The Unique Pure Gaussian State Determined by
the Partial Saturation of the Uncertainty Relations
of a Mixed Gaussian State

Maurice A. de Gosson
University of Vienna
Faculty of Mathematics-NuHAG
Nordbergstr. 15, 1090 Vienna

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Abstract

Let \( \rho \) the density matrix of a mixed Gaussian state. Assuming
that one of the Robertson–Schrödinger uncertainty inequalities is sat-
urated by \( \rho \), e.g. \((\Delta^\rho X_1)^2(\Delta^\rho P_1)^2 = \Delta^\rho(X_1, P_1)^2 + \frac{1}{2} \hbar^2\), we show that
there exists a unique pure Gaussian state whose Wigner distribution is
dominated by that of \( \rho \) and having the same variances and covariance
\( \Delta^\rho X_1, \Delta^\rho P_1, \) and \( \Delta^\rho(X_1, P_1) \) as \( \rho \). This property can be viewed as
an analytic version of Gromov’s non-squeezing theorem in the linear
case, which implies that the intersection of a symplectic ball by a single
plane of conjugate coordinates determines the radius of this ball.

1 Statement of Results

Consider a mixed quantum state, identified with its density matrix \( \rho \), and let

\[ W_\rho(x,p) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p y} \langle x + \frac{1}{2} y | \rho | x - \frac{1}{2} y \rangle dy \]

be its Wigner function. Viewing the latter as a quasi-probability distribution
we assume that it satisfies

\[ \int_{\mathbb{R}^{2n}} (1 + |z|^2) |W_\rho(z)| dz < \infty \]  (1)
we have set $z = (x, p)$ so that the first and second moments of $W_\rho$ exist.

The covariance matrix (CM) of $\rho$ is defined by

$$
\Sigma_\rho = \int_{\mathbb{R}^{2n}} (z - \langle z \rangle)(z - \langle z \rangle)^T W_\rho(z) dz
$$

where $\langle z \rangle = \text{Tr}(z\rho)$ is the mean value vector. It will be convenient to write

$$
\Sigma_\rho = \begin{pmatrix}
\Delta (X, X) & \Delta (X, P) \\
\Delta (X, P) & \Delta (P, P)
\end{pmatrix}
$$

where $\Delta (X, X) = (\Delta (X_i, X_j))_{1 \leq i, j \leq n}$, $\Delta (X, P) = (\Delta (X_i, P_j))_{1 \leq i, j \leq n}$, $\Delta (P, P) = (\Delta (P_i, P_j))_{1 \leq i, j \leq n}$ where $\Delta (X_i, X_j)$, etc., are the covariances.

Setting $\Delta (X_1) = \Delta (X_1, X_1)$ and $\Delta (P_1) = \Delta (P_1, P_1)$ the Robertson–Schrödinger (RS) inequalities

$$
(\Delta (X_1))^2 (\Delta (P_1))^2 \geq \Delta (X_1, P_1)^2 + \frac{1}{4} \hbar^2
$$

hold for $j = 1, ..., n$. We will say that these inequalities are partially saturated if at least one (but not all) of them are equalities.

We now assume that the state $\rho$ is Gaussian; this means that

$$
W_\rho(z) = \left( \frac{1}{2\pi} \right)^{n/2} (\det \Sigma)^{-1/2} \exp \left[ -\frac{1}{2} (z - \langle z \rangle)^T \Sigma^{-1} (z - \langle z \rangle) \right]
$$

(5)

(thus $\Sigma$ is assumed to be invertible, but this is no restriction: see section 2).

We will show that:

**Theorem 1** Suppose that anyone of the (RS) inequalities (4) is saturated, for instance

$$
(\Delta (X_1))^2 (\Delta (P_1))^2 = \Delta (X_1, P_1)^2 + \frac{1}{4} \hbar^2.
$$

Then:

(i) There exists a unique pure Gaussian state $\psi$ such that

$$
\Delta \psi X_1 = \Delta \rho X_1, \Delta \psi P_1 = \Delta \rho P_1, \Delta \psi (X_1, P_1) = \Delta \rho (X_1, P_1)
$$

(ii) That state $\psi$ is the only Gaussian whose Wigner function satisfies the inequality.

$$
W_\psi(x, p) \leq W_\rho(x, p).
$$

(7)
We emphasize that no assumptions are made on the other variances or covariances of the state $\rho$: they may for instance unknown.

