Co-Toeplitz operators and their associated quantization

Stephen Bruce Sontz

Received: 9 October 2019 / Accepted: 10 December 2019 / Published online: 16 January 2020
© Tusi Mathematical Research Group (TMRG) 2020

Abstract
We define co-Toeplitz operators, a new class of Hilbert space operators, in order to define a co-Toeplitz quantization scheme that is dual to the Toeplitz quantization scheme introduced by the author in the setting of symbols that come from a possibly non-commutative algebra with unit. In the present dual setting the symbols come from a possibly non-co-commutative co-algebra with co-unit. However, this co-Toeplitz quantization is a usual quantization scheme in the sense that to each symbol we assign a densely defined linear operator acting in a fixed Hilbert space. Creation and annihilation operators are also introduced as certain types of co-Toeplitz operators, and then their commutation relations provide the way for introducing Planck’s constant into this theory. The domain of the co-Toeplitz quantization is then extended as well to a set of co-symbols, which are the linear functionals defined on the co-algebra. A detailed example based on the quantum group (and hence co-algebra) $SU_q(2)$ as symbol space is presented.

Keywords Co-Toeplitz operator · Co-Toeplitz quantization · Creation and annihilation operators · Second quantization

Mathematics Subject Classification 47B35 · 47B99

1 Introduction
In a series of recent papers the author has introduced a theory of Toeplitz operators having symbols in a not necessarily commutative algebra with a $\ast$-operation (also called a conjugation). See [11] for the general theory and [8,9] and [10] for various
examples of that theory. The associated Toeplitz quantization is also described in those papers. See [2] for Toeplitz operators in Segal-Bargmann analysis, which was my original interest in these topics. Also see [5] for a quite recent review of Berezin–Toeplitz operators and some related topics, including Toeplitz operators. Finally, see [3] for a more general viewpoint of Toeplitz operators in analysis, including Banach space applications.

There are at least three aspects of the theory in [11] that make it relevant to quantum physics. First, the Toeplitz operators are densely defined linear operators, all acting in the same Hilbert space, and so the self-adjoint extensions of the symmetric Toeplitz operators can be interpreted as being physical observables. (A simple sufficient condition is given later in order for a Toeplitz operator to be symmetric). Second, there are creation and annihilation operators that are defined as certain types of Toeplitz operators. Third, the non-zero commutation relations among the creation and annihilation operators allow the introduction of Planck’s constant into the theory.

In this paper we introduce co-Toeplitz operators in order to study the associated dual quantization scheme. This opens up a new area in the well established theory of operators acting in Hilbert space as well as providing a way to quantize new types of ‘symbols’ in a co-algebra. The most fundamental (and dual) property of the co-Toeplitz operators is that their symbols lie in a co-algebra rather than in an algebra as is the case for Toeplitz operators. A related space of ‘co-symbols’ and its quantization are introduced also. This co-Toeplitz quantization is also relevant to quantum physics, since it has the same three aspects as already mentioned in the Toeplitz setting.

Since the co-algebra can be non-co-commutative, the co-Toeplitz quantization is a generalized second quantization, that is, it produces linear operators from symbols coming from an algebraic structure that can lack the appropriate commutativity, which for historical reasons in the case of co-algebras is called co-commutativity. In this regard it is worthwhile to note that P. Dirac was famously known for saying that the essential property of quantum theory is that the observables do not commute. So the lack of the appropriate commutativity of a co-algebra makes it into a quantum object which the co-Toeplitz quantization then quantizes. In this sense we do have a type of second quantization.

Some words are in order to explain the meaning of a quantization or a quantization scheme. I use these two expressions interchangeably. And I do not wish to propose a rigorous mathematical definition. The basic idea is captured in the catch-phrase “operators instead of functions.” By “operators” I mean linear, densely defined operators acting in a Hilbert space, possibly separable. This is a quite conventional interpretation. But by “functions” I merely mean elements in some vector space with some additional algebraic structure, such as an algebra or a co-algebra. This is a far cry from the standard definition of a function, though that is included as a special case. The general properties of a quantization mapping which sends “functions” to operators are left deliberately vague, though linearity might be one.

Due to the novelty of the material of this paper, much of it is devoted to definitions and motivation, while the number of theorems is less than a paper of this size would usually contain. Some possibilities are presented in the Concluding Remarks for research leading to more theorems. But even the definitions might be refined as more examples of co-Toeplitz operators become available. In this regard, see [14].
The paper is organized as follows. In Sect. 2 we review the previously known, general Toeplitz quantization scheme for algebras. Then in Sect. 3 we present the dual co-Toeplitz quantization scheme. We discuss next the role of the co-unit of the co-algebra in co-Toeplitz quantization in Sect. 4 and show how that motivates an extension of this quantization scheme using co-symbols in the dual of the co-algebra. The duality between Toeplitz and co-Toeplitz operators is not as symmetric as one might have expected. This is presented in Sect. 5. Adjoints of the co-Toeplitz operators are studied in Sect. 6. Next the creation and annihilation operators are defined in terms of co-Toeplitz operators in Sect. 7, and then the canonical commutation relations among these operators are defined in Sect. 8 in algebraic terms. At this point Planck’s constant, denoted as \( h \), is introduced into the theory as well as the associated \( \hbar \)-deformed CCR algebras, for which \( \hbar > 0 \), and the classical algebra, for which \( \hbar = 0 \). We continue in Sect. 9 with an example of this new quantization scheme based on the quantum group (and hence co-algebra) \( SU_q(2) \) as symbol space. A Toeplitz quantization of \( SU_q(2) \) has already been presented in [13] using instead its structure as an algebra, but with the same sub-algebra of ‘holomorphic’ elements. Finally, we conclude in Sect. 10 with remarks about possible further developments and alternatives of this theory.

We only consider vector spaces over the field of complex numbers. We use the standard notations \( \mathbb{N} \) for the non-negative integers, \( \mathbb{Z} \) for all the integers, \( \mathbb{R} \) for the real numbers and \( \mathbb{C} \) for the complex numbers. For \( \alpha \in \mathbb{C} \) we let \( \alpha^* \) denote its complex conjugate.

## 2 The Toeplitz quantization

We will introduce the definition of a co-Toeplitz quantization using the Toeplitz quantization as a guide and motivation. Hence, we start with a review in this section of the already known theory of Toeplitz quantization in the setting of possibly non-commuting symbols as is developed by the author in [11].

We let \( \mathcal{A} \) be an associative algebra with identity element \( 1 \equiv 1_{\mathcal{A}} \). This algebra could have a non-commutative multiplication; it will be the symbol space for the Toeplitz quantization. Suppose that \( \langle \cdot, \cdot \rangle_\mathcal{A} \) is a sesquilinear, complex symmetric form on \( \mathcal{A} \). This form could possibly be degenerate. Our convention throughout is that all sesquilinear forms are anti-linear in the first entry and linear in the second. Moreover, suppose that there exists a sub-algebra \( \mathcal{P} \) (not necessarily containing \( 1 \)) of \( \mathcal{A} \) such that the sesquilinear form is positive definite when restricted to \( \mathcal{P} \). Then \( \mathcal{P} \) is a pre-Hilbert space. We let \( \mathcal{H} \) denote a Hilbert space completion of \( \mathcal{P} \) such that \( \mathcal{P} \) is a dense subspace of \( \mathcal{H} \). If we think of \( \mathcal{P} \) as corresponding to a space of holomorphic polynomials, then \( \mathcal{H} \) can be considered as a generalization of the Segal-Bargmann space. See [1].

We let \( \iota : \mathcal{P} \to \mathcal{A} \) denote the inclusion map, which is an algebra morphism. We suppose that there exists a projection map \( P : \mathcal{A} \to \mathcal{P} \), that is, \( P \iota = \text{id}_\mathcal{P} \). While \( P \) is assumed to be linear, it is not assumed to be an algebra morphism. In this abstract formalism the projection \( P \) is rather arbitrary. However, one specific choice for it in several examples is given for \( \phi \in \mathcal{A} \) by
\[ P \phi = \sum_{j \in J} \langle \psi_j, \phi \rangle A \psi_j \]  

(2.1)

where \( \{ \psi_j \mid j \in J \} \) is an orthonormal set in \( \mathcal{P} \) that is an orthonormal basis of \( \mathcal{H} \). Of course, it must be shown that the possibly infinite sum on the right side of (2.1) converges to an element in \( \mathcal{P} \). (This is trivially true if only finitely many of the summands are non-zero.) But be aware that \( P \) defined this way is not necessarily an orthogonal projection, since the form \( \langle \cdot, \cdot \rangle_A \) need not be positive definite and, in fact, is degenerate in some examples.

The operator \( P \) could also be realized more generally as an extension to \( A \) of a reproducing kernel that represents the identity map on the pre-Hilbert space \( \mathcal{P} \). This is what is happening in (2.1) since the right side restricted to \( \mathcal{P} \) is a reproducing kernel for \( \mathcal{P} \). Since the algebra \( \mathcal{P} \) can be non-commutative, the reproducing kernel need not be a function in the usual sense of that word and so will not have all (although some) of the properties of a reproducing kernel function. See [8] for another example of this more general type of reproducing kernel object.

We assume that there is a left action of \( \mathcal{P} \) on \( A \), namely a linear map

\[ \alpha : \mathcal{P} \otimes A \rightarrow A \]

satisfying the standard properties, namely \( 1 \cdot a = a \) if \( 1 = 1_A \in \mathcal{P} \), and \( p_1 \cdot (p_2 \cdot a) = (p_1 p_2) \cdot a \) where \( p \cdot a := \alpha(p \otimes a) \) for \( p, p_1, p_2 \in \mathcal{P} \) and \( a \in A \). Here the juxtaposition \( p_1 p_2 \) means the multiplication of elements in \( \mathcal{P} \). Next, in anticipation of the definition of a left co-action in Sect. 3, we re-write this is terms of the map \( \alpha \) for all \( a \in A \) and \( p_1, p_2 \in \mathcal{P} \) as

\[ \alpha(1 \otimes a) = a \quad \text{and} \quad \alpha(p_1 \otimes \alpha(p_2 \otimes a)) = \alpha(p_1 p_2 \otimes a). \]

The first condition is only required if \( 1 \in \mathcal{P} \).

For example, we could take \( \alpha \) equal to \( \mu_A \) restricted to \( \mathcal{P} \otimes A \), where \( \mu_A : A \otimes A \rightarrow A \) is the multiplication map of \( A \). In short, we could take \( \alpha = \mu_A (\iota \otimes \text{id}) \).

This particular choice for \( \alpha \) is the only place in this theory of Toeplitz operators where we use the multiplication of \( A \). We should emphasize however that this particular choice for \( \alpha \) closely corresponds to what is used in the classical theory of Toeplitz operators acting in function spaces.

Nonetheless, other choices for \( \alpha \), which do not use the multiplicative structure of \( A \), are also possible. In such a case we can drop the assumption that \( A \) is an algebra and instead only assume that it is a vector space. However, we still want to have a \( \ast \)-structure on \( A \) in order to be able to define creation and annihilation operators in Sect. 7. Also a \( \ast \)-structure appropriately compatible with the inner product on \( \mathcal{P} \) gives an easy way to find symmetric operators which then might be extendable to self-adjoint operators representing physical observables. This more general approach is presented in [11].

Given the setting of the previous paragraph we now define Toeplitz operators.
Definition 2.1 Suppose that $g \in \mathcal{A}$ and that $\phi \in \mathcal{P}$. We also introduce the notation
$$\phi g := \alpha(\phi \otimes g) \in \mathcal{A}$$
and define
$$T_g(\phi) := P(\phi g) = P\alpha(\phi \otimes g) \in \mathcal{P}.$$ 
Then $T_g : \mathcal{P} \to \mathcal{P}$ is a linear map, and we say that $T_g$ is the Toeplitz operator with symbol $g$.

The notation $\phi g$ was introduced merely to emphasize the similarity with classical Toeplitz operators. Another handy notation is $\cdot \otimes g$, which is the linear map $\mathcal{P} \to \mathcal{P} \otimes \mathcal{A}$ defined for $g \in \mathcal{A}$ and $\phi \in \mathcal{P}$ by
$$(\cdot \otimes g) \phi := \phi \otimes g.$$ 

Here is the corresponding diagram defining $T_g$ as the composition of these three maps:
$$\mathcal{P} \xrightarrow{\otimes g} \mathcal{P} \otimes \mathcal{A} \xrightarrow{\alpha} \mathcal{A} \xrightarrow{P} \mathcal{P}.$$ (2.2)

Thus the Toeplitz operator $T_g$ is defined for each symbol for $g \in \mathcal{A}$ as
$$T_g := P\alpha(\cdot \otimes g) \in \mathcal{L}(\mathcal{P}).$$ (2.3)

where $\mathcal{L}(\mathcal{P}) := \{A : \mathcal{P} \to \mathcal{P} | A \text{ is linear}\}$.

To bring this more closely into notational accord with the usual definition of a Toeplitz operator in classical analysis, for each $g \in \mathcal{A}$ we define
$$M_g := \alpha(\cdot \otimes g) : \mathcal{P} \to \mathcal{A}.$$ 

We note that $M_g$ is typically not an algebra morphism, even though both $\mathcal{P}$ and $\mathcal{A}$ are algebras. Then $T_g = PM_g$. Moreover, if we take $\alpha$ to be the restriction of the multiplication on $\mathcal{A}$, which as was noted above is a possible case, then $M_g$ is indeed the operation of multiplication by $g$ on the right. (The change to get multiplication by $g$ on the left is easy enough.) However, even the rather general formula $M_g = \alpha(\cdot \otimes g)$ can itself be generalized easily. All that we need is any linear map $\mathcal{A} \ni g \mapsto M_g$, where $M_g : \mathcal{P} \to \mathcal{A}$ is linear, that is, we need a linear map $M : \mathcal{A} \to \text{Hom}_{\text{Vect}}(\mathcal{P}, \mathcal{A})$, where $\text{Hom}_{\text{Vect}}(V, W)$ means the vector space of all linear maps $V \to W$ of the vector spaces $V$ and $W$.

We are using the unconventional notation $\mathcal{L}(\mathcal{P})$ in order to denote the complex vector space of all the linear maps $A : \mathcal{P} \to \mathcal{P}$. Any such map $A$ can be considered as a densely defined linear operator in the Hilbert space $\mathcal{H}$. We note that $A$ may or may not be a bounded operator. However, note that in general there are densely defined linear operators in the Hilbert space $\mathcal{H}$ that do not lie in $\mathcal{L}(\mathcal{P})$. This is so for two reasons: First, the domain of a densely defined operator need not be equal to $\mathcal{P}$; second, the domain need not be mapped to itself under the action of such an operator.

