On dynamical chiral symmetry breaking in quantum electrodynamics

V.E. Rochev
Institute for High Energy Physics
Protvino, Moscow Region, Russia

Abstract

The problem of dynamical chiral symmetry breaking (DCSB) in multidimensional quantum electrodynamics (QED) is considered. It is shown that for six-dimesional QED the phenomenon of DSCB exists in ladder model for any coupling.
Introduction

Dynamical chiral symmetry breaking (DCSB) is an important topic of particle physics. DCSB is a foundation of light hadron theory – chiral dynamics, which is a low-energy limit of quantum chromodynamics. It also plays an important role in generalizations of Standard model.

One of the most studied examples of DCSB is the four-dimensional massless quantum electrodynamics (QED) in the strong coupling regime \[1, 2\]. (A detailed consideration and extensive references can be found in the monograph \[3\].)

In this paper an attempt is made to study the phenomenon of DCSB in the six-dimensional QED. The interest in multidimensional model is stimulated by rather popular investigations of theories with extra dimensions (see \[4\] for review). The mechanism of DCSB for the six-dimensional QED, which is investigated in present paper, can be useful for understanding of multidimensional dynamics.

An investigation of nonperturbative effects, such as DCSB, in the multidimensional QED assumes some model approximation. As such an approximation we will use a leading term of nonperurbative expansion, elaborated in works \[3, 4\]. In the diagram language the leading order equation for electron propagator corresponds to the well-known ladder approximation. Just in the framework of this approximation the phenomenon of DCSB have been investigated firstly in the four-dimensional QED \[1, 2\], and followed investigations have achieved that the approximation quite adequately described this effect. We suppose that in the six-dimensional QED such an approximation is also adequate to the situation.

The principal result of this paper is a conclusion about the existence of DCSB phenomenon for the six-dimensional QED. In contrast to the four-dimensional QED, where a critical coupling constant \(\alpha_c \sim 1\) exists (i.e., at \(\alpha < \alpha_c\) the DCSB phenomenon is absent), for the six-dimensional QED the phenomenon exists at any coupling.

One of the principal problems, which arises in six-dimensional QED studying, is a problem of renormalizability. In the framework of the coupling constant perturbation theory the six-dimensional QED is a non-renormalizable theory. Nevertheless, this fact is not, generally speaking, an obstacle for the existence of renormalized expansions of another type. Thus, for example, renormalized 1/N-expansion exists for some models, which are non-renormalizable in the usual sense of the coupling constant perturbation theory (see, for example, \[7\]). We suppose that similar situation can be realized also for gauge theories: nonperturbative expansions can be sensible for multi-dimensional gauge theory, for which the usual perturbative series does not exist.

1 Schwinger-Dyson equations and iteration scheme

We consider a theory of massless spinor field \(\psi(x)\) (electron) interacting with Abelian gauge field \(A_\mu(x)\) (photon) in \(D\)-dimensional Minkowsky space with a metric \(x^2 \equiv x_\mu x^\mu = x_0^2 - x_1^2 - \cdots - x_{D-1}^2\). (For simplification of notation all vector indices are written as lower ones.)

A Lagrangian (including a gauge fixing term) is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2d_t} (\partial_\mu A_\mu)^2 + \bar{\psi} (i\hat{\partial} + e\hat{A}) \psi.
\]

Here \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), \(\hat{A} \equiv A_\mu \gamma_\mu\); \(\bar{\psi} = \psi^* \gamma_0\), \(e\) is coupling, \(d_t\) is a gauge parameter,
\( \gamma_\mu \) are the Dirac matrices.