One of the crucial steps in the proof is of a topological nature, and is deeply related to Gromov’s [16] symplectic non-squeezing theorem (nicknamed by V.I. Arnol’d [3] the “principle of the symplectic camel”; see [9] for a discussion of that terminology). Let us now call “symplectic ball” the image $S(B_R)$ of a phase space ball $B_R$ with radius $R$ by a linear (or affine) symplectic transformation $S$; the number $R$ is the radius of the symplectic ball. Since symplectic transformations are volume preserving, the volume of $S(B_R)$ is the same as that of the ball $B_R$. More surprising is the following rather unexpected property: the section of the ellipsoid $S(B_R)$ by any plane of conjugate coordinates $x_j, p_j$ passing through its center is always $\pi R^2$. Thus, a single “tomography” of a symplectic ball unambiguously determines its radius! This is counterintuitive, because one would expect that the area of the section of an ellipsoid by different planes yields different results. It turns out that this property (of which we give a proof in the Appendix) is equivalent to the linear version of Gromov’s theorem (see [15] for a review of Gromov’s theory; also [23] for various developments). Its relation with Theorem 1 comes from the fact that the Wigner transform of a Gaussian is a Gaussian of a very special type: its covariance ellipsoid is a symplectic ball, and the state is thus entirely determined by a section of this ellipsoid by a single plane of conjugate variables.

This paper has two forerunners [5] and [6] where we showed that it was possible to associate a unique pure Gaussian state to a mixed state satisfying a certain topological condition related to the uncertainty principle, see (12) in section 2.

Notation. We will use the following notation and terminology: The generic variable of phase space $\mathbb{R}^{2n}$ is $z = (x, p)$ with $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$; in calculations $x, p, z$ will always be viewed as column vectors. $\text{Sp}(2n, \mathbb{R})$ is the symplectic group; it consists of all real $2n \times 2n$ matrices $S$ such that $S^T J S = J$ (or, equivalently, $SJS^T = J$) where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We use Dirac’s notation $\hbar$ for $\hbar/2\pi$ ($\hbar$ Planck’s constant).

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2 Geometry and Uncertainty

In his seminal paper “Geometry and Uncertainty” [21] F. Narcowich (also see [22, 26]) proves, among other things, the following deep result: a real symmetric matrix $\Sigma$ is the covariance matrix of a quantum state if and only if

$$\Sigma + \frac{i\hbar}{2} J \text{ is positive semi-definite}$$

(abbreviated as $\Sigma + \frac{i\hbar}{2} J \succeq 0$). Condition (8) implies ([21], Lemma 2.3) that $\Sigma$ is fact definite positive, and hence invertible. It will be convenient to use the auxiliary matrix

$$M = \frac{\hbar}{2} \Sigma^{-1}$$

in which case (8) becomes

$$M^{-1} + iJ \succeq 0.$$  

(10)

Narcowich’s result seems to have become “folk wisdom” (and is therefore not very much cited); its proof uses arguments from hard harmonic analysis (a symplectic version of Bochner’s theorem on positive measures [17, 19, 20]) and is often stated (most of the time without proof) in the quantum-optical literature; see for instance the references in [1]. A caveat: as we have shown in [13] there exist self-adjoint operators with trace one whose covariance matrix satisfies condition (5), but which are not non-negative, and hence do not represent a quantum state. However, given a covariance matrix $\Sigma$ satisfying (5) one can always construct a quantum state with covariance matrix $\Sigma$, namely the Gaussian state $\rho$ whose Wigner transform is

$$W_{\rho}(z) = \left( \frac{1}{\pi\hbar} \right)^n (\det M) \exp \left( -\frac{1}{\hbar} z^T M z \right).$$

In [7, 8, 9, 15, 10] we have shown that condition (5) is equivalent to a topological statement involving the symplectic capacity of the covariance ellipsoid

$$\Omega = \{ z : \frac{1}{2} z^T \Sigma^{-1} z \leq 1 \} = \{ z : z^T M z \leq \hbar \}.$$  

(11)

In fact, the algebraic condition $M^{-1} + iJ \succeq 0$. is equivalent to the following property:

The symplectic capacity $c(\Omega)$ of the covariance ellipsoid is $\geq \frac{1}{2}\hbar$.  