The Toeplitz quantization that has been defined associates to each symbol $g \in \mathcal{A}$ an operator $T_g \in \mathcal{L}(\mathcal{P})$, which is the Toeplitz operator with symbol $g$. The mapping
$T : \mathcal{A} \to \mathcal{L}(\mathcal{P})$ that is given by $T : g \mapsto T_g$ is called the Toeplitz quantization (scheme). A question that arises naturally is whether the Toeplitz quantization $T$ is injective, that is, if a Toeplitz operator comes from a unique symbol. For example, in a certain context Theorem 4.3 in [10] says that the sesquilinear form on $\mathcal{A}$ being non-degenerate is a necessary and sufficient condition for $T$ to be injective. See [10] for more details.

Even though $\mathcal{A}$ and $\mathcal{L}(\mathcal{P})$ are algebras, the Toeplitz quantization $T$ is not expected nor desired to be an algebra morphism. On the contrary, the deviation of $T$ from being an algebra morphism is some way of measuring the ‘quantum-ness’ of $T$. As an example, we might have elements $g, h \in \mathcal{A}$ satisfying the classical $q$-commutation relation $gh - qhg = 0$ for some $q \in \mathbb{C}$, while the corresponding Toeplitz operators satisfy the quantum $q$-commutation relation $T_g T_h - q T_h T_g = \hbar 1_{\mathcal{P}} \neq 0$. In Sect. 8 the rigorous definitions of classical and quantum relations are given in a related context.

The identity element $1 = 1_{\mathcal{A}}$ in $\mathcal{A}$ has played no essential role so far in this theory. It seems that in the examples the main property of $1$ that arises is $T_1 = 1_{\mathcal{P}}$, the identity map. Nonetheless, we would like to find the dual of this property in the co-Toeplitz setting. To achieve this requires more details about how the Toeplitz setting motivate an important definition there, as we shall see.

Let’s first note that $\text{Hom}_{\text{Vect}}(\mathbb{C}, \mathcal{A}) \cong \mathcal{A}$ in a natural way. Explicitly, a symbol $g \in \mathcal{A}$ corresponds to the linear map $l_g : \mathbb{C} \to \mathcal{A}$ given by $l_g(z) := zg$ for every $z \in \mathbb{C}$. And an arbitrary linear map $l : \mathbb{C} \to \mathcal{A}$ has the form $l = l_g$, where $g := l(1)$ with $1 \in \mathbb{C}$. Then we have that the composition

$$\mathcal{P} \cong \mathcal{P} \otimes \mathbb{C} \xrightarrow{id \otimes l_g} \mathcal{P} \otimes \mathcal{A} \quad \tag{2.4}$$

is equal to $\cdot \otimes g$. So we can use this to re-write (2.3) as $T_g = P \alpha (id \otimes l_g)$. By taking the case where $g = 1_{\mathcal{A}} = 1 \in \mathcal{A}$ we see that $l_1 = \eta : \mathbb{C} \to \mathcal{A}$, the unit map of the algebra $\mathcal{A}$. By further taking $\alpha$ to be the restriction of the multiplication of $\mathcal{A}$, that is $\alpha = \mu_{\mathcal{A}} (\iota \otimes id)$, we easily get $T_1 = 1_{\mathcal{P}}$.

Various examples of this sort of Toeplitz quantization have been worked out in some of the author’s papers. In those examples there is some sort of definition of a ‘holomorphic element’ in the algebra $\mathcal{A}$, which then must actually be a $\ast$-algebra, and $\mathcal{P}$ is the sub-algebra (but not a sub-$\ast$-algebra) of holomorphic elements in $\mathcal{A}$. There is also a concept of ‘anti-holomorphic element’ in $\mathcal{A}$ with its corresponding sub-algebra, defined by $\overline{\mathcal{P}} := \mathcal{P}^\ast$, of the anti-holomorphic elements. Then Toeplitz operators with symbols in $\mathcal{P}$ are defined to be creation operators. On the other hand, Toeplitz operators with symbols in $\overline{\mathcal{P}}$ are defined to be annihilation operators. This aspect of the theory, which includes commutation relations among these operators, gives the theory contact with ideas from the mathematical physics of quantum systems.

It might be worthwhile to recall for the record what a $\ast$-algebra with identity $1$ is. First off, a $\ast$-operation (or conjugation) on a vector space $V$ is an anti-linear map $V \to V$, denoted by $v \mapsto v^\ast$ for $v \in V$, that is also an involution (that is, $v^{\ast \ast} = v$). Then a $\ast$-algebra with identity $1$ is an algebra $\mathcal{A}$ with identity $1$ which also has a $\ast$-operation satisfying $(ab)^\ast = b^\ast a^\ast$ for all $a, b \in \mathcal{A}$ as well as $1^\ast = 1$. 

\begin{flushright} \textcopyright Birkhäuser \end{flushright}
The conjugation in the symbol space $A$ interchanges by definition the holomorphic and anti-holomorphic sub-algebras, namely

$$\mathcal{P}^* = \overline{P} \quad \text{and} \quad (\overline{P})^* = \mathcal{P}.$$  

But the Toeplitz quantization that we have described breaks this symmetry, since the creation and annihilation operators have distinct properties in specific examples. The origin of this has to do with the fact that the Toeplitz operators are acting in the holomorphic space $\mathcal{P}$, even though we could have used the anti-holomorphic space $\overline{\mathcal{P}}$ instead of $\mathcal{P}$. All of the technical details work out if we use $\overline{\mathcal{P}}$. For example, the projection of $A$ onto $\overline{\mathcal{P}}$ is given by the linear operator $P^*$, where $P^*(f) := (P(f^*))^*$ is the standard $*$-operation (but not adjoint) of an operator. Then the Toeplitz quantization (which now produces operators in $L(\overline{\mathcal{P}})$) of the symbols in $\mathcal{P}$ give the annihilation operators, while on the other hand the Toeplitz quantization of the symbols in $\overline{\mathcal{P}}$ give the creation operators. However, this is still to be considered as a type of Toeplitz quantization. The new concept of co-Toeplitz quantization comes in the next section.

It is important to realize that the role played by the sesquilinear form on $A$ is not essential to this theory. However, it does unify three different aspects of it. First, it can be used to define the projection $P$, although that can be done without having a sesquilinear form. Second, it can be used to define the left action, although that can also be defined independently. Third, it restricts to an inner product on $\mathcal{P}$. But one can also define that inner product directly. Given these comments, we see how the sesquilinear form, which does appear in some examples, can be removed from this theory without basically changing it.

### 3 The co-Toeplitz quantization

Now we continue with the dual development of the new theory of co-Toeplitz quantization. This is achieved by reversing most of the arrows in the theory of Toeplitz quantization as outlined in the previous section. This sort of duality is well known in category theory and is called notion duality. We will consider object duality in Sect. 5.

We let $\mathcal{C}$ be a co-associative co-algebra with a co-unit $\epsilon : \mathcal{C} \to \mathbb{C}$ and with $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, a possibly non-co-commutative co-multiplication. The co-algebra $\mathcal{C}$ is the symbol space for the co-Toeplitz quantization. It is important to note that even the co-commutative case is new. For the definition and basic properties of co-algebras see [6].

We suppose next that $\mathcal{C}$ is equipped with a sesquilinear, complex symmetric form denoted by $\langle \cdot, \cdot \rangle_{\mathcal{C}}$. Let $\mathcal{P}$ be a co-associative, co-algebra with co-multiplication $\Delta'$, but not necessarily with a co-unit. Suppose that there also exists a co-algebra morphism $Q : \mathcal{C} \to \mathcal{P}$, dual to $\iota$ in the Toeplitz setting. Also, we suppose that there exists a linear map $j : \mathcal{P} \to \mathcal{C}$, dual to $P$ in the Toeplitz setting, such that

$$Qj = id_{\mathcal{P}}.$$
The injection \( j \) need not be a co-algebra morphism. We suppose that the form on \( \mathcal{C} \) restricts down using \( j \) to a positive definite inner product \( \langle \cdot, \cdot \rangle_{\mathcal{P}} \) on \( \mathcal{P} \), that is to say, \( \langle f, g \rangle_{\mathcal{P}} = \langle j(f), j(g) \rangle_{\mathcal{C}} \) holds for all \( f, g \in \mathcal{P} \). Therefore, \( \mathcal{P} \) is a pre-Hilbert space. We let \( \mathcal{H} \) denote a Hilbert space completion of \( \mathcal{P} \) such that \( \mathcal{P} \) is a dense subspace of \( \mathcal{H} \). Comparing this with the Toeplitz setting, we notice that the arrow of the inclusion map of the pre-Hilbert space \( \mathcal{P} \) into the Hilbert space \( \mathcal{H} \) has not been reversed in the co-Toeplitz setting. So, it still makes intuitive sense to think of \( \mathcal{P} \) as a space of ‘holomorphic polynomials’ and of \( \mathcal{H} \) as a type of generalized Segal-Bargmann space of ‘holomorphic functions’.

The projection map \( Q \) in this setting is quite abstract, although it is required to be a co-algebra morphism while the projection \( P \) in the Toeplitz setting was only required to be linear. Nonetheless, a similar formula using the form \( \langle \cdot, \cdot \rangle_{\mathcal{C}} \) can be used to define \( Q \) in examples. We will see this in the example in Sect. 9.

We also suppose that there is a left co-action of the co-algebra \( \mathcal{P} \) on \( \mathcal{C} \), namely, there exists a linear map \( \beta : \mathcal{C} \to \mathcal{P} \otimes \mathcal{C} \) which has the usual properties dual to those of a left action, namely,

\[
(\varepsilon' \otimes id_{\mathcal{C}}) \beta \cong id_{\mathcal{C}} \quad \text{and} \quad (id_{\mathcal{P}} \otimes \beta) \beta = (\Delta' \otimes id_{\mathcal{C}}) \beta.
\]

Each of these properties can be expressed by a commutative diagram. However, the first property is only required when the co-algebra \( \mathcal{P} \) has a co-unit \( \varepsilon' \). As an example the left co-action \( \beta \) could be the composition

\[
\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{Q \otimes id} \mathcal{P} \otimes \mathcal{C}
\]

as the reader can readily verify by checking that the corresponding diagrams commute. (Hint: The co-associativity of \( \Delta \) is used.) In this case \( \beta \) is a projection of the co-multiplication of \( \mathcal{C} \). In the dual case of Toeplitz operators we had a particular choice of the left action \( \alpha \) given by \( \alpha = \mu_{\mathcal{C}}(\iota \otimes id) \). So this particular choice of \( \beta \) in (3.1) is dual to that choice of \( \alpha \) in the Toeplitz case. Also, much as in the Toeplitz case, this choice of \( \beta \) is the only place in this theory of co-Toeplitz operators where we use the co-multiplicative structure of \( \mathcal{C} \). With other choices of \( \beta \) which do not depend on the co-multiplicative structure of \( \mathcal{C} \) we do not need to assume that \( \mathcal{C} \) is a co-algebra. Rather, we only need to assume that \( \mathcal{C} \) is a vector space equipped with a \( * \)-structure. In the example in Sect. 9 we will use the particular choice (3.1) and so that example will be a co-algebra. It remains for future research work to find non-trivial examples of co-Toeplitz operators in a setting where the symbol space is not a co-algebra.

Given the set-up of the previous paragraph, we now define co-Toeplitz operators.

**Definition 3.1** We take \( g \in \mathcal{C} \), known as a symbol, and then consider the composition, dual to diagram (2.2), of these three linear maps from right to left:

\[
\mathcal{P} \xleftarrow{\pi_g} \mathcal{P} \otimes \mathcal{C} \xleftarrow{\beta} \mathcal{C} \xleftarrow{j} \mathcal{P},
\]

\( \otimes \) Birkhäuser
where the family of linear maps \( \{ \pi_g \mid g \in \mathcal{C} \} \), the dual to the family of linear maps \( \{ \cdot \otimes g \mid g \in \mathcal{C} \} \), has yet to be defined. Then \( C^g := \pi_g \beta j \) is the definition of the (left) co-Toeplitz operator with symbol \( g \).

Clearly, \( C^g : \mathcal{P} \to \mathcal{P} \) is linear or, in other words, \( C^g \in \mathcal{L}(\mathcal{P}) \). In particular, \( C^g \) is a densely defined operator in the Hilbert space \( \mathcal{H} \). By replacing \( \beta \) with a right co-action we get a theory of right co-Toeplitz operators. That quite similar, analogous theory will not be discussed here; we will only concern ourselves with left co-Toeplitz operators.

Next, the possibly non-linear function \( C : \mathcal{C} \to \mathcal{L}(\mathcal{P}) \) defined by \( g \mapsto C^g \) is called the co-Toeplitz quantization. We note in passing that the vector space \( \mathcal{L}(\mathcal{P}) \) is an algebra under the multiplication given by composition of operators, while \( \mathcal{L}(\mathcal{P}) \) does not seem to have a natural co-algebra structure.

As in the Toeplitz setting, it is natural to ask whether the co-Toeplitz quantization map \( C \) is injective. It seems reasonable to conjecture that this will depend on other conditions, much as we already remarked is the case in the Toeplitz setting.

Analogously to the Toeplitz case, we can introduce some notation to help understand better what is going on here. In analogy to \( M^g \) we define

\[
\tilde{M}^g := \pi_g \beta : \mathcal{C} \to \mathcal{P}
\]

for \( g \in \mathcal{C} \). Then \( C^g = \tilde{M}^g j = \pi_g \beta j \in \mathcal{L}(\mathcal{P}) \). Be aware that \( \tilde{M}^g \) maps a co-algebra to a co-algebra, but \( \tilde{M}^g \) is not a co-algebra morphism. This is dual to the Toeplitz setting where \( M^g : \mathcal{P} \to \mathcal{A} \) is a map between algebras, but is not an algebra morphism.

We still have a quite general theory (possibly too general!), since the family \( \{ \pi_g \mid g \in \mathcal{C} \} \) is quite arbitrary in the above discussion. For example, \( \pi_g \) could be independent of \( g \) thereby giving a co-Toeplitz quantization that does not depend on the symbol. This is much more general than we would wish to consider. A more acceptable possibility is to define \( \pi_g : \mathcal{P} \otimes \mathcal{C} \to \mathcal{P} \) by

\[
\pi_g(\phi \otimes f) := \langle g, f \rangle_{\mathcal{C}} \phi
\]

(3.3)

for \( \phi \in \mathcal{P} \) and \( f, g \in \mathcal{C} \). To see that this formula gives a dual to the map \( \cdot \otimes g \) (now defined in the co-Toeplitz setting), consider the following calculation for \( \psi, \phi \in \mathcal{P} \) and \( f, g \in \mathcal{C} \):

\[
\langle (\cdot \otimes g)\psi, \phi \otimes f \rangle_{\mathcal{P} \otimes \mathcal{C}} = \langle \psi \otimes g, \phi \otimes f \rangle_{\mathcal{P} \otimes \mathcal{C}}
\]

\[
= \langle \psi, \phi \rangle_{\mathcal{P}} \langle g, f \rangle_{\mathcal{C}}
\]

\[
= \langle \psi, \langle g, f \rangle_{\mathcal{C}} \phi \rangle_{\mathcal{P}}
\]

\[
= \langle \psi, \pi_g(\phi \otimes f) \rangle_{\mathcal{P}}.
\]

This provides some justification for the formula (3.3) for \( \pi_g \). Note that the second equality here is the standard definition of the sesquilinear form on \( \mathcal{P} \otimes \mathcal{C} \).

