HOW EFFECTIVE IS NEW VARIABLE MODIFIED CHAPLYGIN GAS TO PLAY THE ROLE OF DARK ENERGY- A DYNAMICAL SYSTEM ANALYSIS IN RS II BRANE MODEL

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Motivated by some previous works of Rudra et al we set to explore the background dynamics when dark energy in the form of New Variable Modified Chaplygin gas is coupled to dark matter with a suitable interaction in the universe described by brane cosmology. The main idea is to find out the efficiency of New variable modified Chaplygin gas to play the role of DE. As a result we resort to the technique of comparison with standard dark energy models. Here the RSII brane model have been considered as the gravity theory. An interacting model is considered in order to search for a possible solution of the cosmic coincidence problem. A dynamical system analysis is performed because of the high complexity of the system. The statefinder parameters are also calculated to classify the dark energy model. Graphs and phase diagrams are drawn to study the variations of these parameters and get an insight into the effectiveness of the dark energy model. It is also seen that the background dynamics of New Variable Modified Chaplygin gas is consistent with the late cosmic acceleration. After performing an extensive mathematical analysis, we are able to constrain the parameters of new variable modified Chaplygin gas as $m < n$ to produce the best possible results. Future singularities are studied and it is found that the model has a tendency to result in such singularities unlike the case of generalized cosmic Chaplygin gas. Our investigation leads us to the fact that New Variable Modified Chaplygin gas is not as effective as other Chaplygin gas models to play the role of dark energy.

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I. INTRODUCTION

Recent cosmic acceleration is a well-known and accepted fact in the cosmological society currently[1, 2]. The root cause for this phenomenon is still under research. However of late there have been some speculations regarding the existence of a mysterious negative pressure component which violates the strong energy condition i.e. \( \rho + 3p < 0 \). Because of its invisible nature this energy component is aptly termed as dark energy (DE) [3].

Since the concept of DE flourished in the last decade, cosmologists all over the world started searching for a suitable model of DE. As a result various DE models have come into existence of late. DE represented by a scalar field \(^1[4]\) is often called quintessence. Not only scalar field but also there are other Dark fluid models like Chaplygin gas which plays the role of DE very efficiently. As time passed extensive research was conducted and Chaplygin gas (CG) [5, 6], got modified into Generalized Chaplygin gas (GCG) [7–11] and then to Modified Chaplygin gas (MCG) [12, 13]. In this context it is worth mentioning that dynamics of MCG in Brane world was studied by Rudra et al [14]. Other than these other forms of Chaplygin gas models have also been proposed such as Variable Modified Chaplygin gas (VMCG) [15], New Variable Modified Chaplygin gas (NVMCG) [16], generalized cosmic Chaplygin gas (GCCG) [17]. Dynamics of GCCG in Loop Quantum cosmology was studied by Chowdhury et al in [18]. The dynamics of GCCG in braneworld was studied by Rudra in [19]. Other existing forms of DE are phantom [20], k-essence [21], tachyonic field [22], etc.

The equation of state for NVMCG is given by,

\[
p = A(a)\rho - \frac{B(a)}{\rho^\alpha}, \quad 0 \leq \alpha \leq 1
\]

Here we will consider \( A(a) = A_0 a^{-n} \) and \( B(a) = B_0 a^{-m} \), where \( A_0, B_0, \alpha, m \) and \( n \) are positive constants.

Currently, we live in a special epoch where the densities of DE and DM are comparable. The fact that they have evolved independently from different mass scales makes the fact more interesting. Given their non co-existence in evolution, comparable densities is quite an unexpected phenomenon. This is known as the famous cosmic coincidence problem. Till date several attempts have been made to find a solution to this problem [23, 24]. A suitable interaction between DE and DM provides the best method of solution for this problem. It is obvious that a transition has occurred

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\(^1\) in the presence of a scalar field the transition from a universe filled with matter to an exponentially expanding universe is justified
from a matter dominated universe to dark energy dominated universe, by exchange of energy at an appropriate rate. Now in order to be consistent with the expansion history of the universe as confirmed by the supernovae and CMB data the decay rate has to be fixed such that it is proportional to the present day Hubble parameter. Keeping the fact in mind cosmologists all over the world have studied and proposed a variety of interacting DE models.

Now, dark energy is not the only concept that can demonstrate the present day universe. The left hand side of the Einstein’s field equation can also modified, to obtain suitable results. This modification however gives rise to the famous modified gravity theories, which in their own right can independently give us suitable models for our expanding universe. In this context Brane-gravity was introduced and brane cosmology was developed. A review on brane-gravity and its various applications with special attention to cosmology is available in. In this work we consider a very popular model of brane gravity, namely the RS II brane. The main objective of this work is to examine the nature of the different physical parameters of the DE for the universe around the stable critical points in the brane model in presence of NVMCG. Effectiveness and success of the mathematical formulation of NVMCG will be studied. Impact of any future singularity caused by the DE in the brane world model will also be studied.

