a^2H^2}, \quad \Omega_m = \frac{8\pi G \rho_m}{3H^2}, \quad \Omega_{\text{DE}} = \frac{g_1^2 a^{2p-2}}{H^2}, \]
and thus the first Friedmann equation (3.1) gives rise to the constraint
\[ \Omega_k + \Omega_m + \Omega_{\text{DE}} = 1. \]

Using the above auxiliary variables we can write the cosmological equations in the autonomous form
\[ \frac{d\Omega_k}{d\eta} = -\Omega_k [2p\Omega_{\text{DE}} + (3w_m + 1)(\Omega_{\text{DE}} + \Omega_k - 1)], \]
\[ \frac{d\Omega_{\text{DE}}}{d\eta} = -\Omega_{\text{DE}} [2p(\Omega_{\text{DE}} - 1) + (3w_m + 1)(\Omega_{\text{DE}} + \Omega_k - 1)], \]
where we have used the constraint (3.9) in order to eliminate \( \Omega_m \) and thus reduce the system to dimension two. In these equations, as usual, we define the logarithmic time \( \eta = \ln a \). Hence, the above autonomous system is defined on the compact phase space \( \{(\Omega_k, \Omega_{\text{DE}}) : \Omega_k \geq 0, \Omega_{\text{DE}} \geq 0, \Omega_k + \Omega_{\text{DE}} \leq 1\} \).

Finally, using the auxiliary variables (3.8) we can express the deceleration parameter (3.7) as
\[ q = \frac{1}{2} \left[ 1 + 3w_m \Omega_m - (2p + 1)\Omega_{\text{DE}} - \Omega_k \right]. \]

The scenario of Model I, namely \( f(V) = g_1 V^m \), with zero or negative curvature, admits three physical critical points, corresponding to expanding universe \( (H > 0) \), which are displayed in table 1 along with their existence conditions. In the same table we include the eigenvalues of the involved perturbation matrix, and thus the corresponding stability conditions. Finally, for completeness, we also include the values of the deceleration parameter, calculated through (3.11). Note that the solution associated to \( P_3 \) for \( p \neq 1 \) is the power-law form \( a(t) = [ (1-p)(c_1 + a_1 t) ]^{1-p} \), while for \( p = 1 \) it is just the de Sitter solution \( a(t) = c_1 e^{gt} \), with \( c_1 \) and \( a_1 \) integration constants.

In summary, the scenario at hand admits two stable late-time critical points, namely \( P_2 \) for \( p < 0 \) and \( P_3 \) for \( p > 0 \).

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\(^{1}\)The interest of defining compact phase spaces is that then the flow has well-defined past and future attractors, and this facilitates the drawing of global results for the cosmological scenario [48–55].
Table 1. The physical critical points of the system (3.10) of time asymmetric cosmology of Model I: $f(V) = g_1 V^m$, with zero or negative curvature, and their existence and stability conditions. We have assumed $0 \leq w_m \leq 1$.

3.1.2 Positive curvature

In the case $k = +1$, that is for positive curvature, it is not guaranteed that the Hubble parameter does not change sign during the evolution. This implies that the $H$-normalization that we used in the previous open and flat cases is not a good choice for creating compact variables, since when $H$ crosses zero the dynamical variables would diverge, and moreover when $H$ changes sign our “time” variable $\eta = \ln a$ would change flow. Thus, in the present $k = +1$ case, it is consistent to introduce the auxiliary variables (similarly to the variables introduced in section VI of [39], and in sections 3.3 and 5.3 of [56]) as:

$$\Theta_k = \frac{1}{a^2 D^2}, \quad Q_0 = \frac{H}{D}, \quad \Theta_m = \frac{8\pi G\rho_m}{3D^2}, \quad \Theta_{DE} = \frac{g_1^2 a^{2(p-1)}}{D^2},$$

(3.12)

where $D = \sqrt{H^2 + a^{-2}}$, and which are finite even if $H$ crosses zero. Therefore, the first Friedmann equation (3.1) leads to the constraint

$$\Theta_m + \Theta_{DE} = 1.$$  

(3.13)

Additionally, from the definition of $D$ it follows

$$\Theta_k + Q_0^2 = 1,$$  

(3.14)

while the curvature parameter is expressed as

$$\Omega_k \equiv \frac{1}{a^2 H^2} = \frac{1 - Q_0^2}{Q_0^2}.$$  

Using the above auxiliary variables we can re-write the cosmological equations as

\[
\frac{dQ_0}{d\tau} = \frac{1}{2} (1 - Q_0^2) [2p\Theta_{DE} + (3w_m + 1)(\Theta_{DE} - 1)],
\]

(3.15a)

\[
\frac{d\Theta_{DE}}{d\tau} = -Q_0(2p + 3w_m + 1)(\Theta_{DE} - 1)\Theta_{DE},
\]

(3.15b)

where we have used the constraints (3.13) and (3.14) in order to eliminate $\Theta_m$ and $\Theta_k$ and therefore reduce the system to dimension two. In the above dynamical system, we have introduced the consistent “time” variable $\tau$ through $d\tau = Ddt$, which indeed satisfies the necessary
requirement that it is monotonic even if $H$ change sign. The above autonomous system is defined on the compact phase space $\{(Q_0, \Theta_{DE}) : -1 \leq Q_0 \leq 1, 0 \leq \Theta_{DE} \leq 1\}$. Finally, using the auxiliary variables (3.12) we can express the deceleration parameter (3.7) as

$$q = \frac{1}{2Q_0^2} [1 + 3w_m(1 - \Theta_{DE}) - (2p + 1)\Theta_{DE}] .$$

(3.16)

As we can observe the system (3.15) is symmetric under the transformation

$$(\tau, Q_0, \Theta_{DE}) \to (-\tau, -Q_0, \Theta_{DE}).$$

(3.17)

Thus, it is sufficient to discuss the behavior in one part of the phase space, that is in $\tau \geq 0, Q_0 \geq 0, \Theta_{DE} \geq 0$, and then obtain the dynamics on the other part from (3.17). For example, if a point with coordinates $(Q_0^0, \Theta_{DE}^0), Q_0^0 > 0, \Theta_{DE}^0 > 0$ is a future attractor as $\tau \to +\infty$, then its partner point $(-Q_0^0, \Theta_{DE}^0)$ via (3.17) is a past attractor as $\tau \to -\infty$, and vice versa. Furthermore, we mention that the function

$$M = \frac{1 - \Theta_{DE}}{1 - Q_0^2}, \quad \frac{dM}{d\tau} = -(3w_m + 1)Q_0M,$$

is a monotonic function in the regions $Q_0 < 0$ and $Q_0 > 0$ for $\Theta_{DE} \neq 1$. The points having $Q_0 > 0$ correspond to expansion, while those having $Q_0 < 0$ correspond to contraction. As we will see in the following, the system (3.15) admits a fixed point with $Q_0 = 0$ if we assume $p \geq 0, 0 \leq w_m \leq 1$. However, since for $p < 0, 0 \leq w_m \leq 1$ there are not equilibrium points with $Q_0 = 0$, it follows that $M$ acts as a monotonic function in the interior of the phase space. As a consequence, for $p < 0$ there can be no periodic orbits in the interior of the phase space and global results can be drawn [39]. Additionally, from the definition of $M$ it follows that either $Q_0 \to 1$ or $\Theta_{DE} \to 1$ asymptotically.

Note that the system (3.15) allows for an easy analytical elaboration, leading to

$$Q_0^2(a) = 1 - \frac{c_2a^{3w_m+1}}{a^{2p+3w_m+1} + e^{c_1}}, \quad \Theta_{DE}(a) = 1 - \frac{e^{c_1}}{a^{2p+3w_m+1} + e^{c_1}},$$

(3.19)

with $c_1$ and $a_1$ integration constants.

The scenario at hand admits five physical critical points which are displayed in table 2 along with their existence conditions. In the same table we include the eigenvalues of the corresponding perturbation matrix, and the resulting stability conditions. Finally, we also include the values of the deceleration parameter, calculated through (3.16). Note that the points $P_4$ and $P_5$ have the time reversal behavior of $P_6$ and $P_7$ respectively, due to the symmetry (3.17). Additionally, there exist orbits connecting $P_4$ and $P_5$ with $P_6$ and $P_7$, which implies that $Q_0$ can indeed become zero, i.e. $H = 0$, during the evolution (recall that the points having $Q_0 > 0$ are expanding while those having $Q_0 < 0$ are contracting). Lastly, the system admits a static solution, namely $P_8$, which always behaves as a saddle point.

In summary, the scenario of Model I, namely $f(V) = g_l V^m$, with positive curvature, admits three stable late-time critical points, namely the expanding solution $P_6$ for $p > 0$, the contracting solution $P_6$ for $p > -(3w_m + 1)/2$ and the contracting solution $P_7$ for $p < -(3w_m + 1)/2$.

We close this paragraph with some comments on the auxiliary variables choice. The advantage of using the variable $Q_0$ versus using the variable $\Omega_k$, used in paragraph 3.1.1, is that for closed models the variable $\Omega_k$ would not keep track of the $H$-sign changes, due to
Table 2. The physical critical points of the system (3.15) of time asymmetric cosmology of Model I: $f(V) = g_1 V^m$, with positive curvature, and their existence and stability conditions.

| C.P. | $Q_0$ | $\Theta_{DE}$ | $q$ | Existence | Eigenvalues | Stability |
|------|-------|---------------|----|-----------|-------------|-----------|
| $P_4$ | 1 | 0 | $\frac{3w_m+1}{1}$ | always | $3w_m + 1, 2p + 3w_m + 1$ | saddle for $p < -\frac{3w_m+1}{2}$, unstable for $p > -\frac{3w_m+1}{2}$ |
| $P_5$ | 1 | 1 | $-p$ | always | $-2p, -2p - 3w_m - 1$ | unstable for $p < -\frac{3w_m+1}{2}$, saddle for $-\frac{3w_m+1}{2} < p < 0$, stable for $p > 0$ |
| $P_6$ | -1 | 0 | $\frac{3w_m+1}{1}$ | always | $-(3w_m + 1), -(2p + 3w_m + 1)$ | saddle for $p < -\frac{3w_m+1}{2}$, stable for $p > -\frac{3w_m+1}{2}$ |
| $P_7$ | -1 | 1 | $-p$ | always | $2p, 2p + 3w_m + 1$ | stable for $p < -\frac{3w_m+1}{2}$, saddle for $-\frac{3w_m+1}{2} < p < 0$, unstable for $p > 0$ |

The quadratic dependence on $Q_0$, however these changes may have important cosmological consequences. As we saw, choosing $Q_0$ instead of $\Omega_k$ allows us to to obtain novel features, such as expanding solutions, contracting partners, transition from contracting to expanding cosmologies and vice versa, as well as static solutions (for instance a static solution, where $H = 0$, would obviously not be seen using $H$-normalization). These differences, arising from the possible $H$-sign change in closed models, forbid a unified description of all cases. Such a difference between closed and open/flat geometries, and the implied necessary different normalization, was first observed in [39, 57, 58] despite the fact that closed FRW had been previously studied in [49, 59, 60].

3.2 Model II: $f(V) = g_2 e^{\lambda V}$

In the case where $f(V) = g_2 e^{\lambda V}$, i.e. when $g(a) = g_2 \frac{2}{3} e^{\lambda a^3}$, with $g_2$ a constant and $\lambda$ a parameter, the Friedmann equations (2.11), (2.12) become

$$H^2 + \frac{k}{a^2} = \frac{8\pi G \rho_m}{3} + g_2^2 e^{2\lambda a^3}$$

and thus

$$w_{DE} = -1 - 2\lambda a^3.$$  

Additionally, the “total” equation-of-state parameter reads

$$w_{tot} = -1 - \frac{2\dot{H}}{3H^2} = \frac{8\pi G a^2 w_m \rho_m - 3g_2^2 a^2 e^{2\lambda a^3}(1 + 2\lambda a^3) + k}{8\pi G a^2 \rho_m + 3g_2^2 a^2 e^{2\lambda a^3} - 3k},$$

while the deceleration parameter writes as

$$q = \frac{1 + 3w_{tot}}{2}.$$  

Finally, note that for $g_2 = 0$ we re-obtain standard general relativity.
3.2.1 Zero or negative curvature

In the case $k = 0, -1$, we introduce the density parameters compact auxiliary variables

$$\Omega_k = -\frac{k}{a^2H^2}, \quad \Omega_m = \frac{8\pi G\rho_m}{3H^2}, \quad \Omega_{DE} = \frac{\rho_{DE}a^{-1}e^{2\lambda\Omega_k}}{H^2},$$

(3.25)

and thus the first Friedmann equation (3.20) gives rise to the constraint

$$\Omega_k + \Omega_m + \Omega_{DE} = 1.$$  (3.26)

In order to be able to close the system we need one more auxiliary parameter. Since the corresponding choice proves to be different according to the sign of $\lambda$, we will examine the two cases separately.

