On Jones’ subgroup of R. Thompson group $F$

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Abstract

Recently Vaughan Jones showed that the R. Thompson group $F$ encodes in a natural way all knots and links in $\mathbb{R}^3$, and a certain subgroup $\overrightarrow{F}$ of $F$ encodes all oriented knots and links. We answer several questions of Jones about $\overrightarrow{F}$. In particular we prove that the subgroup $\overrightarrow{F}$ is generated by $x_0x_1, x_1x_2, x_2x_3$ (where $x_i, i \in \mathbb{N}$ are the standard generators of $F$) and is isomorphic to $F_3$, the analog of $F$ where all slopes are powers of 3 and break points are 3-adic rationals. We also show that $\overrightarrow{F}$ coincides with its commensurator. Hence the linearization of the permutational representation of $F$ on $F/\overrightarrow{F}$ is irreducible. Finally we show how to replace 3 in the above results by an arbitrary $n$, and to construct a series of irreducible representations of $F$ defined in a similar way.

1 Introduction

A recent result of Vaughan Jones [7] shows that Thompson group $F$ encodes in a natural way all links. A subgroup of $F$, called by Jones the directed Thompson group $\overrightarrow{F}$, encodes all oriented links. In order to define $\overrightarrow{F}$, Jones associated with every element $D$ of $F$ a graph $T(D)$ using the description of elements of $F$ as pairs of binary trees (see Section 3 for details). The group $\overrightarrow{F}$ is the set of all elements in $F$ for which the associated graph $T(D)$ is bipartite. Jones asked for an abstract description of the subgroup $\overrightarrow{F}$. For example, it is not clear from the definition whether or not $\overrightarrow{F}$ is finitely generated.

We define the graph $T(D)$ in a different (but equivalent) way. By [4] $F$ is a diagram group. For every diagram $\Delta$ in $F$ the graph $T(\Delta)$ is a certain subgraph of $\Delta$. Then, the subgroup of $F$ composed of all reduced diagrams $\Delta$ in $F$ with $T(\Delta)$ bipartite is Jones’ subgroup $\overrightarrow{F}$. Using this definition we give several descriptions of $\overrightarrow{F}$. Recall that for every $n \geq 2$ one can define a “brother” $F_n$ of $F = F_2$ as the group of all piecewise linear increasing homeomorphisms of the unit interval where all slopes are powers of $n$ and all breaks of the derivative occur at $n$-adic fractions, i.e., points of the form $\frac{a}{n^k}$ where $a, k$ are positive integers [1]. It is well known that $F_n$ is finitely presented for every $n$ (a concrete and easy presentation can be found in [1]).

**Theorem 1.** Jones’ subgroup $\overrightarrow{F}$ is generated by elements $x_0x_1, x_1x_2, x_2x_3$ where $x_i, i \in \mathbb{N}$ are the standard generators of $F$. It is isomorphic to $F_3$ and coincides with the smallest subgroup of $F$ which contains $x_0x_1$ and is closed under addition (which is a natural binary operation on $F$, see Section 2.2).

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This theorem implies the following characterization of $\overrightarrow{F}$ which can be found in [7].

**Theorem 2.** Jones’ subgroup $\overrightarrow{F}$ is the stabilizer of the set of dyadic fractions from the unit interval $[0, 1]$ with odd sums of digits, under the standard action of $F$ on the interval $[0, 1]$.

As a corollary from Theorem 2 we get the following statement answering a question by Vaughan Jones.

**Corollary 3.** Jones’ subgroup $\overrightarrow{F}$ coincides with its commensurator in $F$.

As noted in [7], this implies that the linearization of the permutational representation of $F$ on $F/\overrightarrow{F}$ is irreducible (see [8]).

The paper is organized as follows. In Section 2 we give some preliminaries on Thompson group $F$. In Section 3 we give Jones’ definition of the graph associated with an element of $F$, we also give the definition in terms of diagrams and prove the equivalence of the two definitions. In Section 4 we define Jones subgroup $\overrightarrow{F}$ and prove Theorems 1 and 2 and Corollary 3. Section 5 contains generalizations of the previous results for arbitrary $n$. In particular, we show that for every $n \geq 2$, the smallest subgroup of $F$ containing the element $x_0 \cdots x_{n-2}$ and closed under addition, is isomorphic to $F_n$, can be characterized in terms of the graphs $T(\Delta)$, is the intersection of stabilizers of certain sets of binary fractions, and coincides with its commensurator in $F$. Although Theorems 1 and 2 are special cases of the results of Section 5, the direct proofs of these results are much less technical while ideologically similar, and the original questions of Jones concerned the case $n = 3$ only. Thus we decided to keep the proofs of Theorems 1 and 2. In Section 5 we show how to adapt these proofs to the general case.

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## 2 Preliminaries on $F$

### 2.1 $F$ as a group of homeomorphisms

The most well known definition of the R. Thompson group $F$ is this (see [3]): $F$ consists of all piecewise-linear increasing self-homeomorphisms of the unit interval with all slopes powers of 2 and all break points of the derivative dyadic fractions. The group $F$ is generated by two functions $x_0$ and $x_1$ defined as follows.

$$x_0(t) = \begin{cases} 2t & : 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & : \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & : \frac{1}{2} \leq t \leq 1 \end{cases} \quad x_1(t) = \begin{cases} t & : 0 \leq t \leq \frac{1}{2} \\ 2t - \frac{1}{2} & : \frac{1}{2} \leq t \leq \frac{3}{4} \\ t + \frac{1}{4} & : \frac{3}{4} \leq t \leq \frac{5}{4} \\ \frac{t}{2} + \frac{1}{2} & : \frac{5}{4} \leq t \leq 1 \end{cases}$$

One can see that $x_1$ is the identity on $[0, \frac{1}{2}]$ and a shrank by a factor of 2 copy of $x_0$ on $[\frac{1}{2}, 1]$. The composition in $F$ is from left to right.

Equivalently, the group $F$ can be defined using dyadic subdivisions [1]. We call a subdivision of $[0, 1]$ a **dyadic subdivision** if it is obtained by repeatedly cutting intervals in half. If $S_1, S_2$ are dyadic subdivisions with the same number of pieces, they define a piecewise linear map taking each segment of the subdivision $S_1$ linearly to the corresponding segment of $S_2$. We call such a map a **dyadic rearrangement**. The group $F$ consists of all dyadic rearrangements.
2.2 \textit{F} as a diagram group

It was shown in \cite[Example 6.4]{4} that the R. Thompson group \(F\) is a diagram group over the semigroup presentation \(\langle x \mid x = x^2 \rangle\).

Let us recall the definition of a \textit{diagram group} (see \cite{5} for more formal definitions). A (semigroup) \textit{diagram} is a planar directed labeled graph tessellated by cells, defined up to an isotopy of the plane. Each diagram \(\Delta\) has the top path \(\text{top}(\Delta)\), the bottom path \(\text{bot}(\Delta)\), the initial and terminal vertices \(i(\Delta)\) and \(\tau(\Delta)\). These are common vertices of \(\text{top}(\Delta)\) and \(\text{bot}(\Delta)\). The whole diagram is situated between the top and the bottom paths, and every edge of \(\Delta\) belongs to a (directed) path in \(\Delta\) between \(i(\Delta)\) and \(\tau(\Delta)\). More formally, let \(X\) be an alphabet. For every \(x \in X\) we define the \textit{trivial diagram} \(\varepsilon(x)\) which is just an edge labeled by \(x\). The top and bottom paths of \(\varepsilon(x)\) are equal to \(\varepsilon(x)\), the vertices \(i(\varepsilon(x))\) and \(\tau(\varepsilon(x))\) are the initial and terminal vertices of the edge. If \(u\) and \(v\) are words in \(X\), a \textit{cell} \((u \to v)\) is a planar graph consisting of two directed labeled paths, the top path labeled by \(u\) and the bottom path labeled by \(v\), connecting the same points \(i(u \to v)\) and \(\tau(u \to v)\). There are three operations that can be applied to diagrams in order to obtain new diagrams.

1. \textbf{Addition}. Given two diagrams \(\Delta_1\) and \(\Delta_2\), one can identify \(\tau(\Delta_1)\) with \(i(\Delta_2)\). The resulting planar graph is again a diagram denoted by \(\Delta_1 + \Delta_2\), whose top (bottom) path is the concatenation of the top (bottom) paths of \(\Delta_1\) and \(\Delta_2\). If \(u = x_1x_2\ldots x_n\) is a word in \(X\), then we denote \(\varepsilon(x_1) + \varepsilon(x_2) + \cdots + \varepsilon(x_n)\) (i.e. a simple path labeled by \(u\)) by \(\varepsilon(u)\) and call this diagram also \textit{trivial}.

