Existence and Stability of Viscous Shock Profiles for Isentropic MHD with Infinite Electrical Resistivity

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Overview

- Isentropic gas
- Parallel MHD
- MHD with infinite electrical resistivity
- Known instabilities in multi-D problem
- Phase space
- Stability setup
- Numerical approximation
Magnetohydrodynamics (MHD)

\[
\begin{align*}
    v_t - u_{1x} &= 0, \\
    u_{1t} + (p + (1/2\mu_0)(B_x^2 + B_z^2))_x &= (((2\mu + \eta)/\nu)u_{1x})_x, \\
    u_{2t} - ((1/\mu_0)IB_2)_x &= ((\mu/\nu)u_{2x})_x, \\
    u_{3t} - ((1/\mu_0)IB_3)_x &= ((\mu/\nu)u_{3x})_x, \\
    (vB_2)_t - (lu_2)_x &= ((1/\sigma\mu_0\nu)B_{2x})_x, \\
    (vB_3)_t - (lu_3)_x &= ((1/\sigma\mu_0\nu)B_{3x})_x,
\end{align*}
\]

Hyperbolic wherever \( p \) is monotone.
Rescaled Equations

\[
(v, u, \mu_0, x, t, B, a) \rightarrow \left( \frac{v}{\epsilon}, -\frac{u}{\epsilon s}, \epsilon \mu_0, -\epsilon s(x - st), \epsilon s^2 t, -\frac{B}{s}, \frac{a\epsilon^{-\gamma-1}}{s^2} \right)
\]

\[
\begin{align*}
\nu_t + \nu_x - u_{1x} &= 0 \\
\frac{u_{1t} + u_{1x}}{a \nu^{-\gamma} + \left( \frac{1}{2\mu_0} \right) \left( B_2^2 + B_3^2 \right)} &= (2\mu + \eta) \left( \frac{u_{1x}}{v} \right)_x \\
\frac{u_{2t} + u_{2x}}{\frac{1}{\mu_0} I B_2} &= \mu \left( \frac{u_{2x}}{v} \right)_x \\
\frac{u_{3t} + u_{3x}}{\frac{1}{\mu_0} I B_3} &= \mu \left( \frac{u_{3x}}{v} \right)_x \\
(u B_2)_t + (u B_2)_x - (lu_2)_x &= \left( \frac{1}{\sigma \mu_0 v} \right) B_{2x} \\
(u B_3)_t + (u B_3)_x - (lu_3)_x &= \left( \frac{1}{\sigma \mu_0 v} \right) B_{3x}
\end{align*}
\]
Traveling wave solution: 

\[ (v, u_1, u, B)(x, t) = (v, u_1, \tilde{u}, \tilde{B})(x - st) \]

\[ (v, u_1, \tilde{u}, \tilde{B})(\pm \infty) = (v, u_1, \tilde{u}_2, \tilde{B})_{\pm}. \]

\[
\begin{cases}
    v' - u_1' = 0, \\
    u_1' + (p + (1/2\mu_0)|\tilde{B}|^2)' = (((2\mu + \eta)/v)u_1')' \\
    \tilde{u}' - ((1/\mu_0)l\tilde{B})' = ((\mu/v)\tilde{u}')' \\
    (v\tilde{B})' - (l\tilde{u})' = ((1/\sigma\mu_0 v)\tilde{B}')'
\end{cases}
\]
Profile Equations

\[ u = u_1, \ w := \tilde{u}, \ B = \tilde{B} \]

\[(2\mu + \eta)v' = v(v - 1) + v(p - p_-) + \frac{v}{2\mu_0}(B^2 - B_-^2),\]

\[\mu w' = vw - \frac{vl}{\mu_0}(B - B_-),\]

\[\frac{1}{\sigma\mu_0}B' = v^2B - vB_- - lvw,\]

