PIECEWISE ANALYTIC SUBACTIONS FOR ANALYTIC DYNAMICS

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ABSTRACT. We consider a piecewise analytic expanding map $f : [0, 1] \to [0, 1]$ of degree $d$ which preserves orientation, and an analytic positive potential $g : [0, 1] \to \mathbb{R}$.

We address the analysis of the following problem: for a given analytic potential $\beta \log g$, where $\beta$ is a real constant, it is well known that there exists a real analytic eigenfunction $\phi_\beta$ for the Ruelle operator. One can ask: what happens with the function $\phi_\beta$, when $\beta$ goes to infinity? The domain of analyticity can change with $\beta$. The correct question should be: is $\frac{1}{\beta} \log \phi_\beta$ analytic in the limit, when $\beta \to \infty$?

Under a uniqueness assumption, this limit, when $\beta \to \infty$, is in fact a calibrated sub-action $V$.

Denote $m(\log g) = \max_{\nu \text{ an invariant probability for } f} \int \log g(x) \, d\nu(x)$, and $\mu_\infty$, any probability which attains the maximum value. Any one of these probabilities $\mu_\infty$ is called a maximizing probability for $\log g$. The probability $\mu_\infty$ is the limit of the Gibbs states $\mu_\beta$, for the potentials $\beta \log g$. In this sense one case say that $\mu_\infty$ corresponds to the Statistical Mechanics at temperature zero.

We show that when $\mu_\infty$ is unique, has support in a periodic orbit, the analytic function $g$ is generic and satisfies the twist condition, then the calibrated sub-action $V : [0, 1] \to \mathbb{R}$ for the potential $\log g$ is piecewise analytic. We assume the twist condition only in some of the proofs we present here.

An interesting case where the theory can be applied is when $\log g(x) = -\log f'(x)$. In this case we relate the involution kernel to the so called scaling function.

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0. INTRODUCTION

We consider a piecewise analytic expanding map \( f : [0, 1] \to [0, 1] \) of degree \( d \) which preserves orientation and an analytic positive potential \( g : [0, 1] \to \mathbb{R} \). Here analytic means to have a complex analytic extension to a neighborhood of \([0, 1]\) in the complex plane.

We denote

\[
m(\log g) = \max_{\nu \text{ an invariant probability for } f} \int \log g(x) \, d\nu(x),
\]

and \( \mu_\infty \) (or, \( \mu_\infty \log g \)) any probability which attains the maximum value. Any one of these probabilities \( \mu_\infty \) is called a maximizing probability for \( \log g \).

In general these probabilities are not positive in open sets \([9]\).

We refer the reader to \([20\) \([10\) \([19\) \([28\) \([14\) \([22\] \([6\) \([7\] and \([9\] for general references and definitions on Ergodic Optimization.

We show the existence of \( W(w, x) \), an analytic involution kernel for \( A(x) = \log g(x) \), and a relation with the dual potential \( A^*(w) = (\log g)^*(w) \) defined in the Bernoulli space \( \{1, ..., d\}^\mathbb{N} \). In this case \( W : \{1, ..., d\}^\mathbb{N} \times [0, 1] \to \mathbb{R} \) and by analytic we mean: for each \( w \in \{1, ..., d\}^\mathbb{N} \) fixed, the function \( W(w, \cdot) \) has a complex analytic extension to a neighborhood of \([0, 1]\). We refer the reader to \([1\) \([24\] for definitions and properties related to the involution kernel.

An interesting case where the theory can be applied is when \( \log g(x) = -\log f'(x) \). In this case we relate the involution kernel to the so called scaling function \([18\) \([20\].

By definition, a calibrated subaction for \( \log g \) is a function \( V \) such that

\[
\sup_{y \text{ such that } f(y) = x} \{ V(y) + \log g(y) - m(\log g) \} = V(x).
\]

If the maximizing probability is unique the calibrated subaction is unique, up to an additive constant \([8\) (Lemme C) or \([11\) (Proposition 5).

If we denote \( \phi_\beta \) (with the normalization \( \int \phi_\beta \, d\nu_\beta = 1 \), where \( \nu_\beta \) is the eigen-probability, associated to the same eigenvalue as \( \phi_\beta \), of the dual of the Ruelle operator associated to \( \beta A = \beta \log g \)) the main eigenfunction of the Ruelle operator for the potential \( \beta \log g \), it is known (theorem 29 in \([9\] ) that for any convergent (uniform convergence) subsequence of \( \frac{1}{\beta} \log \phi_\beta \) we have that the limit is a calibrated subaction. Under our hypothesis of uniqueness we have that the sequence \( \frac{1}{\beta} \log \phi_\beta \), when \( \beta \to \infty \), converges to a calibrated subaction.

Note that, in the general case, not always the Gibbs invariant probability \( \mu_\beta \), for \( \beta \log(g) \), will converge to a unique (maximizing) probability when \( \beta \to \infty \) \([7\]. This happens if the maximizing probability is unique.

Using the above results we show that under some conditions, when \( \mu_\infty \) is unique, has support in a periodic orbit and \( \log g \) is twist, then there exists a piecewise analytic calibrated sub-action (denoted by \( V : [0, 1] \to \mathbb{R} \) for the potential \( \log g \). Our main result is Theorem 6.2. We will also show that the above conditions are generic on the analytic potential \( g \). Explicit examples of piecewise analytic sub-actions (which are not analytic) for analytic potentials are presented in \([1\) and \([24\].

The ergodic optimization setting has a main difference to the twist maps theory \([13\) or to the Lagrangian Aubry-Mather problem \([27\) \([8\) \([12\) \([25\]): the dynamics of the shift (or the transformation \( f \)) is not defined (via a critical path problem) from the potential \( A \) to be maximized. Sometimes the analogous statements in each theory have to be proved under different techniques. A basic tool in Aubry-Mather theory is the assumption: the Lagrangian is convex on the velocity \( v \). Without this hypothesis the Mather graph theorem (see \([8\) is not true, etc.... In Ergodic Optimization a natural assumption, which in some sense play the role of convexity, is the twist condition on the involution kernel (it is a condition that depends just on \( A \)). Here we will assume this hypothesis which was first considered in this context in \([23\) and \([24\]. Examples of potentials \( A \) such that the corresponding involution kernel satisfies the twist condition appear there. The twist condition is an open property in the variation of the analytic potential \( A = \log g \) defined in a fixed open complex neighborhood of the interval \([0, 1]\). We assume \( f \) preserves orientation in order it can exist potentials \( A = \log g \) which satisfy the twist condition (see \([24\] ). We point out that we do not need this hypothesis for the results of all sections from 2 to 5. In the case \( f \) reverse
orientation, a similar reasoning can be applied, but we have to consider another dynamics (not the shift) on the dual space $\Sigma$. The proof requires a lot of different technicalities and we avoid to consider this case (we address the question to $[24]$).

The main idea of the proof in a short is the following: denote by $I^\ast$ the deviation function of the family of Gibbs probabilities $\mu_{\beta A^\ast}$ which converges to $\mu_{\infty A^\ast}$ (the maximizing probability for $A^\ast$, see [1]), and $V^\ast$ the calibrated subaction for $A^\ast$. From our assumptions the maximizing probability $\mu_{\infty A^\ast}$ is supported in a periodic orbit where $I^\ast$ is zero. Note, however, that $I^\ast$ could be also zero outside this support.

We show in section 5 and 6 that
\[
V(x) = \sup_{w \in \Sigma} [W(w, x) - V^\ast(w) - I^\ast(w)].
\]

Therefore, for each $x$ there exists $w_x$ such that
\[
V(x) = \sup_{w \in \Sigma} [W(w, x) - V^\ast(w) - I^\ast(w)] =
V(x) = [W(w_x, x) - V^\ast(w_x) - I^\ast(w_x)].
\]

For each fixed $w$ we proved in sections 4 and 5 that $W(w, x)$ is analytic in $x$.

As for a fixed $w_x$, $W(w_x, x)$ is analytic on $x$ (see corollary 5.2), a result on piecewise analyticity of $V$ can be obtained if we are able to assume conditions to assure that $w_x$ is a locally constant as a function of $x$.

We need two nice conditions for getting that: 1) the set of $w$ such that $I^\ast(w) < K$, for any given constant $K$ is a finite set.

If this is true, the possible number of $w_x$ is finite.

This is so because $W(w, x)$ is bounded in $\Sigma$ and $V^\ast(w)$ is bounded in $\Sigma$. Then points $w$ with a large value of $I$ can not be a maximizing $w_x$ (otherwise, take $w_x$ in the periodic maximizing probability where $I$ is zero). Note that $I$ is a non-negative lower semi-continuous function (that can attain the value $\infty$).

Condition 1 above will follow if $I^\ast$ is strictly positive on the pre-images of the support of $\mu_{A^\ast}$ (here we have to use a generic condition on $A$, not on $A^\ast$, which is much more easier).

To show this statement is the purpose of section 7 and 8 (see theorem 8.1).

2) For the lexicographic order in $\Sigma$ the function $x \rightarrow w_x$ is monotonous (here we have to use a twist condition for $W$, which is a condition to be checked just on $A$). This is proved in section 6.

Therefore, values $w_x$ can not change up and down when we move $x$ in a monotonous way. From the above we get that $w_x$ is locally constant as a function of $x$ and so $V$ is piecewise analytic.

The proof above is very indirect. Some problems in mathematics are of such kind: you have a primal problem that you are not able to solve, then you take the dual problem, and somehow, you solve the initial problem.

In section 1 we present basic definitions and in section 2 we show the existence of a certain function $h_w(x) = h(w, x)$ which defines by means of $\log(h(w, x))$ an involution kernel for $\log g$. In section 3 we present some basic results in Ergodic Optimization and we describe the main strategy for getting the piecewise analytic sub-action $V$. Section 4 shows the relation of the scaling function (see [30] [18]) with the involution kernel and the potential $\log g = -\log f'$. In fact, we consider in this section a more general setting considering any given potential $\log g$. In section 5 (and also 3) we consider Gibbs states for the potential $\beta \log g$, where $\beta$ is a real parameter. In section 6 we show, under a natural, but technical condition, the existence of the piecewise complex analytic calibrated sub-action, when $\mu_{\infty}$ is unique, has support in a periodic orbit and $A = \log g$ satisfies a twist condition. We also show that our technical condition is true for a generic $g$. In sections 7 and 8 a more general setting for generic properties of potentials is considered. The results about a generic $g$, which were used before, are obtained as a particular application. Finally, in section 9 we present a result of independent interest for the case where the maximizing probability is not a periodic orbit: we consider properties of the involution kernel for a generic $x$.

After this paper was written we discovered that some of the ideas described in section 2 appeared in some form in [29] [18] (but, as far as we can see, not exactly like here).

We would like to thanks R. Leplaideur for a nice example which is described in section 6.
1. ONTO ANALYTIC EXPANDING MAPS

Denote \( I = [0, 1] \). We say that \( f: I \to I \) is an onto map if there exists a finite partition of \( I \) by closed intervals
\[
\{I_i\}_{i \in \{1, 2, \ldots, d\}},
\]
with pairwise disjoint interiors, such that
- For each \( i \) we have that \( f(I_i) = I \),
- \( f_i \) is monotone on each \( I_i \).

We say that \( f \) is expanding if \( f \) is \( C^1 \) on each \( I_i \) and there exists \( \tilde{\lambda} > 1 \) such that
\[
\inf_i \inf_{x \in I_i} |Df(x)| \geq \tilde{\lambda}.
\]

Denote by \( \psi_i: I \to I_i \) the inverse branch of \( f \) satisfying \( \psi_i \circ f(x) = x \) for each \( x \in I_i \).

We will say that an expanding onto map is analytic if there exists a simply connected, precompact open set \( O \subset \mathbb{C} \), with \( I \subset O \), such that, each \( \psi_i \) has a univalent extension \( \psi_i: O \to \psi_i(O) \).

Since \( f \) is expanding, we can choose \( O \) such that
- \( \psi_i \) has a continuous extension \( \psi_i: \overline{O} \to \mathbb{C} \).
- We have \( \psi_i(\overline{O}) \subset O \).
- Moreover
\[
\sup_i \sup_{x \in O} |D\psi_i(x)| \leq \lambda = \frac{\tilde{\lambda}^{-1} + 1}{2} < 1.
\]

Consider a finite word \( \gamma = (i_1, i_2, \ldots, i_k) \), where \( i_j \in \{1, 2, \ldots, d\} \). Denote \( |\gamma| = k \). Define the univalent maps \( \psi_\gamma: O \to \mathbb{C} \) as
\[
\psi_\gamma = \psi_{i_k} \circ \psi_{i_{k-1}} \circ \cdots \circ \psi_{i_1},
\]
We will denote
\[
I_\gamma := \psi_\gamma(I).
\]
Given either an infinite word \( \omega = (i_1, i_2, \ldots, i_k, \ldots) \in \Sigma := \{1, 2, \ldots, d\}^\mathbb{N} \), or a finite word with \( |\omega| \geq k \), define its \( k \)-truncation as
\[
\omega_k = (i_1, i_2, \ldots, i_k).
\]
Note that for \( k \geq 1 \)
\[
\psi_{\omega_k} = \psi_{i_k} \circ \psi_{\omega_{k-1}}.
\]
For every finite word \( \gamma \) we can define the cylinder
\[
C_\gamma = \{\omega \in \{1, 2, \ldots, d\}^\mathbb{N} : \omega|_{|\gamma|} = \gamma\}.
\]
2. Analytic potentials, spectral projections and invariant densities

Some of the results presented in this section extend some of the ones in [26]. We say that a function
g: ∪i int Ii → ℝ
is a complex analytic potential if there are complex analytic functions gi: ψi(O) → ℂ such that
- The functions gi and g coincides in the interior of Ii.
- The functions gi have a continuous extension to ψi(O).
- There exists θ < 1 such that
  0 < \inf_{x \in ψi(O)} |gi(x)| \leq \sup_{x \in ψi(O)} |gi(x)| \leq \theta.
- We have gi(R ∩ ψi(O)) ⊂ ℝ⁺.

Denote \tilde{h}_i(x) = gi(ψi(x)).

For every finite word γ we will define by induction on the lengths of the words the function
\tilde{h}_γ: O → ℂ
in the following way: Let γ = (i₁, i₂, ..., iₖ₊₁). If |γ| = k + 1 = 1 define \tilde{h}_γ(x) = gi₁(ψi₁(x)), otherwise
\tilde{h}_γ(x) = \tilde{h}_{γ_k}(x) \cdot gi_{k+1} \circ ψ_{γ_{k+1}}(x) = \tilde{h}_{γ_k}(x) \cdot \tilde{h}_{i_{k+1}} \circ ψ_{γ_k}(x).

Define the Perron-Frobenius operator
\begin{align*}
P_{\log g} : C(I) &→ C(I), \\
(P_{\log g} q)(x) &= \sum_i \tilde{h}_i(x) \cdot q(ψ_i(x)).
\end{align*}

Note that
\begin{align*}
(P_{\log g}^n q)(x) &= \sum_{|γ| = n} \tilde{h}_γ(x) \cdot q(ψ_γ(x)).
\end{align*}

It is well known that there exists a probability \tilde{µ}, with no atoms and whose support is I, a Holder-
continuous and positive function \nu and α > 0 such that
(3) \quad P_{\log g}^n \nu = α^n \nu, \quad \tilde{µ}(\nu) = 1,

and
\begin{align*}
\tilde{µ}(P_{\log g}^n q) &= α^n \tilde{µ}(q)
\end{align*}

for every q ∈ C(I). Let \nu\tilde{µ} be the measure absolutely continuous with respect to \tilde{µ} and whose Radon-
Nikodym derivative with respect to \tilde{µ} is \nu, that is, for every borelian A we have
\begin{align*}
\nu\tilde{µ}(A) &= \int_A \nu(x) \, d\tilde{µ}(x).
\end{align*}

Then the probability \nu\tilde{µ} is f-invariant. Let ω be either an infinite word ω = (i₁, i₂, ..., iₖ, ...) or a finite word with |ω| ≥ k + n. Then
\begin{align*}
\tilde{µ}(I_{ωk+n}) &= \frac{1}{α^n} \int_{I_{ωk}} \tilde{h}_{ω_{n+k}−ω_k}(x) \, d\tilde{µ}(x),
\end{align*}

where \omega_{n+k} − \omega_k is the word
\begin{align*}
(i_{k+1}, i_{k+2}, ..., i_{k+n}).
\end{align*}

For every finite word γ, define
\begin{align*}
h_γ &= \frac{\tilde{h}_γ}{α^{γ},\tilde{µ}(I_γ)}.
\end{align*}
Note that for $|\omega| \geq k + 1$

\begin{equation}
 h_{\omega_{k+1}}(x) = h_{\omega_k}(x) \cdot g_{i_{k+1}} \circ \psi_{\omega_{k+1}}(x) \cdot \frac{\tilde{\mu}(I_{\omega_k})}{\alpha \tilde{\mu}(I_{\omega_{k+1}})} = h_{\omega_k}(x) \cdot \tilde{h}_{i_{k+1}} \circ \psi_{\omega_k}(x) \cdot \frac{\tilde{\mu}(I_{\omega_k})}{\alpha \tilde{\mu}(I_{\omega_{k+1}})}.
\end{equation}

Let $U \subset \mathbb{C}$ be a pre-compact open set. Consider the Banach space $\mathcal{B}(U)$ of all complex analytic functions

\[ h: U \to \mathbb{C} \]

that have a continuous extension on $\overline{U}$, endowed with the sup norm.

