On 't Hooft’s loop operator

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An explicit realization of 't Hooft’s loop operator in continuum Yang-Mills theory is given.

INTRODUCTION

In a gauge theory all physical information is contained in (all possible) Wilson loops

\[ W[A](C) = P \exp \left( - \oint_C dx_\mu A_\mu(x) \right), \]  

(1)

where \( A \) is the algebra valued gauge field\(^1\). This is the basis of the so-called loop space formulation of gauge theory. The expectation value of a temporal Wilson loop \( \langle W(C) \rangle \) shows an area law in the confined phase, but screening of the color charges carried by the loop in the deconfined phase. Thus the temporal Wilson loop can serve as an order parameter to distinguish the two phases of Yang-Mills theory.

Another order (or more precisely “disorder”) parameter of Yang-Mills theory was introduced by ’t Hooft and is commonly referred to as ’t Hooft loop. This (dis-) order parameter is in a certain sense dual to the Wilson loop and is defined in the following way: Consider Yang-Mills theory in the Hamiltonian formulation (which assumes Weyl gauge \( A_0 = 0 \)), where the spatial components of the gauge field \( \vec{A}(\vec{x}) \) are the coordinates of the theory. (More precisely in this context \( \vec{A}(\vec{x}) \) has to be understood as the quantum mechanical “operator” of the canonical coordinate. In the following we will indicate quantum mechanical operators by a hat “\( \hat{\cdot} \)”). The operator for a spatial Wilson loop \( \hat{W}(C) \) is given by eq. (1) as \( \hat{W}(C) = W[\hat{A}](C) \), where \( C \) is a loop in ordinary three-space \( \mathbb{R}^3 \). The operator of ’t Hooft’s loop \( \hat{V}(C) \) is then defined by its commutation relation with the operator of the Wilson loop (1):

\[ \hat{V}(C_1) \hat{W}(C_2) = Z^{L(C_1,C_2)} \hat{W}(C_2) \hat{V}(C_1). \]  

(2)

Here \( Z \) denotes an element of the center of the gauge group (which is \( Z(N) \) for gauge group \( SU(N) \)) and

\[ L(C_1,C_2) = \frac{1}{4\pi} \oint_{C_1} dx_i \oint_{C_2} dy_j \epsilon_{ijk} \frac{x_k - y_k}{|\vec{x} - \vec{y}|^3} \]  

(3)

is the Gaussian linking number between the two spatial loops \( C_1, C_2 \).

Like the temporal Wilson loop the expectation value of the (spatial) ’t Hooft loop can serve as an order parameter to distinguish the two phases of Yang-Mills theory. As argued by ’t Hooft \( \square \) in the confined phase the expectation value of the spatial ’t Hooft loop operator \( \langle \hat{V}(C) \rangle \) satisfies a perimeter law (at zero temperature), while in the deconfined phase it shows an area law \( \square \). Its behavior is thus dual to the one of the expectation value of a temporal Wilson loop.

The ’t Hooft loop operator, defined by eq. (2), can be interpreted as a center vortex creation operator \( \square \). In statistical physics operators, creating topological excitations, like vortices, are referred to as disorder operators.

\( \square \) supported by DFG - RE 856/5-1
\(^1\) Strictly speaking it would be sufficient to consider the trace of \( W[A](C) \). Note also, that we use anti-hermitian generators of the gauge group.
Their expectation values, referred to as disorder parameters, are related to the free energy of the associated topological excitations. An explicit realization of a vortex creation operator can be easily given on the lattice, where center vortices represent co-closed $D - 2$ dimensional hypersurfaces of plaquettes being equal to a non-trivial center element\(^2\).