- $\lambda > 0$

In this case we define the additional auxiliary variable

$$T = \frac{\lambda a^3}{1 + \lambda a^2}.$$  (3.27)

Since by construction $0 < T < 1$ (since $\lambda > 0$), we can define

$$\frac{d\bar{\eta}}{dt} = H(1 - T)^{-1},$$

(3.28)

which implies $\bar{\eta} = \frac{1}{3} \lambda a(t)^3 + ln[a(t)]$ (modulo an additive constant), and thus $\bar{\eta} \to -\infty$ as $a \to 0$ and $\bar{\eta} \to \infty$ as $a \to \infty$. Hence, using the auxiliary variables (3.25) and (3.27) we can re-write the cosmological equations in their autonomous form, namely

$$\frac{dT}{d\bar{\eta}} = 3T(1 - T)^2,$$  (3.29a)

$$\frac{d\Omega_k}{d\bar{\eta}} = 3(T - 1)(w_m + 1)\Omega_k(\Omega_{DE} + \Omega_k - 1) - 2\Omega_k [T(3\Omega_{DE} + \Omega_k - 1) - \Omega_k + 1],$$  (3.29b)

$$\frac{d\Omega_{DE}}{d\bar{\eta}} = 3(T - 1)(w_m + 1)\Omega_{DE}(\Omega_{DE} + \Omega_k - 1) + 2\Omega_{DE} [\Omega_k - T(3\Omega_{DE} + \Omega_k - 3)],$$  (3.29c)

where we have used the constraint (3.26) in order to eliminate $\Omega_m$. Clearly, the above system is defined on the $\{(T, \Omega_k, \Omega_{DE}) : 0 \leq T \leq 1, \Omega_k \geq 0, \Omega_{DE} \geq 0, \Omega_k + \Omega_{DE} \leq 1\}$ part of the phase space, where we have included the two boundaries $T = 0$ and $T = 1$. Furthermore, note that the invariant subset boundary $T = 1$ corresponds to the asymptotic future, while the invariant subset boundary $T = 0$ is associated asymptotically to the (classical) initial state. Therefore, in this formalism all the fixed points are located at $T = 0$ and $T = 1$ [61]. Lastly, using the auxiliary variables (3.25), (3.27) we can express the deceleration parameter (3.24) as

$$q = \frac{1}{2} \left[ 1 + 3w_m\Omega_m - 3 \left(1 + \frac{T}{1 - T}\right) \Omega_{DE} - \Omega_k \right].$$  (3.30)

Thus, for the critical points having $T = 0$, the expression (3.30) is well-defined and gives $q|_{T=0} = \frac{[1 + 3w_m\Omega_m - 3\Omega_{DE} - \Omega_k]}{2}$, while for the critical points having $T =$$
The physical critical points of the system (3.29) of time asymmetric cosmology of Model II: $f(V) = g_2 e^{\lambda V}$, with zero or negative curvature and $\lambda > 0$, and their existence and stability conditions. We assume $0 \leq w_m \leq 1$.

1. $\Omega_{DE} > 0$ we obtain $q \to -\infty$ as $T \to 1^-$ since $\Omega_m, \Omega_{DE}$ and $\Omega_k$ are bounded. On the other hand, for the critical points having $T = 1, \Omega_{DE} = 0$, $q$ is arbitrary.

The scenario at hand admits five physical critical points, and one curve of critical points (namely $Q_5$), which are summarized in table 3 along with their existence conditions. In the same table we include the eigenvalues of the corresponding perturbation matrix, and the resulting stability conditions. Finally, we also include the values of the deceleration parameter, calculated through (3.30). We mention that for the three nonhyperbolic critical points, the linear analysis is not adequate to determine their stability, and therefore the stability conditions have been extracted applying the center manifold method [62]. The corresponding investigation is performed in appendix A.1.

- $\lambda < 0$

In this case we define the additional auxiliary variable

$$T_1 = -\frac{\lambda a^3}{1 - \lambda a^3}, \quad (3.31)$$

Since $0 < T_1 < 1$ (since $\lambda < 0$), we can define

$$\frac{d\eta}{dt} = H(1 - T_1)^{-1}, \quad (3.32)$$

which implies $\dot{\eta} = -\frac{1}{3} \lambda a(t)^3 + \ln[a(t)]$ (modulo an additive constant), and thus $\dot{\eta} \to -\infty$ as $a \to 0$ and $\dot{\eta} \to \infty$ as $a \to \infty$. Hence, using the auxiliary variables (3.25) and (3.31) we can re-write the cosmological equations in autonomous form as

$$\frac{dT_1}{d\eta} = 3T_1(1 - T_1)^2, \quad (3.33a)$$

$$\frac{d\Omega_k}{d\eta} = 3\Omega_k [(T_1 - 1)w_m - 1]([\Omega_{DE} + \Omega_k - 1] + \Omega_k [T_1(9\Omega_{DE} - 1) - 2] + (T_1 + 2)\Omega_k^2), \quad (3.33b)$$

$$\frac{d\Omega_{DE}}{d\eta} = \Omega_{DE} \{3(\Omega_{DE} - 1) [T_1(w_m + 3) - w_m - 1] + (T_1 - 1)(3w_m + 1)\Omega_k\}. \quad (3.33c)$$

| Label | $\Omega_k$ | $\Omega_{DE}$ | $T$ | $q$ | Existence | Eigenvalues | Stability |
|-------|-----------|---------------|-----|-----|-----------|-------------|-----------|
| $Q_1$ | 0         | 0             | 0   | 0   | always    | $3, 3(1 + w_m), 1 + 3w_m$ | unstable     |
| $Q_2$ | 1         | 0             | 0   | 0   | always    | $3, -3(1 + w_m), 2$ | saddle     |
| $Q_3$ | 0         | 1             | 0   | $-1$ | always    | $3, -3(1 + w_m), -2$ | saddle     |
| $Q_4$ | 0         | 0             | 1   | arbitrary | always | 6, 0, 0 | nonhyperbolic, behaves as unstable |
| $Q_5$ | $\Omega_{kc}$ | 0           | 1   | arbitrary | $\Omega_{kc} \in (0, 1]$ | 6, 0, 0 | nonhyperbolic, behaves as unstable     |
| $Q_6$ | 0         | 1             | 1   | $-\infty$ | always | $-6, -6, 0$ | nonhyperbolic, behaves as stable     |

Table 3. The physical critical points of the system (3.29) of time asymmetric cosmology of Model II: $f(V) = g_2 e^{\lambda V}$, with zero or negative curvature and $\lambda > 0$, and their existence and stability conditions. We assume $0 \leq w_m \leq 1$. 







Table 4. The physical critical points of the system (3.33) of time asymmetric cosmology of Model II: $f(V) = g_2 e^{V}$, with zero or negative curvature and $\lambda < 0$, and their existence and stability conditions. We assume $0 \leq w_m \leq 1$.

| Label | $\Omega_k$ | $\Omega_{DE}$ | $T_1$ | $q$ | Existence | Eigenvalues | Stability |
|-------|------------|--------------|------|-----|-----------|------------|----------|
| $Q_7$ | 0          | 0            | 0    | $3w_m+1$ | always     | $3(1 + w_m), 1 + 3w_m$ | unstable |
| $Q_8$ | 1          | 0            | 0    | 0    | always     | $3, -3(1 + w_m), 2$ | saddle   |
| $Q_9$ | 0          | 1            | 0    | $-1$ | always     | $3, -3(1 + w_m), -2$ | saddle   |
| $Q_{10}$ | 0          | 0            | 1    | arbitrary | always | $-6, 0, 0$ | nonhyperbolic, behaves as stable |
| $Q_{11}$ | $\Omega_{kc}$ | 0   | 1    | arbitrary | $\Omega_{kc} \in (0, 1]$ | $-6, 0, 0$ | nonhyperbolic, behaves as saddle for $\Omega_{kc} \neq 1$. stable for $\Omega_{kc} = 1$. |
| $Q_{12}$ | 0          | 1            | 1    | $+\infty$ | always | $6, 6, 0$ | nonhyperbolic, behaves as unstable |

where we have used the constraint (3.26) in order to eliminate $\Omega_m$. Clearly, the above system is defined on the $\{(T_1, \Omega_k, \Omega_{DE}) : 0 \leq T_1 \leq 1, \Omega_k \geq 0, \Omega_{DE} \geq 0, \Omega_k + \Omega_{DE} \leq 1\}$ part of the phase space, where we have attached the two boundaries $T_1 = 0$ and $T_1 = 1$. Finally, note that in terms of the auxiliary variables (3.25), (3.31) the deceleration parameter (3.24) is given by

$$q = \frac{1}{2} \left[ 1 + 3w_m\Omega_m - 3\left( \frac{1-3T_1}{1-T_1} \right)\Omega_{DE} - \Omega_k \right].$$

(3.34)

Thus, for the critical points having $T_1 = 0$, the expression (3.34) is well-defined and gives $q |_{T_1=0} = [1 + 3w_m\Omega_m - 3\Omega_{DE} - \Omega_k] / 2$, while for the critical points having $T_1 = 1, \Omega_{DE} > 0$ it follows that $q \to +\infty$ as $T_1 \to 1^-$ since $\Omega_m, \Omega_{DE}$ and $\Omega_k$ are bounded. On the other hand, for the critical points having $T_1 = 1, \Omega_{DE} = 0, q$ is arbitrary.

The scenario at hand admits five physical critical points, and one curve of critical points (namely $Q_{11}$), which are summarized in table 4 along with their existence conditions. In the same table we present the eigenvalues of the involved perturbation matrix and the corresponding stability conditions. Lastly, we also include the values of the deceleration parameter, calculated through (3.34). Concerning the three nonhyperbolic critical points the stability conditions have been extracted applying the center manifold method [62]. The corresponding investigation is performed in appendix A.2.

In summary, the scenario of Model II, namely $f(V) = g_2 e^{V}$, with zero or negative curvature admits the following stable late-time solutions: for $\lambda > 0$ the expanding solution $Q_6$ (nonhyperbolic, but with stable center manifold). For $\lambda < 0$ the line of expanding solutions $Q_{10}$ (nonhyperbolic, but with stable center manifold) and a point on the line $Q_{11}$ with $\Omega_{kc} = 1$ (nonhyperbolic, but with stable center manifold) which represent a curvature dominated solution.
3.2.2 Positive curvature

In this case we introduce the density parameters compact auxiliary variables (similarly to the variables introduced in section VI of [39], and in sections 3.3 and 5.3 of [56]):

\[ \Theta_k = \frac{1}{a^2D^2}, \quad Q_0 = \frac{H}{D}, \quad \Theta_m = \frac{8\pi G \rho_m}{3D^2}, \quad \Theta_{DE} = \frac{g_2^2 e^{2a^1\lambda}}{D^2}, \]

(3.35)

with \( D = \sqrt{H^2 + a^{-2}} \), and thus the first Friedmann equation (3.20) gives rise to the constraint

\[ \Theta_m + \Theta_{DE} = 1, \]

(3.36)

and as before we have the restriction

\[ \Theta_k + Q_0^2 = 1. \]

(3.37)

In order to be able to close the system we need one extra auxiliary parameter. Since the corresponding choice is different for different signs of \( \lambda \), we will examine the two cases separately.

- \( \lambda > 0 \)

In this case we define the additional auxiliary variable

\[ T = \frac{\lambda a^3}{1 + \lambda a^5}. \]

(3.38)

Since by construction \( D \geq 0 \) and \( 0 < T < 1 \) (since \( \lambda > 0 \)), we can define

\[ \frac{d\bar{\tau}}{dt} = D(1 - T)^{-1}, \]

(3.39)

which implies \( \bar{\tau} = \int_1^t \frac{(\lambda a(\zeta)^3 + 1)\sqrt{a'(\zeta)^3 + 1}}{a(\zeta)} d\zeta \) (modulo an additive constant). Since \( D(1 - T)^{-1} > 0 \), \( \bar{\tau} \) is a monotonic function of \( t \). Hence, using the auxiliary variables (3.35) and (3.38) we can express the cosmological equations as

\[ \frac{dT}{d\bar{\tau}} = 3Q_0 T(1 - T)^2, \]

(3.40a)

\[ \frac{dQ_0}{d\bar{\tau}} = \frac{1}{2} \left( 1 - Q_0^2 \right) \left[ -3(T - 1)(w_m + 1)(\Theta_{DE} - 1) + T(6\Theta_{DE} - 2) + 2 \right], \]

(3.40b)

\[ \frac{d\Theta_{DE}}{d\bar{\tau}} = 3Q_0(\Theta_{DE} - 1)\Theta_{DE} \left[ T(w_m - 1) - w_m - 1 \right], \]

(3.40c)

where we have used the constraint (3.36) in order to eliminate \( \Theta_m \) and (3.37) to eliminate \( \Theta_k \). The above system is defined on the \( \{(T, Q_0, \Theta_{DE}) : 0 \leq T \leq 1, -1 \leq Q_0 \leq 1, 0 \leq \Theta_{DE} \leq 1 \} \) part of the phase space, where we have included the two boundaries \( T = 0 \) and \( T = 1 \). For \( Q_0 > 0 \), i.e. for expanding cosmologies, the invariant subset boundary \( T = 1 \) corresponds to the asymptotic future, while the invariant subset boundary \( T = 0 \) is associated asymptotically to the (classical) initial state. However, for contracting models \( (Q_0 < 0) \) the roles of the invariant sets \( T = 1 \) and \( T = 0 \) are reversed in time. This arises from the fact that the function

\[ N = \frac{T}{1 - T}, \quad N' = 3(1 - T)Q_0 N, \]

(3.41)
is a monotonic function in the region $Q_0 < 0, 0 < T < 1$, where $N$ is monotonically increasing, and in the region $Q_0 > 0, 0 < T < 1$, where $N$ is monotonically decreasing, and thus it follows that either $T \to 1$ or $T \to 0$ asymptotically. Hence, in this formalism all the fixed points are located at $T = 0$ and $T = 1$ [61].

