Tensor network representations of parton wave functions

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Tensor network states and parton wave functions are two pivotal methods for studying quantum many-body systems. This work connects these two subjects as we demonstrate that a variety of parton wave functions, such as projected Fermi sea and projected fermionic or bosonic paired states, can be represented using tensor networks that consist of sequential operations of matrix product operators on certain initial states. The results can be compressed into matrix product states with moderate bond dimensions so various physical quantities that characterize the parton wave functions can be computed efficiently. For the projected Fermi sea, the single-particle orbitals can be recombined to form maximally localized Wannier orbitals. This greatly reduces the amount of entanglement in intermediate steps and helps to achieve high fidelity compressions. It is demonstrated that our method has a great potential using several paradigmatic parton wave functions for spin-1/2 systems.

Introduction — The complexity of quantum many-body systems has posed considerable challenges for physicists since the dawn of quantum mechanics. One fundamental curse is that the Hilbert space of a composite system grows exponentially with the number of its constituents. While perturbative methods have been very successful in studying weak interactions, the vast arena of strongly correlated quantum matter remain elusive in many aspects. Analytical and numerical progresses have been made along various directions. The subjects of this paper are tensor network states [1–6] and parton wave functions [7–10], which share the common feature of trying to encode quantum many-body states using a moderate amount of resources.

Tensor network states are designed to capture special quantum entanglement patterns that are present in the low-energy eigenstates of many physical Hamiltonians. The wave functions are expressed as contraction of tensors (i.e., multi-index number arrays). If a system is divided into two subsystems, the entanglement entropy of one subsystem is bounded by the number of virtual indices on their boundary. In many cases, the scaling of entanglement entropy ensures that the number of parameters stays constant or grows polynomially, so the approximation is very useful. The success of this approach begins with the invention of the density-matrix renormalization group (DMRG) algorithm [11] and has produced very impressive analytical and numerical results ever since.

The idea of parton was originally conceived to address some phenomena in particle physics but has also been very successful in condensed matter physics. In this approach, the basic constituents such as particles or spins are represented using slave particles (bosons or fermions) which reside in enlarged Hilbert spaces. It is hoped that the strongly correlated states of physical degrees of freedom can be approximated as suitable “mean field” states of the slave particles whose unphysical components in the enlarged Hilbert spaces are removed by some kind of projection. While this method may appear to be ad hoc at first sight, it does provide very valuable insights into many problems. The ground states of some exactly solvable models, such as the Haldane-Shastry model [12, 13] and the Kitaev honeycomb model [14], can be expressed as Gutzwiller projected parton states. In the studies of high- $T_c$ superconductors [15–17], fractional quantum Hall states [18–21], and quantum spin liquids [22–24], parton wave functions have been used extensively as variational ansatz.

It is usually possible to deduce some properties of parton wave functions based on their structures. The low-energy effective field theories can also be constructed in many cases to probe the physics [16, 22]. Nevertheless, numerical results are very much desired for quantitative assessment of parton wave functions. For example, the best variational parameters for the ground state of a system requires energy minimization with respect to the given Hamiltonian. Monte Carlo methods are widely used for computing expectation values such as energy [25–30]. This is relatively simple when the target state is made of fermionic determinants and/or Pfaffians but rather challenging if bosonic permanents are involved. The computation of entanglement entropy and entanglement spectrum [31–35], which have been used extensively to characterize many-body states, is still quite demanding for generic parton wave functions [36–40].

In this work, we prove that generic parton wave functions can be expressed as local tensor networks in a
straightforward manner. The explicit representations of projected Fermi sea and projected fermionic or bosonic paired states correspond to sequential operations of matrix product operators (MPO) on simple initial states. It is possible to compress such tensor networks into matrix product states (MPS) and various physical quantities can be evaluated efficiently using standard MPS techniques. Furthermore, if a parton wave function is a good variational ansatz, its MPS representation may be used as an initial input in DMRG simulations to speedup convergence. For the project Fermi sea, an optimized basis transformation using maximally localized Wannier orbitals is proposed, which greatly reduces the amount of entanglement in intermediate steps and helps to achieve high fidelity compressions.

