Arbitrarily Varying Remote Sources

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Abstract
We study a lossy source coding problem for an arbitrarily varying remote source (AVRS) which was proposed in a prior work. An AVRS transmits symbols, each generated in an independent and identically distributed manner, which are sought to be estimated at the decoder. These symbols are remotely generated, and the encoder and decoder observe noise corrupted versions received through a two-output noisy channel. This channel is an arbitrarily varying channel controlled by a jamming adversary. We assume that the adversary knows the coding scheme as well as the source data non-causally, and hence, can employ malicious jamming strategies correlated to them. Our interest lies in studying the rate distortion function for codes with a stochastic encoder, i.e., when the encoder can privately randomize while the decoder is deterministic. We provide upper and lower bounds on this rate distortion function.

I. INTRODUCTION

The arbitrarily varying remote source (AVRS) model, depicted in Fig. 1, was introduced in [1]. Here a remote source outputs a block of data $X$ comprising symbols generated in an independent and identically distributed (i.i.d.) manner with a fixed distribution $P_X$. $X$ is observed over a noisy arbitrarily varying channel (AVC) $W_{Y,Z|X,J}$ partially controlled by an adversary’s jamming input $J$. The two noisy versions $Y$ and $Z$ of data $X$ are received as input at encoder and side information at decoder respectively. The encoder compresses $Y$ into a message $M$ and transmits it losslessly to the decoder. The decoder then outputs an estimate $\hat{X}$ of $X$. The fidelity of the reconstruction is measured in terms of the average per letter distortion. The adversary knows the coding scheme and $X$, and hence, can leverage this information to design pernicious jamming strategies. Our interest lies in the optimum rate of compression for a given a target distortion under any allowed jamming strategy. In this work, we study this problem when private randomization at the encoder is allowed while the decoder is deterministic.

Practical scenarios in some situations preclude the encoder’s direct access to the exact source realization, unlike in standard lossy source coding [2]. For instance, a remote plant controller may be constrained to initiate control actions based upon a noisy view of plant variables. Dobrushin and Tsybakov [3] introduced the remote source coding problem, where the encoder observes the source through a fixed and known memoryless channel. The authors in [4], [5] extended [3] to the scenario when the decoder too receives correlated side information, thereby generalizing the problem studied by Wyner and Ziv [6]. Unlike in these works where statistics are fixed, the AVRS models a robust scenario where an adversary can induce arbitrary statistics on the observations made at the encoder and decoder jointly. Furthermore, the AVRS model sits at the intersection of several other interesting lossy source coding problems. Apart from the aforementioned works, the AVRS model unifies compound and universal formulations of several interesting problems (cf. [7], [8], [9], and some of the references therein). In addition, some interesting adversarial source coding problems (cf. [10], [11], [12]) can also be modeled as special cases of the AVRS by making appropriate assumptions on the AVC $W_{Y,Z|X,J}$ and the adversary.

It is well known that problems involving adversaries present challenges; results often crucially depend upon the nature of coding (possibility of randomization), adversary’s capabilities, error and/or distortion criteria etc. (see [13] for an excellent survey on problems involving AVCs). In our previous work [1], we studied an AVRS under randomized coding where encoder-decoder share randomness unknown to the adversary. In that work, we gave upper and lower bounds on the randomized rate distortion function. Here we give results when randomization is restricted to the encoder only. Such codes, where the encoder can privately randomize while decoding is deterministic, are generally called codes with a stochastic encoder (cf. [11]). Here is a summary of our main contributions:

- We first extend our result in [1] for the randomized rate distortion function under the ‘average’ (average over all source sequences) distortion criteria, to a stricter ‘maximum’ (maximum over all typical source sequences) distortion criteria.
- We use this ‘strengthening’ of the result for randomized coding along with Ahlswede’s elimination technique [14] to extend our result for codes with a stochastic encoder, i.e., when private encoder-side randomization only is allowed.

Fig. 1. The arbitrarily varying remote source (AVRS) communication setup

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Our proof of the upper bound under randomized coding is along the lines of [1], which uses the refined Markov lemma [15], and involves careful modifications to the proof in [1] necessitated by the stricter distortion criteria. To show our result for codes with a stochastic encoder, the aforementioned ‘strengthening’ vis-à-vis the distortion metric is crucially used (see Remark 5) in an intermediate de-randomization step, where we show the existence of a ‘good’ randomized code with a polynomial-sized ensemble.

The rest of the paper is organized as follows. In Section II, we first introduce the notation and problem setup. We state our main results in Section III. The proofs are presented in Sections IV and V. We make concluding remarks in Section VI.

II. Notation and Problem Setup

A. Notation and Preliminaries

Random variables are denoted by upper case letters (e.g. $X$), the values they take by lower case letters (e.g. $x$) and their alphabets by calligraphic letters (e.g. $\mathcal{X}$). We use boldface notation to denote random vectors (e.g. $X$) and their values (e.g. $x$). All vectors are of length $n$ (e.g. $X = (X_1, X_2, \ldots, X_n)$), where $n$ is the block length of operation. Also, we denote $X^i = (X_1, X_2, \ldots, X_i)$ and $x^i = (x_1, x_2, \ldots, x_i)$ as well as $X^k_{i} = (X_i, X_{i+1}, \ldots, X_k)$ and $x^k_{i} = (x_i, x_{i+1}, \ldots, x_k)$. We use the $l_\infty$ (denoted by $\|\cdot\|_\infty$) norm for vectors. Let $\mathcal{P}(\mathcal{X})$ denote the set of all probability distributions on a set $\mathcal{X}$. Similarly, let $\mathcal{P}(\mathcal{X}|\mathcal{Y})$ be the set of all conditional distributions of a random variable with alphabet $\mathcal{X}$ conditioned on another random variable with alphabet $\mathcal{Y}$. For two random variables $X$ and $Y$, we denote the marginal distribution of $X$ obtained from the joint distribution $P_{X,Y}$ by $[P_{X,Y}|X]$. Distributions corresponding to strategies adopted by the adversary are denoted by $Q$ instead of $P$ for clarity. The set of all conditional distributions $\mathcal{P}(\mathcal{Y}|\mathcal{X})$ is specifically denoted by $\mathcal{Q}$. In cases where the subscripts are clear from the context, we sometimes omit them to keep the notation simple. Deterministic functions will be denoted in lowercase (e.g. $f$). We denote a type of $X$ by $T_X$. Given sequences $x$, $y$, we denote by $T_{x,y}$ the type of $x$, by $T_{x,y}$ the joint type of $(x,y)$ and by $T_{x|y}$ the conditional type of $x$ given $y$. For $\epsilon \in (0,1)$, the set of $\epsilon$-typical set of $x$ sequences for a distribution $P_X$ is $T_\epsilon^n(P_X) = \{ x : \|T_x - P_X \|_\infty \leq \epsilon \}$. In addition, for a joint distribution $P_{X,Y}$ and $x \in \mathcal{X}^n$, the conditionally typical set of $y$ sequences, conditioned on $x$, is defined as $T_\epsilon^n(P_{X,Y}|x) = \{ y : \|T_{x,y} - P_{X,Y} \|_\infty \leq \epsilon \}$.