Furthermore, the system (3.40) is invariant under the symmetry

$$\left(\tilde{\tau}, T, Q_0, \Theta_{DE}\right) \to (-\tilde{\tau}, T, -Q_0, \Theta_{DE}).$$

(3.42)

Thus, it is sufficient to discuss the behavior in one part of the phase space, for instance in $\tilde{\tau} \geq 0, T \geq 0, Q_0 \geq 0, \Theta_{DE} \geq 0$, while the dynamics on the other part is being obtained from (3.42). For example, if a point with coordinates $(T^*, Q_0^*, \Theta_{DE}^*), Q_0^* > 0, \Theta_{DE}^* > 0$ is a future attractor as $\tilde{\tau} \to +\infty$, then, its partner point $(T^*, -Q_0^*, \Theta_{DE}^*)$ via (3.42) is a past attractor as $\tilde{\tau} \to -\infty$, and vice versa.

Finally, using the auxiliary variables (3.35) and (3.38) we can express the deceleration parameter (3.24) as

$$q = \frac{1}{2 \Theta_0} \left[ 1 + 3w_m(1 - \Theta_{DE}) - 3 \left( 1 + T \frac{1 + T}{1 - T} \right) \Theta_{DE} \right].$$

(3.43)

Thus, for the critical points having $T = 0$, the expression (3.43) is well-defined and gives $q \mid_{T=0} = \frac{1}{2 \Theta_0} \left[ 1 + 3w_m(1 - \Theta_{DE}) - 3\Theta_{DE} \right]$, whereas for the critical points having $T = 1, \Theta_{DE} \neq 0$, it is implied that $q \to -\infty$ as $T \to 1^-$. On the other hand, for the critical points having $T = 1, \Theta_{DE} = 0$, $q$ is arbitrary.

Furthermore, we note that the function

$$M = \frac{1 - \Theta_{DE}}{1 - Q_0^2}, \quad M' = -(3w_m + 1)(1 - T)Q_0M,$$

(3.44)

is a monotonic function in the regions $Q_0 < 0, 0 < T < 1$ and $Q_0 > 0, 0 < T < 1$ for $\Theta_{DE} \neq 1$. The points having $Q_0 > 0$ are expanding, and those having $Q_0 < 0$ are contracting. Thus, from the definition of $M$ it follows that either $Q_0^2 \to 1$ or $\Theta_{DE} \to 1$ asymptotically.

| Label | $Q_0$ | $\Theta_{DE}$ | $T$ | $q$ | Existence | Eigenvalues | Stability |
|-------|-------|---------------|-----|-----|-----------|------------|-----------|
| $Q_{13}$ | 1 | 0 | 0 | $\frac{3w_m + 1}{2}$ | always | $3.3(1 + w_m), 1 + 3w_m$ | unstable |
| $Q_{14}$ | 1 | 1 | 0 | $-1$ | always | $3, -3(1 + w_m), -2$ | saddle |
| $Q_{15}$ | 1 | 0 | 1 | arbitrary | always | $6, 0, 0$ | nonhyperbolic, behaves as saddle |
| $Q_{16}$ | $Q_{0c}$ | 0 | 1 | arbitrary | $Q_{0c} \in (0, 1]$ | $6Q_{0c}, 0, 0$ | nonhyperbolic, behaves saddle |
| $Q_{17}$ | 1 | 1 | 1 | $-\infty$ | always | $-6, -6, 0$ | nonhyperbolic, behaves as stable |

Table 5. The physical critical points of the system (3.40) of time asymmetric cosmology of Model II corresponding to expanding cosmologies: $f(V) = g_2 e^{\lambda V}$, with positive curvature and $\lambda > 0$, and their existence and stability conditions. We assume $0 \leq w_m \leq 1$. 


Table 6. The physical critical points of the system (3.40) of time asymmetric cosmology of Model II corresponding to contracting cosmologies: \( f(V) = g_2 e^{\lambda V} \), with positive curvature and \( \lambda > 0 \), and their existence and stability conditions. We assume \( 0 \leq w_m \leq 1 \).

The scenario at hand admits four physical critical points, and one curve of critical points (namely \( Q_{16} \), which is the straight line joining the points \((0,0,1)\) and \((1,0,1)\), with the left point not included) corresponding to expanding cosmologies, which are summarized in table 5 along with their existence conditions. In the same table we include the eigenvalues of the involved perturbation matrix, and the corresponding stability conditions. Moreover, we also include the values of the deceleration parameter, calculated through (3.43). Each of the above critical points have contracting partners via the discrete symmetry (3.42), which are displayed in table 6.

Lastly, there exists a line of static solutions, i.e. neither expanding nor contracting, given by

\[
S_1 : (T,Q_0,\Theta_{DE}) = \left( T_c, 0, \frac{(T_c - 1)(3w_m + 1)}{3T_c(w_m - 1) - 3(w_m + 1)} \right), \quad T_c \in [0,1].
\] (3.45)

Imposing the physical condition \( 0 \leq w_m \leq 1 \), it follows that the above line always satisfies the existence condition \( 0 \leq \Theta_{DE} \leq 1 \). The eigenvalues of the linearization around \( S_1 \) are

\[
0, -\sqrt{(3w_m + 1)(1-T_c)} [(w_m - 1)T_c - w_m - 1][(2w_m + 3T_c^2 - (w_m + 6)T_c - w_m - 1]/(w_m - 1)T_c - w_m - 1,
\]

\[
\sqrt{(3w_m + 1)(1-T_c)} [(w_m - 1)T_c - w_m - 1][(2w_m + 3T_c^2 - (w_m + 6)T_c - w_m - 1)]/(w_m - 1)T_c - w_m - 1,
\]

and thus whenever this line exists these eigenvalues are always real. Since two of them have different sign the whole line behaves as saddle.

We mention that in order to determine the stability of the six nonhyperbolic critical points (expanding and contracting ones) we apply the center manifold method \cite{62}, and the corresponding analysis is performed in appendix A.3.

- \( \lambda < 0 \)

In this case we define the additional auxiliary variable

\[
T_1 = -\frac{\lambda a^3}{1 - \lambda a^5}.
\] (3.46)
Since \(0 < T_1 < 1\) we define
\[
\frac{d\tilde{\tau}}{dt} = D(1 - T_1)^{-1},
\]
so that \(\tilde{\tau}\) is monotonic increasing for \(D(1 - T_1)^{-1} > 0\). Therefore, using the auxiliary variables (3.35) and (3.46) we can re-write the cosmological equations in autonomous form as
\[
\begin{align*}
\frac{dT_1}{d\tilde{\tau}} &= 3Q_0T_1(1 - T_1)^2, \\
\frac{dQ_0}{d\tilde{\tau}} &= -\frac{1}{2} (1 - Q_0^2) [3(T_1 - 1)(w_m + 1)(\Theta_{DE} - 1) + 2(3T_1\Theta_{DE} + T_1 - 1)], \\
\frac{d\Theta_{DE}}{d\tilde{\tau}} &= 3Q_0(\Theta_{DE} - 1)\Theta_{DE}(T_1(w_m + 3) - w_m - 1),
\end{align*}
\]
where we have used the constraint (3.36) in order to eliminate \(\Omega_m\). The above system is defined on the \([T_1, Q_0, \Theta_{DE}); 0 \leq T_1 \leq 1, -1 \leq Q_0 \leq 1, 0 \leq \Theta_{DE} \leq 1\} part of the phase space, where we have attached the invariant boundaries \(T_1 = 0\) and \(T_1 = 1\).

Similarly to the previous section, the function
\[
\tilde{\dot{N}} = \frac{T_1}{1 - T_1}, \quad \tilde{\dot{N}}' = 3(1 - T_1)Q_0\tilde{N},
\]
is a monotonic function in the region \(Q_0 < 0, 0 < T_1 < 1\), where \(\tilde{N}\) is monotonically increasing, and in the region \(Q_0 > 0, 0 < T_1 < 1\), where \(\tilde{N}\) is monotonically decreasing, and it follows that either \(T_1 \to 1\) or \(T_1 \to 0\) asymptotically.

Furthermore, the system (3.48) is invariant under the symmetry
\[
(\tilde{\tau}, T_1, Q_0, \Theta_{DE}) \to (-\tilde{\tau}, T_1, -Q_0, \Theta_{DE}).
\]
Thus, it is sufficient to discuss the behavior in one part of the phase space, that is in \(\tilde{\tau} \geq 0, T_1 \geq 0, Q_0 \geq 0, \Theta_{DE} \geq 0\), while the dynamics on the other part is being obtained from (3.50). For example, if a point with coordinates \((T_1^*, Q_0^*, \Theta_{DE}^*), Q_0^* > 0, \Theta_{DE}^* > 0\) is a future attractor as \(\tilde{\tau} \to +\infty\), then, its partner point \((T_1^*, -Q_0^*, \Theta_{DE}^*)\) via (3.50) is a past attractor as \(\tilde{\tau} \to -\infty\), and vice versa. Additionally, in the invariant set \(T_1 = 0\) we obtain the first integral
\[
\frac{(1 - Q_0^2)^3(1 + w_m)}{(1 - \Theta_{DE})^2Q_{DE}^{1 + 3w_m}} = c,
\]
with \(c\) an integration constant.

Finally, note that in terms of the auxiliary variables (3.35), (3.46) the deceleration parameter (3.24) is given as
\[
q = \frac{1}{2Q_0^2} \left[1 + 3w_m(1 - \Theta_{DE}) - 3 \left(1 - 3\frac{T_1}{1 - T_1}\right)\Theta_{DE}\right].
\]
Hence, for the critical points having \(T_1 = 0\), the expression (3.52) is well-defined and leads to \(q_{|T_1=0} = \frac{1}{2Q_0^2} \left[1 + 3w_m(1 - \Theta_{DE}) - 3\Theta_{DE}\right]\), however for the critical points having \(T_1 = 1, \Theta_{DE} \neq 0\), it follows that \(q \to +\infty\) as \(T_1 \to 1^-\). On the other hand, for the critical points having \(T_1 = 1, \Theta_{DE} = 0, q\) is arbitrary.
has a partner through the symmetry (3.48). Along with their existence conditions. In the same table we present the existence and stability conditions. We assume $0 \leq w_m \leq 1$.

Table 7. The physical critical points of the system (3.48) of time asymmetric cosmology of Model II representing expanding cosmologies: $f(V) = g_2 e^{\lambda V}$, with positive curvature and $\lambda < 0$, and their existence and stability conditions. We assume $0 \leq w_m \leq 1$.

| Label | $Q_0$ | $\Theta_{DE}$ | $T_1$ | $q$ | Existence | Eigenvalues | Stability |
|-------|------|-------------|-----|-----|-----------|-------------|-----------|
| $Q_{18}$ | 1 | 0 | 0 | $\frac{3w_m + 1}{2}$ | always | $3, 3(1 + w_m), 1 + 3w_m$ | unstable |
| $Q_{19}$ | 1 | 1 | 0 | $-1$ | always | $3, -3(1 + w_m), -2$ | saddle |
| $Q_{20}$ | 1 | 0 | 1 | arbitrary | always | $-6, 0, 0$ | nonhyperbolic, behaves as saddle |
| $Q_{21}$ | $Q_{0c}$ | 0 | 1 | arbitrary | $Q_{0c} \in (0, 1)$ | $-6Q_{0c}, 0, 0$ | nonhyperbolic, behaves as saddle |
| $Q_{22}$ | 1 | 1 | 1 | $\infty$ | always | $6, 6, 0$ | nonhyperbolic, behaves as saddle |

The scenario at hand admits four physical critical points, and one curve of critical points (namely $Q_{21}$) representing accelerating solutions ($Q_0 > 0$), which are displayed in table 7 along with their existence conditions. In the same table we present the eigenvalues of the corresponding perturbation matrix, and the resulting stability conditions. Finally, we also include the values of the deceleration parameter, calculated through (3.52). Each point/curve in table 7 has a partner through the symmetry (3.50), representing a contracting cosmology ($Q_0 < 0$), which are displayed in table 8.

Lastly, the system (3.48) admits a line representing static solutions, i.e. neither expanding nor contracting, given by

$$S_2 : (T_1, Q_0, \Theta_{DE}) = \left( T_c, 0, \frac{(3w_m + 1)(T_c - 1)}{3(w_m T_c + 3T_c - w_m - 1)} \right),$$

(3.53)

with eigenvalues

$$0, -\frac{\sqrt{T_c - 1}\sqrt{(3w_m + 1)[(w_m + 3)T_c - w_m - 1][(4w_m + 15)T_c^2 - 5(w_m + 2)T_c + w_m + 1]} - (w_m + 3)T_c - w_m - 1}{(w_m + 3)T_c - w_m - 1},$$

$$\sqrt{T_c - 1}\sqrt{(3w_m + 1)[(w_m + 3)T_c - w_m - 1][(4w_m + 15)T_c^2 - 5(w_m + 2)T_c + w_m + 1]} - (w_m + 3)T_c - w_m - 1.$$

This line exists and is physical, i.e. possessing $0 \leq \Theta_{DE} \leq 1$, for i) $0 \leq T_c < \frac{5w_m - \sqrt{3w_m(3w_m + 8) + 40 + 10}}{8w_m + 30}$, when the eigenvalues are real and $S_2$ behaves as saddle, or

ii) $\frac{5w_m - \sqrt{3w_m(3w_m + 8) + 40 + 10}}{8w_m + 30} < T_c \leq 1$, when two eigenvalues are purely imaginary.

We mention that in order to determine the stability of the six nonhyperbolic critical points we apply the center manifold method [62], and the corresponding analysis is performed in appendix A.4.

