ON CONVERGENCE OF SUBSPACES GENERATED BY DILATIONS OF POLYNOMIALS. AN APPLICATION TO BEST LOCAL APPROXIMATION

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ABSTRACT. We study the convergence of a net of subspaces generated by dilations of polynomials in a finite dimensional subspace. As a consequence, we extend the results given by Zó and Cuenya [Advanced Courses of Mathematical Analysis II (Granada, 2004), 193–213, World Scientific, 2007] on a general approach to the problems of best vector-valued approximation on small regions from a finite dimensional subspace of polynomials.

1. Introduction

Suppose that \( \{a_j\} \) is a data set. These data are values of a function and its derivatives at a point. If we want to approximate these data using a polynomial of degree at most \( l \), which will be the best algorithm to use? A Taylor polynomial of degree \( l \) is probably the most natural procedure to use.

The problem of finding an optimal algorithm to approximate a finite number of data corresponding to a function is developed in the best local approximation theory.

In 1934, Walsh proved in [11] that the Taylor polynomial of degree \( l \) for an analytic function \( f \) can be obtained by taking the limit as \( \epsilon \to 0 \) of the best Chebyshev approximation to \( f \) from \( \Pi^l \) on the disk \( |z| \leq \epsilon \). This paper was the first association between the best local approximation to a function \( f \) from \( \Pi^l \) in \( 0 \) and the Taylor polynomial for \( f \) at the origin. However, the concept of best local approximation has been introduced and developed more recently by Chui, Shisha, and Smith in [2]. Later, several authors [3, 4, 5, 6, 8, 9, 10, 12] have studied this problem.

We consider a family of function seminorms \( \{\|\cdot\|_\epsilon\}_{\epsilon>0} \), acting on Lebesgue measurable functions \( F : B \subset \mathbb{R}^n \to \mathbb{R}^k \), where \( B \) is the unit ball centered at the origin in \( \mathbb{R}^n \). We will use the notation \( F^\epsilon(x) = F(\epsilon x) \) and \( \|F\|_\epsilon^* = \|F^\epsilon\|_\epsilon \). For \( l \in \mathbb{N} \cup \{0\} \),

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we will denote by $\Pi^l_k$ the class of algebraic polynomials in $n$ variables of degree at most $l$, and $\Pi^l_k$ the set \{\(P = (p_1, \ldots, p_k) : p_s \in \Pi^l\)\}.

Let $A$ be a subspace of $\Pi^l_k$ and let \(\{P_\epsilon \}_{\epsilon > 0}\) be a net of best approximants to $F$ from $A$ with respect to $\|\cdot\|_\epsilon$, i.e.,

\[
\|F - P_\epsilon\|_\epsilon \leq \|F - P\|_\epsilon^*, \quad \text{for all } P \in A. \tag{1.1}
\]

If the net \(\{P_\epsilon \}_{\epsilon > 0}\) has a limit in $A$ as $\epsilon \to 0$, this limit is called the best local approximation to $F$ from $A$ in 0. According to (1.1), we observe that $P^\epsilon_\epsilon$ is a polynomial in

\[
A^\epsilon := \{P^\epsilon : P \in A \} \subset \Pi^l_k \tag{1.2}
\]

of best approximation to $F^\epsilon$ by elements of the class $A^\epsilon$, with respect to the seminorm $\|\cdot\|_\epsilon$. We write it briefly as $P^\epsilon_\epsilon \in \mathcal{P}_{A^\epsilon, \epsilon}(F^\epsilon)$. Note that $A^\epsilon$ is a subspace generated by dilations of polynomials in $A$.

From now on, we assume the following properties for the family of seminorms $\|\cdot\|_\epsilon$, $0 \leq \epsilon \leq 1$.

1. For $F = (f_1, \ldots, f_k)$ and $G = (g_1, \ldots, g_k)$, we have $\|F\|_\epsilon \leq \|G\|_\epsilon$, for every $\epsilon > 0$, whenever $|f_s| \leq |g_s|$, $s = 1, \ldots, k$.
2. If 1 is the function $F(x) = (1, \ldots, 1)$, we have $\|1\|_\epsilon < \infty$, for all $\epsilon > 0$.
3. For every $F \in C_k(B)$, we have $\|F\|_\epsilon \to \|F\|_0$, as $\epsilon \to 0$, where $C_k(B)$ is the set of continuous functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$. Moreover, $\|\cdot\|_0$ is a norm on $C_k(B)$.

An important point to note here is that there exist positive constants $C = C(m, k)$ and $\epsilon(m, k)$ such that for every $0 < \epsilon \leq \epsilon(m, k)$,

\[
\frac{1}{C} \|P\|_0 \leq \|P\|_\epsilon \leq C \|P\|_0, \quad \text{for every } P \in \Pi^l_k \tag{1.3}
\]

[13, Proposition 3.1].

In order to give an example of norms $\|\cdot\|_\epsilon$, $0 \leq \epsilon \leq 1$, with the properties (1)–(3), we recall a definition of convergence of measures given in [6]. See also [1] for the notion of weak convergence of measures in general.

**Definition 1.1.** Let $\mu_\epsilon$, $0 \leq \epsilon \leq 1$, be a family of probability measures on $B$. We say that the measures $\mu_\epsilon$ converge weakly in the proper sense to the measure $\mu_0$ if we have

\[
\int_B f(x) \, d\mu_\epsilon(x) \to \int_B f(x) \, d\mu_0(x), \quad f \in C_1(B),
\]

and $\mu_0(B') > 0$ for any ball $B' \subset B$.

The assumption on the measure $\mu_0$ implies that

\[
\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)} = \left(\int_B \|F\|^p \, d\mu_\epsilon\right)^{\frac{1}{p}}
\]

is actually a norm on $C_k(B)$ for $\epsilon = 0$ and $1 \leq p < \infty$, where $\|\cdot\|$ stands for any monotone norm on $R^k$. We use a monotone norm on $R^k$ to ensure property (1) for the family of seminorms $\|\cdot\|_\epsilon$, $0 \leq \epsilon \leq 1$.