**Remark 2.1.** These function spaces have the following remarkable property. If $U, U_1 \subset \mathbb{C}$ are precompact open sets and $\overline{U}_1 \subset U$ then the inclusion $i: \mathcal{B}(U) \to \mathcal{B}(U_1)$ is a compact linear operator. So every bounded sequence $f_n \in \mathcal{B}(U)$ has a subsequence $f_{n_i}$ such that $f_{n_i}$ converges uniformly on $\overline{U}_1$ to a continuous function that is complex analytic in $U_1$. Picking a sequence of open sets $U_n$ such that $\overline{U}_n \subset U$ and $\bigcup_n U_n = U$,

we can use the diagonal argument to show that we can find a subsequence $f_{n_i}$ and a complex analytic function $f$ on $U$ such that $f_{n_i}$ converges uniformly to $f$ on each compact subset of $U$.

**Theorem 2.1.** There exists $K > 0$ with the following property: For every infinite word $\omega$ the sequence $h_{\omega_k}$ is a Cauchy sequence in $\mathcal{B}(O)$. Let $h_\omega$ be its limit. For every $\omega$ and $x \in O$ we have

\[ \frac{1}{K} \leq |h_\omega(x)| \leq K. \]

**Proof.** Indeed since

\[ \psi_{i_{k+1}}(I_{\omega_k}) = I_{\omega_{k+1}}, \]

we have

\begin{equation}
 \alpha \tilde{\mu}(I_{\omega_{k+1}}) = \int_{I_{\omega_k}} g_{i_{k+1}} \circ \psi_{i_{k+1}}(y) \, d\tilde{\mu}(y).
\end{equation}

Since $g_i$ is analytic and
diam $\psi_{\omega_{k+1}}(O) \leq C\lambda^{k+1},$
by Eq. (4) we have that

\[ \frac{g_{i_{k+1}} \circ \psi_{i_{k+1}}(y)}{g_{i_{k+1}} \circ \psi_{i_{k+1}}(x)} = 1 + \delta_{k,x,y}, \]

with

\[ |\delta_{k,x,y}| \leq C\lambda^{k+1}. \]

for every $x, y \in \psi_{\omega_k}(O)$. Here $C$ does not depend on either $x, y \in O$, $k \geq 1$, or $\omega$. In particular

\[ g_{i_{k+1}} \circ \psi_{\omega_{k+1}}(x) \cdot \frac{\tilde{\mu}(I_{\omega_k})}{\alpha \tilde{\mu}(I_{\omega_{k+1}})} = 1 + \tilde{\delta}_{k,x}, \]

with

\[ |\delta_{k,x}| \leq C\lambda^k. \]

for $x \in O$. This implies that for $m > n$

\[ \frac{h_{\omega_m}(x)}{h_{\omega_n}(x)} = 1 + \epsilon_{n,m}, \]

with

\begin{equation}
 |\epsilon_{n,m}| \leq C_1 \lambda^n
\end{equation}

for some $C_1$. Here $C_1$ does not depend on $x, y \in O$, $k \geq 1$, or $\omega$. Let $m_0$ large enough such that $C_1\lambda^{m_0} < 1$. Then

\[ \inf_{y \in O, |y| < m_0} |h_\gamma(y)| \prod_{k=m_0}^{\infty} (1 - C_1\lambda^k) \leq |h_{\omega_k}(x)| \leq \sup_{y \in O, |y| < m_0} |h_\gamma(y)| \prod_{k=m_0}^{\infty} (1 + C_1\lambda^k) \]
for every $x \in O$, infinite word $\omega$ and $k \geq 1$. In particular there exists $K > 0$ such that

$$\frac{1}{K} \leq |h_{\omega_k}(x)| \leq K$$

for every $k \geq 1$, $x \in O$ and infinite word $\omega$. By estimates Eq. (6) and (7) we conclude that $h_{\omega_k}$ converges. Denote

$$h_\omega = \lim_k h_{\omega_k}.$$ 

It follows from Eq. (7) that

$$\frac{1}{K} \leq |h_\omega(x)| \leq K$$

for every $x \in O$ and infinite word $\omega$. □

**Corollary 2.1.** For each $\omega \in \Sigma$ the function $\log h_\omega(\cdot) : I \to \mathbb{R}$ has a complex analytic extension to $O$.

**Proof.** Since $O$ is a simply connected open set and $h_\omega(x) \neq 0$ for every $x \in O$, this follows by classical results of complex analysis. □

We use the notation $h_\omega(x) = h(\omega, x)$, $h_{\omega_k}(x) = h(\omega_k, x)$, for $x \in [0, 1]$ and $\omega \in \{1, 2, \ldots, d\}^N$, according to convenience.

For every $\tilde{\mu}$-integrable function $z : I \to \mathbb{R}$ we can define the signed measure $z\tilde{\mu}$ as

$$(z\tilde{\mu})(A) = \int_A z(x)\tilde{\mu}(x)$$

for every borelian set $A \subset I$.

**Theorem 2.2.** Let

$$z : I \to \mathbb{R}$$

be a positive Holder-continuous function. Then the sequence

$$\rho_z(x) := \lim_k \sum_{|\gamma|=k} h_\gamma(x) \left[(z\tilde{\mu})(I_\gamma)\right] = \lim_k \sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu},$$

converges for each $x \in O$. This convergence is uniform on compact subsets of $O$. Indeed

$$\rho_z(x) = v(x) \int z d\tilde{\mu},$$

where $v$ is the complex analytic extension of the function $v$ defined in $\mathbb{R}$. Furthermore, there exists a borelian probability $\mu$ in the space of infinite words such that

$$v(x) = \rho_v(x) = \int h_\omega(x) d\mu(\omega).$$

**Proof.** Define $\rho^k : O \to \mathbb{C}$ as

$$\rho^k(x) := \sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu}.$$ 

Firstly we will prove that

$$\rho^k(x) \to_k v(x) \int z d\tilde{\mu},$$

for each $x \in I$. Indeed for $x \in I$

$$\sum_{|\gamma|=k} h_\gamma(x) \int_{I_\gamma} z d\tilde{\mu} = \sum_{|\gamma|=k} h_\gamma(x) z(\psi_\gamma(x))(1 + \epsilon_{x,\gamma})\tilde{\mu}(I_\gamma)$$

$$= \sum_{|\gamma|=k} h_\gamma(x) z(\psi_\gamma(x))\tilde{\mu}(I_\gamma) + \tilde{\epsilon}_{x,k}$$
Corollary 2.2. The function \( \rho_z = v(x) \int z \, d\tilde{\mu} \) is an \( \alpha \)-eigenfunction of \( P_{\log g} \)

\[ P_{\log g}(\rho_z) = \alpha \cdot \rho_z. \]
Therefore, any $\rho_2$ is an eigenfunction for the Ruelle operator for $A = \log g$. Later we will consider a real parameter $\beta$ and we will denote by $\phi_\beta(x)$ a specific normalized eigenfunction of the Ruelle operator for $\beta \log g$.

The two results described above are in some sense similar to the ones in [1] and [26]. We explain this claim in a more precise way in the next section.

3. Maximizing probabilities, subactions and the involution kernel

In this section we review some definitions and properties of Ergodic Optimization (see [20] [9] [1] [3]). We compare the setting and notation of [1] with the one described here.

We consider here $f$ an expanding real analytic transformation of degree $d$ on the interval $[0,1]$ with an analytic extension to a small complex neighborhood $B \subset \mathbb{C}$ of $[0,1]$.

By definition, the Bernoulli space is the set $\{1,2,\ldots,d\}^\mathbb{N} = \Sigma$. A general element $w$ in $\Sigma$ is denoted by $w = (w_0,w_1,\ldots,w_n,\ldots)$.

We denote $\Sigma$ the set $\Sigma \times [0,1]$ and $\psi_i$ indicates the $i$-th inverse branch of $f$. We also denote by $\sigma^*$ the shift on $\Sigma$. Finally, $\hat{\sigma}^{-1}$ is the backward shift on $\hat{\Sigma}$ given by $\hat{\sigma}^{-1}(w,x) = (\sigma^*(w),\psi_{w_0}(x))$

Consider $A(x) = \log g(x)$.

A sub-action for $A$ is a function $V : [0,1] \to \mathbb{R}$ such that, for all $x \in [0,1]$ we have

$$V(f(x)) \geq V(x) + A(x) - m(A).$$

A calibrated sub-action $V : [0,1] \to \mathbb{R}$ for the potential $A$, is a function $V$ such that

$$\sup_{y \text{ such that } f(y) = x} \{ V(y) + A(y) - m(A) \} = V(x).$$

A calibrated sub-action is a particular case of sub-action.

If we assume the maximizing probability for $A$ is unique, then there is just one calibrated sub-action up to an additive constant (see [9] [1]).

From [1] it is known that for a certain analytic functions $f$ and $A = \log g$, there is no analytic subaction.

**Definition 3.1.** Consider $A : [0,1] \to \mathbb{R}$ Holder. We say that $W_1 : \hat{\Sigma} \to \mathbb{R}$ is an involution kernel for $A$, if there is a Holder function $A^* : \hat{\Sigma} \to \mathbb{R}$ such that

$$A^*(w) = A \circ \hat{\sigma}^{-1}(w,x) + W_1 \circ \hat{\sigma}^{-1}(w,x) - W_1(w,x).$$

We say that $A^*$ is a dual potential of $A$, or, that $A$ and $A^*$ are in involution.

In [1] it was used the terminology $W$-kernel instead of involution kernel.

We point out that $A^*$ and $W_1$ are not unique. It is also known (proposition 2 in [1]) that two involution kernels for $A$ differ by a function $\varphi(w)$. It always true that

$$m(A) = \max_{\nu \text{ an invariant probability for } f} \int A(x) \, d\nu(x) = \max_{\mu \text{ an invariant probability for } \sigma^*} \int A^*(w) \, d\mu(w) = m(A^*).$$

**Remark:** We point out that in section 5 we are going to consider two specific involution kernels for which we will reserve the notation $H_\infty$ and $W$.

The definition of involution kernel is basically the same as in [1] where it’s considered the Bernoulli space $\{1,2,\ldots,d\}^\mathbb{Z}$. Here the infinite choice of the inverse branches is described by $w$. More precisely, the $y$ in Proposition 3 in [1] is the $w$ here. Moreover, our $x$ is in $[0,1]$ and not in $\{1,2,\ldots,d\}^\mathbb{N}$ as in [1].

The results described here in last section correspond in [26] to the potential $\log g = A = -\log f'$.

In this way $e^{W_1(w,x)}$ coincides with the function $|D\hat{\psi}_w(x)|$ on the variables $(x,w)$ of [26]. Note that in this case for a fixed $w$ the function $W_\varphi(x) = W_1(w,x)$ is analytic on $x$, if $f$ is considered analytic in [26]. We fix from now on a certain $W_1$ as the involution kernel for $A = \log g$. 
**Remark:** We point out that in the present moment we are consider fixed $W_1$ and $A^*$ just for the purpose of explaining the general theory for maximizing probabilities and large deviations. We will consider an specific involution kernel $W$ (and a $A^*$) later, and these ones are obtained in a unique way from the procedure described here.

Given $\beta$ one can consider $\beta A$ and the associated Ruelle operator $P_{\beta A}$. We will be interested here in Thermodynamic Formalism properties for the potential $\beta A$, when $\beta \to \infty$ (the zero temperature case).

The point of view of [1] is the following: it is easy to see that if $W_1$ is a involution kernel for $A$ (we consider $W_1$ fixed as we just said) then $\beta W_1$ is a involution kernel for $\beta A$.

**Remark:** In the notation of last section $\log(h_w(x)) = \log h(w, x) = W_1(w, x)$. This will be shown in the next section. Note that in last section the function $h$ depends on $A = \log g$ in a natural way. In this way, for a given $\beta$ we get in a natural and unique way from $\log(h_{\beta}(w, x))$, which is not necessarily equal to $\beta W_1$, where $W_1 = h$ is fixed and associated to an initial $A = \log g$ (that is, $\beta = 1$). This is main difference from the reasoning in our section 5 to the procedures in [1]. More precisely, If we know that

$$A^*(w) = A \circ \hat{\sigma}^{-1}(w, x) + W_1 \circ \hat{\sigma}^{-1}(w, x) - W_1(w, x),$$

then, an easy way to get an involution kernel for $\beta A$ is just multiply the above equation by $\beta$. Then, in [1] it is used the relation

$$\beta A^*(w) = \beta A \circ \hat{\sigma}^{-1}(w, x) + \beta W_1 \circ \hat{\sigma}^{-1}(w, x) - \beta W_1(w, x).$$

We are not doing this here. In our reasoning in section 5 we are deriving, for each $\beta$, a certain involution kernel $W_\beta$ which is not necessarily equal to $\beta W_1$. In this notation, just when $\beta = 1$ we have $W_1 = 1W_1$.

We describe briefly the main results in [1].

As we said, given $\beta$, one can take the associated involution kernel (to $\beta A$) the function $W_\beta = \beta W_1$. Moreover $(\beta A)^* = \beta A^*$. The normalizing constant

$$c(\beta) = \log \int e^{\beta W_1(w, x)} d\nu_{\beta A^*}(w, x),$$

is such that

$$\phi_{\beta A}(x) = \int e^{\beta W_1(w, x)} - c(\beta) d\nu_{\beta A^*}(w),$$

where $\phi_{\beta A}$ is the normalized eigen-function associated to the Ruelle operator $P_{\beta A}$ and to the maximal eigenvalue $\lambda(\beta)$, and finally $\nu_{\beta A}$ and $\nu_{\beta A^*}$ are the associated eigen-probabilities for the dual of the Ruelle operators $P_{\beta A}$ and $P_{\beta A^*}$ (acting on probabilities) corresponding respectively to $\beta A$ and $\beta A^*$ (see proposition 3 in [1]). We denote by $\mu_{\beta A} = \phi_{\beta A} d\nu_{\beta A}$ and we note that, $\int \phi_{\beta A} d\nu_{\beta A} = 1$. In analogous way $\mu_{\beta A^*} = \phi_{\beta A^*} d\nu_{\beta A^*}$. Here, $P_{\beta A^*}(\phi_{\beta A^*}) = \lambda(\beta) \phi_{\beta A^*}$.

Remember from the corollary of last section that given $A = \beta \log g$, we have $P_{\beta \log g}(\rho_v) = \alpha \rho_v$. Therefore, the expression

$$\rho_v(x) = \int h_\omega(x) d\mu(\omega),$$

obtained in Theorem 1 is similar but slightly different from

$$\phi_{\beta \log g}(x) = \int e^{\beta W_1(w, x)} - c(\beta) d\nu_{(\beta \log g)^*}(w),$$

because $\mu$ is an invariant probability for the shift and $\nu_{(\beta \log g)^*}$ is an eigen-probability (not necessarily invariant for the shift) for $P_{\beta \log g}^*$. This point will be important in the last section.

**Remark:** Note that (using the above notation) $\log h_\omega(x)$ is not necessarily equal to $\beta W_1(w, x) - c(\beta) - \log \phi_{\beta \log g}^*.

It is known that in the analytic setting we consider before, given an analytic potential $A = \log g$, the eigenfunction $\phi_{\beta A}$ for $P_{\beta A}$ is analytic in a neighborhood $C_\beta$ of $[0, 1]$. This can be also derived from the expression above if we know that $W_1(w, x) = W_w(x)$ is analytic on $x$ for any $w$ fixed. A natural question
is: what happen with the domains $C_\beta$ of $\phi_{\beta A}$ when $\beta \to \infty$? The question that makes sense is to ask: is there an analytic limit for

$$\lim_{\beta \to \infty} \frac{\log \phi_{\beta A}}{\beta}$$

Our purpose in this paper is to show that if the maximizing probability is unique and has support in a periodic orbit, then certain subsequences $\beta_n \to \infty$ of above limit will define a piecewise analytic function $V$. The idea is to consider a fixed neighborhood $C$ of $[0, 1]$ on $\mathbb{C}$ and to show that we can select a sequence of bounded complex analytic functions $\frac{\log \phi_{\beta A}}{\beta_n}$. Any of these limits will define a calibrated sub-action (see [2] page 1404)

We assume that the maximizing probability $\mu_\infty$ for $A$ is unique, and so, the maximizing probability for $\mu_\infty^*$ for $A^*$ is also unique (this follows from the cohomological equation for $\hat{\sigma}$). In this case $\lim_{\beta \to \infty} \mu_{\beta A^*} = \mu_\infty^*$ (see [9] [1])

In [1] is shown that for any cylinder $C \in \Sigma$

$$\lim_{\beta \to +\infty} \frac{1}{\beta} \log \mu_{\beta A^*}(C) = - \inf_{w \in C} I^*(w)$$

where

$$I^*(w) = \sum_{n \geq 0} \left( V^* \circ \sigma^* - V^* - (A^* - m^*) \right) \circ (\sigma^*)^n(w), \quad m^* = \int A^* d\mu_\infty^*$$

where $V^*(x)$ is any calibrated subaction of $A^*$.

That is, $A^*$ satisfies for all $\overline{w}$

$$\sup_{w \text{ such that } \sigma^*(w) = \overline{w}} \{ V^*(y) + A^*(y) - m(A) \} = V^*(\overline{w})$$

Adapting the proof of the Varadhan’s Theorem (theorem 4.3.1 in [11]) one can show that for a continuous function $G : \Sigma \to \mathbb{R}$,

$$\lim_{\beta \to +\infty} \frac{1}{\beta} \log \int e^{\beta G(w)} \mu_{\beta A^*}(w) = \sup_{w \in \Sigma} (G(w) - I^*(w))$$

Note in our setting $\mu_{\beta A^*} \to \mu_\infty A^*$, the maximizing probability for $A^*$. In the same way for the case of $A$ (which has a deviation function denoted by $I$) the deviation function $I^*$ can have the value infinity for some points $w$. The function $I^*$ is zero on the support of the maximizing probability for $A^*$. Anyway, in [22] a direct proof of this property is presented.