This paper is organized as follows: Section 2 comprises of the analysis in RS II brane model. In section 3, a detailed graphical analysis for the phase plane is given. In section 4, some details regarding the mathematical construction of NVMCG is provided. In section 5 future singularities arising from the model are studied and finally the paper ends with some concluding remarks in section 6.

II. MODEL 1: RS II BRANE MODEL

Randall and Sundrum proposed a bulk-brane model to explain the higher dimensional theory, popularly known as RS II brane model. According to this model we live in a four dimensional world (called 3-brane, a domain wall) which is embedded in a 5D space time (bulk). All matter fields are confined in the brane whereas gravity can only propagate in the bulk. The consistency of this brane model with the expanding universe has given popularity to this model of late in the field of cosmology.

In RS II model the effective equations of motion on the 3-brane embedded in 5D bulk having $Z_2$-symmetry are given by

$$G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \kappa_4^2 \tau_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}$$  \hspace{1cm} (2)
where

\[ \kappa_4^2 = \frac{1}{6} \lambda \kappa_5^4 \],

(3)

\[ \Lambda_4 = \frac{1}{2} \kappa_5^2 \left( \Lambda_5 + \frac{1}{6} \kappa_5^2 \lambda^2 \right) \],

(4)

and

\[ \Pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau^{\alpha}_{\nu} + \frac{1}{12} \tau \tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2 \]

(5)

and \( E_{\mu\nu} \) is the electric part of the 5D Weyl tensor. Here \( \kappa_5, \Lambda_5, \tau_{\mu\nu} \) and \( \Lambda_4 \) are respectively the 5D gravitational coupling constant, 5D cosmological constant, the brane tension (vacuum energy), brane energy-momentum tensor and effective 4D cosmological constant. The explicit form of the above modified Einstein equations in flat universe are

\[ 3H^2 = \Lambda_4 + \kappa_4^2 \rho + \frac{\kappa_4^2}{2\lambda} \rho^2 + \frac{6}{\lambda \kappa_4^2} U \]

(6)

and

\[ 2\dot{H} + 3H^2 = \Lambda_4 - \kappa_4^2 \rho - \frac{\kappa_4^2}{2\lambda} \rho \rho - \frac{\kappa_4^2}{2\lambda} \rho^2 - \frac{2}{\lambda \kappa_4^2} U \]

(7)

The dark radiation \( U \) obeys

\[ \dot{U} + 4HU = 0 \]

(8)

where \( \rho = \rho_{nvmcg} + \rho_m \) and \( p = p_{nvmcg} + p_m \) are the total energy density and pressure respectively.

As in the present problem the interaction between DE and pressureless DM has been taken into account for interacting DE and DM the energy balance equation will be

\[ \dot{\rho}_{nvmcg} + 3H (1 + \omega_{nvmcg}) \rho_{nvmcg} = -Q, \text{ for NVMCG and} \]

(9)

\[ \dot{\rho}_m + 3H \rho_m = Q, \text{ for the DM interacting with NVMCG.} \]

(10)

where \( Q = 3bH \rho \) is the interaction term, \( b \) is the coupling parameter (or transfer strength) and \( \rho = \rho_{nvmcg} + \rho_m \) is the total cosmic energy density which satisfies the energy conservation equation

\[ \dot{\rho} + 3H (\rho + p) = 0 \]
Since we lack information about the fact, how does DE and DM interact so we are not able to estimate the interaction term from the first principles. However, the negativity of $Q$ immediately implies the possibility of having negative DE in the early universe which is overruled by the necessity of the second law of thermodynamics to be held \[41\]. Hence $Q$ must be positive and small. From the observational data of 182 Gold type Ia supernova samples, CMB data from the three year WMAP survey and the baryonic acoustic oscillations from the Sloan Digital Sky Survey, it is estimated that the coupling parameter between DM and DE must be a small positive value (of the order of unity), which satisfies the requirement for solving the cosmic coincidence problem and the second law of thermodynamics \[42\]. Due to the underlying interaction, the beginning of the accelerated expansion is shifted to higher redshifts. The continuity equations for dark energy and dark matter are given in equations (9) and (10). Now we shall study the dynamical system assuming $\Lambda_4 = U = 0$ (in absence of cosmological constant and dark radiation).

**A. DYNAMICAL SYSTEM ANALYSIS**

In this subsection we plan to analyze the dynamical system. For that firstly we convert the physical parameters into some dimensionless form, given by

$$x = \ln a, \quad u = \frac{\rho_{nvmcg}}{3H^2}, \quad v = \frac{\rho_m}{3H^2}, \quad y = \frac{a}{3H^2}$$  \label{11}

where the present value of the scale factor $a_0 = 1$ is assumed. Using eqns. (1), (6), (7), (9) and (10) into (11) we get the parameter gradients as

$$\frac{du}{dx} = -3b(u + v) - 3\left(u + u\omega^{(RSII)}_{nvmcg}\right) - 6\frac{\dot{H}}{X}u$$ \label{12},

