The Cointegrated VAR without Unit Roots:
Representation Theory and Asymptotics

James A. Duffy∗
Jerome R. Simons†

February 2020

Abstract

It has been known since Elliott (1998) that efficient methods of inference on cointegrating relationships break down when autoregressive roots are near but not exactly equal to unity. This paper addresses this problem within the framework of a VAR with non-unit roots. We develop a characterisation of cointegration, based on the impulse response function implied by the VAR, that remains meaningful even when roots are not exactly unity. Under this characterisation, the long-run equilibrium relationships between the series are identified with a subspace associated to the largest characteristic roots of the VAR. We analyse the asymptotics of maximum likelihood estimators of this subspace, thereby generalising Johansen’s (1995) treatment of the cointegrated VAR with exactly unit roots. Inference is complicated by nuisance parameter problems similar to those encountered in the context of predictive regressions, and can be dealt with by approaches familiar from that setting.

The authors thank G. Bårdsen, V. Berenguer-Rico, P. Boswijk, G. Chevillon, B. Nielsen, S. MAVroeidis, and participants at seminars at Amsterdam, ESSEC (Cergy), NTNU (Trondheim), Southampton and Oxford for helpful comments on earlier drafts of this work.

∗Corpus Christi College and Department of Economics, University of Oxford.
†Nuffield College and Department of Economics, University of Oxford.
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1 Introduction

The cointegrated vector autoregressive (CVAR) model has been widely applied to the modelling of macroeconomic time series – a testament to its ability to account for both the short- and long-run dynamics of these series in a unified way. By allowing for one or more autoregressive roots at unity, the model is able to match two key features of these series: firstly their high degree of persistence, which gives rise to their characteristically ‘random wandering’ behaviour, and secondly the tendency for economically related series to move together, such that certain linear combinations of these series are markedly less persistent than the series themselves. These linear combinations are, of course, the cointegrating relationships between the series.

Cointegrating relations can be efficiently estimated by a variety of methods, such as FM-OLS (Phillips and Hansen, 1990), DOLS (Stock and Watson, 1993), and maximum likelihood estimation of the CVAR itself (Johansen, 1995). However, a shortcoming shared by all these approaches is their reliance on the assumption that the series are generated by a model with some autoregressive roots that are exactly unity. The fragility of inferences to even small violations of this assumption was highlighted in a seminal contribution by Elliott (1998), who showed the possibility of large size distortions when roots lie only within a $O(n^{-1})$ neighbourhood of unity. His findings are particularly disturbing, because data generated by a VAR with roots that are ‘nearly’ unity is essentially indistinguishable from data generated by the same model with exactly unit roots.

The present work is concerned precisely with the problem identified by Elliott – with how one can perform valid inference on the cointegrating relationships implied by a VAR, when the largest characteristic roots may not be exactly unity. In view of the significance of Elliott’s results, it is perhaps surprising that only a few previous contributions have also attempted to address it: most notably Wright (2000), Magdalinos and Phillips (2009), Müller and Watson (2013) and Franchi and Johansen (2017). The approach taken in this paper is quite different from that taken in each of those previous works. While Elliott framed his results in terms of an inferential problem, our view is that the problem is as much one of identification as it is of inference. Indeed, the usual definition of cointegration – in terms of linear combinations of series that eliminate their common integrated components – becomes meaningless as soon as the largest characteristic roots in a VAR depart even slightly from unity.

Our first task is thus to develop a characterisation of cointegration, based on the impulse response function implied by the VAR, that remains meaningfully interpretable in a model with some (distinct) roots near but not necessarily equal to unity. In a $p$-dimensional VAR with $q$ roots ‘near’ but not necessarily equal to unity, one can always identify a $p - q = r$-dimensional subspace $S_r$, such that the decay of the impulse response function in the directions contained in $S_r$ is more rapid than it is in all other directions. We term this the quasi-cointegrating space (QCS). When the roots of the VAR are exactly unity, the QCS coincides exactly with the cointegrating space – and when the largest characteristic roots are modelled as being local to unity (in the sense of lying within a $O(n^{-1})$ neighbourhood of unity), the quasi-cointegrating vectors are those that exactly eliminate the near stochastic trends from the system.

Asymptotic inference on the QCS is complicated by the presence of nuisance parameters related to the proximity of the largest characteristic roots to unity. This problem is similar to
that which arises in predictive regressions when the regressor has an unknown but possibly high degree of persistence, such as has been studied e.g. by Cavanagh, Elliott, and Stock (1995), Campbell and Yogo (2006), Phillips and Lee (2013), Phillips (2014), Kostakis, Magdalinos, and Stamatogiannis (2015) and Elliott, Müller, and Watson (2015). Approaches developed in that literature can accordingly be imported into the present setting, and the asymptotic results developed in this paper are intended to provide the basis for an analysis of such approaches, such as will be developed by the authors in a subsequent paper; here only a very basic Bonferroni-type procedure is outlined.

The remainder of this paper is organised as follows. Section 2 reviews some familiar characterisations of cointegration in a VAR model with unit roots, and extends two of these to develop the notion of quasi-cointegration that is central to this paper. Section 3 develops the asymptotics of likelihood-based inference on the QCS. Auxiliary technical results and proofs of results appearing in the body of the paper are provided in Appendices A–D.

2 ‘Cointegration’ in a VAR without unit roots

2.1 Model and assumptions

The data generating process (DGP) for the observed series \( \{y_t\}_{t=1}^n \) is a kth order vector autoregressive (VAR) model, written in ‘structural’ form as

\[
y_t = \mu + \delta t + x_t = \sum_{i=1}^k \Phi_i x_{t-i} + \varepsilon_t
\]

where \( \varepsilon_t, x_t \) and \( y_t \) are \( p \)-dimensional random vectors. Let \( \Phi(\lambda) := I\lambda^k - \sum_{i=1}^k \Phi_i\lambda^{k-i} \) denote the characteristic polynomial associated to (2.1); we shall refer to any \( \lambda \) for which \( \det \Phi(\lambda) = 0 \) as a ‘root of \( \Phi \).’ Let \( \Phi := (\Phi_1, \Phi_2, \ldots, \Phi_k) \in \mathbb{R}^{p \times kp} \). The following is maintained throughout.

**Assumption DGP.**

- **DGP1** \( \{\varepsilon_t\} \) is i.i.d. with \( \mathbb{E}\varepsilon_t = 0 \) and \( \mathbb{E}\varepsilon_t\varepsilon_t^T = \Sigma \) positive definite.
- **DGP2** \( \det \Phi(\lambda) \neq 0 \) for all \( |\lambda| > 1 \).
- **DGP3** \( x_0 = x_{-1} = \cdots = x_{-k+1} = 0 \).

We say that a \( d_z \)-dimensional process \( \{z_t\} \) is integrated of order zero, denoted \( z_t \sim I(0) \), if there exists a deterministic process \( \{\mu_t\} \) such that \( n^{-1/2} \sum_{s=1}^{[nr]} (z_s - \mu_s) \sim B(r) \), for \( B \) a \( d_z \)-dimensional Brownian motion. Letting \( \Delta^d \) denote the \( d \)-th order temporal differencing operator, we say that \( z_t \) is integrated of order \( d \), denoted \( z_t \sim I(d) \), if \( \Delta^dz_t \sim I(0) \). We say \( \{z_t\} \) is nearly integrated if \( n^{-1/2}(z_{[nr]} - \mu_{[nr]}) \sim \int_0^r e^{C(r-s)}dB(s) \) for some \( C \in \mathbb{R}^{d_z \times d_z} \).

2.2 Cointegration: the model with unit roots

Cointegration analysis is concerned with how linear combinations of \( I(d) \) processes can yield processes that are themselves only \( I(d-b) \) for some \( 0 < b \leq d \); the reduced persistence of the latter being interpreted as evidence of a long-run equilibrium relationship between the original
processes. Here we focus exclusively on the special but practically important case of \( I(1) \) processes having linear combinations that are \( I(0) \), reserving the term ‘cointegration’ exclusively for this case. As is well known, the VAR model (2.1) is able to generate cointegrated \( I(1) \) processes under the following assumption, which defines the \( I(0)/I(1) \) cointegrated VAR (CVAR) model.

**Assumption CV.**

CV1 \( \Phi \) has \( q \) roots at \((\text{real}) \) unity, and all others strictly inside the unit circle.

CV2 \( \text{rk} \Phi(1) = p - q =: r \)

By the Granger–Johansen representation theorem (GJRT; see e.g. Johansen 1995, Thm 4.2 and Cor. 4.3), the preceding is necessary and sufficient for \( y_t \sim I(1) \), and for there to exist a matrix \( \beta \in \mathbb{R}^{p \times r} \) (with \( \text{rk} \beta = r \)) of cointegrating relationships, such that \( \beta^T y_t \sim I(0) \). Clearly \( \beta \) is identified only up to its column space \( \text{CS} := \text{sp} \beta \), termed the **cointegrating space** (CS). Two equivalent characterisations of the cointegrating space, the first of which is definitional and the second of which follows immediately from the GJRT, are:

(i) \( b^T y_t \sim I(0) \) if and only if \( b \in \text{CS} \); and

(ii) \( \text{CS} = \text{sp} \Phi(1)^T = \{ \ker \Phi(1) \}^\perp \).

The object of this paper is to estimate the CS, or at least a subspace that shares some of its key properties, in a setting more general than that of CV. For this purpose, we next recall two further characterisations of the CS that can be extended beyond the setting of CV, in a way that the preceding two cannot. We make no assertions to novelty in formulating these: the contribution of this paper consists rather in the manner in which these characterisations will be exploited once CV has been relaxed. Some similar claims to those that follow have therefore appeared (and been proved) elsewhere, either in textbook presentations of the theory or in the extensive literature concerned with the representation of cointegrated processes (for very general treatments of which, see e.g. the recent papers by Beare and Seo, 2019, and Franchi and Parmulo, 2019). For completeness, formal statements and proofs of the results underlying the discussion that follows (including that of Section 2.3) are given in Appendix A.

Our third characterisation of the CS is in terms of the impulse response function of \( \{ y_t \} \) with respect to the disturbances \( \{ \varepsilon_s \} \), denoted

\[
\text{IRF}_s := \frac{\partial y_{t+s}}{\partial \varepsilon_t} = \frac{\partial x_{t+s}}{\partial \varepsilon_t}.
\]

For a given \( b \in \mathbb{R}^p \), the product \( b^T \text{IRF}_s \) gives the response of the linear combination \( b^T y_{t+s} \) to a shock dated \( s \) periods previously. The rate at which \( b^T \text{IRF}_s \) decays as the horizon \( s \) diverges can be regarded as measure of the persistence of the series \( \{ b^T y_t \} \). Now let \( m < p \), and define \( S_m \subset \mathbb{R}^p \) to be an \( m \)-dimensional linear subspace such that for every \( b \in S_m \) and \( c \notin S_m \),

\[
\lim_{s \to \infty} \frac{\| b^T \text{IRF}_s \|}{\| c^T \text{IRF}_s \|} = 0. \tag{2.2}
\]

When it exists, \( S_m \) collects those \( m \) linear combinations of \( y_t \) that are, in the sense of (2.2), the ‘least persistent’. Under CV we are assured that \( S_r \) exists and is unique, and moreover
(iii) $CS = S_r$

(see Lemma A.3). In other words, the cointegrating space is spanned by the vectors giving the $r$ least persistent linear combinations of $y_t$.

Our final characterisation of the cointegrating space provides the basis for its estimation in settings more general than CV; it derives essentially from the application of an invariant subspace decomposition to the companion form representation of (2.1) (see Lemma A.1). Define

$$L_{LU}^\rho := \{ z \in \mathbb{C} \mid \|z\| \leq 1 \text{ and } |1-z| \leq 1-\rho \}$$

so that for a given $\rho \leq 1$ (close to unity), $L_{LU}^\rho$ defines a neighbourhood of real unity inside the unit circle, and $L_{ST}^\rho$ an open ball of radius $\rho$. Now suppose that $\Phi$ has $q$ roots in $L_{LU}^\rho$ and all others in $L_{ST}^\rho$ for some $\rho \leq 1$; under $CV_1$ this is true with $\rho = 1$ (so that $L_{LU}^1 = \{1\}$). Since $L_{LU}^\rho$ and $L_{ST}^\rho$ are disjoint, there exist real matrices

$$R := \begin{bmatrix} R_{LU} & R_{ST} \end{bmatrix}$$

and

$$L := \begin{bmatrix} L_{LU} & L_{ST} \end{bmatrix}$$

such that: the eigenvalues of $\Lambda_{LU}$ and $\Lambda_{ST}$ correspond to the roots of $\Phi$, and lie in $L_{LU}^\rho$ and $L_{ST}^\rho$ respectively; $(R_{LU}, \Lambda_{LU}, L_{LU})$ satisfy

$$R_{LU} \Lambda_{LU}^k - \sum_{i=1}^k \Phi_i R_{LU} \Lambda_{LU}^{k-i} = 0 \quad \Lambda_{LU}^k L_{LU}^T - \sum_{i=1}^k \Lambda_{LU}^{k-i} L_{LU}^T \Phi_i = 0;$$

and the impulse response function of $y_t$ can be written as

$$\frac{\partial y_{t+s}}{\partial \epsilon_t} = IRF_s = RA^{k-1+s} L^T = R_{LU} \Lambda_{LU}^{k-1+s} L_{LU}^T + R_{ST} \Lambda_{ST}^{k-1+s} L_{ST}^T$$

(see Lemma A.1 and the subsequent remarks). Under $CV$, we have $\Lambda_{LU} = I_q$ and $\text{rk } R_{LU} = \text{rk } L_{LU} = q$ (see Lemma A.3). In particular

$$\lim_{s \to \infty} IRF_s = R_{LU} L_{LU}^T$$

giving our final characterisation of the cointegrating space as

(iv) $CS = (\text{sp } R_{LU})^\perp$.

2.3 ‘Cointegration’ without unit roots

We propose to relax $CV$ by allowing the largest $q$ roots of $\Phi$ to lie in a small neighbourhood of unity, without requiring that these be exactly unity. Relaxing the assumption of exact unit roots is known to create two difficulties. Firstly, if we work with a sequence of models in which $\Lambda_{LU} = I + n^{-1}C$, then standard efficient estimators of the cointegrating relationships (such as FM-OLS, DOLS and ML) will remain consistent but have an asymptotic bias. Associated inferences on the cointegrating relations can be severely size distorted, depending on the magnitude of $C$, which cannot be consistently estimated (Elliott, 1998). This lack of robustness to departures
from exact unit roots is particularly disturbing because it arises in models that cannot be consistently distinguished from those with exact unit roots.

Secondly, there is an even deeper problem of identification. If we instead regard $\Lambda_{LU}$ as being fixed, rather than drifting towards $I_q$, how are we to even define the ‘cointegrating relationships’ among the elements of $y_t$? If all the roots of $\Phi$ are strictly inside the unit circle – as would now be permitted – then all linear combinations of $y_t$ would be $I(0)$, and $\Phi(1)$ would have full rank. The first two characterisations of the cointegrating space given above thus no longer describe something that could be estimated; indeed, both would identify the cointegrating space with the whole of $\mathbb{R}^p$.

Our proposed resolution to both these problems, of non-robustness and non-identification, is to rely instead on our third and fourth characterisations of the cointegrating space as a basis for identifying and estimating the long-run equilibrium relationships among the elements of $y_t$. Consider relaxing CV as follows, so as to allow the VAR to have some roots ‘near’ but not necessarily equal to unity.

**Assumption QC.** Let $\rho \in (0, 1]$ be given.

QC1 $\Phi$ has $q$ roots in $\mathcal{L}_q^\rho$, and all others in $\mathcal{L}_p^\rho$.

Let $\Lambda_{LU}$ denote a real $(q \times q)$ matrix whose eigenvalues correspond to the roots of $\Phi$ that are in $\mathcal{L}_q^\rho$, and let $R_{LU}$ and $L_{LU}$ be $p \times q$ matrices that satisfy (2.4)–(2.6).

QC2 $\text{rk} R_{LU} = \text{rk} L_{LU} = q$ and $\Lambda_{LU}$ is diagonalisable.

QC1 is plainly the analogue of CV1: whereas we previously assumed $q$ roots at unity, this is now relaxed to $q$ roots in the vicinity of unity; indeed it may be shown that CV is a special case of QC with $\rho = 1$ (Lemma A.3). The requirement that $\Lambda_{LU}$ be diagonalisable is required to rule out series that are integrated of order two or higher (see e.g. d’Autume, 1992; such series are also excluded by CV, which implies that $\Lambda_{LU} = I_q$ as noted). For $\rho < 1$ but ‘close’ to unity, a model satisfying QC will thus inherit the main qualitative features of the cointegrated VAR model: the high persistence of $\{y_t\}$, and the lesser persistence of $r$ linear combinations of $\{y_t\}$, where this is understood in terms of (2.2) above. Accordingly, the subspace $S_r$ spanned by the $r$ ‘least persistent’ linear combinations of $y_t$ remains an interesting object in the setting of QC; that it is well defined is guaranteed by the following result, the proof of which is given in Appendix A.

**Proposition 2.1.** Suppose DGP and QC hold. Then $S_r = (\text{sp} R_{LU})^\perp$.

By construction, the vectors in $S_r$ retain similar characteristics to the cointegrating relationships, in the sense that for each $b \in S_r$, $b^T y_t$ will be less persistent than $y_t$ itself. We shall henceforth term the elements of $S_r$ the quasi-cointegrating relationships, and refer to $S_r$ itself as the quasi-cointegrating space, denoted

$$\text{QCS} := S_r = (\text{sp} R_{LU})^\perp.$$  

Henceforth let $\beta \in \mathbb{R}^{p \times r}$ denote a matrix of rank $r$ whose columns span the QCS, and which therefore has the property that $\beta^T R_{LU} = 0$. 

5
It might be argued that the requirement that $\Phi$ have $q$ roots ‘near’ unity – in the sense of lying in $L_{LU}$ – is unnecessary; indeed it is not required for the identification of the QCS, as we have defined it above. However, for the concept of quasi-cointegration to be empirically interesting, in the sense of identifying relationships between series that can be plausibly regarded as ‘long-run equilibrium’ relationships, it would seem necessary that the quasi-cointegrated series should be measurably less persistent than the original series themselves. Thus we are really interested in modelling a situation where there is a high degree of persistence in the data, but where this persistence can be significantly reduced by taking appropriate linear combinations of the series. What is regarded as ‘persistent’ and ‘transitory’ is left to researcher, to be expressed through the chosen value for $\rho$ (see Section 3.4).