with \( u \equiv v - 1.\)
Profile Equations, $\sigma = \infty$

$$
\sigma = \infty
$$

$$(v\tilde{B})' - (I\tilde{u})' = 0,$$

$$(vB)' - (lw)' = 0$$

$$B = \frac{B_- + lw}{v}.$$ 

$$(2\mu + \eta)v' = v(v - 1) + v(p - p_-) + \frac{1}{2\mu_0 v}((B_- + lw)^2 - v^2B_-^2),$$

$$\mu w' = vw - \frac{l}{\mu_0} (B_-(1 - v) + lw).$$
RH conditions

\[ u = u_1, \quad B = (B_2, B_3), \quad w = (u_2, u_3), \]

\[-s[v] = [u],\]

\[-s[u] = - \left[ p + \frac{B^2}{2 \mu_0} \right],\]

\[-s[w] = l \left[ \frac{B}{\mu_0} \right],\]

\[-s[vB] = l[w].\]
Important parameters

\[ J := \frac{(B_{2-})^2}{2\mu_0} \quad \text{and} \quad K := \frac{l^2}{\mu_0}. \]

(Note that, under the rescaling that we used, \( I = -\frac{l}{s} \),
\[ J = \frac{B_{2-}^2}{2\epsilon s^2 \mu_0} = \frac{B_{2-}^2}{2v_- s^2 \mu_0}, \quad K = \frac{(l)^2}{\epsilon s^2 \mu_0} = \frac{(l)^2}{v_- s^2 \mu_0} \]
in the original coordinates.)
$u_+ = v_+ - 1, \quad B_{2-} = \left(\frac{v_+ - K}{1 - K}\right) B_{2+}, \quad w_+ = \frac{K}{l} \left(\frac{1 - v_+}{1 - K}\right) B_{2+},$

$$a = \left(\frac{1 - v_+}{v_+ - \gamma - 1}\right) \left(1 - \frac{B_{2+}^2}{2\mu_0} \frac{(1 + v_+ - 2K)}{(1 - K)^2}\right) = \left(\frac{1 - v_+}{v_+ - \gamma - 1}\right) \left(1 - \frac{J(1 + v_+ - 2K)}{(v_+ - K)^2}\right).$$

This is physically meaningful if and only if $a > 0$, or

$$-1 < v_+ - 1 < 2(K - 1) + (1 - K)^2 \left(\frac{2\mu_0}{B_{2+}^2}\right).$$

(For $K \geq 1/2$, this gives no restriction. For $K < 1/2$, $B_{2+}^2 < \frac{2\mu_0(1-K)^2}{1-2K}$ or $J < \frac{(v_+ - K)^2}{1-2K}$.)
Shock type

Consider a general system of conservation laws

\[ U_t + F(U)_x = (B(U)U_x)_x, \quad U \in \mathbb{R}^n \]

Inviscid shock waves correspond to triples \((U_-, U_+, s)\) satisfying the Rankine–Hugoniot conditions

\[ [F(U)] - s[U] = 0, \]

where \([h] := h(U_+) - h(U_-)\) denotes the jump in quantity \(h\) across the shock. The type of the shock wave is defined by the degree of compressivity

\[ \ell := \dim U(dF(U_-) - sl) + \dim S(dF(U_+) - sl) - n, \]
Figure: Typical phase portrait for MHD with two variables and infinite electric resistivity ($\sigma = \infty$). Parameter values are $\gamma = 2$, $\tau_+ = 0.1$, $I = 0.7$, $B_{2+} = 0.7$, and $\mu_0 = 1$. In Figure (a) we plot level sets of $\phi$ and in Figure (b) we draw the phase portrait.
Figure: Transition to undercompressive profile. Keeping $\tau = 2\mu + \eta = 1$ and letting $\mu \to 0$, we find that the overcompressive family is squeezed to an undercompressive connection somewhere between $\mu = 0.185$ (Figure (a)) and $\mu = 0.17$ (Figure (b)).
Spectral Stability

Linearization about the profile

\[ \nu_t = \left( d\mathcal{F}(\hat{\nu}) - s\partial_x \right) \nu + Q(\nu, \nu_x, \nu_{xx}, \ldots). \]