Unfortunately \'t Hooft did not give an explicit representation of its loop operator in continuum Yang-Mills theory but rather defines this operator by its effect on the eigenstates of the gauge potential: The effect of \(\hat{V}(C)\) is a gauge transformation \(\Omega^{[C]}\), which is singular on the curve \(C\), and if another curve \(C'\) (parametrized by \(s \in [0, 1]\)) winds through \(C\) with \(n\) windings in a certain direction then \(\Omega^{[C]}\) receives a phase:

\[
\Omega^{[C]}(s = 1) = e^{i \frac{2\pi}{N} n} \Omega^{[C]}(s = 0) .
\]  

(4)

This implies that the gauge function \(\Omega^{[C]}(x)\) is multivalued but nevertheless the gauge transformed fields \(A^{[C]}\) will be single valued. It is the purpose of the present paper to provide an explicit realization of \'t Hooft’s loop operator in continuum Yang-Mills theory. In the following we will first construct an operator that generates the singular gauge transformations defined by eq. (4). Second we will show that this operator, when acting on physical (i.e. gauge invariant) states, creates a center vortex. Third we will construct a center vortex creation operator and explicitly show that it satisfies indeed the defining equation (2) for \'t Hooft’s loop operator.

**GENERATOR OF SINGULAR GAUGE TRANSFORMATION**

In the following we explicitly construct the singular gauge transformation defined by eq. (4). A good candidate for this gauge function is

\[
\Omega(\Sigma, x) = e^{-E\omega(\Sigma, x)} .
\]  

(5)

Here \(E = E_a T_a\) denotes a co-weight vector in the Lie algebra, which when exponentiated produces a non-trivial center element

\[
e^{-E} = Z .
\]  

(6)

Furthermore \(\omega(\Sigma, x)\) is the solid angle subtended by the loop \(\partial \Sigma\) as seen from the point \(x\), which can be expressed as

\[
\omega(\Sigma, x) = \int_{\Sigma} d^2 \vec{\sigma} \partial_\sigma D(\vec{x} - \vec{x}(\sigma)) ,
\]  

(7)

where \(D(x)\) denotes the Green’s function of the 3-dimensional Laplacian, satisfying \(-\partial^2 D(x) = \delta^3(x)\). (We have normalized \(\omega(\Sigma, x)\) so that the total solid angle of a 2-sphere \(S^2\) is unity.) A deformation of \(\Sigma\) keeping its boundary \(C = \partial \Sigma\) fixed, leaves \(\omega(\Sigma, x)\) invariant, unless \(x\) crosses \(\Sigma\). When \(x\) crosses the surface \(\Sigma\) the solid angle \(\omega(\Sigma, x)\) jumps by \(\pm 1\) and accordingly the gauge function \(\Omega(\Sigma, x)\) changes by a center element, eq. (6). However, as defined by eq. (5), \(\Omega(\Sigma, x)\) is a single valued function. To obtain \'t Hooft’s multivalued gauge function \(\Omega^{[C]}(x)\) from \(\Omega(\Sigma, x)\), we have to ignore the discontinuity (jump) of \(\omega(\Sigma, x)\) when \(x\) crosses \(\Sigma\) and instead use the smooth (but multivalued) continuation of \(\omega(\Sigma, x)\) \(\rightarrow \omega|_{\partial \Sigma = C}\) (with the jump of \(x\) at \(\Sigma\) ignored), which then depends only on \(\partial \Sigma = C\). Let us now explicitly construct the generator of the gauge transformation \(\Omega\).

The operator, which generates a gauge transformations \(\Omega = e^{-\Theta}, \Theta = \Theta^a T_a\) of the field operators is given by

\[
\mathcal{U}(\Theta) = \exp \left[ i \int d^3 x \left( \hat{D}_{i}^{ab}(x) \Theta^b \right) \Pi_i^a(x) \right] ,
\]  

(8)

\(^2\) Strictly speaking the vortex surfaces leave on the dual lattice.
where $\hat{D}_i = \partial_i + \hat{A}_i$ denotes the covariant derivative with $\hat{A}_i^{ab} = f^{acb}A_i^c$ being the gauge field in the adjoint representation and $f^{abc}$ denotes the structure constants of the gauge group. Furthermore

$$\Pi_i^a(x) = \frac{1}{i} \frac{\delta}{\delta A_i^a(x)} . \tag{9}$$

is the "momentum operator" in Yang-Mills theory, which coincides with the operator of the color electric field and satisfies the (equal time) canonical commutation relation

$$[A_i^a(x), \Pi_i^b(y)] = i\delta^{ab}\delta_{kl}\delta^3(x-y) . \tag{10}$$

Any operator in quantum Hilbert space is gauge transformed by the action of the operator (8). For example, for the coordinate operator we have, by using (10),

$$U(\Theta)A U(\Theta)^\dagger = \Omega A \Omega^\dagger + \Omega \partial \Omega^\dagger = A^\Omega . \tag{11}$$