2.4 Connections to the literature

Extensions of the basic $I(0)/I(1)$ CVAR model, in which the persistence in $y_t$ is generated by some characteristic roots that are not at real unity, have previously been developed in the literature on seasonal cointegration. Johansen and Schaufuberg (1999) consider a VAR with roots at the points $\{z_m | m = 1, \ldots, q\}$ on the complex unit circle, and develop a version of the GJRT in which $y_t$ is decomposed into a sum of persistent nonstationary processes of the form $z_m^t \sum_{s=0}^t z_m^s \varepsilon_s$. They develop likelihood-based inference on the (possibly complex-valued) ‘seasonally cointegrating vectors’ that eliminate the nonstationary component associated to $z_m$, for each $m$ separately. Somewhat related work by Nielsen (2010) considers a VAR with $q$ unit roots and one real explosive root at $\lambda > 1$, and gives a decomposition of $y_t$ into $q$ integrated and one explosive linear process (his Theorem 1), which in the special case of a model initialised at zero with no deterministic terms simplifies to

$$x_t = \frac{1}{1 - \lambda} C_1 \sum_{s=1}^t \varepsilon_s + \frac{1}{\lambda - 1} C_\lambda \sum_{s=1}^t \lambda^{t-s} \varepsilon_s + w_t$$

where $w_t \sim I(0)$, and $C_1, C_\lambda \in \mathbb{R}^{p \times r}$ with $\text{rk} C_1 = q$ and $\text{rk} C_\lambda = p - 1$. His focus is on inference on the $p - 1$ ‘coexplosive vectors’ $\beta_\lambda$ that eliminate that common explosive component, i.e. for which $\beta_\lambda^T C_\lambda = 0$, but which do not necessarily eliminate the common $I(1)$ components. By comparison, the approach taken in the present work would amount to finding the $\beta$ that eliminates both the $I(1)$ and explosive components simultaneously, i.e. for which $\beta^T C_1 = \beta^T C_\lambda = 0$.

As noted in the introduction, there have been relatively few attempts to address the problems identified by Elliott’s (1998) paper: most notably Wright (2000), Magdalinos and Phillips (2009), M"uller and Watson (2013) and Franchi and Johansen (2017). Insofar as they also consider a VAR model with some characteristic roots near unity, the paper by Franchi and Johansen (2017) is perhaps most closely related to the present work. Their setting is a VAR(1) model, written in error correction form as

$$\Delta x_t = (\alpha \beta^T + \alpha_1 \Gamma \beta_1^T) x_{t-1} + \varepsilon_t =: \Pi x_{t-1} + \varepsilon_t$$

(2.8)

where $\alpha, \beta \in \mathbb{R}^{p \times r}$ and $\alpha_1, \beta_1 \in \mathbb{R}^{p \times q}$ have full column rank, and $\Gamma \in \mathbb{R}^{q \times q}$. When $\Gamma = 0$, the model specialises exactly to the CVAR model of Section 2.2 with $q$ unit roots and $\text{CS} = sp \beta$. 


Departures from this in the direction of a model with some roots ‘near’ but not equal to unity are permitted by allowing some elements of $\Gamma$ to be nonzero, with the consequence that $\Pi$ need no longer be of reduced rank. As is acknowledged by the authors, there is an identification problem here if each of $\alpha$, $\beta$, $\alpha_1$, $\beta_1$ and $\Gamma$ are freely varying. They accordingly treat $\alpha_1$ and $\beta_1$ as known, which restores identification and facilitates likelihood-based inference on each of $\alpha$, $\beta$ and $\Gamma$. While a priori knowledge of $\alpha_1$ and $\beta_1$ may indeed be available in certain situations, its unavailability in general is why we have introduced (2.2) as a kind of identifying criterion in the present work. Indeed, unless we work with a drifting sequence of models in which $\Gamma = \Gamma_n \to 0$ (as do Franchi and Johansen), it is not entirely clear how $\beta$ in (2.8) is interpretable in terms of ‘long run’ relationships between the elements of $x_t$.

Magdalinos and Phillips (2009) work with a triangular model of the form

$$x_{1t} = Ax_{2t} + u_{1t}$$
$$x_{2t} = R_n x_{2,t-1} + u_{2t}$$

where $u_t = (u_{1t}^T, u_{2t}^T)^T$ is weakly dependent. When $R_n = I_q$, this model encompasses the $I(0)/I(1)$ CVAR model with $q$ unit roots, but allows for a more general semiparametric treatment of the model’s short-run dynamics; when $R_n = I_q + n^{-1}C$, this is also the framework of Elliott (1998). Beyond certain weak summability conditions (their Assumption LP), the dynamics of $u_t$ are otherwise unrestricted, and it is assumed that $R_n$ drifts towards $I_q$, though possibly at a much slower rate than $n^{-1}$, as $n \to \infty$. Magdalinos and Phillips (2009) show that, under these assumptions, it is possible to obtain an asymptotically mixed normal estimate of $A$, using instruments that are constructed by filtering $x_{2t}$; they term this the ‘IVX’ estimator of $A$. The price of the greater generality afforded by their triangular model is that $R_n \to I_q$ becomes, in a certain sense, necessary for identification of $A$. Indeed, if $R_n$ is fixed with eigenvalues strictly less than unity, it is not clear how $A$ should be defined, since in this case all linear combinations of $x_t$ are weakly dependent (in the sense of their Assumption LP).

Finally, Müller and Watson (2013) consider a very general setting, which goes well beyond the framework of the VAR model, in which the ‘common trends’ in $x_t$ are permitted to belong to a broad family of processes. A consequence of this generality is that these authors conceptualise ‘cointegration’ in terms somewhat different from quasi-cointegration, and the two definitions do not always agree. Essentially, Müller and Watson define $x_t$ to be ‘cointegrated’ with cointegrating relations $\beta \in \mathbb{R}^{p \times r}$, if $n^{-1/2} \sum_{t=1}^{[nr]} \beta^T x_t$ converges weakly to a Brownian motion, while the common trends $n^{-1/2} \beta_{\perp}^T x_{[nr]}$ converge weakly to a cadlag process (where $\beta_{\perp} \in \mathbb{R}^{p \times q}$ has $\text{rk} \beta_{\perp} = q$ and $\beta_{\perp}^T \beta = 0$). In the context of our CVAR model, where QC holds for some $\rho < 1$, $n^{-1/2} \sum_{t=1}^{[nr]} x_t$ converges weakly to a Brownian motion if all the roots are strictly inside the unit circle; so in such a case there is no ‘cointegration’ in the sense of these authors, even though quasi-cointegrating relationships are well defined. On the other hand, if the largest $q$ roots of $\Phi$ are localised to unity at rate $n^{-1}$ (though not more slowly), then it appears that their ‘cointegrating’ vectors coincide with our quasi-cointegrating vectors. Regarding inference on $\beta$, these authors develop and justify an approach that builds a confidence set for $\beta$ by inverting a stationarity test for $\beta^T x_t$. This is similar to an approach originally proposed by Wright (2000), but utilises a test statistic that is deliberately based on only a fixed number of low-frequency
weighted averages of the data.

3 Estimation and inference

3.1 Formulation of the likelihood

As with the CS in a cointegrated VAR model, inference on the QCS in our more general setting will be based on the normal model likelihood (or quasi-likelihood, if \( \varepsilon_t \) is not in fact normally distributed). Recall that the ‘structural’ model (2.1) has the ‘reduced form’

\[
y_t = m + dt + \sum_{i=1}^{k} \Phi_i y_{t-i} + \varepsilon_t.
\]

(3.1)

To allow for a more streamlined exposition, we shall focus on the case where the reduced form model (3.1) is estimated with an unrestricted intercept and trend, while maintaining that the DGP is the structural model (2.1), thus excluding the possibility of a quadratic trend in \( y_t \). A discussion of how our results would be affected by alternative treatments of the deterministic terms is deferred to Section 3.7 below.

Up to irrelevant constants, the concentrated loglikelihood is

\[
\ell_n(\Phi, \Sigma) := -\frac{n}{2} \log \det \Sigma - \min_{m,d} \frac{1}{2} \sum_{t=1}^{n} \left\| y_t - m - dt - \sum_{i=1}^{k} \Phi_i y_{t-i} \right\|^2_{\Sigma^{-1}}
\]

where \( \|x\|^2_W := x^T W x \) for \( x \in \mathbb{R}^p \) and \( W \in \mathbb{R}^{p \times p} \) positive semidefinite. The QCS depends only on \( \Phi \), and the main (asymptotic) results of this paper are not sensitive to the method used to estimate \( \Sigma \), provided that it is estimated consistently. In what follows, we shall generally assume that the unrestricted ML estimator \( \hat{\Sigma}_n \) (i.e. the OLS variance estimator) is used, which simplifies some proofs and the numerical implementation of the inferential procedure outlined in Section 3.5. Henceforth, let \( \ell_n^*(\Phi) := \ell_n(\Phi, \hat{\Sigma}_n) \); for convenience we shall refer to maximisers of \( \ell_n^* \) as ‘maximum likelihood estimators’.

3.2 QCS as a functional of the VAR coefficients

Under QC, the QCS is well defined and has dimension \( q \). Since any basis \( \beta \in \mathbb{R}^{p \times q} \) for the QCS is only identified up to its column space, and has rank \( q \), it is convenient to impose the normalisation

\[
\beta^T = [I_r - A],
\]

(3.2)

so that inference on the QCS reduces to inference on the elements of the matrix \( A \in \mathbb{R}^{r \times q} \). This is not restrictive – i.e. it is indeed merely a ‘normalisation’ – if the QCS does not contain any nonzero vectors whose first \( r \) elements are all zero, as will be the case if the elements of \( y_t \) are ordered appropriately; we shall maintain this throughout the following. Since \( R_{LU} \) has rank \( q \)
\[ \beta^\top R_{LU} = 0, \] 
\eqref{eq:3.2} can be equivalently expressed as
\[ R_{LU} = \begin{bmatrix} A \\ I_q \end{bmatrix}. \quad \tag{3.3} \]

So long as the roots in \( \mathcal{L}^\rho_{LU} \) remain separated from those in \( \mathcal{L}^\rho_{ST} \), the column space of \( R_{LU} \) depends smoothly on the VAR coefficients. To express this rigorously, let \( \lambda_i(\Phi) \) denote the \( i \)th root of the characteristic polynomial associated to the VAR with coefficients \( \Phi \), when these are placed in descending order of modulus, and set \( G_T := [0_{r \times q}, I_q] \). For a given \( \rho \leq 1 \), define \( P \subset \mathbb{R}^{p \times q} \) to be the set of VAR coefficients such that:

(i) \(|\lambda_{q+1}(\Phi)| < |\lambda_q(\Phi)|\); (ii) there exist \( R_{LU} \in \mathbb{R}^{p \times q} \) and \( \Lambda_{LU} \in \mathbb{R}^{q \times q} \) such that the eigenvalues of \( \Lambda_{LU} \) are \( \{\lambda_1(\Phi), \ldots, \lambda_q(\Phi)\} \);

\[ R_{LU} \Lambda_{LU}^k - \sum_{i=1}^{k} \Phi_i R_{LU} \Lambda_{LU}^{k-i} = 0; \quad \tag{3.4} \]

and (iii) \( \text{rk}\{G^\top R_{LU}\} = q \). Then \( \mathcal{P} \) is open, and since \( G^\top R_{LU} \) has full rank, we may choose \( (R_{LU}, \Lambda_{LU}) \) to be additionally consistent with the normalisation \( \text{(3.3)} \). The conditions defining \( \mathcal{P} \), together with \( \text{(3.3)} \), implicitly define smooth (i.e. infinitely differentiable) maps \( R_{LU}(\Phi), A(\Phi), \) and \( \Lambda_{LU}(\Phi) \) on \( \mathcal{P} \) (see Lemma B.1). In this way, inference on the QCS may be rephrased in terms of inference on the parameters \( A = A(\Phi) \) defined by a smooth nonlinear transformation of the VAR coefficients.

### 3.3 Local-to-unity asymptotics

The QCS, and the associated coefficient matrix \( A \), remain identified so long as the roots of \( \Phi \) separate in the manner prescribed by QC. In particular, there is no requirement that the roots in \( \mathcal{L}^\rho_{LU} \) should drift towards unity at any rate, as \( n \to \infty \). However, the distributions of estimators and test statistics will typically be affected by the proximity of those \( q \) largest roots to unity, even in very large samples: we therefore need to work with a sequence of models that allows this dependence to be preserved in the limit. We shall accordingly develop our asymptotics under

**Assumption LOC.** \( \{y_t\} \) is generated under \( \text{(2.1)} \) with \( \Phi = \Phi_n \), where

(i) for some \( C \in \mathbb{R}^{q \times q} \)
\[ \Lambda_{LU}(\Phi_n) = \Lambda_{n,LU} := I_q + n^{-1}C; \quad \tag{3.5} \]

(ii) \( R_{LU}(\Phi_n) = [I_q] \) for some \( A \in \mathbb{R}^{r \times q} \); and

letting \( R_{n,ST}, \Lambda_{n,ST} \) and \( L_n = [L_{n,LU}, L_{n,ST}] \) be such that \( \text{(2.4)}-\text{(2.6)} \) hold for each \( n \):

(iii) \( R_{n,ST} = R_{ST} \) and \( \Lambda_{n,ST} = \Lambda_{ST} \) are fixed, and the eigenvalues of the latter lie strictly inside the complex unit circle.

Under LOC, \( y_t \) can be decomposed into a sum of deterministic, nearly integrated and stationary components. Indeed, we have in general that
\[ y_t - \mu - \delta t = x_t = \Phi_{LU} z_{LU,t-1} + \Phi_{ST} z_{ST,t-1} + \varepsilon_t \quad \tag{3.6} \]
where $\Phi_{LU} = \sum_{i=1}^{k} \Phi_i R_{LU} A_{LU}^{k-i} = R_{LU} A_{LU}^{k}$, and $z_{LU,t} \in \mathbb{R}^q$ and $z_{ST,t} \in \mathbb{R}^{k_p-q}$ follow

$$z_{LU,t} = \Lambda_{LU} z_{LU,t-1} + \varepsilon_{LU,t}$$
$$z_{ST,t} = \Lambda_{ST} z_{ST,t-1} + \varepsilon_{ST,t}$$

(3.7a) (3.7b)

(see Lemma A.4). Under LOC specifically, we have the joint weak convergences

$$n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \xrightarrow{d} E(r)$$
$$n^{-1/2} z_{LU,[nr]} \xrightarrow{d} \int_0^r e^{C(r-s)} L_{LU}^T dE(s) =: Z_C(r),$$

for $E$ a Brownian motion with variance $\Sigma$, and $L_{LU} = \lim_{n\to\infty} L_{n,LU}$; thus

$$n^{-1/2} x_{[nr]} = \Phi_{n,LU} n^{-1/2} z_{LU,[nr]} + o_p(1) =_{d} \Phi_{n,LU} Z_C(r) + o_p(1)$$

so that $z_{LU,t}$ and $x_t$ are nearly integrated. Although $\Phi_{n,LU} = R_{LU} A_{LU}^{k}$ depends on $n$, its column space does not, and

$$\beta^T x_t = \beta^T \Phi_{ST} z_{ST,t-1} + \beta^T \varepsilon_t \sim I(0).$$

(3.9)

Thus, analogously to the GJRT, (3.6) decomposes $x_t$ (and therefore also $y_t$, upon detrending) into the sum of a nearly integrated component and an $I(0)$ component; the quasi-cointegrating relations are precisely those that eliminate the nearly integrated common trends from $y_t$.

For developing the asymptotics of likelihood-based inference on $(A)$ and $(\Lambda_{LU})$, it is convenient to reparametrise the model the model in terms of $(\Phi_{LU}, \Phi_{ST})$, which isolates the nearly integrated and $I(0)$ components of $y_t$. The analysis performed in Appendix C shows that the information matrix in terms of (vectorised) $\Phi_{LU}$ and $\Phi_{ST}$ is asymptotically diagonal, with the ML estimator $\hat{\Phi}_{n,LU}$ converging at rate $n^{-1}$. Locally to the true parameters $\Phi_n$, the functionals $A(\Phi)$ and $\Lambda_{LU}(\Phi)$ depend only on perturbations of $\Phi_{LU}$ (see Lemma B.2), and thus the estimators of these quantities inherit this elevated rate of convergence.

We thus have the following theorem, whose proof appears in Appendix D. In order to state it, let $Z_C(r)$ denote the residual of an $L^2[0,1]$ projection of each sample path of $Z_C$ onto a constant and linear trend. We say that a random vector $\eta$ is mixed normal with mean zero and conditional variance $V$, denoted $\eta \sim MN[0, V]$, if $E e^{r^T \eta} = E e^{-\frac{1}{2} r^T V r}$. Denote the unrestricted and restricted estimators, when $\Lambda_{LU}(\Phi) = \Lambda_0 \in \mathbb{R}^{r \times q}$ is imposed, by

$$\hat{\Phi}_n := \text{argmax}_{\Phi \in \mathbb{R}^{p \times k_p}} \ell^*_n(\Phi)$$
$$\hat{\Phi}_n|\Lambda_0 := \text{argmax}_{\{\Phi \in \mathbb{R}^{p \times k_p} | \Lambda_{LU}(\Phi) = \Lambda_0\}} \ell^*_n(\Phi).$$

(For a discussion of how to compute the restricted estimates in practice, see Section 3.6 below.) With probability approaching one, $\hat{\Phi}_n$ will lie in $\mathcal{P}$, in which case the estimators $\hat{A}_n := A(\hat{\Phi}_n)$ and $\hat{\Lambda}_{n,LU} := \Lambda_{LU}(\hat{\Phi}_n)$ are well defined. Let $\hat{A}_n|\Lambda_0 := A(\hat{\Phi}_n|\Lambda_0)$ denote the estimate of $A$ implied by the restricted estimator $\hat{\Phi}_n|\Lambda_0$. Define $L_{LU} := \lim_{n\to\infty} L_{LU}(\Phi_n)$ and $L_{LU,\perp}$ to be any $p \times r$ matrix spanning $\text{sp} L_{LU}^\perp$; one possible choice is $\alpha := \lim_{n\to\infty} \Phi_n(1) \beta (\beta^T \beta)^{-1}$. 
Theorem 3.1. Suppose DGP and LOC hold. Then

(i) \( \hat{A}_n := A(\hat{\Phi}_n) \) and \( \hat{A}_{n,LU} := \Lambda_{LU}(\hat{\Phi}_n) \) satisfy

\[
\begin{align*}
\frac{1}{n} \left[ \hat{A} - A - \Lambda_{n,LU} \right] & \overset{d}{\to} \beta^T R_{ST}(I - \Lambda_{ST})^{-1} L_{ST}^T, \\
\end{align*}
\]

(ii) \( n \text{vec}(\hat{A}_{n,|\Lambda_{n,LU}} - A) \overset{d}{\to} \text{MN}[0, V_{zz} \otimes V_{\varepsilon \varepsilon}] \), where

\[
V_{zz} \otimes V_{\varepsilon \varepsilon} := \left( \int Z_C Z_C^T \right)^{-1} \otimes J L_{LU,\perp} (L_{LU,\perp}^{-1} \Sigma_{LU,\perp}^{-1} L_{LU,\perp})^{-1} L_{LU,\perp}^T \otimes (\alpha^T \Sigma^{-1} \alpha)^{-1}
\]

for \( J := \beta^T R_{ST}(I - \Lambda_{ST})^{-1} L_{ST}^T \).