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Let $F$ be in $C_k(B)$; it is readily seen, by using the definition of weak convergence of measures, that there exists $\epsilon_0 = \epsilon_0(F) > 0$ such that if $\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)} = 0$, for some $0 < \epsilon \leq \epsilon_0$, then $F = 0$. Moreover we have that $\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)}$ converges as $\epsilon \to 0$ to the norm $\|F\|_0 = \|F\|_{L^p(\mu_0)}$ if $F \in C_k(B)$.

For more examples of nets of seminorms fulfilling conditions (1)–(3), we refer the reader to [13].

We say that $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ has a Taylor polynomial of degree $m$ at $0$ if there exists $P \in \Pi^m_k$ such that

$$\|F - P\|_\epsilon = o(\epsilon^m), \quad \text{as } \epsilon \to 0.$$

It is well known that if a Taylor polynomial exists, it is unique [13, Proposition 3.3]; we denote it by $T_m = T_m(F)$. We write $F \in t^m$ if the function $F$ has the Taylor polynomial of degree $m$ at $0$. Moreover, if $F \in t^m$ and $T_m(F) = \sum_{|\alpha| \leq m} C_\alpha x^\alpha$, then the Taylor polynomial of degree $l \leq m$ for $F$ at $0$ is given by $T_l(F) = \sum_{|\alpha| \leq l} C_\alpha x^\alpha$ [13, Proposition 3.5], where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We set $\partial^\alpha F(0)$ for the vector $\alpha! C_\alpha$ with $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$.

The problem of best local approximation with a family of function seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon > 0}$ satisfying (1)–(3) was considered in [13] for two types of approximation class $A$ fulfilling $\Pi^m_k \subset A \subset \Pi^t_k$ and

(c1) $A^\epsilon = A$, for each $\epsilon > 0$, or
(c2) if $P \in A$ and $T_{m+1}(P) = 0$, then $P = 0$.

Firstly, the authors studied the asymptotic behavior of a normalized error function as $\epsilon \to 0$ [13, Theorems 4.2 and 4.5]. Secondly, they showed that there exists the best local approximation to $F$ in $0$ and is associated with a Taylor polynomial for $F$ in $0$ [13, Theorem 5.1]. In particular, if $A = \Pi^m_k$ and $F \in t^m$, they proved that $P_\epsilon \to T_m(F)$ as $\epsilon \to 0$ [13, Theorem 3.1].

In this work we generalize the results found in [13], without the restrictions (c1) or (c2) given above. For this, it is essential to study the convergence of the net $\{A^\epsilon\}$ as $\epsilon \to 0$.

This paper is organized as follows. In Section 2, we investigate the asymptotic behavior of $\{A^\epsilon\}$. In Section 3, we study the asymptotic behavior of the error function $\epsilon^{-m-1}(F_\epsilon - P_\epsilon)^\epsilon$ for a suitable integer, and we show some results about the best local approximation in the origin which generalizes those of [13].

2. Asymptotic Behavior of the Net $\{A^\epsilon\}$

In this section, we study the asymptotic behavior of the net $\{A^\epsilon\}$ given in [12]. We begin with the following definition.

**Definition 2.1.** Let $A \subset \Pi^t_k$ be a subspace. We say that $P \in \lim A^\epsilon$ if there exists a net $\{P_\epsilon\} \subset A$ such that $\lim_{\epsilon \to 0} \|P - P_\epsilon\|_\epsilon = 0$. We denote $B = \lim A^\epsilon$.

**Remark 2.2.** If $A \subset \Pi^t_k$ is a subspace, then the sets $A^\epsilon$ and $B$ are also subspaces of $\Pi^t_k$. Furthermore, if $A^\epsilon = A$, for all $\epsilon > 0$, we have that $B = A$. 

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Next, we show a simple example of $\mathcal{A}^c = \mathcal{A}$.

**Example 2.3.** Set $n = 3$ and $\mathcal{A} = \text{span}\{ (x_1, x_1 + x_2 + x_3, x_1^2 + x_2^2) \}$. Then, clearly we obtain $\mathcal{A}^c = \text{span}\{ (\epsilon x_1, \epsilon(x_1 + x_2 + x_3), \epsilon^2(x_1^2 + x_2^2)) \} = \mathcal{A}$.

**Proposition 2.4.** Let $\mathcal{A}$ be a subspace of polynomials such that $\Pi_k^m \subset \mathcal{A}$ for some $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Then $\Pi_k^m \subset \mathcal{A}^c$ for all $0 > \epsilon$. Moreover, $\Pi_k^m \subset \mathcal{B}$.

Proof. Set $R_{\alpha,i}(x) = x^\alpha e_i$, $|\alpha| \leq m$, $1 \leq i \leq k$, where $\{e_i\}_{i=1}^k$ is the canonical basis of $\mathbb{R}^k$. Then

$$\{R_{\alpha,i} : |\alpha| \leq m, 1 \leq i \leq k\} \quad (2.1)$$

is a basis of the space $\Pi_k^m$. Since $\mathcal{A}$ is a subspace, we have $R_{\alpha,i} = \frac{1}{\epsilon^{\alpha}} R_{\alpha,i}^\epsilon \in \mathcal{A}^c$, and so $\Pi_k^m \subset \mathcal{A}^c$, for all $0 > \epsilon$. Finally, using the definition of $\mathcal{B}$, we obtain $\Pi_k^m \subset \mathcal{B}$. \hfill \Box

From now on, for any Lebesgue measurable function $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ we denote $T_{-1}(F) = 0$.

**Proposition 2.5.** Let $\mathcal{A}$ be a subspace of $\Pi_k^1$ and let $0 \leq s + 1 \leq l$ be an integer. If $P \in \mathcal{A}$ satisfies $T_s(P) = 0$ and $T_{s+1}(P) \neq 0$, then $T_{s+1}(P) \in \mathcal{B}$.