Now given our convention for sesquilinear forms, \( \pi_g \) is a linear map, but in this case the co-Toeplitz quantization mapping \( C : g \mapsto C^g \) is anti-linear. It seems to be some
sort of tradition in mathematical physics that a quantization map should be linear. To avoid this slight unpleasantness we could have defined \( \pi_g \) by

\[
\pi_g(\phi \otimes f) = \langle g^*, f \rangle_C \phi
\]

for \( \phi \in \mathcal{P} \) and \( f, g \in \mathcal{C} \). Of course, to have this make sense we must assume that \( \mathcal{C} \) is a \( \ast \)-co-algebra, which we will do anyway later. But, we rather prefer to let the quantization mapping be anti-linear.

Again for the record let us recall that a \( \ast \)-co-algebra \( \mathcal{C} \) is a co-algebra with a \( \ast \)-operation such that the co-multiplication map \( \Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \) is a \( \ast \)-morphism, namely, \( \Delta(g^*) = (\Delta(g))^\ast \). Since the co-algebra \( \mathcal{C} \) has a co-unit \( \varepsilon : \mathcal{C} \to \mathbb{C} \), we also require that \( \varepsilon \) is a \( \ast \)-morphism, namely, \( \varepsilon(g^*) = (\varepsilon(g))^\ast \). Note that the \( \ast \)-operation of \( \mathcal{C} \otimes \mathcal{C} \) is determined by \( (g \otimes h)^\ast = g^* \otimes h^* \) for \( g, h \in \mathcal{C} \). Be aware that this is not exactly dual to the definition of a \( \ast \)-algebra, where the multiplication is required to be an anti-\( \ast \)-morphism.

Given the definition (3.3) for \( \pi_g \) we can write down more explicit expressions for \( \tilde{M}_g \) and \( C_g \). So we take \( f \in \mathcal{C} \) and then in Sweedler’s notation for a co-action (see Appendix B in [12]) we have

\[
\beta(f) = f^{(0)} \otimes f^{(1)} \in \mathcal{P} \otimes \mathcal{C}.
\]

It follows for \( g \in \mathcal{C} \) that

\[
\tilde{M}_g(f) = \pi_g \beta(f) = \pi_g(f^{(0)} \otimes f^{(1)}) = \langle g, f^{(1)} \rangle_C f^{(0)}.
\]

For \( C_g \) we simply note that for \( \phi \in \mathcal{P} \) we have that

\[
C_g(\phi) = \tilde{M}_g j(\phi) = \langle g, f^{(1)} \rangle_C f^{(0)},
\]

where now \( f = j(\phi) \). If we use the injection \( j \) to identify \( \mathcal{P} \) as a subspace of \( \mathcal{C} \), then the previous expression simplifies to

\[
C_g(\phi) = \langle g, \phi^{(1)} \rangle_C \phi^{(0)}.
\]

The co-action \( \beta \) is a basic operation in these expressions. However, \( \beta \) is hidden inside Sweedler’s notation. For example, as we have noted earlier, we could take \( \beta = (Q \otimes \text{id}) \Delta_C : \mathcal{C} \to \mathcal{P} \otimes \mathcal{C} \). Then for \( f \in \mathcal{C} \) we have

\[
\beta(f) = Q(f^{(1)}) \otimes f^{(2)} \quad \text{and} \quad C_g(\phi) = \langle g, f^{(2)} \rangle_C Q(f^{(1)}),
\]

where we are using Sweedler’s notation for the co-multiplication, that is, \( \Delta_C(f) = f^{(1)} \otimes f^{(2)} \in \mathcal{C} \otimes \mathcal{C} \). Be aware please that this is not Sweedler’s notation \( f^{(0)} \otimes f^{(1)} \) introduced above for the co-action \( \beta \).

To maintain contact with physics ideas we only consider the case when \( \mathcal{C} \) is a \( \ast \)-co-algebra. But, in that case we do not require \( \mathcal{P} \) to be a sub-\( \ast \)-co-algebra. Rather we think of the elements in \( \mathcal{P} \) as being holomorphic variables, while those in \( \mathcal{P}^* \) are
anti-holomorphic variables. Then the creation operators are defined to be those of the form $C_g$ where $g \in P^*$, while annihilation operators are those of the form $C_g$ where $g \in P$. What relation holds between the operators $(C_g)^*$, the adjoint of $C_g$, and $C_g^*$ for a symbol $g \in C$ is a question that we will consider later.

A possible relation between the sesquilinear form and the $*$-operation is given in the next definition. This property was already described in the Toeplitz setting in [13], but it was not given its own name there.

**Definition 3.2** If for all $f, g \in C$ the identity

$$\langle f^*, g^* \rangle_C = \langle f, g \rangle^*_C$$  \hspace{1cm} (3.4)

holds, then we say that the sesquilinear form $\langle \cdot, \cdot \rangle_C$ is $*$-symmetric.

As in the Toeplitz setting it is important to understand the role of the sesquilinear form in the co-Toeplitz setting, where it has the same three aspects mentioned earlier as in the Toeplitz setting plus a new aspect, which is that it appears in the definition (3.3) of $\pi_g$. This seems to be a more essential role since $\pi_g$ so defined is dual to the map $\cdot \otimes g$ in the Toeplitz setting.

### 4 The co-unit and co-symbols

So far the co-unit has not played a role in this theory of co-Toeplitz operators. To achieve this we now will dualize the theory from the Toeplitz setting. Since the co-unit $\varepsilon : C \to C$ is a linear map, we consider how to deal with an arbitrary linear map $\lambda : C \to C$ in a way that is dual to the linear maps $l : C \to A$ which appeared in the Toeplitz setting. The dual construction, starting with $\lambda$ instead of with some symbol $g \in C$, gives us a more general type of co-Toeplitz operator defined as the composition from right to left as follows:

$$\mathcal{P} \cong \mathcal{P} \otimes C \overset{id \otimes \lambda}{\leftarrow} \mathcal{P} \otimes C \overset{\beta}{\leftarrow} C \overset{i}{\leftarrow} \mathcal{P}$$  \hspace{1cm} (4.1)

However, the linear functional $\lambda$ lies in $\text{Hom}_{\text{Vect}}(\mathcal{C}, C)$ which, quite unlike its dual $\text{Hom}_{\text{Vect}}(\mathcal{C}, \mathcal{C})$, is *not* naturally isomorphic in general to $\mathcal{C}$. Of course, for every symbol $g \in C$ each

$$e_g := \langle g, \cdot \rangle_C$$  \hspace{1cm} (4.2)

lies in $\text{Hom}_{\text{Vect}}(\mathcal{C}, C)$. Moreover, if we take $\lambda = e_g$ in diagram (4.1), we readily see that

$$id \otimes e_g = \pi_g$$

and so we do have the co-Toeplitz operators as defined above as a special case of the more general definition

$$C_\lambda := (id \otimes \lambda) \beta j$$
for \( \lambda \in \text{Hom}_{\text{Vect}}(\mathcal{C}, \mathbb{C}) \). Having this definition in hand, it now makes sense to study the co-Toeplitz operator \( C_\varepsilon \), where \( \varepsilon : \mathcal{C} \to \mathbb{C} \) is the co-unit of the co-algebra \( \mathcal{C} \).

In the Toeplitz setting we had that \( T_1 = I_\mathcal{P} \) in the special case when the left action was the restriction of the multiplication of \( \mathcal{A} \). So in the present co-Toeplitz setting we expect a similar result when the left co-action \( \beta \) is the projection of the co-multiplication, that is, when we have \( \beta = (Q \otimes id_{\mathcal{C}}) \Delta_{\mathcal{C}} \). In this case for \( \phi \in \mathcal{P} \) we compute that

\[
C_\varepsilon \phi = (id_\mathcal{P} \otimes \varepsilon) \beta j \phi = (id_\mathcal{P} \otimes \varepsilon) (Q \otimes id_{\mathcal{C}}) \Delta_{\mathcal{C}} \phi \\
= (Q \otimes id_{\mathcal{C}})(id_{\mathcal{C}} \otimes \varepsilon) \Delta_{\mathcal{C}} \phi = (Q \otimes id_{\mathcal{C}})(\phi \otimes 1) \\
\cong Q \phi = \phi,
\]

where in the last equality we used that \( \phi \in \mathcal{P} \). Also, 1 here means the number 1 \( \in \mathbb{C} \).

This discussion, which seemed at the start to be a minor side issue, has given rise to a new definition which we now explicitly state.

**Definition 4.1** Let \( \lambda \in C' := \text{Hom}_{\text{Vect}}(\mathcal{C}, \mathbb{C}) \) be a linear functional on the co-algebra \( \mathcal{C} \). Then we define the (generalized) co-Toeplitz operator with co-symbol \( \lambda \) to be the linear map

\[
C_\lambda := (id_\mathcal{P} \otimes \lambda) \beta j \in \mathcal{L}(\mathcal{P}).
\]

Much as before, we define the (generalized) co-Toeplitz quantization to be the map \( C : C' \to \mathcal{L}(\mathcal{P}) \) given by \( \lambda \mapsto C_\lambda \) for \( \lambda \in C' \).

We sometimes omit the word ‘generalized’ when speaking of these new objects, since the fact that we are using co-symbols in \( C' \) rather than symbols in \( C \) suffices to remove any ambiguity. Note that the notation in this definition gives us the strange looking identity

\[
C_\varepsilon = C_{e_\varepsilon}
\]

for any \( g \in \mathcal{C} \), where on the left side there is a co-Toeplitz operator with symbol \( g \) and on the right side there is a generalized co-Toeplitz operator with co-symbol \( e_\varepsilon \) as defined in (4.2).

So, given this definition, we have proved above the following result, which is dual to the result that \( T_1 = I_\mathcal{P} \) in the Toeplitz setting.

**Proposition 4.1** Let the left co-action be

\[
\beta = (Q \otimes id_{\mathcal{C}}) \Delta_{\mathcal{C}}.
\]

Then the co-Toeplitz quantization of the co-unit \( \varepsilon \) of the co-algebra \( \mathcal{C} \) is

\[
C_\varepsilon = I_\mathcal{P},
\]

the identity operator on \( \mathcal{P} \).
So an important point here is that the set of co-symbols can be strictly larger than the set of co-symbols of the form $e_g$ for $g \in \mathcal{C}$. Recall that the sesquilinear form on $\mathcal{C}$ could be degenerate, and so the Riesz representation theorem need not apply here. What we do see however is that the theory of co-Toeplitz operators with co-symbols can possibly admit more operators than the original co-Toeplitz operator theory with symbols only in $\mathcal{C}$.

A more important point is that the dual space $\mathcal{C}'$ of a co-algebra with co-unit has a canonical structure as an algebra with unit, where the multiplication of the elements $\alpha, \beta \in \mathcal{C}'$ is defined as the composition

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\alpha \otimes \beta} \mathcal{C} \otimes \mathcal{C} \cong \mathcal{C}$$

and the unit is the linear map $\eta : \mathbb{C} \to \mathcal{C}'$ defined by $\eta(z) := z \varepsilon$ for all $z \in \mathbb{C}$, where $\varepsilon \in \mathcal{C}'$ is the co-unit of $\mathcal{C}$. So the moral of this story is that the generalized co-Toeplitz quantization with co-symbols in $\mathcal{C}'$ is a map from the algebra $\mathcal{C}'$ to the algebra $\mathcal{L}(\mathcal{P})$ of linear operators. Of course, we do not expect this map $C$ to be an algebra morphism. Rather, as we have remarked earlier in the Toeplitz setting, the discrepancy that $C$ has from being an algebra morphism is an indication of the ‘quantum-ness’ of the generalized co-Toeplitz quantization $C$.

A particular case of this multiplication occurs by taking $\alpha = e_g$ and $\beta = e_h$ for $g, h \in \mathcal{C}$. Then for all $\phi \in \mathcal{C}$ we get

$$(e_g e_h)(\phi) = (e_g \otimes e_h)(\phi^{(1)} \otimes \phi^{(2)})$$

$$= e_g(\phi^{(1)}) e_h(\phi^{(2)})$$

$$= \langle g, \phi^{(1)} \rangle e_h(\phi^{(2)}) = \langle g, \phi \rangle e_h(\phi) = e_g(\phi) e_h(\phi).$$

Furthermore, if $\phi$ is a group-like element (that is, $\Delta(\phi) = \phi \otimes \phi$), then this simplifies to

$$(e_g e_h)(\phi) = \langle g, \phi \rangle e_h(\phi) = e_g(\phi) e_h(\phi).$$

The family $\{e_g \mid g \in \mathcal{C}\}$ plays an important role in this theory.

**Definition 4.2** We define $e : \mathcal{C} \to \mathcal{C}'$ by $e(g) := e_g$ for all $g \in \mathcal{C}$.

Note that $e$ is an anti-linear map which need not be injective nor surjective. Moreover, the range of $e$ need not be a sub-algebra of $\mathcal{C}'$. However, we do have the following nice property.

**Theorem 4.1** Suppose the sesquilinear form is $\ast$-symmetric. Then the range $\text{Ran } e$ of $e$ is closed under the $\ast$-operation of $\mathcal{C}'$. More specifically, $(e_g^*) = e_g^*$ holds for all $g \in \mathcal{C}$, that is, $e$ is a $\ast$-morphism.

**Remark 4.1** The $\ast$-operation of $\mathcal{C}'$ is defined by $\lambda^*(g) := (\lambda(g^*))^\ast$ for $\lambda \in \mathcal{C}'$ and $g \in \mathcal{C}$.
Proof We calculate for $g, h \in \mathcal{C}$ that
\[(e^*_g)(h) = (e_g(h^*))^* = \langle g, h^* \rangle_{\mathcal{C}}^* = \langle g^*, h \rangle_{\mathcal{C}} = e^*_g(h),\]
where we used the $*$-symmetry in the third equality. This shows the second assertion of the theorem from which the first assertion follows directly. \qed

It is a quite general fact that the Toeplitz quantization map does not preserve multiplication, even though it is a map between algebras. The co-Toeplitz quantization map does not preserve co-multiplication ever, since it maps into a vector space with no natural co-multiplication even though its domain is a co-algebra. But the generalized co-Toeplitz quantization is a map from the algebra $\mathcal{C}'$ to the algebra $\mathcal{L}(\mathcal{P})$. And this map sends the identity element $\varepsilon$ of $\mathcal{C}'$ to the identity element $I_{\mathcal{P}}$ of $\mathcal{L}(\mathcal{P})$ when (4.3) holds. But what is the relation of the generalized co-Toeplitz quantization map with the multiplication? The next result may come as a surprise.