Tensor network representation — The method proposed here can be applied to any spin, bosonic, or fermionic systems, but we shall use spin-1/2 lattice models to illustrate it [see Fig. 1 (a)]. The lattice sites are labeled by \( j \in [1, N] \) and the spin operators are denoted as \( S_j^a \) (\( a = x, y, z \)). In the Abrikosov fermion representation,

\[
S_j^a = \frac{1}{2} \sum_{\alpha \beta} c_{j \alpha}^\dagger \sigma^a_{\alpha \beta} c_{j \beta},
\]

where \( c_{j \alpha}^\dagger \) (\( c_{j \alpha} \)) are fermionic creation (annihilation) operators at site \( j \), \( \alpha = \uparrow, \downarrow \) is the spin index, and \( \sigma^a \) are Pauli matrices. This is an overcomplete representation with unphysical states (empty and doubly occupied) that need to be removed by the single-occupancy constraint \( \sum_\alpha c_{j \alpha}^\dagger c_{j \alpha} = 1 \). The Schwinger boson representation is very similar, where the fermionic operators are replaced by their bosonic counterparts.

The projected Fermi sea is a popular class of trial wave functions for quantum spin models. It is defined as

\[
|\Psi\rangle = P_G \prod_{m=1}^N d_m^\dagger \langle 0 |,
\]

where \( |0\rangle \) is the vacuum, the \( d_m^\dagger \)'s are single-particle orbitals of the partons, \( P_G = \prod_{j=1}^N P_j \) is a product of projectors on each site that impose the single-occupancy constraints. In general, the single-particle orbitals can be written as

\[
d_m^l = \sum_{j=1}^N \sum_{\alpha = \uparrow, \downarrow} A_{m,j,\alpha} c_{j \alpha}^\dagger = \sum_{l=1}^{2N} A_{ml} c_{l}^\dagger,
\]

where \( l = (j, \alpha) \) is introduced for ease of notation. The \( 2N \) degrees of freedom labeled by \( l \) are called interleaved sites. The \( N \times 2N \) matrix \( A_{ml} \) that parametrizes the occupied orbitals is usually obtained by solving some “mean-field” Hamiltonians that are quadratic in the parton operators.

The central result of this paper is that the projected Fermi sea in Eq. (2) has a very natural tensor network representation. More importantly, it can be compressed into MPS with moderate bond dimensions, which allows for efficient computation of various quantities such as variational energy, correlation functions, and entanglement measures. The key observation that leads to the tensor network representation is that the single-particle orbital \( d_m^l \) can be converted to an MPO with bond dimension \( D = 2 \) as

\[
d_m^l = \begin{pmatrix} 0 & 1 \end{pmatrix} \left[ \prod_{l=1}^{2N} \begin{pmatrix} 1 & 0 \\ A_{ml} c_l^\dagger & 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

One dummy column and one dummy row are appended in the expression to ensure that the MPOs on all lattice sites have the same form. If the dummy vectors are multiplied with their neighboring matrices, we recover a usual MPS with open boundary condition. It is then straightforward to find the tensor network representation of Eq. (2): 1) apply the \( N \) MPOs corresponding to the \( d_m^l \)'s to the fermionic vacuum; 2) apply the projector \( P_G \) to the Fermi sea with each term \( P_j \) acting on two neighboring interleave sites. This two-step procedure is illustrated in Fig. 1 (b). Further technical details that are useful in practical calculations are discussed in Ref. [41].

In the same spirit, tensor network representations of projected fermionic or bosonic paired states can be obtained using MPOs that create fermionic or bosonic pairs [41]. It should be emphasized that our prescription converts parton wave functions to rectangular-shaped two-dimensional tensor networks, regardless of the dimensionality and geometry of the original system. This is in sharp contrast to previous works that construct (possibly non-local) tensor networks for parton wave functions [42] or their norms [43] on the original lattice.

Compressing into MPS — Although the representation derived above is exact, physical quantities cannot be computed simply in such a tensor network. In fact, it is well known that the exact contraction of a two-dimensional tensor network with closed loops is exponentially difficult [2, 5]. This makes it imperative to develop an
approximation scheme that would enable actual calculations. An obvious choice is to sequentially act the MPOs on the MPS (with fermionic vacuum as the initial input) to generate another MPS. However, the bond dimension of the MPS increases exponentially with the number of MPOs, so it is impossible to carry out the procedure for more than ~ 12 MPOs. To this end, we need to truncate the MPS at intermediate steps such that its bond dimension $D$ never exceeds some fixed values. The simplest truncation method is the singular value decomposition (SVD), where one converts the MPS into the so-called mixed canonical form and small singular values are discarded [3]. The efficiency of this method is determined by the entanglement properties of the target state and its error is quantified by the norm of the discarded singular values.