2. \textbf{Multiplication}. If the label of the bottom path of \(\Delta_1\) coincides with the label of the top path of \(\Delta_2\), then we can \textit{multiply} \(\Delta_1\) and \(\Delta_2\), identifying \(\text{bot}(\Delta_1)\) with \(\text{top}(\Delta_2)\). The new diagram is denoted by \(\Delta_1 \circ \Delta_2\). The vertices \(i(\Delta_1 \circ \Delta_2)\) and \(\tau(\Delta_1 \circ \Delta_2)\) coincide with the corresponding vertices of \(\Delta_1, \Delta_2\), \(\text{top}(\Delta_1 \circ \Delta_2) = \text{top}(\Delta_1), \text{bot}(\Delta_1 \circ \Delta_2) = \text{bot}(\Delta_2)\).

3. \textbf{Inversion}. Given a diagram \(\Delta\), we can flip it about a horizontal line obtaining a new diagram \(\Delta^{-1}\) whose top (bottom) path coincides with the bottom (top) path of \(\Delta\).

![Figure 2.1: The multiplication and addition of diagrams.](image)

\textbf{Definition 2.1.} A diagram over a collection of cells (i.e., a semigroup presentation) \(P\) is any planar graph obtained from the trivial diagrams and cells of \(P\) by the operations of addition, multiplication and inversion. If the top path of a diagram \(\Delta\) is labeled by a word \(u\) and the bottom path is labeled by a word \(v\), then we call \(\Delta\) a \((u,v)\)-diagram over \(P\).

Two cells in a diagram form a \textit{dipole} if the bottom part of the first cell coincides with the top part of the second cell, and the cells are inverses of each other. Thus a dipole is a
subdiagram of the form $\pi \circ \pi^{-1}$ where $\pi$ is a cell. In this case, we can obtain a new diagram by removing the two cells and replacing them by the top path of the first cell. This operation is called elimination of dipoles. The new diagram is called equivalent to the initial one. A diagram is called reduced if it does not contain dipoles. It is proved in [4, Theorem 3.17] that every diagram is equivalent to a unique reduced diagram.

If the top and the bottom paths of a diagram are labeled by the same word $u$, we call it a spherical $(u,u)$-diagram. Now let $\mathcal{P} = \{c_1, c_2, \ldots\}$ be a collection of cells. The diagram group $DG(\mathcal{P}, u)$ corresponding to the collection of cells $\mathcal{P}$ and a word $u$ consists of all reduced spherical $(u,u)$-diagrams obtained from the cells of $\mathcal{P}$ and trivial diagrams by using the three operations mentioned above. The product $\Delta_1 \Delta_2$ of two diagrams $\Delta_1$ and $\Delta_2$ is the reduced diagram obtained by removing all dipoles from $\Delta_1 \circ \Delta_2$. The fact that $DG(\mathcal{P}, u)$ is a group is proved in [4].

Lemma 2.2 (See [4]). If $X$ consists of one letter $x$ and $\mathcal{P}$ consists of one cell $x \to x^2$, then the group $DG(\mathcal{P}, x)$ is the R. Thompson group $F$.

Since $X$ consists of one letter $x$, we shall omit the labels of the edges of diagrams in $F$. Since all edges are oriented from left to right, we will not indicate the orientation of edges in the pictures of diagrams from $F$.

Thus in the case of the group $F$, the set of cells $\mathcal{P}$ consists of one cell $\pi$ of the form

![Diagram showing the cell defining the group F](image)

Figure 2.2: The cell defining the group $F$.

The role of 1 in the group $F$ is played by the trivial diagram $\varepsilon(x)$ which will be denote by 1.

Using the addition of diagrams one can define a useful operation of addition on the group $F$: $\Delta_1 \oplus \Delta_2 = \pi \circ (\Delta_1 + \Delta_2) \circ \pi^{-1}$. Note that if $\Delta_1, \Delta_2$ are reduced, then so is $\Delta_1 \oplus \Delta_2$ unless $\Delta_1 = \Delta_2 = 1$ in which case $1 \oplus 1 = 1$.

The following property of $\oplus$ is obvious:

Lemma 2.3. For every $a, b, c, d \in F$ we have

\[(a \oplus b)(c \oplus d) = ac \oplus bd.\] (2.1)

In particular $a \oplus b = (a \oplus 1)(1 \oplus b)$.

Remark 2.4. Note that the sum $\oplus$ is not associative but “almost” associative, that is, there exists an element $g \in F$ such that for every $a, b, c \in F$, we have

\[((a \oplus b) \oplus c)^g = a \oplus (b \oplus c).\]

Figure 2.4 below shows that in fact $g = x_0$.

Thus $F$ can be considered as an algebra with two binary operations: multiplication and addition. It is a group under multiplication and satisfies the identity (2.1). Algebras with two binary operations satisfying these conditions form a variety, which we shall call the variety of Thompson algebras. Note that every group can be turned into a Thompson algebra in a trivial way by setting $a \oplus b = 1$ for every $a, b$. Our main result shows, in particular, that $F_3$ has a non-trivial structure as a Thompson algebra.
The Thompson group $F$ has an obvious involutary automorphism that flips a diagram about a vertical line (it is also an anti-automorphism with respect to addition). Thus every statement about $F$ has its left-right dual.

Note that for every $n \geq 2$, the group $F_n$ is a diagram group over the semigroup presentation $\langle x \mid x = x^n \rangle$ [8]. This was used in [8] to find a nice presentation of $F_n$ for every $n$. We will use this presentation below.

2.3 A normal form of elements of $F$

Let $x_0, x_1$ be the standard generators of $F$. Recall that $x_{i+1}, i \geq 1$, denotes $x_0^{-1}x_1x_0$. In these generators, the group $F$ has the following presentation $\langle x, i \geq 0 \mid x_i^{x_{j}} = x_{i+1} \text{ for every } j < i \rangle$ [3].

There exists a clear connection between representation of elements of $F$ by diagrams and the normal form of elements in $F$. Recall [3] that every element in $F$ is uniquely representable in the following form:

$$x_{i_1}^{s_1} \ldots x_{i_m}^{s_m} x_{j_1}^{-t_1} \ldots x_{j_n}^{-t_n},$$  \hspace{1cm} (2.2)

where $i_1 \leq \cdots \leq i_m \neq j_n \geq \cdots \geq j_1; s_1, \ldots, s_m, t_1, \ldots, t_n \geq 0$, and if $x_i$ and $x_i^{-1}$ occur in (2.2) for some $i \geq 0$ then either $x_{i+1}$ or $x_{i+1}^{-1}$ also occurs in (2.2). This form is called the normal form of elements in $F$.

We say that a path in a diagram is positive if all the edges in the path are oriented from left to right. Let $\mathcal{P}$ be the collection of cells which consists of one cell $x \to x^2$. It was noticed in [3] that every reduced diagram $\Delta$ over $\mathcal{P}$ can be divided by its longest positive path from its initial vertex to its terminal vertex into two parts, positive and negative, denoted by $\Delta^+$ and $\Delta^−$, respectively. So $\Delta = \Delta^+ \circ \Delta^−$. It is easy to prove by induction on the number of cells that $\Delta^+$ are $(x, x^2)$-cells and all cells in $\Delta^−$ are $(x^2, x)$-cells.

Let us show how given an $(x, x)$-diagram over $\mathcal{P}$ one can get the normal form of the element represented by this diagram. This is the left-right dual of the procedure described in [3] Example 2 and after Theorem 5.6.41 in [9].

**Lemma 2.5.** Let us number the cells of $\Delta^+$ by numbers from 1 to $k$ by taking every time the “leftmost” cell, that is, the cell which is to the left of any other cell attached to the bottom path of the diagram formed by the previous cells. The first cell is attached to the top path of $\Delta^+$ (which is the top path of $\Delta$). The $i$th cell in this sequence of cells corresponds to an edge of the Squier graph $\Gamma(\mathcal{P})$, which has the form $(x^{r_i}, x \to x^{r_i})$, where $r_i (r_j)$ is the length of the path from the initial (resp. terminal) vertex of the diagram (resp. the cell) to the initial (resp. terminal) vertex of the cell (resp. the diagram), such that the path is contained in the bottom path of the diagram formed by the first $i−1$ cells. If $r_i = 0$ then we label this cell by 1. If $r_i \neq 0$ then we label this cell by the element $x_{r_i}$ of $F$. Multiplying the labels of all cells, we get the “positive” part of the normal form. In order to find the “negative” part of the normal form, consider $(\Delta^−)^{−1}$, number its cells as above and label them as above. The normal form of $\Delta$ is then the product of the normal form of $\Delta^+$ and the inverse of the normal form of $(\Delta^−)^{−1}$.

For example, applying the procedure from Lemma 2.5 to the diagram on Figure 2.3 we get the normal form $x_0x_1^3x_4(x_0^{-1}x_1x_2^2x_3)^{-1}$.

Diagrams for the generators of $F$, $x_0, x_1$ are on Figure 2.4.

**Lemma 2.5** immediately implies

**Lemma 2.6.** If $u$ is the normal form of $\Delta$, then the normal form of $1 \oplus \Delta$ is obtained from $u$ by increasing every index by 1.