**Theorem 4.2** Suppose that a co-Toeplitz quantization satisfies:

- The left co-action $\beta$ is given by (4.3).
- $\mathcal{P}$ is a sub-co-algebra of $\mathcal{C}$, that is, $\Delta_{\mathcal{P}} = \Delta_{\mathcal{C}} \restriction_{\mathcal{P}}$.

Then the generalized co-Toeplitz quantization map $C : \mathcal{C}' \to \mathcal{L}(\mathcal{P})$ is an algebra morphism.

**Remark 4.2** This tells us that under the given hypotheses the generalized co-Toeplitz quantization map is just too nice. For physical reasons we want to have a quantization that is not quite so nice. After all, Dirac has taught us that the distinguishing characteristic of quantum theory is that the observables do not commute. In this more general context Dirac’s insight can be extended to say that the range of a quantization mapping should be less commutative that its domain. So, in the favorable case when $C$ is injective, we do not want $C$ to be an algebra morphism. Therefore, I consider this to be a No Go theorem. Now the hypothesis on $\beta$ seems reasonable, since it is the dual of the commonly used condition on the left action in the Toeplitz setting. But the second hypothesis is dual to assuming in the Toeplitz setting that the projection $P : \mathcal{A} \to \mathcal{P}$ is an algebra morphism. And that is a condition which we do not wish to impose. Hence, the second hypothesis is something which we want to not hold in examples and in the future development of this theory. That hypothesis does not hold in the example in Section 9. Of course, a No Go theorem is a theorem and is worth knowing.

**Proof** Take $\lambda, \mu \in \mathcal{C}'$ and $\phi \in \mathcal{P}$. Throughout the proof we use the notation $\Delta := \Delta_{\mathcal{P}} = \Delta_{\mathcal{C}} \restriction_{\mathcal{P}}$, which comes from the second hypothesis. We also use the iterated Sweedler notation as explained, for example, in [12]. Then we calculate as follows.
\[
C_{\lambda} C_{\mu} \phi = (id \otimes \lambda)(Q \otimes id) \Delta (id \otimes \mu)(Q \otimes id) \Delta \phi
= (id \otimes \lambda)(Q \otimes id) \Delta (id \otimes \mu)(Q\phi^{(1)} \otimes \phi^{(2)})
= \mu(\phi^{(2)})(id \otimes \lambda)(Q \otimes id) \Delta \phi^{(1)}
\]
\[
\begin{align*}
&= \mu(\phi^{(2)})(id \otimes \lambda)(Q \otimes id)(\phi^{(11)} \otimes \phi^{(12)}) \\
&= \mu(\phi^{(2)}) \lambda(\phi^{(12)}) \phi^{(11)} \\
&= \mu(\phi^{(22)}) \lambda(\phi^{(21)}) \phi^{(1)} \\
&= ((\lambda \otimes \mu) \Delta \phi^{(2)}) \phi^{(1)} \\
&= (\lambda \mu) \phi^{(2)} Q \phi^{(1)} \\
&= (id \otimes \lambda \mu)(Q \otimes id) \Delta \phi \\
&= C_{\lambda \mu} \phi.
\end{align*}
\]

Here we used \(\Delta \phi = \phi^{(1)} \otimes \phi^{(2)} \in \mathcal{P} \otimes \mathcal{P}\), the fact that \(Q\) acts as the identity on \(\mathcal{P}\), the co-associativity of \(\Delta\), the definition of the product \(\lambda \mu\) and the definition of the co-Toeplitz quantization mapping \(C\). The first hypothesis was used in the first and last equalities.

The question naturally arises whether the generalized co-Toeplitz quantization map is injective. Using Definition 4.2 and Eq. (4.2), we see that a necessary condition for this injectivity is that \(e: C \to C'\) is injective, which itself is equivalent to the sesquilinear form on \(C\) being non-degenerate.

The extension of the co-Toeplitz quantization from the domain of symbols to the domain of co-symbols leads one to wonder if there is a corresponding extension of the domain of the Toeplitz quantization. Now the symbol \(g \in \mathcal{A}\) in the Toeplitz setting was used there to define a linear map \(l_g: C \to \mathcal{A}\) (cp. (2.4)). And this map \(l_g\) was all that we needed to define the Toeplitz operator with symbol \(g\). However, the generalization given by replacing \(l_g\) with an arbitrary linear map \(l: C \to \mathcal{A}\) is no generalization at all because, as noted earlier, any such map \(l\) is equal to \(l_g\) for a unique symbol \(g \in \mathcal{A}\). So the co-Toeplitz quantization shows a bit of flexibility, let’s say, that is not present in the Toeplitz quantization. This is an indication of a lack of symmetry between the Toeplitz and co-Toeplitz quantizations, a topic that we will consider in more detail in the next section.

### 5 Duality

We now discuss in detail in what sense the theories of Toeplitz and co-Toeplitz quantization are duals of each other. The duality behind the definition of co-Toeplitz operators comes about simply by reversing the direction of all the arrows (i.e., morphisms) in the definition of a Toeplitz operator. This sort of duality comes from category theory and is seen in the formulation of the basic concepts of non-commutative geometry, for example. It is called notion duality. This is exactly what we see in the relation between the definitions (2.2) and (3.2) of Toeplitz operators and of co-Toeplitz operators, respectively.

However, another sort of duality (called object duality) arises from applying the duality contravariant functor \(V \mapsto V' \equiv \text{Hom}_{\text{Vect}}(V, \mathbb{C})\) for \(V\) a complex vector space and the corresponding pull-back definition \(T \mapsto T': W' \to V'\) for a morphism (i.e., linear map) \(T: V \to W\) of vector spaces \(V\) and \(W\). Specifically, \(T'(\lambda) := \)
\( \lambda \circ T \in V' \) for \( \lambda \in W' \). So the question arises as to what happens to (2.2) and (3.2) when we apply this duality contravariant functor to each of them. Of course, we do get some operator. The question is what type of operator it is and whether it has a simple formula.

One nice property is that a \(*\)-operation on \( V \) induces a \(*\)-operation on \( V' \) defined by \( \lambda^*(v) := (\lambda(v^*))^* \) for \( \lambda \in V' \) and \( v \in V \). Let us also recall from the last section that the dual \( C' \) of a co-algebra \( C \) is always an algebra. On the other hand, the dual \( \mathcal{A}' \) of an algebra \( \mathcal{A} \) is not necessarily a co-algebra. Briefly, the point is that in general the duality contravariant functor is only sub-multiplicative with respect to the tensor product, namely, \( V' \otimes W' \subset (V \otimes W)' \). However, if either \( V \) or \( W \) is finite dimensional, then the duality contravariant functor is multiplicative, \( V' \otimes W' = (V \otimes W)' \). To get multiplicativity in the full infinite dimensional setting requires changing either the definition of the duality contravariant functor or the definition of the tensor product (or of both). See [6] for more details. A rather similar analysis, which we leave to the interested reader, shows that the dual of a co-action is always an action, while the dual of an action is not necessarily a co-action.

But the dual of a vector space with a sesquilinear form does not in general have a naturally defined sesquilinear form. So, we will not look for a full duality between Toeplitz and co-Toeplitz operators using this duality contravariant functor. Thus, we will mainly consider the duality relation between the diagrams (2.2) and (3.2) considered as diagrams of vector spaces which are the definitions of Toeplitz and co-Toeplitz operators, respectively. Similarly, we take (4.1) to be the diagram of vector spaces which defines a generalized co-Toeplitz operator. But we will comment on other algebraic aspects of this duality contravariant functor as they arise in specific contexts. Given this situation, it seems more feasible for us to first consider the dual of a co-Toeplitz operator as defined in (3.2) with symbol \( g \in C \), a co-algebra, which gives us this diagram dual to (3.2):

\[
\begin{array}{ccc}
P' & \xrightarrow{\pi'_g} & (P \otimes C)' \\
& \xrightarrow{\beta'} & C' \\
& \xrightarrow{j'} & P',
\end{array}
\]

To understand this diagram we evaluate \( \pi'_g \). So for \( \lambda \in P' \), \( \phi \in P \) and \( f, g \in C \) we have

\[
\pi'_g(\lambda)(\phi \otimes f) = (\lambda \circ \pi_g)(\phi \otimes f) = \lambda(\langle g, f \rangle_C \phi) = \langle g, f \rangle_C \lambda(\phi) = e_g(f)\lambda(\phi) = (\lambda \otimes e_g)(\phi \otimes f),
\]

which implies that \( \pi'_g(\lambda) = \lambda \otimes e_g \in P' \otimes C' \) and hence

\[
\pi'_g = \cdot \otimes e_g : P' \to P' \otimes C' \subset (P \otimes C)'.
\]

Then (5.1) becomes

\[
\begin{array}{ccc}
P' & \xrightarrow{\otimes e_g} & P' \otimes C' \\
& \xrightarrow{\beta'} & C' \\
& \xrightarrow{j'} & P',
\end{array}
\]

\( \otimes \) Birkhäuser
This is a Toeplitz operator as defined by \((2.2)\) with symbol \(e_g\) in the algebra \(C'\). Moreover, \(\beta'\) is a left action and \(j'\) is a projection. Also \(Q' : \mathcal{P}' \to \mathcal{C}'\) is a unital algebra morphism. We have the following.

**Theorem 5.1** If \(C_g \in \mathcal{L}(\mathcal{P})\) is a co-Toeplitz operator with symbol \(g\) in the co-algebra \(\mathcal{C}\), then \((C_g)' = T_{e_g} \in \mathcal{L}(\mathcal{P}')\) is a Toeplitz operator with symbol \(e_g\) in the algebra \(C'\).

If \(C_\mu \in \mathcal{L}(\mathcal{P})\) is a generalized co-Toeplitz operator with co-symbol \(\mu\) in the algebra \(\mathcal{C}'\), then \((C_\mu)' = T_\mu \in \mathcal{L}(\mathcal{P}')\) is a Toeplitz operator with symbol \(\mu\) in the algebra \(\mathcal{C}'\).

**Remark 5.1** We can also write the result of the first part as \((C_{e_g})' = T_{e_g}\).

**Proof** We have already proved the first assertion above. As for the second assertion we note that in the above argument the symbol \(g\) is used to define the linear functional \(e_g \in C'\), which is the only occurrence of \(g\) in \((5.2)\). So we replace \(e_g\) with the co-symbol \(\mu\) in that argument to obtain \((\cdot \otimes \mu) : \mathcal{P}' \to \mathcal{P}' \otimes \mathcal{C}'\) in \((5.2)\), and the second result follows immediately.

On the other hand, the dual of a Toeplitz operator is not necessarily a co-Toeplitz operator. To see this we examine the dual of diagram \((2.2)\), which is

\[
\mathcal{P}' \leftarrow (\mathcal{P} \otimes \mathcal{A})' \xleftarrow{\alpha'} \mathcal{A}' \xleftarrow{\beta'} \mathcal{P}'.
\]

(5.3)

Here neither \(\mathcal{A}'\) nor \(\mathcal{P}'\) need be a co-algebra although each does have a \(*\)-operation. Consequently, it need not make sense in general to require \(P'\) to be a co-algebra morphism. Recall that ‘co-Toeplitz operator’ (resp., ‘generalized co-Toeplitz operator’) now means the composition of the maps of vector spaces in diagram \((3.2)\) (resp., diagram \((4.1)\)).

Even if \(\mathcal{P}'\) is a co-algebra, \(\alpha'\) need not be a left co-action on \(\mathcal{A}'\), since

\[
\mathcal{P}' \otimes \mathcal{A}' \subset (\mathcal{P} \otimes \mathcal{A})'
\]

can be a proper inclusion. But we do have the following result.

**Theorem 5.2** If \(T_g \in \mathcal{L}(\mathcal{P})\) is a Toeplitz operator with symbol \(g\) in the algebra \(\mathcal{A}\) and the left action \(\alpha : \mathcal{P} \otimes \mathcal{A} \to \mathcal{A}\) (used to define the Toeplitz operator) satisfies \(\text{Ran} \; \alpha' \subset \mathcal{P}' \otimes \mathcal{A}'\), then

\[
(T_g)' = C_{\text{ev}_g} \in \mathcal{L}(\mathcal{P}')
\]

is a generalized co-Toeplitz operator with co-symbol \(\text{ev}_g \in \mathcal{A}''\). (We will define \(\text{ev}_g\) in the course of the proof.)

**Proof** We take \(g \in \mathcal{A}, \phi \in \mathcal{P}, \lambda \in \mathcal{P}'\) and \(\omega \in \mathcal{A}'\). Then we calculate

\[
((\cdot \otimes g)'(\lambda \otimes \omega))(\phi) = (\lambda \otimes \omega)((\cdot \otimes g)(\phi)) = (\lambda \otimes \omega)(\phi \otimes g)
= \lambda(\phi)\omega(g) = (\omega(g)\lambda)(\phi),
\]

\(\Box\) Birkhäuser
which implies $(· \otimes g)'(\lambda \otimes \omega) = \omega(g) \lambda = (id \otimes ev_g)(\lambda \otimes \omega)$, where $ev_g(\omega) := \omega(g)$ defines the evaluation functional $ev_g$ at $g$. Let’s note that $ev_g \in \mathcal{A}''$ does hold. Therefore we have arrived at

$$(· \otimes g)' = id \otimes ev_g.$$ 

So (5.3) becomes

$$\mathcal{P}' \xleftarrow{id \otimes ev_g} \mathcal{P}' \otimes \mathcal{A}' \xleftarrow{ev} \mathcal{A}' \xleftarrow{P'} \mathcal{P},$$

where we also used the hypothesis on the range of $\alpha'$. And so we have shown that $(T_g)'$ is the generalized co-Toeplitz operator $C_{ev_g}$.

These two theorems show an asymmetry in this duality, namely, the dual of a co-Toeplitz operator is always a Toeplitz operator while for a Toeplitz operator we used an extra technical hypothesis in order to show that its dual is a co-Toeplitz operator. Of course, this opens the door to the possibility of altering the definition of Toeplitz operator (and maybe of co-Toeplitz operator as well) in the infinite dimensional case in order to obtain a more precise duality.

We are now in a position to evaluate the double duals of Toeplitz and co-Toeplitz operators. It is an elementary fact that the double dual always exists. What we want to do is describe it explicitly. Here is some well known material that we are going to use in order to study double duals.

**Definition 5.1** Suppose that $V$ is a vector space and $v \in V$. Then we define $ev_v^V \in V''$, the evaluation at $v$, by

$$ev_v^V(f) := f(v)$$

for all $f \in V'$. We also define the evaluation map

$$ev \equiv ev^V : V \to V''$$

by $ev^V(v) := ev_v^V$ for all $v \in V$. We sometimes write $ev$ instead of $ev^V$ when the context indicates what the vector space $V$ is.

We state the next elementary result without proof.

