ON THE BLOW-UP FOR CRITICAL SEMILINEAR WAVE EQUATIONS WITH DAMPING IN THE SCATTERING CASE

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ABSTRACT. We consider the Cauchy problem for semilinear wave equations with variable coefficients and time-dependent scattering damping in $\mathbb{R}^n$, where $n \geq 2$. It is expected that the critical exponent will be Strauss’ number $p_0(n)$, which is also the one for semilinear wave equations without damping terms.

Lai and Takamura [7] have obtained the blow-up part, together with the upper bound of lifespan, in the sub-critical case $p < p_0(n)$. In this paper, we extend their results to the critical case $p = p_0(n)$. The proof is based on [16], which concerns the blow-up and upper bound of lifespan for critical semilinear wave equations with variable coefficients.

1. Introduction

We study the blow-up problem for critical semilinear wave equations with variable coefficients and scattering damping depending on time. The perturbations of Laplacian are uniformly elliptic operators

$$\Delta_g = \sum_{i,j=1}^{n} \partial_{x_i} g_{ij}(x) \partial_{x_j} g_{ij}(x)$$

whose coefficients satisfy, with some $\alpha > 0$, the following:

(1.1) $g_{ij} \in C^1(\mathbb{R}^n)$, $|\nabla g_{ij}(x)| + |g_{ij}(x) - \delta_{ij}| = O(e^{-\alpha|x|})$ as $|x| \to \infty$.

The admissible damping coefficients are $a \in C([0, \infty))$, such that

(1.2) $\forall t \geq 0 \quad a(t) \geq 0$ and $\int_0^\infty a(t)dt < \infty$.

For $n \geq 2$ and $p > 1$, we consider the Cauchy problem

(1.3) $u_{tt} - \Delta_g u + a(t)u_t = |u|^p$, $x \in \mathbb{R}^n$, $t > 0$,

(1.4) $u|_{t=0} = \varepsilon u_0$, $u_t|_{t=0} = \varepsilon u_1$, $x \in \mathbb{R}^n$,

where $u_0$, $u_1 \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter. Our results concern only the critical case $p = p_0(n)$ with Strauss’ exponent defined in (1.5) below.

Let us briefly review previous results concerning (1.3) with $g_{ij} = \delta_{ij}$ and various types of damping $a$. When $a(t) = 1$, Todorova and Yordanov [13] showed that the solution of (1.3) blows up in finite time if $1 < p < p_F(n)$, where $p_F(n) = 1 + 2/n$ is the Fujita exponent known to be the critical exponent for the semilinear heat equation. The same work also obtained global existence for $p > p_F(n)$. Finally, Zhang [20] established the blow-up in the critical case $p = p_F(n)$.

The other typical example of effective damping is $a(t) = \mu/(1 + t)^\beta$ with $\mu > 0$ and $\beta \in \mathbb{R}$. When $-1 < \beta < 1$, Lin, Nishihara and Zhai [9] obtained the expected

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blow-up result, if $1 < p \leq p_F(n)$, and global existence result, if $p > p_F(n)$; see also D’Abbicco, S.Lucente and M.Reissig [2].

In the case of critical decay $\beta = 1$, there are several works about finite time blow-up and global existence. Wakasugi [17] showed the blow-up, if $1 < p \leq p_F(n)$ and $\mu > 1$ or $1 < p \leq 1 + 2/(n + \mu - 1)$ and $0 < \mu \leq 1$. Moreover, D’Abbicco [11] verified the global existence, if $p > p_F(n)$ and $\mu$ satisfies one of the following: $\mu \geq 5/3$ for $n = 1$, $\mu \geq 3$ for $n = 2$ and $\mu \geq n + 2$ for $n \geq 3$. An interesting observation is that the Liouville substitution $w(x, t) := (1 + t)^{n/2}u(x, t)$ transforms the damped wave equation (1.3) into the Klein-Gordon equation

$$w_{tt} - \Delta w + \frac{\mu(2 - \mu)}{4(1 + t)^2}w = \frac{|w|^p}{(1 + t)^{\alpha(p - 1)/2}}.$$  

Thus, one expects that the critical exponent for $\mu = 2$ is related to that of the semilinear wave equation. D’Abbicco, Lucente and Reissig [3] have actually obtained the corresponding blow-up result, if $1 < p < p_c(n) := \max\{p_F(n), p_0(n + 2)\}$ and

$$(1.5)\quad p_0(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}$$

is the so-called Strauss exponent, the positive root of the quadratic equation

$$(1.6)\quad \gamma(p, n) := 2 + (n + 1)p - (n - 1)p^2 = 0.$$  

Their work also showed the existence of global classical solutions for small $\varepsilon > 0$, if $p > p_c(n)$ and either $n = 2$ or $n = 3$ and the data are radially symmetric. Finally, we mention that our original equations (1.3) is related to semilinear wave equations in the Einstein-de Sitter spacetime considered by Galstian & Yagdjian [4].