(12)

The symplectic capacity $c(\Omega)$ is defined as follows: if $n = 1$ it is just the area of the ellipse $\Omega$; in higher dimensions it is the supremum of all numbers
$\pi R^2$ such that a symplectic ball $S(B_R)$ is contained in $\Omega$. In [12] we have called a symplectic ball with radius $\sqrt{\hbar}$ a “quantum blob”.

Both conditions (8), (12) are, for a given quantum system, equivalent to the Robertson–Schrödinger (RS) inequalities

\[
(\Delta^\rho X_j)^2(\Delta^\rho P_j)^2 \geq \Delta^\rho (X_j, P_j)^2 + \frac{1}{4} \hbar^2, \quad 1 \leq j \leq n.
\]  

(13)

Using either (8) or (12) it is easy to show that the RS inequalities are covariant (i.e. they retain their form) under linear or affine symplectic transformations. If we set $(X', P') = S(X, P)$ where $S$ is a symplectic matrix, then

\[
(\Delta^\rho X'_j)^2(\Delta^\rho P'_j)^2 \geq \Delta^\rho (X'_j, P'_j)^2 + \frac{1}{4} \hbar^2, \quad 1 \leq j \leq n
\]

(14) since the new covariance matrix is $\Sigma' = S\Sigma S^T$ and

\[
\Sigma' + \frac{i\hbar}{2} J = S \left( \Sigma + \frac{i\hbar}{2} J \right) S^T \geq 0
\]

because $SJ\Sigma S^T J = J$ as $S$ is symplectic.

The key to the proof of the equivalences $(8) \iff (12) \iff (13)$ is the following well-known symplectic diagonalization result (Williamson’s [25] theorem, of which [24] gives an elementary proof). Let $M$ be a (real) symmetric positive-definite matrix of size $2n$. There exists $S \in \text{Sp}(2n, \mathbb{R})$ and positive numbers $\lambda_1, \ldots, \lambda_n$ such that

\[
S^T MS = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).
\]

(15)

The numbers $\lambda_1, \ldots, \lambda_n$ are called the symplectic eigenvalues of $M$ (or sometimes Williamson invariants); they are written in decreasing order: $\lambda_1 \geq \cdots \geq \lambda_n$ and the array

\[
\text{Spec}_\sigma(M) = (\lambda_1, \ldots, \lambda_n)
\]

(16)

is the symplectic spectrum of $M$. The symplectic eigenvalues are calculated as follows: consider the product $JM$ of $M$ by the standard symplectic matrix and let $M^{1/2}$ be the positive square root of $M$; the matrices $M^{1/2}JM^{1/2}$ and $JM$ are equivalent, and hence have the same eigenvalues. Since $M^{1/2}JM^{1/2}$ is antisymmetric (because $J$ is), these eigenvalues occur in pairs $(i\lambda_j, -i\lambda_j)$, $\lambda_j > 0$, and the $\lambda_j$ are precisely the symplectic eigenvalues of $M$.

The symplectic spectrum has the following straightforward properties: if $c > 0$

\[
\text{Spec}_\sigma(cM) = (c\lambda_1, \ldots, c\lambda_n)
\]

(17)
\[ \text{Spec}_\sigma(M^{-1}) = (\lambda_n^{-1}, ..., \lambda_1^{-1}). \]  

(18)

The relation between symplectic spectrum and symplectic capacity is essential: let \( M \) be a symmetric positive definite matrix and consider the phase space ellipsoid \( \Omega : z^TMz \leq \hbar \). We have

\[ c(\Omega) = \frac{\pi \hbar}{\lambda_1} \text{ if Spec}_\sigma(M) = (\lambda_1, ..., \lambda_n). \]  

(19)

3 The Wigner function of a Gaussian

We will consider (normalized) Gaussians of the type

\[ \psi_{X,Y}(x) = \left(\frac{\pi}{\hbar}\right)^{-n/4}(\det X)^{1/4}e^{-\frac{1}{2\hbar}x^T(X+iy)x} \]  

(20)

where \( X \) and \( Y \) are real symmetric \( n \times n \) matrices, \( X \) positive definite.