$$\frac{dv}{dx} = 3b(u + v) - 3v - 6\frac{\dot{H}}{X}v$$ \label{13}

and

$$\frac{dy}{dx} = y\left(1 - 6\frac{\dot{H}}{X}\right)$$ \label{14}

where $\omega^{(RSII)}_{nvmcg}$ is the EoS parameter for NVMCG determined as

$$\omega^{(RSII)}_{nvmcg} = p_{nvmcg} = \frac{1}{y^n X^n} \left(A - \frac{By^{n-m}}{u^{\alpha+1}X^{m-n+\alpha+1}}\right),$$ \label{15}
\[ \dot{H} = \frac{\lambda}{u + v} \left( \kappa^2 - \frac{1}{u + v} \right) \left[ u \omega_{\text{nvmcg}}^{(RSHI)} + \left\{ \left( \frac{1}{\kappa^2 (u + v)} - 1 \right) (u \omega_{\text{nvmcg}}^{RSHI} + 2v) \right\} \right] \]  

and

\[ X = \frac{2\lambda}{u + v} \left[ \frac{1}{\kappa^2 (u + v)} - 1 \right] \]  

1. CRITICAL POINTS

The critical points of the above system are obtained by putting \( \frac{du}{dx} = \frac{dv}{dx} = \frac{dy}{dx} = 0 \). But due to the complexity of these equations, it is not possible to find a solution in terms of all the involved parameters. So we find a solution for the above system, by putting the following numerical values to some of the parameters appearing in the system. We take,

\[ \alpha = 0.5, \quad b = 0.5, \quad n = 1, \quad m = 1 \]

and obtain the following critical point,

\[ u_c = \frac{0.3125}{\kappa^2}, \quad v_c = \frac{0.1875}{\kappa^2}, \quad y_c = \frac{1.02337 \times 10^{-20}}{\lambda^5} \left( \frac{8.19379 \times 10^{18} B \lambda^2}{\kappa^2} - \frac{1.14512 \times 10^{19} A \lambda^4}{\kappa^2} \right) \]

The critical point corresponds to the era dominated by DM and NVMCG type DE. For the critical point \((u_c, v_c)\), the equation of state parameter given by equation (15) of the interacting DE takes the form

\[ \omega_{\text{nvmcg}}^{(RSHI)} = \frac{p_{\text{nvmcg}}}{\rho_{\text{nvmcg}}} = \frac{1}{y_c^n X^n} \left( A - \frac{B y_c^{n-m}}{u_c^{\alpha+1} X^{m-n+\alpha+1}} \right), \]

where

\[ X = \frac{2\lambda}{u_c + v_c} \left[ \frac{1}{\kappa^2 (u_c + v_c)} - 1 \right] \]

Fig 1: The dimensionless density parameters \(u, v\) and \(y\) are plotted against each other in a 3D-scenario. Other parameters are fixed at \(\alpha = 0.5, b = 0.5, A_0 = 1/3, B_0 = 3, n = 2, m = 1, \lambda = 1.5,\) and \(\kappa = 0.2\).

Fig 2: The dimensionless density parameters are plotted against e-folding time. The initial conditions are \(v(1.1) = 0.05, u(1.1) = 2.5\) and \(y(1.1) = 2.8\). Other parameters are fixed at
$\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5, \text{ and } \kappa = 0.2.$

Fig 3 : The dimensionless density parameters are plotted against e-folding time. The initial conditions are $v(1.1) = 0.05$, $u(1.1) = 2.5$ and $y(1.1) = 2.8$. Other parameters are fixed at $\alpha = 0.5, b = 0.1, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5, \text{ and } \kappa = 0.2.$

Fig 4 : The dimensionless density parameters are plotted against e-folding time. The initial conditions are $v(1.1) = 0.05$, $u(1.1) = 2.5$ and $y(1.1) = 2.8$. Other parameters are fixed at $\alpha = 0.5, b = 0.5, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5, \text{ and } \kappa = 0.2.$
Fig 5 : The phase diagram of the parameters $u(t)$ and $v(t)$ depicting an attractor solution. The initial conditions chosen are $v(1) = 0.05, u(0) = 2.5, y(1) = 1.8$ (green); $v(1) = 0.06, u(1) = 2.6, y(1) = 1.9$ (blue); $v(1) = 0.07, u(1) = 2.7, y(1) = 2$ (red); $v(1) = 0.08, u(1) = 2.8, y(1) = 2.1$ (gold). Other parameters are fixed at $\alpha = 0.5, b = 0.01, A = 1/3, B = 3, n = 5, m = 1, \lambda = 1.5$ and $\kappa = 0.2$.

Fig 6 : The dimensionless density parameters $u(t)$ and $y(t)$ are plotted against e-folding time. The initial conditions are $v(1.1) = 0.05, u(1.1) = 2.5$ and $y(1.1) = 2.8$. Other parameters are fixed at $\alpha = 0.5, b = 0.01, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5$, and $\kappa = 0.2$.

