LARGE DEVIATIONS FOR EMPIRICAL MEASURES OF MEAN FIELD GIBBS MEASURES

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Abstract. In this paper, we establish the large deviation principles, with respect to the weak convergence topology and the stronger Wasserstein metrics, for the empirical measure under the mean field Gibbs measure, under the strong exponential integrability condition for the negative part of the interaction potential. This is proved without any continuity or boundedness condition on the interaction potential existed in the known results. The proof relies mainly on the law of large numbers and the exponential decoupling inequality of $U$-statistics.

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1. Introduction

In this paper, we consider the configurations of $n$ particles at positions $x_1, \ldots, x_n$ in a separable and complete metric space $(S, \rho)$ (say Polish space), subject to an external force consisting of a confining potential $V : S \to (-\infty, +\infty]$ acting on each particle and an interaction potential $W : S \times S \to (-\infty, +\infty]$ acting on each pair of particles. The function $W$ is assumed to be measurable and symmetric, i.e., $W(x, y) = W(y, x)$ for all $x, y \in S$. The mean field Hamiltonian or energy functional $H_n : S^n \to (-\infty, +\infty]$ corresponding to the configuration $(x_1, \ldots, x_n)$ is given by

$$H_n(x^n) \equiv H_n(x_1, \ldots, x_n) := \sum_{i=1}^{n} V(x_i) + \frac{1}{n-1} \sum_{1 \leq i \neq j \leq n} W(x_i, x_j). \quad (1.1)$$

The mean field Gibbs probability measure $P_n$ on $S^n$ is defined by

$$dP_n(x_1, \ldots, x_n) := \frac{1}{Z_n} \exp(-H_n(x_1, \ldots, x_n)) m(dx_1) \cdots m(dx_n), \quad (1.2)$$

where $m$ is some nonnegative $\sigma$-finite measure on $S$ equipped with the Borel $\sigma$-field $\mathcal{B}$, and

$$Z_n := \int_S \cdots \int_S \exp(-H_n(x_1, \ldots, x_n)) m(dx_1) \cdots m(dx_n) \quad (1.3)$$

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is the normalizing constant, called the partition function. The main objective of this paper is to study the large deviations of the empirical measure

$$L_n(x^n; \cdot) := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}(\cdot)$$

under the mean field measure $P_n$. We will simply denote $L_n(x^n; \cdot)$ by $L_n$ in absence of confusion.

In the case $S = \mathbb{R}^d$, $P_n$ is just the equilibrium state (or the invariant probability measure) of the system of $n$ interacting particles below:

$$dX^i_n(t) = dB^i_t - \frac{1}{2} \nabla V(X^i_n(t)) + \frac{1}{n-1} \sum_{j:j \neq i} \nabla x W(X^i_n(t), X^j_n(t))dt, \quad i = 1, \cdots, n, \quad (1.4)$$

or simply

$$dX^n(t) = dB_t - \frac{1}{2} \nabla H_n(X^n(t))dt, \quad (1.5)$$

where $X^n(t) := (X^1_n(t), \cdots, X^n_n(t))^T$ takes values in $(\mathbb{R}^d)^n$, $B^1_t, \cdots, B^n_t$ are $n$ independent Brownian motions taking values in $\mathbb{R}^d$, and $B_t = (B^1_t, \cdots, B^n_t)$. It is well known that when $n$ goes to infinity, $L_n(X^n(t); \cdot)$ converges to the solution of the nonlinear McKean-Vlasov equation (the so called propagation of chaos), under quite general condition ([15]).

A classical problem is to establish conditions for the existence of a macroscopic limit of the empirical measures $L_n$ as the number of the particles $n \to +\infty$. It is well-known that the large deviation principle (LDP in short) provide a strong exponential concentration with the speed $n$ in terms of some explicit rate functional, which is very useful to study the macroscopic limit and microscopic phenomenon in statistical mechanics. Léonard established for the first in [12, 1987] the LDP for the empirical measure $L_n$ under the Gibbs measure $P_n$ in the weighted weak convergence topology, when $\nu \to \int \int W(x, y)\nu(x)\nu(y)$ is continuous in some appropriate topology and bounded by some weighted function satisfying the strong exponential integrability condition. By means of the weak convergence approach developed in Dupuis et Ellis [7], Dupuis et al. established in [9] an LDP by assuming that $W$ is lower bounded and lower semi-continuous (l.s.c. in short), which generalized the result obtained in [12]. For more results in this field, the reader is referred to [2, 3, 4, 13, 20] and the references therein.

