Criticality of environmental information obtainable by dynamically controlled quantum probes

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A universal approach to decoherence control combined with quantum estimation theory reveals a critical behavior, akin to a phase transition, of the information obtainable by a qubit probe concerning the memory time of environmental fluctuations of generalized Ornstein-Uhlenbeck processes. The criticality is intrinsic to the environmental fluctuations but emerges only when the probe is subject to suitable dynamical control aimed at inferring the memory time. A sharp transition is anticipated between two dynamical phases characterized by either a short or long memory time compared to the probing time. This phase transition of the environmental information is a fundamental feature that characterizes open quantum-system dynamics and is important for attaining the highest estimation precision of the environment memory time under experimental limitations.

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I. INTRODUCTION

A quantum probe, such as a qubit, is capable of extracting information on the environment dynamics and its space-time fluctuations through the spectrum of the dephasing noise the probe is subjected to [1–14]. This information is the subject of an emerging field of research dubbed environmental quantum-noise spectroscopy [5,6]. Its most straightforward implementation is by monitoring the free-induction decay (FID) of an initially prepared qubit-probe coherence and inferring the dephasing characteristics from this decay [15–17]. A more promising option is to exert control (driving) fields, whether pulsed or continuous wave (CW), on the qubit probe and study its dephasing as a function of the control-field characteristics [4–6,18]. Pulsed control of qubit dephasing is commonly described by dynamical decoupling [19–23]. However, for the purpose of environment-noise spectroscopy it is useful to resort to the universal formula for the rate of decoherence under dynamical control [24–27], which is at the heart of the unified theory of dynamically controlled open quantum-systems [28–30]. This formula allows design of control fields or pulse-sequences that through the choice of a spectral filter function are optimally tailored to specific environment-noise spectrums and the task at hand [27,31]: decoherence control [24,25,28,29,32–36], state transfer [37,38], or storage [38–40] in a fluctuating environment. Here, the filter function will be adapted to the task of probing parameters of the environment-noise fluctuation spectrum obtained by this approach may exhibit critical behavior as a function of the memory-time parameter. This critical behavior, akin to a phase transition, is intrinsic to the environmental noise spectrum and is only revealed under dynamical control: it defines a sharp boundary between the short- and long-time regimes of the probe decoherence corresponding to long and short memory of the environment, respectively. By contrast, the FID of the probe coherence undergoes the usual smooth transition between the two dynamical regimes, thus conforming to the gradual change from non-Markovianity to Markovianity that has been previously analyzed [54,55]. The criticality or phase transition of the environmental information revealed here is a fundamental feature that serves as a means of characterizing dynamical behavior. Moreover, it is important for attaining the highest estimation precision of the environment memory time under experimental limitations.

II. CONTROLLED QUBIT PROBE AS A SENSOR OF THE ENVIRONMENTAL FLUCTUATIONS

We consider a dynamically controlled qubit probe experiencing pure dephasing due to the probe-environment interaction $H_B = g\sigma B$, where $\sigma$ is the Pauli operator for the probe and $B$ is the environment operator. In the weak-coupling probe-environment regime [Fig. 1(a)], its dephasing is characterized by the attenuation (decay) factor $J(\bar{x}_B, t)$ of the qubit coherence (Appendix A)

$$\langle \sigma_x(t) \rangle = \langle \sigma_x(0) e^{-J(\bar{x}_B, t)} \rangle,$$

where $\bar{x}_B$ is a set of parameters describing the environment and $J(\bar{x}_B, t)$ obeys the universal formula [24–31]

$$J(\bar{x}_B, t) = \int_\infty^{\infty} dw F_\omega G(\bar{x}_B, \omega).$$

Here $G(\bar{x}_B, \omega)$ is the coupling spectrum (spectral density) of the environment noise (the Fourier transform of its autocorrelation function). Explicitly, $\bar{x}_B = [g, \tau_c, \beta]$, with $\tau_c$ as the correlation or memory time of the environment noise, i.e., the inverse width of its spectral density, $g$ as the...
The power-law regime $\propto \omega^{-\beta}$ of the spectral density, obtained for $\omega \tau_c \gg 1$, is the spectral range with the strongest dependence on the frequency $\omega$, describing the short-time behavior of the probe-qubit dephasing. We define this limit as the long-memory (LM) regime. In the opposite limit $\omega \tau_c \ll 1$, associated with long times, the spectral density becomes independent of the frequency, and the attenuation factor $\mathcal{J}(\chi_B, t)$ is given by the Fermi golden rule. We dub this limit the short-memory (SM) regime.

