Algorithm for parametric identification of mathematical models of viscoelastic materials

S Erokhin
Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia

Abstract. Mathematical models containing fractional derivatives for viscoelastic materials and parametric identification problems are being considered in the paper. Specific attention is paid to statistical methods of data processing which allow increasing the accuracy of identification. A description of software implementing the presented techniques and algorithms is also given in the passage.

1. Introduction. Parametric identification techniques
The use of fractional derivatives [1] is considered as one of the most effective methods for modeling viscoelasticity. Fractional calculus has been used successfully to describe viscoelastic materials for almost 100 years [2,3]. The papers [4,5] can be considered fundamental for modern research. For example, the relationship between stress and strain of polymer films is described by the formula [6]:

\[ \sigma(t) = E D^\beta \varepsilon(t) , \]  
(1)

where \( \sigma(t) \) – stress, \( \varepsilon(t) \) – strain, \( E, \beta \) – material’s parameters, \( D^\beta \) – the Caputo fractional differentiation operator [7]

\[ D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f'(\tau)d\tau}{(x-\tau)^\beta} . \]  
(2)

Models using a Riemann-Liouville derivative are described in the paper [8].
To model the creep of some types of concrete and woven fiberglass, the following model is used [9]

\[ \sigma_0 = E \varepsilon(t) + \eta D^\beta \varepsilon(t) , \]  
(3)

where \( E, \beta, \eta \) – material’s parameters, \( \sigma_0 \) – the constant load on the specimen.
The following model was used to describe relaxation (the property of materials to reduce internal stress with rigidly fixed initial strain) [10]
\[ \frac{1}{E} D^\beta \sigma(t) + \frac{1}{\eta} \sigma(t) = 0, \]
\[ \sigma(0) = \sigma_0 = E \epsilon_0. \]  \hfill (4)

For the considered models, the most important problem is parametric identification - the determination of their parameters using experimental data [11,12]. Without knowledge of the model parameters, it’s impossible to work out qualitative predictions, for example, on the life span of building structures made of modern materials. The values of these parameters depend on the properties of the materials only.

Parametric identification techniques for models (1), (3), and (4) were presented in [13]. If three experimental points of the stress-strain state of the specimen \((e_0, \sigma_0), (e_1, \sigma_1), (e_2, \sigma_2)\), for the model (1) are known (Figure 1), then the order of the fractional derivative can be calculated by the formula

\[ \beta = 1 - \frac{\ln \Delta \sigma_2}{\ln \Delta \sigma_1}, \]  \hfill (5)

where \(\epsilon_1 = e_1 - e_0, \Delta \epsilon_2 = \epsilon_2 - \epsilon_0, \Delta \sigma_1 = \sigma_1 - \sigma_0, \Delta \sigma_2 = \sigma_2 - \sigma_0.\)

For the model (3), the problem of determining the parameter \(\beta\) is reduced to a numerical solution of the equation

\[ \frac{\epsilon_2 t_2^\beta - \epsilon_1 t_1^\beta}{\epsilon_2 t_1^{2\beta} - \epsilon_1 t_2^{2\beta}} = \frac{\epsilon_3 t_3^\beta - \epsilon_1 t_1^\beta}{\epsilon_3 t_1^{2\beta} - \epsilon_1 t_3^{2\beta}} \]  \hfill (6)

where \((t_1, \epsilon_1), (t_2, \epsilon_2), (t_3, \epsilon_3)\) – are the experimental points on the graph of the change in the specimen strain (Figure 2). The other parameters of the model are calculated by the formulas
\[ E = r = \frac{\Gamma(2\beta + 1) \varepsilon_3 t_1^\beta - \varepsilon_1 t_3^\beta}{\Gamma(\beta + 1) \varepsilon_3 t_1^{2\beta} - \varepsilon_1 t_3^{2\beta}} \]  \hspace{1cm} (7)

\[ \frac{\sigma_0}{\eta} = k = \frac{\varepsilon_1}{t_1^\beta (\Gamma(r + 1) - r \Gamma(2r + 1))} - r \frac{t_1^{2\beta}}{r^2 \varepsilon_0} \]  \hspace{1cm} (8)

**Figure 2.** Determination of creep model parameters

For parametric identification in the model of relaxation (4), it is necessary to know two points of stress dependence on time \((t_1, \sigma_1), (t_2, \sigma_2)\), and these points should be near the initial point \(\sigma(0) = \sigma_0\). Then the parameter \(\beta\) is calculated by the formula

\[ \beta_0 \approx \frac{\sigma_0 - \sigma_1}{t_1^{\beta_0} - t_2^{\beta_0}} \]  \hspace{1cm} (9)

Via this formula, the frozen soils and woven fiberglass parameters were calculated (for various temperatures), and the received model dependences showed good conformity with experimental ones [10].

