LATTICE REPRESENTATIONS OF HEISENBERG GROUPS

JAE-HYUN YANG

1 Introduction

For any positive integers $g$ and $h$, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} := \left\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu^t \lambda \text{ symmetric} \right\}.$$  

Recall that the multiplication law is

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') := (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda'^t \mu' - \mu^t \lambda').$$

Here $\mathbb{R}^{(h,g)}$ (resp. $\mathbb{R}^{(h,h)}$) denotes the set of all $h \times g$ (resp. $h \times h$) real matrices.

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded to the symplectic group $Sp(g + h, \mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & t \mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -^t \lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g + h, \mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactifications of Siegel moduli spaces. In fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g + h, \mathbb{R})$ associated with the rational boundary component $F_g$ (cf. [F-C] p. 123 or [N] p. 21). For the motivation of the study of this Heisenberg group we refer to [Y4]-[Y8] and [Z]. We refer to [Y1]-[Y3] for more results on $H_{\mathbb{R}}^{(g,h)}$.

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In [C], P. Cartier stated without proof that for $h = 1$, the lattice representation of $H_{\mathbb{R}}^{(g,1)}$ associated to the lattice $L$ is unitarily equivalent to the direct sum of $[L^* : L]^\frac{1}{2}$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,1)}$, where $L^*$ is the dual lattice of $L$ with respect to a certain nondegenerate alternating bilinear form. R. Berndt proved that the above fact for the case $h = 1$ in his lecture notes [B]. In this paper, we give a complete proof of Cartier’s theorem for $H_{\mathbb{R}}^{(g,h)}$.

**Main Theorem.** Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree $h$ and $L$ be a self-dual lattice in $\mathbb{C}^{(h,g)}$. Then the lattice representation $\pi_\mathcal{M}$ of $H_{\mathbb{R}}^{(g,h)}$ associated with $L$ and $\mathcal{M}$ is unitarily equivalent to the direct sum of $(\det 2\mathcal{M})^g$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,h)}$. For more details, we refer to Section 3.

The paper is organized as follows. In Section 2, we review the Schrödinger representations of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$. In Section 3, we prove the main theorem. In the final section, we provide a relation between lattice representations and theta functions.

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**Notations:** We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. The symbol $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers $z$ with $|z| = 1$, and the symbol $Sp(g, \mathbb{R})$ the symplectic group of degree $g$, $H_g$ the Siegel upper half plane of degree $g$. The symbol “$:=”$ means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring $F$. For any $M \in F^{(k,l)}$, $tM$ denotes the transpose matrix of $M$. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of $A$. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = tABA$. We denote the identity matrix of degree $k$ by $E_k$. For a positive integer $n$, $\text{Symm}(n, K)$ denotes the vector space consisting of all symmetric $n \times n$ matrices with entries in a field $K$.

## 2 Schrödinger Representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy
to see that the inverse of an element \((\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}\) is given by
\[
(\lambda, \mu, \kappa)^{-1} = (-\lambda, -\mu, -\kappa + \lambda^t \mu - \mu^t \lambda).
\]

Now we set
\[
(2.1) \quad [\lambda, \mu, \kappa] := (0, \mu, \kappa) \circ (\lambda, 0, 0) = (\lambda, \mu, \kappa - \mu^t \lambda).
\]

Then \(H_{\mathbb{R}}^{(g,h)}\) may be regarded as a group equipped with the following multiplication
\[
(2.2) \quad [\lambda, \mu, \kappa] \cdot [\lambda_0, \mu_0, \kappa_0] := [\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda^t \mu_0 + \mu_0^t \lambda].
\]

The inverse of \([\lambda, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)}\) is given by
\[
[\lambda, \mu, \kappa]^{-1} = [-\lambda, -\mu, -\kappa + \lambda^t \mu + \mu^t \lambda].
\]

We set
\[
(2.3) \quad K := \left\{ [0, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)} \mid \mu \in \mathbb{R}^{(h,g)}, \kappa = t^t \kappa \in \mathbb{R}^{(h,h)} \right\}.
\]

Then \(K\) is a commutative normal subgroup of \(H_{\mathbb{R}}^{(g,h)}\). Let \(\hat{K}\) be the Pontrajagin dual of \(K\), i.e., the commutative group consisting of all unitary characters of \(K\). Then \(\hat{K}\) is isomorphic to the additive group \(\mathbb{R}^{(h,g)} \times \text{Symm}(h, \mathbb{R})\) via
\[
(2.4) \quad < a, \hat{a} > := e^{2\pi i \sigma(\hat{\mu}^t \mu + \hat{\kappa} \kappa)}, \quad a = [0, \mu, \kappa] \in K, \quad \hat{a} = (\hat{\mu}, \hat{\kappa}) \in \hat{K}.
\]

We put
\[
(2.5) \quad S := \left\{ [\lambda, 0, 0] \in H_{\mathbb{R}}^{(g,h)} \mid \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.
\]

Then \(S\) acts on \(K\) as follows:
\[
(2.6) \quad \alpha_\lambda([0, \mu, \kappa]) := [0, \mu, \kappa + \lambda^t \mu + \mu^t \lambda], \quad [\lambda, 0, 0] \in S.
\]

It is easy to see that the Heisenberg group \((H_{\mathbb{R}}^{(g,h)}, \cdot)\) is isomorphic to the semi-direct product \(S \rtimes K\) of \(S\) and \(K\) whose multiplication is given by
\[
(\lambda, a) \cdot (\lambda_0, a_0) := (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S, \quad a, a_0 \in K.
\]

On the other hand, \(S\) acts on \(\hat{K}\) by
\[
(2.7) \quad \alpha_\lambda^*(\hat{a}) := (\hat{\mu} + 2\hat{\kappa} \lambda, \hat{\kappa}), \quad [\lambda, 0, 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \hat{K}.
\]

Then, we have the relation \(< \alpha_\lambda(a), \hat{a} > = < a, \alpha_\lambda^*(\hat{a}) >\) for all \(a \in K\) and \(\hat{a} \in \hat{K}\).
We have two types of $S$-orbits in $\hat{K}$.