If the $A_m l$'s in $d_m^l$ have similar magnitudes, it would substantially modify the matrices on all lattice sites when acting on an MPS, then the truncation is likely to introduce considerable errors. This is often the case when $d_m^l$ are eigenmodes of parton “mean-field” Hamiltonians, where the $A_m l$'s describe spatially extended Bloch waves (for periodic boundary conditions) or standing waves (for open boundary conditions). The maximally localized Wannier orbitals [44–48] is adopted to facilitate the truncation. The basic idea is to convert the wave function in Eq. (2) to

$$|\Psi\rangle = P_G \prod_{r=1}^{N} \zeta^l_r |0\rangle,$$  \hspace{1cm} (5)

where the $\zeta^l_r$'s are linear combinations of the $d_m^l$'s. The entanglement entropy grows much slower when using the MPOs built from the $\zeta^l_r$'s because each one of them only causes appreciable changes (i.e., entanglement increase) in the vicinity of a particular lattice site. This is possible when the $\zeta^l_r$'s are designed to mimic the maximally localized Wannier orbitals. To be specific, the position operator $X = \sum_{j=1}^{N} \sum_{\alpha=\uparrow,\downarrow} \hat{c}^\dagger_{j\alpha} \hat{c}_{j\alpha}$ is expressed as a matrix [49]

$$\overline{X}_{mn} = \langle 0 | d_m^l X d_n^l | 0 \rangle$$ \hspace{1cm} (6)

in the subspace spanned by the $d_m^l$'s. Its eigenvectors are denoted using a matrix $B$ such that $B^\dagger \overline{X} B$ is diagonal. The transformed orbital $\zeta^l_r$ is defined using $B_{mr}$ as

$$\zeta^l_r = \sum_{m=1}^{N} B_{mr} d_m^l = \sum_{i=1}^{2N} (B^T A)_{ri} \hat{c}_i^l.$$ \hspace{1cm} (7)

The parton wave function is unchanged because the $\zeta^l_r$'s are just linear combinations of the same set of orbitals. In many cases, the $\zeta^l_r$'s do not mix partons with different spins, so they can be separated to two groups that are transformed using the spin-up and spin-down position operators, respectively. As the order of the $\zeta^l_r$'s in

| $D$ | energy deviation |
|-----|-------------------|
| 1000 | $-41.14354115$ | $3.7 \times 10^{-4}$ |
| 2000 | $-41.14387956$ | $3.4 \times 10^{-5}$ |
| 3000 | $-41.14390614$ | $7.2 \times 10^{-6}$ |
| 4000 | $-41.14391112$ | $2.2 \times 10^{-6}$ |
| 5000 | $-41.14391248$ | $8.6 \times 10^{-7}$ |

TABLE I. Energy of the MPO-MPS results for the Haldane-Shastry model with $N = 100$ at several different bond dimensions. The deviation is computed with respect to the exact ground state energy $-41.14391334$.

FIG. 2. (a) The absolute difference $F$ between the numerical and exact values of the spin-spin correlation function in the $N = 100$ system. (b) The evolution of the von Neumann entanglement entropy $S_e$ at the center of the $N = 100$ system during the calculation. Three methods are compared: (1) use the original modes in Eq. (9) (red dots); (2) use the Wannier transformed modes from left to right (blue squares); (3) use the Wannier transformed modes and the left-meet-right strategy (magenta hexagons).

Eq. (5) does not matter, the truncation error can be further reduced by a “left-meet-right” strategy: alternately act the operator localized at the left or right edge and gradually move toward the center.

**Numerical results 1** — The first example that we have investigated is the Haldane-Shastry model [12, 13] with the Hamiltonian

$$H_{HS} = \sum_{p<q} \frac{\pi^2}{N^2} S_p \cdot S_q \sin \frac{\pi}{N} (p-\pi),$$ \hspace{1cm} (8)

Its ground state for even $N$ is a Gutzwiller projected half-filled Fermi sea

$$|\Psi_{HS}\rangle = P_G \prod_{m\alpha=\uparrow,\downarrow} d_m^\dagger \prod_{n\alpha=\uparrow,\downarrow} d_n^l |0\rangle,$$ \hspace{1cm} (9)

where $d_m^\dagger = N^{-1/2} \sum_{j=1}^{N} e^{-i(jm)} \hat{c}_{j\alpha}^\dagger$ is the creation operator in momentum space and the occupied momenta are $m = \frac{2\pi s}{N} \alpha$ with

$$s = \begin{cases} 0, \pm 1, \ldots, \pm \left(\frac{N-1}{4}\right), \frac{N}{4} & \text{if } N \text{ mod } 4 = 0 \\ 0, \pm 1, \ldots, \pm \frac{N-2}{4} & \text{if } N \text{ mod } 4 = 2 \end{cases}.$$ \hspace{1cm} (10)