Proof. For each $\epsilon > 0$ we define $Q_\epsilon = \frac{P}{\epsilon^{s+1}} \in \mathcal{A}$. Since $T_s(P) = 0$, it follows that $\|\|T_{s+1}(P) - Q_\epsilon^\epsilon\|_0 = \|\|(T_{s+1}(P) - Q_\epsilon^\epsilon)\|_0$. So $\|T_{s+1}(P) - Q_\epsilon^\epsilon\|_0 = o(1)$ as $\epsilon \to 0$, and thus $T_{s+1}(P) \in \mathcal{B}$. \hfill \Box

The following sets will be needed throughout the paper. Let $\mathcal{A}$ be a non-zero subspace of $\Pi_k^1$. We define

$$A_{-1} := \mathcal{A} \quad \text{and} \quad A_j := \{ P \in \mathcal{A} : T_j(P) = 0 \}, \quad \text{for } 0 \leq j \leq l. \quad (2.2)$$

We note that

$$A_j \subset A_i, \quad \text{whenever } i < j.$$

Since $A_i \subset \{ P \in \Pi_k^m : T_i(P) = 0 \} = \{0\}$, we have

$$\{ j : 0 \leq j \leq l \text{ and } A_j \neq \mathcal{A} \} = \emptyset \quad \text{and} \quad \{ j : 0 \leq j \leq l \text{ and } A_j = \{0\} \} \neq \emptyset.$$

Set

$$s_0 = \min \{ j : 0 \leq j \leq l \text{ and } A_j \neq \mathcal{A} \}$$

and

$$r_0 = \min \{ j : 0 \leq j \leq l \text{ and } A_j = \{0\} \}.$$

It is easy to see that $0 \leq s_0 \leq r_0 \leq l$, and

$$s_0, r_0 \in \{ j : s_0 \leq j \leq r_0 \text{ and } A_j \subset A_{j-1} \} = : J.$$

We can now formulate our main result which describes the limit set $\mathcal{B}$.

**Theorem 2.6.** Let $\mathcal{A}$ be a non-zero subspace of $\Pi_k^1$. Then $\mathcal{B}$ is a subspace of $\Pi_k^0$ isomorphic to $\mathcal{A}$. Furthermore, under the above notation the following holds:

(a) if $s_0 < r_0$ and $J \setminus \{ r_0 \} = \{ s_0, \ldots , s_N \}$ with $s_i < s_{i+1}$ for $N > 0$, then $\mathcal{B} = T_{r_0}(A_{s_0}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0})$, where $A_{s_i} \oplus S_{s_i} = A_{s_{i-1}}$, $0 \leq i \leq N;$

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(b) if \( s_0 = r_0 \), then \( \mathcal{B} = T_{r_0}(A) \).

Proof. (a) Assume \( s_0 < r_0 \). Since every subspace of \( A_{s_i-1} \), \( 0 \leq i \leq N \), has a complement, there exists a subspace \( S_{s_i} \subset A_{s_i-1} \) such that
\[
A_{s_i} \oplus S_{s_i} = A_{s_i-1}, \quad 0 \leq i \leq N. \tag{2.3}
\]
In consequence,
\[
A = A_{s_N} \oplus S_{s_N} \oplus S_{s_{N-1}} \oplus \cdots \oplus S_{s_0}. \tag{2.4}
\]
As \( S_{s_i} \subset A_{s_i-1} \), \( 0 \leq i \leq N \), and \( A_{r_0-1} = A_{s_N} \), we obtain
\[
Q(x) = \begin{cases} 
\sum_{|\alpha| \geq s_i} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in S_{s_i}, \ 0 \leq i \leq N; \\
\sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in A_{s_N},
\end{cases} \tag{2.5}
\]
where \( s_{N+1} = r_0 \). Let \( T_i : S_{s_i} \to \Pi_k^{s_i} \) be a linear operator defined by \( T_i(P) = T_{s_i}(P), \ 0 \leq i \leq N \), and \( T_{N+1} : A \to \Pi_k^{s_{N+1}} \) be the linear operator given by \( T_{N+1}(P) = T_{s_{N+1}}(P) \). We claim that
(i) \( T_i \) is an injective operator, \( 0 \leq i \leq N + 1 \).
(ii) \( T_{s_{N+1}}(A_{s_N}) \cap \sum_{i=0}^{N} T_i(S_{s_i}) = \{0\} \).
(iii) If \( N > 0 \) then \( T_{s_i}(S_{s_i}) \cap \left( T_{s_{N+1}}(A_{s_N}) + \sum_{i=0, i \neq l}^{N} T_{s_i}(S_{s_i}) \right) = \{0\} \) whenever \( l \neq i \).

Indeed, let \( 0 \leq i \leq N \). If \( T_i(P) = T_{s_i}(Q) \) for some \( P, Q \in S_{s_i} \), then \( P - Q \in A_{s_i} \cap S_{s_i} \). So [2.3] implies that \( P = Q \). On the other hand, if \( T_{s_{N+1}}(P) = T_{s_{N+1}}(Q) \) with \( P, Q \in A \), then \( P - Q \in A_{s_{N+1}} = \{0\} \), which proves (i). To prove (ii) we consider \( Q_{N+1} \in A_{s_N} \) and \( Q_i \in S_{s_i} \) such that \( P = T_{s_{N+1}}(Q_{N+1}) = \sum_{i=0}^{N} T_{s_i}(Q_i) \).