We recall that $p_0(n)$ in (1.5) is the critical exponent for the semilinear wave equation conjectured by Strauss [11]. The hypothesis has been verified in several cases; see [10] and the references therein. A related problem is to estimate the lifespan, or the maximal existence time $T_\varepsilon$ of solutions to (1.3), (1.4) in the energy space $C([0, T_\varepsilon), H^1(\mathbb{R}^n)) \cap C^1([0, T_\varepsilon), L^2(\mathbb{R}^n))$.

Lai, Takamura and Wakasa [8] have obtained the blow-up part of Strauss’ conjecture, together with an upper bound of the lifespan $T_\varepsilon$, for (1.3), (1.4) in the case $n \geq 2$, $0 < \mu < (n^2 + n + 2)(n + 2)$ and $p_F(n) \leq p < p_0(n + 2\mu)$. Later, Ikeda and Sobajima [3] were able to replace these conditions by less restrictive $0 < \mu < (n^2 + n + 2)/(n + 2)$ and $p_F(n) \leq p \leq p_0(n + \mu)$. In addition, they have derived an upper bound on the lifespan. Tu and Lin [14], [15] have improved the estimates of $T_\varepsilon$ in [5] recently.

For $\beta \leq -1$, the long time behavior of solutions to (1.3), (1.4) is quite different. When $\beta = -1$, Wakasugi [13] has obtained the global existence for exponents $p_F(n) < p < n/[n - 2]_+$, where

$$[n - 2]_+ := \begin{cases} \infty & \text{for } n = 1, 2, \\ m/(n - 2) & \text{for } n \geq 3. \end{cases}$$

Ikeda and Wakasugi [6] have proved that the global existence actually holds for any $p > 1$ when $\beta < -1$.

For $\beta > 1$, we expect the critical exponent to be exactly the Strauss exponent. In fact, Lai and Takamura [7] have shown that certain solutions of (1.3), (1.4) blow up in finite time when $1 < p < p_0(n)$. Moreover, Liu and Wang [10] have just obtained the global existence results for $n = 3, 4$ and $p > p_0(n)$ on asymptotically Euclidean manifolds.
If \( T_\varepsilon \) denotes the lifespan of these solutions, then \([7]\) have also given the upper bound \( T_\varepsilon \leq C\varepsilon^{-2p(p-1)/\gamma(p,n)} \) for \( n \geq 2 \) and \( 1 < p < p_0(n) \). This result is probably sharp, since Takamura \([12]\) proved the same type of estimate in the sub-critical case of Strauss’ conjecture for semilinear wave equations without damping. However, both the conjecture and lifespan bound remained open problems in the critical case \( p = p_0(n) \).

The purpose of this paper is to verify the blow-up for \( p = p_0(n) \) and to give a proof that extends to more general damping, including \( a(t) \sim (1+t)^{-\beta} \) with \( \beta > 1 \).

We also succeed to derive an exponential type upper bound on the lifespan \( T_\varepsilon \), which is the same as that of the Strauss conjecture in the conservative critical case.

Such results are consistent with our knowledge of the linear problem corresponding to (1.2), (1.3); Wirth \([19]\) has shown that energy space solutions scatter, that is approach solutions to the free wave equations, as \( t \to \infty \).

**Theorem 1.1.** Let \( n \geq 2 \), \( p = p_0(n) \) and \( a(t) \) satisfy (1.2). Assume that both \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \) are nonnegative, do not vanish identically and have supports in the ball \( \{ x \in \mathbb{R}^n : |x| \leq R_0 \} \), where \( R_0 > 1 \).