The limiting distribution of the unrestricted ML estimator of \( A \) thus depends on \( C \), which cannot be consistently estimated. However, if the correct value of \( \Lambda_{LU} \) is imposed, then the restricted ML estimator \( \hat{A}_{n,|\Lambda_{n,LU}} \) is asymptotically mixed normal; a result that generalises those obtained in the special case when \( \Lambda_{LU} = I_q \) is correctly imposed. (See e.g. Johansen, 1995, Thm. 13.3, noting the differences between that result and (3.11) are entirely a consequence of the different assumptions made on the deterministic terms in the VAR.) Though we shall not give a formal proof here, it may be shown that the model likelihood is locally asymptotically mixed normal (LAMN), so that \( \hat{A}_{n,|\Lambda_{n,LU}} \) also inherits the large-sample efficiency properties familiar from the case of exact unit roots (Phillips, 1991).

Though part (i) of the preceding result could be used as the basis for inference on \( A \) using Wald-type statistics, there are some difficulties with this approach in practice, due to there being no guarantee that the characteristic roots of the unrestrictedly estimated VAR will ‘separate’ in the manner desired. Since these roots come in conjugate pairs, it may well be the case that when ordered in terms of their complex modulus (or proximity to real unity), the \( q \)th and \( (q + 1) \)th roots will be complex conjugates, preventing us from isolating the ‘first’ \( q \) roots from the rest – a problem exacerbated by typically imprecise estimation of these roots (see Onatski and Uhlig, 2012). Our preferred approach therefore utilises (quasi-) likelihood ratio (LR) tests to perform inference on both \( \Lambda_{LU} \) and \( A \); specifically the statistics

\[
\begin{align*}
\mathcal{L}_R(n)(\Lambda_0) & := 2 \max_{\{\Phi \in \mathcal{P}_{|\Lambda_{LU}(\Phi) = \Lambda_0}\}} \ell^*_n(\Phi) - \max_{\{\Phi \in \mathcal{P}_{|\Lambda_{LU}(\Phi) = \Lambda_0}\}} \ell^*_n(\Phi), \\
\mathcal{L}_R(n)(a_0; \Lambda_0) & := 2 \max_{\{\Phi \in \mathcal{P}_{|\Lambda_{LU}(\Phi) = \Lambda_0}\}} \ell^*_n(\Phi) - \max_{\{\Phi \in \mathcal{P}_{|\Lambda_{LU}(\Phi) = \Lambda_0, a_0}\}} \ell^*_n(\Phi)
\end{align*}
\]

where \( \mathcal{P} \) denotes an appropriate parameter space for \( \Lambda_{LU} \) (to be discussed in Section 3.4 below). \( \mathcal{L}_R(n)(\Lambda_0) \) is thus the usual LR test for \( H_0 : \Lambda_{LU}(\Phi) = \Lambda_0 \), while \( \mathcal{L}_R(n)(a_0; \Lambda_0) \) corresponds to the LR test of \( H_0 : a_{ij}(\Phi) = a_0 \), when \( \Lambda_{LU}(\Phi) = \Lambda_0 \) is maintained under both the null and the

\[
1\text{Recall } \bar{Z}_C(r) = Z_C(r) - \mu_0 - \mu_1 r, \text{ for } \mu_0 := \int_0^q (4 - 6s)B(s) \, ds \text{ and } \mu_1 = \int_0^q (-6 + 12s)B(s) \, ds \text{ (see e.g. Elliott, 1998, p. 151). Since } \bar{Z}_C \text{ is not adapted, an expression such as } \int \bar{Z}_C(dE)^T \text{ should be understood as a convenient shorthand for } \int \bar{Z}_C(dE)^T - \mu \int (dE)^T - \mu_1 \int r(dE)^T.
\]

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alternative. The asymptotic distributions of these test statistics are given by the next result; for given $C \in \mathbb{R}^{p \times q}$, let
\[
C_* := (L_{LU}^T \Sigma L_{LU})^{-1/2} C (L_{LU}^T \Sigma L_{LU})^{1/2}
\]
where $M^{1/2}$ denotes the principal square root of the positive semi-definite matrix $M$.

**Theorem 3.2.** Suppose DGP and LOC hold. Then
\[
\mathcal{L}R_n(\Lambda_{n,LU}) \sim \text{tr} \left\{ \int (dW_*) \bar{Z}_C (\int \bar{Z}_C (dW_*)^T) \right\} \tag{3.13}
\]
where $W_* \sim \text{BM}(I_q)$, $\bar{Z}_C$ is the residual from an $L^2[0,1]$ projection of the sample paths of $Z_C(r) := \int_0^r e^{C_+ (r-s)} dW_*(s)$ onto a constant and linear trend; and
\[
\mathcal{L}R_n[a_{ij}(\Phi_n); \Lambda_{n,LU}] \sim \chi^2_1. \tag{3.14}
\]

### 3.4 Parameter space for $\Lambda_{LU}$

Theorem 3.2 leads naturally to Bonferroni-based inference on $A$; there is an analogy here with predictive regression, if we regard $A$ and $\Lambda_{LU}$ as corresponding to the regression coefficients and the autoregressive matrix of the regressor process; indeed the inferential procedure outlined in Section 3.5 below is closely related to the $Q$-test Bonferroni procedure of Campbell and Yogo (2006). However, this analogy is imperfect, because in a predictive regression there is no reason to place any restrictions on the parameter space $\mathcal{L}$ for $\Lambda_{LU}$, beyond perhaps requiring that $\Lambda_{LU}$ should have all its eigenvalues less than unity. Whereas in the present setting, the eigenvalues of $\Lambda_{LU}$ should also be bounded from below, if the model is to be consistent with $r = p - q$ linear combinations of the original series being measurably less persistent than the series themselves. Since the specification of $\mathcal{L}$ is of critical importance to the performance of any inferential procedure, we first provide a discussion this issue.

When $q = 1$ we have $\mathcal{L} = [\rho, 1]$, and we need only to choose a lower bound for the largest root of $\Phi$. Via the impulse response function (2.6), $\rho$ can be readily interpreted in terms of the minimum half-life of the most persistent shocks $(L_{LU}^T \varepsilon_t)$ driving $y_t$, as $h := -\log 2 / \log \rho$ periods. $h$ may itself be chosen with reference to the extent of robustness that is deemed desirable for the application at hand. For example, in a macroeconomic context, it seems appropriate to allow that the most persistent shocks to $y_t$ may not have permanent effects, but still have a half-life somewhat longer than the average duration of the business cycle: with postwar US data of annual frequency, this might justify setting $h = 8$ or $10$ and thus $\rho = 2^{-1/h} = 0.917$ or 0.933.

When $q \geq 2$, $\mathcal{L}$ is some set of matrices with eigenvalues lying in the interval $[\rho, 1]$. In this case, the same considerations as when $q = 1$ should inform the choice of $\rho$, but this does not fully determine $\mathcal{L}$. One possibility is to take $\mathcal{L}$ to be the subset $\mathcal{L}_d$ of real, diagonalisable $q \times q$ matrices. A potential difficulty with $\mathcal{L}_d$ is that is that some non-diagonalisable matrices are in

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2Phillips (2014) demonstrates that the Campbell and Yogo’s procedure does not have the correct asymptotic size in the stationary region, because their confidence intervals for the largest autoregressive root (of the regressor) are constructed by inverting a $t$ test that is centred on unity, rather than on the null value of the root (see also Mikusheva, 2007). Since the likelihood ratio statistic (3.12a) is centred on the null value of $\Lambda_{LU}$, these difficulties do not arise in the present setting.
its closure, as can be seen e.g. by taking the limit of \[ \begin{bmatrix} \lambda + \epsilon & 1 \\ 0 & \lambda - \epsilon \end{bmatrix} \] as \( \epsilon \to 0 \). This in effect permits departures from the I(1)/I(2) cointegrated VAR model in the direction of a model with some I(0) components – whereas the concern of this paper is with departures from that model in the direction of stationarity. We therefore regard either the subsets of \( L_d \) consisting of the normal \((L_n)\) or symmetric \((L_s)\) matrices as being more appropriate choices, the only difference between the two being that the former allows for complex eigenvalues.3 (Of course, in both cases \( \Lambda_{LU} \) is itself a real matrix.)

### 3.5 Point estimates and confidence intervals

Having specified \( \mathcal{L} \), ‘unrestricted’ point estimates for the model parameters can be computed as \( \hat{\Phi}_n|\mathcal{L} := \arg\max_{\Phi \in \mathcal{P}} \{ \ell^*_n(\Phi) \} \), and the implied estimates for \( A \) recovered by applying an invariant subspace decomposition to \( \hat{\Phi}_n|\mathcal{L} \). Details on the numerical implementation of this calculation are given in the following section. We may also use Theorem 3.2 to develop Bonferroni-based inference on a given element \( a_{ij} \) of \( A \). Let

\[
\begin{align*}
C_\Lambda(\alpha_1) &:= \{ \Lambda_0 \in \mathcal{L} \mid LR_n(\Lambda_0) \leq c_{1-\alpha_1}[n(\Lambda_0 - I_q)] \} \\
C_{a_{ij}|\Lambda_0}(\alpha_2) &:= \{ a_0 \in \mathbb{R} \mid LR_n(a_0; \Lambda_0) \leq \chi^2_{1,1-\alpha_2} \}
\end{align*}
\]

denote a \( 1 - \alpha_1 \) confidence set for \( \Lambda \), and a \( 1 - \alpha_2 \) confidence set for \( a_{ij} \) conditional on an imposed \( \Lambda_0 \in \mathcal{L} \); here \( c_\tau \) and \( \chi^2_{1,\tau} \) denote the \( \tau \)th quantiles of the distribution in (3.13) and a \( \chi^2_{1} \) distribution, respectively. As is well known, a Bonferroni-based confidence interval for \( a_{ij} \), with level \( 1 - \alpha \), can then be constructed as

\[
C_B(\alpha_1, \alpha_2) := \bigcup_{\Lambda_0 \in C_\Lambda(\alpha_1)} C_{a_{ij}|\Lambda_0}(\alpha_2),
\]

by taking \( \alpha_1 + \alpha_2 = \alpha \). Since this yields inferences on \( a_{ij} \) that are necessarily conservative, refinements along lines proposed by Cavanagh, Elliott, and Stock (1995) and Campbell and Yogo (2006) in the context of predictive regression (an approach that has since been further extended by McCloskey, 2017), will be considered by the authors in a subsequent paper, and their finite-sample performance evaluated in comparison with that of other possible approaches (and with ‘conventional’ approaches that impose \( \Lambda_{LU} = I_q \)).

### 3.6 Numerical implementation

Computation of \( C_B(\alpha_1, \alpha_2) \) involves maximising \( \ell^*_n(\Phi) \) subject to the restrictions that \( \Phi \in \mathcal{P} \) and \( \Lambda_{LU}(\Phi) = \Lambda_0 \) for some specified \( \Lambda_0 \in \mathcal{L} \), and possibly also that \( a_{ij}(\Phi) = a_0 \) for some

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3It might be asked why we do not also consider the class of diagonal (as distinct from diagonalisable) matrices. Because we impose the normalisation (3.3) on \( R_{LU} \), this would amount to a substantive restriction on \( \Phi \), and not one that we believe would be appropriate. In particular, it would imply under (3.5) that the last \( q \) elements of \( y_t = (y_{t1}, \ldots, y_{tp})^T \) would each be dependent on only a single (and distinct) element of the near integrated process \( z_{LU,t} \) (see (3.7a) above).
To implement this numerically, we suggest introducing the constraint
\[
\begin{bmatrix}
A \\
I_q
\end{bmatrix}
A_0^k - \sum_{i=1}^{k} \Phi_i \begin{bmatrix}
A \\
I_q
\end{bmatrix} A_0^{k-i} = 0,
\]
which is derived from (2.5) and (3.3) above: it forces \( \Phi(\lambda) \) to have roots at the eigenvalues of \( A_0 \), and the associated \( R_{LU} \) matrix to respect the normalisation (3.3). Then proceed as follows:

(i) Given \( A \in \mathbb{R}^{r \times q} \) and \( A_0 \in \mathcal{L} \), maximise \( \ell_n^*(\Phi) \) over \( \Phi \in \mathbb{R}^{p \times kp} \), subject to (3.15), to obtain the restricted MLE \( \hat{\Phi}_{n|A,A_0} \). (A straightforward calculation, since (3.15) is a linear restriction on \( \Phi \): see Lütkepohl, 2007, Ch. 7.)

(ii) As the ‘outer loop’ of the optimisation procedure, compute

\[
\max_{A \in \mathbb{R}^{r \times q}} \ell_n^*(\hat{\Phi}_{n|A,A_0}).
\]

The maximum of \( \ell_n^*(\Phi) \) subject to \( \Lambda_{LU}(\Phi) = \Lambda_0 \) and \( a_{ij}(\Phi) = a_0 \) can be computed similarly, by holding \( a_{ij} \) constant in (3.16). Point estimates of \( \Phi \) can be calculated by maximising \( \Phi_{n|A,A_0} \) over both \( A \) and \( A_0 \).

When \( q = 1 \), and in the special case where \( A_0 = \lambda_0 I_q \), (3.16) can be even more simply calculated. In this case, we may rewrite the reduced form model (3.1) as

\[
\Delta_{\lambda_0} y_t = m + dt - \Phi(\lambda_0) y_{t-1} + \sum_{i=1}^{p-1} \Psi_i \Delta_{\lambda_0} y_{t-i}
\]

where \( \Delta_{\lambda_0} y_t := y_t - \lambda_0 y_{t-1} \) denotes a quasi-difference (see Theorem 1 and Corollary 2 of Johansen and Schaumburg, 1999, which hold even if \( \lambda_0 \) does not lie on the unit circle). Since \( \Phi(\lambda_0) \) has rank \( p-q = r \), \( \ell_n^*(\Phi) \) can then be efficiently maximised, subject to \( \Lambda_{LU}(\Phi) = \lambda_0 I_q \), via a reduced rank regression, exactly as in Johansen (1995, Ch. 6).

When \( q \geq 2 \), some care needs to be taken with the parametrisation of \( \mathcal{L} \). If we take this to be the set of real normal (\( \mathcal{L}_n \)) or symmetric (\( \mathcal{L}_s \)) matrices, then each \( \Lambda_{LU} \in \mathcal{L} \) can be expressed as \( \Lambda_{LU} = QD_{LU}Q^T \), where \( Q \in \mathbb{R}^{q \times q} \) is an orthogonal matrix (\( Q^T Q = I_q \)) and \( D_{LU} \) is a block diagonal, with blocks that are either: \( 1 \times 1 \) and equal to each of the real eigenvalues of \( \Lambda_{LU} \), or \( 2 \times 2 \) and of the form \( \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \), if \( \Lambda_{LU} \) has a pair of complex eigenvalues at \( \lambda = a \pm ib \) (Horn and Johnson, 2013, Thm. 2.5.6 and 2.5.8). Since \( Q \) can be constructed from \( q \) plane rotations (Horn and Johnson, 2013, Prob. 2.1.P29), both \( \mathcal{L}_n \) and \( \mathcal{L}_s \) can thus be expressed in terms of of \( q(q+1)/2 \) free parameters lying in a compact set.

### 3.7 Deterministic terms

For the cointegrated VAR with exact unit roots, Johansen (1995, Sec. 5.7) develops a hierarchy of models – in his notation, \( H_2 \subset H_1^* \subset H_1 \subset H^* \subset H \) – ordered according to their treatment of the deterministic terms in the reduced form model (3.1). In our more general setting where \( \Lambda_{LU} = I_q \) is not required, these models take on a slightly altered expression, and not all are realisable through restrictions on the model parameters.
To discuss how the deterministic terms might be treated, and possible the consequences of this for inference, we first recall that the mapping from the ‘structural’ VAR (2.1) to the ‘reduced form’ VAR (3.1) implies that

\[ m = \Phi(1) \mu + \Psi \delta \]
\[ d = \Phi(1) \delta \]

where \( \Psi := \sum_{i=1}^{k} i \Phi_i \). Three important cases are the following:

(i) \( \mu, \delta \) unrestricted. The reduced form VAR (3.1) should estimated with \( (m, d) \) unrestricted (as per Johansen’s model \( H \)). Our asymptotics assume that the DGP is the structural VAR (2.1), so that \( d = \Phi(1) \delta \) holds even though this is not imposed in estimation. Indeed, it would not be possible to impose the restriction \( d \in \text{sp} \Phi(1) \) (as per Johansen’s \( H^* \)) in the present setting, because whenever the largest roots of \( \Phi \) are not exactly unity, \( \Phi(1) \) has full rank, and so \( d \) is unrestricted – and thus a model with exact unit roots and \( d \notin \Phi(1) \) lies in the closure of the parameter space.

(ii) \( \mu \) unrestricted, \( \delta = 0 \). The VAR (3.1) should be estimated with only a constant (as in Johansen’s model \( H_1 \)). Under the assumption that the DGP is the structural VAR with \( \delta = 0 \), \( y_t \) has no drift. The asymptotic distributions given in Theorems 3.1 and 3.2 must be amended in this case, by replacing each instance of \( \bar{Z}_C \) with the demeaned diffusion process \( Z_C(r) - \int_0^1 Z_C(s) \, ds \). (Imposing the restriction that \( m \in \text{sp} \Phi(1) \), as per Johansen’s model \( H^*_1 \), is impossible in our setting.)