B. The Problem Setup

Consider the communication setup depicted in Fig. 1. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{J}$ and $\bar{X}$ denote finite sets. We consider a lossy source coding problem for an independent and identically distributed (i.i.d.) source with a distribution $P$ on $\mathcal{X}$. We assume without loss of generality that $P_X(x) > 0$, $\forall x \in \mathcal{X}$. A length-$n$ block of data is sent over an arbitrarily varying channel (AVC). The AVC has two inputs, and two outputs. The two inputs comprise the data input $X \in \mathcal{X}$ and adversary’s jamming input $J \in \mathcal{J}$, while the two outputs are $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$. We assume that the adversary knows $X$ non-causally and can randomize its jamming input $J$. We denote its jamming strategy by $Q_{J|X}$. The channel outputs $Y$ and $Z$ are observed at the encoder and decoder respectively. The channel behaviour is given by the conditional distribution $W_{Y,Z|X,J}$. Thus, given inputs $x$ and $j$, channel outputs $y$ and $z$ are observed with probability given by $P \{ Y = y, Z = z | X = x, J = j \} = \sum_{i=1}^{n} W_{Y,Z|X,J}(y_i, z_i | x_i, j_i)$. Upon observing $Y$, the encoder compresses it into a message $x$ and send it losslessly to the decoder. Given $M$ and $Z$, the decoder then outputs an estimate $\hat{X}$ of the source data $X$. We assume that the adversary knows the coding scheme. The quality of the estimate is given in terms of the average per-letter distortion $d(X, \hat{X}) = \frac{1}{n} \sum_{i=1}^{n} d(X_i, \hat{X}_i)$, where $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$ denotes a single-letter distortion measure with $d_{\max} = \max_{x \in \mathcal{X}, \hat{x} \in \mathcal{X}} d(x, \hat{x}) < \infty$.

An $(n, R)$ deterministic code of block length $n$ and rate $R$ is a pair $(\psi, \phi)$ of mappings, which consists of an encoder map $\psi : \mathcal{Y}^n \rightarrow \{ 1, 2, \ldots, 2^{nR} \}$, and a decoder map $\phi : \{ 1, 2, \ldots, 2^{nR} \} \times \mathcal{Z}^n \rightarrow \mathcal{X}^n$. The encoder’s output $M = \psi(Y)$. An $(n, R)$ randomized code of block length $n$ and rate $R$ is a random variable which takes values in the set of all $(n, R)$ deterministic codes. We denote the encoder-decoder pair for this $(n, R)$ randomized code by $(\psi, \Phi)$. This also forms the randomness $\Theta$ shared between the encoder-decoder, but unknown to the adversary. The message sent losslessly to the decoder is $M = \psi(Y)$. For this $(n, R)$ randomized code, the maximum expected distortion or maximum distortion $D^{(n)}$ is given by

$$D^{(n)} = \max_{x \in \mathcal{T}_0^n(P_X) \cap Q_{J|X}} \max_{\Phi} \mathcal{E}[d(x, \Phi(\psi(Y), Z))],$$

where the expectation is over the channel and the adversary’s jamming action and the shared randomness. Note that the first maximization above is over source sequences $x$ which are $\delta_0$-typical according to distribution $P_X$. Here $\delta_0 = \delta_0(n)$ is a fixed sequence which depends on $n$. Our distortion criterion here differs from the usual average distortion criterion (cf. [16])

$$D^{(n)} = \max_{Q_{J|X}} \mathcal{E}[d(X, \Phi(\psi(Y), Z))],$$

where distortion is averaged over all $x$ sequences under the i.i.d. distribution $P_X$. Our earlier work in [1] considered this criterion. Note that the criterion in [1], which has also been considered in other works (cf. [11]), is stronger than the one in [2], and proves crucial in our proof of Theorem 2 (see Remark 5). In a manner similar to [1], we assume here that $\delta_0(n) \rightarrow 0$ and $\sqrt{n}\delta_0(n) \rightarrow \infty$ as $n \rightarrow \infty$. To keep the notation simple, we henceforth suppress the dependence of $\delta_0$ on $n$.
An \((n, R)\) code with a stochastic encoder of block length \(n\) and rate \(R\) is a pair \((\Psi, \phi)\) of mappings consisting of a randomized encoding map \(\Psi : \mathcal{Y}^n \to \mathcal{P}(\{1, 2, \ldots, 2^{nR}\})\), and a deterministic decoding map \(\phi : \{1, 2, \ldots, 2^{nR}\} \times \mathcal{Z}^n \to \mathcal{X}^n\). The encoder’s output message \(M = \Psi(Y)\) is received error-free at the decoder. For this \((n, R)\) code with a stochastic encoder, the maximum distortion \(D^{(n)}\) is given by

\[
D^{(n)} = \max_{x \in T_0^n(P_X)} \max_{Q_{J|X}} \mathbb{E}[d(x, \phi(\Psi(Y), Z))],
\]

where the expectation is over the channel, the adversary’s jamming action and the encoding.

Given a target distortion \(D\), a rate \(R\) is achievable under randomized coding if for any \(\epsilon > 0\) there exists an \(n_0(\epsilon)\) such that for every \(n \geq n_0(\epsilon)\) there exists an \((n, R)\) randomized code with the resulting maximum distortion \(D^{(n)} \leq D + \epsilon\). We define the adversarial rate distortion function under randomized coding \(R_r(D)\) as the infimum of all achievable rates. These definitions for achievable rate and rate distortion function can be analogously stated for codes with a stochastic encoder. We denote by \(R_s(D)\) the rate distortion function for codes with a stochastic encoder. Our aim in this work is to obtain upper and lower bounds on the adversarial rate distortion functions under randomized coding as well as for codes with a stochastic encoder under the distortion criteria stated in (1) and (3) respectively.