In particular, $x_1 = 1 \oplus x_0$, and, in general, $x_{i+1} = 1 \oplus x_i$, $i \geq 0$. Thus we get

**Proposition 2.7.** As a Thompson algebra, $F$ is generated by one element $x_0$.  

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2.4 From diagrams to homeomorphisms

There is a natural isomorphism between Thompson group $F$ defined as a diagram group and $F$ defined as a group of homeomorphisms. Let $\Delta$ be a diagram in $F$. The positive subdiagram $\Delta^+$ describes a binary subdivision of $[0, 1]$ in the following way. Every edge of $\Delta^+$ corresponds to a dyadic sub-interval of $[0, 1]$; that is, an interval of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ for integers $n \geq 0$ and $k = 0, \ldots, 2^n - 1$. The top edge $\text{top}(\Delta)$ corresponds to the interval $[0, 1]$.

For each cell $\pi$ of $\Delta^+$ (hence an $(x, x^2)$-cell), if $\text{top}(\pi)$ corresponds to an interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, then the left bottom edge of $\pi$ corresponds to the left half of the interval, $[\frac{k}{2^n}, \frac{k+1}{2^n} + \frac{1}{2^n}]$, and the right bottom edge of $\pi$ corresponds to the right half of the interval, $[\frac{k}{2^n} + \frac{1}{2^n}, \frac{k+1}{2^n}]$.

Thus, if the bottom path of $\Delta^+$ consists of $n$ edges, $\Delta^+$ describes a binary subdivision composed of $n$ intervals. Similarly, the negative subdiagram $\Delta^-$ describes a binary subdivision with $n$ intervals as well. The diagram $\Delta$ corresponds to the dyadic rearrangement mapping the top subdivision (associated with $\Delta^+$) to the bottom subdivision (associated with $\Delta^-$).

Thinking of $\Delta^+$ as a subdivision of $[0, 1]$, the inner vertices of $\Delta$ (that is, the vertices other than $\iota(\Delta)$ and $\tau(\Delta)$) correspond to the break points of the subdivision. Thus, in the diagram $\Delta$ every inner vertex is associated with two break points; one break point of the top subdivision and one break point of the bottom subdivision.
2.5 From diagrams to pairs of binary trees

Thompson group \( F \) can be defined in terms of reduced pairs of binary trees. Let \( \Delta \) be a diagram in \( F \) with \( n+1 \) vertices. It is possible to put a vertex in the middle of every edge of the diagram. Then, for each cell \( \pi \) in \( \Delta^+ \) (hence an \((x,x^2)\)-cell) one can draw edges from the vertex on top(\( \pi \)) to the vertices on the bottom edges of \( \pi \). We get a binary tree \( T_+ \) with the root lying on top(\( \Delta \)) and \( n \) leaves lying on bot(\( \Delta^+ \)) = top(\( \Delta^- \)). A similar construction in \( \Delta^- \) gives a second binary tree \( T_- \), lying upside down, such that the leaves of \( T_+ \) and \( T_- \) coincide. Whenever we speak of a pair of binary trees \((T_+,T_-)\) we assume that they have the same number of leaves, and that \( T_- \) is drawn upside down so that the leaves of \( T_+ \) and \( T_- \) coincide.

Let \( T \) be a binary tree. We call a vertex with two children a caret in the tree. We say that a pair of binary trees \((T_+,T_-)\) have a common caret if for some \( i \), the \( i \) and \( i+1 \) leaves have a common father both in \( T_+ \) and in \( T_- \). We call a pair of trees \((T_+,T_-)\) reduced if it has no common carets. The construction above maps every reduced diagram \( \Delta \) to a reduced pair of binary trees. The correspondence is one to one and enables to view Thompson group \( F \) as a group of reduced pairs of binary trees with the proper multiplication.

3 The Thompson graphs

3.1 The definition in terms of pairs of trees

Jones \[7\] defined for each element of \( F \), viewed as a reduced pair of binary trees, an associated graph, which he called the Thompson graph of that element. If \((T_+,T_-)\) is a reduced pair of binary trees, we call an edge \( t \) in \( T_+ \) or \( T_- \) a left edge if it connects a vertex to its left child.

**Definition 3.1** (Jones \[7\]). Let \( D \) be an element of \( F \) and \((T_+,T_-)\) the corresponding reduced pair of binary trees. If \( T_+,T_- \) have \( n \) common leaves, enumerated \( l_1, \ldots, l_n \) from left to right, we can assume that all of them lie on the same horizontal line. We define a graph \( J(D) \) as follows. The graph \( J(D) \) has \( n \) vertices \( v_0, \ldots, v_{n-1} \). The vertex \( v_0 \) lies to the left of the first leaf \( l_1 \) and for all \( i = 1, \ldots, n-1 \), the vertex \( v_i \) lies between \( l_i \) and \( l_{i+1} \) on the horizontal line. The edges of \( J(D) \) are defined as follows. For every left edge \( t \) of \((T_+,T_-)\), we draw a single edge \( e \) in \( J(D) \) which crosses the edge \( t \) and no other edge of \((T_+,T_-)\). Note that this property determines the end vertices of the edge \( e \).

For example, the graph \( J(x_1) \), associated with \( x_1 \), is depicted in Figure 3.5

3.2 The definition in terms of diagrams

**Definition 3.2.** Let \( \Delta \) be a (not necessarily reduced) diagram over the presentation \( \mathcal{P} = \langle x \mid x = x^2 \rangle \). The Thompson graph \( T(\Delta) \) is a “subgraph” of the diagram \( \Delta \), defined as follows. The vertex set of \( T(\Delta) \) is the vertex set of \( \Delta \) minus the terminal vertex \( \tau(\Delta) \). For every inner vertex \( v \) of \( \Delta \) the only incoming edges of \( v \) which belong to \( T(\Delta) \) are the top-most and bottom-most incoming edges of \( v \) in \( \Delta \). If the top-most and bottom-most incoming edges of \( v \) coincide we shall consider them as two distinct edges (hence the quotation marks on the word subgraph). An edge of \( T(\Delta) \) will be called an upper (lower) edge if it is the top-most (bottom-most) incoming edge of an inner vertex in \( \Delta \).

The subgraph \( T(x_0,x_1) \) of the diagram \( x_0x_1 \) of Thompson group \( F \) is depicted in Figure 3.6. The upper edges in the graph are colored red, while the lower edges are colored blue. Note that the graph is bipartite.

**Remark 3.3.** Let \( \Delta \) be a reduced diagram in \( F \). Consider the positive subdiagram \( \Delta^+ \). Every cell \( \pi \) in \( \Delta^+ \) is an \((x,x^2)\)-cell. As such, it has a unique bottom vertex separating the
left bottom edge of $\pi$ from its right bottom edge (see Figure 2.2). Conversely, every inner vertex $v$ of $\Delta$ is the bottom vertex of a unique cell $\pi_v$ in $\Delta^+$. Clearly, the top-most incoming edge of $v$ in $\Delta$ is the left bottom edge of the cell $\pi_v$. Thus, the upper edges in the graph $T(\Delta)$ are exactly the left bottom edges of the cells in $\Delta^+$. Similarly, the lower edges of the graph $T(\Delta)$ are exactly the left top edges of the cells in $\Delta^-$.  

3.3 The equivalence of the two definitions

**Proposition 3.4.** Let $\Delta$ be a reduced diagram in $F$. Then, the associated graphs $T(\Delta)$ (from Definition 3.2) and $J(\Delta)$ (from Definition 3.1) are isomorphic.

**Proof.** Let $(T_+, T_-)$ be the reduced pair of binary trees associated with $\Delta$. We can assume that the pair $(T_+, T_-)$ is drawn inside the diagram $\Delta$ as described in Section 2.5. Let $T(\Delta)$ be the subgraph associated with $\Delta$. It is possible to stretch each of the upper edges of $T(\Delta)$ up as follows. By Remark 3.3, $e$ is an upper edge in $T(\Delta)$, if and only if $e$ is a left bottom edge of some cell in $\Delta^+$. Let $v$ be the vertex of $T_+$ which lies on $e$ and $t$ the edge of $T_+$ connecting $v$ to its father. Clearly, $t$ is a left edge in the tree $T_+$. We stretch the edge $e$ slightly so that instead of crossing the vertex $v$ it crosses the edge $t$ of $T_+$ (and no other edges of the tree). Similarly, we stretch every bottom edge of $T(\Delta)$ down so that instead of crossing a vertex of the tree $T_-$, it crosses the edge of the tree connecting the vertex to its father. The process is illustrated in Figure 3.7.