Moreover, for any $x$ is true

$$\phi_{\beta A}(x) = \int e^{\beta W_1(w, x) - e(\beta) - \log \phi_{\beta A^*}(w)} \phi_{\beta A^*}(w) d\nu_{\beta A^*}(w) =$$

$$\int e^{\beta W_1(w, x) - \frac{1}{\beta} e(\beta) - \frac{1}{\beta} \log \phi_{\beta A^*}(w)} d\mu_{\beta A^*}(w),$$

where $\mu_{\beta A^*}$ is the invariant probability which maximizes the pressure $P(\beta A^*)$ and $c(\beta)$ is the corresponding normalizing constant to such $\beta W_1$. It is known that there exists $\gamma$, such that $\frac{c(\beta)}{\beta} \to \gamma$, as $\beta \to \infty$. All these results are described in [1].

**Remark:** We point out that we will not follow the above strategy here because we have a procedure that defines an involution kernel $W_\beta$ in a unique way (and it is not equal to $\beta W_1$).
4. Scaling functions and dual potentials

Given a finite word $\gamma = (i_1, i_2, \ldots, i_k)$, $k > 1$, define $\sigma^*(\gamma) = (i_2, \ldots, i_k)$. For infinite words we define $\sigma^*$ as the usual shift function. The scaling function $s : \Sigma \rightarrow \mathbb{R}$ of the potential $g$ is defined as

$$s(\omega) = \lim_{k \rightarrow \infty} \frac{\tilde{\mu}(I_{\sigma_k})}{\tilde{\mu}(I_{\sigma^*(\omega_k)})}.$$ 

This definition is the natural generalization of the one in [30] and [18]. When $\log g = -\log f'$ we get the usual one. In this section we show the existence of a natural involution kernel which provides a co-homology between the scaling function $[\log(\alpha s)](\omega)$ and $\log g(x) = -\log f'(x)$. The constant $\alpha$ is the eigenvalue defined before in section 1.

To verify that the above limit indeed exists, note that by Eq. (5) and since $g$ is a Holder-continuous function we have that

$$\frac{\tilde{\mu}(I_{\omega_k})}{\tilde{\mu}(I_{\sigma^*(\omega_k)})} = \frac{\int_{I_{\omega_k}} g \circ \psi_{i_{k+1}}(y) \, d\tilde{\mu}(y)}{\int_{I_{\sigma^*(\omega_k)}} g \circ \psi_{i_{k+1}}(y) \, d\tilde{\mu}(y)} = (1 + \epsilon_k) \frac{\tilde{\mu}(I_{\omega_k})}{\tilde{\mu}(I_{\sigma^*(\omega_k)})},$$

where $|\epsilon_k| \leq C\lambda^k$. So $s(\omega)$ is well defined.

Note that, since $v > 0$ is a Holder function and $I_{\omega_k} \subset I_{\sigma(\omega_k)}$,

$$s(\omega) = \lim_{k \rightarrow \infty} \frac{(v\tilde{\mu})(I_{\omega_k})}{(v\tilde{\mu})(I_{\sigma^*(\omega_k)})} = \lim_{k \rightarrow \infty} \frac{\mu(C_{\omega_k})}{\mu(C_{\sigma^*(\omega_k)})},$$

so the scaling function $s$ is the Jacobian of the measure $\mu$.

The dual potential $g^*$ is defined as

$$g^*(\omega) := \alpha s(\omega).$$

For every $\omega = (i_0, i_1, \ldots, i_k, \ldots)$ and $x \in I$, define

$$\sigma^{-1}(\omega, x) := (\sigma^*(\omega), \psi_{i_0}(x)).$$

**Proposition 4.1.** We have

$$\frac{g^*(\omega)}{g(\psi_{i_0}(x))} = \frac{h(\sigma^*(\omega), \psi_{i_0}(x))}{h(\omega, x)}.$$

**Proof.** Indeed

$$\frac{h(\sigma^*(\omega), \psi_{i_0}(x))}{h(\omega, x)} = \lim_k \frac{h(\sigma^*(\omega_k), \psi_{i_0}(x))}{h(\omega_k, x)}.$$

$$\lim_k \frac{\tilde{h}(\sigma^*(\omega_k), \psi_{i_0}(x))}{\tilde{h}(\omega_k, x)} \frac{\alpha^k \tilde{\mu}(I_{\omega_k})}{\alpha^{k-1} \tilde{\mu}(I_{\sigma^*(\omega_k)})} = \alpha \frac{\tilde{\mu}(I_{\omega_k})}{g(\psi_{i_0}(x)) \tilde{\mu}(I_{\sigma^*(\omega_k)})} = \frac{\alpha}{\tilde{\mu}(\psi_{i_0}(x))} s(\omega).$$

We finally get the following result:

**Proposition 4.2.** $\log h_{\omega}(x) = \log h(\omega, x)$ is well defined and is an involution kernel for $\log g$. For $\omega$ fixed, the function $\log h(\omega, \cdot)$ has a complex analytic extension to a complex neighborhood $O$ of $[0, 1]$.

The dual of $A = \log g$ is naturally associated to the scaling function $s$.

We will need an special involution kernel $H_\omega$ later, not this one $\log h_{\omega}(x)$. The reason is that we have to consider a variable parameter $\beta$ and moreover $\beta \log g$. The complex neighborhood $O$ of $[0, 1]$ could change, in principle, very much with $\beta$. 
5. When $\beta \to \infty$ we get an involution kernel which is analytic on $x$ for $w$ fixed

Given an analytic potential $g$, let $g_i : O \to \mathbb{R}$, $i = 1, \ldots, d$, be complex analytic functions satisfying the properties described in Section 2. Since $O$ is simply connected and $g_i(x) \neq 0$ for $x \in O$, reducing $O$ a little bit, if necessary, we have that $\log g_i$ is a well defined continuous function in $\overline{O}$ and complex analytic in $O$. For each real value $\beta$, we consider the analytic potential $g^\beta$ and the corresponding the functions $g^\beta_i(x) := \exp(\beta \log g_i(x))$. Then for each $\beta$ these functions also satisfies the assumptions of Section 2 so we can construct functions $h_\beta(\omega, x)$, as in Theorem 2.1 corresponding scaling functions $s_\beta$ and dual potentials $g^*_\beta = \alpha_\beta s_\beta$ as in Section 3. The main goal of this section it to study how these functions are perturbed when we move $\beta$.

Note that we are going to consider $g$ fixed, and for a variable $\beta$ the potential $g^\beta$. We point out that, in principle, the corresponding dual potential $g^*_\beta$ does not satisfy necessarily $g^*_\beta = (g^*)^\beta$, where $g^*$ corresponds to $g$ by the procedure of last section.

Note also that if $g^*$ is the one associated to $g$, then

$$(\log g)^*(w) = \log g \circ \sigma^{-1}(w,x) + W_1 \circ \sigma^{-1}(w,x) - W_1(w,x),$$

Therefore, given a real value $\beta$ we have

$$\beta (\log g)^*(w) = \beta \log g \circ \sigma^{-1}(w,x) + \beta W_1 \circ \sigma^{-1}(w,x) - \beta W_1(w,x).$$

Therefore, $\beta W_1$ is a involution kernel for $\beta \log g$. This point was very important in [1].

We consider here a new procedure that gives in a unique way (for each value $\beta$) an involution kernel $W_\beta = \log h_\beta$ for $\beta \log g$.

The important point we would like to stress here is that in [1] it is consider first ca fixed $W_1$. Then, it is known that one can get the main eigenfunction for the Ruelle operator for $\beta A$ as

$$\phi_{\beta A}(x) = \int e^{\beta W_1(w,x) - c(\beta)} d\nu_{\beta A^*}(w).$$

Here the procedure is different: we will get (for each value $\beta$ another away different from [1])

$$\phi_{\beta A}(x) = \int e^{W_\beta(w,x)} d\nu_{\beta A^*}(w) = \int h_\beta(w,x) d\nu_{\beta A^*}(w),$$

for a $W_\beta$ which depends of the variable $\beta$ (see the third remark of last section).

$h_\beta$ acting on the variable $(w,x)$ is an integral kernel that transforms eigen-probabilities of the dual of the Ruelle operator for $\beta A^*$ in eigen-functions of the Ruelle operator for $\beta A$.

First we want to show that there exists $H_\infty(w,x)$ (complex analytic on $x$) such that $h_\beta(w,x) \sim e^{\beta H_\infty(w,x)}$ (in the sense that $\lim_{\beta \to \infty} \frac{1}{\beta} \log h_\beta(w,x) = H_\infty(w,x)$). In other words, we want to replace $W_\beta$ by a $\beta H_\infty$ (in the notation that will be followed later). This will be useful to apply Varadhan’s Theorem later from an expression in the form

$$\phi_{\beta A}(x) = \int e^{\beta H_\infty(w,x) - c(\beta)} d\nu_{\beta A^*}(w) = \int e^{\beta H_\infty(w,x) - c(\beta)} \frac{1}{\phi_{\beta A^*}(x)} d\mu_{\beta A^*}(w).$$

Remember that for a given $w \in \Sigma$, we have $h_\omega = \lim_k h_{\omega_k}$.

**Proposition 5.1.** Let $K \subset O$ be a compact. There exists $C$ such that the following holds:

A. For every $\beta \geq 1$ and $x \in K$, $\omega \in \Sigma$, we have

$$e^{-\beta C} \leq |h_\beta(\omega_1, x)| \leq e^{\beta C}$$

B. For every $\beta \geq 1$, $x \in K$, $\omega \in \Sigma$ and $k \geq 1$ we have

$$e^{-C_{\beta} \lambda^k} \leq \left| \frac{h_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} \right| \leq e^{C_{\beta} \lambda^k}.$$
that is holomorphic on $x$, real valued for $x \in \mathbb{R}$ and which does not depend on $K$, such that for every $x \in O$, $\beta \geq 1$, $\omega \in \Sigma$ we have

(12) \[ h_\beta(\omega_{k+1}, x) = e^{q_{k+1}(\beta,x)}. \]

Furthermore

(13) \[ |q_\omega(\beta, x)| \leq C_\beta \]

and

(14) \[ |q_{\omega_{k+1}}(\beta, x) - q_{\omega_k}(\beta, x)| \leq C_\beta \lambda^k \]

for every $\beta \geq 1$, $x \in K$, $\omega \in \Sigma$ and $k \geq 1$.

Proof of Claim A. Recall that for $i \in \{1, \ldots, d\}$

(15) \[ h_\beta(i, x) = \frac{g_i(\psi_i(x))}{\alpha_\beta(I_i)} = \frac{g_i(\psi_i(x))}{\int_I g_i(\psi_i(y)) \mu_\beta(y)}, \]

so

\[ |h_\beta(i, x)| = \frac{1}{\int_I g_i(\psi_i(y)) \mu_\beta(y)}. \]

Since $g_i$ are holomorphic on $\psi_i(O)$, $g_i \neq 0$ in $\psi_i(O)$, for every compact $K \subset O$ there exists $C$ such that

(16) \[ e^{-C} \leq \frac{|g_i(\psi_i(x))|}{|g_i(\psi_i(y))|} \leq e^C \]

for every $x, y \in K$ and $i$. Since $\mu_\beta(I) = 1$, it is now easy to obtain Eq. \textbf{10}. \hfill \Box

Proof of Claim B. Since $g_i$ are holomorphic on $\psi_i(O)$, $g_i \neq 0$ in $\psi_i(O)$, for every compact $K \subset O$ there exists $C$ such that

(17) \[ e^{-C|x-y|} \leq \frac{|g_i(\psi_i(x))|}{|g_i(\psi_i(y))|} \leq e^{C|x-y|} \]

for every $x, y \in K$ and $i$. Note that every such compact is contained in a larger compact set $\tilde{K} \subset K$ for every $i$, so we can assume that $K$ has this property. Let $x \in K$. By Eq. \textbf{5}

(18) \[ \frac{h_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} = \frac{\tilde{h}_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} \frac{\alpha_\beta^k \tilde{\mu}_\beta(I_{\omega_k})}{\mu_\beta(I_{\omega_{k+1}})} \]

\[ = \frac{g_{i_{k+1}}(\psi_{\omega_{k+1}}(x))}{\alpha_\beta} \frac{\alpha_\beta \tilde{\mu}_\beta(I_{\omega_k})}{\tilde{\mu}_\beta(I_{\omega_{k+1}})} \]

\[ = \frac{\alpha_\beta}{\int_{I_{\omega_k}} g_{i_{k+1}}(x) \psi_{\omega_{k+1}}(y) \tilde{\mu}_\beta(y) \mu_\beta(I_{\omega_k})} \]

\[ = \frac{\alpha_\beta}{\int_{I_{\omega_k}} g_{i_{k+1}}(x) \psi_{\omega_{k+1}}(y) \tilde{\mu}_\beta(y) \mu_\beta(I_{\omega_k})} \]

\[ = \frac{\alpha_\beta}{\int_{I_{\omega_k}} g_{i_{k+1}}(x) \psi_{\omega_{k+1}}(y) \tilde{\mu}_\beta(y) \mu_\beta(I_{\omega_k})} \]

\[ = \frac{\alpha_\beta}{\int_{I_{\omega_k}} g_{i_{k+1}}(x) \psi_{\omega_{k+1}}(y) \tilde{\mu}_\beta(y) \mu_\beta(I_{\omega_k})} \]

In particular

\[ \left| \frac{h_\beta(\omega_{k+1}, x)}{h_\beta(\omega_k, x)} \right| = \frac{\tilde{\mu}_\beta(I_{\omega_k})}{\int_{I_{\omega_k}} g_{i_{k+1}}(x) \psi_{\omega_{k+1}}(y) \tilde{\mu}_\beta(y) \mu_\beta(I_{\omega_k})} \]

For every $y \in I_{\omega_k}$ we have

$\psi_{i_{k+1}}(y), \psi_{\omega_{k+1}}(x) \in \psi_{\omega_{k+1}}(O)$
From Eq. (17) we obtain
\[ e^{-C \beta \lambda^k} \leq e^{-C \beta \text{diam } \psi_{\omega_{k+1}}(O)} \leq \frac{g_{\beta k+1} \circ \psi_{\omega_{k+1}}(y)}{|g_{\beta k+1}(\psi_{\omega_{k+1}}(x))|} \leq e^{C \beta \text{diam } \psi_{\omega_{k+1}}(O)} \leq e^{C \beta \lambda^k} \]

So
\[ e^{-C \beta \lambda^k} \leq \frac{|h_{\beta}(\omega_{k+1}, x)|}{|h_{\beta}(\omega_k, x)|} \leq e^{C \beta \lambda^k}. \]

\[ \square \]

**Proof of Claim C.** Since \( g_t \circ \psi : O \rightarrow \mathbb{C} \) does not vanish and \( O \) is a simply connected domain, there exists a (unique) function \( r_i : O \rightarrow \mathbb{C} \) such that \( g_t \circ \psi \) is \( e^t \) on \( O \) and \( \text{Im } r_i(x) = 0 \) for \( x \in \mathbb{R} \). Since \( \psi_{\gamma_i}(O) \cap I \neq \emptyset \) and \( \text{diam } \psi_{\gamma_i}(O) \leq \lambda_i |\gamma_i| \) we have that
\[ (20) \quad |\text{Im } r_i(\psi_{\gamma_i}(x))| \leq C \lambda_i |\gamma_i| \]

for every \( x \in O \) and every finite word \( \gamma_i \).