**Proposition 5.1** The map $ev^V$ is linear and injective. For any linear map $T : V \to W$ between vector spaces $V$ and $W$ we have that this diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{ev^V} & V'' \\
T \downarrow & & \downarrow T'' \\
W & \xleftarrow{ev_W} & W''
\end{array}
$$

Using $ev^V$ to identify $V$ as a subspace of $V''$ (and similarly for $W$), we can read this diagram as saying that the restriction of $T''$ to the subspace $V$ is $T$, that is, $T''|_V = T$. Equivalently, $T''$ can be viewed as an extension of $T$. 

\(\copyright\) Birkhäuser
We now proceed to the theorem about double duals.

**Theorem 5.3** There are three cases of a double dual.

- Let \( g \in \mathcal{C} \) be a symbol and \( C_g \in \mathcal{L}(\mathcal{P}) \) be its associated co-Toeplitz operator. If the map \( \beta \) used in defining \( C_g \) satisfies \( \text{Ran} \beta'' \subset \mathcal{P}'' \otimes \mathcal{C}'' \), then \( (C_g)'' = C_{\text{ev}_g} \in \mathcal{L}(\mathcal{P}'') \).

- Let \( \mu \in \mathcal{C}' \) be a co-symbol and \( C_\mu \in \mathcal{L}(\mathcal{P}) \) be its associated generalized co-Toeplitz operator. If \( \beta \) satisfies the condition in the previous part of this theorem, then \( (C_\mu)'' = C_{\text{ev}_\mu} \in \mathcal{L}(\mathcal{P}'') \).

- Let \( g \in \mathcal{A} \) be a symbol and \( T_g \in \mathcal{L}(\mathcal{P}) \) be its associated Toeplitz operator. Suppose that the left action \( \alpha \) used in the definition of \( T_g \) satisfies the technical condition in Theorem 5.2. Then \( (T_g)'' = T_{\text{ev}_g} \in \mathcal{L}(\mathcal{P}'') \).

**Remark 5.2** By Proposition 5.1 in each of these three cases the double dual of the initially given operator is necessarily an extension of that operator. The question is whether the double dual of a Toeplitz (resp., co-Toeplitz) operator is again a Toeplitz (resp., co-Toeplitz) operator and, if so, what is the formula for the double dual. This theorem answers that question provided a specific technical condition holds.

**Proof** By Theorem 5.1 we have \( (C_g)' = T_{\text{ev}_g} \) for \( g \in \mathcal{C} \). Taking the dual of this using Theorem 5.2 gives

\[
(C_g)'' = (T_{\text{ev}_g})' = C_{\text{ev}(\text{ev}_g)}
\]

using the hypothesis on \( \beta \). This shows the first part of the theorem.

For the second part we have from Theorem 5.1 that \( (C_\mu)' = T_\mu \) for a co-symbol \( \mu \) in the algebra \( \mathcal{C}' \). Then by Theorem 5.2 we obtain

\[
(C_\mu)'' = (T_\mu)' = C_{\text{ev}_\mu}
\]

where we again use the same hypothesis on \( \beta \).

For the last part from Theorem 5.2 we have \( (T_g)' = C_{\text{ev}_g} \), using the hypothesis on \( \alpha \). Then applying Theorem 5.1 we immediately get

\[
(T_g)'' = (C_{\text{ev}_g})' = T_{\text{ev}_g}.
\]

This concludes the proof. \( \Box \)

A consequence of this section is that the dual of a co-Toeplitz operator is a Toeplitz operator and has a relatively simple formula. However, the corresponding result for the dual of a Toeplitz operator required an extra hypothesis. So this is an asymmetry in this duality. Another question is whether every Toeplitz (resp., co-Toeplitz) operator is the dual of a co-Toeplitz (resp., Toeplitz) operator. This question remains as an open problem.
6 Adjoinst

We next examine the relation between the operator adjoint \((C_g)^*\) of a co-Toeplitz operator \(C_g\) with symbol \(g \in \mathbb{C}\) and the co-Toeplitz operator \(C_g^\star\). Since \(C_g : \mathcal{P} \rightarrow \mathcal{P}\) and the vector space \(\mathcal{P}\) does not in general have a \(\star\)-operation on it, there should be no confusion with the adjoint notation \((C_g)^*\) and the previously defined \(\star\)-operation of an operator that maps between vector spaces with a \(\star\)-operation.

As one would expect, to get a result we need to assume some sort of a relation between the inner product on the pre-Hilbert space \(\mathcal{P}\), used to define \((C_g)^*\), and the \(\star\)-operation in the symbol space, used to define \(C_g^\star\). In the Toeplitz case the relation needed is easily seen to be

\[
\langle Mg^\star \phi, \psi \rangle_{\mathcal{P}} = \langle \phi, Mg \psi \rangle_{\mathcal{P}} \quad \text{or} \quad \langle \phi g^\star, \psi \rangle_{\mathcal{A}} = \langle \phi, g \psi \rangle_{\mathcal{A}} \quad (6.1)
\]

for \(\phi, \psi \in \mathcal{P}\) and \(g \in \mathcal{A}\). This translates directly into \(T_g^\star \subset (T_g)^*\), an inclusion of densely defined operators acting in \(\mathcal{H}\). For more details, including examples, see [13].

For the co-Toeplitz case with symbol \(g \in \mathcal{C}\), a co-algebra, we do two straightforward calculations using the formula \(C_g = \tilde{M}_g j\). In the following we take \(\phi, \psi \in \mathcal{P}\) and \(g \in \mathcal{C}\). First we have

\[
\langle \phi, C_g \psi \rangle_{\mathcal{P}} = \langle \phi, (\tilde{M}_g j) \psi \rangle_{\mathcal{P}} = \langle \phi, \tilde{M}_g \psi \rangle_{\mathcal{P}}.
\]

On the other hand we get

\[
\langle C_g^\star \phi, \psi \rangle_{\mathcal{P}} = \langle (\tilde{M}_g^\star j) \phi, \psi \rangle_{\mathcal{P}} = \langle \tilde{M}_g^\star \phi, \psi \rangle_{\mathcal{P}}.
\]

So the condition we impose now and for the rest of this paper is

\[
\langle \tilde{M}_g^\star \phi, \psi \rangle_{\mathcal{P}} = \langle \phi, \tilde{M}_g \psi \rangle_{\mathcal{P}} \quad (6.2)
\]

for all \(\phi, \psi \in \mathcal{P}\) and \(g \in \mathcal{C}\). We have shown the next result.

**Theorem 6.1** Assume (6.2) holds. Then we have this inclusion of densely defined operators acting in \(\mathcal{H}\):

\[
C_g^\star \subset (C_g)^* \quad (6.3)
\]

In particular, the adjoint of \(C_g\) restricted to \(\mathcal{P}\) is exactly \(C_g^\star\).

So far the argument closely follows the Toeplitz case. Replacing \(g\) with \(g^\star\) in (6.3) we obtain \(C_g \subset (C_g^\star)^*\), which implies by functional analysis that \(C_g\) is a closable operator. Also, for \(g\) real, that is \(g^\star = g\), we see directly from (6.3) that \(C_g\) is a symmetric operator, in which case it then becomes relevant to analyze its self-adjoint extensions, if such extensions exist. In particular, it would be interesting to know if \(C_g\) is essentially self-adjoint under certain hypotheses.

The condition (6.2) can be expanded out in various special cases. We use the special case for \(\beta\) given in (3.1) and the definition of \(\pi_g\) in (3.3). In the following calculations...
we take \( \phi, \psi \in \mathcal{P} \) and \( g \in \mathcal{C} \). So, on the one hand we have

\[
\langle \phi, \tilde{M}_g \psi \rangle_{P} = \langle \phi, \pi g \beta \psi \rangle_{P} \\
= \langle \phi, \pi g (Q \otimes \text{id}) \Delta_C \psi \rangle_{P} \\
= \langle \phi, \pi g (Q \psi^{(1)} \otimes \psi^{(2)}) \rangle_{P} \\
= \langle \phi, (g, \psi^{(2)})_C Q \psi^{(1)} \rangle_{P} \\
= \langle g, \psi^{(2)} \rangle_C \langle \phi, Q \psi^{(1)} \rangle_{P}.
\] (6.4)

On the other hand, using this result (6.4), we see that

\[
\langle \tilde{M}_g^* \phi, \psi \rangle_{P} = \langle \psi, \tilde{M}_g^* \phi \rangle^*_{P} \\
= (\langle g^*, \phi^{(2)} \rangle_C \langle \psi, Q \phi^{(1)} \rangle_{P})^* \\
= \langle \phi^{(2)}, g^* \rangle_C \langle Q \phi^{(1)}, \psi \rangle_{P}.
\]

So we have obtained the following result.

**Theorem 6.2** With the above choices for \( \beta \) and \( \pi g \) we get that the symmetry condition (6.2) is equivalent to

\[
\langle g, \psi^{(2)} \rangle_C \langle \phi, Q \psi^{(1)} \rangle_{P} = \langle \phi^{(2)}, g^* \rangle_C \langle Q \phi^{(1)}, \psi \rangle_{P}
\]

for all \( \phi, \psi \in \mathcal{P} \) and \( g \in \mathcal{C} \).

The condition in this theorem does not seem to be the dual of the condition (6.1) in the Toeplitz setting, although it actually is.

### 7 Creation and annihilation operators

We now come back to one of the most important aspects of this theory. First, we give the basic definition.

**Definition 7.1** Let \( g \in \mathcal{P}^\circ \) (or, equivalently, \( g^* \in \mathcal{P} \)) be given. Then we define

\[
A^\dagger(g) := C_g \in \mathcal{L}(\mathcal{P}),
\]

the creation operator (associated to the anti-holomorphic symbol \( g \)).

Let \( g \in \mathcal{P} \) be given. Then we define

\[
A(g) := C_g \in \mathcal{L}(\mathcal{P}),
\]

the annihilation operator (associated to the holomorphic symbol \( g \)).

**Remark 7.1** One way to extend this definition to include the generalized co-Toeplitz operators is to extend to the co-symbols the definitions of holomorphic and anti-holomorphic elements. We leave this topic for future investigation. We also bring to the

\( \text{Birkhäuser} \)
reader’s attention that in the Toeplitz setting the holomorphic (resp., anti-holomorphic) symbols give the creation (resp., annihilation) operators. These relations are inverted in the co-Toeplitz setting. The motivation for this reversal comes from the example in Sect. 9.

These definitions are originally motivated by the definitions in Segal-Bargmann analysis and its generalizations. See Bargmann’s paper [1] where creation and annihilation operators were realized for the first time as adjoints of each other, which is basically the case here when (6.2) holds. In this formulation the annihilation operators could have been defined without a ∗-structure, while the creation operators use explicitly the ∗-structure. This is just a consequence of using P as the pre-Hilbert space. If the sesquilinear form is ∗-symmetric (see (3.4)), then P∗ is a pre-Hilbert space with inner product given by restricting the sesquilinear form ⟨·, ·⟩C to P∗. This is so, since for all f, g ∈ P∗ the identity (3.4) implies

\[ \langle f, g \rangle_{P^*} = \langle f, g \rangle_C = \langle g^*, f^* \rangle_{P^*} = \langle g, f \rangle_P, \]

which shows that we do get a positive definite inner product on P∗. Then the completion of the pre-Hilbert space P∗ is denoted as H∗. We can think of these as the space of anti-holomorphic polynomials P∗ and as the anti-holomorphic Segal-Bargmann space H∗. The identity (7.1) can be re-written as

\[ \langle f, g \rangle_{P} = \langle g^*, f^* \rangle_{P^*} \]

which says that the anti-linear bijective map VP : P → P∗ given by VP f := f∗ is anti-unitary. Also, V−1 P = VP∗. Therefore, we next define the co-Toeplitz operator Ĉg ∈ L(P∗) for g ∈ C by Ĉg := VP Cg V−1 P. This gives us essentially the same set-up as we had above, except now with the co-Toeplitz operators acting in a dense subspace of an anti-holomorphic Hilbert space. In this new set-up an annihilation operator is defined as Ĉg for g ∈ P∗, that is, the conjugation of a creation operator acting in the holomorphic Hilbert space H. Similarly, we define a creation operator acting in the anti-holomorphic Hilbert space as Ĉg for g ∈ P, the conjugation by V of an annihilation operator acting in the holomorphic Hilbert space.

Some related structures are defined next.

**Definition 7.2** The unital sub-algebra of L(P) generated by all of the creation and annihilation operators is called the **canonical commutation relations (CCR) algebra** and is denoted as CCR.

The unital sub-algebra of L(P) generated by all of the co-Toeplitz operators with symbols in C is called the **co-Toeplitz algebra**.

Finally, the unital sub-algebra of L(P) generated by all of the generalized co-Toeplitz operators with co-symbols in C′ is called the **generalized co-Toeplitz algebra**.

Creation and annihilation operators have a multitude of applications in physics. The CCR algebra also arises in many parts of physics. However, the newly introduced co-Toeplitz algebra and the generalized co-Toeplitz algebra are objects that are of more interest in the area of operator theory in mathematics. While all of these algebras have
their importance, it seems that very little can be said about them in general. However, they all can be studied in specific examples of this theory.

8 Canonical commutation relations

The algebra $\mathcal{CCR}$ defined here can be studied in much the same way as the canonical commutation algebra is studied in [13] in the Toeplitz setting. The upshot is that Planck’s constant $\hbar$ and $\hbar$-deformed CCR algebras will be introduced into the theory as well as a dequantized (or classical) algebra. To make this paper more self-contained we review how the relevant material of [13] applies in the co-Toeplitz setting.

Note that we have already defined the algebra $\mathcal{CCR}$. It still remains to define the canonical commutation relations themselves. In physics one usually defines the algebra of canonical commutation relations by explicitly using generators and their relations, where these relations are by very definition the canonical commutation relations. In this setting we do the opposite by starting with $\mathcal{CCR}$, then writing it as the quotient of a free algebra $F$ and next identifying the kernel of the quotient map $p : F \to \mathcal{CCR}$ as the ideal of canonical commutation relations. Finally, any minimal set of generators of this ideal serves as canonical commutation relations associated to $\mathcal{CCR}$.

To achieve this we define $F$ to be the free unital algebra generated by the abstract set $F = \{G_f \mid f \in P \cup P^* \subset \mathbb{C}\}$ in bijective correspondence with the set $P \cup P^*$. The unital algebra morphism $p : F \to \mathcal{CCR}$ is then defined on the algebra generators $G_f$ of $F$ by $p(G_f) := C_f$ for all $f \in P \cup P^*$. By the universal property of the free algebra $F$ this uniquely defines the unital algebra morphism $p$. And since by definition the elements $C_f$ for $f \in P \cup P^*$ generate $\mathcal{CCR}$ as a unital algebra, we see that $p$ is surjective.

**Definition 8.1** We define the ideal of the canonical commutation relations (CCR) of the co-Toeplitz quantization $C$ to be $\mathcal{R} := \ker p$.