From (2.5), we see that
\[
T_{s_{N+1}}(Q_{N+1})(x) = \sum_{|\alpha| = s_{N+1}} \frac{\partial^\alpha Q_N(0)}{\alpha!} x^\alpha \quad \text{and} \quad \sum_{i=0}^{N} T_{s_i}(Q_i) = \Pi_k^{s_{N+1}}. \tag{2.6}
\]
Therefore \( P = 0 \). Now, let \( Q_{N+1} \in A_{s_N} \) and \( Q_i \in S_{s_i} \) be such that
\[
P = T_{s_i}(Q_i) = T_{s_{N+1}}(Q_{N+1}) + \sum_{i=0, i \neq l}^{N} T_{s_i}(Q_i). \tag{2.7}
\]
From (2.5) it follows that
\[
T_{s_i}(Q_i) = \sum_{|\alpha| = s_i} \frac{\partial^\alpha Q_i(0)}{\alpha!} x^\alpha, \quad 0 \leq i \leq N.
\]
According to (2.6) and (2.7), we have \( P = 0 \), and (iii) is proved. Using (i)–(iii), we deduce that the subspace
\[
T_{s_{N+1}}(A_{s_N}) + T_{s_N}(S_{s_N}) + T_{s_{N-1}}(S_{s_{N-1}}) + \cdots + T_{s_0}(S_{s_0})
\]
is a direct sum isomorphic to \( A \). The proof concludes by proving
\[
\mathcal{B} = T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0}).
\]
We observe that if $P \in S_{s_i} \setminus \{0\}$, then $T_{s_i}(P) \neq 0$ and $T_{s_i-1}(P) = 0$ by (2.3). So, Proposition 2.5 implies that $T_{s_i}(P) \in \mathcal{B}$. On the other hand, if $P \in A_{s_N} \setminus \{0\}$, we get $T_{s_N}(P) = 0$. Moreover, we have $T_{s_{N+1}}(P) \neq 0$. In fact, on the contrary, we see that $P \in A_{s_{N+1}} = \{0\}$. Proposition 2.5 now gives $T_{s_{N+1}}(P) \in \mathcal{B}$. Therefore,

$$T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_s}) \oplus T_{s_{N-1}}(S_{s_{s-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0}) \subset \mathcal{B}.$$

On the other hand, if $P \in \mathcal{B}$, there exists $\{P_i\} \subset \mathcal{A}$ such that

$$\lim_{\epsilon \to 0} \|P - P_\epsilon\|_0 = 0. \quad (2.8)$$

Let $d_{N+1} = \dim(A_{s_N})$ and $d_i = \dim(S_{s_i})$, $0 \leq i \leq N$. We take $\{v_i\}_{i=1}^{d_{N+1}}$ and $\{w_{ir}\}_{r=1}^{d_i}$ bases of $A_{s_N}$ and $S_{s_i}$, respectively. It is easy to check that for each $0 < \epsilon \leq 1$, $\{e^{-s_{N+1}}v_i\}_{i=1}^{d_{N+1}}$ is a basis of $A_{s_N}$ and $\{e^{-s_i}w_{ir}\}_{r=1}^{d_i}$ is a basis of $S_{s_i}$, $0 \leq i \leq N$. According to (2.4), we have that there exist real numbers $D_{i,\epsilon}$ and $C_{i,r,\epsilon}$ such that

$$P_\epsilon = \sum_{i=1}^{d_{N+1}} e^{-s_{N+1}}D_{i,\epsilon}v_i + \sum_{i=0}^{N} \sum_{r=1}^{d_i} e^{-s_i}C_{i,r,\epsilon}w_{ir}.$$

From (2.5) it follows that

$$v_i(x) = \sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha v_i(0)}{\alpha!} x^\alpha \quad \text{and} \quad w_{ir}(x) = \sum_{|\alpha| \geq s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha. \quad (2.9)$$

So,

$$P_\epsilon^\alpha(x) = \sum_{i=1}^{d_{N+1}} D_{i,\epsilon}e^{-s_{N+1}}v_i^\alpha(x) + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon}e^{-s_i}w_{ir}^\alpha(x)$$

$$= \sum_{i=1}^{d_{N+1}} D_{i,\epsilon} \sum_{|\alpha| = s_{N+1}} \frac{\partial^\alpha v_i(0)}{\alpha!} x^\alpha + \sum_{i=1}^{d_{N+1}} D_{i,\epsilon} \sum_{|\alpha| > s_{N+1}} \frac{\partial^\alpha v_i(0)}{\alpha!} x^\alpha$$

$$+ \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| = s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| > s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha.$$

Consequently

$$T_{s_j}(P_\epsilon^\alpha)(x) = \sum_{i=0}^{j} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| = s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha$$

$$+ \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_i < |\alpha| \leq s_j} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha.$$
if \(0 \leq j \leq N\), and

\[
T_{s_{N+1}}(P^\varepsilon_{\ell})(x) = \sum_{l=1}^{d_{N+1}} D_{l,\varepsilon} \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\varepsilon} \sum_{|\alpha|=s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha
\]

\[+ \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\varepsilon} \sum_{s_i < |\alpha| \leq s_{N+1}} \varepsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha.
\]

From (2.9) it follows that

\[
T_{s_{N+1}}(v_l)(x) = \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \quad \text{and} \quad T_{s_j}(w_{j,r})(x) = \sum_{|\alpha|=s_j} \frac{\partial^\alpha w_{j,r}(0)}{\alpha!} x^\alpha.
\]

Thus, a straightforward computation yields

\[
T_{s_0}(P^\varepsilon_{\ell})(x) = \sum_{r=1}^{d_0} C_{0,r,\varepsilon} T_{s_0}(w_{0,r})(x),
\]

\[
T_{s_j}(P^\varepsilon_{\ell})(x) = T_{s_{j-1}}(P^\varepsilon_{\ell})(x) + \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} C_{i,r,\varepsilon} \sum_{s_{j-1} < |\alpha| \leq s_j} \varepsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha
\]

\[+ \sum_{r=1}^{d_j} C_{j,r,\varepsilon} T_{s_j}(w_{j,r})(x)
\]

if \(1 \leq j \leq N\), and

\[
T_{s_{N+1}}(P^\varepsilon_{\ell})(x) = T_{s_N}(P^\varepsilon_{\ell})(x) + \sum_{l=1}^{d_{N+1}} D_{l,\varepsilon} T_{s_{N+1}}(v_l)(x)
\]

\[+ \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\varepsilon} \sum_{s_i < |\alpha| \leq s_{N+1}} \varepsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha.
\]

