Interpreting the two variable Distance enumerator of the Shi hyperplane arrangement

Sivaramakrishnan Sivasubramanian
Institute of Information and Practical Mathematics
Christian-Albrechts-University
Kiel, Germany
email: ssi@informatik.uni-kiel.de

February 4, 2022

Abstract

We give an interpretation of the coefficients of the two variable refinement $D_{S_n}(q,t)$ of the distance enumerator of the Shi hyperplane arrangement $S_n$ in $n$ dimensions. This two variable refinement was defined by Stanley [St-98] for the general $r$-extended Shi hyperplane arrangements. We give an interpretation when $r = 1$.

We define three natural three-dimensional partitions of the number $(n+1)^{n-1}$. The first arises from parking functions of length $n$, the second from special posets (we call them tree-posets) on $n$ vertices defined by Athanasiadis [At-97] and the third from spanning trees on $n+1$ vertices. We call the three partitions as the parking partition, the tree-poset partition and the spanning-tree partition respectively. We show that one of the parts of the parking partition is identical to the number of edge-labelled trees with label set $\{1,2,\ldots,n\}$ on $n+1$ unlabelled vertices. We prove that the parking partition majorises the tree-poset partition and conjecture that the spanning-tree partition also majorises the tree-poset partition.

1 Introduction

Let $r \geq 1$ and $n \geq 2$. The $r$-extended Shi hyperplane arrangement in $n$ dimensions is denoted $S^n_r$. It is given by the following hyperplanes in $\mathbb{R}^n$.

$$x_i - x_j = -r + 1, -r + 2, \ldots, r,$$

for $1 \leq i < j \leq n$.

When $r = 1$, the arrangement is called the Shi hyperplane arrangement in $n$ dimensions and denoted $S_n$. Stanley [St-98] defined a two variable distance enumerator of the Shi hyperplane arrangement with respect to a base region $B$. Let $\mathcal{R}(S_n)$ be the set of regions of the Shi hyperplane arrangement. Each region $R \in \mathcal{R}(S_n)$ is separated from $B$ by a set
of hyperplanes and let $a$ be the number of separating hyperplanes of the form $x_i - x_j = 0$ and $b$ the number of separating hyperplanes of the form $x_i - x_j = 1$. The two variable distance enumerator is defined as $D_{S_n}(q, t) = \sum_{R \in R(S_n)} q^a t^b$. We denote the coefficient of $q^a t^b$ of $D_{S_n}(q, t)$ as $\text{Dist}_a(k, \ell)$.

Fix $n$ and let $\Pi_k$ be the set of permutations on $[n]$ which have exactly $k$ non-inversions. For a permutation $\pi \in \Pi_k$, let $\text{IP}_\pi$ be a poset of its inversions ordered by containment, (ie if $g = (\pi_i, \pi_j)$ where $i < j$, and $h = (\pi_a, \pi_b)$ where $a < b$, are inversions, then $g \leq_{\text{IP}_\pi} h$ iff $a \leq i < j \leq b$. For example, when $\pi = 623415$, the poset $\text{IP}_{623415}$ is shown in Figure 7. For $\pi \in \Pi_k$, let the number of ideals of $\text{IP}_\pi$ with \( \binom{n}{k} - k - \ell \) elements be $\text{IP}_\pi(\ell)$.

**Theorem 1** $\text{Dist}_a(k, \ell) = \sum_{\pi \in \Pi_k} \text{IP}_\pi(\ell)$

Theorem 1 gives a two variable generalisation to the equality (see Page 96, [EC2])

$$
\sum_{\pi \in S_n} F(J(\text{NIP}_\pi), q) = I_{n+1}(q)
$$

where $S_n$ is the set of permutations on $n$ distinct alphabets, $F(J(\text{NIP}_\pi), q)$ is the rank generating function of the lattice of order ideals of the poset of non-inversion NIP\_\pi which is similar to IP\_\pi, the only difference being that we order non-inversions of $\pi$ instead of its inversions (please see Remark 2). $I_{n+1}(q)$ is the inversion enumerator of spanning trees on a vertex set of size $n + 1$.