A generating functional of Green functions (vacuum expectation values of \( T \)-products of fields) can be represented as the functional integral

\[
G(J, \eta) = \int D(\psi, \bar{\psi}, A) \exp \{ i \int dx (L + J_\mu(x) A_\mu(x)) - \int dx dy \bar{\psi}(y) \eta^\beta(y, x) \psi^\alpha(x) \}. \tag{2}
\]

Here \( J_\mu(x) \) is a gauge field source, \( \eta^\alpha(y, x) \) is a bilocal source of spinor field (\( \alpha \) and \( \beta \) are spinor indices). Normalization constant is omitted. Functional derivatives of \( G \) over sources are vacuum expectation values:

\[
\frac{\delta G}{\delta J_\mu(x)} = i < 0 \mid A_\mu(x) \mid 0 >, \quad \frac{\delta G}{\delta \eta^\alpha(y, x)} = i < 0 \mid T \{ \psi^\alpha(x) \bar{\psi}^\beta(y) \} \mid 0 >. \tag{3}
\]

Functional-derivative Schwinger-Dyson equations (SDEs) for generating functional \( G \) read as follows

\[
(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu + \frac{1}{dt} \partial_\mu \partial_\nu) \frac{1}{i \delta J_\nu(x)} \frac{\delta G}{\delta J_\mu(x)} + i e \gamma_\mu \frac{\delta G}{\delta \eta^\alpha(x, x)} + J_\mu(x)G = 0, \tag{4}
\]

\[
\delta(x - y)G + i \delta \frac{\delta G}{\delta \eta^\alpha(x, y)} + \frac{e}{i} \gamma_\mu \frac{\delta^2G}{\delta J_\mu(x) \delta \eta(y, x)} - \int dx' \eta(x', x') \frac{\delta G}{\delta \eta(y, x')} = 0. \tag{5}
\]

Let resolve SDE (4) with respect to the first derivative of generating functional over \( J_\mu \):

\[
\frac{1}{i \delta J_\mu(x)} = - \int dx_1 D^c_{\mu\nu}(x - x_1) \{ J_\nu(x_1)G + ie \gamma_\nu \frac{\delta G}{\delta \eta(x_1, x_1)} \} \tag{6}
\]

and put the result into the second SDE (5). (Here \( D^c_{\mu\nu} = [g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu + \frac{1}{dt} \partial_\mu \partial_\nu]^{-1} \) is a free photon propagator.) As a result we obtain ”integrated over \( A_\mu \) ” equation

\[
\delta(x - y)G + i \delta \frac{\delta G}{\delta \eta^\alpha(x, y)} + \frac{e^2}{i} \int dx_1 D^c_{\nu\mu}(x - x_1) \gamma_\mu \frac{\delta}{\delta \eta(y, x_1)} \gamma_\nu \frac{\delta G}{\delta \eta(x_1, x_1)} = \int dx_1 \{ \eta(x_1, x_1) \frac{\delta G}{\delta \eta(y, x_1)} + e D^c_{\nu\mu}(x - x_1) J_\nu(x_1) \gamma_\mu \frac{\delta G}{\delta \eta(y, x_1)} \}. \tag{7}
\]

Exploiting fermi-symmetry condition, one can rewrite equation (7) in the form

\[
\delta(x - y)G + i \delta \frac{\delta G}{\delta \eta^\alpha(x, y)} + ie^2 \int dx_1 D^c_{\nu\mu}(x - x_1) \gamma_\mu \frac{\delta}{\delta \eta(x_1, x)} \gamma_\nu \frac{\delta G}{\delta \eta(y, x_1)} = \int dx_1 \{ \eta(x_1, x_1) \frac{\delta G}{\delta \eta(y, x_1)} + e D^c_{\nu\mu}(x - x_1) J_\nu(x_1) \gamma_\mu \frac{\delta G}{\delta \eta(y, x_1)} \}. \tag{8}
\]

For solution of SDE (8) we shall use the iteration scheme proposed in works \([5], [6]\). A general idea of this scheme is an approximation of functional-differential equation (8) by an equation with ”constant”, i.e. independent of the sources \( J_\mu \) and \( \eta \), coefficients. Thus we approximate the functional-differential SDE near the point \( J_\mu = 0, \eta = 0 \). Since the objects of calculations are Green functions, i.e., derivatives of \( G \) in zero, such an approximation seems to be quite natural.

In every order of this scheme the Green functions can be found as solutions of a closed system of equations.
In correspondence with aforesaid, we choose as the leading approximation to equation (8) the following equation:

\[
\delta(x - y)G^{(0)} + i\hat{\partial}\frac{\delta G^{(0)}}{\delta \eta(y, x)} + ie^2 \int dx_1 D_{\mu\nu}^c(x - x_1)\gamma_\mu \frac{\delta}{\delta \eta(x_1, x)} \gamma_\nu \frac{\delta G^{(0)}}{\delta \eta(y, x_1)} = 0. \tag{9}
\]

A solution of this equation is the functional

\[
G^{(0)} = \exp \left\{ \text{Tr}(S \star \eta) \right\}. \tag{10}
\]

(The sign \(\star\) denotes a multiplication in operator sense, and the sign Tr denotes a trace in operator sense.)