**Type I.** Let $\hat{\kappa} \in \text{Symm}(h, \mathbb{R})$ with $\hat{\kappa} \neq 0$. The $S$-orbit of $\hat{a}(\hat{\kappa}) := (0, \hat{\kappa}) \in \hat{K}$ is given by

$$\hat{O}_{\hat{\kappa}} := \{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \hat{K} \mid \lambda \in \mathbb{R}^{(h,g)} \} \cong \mathbb{R}^{(h,g)}.$$  

**Type II.** Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The $S$-orbit $\hat{O}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$\hat{O}_{\hat{y}} := \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}).$$

We have

$$\hat{K} = \left( \bigcup_{\hat{\kappa} \in \text{Symm}(h, \mathbb{R})} \hat{O}_{\hat{\kappa}} \right) \bigcup \left( \bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{O}_{\hat{y}} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of $S$ at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{ 0 \}.$$  

And the stabilizer $S_{\hat{y}}$ of $S$ at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \{ [\lambda, 0, 0] \mid \lambda \in \mathbb{R}^{(h,g)} \} = S \cong \mathbb{R}^{(h,g)}.$$  

From now on, we set $G := H_{\mathbb{R}}^{(g,h)}$ for brevity. It is known that $K$ is a closed, commutative normal subgroup of $G$. Since $(\lambda, \mu, \kappa) = (0, \mu, \kappa + \mu^t \lambda) \circ (\lambda, 0, 0)$ for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X := K \setminus G$ can be identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$  

We observe that $G$ acts on $X$ by

$$ (Kg) \cdot g_0 := K(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0, $$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$k_g = (0, \mu, \kappa + \mu^t \lambda), \quad s_g = (\lambda, 0, 0)$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf. [M]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda^t \mu_0).$$
and so

\[(2.15)\]
\[k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0^t \lambda_0 + \lambda^t \mu_0 + \mu_0^t \lambda).\]

For a real symmetric matrix \(c = t_c \in \mathbb{R}^{(h,h)}\) with \(c \neq 0\), we consider the one-dimensional unitary representation \(\sigma_c\) of \(K\) defined by

\[(2.16)\]
\[\sigma_c ((0, \mu, \kappa)) := e^{2\pi i \sigma(c \kappa)} I, \quad (0, \mu, \kappa) \in K,\]

where \(I\) denotes the identity mapping. Then the induced representation \(U(\sigma_c) := \text{Ind}_K \sigma_c\) of \(G\) induced from \(\sigma_c\) is realized in the Hilbert space \(\mathcal{H}_{\sigma_c} = L^2(X, dg, \mathbb{C}) \cong L^2(\mathbb{R}^{(h,g)}, d\xi)\) as follows. If \(g_0 = (\lambda_0, \mu_0, \kappa_0) \in G\) and \(x = Kg \in X\) with \(g = (\lambda, \mu, \kappa) \in G\), we have

\[(2.17)\]
\[(U_{g_0}(\sigma_c)f)(x) = \sigma_c (k_{sg \circ g_0}) (f(xg_0)), \quad f \in \mathcal{H}_{\sigma_c}.\]

It follows from (2.15) that

\[(2.18)\]
\[(U_{g_0}(\sigma_c)f)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0^{\lambda_0} + 2\lambda_0^t \mu_0)\}} f(\lambda + \lambda_0).\]

Here, we identified \(x = Kg\) (resp. \(xg_0 = Kgg_0\)) with \(\lambda\) (resp. \(\lambda_0 + \lambda_0\)). The induced representation \(U(\sigma_c)\) is called the Schrödinger representation of \(G\) associated with \(\sigma_c\). Thus \(U(\sigma_c)\) is a monomial representation.

Now, we denote by \(\mathcal{H}^{\sigma_c}\) the Hilbert space consisting of all functions \(\phi : G \rightarrow \mathbb{C}\) which satisfy the following conditions:

1. \(\phi(g)\) is measurable with respect to \(dg\),
2. \(\phi((0, \mu, \kappa) \circ g)) = e^{2\pi i \sigma(c \kappa) \phi(g)}\) for all \(g \in G\),
3. \(\|\phi\|^2 := \int_X |\phi(g)|^2\ dg < \infty, \quad g = Kg,\)

where \(dg\) (resp. \(dg\)) is a \(G\)-invariant measure on \(G\) (resp. \(X = K \backslash G\)). The inner product \((,\)\) on \(\mathcal{H}^{\sigma_c}\) is given by

\[(\phi_1, \phi_2) := \int_G \phi_1(g) \overline{\phi_2(g)} dg \quad \text{for} \ \phi_1, \phi_2 \in \mathcal{H}^{\sigma_c}.\]

We observe that the mapping \(\Phi_c : \mathcal{H}_{\sigma_c} \rightarrow \mathcal{H}^{\sigma_c}\) defined by

\[(2.19)\]
\[(\Phi_c(f))(g) := e^{2\pi i \sigma\{c(\kappa + \mu^t \lambda)\}} f(\lambda), \quad f \in \mathcal{H}_{\sigma_c}, \ g = (\lambda, \mu, \kappa) \in G\]

is an isomorphism of Hilbert spaces. The inverse \(\Psi_c : \mathcal{H}^{\sigma_c} \rightarrow \mathcal{H}_{\sigma_c}\) of \(\Phi_c\) is given by

\[(2.20)\]
\[(\Psi_c(\phi))(\lambda) := \phi((\lambda, 0, 0)), \quad \phi \in \mathcal{H}^{\sigma_c}, \ \lambda \in \mathbb{R}^{(h,g)}.\]
The Schrödinger representation $U(\sigma_c)$ of $G$ on $H^{\sigma_c}$ is given by

$$(U_{g_0}(\sigma_c)\phi)(g) = e^{2\pi i \sigma \{c(\kappa_0 + \mu_0 \iota \lambda_0 + \lambda^t \mu_0 - \lambda_0 \mu_0)\}} \phi((\lambda_0, 0, 0) \circ g),$$

where $g_0 = (\lambda_0, \mu_0, \kappa_0)$, $g = (\lambda, \mu, \kappa) \in G$ and $\phi \in H^{\sigma_c}$. (2.21) can be expressed as follows.