If the graph $T'(\Delta)$ results from stretching the edges of $T(\Delta)$ as described, then there is a one to one correspondence between the left edges of the pair of trees $(T_+, T_-)$ and the edges of $T'(\Delta)$. Indeed, every edge of $T'(\Delta)$ crosses a single left edge of $(T_+, T_-)$ and every left edge of $(T_+, T_-)$ is crossed by an edge of $T'(\Delta)$. Note, that if $(T_+, T_-)$ have $n$ common leaves, then $T(\Delta)$ (hence $T'(\Delta)$) has $n$ vertices; one to the left of the left most leaf of $T_+$ and one between any pair of consecutive leaves of $T_+$. It follows that the graph $T'(\Delta)$ is
Figure 3.6: The graph $T(x_0x_1)$.

Figure 3.7: Stretching the edges of $T(\Delta)$.

isomorphic to the graph $J(\Delta)$. Since $T(\Delta)$ and $T'(\Delta)$ are clearly isomorphic as graphs we get the result.

4 The Jones’ subgroup $\overrightarrow{F}$ and its properties

4.1 The definition of $\overrightarrow{F}$

Lemma 4.1. Suppose that a diagram $\Delta$ is obtained from a diagram $\Delta'$ by removing a dipole. Suppose that $T(\Delta')$ is bipartite. Then $T(\Delta)$ is bipartite.

Proof. The dipole can be of type $\pi \circ \pi^{-1}$ or of type $\pi^{-1} \circ \pi$ where $\pi$ is the cell on Figure 2.2. In the first case to get the graph $T(\Delta)$ we remove from $T(\Delta')$ a vertex with exactly two edges connecting it to another vertex of $T(\Delta')$, and the statement is obvious. In the second case, since $T(\Delta')$ is bipartite, we can label the vertices of $\Delta'$ by "+" and "-", so that every two incident vertices have opposite signs. Consider the four vertices of the dipole. In $T(\Delta')$...
the top and the bottom vertices of the dipole are incident to the left vertex. Hence the top and the bottom vertices have the same label. Note that the edge of the dipole connecting the left vertex with the right vertex is not an edge of \( T(\Delta') \). Thus, the effect of removing the dipole on \( T(\Delta') \) amounts to identifying the top and the bottom vertices of the dipole and erasing the lower edge connecting the left vertex with the top vertex and the upper edge connecting the left vertex with the bottom one. Since the top and bottom vertices have the same label, the Thompson graph \( T(\Delta) \) is bipartite.

**Definition 4.2.** Jones’ subgroup \( \overrightarrow{F} \) is the set of all reduced diagrams \( \Delta \) in \( F \) for which the associated graph \( T(\Delta) \) is bipartite.

**Proposition 4.3.** Jones’ subgroup \( \overrightarrow{F} \) is indeed a subgroup of \( F \).

**Proof.** Suppose that \( \Delta_1, \Delta_2 \) belong to \( \overrightarrow{F} \). The Thompson graph \( T(\Delta_1 \circ \Delta_2) \) is the union of \( T(\Delta_1) \) and \( T(\Delta_2) \) with vertices \( \iota(\Delta_1) \) and \( \iota(\Delta_2) \) identified. Hence \( T(\Delta_1 \circ \Delta_2) \) is bipartite. By Lemma 4.4 the graph \( T(\Delta_1 \Delta_2) \) is bipartite as well.

**4.2 The subgroup \( \overrightarrow{F} \) is isomorphic to \( F_3 \)**

**Lemma 4.4.** The Jones’ subgroup \( \overrightarrow{F} \) coincides with the subgroup \( H \) which is the smallest subgroup of \( F \) that contains \( x_0 x_1 \) and closed under addition.

**Proof.** Clearly \( H \) is inside \( \overrightarrow{F} \). Also from Lemma 2.5 it follows that if we add the trivial diagram \( 1 \) on the right to the reduced diagram representing \( x_0 x_1 \), we get the diagram corresponding to the normal form \( x_0 x_0 x_1 (x_0 x_1 x_2)^{-1} = x_0 x_0 x_1 x_2^{-1} x_0 x_0 \). Hence \( x_0 x_0 x_0 x_0 x_1 x_2^{-1} \) also belongs to \( H \). If we add to this element the diagram \( 1 \) on the right, we get \( (x_0 x_0 x_0 x_0 x_0 x_0 x_0 x_0 x_1 x_2)^{-1} \), and so the element \( x_0 x_0 x_0 x_1 x_2^{-1} \) belongs to \( H \). By induction, we see that all elements \( x_0 x_i x_1 x_2^{-n+1} \) belong to \( H \). Now consider an arbitrary reduced diagram \( \Delta \) in \( \overrightarrow{F} \). Let us enumerate the vertices of \( \Delta \) from left to right: \( 0, 1, \ldots, s \) so that \( \iota(\Delta) = 0, \tau(\Delta) = s \).

If the Thompson graph \( T(\Delta) \) does not contain top nor bottom edges connecting \( j > 1 \) with \( 0 \), then \( \Delta \) is a sum of \( \Delta' \) and the trivial diagram \( 1 \) (added on the left). The diagram \( \Delta' \) also belongs to \( \overrightarrow{F} \) (its graph \( T(\Delta') \) is bipartite), so by induction \( \Delta' \) is in \( H \), and \( \Delta \) is in \( H \) also. So assume that \( T(\Delta) \) contains an edge \((0, j), j > 1 \). Without loss of generality we can assume that this is an upper edge, that is, it belongs to the positive part of the diagram. Therefore the positive part of the normal form for \( \Delta \) in the generators \( x_0, x_1, x_2, \ldots \) starts with \( x_0 \). Let it start with \( x_0^a x_i, i > 0 \). The bottom-most cell corresponding to the prefix \( x_0^a \) cannot have both its bottom edges on the maximal positive path of \( \Delta \), i.e. its top edge cannot connect 0 with 2. Indeed if it connects 0 with 2, then 2 and 0 are in different parts of the bipartite graph \( T(\Delta) \). The vertex 1 then belongs to the same part as 2. Then 2 cannot be connected with 1 by a lower edge from \( T(\Delta) \), so it has to be connected with 0, which means that the diagram \( \Delta \) is not reduced, a contradiction. Therefore there is an edge connecting 1 with \( j' \leq j \). Hence the normal form corresponding to \( \Delta \) starts with \( x_0^a x_1 \). If \( n = 1 \), then we can divide by \( x_0 x_1 \) (on the left) and get an element from \( \overrightarrow{F} \) with a shorter normal form. Hence \( \Delta \) is in \( H \) since \( x_0 x_1 \) is in \( H \). If \( n > 1 \), then we can replace \( x_0^a x_1 \) by \( x_2^{-n+1} \), and the resulting element would still be in \( \overrightarrow{F} \). Since its normal form is shorter than that of \( \Delta \), it is in \( H \), and so \( \Delta \) is in \( H \).

The proof of Lemma 4.4 proves the following.

**Lemma 4.5.** The subgroup \( \overrightarrow{F} \) is generated by two sets \( X = \{x_i x_{i+1}, i \geq 0 \} \) and \( X' = \{x_i^{-n+1} x_{i+1} x_{i+2}, i \geq 0, n \geq 1 \} \).
Proof. Indeed, in the proof of Lemma 4.4 we proved that \(X\) and \(X'\) are inside \(\vec{F}\), and every element of \(\vec{F}\) is either a sum of the trivial diagram \(1\) and some other diagram from \(\vec{F}\) or is a product of an element from \(X \cup X'\) and a reduced diagram from \(\vec{F}\) with a shorter normal form or is the inverse of such an element. Since by Lemma 2.6 the subgroup generated by \(X \cup X'\) is closed under addition of the trivial diagram \(1\) on the left, the lemma is proved. \(\square\)

Lemma 4.6. The subgroup \(\vec{F}\) is generated by three elements \(x_0x_1, x_1x_2, x_2x_3\).

Proof. Let \(X\) and \(X'\) be the sets from Lemma 4.4. It is obvious that for every \(j \geq 0\) the element \(x_jx_{j+1}\) is equal to \((x_jx_{j+2})x_{j+1}\). Thus the set \(X = \{x_jx_{j+1}, j \geq 0\}\) is contained in the subgroup \(\langle x_0x_1, x_1x_2, x_2x_3 \rangle\). It remains to show that \(X' \subseteq (X)\). Note that

\[
x_0^n x_1 x_2^{-1} = x_0 x_1 x_2^{-1} x_0 x_1 x_2^{-1} = x_0 x_1 x_0 x_2^{-1} x_1 x_2^{-1} = x_0 x_1 x_0 x_1 x_3^{-1} x_2^{-1} = (x_0 x_1)^2 (x_2 x_3)^{-1}.
\]

Similarly,

\[
x_0^{n+1} x_1 x_2^{-n} = (x_0 x_1)^{n+1} (x_2 x_3)^{-n}
\]

for every \(n \geq 1\). Adding several trivial diagrams \(1\) on the left, we get

\[
x_j^{n+1} x_{j+1} x_{j+2}^{-n} = (x_j x_{j+1})^{n+1} (x_{j+2} x_{j+3})^{-n}
\]

for every \(j \geq 0\). \(\square\)

Lemma 4.7. The subgroup \(\vec{F}\) is isomorphic to \(F_3\).

