POLYNOMIAL EQUATIONS IN SUBGROUPS AND APPLICATIONS

SERGEI V. KONYAGIN, IGOR E. SHPARLINSKI, AND ILYA V. VYUGIN

Abstract. We obtain a new bound for the number of solutions to polynomial equations in cosets of multiplicative subgroups in finite fields, which generalises previous results of P. Corvaja and U. Zannier (2013). We also obtain a conditional improvement of recent results of J. Bourgain, A. Gamburd and P. Sarnak (2016) and S. V. Konyagin, S. V. Makarychev, I. E. Shparlinski and I. V. Vyugin (2019) on the structure of solutions to the reduction of the Markoff equation \( x^2 + y^2 + z^2 = 3xyz \) modulo a prime \( p \).

1. Introduction

1.1. Background and motivation. Bourgain, Gamburd and Sarnak [2, 3] have recently initiated the study of reductions modulo \( p \) of the set \( \mathcal{M} \) of Markoff triples \( (x, y, z) \in \mathbb{N}^3 \) which are positive integer solutions to the Diophantine equation

\[
(1.1) \quad x^2 + y^2 + z^2 = 3xyz, \quad (x, y, z) \in \mathbb{Z}^3.
\]

Simple computation shows that the map

\[
\mathcal{R}_1 : (x, y, z) \mapsto (3yz - x, y, z)
\]

and similarly defined maps \( \mathcal{R}_2, \mathcal{R}_3 \) (which are all involutions), send one Markoff triple to another. Due to the symmetry of (1.1), the set \( \mathcal{M} \) is also invariant under permutations \( \Pi \in S_3 \) of the components of \( (x, y, z) \in \mathcal{M} \). It is also easy to check that the transformations \( \mathcal{R}_i, i = 1, 2, 3 \) and permutations \( \Pi \) generate a group of transformations acting on \( \mathcal{M} \).

A celebrated result of Markoff [17, 18] asserts that all integer solutions to (1.1) can be generated from the solution \((1, 1, 1)\) by using sequences of the above transformations.

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This naturally leads to the notion of the functional graph on Markoff triples with the “root” \((1, 1, 1)\), and edges \((x_1, y_1, z_1) \to (x_2, y_2, z_2)\), provided that \((x_2, y_2, z_2) = T(x_1, y_1, z_1)\), where

\[
(1.2) \quad T = \{R_1, R_2, R_3\} \cup S_3.
\]

In this terminology the result of Markoff \([17, 18]\) asserts then this graph is connected.

Baragar [1, Section V.3] and, more recently, Bourgain, Gamburd and Sarnak \([2, 3]\) conjecture that this property is preserved modulo all sufficiently large primes and the set of non-zero solutions \(\mathcal{M}_p\) to (1.1) considered modulo \(p\). In particular, this means that \(\mathcal{M}_p\) can be obtained from the set of Markoff triples \(\mathcal{M}\) reduced modulo \(p\).

This conjecture, which we can also as write \(\mathcal{M}_p = \mathcal{M} \pmod{p}\), means that the functional graph \(\mathcal{X}_p\) associated with the transformation (1.2) remains connected.

Accordingly, if we define by \(\mathcal{C}_p \subseteq \mathcal{M}_p\) the set of the triples in the largest connected component of the above graph \(\mathcal{X}_p\), then we can state:

**Conjecture 1.1** (Baragar [1]; Bourgain, Gamburd and Sarnak [2, 3]).

*For every prime \(p\) we have \(\mathcal{C}_p = \mathcal{M}_p\).*

Bourgain, Gamburd and Sarnak \([2, 3]\) have obtained several major results towards Conjecture 1.1, see also \([4, 6, 7, 10]\). For example, by \([2, \text{Theorem 1}]\) we have

\[
(1.3) \quad \# (\mathcal{M}_p \setminus \mathcal{C}_p) = p^{o(1)}, \quad \text{as } p \to \infty,
\]

and also by \([2, \text{Theorem 2}]\) we know that Conjecture 1.1 holds for all but maybe at most \(X^{o(1)}\) primes \(p \leq X\) as \(X \to \infty\).

The bound (1.3) has been improved in \([15, \text{Theorem 1.2}]\) as

\[
(1.4) \quad \# (\mathcal{M}_p \setminus \mathcal{C}_p) \leq \exp \left( (\log p)^{2/3+o(1)} \right), \quad \text{as } p \to \infty
\]

Furthermore, Bourgain, Gamburd and Sarnak \([2, 3]\) have also proved that the size of any connected component of the graphs \(\mathcal{X}_p\) is at least

\[
(1.5) \quad \# \mathcal{X}_p \geq c (\log p)^{1/3},
\]

for some absolute constant \(c > 0\). In turn, the bound (1.5) has been improved in \([15, \text{Theorem 1.3}]\) as

\[
(1.6) \quad \# \mathcal{X}_p \geq c (\log p)^{7/9}.
\]

The improvements in (1.4) and (1.6) are based on a bound of Corvaja and Zannier \([8, \text{Corollary 2}]\), on the number of solutions to the equation

\[
P(u, v) = 0, \quad (u, v) \in \mathcal{G}_1 \times \mathcal{G}_2
\]
where $P$ is a bivariate absolutely irreducible polynomial over the finite field $\mathbb{F}_p$ of $p$ elements and $G_1, G_2 \subseteq \mathbb{F}_p$ are multiplicative groups in the algebraic closure of $\mathbb{F}_p$, see also [11, 13, 16, 19] for some related results.

Motivated by the above results and connections, here we

- derive a new bound on the number of solutions in subgroups to a systems of several polynomials which covers under a unified setting the results of [8, 16, 19];
- obtain an improvement of (1.4) under a very plausible conjecture on the number of solutions in subgroups of some particular equation over $\mathbb{F}_p^*$. 

1.2. New results. As before, for a prime $p$ we use $\overline{\mathbb{F}}_p$ to denote the algebraic closure of the finite field $\mathbb{F}_p$ of $p$ elements.

For a bivariate irreducible polynomial

$$P(X, Y) = \sum_{i+j \leq d} a_{ij} X^i Y^j \in \mathbb{F}_p[X, Y]$$

of total degree $\deg P \leq d$, we define $P^d(X, Y)$ as the homogeneous polynomial of degree $d^p = \min\{i + j : a_{ij} \neq 0\}$ given by

$$P^d(X, Y) = \sum_{i+j=d^p} a_{ij} X^i Y^j.$$ 

We also consider the set of polynomials $\mathcal{P}$:

$$\mathcal{P} = \{P(\lambda X, \mu Y) \mid \lambda, \mu \in \overline{\mathbb{F}}_p\}.$$ 

Define $g$ as the greatest common divisor of the following set of differences

$$g = \gcd\{i_1 + j_1 - i_2 - j_2 : a_{i_1,j_1,a_{i_2,j_2}} \neq 0\}.$$ 

Given a multiplicative subgroup $G \subseteq \overline{\mathbb{F}}_p$, we say that two polynomials $P, Q \in \overline{\mathbb{F}}_p[X, Y]$ are $G$-independent if there is no $(u, v) \in G^2$ and $\gamma \in \overline{\mathbb{F}}_p^*$ such that polynomials $P(X, Y)$ and $\gamma Q(uX, vY)$ coincide.

