Reconstructing the CMB power spectrum

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Abstract: We discuss a method for a model independent reconstruction of the CMB temperature fluctuation power spectrum on small and intermediate scales that is geared to individual experiments. The importance of off-diagonal correlations for determining the shape of the power spectrum is emphasized and some examples of a reconstruction method are given. By using this method to “map” the power spectrum on the scales to which they are sensitive several experiments could be combined to map out the full power spectrum, and test consistency of observed features. For example we find that the GUM scan of the 3rd flight of the MAX experiment prefers a positive slope to the power spectrum near $\ell \approx 160$, which provides weak evidence for the presence of a Doppler peak.

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1. Introduction

The study of anisotropies in the Cosmic Microwave Background (CMB) radiation has now reached the level where experiments on a wide range of scales are reporting detections or significant upper limits. It seems appropriate therefore to consider ways of analyzing the data, with a view to reconstructing the power spectrum of temperature fluctuations. One possibility is to give a model (e.g. CDM) of the radiation power spectrum in terms of a few parameters and then use many experiments to constrain the model parameters. In this letter we will focus on the complimentary, model independent, approach in which a parameterization of the power spectrum is developed on a per-experiment basis and individual experiments are used to constrain that part of the power spectrum to which they are most sensitive. Since on large scales the inversion of the sky-map into a power spectrum is a well studied problem we will focus here on individual experiments on smaller angular scales. Given that experiments now cover the full range of the power spectrum from degree scales up (see e.g. Bond 1993, White, Scott & Silk 1994), this in principle allows a reconstruction of the power spectrum.

2. Power spectrum and window functions

In this letter we will assume that the CMB temperature fluctuations are gaussian distributed so that all of the information resides in the correlation matrix or power spectrum. It is conventional to expand the temperature fluctuations in spherical harmonics $T = \sum_{\ell m} a_{\ell m} Y_{\ell m}$ and denote the power per mode by $C_{\ell}$:

$$\langle a_{\ell m}^* a_{\ell' m'} \rangle_{\text{ens}} \equiv C_{\ell} \delta_{\ell \ell'} \delta_{m m'}$$

where the angled brackets represent an average over the ensemble of temperature fluctuations. The Harrison-Zel’dovich power spectrum corresponds to constant $\ell(\ell + 1)C_{\ell}$ or $C_{\ell}^{-1} \propto \ell(\ell + 1)$. In general one expects the $C_{\ell}$ to be a power-law for small $\ell$ with more structure at higher $\ell$ (smaller angular scales). The spectrum for CDM for example, viewed as $\ell(\ell + 1)C_{\ell}$ vs $\ln \ell$, is a flat line until $\ell \sim 100$ when it rises into two “Doppler” peaks before falling off exponentially at large $\ell \sim 1000$ (for a discussion of power spectra in various models see Bond & Efstathiou 1987, Holtzman 1989, Vittorio & Silk 1992, Sugiyama & Gouda 1992, Dodelson & Jubas 1993, Stompor 1993, Crittenden et al. 1993, Bond et al. 1993).
When fitting model parameters to data one is interesting in computing the correlation matrix of the temperature fluctuations and from this a likelihood function (see e.g. Readhead & Lawrence 1992). The correlation matrix is a sum of two parts: the experimental error matrix (usually assumed diagonal: $\sigma_i \delta_{ij}$) and a “theoretical” correlation matrix

$$C_{\text{th}}(\mathbf{n}_i, \mathbf{n}_j) = \langle T(\mathbf{n}_i) T(\mathbf{n}_j) \rangle_{\text{ens}} = \frac{1}{4\pi} \sum_{\ell=1}^{\infty} (2\ell + 1) C_{\ell} W_{\ell}(\mathbf{n}_i, \mathbf{n}_j), \tag{2}$$

Here $T(\mathbf{n})$ is the measured temperature assigned to direction $\mathbf{n}$ and $W_{\ell}$ is the window function for the experiment (e.g. see Srednicki & White 1994 for a discussion of window functions and Bond 1993, White, Scott & Silk 1994 for plots of window functions for existing experiments). The method proposed below does not rely on being able to construct $W_{\ell}(\mathbf{n}_i, \mathbf{n}_j)$ and so is more general than just Eq. (2). However for simplicity we will phrase our discussion as if $W_{\ell}(\mathbf{n}_i, \mathbf{n}_j)$ were known, as it is for most small-scale experiments. For a given $C_{\text{th}}$ the likelihood function is expressed in terms of $C_{ij} = C_{\text{th}}(\mathbf{n}_i, \mathbf{n}_j) + \sigma_i \delta_{ij}$ as

$$L \propto \frac{1}{\sqrt{\det C}} \exp \left[-\frac{1}{2} T_i C_{ij}^{-1} T_j \right] \tag{3}$$

and provides a convenient way of using all the available experimental information to constrain the model (in this case $C_{\text{th}}$).