By putting $\Theta = E\omega(\Sigma, x)$, we obtain the operator which generates the gauge transformation (5)

$$U(E\omega) = \exp \left[ i \int d^3x E^a \left( \partial_i \omega \Pi_i^a(x) + \omega \hat{A}_i^{ab} E^b \Pi_i^a(x) \right) \right] . \tag{12}$$

This operator leaves physical states invariant. To see this perform a partial integration in the exponent of eq. (12)

$$U(E\omega) = \exp \left[ i \int d^3x \left( \partial_i ((E\omega)^a \Pi_i^a(x)) - \omega E^a \hat{D}_i^{ab} \Pi_i^b(x) \right) \right] . \tag{13}$$

The first term in the exponent can be converted into a surface integral, which vanishes as the solid angle $\omega(\Sigma, x)$ vanishes for $\vec{x}^2 \to \infty^3$. The second term in the exponent of eq. (13) vanishes, when acting on gauge invariant states by Gauss’ law $\hat{D} \Pi \Psi = 0$. Thus, when taking the operator $U(E\omega)$ literally it has no effect on physical states. However, this operator generates precisely the desired multivalued gauge transformation (4), which defines the effect of ’t Hooft’s loop operator. Thus eq. (12), ignoring the singular part of $\partial_i \omega$, which arises from the jump of $\omega(\Sigma, x)$ by $\pm 1$, when $x$ crosses the surface $\Sigma$. Then this operator generates precisely the desired multivalued gauge transformation (4), which defines the effect of ’t Hooft’s loop operator. Thus eq. (12), ignoring the singular part of $\partial_i \omega$, should give an explicit realization of ’t Hooft’s loop operator. In fact such an operator (12) was considered in ref. [4] for planar loops as an explicit realization of the ’t Hooft loop operator, but it was not shown that this operator satisfies the defining eq. (2). In this paper we will explicitly show, that the so defined operator (12) (again with the singular piece of $\partial_i \omega$ omitted) is gauge equivalent to a center vortex creation operator and that the latter will satisfy ’t Hooft’s loop algebra (2).

**CENTER VORTEX CREATION OPERATOR**

Let us now explicitly work out the effect of ignoring the singular part of $\partial \omega$. One can show that

$$\Omega(\Sigma, x)\partial \Omega^\dagger(\Sigma, x) = E\partial \omega(\Sigma, x) = a(\partial \Sigma, x) - A(\Sigma, x) , \tag{14}$$

where

$$A_i(\Sigma, x) = E \int_\Sigma d^2 \sigma \delta^3(x - \bar{x}(\sigma)) \tag{15},$$

$$a_i(\partial \Sigma, x) = E \int_{\partial \Sigma} d\bar{x} k_{ij} \partial_i \partial_j D(x - \bar{x}) . \tag{16}$$

3 In fact, this term can be shown to vanish for all topologically trivial gauge transformation with zero winding number $n[\Omega] \in \Pi_3(S_3)$.
Here \( x_1(\sigma) \) denotes a parameterization of the 2-dimensional open surface \( \Sigma \). Furthermore \( a(\partial \Sigma, x) \) and \( A(\Sigma, x) \) represent the smooth and the singular part of \( E \partial \omega \), respectively. The singular part \( A(\Sigma, x) \) arises precisely from the jump of \( \omega(\Sigma, x) \), when \( x \) crosses \( \Sigma \). From the physical point of view \( a(\partial \Sigma, x) \) and \( A(\Sigma, x) \) are different gauge representations of one and the same center vortex located at \( C = \partial \Sigma \). An ideal center vortex field \( A(C_1) \), whose flux is located at a loop \( C_1 \) is defined by its property that it produces a non-trivial center element for Wilson loops \( C_2 \) non-trivially (modulo \( N \)) linked to the loop \( C_1 \), i.e.