### 1.1 Three 3d partitions of the regions of $S_n$

For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$ and let $[n_0] = \{0\} \cup [n]$. Let $T$ be a spanning tree on the set $[n_0]$. We call the vertex 0 as the “root” of $T$ and call such trees 0-rooted spanning trees.

From the bijection between $R(S_n)$ and spanning trees on $(n+1)$ vertices $[n_0] = \{0, 1, \ldots, n\}$ (see [St-06]), we can view the regions alternatively as 0-rooted spanning trees on $[n_0]$. Likewise, we can also view the regions as indexed by Parking Functions of length $n$. We recall the definition of an $n$ length parking function. There are $n$ parking spaces 0, 1, $n - 1$ in a one-way street. $n$ cars $C_1, C_2, \ldots, C_n$ enter the street in that order. $C_i$ has a preferred space $a_i$ and proceeds directly to slot $a_i$. If slot $a_i$ is occupied, it will try to park in the next available space. If a car leaves the street without parking then the process fails. $\overline{\pi} = (a_1, a_2, \ldots, a_n)$ is an $n$-length parking function if all cars can park with $a_i$ being their respective choices. The set of all parking functions of length $n$ is denoted $\text{PF}_n$. It is known that $\overline{\pi} = (a_1, a_2, \ldots, a_n)$ is a parking function iff the weakly increasing permutation $\overline{b} = (b_1, b_2, \ldots, b_n)$ of $\overline{\pi}$ satisfies $b_i < i$ (see [EC2]).

#### 1.1.1 Parking Partition

Let $\overline{\pi} = (a_1, a_2, \ldots, a_n) \in \text{PF}_n$. It is simple to check that any permutation of $\overline{\pi}$ is yields a valid parking function. We partition $\text{PF}_n$ into the following three parts: those with $a_1 > a_2,$
with \( a_1 = a_2 \) and with \( a_1 < a_2 \). We call the number of such \( n \)-length parking functions as \( gt_n \), \( eq_n \) and \( lt_n \) respectively. It is clear that we could have chosen any indices \( i \neq j \) and partitioned \( PF_n \) into three parts as above depending on the relation between \( a_i \) and \( a_j \) and still obtained the same numbers. Below we tabulate the numbers \( gt_n \), \( eq_n \) and \( lt_n \) for small values of \( n \).

1.1.2 Tree-poset partitions

We define the tree-poset partition next. Consider the hyperplane \( x_1 - x_2 = \alpha \) for \( \alpha = 0, 1 \); and let \( R \in \mathcal{R}(S_n) \). Let \( \pi_R = (a_1, a_2, \ldots, a_n) \) be any point in \( R \). Clearly, the value \( a_1 - a_2 \) is either \( < 0 \), strictly between \( 0 \) and \( 1 \), or \( > 1 \) and this condition is independent of the point \( \pi_R \). Thus each region \( R \) with respect to the dimensions \( x_1 \) and \( x_2 \) satisfies one of the three properties: all points \( \pi_R \in R \) either have \( a_1 - a_2 < 0 \), or \( 0 < a_1 - a_2 < 1 \) or \( a_1 - a_2 > 1 \).

Let \( R_n^{<0} \), \( R_n^{0<1} \) and \( R_n^{>1} \) respectively denote the number of regions satisfying the above three conditions. The main reason for this definition is to understand how \( \mathcal{R}(S_n) \) gets partitioned by the parallel hyperplanes \( x_1 - x_2 = 0, 1 \).

Below we tabulate the numbers \( R_n^{<0} \), \( R_n^{0<1} \) and \( R_n^{>1} \). For this definition, the numbers are not necessarily independent of the choices 1 and 2.