### III. The Main Result

Recall that \(\mathcal{Q} = \mathcal{P}(J|X)\) denotes the set of all conditional distributions of \(J\) given \(X\). For any distribution \(Q_{J|X} \in \mathcal{Q}\), consider the single-letter joint distribution \(P_XQ_{J|X}W_{Y,Z|X,J}\). Let \(D_0 := \min_{P_{X|Y,\mathcal{A}}} \max_{Q_{J|X} \in \mathcal{Q}} \mathbb{E}[d(X, \hat{X})]\) and \(D_1 := \min_{P_{X|Z}} \max_{Q_{J|X} \in \mathcal{Q}} \mathbb{E}[d(X, \hat{X})]\). Here \(D_0\) is the minimax expected distortion when \(X\) is jointly estimated from \(Y\) and \(Z\), while \(D_1\) is the minimax expected distortion when \(X\) is estimated from \(Z\) only. Consider an auxiliary random variable \(U\) with a finite alphabet \(\mathcal{U}\) distributed according to \(P_{U|Y}\), such that \((X, J, Z) \leftrightarrow Y \leftrightarrow U\) forms a Markov chain. The joint distribution of \((X, J, Y, Z, U)\) is then given by \(P_XQ_{J|X}W_{Y,Z|X,J}P_{U|Y}\). Let us now define the following.

\[
R_r^*(D) := \begin{cases} \min_{P_{U|Y}, Q_{J|X} \in \mathcal{Q}} \max_{\mathcal{A}} I(U; Y|Z), & \text{if } D \in [D_0, D_1] \\ 0, & \text{if } D > D_1. \end{cases}
\]

(4)

The minimization above is over \(P_{U|Y} \in \mathcal{P}(\mathcal{U}|\mathcal{Y})\) and \(Q_{J|X} \in \mathcal{Q}\). Note that the cardinality \(|\mathcal{U}|\) of \(\mathcal{U}\) can be restricted to \(|\mathcal{U}| \leq |X|^{2R}\), which is the number of possible functions from \(Z\) to \(\tilde{X}\).

\[
R_r^*(D) := \begin{cases} \min_{P_{U|Y}, Q_{J|X} \in \mathcal{Q}} \max_{\mathcal{A}} I(U; Y|Z), & \text{if } D \in [D_0, D_1] \\ 0, & \text{if } D > D_1, \end{cases}
\]

(5)

where the minimization is over \(P_{U|Y} \in \mathcal{P}(\mathcal{U}|\mathcal{Y})\) and \(Q_{J|X} \in \mathcal{Q}\). In a manner similar to (6), we can restrict the cardinality of \(U\) to \(|\mathcal{U}| \leq |\mathcal{Y}| + 1\). Next, we state our results.

**Theorem 1.** The adversarial rate distortion function \(R_r(D)\) under randomized coding is such that

\[
R_r^*(D) \leq R_r(D) \leq R_r^*(D).
\]

The proof uses the approach in (1); see Section IV for details. We now state our main result which shows that the adversarial rate distortion function under codes with a stochastic encoder equals that under randomized codes.

**Theorem 2.** The adversarial rate distortion function under codes with a stochastic encoder is equal to that under randomized coding, i.e.,

\[
R_s(D) = R_r(D),
\]

and hence,

\[
R_r^*(D) \leq R_s(D) \leq R_r^*(D).
\]

The proof is given in Section IV.

**Remark 3.** The results in Theorem 2 continue to hold for codes with a stochastic encoder under the ‘usual’ average distortion criterion. Note that for this criterion, we replace the maximum (over all \(\delta_0\)-typical \(x\) sequences) in (3) by an average (over all \(x\) sequences). The converse directly follows from that in Theorem 2. The assertion now follows by noting that our achievability proof under the ‘stronger’ criterion in (3) guarantees achievability under the ‘average’ distortion criterion.
A. Achievability

We use the approach in \[1\], \[17\]. Note that while the analysis proceeds along similar lines, there are important differences. These result from the ‘stricter’ requirement of guaranteeing that \(\mathbb{E}[d(x, \bar{X})] \leq D + \epsilon, \forall x \in T^n_0(P_X)\), unlike in \[17\], where we required \(\mathbb{E}[d(X, \bar{X})] \leq D + \epsilon\). In particular, we establish more general versions of several claims in \[17\] by careful modifications (see, for instance, Claim 9 and other associated claims where, unlike in \[17\], there is no averaging over \(P_X\) and the vector \(x \in T^n_0(P_X)\) is now fixed) necessitated by the aforementioned requirement.

We first present a brief outline of the coding scheme and the analysis. The detailed proof can be found in Appendix A. First, note that for \(D > D_1\), we can directly estimate \(X\) from the side information \(Z\) using a possibly randomized estimator \(P_{\hat{X}|Z}\). Hence, we have \(R(D) = 0\) for \(D > D_1\). Let us fix \(D_1 \geq D \geq D_0\). We now fix \(P_U|Y\) and \(\zeta(u, z)\) given by \(4\), and prove the achievability of the rate

\[
R^{(P_U|Y, \zeta)} := \max_{Q_{J|X} \in \mathcal{Q}} \left( I(U; Y) - I(U; Z) \right).
\]

Following the analysis in \[1\], we can express \(R^{(P_U|Y, \zeta)}\) as

\[
R^{(P_U|Y, \zeta)} \geq \max_{P_Y \in \mathcal{P}(Y)} \left[ I_{P_Y}(U; Y) - \min_{Q_{J|X} \in \mathcal{Q}} I_{Q_{J|X}}(U; Z) \right] - \frac{\epsilon}{4}. \tag{6}
\]

Let us now define for every type \(T_Y \in \mathcal{P}(Y)\),

\[
R_U(T_Y) := I_{T_Y}(U; Y) + \frac{\epsilon}{4} \tag{7}
\]

\[
\hat{R}(T_Y) := \min_{P_Y \simeq T_Y} \left( I_{Q_{J|X}}(U; Z) - \frac{\epsilon}{4} \right). \tag{8}
\]

Code Construction:

- The random code generation is as follows. Here we assume that the entire ensemble of all possible codes is shared between the encoder and decoder, and they jointly select, at random, a code from the ensemble using the shared randomness \(\Theta\). Note that this process is equivalent to generating the code randomly and then sharing it with the encoder-decoder.
- Fix type \(T_Y \in \mathcal{P}(Y)\). Generate a codebook \(C(T_Y)\) with \(2^{nR_U(T_Y)}\) vectors i.i.d. \(\sim P_U\), where \(P_U := [T_Y P_{U|Y}]_{U}\). \(C(T_Y)\) is randomly partitioned into \(2^n(R_U(T_Y) - \hat{R}(T_Y))\) bins. Do this for every \(T_Y \in \mathcal{P}(Y)\).
- We share between the encoder and decoder the entire list of binned codebooks for all \(T_Y \in \mathcal{P}(Y)\).