In summary, the scenario of Model II, namely $f(V) = g_2 e^{\lambda V}$, with zero or negative curvature admits the following stable late-time solutions: for $\lambda > 0$ the contracting solution $R_{13}$ and the expanding solution $Q_{17}$ (nonhyperbolic but with stable center manifold), while for $\lambda < 0$ the contracting solution $R_{18}$. 

\[ -17 - \]
Table 8. The physical critical points of the system (3.48) of time asymmetric cosmology of Model II representing contracting cosmologies: \( f(V) = g_2 e^{\lambda V} \), with positive curvature and \( \lambda < 0 \), and their existence and stability conditions. We assume \( 0 \leq w_m \leq 1 \).

4 Physical implications

Having performed a complete dynamical analysis of cosmological scenarios governed by time asymmetric extensions of general relativity, we can now proceed to the discussion of the physical implications. In particular, we focus on the stable late-time solutions, since these solutions can attract the universe at late times, independently of the specific initial conditions and the specific intermediate evolution.

4.1 Model I: \( f(V) = g_1 V^m \)

In the case where \( f(V) = g_1 V^m \), i.e. when \( g(a) = g_1 a^p \) (with \( p = 3m + 1 \)), with \( g_1 \) a constant and \( p \) a parameter, and with open or zero curvature, the scenario at hand exhibits the three critical points presented in table 1. Point \( P_1 \) corresponds to a dark-matter dominated universe (\( \Omega_m = 1 \)), that is non-accelerating (\( q > 0 \)), however it is never stable and thus it cannot attract the universe at late times. Point \( P_2 \) corresponds to a universe governed by the curvature term (\( \Omega_k = 1 \)), which is neither accelerating nor decelerating (this is typical for curvature dominated solutions [63]). For \( p < 0 \) it can be stable, and thus it can attract the universe at late times (this is actually expected since for \( p < 0 \) the effective dark-energy term decreases faster than the curvature term, and hence the latter dominates). However, its observational features are disfavored by observations. Point \( P_3 \) is stable for \( p > 0 \) and thus it can be the stable late-time state of the universe. It corresponds to a dark-energy dominated, accelerating universe, where the dark-energy equation-of-state parameter (3.5), namely \( w_{DE} = -(1 + 2p)/3 \), can lie either in the quintessence regime (for \( 0 < p < 1 \)), or in the phantom one (for \( 1 < p \)), or either behave as an effective cosmological constant (for \( p = 1 \)) giving rise to a de Sitter universe. These features make it a good candidate for the description of the universe, especially if \( 0.9 \lesssim p \lesssim 1.1 \), in which case \(-1.07 \lesssim w_{DE} \lesssim -0.93 \) in agreement with observations [64]. We mention that the above behavior is obtained without the addition of an explicit cosmological constant term in the action, i.e. it is a pure effect of the novel, time-asymmetric theory. Finally, note that even when the effective dark energy lies in the phantom regime, the universe does not end in a Big Rip [65–68], or any other type of singularity [69], at finite time.

In order to present the above behavior in a more transparent way, we evolve numerically the cosmological equations and in figure 1 we depict the corresponding phase-space behavior.
The unphysical part of the phase space (in which the density parameters exceed one) is marked by the shadowed region. As we can see, in this specific example the universe results in the dark-energy dominated, accelerating solution $P_3$.

In the case of positive curvature, the model possesses five critical points, displayed in table 2. Amongst them, the points $P_5$ and $P_6$ can be stable, and thus they can attract the universe at late times. $P_5$ corresponds to an accelerating, dark-energy dominated universe ($\Omega_{DE} = \Theta_{DE} = 1$ since for this point $D \to H$ in (3.12)), in which the dark-energy equation-of-state parameter can lie either in the quintessence or in the phantom regime, or behave like an effective cosmological constant. Hence, it can be a good candidate for the description of the universe. On the other hand, $P_6$ corresponds to a matter dominated, contracting solution, and as we mentioned before it could be an attractor too, but it does not describe accurately the universe at late times. Moreover, point $P_7$ has the reverse dynamical behavior of $P_5$ due to (3.17), i.e. it corresponds to a contracting ($Q_0 < 0$), dark-energy dominated universe ($\Omega_{DE} = \Theta_{DE} = 1$), and it can be an attractor too. Similarly, point $P_4$ presents the time reversal behavior of $P_6$. Note that there exist orbits connecting $P_4$ and $P_5$ with $P_6$ and $P_7$, which implies that $Q_0$ can cross zero, i.e. $H = 0$, during the evolution, and thus the universe exhibits a bounce or a cosmological turnaround. Finally, the system admits an static solution, $P_8$, which always behaves as a saddle point, hence it cannot represent the late-time universe. We mention that the above features are not obtained in the flat or open curvature, where $H$ cannot change sign.

In order to present the above behavior in a more transparent way, we evolve numerically the cosmological equations and in figure 2 we depict the corresponding phase-space behavior. As we can see, in this specific example the universe results in the dark-energy dominated, accelerating solution $P_5$ or in the matter dominated, contracting solution $P_6$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_space.png}
\caption{The phase-space behavior of time asymmetric cosmology of Model I: \(f(V) = g_1V^m\), with negative curvature, \(p = 0.9\) (i.e. \(m = -0.033\)), and \(w_m = 0\). The shadowed region marks the unphysical part of the phase space. In this specific example the universe is led to the the dark-energy dominated, accelerating solution $P_3$.}
\end{figure}
Figure 2. The phase-space behavior of time asymmetric cosmology of Model I: \( f(V) = g_1 V^m \), with positive curvature, \( p = 0.9 \) (i.e. \( m = -0.033 \)), and \( w_m = 0 \). In this specific example the universe is led to either (a) the dark-energy dominated, accelerating solution \( P_5 \) or (b) the matter dominated, contracting solution \( P_6 \).

4.2 Model II: \( f(V) = g_2 e^{\lambda V} \)

In the case where \( f(V) = g_2 e^{\lambda V} \), i.e. when \( g(a) = g_2 \frac{a}{C} e^{\lambda a^3} \), with \( g_2 \) a constant and \( \lambda \) a parameter, with open or zero curvature, and \( \lambda > 0 \), the scenario at hand exhibits five isolated critical points and one curve of critical points presented in table 3. Amongst them, only point \( Q_6 \) behaves like a stable one (although nonhyperbolic) and thus it can be the late-time state of the universe. It corresponds to a dark-energy dominated universe, in which the dark-energy equation-of-state parameter lies in the phantom regime. Note however that as the universe approaches this point, the deceleration parameter \( q \) decreases monotonically, resulting to a divergence at the critical point. In particular, as the scale factor increases and the dark energy term becomes dominant, we can obtain an approximate solution for the scale factor, namely the inverse of \( t - t_0 = \frac{Ei(-a^3\lambda)}{3g_2} = e^{-a^3\lambda}O\left(\left(\frac{1}{a}\right)^3\right) \), where \( t_0 = -c_1/g_2 \), with \( Ei(z)_z < 0 \), the exponential integral function and \( c_1 \) an integration constant, and we can immediately see that the scale factor diverges at a finite time, which is the realization of a Big Rip [69]. This behavior was expected, since for \( \lambda > 0 \) the extra, time-asymmetric, term that constitutes the effective dark energy sector increases monotonically. Hence, for these parameter choices, the scenario at hand does not correspond to the usual classes of cosmological models, and thus it should not be considered as a successful one. In figure 3 we depict the phase-space behavior of such a scenario, arising from numerical elaboration. As we observe, in this example the universe results in the dark-energy dominated, accelerating solution \( Q_6 \).

In the case of zero or open curvature and \( \lambda < 0 \), the model exhibits five isolated critical points and one curve of critical points, displayed in table 4. Amongst them, point \( Q_{10} \) behaves
Figure 3. The phase-space behavior of time asymmetric cosmology of Model II: \( f(V) = g_2 e^{\lambda V} \), with negative curvature, \( w_m = 0 \) and \( \lambda > 0 \) (the specific value of \( \lambda \) is not relevant, only its sign, since it has been absorbed into the auxiliary variable \( T \) according to (3.27)). In this specific example the universe is led to the dark-energy dominated, accelerating solution \( Q_6 \). The bold dashed line named \( Q_5 \) in general presents saddle behavior, however it is a local source for all the orbits (which have the shape of straight lines connecting it with \( Q_6 \)) located at the invariant set \( T = 1 \).

as stable for the flat models, and thus it can attract the universe at late times. However, it corresponds to a dark-matter dominated universe, and thus it is not favored by observations. This was expected, since for \( \lambda < 0 \) the effective dark-energy terms are redshifted away in a much faster way (due to the exponential) than the matter contribution, leaving the universe matter dominated. Nevertheless, one could improve this behavior by the addition of an explicit cosmological constant, in which case he could get the correct thermal history, namely the succession of matter and dark-energy eras. However, since in this work we are interested in investigating the effects of the pure time-asymmetric cosmology, without the explicit presence of a cosmological constant, we do not examine such a possibility further. Additionally, as we describe in detail in appendix A.2, the nonhyperbolic curve of critical points \( Q_{11} \) (with the exception of its endpoint with \( \Omega_k = 1 \)) behaves as saddle. For \( 0 < \Omega_k < 1 \), it corresponds to a universe with \( \Omega_{DE} = 0 \), however not completely matter-dominated, since the curvature contribution remains non-zero. Another interesting point located on the curve \( Q_{11} \) is the one corresponding to complete curvature domination, namely with \( \Omega_k = 1 \). This point is indeed a stable late-time state of the universe. Similarly to \( Q_{10} \), the above features are not favored by observations to be the late-time state of the universe, however these curves of points could be a good candidate for the description of its intermediate phases, especially under the addition of an explicit cosmological constant. In figure 4, through a numerical elaboration, we present the phase-space behavior of this model. As we see, in this example if the universe starts with \( \Omega_k = 0 \) it results in the dark-matter dominated solution \( Q_{10} \). On the other hand, if \( \Omega_k > 0 \) initially then the universe results in the curvature-dominated solution \((\Omega_k, \Omega_{DE}, T_1) = (1, 0, 1)\) located on the bold dashed line \( Q_{11} \).
Figure 4. The phase-space behavior of time asymmetric cosmology of Model II: $f(V) = g_2 e^{\lambda V}$, with negative curvature, $w_m = 0$ and $\lambda < 0$ (the specific value of $\lambda$ is not relevant, only its sign, since it has been absorbed into the auxiliary variable $T_1$ according to (3.31)). In this specific example the universe is led to the dark-matter dominated solution $Q_{10}$ (if $\Omega_k = 0$ at the initial state), or to the curvature-dominated solution located on one endpoint of line $Q_{11}$, namely $(\Omega_k, \Omega_{DE}, T_1) = (1, 0, 1)$ (if $\Omega_k > 0$ at the initial state). All other points of the curve $Q_{11}$, which is represented by a bold dashed line, behave as saddle.

In the case of positive curvature and $\lambda > 0$ the scenario at hand exhibits four physical critical points, and one curve of critical points, namely $Q_{16}$, which is the straight line joining the points $(0, 0, 1)$ and $(1, 0, 1)$ (with the left endpoint not included), corresponding to expanding cosmologies, which are summarized in table 5. All these critical points have contracting partners via the discrete symmetry (3.42), which are displayed in table 6. Additionally, there exists a line of static solutions namely $S_1$, however since they are saddle they cannot attract the universe at late times. Amongst all these points, the late-time attractors are the expanding solution $Q_{17}$ and the contracting $R_{13}$. In particular, $Q_{17}$ corresponds to a dark-energy dominated universe in which the dark-energy equation-of-state parameter is phantom-like. Note however that as the universe approaches this point, the deceleration parameter $q$ decreases monotonically, resulting to a divergence at the critical point. Using similar arguments as for point $Q_6$ for the open or zero curvature case, it can be shown that it is of a finite-time type, namely a Big Rip [69]. Similarly to the open or zero curvature case, this behavior was expected, since for $\lambda > 0$ the extra, time-asymmetric, term that constitutes the effective dark energy sector increases monotonically. Additionally, there is another stable late-time solution, namely the matter-dominated point $R_{13}$ which ends in a Big-Crunch. In figure 5 we depict the phase-space behavior of this scenario. As we see, in this example the universe results in the dark-energy dominated, accelerating solution $Q_{17}$ or in the Big-Crunch singularity $R_{13}$.

In the case of positive curvature and $\lambda < 0$, the model exhibits four isolated critical points and one curve of critical points, corresponding to expanding cosmologies, displayed in table 7. Each of the above critical points have contracting partners via the discrete
Figure 5. The phase-space behavior of time asymmetric cosmology of Model II: \( f(V) = g_2 e^{\lambda V} \), with positive curvature, \( w_m = 0 \) and \( \lambda > 0 \) (the specific value of \( \lambda \) is not relevant, only its sign, since it has been absorbed into the auxiliary variable \( T \) according to (3.27)). In this specific example the universe is led to either the dark-energy dominated, accelerating solution \( Q_{17} \), or to the contracting solution \( R_{13} \). The dot-dashed (red) line represents the curve of static solutions \( S_1 \). Notice the presence of orbits crossing the line \( Q = 0 \), i.e. \( H = 0 \), which correspond to transitions from expanding to contracting cosmologies and vice versa, that is to cosmological turnarounds and bounces.

symmetry (3.42), which are displayed in table 8. Amongst them, point \( R_{18} \) behaves as stable, and thus it can be the late-time state of the universe. However, it corresponds to a contracting dark-matter dominated universe, and therefore it is not favored by observations. Similarly to the open or flat case, this was expected since for \( \lambda < 0 \) the effective dark-energy terms are redshifted away in a much faster way than the matter contribution. Additionally, there exists a line of static solutions which are saddle, while, as we describe in detail in appendix A.4, the nonhyperbolic curves of critical points \( Q_{20} \) and \( Q_{21} \) behave typically as saddle. In figure 6 we present the phase-space behavior for the model at hand, where we observe that the late-time attractor is the contracting solution \( R_{18} \). Additionally, the figure shows orbits exhibiting the crossing of the \( Q_0 = 0 \) line, which correspond the transition from contracting to expanding cosmologies and vice versa.