Define
\[ q_i(\beta, x) = \text{Re } r_i(\beta, x) + \log \int_I g_t(\beta) \circ \psi (y) \, d\mu(\beta)(y), \]

and \( q_{\gamma_i} \), with \( \gamma_i = (i_1, \ldots, i_{k+1}) \), by induction on \( k \), as
\[ q_{\gamma}(\beta, x) = q_{\gamma_k}(\beta, x) + \beta r_{i_{k+1}}(\psi_{\gamma_k}(x)) + \log \frac{\mu(\gamma_k)}{\int_{I_{\gamma_k}} g_{\beta i_{k+1}} \circ \psi_{\omega_{i_{k+1}}}(y) \, d\mu_{\beta}(y)}. \]

It follows from Eq. (18) that \( q_{\gamma} \) satisfies Eq. (12), so
\[ \text{Re } q_{\omega_1}(\beta, x) = \log |h_{\beta}(\gamma, x)|, \]

in particular by Eq. (10) e (11) we have
\[ (21) \quad |\text{Re } q_{\omega_1}(\beta, x)| \leq C \beta \]

and
\[ (22) \quad |\text{Re } q_{\omega_{k+1}}(\beta, x) - \text{Re } q_{\omega_k}(\beta, x)| \leq C \beta \lambda^k \]

for \( k \geq 1 \). Furthermore for every \( \beta \in \mathbb{R}, \omega \in \Sigma \) and \( k \geq 1 \)
\[ |\text{Im } q_{\omega_{k+1}}(\beta, x) - \text{Im } q_{\omega_k}(\beta, x)| = |\beta |\text{Im } r_{i_{k+1}}(\psi_{\omega_k}(x))| \leq C |\beta| \lambda^k. \]

Moreover for \( \beta > 0 \) we have
\[ |\text{Im } q_i(\beta, x)| = |\beta| |\text{Im } r_i(\psi_{\omega_k}(x))| \leq C |\beta|. \]

\[ \square \]

For every \( x \in O \) define
\[ H_{\beta, k}(\omega, x) := \frac{1}{\beta} q_{\omega_k}(\beta, x). \]

In particular, if \( x \in I \) we have that \( h_{\beta}(\omega_k, x) \) is a nonnegative real number by our choice of the branches \( r_i \), so
\[ H_{\beta, k}(\omega, x) = \frac{1}{\beta} \log h_{\beta}(\omega_k, x) \]

for \( x \in I \). It follows from Proposition $[5]$ that for every compact \( K \subset O \) there exists \( C \) such that
\[ (23) \quad |H_{\beta, 1}(\omega, x)| \leq C, \]
\[ (24) \quad |H_{\beta, k+1}(\omega, x) - H_{\beta, k}(\omega, x)| \leq C \lambda^k \]

for \( x \in K \), and every \( k \) and \( \omega \). So there exists some constant \( C \) such that
\[ |H_{\beta, k}(\omega, x)| \leq C. \]
for every $k, \omega, x \in K$. This implies that the family of functions

$$F_1 = \{H_{\beta,k}(\omega, \cdot)\}_{k, \omega, \beta \geq 1}$$

is a normal family on $O$, that is, every sequence of functions in this family admits a subsequence that converges uniformly on every compact subset of $O$. In Theorem 2.1 we showed that for every $x \in I$ we have

$$\lim_{k} h_{\beta}(\omega_k, x) = h_{\beta}(\omega, x) > 0,$$

so

$$\lim_{k} H_{\beta,k}(\omega, x) = \frac{1}{\beta} \log h_{\beta}(\omega, x),$$

for $x \in I$. It follows from the normality of the family $F$ that the limit

$$H_{\beta}(\omega, x) := \lim_{k} H_{\beta,k}(\omega, x)$$

exists for every $x \in O$ and that this limit is uniform on every compact subset of $O$. Moreover

$$F_2 = \{H_{\beta}(\omega, \cdot)\}_{\omega, \beta \geq 1}$$

is also a normal family on $O$.

We consider in $\Sigma$ the metric $d$, such that $d(\omega, \gamma) = 2^{-n}$, where $n$ is the position of the first symbol in which $\omega$ and $\gamma$ disagree.

**Corollary 5.1.** For every compact $K \subset O$ there exists $C$ such that

(25) $$|H_{\beta}(\omega, x) - H_{\beta}(\omega, y)| \leq C|x - y| + Cd(\omega, \gamma)$$

for every $x, y \in K$.

**Proof.** Since the family $F_2$ is uniformly bounded on each compact set $K \subset O$, we have that the family of functions

$$F_3 := \{H'_{\beta}(\omega, \cdot)\}_{\omega, \beta \geq 1}$$

has the same property, so it is easy to see that for every compact $K \subset O$ there exists $C$ such that

$$|H_{\beta}(\omega, x) - H_{\beta}(\omega, y)| \leq C|x - y|.$$ 

Note also that Eq. (24) implies

$$|H_{\beta}(\omega, x) - H_{\beta}(\omega, y)| \leq C \lambda^k,$$

Let $k + 1 = \log(d(\gamma, \omega))/\log \lambda$. Then $\gamma_k = \omega_k$ and we have

$$|H_{\beta}(\omega, y) - H_{\beta}(\gamma, y)| \leq |H_{\beta}(\omega, y) - H_{\beta}(\omega, y)| + |H_{\beta}(\gamma_k, y) - H_{\beta}(\gamma, y)| \leq Cd(\omega, \gamma).$$

$$\square$$

**Corollary 5.2.** There exists a sequence $\beta_n > 0$ satisfying $\beta_n \to \infty$ when $n \to \infty$ such that the limit

(26) $$H_{\infty}(\omega, x) = \lim_{n \to \infty} H_{\beta_n}(\omega, x),$$

exists for every $(\omega, x)$ in

$$\{1, \ldots, d\}^N \times O.$$ 

Moreover for every compact $K \subset O$ there exist $C$ such that

(27) $$|H_{\infty}(\omega, x) - H_{\infty}(\gamma, y)| \leq C|x - y| + Cd(\omega, \gamma)$$

and the limit in Eq. (26) is uniform with respect to $(\omega, x)$ on

(28) $$\{1, \ldots, d\}^N \times K$$

In particular for each $\omega$ we have that $x \to H_{\infty}(\omega, x)$ is holomorphic on $O$.
Proposition 5.2. The function $H_{\infty}(w, x)$ is analytic on $x.$

Proof. Consider $g$ fixed. Let $\beta_n$ be a sequence as in Corollary 5.2. For any $\beta_n$ we have

$$
\frac{g_{\beta_n}^3(\omega)}{g_{\beta_n}^3(\psi_{i_0}(x))} = \frac{h_{\beta_n}(\sigma^*(\omega), \psi_{i_0}(x))}{h_{\beta_n}(\omega, x)}.
$$

Taking $\frac{1}{\beta_n}$ log in both sides and taking the limit $n \to +\infty$ we get that

$$
g(\hat{\sigma}^{-1}(\omega, x)) + H_{\infty}(\hat{\sigma}^{-1}(\omega, x)) - H_{\infty}(\omega, x)
$$

depends only in the variable $w$.

Therefore, $H_{\infty}(w, x)$ is an involution kernel. \hfill \Box

Given the analytic involution kernel $H_{\infty}(w, x)$ and a fixed calibrated $V^*$ (unique up to additive constant) define $W(w, x) = H_{\infty}(w, x) - V^*(w).$ We point out that $W$ is also analytic on the variable $x \in (0, 1)$ for each $w$ fixed.

The reason for the introduction of such $W$ (and not $H_{\infty}$) is that, in next section, instead of

$$
V(x) = \sup_{w \in \Sigma} [H_{\infty}(w, x) - I^*(w)],
$$

it is more convenient the expression

$$
V(x) = \sup_{w \in \Sigma} [W(w, x) - V^*(w) - I^*(w)].
$$
6. The subaction is piecewise analytic when the potential $A = \log g$ is twist and $g$ is generic

We sometimes denote $\sigma^*$ by $\sigma$. We believe will be no confusion: $f$ acts on points $x \in [0,1]$ and $\sigma^*$ (or, $\sigma$) acts on $w$ on $\Sigma = \{0,...,d-1\}^\mathbb{N}$. This avoids using * all the time. In any case $\hat{\sigma}$ acts on $\Sigma \times [0,1] = \hat{\Sigma}$.

We suppose in this section that the maximizing probability for $A^*$ is unique (see [24]) in order we can define the deviation function $I^*$. This property will follow from the uniqueness of the maximizing probability for $A = \log g$ (which implies the same for $A^*$).

Adapting Varadhan’s Theorem one can show that that

$$V(x) = \sup_{w \in \Sigma} \{ W(w,x) - V^*(w) - I^*(w) \}.$$ 

See also [24] for a direct proof of this result.

For each $x$ we get one (or, more) $w_x$ such attains the supremum above. Therefore,

$$V(x) = W(w_x, x) - V^*(w_x) - I^*(w_x).$$

The main strategy in the present section is to find suitable hypothesis in such way that $w_x$ is unique and locally constant on $x$. Remember that for a fixed $w$, we have that $W(w,x)$ is analytic on $x$. It seems difficult to us to imagine how one could be able to show that $V(x)$ is locally analytic using a different procedure. But, we may be wrong.

One can consider on $\Sigma = \{0,...,d-1\}^\mathbb{N}$ the lexicographic order. We will consider, by technical reasons, the case where $f : (0,1) \rightarrow (0,1)$ has positive derivative. In the most of the cases we will consider, $d = 2$, in order to avoid unnecessary complex notation.

Following [24] we define:

**Definition 6.1.** We say a continuous $G : \hat{\Sigma} = \Sigma \times [0,1] \rightarrow \mathbb{R}$ satisfies the twist condition on $\hat{\Sigma}$, if for any $(a,b) \in \hat{\Sigma} = \Sigma \times [0,1]$ and $(a',b') \in \Sigma \times [0,1]$, with $a' > a$, $b' > b$, we have

$$G(a,b) + G(a',b') < G(a,b') + G(a',b).$$

(29)

**Definition 6.2.** We say a continuous $A : [0,1] \rightarrow \mathbb{R}$ satisfies the twist condition, if some (all) of its involution kernels satisfies the twist condition.

Note that $W$ satisfies the twist condition, if, and only if, $W - V^*$ (or, $W(w,x) - V^*(w) - I^*(w)$) satisfies the twist condition.

We will assume later that $A = \log g$ satisfies the twist condition. We point out that in order to check that $W$ is twist we just have to check properties of the potential $A$ (see [24]). The property of been twist is stable by perturbations. Examples of twist potentials $A$ are presented in [24].

We point out that in the case $f$ reverse orientation (like $-2x$ (mod 1)), then there is no potential $A = \log g$ which is twist for the dynamics on $\Sigma \times [0,1]$. A careful analysis (for different types of Baker maps) of when it is possible for $A$ to be twist for a given dynamics $f$ is presented in [24]. We will not consider this case here.

Proposition 5 in [1] claims that if $\hat{\mu}_{max}$ is the natural extension of the maximizing probability $\mu_\infty$, then for all $(p,p^*)$ in the support of $\hat{\mu}_{max}$ we have

$$V(p) + V^*(p^*) = W(p,p^*) - \gamma.$$

From this follows that if $(p,p^*)$ in the support of $\hat{\mu}_{max}$ (then, $p \in [0,1]$ is in the support of $\mu_\infty$ and $p^* \in \Sigma$ is in the support of $\mu_\infty^*$), then

$$V(p) = \sup_{w \in \Sigma} \{ W(w,p) - \gamma - V^*(w) - I^*(w) \} =$$

$$(W(p^*,p) - \gamma - V^*(p^*) - I^*(p^*)) = (W(p^*,p) - \gamma - V^*(p^*)).$$

If the potential $\log g$ is twist, then for any given $p$ in the support of $\mu_\infty$, there is only one $p^*$, such that $(p,p^*)$ is in the support of $\hat{\mu}_{max}$ (see [24]).

In principle could exist another $\bar{w} \in \Sigma$ such that for such $p$ we have

$$V(p) = W(\bar{w},p) - \gamma - V^*(\bar{w}) - I^*(\bar{w}).$$
The calibrated subaction will be analytic, if there exists \( \hat{w} \) such that for all \( x \)

\[
V(x) = \sup_{w \in \Sigma} (H_\infty(w, x) - I^*(w)) = H_\infty(\hat{w}, x) - I^*(\hat{w}) = W(\hat{w}, x) - V^*(\hat{w}) - I^*(\hat{w}).
\]

This will not be always the case.

Let’s consider for a moment the general case \((A \text{ not necessarily twist})\).

We denote by \( M \) the support of \( \mu_\infty^* \).

As \( I^* \) is lower semicontinuous and \( W - V^* \) is continuous, then for each fixed \( x \), the supremum of \( H_\infty(w, x) - I(w) \) in the variable \( w \) is achieved, and we denote (one of such \( w \)) it by \( w_x \). In this case we say \( w_x \) is optimal for \( x \). One can ask if this \( w_x \) is independent of \( x \), and equal to a fixed \( \hat{w} \). This would imply that \( V \) is analytic. If for all \( x \) in a certain open interval \((a, b)\), the \( w_x \) is the same, then \( V \) is analytic in this interval. We will show under some restrictions that given any \( x \) we can find a neighborhood \((a, b)\) of \( x \) where this is the case.

Given \( x \), this maximum at \( w_x \) can not be realized where \( I(w) \) is infinity. Moreover, as \( W - V^* \) is bounded, there exists a constant \( K \), such that, we know a priori that \( w_x \) is such that \( I(w_x) < K \).

Consider for any \( x \)

\[
K(x) = \max_w H_\infty(x, w) - \min_w H_\infty(x, w)
\]

Then, \( K = \sup_x K(x) \).

**Remark:** We just have to consider \( w \) such that \( I(w) < K \).

In order to simplify the notation we assume that \( m(A^*) = 0 \).

If we denote \( R^*(w) = V^* \circ \sigma(w) - V^*(w) - A^*(w) \), then we know that \( R^* \geq 0 \).

Consider the compact set of points \( P = \{ w \in \Sigma, \text{such that } \sigma^*(w) \in M, \text{and } w \text{ is not on } M \} \).

**Assumption:** We say that \( R^* \) is good for \( A^* \), if for each \( w \in P \), we have that \( R^*(w) > 0 \).

We point out that there are examples of potentials \( A^* \) (with a unique maximizing probability) where the corresponding \( R^* \) is not good (see example 2 in the end of the present section).

We will assume first that \( R^* \) is good, present some main results, and later we show that generically on the analytic function \( g \) (not generically on \( A^* \) which is much more easy) we have that the corresponding \( R^* \) satisfies the assumption.

**Example 1.** We point out that in the example described in [24], for the potential \( A = -(1 - x)^2 \), and the transformation \( T(x) = -2x \ (\text{mod } 1) \), we have that the maximizing probability \( \mu_\infty \) for \( A \) has support on \( x_0 = 2/3 \). The pre-image of \( 2/3 \) outside the support of \( \mu_\infty \) is \( x_1 = 1/6 \). That is, \( P = \{ x \in [0, 1] - \{ 2/3 \}, \text{such that } T(x) \text{ is in the support of } \mu_\infty \} = \{ 1/6 \} \). The explicit value of the calibrated sub-action is \( V(x) = -1/2x^2 + 2/9x^2 \). In this case \( R(1/6) = V(2/3) - V(1/6) - (A(1/6) - m(A)) = 0.665.. > 0 \).

Therefore, the \( R \) corresponding to such \( A \) (not \( A^* \)) satisfies the property of been good for \( A \).

This potential \( A \) is not twist when we consider the question in \( \Sigma \times [0, 1] \). If we consider instead a different kind of Baker map \( \hat{F} \), like the one that can be naturally defined \( \hat{F} : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1] \), which satisfies \( \hat{F}(x, f(y)) = (f(x), y), \forall (x, y) \in [0, 1] \times [0, 1], \) then the potential is twist (see [24] for the appropriate definition). All results we present in this section also applied to this last situation.

Remember that,

\[
I^*(w) = \sum_{n \geq 0} (V^* \circ \sigma - V^* - A^*) \circ \sigma^n(w) = \sum_{n \geq 0} R^*(\sigma^n(w)).
\]

In [24] section 5 it is shown that if \( I^*(w) \) is finite, then

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{(\sigma^j)(w)} \rightarrow \mu_\infty^*.
\]

Our main assumption says that \( R^* \) is positive in the compact set \( P \).
We consider in $\Sigma$ the metric $d$, such that $d(w_1, w_2) = \frac{1}{\|w\|}$, where $n$ is the first symbol in which $w_1$ and $w_2$ disagree.

There exist a fixed $0 < \delta < 2^{-(p+1)}$, (in the case $M$ is a periodic orbit, the $p$ can be taken the period) for some $p > 0$, such that, if

$$\Omega_\delta = \{ w \in \Sigma \mid d(w, P) < \delta \},$$

then

$$c_\delta = \min_{w \in \Omega_\delta} R^*(w) > 0.$$

Consider a small neighborhood $A_\delta$ of the set $M$ such that $\sigma^*(\Omega_\delta) = A_\delta$.

We can assume the above $\delta$ is such that any point in $A_\delta$ has a distance smaller that $2^{-p}$ to a point of $M$, where $p$ is the period.

Note that in order that the orbit of point $w$ by $\sigma^*$ enter (a new time) the set $A_\delta$, it has to pass before by $\Omega_\delta$.

As $\mu^*_{\infty}(M) > 0$, then considering the continuous function $\chi_{A_\delta}$ (indicator of $A_\delta$), we have that, if $I^*(w) < \infty$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{A_\delta}((\sigma^*)^j(w)) > 0.$$

Therefore, $((\sigma^*)^j(w)$ visits $A_\delta$ for infinitely many values of $j$.

Given $w$, suppose there exist a $N > 0$, such that for all $j > N$, we have that $((\sigma^*)^j(w) \in A_\delta$. In this case, there exist a $k$ such that $(\sigma^*)^k(w) \in M$.

Now, we consider the other case.

Denote by $m_1$ the total amount of time the orbit $(\sigma^*)^k(w)$ remains in $A_\delta$ for the first time, then the trajectory goes out of $A_\delta$, and $m_2$ is the total amount of time the orbit $(\sigma^*)^k(w)$ remains in $A_\delta$ for the second time it returns to $A_\delta$, and so on...

We suppose from now on that the maximizing probability for $A^*$ has support in a unique periodic orbit of period $p$ denoted by $M = \{ \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_p \} \subset \Sigma$.

We have two possibilities:

a) The times $m_n, n \in \mathbb{N}$, of visits to $A_\delta$, satisfies $2^{-m_n} < \delta$, for infinitely many values of $n$. In this case, the orbit visits $\Omega_\delta$ an infinite number of times, and $I^*(w) = \infty$, and we reach a contradiction.

b) The times $m_n, n \in \mathbb{N}$, are bounded by a constant $N$. We can consider now a new set $A_{\tilde{\delta}}$, which is a smaller neighborhood of $M$, in such way that any point in $A_{\tilde{\delta}}$ has a distance smaller that $2^{-N}$, to a point of $M$.

As,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{A_{\tilde{\delta}}}((\sigma^*)^j(w)) > 0,$$

we reach a contradiction.