Encoder operations:

- The encoder observes \(y\) and computes its type \(T_Y\). It finds if there exists at least one codeword in \(C(T_Y)\) which is jointly typical with \(y\) with respect to (w.r.t.) the distribution \(T_Y P_{U|Y}\). If there exists at least one possible codeword, it selects one from amongst them at random and sends its bin index in \(C(T_Y)\) along with \(T_Y\) to the decoder.

Decoder operations:

- The decoder observes side information \(Z = z\), and receives \(T_Y\) and the bin index. It first identifies the following set \(\mathcal{Q}^{(n)}(T_Y)\) of valid conditional types corresponding to block length \(n\)

\[
\mathcal{Q}^{(n)}(T_Y) := \{ T_{J|X} \in \mathcal{Q} : [P_X T_{J|X} W_{Y,Z|X,J}]_{Y} f(e) \simeq T_Y \}. \tag{9}
\]

Here every \(T_{J|X} \in \mathcal{Q}^{(n)}(T_Y)\) induces a \(Y\)-marginal distribution ‘close’ to \(T_Y\).}

\[\text{\footnotesize{1Here we indicate} I(U; Y) as \text{\footnotesize{a} function of only} P_Y as \text{\footnotesize{in our proof of achievability we have fixed} P_{U|Y}. For the same reason, we indicate I(U; Z) only as \text{\footnotesize{a} function of Q_{J|X}, as the other distributions} P_X, P_{U|Y}, \text{\footnotesize{and} W_{Y,Z|X,J} are fixed in our discussion.}}\]

By the notation \(P_Y f(e) \simeq P_Y\), we mean that \(\|P_Y - P_Y\|_\infty \leq f(e)\) for an appropriate \(f(e) > 0\), where \(f(e) \to 0\) as \(\epsilon \to 0\). See \[1\] for details.
Next, the decoder checks within the bin if there is a codeword $u \in C(T_y)$ such that $(u, z)$ is jointly typical w.r.t. the distribution $[P_X T_{Y|X} W_{Y,Z|X,Y} P_{U|Y}]_{U,Z}$ for some $T_{Y|X} \in \mathcal{Q}^{(n)}(T_y)$. It chooses that codeword if unique, otherwise it chooses an arbitrary codeword $u$ from the bin. It then outputs $\hat{x}$, where $\hat{x}_i = \zeta(u_i, z_i)$, $i = 1, 2, \ldots, n$, using the chosen codeword $u$ and $z$.

**Maximum distortion analysis:**

We fix a source sequence $x \in T^n_{\delta_0}(P_X)$, and show that under any feasible jamming strategy $Q_{J|X=x}$, the resulting distortion is at most $D + \epsilon$. As this distortion does not depend on the chosen sequence $x \in T^n_{\delta_0}(P_X)$, it follows that the overall distortion $D^{(n)} \leq D + \epsilon$, where $D^{(n)}$ is given by (1). The detailed proof follows the approach in [1] and uses the refined Markov lemma [1, Lemma 8]. Refer Appendix A for details.

**B. The proof of the lower bound**

Our converse follows from the converse in [1]. There is shown that any achievable rate is lower bounded by the maximin expression in [5], though under the usual distortion criterion given in [1]. However, the distortion criterion in (2) is a ‘stronger’ criterion than $R$ which is not achievable under (2) will also not be achievable under (1).

**V. PROOF OF THEOREM 2**

Clearly, $R_s(D) \geq R_r(D)$. We now give the proof of achievability to show that $R_s(D) = R_r(D)$; the bounds on $R_s(D)$ then directly follow from those given for $R_r(D)$ in Theorem 1. This proof uses the approach in [14] and has two parts, viz., part (a) and part (b). In part (a), we show that given any randomized code of rate $R \geq R_s(D)$, there exists a randomized code with the same rate $R$ but with an ensemble size $n^2$. In part (b), we construct a code with a stochastic encoder with rate arbitrarily close to $R$, thereby completing the proof.

**Part (a):** Let $\epsilon > 0$. Consider any randomized code, say $C = (\Psi, \Phi)$, of rate $R \geq R_s(D)$ for which the resulting maximum distortion is $D^{(n)} \leq D + \epsilon$. Consider $K$ independent repetitions of a random experiment of deterministic codebook selection from the randomized code $C$. We denote the $K$ outcomes, i.e., deterministic codes, by $C_i := (\psi_i, \phi_i)$, $i = 1, 2, \ldots, K$, where $(\psi_i, \phi_i)$ denote the encoder-decoder pair for code $C_i$. Given any $x \in T^n_{\delta_0}(P_X)$ and jamming input $j \in \mathcal{J}^n$, let the corresponding distortion under code $C_i$ be $D^{(n)}(x, j, C_i) := \mathbb{E}[d(x, \tilde{X})]$, where the expectation is over the channel $W_{Y,Z|X,J}$.

Note that $\forall x \in T^n_{\delta_0}(P_X)$ and $\forall j \in \mathcal{J}^n$, we have

$$\mathbb{E}_C[D^{(n)}(x, j, C_i)] \leq D + \epsilon.$$ 

We now state the following useful result.

**Lemma 4** ([18, pg. 16]). Let $Z_i$, $i = 1, 2, \ldots, N$, be a sequence of discrete independent random variables that take values in $[−b, b]$, where $b \in (0, \infty)$. Then, for any $\mu > 0$, there exists $0 < \alpha \leq \min \{\frac{1}{2} e^{-2b}\}$ such that

$$\mathbb{P}\left(\frac{1}{N} \sum_{i=1}^{N} (Z_i - \mathbb{E}[Z_i]) \geq \mu\right) \leq e^{-\left(\alpha \mu + \alpha^2 b^2\right)N}.$$ 

Observe that $D^{(n)}(x, j, C_i)$, $i = 1, 2, \ldots, K$ are i.i.d.. Further, $\forall i$, $D^{(n)}(x, j, C_i) \in [0, d_{\text{max}})$, where $d_{\text{max}} < \infty$. Hence, $D^{(n)}(x, j, C_i)$, $\forall i$ are bounded. Let us define

$$\mathbb{D}^{(n)}(x, j, C_i) := D^{(n)}(x, j, C_i) - \mathbb{E}[D^{(n)}(x, j, C_i)].$$