5 Conclusions

In this work we studied the cosmological behavior in a universe governed by time asymmetric extensions of general relativity. This novel modified gravity is based on the addition on the Hamiltonian framework of new, time-asymmetric, terms, in a way that the algebra of constraints and local physics remain unchanged [38]. However, at cosmological scales these new terms can have significant effects that can alter the universe evolution, both at early and late times. In particular, assuming that the new terms in the Hamiltonian are proportional to an arbitrary function of the spatial volume, we finally obtain modifications of the Friedmann
Figure 6. The phase-space behavior of time asymmetric cosmology of Model II: \( f(V) = g_2 e^{\lambda V} \), with positive curvature, \( w_m = 0 \) and \( \lambda < 0 \) (the specific value of \( \lambda \) is not relevant, only its sign, since it has been absorbed into the auxiliary variable \( T_1 \) according to (3.31)). In this specific example the universe is led to the contracting solution \( R_{18} \). Additionally, the figure shows orbits exhibiting the crossing of the \( Q_0 = 0 \) line, which correspond the transition from contracting to expanding cosmologies and vice versa.

equations depending on an arbitrary function of the scale factor. Definitely, the capabilities of such cosmological constructions are huge.

We considered two basic ansatzes for the aforementioned modification, namely a power law and an exponential one. We mention that we did not consider an explicit cosmological constant, since we desired to investigate the pure effects of the new terms. In order to bypass the complexity of the equations, we applied the dynamical systems method, which allows to reveal the global behavior of time asymmetric cosmology, independently of the details of the evolution and the specific initial conditions. In particular, we extracted the critical points of the scenario and we examined which of them are stable and thus they can be the late-time state of the universe, calculating also the corresponding observables, such as the various density parameters and the deceleration parameter.

For the power-law ansatz we found that the universe can result in a dark-energy dominated, accelerating universe, where the dark-energy equation-of-state parameter \( w_{DE} \) can lie either in the quintessence or in the phantom regime, or even behave as an effective cosmological constant giving rise to a de Sitter universe. Moreover, by suitably choosing the model parameter, one can obtain a \( w_{DE} \) in agreement with observations.

For the exponential ansatz we showed that for positive exponential coefficient at late times the universe is attracted by a dark-energy dominated universe, in which \( w_{DE} \) lies in the phantom regime, resulting finally to a finite-time Big-Rip singularity (due to the exponential increase of the novel terms). On the other hand, for negative exponential coefficient the universe results to a dark-matter dominated universe (due to the exponential decrease of the novel terms comparing to the matter sector), which is not favored by observations. Nevertheless, one could improve this behavior by the addition of an explicit cosmological...
constant, in which case he could get the correct thermal history, namely the succession of matter and dark-energy eras. Finally, note that in the case of closed curvature, the universe may experience a cosmological bounce or turnaround, or even cyclic behavior.

Concerning phenomenology, we should mention that in the scenario at hand the left-handed neutrinos propagate differently than the photons [38], since the latter propagate according to the usual connection of the spacetime metric, while the former propagate according to the Ashtekar connection and geometry. Hence, if one desires to be in agreement with observations, for instance with the data from SN1987A supernova which show that massless neutrinos propagate similarly to photons with an error less than $10^{-9}$ [70–72], then he should impose the new time-asymmetric modifications to be small, as expected. Interestingly enough, even if one considers the extreme realization of the above requirement, namely to assume that the new terms tend asymptotically to zero (instead of being increasing) as the universe expands, one can still have significant effects at large scales, that can radically alter the universe behavior (for instance in the power-law modification with $f(V) = g_1 V^m$ and $p = 3m + 1$, for the parameter window $0 > m > -1/3$ one has an asymptotically vanishing modification term which is nevertheless able to drive late-time acceleration (since $1 > p > 0$) since its tends to zero as $a^{(2p−2)}$ i.e slower than the matter and curvature contributions in the Friedmann equation). Hence, one can easily pass all the cosmological tests, and definitely all the Solar System ones. An interesting study would be to examine the bounce realization, since in such a case one would expect the time asymmetry to lead to distinguishable signatures on observations, especially having in mind the different behavior of the spacetime and Ashtekar related quantities. Additionally, an important and necessary investigation would be to examine the cosmological perturbations and their relation to various observables, either at early, inflationary times, or at late epochs. Since both these studies lie beyond the scope of the present work they are left for future projects.

In summary, the cosmological application of time asymmetric extensions of general relativity has many capabilities and thus it can be a good candidate for the description of the universe, that is worthy to be studied further.

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A Stability of the nonhyperbolic critical points of Model II: $f(V) = g_2 e^{λV}$

In this appendix we investigate the stability of the nonhyperbolic critical points that appear in the analysis of Model II in subsection 3.2, using the center manifold method [62], since in this case the simple linear analysis is not adequate.

A.1 Zero or negative curvature and $λ > 0$

In the case of zero or negative curvature and $λ > 0$, we extract two isolated nonhyperbolic critical points, and a curve of nonhyperbolic critical points, displayed in table 3. Since point
$Q_4$ and the curve $Q_5$ have at least one unstable eigen-direction they will definitely be non-stable (i.e. saddle or unstable), and hence we do not need to perform the center manifold analysis, since in this work we are interested in the stable late-time solutions. Thus, we restrict our analysis in the case of $Q_6$.

We introduce the new variables

$$\epsilon = 1 - T, \quad x = \Omega_k, \quad y = 1 - \Omega_{DE},$$

in order to translate $Q_6$ to the origin, and thus we obtain the system

$$\frac{d\epsilon}{d\bar{\eta}} = 3(\epsilon - 1)\epsilon^2, \quad \frac{dx}{d\bar{\eta}} = -x \{\epsilon(3w_m x + x - 4) - 3y [(w_m - 1)\epsilon + 2] + 6\}, \quad \frac{dy}{d\bar{\eta}} = (1 - y) \{(3w_m + 1)x\epsilon - 3y [(w_m - 1)\epsilon + 2]\},$$

where the local center manifold of the origin $(\epsilon, x, y) = (0, 0, 0)$ is tangent to the $\epsilon$-axis. Hence, it can be written locally as the graph

$$\{(\epsilon, x, y) : x = h_1(\epsilon), y = h_2(\epsilon), h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0, |\epsilon| < \delta\},$$

where $\delta$ is a suitably small number. The functions $h_1$ and $h_2$ must satisfy the quasilinear system of differential equations

$$3(\epsilon - 1)\epsilon^2 h_1'(\epsilon) + h_1(\epsilon) \{\epsilon [3w_m h_1(\epsilon) + h_1(\epsilon) - 4] - 3h_2(\epsilon) [(w_m - 1)\epsilon + 2] + 6\} = 0, \quad [1 - h_2(\epsilon)]h_1(\epsilon)(3w_m + 1)\epsilon + 3h_2(\epsilon) [(w_m - 1)\epsilon + 2] - 3(\epsilon - 1)\epsilon^2 h_2'(\epsilon) = 0.$$

This system admits the following solutions:

1. the point:

$$h_1(\epsilon) = 0, \quad h_2(\epsilon) = 0,$$

2. the 1-parameter solution:

$$h_1(\epsilon) = 0, \quad h_2(\epsilon) = \begin{cases} \frac{e^{\epsilon w_m+1}}{e^{\epsilon + \frac{2}{3}(1-\epsilon)w_m + \frac{2}{3} + c_1 e^{2/3(1-\epsilon)w_m + 2}}}, & \epsilon \neq 0 \\ 0, & \epsilon = 0 \end{cases},$$

3. the 2-parameter solution:

$$h_1(\epsilon) = \begin{cases} \frac{e^{\frac{2}{3}e^{2/3(1-\epsilon)w_m + 2}}}{e^{\frac{2}{3}e^{2/3(1-\epsilon)w_m + 2} + c_1 e^{2/3(1-\epsilon)w_m + 2}}}, & \epsilon \neq 0 \\ 0, & \epsilon = 0 \end{cases},$$

$$h_2(\epsilon) = \begin{cases} \frac{e^{\frac{2}{3}e^{2/3(1-\epsilon)w_m + 2}}}{e^{\frac{2}{3}e^{2/3(1-\epsilon)w_m + 2} + c_1 e^{2/3(1-\epsilon)w_m + 2}}}, & \epsilon \neq 0 \\ 0, & \epsilon = 0 \end{cases}.$$
These three classes of solutions satisfy the smoothness conditions required in order to obtain the center manifold of the origin (note that the expression for the center manifold is not unique). Thus, we conclude that the evolution on the center manifold is given by the equation

\[ \frac{d\epsilon}{d\bar{\eta}} = -3(1 - \epsilon)\epsilon^2, \]  
(A.11)

which admits the solution

\[ \bar{\eta} = c_1 + \frac{1}{3} \left[ \frac{1}{\epsilon} + 2 \tanh^{-1}(1 - 2\epsilon) \right] = c_1 + \frac{1}{3\epsilon} - \frac{\log(\epsilon)}{3} - \frac{\epsilon}{3} + O(\epsilon^2), \]  
(A.12)

and therefore by inverting the above expression we find \( \epsilon(\bar{\eta}) \). It is easy to see that \( \epsilon \to 0 \) as \( \bar{\eta} \to \infty \) and that \( \epsilon \to 1 \) as \( \bar{\eta} \to -\infty \). Hence, we deduce that the center manifold of \( Q_6 \) is stable [62].

A.2 Zero or negative curvature and \( \lambda < 0 \)

In the case of zero or negative curvature and \( \lambda < 0 \), we extract two isolated nonhyperbolic critical points, and a curve of nonhyperbolic critical points, which are presented in table 4. Since point \( Q_{12} \) has at least two unstable eigen-directions it will definitely be non-stable, and hence we do not investigate it further.

In order to examine the stability of \( Q_{10} \) using the center manifold theorem we introduce the variables

\[ \epsilon = 1 - T_1, \quad u = \Omega_{\text{DE}}, \quad v = \Omega_k, \]  
(A.13)

with evolution equations given by

\[ \frac{d\epsilon}{d\bar{\eta}} = 3(\epsilon - 1)\epsilon^2, \]  
(A.14a)

\[ \frac{du}{d\bar{\eta}} = -u \left\{ \epsilon [3w_m(u + v - 1) + 9u + v] - 6u - 9\epsilon + 6 \right\}, \]  
(A.14b)

\[ \frac{dv}{d\bar{\eta}} = -v \left\{ 3u [(w_m + 3)\epsilon - 2] + (v - 1)(3w_m + 1)\epsilon \right\}. \]  
(A.14c)

The center subspace of the origin of (A.14) is spanned by the vectors \((1, 0, 0)\) and \((0, 1, 0)\), which implies that the local center manifold of the origin can be written locally as the graph \( \{ (\epsilon, u, v) : v = h(\epsilon, u), h(0, 0) = 0, Dh(0, 0) = 0, ||(\epsilon, u)|| < \delta \} \), where \( Dh \) is the matrix of derivatives, \( \delta \) is a suitably small constant, and \( h(\epsilon, u) \) satisfies the quasilinear partial differential equation

\[ u \frac{\partial h}{\partial u} \left\{ (3w_m + \epsilon)h + 3(u - 1) [(w_m + 3)\epsilon - 2] \right\} - 3(\epsilon - 1)\epsilon^2 u \frac{\partial h}{\partial \epsilon} \\
- h \left\{ (3w_m + 1)\epsilon(h - 1) + 3u [(w_m + 3)\epsilon - 2] \right\} = 0. \]  
(A.15)

Assuming that \( h(\epsilon, u) = uf(\epsilon) \) and \( \lim_{\epsilon \to 0} f(\epsilon) = \lim_{\epsilon \to 0} f'(\epsilon) = 0 \), and substituting in (A.15), we obtain

\[ 3(\epsilon - 1)\epsilon^2 f'(\epsilon) + (8\epsilon - 6)f(\epsilon) = 0, \]  
(A.16)

which has the general solution

\[ f(\epsilon) = \frac{c_1 \epsilon^{2/3} \epsilon^{2/3}}{(1 - \epsilon)^{2/3}}, \]  
(A.17)
and the trivial solution \( f(u) = 0 \). However, the general solution leads to \( \lim_{\epsilon \to 0} f(\epsilon) = \text{sgn}(c_1)\infty, \lim_{\epsilon \to 0} f(\epsilon) = -\text{sgn}(c_1)\infty \), and hence it does not satisfy the imposed limits. Thus, the only accepted solution is the trivial one, which implies \( h(\epsilon, u) \equiv 0 \). Hence, for this case the dynamics on the center manifold is governed by

\[
\begin{align*}
\frac{de}{d\tilde{\eta}} &= 3(\epsilon - 1)\epsilon^2, \\
\frac{du}{d\tilde{\eta}} &= -3(u - 1)u \left[(w_m + 3)\epsilon - 2\right].
\end{align*}
\]

Eliminating time and integrating out we finally acquire

\[
\frac{1}{\epsilon^{c_1 + \frac{2}{3}w_m + 1} (1 - \epsilon)^{w_m - 1} + 1},
\]

which satisfies \( u \to 0 \) as \( \epsilon \to 0 \). This feature implies that \( Q_{10} \) attracts the orbits contained in its center manifold (that is the 2D set \( T_{DE} \)), and thus this nonhyperbolic point behaves as stable.