(iii) \( \mu = \delta = 0 \). The VAR (3.1) should be estimated with \( m = d = 0 \) (as per Johansen’s model \( H_2 \)); in Theorems 3.1 and 3.2, \( \bar{Z}_C \) is replaced by \( Z_C \).

Thus our recommendation is to estimate the model with an unrestricted intercept and trend if there is a discernable linear drift in the data, and to otherwise estimate the model with only an intercept.

There is a fourth important case, which sits in between the first two, in which a linear trend is present in \( y_t \) but is assumed to be eliminated by the quasi-cointegrating relationships, whence \( \beta^\top \delta = 0 \). Since \( \beta \) spans the orthocomplement of \( R_{LU} \), this is equivalent to requiring \( d \in \text{sp} \Phi(1) R_{LU} \). If we assume exact unit roots, then \( \Phi(1) R_{LU} = 0 \) (from (2.5) above) and this restriction can be imposed simply by estimating the reduced form VAR without a trend (as in Johansen’s model \( H_1 \)). However, in our setting with non-unit roots this restriction cannot be so simply expressed, because \( \Phi(1) \) may have full rank; all that can be said is that \( d \in \text{sp} \Phi(1) R_{LU} \).

Estimation under this restriction is accordingly more involved, and we leave the development of the asymptotics of our procedure in this case for future work.

4 Conclusion

This paper has developed a characterisation of cointegration that extends naturally to a VAR with non-unit roots, under which the long-run equilibrium relationships between the series are identified with those directions in which the implied impulse responses decay most rapidly. The subspace spanned by those directions, which we have termed the quasi-cointegrating space,
can be estimated by maximum likelihood. Likelihood-based inference faces similar challenges
to inference in predictive regressions, and the performance of procedures developed in that
context, modified so as to be applicable to the present setting, will be evaluated by the authors
in a subsequent paper.

5 References

Beare, B. K., and W.-K. Seo (2019): “Representation of I(1) and I(2) autoregressive Hilbertian processes,” arXiv:1701.08149v3.

Campbell, J. Y., and M. Yogo (2006): “Efficient tests of stock return predictability,” Journal of Financial Economics, 81(1), 27–60.

Cavanagh, C. L., G. Elliott, and J. H. Stock (1995): “Inference in models with nearly integrated regressors,” Econometric Theory, 11(5), 1131–1147.

d’Autume, A. (1992): “Deterministic dynamics and cointegration of higher orders,” Discussion paper, Université de Paris I.

Elliott, G. (1998): “On the robustness of cointegration methods when regressors almost have unit roots,” Econometrica, 66(1), 149–158.

Elliott, G., U. K. Müller, and M. W. Watson (2015): “Nearly optimal tests when a nuisance parameter is present under the null hypothesis,” Econometrica, 83(2), 771–811.

Franchi, M., and S. Johansen (2017): “Improved inference on cointegrating vectors in the presence of a near unit root using adjusted quantiles,” Econometrics, 5(2), 25–44.

Franchi, M., and P. Paruolo (2019): “A general inversion theorem for cointegration,” Econometric Reviews, 38(10), 1176–1201.

Gohberg, I., S. Lancaster, and L. Rodman (1982): Matrix Polynomials. Academic Press, London (UK).

Hall, P., and C. C. Heyde (1980): Martingale Limit Theory and Its Application. Academic Press, New York (USA).

Horn, R. A., and C. R. Johnson (2013): Matrix Analysis. C.U.P., New York (USA), 2nd edn.

Johansen, S. (1995): Likelihood-based Inference in Cointegrated Vector Autoregressive Models. O.U.P.

Johansen, S., and E. Schaumburg (1999): “Likelihood analysis of seasonal cointegration,” Journal of Econometrics, 88, 301–339.

Kostakis, A., T. Magdalinos, and M. P. Stamatogiannis (2015): “Robust econometric inference for stock return predictability,” Review of Financial Studies, 28(5), 1506–1553.

Lang, S. (1993): Real and Functional Analysis. Springer, New York (USA), 3rd edn.

Lütkepohl, H. (2007): New Introduction to Multiple Time Series Analysis. Springer, 2nd edn.

Magdalinos, T., and P. C. B. Phillips (2009): “Econometric inference in the vicinity of unity,” CoFie Working Paper 7, Singapore Management University.
Magnus, J. R., and H. Neudecker (2007): *Matrix Differential Calculus with Applications in Statistics and Econometrics*. Wiley, Chichester (UK), 3rd edn.

McCloskey, A. (2017): “Bonferroni-based size-correction for nonstandard testing problems,” *Journal of Econometrics*, 200(17–35).

Mikusheva, A. (2007): “Uniform inference in autoregressive models,” *Econometrica*, 75(5), 1411–1452.

Müller, U. K., and M. W. Watson (2013): “Low-frequency robust cointegration testing,” *Journal of Econometrics*, 174(2), 66–81.

Nielsen, B. (2010): “Analysis of coexplosive processes,” *Econometric Theory*, 26, 882–915.

Onatski, A., and H. Uhlig (2012): “Unit roots in white noise,” *Econometric Theory*, 28, 485–508.

Phillips, P. C. B. (1988): “Regression theory for near-integrated time series,” *Econometrica*, 56(5), 1021–1043.

Phillips, P. C. B. (1991): “Optimal inference in cointegrated systems,” *Econometrica*, 59(2), 283–306.

Phillips, P. C. B. (2014): “On confidence intervals for autoregressive roots and predictive regression,” *Econometrica*, 82(3), 1177–1195.

Phillips, P. C. B., and B. E. Hansen (1990): “Statistical inference in instrumental variables regression with I(1) processes,” *Review of Economic Studies*, 57(1), 99–125.

Phillips, P. C. B., and J. H. Lee (2013): “Predictive regression under various degrees of persistence and robust long-horizon regression,” *Journal of Econometrics*, 177(2), 250–264.

Stewart, G. W., and J.-g. Sun (1990): *Matrix Perturbation Theory*. Academic Press, Boston (USA).

Stock, J. H., and M. W. Watson (1993): “A simple estimator of cointegrating vectors in higher order integrated systems,” *Econometrica*, 61(4), 783–820.

Sun, J.-g. (1991): “Perturbation expansions for invariant subspaces,” *Linear Algebra and its Applications*, 153, 85–97.

Wright, J. H. (2000): “Confidence sets for cointegrating coefficients based on stationarity tests,” *Journal of Business and Economic Statistics*, 18(2), 211–222.
Appendices

Notation. For \( x \in \mathbb{R}^p \) and \( A \in \mathbb{R}^{p \times p} \), \( \|x\| \) denotes the Euclidean norm and \( \|A\| := \sup_{\|x\|=1} \|Ax\| \) the induced matrix norm.

A Representation theory

This section provides results that support some of the assertions made in the course of Sections 2 and 3, and which are auxiliary to results proved in the following appendices. Some are well known, but are collected here for ease of reference. Proofs follow at the end of this appendix.

For VAR coefficients \( \Phi := (\Phi_1, \ldots, \Phi_k) \in \mathbb{R}^{p \times kp} \), let

\[
F := F(\Phi) := \begin{bmatrix}
\Phi_1 & \Phi_2 & \cdots & \Phi_{k-1} & \Phi_k \\
I & 0 & \cdots & 0 & 0 \\
0 & I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & 0
\end{bmatrix}
\tag{A.1}
\]

denote the associated companion form matrix. For a collection of \( m \times n \) matrices \( Z_1, \ldots, Z_k \), let

\[
\text{col}\{Z_i\}_{i=1}^k := \begin{bmatrix}
Z_1 \\
\vdots \\
Z_k
\end{bmatrix}
\]

so that taking \( x_t := \text{col}\{x_{t-i}\}_{i=0}^{k-1} \), we may write (2.1) as

\[
x_t = Fx_{t-1} + \begin{bmatrix}
\varepsilon_t \\
0_{(k-1)p \times 1}
\end{bmatrix}
\tag{A.2}
\]

Let \( \lambda_i(\Phi) \) denote the \( i \)th root of the characteristic polynomial associated to \( \Phi \), when these are placed in descending order of modulus.

Lemma A.1. Suppose that \( |\lambda_q(\Phi)| > |\lambda_{q+1}(\Phi)| \) for some \( q \in \{1, \ldots, p\} \). Then there exist there matrices \( R \in \mathbb{R}^{p \times kp} \), \( \Lambda \in \mathbb{R}^{kp \times kp} \) and \( L \in \mathbb{R}^{p \times kp} \) such that:

(i) \( \Lambda = \text{diag}\{\Lambda_{LU}, \Lambda_{ST}\} \), where the eigenvalues of \( \Lambda_{LU} \in \mathbb{R}^{q \times q} \) and \( \Lambda_{ST} \) are \( \{\lambda_i(\Phi)\}_{i=1}^q \) and \( \{\lambda_i(\Phi)\}_{i=q+1}^{kp} \) respectively;

(ii) the following hold:

\[
RA^k - \sum_{i=1}^k \Phi_i RA^{k-i} = 0 \quad \Lambda^k L^T - \sum_{i=1}^k \Lambda^{k-i} L^T \Phi_i = 0. \tag{A.3}
\]

(iii) \( R := \text{col}\{RA^{k-1}\}_{i=1}^k \) is invertible, and \( L \) equals the first \( p \) rows of \( L := (R^{-1})^T \);

(iv) \( F(\Phi) = RAL^T \); and
Further, the matrices $R^* \in \mathbb{R}^{p \times kp}$, $A^* \in \mathbb{R}^{kp \times kp}$ and $L^* \in \mathbb{R}^{p \times kp}$ satisfy conditions (i)–(v) if and only if there exists an invertible $kp \times kp$ matrix $Q = \text{diag}\{Q_{LU}, Q_{ST}\}$, where $Q_{LU} \in \mathbb{R}^{q \times q}$, such that $R^* = RQ$, $A^* = Q^{-1}AQ$ and $L^* = L(Q^T)^{-1}$.

For a given $\Phi$, and its associated companion form $F = F(\Phi)$, we shall routinely partition the matrices appearing in Lemma A.1 as

$$R := [R_{LU}, R_{ST}] \quad R := [R_{LU}, R_{ST}] \quad L := [L_{LU}, L_{ST}] \quad L := [L_{LU}, L_{ST}] \quad (A.4)$$

where each of $R_{LU}$, $R_{ST}$, $L_{LU}$ and $L_{ST}$ have $q$ columns, i.e. the partitioning is conformable with that of $\Lambda = \text{diag}\{\Lambda_{LU}, \Lambda_{ST}\}$. This partitioning, in conjunction with parts (ii) and (v) of the preceding lemma, yields (2.5) and (2.6) above. Moreover, we may write part (iv) as

$$F = R\Lambda L^T = R_{LU}\Lambda_{LU}L_{LU}^T + R_{ST}\Lambda_{ST}L_{ST}^T \quad (A.5)$$

which decomposes $F$ with respect to the invariant subspaces associated to the eigenvalues of $\Lambda_{LU}$ and $\Lambda_{ST}$.

**Lemma A.2.** Suppose that $|\lambda_q(\Phi)| > |\lambda_{q+1}(\Phi)|$ for some $q \in \{1, \ldots, p\}$, the eigenvalues of $\Lambda_0 \in \mathbb{R}^{q \times q}$ are all greater than $|\lambda_{q+1}(\Phi)|$ in modulus, and $R_0 \in \mathbb{R}^{p \times q}$ is a full column rank matrix such that

$$R_0 \Lambda_0^k - \sum_{i=1}^k \Phi_i R_0 \Lambda_0^{k-i} = 0. \quad (A.6)$$

Then there exist matrices $R = [R_{LU}, R_{ST}]$, $\Lambda = \text{diag}\{\Lambda_{LU}, \Lambda_{ST}\}$ and $L$ satisfying the conditions of Lemma A.1, with $R_{LU} = R_0$ and $\Lambda_{LU} = \Lambda_0$.

For the next result, recall the definition of $S_r$ given in the context of (2.2) above.

**Lemma A.3.** Suppose DGP holds.

(i) If QC holds for some $\rho \in (0, 1]$, then $S_r = (\text{sp}\, R_{LU})^\perp$.

(ii) If CV holds, then CS = $S_r = (\text{sp}\, R_{LU})^\perp$, and QC holds with $\rho = 1$.

**Lemma A.4.** Suppose DGP and QC hold. Let $\Lambda = \text{diag}\{\Lambda_{LU}, \Lambda_{ST}\}$, $R = [R_{LU}, R_{ST}]$ and $L = [L_{LU}, L_{ST}]$ be as in Lemma A.1 and (A.4). Then (3.6)–(3.7) hold with $\Phi \Lambda = R_{LU}\Lambda_{LU}^k$, $\Phi \Lambda = R_{ST}\Lambda_{ST}^k$, $z_{LU,t} := L_{LU}^T x_t$ and $z_{ST,t} := L_{ST}^T x_t$.

**Proof of Lemma A.1.** Let $J$ denote a $(kp \times kp)$ real Jordan matrix similar to $F$, each of whose diagonal blocks correspond to roots of $\Phi(-)$, so that $P^{-1}FP = J$ for some $P \in \mathbb{R}^{kp \times kp}$. We may take the diagonal blocks of $J$ to be ordered such that $J = \text{diag}\{J_{LU}, J_{ST}\}$, where $J_{LU} \in \mathbb{R}^{q \times q}$ has all its eigenvalues in $\mathcal{L}_{LU}^p$. Letting

$$X := [0_p \quad \cdots \quad 0_p \quad I_p] P$$
we have by Gohberg, Lancaster, and Rodman (1982, Thm. 1.24 and 1.25) that the matrices \((X, J)\) form a standard pair for \(\Phi()\).\(^4\) Therefore,

\[ X J^k - \sum_{i=1}^{k} \Phi_i X J^{k-i} = 0, \]

and \(\col\{ X J^{k-i} \}_{i=1}^{k} = P\) is invertible, so that the matrix

\[ Y := [I_p \cdots 0_p 0_p](P^T)^{-1} \]

is well defined. By Gohberg, Lancaster, and Rodman (1982, Prop. 2.1), \((Y, J)\) satisfy

\[ J^k Y^T - \sum_{i=1}^{k} J^{k-i} Y^T \Phi_i = 0. \]

Parts (i)–(iv) of the lemma are thus satisfied with \((R, \Lambda, L^T, R, L^T) = (X, J, Y^T, P, P^{-1})\). It further follows by recursive substitution that

\[ \text{IRF}_s = [F^s]_{11} = [RL^sL^T]_{11} = RL^{k-1+s}L^T \]

where \([A]_{11}\) denotes the upper left \(p \times p\) block of the matrix \(A\); thus part (v) is proved.

Finally, let \(Q = \text{diag}\{Q_{LU}, Q_{ST}\}\) be as in the final part of the lemma. It is easily verified that

\[ \Lambda_\ast := \text{diag}\{Q_{LU}^{-1}L_{LU}Q_{LU}, Q_{ST}^{-1}L_{ST}Q_{ST}\} = Q^{-1}\Lambda Q, \]

\(R_\ast := RQ\) and \(L_\ast := L(Q^T)^{-1}\) have the required properties. Conversely, if both \((R, \Lambda, L)\) and \((R_\ast, \Lambda_\ast, L_\ast)\) satisfy conditions (i)–(v), then both \(\Lambda\) and \(\Lambda_\ast\) are block diagonal matrices similar to \(J = \text{diag}\{J_{LU}, J_{ST}\}\), whence there exists \(Q = \text{diag}\{Q_{LU}, Q_{ST}\}\) such that \(\Lambda_\ast = Q^{-1}\Lambda Q\), etc. \(\square\)

**Proof of Lemma A.2.** \(R_0 := \col\{R_0 \Lambda_0^{k-1} \}_{j=1}^{k} \in \mathbb{R}^{kp \times kp}\) has rank \(q\), and (A.6) implies that \(FR_0 = R_0 \Lambda_0\), for \(F := F(\Phi)\). Since the remaining \(kp - q\) eigenvalues of \(F\) are distinct from the eigenvalues of \(\Lambda_0\), \(R_0\) is a simple invariant subspace of \(F\) (Stewart and Sun, 1990, Defn V.1.2). Hence there exist \(R, \Lambda, L \in \mathbb{R}^{kp \times kp}\) such that \(F = R\Lambda L^T\) and \(L^T R = I_{kp}\), and \(R\) and \(\Lambda\) can be partitioned as \(R = [R_0, R_{ST}]\) and \(\Lambda = \text{diag}\{\Lambda_0, \Lambda_{ST}\}\) (Stewart and Sun, 1990, Thm V.1.5). Since \(\Lambda_0\) and \(\Lambda_{ST}\) must be similar to the blocks \(J_{LU}\) and \(J_{ST}\) of the real Jordan form of \(F\), as introduced in the proof of Lemma A.1, the result then follows by the same arguments as were given in that proof. \(\square\)

**Proof of Lemma A.3.** (i). By Lemma A.1(v), for any \(b \in \mathbb{R}^p\),

\[ b^T \text{IRF}_s = b^T R\Lambda^{k-1+s}L^T = b^T R_{LU} L_{LU}^{k-1+s}L_{LU}^T + b^T R_{ST} L_{ST}^{k-1+s}L_{ST}^T. \]

(A.8)

Since the spectral radius of \(\Lambda_{ST}\) is strictly less than \(\rho\), we have by Horn and Johnson (2013,\(^4\)Note that the ‘first companion form’ matrix defined by these authors \((C_1\) on p. 13 of that work) equals \(F\) with the ordering of its rows and columns reversed, so our definitions of \(X\) (and below, \(Y\)) differ from theirs.

