RANDOM SELF-SIMILAR TREES AND
A HIERARCHICAL BRANCHING PROCESS

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ABSTRACT. We study self-similarity in random binary rooted trees. In a well-understood case of Galton-Watson trees, a distribution is said to be self-similar if it is invariant with respect to the operation of pruning, which cuts the tree leaves. This only happens in the critical case (a constant process progeny), which also exhibits other special symmetries. We extend the prune-invariance set-up to trees with edge lengths with non-Markovian growth. In this general case the class of self-similar processes becomes much richer and covers a variety of practically important situations. The main result is construction of the hierarchical branching processes that satisfy various self-similarity definitions (distributional, mean, in edge-lengths) depending on the process parameters. Taking the limit of averaged stochastic dynamics, as the number of trajectories increases, we obtain a deterministic system of differential equations that describes the process evolution. This system is used to establish a phase transition that separates fading and explosive behavior of the average process progeny. We describe a class of critical Tokunaga processes that happen at the phase transition boundary. They enjoy multiple additional symmetries and include the celebrated critical binary Galton-Watson tree with independent exponential edge length as a special case. Finally, we discuss a duality between trees and continuous functions, and introduce a class of extreme-invariant processes, constructed as the Harris paths of the self-similar hierarchical branching process, whose local minima has the same (linearly scaled) distribution as the original process.

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1. INTRODUCTION

Nature commonly exhibits dendritic structures, both static and dynamic, that can be represented by tree graphs [11, 26, 17]. Examples from diverse applications, together with a review of related coalescence and branching models can be found in Aldous [1], Berestycki [2], Bertoin [3], Le Gall [14], and Pitman [19]. Despite their apparent diversity, a number of rigorously studied dendritic structures possess structural self-similarity, which often allows a low-dimensional parameterization [18, 17, 25, 10]. An illuminating example is the combinatorial structure of river networks, which is closely approximated by a two-parametric Tokunaga self-similar model with parameters that are independent of river’s geographic location [24, 18, 6, 28]. The tree self-similarity has been studied primarily in terms of the average values of selected branch statistics, and rigorous results have been obtained only for a very special classes of Markov trees (e.g., binary Galton-Watson trees with no edge lengths, as in [4]). At the same time, solid empirical evidence motivates the search for a flexible class of self-similar models that would encompass a variety of observed combinatorial and metric structures and extend beyond the Markov constraint. We introduce here a general concept of distributional self-similarity that accounts for both combinatorial and metric tree structure (Sec. 3.5, Def. 10) and describe a model (Sect. 5), called hierarchical branching process, that generates a broad range of self-similar trees (Thm. 4) and includes the critical binary Galton-Watson tree with exponential edge lengths as a special case (Thm. 8). We study time-invariant tree distributions, which is a convenient generalization of Markov growth (Thm. 7). We also introduce a class of critical self-similar Tokunaga processes (Sect. 5.7) that enjoy additional symmetries — their edge lengths are i.i.d. random variables (Prop. 8), and sub-trees of the large Tokunaga trees reproduce the probabilistic structure of the entire random tree space (Props. 10, 11). The duality between planar trees and continuous functions [8, 19, 27] allows us using the hierarchical branching process to construct a novel class of time series that satisfy the extreme-invariance property: the distribution of their local minima is the same as that of the original series (Sect. 4).

The paper is organized as follows. Section 2 introduces the main definitions, including the Horton-Strahler order of a tree, tree pruning, and a related concept of prune-invariance. Self-similarity for trees with edge lengths is defined in Sect. 3. The duality between trees
and continuous functions is reviewed in Sect. 4. In particular, we define here extreme-invariant processes that are equivalent to self-similar trees. The main results are presented in Sect. 5. Sect. 5.1 introduces a hierarchical branching process that generates a rich collection of self-similar trees. The hydrodynamic limit for dynamics of the average numbers of Horton-Strahler branches is established in Sect. 5.2. The properties of criticality and time-invariance are defined in Sect. 5.3 and explored in a self-similar processes in Sect. 5.4. Critical Galton-Watson process and critical Tokunaga processes, which generate the most intriguing examples of self-similar trees, are discussed in Sects. 5.6, 5.7. Section 6 concludes with two open problems.

2. Random Trees

The focus of this paper is on finite unlabeled rooted reduced binary trees with no planar embedding. The space of such trees, which includes the empty tree $\phi$ comprised of a single root vertex and no edges, is denoted by $T$. The existence of the root vertex imposes the parent-offspring relationship between each pair of the connected vertices in a tree $T \in T$: the one closest to the root is called parent, and the other – offspring. The absence of planar embedding in this context means the absence of order between the two offspring of the same parent. A tree is called reduced if it has no vertices of degree 2; such trees are also called full binary trees. There are three types of vertices in a reduced rooted binary tree from $T \setminus \{\phi\}$: internal vertices of degree 3, leaves (degree 1) and the root (degree 1). The operation of series reduction removes each degree-two vertex of a binary tree by merging its adjacent edges into one. Series reduction turns a rooted binary tree into a reduced rooted binary tree.

The edges of a tree from $T$ may be assigned positive lengths. The space of trees from $T$ with edge lengths is denoted by $L$.

Any tree from $T$ or $L$ can be embedded (and represented graphically) in a plane by selecting an order for each pair of offspring of the same parent. The space of embedded trees from $T$ (and respectively $L$) is denoted $T_{\text{plane}}$ (and respectively $L_{\text{plane}}$). Choosing different embeddings for the same tree $T \in T$ (or $T \in L$) leads, in general, to different trees from $T_{\text{plane}}$ (or $L_{\text{plane}}$). Sometimes we focus on the combinatorial tree $\text{SHAPE}(T) \in T$, which retains the branching structure of $T$ while omitting its edge lengths and embedding.

2.1. Tree pruning and related concepts. The concept of self-similarity is related to the pruning operation [18, 4, 10]. Pruning of a tree is an onto function $\mathcal{R}: T \rightarrow T$, whose value $\mathcal{R}(T)$ for a tree $T \neq \phi$ is obtained by removing the leaves and their parental edges from $T$, followed by series reduction. We also set $\mathcal{R}(\phi) = \phi$.

The pruning is also well defined for trees with edge lengths ($L$), where series reduction adds the lengths of merging edges, and for planar trees ($T_{\text{plane}}, L_{\text{plane}}$), where the embedding of the remaining part of a tree is unaffected by pruning. Pruning is illustrated in Fig. 1.

Pruning induces a contracting flow on $T$. The trajectory of each tree $T$ under $\mathcal{R}(\cdot)$ is uniquely determined and finite:

$$T \equiv \mathcal{R}^0(T) \rightarrow \mathcal{R}^1(T) \rightarrow \cdots \rightarrow \mathcal{R}^k(T) = \phi,$$
with the empty tree $\phi$ as the (only) fixed point. The pre-image $R^{-1}(T)$ of any non-empty tree $T$ consists of an infinite collection of trees. It is natural to think of the distance to $\phi$ under the pruning flow and introduce the respective notion of tree order \cite{11, 23} (see Fig. 1).

**Definition 1 (Horton-Strahler orders).** The Horton-Strahler order $k(T) \in \mathbb{Z}_+$ of a tree $T \in \mathcal{T}$ is defined as the minimal number of prunings necessary to eliminate the tree:

$$k(T) = \min_{k \geq 0} \left( R^k(T) = \phi \right).$$

![Figure 1. Example of pruning and Horton-Strahler ordering. The Horton-
Strahler orders are shown next to each vertex of the initial tree $T$. The figure
shows the two stages of each pruning – cutting the leaves (top row), and
consecutive series reduction (bottom row). The order of the tree is $k(T) = 3$
with $N_1 = 10$, $N_2 = 3$, $N_3 = 1$, and $N_{1,2} = 3$, $N_{1,3} = 1$, $N_{2,3} = 1$.](image)

Pruning partitions the tree space into exhaustive and mutually exclusive set of subspaces $\mathcal{H}_K$ of trees of order $K \geq 1$ such that $R(\mathcal{H}_{K+1}) = \mathcal{H}_K$.

**Definition 2 (Horton-Strahler terminology).** We introduce the following definitions related to the Horton-Strahler order of a tree (see Fig. 2):

1. For any non-root vertex $v$ in $T \in \mathcal{T}\setminus\{\phi\}$, a *sub-tree* $T_v$ of $T$ is defined as the only sub-tree in $T$ rooted at the parental vertex $p(v)$ of $v$, and comprised by $v$ and all its descendant vertices together with their parental edges (Fig. 2b).
2. The Horton-Strahler order $k(v)$ of a vertex $v \in T$ coincides with the order of the sub-tree $T_v \subset T$ (Fig. 2b).
The parental edge of a vertex has the same order as the vertex.

A connected sequence of vertices of the same order together with their parental edges is called a branch (Fig. 2a).

The branch vertex closest to the root is called the initial vertex of the branch (Fig. 2a).

For the initial vertex $v$ of a branch of order $K$, a sub-tree of $T_v$ is called a complete sub-tree of order $K$ (Fig. 2b). The single complete sub-tree of order $k(T)$ coincides with $T$. (All sub-trees of order $k = 1$ are complete.)

**Remark 1.** Equivalently, the Horton-Strahler ordering can be done by hierarchical counting. In this approach, each leaf is assigned order $k(\text{leaf}) = 1$. An internal vertex $p$ whose children have orders $i$ and $j$ is assigned the order

$$k(p) = \max(i, j) + \delta_{ij} = \lfloor \log_2(2^i + 2^j) \rfloor,$$

where $\delta_{ij}$ is the Kronecker’s delta and $\lfloor x \rfloor$ denotes the maximal integer less than or equal to $x$. The parental edge of a vertex has the same order as the vertex. The Horton-Strahler order of the tree is $k(T) = \max_v k(v)$, where the maximum is taken over all non-root vertices of $T \in T\setminus\{\phi\}$.

### Figure 2

Illustration of the Horton-Strahler terminology (Def. 2) in a tree $T$ of order $k(T) = 3$. (a) A tree root, a branch, and an initial vertex of a branch. The numbers indicate the Horton-Strahler orders of the vertices. (b) Examples of a complete ($T_v$) and incomplete ($T_u$) sub-trees. The sub-tree $T_u$ is incomplete since it roots not at the initial vertex of a branch. This tree has four complete sub-trees of order $k \geq 2$: $T_v$, $T_b$, $T_c$, and $T_u = T$.

### 2.2. Labeling tree vertices

Sometimes we will need to label the vertices and edges of a tree (e.g., for selecting a branch or vertex uniformly). The vertices of a planar tree can be labeled by numbers $1, \ldots, \#T$ ($\#T$ denoting the total number of vertices in $T$) in order of depth-first search. We also assume that label of the parental edge for each vertex is taken from that vertex.

For a tree with no embedding, labeling is done by selecting a suitable embedding and then using the depth-first search labeling as above. Such embedding should be properly aligned with the pruning operation, as we describe in the following definition.
Definition 3 (Proper embedding). An embedding function \( \text{MBED} : \mathcal{T} \to \mathcal{T}_{\text{plane}} \) is called proper if for any \( T \in \mathcal{T} \) (\( T \in \mathcal{L} \))
\[
\mathcal{R} \left( \text{MBED}(T) \right) = \text{MBED} \left( \mathcal{R}(T) \right),
\]
where the pruning on the left-hand side is in \( \mathcal{T}_{\text{plane}} \) (\( \mathcal{L}_{\text{plane}} \)) and pruning on the right-hand side is in \( \mathcal{T} \) (\( \mathcal{L} \)).