Usually when people discuss “the” window function for a particular experiment they are referring to $W_{\ell}(\mathbf{n}, \mathbf{n})$ (the window function at zero-lag) appropriate for computing the diagonal entries of $C_{\text{th}}$. This window function defines a “bandpass” outside of which the experiment has no sensitivity: the small $\ell$ cutoff is set by the amplitude of the chop and the high $\ell$ cutoff is controlled by the beam width (see e.g. Bond 1993). Integrating the power through the bandpass gives $\Delta T_{\text{rms}}$, a fact which has been exploited by several authors (see e.g. Bond 1993, White, Scott & Silk 1994) to estimate the amplitude of the power spectrum measured by the many experiments on degree scales. It has long been realized that separating the scales of the beam chop and width increases the area under the window function or the sensitivity of the experiment.

Such an increase in the width of $W_{\ell}$ doesn’t necessarily mean less resolving power for $C_{\ell}$ however. Off-diagonal entries of $C_{\text{th}}$ (i.e. $\mathbf{n}_i \neq \mathbf{n}_j$) potentially contain more information than just the integrated power. They provide a lever arm which gives sensitivity to the shape of the power spectrum, as is reflected in the shape of $W_{\ell}(\mathbf{n}_i, \mathbf{n}_j)$. In figure 1 we show $W_{\ell}(\theta = 0^\circ)$ and $W_{\ell}(\theta = 2^\circ)$ ($\cos \theta = \mathbf{n}_i \cdot \mathbf{n}_j$, with points separated parallel to the chop
direction) for a hypothetical experiment with 2-beam, square-wave chop of amplitude 2° and gaussian beam of FWHM 1°. We choose \( \theta = 2° \) since \( W_{\ell}(2°) \) is approximately the derivative of \( W_{\ell}(0°) \). This derivative or “dipole-like” form provides maximum sensitivity to the slope of the power spectrum through the bandpass (to which \( C_{\text{th}}(0) \) is mostly insensitive).

The diagonal elements of \( C_{\text{th}} \) coming from \( W_{\ell}(n,n) \) allow an estimate of the power through the bandpass. From the above we see that without losing sensitivity to the amplitude, the off-diagonal elements provide (model independent) information on the shape of the power spectrum over the same range of \( \ell \).

3. Parameterizing the power spectrum

We wish to parameterize the \( C_\ell \) in a simple way that is tailored to the experiment under consideration. Start by defining \( D_\ell \equiv \ell(\ell + 1)C_\ell \) and \( x \equiv \ln \ell \) and extend \( D_\ell \) to a smooth function \( D(x) \) in the obvious manner. Since on general physical grounds we expect the power spectrum to be smooth (the neighbouring \( C_\ell \) are highly “correlated”) we can expand \( D(x) \) in a Taylor series about the peak of the window function, \( x_0 \), as

\[
D(x) = D(x_0) \left[ 1 + m(x - x_0) + \frac{1}{2}m'(x - x_0)^2 + \cdots \right]
\]

(4)

where \( m \) and \( m' \) are related to the derivatives of the power spectrum at \( x_0 \). Due to the finite beam width and chopping, any experiment is sensitive to the power spectrum only over a limited (and usually small) range of \( x \), defined by the window function at zero-lag. So for the purposes of each experiment, Eq. (4) should be a good expansion. In what follows we will take as given that our expansion of \( D(x) \) is not to be extrapolated outside of the range of sensitivity of the experiment.

One problem with this expansion is that it does not enforce the physical condition that \( D(x) \geq 0 \) for all \( x \). Thus the allowed region of \( m, m' \ldots \) is constrained in a non-trivial (but calculable) way. In principle one could get around this by expanding not \( D(x) \) but \( \ln D(x) \) in a power series, but this has the technical disadvantage that the expansion is no longer linear in \( D(x) \). An expansion in which \( C_{\text{th}} \propto C_{\text{th}}^{(0)}(1 + mC_{\text{th}}^{(1)} + \cdots) \) is more computationally efficient when performing fits to the data because \( C_{\text{th}}^{(0)}, C_{\text{th}}^{(1)} \ldots \) only have to be computed once. This has motivated us to stick with Eq. (4). If a Monte-Carlo analysis of the data is used to constrain the parameters, the advantage of linearity could be overlooked in favor
of direct imposition of the constraint $D \geq 0$. Eventually the data should constrain the parameters sufficiently that this question will not be an issue.