Here

\[
S^{-1}(x) = -i\hat{\partial}\delta(x) - ie^2 D_{\mu\nu}^c(x)\gamma_\mu S(x)\gamma_\nu. \tag{11}
\]

Equation (11) is an equation for electron propagator in the leading approximation of the iteration scheme.

In correspondence with (8) and (9) iteration equation is

\[
\delta(x - y)G^{(i)} + i\hat{\partial}\frac{\delta G^{(i)}}{\delta \eta(y, x)} + ie^2 \int dx_1 D_{\mu\nu}^c(x - x_1)\gamma_\mu \frac{\delta}{\delta \eta(x_1, x)} \gamma_\nu \frac{\delta G^{(i)}}{\delta \eta(y, x_1)} = \\
= \int dx_1 \left\{ \eta(x, x_1) \frac{\delta G^{(i-1)}}{\delta \eta(y, x_1)} + eD_{\mu\nu}^c(x - x_1)J_\nu(x_1)\gamma_\mu \frac{\delta G^{(i-1)}}{\delta \eta(y, x)} \right\}. \tag{12}
\]

Equation (11) and equations for the higher Green functions, which followed from (12), in diagram language correspond to the well-known ladder approximation. In our treatment these equations are a consistent part of the iteration scheme.

2 Asymptotic solution of electron propagator equation and dynamical chiral symmetry breaking

In transverse Landau gauge \(d_l = 0\) electron propagator equation (11) has a simple solution

\[
S_0 = -1/i\hat{\partial}. \tag{13}
\]

Really, in coordinate space

\[
D_{\mu\nu}^c(x) = e^{-i\pi D/2}\Gamma(D/2 - 1) \left[ \frac{1 + d_l}{2} g_{\mu\nu} + (1 - d_l)(D/2 - 1) \frac{x_\mu x_\nu}{x^2 - i0} \right]. \tag{14}
\]

At \(d_l = 0\) the function \(D_{\mu\nu}^c(x)\) possesses an important property (”\(\hat{x}\)-transversality”)

\[
D_{\mu\nu}^c(x)\gamma_\mu \hat{x}\gamma_\nu = 0, \tag{15}
\]

from which the existence of solution (13) follows immediately, since \(S_0(x) \sim \hat{x}\).

At \(D\) even one can define chiral components of spinor fields and corresponding chiral transformations. Lagrangian (1) of massless electrodynamics is invariant in respect of the
chiral transformations. Solution ([13]) is the chiral-symmetric one. The existence of non-chiral-symmetric solutions of equation ([11]) with \( \text{tr} \, S \neq 0 \) denotes dynamical chiral symmetry breaking (DCSB) in this model.

Equation ([11]) is a nonlinear equation. This fact leads to essential difficulties for its investigation. However, experience of investigation of this equation at \( D = 4 \) (see [3] and refs. therein) demonstrates that an ultraviolet behaviour of its solution is defined by the linear approximation. Since for QED the nonperturbative region is the ultraviolet region, then a linearized version of this equation is quite enough for description of nonperturbative effects (such as DCSB). At \( D = 4 \) this question was investigated in detail (see [3]). We accept this supposition also for \( D > 4 \) and shall investigate a linearization which is known as the bifurcation approximation [8].

Let introduce mass operator

\[
\Sigma = S^{-1} - S_0^{-1}.
\]

The linearization procedure consists in the following approximation

\[
S = [S_0^{-1} + \Sigma]^{-1} \approx S_0 - S_0 \ast \Sigma \ast S_0.
\]

For the mass operator in transverse gauge we obtain equation

\[
\Sigma = ie^2 D_\mu \gamma_\mu (S_0 \ast \Sigma \ast S_0) \gamma_\nu.
\]

(16)

Due to construction above it is evident that a region of applicability of this linearized version is an asymptotic ultraviolet region, i.e. the momentum region \( p^2 \gg \lambda^2 \), where \( \lambda \) is a mass parameter which plays a role of infrared cutoff. As a particular consequence one gets that in the region of applicability the solutions of equation (16) should fulfil the condition \( \Sigma^2 \leq p^2 \).

