A minimal twist for the Standard Model in noncommutative geometry I: the field content

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Abstract

Noncommutative geometry provides both a unified description of the Standard Model of particle physics together with Einstein-Hilbert action (in euclidean signature) and some tools to go beyond the Standard Model. In this paper, we extend to the full noncommutative geometry of the Standard Model the twist (in the sense of Connes-Moscovici) initially worked out for the electroweak sector and the free Dirac operator only. Namely, we apply the twist also to the strong interaction sector and the finite part of the Dirac operator. To do so, we are forced to take into account a violation of the twisted first-order condition. As a result, we still obtain the extra scalar field required to stabilise the electroweak vacuum and fit the Higgs mass, but it now has two chiral components. We also get the additive field of 1-forms already pointed out in the electroweak model, but with a richer structure. Finally, we obtain a pair of Higgs doublets, which are expected to combine into a single Higgs doublet in the action formula, as will be investigated in the second part of this work.

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1 Introduction

Noncommutative geometry [11] (see [14] for a recent review of the various aspects of the field) provides a mathematical framework in which a single action formula yields both the lagrangian of the Standard Model of fundamental interactions and the Einstein-Hilbert action (in euclidean signature). As an added value, the Higgs field is obtained on the same footing as the other gauge bosons – as a connection 1-form – but a connection that lives on a slightly generalised notion of space, where points come equipped with an internal structure. Such “spaces” are described by spectral triples

$$\mathcal{A}, \mathcal{H}, D$$  

consisting in an algebra $\mathcal{A}$, acting on an Hilbert space $\mathcal{H}$ together with an operator $D$ on $\mathcal{H}$ which satisfies a set of axioms [12] guaranteeing that – in case $\mathcal{A}$ is commutative and unital – then there exists a (closed) riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A}$ coincides with the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$. In other terms, a spectral triple with $\mathcal{A}$ commutative does encode all the geometrical information of a (closed) riemannian spin manifold [13]. These axioms still make sense when $\mathcal{A}$ is noncommutative, and provide then a definition of a noncommutative geometry as a spectral triple in which the algebra is non necessarily commutative.

The spectral triple of the Standard Model [7] is built upon an “almost-commutative algebra”,

$$C^\infty(\mathcal{M}) \otimes \mathcal{A}_\text{SM}$$  

where $\mathcal{M}$ is an even dimensional closed riemannian spin manifold and $\mathcal{A}_\text{SM}$ a noncommutative matrix algebra that encodes the gauge degrees of freedom of the Standard Model. As explained in [12], this non-commutative algebra provides the points of $\mathcal{M}$ with an internal structure, in such a way that the Standard Model is actually nothing but a pure theory of gravity, on a space that is made slightly noncommutative by multiplying the (infinite dimensional) commutative algebra $C^\infty(\mathcal{M})$ with the finite dimensional noncommutative $\mathcal{A}_\text{SM}$.

After the discovery of the Higgs boson in 2012, it has been noticed in [6] that an extra scalar field – usually denoted $\sigma$ – proposed by particle physicist to cure the instability of the electroweak vacuum due to the “low mass of the Higgs”, also makes the computation of the Higgs mass (which is not a free parameter in the noncommutative description of the Standard Model) compatible with its experimental value. Various scenarios have been proposed to make this extra scalar field emerge from the mathematical framework of noncommutative geometry, all of them consisting in some modification of one of the axioms, the first order condition (e.g. [9, 8, 4, 5, 3, 2, 1], see [10] for a recent review).

In this paper, we push forward one of these scenarios, consisting in twisting the spectral triple of the Standard Model. Twists have been introduced by Connes and Moscovici in [16] with purely mathematical motivations. Later, it has been discovered in [19] that a very simple twist of the Standard Model not only produces the extra scalar field $\sigma$, but also an additive
field of 1-form $X_\mu$ which turns out to be related with Wick rotation and the transition from the euclidean to the lorentzian signature \cite{17, 24}. However, in \cite{19} the twist was applied only to the part of the spectral triple that yields the field $\sigma$, namely the subalgebra of $\mathcal{A}_{\text{SM}}$ describing the electroweak interaction and the part of the operator $D$ that contains the Majorana mass of the neutrinos. For simplicity, the subalgebra of $\mathcal{A}_{\text{SM}}$ describing the strong interaction was left untouched, and the part of $D$ containing the Yukawa coupling of fermions was not taken into account. In this paper, we extend the twisting procedure to the whole spectral triple of the Standard Model, according to the following lines.

The twist of gauge theories have been investigated in a systematic way in \cite{22, 23}, where the twisted version of the first-order condition – introduced by imitation of the non-twisted case in \cite{19} – has been put onto solid mathematical bases. A notion of minimal twist of a spectral triple has also been defined, which consists in making several copies of $\mathcal{A}$ act on $\mathcal{H}$, letting $D$ untouched. By doing so, one produces models with new bosonic fields, keeping the fermionic content untouched, in agreement with the state of the art of the Standard Model (indeed the metastability of the electroweak vacuum points towards new scalar fields, but there are no indications of new fermions). A procedure for minimally twisting any real spectral triple is to make two copies of the algebra act independently on the eigenspaces of the grading operator. However, applied to the Standard Model, this does not produce any extra-scalar field, as explained in \cite{21}.

That is why in this paper we investigate another minimal twist of the Standard Model, that does produce an extra scalar $\sigma$. The price to pay is a violation of the twisted first order condition, which is taken into account following the way pioneered in \cite{9} and adapted to the twisted case in \cite{25}.

Besides the field content of the Standard Model, we find that the extra scalar $\sigma$ actually decomposes into two chiral components $\sigma_r, \sigma_l$ (proposition 4.6) which are invariant under a gauge transformation (proposition 6.6). We also work out the structure of the 1-form field $X_\mu$ (proposition 5.5), and study how it behaves under a gauge transformation (proposition 6.2). In brief, imposing the same unimodular condition as in the non-twisted case, we find that the antselfadjoint part of the (generalised) 1-form generated by the free Dirac operator $\partial$ yields exactly the bosonic content of the Standard Model, as in the non-twisted case. But there is also a selfadjoint part made of two real 1-form fields and one selfadjoint $M_3(\mathbb{C})$-value 1-form field. Altogether these three fields compose the 1-form field $X_\mu$.

The complete understanding of the physical meaning of these fields passes through the computation of the fermionic and spectral actions, and will be the object of a second paper \cite{21}.

The paper is organised as follows. In section 2 we recall the basics of the spectral triple of the Standard Model (§2.1), make explicit the tensorial notations employed all along the paper (§2.2) and use them to write explicitly the Dirac operator, the grading and the real structure (§2.3). Section 3 deals with the twist. After recalling the procedure of minimal twisting defined in \cite{22}, we apply it to the spectral triple of the Standard Model: the algebra is doubled so as to act independently on the left and right components of Dirac spinors (§3.1). The grading and the real structure are the same as in the non-twisted case, and we check explicitly that one of the axioms (the order zero condition) still holds in the twisted case (§3.2), as expected from the general result of \cite{22}. Paragraph 3.3 is a brief recalling about twisting fluctuations, that is the way to generate the bosonic fields. The detail computation of this fluctuations is the object of section 4 and 5, which contain the main results of this paper. We first work out the Higgs sector in §4.1. The main result is proposition 4.4 in which we find two Higgs doublets. The extra scalar field $\sigma$ is generated in §4.2. Its structure as a doublet of real scalar fields $\sigma_r, \sigma_l$ is established in proposition 4.6. In section 5 we compute the twisted fluctuation of the free part.
of the Dirac operator. Useful properties of the Dirac matrices with respect to the twist are worked out in §5.1. The generalised twisted 1-forms generated by the free Dirac operator are computed in §5.2, and the physical degrees of freedom are identified in §5.3. The structure of the 1-form field $X_\mu$ is summarised in proposition 5.5 and yields, in §5.4, the explicit form of the twisted fluctuation of the free Dirac operator. In section 6 we study how all these fields behave under a gauge transformation. After recalling the basics of gauge transformation for twisted spectral triple (as stabilised in [23]), we apply these techniques to the gauge and the 1-form fields in §6.1 and to the scalar fields in §6.2. We show in proposition 6.2 that the bosonic fields transform in the correct way, while the 1-form field is invariant, up to a unitary transformation on the $M_3(\mathbb{C})$-value part. The Higgs doublets as well transform as expected (proposition 6.5), while the extra scalar field $\sigma$ is gauge invariant, as shown in proposition 6.6.

The first appendix contains notations and generalities on Dirac matrices. In the second one, we write explicitly the components of the twisted fluctuation in terms of the gauge fields (this will be useful in the second part of the paper, to compute the action). In the last appendix, we check that the twisted first-order condition is only partially verified.

Notations and important comments regarding the literature:

- In the first version of this paper, we erroneously thought the twist we were using was “by grading”, and assumed the twisted first-order condition. Actually the latter is violated only by the off-diagonal part of the internal Dirac operator, and this does not modify the extra-scalar field, as explained before remark 4.7 neither the gauge invariance of the fermionic action, as explained before proposition 6.6.

- We work with one generation of fermions (electron $e$, neutrino $\nu_e$, quarks up $u$ and down $d$). The extension to three generations will be discussed in the second part of the work [20].

- All along the paper, we apply the usual rule of contractions of indices in alternate up/down positions. Typically the greek indices label the coordinates of the manifold.

2 The non-twisted case

As a preparation to the twisting, we recall in this section the main features of the spectral description of the Standard Model. Besides the original papers (recalled in the text), the details are extensively discussed in the books [15] and [26] (for a more physics-oriented presentation).

2.1 The spectral triple of the Standard Model

The usual spectral triple of the Standard Model [7] is the product of the canonical triple of a (closed) riemannian spin manifold $\mathcal{M}$ of even dimension $m$, with the finite dimensional spectral triple (called internal)

$$\mathcal{A}_{SM} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{32n}, \quad D_F$$

that describes the gauge degrees of freedom of the Standard Model. In (2.1), $C^\infty(\mathcal{M})$ denotes the algebra of smooth functions on $\mathcal{M}$, that acts by multiplication on the Hilbert space $L^2(\mathcal{M}, S)$ of square integrable spinors as

$$(f\psi)(x) = f(x)\psi(x) \quad \forall f \in C^\infty(\mathcal{M}), \psi \in L^2(\mathcal{M}, S), \ x \in \mathcal{M},$$

(2.3)
\[ \partial = -i \gamma^\mu \nabla_\mu \quad \text{with} \quad \nabla_\mu = \partial_\mu + \omega_\mu \] (2.4)

is the Dirac operator on \( L^2(\mathcal{M}, S) \) associated with the spin connection \( \omega_\mu \) and the \( \gamma^\mu \)'s are the Dirac matrices associated with the Riemannian metric \( g \) on \( \mathcal{M} \):

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{I} \quad \forall \mu, \nu = 0, m - 1 \] (2.5)

(\( \mathbb{I} \) is the identity operator on \( L^2(\mathcal{M}, S) \) and we label the coordinates of \( \mathcal{M} \) from 0 to \( m - 1 \)).

In (2.2), \( n \) is the number of generations of fermions, and \( D_F \) is a \( 32n \) square complex matrix whose entries are the Yukawa couplings of fermions and the coefficients of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix of quarks and of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix of neutrinos. Details are given in §2.3, and the representation of \( \mathcal{A}_{SM} \) on \( \mathcal{H}_F \) is in §2.2.

The product spectral triple is

\[ C^\infty(\mathcal{M}) \otimes \mathcal{A}_{SM}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial \otimes \mathbb{I}_F + \gamma_M \otimes D_F \] (2.6)

with \( \gamma_M \) the product of the Euclidean Dirac matrices (appendix 6.2) and \( \mathbb{I}_F \) the identity on \( \mathcal{H}_F \).

A spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) is \emph{graded} when the Hilbert space comes equipped with a grading (that is a selfadjoint operator that squares to \( \mathbb{I} \)) which anticommutes with \( D \). The spectral triple (2.1) is graded with grading \( \gamma_M \). The internal spectral triple (2.2) is graded, with grading the operator \( \gamma_F \) on \( \mathcal{H}_F \) that takes value +1 on right particles & left antiparticles, −1 on left particles & right antiparticles. The product spectral triple (2.6) is graded, with grading

\[ \Gamma = \gamma_M \otimes \gamma_F. \] (2.7)

Another important ingredient is the \emph{real structure}, that is an anti-linear operator that squares to \( \pm \mathbb{I} \) and commutes or anticommutes with the grading and the operator \( D \) (the possible choices define the so-called KO-dimension of the spectral triple). For a manifold, the real structure \( J \) is given by the charge conjugation operator. In dimension \( m = 4 \), it satisfies

\[ J^2 = -\mathbb{I}, \quad J \partial = \partial J, \quad J \gamma_M = \gamma_M J. \] (2.8)

The real structure of the internal spectral triple (2.2) is the anti-linear operator \( J_F \) that exchanges particles with antiparticles on \( \mathcal{H}_F \). It satisfies

\[ J_F^2 = \mathbb{I}, \quad J_F D_F = D_F J_F, \quad J_F \gamma_F = -\gamma_F J_F. \] (2.9)

The real structure for the product spectral triple (2.6) is

\[ J = J \otimes J_F. \] (2.10)

For a manifold of dimension \( m = 4 \), it is such that

\[ J^2 = -\mathbb{I}, \quad JD = DJ, \quad J\Gamma = -\Gamma J. \] (2.11)

The real structure implements an action of the opposite algebra \( \mathcal{A}^0 \) on \( \mathcal{H} \), identifying \( a^0 \in \mathcal{A}^0 \) with \( Ja^0 J^{-1} \). This action is asked to commute with the one of \( \mathcal{A} \), yielding the \emph{order zero condition}

\[ [a, b^0] = 0 \quad \forall a \in \mathcal{A}, b \in \mathcal{A}^0. \] (2.12)

Among the properties of a spectral triple, one particularly relevant for physical models is the \emph{first order condition}

\[ [[D, b], a^0] = 0 \quad \forall a, b \in \mathcal{A}. \] (2.13)
2.2 Representation of the algebra

To describe the action of \( A_{\text{SM}} \otimes C^\infty(\mathcal{M}) \) on \( \mathcal{H} \) in (2.6), it is convenient to label the 32\( n \) degrees of freedom of the finite dimensional Hilbert space \( \mathcal{H}_F \) by a multi-index \( CI\alpha \) defined as follows.

- \( C = 0, 1 \) is for particle \( (C = 0) \) or anti-particle \( (C = 1) \);
- \( I = 0; i \) with \( i = 1, 2, 3 \) is the lepto-colour index: \( I = 0 \) means lepton, while \( I = 1, 2, 3 \) are for the quark, which exists in three colours;
- \( \alpha = 1, 2; a \) with \( a = 1, 2 \) is the flavour index:

\[
\begin{align*}
\hat{1} &= \nu_R, \hat{2} = e_R, 1 = \nu_L, 2 = e_L & \text{for leptons} \ (I = 0), \\
\hat{1} &= u_R, \hat{2} = d_R, 1 = q_L, 2 = d_L & \text{for quarks} \ (I = i).
\end{align*}
\]  

(2.14)

(2.15)

We sometimes use the shorthand notation \( \ell^n_I = (\nu_L, e_L) \) for the left handed neutrino and the associated lepton, and \( q^n_I = (u_L, d_L) \) for the pair of left-handed quarks.

There are \( 2 \times 4 \times 4 = 32 \) choices of triplet of indices \( (C, I, \alpha) \), which is the number of fermions per generation. One should also take into account an extra index \( n = 1, 2, 3 \) for the generations, but in this paper we work with one generation only and we omit it (we will discuss the number of generations in the computation of the action [21]). So from now on

\[ \mathcal{H}_F = \mathbb{C}^{32}. \]  

(2.16)

An element \( \psi \in \mathcal{H} = C^\infty(\mathcal{M}) \otimes \mathcal{H}_F \) is thus a 32 dimensional column-vector, in which each component \( \psi_{CI\alpha} \) is a Dirac spinor in \( L^2(\mathcal{M}, S) \).