Higher order

The eigenvalue problem is a BVP.

\[ L\nu = \lambda \nu, \quad \nu^{(k)}(\pm\infty) = 0. \]

Spectral Stability when \( \sigma(L) \cap P = \emptyset \),

where \( P = \{ \lambda \mid \Re(\lambda) \geq 0 \} \setminus \{0\} \).
## Artificial Viscosity

**Existence and Asymptotic Behavior**  
Madja and Pego (83)

**Small-Amplitude Spectral Stability**  
Goodman (86)

**Nonlinear Stability**  
T.P. Liu (86), partial result  
Szepessy and Xin (92)

**Pointwise Greens Function Bounds**  
T.P. Liu (97)

**Spectral Stability → Nonlinear Stability**  
Zumbrun and Howard (98,00)

## Real Viscosity

**Existence and Asymptotic Behavior**  
Pego (82)

**Dissipativity Condition**  
Kawashima (83)

**Small-Amplitude Spectral Stability (Navier Stokes)**  
Matsumura and Nishihara (Isentropic)(84)  
Kawashima, Matsumura and Nishihara (86)

**Small-Amplitude Spectral Stability**  
Humpherys and Zumbrun (02)

**Spectral Stability → Nonlinear Stability**  
Mascia and Zumbrun (01,04,06)
Write the eigenvalue problem

$$\lambda v = L v, \quad -\infty < x < \infty,$$

as a first-order system

$$\begin{cases} W' = A(x, \lambda) W, \quad W \in \mathbb{C}^n \\
W(\pm\infty) = 0. \end{cases}$$

- Assume $A(x, \lambda)$ is consistently split in $x$.
- Assume $A(x, \lambda)$ is asymptotically constant in $x$, that is, $\lim_{x \to \pm\infty} A(x, \lambda) = A_{\pm}(\lambda)$. 

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MHD with Infinite Electrical Resistivity
Evans Function

Define at $x = 0$

$$D(\lambda) = \left\{ W_1^+ \wedge \ldots \wedge W_k^+ \right\} \left\{ S^+(\lambda) \right\} \left\{ W_{k+1}^- \wedge \ldots \wedge W_n^- \right\} \left\{ U^-(\lambda) \right\}$$

where $\{W_i^+\}_{i=1}^k$ and $\{W_j^-\}_{j=k+1}^n$ are analytic bases of the stable/unstable manifolds of $x = \pm \infty$, respectively.

**Theorem (Alexander, Gardner, Jones)**

The Evans Function enjoys the following properties:

- $D(\lambda)$ is analytic to the right of the essential spectrum.
- Degree of roots of $D(\lambda)$ match multiplicity of $\sigma_P(L)$.

**Remark**

Since $D(\lambda)$ is analytic in right half plane we can use winding number computations to test for spectral stability.
Linearized equations

\[(\hat{v}, \hat{u}_1, \hat{u}_2, \hat{B}_2)\]

\[
v_t + v_x - u_{1x} = 0
\]

\[
u_{1t} + u_{1x} + (-a\gamma\hat{v}^{-\gamma-1}v + (1/\mu_0)(\hat{B}_2B_2))_x = \tau(\hat{u}_1v - \hat{u}_{1x}v/\hat{v}^2)_x
\]

\[
u_{2t} + u_{2x} - (1/\mu_0)B_{2x} = \mu(\hat{u}_2v - \hat{u}_{2x}v/\hat{v}^2)_x
\]

\[
\tilde{\alpha}_t + \tilde{\alpha}_x - lu_{2x} = (\sigma\mu_0)^{-1}(B_{2x}/\hat{v} - \hat{B}_{2x}v/\hat{v}^2)_x,
\]

where \(\tilde{\alpha} = \hat{v}B_2 + v\hat{B}_2\), so that \(B_2 = (\tilde{\alpha} - \hat{B}_2v)/\hat{v}\).
Eigenvalue problem