A set of canonical commutation relations (CCR) of the co-Toeplitz quantization $C$ is defined to be any minimal subset of ideal generators of the two-sided ideal $\mathcal{R}$.

Notice that not only is a set of canonical commutation relations not unique in general, even its cardinality in general will not be uniquely determined by the given co-Toeplitz quantization.

The free algebra $F$ has a natural grading $\text{deg}(G_{f_1} \cdots G_{f_n}) := n$ for integer $n \geq 1$ and $f_1, \ldots, f_n \in P \cup P^*$. We also put $\text{deg}(1) := 0$, where $1 \in F$ is the identity element. This leads to an important definition.

**Definition 8.2** A homogeneous element with respect to this grading in $\mathcal{R}$ is called a classical relation, and a non-homogeneous element in $\mathcal{R}$ is called a quantum relation.

The motivation for the previous definition is given in [13]. While this definition applies to any element in $\mathcal{R}$, its main intent is to divide the elements in a set of CCR into two disjoint subsets.

It turns out that a logically possible, though physically anomalous, situation happens when $\mathcal{R} = \ker p = 0$, in which case $p$ is an algebra isomorphism and the (unique!) set of CCRs is empty. In this strange case the quantization is over-quantized in the sense
that there are no pairs \( f_1 \neq f_2 \in \mathcal{P} \cup \mathcal{P}^* \) with the classical (or trivial) commutation relation \( C_{f_1} C_{f_2} - C_{f_2} C_{f_1} = 0 \), and then, as we will see momentarily, we can not introduce Planck’s constant \( \hbar \) into the theory. Also, despite Dirac’s insistence on the importance of non-commuting observables, some non-trivial and useful classical commutation relations are always present in quantum theory.

The next definition is also motivated in the discussion in [13].

**Definition 8.3** Let \( R \in \mathcal{B} \) be a non-zero relation. Then we write \( R \) uniquely as

\[
R = R_0 + R_1 + \cdots + R_n, \tag{8.1}
\]

where \( R_i \) is homogeneous with \( \deg R_i = i \) (for all \( i = 0, 1, \ldots, n \)) and \( R_n \neq 0 \).

Then we say that \( R_n \) is the classical relation associated to \( R \).

Note that \( R_n \) is indeed a non-zero classical relation. Based on what is true in the Toeplitz setting as is presented in [13], I conjecture that both of the cases \( R_n \in \mathcal{B} \) and \( R_n \notin \mathcal{B} \) can occur. The intuition here is that the terms \( R_0, R_1, \ldots, R_{n-1} \) are ‘quantum corrections’ to the classical relation \( R_n \). To see what that means let us now define the \( \hbar \)-deformation of a non-zero relation \( R \in \mathcal{B} \) to be

\[
R(\hbar) := \hbar^{n/2} R_0 + \hbar^{(n-1)/2} R_1 + \cdots + \hbar^{1/2} R_{n-1} + R_n, \tag{8.2}
\]

where \( \hbar^{1/2} \in \mathbb{C} \) is arbitrary, \( \hbar = (\hbar^{1/2})^2 \) and \( R \) is given as in (8.1). Note that \( R(0) = R_n \). This says that the classical case \( \hbar = 0 \) gives us the classical relation associated to \( R \).

In physics we take \( \hbar^{1/2} > 0 \), but for now there is no need to impose that restriction. We use these definitions to define some more two-sided ideals in \( \mathcal{F} \) and their associated quotient algebras.

**Definition 8.4** Let \( \mathcal{B}_{cl} \) denote the two-sided ideal in \( \mathcal{F} \) generated by all the classical relations arising from \( \mathcal{B} \) and with degree \( \geq 1 \).

The dequantized (or classical) algebra of the co-Toeplitz quantization is defined as:

\[
\mathcal{A}_{cl} = DQ := \mathcal{F} / \mathcal{B}_{cl}.
\]

Let \( \mathcal{B}_\hbar \) denote the two-sided ideal in \( \mathcal{F} \) generated by all the relations \( R(\hbar) \) using (8.1) and (8.2) with \( n \geq 1 \).

Then the \( \hbar \)-deformed CCR algebra associated with the co-Toeplitz quantization is defined as:

\[
CCR_\hbar := \mathcal{F} / \mathcal{B}_\hbar.
\]

By the above remarks we see that \( DQ = CCR_0 \). Also, we have \( CCR_\hbar = CCR_1 \). There seems to be no reason why the dequantized (or classical) algebra \( DQ \) should be commutative, and so I conjecture that there are examples where it is not.

The algebras \( CCR_\hbar \) may have limiting properties as \( \hbar > 0 \) tends to zero. These would be the semi-classical properties of the co-Toeplitz quantization. And properties
of the algebra $DQ$ would be the classical properties of the co-Toeplitz quantization. In short, this gives us a framework for analyzing semi-classical as well as classical aspects of this theory. However, it seems difficult to delve into all this in greater detail at the present abstract level, though these considerations can be brought to bear on specific examples. The reader can consult [13] for more details, including motivation, for the topics of this section.

Let me emphasize that the approach here is the opposite of the usual approach in mathematical physics, where one takes certain interesting commutation relations to be the given CCRs, and then representations of those same commutation relations are realized by operators acting in some Hilbert space, often a Fock space of some sort. This more usual approach is found in the recent paper [4] and many of the papers in its list of references. Here, on the other hand, we start with a Hilbert space and then define the creation and annihilations operators acting in it. Only after this do we finally arrive at a definition of the CCRs.

9 An example: $SU_q(2)$

This general theory of co-Toeplitz quantization should be fleshed out with specific examples. We now proceed with such an example.

We let $\mathcal{C} = SU_q(2)$ for $0 \neq q \in \mathbb{R}$. To avoid technicalities we assume as well that $q \neq -1$. Then $SU_q(2)$ is a Hopf $\ast$-algebra, and so in particular it is a $\ast$-co-algebra. We first review some of the well-known facts concerning the quantum group $SU_q(2)$. For these and many more details see [6].

$SU_q(2)$ can be defined as the universal $\ast$-algebra with the identity element 1 generated by elements $a$ and $c$ satisfying these relations:

\[
ac = qa \ c, \quad ac^* = q^* c a, \quad cc^* = c \ c^*,
\]
\[
a^* a + c^* c = 1, \quad a a^* + q^2 c^* c = 1. \quad (9.1)
\]

The co-multiplication $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ of this co-algebra is the unique $\ast$-algebra morphism determined by

\[
\Delta_{\mathcal{C}}(a) = a \otimes a - q \ c^* \otimes c,
\]
\[
\Delta_{\mathcal{C}}(c) = c \otimes a + a^* \otimes c.
\]

The co-unit $\varepsilon : \mathcal{C} \to \mathbb{C}$ is the unique $\ast$-algebra morphism determined by

$$\varepsilon(a) = 1 \quad \text{and} \quad \varepsilon(c) = 0.$$ 

Even though only the $\ast$-co-algebra structure of $SU_q(2)$ will be used, for completeness we also note that the antipode, denoted by $S$, is the unique unit preserving, anti-multiplicative algebra morphism (but not $\ast$-morphism) determined by

\[
S(a) = a^*, \quad S(a^*) = a, \quad S(c) = -qc, \quad S(c^*) = -q^{-1} c^*.
\]
While $SU_q(2)$ is generated by two elements as a $*$-algebra, it is an infinite dimensional vector space. A Hamel basis of $SU_q(2)$ is given by $\{\varepsilon_{k lm} | k \in \mathbb{Z} \text{ and } l, m \in \mathbb{N}\}$, where

$$\varepsilon_{k lm} = a^k c^l (c^*)^m \quad \text{if } k \geq 0,$$

$$\varepsilon_{k lm} = (a^*)^{-k} c^l (c^*)^m \quad \text{if } k < 0.$$  

We define a sesquilinear form on $C = SU_q(2)$ by requiring

$$\langle \varepsilon_{k lm}, \varepsilon_{r st} \rangle_C = w(k, l - m) \delta_{k, r} \delta_{l - m, s - t} \quad (9.2)$$

and then extending anti-linearly in the first entry and linearly in the second entry. Here $w : \mathbb{Z} \times \mathbb{Z} \to (0, \infty)$ is some strictly positive weight function, and $\delta_{i, j}$ is the Kronecker delta function for $i, j \in \mathbb{Z}$. See [9] for motivation for how such a formula is related with the inner product defined in the holomorphic Hilbert space in Bargmann’s paper [1].

While [1] was the original motivation for (9.2), there is another way of understanding this, which we now sketch. See [6] for more details and background. It turns out that there is an algebraic direct sum decomposition

$$C = \bigoplus_{(m, n)} A[m, n], \quad (9.3)$$

where the direct sum is over $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. This is defined in terms of two co-actions on $C$ of the diagonal quantum group $\mathcal{H} = \mathbb{C}[t, t^{-1}]$, the algebra of Laurent polynomials in the variable $t$. One realizes $\mathcal{H}$ (which actually is a Hopf $*$-algebra) as a quantum subgroup of $C$ via the surjection $\pi : C \to \mathcal{H}$ which is defined to be the algebra morphism determined by $\pi(a) = t$, $\pi(a^*) = t^{-1}$ and $\pi(c) = \pi(c^*) = 0$. Then a left co-action $L_{\mathcal{H}}$ of $\mathcal{H}$ on $C$ is defined as the composition

$$C \xrightarrow{\Delta_{\mathcal{H}}} C \otimes C \xrightarrow{\pi \otimes id} \mathcal{H} \otimes C.$$  

Similarly, a right co-action $R_{\mathcal{H}}$ of $\mathcal{H}$ on $C$ is defined as the composition

$$C \xrightarrow{\Delta_{\mathcal{H}}} C \otimes C \xrightarrow{id \otimes \pi} C \otimes \mathcal{H}.$$  

Using these co-actions, for every pair $m, n \in \mathbb{Z}$ we define

$$A[m, n] := \{x \in C \mid L_{\mathcal{H}}(x) = t^m \otimes x \quad \text{and} \quad R_{\mathcal{H}}(x) = x \otimes t^n\},$$

the vector subspace of bi-homogeneous elements with respect to these co-actions. For such a bi-homogeneous element $x \in A[m, n]$ we write $\text{bideg}(x) = (m, n) \in \mathbb{Z} \times \mathbb{Z}$, an additive group. One can show that this bi-grading is compatible with the multiplication in $C$ in the sense that

$$A[m, n] A[p, q] \subset A[m + p, n + q] \quad (9.4)$$
for $m, n, p, q \in \mathbb{Z}$, since $L_{\mathcal{K}}$ and $R_{\mathcal{K}}$ are algebra morphisms. This can alternatively be written as

\[ \text{bideg}(xy) = \text{bideg}(x) + \text{bideg}(y) \]

for all bi-homogeneous elements $x$ and $y$. We also have that $a \in A[1, 1]$ and $c \in A[-1, 1]$. Moreover, $x \in A[m, n]$ implies that $x^* \in A[-m, -n]$. Another fact is that $A[m, n] = 0$ if and only if $m - n$ is odd.

From (9.4) we can see that $A[0, 0]$ is a sub-algebra of $\mathcal{C}$ and then that each $A[m, n]$ is an $A[0, 0]$-bimodule. One has that $A[0, 0] = \mathbb{C}[\zeta]$, the polynomial algebra in the variable $\zeta = q^{2cc}$. (The coefficient $q^2$ makes this notation conform with that in [6]). Furthermore, each subspace $A[m, n]$ with $m - n$ even is a free left (respectively, right) $\mathbb{C}[\zeta]$-module on one generator denoted as $e_{m,n}$ in the notation of [6].

The basis elements $\epsilon_{klm}$ of $\mathcal{C}$ turn out to be bi-homogeneous with $\text{bideg}(\epsilon_{klm}) = (k - l + m, k + l - m)$ for all $k \in \mathbb{Z}$ and $l, m \in \mathbb{N}$. Since the weight function in (9.2) is strictly positive we see that $\langle \epsilon_{klm}, \epsilon_{rst}\rangle_{\mathcal{C}} \neq 0$ if and only if both $k = r$ and $l - m = s - t$. But this last condition is equivalent to $k - l + m = r - s + t$ and $k + l - m = r + s - t$, which is the same as $\text{bideg}(\epsilon_{klm}) = \text{bideg}(\epsilon_{rst})$. This shows that (9.3) is a direct sum compatible with the sesquilinear form (9.2), even though this property was not being considered when I first defined (9.2). However, this same analysis shows that the Hamel basis $\{\epsilon_{klm}\}$ is not an orthogonal basis, since for given indices $k, l, m$ we have $\langle \epsilon_{klm}, \epsilon_{kst}\rangle_{\mathcal{C}} \neq 0$ for all pairs $s, t \in \mathbb{N}$ satisfying $s - t = l - m$. And there are infinitely many such pairs. It is known that there is another natural sesquilinear form on $\mathcal{C}$ for which (9.3) is an orthogonal direct sum. In fact, this is done using the (unique!) Haar state of $SU_q(2)$ and so is more closely related to the structure of $SU_q(2)$ as a compact quantum group. Again, see [6] for more details.

We define $\mathcal{P} := \text{alg}(a, c)$, the sub-algebra (but not sub-$*$-algebra) of $SU_q(2)$ generated by $a$ and $c$. This is a sub-algebra of ‘holomorphic’ elements. This is the same sub-algebra that was used in [13] for a Toeplitz quantization of $SU_q(2)$ as an algebra. We can identify $\mathcal{P}$ as the free algebra generated by $a$ and $c$, modulo the relation $ac = qa$, and so (as an algebra) $\mathcal{P}$ is the complex Manin quantum plane, which is denoted by $A_q^{2|0}$ in [7].

A Hamel basis of $\mathcal{P}$ is given by the monomials $a^kc^l = \epsilon_{kl0}$ for $k, l \in \mathbb{N}$. Since

\[ \langle \epsilon_{kl0}, \epsilon_{rs0}\rangle_{\mathcal{C}} = w(k, l) \delta_{k,r} \delta_{l,s}, \]

we have that $\{a^kc^l \mid k, l \in \mathbb{N}\}$ is an orthogonal basis of $\mathcal{P}$ and that the sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{C}}$ when restricted to $\mathcal{P}$ is a positive definite inner product. Clearly,

\[ \phi_{kl} := \frac{1}{\sqrt{w(k, l)}} a^kc^l = \frac{1}{\sqrt{w(k, l)}} \epsilon_{kl0} \quad \text{for } k, l \geq 0 \]

is an orthonormal basis of $\mathcal{P}$. Thus $\mathcal{P}$ is a pre-Hilbert space whose completion we denote by $\mathcal{H}$. With no loss of generality we can assume that $\mathcal{P}$ is a dense subspace of $\mathcal{H}$.
The injection \( j : \mathcal{P} \to \mathcal{C} \) is defined to be the inclusion map. The quotient map \( Q : \mathcal{C} \to \mathcal{P} \) is defined as in (2.1) by

\[
Q(f) := \sum_{i,j \geq 0} \langle \phi_{ij}, f \rangle \phi_{ij}
\]

for \( f \in \mathcal{C} \). The infinite sum on the right side here has only finitely many non-zero terms. It is now an easy exercise to prove that \( Q(a) = a \), \( Q(c) = c \) and \( Q(a^*) = Q(c^*) = 0 \), these being results needed to prove some of the statements in the next paragraph. We will discuss the action of \( Q \) on the basis elements \( \varepsilon_{klm} \) a little later on.