\[
W[A(C_1)](C_2) = Z^{L(C_1, C_2)},
\]

where \( L(C_1, C_2) \) is the Gaussian linking number \( \mathbb{B} \). Indeed the two potentials \( \mathbb{I} \), \( \mathbb{O} \) satisfy

\[
\int_{C} dx_i A_i(\Sigma, x) = EI(C, \Sigma)
\]

\[
\int_{C} dx_i \partial \omega_i(\Sigma, x) = EL(C, \partial \Sigma),
\]

where \( I(C, \Sigma) \) is the intersection number between the loop \( C \) and the open surface \( \Sigma \). Since \( I(C, \Sigma) = L(C, \partial \Sigma) \) both potentials satisfy eq. \( \mathbb{I} \). Note, that \( A(\Sigma, x) \) represents the continuum analog of the ideal center vortices \( \mathbb{B} \) arising on the lattice after center projection \( \mathbb{I} \), which in \( D = 3 \) gives rise to open surfaces \( \Sigma \) of links being equal to a non-trivial center element \( Z \in Z(N) \). The boundaries \( C = \partial \Sigma \) of these surfaces define closed loops of center flux on the dual lattice. \( a(\partial \Sigma, x) \) \( \mathbb{O} \) represents the same center vortex as \( A(\Sigma, x) \) \( \mathbb{I} \), but depends only on the vortex loop \( C = \partial \Sigma \), and is transversal \( \partial \omega(\partial \Sigma, x) = 0 \).

Neglecting the singular part of \( \partial \omega(\Sigma, x) \) in the exponent of eq.\( \mathbb{I} \), i.e. neglecting \( A(\Sigma, x) \) in eq. \( \mathbb{O} \), we obtain the operator

\[
\hat{U}(E \omega) = e^{i \int d^3 x (a_i^a \Pi_i^a - \omega E^a \hat{A}^{ab} \Pi^b) },
\]

which precisely generates the singular gauge transformation \( \mathbb{I} \), i.e. \( \hat{U}(E \omega) \hat{U}(E \omega) = A^0(\Sigma) \). For later use we rewrite this operator by adding a “zero” in the exponent and using eq. \( \mathbb{O} \)

\[
\hat{U}(E \omega) = \exp \left[ i \int d^3 x \left( (a_i^a - A_i^a) \Pi_i^a - \omega E^a \hat{A}^{ab} \Pi^b + A_i^a \Pi^a_i(x) \right) \right] = \exp \left[ i \int d^3 x \left( A_i^a(x) \Pi_i^a(x) + \left( \hat{D}_i \omega E \right)^a \Pi_i^a \right) \right].
\]

Now observe that the two operators in the exponent commute

\[
\left[ A_k^a \Pi_k^b, \left( \hat{D} \omega \right)^b \Pi^b \right] = A_k^a \Pi_k^b, \left[ \hat{A} \omega \right]^b \Pi^b = A_k^a f^{bac} E_c \omega \Pi_k^b = 0 .
\]

The last relation follows since \( A_k^a \sim E^a \) (see eq. \( \mathbb{O} \)). We can therefore rewrite the operator \( \mathbb{I} \) as

\[
\hat{U}(E \omega) = \exp \left[ i \int d^3 x A_i^a(\Sigma, x) \Pi_i^a(x) \right] \hat{U}(E \omega),
\]

where \( \hat{U}(E \omega) \) is the generator \( \mathbb{I} \) of the (single valued) gauge transformation \( \Omega(\Sigma, x) \) \( \mathbb{I} \). When acting on physical states \( | \Psi > \) this operator becomes unity \( \langle \hat{U}(E \omega) | \Psi > = | \Psi > \) and the operator \( \hat{U}(E \omega) \) can be replaced by

\[
\hat{V}(\Sigma) = \exp \left[ i \int d^3 x A_i^a(\Sigma, x) \Pi_i^a(x) \right] .
\]