\[
\begin{align*}
\lambda v + v' - u'_1 &= 0 \\
\lambda u_1 + u'_1 - (h(\hat{v})v/\hat{v}^{\gamma+1})' &= -(\hat{B}_2(\hat{\alpha} - \hat{B}_2 v)/(\mu_0 \hat{v}))' + \tau(u'_1/\hat{v})' \\
\lambda u_2 + u'_2 - (I/\mu_0)(\hat{\alpha}/\hat{v} - \hat{B}_2 v/\hat{v})' &= \mu(u'_2/\hat{v} - \hat{u}_2 v/\hat{v}^2)' \\
\lambda \hat{\alpha} + \hat{\alpha}' - Iu'_2 &= (\sigma \mu_0)^{-1}(\hat{v}^{-1}(\hat{\alpha}/\hat{v} - \hat{B}_2 v/\hat{v})' - \hat{B}'_2 v/\hat{v}^2)',
\end{align*}
\]

\[
h(\hat{v}) = -\hat{v}^{\gamma+1}(\tau \hat{u}'_1/\hat{v}^2 - a\gamma \hat{v}^{-\gamma-1})
\]

\[
= -\hat{v}^{\gamma+1}(\tau \hat{v}'/\hat{v}^2 - a\gamma \hat{v}^{-\gamma-1})
\]

\[
= -\hat{v}^{\gamma+1}(\hat{v}^{-2}(\hat{v}(\hat{v} - 1) + a\hat{v}^{1-\gamma} - a\hat{v} + (2\mu_0 \hat{v})^{-1}((B_{2-} + I\hat{u}_2)^2 - \hat{v}^2 B_{2-}^2)) - a\gamma \hat{v}^{-\gamma-1}).
\]
Integrated coordinates

\[ u(x) = \int_{-\infty}^{x} u_1(z) dz, \quad w = \int_{-\infty}^{x} u_2(z) dz, \]

\[ V = \int_{-\infty}^{x} v(z) dz, \quad \text{and} \quad \alpha = \int_{-\infty}^{x} \tilde{\alpha}(z) dz \]

\[ \lambda V' + V'' - u'' = 0 \]

\[ \lambda u' + u'' - (h(\hat{v})V'/\hat{v}^{\gamma+1})' = -(\hat{B}_2(\alpha' - \hat{B}_2 V')/(\mu_0 \hat{v}))' + \tau(u''/\hat{v})' \]

\[ \lambda w' + w'' - (1/\mu_0)(\alpha'/\hat{v} - \hat{B}_2 V'/\hat{v})' = \mu(w''/\hat{v} - \hat{u}_2' V'/\hat{v}^2)' \]

\[ \lambda \alpha' + \alpha'' - lw''' = (\sigma \mu_0)^{-1}(\hat{v}^{-1}((\alpha' - \hat{B}_2 V')/\hat{v})' - \hat{B}_2' V'/\hat{v}^2)' \].
Integrating from $-\infty$ to $x$ we obtain

\[
\begin{align*}
\lambda V + V' - u' &= 0 \\
\lambda u + u' - h(\hat{\nu})V'/\hat{\nu}^{\gamma+1} &= -\hat{B}_2(\alpha' - \hat{B}_2 V')/\mu_0 + \tau (u''/\hat{\nu}) \\
\lambda w + w' - (1/\mu_0)(\alpha'/\hat{\nu} - \hat{B}_2 V'/\hat{\nu}) &= \mu(w''/\hat{\nu} - \hat{\nu}' V'/\hat{\nu}^2) \\
\lambda \alpha + \alpha' - lw' &= (\sigma \mu_0)^{-1}(\hat{\nu}^{-1}(\alpha'/\hat{\nu} - \hat{B}_2 V'/\hat{\nu})' - \hat{B}_2' V'/\hat{\nu}^2).
\end{align*}
\]
Eigenvalue problem, $\sigma = \infty$