A proper embedding for a tree with no edge lengths can be done using the following induction construction. A tree of order \( k = 1 \) assumes a unique embedding. A tree of order \( k = 2 \) is embedded by branching all its side-branches of order 1 to the right. Assuming there exists a proper embedding for trees of order \( k \leq K \), we construct the labeling for a tree of order \( K + 1 \). All its side-branches (of any order) branch to the right. To embed the (only) two merging complete sub-trees, \( \tau_1 \neq \tau_2 \), of order \( K \), we consider their farthest non-identical pruning descendants: trees \( d_i = \mathcal{R}^k(\tau_i) \), \( i = 1, 2 \) obtained by the maximal possible number \( k \) of pruning iterations such that \( d_1 \neq d_2 \). The number \( 0 \leq k \leq K - 2 \) is well defined since all trees of order 1, which is the ultimate pruning limit, coincide. By construction, the trees \( d_i \) differ only by the number of side-branches of order 1 attached to the tree \( d_0 = \mathcal{R}^{k+1}(\tau_0) \), which already has proper embedding. Consider the numbers of order-1 side-branches within each edge of \( d_0 \), in the order of its labeling: \( (n_1^{(0)}, \ldots, n_{d_0}^{(0)}) \). The tree whose sequence has the smallest first non-coinciding number, will branch to the right.

A proper embedding for a tree \( T \in \mathcal{L} \) with edge length is constructed in the same fashion, with the only correction. From the two merging complete sub-trees of order \( K \) with the same combinatorial structure, the one with the shortest root edge branches to the right. This definition covers the situation of atomless length distribution, which is of primary interest to us.

3. Tree Self-Similarity

This section defines self-similarity for the combinatorial (Sects. 3.1, 3.2) and metric (Sect. 3.5) tree structure.

3.1. Distributional self-similarity of a combinatorial tree.

Definition 4 (Prune-invariance). Consider a probability measure \( \mu \) on \( \mathcal{T} \) such that \( \mu(\phi) = 0 \). Let \( \nu(T) = \mu \circ \mathcal{R}^{-1}(T) = \mu(\mathcal{R}^{-1}(T)) \). (Note that \( \nu(\phi) > 0 \).) Measure \( \mu \) is called invariant with respect to the pruning operation if for any tree \( T \in \mathcal{T} \) we have
\[
\nu \left( T \mid T \neq \phi \right) = \mu(T).
\]

Let \( \mathcal{H}_K \subset \mathcal{T} \) be the subspace of trees of Horton-Strahler order \( K \geq 1 \). Naturally, \( \mathcal{H}_K \cap \mathcal{H}_{K'} = \emptyset \) if \( K \neq K' \), and \( \bigcup_{K \geq 1} \mathcal{H}_K = \mathcal{T} \setminus \{ \phi \} \). Consider a set of conditional probability measures \( \{ \mu_K \}_{K \geq 1} \) on \( \mathcal{H}_K: \mu_K(T) = \mu(T \mid T \in \mathcal{H}_K) \). Then
\[
\mu = \sum_{K=1}^{\infty} \mu(\mathcal{H}_K) \mu_K.
\]

Proposition 1. Let \( \mu \) be a prune-invariant measure on \( \mathcal{T} \). Then the distribution of orders is geometric:
\[
\mu(\mathcal{H}_K) = \mu(\mathcal{H}_1) \left( 1 - \mu(\mathcal{H}_1) \right)^{K-1},
\]
and for any $T \in \mathcal{H}_K$

$$
\mu_{K+1}(\mathcal{R}^{-1}(T)) = \mu_K(T).
$$

\textit{Proof.} Pruning is a shift operator on the sequence of subspaces $\{\mathcal{H}_k\}$:

(5)  
$$
\mathcal{R}^{-1}(\mathcal{H}_{K-1}) = \mathcal{H}_K, \ K \geq 2.
$$

The only tree eliminated by pruning is the tree of order 1: $\{\tau : \mathcal{R}(\tau) = \phi\} = \mathcal{H}_1$. This means that we can rewrite (2) for any $T \neq \phi$ as

(6)  
$$
\mu(\mathcal{R}^{-1}(T)) = \mu(T)(1 - \mu(\mathcal{H}_1)).
$$

Combining (5) and (6) we find for any $K \geq 2$

(7)  
$$
\mu(\mathcal{H}_K) \text{ by (4)} \mu(\mathcal{R}^{-1}(\mathcal{H}_{K-1})) \text{ by (6)} (1 - \mu(\mathcal{H}_1)) \mu(\mathcal{H}_{K-1}),
$$

which establishes (3). Next, for any tree $T \in \mathcal{H}_K$ we have

$$
\mu(T) = \mu(\mathcal{H}_1)(1 - \mu(\mathcal{H}_1))^{K-1} \mu_K(T),
$$

$$
\mu(\mathcal{R}^{-1}(T)) = \mu(\mathcal{H}_1)(1 - \mu(\mathcal{H}_1))^K \mu_{K+1}(\mathcal{R}^{-1}(T)).
$$

Together with (4) this implies (4). \hfill \Box

Proposition 1 shows that a prune-invariant measure $\mu$ is completely specified by its conditional measures $\mu_K$ and the mass $\mu(\mathcal{H}_1)$ of the single-leaf tree. The same result was obtained for Galton-Watson trees in [4, Thm. 3.5].

We assume that complete sub-trees of the same order have a common distribution, although they can be dependent, as discussed below. Consider the following process of selecting a uniform random complete sub-tree subtree$_K$ of order $K$ from $T \in \mathcal{H}_{\geq K} = \bigcup_{k \geq K} \mathcal{H}_k$.

First, select a random tree $T$ according to conditional measure

$$
\mu_{\geq K}(\cdot) = \mu(\cdot | T \in \mathcal{H}_{\geq K}).
$$

Label all complete sub-trees of order $K$ in $T$ in order of proper labeling of Sect. 2.2 and select a uniform random sub-tree.

\textbf{Definition 5 (Coordination).} A probability measure $\mu$ on $\mathcal{T}$ is called \textit{coordinated} if for any $K \geq 1$ a uniform random complete sub-tree of order $K$ has distribution $\mu_K$:

$$
P(\text{subtree}_K = T) = \mu_K(T) \quad \forall T \in \mathcal{H}_{\geq K}.
$$

\textbf{Example 1.} The space of finite Galton-Watson binary trees has the coordination property. Recall that a random Galton-Watson binary tree starts with a single progenitor (root) and increases its depth in discrete steps: at every step each existing vertex can either split in two with probability $p_2$ or become a leaf (disappear) with probability $p_0 = 1 - p_2$. This generation mechanism creates complete sub-trees of the same structure, independently of the other complete sub-trees of the same order (coordination).

\textbf{Definition 6 (Distributional self-similarity).} A probability measure $\mu$ on $\mathcal{T}$ is called \textit{self-similar} if and only if it is coordinated and prune-invariant.

It is readily seen that there exist coordinated measures that are not prune-invariant (e.g., non-critical Galton-Watson trees), as well as prune-invariant measures with no coordination.
3.2. **Mean self-similarity of a combinatorial tree.** This section describes a weaker type of self-similarity, introduced in [10], that only consider the average values of branch statistics.

We write $E_K(\cdot)$ for the mathematical expectation with respect to $\mu_K$. Let $N_k = N_k[T]$ denotes the number of branches of order $k$ in a tree $T \in \mathcal{T}$. We define the *average Horton numbers* for subspace $\mathcal{H}_K$ as

$$N_k[K] = E_K(N_k), \quad 1 \leq k \leq K, \quad K \geq 1.$$ 

Let $N_{i,j} = N_{i,j}[T]$ denote the number of instances when an order-$i$ branch merges with an order-$j$ branch, $1 \leq i < j$, in a tree $T$. Such branches are referred to as *side-branches* of order $(i,j)$. Consider the respective expectation $N_{i,j}[K] := E_K(N_{i,j})$. The Tokunaga coefficients $T_{i,j}[K]$ for subspace $\mathcal{H}_K$ are defined as

$$(8) \quad T_{i,j}[K] = \frac{N_{i,j}[K]}{N_j[K]}, \quad 1 \leq i < j \leq K.$$ 

**Definition 7 (Mean coordination).** A set of measures $\{\mu_K\}_{K \geq 1}$ on $\{\mathcal{H}_K\}_{K \geq 1}$ is called *mean-coordinated* if $T_{i,j} := T_{i,j}[K]$ for all $K \geq 2$ and $1 \leq i < j \leq K$.

For a set $\{\mu_K\}$ of coordinated measures, the Tokunaga matrix $T_K$ is a $K \times K$ matrix

$$T_K = \begin{bmatrix}
0 & T_{1,2} & T_{1,3} & \cdots & T_{1,K} \\
0 & 0 & T_{2,3} & \cdots & T_{2,K} \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & T_{K-1,K} \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix},$$

which coincides with the restriction of any larger-order Tokunaga matrix $T_M, M > K$, to the first $K \times K$ entries.

**Definition 8 (Mean self-similarity).** A collection of mean-coordinated probability measures $\{\mu_K\}$ on $\{\mathcal{H}_K\}$ is called *mean self-similar* if $T_{i,j} = T_{j,i}$ for some sequence $T_k \geq 0$, $k = 1, 2, \ldots$. The elements of the sequence $T_k$ are also referred to as Tokunaga coefficients, which does not create confusion with $T_{i,j}$.

For a mean self-similar collection of measures the Tokunaga matrix becomes Toeplitz:

$$T_K = \begin{bmatrix}
0 & T_1 & T_2 & \cdots & T_{K-1} \\
0 & 0 & T_1 & \cdots & T_{K-2} \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & T_1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}.$$ 

Pruning $\mathcal{R}$ decreases the Horton-Strahler order of each vertex (and hence of each branch) by unity; in particular

$$(9) \quad N_k[T] = N_{k-1}[\mathcal{R}(T)], \quad k \geq 2,$$

$$(10) \quad N_{i,j}[T] = N_{i-1,j-1}[\mathcal{R}(T)], \quad 2 \leq i < j.$$ 

Consider measure $\mu_K^\mathcal{R}$ induced on $\mathcal{H}_K$ by the pruning operator:

$$\mu_K^\mathcal{R}(A) = \mu_{K+1}(\mathcal{R}^{-1}(A)) \quad \forall A \subset \mathcal{H}_K.$$
The Tokunaga coefficients computed on $\mathcal{H}_K$ using the induced measure $\mu^R_K$ are denoted by $T_{i,j}^R[K]$.

**Definition 9 (Mean self-similarity).** A collection of mean-coordinated probability measures $\{\mu_K\}$ on $\{\mathcal{H}_K\}$ is called mean self-similar if $T_{i,j}[K] = T_{i,j}^R[K]$ for any $K \geq 2$ and all $1 \leq i < j \leq K$.

It is shown in [10] that the Defs. 8 and 9 are equivalent. The Def. 9 is a direct analog of the prune-invariance of Def. 3 expressed in terms of the means.