Proof. The elements \(x_0x_1, x_1x_2, x_2x_3\) satisfy the defining relations of \(F_3\) (see [4, page 54]). All proper homomorphic images of \(F_3\) are Abelian [1] Theorem 4.13. Since \(x_0x_1, x_1x_2\) do not commute, the natural homomorphism from \(F_3\) onto \(\vec{F}\) is an isomorphism. \(\square\)

As an immediate corollary of Theorem 1 we get

Proposition 4.8 (Compare with Proposition 2.7). \(\vec{F}\) is a subalgebra of the Thompson algebra \(F\) generated by one element \(x_0x_1\).

4.3 \(\vec{F}\) is the stabilizer of the set of dyadic fractions with odd sums of digits

Let \(\Delta\) be a reduced diagram in \(F\) and \(T(\Delta)\) the associated graph. Let \(\Delta^+\) be the positive subdiagram. Recall (Section 2.4) that \(\Delta^+\) describes a binary subdivision of \([0, 1]\). Every edge \(e\) in \(\Delta^+\) corresponds to a dyadic interval of length \(\frac{1}{2^m}\) for some integer \(m \geq 0\). We will call this length the weight of the edge \(e\) and denote it by \(\omega(e)\). The inner vertices of \(\Delta^+\) correspond to the break points of the subdivision. Note that if \(v\) is an inner vertex of \(\Delta^+\), the dyadic fraction \(f\) is the corresponding break point of the subdivision and \(p\) is a positive path in \(\Delta^+\) from \(v(\Delta)\) to \(v\), then the weight of the path \(p\) (that is, the sum of weights of its edges) is equal to the fraction \(f\).

Lemma 4.9. Let \(v\) be an inner vertex of \(\Delta\). Let \(e\) be the (unique) upper incoming edge of \(v\) in \(T(\Delta)\) and \(e_1\) be any outgoing upper edge of \(v\) in \(T(\Delta)\), then \(\omega(e_1) < \omega(e)\).

Proof. By Remark 3.3 \(e\) is the left bottom edge of some cell \(\pi\) in \(\Delta^+\). Let \(e'\) be the right bottom edge of the cell \(\pi\). Clearly, the edge \(e'\) is the top-most outgoing edge of \(v\) in \(\Delta\). Since \(e'\) is not the left bottom edge of any cell in \(\Delta^+\), by Remark 3.3 the edge \(e'\) does not belong to \(T(\Delta)\). Hence, the edge \(e_1 \neq e'\) so that \(e_1\) lies beneath \(e'\) in \(\Delta^+\). From the construction in Section 2.4 it is obvious that \(\omega(e_1) \leq \frac{1}{2} \omega(e') = \frac{1}{2} \omega(e) < \omega(e)\). \(\square\)
Definition 4.10. Let $\Delta$ be a (not necessarily reduced) diagram. Let $v$ be an inner vertex of $\Delta$. Since every inner vertex of $\Delta$ has a unique incoming upper edge in $T(\Delta)$, there is a unique positive path in $T(\Delta)$ from $\iota(\Delta)$ to $v$, composed entirely of upper edges. We call this path the top path from $\iota(\Delta)$ to $v$.

Lemma 4.11. Let $\Delta$ be a reduced diagram in $F$. Let $v$ be an inner vertex of $\Delta$ and $f$ the corresponding break point of the subdivision associated with $\Delta^+$. Let $p$ be the top path from $\iota(\Delta)$ to $v$. Then, the length of $p$ is equal to the sum of digits in the binary form of $f$.

Proof. Let $e_1, \ldots, e_k$ be the edges of $p$. For each $i = 1, \ldots, k$, the weight $\omega(e_i) = \frac{1}{2^{m_i}}$ for some positive integer $m_i$. By Lemma 4.9, $\omega(e_1) > \cdots > \omega(e_k)$. Thus, the weight of $p$ is $\omega(p) = \sum_{i=1}^{k} \frac{1}{2^{m_i}}$ where $\frac{1}{2^{m_1}} > \cdots > \frac{1}{2^{m_k}}$. Clearly, the sum of digits of $\omega(p)$ in binary form is $k$. Therefore, the sum of digits of $f = \omega(p)$ is equal to the number of edges in $p$. \qed

Viewed as a dyadic rearrangement, a reduced diagram $\Delta$ takes the break points of the top subdivision (associated with $\Delta^+$) to the break points of the bottom subdivision (associated with $\Delta^-$). Reflecting the diagram $\Delta$ about a horizontal line shows that the analogue of Lemma 4.11 holds for the bottom subdivision: If $v$ is an inner vertex of $\Delta$ and $p$ is the bottom path from $\iota(\Delta)$ to $v$, composed entirely of lower edges of $T(\Delta)$, then the length of $p$ is equal to the sum of digits of the break point corresponding to $v$ in the bottom subdivision.

Corollary 4.12. Let $S$ be the set of dyadic fractions with odd sums of digits. Let $\Delta$ be a reduced diagram which stabilizes $S$ and $v$ be an inner vertex of $\Delta$. Then, the length of the top path from $\iota(\Delta)$ to $v$ and the length of the bottom path from $\iota(\Delta)$ to $v$ have the same parity.

Proof. Let $f^+, f^-$ be the break points corresponding to $v$ in the top and bottom subdivisions respectively. Since $\Delta$ takes $f^+$ to $f^-$, the sum of digits of $f^+$ and the sum of digits of $f^-$ have the same parity. The result follows from Lemma 4.11 and its stated analogue. \qed

Now we are ready to prove Theorem 2. Jones’ subgroup $\bar{F}$ is the stabilizer of the set of dyadic fractions from the unit interval $[0, 1]$ with odd sums of digits, under the standard action of $F$ on the interval $[0, 1]$.

Proof. Let $S$ be the set of dyadic fractions with odd sums of digits. Let $\Delta$ be a reduced diagram which stabilizes $S$. By Corollary 4.12, for every inner vertex $v$, the length of the top path from $\iota(\Delta)$ to $v$ and the length of the bottom path from $\iota(\Delta)$ to $v$ have the same parity. It is possible to use this parity to assign to every vertex of $T(\Delta)$ a label "0" or "1". Since for every vertex there is a unique top path and a unique bottom path from $\iota(\Delta)$ to the vertex, neighbors in $T(\Delta)$ have different labels and $T(\Delta)$ is bipartite.

The other direction follows from Theorem 1. On finite binary fractions $x_0x_1$ acts as follows:

$$x_0x_1(t) = \begin{cases} t = 0.00\alpha & \rightarrow t = 0.0\alpha \\ t = 0.01\alpha & \rightarrow t = 0.10\alpha \\ t = 0.11\alpha & \rightarrow t = 0.110\alpha \\ t = 0.1\alpha & \rightarrow t = 0.111\alpha \end{cases}$$

In particular, $x_0x_1$ stabilizes the set $S$. If $\Delta$ and $\Delta'$ are diagrams in $F$, then viewed as maps from $[0, 1]$ to itself, the sum $\Delta \oplus \Delta'$ is defined as

$$(\Delta \oplus \Delta')(t) = \begin{cases} \frac{\Delta(t)}{2} & : t \in [0, \frac{1}{2}] \\ \frac{\Delta'(t - \frac{1}{2})}{2} + \frac{1}{2} & : t \in [\frac{1}{2}, 1] \end{cases}$$

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It is easy to see that if $\Delta$ and $\Delta'$ stabilize $S$, then $\Delta \oplus \Delta'$ stabilize $S$ as well. Since by Theorem\[1\] $F^\alpha$ is the smallest subgroup of $F$ containing $x_0x_1$ and closed under sums, the inclusion $F \subset \text{Stab}(S)$ follows. 

In order to prove Corollary\[3\] we will need the following observations.

**Remark 4.13.** Let $c = (x_0x_1)^{-1} \in F$. On finite binary fractions $c$ acts as follows:

$$c(t) = \begin{cases} 
  t = .0\alpha & \rightarrow t = .00\alpha \\
  t = .10\alpha & \rightarrow t = .010\alpha \\
  t = .110\alpha & \rightarrow t = .011\alpha \\
  t = .111\alpha & \rightarrow t = .1\alpha 
\end{cases}$$

In particular, if $t$ is a finite binary fraction and $m \in \mathbb{N}$ then for any large enough $n \in \mathbb{N}$, the first $m$ digits in the binary form of $c^n(t)$ are zeros. That is, $c^n(t) < \frac{1}{2^m}$.

**Proof.** If the first digit of $t$ is 0 then each application of $c$ adds another 0 to the leading sequence of zeros in the binary form of $t$ so for every $n \geq m$ we get the result. If the first digit of $t$ is 1, then $t$ starts with a sequence of ones followed by a 0. Let $l$ be the length of this sequence. If $l > 3$ then each application of $c$ reduces the length of the sequence of ones by 2. Thus, possibly after several applications of $c$, we can assume that $l = 1$ or $l = 2$. In both cases, one application of $c$ yields $c(t)$ which starts with 0 and we are done by the previous case.