## III. Identifying the Dynamical Regimes’ Criticality by Dynamically Controlled Probes

Under FID, the LM and SM dynamical regimes are attained at times $t \ll \tau_c$ and $t \gg \tau_c$, respectively. The respective attenuation (decay) factors are $\mathcal{J}^{LM}_t \propto \omega^{-1}$ (independent of $\tau_c$) and $\mathcal{J}^{SM}_t \propto \omega^0$ (Appendix B1). The transition from the LM to the SM regime is smooth [Fig. 1(b)] as the ratio $\frac{\tau_c}{\tau}$ is varied and does not depend on $g$. Invariably, $\frac{\partial \mathcal{J}^{SM}}{\partial \tau_c} \equiv 0$, without sign change.

Consider now the change that may arise in the character of this transition under dynamical control. An example is a decoupling control sequence of $N \gg 1$ equidistant $\pi$ pulses (known as CPMG) [16,67,68]. The filter function $F_t(\omega)$ [4,24–31] then converges to a sum of delta functions (narrowband filters) centered at the harmonics of the inverse modulation period, $k\omega_{ctrl} = k\pi N / t$ with $k = 1, 2, 3, \ldots$ [5,36]. Another suitable control is CW qubit driving, which has a single frequency component ($k = 1$). Under such controls, the LM and SM dynamical regimes are attained for $\omega_{ctrl} \tau_c \gg 1$ and $\omega_{ctrl} \tau_c \ll 1$, respectively. The corresponding decay factors considering the dominant filter frequency $\mathcal{J} \propto F_t(\omega_{ctrl})G(\omega_{ctrl})$ in Eq. (2), are [Fig. 1(b)] $\mathcal{J}^{LM} \propto \omega^{-1}$ and $\mathcal{J}^{SM} \propto \omega^0$, respectively (Appendix B2). This reflects the effect of narrow-band filters $F_t(\omega_{ctrl})$ that may be used to scan the spectral density $G(\omega)$ [4–6], upon varying the modulating frequency (pulse rate or Rabi frequency) $\omega_{ctrl}$ of the control field, all the way from the frequency-independent regime $G \propto \tau_c$ for $\omega \tau_c \ll 1$ to the power-law regime $G \propto \omega^{-1} \tau_c^{-1}$ for $\omega \tau_c \gg 1$ [Fig. 2(a)]. In the limit of extremely narrow spectral filters, i.e., $N \rightarrow \infty$, with $\omega_{ctrl} = \pi N / t$, we have

$$\frac{\partial \mathcal{J}}{\partial \tau_c} \bigg|_{\omega_{ctrl} \sim \omega_0} \propto \frac{\partial G}{\partial \tau_c} \bigg|_{\omega_{ctrl} \sim \omega_0} \propto \omega_{ctrl} - \omega_0.$$  

Here $\omega_0 = \tau_c^{-1}(\beta - 1)^{-\frac{1}{2}}$ is the probing frequency at which the dephasing rate is maximal for a given $\tau_c$ under the assumption of a narrow-band filter (Appendix C).

An abrupt change [Fig. 2(a)] is then revealed in the parametric sensitivity of the attenuation factor, defined by the derivative $\frac{\partial \mathcal{J}}{\partial \tau_c}$, through its change of sign: $\frac{\partial \mathcal{J}}{\partial \tau_c} \propto -(\beta - 1)^{-1} \mathcal{J}^{LM} < 0$ for LM and $\frac{\partial \mathcal{J}}{\partial \tau_c} \propto \tau_c^{-1} \mathcal{J}^{SM} > 0$ for SM, implying that (Appendix C)

$$\frac{\partial \mathcal{J}}{\partial \tau_c} \bigg|_{\omega_{ctrl} \sim \omega_0} = 0,$$

at a value dependent on the control frequency, when $\omega_{ctrl} \approx \omega_0$. 

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The effective probe-environment coupling strength, and $\beta$ as a power-law exponent that defines the type of stochastic (noise) process. The filter function $F_t(\omega)$ explicitly depends upon the dynamical control of the probe during time $t$. The information about the unknown environment parameters $x_B$ is encoded by the probabilities $p$ of finding the qubit probe in the $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ (symmetric) or $|−\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle − |\downarrow\rangle)$ (antisymmetric) superposition of the qubit energy states when measuring $\sigma_z$. These probabilities obey

$$p_{±}(x_B, t) = \frac{1}{2} (1 ± e^{−2J(x_B,t)}).$$

As a model to describe the memory timescales of the environment, we consider a generalized Ornstein-Uhlenbeck spectral density

$$G_β(g, \tau_c, \beta, \omega) = g^2 \frac{A_β \tau_c}{1 + \omega^β \tau_c^β},$$