We are interested only in the case the support is periodic orbit. The shift is expanding, then by the shadowing property there is an $\epsilon$ such if the corresponding forward orbits of two points are $\epsilon$ close, for all $n$, then the points are the same. From this it follows that the orbit we are considering (which eventually remains indefinitely within $A_\delta$) should be eventually periodic.

Therefore, if $w$ is such that $I^*(w) < \infty$, then, there exists a $k$ such that $((\sigma^*)^k = \tilde{w} \in M$.

From [1], [2] it follows that the support of the maximizing probability for $A$ is a periodic orbit $M_1 = \{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p \} \subset [0, 1]$.

We are going to show in this case that if $R^*$ is good for $A^*$ and the twist condition for $A^*$ is also true, then the subaction $V$ is piecewise analytic.

Remark: The function $I^*$ is lower semi-continuous, that is, if $w_n \rightarrow w$, then $\liminf I(w_n) \geq I(w)$. From this follows that given $K > 0$, if $I(w_n) < K$, and $w_n \rightarrow w$, then $I(w) \leq K$.

We claim that if $R^*$ is good for $A^*$, then given $K > 0$ there exist just a finite number of points $\overline{w}$ with $I(\overline{w}) < K$. This is so, because the times to arrival in the set $M$ are bounded. Indeed, if there was
an infinite number of such \( w_n \) they would accumulate in a point \( w \), such that \( I(w) < K \), but this point cannot reach the set \( M \) by forward iteration in a finite number of steps.

In this way, the above claim, applied to the situation we consider here, says that the set of all possible \( w_x \) is a finite set (all points in in the pre-orbit of the periodic maximizing probability), when we consider all the possible \( x \in [0, 1] \).

**Remark:** We point out that if \( A^* \) depends on a finite number of coordinates and the maximizing probability is unique, then \( R^* \) is good for \( A^* \).

For each \( w \) such that is in the pre-orbit of a point of \( M \), denote by \( k(w) \), the smaller non-negative integer such that \( (\sigma^*)^{k(w)}(w) \in M \). Denote by \( o(w) \), this point in \( M \), such that \( (\sigma^*)^{k(w)}(w) = o(w) \). As we said before, the possible \( k(w) \) are uniformly bounded by a uniform constant \( N \).

**Remark:** We point out that the above property is not necessarily true if we do not assume that \( R^* \) is good for \( A^* \).

The conclusion is that if \( R^* \) is good for \( A^* \), then

\[
V(x) = \sup_{w \in \Sigma, \sigma(w) \in M, \text{for some } 0 \leq j \leq N} (H_\infty(w, x) - I^*(w)).
\]

For such kind of \( w \) we have

\[
I^*(w) = \sum_{n \geq 0} (V^* \circ \sigma - V^* - A^*) \circ \sigma^n(w) = \sum_{n \geq 0} R^*(\sigma^n(w)) = \sum_{n=0}^{k(w)-1} (V^* \circ \sigma - V^* - A^*) \circ \sigma^n(w) = \sum_{n=0}^{k(w)-1} R^*(\sigma^n(w)) = \left[ V^*(o(w)) - V^*(w) \right] - (A^*(w) + A^*(\sigma(w)) + \ldots + A^*(\sigma^{k(w)-1}(w)))).
\]

In this way, for \( w \) satisfying \( \sigma^k(w) = o(w) \in M \) (where \( k \) is the smallest possible) we have that

\[
H_\infty(w, x) - I^*(w) = W(w, x) - V^*(w) - I^*(w) = (W(w, x) - V^*(o(w))) + (A^*(w) + \ldots + A^*(\sigma^{k(w)-1}(w))).
\]

The above expression is the main reason for considering \( W - V^* \) instead of \( H_\infty \).

The \( k \) above could be eventually equal to zero when \( w \in M \). In this particular case \( H_\infty(w, x) - I(w) = W(w, x) - V^*(w) \).

We assume from now on that \( A = \log g \) satisfies the twist condition.

It is known (see [2] [23]) that \( x \to w_x \) is monotonous decreasing.

Indeed, as

\[
V(x) = \sup_{w \in \Sigma} (W(w, x) - V^*(w) - I^*(w)) = W(w_x, x) - V^*(w_x) - I^*(w_x),
\]

then

\[
W(w, x) - V^*(w) - I^*(w) \leq W(w_x, x) - V^*(w_x) - I^*(w_x)
\]

for any \( w \), and we also have that

\[
V(x') = \sup_{w \in \Sigma} (W(w, x') - V^*(w) - I^*(w)) = W(w_{x'}, x') - V^*(w_{x'}) - I^*(w_{x'}).\]

Therefore,

\[
W(w, x') - V^*(w) - I^*(w) \leq W(w_{x'}, x') - V^*(w_{x'}) - I^*(w_{x'}).\]

for any \( w \).

Suppose, \( x < x' \). Substituting \( w_{x'} \) in the first one, and \( w_x \) in the second one we get

\[
\Delta(x, x', w) \leq \Delta(x, x', w_x),
\]

where \( W(x, w) - W(x', w) = \Delta(x, x', w) \). So the twist property implies that \( w_{x'} \leq w_x \).
Theorem 6.1. Consider the transformation $T(x) = 2x$. Suppose $A$ satisfies the twist condition, $R^*$ is good for $A^*$, and the maximizing probability for $A$ has support in a periodic orbit, then the subaction $V$ for $A$ is piecewise analytic.

Proof. Consider a point $x_0 \in [0,1]$ and a variable $x$ in a small interval $(x_0 - \epsilon, x_0)$ on the left of $x_0$. Note that $x \rightarrow w_x$ is monotonous decreasing and can reach just a finite number of values.

Remember that from a previous remark the possible values of optimal $w_x$ are in a finite set.

This shows that $w_x$ is constant for a certain interval $(x_0 - \epsilon, x_0)$, with $\epsilon > 0$. Moreover, the above argument shows that there exist $t > 0$ and a certain finite number of points $z_j$, such that $0 = z_1 < z_2 < z_3 < \ldots < z_t = 1$, $t \in \mathbb{N}$, such that $w_x$ is constant in each interval $(z_j, z_{j+1})$. Furthermore,

$$V(x) = H_\infty(w_x, x) - I(w_x),$$

is analytic, for $x \in (z_j, z_{j+1})$, $j \in \{1, \ldots, t - 1\}$.

It is easy to see from the above that if $A$ is monotonous increasing on $x$, then the maximizing probability is in the fixed point 1 and $V(x)$ is analytic.

Examples in which the hypothesis of the above theorem are true appear in [24].

The proof of the generic this result will be done in Theorem 8.1.

Theorem 6.2. For a fixed transformation $f$ and for a generic analytic potential $g$ (with a unique maximizing periodic probability) the corresponding $R^*$ is good for $A^*$, where $A^*$ is the dual potential of $A = \log g$. Then, in this case, the subaction $V$ for $A = \log g$ is piecewise analytic.

The proof of the generic this result will be done in Theorem 8.1.

The final conclusion is that for a generic $A = \log g$ satisfying the twist condition, if the maximizing probability is supported in a unique periodic orbit, then the corresponding subaction for $A$ is piecewise analytic.

In the next sections we will show the proof of some results we used before.

Now we will provide a counterexample.

Example 2. The following example is due to R. Leplaideur.

We will show an example on the shift where the maximizing probability for a certain Lipschitz potential $A^* : \{0,1\}^\mathbb{N} \rightarrow \mathbb{R}$ is a unique periodic orbit $\gamma$ of period two, denoted by $p_0 = (01010101\ldots), p_1 = (10101010\ldots)$, but for a certain point, namely, $w_0 = (11010101\ldots)$, which satisfies $\sigma(w_0) = p_1$, we have that $R^*(w_0) = 0$.

The potential $A^*$ is given by $A^*(w) = -d(w, \gamma \cup \Gamma)$, where $d$ is the usual distance in the Bernoulli space. The set $\Gamma$ is described later.

For each integer $n$, we define a $2n + 3$-periodic orbit $z_n, \sigma(z_n), \ldots, \sigma^{2n+2}(z_n)$ as follows: we first set

$$b_n = (01010101\ldots01101)^{2n},$$

and the point $z_n$ is the concatenation of the word $b_n$: $z_n = (b_n, b_n, \ldots)$.

The main idea here is to get a sequence of periodic points which spin around the periodic orbit $\{p_0, p_1\}$ during the time $2n$, and then pass close by $w_0$ (note that $d(\sigma^{2n}(z_n), w_0) = 2^{-2(n+1)}$).

Denote $\gamma_n$ the periodic orbit $\gamma_n = \{z_n, \sigma(z_n), \sigma^2(z_n), \ldots, \sigma^{2n+2}(z_n)\}$.

Consider the sequence of Lipschitz potentials $A_n^*(w) = -d(w, \gamma_n \cup \gamma)$. The support of the maximizing probability for $A_n^*$ is $\gamma_n \cup \gamma$. Moreover

$$0 = m(A_n^*) = \max_{\nu \text{ an invariant probability for } \sigma} \int A_n^*(w) \, d\nu(w).$$
Denote by $V^*_n$ a Lipchitz calibrated subaction for $A^*_n$ such that $V^*_n(w_0) = 0$. In this way, for all $w$

$$R^*_n(w) = (V^*_n \circ \sigma - V^*_n - A^*_n)(w) \geq 0,$$

and for $w \in \gamma_n \cup \gamma$ we have that $R^*_n(w) = 0$.

We know that $R^*_n$ is zero on the orbit $\gamma_n$, because $\gamma_n$ is included in the Masur set.

Note that we not necessarily have $R^*_n(w_0) = 0$.

By construction, the Lipchitz constant for $A^*_n$ is 1. This is also true for $V^*_n$. Hence the family of subactions $(V^*_n)$ is a family of equicontinuous functions. Let us denote by $V^*$ any accumulation point for $(V^*_n)$ for the $C^0$-topology. Note that $V^*$ is also 1-Lipschitz continuous. For simplicity we set

$$V^* = \lim_{k \to \infty} V^*_{n_k}.$$

We denote by $\Gamma$ the set which is the limit of the sets $\gamma_n$ (using the Hausdorff distance). $\gamma \cup \Gamma$ is a compact set. Note that $\Gamma$ is not a compact set, but the set of accumulation points for $\Gamma$ is the set $\gamma$. We now consider $A^*(w) = -d(w, \gamma \cup \Gamma)$.

As any accumulation point of $\Gamma$ is in $\gamma$, any maximizing probability for the potential $A^*$ has support in $\gamma$. On the contrary, the unique $\sigma$-invariant measure with support in $\gamma$ is maximizing for $A^*$.

Remember that for any $n$ we have $V^*_n(w_0) = 0$. We also claim that we have $A^*_{n_k}(w_0) \to 0$ and $V^*_n(\sigma(w_0)) \to 0$, as $k \to \infty$.

For each fixed $w$ we set

$$R^*_n(w) = (V^*_n \circ \sigma - V^*_n - A^*_n)(w) \geq 0.$$ 

The right hand side terms converge (for the $C^0$-topology) as $k$ goes to $+\infty$. Then $R^*_n$ converge, and we denote by $R^*$ its limit. Then for every $w$ we have:

$$R^*(w) = (V^* \circ \sigma - V^* - A^*)(w) \geq 0.$$ 

This shows that $V^*$ is a subaction for $A^*$. Note that $R^*(w_0) = 0$. From the uniqueness of the maximizing probability for $A^*$ we know that there exists a unique calibrated subaction for $A^*$ (up to an additive constant).

Consider a fixed $w$ and its two preimages $w_a$ and $w_b$. For any given $n$, one of the two possibilities occur $R^*_n(w_a) = 0$ or $R^*_n(w_b) = 0$, because $V^*_n$ is calibrated for $A^*_n$.

Therefore, for an infinite number of values $k$ either $R^*_n(w_a) = 0$ or $R^*_n(w_b) = 0$.

In this way the limit of $V^*_n$ is unique (independent of the convergent subsequence) and equal to $V^*$, the calibrated subaction for $A^*$ (such that $V^*(w_0) = 0$).

Therefore,

$$R^*(w_0) = (V^* \circ \sigma - V^* - A^*)(w_0) = 0,$$

and $V^*$ is a calibrated subaction for $A^*(w) = d(w, \gamma \cup \Gamma)$.

7. Generic continuity of the Aubry set.

In sections 7 and 8 we will present the proof of the generic properties we mention before.

We will present our main results in great generality. First, in this section, we analyze the main properties of sub-actions and its dependence on the potential $\Lambda$.

We refer the reader to [23] for related results in (eventually) different settings.

First we will present the main definitions we will consider here.

We denote by $\mathbb{K}$ a compact metric space and $T : \mathbb{K} \to \mathbb{K}$ continuous expandig map (that is, there exists $\epsilon > 0$, and $1 > \lambda > 0$, such that, if $d(x, y) < \epsilon$, then $d(T(x), T(y)) > 1/\lambda$) such that sup$_{x \in \mathbb{K}} T^{-1} \{x\} < \infty$ (that is, each point has a finite number of pre-images by $T$). $\lambda$ is called the contraction constant for $T$.

Typical examples are:
1) the shift transformation acting on the Bernoulli space $\{1, 2, \ldots, d\}^\mathbb{N}$,

and
2) $T : S^1 \to S^1$, of class $C^{1+\alpha}$, such that there exists $\lambda > 1$, such that, $|T'(x)| > \lambda$, for all $x \in S^1$. 
A function $g(x)$, defined locally on an open set $Z$, such that $T(g(z)) = z, \forall z \in Z$, is called an inverse branch. Most of the time we consider maximal inverse branches.

The constant $\lambda$ gives an upper bound for the rate of contraction in each inverse branch.

$\mathcal{F} \subset C^0(\mathbb{K}, \mathbb{R})$ denotes a complete metric space with a (topology finer than) metric larger than $d_{C^0}(f, g) = \|f - g\|_0 := \sup_{x \in \mathbb{K}} |f(x) - g(x)|$; (for instance, Hölder functions, Lispchitz functions or $C^k(\mathbb{K}, \mathbb{R})$) AND such that

$$\forall K \subset \mathbb{K} \text{ compact}, \exists \psi \in \mathcal{F} \text{ s.t. } \psi \leq 0, \ {\psi} = 0 = \{x \mid \psi(x) = 0\} = K.$$ (30)

Property (30) does not hold for real analytic functions on $[0, 1]$ unless $K$ is a finite set, like, for instance, a periodic orbit.

We will assume in the rest of the paper at least that $F \in C^\alpha(\mathbb{K}, \mathbb{R})$, unless we explicit something else less.

When the Aubry set (see definition bellow) is one periodic orbit, the arguments below should apply for $\mathcal{F} = C^\alpha([0, 1], \mathbb{R})$ (real analytic functions) with the $C^0$ topology. This will be enough for the purpose of our main result on piecewise analytic subactions which was stated before.

Given $A \in \mathcal{F}$, a calibrated sub-action for $A$ is $F : \mathbb{K} \to \mathbb{R}$ continuous such that

$$F(x) = \max_{y \in F^{-1}(x)} [F(y) + A(y) - m_A] \quad \forall x \in \mathbb{K}$$

where

$$m_A := \sup_{\mu \in M(T)} \int A \, d\mu, \quad M(T) := \text{Borel } T\text{-invariant probabilities;}$$

its error is denoted by $R = R_A : \mathbb{K} \to [0, +\infty]$: $R(x) := F(T(x)) - F(x) - A(x) + m_A \geq 0$.

$S(A)$ denotes the set of Hölder calibrated sub-actions (it is not empty $\mathcal{F}$)

The expression

$$F(T(x)) - F(x) + m_A \geq A(x), \quad \forall x \in K,$$

is the discrete time version of the sub-solution of Hamilton-Jacobi equations $\mathcal{F}$. The next definitions were also consider and explored in these references. We denote by

The Mañé action potential

$$\mathcal{S}_A(x, y) := \lim_{\varepsilon \to 0} \left[ \sup \left\{ \sum_{i=0}^{n-1} \left[ A(T^i(z)) - m_A \right] \mid n \in \mathbb{N}, T^n(z) = y, \, d(z, x) < \varepsilon \right\} \right].$$

Given $x$ and $y$ the above value describe the $A$-cost of going from $x$ to $y$ following the dynamics.

The Aubry set is $\mathcal{A}(A) := \{x \in \mathbb{K} \mid \mathcal{S}_A(x, x) = 0\}$.

The terminology is borrowed from the Aubry-Mather Theory $\mathcal{F}$.

For any $x \in \mathcal{A}(A)$, we have that $\mathcal{S}_A(x, .)$ is a sub-action (in particular, in this case, $\mathcal{S}_A(x, y) > -\infty$, for any $y$), see Proposition 23 in $\mathcal{F}$.

The set of maximizing measures is

$$\mathcal{M}(A) := \{ \mu \in M(T) \mid \int A \, d\mu = m_A \}.$$

If $F \in C^\alpha(\mathbb{K}, \mathbb{R})$ is a Hélder function define

$$\|F\|_\alpha := \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x, y)^\alpha}.$$ Define the Mañé set as

$$\mathcal{N}(A) := \bigcup_{F \in S(A)} I_F^{-1}\{0\},$$

where the union is among all the $\alpha$-Hölder callibrated sub-actions $F$ for $A$ and

$$I_F(x) = \sum_{i=0}^{\infty} R_F(T^i(x)).$$
Lemma 7.1. \( I_F(x) \) is the deviation function we considered before. For \( A \in \mathcal{F} \) define the Mather set as

\[
M(A) := \bigcup_{\mu \in M(A)} \text{supp}(\mu).
\]

The Peierls barrier is

\[
h_A(x, y) := \lim_{\varepsilon \to 0} \limsup_{k \to +\infty} S_A(x, y, k, \varepsilon),
\]

where \( S_A(x, y, k, \varepsilon) := \sup \left\{ \sum_{i=0}^{n-1} [A(T^i(x)) - m_A] \mid n \geq k, T^n(z) = y, d(z, x) < \varepsilon \right\} \).

