EXISTENCE AND UNIQUENESS OF SOLUTION
to a functional integro-differential
fractional equation

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Abstract. We prove, using a fixed point theorem in a Banach algebra, an existence result for a fractional functional differential equation in the Riemann–Liouville sense. Dependence of solutions with respect to initial data and an uniqueness result are also derived.

1. Introduction

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject has its origin in the 1600s. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. In the last few decades, however, fractional differentiation proved very useful in various fields of applied sciences and engineering: physics (classic and quantum mechanics), chemistry, biology, economics, signal and image processing, calculus of variations, control theory, electrophoresis, viscoelasticity, feedback amplifiers, and electrical circuits [4, 6, 8, 9, 15, 18, 20, 22, 23]. The “bible” of fractional calculus is the book of Samko, Kilbas and Marichev [20].

Several definitions of fractional derivatives are available in the literature, including the Riemann–Liouville, Grunwald–Letnikov, Caputo, Riesz, Riesz–Caputo, Weyl, Hadamard, and Chen derivatives [11, 12, 13, 16, 17, 20]. The most common used fractional derivative is the Riemann–Liouville [10, 17, 20, 21], which we adopt here. It is worth to mention that functions that have no first order derivative might have Riemann–Liouville fractional derivatives of all orders less than one [17]. Recently, the physical meaning of the initial conditions to fractional differential equations with Riemann–Liouville derivatives has been discussed [7, 14, 16].

Using a fixed point theorem, like Schauder’s fixed point theorem, and the Banach contraction mapping principle, several results of existence have been obtained in the literature to linear and nonlinear equations, and recently also to fractional differential equations. The interested reader is referred to [1, 2, 3, 19].

Let \( I_0 = [-\delta, 0] \) and \( I = [0, T] \) be two closed and bounded intervals in \( \mathbb{R} \); \( B(I, \mathbb{R}) \) be the space of bounded real-valued functions on \( I \); and \( C = C(I_0, \mathbb{R}) \) be the space of continuous real-valued functions on \( I_0 \). Given a function \( \phi \in C \), we consider the
functional integro-differential fractional equation
\[
\frac{d^\alpha}{dt^\alpha} \left[ \frac{x(t)}{f(t, x(t))} \right] = g \left( t, x_t, \int_0^t k(s, x_s) \, ds \right) \quad \text{a.e.,} \quad t \in I,
\]
subject to
\[
x(t) = \phi(t), \quad t \in I_0,
\]
where \( d^\alpha/dt^\alpha \) denotes the Riemann–Liouville derivative of order \( \alpha, 0 < \alpha < 1 \), and \( x_t : I_0 \to \mathbb{C} \) is the continuous function defined by \( x_t(\theta) = x(t + \theta) \) for all \( \theta \in I_0 \), under suitable mixed Lipschitz and other conditions on the nonlinearities \( f \) and \( g \). For a motivation to study such type of problems we refer to [1]. Here we just mention that problems of type (1.1)–(1.2) seem important in the study of dynamics of biological systems [1].

Our main aim is to prove existence of solutions for (1.1)–(1.2). This is done in Section 3 (Theorem 3.1). Our main tool is a fixed point theorem that is often useful in proving existence results for integral equations of mixed type in Banach algebras [5], and which we recall in Section 2 (Theorem 2.2). We end with Section 4 by proving dependence of the solutions with respect to their initial values (Theorem 4.1) and, consequently, uniqueness to (1.1)–(1.2) (Corollary 4.2).

2. Preliminaries

In this section we give the notations, definitions, hypotheses and preliminary tools, which will be used in the sequel. We deal with the (left) Riemann–Liouville fractional derivative, which is defined in the following way.