According to the general theory of Sect. 3, the projection \( Q \) should be a co-algebra morphism, meaning a linear map intertwining the two co-multiplications. While \( Q \) is clearly linear, we have not specified a co-multiplication \( \Delta \mathcal{P} \) on the Manin quantum plane \( \mathcal{P} \). To do this we require that \( \Delta \mathcal{P} \) is the unique algebra morphism \( \mathcal{P} \to \mathcal{P} \otimes \mathcal{P} \) satisfying

\[
\Delta \mathcal{P}(a) := a \otimes a \quad \text{and} \quad \Delta \mathcal{P}(c) := c \otimes a.
\]

To see that this does make sense, one first defines the algebra morphism \( \Delta \mathcal{P} \) on the free algebra generated by \( a \) and \( c \) by using the previous formulas, and then one shows that \( \Delta \mathcal{P}(ac - q ca) = 0 \). Hence \( \Delta \mathcal{P} \) passes to the quotient algebra \( \mathcal{P} \). It is straightforward to show that \( \Delta \mathcal{P} \) so defined is co-associative. However, no linear map \( l : \mathcal{P} \to \mathbb{C} \) can be the co-unit for this co-multiplication, since

\[
(l \otimes \text{id}) \Delta \mathcal{P}(c) = (l \otimes \text{id})(c \otimes a) = l(c)a \neq c.
\]

So, \( \mathcal{P} \) is a co-algebra without co-unit, which is allowed in the general theory. Finally, one can readily prove that \( Q : \mathcal{C} \to \mathcal{P} \) is a co-algebra morphism and that \( \mathcal{P} \) is not a sub-co-algebra of \( \mathcal{C} \).

We now calculate the action of \( Q \) on the basis elements \( \varepsilon_{klm} \) of the co-algebra \( \mathcal{C} = SU_q(2) \):

\[
Q(\varepsilon_{klm}) = \sum_{i,j \geq 0} \langle \phi_{ij}, \varepsilon_{klm} \rangle \phi_{ij} = \sum_{i,j \geq 0} \frac{1}{w(i, j)} \langle \varepsilon_{ij0}, \varepsilon_{klm} \rangle \varepsilon_{ij0}
\]

\[
= \sum_{i,j \geq 0} \frac{1}{w(i, j)} w(i, j) \delta_{i,k} \delta_{j,l-m} \varepsilon_{ij0} = \sum_{i,j \geq 0} \delta_{i,k} \delta_{j,l-m} \varepsilon_{ij0}
\]

\[
= \varepsilon_{k,l-m,0},
\]

if \( k \geq 0 \) and \( l \geq m \). Otherwise, \( Q(\varepsilon_{klm}) = 0 \). Summarizing, we have shown the following:

**Proposition 9.1** The action of the projection \( Q \) on the basis elements \( \varepsilon_{klm} \) is given by

\[
Q(\varepsilon_{klm}) = \varepsilon_{k,l-m,0} \neq 0 \quad \text{if} \ k \geq 0, \ l \geq m,
\]

\[
Q(\varepsilon_{klm}) = 0 \quad \text{otherwise}.
\]
In the case $k \geq 0$, one can interpret these formulas for $Q(\varepsilon_{klm})$ as saying that all the $c^*$’s disappear and each one of them also ‘kills off’ exactly one of the $c$’s. The condition $l < m$ means that the monomial $\varepsilon_{klm}$ has strictly more occurrences of $c^*$’s than of $c$’s, in which case all the $c$’s get ‘killed off,’ as does everything else, and the result is 0. Finally, if $k < 0$, then there are occurrences of $a^*$ but none of $a$, and this in itself suffices to give 0. This last fact has a handy generalization, which we now present.

**Proposition 9.2** Let $w \in \mathcal{C}$ be a finite word in the alphabet with these four letters: $a, a^*, c, c^*$. If $w$ has strictly more occurrences of the letter $a^*$ than of the letter $a$, then $Q(w) = 0$.

**Remark 9.1** The hypothesis implies that the number of occurrences of $a^*$ is strictly larger than zero.

**Proof** Using the defining relations (9.1) we can push all occurrences of $c$ and $c^*$ to the right, thereby getting $w = q^n w' c^l (c^*)^m$, where $l, m \in \mathbb{N}, n \in \mathbb{Z}$ and $w'$ is a word with only occurrences of $a, a^*$. The number of occurrences of $a$ (resp., $a^*$) in $w'$ is equal to the number of occurrences of $a$ (resp., $a^*$) in $w$. Let $j$ be the number of occurrences of $a^*$. We proceed by using induction on $k$, the number of occurrences of $a$ in $w$.

First, we consider the case $k = 0$. Then we have $w = q^n \varepsilon_{j,l,m}$, where $j \geq k + 1 = 1$ is the number of occurrences of $a^*$ in $w$. So, $Q(w) = 0$ by Proposition 9.1.

For the induction step we assume that the assertion $Q(w) = 0$ is true for some $k \geq 0$, and then we will prove it for $k + 1$. So, let $w$ be a word with $k + 1 \geq 1$ occurrences of $a$. Then by hypothesis $j > k + 1$. We again have $w = q^n w' c^l (c^*)^m$ as above. Since $w'$ has a non-zero number of occurrences of both $a$ and $a^*$, there must be at least one juxtaposition in $w'$ of $a$ and $a^*$. So, we can write $w'$ in at least one of these two forms:

$$w' = u (aa^*) v \quad \text{or} \quad w' = u (a^* a) v,$$

where $u$ and $v$ are words (possibly empty) with occurrences of $a$ and $a^*$ only. In the first case we see for example that

$$Q(w' c^l (c^*)^m) = Q(u (aa^*) v c^l (c^*)^m) = Q(u (1 - q^2 cc^*) v c^l (c^*)^m) = Q(u v c^l (c^*)^m) - q^l Q(u c^l (c^*)^m) = Q(u v c^l (c^*)^m) - q^r Q(u v c^{l+1} (c^*)^{m+1}) = 0 - 0 = 0.$$

Here the exponent $r \in \mathbb{N}$ arises from pushing the factor $cc^*$ to the right through $v$. The next to the last equality follows from the induction hypothesis and the fact that the word $u v$ has $k$ occurrences of $a$ and $j - 1 > k \geq 0$ occurrences of $a^*$.

The proof for the second form of $w'$ is quite similar and so is left to the reader. And that finishes the proof. □

This result can also be proved by evaluating the bi-degree of a word with more $a^*$’s than $a$’s and showing that it is not equal to the bi-degree of any $\varepsilon_{rs0}$ with $r, s \geq 0$. 

© Birkhäuser
We have a result similar to Proposition 9.2 for $c$ and $c^\ast$.

**Proposition 9.3** Let $w \in \mathcal{C}$ be a finite word in the alphabet with these four letters: $a, a^\ast, c, c^\ast$. If $w$ has strictly more occurrences of the letter $c^\ast$ than of the letter $c$, then $Q(w) = 0$.

**Proof** Here is a proof using bi-degrees instead on a similar induction argument, which could also be made. Suppose that $w$ has $j, k, l, m$ occurrences of $a, a^\ast, c, c^\ast$ respectively. Then, independent of the order of these occurrences, we have that

$$\text{bideg}(w) = j(1, 1) + k(-1, -1) + l(-1, 1) + m(1, -1),$$

while for $r, s \geq 0$ we have $\text{bideg}(ar^r c^s) = (r - s, r + s)$. The difference of the two entries in $\text{bideg}(w)$ is $-2l + 2m > 0$, since $m > l$ by hypothesis. However, the corresponding difference for $\text{bideg}(ar^r c^s)$ is $-2s \leq 0$. This implies that $\text{bideg}(w) \neq \text{bideg}(ar^r c^s)$ and therefore $\langle ar^r c^s, w \rangle_{Q} = 0$ for all $r, s \geq 0$, which in turn implies that $Q(w) = 0$. \qed

We have now on hand enough formulas to calculate the action of the co-Toeplitz operators $C_{\varepsilon_{klm}}$. This is sufficient information, since $C_g$ for any symbol $g \in SU_q(2)$ can be written as a finite linear combination with complex coefficients of the co-Toeplitz operators $C_{\varepsilon_{klm}}$. Moreover, it suffices to calculate $C_{\varepsilon_{klm}}$ acting on the elements $\phi_{r,s}$ in the standard orthonormal basis of $\mathcal{P}$, where $r, s \in \mathbb{N}$. We recall that the co-Toeplitz operator with symbol $g$ was defined as $C_g = \pi_g \beta$. Since $j$ is simply the inclusion map, we have

$$C_{\varepsilon_{klm}}(\phi_{r,s}) = \pi_{\varepsilon_{klm}} \beta(\phi_{r,s}).$$

We will take the co-action map $\beta : \mathcal{C} \to \mathcal{P} \otimes \mathcal{C}$ to be of the form (4.3), namely

$$\mathcal{C} \xrightarrow{\Delta_{\mathcal{C}}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{Q \otimes \text{id}} \mathcal{P} \otimes \mathcal{C},$$

where $\Delta_{\mathcal{C}}$ is the co-multiplication of $\mathcal{C}$. Dropping the normalization constant for the moment, we calculate with the monomial $ar^r c^s$ instead of with $\phi_{r,s}$. We then see that

$$\beta(ar^r c^s) = (Q \otimes \text{id}) (\Delta_{\mathcal{C}}(ar^r c^s)) = (Q \otimes \text{id}) (\Delta_{\mathcal{C}}(a)^r \Delta_{\mathcal{C}}(c)^s)$$

$$= (Q \otimes \text{id}) \left( (a \otimes a - q c^\ast \otimes c)^r (c \otimes a + a^\ast \otimes c)^s \right).$$

We will use the standard binomial theorem on the second factor, since $c \otimes a$ and $a^\ast \otimes c$ commute, as follows from (9.1). To continue with the first factor we will use the $q$-binomial theorem (see [6]), which states that if variables $v, w$ satisfy the commutation relation $vw = qwv$ for $0 \neq q \in \mathbb{C}$, then for any integer $n \geq 0$ one has

$$(v + w)^n = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right]_{q^{-1}} v^m w^{n-m},$$
where the coefficient is an explicitly given deformation of the standard binomial coefficient. This is applicable in this situation, since

\((-qc^* \otimes c)(a \otimes a) = -qc^*a \otimes ca,\)

and hence for \(v = a \otimes a\) and \(w = -qc^* \otimes c\) by using the relations (9.1) again we obtain

\[vw = (a \otimes a) (-qc^* \otimes c) = -qac^* \otimes ac\]
\[= q^2(-qc^*a \otimes ca) = q^2wv.\]

Next, to simplify somewhat the rather cumbersome binomial-type notation, we introduce

\[B_{n,q} := \left[\begin{array}{c} r \\ n \end{array}\right]_{q-2},\]

which also suppresses the variable \(r\). We also use \(B_{p,1} := \left(\begin{array}{c} s \\ p \end{array}\right)\), a standard binomial coefficient (which suppresses the variable \(s\)). We will use this material in the next and subsequent calculations. The reader can consult [6] for more details about the so-called \(q\)-calculus.

Then for \(r, s \in \mathbb{N}\) we have

\[\beta(a^r c^s) = (Q \otimes id)\left( (a \otimes a - q c^* \otimes c)^r (c \otimes a + a^* \otimes c)^s \right)\]

\[= (Q \otimes id) \sum_{n,p=0}^{r,s} B_{n,q}(a \otimes a)^{r-n} (-q)^n (c \otimes a)^n B_{p,1}(c \otimes a)^{s-p} (a^* \otimes c)^p\]

\[= (Q \otimes id) \left( \sum_{n,p=0}^{r,s} (-q)^n B_{n,q} B_{p,1} a^{r-n} (c^*)^n c^{s-p} a^* p \otimes a^{r-n} c^n a^{s-p} c^p \right)\]

\[= \sum_{n,p=0}^{r,s} (-q)^n B_{n,q} B_{p,1} Q(a^{r-n} (c^*)^n c^{s-p} a^* p) \otimes a^{r-n} c^n a^{s-p} c^p\]

\[= \sum_{n,p=0}^{r,s} \phi \otimes a^{r-n} c^n a^{s-p} c^p.\]

To simplify notation we have put

\[\phi = \phi_{nprs} = (-q)^n B_{n,q} B_{p,1} Q(a^{r-n} (c^*)^n c^{s-p} a^* p) \in \mathcal{P}.\] (9.5)

By Propositions 9.2 and 9.3 we see that if \(p > r - n \) or \(n > s - p\), then \(\phi = 0\). In the contrary case the calculation of \(\phi\) is a bit more complicated. The contrary case occurs when \(p \leq r - n\) and \(n \leq s - p\), that is, \(n + p \leq r\) and \(n + p \leq s\). This condition is then equivalent to \(n + p \leq \min(r, s)\), which we will assume to hold throughout the following. The summation indices \(n\) and \(p\) also satisfy \(0 \leq n \leq r\) and \(0 \leq p \leq s\). To do this calculation we will use the identity

\[\otimes\]