Due to the condition of \( \hat{x} \)-transversality ([13]) a spinor structure of solution of equation (16) is trivial:

\[
\Sigma_{\alpha\beta} = I_{\alpha\beta} \cdot \Sigma,
\]

and therefore

\[
F_{\alpha\beta} \equiv (S_0 \ast \Sigma \ast S_0)_{\alpha\beta} = I_{\alpha\beta} \cdot F,
\]

and finally we obtain for the mass operator the following equation in \( x \)-space:

\[
\Sigma(x^2) = \alpha \frac{(D - 1)e^{-i\pi D/2} \Gamma(D/2 - 1)}{\pi(x^2 - i0)^{D/2 - 1}} \cdot F(x^2).
\]

Here \( \alpha = e^2/4\pi \).

Multiplying this equation by \( (x^2)^{D/2 - 1} \) and passing to \( p \)-space, we obtain the differential equation

\[
(\partial^2)^{D/2 - 1} \Sigma(p^2) = -\alpha \frac{(D - 1) \Gamma(D/2 - 1)}{\pi^{D/2 - 1}(p^2 + i0)} \cdot \Sigma(p^2).
\]

This is an equation for the mass operator in the pseudo-Euclidean Minkowsky space. Performing Euclidean rotation \( \partial^2 \rightarrow -\partial^2, \ p^2 \rightarrow -p^2 \), we obtain the following equation for the mass operator in the Euclidean momentum space:

\[
(-\partial^2)^k \Sigma(p^2) = \alpha \frac{(2k + 1) \Gamma(k)}{\pi^k p^2} \cdot \Sigma(p^2),
\]

(17)

\[\text{Such a multiplication is, in essence, some regularization of singular product } (x^2 - i0)^{1-D/2} \cdot F(x).\]
where $k = D/2 - 1$.

Consider firstly the four-dimensional case ($k = 1$). In this case at $\alpha < \pi/3$ equation (17) has asymptotic solution

$$\Sigma = C(p^2)^a,$$

where

$$a = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 3\alpha/\pi}. \tag{18}$$

At $\alpha \geq \pi/3$ a solution of the equation is

$$\Sigma = \frac{C}{\sqrt{p^2}} \sin\left(\frac{\omega}{2} \log\frac{p^2}{M^2}\right), \tag{19}$$

where $\omega = \sqrt{3\alpha/\pi - 1}$. Here $C$ and $M$ are real-valued constants. (At $\omega \to 0$ the solution is $\Sigma = \frac{C}{\sqrt{p^2}} \log\frac{p^2}{M^2}$.)

At critical point $\alpha_c = \pi/3$ the type of the asymptotics changes – it becomes oscillating. This change of behaviour means a phase transition to the state with dynamically broken chiral symmetry (see [3]). By other words, at $\alpha < \alpha_c$ only trivial solution $\Sigma \equiv 0$ exists, and at $\alpha \geq \alpha_c$ a non-trivial solution arises, which corresponds to DCSB. To illustrate this thesis consider a procedure of normalization of the solution in the pseudo-Euclidean Minkowski space. In the pseudo-Euclidean space a mass operator should satisfy the normalization condition

$$\Sigma(m^2) = m. \tag{20}$$

An analytical continuation into the pseudo-Euclidean space $p^2 \to -p^2 - i0$ is performed with well-known formulae

$$(-p^2 - i0)^a = e^{-ia} (p^2 + i0)^a, \quad \log(-p^2 - i0) = \log(p^2 + i0) - i\pi. \tag{21}$$

Taking into account these formulae it is easy to see that for solutions at $\alpha < \pi/3$ the normalization condition contradicts to reality condition for $C$ and $m$ (at non-zero values of these quaities). At the same time in region $\alpha \geq \pi/3$ a normalized solution with $C \neq 0$ exists, which corresponds to DCSB. It has the form (in the Euclidean space):

$$\Sigma(p^2) = m^2 \frac{\sinh(\frac{\omega}{2})}{\sinh(\frac{\omega}{2})\sqrt{p^2}} \sin\left(\frac{\omega}{2} \log\frac{p^2}{m^2}\right). \tag{22}$$