A variety of mean self-similar measures can be constructed for an arbitrary sequence of Tokunaga coefficients $T_k \geq 0$, $k \geq 1$. Next, we give a natural example [10].

### 3.3. Example: Independent random attachment.

The subspace $\mathcal{H}_1$, which consists of a single-leaf tree, possesses a trivial unity mass measure. To construct a random tree from $\mathcal{H}_2$, we select a discrete probability distribution $P_{1,2}(n)$, $n = 0, 1, \ldots$, with the mean value $T_1$. A random tree $T \in \mathcal{H}_2$ is obtained from the single-leaf tree $\tau_1$ of order 1 via the following two operations. First, we attach two offspring vertices to the leaf of $\tau_1$. This creates a tree of order 2 with no side-branches – one internal vertex of degree 3, two leaves, and the root. Second, we draw the number $\tilde{N}_{1,2}$ from the distribution $P_{1,2}$, and attach $\tilde{N}_{1,2}$ vertices to this tree so that they form side-branches of order $\{1, 2\}$.

In general, to construct a random tree $T \in \mathcal{H}_K$ of order $K \geq 2$ we select a set of discrete probability distributions $P_{k,K}(n)$, $k = 1, \ldots, K-1$, on $\mathbb{Z}_+$ with the respective mean values $T_k$. A random tree $T \in \mathcal{H}_K$ is constructed by adding branches of order 1 (leaves) to a random tree $\tau \in \mathcal{H}_{K-1}$. First, we add two new child vertices to every leaf of $\tau$ hence producing a tree $\tilde{T}$ of order $K$ with no side-branches of order 1. Second, for each branch $b$ of order $2 \leq j \leq K$ in $\tilde{T}$ we draw a random number $\tilde{N}_{1,j}(b)$ from the distribution $P_{j-1,K}$ and attach $\tilde{N}_{1,j}(b)$ new child vertices to this branch so that they form side-branches of order $\{1, j\}$. Each new vertex is attached in a random order with respect to the existing side-branches. Specifically, we notice that $m \geq 0$ side-branches attached to a branch of order $j$ are uniquely associated with $m+1$ edges within this branch. The attachment of the new $\tilde{N}_{1,j}(b)$ vertices among the $m+1$ edges is given by the equiprobable multinomial distribution with $m+1$ categories and $\tilde{N}_{1,j}(b)$ trials.

Observe that

$$N_{i,j} = \sum_{b_i=1}^{N_j} \tilde{N}_{1,j-i+1}(b_i),$$

(11) $$\mathcal{N}_{i,j}[K] = E_K(N_{i,j}) = E_K(E_K(N_{i,j}|N_j)) = E_K(N_jT_{j-i}) = T_{j-i}E_K(N_j) = T_{j-i}N_j[K],$$

and hence $T_{i,j}[K] = \mathcal{N}_{i,j}[K]/\mathcal{N}_j[K] = T_{j-i}$, so the tree is mean self-similar, according to Defs. 8, 9.

### 3.4. Horton law in self-similar trees.

We say that a random tree $T$ satisfies a strong Horton law if the respective sequence $\mathcal{N}_k[K]$ of branch numbers decays in geometric fashion. Specifically, we require

$$\lim_{K \to \infty} \frac{\mathcal{N}_k[K]}{\mathcal{N}_1[K]} = R^{1-k}, \text{ for any } k \geq 1.$$
Horton law and its ramifications, which epitomize scale-invariance of dendritic hierarchical
structures, play an important role in hydrology (e.g., [21, 18, 6]) and have been reported in
biology and other areas (e.g., [17]). It has been shown in [11] that the tree of Kingman’s
coalescent process with \( N \) particles obeys a weaker version of Horton law as \( N \to \infty \), and
that the first pruning of this tree is equivalent to a level set tree of a white noise (see Sect. 4
for definitions).

A necessary and sufficient condition for strong Horton law in a mean self-similar tree has
been established in [10]:

\[
\limsup_{k \to \infty} T_{k}^{1/k} < \infty.
\]

The Horton exponent \( R \) in this case is given by

\[
R \equiv 1 - 1 + 2z + \sum_{k=1}^{\infty} z^k T_k
\]

within the interval \((0, 1/2]\). Informally this means that any mean self-similar tree with a
“tamed” sequence of Tokunaga coefficients satisfies a strong Horton law.

3.5. Distributional self-similarity of trees with edge lengths. Consider a tree \( T \in \mathcal{L} \)
with edge lengths given by a positive vector \( l_T = (l_1, \ldots, l_{\#T}) \) and let \( \text{LENGTH}(T) = \sum_i l_i \).
We assume that the edges are labeled in a proper way as described in Sect. 2.2. A tree is
completely specified by its combinatorial shape \( \text{SHAPE}(T) \) and edge length vector \( l_T \). The
degree length vector \( d_T \) can be specified by distribution \( \chi(\cdot) \) of a point \( x_T = (x_1, \ldots, x_{\#T}) \) on
the simplex \( \sum_i x_i = 1, \) \( 0 < x_i \leq 1 \), and conditional distribution \( F(\cdot | x_T) \) of the tree length
\( \text{LENGTH}(T) \), where

\[
l_T = x_T \times \text{LENGTH}(T).
\]

A measure \( \eta \) on \( \mathcal{L} \) is a joint distribution of tree’s combinatorial shape and its edge lengths
with the following component distributions

\[
\mu(\tau) = \text{Law} (\text{SHAPE}(T) = \tau),
\]

\[
\chi(\bar{x}) = \text{Law} (x_T = \bar{x} | \text{SHAPE}(T) = \tau),
\]

\[
F_{\tau,\bar{x}}(\ell) = \text{Law} (\text{LENGTH}(T) = \ell | x_T = \bar{x}, \text{SHAPE}(T) = \tau).
\]

We denote by \( \mu^K, \chi^K, \) and \( F^K \) the respective measures for a uniform complete sub-tree
\( \text{subtree}_K \). We also consider the distribution of edge lengths after pruning:

\[
\Xi(\bar{x}) = \text{Law} (x_{\mathcal{R}(T)} = \bar{x} | \text{SHAPE}(\mathcal{R}(T)) = \tau).
\]

and

\[
\Phi_{\tau,\bar{x}}(\ell) = \text{Law} (\text{LENGTH}(\mathcal{R}(T)) = \ell | x_{\mathcal{R}(T)} = \bar{x}, \text{SHAPE}(\mathcal{R}(T)) = \tau).
\]

Definition 10 (Distributional self-similarity for trees with edge lengths). We call
a measure \( \eta \) on \( \mathcal{L} \) self-similar if and only if the following conditions hold

(i) The measure is coordinated in shapes and lengths:

\[
\mu^K(\tau) = \mu_K(\tau), \quad \chi^K(\bar{x}) = \chi^K(\bar{x}) \quad \text{and} \quad F^K_{\tau,\bar{x}}(\ell) = F_{\tau,\bar{x}}(\ell).
\]
The measure is prune-invariant in shapes and lengths. This means that for $\nu = \mu \circ R^{-1}$ we have

$$\mu(\tau) = \nu(\tau | \tau \neq \phi), \quad \Xi_\tau(\bar{x}) = \chi_\tau(\bar{x})$$

and there exists a self-similarity constant $\zeta > 0$ such that for any combinatorial tree $\tau \in \mathcal{T}$ we have

$$\Phi_{\tau,\bar{x}}(\ell) = \zeta^{-1} F_{\tau,\bar{x}} \left( \frac{\ell}{\zeta} \right).$$

Section 5 below introduces a rich class of measures that satisfy the above definition.

4. Tree Representation of Continuous Functions

We review here the results of [13, 16, 19, 27] on tree representation of continuous functions. This allows us to apply the self-similarity concepts to time series and motivates discussion in Sects. 5.6, 5.7 below.

4.1. Harris path. For any embedded tree with edge lengths $T \in \mathcal{L}_{\text{plane}}$ the Harris path is defined as a piece-wise linear function [8, 19]

$$H_T(t) : [0, 2\text{LENGTH}(T)] \rightarrow \mathbb{R}$$

that equals the distance from the root traveled along the tree $T$ in the depth-first search, as illustrated in Fig. 3. For a tree $T$ with $n$ leaves, the Harris path $H_T(t)$ is a piece-wise linear positive excursion that consists of $2n$ linear segments with alternating slopes $\pm 1$ [19].

![Figure 3](image_url)

**Figure 3.** (a) Tree $T$ and its depth-first search illustrated by dashed arrows. (b) Harris path $H_T(t)$ for the tree $T$ of panel (a).

4.2. Level set tree. Consider a continuous function $X_t, t \in [a, b]$ with a finite number of distinct local minima. The level set $\mathcal{L}_\alpha(X_t)$ is defined as the pre-image of the function values above $\alpha$:

$$\mathcal{L}_\alpha(X_t) = \{ t : X_t \geq \alpha \}.$$ 

The level set $\mathcal{L}_\alpha$ for each $\alpha$ is a union of non-overlapping intervals; we write $|\mathcal{L}_\alpha|$ for their number. Notice that $|\mathcal{L}_\alpha| = |\mathcal{L}_\beta|$ as soon as the interval $[\alpha, \beta]$ does not contain a value of local extrema of $X_t$ and $0 \leq |\mathcal{L}_\alpha| \leq n$, where $n$ is the number of the local maxima of $X_t$. 
The level set tree \( \text{LEVEL}(X_t) \in \mathcal{L}_{\text{plane}} \) is a tree that describes the topology of the level sets \( \mathcal{L}_\alpha \) as a function of threshold \( \alpha \), as illustrated in Fig. 4. Specifically, there are bijections between (i) the leaves of \( \text{LEVEL}(X_t) \) and the local maxima of \( X_t \), (ii) the internal (parental) vertices of \( \text{LEVEL}(X_t) \) and the local minima of \( X_t \) (excluding possible local minima at the boundary points), and (iii) the pair of sub-trees of \( \text{LEVEL}(X_t) \) rooted at a local minima \( X_t^r \) and the first positive excursions (or meanders bounded by \( t = a \) or \( t = b \)) of \( X_t - X_t^r \) to right and left of \( t^r \). Every edge in the tree is assigned a length equal the difference of the values of \( X_t \) at the local extrema that correspond to the vertices adjacent to this edge according to the bijections (i) and (ii) above. If the global minimum of \( X_t \) is not reached at the boundary, the minimum value will correspond to the first descendant of the root, as in Fig. 4. We refer to [27] for discussion of some subtleties related to this construction as well as for further references.

By construction, the level set tree \( \text{LEVEL}(X_t) \) is completely determined by the sequence of the values of local extrema of \( X_t \). Specifically, if \( g(t) \) is continuous and monotone increasing on \([a, b]\), and \( X_t^{\text{lin}} \) is a linear interpolation of the local extrema of \( X_t \), then

\[
\text{LEVEL}(X_t) = \text{LEVEL} \left( X_t^{\text{lin}}(g(t)) \right).
\]

Hence, without loss of generality we can focus on the level set trees of continuous functions with alternating slopes \( \pm 1 \). To ensure that the level set tree of a function is binary, we need to eliminate the instances of consecutive local minima with the same values. This, for instance, is achieved if the distribution of lengths of linear segments has no atoms (this condition is sufficient, but not necessary). The space of piece-wise linear continuous functions on \([a, b]\) with alternating slopes \( \pm 1 \) and atomless segment length distribution is denoted \( \mathcal{E}([a, b]) \).