$u_3 \equiv B_3 \equiv 0, \sigma = \infty,$

$$
\begin{align*}
\lambda V + V' - u' &= 0 \\
\lambda u + u' - h(\hat{\nu}) V'/\hat{\nu}^{\gamma+1} &= -\hat{B}_2 (\alpha' - \hat{B}_2 V')/(\mu_0 \hat{\nu}) + \tau (u''/\hat{\nu}) \\
\lambda w + w' - (I/\mu_0)(\alpha'/\hat{\nu} - \hat{B}_2 V'/\hat{\nu}) &= \mu (w''/\hat{\nu} - \hat{u}_2 V'/\hat{\nu}^2) \\
\lambda \alpha + \alpha' - lw' &= 0.
\end{align*}
$$
Eigenvalue problem, $\sigma = \infty$

\[(u, v, v', w, \mu w', \alpha)^T, \quad u'' = \lambda V' + V'', \quad K = I^2/\mu_0\]

\[u' = \lambda V + V'\]

\[V'' = \frac{\lambda \hat{v} V}{\tau} + \left( -\frac{h(\hat{v})}{\tau \hat{v} \gamma} - \lambda + \frac{\hat{v}}{\tau} - \frac{\hat{B}_2^2}{\mu_0 \tau} \right) V' + \frac{\lambda \hat{v} u}{\tau} + \frac{I \hat{B}_2 w'}{\mu_0 \tau} - \frac{\lambda \hat{B}_2 \alpha}{\mu_0 \tau}\]

\[w'' = \left( \frac{\hat{u}_2'}{\hat{v}} + \frac{I \hat{B}_2}{\mu_0 \mu} \right) V' + \frac{\lambda \hat{v} w}{\mu} + \frac{\hat{v} w'}{\mu} - \frac{K w'}{\mu} + \frac{\lambda I \alpha}{\mu_0 \mu}\]

\[\alpha' = I w' - \lambda \alpha.\]
First order system, $\sigma = \infty$

$W' = A(x, \lambda) W$

$$A(x, \lambda) = \begin{pmatrix} 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\lambda \hat{v}}{\tau} & \frac{\lambda \hat{v}}{\tau} & f(\hat{v}) - \lambda - \frac{\hat{B}_2^2}{\mu_0 \tau} & 0 & \frac{I \hat{B}_2}{\mu_0 \mu \tau} & -\frac{\lambda \hat{B}_2}{\mu_0 \tau} \\ 0 & 0 & 0 & 0 & \frac{I}{\mu} & 0 \\ 0 & 0 & \frac{\mu \hat{u}'}{\hat{v}} + \frac{I \hat{B}_2}{\mu_0} & \lambda \hat{v} & \frac{\hat{v} - K}{\mu} & \frac{\lambda I}{\mu_0} \\ 0 & 0 & 0 & 0 & \frac{I}{\mu} & -\lambda \end{pmatrix},$$

$W = (u, V, V', w, \mu w', \alpha)^T$, $f(\hat{v}) = \tau^{-1}(\hat{v} - \hat{v} - \gamma h(\hat{v}))$. 
Numerical stability investigation: profile approximation

- STABLAB (quite general)
- bvp4c, bvp5c, bvp6c (Lobatto quadrature scheme)
- finite computational domain \([-L, L]\)
- \(L_{\pm}\) chosen experimentally, \(|U(\pm L_{\pm}) - U_{\pm}| < \text{error}\)
- projective boundary conditions \(M_{\pm}(U - U_{\pm}) = 0\)
Approximation of the Evans function