Fig 7 : The dimensionless density parameters $v(t)$ and $y(t)$ are plotted against e-folding time. The initial conditions are $v(1.1) = 0.05, u(1.1) = 2.5$ and $y(1.1) = 2.8$. Other parameters are fixed at $\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5$, and $\kappa = 0.2$.

Fig 8 : A 3D-phase portrait of the dimensionless density parameters $u(t)$, $v(t)$ and $y(t)$ is plotted against each other. The initial conditions are $v(1.1) = 0.05, u(1.1) = 2.5$ and $y(1.1) = 2.8$. Other parameters are fixed at $\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5$, and $\kappa = 0.2$. 
Fig 9: The deceleration parameter is plotted against the EoS parameter. Other parameters are fixed at $\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5$, and $\kappa = 0.2$.

Fig 10: The statefinder parameter $r$ is plotted against the EoS parameter. Other parameters are fixed at $\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5$, and $\kappa = 0.2$. 
Fig 11 : The statefinder parameter $s$ is plotted against the EoS parameter. Other parameters are fixed at $\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5, \text{and } \kappa = 0.2$.

Fig 12 : The ratio of density parameters is shown against e-folding time. The initial conditions chosen are $v(1) = 0.05, u(1) = 2.5, y(1) = 1.8$. Other parameters are fixed at $\alpha = 0.5, b = 0.001, A_0 = 1/3, B_0 = 3, n = 5, m = 1, \lambda = 1.5, \text{and } \kappa = 0.2$. 
2. STABILITY AROUND CRITICAL POINT

Now we check the stability of the dynamical system (eqs. (12) and (13) and (14)) about the critical point. In order to do this, we linearize the governing equations about the critical point i.e.,

\[
\begin{align*}
    u &= u_c + \delta u, \quad v = v_c + \delta v, \quad y = y_c + \delta y \\
\end{align*}
\]  

(21)

Now if we assume \( f = \frac{du}{dx} \), \( g = \frac{dv}{dx} \) and \( h = \frac{dy}{dx} \) then we may obtain

\[
\begin{align*}
    \delta \left( \frac{du}{dx} \right) &= \left[ \partial_u f \right]_c \delta u + \left[ \partial_v f \right]_c \delta v + \left[ \partial_y f \right]_c \delta y \\
\end{align*}
\]  

(22)

\[
\begin{align*}
    \delta \left( \frac{dv}{dx} \right) &= \left[ \partial_u g \right]_c \delta u + \left[ \partial_v g \right]_c \delta v + \left[ \partial_y g \right]_c \delta y \\
\end{align*}
\]  

(23)

and

\[
\begin{align*}
    \delta \left( \frac{dy}{dx} \right) &= \left[ \partial_u h \right]_c \delta u + \left[ \partial_v h \right]_c \delta v + \left[ \partial_y h \right]_c \delta y \\
\end{align*}
\]  

(24)

where
\[ \partial_u f = -3 \frac{2^{-1-m-n-\alpha} u^{-1-\alpha} y^{-m-n}}{\lambda(u + v)^2 (-1 + (u + v)\kappa^2)^2} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \times \left[ -2^n Bv(u + v)^2 y^n \kappa^2 (u + 2mu + u\alpha - v\alpha - (u + v)(mu - v\alpha)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n + 2^{1+m+\alpha} u^{\alpha} y^m \right] \]

\[ \times (-1 + (u + v)\kappa^2) \left( 1 + \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{m+\alpha} \left( Av(-2nu - v + (u + v)(nu + v)\kappa^2) + 2^n(u + v)^2 y^n \right) \]

\[ \times (-1 + (u + v)\kappa^2)(-1 + b + 2(2u + v)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]  

\[ \partial_v f = -3 \frac{2^{-1-m-n-\alpha} u^{-\alpha} y^{-m-n}}{\lambda(u + v)^2 (-1 + (u + v)\kappa^2)^2} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \times \left[ 2^n B(u + v)^2 y^n \kappa^2 (-u - 2v(1 + m + \alpha) + (u + v)(u + v(1 + m + \alpha))\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n + 2^{1+m+\alpha} u^{\alpha} \right] \]

\[ \times y^m(-1 + (u + v)\kappa^2) \lambda \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{m+\alpha} \left( Av(-u - 2nv + (u + v)(nu + v)\kappa^2) + 2^n(u + v)^2 y^n \right) \]

\[ \times (b + 2u\kappa^2)(-1 + (u + v)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \]  

\[ \partial_y f = \frac{2^{-n} 3Anuvy^{1-n} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-n}}{u + v} + \frac{2^{1-m-\alpha} 3Bmu^{-\alpha} v(u + v)y^{-1-m}\kappa^2 \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-\alpha}}{(-1 + (u + v)\kappa^2)\lambda} \]

\[ \partial_y g = 3 \frac{2^{-1-m-n-\alpha} u^{-1-\alpha} y^{-m-n}}{\lambda(u + v)^2 (-1 + (u + v)\kappa^2)^2} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \times \left[ -2^n Bv(u + v)^2 y^n \kappa^2 (u + 2mu + u\alpha - v\alpha - (u + v)(mu - v\alpha)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n + 2^{1+m+\alpha} u^{1+\alpha} y^m \right] \]
\[ \times (-1 + (u + v)\kappa^2)\lambda \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{m+\alpha} \]