By construction, the level set tree and Harris path are reciprocal to each other as described in the following statement.

**Proposition 2 (Reciprocity of Harris path and level set tree).** Consider the space \( \mathcal{E}^{\text{ex}}([a, b]) \) of positive excursions from \( \mathcal{E}([a, b]) \). The Harris path \( H_T : \mathcal{L}_{\text{plane}} \to \mathcal{E}^{\text{ex}} \) and the level set tree \( \text{LEVEL}(X) : \mathcal{E}^{\text{ex}} \to \mathcal{L}_{\text{plane}} \) are reciprocal to each other. Specifically, for any \( T \in \mathcal{L}_{\text{plane}} \) we have \( \text{LEVEL}(H_T) \equiv T \), and for any \( X \in \mathcal{E}^{\text{ex}} \) we have \( H_{\text{LEVEL}(X)} \equiv X \).
4.3. **Pruning of continuous functions.** In the space of continuous functions, the pruning corresponds to coarsening the function resolution by removing (smoothing) the local maxima. An iterative pruning corresponds to iterative transition to the local minima.

**Proposition 3 (Pruning of a function, [27]).** The transition from a function $X \in C([a, b])$ with a finite number of distinct local minima to the linearly interpolated function $X^{(1)} \in C([a, b])$ of its local minima corresponds to the pruning of the level set tree $\text{level}(X)$. Formally,

$$\text{LEVEL} \left( X^{(m)} \right) = R^m \left( \text{LEVEL}(X) \right), \forall m \geq 1,$$

where $X^{(m)}$ is obtained from $X$ by iteratively taking local minima $m$ times (i.e., local minima of local minima and so on.)

According to Prop. 3, the problem of finding self-similar trees with edge lengths is equivalent to finding extreme-invariant processes

(13)

$$X_k \overset{d}{=} \zeta X_k^{\min} \quad \text{for some } \zeta > 0,$$

where $X_k, k \in \mathbb{Z}_+$, is a discrete-time process with an atomless value distribution such that $X_0 = 0$ and $X_k^{\min}$ is the discrete-time process of the local minima of $X_k$: the $i$-th local minimum of $X_k$ corresponds to $X_i^{\min}$. If $X_k$ satisfies (13), the level set tree $T = \text{level}(X)$ of its continuous linear interpolation, considered as an element of $L$, is self-similar according to Def. 10.

4.4. **Self-similarity of random walks on $\mathbb{R}$.** Consider a random walk $\{X_t\}_{t \in \mathbb{Z}}$ on $\mathbb{R}$ with a homogeneous transition kernel $p(x, y) \equiv p(x - y)$, for any $x, y \in \mathbb{R}$, where $p(x)$ is an atomless density function. Extending to $\mathbb{R}$, we let $\{X_t\}_{t \in \mathbb{R}}$ denote a function from $C(-\infty, \infty)$ obtained by linearly interpolating the values of the original time series $\{X_t\}_{t \in \mathbb{Z}}$. A homogeneous random walk is called symmetric if $p(x) = p(-x)$ for all $x \in \mathbb{R}$.

**Lemma 1 (Pruning random walks, [27]).** The following statements hold.

- **a:** The local minima of a homogeneous random walk $\{X_t\}_{t \in \mathbb{Z}}$ form a homogeneous random walk (with a different transition kernel in general).
- **b:** The local minima of a symmetric homogeneous random walk $\{X_t\}_{t \in \mathbb{Z}}$ form a symmetric homogeneous random walk (with a different transition kernel in general).

The transition kernel of a symmetric random walk can be represented as

$$p(x) = \frac{f(x) + f(-x)}{2},$$

where $f(x)$ is a probability density function with support in $\mathbb{R}_+$. The following result describes the solution of the problem [13] in terms of characteristic function of $f(x)$.

**Proposition 4 (Distributional self-similarity of symmetric homogeneous random walks on $\mathbb{R}$, [27]).** The local minima of a symmetric homogeneous random walk $\{X_t\}_{t \in \mathbb{Z}}$ on the real line with a transition kernel $p(x) = \frac{f(x) + f(-x)}{2}$ form a symmetric homogeneous random walk with a transition kernel

$$p^{(1)}(x) = \zeta^{-1} p(x/\zeta), \quad c > 0$$
if and only if $\zeta = 2$ and

\begin{equation}
\mathbb{R} \left[ \hat{f}(2s) \right] = \left| \frac{\hat{f}(s)}{2 - \hat{f}(s)} \right|^2,
\end{equation}

where $\hat{f}(s)$ is the characteristic function of $f(x)$ and $\mathbb{R}[z]$ stays for the real part of $z \in \mathbb{C}$.

A solution to (14) is given for example by an exponential density $f(x) = \phi_\lambda(x)$ of (16) for any $\lambda > 0$; a detailed discussion of exponential kernels is given in Sect. 4.5. A weaker, mean self-similarity of Defs. 8, 9 is satisfied in any symmetric random walk, as discussed in the following statement.

**Theorem 1 (Mean self-similarity of symmetric homogeneous random walks on $\mathbb{R}$, [27]).** The combinatorial level set tree $T = \text{shape(level}(X_t))$ of a finite symmetric homogeneous random walk $X_t$ with $t = 1, \ldots, N$ is mean self-similar. Specifically, for any complete sub-tree $\tau \in T$ of order $K < k(T)$ the numbers $\tau_{i,j}^{(r)}$ of side-branches of order $i$ that merge the $r$-th branch of order $j$ with $2 \leq j \leq K$ in $\tau$ are independent identically distributed random variables. If $\tau_{i,j}$ is a random variable such that $\tau_{i,j} \overset{d}{=} \tau_{i,j}$, then

\begin{equation}
T_{i,j} := \mathbb{E} [\tau_{i,j}] = 2^{j-i-1} =: T_{j-i}.
\end{equation}

Moreover, by the strong law of large numbers $k(T) \overset{a.s.}{\to} \infty$ and for any $i, j \geq 1$ we have

\begin{equation}
T_{i,j} \overset{a.s.}{\to} T_{j-i} = 2^{j-i-1}, \quad \text{as } N \to \infty,
\end{equation}

where $T_{i,j}$ can be computed over the entire $T$.

**4.5. Exponential random walks.** We call a symmetric homogeneous random *exponential* if its kernel is a mixture of exponential jumps constructed as follows

\begin{equation}
p(x) = p \phi_{\lambda_u}(x) + (1 - p) \phi_{\lambda_d}(-x), \quad 0 \leq p \leq 1, \quad \lambda_u, \lambda_d > 0,
\end{equation}

where $\phi_\lambda$ is the exponential density with parameter $\lambda > 0$,

\begin{equation}
\phi_\lambda(x) = \begin{cases} 
\lambda e^{-\lambda x}, & x \geq 0, \\
0, & x < 0.
\end{cases}
\end{equation}

We refer to an exponential random walk by its parameter triplet \{p, \lambda_u, \lambda_d\}. Each exponential random walk with parameters \{p, \lambda_u, \lambda_d\} corresponds to a piece-wise linear function whose rises and falls have independent exponential lengths with parameters $(1 - p)\lambda_u$ and $p\lambda_d$, respectively.

**Theorem 2 (Distributional self-similarity of exponential random walks, [27]).** Let $X_t$ be an exponential random walk with parameters \{p, \lambda_u, \lambda_d\}. Then

- **a:** The local minima of $X_t$ form a exponential random walk with parameters \{p*, \lambda_u^*, \lambda_d^*\} such that

\begin{equation}
p^* = \frac{p \lambda_d}{p \lambda_d + (1 - p) \lambda_u}, \quad \lambda_u^* = p \lambda_d, \quad \text{and} \quad \lambda_d^* = (1 - p) \lambda_u.
\end{equation}

- **b:** The walk $X_t$ satisfies the distributional self-similarity [13] if and only if $p = 1/2$ and $\lambda_u = \lambda_d$.

- **c:** The self-similarity [13] is achieved after the first pruning, for the chain $X_t^{(1)}$ of the local minima, if and only if the walk’s increments have zero mean, $p \lambda_d = (1 - p) \lambda_u$. 

Recall that a binary Galton-Watson tree describes trajectory of the Galton-Watson branching process. The process starts with a single progenitor (tree root) at time $t = 0$. At each discrete time step every existing population member terminates and produces either no offspring with probability $p_0$ or two offspring with probability $p_2 = 1 - p_0$, independently of the other members. We denote the resulting tree distribution on $\mathcal{T}$ by $\mathcal{GW}(p_0, p_2)$.

**Definition 11 (Exponential binary Galton-Watson tree, [19]).** We say that a random embedded binary tree $T \in \mathcal{L}_{\text{plane}}$ is an exponential binary Galton-Watson tree $\mathcal{GW}(\lambda', \lambda)$, for $0 \leq \lambda' < \lambda$, if $\text{shape}(T)$ is a binary Galton-Watson tree $\mathcal{GW}(p_0, p_2)$ with
\[
p_0 = \frac{\lambda + \lambda'}{2\lambda}, \quad p_2 = \frac{\lambda - \lambda'}{2\lambda},
\]
and given $\text{shape}(T)$, the edges of $T$ are sampled as independent exponential random variables with parameter $2\lambda$, i.e., with density $\phi_{2\lambda}(x)$.

A connection between exponential random walks and Galton-Watson trees is given by the following well known result.

**Theorem 3.** [19] Lemma 7.3, [13] [16] Consider a random walk $X_t \in \mathcal{E}^{\text{ex}}([a, b])$. The level set tree $\text{level}(X_t)$ is an exponential binary Galton-Watson tree $\mathcal{GW}(\lambda', \lambda)$ if and only if the rises and falls of $X_t$, excluding the last fall, are distributed as independent exponential random variables with parameters $(\lambda + \lambda')$ and $(\lambda - \lambda')$, respectively, for some $0 \leq \lambda' < \lambda$. In other words, a level set tree of a homogeneous random walk is a binary Galton-Watson tree $\mathcal{GW}(\lambda', \lambda)$ if and only if the walk is exponential with parameters $\{p, \lambda_u, \lambda_p\}$ such that $(1 - p)\lambda_u = \lambda + \lambda'$ and $p\lambda_p = \lambda - \lambda'$.

We emphasize the following direct consequence of Thms. 2(a) and 3.

**Corollary 1.** Consider a critical binary Galton-Watson tree with independent exponential lengths, $T = \mathcal{GW}(0, \gamma)$. The following statements hold:

- **a:** The Harris path of $\mathcal{R}^k(T)$ for any $0 \leq k < k(T)$ is a positive excursion of a symmetric exponential random walk with parameters $\{\frac{1}{2}, 2^{1-k}\gamma, 2^{1-k}\gamma\}$, or, equivalently, $\mathcal{R}^k(T) = \mathcal{GW}(0, 2^{-k}\gamma)$.

- **b:** The length of any branch of order $j \geq 1$ in $T$ has exponential distribution with parameter $2^{2-j}\gamma$. The lengths of branches (of all orders) are independent.