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Cor. 5.6.13) that
\[
\Lambda^t_{ST}/\rho^t \rightarrow 0 \quad (A.9)
\]
as \( t \to \infty \). Since \( \Lambda_{LU} \) is diagonalisable under \( QC2 \), by Lemma A.1 we may choose \((R_{LU}, \Lambda_{LU}, L_{LU})\) such that \( \Lambda_{LU} \) is a real Jordan block diagonal matrix (as in Corollary 3.4.1.10 of Horn and Johnson, 2013). The eigenvalues of \( \Lambda_{LU}^T \Lambda_{LU} = \Lambda_{LU} \Lambda_{LU}^T \) are therefore of the form \( |\lambda|^2 \), for \( \lambda \) an eigenvalue of \( \Lambda_{LU} \), and thus \( \lambda_{\min}(\Lambda_{LU}^T \Lambda_{LU}) \geq \rho^2 \), where \( \lambda_{\min}(M) \) denotes the smallest eigenvalue of a positive-definite matrix \( M \). Therefore letting \( x := R_{LU}^T b \),
\[
\|x^T \Lambda_{LU}^t L_{LU}^T \|^2 \geq \lambda_{\min}(L_{LU}^T L_{LU}) \|x^T \Lambda_{LU}^t \|^2 \\
\geq \rho \lambda_{\min}(L_{LU}^T L_{LU}) \|x^T \Lambda_{LU}^t - 1 \|^2 \geq \cdots \geq \rho^{2t} \lambda_{\min}(L_{LU}^T L_{LU}) \|x\|^2.
\]
\( \lambda_{\min}(L_{LU}^T L_{LU}) > 0 \), since \( L_{LU} \) has full column rank under \( QC2 \). Deduce that if \( b^T R_{LU} \neq 0 \), then
\[
\lim_{t \to \infty} \|b^T R_{LU} \Lambda_{LU}^t L_{LU}^T \|/\rho^t > 0.
\]
It follows from (A.8)–(A.10) that \( b^T \text{IRF}s/\rho^s \rightarrow 0 \) as \( s \to \infty \) if and only if \( b \perp \text{sp} R_{LU} \). Thus \((\text{sp} R_{LU})^\perp\) gives the unique \( r \)-dimensional subspace of \( \mathbb{R}^p \) satisfying the definition of \( S_r \).

(ii). Since \( \text{rk} \Phi(1) = p - q \) under \( CV2 \), there exists \( R_{LU} \in \mathbb{R}^{p \times q} \) having rank \( q \) such that
\[
0 = \Phi(1) R_{LU} = R_{LU} - \sum_{i=1}^k \Phi_i R_{LU} =_{(1)} R_{LU} \Lambda_{LU}^k - \sum_{i=1}^k \Phi_i R_{LU} \Lambda_{LU}^{k-i} \quad (A.11)
\]
where \( =_{(1)} \) follows by taking \( \Lambda_{LU} = I_q \). By a similar argument, here exists a \( L_{LU} \in \mathbb{R}^{p \times q} \) with \( \text{rk} \ L_{LU} = q \) and \( L_{LU}^T \Phi(1) = 0 \). \( CV \) is thus a special case of \( QC \) with \( \rho = 1 \). \( S_r = (\text{sp} R_{LU})^\perp \) therefore follows immediately from part (i) of the lemma. Finally, recall from the second characterisation of the cointegrating space given in Section 2.2 that \( CS = \{\ker \Phi(1)\}^\perp \). By (A.11) this also coincides with \((\text{sp} R_{LU})^\perp\).

**Proof of Lemma A.4.** By (A.2) and Lemma A.1,
\[
L^T x_t = L^T F x_{t-1} + L^T \epsilon_t = \lambda L^T x_{t-1} + L^T \epsilon_t.
\]
Since \( \Lambda = \text{diag}\{\Lambda_{LU}, \Lambda_{ST}\} \), it is clear that (3.7) holds for \( z_{LU,t} \) and \( z_{ST,t} \) as defined in the lemma. Further, taking the first \( p \) rows of (A.2) and using Lemma A.1 again yields
\[
x_t = R \Lambda^k L^T x_{t-1} + \epsilon_t = R_{LU} \Lambda_{LU}^k L_{LU}^T x_{t-1} + R_{ST} \Lambda_{ST}^k L_{ST}^T x_{t-1} + \epsilon_t.
\]

**Proof of Proposition 2.1.** This is an immediate corollary of Lemma A.3.
B Perturbation theory

Recall the definition of $\mathcal{P} \subset \mathbb{R}^{p \times kp}$ given in Section 3.2. The normalisation (3.3) entails that

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I}_q \end{bmatrix} \mathbf{A}^k_{LU} - \sum_{i=1}^k \Phi_i \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_q \end{bmatrix} \mathbf{A}^{k-i}_{LU} = 0 \quad (B.1)$$

which by Lemmas A.1 and A.2 uniquely determines $\mathbf{R}_{LU} = [\mathbf{A}^i_q]$ and $\mathbf{A}_{LU}$ as a function of $\Phi \in \mathcal{P}$. As in Section 3.2, we shall denote the implied mappings by $\mathbf{R}_{LU}(\Phi)$, $\mathbf{A}(\Phi)$, $\mathbf{A}_{LU}(\Phi)$, and $\mathbf{R}_{LU}(\Phi) := \text{col}\{\mathbf{R}_{LU}(\Phi)\mathbf{A}^{k-i}_{LU}(\Phi)\}_{i=1}^k$. Our first result is that these are smooth (i.e. infinitely differentiable); its proof and those of the subsequent lemmas appear at the end of this appendix.

**Lemma B.1.** $\mathcal{P}$ is open; and $\mathbf{A}(\Phi)$ and $\mathbf{A}_{LU}(\Phi)$ are smooth on $\mathcal{P}$.

Our next result gives the first derivatives of the maps $\mathbf{A}(\Phi)$ and $\mathbf{A}_{LU}(\Phi)$; it is closely related to Theorem 2.1 in Sun (1991). To express these derivatives more concisely, let

$$\mathbf{B}(\Phi) := (\mathbf{I}_q \otimes \mathbf{R}_{ST})[(\mathbf{A}^T_{LU} \otimes \mathbf{I}_{kp-q}) - (\mathbf{I}_q \otimes \mathbf{A}_{ST})]^(-1)(\mathbf{I}_q \otimes \mathbf{L}^T_{ST}), \quad (B.2)$$

where we have suppressed the dependence of each of the r.h.s. quantities on $\Phi$. The matrix in square brackets on the r.h.s. has eigenvalues of the form $\lambda - \mu$, where $\lambda$ and $\mu$ are eigenvalues of $\mathbf{A}_{LU}$ and $\mathbf{A}_{ST}$ respectively; it is thus invertible for all $\Phi \in \mathcal{P}$. Under the normalisation implied by (B.1), $\mathbf{B}$ is uniquely determined by $\Phi \in \mathcal{P}$, even though $\mathbf{R}_{ST}$, $\mathbf{A}_{ST}$ and $\mathbf{L}_{ST}$ individually are not (as follows from the final part of Lemma A.1).

**Lemma B.2.** Let $\Phi_0 \in \mathcal{P}$, $\mathbf{A}_0 := \mathbf{A}(\Phi_0)$, $\mathbf{A}_{0,LU} := \mathbf{A}_{LU}(\Phi_0)$, $\mathbf{R}_{0,LU} := [\mathbf{A}^i_q]$ and $\mathbf{R}_{0,LU} := \text{col}\{\mathbf{R}_{0,LU}\mathbf{A}^{k-i}_{LU}\}_{i=1}^k$. Then

(i) $\mathbf{A}_0 = \mathbf{A}(\Phi)$ and $\mathbf{A}_{0,LU} = \mathbf{A}_{LU}(\Phi)$ for all $\Phi \in \mathcal{P}$ such that $(\Phi - \Phi_0)\mathbf{R}_{0,LU} = 0$ and $|\lambda_{q+1}(\Phi)| < |\lambda_q(\Phi_0)|$;

(ii) the first differentials of $\mathbf{A}(\cdot)$ and $\mathbf{A}_{LU}(\cdot)$ at $\Phi = \Phi_0$ satisfy

$$\begin{bmatrix} \text{vec}(d \mathbf{A}) \\ \text{vec}(d \mathbf{A}_{LU}) \end{bmatrix} = \begin{bmatrix} J_{\mathbf{A}}(\Phi_0) \\ J_{\mathbf{A}_{LU}}(\Phi_0) \end{bmatrix} \text{vec}(\{d \Phi\} \mathbf{R}_{0,LU})$$

where

$$J(\Phi) := \begin{bmatrix} J_{\mathbf{A}}(\Phi) \\ J_{\mathbf{A}_{LU}}(\Phi) \end{bmatrix} := \begin{bmatrix} (\mathbf{I}_q \otimes \beta^T)\mathbf{B} \\ (\mathbf{I}_q \otimes \mathbf{A}_{LU})((\mathbf{I}_q \otimes \mathbf{L}^T_{LU}) - (\mathbf{I}_q \otimes \mathbf{A}_{LU}))\mathbf{B} + (\mathbf{I}_q \otimes \mathbf{L}^T_{LU}) \end{bmatrix} \quad (B.3)$$

for $\mathbf{G}^T := [0_{q \times r}, \mathbf{I}_q]$, $\beta^T = [\mathbf{I}_r, -\mathbf{A}]$, and $\mathbf{A}_{LU} = \mathbf{A}_{LU}(\Phi)$, etc.; and

(iii) $J(\Phi)$ is continuous.

When $\mathbf{A}_{LU}(\Phi) = \mathbf{I}_q$, the $pq \times pq$ matrix $J(\Phi)$ simplifies as follows.

---

For a more compact notation, here and subsequently we express matrix derivatives in terms of differentials, in the manner of Magnus and Neudecker (2007).
Lemma B.3. Suppose $\Phi \in \mathcal{P}$ with $\Lambda_{LU}(\Phi) = I_q$. Then $J(\Phi)$ is nonsingular, and

$$
\begin{bmatrix}
J_A(\Phi) \\
J_A(\Phi)
\end{bmatrix} = \begin{bmatrix}
I_q \otimes \beta^T R_{ST}(I_{kp - q} - \Lambda_{ST})^{-1} L_{ST}^T \\
I_q \otimes L_{LU}^T
\end{bmatrix}.
$$

Proof of Lemma B.1. We first prove $\mathcal{P}$ is open. For $F \in \mathbb{R}^{kp \times kp}$, let $\lambda_i(F)$ denote the $i$th eigenvalue of $F$, when these are placed in descending order of modulus. Let $\mathcal{F}$ denote the set of $kp \times kp$ matrices such that

(i) $|\lambda_{q+1}(F)| < |\lambda_q(F)|$; and

there exist $\Lambda_{LU} \in \mathbb{R}^{q \times q}$ and $R_{LU} \in \mathbb{R}^{kp \times q}$ such that

(ii) the eigenvalues of $\Lambda_{LU}$ are $\{\lambda_i(F)\}_{i=1}^q$, $FR_{LU} = R_{LU} \Lambda_{LU}$; and

(iii) $\text{rk}\{G^T R_{LU}\} = q$, where $G^T \in [0_{q \times (kp-q)}, I_q] = [0_{q \times k(p-1)}, G^T]$.

In view of Lemma A.1, $\Phi \in \mathcal{P}$ if and only if the companion form matrix $F(\Phi)$ is in $\mathcal{F}$. Since $F(\cdot)$ is continuously trivial, it suffices to show that $\mathcal{F}$ is open.

To that end, fix $F_0 \in \mathcal{F}$, and let $R_{0,LU}$ and $\Lambda_{0,LU}$ denote matrices satisfying (ii) and (iii) above. By the continuity of eigenvalues and simple invariant subspaces (Theorems IV.1.1 and V.2.8 in Stewart and Sun, 1990), for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|F - F_0\| < \delta$, $F$ satisfies requirements (i) and (ii) above, with associated $R_{LU}$ such that $\|R_{LU} - R_{0,LU}\| < \epsilon$. Since the set of full rank matrices is open, we may take $\epsilon > 0$ sufficiently small such that (iii) also holds. Thus $F \in \mathcal{F}$, and so $F_0$ is an interior point of $\mathcal{F}$; deduce $\mathcal{F}$ is open.

We turn next to the smoothness of $A(\Phi)$ and $\Lambda_{LU}(\Phi)$. For $F_0 \in \mathcal{F}$ we have the invariant subspace decomposition (as per (A.5) above)

$$F_0 = R_{0,LU} \Lambda_{0,LU} L_{0,LU}^T + R_{0,ST} \Lambda_{0,ST} L_{0,ST}^T$$

(B.4)

where $R_{0,LU}$ and $\Lambda_{0,LU}$ satisfy (ii)–(iii) above. Since (iii) holds, we may choose $R_{0,LU}$ such that $G^T R_{0,LU} = I_q$; note that $L_{0,LU}^T R_0 = I_{kp}$ (as per Lemma A.1(iii)) implies $L_{0,LU}^T R_{0,LU} = I_q$. Define the maps

$$H(R_{LU}, \Lambda_{LU}; F) := \begin{bmatrix}
R_{LU} \Lambda_{LU} - FR_{LU} \\
G^T R_{LU} - I_q
\end{bmatrix}$$

(B.5a)

$$H^*(R_{LU}, \Lambda_{LU}; F) := \begin{bmatrix}
R_{LU} \Lambda_{LU} - FR_{LU} \\
L_{0,LU}^T R_{LU} - I_q
\end{bmatrix},$$

(B.5b)

so that $H(R_{0,LU}, \Lambda_{0,LU}; F_0) = H^*(R_{0,LU}, \Lambda_{0,LU}; F_0) = 0$; but note that these maps need not otherwise agree, since they impose distinct normalisations on $R_{LU}$. Once we have shown that the Jacobian of $H^*$ with respect to $(R_{LU}, \Lambda_{LU})$ is nonsingular at $(R_{0,LU}, \Lambda_{0,LU}; F_0)$, it will follow by the implicit mapping theorem (Lang, 1993, Thm. XIV.2.1) that there exists a neighbourhood $N \subset \mathcal{F}$ of $F_0$ and smooth functions $R_{LU}^*: N \rightarrow \mathbb{R}^{kp \times q}$, $\Lambda_{LU}^*: N \rightarrow \mathbb{R}^{q \times q}$ such that

$$H^*[R_{LU}^*(F), \Lambda_{LU}^*(F); F] = 0$$

for all $F \in N$; by the continuity of $R_{LU}^*(\cdot)$, we may choose $N$ such that $\text{rk}\{G^T R_{LU}^*(F)\} = q$ for
all $F \in N$. Thus

\[
\begin{align*}
\mathbf{R}_{LU}(F) &:= \mathbf{R}_{LU}^*(F)|G^T\mathbf{R}_{LU}^*(F)|^{-1} \\
\Lambda_{LU}(F) &:= [G^T\mathbf{R}_{LU}^*(F)]\Lambda_{LU}^*(F)[G^T\mathbf{R}_{LU}^*(F)]^{-1}
\end{align*}
\]

(B.6) (B.7)

are well defined for all $F \in N$, and have the property that

\[
H[\mathbf{R}_{LU}(F), \Lambda_{LU}(F); F] = 0
\]

for all $F \in N$. Since the $(\mathbf{R}_{LU}, \Lambda_{LU})$ satisfying $H(\mathbf{R}_{LU}, \Lambda_{LU}; F) = 0$ is unique, repeating this construction for every $F_0 \in \mathcal{F}$ allows the smooth maps $\mathbf{R}_{LU}(F)$ and $\Lambda_{LU}(F)$ to be extended to the whole of $\mathcal{F}$. The smoothness of $\Lambda_{LU}(\Phi) := \Lambda_{LU}[F(\Phi)]$ and $\mathbf{R}_{LU}(\Phi) := \mathbf{R}_{LU}[F(\Phi)]$ follows immediately, and that of $A(\Phi)$ by noting that it corresponds to rows $(k-1)p+1$ to $(k-1)p+r$ of $\mathbf{R}_{LU}(\Phi)$.

It thus remains to verify that the Jacobian of $H^*$ with respect to $(\mathbf{R}_{LU}, \Lambda_{LU})$ is nonsingular at $(\mathbf{R}_{0,LU}, \Lambda_{0,LU}; F_0)$. Matrix differentiation gives

\[
dH^* = \left[\mathbf{R}_{0,LU}(d\Lambda_{LU}) + (d\mathbf{R}_{LU})\Lambda_{0,LU} - F_0(d\mathbf{R}_{LU}); \quad \mathbf{L}_{0,LU}^T(d\mathbf{R}_{LU})\right] := \left[dH_1^*; \quad dH_2^*\right]
\]

The Jacobian is nonsingular if $dH^* = 0$ implies $d\mathbf{R}_{LU} = 0$ and $d\Lambda_{LU} = 0$. To that end, suppose $dH^* = 0$. Then $0 = dH_2^* = \mathbf{L}_{0,LU}^T(d\mathbf{R}_{LU})$, and

\[
d\mathbf{R}_{LU} = (\mathbf{R}_0\mathbf{L}_0^T)d\mathbf{R}_{LU} = (\mathbf{R}_{0,LU}\mathbf{L}_{0,LU}^T + \mathbf{R}_{0,ST}\mathbf{L}_{0,ST}^T)d\mathbf{R}_{LU} = (\mathbf{R}_{0,ST}\mathbf{L}_{0,ST}^T)d\mathbf{R}_{LU}
\]

and similarly, by (B.4) above,

\[
F_0(d\mathbf{R}_{LU}) = (\mathbf{R}_{0,LU}\mathbf{L}_{0,LU}^T + \mathbf{R}_{0,ST}\mathbf{L}_{0,ST}^T)d\mathbf{R}_{LU} = \mathbf{R}_{0,ST}\mathbf{L}_{0,ST}^T(d\mathbf{R}_{LU}).
\]

Hence

\[
dH_1^* = \mathbf{R}_{0,LU}(d\Lambda_{LU}) + \mathbf{R}_{0,ST}[\mathbf{L}_{0,ST}^T(d\mathbf{R}_{LU})\Lambda_{0,LU} - \Lambda_{0,ST}\mathbf{L}_{0,ST}^T(d\mathbf{R}_{LU})] = \left[\begin{array}{c}
\mathbf{R}_{0,LU} \\
\mathbf{R}_{0,ST}
\end{array}\right]
\begin{bmatrix}
\begin{array}{c}
d\Lambda_{LU} \\
\mathcal{T}^T[d\mathbf{R}_{LU}]
\end{array}
\end{bmatrix},
\]

where $\mathcal{T}(M) := MA_{0,LU} - \Lambda_{0,ST}M$. Since $\mathbf{R}_0$ is nonsingular, $dH_1^* = 0$ implies that $d\Lambda_{LU} = 0$ and $\mathcal{T}[\mathbf{L}_{0,ST}^T(d\mathbf{R}_{LU})] = 0$; but since $\Lambda_{0,LU}$ and $\Lambda_{0,ST}$ have no common eigenvalues, $\mathcal{T}(M) = 0$ if and only if $M = 0$ (Stewart and Sun, 1990, Thm V.1.3). Thus $\mathbf{L}_{0,ST}^T(d\mathbf{R}_{LU}) = 0$, whence

\[
\begin{bmatrix}
\mathbf{L}_{0,LU}^T \\
\mathbf{L}_{0,ST}^T
\end{bmatrix}
\begin{bmatrix}
d\mathbf{R}_{LU}
\end{bmatrix} = 0
\]

from which it follows that $d\mathbf{R}_{LU} = 0$, since $\mathbf{L}_0$ is nonsingular. \qed

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Proof of Lemma B.2. (i). We have
\[ R_{0,LU} A^{k}_{0,LU} - \sum_{i=1}^{k} \Phi_i R_{0,LU} A^{k-i}_{0,LU} = \Phi R_{0,LU} = (1) \Phi_0 R_{0,LU} = R_{0,LU} A^{k}_{0,LU} - \sum_{i=1}^{k} \Phi_0 i R_{0,LU} A^{k-i}_{0,LU} = (2) 0 \]
where \((1)\) is by hypothesis, and \((2)\) by Lemma A.1. Since \(|\lambda_{q+1}(\Phi)| < |\lambda_q(\Phi)| = |\lambda_q(A_{0,LU})|\) and \(\Phi \in \mathcal{P}\), the result then follows by Lemma A.2.

