A Riemannian Primal-dual Algorithm Based on Proximal Operator and its Application in Metric Learning

1st Shijun Wang  
Dept. of Artificial Intelligence  
Ant Financial Services Group  
Seattle, USA  
shijun.wang@alibaba-inc.com

2nd Baocheng Zhu  
Dept. of Artificial Intelligence  
Ant Financial Services Group  
Shanghai, China  
baocheng.zbc@antfin.com

3rd Lintao Ma  
Dept. of Artificial Intelligence  
Ant Financial Services Group  
Shanghai, China  
lintao.mlt@antfin.com

4th Yuan Qi  
Dept. of Artificial Intelligence  
Ant Financial Services Group  
Hangzhou, China  
yuan.qi@antfin.com

Abstract—In this paper, we consider optimizing a smooth, convex, lower semicontinuous function in Riemannian space with constraints. To solve the problem, we first convert it to a dual problem and then propose a general primal-dual algorithm to optimize the primal and dual variables iteratively. In each optimization iteration, we employ a proximal operator to search optimal solution in the primal space. We prove convergence of the proposed algorithm and show its non-asymptotic convergence rate. By utilizing the proposed primal-dual optimization technique, we propose a novel metric learning algorithm which learns an optimal feature transformation matrix in the Riemannian space of positive definite matrices. Preliminary experimental results on an optimal fund selection problem in fund of funds (FOF) management for quantitative investment showed its efficacy.

I. INTRODUCTION

Many machine learning problems can be solved by optimization algorithms which minimize or maximize a predefined objective function under certain constraints if there are any. In the past decades, searching optimal variables in Euclidean space is a mainstream direction for optimization techniques. In recent years, there is a shift from Euclidean space to Riemannian space due to manifold structures existed in many machine learning problems [1]–[6]. To solve optimization problems in the Riemannian space, a straightforward method is to generalize optimization algorithms developed in the Euclidean space to the Riemannian space with consideration of manifold constraints on the variables to be optimized. Gradient descent methods, Newton’s methods and conjugate gradient methods can be natural extended from the Euclidean space to the Riemannian space, see [7]–[10] and references therein. Studies on Riemannian accelerated gradient methods, quasi-Newton algorithms like BFGS and adaptive optimization methods can be found in [11]–[13]. P. -A. Absil et al. proposed a trust-region approach for optimizing a smooth function on a Riemannian manifold in which the trust-region subproblems are solved using a truncated conjugate gradient algorithm [2]. Furthermore, Qi and Agarwal generalized the adaptive regularization with cubics algorithm to Riemannian manifold, and obtain an upper bound on the iteration complexity which is optimal compared to the complexity of steepest descent and trust-region methods [14], [15]. In recent years, variance reduction techniques drew tremendous attention for optimizing finite-sum problems [16]–[18]. Extending the idea of variance reduction for optimizing finite sums of geodesically smooth functions on Riemannian manifolds can be found in [19]–[21].

Although intensive studies on Riemannian manifold optimization, all the above works have only considered problems on unconstrained manifold, the only constraint is that the solution has to lie on the manifold. This severely limits the scope of possible applications with those methods. In many problems, additional equality or inequality constraints need to be imposed. There are few works addressed this problem, Hauswirth et al. extended the projected gradient descent algorithm to Riemannian manifold with inequality constraints and show its well-behaved convergent behaviour, but the manifold is restricted to submanifold of Euclidean space [22]. Zhang et al. developed an ADMM-like primal-dual approach to solve nonconvex and nonsmooth multi-block optimization over Riemannian manifold with coupled linear equality constraints [23].

In this paper, we consider the following general nonlinear primal problem which is constrained on a complete Riemannian manifold $M$:

$$
\min_{x \in M} f(x),
$$

subject to constraints: $h(x) \preceq 0$, where $f \in C^2(M, R)$ and $h = (h_1, h_2, ..., h_m) \in C^2(M, R)$ are closed proper, convex, lower semicontinuous (l.s.c.) and real-valued functions. A function is of class $C^2$ if its first and second derivatives both exist and are continuous.

A. Previous Works

Khuzani and Li studied stochastic primal-dual method on the Riemannian manifolds with bounded sectional curvature [24]. They proved non-asymptotic convergence of the primal-dual method and established a connection between convergence rate and sectional curvature lower bound. In their algorithm, standard gradient descent method followed by exponential map to search optimal variable in each iteration. In recent
years, proximal algorithms emerge from many machine learning applications due to their capability on handling nonsmooth, constrained, large-scale or distributed problems [25, 26]. Ferreira and Oliveira considered minimization problem on a Riemannian manifold with nonpositive sectional curvature. They solved the problem by extending the proximal method in the Euclidean space to the Riemannian space [27]. Recent advances on Riemannian proximal point method can be found in [28, 29].

B. Our Contributions

Previous works concentrate on either constrained Euclidean space optimization or unconstrained Riemannian space optimization, we propose a novel algorithm to solve constrained optimization problem (1) over the Riemannian manifold. We first convert it to a dual problem and then use a general primal-dual algorithm to optimize the primal and dual variables iteratively. In each optimization iteration, we employ the proximal point algorithm and gradient ascend method alternatively to search optimal solution in the primal and dual space. We prove convergence of the proposed algorithm and show its non-asymptotic convergence rate. To show the efficacy of the proposed primal-dual algorithm for optimizing nonlinear l.s.c. functions in $C^2$ space with constraints, we propose a novel metric learning algorithm and solved it using the proposed Riemannian primal-dual method.