The purpose of this paper is to establish the LDP for the empirical measures $L_n$ under a more general condition that $W$ is only measurable and its negative part $W^-$ satisfies the strong exponential integrability condition, which generalizes the previous results in [9] and [12]. We first obtain the LDP with respect to the weak convergence topology, then Wasserstein metric $W_\rho$ by using the Sanov’s theorem in the Wasserstein distance established by the second author et. al in [18]. Our main approach is the law of large numbers(LLL in short) and the exponential approximation for $U$-statistics. Our result can also be extended to many bodies interactions case.

The paper is organized as follows. In the next section, we will first briefly introduce some notations and definitions about LDP, and then present our main results. We give
the proof in the third section. The last section is devoted to the case of many bodies interactions.

2. Main result

2.1. Preliminaries. We recall the definition of a rate function on a Polish space $S$ and the LDP for a sequence of probability measures on $(S, \mathcal{B}(S))$.

**Definition 2.1** (Rate function). $I$ is said to be a rate function on $S$ if it is a lower semi-continuous function from $S$ to $[0, \infty]$ (i.e., for all $L \geq 0$, the level set $[I \leq L]$ is closed). $I$ is said to be a good rate function if it is inf-compact, i.e. $[I \leq L]$ is compact for any $L \in \mathbb{R}$.

A consequence of a rate function being good is that its infimum is achieved over any non-empty closed set.

We denote by $\mathcal{M}_1(S)$ the space of probability measures on $S$.

**Definition 2.2** (LDP). Let $\{\nu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on $\mathcal{M}_1(S)$.

(a) $\{\nu_n\}_{n \in \mathbb{N}}$ is said to satisfy the large deviation lower bound with the speed $n$ and a rate function $I$ if for any open subset $G \in \mathcal{B}(S)$,

$$l(G) := \liminf_{n \to +\infty} \frac{1}{n} \log \nu_n(G) \geq - \inf_{\nu \in G} I(\nu); \quad (2.1)$$

(b) $\{\nu_n\}_{n \in \mathbb{N}}$ is said to satisfy the large deviation upper bound with the speed $n$ and a rate function $I$ if for any closed subset $F \in \mathcal{B}(S)$,

$$u(F) := \limsup_{n \to +\infty} \frac{1}{n} \log \nu_n(F) \leq - \inf_{\nu \in F} I(\nu); \quad (2.2)$$

(c) $\{\nu_n\}_{n \in \mathbb{N}}$ is said to satisfy the large deviation principle with the speed $n$ and a rate function $I$ if both (a) and (b) hold, and $I$ is good.

The LDP characterizes the exponential concentration behavior, as $n \to +\infty$, of a sequence of probability measures $\{\nu_n\}_{n \in \mathbb{N}}$ in terms of a rate function. This characterization is via asymptotic upper and lower exponential bounds on the values that $\nu_n$ assigns to measurable subsets of $S$.

**Definition 2.3** ($U$-statistics). Let $X_1, X_2, \cdots$ be a sequence of independent random variables taking values in a measurable space $(S, \mathcal{S})$. A $U$-statistics of order 2 is defined as follows

$$U_n(\Phi) = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \Phi(X_i, X_j), \quad (2.3)$$

where $\Phi : S \times S \to \mathbb{R}$ is a symmetric function of two variables. The function $\Phi$ is called the kernel of the $U$-statistics.
2.2. Main results. Throughout this paper, we assume that

\[ C := \int_S \exp(-V(x))m(dx) < +\infty. \quad (2.4) \]

Let

\[ \alpha(dx) := \frac{1}{C} \exp(-V(x))m(dx) \quad (2.5) \]

be the probability measure on \( S \), then the mean field Gibbs probability measure \( P_n \) can be rewritten as

\[ dP_n(x_1, \cdots, x_n) = \frac{1}{\tilde{Z}_n} \exp(-nU_n(W))\alpha \otimes n(dx_1, \cdots, dx_n), \quad (2.6) \]

where \( \tilde{Z}_n := \frac{Z_n}{C^n} \). Without interaction (i.e. \( W = 0 \)), \( P_n = \alpha \otimes n \), i.e. the \( n \) particles are free, identically distributed as \( \alpha \).