Now for any $\mu > 0$, there exists $\alpha \in (0, \min \{\frac{1}{2} e^{-2b}\})$, where $b := d_{\text{max}} \in (0, \infty)$, such that

$$\mathbb{P}_C\left(\frac{1}{K} \sum_{i=1}^{K} \mathbb{D}^{(n)}(x, j, C_i) \geq \mu\right) \leq e^{-\left(\alpha \mu + \alpha^2 b^2\right)K}.$$ 

where we have used (9) and Lemma 4. Now from the union bound, we get

$$\mathbb{P}_C\left(\exists x \in T^n_{\delta_0}(P_X), j \in \mathcal{J}^n : \frac{1}{K} \sum_{i=1}^{K} \mathbb{D}^{(n)}(x, j, C_i) \geq \mu\right) \leq |\mathcal{X}|^n |\mathcal{J}|^n e^{-\left(\alpha \mu + \alpha^2 b^2\right)K} = e^{-\left(K(\alpha \mu + \alpha^2 b^2) - n \log(|\mathcal{J}||\mathcal{X}|)\right)},$$

which is vanishing as $n \to \infty$ when $K = n^2$. Thus, for $\epsilon > 0$, there exists a randomized code with ensemble size $K = n^2$ such that for $n$ sufficiently large, the corresponding expected distortion $D^{(n)}(x, j) \leq D + \epsilon$, $\forall x \in T^n_{\delta_0}(P_X)$ and $j \in \mathcal{J}^n$. As $\epsilon > 0$ is arbitrary, this completes the proof of this part.
Remark 5. The choice of the distortion criteria in (1) is crucial for the application of the elimination technique in our proof above. If the ‘usual’ distortion criterion, viz., the average (averaged over source) distortion criterion (2), had been chosen, then it would have necessitated taking a union bound in (10) over all functions $f : X^n \to \mathcal{Y}^n$, instead of pairs of sequences $(x,\bar{x})$. However, the number of such functions $f : X^n \to \mathcal{Y}^n$ grows as $|\mathcal{Y}^n|^{|X^n|}$, i.e., doubly-exponentially in $n$. Thus, guaranteeing a vanishing probability of error in (10) under a polynomial-sized code collection would not have been possible using this approach.

Part (b): For this part, let us first denote the randomized code of ensemble size $n^2$ in part (a) by $(\Psi, \Phi) := \{\psi_k, \phi_k\}$, where $k = 1, 2, \ldots, n^2$. Recall that the rate of this code $(\Psi, \Phi)$ is $R \geq R_\epsilon(D)$, and given any $\epsilon > 0$ and $n$ large enough, its corresponding maximum distortion $D^{(n)} \leq D + \epsilon$. For our code with a stochastic encoder, denoted by $(\tilde{\Psi}, \tilde{\phi})$, the encoder first chooses, uniformly at random and privately, a deterministic code, say $(\psi_1, \phi_1)$ from the $n^2$-sized ensemble of code $(\Psi, \Phi)$. The encoder then sends this index $I \in [1, 2, \ldots, n^2]$ to the decoder using $\log(n^2)$ bits, followed by the corresponding codeword $\psi_I(y)$. The informed decoder then outputs the estimate $\hat{\phi}_I(\psi_I(y))$. Note that the overall distortion of this code $(\tilde{\Psi}, \tilde{\phi})$ constructed from $(\Psi, \Phi)$ is $D + \epsilon$. The rate $\tilde{R}$ of this code $(\tilde{\Psi}, \tilde{\phi})$ is $\tilde{R} = R + \frac{\log(n^2)}{n}$. However, as $\frac{2\log(n)}{n} \to 0$ as $n \to \infty$, we have $\tilde{R} \to R$ as $n \to \infty$. Thus, by choosing a sufficiently large $n$, we get a code with a stochastic encoder with rate arbitrarily close to $R$ and maximum distortion $D^{(n)} \leq D + \epsilon$. This completes the proof of part (b), and thus, the proof of the theorem.

VI. Conclusion

We studied lossy source coding for an arbitrarily varying remote source. Here statistics of the noisy source observations at the encoder and decoder are controlled by an adversary and vary arbitrarily across time. The adversary knows the coding scheme and the source data non-causally, and hence, can employ malicious strategies. We studied the rate distortion function when the encoder can privately randomize, i.e., for codes with a stochastic encoder. Toward this, we first extended an earlier result for randomized coding under the usual average (averaged over all sequences) distortion criterion to a ‘stronger’ maximum (over all typical source sequences) distortion criterion. Using this ‘strengthening’ of the result under randomized coding, we then showed that the result remains unchanged if randomization is restricted to the encoder only.

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APPENDIX A

PROOF OF ACHIEVABILITY

In this detailed proof of achievability, we begin with the description of our randomized coding scheme.

Code Construction:

- As discussed in the outline, the random code $C$ is a list of individual codes $C(T_Y)$ for every type $T_Y \in \mathcal{F}^{(n)}(Y)$. This list of codes is shared as the common randomness $\Theta$ between the encoder-decoder.

- For a fixed type $T_Y \in \mathcal{F}^{(n)}(Y)$, our code $C(T_Y)$ is a binned codebook comprising $2^{nR_U(T_Y)} = 2^{n(R(T_Y) + \tilde{R}(T_Y))}$ vectors $U_{j,k}$, where $j = 1, 2, \ldots, 2^{nR(T_Y)}$ and $k = 1, 2, \ldots, 2^{n\tilde{R}(T_Y)}$. Here $R_U(T_Y)$ and $\tilde{R}(T_Y)$ are as given in (7) and (8) respectively, and $\tilde{R}(T_Y) = R_U(T_Y) - \tilde{R}(T_Y)$. Every codeword $U_{j,k}$ is chosen i.i.d. $P_U$, where $P_U := |P_{U|Y}T_Y|_U$. There are $2^{n\tilde{R}(T_Y)}$ bins indexed by $j$, with each bin containing $2^{n\tilde{R}(T_Y)}$ codewords indexed by $k$. Let $B_{m,T_Y}^{(n)}$ denote the bin with index $m$. Thus, our code $C$ is the list containing $C(T_Y); T_Y \in \mathcal{F}^{(n)}(Y)$.

Encoding:

- Given input $Y$, the encoder determines its type $T_Y$ to identify $C(T_Y)$. In $C(T_Y)$, it finds a codeword $U_{m,l}$, where $m \in \{1, 2, \ldots, 2^{n\tilde{R}(T_Y)}\}$ and $l \in \{1, 2, \ldots, 2^{nR(T_Y)}\}$, such that

$$\|T_{U_{m,l},Y} - P_{U|Y}T_Y\|_{\infty} \leq \delta_2(\delta).$$

(11)

Here $\delta_2(\delta) > 0$ is a fixed constant (the choice of $\delta_2(\delta)$ is indicated in Lemma[7].) This implies that $U_{m,l}$ and $Y$ are jointly typical according to the distribution $P_{U|Y}T_Y$. If no such $U_{m,l}$ is found, then the encoder chooses $U_{1,1}$. If more than one $U_{m,l}$ satisfying (11) exist, then the encoder chooses one uniformly at random from amongst them. Let $U = U_{M,L}$ denote the chosen codeword.