Regarding the algebra, unless necessary we omit the symbol of the representation and identify an element \( a = (c, q, m) \) in \( C^\infty(\mathcal{M}) \otimes A_{\text{SM}}, \) where

\[ c \in C^\infty(\mathcal{M}, \mathbb{C}), \quad q \in C^\infty(\mathcal{M}, \mathbb{H}), \quad m \in C^\infty(\mathcal{M}, \mathbb{M}_3(\mathbb{C})), \]  

(2.17)

with its representation as bounded operator on \( \mathcal{H} \), that is a 32 square matrix whose components*

\[ a^{D\beta}_{CI\alpha} \]  

(2.18)

are smooth functions acting by multiplication on \( L^2(\mathcal{M}, S) \) as in (2.3). Explicitly†

\[ a = \begin{pmatrix} Q \\ M \end{pmatrix}_C^D \]  

(2.19)

where the 16 \( \times \) 16 square matrices \( Q, M \) have components

\[ Q_{I\alpha}^{J\beta} = \delta^I_J Q^\beta_\alpha, \quad M_{I\alpha}^{J\beta} = \delta^\alpha_\beta M^J_I, \]  

(2.20)

where

\[
Q^\beta_\alpha = \begin{pmatrix} c & \bar{c} \\ q & \bar{q} \end{pmatrix}^\beta_\alpha, \quad M^J_I = \begin{pmatrix} c & m \end{pmatrix}_I^J.
\]  

(2.21)

Here, the over-bar \( \bar{\cdot} \), denotes the complex conjugate, \( m \) (evaluated at the point \( x \)) identifies with its usual representation as \( 3 \times 3 \) complex matrices and the quaternion \( q \) (evaluated at \( x \)) acts through its representation as \( 2 \times 2 \) matrices:

\[ \mathbb{H} \ni q(x) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}. \]  

(2.22)

* \( D,J,\beta \) are column indices with the same range as the line indices \( C,I,\alpha \) (the position of the indices was slightly different in [19], the one adopted here makes the tensorial computation more tractable).
† The indices after the closing parenthesis are here to recall that the block-entries of \( A \) are labelled by the \( C,D \) indices, that is \( a_1^1 = Q, a_2^2 = M, a_3^3 = a_4^4 = 0 \).
2.3 Finite dimensional Dirac operator, grading and real structure

With respect to the particle/antiparticle index $C$, the internal Dirac operator

\[ D_F = D_Y + D_M \quad (2.23) \]

decomposes into a diagonal and an off-diagonal part

\[ D_Y = \begin{pmatrix} D_0 & D^\ell_0 \\ D_0^\dagger & D_0^\dagger \end{pmatrix}_C, \quad D_M = \begin{pmatrix} 0 & D_R \\ D_R^\dagger & 0 \end{pmatrix}_C \quad (2.24) \]

containing respectively the Yukawa couplings of fermions and the Majorana mass of the neutrino.

The $16 \times 16$ matrices $D_0$ and $D_R$ are block-diagonal with respect to the lepto-colour index $I$

\[ D_0 = \begin{pmatrix} D_0^\ell & D_0^q \\ D_0^q & D_0^\dagger \end{pmatrix}_I \quad D_R = \begin{pmatrix} D_R^\ell & 0 \\ 0 & D_R \end{pmatrix}_I \quad (2.25) \]

where we write $\ell$ for $I = 0$ and $q$ for $I = 1, 2, 3$. Each $D_0^I$ is a $4 \times 4$ matrix (in the flavour index $\alpha$),

\[ D_0^I = \begin{pmatrix} 0 & k^I_\alpha \\ k^I_\alpha & 0 \end{pmatrix}, \quad k^I = \begin{pmatrix} k^I_u & 0 \\ 0 & k^I_d \end{pmatrix}_\alpha, \quad (2.26) \]

whose entries are the Yukawa couplings of elementary fermions

\[ k^I_u = (k_{\nu}, k_u, k_u, k_u) \quad k^I_d = (k_e, k_d, k_d, k_d) \quad (2.27) \]

(three of them are equal because the Yukawa coupling of quarks does not depend on the colour). Similarly, $D_R^I$ is a $4 \times 4$ matrix (in the flavour index),

\[ D_R^I = \begin{pmatrix} k_R^I & 0 \\ 0 & 0 \end{pmatrix}_\alpha \quad (2.28) \]

whose only non-zero entry is the Majorana mass of the neutrino.

In tensorial notations, one has

\[ D_R = k_R^I \Xi_I^{\beta\alpha} \quad (2.29) \]

where

\[ \Xi_\alpha^\beta := \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\alpha^\beta, \quad \Xi_I^J := \begin{pmatrix} 1 \\ 0 \end{pmatrix}_I^J \quad (2.30) \]

and $\Xi_I^{\beta\alpha}$ is a shorthand notation for the tensor $\Xi_I^J \Xi_J^{\beta\alpha}$. Similarly, the internal grading is

\[ \gamma_F = \begin{pmatrix} I_8 & -I_8 \\ -I_8 & I_8 \end{pmatrix} = \eta^{\beta\alpha}_C \delta^I_J \quad (2.31) \]

where the blocks in the matrix act respectively on right/left particles, then right/left antiparticles, and we define

\[ \eta_\alpha^\beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}_\alpha^\beta, \quad \eta_C^D := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_C^D \quad (2.32) \]
and $\eta_C^{D\beta}$ holds for $\eta_C^{D\beta}_i$. The internal real structure is

$$J_F = \left( \begin{array}{cc} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{array} \right) \quad \text{where} \quad \xi_C^D := \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$$

(2.33)

where $cc$ denotes the complex conjugation and we define

$$\xi_C^D := \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$$

(2.34)

3 Minimal twist of the Standard Model

In the noncommutative geometry description of the Standard Model, the bosonic degrees of freedom are obtained by a so-called fluctuation of the metric, that is the substitution of the operator $D$ with $D + A + JAJ^{-1}$ where

$$A = \sum_i a_i [D, b_i] \quad a_i, b_i \in \mathcal{A}$$

(3.1)

is a generalised 1-form (see [12] for details and the justification of the terminology).

As already noticed in [7, 15], the Majorana mass of the neutrino does not contribute to the bosonic content of the model, for $D_M$ commute with algebra:

$$[\gamma^5 \otimes D_M, a] = 0 \quad \forall a \in \mathcal{A}$$

(3.2)

However, in order to generate the $\sigma$ field proposed in [6] to cure the electroweak vacuum instability and solve the problem of the computation of the Higgs mass, one precisely needs to make $D_M$ contribute to the fluctuation.

To do this, a possibility consists in substituting the commutator $[D, a]$ with a twisted commutator

$$[D, a]_{\rho} := Da - \rho(a)D$$

(3.3)

where $\rho$ is a fixed automorphism of $\mathcal{A}$. This substitution is the base of the definition of twisted spectral triple [16] where, instead of asking that $[D, a]$ be bounded for any $a$ (which is one of the axioms of a spectral triple), one requires that there exists an automorphism $\rho$ such that the twisted-commutator $[D, a]_{\rho}$ is bounded for any $a \in \mathcal{A}$. As shown in [22], starting with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is almost commutative as in (1.2), then the only ways to build a twisted spectral triple with the same Hilbert space and Dirac operator (which, from a physics point of view, means that one looks for models with the same fermionic content as the Standard Model) are to double the algebra and make them act independently on the left and right components of spinors (following actually an idea of [18]). All this is detailed in the next section.

3.1 Algebra and Hilbert space

The algebra $\mathcal{A}$ of the twisted spectral triple of the Standard Model is twice the algebra (2.6),

$$\mathcal{A} = (C^\infty(M) \otimes \mathcal{A}_{\text{SM}}) \otimes \mathbb{C}^2,$$

(3.4)

which is isomorphic to

$$(C^\infty(M) \otimes \mathcal{A}_{\text{SM}}) \oplus (C^\infty(M) \otimes \mathcal{A}_{\text{SM}}).$$

(3.5)

It acts on the same Hilbert space $\mathcal{H}$ as in the non-twisted case, but now the two copies of $C^\infty(M) \otimes \mathcal{A}_{\text{SM}}$ act independently on the right and left components of spinors. To write this action, it is convenient to view an element of $\mathcal{H}$ as a column vector with $4 \times 32 = 128$ components (4 being the number of components of a usual spinor in $L^2(M, S)$ for $m = 4$). To this aim, one introduce two extra-indices to label the degrees of freedom of $L^2(M, S)$:
• $s = r, l$ is the chirality index;

• $s = \hat{0}, \hat{1}$ denotes particle ($\hat{0}$) or anti-particle part ($\hat{1}$).

An element $a$ of (3.5) is a pair of elements of (2.6), namely

$$a = (c, c', q, q', m, m')$$

with

$$c, c' \in C^\infty(\mathcal{M}, \mathbb{C}) \quad q, q' \in C^\infty(\mathcal{M}, \mathbb{H}) \quad m, m' \in C^\infty(\mathcal{M}, M_3(\mathbb{C})).$$

We make $(c, q, m)$ act on the chiral subspace $\mathcal{H}_c$ of $\mathcal{H}$, consisting in particles and antiparticles whose chirality as Dirac spinors coincides with chirality in the internal space; whereas $(c', q', m')$ acts on the anti-chiral subspace $\mathcal{H}_a$ consisting in particles and particles whose Dirac and internal chiralities do not coincide. The chiral subspace $\mathcal{H}_c$ is the subspace of $\mathcal{H}$ spanned by $r, \alpha = \hat{0}, \hat{1}$, and $l, \alpha = \hat{1}, \hat{2}$ while $\mathcal{H}_a$ is spanned by $l, \alpha = \hat{1}, \hat{2}$ and $r, \alpha = \hat{1}, \hat{2}$ (in both cases, $C$ takes both values $1, 0$). In other terms, $a \in \mathcal{A}$ acts as in (2.19), but now the two $64 \times 64$ matrices $Q, M$ are tensor fields of components

$$Q_s^{\hat{t}J_s} = \delta_s^{\hat{t}} Q_s^{\hat{j}J_s}, \quad M_s^{\hat{t}J_s} = \delta_s^{\hat{t}} M_s^{\hat{j}J_s}$$

where $\delta_s^{\hat{t}}$ denotes the product of the two Kronecker symbols $\delta_s^{t}, \delta_s^{j}$. Both $Q$ and $M$ stills act trivially (i.e. as the identity) on the indices $\hat{t}s$, but no longer on the chiral indices $st$. On the latter, the action is given by

$$Q_s^{\hat{t}J_s} = \left((Q_r)^{\hat{t}J_s}_{\alpha} (Q_l)^{\hat{t}J_s}_{\alpha}\right)^s, \quad M_s^{\hat{t}J_s} = \left((M_r)^{\hat{t}J_s}_{\alpha I} (M_l)^{\hat{t}J_s}_{\alpha I}\right)^s,$$

with

$$Q_r = \begin{pmatrix} c \\ q' \end{pmatrix}_\alpha, \quad Q_l = \begin{pmatrix} c' \\ q \end{pmatrix}_\alpha,$$

and

$$M_r = \begin{pmatrix} m \otimes \mathbb{I}_2 & 0 \\ 0 & m' \otimes \mathbb{I}_2 \end{pmatrix}, \quad M_l = \begin{pmatrix} m' \otimes \mathbb{I}_2 & 0 \\ 0 & m \otimes \mathbb{I}_2 \end{pmatrix},$$

where we denote

$$c := \begin{pmatrix} c \\ \bar{c} \end{pmatrix}, \quad m := \begin{pmatrix} c \\ m \end{pmatrix}, \quad c' := \begin{pmatrix} c' \\ \bar{c}' \end{pmatrix}, \quad m' := \begin{pmatrix} m' \\ m' \end{pmatrix}.$$

Compared to the usual spectral triple of the Standard Model, $M_{r,l}$ are no longer trivial in the flavour index $\alpha$.

**Remark 3.1.** If we were using the twist-by grading, we should permute $m$ with $m'$ in (3.11), for on the antiparticles subspace – i.e. $C = 1$ – then $\mathcal{H}_c$ is a subspace of the $-1$-eigenspace of the grading (see also appendix 6.2 regarding the twist used in [19]).

The twist $\rho$ is the automorphism of $\mathcal{A}$ that exchanges the two components of $\mathcal{A}_{SM}$, namely

$$\rho(c, c', q, q', m, m') = (c', c, q', q, m', m).$$
In terms of the representation, one has

$$\rho(a) = \left( \rho(Q), \rho(M) \right)_C^D$$

with

$$\rho(Q)^{ij}_{\alpha\beta} = \delta^i_j \rho(Q)_{\alpha\beta}^{\beta}, \quad \rho(M)^{ij}_{\alpha\beta} = \delta^i_j \rho(M)_{\alpha\beta}^{\beta}$$

where

$$\rho(Q)_{\alpha\beta}^{\beta} = \left( (Q_{\alpha})_\beta, (Q_{\beta})_\alpha \right)_s^t, \quad \rho(M)_{\alpha\beta}^{\beta} = \left( (M_{\alpha})_\beta, (M_{\beta})_\alpha \right)_s^t.$$

In short, the twist amounts to flipping the left/right indices $l/r$.

### 3.2 Grading and real structure

The operators $\Gamma$ in (2.7) and $J$ in (2.10) are the grading and the real structure for the twisted spectral triple, in the sense defined in [19, 22] (the rule of signs defining the $KO$-dimension is not affected by the twist; that $\Gamma$ commutes with the representation (3.9) follows from the latter being diagonal but on the $\alpha$ and $I$ indices, where $\Gamma$ is (block)-diagonal). In particular, as in the non twisted case, the real structure implements an action of the opposite algebra $A^\circ$ on $H$, that commutes with the one of $A$. To check this, let us first write down the representation of the opposite algebra.

**Proposition 3.2.** For $a \in A$ as in (2.19), one has (for $\mathcal{M}$ of dimension 4)

$$JaJ^{-1} = -\left( \begin{array}{cc} \bar{M} & 0 \\ 0 & Q \end{array} \right)_C^D.$$

**Proof.** From (2.10) and (2.33) one has

$$J = \left( \begin{array}{cc} 0 & \mathcal{J} \otimes \mathbb{1}_{16} \\ \mathcal{J} \otimes \mathbb{1}_{16} & 0 \end{array} \right)_C^D. \quad (3.18)$$

Since $J^{-1} = -J$ by (2.11), using the representation (2.19) of $a$ one obtains (omitting $\mathbb{1}_{16}$)

$$JaJ^{-1} = -JaJ = \left( \begin{array}{cc} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{array} \right)_C^E \left( \begin{array}{cc} Q & 0 \\ 0 & M \end{array} \right)_F^E \left( \begin{array}{cc} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{array} \right)_F^D = -\left( \mathcal{J}M \mathcal{J} \right)_C^D \left( \begin{array}{cc} 0 & 0 \\ 0 & \mathcal{J}Q \mathcal{J} \end{array} \right)_C^D. \quad (3.19)$$

In addition, $\mathcal{J}$ commutes with the grading $\gamma_\mathcal{M}$ (see (2.8)), so it is of the form

$$\mathcal{J} = \left( \begin{array}{cc} \mathcal{J}_r & 0 \\ 0 & \mathcal{J}_l \end{array} \right)_{cc} \quad (3.20)$$

where $\mathcal{J}_{r/l}$ are $2 \times 2$ matrices carrying the $s, t$ indices, such that $\mathcal{J}_r \mathcal{J}_r = \mathcal{J}_l \mathcal{J}_l = -\mathbb{1}_2$. From the explicit form (3.8) of $Q$ and $M$, one gets (still omitting the indices $\alpha, I$ in which $J$ is trivial)

$$\mathcal{J}Q \mathcal{J} = \left( \begin{array}{cc} \mathcal{J}_r(\delta^j_s \bar{Q}_r) \mathcal{J}_r & 0 \\ \mathcal{J}_l(\delta^j_s \bar{Q}_l) \mathcal{J}_l & 0 \end{array} \right)_s^t \left( \begin{array}{cc} -\delta^j_s \bar{Q}_r & 0 \\ 0 & -\delta^j_s \bar{Q}_l \end{array} \right)_s^t = -\bar{Q}, \quad (3.21)$$

$$\mathcal{J}M \mathcal{J} = \left( \begin{array}{cc} \mathcal{J}_r(\delta^j_s \bar{M}_r) \mathcal{J}_r & 0 \\ \mathcal{J}_l(\delta^j_s \bar{M}_l) \mathcal{J}_l & 0 \end{array} \right)_s^t \left( \begin{array}{cc} -\delta^j_s \bar{M}_r & 0 \\ 0 & -\delta^j_s \bar{M}_l \end{array} \right)_s^t = -\bar{M}, \quad (3.22)$$

hence the result. 