Given a measure \( \mu \in \mathcal{M}_1(S) \), the relative entropy of \( \nu \) with respect to \( \mu \) is defined by

\[ H(\nu | \mu) = \left\{ \begin{array}{ll} \int_S \frac{d\nu}{d\mu}(x) \log \frac{d\nu}{d\mu}(x) \mu(dx), & \text{if } \nu \ll \mu; \\ +\infty, & \text{otherwise}. \end{array} \right. \quad (2.7) \]

For any measure \( \nu \in \mathcal{M}_1(S) \) such that \( W^- := (-W) \vee 0 \in L^1(\nu \otimes 2) \), define

\[ W(\nu) := \int_{S \times S} W(x, y) d\nu(x) d\nu(y) \in (-\infty, +\infty], \quad (2.8) \]

and

\[ H_W(\nu) := \left\{ \begin{array}{ll} H(\nu|\alpha) + W(\nu), & \text{if } H(\nu|\alpha) < +\infty, \text{ and } W^- \in L^1(\nu \otimes 2); \\ +\infty, & \text{otherwise}. \end{array} \right. \quad (2.9) \]

We make the following assumption on the interaction potential \( W \):

(A1). The function \( W : S \times S \to (-\infty, +\infty] \) is measurable, symmetric; its positive part \( W^+ \) satisfies

\[ H(\nu|\alpha) + \int_{S \times S} W^+(x, y) d\nu(x) d\nu(y) < +\infty \quad \text{for some } \nu \in \mathcal{M}_1(S) \quad (2.10) \]

and its negative part \( W^- \) satisfies the following strong exponential integrability condition

\[ \mathbb{E}[\exp(\lambda W^-(X, Y))] < +\infty, \forall \lambda > 0 \quad (2.11) \]

where \( X \) and \( Y \) are two independent random variables with the same probability distribution \( \alpha \) defined in (2.5).

Remark 2.4. The simplest condition for (2.10) is: there is some closed subset \( F \) with \( \alpha(F) > 0 \) such that \( 1_{F^2}W^+ \) is \( \alpha \otimes 2 \)-integrable. In fact one can take \( \nu = h\alpha \) with the density \( h : S \to \mathbb{R}^+ \) which is bounded with support contained in \( F \).
Under (2.11), if $H(\nu|\alpha) < +\infty$, then $W^- \in L^1(\nu^{\otimes 2})$. In fact, for any $\lambda > 0$, by Donsker-Varadhan’s variational formula,

$$
\lambda \iint_{S \times S} W^-(x, y) d\nu(x) \nu(y) \leq H(\nu^{\otimes 2}|\alpha^{\otimes 2}) + \log \iint_{S \times S} e^{\lambda W^-(x, y)} d\alpha(x) \alpha(y)
$$

$$
= 2H(\nu|\alpha) + \log \iint_{S \times S} e^{\lambda W^-(x, y)} d\alpha(x) \alpha(y) < +\infty.
$$

When $S = \mathbb{R}^d$, some interesting examples satisfying (A1) are

1) $W(x, y) = \frac{b}{|x-y|^\beta}$ with $\beta < d$, $b > 0$ (Coulomb potential if $\beta = 1$), $\forall x, y \in \mathbb{R}^d$;

2) $W(x, y) = -b \log |x-y|$ (the potential appearing in random matrices) and $\int |x|^p d\alpha < +\infty$ for all $p > 1$, $\forall x, y \in \mathbb{R}^d$ (the result in [9] does not apply for this example).

Now we present our main result, whose proof is given in the next section.

**Theorem 2.5.** Assume $\int_S \exp(-V(x)) m(dx) < +\infty$, and the assumption (A1). Then $H_W$ is inf-compact on $\mathcal{M}_1(S)$, and

$$
-\infty < \inf_{\mu \in \mathcal{M}_1(S)} H_W(\mu) < +\infty
$$

(2.12)

and the sequence of probability measures $\{P_n(L_n)\}_{n \geq 2}$ satisfies the LDP on $\mathcal{M}_1(S)$ equipped with the weak convergence topology, with speed $n$ and the good rate function

$$
I_W(\nu) := H_W(\nu) - \inf_{\mu \in \mathcal{M}_1(S)} H_W(\mu), \nu \in \mathcal{M}_1(S).
$$

(2.13)

Moreover, for any $\nu$ such that $\nu \ll \alpha$ and $W^- \in L^1(\nu^{\otimes 2})$,

$$
I_W(\nu) = \lim_{n \to +\infty} \frac{1}{n} H(\nu^{\otimes n}|P_n).
$$

(2.14)

**Remark 2.6.** (1) Notice that we have removed the lower semi-continuity and lower boundness conditions on $W$ in [9]. Our result generalizes the known results in Léonard [12], Dupuis et al. [9].

