Poisson modules and generalized geometry

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Dedicated to Shing-Tung Yau on the occasion of his 60th birthday

1 Introduction

Generalized complex structures were introduced as a common format for discussing both symplectic and complex manifolds, but the most interesting examples are hybrid objects – part symplectic and part complex. One such class of examples consists of holomorphic Poisson surfaces, but in [5], [6] Cavalcanti and Gualtieri also construct generalized complex 4-manifolds with similar features which are globally neither complex nor symplectic.

In Gualtieri’s development of the subject [9], [10] he introduced generalized analogues of a number of familiar concepts in complex geometry, and notably the idea of a generalized holomorphic bundle. In the symplectic case this is simply a flat connection, but in the Poisson case it is more interesting and coincides with the notion of Poisson module: a locally free sheaf with a Poisson action of the sheaf of functions on it. These play a significant role in Poisson geometry, and can be thought of as semiclassical limits of noncommutative bimodules.

A Poisson structure on a complex surface is a section $\sigma$ of the anticanonical bundle. It vanishes in general on an elliptic curve. We begin this paper by using algebraic geometric methods to construct rank two Poisson modules on such a surface. The data for this particular construction is located on the elliptic curve: a line bundle together with a pair of sections with no common zero.

Now if we consider the surface as a generalized complex manifold, then where $\sigma \neq 0$, $\sigma^{-1} = B + i\omega$ and we regard the generalized complex structure as being defined by the symplectic form $\omega$ transformed by a B-field $B$. Where $\sigma = 0$ it is the transform
of a complex structure. The nonholomorphic examples in [5], [6] also have a 2-torus on which the generalized complex structure changes type from symplectic to complex. Moreover the torus acquires the structure of an elliptic curve. This provokes the natural question of whether a holomorphic line bundle on this curve with a pair of sections can generate a generalized holomorphic bundle in analogy with the holomorphic Poisson case. This we answer in the rest of the paper, and in fact conclude from the general result that a line bundle with a single section is sufficient. The proof entails replacing the algebraic geometry of the Serre construction by a differential geometric version, and using this as a model in the generalized case.

As an application, we adapt a construction of Polishchuk in [12] to define generalized complex structures on $\mathbb{P}^1$-bundles over the examples of Cavalcanti and Gualtieri.

## 2 Poisson modules

### 2.1 Definitions

Let $M$ be a holomorphic Poisson manifold, defined by a section $\sigma$ of $\Lambda^2 T$, then the Poisson bracket of two locally defined holomorphic functions $f, g$ is $\{f, g\} = \sigma(df, dg)$. Algebraically, a Poisson module is a locally free sheaf $\mathcal{O}(V)$ with an action $s \mapsto \{f, s\}$ of the structure sheaf with the properties

- $\{f, gs\} = \{f, g\} s + g \{f, s\}$
- $\{(f, g), s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}$.

The first equation defines a first order linear differential operator

$$D : \mathcal{O}(V) \to \mathcal{O}(V \otimes T)$$

where $\{f, s\} = \langle Ds, df \rangle$. This is simply a holomorphic differential operator whose symbol is $1 \otimes \sigma : V \otimes T^* \to V \otimes T$.

The second equation is a zero curvature condition. Relative to a local basis $s_i$ of $V$, $D$ is defined by a “connection matrix” $A$ of vector fields:

$$Ds_i = \sum_j s_j \otimes A_{ji}$$

and the condition becomes $\mathcal{L}_A \sigma = A^2 \in \text{End}(V \otimes \Lambda^2 T)$. When $\sigma$ is non-degenerate and identifies $T$ with $T^*$ then this is a flat connection.
**Example:** If $X = \sigma(df)$ is the Hamiltonian vector field of $f$ then the Lie derivative $\mathcal{L}_X$ acts on tensors but the action in general involves the second derivative of $f$. However for the canonical line bundle $K = \Lambda^nT^*$ we have

$$\mathcal{L}_X(dz_1 \wedge \ldots \wedge dz_n) = \sum_i \frac{\partial X_i}{\partial z_i}(dz_1 \wedge \ldots \wedge dz_n)$$

and

$$\sum_i \frac{\partial X_i}{\partial z_i} = \sum_{i,j} \frac{\partial}{\partial z_i} (\sigma^{ij} \frac{\partial f}{\partial z_j}) = \sum_{i,j} \frac{\partial \sigma^{ij}}{\partial z_i} \frac{\partial f}{\partial z_j}$$

which involves only the first derivative of $f$. Thus

$$\{f, s\} = \mathcal{L}_X s = \langle Ds, df \rangle$$

defines a first order operator. The second condition follows from the integrability of the Poisson structure: since $\sigma(df) = X, \sigma(dg) = Y$ implies $\sigma(d\{f, g\}) = [X, Y]$, it follows that

$$\{\{f, g\}, s\} = \mathcal{L}_{[X,Y]} s = [\mathcal{L}_X, \mathcal{L}_Y]s = \{f, \{g, s\}\} - \{g, \{f, s\}\}.$$

This clearly holds for any power $K^m$.

