Scalars convected by a 2D incompressible flow

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1 Abstract

We provide a test for numerical simulations, for several two dimensional incompressible flows, that appear to develop sharp fronts. We show that in order to have a front the velocity has to have uncontrolled velocity growth.

2 Introduction

The aim of this paper is to study the possible formation of sharp fronts in finite time for a scalar convected by a two dimensional divergence-free velocity field, with $x = (x_1, x_2) \in \mathbb{R}^2$ or $\mathbb{R}^2/\mathbb{Z}^2$, and $t \in [0, T)$ with $T \leq \infty$. The

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scalar function $\theta(x,t)$ and the velocity field $u(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ satisfy the following set of equations

$$
(\partial_t + u \cdot \nabla) \theta = 0
$$

$$\nabla^\perp \psi = u,
$$

where $\nabla^\perp_x f = (-\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1})$ for scalar functions $f$. The function $\psi$ is the stream function.

There are many physical examples where the solutions satisfy the equations above, with an extra equation or operator that relates $\theta$ with the velocity field. Examples include: Passive scalars, Unsteady Prandtl equations, 2D incompressible Euler equations, Boussinesq, 2D Ideal Magnetohydrodynamics and the Quasi-geostrophic equation.

In the literature on numerical simulations for the 2D Ideal Magnetohydrodynamics (MHD) a standard candidate for a current sheet formation (see Fig. 1) is when the level sets of the magnetic stream function (represented in (1) by $\theta$) contain a hyperbolic saddle (an X-point configuration). The front is formed when the hyperbolic saddle closes, and becomes two Y-points configuration joined by a current sheet. (See Parker [12], Priest-Titov-Rickard [13], Friedel-Grauer-Marliani [10] and Cordoba-Marliani [8].)

\[ t < T \quad \text{Fig. 1. Level curves of } \theta. \quad t = T \]

The same configuration was observed in numerical simulations for the Quasi-geostrophic equation (QG). In this case the geometry of the level sets of the temperature has a hyperbolic structure (See Constantin-Majda-Tabak [4], Okhitani-Yamada [11], Cordoba [6] and Constantin-Nie-Schorghofer [5]). The QG literature discusses X-points, but not Y-points. In the case of Boussinesq there is no mention, on any numerical simulation study, that a possible singularity is due to the closing of a hyperbolic saddle. In the work of Pumir-Siggia [14] there has been observed evidence for a formation of a front in finite time, across which $\theta$ varies dramatically, on a cap of a symmetric rising bubble. E-Shu [9] performed numerical simulations with the same initial data.
as in [14], which suggest that the thickness of the bubble decreases only exponentially.

The equations for MHD, QG and Boussinesq are as follows:

**MHD:**

\[
\begin{align*}
(\partial_t + u \cdot \nabla) \theta &= 0 \\
(\partial_t + u \cdot \nabla) \omega &= \nabla^\perp \theta \cdot \nabla (\Delta \theta) \\
u &= \nabla^\perp \psi
\end{align*}
\]

and initial conditions \( \theta(x, 0) = \theta_0 \) and \( u(x, 0) = u_0 \). The \( \nabla^\perp \theta \) represents the magnetic field, \( \Delta \theta \) represents the current density and \( \omega = -\Delta \psi \) the vorticity.

**QG:**

\[
\begin{align*}
(\partial_t + u \cdot \nabla) \theta &= 0 \\
u &= \nabla^\perp \psi \quad \text{where} \quad \theta = -(\Delta)^{\frac{3}{2}} \psi
\end{align*}
\]

and initial condition \( \theta(x, 0) = \theta_0 \). The temperature is represented by \( \theta \).

**Boussinesq:**

\[
\begin{align*}
(\partial_t + u \cdot \nabla_x) \theta &= 0 \\
(\partial_t + u \cdot \nabla_x) \omega &= -\theta_{x_1} \\
u &= \nabla^\perp \psi
\end{align*}
\]

Again, \( \theta \) and \( u \) are specified at time \( t=0 \).

### 3 Criterion

A singularity can be formed by collision of two particle trajectories. A trajectory \( X(q,t) \) is obtained by solving the following ordinary differential equation

\[
\frac{dX(q,t)}{dt} = u(X(q,t), t)
\]

\[
X(q,0) = q
\]

Therefore,

\[
(X(q,t) - X(p,t))_t \leq |X(q,t) - X(p,t)| |\nabla u|_{L^\infty}
\]

3
\[ |X(q, t) - X(p, t)| \geq |X(q, 0) - X(p, 0)|e^{-\int_0^t |\nabla u|_{L^\infty} ds} \]

By this trivial argument, in order to have a collision the quantity \( \int_0^t |\nabla u|_{L^\infty} ds \) has to diverge.