The ground-state energy is $-\pi^2 (N + 5N^{-1})/24$ and the spin-spin correlation function in the ground state is [50, 51]

$$\langle S_p \cdot S_{p+q} \rangle = \frac{\sum_{\alpha=1}^{N/2} 3(-1)^\alpha \sin \left(\frac{\pi}{N}(2\alpha - 1)q\right)}{2N \sin \frac{\pi q}{N}}.$$ \hspace{1cm} (11)
The parton state Eq. (9) has been constructed using our MPO-MPS method in the $N = 100$ system for bond dimension $D$ up to 5000. The comparison between the energy values in Table 1 and the spin-spin correlation function in Fig. 2 (a) clearly demonstrates the success of our method. This level of accuracy is very difficult to achieve using Monte Carlo methods. The evolution of the von Neumann entanglement entropy at the center of the system during the calculation is presented in Fig. 2 (b). It is apparent that the Wannier mode transformation and the left-meet-right strategy are both very useful as they can significantly reduce the amount of entanglement. The Haldane-Shastry model is difficult to study using direct DMRG method due to its gapless nature and the long-range interaction. Indeed, we have checked that such calculations will take much longer time (and produce more accurate results at the same bond dimension) than the MPO-MPS method.

Numerical results 2 — The second example that we have investigated is a chiral spin liquid model [36] that has the same topological order as the $\nu = 1/2$ Laughlin quantum Hall state [52, 53]. It is defined on a square lattice with $N_x = 16$ and $N_y = 10$ on the cylinder with $\Theta_y = \pi$. There are two exact zero modes $d_{L\alpha}^\dagger$ and $d_{R\alpha}^\dagger$ for each spin that are localized at the left and right edges. The entanglement spectrum of $|\Psi_1\rangle$ and $|\Psi_2\rangle$. The dashed lines indicate two sets of conformal towers in the two panels.

FIG. 3. (a) Schematics of the parton Hamiltonian of the chiral spin liquid model. Each unit cell contains two lattice sites labeled as $A$ and $B$. The signs of $t_{jk}$ are indicated using $\pm$ along the bonds. The signs of $\Delta_{jk}$ are indicated using arrows along the colored lines. (b) The parton energy spectrum of the system with $N_x = 16$ and $N_y = 10$ on the cylinder with $\Theta_y = \pi$. There are two exact zero modes $d_{L\alpha}^\dagger$ and $d_{R\alpha}^\dagger$ for each spin that are localized at the left and right edges. (c,d) The entanglement spectrum of $|\Psi_1\rangle$ and $|\Psi_2\rangle$. The dashed lines indicate two sets of conformal towers in the two panels.

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The overlaps between them have been computed at $D = 8000$ and the only appreciable one is $|\langle \Psi_1 | \Psi_1 \rangle | = 0.9237$ (others are smaller than $10^{-8}$). The MPO-MPS results are consistent with the previous claim that $|\Psi_1\rangle = |\Psi_2\rangle$ [58], and further suggest that there are three rather than two linearly independent states. This makes the choice of MES a subtle issue, but it turns out that either $|\Psi_{1,2}\rangle$ or $|\Psi_{1,3}\rangle$ can be used as the two MESs [41]. The entanglement spectra of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ at $D = 9000$ are shown in Fig. 3 (c) and (d). The accuracy of an MPS...
is quantified by its many-body momentum $K_y$. If there is no truncation error, $\exp[iK_y N_y/(2\pi)]$ would be 1 for all the $|\Psi_i\rangle$’s. The numerical value of $\exp[iK_y N_y/(2\pi)]$ is 0.9714 for $|\Psi_1\rangle$ and 0.9955 for $|\Psi_2\rangle$ at $D = 9000$. The good quantum numbers for the entanglement levels are the $z$-component spin $S_z^L$ and the momentum $K_y^L$ of the left half. The characteristic chiral boson counting 1, 1, 2, 3, 5, . . . are observed in all cases. The lowest entanglement eigenvalue of $|\Psi_1\rangle$ is smaller than that of $|\Psi_2\rangle$, so the former is the identity sector and the latter is the semion sector. The total countings agree with those of the SU(2)$_1$ Wess-Zumino-Witten model: 1, 3, 4, 7, . . . in the identity sector and 2, 2, 6, 8, . . . in the semion sector [61]. The topological spin $h$ of the semion can be computed using