Several properties of the Mañé potential and the Peierls barrier are similar (but not all, see section 4 in [16]). We will present proofs for one of them and the other case is similar.

**Lemma 7.1.**

1. If \( \mu \) is a minimizing measure then

\[
\text{supp}(\mu) \subset A(A) = \{ x \in \mathcal{K} \mid S_A(x, x) = 0 \}.
\]

2. \( S_A(x, x) \leq 0 \) for every \( x \in \mathcal{K} \).

3. For any \( z \in \mathcal{K} \) the function \( F(y) = h_A(z, y) \) is Hölder continuous.

4. If \( a \in A(A) \) then \( h_A(a, x) = S_A(a, x) \) for all \( x \in \mathcal{K} \). In particular, \( F(y) = S_A(a, x) \) is continuous if \( a \in A(A) \).

5. If \( S_A(w, y) = h_A(w, y) \) then the function \( F(y) = S_A(w, y) \) is continuous at \( y \).

6. If \( S(x_0, T^{n_k}(x_0)) = \sum_{j=0}^{n_k-1} A(T^j(x_0)), \lim_k T^{n_k}(x_0) = b \) and \( \lim_k n_k = +\infty \), then

\[
\lim_k S(x_0, T^{n_k}(x_0)) = S(x_0, b).
\]

Item (1) follows from Atkinson-Mañé’s lemma which says that if \( \mu \) is ergodic for \( \mu \)-almost every \( x \) and every \( \varepsilon > 0 \), the set

\[
N(x, \varepsilon) := \left\{ n \in \mathbb{N} \mid \left| \sum_{j=0}^{n-1} A(T^j(x)) - n \int A \, d\mu \right| \leq \varepsilon \right\}
\]

is infinite (see Lemma 2.2 [25] (which consider non-invertible transformation, [8], [9] or [15] for the proof). We will show bellow just the items which are not proved in the mentioned references.

The problem with the discontinuity of \( F(y) = S_A(w, y) \) is when the maximum is obtained at a finite orbit segment (i.e. when \( S_A(w, y) \neq h_A(w, y) \)), the hypothesis in item (3).

**Proof.**

By adding a constant we can assume that \( m_A = 0 \).

Let \( F \) be a continuous sub-action for \( A \). Then

\[-R_F = A + F - F \circ T \leq 0.\]

Given \( x_0 \in \mathcal{K} \), let \( x_k \in \mathcal{K} \) and \( n_k \in \mathbb{N} \) be such that \( T^{n_k}(x_k) = x_0 \), \( \lim_k x_k = x_0 \) and

\[
S(x_0, x_0) = \lim_k \sum_{j=0}^{n_k-1} A(T^j(x_k)).
\]

We have

\[
\sum_{j=0}^{n_k-1} A(T^j(x_k)) = \left[ \sum_{j=0}^{n_k-1} (A + F - F \circ T)(T^j(x_k)) \right] + F(x_0) - F(x_k)
\]

\[
\leq F(x_0) - F(x_k).
\]
Then
\[ S(x_0, x_0) = \lim_k \sum_{j=0}^{n_k-1} A(T^j(x_k)) \leq \lim_k [F(x_0) - F(x_k)] = 0. \]

The proofs of (3) (4) (5) can be found in [9] [15].

Let \( \sigma_k \) be the branch of the inverse of \( T^{n_k} \) such that \( \sigma_k(T^{n_k}(x_0)) = x_0 \). Let \( b_k = \sigma_k(b) \) for \( k \) sufficiently large. Then
\[
d(x_0, b_k) \leq \lambda^{n_k} d(T^{n_k}(x_0), b) \xrightarrow{k \to \infty} 0.
\]

Write \( Q := \frac{\|A\|_a}{1 - \lambda^a} \), then
\[
S(x_0, b) \geq \limsup_k \sum_{i=0}^{n_k-1} A(T^i(b_k))
\]
\[
\geq \limsup_k S(x_0, T^{n_k}(x_0)) - Q d(T^{n_k}(x_0), b)^a
\]
\[
\geq \limsup_k S(x_0, T^{n_k}(x_0)).
\]

Now for \( \ell \in \mathbb{N} \) let \( b_\ell \in \mathbb{K} \) and \( m_\ell \in \mathbb{N} \) be such that \( \lim b_\ell = x_0, T^{m_\ell}(b_\ell) = b \)
and
\[
\lim_\ell \sum_{j=0}^{m_\ell-1} A(T^j(b_\ell)) = S(x_0, b).
\]

Let \( \overline{\sigma}_\ell \) be the branch of the inverse of \( T^{m_\ell} \) such that \( \overline{\sigma}_\ell(b) = b_\ell \). Let \( x_\ell := \overline{\sigma}_\ell(T^{n_k}(x_0)) \). Then
\[
d(x_\ell, x_0) \leq d(x_0, b_\ell) + d(b_\ell, x_\ell)
\]
\[
\leq d(x_0, b_\ell) + \lambda^\ell d(T^{n_k}(x_0), b) \xrightarrow{\ell \to \infty} 0.
\]

Since \( x_\ell \to x_0 \) and \( T^{m_\ell}(x_\ell) = T^{n_k}(x_0) \), we have that
\[
S(x_0, T^{n_k}(x_0)) \geq \limsup_\ell \sum_{j=0}^{m_\ell-1} A(T^j(x_\ell))
\]
\[
\geq \limsup_\ell \sum_{j=0}^{m_\ell-1} A(T^j(b_\ell)) - Q d(T^{n_k}(x_0), b)^a
\]
\[
\geq S(x_0, b) - Q d(T^{n_k}(x_0), b)^a.
\]

And hence
\[
\liminf_k S(x_0, T^{n_k}(x_0)) \geq S(x_0, b).
\]

\[ \square \]

**Proposition 7.1.** The Aubry set is
\[ \mathcal{A}(A) = \bigcap_{F \in S(A)} I^{-1}_F \{0\}, \]

where the intersection is among all the \( \alpha \)-Hölder calibrated sub-actions for \( A \).
Proof.
By adding a constant we can assume that \( m_\mathcal{A} = 0 \).
We first prove that \( \mathcal{A}(A) \subseteq \bigcap_{F \in \mathcal{S}(A)} I_F^{-1}\{0\} \).

Let \( F \in \mathcal{S}(A) \) be a Hölder sub-action and \( x_0 \in \mathcal{A}(A) \). Since \( \mathcal{S}_A(x_0, x_0) = 0 \) then there is \( x_k \rightarrow x_0 \) and \( n_k \uparrow \infty \) such that \( \lim_k T^{n_k}(x_k) = x_0 \) and \( \lim_k \sum_{j=0}^{n_k-1} A(T^j(x_k)) = 0 \). If \( m \in \mathbb{N} \) we have that

\[
F(T^{m+1}(x_0)) \geq F(T^m(x_0)) + A(T^m(x_0))
\]

\[
\geq F(T^{m+1}(x_k)) + \sum_{j=m+1}^{n_k+m-1} A(T^j(x_k)) + A(T^m(x_0))
\]

\[
\geq F(T^{m+1}(x_k)) + \sum_{j=0}^{n_k-1} A(T^j(x_k)) - \sum_{j=0}^{m} |A(T^j(x_k)) - A(T^j(x_0))|
\]

(31)

When \( k \rightarrow \infty \) the right hand side of (31) converges to \( F(T^{m+1}(x_0)) \), and hence all those inequalities are equalities. Therefore \( R_F(T^m(x_0)) = 0 \) for all \( m \) and hence \( I_F(x_0) = 0 \).

Now let \( x_0 \in \bigcap_{F \in \mathcal{S}(A)} I_F^{-1}\{0\} \). Since \( \mathcal{K} \) is compact there is \( n_k \rightarrow +\infty \) such that the limits \( b = \lim_k T^{n_k}(x_0) \in \mathcal{K} \) and \( \mu = \lim_k \mu_k \in \mathcal{M}(T) \). \( \mu_k := \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{T^i(x_0)} \) exist and \( b \subseteq \text{supp}(\mu) \). Let \( G \) be a Hölder calibrated sub-action. For \( m \geq n \) we have

\[
G(T^n(x_0)) + S_A(T^n(x_0), T^m(x_0)) \geq G(T^n(x_0)) + \sum_{j=n}^{m-1} A(T^j(x_0))
\]

\[
= G(T^n(x_0)) \quad \text{[because } I_G(x_0) = 0 \text{]}
\]

\[
\geq G(T^n(x_0)) + S_A(T^n(x_0), T^m(x_0)).
\]

Then they are all equalities and hence for any \( m \geq n \)

\[
S_A(T^n(x_0), T^m(x_0)) = \sum_{j=n}^{m-1} A(T^j(x_0)).
\]

Since

\[
0 = \lim_k \frac{1}{n_k} S_A(T^n(x_0), T^m(x_0)) = \lim_k \frac{1}{n_k} \sum_{j=n}^{m-1} A(T^j(x_0)) = \int A \, d\mu,
\]

\( \mu \) is a minimizing measure. By lemma \( \square \) \( b \in \mathcal{A}(A) \).

Let \( F : \mathcal{K} \to \mathbb{R} \) be \( F(x) := S_A(b, x) \). Then \( F \) is a Hölder calibrated sub-action. By hypothesis \( I_F(x_0) = 0 \) and then

\[
F(T^n(x_0)) = F(x_0) + S_A(x_0, T^{n_k}(x_0)).
\]

\[
S_A(b, T^{n_k}(x_0)) = S_A(b, x_0) + S_A(x_0, T^{n_k}(x_0)).
\]

By lemma \( \square \) and Lemma \( \square \), taking the limit on \( k \) we have that

\[
0 = S_A(b, b) = S_A(b, x_0) + S_A(x_0, b) = 0.
\]

\[
0 \leq S_A(x_0, x_0) \leq S_A(x_0, b) + S_A(b, x_0) = 0.
\]

Therefore \( x_0 \in \mathcal{A}(A) \). \( \square \)

We want to show the following result which will require several preliminary results.

**Theorem 7.1.** The set

\[
\mathcal{R} := \{ A \in \mathcal{C}^0(\mathcal{K}, \mathbb{R}) \mid \mathcal{M}(A) = \{ \mu \}, \mathcal{A}(A) = \text{supp}(\mu) \}
\]

contains a residual set in \( \mathcal{C}^0(\mathcal{K}, \mathbb{R}) \).

The proof of the below lemma (Atkinson-Mañé) can be found in [25] and [8].
Lemma 7.2. Let \((X, \mathcal{B}, \nu)\) be a probability space, \(f\) an ergodic measure preserving map and \(F : X \to \mathbb{R}\) an integrable function. Given \(A \in \mathcal{B}\) with \(\nu(A) > 0\) denote by \(\hat{A}\) the set of points \(p \in A\) such that for all \(\varepsilon > 0\) there exists an integer \(N > 0\) such that \(f^{N}(p) \in A\) and
\[
\left| \sum_{j=0}^{N-1} F(f^{j}(p)) - N \int F d\nu \right| < \varepsilon.
\]
Then \(\nu(\hat{A}) = \nu(A)\).

Corollary 7.3. If besides the hypothesis of lemma 7.2 \(X\) is a complete separable metric space, and \(\mathcal{B}\) is its Borel \(\sigma\)-algebra, then for a.e. \(x \in X\) the following property holds: for all \(\varepsilon > 0\) there exists \(N > 0\) such that \(d(f^{N}(x), x) < \varepsilon\) and
\[
\left| \sum_{j=0}^{N-1} F(f^{j}(x)) - N \int F d\nu \right| < \varepsilon.
\]
Proof. Given \(\varepsilon > 0\) let \(\{V_{n}(\varepsilon)\}\) be a countable basis of neighbourhoods with diameter \(\varepsilon\) and let \(\hat{V}_{n}\) be associated to \(V_{n}\) as in lemma 7.2. Then the full measure subset \(\bigcap_{m} \bigcup_{n} \hat{V}_{n}(\frac{1}{m})\) satisfies the required property. \(\square\)

Lemma 7.4. Let \(\mathcal{R}\) be as in Theorem 7.1. Then if \(A \in \mathcal{R},\ F \in \mathcal{S}(A)\) we have
1. If \(a, b \in \mathcal{A}(A)\) then \(S_{A}(a, b) + S_{A}(b, a) = 0\).
2. If \(a \in \mathcal{A}(A) = \text{supp}(\mu)\) then \(F(x) = F(a) + S_{A}(a, x)\) for all \(x \in \mathcal{K}\).

Proof. \(\Box\). Let \(a, b \in \mathcal{A}(A) = \text{supp}(\mu)\). Since \(\mu\) is ergodic, by Corollary 7.3 there are sequences \(\alpha_{k} \in \mathbb{K}, m_{k} \in \mathbb{N}\) such that \(\lim_{k} m_{k} = \infty, \lim_{k} \alpha_{k} = a, \lim_{k} d(T^{m_{k}}(\alpha_{k}), \alpha_{k}) = 0\),
\[
\sum_{j=0}^{m_{k}-1} A(T^{j}(\alpha_{k})) \geq \frac{1}{k}, \quad \text{and writing} \quad \mu_{k} := \frac{1}{m_{k}} \sum_{j=1}^{m_{k}-1} \delta_{T^{m_{k}}(\alpha_{k})}, \quad \lim_{k} \mu_{k} = \mu.
\]
Since \(b \in \text{supp}(\mu)\) there are \(n_{k} \leq m_{k}\) such that \(\lim_{k} T^{m_{k}}(\alpha_{k}) = b\).
Let \(\sigma_{k}\) be the branch of the inverse of \(T^{m_{k}}\) such that \(\sigma_{k}(T^{m_{k}}(\alpha_{k})) = \alpha_{k}\). Let \(b_{k} := \sigma_{k}(b)\). Then \(T^{n_{k}}(b_{k}) = b\) and
\[
d(b_{k}, a) \leq d(b_{k}, \alpha_{k}) + d(\alpha_{k}, a) \\
\leq \lambda^{n_{k}} d(b, T^{m_{k}}(\alpha_{k})) + d(\alpha_{k}, a) \\
\leq d(b, T^{n_{k}}(\alpha_{k})) + d(\alpha_{k}, a) \xrightarrow{k} 0.
\]
We have that
\[
\left| \sum_{j=0}^{n_{k}-1} A(T^{j}(b_{k})) - \sum_{j=0}^{n_{k}-1} A(T^{j}(\alpha_{k})) \right| \leq \frac{\|A\|_{\alpha}}{1 - \lambda^{\alpha}} d(T^{n_{k}}(\alpha_{k}), b)^{\alpha}.
\]
\[
S(a, b) \geq \limsup_{k} \sum_{j=0}^{n_{k}-1} A(T^{j}(b_{k})) \\
\geq \limsup_{k} \sum_{j=0}^{n_{k}-1} A(T^{j}(\alpha_{k})) - Q d(T^{n_{k}}(\alpha_{k}), b)^{\alpha}.
\]

Let \( \tau_k \) be the branch of the inverse of \( T^{m_k-n_k} \) such that \( \tau_k(T^{m_k}(\alpha_k)) = T^{n_k}(\alpha_k) \). Let \( a_k := \tau_k(a) \) Then \( T^{m_k-n_k}(a_k) = a \) and

\[
d(b, a_k) \leq d(b, T^{m_k}(\alpha_k)) + d(T^{n_k}(\alpha_k), a_k) \\
\leq d(b, T^{n_k}(\alpha_k)) + \lambda^{m_k-n_k} d(T^{m_k}(a_k), a) \\
\leq d(b, T^{n_k}(\alpha_k)) + d(T^{m_k}(a_k), a) \rightarrow 0.
\]

Also

\[
\left| \sum_{j=0}^{m_k-n_k-1} A(T^j(a_k)) - \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) \right| \leq \frac{\|A\|_\alpha}{1-\lambda} d(a, T^{m_k}(\alpha_k)).
\]

\[
S(a, b) \geq \limsup_k \sum_{j=0}^{m_k-n_k-1} A(T^j(a_k)) \\
\geq \limsup_k \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) - Q d(a, T^{n_k}(\alpha_k))
\]

Therefore

\[
0 \geq S(a, a) \geq S(a, b) + S(b, a) \\
\geq \limsup_k \sum_{j=0}^{n_k-1} A(T^j(\alpha_k)) + \limsup_k \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) \\
\geq \limsup_k \left[ \sum_{j=0}^{n_k-1} A(T^j(\alpha_k)) + \sum_{j=n_k}^{m_k-1} A(T^j(\alpha_k)) \right] \\
\geq \limsup_k \frac{1}{k} \\
\geq 0.
\]

(2). We first prove that if for some \( x_0 \in \mathbb{K} \) and \( a \in \mathbb{A}(A) \) we have

\[(33) \quad F(x_0) = F(a) + S_A(a, x_0),\]

then equation \((33)\) holds for every \( a \in \mathbb{A}(A) \). If \( b \in \mathbb{A}(A) \), using item \( \text{H} \) we have that

\[
F(x_0) \geq F(b) + S(b, x_0) \\
\geq F(a) + S_A(a, b) + S_A(b, x_0) \\
\geq F(a) + S_A(a, b) + S_A(b, a) + S_A(a, x_0) \\
= F(a) + S_A(a, x_0) \\
= F(x_0).
\]

Therefore \( F(x_0) = F(b) + S(b, x_0) \).