1.1.3 Spanning-tree partitions

Lastly, we define the spanning-tree partition. Let \( v_1, v_2 \in [n], v_1 \neq v_2 \) be two fixed vertices, and let \( T \) be a 0-rooted spanning tree on \([n] \), \( n \) be the number of edge labelled trees with label set \( \{1, 2, \ldots, n\} \) on \( n + 1 \) unlabelled vertices. It is known (see [EC2]) that \( UT_n = (n + 1)^{n-2} \).

Theorem 2 For all \( n \geq 1 \), \( eq_n = UT_n \).

We show the following majorisation theorem.

Theorem 3 For \( n \geq 2 \), the largest part of the parking partition is equal to the largest part of the tree-poset partition. Hence, the sorted parking partition majorises the sorted tree-poset partition.
2 Two variable distance enumerator: an interpretation

In this section, we prove Theorem 1. We use a poset representation for each region \( R \in \mathcal{R}(S_n) \). This representation was defined by Athanasiadis [At-97].

2.1 The posets of Athanasiadis

Let \( \sigma_R \) be a point of \( R \in \mathcal{R}(S_n) \). Represent each of the three possibilities \( a_1 - a_2 < 0 \), \( 0 < a_1 - a_2 < 1 \) and \( a_1 - a_2 > 1 \) Figure 1 (the dotted lines in the second figure represent an incomparability relation between the vertices \( i \) and \( j \)).

We call arcs of the form \((i, j)\) where \( i < j \) as forward arcs and those of the form \((j, i)\) where \( i < j \) as backward arcs.

Athanasiadis [At-97] showed that this representation yields a poset on \([n]\) and that such posets do not have the three subposets shown in Figure 2. Athanasiadis also proved that any poset without these three “forbidden” subposets arose from a region thereby characterising such posets. We refer to such posets as “tree-posets”.

\[
\begin{align*}
\text{\(a_1 - a_j < 0\):} & \quad \begin{array}{c}
\text{\(i\)} \quad \overset{\text{forbidden}}{\longrightarrow} \quad \text{\(j\)}
\end{array} \\
\text{\(0 < a_i - a_j < 1\):} & \quad \begin{array}{c}
\text{\(i\)} \quad \overset{\text{forward}}{\longrightarrow} \quad \text{\(j\)}
\end{array} \\
\text{\(a_i - a_j > 1\):} & \quad \begin{array}{c}
\text{\(i\)} \quad \overset{\text{backward}}{\longrightarrow} \quad \text{\(j\)}
\end{array}
\end{align*}
\]

Figure 1: Representing the three possibilities, where \( i < j \).

\[
\begin{align*}
\text{\(i\)} \quad \overset{\text{forward}}{\longrightarrow} \quad \text{\(j\)} \quad \overset{\text{forbidden}}{\longrightarrow} \quad \text{\(k\)} \\
\text{\(i\)} \quad \overset{\text{forbidden}}{\longrightarrow} \quad \text{\(j\)} \quad \overset{\text{forward}}{\longrightarrow} \quad \text{\(k\)} \\
\text{\(i\)} \quad \overset{\text{forward}}{\longrightarrow} \quad \text{\(j\)} \quad \overset{\text{forbidden}}{\longrightarrow} \quad \text{\(k\)}
\end{align*}
\]

Figure 2: The three forbidden subposets where \( i < j < k \).

**Lemma 1** \( \text{Dist}_n(k, \ell) \) is equal to the number of tree posets on \([n]\) which have \( k \) forward arcs and \( \ell \) backward arcs.