**Lemma 4.14.** Let $g$ be an element of $R$. Thompson group $F$. Then, there exists $m \in \mathbb{N}$ such that for any finite binary fraction $t < \frac{1}{2^m}$ the sum of digits of $t$ in binary form is equal to the sum of digits of $g(t)$.

**Proof.** The element $g$ maps some binary subdivision $B_1$ onto a subdivision $B_2$. The first segment of $B_1$ is of the form $J = [0, \frac{1}{2^r}]$ for some positive integer $r$. Since 0 is mapped to 0, on the segment $J$ the function $g$ is defined as a linear function with slope $2^r$ for some $l \in \mathbb{Z}$ and with constant number 0. That is, for every $t \in J$ we have $g(t) = 2^rt$. If $l \leq 0$, then for any binary fraction $t \leq \frac{1}{2^r}$, the application of $g$ adds $l$ zeros to the beginning of the binary form of $g$, which does not affect the sum of digits of $t$. In that case, taking $m = r$ would do. If $l > 0$ it is possible to take $m = \max\{r,l\}$. Since $m \geq r$, the binary fraction $t \in J$. Since $m \geq l$, the binary form of $t$ starts with at least $l$ zeros. The application of $g$ erases the first $l$ zeros and thus does not affect the sum of digits of $t$. 

Corollary\[3\] follows immediately from the following.

**Theorem 4.15.** Let $h$ be an element of $F$ which does not belong to $F^\alpha$. Then the index $[F : F \cap hFh^{-1}]$ is infinite.

**Proof.** Let $h \notin F^\alpha$. If the index $[F : F \cap hFh^{-1}]$ is finite then there exists $r \in \mathbb{N}$ such that for every $g \in F$, we have $g^r \in hFh^{-1}$. That is, $h^{-1}g^rh \in F$. In particular, for every $k \in \mathbb{N}$ we have $h^{-1}g^rh \in F$. Let $g = (x_0x_1)^{-1} \in F$. We will show that for every $n$ large enough, $h^{-1}g^nfh \notin F$ and get the required contradiction.

Let $S$ be the set of finite binary fractions with odd sums of digits. Since $h^{-1} \notin F$, there exists $t \in S$ such that $h^{-1}(t) \notin S$. Let $t_1 = h^{-1}(t)$. By Lemma 4.14 there exists $m$ for which the sum of digits of every binary fraction $< \frac{1}{2^m}$ is preserved by $h$. By Remark 4.13 for every $n$ large enough, $g^n(t_1) < \frac{1}{2^m}$. Since $g^n \in F$ and $t_1 \notin S$, the binary fraction $g^n(t_1) \notin S$. Since $g^n(t_1) < \frac{1}{2^m}$, the sum of digits of $h(g^n(t_1))$ is equal to the sum of digits
of $g^n(t_1)$. Thus $h(g^n(t_1)) \notin S$. Therefore, (recall that the composition in $F$ is from left to right), $h^{-1}g^nh(t) = h(g^n(h^{-1}(t))) = h(g^n(t_1)) \notin S$. The element $t$ belonging to $S$ implies that $h^{-1}g^nh$ does not stabilize $S$ and in particular $h^{-1}g^nh \notin \overrightarrow{F}$.

\[\square\]

5 The subgroup $\overrightarrow{F}_n$

In this section we generalize results of the previous sections from 2 and 3 to arbitrary $n$. It turns out that the generalization is quite natural. The proofs follow the same paths and for some theorems, the proof for arbitrary $n$ is almost identical to the proof of the particular case considered before.

5.1 The definition of the subgroup

Definition 5.1. Let $\Delta$ be a (not necessarily reduced) diagram over the presentation $\mathcal{P} = \langle x : x = x^2 \rangle$ and let $n \in \mathbb{N}$. The diagram $\Delta$ is said to be $n$-good if for every inner vertex $v$ the lengths of the top and bottom paths from $i(\Delta)$ to $v$ are equal modulo $n$.

Note that if $n = 2$, then being 2-good is formally weaker than being bipartite. Lemma \[5.0\] shows that these conditions are in fact equivalent for reduced diagrams.

The proof of the following lemma is completely analogous to the proof of Lemma \[4.1\] so we leave the proof to the reader.

Lemma 5.2. Suppose that a diagram $\Delta$ is obtained from a diagram $\Delta'$ by removing a dipole. Suppose that $\Delta'$ is $n$-good for some $n \in \mathbb{N}$. Then $T(\Delta)$ is $n$-good as well.

Definition 5.3. Let $n \in \mathbb{N}$. Jones’ $n$-subgroup $\overrightarrow{F}_n$ is the set of all reduced $n$-good diagrams $\Delta$ in $F$.

In particular Jones’ 1-subgroup is the entire Thompson group $F$. Jones’ 2-subgroup coincides with Jones’ subgroup $\overrightarrow{F}$.

The proof of the following proposition is identical to that of Proposition \[4.3\]

Proposition 5.4. For every $n \in \mathbb{N}$, Jones’ $n$-subgroup $\overrightarrow{F}_n$ is indeed a subgroup of $F$.

5.2 The subgroup $\overrightarrow{F}_{n-1}$ is isomorphic to $F_n$

Let $n \geq 2$. We denote by $H_n$ the Thompson subalgebra of $F$ generated by $x_0 \cdots x_{n-2}$, i.e., the smallest subgroup of $F$ containing $x_0 \cdots x_{n-2}$ and closed under $\oplus$.

Lemma 5.5. Let $i, d \in \mathbb{N} \cup \{0\}$. Let $(m_0, \ldots, m_d)$ be a sequence of positive integers such that $m_d \geq 2$ if $d > 0$. Then,

\[
\prod_{k=0}^{d} x_{i+k}^{m_k} \prod_{k=1}^{n-2} x_{i+d+k} \left[ \prod_{k=0}^{d-1} x_{i+n-1+k}^{m_k} \right]^{-1} x_{i+n-1+d}^{m_d-1} \in H_n.
\]

Proof. We first prove the Lemma for $i = 0$ by induction on $d$. For $d = 0$, the argument is similar to the one in the proof of Lemma \[4.4\]. If we add the trivial diagram 1 on the right to the reduced diagram representing $\prod_{k=0}^{d-1} x_k$, we get the diagram corresponding to the normal form $x_0^3 \prod_{k=1}^{n-2} x_k (\prod_{k=0}^{n-1} x_k)^{-1} = x_0^2 (\prod_{k=1}^{n-2} x_k) x_{n-1}^{-1} (\prod_{k=0}^{n-2} x_k)^{-1}$. Hence $x_0^2 (\prod_{k=1}^{n-2} x_k) (x_{n-1})^{-1}$ also belongs to $H_n$. If we add to this element the diagram 1 on the
right, we get \( x_0^n \prod_{k=1}^{n-2} x_k (\prod_{k=0}^{n-2} x_k x_{n-1})^{-1} = x_0^n (\prod_{k=1}^{n-2} x_k (x_{n-1})^{-1} (\prod_{k=0}^{n-2} x_k)^{-1}. \) Multiplying on the right by \( \prod_{k=0}^{n-2} x_k \) we get that \( x_0^n (\prod_{k=1}^{n-2} x_k (x_{n-1})^{-1} (\prod_{k=0}^{n-2} x_k)^{-1} \) belongs to \( H_n. \)

Note the effect of the addition of 1 on the right to the element \( x_0^n (\prod_{k=1}^{n-2} x_k (x_{n-1})^{-1} \) The negative part of the normal form got multiplied on the left by \( \prod_{k=0}^{n-2} x_k \), when originally it started with \( x_{n-1} \). Similarly, the positive part of the normal form was multiplied by \( x_0 \), when originally it started with \( x_0 \). In particular the exponents of both \( x_0 \) and \( x_{n-1} \) were increased by one. By repeating the process of adding 1 on the right and multiplying by \( \prod_{k=0}^{n-2} x_k \) (on the right) we get that for every positive integer \( m_0 \), the element \( x_0^{m_0} (\prod_{k=1}^{n-2} x_k (x_{n-1})^{-1} \) belongs to \( H_n \) as required.