4. Two examples of the method

As an illustrative example of these ideas we have taken Eq. (4) and truncated at linear order (which is all the current sensitivities merit). Converting to the conventional notation we assume

$$\ell(\ell + 1)C_\ell = \ell_0(\ell_0 + 1)C_{\ell_0} \left[ 1 + m \ln \frac{\ell}{\ell_0} \right]$$

(5)

where $\ell_0$ is the peak of $W_\ell(0^\circ)$. We computed $C_{\text{th}}(0^\circ, m)$ and $C_{\text{th}}(2^\circ, m)$ using Eq. (5) and the window functions of Figure 2. We find $C_{\text{th}}(0^\circ)$ (the integral through the bandpass) is mostly insensitive to $m$ while $C_{\text{th}}(2^\circ)$ has good discriminating power between different slopes. We have, normalizing at $m = 0$,

$$C_{\text{th}}(0^\circ; m)/C_{\text{th}}(0^\circ; m = 0) = 1 - 0.2m$$

$$C_{\text{th}}(2^\circ; m)/C_{\text{th}}(2^\circ; m = 0) = 1 - 1.7m$$

(6)

Obviously the more off-diagonal elements one has with good signal-to-noise, the more information one has about the shape of the power spectrum through the bandpass, and the more one can constrain higher orders in the expansion (the “shape” of the power spectrum). In the above $C_{\text{th}}(2^\circ) \simeq 0.2C_{\text{th}}(0^\circ)$ at $m = 0$ which sets the scale of sensitivity needed to extract the full $m$-dependence from the signal. If noise is a problem one could increase $C_{\text{th}}(\theta)$ by measuring correlations at other separations, but these would have reduced sensitivity to the slope of the power spectrum.

Clearly an experiment which observes only at widely separated points will have no non-vanishing off-diagonal correlations and sensitivity only to the integral of the power spectrum though the bandpass. Alternatively an experiment which samples too closely will waste time recording “redundant” information and not reduce the experimental uncertainty on the data points sufficiently to constrain the higher moments of the power spectrum. The question thus arises as to what observing strategy maximizes the constraints on the power spectrum, analyzed in this manner. Such a question cannot be answered without also considering other experimental limitations, which will clearly vary from one experiment to another. The question can be analyzed by the individual groups in preparing their
observing strategy. In general one gains information on the power spectrum by working with a strategy which (as much as possible) provides strong off-diagonal signals.

As an application of this method to a ‘realistic’ scenario we have computed the likelihood function in \((C_{\ell_0}, m)\) space, assuming Eq. (5), for the GUM (Meinhold et al. 1993) scan of the 3rd flight of the MAX experiment. The data consists of 165 points in a two-dimensional “bow-tie” pattern, with varying spacing and a strongly anisotropic correlation matrix. The fitting procedure and data used are as discussed in Srednicki et al. (1994). For the parameters appropriate to MAX3 \(W_\ell(n, n)\) peaks at \(\ln \ell_0 \simeq 5\) \((\ell_0 \simeq 160)\) with a FWHM of \(\ln \ell \simeq 1.4\). Due to the large number of points and the wide range of correlations available we already find a weakly significant constraint. [The MuPeg data set (Gundersen et al. 1993) from the same flight consisted of relatively few, equally (“widely”) spaced data points at constant elevation, and we find no significant constraint on \(m\).] This analysis provides weak evidence that GUM prefers a positive slope over its range of sensitivity. The contours of the likelihood function are shown in figure 2 (we have restricted \(m \in (-1, 1)\) to enforce \(C_\ell \geq 0\) over the range of \(\ell\) to which MAX is sensitive). For comparison, if one weights by \((2\ell + 1)^2 W_\ell^2\) the \(C_\ell\) for CDM (see Eq. (2)) with \(h = \frac{1}{2}\) and \(\Omega_B = 0.01-0.10\), the best fits to the form of Eq. (5) are \(m = 0.5-0.6\). The low values of the slope come from the fact that the MAX window function is centered near the maximum of the first “Doppler” peak for the models considered here.

The analysis presented above neglects the important issue of foreground contamination, and the data are clearly not at the level where \(m\) is severely restricted. However it serves to show that the method works in principle. As sensitivities increase, a more detailed analysis along these lines is sure to provide information on the shape of the power spectrum. A fit including a quadratic component, for example, would allow one to test whether the power spectrum exhibits a peak in the range of \(\ell\) probed (at present the likelihood function for MAX is very broad in \(m'\) and no conclusion can be drawn).

5. Conclusions

In the future, with reduced experimental errors and more coverage, the method presented in this letter could be used to map the power spectrum, over the range of sensitivity of individual experiments, in a way which provides more and more shape information as higher off-diagonal correlations are observed above the noise. Pasting together constraints
of this form from many experiments should allow one to reconstruct the full power spectrum in a model independent manner.

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Figure Caption

Fig. 1. The window functions $W_{\ell}(\theta = 0^\circ)$ (solid) and $W_{\ell}(\theta = 2^\circ)$ (dashed) for a hypothetical experiment with 2-beam, square wave chop of amplitude $2^\circ$ and a gaussian beam of FWHM $1^\circ$.

Fig. 2. The contours of the likelihood function in $(C_{\ell_0}, m)$ for the MAX-GUM data. The contours are in units of $\frac{1}{2}\sigma$. 

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