**Proof:** We use the Pak and Stanley method of starting from a base region \( B \), assigning a poset \( P_B \) to it and use their rules to get a poset for adjacent regions. We recall that the base
region is the region bounded by the hyperplanes $x_1 > x_2 > \cdots > x_n > x_1 - 1$. Assign the $n$ element antichain to this region. When region $R'$ is separated from $R$ by the hyperplane $x_i - x_j = 0$ (for $i < j$), set $P_{R'} = P_R \cup (i,j)$, ie add the forward arc $(i,j)$. Similarly, when region $R'$ is separated from $R$ by the hyperplane $x_i - x_j = 1$ (and $i < j$), then set $P_{R'} = P_R \cup (j,i)$, ie add the backward arc $(j,i)$. We note that whenever we cross from region $R$ in this manner, we always cross a hyperplane $x_i - x_j = 0,1$ such that, in $P_R$ the vertices $i$ and $j$ are incomparable. Thus, the above two cases are exhaustive. The proof of Pak and Stanley shows that this algorithm is well defined over different shortest paths from the base region $B$ to any other region $R$.

This is the same labelling given by Athanasiadis [At-97], though the algorithm explains the proof better. From the above algorithm, we see the following invariant: when we cross $R$ to observe that such posets $P$ even when the poset is not a permutation (ie has incomparable elements). It is also simple to see that backward arcs in permutations correspond to its inversions. We use both inversions and backward arcs interchangeably even when the poset is not a permutation (ie has incomparable elements). It is also simple to observe that such posets $P_R$ with no inversions (ie only forward arcs and incomparability relations) are the “nearest” regions to the base region $B$ in the regions of $B_n$ (the Braid hyperplane arrangement).

A reverse propagation shown below proves that we can start from $(k,\ell+1)$-tree posets and by converting an inversion into an incomparability relation, obtain all $(k,\ell)$-tree posets.

**Lemma 2** All $(k,\ell)$-tree posets can be obtained from $(k,\ell+1)$-tree posets by converting a backward arc into an incomparability relation.

**Proof:** Let $P$ be a $(k,\ell)$-tree poset on $[n]$. We prove this by induction on $\binom{n}{2} - (k+\ell)$. The base case when $k+\ell = \binom{n}{2}$ ie when $P$ is a permutation, is simple.

Let $P$ be a $(k,\ell)$-tree poset with $k+\ell < \binom{n}{2}$. We exhibit a $(k,\ell+1)$-tree poset $Q$ and identify an inversion $i_P$ in $Q$ such that $P = Q - \{i_P\}$. (ie We convert one incomparability relation $i_P$, (we also call these as non-arcs) in $P$ into a backward arc and get a poset $Q$ which does not have any of the three forbidden subposets.) Clearly $P = Q - \{i_P\}$ and by induction, we will be done.

Let $P$ be a $(k,\ell)$-tree poset. Consider the linear extension $\sigma_P$ of $P$ where we break ties when they exist, by the “largest” vertex first rule (see Figure 1 for an example). Order the vertices (ie $[n]$) according to $\sigma_P$ so that all arcs of $P$ are directed towards the right and let $g = (\sigma_r, \sigma_s)$ for $r < s$ be a non-arc in $P$. Define the “length” $l(g)$ of $g$ to be the number of non-arcs contained within it (including itself) in $\sigma_P$ (ie the number of non-arcs in the subpermutation $\sigma[r,s] = (\sigma_r, \sigma_{r+1}, \ldots, \sigma_s)$. Let $i_P$ be a non-arc of maximum length. We claim that we can convert $i_P$ into a backward arc and obtain a $(k,\ell+1)$-tree poset, $Q$. 

5
σₚ = 54312

Figure 3: Linear extensions with the “largest” vertex first and lengths l(g).

To prove this, we need to show that none of the three forbidden subposets appear in Q. Since they do not appear in P, iₚ must be involved in any forbidden subposet. Suppose the second or the third forbidden subposet of Figure 2 appears in Q. Then, iₚ must be the backward-arc in a forbidden subposet and we have a situation shown in Figure 4. In both cases, let iₚ = (a, b). We note that a > b. In the first case, there is a non arc (b, c) and since b appears to the left of c in σₚ, b > c. Thus we have a > b > c and this violates the maximality of iₚ = (a, b). In the second case, similarly there are three vertices c > a > b and the maximality of iₚ is again violated. The first forbidden subposet cannot appear in Q as we need a forward arc for it and we convert iₚ into a backward arc. Thus Q has one less non-arc, has no forbidden subposets and (hence by the Theorem 1.1 [At-97]) corresponds to a region of Sₙ. This completes the proof.