Let turn to the six-dimensional case. Equation (17) at $k = 2$ can be rewritten as the Mejer equation [8]

$$\left(z \frac{d}{dz} + 2\right)\left(z \frac{d}{dz} + 1\right)\left(z \frac{d}{dz} - 1\right)z \frac{d\Sigma}{dz} - z\Sigma = 0, \tag{23}$$

where $z = \left(\frac{5\alpha}{4\pi}\right)p^2$. A real-valued fundamental system of solutions near the infinite point $z = \infty$ has the form

$$u_1(z) = G_{04}^{40}(ze^{-4\pi i} | -2, -1, 0, 1),$$

$$u_2(z) = G_{04}^{40}(z | -2, -1, 0, 1),$$

$$u_{3,4}(z) = e^{i\phi} G_{04}^{40}(ze^{-2\pi i} | -2, -1, 0, 1) + e^{-i\phi} G_{04}^{40}(ze^{2\pi i} | -2, -1, 0, 1).$$
Here $G_{64}^{40}$ is the Mejer function, $\phi$ is a real number. Asymptotics of functions $u_l$ at $z \to \infty$ are

$$
\begin{align*}
    u_1(z) &\sim z^{-7/8} \exp(4z^{1/4}), \\
    u_2(z) &\sim z^{-7/8} \exp(-4z^{1/4}), \\
    u_{3,4}(z) &\sim z^{-7/8} \cos(4z^{1/4} + \phi).
\end{align*}
$$

Since equation (17) itself has the asymptotical character, we can consider these asymptotics as solutions of our problem. The exponentially rising solution is not satisfy to the condition $\Sigma^2 \leq p^2$ and should be ignored. Hence, the leading asymptotic solution is

$$
\Sigma \approx Cz^{-7/8} \cos(4z^{1/4} + \phi).
$$

On taking into account the analytic continuation formulae (21) normalization condition (20) in the pseudo-Euclidean space and the reality condition for the constants $C$ and $\phi$ give us equations which connect $C$ and $\phi$ with the mass $m$. Resolving these equations we obtain in the Euclidean space the following normalized asymptotic solution:

$$
\Sigma(p^2) \approx 2m \left( \frac{m^2}{p^2} \right)^{7/8} \exp\left\{ (5\alpha)^{1/4} \sqrt{\frac{2m}{\pi}} \right\} \cos\left\{ \frac{2}{\sqrt{\pi}} (5\alpha)^{1/4} \left( \sqrt{|p|} - \sqrt{\frac{m}{2}} \right) - \frac{7}{8} \pi \right\}.
$$

(24)

Here $|p| \equiv \sqrt{p^2}$.

3 Dynamical chiral symmetry breaking in ultraviolet cutoff scheme

The construction given above is based on asymptotic solutions of differential equation (17) and normalization condition (20) and is, in essence, nothing else than heuristic consideration.

For more complete motivation of our principal position on existence of DCSB phase in the multidimensional electrodynamics we use the general Bogoliubov method for elaborating models with spontaneous symmetry breaking. In accordance with the method we shall consider the problem with explicit breakdown of chiral symmetry. For this purpose we introduce a mass term $m_0 \bar{\psi}\psi$ with "seed" mass $m_0$ into Lagrangian (1), and, after solution of the corresponding asymptotic boundary problem, go to chiral limit, i.e. tend $m_0$ to zero. DCSB criterion will be non-zero value of mass operator in such a chiral limit:

$$
\lim_{m_0 \to 0} \Sigma \neq 0.
$$

Introducing of the seed mass $m_0$ results in modification of the inhomogeneous term in equation (16): $-i\hat{\partial} \to (m_0 - i\hat{\partial})$, but does not change differential equation (17), since the inhomogeneous term disappears after the multiplication by $(x^2)^{D/2-1}$. The role of the seed mass consists in a modification of boundary conditions. To derive and take into account these boundary conditions it is necessary to turn to an integral equation in the momentum space. The integral equation for mass operator in the Euclidean momentum space for the linearized version of the model under consideration has the form:

$$
\Sigma(p^2) = m_0 + e^2 \frac{D-1}{2\pi^D} \int d^D q \frac{\Sigma(q^2)}{q^2} \frac{1}{(p-q)^2}.
$$

(25)
Here $\Sigma$ is the renormalized mass operator, $e^2$ is the renormalized coupling. The seed mass $m_0$ is a function of regularization parameter. In the definition of this mass a wave function renormalization constant and a counterterm of mass renormalization are included (see [3] for more detail).