Polar coordinate method (Humpherys-Zumbrun)

\[ \mathcal{W} = r \Omega, \]

\[ \mathcal{W} = \mathcal{W}_1 \wedge \cdots \wedge \mathcal{W}_k \]

\[ \Omega = \omega_1 \wedge \cdots \wedge \omega_k \]

\[ D(\lambda) = \mathcal{W}^- \wedge \mathcal{W}^+ |_{x=0} = \det(\mathcal{W}_1^-, \ldots, \mathcal{W}_k^-, \mathcal{W}_{k+1}^+, \ldots, \mathcal{W}_N) |_{x=0}. \]

\[ \mathcal{W}^- (-L_-) \sim e^{-\mu L_-} (R_1^- \wedge \cdots \wedge R_k^-) \]

\[ \mu = \text{Trace} A_- |_{U(A_-)} \]
Shooting and initialization

- ode45 (adaptive 4th-order Runge-Kutta-Fehlberg)
- Error tolerance: AbsTol=1e-8, RelTol=1e-6
- Kato’s ODE
- Second-order algorithm (Zumbrun)
Winding number computation

\[ S := \partial \left( B(0, R) \cap \{ \Re \lambda \geq 0 \} \right) \]

- 20 points taken quadratic in modulus
- winding number calculation
- require relative error be less than 0.2
- Rouché’s Theorem
Proposition

\[
\lim_{{|\lambda| \to \infty}} \frac{\tilde{D}(\lambda)}{e^{\alpha \lambda^{1/2}}} = C \text{ uniformly on } \Re \lambda \geq 0,
\]

where \( \alpha \) and \( C \) are constants.

\[
\log \tilde{D}(\lambda) = \log C + \alpha \lambda^{1/2} \quad |\lambda| \gg 1.
\]
Transverse Evans function

Figure: Typical transverse Evans function output, parameter values $\gamma = 5/3, \ I = 0.6, \ B_+ = 1.4, \text{and} \ \mu_0 = 1$. In Figure (a) we display the nonmonotone profile. In Figure (b) we display the winding number computation.
**Figure:** Large-amplitude limits, parameters $K = 2$, $J = 1$, $\gamma = 5/3$. In Figure (a), we display the image of the semicircle under $\tilde{D}$ for a Lax 2-shock in two-rest point configuration in the $a \to 0$ limit, $a = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$. Where $a = 10^{-8}$ corresponds to Mach number $\approx 10,954$. Convergence of contours appears to occur at $a \sim 10^{-6}$. or Mach number $\approx 1,095$. In Figure (b), for the same sequence of $a$-values, we display the images under the transverse Evans function, again suggestive of convergence.
Large amplitude limits

Figure: Large amplitude limits, parameters $K = 0.7$, $J = 0.5$, and $a = 10^{-1}, 10^{-2}, \ldots, 10^{-k}$, getting smaller as necessary to see what appear to be convergence to a limit. (a). Lax 2-shock, $v_2$ to $v_1$. (b). Intermediate Lax 2-shock, $v_3$ to $v_1$. (c). overcompressive 1-2 shock, $v_4$ to $v_1$. (d). Transverse Evans study for (c). In each case, we appear to obtain convergence at $a = 10^{-7}$, corresponding to Mach number $\approx 3, 817$. 

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Experiments

\[(\gamma, K, J, v_+, \mu_0) \in \{7/5, 5/3\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 1.05, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\} \times \{0.1\} \times \{1.0\}.

\[(\gamma, v_+, l, B_{2+}, \mu_0) \in \{7/5, 5/3\} \times \{0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 10^{-1}, 10^{-2}\} \times \{0.2, 0.4, 0.6, 0.8, 1.2, 1.4, 1.6, 1.8, 2.0\} \times \{0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0\} \times \{1.0\}.

Typical computation time around 1 minute
Experiments: overcompressive family

\((\gamma, K, J, a, \mu_0) \in \{7/5, 5/3\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\} \times \{a_1, a_2, a_3, a_4, a_5\} \times \{1.0\}.
Experiments: undercompressive

\((\gamma, \nu_+, K, J) \in \{7/5, 5/3\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}\)
The take away

- The system appears stable (known instabilities for multi-D model)
- Organizing the phase space important (knowing way around)
- adaptive mesh
- parallel again in future
Interesting directions

- Finite electrical resistivity
- Three dimensional profiles
- Full nonisentropic case
- Multidimensional stability (in $x$)