**Definition 2.1** ([20]). The fractional integral of order \( \alpha \in (0, 1) \) of a function \( f \in L^1[0,T] \) is defined by
\[
I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds,
\]
t \( \in [0,T] \), where \( \Gamma \) is the Euler gamma function. The Riemann–Liouville fractional derivative operator of order \( \alpha \) is then defined by
\[
\frac{d^\alpha}{dt^\alpha} := \frac{d}{dt} \circ I^{1-\alpha}.
\]

Along the paper, \( X \) denotes a Banach algebra with norm \( \| \cdot \| \). The space \( C(I, \mathbb{R}) \) of all continuous functions endowed with the norm \( \| x \| = \sup_{t \in I} |x(t)| \) is a Banach algebra. To prove the existence result for (1.1)–(1.2), we shall use the following fixed point theorem.

**Theorem 2.2** ([5]). Let \( B_r(0) \) and \( \overline{B_r(0)} \) be, respectively, open and closed balls in a Banach algebra \( X \) centered at origin \( 0 \) and of radius \( r \). Let \( A, B : \overline{B_r(0)} \to X \) be two operators satisfying:

(a) \( A \) is Lipschitz with Lipschitz constant \( L_A \),
(b) \( B \) is compact and continuous, and
(c) \( L_A M < 1 \), where \( M = \| B(\overline{B_r(0)}) \| := \sup \{ \| Bx \| ; x \in \overline{B_r(0)} \} \).

Then, either

(i) the equation \( \lambda [Ax]Bx = x \) has a solution for \( \lambda = 1 \), or
(ii) there exists \( x \in X \) such that \( \| x \| = r \), \( \lambda [Ax]Bx = x \) for some \( 0 < \lambda < 1 \).

Throughout the paper, we assume the following hypotheses:
In other terms, the integral equation (3.2) is equivalent to the operator equation (3.1).

To continue the proof of Theorem 3.1, we make use of two technical lemmas.

Lemma 3.2. Suppose that hypotheses (H1)–(H5) hold. Assume there exist $s$ a real number $r > 0$ such that for all $s$, $x$, where $A$, $B$, $L$, $L_1$, $I$, $C$, $R$, $K$, $\Gamma$, $\alpha$, $\beta$, $\psi$, $F$, $t$, $0$, $1$, $\sup$, $\gamma$, and $\forall$.

3. Existence of solution

We prove existence of a solution to (1.1)–(1.2) under hypotheses (H1)–(H5).

Theorem 3.1. Suppose that hypotheses (H1)–(H5) hold. Assume there exists a real number $r > 0$ such that

$$r > \frac{LT^\alpha}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s)(1 + \|L\|_1)\psi(r).$$

(3.1)

where

$$1 - \frac{LT^\alpha}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s)(1 + \|L\|_1)\psi(r) > 0, \quad F = \sup_{t \in [0, T]} |f(t, 0)|.$$

Then, problem (1.1)–(1.2) has a solution on $I$.

Proof. Let $X = C(I, \mathbb{R})$. Define an open ball $B_r(0)$ centered at origin and of radius $r > 0$, which satisfies (3.1). It is easy to see that $x$ is a solution to (1.1)–(1.2) if and only if it is a solution of the integral equation

$$x(t) = f(t, x(t)) \int_0^t k(s, x_s)ds.$$

In other terms,

$$x(t) = f(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_s, \int_0^s k(\tau, x_\tau)d\tau)ds.$$

(3.2)

Integral equation (3.2) is equivalent to the operator equation $Ax(t)Bx(t) = x(t)$, $t \in I$, where $A, B : B_r(0) \rightarrow X$ are defined by

$$Ax(t) = f(t, x(t)) \quad \text{and} \quad Bx(t) = \int_0^t k(s, x_s)ds.$$

We need to prove that the operators $A$ and $B$ verify the hypotheses of Theorem 2.2.

To continue the proof of Theorem 3.1, we make use of two technical lemmas.

Lemma 3.2. The operator $A$ is Lipschitz on $X$.

Proof. Let $x, y \in X$ and $t \in I$. By (H1) we have

$$|Ax(t) - Ay(t)| = |f(t, x) - f(t, y)| \leq L|x(t) - y(t)| \leq L\|x - y\|.$$

Then, $\|Ax - Ay\| \leq L\|x - y\|$ and it follows that $A$ is Lipschitz on $X$ with Lipschitz constant $L$. \qed
Lemma 3.3. The operator $B$ is completely continuous on $X$.