$$h = \frac{\xi_{0,s} - \xi_{0,1}}{\Delta},$$

(14)

where $\xi_{0,1}$ ($\xi_{0,s}$) is the lowest entanglement eigenvalue in the identity (semion) sector and $\Delta$ is the spacing between the first two entanglement levels in the identity sector [see Fig. 3 (c)]. Its numerical value 0.2617 is reasonably close to the theoretical value 1/4.

Conclusion and discussion — In summary, we have constructed an exact tensor network representation for generic parton wave functions. This tensor network representation takes the form of sequential operations of matrix product operators on product states and can be conveniently compressed into matrix product states. This allows to characterize parton wave functions using powerful MPS techniques and greatly expands the utility of parton wave functions as variational ansatz. The parton wave functions studied in this paper have no free parameters. An immediate next step is to consider parton wave functions with variational parameters and search for their optimal values. The tensor network automatic differentiation method is well-adapted for this purpose [62–64]. The parton wave functions could be supplied as initial inputs to speedup DMRG simulations. Besides the ground states, our method is also capable for studying excitations. The numerical prospect of parton wave functions in the age of tensor networks deserves further investigations and we hope to report other interesting results in future works.

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See the Appendices for some technical details about numerical implementations, the tensor network representations of projected fermionic or bosonic paired states, additional discussions about the MES, and how to compute permanents using tensor network states.

Z.-C. Gu, F. Verstraete, and X.-G. Wen, arXiv:1004.2563 (2010).

B. Béri and N. R. Cooper, Phys. Rev. Lett. 106, 156401 (2011).

[47] X.-L. Qi, Phys. Rev. Lett. 107, 126803 (2011).

[48] N. Marzari, A. A. Mostofi, J. R. Yates, I. Souza, and D. Vanderbilt, Rev. Mod. Phys. 84, 1419 (2012).

[49] It is assumed that the $d_m$'s have been orthogonalized to form a set of orthonormal modes with $\{d_m, d_n^\dagger\} = \delta_{mn}$.

[50] Y. Kuramoto and Y. Kato, Dynamics of one-dimensional quantum systems: inverse-square interaction models (Cambridge University Press, New York, 2009).

[51] A. E. B. Nielsen, J. I. Cirac, and G. Sierra, J. Stat. Mech. 2011, P11014 (2011).

[52] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).

[53] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987).

[54] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, Phys. Rev. B 85, 235151 (2012).

[55] E. Keski-Vakkuri and X.-G. Wen, Int. J. Mod. Phys. B 07, 4227 (1993).

[56] L. Cincio and G. Vidal, Phys. Rev. Lett. 110, 067208 (2013).

[57] M. P. Zaletel, R. S. K. Mong, and F. Pollmann, Phys. Rev. Lett. 110, 236801 (2013).

[58] H.-H. Tu, Y. Zhang, and X.-L. Qi, Phys. Rev. B 88, 195412 (2013).

[59] W. Zhu, S. S. Gong, F. D. M. Haldane, and D. N. Sheng, Phys. Rev. B 92, 165106 (2015).

[60] Z. Liu, E. J. Bergholtz, H. Fan, and A. M. Läuchli, Phys. Rev. B 85, 045119 (2012).

[61] P. D. Francesco, P. Mathieu, and D. Sénéchal, Conformal Field Theory (Springer-Verlag, New York, 1997).

[62] H.-J. Liao, J.-G. Liu, L. Wang, and T. Xiang, Phys. Rev. X 9, 031041 (2019).

[63] C. Hubig, arXiv:1907.13422.

[64] Z.-Q. Wan and S.-X. Zhang, arXiv:1909.02659.