In order to examine the stability of the curve of critical points \( Q_{11} \) (with \( \Omega_{kc} \in (0, 1] \)), using the center manifold theorem, we introduce the variables

\[
\begin{align*}
\epsilon &= 1 - T_1, \\
u &= \Omega_{kc}(1 - \Omega_{DE}) - \Omega_k, \\
v &= \Omega_{DE},
\end{align*}
\]

which satisfy the evolution equations

\[
\begin{align*}
\frac{de}{d\tilde{\eta}} &= -3(1 - \epsilon)\epsilon^2, \\
\frac{du}{d\tilde{\eta}} &= \epsilon \left\{ u^2(3w_m + 1) + u \left[-3v(w_m + 3) + 3w_m + 1\right] \right. \\
&\quad + \Omega_{kc} \epsilon \left[ u \left[(v - 2)(3w_m + 1)\right] + (1 - \Omega_{kc})(v - 1)(3w_m + 1)\right] + 6uv, \\
\frac{dv}{d\tilde{\eta}} &= \epsilon \left[ uv(3w_m + 1) + 3v(1 - v)(w_m + 3) \right] \\
&\quad + \Omega_{kc} \epsilon \left[ v(v - 1)(3w_m + 1) + 6v(v - 1)\right].
\end{align*}
\]

Since the center subspace of the origin of (A.21) is spanned by the vectors \((1, 0, 0)\) and \((0, 1, 0)\), we deduce that the local center manifold of the origin can be written locally as the graph \( \{(\epsilon, u, v) : v = h(\epsilon, u), h(0, 0) = 0, Dh(0, 0) = 0, ||(\epsilon, u)|| < \delta\} \), with \( \delta \) a suitably small constant, and where the function \( h(\epsilon, u) \) that defines the center manifold must satisfy the quasilinear partial differential equation

\[
\frac{\partial h}{\partial u} \left\{ h \left[(3w_m + 1)(\Omega_{kc} - 1)\Omega_{kc} + u \left(\epsilon [3w_m(\Omega_{kc} - 1) + \Omega_{kc} - 9] + 6\right)\right] \\
- \epsilon(3w_m + 1)(u - \Omega_{kc})(u - \Omega_{kc} + 1) \right\} \\
- 3(\epsilon - 1)\epsilon^2 \frac{\partial h}{\partial \epsilon} + \left\{ \epsilon [3w_m(\Omega_{kc} - 1) + \Omega_{kc} - 9] + 6\right\} h^2 \\
+ h \left\{ (u - \Omega_{kc})(3w_m + 1) + 3(w_m + 3)\right\} - 6\} = 0.
\]

For \( \Omega_{kc} \neq 1, w_m \neq -1/3 \) the above equation should be integrated numerically.
We will proceed using Taylor expansion. In particular, the solution $h(\epsilon, u)$ must satisfy the conditions $h(0, 0) = 0, Dh(0, 0) = 0$, that is it must be at least of second order in the variables $\epsilon$ and $u$. Hence, we assume that $h(\epsilon, u) = a_{11}\epsilon^2 + a_{12}\epsilon u + a_{22}u^2 + \mathcal{O}(3)$, where $\mathcal{O}(3)$ denotes terms of third order on the vector norm, i.e. terms like $\epsilon^2 u, \epsilon^3, u^3$. These terms and higher-order terms neglected in the approximation scheme. Substituting back this expression for $h$, neglecting third-order terms, comparing terms of the same power, equating to zero the coefficients, and assuming that $\Omega_{kc} \neq 1, w_m \neq -1/3$, we obtain that a good approximation of the center manifold is given by

$$h(\epsilon, u) = \frac{1}{6}a_{12}(1 + 3w_m)(1 - \Omega_{kc})\Omega_{kc}\epsilon^2 + a_{12}\epsilon u + \frac{3a_{12}}{(1 + 3w_m)(1 - \Omega_{kc})}\Omega_{kc}^{-1}u^2.$$  \hspace{0.5cm} (A.23)

Therefore, we deduce that the dynamics on the center manifold is determined up to third order by

$$\frac{de}{d\eta} = -3\epsilon^2$$ \hspace{0.5cm} (A.24a)

$$\frac{du}{d\eta} = (3w_m + 1)\epsilon [(\Omega_{kc} - 1)\Omega_{kc} - u(2\Omega_{kc} - 1)].$$ \hspace{0.5cm} (A.24b)

The system (A.24) admits the general solution

$$\epsilon(\eta) = \frac{1}{3\eta - c_1},$$  \hspace{0.5cm} (A.25)

$$u(\eta) = c_2(3\eta - c_1) - \frac{1}{3}(3w_m + 1)(2\Omega_{kc} - 1) + \frac{(\Omega_{kc} - 1)\Omega_{kc}}{2\Omega_{kc} - 1}.$$  \hspace{0.5cm} (A.26)

Observe that as $\eta \to +\infty$, $\epsilon \to 0$, but $u$ departs from zero and becomes unbounded in the case $\Omega_{kc} \leq \frac{1}{2}$, or tends to $\frac{(\Omega_{kc} - 1)\Omega_{kc}}{2\Omega_{kc} - 1}$ for $\Omega_{kc} > \frac{1}{2}$ as $\tau \to +\infty$, which is nonzero since $\Omega_{kc} \notin \{0, 1\}$. Thus, the origin is unstable along both $u$-axis and stable along the $\epsilon$-axis. Summarizing, the line of fixed points $Q_{11}$ behaves as saddle, provided that $\Omega_{kc} \neq 1, w_m \neq -1/3$.

Let us mention that the above analysis is essentially an approximation. Nevertheless, there is a special point of the curve $Q_{11}$, namely $(\Omega_k, \Omega_{DE}, T_1) = (1, 0, 1)$, for which the above procedure is not valid, that allows for an analytical application of the center manifold analysis. It corresponds to $\Omega_{kc} = 1$ in (A.21). Setting $\Omega_{kc} = 1$ in (A.22) we obtain the simpler quasilinear partial differential equation

$$-u[(6 - 8\epsilon)h + (u - 1)(3w_m + 1)\epsilon] \frac{\partial h}{\partial u} - 3(\epsilon - 1)\epsilon^2 \frac{\partial h}{\partial \epsilon} + h[(6 - 8\epsilon)h + \epsilon(3uw_m + u + 8) - 6] = 0.$$  \hspace{0.5cm} (A.27)

Given the solution $v = h(\epsilon, u)$, the dynamics on the center manifold is determined by

$$\frac{de}{d\tau} = -3(1 - \epsilon)\epsilon^2$$ \hspace{0.5cm} (A.28a)

$$\frac{du}{d\tau} = u[(u - 1)(3w_m + 1)\epsilon + h(\epsilon, u)(6 - 8\epsilon)].$$ \hspace{0.5cm} (A.28b)

Assuming that $h(\epsilon, u) = uf(\epsilon)$ and $\lim_{\epsilon\to 0} f(\epsilon) = \lim_{\epsilon\to 0} f'(\epsilon) = 0$, and substituting into (A.27), we obtain

$$u \{f(\epsilon) [(w_m + 3)\epsilon - 2] - (\epsilon - 1)\epsilon^2 f'(\epsilon) \} = 0,$$  \hspace{0.5cm} (A.29)
which has the general solution
\[
    f(\epsilon) = \begin{cases} 
        c_1 e^{-2/\epsilon}(1 - \epsilon)^{w_m+1} \epsilon^{-w_m-1}, & \epsilon \neq 0 \\
        0, & \epsilon = 0 
    \end{cases},
\]
which indeed satisfies the imposed limits. Hence, the dynamics on the center manifold is
governed by the evolution equations
\[
    \frac{dc}{d\tilde{\tau}} = -3(1 - \epsilon)\epsilon^2 
    \quad \text{(A.31a)}
\]
\[
    \frac{du}{d\tilde{\tau}} = u \left[ c_1 u e^{-2/\epsilon}(6 - 8\epsilon)(1 - \epsilon)^{w_m+1} \epsilon^{-w_m-1} + (u - 1)(3w_m + 1)\epsilon \right].
\]
(A.31b)

Eliminating the time variable we find that the system (A.31) can be expressed as
\[
    3(1 - \epsilon)\epsilon \frac{du(\epsilon)}{de} = (3w_m + 1)u(\epsilon) + \mu(\epsilon)u(\epsilon)^2
\]
with \(\mu(\epsilon) = 2c_1 e^{-2/\epsilon}(4\epsilon - 3)(1 - \epsilon)^{w_m+1} \epsilon^{-w_m-2} - 3w_m - 1\), which admits the quadrature
\[
    u(\epsilon) = \frac{(1 - \epsilon)^{-w_m-\frac{1}{3}} c_2^{-\frac{1}{3}} - \int \frac{1}{3} \mu(\epsilon)(1 - \epsilon)^{-w_m-\frac{1}{3}} \epsilon^{-w_m-\frac{2}{3}} d\epsilon}{c_2 - \int \frac{1}{3} \mu(\epsilon)(1 - \epsilon)^{-w_m-\frac{1}{3}} \epsilon^{-w_m-\frac{2}{3}} d\epsilon}.
\]
(A.33)

Since \(\mu(\epsilon) \to -3w_m - 1\) as \(\epsilon \to 0\), we can integrate the above quadrature in the approximation \(\epsilon \to 0\), obtaining
\[
    u(\epsilon) \approx \frac{1}{c_2(1 - \epsilon)^{w_m+\frac{1}{3}} \epsilon^{-w_m-\frac{1}{3}} + 1},
\]
which tends to zero as \(\epsilon \to 0\), for \(w_m > -\frac{1}{3}\). Hence, we deduce that the center manifold associated to the point \((1, 0, 1)\) is stable. Indeed, this behavior is the typical one for \(w_m > -\frac{1}{3}\), as can be verified by figure 4.

### A.3 Positive curvature and \(\lambda > 0\)

In the case of positive curvature and \(\lambda > 0\), we extract two isolated nonhyperbolic critical points, and a curve of nonhyperbolic critical points corresponding to expansion, which are presented in table 5. Each of the above points/curve in table 5 has a partner through the symmetry (3.42), which represents a contracting cosmology, and are displayed in table 6. Amongst them in this appendix we analyze only the nonhyperbolic fixed points that might be late-time attractors (for instance points like \(Q_{15}\) and the curve of critical points \(Q_{16}\) that have at least one unstable eigen-direction will definitely be either unstable or saddle and thus we do not investigate them further). These are the expanding solution \(Q_{17}\) and the contracting solutions \(R_{15}\) and \(R_{16}\). We remind that the points \(Q_i\) and their contracting partners points \(R_i\) through the symmetry (3.42), exhibit opposite dynamical behaviors, and thus from the following analysis we also obtain information for the contracting solution \(R_{17}\) and the expanding ones \(Q_{15}\) and \(Q_{16}\).

In order to examine the stability of the contracting solution \(R_{15}\) we introduce the variables
\[
    \epsilon = 1 - T, \quad x = 1 + Q_0, \quad y = \Theta_{DE},
\]
(A.35)
The center manifold of the origin of \((A.36a)\) is spanned by the vectors \((1, 0, 0)\) and \((0, 1, 0)\), which implies that the local center manifold of the origin can be written locally as the graph \(\{(\epsilon, x, y) : y = h(\epsilon, x), h(0, 0) = 0, Dh(0, 0) = 0, ||(\epsilon, u)|| < \delta\}\), with \(\delta\) a suitably small constant and \(Dh\) the matrix of derivatives. The function \(h(\epsilon, x)\) satisfies the quasilinear partial differential equation

\[
\frac{1}{2} (x - 2) x \frac{\partial h}{\partial x} \left[ 3(w_m - 1) \epsilon + 2 h - (3w_m + 1) \epsilon \right] - 3(x - 1)(\epsilon - 1) x^2 \frac{\partial h}{\partial \epsilon} - 3(x - 1)(w_m - 1) \epsilon + 2(h - 1) h = 0.
\]

The equation \((A.37)\) admits the solutions:

1. The trivial solution \(h(\epsilon, x) = 0\),
2. the one-parameter solution \(h(\epsilon, x) = \left\{ \begin{array}{ll} e^{x/(\epsilon - 1) w_m} & \epsilon \neq 0 \\ 1 & \epsilon = 0 \end{array} \right. \).