From (2.8) and (2.10), we deduce that \(T_{s_0}(P^\varepsilon_{\ell})(x) = \sum_{r=1}^{d_0} C_{0,r,\varepsilon} T_{s_0}(w_{0,r})(x)\) is convergent as \(\varepsilon \to 0\). Since \(\{T_{s_0}(w_{0,r})\}_{r=1}^{d_0}\) is a basis of \(T_{s_0}(S_{s_0})\), there are real numbers \(C_{0,r,\varepsilon}, 1 \leq r \leq d_0\), such that \(C_{0,r,\varepsilon} \to C_{0,r}\) as \(\varepsilon \to 0\). According to (2.8) and (2.11) it follows that \(\sum_{r=1}^{d_1} C_{1,r,\varepsilon} T_{s_1}(w_{1,r})(x)\) is convergent as \(\varepsilon \to 0\). Hence, there are real numbers \(C_{1,r,\varepsilon}, 1 \leq r \leq d_1\), such that \(C_{1,r,\varepsilon} \to C_{1,r}\) as \(\varepsilon \to 0\), because \(\{T_{s_1}(w_{1,r})\}_{r=1}^{d_1}\) is a basis of \(T_{s_1}(S_{s_1})\). Similarly, as \(\{T_{s_{N+1}}(v_l)\}_{l=1}^{a}\) is a basis of \(T_{s_{N+1}}(A_{s_N})\) and \(\{T_{s_i}(w_{i,r})\}_{r=1}^{d_i}\) is a basis of \(T_{s_i}(S_{s_i})\), 0 \(\leq i \leq N\), (2.8) and (2.10)–(2.12) show that there are real numbers \(D_l\) and \(C_{i,r,\varepsilon}\) such that \(D_{l,\varepsilon} \to D_l\) and \(C_{i,r,\varepsilon} \to C_{i,r}\) as \(\varepsilon \to 0\). In consequence,

\[
P = \sum_{l=1}^{a} D_l T_{s_{N+1}}(v_l) + \sum_{i=0}^{N} \left( \sum_{r=1}^{d_i} C_{i,r} T_{s_i}(w_{i,r}) \right),
\]

and so \(P \in T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0})\).
(b) Now assume \( s_0 = r_0 \), i.e., \( A_{s_0} = \{0\} \). Then \( A \) has the form \( (2.4) \) with \( N = 0 \), \( A_{s_0} = \{0\} \) and \( S_{s_0} = A \). An analysis similar to the proof of (a) shows that \( T_{r_0} \) is an isomorphism and \( B = T_{s_0}(S_{s_0}) = T_{r_0}(A) \).

The following corollary follows immediately from the proof of Theorem 2.6.

**Corollary 2.7.** Let \( A \) be a non-zero subspace of \( \Pi_k \). Then \( \lim_{n \to \infty} A^{*n} = B \) for any sequence \( \{\epsilon_n\} \) of the net \( \epsilon \downarrow 0 \).

**Remark 2.8.** \( B \) is isomorphic to \( T_{r_0}(A) \).

**Corollary 2.9.** Let \( s \geq m+1 \) and let \( A = \Pi_k^m \oplus A_{s-1} \) be such that \( A_s = \{0\} \). Then \( B = \Pi_k^m \oplus T_s(A_{s-1}) \) and the linear operator \( T : A \to \Pi_k^m \) given by \( T(P) = T_s(P) \) defines an isomorphism between \( A \) and \( B \).

**Proof.** We first claim that \( T \) is an injective operator. Indeed, if \( T(P) = T(Q) \) for \( P, Q \in A \), then \( T_s(P - Q) = 0 \) and so \( P - Q \in A_s \). Since \( A_s = \{0\} \), we have \( P = Q \).

Since \( A \) is isomorphic to \( T(A) \), the proof concludes by proving \( B = \Pi_k^m \oplus T_s(A_{s-1}) = T_s(A) \).

Let \( A_j \) be the sets defined in (2.2). Since

\[
\{0\} = A_s \subseteq A_{s-1} \subseteq \cdots \subseteq A_m \subseteq A_{m-1} \subseteq \cdots \subseteq A_0 \subseteq A,
\]

then \( A = A_{s-1} \oplus B_m \oplus B_{m-1} \oplus \cdots \oplus B_0 \), where \( A_i \oplus B_i = A_{i-1} \), \( 0 \leq i \leq m \). Therefore \( \Pi_k^m \) is isomorphic to \( B_m \oplus \cdots \oplus B_0 \). On the other hand, since \( s_0 = 0 \), \( r_0 = s \) and \( J \setminus \{r_0\} = \{0, 1, \ldots, m\} \), by Proposition 2.6 (a),

\[
B = T_s(A_{s-1}) \oplus T_m(B_m) \oplus \cdots \oplus T_0(B_0).
\]

From the proof of Theorem 2.6 we obtain that \( B_m \oplus \cdots \oplus B_0 \) is isomorphic to \( T_m(B_m) \oplus \cdots \oplus T_0(B_0) \), and consequently \( \Pi_k^m \) is isomorphic to \( T_m(B_m) \oplus \cdots \oplus T_0(B_0) \subseteq \Pi_k^m \). Hence, \( T_m(B_m) \oplus \cdots \oplus T_0(B_0) = \Pi_k^m \) and so \( B = T_s(A_{s-1}) \oplus \Pi_k^m = T_s(A_{s-1}) \oplus T_s(\Pi_k^m) = T_s(A) \).

\[\square\]

3. An Application to Best Local Approximation

Let \( \{P_\epsilon\} \) be a net of best approximants to \( F \) from \( A \) with respect to \( \| \cdot \|_e^* \), and let \( E_\epsilon \) be the error function

\[
E_\epsilon(F) = \frac{F_\epsilon - P_\epsilon}{\epsilon^{m+1}}.
\]