(ii). Analogously to (B.5) above, define
\[ H(R_{LU}, A_{LU}; \Phi) := \left[ R_{LU} A_{LU} - F(\Phi) R_{LU}; \quad G^T R_{LU} - I_q \right] \]
\[ H^*(R_{LU}, A_{LU}; \Phi) := \left[ R_{LU} A_{LU} - F(\Phi) R_{LU}; \quad L_{0,LU}^T R_{LU} - I_q \right] \]

By the argument given in the proof of Lemma B.1, there are smooth maps \(R_{LU}(\Phi), R_{LU}^*(\Phi), A_{LU}(\Phi)\) and \(A_{LU}^*(\Phi)\) such that \(H[R_{LU}(\Phi), A_{LU}(\Phi); \Phi] = 0\) and \(H^*[R_{LU}^*(\Phi), A_{LU}^*(\Phi); \Phi] = 0\) for all \(\Phi \in \mathcal{P}\). Since \(G^T R_{0,LU} = I_q\) implies that \(G^T R_{0,LU} = I_q\), we have \(R_{LU}(\Phi) = R_{LU}^*(\Phi) = R_{0,LU}\) and \(A_{LU}(\Phi) = A_{LU}^*(\Phi) = A_{0,LU}\) when \(\Phi = \Phi_0\), but otherwise these maps need not agree. Since the maps \(R_{LU}(\Phi)\) and \(A_{LU}^*(\Phi)\) are easier to work with, we first obtain the derivatives of these, and subsequently those of \(A(\Phi)\) and \(A_{LU}(\Phi)\) via renormalisation, analogously to (B.6)–(B.7).

Setting the total differential of \(H^*\) to zero gives
\[ 0 = dH^* = \left[ R_{0,LU}(dA_{LU}^*) + (dR_{LU}^*) A_{0,LU} - F_0(dR_{LU}^*) - F(d\Phi) R_{0,LU}; \quad L_{0,LU}^T (dR_{LU}^*) \right] \]  
where \(F_0 := F(\Phi)\), whence by similar arguments as were given in the proof of Lemma B.1,
\[ F(d\Phi) R_{0,LU} = R_{0,LU}(dA_{LU}^*) + R_{0,st} L_{0,st}^T (dR_{LU}^*) A_{0,LU} - R_{0,st} A_{0,LU} L_{0,st}^T (dR_{LU}^*). \]  

Vectorising gives
\[ \text{vec}[F(d\Phi) R_{0,LU}] = (I_q \otimes R_{0,LU}) \text{vec}(dA_{LU}^*) + M \text{vec}(dR_{LU}^*) \]  
for \(M := (I_q \otimes R_{0,st})[(A_{LU}^T \otimes I_{kp-q}) - (I_q \otimes A_{0,LU})] (I_q \otimes L_{0,LU}^T)\). Since \(L_{0,st}^T R_{0,LU} = 0\) and \(L_{0,st}^T R_{0,LU} = I_{kp-q}\), setting
\[ M^\dagger := (I_q \otimes R_{0,st})[(A_{LU}^T \otimes I_{kp-q}) - (I_q \otimes A_{0,LU})]^{-1} (I_q \otimes L_{0,LU}^T) \]
we have \(M^\dagger (I_q \otimes R_{0,LU}) = 0\) and \(M^\dagger M = I_q \otimes R_{0,LU} L_{0,LU}^T\). Since \(L_{0,LU}^T (dR_{LU}^*) = 0\) by (B.8), it follows that
\[ dR_{LU}^* = (R_{0,LU} L_{0,LU}^T + R_{0,st} L_{0,st}^T) dR_{LU}^* = (R_{0,st} L_{0,st}^T) dR_{LU}^* \]
whence \(M^\dagger M \text{vec}(dR_{LU}^*) = \text{vec}(dR_{LU}^*)\), and so premultiplying (B.10) by \(M^\dagger\) gives
\[ \text{vec}(dR_{LU}^*) = M^\dagger \text{vec}[F(d\Phi) R_{0,LU}]. \]

By the structure of the companion form matrix, \(L_{0,st}^T F(d\Phi) R_{0,LU} = L_{0,st}^T (d\Phi) R_{0,LU}\). Since \(R\)
Thus (B.11), (B.13) and (B.14) give the second part of (B.3).

Recognising that for \( \beta \)

the Jacobian of \( \Lambda_{LU} \) is given in (B.11) above. To obtain \( d\Lambda_{LU}^* \) matrix

we have \( \Phi(\Phi_0) = \Phi_0 \) and \( \Phi_0^T \Phi_0 = I_q \) and \( \Phi_0^T \Phi_0 = A_0 \), it follows that at \( \Phi = \Phi_0 \)

we have \( A(\Phi) = G_{LU}^T R_{LU}^*(\Phi) = G_{LU}^T R_{LU}^*(\Phi)[G_{LU}^T R_{LU}^*(\Phi)]^{-1} \). From \( R_{LU}^*(\Phi_0) = R_{0,LU}, G_{LU}^T R_{0,LU} = G_{LU}^T R_{0,LU} = A_0 \), it follows that at \( \Phi = \Phi_0 \)

\[
\begin{align*}
    \text{vec}(dR_{LU}^*) &= (I_q \otimes I_{k_p-q}) - (I_q \otimes \Lambda_{0,LU})]^{-1}(I_q \otimes L_{0,ST}^T)\text{vec}((d\Phi)R_{0,LU}) \\
    &= B(\Phi_0)\text{vec}((d\Phi)R_{0,LU}).
\end{align*}
\]

To compute the Jacobian of \( A(\Phi) \), note that by partitioning the \( p \times p \) identity matrix as

\[
\begin{bmatrix} G_{LU} & G \end{bmatrix} := \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}
\]

we have \( A(\Phi) = G_{LU}^T R_{LU}(\Phi) = G_{LU}^T R_{LU}(\Phi)[G_{LU}^T R_{LU}(\Phi)]^{-1} \). From \( R_{LU}^*(\Phi_0) = R_{0,LU}, G_{LU}^T R_{0,LU} = G_{LU}^T R_{0,LU} = A_0 \), it follows that at \( \Phi = \Phi_0 \)

\[
\begin{align*}
    dA = G_{LU}^T (dR_{LU}^*) - (G_{LU}^T R_{LU}) (dR_{LU}^*) = (G_{LU}^T - A_0 G_{LU}^T) dR_{LU}^* = \beta_0^T dR_{LU}^*
\end{align*}
\]

for \( \beta_0^T = \begin{bmatrix} I_p & -A_0 \end{bmatrix} \). The first part of (B.3) follows immediately from (B.11) and (B.12). For the Jacobian of \( \Lambda_{LU}(\Phi) \), note that (as per (B.7) above)

\[
\Lambda_{LU}(\Phi) = [G_{LU}^T R_{LU}(\Phi)]^T \Lambda_{LU}(\Phi) [G_{LU}^T R_{LU}(\Phi)]^{-1}
\]

whence at \( \Phi = \Phi_0 \),

\[
\begin{align*}
    d\Lambda_{LU} &= G_{LU}^T (dR_{LU}^*) \Lambda_{0,LU} + d\Lambda_{LU}^* - \Lambda_{0,LU} G_{LU}^T (dR_{LU}^*). \\
    \text{vec}(d\Lambda_{LU}) &= \{(A_{0,LU}^T \otimes I_q) - (I_q \otimes \Lambda_{0,LU})\} (I_q \otimes G_{LU}^T) \text{vec}(dR_{LU}^*) + \text{vec}(d\Lambda_{LU}^*).
\end{align*}
\]

Recoginsing that \( G_{LU}^T (dR_{LU}^*) = G_{LU}^T (dR_{LU}^*) \) and vectorising, we have

\[
\begin{align*}
    \text{vec}(d\Lambda_{LU}) &= \{(A_{0,LU}^T \otimes I_q) - (I_q \otimes \Lambda_{0,LU})\} (I_q \otimes G_{LU}^T) \text{vec}(dR_{LU}^*) + \text{vec}(d\Lambda_{LU}^*). \quad (B.13)
\end{align*}
\]

\( dR_{LU}^* \) is given in (B.11) above. To obtain \( d\Lambda_{LU}^* \), note that premultiplying (B.9) by \( L_{0,LU}^T \) yields

\[
\begin{align*}
    d\Lambda_{LU}^* &= L_{0,LU}^T F(d\Phi)R_{0,LU} = L_{0,LU}^T (d\Phi)R_{0,LU}.
\end{align*}
\]

Thus (B.11), (B.13) and (B.14) give the second part of (B.3).

\( \text{(iii). Continuity of } J(\Phi) \text{ is immediate from } A(\Phi) \text{ and } \Lambda_{LU}(\Phi) \text{ being smooth.} \)

\( \Box \)

**Proof of Lemma B.3.** The stated expression for \( J(\Phi) \) is immediate from (B.2), Lemma B.2, and \( \Lambda_{LU}(\Phi) = I_q \). That \( J(\Phi) \) is nonsingular will follow once we have shown that the \( (p \times p) \) matrix

\[
K := \begin{bmatrix} \beta^T R_{ST}(I_{k_p-q} - \Lambda_{ST})^{-1} L_{ST}^T \\ L_{LU}^T \end{bmatrix}
\]

is nonsingular. We first note the following facts. Since \( \Phi \in \mathcal{P} \) with \( \Lambda_{LU}(\Phi) = I_q \), it follows from (B.1) that \( \text{rk } \Phi(1) \leq p - q \). Since \( \Phi(\cdot) \) has exactly \( q \) roots at unity, the reverse inequality holds by Corollary 4.3 of Johansen (1995), whence \( \text{rk } \Phi(1) = p - q \). Thus cv holds: this implies that \( \text{sp } \beta = \text{sp } \Phi(1)^T \) and \( \text{rk } L_{LU} = q \) (see Lemma A.3 and the characterisation of the CS discussed in Section 2.2).
Now let \( c \in \mathbb{R}^p \) be such that \( Kc = 0 \), so that in particular \( L_{LU}^T c = 0 \). Since \( \text{rk} \Phi(1) + \text{rk} L_{LU} = p \), while (2.5) with \( \Lambda_{LU} = I_q \) implies \( L_{LU}^T \Phi(1) = 0 \), it follows that \( c \in \text{sp} \Phi(1) \), i.e. \( c = \Phi(1)b \) for some \( b \in \mathbb{R}^p \). By Gohberg, Lancaster, and Rodman (1982, Thm 2.4), \( \Phi(\mu)^{-1} = (\mu I - \Lambda)^{-1} L_{LU}^T \) for any \( \mu \) not a root of \( \Phi(\cdot) \). Since the columns of the quasi-cointegrating matrix \( \beta \) are orthogonal to \( R_{LU} \), we have

\[
\beta^T = \beta^T R_{ST}(\mu I_{kp-q} - \Lambda_{ST})^{-1} L_{ST}^T \Phi(\mu) \rightarrow \beta^T R_{ST}(I_{kp-q} - \Lambda_{ST})^{-1} L_{ST}^T \Phi(1) \quad (B.15)
\]

by the continuity of the r.h.s., as \( \mu \to 1 \), since \( \Lambda_{ST} \) has no eigenvalues at unity. Hence

\[
0 = Kc = \begin{bmatrix} \beta^T R_{ST}(I_{kp-q} - \Lambda_{ST})^{-1} L_{ST}^T \Phi(1) \vspace{1em} 0 \end{bmatrix} = \begin{bmatrix} \beta^T b \vspace{1em} 0 \end{bmatrix}
\]

implying \( \beta^T b = 0 \). But \( \text{sp} \beta = \text{sp} \Phi(1)^T \), so we must have \( \Phi(1)b = 0 \). Thus \( c = 0 \), from which it follows that \( K \) is nonsingular. \( \square \)

### C Asymptotics

The assumptions DGP and LOC are maintained throughout this appendix. We first recall some notation. Let \( \Phi_0 := \lim_{n \to \infty} \Phi_n \), where \( \{\Phi_n\} \) is the sequence specified by LOC. Let \( R_n := [R_{LU}(\Phi_n), R_{ST}] \) and \( \Lambda_n := \text{diag}\{\Lambda_{LU}, \Lambda_{ST}\} \) be as in LOC. Take \( R_n := \text{col}\{R_n \Lambda_n^{k-i}\}_{i=1}^k \) and \( L_n := (R_n^T)^{-1} \) as in Lemma A.1, and partition these as \( R_n = [R_{n,LU}, R_{n,ST}] \) and \( L_n = [L_{n,LU}, L_{n,ST}] \) (as per (A.4)); note that both these matrices are convergent.

Let \( z_{LU,t} := L_{n,LU}^T x_t \) and \( z_{ST,t} := L_{n,ST}^T x_t \) be as in Lemma A.4 (for \( \Phi = \Phi_n \)); these follow the autoregressions given in (3.7). Recall \( E \sim \text{BM}(\Sigma) \) and \( Z_C(r) := \int_0^r e^{C(r-s)} L_{LU}^T dE(s) \) from (3.8). For \( i \in \{LU, ST\} \), let \( \bar{z}_{i,t} \) denote the residual from an OLS regression of \( \{z_{LU,t}, z_{ST,t}\}_t \) onto a constant and linear trend. Recall that \( \bar{Z}_C \) denotes the residual from an \( L_2(0,1) \) projection of each sample path of \( Z_C \) onto a constant and linear trend. As in Section 3.1, let \( \hat{\Sigma}_n \) denote the unrestricted MLE for \( \Sigma \), i.e. the OLS residual variance matrix estimator.

Proofs of the following results appear at the end of this section.

**Lemma C.1.** The following hold jointly:

(i) \( n^{-1/2} \sum_{t=1}^{[nr]} \varepsilon_t \sim E(r) \)

(ii) \( n^{-1/2} z_{LU,[nr]} \sim Z_C(r) \)

(iii) \( n^{-1/2} z_{ST,[nr]} \sim \bar{Z}_C(r) \)

as weak convergences on the space of right-continuous functions \([0,1] \to \mathbb{R}^m \) (with respect to the uniform topology); and

(iv) \( n^{-1} \sum_{t=1}^{n} (z_{LU,t-1} \otimes \varepsilon_t) \sim \int_0^1 [\bar{Z}_C(r) \otimes dE(r)] \)

(v) \( n^{-1/2} \sum_{t=1}^{n} (z_{ST,t-1} \otimes \varepsilon_t) \sim \xi \sim N(0, \Omega \otimes \Sigma) \)

(vi) \( \hat{\Sigma}_n \overset{p}{\rightarrow} \Sigma \),

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where \( \Omega := \lim_{n \to \infty} \text{var}(z_{ST,n}) \) and \( \xi \) is independent of \( E \).

Now define the reparametrisation \( \Phi \mapsto \varphi \) by

\[
\varphi := \begin{bmatrix} \varphi_{LU} \\ \varphi_{ST} \end{bmatrix} = \begin{bmatrix} \text{vec}\{(\Phi - \Phi_n)R_{n,LU}\} \\ \text{vec}\{(\Phi - \Phi_n)R_{n,ST}\} \end{bmatrix} = \text{vec}\{(\Phi - \Phi_n)R_n\},
\]

which is reversed by setting \( \Phi = \Phi_n + \text{vec}^{-1}(\varphi)L^T_n \), where \( \text{vec}^{-1}(x) \) maps \( x \in \mathbb{R}^{kp} \) to the matrix \( X \in \mathbb{R}^{p \times kp} \) for which \( \text{vec}(X) = x \). The parameter space for \( \varphi \) is the open set

\[
P_n := \{ \text{vec}\{(\Phi - \Phi_n)R_n\} \mid \Phi \in \mathcal{P} \},
\]

and the true parameters correspond to \( \varphi = 0 \). Although \( P_n \) depends on \( n \), since \( \Phi_n \to \Phi_0 \in \mathcal{P} \) and \( \mathcal{P} \) is open (Lemma B.1), there is an \( \epsilon > 0 \) such that \( P_n \) contains a ball of radius \( \epsilon \) centred at the origin, for all \( n \) sufficiently large. Let

\[
\ell_n^*(\varphi) := \ell_n[\Phi_n + \text{vec}^{-1}(\varphi)L^T_n, \hat{\Sigma}_n].
\]

Define \( D_n := \text{diag}\{nI_{\#LU}, n^{1/2}I_{\#ST}\} \), where \( \#LU := pq \) and \( \#ST := p(kp - q) \) correspond to the dimensions of the vectors \( \varphi_{LU} \) and \( \varphi_{ST} \) respectively.

**Lemma C.2.** There exist \( S_n \) and \( H_n \) such that for all \( \varphi \in P_n \),

\[
\ell_n^*(\varphi) - \ell_n^*(0) = S_n^T(D_n\varphi) - \frac{1}{2}(D_n\varphi)^TH_n(D_n\varphi)
\]

where

\[
S_n \sim \int_0^1 [\tilde{Z}_C(r) \otimes \Sigma^{-1}dE(r)] =: \begin{bmatrix} S_{LU} \\ S_{ST} \end{bmatrix} =: S,
\]

\[
H_n \sim \int \tilde{Z}_C \tilde{Z}_C^T \otimes \Sigma^{-1} =: \begin{bmatrix} H_{LU} & 0 \\ 0 & H_{ST} \end{bmatrix} =: H,
\]

for \( \xi \) as in Lemma C.1.