We will need the two following properties of the Wigner function

\[ W(\hat{S}\psi)(z) = W(\psi)(z). \]  

(22)

Recall that the metaplectic operators are defined as follows: the symplectic group \( \text{Sp}(2n, \mathbb{R}) \) has a covering group of order two, the metaplectic group \( \text{Mp}(2n, \mathbb{R}) \). That group consists of unitary operators on \( L^2(\mathbb{R}^n) \), and is generated by the following elementary unitary transformations:

- The modified Fourier transform
  \[ F\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}x^TP^Txx'}\psi(x')dx' \]
  (with the convention \( i^{1/2} = e^{i\pi/4} \));

- "Chirps"
  \[ V_P\psi(x) = \exp\left(-\frac{i}{2\hbar}x^TPx\right), \quad P = P^T; \]

(21)
• Dilations

\[ M_L \psi(x) = \sqrt{\det L} \psi(Lx) \quad \text{det } L \neq 0 \]

where \( \sqrt{\det L} = i^m \sqrt{|\det L|} \), \( m = 0 \) or \( 2 \) if \( \text{det } L > 0 \) and \( m = 1 \) or \( 3 \) if \( \text{det } L < 0 \).

For a detailed study of the metaplectic group \( \text{Mp}(2n, \mathbb{R}) \) see [7].

**Transformation of Gaussians.** The Wigner transform of a Gaussian is itself a Gaussian. In fact

\[ W_{\psi_{X,Y}}(z) = (\pi \hbar)^{-n} e^{-\frac{1}{2} z^T G z} \quad (23) \]

where \( G \) is the real \( 2n \times 2n \) matrix

\[ G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1}X^{-1}Y \\ X^{-1}Y & X^{-1} \end{pmatrix} \quad (24) \]

It turns out that the matrix \( G \) defined in the formula above is both positive-definite and symplectic: we have \( G = S^T S \) where

\[ S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \quad (25) \]

and \( S \) is obviously in \( \text{Sp}(2n, \mathbb{R}) \). The argument can be reversed: given a positive-definite symplectic matrix \( G \), we can always find \( S \in \text{Sp}(2n, \mathbb{R}) \) such that \( G = S^T S \) where \( S \) has the form (25); this determines \( X \) and \( Y \) and hence a Gaussian state \( \psi_{X,Y} \) satisfying (23).

### 4 Proof of Theorem 1

It is sufficient to assume that \( \rho \) is centered at the origin, that is \( \langle z \rangle = 0 \).

**First step.** We begin by noting that the saturation of one of the RS inequalities, for instance

\[ (\Delta^p X_1)^2 (\Delta^p P_1)^2 = \Delta^p (X_1, P_1)^2 + \frac{1}{4} \hbar^2, \quad (26) \]

implies that \( c(\Omega) = \frac{1}{2} \hbar \). Let us prove this by *reductio ad absurdum*. Suppose that \( c(\Omega) > \frac{1}{2} \hbar \) and let \( D = S^T M S \) be a symplectic diagonalization \((15)\) of \( M = \frac{\hbar}{2} \Sigma^{-} \). Since \( c(\Omega) \) is a symplectic invariant (i.e. \( c(S(\Omega)) = c(\Omega) \) for every \( S \in \text{Sp}(2n, \mathbb{R}) \)), our assumption can be rewritten \( c(\Delta) > \frac{1}{2} \hbar \) where \( \Delta \) is the ellipsoid defined by \( z^T D z \leq \hbar \). In view of formula \((19)\) we have \( c(\Delta) = \pi \hbar / \lambda_1 \), recalling that the \( \lambda_j \) form a decreasing sequence.
The assumption \( c(\Delta) > \frac{1}{2} \hbar \) is thus equivalent to \( \lambda_1 < 1 \) and hence to \( \lambda_j < 1 \) for all \( j = 1, ..., n \). Consider now the matrix
\[
D^{-1} + iJ = \begin{pmatrix} \Lambda^{-1} & I \\ -I & \Lambda^{-1} \end{pmatrix} \succeq 0.
\]
Its eigenvalues are the roots of the polynomial
\[
P(t) = \prod_{j=1}^{n} \left[ \left( \lambda_j^{-1} - t \right)^2 - 1 \right]
\]
and are thus the real numbers \( t_j = \lambda_j^{-1} \pm 1 \). Since \( \lambda_j < 1 \) for every \( j \) this means that \( t_j > 0 \) for every \( j \) and hence \( D^{-1} + iJ > 0 \); returning to the covariance matrix, it follows that we also have \( \Sigma + \frac{i \hbar}{2} J > 0 \). But this leads to a contradiction: in view of Sylvester’s criterion for definite positiveness the condition \( \Sigma + \frac{i \hbar}{2} J > 0 \) implies the following condition on the minors of \( \Sigma + \frac{i \hbar}{2} J \):
\[
\left| \frac{(\Delta^\rho X_j)^2}{\Delta^\rho(X_j, P_j)} - \frac{i \hbar}{2} \right| > 0
\]
that is
\[
(\Delta^\rho X_j)^2(\Delta^\rho P_j)^2 > \Delta^\rho(X_j, P_j)^2 + \frac{1}{4} \hbar^2
\]
for all \( j = 1, ..., n \).