II. NOTATION AND PRELIMINARIES

Let $M$ be a connected and finite dimensional manifold with dimensionality of $m$. We denote by $T_p M$ the tangent space of $M$ at $p$. Let $M$ be endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$, with corresponding norm denoted by $\| \cdot \|$, so that $M$ is a Riemannian manifold [30]. We use $L(\gamma) = \int_0^1 \| \gamma'(t) \| \, dt$ to denote the length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining $x'$ to $x$, i.e., such that $\gamma(a) = x'$ and $\gamma(b) = x$. Minimizing this length functional over the set of all piecewise smooth curves passing $x'$ and $x$ we get a Riemannian distance $d(x', x)$ which induces the original topology on $M$. Take $x \in M$, the exponential map $\text{exp}_x : T_x M \to M$ is defined by $\text{exp}_x v = \gamma_v(1, x)$ which maps a tangent vector $v$ at $x$ to $M$ along the curve $\gamma$. For any $x' \in M$ we define the exponential inverse map $\text{exp}^{-1}_x : M \to T_x M$ which is $C^\infty$ and maps a point $x'$ on $M$ to a tangent vector at $x$ with $d(x', x) = \| \text{exp}^{-1}_x x' \|$. We assume $(M, d)$ is a complete metric space, bounded and all closed subsets of $M$ are compact. For a given convex function $f : M \to R$ at $x' \in M$, a vector $s \in T_{x'} M$ is called subgradient of $f$ at $x' \in M$ if $f(x) \geq f(x') + \langle s, x - x' \rangle$, for all $x \in M$. The set of all subgradients of $f$ at $x' \in M$ is called subdifferential of $f$ at $x' \in M$ which is denoted by $\partial f(x')$. If $M$ is a Hadamard manifold which is complete, simply connected and has everywhere non-positive sectional curvature, the subdifferential of $f$ at any point on $M$ is nonempty [27].

III. THE ALGORITHM

By employing duality, we convert original optimization problem (1) to an augmented Lagrangian function (generic saddle-point problem):

$$L(x, \lambda) = f(x) + \langle \lambda, h(x) \rangle - \frac{\alpha}{2} \| \lambda \|^2,$$

(2)
where $\lambda \in \mathbb{R}_+^n$ is Lagrangian dual vector for inequality constraints, and $\alpha > 0$ is a regularization parameter which weights norm of the dual variables. The norm of the Lagrangian dual variables $\| \lambda \|$ is upper bounded and so is gradient of the deterministic Lagrangian function $L(x, \lambda_i)$.

To solve the generic primal-dual problem shown in eq. (2), we propose a primal-dual algorithm based on proximal operator (see Algorithm 1).

Algorithm 1 Riemannian Primal-dual Algorithm Based on Proximal Operator

1. initialize: set initial point $x_0 \in M$, $\lambda_0 = 0$, and step size sequence $\{ \eta_t \}_{t=0}^\infty$ which is decreasing and $\eta_t \in \mathbb{R}_+$.
2. for $t = 0, 1, 2, ..., T$ do:
   $$x_{t+1} = \text{prox}_L(x_t) = \arg\min_{x \in M} \left( L(x, \lambda_t) + \frac{1}{2\eta_t}d^2(x, x_t) \right),$$
   $$\lambda_{t+1} = [\lambda_t + \eta_t \text{grad}_x L(x_{t+1}, \lambda_t)]_+,$$
where $\text{grad}_x L(x_{t+1}, \lambda) = h(x_{t+1}) - \alpha \lambda$, and $[x]_+$ means for a vector $x$, change every negative entry of $x$ to zero.

Assumption 1. Assume $M$ is a compact Riemannian manifold with finite diameter $R = \sup_{x, y \in M} d(x, y)$ and nonpositive sectional curvature. For any $x \in M$, the following gradients are bounded by $\| \text{grad}_x f(x) \| \leq C$, $\| \text{grad}_x h_k(x) \| \leq C$, and $| h_k(x) | \leq G$, $k = 1, 2, ..., m$.

With Assumption 1, we have the following theorem.

Theorem 1. Assume problem (3) has a saddle-point $(x_*, \lambda_*)$ and Assumption 1 hold. Let $\{ x_t \}_{t=0}^T$ be a finite sequence generated by Algorithm 1 iteratively, and step size $\eta_t \alpha \leq 1$, $t \in [T] = \{ 0, 1, 2, ..., T - 1 \}$. Then,

$$\min_{t \in [T]} f(x_{t+1}) - f(x_*) \leq \frac{1}{\sum_{t=0}^{T-1} \eta_t} \left( \frac{1}{2}d^2(x_0, x_0) + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2) \right)$$

(5)
for all $x_t \in M, t \in [T]$. Proof of Theorem 1 is shown in appendix A.