Proof. We prove that $B(\overline{B_r(0)})$ is an uniformly bounded and equicontinuous set in $X$. Let $x$ be arbitrary in $B_r(0)$. By hypotheses (H2)–(H4) we have

$$|Bx(t)| \leq I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) \right)$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) \psi \left( \|x_s\| + \int_0^s \beta(\tau) \|x_\tau\| d\tau \right) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) \left( 1 + \|\beta\|_{L^1} \right) \psi(r) ds$$

$$\leq \frac{\sup_{s \in [0,T]} \gamma(s)}{\Gamma(\alpha + 1)} \left( 1 + \|\beta\|_{L^1} \right) \psi(r) T^\alpha.$$

Taking the supremum over $t$, we get $\|Bx\| \leq M$ for all $x \in \overline{B_r(0)}$, where

$$M = \frac{\sup_{s \in [0,T]} \gamma(s)}{\Gamma(\alpha + 1)} \left( 1 + \|\beta\|_{L^1} \right) \psi(r) T^\alpha.$$

It results that $B(\overline{B_r(0)})$ is an uniformly bounded set in $X$. Now, we shall prove that $B(\overline{B_r(0)})$ is an equicontinuous set in $X$. For $0 \leq t_1 \leq t_2 \leq T$ we have

$$|Bx(t_2) - Bx(t_1)|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_2} (t_2-s)^{\alpha-1} g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds \right. - \left. \int_0^{t_1} (t_1-s)^{\alpha-1} g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} (t_2-s)^{\alpha-1} g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds \right. + \left. \int_0^{t_2} (t_2-s)^{\alpha-1} g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds \right. - \left. \int_0^{t_1} (t_1-s)^{\alpha-1} g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds$$

$$\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) ds.$$

On the other hand,

$$\left| g \left( s, x_s, \int_0^s k(\tau, x_\tau) d\tau \right) \right| \leq \gamma(s) \psi \left( \left\| x \right\| + \int_0^s \left| k(\tau, x_\tau) \right| d\tau \right).$$
\[ |Bx(t_2) - Bx(t_1)| \leq \frac{c}{\Gamma(\alpha)} \int_0^1 |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \, ds + \frac{c}{\Gamma(\alpha + 1)} |t_2 - t_1 - 2(t_2 - t_1)^\alpha|.
\]

Because the right hand side of the above inequality doesn’t depend on \( x \) and tends to zero when \( t_2 \to t_1 \), we conclude that \( B(B_r(0)) \) is relatively compact. Hence, \( B \) is compact by the Arzela–Ascoli theorem. It remains to prove that \( B \) is continuous. For that, let us consider a sequence \( x^n \) converging to \( x \). Then,

\[ |F x^n(t) - Fx(t)| \leq \left| f(t, x^n) \int_0^t k(s, x^n_s) \, ds \right| - f(t, x) I^\alpha \left( g \left( t, x^n, \int_0^t k(s, x_s) \, ds \right) \right) + |f(t, x)| I^\alpha \left( g \left( t, x^n, \int_0^t k(s, x^n_s) \, ds \right) \right) - I^\alpha \left( g \left( t, x^n, \int_0^t k(s, x^n_s) \, ds \right) \right) - g \left( t, x^n, \int_0^t k(s, x_s) \, ds \right) - g \left( t, x, \int_0^t k(s, x_s) \, ds \right) |.
\]

On the other hand,

\[ I^\alpha \left( g \left( t, x^n, \int_0^t k(s, x^n_s) \, ds \right) - g \left( t, x^n, \int_0^t k(s, x^n_s) \, ds \right) \right) \leq I^\alpha \left( L_1 |x^n - x| + L_2 \int_0^t k(s, x^n_s) - k(s, x_s) \, ds \right) \leq I^\alpha \left( L_1 |x^n - x| + L_2 \int_0^t |x^n_s - x_s| \, ds \right) \leq I^\alpha \left( L_1 \|x^n - x\| + L_2 \|x^n - x\| \right) \leq (L_1 + L_2 T) \|x^n - x\| \leq \frac{1}{\Gamma(\alpha + 1)} (L_1 + L_2 T) \|x^n - x\|.
\]
Taking the norm, \( \| Fx^n - Fx \| \leq L \| x^n - x \| + \frac{1}{\Gamma(\alpha+1)} |f(t, x)| (L_1 + L_2 L_k T) \| x^n_t - x_t \| \).