To check the order zero condition, we denote
\[ b = (d, d', p, p', n, n') \] (3.23)
another element of \( \mathcal{A} \) with \( d, d' \in C^\infty(\mathcal{M}, \mathbb{C}) \), \( p, p' \in C^\infty(\mathcal{M}, \mathbb{H}) \), \( n, n' \in C^\infty(\mathcal{M}, M_3(\mathbb{C})) \). It acts on \( \mathcal{H} \) by (3.24) as
\[ b = \begin{pmatrix} R \\ N \end{pmatrix}_D^C \] (3.24)
where \( R, N \) are defined as \( Q, M \) in (3.8), with
\[ R_r = \left( d \oplus p \right)_\alpha^{\beta}, \quad R_l = \left( d' \oplus p' \right)_\alpha^{\beta}, \quad N_r = \left( n \otimes I_2 \otimes n' \otimes I_2 \right)_\alpha^{\beta}, \quad N_l = \left( n' \otimes I_2 \otimes n \otimes I_2 \right)_\alpha^{\beta}. \] (3.25)

**Corollary 3.2.1.** The order-zero condition (2.12) holds.

**Proof.** By Prop. 3.2, the order zero condition \( [a, JbJ^{-1}] = 0 \) for all \( a, b \in \mathcal{A} \) is equivalent to \( [R, \bar{M}] = 0 \) and \( [N, \bar{Q}] = 0 \). By (3.8) and (3.9), one gets (omitting the indices \( \dot{s} \) on which all actions are trivial)
\[ [R, M] = \begin{pmatrix} [\delta^J I R, M] & 0 \\ 0 & [\delta^J I R, M] \end{pmatrix}^t_s. \] (3.26)
By (3.11), one has
\[ [\delta^J I R, M] = \begin{pmatrix} [\delta^J I d, m \otimes I_2] & 0 \\ 0 & [\delta^J I p', m' \otimes I_2] \end{pmatrix}_\alpha^{\beta}, \] (3.27)
which is zero, as can be seen writing \( \delta^J I d = I_4 \otimes d \) and similarly for \( [\delta^J I p', m' \otimes I_2] \). The same holds true for \( [\delta^J I R, M] \).

### 3.3 Twisted fluctuation

In the twisted context, fluctuations are similar to (3.1), replacing the commutator for a twisted one [23]. In addition, if the twisted first-order condition does not hold, one should add a non-linear term [9, 25]. We thus consider the twisted-covariant Dirac operator
\[ D_A = D + A_{(1)} + \hat{A}_{(1)} + A_{(2)} \] (3.28)
where
\[ A_{(1)} = \sum_i a_i [D, b_i]_\rho, \quad a_i, b_i \in \mathcal{A} \] (3.29)
is a twisted (generalised) 1-form, \( \hat{A}_{(1)} := JA_{(1)}J^{-1} \) is its image by the conjugation with the real structure, while
\[ A_{(2)} = \sum_i \hat{a}_i \left[ A_{\rho}, \hat{b}_i \right]_{\rho^\circ} \] with \( \hat{a}_i := Ja_iJ^{-1} = (a_i^*)^\circ, \hat{b}_i := JB_iJ^{-1} = (b_i^*)^\circ \) (3.30)
and \( \rho^\circ \) denotes the automorphism of the opposite algebra defined as
\[ \rho^\circ(a^\circ) := (\rho^{-1}(a))^\circ. \] (3.31)

The term \( A_{(2)} \) breaks the linearity of the map \( A_{(1)} \to D + A_{(1)} + JA_{(1)}J^{-1} \) and vanishes when the twisted first-order condition (A.16) holds (this is a straightforward adaptation to the twisted
context of the result of [9]). We need to take it into account for, as explained in §6.2, the twisted first-order condition only holds partially.

The twisted 1-form decomposes as the sum \( A^{(1)} = A_F + \hat{A} \) of two pieces: one that we call the *finite part* of the fluctuation because it comes from the finite dimensional spectral triple, namely

\[
A_F = \sum_i a_i [\gamma_M \otimes D_F, b_i]_\rho \quad a_i, b_i \in \mathcal{A};
\]

another one coming from the manifold part of the spectral triple

\[
\hat{A} = \sum_i a_i [\hat{D}, b_i]_\rho \quad a_i, b_i \in \mathcal{A}
\]

that we call *gauge part* in the following (terminology will become clear later).

To guarantee that the twisted covariant operator (3.28) is selfadjoint, one assumes that the twisted 1-form \( A^{(1)} \) is selfadjoint [25, Prop.3.8] (actually this is not a necessary condition, but requiring \( A^{(1)} \) to be selfadjoint makes sense viewing the fluctuation \( D \to D_A \) as a three steps process

\[
D \to D + A^{(1)} \to D + A^{(1)} + \hat{A}^{(1)} \to D_A
\]

such that selfadjointness is preserved at each step). This means that for physical models, we assume that both the gauge \( \hat{A} \) and the finite \( A_F \) parts are selfadjoint.

So far, the construction works for any even dimension manifold \( \mathcal{M} \). To build explicitly the Standard Model, from now on one fixes the dimension of \( \mathcal{M} \) to \( m = 4 \). The grading and the real structure are

\[
\gamma_M = \gamma^5 = \gamma^0_E \gamma^1_E \gamma^2_E \gamma^3_E = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}^t_s = \eta_D^\delta_s^t
\]

and

\[
J = i \gamma^0_E \gamma^2_E cc = i \begin{pmatrix} \bar{\sigma}^2 & 0_2 \\ 0_2 & \sigma^2 \end{pmatrix}_{st} cc = -i \eta^{\delta_s^t}_s \tau^{i}_s cc,
\]

where \( cc \) denotes the complex conjugation and we define

\[
\tau^i_s := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_s^i, \quad \eta^t_s := \begin{pmatrix} 1 & -1 \end{pmatrix}_s^t.
\]

For the internal spectral triple, one has

\[
\gamma_F = \begin{pmatrix} I_8 & -I_8 \\ -I_8 & -I_8 \end{pmatrix} = \eta^{D\delta}_{CA} \delta^I_s, \quad J_F = \begin{pmatrix} 0 & I_{16} \\ I_{16} & 0 \end{pmatrix}^D_C cc = \xi^{D}_{J\delta_{1a}^\beta}
\]

where the matrix \( \gamma_F \) is written in the basis left/right particles then left/right antiparticles, and we define

\[
\eta^\beta_{\alpha} := \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}_\alpha^\beta, \quad \eta_{CA}^D := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^D_C, \quad \xi_{J\delta_{1a}}^D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^D_C
\]

with \( \eta^{D\delta}_{CA} \) holding for \( \eta^{D}_{CA} \eta^\beta_{\alpha} \). Thus

\[
\Gamma = \gamma_M \otimes \gamma_F = \eta^{D\delta}_{CA}^t_s \delta^t_s \quad \text{and} \quad J = J_M \otimes J_F = -i \eta^t_s \tau^i_s \xi_{J\delta_{1a}^\beta} cc.
\]
4 Scalar part of the twisted fluctuation

The scalar sector of the twisted Standard Model is obtained from the finite part (3.32) of the twisted 1-form, which in turns decomposes into a diagonal part (determined by the Yukawa couplings of fermions)

\[ A_Y = \sum_i a_i [\gamma^5 \otimes D_Y, b_i]_\rho, \] (4.1)

and an off-diagonal part (determined by the Majorana mass of the neutrino)

\[ A_M = \sum_i a_i [\gamma^5 \otimes D_M, b_i]_\rho. \] (4.2)

As shown below, the former produces the Higgs sector, the latter a pair of extra scalar fields.

4.1 The Higgs sector

We begin with the diagonal part (4.1). We first notice that the \( M_\beta(\mathbb{C}) \) part of the algebra (3.4) twist-commutes with \( \gamma^5 \otimes D_Y \).

**Lemma 4.1.** For any \( b \in \mathcal{A} \) as in (3.24), one has

\[ [\gamma^5 \otimes D_Y, b]_\rho = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix} C \] (4.3)

where \( S \) has components

\[ S^{t_{ij}b}_{ss_{fa}} = \delta^t_s \left( \eta^u_s(D_0)^{K}_{fa} R^{t_{ij}b}_{K\gamma} - \rho(R)^{u\gamma}_{sa} \eta^s_u(D_0)^{t_{ij}b}_{K\gamma} \right). \] (4.4)

**Proof.** From the explicit forms (2.24) of \( D_Y \) and (3.24) of \( b \), one has

\[ [\gamma^5 \otimes D_Y, b]_\rho = \begin{pmatrix} [\gamma^5 \otimes D_0, R]_\rho \\ [\gamma^5 \otimes D_0, N]_\rho \end{pmatrix}. \]

In the tensorial notation, \( S := [\gamma^5 \otimes D_0, R]_\rho \) has components

\[ S^{t_{ij}b}_{ss_{fa}} = \eta^u_s(D_0)^{K}_{fa} \delta^t_s \left( \eta^v_s(D_0)^{K}_{fa} R^{t_{ij}b}_{K\gamma} - \delta^v_s \rho(R)^{v\gamma}_{sa} \eta^v_u(D_0)^{t_{ij}b}_{K\gamma} \right), \] (4.5)

\[ = \delta^t_s \left( \eta^u_s(D_0)^{K}_{fa} R^{t_{ij}b}_{K\gamma} - \rho(R)^{u\gamma}_{sa} \eta^s_u(D_0)^{t_{ij}b}_{K\gamma} \right), \] (4.6)

which shows (4.4). To show that

\[ [\gamma^5 \otimes D_0^1, N]_\rho = 0, \] (4.7)

let us denote \( T \) the left-hand side of the equation above. It has components

\[ T^{t_{ij}b}_{ss_{fa}} = \eta^u_s(D_0)^{K}_{fa} \delta^t_s \left( \eta^v_s(D_0)^{K}_{fa} N^{t_{ij}b}_{K\gamma} - \delta^v_s \rho(N)^{v\gamma}_{sa} \eta^v_u(D_0)^{t_{ij}b}_{K\gamma} \right), \] (4.8)

\[ = \delta^t_s \left( \eta^u_s(D_0)^{K}_{fa} N^{t_{ij}b}_{K\gamma} - \rho(N)^{u\gamma}_{sa} \eta^s_u(D_0)^{t_{ij}b}_{K\gamma} \right), \] (4.9)

\[ = \delta^t_s \left( (D_0)^{K}_{fa} \gamma^\beta_{\gamma K} - (N_i)^{K}_{\alpha l} (D_0)^{t_{ij}b}_{K\gamma} \right) \]

\[ - (D_0^1)^{K}_{fa} \gamma^\beta_{\gamma K} + (N_i)^{K}_{\alpha l} (D_0^1)^{t_{ij}b}_{K\gamma} \right). \] (4.10)
Since \((D^0_1)^I = \delta_I^J(D^0_0)\) and \((D^0_0)^I_J = \delta_I^J(D^0_0)^I_J\) (with no summation on \(I\) and \(J\)), the upper-left term in (4.10) is
\[
(D^0_0)^\beta_\gamma (N^\alpha_0)^\delta_\beta_J - (N^\alpha_0)^\delta_\beta_J (D^0_0)^\gamma_\beta = \left( \begin{array}{cc} 0 & \vec{k}^I(n' \otimes I_2) \\ k^I(n \otimes I_2) & 0 \end{array} \right) - \left( \begin{array}{cc} 0 & (n \otimes I_2)k^J \\ (n' \otimes I_2)k^J & 0 \end{array} \right)
\]
where we omitted the \(I, J\) indices on \(n\). One has
\[
k^I(n \otimes I_2) = \left( \begin{array}{c} k^I_u(n) \\ k^I_d(n) \end{array} \right), \quad (n \otimes I_2)k^J = \left( \begin{array}{c} n^J_u \\ nk^J_d \end{array} \right),
\]
and similarly for the terms in \(n'\). Restoring the indices, one has
\[
k^I_u n^J_J = \left( \begin{array}{c} d_{k^I_u} \\ k^I_u \end{array} \right), \quad n^J_J k^J_u = \left( \begin{array}{c} d_{k^J_u} \\ nk^J_d \end{array} \right)
\]
where we write \(k^I_u = 0\) for the lepton, and \(k^I_u = 1, 2, 3\) for the coloured quarks. Again, in the expression above, there is no summation on \(I\) and \(J\): \(k^I_u n^J_J\) means the matrix \(n\) in which the \(I\)th line is multiplied by \(k^I_u\), while in \(n^J_J k^J_u\) this is the \(J\)th column of \(n\) which is multiplied by \(k^J_u\). Therefore
\[
k^I(n \otimes I_2) - (n \otimes I_2)k^J = 0.
\]
Similarly \(\vec{k}^I(n' \otimes I_2) - (n' \otimes I_2)k^J = 0\), so that (4.11) - that is the upper-left term in (4.10) - is zero. The proof that the lower-right term is zero is similar. Hence (4.7) and the result.

A similar result holds in the non-twisted case (the computation is similar as above, with \(n' = n\), so that everything boils down to the single equation (4.14)). The result however is not true if one genuinely generalises the twist used in [19]. As explained below, this yields an additional violation of the twisted first order condition, besides the one required to generate the field \(\sigma\). That is why we do not use this genuine twist, but rather the one presented in section 3.

**Remark 4.2.** The twist in [19] was not applied to the \(M_3(\mathbb{C})\) part of the algebra. Only \(\mathbb{C} \oplus \mathbb{H}\) was doubled and this yielded an action similar as the one used on the present paper (modulo a change of notations, the representation (4.7) of [19] coincides with (3.9)). A genuine generalisation of this twist consists in making two copies of \(M_3(\mathbb{C})\) acting independently on the left and right components of spinors, namely \(a \in A\) acts as in (3.8), but now \(M_{r,l}\) are given by
\[
M_r = (m \otimes I_4)^\beta_\alpha, \quad M_l = (m' \otimes I_4)^\beta_\alpha.
\]
Then Lemma 4.1 no longer holds for the lower right term \(T\) is non necessarily zero (on the r.h.s. of (4.11) the first parenthesis now contains only \(n\), and the second only \(n'\), so that the cancellation (4.14) is no longer true).

We now compute the 1-forms generated by the Yukawa couplings of the fermions. In order to do so, we extend the action of the automorphism \(\rho\) to any polynomial in \(q, q', p, p', c, c', d, d'\). Namely \(\rho\) “primis” what is un-primed, and vice-versa. For instance \(\rho(qp' - cd) = q'p - cd\).