\[ \times (Av(-2nu - v + (u + v)(nu + v)\kappa^2) + 2n(u + v)^2y^n) \]

\[ \times (b - 2n\kappa^2)(-1 + (u + v)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \right] \]

\[ \partial_v g = 3 \left[ \frac{2^{-1-m-n-\alpha}u^{-\alpha}y^{-m-n}}{\lambda(u + v)^2(-1 + (u + v)\kappa^2)^2} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \times \left[ 2^n B(u + v)^2y^n\kappa^2(-u - 2v(1 + m + \alpha) + (u + v)(u + v(1 + m + \alpha))\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n - 2^{1+m+\alpha}u^\alpha \right] \]

\[ \partial_y \left[ 2^n B(u + v)^2y^n\kappa^2(-u - 2v(1 + m + \alpha) + (u + v)(u + v(1 + m + \alpha))\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n - 2^{1+m+\alpha}u^\alpha \right] \]

\[ \times (-1 - b + 2(u + 2v)\kappa^2)(-1 + (u + v)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \right] \]

\[ \partial_y = \frac{2^{-n}Auvy^{-1-n} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-n} - 2^{-1-m-\alpha}Bu^{-\alpha}v(u + v)y^{-1-m}\kappa^2 \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-\alpha}}{u + v} \]

\[ \times \frac{(-1 + (u + v)\kappa^2)\lambda}{(-1 + (u + v)\kappa^2)\lambda} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \right] \]

\[ \partial_u h = -3 \left[ \frac{2^{-1-m-n-\alpha}u^{-\alpha}y^{-m-n}}{\lambda(u + v)^2(-1 + (u + v)\kappa^2)^2} \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \times \left[ -2^n B(u + v)^2y^n\kappa^2(-u - 2mu - u\alpha + v\alpha + (u + v)(mu - v\alpha)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n + 2^{1+m+\alpha}u^{1+\alpha}y^m \right] \]

\[ \times (-1 + (u + v)\kappa^2)\lambda \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{m+\alpha} \]

\[ \times (-A(2nu + v) + A(u + v)(nu + v)\kappa^2 - 2^{1+n}(u + v)^2y^n\kappa^2) \]

\[ \times (-1 + (u + v)\kappa^2) \left( \frac{\lambda(-u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \right] \]
\[ \partial_v h = -3 \frac{2^{-1-m-n-\alpha}u^{-\alpha}y^{1-m-n}}{\lambda(u^2 + (u+v)\kappa^2)^2} \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \times \left[ -2^n B(u + v)^2 y^n \kappa^2 (-1 - 2m - 2\alpha + (u + v)(m + \alpha)) \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n + 2^{1+m+\alpha}u^\alpha y^m \right] \]

\[ \times (-1 + (u + v)\kappa^2) \lambda \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{m+\alpha} \] \[ (Au(-1 + 2n - (n - 1)(u + v)\kappa^2) + 2^{1+n}(u + v)^2 y^n \kappa^2 \]

\[ \times (-1 + (u + v)\kappa^2) \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \right] \]

\[ \partial_y h = \frac{1}{(u + v) (-1 + (u + v)\kappa^2)} \lambda^{2^{-1-m-n-\alpha}u^{-\alpha}y^{-m-n}} \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{-m-n-\alpha} \]

\[ \left[ -3 \times 2^n B (m - 1) (u + v)^2 y^n \kappa^2 \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n - 2^{1+m+\alpha}u^\alpha y^m \left( -1 + (u + v)\kappa^2 \right) \lambda \times \]

\[ \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^{m+\alpha} \left\{ 3A(n - 1) u + 2^n (u + v) y^n (6(u + v)\kappa^2 - 7) \left( \frac{\lambda(u - v + \frac{1}{\kappa^2})}{(u + v)^2} \right)^n \right\} \]

The Jacobian matrix of the above system is given by,

\[ J_{(u,v)}^{(RSI)} = \begin{pmatrix} \frac{\delta f}{\delta u} & \frac{\delta f}{\delta v} & \frac{\delta f}{\delta y} \\ \frac{\delta g}{\delta u} & \frac{\delta g}{\delta v} & \frac{\delta g}{\delta y} \\ \frac{\delta h}{\delta u} & \frac{\delta h}{\delta v} & \frac{\delta h}{\delta y} \end{pmatrix} \]

The eigen values of the above matrix are calculated at the critical point \((u_c, v_c)\) and are found to be \(\lambda_1 = 5.99118, \; \lambda_2 = -3, \; \lambda_3 = -2.67056\). Hence it is a Saddle point.