Hence, the right hand side of the above inequality tends to zero whenever \( x^n \to x \).

Therefore, \( Fx^n \to Fx \). This proves the continuity of \( F \).

Using Theorem 2.2, we obtain that either the conclusion (i) or (ii) holds. We show that item (ii) of Theorem 2.2 cannot be realizable. Let \( x \in X \) be such that \( \| x \| = r \) and \( x(t) = \lambda f(t, x(t)) I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) \right) \) for any \( \lambda \in (0, 1) \) and \( t \in I \). It follows that

\[
|x(t)| \leq \lambda (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) \right)
\leq \lambda (L \| x \| + F) I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) \right)
\leq \lambda (L \| x \| + F) I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) \right)
\leq (L \| x \| + F) I^\alpha \left( \gamma(t) \psi \left( \| x \| + \left| \int_0^t k(s, x_s) ds \right| \right) \right)
\leq (L \| x \| + F) I^\alpha \left( \gamma(t) \psi \left( \| x \| + \| \beta \|_{L^1} \| x \| \right) \right)
\leq \frac{L \| x \| + F}{\Gamma(\alpha)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right) \int_0^t (t - s)^{\alpha - 1} ds
\leq \frac{L \| x \| + F}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right) T^\alpha
\leq \left( \frac{LT^\alpha}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right) \right) \| x \|
\quad + \frac{FT^\alpha}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right) .
\]

Passing to the supremum in the above inequality, we obtain

\[
\| x \| \leq \frac{FT^\alpha}{1 - \frac{L \Gamma(\alpha)}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right) .
\] (3.3)

If we replace \( \| x \| = r \) in (3.3), we have

\[
r \leq \frac{FT^\alpha}{1 - \frac{L \Gamma(\alpha)}{\Gamma(\alpha + 1)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right)} \sup_{s \in (0, T)} \gamma(s) (1 + \| \beta \|_{L^1}) \psi \left( \| x \| \right),
\]

which is in contradiction to (3.1). Then the conclusion (ii) of Theorem 2.2 is not possible. Therefore, the operator equation \( Ax Bx = x \) and, consequently, problem (1.1)–(1.2), has a solution on \( I \). This ends the proof of Theorem 3.1.

Let us see an example of application of our Theorem 3.1. Let \( I_0 = [-\pi, 0] \) and \( I = [0, \pi] \). Consider the integro-differential fractional equation

\[
\frac{d^\alpha}{dt^\alpha} \left( \frac{x(t)}{1 + \sin \frac{t}{2} |x(t)|} \right) = g \left( t, x_t, \int_0^t k(s, x_s) ds \right), \quad t \in I,
\] (3.4)
Let $t > 0$. Let $\psi$ be a solution of (1.1)–(1.2) subject to $y(t,x) = r(t,x)$ for all $r \in \mathbb{R}^+$. By Theorem 3.1, $r$ satisfies

$$
\frac{12 - FB}{LB} \approx 0, 26 \leq r \leq \frac{12}{LB} \approx 18, 34,
$$

where $B = \frac{\Gamma(\alpha + 1)}{T(\alpha + 1)} \sup_{s \in (0,T)} \gamma(s) (1 + \|\beta\|_L^t)$. We conclude that if $r = 2$, then (3.2) has a solution in $B_2(0)$.

4. Dependence on the data and uniqueness of solution

In this section we derive uniqueness of solution to (1.1)–(1.2).