where $A_β$ is a normalization factor depending on the power law $\beta > 2$. These types of bath spectra are ubiquitous in solid-state, liquid, or gas phases [5,8,41,47,61–64], where they are associated with collisional or diffusion processes: e.g., molecular diffusion in biological systems [8,42,43], charge diffusion in conducting crystals [44], or spin diffusion in complex spin networks [5,45–47]. This model is also a building block of universal lineshapes: it may characterize the memory-time of arbitrary bosonic baths, by assuming that a chosen harmonic-oscillator mode constitutes an interface between the qubit probe and the environment’s modes [65]. The combined spectrum of any environment plus the interface mode is reshaped, or “filtered,” according to the chosen oscillator-mode frequency and its coupling strength with the probe, resulting in a skewed-Lorentzian lineshape [65,66].
FIG. 2. Criticality of the probe-extracted information on the environmental correlation (memory) time \( \tau_c \). (a) Spectral density for the Ornstein-Uhlenbeck process (Lorentzian spectrum, \( \beta = 2 \): red solid-line). The spectrum’s derivative \( \frac{dG(\omega, \tau_c)}{d\tau_c} \) exhibits a critical behavior \( \frac{dG(\omega, \tau_c)}{d\tau_c} \propto |\omega - \omega_0| \) at \( \omega_0 = \tau_c^{-1}(\beta - 1)^{-1/2} \) (orange, lighter solid-line). Then the two dynamical regimes occur when the narrow filters probe frequency components of \( G(\omega) \) on both sides of the critical point. Two typical CW filter functions \( F_\omega(\omega) \) (green sinc function curves, in linear scale) scan the spectrum on both sides of the transition \( (N = 20) \). (b) The attainable relative-error \( \epsilon_F^\text{ctrl}(\tau_c,t) \) on \( \tau_c \) by the qubit probe under CW control \( (\sqrt{2Ng}\tau_c = 1, \beta = 2, \) and \( \omega_{\text{ctrl}} = \frac{\pi N}{4} \), see Appendices B 2-E). The divergence \( \propto |\omega_{\text{ctrl}} - \omega_0|^{-1} \) at the critical point evidences the critical behavior of \( \frac{dG}{d\tau_c} \).

Equation (6) signifies the vanishing of the quantum Fisher information (QFI) \[56,57\], which quantifies the attainable precision per measurement, \( \epsilon_F^\text{ctrl}(\tau_c,t) \), to the best tradeoff between a signal amplitude contrast, such criticality does not arise under FID, for which the filter function, \( F_Q^{\text{FID}}(\omega) \), is a much broader sinc function centered around \( \omega = 0 \) (Appendix A 1). The critical point divides the noise spectrum into distinct regimes of dephasing dynamics. This feature can serves as a means of characterizing the noise environment. Any spectral density of the noise that gives rise to a change of sign of \( \frac{dG}{d\tau_c} \) for a finite value of \( \omega_{\text{ctrl}} \), thus implying Eq. (6), will exhibit critical behavior.

IV. CRITICAL BEHAVIOR OF THE MAXIMAL ESTIMATION PRECISION OF \( \tau_c \)

Another central result in this paper, with practical implications, is that the critical behavior shown above is also manifest, under the same control on the probe, for the maximum estimation precision, i.e., the smallest possible minimal relative error in the estimation of \( \tau_c \) in Eq. (8),

\[
\epsilon_F^\text{ctrl}(\tau_c,t_{\text{opt}}) = \min_{t} \epsilon_F^\text{ctrl}(\tau_c,t).
\]

The error minimization is the outcome of selecting the optimal time \( t_{\text{opt}} \) at which the measurement [cf. Eqs. (1) and (2)] is performed on the probe, its dephasing under the control we have applied.

Figure 3 shows the critical behavior of the maximum precision per measurement, \( \epsilon_F^\text{ctrl}(\tau_c,t_{\text{opt}}) \), for the Lorentzian spectrum \( (\beta = 2) \) following CW control of the qubit probe as a function of \( \sqrt{2Ng}\tau_c \) (Appendices D and E).

The critical point \( \sqrt{2Ng}\tau_c \approx 1 \)

separates two regions characterized by different scaling laws of the minimal relative error as a function of \( \sqrt{2Ng}\tau_c \) [Fig. 3(a)]. These scaling laws are dictated by the different dynamical regimes for the attenuation-factor shown in Figs. 3(c) and 3(d).