Let n be fixed and for 0 ≤ k ≤ \(\binom{n}{2}\), let Πₖ be the set of permutations π on [n] having \(\binom{n}{2} - k\) inversions. For π ∈ Πₖ and 0 ≤ ℓ ≤ \(\binom{n}{2} - k\), let the number of (k, ℓ)-tree posets obtained from π (ie those obtained from π by deleting \(\binom{n}{2} - k - ℓ\) inversions) be denoted π(k, ℓ).

Corollary 1 With the above notation, Distₙ(k, ℓ) = \(\sum_{\pi \in \Pi_k} \pi(k, \ell)\)

Proof: It is simple to check that (k, ℓ)-tree posets obtained from σ ≠ π, π, σ ∈ Πₖ are different and that we can just add up the numbers π(k, ℓ) over different π ∈ Πₖ. Thus to get (k, ℓ)-tree posets, we could start from π ∈ Πₖ and delete \(\binom{n}{2} - k - ℓ\) inversions such that the three forbidden posets do not occur.

Example 1 We show an example of the inversion-deletion process described above. Let π = 623415. We can order the (k, ℓ)-tree posets we obtain by containment. This poset
of $(7, \ell)$-tree posets obtained from $\pi$ is shown in Figure 5. The edges of the poset are all oriented rightwards and the edge labels are the inversions converted into non-arcs from the (poset corresponding to the) previous vertex.

$\pi = 623415$

![Diagram of the poset](image)

$\pi(7, \ell) = 1 2 3 4 3 2 1 1 1$

Remark 1 All $(k, \ell)$-tree posets arising from $\pi \in \Pi_k$ have $\pi$ as a linear extension and when we break ties due to incomparability using the “largest vertex” first rule, these posets have $\pi$ as the linear extension. Further, all points $\alpha_R = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in a region $R$ corresponding to a $(k, \ell)$-tree poset from $\pi \in \Pi_k$ have $\pi$ as the permutation when the $\alpha_i$’s are sorted in increasing order.

2.2 Ideals of the inversion poset of a permutation

We give an alternate interpretation of the number of $(k, \ell)$-tree posets on $[n]$. It is clear that inversions of permutations $\pi \in \Pi_k$ are to be deleted in some sequence to obtain $(k, \ell)$-tree posets. Such sequences are described below.

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi_k$. Let the sub-permutation between two indices $i < j$ be denoted $\pi[i, j]$, i.e., $\pi[i, j] = (\pi_i, \pi_{i+1}, \ldots, \pi_j)$. Let $p = (\pi_i, \pi_j)$ be an inversion. Let $\text{invs}_p$ be the number of inversions in $\pi[i, j]$.

**Lemma 3** Let $\pi \in \Pi_k$ and let $g$ be an inversion in $\pi$. In the deletion process described above, $g$ can be converted into a non-arc only after all the inversions strictly within it have been converted.
Figure 6: Inner inversions must be deleted earlier

**Proof:** Let $P$ be a poset obtained from $\pi$ and suppose we could convert an inversion $(a, b)$ to a non-arc while an inner inversion $(c, d)$ remained (ie if $\pi_w = a, \pi_x = b, \pi_y = c$ and $\pi_z = d$, we have $w < y < z < x$, for example, see Fig 6).