$$(U_{g_0}(\sigma_c)\phi)(g) = e^{2\pi i \sigma \{c(\kappa_0 + \kappa + \mu_0 \iota \lambda_0 + \mu^t \lambda + 2 \lambda^t \mu_0)\}} \phi((\lambda_0 + \lambda, 0, 0)).$$

**Theorem 2.1.** Let $c$ be a positive symmetric half-integral matrix of degree $h$. Then the Schrödinger representation $U(\sigma_c)$ of $G$ is irreducible.

*Proof.* The proof can be found in [Y1], theorem 3. □

### 3 Proof of the Main Theorem

Let $L := \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ be the lattice in the vector space $V \cong \mathbb{C}^{(h,g)}$. Let $B$ be an alternating bilinear form on $V$ such that $B(L, L) \subset \mathbb{Z}$, that is, $\mathbb{Z}$-valued on $L \times L$. The dual $L_B^*$ of $L$ with respect to $B$ is defined by

$$L_B^* := \{ v \in V \mid B(v, l) \in \mathbb{Z} \text{ for all } l \in L \}.$$ 

Then $L \subset L_B^*$. If $B$ is nondegenerate, $L_B^*$ is also a lattice in $V$, called the *dual lattice* of $L$. In case $B$ is nondegenerate, there exist a $\mathbb{Z}$-basis $\{\xi_{11}, \xi_{12}, \ldots, \xi_{hg}, \eta_{11}, \eta_{12}, \ldots, \eta_{hg}\}$ of $L$ and a set $\{e_{11}, e_{12}, \ldots, e_{hg}\}$ of positive integers such that $e_{11}|e_{12}, e_{12}|e_{13}, \ldots, e_{h,g-1}|e_{hg}$ for which

$$
\begin{pmatrix}
B(\xi_{ka}, \xi_{lb}) & B(\xi_{ka}, \eta_{lb}) \\
B(\eta_{ka}, \xi_{lb}) & B(\eta_{ka}, \eta_{lb})
\end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix},
$$

where $1 \leq k, l \leq h$, $1 \leq a, b \leq g$ and $e := \text{diag}(e_{11}, e_{12}, \ldots, e_{hg})$ is the diagonal matrix of degree $hg$ with entries $e_{11}, e_{12}, \ldots, e_{hg}$. It is well known that $[L_B^* : L] = (\det e)^2 = (e_{11}e_{12} \cdots e_{hg})^2$ (cf. [I] p. 72). The number $\det e$ is called the *Pfaffian* of $B$.

Now, we consider the following subgroups of $G$:

$$(3.1) \Gamma_L := \left\{ (\lambda, \mu, \kappa) \in G \mid (\lambda, \mu) \in L, \kappa \in \mathbb{R}^{(h,h)} \right\}$$

and

$$(3.2) \Gamma_{L_B^*} := \left\{ (\lambda, \mu, \kappa) \in G \mid (\lambda, \mu) \in L_B^*, \kappa \in \mathbb{R}^{(h,h)} \right\}.$$
Then both $\Gamma_L$ and $\Gamma_B^*$ are normal subgroups of $G$. We set

\begin{equation}
Z_0 := \left\{ (0, 0, \kappa) \in G \mid \kappa = t_\kappa \in \mathbb{Z}^{(h,h)} \text{ integral} \right\}.
\end{equation}

It is easy to show that

\begin{equation}
\Gamma_B^* = \left\{ g \in G \mid g\gamma g^{-1}\gamma^{-1} \in Z_0 \text{ for all } \gamma \in \Gamma_L \right\}.
\end{equation}

We define

\begin{equation}
Y_L := \{ \varphi \in \text{Hom} (\Gamma_L, \mathbb{C}^\times) \mid \varphi \text{ is trivial on } Z_0 \}
\end{equation}

and

\begin{equation}
Y_{L,S} := \{ \varphi \in Y_L \mid \varphi(\kappa) = e^{2\pi i \sigma(S\kappa)} \text{ for all } \kappa = t_\kappa \in \mathbb{R}^{(h,h)} \}
\end{equation}

for each symmetric real matrix $S$ of degree $h$. We observe that, if $S$ is not half-integral, then $Y_L = \emptyset$ and so $Y_{L,S} = \emptyset$. It is clear that, if $S$ is symmetric half-integral, then $Y_{L,S}$ is not empty.

Thus we have

\begin{equation}
Y_L = \bigcup_M Y_{L,M},
\end{equation}

where $M$ runs through the set of all symmetric half-integral matrices of degree $h$.

**Lemma 3.1.** Let $M$ be a symmetric half-integral matrix of degree $h$ with $M \neq 0$. Then any element $\varphi$ of $Y_{L,M}$ is of the form $\varphi_{M,q}$. Here $\varphi_{M,q}$ is the character of $\Gamma_L$ defined by

\begin{equation}
\varphi_{M,q}((l, \kappa)) := e^{2\pi i \sigma(M\kappa)} \cdot e^{\pi i q(l)} \quad \text{for } (l, \kappa) \in \Gamma_L,
\end{equation}

where $q : L \rightarrow \mathbb{R}/2\mathbb{Z} \cong [0, 2)$ is a function on $L$ satisfying the following condition:

\begin{equation}
q(l_0 + l_1) \equiv q(l_0) + q(l_1) - 2\sigma(M(\lambda_0^t M_1 - \mu_0^t M_1)) \pmod{2}
\end{equation}

for all $l_0 = (\lambda_0, \mu_0) \in L$ and $l_1 = (\lambda_1, \mu_1) \in L$.