The optimal probing (measurement) and control time \( t_{\text{opt}} \) also undergoes a sudden transition at the critical point \( (11) \), as shown in Fig. 3(b). This optimal time corresponds to the best tradeoff between a signal amplitude contrast, \( e^{-2J/(1-e^{-2J})} \), and the parametric sensitivity of the signal attenuation factor, \( (\frac{\partial J}{\partial \tau_c})^2 \). The optimal tradeoff occurs [Fig. 3(b)] at either a long time compared to \( \tau_c \), \( t_{\text{opt}}^{LM} \) (red circle), corresponding to a linear attenuation factor \( J_{\text{LM}} \propto t \) [Fig. 3(c)], or at a short-time, \( t_{\text{opt}}^{SM} \) (blue circle), corresponding to \( J_{\text{SM}} \propto t^{\beta + 1} \) [Figs. 3(d)]. These optimal control times in the two regimes are situated on both sides of the critical value \( \sqrt{2Ng}\tau_c \) [Fig. 3(b)], \( t_{\text{opt}}^{LM} < t_0 < t_{\text{opt}}^{SM} \). These critical behaviors can be characterized by two local minima as a function of the parameter \( \sqrt{2Ng}\tau_c \) [Fig. 3(b)], resembling the behavior of...
The optimal scaled measurement time \( \tau_c \) for a Lorentzian environmental spectrum under dynamical control. Practical limitations on the number of pulses of a CPMG sequence, such that \( N \leq N_{\text{max}} \), may prevent attaining the ultimate bound (dashed line). A sudden change of the dynamical control strategy as a function of \( g_\tau \) may help: For \( g_\tau \) lower than the critical value, the highest precision is achieved by the single-pulse Hahn echo (\( N = 1 \)). However, if \( g_\tau \) is larger than the critical value, the CPMG sequence with \( N = N_{\text{max}} \) is optimal. This dynamical control strategy under practical limitations reduces the minimal error represented by the shaded area, which is determined by the optimal control on each side of the intersection (critical value) of the Hahn and the CPMG curves.

**FIG. 4.** Minimal relative error per measurement in the estimation of \( \tau_c \) as a function of \( g_\tau \) for a Lorentzian environmental spectrum under dynamical control. Practical limitations on the number of pulses of a CPMG sequence, such that \( N \leq N_{\text{max}} \), may prevent attaining the ultimate bound (dashed line). A sudden change of the dynamical control strategy as a function of \( g_\tau \) may help: For \( g_\tau \) lower than the critical value, the highest precision is achieved by the single-pulse Hahn echo (\( N = 1 \)). However, if \( g_\tau \) is larger than the critical value, the CPMG sequence with \( N = N_{\text{max}} \) is optimal. This dynamical control strategy under practical limitations reduces the minimal error represented by the shaded area, which is determined by the optimal control on each side of the intersection (critical value) of the Hahn and the CPMG curves.

conventional phase transitions. At the critical point, both local minima are equal, as displayed in Fig. 2(b), which leads to the fastest decay in the narrow-band limit as seen in Figs. 3(c) and (d) by the green solid line.

**V. DISCUSSION**

We demonstrated a critical behavior of information (estimation precision) on the environment fluctuation (noise) spectrum of a generalized Ornstein-Uhlenbeck process, extracted by a probe subject to appropriate dynamical control as a function of the ratio between the probing time and the environment memory time \( t/\tau_c \). This finding applies to any environment characterized by such spectra that are ubiquitous in solid-state, liquid, or gas phases [5,8,44–47,61–64].

We have shown that similar critical behavior is manifest for the maximal estimation precision of \( \tau_c \). At the critical point there is a massive loss of information on \( \tau_c \). Near this point, the optimal time for measuring and controlling the quantum probe is either very short, corresponding to little parametric sensitivity, or very long, corresponding to a significant decay of the probe signal.

The critical behavior of the maximal estimation precision of \( \tau_c \) has paramount practical implications:

(i) **Complete dynamical behavior characterization:** Rather than mapping out the long- and short-memory probe dynamics regimes by varying the probing time, the critical behavior demonstrated here allows one to characterize the complete dynamics as consisting of two distinct dynamical phases (regimes) according to the maximal information they yield about the environment memory time. These two dynamical phases are sharply separated by the critical point (11).

(ii) **Sudden change of the optimal dynamical control sequence:** The critical point depends on the control scheme; thus, for CPMG control [16,67,68] \( g_\tau \approx 1/\sqrt{2N} \) when probing an Ornstein-Uhlenbeck process (Lorentzian spectra). This fact highlights the importance of optimizing the number of pulses \( N \) to improve the estimation precision, if \( N \) is bounded by \( N_{\text{max}} \) due to practical limitations on the power deposition and/or on the pulse length. Under these conditions, the ultimate bound on the estimation precision found in Ref. [41], \( \xi_F(\tau_c, t) \geq \xi_0 \approx 2.48 \), may not be attained for \( g_\tau \leq 1/\sqrt{2N_{\text{max}}} \). A sudden change of \( N \) should be undertaken as a function of \( g_\tau \) to optimize the estimation: For \( g_\tau \) lower than a certain critical value shown in Fig. 4, the best precision is achieved by the single-pulse Hahn echo (\( N = 1 \)). However, if \( g_\tau \) is larger than this critical value, the CPMG sequence with \( N = N_{\text{max}} \) is optimal. Qualitatively similar considerations apply for generalized Ornstein-Uhlenbeck processes.