### 2.2 A construction

If a rank $m$ vector bundle $V$ is a Poisson module, then so is the line bundle $\Lambda^mV$. Now let $M$ be a complex surface and $V$ a rank 2 holomorphic vector bundle with $\Lambda^2V \cong K^*$ (a line bundle which, as noted above, is a Poisson module for any Poisson structure). Suppose $V$ has two sections $s_1, s_2$ which are generically linearly independent. Then $s_1 \wedge s_2$ is a holomorphic section $\sigma$ of $\Lambda^2V \cong K^*$ and so defines a Poisson structure. It vanishes on a curve $C$. Moreover, where $\sigma \neq 0$, $s_1, s_2$ are linearly independent and define a trivialization of $V$ and hence a flat connection.

**Proposition 1** The flat connection extends to a Poisson module structure on $V$.

**Proof:** Let $(u_1, u_2)$ be a local holomorphic basis for $V$ in a neighbourhood of a point of $C$, then

$$s_i = \sum_j P_{ji} u_j$$
and det $P = 0$ is the local equation for $C$. Now a connection matrix for $D$ with $Ds_i = 0$ in the basis $(u_1, u_2)$ is given by a matrix $A$ of vector fields such that

$$0 = Ds_i = D\left(\sum_j P_{ji}u_j\right) = \sum_j \sigma(dP_{ji})u_j + \sum_{jk} P_{ji}A_{kj}u_k$$

which has solution

$$A = -\sigma(dP)P^{-1} = -\sigma(dP)\frac{\text{adj} P}{\text{det} P}.$$ 

This is smooth since det $P$ divides $\sigma$.

Finding rank 2 bundles with two sections is a problem that has been considered before, most notably in the study of charge 2 instanton bundles on $\mathbb{P}^3$ [11]. Our case is similar but one dimension lower. The choice of two sections defines an extension of sheaves:

$$0 \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(V) \to \mathcal{O}_C(L^*K_M^*) \to 0$$

where $L$ is the line bundle on the elliptic curve $C$ where the two, now linearly dependent, sections take their value. Such an extension is classified by an element of global Ext$^1(\mathcal{O}_C(L^*K_M^*), \mathcal{O}) \otimes \mathbb{C}^2$ but local duality gives an isomorphism of sheaves Ext$^1(\mathcal{O}_C, K_M) \cong \mathcal{O}_C(K_C)$. Hence

$$\text{Ext}^1(\mathcal{O}_C(L^*K_M^*), \mathcal{O}) \otimes \mathbb{C}^2 \cong H^0(C, L) \otimes \mathbb{C}^2$$

and we are looking for a pair of sections of the line bundle $L$ on $C$. To get a locally free sheaf we need the pair to have no common zeros.

To relate this data to the vector bundle, note that any one of the sections is non-vanishing outside $C$ and so the number of zeros is the same as the number of zeros of a section of $L$. Counting multiplicities, this means that

$$c_1(L) = c_2(V).$$

Another way of recording the information is to consider the meromorphic function $s_1/s_2$ on $C$.

### 3 The Serre construction

#### 3.1 The algebraic approach

To obtain an analogue of the construction above when $M$ is generalized complex, we have to replace the sheaf theory by a more analytic method. To prepare for this we
consider next the Serre construction of rank 2 holomorphic vector bundles. This is the question of constructing a vector bundle with at least one section and not two as above. For surfaces, a good source is Griffiths and Harris [8]. We begin with a sheaf-theoretic approach.

The problem is this: given \( k \) points \( x_i \in M \), find a rank 2 vector bundle \( V \) with a section \( s \) which vanishes non-degenerately at the \( k \) points. We shall assume, as previously, that \( \Lambda^2 V \cong K^* \). The derivative of \( s \) at a zero \( x \) is a well defined element of \( (V \otimes T^*)_x \) and we can take

\[
\det ds(x) \in (\Lambda^2 V \otimes \Lambda^2 T^*)_x
\]

which is canonically the complex numbers thanks to the isomorphism \( \Lambda^2 V \cong K^* \). It is non-zero since the zero is nondegenerate. So at each zero \( x_i \), \( s \) has a residue

\[
(\det ds(x_i))^{-1} = \lambda_i \neq 0.
\]

Then, as in [8] Chapter 5:

**Proposition 2** Given \( k \) points \( X = \{x_1, \ldots, x_k\} \in M \), and \( \lambda_i \in \mathbb{C}^* \) such that \( \lambda_1 + \ldots + \lambda_k = 0 \), there exists a rank 2 vector bundle \( V \) with \( \Lambda^2 V = K^* \) and a section \( s \) such that \( s(x_i) = 0 \) and \( \det ds(x_i) = \lambda_i^{-1} \).