If \( F \in \mathfrak{m}^{m+1} \), then

\[
F_\epsilon = T_{m+1}^{m+1} \epsilon^{m+1} R_{m+1}^{\epsilon},
\]

where \( R_{m+1} = \frac{F - T_{m+1}^{m+1}}{\epsilon^{m+1}} \), \( \|R_{m+1}^\epsilon\|_e = o(1) \), and \( T_{m+1} \) is the Taylor polynomial of \( F \) of degree \( m+1 \) at 0. Moreover,

\[
\lambda P_\epsilon^e \in \mathcal{P}_{A^e}(\lambda F_\epsilon^e) \quad \text{and} \quad P_\epsilon^e + P_\epsilon^e \in \mathcal{P}_{A^e}((P + F)^e), \quad \text{for} \ P \in A.
\]

The following proposition may be proved in much the same way as [13, Proposition 4.1]. However, we repeat the proof for completeness.
Proposition 3.1. Let $A$ be a non-zero subspace of $\Pi_k^l$ with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of $F$ from $A$ with respect to $\| \cdot \|_\epsilon^*$. If $F \in t^{m+1}$, $T_m \in A$ and $\phi_{m+1} = T_{m+1} - T_m$, then

$$E_\epsilon(F) = \phi_{m+1} + R_{m+1}^\epsilon - P_{A^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon),$$

where $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$ as $\epsilon \to 0$.

Proof. Since $R_{m+1}^\epsilon = \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}}$, then

$$\phi_{m+1} + R_{m+1}^\epsilon = T_{m+1} - T_m + \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}} = \frac{T_{m+1} - T_m}{\epsilon^{m+1}} + \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}}.$$

As $T_m \in A$, we have

$$\phi_{m+1} + R_{m+1}^\epsilon - P_{A^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon) = \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}} - P_{A^\epsilon, \epsilon}\left(\frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}}\right) = \frac{F^\epsilon - P_{A^\epsilon, \epsilon}^\epsilon}{\epsilon^{m+1}} = E_\epsilon(F).$$

□

Next, we give a new result about the asymptotic behavior of the error without the conditions (c1) or (c2), which generalizes Theorems 4.2 and 4.5 of [13].

Theorem 3.2. Let $A$ be a non-zero subspace of $\Pi_k^l$ with $l > m$. If $F \in t^{m+1}$, $T_m \in A$ and $\phi_{m+1} = T_{m+1} - T_m$, then

$$\|E_\epsilon(F)\|_\epsilon \to \inf_{P \in B} \|\phi_{m+1} - P\|_0, \quad \text{as} \ \epsilon \to 0.$$

Proof. By Proposition 3.1

$$E_\epsilon(F) = \phi_{m+1} + R_{m+1}^\epsilon - P_{A^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon),$$

where $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$ as $\epsilon \to 0$. We first prove

$$\lim_{\epsilon \to 0} \|E_\epsilon(F)\|_\epsilon \leq \inf_{P \in B} \|\phi_{m+1} - P\|_0. \quad (3.2)$$

In fact, let $P \in B$. By the definition of $B$, there exists a net $\{Q_\epsilon\} \subset A$ such that $\|P - Q_\epsilon\|_0 \to 0$, as $\epsilon \to 0$. In consequence, $\|P - Q_\epsilon\|_\epsilon = o(1)$, as $\epsilon \to 0$, by (1.3). Since $Q_\epsilon \in A^\epsilon$ and $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$, from (3.1) we obtain

$$\|E_\epsilon(F)\|_\epsilon \leq \|\phi_{m+1} + R_{m+1}^\epsilon - Q_\epsilon\|_\epsilon \leq \|\phi_{m+1} - Q_\epsilon\|_\epsilon + o(1), \quad \text{as} \ \epsilon \to 0. \quad (3.3)$$

By Property (3), $\|\phi_{m+1} - P\|_\epsilon \to \|\phi_{m+1} - P\|_0$, as $\epsilon \to 0$. Hence, using the triangle inequality we have

$$\|\phi_{m+1} - Q_\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_0 \leq \|\phi_{m+1} - Q_\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_\epsilon$$

$$+ \|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0 \leq \|P - Q_\epsilon\|_\epsilon + \|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0 = o(1)$$

as $\epsilon \to 0$. Now, according to (3.3), we get (3.2).
We observe that
\[ \lim_{\epsilon \to 0} \| E_\epsilon(F) \|_\epsilon \geq \inf_{P \in B} \| \phi_{m+1} - P \|_0. \] (3.4)

Let \( \epsilon \downarrow 0 \) be a sequence such that \( \lim_{\epsilon \to 0} \| E_\epsilon(F) \|_\epsilon = \lim_{\epsilon \to 0} \| E_\epsilon(F) \|_\epsilon. \) We consider
\( P_\epsilon \in P_{A^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon) \). We claim that there exist constants \( M, \epsilon_0 > 0 \) such that
\[ \| P_\epsilon \|_0 \leq M, \quad 0 < \epsilon \leq \epsilon_0. \] (3.5)

Indeed, as \( 0 \in A^\epsilon \) we get
\[
\| P_\epsilon \|_\epsilon \leq \| P_\epsilon - (\phi_{m+1} + R_{m+1}^\epsilon) \|_\epsilon + \| \phi_{m+1} + R_{m+1}^\epsilon \|_\epsilon \\
\leq 2\| \phi_{m+1} + R_{m+1}^\epsilon \|_\epsilon \leq 2\| \phi_{m+1} + R_{m+1}^\epsilon \|_\epsilon,
\]
for \( 0 < \epsilon \leq 1 \). By Proposition 3.1 and Property (3), we see that \( 2\| \phi_{m+1} \|_\epsilon + 2\| R_{m+1}^\epsilon \|_\epsilon \to 2\| \phi_{m+1} \|_0 \), as \( \epsilon \to 0 \). So, from (1.3) and (3.6), we obtain (3.5).