5. **Hierarchical branching process**

The results of previous section concern a very narrow class of mean self-similar trees – those with $T_k = 2^{k-1}$. Among such trees, the distributional self-similarity is established only for the critical Galton-Watson binary tree $\mathcal{GW}(0, \gamma)$ with independent exponential edge lengths; this case corresponds to the length self-similarity constant $\zeta = 2$. Here we construct a branching process that generates distributionally self-similar trees for an arbitrary sequence $T_k \geq 0$ and for any $\zeta > 0$; it includes the critical Galton-Watson tree as a special case.

5.1. **Definition, branch structure, self-similarity.** Consider a probability mass function $\{p_K\}_{K \geq 1}$, and positive sequences $\{T_k\}_{k \geq 1}$, $\{\lambda_j\}_{j \geq 1}$. A multi-type branching process $S(t)$ starts with a root edge of hierarchical order $K \geq 1$ with probability $p_K$. Every branch of order $j \leq K$ produces offspring of order $i < j$ with rate $\lambda_i T_{j-i}$. A branch of order $j$ terminates with rate $\lambda_j$. After termination, a branch of order $j \geq 2$ splits into two branches of order...
$j - 1$. A branch of order $j = 1$ terminates without leaving offspring. The branching history of $S(t)$ creates a random binary tree $T[S]$ in the space $L$ of binary trees with edge lengths and no planar embedding. The process is uniquely specified by the triplet

$$S(t) = (\{T_k\}, \{\lambda_j\}, \{p_K\}).$$

**Proposition 5 (Side-branching in hierarchical branching process).** Consider a hierarchical branching process $S(t) = (\{T_k\}, \{\lambda_j\}, \{p_K\})$. For any branch $b \subset T[S]$ of order $K + 1 \geq 2$, let $m_i := m_i(b) \geq 0$ be the number of its side branches of order $i = 1, \ldots, K$, and $m := m(b) = m_1 + \cdots + m_K$ be the total number of the side branches. Let $l_i := l_i(b)$ be the lengths of $m + 1$ edges within $b$, counted sequentially from the initial vertex, and $l := l(b) = l_1 + \cdots + l_{m+1}$ be the total branch length. Then the following statements hold:

1. The total numbers $m(b)$ of side branches within different branches of order $K + 1$ are i.i.d. random variables with a common geometric distribution:

$$P(m = \kappa) = q(1 - q)^\kappa \quad \text{with} \quad q = \frac{1}{1 + T_1 + \cdots + T_K}, \quad \kappa = 0, 1, \ldots.$$

2. The number $m_i$ of side branches of order $i$ has geometric distribution:

$$P(m_i = \kappa) = q_i(1 - q_i)^\kappa \quad \text{with} \quad q_i = \frac{1}{1 + T_{K-i+1}}, \quad \kappa = 0, 1, \ldots.$$

3. Conditioned on the total number $m$ of side branches, the distribution of $\{m_i\}$ is multinomial with $m$ trials and success probabilities

$$P(\text{side branch has order } i) = \frac{T_{K-i+1}}{T_1 + \cdots + T_K}.$$

The side branch order vector $k = (k_1, \ldots, k_m)$, where the side branches are labeled sequentially starting from the initial vertex of $b$, is obtained from the sequence

orders = $\begin{cases} 1, \ldots, 1, 2, \ldots, 2, \ldots, K, \ldots, K \end{cases}$

$m_1$ times \hspace{1cm} $m_2$ times \hspace{1cm} $m_K$ times

by a uniform random permutation $\sigma_m$ of indices $\{1, \ldots, m\}$:

$$k = \text{orders} \circ \sigma_m.$$

4. The branch length $l$ has exponential distribution with rate $\lambda_{K+1}$, independent of the lengths of any other branch (of any order). The corresponding edge lengths $l_i$ are i.i.d. random variables; they have a common exponential distribution with rate

$$\lambda_{K+1}(1 + T_1 + \cdots + T_K).$$

**Proof.** All the properties readily follow from process construction. \qed

Proposition 5 provides an alternative definition of the hierarchical branching process, and its construction – via parts (1), (3), and (4) – that does not require time-dependent simulations.

**Theorem 4 (Self-similarity of hierarchical branching process).** Consider a hierarchical branching process $S(t) = (\{T_k\}, \{\lambda_j\}, \{p_K\})$ and let $T := T[S]$ be the tree generated by $S(t)$. The following statements hold.
(1) The combinatorial part of $T$ is mean self-similar (according to Def. 8,9) with Tokunaga coefficients $T_k$.
(2) The combinatorial part of $T$ is distributionally self-similar (according to Def. 6) with Tokunaga coefficients $T_k$ if and only if

$$p_K = p(1 - p)^{K-1}$$

for some $0 < p < 1$.
(3) The tree $T$ is distributionally self-similar (according to Def. 10) with length self-similarity constant $\zeta > 0$ if and only if

$$p_K = p(1 - p)^{K-1} \quad \text{and} \quad \lambda_j = \gamma \zeta^{-j}$$

for some positive $\gamma$ and $0 < p < 1$.

**Proof.** By process construction, the tree $T$ is coordinated in shapes and length (according to Def. 10), with independent complete sub-trees.

(1) Proposition 5, part (3) implies that the expected value of the number $\tilde{N}_{i,j}$ of side branches of order $i \geq 1$ within a branch of order $j > i$ is given by $E(\tilde{N}_{i,j}) = T_{j-i}$. The mean self-similarity of Def. 8 with coefficients $T_k$ immediately follows, using a conditional argument as in (11).

(2) Assume that SHAPE($T$) is distributionally self-similar. A geometric distribution of orders is then established in Prop. 1. Inversely, a geometric distribution of orders ensures that the total mass $\mu(H_K)$, $K \geq 1$, is invariant with respect to pruning. The conditional distribution of trees of a given order is completely specified by the side branch distribution, described in Proposition 5 parts (1)-(3). Consider a branch of order $K + 1$, $K \geq 1$. Pruning decreases the orders of this branch, and all its side branches, by unity. Pruning eliminates a random geometric number $m_1$ of side-branches of order 1 from the branch. It acts as a thinning (with removal probability $T_K/(T_1 + \cdots + T_K)$) on the total side branch count $m$. Accordingly, the total side branch count after pruning has geometric distribution with success probability

$$q^R = \frac{1}{1 + T_1 + \cdots + T_{K-1}}.$$  

The order assignment among the remaining side branches of orders $i = 1, \ldots, K - 1$ is done according to multinomial distribution with probabilities proportional to $T_{K-i}$. This coincides with the side branch structure in the original tree, hence completing the proof of (2).

(3) Having proven (2), it remains to prove the statement for the length structure of the tree. Assume that $T$ is distributionally self-similar with length self-similarity constant $\zeta$. The branches of order $j \geq 2$ become branches of order $j - 1$ after pruning, which necessitates $\lambda_j = \zeta \lambda_{j-1}$. Inversely, pruning acts as a thinning on the side branches within a branch of order $K + 1$, eliminating the side branches of order $k = 1$. Accordingly, the spacings between the remaining side branches are exponentially distributed with a decreased rate

$$\lambda_{K+1}(1 + T_1 + \cdots + T_{K-1}) = \zeta \lambda_K(1 + T_1 + \cdots + T_{K-1}).$$

Comparing this with (21), and recalling the distributional self-similarity of SHAPE($T$), we conclude that Def. 10 is satisfied with self-similarity constant $\zeta$. □
5.2. **Hydrodynamic limit.** Here we analyze the average numbers of branches of different orders in a hierarchical branching process, using a hydrodynamic limit. Specifically, let \( n x_j^{(n)}(s) \) be the number of branches of order \( j \) at time \( s \) observed in \( n \) independent copies of the process \( S \). Let \( N_j(s) \) be the number of branches of order \( j \geq 1 \) in the process \( S \) at instant \( s \geq 0 \). We observe that, by the law of large numbers,

\[
x_j^{(n)}(s) \xrightarrow{a.s.} E(N_j(s)) =: x_j(s).
\]

**Theorem 5 (Hydrodynamic limit for branch dynamics).** Suppose that the following conditions are satisfied:

(22) \[
L := \lim_{k \to \infty} T_k^{1/k} < \infty,
\]

and

(23) \[
\sup_{j \geq 1} \lambda_j < \infty, \quad \lim_{j \to \infty} \lambda_j^{1/j} \leq 1/L.
\]

Then, for any given \( T > 0 \), the empirical process

\[
x^{(n)}(s) = \left( x_1^{(n)}(s), x_2^{(n)}(s), \ldots \right)^T, \quad s \in [0, T],
\]

converges almost surely, as \( n \to \infty \), to the process

\[
x(s) = \left( x_1(s), x_2(s), \ldots \right)^T, \quad s \in [0, T],
\]

that satisfies

(24) \[
\dot{x} = GAx \quad \text{with the initial conditions} \quad x(0) = \pi := \sum_{K=1}^\infty p_Ke_K,
\]

where \( \Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots\} \) is a diagonal operator with the entries \( \lambda_1, \lambda_2, \ldots, e_i \) are the standard basis vectors, and

(25) \[
G := \begin{bmatrix}
-1 & T_1 + 2 & T_2 & T_3 & \cdots \\
0 & -1 & T_1 + 2 & T_2 & \cdots \\
0 & 0 & -1 & T_1 + 2 & \cdots \\
0 & 0 & 0 & -1 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}.
\]

**Proof.** The process \( x^{(n)}(s) \) evolves according to the transition rates

\[
q^{(n)}(x, x + \ell) = n\beta_\ell \left( \frac{1}{n} \right)
\]

with

\[
\beta_\ell(x) = \begin{cases} 
\lambda_1x_1 & \text{if } \ell = -e_1, \\
\lambda_i\ell x_{i+1} & \text{if } \ell = 2e_i - e_{i+1}, i \geq 1, \\
\sum_{j=i+1}^\infty \lambda_j T_{j-i} x_j & \text{if } \ell = e_i, i \geq 1.
\end{cases}
\]

Here the first term reflects termination of branches of order 1; the second term reflects termination of branches of orders \( i + 1 > 1 \), each of which results in creation of two branches.
of order \( i \); and the last term reflects side-branching. Thus, the infinitesimal generator of the stochastic process \( x^{(n)}(s) \) is
\[
L_n f(x) = n \lambda_1 x_1 \left[ f \left( x - \frac{1}{n} e_1 \right) - f(x) \right] + \sum_{i=1}^{\infty} n \lambda_{i+1} x_{i+1} \left[ f \left( x - \frac{1}{n} e_{i+1} + \frac{2}{n} e_i \right) - f(x) \right] + \sum_{i=1}^{\infty} \left( \sum_{j=i+1}^{\infty} n \lambda_j T_{j-i} x_j \right) \left( f \left( x + \frac{1}{n} e_i \right) - f(x) \right).
\]
(26)