Define the constraint maps

\[
\theta_n(\varphi) := \text{vec}\{\Lambda_{LU}[\Phi_n + \text{vec}^{-1}(\varphi)L_n^T] - (I_q + C/n)\}
\]

\[
\gamma_n(\varphi) := a_{ij}[\Phi_n + \text{vec}^{-1}(\varphi)L_n^T] - a_{ij}(\Phi_n),
\]

and the associated restricted parameter spaces

\[
P_n|\theta := \{ \varphi \in P_n \mid \theta_n(\varphi) = 0 \}
\]

\[
P_n|\theta,\gamma := \{ \varphi \in P_n \mid \theta_n(\varphi) = 0 \text{ and } \gamma_n(\varphi) = 0 \}.
\]

Let \( \hat{\varphi}_n, \hat{\varphi}_n|\theta \) and \( \hat{\varphi}_n|\theta,\gamma \) denote exact maximisers of \( \ell_n^*(\varphi) \) over the sets \( P_n, P_n|\theta \) and \( P_n|\theta,\gamma \) respectively: which may be shown to exist at least with with probability approaching one (w.p.a.1), and may be arbitrarily defined otherwise.
Lemma C.3. Each of $D_n\hat{\varphi}_n$, $D_n\hat{\varphi}_n[\theta]$ and $D_n\hat{\varphi}_n[\theta, \gamma]$ are $O_p(1)$.

Let $\nabla_{\varphi} g(\hat{\varphi}_0)$ denote the gradient of $g : \mathcal{P} \to \mathbb{R}^{d_\varphi}$ at $\varphi = \varphi_0$. The derivatives of the maps $\theta_n$ and $\gamma_n$ can be inferred from Lemma B.2. Part (ii) of that result gives the derivatives with respect to $\varphi_{LU}$, and part (i) implies that when $\varphi_{LU} = 0$, the the first (and higher order) derivatives with respect to $\varphi_{ST}$ are identically zero. Now letting $e_{d,i} \in \mathbb{R}^d$ denote a vector with zero everywhere except for a 1 in the $d$th position, define

$$\Pi := [\Theta; \Gamma] := [I_q \otimes L_{LU}; \ e_{q,j} \otimes L_{ST}(I_{kp-q} - \Lambda_{ST}^T)^{-1}R_{ST}^T\beta_{r,i}],$$

which by Lemma B.3 has full column rank, and

$$\Theta := \begin{bmatrix} \Theta \\ 0_{\#ST \times q^2} \end{bmatrix}, \quad \Pi := \begin{bmatrix} \Pi \\ 0_{\#ST \times (q^2+1)} \end{bmatrix}.$$

Lemma C.4.

(i) Let $\{\hat{\varphi}_n\}$ denote a random sequence in $\mathcal{P}_n$ with $\hat{\varphi}_n \xrightarrow{p} 0$. Then

$$\nabla_{\varphi}\theta_n(\hat{\varphi}_n) \xrightarrow{p} \Theta \quad \quad \nabla_{\varphi}\gamma_n(\hat{\varphi}_n) \xrightarrow{p} \Gamma.$$

(ii) Let $Q_{\Theta, \perp}$ and $Q_{\Pi, \perp}$ denote orthogonal projections from $\mathbb{R}^{kp^2}$ onto the subspaces orthogonal to the the columns of $\Theta$ and $\Pi$. Then

$$D_n\hat{\varphi}_n[\theta] = Q_{\Theta, \perp}D_n\hat{\varphi}_n[\theta] + o_p(\|D_n\hat{\varphi}_n[\theta]\|)$$

$$D_n\hat{\varphi}_n[\theta, \gamma] = Q_{\Pi, \perp}D_n\hat{\varphi}_n[\theta, \gamma] + o_p(\|D_n\hat{\varphi}_n[\theta, \gamma]\|).$$

Let $\Theta_{\perp} \in \mathbb{R}^{pq \times qr}$ and $\Pi_{\perp} \in \mathbb{R}^{pq \times (qr-1)}$ denote matrices having full column rank, such that $\Theta_{\perp}^T\Theta = 0$ and $\Pi_{\perp}^T\Pi = 0$. We may take $\Theta_{\perp} = I_q \otimes L_{LU, \perp}$, for $L_{LU, \perp}$ a $p \times r$ matrix having rank $r$ and for which $L_{LU, \perp}^T L_{LU} = 0$. Since $\Pi = [\Theta, \Gamma]$ there exists a full column rank matrix $\Xi \in \mathbb{R}^{qr \times (qr-1)}$ for which $\Pi_{\perp} := \Theta_{\perp} \Xi$.

Proposition C.1.

(i) $D_n\hat{\varphi}_n = \begin{bmatrix} n\hat{\varphi}_{n,LU} \\ n/2\hat{\varphi}_{n,ST} \end{bmatrix} \xrightarrow{\Delta} \begin{bmatrix} H_{LU}^{-1}S_{LU} \\ H_{ST}^{-1}S_{ST} \end{bmatrix},$

(ii) $D_n\hat{\varphi}_n[\theta] = \begin{bmatrix} n\hat{\varphi}_{n,LU}[\theta] \\ n/2\hat{\varphi}_{n,ST}[\theta] \end{bmatrix} \xrightarrow{\Delta} \begin{bmatrix} \Theta_{\perp}(\Theta_{\perp}^T H_{LU,\perp}\Theta_{\perp})^{-1}\Theta_{\perp}^T S_{LU} \\ H_{ST}^{-1}S_{ST} \end{bmatrix},$

(iii) $2[\ell_n(\hat{\varphi}_n) - \ell_n(\hat{\varphi}_n[\theta])] \xrightarrow{\Delta} \begin{bmatrix} \Theta_{\perp} H_{LU,\perp}\Theta_{\perp}^{-1}\Theta_{\perp}^T S_{LU} \\ H_{ST}^{-1}S_{ST} \end{bmatrix}.$

Let $H_{\Theta_{\perp}, \perp} := \Theta_{\perp}^T H_{LU,\perp} \Theta_{\perp}$, and $\mathcal{Q} \in \mathbb{R}^{qr \times qr}$ denote the orthogonal projection onto $\text{sp} H_{\Theta_{\perp}, \perp}^{1/2}$. Then

(iv) $2[\ell_n(\hat{\varphi}_n[\theta]) - \ell_n(\hat{\varphi}_n[\theta, \gamma])] \xrightarrow{\Delta} (H_{\Theta_{\perp}, \perp}^{-1/2}\Theta_{\perp}^T S_{LU})^T[I_{qr} - \mathcal{Q}](H_{\Theta_{\perp}, \perp}^{-1/2}\Theta_{\perp}^T S_{LU}).$

The preceding gives the limiting distribution of $\hat{\Phi}_n$ under the reparametrisation (C.1); the limiting distributions of estimators of $A$ and $\Lambda_{LU}$ will then follow by an application of the delta method, as per

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Proposition C.2. Let \{\Phi_n\} be as in LOC, \Phi_0 := \lim_{n \to \infty} \Phi_n \in \mathcal{P}, and \{\tilde{\Phi}_n\} a random sequence in \mathcal{P} with \tilde{\Phi}_n = \Phi_n + o_p(1). Then
\[
\begin{pmatrix}
\text{vec}\{A(\tilde{\Phi}_n) - A(\Phi_n)\} \\
\text{vec}\{A_{LU}(\tilde{\Phi}_n) - A_{LU}(\Phi_n)\}
\end{pmatrix} = \begin{pmatrix}
J_A(\Phi_0) \\
J_A(\Phi_0)
\end{pmatrix} + o_p(1)
\text{vec}\{(\tilde{\Phi}_n - \Phi_n)R_{n,LU}\}
\]
where \(R_{n,LU} := R_{LU}(\Phi_n)\).

Proof of Lemma C.1. (i)–(iv) follow by Donsker’s theorem for partial sums, Lemma 3.1 in Phillips (1988) and the continuous mapping theorem; (v) by the martingale central limit theorem (Hall and Heyde, 1980, Thm. 3.2); and (vi) by arguments similar to those given in Section 3.2.2 of Lütkepohl (2007).

Proof of Lemma C.2. Let \(\Phi_i := \Phi R_{n,i}\) and \(\Phi_{n,i} := \Phi_n R_{n,i}\) for \(i \in \{\text{l}, \text{st}\}\). Then
\[
\ell_n(\Phi, \Sigma) = -\frac{n}{2} \log \det \Sigma - \min_{m,d} \frac{1}{2} \sum_{t=1}^{n} \|y_t - m - dt - \Phi y_{t-1}\|_{\Sigma^{-1}}^2
\]
\[
= -\frac{n}{2} \log \det \Sigma - \min_{m,d} \frac{1}{2} \sum_{t=1}^{n} \|x_t - m - dt - \Phi x_{t-1}\|_{\Sigma^{-1}}^2
\]
\[
= -\frac{n}{2} \log \det \Sigma - \frac{1}{2} \sum_{t=1}^{n} \|\bar{z}_t - \Phi_{LU}\bar{z}_{LU,t-1} - \Phi_{ST}\bar{z}_{ST,t-1}\|_{\Sigma^{-1}}^2
\]
Twice differentiating the r.h.s. (as in Lütkepohl 2007, Sec. 3.4) with respect to \(\Phi_{LU}\) and \(\Phi_{ST}\), and noting that \(\varphi_i = \text{vec}(\Phi_i - \Phi_{n,i})\), we thus have
\[
\ell_n^*(\varphi) - \ell_n^*(0) = \ell_n(\Phi_{LU}, \hat{\Sigma}_n) - \ell_n(\Phi_n, \hat{\Sigma}_n) = S_n^T(D_n \varphi) - \frac{1}{2}(D_n \varphi)^T H_n(D_n \varphi)
\]
where
\[
S_n := \begin{bmatrix}
\sum_{t=1}^{n} (\bar{z}_{LU,t-1} \otimes \hat{\Sigma}_n^{-1} \bar{z}_t) \\
\sum_{t=1}^{n} (\bar{z}_{ST,t-1} \otimes \hat{\Sigma}_n^{-1} \bar{z}_t)
\end{bmatrix}
\]
\[
H_n := \begin{bmatrix}
\frac{1}{n} \sum_{t=1}^{n} (\hat{z}_{LU,t-1} \otimes \hat{\Sigma}_n^{-1} \bar{z}_t) \\
\frac{1}{n} \sum_{t=1}^{n} (\hat{z}_{ST,t-1} \otimes \hat{\Sigma}_n^{-1} \bar{z}_t)
\end{bmatrix}
\]
and \(\bar{z}_t\) denotes the residual from an OLS regression of \(\{z_t\}_{t=1}^{n}\) on a constant and a linear trend; \(= (1)\) holds because each element of \(\bar{z}_{LU,t-1}\) and \(\bar{z}_{ST,t-1}\) is orthogonal to a constant and linear trend. The stated convergences of \(S_n\) and \(H_n\) then follow by Lemma C.1 and the continuous mapping theorem.

Proof of Lemma C.3. By Lemma C.2, we have
\[
\ell_n^*(\varphi) - \ell_n^*(0) = \|D_n \varphi\| \|S_n\| - \frac{1}{2} \lambda_{\min}(H_n) \|D_n \varphi\|.
\]
Let \(M < \infty\) and \(\epsilon > 0\). Since \(D_n = \text{diag}\{nI_{\#LU}, n^{1/2}I_{\#ST}\}\), \(S_n = O_p(1)\) and \(H_n \sim H\) is positive.
definite w.p.a.1, it is evident that
\[ P \left\{ \sup_{\varphi \in \mathcal{P}_n, ||D_n \varphi|| \geq M} [\ell_n^*(\varphi) - \ell_n^*(0)] \leq -\varepsilon \right\} \geq P \left\{ M[||S_n|| - \frac{1}{2} \lambda_{\text{min}}(H_n)M] \leq -\varepsilon \right\} \]
and
\[ \limsup_{n \to \infty} P \left\{ M[||S_n|| - \frac{1}{2} \lambda_{\text{min}}(H_n)M] \leq -\varepsilon \right\} \leq P \left\{ M[||S|| - \frac{1}{2} \lambda_{\text{min}}(H)M] \leq -\varepsilon \right\} \to 1 \]
as \( M \to \infty \). Deduce that \( D_n \hat{\varphi}_n = O_p(1) \). Since \( \mathcal{P}_{n|\vartheta, \gamma} \subset \mathcal{P}_n \) for all \( n \) sufficiently large, and \( \mathcal{P}_{n|\vartheta, \gamma} \), that \( D_n \hat{\varphi}_n|\vartheta \) and \( D_n \hat{\varphi}_n|\vartheta, \gamma \) are stochastically bounded follows by the same argument.

**Proof of Lemma C.4.** (i). Since \( \Phi_n \to \Phi_0, L_n \to L_0 \) and \( \Lambda_{LU}(\cdot) \) is continuously differentiable (Lemma B.1),
\[ \nabla \varphi \theta_n(\hat{\varphi}_n) \xrightarrow{p} \nabla \varphi \text{vec}\{\Lambda_{LU}[\Phi_0 + \text{vec}^{-1}(\varphi)L_0^T]\} = (1) \left[ I_q \otimes L_{LU} \right]_{0 \#ST \times q^2} = \Theta \]
where \((1)\) follows by Lemmas B.2 and B.3. The probability limit of \( \nabla \varphi \gamma_n(\hat{\varphi}_n) \) follows similarly.

(ii). By Lemma C.3 and the remarks following (C.2), there exists a ball \( B(0, \varepsilon) \) of radius \( \varepsilon > 0 \), centred on the origin, such that \( B(0, \varepsilon) \subset \mathcal{P}_n \) for all \( n \) sufficiently large, and \( P\{\hat{\varphi}_n|\vartheta \in B(0, \varepsilon)\} \to 1 \). We may take \( \varepsilon \) sufficiently small that \( \Phi_\varphi := \Phi_n + \text{vec}^{-1}(\varphi)L_n^T \) has \( |\lambda_{\varphi+1}(\Phi_\varphi)| < |\lambda_\varphi(\Phi_n)| \) for all \( n \) sufficiently large, for all \( \varphi \in B(0, \varepsilon) \). In particular, suppose \( \varphi_{LU} = 0 \); then \( (\Phi_\varphi - \Phi_n)R_{n,LU} = 0 \) and we have by Lemma B.2(i) that \( \Lambda_{LU}(\Phi_\varphi) = \Lambda_{LU}(\Phi_n) = C/n \). It follows that \( \theta_n(0, \hat{\varphi}_{n,ST}|\vartheta) = 0 \) w.p.a.1., whence
\[ 0 = \theta_n(\hat{\varphi}_{n,LU}|\vartheta, \hat{\varphi}_{n,ST}|\vartheta) = \theta_n(\hat{\varphi}_{n,LU}|\vartheta, \hat{\varphi}_{n,ST}|\vartheta) - \theta_n(0, \hat{\varphi}_{n,ST}|\vartheta) = [\Theta + o_p(1)]^T \hat{\varphi}_{n,LU}|\vartheta = \Theta^T \hat{\varphi}_{n,LU}|\vartheta + o_p(||\hat{\varphi}_{n,LU}|\vartheta||) \]
by part (i) of the lemma and a mean value expansion. Hence, letting \( Q_{\Theta} \) and \( Q_{\Theta,\perp} \) denote the matrices that orthogonally project from \( \mathbb{R}^{\#LU} \) onto sp \( \Theta \) and (sp \( \Theta \))\( \perp \) respectively, we have
\[ D_n \hat{\varphi}_n|\vartheta = \begin{bmatrix} ni_{LU} \otimes \Theta + Q_{\Theta,\perp} & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} \hat{\varphi}_{n,LU}|\vartheta \\ \hat{\varphi}_{n,ST}|\vartheta \end{bmatrix} + o_p(||\hat{\varphi}_{n,LU}|\vartheta||) = Q_{\Theta,\perp} D_n \hat{\varphi}_n|\vartheta + o_p(||D_n \hat{\varphi}_n|\vartheta||). \]

**Proof of Proposition C.1.** (i). Immediate from Lemma C.2.

(ii). As in the proof of Lemma C.4(ii), we may take \( \varepsilon > 0 \) such that \( B(0, \varepsilon) \subset \mathcal{P}_n \) for all \( n \) sufficiently large, and \( P\{\hat{\varphi}_n|\vartheta \in B(0, \varepsilon)\} \to 1 \). Hence w.p.a.1., \( \hat{\varphi}_n|\vartheta \) satisfies the first-order conditions for a constrained interior maximum,
\[ \nabla \varphi \ell_n^*(\hat{\varphi}_n|\vartheta) = D_n S_n - D_n H_n(D_n \hat{\varphi}_n|\vartheta) = \nabla \varphi \theta_n(\hat{\varphi}_n|\vartheta) \mu_n, \]
where \( \mu_n \in \mathbb{R}^{n^2} \) is a vector of Lagrange multipliers; whence

\[
S_n - H_n(D_n\hat{\varphi}_n|\theta) = (nD_n^{\frac{1}{n}}) \nabla \varphi \theta_n(\hat{\varphi}_n|\theta)(n^{-1}\mu_n) =: \Theta_n(n^{-1}\mu_n)
\]

w.p.a.1. By a similar argument as given in the proof of Lemma C.4(ii), it follows from Lemma B.2(i) that \( \nabla \varphi ST \theta_n(0, \hat{\varphi}_{nST}|\theta) = 0 \) w.p.a.1, and so by a a mean value expansion and Lemma C.3,

\[
\nabla \varphi ST \theta_n(\hat{\varphi}_n) = \nabla \varphi ST \theta_n(\hat{\varphi}_{nLU}|\theta, \hat{\varphi}_{nST}|\theta) - \nabla \varphi ST \theta_n(0, \hat{\varphi}_{nST}|\theta) = O_p(||\hat{\varphi}_{nLU}|\theta||) = O_p(n^{-1}).
\]

Deduce from the preceding and Lemma C.4(i) that

\[
\Theta_n = (nD_n^{\frac{1}{n}}) \nabla \varphi \theta_n(\hat{\varphi}_n|\theta) = \begin{bmatrix} \nabla \varphi LU \theta_n(\hat{\varphi}_n|\theta) \\ n^{1/2} \nabla \varphi ST \theta_n(\hat{\varphi}_n|\theta) \end{bmatrix} \overset{p}{\rightarrow} \Theta,
\]

which has full column rank. Let \( \Theta_\perp := \text{diag}\{\Theta_\perp, I_{\#ST}\} \), a full column rank matrix for which \( \Theta_\perp^T \Theta_\perp = 0 \); then \( \Theta_{n,\perp} := [I_{kp^2} - \Theta_n(\Theta_n^T \Theta_n)^{-1} \Theta_n^T \Theta_\perp] \overset{p}{\rightarrow} \Theta_\perp \) and \( \Theta_{n,\perp}^T \Theta_n = 0 \) for all \( n \). Hence w.p.a.1

\[
0 = (1) \Theta_{n,\perp}^T S_n - \Theta_{n,\perp}^T H_n(D_n\hat{\varphi}_n|\theta)
= (2) \Theta_{n,\perp}^T S_n - \Theta_{n,\perp}^T H_n(\Theta_\perp(\Theta_\perp^T \Theta_\perp)^{-1} \Theta_\perp^T (D_n\hat{\varphi}_n|\theta) + o_p(||D_n\hat{\varphi}_n|\theta||))
\]

where \( = (1) \) follows from premultiplying (C.5) by \( \Theta_{n,\perp}^T \), and \( = (2) \) from Lemma C.4(ii). A further appeal to that result and rearranging the preceding yields

\[
D_n\hat{\varphi}_n|\theta = Q\Theta_\perp D_n\hat{\varphi}_n|\theta + o_p(||D_n\hat{\varphi}_n|\theta||) = \Theta_\perp(\Theta_{n,\perp}^T H_n \Theta_\perp)^{-1} \Theta_{n,\perp}^T S_n + o_p(1 + ||D_n\hat{\varphi}_n|\theta||).
\]

The result then follows by Lemmas C.2 and C.3.