**Proof.** (3.8) follows immediately from the fact that $\varphi_{M,q}$ is a character of $\Gamma_L$. It is obvious that any element of $Y_{L,M}$ is of the form $\varphi_{M,q}$. \qed

**Lemma 3.2.** An element of $Y_{L,0}$ is of the form $\varphi_{k,l} (k, l \in \mathbb{R}^{(h,g)})$. Here $\varphi_{k,l}$ is the character of $\Gamma_L$ defined by

\begin{equation}
\varphi_{k,l}(\gamma) := e^{2\pi i \sigma(k^t \lambda + l^t \mu)}, \quad \gamma = (\lambda, \mu, \kappa) \in \Gamma_L.
\end{equation}

**Proof.** It is easy to prove and so we omit the proof. \qed
Lemma 3.3. Let $M$ be a nonsingular symmetric half-integral matrix of degree $h$. Let $\varphi_{M,q_1}$ and $\varphi_{M,q_2}$ be the characters of $\Gamma_L$ defined by (3.7). The character $\varphi$ of $\Gamma_L$ defined by $\varphi := \varphi_{M,q_1} \cdot \varphi_{M,q_2}^{-1}$ is an element of $Y_{L,0}$.

Proof. It follows from the existence of an element $g = (M^{-1} \lambda, M^{-1} \mu, 0) \in G$ with $(\lambda, \mu) \in V$ such that

$$\varphi_{M,q_1}(\gamma) = \varphi_{M,q_2}(g \gamma g^{-1}) \text{ for all } \gamma \in \Gamma_L.$$ 

□

For a unitary character $\varphi_{M,q}$ of $\Gamma_L$ defined by (3.7), we let

$$(3.10) \quad \pi_{M,q} := \text{Ind}_G^G \varphi_{M,q}$$

be the representation of $G$ induced from $\varphi_{M,q}$. Let $\mathcal{H}_{M,q}$ be the Hilbert space consisting of all measurable functions $\phi : G \rightarrow \mathbb{C}$ satisfying

(L1) $\phi(\gamma g) = \varphi_{M,q}(\gamma) \phi(g)$ for all $\gamma \in \Gamma_L$ and $g \in G$.

(L2) $\|\phi\|_{\mathcal{H}_{M,q}}^2 := \int_{\Gamma_L \backslash G} |\phi(\bar{g})|^2 d\bar{g} < \infty$, $\bar{g} = \Gamma_L g$.

The induced representation $\pi_{M,q}$ is realized in $\mathcal{H}_{M,q}$ as follows:

$$(3.11) \quad \left( \pi_{M,q}(g_0) \phi \right)(g) := \phi(g_0 g), \quad g_0, g \in G, \phi \in \mathcal{H}_{M,q}.$$ 

The representation $\pi_{M,q}$ is called the lattice representation of $G$ associated with the lattice $L$.

Main Theorem. Let $M$ be a positive definite, symmetric half integral matrix of degree $h$. Let $\varphi_{M}$ be the character of $\Gamma_L$ defined by $\varphi_M((\lambda, \mu, \kappa)) := e^{2\pi i \sigma(M \kappa)}$ for all $(\lambda, \mu, \kappa) \in \Gamma_L$. Then the lattice representation

$$\pi_M := \text{Ind}_G^G \varphi_M$$

induced from the character $\varphi_{M}$ is unitarily equivalent to the direct sum

$$\bigoplus U(\sigma_M) := \bigoplus \text{Ind}_K^G \sigma_M \quad (\text{det } 2M)^g\text{-copies}$$

of the Schrödinger representation $\text{Ind}_K^G \sigma_M$. 

Proof. We first recall that the induced representation $\pi_M$ is realized in the Hilbert space $H_M$ consisting of all measurable functions $\phi : G \rightarrow \mathbb{C}$ satisfying the conditions
\begin{align}
\phi((\lambda_0, \mu_0, \kappa_0) \circ g) &= e^{2\pi i \sigma(M\kappa_0)} \phi(g), \quad (\lambda_0, \mu_0, \kappa_0) \in \Gamma_L, \; g \in G
\end{align}
and
\begin{align}
\|\phi\|_{\pi_M}^2 := \int_{\Gamma_L \setminus G} |\phi(\bar{g})|^2 \, d\bar{g} < \infty, \quad \bar{g} = \Gamma_L \circ g.
\end{align}
Now, we write $g_0 = [\lambda_0, \mu_0, \kappa_0] \in \Gamma_L$ and $g = [\lambda, \mu, \kappa] \in G$.