**Second step.** Let us show that if \( \Omega \) is a phase space ellipsoid such that \( c(\Omega) = \frac{1}{2} \hbar \), then \( \Omega \) contains a unique quantum blob (i.e. a symplectic ball with radius \( \sqrt{\hbar} \)); we do not assume explicitly that \( \Omega \) is a covariance ellipsoid: the argument is quite general. We begin by noting that the condition \( c(\Omega) = \frac{1}{2} \hbar \) implies, by definition of a symplectic capacity, that \( \Omega \) contains the image by some \( S \in \text{Sp}(2n, \mathbb{R}) \) of a ball \( B_{\sqrt{\hbar}} \) (and no symplectic ball with larger radius). This symplectic ball can be explicitly constructed: assume that \( \Omega \) is given by \( z^T M z \leq \hbar, M > 0 \). In view of Williamson’s diagonalization theorem, there exists \( S \in \text{Sp}(2n, \mathbb{R}) \) such that
\[
S^T M S = D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = \text{Spec}_\sigma(M) = (\lambda_1, ..., \lambda_n).
\tag{27}
\]
(cf. [15]). The inequality \( z^T M z \leq \hbar \) is equivalent to
\[
\sum_{j=1}^{n} \lambda_j (x_j^2 + p_j^2) \leq \hbar, \quad (x', p') = S(x, p)
\tag{28}
\]
and we have \( c(\Omega) = \pi \hbar / \lambda_1 = \frac{1}{2} \hbar \) so that \( \lambda_1 = 1 \); it follows that the ellipsoid (28) contains \( B_{\sqrt{\hbar}} \), hence \( \Omega \) contains the symplectic ball \( S(B_{\sqrt{\hbar}}) \). There remains to prove the uniqueness of the a quantum blob contained in \( \Omega \). Using if necessary a phase space translation it is sufficient to consider the case where \( B_{\sqrt{\hbar}} : |z| \leq \hbar \) (in view of elementary geometric considerations, the largest symplectic ball contained in \( \Omega \) must be centered at the origin: see [7], §8.4). Let us assume that there exists \( S' \in \text{Sp}(2n, \mathbb{R}) \) such that \( S'(B_{\sqrt{\hbar}}) \) is also contained in the ellipsoid \( \Omega \); we are going to show that \( S'(B_{\sqrt{\hbar}}) = S(B_{\sqrt{\hbar}}) \), following the argument in de Gosson [5, 6, 15]. Conjugating if necessary \( S \) and \( S' \) with an adequately chosen symplectic matrix we may assume, using Williamson’s theorem, that the \( \lambda_j \) are the symplectic eigenvalues of the matrix of \( \Omega \). The condition \( c(\Omega) = \frac{1}{2} \hbar \) then means that \( \lambda_1 = 1 \) in view of formula (19). Let us show that \( U = S'S^{-1} \) belongs to the group of symplectic rotations \( \text{U}(n) = \text{Sp}(2n, \mathbb{R}) \cap \text{O}(2n, \mathbb{R}) \); (29) the claim will follow since then \( U(B_{\sqrt{\hbar}}) = B_{\sqrt{\hbar}} \) so that \( S(B_{\sqrt{\hbar}}) = S'(B_{\sqrt{\hbar}}) \). Clearly \( U \in \text{Sp}(2n, \mathbb{R}) \) so it suffices to show that \( U \) is in addition a rotation; for this it is sufficient to check that \( UJ = JU \). Set \( R = D^{1/2}UD^{-1/2} \) where \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \); we have \( U^T DU = D \) hence \( R^T R = I \) so that \( R \) is orthogonal. Let us prove that \( R \) is in addition symplectic. Since \( J \) commutes with every power of the diagonal matrix \( D \) we have, taking into account the relation \( JU = (U^T)^{-1}J \) (because \( U \) is symplectic):
\[
JR = D^{1/2}JUD^{-1/2} = D^{1/2}(U^T)^{-1}JD^{-1/2} \\
= D^{1/2}(U^T)^{-1}D^{-1/2}J = (R^T)^{-1}J
\]