From Theorem 1, we could derive the following corollary:

Corollary 1. (Non-asymptotic Convergence) By choosing step size $\eta_t = \frac{1}{\sqrt{T+1}}$, and $\alpha \eta_t \leq 1$ for all $t \in [T]$, the sequence $\{ x_t \}, t \in [T]$ generated by Algorithm 1 converges at rate:

$$\min_{t \in [T]} f(x_{t+1}) - f(x_*) = O \left( \frac{\log(T)}{\sqrt{T - 1}} \right).$$

(6)
Proof of Corollary 1 is shown in appendix B.
IV. APPLICATION IN METRIC LEARNING

Metric learning is a technique to learn a distance metric in data feature space, and finds application in various machine learning tasks relying on distances or similarities measure, like classification, clustering, dimensionality reduction and domain adaptation, to name a few [31]–[38]. Most methods learn the metric (positive definite matrix $W$) in a weakly-supervised way from pairwise or triplet constraints of data points. In general, metric learning can be formulated as an optimization problem that shares the same form as standard regularized empirical risk minimization:

$$
\min_{W} \mathcal{L}(W, X) + \lambda \Omega(W),
$$

(7)

where $X$ denotes training samples, $\mathcal{L}$ is the loss function associated with sample constraints and $\Omega$ is the regularizer, $\lambda$ is the trade-off parameter. Many methods are specified as a constrained optimization problem by writing down $\mathcal{L}$ explicitly as inequality constraints $h(W, X) \leq 0$, although we can always transform it into an unconstrained problem using hinge loss or other tricks [31], [37].

Some techniques are developed to solve metric learning optimization problem eq. (7). Projected gradient descent and its stochastic version use traditional (stochastic) gradient descent followed by an orthogonal projection onto the positive semi-definite cone [33], [39]–[41]. Bregman projections update based on one single constraint at each iteration, and perform a general non-orthogonal projection so that the chosen constraint is satisfied. After projecting, an appropriate correction is employed [35].

However, these methods do not fully use the intrinsic manifold structure of the problem, i.e. the learned metric must lie in a Riemannian space of positive definite matrices. So it is naturally an optimization problem on Riemannian manifold rather than Euclidean space. In this section, we apply the proposed method to metric learning problem and illustrate how to optimize a convex target function in a Riemannian manifold.

A. Metric Learning Problem Formulation

We consider the following convex metric learning problem with $W$ in the Riemannian space $\mathbb{S}^n_+$ of $n \times n$ positive definite matrices:

$$
\min_{W \in \mathbb{S}^n_+} \Omega(W),
$$

(8)

s.t.

$$(x_i - x_j) W (x_i - x_j)^\top \leq u, \forall (i, j) \in C^+,$$

$$(x_i - x_j) W (x_i - x_j)^\top \geq l, \forall (i, j) \in C^-,$$

where $\Omega(W) = \frac{1}{2} d^2(W, W_0)$, $d^2(W, W_0) = tr(W W_0^{-1} - \logdet(W W_0^{-1}) - n$ is the LogDet divergence which is a scale-invariant distance measure on Riemannian metrics manifold [35]. $W_0$ is a target transformation matrix initialized to identity (corresponds to the Euclidean distance) or inverse of data covariance matrix (corresponds to the Mahalanobis distance), $C^+/C^-$ is set of all sample pairs with the same/different labels, $u$ and $l$ are the upper/lower distance bound of similar/dissimilar pairs of points and are set to 5-th/95-th percentiles of the observed distribution of distances in the following experiments. It is known that space of all $n \times n$ positive definite Hermitian matrices is a Cartan-Hadamard manifold which is a simply connected complete Riemannian manifold with non-positive sectional curvature.

B. Optimization by the Proposed Riemannian Primal-dual Algorithm

By introducing relaxation variables $\xi$, we have

$$
\min_{W \in \mathbb{S}^n_+} \Omega(W) + \frac{C_1}{2} \parallel \xi \parallel^2_2,
$$

(9)

s.t.

$$(x_i - x_j) W (x_i - x_j)^\top \leq u(1 + \xi_{ij}), \forall (i, j) \in C^+,$$

$$(x_i - x_j) W (x_i - x_j)^\top \geq l(1 - \xi_{ij}), \forall (i, j) \in C^-.$$

Let’s define $h_+ (W) = \text{diag} (X_+ W X_+^\top) - u(e + \xi_+) \leq 0$ ($e$ is a vector whose entries are all one, $\text{diag}(X)$ extracts diagonal elements of an input matrix $X$ and write them to a vector), and $h_- (W) = -\text{diag} (X_- W X_-^\top) + l(e - \xi_-) \leq 0$, where $X_+/X_-.$ are matrices composed by sample pairs with the same / different labels (shape: number of samples by feature dimensions), and $\xi_+/-\xi_-$ are corresponding relaxation vectors which are greater than or equal to zero. So we have

$$
\min_{W \in \mathbb{S}^n_+} \Omega(W) + \frac{C_1}{2} \parallel \xi_+ \parallel^2_2 + \frac{C_1}{2} \parallel \xi_- \parallel^2_2.
$$

(10)

s.t.

$$
\begin{align*}
&h_+ (W) = \text{diag} (X_+ W X_+^\top) - u(e + \xi_+) \leq 0, \\
&h_- (W) = -\text{diag} (X_- W X_-^\top) + l(e - \xi_-) \leq 0, \\
&-\xi_+ \leq 0, \\
&-\xi_- \leq 0.
\end{align*}
$$

Further more, we define $h (W) = [h_+ (W), h_- (W)]^\top$, and $\xi = [\xi_+, \xi_-]$ , then

$$
\min_{W \in \mathbb{S}^n_+} \Omega(W) + \frac{C_1}{2} \parallel \xi \parallel^2_2,
$$