For $\phi \in H_M$, we have
\begin{align}
\phi(g_0 \circ g) &= \phi([\lambda_0 + \lambda, \mu_0 + \mu, \kappa_0 + \kappa + \lambda_0^t \mu + \mu^t \lambda_0]).
\end{align}
On the other hand, we get
\begin{align}
\phi(g_0 \circ g) &= \phi((\lambda_0, \mu_0, \kappa_0 - \mu_0^t \lambda_0) \circ g) \\
&= e^{2\pi i \sigma(M(\kappa_0 - \mu_0^t \lambda_0))} \phi(g) \\
&= e^{2\pi i \sigma(M\kappa_0)} \phi(g) \quad \text{(because $\sigma(M\lambda_0^t \lambda_0) \in \mathbb{Z}$)}
\end{align}
Thus, putting $\kappa' := \kappa_0 + \lambda_0^t \mu + \mu^t \lambda_0$, we get
\begin{align}
\phi([\lambda_0 + \lambda, \mu_0 + \mu, \kappa + \kappa']) &= e^{2\pi i \sigma(M\kappa')} \cdot e^{-4\pi i \sigma(M\lambda_0^t \mu)} \phi([\lambda, \mu, \kappa]).
\end{align}
Putting $\lambda_0 = \kappa_0 = 0$ in (3.16), we have
\begin{align}
\phi([\lambda, \mu, \mu_0 + \kappa]) = \phi([\lambda, \mu, \kappa]) \quad \text{for all $\mu_0 \in \mathbb{Z}^{(h,g)}$ and $[\lambda, \mu, \kappa] \in G$}.
\end{align}
Therefore if we fix $\lambda$ and $\kappa$, $\phi$ is periodic in $\mu$ with respect to the lattice $\mathbb{Z}^{(h,g)}$ in $\mathbb{R}^{(h,g)}$. We note that
\begin{align}
\phi([\lambda, \mu, \kappa]) = \phi([0, 0, \kappa] \circ [\lambda, \mu, 0]) = e^{2\pi i \sigma(M\kappa)} \phi([\lambda, 0, 0])
\end{align}
for $[\lambda, \mu, \kappa] \in G$. Hence, $\phi$ admits a Fourier expansion in $\mu$:
\begin{align}
\phi([\lambda, \mu, \kappa]) = e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda) e^{2\pi i \sigma(N^t \mu)}.
\end{align}
If $\lambda_0 \in \mathbb{Z}^{(h,g)}$, then we have

$$
\phi([\lambda + \lambda_0, \mu, \kappa]) = e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda + \lambda_0) e^{2\pi i \sigma(N^t \mu)}
$$

$$
= e^{-4\pi i \sigma(M\lambda_0^t \mu)} \phi([\lambda, \mu, \kappa]) \quad \text{(by (3.16))}
$$

$$
= e^{-4\pi i \sigma(M\lambda_0^t \mu)} e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda) e^{2\pi i \sigma(N^t \mu)},
$$

$$
= e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda) e^{2\pi i \sigma((N-2M\lambda_0)^t \mu)} \quad \text{(by (3.18))}
$$

So we get

$$
\sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda + \lambda_0) e^{2\pi i \sigma(N^t \mu)}
$$

$$
= \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda) e^{2\pi i \sigma((N-2M\lambda_0)^t \mu)}
$$

$$
= \sum_{N \in \mathbb{Z}^{(h,g)}} c_{N+2M\lambda_0}(\lambda) e^{2\pi i \sigma(N^t \mu)}.
$$

Hence, we get

$$
c_N(\lambda + \lambda_0) = c_{N+2M\lambda_0}(\lambda) \quad \text{for all } \lambda_0 \in \mathbb{Z}^{(h,g)} \text{ and } \lambda \in \mathbb{R}^{(h,g)}.
$$

Consequently, it is enough to know only the coefficients $c_\alpha(\lambda)$ for the representatives $\alpha$ in $\mathbb{Z}^{(h,g)}$ modulo $2M$. It is obvious that the number of all such $\alpha$’s is $(\det 2M)^g$. We denote by $\mathcal{J}$ a complete system of such representatives in $\mathbb{Z}^{(h,g)}$ modulo $2M$.

Then, we have

$$
\phi([\lambda, \mu, \kappa]) = e^{2\pi i \sigma(M\kappa)} \left\{ \sum_{N \in \mathbb{Z}^{(h,g)}} c_{\alpha+2MN}(\lambda) e^{2\pi i \sigma((\alpha+2MN)^t \mu)} + \sum_{N \in \mathbb{Z}^{(h,g)}} c_{\beta+2MN}(\lambda) e^{2\pi i \sigma((\beta+2MN)^t \mu)} \right. \\
\left. \quad \cdots \right. \\
\left. + \sum_{N \in \mathbb{Z}^{(h,g)}} c_{\gamma+2MN}(\lambda) e^{2\pi i ((\gamma+2MN)^t \mu)} \right\},
$$

where $\{\alpha, \beta, \cdots, \gamma\}$ denotes the complete system $\mathcal{J}$. 
For each \( \alpha \in J \), we denote by \( \mathcal{H}_{M,\alpha} \) the Hilbert space consisting of Fourier expansions

\[
e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}(h,g)} c_{\alpha+2MN}(\lambda) e^{2\pi i \sigma((\alpha+2MN)^t\mu)}, \quad (\lambda, \mu, \kappa) \in G,
\]

where \( c_N(\lambda) \) denotes the coefficients of the Fourier expansion (3.18) of \( \phi \in \mathcal{H}_M \) and \( \phi \) runs over the set \( \{ \phi \in \pi_M \} \). It is easy to see that \( \mathcal{H}_{M,\alpha} \) is invariant under \( \pi_M \). We denote the restriction of \( \pi_M \) to \( \mathcal{H}_{M,\alpha} \) by \( \pi_{M,\alpha} \). Then we have

\[
(3.20) \quad \pi_M = \bigoplus_{\alpha \in J} \pi_{M,\alpha}.
\]

Let \( \phi_\alpha \in \pi_{M,\alpha} \). Then for \( [\lambda, \mu, \kappa] \in G \), we get

\[
(3.21) \quad \phi_\alpha([\lambda, \mu, \kappa]) = e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}(h,g)} c_{\alpha+2MN}(\lambda) e^{2\pi i \sigma((\alpha+2MN)^t\mu)}.
\]

We put

\[
I_\lambda := [0, 1] \times [0, 1] \times \cdots \times [0, 1] \subset \{ [\lambda, 0, 0] \mid \lambda \in \mathbb{R}(h,g) \}
\]

and

\[
I_\mu := [0, 1] \times [0, 1] \times \cdots \times [0, 1] \subset \{ [0, \mu, 0] \mid \mu \in \mathbb{R}(h,g) \}.
\]