If (1.2) has a solution \( (u, u_1) \in C([0,T_\varepsilon), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \), such that
\[
(1.7) \quad \text{supp}(u, u_1) \subset \{(x, t) \in \mathbb{R}^n \times [0, T_\varepsilon) : |x| \leq t + R \},
\]
with \( R \geq R_0 \), then \( T_\varepsilon < \infty \). Moreover, there exist constants \( \varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, R, a) \) and \( K = K(u_0, u_1, n, p, R, a) \), such that
\[
(1.8) \quad T_\varepsilon \leq \exp(K\varepsilon^{-p(p-1)}) \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon_0.
\]

**Remark 1.2.** The lifespan estimates is the same as that of the Strauss conjecture in the critical case of semilinear wave equations without damping. For details, see the introduction in [16]. We also note that Liu and Wang \([10]\) have obtained the sharp lower bound of the lifespan, \( T_\varepsilon \geq \exp(c\varepsilon^{-2}) \) if \( n = 4 \) and \( p = p_0(4) = 2 \).

Our proof is based on the approach of Wakasa and Yordanov \([16]\). Averaging the solution with respect to a suitable test function, we derive a second-order dissipative ODE which corresponds to equation (1.2). The key point is to establish lower bounds for the fundamental system of solutions to this ODE; see Lemma 2.3. As a consequence, we can follow \([16]\) and obtain the same nonlinear integral inequality. The final blow-up argument also repeats the iteration argument of \([16]\).

## 2. Test Functions

Similarly to the proof of \([16]\), we first consider the following elliptic problem:
\[
(2.1) \quad \Delta_y \varphi_\lambda = \lambda^2 \varphi_\lambda, \quad x \in \mathbb{R}^n,
\]
where \( \lambda \in (0, \alpha/2] \). As \( \lambda |x| \to \infty \), these \( \varphi_\lambda(x) \) are asymptotically given by \( \varphi(\lambda x) \), with \( \varphi \) being the standard radial solution to the unperturbed equation \( \Delta \varphi = \varphi \):
\[
(2.2) \quad \varphi(x) = \int_{S^{n-1}} e^{x \cdot \omega} dS_\omega \sim c_n |x|^{-(n-1)/2} e^{c|x|}, \quad |x| \to \infty.
\]

We recall the following result about the existence and main properties of \( \varphi_\lambda \).

**Lemma 2.1.** Let \( n \geq 2 \). There exists a solution \( \varphi_\lambda \in C^\infty(\mathbb{R}^n) \) to (2.1), such that
\[
(2.3) \quad |\varphi_\lambda(x) - \varphi(\lambda x)| \leq C_\alpha \lambda^\theta, \quad x \in \mathbb{R}^n, \quad \lambda \in (0, \alpha/2],
\]
where \( \theta \in (0, 1] \) and \( \varphi(x) = \int_{S^{n-1}} e^{x \cdot \omega} dS_\omega \sim c_n |x|^{-(n-1)/2} e^{c|x|}, c_n > 0, \) as \( |x| \to \infty \).
Moreover, $\varphi(\cdot) - \varphi(\lambda \cdot)$ is a continuous $L^\infty(\mathbb{R}^n)$ valued function of $\lambda \in (0, \alpha/2]$ and there exist positive constants $D_0$, $D_1$ and $\lambda_0$, such that

\begin{align}
D_0 |x|^{-\alpha/2} e^{\lambda|x|} \leq \varphi(x) \leq D_1 |x|^{-\alpha/2} e^{\lambda|x|}, \quad x \in \mathbb{R}^n,
\end{align}

holds whenever $0 < \lambda \leq \lambda_0$.

**Proof.** See Lemma 2.2 in [10]. \hfill \square

Given $\lambda_0 \in (0, \alpha/2]$ and $q > 0$, we also introduce the auxiliary functions

\begin{align}
\xi_q(x,t) &= \int_0^{\lambda_0} e^{-\lambda(t+R)} \cosh \lambda x \lambda^d \lambda, \\
\eta_q(x,t,s) &= \int_0^{\lambda_0} e^{-\lambda(t+R)} \frac{\sinh \lambda(t-s)}{\lambda(t-s)} \varphi(x) \lambda^d \lambda,
\end{align}

for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $s \in \mathbb{R}$. Useful estimates are collected in the next lemma.