(11)

s.t.

$$
\begin{align*}
&h_+ (W) \leq 0, \\
&-\xi \leq 0.
\end{align*}
$$

By employing duality, we have the following augmented Lagrangian function:

$$
L(W, \xi, \lambda, \gamma) = \Omega(W) + \frac{C_1}{2} \parallel \xi \parallel^2_2 + < \lambda, h(W) > + < \gamma, -\xi > - \frac{C_1}{2} \parallel \xi \parallel^2_2 - \frac{C_1}{2} \parallel \gamma \parallel^2.
$$

Now let’s solve the above Lagrange function using Algorithm 1. At each step $t+1$, we have the following updates:

$$
W_{t+1} = \arg \min_{W \in \mathbb{S}^n_+} \left\{ L(W, \xi_t, \lambda_t, \gamma_t) + \frac{1}{2\eta_t} d^2(W_t, W) \right\},
$$

$$
\xi_{t+1} = \arg \min_{\xi \in [0, \infty)^n} \left\{ L(W, \xi, \lambda_t, \gamma_t) + \frac{1}{2\eta_t} d^2(\xi_t, \xi) \right\},
$$

$$
\lambda_{t+1} = [\lambda_t + \nabla_{\lambda_t} L(W_{t+1}, \xi_{t+1}, \lambda_t, \gamma_t)],
$$

$$
\gamma_{t+1} = [\gamma_t + \nabla_{\gamma_t} L(W_{t+1}, \xi_{t+1}, \lambda_t, \gamma_t)].
$$

So,

$$
W_{t+1} = \arg \min_{W \in \mathbb{S}^n_+} \left\{ \frac{1}{2} d^2(W, W_0) + < \lambda_t, h(W) > + \frac{1}{2\eta_t} d^2(W_t, W) \right\},
$$

$$
\xi_{t+1} = \arg \min_{\xi \in [0, \infty)^n} \left\{ \frac{C_1}{2} \parallel \xi \parallel^2_2 + < \lambda_t, h(W, \xi) > + < \gamma_t, -\xi > + \frac{1}{2\eta_t} d^2(\xi_t, \xi) \right\}.
$$
\[ \lambda_{t+1} = [\lambda_t + \eta_t \gamma_t] < \lambda, h(\mathbf{W}_{t+1}) > - \frac{\gamma_t}{2} \| \lambda_t \|^2  \]_

\[ \gamma_{t+1} = [\gamma_t + \eta_t \gamma_t] < \gamma_t, -\xi_{t+1} > - \frac{\gamma_t}{2} \| \gamma_t \|^2  \]_

In the following paragraphs, we will show how to update primal and dual variables in each iteration.

1. We employ Riemannian gradient decent method to search optimal \( \mathbf{W}_{t+1} \). Define

\[ J_W = L(\mathbf{W}, \xi_t, \lambda_t, \gamma_t) = \frac{1}{2} d^2(\mathbf{W}, \mathbf{W}_0) + < \lambda, h(\mathbf{W}) >. \]

We have \( \mathbf{W}_{t+1} = R_W(-\eta_t \gamma_t, \mathbf{W}_t) \), where \( \gamma_t, \xi_t, \lambda_t \) are the Riemannian gradient and operator \( R_W \) means retraction. See Appendix C for the full procedure.

2. Define \( J_\xi = \frac{\gamma}{2} \| \xi \|^2 + < \lambda_t, h(\mathbf{W}, \xi) > + < \gamma_t, -\xi > + \frac{1}{2\eta_t} d^2(\xi_t, \xi) \) and \( d^2(\xi_t, \xi) = \| \xi - \xi_t \|^2 \).

\[ grad_{\xi} J_\xi = C_1 \xi - < \lambda_t, (\gamma_t) > - \gamma_t + \eta_t (\xi - \xi_t) = (C_1 + \eta_t) \xi - \eta_t \xi_t - \gamma_t < \lambda_t, (\gamma_t) > \]

\[ \xi_{t+1} = \frac{1}{C_1 + \eta_t} (\eta_t \xi_t + \gamma_t + < \lambda_t, (\gamma_t) >) \]_

3. \[ \lambda_{t+1} = (1 - C_2 \eta_t) \lambda_t + \eta_t h(\mathbf{W}_{t+1}) \]_

4. \[ \gamma_{t+1} = (1 - C_2 \eta_t) \gamma_t - \xi_{t+1} \]_

C. Experimental Results

Both investors and machine learning researchers showed great interests on applying machine learning to finance area in recent years. The Holy Grail of quantitative investment is selection of high-quality financial assets with good timing to achieve higher returns with less risk [42]. To measure quality and trend of financial assets, technical, fundamental, and macroeconomic factors or features are developed. Usually multi-factor regression models [43] are deployed to find the most effective features to achieve higher asset return. However, when the number of features is large, or heterogeneity/multicollinearity exists in these features, traditional factor-oriented asset selection models tend to fail and may not give encouraging results.