Then, we obtain

\[
(3.22) \quad \int_{I_\lambda} \phi_\alpha([\lambda, \mu, \kappa]) e^{-2\pi i \sigma(\lambda^t\mu)} d\mu = e^{2\pi i \sigma(M\kappa)} c_\alpha(\lambda), \quad \alpha \in J.
\]

Since \( \Gamma_L \setminus G \cong I_\lambda \times I_\mu \), we get

\[
\|\phi_\alpha\|^2_{\pi,\mathcal{M},\alpha} : = \|\phi_\alpha\|^2_{\pi,\mathcal{M}} = \int_{\Gamma_L \setminus G} |\phi_\alpha(\bar{g})|^2 \, d\bar{g}
\]

\[
= \int_{I_\lambda} \int_{I_\mu} |\phi_\alpha(\bar{g})|^2 \, d\lambda d\mu
\]

\[
= \int_{I_\lambda \times I_\mu} \left| \sum_{N \in \mathbb{Z}(h,g)} c_{\alpha+2MN}(\lambda) e^{2\pi i \sigma((\alpha+2MN)^t\mu)} \right|^2 \, d\lambda d\mu
\]

\[
= \int_{I_\lambda} \sum_{N \in \mathbb{Z}(h,g)} |c_{\alpha+2MN}(\lambda)|^2 \, d\lambda
\]

\[
= \int_{I_\lambda} \sum_{N \in \mathbb{Z}(h,g)} |c_\alpha(\lambda + N)|^2 \, d\lambda \quad \text{(by (3.19))}
\]

\[
= \int_{\mathbb{R}(h,g)} |c_\alpha(\lambda)|^2 \, d\lambda.
\]
Since $\phi_\alpha \in \pi_{M,\alpha}$, $\|\phi_\alpha\|_{\pi_{M,\alpha}} < \infty$ and so $c_\alpha(\lambda) \in L^2(\mathbb{R}^{(h,g)}, d\xi)$ for all $\alpha \in J$.

For each $\alpha \in J$, we define the mapping $\theta_{M,\alpha}$ on $L^2(\mathbb{R}^{(h,g)}, d\xi)$ by

$$
(\theta_{M,\alpha} f)([\lambda, \mu, \kappa]) := e^{2\pi i \sigma(M\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} f(\lambda + N) e^{2\pi i \{\alpha t + 2MN\} \mu} ,
$$

where $f \in L^2(\mathbb{R}^{(h,g)}, d\xi)$ and $[\lambda, \mu, \kappa] \in G$.

**Lemma 3.4.** For each $\alpha \in J$, the image of $L^2(\mathbb{R}^{(h,g)}, d\xi)$ under $\theta_{M,\alpha}$ is contained in $\mathcal{H}_{M,\alpha}$. Moreover, the mapping $\theta_{M,\alpha}$ is a one-to-one unitary operator of $L^2(\mathbb{R}^{(h,g)}, d\xi)$ onto $\mathcal{H}_{M,\alpha}$ preserving the norms. In other words, the mapping

$$
\theta_{M,\alpha} : L^2(\mathbb{R}^{(h,g)}, d\xi) \longrightarrow \mathcal{H}_{M,\alpha}
$$

is an isometry.

**Proof.** We already showed that $\theta_{M,\alpha}$ preserves the norms. First, we observe that if $(\lambda_0, \mu_0, \kappa_0) \in \Gamma_L$ and $g = [\lambda, \mu, \kappa] \in G$,

$$
(\lambda_0, \mu_0, \kappa_0) \circ g = [\lambda_0, \mu_0, \kappa_0 + \mu_0^t \lambda_0] \circ [\lambda, \mu, \kappa] = [\lambda_0 + \lambda, \mu_0 + \mu, \kappa + \kappa_0 + \mu_0^t \lambda_0 + \lambda_0^t \mu + \mu^t \lambda_0].
$$

Thus we get

$$
(\theta_{M,\alpha} f)((\lambda_0, \mu_0, \kappa_0) \circ g)
= e^{2\pi i \sigma(M\kappa)} \cdot e^{2\pi i \sigma(M\kappa)} \cdot e^{2\pi i \sigma(\alpha \mu_0)} \sum_{N \in \mathbb{Z}^{(h,g)}} f(\lambda + N) e^{2\pi i \{\alpha t + 2MN\} \{\mu_0 + \mu\}}
= e^{2\pi i \sigma(M\kappa)} \cdot f(\bar{g}).
$$

Here, in the above equalities we used the facts that $2\sigma(MN^t \mu_0) \in \mathbb{Z}$ and $\alpha^t \mu_0 \in \mathbb{Z}$. It is easy to show that

$$
\int_{\Gamma_L \setminus G} |\theta_{M,\alpha} f(\bar{g})|^2 d\bar{g} = \int_{\mathbb{R}^{(h,g)}} |f(\lambda)|^2 d\lambda = \|f\|_2^2 < \infty.
$$

This completes the proof of Lemma 3.4.
Finally, it is easy to show that for each $\alpha \in J$, the mapping $\vartheta_{\mathcal{M},\alpha}$ intertwines the Schrödinger representation $(U(\sigma_{\mathcal{M}}), L^2(\mathbb{R}^{(h,g)}, d\xi))$ and the representation $(\pi_{\mathcal{M},\alpha}, \mathcal{H}_{\mathcal{M},\alpha})$. Therefore, by Lemma 3.4, for each $\alpha \in J$, $\pi_{\mathcal{M},\alpha}$ is unitarily equivalent to $U(\sigma_{\mathcal{M}})$ and so $\pi_{\mathcal{M},\alpha}$ is an irreducible unitary representation of $G$. According to (3.20), the induced representation $\pi_{\mathcal{M}}$ is unitarily equivalent to

$$\bigoplus U(\sigma_{\mathcal{M}}) \quad (\text{(det } 2\mathcal{M})^g\text{-copies}) .$$