A classic criterion for formation of singularities in fluid flows is the theorem of Beale-Kato-Majda (BKM); (see [1]), which improves the estimate described above, and deals with arbitrary singularities, not just collisions. Analogues of the BKM theorem for the above 2-dimensional equations include the following results.

For MHD, a singularity cannot develop at a finite time \( T \), unless we have

\[ \int_0^T \sup_x |\omega(x, t)| + \sup_x |\Delta_x \theta(x, t)|dt = \infty, \]

where \( \omega \) denotes the vorticity. (See Caflisch-Klapper-Steele [2].)

For QG, a singularity cannot develop at a finite time \( T \), unless we have

\[ \int_0^T \sup_x |\nabla_x \theta(x, t)|dt = \infty, \]

(See Constantin-Majda-Tabak [4]).

For Boussinesq, if a singularity develops at a finite time \( T \) then

\[ \int_0^T \sup_x |\omega(x, t)|dt = \infty \quad \text{and} \quad \int_0^T \int_0^t \sup_x |\nabla_x \theta(x, s)|dsdt = \infty. \]

(See E-Shu [8].)

See also Constantin-Majda-Tabak [4] and Constantin-Fefferman-Majda [3] for other conditions involving direction fields, that rule out formation of singularities in fluids.

In the case of 2D Euler, a singularity cannot develop at a finite time. From the BKM viewpoint this follows from the fact that \( \omega \) is advected by the fluid, and therefore \( \sup_x |\omega(x, t)| \) is independent of \( t \). (See BKM [1].)

Instead of looking at particle trajectories we look at level curves. Because the scalar function \( \theta \) is convected by the flow, that implies that the level curves are transported by the flow. A possible singular scenario is due to level curves approaching each other very fast which will lead to a fast growth of the gradient of the scalar function. In this paper we present a variant of the BKM criterion for sharp front formation. We provide a test for numerical simulations that appear to develop sharp fronts. The BKM Theorem shows
that the vorticity grows large if any singularity forms; our Theorem 1 shows that the velocity grows large if a sharp front forms.

The theorem we present in this paper was announced in [7].

4 Sharp Fronts

The scalar function \( \theta \) is convected by the flow, therefore the level curves move with the flow. A sharp front forms when two of these level curves collapse on a single curve. We define two level curves to be two distinct time-dependent arcs \( \Gamma_{\pm}(t) \) that move with the fluid and collapse at finite time into a single arc \( \Gamma \). More precisely, suppose the arcs are given by

\[
\Gamma_{\pm} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = f_{\pm}(x_1, t), x_1 \in [a, b]\} \quad \text{for} \quad 0 \leq t < T, \quad (2)
\]

with

\[
f_{\pm} \in C^1([a, b] \times [0, T)) \quad (3)
\]

and

\[
f_-(x_1, t) < f_+(x_1, t) \quad \text{for all} \quad x_1 \in [a, b], \quad t \in [0, T). \quad (4)
\]

We call the length \( b-a \) of the interval \([a, b]\) the length of the front. The assumption that \( \Gamma_{\pm}(t) \) move with the fluid means that

\[
u_2(x_1, x_2, t) = \frac{\partial f_{\pm}}{\partial x_1}(x_1, t) \cdot u_1(x_1, x_2, t) + \frac{\partial f_{\pm}}{\partial t}(x_1, t) \quad \text{at} \quad x_2 = f_{\pm}(x_1, t). \quad (5)
\]

This holds in particular for level curves of scalar functions \( g(x, t) \) that satisfy

\[
(\partial_t + u \cdot \nabla_x) g = 0.
\]

The collapse of \( \Gamma_{\pm}(t) \) into a single curve \( \Gamma \) at time \( T \) means here simply that

\[
\lim_{t \to T^-} (f_+(x_1, t) - f_-(x_1, t)) = 0 \quad \text{for all} \quad x_1 \in [a, b]. \quad (6)
\]

and \( f_+(x_1, t) - f_-(x_1, t) \) is bounded for all \( x_1 \in [a, b], t \in [0, T) \).

When (2), (3), (4), (5) and (6) hold, then we say that the fluid forms a **sharp front** at time \( T \).

The standard candidates for a singularity for MHD and QG are described by the definition given for a sharp front. We investigate the possible formation of a sharp front.
The following assumption will allow us to rule out formation of sharp fronts. We say that the fluid has **controlled velocity growth** if we have
\[ \int_0^T \sup \{ |u(x_1, x_2, t)| : x_1 \in [a, b], f_-(x_1, t) \leq x_2 \leq f_(x_1, t) \} dt < \infty. \] (7)
If (7) fails, then we say that the fluid has **uncontrolled velocity growth**.