We now fix $h$ polynomials

$$P_k(X, Y) = P(\lambda_k X, \mu_k Y) \in \mathcal{P}, \quad k = 1, \ldots, h,$$

which are $G$-independent.

The following result generalises a series of previous estimates of a similar type, see [8, 11, 13, 16, 19] and references therein.
Theorem 1.2. Suppose that $P$ is irreducible,
\[ \deg_X P = m \quad \text{and} \quad \deg_Y P = n \]
and also that $P^*(X,Y)$ consists of at least two monomials. There exists a constant $c_0(m,n)$, depending only on $m$ and $n$, such that for any multiplicative subgroup $G \subseteq \mathbb{F}_p$ of order $t = \#G$ satisfying
\[ \frac{1}{2}p^{3/4}h^{-1/4} \geq t \geq \max\{h^2, c_0(m,n)\}, \]
and $G$-independent polynomials (1.10) we have
\[ \sum_{i=1}^{h} \# \{(u,v) \in G^2 : P_i(u,v) = 0\} < 12mn(m + n)gh^{2/3}t^{2/3}. \]

Our next result is conditional on the following:

Conjecture 1.3. There exist constants $\varepsilon_0 > 0$ and $A$ such that for any prime $p$, any subgroup $G \subseteq \mathbb{F}_p$ with $\#G \leq p^{\varepsilon_0}$, and any elements $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2} \in \mathbb{F}_p$ satisfying
\[ \alpha_{1,1} \neq 0, \quad \alpha_{1,2} \neq 0, \quad \alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1} \neq 0, \]
the equation
\[ \frac{\alpha_{1,1}u - \alpha_{1,2}}{\alpha_{2,1}u - \alpha_{2,2}} = v \]
has at most $A$ solutions in $u, v \in G$.

Remark 1.4. It is likely that the constant $A$ in Conjecture 1.3 cannot be taken less than 9, even for $G \subseteq \mathbb{F}_p$ rather than for $G \subseteq \mathbb{F}_p$, see some heuristic arguments in Section 6. It is possible that this is optimal and Conjecture 1.3 holds with $A = 9$. Also we have $\varepsilon_0 \leq 1/2$, see Section 6.

Remark 1.5. It is easy to see that using the bound (1.4) instead of (1.3) in the argument of the proof of Theorem 1.6 immediately allows us to relax the condition of Conjecture 1.3 to counting solutions in subgroups $G \subseteq \mathbb{F}_p^2$ of order $\#G \leq \exp \left( (\log p)^{2/3+\varepsilon_0} \right)$. However we believe Conjecture 1.3 holds as stated.

Theorem 1.6. If Conjecture 1.3 holds for some $\varepsilon_0$ and $A$, then for sufficiently large $p$ we have
\[ \#(M_p \setminus C_p) \leq (\log p)^B, \]
where $B = 16 \log A + c$ for an absolute constant $c$. 
2. Solutions to polynomial equations in subgroups of finite fields

2.1. Stepanov’s method. Consider a polynomial \( \Phi \in \mathbb{F}_p[X, Y, Z] \) such that

\[
\deg_X \Phi < A, \quad \deg_Y \Phi < B, \quad \deg_Z \Phi < C,
\]

that is,

\[
\Phi(X, Y, Z) = \sum_{0 \leq a < A} \sum_{0 \leq b < B} \sum_{0 \leq c < C} \omega_{a,b,c} X^a Y^b Z^c.
\]

We assume

\[ A < t \]

where \( t = \#\mathcal{G} \) is the order of the subgroup \( \mathcal{G} \subseteq \mathbb{F}_p^* \), and consider the polynomial

\[
\Psi(X, Y) = Y^t \Phi(X/Y, X^t, Y^t).
\]

Clearly

\[
\deg \Psi \leq t + t(B - 1) + t(C - 1) = (B + C - 1)t.
\]

We now fix some \( \mathcal{G} \)-independent polynomials (1.10) and define the sets

\[
(2.1) \quad \mathcal{F}_i = (\lambda_i^{-1} \mathcal{G} \times \mu_i^{-1} \mathcal{G}), \quad i = 1, \ldots, h, \quad \text{and} \quad \mathcal{E} = \bigcup_{i=1}^{h} \mathcal{F}_i.
\]

We also consider the locus of singularity

\[
\mathcal{M}_{\text{sing}} = \{ (X, Y) \mid XY = P(X, Y) = 0 \text{ or } \frac{\partial}{\partial Y} P(X, Y) = P(X, Y) = 0 \}.
\]

Lemma 2.1. Let \( P(X, Y) \) be an irreducible polynomial of bi-degree

\[(\deg_X P, \deg_Y P) = (m, n)\]

and let \( n \geq 1 \). Then for the cardinality of the set \( \mathcal{M}_{\text{sing}} \) the following holds:

\[
\#\mathcal{M}_{\text{sing}} \leq (m + n)^2.
\]

Proof. If the polynomial \( P(X, Y) \) is irreducible, then the polynomials \( P(X, Y) \) and \( \frac{\partial P}{\partial Y}(X, Y) \) are relatively prime. Thus the Bézout theorem yields the bound \( L \leq (m + n)(m + n - 1) \), where \( L \) is the number of roots of the system

\[
\frac{\partial}{\partial Y} P(X, Y) = P(X, Y) = 0.
\]
Actually, the number of \( X \) with \( P(X,0) = 0 \) is less than or equal to \( \deg_X P(X,Y) = m \), the number of pairs \( (0,Y) \) on the curve
\[
P(X,Y) = 0
\] (2.2)
where \( P \) is given by (1.7), is less than or equal to \( \deg_Y P(X,Y) = n \). The total numbers of such pairs is at most \( L + m + n \leq (m+n)^2 \). \( \square \)

Assume that the polynomial \( \Psi \) and the \( \mathcal{G} \)-independent polynomials (1.10) satisfy the following conditions:

- all pairs in the set
\[
\{(X,Y) \in \mathcal{E} \setminus \mathcal{M}_{\text{sing}} \mid P(X,Y) = 0\}
\]
are zeros of orders at least \( D \) of the function \( \Psi(X,Y) \) on the curve (2.2);
- the polynomials \( \Psi(X,Y) \) and \( P(X,Y) \) are relatively prime.