**Proposition 4.3.** The diagonal part (4.1) of a twisted 1-form is
\[
A_Y = \left( \begin{array}{c} A \\ 0 \end{array} \right)^D_C \text{ where } A = \delta^I_J \left( \begin{array}{c} A_r \\ A_l \end{array} \right)^S_t
\]
with
\[
A_r = \left( H_2 k^I \vec{k}^I H_1 \right)^\beta_\alpha, \quad A_l = -\left( H_2' k^I \vec{k}^I H_1' \right)^\beta_\alpha,
\]
where \(H_{1,2} = \rho(H_{1,2})\) are quaternionic fields.
Proof. From (2.19) and lemma 4.1, one has $a[\gamma^5 \otimes D_Y,b]_\rho = Q S$. In components, this gives (using the explicit forms (3.8) of $Q,R$):

$$A^{\dot{t}l}_{s;i\alpha} = Q^{\dot{u}i}_{s;i\alpha} \delta^\dot{u}_a \left[ \eta^\nu_a (D_0)^\nu_{\delta\gamma} R_{\nu\delta}^{\dot{b}\beta} - \rho(R)^{\nu\delta}_{a\gamma} \eta^\nu_b (D_0)^{\dot{b}\beta}_{\delta\gamma} \right]$$ \hspace{1cm} (4.18)

$$= \delta^\dot{u}_a Q^{\dot{u}i}_{s;i\alpha} \eta^\nu_a (D_0)^\nu_{\delta\gamma} R_{\nu\delta}^{\dot{b}\beta} - \rho(R)^{\nu\delta}_{a\gamma} \eta^\nu_b (D_0)^{\dot{b}\beta}_{\delta\gamma}$$ \hspace{1cm} (4.19)

where we use $\delta^K(D_0)^K_{\delta\gamma} = \delta^I_l (D_0)^I_l$ (with no summation on $I$ in the last expression). Since $Q$ is diagonal on the chiral indices $s$, the only non-zero components of $A$ are for $s = t = r$ and $s = t = l$, namely

$$A^{r;l}_{s;i\alpha} = \delta^I_{si} \left( A^I_r \right)_{\alpha} \text{ with } (A^I_r)_{\alpha} = (Q^r)_{\gamma} \left[ (D_0)^I_{\gamma} (R_r)^{\beta\gamma} - (R_l)^I_{\gamma} (D_0)^{\beta\gamma}_{\delta\gamma} \right]$$ \hspace{1cm} (4.20)

$$A^{l;l}_{s;i\alpha} = \delta^I_{si} \left( A^I_r \right)_{\alpha} \text{ with } (A^I_r)_{\alpha} = (Q^r)_{\gamma} \left[ - (D_0)^I_{\gamma} (R_l)^{\beta\gamma} + (R_r)^I_{\gamma} (D_0)^{\beta\gamma}_{\delta\gamma} \right].$$ \hspace{1cm} (4.21)

From the explicit expression (3.10), (3.25), (2.26) of $Q_{r/t}, R_{r/t}$ and $D_0^I$ one gets

$$Q_r D_0^I R_r = \left( q' k_l q \right)^{\beta}_{\alpha}, Q_r R_l D_0^I = \left( q' p k_l \right)^{\beta}_{\alpha},$$ \hspace{1cm} (4.22)

$$Q_l D_0^I R_l = \left( q k_l q' \right)^{\beta}_{\alpha}, Q_l R_r D_0^I = \left( q p' k_l \right)^{\beta}_{\alpha}.$$ \hspace{1cm} (4.23)

Using that $c,c',d,d'$ commute with $k_l$, one has

$$Q_r (D_0^I R_r - R_l D_0^I) = \left( H_2 k_l \tilde{k}^l H_1 \right)^{\beta}_{\alpha}, -Q_l D_0^I R_l + Q_l R_r D_0^I = - \left( H_2' q' k_l \tilde{k}^l H_1' \right)^{\beta}_{\alpha}$$ \hspace{1cm} (4.24)

where

$$H_1 := c(p' - d'), \quad H_2 := q'(d - p), \quad H_1' := c'(p - d), \quad H_2' := q(d' - p').$$ \hspace{1cm} (4.25)

This shows the result. \hfill $\Box$

Imposing now selfadjointness as stressed before (3.34) at the beginning of this section, we get the

**Corollary 4.3.1.** A selfadjoint diagonal twisted 1-form (4.1) is parametrized by two independent scalar quaternionic field $H_r, H_l$.

Proof. The twisted 1-form (4.16) is selfadjoint if and only if

$$H_2 = H_1^1 =: H_r \text{ and } H_2' = H_1'^1 =: H_l.$$ \hspace{1cm} (4.26)

They are independent as follows from their definition (4.25). \hfill $\Box$

Since $\gamma^5 \otimes D_Y$ satisfies the twisted first-order condition (Prop. A.8), it does not contribute to the non-linear term $A^{1}_2$ of the twisted fluctuation. Gathering the results of this section, one thus works out the fields induced by the Yukawa coupling of fermions via a twisted fluctuation of the metric.
Proposition 4.4. A selfadjoint diagonal fluctuation is

\[ D_{AY} = \gamma^5 \otimes D_Y + A_Y + \tilde{A}_Y = \left( \eta_s^l \delta_s^l D_0 + A \right) \eta_s^l \delta_s^l D_0^t + \tilde{A} \right)^D \]

(4.27)

where \( A = \delta_{sI}^l \left( A_r \right)^t \) is generated by two quaternionic fields \( H_r, H_t \) as

\[ A_r = \left( H_r k^l \tilde{k}^l H^t_r \right)^{\beta}_{\alpha}, \quad A_l = \left( H_l k^l \tilde{k}^l H^t_l \right)^{\beta}_{\alpha}. \]

(4.28)

Proof. Remembering that \( J^{-1} = -J \), proposition 4.3 yields

\[ \tilde{A}_Y = J A_Y J^{-1} = \left( \begin{array}{cc} 0 & J \\ 0 & 0 \end{array} \right)^D \left( \begin{array}{cc} A & 0 \\ 0 & -J \end{array} \right)^D \left( \begin{array}{cc} 0 & -J A J^{-1} \end{array} \right)^D. \]

(4.29)

From the explicit form (3.36) of \( J = -J \) and (4.16) of \( A \), one obtains (omitting the \( IJ \) and \( \alpha\beta \) indices in which the real structure \( J \) is trivial)

\[ J A J^{-1} = \eta_s^l \tilde{r}_s \tilde{A}_{\alpha\beta} \eta_{sI}^{\alpha} \eta_{sI}^{\beta} = \eta_s^l \tilde{r}_s \tilde{A}_{\alpha\beta}^l \eta_s^l \tilde{r}_s = \eta_s^l \tilde{r}_s \tilde{A}_{\alpha\beta} = \left( \begin{array}{cc} A_r & 0 \\ 0 & -J \end{array} \right)_{\alpha\beta} \]

(4.30)

where we used (4.16) and write \( \tau_s^l \delta_s^l \tilde{r}_s = -\delta_s^l \).

The result follows summing (4.29) with \( A_Y \) given in Prop. 4.3 and \( D_Y \) given in (2.24), then using corollary 4.3.1 to rename \( H_r \) and \( H_t \).

In the non-twisted case, the primed and unprimed quantities are equal, so that one obtains only one quaternionic field \( H_r = H_t \), which combines in the action as

\[ H := H_r + H_t = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right), \]

(4.31)

whose complex components \( \phi_1, \phi_2 \) identify with the Higgs doublet. In the twisted case, the complex components \( \phi_{1,2}^l, \phi_{1,2}^t \) of \( H_r, H_t \) define two scalar doublets

\[ \Phi_r := \left( \begin{array}{c} \phi_r^l \\ \phi_r^t \end{array} \right), \quad \Phi_t := \left( \begin{array}{c} \phi_t^l \\ \phi_t^t \end{array} \right), \]

(4.32)

which act respectively on the right and on the left part of the Dirac spinors. However, similar to (4.31) they only appear in the fermionic action through their linear combination \( H_r + H_t \) [21], therefore there is actually only one physical Higgs doublet in the twisted case as well.

4.2 The extra scalar field

The computation of the off-diagonal term (4.2) of the finite part of the twisted 1-form is easier than for the diagonal part, because \( D_M \) has only one non-zero component.

Proposition 4.5. The off-diagonal part (4.2) of a twisted 1-form is

\[ A_M = \left( \begin{array}{c} D \\ C \end{array} \right)^D \]

(4.33)
where
\[ C = k_R \delta^i_s \begin{pmatrix} C_r \\ C_l \end{pmatrix}_s^t, \quad D = \bar{k}_R \delta^i_s \begin{pmatrix} D_r \\ D_l \end{pmatrix}_s^t \]  
(4.34)
with
\[ C_r = D_r = \Xi^{j\beta}_{1\alpha} \sigma, \quad C_l = D_l = -\Xi^{j\beta}_{1\alpha} \sigma' \]  
(4.35)
where \( \sigma \) and \( \sigma' \) are complex fields.

**Proof.** Using the explicit form (2.24) of \( D_M \), for \( a \) in (2.19) and \( b \) in (3.24) one gets
\[ a \left[ \gamma^5 \otimes D_M, b \right]_\rho = \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \left( \gamma^5 \otimes D_R \right) & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}^\dagger \left( \gamma^5 \otimes D_R \right)_\rho = \left( M \left( \gamma^5 \otimes D_R^\dagger \right) R - \rho \left( N \left( \gamma^5 \otimes D_R \right) \right) \right)_r^C \]  
(4.36)
With \( D_R \) given in (2.29), one computes the upper-right component \( C \) of the matrix above:
\[ C_{\alpha l\beta} = Q_{\alpha l} K_{\gamma} \left[ k_R \eta^{\alpha}_{\nu} \delta_{\nu}^{\beta} \Xi_{\delta \beta}^{L\delta} - k_R \rho \left( R_{\nu} \right)^{\nu L\delta}_{\eta} \eta^{\nu}_{\beta} \Xi_{\delta \beta}^{L\delta} \right] . \]  
(4.37)
Since \( Q, N \) are diagonal in the index and proportional to \( \delta^i_s \), the non-zero components of \( C \) are
\[ (C_r)_{1\alpha} = k_R \delta^i_s \left( Q_r \right)_{1\alpha} K_{\gamma} \left[ -\Xi_{\delta \beta}^{L\delta} \left( R_{\nu} \right)^{\nu \beta}_{\delta} \right], \]  
(4.38)
\[ (C_l)_{1\alpha} = k_R \delta^i_s \left( Q_l \right)_{1\alpha} \left[ -\Xi_{\delta \beta}^{L\delta} \left( R_{\nu} \right)^{\nu \beta}_{\delta} + \left( R_{\nu} \right)^{\nu \beta}_{L\delta} \right]. \]  
(4.39)
Explicitly, from the formula (3.10) for \( Q_{r/l} \) and (3.25) of \( R_{r/l} \), one gets
\[ Q_r \left( \Xi N_r - R_l \Xi \right) = \left( c \delta^j_{l} \right)_{\beta} \left( \left( \Xi^j_{l} \right)_{\alpha} \right) \left( n \otimes \mathbb{I}_2 \right)_{\alpha} \left( n' \otimes \mathbb{I}_2 \right)_{\alpha} \left( d' \delta^j_{l} \right)_{\beta} \left( \Xi^j_{l} \right)_{\alpha} = \left( c \left( d - d' \right) \Xi^j_{l} \right)_{\alpha} \]  
and similarly
\[ Q_l \left( -\Xi N_l + R_r \Xi \right) = -\sigma' \Xi^{j\beta}_{\alpha l} \]  
(4.40)
where we define the scalar fields
\[ \sigma := c \left( d - d' \right), \quad \sigma' := c' \left( d' - d \right). \]  
(4.41)
Similarly, one computes that the lower left component \( D \) of (4.36) has non zero components
\[ D_r = \bar{k}_R \delta^i_s M_r \left( \Xi R_r - N_l \Xi \right) = \bar{k}_R \delta^i_s \Xi^{j\beta}_{l \alpha} c \left( d - d' \right) = \bar{k}_R \delta^i_s \Xi_{\alpha l}^{j\beta} \sigma, \]  
(4.42)
\[ D_l = \bar{k}_R \delta^i_s M_l \left( -\Xi R_l + N_r \Xi \right) = \bar{k}_R \delta^i_s \Xi^{j\beta}_{l \alpha} c' \left( d' - d \right) = -\bar{k}_R \delta^i_s \Xi_{\alpha l}^{j\beta} \sigma'. \]  
(4.43)
The part of the twisted fluctuation induced by the Majorana mass of the neutrino is then easily obtained, taking however into account the contribution of \( D_M \) to the non-linear term \( A_{(2)} \), since \( \gamma^5 \otimes D_M \) violates the twisted first-order condition (cf. Prop. A.8).
Proposition 4.6. A off-diagonal fluctuation is parametrised by two independent real scalar fields $\sigma_r, \sigma_l$:

$$D_{AM} = \gamma^5 \otimes D_M + A_M + \overline{A}_M + A_{M(2)} = \delta^i_j \left( \eta^i_D 0 \begin{pmatrix} \eta^j D_0 + k R \Xi_{\overline{I}a} \Sigma_s^t & 0 \\ 0 & 0 \end{pmatrix} \right)^D_C.$$ \hspace{1cm} (4.44)

where

$$\Sigma = \begin{pmatrix} \sigma_r \\ \sigma_l \end{pmatrix}^t_s.$$ \hspace{1cm} (4.45)

Proof. As in the proof of proposition 4.4, one has

$$\overline{A}_M = J A_M J^{-1} = \begin{pmatrix} 0 & -J D \overline{J} \\ -JC \overline{J} & 0 \end{pmatrix}^D_C$$ \hspace{1cm} (4.46)

with

$$JC \overline{J}^{-1} = \eta^u v_\alpha \overline{C}_{\alpha \alpha} \eta^I_j = -\overline{C}$$ \hspace{1cm} (4.47)

and similarly for $D$. Hence

$$A_M + \overline{J} A_M = \begin{pmatrix} 0 & C + \overline{D} \\ C + D & 0 \end{pmatrix}^D_C.$$ \hspace{1cm} (4.48)

The non linear term is (omitting the summation index)

$$A_{M(2)} = \hat{a} \left[ A_M, \hat{b} \right]_{\rho^\circ}.$$ \hspace{1cm} (4.49)

By proposition 3.2 and the explicit form (4.33) of $A_M$ one gets

$$\hat{a} \left[ A_M, \hat{b} \right]_{\rho^\circ} = -\left( M 0 \begin{pmatrix} 0 \\ \overline{Q} \end{pmatrix}^D_C \left( \frac{\overline{\rho(N)} C - C \overline{R}}{\rho(\overline{R}) D - \overline{D} \overline{N}} \right)^D_C \right),$$ \hspace{1cm} (4.50)

where we use $\rho^\circ(\hat{b}) = \rho^\circ((b^*)^\circ) = (\rho^{-1}(b^*))^\circ = \rho(\overline{b})$ which follows from the definition (3.31) of $\rho^\circ$ together with the regularity condition $\rho(a^*) = (\rho^{-1}(a))^*$ satisfied by $\rho$. From (4.34) and (3.24)

$$\overline{\rho(N)} C = k R \delta^i_j \Xi_{\overline{I}a} \left( \overline{d} \sigma - \overline{d} \sigma' \right)^t_s,$$ \hspace{1cm} (4.51)

$$\frac{\overline{\rho(N)} C}{\rho(\overline{R}) D} = k R \delta^i_j \Xi_{\overline{I}a} \left( \overline{d} \sigma - \overline{d} \sigma' \right)^t_s,$$ \hspace{1cm} (4.52)

Remembering (4.41), one obtains

$$- M \left( \overline{\rho(N)} C - C \overline{R} \right) = k R \delta^i_j \Xi_{\overline{I}a} \left( \overline{c} \left( \overline{d} - \overline{d} \right) \sigma \right)^t_s = k R \delta^i_j \Xi_{\overline{I}a} \left( |\sigma|^2 - |\sigma'|^2 \right)^t_s,$$ \hspace{1cm} (4.53)

$$- \overline{Q} \left( \overline{R} D - \overline{D} \overline{N} \right) = k R \delta^i_j \Xi_{\overline{I}a} \left( \overline{c} \left( \overline{d} - \overline{d} \right) \sigma \right)^t_s = \left( |\sigma|^2 - |\sigma'|^2 \right)^t_s.$$ \hspace{1cm} (4.54)

Hence

$$A_{(2)} = \delta^i_j \Xi_{\overline{I}a} \left( \begin{pmatrix} 0 \\ k R \end{pmatrix}^D_C \left( |\sigma|^2 - |\sigma'|^2 \right)^t_s. \right.$$ \hspace{1cm} (4.55)

The explicit form of $\Sigma$ follows from (4.34)-(4.35), defining $\sigma_r = \sigma + |\sigma|^2$ and $\sigma_l = -\sigma' - |\sigma'|^2$.

\[ \square \]
The non-linear term does not modify the nature of the extra-scalar field $\sigma$. It simply modifies the relation between the components $\sigma_r, \sigma_l$ and the elements of the algebra defining the twisted 1-form, introducing the terms $|\sigma|^2, |\sigma'|^2$ in the equation above.