In this case the bundle is given by an extension of sheaves:

\[
0 \to \mathcal{O} \to \mathcal{O}(V) \to \mathcal{I}_X \otimes K^* \to 0.
\]

To link things up with the previous section we ask when there is a second section. The exact cohomology sequence gives:

\[
0 \to H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}(V)) \to H^0(M, \mathcal{I}_X \otimes K^*) \to H^1(M, \mathcal{O}) \to
\]

Now if the points lie on the zero set of a Poisson structure \( \sigma \), then \( \sigma \) lies in the space \( H^0(M, \mathcal{I}_X \otimes K^*) \). If \( H^1(M, \mathcal{O}) = 0 \) then \( \sigma \) pulls back to a second section.

**Remarks:**

1. Our assumption that \( \sigma \) vanishes nondegenerately on a single elliptic curve actually implies that \( H^1(M, \mathcal{O}) = 0 \) when the surface is algebraic. This follows from the classification of [2].

2. In the previous section, the data for the construction of the bundle was a pair of sections \( s_1, s_2 \) on \( C \). One might ask where the \( \lambda_i \) come from if we choose just one of these, \( s_1 \). In fact \( d\sigma \) restricted to \( C \) gives an isomorphism between the normal bundle \( N \) and \( K^* \). But \( K^* \cong NK_C^* \), so we get a canonical non-vanishing vector field on \( C \) (the so-called modular vector field). Its inverse is a nonvanishing differential \( \alpha \) and then we can take \( \lambda_i \) to be the residue of the differential \( (s_2/s_1)\alpha \) at \( x_i \in C \). The sum of the residues of a differential on a curve is of course zero.
3.2 The analytical approach

We now reformulate the Serre construction in Dolbeault terms (see also Chapter 10.2 in [7]). First consider the sequence of sheaves:

\[ 0 \to \mathcal{O} \to \mathcal{O}(V) \xrightarrow{\pi} \mathfrak{I}_X \otimes K^* \to 0. \]

This is an extension of line bundles outside the points \( x_i \), and the standard way to obtain a Dolbeault representative for this is to choose a Hermitian metric on \( V \), and form the orthogonal complement of the trivial subbundle. Restrict \( \bar{\partial} \) to this line bundle. Since the homomorphism \( \pi \) in the complex is holomorphic, we obtain a \( \bar{\partial} \)-closed \((0,1)\)-form with values in \( \text{Hom}(K^*, \mathcal{O}) \cong K \). In other words a \((2,1)\)-form.

In our case this will acquire singularities at the points \( x_i \), so let us consider the local model. Let \( x_i \) be given by the origin in coordinates \( z_1, z_2 \), then the two maps in the complex are represented by \( 1 \mapsto (z_1, z_2) \) and \( (u, v) \mapsto -z_2 u + z_1 v \). Using the trivial Hermitian structure we take the orthogonal complement of \( (z_1, z_2) \) to give

\[ \bar{\partial} \left( \frac{1}{r^2}(-\bar{z}_2, \bar{z}_1) \right) = \frac{1}{r^4}(\bar{z}_2 dz_1 - \bar{z}_1 d\bar{z}_2)(z_1, z_2) \]

and then

\[ A_0 = \frac{1}{r^4} dz_1 \wedge dz_2 \wedge (\bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2) \quad (1) \]

is the required \((2,1)\) form.

Now, using the flat metric on \( \mathbb{C}^2 \), we calculate

\[ *d \left( \frac{1}{r^2} \right)^{2,1} = \frac{1}{4r^4} dz_1 \wedge dz_2 \wedge (\bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2) \]

and in four dimensions \( 1/r^2 \) is, up to a universal constant, the fundamental solution of the Laplacian. It follows that

\[ \bar{\partial} \left( \frac{1}{r^4} dz_1 \wedge dz_2 \wedge (\bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2) \right) = c\delta(0). \]

Thus, a distributional \((2,1)\) form \( A \) which has the form \( (1) \) at each point \( x_i \) and is smooth elsewhere, is an analytical way of defining the vector bundle \( V \) with a section.

**Remark:** Near \( x_i \) we have a non-holomorphic basis for \( V \) defined by \( (z_1, z_2), (-\bar{z}_2, \bar{z}_1) \). This is obtained from a trivialization which extends to \( x_i \) by the gauge transformation on \( \mathbb{C}^2 \setminus \{0\} \)

\[ \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \]
For example the flat trivialization of the spinor bundle on $\mathbb{R}^4$, the complement of a point in $S^4$, extends this way.

The proof of Proposition 2 now goes as follows. Consider the distributional form (or current) $T$ defined by taking delta functions at the points $x_i$:

$$T = \sum_i \lambda_i \delta(x_i) \in \Omega^{2,2}.$$  

This defines a class in $H^2(M, K)$. Since $H^2(M, K)$ is dual to $H^0(M, \mathcal{O}) \cong \mathbb{C}$, this class is determined by evaluating it on the function 1. But

$$\langle T, 1 \rangle = \sum_i \lambda_i$$

so if the $\lambda_i$ sum to zero the class is zero.