3. NATURE OF COSMOLOGICAL PARAMETERS

Deceleration Parameter:
In this RSII model, the deceleration parameter $q$ can be obtained as

$$q^{(RSII)} = -1 - \frac{3}{2} \left\{ \frac{\rho^{(RSII)} w_{nvmcg}^{(RSII)} - 2}{\rho} - \left( 1 + w_{nvmcg}^{(RSII)} \frac{\rho}{\rho} \right) \right\} \left( 1 + \frac{\rho}{\rho} \right)$$

which can be written in terms of dimensionless density parameter $\Omega_{nvmcg} = \frac{\rho_{nvmcg}}{\rho}$ as in the following

$$q^{(RSII)} = -1 + 3 \left\{ \frac{\left( \frac{\rho}{\rho} - 1 \right) w_{nvmcg}^{(RSII)} \Omega_{nvmcg} + (1 + \frac{\rho}{\rho})}{\left( 1 + \frac{\rho}{\rho} \right)} \right\}$$

Now since $\Omega_{nvmcg} = \frac{\rho_{nvmcg}}{\rho} = \frac{u}{u+v}$ and assuming $\frac{\rho}{\rho} = \epsilon^{(RSII)}$ we get,

$$q^{(RSII)} = -1 + \frac{3}{2} \left\{ \frac{\left( 1 - \epsilon^{(RSII)} \right) w_{nvmcg}^{(RSII)} u}{u+v} + (1 + \epsilon^{(RSII)}) \right\} \left( 1 + \frac{\epsilon^{(RSII)}}{\rho} \right)$$

Considering only the first stable critical point, such that $(u, v) \rightarrow (u_{1c}, v_{1c})$, using (35) we get,

$$q^{c(RSII)} = -1 + \frac{3}{2} X^{(RSII)}$$

where

$$X^{(RSII)} = \frac{\left( 1 - \epsilon^{(RSII)} \right) w_{nvmcg}^{(RSII)} u_{1c}}{u_{1c} + v_{1c}} + (1 + \epsilon^{(RSII)}) \left( 1 + \frac{\epsilon^{(RSII)}}{\rho} \right)$$

If $\epsilon^{(RSII)} = -2 \left[ \frac{1 + w_{nvmcg}^{(RSII)}}{2 - w_{nvmcg}^{(RSII)}} u_{1c} + v_{1c} \right]$, $X^{(RSII)} = 0$, we have $q = -1$, which confirms the accelerated expansion of the universe. When $\epsilon^{(RSII)} = -2$ we have $q = -\infty$. Therefore we have super accelerated expansion of the universe.

In this scenario, the Hubble parameter can be obtained as,

$$H = \frac{2}{3X^{(RSII)} t}$$

where the integration constant has been ignored. Integration of (37) yields

$$a(t) = a_0 t^{\frac{2}{3X^{(RSII)}}}$$

which gives the power law form of expansion of the universe. In order to have an accelerated expansion of universe in RSII brane we must have $0 < X^{(RSII)} < \frac{2}{3}$. Using this range of $X^{(RSII)}$ in the equation $q^{c(RSII)} = -1 + \frac{3}{2} X^{(RSII)}$, we get the range of $q^{c(RSII)}$ as $-1 < q^{c(RSII)} < 0$. This is again consistent with the accelerated expansion of the universe.

2. Statefinder Parameters

As so many cosmological models have been developed, so for discrimination between these contenders, Sahni et al. [43] proposed a new geometrical diagnostic named the statefinder pair
\{r, s\}. The statefinder parameters are defined as follows,

\[ r \equiv \frac{\ddot{a}}{aH^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)} \]  \hspace{1cm} (39)

where \(a\) is the scale factor of the universe, \(H\) is the Hubble parameter, a dot denotes differentiation with respect to the cosmic time \(t\).

Here we calculate the statefinder parameters \(\{r, s\}\) in order to get relevant information of DE and DM in the context of background geometry only without depending on the theory of gravity. The expressions of the statefinder pair eq. (39) in the RSII model can be obtained in the form

\[ r_{(\text{RSII})} = \left(1 - \frac{3X_{(\text{RSII})}}{2}\right)(1 - 3X_{(\text{RSII})}). \]  \hspace{1cm} (40)

and

\[ s_{(\text{RSII})} = X_{(\text{RSII})} \]  \hspace{1cm} (41)

III. GRAPHICAL ANALYSIS

Graphs are obtained and phase diagrams are drawn in order to determine the type of critical point obtained in this model. Below we discuss the results obtained in detail:

The dimensionless density parameters \(u\) and \(v\) and \(y\) are plotted against each other in figure 1. From the figure we see that \(v\) decreases, and \(u\) increases during evolution of the universe. In figs. 2, 3 and 4, the density parameters are plotted against time. It is evident from the figures that the density of DM decreases while the density of DE increases as the universe evolves with an increase in scale factor \(a\). So this result is consistent with the well known idea of an energy dominated universe. In these figures it is seen that with an increase in the value of interaction the values of \(u\) and \(v\) become more and more comparable to each other, which is quite an expected result. So we have a possible solution of the cosmic coincidence problem. A comparative study between [14, 18, 19] and the current work reveals that DE domination over DM is much less pronounced in case of NVMCG than in case of other Chaplygin gas models like, MCG or GCCG. This is a very interesting feature in the character of NVMCG. Moreover a comparative study of the current work with [18] reveals the fact that density of DM is much more comparable to the density of GCCG than to that of NVMCG. Hence it can be speculated that NVMCG is perhaps less effective to play the role of DE in comparison to MCG and GCCG as well.