**Remark 4.7.** The field $\sigma$ is chiral, in the sense it has two independent components $\sigma_r, \sigma_l$. The one initially worked out in $[19]$ was not chiral. This is because in the latter case, one does not double $M_3(\mathbb{C})$ and identifies the complex component of $m$ with the complex component of $Q_r$. This means that the component $d'$ of $N_l$ identifies with the component $d$ of $R_r$, so that (4.40) and (4.42) vanish, that is $C_l = D_r = 0$. Similarly, the component $c'$ of $M_l$ becomes $c$, so that $D_l = C_r$. One thus retrieves the formula (4.32) of $[19]$ (in which the role of $c$ and $d$ have been interchanged). However, forcing the identification of the (non-doubled) component of $\mathbb{C}$ is actually not compatible with the twist, as explained in greater details in $[21]$. This problem is resolved in the present paper, where $M_3(\mathbb{C})$ is doubled and there is a minimal violation of the twisted first-order condition.

As an illustration that the selfadjointness of the 1-form is not necessary to get a selfadjoint twisted fluctuation (see § 3.3), notice that in the proposition above $D_{A_M}$ is selfadjoint regardless of the selfadjointness of $A_M$. As well, one does not need to assume that $A_M$ is selfadjoint to ensure that the fields $\sigma_r, \sigma_l$ are real.

## 5 Gauge part of the twisted fluctuation

In this section, we compute the twisted fluctuation induced by the free part $\bar{D} = \bar{\partial} \otimes \mathbb{I}_F$ of the Dirac operator (2.6), that is

$$\bar{D} + \bar{A} + J\bar{A}J^{-1}$$

(5.1)

where $\bar{A}$ is the twisted 1-form (3.33) induced by $\bar{D}$, that we call in the following a free 1-form.

As will be checked in section 6, the components of this form are the gauge fields of the model. There is no non-linear term $\bar{A}(2)$, for $\bar{D}$ does verify the twisted first-order condition, as shown in proposition A.8.

### 5.1 Dirac matrices and twist

We begin by recalling some useful relations between the Dirac matrices and the twist.

**Lemma 5.1.** If an operator $\mathcal{O}$ on $L^2(\mathcal{M}, S)$ twist-commutes with the Dirac matrices,

$$\gamma^\mu \mathcal{O} = \rho(\mathcal{O}) \gamma^\mu \quad \forall \mu$$

(5.2)

for some automorphism $\rho$ of $B(\mathcal{H})$, and commutes the spin connection $\omega_\mu$, then

$$[\bar{\partial}, \mathcal{O}]_\rho = -i\gamma^\mu \partial_\mu \mathcal{O}.$$  

(5.3)

**Proof.** One has

$$[\gamma^\mu \nabla_\mu, \mathcal{O}]_\rho = [\gamma^\mu \partial_\mu, \mathcal{O}]_\rho + [\gamma^\mu \omega_\mu, \mathcal{O}]_\rho.$$  

(5.4)

On the one side, the Leibniz rule for the differential operator $\partial_\mu$ together with (5.2) yields

$$[\gamma^\mu \partial_\mu, \mathcal{O}]_\rho \psi = \gamma^\mu \partial_\mu \mathcal{O} \psi - \rho(\mathcal{O}) \gamma^\mu \partial_\mu \psi = \gamma^\mu (\partial_\mu \mathcal{O}) \psi + \gamma^\mu \partial_\mu \mathcal{O} \psi - \rho(\mathcal{O}) \gamma^\mu \partial_\mu \psi = \gamma^\mu (\partial_\mu \mathcal{O}) \psi.$$

On the other side, by (5.2),

$$[\gamma^\mu \omega_\mu, \mathcal{O}]_\rho = \gamma^\mu \omega_\mu \mathcal{O} - \rho(\mathcal{O}) \gamma^\mu \omega_\mu = \gamma^\mu [\omega_\mu, \mathcal{O}]$$

(5.5)

vanishes by hypothesis. Hence the result. □
This lemma applies in particular to the components $Q$ and $M$ of the representation of the algebra $A$ in (2.19). The slight difference is that these components do not act on $L^2(M, S)$, but on $L^2(M, S) \otimes \mathbb{C}^{32}$. With a slight abuse of notation, we write

$$\gamma^\mu Q := (\gamma^\mu \otimes I_{16}) Q, \quad \partial_\mu Q := (\partial_\mu \otimes I_{16}) Q$$

(5.6)

and similarly for $M$.

**Corollary 5.1.1.** One has

$$\gamma^\mu Q = \rho(Q) \gamma^\mu, \quad [\partial_\mu Q]_\rho = -i \gamma^\mu \partial_\mu Q,$$

$$\gamma^\mu M = \rho(M) \gamma^\mu, \quad [\partial_\mu M]_\rho = -i \gamma^\mu \partial_\mu Q.$$  

(5.7) (5.8)

**Proof.** From (3.9) and omitting the internal indices (on which the action of $\gamma^\mu \otimes I_{16}$ is trivial), one checks from the explicit form (A.2) of the euclidean Dirac matrices that

$$\gamma^\mu_E Q - \rho(Q) \gamma^\mu_E = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}_s \begin{pmatrix} Q_r \cr 0 \end{pmatrix}_t - \begin{pmatrix} Q_l \cr 0 \end{pmatrix}_t \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}_s = 0.$$  

(5.9)

The same holds true for the curved Dirac matrices (A.4), by linear combination.

The commutation with the spin connection follows remembering that the latter is $\omega_\mu = \Gamma^\rho_\mu \gamma_\rho \gamma_\nu = \Gamma^\rho_\mu \left( \sigma_\mu \tilde{\sigma}_\nu \right)_s = 0$.  

(5.10)

and so commutes with $Q$, which is diagonal in the $s, t$ indices and trivial in the $\tilde{s}, \tilde{t}$ indices.

5.2 Free 1-form

With the previous results, it is not difficult to compute a free 1-form (3.33).

**Lemma 5.2.** A free 1-form is

$$\mathcal{A} = -i \gamma^\mu A_\mu \quad \text{with} \quad A_\mu = \begin{pmatrix} Q_\mu & 0 \\ 0 & M_\mu \end{pmatrix}^D_C,$$

(5.11)

where we use notations similar to (5.6), with

$$Q_\mu := \sum_i \rho(Q_i) \partial_\mu R_i, \quad M_\mu := \sum_i \rho(M_i) \partial_\mu N_i$$

(5.12)

for $Q_i, M_i$ and $R_i, N_i$ the components of $a_i, b_i$ as in (2.19, 3.24).

**Proof.** Omitting the summation index $i$, one has

$$\mathcal{A} = a \left[ [\partial_\mu, b]_\rho \right] = \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}^D_C \begin{pmatrix} [\partial_\mu R]_\rho & 0 \\ 0 & [\partial_\mu N]_\rho \end{pmatrix}^D_C = -i \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}^D_C \begin{pmatrix} \gamma^\mu \partial_\mu R & 0 \\ 0 & \gamma^\mu \partial_\mu N \end{pmatrix}^D_C = -i \gamma^\mu \begin{pmatrix} \rho(Q) \partial_\mu R & 0 \\ 0 & \rho(M) \partial_\mu N \end{pmatrix}^D_C,$$

(5.13)

where the last equalities follow from corollary 5.1.1. Restoring the index $i$, one gets the result.

By computing explicitly the components of $\mathcal{A}$, one finds that a free 1-form is parametrised by two complex fields $c^i_\mu, c^i_\mu$, two quaternionic fields $q^r_\mu, q^l_\mu$ and two $M_3(\mathbb{C})$-valued fields $m^r_\mu, m^l_\mu$. 

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Proposition 5.3. The components $Q_{\mu}, M_{\mu}$ of $\mathcal{A}$ in (5.11) are

$$Q_{\mu} = \delta^i_j \left( Q^r_{\mu} \right)^t, \quad M_{\mu} = \delta^i_j \left( M^r_{\mu} \right)^t$$

(5.14)

where

$$Q^r_{\mu} = \left( c^r_{\mu}, q^r_{\mu} \right)^{\beta}_\alpha, \quad Q^l_{\mu} = \left( c^l_{\mu}, q^l_{\mu} \right)^{\beta}_\alpha$$

(5.15)

for $c^r_{\mu} = \left( c^r_{\mu}, c^r_{\mu} \right), \ c^l_{\mu} = \left( c^l_{\mu}, c^l_{\mu} \right)$ and

$$M^r_{\mu} = \left( m^r_{\mu} \otimes I_2 \ 0 \ m^r_{\mu} \otimes I_2 \right)^{\beta}_\alpha, \quad M^l_{\mu} = \left( m^l_{\mu} \otimes I_2 \ 0 \ m^l_{\mu} \otimes I_2 \right)^{\beta}_\alpha$$

(5.16)

for $m^r_{\mu} = \left( c^r_{\mu}, m^r_{\mu} \right)^J_I, \ m^l_{\mu} = \left( c^l_{\mu}, m^l_{\mu} \right)^J_I$.

The complex, quaternionic and $M_3(\mathbb{C})$-value fields $c^r_{\mu}, q^r_{\mu}, m^r_{\mu}$ are defined in the proof.

Proof. The form (5.14)-(5.15) of the components of $\mathcal{A}$ follows calculating explicitly (5.12) using (3.9)-(3.12) for $Q_i, M_i$ and (3.25) for $R_i, N_i$. Omitting the $i$ index, one finds

$$Q^r_{\mu} = Q_i \partial_\mu R_r, \quad Q^l_{\mu} = Q_i \partial_\mu R_l, \quad M^r_{\mu} = M_i \partial_\mu N_r, \quad M^l_{\mu} = M_i \partial_\mu N_l.$$  

(5.17)

The first two equations yield (5.15) with

$$c^r_{\mu} = c^r \partial_\mu d, \quad c^l_{\mu} = c^l \partial_\mu d^t, \quad q^r_{\mu} = q \partial_\mu p^t, \quad q^l_{\mu} = q^l \partial_\mu p,$$

(5.18)

the last two ones yields $m^r_{\mu} = m' \partial_\mu n$, $m^l_{\mu} = m' \partial_\mu n'$, from which (5.16) follows with

$$m^r_{\mu} = m' \partial_\mu n, \quad m^l_{\mu} = m' \partial_\mu n'.$$

Corollary 5.3.1. A free 1-form $\mathcal{A}$ is selfadjoint if and only if

$$c^l_{\mu} = -c^r_{\mu}, \quad q^l_{\mu} = -(q^r_{\mu})^\dagger, \quad m^l_{\mu} = -(m^r_{\mu})^\dagger.$$  

(5.19)

Proof. From lemma 5.2 and corollary 5.1.1, using that $\rho$ is a $*$-automorphism, one has

$$\mathcal{A}^\dagger = i(A^\dagger)^\dagger \gamma^\dagger = i\gamma \rho(A^\dagger),$$

(5.20)

so $\mathcal{A}$ is selfadjoint if and only if $\gamma^\dagger \rho(A_{\mu})^\dagger + A_{\mu} = 0$. Since $A_{\mu}$ is diagonal the $s, t$ indices, the sum $\Delta_{\mu} := \rho(A_{\mu})^\dagger + A_{\mu}$ is also diagonal with components $\Delta^r_{\mu}$. Thus

$$\gamma^\dagger \Delta_{\mu} = \left( 0 \ \sigma^\mu \Delta^r_{\mu} \right)^t_s.$$

(5.21)

If this is zero, then for any $\gamma^\nu$

$$\gamma^\nu \gamma^\mu \Delta_{\mu} = \left( \sigma^\nu \hat{\sigma}^\mu \Delta^r_{\mu} \ 0 \ \hat{\sigma}^\nu \sigma^\mu \Delta^l_{\mu} \right)^t_s = 0.$$

(5.22)

\[ ^1\text{In a twisted spectral triple the automorphism is not necessarily involutive. What is asked is the regularity condition } \rho(a^*) = (\rho^{-1}(a))^\dagger. \text{ In our case since } \rho^{-1} = \rho, \text{ the latter is equivalent to } \rho \text{ being a } *\text{-autormorphism.} \]
Being $A_\mu$ – hence $\Delta_\mu$ – trivial in $\dot{s}, \dot{t}$, and since $\text{Tr} \hat{\sigma}^{\mu} \sigma^\nu = 2 \delta_{\mu\nu}$, the partial trace on the $\dot{s}, \dot{t}$ indices of the expression above yields $\Delta_{\mu}^f = \Delta_{\mu}^l = 0$. Therefore $\gamma^\mu (\rho(A_\mu))^\dagger + A_\mu) = 0$ implies
\[ \rho(A_\mu)^\dagger = -A_\mu. \] (5.23)

The converse is obviously true. Consequently, $A_\mu$ is selfadjoint if and only if (5.23) holds true.

From (5.11), this is equivalent to $\rho(Q_\mu)^\dagger = -Q_\mu$ and $\rho(M_\mu)^\dagger = -M_\mu$ that is, from (5.14),
\[ (Q_\mu)^\dagger = -Q_\mu \quad \text{and} \quad (M_\mu)^\dagger = -M_\mu. \] (5.24)

This is equivalent to (5.19).

### 5.3 Identification of the physical degrees of freedom

To identify the physical fields, one follows the non twisted case [7] and separates the real from the imaginary parts. We thus define two real fields $a_\mu = \text{Re} c_\mu^r$ and $B_\mu = -\frac{i}{2} \text{Im} c_\mu^l$ (with $g_1$ a real constant and the signs are such to match the notations of [15], see remark 5.6), so that
\[ c_\mu^r = a_\mu - i \frac{g_1}{2} B_\mu, \quad c_\mu^l = -c_\mu^r = -a_\mu - i \frac{g_1}{2} B_\mu. \] (5.25)

Moreover, we denote $w_\mu$ and $-\frac{g_2}{2} W^k$ for $k = 1, 2, 3$ the real components of the quaternionic field $q_\mu^r$ on the basis $\{i_2, i_0 j\}$ of the (real) algebra of quaternions (with $g_2$ another real constant), so that
\[ q_\mu^r = w_\mu i_2 - i \frac{g_2}{2} W^k \sigma_k, \quad q_\mu^l = -(q_\mu^r)^\dagger = -w_\mu i_2 - i \frac{g_2}{2} W^k \sigma_k. \] (5.26)

Finally, we write $m_\mu^r$ as the sum of a selfadjoint part $g_3 = \frac{1}{2}(m_\mu^r + m_\mu^l \dagger)$ and an antiaessential part $\frac{1}{2}(m_\mu^r - m_\mu^l \dagger)$. We denote $V_\mu^0, \frac{2}{2} V_\mu^m$ the real-field components of the latter on the basis $\{i_3, i_\lambda m\}$ of the (real) vector space of antiselfadjoint $3 \times 3$ complex matrices (with $\{\lambda m, m = 1 \ldots 8\}$ the Gell-Mann matrices and $g_3$ a real constant), so that
\[ m_\mu^r = g_\mu + i V_\mu^0 i_3 + i \frac{g_3}{2} V_\mu^m \lambda_m, \] (5.27)
\[ m_\mu^l = -(m_\mu^r)^\dagger = -g_\mu + i V_\mu^0 i_3 + i \frac{g_3}{2} V_\mu^m \lambda_m. \] (5.28)

The cancellation of anomalies is imposed requiring the the unimodularity condition
\[ \text{Tr} A_\mu = 0. \] (5.29)

This yields the same condition as in the non-twisted case.