\Birkhäuser
\[ a^m (a^*)^m = \sum_{i=0}^{m} \binom{m}{i} q^{-2} (-1)^i q^{i+2im-i^2} c^i (c^*)^i, \quad (9.6) \]

for integer \( m \geq 0 \). (Cp. [6], p. 100, Eq. (13). Or prove it yourself by induction on \( m \).) Note that this identity is not surprising, since \( \text{bideg}(a^m (a^*)^m) = (0, 0) \) and \( A[0, 0] \) is the polynomial algebra in the variable \( cc^* \). (Recall that \( c \) and \( c^* \) commute so that \( c^i (c^*)^j = (cc^*)^{i+j} \).) What the identity (9.6) tells us more specifically is that \( a^m (a^*)^m \) is a polynomial in \( A[0, 0] \) of degree \( m \) and what its coefficients are exactly. Then using this identity we have

\[
a^{r-n} (c^*)^n c^{s-p} a^{*p} = q^{p(s-p)+pn} a^{r-n} a^{*p} (c^*)^n c^{s-p} \\
= q^{p(s-p+n)} a^{r-n-p} a^{*p} (c^*)^n c^{s-p} \\
= q^{p(s-p+n)} a^{r-n-p} \sum_{i=0}^{p} \left[ \binom{p}{i} q^{-2} (-1)^i q^{i+2ip-i^2} c^i (c^*)^i \right] (c^*)^n c^{s-p} \\
= \sum_{i=0}^{p} \left[ \binom{p}{i} q^{-2} (-1)^i q^A a^{r-n-p} c^{i+s-p} (c^*)^i \right] + n, \\
= \sum_{i=0}^{p} \left[ \binom{p}{i} q^{-2} (-1)^i q^A \varepsilon_{r-n-p,i+s-p,i+n} \right], \\
\]

where \( A = p(s - p + n) + i + 2ip - i^2 \). Continuing, we see that

\[
\phi_{nprs} = (-q)^n B_{n,q} B_{p,1} Q(a^{r-n} (c^*)^n c^{s-p} a^{*p}) \\
= (-q)^n B_{n,q} B_{p,1} \left( \sum_{i=0}^{p} \left[ \binom{p}{i} q^{-2} (-1)^i q^A \varepsilon_{r-n-p,i+s-p,i+n} \right] \right) \\
= (-q)^n B_{n,q} B_{p,1} \left( \sum_{i=0}^{p} \left[ \binom{p}{i} q^{-2} (-1)^i q^A \right] \varepsilon_{r-n-p,s-n-p,0} \right) \\
= D_{nprs} \varepsilon_{r-(n+p),s-(n+p),0}, \\
\]

where the real number \( D_{nprs} \) has the obvious definition. Here we also used Proposition 9.1, which has the fortuitous virtue of changing the scope of the sum on \( i \). Notice that this shows that \( \phi \) is proportional to an element in the basis \( \{ \varepsilon_{kli0} | k, l \geq 0 \} \) of \( \mathcal{P} \). The bi-degree of the bi-homogeneous element \( \phi \) is easily seen to be given by

\[
\text{bideg}(\phi) = \text{bideg}(\varepsilon_{r-(n+p),s-(n+p),0}) = (r - s, r + s - 2(n + p)). \quad (9.7)\]
Next, for $r, s \in \mathbb{N}$ we obtain

$$C_{\varepsilon_{klm}}(a^r c^s) = \pi_{\varepsilon_{klm}} \beta(a^r c^s) = \pi_{\varepsilon_{klm}} \sum_{n+p=0}^{\min(r,s)} \phi \otimes a^{r-n}c^n a^{s-p}c^p$$

$$= \sum_{n+p=0}^{\min(r,s)} \langle \varepsilon_{klm}, a^{r-n}c^n a^{s-p}c^p \rangle \phi$$

$$= \sum_{n+p=0}^{\min(r,s)} q^{n(s-p)} \langle \varepsilon_{klm}, a^{r+s-(n+p)}c^{n+p} \rangle \phi_{nprs}$$

$$= \sum_{n+p=0}^{\min(r,s)} q^{n(s-p)} \langle \varepsilon_{klm}, a^{r+s-(n+p)}c^{n+p} \rangle D_{nprs} \varepsilon_{r-(n+p),s-(n+p),0}. \quad (9.8)$$

Note that the condition $0 \leq n + p \leq \min(r, s)$ means according to (9.7) that (9.8) is in general a sum of bi-homogeneous elements with different bi-degrees. However, the coefficients of these summands will be non-zero only if the inner product in the expression (9.8) is non-zero which is equivalent to

$$\text{bideg}(\varepsilon_{klm}) = \text{bideg}(a^{r+s-(n+p)}c^{n+p}),$$

which itself is equivalent to

$$(k - l + m, k + l - m) = (r + s - 2(n + p), r + s).$$

The indices $k, l, m, r, s$ are given and the ‘unknowns’ are the summation indices $n$ and $p$. The previous equality is equivalent to

$$n + p = r + s - k = l - m. \quad (9.9)$$

If this holds for some pair $n, p$ satisfying $n + p \leq \min(r, s), 0 \leq n \leq r$ and $0 \leq p \leq s$, then (9.8) is a (possibly zero) multiple of

$$\varepsilon_{r-(n+p),s-(n+p),0} = \varepsilon_{r-(l-m),s-(l-m),0};$$

otherwise, (9.8) is 0. ($D_{nprs} \in \mathbb{R}$ can be positive or negative.) In order that there exists at least one solution of (9.9) for a pair $n \geq 0, \ p \geq 0$ it is necessary and sufficient that the five indices $k, l, m, r, s$ satisfy

$$k \leq r + s \quad \text{and} \quad m \leq l. \quad (9.10)$$

And in that case the co-Toeplitz operator $C_{\varepsilon_{klm}}$ lowers the degree of each variable $a, c$ by $l - m \geq 0$. Alternatively, we note that $C_{\varepsilon_{klm}}$ maps $a^r c^s = \varepsilon_{r,s,0}$ of bi-degree
In other words on this scale the co-Toeplitz operator $C_{\varepsilon klm}$ can be understood as an operator having bi-degree $(0, -2(l - m))$. In physics terminology, these co-Toeplitz operators are not creation operators in the sense that the degree of the powers of monomials is strictly increased. Similarly, the bi-degree also is not strictly increased.

We have shown the following.

**Theorem 9.1** Suppose that $k \in \mathbb{Z}$ and $l, m, r, s \in \mathbb{N}$ satisfy $r + s - k = l - m$ and that $0 \leq l - m \leq \min(r, s)$. Suppose that this set is non-empty:

$$\{(n, p) \mid n + p = l - m, \ 0 \leq n \leq r, \ 0 \leq p \leq s\}.$$

Then $C_{\varepsilon klm}(ar^s c^s) = Ka^{r-(l-m)}c^{r-(l-m)}$ for some real number $K$.

In terms of basis elements $C_{\varepsilon klm}(\varphi_{rs}) = K'\varphi_{r-(l-m), s-(l-m)}$ for some real number $K'$. And $K' \neq 0$ if and only if $K \neq 0$.

Otherwise, we have $C_{\varepsilon klm}(ar^s c^s) = 0$.

Here are some special cases of this theorem. First, we consider the case $l = m$. In this case $C_{\varepsilon kl}$ maps $ar^s c^s$ to a multiple of $ar^s c^s$ for any value of $k \in \mathbb{Z}$. Notice that the multiplicative constant depends on $k$ and can be $0$. In physics terminology this is a preservation operator, which simply means mathematically that it preserves degrees. The sub-case $l = m = 0$ is the co-Toeplitz operator with ‘holomorphic’ symbol $a^k$ if $k \geq 0$ or with ‘anti-holomorphic’ symbol $(a^*)^{-k}$ if $k < 0$.

The next case is $l > 0$, $m = 0$. In this case $C_{\varepsilon kl0}$ maps $ar^s c^s$ to some multiple of $ar^{-l}c^{s-l}$. In usual physics terminology this is called an annihilation operator, which simply means mathematically that it lowers degrees. We remark that $\varepsilon_{kl0}$ is the most general holomorphic monomial in the variables $a$ and $c$. It is because of this particular case that we have defined co-Toeplitz operators with holomorphic symbols to be annihilation operators. (See Definition 7.1.)

In the case $l = 0$ we have that $m = 0$ must hold as well. And so this case was already considered as part of the first case. Or in other words, the case $l = 0$ and $m > 0$ gives a zero co-Toeplitz operator.

This leads us up to the analysis of the co-Toeplitz operators whose symbols are one of the four algebra generators, $a, a^*, c, c^*$, of $SU_q(2)$. For the symbol $c$ we have $k = m = 0, l = 1$ and so $C_c$ is an annihilation operator that maps $ar^s c^s$ to a multiple of $ar^{-1}c^{s-1}$.

For the symbol $c^*$ we have $k = l = 0, m = 1$ and so $C_{c^*} = 0$, since $m > l$ holds. The same reasoning applies to the ‘anti-holomorphic’ symbol $(a^*)^k(c^*)^m$ for $m > 0$, since $m > l = 0$. So, $C_{(a^*)^k(c^*)^m} = 0$.

For the symbol $a^*$ we have $l = m = 0, k = -1$. Now $n + p = l - m = 0$ implies that $n = p = 0$ and therefore that $r + s = k = -1$, which has no solutions $r \geq 0, s \geq 0$. Thus, $C_{a^*} = 0$.

For the symbol being $a$ we have $l = m = 0, k = 1$, and thus $C_a$ is a preservation operator. But $n + p = l - m = 0$ implies that $n = p = 0$. So there is only one term in the sum (9.8). We note that $q^{n(s-p)} = q^0 = 1$ and $D_{00rs} = 1$. But the coefficient in that unique term is
\[ \langle \epsilon_{100}, a \rangle = \langle a, a \rangle = w(1, 0) > 0. \]

Consequently, \( C_a \) is a non-zero multiple of the identity operator. In particular, \( C_a \) \( \neq 0 \) and \( C_a^* \) \( = 0 \) are not adjoints of each other. So the condition (6.2) does not hold for our choice (9.2) for the sesquilinear form.

In this example, the creation and annihilation operators have strange properties from the point of view of quantum physics. This is in part a consequence of the choice of the sesquilinear form for this example. As I have emphasized elsewhere, the study of more examples of the co-Toeplitz quantization scheme is really needed for getting a better understanding of the general theory. A similar example for the Toeplitz quantization of \( SU_q (2) \) in [13] gave creation and annihilation operators which are more intuitive physically. This goes to show that co-Toeplitz quantization has new, rather curious properties, even though it is dual in the sense of notion duality to Toeplitz quantization.

This example depends on more than the choice of the co-algebra \( SU_q (2) \). We have to choose also the sesquilinear form and the subspace \( P \). We could continue with the same family of sesquilinear forms, where that family is parameterized by the weight function. Instead, we could use a different subspace, say for example:

\[ P' := \text{span} \{ \epsilon_{kl0} = a^k c^l, \epsilon_{km} = a^k (c^*)^m \mid k, l \geq 0, m > 0 \}. \]

Since no two elements in this set of generators have the same bi-degree, we have that this is an orthogonal basis of \( P' \). So an orthonormal basis of \( P' \) is given by

\[ \phi_{kl} = \frac{1}{\sqrt{w(k, l)}} \epsilon_{kl0} \quad \text{and} \quad \psi_{km} := \frac{1}{\sqrt{w(k, -m)}} \epsilon_{km}, \]

for \( k, l \geq 0 \) and \( m > 0 \), where we continue to use the notation \( \phi_{kl} \) introduced earlier. Thus \( P' \) is a pre-Hilbert space.

The injection \( j' : P' \to C \) is defined to be the inclusion map. The quotient map \( Q' : C \to P' \) is defined for \( f \in C \) as

\[ Q'(f) := \sum_{i, j \geq 0} \langle \phi_{ij}, f \rangle C \phi_{ij} + \sum_{i \geq 0, j > 0} \langle \psi_{ij}, f \rangle C \psi_{ij}, \]

where the two infinite sums on the right side have only finitely many non-zero terms. This shows just one possible way of giving another example based on the co-algebra \( SU_q (2) \).

Another possible modification of this example is to use the positive definite inner product defined for \( x, y \in SU_q (2) \) by \( \langle x, y \rangle := h(x^* y) \), where \( h : SU_q (2) \to C \) is the unique Haar state on \( SU_q (2) \) (see [6]), instead of the sesquilinear form defined in (9.2). This is an approach that is better attuned to the Hopf \( * \)-algebra structure of \( SU_q (2) \). These two alternatives as well as examples of co-Toeplitz quantizations of other co-algebras will be the subject of forthcoming research work. It is worthwhile noting that there are many interesting co-algebras that arise in various areas of mathematics and its applications.
10 Concluding remarks

This paper begins the new theory of co-Toeplitz operators and their associated quantization, as the title indicates. On the other hand, the theory of Toeplitz operators is over one hundred years old. Obviously, one strategy is to use the ideas and results in the Toeplitz setting to inspire research in this new theory. However, I hope that there will be more new ideas arising in the co-Toeplitz setting and that some of these may even shed light on the well-known Toeplitz setting. To bring this theory to maturity requires more than anything a reasonable quantity of illuminating examples, which could help in fine tuning definitions and in providing insights into relations among the various structures introduced here. Also, bi-algebras can now be quantized either by using their algebra structure or their co-algebra structure. So it would be interesting to understand how those two quantizations might be related. In the more specific case of Hopf algebras (or quantum groups) one would like to know what the role of the antipode is. One might also be able to introduce into this setting such structures as a symplectic form, Poisson brackets or coherent states, just to name a few possibilities. Finally, other types of quantization schemes may also be extended to theories based on arbitrary algebras or co-algebras. This is a broad outline of possible future research in this area.

Acknowledgements I thank Micho Durdevich and Jean-Pierre Gazeau for providing me insights from rather complementary points of view of mathematical physics. I can not imagine how I could ever have possibly written this paper without their generosity in sharing ideas with me.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

References

1. Bargmann, V.: On a Hilbert space of analytic functions and its associated integral transform. I. Commun. Pure Appl. Math. 14, 187–214 (1961)
2. Berger, C.A., Coburn, L.A.: Toeplitz operators on the Segal-Bargmann space. Trans. Am. Math. Soc. 301, 813–829 (1987)
3. Böttcher, A., Silbermann, B.: Analysis of Toeplitz Operators, 2nd edn. Springer, Berlin (2006)
4. Bożejko, M., et al.: Fock representations of $Q$-deformed commutation relations. J. Math. Phys. 58(7), 073501 (2017). (19 pp)
5. Englis, M.: An Excursion into Berezin–Toeplitz Quantization and Related Topics. In: Bahns, D. (ed.) Quantization, PDEs, and Geometry, (Operator Theory: Advances and Applications, vol. 251, pp. 69–115. Birkhäuser, Cham (2016)
6. Klimyk, A., Schmudgen, K.: Quantum Groups and Their Representations. Springer, Berlin (1997)
7. Manin, Yu.I.: Topics in Noncommutative Geometry. Princeton University Press, Princeton (1991)
8. Sontz, S.B.: Paragrassmann Algebras as Quantum Spaces, Part I: Reproducing Kernels. In: Kielanowski, P., et al. (eds.) Geometric Methods in Physics. XXXI Workshop 2012. Birkhäuser, Cham (2013)
9. Sontz, S.B.: A reproducing kernel and Toeplitz operators in the quantum plane. Commun. Math. 21, 137–160 (2013)
10. Sontz, S.B.: Paragrassmann algebras as quantum spaces, part II: Toeplitz operators. J. Oper. Theory 71, 411–426 (2014)
11. Sontz, S.B.: Proceedings of: Geometric Methods in Physics. XXXII Workshop 2013. Trends in Mathematics. Toeplitz Quantization without Measure or Inner Product, pp. 57–66. Birkhäuser, Berlin (2014)
12. Sontz, S.B.: Principal bundles, the quantum case. Springer, Berlin (2015)
13. Sontz, S.B.: Toeplitz Quantization for Non-commutating Symbol Spaces such as $SU_q(2)$. Communications in Mathematics 24, 43–69 (2016)
14. Sontz, S.B.: Co-Toeplitz Quantization: A Simple Case, accepted for publication in Proceedings of the Workshop on Geometric Methods in Physics. WGMP) XXXVII, Białowieża, Poland (2018)