If these conditions are satisfied then the Bézout theorem gives us the upper bound \( D^{-1} \deg \Psi \deg P + \# \mathcal{M}_{\text{sing}} \) for the number of roots \((x,y)\) of the system
\[
\Psi(X,Y) = P(X,Y) = 0, \quad (X,Y) \in \mathcal{G}.
\]
Since the polynomials \( P_k \) are \( \mathcal{G} \)-independent, the sets \( \mathcal{F}_k \) are disjoint and also there is a one-to-one correspondence between the zeros:
\[
P_k(X,Y) = 0, \ (X,Y) \in \mathcal{G}^2,
\]
\[
\iff P(u,v) = 0, \ (u,v) = (\lambda_k^{-1}X,\mu_k^{-1}Y) \in \mathcal{F}_k.
\]
Therefore, we obtain the bound
\[
N_h \leq \frac{\deg \Psi \cdot \deg P}{D} + \# \mathcal{M}_{\text{sing}}
\leq \frac{(m+n)(B+C-1)t}{D} + \# \mathcal{M}_{\text{sing}}
\] (2.3)
on the total number of zeros of \( P_k \) in \( \mathcal{G}^2 \), \( k = 1, \ldots, h \):
\[
N_h = \sum_{k=1}^{h} \# \{(u,v) \in \mathcal{G}^2 : P_k(u,v) = 0\}.
\]

For completeness, we present proofs of several results from [16] which we use here as well.
2.2. Some divisibilities and non-divisibilities. We begin with some simple preparatory results on the divisibility of polynomials.

**Lemma 2.2.** Suppose that \( Q(X,Y) \in \mathbb{F}_p[X,Y] \) is an irreducible polynomial such that
\[
Q(X,Y) \mid \Psi(X,Y)
\]
and \( Q^\sharp(X,Y) \) consists of at least two monomials. Then
\[
Q^\sharp(X,Y)^{\lceil t/e \rceil} \mid \Psi^\sharp(X,Y),
\]
where \( Q^\sharp(X,Y) \) and \( \Psi^\sharp(X,Y) \) are defined as in (1.8) and \( e \) is defined as \( g \) in (1.9), with respect to \( Q(X,Y) \) instead of \( P(x,y) \).

**Proof.** Consider \( \rho \in G \) and substitute \( X = \rho \tilde{X} \) and \( Y = \rho \tilde{Y} \) in the polynomials \( Q(X,Y) \) and \( \Psi(X,Y) \). Then
\[
Q(X,Y) \mapsto Q_\rho(\tilde{X},\tilde{Y}) = Q(\rho \tilde{X}, \rho \tilde{Y}),
\]
and
\[
\Psi(X,Y) = \Psi(\rho \tilde{X}, \rho \tilde{Y})
\]
\[
= (\rho \tilde{Y})^t \Phi((\rho \tilde{X})/(\rho \tilde{Y}), (\rho \tilde{X})^t, (\rho \tilde{Y})^t) = \Psi(\tilde{X}, \tilde{Y}),
\]
because \( \rho^t = 1 \). Hence for any \( \rho \in G \) we have
\[
Q_\rho(X,Y) \mid \Psi(X,Y),
\]
and we also note that \( Q_\rho(X,Y) \) is irreducible.

Since \( Q^\sharp(X,Y) \) contains at least two monomials \( e \geq 1 \) is a correctly defined and there exist at least \( s = \lceil t/e \rceil \) elements \( \rho_1, \ldots, \rho_s \in G \) such that
\[
Q_{\rho_i}(X,Y)/Q_{\rho_j}(X,Y) \notin \mathbb{F}_p, \quad 1 \leq i < j \leq s.
\]
Obviously the polynomials \( Q_{\rho_1}(X,Y), \ldots, Q_{\rho_s}(X,Y) \) are pairwise relatively prime, because they are irreducible and satisfy (2.4). Polynomials \( Q_{\rho_i}^\sharp(X,Y) \) are homogeneous of degree \( d^\sharp \) and the following holds
\[
Q^\sharp(X,Y) = \rho_1^{-d^\sharp} Q_{\rho_1}^\sharp(X,Y) = \ldots = \rho_s^{-d^\sharp} Q_{\rho_s}^\sharp(X,Y).
\]
So, we have
\[
Q_{\rho_1}(X,Y) \cdot \ldots \cdot Q_{\rho_s}(X,Y) \mid \Psi(X,Y),
\]
consequently,
\[
Q_{\rho_1}^\sharp(X,Y) \cdot \ldots \cdot Q_{\rho_s}^\sharp(X,Y) \mid \Psi^\sharp(X,Y).
\]
Since
\[
Q_{\rho_1}^\sharp(X,Y) \cdot \ldots \cdot Q_{\rho_s}^\sharp(X,Y) = (\rho_1 \cdot \ldots \cdot \rho_s)^{d^\sharp} Q^\sharp(X,Y)^s
\]
we obtain the desired result. \( \Box \)
Lemma 2.3. Let $G(X,Y), H(X,Y) \in \mathbb{F}_p[X,Y]$ be two homogeneous polynomials. Also suppose that $G(X,Y)$ consists of at least two nonzero monomials, $\deg H < p$ and the number of monomials of the polynomial $H(X,Y)$ does not exceed $s$ for some positive integer $s < p$. Then

$$G(X,Y)^s \nmid H(X,Y).$$

Proof. Let us put $Y = 1$. If $G(X,Y)^s \mid H(X,Y)$ then $G(X,1)^s \mid H(X,1)$. The polynomial $G(X,1)$ has at least one nonzero root. It has been proved in [13, Lemma 6] that such a polynomial $H(X,1)$ cannot have a nonzero root of order $s$ and the result follows. $\square$

Lemma 2.4. If $AB < t/g$ and $\deg \Psi < p$ then for the polynomial $P(X,Y)$ given by (1.7) we have

$$P(X,Y) \nmid \Psi(X,Y).$$

2.3. Derivatives on some curves. There we study derivatives on the algebraic curve and define some special differential operators. Through this section we use

$$\frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \quad \text{and} \quad \frac{d}{dX}$$

for standard partial derivatives with respect to $X$ and $Y$ and for a derivative with respect to $X$ along the curve (2.2). In particular

$$(2.5) \quad \frac{d}{dX} = \frac{\partial}{\partial X} + \frac{dY}{dX} \frac{\partial}{\partial Y},$$

where by the implicit function theorem from the equation (2.2) we have

$$\frac{dY}{dX} = -\frac{\partial P}{\partial X}(X,Y) \frac{\partial P}{\partial Y}(X,Y).$$

We also define inductively

$$\frac{d^k}{dX^k} = \frac{d}{dX} \frac{d^{k-1}}{dX^{k-1}}$$

the $k$-th derivative on the curve (2.2).