Only the trivial solution satisfies the conditions \(h(0, 0) = 0, Dh(0, 0) = 0\). Henceforth, the dynamics on the center manifold is governed by

\[
\begin{align*}
\frac{d\epsilon}{d\tilde{\tau}} &= -3(1 - x)(\epsilon - 1) \epsilon^2, \\
\frac{dx}{d\tilde{\tau}} &= \frac{1}{2} (x - 2) x (3w_m(y - 1) - 3y - 1) + 6y, \\
\frac{dy}{d\tilde{\tau}} &= -3(1 - x)(1 - y)y[(w_m - 1) \epsilon + 2].
\end{align*}
\]

Eliminating the time variable, \(\tilde{\tau}\), and using the chain rule for derivatives we find that the orbits on the invariant manifold satisfy

\[
x'(\epsilon) = \frac{(3w_m + 1)[x(\epsilon) - 2|x(\epsilon)|]}{6(\epsilon - 1)\epsilon[x(\epsilon) - 1]},
\]

which admits the general solutions

\[
x(\epsilon) = 1 \pm e^{-w_m - \frac{1}{3}} \sqrt{e^{w_m + \frac{1}{3}} - e^{2\epsilon}} (1 - \epsilon)^{w_m + \frac{1}{3}}.
\]

None of these solutions satisfy the condition \(x(0) = 0\), indeed \(x\) becomes infinity as \(\epsilon \to 0\). Thus, any solution starting with \(\epsilon \neq 0\) and \(x \neq 0\) departs from the origin along the \(x\)-direction, which implies that \(R_{15}\) behaves as a saddle.

In order to examine the stability of the contracting solution \(R_{16}\) we introduce the variables

\[
u_1 = Q_0 - \frac{(Q_{0c} - 1) \Theta_{DE}}{2Q_{0c}} + Q_{0c}, \quad \nu_2 = \frac{1}{2} (Q_{0c} - 1)(1 - T)(3w_m + 1), \quad v = \Theta_{DE}, \quad (A.41)
\]
where \( Q_{0c} \) is a constant \( Q_{0c} \in (0, 1) \), which satisfy the equations

\[
\frac{du}{d\tau} = \frac{3(v-1)v}{2Q_{0c}^2(3w_m+1)} \left[ Q_{0c}^2(v-2) + 2Q_{0c}u_1 - v \right] \left[ w_m \left( 3Q_{0c}^2 + u_2 - 3 \right) + Q_{0c}^2 - u_2 - 1 \right] \\
- \frac{1}{2} \left[ \left( \frac{Q_{0c}v}{2} + \frac{v}{2Q_{0c}} + Q_{0c} - u_1 \right)^2 - 1 \right] \left[ \frac{2u_2(3v(w_m-1)-3w_m-1)}{(Q_{0c}^2-1)(3w_m+1)} + 6v \right], \quad (A.42a)
\]

\[
\frac{du}{d\tau} = \frac{3u_2^2}{Q_{0c}(Q_{0c}^2-1)(3w_m+1)^2} \left[ Q_{0c}^2(v-2) + 2Q_{0c}u_1 - v \right] \left[ Q_{0c}^2(-3w_m+1) + 2u_2 + 3w_m + 1 \right], \quad (A.42b)
\]

\[
\frac{dv}{d\tau} = 3(v-1)v \left[ -\frac{Q_{0c}v}{2} + \frac{v}{2Q_{0c}} + Q_{0c} - u_1 \right] \left[ \frac{2u_2(w_m-1)}{(Q_{0c}^2-1)(3w_m+1)} + 2 \right]. \quad (A.42c)
\]

Note that by definition \( u_2 \leq 0 \).

The center subspace of the origin of (A.42) is spanned by the vectors \((1,0,0)\) and \((0,1,0)\), which implies that the local center manifold of the origin can be written locally as the graph \( \{(u_1, u_2, v) : v = h(u_1, u_2), h(0,0) = 0, \|Dh(0,0)\| < \delta \} \), with \( \delta \) a suitably small constant and \( Dh \) the matrix of derivatives. The function \( h(u_1, u_2) \) satisfies the quasilinear partial differential equation

\[
3u_2^2 \left[ 3 \left( Q_{0c}^2 - 1 \right) w_m + Q_{0c}^2 - 2w_2 - 1 \right] \left[ \left( Q_{0c}^2 - 1 \right) h + 2Q_{0c}(u_1 - Q_{0c}) \right] \frac{\partial h}{\partial u_2} + \frac{1}{2} \frac{\partial h}{\partial u_1} \left\{ \left[ \frac{(Q_{0c}^2 - 1) h}{2Q_{0c}} - Q_{0c} + u_1 \right]^2 - 1 \right\} \left[ \frac{2u_2(3w_m-1)h-3w_m-1}{(Q_{0c}^2-1)(3w_m+1)} + 6h \right] \\
- 3 \left[ w_m \left( 3Q_{0c}^2 + u_2 - 3 \right) + Q_{0c}^2 - u_2 - 1 \right] (h-1)h \left[ (Q_{0c}^2-1)h + 2Q_{0c}(u_1 - Q_{0c}) \right] \left[ \frac{2u_2(w_m-1)}{(Q_{0c}^2-1)(3w_m+1)} + 2 \right] (h-1)h \left[ \frac{(Q_{0c}^2-1)h}{2Q_{0c}} + Q_{0c} - u_1 \right] = 0. \quad (A.43)
\]

Assuming that \( h \) is locally given by \( h = a_{11}u_1^2 + a_{12}u_1u_2 + a_{22}u_2^2 + \mathcal{O}(3) \), where \( \mathcal{O}(3) \) denotes terms of third order in the vector norm, plugging back into (A.43), comparing equal powers in the variables \( u_1 \) and \( u_2 \) and equating to zero the corresponding coefficients, we obtain up to third order that \( a_{11} = -3a_{12}Q_{0c} \) and \( a_{22} = -\frac{a_{12}}{6Q_{0c}} \). That is, the graph of the center manifold is given up to third order by \( v = -3a_{12}Q_{0c}u_1^2 + a_{12}u_1u_2 - \frac{a_{12}}{6Q_{0c}}u_2^2 \). Therefore, by neglecting the third-order terms we find that the dynamics on the center manifold is governed by equations

\[
\frac{du_1}{d\tau} = u_2 + \frac{2Q_{0c}u_1u_2}{1 - Q_{0c}^2}, \quad (A.44a)
\]

\[
\frac{du_2}{d\tau} = -\frac{6Q_{0c}u_2^2}{(1 - Q_{0c}^2)(3w_m+1)}. \quad (A.44b)
\]
Integrating (A.44) it follows

\[
\begin{align*}
\frac{du}{d\tau} &= \frac{2c_2Q_{0c}}{2Q_{0c}} \left[ c_1 \left( Q_{0c}^2 - 1 \right) (3w_m + 1) + 6Q_{0c}\tau \right]^{w_m + \frac{1}{2}} + Q_{0c}^2 - 1, \\
\frac{dv}{d\tau} &= -\frac{(Q_{0c}^2 - 1)(3w_m + 1)}{c_1 \left( Q_{0c}^2 - 1 \right) (3w_m + 1) + 6Q_{0c}\tau}.
\end{align*}
\]  

(A.45a) (A.45b)

Taking the limit as \( \tau \to \infty \) in the above expressions we obtain \((u_1, u_2) \to (c_2, 0), c_2 \neq 0, u_2 \neq 0 \). In the special case \( c_2 = 0 \) we obtain the limits \((u_1, u_2) \to \left( \frac{Q_{0c}^2 - 1}{2Q_{0c}}, 0 \right) \). In both cases the origin is unstable along the \( u_1 \)-axis. Since it is stable along the \( u_2 \)-axis, it follows that \( R_{16} \) is a saddle.

Finally, let us examine the stability of \( Q_{17} \) using the center manifold theorem. We introduce the variables

\[ \epsilon = 1 - T, \quad x = 1 - Q_0, \quad y = 1 - \Theta_{DE}, \]  

(A.46)

and therefore the autonomous system (3.40) is equivalent to the system

\[
\begin{align*}
\frac{de}{d\tau} &= -3(1 - x)(1 - \epsilon)^2, \\
\frac{dx}{d\tau} &= -\frac{1}{2}(x - 2)x [3y ((w_m - 1)\epsilon + 2) + 4\epsilon - 6], \\
\frac{dy}{d\tau} &= -3(1 - x)(1 - y)y [(w_m - 1)\epsilon + 2].
\end{align*}
\]  

(A.47a) (A.47b) (A.47c)

The local center manifold of the origin \((\epsilon, x, y) = (0, 0, 0)\) is tangent to the \( \epsilon \)-axis. Thus, it can be written locally as the graph \( \{ (\epsilon, x, y) : x = h_1(\epsilon), y = h_2(\epsilon), h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0, |\epsilon| < \delta \} \), with \( \delta \) a suitably small number. The functions \( h_1 \) and \( h_2 \) must satisfy the quasilinear system of differential equations

\[
\begin{align*}
[h_1(\epsilon) - 2]h_1(\epsilon)[3h_2(\epsilon)((w_m - 1)\epsilon + 2) + 4\epsilon - 6] - 6(\epsilon - 1)\epsilon^2(h_1(\epsilon) - 1)h_1'(\epsilon) = 0, \\
[h_1(\epsilon) - 1] \left[ (h_2(\epsilon) - 1)h_2(\epsilon)((w_m - 1)\epsilon + 2) - (\epsilon - 1)\epsilon^2h_2'(\epsilon) \right] = 0,
\end{align*}
\]  

(A.48a) (A.48b)

which admits the general solution satisfying the conditions \( h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0 \), namely:

\[
\begin{align*}
h_1(\epsilon) &= \begin{cases} 
1 - \sqrt{\epsilon^{w_{m+1}} + \epsilon^{w_{m+1}} + 1} \left( 1 - \epsilon \right)^{w_m + \frac{1}{2}} \left( 1 - \epsilon \right)^{w_m + 1} \left( 1 - \epsilon \right)^{w_m + w_m + 1}, & \epsilon \neq 0, \\
0, & \epsilon = 0
\end{cases}, \\
h_2(\epsilon) &= \begin{cases} 
\epsilon^{w_{m+1}} + 1, & \epsilon \neq 0, \\
0, & \epsilon = 0
\end{cases}.
\end{align*}
\]  

(A.49a) (A.49b)

where \( c_1 \) and \( c_2 \) are integration constants, as well as the trivial solution \( h_1(\epsilon) = 0, h_2(\epsilon) = 0 \). Note that the expression for the center manifold of the origin is not necessarily unique.

For the expression of the center manifold of the origin given by (A.49), the dynamics on it is given by

\[
\frac{de}{d\tau} = -3f(\epsilon)(1 - \epsilon)^2,
\]  

(A.50)
where
\[ f(\epsilon) = \sqrt{\frac{e^{2c_2}e^{2/3}(1-\epsilon)^{w_m+1/3} + e^{c_1+\frac{2}{3}}(1-\epsilon)^{w_m+1} + \epsilon w_m+1}}{e^{c_1+\frac{2}{3}}(1-\epsilon)^{w_m+1} + \epsilon w_m+1}. \] \tag{A.51}

Since \( f(\epsilon) > 0 \), the flow of (A.50) is equivalent to the flow of
\[ \frac{de}{d\xi} = -3(1-\epsilon)e^2, \] \tag{A.52}
where we have introduced a time rescaling. The general solution of (A.52) reads
\[ \xi = c_3 + \frac{1}{3} \left[ \frac{1}{\epsilon} + 2 \tanh^{-1}(1-2\epsilon) \right] = c_3 + \frac{1}{3\epsilon} - \frac{\log(\epsilon)}{3} - \frac{\epsilon}{3} + O(\epsilon^2). \tag{A.53}\]

Since \( \epsilon \to 0 \) as \( \xi \to \infty \), the center manifold of \( Q_{17} \) is stable, and it corresponds to the late-time attractor. Additionally \( \epsilon \to 1 \) as \( \xi \to -\infty \). Note that the relation with the original time variable is obtained through the quadrature
\[ \tilde{\tau} = \int \frac{\xi'(\epsilon)d\epsilon}{f(\epsilon)} = \frac{1}{3} \int \frac{\sqrt{e^{c_1+\frac{2}{3}}(1-\epsilon)^{w_m+1} + \epsilon w_m+1}}{(\epsilon-1)e^{2\epsilon^{2/3}(1-\epsilon)^{w_m+1/3} + e^{c_1+\frac{2}{3}}(1-\epsilon)^{w_m+1} + \epsilon w_m+1}} d\epsilon. \] \tag{A.54}\]

If the center manifold is given by the trivial solution \( h_1(\epsilon) = 0, h_2(\epsilon) = 0 \) we deduce that the evolution on it is dictated by
\[ \frac{de}{d\tilde{\tau}} = -3(1-\epsilon)e^2, \] \tag{A.55}\]
which admits the solution
\[ \tilde{\tau} = c_1 + \frac{1}{3} \left[ \frac{1}{\epsilon} + 2 \tanh^{-1}(1-2\epsilon) \right] = c_1 + \frac{1}{3\epsilon} - \frac{\log(\epsilon)}{3} - \frac{\epsilon}{3} + O(\epsilon^2). \tag{A.56}\]

Since \( \epsilon \to 0 \) as \( \tilde{\tau} \to \infty \), the center manifold of \( Q_{17} \) is stable, and it corresponds to the late-time attractor.

A.4 Positive curvature and \( \lambda < 0 \)

In the case of positive curvature and \( \lambda < 0 \), we extract two isolated nonhyperbolic critical points, and a curve of nonhyperbolic critical points, representing expanding solutions, which are presented in table 7. Each of the above points/curve in table 7 has a partner through the symmetry (3.50), which represents a contracting cosmology, and are displayed in table 8. Amongst them we analyze only the nonhyperbolic fixed points that might be late-time attractors (for instance points having at least one unstable eigen-direction, like \( Q_{22} \), are excluded from the analysis). These are the expanding solutions \( Q_{20} \) and \( Q_{21} \) and the contracting one \( R_{22} \). We remind that the points \( Q_i \) and their contracting partners points \( R_i \) through the symmetry (3.50), exhibit opposite dynamical behaviors, and hence from the following analysis we also obtain information for the contracting solutions \( R_{20}, R_{21} \) and the expanding one \( Q_{22} \).