**Lemma 2.2.** Let $n \geq 2$. There exists $\lambda_0 \in (0, \alpha/2]$, such that the following hold:

(i) if $0 < q$, $|x| \leq R$ and $0 \leq t$, then

$$\xi_q(x,t) \geq A_0,$$

$$\eta_q(x,t,0) \geq B_0(t)^{-1};$$

(ii) if $0 < q$, $|x| \leq s + R$ and $0 \leq s < t$, then

$$\eta_q(x,t,s) \geq B_1(t)^{-1}(s)^{-q};$$

(iii) if $(n-3)/2 < q$, $|x| \leq t + R$ and $0 < t$, then

$$\eta_q(x,t,t) \leq B_2(t)^{(n-3)/2} |x|^{(n-3)/2-q}.$$

Here $A_0$ and $B_k$, $k = 0, 1, 2$, are positive constants depending only on $\alpha$, $q$ and $R$, while $(s) = 3 + |s|$.

**Proof.** See Lemma 3.1 in [10]. \hfill \square

The following lemma plays a key role in the proof of Theorem 1.1.

**Lemma 2.3.** Let $\lambda > 0$ and introduce the ordinary differential operators

$$L_a = \partial^2_t + a(t) \partial_t - \lambda^2, \quad L^*_a = \partial^2_s - \partial_s a(s) - \lambda^2.$$

The fundamental system of solutions $\{y_1(t,s;\lambda), y_2(t,s;\lambda)\}$, defined through

$$L_a y_1(t,s;\lambda) = 0, \quad y_1(s,s;\lambda) = 1, \quad \partial_t y_1(s,s;\lambda) = 0,$$

$$L_a y_2(t,s;\lambda) = 0, \quad y_2(s,s;\lambda) = 0, \quad \partial_t y_2(s,s;\lambda) = 1,$$

depends continuously on $\lambda$ and satisfies the following estimates, for $t \geq s \geq 0$:

\begin{align}
(i) \quad y_1(t,s;\lambda) \geq e^{-\|a\|_{L^1}} \cosh \lambda(t-s), \\
(ii) \quad y_2(t,s;\lambda) \geq e^{-2\|a\|_{L^1}} \frac{\sinh \lambda(t-s)}{\lambda}.
\end{align}

Moreover, the conjugate equations and initial conditions hold:

\begin{align}
(iii) \quad L^*_a y_2(t,s;\lambda) = 0, \\
(iv) \quad y_1(t,0;\lambda) = a(0) y_2(t,0;\lambda) - \partial_s y_2(t,0;\lambda).
\end{align}

**Proof.** See Section 4. \hfill \square
3. Proof of Theorem 1.1

Let $u$ be a weak solution to problem \[[1.3]\], defined below, and $\eta_q(x,t)\eta_q(x,t)$ be a test function, defined in Section 2, with $q > -1$. We will show that

\[ F(t) = \int_{\mathbb{R}^n} u(x,t)\eta_q(x,t)dx \]

satisfies a nonlinear integral inequality which implies finite time blow-up. Our definition of weak solutions is standard: \((u, u_t) \in C([0,T_e), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\) and \(\forall \phi \in C^\infty_0(\mathbb{R}^n \times [0,T_e))\) and \(t \in (0, T_e)\)

\[
\int u_s(x,t)\phi(x,t)dx - \int u_s(x,0)\phi(x,0)dx \\
- \int_0^t \int (u_s(x,s)\phi_s(x,s) - g(x)\nabla u(x,s) \cdot \nabla \phi(x,s) - a(s)u_s(x,s)\phi(x,s))dxds \\
= \int_0^\infty \int |u(x,s)|^p\phi(x,s)dxds.
\]

In the next result, however, it will be more convenient to work with

\[
\int (u_s(x,t)\phi(x,t) - u(x,t)\phi_s(x,t) + a(t)u(x,t)\phi(x,t))dx \\
- \int (u_s(x,0)\phi(x,0) - u(x,0)\phi_s(x,0) + a(0)u(x,0)\phi(x,0))dx \\
+ \int_0^t \int u(x,s)(\phi_{ss}(x,s) - \Delta_y \phi(x,s) - (a(s)\phi(x,s))_s)dxds \\
= \int_0^\infty \int |u(x,s)|^p\phi(x,s)dxds,
\]

which follows from integration by parts. We can also use \(\phi \in C^\infty(\mathbb{R}^n \times [0,T_e))\), since \(u(\cdot,s)\) is compactly supported for every \(s\).