Consider the polynomials $q_k(X,Y)$ and $r_k(X,Y)$, $k \in \mathbb{N}$, which are defined inductively as

$$q_1(X,Y) = -\frac{\partial}{\partial X} P(X,Y), \quad r_1(X,Y) = \frac{\partial}{\partial Y} P(X,Y),$$
and
\[ q_{k+1}(X, Y) = \partial q_k \frac{\partial P}{\partial Y} \left( \frac{\partial P}{\partial Y} \right)^2 - \partial q_k \frac{\partial P}{\partial Y} \frac{\partial P}{\partial X} \frac{\partial P}{\partial Y} - (2k - 1)q_k(X, Y) \frac{\partial^2 P}{\partial X \partial Y} \frac{\partial P}{\partial Y} + (2k - 1)q_k(X, Y) \frac{\partial^2 P}{\partial Y^2} \frac{\partial P}{\partial X}, \]
\[ r_{k+1}(X, Y) = r_k(X, Y) \left( \frac{\partial P}{\partial Y} \right)^2 = \left( \frac{\partial P}{\partial Y} \right)^{2k+1}. \]

(2.6)

We now show by induction that
\[ \frac{d^k}{dX^k} Y = \frac{q_k(X, Y)}{r_k(X, Y)}, \quad k \in \mathbb{N}. \]

(2.7)

The base of induction is
\[ \frac{d}{dX} Y = -\frac{\partial}{\partial X} P(X, Y) = \frac{q_1(X, Y)}{r_1(X, Y)}. \]

One can now easily verifies that assuming (2.7) and (2.5) we have
\[ \frac{d^{k+1}}{dX^{k+1}} Y = \frac{d}{dX} \frac{d^k}{dX^k} Y = \frac{d}{dX} \frac{q_k(X, Y)}{r_k(X, Y)} = \frac{q_{k+1}(X, Y)}{r_{k+1}(X, Y)}, \]
where \( q_{k+1} \) and \( r_{k+1} \) are given by (2.6), which concludes the induction and proves the formula (2.7).

The implicit function theorem gives us the derivatives \( \frac{d^{k+1}}{dX^{k+1}} Y \) at a point \((X, Y)\) on the algebraic curve \((2.2)\), if the denominator \( r_k(X, Y) \) is not equal to zero. Otherwise \( r_k(X, Y) = 0 \) if and only if the following system holds
\[ \frac{\partial}{\partial Y} P(X, Y) = P(X, Y) = 0. \]

Let us give the following estimates

**Lemma 2.5.** For all integers \( k \geq 1 \), the degrees of the polynomials \( q_k(X, Y) \) and \( r_k(X, Y) \) satisfy the bounds
\[
\deg_X q_k \leq (2k - 1)m - k, \quad \deg_Y q_k \leq (2k - 1)n - 2k + 2, \\
\deg_X r_k \leq (2k - 1)m, \quad \deg_Y r_k \leq (2k - 1)(n - 1).
\]

**Proof.** Direct calculations show that
\[
\deg_X q_1 \leq m - 1 \quad \text{and} \quad \deg_Y q_1 \leq n,
\]
and using (2.6) (with \( k - 1 \) instead of \( k \)) and examining the degree of each term, we obtain the inequalities

\[
\begin{align*}
\deg_X q_k & \leq \deg_X q_{k-1} + 2m - 1 \leq (2k - 1)m - k, \\
\deg_Y q_k & \leq \deg_Y q_{k-1} + 2n - 2 \leq (2k - 1)n - 2k + 2.
\end{align*}
\]

We now obtain the desire bounds on \( \deg_X q_k \) and \( \deg_Y q_k \) by induction.

For the polynomials \( r_k \) the statement is obvious. \( \square \)

**Lemma 2.6.** Let \( Q(X,Y) \in \mathbb{F}_p[X,Y] \) be a polynomial such that

\[
(2.8) \quad \deg_X Q(X,Y) \leq \mu, \quad \deg_Y Q(X,Y) \leq \nu,
\]

and \( P(X,Y) \in \mathbb{F}_p[X,Y] \) be a polynomial such that

\[
(2.9) \quad \deg_X P(X,Y) \leq m, \quad \deg_Y P(X,Y) \leq n.
\]

Then the divisibility condition

\[
P(X,Y) \mid Q(X,Y)
\]

on the coefficients of the polynomial \( Q(X,Y) \) is equivalent to a certain system of not more than \((\mu + \nu + 1)mn\) homogeneous linear algebraic equations in coefficients of \( Q(X,Y) \) as variables.

**Proof.** The dimension of the vector space \( \mathcal{L} \) of polynomials \( Q(X,Y) \) that satisfy (2.8) is equal to \((\mu + 1)(\nu + 1)\). Let us call the vector subspace of polynomials \( Q(X,Y) \) that satisfy (2.8) and (2.9) by \( \tilde{\mathcal{L}} \). Because \( Q(X,Y) = P(X,Y)R(X,Y) \) where the polynomial \( R(X,Y) \) is such that

\[
(2.10) \quad \deg_X R(X,Y) \leq \mu - m \quad \text{and} \quad \deg_Y R(X,Y) \leq \nu - n,
\]

then the vector space \( \tilde{\mathcal{L}} \) is isomorphic to the vector space of the coefficients of the polynomials \( R(x,y) \) satisfying (2.10). The dimension of the vector space \( \tilde{\mathcal{L}} \) is equal to

\[
\dim \tilde{\mathcal{L}} = (\mu - m + 1)(\nu - n + 1).
\]

It means that the subspace \( \tilde{\mathcal{L}} \) of the space \( \mathcal{L} \) is given by a system of

\[
(\mu + 1)(\nu + 1) - (\mu - m + 1)(\nu - n + 1) = \mu n + \nu m - mn + m + n + 1 \leq (\mu + \nu + 1)mn
\]

homogeneous linear algebraic equations. \( \square \)

As in [16], we now consider the differential operators:

\[
(2.11) \quad D_k = \left( \frac{\partial P}{\partial Y} \right)^{2k-1} X^k Y^k \frac{d^k}{dX^k}, \quad k \in \mathbb{N},
\]
where, as before, \( \frac{d^k}{dX^k} \) denotes the \( k \)-th derivative on the algebraic curve (2.2) with the local parameter \( X \). We note now that the derivative of a polynomial in two variables along a curve is a rational function. As one can see from the inductive formula for \( \frac{d^k}{dX^k} \), the result of applying any operator \( D_k \) to a polynomial in two variables is again a polynomial in two variables.

Consider non-negative integers \( a, b, c \) such that \( a < A, b < B, c < C \). From the formulas (2.7) for derivatives on the algebraic curve (2.2) we obtain by induction the following relations

\[
D_k \left( \frac{X}{Y} \right)^a X^{bl} Y^{(c+1)t} = R_{k,a,b,c}(X,Y) \left( \frac{X}{Y} \right)^a X^{bl} Y^{(c+1)t},
\]

\[
D_k \psi(X,Y) \big|_{X,Y \in \mathcal{F}_i} = R_{k,i}(X,Y) \big|_{X,Y \in \mathcal{F}_i},
\]

where \( \mathcal{F}_i \) from formula (2.1),

\[
R_{k,i}(X,Y) = \sum_{0 \leq a < A} \sum_{0 \leq b < B} \sum_{0 \leq c < C} \omega_{a,b,c} R_{k,a,b,c}(X,Y) \left( \frac{X}{Y} \right)^a \lambda_i^{-bt} \mu_i^{-(c+1)t}
\]

for some coefficients \( \omega_{a,b,c} \in \mathbb{F}_p, a < A, b < B, c < C \), and \( \lambda_i, \mu_i \) from (2.1).