**Proposition 5.4.** The unimodularity condition for a selfadjoint free 1-form yields
\[ V_\mu^0 = \frac{g_1}{6} B_\mu. \] (5.30)

**Proof.** From proposition 5.3 one gets $\text{Tr} A_\mu = \text{Tr} Q_\mu + \text{Tr} M_\mu$. On the one side (neglecting the $\dot{s}$ and $I$ indices)
\[ \text{Tr} Q_\mu = \text{Tr} Q_\mu^r + \text{Tr} Q_\mu^l = c_\mu^r + c_\mu^r + \text{Tr} q_\mu^r + c_\mu^l + c_\mu^l + \text{Tr} q_\mu^l \] (5.31)

vanishes by (5.19), when one notices that $\text{Tr} q^l = \text{Tr} q$ for any quaternion $q$. On the other side
\[ \text{Tr} M_\mu = \text{Tr} M_\mu^r + \text{Tr} M_\mu^l = 4 \text{Tr} \{m\}^r_\mu + 4 \text{Tr} \{m\}^l_\mu = 4(c_\mu^r + \text{Tr} m_\mu^r + c_\mu^l + \text{Tr} m_\mu^l) = 4(-ig_1 B_\mu + 6i V_\mu^0) \] (5.32)

where we use $c_\mu^r + c_\mu^l = -ig_1 B_\mu$ and $m_\mu^r + m_\mu^l = 2i V_\mu^0 i_3 + 2i g_3 V_\mu^m \lambda_m$, remembering then that the Gell-Mann matrices are traceless. Hence (5.29) is equivalent to (5.30).
Let us summarise the results of this section in the following

**Proposition 5.5.** A unimodular selfadjoint free 1-form $A$ is parametrised by

- two real 1-form fields $a_\mu$, $w_\mu$ and a selfadjoint $M_3(\mathbb{C})$-value field $g_\mu$,
- a $\mathfrak{u}(1)$-value field $iB_\mu$, a $\mathfrak{su}(2)$-value field $iW_\mu$ and a $\mathfrak{su}(3)$-value field $iV_\mu$.

**Proof.** Collecting the previous results, denoting $W_\mu := W_\mu^k \sigma_k$ and $V_\mu := V_\mu^m \lambda_m$, one has

$$
\begin{align*}
c_\mu^r &= a_\mu - i \frac{q_1}{2} B_\mu, & c_\mu^l &= -a_\mu - i \frac{q_1}{2} B_\mu, \\
q_\mu^r &= w_\mu \mathbb{I}_2 - i \frac{q_2}{2} W_\mu, & q_\mu^l &= -w_\mu \mathbb{I}_2 - i \frac{q_2}{2} W_\mu, \\
m_\mu^r &= g_\mu + i \left( \frac{q_1}{6} B_\mu \mathbb{I}_3 + \frac{q_3}{2} V_\mu \right), & m_\mu^l &= -g_\mu + i \left( \frac{q_1}{6} B_\mu \mathbb{I}_3 + \frac{q_3}{2} V_\mu \right).
\end{align*}
$$

On the one side, $a_\mu$, $w_\mu$ are in $C^\infty(\mathcal{M}, \mathbb{R})$ and $g_\mu = g_\mu^\dagger$ is in $C^\infty(\mathcal{M}, M_3(\mathbb{C}))$. On the other side, since $B_\mu$ is real, $iB_\mu \in C^\infty(\mathcal{M}, i\mathbb{R})$ is a $\mathfrak{u}(1)$-value field. The Pauli matrices span the space of traceless $2 \times 2$ selfadjoint matrices, thus the field $iW_\mu$ takes value in the set of antiselfadjoint such matrices, that is $\mathfrak{su}(2)$. Finally, the real span of the Gell-Mann matrices is the space of traceless selfadjoint elements of $M_3(\mathbb{C})$, hence $iV_\mu$ is a $\mathfrak{su}(3)$-value field.

In the non-twisted case, the primed and unprimed quantities in (5.18) and the next equation are equal, meaning that the right and left components of the fields (5.33)-(5.35) are equal, hence

$$
a_\mu = w_\mu = g_\mu = 0. 
$$

That the twisting produces some extra 1-form fields has already been pointed out for manifolds in [22], and for electrodynamic in [24]. Actually, such a field (improperly called vector field) appeared initially in the twisted version of the Standard Model presented in [19], but its precise structure – a collection of three selfadjoint fields $a_\mu$, $w_\mu$, $g_\mu$, each associated with a gauge field of the Standard Model – had not been worked out there.

In the minimal twist of electrodynamics, there is only one such field (associated with the $U(1)$ gauge symmetry). By studying the fermionic action, it gets interpreted as energy-momentum 4-vector in lorentzian signature. Whether such an interpretation still holds for $a_\mu$, $w_\mu$, $g_\mu$ will be investigated in a forthcoming paper [21].

**Remark 5.6.** In the non-twisted case, the fields $B_\mu$, $W_\mu$ and $V_\mu$ coincide with those of the spectral triple of the Standard Model. More precisely, within the conditions of (5.36), then

- our $c_\mu^r = c_\mu^l$ coincides with $-i \Lambda_\mu$ of [15, §15.4]8. The selfadjointness condition (5.19) then implies that $\Lambda_\mu$ is real, in agreement with [15]. Then $B_\mu = \frac{2}{9} \Lambda_\mu$ as defined in [15, 1.729] coincides with our $B_\mu = -i \frac{2}{9} c_\mu^r = -i \frac{2}{9} c_\mu^l$ as defined in (5.25).

- our $q_\mu^r = q_\mu^l$ coincides with $-i Q_\mu$ of [15, §15.4]. The selfadjointness condition (5.19) then implies that $Q_\mu$ is selfadjoint, in agreement with [15]. Then $W_\mu = \frac{2}{9} Q_\mu$ as defined in [15, 1.739] coincides with our $W_\mu = W_\mu^k \sigma_k = i \frac{2}{9} q_\mu^r = i \frac{2}{9} q_\mu^l$ in (5.26).

- the identification of our $V_\mu$ with the one of the non-twisted case is made after proposition 5.8.

**Remark 5.7.** If one does not impose the selfadjointness of $A$, then one obtains two copies of the bosonic contents of the Standard Model, acting independently on the right and left components of Dirac spinors. Whether this may yield physically meaningful models should be investigated elsewhere (considering to remove also the selfadjointness of the finite part of the fluctuation).

---

8 Beware that $\Phi_\mu$ in the formula of $\Lambda$ is $i \gamma^\nu \partial_\mu$ [15, 1.580], so that $\Lambda = \Lambda_\mu \gamma^\mu$ is the $U(1)$ part of $-A$, meaning that $\Lambda_\mu$ is the $U(1)$ part of $iA_\mu$. 

23
5.4 Twisted fluctuation of the free Dirac operator

We now compute the free part (5.1) of the twisted fluctuation.

**Proposition 5.8.** A twisted fluctuation of the free Dirac operator $\mathcal{D}$ is $D_Z = \mathcal{D} + Z$ where

$$Z = \mathcal{A} + J\mathcal{A}J^{-1} = -i\gamma^\mu \left( \begin{array}{cc} Z^\mu & 0 \\ 0 & Z^{\mu*} \end{array} \right)_C^D \quad \text{with} \quad Z^\mu = \gamma^5 \otimes X^\mu + \mathbb{I}_4 \otimes iY^\mu, \quad (5.37)$$

in which $X^\mu$ and $Y^\mu$ are selfadjoint $\mathcal{A}_{SM}$-value tensor fields on $\mathcal{M}$ with components

$$(X^\mu)_{1J} = (X^\mu)_{2J} = \left( 2a_\mu a_\mu \mathbb{I}_3 + g_\mu \right)_I^J, \quad (5.38)$$

and

$$(Y^\mu)_{1J} = \left( 0 - \frac{2a_\mu}{B_\mu} \mathbb{I}_3 - \frac{g_\mu}{2} \mathbb{V}_\mu \right)_I^J, \quad (Y^\mu)_{1J} = \left( \frac{g_1 B_\mu}{B_\mu} \mathbb{I}_3 - \frac{g_2}{2} \mathbb{V}_\mu \right)_I^J \quad (5.40)$$

$$(X^\mu)_{aJ} = \left( \frac{\delta^b_a (w_\mu - a_\mu)}{\delta^b_a w_\mu \mathbb{I}_3 - g_\mu} \right)_I^J, \quad (5.41)$$

$$(Y^\mu)_{aJ} = \left( \frac{\delta^b_a (\frac{g_1}{2} B_\mu - \frac{g_2}{2} W_\mu)}{\frac{g_1}{2} B_\mu \mathbb{I}_3 + \frac{g_2}{2} \mathbb{V}_\mu} \right)_I^J. \quad (5.42)$$

**Proof.** With $J = -J^{-1}$ as defined in (3.18) one has

$$J\mathcal{A}J^{-1} = -J(-i\gamma^\mu A_\mu)J^{-1} = -iJ\gamma^\mu A_\mu J^{-1} = i\gamma^\mu J A_\mu J^{-1} = = i\gamma^\mu \left( \mathcal{J} M_\mu J^{-1} \right)_C^D \quad (5.43)$$

where we use that $J$ is anti-linear and anticommutates with $\gamma^\mu$ (lemma A.7). Noticing that $\mathcal{J} M_\mu J^{-1} = -M_\mu$ and $\mathcal{J} Q_\mu J^{-1} = -Q_\mu$ (this is shown as in (3.22), (3.21)), one obtains

$$Z^\mu = Q^\mu + M^\mu. \quad (5.43)$$

Explicitly, $Z^\mu = \left( \begin{array}{cc} Z^\mu_1 \\ Z^\mu_2 \end{array} \right)$ where, using the explicit forms (5.15) and (5.16) of $Q^\mu_1$ and $M^\mu_1$,

$$Z^\mu_1 = \delta^i_2 J Q^\mu + \delta^i_3 \mathcal{M}^\mu = \delta^i_4 \left( c^\mu_\alpha \delta^i_j + \delta^b_\alpha \overline{m}^\mu \right) \delta^i_4 (\overline{q}^\mu_\alpha \delta^i_j + \delta^b_\alpha \overline{m}^\mu) \quad (5.44)$$

and

$$Z^\mu_2 = \delta^i_2 J Q^\mu + \delta^i_3 \mathcal{M}^\mu. \quad (5.45)$$

The components of the matrix in the r.h.s. of (5.44) are

$$(Z^\mu_2)_{aJ} = c^\mu_\alpha \delta^i_j + \delta^b_\alpha \overline{m}^\mu = \left( c^\mu_\alpha \delta^i_j + \overline{m}^\mu \right) \delta^i_4 \delta^i_4 \frac{b}{a} \quad (5.46)$$

with $(Z^\mu_2)_{1J} = (Z^\mu_2)_{2J} = 0$ and, using Proposition 5.5,

$$(Z^\mu_1)_{1J} = c^\mu_\alpha \delta^i_j + \overline{m}^\mu = \left( 2a_\mu (a_\mu - \frac{g_1}{2} B_\mu) \mathbb{I}_3 + g_\mu - \frac{g_2}{2} \mathbb{V}_\mu \right)_I^J = \left( X^\mu_{1J} + i (Y^\mu_{1J}) \right)_I^J \quad (5.40)$$

$$(Z^\mu_1)_{2J} = c^\mu_\alpha \delta^i_j + \overline{m}^\mu = \left( 2a_\mu - ig_1 B_\mu (a_\mu + \frac{g_1}{2} B_\mu) \mathbb{I}_3 + g_\mu - \frac{g_2}{2} \mathbb{V}_\mu \right)_I^J = \left( X^\mu_{2J} + i (Y^\mu_{2J}) \right)_I^J \quad (5.43)$$
and

\[(Z^r_\mu)^{b,l} = q^r_\mu \delta^l_j + \delta^b_\mu \overline{m}_\mu = \begin{pmatrix} (q^r_\mu)^1 \delta^l_1 + \overline{m}_\mu^1 & (q^r_\mu)^2 \delta^l_2 \\ (q^r_\mu)^1 \overline{m}_\mu^2 & (q^r_\mu)^2 + \overline{m}_\mu \end{pmatrix}_a \]  

(5.47)

with

\[(Z^r_\mu)^a_j = (q^r_\mu)^a \delta^j_l + \overline{m}_\mu^a = \begin{pmatrix} \overline{w}_\mu - i \frac{g}{2}(W_\mu)^a_\mu - a_\mu + i \frac{a_\mu}{2} g \delta^j_3 - \frac{g}{2}(W_\mu)^a_\mu \delta^j_3 & -i \frac{g}{2}(W_\mu)^a_\mu \delta^j_3 \\ \delta^j_3 \end{pmatrix}_I^J,\]

\[(Z^r_\mu)^{b \neq a}_j = (q^r_\mu)^b \delta^j_l = \begin{pmatrix} -i \frac{g}{2}(W_\mu)^b_\mu & -i \frac{g}{2}(W_\mu)^b_\mu \delta^j_3 \\ \delta^j_3 \end{pmatrix}_I^J = (X^r_\mu)^{b,j}_I + i(Y^r_\mu)^{b,j}_I.\]

The matrices \(X^r_\mu\) and \(Y^r_\mu\) defined by the equations above are selfadjoint (notice that \(W_\mu\) as defined in Prop. 5.5 is selfadjoint) and such that

\[Z^r_\mu = X^r_\mu + iY^r_\mu.\]

(5.48)

The selfadjointness condition 5.24 applied to (5.45) yields

\[Z^l_\mu = -(Z^r_\mu)^\dagger = -X^r_\mu + iY^r_\mu.\]

(5.49)

In other terms, \(Z^l_\mu = X^l_\mu + iY^l_\mu\) with

\[X^l_\mu = -X^r_\mu, \quad Y^l_\mu = Y^r_\mu.\]

(5.50)

Redefining \(X_\mu := X^r_\mu = -X^l_\mu, \quad Y_\mu := Y^r_\mu = Y^l_\mu\), one obtains the result. \(\square\)

We collect the components of \(Z\) in appendix 6.2. There, we also make explicit that \(iY_\mu\) coincides exactly with the gauge fields of the Standard Model (including the \(su(3)\) gauge field \(V_\mu\)). Thus the twist does not modify the gauge content of the model. What it does is to add the selfadjoint part \(X_\mu\) whose action on spinors breaks chirality. As shown in the next section, this field is invariant under a gauge transformation.

### 6 Gauge Transformations

A gauge transformation is implemented by an action of the group \(U(A)\) of unitary elements of \(A\), both on the Hilbert space and on the Dirac operator. On a twisted spectral triple, these actions have been worked out in [17, 23] and consist in a twist of the original formula of Connes [12], later generalised without the first order condition in [9]. Explicitly, on the Hilbert space, the fermion fields transform under the adjoint action of \(U(A)\) induced by the real structure, namely

\[\psi \to \text{Ad } u \psi := u \psi u = u \psi u^* \psi = u J u^* J^{-1} \psi, \quad u \in U.\]

(6.1)

On the other hand, the twisted-covariant Dirac operator \(D_A\) (3.28) transforms under the twisted conjugate action of \(\text{Ad } u\),

\[D_A \to \text{Ad } \rho(u) D_A \text{Ad } u^*.\]

(6.2)

By [25, Prop.4.2], the operator \(D_A\), viewed as a function of the components \(a_i, b_i\) of the twisted 1-form \(A = A_{(1)} = \sum_i a_i[D, b_i]\), transforms under a gauge transformation in the operator \(D_A^*\), where

\[A^u := \rho(u) [D, u^*]_\rho + \rho(u) A u^*.\]

(6.3)
This is the twisted version of the law of transformation of generalised 1-forms in ordinary spectral triples, which in turn is a non-commutative generalisation of the law of transformation of the gauge potential in ordinary gauge theories.