Now choose a Hermitian metric on $M$, flat near the $x_i$. From harmonic theory there is a current $S \in \Omega^{2,2}$ such that $\bar{\partial} \bar{\partial}^* S = T$ and then we take

$$A = \bar{\partial}^* S$$

to be the distribution defining the extension. Near $x_i$,

$$\bar{\partial} \bar{\partial}^* (S - k \ast \frac{1}{r^2}) = 0$$

so by elliptic regularity the difference is smooth and $A$ defines a holomorphic structure on the bundle obtained as in the Remark above.

**Remark:** The 't Hooft construction of $SU(2)$ instantons on $\mathbb{R}^4 = \mathbb{C}^2$ defines an anti-self-dual connection (and a fortiori a holomorphic structure) from a harmonic function of the form

$$\phi = \sum_i \frac{1}{|x - x_i|^2}.$$  

Its twistor interpretation is the Serre construction for the lines in $\mathbb{P}^3$ corresponding to the points $x_i \in \mathbb{R}^4$ (see [1]). This is a model for the above reformulation.

### 3.3 The second section

We now look analytically for a second section of the vector bundle $V$. The bundle outside of $X$ is a direct sum $1 \oplus K^*$ with $\bar{\partial}$-operator defined by

$$\bar{\partial}(u, v) = (\bar{\partial} u + Au, \bar{\partial} v).$$
In this description, the section coming from the Serre construction is \( s_1 = (1, 0) \), and we want a second one \( s_2 \) such that \( s_1 \wedge s_2 = \sigma \), so we write \( s_2 = (u, \sigma) \). For holomorphicity we need
\[
\bar{\partial}u + A\sigma = 0.
\]
Now
\[
T = \sum_i \lambda_i \delta(x_i) \in \Omega^{2, 2}
\]
so consider \( \sigma T \in \Omega^{0, 2} \). Evaluating \( \sigma T \) on a \((2, 0)\) form \( \alpha \) gives
\[
\sum_i \lambda_i \sigma(x_i) \alpha(x_i) = 0
\]
if the points \( x_i \) lie on the zero set of \( \sigma \). Thus \( \sigma T = 0 \).

We have \( \bar{\partial}A = T \) and \( \sigma \) is holomorphic so \( \bar{\partial}(A\sigma) = 0 \). But if \( H^1(M, \mathcal{O}) = 0 \) then this implies that \( A\sigma = -\bar{\partial}u \) for the required distributional section \( u \).

## 4 Generalized geometry

### 4.1 Basic features

Before we adapt this method to generalized complex manifolds, we review here the basic features. For more details see [9], [10], [4]. The key idea is to replace the tangent bundle \( T \) by \( T \oplus T^* \) with its natural indefinite inner product \( (X + \xi, X + \xi) = i_X \xi \) and the Lie bracket by the Courant bracket
\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d(i_X \eta - i_Y \xi).
\]
If \( B \) is a closed 2-form the action \( X + \xi \mapsto X + \xi + i_X B \) preserves both the inner product and the Courant bracket and is called a B-field transform.

A generalized complex structure is an orthogonal transformation \( J : T \oplus T^* \to T \oplus T^* \) with \( J^2 = -1 \) which satisfies an integrability condition which can be expressed in various ways, all analogous to the integrability condition for a complex structure but using the Courant bracket instead of the Lie bracket. The simplest is to take the isotropic subbundle \( E \) of the complexification \((T \oplus T^*)^c\) on which \( J = i \) and say that sections of \( E \) are closed under the Courant bracket. The standard examples are complex structures where \( E \) is spanned by \((0, 1)\) vector fields and \((1, 0)\)-forms, or symplectic structures where \( E \) consists of objects of the form \( X - i_X \omega \) where \( X \) is
a vector field and \( \omega \) the symplectic form. A holomorphic Poisson manifold defines a
generalized complex structure where \( E \) is spanned by \((0, 1)\) vector fields and objects
of the form \( \sigma(\alpha) + \alpha \) where \( \alpha \) is a \((1, 0)\)-form.

One of the key aspects of generalized geometry is that differential forms are interpreted
as spinors – the Clifford action of \( T \oplus T^* \) on the exterior algebra of forms \( \Lambda^* \) is
\( (X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi \). Then the action of a 2-form \( B \) is \( \varphi \mapsto e^{-B} \varphi \) using exterior
multiplication. There is an invariant pairing, the Mukai pairing, on forms with values
in the top degree defined by \( \langle \varphi, \psi \rangle = [\varphi \wedge s(\psi)]_n \) where \( s(\psi) = \psi_0 - \psi_1 + \psi_2 - \ldots \),
expanding by degree.