To sum up, the critical behavior of the environmental information revealed here is a fundamental feature that characterizes open quantum-system dynamics and is important for attaining the highest estimation precision of the environment memory time under practical limitations. It represents an alternative characterization of the probe-qubit dynamics under suitable control or observation that leads to a phase transition on the dynamical behavior [46,71–76]. Intriguingly,
the absence of information on $\tau$, has been shown to provide a distinctive signature of the environmental noise estimation.

Such information may be useful, e.g., for studying molecular diffusion at the nanoscale and thereby characterizing biological systems [8,41–43] or chemical-identities [7], charge diffusion in conducting crystals [44], or spin diffusion in complex spin networks [5,45–47,77]. Knowledge of the memory time may also be important for studying fundamental effects, such as quantum phase-transitions in a spin environment [48,49] or nonlocal correlations within a composite environment [50–53]. We envisage that this critical behavior may be exploited to characterize noises/baths with multiple memory-times as in the case of spectral densities described by sums of Lorentzians [42,63,78]. Multiple peaks of the relative-error associated with such a generalized memory-time may also be important for studying fundamental fluctuations.

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APPENDIX A: QUBIT PROBE UNDER DEPHASING

Here we provide the background necessary for Sec. II, “Controlled qubit probe as a sensor of the environmental fluctuations.”

1. Weak coupling regime description

A single qubit probe (denoted here as system) experiences dephasing due to its interaction with the environment (bath) and undergoes control. They are described by the Hamiltonian

$$ H = H_S(t) + H_B + H_{SB}, $$

where $S$ and $B$ label the system and environment respectively, and

$$ H_S = \omega_c \sigma_z + V_s(t) \sigma_x, \quad H_{SB} = S \otimes B = \sigma_z g B, $$

$$ \sigma_{x,y,z} \text{ being the Pauli operators of the qubit, } \{ I + \sigma_z \}/2 = |\uparrow\rangle \langle \uparrow | \text{ being } |\uparrow\rangle \text{ the upper (excited) state of the qubit probe, } \omega_c \text{ its resonant frequency, } V_s(t) = V(t)e^{-i\omega_c t} + \text{c.c.} \text{ the control acting on the qubit, and } g \text{ the qubit-bath coupling strength.} $$

The actual form of the environment Hamiltonian $H_B$ can be in general arbitrary, and it is not relevant for the present discussion.

A non-Markovian master equation for the density matrix of the system $\rho_S(t)$ can be derived in the interaction picture. Under the Born approximation, also known as the weak-coupling regime, the system-environment coupling strength $g$ is assumed to be weak enough for the influence of the system on the density matrix of the environment $\rho_B$ to be negligible.

As a result, the density matrix of the total system at a time $t$ can be expressed as $\rho(t) \approx \rho_S(t) \otimes \rho_B$ [79], yielding the non-Markovian master equation [24–26,28,80]

$$ \dot{\rho}_S(x_B,t) = \int_0^t dt' [g^2 \Phi(x_B,t-t')[S(t'),S(t)] + H.c]. $$

(A3)

Here $\Phi(x_B,t'-t'') = \text{Tr}_B \{ B(t''-t')B(0)\rho_B(0) \}$ is the environment autocorrelation function, $x_B$ is a parameter that characterizes the environment, and, in the interaction picture,

$$ S(t) = U_S^\dagger(t)SU_S(t), \quad U_S(t) = \text{exp}(-i \int_0^t dt' H_S(t')). $$

(B(t) = U_B^\dagger(t)BU_B(t), \quad U_B(t) = e^{-iH_B t}. $$

(A4)

2. The attenuation factor of the qubit-probe dephasing

The phase due to the unperturbed energy difference $\omega_c$ is irrelevant for describing the dephasing experienced by the qubit probe. This phase dependence can be eliminated by transforming to the rotating frame that can be described by the time-dependent basis

$$ |p_\pm \rangle = \frac{1}{\sqrt{2}} (e^{-i\omega_c t} |\uparrow \rangle \pm |\downarrow \rangle), $$

(A5)

where $|\downarrow \rangle$ and $|\uparrow \rangle$ are the lower and upper states of the qubit probe in the laboratory frame, respectively. The system Hamiltonian of Eq. (A1) tilted in this basis becomes

$$ \tilde{H}_S = \frac{V(t)}{2} \sigma_z, \quad \tilde{\sigma}_z = |p_+ \rangle \langle p_+ | - |p_- \rangle \langle p_- |. $$