[65] C. Moore and S. Mertens, The nature of computation (Oxford University Press, Oxford, 2011).
APPENDIX A: FURTHER TECHNICAL DETAILS

This section provides more technical details that are helpful in practical calculations. The fermionic creation and annihilation operators are not convenient to handle in tensor networks due to their anticommutation relation. To this end, the fermionic orbitals are converted to spin-1/2 degrees of freedom $\sigma_i$ (not to be confused with the original spin-1/2’s) using the Jordan-Wigner transformation $c_i^\dagger = \sigma_i^+ \cdots \sigma_{i-1}^+$, where $\sigma_i^\pm = \frac{1}{2}([c_i, \sigma_i] \pm 1)$. The MPO for $d_m^\dagger$ becomes

$$d_m^\dagger = (0 1) \left[ \prod_{l=1}^{2N} \left( \begin{array}{cc} 1 & 0 \\ A_{ml} \sigma_l^+ & 1 \end{array} \right) \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right),$$

(A1)

and the fermionic vacuum changes to $|\downarrow \downarrow \ldots \downarrow\rangle$ where all spins point down. In the projected Fermi sea state, the number of partons is a good quantum number and it translates to the total $z$-component spin after the Jordan-Wigner transformation. The vacuum has no parton and acting one $d_m^\dagger$ increases the number of partons by 1. For the two examples studied in the main text, the spin-up and spin-down modes are not mixed in any step, so the intermediate states have a $U(1) \times U(1)$ symmetry corresponding to the numbers of partons with spin-up and spin-down. This symmetry can be exploited to significantly improve the computational speed. In principle, the Gutzwiller projection should be implemented after the whole unprojected state has been generated. However, if only the ground state is concerned (as in the main text), all double occupancy in the intermediate steps can be removed immediately since they would not survive in the final projection. The spin-up and spin-down partons were placed on neighboring sites in the interleave representation, but the removal of doubly occupied states can be performed most easily without using such a representation. In fact, one can merge two neighboring interleaved sites such that each site hosts one spin-up mode and one spin-down mode. The local Hilbert space dimension is 4 if there is no constraint. One can simply discard the doubly occupied state and reduce the dimension to 3.

APPENDIX B: PROJECTED FERMIONIC OR BOSONIC PAIRED STATES

Another important class of parton wave functions is projected fermionic or bosonic paired states. The bosonic paired state has the general form

$$|\Psi\rangle = P_G \prod_{k=1}^{2N} \prod_{l=1}^{2N} (1 + g_{kl} b_k^\dagger b_l^\dagger) |0\rangle,$$

(A2)

where $k, l$ denote interleave sites, $g_{kl}$ is the pairing function between them, and $b_k^\dagger$ is the creation operator for the $k$-th bosonic mode. As for the projected Fermi sea in Eq. (2) of the main text, the unprojected paired states in Eq. (A2) are usually obtained by solving some “mean-field” Hamiltonians of the partons with pairing terms.

The state in Eq. (A2) also has a natural tensor network representation. We again consider the spin-1/2 case for illustrating the method. Because of the single-occupancy constraint imposed by the Gutzwiller projector, Eq. (A2) can be rewritten as

$$|\Psi\rangle = P_G \prod_{k=1}^{2N} \prod_{l=1}^{2N} (1 + g_{kl} b_k^\dagger b_l^\dagger) |0\rangle = P_G \prod_{k=1}^{2N} W_k |0\rangle,$$

(A3)

where

$$W_k = 1 + \sum_{l=1}^{2N} g_{kl} b_k^\dagger b_l^\dagger = (1 0) \left[ \prod_{l=1}^{k-1} \left( \begin{array}{cc} 1 & g_{kl} b_l^\dagger b_l^\dagger \\ 0 & 1 \end{array} \right) \right] \left( \begin{array}{c} 1 \\ b_k^\dagger \\ 0 \end{array} \right) \left[ \prod_{l=k+1}^{2N} \left( \begin{array}{cc} 1 & 0 \\ g_{kl} b_l^\dagger b_l^\dagger & 1 \end{array} \right) \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

(A4)

is explicitly expressed as an MPO with bond dimension 2. The projected bosonic paired state can be obtained by successively applying $2N$ MPOs to the bosonic vacuum and performing Gutzwiller projection at the end. It is worth noting that the bosonic operators in Eq. (A4) actually create hardcore bosons due to the presence of $P_G$. The role of $W_k$ is to create valence-bond singlets between site $k$ and other sites.