Assume that the lemma holds for every non negative integer \( \leq d \). Let \( (m_0, \ldots, m_{d+1}) \) be a sequence of positive integers such that \( m_{k+1} \geq 2. \) Let \( j = \min\{1, \ldots, d+1 \} \) such that \( m_j \geq 2. \) For all \( r = 1, \ldots, d+1 - j \) let \( n_r = m_{r+j} \). Let \( n_0 = m_j - 1. \) We use the induction hypothesis on \( d + 1 - j \) with the sequence \( (n_0, \ldots, n_{d+1-j}) \). Thus, we have that

\[
\prod_{k=0}^{d+1-j} x_k^{n_k} \prod_{k=1}^{n-2} x_{d+1-j+k} \left( \prod_{k=0}^{d+1-j} x_k^{n_{d+1-j+k}} \right)^{-1} \in H_n
\]

Adding the trivial diagram 1 \( j \) times on the left we get by Lemma 2.6 the element

\[
\prod_{k=0}^{d+1-j} x_k^{n_k} \prod_{k=1}^{n-2} x_{d+1+k} \left( \prod_{k=0}^{d+1-j} x_k^{n_{d+1-j+k}} \right)^{-1} \in H_n.
\]

We assume that \( j < d + 1, \) the other case being similar. Then, adding 1 on the right results in the following element. Note that as in the case \( d = 0 \) the positive part of the normal form gets multiplied by \( \prod_{k=0}^{n-1} x_k \) (it currently starts with \( x_j \)). Similarly, the negative part of the normal form is multiplied by \( \prod_{k=0}^{n-1} x_k \).

\[
\prod_{k=0}^{j-1} x_k^{n_k} \prod_{k=1}^{d+1-j} x_{j+k}^{m_{j+k}} \prod_{k=1}^{n-2} x_{d+1+k} \left( \prod_{k=0}^{d+1-j} x_k^{m_{d+1-j+k}} \right)^{-1} \in H_n
\]

Substituting \( n_k \) by \( m_{k+j} \) and \( n_0 + 1 \) by \( m_j \) we get

\[
\prod_{k=0}^{j-1} x_k^{m_k} \prod_{k=1}^{d+1-j} x_{j+k}^{m_{j+k}} \prod_{k=1}^{n-2} x_{d+1+k} \left( \prod_{k=0}^{d+1-j} x_k^{m_{d+1-j+k}} \right)^{-1} \in H_n
\]

Then, shifting the indexes we get that

\[
\prod_{k=0}^{j-1} x_k^{m_k} \prod_{k=1}^{d+1-j} x_{j+k}^{m_{j+k}} \prod_{k=1}^{n-2} x_{d+1+k} \left( \prod_{k=0}^{d+1-j} x_k^{m_{d+1-j+k}} \right)^{-1} \in H_n
\]

Multiplying on the right by \( \prod_{k=0}^{n-2} x_k \) cancels a prefix of the negative part of the normal form so we have that

\[
\prod_{k=0}^{j-1} x_k^{m_k} \prod_{k=1}^{d+1-j} x_{j+k}^{m_{j+k}} \prod_{k=1}^{n-2} x_{d+1+k} \left( \prod_{k=0}^{d+1-j} x_k^{m_{d+1-j+k}} \right)^{-1} \in H_n
\]

Since \( m_k = 1 \) for all \( k = 1, \ldots, j - 1 \) we have,

\[
x_0 \prod_{k=1}^{d+1-j} x_k^{m_k} \prod_{k=1}^{n-2} x_{d+1+k} \left( \prod_{k=1}^{d} x_k^{m_{d+1-j+k}} \right)^{-1} \in H_n
\]
To get the result for \( m_0 \geq 1 \) it is possible to increase the exponents of \( x_0 \) and \( x_{n-1} \) simultaneously by repeatedly adding \( 1 \) on the right and multiplying by \( \prod_{k=0}^{n-2} x_k \). Thus, we have that
\[
\prod_{k=0}^{d+1} x_k^{m_k} \prod_{k=1}^{n-2} x_{d+1+k}^{m_{d+1+k}} \left[ \left( \prod_{k=0}^{d} x_{m_{n-1+k}}^{m_{d+1+k}} \right) x_{m_{d+1}+1}^{-1} \right]^{-1} \in H_n
\]
as required. If \( i \neq 0 \) then by Lemma 2.6 adding the trivial diagram \( i \) times on the left, gives the result. \( \square \)

**Lemma 5.6.** Let \( n \geq 2 \). Then, the subgroup \( \mathbf{F}_{n-1} \rightarrow \mathbf{F}_{n-1} \) coincides with the Thompson subalgebra \( H_n \).

**Proof.** Clearly, \( H_n \) is inside \( \mathbf{F}_{n-1} \). Let \( \Delta \) be a reduced diagram in \( \mathbf{F}_{n-1} \). We enumerate the vertices of \( \Delta \) from left to right: \( 0, 1, \ldots, s \) so that \( \iota(\Delta) = 0, \tau(\Delta) = s \). We shall need the following lemma.

**Lemma 5.7.** Let \( r \) be the left-most vertex such that there exists \( \ell < r - 1 \) such that \( \ell \) and \( r \) are connected by an edge in \( T(\Delta) \). Let \( \ell \) be the left-most vertex such that \( (\ell, r) \) is an upper or lower edge in \( T(\Delta) \). Then \( r - \ell \geq n \). That is, there are at least \( n - 1 \) inner vertices of \( \Delta \) between the vertices \( \ell \) and \( r \).

**Proof.** We assume that the edge \( (\ell, r) \) is an upper edge in \( T(\Delta) \). Otherwise, we look at \( \Delta^{-1} \) instead. Note that a priori, there might also be a lower edge in \( T(\Delta) \) connecting the vertices \( \ell \) and \( r \). Let \( \Delta_1 \) be the subdiagram of \( \Delta^{+} \) bounded from above by the edge \( (\ell, r) \) and from below by (part of) the path \( \text{bot}(\Delta^{+}) \). Since every inner vertex \( i < r \) has no incoming edges in \( \Delta \) other than \( (i-1, i) \), every inner vertex of \( \Delta_1 \) is connected by an edge in \( \Delta_1 \) to the terminal vertex \( r \). It follows that the vertices \( \ell \) and \( r \) are not connected by a lower edge in \( T(\Delta) \). Otherwise the same argument for the subdiagram \( \Delta_2 \), bounded from above by \( \text{top}(\Delta^{-}) = \text{bot}(\Delta^{+}) \) and from below by the lower edge from \( \ell \) to \( r \), would show that the diagram \( \Delta_2 \) is the inverse of \( \Delta_1 \), by contradiction to the diagram \( \Delta \) being reduced.

Since the length of the top path from \( \iota(\Delta) \) to \( r \) is \( \ell + 1 \), the length of the bottom path from \( \iota(\Delta) \) to \( r \) is equal to \( \ell + 1 \) modulo \( n - 1 \). From the minimality of \( r \), there exists some \( \ell' < r \) such that the bottom path from \( \iota(\Delta) \) to \( r \) is composed of the lower edges \( (0, 1), (1, 2), \ldots, (\ell'-1, \ell') \) and \( (\ell', r) \). Therefore, its length \( \ell' + 1 \equiv \ell + 1 \mod n - 1 \). From the minimality of \( \ell \) and the absence of a lower edge from \( \ell \) to \( r \), it follows that \( \ell' > \ell \). Therefore, \( r - \ell > \ell' - \ell \equiv 0 \mod n - 1 \) implies that \( r - \ell > n - 1 \). \( \square \)

Now we can finish the proof of Theorem 5.6. Consider the subdiagram \( \Delta_1 \) of the diagram \( \Delta^{+} \), bounded from above by the upper edge \( (\ell, r) \) and from below by the bottom path \( \text{bot}(\Delta^{+}) \). The diagram \( \Delta_1 \) has at least \( n - 1 \) inner vertices, the first \( n - 2 \) of which are connected by arcs (i.e., edges which do not lie on \( \text{bot}(\Delta^{+}) \)) to the terminal vertex \( r \).

Let \( x_i \) be the leading term of the positive part of the normal form of \( \Delta \) (since \( (\ell, r) \) is an upper edge, such \( x_i \) exists). Let \( x_i^{m_0} x_i^{m_1} \ldots x_i^{m_p} \) be the longest sequence of positive powers of consecutive letters \( x_j \) which forms a prefix of the normal form of \( \Delta \).

Each positive letter \( x_j \) in the normal form of \( \Delta \) has a corresponding edge in \( \Delta^{+} \). Namely, the top edge of the cell “labeled by \( x_j \)" in the algorithm described in Lemma 2.3. From Lemma 2.3 it follows that if for all \( j = 0, \ldots, p \) the exponent \( m_j = 1 \), then the edge corresponding to the first letter in the sequence (i.e. to \( x_i \)) is in fact the edge \( (\ell, r) \). In this case, the \( n - 2 \) mentioned arcs from vertices beneath the edge \( (\ell, r) \) to \( r \) show that the normal form of \( \Delta \) starts with \( \prod_{k=0}^{n-2} x_i^{m_k} \) which belongs to \( H_n \). Dividing the normal form of \( \Delta \) from the left
by \( \prod_{k=0}^{n-2} x_{i+k} \) gives an element of \( \overline{F}_{n-1} \) with a shorter normal form and we are done by induction.