(A6)

while $H_B$ and $H_{SB}$ remain invariant. We now go to the interaction picture as described in the previous section and write Eqs. (A4) as

$$ U_S(t) = \Omega_0(t)|p_+ \rangle \langle p_+ | + \Omega_0^\dagger(t)|p_- \rangle \langle p_- |, \quad S(t) = \Omega(t)|p_+ \rangle \langle p_- | + \Omega^\dagger(t)|p_- \rangle \langle p_+ |, $$

(A7)

where

$$ \Omega_0(t) = \exp\left(-i \int_0^t \frac{V'(t)}{2} dt \right), $$

$$ \Omega(t) = \exp\left(i \int_0^t V(t') dt \right) $$

(A8)

(A9)

allows us to express the quantum master equation of Eq. (A3) as [24–26,28,80]

$$ \frac{d}{dt} \Delta\rho_{S,\pm} = - \frac{d}{dt} J \Delta\rho_{S,\pm}. $$

(A10)

where $\Delta\rho_{S,\pm} = \rho_{S,++} - \rho_{S,--}$ represents the qubit-coherence in the basis $|\downarrow \rangle,|\uparrow \rangle$ at time $t$,

$$ \langle \sigma_z(t) \rangle = \langle \tilde{\sigma}_z(t) \rangle = \Delta\rho_{S,\pm} = e^{-\mathcal{J}(x_B,t)} \langle \sigma_z(0) \rangle, $$

(A11)

which is characterized by the attenuation factor [24–26,28,80]

$$ \mathcal{J}(x_B,t) = \text{Re} \left[ \int_0^t dt' \int_0^t dt'' g^2 \Phi(x_B,t'-t'') \Omega(t') \Omega^\dagger(t'') \right]. $$

(A12)
In the spectral representation, this attenuation factor becomes
\[ \mathcal{J}(x_B,t) = \int_{-\infty}^{\infty} d\omega F_i(\omega) G(x_B,\omega), \] (A13)
where
\[ G(x_B,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, g^2 \Phi(x_B,t)e^{i\omega t} \] (A14)
is the environment-coupling spectrum given by the Fourier transform of the environmental-correlation function which is normalized in the frequency domain,
\[ \int_{-\infty}^{\infty} d\omega \, G(x_B,\omega) = g^2, \] (A15)
and
\[ F_i(\omega) = \frac{1}{2\pi} \int_0^t dt' \Omega(t')e^{i\omega t'} \] (A16)
is the control-field filter function. It is determined by the finite-time Fourier transform of the dynamical control on the qubit probe which satisfies the following sum rule:
\[ \int_{-\infty}^{\infty} \frac{d\omega}{\pi} F_i(\omega) = 1. \] (A17)

### APPENDIX B: DEPHASING ATTENUATION FACTORS FOR GENERALIZED ORNSTEIN-UHLENBECK SPECTRA

In the following we derive the dephasing attenuation factor, Eq. (A13) with \( x_B = \tau_c \), presented in Sec. III. “Identifying the dynamical regimes’ criticality by dynamically controlled probes” of the main text, and plotted in Fig. 1(b). There, the qubit probe experiences dephasing due to the interaction with an environment, Eq. (A14), described by the generalized Ornstein-Uhlenbeck spectral density
\[ G(\tau_c,\omega) = g^2 \frac{A_B \tau_c}{1 + \omega^2 \tau_c^2} \] (B1)
where \( A_B = \frac{\beta}{\pi} \sin^{\frac{\beta}{2}} \) is a normalization factor in accordance with Eq. (A15). Here the actual form of \( H_B \) has to lead to an autocorrelation function \( \Phi(x_B,t) \) whose Fourier transform, Eq. (A14), gives Eq. (B1). These types of bath spectra are ubiquitous in solid-state, liquid, or gas phases [5,8,41,47,61–64], where they are associated with collisional or diffusion processes, e.g., molecular diffusion in biological systems [8,42,43], charge diffusion in conducting crystals [44], or spin diffusion in complex spin networks [5,45–47].

1. Dephasing attenuation factor under free evolution of the quantum probe

The filter function (A16) for a freely evolving qubit probe over a total time \( t \), the free-induction decay, is given by
\[ F^{\text{free}}_i(\omega) = \left| \frac{e^{-\omega t} \sin \left( \frac{\omega t}{2} \right)}{\omega t/2} \right|^2 = \frac{t^2}{2} \sin^2 \left( \frac{\omega t}{2} \right). \] (B2)
Then, we obtain the dephasing attenuation factor for a freely evolving qubit probe by introducing Eqs. (B1) and (B2) in Eq. (A13) and by considering the short-memory (SM) or long-memory (LM) regimes.