(2) The LDP result in Theorem 2.5 can be generalized to the case of many bodies interactions, which will be presented in the fourth section.

**Remark 2.7.** To see the main difficulty in this LDP, let us proceed naively: when $W(x, y)$ is bounded and continuous, the $U$-statistics

$$
U_n(W) = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} W(x_i, x_j)
$$

is very close to $\iint W(x, y) L_n(dx) L_n(dy)$ which is continuous in $L_n$ in the weak convergence topology, because one can approach $W(x, y)$ by a sequence of uniformly bounded functions of type $\sum_k c_k f_k(x) g_k(y)$ with $f_k, g_k \in C_b(S)$, uniformly over compacts of $S^2$. So in that case the LDP follows from the LDP of $L_n$ under $\alpha^{\otimes \mathbb{N}}$ (Sanov theorem) and Varadhan’s Laplace lemma. When $W$ is bounded and only measurable, we do not know whether the functional $\nu \to \iint W(x, y) \nu(dx) \nu(dy)$ is continuous in the (non-metrizable) $\tau$-topology, whereas the Sanov theorem still holds in the $\tau$-topology. The continuity of the last functional is a basic assumption in Léonard [12].
Remark 2.8. Since \( H_W \) is inf-compact by Theorem 2.5, there is at least one minimizer. From the view of statistical physics, \( H_W \) is an entropy or free energy associated to the nonlinear McKean-Vlasov equation. The uniqueness of the minimizer means that there is no phase transition for the particles system.

For the uniqueness of the minimizer, it is sufficient to prove that \( H_W \) is strictly convex along some path \((\nu_t)_{t\in[0,1]}\) connecting \( \nu_0 \) to \( \nu_1 \), for any two probability measures \( \nu_0, \nu_1 \). Let \( \nu_* \) be a minimizer of \( H_W \) satisfying that \( \nu_* \ll \alpha \). Then the critical equation for the minimizer is

\[
\nu_*(dx) = e^{-2\pi \nu_* W(x) - V(x)} m(dx)/C, \tag{2.15}
\]

where \( \pi_{\nu_*} W(x) := \int_S W(x, y) \nu_*(dy) \), and \( C := \int_S e^{-2\pi \nu_* W(x) - V(x)} m(dx) \) is the normalized constant.

The critical equation above is equivalent to the following stationary equation of the nonlinear McKean-Vlasov equation:

\[
\Delta \nu_* + \nabla \cdot (\nu_* \nabla V) + \nabla \cdot ((\nabla W \ast \nu_*) \cdot \nu_*) = 0, \tag{2.16}
\]

where the symbols \( \nabla \) and \( \nabla \cdot \) denote the gradient operator and divergence operator respectively. For the uniqueness of the solution of (2.16), the reader is referred to McCann \cite{14} and Carrilo et al. \cite{5}. These authors showed that \( H_W \) is strictly displacement convex (i.e. along the \( W_2 \)-geodesic) under various sufficient conditions on the confinement potential \( V \) and interaction potential \( W \).

We also consider \( M_1(S) \) equipped with the Wasserstein topology, which is much stronger than the weak convergence topology. The \( L^p \)-Wasserstein distance \((p \geq 1)\) with respect to the metric \( \rho \), between any two probability measures \( \mu \) and \( \nu \) on \( S \), is defined by

\[
W_p(\mu, \nu) = \inf_{\xi \in \Pi(\mu, \nu)} \left( \int_S \int_S \rho^p(x, y) \xi(dx, dy) \right)^{1/p}, \tag{2.17}
\]

where \( \Pi(\mu, \nu) \) is the set of all probability measures on \( S \times S \) with marginal distribution \( \mu \) and \( \nu \) respectively (say couplings of \( \mu \) and \( \nu \)).

The Wasserstein space of order \( p \) is defined as

\[
M^p_1(S) = \left\{ \mu \in M_1(S); \int_S \rho^p(x, x_0) \mu(dx) < +\infty \right\},
\]

where \( x_0 \) is some fixed point of \( S \). It is known that \( W_p \) is a finite distance on \( M^p_1(S) \) and \((M^p_1(S), W_p)\) is a Polish space (see Villani \cite{16} \cite{17}).