We now define

\[
\tilde{R}_{k,i}(X,Y) = Y^{A-1} R_{k,i}(X,Y).
\]

**Lemma 2.7.** The rational functions \( R_{k,a,b,c}(X,Y) \) and \( \tilde{R}_{k,i}(X,Y) \), given by (2.12) and (2.14), are polynomials of degrees

\[
\text{deg}_X R_{k,a,b,c} \leq 4km, \quad \text{deg}_Y R_{k,a,b,c} \leq 4kn,
\]

and

\[
\text{deg}_X \tilde{R}_{k,i} \leq A + 4km, \quad \text{deg}_Y \tilde{R}_{k,i} \leq A + 4kn.
\]

**Proof.** We have

\[
\frac{d^k}{dX^k} X^{a+bl} Y^{(c+1)t} = \sum_{(\ell_1, \ldots, \ell_s)} C_{\ell_1, \ldots, \ell_s} X^{a+bl-k+s \sum_{i=1}^s \ell_i} Y^{(c+1)t-a-s} \left( \frac{d^{\ell_1} Y}{dX^{\ell_1}} \right) \cdots \left( \frac{d^{\ell_s} Y}{dX^{\ell_s}} \right),
\]

where \( (\ell_1, \ldots, \ell_s) \) runs through the all \( s \)-tuples of positive integers with \( \ell_1 + \ldots + \ell_s \leq k \), \( s = 0, \ldots, k \) and \( C_{\ell_1, \ldots, \ell_s} \) are some constant coefficients.
By the formula (2.15) and the form of the operator (2.11) we obtain that \( R_{k,a,b,c}(x,y) \) are polynomials and \( R_{k,i}(x,y) \) are rational functions. Actually, from the formulas (2.15) and (2.7) we easily obtain that the denominator of
\[
\frac{d^k}{dX^k} \left( \frac{X}{Y} \right)^a X^{bt} Y^{(c+1)t}
\]
dives \((\frac{\partial P}{\partial Y}(X,Y))^{2k-1}\). We obtain that \( R_{k,a,b,c}(X,Y) \) are polynomials. From the formula (2.13) we obtain that \( R_{k,i} \) is a rational function with denominator divided by \( Y^{A-1} \). Consequently, \( \tilde{R}_{k,i} \) are polynomials.

The result now follows from Lemma 2.5 and the formulas (2.11) and (2.12). 

2.4. Multiplicities points on some curves.

Lemma 2.8. If \( P(X,Y) \mid \Psi(X,Y) \) and \( P(X,Y) \mid D_j\Psi(X,Y) \), \( j = 1, \ldots, k-1 \), then at least one of the following alternatives holds:

- either \((x,y)\) is a root of order at least \( k \) of \( \Psi(X,Y) \) on the algebraic curve (2.2);
- or \((x,y) \in M_{\text{sing}}\).

Proof. If \( D_j\Psi(X,Y) \) vanishes on the curve \( P(X,Y) = 0 \), then either
\[
\frac{d^j}{dX^j} \Psi(x,y) = 0,
\]
where, as before, \( \frac{d^j}{dX^j} \) is \( j \)-th derivative on the algebraic curve (2.2) with the local parameter \( X \), or
\[
xy = 0,
\]
or
\[
\frac{\partial P}{\partial Y}(x,y) = 0,
\]
on the curve (2.2).

If we have (2.16) for \( j = 1, \ldots, k-1 \) and also \( \Psi(x,y) = 0 \) then the pair \((x,y)\) satisfies the first case of conditions of Lemma 2.8.

If we have (2.17) or (2.18) on the curve (2.2) then the pair \((x,y)\) satisfies the second case of conditions of Lemma 2.8. \(\square\)
3. Small divisors of integers

3.1. Smooth numbers. As usual, we say that a positive integer is $y$-smooth if it is composed of prime numbers up to $y$. Then we denote by $\psi(x, y)$ the number of $y$-smooth positive integers that are up to $x$. Among a larger variety of bounds and asymptotic formulas for $\psi(x, y)$, see [12, 14, 21], the most convenient for our applications bound is given by [21, Theorem 5.1].

**Lemma 3.1.** There is an absolute constant $c_0$ such that for any fixed real positive $x \geq y \geq 2$ we have

$$\psi(x, y) \leq c_0 e^{-u/2} x$$

where

$$u = \frac{\log x}{\log y}.$$  

3.2. Number of small divisors of integers. For a real $z$ and an integer $n$ we use $\tau_z(n)$ to denote the number of integer positive divisors $d \mid n$ with $d \leq z$. We present a bound on $\tau_z(n)$ for small values of $z$ (which we put in a slightly more general form than we need for our applications).

**Lemma 3.2.** There is an absolute constant $C_0$ such that for any fixed real positive $\varepsilon < 1$ there is $n(\varepsilon)$ such that if $n \geq n(\varepsilon, b)$ and $z \geq (\log n)^{2\log(1/\varepsilon)}$ then

$$\tau_z(n) \leq C_0 \varepsilon z.$$  

**Proof.** Let $s$ be the number of all distinct prime divisors of $n$ and let $p_1, \ldots, p_s$ be the first $s$ primes. We note that

(3.1) $$\tau_z(n) \leq \psi(z, p_s).$$

By the prime number theorem we have $n \geq p_1 \ldots p_s = \exp(p_s + o(p_s))$ and thus

(3.2) $$p_s \ll \log n \leq z^{1/b}.$$  

where $b = 2 \log(1/\varepsilon)$. Combining Lemma 3.1 with (3.1) and (3.2) we see that

$$\tau_z(n) \leq \psi(z, z^{1/b} + o(1)) \leq c_0 e^{-b/2 + o(1)} z = (c_0 + o(1)) e^{-b/2} z \leq C_0 \varepsilon z$$

for any $C_0 > c_0$ (where $c_0$ is as in Lemma 3.1), provided that $n$ and thus $z$ are large enough. □
4. Proof of Theorem 1.2

4.1. Preliminary estimates. We define the following parameters:

\[ A = \left\lfloor \frac{t^{2/3}}{gh^{1/3}} \right\rfloor, \quad B = C = \left\lfloor \frac{h^{1/3}t^{1/3}}{2} \right\rfloor, \quad D = \left\lfloor \frac{t^{2/3}}{4gh^{1/3}mn} \right\rfloor \]

The exact values of \( A, B, C \) and \( D \) play no role until the optimization step at the very end of the proof. However, it is important to note that their choice ensures (4.4) and (4.5) below.