To integrate over angles we use formula

$$J_D \equiv \int d^D q \frac{f(q^2)}{(p - q)^2} = \frac{\pi^{D-1}}{\Gamma(D/2)} \int dq^2 (q^2)^{D/2-1} f(q^2) \int_0^\pi d\theta \frac{\sin^{D-2}\theta}{p^2 + q^2 - 2|p||q|\cos\theta}. \tag{26}$$

At $D = 4$:

$$J_4 = \pi^2 \int dq^2 f(q^2) \left( \frac{1}{p^2} \theta(p^2 - q^2) + \frac{1}{q^2} \theta(q^2 - p^2) \right),$$

and at $D = 6$:

$$J_6 = \frac{\pi^3}{6} \int dq^2 (q^2)^2 f(q^2) \left( \frac{1}{p^2} \left( 3 - \frac{q^2}{p^2} \right) \theta(p^2 - q^2) + \frac{1}{q^2} \left( 3 - \frac{p^2}{q^2} \right) \theta(q^2 - p^2) \right).$$

Consider firstly the four-dimensional case. In a scheme with ultraviolet cutoff the integral equation for the mass operator at $D = 4$ has the form

$$\Sigma(p^2) = m_0 + \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} \frac{d^2\Sigma(q^2)}{dq^2} \left( \frac{1}{p^2} \theta(p^2 - q^2) + \frac{1}{q^2} \theta(q^2 - p^2) \right). \tag{26}$$

This equation leads to a boundary condition at $p^2 = \Lambda^2$ (which we shall name as the ultraviolet condition):

$$\left. \frac{d}{dp^2} (p^2 \Sigma(p^2)) \right|_{p^2=\Lambda^2} = m_0 \tag{27}$$

Another boundary condition (at small $p^2$) does not contain the parameter $m_0$ and does not play any part in our construction. Integral equation (26) is reduced to the differential equation

$$\frac{d^2}{d(p^2)^2} \left( p^2 \Sigma(p^2) \right) = -\frac{3\alpha \Sigma(p^2)}{4\pi p^2}. \tag{28}$$

It is easy to see that this equation is the same as equation (17) at $k = 1$ ($D = 4$).

Consider a pre-critical case $\alpha < \pi/3$. In this case a general solution of equation (28) is

$$\Sigma = C_1(p^2)^a + C_2(p^2)^{-a-1},$$

Here $-1/2 < a < 0$ (see eq. (18)).

Suppose firstly $C_1 \neq 0$. Then from condition (27) we see, to make the solution independent of the cutoff parameter $\Lambda$ it is necessary to renormalize the seed mass

$$m_0(\Lambda) = \mu \left( \frac{\Lambda^2}{\mu^2} \right)^a, \tag{29}$$

which gives the value $C_1 = \frac{\mu^{1-2a}}{a+1}$. From (28) it follows that at the chiral limit we have $\mu \to 0$, and, consequently, $C_1 = 0$. If $C_1 = 0$, the renormalization of the seed mass is produced with the formula

$$m_0(\Lambda) = \mu \left( \frac{\Lambda^2}{\mu^2} \right)^{-a-1}$$
and condition (27) give us $C_2 = -\mu^{3+2a}/a$, and again in the chiral limit we have $\mu \rightarrow 0$, and, consequently, $C_2 = 0$. So, taking into account ultraviolet boundary condition (27) results in absence of nontrivial solutions in the chiral limit.