If both $(a, c)$ and $(d, b)$ were arcs, then the posetness of $Q$ would be violated. Since all arcs go rightwards, we assume $(a, c)$ is an incomparable pair ie that $a > c$ ie this was an inversion that got converted. Since $(c, d)$ is an inversion, $c > d$. Thus $a > d$ and this inversion either stays as an inversion or has been converted into a non-arc. If it is a non-arc, we have a forbidden subposet on $a, c, d$ and hence $(a, d)$ remains as an arc. If $(d, b)$ exists, then again we violate posetness and hence $(d, b)$ is a non-arc and this induces a forbidden subposet on $a, d, b$. The argument is identical if we had started with $(d, b)$ being a non-arc.

![Diagram](image)

**Figure 7:** An example of the poset $\mathcal{I}P_{\pi}$

**Proof:** (Of Theorem 1) From Lemma 3, we see that for $\pi \in \Pi_k$, the ideals of $\mathcal{I}P_{\pi}$ with $(n \choose 2) - k - \ell$ elements are precisely the elements constituting $\pi(k, \ell)$. Lemma 4 completes the proof.

We note that the earlier poset obtained by inversion-deletion is actually a distributive lattice and that it is isomorphic to the lattice of order ideals $J(\mathcal{I}P_{\pi})$ (where $\pi$ is the starting permutation).

**Remark 2** We are essentially assigning two values to each “embroidered permutation” (see Page 81, [St-06]), though we use inversions instead of non-inversions. The region of $\mathcal{R}(S_n)$ that $(\pi, \mathcal{C})$ represents is slightly different for us. Suppose $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, then the region corresponding to this embroidered permutation is $x_{\pi_1} < x_{\pi_2} < \cdots < x_{\pi_n}$ and $\forall g = (i, j) \in \mathcal{C}$, $0 < x_j - x_i < 1$. This is why we need $g$ to be an inversion. We are assigning two parameters $(a, b)$ to each embroidered permutation $(\pi, \mathcal{C})$ where $a$ is the number of non-inversions of $\pi$ and $b$ is the total number of inversions contained in the family $\mathcal{C}$.
Remark 3 Let \( \pi \in \Pi_k \). Because there is a single hyperplane separating regions corresponding to \((k, \ell)\)-tree posets and \((k, \ell - 1)\)-tree posets, the lattice \( J(\IP_{\pi}) \) when treated as a graph is the subgraph of distance graph of \( R(S_n) \) with respect to the base region \( B \) consisting of those regions of \( R(S_n) \) which sit inside a given region of \( B_n \) (the Braid arrangement).

3 Results on the 3d partitions

We collect some properties of each of the partitions below.

3.1 Properties of the partitions

We prove some properties about the order of the components of the three 3d partitions.

Lemma 4 For \( n \geq 2 \), the parking partition satisfies \( gt_n = lt_n \geq eq_n \).

Proof: It is known that \( \overline{a} = (a_1, a_2, \ldots, a_n) \) is a parking function iff its weakly increasing permutation \( \overline{b} = (b_1, b_2, \ldots, b_n) \) satisfies the relation \( b_i < i \). Let \( \overline{a} = (a_1, a_2, \ldots, a_n) \in \PF_n \) with \( a_1 > a_2 \). Clearly, \( \overline{\pi}' = (a_2, a_1, \ldots, a_n) \) obtained from \( \overline{\pi} \) by swapping the first two coordinates is also a valid parking function, and has \( a_1' < a_2' \). The argument is reversible and this bijection proves that \( gt_n = lt_n \).

We show that \( lt_n \geq eq_n \). Let \( \overline{\pi} \in eq_n \). Let \( \overline{b} = (a_1, a_2 + 1, a_3, \ldots, a_n) \) and \( \overline{\pi} = (c_1, c_2, \ldots, c_n) \) be a weakly increasing permutation of \( \overline{b} \). We show that \( \overline{b} \in lt_n \). We only need to check that \( \overline{b} \in \PF_n \). Suppose not, then there is an index \( k \) such that \( c_k \neq k \). Since we changed only one coordinate to obtain \( \overline{b} \) from \( \overline{\pi} \), \( c_k = a_2 + 1 \). But then \( a_1 = a_2 - 1 \) will be \( c_{k-x} \) for \( x \geq 1 \) and thus we get \( \overline{\pi} \notin \PF_n \) which is a contradiction.