If \( P_i(x, y) = 0 \) for at least one \( i = 1, \ldots, h \), then

\[ D_k \Psi(x, y) = 0, \quad (x, y) \in \bigcup_{i=1}^h \mathcal{F}_i, \]

with the operators (2.11), where the sets \( \mathcal{F}_i \) are as in (2.1). The condition (4.1) is given by a system of linear homogeneous algebraic equations in the variables \( \omega_{a,b,c} \). The number of equations can be calculated by means of Lemmas 2.6 and 2.7. To satisfy the condition (4.1) for some \( k \) we have to make sure that the polynomials \( \tilde{R}_{k,i}(X, Y), i = 1, \ldots, h \), given by (2.14), vanish identically on the curve (2.2). The bi-degree of \( \tilde{R}_{k,i}(X, Y) \) is given by Lemma 2.7:

\[ \deg_X \tilde{R}_{k,i} \leq A + 4km, \quad \deg_Y \tilde{R}_{k,i} \leq A + 4kn. \]

The number of equations on the coefficients that give us the vanishing of polynomial \( \tilde{R}_{k,i}(X, Y) \) on the curve (2.2) is given by Lemma 2.6 and is equal to \((\mu + \nu + 1)mn\), where \( \mu, \nu \) are as Lemma 2.6 and

\[ \mu \leq A + 4km, \quad \nu \leq A + 4kn. \]

Finally, the condition (4.1) for some \( k \) is given by \( h(\mu + \nu + 1)mn \leq mnh(2A + 4k(m + n)) \) linear algebraic homogeneous equations. Consequently, the condition (4.1) for all \( k = 0, \ldots, D - 1 \) is given by the system of

\[ L = hmn \sum_{k=0}^{D-1} (4k(m + n) + 2A + 1) \]

linear algebraic homogeneous equations in variables \( \omega_{a,b,c} \). Now it is easy to see that

\[ L = h ((2A + 1)Dmn + 2nm(m + n)D(D - 1)) \leq 2hADmn + 2hmn(m + n)D^2 = 2hmn(AD + (m + n)D^2). \]
4.2. **Optimization of parameters.** The system has a nonzero solution if the number of equations is less than to the number of variables, in particular, if

\[ 2hmn(AD + (m + n)D^2) < ABC, \]

as we have \( ABC \) variables. It is easy to get an upper bound for the left hand side of (4.2). For sufficiently large \( t > c_0(m,n) \), where \( c_0(m,n) \) is some constant depending only on \( m \) and \( n \), we have

\[
2hmn(AD + (m + n)D^2) < 2hmn \left( \frac{h^{-1/3}t^{2/3}}{g} + \frac{h^{-1/3}t^{2/3}}{4mng} + (m + n) \frac{h^{-2/3}t^{4/3}}{16m^2n^2g^2} \right) < \frac{3h^{1/3}t^{4/3}}{4g^2}.
\]

Assuming that \( c_0(m,n) \) is large enough, we obtain

\[ ABC = \left\lfloor \frac{h^{-1/3}t^{2/3}}{g} \right\rfloor \left\lfloor \frac{h^{1/3}t^{1/3}}{g} \right\rfloor^2 > \frac{3h^{1/3}t^{4/3}}{4g^2}, \]

which together with (4.3) implies (4.2).

It is clear that

\[ gAB \leq t. \]

We also require that the degree of the polynomial \( \Psi(x,y) \) should be less than \( p \),

\[ \deg \Psi(x,y) \leq (B - 1)t + Ct < p. \]

Actually, the inequality \((B - 1)t + Ct < 2h^{1/3}t^{4/3} < p\) is satisfied because \( t < \frac{1}{2}p^{3/4}h^{-1/4} \).

Finally, recalling Lemmas 2.2, 2.3 and 2.4 and also the irreducibility of the polynomial \( P(x,y) \), we see that \( P_k(X,Y) \) and \( \Psi(X,Y) \) are co-prime. Hence, by Lemmas 2.1 and 2.8 and the inequality (2.3) we obtain that \( N_h \) satisfies the inequality

\[
N_h \leq \#\mathcal{M}_{sing} + (m+n) \frac{(B+C-1)t}{D} < (m+n)^2 + (m+n) \frac{2h^{1/3}t^{4/3}}{[h^{-1/3}t^{2/3}/(4mng)]} < 12mn(m+n)gh^{2/3}t^{2/3}
\]

for sufficiently large \( t > c_0(m,n) \), which concludes the proof.
5. Proof of Theorem 1.6

5.1. Outline of the proof. Before giving technical details we first outline the sequence of steps

- We consider the set $\mathcal{R} = \mathcal{M}_p \setminus \mathcal{C}_p$ and show that if it is large then by Lemma 3.2 there is a large set $\mathcal{L} \subseteq \mathcal{R}$ elements of large orders.
- Each element $x \in \mathcal{L}$ has an orbit of size $\geq t(x)/2$ which is also in $\mathcal{R}$.
- Using Conjecture 1.3, we estimate the size of intersections of these orbits for distinct elements $x_1, x_2 \in \mathcal{L}$.
- We conclude that all intersections together are small and so to fit them all in $\mathcal{R}$ the size of $\mathcal{R}$ must be even larger than we initially assumed.

5.2. Formal argument. We always assume that $p$ is large enough. Define the mapping

$$\mathcal{T}_0 (x, y, z) \mapsto (x, z, 3xz - y)$$

where $\mathcal{T}_0 = \Pi_{1,3,2} \circ \mathcal{R}_2$ is the composition of the permutations

$$\Pi_{1,3,2} = (x, y, z) \mapsto (x, z, y)$$

and the involution

$$\mathcal{R}_2 : (x, y, z) \mapsto (x, 3xz - y, z)$$

as in the above.

Therefore the orbit $\Gamma(x, y, z)$ of $(x, y, z)$ under the above group of transformations $\Gamma$ contains, in particular the triples $(x, u_n, u_{n+1})$, $n = 1, 2, \ldots$, where the sequence $u_n$ satisfies a binary linear recurrence relation

$$u_{n+2} = 3xu_{n+1} - u_n, \quad n = 1, 2, \ldots,$$

with the initial values, $u_1 = y, u_2 = z$. This also means that $\Gamma(x, y, z)$ contains all triples obtained by the permutations of the elements in $(x, u_n, u_{n+1})$.

Let $\xi, \xi^{-1} \in \mathbb{F}_p^*$ be the roots of the characteristic polynomial $Z^2 - 3xZ + 1$ of the recurrence relation (5.1). In particular $3x = \xi + \xi^{-1}$. Then, it is easy to see that unless $(x, y, z) = (0, 0, 0)$, which we eliminate from the consideration, the sequence $u_n$ is periodic with period $t(x)$ which is the order of $\xi$ in $\mathbb{F}_p^*$.

Let $B$ be a fixed positive number to be chosen later. We denote

$$M_0 = (\log p)^B, \quad M_1 = M_0^{1/4} / 3 = (\log p)^{B/4} / 3.$$
Assume that the remaining set of nodes $\mathcal{R} = \mathcal{M}_p \setminus C_p$ is of size $\#\mathcal{R} > M_0$. Note that if $(x, y, z) \in \mathcal{R}$ then also $(y, x, z) \in \mathcal{R}$ and for any $x, y$ there are at most two values of $z$ such that $(x, y, z) \in \mathcal{R}$. Therefore, there are more than $(M_0/2)^{1/2}$ elements $x \in \mathbb{F}_p^n$ with $(x, y, z) \in \mathcal{R}$ for some $y, z \in \mathbb{F}_p$.

Since there are obviously at most $T(T + 1)/2$ elements $\xi \in \mathbb{F}_p^*$ of order at most $T$ we conclude that there is a triple $(x^*, y^*, z^*) \in \mathcal{R}$ with

$$t(x^*) > \sqrt{(M_0/2)^{1/2}} > 2M_1,$$

where $t(x^*)$ is the period of the sequence $u_n$ which is defined as in (5.1) with respect to $(x^*, y^*, z^*)$.