Each asset can be represented by a data point in high-dimensional feature space, then a good distance metric in the space is crucial for more accurate similarity measure between assets. In our treating, assets selection can be regarded as a classification problem, assets are divided into two groups, which is exactly what \( \Theta \) formulated. Using metric learning approach to asset selection, above mentioned factor model problems can be largely alleviated. In following sections, we apply the proposed Riemannian primal-dual metric learning (RDPML) algorithm to fund of funds (FOF) management problem. FOF is a multi-manager investment strategy whose portfolios are composed by mutual or hedge funds which invest directly in stocks, bonds or other financial assets.

D. Data

In this research, we consider Chinese mutual funds that were publicly traded for at least 12 consecutive months in the period 2012-01-01 to 2018-12-31. We select a total of 697 funds with capital size larger than 100 million RMB. Fund features consist of totally 70 technical factors with different rolling windows (10, 14, 21, 28, 42, 63 and 90 trading days respectively), including ROC, EMA, MDD, STDDEV, Sharpe ratio, Sortino ratio, Calmar ratio, RSI, MACD, and Stability [https://www.investopedia.com/]. All features are normalized to have zero mean and unit variance.

E. Backtest Protocol

For mutual fund management, typical length of rebalancing interval is one quarter of a year. So we split original sequential fund data into segments of quarters. We use a rolling window prediction schema, in the training set, we learn a distance metric from 70 technical factors from previous quarter with quarterly return of current quarter as target. In the test set, features from current quarter are used to predict quarterly return of next quarter. To validate learned metric, a simple k-nearest neighbors (k-NN) algorithm is employed, the predict return of next quarter for each fund is based on the learned metric from training set. The hyper-parameter k of k-NN is set as 10. To avoid overfitting, rolling data before 2017-01-01 are used as validation and data from 2017-01-01 to 2018-12-31 as test. For convenience, following results are based on both validation and test set.

For each fund, with k nearest neighbor funds predicted, we use average of returns of the k neighbor funds in current quarter as prediction of the fund’s return in the next quarter. At each quarterly rebalance day (the last trading day of each quarter), a top 10 buy trading strategy is used based on the prediction of each fund. In this strategy, we rank all funds based on their predicted quarterly return and select the top 10 funds for portfolio construction and rebalancing with equal weight.

We compare RDPML with the following four distance metrics and a baseline fund index:

- Euclidean distance metric.
- Mahalanobis distance metric, which is computed as the inverse of covariance matrix of training samples.
- LMNN, a distance metric learning algorithm based on the large margin nearest neighbor classification [34].
- ITML, a distance metric learning algorithm based on information geometry [35].
- GMML, a distance metric learning algorithm, the learned metric can be viewed as “matrix geometric mean” on the Riemannian manifold of positive definite matrices [37].

- Baseline fund index, CSI Partial Equity Funds Index [http://www.csindex.com.cn/en/indices/index-detail/930950]

F. Result

In the FoF setting, we care more about the order of predicted return than the absolute value. So we calculate the Spearman’s rank-order correlation of the predicted return to the true return for different algorithms, a higher correlation means better predictive power. In financial community, this correlation is often called Information coefficient (IC). The calculation is done rollingly, and we show the mean and standard deviation of IC in Table 1. We can see that RDPML achieves highest...
mean correlation. The backtest performance using different prediction models is shown in Figure 1. We also show CSI Partial Equity Funds Index (dashed curves) as baseline fund index which reflects overall performance of all partial equity funds in China’s financial market. From the top panel, we observed that all the experimental algorithms outperform baseline index, and among them, RPDML achieved the best performance, with total accumulated return of 148% in the whole backtest period, while the worst portfolio with Euclidean distance metric only achieved 25%, even less than CSI Partial Equity Funds Index. We also plot one year rolling maximum drawdown (MDD) in the bottom panel, MDD of the RPDML algorithm is quite low which reflects overall performance of all partial equity funds potential Equity Funds Index (dashed curves) as baseline fund index.

In Figure 2, we show annual returns of FOF of each algorithm. We can see that the proposed algorithm PRDML achieved highest returns in most years. Besides, we also notice that LMNN performed quite well in some years.

| Algorithm   | Euclidean | Mahalanobis | LMNN | ITML | GMML | RPDML | Fund Index |
|-------------|-----------|-------------|------|------|------|--------|-------------|
| IC          | 0.018 ± 0.138 | 0.050 ± 0.071 | 0.017 ± 0.201 | 0.030 ± 0.071 | 0.068 ± 0.086 | 0.069 ± 0.187 |          |

Table I: Spearman’s correlation/information coefficient for different algorithms.

![Fig. 1. Portfolio performance comparison of each metric learning algorithm and a baseline fund index. (Top) Accumulate rate of return, (Bottom) Maximum drawdown.](image)

In Figure 2, we show annual return of FOF for each algorithm.

**V. CONCLUSION**

In this paper, we propose a Riemannian primal-dual algorithm based on proximal operator for optimizing a smooth, convex, lower semicontinuous function on Riemannian manifolds with constraints. We prove convergence of the proposed algorithm and show its non-asymptotic rate. By utilizing the proposed primal-dual optimization technique, we propose a novel metric learning algorithm which learns an optimal feature transformation matrix in the Riemannian space of positive definite matrices. Preliminary experimental results on an optimal fund selection problem in FOF management for quantitative investment showed its efficacy.