Generalized complex structures are defined by maximal isotropic subbundles \( E \subset (T \oplus T^*)^c \) and the annihilator under Clifford multiplication of any spinor is isotropic.
If a complex form \( \rho \) is closed and its annihilator is maximal isotropic (i.e. it is a pure
spinor) with \( E \cap \bar{E} = 0 \) (equivalently \( \langle \rho, \bar{\rho} \rangle \neq 0 \)) then \( \rho \) defines a generalized complex
structure. An example is a symplectic structure where \( \rho = e^{i\omega} \). The more general
condition is that \( E \) is integrable if
\[
d\rho = (X + \xi) \cdot \rho
\]
for some local section \( X + \xi \) of \((T \oplus T^*)^c\).

4.2 Generalized Dolbeault operators

If \( f \) is a function on a generalized complex manifold \((M, J)\) we have
\[
df \in T^* \in (T \oplus T^*)^c = E \oplus \bar{E}
\]
and we define \( \bar{\partial}_J f \) to be the \( \bar{E} \) component. For a complex structure this is the usual
\( \bar{\partial} f \) and for a symplectic structure \( \omega \), \( \bar{\partial}_J f = (iX + df)/2 \) where \( X \) is the Hamiltonian
vector field of \( f \). For a holomorphic Poisson structure \( \sigma \) we obtain (where in the
formula we use the standard meaning of \( \bar{\partial} f \) and \( \partial f \)):
\[
\bar{\partial}_J f = \bar{\partial} f - \sigma(\partial f) + \bar{\sigma}(\bar{\partial} f)
\]
(2)

The \( \bar{\partial}_J \) operator can be extended to a generalized Dolbeault complex
\[
\cdots \to C^\infty(\Lambda^p \bar{E}) \xrightarrow{\bar{\partial}} C^\infty(\Lambda^{p+1} \bar{E}) \to \cdots
\]
(where for simplicity we suppress the subscript). This is purely analogous to the usual
Dolbeault operator and it is well-defined because sections of the bundle \( E \) are closed
under Courant bracket and $E$ is isotropic. It forms a complex for the same reason: the term $([A, B], C) + ([B, C], A) + ([C, A], B)$, whose derivative obstructs the Jacobi identity, vanishes.

This motivates the definition of a \textit{generalized holomorphic structure} on a vector bundle $V$ over a generalized complex manifold. This consists of a differential operator

$$\bar{\partial}_V : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V \otimes E)$$

with the properties

- $\bar{\partial}_V(fs) = \bar{\partial}fs + f\bar{\partial}_V s$
- $\bar{\partial}^2_V = 0$

where the last condition involves the bundle-valued extension of the generalized Dolbeault operator. If in a local basis the operator is defined by a matrix valued section $A$ of $\bar{E}$, then this condition is $\bar{\partial}A + A \cdot A = 0$. (Note that since $\bar{E}$ is isotropic, for $e, e' \in \bar{E}$, $e \cdot e' = -e' \cdot e$ so that this is essentially an exterior product $\bar{\partial}A + A \wedge A = 0$.)

The Dolbeault complex is also related to the decomposition of forms on a generalized complex manifold. The endomorphism $J$ of $T \oplus T^*$ is skew adjoint and we can consider its Lie algebra action on spinors (which of course are differential forms). If the manifold has (real) dimension $2m$, then the forms are decomposed into eigenbundles with eigenvalues $ik$ ($-m \leq k \leq m$).

$$U_{-m}, U_{-m+1}, \ldots, U_0, U_1, \ldots, U_m.$$ 

\textbf{Example:} For a complex structure

$$U_k = \bigoplus_{p-q=k} \Lambda^{p,q}.$$ 

The integrability of the generalized complex structure means that the exterior derivative $d$ maps sections of $U_k$ to $U_{k-1} \oplus U_{k+1}$. The two parts are closely related to the $\bar{\partial}$ operators above, but we need to consider in more detail one of these eigenbundles first, namely $U_m$.

\textbf{4.3 The canonical bundle}

The maximal isotropic subbundle $E \subset (T \oplus T^*)^c$ is the annihilator of a spinor, but a spinor only defined up to a scalar multiple so this defines a distinguished line bundle in $\Lambda^*$ called the \textit{canonical bundle} $K$. 

\textbf{10}
Example: On a complex \( m \)-dimensional manifold, \( dz_1 \wedge dz_2 \wedge \ldots \wedge dz_m \) is annihilated by interior product with a \((0, 1)\) vector and exterior product with a \((1, 0)\) form, so \( K \) is the usual canonical bundle of a complex manifold. For a symplectic manifold, \( e^{i\omega} \) trivializes the canonical bundle.