2. **Statistical data processing and software**

Due to the simplicity of the methods of parametric identification considered above, they have an advantage over nomographic methods (comparing processed experimental graphs with reference nomograms [14]). However, measurement errors and other errors in obtaining experimental data can significantly increase the amount of errors of the parameters identification. For example to solve this problem iterative identification methods, in which the values obtained by formulas (6)-(8) acted as the initial approximation, were proposed in [15]. In this paper, we will consider statistical methods that are applicable when at least 10-15 experimental points are known. No doubt, the availability of additional information can significantly improve the accuracy of parametric identification.

The implementation of statistical techniques requires the development of special software. These programs were developed and now are registered in the Sectoral Fund of Electronic Resources “Science and Education”. The program “Parametric identification of models with fractional derivatives” is intended to solve the problem of determining the parameters of the stress-strain state of a material from experimentally gained data. Three equations simulating the stress-strain state of a material are available: viscoelasticity, creep, and relaxation. The appearance of the program is shown in Figure 3.
It is necessary to enter experimental data for the selected model (at least three experimentally known points of the stress-strain state of the material). Using them, the program will calculate all the parameters of the model, including the used fractional derivatives. In addition to determining the parameters, a model graph is displayed on the screen. The block diagram of the program is presented in Figure 3.

Data input could be done both ways, manually and automatically (from a file). For the correct work of the parametric identification algorithm, at least 4 experimental points are required. Figure 4 shows the structural diagram of the program.

If more than 10 points are input, then the program performs the procedure of generating a data sample: triples of experimental points are selected from the entire array, the model parameters are calculated for each triple. For the accuracy increase, 10% of the outer results on each side are discarded, the remaining results are averaged. For example, 120 triples will be processed for 100 points, 96 of which will form the final result.

If the discrepancies between the values of the calculated parameters are too noticeable, the program will issue a warning and offer to correct the initial data.

3. Testing statistical technique. Results and conclusion

Using the described statistical technique, the parameters for five specimens of polyester films were calculated. The exact values of the identifiable parameters were known in advance for each of them [16], which made it possible to evaluate the effectiveness of the technique. The results are presented in the table 1.

Table 1. Results of parametric identification of samples of polyester films

| Material                  | The number of experimental points | The calculated value of the parameter ($\beta$) | Exact parameter value |
|---------------------------|----------------------------------|-----------------------------------------------|-----------------------|
| Tetrachlordian            | 10                               | 0.62±0.015                                    | 0.63                  |
| Phenolphthalein           | 15                               | 0.41±0.01                                     | 0.42                  |
| 1,1-dichloro-2,2-diethylene | 12                               | 0.49±0.012                                    | 0.49                  |
| Diane                     | 10                               | 0.82±0.015                                    | 0.81                  |
| Dioxidiphenylsulfone      | 15                               | 0.45±0.01                                     | 0.46                  |
Figure 4. Block diagram of the program “Parametric identification of models with fractional derivatives”
Comparison of the identification results for different sample objects enables to increase the accuracy of parameter estimates and determine the variance (calculation error) up to 30%. The program can be used for modelling, engineering computations, predictions of real materials performance (polymers, concrete, soil, glass, enamel, etc.)

References

[1] Gorenflo R, Mainardi F, 1997, Fractals and Fractional Calculus in Continuum Mechanics, CISM Courses and Lectures, 378, Springer, Wien, 223–276.
[2] Gement A 1938 Philosophical Magazine 25 540-549.
[3] Gement A 1936 J. Appl. Phys. 7 311–317.
[4] Bagley R L, Torvik P J 1983 J. Rheolog., 27 №3 201 – 203.
[5] Bagley R L, Torvik P J 1983 AIAA Journal 21 №5 741 – 748.
[6] Erokhin S V, Gachaev A M 2011 Structural Mechanics of Engineering Constructions and Buildings 1 36-39.
[7] Kilbas A A, Srivastava H M, Trujillo J J 2006 Theory and applications of fractional differential equations. Amsterdam: Elsevier.
[8] Ogorodnikov E N, Radchenko V P, Ungarova L G 2016 Journal of Samara State Technical University. Ser. Physical and Mathematical Sciences 20 1 167-194.
[9] Erokhin S V 2014 Structural Mechanics of Engineering Constructions and Buildings 6 35-39.
[10] Erokhin S V, Semenov S V, Roshka O O 2017 Scientific and technical Volga region bulletin. 6 190-192.
[11] Aleroev T, Erokhin S, Kekharsaeva E 2018 IOP Conf. Series: Materials Science and Engineering 365 (2018) 032004.
[12] Aleroev T, Erokhin S 2019 International Journal of Modeling, Simulation, and Scientific Computing 10 01 1941002 (2019).
[13] Erokhin S V, Aleroev T S, Frishter L Iu, Kolesnichenko A V 2015 International Journal for Computational Civil and Structural Engineering 11 3 82-85.
[14] Man'kovskii V A, Sapunov V T 2000 Factory laboratory. Material diagnostics 66 3 45-50.
[15] Ogorodnikov E N, Radchenko V P, Ungarova L G 2018 PNRPU Mechanics Bulletin 2 147-161.
[16] Kekharsaeva E R, Aleroev T S 2001 Plastic materials 3 35.