**VI. APPENDIX**

**A. Proof of Theorem 1**

Due to convexity of $L(x, \lambda)$, for any $x \in M$ we have

$$L(x, \lambda_t) \geq L(x_{t+1}, \lambda_t) + <s, \exp^{-1}_{x_{t+1}} x>,$$  \hspace{1cm} (12)

where $s \in \partial L(x_{t+1}, \lambda_t)$ and $\exp^{-1}_{x_{t+1}} x \in T_{x_{t+1}} M$.

Because $M$ is a Hadamard manifold which has non-positive curvature, then

$$d^2(x, x_t) \geq d^2(x, x_{t+1}) + d^2(x_{t+1}, x_t) - 2 <\exp^{-1}_{x_{t+1}} x, \exp^{-1}_{x_{t+1}} x>. \hspace{1cm} (13)$$

[ref. [44], Proposition 1]

Multiplying eq. (13) by $\frac{1}{2\eta_t}$ and summing the result with eq. (12), we get the following inequality:

$$L(x, \lambda_t) + \frac{1}{2\eta_t} d^2(x, x_t) \geq L(x_{t+1}, \lambda_t) + \frac{1}{2\eta_t} d^2(x, x_{t+1}) + \frac{1}{2\eta_t} d^2(x_{t+1}, x_t) + \frac{1}{2\eta_t} d^2(x_{t+1}, x_t)$$

$$+ <s - \eta_t \exp^{-1}_{x_{t+1}} x_t, \exp^{-1}_{x_{t+1}} x>. \hspace{1cm} (14)$$

From eq. (13), we have $0 \in \partial L(x_{t+1}, \lambda_t) + \eta_t \exp^{-1}_{x_{t+1}} x_t$. So

$$L(x, \lambda_t) + \frac{1}{2\eta_t} d^2(x, x_t) \geq L(x_{t+1}, \lambda_t) + \frac{1}{2\eta_t} d^2(x, x_{t+1}) + \frac{1}{2\eta_t} d^2(x_{t+1}, x_t).$$

Since $\frac{1}{2\eta_t} d^2(x_{t+1}, x_t)$ is zero or positive, by taking out it,

$$L(x_{t+1}, \lambda_t) - L(x, \lambda_t) \leq \frac{1}{2\eta_t} d^2(x, x_t) - \frac{1}{2\eta_t} d^2(x, x_{t+1}).$$
Let \((x_*, \lambda_*)\) be the saddle (min-max) point which satisfies 
\[ L(x_*, \lambda) \leq L(x_*, \lambda_*) \leq L(x, \lambda_*) \]. By choosing \(x = x_*\), we have
\[ L(x_{t+1}, \lambda_t) - L(x_*, \lambda_t) \leq \frac{1}{2\eta_t} d^2(x_*, x_t) - \frac{1}{2\eta_t} d^2(x_*, x_{t+1}) . \] (15)

Multiplying eq. (15) with \(\eta_t\) and summing over \(t = 0, 1, 2, ..., T - 1\),
\[ \sum_{t=0}^{T-1} \eta_t \left( L(x_{t+1}, \lambda_t) - L(x_*, \lambda_t) \right) \leq \frac{1}{2} \sum_{t=0}^{T-1} \left( d^2(x_*, x_t) - d^2(x_*, x_{t+1}) \right) . \] (16)

For any dual variable \(\lambda \in \mathbb{R}^m_+\), we have
\[ \| \lambda_{t+1} - \lambda \|^2 = \| \lambda_t + \eta_t \nabla \lambda_t \|_{\lambda} - \lambda \|^2 \leq \| \lambda_t - \lambda \|^2 + 2\eta_t \left( \| \nabla \lambda_t \|_{\lambda} \right) \]
By summing over \(t = 0, 1, 2, ..., T - 1\), and using the telescoping sum series, we have
\[ \sum_{t=0}^{T-1} \left( \eta_t^2 \| \nabla \lambda_t \|_{\lambda} \right) = \frac{1}{2} \sum_{t=0}^{T-1} \left( d^2(x_*, x_t) - d^2(x_*, x_{t+1}) \right) . \]
(17)

Because \(L(x_{t+1}, \lambda)\) is concave with respect to \(\lambda\),
\[ \nabla \lambda_t \|_{\lambda} \leq L(x_{t+1}, \lambda) - L(x_{t+1}, \lambda_t) \]
By replacing \(\nabla \lambda_t \|_{\lambda} > \lambda - \lambda_t \) in eq. (17), we have
\[ \sum_{t=0}^{T-1} \left( \eta_t^2 \| \nabla \lambda_t \|_{\lambda} \right) \]
(18)

Combine eq. (16) and eq. (18).
\[ \sum_{t=0}^{T-1} \eta_t \left( L(x_{t+1}, \lambda) - L(x_*, \lambda_t) \right) \]
\[ \leq \frac{1}{2} \sum_{t=0}^{T-1} \left( d^2(x_*, x_t) - d^2(x_*, x_{t+1}) \right) + \frac{1}{2} \| \lambda \|^2 . \] (19)

To bound the gradient item \(\| \nabla \lambda_t \|_{\lambda}^2\) in eq. (19), we employ Lemma 13 from Ref. [24].
\[ \sum_{t=0}^{T-1} \eta_t \left( L(x_{t+1}, \lambda) - L(x_*, \lambda_t) \right) \]
\[ \leq \frac{1}{2} \sum_{t=0}^{T-1} \left( d^2(x_*, x_t) - d^2(x_*, x_{t+1}) \right) + \frac{1}{2} \| \lambda \|^2 + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2) . \] (20)