This completes the proof of the Main Theorem. \qed

4 Relation of Lattice Representations to Theta Functions

In this section, we state the connection between lattice representations and theta functions. As before, we write $V = \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \cong \mathbb{C}^{(h,g)}$, $L = \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ and $\mathcal{M}$ is a positive symmetric half-integral matrix of degree $h$. The function $q_{\mathcal{M}} : L \rightarrow \mathbb{R}/2\mathbb{Z} = [0, 2)$ defined by

$$(4.1) \quad q_{\mathcal{M}}((\xi, \eta)) := 2(\mathcal{M}\xi^t\eta), \quad (\xi, \eta) \in L$$

satisfies Condition (3.8). We let $\varphi_{\mathcal{M},q_{\mathcal{M}}} : \Gamma_L \rightarrow \mathbb{C}^\times$ be the character of $\Gamma_L$ defined by

$$\varphi_{\mathcal{M},q_{\mathcal{M}}}((l, \kappa)) = e^{2\pi i \sigma((\mathcal{M}\kappa))} e^{\pi i q_{\mathcal{M}}(l)}, \quad (l, \kappa) \in \Gamma_L .$$

We denote by $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ the Hilbert space consisting of measurable functions $\phi : G \rightarrow \mathbb{C}$ which satisfy Condition (4.2) and Condition (4.3):

$$(4.2) \quad \phi((l, \kappa) \circ g) = \varphi_{\mathcal{M},q_{\mathcal{M}}}((l, \kappa)) \phi(g) \quad \text{for all } (l, \kappa) \in \Gamma_L \text{ and } g \in G .$$

$$(4.3) \quad \int_{\Gamma_L \backslash G} \| \phi(\hat{g}) \|^2 d\hat{g} < \infty, \quad \hat{g} = \Gamma_L \circ g .$$

Then the lattice representation

$$\pi_{\mathcal{M},q_{\mathcal{M}}} := \text{Ind}_{\Gamma_L}^G \varphi_{\mathcal{M},q_{\mathcal{M}}}$$

of $G$ induced from the character $\varphi_{\mathcal{M},q_{\mathcal{M}}}$ is realized in $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ as

$$(\pi_{\mathcal{M},q_{\mathcal{M}}}(g_0) \phi)(g) = \phi(gg_0), \quad g_0, g \in G, \ \phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}} .$$

Let $\mathbb{H}_{\mathcal{M},q_{\mathcal{M}}}$ be the vector space consisting of measurable functions $F : V \rightarrow \mathbb{C}$ satisfying Conditions (4.4) and (4.5):

$$(4.4) \quad F(\lambda + \xi, \mu + \eta) = e^{2\pi i \sigma(\mathcal{M}^t(\eta^t\xi + \mu^t\eta - \mu^t\xi))} F(\lambda, \mu) \quad \text{for all } (\lambda, \mu) \in V \text{ and } (\xi, \eta) \in L .$$

$$(4.5) \quad \int_{L \backslash V} \| F(\tilde{v}) \|^2 d\tilde{v} = \int_{I_x \times I_\mu} \| F(\lambda, \mu) \|^2 d\lambda d\mu < \infty .$$

Given $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ and a fixed element $\Omega \in H_g$, we put
\[
(4.6) \quad E_\phi(\lambda, \mu) := \phi((\lambda, \mu, 0)), \quad \lambda, \mu \in \mathbb{R}^{(h, g)},
\]
\[
(4.7) \quad F_\phi(\lambda, \mu) := \phi([\lambda, \mu, 0]), \quad \lambda, \mu \in \mathbb{R}^{(h, g)},
\]
\[
(4.8) \quad F_{\Omega, \phi}(\lambda, \mu) := e^{-2\pi i \sigma(M\lambda\Omega^t\lambda)} F_\phi(\lambda, \mu), \quad \lambda, \mu \in \mathbb{R}^{(h, g)}.
\]

In addition, we put for \( W = \lambda\Omega + \mu \in \mathbb{C}^{(h, g)} \),
\[
(4.9) \quad \vartheta_{\Omega, \phi}(W) := \vartheta_{\Omega, \phi}(\lambda\Omega + \mu) := F_{\Omega, \phi}(\lambda, \mu).
\]

We observe that \( E_\phi, F_\phi \) and \( F_{\Omega, \phi} \) are functions defined on \( V \) and \( \vartheta_{\Omega, \phi} \) is a function defined on \( \mathbb{C}^{(h, g)} \).

**Proposition 4.1.** If \( \phi \in H_{M,q}, (\xi, \eta) \in L \) and \( (\lambda, \mu) \in V \), then we have the formulas

\[
(4.10) \quad E_\phi(\lambda + \xi, \mu + \eta) = e^{2\pi i \sigma(M(\xi^t\eta + \lambda^t\eta - \mu^t\xi))} E_\phi(\lambda, \mu).
\]

\[
(4.11) \quad F_\phi(\lambda + \xi, \mu + \eta) = e^{-4\pi i \sigma(M\xi^t\mu)} F_\phi(\lambda, \mu).
\]

\[
(4.12) \quad F_{\Omega, \phi}(\lambda + \xi, \mu + \eta) = e^{-2\pi i \sigma(M(\xi^t\Omega^t\xi + 2\lambda^t\xi + 2\mu^t\xi))} F_{\Omega, \phi}(\lambda, \mu).
\]

If \( W = \lambda\Omega + \eta \in \mathbb{C}^{(h, g)} \), then we have
\[
(4.13) \quad \vartheta_{\Omega, \phi}(W + \xi\Omega + \eta) = e^{-2\pi i \sigma(M(\xi^t\Omega^t\xi + 2W^t\xi))} \vartheta_{\Omega, \phi}(W).
\]

Moreover, \( F_\phi \) is an element of \( H_{M,q} \).