hence \( R^T JR = J \) so that \( R \) is in \( \text{Sp}(2n, \mathbb{R}) \). Since \( R \) is also a rotation we have \( R \in \text{U}(n) \) and thus \( JR = RJ \). Since \( U = D^{-1/2}RD^{1/2} \) we have
\[
JU = JD^{-1/2}RD^{1/2} = D^{-1/2}JRD^{1/2} \\
= D^{-1/2}RD^{1/2}J = UJ
\]

proving our claim.

**Third step.** We now define a normalized Gaussian state \( \psi \) by specifying its Wigner transform:
\[
W\psi(z) = (\pi \hbar)^{-n} \exp \left[ -\frac{1}{\hbar} z^T (SS^T)^{-1} z \right]
\]
where $S \in \text{Sp}(2n, \mathbb{R})$ is defined as above, i.e. $S(B_{\sqrt{n}})$ is the largest quantum blob contained in $\Omega = \{ z : z^T M z \leq h \}$. We note that the choice of $S$ is irrelevant, because we have seen above that if $S(B_{\sqrt{n}}) = S'(B_{\sqrt{n}})$ then $S' = SU$ with $U \in U(n)$ so that $S'S'^T = SS^T$. Since $S(B_{\sqrt{n}}) \subset \Omega$ we have $z^T M z \leq z^T (SS^T)^{-1} z$ for all $z \in \mathbb{R}^{2n}$ and hence, taking definition (6) of $W_\rho(z)$ into account,

$$W_\psi(z) \leq W_\rho(z)$$

which is the inequality (7) in Theorem 1. That $\psi$ is the only Gaussian state satisfying this inequality follows at once from the uniqueness of the quantum blob $S(B_{\sqrt{n}})$. Let us finally show that the states $\psi$ and $\rho$ have the same covariances $(\Delta^\rho X_1)^2$, $(\Delta^\rho P_1)^2$, and $\Delta^\rho (X_1, P_1)$. The covariance matrix of $\psi$ is $\Sigma_\psi = \frac{\hbar}{2} SS^T$ and the inclusion $S(B_{\sqrt{n}}) \subset \Omega$ implies that we have $\Omega^* \subset S(B_{\sqrt{n}})^*$ where $\Omega^*$ and $S(B_{\sqrt{n}})^*$ are the dual ellipsoids of $\Omega$ and $S(B_{\sqrt{n}})$, obtained by using a Legendre transformation (for a review of the latter see [27]). Since the dual ellipsoid of an ellipsoid $\frac{1}{2} z^T \Sigma z \leq 1$ is $\frac{1}{2} z^T \Sigma^{-1} z \leq 1$ this implies that $S(B_{\sqrt{n}})^* = (S^T)^{-1} B_{2/\sqrt{n}}$, so the points of these dual ellipsoids satisfy

$$\frac{\hbar}{4} z^T SS^T z \leq \frac{1}{2} z^T \Sigma_\rho z \leq 1.$$  

Let us cut $S(B_{\sqrt{n}})^* = (S^T)^{-1} B_{2/\sqrt{n}}$ and $\Omega^* = \{ z : \frac{1}{2} z^T \Sigma_\rho z \leq 1 \}$ by the plane $P_{1,1}$ of coordinates $x_1, p_1$. This section consists of two ellipses: $P_{1,1} \cap (S^T)^{-1} B_{2/\sqrt{n}}$, which has area $4\pi/\hbar$, and

$$P_{1,1} \cap \Omega^* = \frac{1}{2} \begin{pmatrix} x_1 & p_1 \end{pmatrix} \begin{pmatrix} (\Delta^\rho X_1)^2 & \Delta^\rho (X_1, P_1) \\ \Delta^\rho (X_1, P_1) & (\Delta^\rho P_1)^2 \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} \leq 1;$$

taking the saturation of the first RS inequality (6) into account, $P_{1,1} \cap \Omega^*$ also has area

$$2\pi [(\Delta^\rho X_1)^2 (\Delta^\rho P_1)^2 - \Delta^\rho (X_1, P_1)^2]^{-1/2} = 4\pi/\hbar.$$ 