Now let's expand the left side of eq. (20).
\[ \sum_{t=0}^{T-1} \eta_t \left( L(x_{t+1}, \lambda) - L(x_*, \lambda_t) \right) = \frac{1}{2} \sum_{t=0}^{T-1} \left( d^2(x_*, x_t) - d^2(x_*, x_{t+1}) \right) + \frac{1}{2} \| \lambda \|^2 + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2) . \]

Since \(h(x_*) \leq 0, \alpha > 0, \lambda_1 \geq 0\) and \(\| \lambda_t \|^2 \geq 0\), by removing positive terms \(\leq \lambda, h(x_*)\) and \(\alpha \| \lambda \|^2\), we have the following inequality
\[ \sum_{t=0}^{T-1} \eta_t \left( f(x_{t+1}) - f(x_*) + \lambda, h(x_{t+1}) > -\alpha \| \lambda \|^2 \right) \]
\[ \leq \frac{1}{2} d^2(x_*, x_0) + \frac{1}{2} \| \lambda \|^2 + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2) . \] (21)

By moving \(\frac{1}{2} \| \lambda \|^2\) from r.h.s. of eq. (21) to l.h.s. of eq. (21), we have
\[ \sum_{t=0}^{T-1} \eta_t \left( f(x_{t+1}) - f(x_*) \right) \]
\[ + \left( \lambda, \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) > -\alpha \| \lambda \|^2 \right) \]
\[ \leq \frac{1}{2} d^2(x_*, x_0) + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2) . \] (22)

By maximizing \(\left( \lambda, \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) \right) \geq \frac{\alpha \sum_{t=0}^{T-1} \eta_t}{2} \| \lambda \|^2\)
, we have \(\lambda_{max} = \left( \left( \alpha \sum_{t=0}^{T-1} \eta_t \right) + 1 \right)^{-1} \left[ \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) \right]_+ \).
and
\[ \max \left( \left( \lambda, \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) > -\frac{1}{2} \left( \alpha \sum_{t=0}^{T-1} \eta_t \right) + 1 \right) \| \lambda \|^2 \right) \]
\[ = \frac{1}{2} \left( \left( \alpha \sum_{t=0}^{T-1} \eta_t \right) + 1 \right)^{-1} \left( \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) \right)_+ \| \lambda \|^2 . \]
Since \( \lambda \in \mathbb{R}^m_+ \) could be any value in \( \mathbb{R}^m_+ \), so
\[
\sum_{t=0}^{T-1} \eta_t (f(x_{t+1}) - f(x_t)) + \left( \frac{1}{\lambda} \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) + \frac{\alpha \left( \sum_{t=0}^{T-1} \eta_t \right)}{2} \right) \| \lambda \|^2 \\
\leq \sum_{t=0}^{T-1} \eta_t (f(x_{t+1}) - f(x_t)) + \frac{1}{2} \left( \left( \frac{1}{\lambda} \sum_{t=0}^{T-1} \eta_t \right) + 1 \right) \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) \left( \sum_{t=0}^{T-1} \eta_t h(x_{t+1}) \right)^2 \\
\leq \frac{1}{2} d^2(x_0, x_T) + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2). \]

By removing the maximum term which is positive on the l.h.s. of above equation, we have
\[
\sum_{t=0}^{T-1} \eta_t (f(x_{t+1}) - f(x_t)) \leq \frac{1}{2} d^2(x_0, x_T) + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2). \]

Since
\[
\sum_{t=0}^{T-1} \eta_t (f(x_{t+1}) - f(x_t)) \geq \sum_{t=0}^{T-1} \eta_t \left( \min_{t \in [T]} f(x_{t+1}) - f(x_t) \right) = \left( \min_{t \in [T]} f(x_{t+1}) - f(x_t) \right) \sum_{t=0}^{T-1} \eta_t, \]
we have
\[
\left( \min_{t \in [T]} f(x_{t+1}) - f(x_t) \right) \leq \frac{1}{\sum_{t=0}^{T-1} \eta_t} \left( \frac{1}{2} d^2(x_0, x_T) + 2mG^2 \sum_{t=0}^{T-1} (\eta_t^2) \right). \tag{23} \]
End of proof.

B. Proof of Corollary 1

The proof simply follows the proof of Corollary 8 in ref. [24].

First we have the following bounds:
\[
\sum_{t=0}^{T-1} (\eta_t) = \sum_{t=0}^{T-1} \frac{1}{\sqrt{t+1}} \geq \int_{0}^{T-1} \frac{1}{\sqrt{t+1}} dt = 2\sqrt{T} - 1 + \frac{1}{2} T \leq 1 + \frac{1}{T} \leq 1 + \log (T). \]
\[
\sum_{t=0}^{T-1} (\eta_t^2) = \sum_{t=0}^{T-1} \frac{1}{t+1} \leq 1 + \int_{0}^{T-1} \frac{1}{t+1} dt = 1 + \log (T). \]

By using the fact that \( d^2(x_0, x_T) \leq R^2 \), we have
\[
\left( \min_{t \in [T]} f(x_{t+1}) - f(x_t) \right) \leq \frac{1}{2} \left( \frac{1}{\sqrt{T} - 1} \right) \left( \frac{1}{2} R^2 + 2mG^2 (1 + \log (T)) \right) = O \left( \frac{\log (T)}{\sqrt{T} - 1} \right). \]