**Theorem 4.1.** Let $x$ and $y$ be two solutions to the nonlocal fractional equation (1.1) subject to (1.2) with $\phi = \phi_1$ and $\phi = \phi_2$, respectively. Then, we have

$$
\|x - y\| \leq \frac{(L\|y\| + F)(L_1 + L_2L_kT)\frac{\Gamma(\alpha + 1)}{T(\alpha + 1)}}{1 - L} \|\phi_1 - \phi_2\|.
$$

**Proof.** Let $x$ and $y$ be two solutions of (1.1). Then, from (3.2), one has

$$
|x(t) - y(t)| \leq \left| f(t, x) - f(t, y) \right| I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) \right).
$$

for $t > 0$. On the other hand, we have

$$
g \left( t, x_t, \int_0^t k(s, x_s) ds \right) = g \left( t, y_t, \int_0^t k(s, y_s) ds \right)
$$

$$
\leq L_1 |x_t - y_t| + L_2 \int_0^t |k(s, x_s) - k(s, y_s)| ds
$$

$$
\leq L_1 |\phi_1(t) - \phi_2(t)| + L_2L_kT \|\phi_1 - \phi_2\|
$$

$$
\leq (L_1 + L_2L_kT) \|\phi_1 - \phi_2\|.
$$

Then,

$$
|f(t, y)| I^\alpha \left( g \left( t, x_t, \int_0^t k(s, x_s) ds \right) - g \left( t, y_t, \int_0^t k(s, y_s) ds \right) \right)
$$

$$
\leq |f(t, y)| (L_1 + L_2L_kT) \|\phi_1 - \phi_2\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds.
$$

where $\alpha = \frac{1}{2}$ and $f : I \times \mathbb{R} \rightarrow \mathbb{R}^+ \setminus \{0\}$, $g : I \times C \times \mathbb{R} \rightarrow \mathbb{R}$ and $k : I \times C \rightarrow \mathbb{R}$ are given by

$$
f(t, x) = 1 + \sin \frac{t}{12} (|x(t)|), \quad g(t, x, y) = \gamma(t)(|x| + |y|), \quad k(t, x) = \frac{x}{4\pi},
$$

with $\gamma(t) = t, \beta(t) = \frac{1}{T}, T = \pi, L = \frac{1}{T}, F = \sup_{t \in [0,\pi]} f(t, 0) = 1, \|\beta\|_L^t = \frac{1}{2}$ and $\sup_{t \in [0,\pi]} \gamma(t) = \pi$. It is easy to see that all hypotheses (H1)–(H5) are satisfied with $\psi(r) = \frac{r}{12}$ for all $r \in \mathbb{R}^+$. By Theorem 4.1, $r$ satisfies
\[ \leq |f(t, y)| (L_1 + L_2 L_k T) \| \phi_1 - \phi_2 \| \frac{T^\alpha}{\Gamma(\alpha + 1)}. \]

Taking the supremum, we conclude that
\[ \| x - y \| \leq L \| x - y \| + |f(t, y)| (L_1 + L_2 L_k T) \| \phi_1 - \phi_2 \| \frac{T^\alpha}{\Gamma(\alpha)} \]
\[ \leq L \| x - y \| + (|f(t, y) - f(t, 0)| + |f(t, 0)|) (L_1 + L_2 L_k T) \| \phi_1 - \phi_2 \| \frac{T^\alpha}{\Gamma(\alpha + 1)} \]
\[ \leq L \| x - y \| + (L \| y \| + F) (L_1 + L_2 L_k T) \| \phi_1 - \phi_2 \| \frac{T^\alpha}{\Gamma(\alpha + 1)} \]
\[ \leq L \| x - y \| + (L \| y \| + F) (L_1 + L_2 L_k T) \| \phi_1 - \phi_2 \|. \]

Therefore,
\[ \| x - y \| \leq \frac{(L \| y \| + F)(L_1 + L_2 L_k T)T^\alpha}{1 - L} \| \phi_1 - \phi_2 \|. \]

\[ \square \]

**Corollary 4.2.** The solution predicted by Theorem 3.1 is unique.

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