In consequence, there exists a subsequence of \( \{ P_\epsilon \} \), which is denoted in the same way, and \( P_0 \in \Pi_k^l \) such that \( P_\epsilon \to P \) uniformly on \( B \), as \( \epsilon \to 0 \). Since
\[
\| \phi_{m+1} - P_0 \|_\epsilon - \| \phi_{m+1} - P \|_0 \leq \| \phi_{m+1} - P_\epsilon \|_\epsilon - \| \phi_{m+1} - P \|_\epsilon + \| \phi_{m+1} - P \|_\epsilon - \| \phi_{m+1} - P \|_0 \leq \| P - P_\epsilon \|_\epsilon + \| \phi_{m+1} - P \|_\epsilon - \| \phi_{m+1} - P \|_0 ,
\]
we get
\[ \| \phi_{m+1} - P \|_0 = \| \phi_{m+1} - P_\epsilon \|_\epsilon + o(1), \quad \text{as} \ \epsilon \to 0. \]

We observe that \( P \in B \) by Corollary 2.7. Therefore, by Proposition 3.1
\[
\inf_{Q \in B} \| \phi_{m+1} - Q \|_0 \leq \| \phi_{m+1} - P \|_0 = \| \phi_{m+1} - P_\epsilon \|_\epsilon + o(1) \\
\leq \| \phi_{m+1} + R_{m+1}^\epsilon - P_\epsilon \|_\epsilon + \| R_{m+1}^\epsilon \|_\epsilon \\
= \| E_\epsilon(F) \|_\epsilon + \| R_{m+1}^\epsilon \|_\epsilon.
\]
So, \( \inf_{Q \in B} \| \phi_{m+1} - Q \|_0 \leq \lim_{\epsilon \to 0} \left( \| E_\epsilon(F) \|_\epsilon + \| R_{m+1}^\epsilon \|_\epsilon \right) = \lim_{\epsilon \to 0} \| E_\epsilon(F) \|_\epsilon, \) and (3.4) is proved.

\[ \square \]

The following result provides us with a useful and important property for a net of best approximants to \( F \) from \( A \).

**Theorem 3.3.** Let \( A \) be a non-zero subspace of \( \Pi_k^l \) with \( l > m \), and let \( \{ P_\epsilon \} \) be a net of best approximants of \( F \) from \( A \) with respect to \( \| \cdot \|_\epsilon \). Assume \( F \in \Pi_{m+1}^l, T_m \in A \) and \( \phi_{m+1} = T_{m+1} - T_m \). If \( C \) is the cluster point set of the net \( \{ (P_\epsilon - T_m)^\epsilon \} \), as \( \epsilon \to 0 \), then \( C \neq \emptyset \). Moreover, each polynomial in \( C \) is a solution of the minimization problem
\[ \min_{P \in B} \| \phi_{m+1} - P \|_0. \] (3.7)
Proof. We observe that
\[
E_\epsilon(F) = \frac{(F - P_\epsilon)^\epsilon}{\epsilon^{m+1}} = \frac{m+1}{\epsilon^{m+1}} + \frac{(F - T_{m+1})^\epsilon}{e^{m+1}}
\]
\[
= \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} + \frac{(F - T_{m+1})^\epsilon}{e^{m+1}}.
\]
Then
\[
\|\phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}}\|_\epsilon \leq \|E_\epsilon(F)\|_\epsilon \leq \|\phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}}\|_\epsilon + \|(F - T_{m+1})^\epsilon\|_\epsilon,
\]
and consequently,
\[
\|E_\epsilon(F)\|_\epsilon = \|\phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}}\|_\epsilon + o(1), \quad \text{as } \epsilon \to 0,
\]
since \(F \in t^{m+1}\). By Theorem 3.2,
\[
\inf_{P \in B} \|\phi_{m+1} - P\|_0 = \lim_{\epsilon \to 0} \|\phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}}\|_\epsilon.
\]
According to (1.3), there exist constants \(\epsilon_0, M > 0\) such that
\[
\|\phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}}\|_0 \leq M,
\]
for all \(0 < \epsilon \leq \epsilon_0\). The equivalence of the norms in \(\Pi_k\) implies that the net \(\{(P_\epsilon - T_m)^\epsilon\}_{\epsilon \in (0, \epsilon_0]}\) is uniformly bounded on \(B\). So, there exists a subsequence of \(\{(P_\epsilon - T_m)^\epsilon\}_{\epsilon \in (0, \epsilon_0]}\), which is denoted in the same way, and a polynomial \(P_0\) such that
\[
\frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \text{ converges to } P_0, \text{ uniformly on } B, \text{ as } \epsilon \to 0.
\]
In consequence, \(C \neq \emptyset\).

On the other hand, if \(P_0 \in C\), there is a sequence \(\epsilon \downarrow 0\) such that \(\frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \to P_0\). Since \(T_m \in A\), we have \(P_\epsilon - T_m \in A\), and so \(P_0 \in B\) by Corollary 2.7. Finally, from Property (3) and (3.8), we conclude that
\[
\inf_{P \in B} \|\phi_{m+1} - P\|_0 = \lim_{\epsilon \to 0} \|\phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}}\|_\epsilon = \|\phi_{m+1} - P_0\|_0,
\]
i.e., \(P_0\) is a solution of (3.7).

The following theorem is an extension of [13, Theorem 5.1].
Theorem 3.4. Let $A$ be a non-zero subspace of $\Pi_k^m$ with $l > m$, and let \{P_\epsilon\} be a net of best approximants of $F$ from $A$ with respect to $\| \cdot \|_\epsilon$. Assume $m + 1 = \min \{ j : 0 \leq j \leq l$ and $A_j = \{0\} \}$, $F \in t^{m+1}$ with $T_m \in A$, and set $\phi_{m+1} = T_{m+1} - T_m$. If the minimization problem $\min_{P \in A}$ implies that there exists a unique

\[ \left(3.9\right) \] implies that there exists a unique $P_0$, then $P_\epsilon \to T_m + P$, where $P \in A$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.