Since $P_{1,1} \cap \Omega^* \subset P_{1,1} \cap S(B_{\sqrt{n}})^*$ equality of areas implies that the ellipses $P_{1,1} \cap \Omega^*$ and $P_{1,1} \cap S(B_{\sqrt{n}})^*$ are identical, hence the covariances $(\Delta^\rho X_1)^2$, $(\Delta^\rho P_1)^2$, and $\Delta^\rho (X_1, P_1)$ are the same for both states $\psi$ and $\rho$. QED.

**Appendix: The Linear Gromov Theorem**

Let us prove the following linear version of Gromov’s non-squeezing theorem. We are following de Gosson [11], §5.1.2; we give two independent proofs.
Proposition 2 Let $S \in \text{Sp}(2n, \mathbb{R})$ and $B_R : |z| \leq R$ The intersection of $S(B_R)$ by a plane of conjugate coordinates $x_j, p_j$ is an ellipse with area $\pi R^2$.

First proof. It relies on the fact that the form $pdx = \sum_j p_j dx_j$ is a relative symplectic integral invariant, that is: if $\phi$ is a symplectomorphism of $\mathbb{R}^{2n}$ and $\gamma$ a loop in $\mathbb{R}^{2n}$, then
\[
\oint_{\gamma} pdx = \oint_{\phi(\gamma)} pdx \tag{32}
\]
(see for instance Arnol’d [2], §44, p.239). We claim that the ellipse $\Gamma_j = S(B_R) \cap P_j$, intersection of the ellipsoid $S(B_R)$ with any plane $P_j$ of conjugate coordinates $x_j, p_j$ has area $\pi R^2$; the proposition immediately follows from this property. Let $\gamma_j$ be the curve bounding the ellipse $\Gamma_j$ and orient it positively; the area it encloses is
\[
\text{Area}(\Gamma_j) = \oint_{\gamma_j} pdx = \oint_{S^{-1}(\gamma_j)} pdx = \pi R^2 \tag{33}
\]
(because $S^{-1}(\gamma_j)$ is a big circle of $B_R$); notice that the assumption that $P_j$ is a plane of conjugate coordinates $x_j, p_j$ is essential for (33) to hold, making the use of the formula (32) possible [more generally, the argument works when $P_j$ is replaced by any symplectic plane].

Second proof. With the same notation as above we note that the set
\[
S^{-1} [S(B_R) \cap P_j]
\]
is a big circle of $B_R$, and hence encloses a surface with area $\pi R^2$. Now, $P_j$ is a symplectic space when equipped with the two-form $\sigma_j = dp_j \wedge dx_j$ and the restriction of $S$ to $P_j$ is a linear symplectomorphism from $(P_j, \sigma_j)$ to the symplectic plane $S(P_j)$ equipped with the restriction of the symplectic form $\sigma$. Symplectomorphisms being volume (here: area) preserving, it follows that $S(B_R) \cap P_j$ also has area $\pi R^2$.

Remark. It would certainly be interesting to generalize the first proof to arbitrary symplectomorphisms. The difficulty comes from the following fact: the key to the proof in the linear case is the fact that we were able to derive the equality
\[
\int_{\gamma_R} p_j dx_j = \pi R^2
\]
by exploiting the fact the inverse image of the $x_j, p_j$ plane by $S$ was a plane cutting $B_R$ along a big circle, which thus encloses an area equal to
\[ \pi R^2. \] When one replaces the linear transformation \( S \) by a non-linear one, the inverse image of the \( x_j, p_j \) plane will not generally be a plane, but rather a 2-dimensional symplectic manifold. If the following property holds: The section of \( B_R \) by any 2-dimensional symplectic manifold containing the center of \( B_R \) has an area at least \( \pi R^2 \) then we would have, by the same argument as above

\[ \int_{\gamma_R} p_j dx_j \geq \pi R^2. \]

We do not know any proof of this property; nor do we know whether it is true!