Theorem 2.9. Assume \( \int_S \exp(-V(x)) m(dx) < +\infty \), and

\[
\int_S \exp\{\lambda \rho^p(x, x_0)\} \alpha(dx) < +\infty, \quad \forall \lambda > 0, \tag{2.18}
\]

for some (hence for any) \( x_0 \in S \). Under the assumption \( (A1) \), the sequence of probability measures \( \{P_n(L_n \in \cdot)\}_{n\geq2} \) satisfies the LDP on \((M^p_1(S), W_p)\) with speed \( n \) and the good rate function \( I_W \) defined in (2.13).
Remark 2.10. In [9], Dupuis et al. imposed the following non-explicit condition for the LDP result above when \( S = \mathbb{R}^d \): there exists a lower-semicontinuous function \( \phi : \mathbb{R}_+ \to \mathbb{R} \) with

\[
\lim_{s \to +\infty} \frac{\phi(s)}{s} = +\infty,
\]

such that for every \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d} \phi(|x|^p) \mu(dx) \leq \inf_{\xi \in \Pi(\mu, \mu)} \left\{ H(\xi|\alpha^{\otimes 2}) + \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x, y) \xi(dx, dy) \right\}.
\]

3. Proof of the main results

Let \( P_n^* \) be the measure by removing the normalizing constant \( \tilde{Z}_n \) from \( P_n \) presented in (2.6), i.e.,

\[
dP_n^*(x_1, \ldots, x_n) := \exp(-nU_n(W))\alpha^{\otimes n}(dx_1, \ldots, dx_n).
\]  (3.1)

To establish the LDP for \( L_n \) under \( P_n \), it suffices to establish the LDP for \( L_n \) under the non-probability \( P_n^* \). The proof of Theorem 2.5 will be divided into several steps.

3.1. LDP lower bound. First we present the law of large numbers of the \( U \)-statistics (see [11, Theorem 3.1]). Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables in a measurable space \( (S, \mathcal{B}(S)) \).

Lemma 3.1. [11, Koroljuk et Borovskich] Assume that

\[
\mathbb{E}|\Phi(X_1, X_2)| < +\infty,
\]  (3.2)

then

\[
U_n(\Phi) \to \mathbb{E}\Phi(X_1, X_2)
\]  (3.3)

as \( n \to +\infty \) with probability 1.

Proof. For the sake of completeness, we re-present the simple proof in [11].

Let \( \Pi_n \) be the set of all the permutations \((i_1, \ldots, i_n)\) on \( \{1, \ldots, n\} \) and \( \mathcal{B}_n \) the \( \sigma \)-algebra defined by

\[
\mathcal{B}_n := \sigma \left\{ B_n \times C_n | C_n \in \mathcal{B}(S^{n+1, +\infty}), B_n \in \mathcal{B}(S^n), \pi 1_{B_n} = 1_{B_n}, \forall \pi \in \Pi_n \right\}.
\]

The \( \sigma \)-algebra \( \mathcal{B}_n \) remains unchanged under any permutation in \( \Pi_n \), and \( \mathcal{B}_n \supseteq \mathcal{B}_{n+1} \) for every \( n \geq 1 \).

For every \( 1 \leq i < j \leq n \), by (3.2) we have

\[
\mathbb{E}[\Phi(X_i, X_j)|\mathcal{B}_n] = \mathbb{E}[\Phi(X_1, X_2)|\mathcal{B}_n],
\]

which yields

\[
U_n(\Phi) = \mathbb{E}[\Phi(X_1, X_2)|\mathcal{B}_n].
\]

According to the limit theorem for reversed martingales and the 0-1 law for \( \mathcal{B}_\infty = \bigcap_{n \geq 1} \mathcal{B}_n \),

\[
U_n(\Phi) \xrightarrow{n \to \infty} \mathbb{E}[\Phi(X_1, X_2)|\mathcal{B}_\infty] = \mathbb{E}\Phi(X_1, X_2).
\]

We have the following LDP lower bound for the empirical measure \( L_n \) under \( P_n^* \).
**Proposition 3.2.** Without any integrability condition on \(W\), \(\{P^n_n \{L_n \in \cdot \}\}_{n \geq 2}\) satisfies the following large deviation lower bound: for any open subset \(G \in \mathcal{B}(\mathcal{M}_1(S))\),

\[
l^*(G) := \liminf_{n \to +\infty} \frac{1}{n} \log P^n_n \{L_n \in G\} \geq -\inf \{H(\nu|\alpha) + \mathcal{W}(\nu)|\nu \in G, W \in L^1(\nu^{\otimes 2})\}. (3.4)
\]