To write down the transformation $A \to A^u$, we need the explicit form of a unitary $u$ of $A$. The latter is a pair of functions on $M$ with value in

$$U(\mathbb{C}) \times U(\mathbb{H}) \times U(M_3(\mathbb{C})) \simeq U(1) \times SU(2) \times U(3).$$

Namely

$$u = (e^{i\alpha}, e^{i\alpha'}, q, q', m, m')$$

with

$$\alpha, \alpha' \in C^\infty(M, \mathbb{R}), \quad q, q' \in C^\infty(M, SU(2)), \quad m, m' \in C^\infty(M, U(3)).$$

It acts on $\mathcal{H}$ as

$$u = \left( \begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array} \right)^D_C$$

where, following (3.8)-(3.12), one has $\mathcal{A}^{iJ}_{sIa} = \delta_{sI}^{iJ} \mathcal{A}^{\beta}_{sa}$ and $\mathcal{B}^{iJ}_{sIa} = \delta_{sI}^{iJ} \mathcal{B}^{\beta}_{sa}$ with

$$\mathcal{A}^{\beta}_{sa} = \left( \begin{array}{c} (\mathcal{A}_r)^\beta_{\alpha} \\ (\mathcal{A}_l)^\beta_{\alpha} \end{array} \right)^t_s, \quad \mathcal{B}^{iJ}_{sa} = \left( \begin{array}{c} (\mathcal{B}_r)^{iJ}_{\alpha I} \\ (\mathcal{B}_l)^{iJ}_{\alpha I} \end{array} \right)^t_s,$$

in which

$$\mathcal{A}_r = \left( \begin{array}{c} \alpha \\ q' \end{array} \right)^\beta_{\alpha}, \quad \mathcal{A}_l = \left( \begin{array}{c} \alpha' \\ q \end{array} \right)^\beta_{\alpha},$$

and

$$\mathcal{B}_r = \left( \begin{array}{cc} m \otimes I_2 & 0 \\ 0 & m' \otimes I_2 \end{array} \right)^\beta_{\alpha}, \quad \mathcal{B}_l = \left( \begin{array}{cc} m' \otimes I_2 & 0 \\ 0 & m \otimes I_2 \end{array} \right)^\beta_{\alpha},$$

where we denote

$$\alpha := \left( \begin{array}{c} e^{i\alpha} \\ e^{-i\alpha} \end{array} \right), \quad m := \left( \begin{array}{c} e^{i\alpha} \\ m \end{array} \right)^J_I, \quad \alpha' := \left( \begin{array}{c} e^{i\alpha'} \\ e^{-i\alpha'} \end{array} \right), \quad m' := \left( \begin{array}{c} e^{i\alpha'} \\ m' \end{array} \right)^J_I.$$

### 6.1 Gauge sector

A twisted gauge transformation (6.2) does not necessarily preserve the selfadjointness of the Dirac operator (because the action of the unitary is twisted on the left, not on the right). Equivalently, $A^u$ in (6.3) is not necessarily selfadjoint, even though one starts with a selfadjoint $A$.

This may seem as a weakness of the twisted case, since in the non-twisted case selfadjointness is preserved. Actually the possibility to lose selfadjointness allows to implement Lorentz symmetry and yields – at least for electrodynamics [24] – an interesting interpretation of the component $X_\mu$ of the free fluctuation $Z$ of proposition 5.8 as a four-vector energy-impulsion.

However, regarding the gauge part of the Standard Model which – as shown below – is fully encoded in the component $iY_\mu$ of $Z$, it is rather natural to ask the selfadjointness of the free 1-form $A$ to be preserved. This reduces the choice of unitaries to pair of elements of (6.4) equal up to a constant.
Proposition 6.1. A unitary $u$ whose action (6.3) preserves the selfadjointness of any unimodular selfadjoint free 1-form $\mathcal{A}$ is given by (6.5) with
\[
\alpha' = \alpha + K, \quad q = q', \quad m' = m. \tag{6.12}
\]

The components (5.11) of $\mathcal{A}$ then transform as
\[
c' \to c' - i\partial c', \quad c' \to c' - i\partial c', \tag{6.13}
\]
\[
q' \to q' q' q + q \left( \partial q q' \right), \quad q' \to q' q' q + q \left( \partial q q' \right), \tag{6.14}
\]
\[
m' \to m' + m \left( \partial m \right), \quad m' \to m' + m \left( \partial m \right). \tag{6.15}
\]

Proof. From corollary 5.1.1 one has (with the same abuse of notations (5.6), now with $\mathbb{I}$)
\[
\mathcal{A}^u = \rho(u) \left( [\mathcal{D}, u^*]_\mu + \mathcal{A} u^* \right) = -i\gamma^\mu \left( u (\partial \mu u^*) + u A^\mu u^* \right). \tag{6.16}
\]

Using the explicit forms (6.7) of $u$ and (5.11) of $A^\mu$, one finds
\[
\mathcal{A}^u = -i\gamma^\mu \left( \mathfrak{A} (\partial \mu \mathfrak{A}^\dagger) + \mathfrak{A} Q^\mu \mathfrak{A}^\dagger \right. \left. 0 \mathfrak{B} (\partial \mu \mathfrak{B}^\dagger) + \mathfrak{B} M^\mu \mathfrak{B}^\dagger \right) \right)^D \tag{6.17}
\]
meaning that a gauge transformation is equivalent to the transformation
\[
Q^\mu \to \mathfrak{A} \left( \partial \mu \mathfrak{A}^\dagger \right) + \mathfrak{A} Q^\mu \mathfrak{A}^\dagger, \quad M^\mu \to \mathfrak{B} \left( \partial \mu \mathfrak{B}^\dagger \right) + \mathfrak{B} M^\mu \mathfrak{B}^\dagger. \tag{6.18}
\]

From (5.15) and (5.16), these equations are equivalent to
\[
c' \to e^{i\alpha} \partial c e^{-i\alpha} + c' = c' - i\partial c', \quad c' \to c' - i\partial c', \tag{6.19}
\]
\[
q' \to q' q' q + q \left( \partial q q' \right), \quad q' \to q' q' q + q \left( \partial q q' \right), \tag{6.20}
\]
\[
m' \to m' + m \left( \partial m \right), \quad m' \to m' + m \left( \partial m \right). \tag{6.21}
\]
For any unitary operator $q$, one has that $q \left( \partial q \right) = q \left[ \partial q \right]$ is anti-hermitian (being $\partial q$ anti-hermitian as well). Hence, beginning with a selfadjoint $\mathcal{A}$ as in (5.19), requiring that $\mathcal{A}^u$ be selfadjoint is equivalent to
\[
\partial \mu \alpha' = \partial \mu \alpha, \tag{6.22}
\]
\[
q q' q + q \left( \partial q q' \right) = q q' q q' + q \left( \partial q q' \right), \tag{6.23}
\]
\[
m' m' + m' \left( \partial m \right) = m m' + m \left( \partial m \right). \tag{6.24}
\]
In particular, for $q' = I$, the identity, the second of these equations yields $q \left( \partial q \right) = q' \left( \partial q q' \right)$ for any $q, q'$. Hence for any $q' = I$, one has $q q' q q' = q q' q q'$, and this means that $q q'$ is in the centre of $\mathbb{H}$. Being a unitary, $q q'$ is thus the identity. So $q = q'$. Similarly, one gets that $m m$ is in the centre of $\mathbb{M} \mathbb{C}$, that is a multiple of the identity. Being unitary, $m m$ can only be the identity, hence $m' = m$. Thus (6.19-6.21) yield the result.

These transformations of the components of the free 1-form induce the following transformations of the physical fields defined in (5.33)-(5.35).
Proposition 6.2. Under a twisted gauge transformation that preserve the selfadjointness of a unimodular free 1-form, the physical fields $a_\mu$ and $w_\mu$ are invariant, $g_\mu$ undergoes an algebraic (i.e. non-differential) transformation

$$g_\mu \rightarrow ng_\mu n^\dagger$$

(6.25)

and the gauge fields transform as in the Standard Model

$$B_\mu \rightarrow B_\mu + \frac{2}{g_1} \partial_\mu \alpha,$$  

(6.26)

$$W_\mu \rightarrow q W_\mu q^\dagger + \frac{2i}{g_2} q (\partial_\mu q^\dagger),$$  

(6.27)

$$V_\mu \rightarrow n V_\mu n^\dagger - \frac{2i}{g_3} n (\partial_\mu n^\dagger),$$  

(6.28)

where $n = (\det m)^{-\frac{1}{3}}$ is the SU(3) part of $m$.

Proof. Applying the gauge transformations (6.13)-(6.15) to the physical fields defined through (5.33)-(5.35), one obtains

$$\pm a_\mu - ig_1 \frac{i}{2} B_\mu \rightarrow \pm a_\mu - i \left( \frac{g_1}{2} B_\mu + \partial_\mu \alpha \right),$$  

(6.29)

$$\pm w_\mu \rightarrow \pm w_\mu - i \left( \frac{g_2}{2} W_\mu + i q (\partial_\mu q^\dagger) \right),$$  

(6.30)

$$\pm g_\mu \rightarrow \pm g_\mu \pm i \left( \frac{g_1}{6} B_\mu \beta_3 + \frac{g_3}{2} V_\mu \right),$$  

(6.31)

where the anti-selfadjointness of $q (\partial_\mu q^\dagger)$ and $m (\partial_\mu m^\dagger)$ guarantee that the r.h.s. of (6.30) and (6.31) is split into a selfadjoint and anti-selfadjoint part. The first two equations above yield (6.26)-(6.27) Writing $m = e^{i\theta} n$ with $e^{i\theta} = (\det m)^{\frac{1}{3}}$ and $n \in SU(3)$, then the right hand side of (6.31) becomes

$$\pm ng_\mu n^\dagger + i \left( \frac{g_1}{6} B_\mu - \partial_\mu \theta \right) \beta_3 + \frac{g_3}{2} n V_\mu n^\dagger - im (\partial_\mu m^\dagger),$$  

(6.32)

where we use $m \partial_\mu m^\dagger = -i \partial_\mu \theta + n \partial_\mu n^\dagger$. Requiring the unimodularity condition to be gauge invariant forces to identify $-\theta$ with $\frac{\alpha}{3}$, thus reducing the gauge group $U(3)$ to $SU(3)$. This yields (6.25) and (6.28).

Remark 6.3. If one does not impose that the twisted gauge transformation preserves selfadjointness, then the left and right components of spinors transform independently. As explained in remark 5.7, the viability of such models should be explore elsewhere.

6.2 Scalar sector

We now study the gauge transformation 6.3 of the scalar part of the twisted $A_Y + A_M$ of the twisted 1-form computed in section 4, beginning with the Yukawa part $A_Y$ in (4.1).

Lemma 6.4. Let $u$ be a unitary of $A$ as in (6.5). One has

$$A^u_Y = \rho(u) \left[ \gamma^5 \otimes D_Y, u^\dagger \right]_\rho + \rho(u) A_Y u^\dagger = \left( A^u \right)^D \frac{0}{C}$$  

(6.33)

where

$$A^u = \delta^i_4 \left( \left( A^u \right)_r, (A^u)_{l} \right)^i_s,$$  

(6.34)
with

\( (A^u)_r = \begin{pmatrix} 0 & \tilde{k}^l (\alpha' (H_1 + \mathbb{1}) q'^\dagger - \mathbb{1}) \\ (q (H_2 + \mathbb{1}) \alpha'^\dagger - \mathbb{1}) k^l & 0 \end{pmatrix} \),

\( (A^u)_l = - \begin{pmatrix} 0 & \tilde{k}^l (\alpha (H_1' + \mathbb{1}) q'^\dagger - \mathbb{1}) \\ (q' (H_2' + \mathbb{1}) \alpha'^\dagger - \mathbb{1}) k^l & 0 \end{pmatrix} \) \hspace{1cm} (6.35)

where \( H_{1,2} \) are the components of \( A_Y \), and \( \alpha, \alpha', q, q' \) those of \( u \).

**Proof.** From the formula (4.16) of \( A_Y \) and (6.7)-(6.8) of \( u \), one gets

\[
\rho(u) A_Y u^\dagger = \left( \rho(\mathfrak{A}) A \mathfrak{A}^\dagger \right)_D \bigg|_C \text{ where } \rho(\mathfrak{A}) A \mathfrak{A}^\dagger = \delta_{sl}^{ij} \left( \mathfrak{A}_l A \mathfrak{A}_r^\dagger \right)_s,
\]

\[
(6.37)
\]

where, using (4.17) and (6.9),

\[
\mathfrak{A}_l A \mathfrak{A}_r^\dagger = (\alpha' \ q) \left( H_2 k^l \tilde{k}^l H_1 \right) \left( \alpha'^\dagger \ q'^\dagger \right) = \left( q H_2 \alpha'^\dagger k^l \tilde{k}^l \alpha' H_1 q'^\dagger \right)_{\alpha} \beta,
\]

\[
(6.38)
\]

\[
\mathfrak{A}_r A \mathfrak{A}_l^\dagger = - (\alpha \ q) \left( H_2 k^l \tilde{k}^l H_1 \right) \left( \alpha'^\dagger \ q'^\dagger \right) = - \left( q' H_2' \alpha'^\dagger k^l \tilde{k}^l \alpha' H_1 q'^\dagger \right)_{\alpha} \beta,
\]

\[
(6.39)
\]

where we used that \( k^l, \tilde{k}^l \) commute with \( \alpha, \alpha' \) and their conjugates.

The computation of the twisted commutator part in (6.33) is similar to that of \( A_Y \) in proposition 4.3, with \( a_i = \rho(u) \) and \( b_i = u^\dagger \) for \( u \) as in (6.5), that is

\[
\rho(u) \left[ \gamma^5 \otimes D_Y, u^\dagger \right]_p = \left( \mathfrak{U} \right)_D \bigg|_C
\]

\[
(6.40)
\]

where

\[
\mathfrak{U} = \delta_{sl}^{ij} \left( \mathfrak{U}_l \mathfrak{U}_r \right)_s \text{ with } \mathfrak{U}_r = \left( \mathfrak{S}_2 k^l \tilde{k}^l \mathfrak{S}_1 \right)_{\alpha} \beta, \ \mathfrak{U}_l = - \left( \mathfrak{S}_2' k^l \tilde{k}^l \mathfrak{S}_1' \right)_{\alpha} \beta,
\]

\[
(6.41)
\]

in which \( \mathfrak{S}_{1,2} \) and \( \mathfrak{S}_{1,2}' = \rho(\mathfrak{S}_{1,2}) \) are given by (4.25) with (remembering (6.11))

\[
c = \alpha', \ c' = \alpha, \ q = q', \ q' = q \text{ and } d = \alpha^\dagger, \ d' = \alpha'^\dagger, \ p = q^\dagger, \ p' = q'^\dagger;
\]

\[
(6.42)
\]

that is

\[
\mathfrak{S}_1 = \alpha' (q'^\dagger - \alpha'^\dagger), \ \mathfrak{S}_2 = q (\alpha'^\dagger - q'^\dagger) \text{ and } \mathfrak{S}_1' = \alpha (q'^\dagger - \alpha'^\dagger), \ \mathfrak{S}_2' = q' (\alpha'^\dagger - q'^\dagger).
\]

(6.43)

Thus one obtains (6.33) with

\[
A^u = \mathfrak{U} + \rho(\mathfrak{A}) A \mathfrak{A}^\dagger = \delta_{sl}^{ij} \left( \mathfrak{U}_r + \mathfrak{A}_l A \mathfrak{A}_r^\dagger \right)_s \bigg|_C
\]

\[
(6.44)
\]

From (6.38) and (6.41) one obtains the explicit forms of \( (A^u)_r \) and \( (A^u)_l \)

\[
(A^u)_r := \mathfrak{U}_r + \mathfrak{A}_l A \mathfrak{A}_r^\dagger = \left( \mathfrak{S}_2 + \mathfrak{Q} H_2 \alpha'^\dagger \right) k^l \tilde{k}^l (\mathfrak{S}_1 + \alpha' H_1 q'^\dagger),
\]

\[
(6.45)
\]

\[
(A^u)_l := \mathfrak{U}_l + \mathfrak{A}_r A \mathfrak{A}_l^\dagger = - \left( \mathfrak{S}_2' + \mathfrak{Q}' H_2' \alpha'^\dagger \right) k^l \tilde{k}^l (\mathfrak{S}_1' + \alpha H_1' q'^\dagger).
\]

\[
(6.46)
\]

The final result follow substituting \( \mathfrak{S}_{1,2} \) with their explicit formulas (6.43). \( \square \)
A unitary $u$ that preserves the selfadjointness of the unimodular free 1-form (Prop. 6.1) also preserves the selfadjointness of $A_Y$ if, and only if, $K = 0$. Indeed, in that case $u$ is twist-invariant (i.e. $q' = q$ and $\alpha' = \alpha$) and one easily checks that for a selfadjoint $A_Y$ (that is $H_1^\dagger = H_2 = H_r$ and $H_1^\dagger = H_2 = H_l$ by corollary 4.3.1) then $A_Y u$ is selfadjoint as well. If $K \neq 0$, then $H_1$ and $H_2$ undergo different gauge transformations, forbidding $A_Y u$ to be selfadjoint. For this reason, from now on we take $K = 0$. With this caveat, the gauge transformation of lemma 6.4 then reads as a law of transformation of the complex components (4.32) of the quaternionic fields $H_r$ and $H_l$.