Lemma 5 For \( n \geq 2 \), the tree-poset partition satisfies \( R_n^{<0} \geq \max(R_n^{>1}, R_n^{0<1}) \).

Proof: We first prove that \( R_n^{<0} \geq R_n^{>1} \). To do this, we note that by Theorem \( \Pi \) the regions \( R_n^{<0} \) are those which have \((1, 2)\) as a forward arc and the regions of \( R_n^{>1} \) are those which have \((1, 2)\) as a backward arc, with the condition that \((1, 2)\) has not been converted into an incomparability relation.

We will show a slightly stronger property: consider all permutations \( \pi \) of \([n]\) in which 1 precedes 2 (ie \((1, 2)\) is a forward arc). Such permutations contribute \( |J(\IP_{\pi})| \) elements to \( R_n^{<0} \) and only such permutations contribute to \( R_n^{>1} \).

For each such \( \pi \), let \( \pi' \) be the permutation obtained by inverting the position of the elements 1 and 2. Similar to the above argument, every region of \( R_n^{>1} \) occurs from \( \pi' \) and an ideal of \( \IP_{\pi'} \) which does not contain the inversion \( \{2, 1\} \) (and hence all elements \( X = \{ x \geq_{\IP_{\pi'}} \{2, 1\} \} \)). Let \( \IP_{\pi'}(21) \) denote the subposet \( \IP_{\pi'} - X \). It is simple to see that the \( \IP_{\pi'}(21) \) is a subposet of \( \IP_{\pi} \) as well. Thus the number of order ideals is smaller for each \((\pi', \pi)\) pair and summing over these pairs completes the proof.
An almost identical proof works to show that \( R_{n<0}^0 \geq R_{n<0<1}^0 \). We note that \( R_{n<0<1}^0 \) is the number of \((\pi', \mathcal{I})\) pairs where \( \pi' \) is a permutation with 2 preceding 1 and \( \mathcal{I} \) is an ideal of \( IP_{\pi'} \) such that the inversion \((2,1) \in \mathcal{I} \). Thus \( X = \{x| x <_{IP_{\pi'}} (2,1) \} \in \mathcal{I} \) as well. Let \( IP_{\pi'}(2,1) = IP_{\pi'} - X \). The remaining argument is identical. 

**Lemma 6** For \( n \geq 2 \), the spanning-tree partition satisfies \( T_n^{\text{disj}} \geq T_n^v = T_n^v \).

**Proof:** We first prove that \( T_n^v = T_n^v \). Let \( T \in T_n^v \). Thus \( T \) is a 0-rooted spanning tree on \([n]_0\) and \( v\) is on the unique \( v - 0 \) path. By swapping the vertices \( v \) and \( v\), we get a tree \( T' \in T_n^v \). The equality part of the Lemma is thus proved.

To show that \( T_n^{\text{disj}} \geq T_n^v \), let \( T \in T_n^v \) as before. Let \( T'' \) be obtained from \( T \) by swapping \( v \) and 0. Clearly \( T'' \in T_n^{\text{disj}} \).

3.2 Properties among the partitions

In this section, we prove Theorems 2 and 3. We recall that \( eq_n \) is the number of \( \pi \in PF_n \) which satisfy \( a_1 = a_2 \).

**Proof:** (Of Theorem 2) The proof of Pollack given in [St-06](Page 92) to count the number of \( n \)-length Parking functions carries over exactly. 

**Proof:** (Of Theorem 3) We first show that the largest elements the parking partition is equal to the largest element of the tree-poset partition. Since the partitions are 3-dimensional, this is sufficient to prove the majorisation result.