Figs. 5, 6 and 7 shows the phase portrait of the density parameters of DE and DM. In fig. 5, we see a phase diagram between density parameters \(u(t)\) and \(v(t)\). Figs. 6 and 7 shows the
phase diagram between \( u(t) \), \( y(t) \) and \( y(t), v(t) \) respectively. Finally in fig. 8, a 3D-phase diagram between all the three density parameters is obtained in a single system. As already stated before that the critical point obtained in this system is a Saddle point and hence there always remains a question on the stability of the system. In fig. 9, a plot of deceleration parameter, \( q \) is obtained against the EoS parameter, \( \omega \). It is seen that \( q \) remains in the negative level thus confirming the recent cosmic acceleration. Figs. 10 and 11 show the plot of the statefinder parameters \( r \) and \( s \) respectively against the EoS parameter, \( \omega \). It is known that in case of \( \Lambda \)CDM model \( r = 1 \) and \( s = 0 \). Here we see that in fig. 11 except at \( \omega = 0 \), the values of \( s \) correspond to that of the \( \Lambda \)CDM model. In fig. 10, we see that the values of \( r \) is quite different from 1 corresponding to the values of \( \omega \) when \( s = 0 \) in fig. 11.

This gives the deviation of the model from the \( \Lambda \)CDM model. Finally in fig. 12, the ratio \( v/u \) is plotted against \( x = \ln a \). The decreasing trajectory confirms the existence of an energy dominated universe with progressive values of scale factor.

IV. SOME NOTES ON THE MATHEMATICAL CONSTRUCTION OF NEW VARIABLE MODIFIED CHAPLYGIN GAS

We know that NVMCG is basically an extension of modified Chaplygin gas, and its mathematical formulation is based on the modification of the EoS of MCG in a way that makes it more suitable as a candidate to play the role of DE. Our motive is to investigate how far successful is NVMCG over MCG as a DE and consequently how far essential or justified is this modification. So in this section we consider modified Chaplygin gas as a standard model of dark energy and perform a comparative study with NVMCG. The EoS for MCG is given by,

\[
p = A\rho_{mcg} - \frac{B}{\rho_{mcg}^\alpha}
\]

(42)

where \( A \), \( B \) and \( \alpha \) are positive constants. For negative pressure we should have

\[
\rho_{mcg} < \left( \frac{B}{A} \right)^\frac{1}{\alpha+1}
\]

(43)

Now to get negative pressure in case of NVMCG we should have,

\[
\rho_{nvmcg} < \left( \frac{B_0}{A_0} \right)^\frac{1}{\alpha+1} a^{-\frac{\alpha+1}{\alpha+1}}
\]

(44)

**Case I: for \( m = n \)**

From relation (43), we have, \( \rho_{nvmcg} < \left( \frac{B_0}{A_0} \right)^\frac{1}{\alpha+1} \). The above value of \( \rho_{nvmcg} \) coincides with that of MCG for \( B_0 = B \) and \( A_0 = A \).
Case II: for $m > n$
Let $m - n = m_1$, where $m_1 > 0$. Therefore relation (43) becomes, $\rho < \left( \frac{B_0}{A_0} \right)^{\frac{1}{\alpha+1}} a^{m_1 \alpha+1}$. Now since $a > 1$ in an accelerating universe, so $a^{m_1 \alpha+1} > 1$. Let $a^{m_1 \alpha+1} = m_2$. So we get $\rho_{\text{nvmcg}} < \left( \frac{B_0}{A_0} \right)^{\frac{1}{\alpha+1}} m_2$.

Now since $m_2 > 1$, $\rho_{\text{mcg}} < \rho_{\text{nvmcg}}$ if $A_0 = A$ and $B_0 = B$. Here it can be seen that the density of NVMCG increases due to the introduction of scale factor, $a$ in its EoS. It can be seen from the EoS of NVMCG that this effect increases the magnitude of the first term and decreases the magnitude of the second term, which eventually increases the the value of pressure as a whole. This push towards the towards the positive region in the value of pressure can be speculated as a basic flaw in the mathematical construction of NVMCG, since it reduces its efficiency as a DE. Hence $m > n$ is not at all acceptable as far as the concept of dark energy is concerned.

Case III: for $m < n$
In this case $m - n = -m_1$. Using this we get, $\rho < \left( \frac{B_0}{A_0} \right)^{\frac{1}{\alpha+1}} a^{m_1 \alpha+1}$. For an accelerating universe $a > 1$, which implies $a^{m_1 \alpha+1} > 1$. Therefore $\rho < \left( \frac{B_0}{A_0} \right)^{\frac{1}{\alpha+1}} m_2$. Now since $m_2 > 1$, which implies $\rho_{\text{mcg}} > \rho_{\text{nvmcg}}$, if $A_0 = A$ and $B_0 = B$. So there is a relative decrease in the value of density of NVMCG compared to its counterpart MCG. This in fact pushes the value of pressure towards the more negative region, thus enhancing the ability of NVMCG to play the role of DE. This is the most acceptable case and the EoS of NVMCG should be constrained by $m < n$.