The bosonic partons in Eq. (A2) can be replaced by fermionic partons and the result can also be converted to a tensor network. For the fermionic paired state, we have the operator

$$W_k = (1 0) \left[ \prod_{l=1}^{k-1} \left( \begin{array}{cc} 1 & g_{kl} c_l^\dagger c_l^\dagger \\ 0 & 1 \end{array} \right) \right] \left( \begin{array}{c} 1 \\ c_k^\dagger \\ 0 \end{array} \right) \left[ \prod_{l=k+1}^{2N} \left( \begin{array}{cc} 1 & 0 \\ g_{kl} c_l^\dagger c_l^\dagger & 1 \end{array} \right) \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

(A5)
for creating fermionic valence bonds. As for the projected Fermi sea, it is also convenient to perform a Jordan-Wigner transformation and use

$$W_k = \begin{pmatrix} 1 & 0 \\ \mathbf{g}_{kl} \mathbf{\sigma}_k^+ \mathbf{\sigma}_l^- \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{\sigma}_k^+ \\ -\mathbf{\sigma}_k^- & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{g}_{kl} \mathbf{\sigma}_k^+ \mathbf{\sigma}_l^- \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}. \quad (A6)$$

APPENDIX C: MINIMALLY ENTANGLLED STATES ON THE CYLINDER

This section provides more details about the minimally entangled states (MESs) of the chiral spin liquid on the cylinder. For topologically ordered systems, the ground state is not unique on certain mainfolds. This brings out the problem of choosing a suitable basis for multiple degenerate ground states. The MESs constitute an important basis because they minimize the entanglement entropy of non-contractible regions on the manifold [54]. In the present setup, the region is half of the cylinder.

As we have explained in the main text, filling the boundary zero modes in the parton energy spectrum results in three orthogonal states

$$|\Psi_1\rangle = P_G d_{L\alpha}^\dagger d_{R\alpha}^\dagger |\Phi\rangle, \quad |\Psi_2\rangle = P_G d_{L\alpha}^\dagger d_{R\alpha}^\dagger d_{L\alpha}^\dagger d_{R\alpha}^\dagger |\Phi\rangle, \quad |\Psi_3\rangle = P_G d_{L\alpha}^\dagger d_{R\alpha}^\dagger |\Phi\rangle. \quad (A7)$$

The orthogonality has been checked by computing their overlaps using standard MPS techniques. It is easy to see that $|\Psi_1\rangle$ is a spin-singlet. In contrast, $|\Psi_2\rangle$ and $|\Psi_3\rangle$ do not have definite total spins. One can use their linear combinations to form a spin-singlet

$$|\tilde{\Psi}_s\rangle = P_G (d_{L\alpha}^\dagger d_{R\alpha}^\dagger + d_{L\alpha}^\dagger d_{R\alpha}^\dagger) |\Phi\rangle \quad (A8)$$

and a spin triplet (within the $S_z = 0$ subspace)

$$|\tilde{\Psi}_t\rangle = P_G (d_{L\alpha}^\dagger d_{R\alpha}^\dagger - d_{L\alpha}^\dagger d_{R\alpha}^\dagger) |\Phi\rangle. \quad (A9)$$

If the cylinder is wrapped to be a torus, $|\Psi_1\rangle$ and $|\tilde{\Psi}_s\rangle$ would adiabatically evolve to the two degenerate ground states. The physical picture for $|\tilde{\Psi}_s\rangle$ is that $d_{L\alpha}^\dagger (d_{R\alpha}^\dagger)$ creates a semion with spin projection $\alpha$ at the left (right) boundary, which ensures that $|\tilde{\Psi}_s\rangle$ has a well-defined semion flux inside the cylinder.

For a system with chiral topological order, Ref. [34] found that the reduced density matrix of an MES on the half-cylinder is a thermal state of a chiral conformal field theory (CFT)

$$\rho_{\alpha} \propto e^{-H_{\text{CFT}}} |\alpha\rangle. \quad (A10)$$

The subscript $\alpha$ labels the anyon flux in the MES as well as the CFT primary field associated with the anyon. This result helps us to identify the MESs of the chiral spin liquid. As shown in Fig. 3 (c) of the main text, the level counting in the entanglement spectrum of $|\Psi_1\rangle$ agrees with the identity sector of the SU(2) Wess-Zumino-Witten (WZW) model, so we conclude that $|\Psi_1\rangle$ is the MES in the identity sector.

For the semion sector, some subtle issues arise when we try to find the MES. The entanglement spectrum of $|\tilde{\Psi}_s\rangle$ does not exhibit the conformal towers of the spin-1/2 primary of the SU(2) WZW model but turns out to be a tensor product of two identical copies of that. This is somewhat confusing at first sight but not really surprising. The two semions at the left and right edges in $|\tilde{\Psi}_s\rangle$ have opposite spins and form a singlet. However, the MES in the semion sector should break this nonlocal entanglement by fixing the $S_z$ quantum number of the distant semions. This can only be achieved by mixing $|\tilde{\Psi}_s\rangle$ and $|\tilde{\Psi}_t\rangle$ properly to break down their SU(2) symmetry to U(1) symmetry. The MES in the semion sector can be chosen as $|\Psi_2\rangle$ or $|\Psi_3\rangle$. As shown in Fig. 3 (d) of the main text, the level counting in the entanglement spectrum of $|\Psi_2\rangle$ agrees with the semion sector of the SU(2) WZW model. This is also the case for $|\Psi_3\rangle$.