Thus we can assume that there exists \( d = \max\{j = 0, \ldots, p \mid m_d \geq 2\} \). Again, from Lemma 2.5 it follows that the edge corresponding to the last letter of the power \( x_d^{m_d} \) is the edge \((\ell, r)\). The arcs beneath \((\ell, r)\) show that the prefix \( x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \) of the normal form of \( \Delta \) is followed by at least \( n - 2 \) letters forming the product \( \prod_{k=1}^{n-2} x_{i+n-1+k} \). By Lemma 5.5, it is possible to replace \( \prod_{k=0}^{d} x_{i+k} \prod_{k=1}^{n-2} x_{i+n-1+k} \) by \( \left( \prod_{k=0}^{d-1} x_{i+n-1+k} \right) x_{i+n-1+\ell} \) and the resulting element would still be in \( \overline{F}_{n-1} \). Since its normal form is shorter than that of \( \Delta \) we are done by induction. \( \square \)

**Lemma 5.8.** For all \( n \geq 2 \), the Thompson subalgebra \( H_n \) is generated as a group by the set \( A = \{ x_j \cdots x_{j+n-2} \mid j \geq 0 \} \).

**Proof.** Let \( G \) be the subgroup of \( F \) generated by \( A \). It suffices to prove that \( G \) is closed under sums. Let \( a \) and \( b \) be elements of \( G \). In particular, \( a = y_1 \cdots y_t \) and \( b = z_1 \cdots z_s \) where \( y_1, \ldots, y_t \) and \( z_1, \ldots, z_s \) are elements in \( A^{+1} \). By Lemma 2.5 \( a \oplus b = (y_1 \oplus 1) \cdots (y_t \oplus 1)(1 \oplus z_1) \cdots (1 \oplus z_s) \). Since for any diagram \( z \), we have \((1 \oplus z^{-1}) = (1 \oplus z)^{-1}\) and \( A \) is closed under addition of \( 1 \) from the left, \((1 \oplus z^{-1}) \in G \) for all \( m = 1, \ldots, s \). Thus it suffices to prove that for all \( j \geq 0 \), the element \( x_j \cdots x_{j+n-2} \oplus 1 \in G \).

From Lemma 2.5 it follows that

\[
\left( \prod_{k=0}^{n-2} x_{j+k} \right) \oplus 1 = \prod_{k=0}^{j} x_k \prod_{k=0}^{n-2} x_{j+k} \left( \prod_{k=0}^{j+n-1} x_k \right)^{-1}.
\]

Let \( r \in \{0, \ldots, n-2\} \) be the residue of \( j \) modulo \( n-1 \). Inserting \( \prod_{k=0}^{j+n-2} x_k \) and its inverse to the right side of the above equation we get that

\[
\left( \prod_{k=0}^{n-2} x_{j+k} \right) \oplus 1 = \prod_{k=0}^{j} x_k \prod_{k=0}^{j+n-2} x_k \left( \prod_{k=0}^{j+n-2} x_k \right)^{-1}.
\]

Since in Thompson group \( F \) for all \( \ell > r \) we have \( x_{\ell}^{-1} x_r = x_r x_{\ell+1}^{-1} \), it is possible to replace

\[
\left( \prod_{k=j+1}^{j+n-2} x_k \right)^{-1} \prod_{k=0}^{j+n-2} x_{j+k} \text{ by } \prod_{k=0}^{j+n-2} x_{j+k} \left( \prod_{k=j+1}^{j+n-2} x_k \right)^{-1}.
\]

Thus,

\[
\left( \prod_{k=0}^{n-2} x_{j+k} \right) \oplus 1 = \prod_{k=0}^{j} x_k \prod_{k=0}^{j+n-2} x_k \prod_{k=0}^{j+n-2} x_{j+k} \left( \prod_{k=j+1}^{j+n-2} x_{k+n-1} \right)^{-1}.
\]

Merging adjacent products, we get that

\[
\left( \prod_{k=0}^{n-2} x_{j+k} \right) \oplus 1 = \prod_{k=0}^{j+n-3} x_k \prod_{k=0}^{n-2} x_{j+k} \left( \prod_{k=0}^{j+n-3} x_k \right)^{-1}.
\]

Since the length of each of the products \( \prod_{k=0}^{j+n-2} x_k \), \( \prod_{k=0}^{n-2} x_{j+k} \) and \( \prod_{k=0}^{j+n-3} x_k \) is divisible by \( n-1 \), it is possible to present each of them as a product of elements of \( A \). The result clearly follows. \( \square \)

**Corollary 5.9.** For all \( n \geq 2 \), the Thompson subalgebra \( H_n \) is generated as a group by the set \( \{ x_j \cdots x_{j+n-2} \mid j = 0, \ldots, n-1 \} \).
Proof. For every \( j \geq n \), the element \( x_j \ldots x_{j+n-2} \) is equal to \( (x_{j-n+1} \ldots x_{j-1})^{x_0 \ldots x_{n-2}} \). \qed

Lemma 5.10. For all \( n \geq 2 \), the subgroup \( \overline{F}_{n-1}^2 \) is isomorphic to \( F_n \).

Proof. The elements \( x_j \ldots x_{j+n-2} \), for \( j = 0, \ldots, n-1 \) satisfy the defining relations of \( F_n \) (see [4, page 54]). Since all proper homomorphic images of \( F_n \) are Abelian [4, Theorem 4.13] and \( x_j \ldots x_{j+n-2} \), for \( j = 0, 1 \) do not commute, we get the result. \qed

Finally, the proofs of the following theorems are almost identical to the proofs of Theorem 2 and Corollary 3 so we leave them to the reader.

Theorem 5.11. Let \( n \geq 1 \). For each \( i = 0, \ldots, n-1 \), let \( S_i \) be the set of all finite binary fractions from the unit interval \([0,1]\) with sums of digits equal to \( i \) modulo \( n \). Then, the subgroup \( \overline{F}_n \) is the intersection of the stabilizers of \( S_i, i = 0, \ldots, n-1 \) under the natural action of \( F \) on \([0,1]\).

Theorem 5.12. For all \( n \geq 1 \), the subgroup \( \overline{F}_n \) coincides with its commensurator in \( F \), hence the linearization of the permutational representation of \( F \) on \( F/\overline{F}_n \) is irreducible.

5.3 The embeddings of \( F_n \) into \( F \) are natural

Finally let us note that the embeddings of \( F_n \) into \( F \) studied in this paper are natural in the terminology of [4]. Suppose that \( G_i = \text{DG}(\mathcal{P}_i, u_i) \), \( i = 1, 2 \), are diagram groups, and we can tessellate every cell from \( \mathcal{P}_1 \) by cells from \( \mathcal{P}_2 \), which turns every cell from \( \mathcal{P}_1 \) into a diagram over \( \mathcal{P}_2 \). Then we can define a map from \( G_1 \) to \( G_2 \) which sends every diagram \( \Delta \) from \( G_1 \) to a diagram obtained by replacing every cell from \( \Delta \) by the corresponding diagram over \( \mathcal{P}_2 \). This map is obviously a homomorphism. Such homomorphisms were called natural in [6].

Instead of giving a more precise general definition of a natural homomorphism from one diagram group into another, let us define it in the case of \( F_n = \text{DG}(\{x \mid x = x^n\}, x) \). Let \( \pi_n \) be the cell \( x \rightarrow x^n \). It has \( n+1 \) vertices \( 0, \ldots, n \). Suppose that \( n \geq 3 \). Connect each vertex \( 1, \ldots, n-2 \) with the vertex \( n \) by an arc. The result is an \( (x, x^n) \)-diagram \( \Delta(\pi_n) \) over the presentation \( \langle x \mid x = x^2 \rangle \). Now for every \( (x, x) \)-diagram \( \Delta \) from \( F_n \), consider the diagram \( \phi(\Delta) \) obtained by replacing every cell \( \pi_n \) (resp. \( \pi_n^{-1} \)) by a copy of \( \Delta(\pi_n) \) (resp. \( \Delta(\pi_n)^{-1} \)). The resulting diagram belongs to \( F = \text{DG}(\{x \mid x = x^2\}, x) \). It is easy to check that the map \( \phi \) is a homomorphism.

A set of generators of \( F_n \) is described in [4, Page 54]. Their left-right duals also generate \( F_n \). If we apply \( \phi \) to these generators, we get (using Lemma 2.5) elements \( x_j \ldots x_{j+n-2}, j = 0, \ldots, n-1 \), which are generators of \( \overline{F}_{n-1}^2 \) by Corollary 5.9. Since \( F_n \) does not have a non-injective homomorphism with a non-Abelian image, \( \phi \) is an isomorphism between \( F_n \) and \( \overline{F}_{n-1}^2 \). Figure 5.8 below shows the left-right duals of generators of \( F_3 \) from [4], and Figure 5.9 shows the images of these generators under \( \phi \). Note that \( \phi \) is also an example of a caret replacement homomorphism from \( F_p \) to \( F_q \) studied in [2]. It is proved in [2] that the image of \( F_p \) under \( \phi \) is undistorted in \( F \).
Figure 5.8: Generators of $F_3$.

Figure 5.9: Images of generators of $F_3$ under $\phi$. Red arcs are added by $\phi$.

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