**References**

[1] G. Adesso and F. Illuminati. Entanglement in continuous-variable systems: recent advances and current perspectives. J. Phys. A: Math. Theor. 40, 7821–7880 (2007)

[2] V.I. Arnold. Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics, 2nd edition, Springer-Verlag, 1989

[3] V.I. Arnold. First steps in symplectic topology. Uspekhi Mat. Nauk 41:6, 3–18 (1986); translation: Russian Math. Surveys 41:6, 1–21 (1986),

[4] M. de Gosson. Phase space quantization and the uncertainty principle. Phys. Lett. A 317, 365–369 (2003)

[5] M. de Gosson. The optimal pure Gaussian state canonically associated to a Gaussian quantum state. Phys. Lett. A 330, 161–167(2004)

[6] M. de Gosson. Cellules quantiques symplectiques et fonctions de Husimi–Wigner. Bull. Sci. Math. 129, 211–226 (2005)

[7] M. de Gosson. Symplectic Geometry and Quantum Mechanics, series “Operator Theory: Advances and Applications” Vol. 166, Birkhäuser, Basel (2006)

[8] M. de Gosson. Uncertainty Principle, Phase Space Ellipsoids and Weyl Calculus. Operator Theory: Advances and applications. Vol. 164, 121–132 (Birkhäuser Verlag Basel, 2006)
[9] M. de Gosson. The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? Found. Phys. 99, 194–214 (2009)

[10] M. de Gosson. On the use of minimum volume ellipsoids and symplectic capacities for studying classical uncertainties for joint position-momentum measurements. J. Stat. Mech. P11005 (2010)

[11] M. de Gosson. Symplectic Methods in Harmonic Analysis and in Mathematical Physics. Pseudo-Differential Operators. Theory and Applications 7, Birkhäuser/Springer Basel AG, Basel (2011)

[12] M. de Gosson. Quantum Blobs. To appear in Found. Phys. (2012). Preprint: arXiv:1106.5468v1 [quant-ph]

[13] M. de Gosson and F. Luef. Remarks on the fact that the uncertainty principle does not characterize the quantum state. Phys. Lett. A. 364, 453–457 (2007)

[14] M. de Gosson and F. Luef. Principe d’Incertitude et Positivité des Opérateurs à Trace ; Applications aux Opérateurs Densité. Ann. Inst. Henri Poincaré 9(2), 329–346 (2008)

[15] M. de Gosson and F. Luef. Symplectic capacities and the geometry of uncertainty: The irruption of symplectic topology in classical and quantum mechanics. Phys. Reps. 484(5), 131–179 (2009)

[16] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. Inventiones Mathematica 82, 307–347 (1985)

[17] D. Kastler. The C*-Algebras of a Free Boson Field, Commun. math. Phys. 1, 14–48 (1965)

[18] R.G. Littlejohn. The semiclassical evolution of wave packets, Phys. Reps. 138(4–5), 193–291 (1986)

[19] G. Loupias and S. Miracle-Sole. C*-Algèbres des systèmes canoniques, I, Commun. math. Phys. 2, 31–48 (1966)

[20] G. Loupias and S. Miracle-Sole. C*-Algèbres des systèmes canoniques, II, Ann. Inst. Henri Poincaré 6(1), 39–58 (1967)

[21] F.J. Narcowich. Geometry and uncertainty, Journal of Mathematical Physics, 31(2), 354–364 (1990)
[22] F.J. Narcowich and R.F. O’Connell. Necessary and sufficient conditions for a phase-space function to be a Wigner distribution, Phys. Rev. A 34(1), 1–6 (1986)

[23] L. Polterovich. The Geometry of the Group of Symplectic Diffeomorphisms. Lectures in Mathematics, Birkhäuser (2001)

[24] R. Simon, S. Chaturvedi, and V. Srinivasan. Congruences and canonical forms for a positive matrix: Application to the Schweinler–Wigner extremum principle. J. Math. Phys. 40, 3632–3642 (1999)

[25] J. Williamson. On the algebraic problem concerning the normal forms of linear dynamical systems, Amer. J. of Math. 58, 141–163 (1936)

[26] H.P. Yuen. Multimode two-photon coherent states, in International Symposium on Spacetime Symmetries, edited by Y.S. Kim and W.W. Zachary North-Holland, Amsterdam, 309-313 (1989)

[27] R.K.P. Zia, E.F. Redish, and S.R. McKay. Making sense of the Legendre transform. Am. J. Phys. 77(7), 614–622 (2009)