In the eigenspace decomposition, \( K \) is the subbundle of forms \( U_m \). Moreover,

\[
U_{m-k} \cong K \otimes \Lambda^k \bar{E},
\]

essentially generated by the Clifford products of \( k \) elements of \( \bar{E} \) acting on \( K \). Now, as remarked above, \( d \) maps sections of \( U_k \) to \( U_{k-1} \oplus U_{k+1} \), and so takes sections of \( U_m = K \) to \( U_{m-1} = K \otimes \bar{E} \) since \( U_{m+1} = 0 \). This defines a generalized holomorphic structure on \( K \).

Remark: For a symplectic manifold, \( e^{i\omega} \) trivializes \( K \) and is closed and hence holomorphic in this generalized sense – hence the appropriate terminology *generalized Calabi-Yau manifold* for such a manifold.

When the canonical bundle is an even form there is a tautological section \( \tau \) of its dual bundle \( K^* \). This is just the projection from \( \Lambda^* \) to \( \Lambda^0 = \mathbb{C} \), restricted to \( K \). It is holomorphic in the generalized sense. The section \( \tau \) may be identically zero, but it is non-zero clearly at points where the generalized complex structure is the B-field transform of a symplectic structure, for

\[
e^B e^{i\omega} = 1 + (B + i\omega) + \ldots
\]

Example: For a holomorphic Poisson structure \( \sigma \) on a surface, where \( \sigma \neq 0 \) the generalized complex structure is the B-field transform of a symplectic structure \((\sigma^{-1} = B + i\omega)\). The tautological section vanishes on the elliptic curve \( C \).

Returning to \( d : C^\infty(U_k) \to C^\infty(U_{k-1} \oplus U_{k+1}) \) we write the projection to \( U_{k-1} \) as \( \bar{\partial} \) and to \( U_{k+1} \) as \( \partial \). The notation is consistent with the previous one in the sense that \( U_{m-1} = K \otimes \bar{E} \) and the operator is the \( \bar{\partial}_K \) operator for the tautological generalized holomorphic structure on \( K \).

We have then a natural elliptic complex which we can write as either

\[
\cdots \to C^\infty(K \otimes \Lambda^p \bar{E}) \overset{\bar{\partial}}{\to} C^\infty(K \otimes \Lambda^{p+1} \bar{E}) \to \cdots
\]

or

\[
\cdots \to C^\infty(U_{m-p}) \overset{\bar{\partial}}{\to} C^\infty(U_{m-p-1}) \to \cdots
\]
5 A generalized construction

5.1 The problem

Suppose now that $M$ is a 4-manifold with a generalized complex structure such that the tautological section $\tau$ of the canonical bundle has a connected nondegenerate zero-set. As shown in [5] this is a 2-torus with a complex structure, hence an elliptic curve $C$. We shall construct a rank 2 bundle on $M$ with a generalized holomorphic structure, given a set of points on $C$.

We imitate the analytical approach to the Serre construction and take the bundle $1 \oplus K^*$ where $K$ is the canonical bundle and find a distributional section $A$ of $K \otimes \bar{E}$ to define a generalized holomorphic structure by

$$\bar{\partial}(u, v) = (\bar{\partial}u + Av, \bar{\partial}v).$$

We are in the case $m = 2$, so $K \otimes \bar{E} \cong U_{2-1} = U_1$. We start with a set of points $x_i$ and look at the distributional form

$$T = \sum_i \lambda_i \delta(x_i).$$

If we are to solve $\bar{\partial}A = T$ then $T$ must take values in $U_0$. There are then two questions that need to be answered:

1. When does $T$ lie in $U_0$?
2. When is $T = \bar{\partial}A$ for $A$ in $U_1$?

The first question needs a little more generalized geometry.

5.2 Generalized complex submanifolds

Given a submanifold $Y \subset M$, there is a distinguished subbundle

$$TY \oplus N^* \subset (T \oplus T^*)|_Y$$

where $N^*$ is the conormal bundle. A submanifold is called generalized complex if $TY \oplus N^*$ is preserved by $J$. 
Example: For a complex manifold, this gives the usual notion of complex submanifold, for a symplectic manifold a Lagrangian submanifold. Applying a B-field to a symplectic structure can give new types of generalized complex submanifold but a point is never a generalized complex submanifold. Indeed a point $x$ is complex if the cotangent space $T^*_x \subset (T \oplus T^*)_x$ is preserved by $J$. But that means there are complex cotangent vectors in $E$. However, $E$ is spanned by terms $X + i_X(B + i\omega)$ and so $X$ is never zero.

Now a compact oriented submanifold $Y^k$ defines a current $\Delta_Y$ in $\Omega^{n-k}$ by

$$\langle \Delta_Y, \alpha \rangle = \int_Y \alpha.$$ 

We then have

**Proposition 3** $\Delta_Y$ lies in $U_0$ if and only if $Y$ is a generalized complex submanifold.

Proof: Consider the top exterior power $\Lambda^{2m-k}N^*$. Since $N^* \subset T$ is the annihilator of $TY$, if $\nu \in \Lambda^{2m-k}N^*$ is a generator, then $i_X\nu = 0$ if and only if $X \in TY$. Similarly $\xi \wedge \nu = 0$ if and only if $\xi \in N^*$. Thus $TY \oplus N^*$ is the annihilator under Clifford multiplication of $\nu$.