In the critical region $\alpha \geq \pi/3$ a general solution of equation (28) is given by formula (19). Boundary condition (27) in this case results in the following formula of the mass renormalization:

$$m_0(\Lambda) = \frac{\mu^2}{2\Lambda} \left( \sin\left(\frac{\omega}{2} \log \frac{\Lambda^2}{M^2}\right) + \omega \cos\left(\frac{\omega}{2} \log \frac{\Lambda^2}{M^2}\right) \right).$$

If the following condition fulfills

$$\tan\left(\frac{\omega}{2} \log \frac{\Lambda^2}{M^2}\right) = -\omega,$$ (30)

then at the chiral limit a nontrivial solution exists:

$$\Sigma = \frac{\mu^2}{\sqrt{p^2}} \sin\left(\frac{\omega}{2} \log \frac{p^2}{M^2}\right),$$

which corresponds to DCSB phase. After normalization of this solution in the pseudo-Euclidean space on the physical mass $m$ we come back to normalized solution (22).

We see, that oscillating character of solution in critical region $\alpha > \pi/3$ is the major property ensuring the existence of DCSB.

For the six-dimensional space the integral equation for the mass operator has the form

$$\Sigma(p^2) = m_0 + \frac{5\alpha}{6(4\pi)^2} \int_{\Lambda^2} dq^2 q^2 \Sigma(q^2) \left( \frac{1}{p^2} \left( 3 - \frac{q^2}{p^2} \right) \theta(p^2 - q^2) + \frac{1}{q^2} \left( 3 - \frac{q^2}{p^2} \right) \theta(q^2 - p^2) \right).$$ (31)

Ultraviolet boundary conditions, which follow from this integral equation, are

$$\frac{d^2}{d(p^2)^2} \left( (p^2)^2 \Sigma(p^2) \right) \bigg|_{p^2 = \Lambda^2} = 2m_0,$$ (32)

$$\frac{d^3}{d(p^2)^3} \left( (p^2)^2 \Sigma(p^2) \right) \bigg|_{p^2 = \Lambda^2} = 0.$$ (33)

A differential equation, which follows from (31), coincides with equation (23). On taking into account condition $\Sigma^2 \leq p^2$, its solution is

$$\Sigma \approx (p^2)^{-7/8} \left( C_1 \cos(\kappa \sqrt{|p|} + \phi) + C_2 \exp(-\kappa \sqrt{|p|}) \right),$$

where $\kappa = 2(5\alpha/\pi^2)^{1/4}$.

Boundary conditions (32) and (33) give relations

$$C_2 = C_1 e^{\kappa \sqrt{\Lambda}} \sin(\kappa \sqrt{\Lambda} + \phi)$$ (34)

and

$$\left(\frac{\kappa}{4}\right)^{3/4} \Lambda^{-3/4} C_1 (\sin(\kappa \sqrt{\Lambda} + \phi) - \cos(\kappa \sqrt{\Lambda} + \phi)) = 2m_0.$$
Consequently, the mass renormalization is made with the formula

\[ m_0 = \mu \left( \frac{\mu}{\Lambda} \right)^{3/4} \left( \sin(\kappa \sqrt{\Lambda} + \phi) - \cos(\kappa \sqrt{\Lambda} + \phi) \right), \]

and, under condition

\[ \tan(\kappa \sqrt{\Lambda} + \phi) = 1, \]

as well as for the critical region of four-dimensional theory, at the chiral limit \( m_0 = 0 \) a nontrivial solution exists, which corresponds to DCSB.

An analytic continuation into the pseudo-Euclidean region and normalization condition (20) give relations, which connect the constants \( C_1, C_2 \) and \( \phi \) with the physical mass \( m \). These relations are

\[
\begin{align*}
\frac{1}{2} C_1 e^{\kappa \sqrt{\frac{m}{2}}} \cos(\kappa \sqrt{\frac{m}{2}} + \phi) + C_2 e^{-\kappa \sqrt{\frac{m}{2}}} \cos(\kappa \sqrt{\frac{m}{2}}) &= m^{11/4} \cos \frac{7\pi}{8}, \\
\frac{1}{2} C_1 e^{\kappa \sqrt{\frac{m}{2}}} \sin(\kappa \sqrt{\frac{m}{2}} + \phi) + C_2 e^{-\kappa \sqrt{\frac{m}{2}}} \sin(\kappa \sqrt{\frac{m}{2}}) &= -m^{11/4} \sin \frac{7\pi}{8}.
\end{align*}
\]