Let
\[
F(x) := \sum_{i} \beta_i(x) = -\lambda_1 x_1 e_1 + \sum_{i=1}^{\infty} \lambda_{i+1} x_{i+1} (2e_i - e_{i+1}) + \sum_{i=1}^{\infty} \left( \sum_{j=i+1}^{\infty} \lambda_j T_{j-i} x_j \right) e_i.
\]
The convergence result of Kurtz ([7, Theorem 2.1, Chapter 11], [12, Theorem 8.1]) extends (without changing the proof) to the Banach space \( \ell^1(\mathbb{R}) \) provided the same conditions are satisfied for \( \ell^1(\mathbb{R}) \) as for \( \mathbb{R}^d \) in the theorem of Kurtz. Specifically, we require that for a compact set \( C \) in \( \ell^1(\mathbb{R}) \),
\[
\sum_{\ell} \|\ell\|_1 \sup_{x \in C} \beta_\ell(x) < \infty,
\]
and there exists \( M_C > 0 \) such that
\[
\|F(x) - F(y)\|_1 \leq M_C \|x - y\|_1, \quad x, y \in C.
\]
(27)
Here the condition [27] follows from
\[
\sum_{i} \sup_{x \in C} |\lambda_i x_i| < \infty \quad \text{and} \quad \sum_{i} \sup_{x \in C} \sum_{j=i+1}^{\infty} |\lambda_j T_{j-i} x_j| < \infty,
\]
which in turn follow from conditions [23]. Similarly, Lipschitz conditions [28] are satisfied in \( C \) due to conditions (23). Thus, by Kurtz ([7, Theorem 2.1, Chapter 11], [12, Theorem 8.1]), the process \( x^{(n)}(s) \) converges almost surely to \( x(s) \) that satisfies \( \dot{x} = F(x) \), which expands as the following system of ordinary differential equations:
\[
\begin{cases}
x_1'(s) = -\lambda_1 x_1 + \lambda_2 (T_1 + 2) x_2 + \lambda_3 T_2 x_3 + \ldots \\
x_2'(s) = -\lambda_2 x_2 + \lambda_3 (T_1 + 2) x_3 + \lambda_4 T_2 x_4 + \ldots \\
\vdots \\
x_k'(s) = -\lambda_k x_k + \lambda_{k+1} (T_1 + 2) x_{k+1} + \lambda_{k+2} T_2 x_{k+2} + \ldots \\
\end{cases}
\]
(29)
with the initial conditions \( x(0) = \lim_{n \to \infty} x^{(n)}(0) = \pi := \sum_{K=1}^{\infty} p_K e_K \) by the law of large numbers.

Finally, we observe that \( \|\pi\|_1 = 1 \), and conditions [23] imply that \( GA \) is a bounded operator in \( \ell^1(\mathbb{R}) \). \( \square \)
5.3. Criticality and time invariance. Assume that the hydrodynamic limit \( x(s) \), and hence the averages \( x_j(s) \), exist. Let \( \pi = \sum_{K=1}^{\infty} p_K e_K \). Then one can consider the average progeny of the process, that is the average number of branches of any order alive at instant \( s \):

\[
C(s) = \sum_{j=1}^{\infty} x_j(s) = \| e^{G_A s} \pi \|_1 .
\]

In hydrological literature, an empirical version of the process \( C(s) \) is called the width function of a tree \( T[S] \).

**Definition 12.** A hierarchical branching process \( S(s) \) is said to be critical if and only if the width function \( C(s) = 1 \) for all \( s \geq 0 \).

**Definition 13.** A hierarchical branching process \( S(s) \) is said to be time-invariant if and only if

\[
ed^{G_A s} \pi = \pi \quad \text{for all} \quad s \geq 0 .
\]

**Proposition 6.** Suppose that the hydrodynamic limit \( x(s) \) exists, and \( \pi \) is time-invariant. Then the process \( S(s) \) is critical.

**Proof.**

\[
C(s) = \| x(s) \|_1 = \| e^{G_A s} \pi \|_1 = \| \pi \|_1 = 1 .
\]

Let \( \hat{\iota}(z) = -1 + 2z + \sum_j z^j T_j \) for \(|z| < 1/L\), where \( L \) is defined in (22). Observe that there is a unique real root \( w_0 \) of \( \hat{\iota}(z) \) within \((0, \frac{1}{2}]\). We formulate our results in terms of the Horton exponent \( R := w_0^{-1} \) (e.g., [18, 10]).

**Proposition 7.** Suppose \( \Lambda \pi \) is a constant multiple of the geometric vector \( v_0 = \sum_{K=1}^{\infty} R^{-K} e_K \).

Then the process \( S(s) \) is time-invariant.

**Proof.**

Observe that since \( \hat{\iota}(R^{-1}) = 0 \) and \( G \) is a Toeplitz operator,

\[
G v = \hat{\iota}(w) v \quad \text{for} \quad v = \sum_{K=1}^{\infty} w^K e_K , \ |w| < L .
\]

and

\[
G v_0 = \hat{\iota}(R^{-1}) v_0 = 0 \quad \text{for} \quad v_0 := \sum_{K=1}^{\infty} R^{-K} e_K .
\]

Hence \( G \Lambda \pi = \hat{\iota}(R^{-1}) \Lambda \pi = 0 \) and

\[
ed^{G_A s} \pi = \pi + \sum_{m=1}^{\infty} S^m \frac{1}{m!} (G \Lambda)^m \pi = \pi .
\]

**Remark 2.** Proposition 6 states that the condition

\[
\lambda_K p_K = b R^{-K} , \ K \geq 1
\]

is sufficient for time-invariance, for any proportionality constant \( b > 0 \). This implies that a time-invariant process can be constructed for
(i) an arbitrary sequence of Tokunaga coefficients \( \{T_k\} \) satisfying (22) – by selecting \( \lambda_K p_K = b R^{-K} \);
(ii) arbitrary sequences \( \{T_k\} \) satisfying (22) and \( \{p_K\} \) – by selecting \( \lambda_K = b R^{-K} p_K^{-1} \);
(iii) arbitrary sequences \( \{T_k\} \) satisfying (22) and \( \{\lambda_K\} \) – by selecting \( p_K = b R^{-K} \lambda_K^{-1} \).

At the same time, arbitrary sequences \( \{\lambda_K\}, \{p_K\} \) will not, in general, satisfy (31) and hence will not correspond to a time-invariant process.

5.4. Criticality and time-invariance in a self-similar process. A convenient characterization of criticality can be established for self-similar hierarchical branching processes. Recall that by Theorem 4, part (3), a self-similar process \( S(s) \) is specified by parameters \( \gamma > 0, 0 < p < 1 \) and length self-similarity constant \( \zeta > 0 \) such that \( p_K = p(1-p)^K \) and \( \lambda_j = \gamma \zeta^j \). We refer to a self-similar process by its parameter triplet, \( S_{p, \gamma, \zeta} \), and denote the respective width function by \( C_{p, \gamma, \zeta} \).

Observe that in the self-similar case the first of the conditions (23) is equivalent to \( \zeta \geq 1 \), and the second is equivalent to \( \zeta \geq L \).

Hence, the conditions (23) are equivalent to \( \zeta \geq 1 \lor L \).

Theorem 6 (Width function of a self-similar process). Consider a self-similar process \( S_{p, \gamma, \zeta}(s) \) with \( 0 < p < 1, \gamma > 0 \). Suppose that (22) is satisfied and \( \zeta \geq 1 \lor L \). Then

\[
C_{p, \gamma, \zeta}(s) = \begin{cases} 
\text{decreases} & \text{if } p > 1 - \frac{\zeta}{R}, \\
1 & \text{if } p = 1 - \frac{\zeta}{R}, \\
\text{increases} & \text{if } p < 1 - \frac{\zeta}{R}.
\end{cases}
\]

Proof. The choice of the limits for \( \zeta \) ensures that the conditions (23) are satisfied and hence, by Theorem 5, the hydrodynamic limit \( x(s) \) exists and the width function \( C_{p, \gamma, \zeta}(s) \) is well defined. Now we have

\[
\Lambda \pi = \frac{\gamma p}{1-p} \sum_{K=1}^{\infty} \left( \zeta^{-1}(1-p) \right)^K e_K,
\]

and therefore

\[
(\mathcal{G} \Lambda)^2 \pi = \hat{t}(\zeta^{-1}(1-p)) \mathcal{G} \Lambda^2 \pi = \hat{t}(\zeta^{-1}(1-p)) \hat{t}(\zeta^{-2}(1-p)) \Lambda^2 \pi,
\]

and in general,

\[
(\mathcal{G} \Lambda)^m \pi = \hat{t}(\zeta^{-1}(1-p)) \mathcal{G} \Lambda^m \pi = \left[ \prod_{i=1}^{m} \hat{t}(\zeta^{-i}(1-p)) \right] \Lambda^m \pi.
\]

Thus, taking \( x(0) = \pi \),

\[
x(s) = e^{\mathcal{G} \Lambda s} \pi = \pi + \sum_{m=1}^{\infty} \frac{s^m}{m!} \left[ \prod_{i=1}^{m} \hat{t}(\zeta^{-i}(1-p)) \right] \Lambda^m \pi.
\]
The width function for the given values of \( p \in (0,1) \), \( \gamma > 0 \) and \( \zeta \geq 1 \) can therefore be expressed as

\[
C_{p,\gamma,\zeta}(s) = \sum_{j=1}^{\infty} x_j(s) = 1 + \sum_{m=1}^{\infty} \frac{s^m}{m!} \left[ \prod_{i=1}^{m} \hat{\ell}(\zeta^{-i}(1-p)) \right] \sum_{j=1}^{\infty} (\Lambda^m \pi)_j
\]

\[(34)\]

\[
= 1 + \sum_{m=1}^{\infty} \frac{(s\gamma/\zeta)^m}{m!} \left[ \prod_{i=1}^{m} \hat{\ell}(\zeta^{-i}(1-p)) \right] \frac{p}{1-\zeta^{-m}(1-p)}
\]

as \( \sum_{j=1}^{\infty} (\Lambda^m \pi)_j = \sum_{j=1}^{\infty} \lambda_j^m \pi_j = \sum_{j=1}^{\infty} \gamma^m \zeta^{-jm} p(1-p)^{j-1} = \gamma^m \zeta^{-m} \frac{p}{1-\zeta^{-m}(1-p)}. \)

Next, notice that by letting \( p' = 1 - \zeta^{-1}(1-p) \), we have from (34) and the uniform convergence of the corresponding series for any fixed \( M > 0 \) and \( s \in [0,M] \), that

\[(35)\]

\[
\frac{d}{ds} C_{p,\gamma,\zeta}(s) = \frac{\gamma}{\zeta} \hat{\ell}(1-p') C_{p',\gamma,\zeta}(s) \quad \text{with} \quad C_{p,\gamma,\zeta}(0) = C_{p',\gamma,\zeta}(0) = 1.
\]

Observe that \( \zeta \geq 1 \) implies \( p' \geq p \) and \( C_{p',\gamma,\zeta}(s) \leq C_{p,\gamma,\zeta}(s) \). Also, observe that

\[
\hat{\ell}(1-p') \begin{cases} 
< 0 & \text{if } p > 1 - \frac{\zeta}{R} \\
= 0 & \text{if } p = 1 - \frac{\zeta}{R} \\
> 0 & \text{if } p < 1 - \frac{\zeta}{R}
\end{cases}
\]

as \( \hat{\ell} \) is an increasing function on \([0,\infty)\) and \( \hat{\ell}(1/R) = 0 \). This leads to the statement of the theorem. \(\square\)

**Remark 3.** If \( \zeta = 1 \), equation (35) implies \( C_{p,1,1}(s) = \exp \{ s\gamma \hat{\ell}(1-p) \} \) and hence

\[
C_{p,1,1}(s) \begin{cases} 
\text{exponentially decreases} & \text{if } p > 1 - R^{-1} \\
= 1 \text{ for all } s \geq 0 & \text{if } p = 1 - R^{-1} \\
\text{exponentially increases} & \text{if } p < 1 - R^{-1}
\end{cases}
\]

This case is further examined in Sect. 5.5. In general, the width function \( C_{p,\gamma,\zeta}(s) \) may increase sub-exponentially for \( p < 1 - \frac{\zeta}{R} \). For example, if there is a nonnegative integer \( d \) such that \( \zeta^{d+1} < R \), then for \( p = 1 - \frac{\zeta}{R} \) we have \( \hat{\ell}(\zeta^{-d-1}(1-p)) = 0 \). Hence, (33) implies that \( C_{p,\gamma,\zeta}(s) \) is a polynomial of degree \( d \).