In the next section we will show, that this operator indeed satisfies the defining equation \( \mathbb{I} \) for ‘t Hooft’s loop operator. Before giving that proof let us show that this operator generates a center vortex located at \( \partial \Sigma \).
A vortex creation operator should add a center vortex field to any given gauge field $\vec{A}(x)$. Let us recall, that in the canonical formulation of continuum Yang-Mills theory the gauge field $\vec{A}(x)$ represents the quantum mechanical coordinate. Let $|A>$ denote an eigenstate of the quantum mechanical operator $\hat{A}$ of the Yang-Mills coordinate with “eigenvalue” $\vec{A}(x)$ i.e.

$$\hat{A}(x)|A> = \vec{A}(x)|A>.$$  

(24)

A vortex creation operator should then satisfy

$$\hat{V}(C)|A> = |A + A(C)>,$$

where $A(C)$ denotes the gauge potential of a center vortex located at the loop $C$. In quantum mechanics the operator, which shifts the coordinate, say $x$, by an amount $x_0$

$$T(x_0)|x> = |x + x_0>.$$  

(26)

given by

$$T(x_0) = e^{ix_0\hat{p}},$$  

(27)

where $\hat{p} = \frac{i}{\hbar}\frac{\partial}{\partial x}$ is the usual momentum operator. The operator (27) obviously generates a translation in coordinate space. Analogously the operator (23) creates a center vortex field $A(\Sigma)$. Indeed from the commutation relation (14) follows

$$\hat{V}(C)\hat{A}\hat{V}(C)^\dagger = \hat{A} + A(C)$$  

(28)

and by Taylor expansion one finds for any functional of the gauge potential

$$\hat{V}(C)f(\hat{A})\hat{V}(C)^\dagger = f(\hat{A} + A(C)),$$

(29)

which proves eq. (25). Eq. (23) represents the continuum version of the center vortex analog of the monopole creation operator introduced by DiGiacomo at al. on the lattice [6]. Inserting the explicit representation (15) into eq. (23) ‘t Hooft’s loop operator becomes

$$\hat{V}(C) = e^{i\int_{\Sigma(C)} d^2\sigma E^a(\vec{x}(\sigma))\Pi_a(x)}.$$  

(30)

Since $\Pi_a(x)$ represents the operator of the electric field it is seen that the ‘t Hooft loop operator $\hat{V}(C)$ measures the electric flux through the loop $C$. In this sense the spatial ‘t Hooft loop is indeed dual to the spatial Wilson loop, which measures the magnetic flux.

PROVING ‘T HOOFT’S LOOP ALGEBRA

Let us now show that the vortex creation operator defined by eqs. (23), (1) and (15), indeed, satisfies the defining eq. (2) for the ‘t Hooft loop operator.

From eq. (29) follows

$$\hat{V}(C_1) W[\hat{A}(C_2)]\hat{V}(C_1)^\dagger = W[\hat{A} + A(C_1)](C_2).$$  

(31)

Note, that $\hat{V}(C)$ is an operator in quantum Hilbert space but not in color space (it does not contain the generators of the gauge group). Hence, this operator does not interfer with the path ordering in the Wilson loop. However, in the resulting Wilson loop $W[\hat{A} + A(C_1)](C_2)$ appearing on the right-hand side of eq.(31) the two gauge potentials $\hat{A}$ and $A(C_1)$ are both Lie algebra valued gauge fields, which in general do not commute.
To simplify the expression for the Wilson loop $W[A + A(C_1)](C_2)$ let us introduce a parameterization of the loop $C_2 : \vec{x}(s), s \in [0, 1]$. Then the Wilson loop

$$W[A](C_2) = Pe^{-\int_0^1 ds \vec{A}(\vec{x}(s))\overline{A}(\vec{x}(s))}$$

(32)