We use the bijection of Pak and Stanley [St-98], coupled with the forbidden subposets of Athanasiadis [At-97]. By Lemma 6 \( R_{n<0} \) is the largest part of the tree-poset partition. We recall that the posets \( P_R \) of such a region \( R \) has a forward arc \((1,2)\) between vertices \( 1 \) and \( 2 \).

We first show that when the poset \( P_R \) of a region \( R \) has \((1,2)\) as a forward arc, then the corresponding parking function \( \pi_R \) of \( R \) under the bijection of Pak and Stanley has the property \( a_1 > a_2 \). Since \((1,2)\) is a forward arc, \( R \) is on the “less than” side of the hyperplane \( x_1 - x_2 = 0 \). Since the base region \( B \) has \( x_1 > x_2 \), we must cross the hyperplane \( x_1 - x_2 = 0 \) at some point in any shortest distance path from \( B \) to \( R \). This crossover will contribute a 1 to \( a_1 \), the first component of the parking function \( \pi \) and 0 to \( a_2 \). It is simple to check that the only way to increase \( a_2 \) is to cross the hyperplane \( x_2 - x_v = 0 \) for some \( v \in [n] \) on a path from \( B \) to \( R \). All such crossovers are recorded by a forward arc \((2,v)\) in the poset representation of \( R \). For such vertices \( v \), since \((2,v)\) and \((1,2)\) are forward arcs, by transitivity of the poset, \((1,v)\) is also a forward arc and this means we contribute a 1 to \( a_1 \) as well. This completes the proof of one half of the bijection.

For the other half, let \( \pi \in gt_n \). We claim that its corresponding region \( R \) under the bijection of Pak and Stanley has \((1,2)\) as a forward arc. As before, if \( a_2 = k \), there exists a set \( S \) with \( |S| = k \) such that for all \( v \in S \), \((2,v)\) is a forward arc. Similarly, when \( a_1 = k + x \) for \( x > 0 \), there is a set \( T \) such that for all \( v \in T \), \((1,v)\) is a forward arc. We claim that \( 2 \in T \). Suppose not, then there is a vertex \( v \in T - S \), \( v \neq 2 \) such that \((1,v)\) is a forward arc and \((2,v)\) is not (see Figure 8). Thus there are two cases for the relation between 2 and \( v \).
• When \((v, 2)\) is a forward arc: As \((1, v)\) and \((v, 2)\) are forward arcs, by transitivity \((1, 2)\) too is.

• When \((2, v)\) is an incomparability: If \((1, 2)\) is a backward arc, then transitivity among these three vertices would be violated. If \((1, 2)\) were an incomparability relation, then we would get the first forbidden subposet of Figure 2 on the vertices 1, 2, \(v\).

This completes the proof of the theorem.

Conjecture 1 Similar to Theorem 3, the smallest parts of spanning-tree partition and the tree-poset partition are equal. For \(n \geq 2\), the sorted spanning-tree partition majorises the sorted tree-poset partition.

Conjecture 2 For fixed \(n, k\), the numbers \(\text{Dist}_n(k, \ell)\) as \(\ell\) increases are unimodal.

Question 1 Is there a recurrence or a generating function for the numbers occurring in the spanning tree partition?

References

[At-97] Christos A. Athanasiadis. A Class of Labelled Posets and the Shi Arrangement of Hyperplanes. J Comb. Theory Ser. A, 80, 1997, pp 158–162.

[EC2] R. P. Stanley. Enumerative Combinatorics, vol 2. Cambridge University Press, 1999.

[St-98] R. P. Stanley. Hyperplane arrangements, parking functions and tree inversions. in Mathematical Essays in Honor of Gian-Carlo Rota (B. Sagan and R. Stanley, eds.), Birkhauser, Boston/Basel/Berlin, 1998, pp. 359-375.

[St-06] R. P. Stanley. An Introduction to Hyperplane arrangements. Lecture notes, available at http://www-math.mit.edu/~rstan