C. Riemannian Gradient with Retraction

We can write Euclidean gradient as \( \text{grad}_{W} J_{W} = \partial_{W} (\frac{1}{2} d^2(W_t, W) + < \lambda_t, h(W) >) \)
\[
= \frac{1}{2} (W_0 - W)^{-1} + < \lambda_t, \partial_{W} (\eta_t \partial_{W}) > \]
\[
\frac{\partial_{W} (\eta_t \partial_{W})}{\partial_{W}} = \frac{\partial_{W} \left( [h_+(W), h_-(W)]^T \right)}{\partial_{W}} = \left[ \text{diag} \left( \text{diag} (X_0^T), \text{diag} (X_1^T) \right), -\text{diag} \left( \text{diag} (X_0^T), \text{diag} (X_1^T) \right) \right]^T, \]
where \( \text{diag} \left( \text{diag} (X_0^T), \text{diag} (X_1^T) \right) \) means constructing a block diagonal matrix whose block diagonal elements are columns / rows of \( X_0^T, X_1^T \). In another word, each element of \( \partial_{W} (\eta_t \partial_{W}) \) is a covariance matrix \( x_i^T \times x_i \) of a sample pair vector \( x_i \), \( i = 1, 2, ..., N \) (assume the expectation of \( x_i \), \( i = 1, 2, ..., N \), is zero).
\[
< \lambda_t, \frac{\partial_{W} (\eta_t \partial_{W})}{\partial_{W}} > \]
is tensor dot product between \( \lambda_t \) and a vector of size \( N \) whose element is covariance matrix of the \( i \)-th sample pair.

1) Riemannian Gradient

First, we show that in our case, \( \text{Grad}_{W} J_{W} = \text{grad}_{W} J_{W} \).
Let’s define \( M_{\lambda} := \{ W = U \Sigma V^T \} \), s.t. \( U \in \mathbb{R}^{m \times d} \), \( V \in \mathbb{R}^{d \times n} \), and \( \Sigma = \text{diag} (\sigma_1, \sigma_2, ..., \sigma_d) \), with \( \sigma_i > 0 \), \( \forall i \). \( M_{\lambda} \) is the Stiefel manifold of \( m \times d \) real, orthonormal matrices. \( M_{\lambda} \) is a Riemannian manifold with tangent space \( \mathbb{R}^{m \times d} \).

For a given objective function \( J \) which depends on input matrix \( W \) of size \( m \times n \), we use \( \text{grad}_{W} J_{W} \in \mathbb{R}^{m \times n} \) to represent Euclidean gradient of \( J \) w.r.t \( W \); and denote \( \text{Grad}_{W} J_{W} \) as its Riemannian gradient by projecting the Euclidean gradient onto the tangent space of \( M_{\lambda} \):

\[
\text{Grad}_{W} J_{W} = P_{U} \text{grad}_{W} J_{W} P_{U}^T + P_{V} \text{grad}_{W} J_{W} P_{V}^T + \text{grad}_{W} J_{W} P_{U}^T + \text{grad}_{W} J_{W} P_{V}^T, \]
where \( P_{U} := UU^T, P_{U}^T := I - UU^T, P_{V} := VV^T, P_{V}^T := I - VV^T \), and \( \text{grad}_{W} J_{W} \) is the projection of the Euclidean gradient \( \text{grad}_{W} J_{W} \) onto the tangent space of \( M_{\lambda} \).

For our problem, since \( W \) is a real symmetric positive definite matrix, we have \( M_{\lambda} := \{ W = Q\Lambda Q^T \} \), and \( T_{W} M_{\lambda} := \{ QMQ^T \} \), where \( Q \) is an orthogonal matrix, \( \Lambda \) is a diagonal matrix whose entries are the eigenvalues of \( W \) and greater than or equal to zero.

Since \( P_{U} := UU^T = QQ^T = I, P_{U}^T := I - UU^T = 0, P_{V} := VV^T = I, P_{V}^T := I - VV^T = 0 \), the projection of the Euclidean gradient \( \text{grad}_{W} J_{W} \) onto the tangent space of \( M_{\lambda} \) is

\[
\text{Grad}_{W} J_{W} := P_{U} \text{grad}_{W} J_{W} P_{U}^T + \text{grad}_{W} J_{W} P_{U}^T + \text{grad}_{W} J_{W} P_{V}^T = \text{grad}_{W} J_{W}. \]

2) Retraction

With \( W_t \) and \( \text{Grad}_{W} J_{W} \) shown above, we would like to calculate \( W_{t+1} \) using retraction. From Ref. [3], for any tangent vector \( \eta \in T_{W} M_{\lambda} \), its retraction \( R_{W} (\eta) := \text{argmin} \| W + \eta - X \| \) for any \( X \) in \( M_{\lambda} \). For our case \( R_{W} (\eta) \text{Grad}_{W} J_{W} = \sum_{i=1}^{n} \sigma_i q_i q_i^T \), where \( \sigma_i \) and \( q_i \) are the \( i \)-th eigenvalues and eigenvector of matrix \( W \), \( \text{Grad}_{W} J_{W} \).

The calculation and retraction shown above are repeated until \( J_{W} \) converges.