(iii). From parts (i) and (ii) and Lemma C.2 we have

\[
2[\ell_n^s(\hat{\varphi}_n) - \ell_n^s(0)] \sim S_{LU}^T H_{LU}^{-1} S_{LU} + S_{ST}^T H_{ST}^{-1} S_{ST}
\]

\[
2[\ell_n^s(\hat{\varphi}_n|\theta) - \ell_n^s(0)] \sim S_{LU}^T \Theta_\perp (\Theta_\perp^T H_{LU} \Theta_\perp)^{-1} \Theta_\perp^T S_{LU} + S_{ST}^T H_{ST}^{-1} S_{ST}
\]

whence the result follows by subtracting (C.7) from (C.6) and noting that

\[
H_{LU}^{-1/2} \Theta(\Theta^T H_{LU}^{-1} \Theta)^{-1} \Theta^T H_{LU}^{-1/2} + H_{LU}^{-1/2} \Theta_\perp (\Theta_\perp^T H_{LU} \Theta_\perp)^{-1} \Theta_\perp^T H_{LU}^{-1/2} = I_{pq}
\]

since the columns of \( H_{LU}^{-1/2} \Theta \) and \( H_{LU}^{-1/2} \Theta_\perp \) are mutually orthogonal, and collectively span the whole of \( \mathbb{R}^{pq} \).

(iv). The same argument as which yielded (C.7) also gives

\[
2[\ell_n^s(\hat{\varphi}_n|\theta, \gamma) - \ell_n^s(0)] = S_{LU}^T \Pi_\perp (\Pi_\perp^T H_{LU} \Pi_\perp)^{-1} \Pi_\perp^T S_{LU} + S_{ST}^T H_{ST}^{-1} S_{ST}
\]

so that subtracting (C.8) from (C.7), and recalling \( \Pi_\perp = \Theta_\perp \Xi \), yields

\[
2[\ell_n^s(\hat{\varphi}_n|\theta) - \ell_n^s(\hat{\varphi}_n|\theta, \gamma)] = S_{LU}^T \Theta_\perp (\Theta_\perp^T H_{LU} \Theta_\perp)^{-1} \Theta_\perp^T S_{LU} - S_{LU}^T \Pi_\perp (\Pi_\perp^T H_{LU} \Pi_\perp)^{-1} \Pi_\perp^T S_{LU}
\]
Proof of Proposition C.2. Recall the definitions of $\mathbf{R}_n = [\mathbf{R}_{n,LU}, \mathbf{R}_{n,ST}]$ and $\mathbf{l}_n = [\mathbf{l}_{n,LU}, \mathbf{l}_{n,ST}]$ given at the beginning of this appendix. Since $I_{kp} = \mathbf{R}_{n,LU}^T \mathbf{l}_{n,LU} + \mathbf{R}_{n,ST} \mathbf{l}_{n,ST}$, we may write

$$
\tilde{\Phi}_n = \Phi_n + [(\tilde{\Phi}_n - \Phi_n)^T \mathbf{R}_{n,LU} + (\tilde{\Phi}_n - \Phi_n)^T \mathbf{R}_{n,ST}] \mathbf{l}_{n,ST} = \Phi_n + \tilde{\Delta}_{n,LU} + \tilde{\Delta}_{n,ST}.
$$

Since $\tilde{\Delta}_{n,LU} = o_p(1)$ and $\Phi_n \rightarrow \Phi_0$, we have $|\lambda_{q+1}(\Phi_n + \tilde{\Delta}_{n,ST})| < |\lambda_q(\Phi_n)|$ w.p.a.1, and so by Lemma B.2(i)

$$
A(\tilde{\Phi}_n) - A(\Phi_n) = A(\Phi_n + \tilde{\Delta}_{n,ST} + \tilde{\Delta}_{n,LU}) - A(\Phi_n + \tilde{\Delta}_{n,ST})
$$

w.p.a.1. Since $A(\cdot)$ is smooth, a second-order Taylor series expansion and Lemma B.2(ii) yield

$$
\text{vec}\{A(\Phi_n + \tilde{\Delta}_{n,ST} + \tilde{\Delta}_{n,LU}) - A(\Phi_n + \tilde{\Delta}_{n,ST})\} = [J_A(\Phi_n + \tilde{\Delta}_{n,ST}) + o_p(1)] \text{vec}\{\tilde{\Delta}_{n,LU} \mathbf{R}_{n,LU}(\Phi_n + \tilde{\Delta}_{n,ST})\}
$$

$$
= [J_A(\Phi_0) + o_p(1)] \text{vec}\{\tilde{\Delta}_{n,LU} \mathbf{R}_{n,LU}\}.
$$

where the second equality holds w.p.a.1, and follows from the continuity of $J_A$ (Lemma B.2(iii)), $\Phi_n + \tilde{\Delta}_{n,ST} = \Phi_0 + o_p(1)$, and $\mathbf{R}_{LU}(\Phi_n + \tilde{\Delta}_{n,ST}) = \mathbf{R}_{LU}(\Phi_n) = \mathbf{R}_{LU}$ (w.p.a.1, as implied by Lemma B.2(i)). Finally, since

$$
\tilde{\Delta}_{n,LU} \mathbf{R}_{n,LU} = [(\tilde{\Phi}_n - \Phi_n)^T \mathbf{R}_{n,LU}] \mathbf{l}_{n,LU} \mathbf{R}_{n,LU} = (\tilde{\Phi}_n - \Phi_n) \mathbf{R}_{n,LU},
$$

the first part of (C.4) follows from (C.9)–(C.11). The proof of the second part is analogous. \hfill \Box

D Proofs of theorems

Proof of Theorem 3.1. (i). In the notation of Appendix C, $\tilde{\varphi}_{n,LU} = \text{vec}\{(\tilde{\Phi}_n - \Phi_n)^T \mathbf{R}_{n,LU}\}$. By Proposition C.1(i)

$$
n \text{vec}\{(\tilde{\Phi}_n - \Phi_n)^T \mathbf{R}_{n,LU}\} \Rightarrow \left[\left(\int \tilde{Z}_C \tilde{Z}_C^T \right)^{-1} \otimes I_p \right] \int_0^1 \left[\tilde{Z}_C(r) \otimes dE(r)\right]
$$

$$
= \text{vec}\left\{ \int (dE) \tilde{Z}_C \left(\int \tilde{Z}_C \tilde{Z}_C^T \right)^{-1} \right\},
$$

and so by Proposition C.2

$$
\begin{bmatrix} \text{vec}\{A(\tilde{\Phi}_n) - A(\Phi_n)\} \\ \text{vec}\{\Lambda_{LU}(\tilde{\Phi}_n) - \Lambda_{LU}(\Phi_n)\} \end{bmatrix} \Rightarrow \begin{bmatrix} J_A(\Phi_0) \\ J_A(\Phi_0) \end{bmatrix} \text{vec}\left\{ \int (dE) \tilde{Z}_C \left(\int \tilde{Z}_C \tilde{Z}_C^T \right)^{-1} \right\}.
$$

(D.1)
Since $\Phi_n \to \Phi_0$ with $\Lambda_{LU}(\Phi_0) = I_q$ under LOC, we have by Lemma B.3 that
\[
\begin{bmatrix}
J_A(\Phi_0) \\
J_A(\Phi_0)
\end{bmatrix} = \begin{bmatrix}
I_q \otimes \beta^TR_{ST}(I_{kp-q} - \Lambda_{ST})^{-1}L_{ST}^T \\
I_q \otimes L_{LU}^T
\end{bmatrix}.
\] (D.2)
The result then follows from (D.1) and (D.2), by reversing the vectorisation.

(ii). In the notation of Appendix C, maximising $\ell_n^r(\Phi)$ subject to $\Lambda_{LU}(\Phi) = \Lambda_{n,LU} = I_q + C/n$ corresponds to maximising $\ell_n(\varphi)$ subject to $\theta_n(\varphi) = 0$. Thus $\hat{\varphi}_{n,LU}|\theta = \text{vec}\{(\hat{\Phi}_n|\Lambda_{n,LU} - \Phi_n)R_{n,LU}\}$, and so by Proposition C.1(ii)

\[
n \text{vec}\{(\hat{\Phi}_n|\Lambda_{n,LU} - \Phi_n)R_{n,LU}\} \sim \Theta_{\perp}(\Theta_{\perp}^TH_{LU}\Theta_{\perp})^{-1}\Theta_{\perp}^TS_{LU}
\]
where $\Theta_{\perp} = I_q \otimes L_{LU,\perp}$. Hence by Proposition C.2,

\[
\text{vec}\{A(\hat{\Phi}_n|\Lambda_{n,LU}) - A(\Phi_n)\} \sim J_A(\Phi_0)\Theta_{\perp}(\Theta_{\perp}^TH_{LU}\Theta_{\perp})^{-1}\Theta_{\perp}^TS_{LU}.
\]

To determine the distribution of the r.h.s., we note that
\[
\Theta_{\perp}^TS_{LU} = \int_0^1 [\bar{Z}_C(r) \otimes L_{LU,\perp}^T\Sigma^{-1}dE(r)] =: \int_0^1 [\bar{Z}_C(r) \otimes dU(r)].
\] (D.3)
Recall that $\bar{Z}_C$ is a function only of $Z_C$, which from (3.8) is given by
\[
Z_C(r) = \int_0^r e^{C(r-s)L_{LU,\perp}^tdE(s)} =: \int_0^r e^{C(r-s)dV(s)}.
\] (D.4)
$(U, V) = (L_{LU,\perp}^T\Sigma^{-1}E, L_{LU}^T E)$ is a pair of vector Brownian motions, with covariance
\[
\mathbb{E}U(1)V^T = L_{LU,\perp}^T\Sigma^{-1}\mathbb{E}[E(1)E(1)^T]L_{LU} = L_{LU,\perp}^T L_{LU} = 0;
\]
whence $U$ and $V$ are independent. In particular, we have from (D.4) that $U$ is independent of $\bar{Z}_C$. This, combined with the fact that
\[
J_A(\Phi_0)\Theta_{\perp}(\Theta_{\perp}^TH_{LU}\Theta_{\perp})^{-1} = \left(\int \bar{Z}_C \bar{Z}_C^T\right)^{-1} \otimes JL_{LU,\perp}(L_{LU,\perp}^T\Sigma^{-1}L_{LU,\perp})^{-1}
\]
depends only on $\bar{Z}_C$, implies $J_A(\Phi_0)\Theta_{\perp}(\Theta_{\perp}^TH_{LU}\Theta_{\perp})^{-1}\Theta_{\perp}^TS_{LU}$ is mixed normal with variance
\[
\left(\int \bar{Z}_C \bar{Z}_C^T\right)^{-1} \otimes JL_{LU,\perp}(L_{LU,\perp}^T\Sigma^{-1}L_{LU,\perp})^{-1}L_{LU,\perp}^TJ^T,
\]
which proves (3.10).

Finally, note that the preceding holds for any choice of $L_{LU,\perp} \in \mathbb{R}^{p \times r}$ having full column rank and $L_{LU,\perp}^T L_{LU} = 0$. Let $\alpha := \Phi_0(1)\beta(\beta^T \beta)^{-1} \in \mathbb{R}^{p \times r}$, where $\Phi_0(1) := \lim_{n \to \infty} \Phi_n(1)$; then
\[
L_{LU}^T \alpha = L_{LU}^T \Phi_0(1)\beta(\beta^T \beta)^{-1} = 0
\]
by (2.5) with $\Lambda_{LU} = \Lambda_{LU}(\Phi_0) = I_q$. Further, $\text{rk} \alpha = r$ since $\text{sp} \Phi_0(1) = \text{sp} \beta$, and thus we may
indeed choose $L_{LU,\perp} = \alpha$. In this case,

$$\mathcal{J}L_{LU,\perp} = \beta^T R_{st}(I_{kp-q} - \Lambda_{st})^{-1}L_{st}^T \Phi_0(1)\beta(\beta^T \beta)^{-1} = (1) \beta^T \beta(\beta^T \beta)^{-1} = I_r,$$

where $= (1)$ follows from (B.15) above. Thus (3.11) is proved.

**Proof of Theorem 3.2.** We first prove (3.13). In the notation of Appendix C, $\mathcal{L}\mathcal{R}_n(\Lambda_{n,LU}) = 2[\ell_n^*(\varphi_n) - \ell_n^*(\hat{\varphi}_n)]$. By Proposition C.1(iii),

$$\mathcal{L}\mathcal{R}_n(\Lambda_{n,LU}) \sim S_{LU}^T H_{LU}^{-1}\Theta(\Theta^T H_{LU}^{-1}\Theta)^{-1}\Theta^T H_{LU}^{-1}S_{LU} =: \mathcal{L}\mathcal{R},$$

where $\Theta = I_q \otimes L_{LU}$, $S_{LU} = \int [\bar{Z}_C(r) \otimes \Sigma^{-1}dE]$, and $H_{LU} = \int \bar{Z}_C \bar{Z}_C^T \otimes \Sigma^{-1}$. To obtain the claimed expression for $\mathcal{L}\mathcal{R}$, note that

$$S_{LU} = \int [\bar{Z}_C(r) \otimes \Sigma^{-1}dE] = \text{vec } \left\{ \Sigma^{-1} \int (dE) \bar{Z}_C^T \right\}$$

and

$$H_{LU}^{-1}\Theta(\Theta^T H_{LU}^{-1}\Theta)^{-1}\Theta^T H_{LU}^{-1} = \left( \int \bar{Z}_C \bar{Z}_C^T \right)^{-1} \otimes \Sigma L_{LU}(L_{LU}^T \Sigma L_{LU})^{-1} L_{LU}^T \Sigma$$

whence, using $\text{vec}(A)^T \text{vec}(B) = \text{tr}(A^T B)$,

$$\mathcal{L}\mathcal{R} = \text{tr } \left\{ \Delta^{-1/2} L_{LU}^T \int (dE) \bar{Z}_C^T \left( \int \bar{Z}_C \bar{Z}_C^T \right)^{-1} \int \bar{Z}_C (dE)^T L_{LU} \Delta^{-1/2} \right\}$$

(D.5)

where $\Delta := L_{LU}^T \Sigma L_{LU}$. To simplify this further, note that $L_{LU}^T E$ is a $q$-dimensional Brownian motion with variance $\Delta$, and so for $W_*(r) := \Delta^{-1/2} L_{LU} E(r) \sim \text{BM}(I_q)$, we have

$$Z_C(r) = \int_0^r e^{C(r-s)} L_{LU}^T dE(s) = \int_0^r e^{C(r-s)} \Delta^{1/2} dW_*(s)$$

$$= (1) \Delta^{1/2} \int_0^r e^{C_*(r-s)} dW_*(s) =: \Delta^{1/2} Z_{C_*}(r)$$

where $C_* := \Delta^{-1/2} C \Delta^{1/2}$ is as in the statement of the theorem, and $= (1)$ follows from $e^C D = De^{D^{-1}CD}$ for any nonsingular $D$. Hence $\bar{Z}_C(r) = \Delta^{1/2} \bar{Z}_{C_*}(r)$, whereupon (3.13) follows from (D.5) and the definition of $W_*$.

We next prove (3.14). Maximisation of $\ell_n^*(\Phi)$ subject to $\Lambda_{LU}(\Phi) = I_q + C/n$ and $a_{ij}(\Phi) = a_0$ corresponds, in the notation of Appendix C, to maximisation of $\ell_n(\varphi)$ subject to $\theta_n(\varphi) = 0$ and $\gamma_n(\varphi) = 0$. Therefore by Proposition C.1(iv),

$$\mathcal{L}\mathcal{R}_n[a_{ij}(\Phi_n); \Lambda_{n,LU}] = 2[\ell_n(\hat{\varphi}_n) - \ell_n(\varphi)] \sim (H_{\Theta,\perp}^{-1/2} \Theta_{\perp}^T S_{LU})^T [I_{qr} - Q](H_{\Theta,\perp}^{-1/2} \Theta_{\perp}^T S_{LU}).$$

Recall from (D.3) and the subsequent arguments that

$$\text{vec } (\Theta_{\perp}^T S_{LU}) = d \left( \int \bar{Z}_C \bar{Z}_C^T \otimes L_{LU,\perp} \Sigma^{-1} L_{LU,\perp} \right)^{1/2} \eta$$

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for \( \eta \sim N[0, I_{qr}] \) independent of \( \bar{Z}_C \), and therefore also of

\[
H_{\Theta,\perp} = \Theta_{\perp}^T H_{LU} \Theta_{\perp} = \int \bar{Z}_C \bar{Z}_C^T \otimes L_{LU,\perp}^T \Sigma^{-1} L_{LU,\perp}.
\]

Thus \( \text{vec} \{ H_{\Theta,\perp}^{-1/2} \Theta_{\perp}^T S_{LU} \} \sim N[0, I_{qr}] \) is independent of \( H_{LU} \), and therefore also of \( Q \). The result follows by noting that \( H_{\Theta,\perp}^{1/2} \Xi \) has rank \( qr - 1 \) a.s., whence \( I_{qr} - Q \) projects orthogonally onto a subspace of dimension 1, a.s. \( \square \)