**Theorem 7 (Criticality of a self-similar process).** Consider a self-similar process \( S_{p,\gamma,\zeta}(s) \) with \( 0 < p < 1 \), \( \gamma > 0 \). Suppose that (22) is satisfied and \( \zeta \geq 1 \lor L \). Then the following conditions are equivalent:

(i) The process is critical.

(ii) The process is time-invariant.

(iii) The following relations hold: \( \zeta < R \) and \( p = p_c := 1 - \frac{\zeta}{R} \).

*Proof.* (i)\(\leftrightarrow\)(iii) is established in Theorem 6, (ii)\(\rightarrow\)(i) is established in Prop 6, (iii)\(\rightarrow\)(ii): Observe that \( \hat{\ell}(\zeta^{-1}(1-p)) = \hat{\ell}(R^{-1}) = 0 \). Time invariance now follows from (33). \(\square\)

**Remark 4.** In a self-similar process the sequences \( \lambda_K \) and \( p_K \) are geometric such that (Thm. 4)

\[
\lambda_K p_K = \frac{\gamma p}{1-p} \left( \frac{1-p}{\zeta} \right)^K
\]
for some $0 < p < 1$, $\gamma > 0$, and $\zeta > 1 \lor L$. Hence, a time-invariant process can be constructed, according to Prop. 7 and (31), by selecting any sequence $\{T_k\}$ that corresponds to

$$R = \zeta (1 - p)^{-1}.$$ 

Theorem 7 states that this is the only possible way to construct a time-invariant process, given that the process is self-similar.

5.5. A closed form solution for the case of equally distributed branch lengths. Observe that if $\Lambda = \gamma I$, then $x(s) = e^{\gamma (1 - p) s}$ and $C(s) = \|x(s)\|_1 = e^{\gamma (1 - p) s}$.

Consider a hierarchical branching process with $\Lambda = I$ and $x(0) = e_K$ for a given integer $K \geq 1$. Here the system of equation (29) is finite dimensional,

$$\begin{cases}
  x'_1(s) &= -x_1 + (T_1 + 2)x_2 + T_2x_3 + \ldots + T_{K-1}x_K \\
  x'_2(s) &= -x_2 + (T_1 + 2)x_3 + T_2x_4 + \ldots + T_{K-2}x_K \\
  \vdots \\
  x'_{K-1}(s) &= -x_{K-1} + (T_1 + 2)x_K \\
  x'_K(s) &= -x_K
\end{cases}$$

(36)

with the initial conditions $x(0) = e_K$.

Define a sequence $t(j)$ as

$$t(0) = -1, \quad t(1) = T_1 + 2, \quad \text{and} \quad t(j) = T_j \quad \text{for} \quad j \geq 2,$$

and let $y(s) = e^sx(s)$. Then (36) becomes

$$\begin{cases}
  y'_1(s) &= t(1)y_2 + t(2)y_3 + \ldots + t(K - 1)y_K \\
  y'_2(s) &= t(1)y_3 + t(2)y_4 + \ldots + t(K - 2)y_K \\
  \vdots \\
  y'_{K-2}(s) &= t(1)y_{K-1} + t(2)y_K \\
  y'_{K-1}(s) &= t(1)y_K \\
  y'_K(s) &= 0
\end{cases}$$

(37)

with the initial conditions $y(0) = e_K$. The ODEs (37) can be solved recursively in a reversed order of equations in the system obtaining for $m = 1, \ldots, K - 1$,

$$y_{K-m}(s) = \sum_{n=1}^{m} \left( \sum_{i_1, \ldots, i_n \geq 1, i_1 + \ldots + i_n = m} t(i_1) \cdot \ldots \cdot t(i_n) \right) \frac{s^n}{n!}.$$ 

Let $\delta_0(j) = I_{(j=0)}$ be the Kronecker delta function. Then we arrive with the closed form solution

$$x_{K-m}(s) = e^{-s}y_{K-m}(s) = e^{-s} \sum_{n=1}^{\infty} \left( t + \delta_0 \right) \ast \left( t + \delta_0 \right) \ast \ldots \ast \left( t + \delta_0 \right) (m) \frac{s^n}{n!}.$$ 

(38)
Observe that if we randomize the orders of trees by assigning an order $K$ to a tree with geometric probability $p_K = p(1 - p)^{K - 1}$, then the above closed form expression \( (38) \) would yield an expression for the width function that was observed in Remark 3 of this section:

$$\begin{equation}
C(s) = e^{-s} + e^{-s} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (1 - p)^m \left( t + \delta_0 \right) \cdots \left( t + \delta_0 \right)(m) \frac{s^n}{n!} \left( t + \delta_0 \right) \cdots \left( t + \delta_0 \right)
\end{equation}$$

$$= e^{-s} + e^{-s} \sum_{n=1}^{\infty} \left( \hat{t}(1 - p) + 1 \right) \frac{s^n}{n!} = \exp \left\{ t \left( 1 - p \right) \right\}.$$  

5.6. **Critical Galton-Watson process.** The critical binary Galton-Watson process plays an important role in theory and applications because of its multiple symmetries. Burd, Waymire and Winn \([4]\) have shown that the following three properties are equivalent for the binary Galton-Watson distributions $\mathcal{GW}(p_0, p_2)$: (i) A distribution is prune-invariant; (ii) A distribution is mean self-similar with $E(T_{i,j}) = T_{j-i} = 2^{j-i-1}$; and (iii) A distribution is critical: $p_0 = p_1 = 1/2$. The Markov structure of the critical Galton-Watson trees ensures the existence of two other special properties: (iv) Time-invariance (in discrete time): the forest of trees, obtained by removing the edges and the vertices below depth $d$, has the same frequency structure as the original space $\mathcal{GW}(1/2, 1/2)$; and (v) The forest of trees obtained by considering sub-trees rooted at every vertex of a random tree $T$ approximates the frequency structure of the entire space of trees when the order of $T$ increases.

The next results shows that the critical binary Galton-Watson tree is a special case of the hierarchical branching process.

**Theorem 8 (Critical Galton-Watson tree).** A hierarchical branching process with parameters

$$\lambda_j = \gamma 2^{-j}, \quad p_K = 2^{-K}, \quad \text{and} \quad T_k = 2^{k-1} \text{ for any } \gamma > 0$$

is distributionally equivalent to the critical binary Galton-Watson tree $\mathcal{GW}(0, \gamma)$ with i.i.d. edge lengths that have a common exponential distribution with rate $2\gamma$. This is a self-similar, critical, and time-invariant process with $R = 4, \quad L = 2, \quad \text{and} \quad \zeta = 2$.

**Proof.** Consider a tree $T = \mathcal{GW}(0, \gamma) \in \mathcal{L}$. By Corollary \(1\) each branch of order $j$ in $T$ is exponentially distributed with parameter $\lambda_j = \gamma 2^{-j}$, which matches the branch length distribution in the hierarchical branching process \(39\). Furthermore, conditioned on $\mathcal{R}^i(T) \neq \phi$ (which happens with a positive probability), we have $\mathcal{R}^i(T) = \mathcal{GW}(0, 2^{-i}\gamma)$. This means that the space $\mathcal{R}^i(\mathcal{L})$ of pruned trees is a linearly scaled version of the original space $\mathcal{L}$ (the same combinatorial structure, linearly scaled edge lengths). Burd et al. \(4\) have shown that the total number of sub-branches within a branch of order $j \geq 2$ in $T$ is geometrically distributed over $\mathbb{Z}_+$ with mean $T_{i,j} = T_{j-i} = 2^{i-1} - 1$ (that is $P(m) = 2^{i-j} (1 - 2^{j-i})^m$ for $m \in \mathbb{Z}_+$), where $T_{i,j} = T_{j-i} = 2^{j-i-1}$. The assignment of orders among the $m$ side-branches is done according to the multinomial distribution with $m$ trials and success probabilities $T_i / (T_1 + \cdots + T_{j-1}), \ i = 1, \ldots, j - 1$. This implies that, conditioning on a particular implementation of the pruned tree $\mathcal{R}(T)$, the leaves of the original tree merge into every branch of the pruned tree as a Poisson point process with intensity $\gamma = \lambda_j T_{j-1}$. Iterating this pruning argument, conditioning on the particular implementation of $\mathcal{R}^i(T) = \mathcal{GW}(0, 2^{-i}\gamma)$,
the branches of order $i$ merge into any branch of the pruned tree $R^i(T)$ as a Poisson point process with intensity $\gamma 2^{1-i} = \lambda_j T_{j-i}$ for every $j > i$. Finally, the critical binary Galton-Watson space has $p_K = 2^{-K}$ \[39\]. We, hence, conclude that a $\text{GW}(0, \gamma)$ tree is distributionally identical to the hierarchical branching process with parameters \[39\].

By Thm. \[4\], the process \[39\] is self-similar with the length self-similarity constant $\zeta = 2$. Criticality and time-invariance follow from Prop. \[7\].

5.7. **Critical Tokunaga processes.** We introduce here a class of processes that extends the symmetries observed in the critical binary Galton-Watson tree with exponential edge lengths (where $\zeta = 2$) to the general case of $\zeta \geq 1$. Specifically, consider a hierarchical branching process $S^{\text{Tok}}(t; c, \gamma)$, which we call the **critical Tokunaga branching process**, with parameters

$$\lambda_j = \gamma c^{2-j}, \quad p_K = 2^{-K}, \quad \text{and} \quad T_k = (c-1)c^{k-1} \text{ for any } \gamma > 0, \ c \geq 1.$$ \[40\]

**Proposition 8.** The process $S^{\text{Tok}}(t; c, \gamma)$ is a distributionally self-similar critical time invariant process. Independently of the process combinatorial shape, its edge lengths are i.i.d. exponential random variables with rate $\gamma c$. In addition, we have

$$\hat{t}(z) = \frac{(1-2cz)(z-1)}{1-cz}, \quad R = w_0^{-1} = 2c, \quad \zeta = L = c, \quad \text{and} \quad p_c = 2^{-1}.$$ \[414\]

**Proof.** Self-similarity follows from Thm. \[4\] Criticality and time-invariance are established in Prop. \[7\]. The edge lengths property follows from Prop. \[5\] part(4).

\[ \square \]

**Remark 5.** The condition $T_{i,i+k} = T_k = a c^{k-1}$ was first introduced in hydrology by Eiji Tokunaga \[24\] in a study of river networks, hence the process name. The additional constraint $a = c-1$ is necessitated here by the self-similarity of tree lengths, which requires the sequence $\lambda_j$ to be geometric. The sequence of the Tokunaga coefficients then also has to be geometric, and satisfy $a = c-1$, to ensure identical distribution of the edge lengths, see Prop. \[5\] part(4). Interestingly, the constraint $a = c-1$ appears in the Random Self-similar Network (RSN) model introduced by Veitzer and Gupta \[25\], which uses a purely topological algorithm of recursive local replacement of the network generators to generate self-similar random trees. The necessity of the constraint $a = c-1$ in a combinatorial situation is explained by the following discussion and Propositions \[91011\].