V. STUDY OF FUTURE SINGULARITIES

We know that any energy dominated model of the universe will result in a future singularity. As a result the study of dynamical model of universe in the presence of DE and DM is in fact incomplete without the study of these singularities, which are the ultimate fate of the universe. It is known that the universe dominated by phantom energy ends with a future singularity known as Big Rip [44], due to the violation of dominant energy condition (DEC). But other than this there are other types of singularities as well. Nojiri et al [4] studied the various types of singularities that can result from an phantom energy dominated universe. These possible singularities are characterized by the growth of energy and curvature at the time of occurrence of the singularity. It is found that near the singularity quantum effects becomes very dominant which may alleviate or even prevent these singularities. So it is extremely necessary to study these singularities and classify them accordingly so that we can search for methods to eliminate them. The appearance of all four types of future singularities in coupled fluid dark energy, $F(R)$ theory, modified Gauss-

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Bonnet gravity and modified $F(R)$ Horava-Lifshitz gravity was demonstrated in [20]. The universal procedure for resolving such singularities that may lead to bad phenomenological consequences was proposed. In Rudra et al [14] it has been shown that in case of Modified Chaplygin gas (MCG), both Type I and Type II singularities are possible. However in [19] it was shown that GCCG does not result in any type of future singularity.

A. TYPE I Singularity (Big Rip singularity)

If $\rho \to \infty$, $|p| \to \infty$ when $a \to \infty$ and $t \to t_s$. Then the singularity formed is said to be the Type I singularity.

In the present case by considering the NVMCG equation of state from equation (1) we see that when $a \to \infty$, $|p| \to 0$. Therefore we see that there is no possibility for Type I singularity, i.e., Big Rip singularity, in case of NVMCG.

B. TYPE II Singularity (Sudden singularity)

If $\rho \to \rho_s$ and $\rho_s \sim 0$, then $|p| \to -\infty$ for $t \to t_s$ and $a \to a_s$, then the resulting singularity is called the Type II singularity.

Considering the equation of state for NVMCG, We see that if $\rho_s \sim 0$, then $|p| \to -\infty$ for $t \to t_s$ and $a \to a_s$. Hence there is a strong possibility of the type II singularity or the sudden singularity in case of NVMCG.

C. TYPE III Singularity

For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$. Then the resulting singularity is Type III singularity. It is quite evident from the equation of state of NVMCG that it supports this type of singularity.

D. TYPE IV Singularity

For $t \to t_s$, $a \to a_s$, $\rho \to 0$ and $|p| \to 0$. Then the resulting singularity is Type IV singularity. Investigation shows that this type of singularity is not supported by NVMCG type DE.

As a remark, one should stress that our consideration is totally classical. Nevertheless, it is expected that quantum gravity effects may play significant role near the singularity. It is clear
that such effects may contribute to the singularity occurrence or removal too. Unfortunately, due to the absence of a complete quantum gravity theory only preliminary estimations may be done.

VI. CONCLUSION

In this work we have considered New variable modified Chaplygin gas and tried to determine its efficiency to play the role of dark energy in an universe described by RSII brane. A numerical system study was carried out in order to throw some light on the dynamics of the dark energy. The system was formed and a solution was obtained. An eigen value analysis of the system at the critical point showed that the system was far from attaining stability since it produced a saddle point. Plots were obtained to get a clear idea about the result of our analysis in a both qualitative and quantitative aspect. It was found that NNMCG in RSII brane model is perfectly consistent with the idea of an energy dominated universe, but the DE domination over DM is much less pronounced in case of NVMCG when compared to that with Modified Chaplygin gas or generalized cosmic Chaplygin gas. This is an important result indeed. It was also discovered from the plots that with the increase in the magnitude of interaction the values of DE and DM became more and more comparable to each other thus providing a solution of the cosmic coincidence problem at higher interaction scales. The plot of the deceleration parameter revealed that the model is perfectly consistent with the notion of an energy dominated universe. The unique trajectories in the plots of statefinder parameters differentiated the model from other DE models. An extensive study regarding the mathematical formulation of NVMCG was performed, and it was found that $m < n$ best suited the nature of NVMCG as a DE. Hence we have been able to constrain the parameters of NVMCG to give the best possible results. Finally the future singularities were studied and the model was found to be affected by some of them, quite unlike the generalized cosmic Chaplygin gas. In a nutshell it can be said that the performance of NVMCG as a dark energy is quite moderate. When compared with other DE models like MCG or GCCG, the weakness of NVMCG is quite visible. Hence as a conclusion we speculate that NVMCG is not quite the brightest contender to play the role of DE and there is a lot of scope for improvement as far as the mathematical aspect of the model is concerned.

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