can be interpreted as a “time evolution operator” in color space. Adopting for this evolution operator the familiar “interaction picture”, interpreting $\vec{A}(\vec{x}(s))\overline{A}(C_1, \vec{x}(s))$ and $\vec{A}(\vec{x}(s))\overline{A}(x(s))$ as “free” and “interaction” parts, respectively, of the “Hamiltonian” we find

$$W[A + A(C_1)](C_2) = W[A(C_1)](C_2) W[A^I(C_1)](C_2),$$

(33)

where

$$A^I(C_1 : x(s)) = U^I(C_1, s) A(x(s)) U(C_1, s),$$

(34)

$$U(C_1, s) = U[A(A_1)](s) = P \exp \left( -\int_0^s ds' \vec{A}(\vec{x}(s'), \overline{A}(C_1, x(s'))) \right)$$

(35)

is the interaction representation of the color matrix $A(x) = A^a(x)T_a$. Note, that in the “free” time evolution operator $U(C_1, s)$ defined by eq. (35) the path ordering is irrelevant, since the vortex gauge potential $\overline{A} = A^aT_a$ defined by eq. (13) lives in the Cartan subalgebra. (For later use we have, however, kept the path ordering operator.) With the explicit expression for the vortex gauge potential, eq. (15), we find for the “free” time evolution operator (32),

$$U(C_1: s) = e^{-EI[L_x(0)\to x(s); \Sigma(C_1)]},$$

(36)

where

$$I[L_x(0)\to x(s); \Sigma] = \int_{\vec{x}(0)}^{\vec{x}(s)} d\vec{x} \int d\sigma d^3x (x - \vec{x}(\sigma))$$

(37)

is the intersection number between the open path segment $L_x(0)\to x(s)$ of the loop $C_2$ and the open surface $\Sigma(C_1)$ bounded by $C_1$. Note, the intersection number, eq. (27), depends on the particularly chosen surface $\Sigma(C_1)$. Different choices of $\Sigma(C_1)$ keeping the boundary $C_1 = \partial \Sigma(C_1)$ fixed corresponds to different choices of gauges related by a center gauge transformations, see ref. 3. However, for any surface $\Sigma(C_1)$ (i.e. center gauge) choosen the intersection number $I(L, \Sigma(C_1))$ is an integer, and accordingly the evolution operator eq. (33) is equal to a center element (3)

$$U(C_1, x(s)) = Z^I[L_x(0)\to x(s); \Sigma(C_1)]$$

(38)

and, hence, commutes with any gauge potential. This implies, that the “interaction representation” $A^I(C_1, x)$, eq. (34), coincides with the gauge potential itself, that is $A^I(C_1, x) = A(x)$ and eq. (38) simplifies to

$$W[A + A(C_1)](C_2) = W[A(C_1)](C_2) W[A](C_2).$$

(39)

Using this relation and, furthermore, eq. (17) one finds from eq. (13) immediately the desired result (2).

A comment is here in order: Strictly speaking ’t Hooft’s loop operator is defined by eq. (1) with $W(C_2)$ replaced by its trace, $trW(C_2)$. However, taking the trace of eq. (1) and taking into account that $\overline{A}(C), (23)$ is a unit matrix in color space precisely replaces $W(C)$ by $tr\overline{W}(C)$. 

GAUGE INVARIANCE

Under (non-singular) gauge transformation $\Omega : A \rightarrow A^\Omega = \Omega A \Omega^\dagger + \Omega \partial \Omega^\dagger$ the operator of the electric field transforms homogeneously

$$\Pi = \Pi^a T_a \rightarrow \Pi^\Omega = \Omega \Pi \Omega^\dagger.$$  \hspace{1cm} (40)

Thus a gauge transformation de facto replaces the center vortex field $A = A_a T_a$ in the 't Hooft loop operator $V(C)$ \cite{23}, by the color rotated one $\Omega^\dagger A \Omega$. Thus in order to prove that the result of the action of the 't Hooft operator \cite{23} is independent of the chosen gauge\cite{4} we have to show that the 't Hooft algebra \cite{2} remains valid when the center vortex field $A(C, x)$ is replaced by $\Omega^\dagger A(C, x) \Omega$. For this we have to show that $A(C)$ and $\Omega^\dagger A(C) \Omega$ give rise to the same evolution operator \cite{35}.