Proof. Since $\left(3.7\right)$ has a unique solution $P_0$, Theorem 3.3 implies that

\[ \lim_{\epsilon \to 0} \frac{(P_\epsilon - T_m)\epsilon}{\epsilon^{m+1}} = P_0. \]

In consequence, $\partial^\alpha(P_\epsilon - T_m)(0) \to 0$, $|\alpha| \leq m$, and $\partial^\alpha(P_\epsilon - T_m)(0) \to \partial^\alpha P_0(0)$, $|\alpha| = m + 1$, as $\epsilon \to 0$. Therefore

\[ T_{m+1}(P_\epsilon - T_m)(x) \to \sum_{|\alpha|=m+1} \frac{\partial^\alpha P_0(0)}{\alpha!} x^\alpha =: R(x), \quad x \in B, \text{ as } \epsilon \to 0. \quad \left(3.9\right) \]

Let $T : A \to \Pi_k^{m+1}$ be the linear operator defined by $T(P) = T_{m+1}(P)$. As $A_{m+1} = \{0\}$, an analysis similar to that in the proof of Corollary 2.9 shows that $T$ is an injective operator. Since $T(A)$ is a closed subspace and $\{T_{m+1}(P_\epsilon - T_m)\} \subset T(A)$, $\left(3.9\right)$ implies that there exists a unique $P \in A$ such that $T_{m+1}(P) = R$. Hence $T_{m+1}(P_\epsilon - T_m - P) \to 0$ as $\epsilon \to 0$. As $A_{m+1} = \{0\}$ we see that $\|Q\| := \|T_{m+1}(Q)\|_0$ is a norm on $A$, and so $P_\epsilon \to T_m + P$ as $\epsilon \to 0$. Finally, by Theorem 2.6 $B \subset \Pi_k^{m+1}$, and consequently $P_0 - T_m(P_0) = T_{m+1}(P_0) - T_m(P_0) = R$. The proof is complete.

Remark 3.5. If $A$ satisfies the condition (c2), then $A = \Pi_k^m \oplus A_m$ with $A_{m+1} = \{0\}$. By Corollary 2.9 $B = \Pi_k^m \oplus T_{m+1}(A_m)$ and each element $P \in A$ is uniquely determined by $T_{m+1}(P)$. So, we can rewrite the problem $\left(3.7\right)$ in the following (equivalent) form:

\[ \min_{Q + U \in \Pi_k^m \oplus A_m} \|\phi_{m+1} - (Q + T_{m+1}(U))\|_0. \quad \left(3.10\right) \]

The following result has been proved in [13, Theorem 5.1] and it is a consequence of Theorem 3.4.

Corollary 3.6. Let $\Pi_k^m \subset A \subset \Pi_k^l$ be a non-zero subspace that satisfies the condition (c2) and let \{P_\epsilon\} be a net of best approximants of $F$ from $A$ with respect to $\| \cdot \|_\epsilon$. Assume $F \in t^{m+1}$. If the minimization problem $\min_{P \in A}$ has a unique solution $P_0$, then $P_\epsilon \to T_m + P$, where $P \in A$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.

In the following example we present a function $F \in \bigcap_{m=0}^\infty t^m$ such that $T_2(F) \notin A$ and the net \{T_i(P_i)\} does not converge for the same $i > m + 1.$
Example 3.7. Set $B = [-1, 1]$, $\|G\|_e = \left( \frac{1}{-1} \int |G(x)|^2 \, dx \right)^{\frac{1}{2}}$, $A = \text{span}\{1, x^2, x^3\}$, and $F(x) = x$. So
\[
\|G\|_e^* = \left( \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} |G(x)|^2 \, dx \right)^{\frac{1}{2}},
\]
and $A_0 = A_1 = \text{span}\{x^2, x^3\}$, $A_2 = \text{span}\{x^3\}$ and $A_3 = \{0\}$. Since $T_1(x^2) = 0$, we observe that the subspace $A$ does not satisfy the condition (c2). Moreover, an straightforward computation shows that
\[
\frac{\|F - T_0\|_e^*}{\epsilon^0} = \frac{\sqrt{6}}{3} \epsilon \quad \text{and} \quad \frac{\|F - T_s\|_e^*}{\epsilon^s} = 0, \quad s \in \mathbb{N},
\]
where $T_0(x) = 0$ and $T_s(x) = x$. In consequence, $F \in t^m$ for all $m \in \mathbb{N} \cup \{0\}$, and $T_2(F) \notin A$. Since $\int_{-\epsilon}^{\epsilon} (x - \frac{7}{5\epsilon^2} x^3) x^i \, dx = 0$, $i = 0, 2, 3$, then $P_\epsilon(x) = \frac{7}{5\epsilon^2} x^3$ is the best approximant to $F$ from $A$ with respect to $\| \cdot \|_e^*$. Therefore $T_i(P_\epsilon)(x) \to 0$, for $i = 0, 1, 2$, but $T_3(P_\epsilon)(x)$ does not converge, as $\epsilon \to 0$. So, the best local approximation to $F$ from $A$ in 0 does not exist, and
\[
\|E_\epsilon(F)\|_e = \frac{\|F - P_\epsilon\|_e^*}{\epsilon^3} = \frac{2\sqrt{6}}{15\epsilon^2} \to \infty, \quad \text{as} \quad \epsilon \to 0.
\]

We now give another example which shows that the condition $T_m \in A$ is not necessary for the existence of the best local approximation.

Example 3.8. Set $B$, $\| \cdot \|_e^*$ and $F$ as in Example 3.7 and we consider the subspace $A = \text{span}\{1, x^2\}$. It is clear that $A_0 = A_1 = \text{span}\{x^2\}$, $A_2 = \{0\}$ and $B = A$. Moreover, we have $F \notin t^2$, $T_1 \notin A$, and $A$ does not satisfy the condition (c2) since $T_1(x^2) = 0$. As $\int_{-\epsilon}^{\epsilon} (x - 0) x^i \, dx = 0$, $i = 0, 2$, then $P_\epsilon(x) = 0$ is the best approximant to $F$ from $A$ with respect to $\| \cdot \|_e^*$. Therefore, the polynomial 0 is the best local approximation to $F$ from $A$ in 0.

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