Exploiting condition (35) and formula (34) in the region of applicability of our constructions one can prove the following inequality: \( C_2 e^{-\kappa \sqrt{\frac{m}{2}}} \ll C_1 e^{\kappa \sqrt{\frac{m}{2}}} \). Really, taking into account equations (35) and (34), one can exclude the coefficients \( C_1 \) and \( C_2 \) from the above relations and obtains the following equation on the phase factor:

\[ \sin x = -\sqrt{2} \sin x_0 \ e^{-x_0 - x} \] (36)

Here the following notations are introduced:

\[ x_0 = \kappa \sqrt{\frac{m}{2}} - \frac{\pi}{8}, \]
\[ x = \phi + x_0 + \pi l, \]

where \( l \) is an entire number, which is produced by a solution of condition (35): \( \kappa \sqrt{\Lambda} + \phi = \pi/4 + \pi l \). In the asymptotic region the solution of equation (36) is

\[ x \approx \pi n, \]

where \( n \) is an entire positive number. Taking into account (33) and (34), we obtain

\[ C_2 e^{-\kappa \sqrt{\frac{m}{2}}} \approx \frac{C_1}{\sqrt{2}} e^{\pi/8 - \pi n} \cos \pi l \ll C_1 e^{\kappa \sqrt{\frac{m}{2}}}. \]

Neglecting in correspondence with proven inequality the terms with \( C_2 \) in the normalization condition, we go again to formula (24) for the mass operator in the Euclidean space.

**Conclusion**

The principal result of this paper is a nonperturbative model motivation for the existence of DCSB phenomenon in the six-dimensional QED. This result is in want of further specification and investigation. Thus, for example, it is not quite clear what is happen at a cutoff removing
for the six-dimensional case. In the four-dimensional ladder QED, in correspondence with results of investigations summarized in monograph [3] we have

\[ \alpha \to \alpha_c \]

at the cutoff removing in the critical region, i.e. the renormalized QED in the strong coupling regime exists at the critical coupling only. If one will proceed by analogy with the four-dimensional theory in the six-dimensional case, it may be expected, that \( \alpha \to 0 \) at the cutoff removing (though DCSB phenomenon is left, i.e. an electron get a mass). In this connection unavoidable question about triviality arises (as in the four-dimensional theory, though). A solution of these problems requires an investigation of the Bethe-Salpeter equation for bound states, i.e., in terms of our expansion, an investigation of equations of the following iteration step.

In conclusion let us touch on the multidimensional QED with a dimension greater than six. Equation (17) has the oscillating asymptotic solution at any even \( D \geq 6 \). This fact gives rise supposition that DCSB phenomenon exists at any even dimension greater than four.

Author is grateful to G.G. Volkov for stimulating discussion and P.A. Saponov for reading the manuscript.

References

[1] Maskawa T. and Nakajima H.: Prog.Theor.Phys. 52 (1974) 1326

[2] Fomin P.I., Gusynin V.P., Miransky V.A. and Sitenko Yu.A.: Riv.Nuo.Cim. 6 (1983) 1

[3] Miransky V.A.: ”Dynamical Symmetry Breaking in Quantum Field Theories”, Singapore, World Scientific, 1993

[4] Rubakov V.A.: hep-ph/0104152

[5] Rochev V.E.: J.Phys. A 30 (1997) 3671; hep-th/9606155

[6] Rochev V.E.: J.Phys. A 33 (2000) 7379; hep-ph/9907534

[7] Zinn-Justin J.: ”Quantum Field Theory and Critical Phenomena”, Oxford, Clarendon Press, 1993

[8] Atkinson D. and Johnson P.W.: J.Math.Phys. 28 (1987) 2488

Gusynin V.P.: Mod.Phys.Letters A5 (1990) 133

[9] Meijer C.S.: Proc.Kon.Nederl.Akad.v.Wetensch. A49 (1946) 227, 344, 457, 632, 765, 936, 1063, 1165

Bateman H. and Erdelyi A.: ”Higher Transcendental Functions”, McGraw-Hill, 1953, Vol.1