In order to calculate the center manifold of \( Q_{20} = (1,0,1) \) for the system (3.48) we introduce the variables
\[ \epsilon = 1 - T_1, \quad u = 1 - Q_0, \quad v = \Theta_{DE}, \] \tag{A.57}\]
which satisfy the evolution equations

\[ \frac{de}{d\tilde{\tau}} = -3(1 - u)(1 - \epsilon)e^2, \quad (A.58a) \]
\[ \frac{du}{d\tilde{\tau}} = -\frac{1}{2}(2 - u)u [e(3v(w_m + 3) - 3w_m - 1) - 6v], \quad (A.58b) \]
\[ \frac{dv}{d\tilde{\tau}} = 3(1 - u)(1 - v)v [(w_m + 3)\epsilon - 2]. \quad (A.58c) \]

The center subspace of the origin of (A.58) is spanned by the vectors \((1, 0, 0)\) and \((0, 1, 0)\), which implies that the local center manifold of the origin can be written locally as the graph \(\{(\epsilon, u, v) : v = h(\epsilon, u), h(0,0) = 0, ||(\epsilon, u)|| < \delta\}\), with \(\delta\) a suitably small constant and \(Dh\) the matrix of derivatives. The function \(h(\epsilon, u)\) satisfies the quasilinear partial differential equation

\[ -\frac{1}{2}(u - 2)u \frac{\partial h}{\partial u} [3 \{(w_m + 3)\epsilon - 2\} h - (3w_m + 1)\epsilon] + 3(u - 1)\epsilon h \frac{2\partial h}{\partial \epsilon} + 3(u - 1)\{(w_m + 3)\epsilon - 2\}(h - 1) h = 0, \quad (A.59) \]

which admits the formal general solution

\[ F \left( \frac{1}{2} \ln \left[ \frac{(u - 2)u(1 - \epsilon)^{2/3}}{\epsilon^{2/3} h(\epsilon, u)} \right], \ln \left[ \frac{e^{-2/3}(\epsilon - 1)(1 - \epsilon)^{w_m} e^{-w_m - 1}(h(\epsilon, u) - 1)}{h(\epsilon, u)} \right] \right) = 0, \quad (A.60) \]

and the trivial solution \(h \equiv 0\). Nevertheless, in order to complete the analysis numerical investigation is required. Using Taylor expansion we obtain that, up to third order in the vector norm, the solution of (A.59) is the trivial solution \(h \equiv 0\). Thus, the dynamics on the center manifold is given up to the same order by

\[ \frac{de}{d\tilde{\tau}} = -3(1 - u)e^2 + O(3), \quad (A.61a) \]
\[ \frac{du}{d\tilde{\tau}} = u(3w_m + 1)e + O(3). \quad (A.61b) \]

Neglecting the error terms, and eliminating the time variable, we obtain the equation

\[ \epsilon'(u) = \frac{3(u - 1)\epsilon(u)}{u(3w_m + 1)}, \quad (A.62) \]

which admits the solution

\[ \epsilon(u) = c_1 e^{\frac{3(u - \ln(u))}{3w_m + 1}}, \quad (A.63) \]

that satisfy \(\lim_{u \to 0} = c_1 \infty\), and hence it is infinity unless \(c_1 = 0\). Therefore, \(Q_{20}\) is a saddle for \(w_m \geq 0\), as shown in figure 6. Finally, by symmetry, \(R_{20}\) behaves as saddle too.

In order to examine the stability of the curve of critical points \(Q_{21}\), using the center manifold theorem, we introduce the variables

\[ u_1 = Q_{0c} - Q_0 + \frac{(1 - Q_{0c}^2)}{2Q_{0c}} \Theta_{DE}, \quad u_2 = \frac{1}{2} (1 - Q_{0c}^2) (1 - T_1)(1 + 3w_m), \quad v = \Theta_{DE}, \quad (A.64) \]

\[ 35 \]
where $Q_{0c} \in (0,1)$ is a constant, which satisfy the evolution equations

\begin{equation}
\frac{du_1}{d\tau} = \frac{3v(v-1) \left[ Q_{0c}^2 (v-2) + 2Q_{0c}u_1 - v \right] \left[ w_m (3Q_{0c}^2 + u_2 - 3) + Q_{0c}^2 + 3u_2 - 1 \right]}{2Q_{0c}^2 (3w_m + 1)} + \left[ \frac{(-Q_{0c}v}{2} + \frac{v}{2Q_{0c}} + Q_{0c} - u_1 \right]^2 - 1 \right] \left[ u_2 \left[ -3v(w_m + 3) + 3w_m + 1 \right] \right] \left( Q_{0c}^2 - 1 \right) \left( 3w_m + 1 \right) - 3v}, \quad (A.65a)
\end{equation}

\begin{equation}
\frac{du_2}{d\tau} = -\frac{3u_2^2 \left[ Q_{0c}^2 (v-2) + 2Q_{0c}u_1 - v \right] \left[ 3 \left( Q_{0c}^2 - 1 \right) w_m + Q_{0c}^2 + 2u_2 - 1 \right]}{Q_{0c}^2 \left( Q_{0c}^2 - 1 \right)^2 (3w_m + 1)^2}, \quad (A.65b)
\end{equation}

\begin{equation}
\frac{dv}{d\tau} = -3v(v-1) \left( -\frac{Q_{0c}v}{2} + \frac{v}{2Q_{0c}} + Q_{0c} - u_1 \right) \left[ -\frac{2u_2 (w_m + 3) + 3w_m + 1}{Q_{0c}^2 - 1} - 2 \right]. \quad (A.65c)
\end{equation}

The center subspace of the origin of \((A.65)\) is spanned by the vectors \((1,0,0)\) and \((0,1,0)\), and hence the local center manifold of the origin can be written locally as the graph \(\{(u_1, u_2, v) : v = h(u_1, u_2), h(0,0) = 0, Dv(0,0) = 0, ||(u_1, u_2)|| < \delta\}\), with \(\delta\) a suitably small constant, and where the function \(h\) satisfies the quasilinear partial differential equation

\begin{equation}
\frac{3u_2^2}{2Q_{0c}^2 (3w_m + 1)^2} \left[ (Q_{0c}^2 - 1) w_m + Q_{0c}^2 + 2u_2 - 1 \right] \left( Q_{0c}^2 - 1 \right) h \left( \frac{Q_{0c}^2 - 1}{2Q_{0c}} + Q_{0c} - u_1 \right) \partial h \frac{3 \left[ w_m (3Q_{0c}^2 + u_2 - 3) + Q_{0c}^2 + 3u_2 - 1 \right] (h-1)h \left( \frac{Q_{0c}^2 - 1}{2Q_{0c}} + Q_{0c} - u_1 \right) \partial h}{2Q_{0c}^2 (3w_m + 1)} + \left( \frac{Q_{0c}^2 - 1}{2Q_{0c}} - Q_{0c} + u_1 \right)^2 - 1 \right] \left[ u_2 \left[ -3(w_m + 3)h + 3w_m + 1 \right] \left( Q_{0c}^2 - 1 \right) (3w_m + 1) - 3h \right] \partial h \frac{2u_2 (w_m + 3) + 3w_m + 1}{(Q_{0c}^2 - 1) (3w_m + 1)} (h-1)h \left( \frac{Q_{0c}^2 - 1}{2Q_{0c}} + Q_{0c} - u_1 \right) = 0. \quad (A.66)
\end{equation}

Assuming that \(h(u_1, u_2) = a_{11}u_1^2 + a_{12}u_1u_2 + a_{22}u_2^2 + O(3)\) and evaluating up to third order, we find a good approximation of the center manifold given by

\begin{equation}
h(u_1, u_2) = -3a_{12}Q_{0c}u_1^2 + a_{12}u_1u_2 - \frac{a_{12}}{6Q_{0c}} u_2^2 + O(3). \quad (A.67)
\end{equation}

Therefore, neglecting the \(O(3)\)-terms, the evolution on the center manifold is governed by the equations

\begin{equation}
\frac{du_1}{d\tau} = u_2 + \frac{2Q_{0c}u_1u_2}{1 - Q_{0c}^2}, \quad (A.68a)
\end{equation}

\begin{equation}
\frac{du_2}{d\tau} = -\frac{6Q_{0c}u_2^2}{(1 - Q_{0c}^2) (3w_m + 1)}, \quad (A.68b)
\end{equation}

where by definition, \(u_2 \geq 0\).

Integrating \((A.68)\) we acquire

\begin{equation}
u_1(\tau) = \frac{2c_2Q_{0c} \left[ c_1 (Q_{0c}^2 - 1) (3w_m + 1) + 6Q_{0c} \right] w_m + 1}{2Q_{0c}}, \quad (A.69a)
\end{equation}

\begin{equation}
u_2(\tau) = -\frac{(Q_{0c}^2 - 1) (3w_m + 1)}{c_1 (Q_{0c}^2 - 1) (3w_m + 1) + 6Q_{0c} \tau}. \quad (A.69b)
\end{equation}
Taking the limit $\tau \to \infty$ in the above expressions we obtain $(u_1, u_2) \to (c_2 \infty, 0), c_2 \neq 0, u_2 \neq 0$. In the special case where $c_2 = 0$ we obtain the limits $(u_1, u_2) \to \left(\frac{Q_0^2}{2Q_0}, 0\right)$. In both cases the origin is unstable along the $u_1$-axis. Since it is stable along the $u_2$-axis, it follows that $Q_{21}$ is a saddle.

Another nonhyperbolic point that can be a late-time attractor is the contracting point $R_{22}$, which has a 2D stable manifold. Introducing the new variables

$$\epsilon = 1 - T_1, x = Q_0 + 1, y = 1 - \Theta_{DE},$$

we obtain the equivalent dynamical system

\begin{align}
\frac{d\epsilon}{d\tau} &= 3(x - 1)(\epsilon - 1)e^2, \quad (A.71a) \\
\frac{dx}{d\tau} &= \frac{1}{2}(x - 2)x\{3y[(w_m + 3)\epsilon - 2] - 8\epsilon + 6\}, \quad (A.71b) \\
\frac{dy}{d\tau} &= 3(x - 1)(y - 1)y[(w_m + 3)\epsilon - 2], \quad (A.71c)
\end{align}

where the local center manifold of the origin $(\epsilon, x, y) = (0, 0, 0)$ is tangent to the $\epsilon$-axis. Hence, it can be written locally as the graph

$$\{(\epsilon, x, y) : x = h_1(\epsilon), y = h_2(\epsilon), h_1(0) = 0, h_2(0) = 0, h_1'(0) = 0, h_2'(0) = 0, |\epsilon| < \delta\}, \quad (A.72)$$

where $\delta$ is a suitably small number. The functions $h_1$ and $h_2$ must satisfy the quasilinear system of differential equations

\begin{align}
(h_1 - 2)h_1 \{3h_2[(w_m + 3)\epsilon - 2] - 8\epsilon + 6\} - 6(\epsilon - 1)e^2(h_1 - 1)h_1' = 0, \quad (A.73a) \\
(h_1 - 1) \{(h_2 - 1)h_2[(w_m + 3)\epsilon - 2] - (\epsilon - 1)e^2h_1\} = 0. \quad (A.73b)
\end{align}

These equations admit the following classes of solutions:

1. The trivial solution $h_1 \equiv 0, h_2 \equiv 0$,

2. the one-parameter class of solutions

$$h_1 = 1 \pm \sqrt{1 - \frac{e^{2/3}e^{2(c_1 + \frac{1}{2})}}{(1 - \epsilon)^{2/3}}}, \quad (A.74a)$$

$$h_2 = 0, \quad (A.74b)$$

3. the one-parameter class of solutions

$$h_1 = 0, \quad (A.75a)$$

$$h_2 = \frac{1}{e^{c_1 - \frac{1}{2}}(1 - \epsilon)^{w_m + 1}e^{-w_m - 1} + 1}, \quad (A.75b)$$

4. the 2-parameter class of solutions

$$h_1 = 1 \pm \sqrt{\frac{e^{2c_2}e^{2/3(1 - \epsilon)^{w_m + \frac{3}{2}}}}{e^{c_1(1 - \epsilon)^{w_m + 1} + e^{2/3}e^{w_m + 1}} + 1}}, \quad (A.76a)$$

$$h_2 = \frac{1}{e^{c_1 - \frac{1}{2}}(1 - \epsilon)^{w_m + 1}e^{-w_m - 1} + 1}. \quad (A.76b)$$
From all the above solutions the only one that satisfies the conditions $h_1(0) = 0, h_2(0) = 0, h'_1(0) = 0, h'_2(0) = 0$ is the trivial one. Thus, the dynamics on the center manifold is given by

$$\frac{d\epsilon}{d\check{\tau}} = 3(1 - \epsilon)\epsilon^2,$$

(A.77)

that corresponds to a gradient-like equation for the potential $U(\epsilon) = 1/4\epsilon^3(-4 + 3\epsilon)$ for which the origin is a local maximum. Thus, the center manifold of the origin is unstable, and hence $R_{22}$ is a saddle. Integrating out the above equation we extract the solution

$$\check{\tau}(\epsilon) = c_1 - \frac{1}{3\epsilon} - \frac{2}{3} \tanh^{-1}(1 - 2\epsilon),$$

(A.78)

for which the origin is approached as $\check{\tau} \to -\infty$ and not as $\check{\tau} \to +\infty$, that indeed confirms the above statements.

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