**Proposition 3.1.** Let the assumptions in Theorem \[[14]\] be fulfilled and $q > -1$.

\[
\int_{\mathbb{R}^n} u(x,t)\eta_q(x,t)dx \\
\geq \varepsilon e^{-\|a\|_L^1} \int_{\mathbb{R}^n} u_0(x)\xi_0(x,t)dx + \varepsilon e^{-2\|a\|_L^1} \int_{\mathbb{R}^n} u_1(x)\eta_q(x,t,0)dx \\
+ e^{-2\|a\|_L^1} \int_0^t (t-s) \int_{\mathbb{R}^n} |u(x,s)|^p\eta_q(x,t,s)dxds
\]

for all $t \in (0, T_e)$.

**Proof.** We will apply \[[3.2]\] to \(\phi(x,s) = \varphi_\lambda(x)y_2(t,s;\lambda)\), which satisfies

\[
\phi_{ss}(x,s) - \Delta_y \phi(x,s) - (a(s)\phi(x,s))_s = 0, \\
a(0)\phi(x,0) - \phi_s(x,0) = \varphi_\lambda(x)y_1(t,0;\lambda),
\]
from Lemma 2.3 (iii) and (iv), respectively. Then we obtain

$$\int u(x,t)\varphi_\lambda(x)dx = \varepsilon y_1(t,0;\lambda) \int u_0(x)\varphi_\lambda(x)dx$$

$$+ \varepsilon y_2(t,0;\lambda) \int u_1(x)\varphi_\lambda(x)dx$$

$$+ \int_0^t y_2(t,s;\lambda) \left( \int |u(x,s)|^p \varphi_\lambda(x)dx \right) ds,$$

where the initial conditions are determined by (1.4) and the pair \{y_1, y_2\} is defined in Lemma 2.3. Making use of estimates (i) and (ii) in this lemma, we have that

$$\int u(x,t)\varphi_\lambda(x)dx \geq \varepsilon e^{-\|a\|_{L^1}} \cosh(\lambda t) \int u_0(x)\varphi_\lambda(x)dx$$

$$+ \varepsilon e^{-2\|a\|_{L^1}} \frac{\sinh \lambda t}{\lambda} \int u_1(x)\varphi_\lambda(x)dx$$

$$+ e^{-2\|a\|_{L^1}} \int_0^t \frac{\sinh \lambda(t-s)}{\lambda} \left( \int |u(x,s)|^p \varphi_\lambda(x)dx \right) ds.$$
and \( \lambda \to 0 \). Our proof gives two-sided bounds and relies only on three identities:

(4.1) \((y_1' e^{A(t)})' = \lambda^2 y_1 e^{A(t)}\), where \( A(t) = \int_{0}^{t} a(r) dr \),

(4.2) \( \left( y_1 e^{A(t)} - \int_{s}^{t} a(r) y_1 e^{A(r)} dr \right)'' = \lambda^2 y_1 e^{A(t)} \),

(4.3) \( y_2 y_1 - y_2 y_1' = e^{A(s) - A(t)} \) or \( \left( \frac{y_2}{y_1} \right)' = \frac{e^{A(s) - A(t)}}{y_1^2} \).

To verify claim (i), we observe that \( y_1(t_0, s; \lambda) = 0 \) at some \( t_0 > s \) leads to a contradiction: if \( t_0 \) is the first such number, then \( y_1(t, s; \lambda) \geq 0 \) for \( t \in [s, t_0] \) and (4.1) imply that

\[ y_1'(t, s; \lambda)e^{A(t)} = \lambda^2 \int_{s}^{t} y_1(r, s; \lambda)e^{A(r)} dr \geq 0, \text{ so } y_1'(t, s; \lambda) \geq 0 \text{ for } t \in [s, t_0]. \]

Hence, \( y_1(t, s; \lambda) \) is increasing on \([s, t_0]\) and \( 0 = y_1(t_0, s; \lambda) \geq y_1(s, s; \lambda) = 1 \) can not hold. The positivity of \( y_1(t, s; \lambda) \) also yields, through (4.1), the positivity of its derivative: \( y_1'(t, s; \lambda) \geq 0 \) for all \( t \geq s \).