In the LM regime, i.e., \( t \ll \tau_c \), the relevant part of the filter function (B2) becomes frequency independent,
\[ F^{\text{free}}_i \approx \frac{t^2}{2}, \] (B3)
leading, along with Eq. (A15), to the attenuation factor
\[ \mathcal{J}^{\text{free}}(\tau_c,t) \approx \int_{-\infty}^{\infty} d\omega \, g^2 \frac{\tau_c}{2} \frac{1}{2} g^2 \tau_c^2. \] (B4)

In the SM regime, i.e., \( t \gg \tau_c \), the filter function from Eq. (B2) is a narrow sinc function centered at \( \omega = 0 \). There, the spectral density (B1) is well approximated as \( G(\tau_c,\omega) \approx G(\tau_c,0) \), leading to an attenuation factor
\[ \mathcal{J}^{\text{SM}}(\tau_c,t) \approx \int_{-\infty}^{\infty} d\omega \, F^{\text{free}}_i(\omega) G(\tau_c,\omega) = g^2 \pi A_B \tau_c t, \] (B5)
where we invoke the filter function property (A17).

2. Dephasing attenuation factor for dynamically controlled quantum probe under the narrow-band filter approximation

The filter functions (A16) for a dynamically controlled qubit probe under CPMG and Hahn (spin-echo) sequences or CW driving over a total time \( t \) are given respectively by [4,36]
\[ F^{\text{CPMG}}_i(\omega,N) = \frac{i2e^{-\omega^2/2}}{\cos \left( \frac{\omega N^2}{2} \right)} \int_{-\infty}^{\infty} dt g^2 \beta \tau_c^t \] (B6)
for a CPMG sequence of \( N \pi \) pulses,
\[ F^{\text{Hahn}}_i(\omega) = \left| \frac{e^{-\omega^2/4} \sin^{\frac{\omega}{4}}}{\omega N^2} \right|^2. \] (B7)
for a Hahn sequence obtained by setting \( N = 1 \) in Eq. (B6) and
\[ F^{\text{CW}}_i(\omega,N) = \frac{T^2}{2} \left[ \sin^2(\nu_+) + \sin^2(\nu_-) \right], \quad \nu_{\pm} = \pi N \pm \omega T, \] (B8)
for CW \( N \)-period driving.

The attenuation factor in Eq. (A13) for CPMG or CW dynamical controls acting on the qubit probe under the narrow-band filter approximation [5,36] is
\[ \mathcal{J}(\tau_c,t) = \sum_{k=1}^{\infty} F_i(k\omega_{\text{ctrl}}) G(\tau_c,k\omega_{\text{ctrl}}), \] (B9)
where \( k\omega_{\text{ctrl}} \) are the harmonics of the CPMG or CW modulation functions. In keeping with the main text, here we consider the generalized Ornstein-Uhlenbeck spectra in Eq. (B1).

In the long-memory (LM) regime, where \( k\omega_{\text{ctrl}} \tau_c \gg 1 \), the attenuation factor becomes
\[ \mathcal{J}^{\text{LM}}(\tau_c,t) \approx g^2 \sum_{k=1}^{\infty} F_i \left( \frac{\pi k N \tau_c}{t} \right) \frac{A_B \tau_c}{\pi k N \tau_c} \left( \frac{k N \tau_c}{t} \right)^B \approx \frac{c_B g^2 \beta^{B+1}}{N^B \tau_c^{B-1}}, \] (B10)
ζ is the zeta function defined for Re(ω) provided by suitable CW control, Eq. (C1) becomes dependent with CRITICALITY OF ENVIRONMENTAL INFORMATION . . . PHYSICAL REVIEW A 94

Near this zero probe generates a filter function centered at frequency

for CPMG control (only odd k harmonics are nonzero), where ζ is the zeta function defined for Re(ω) > 1 as ζ(ω) = Σ∞i=1 1iω. It turns out that Cw ≈ CFMG.

In the short-memory (SM) regime, ωctrlτc ≪ 1, the attenuation factor is given by the Fermi golden rule (B5),

which holds for both CW and CPMG controls.

APPENDIX C: DERIVATIVE OF THE ATTENUATION FACTOR UNDER THE NARROW-BAND APPROXIMATION

Here we describe the maximum dephasing rate for a given τc discussed between Eqs. (5) and (6) in the main text.