- The encoder transmits $T_Y$ and the bin index $M$ losslessly to the decoder.

\[\text{Here } \delta > 0 \text{ is a function of } \epsilon, \text{ such that } \delta \to 0 \text{ as } \epsilon \to 0 \text{ and it is such that (15) holds.} \]
Decoding:

- Let the bin index $m$ and side information $z$ be received at the decoder. In addition, the decoder knows the type $T_Y$ of the encoder’s input $y$, and so the code $C(T_Y)$ used by the encoder.
- For some fixed parameter $\gamma(\delta) > 0$ (the choice of $\gamma(\delta)$ is indicated in Lemma 8), the decoder determines the set of codewords

\[
\mathcal{L}_n(\delta)(m, z) = \left\{ u \in \mathbb{B}_m^{(T_y)} : \|P_X T_{J|X} W_{Y,Z|X,J} P_{U|Y} u, z\|_\infty \leq \gamma(\delta), \text{ for some } T_{J|X} \in \mathcal{D}(T_y) \right\},
\]

(12)

Here $\mathcal{D}(T_Y) := \{T_{J|X} : [P_X T_{J|X} W_{Y,Z|X,J} Y \rightarrow T_Y] \},$ where recall that by the notation $P_Y f(\epsilon) \approx P_Y,$ we mean that $\|P_Y - P_Y f(\epsilon)\|_\infty \leq \epsilon$ for an appropriate $f(\epsilon) > 0,$ where $f(\epsilon) \to 0$ as $\epsilon \to 0.$
- If $\mathcal{L}_n(\delta)(m, z)$ contains exactly one codeword, then the decoder chooses it. Otherwise it chooses $u_{m,1}.$ Let the chosen codeword be $u_{m,l}.$
- The decoder outputs $\tilde{x}_i,$ where $\tilde{x}_i = \eta(u_i(m, \tilde{t}), z_i).$

Maximum distortion analysis: To begin, let us fix $x \in T^n_{\delta}(P_X).$ We first analyze the error in decoding the codeword $U = U_{M,L}$ chosen by the encoder. The decoder makes an error if one or more of the following events occur.

\[
E_{\text{enc}} = \{(U_{j,k}, Y) \notin T^n_{\delta}(P_{U|Y} T_X), \forall j, k\}
\]

\[
E_{\text{dec},1} = \{(U, Z) \notin \mathcal{L}_n(\delta)(M, Z)\}
\]

\[
E_{\text{dec},2} = \{(U_{M,l'}, Z) \in \mathcal{L}_n(\delta)(M, Z) \text{ for some } l' \neq L\},
\]

Then, using the union bound we can express the probability of decoding error by

\[
P(E) \leq P(E_{\text{enc}}) + P(E_{\text{dec},1}|E_{\text{enc}}) + P(E_{\text{dec},2}|E_{\text{enc}}).
\]

(13)

We will show that for every $\epsilon > 0$ there exists small enough $\delta > 0$ such that $P(E) \to 0$ as $n \to \infty.$ We first make the following obvious claim.

Claim 6. Let $U$ be generated i.i.d. $P_U.$ Then, with probability at least $1 - |\mathcal{U}|e^{-2n\delta^2},$ $U \in T^n_{\delta}(P_U).$

Let us define this “good” event as $A_U := \{U \in T^n_{\delta}(P_U)\}.$ We now state the following lemma which guarantees that the first term in (13) is vanishingly small.

Lemma 7. Under the event $A_U,$ there exist $\delta_2(\delta), f_1(\delta, \epsilon) > 0,$ where $\delta_2(\delta), f_1(\delta, \epsilon) \to 0$ as $\delta, \epsilon \to 0,$ such that the encoder finds a codeword $U$ with probability at least $1 - 2^{-2n f_1(\delta, \epsilon)}$ such that $(Y, U) \in T^n_{\delta}(P_{U|Y} T_X).$

The proof of this lemma follows from the covering lemma [16] Lemma 3.3. Note that this lemma specifies the $\delta_2(\delta)$ parameter which appears in the definition of the encoder in (11). This lemma implies $P(E_{\text{enc}}) \to 0$ as $n \to \infty.$ Our next lemma addresses the remaining two terms in the RHS of (13).

Lemma 8. Let the codeword chosen be $U$ (where $U \in T^n_{\delta}(P_U))$ and let the output on the channel $W_{Y,Z|X,J} \text{ be } (Y, Z).$ Then,

(a) there exists $\gamma(\delta) > 0,$ where $\gamma(\delta) \to 0$ as $\delta \to 0,$ such that except for an exponentially small probability, $U \in \mathcal{L}_n(\delta)(M, Z).$

(b) there exists $f_2(\delta, \epsilon) > 0,$ where $f_2(\delta, \epsilon) \to 0$ as $\delta, \epsilon \to 0,$ such that

\[
P(U_{M,l'} \in \mathcal{L}_n(\delta)(M, Z), \text{ for some } l' \neq L) \leq 2^{-n f_2(\delta, \epsilon)}.
\]

(14)

The proof of this lemma can be found in Appendix B. This lemma specifies the parameter $\gamma(\delta)$ which appears in the decoder operation in (12). Lemma 8 implies that $P(E_{\text{dec},1}|E_{\text{enc}}), P(E_{\text{dec},2}|E_{\text{enc}}) \to 0$ as $n \to \infty.$ Hence, we can conclude that $P(E) \to 0$ as $n \to \infty.$

We now get a bound on the expected distortion for $x.$ Toward this, we first make the following claim.

Claim 9. There exists $r(\delta), f_3(\delta, \epsilon) > 0,$ where $r(\delta), f_3(\delta, \epsilon) \to 0$ as $\delta, \epsilon \to 0,$ such that $P \left( (x, \tilde{X}) \in T^n_{\delta}(P_{X,\tilde{X}}) \right) \geq 1 - 2^{-n f_3(\delta, \epsilon)}.$

Proof: By Claim 17 in Appendix B, with high probability, $(x, J, Y, Z, U)$ is $\delta_4$-typical according to the joint distribution $P_X T_{J|X} W_{Y,Z|X,J} P_{U|Y},$ as $\tilde{X}$ is a deterministic function of $(U, Z),$ it follows by the conditional typicality lemma (see Lemma 11 in Appendix B) that with probability at least $(1 - |\mathcal{X}||\mathcal{Y}|\mathbb{E}[|Z||U||\tilde{X}|^2 - n \delta^2]),$ the tuple $(x, J, Y, Z, U, \tilde{X})$ is $3\delta_4$-typical, and hence $(x, \tilde{X})$ is $r(\delta)$-typical, where $r(\delta) := 3|\mathcal{X}|^2|\mathcal{Y}|^2 \delta_4.$ This completes the proof.