The situation encountered here is reminiscent of the spin-1 AKLT model on an open chain with finite length. This system has four degenerate ground states (one singlet and one triplet) due to two emergent spin-1/2 edge states. The entanglement spectrum of the singlet state on a half-chain has four quasi-degenerate levels, which is twice as much as the two-fold degeneracy expected for the spin-1 Haldane phase. To reveal the two-fold degeneracy, we need to project the two edge states to subspaces with fixed $S_z$ quantum numbers and then compute the entanglement spectrum. This can be done using a linear combination of the singlet and the triplet with $S_z = 0$. 
APPENDIX D: PERMANENTS FROM TENSOR NETWORK STATES

As a byproduct of our method, a tensor network representation of permanents can be designed. The permanent of a $N \times N$ matrix $A$ can be encoded using $N$ bosonic modes described by creation (annihilation) operators $b_k$ ($b_k$) with $k = 1, 2, \cdots, N$. Let us consider the many-body wave function

$$|\Psi\rangle = d_1^\dagger d_2^\dagger \cdots d_N^\dagger |0\rangle,$$

(A11)

where

$$d_m^\dagger = \sum_{j=1}^{N} A_{m,j} b_j^\dagger, \quad (1 \leq m \leq N).$$

(A12)

The permanent $\text{Per}(A)$ is the overlap

$$\langle 0 | b_N b_{N-1} \cdots b_1 | \Psi \rangle = \langle 0 | b_N b_{N-1} \cdots b_1 d_1^\dagger d_2^\dagger \cdots d_N^\dagger |0\rangle.$$  

(A13)

It is apparent that the one-mode overlap $\langle 0 | b_j d_m^\dagger |0\rangle = A_{m,j}$. The Wick’s theorem tells us that the right hand side of Eq. (A13) can be expressed using the one-mode overlap as

$$\sum_{p \in S_N} A_{1,p(1)} A_{2,p(2)} \cdots A_{N,p(N)}.$$  

(A14)

It consists of all possible combinations of the $b_j$’s and the $d_m^\dagger$’s ($S_N$ is the permutation group of $N$ elements), which is precisely the definition of the permanent of $A$. The bosonic mode $d_m^\dagger$ can be converted to an MPO as we have done for the fermionic mode in the main text. This helps us to find the tensor network representation of $|\Psi\rangle$. The overlap is the contraction of $|\Psi\rangle$ with the MPS $\langle 0 | b_N b_{N-1} \cdots b_1 | \Psi \rangle$. The hardcore condition can be imposed on each site because the configurations with more than one boson on any site do not contribute to the permanent. This means that the physical legs of the MPO always have dimension 2.

The tensor network representation of permanents has an appealing geometric picture that reveals its connection to the counting of perfect matching [65]. As shown in Fig. A1 (a), the local tensor has only five nonzero elements, whose values are either 1 or $A_{m,j}$. The $m, j$ indices label the location of the tensor in the network. The binary indices are denoted using red solid and blue dashed lines, so they acquire geometric meaning of world lines. The tensor network for $\text{Per}(A) = \langle 0 | b_N b_{N-1} \cdots b_1 | \Psi \rangle$ assembles local tensors into the form of Fig. A1(b). One recognizes that the tensor contraction amounts to count the weighted sum of perfect matchings. In cases where all matrix elements are non-negative, there can be efficient stochastic algorithm to estimate the permanent by sampling the permutation of the world lines. The tensor network method can even deal with the cases where $A$ is a complex matrix. Moreover, suppose that one changes the tensor element of the first diagram in Fig. A1(a) from 1 to $-1$, the tensor contraction would then evaluate the matrix determinant instead of permanent. This is also intuitive from the picture since the tensor element corresponds to the crossing of the world lines.
After extensive numerical experiments, we conclude that this method is not as fast as the Ryser’s algorithm with gray code. Nevertheless, we hope that this observation could be useful in some different settings, such as designing efficient algorithms for approximating permanents.