**Proof.** We note that
\[
(\lambda + \xi, \mu + \eta, 0) = (\xi, \eta, -\xi^t\mu + \eta^t\lambda) \circ (\lambda, \mu, 0).
\]

Thus we have
\[
E_\phi(\lambda + \xi, \mu + \eta) = \phi((\lambda + \xi, \mu + \eta, 0))
\]
\[
= \phi((\xi, \eta, -\xi^t\mu + \eta^t\lambda) \circ (\lambda, \mu, 0))
\]
\[
= e^{2\pi i \sigma(M(\xi^t\eta + \lambda^t\eta - \mu^t\xi))} \phi((\lambda, \mu, 0))
\]
\[
= e^{2\pi i \sigma(M(\xi^t\eta + \lambda^t\eta - \mu^t\xi))} E_\phi(\lambda, \mu).
\]

This proves Formula (4.10). We observe that
\[
[\lambda + \xi, \mu + \eta, 0] = (\xi, \eta, -\xi^t\mu - \mu^t\xi - \eta^t\xi) \circ [\lambda, \mu, 0].
\]
Thus we have
\[
F_\phi(\lambda + \xi, \mu + \eta) = \phi([\lambda + \xi, \mu + \eta, 0]) \\
= e^{-2\pi i \sigma \{ M(\xi^t \mu + \mu^t \xi + \eta^t \xi) \}} \\
\times e^{2\pi i \sigma (M \xi \eta) \phi([\lambda, \mu])} \\
= e^{-4\pi i \sigma (M \xi \mu) \phi([\lambda, \mu, 0])} \\
= e^{-4\pi i \sigma (M \xi \mu) F_\phi(\lambda, \mu)}.
\]

This proves Formula (4.11). According to (4.11), we have
\[
F_{\Omega, \phi}(\lambda + \xi, \mu + \eta) = e^{-2\pi i \sigma \{ M(\lambda+\xi) \Omega^t (\lambda+\xi) \}} F_\phi(\lambda + \xi, \mu + \eta) \\
= e^{-2\pi i \sigma \{ M(\lambda+\xi) \Omega^t (\lambda+\xi) \}} \\
\times e^{-4\pi i \sigma (M \xi \mu) F_\phi(\lambda, \mu)} \\
= e^{-4\pi i \sigma (M(\xi^t \xi + 2\lambda^t \Omega^t \xi + 2\mu^t \xi) \Omega)} F_\phi(\lambda, \mu) \\
= e^{-2\pi i \sigma \{ M(\xi^t \xi + 2\lambda^t \xi + 2\mu^t \xi) \}} F_{\Omega, \phi}(\lambda, \mu).
\]

This proves Formula (4.12). Formula (4.13) follows immediately from Formula (4.12). Indeed, if \( W = \lambda \Omega + \mu \) with \( \lambda, \mu \in \mathbb{R}^{(h,g)} \), we have
\[
\vartheta_{\Omega, \phi}(W + \xi \Omega + \eta) = F_{\Omega, \phi}(\lambda + \xi, \mu + \eta) \\
= e^{-2\pi i \sigma \{ M(\xi^t \xi + 2(\lambda \Omega + \mu)^t \xi) \}} F_{\Omega, \phi}(\lambda, \mu) \\
= e^{-2\pi i \sigma \{ M(\xi^t \xi + 2W^t \xi) \}} \vartheta_{\Omega, \phi}(W).
\]

\[\square\]

Remark 4.2. The function \( \vartheta_{\Omega, \phi}(W) \) is a theta function of level \( 2M \) with respect to \( \Omega \) if \( \vartheta_{\Omega, \phi} \) is holomorphic. For any \( \phi \in \mathcal{H}_{M,qM} \), the function \( \vartheta_{\Omega, \phi} \) satisfies the well known transformation law of a theta function. In this sense, the lattice representation \( (\pi_{M,qM}, \mathcal{H}_{M,qM}) \) is closely related to theta functions.

References

[B] R. Berndt, *Darstellungen der Heisenberggruppe und Thetafunktionen. Vorlesungsausarbeitung*, Hamburg, 1988.

[C] P. Cartier, *Quantum Mechanical Commutation Relations and Theta Functions*, Proc. of Symp. Pure Mathematics, 9, Amer. Math. Soc., 1966, pp. 361-383.
[F-C] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, EMG, Band 22, Springer-Verlag, New York/Berlin, 1990.

[I] J. Igusa, Theta functions, Springer-Verlag, New York/Berlin, 1972.

[M] G. W. Mackey, Induced Representations of Locally Compact Groups I, Ann. of Math. 55 (1952), 101-139.

[N] Y. Namikawa, Toroidal Compactification of Siegel Spaces, Lect. Notes in Math. 812, Springer-Verlag, New York /Berlin, 1980.

[Y1] J.-H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, Nagoya Math. J. 123 (1991), 103-117.

[Y2] ______, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II, J. Number Theory 49 (1994), 63-72.

[Y3] ______, A decomposition theorem on differential polynomials of theta functions of high level, Japanese J. Math., Math. Soc. Japan, New Series 22 (1996), 37-49.

[Y4] ______, The Siegel-Jacobi Operator, Abh. Math. Sem. Univ. Hamburg 63 (1993), 135-146.

[Y5] ______, Remarks on Jacobi forms of higher degree, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, PIMS (1993), 33-58.

[Y6] ______, Singular Jacobi Forms, Trans. Amer. Math. Soc. 347 (1995), 2041-2049.

[Y7] ______, Construction of Vector-Valued Modular Forms from Jacobi Forms, Canadian J. Math. 47 (1995), 1329-1339.

[Y8] ______, A geometrical theory of Jacobi forms of higher degree, Proc. of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147.

[Z] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg 59 (1989), 191-224.

Max-Planck Institut für Mathematik
Gottfried-Claren-Strasse 26
D-53225 Bonn
Germany

The present address is

Department of Mathematics
Inha University
Inchon 402-751
Republic of Korea

E-MAIL : JHYANG@INHA.AC.KR