We now show that the maximum distortion $D^n(\alpha)$ for the code $C$ can be made arbitrarily close to $D.$ Let $\tilde{E} := \{(x, \tilde{X}) \notin T^n_{\delta}(P_{X,\tilde{X}})\}. From Claim 9 we know that $P(E) \to 0$ as $n \to \infty.$ Then,

\[
\mathbb{E}[d(x, \tilde{X})] = P(E)\mathbb{E}[d(x, \tilde{X})|E] + P(\tilde{E})\mathbb{E}[d(x, \tilde{X})|\tilde{E}]
\]

\[
\leq P(E)\mathbb{E}[d(x, \tilde{X})|E] + \mathbb{E}[d(x, \tilde{X})|\tilde{E}]|	ilde{E}.
\]
Recall that $d_{\text{max}} < \infty$. In addition, from the typical average lemma [16 pg. 26] we know that $\mathbb{E}[d(x, \tilde{X})|\tilde{E}] \leq D + h(\delta)$, where $h(\delta) > 0$ and $h(\delta) \to 0$ as $\delta \to 0$. Thus,

$$\mathbb{E}[d(x, \tilde{X})] \leq \mathbb{P}(\tilde{E})d_{\text{max}} + D + h(\delta)$$

As $\mathbb{P}(\tilde{E}) \to 0$ as $n \to \infty$, we choose a large enough $n$ and a small enough $\delta > 0$ to get (a). It follows from (15) that the expected distortion for any $x \in \mathcal{T}_n(U)\cap(P_X)$ under any randomized $J$ (i.e., any $Q_{J|x}\approx\delta$) is bounded by $D + \epsilon$, and hence, the maximum distortion $D(n)$ can be made arbitrarily close to $D$. We have, thus, shown that for any $\epsilon > 0$, the rate $R \leq \max_{Q_{J|x}} I(U;Y) - I(U;Z) + \epsilon$ is achievable. This completes the proof of achievability.

**APPENDIX B**

**Proof of Lemma 3**

Let us define $\delta_0 = \delta/2$. Consider the “good” encoder event $E_{\text{enc}} = \{(Y, U) \in \mathcal{T}_n(U|Y)|\tilde{Y}\}$. We now state and prove some necessary claims.

**Claim 10.** Given $x \in \mathcal{T}_n(P_X)$, and for any $j \in \mathcal{J}$, $(x, j) \in \mathcal{T}_n(P_X)$.

Recall from earlier (cf. [1]) our assumption on $\delta_0 = \delta_0(n)$. Here as $\delta \to 0$, we can make $\delta_0 \to 0$ by making $n \to \infty$.

**Lemma 11** (Conditional typicality lemma). Let $s \in \mathcal{T}_n(P_S)$ and $T$ be generated from $s$ using the memoryless distribution $W_T|S$. Then,

$$\mathbb{P}\left(\{s, T\} \in \mathcal{T}_n(P_S W_{T|S})\right) \geq 1 - |S||T|e^{-2n\delta_0}.$$  \hspace{1cm} (16)

**Proof:** We need to show that

$$\mathbb{P}\left(\{T_{s,T}(s, t) - P_S(s)W_{T|S}(t|s) > 2\delta_0\right)$$

is exponentially small for all $s, t$. We consider two cases.

**Case I:** $T_s(s) \leq \delta_0$. As $s \in \mathcal{T}_n(P_S)$, this implies that $P_S(s) \leq T_s(s) + \delta_0 \leq 2\delta_0$. Then, $\forall(s, t),$

$$|T_{s,T}(s, t) - P_S(s)W_{T|S}(t|s)| = |T_{s,T}(s, t) - P_S(s)W_{T|S}(t|s)| \leq \max(T_{s,T}(s, t) - P_S(s)W_{T|S}(t|s), P_S(s)W_{T|S}(t|s)) \leq 2\delta_0 + 1$$

Thus, for such $s$, $\mathbb{P}\left(\{T_{s,T}(s, t) - P_S(s)W_{T|S}(t|s) > 2\delta_0\right) = 0$.

**Case II:** $T_s(s) > \delta_0$. Using Chernoff-Hoeffding’s theorem [19 Theorem 1] for each $t \in T$, we have

$$\mathbb{P}(\{|W_{T|S}(t|s) - P_{T|s}(t|s)| > \delta_0, \text{ for any } t \leq |T|e^{-2n\delta_0}.$$  \hspace{1cm} (1)

Now, it can be easily checked that $|W_{T|S}(t|s) - P_{T|s}(t|s)| \leq \delta_0$ and $|P(s) - T_s(s)| \leq \delta_0$ together imply

$$|T_s(s)P_{T|s}(t|s) - P_{S}(s)W_{T|S}(t|s)| \leq 2\delta_0 + 2\delta_0^2$$

Hence, (16) follows by taking union bound over all $s \in S$. \hfill $\blacksquare$

Let us denote by $A_{x,j}$, the event that the given $x \in \mathcal{T}_n(P_X)$ and the (possibly random) jamming signal $J$ are $\delta_0$-typical w.r.t. $P_X T_{j|x}$. Note that it follows from Claim 10 that $\mathbb{P}(A_{x,j}) = 1$ for the specified $x \in \mathcal{T}_n(P_X)$.

**Claim 12.** Conditioned on the event $A_{x,j}$, with probability at least $(1 - |X||J||J||Y||Z|e^{-2\delta_0})$, we have $(x, j, Y, Z) \in \mathcal{T}_{3\delta_0}(P_X T_{j|x} W_{Y,Z,X})$.

The proof of this result follows from Lemma 11. We now consider this “good” event $A_{x,j,Y,Z}$, where $A_{x,j,Y,Z} := \{(x, j, Y, Z) \in \mathcal{T}_{3\delta_0}(P_X T_{j|x} W_{Y,Z,X})\}$.

**Claim 13.** Under the event $A_{x,j,Y,Z}$, $Y$ is $\delta_1$-typical w.r.t. $P_Y = [P_X T_{j|x} W_{Y,Z,X} Y]$, where $\delta_1(\delta) := 3|X||J||Z|\delta_0(\delta) \to 0$ as $\delta \to 0$. That is, $\|T_{Y} - P_{Y}\|\infty \leq 3|X||J||J||Z|\delta_0$.

The proof is straightforward, and hence, omitted. The above claim implies that, except for an exponentially small probability, the decoder considers the conditional type $T_{j|x}$ for decoding.