$$U[\Omega^\dagger A(C) \Omega] (s) = U[A(C)] (s).$$  \hspace{1cm} (41)

Since for $s = 1$ the evolution operator $U[A(C)] (s)$ \cite{24} becomes the Wilson loop $W[A(C)] (C_2)$, eq. \cite{4}, from eq. \cite{41} follows that $\Omega^\dagger A(C, x) \Omega$ and $A(C, x)$ represent the same center vortex flux, i.e.

$$W[\Omega^\dagger A(C_1) \Omega] (C_2) = W[A(C_1)] (C_2) = Z^{L(C_1, C_2)}.$$  \hspace{1cm} (42)

To prove eq. \cite{41} we write $\Omega^\dagger A \Omega$ as gauge transformed of $A^\dagger + \Omega \partial \Omega^\dagger$ i.e.

$$\Omega^\dagger A \Omega = (A + \Omega \partial \Omega^\dagger)^\Omega^\dagger.$$  \hspace{1cm} (43)

From the definition \cite{23} follows

$$U[A^\Omega] (s) = \Omega(s) U[A] (s) \Omega^\dagger(0).$$  \hspace{1cm} (44)

Hence we have

$$U[\Omega^\dagger A \Omega] (s) = U[(A + \Omega \partial \Omega^\dagger)^\Omega^\dagger] = \Omega^\dagger(s) U[\Omega^\dagger A + \Omega \partial \Omega^\dagger] (s)(0).$$  \hspace{1cm} (45)

For the evolution operator $U[A(C) + \Omega \partial \Omega^\dagger]$ we use again the “interaction representation” with $A(C)$ and $B = \Omega \partial \Omega^\dagger$ being the “free” and “interaction”, respectively, parts

$$U[A(C)] (s) = U[A(C)] (s) U[B^\dagger(C)] (s),$$  \hspace{1cm} (46)

where the interaction representation $B^\dagger(C)$ is defined by eqs. \cite{24}, \cite{32}. Using the fact that $U[A(C)] (s)$ represents a center element, see eq. \cite{38}, we find that $B^\dagger(C) = B = \Omega \partial \Omega^\dagger$, so that eq. \cite{41} becomes

$$U[A(C) + \Omega \partial \Omega^\dagger] (s) = U[A(C)] (s) U[\Omega \partial \Omega^\dagger] (s).$$  \hspace{1cm} (47)

Using eq. \cite{44} we have

$$U[\Omega \partial \Omega^\dagger] (s) = \Omega(s) U[0](s) \Omega^\dagger(0),$$  \hspace{1cm} (48)

with $U[0](s) = 1$, so that we find from \cite{41} and \cite{47}

$$U[\Omega^\dagger A \Omega] (s) = \Omega^\dagger(s) (U[A(C)] (s) \Omega(s) \Omega^\dagger(0)) \Omega(0) = U[A(C)] (s),$$  \hspace{1cm} (49)

where we have again used that $U[A(C)] (s)$ represents a center element (see eq. \cite{36}). This proves that the vortex generation operator $V(C)$ \cite{23} does not change when the center vortex field $A(C)$ is replaced by the color rotated one $\Omega^\dagger A(C) \Omega$, which proves the gauge invariance of the action of $V(C)$.

\footnote{From eq. \cite{23} and \cite{22} it is seen that 't Hooft's loop operator shifts the canonical coordinate $\vec{A}$ by the center vortex field. Requiring the shifted field variable $\vec{A}' = \vec{A} + \vec{A}[C]$ to transform under gauge transformations in the same way as the unshifted one, $\vec{A}$, (see eq. \cite{22}) implies to demand the “background” field $\mathcal{A}(C)$ to transform homogeneously, $\mathcal{A}(C) \rightarrow \mathcal{A}(C) \Omega^\dagger$. Then the 't Hooft operator would be manifestly gauge-invariant.}
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