Then the orbit $\Gamma(x^*, y^*, z^*)$ of this triple has at least $2M_1$ elements. Let $M$ be the cardinality of the set $\mathcal{X}$ of projections along the first components of all triples $(x, y, z) \in \Gamma(x^*, y^*, z^*)$. Since the orbits are closed under the permutation of coordinates, and permutations of the triples $(x^*, u_n, u_{n+1})$, $n = 1, \ldots, t(x^*)$, where as above the sequence $u_n$ is defined as in (5.1) with respect to $(x^*, y^*, z^*)$ and $t(x^*)$ is its period, produce the same projection no more than twice we obtain

$$M \geq \frac{1}{2} t(x^*).$$

Recalling (5.2), we obtain

$$M > M_1 = (\log p)^{B/4}/3.$$

Using that $(x, y, z) \notin \mathcal{M}_p$, we notice, that by the bound (1.3)

$$M = p^{\omega(1)}.$$

For $t \mid p^2 - 1$ we denote $g(t)$ the number of $x \in \mathcal{X}$ for which the period of the sequence $u_n$ defined as in (5.1) satisfies $t(x) = t$. Observe that

$$\sum_{t \mid p^2 - 1} g(t) = M.$$

The same argument as used in the bound (5.3) implies that

$$g(t) = 0 \quad \text{for} \quad t > 2M.$$

We apply Lemma 3.2 with

$$\varepsilon = \frac{1}{40AC_0},$$

and

$$t(x^*) > \sqrt{(M_0/2)^{1/2}} > 2M_1.$$
where $A$ is a bound from Conjecture 1.3 and $C_0$ is as in Lemma 3.1.

Take

$$B = 16 \log(1/\varepsilon) + 1.$$  

Since $g(t) < t$ for any $t$ and also since due to (5.4) we have

$$4\sqrt{AM} > (\log p)^{B/8} \geq (\log(p^2 - 1))^{2\log(1/\varepsilon)},$$

by Lemma 3.2,

$$\sum_{t \leq 4\sqrt{AM}} g(t) < \sum_{t \leq 4\sqrt{AM}} t \leq 4\sqrt{AM}\tau_{4\sqrt{AM}}(p^2 - 1)$$

$$\leq C_0 \varepsilon (4\sqrt{AM})^2 = 0.4M.$$  

Hence, we conclude that

$$\sum_{t > 4\sqrt{AM}} g(t) \geq 0.6M.$$  

Let $\mathcal{L}$ be the set of $x \in X$ with $t(x) > 4\sqrt{AM}$. We have shown that

$$(5.9) \quad \#\mathcal{L} \geq 0.6M.$$  

For each $x \in \mathcal{L}$ we fix some $y, z \in \mathbb{F}_p$ such $(x, y, z) \in \Gamma(x^*, y^*, z^*)$ and again consider the sequence $u_n, n = 1, 2, \ldots$, given by (5.1) and of period $t(x) = t_0$, so we consider the set

$$\mathcal{Z}(x) = \{u_n : n = 1, \ldots, t_0\}.$$  

Let $\mathcal{H}_x$ be the subgroup of $\mathbb{F}_{p^2}^*$ of order $t(x)$, and $\xi(x)$ satisfy the equation $3x = \xi(x) + \xi(x)^{-1}$. One can easily check, using an explicit expression for binary recurrence sequences via the roots of the characteristic polynomial, that

$$\mathcal{Z}(x) = \left\{\frac{r(x)}{\alpha(x)} : u \in \mathcal{H}_x\right\},$$

where

$$r(x) = \frac{(\xi(x)^2 + 1)^2}{9(\xi(x)^2 - 1)^2},$$

and $\alpha(x) \in \mathbb{F}_{p^2}^*$. If $\xi = \xi_0$ satisfies the equation

$$r = \frac{(\xi^2 + 1)^2}{9(\xi^2 - 1)^2},$$

then other solutions are $-\xi_0, 1/\xi_0, -1/\xi_0$. Moreover, $3x = \xi + \xi^{-1}$ can take, for a fixed $r$, at most two values whose sum is 0. Since every
value is taken at most twice among the elements of the sequence \( u_n \), \( n = 1, \ldots, t(x) \), we have

\[(5.10) \quad \#Z(x) \geq \frac{1}{2} t(x) > 2\sqrt{AM}.\]

Now we construct a set \( L^* \subseteq L \). If \( x, x^* \in L \) and \( x + x^* = 0 \), then we put one of the elements \( x, x^* \) in \( L^* \). If \( x \in L \) and \( -x \notin L \), then we set \( x \in L^* \). Due to (5.9), we get

\[(5.11) \quad \#L^* \geq 0.3M.\]

Moreover, for any distinct \( x, x^* \in L^* \) we have \( x + x^* \neq 0 \) and, hence, \( r(x) \neq r(x^*) \).

We claim that under Conjecture 1.3 for any distinct \( x, x^* \in L^* \) the inequality

\[(5.12) \quad \# \left( Z(x) \cap Z(x^*) \right) \leq 2A\]

holds.

Indeed, take distinct elements \( x, x^* \in L^* \). By \( G \) we denote the subgroup of \( F_{p^2}^* \) generated by \( H_x \) and \( H_{x^*} \). Notice that due to (5.5) and (5.6) we have

\[(5.13) \quad \#G = p^{\rho(1)}.\]

Next, \( \#(Z(x) \cap Z(x^*)) \) is the number of solutions to the equation

\[\alpha(x)u + \frac{r(x)}{\alpha(x)u} = \alpha(x^*)v + \frac{r(x^*)}{\alpha(x^*)v}, \quad (u, v) \in H_x \times H_{x^*},\]

as in the above or, equivalently,

\[P_{x,x^*}(u, v) = 0, \quad (u, v) \in H_x \times H_{x^*},\]

where

\[P_{x,x^*}(X, Y) = \alpha(x)^2 \alpha(x^*) X^2 Y - \alpha(x) \alpha(x^*)^2 X Y^2 - \alpha(x) r(x^*) X + \alpha(x^*) r(x) Y.\]

The number of solutions to the last equation in \((u, v) \in H_x \times H_{x^*}\) does not exceed the number of solutions in \((u, v) \in G^2\). Let \( Z = X/Y \). Then the equation is reduced to

\[(5.14) \quad \frac{\alpha(x)^2 \alpha(x^*) Z - \alpha(x) \alpha(x^*)^2}{\alpha(x) r(x^*) Z - \alpha(x^*) r(x)} = U,\]

where \( U = Y^{-2} Z^{-1} \).

Now we are in position to use Conjecture 1.3. The conditions (1.11) on the coefficients of linear functions in the numerator and in the
denominator of the fraction in (5.14) are satisfied since \( \alpha(x) \neq 0 \), \( \alpha(x^*) \neq 0 \), and \( r(x) \neq r(x^*) \).

Also, for large \( p \) we have \( \#\mathcal{G} \leq p^{\varepsilon_0} \) due to (5.13). By Conjecture 1.3, equation (5.14) has at most \( A \) solutions in \( Z,Y \). For each solution thewre are at most two possible values of \( Y \). Fixing \( Y \), we determine \( X \). So, the inequality (5.12) holds.