**Claim 14.** Under $E_{\text{enc}}$ and $A_{x,j,Y,Z}$, $(Y, U)$ are jointly $\delta_3$-typical according to the distribution $P_Y P_{U|Y}$, where $P_Y = [P_X T_{j|x} W_{Y,Z,X} Y]$ and $\delta_3(\delta) := 3|X||Y||Z|\delta_0(\delta) + 2\delta(\delta) \to 0$ as $\delta \to 0$. 
Proof: Note that
\[
\| P_Y P_{U|Y} - T_{U|Y} \|_\infty \leq \| P_Y P_{U|Y} - T_Y P_{U|Y} \|_\infty + \| T_Y P_{U|Y} - T_{U|Y} \|_\infty
\]
\[
\leq 3\|X\|\|Y\|Z|\delta_0 + \delta_2 \quad \text{(using } A_{x,i,Y,Z} \text{ and } E_{enc})
\]
\[
= \delta_3,
\]
where \( \delta_3 = 3\|X\|\|Y\|Z|\delta_0 + \delta_2 \).

Claim 15. There exists \( g(\delta) > 0 \), where \( g(\delta) \to 0 \) as \( \delta \to 0 \), such that \( \forall u \in T_{\delta_0}^n(P_{U|Y} P_Y|y), \)
\[
P_U(U = u|Y = y) \leq 2^{-n(H(U|Y) - g(\delta))},
\]
where \( H(U|Y) \) is computed with the distribution \( P_{U|Y} P_Y \).

The proof of this result directly follows from [17, Claim 13]. We now state the following result from [15] (see also [17, Lemma 14]).

Lemma 16 (Refined Markov Lemma [15]). Suppose \( X \to Y \to Z \) is a Markov chain, i.e., \( P_{X,Y,Z} = P_Y P_{X|Y} P_{Z|Y} \). Let \((x, y) \in T_{\delta_0}^n(P_{X,Y}) \) and \( Z \sim P_Z \) be such that
(a) \( P((x, y) \notin T_{\delta_0}^n(P_{Y,Z})) \leq \epsilon \), where \( \epsilon > 0 \),
(b) for every \( z \in T_{\delta_0}^n(P_{Y,Z}|y), \)
\[
P_Z(z) \leq 2^{-n(H(Z|Y) - g(\delta_0))},
\]
for some \( g : \mathbb{R}^+ \to \mathbb{R}^+ \), where \( g(\delta_0) \to 0 \) as \( \delta_0 \to 0 \).

Then, there exists \( \delta : \mathbb{R}^+ \to \mathbb{R}^+ \), where \( \delta(\delta_0) \to 0 \) as \( \delta_0 \to 0 \), such that
\[
P\left((x, y, Z) \notin T_{\delta_0}^n(P_{X,Y,Z})\right) \leq 2\|X\|\|Y\|Z|\epsilon^{-nK} + \epsilon.
\]

Here \( K > 0 \) and \( K \) does not depend on \( P_{X,Y}, P_Z \) or \((x, y)\) but does depend on \( \delta_0, g \) and \( P_{Z|Y} \). Further, the \( \delta \) function does not depend on \((x, y), P_{X,Y} \) or \( P_Z \).

The proof of this result follows through careful though minor modifications of the proof in [15], and hence, is omitted. We now use the above lemma to prove the following claim.

Claim 17. There exists \( \delta_4(\delta) > 0 \), where \( \delta_4(\delta) \to 0 \) as \( \delta \to 0 \), such that except for a small probability, \((x, J, Z, Y, U)\) is jointly \( \delta_4 \)-typical w.r.t. \( P_X T_{J,X} W_{Y,Z|X,J} P_{U|Y} \).

Proof: Let us assume that \( A_{x,i,Y,Z} \) is true. Now we use the refined Markov lemma (Lemma 16) on the Markov chain \((X, J, Z) \to Y \to U\). Then, by Claims 13, 14, 15 and Lemma 7, \( U \) is chosen such that both conditions (a) and (b) in Lemma 16 are satisfied. Thus, the claim follows.

We define this “good” event as \( A_{x,i,Y,Z,U} := \{(x, J, Z, Y, U) \in T_{\delta_0}^n(P_X T_{J,X} W_{Y,Z|X,J} P_{U|Y})\} \).

Claim 18. There exists \( \gamma(\delta) > 0 \), where \( \gamma(\delta) \to 0 \) as \( \delta \to 0 \), such that
\[
\gamma(\delta) = |X||J||Y|\delta_4(\delta).
\]
Thus, the claim follows from Claim 17.

This completes the proof of the first part of the lemma. The proof of the second part directly follows from the following claim.

Claim 19. There exists \( f_2(\delta, \epsilon) > 0 \), where \( f_2(\delta, \epsilon) \to 0 \) as \( \delta, \epsilon \to 0 \), such that
\[
P\left(U_{M,L'} \in L_{\gamma(\delta)}(M, Z), \text{ for some } L' \neq L\right) \leq 2^{-n f_2(\delta, \epsilon)}.
\]

Proof: Note that the codewords \( \{U_{M,L'}\}_{L' \neq L} \) are independently generated, and hence, \( \{U_{M,L'}\}_{L' \neq L} \) and \( Z \) are independent. Consider a fixed conditional type \( T_{J,X} \in \mathcal{Q}(T_Y) \), and let the resulting distribution \( P_{U,Z} = [P_X T_{J,X} W_{Y,Z|X,J} P_{U|Y}]_{U,Z} \). Then,
\[
P\left(\exists L' \neq L : (U_{M,L'}, Z) \in T_{\gamma(\delta)}(P_{U,Z})\right) \leq 2^{-n f_2(\delta, \epsilon)}
\]
for some \( f_2(\delta, \epsilon) \to 0 \) as \( \delta, \epsilon \to 0 \). This follows from the packing lemma [16, Lemma 3.1]. By taking the union bound over

3 In the refined Markov lemma presented in [15, 12], \( K \) depends on \( \delta_0 \) but does not depend on the block length \( n \). Recall that our choice of \( \delta_0 \) here depends on \( n \), which further results in \( K \) also depending on \( n \). However, it can be easily verified (see the detailed proof in [15]) that the first term in the RHS of (19) is vanishing in \( n \), when \( \delta_0 = \delta_0(n) \to 0 \) and \( \sqrt{n} \delta_0(n) \to \infty \), as is the case here.
all conditional types $T_{J|X} \in \mathcal{P}(T_Y)$ (the number of such types is at most polynomial in $n$), we get

$$
\mathbb{P}\left( \exists l' \neq L : (U_{M,l'}, Z) \in T^0_{\gamma(\delta)}(P_U, Z) \text{ for some } T_{J|X} \in \mathcal{P}(T_Y) \right) \leq (n + 1)^{|U||Z|} 2^{-n f_2(\delta, \epsilon)} \leq 2^{-n f_2(\delta, \epsilon)}.
$$

This completes the proof of the lemma.

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