**Proposition 6.5.** Let $A_Y$ be a selfadjoint diagonal 1-form parametrised by two quaternionic field $H_r, H_l$. Under a gauge transformation induced by a twist-invariant unitary $u = (\alpha, \alpha, q, q, m, m)$, the components $\phi_{1,2}^r, \phi_{1,2}^l$ of $H_r, H_l$ transform as

$$
\begin{aligned}
(\phi_1^r + 1) &\rightarrow q (\phi_1^r + 1) e^{-i\alpha}, \\
(\phi_2^r + 1) &\rightarrow q (\phi_2^r + 1) e^{-i\alpha}.
\end{aligned}
$$

(6.47)

Proof. $A_Y$ being selfadjoint means that (4.26) holds. A twist-invariant unitary satisfies (6.12) with $K = 0$. Under these conditions, one easily checks that for a selfadjoint $A_Y$ (that is $H_1^\dagger = H_2 = H_r$ and $H_1^\dagger = H_2 = H_l$ by corollary 4.3.1) then $A_Y u$ is selfadjoint as well. If $K \neq 0$, then $H_1$ and $H_2$ undergo different gauge transformations, forbidding $A_Y u$ to be selfadjoint. For this reason, from now on we take $K = 0$. With this caveat, the gauge transformation of lemma 6.4 then reads as a law of transformation of the complex components (4.32) of the quaternionic fields $H_r$ and $H_l$.

**Proposition 6.6.** Under a gauge transformation induced by a twist-invariant unitary $u$, the real fields $\sigma_r, \sigma_l$ parameterising a self-adjoint off-diagonal fluctuation (proposition 4.6) are invariant.

Proof. The result amounts to showing that $A_M$ is invariant under (6.50). Since $u = \rho(u)$ by hypothesis, the twisted-commutator in (6.3) coincides with the usual one $[\gamma^5 \otimes D_M, u^\dagger]$ which is zero by (3.2). The explicit forms (4.33) of $A_M$ and (6.7) of $u$ yields

$$
u A_M u^\dagger = (B \bar B \bar A \bar C \bar B \bar A)^\dagger.
$$

(6.51)

From (4.35), one checks that $\bar A C \bar B \bar A \bar C \bar B \bar A$ has components (omitting the global factor $k_R \delta^\dagger_k$ and $\delta^\dagger_l$ per $A_r/l$)

$$
\begin{aligned}
\bar A_r C_r \bar B_r^\dagger = &\sigma \bar A_r \Xi_{j \alpha}^{j \beta} \bar B_r^\dagger = \sigma \Xi_{j \alpha}^{j \beta}, \\
\bar A_l C_l \bar B_l^\dagger = &-\sigma' \bar A_l \Xi_{j \alpha}^{j \beta} \bar B_l^\dagger = -\sigma' \Xi_{j \alpha}^{j \beta},
\end{aligned}
$$

(6.52)

(6.53)

where we use the explicit forms (6.9)-(6.11) of $\bar A \bar B$ to get $\bar A_r \Xi_{j \alpha}^{j \beta} \bar B_r^\dagger = \epsilon^{\alpha \beta} \Xi_{j \alpha}^{j \beta} e^{-i\alpha} = \Xi_{j \alpha}^{j \beta}$ and similarly for (6.53). Hence $u A_M u^\dagger = A_M$, and the result.

30
Conclusion

We have worked out the field content of a twisted version of the spectral triple of the Standard Model. The physical meaning of these fields will be made precise by the computation of the fermionic action in the second part of this work [20], as well as the possibility of gauge transformations induced by non twist-invariant unitaries, and their relation with lorentzian signature.

As shown in [21], the twisted first-order condition needs to be violated in order to generate the extra scalar field $\sigma$. This forbids to apply the twist-by-grading of [23], since the latter always preserves this condition. However, this violation has no real importance, being reabsorbed in the definition of the components of $\sigma$. In this sense, the model presented here is the one that minimally violates the twisted first-order condition.

Appendix

A.1 Dirac matrices and real structure

Let $\sigma_{j=1,2,3}$ be the Pauli matrices:

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(A.1)

In four-dimensional euclidean space, the Dirac matrices (in chiral representation) are

$$
\gamma^\mu_E = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5_E := \gamma^1_E \gamma^2_E \gamma^3_E \gamma^0_E = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},
$$

(A.2)

where, for $\mu = 0, j$, we define

$$
\sigma^\mu := \{I_2, -i\sigma_j\}, \quad \tilde{\sigma}^\mu := \{I_2, i\sigma_j\}.
$$

(A.3)

On a (non-necessarily flat) riemannian spin manifold, the Dirac matrices are linear combinations of the euclidean ones,

$$
\gamma^\mu = e^\alpha_\mu \gamma^\alpha_E
$$

(A.4)

where $\{e^\alpha_\mu\}$ are the vierbein, which are real fields on $\mathcal{M}$. These Dirac matrices are no longer constant on $\mathcal{M}$. This is a general result of spin geometry that the charge conjugation commutes with the spin derivative (see e.g. [26, Prop. 4.18]). For sake of completeness, we check it explicitly for a four dimensional riemannian manifold:

Lemma A.7. The real structure satisfies

$$
\mathcal{J} \gamma^\mu = -\gamma^\mu \mathcal{J}, \quad \mathcal{J} \sigma^s = \sigma^s \mathcal{J}, \quad \mathcal{J} \nabla^s_\mu = +\nabla^s_\mu \mathcal{J}.
$$

(A.5)

Proof. Let us first show that $\mathcal{J}$ anticommutes with the euclidean Dirac matrices,

$$
\{ \mathcal{J}, \gamma^\mu_E \} = 0.
$$

(A.6)

From the explicit forms (2.33) of $\mathcal{J}$, this is equivalent to

$$
\gamma^\mu_E \gamma^0_E \gamma^2_E = -\gamma^\mu_E \gamma^0_E \gamma^2_E
$$

(A.7)

which is true for $\mu = 0, 2$ since then $\tilde{\gamma}^\mu_E \gamma^0_E \gamma^2_E = \gamma^\mu_E$ anticommutes with $\gamma^0_E \gamma^2_E$, and is also true for $\mu = 1, 2$ in which case $\gamma^\mu_E = -\gamma_E$ commutes with $\gamma^0_E \gamma^2_E$. 

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Since the spin connection is a real linear combination of products of two euclidean Dirac matrices, it commutes with $\mathcal{J}$. The latter, having constant components, commutes with $\partial_\mu$, hence also with the spin covariant derivative $\nabla_\mu$.

These results hold as well in the curved case, for then one has from (A.4)
\[
\{\mathcal{J}, \gamma^\mu\} = e_\mu^\alpha \{\mathcal{J}, \gamma^\alpha_E\} = 0.
\] (A.8)

\[\Box\]

A.2 Components of the gauge sector of the twisted fluctuation

The components of the free twisted fluctuation of proposition 5.8 are $Z^\mu_\alpha = \delta^i_\alpha (Z^\mu_i)$, given by (we invert the order of the lepto-colour and flavour indices in order to make the comparison with the non-twisted case easier)

\[
\begin{align*}
(Z^\mu_{0\alpha})^1 &= 2a_\mu, \\
(Z^\mu_{0\alpha})^2 &= 2a_\mu + ig_1 B_\mu, \\
(Z^\mu_{0\alpha})^b &= \delta^b_a (w_\mu - a_\mu) + i \left( \delta^b_a g_1 B_\mu - \frac{g_2}{2} (W_\mu)^b_a \right), \\
(Z^\mu_{j\alpha})^i &= \left( a_\mu \delta^i_j + (g_\mu)_i^j \right) - i \left( \frac{2g_1 B_\mu}{3} \delta^i_j + \frac{g_3}{2} (V_\mu)_i^j \right), \\
(Z^\mu_{j\alpha})^j &= \left( a_\mu \delta^j_i + (g_\mu)_i^j \right) + i \left( \frac{g_1 B_\mu}{3} \delta^j_i - \frac{g_3}{2} (V_\mu)_j^i \right), \\
(Z^\mu_{j\alpha})^b &= \left( \delta^b_a w_\mu \delta^j_i - (g_\mu)_i^j \right) - i \left( \delta^b_a \left( \frac{g_1 B_\mu}{6} \delta^j_i + \frac{g_3}{2} (V_\mu)_j^i \right) + \frac{g_2}{2} (W_\mu)^b_a \delta^j_i \right), \\
(Z^\mu_{j\alpha})^b &= \left( Z^\alpha_i \right)^b_{j1} = 0.
\end{align*}
\] (A.9) (A.10) (A.11) (A.12) (A.13) (A.14) (A.15)

One then checks that

\[
i \begin{pmatrix}
(Y^\mu)^{i1}_{\alpha1} \\
(Y^\mu)^{i2}_{\alpha1} \\
(Y^\mu)^{j1}_{\alpha1}
\end{pmatrix}_{i1}^{\beta} = \begin{pmatrix}
-\left( \frac{2g_1 B_\mu}{3} \delta^i_j + \frac{g_3}{2} (V_\mu)_i^j \right) \\
i \left( \frac{g_1 B_\mu}{3} \delta^i_j - \frac{g_3}{2} (V_\mu)_j^i \right) \\
i \left( \delta^b_a \left( \frac{g_1 B_\mu}{6} \delta^j_i + \frac{g_3}{2} (V_\mu)_j^i \right) + \frac{g_2}{2} (W_\mu)^b_a \delta^j_i \right)
\end{pmatrix}^{\alpha}_{\beta}
\]

coincides with the matrix $\mathbf{A}^\mu_\alpha$ of the non-twisted case [15, eq. 1.733], while

\[
i \begin{pmatrix}
(Y^\mu)^{01}_{\alpha0} \\
(Y^\mu)^{02}_{\alpha0} \\
(Y^\mu)^{0b}_{\alpha0}
\end{pmatrix}_{i0}^{\beta} = \begin{pmatrix}
0 \\
ig_1 B_\mu \\
i \left( \delta^b_a \frac{g_1 B_\mu}{2} - \frac{g_2}{2} (W_\mu)^b_a \right)
\end{pmatrix}^{\alpha}_{\beta}
\]

coincides with the matrix $\mathbf{A}^i_\mu$ [15, eq. 1.734].

A.3 Twisted first-order condition

For a twisted spectral triple, there is a natural twisted version of the first order condition 2.13 that was introduced in [19] and whose mathematic pertinence has been investigated in details in [22, 23], namely

\[
[[D, b], a^\rho] = 0 \quad a, b \in \mathcal{A}
\] (A.16)

where $\rho^\circ \in \text{Aut} \mathcal{A}^\circ$ is the automorphism of the opposite algebra $\mathcal{A}^\circ$ induces in (3.31) by the twisting automorphism $\rho \in \text{Aut} \mathcal{A}$. 

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Proposition A.8. The free part $\tilde{\phi} \otimes I_F$ and the diagonal part $\gamma^5 \otimes D_Y$ of the Dirac operator satisfy the twisted first-order condition (A.16), while the off-diagonal part $\gamma^5 \otimes D_M$ violates it.

Proof. For $\tilde{\phi} \otimes I_F$, using (3.17) and corollary 5.1.1 one gets
\[
\left[ \left[ \partial_{\mu} \tilde{\phi} \otimes I_F, b \right] \left( J a J^{-1} \right) \right]_{\rho^\mu} = i \gamma^\mu \begin{pmatrix} \left[ \partial_{\rho} R, M \right] \left( J a J^{-1} \right) \end{pmatrix}_{C}.
\]
(A.17)
The top-left entry reads (omitting the $s, \bar{s}$ indices for simplicity)
\[
\begin{pmatrix} \partial_{\rho} R, M \end{pmatrix} = \begin{pmatrix} \partial_{\rho} d \left[ I_4, \bar{M} \right]_J \\ \partial_{\rho} p \left[ I_4, \bar{M} \right]_J \end{pmatrix} = 0.
\]
(A.18)
Similarly one shows that $[\partial_{\rho} N, \bar{Q}] = 0$, hence $\tilde{\phi} \otimes I_F$ satisfies the twisted first-order condition.

For the diagonal part $\gamma^5 \otimes D_Y$, lemma 4.1 and (3.17) yield
\[
\left[ \gamma^5 \otimes D_Y, b \right] \left( J a J^{-1} \right) = - \begin{pmatrix} \left[ S, M \right] \left( J a J^{-1} \right) \end{pmatrix}_{C}.
\]
(A.19)
In tensorial notations
\[
\left[ [S, M],_{\rho^\mu} \right] = \delta^{l}_s \left[ \eta^{a}_s \left( D_0 \right)_{a\gamma} R_{ij}, M_{s\alpha l} \right]_{\rho^\mu} - \delta^r \left[ \rho \left( R_{s\alpha l} \eta^{a}_s \left( D_0 \right)_{a\gamma} R_{ij}, M_{s\alpha l} \right) \right]_{\rho^\mu}.
\]
(A.20)
The right hand side of (A.20) is (omitting the indices $s, \alpha, l$)
\[
\begin{pmatrix} D_0 R_r - D_0 R_l \\ - D_0 R_l \end{pmatrix}^t \begin{pmatrix} \bar{M}_r \bar{M}_l \end{pmatrix}^s \begin{pmatrix} D_0 R_r \\ - D_0 R_l \end{pmatrix}^t - \begin{pmatrix} R_l D_0 \\ - R_r D_0 \end{pmatrix}^t \begin{pmatrix} \bar{M}_r \bar{M}_l \end{pmatrix}^s \begin{pmatrix} R_l D_0 \\ - R_r D_0 \end{pmatrix}^t ;
\]
\[
\begin{pmatrix} D_0 R_r \bar{M}_r - M_r D_0 R_r - R_l D_0 \bar{M}_r + M_r R_l D_0 \\ - D_0 R_l \bar{M}_r + M_l D_0 R_r + R_r D_0 \bar{M}_l - M_r R_r D_0 \end{pmatrix}.
\]
(A.21)
From the explicit form (2.26) of $D_0$, (3.11) of $M_{l/r}$ and (3.25) of $R_{r/l}$, one checks that
\[
D_0 R_r \bar{M}_r = \bar{M}_l D_0 R_r = \begin{pmatrix} 0 & \bar{k} p m' \end{pmatrix}, \quad R_l D_0 \bar{M}_r = \bar{M}_l R_l D_0 = \begin{pmatrix} 0 \\ \bar{k} m p \end{pmatrix},
\]
so that the upper left term in (A.21) is zero. The same is true for the l.r.t., hence $[S, \bar{M}] = 0$.

This shows that (A.19) vanishes, which is equivalent to the proposition.

For the off-diagonal part $\gamma^5 \otimes D_M$, one has (omitting the $s, \bar{s}$ indices for simplicity)
\[
\left[ \gamma^5 \otimes D_M, b \right]_{\rho^\mu} = \begin{pmatrix} 0 \\ \gamma^5 \Xi_{la}^{J\gamma} R_{K\gamma} (d - d') \end{pmatrix}_{C}.
\]
(A.23)
hence
\[
\left[ [\gamma^5 \otimes D_M, b], J a J^{-1} \right]_{\rho^\mu} = - \begin{pmatrix} 0 & \gamma^5 \Xi_{la}^{K\gamma} R_{K\gamma} (d - d') & \rho \left( M_{l\alpha}^{K\gamma} \right) \gamma^5 \Xi_{K\gamma}^{J\beta} R_{l\beta} (d - d') \end{pmatrix}_{C}.
\]
(A.24)
whose top-right entry reads
\[
\gamma^5 \Xi_{la}^{K\gamma} R_{K\gamma} (d - d') \rho \left( M_{l\alpha}^{K\gamma} \right) \gamma^5 \Xi_{K\gamma}^{J\beta} R_{l\beta} (d - d') = k R \delta^{l}_s \Xi_{la}^{J\beta} \left( \sigma + \sigma' - (\sigma + \sigma') \right),
\]
(A.25)
and is non-zero.

\[\square\]
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