**Lemma 1.** Let \( \theta \) be a smooth solution of Eq.1 defined for \( t \in [0, T) \). Assume there is a sharp front at time \( T \). Then

\[ \left( \frac{d}{dt} \right) \left( \int_a^b [f_+(x_1, t) - f_-(x_1, t)] dx_1 \right) = \psi(a, f_+(a, t), t) - \psi(a, f_-(a, t), t) \]
\[ + \psi(b, f_-(b, t), t) - \psi(b, f_+(b, t), t). \] (8)

**Proof:** Take the derivative of the stream function with respect to \( x_1 \) along an arc \( \Gamma_\pm(t) \)
\[ \frac{\partial \psi(x_1, f_\pm(x_1, t), t)}{\partial x_1} = u_2(x_1, f_\pm(x_1, t), t) - \frac{\partial f_\pm}{\partial x_1} u_1(x_1, f_\pm(x_1, t), t) \] (9)
by combining (9) and (10) we obtain
\[ \frac{\partial \psi(x_1, f_\pm(x_1, t), t)}{\partial x_1} = \frac{\partial f_\pm}{\partial t}(x_1, t) \] (10)
Expression (8) follows from integrating (10) with respect to \( x_1 \) between \( a \) and \( b \).

**Theorem 1.** Let \( u(x, t) \) be a divergence-free velocity field, with controlled velocity growth. Then a **sharp front** cannot develop at time \( T \).

**Proof:** Assume there is a sharp front at time \( T \). We define
\[ A(t) = \int_{\tilde{a}(t)}^{\tilde{b}(t)} [f_+(x_1, t) - f_-(x_1, t)] dx_1 \]
where
\[ \tilde{a}(t) = a + \int_t^T \sup \{ |u(x_1, x_2, s)| : x_1 \in [a, b], f_-(x_1, s) \leq x_2 \leq f_+(x_1, s) \} ds \]
\[
\tilde{b}(t) = b - \int_{t}^{T} \sup\{|u(x_1, x_2, s)| : x_1 \in [a, b], f_-(x_1, s) \leq x_2 \leq f_+(x_1, s)\} \, ds
\]

There is controlled velocity growth, therefore there exists \( t^* \in [0, T) \) such that \( \tilde{a}(t) \in [a, b] \) and \( \tilde{b}(t) \in [a, b] \) for all \( t \in [t^*, T) \).

We take the derivative of \( A(t) \) with respect to time

\[
\frac{dA(t)}{dt} = \sup|u| \cdot \delta(b, t) + \sup|u| \cdot \delta(a, t) + \int_{\tilde{a}(t)}^{\tilde{b}(t)} \frac{\partial}{\partial t} [f_+(x_1, t) - f_-(x_1, t)] \, dx_1.
\]

where \( \sup|u| = \sup\{|u(x_1, x_2, t)| : x_1 \in [a, b], f_-(x_1, t) \leq x_2 \leq f_+(x_1, t)\} \) and
\[
\delta(z, t) = f_+(z, t) - f_-(z, t).
\]

Using the definition of the stream function, the mean value theorem and \( \delta \), it is easy to check that \( \frac{dA(t)}{dt} > 0 \) for \( t > t^* \). This contradicts (6) by the dominated convergence theorem.

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**References**

[1] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3D Euler equations. *Comm. Math. Phys.*, 94:61–64, 1984.

[2] R.E. Caflisch, I. Klapper, G. Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 184:443-455, 1997.

[3] P. Constantin, C. Fefferman, and A. J. Majda. Geometric constraints on potentially singular solutions for the 3-D Euler equations. *Commun. Part. Diff. Eq.*, 21:559–571, 1996.

[4] P. Constantin, A. J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7:1495–1533, 1994.
[5] P. Constantin, Q. Nie and N. Schorghofer. Nonsingular surface-quasi-geostrophic flow *Phys. Lett. A*, 24:168-172.

[6] D. Cordoba. Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. *Ann. of Math.*, 148(3), 1998.

[7] D. Cordoba and C. Fefferman. Behavior of several 2D fluid equations in singular scenarios. *submitted to Proc. Natl. Acad. Sci. USA*

[8] D. Cordoba and C. Marliani. Evolution of current sheets and regularity of ideal incompressible magnetic fluids in 2D. *Comm. Pure Appl.Math*, 53(4):512-524, 2000.

[9] W. E and C-H. Shu. *Phys. Fluids*, 1:49-58.

[10] H. Friedel, R. Grauer, and C. Marliani. Adaptive mesh refinement for singular current sheets in incompressible magnetohydrodynamic flows. *J. Comput. Phys.*, 134:190–198, 1997.

[11] K. Ohkitani and M. Yamada. Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow. *Phys. Fluids*, 9:876-882.

[12] E.N. Parker.

[13] E. Priest and V.S. Titov. *Phil. Trans. R. Soc. Lond. A*, 351:1-37.

[14] A. Pumir and E.D. Siggia. *Phys. Fluids A*, 4:1472-1491.