We can now derive an upper bound on \( y_1 \) using that \( y_1'' = \lambda^2 y_1 - a y_1' \leq \lambda^2 y_1 \) and \( y_1(s, s; \lambda) = 1, y_1'(s, s; \lambda) = 0 \):

(4.4) \[ y_1(t, s; \lambda) \leq \cosh \lambda(t - s), \quad t \geq s. \]

The lower bound on \( y_1 \) is a consequence of (4.2) and the positivity of \( y_1(t, s; \lambda) \):

\[ \left( y_1 e^{A(t)} - \int_{s}^{t} a(r) y_1 e^{A(r)} dr \right)'' \geq \lambda^2 \left( y_1 e^{A(t)} - \int_{s}^{t} a(r) y_1 e^{A(r)} dr \right). \]

Combining this inequality with the initial values at \( t = s \),

\[ \left. \left( y_1 e^{A(t)} - \int_{s}^{t} a(r) y_1 e^{A(r)} dr \right) \right|_{t=s} = e^{A(s)}, \]

\[ \frac{d}{dt} \left. \left( y_1 e^{A(t)} - \int_{s}^{t} a(r) y_1 e^{A(r)} dr \right) \right|_{t=s} = 0, \]

we obtain that

\[ y_1(t, s; \lambda)e^{A(t)} - \int_{s}^{t} a(\tau) y_1 e^{A(\tau)} d\tau \geq e^{A(s)} \cosh \lambda(t - s). \]

After simplifying,

(4.5) \[ y_1(t, s; \lambda) \geq e^{A(s) - A(t)} \cosh \lambda(t - s), \quad t \geq s. \]

Since \( A(s) - A(t) \geq -\|a\|_{L^1} \) and (4.1) holds, claim (i) follows:

(4.6) \[ \cosh \lambda(t - s) \geq y_1(t, s; \lambda) \geq e^{-\|a\|_{L^1}} \cosh \lambda(t - s). \]

To check claim (ii), we combine (4.4), (4.5) and identity (4.3):

\[ \frac{e^{A(t) - A(s)}}{\cosh^2 \lambda(t - s)} \geq \left( \frac{y_2}{y_1} \right)' \geq \frac{e^{A(s) - A(t)}}{\cosh^2 \lambda(t - s)}. \]

Using \( A(s) - A(t) \geq -\|a\|_{L^1} \) and integration on \([s, t]\), we have that

\[ \frac{e^{-\|a\|_{L^1}}}{\lambda} \tanh \lambda(t - s) \geq \frac{y_2(t, s; \lambda)}{y_1(t, s; \lambda)} \geq \frac{e^{-\|a\|_{L^1}}}{\lambda} \tanh \lambda(t - s). \]
The final result follows from (4.6):
\[
\frac{\sinh \lambda(t-s)}{\lambda} \geq y_2(t,s;\lambda) \geq e^{-2\|a\|_{L^1}} \frac{\sinh \lambda(t-s)}{\lambda}.
\]

Finally, we will show the equalities in (iii) and (iv). Set \( y_1(t) := y_1(t,0;\lambda) \) and 
\( y_2(t) := y_2(t,0;\lambda) \). It easy to see that 
\[
y_2(t,s;\lambda) = \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1(s)y_2(s) - y_1(s)y_2(t)} = (y_1(s)y_2(t) - y_1(t)y_2(s))e^{A(s)}.
\]

Thus, we can calculate \( \partial_s^i y_2(t,s;\lambda) \) with \( i = 1, 2 \) as follows:
\[
\begin{align*}
\partial_s y_2(t,s;\lambda) &= (y_1'(s)y_2(t) - y_1(t)y_2'(s))e^{A(s)} + a(s)y_2(t,s;\lambda),
\end{align*}
\]
(4.7)
\[
\begin{align*}
\partial_s^2 y_2(t,s;\lambda) &= (y_1''(s)y_2(t) - y_1(t)y_2''(s))e^{A(s)} \\
&\quad + a(s)(y_1'(s)y_2(t) - y_1(t)y_2'(s))e^{A(s)} + \partial_s(a(s)y_2(t,s;\lambda)).
\end{align*}
\]

Noticing that \( y_i(s) \) with \( i = 1, 2 \) satisfy the differential equation \( L_0 y_i(s) = 0 \), we get (iii).

To derive (iv), we just set \( s = 0 \) in (4.7) for \( \partial_s y_2(t,s;\lambda) \) and use the initial conditions for \( y_i(s) \). The proof is complete.

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