It is enough to prove that given any \( x_0 \in \mathbb{K} \) there is \( a \in \mathbb{A}(A) \) such that the equality \((33)\) holds. Since \( F \) is calibrated there are \( x_k \in \mathbb{K} \) and \( n_k \in \mathbb{N} \) such that \( T^{n_k}(x_k) = x_0 \), \( \exists \lim_k x_k = a \) and for every \( k \in \mathbb{N} \),

\[
F(x_0) = F(x_k) + \sum_{j=0}^{n_k-1} A(T^j(x_k)).
\]
We have that
\[ S_A(a, x_0) \geq \limsup_k \sum_{j=0}^{n_k-1} A(T^j(x_k)) \]
\[ = \limsup_k F(x_0) - F(x_k) \]
\[ = F(x_0) - F(a) \]
\[ \geq S(a, x_0). \]

Therefore equality (33) holds.

It remains to prove that \( a \in A(A) \), i.e. that \( S_A(a, a) = 0 \). We can assume that the sequence \( n_k \) is increasing. Let \( m_k = n_{k+1} - n_k \). Then \( T^{m_k}(x_{k+1}) = x_k \). Let \( \sigma_k \) be the branch of the inverse of \( T^{m_k} \) such that \( \sigma_k(x_k) = x_{k+1} \) and \( a_{k+1} = \sigma_k(a) \). We have that
\[ \left| \sum_{j=0}^{m_k-1} A(T^j(a_{k+1})) - \sum_{j=0}^{m_k-1} A(T^j(x_{k+1})) \right| \leq \frac{\|A\|}{1 - \lambda^\alpha} d(a, x_k)^\alpha. \]

Since \( x_k \to a \) we have that
\[ d(a_{k+1}, a) \leq d(a_{k+1}, x_{k+1}) + d(x_{k+1}, a) \]
\[ \leq \lambda^{m_k} d(x_k, a) + d(x_{k+1}, a) \]
\[ \leq d(x_k, a) + d(x_{k+1}, a) \xrightarrow{k \to \infty} 0. \]

Therefore
\[ 0 \geq S_A(a, a) \geq \limsup_k \sum_{j=0}^{m_k-1} A(T^j(a_{k+1})) \]
\[ \geq \limsup_k \sum_{j=0}^{m_k-1} A(T^j(x_{k+1})) - Q d(a, x_k)^\alpha \]
\[ = \limsup_k F(x_k) - F(x_{k+1}) - Q d(a, x_k)^\alpha \]
\[ = 0. \]

The above result (2) is true for \( F \) only continuous.

**Corollary 7.5.** Let \( R \) be as in Theorem 7.1. Then if \( A \in R \), \( F \in S(A) \) we have

1. If \( x \notin \mathcal{A}(A) \) then \( I_F(x) > 0 \).
2. If \( x \notin \mathcal{A}(A) \) and \( T(x) \in \mathcal{A}(A) \) then \( R_F(x) > 0 \).

**Proof.**

1. By lemma 7.4, modulo adding a constant there is only one Hölder calibrated sub-action \( F \) in \( S(A) \). Then by proposition 7.1 \( \mathcal{A}(A) = [I_F = 0] \). Since \( I_F \geq 0 \), this proves item 1.

2. Since \( T(x) \in \mathcal{A}(A) \)
\[ I_F(x) = \sum_{n \geq 1} R_F(T^n(x)) = 0. \]

Since \( x \notin \mathcal{A}(A) \), by item 2 and proposition 7.1 \( \mathcal{A}(A) = [I_F = 0] \). Then
\[ I_F(x) = \sum_{n \geq 0} R_F(T^n(x)) > 0. \]

Hence \( R_F(x) > 0 \).

□
Lemma 7.6.

(1) \( A \mapsto m_A \) has Lipschitz constant 1.
(2) Fix \( x_0 \in X \). The set \( S(A) \) of \( \alpha \)-Hölder calibrated sub-actions \( F \) for \( A \) with \( F(x_0) = 0 \) is an equicontinuous family. In fact

\[
\sup_{F \in S(A)} \| F \|_{\alpha} < \infty.
\]

(3) The set \( S(A) \) of \( \alpha \)-Hölder continuous calibrated sub-actions is closed under the \( C^0 \) topology.
(4) If \( \# \mathcal{M}(A) = 1 \), \( A_n \xrightarrow{\mathcal{N}} A \) uniformly, \( \sup_n \| A_n \|_{\alpha} < \infty \) and \( F_n \in S(A) \), then \( \lim_n F_n = F \) uniformly.
(5) \( A \leq B \) & \( m_A = m_B \implies S_A \leq S_B \).
(6) \( \limsup_{B \to A} \mathbb{N}(B) \subseteq \mathbb{N}(A) \), where

\[
\limsup_{B \to A} \mathbb{N}(B) = \{ \lim_{n} x_n \mid x_n \in \mathbb{N}(B_n), \ B_n \xrightarrow{\mathcal{N}} A, \ \exists \lim x_n \}
\]

(7) If \( A \in \mathcal{R} \) then

\[
\lim_{B \to A} d_H(\mathcal{H}(B), \mathcal{H}(A)) = 0,
\]

where \( d_H \) is the Hausdorff distance.
(8) If \( A \in \mathcal{R} \) with \( \mathcal{M}(A) = \{ \mu \} \) and \( \nu_B \in \mathcal{M}(B) \) then

\[
\lim_{B \to A} d_H(\text{supp}(\nu_B), \text{supp}(\mu)) = 0.
\]
(9) If \( A \in \mathcal{R} \) then

\[
\lim_{B \to A} d_H(\mathcal{M}(B), \mathcal{M}(A)) = 0.
\]

If \( X, Y \) are two metric spaces and \( F : X \to 2^Y = \mathcal{P}(Y) \) is a set valued function, define

\[
\limsup_{x \to x_0} F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\delta > 0} \bigcup_{d(x, x_0) < \delta} V_{\varepsilon}(F(x)),
\]

\[
\liminf_{x \to x_0} F(x) = \bigcup_{\varepsilon > 0} \bigcup_{\delta < \varepsilon} \bigcap_{d(x, x_0) < \delta} V_{\varepsilon}(F(x)),
\]

where

\[
V_{\varepsilon}(C) = \bigcup_{y \in C} \{ z \in Y \mid d(z, y) < \varepsilon \}.
\]

Proof.

(1). We have that \( A \leq B + \| A - B \|_0 \), then

\[
\int A \, d\mu \leq \int B \, d\mu + \| A - B \|_0, \quad \forall \mu \in \mathcal{M}(T),
\]

\[
\int A \, d\mu \leq \sup_{\mu \in \mathcal{M}(T)} \int B \, d\mu + \| A - B \|_0 = m_B + \| A - B \|_0,
\]

\[
m_A \leq m_B + \| A - B \|_0.
\]

Similarly \( m_B \leq m_A + \| A - B \|_0 \) and then \( |m_A - m_B| \leq \| A - B \|_0 \).

See also [20] and [9] for a proof.

(2). Let \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) be such that for any \( x \in X \) there is an inverse branch \( \tau \) of \( T \) which is defined on the ball \( B(T(x), \varepsilon) := \{ z \in X \mid d(z, T(x)) < \varepsilon \} \), has Lipschitz constant \( \lambda \) and \( \tau(T(x)) = x \).

Let \( F \in S(A) \). Let

\[
K := \| F \|_{\alpha} := \sup_{d(x, y) < \varepsilon} \frac{|F(x) - F(y)|}{d(x, y)^{\alpha}}, \quad a := \| A \|_{\alpha} := \sup_{d(x, y) < \varepsilon} \frac{|A(x) - A(y)|}{d(x, y)^{\alpha}}.
\]
be Hölder constants for $F$ and $A$. Given $x, y \in \mathbb{K}$ with $d(x, y) < \varepsilon$ let $\tau_i, i = 1, \ldots, m(x) \leq M$ be the inverse branches for $T$ about $x$ and let $x_i = \tau_i(x), y_i = \tau_i(y)$. We have that

$$|F(x_i) - F(y_i)| \leq (K \alpha d(x, y) + (K + a) \lambda d(x, y)\alpha,$$

$$F(x_i) + A(x_i) \leq F(y_i) + A(y_i) + (K + a) \lambda d(x, y)\alpha,$$

$$\max_i [F(x_i) + A(x_i) - m_A] \leq \max_i [F(y_i) + A(y_i) - m_A] + (K + a) \lambda d(x, y)\alpha,$$

$$F(x) \leq F(y) + (K + a) \lambda d(x, y)\alpha,$$

Then $\|F\|_\alpha \leq \lambda \alpha (\|F\|_\alpha + \|A\|_\alpha)$ and hence

$$\|F\|_\alpha \leq \frac{\lambda \alpha}{(1 - \lambda \alpha)} \|A\|_\alpha.$$  

This implies the equicontinuity of $\mathcal{S}(A)$.

The proof of the above result could be also get if we just assume that $F$ is continuous.

It is easy to see that uniform limit of calibrated sub-actions is a sub-action, and it is calibrated because the number of inverse branches of $T$ is finite, i.e. sup$_{y \in \mathbb{K}} \# T^{-1}\{y\} < \infty$. By (2) all $C^0$ calibrated sub-actions have a common Hölder constant, the uniform limits of them have the same Hölder constant. Also the family $\{F_n\}$ satisfies $F_n(x_0) = 0$ and by inequality (34)

$$\|F_n\|_\alpha < \frac{\lambda \alpha}{(1 - \lambda \alpha)} \sup_n \|A_n\|_\alpha \leq \infty.$$  

Hence $\{F_n\}$ is equicontinuous. By Arzelá-Ascoli theorem it is enough to prove that there is a unique $F(x) = S_A(x_0, x)$ which is the limit of any convergent subsequence of $\{F_n\}$. Since $\sup \|A_n\|_\alpha < \infty$, by inequality (34), any such limit is $\alpha$-Hölder. Since by lemma (7.4)[2], $\mathcal{S}(A) \cap \{F(x_0) = 0\} = \{F(x) = S_A(x_0, x)\}$, it is enough to prove that any limit of a subsequence of $\{F_n\}$ is a calibrated sub-action. But this follows form the continuity of $A \mapsto m_A$, the equality

$$F_n(x) = \max_{T(y) = x} F_n(y) + A_n(y) - m_A,$$

and the fact $\sup_{x \in \mathbb{K}} \#(T^{-1}\{x\}) < \infty$.

The proof follows from the expression

$$S_A(x, y) := \lim_{\varepsilon \to 0} \left[ \sup_{\varepsilon > 0} \left\{ \sum_{i=0}^{n-1} \left[ A(T^i(z)) - m_A \right] \left| n \in \mathbb{N}, T^n(z) = y, d(z, x) < \varepsilon \right. \right\} \right].$$

Let $x_n \in B_n \to A$ be such that $x_n \to x_0$. Let $F_n \in \mathcal{S}(A)$ be such that $I_{F_n}(x_n) = 0$. Adding a constant we can assume that $F_n(x_0) = 0$ for all $n$. By (2), taking a subsequence we can assume that $\exists F = \lim_n F_n$ in the $C^0$ topology. Then $F$ is a $C^0$ calibrated sub-action for $A$. Also $R_{F_n} \to R_F$ uniformly and there is a common Hölder constant $C$ for all the $R_{F_n}$. We have that

$$|R_{F_n}(T^k(x_n)) - R_F(T^k(x_0))| \leq |R_{F_n}(T^k(x_n)) - R_{F_n}(T^k(x_0))| + |R_{F_n}(T^k(x_0)) - R_F(T^k(x_0))| \leq C d(T^k(x_n), T^k(x_0)) \leq \frac{\lambda \alpha}{(1 - \lambda \alpha)} \|F_n - F\|_\alpha \to 0$$  

Since for all $n, k, F_n(T^k(x_n)) = 0$, we have that $R_F(T^k(x_0)) = 0$ for any $k$. Hence $I_F(x_0) = 0$ and then $x_0 \in \mathcal{N}(A)$.

By Lemma (7.4), there is only one calibrated sub-action modulo adding a constant. Then by Proposition (7.4), $A(A) = \mathcal{N}(A)$. Then by (8) $\limsup B \to A A(B) \subset A(A)$. It is enough to prove that for any $x_0 \in A(A)$ and $B_n \to A$, there is $x_n \in A(B_n)$ such that $\lim_n x_n = x_0$. Let $\mu_n \in \mathcal{M}(B_n)$. Then $\lim_n \mu_n = \mu$ in the weak* topology. Given $x_0 \in A(A) = \supp(\mu)$ we have that

$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) > 0 \quad \forall n \geq N : \quad \mu_n(B(x_0, \varepsilon)) > 0.$$
We can assume that for all $m \in \mathbb{N}$, $N(\frac{1}{m}) < N(\frac{1}{m+1})$. For $N(\frac{1}{m}) \leq n \leq N(\frac{1}{m+1})$ choose $x_n \in \text{supp}(\mu_n) \cap B(x_0, \frac{1}{m})$. Then $x_n \in A(B_n)$ and $\lim_n x_n = x_0$.

(8). For any $B \in \mathcal{F}$ we have that
\[
\text{supp}(\nu_B) \subseteq A(B) \subseteq \mathbb{N}(B).
\]
By item (7)
\[
\limsup_{B \to A} \text{supp}(\nu_B) \subseteq \mathbb{A}(A) = \text{supp}(\mu).
\]
It remains to prove that
\[
\liminf_{B \to A} \text{supp}(\nu_B) \supseteq \text{supp}(\mu).
\]
But this follows from the convergence $\lim_{B \to A} \nu_B = \mu$ in the weak* topology.

(9). Write $\mathcal{M}(A) = \{ \mu \}$. By items (7) and (8) we have that
\[
\limsup_{B \to A} \mathbb{M}(B) \subseteq \limsup_{B \to A} \mathbb{A}(B) \subseteq \mathbb{A}(B),
\]
\[\mathbb{A}(A) = \text{supp}(\mu) \subseteq \liminf_{B \to A} \mathbb{M}(B).
\]

\[\square\]

**Proof of Theorem 7.1**

The set
\[\mathcal{D} := \{ A \in \mathcal{F} \mid \#\mathcal{M}(A) = 1 \}\]

is dense (c.f. [9]). We first prove that $\mathcal{D} \subseteq \overline{\mathcal{R}}$, and hence that $\mathcal{R}$ is dense.

Given $A \in \mathcal{D}$ with $\mathcal{M}(A) = \{ \mu \}$ and $\varepsilon > 0$, let $\psi \in \mathcal{F}$ be such that $\|\psi\|_0 + \|\psi\|_{\alpha} < \varepsilon$, $\psi \leq 0$, $[\psi = 0] = \text{supp}(\mu)$. It is easy to see that $\mathcal{M}(A + \psi) = \{ \mu \} = \mathcal{M}(A)$. Let $x_0 \notin \text{supp}(\mu)$. Given $\delta > 0$, write
\[S_A(x_0, x_0; \delta) := \sup \left\{ \sum_{k=0}^{n-1} A(T^k(x_n)) \mid T^n(x_n) = x_0, \ d(x_n, x_0) < \delta \right\}.
\]
If $T^n(x_n) = x_0$ is such that $d(x_n, x_0) < \delta$ then
\[
\sum_{k=0}^{n-1} (A + \psi)(T^k(x_n)) \leq S_A(x_0, x_0; \delta) + \sum_{k=0}^{n-1} \psi(T^k(x_n)) \\
\leq S_A(x_0, x_0; \delta) + \psi(x_n).
\]
Taking $\limsup_{\delta \to 0}$,
\[S_{A+\psi}(x_0, x_0) \leq S_A(x_0, x_0) + \psi(x_0) \leq \psi(x_0) < 0.
\]
Hence $x_0 \notin A(A + \psi)$. Since by lemma 7.1, $\text{supp}(\mu) \subseteq A(A + \psi)$, then $A(A + \psi) = \text{supp}(\mu)$ and hence $A + \psi \in \mathcal{R}$.

Let
\[\mathcal{U}(\varepsilon) := \{ A \in \mathcal{F} \mid d_H(\mathbb{A}(A), \mathbb{M}(A)) < \varepsilon \}.
\]
From the triangle inequality
\[d_H(\mathbb{A}(B), \mathbb{M}(B)) \leq d_H(\mathbb{A}(B), \mathbb{A}(A)) + d_H(\mathbb{A}(A), \mathbb{M}(B))
\]
and items (7) and (9) of lemma 7.6 we obtain that $\mathcal{U}(\varepsilon)$ contains a neighbourhood of $\mathcal{D}$. Then the set
\[\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{U}(\frac{1}{n})
\]
contains a residual set.

\[\square\]
8. Duality.

In this section we have to consider properties for $A^*$ which depends on the initial potential $A$. This is a main step in the reasoning of the main section 6.

We will consider now the specific example described before. We point out that the results presented below should hold in general for natural extensions.

We will assume that $T$ and $\sigma$ are topologically mixing.