If $Y$ is a generalized complex submanifold, then this annihilator is $J$-invariant, which means that the real form $\nu$ is in the zero eigenspace of the Lie algebra action of $J$, i.e. $\nu \in U_0$. Conversely if $\nu \in U_0$, $Y$ is complex.

Now consider a form $\alpha$. To evaluate $\Delta_Y$ on this we take the degree $k$ component and integrate over $Y$. Now $\Lambda^{2m}T^*$ is canonically $\Lambda^kT^*Y \otimes \Lambda^{2m-k}N^*$. The Mukai pairing takes values in $\Lambda^{2m}T^*$, so $\nu \mapsto \langle \alpha, \nu \rangle$ defines a homomorphism from $\Lambda^{2m-k}N^*$ to $\Lambda^kT^*Y \otimes \Lambda^{2m-k}N^*$, or equivalently an element of $\Lambda^kT^*Y$. It is straightforward to see that, up to a sign, this is the degree $k$ component of $\alpha$ restricted to $Y$.

If $J\alpha = ik\alpha$, then

$$ik\langle \alpha, \nu \rangle = \langle J\alpha, \nu \rangle = -\langle \alpha, J\nu \rangle = 0$$

Hence $\Delta_Y$ evaluated on $U_k$ for $k \neq 0$ is zero, and hence $\Delta_Y$ lies in $U_0$. \qed

Example: The current defined by a complex submanifold of a complex manifold is of type $(p, p)$.

Returning to our question we see that $T$ lies in $U_0$ if and only if each point $x_i$ is a generalized complex submanifold. Outside the elliptic curve $C$ the generalized
complex structure is the B-field transform of a symplectic one, and as we have seen, points here are not complex. In four dimensions, if $\tau = 0$, the generalized complex structure is the stabilizer of a spinor of the form $e^B \alpha_1 \wedge \alpha_2$. This is the B-field transform of an ordinary complex structure. Since $T^*$ is preserved by $J$ for an ordinary complex structure, and the B-field acts trivially on $T^*$ we see that any point on $C$ is a generalized complex submanifold. So we have an answer to the first question:

**Proposition 4** $T$ lies in $U_0$ if and only if the points $x_i$ lie on the elliptic curve $C$.

### 5.3 The construction

We now address the second question: suppose $T$ lies in $U_0$, when is it of the form $\bar{\partial}A$ for $A$ in $U_1$? In the standard case we used Serre duality to say that the Dolbeault cohomology class of $T$ is trivial if we evaluate on the generator 1 of $H^0(M, \mathcal{O})$. For the generalized $\bar{\partial}$ operator Serre duality consists of the non-degeneracy of the natural Mukai pairing of $U_k$ and $U_{-k}$ at the level of cohomology. A proof can be found in [4].

In our case, it means that $T$ in $U_0$ is cohomologically trivial if evaluation on all $\bar{\partial}$-closed forms in $U_0$ is zero. So suppose $\alpha$ is a section of $U_0$ with $\bar{\partial}\alpha = 0$. The distribution $T$ is a sum of delta functions of points $x_i$, which lie on the curve $C$, so we need to know $U_0$ here. But the generalized complex structure on $C$ is, as we have seen, the B-field transform of a complex structure. Now for a complex structure,

$$U_0 = \bigoplus_p \Lambda^{p,p}$$

so $U_0|_C = e^B(\Lambda^{0,0} \oplus \Lambda^{1,1} \oplus \Lambda^{2,2})$, where $B$ is possibly locally defined. Hence we can locally write

$$\alpha = e^B(a_0, a_1, a_2).$$

However, $B$ leaves the degree zero part invariant, so in this local expression $a_0$ is the restriction of a globally defined function on $C$.

Now, as shown in [6], a normal form (up to diffeomorphism and B-field transform) for a neighbourhood of a nondegenerate complex locus in four dimensions is provided by the holomorphic Poisson structure

$$\sigma = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}.$$

From this and [2] one can see that $\bar{\partial}\alpha = 0$ implies that the degree zero term $a_0$ is holomorphic on the compact elliptic curve $C$, and hence constant.
Now $T$ evaluates at points $x_i \in C$ and involves just the degree zero component of $\alpha$. It follows that the condition on $T$ to be cohomologically trivial is

$$\langle T, \alpha \rangle = \text{const.} \sum_i \lambda_i = 0$$

as before.