Consider a tree $T[S]$ generated by a combinatorially self-similar hierarchical branching process $S$ with Tokunaga sequence $\{T_k\}$ and initial distribution $p_K = p(1-p)^{K-1}$. Let random variable $K$ be the order of the tree $T$, and, conditioned on $K > 1$, let $K_a, K_b$ be the orders of its two sub-trees, $T_a$ and $T_b$, rooted at the internal vertex closest to the root, randomly and uniformly permuted. We call $T_a$ and $T_b$ the **principal** sub-trees of $T$. Observe that the pair $K_a, K_b$ uniquely defines the tree order $K$:

$$K = \begin{cases} K_a \vee K_b, & \text{if } K_a \neq K_b, \\ K_a + 1, & \text{if } K_a = K_b. \end{cases}$$

Let $K_1 \leq K_2$ be the order statistics of $K_a, K_b$. The joint distribution of $(K_1, K_2)$ is given by

$$\mathbb{P}(K_1 = j, K_2 = m | K = k) = \begin{cases} 1 & \text{if } j = m = k - 1 \\ \frac{1 + T_1 + \cdots + T_{k-1}}{1 + T_1 + \cdots + T_{k-1}} & \text{if } j < m = k \end{cases}$$ \[41\]
where
\[ P(K = k | K > 1) = (1 - p)p^{k-2}. \]

**Proposition 9.** Consider a critical Tokunaga process \( S^{\text{Tok}}(t; c, \gamma) \). Then, conditioned on \( K > 1 \), the marginal order distribution of \( K_a \) coincides with that of \( K \):
\[ P(K_a = m | K > 1) = 2^{-m} = p_m \quad \text{for} \quad m \geq 1. \]

At the same time, the joint distribution of \( (K_a, K_b) \) equals the product of the marginals,
\[ P(K_a = m, K_b = j | K > 1) = P(K_a = m | K > 1)P(K_b = j | K > 1), \]
if and only if \( c = 2 \).

**Proof.** Observe that for \( k > 1 \),
\[
P(K_a = m | K = k) = \begin{cases} 
\frac{1}{2} \sum_{j : j < k} P(K_1 = j, K_2 = k | K = k) & \text{if } m = k, \\
\frac{1}{2} P(K_1 = k - 1, K_2 = k | K = k) + \frac{1}{2} P(K_1 = k, K_2 = k | K = k) & \text{if } m = k - 1, \\
\frac{1}{2} P(K_1 = m, K_2 = k | K = k) & \text{if } m < k - 1,
\end{cases}
\]
\[
= \begin{cases} 
\frac{1}{2} \left( \frac{T_1 + \cdots + T_{k-1}}{1 + \frac{1}{2}T_1} \right) = \frac{1-c}{2} & \text{if } m = k, \\
\frac{1}{2} \left( \frac{T_1 + \cdots + T_{k-1}}{1 + \frac{1}{2}T_{k-m}} \right) = \frac{c+1}{2}c^{1-k} & \text{if } m = k - 1, \\
\frac{1}{2} \left( \frac{T_{k-m}}{1 + T_1 + \cdots + T_{k-1}} \right) = \frac{(c-1)c^{-m}}{2} & \text{if } m < k - 1.
\end{cases}
\]

Thus,
\[
P(K_a = m | K > 1) = (1 - c^{-m})2^{-m} + c^{-m}2^{-m} + \frac{(c-1)c^{-m}}{2} \sum_{k=m+1}^{\infty} 2^{1-k} = 2^{-m}.
\]

At the same time,
\[
P(K_a = m, K_b = j | K > 1) = \begin{cases} 
(c-1)c^{-j}2^{-m} & \text{if } j < m, \\
c^{-m}2^{-m} & \text{if } j = m.
\end{cases}
\]

Hence, (43) holds if and only if \( c = 2 \). \( \square \)

**Remark 6.** Proposition 9 asserts that the principal sub-trees in a random critical Tokunaga tree are dependent, except the critical Galton-Watson case. This implies that, in general, non-overlapping sub-trees within a critical Tokunaga tree are dependent. Accordingly, the increments of the Harris path \( H \) of a critical Tokunaga process have (long-range) dependence. The only exception is the case \( c = 2 \) that was discussed in Sect. 4.5. The structure of \( H \) is hence reminiscent of a stable random walk \([15, 20]\). Establishing the correlation structure of the Harris paths of critical Tokunaga processes is an interesting open problem (see Sect. 6).

The critical Tokunaga trees introduced in Prop. 8 have an additional important property: the frequencies of vertex orders in a large-order tree approximate the frequencies of orders in the entire space \( T \). To formalize this observation, let \( \mu \) be the measure on \( T \) induced by \( S^{\text{Tok}}(t; c, \gamma) \), i.e. \( \mu(T) = P(\text{shape}(T[S^{\text{Tok}}]) = T) \). Next, for a fixed \( K \geq 1 \), let \( \mu_K(T) = \)
\(\mu(T \mid T \in H_K)\). Let \(V_k[K]\) denote the number of vertices of order \(k \in \{1, \ldots, K\}\) in a tree generated by \(\mu_K\), and let \(V_k[K] = E(V_k[K])\). Finally, we denote by \(V[K] = \sum_{k=1}^{K} V_k[K]\) the total number of non-root vertices, and notice that \(V[K] = 2V_1[K] - 1\). Thus, \(V[K] := E(V[K]) = 2V_1[K] - 1\).

**Proposition 10.** Let \(S^{Tok}(t; c, \gamma)\) be a critical Tokunaga branching process, then

\[
\lim_{K \to \infty} \frac{V_k[K]}{V_1[K]} = 2^{-k}.
\]

Let \(T = \text{shape}(T[S^{Tok}]) \in H_K\) be a tree generated by \(\mu_K\), and let \(v\) be a vertex selected by uniform random drawing from the non-root vertices of \(T\). Then

\[
\lim_{K \to \infty} P(v \text{ has order } k) = p_k = 2^{-k}.
\]

**Proof.** It has been shown in [10] that the mean self-similar trees satisfy the strong Horton law:

\[
\lim_{K \to \infty} \frac{N_k[K]}{N_1[K]} = R^{1-k}, \text{ for any } k \geq 1.
\]

Observe now that for any \(T \in H_K\) we have

\[
V_k(T) = \sum_{i=1}^{N_k(T)} (1 + m_i(T)),
\]

where \(m_i(T)\) is the number of sub-branches that merge the \(i\)-th branch of order \(k \in \{1, \ldots, K\}\) in \(T\), according to the proper branch labeling of Sect. 2.2. Proposition 5 gives

\[
V_k[K] = N_k[K](1 + T_1 + \cdots + T_{k-1}).
\]

For the process \(S^{Tok}(t; c, \gamma)\) this implies

\[
\lim_{K \to \infty} \frac{V_k[K]}{V_1[K]} = \lim_{K \to \infty} \frac{N_k[K](1 + T_1 + \cdots + T_{k-1})}{N_1[K]} = R^{1-k}c^{k-1} = 2^{1-k}.
\]

The statement (45) is an immediate consequence of (44), since

\[
\lim_{K \to \infty} \frac{V_k[K]}{V[K]} = \lim_{K \to \infty} \frac{V_k[K]}{2V_1[K] - 1} = 2^{-k}
\]

as \(V_1[K] \geq 2^{K-1}\). \(\square\)

Finally, we notice that a tree \(T \in \mathcal{L}\) can be considered a metric space with distance \(d(a, b)\) between two points \(a, b \in T\) defined as the length of the shortest path within \(T\) connecting them; see [19, Sect. 7.3] for details.

**Proposition 11.** Consider a random tree \(T = T[S^{Tok}] \in \mathcal{L}\) generated by a critical Tokunaga branching process \(S^{Tok}(t; c, \gamma)\) conditioned on the order \(k(T) = K\). Let point \(u \in T\) be sampled from a uniform density function on the metric space \(T\), and let \(r_u[K]\) denote the order of the edge to which the point \(u\) belongs. Then

\[
\lim_{K \to \infty} P(r_u[K] = k) = p_k = 2^{-k}.
\]
Proof. Proposition \([8]\) establishes that the edge lengths in \(T\) are i.i.d. exponential random variables. Thus we can generate \(T\) by first sampling the combinatorial tree \(\text{shape}(T)\) from \(\mu_K(T) = \mu(T|T \in \mathcal{H}_K)\) over \(\mathcal{H}_K\), and then assigning i.i.d. exponential edge lengths. Thus, provided that we already sampled \(\text{shape}(T)\), selecting the i.i.d. edge lengths and then selecting the point \(u \in T\) uniformly at random, and marking the edge that \(u\) belongs to, is equivalent to selecting a random edge uniformly from the edges of \(\text{shape}(T)\) as enumerated (labeled) in Sect. 2.2. Consequently, the order \(r_u[K]\) is uniquely determined by the edge to which \(u\) belongs, and is independent of the position of \(u\) within the edge. The statement now follows immediately from Prop. \([10]\). \(\square\)

6. Open problems

We conclude with two open problems, which refer to extending selected properties of the critical Galton-Watson tree with independent exponential edge lengths, \(GW(0, \gamma)\), which is a special case of the hierarchical branding process (see Thm. \([8]\)), to a general case. Here \(S(t) = (\{T_k\}, \{\lambda_j\}, \{p_K\})\) is a distributionally self-similar hierarchical branching process with

\[
L = \lim \sup_{k \to \infty} T_k^{1/k} < \infty, \quad p_K = p(1 - p)^{K-1}, \quad \lambda_j = \gamma \zeta^{-j}
\]

for some positive \(0 < p < 1, \gamma > 0, \text{and } \zeta \geq 1 + L\).

**Open Problem 1.** Describe the correlation structure of the Harris path of \(S(t)\). (The critical binary Galton-Watson tree with independent exponential edge lengths \(GW(0, \gamma)\) corresponds to a symmetric Markov chain with exponential jumps \(\{\frac{1}{2}, 2\gamma, 2\gamma\}\), see Thm. \([8]\).)

**Open Problem 2.** Establish a proper infinite-tree limit of \(S(t)\), where the edge lengths go to zero and the tree length increases to infinity, that preserves the suitably defined limit version of the self-similarity property. Describe the respective limit Harris path processes. (The Harris path of the critical binary Galton-Watson tree \(GW(0, \gamma)\) can be rescaled to converge to excursion of the standard Brownian motion [13, 16].)

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