So we take, $\mathbb{K} = [0, 1]$, $T(x) = 2x \mod 1$, $\Sigma = \Pi_{n \in \mathbb{N}} \{0, 1\}$, $\sigma : \Sigma \leftrightarrow$ the shift map $\sigma(x)_n = x_{n+1}$ and $T : \mathbb{K} \times \Sigma \rightarrow \mathbb{K} \times \Sigma$.

\[
T(x, \omega) = (T(x), \tau_x(\omega)), \quad T^{-1}(x, \omega) = (\tau_x(\omega), \sigma(\omega))
\]

\[
\tau_x(\omega) = (\nu(x), \omega) \in \Sigma,
\]

\[
\nu(x) = \begin{cases} 
0 & x \in [0, \frac{1}{2}], \\
1 & x \in [\frac{1}{2}, 1].
\end{cases}
\]

Given $A \in \mathcal{F}$ define $\Delta_A : \mathbb{K} \times \mathbb{K} \times \sigma \rightarrow \mathbb{R}$ as

\[
\Delta_A(x, y, \omega) := \sum_{n \geq 0} A(\tau_{n,\omega}(x)) - A(\tau_{n,\omega}(y))
\]

where

\[
\tau_{n,\omega}(x) = \tau_{\sigma^n \omega} \circ \tau_{\sigma^{n-1} \omega} \circ \cdots \circ \tau_{\omega}(x).
\]

Fix $\pi \in \mathbb{K}$ and $\bar{\pi} \in \Sigma$.

Define the involution $W$-kernel as $W_A : \mathbb{K} \times \Sigma \rightarrow \mathbb{R}$, $W_A(x, \omega) = \Delta_A(x, \pi, \omega)$. Writing $A := A \circ \pi_1 : \mathbb{K} \times \Sigma \rightarrow \mathbb{R}$, we have that

\[
W(x, \omega) = \sum_{n \geq 0} A(T^{-n}(x, \omega)) - A(T^{-n}(\pi, \omega))
\]

Define the dual function $A^* : \Sigma \rightarrow \mathbb{R}$ as

\[
A^*(\omega) := (W_A \circ T^{-1} - W_A + A \circ \pi_1)(x, \omega).
\]

Define a metric on $\Sigma$ by

\[
d(\omega, \nu) := \lambda_N, \quad N := \min\{k \in \mathbb{N} \mid \omega_k \neq \nu_k\}
\]

Then $\lambda$ is a Lipschitz constant for both $\tau_x$, and $\tau_\omega$ and also for $T_{\{x\} \times \Sigma}$ and $T^{-1}_{\{x\} \times \{\omega\}}$.

Write $\mathcal{F} := C^\alpha(\mathbb{K}, \mathbb{R})$ and $\mathcal{F}^* := C^\alpha(\Sigma, \mathbb{R})$. Let $\mathcal{B}$ and $\mathcal{B}^*$ be the set of coboundaries

\[
\mathcal{B} := \{ u \circ T - u \mid u \in C^\alpha(\mathbb{K}, \mathbb{R}) \},
\]

\[
\mathcal{B}^* := \{ u \circ \sigma - u \mid u \in C^\alpha(\Sigma, \mathbb{R}) \}.
\]

Define

\[
[z]_\alpha := \|z\|_0 + \|z\|_\alpha.
\]

Lemma 8.1.

(1) $z \in \mathcal{B}$ $\iff$ $z \in C^\alpha(\mathbb{K}, \mathbb{R})$ $\&$ $\forall \mu \in \mathcal{M}(T)$ $\int z \, d\mu = 0$.

(2) The linear subspace $\mathcal{B} \subseteq C^\alpha(\mathbb{K}, \mathbb{R})$ is closed.

(3) The function

\[
[z + \mathcal{B}]_\alpha = \inf_{b \in \mathcal{B}} [z + b]_\alpha
\]

is a norm in $\mathcal{F}/\mathcal{B}$.

Proof.

(1). This follows\footnote{Theorem 1.28 of R. Bowen\cite{5} asks for $T$ to be topologically mixing.} from \cite{5}, Theorem 1.28 (ii) $\implies$ (iii).
Lemma 8.2. We prove that the complement $B^c$ is open. If $z \in C^\alpha(K, \mathbb{R}) \setminus B$, by item (1), there is $\mu \in M(T)$ such that $\int z \, d\mu \neq 0$. If $u \in C^\alpha(K, \mathbb{R})$ is such that

$$\|u - z\|_0 < \frac{1}{2} \left| \int z \, d\mu \right|$$

then $\int u \, d\mu \neq 0$ and hence $u \notin B$.

(3). This follows from item (2).

\[ \Box \]

(2). If $A$ is $C^\alpha$ then $A^*$ is $C^\alpha$.

(3). The linear map $L : C^\alpha(K, \mathbb{R}) \to C^\alpha(\Sigma, \mathbb{R})$ given by $L(A) = A^*$ is continuous.

(4). The induced linear map $L : F/B \to F^*/B^*$ is continuous.

(5). Similarly the corresponding linear map $L^* : F^*/B^* \to F$, given by

$$L^*(\psi) = W^*_\psi \circ \tau - W_\psi + \psi \circ \pi_2$$

$$= \sum_{n \geq 0} \psi(T^n(x, \omega)) - \Psi(T^n(Tx, \omega))$$

$$= \Psi(x, \omega) + \sum_{n \geq 0} \Psi(T^n(Tx, \tau_x \omega)) - \Psi(T^n(Tx, \omega)),$$

with $\Psi = \psi \circ \pi_2$, is continuous and induces a continuous linear map $L^* : F^*/B^* \to F/B$, which is the inverse of $L : F/B \to F^*/B^*$.

Proof.

(1) and (2). We have that

$$A^*(\omega) = \sum_{n \geq 0} \lambda^n (T^{-n}(\tau, \omega)) - \lambda^n (T^{-n}(\tau, \sigma \omega))$$

$$= A(\tau) + \sum_{n \geq 0} \lambda^n (T^{-n}(\tau, \omega), \sigma \omega)) - \lambda^n (T^{-n}(\tau, \sigma \omega))$$

Since $d(T^{-n}(\tau, \omega), T^{-n}(\tau, \sigma \omega)) \leq \lambda^n d(\tau, \omega) \leq \lambda^n$ and $\|A\|_\alpha = \|A \circ \pi_1\|_\alpha = \|A\|_\alpha$, we have that

$$\|A^*\|_0 \leq \|A\|_0 + \frac{\|A\|_\alpha}{1 - \lambda^\alpha}.$$

Also if $m := \min \{ k \geq 0 \mid w_k \neq \nu_k \}$

$$A^*(\omega) - A^*(\nu) = \sum_{n \geq m-1} \lambda^n (T^{-n}(\tau, \omega), \sigma \omega)) - \lambda^n (T^{-n}(\tau, \sigma \omega))$$

$$- \sum_{n \geq m-1} \lambda^n (T^{-n}(\tau, \sigma \nu)) - \lambda^n (T^{-n}(\tau, \sigma \nu))$$

$$|A^*(\omega) - A^*(\nu)| \leq 2 \|A\|_\alpha \frac{\lambda^{(m-1)\alpha}}{1 - \lambda^\alpha} = \frac{2 \|A\|_\alpha \lambda^{-\alpha}}{1 - \lambda^\alpha} d(\omega, \nu)^\alpha.$$

$$\|A^*\|_\alpha \leq \frac{2 \|A\|_\alpha}{\lambda^\alpha(1 - \lambda^\alpha)}.$$

(3). If $u \in F$ and $U := u \circ \pi_1$ from the formula for $L$ (in the proof of item (1)) we have that

$$L(u \circ T - u) = U(T(\tau, \omega)) - U(T(\tau, \sigma \omega))$$

$$= u(T\tau) - u(T\tau) = 0.$$

(4). Item (4) follows from items (2) and (3).
We only prove that for any \( A \in \mathcal{F}, L^*(L(A)) \in A + \mathcal{B}. \) Write
\[
L^*(L(A)) = (W_A^* \circ \mathcal{T} - W_A^* + A^*)(\cdot, \overline{w}) \\
= (W_A^* \circ \mathcal{T} - W_A^* + W_A \circ \mathcal{T}^{-1} - W_A) + A
\]
Write
\[
B := W_A^* \circ \mathcal{T} - W_A^* + W_A \circ \mathcal{T}^{-1} - W_A.
\]
Since \( A, L^*(L(A)) \in \mathcal{F} = C^\alpha(\mathbb{K}, \mathbb{R}), \) then \( B \in C^\alpha(\mathbb{K}, \mathbb{R}). \)

Following Bowen, given any \( \mu \in \mathcal{M}(\mathbb{T}) \) we construct an associated measure \( \nu \in \mathcal{M}(\mathbb{T}). \) Given \( z \in C^0(\mathbb{K} \times \Sigma, \mathbb{R}) \) define \( z^\sharp \in C^0(\mathbb{K}, \mathbb{R}) \) as \( z^\sharp(x) := z(x, \overline{w}). \) We have that
\[
\| (z \circ T^n)^\sharp \circ T^m - (z \circ T^{n+m})^\sharp \|_0 \leq \var_n z_n \to 0,
\]
where
\[
\var_n z = \sup \{|z(a) - z(b)| \mid \exists x \in \mathbb{K}, a, b \in T^n(\{x\} \times \Sigma)\} \\
\leq \sup \{|z(a) - z(b)| \mid d_{\mathbb{K} \times \Sigma}(a, b) \leq \lambda^n \}, \to 0,
\]

Then
\[
|\mu((z \circ T^n)^\sharp) - \mu((z \circ T^{n+m})^\sharp)| = |\mu((z \circ T^n)^\sharp \circ T^m) - \mu((z \circ T^{n+m})^\sharp)| \leq \var_n z.
\]
Therefore \( \mu((z \circ T^n)^\sharp) \) is a Cauchy sequence in \( \mathbb{R} \) and hence the limit
\[
\nu(z) := \lim_n \mu((z \circ T^n)^\sharp)
\]
exists. By the Riesz representation theorem \( \nu \) defines a Borel probability measure in \( \mathbb{K} \times \Sigma, \) and it is invariant because
\[
\nu(z \circ T) = \lim_n \mu((z \circ T^{n+1})^\sharp) = \nu(z).
\]

Now let \( B := L^*(L(A)) - A \) and \( \mathcal{B} := B \circ \pi_1. \) By formula (35) we have that \( \mathcal{B} \) is a coboundary in \( \mathbb{K} \times \Sigma. \) Since \( \pi_1 \circ T^n = T^n \) we have that
\[
0 = \nu(\mathcal{B}) = \lim_n \mu((\mathcal{B} \circ T^n)^\sharp) \\
= \lim_n \mu(B \circ T^n) \\
= \mu(B).
\]

Since this holds for every \( \mu \in \mathcal{M}(\mathbb{T}), \) by lemma 8.1, \( \mathcal{B} \in \mathcal{B} \) and then \( (L^+ \circ L)(A + \mathcal{B}) \subset A + \mathcal{B}. \)

\( \square \)

**Theorem 8.1.**

There is a residual subset \( \mathcal{Q} \subset C^\alpha(\mathbb{K}, \mathbb{R}) \) such that if \( A \in \mathcal{Q} \) and \( A^* = L(A) \) then
\[
\mathcal{M}(A) = \{\mu\}, \quad \mathcal{A}(A) = \text{supp}(\mu),
\]
\[
\mathcal{M}(A^*) = \{\mu^*\}, \quad \mathcal{A}(A^*) = \text{supp}(\mu^*).
\]

In particular
\[
I_A(x) > 0 \quad \text{if} \quad x \notin \text{supp}(\mu),
\]
\[
I_A(\omega) > 0 \quad \text{if} \quad \omega \notin \text{supp}(\mu^*).
\]

and
\[
R_A(x) > 0 \quad \text{if} \quad x \notin \text{supp}(\mu) \quad \text{and} \quad T(x) \in \text{supp}(\mu),
\]
\[
R_A(\omega) > 0 \quad \text{if} \quad \omega \notin \text{supp}(\mu^*) \quad \text{and} \quad \sigma(\omega) \in \text{supp}(\mu).
\]
Proof. Observe that the subset $\mathcal{R}$ defined in (32) in theorem 7.1 is invariant under translations by coboundaries, i.e. $\mathcal{R} = \mathcal{R} + B$. Indeed if $B = u \circ T - u \in B$, we have that
\[
\int (A + B) \, d\mu = \int A \, d\mu, \quad \forall \mu \in B,
\]
Then the Aubry set and the set of minimizing measures are unchanged:
\[
\mathcal{M}(A + B) = \mathcal{M}(A), \quad \mathcal{A}(A + B) = \mathcal{A}(A).
\]
For the dynamical system $(\Sigma, \sigma)$ let
\[
\mathcal{R}^* = \{ \psi \in C^\alpha(\Sigma, \mathbb{R}) | \mathcal{M}(\psi) = \{ \mu \}, \mathcal{A}(\psi) = \text{supp}(\nu) \}
\]
By theorem 7.1, the subset $\mathcal{R}^*$ contains a residual set in $\mathcal{F}^* = C^\alpha(\Sigma, \mathbb{R})$ and it is invariant under translations by coboundaries: $\mathcal{R}^* = \mathcal{R}^* + B^*$.

By lemma 8.2 the linear map $L : \mathcal{F}/B \to \mathcal{F}^*/B^*$ is a homeomorphism with inverse $L^*$. Then the set $Q := \mathcal{R} \cap L^{-1}(\mathcal{R}^*) = \mathcal{R} \cap (L^*(\mathcal{R}^*) + B)$ contains a residual subset and satisfies (36).

By Corollary 7.5 the other properties are automatically satisfied.

From this last theorem it follows our main result about the generic analytic $g$, by adapting the proof and taking $T = f$, where $f$ is the transformation defined in section 2.

9. The optimal solution when the maximizing probability is not a periodic orbit

We are going to analyze now the variation of the optimal point when the support of the maximizing probability is not necessarily a periodic orbit. What can be said in the general case?

Consider the subaction defined by,
\[
V(x) = \sup_{w \in \Sigma} (H_{\infty}(w, x) - I^*(w))
\]
Remember that as $I^*$ is lower semicontinuous and $H_{\infty} = W - V^*$ is continuous, then for each fixed $x$, the supremum of $H_{\infty}(w, x) - I(w)$ in the variable $w$ is achieved, and we denote (one of such $w$) it by $w_x$. In this case we say $w_x$ is an optimal point for $x$.

We want to show that $w_x$ is unique for the generic $x$.

Define the multi-valuated function $U : [0, 1] \to \Sigma$ given by:
\[
U(x) = \{ w_x | x \in [0, 1] \}
\]
As graph$(U)$ is closed in each fiber, and $\Sigma$ is compact we can define:
\[
u^+(x) = \max U(x), \quad \nu^-(x) = \min U(x).
\]

Since the potential $A$ is twist we know that $U$ is a monotone not-increasing multi-valuated function, that is,
\[
u^-(x) \geq \nu^+(x + \delta),
\]
when $x < x + \delta$. In particular are monotone not-increasing single-valuated functions.
We claim that $u^+$ is left continuous. In order to conclude that, take a sequence $x_n \to x$ on the left side. Consider, the sequence $u^+(x_n) \in \Sigma$, so its set of accumulation points is contained in $U(x)$. Indeed, suppose $\liminf u^+(x_n) \to \tilde{w} \in \Sigma$. In one hand, we have, $V(x_n) = H_\infty(u^+(x_n), x_n) - I^*(u^+(x_n))$. Taking limits on this equation and using the continuity of $V$ and $H_\infty$ and the lower semicontinuity of $I^*$ we get,

$$V(x) \leq H_\infty(\tilde{w}, x) - I^*(\tilde{w}).$$

Because $\liminf I^*(u^+(x_n)) \geq I^*(\tilde{w})$. So $\tilde{w} \in U(x)$. On the other hand, $u^+$ is monotone not-increasing, so $u^+(x_n) \geq u^+(x)$. From the previous we get

$$\limsup_{x_n \to x^-} u^+(x_n) \geq u^+(x) \geq \tilde{w} = \liminf u^+(x_n),$$

that is,

$$\lim_{x_n \to x^-} u^+(x_n) = u^+(x).$$

Now consider a sequence $x_n \to x$ on the right side. Take, the sequence $u^+(x_n) \in \Sigma$, so its set of accumulation points is not necessarily contained in $U(x)$. However it is the case. Let $x_{n_k}$ be a subsequence such that, $u^+(x_{n_k}) \to \check{w}$.

We know that $V(x_{n_k}) = H_\infty(u^+(x_{n_k}), x_{n_k}) - I^*(u^+(x_{n_k}))$. Taking limits on this equation and using the uniform continuity of $V$ and $H_\infty$ we get

$$I^*(\check{w}) \leq \liminf_{k \to \infty} I^*(u^+(x_{n_k})) =$$

$$= \liminf_{k \to \infty} H_\infty(u^+(x_{n_k}), x_{n_k}) - V(x_{n_k}) = H_\infty(\check{w}, x) - V(x).$$

In other words, $V(x) \leq H_\infty(\check{w}, x) - I^*(\check{w})$, that is, $\check{w} \in U(x)$. So

$$\text{cl}(u^+(x_n)) \subseteq U(x).$$

Since $u^+$ is monotone not-increasing, $u^+(x_n) \leq u^+(x)$, thus

$$\limsup_{x_n \to x} u^+(x_n) \leq u^+(x),$$

that is, $u^+$ is right upper-semicontinuous.

It is known that for any USC function defined in a complete metric space the set of points of continuity is generic.

Therefore, we get that:

**Theorem 9.1.** For a generic $x$ we have that $U(x) = \{ u^+(x) = u^-(x) \}$ and $w_x$ is unique.

**Proof.** Indeed, suppose that there is a point in the set of continuity of $u^+(x)$ such that, $u^+(x) > u^-(x)$ so the monotonicity of $U$ implies that

$$u^+(x) > u^-(x) \geq u^+(x + \delta),$$

for all $\delta > 0$. Contradicting the continuity. \qed
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