We conclude:

**Theorem 5** Let $M$ be a 4-manifold with a generalized complex structure such that the tautological section $\tau$ of the canonical bundle has a connected nondegenerate zero-set $C$. A set of $k$ distinct points $x_i \in C$ and $\lambda_i \in \mathbb{C}^*$ with $\lambda_1 + \ldots + \lambda_k = 0$ defines a rank 2 generalized holomorphic bundle $V$ with a generalized holomorphic section vanishing at the points $x_i$.

**Remark:** Note that here we have no condition for a second section, but neither have we attempted to find bundles with two generalized holomorphic sections: the construction in Section 2.2 was a simple way to get Poisson modules, but they are more special than they need to be. So Theorem 5 tells us that the Serre construction where the points are taken on a smooth anticanonical divisor gives us a Poisson module – we don’t need two sections of the line bundle on the elliptic curve. What happens is that the flat connection on $M \setminus C$ has upper-triangular rather than trivial holonomy.

6 An application

It is observed in [12] that if $V$ is a rank 2 Poisson module on a complex Poisson manifold, then the projective bundle $P(V)$ acquires a naturally induced Poisson structure. Here we prove a generalized version:

**Proposition 6** Let $V$ be a rank two generalized holomorphic bundle over a generalized complex manifold $M$. Then $P(V)$ has a natural generalized complex structure.

**Proof:** First consider the generalized complex structure which is the product of the standard complex structure on $\mathbb{P}^1$ and the given generalized complex structure on $M$. If $\rho$ is a local non-zero section of the canonical bundle of $M$ then, using an affine coordinate $z$ on $\mathbb{P}^1$, $dz \wedge \rho$ is a section of the canonical bundle for the structure on the product.
Now over an open set $U \subseteq M$ apply the diffeomorphism of $\mathbb{P}^1 \times U$ defined by a map $a$ from $U$ to $SL(2, \mathbb{C})$:

$$\tilde{z} = \frac{a_{11} z + a_{12}}{a_{21} z + a_{22}}.$$ 

Then

$$(a_{21} z + a_{22})^2 d\tilde{z} = dz + A_{12} + (A_{11} - A_{22})z - A_{21}z^2 = dz + \theta$$

where $A = a^{-1} da$. Using $dA + A^2 = 0$, we have

$$d(dz + \theta) = (dz + \theta) \wedge (2zA_{21} - (A_{11} - A_{22})) = (dz + \theta) \wedge \alpha.$$  \hspace{1cm} (3)

Consider $(dz + \theta) \wedge \rho$, or equivalently the Clifford product $(dz + \theta) \cdot \rho$, since $dz + \theta$ is a one-form. Now $\rho$ is annihilated by $E \subset (T \oplus T^*)^c$, so it is only the $\bar{E}$ component of $\theta$ (denote it $\theta^{01}$) which contributes. This defines an integrable generalized complex structure trivially since we simply transformed the product by a diffeomorphism, but a direct check of integrability goes as follows: we need to show that locally

$$d((dz + \theta^{01}) \cdot \rho) = \beta \cdot (dz + \theta^{01}) \cdot \rho$$

for some section $\beta$ of $(T \oplus T^*)^c$. But from (3)

$$d((dz + \theta^{01}) \wedge \rho) = -\alpha \wedge (dz + \theta^{01}) \wedge \rho - \theta^{01} \wedge \gamma \cdot \rho = -\alpha \cdot (dz + \theta^{01}) \cdot \rho - (dz + \theta^{01}) \cdot \gamma \cdot \rho$$

using the integrability $d\rho = \gamma \cdot \rho$ of the structure on $M$, where again we can take $\gamma$ to be in $\bar{E}$. Now since $\bar{E}$ is isotropic, two sections anticommute under the Clifford product, so

$$d((dz + \theta^{01}) \wedge \rho) = (\gamma - \alpha) \cdot (dz + \theta^{01}) \wedge \rho$$

which is the required integrability.

Now suppose $V$ is a rank 2 bundle with a generalized holomorphic structure, and in a local trivialization $\bar{\partial}_V$ is defined by a “connection matrix” $A$ with values in $\bar{E}$. Then define an almost generalized complex structure by

$$(dz + A_{12} + (A_{11} - A_{22})z - A_{21}z^2) \cdot \rho.$$ 

In the argument for integrability above, we only needed the vanishing of the $\Lambda^2 \bar{E}$ component of $dA + A^2 = 0$ and from the definition of a generalized holomorphic structure, we have $\bar{\partial}A + A \cdot A = 0$, so this provides the ingredient to prove integrability for the generalized holomorphic structure. \hfill $\Box$

As a consequence, if we use our construction to generate rank 2 vector bundles with generalized holomorphic structure on the Cavalcanti-Gualtieri 4-manifolds, we can find six-dimensional generalized complex examples on their projective bundles. These have a structure which is complex in the fibre directions. In [3] it is shown that a symplectic bundle over a generalized complex base has a generalized complex structure which is symplectic along the fibres.
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