Denote
\[
h = \left\lceil \sqrt{M/A} \right\rceil + 1.
\]
Due to (5.4) and (5.11) we have \( \#\mathcal{L}^* \geq h \) provided that \( p \) is large enough. We choose \( h \) elements \( x_1, \ldots, x_h \) from \( \mathcal{L}^* \). It follows from (5.12) that \( j = 1, \ldots, h \) we have
\[
\sum_{i=1}^{j-1} \# \left( Z(x_j) \cap Z(x_i) \right) \leq 2(j-1)A.
\]
which implies, by (5.10)
\[
\# \left( Z(x_j) \setminus \bigcup_{i=1}^{j-1} Z(x_i) \right) \geq 2\sqrt{AM} - 2(j-1)A.
\]
Observe that
\[
\# \left( \bigcup_{j=1}^{h} Z(x_j) \right) = \sum_{j=1}^{h} \# \left( Z(x_j) \setminus \bigcup_{i=1}^{j-1} Z(x_i) \right).
\]
Hence,
\[
\# \left( \bigcup_{j=1}^{h} Z(x_j) \right) > 2\sqrt{AM}h - (h-1)hA
\]
\[
= (2\sqrt{AM} - (h-1)A)h
\]
\[
> (2\sqrt{AM} - \sqrt{AM})\sqrt{M/A} > M,
\]
but this inequality contradicts the definition of \( M \). Together with the choice of \( B \) given by (5.7) and (5.8), this concludes the proof.

6. Comments

Let \( P(n) \) be the largest primitive prime divisor of \( 2^n - 1 \), that is, the largest prime which divides \( 2^n - 1 \), but does not divide any of the numbers \( 2^d - 1 \) for \( 1 \leq d < n \). Note that \( P(n) \equiv 1 \pmod{n} \). By a striking result of Stewart [20, Theorem 1.1] we have
\[
P(n) \geq n \exp \left( \frac{\log n}{104 \log \log n} \right)
\]
provided that \( n \) is large enough. It is also natural to assume that
\[
\log P(n) / \log n \rightarrow \infty \text{ for } n \rightarrow \infty.
\]
However for us a weaker assumption is sufficient. Namely as assume that
\[
\lim \sup \frac{\log P(24m)}{\log m} = \infty.
\]
We then take \( n = 24m, m \in \mathbb{N} \), and \( p = P(n) \) such that \( n = p^{\alpha(1)} \).
Then \( p \equiv 1 \pmod{24} \). Since 2 is a quadratic residue modulo \( p \), we can take \( \xi \in \mathbb{F}_p \) such that \( \xi^2 = 2 \). We consider a group \( \mathcal{G} \) generated by \( \xi \). Note that \( \# \mathcal{G} = 2n = p^{\alpha(1)} \) as \( n \rightarrow \infty \). The group \( \mathcal{G} \) contains an element \( \zeta_4 \) of order 4 and an element \( \zeta_6 \) of order 6. It is easy to check that
\[
((\pm \zeta_4 \pm 1) / \xi)^8 = 1.
\]
Thus
\[
(\pm \zeta_4 \pm 1)^{2n} = \xi^{6n} = 1.
\]
Hence \( \pm \zeta_4 \pm 1 \in \mathcal{G} \). Also,
\[
(\pm \zeta_6 - 1)^3 = 1.
\]
Hence, similarly \( \pm \zeta_6 - 1 \in \mathcal{G} \). Consider a set \( \mathcal{D} \) consisting of 9 elements
\[
\mathcal{D} = \{(p - 1/2), 1, -2, \zeta_4, -\zeta_4, \zeta_4 - 1, -\zeta_4 - 1, \zeta_6 - 1, -\zeta_6 - 1\}.
\]
Clearly, \( x \in \mathcal{G}, x + 1 \in \mathcal{G} \) for any \( x \in \mathcal{D} \). This shows that probably \( A \) in Conjecture 1.3 should be at least 9.

We also observe that in Conjecture 1.3 the value of \( \varepsilon_0 \) cannot be taken greater than \( 1/2 \).

Indeed, suppose that \( p \) is a prime and \( p - 1 \) has a divisor \( t = p^{\varepsilon_0 + o(1)} \), as \( p \rightarrow \infty \) with a fixed \( \varepsilon_0 > 1/2 \) (the infinitude of such primes follows instantly from [9, Theorem 7]).

Let us fix any \( \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2} \in \mathbb{F}_p \). Clearly the equation (1.12) has \( N = p + O(1) \) of solutions \( (u, v) \in (\mathbb{F}_p^*)^2 \). Let \( \mathcal{G} \subseteq \mathbb{F}_p^* \) be a subgroup of order \( t \). Since \( \mathbb{F}_p^* \) is the union of \( (p - 1)/t \) cosets \( a\mathcal{G} \) of \( \mathcal{G} \), the direct product \( \mathbb{F}_p^* \times \mathbb{F}_p^* \) is the union of \( (p - 1)^2/t^2 \) products of cosets of \( \mathcal{G} \). By the Dirichlet principle that there is at least one product \( a\mathcal{G} \times b\mathcal{G} \) such that the number of solutions \( (u, v) \in a\mathcal{G} \times b\mathcal{G} \) (with some \( a, b \in \mathbb{F}_p^* \)) is not less than
\[
\frac{N}{(p - 1)^2/t^2} \geq (1 + o(1))t^2/p \geq p^{2\varepsilon_0 - 1 + o(1)}
\]
and hence is not bounded as \( p \to \infty \). Changing the variables \( \tilde{u} = a^{-1}u \), \( \tilde{v} = b^{-1}v \) in (1.12) we obtain another equation of the same type

\[
\frac{\alpha_{1,1}ab^{-1}\tilde{u} - \alpha_{1,2}b^{-1}}{\alpha_{2,1}a\tilde{u} - \alpha_{2,2}} = \tilde{v}
\]

with an unbounded number of solutions \((\tilde{u}, \tilde{v}) \in G^2\).

Finally, we note that using [5, Theorem 1.2] one concludes that Conjecture 1.3 holds (in much stronger and general form) for a sequence of primes of relative density 1. However this does not give any new results for the sets \( \mathcal{M}_p \) because, as we mentioned, Bourgain, Gamburd and Sarnak [2, Theorem 2] have already shown that Conjecture 1.1 holds for an overwhelming majority of primes \( p \leq X \) as \( X \to \infty \).

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Steklov Mathematical Institute, 8, Gubkin Street, Moscow, 119991, Russia

E-mail address: konyagin@mi.ras.ru

Department of Pure Mathematics, University of New South Wales, 2052 NSW, Australia

E-mail address: igor.shparlinski@unsw.edu.au

Institute for Information Transmission Problems RAS, 19, Bolshoy Karetny per., Moscow, 127051, Russia, and, Department of Mathematics, National Research University Higher School of Economics, 6, Usacheva Street, Moscow, 119048, Russia, and, Steklov Mathematical Institute,, 8, Gubkin Street, Moscow, 119991, Russia

E-mail address: vyugin@gmail.com