The derivative of the attenuation factor, Eq. (A13), with respect to τc is

For a generalized Ornstein-Uhlenbeck spectral density, Eq. (B1), the argument of the integral has a zero at

Near this zero

Under the narrow-band filter approximation in Eq. (B9) provided by suitable CW control, Eq. (C1) becomes dependent on a single frequency ωctrl = τc. This implies that when ωctrl ≈ ω0

and

Remarkably, the zero in (C5) is only observed under suitable control, as we described. By contrast, a freely evolving qubit probe generates a filter function centered at frequency ω = 0, Eq. (B2), that may overlap not only with ω0 but also with the entire spectrum |ω| < ω0, preventing us from distilling the contribution of ω0 related to the criticality of the environmental information.

APPENDIX D: QUANTUM FISHER INFORMATION ON THE ENVIRONMENT MEMORY TIME

The quantum Fisher information on the environment memory time τc is extractable from the probe qubit state through the expression [41,56,57,59,69,81]

where

are the respective probabilities of finding the qubit probe in the states |±⟩ = 1/√2(|↑⟩ ± |↓⟩), but also the eigenvalues of the qubit-probe density matrix in the |p±⟩ basis [Eq. (A5)] [28,80].

The measurement on the qubit probe that provides more information is effected by projections onto the eigenstates |p±⟩ of σx in the rotating frame. When the quantum Fisher information $F_Q$ coincides with its classical counterpart, the measurement is said to be optimal [56,57,59,81]. This is the case here, under pure dephasing, when the last term in (D1) is null, given that $\frac{\partial |\theta|}{\partial \tau_c} = 0$.

Then, Eq. (D1) becomes

for the optimal initial probe state |+⟩. For an arbitrary initial state, (cos(θ)|↑⟩ + i sin(θ)|↓⟩), 0 < θ < π/2, $F_Q(\tau_c,t) \propto \sin^2(\theta)$ [59]. Under this condition the optimal initial state maximizes the quantum Fisher information when $\theta = \pi/2$.

APPENDIX E: ANALYTICAL ATTENUATION FACTORS FOR ORNSTEIN-UHLENBECK PROCESSES

Here we provide analytical expressions for the dephasing attenuation factor in Ornstein-Uhlenbeck processes [i.e., Lorentzian spectrum $\beta = 2$ in Eq. (B1)] supporting the numerical simulations presented in Figs. 1–4. Analytical expressions for the attenuation factors under CPMG [8] and CW control sequences can be derived in the form

and

where

and

Remarkably, the zero in (C5) is only observed under suitable control, as we described. By contrast, a freely evolving qubit probe generates a filter function centered at frequency ω = 0, Eq. (B2), that may overlap not only with ω0 but also with the entire spectrum |ω| < ω0, preventing us from distilling the contribution of ω0 related to the criticality of the environmental information.
The CPMG sequence with a single pulse $N = 1$ is the well-known Hahn sequence. By evaluating $N = 1$ in Eq. (E2), we obtain the corresponding attenuation factor
\[
\mathcal{J}_{\text{Hahn}}(\tau_c, t) = -g^2 \tau_c t \left[ 1 - \frac{\tau_c}{t} (3e^{-\pi t} - 4e^{-\pi \tau_c}) \right].
\] (E3)

Upon inserting Eqs. (E1) and (E2) in Eq. (D3), we find an analytical expression for the quantum Fisher information that corresponds to the analytical expression of the Cramer-Rao bound [56,57,70], as per Eq. (8) in the main text,
\[
\varepsilon_f(\tau_c, t) = \frac{1}{\tau_c \sqrt{\mathcal{J}_g(\tau_c, t)}}
\] (E4)

that was displayed in Fig. 2(b), and in the insets of Figs. 3(a) and 3(b).

The optimal times $t_{\text{opt}}$ [displayed in Figs. 3(b)–3(d)] that globally minimize the minimum relative error $\varepsilon_f(\tau_c, t_{\text{opt}}) = \min_i \varepsilon_f(\tau_c, t_i)$ [displayed in Figs. 3(a) and 4] were numerically obtained from the analytical expression of $\varepsilon_f(\tau_c, t)$ using Eqs. (E1), (E2), and (D3).

A good approximation for the optimal time under CPMG control for the entire SM regime is
\[
t_{\text{opt}} \approx \frac{W(-2e^{-2(4N+1)^2\tau_c/2}) + 2 + (8N + 4)g^2\tau_c^2}{2g^2\tau_c},
\] (E5)

which corresponds to $\sqrt{2Ng}\tau_c < 1$ in Figs. 3 and 4 in the main text.

For the LM regime
\[
t_{\text{opt}} \approx \tau_c \sqrt{\frac{N^2\mathcal{J}_0}{c_gg^2\tau_c^2}},
\] (E6)

provided $\sqrt{2Ng}\tau_c > 1$, under CPMG and CW controls.

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