Furthermore, we have

\[
\liminf_{n \to +\infty} \frac{1}{n} \log \tilde{Z}_n \geq -\inf_{\nu \in \mathcal{M}_1(S); W \in L^1(\nu^{\otimes 2})} \{\mathcal{W}(\nu) + H(\nu|\alpha)\}. (3.5)
\]

**Proof.** Since (3.5) is obtained just by taking \(G\) as \(\mathcal{M}_1(S)\) in (3.4), we only need to prove (3.4). For (3.4), it is enough to show that for any \(\nu \in G\) such that \(W \in L^1(\nu^{\otimes 2})\) and \(\nu|\alpha < +\infty\),

\[
l^*(G) \geq -(H(\nu|\alpha) + \mathcal{W}(\nu)).
\]

Let \(\mathcal{N}(\nu, \delta)\) be the neighborhood of \(\nu\) in \(\mathcal{M}_1(S)\) with radius \(\delta\) in the metric \(d_w\) such that \(\mathcal{N}(\nu, \delta) \subset G\). Denote by \(A_n\) the event \(\{x^n = (x_1, \cdots, x_n) \in S^n|L_n = L_n(x^n, \cdot) \in \mathcal{N}(\nu, \delta)\}\), then we have for any \(\varepsilon > 0\),

\[
P^n_n \{L_n \in \mathcal{N}(\nu, \delta)\} \geq \int_{A_n} \left(\frac{d\nu^{\otimes n}_{\nu|x}}{dP^n_{\nu|x}}(x_1, \cdots, x_n)\right)^{-1} d\nu^{\otimes n}(x_1, \cdots, x_n)
\]

\[
= \int_{A_n} \exp \left(-\sum_{i=1}^{n} \log \frac{d\nu}{d\alpha}(x_i)\right) \exp(-nU_n(W))\nu^{\otimes n}(dx_1, \cdots, dx_n),
\]

\[
\geq \nu^{\otimes n}(A_n \cap B_n \cap C_n) \exp \left(-n(H(\nu|\alpha) + \varepsilon) - n(U_n(W) + \varepsilon)\right)
\]

\[
= \nu^{\otimes n}(A_n \cap B_n \cap C_n) \exp \left(-n(H(\nu|\alpha) + U_n(W)) - 2n\varepsilon\right),
\]

where \(B_n\) and \(C_n\) are defined by

\[
B_n := \{(x_1, \cdots, x_n)|\frac{1}{n} \sum_{i=1}^{n} \log \frac{d\nu}{d\alpha}(x_i) \leq H(\nu|\alpha) + \varepsilon\},
\]

\[
C_n := \{(x_1, \cdots, x_n)|U_n(W) \leq \mathcal{W}(\nu) + \varepsilon\}.
\]

We claim that \(\lim_{n \to +\infty} \nu^{\otimes n}(A_n \cap B_n \cap C_n) = 1\). Indeed, by the LLNs, it is obvious that \(\nu^{\otimes n}(A_n) \to 1\) and \(\nu^{\otimes n}(B_n) \to 1\) as \(n \to +\infty\).

Again by the LLN of \(U\)-statistics in Lemma 3.1, we also have

\[
\lim_{n \to +\infty} \nu^{\otimes n}(C_n) = 1.
\]

With this claim in hand, we immediately get from (3.6) that

\[
l^*(G) \geq \liminf_{n \to +\infty} \frac{1}{n} \log P^n_n \{L_n \in \mathcal{N}(\nu, \delta)\} \geq -[H(\nu|\alpha) + \mathcal{W}(\nu)] - 2\varepsilon,
\]

where the desired result follows since \(\varepsilon > 0\) is arbitrary. \(\Box\)
3.2. Decoupling inequality of de la Peña and the key lemma. We state here the decoupling inequality of de la Peña [6, 1992], which will be used in the proof of our key lemma. For $n \geq k \geq 1$, we denote the set $\{(i_1, i_2, \ldots, i_k) \in \mathbb{N}^k \mid i_1, \ldots, i_k \text{ are different, } 1 \leq i_1, \ldots, i_k \leq n\}$ by $I^k_n$.

**Proposition 3.3.** [6] de la Peña] Let $\{X_i\}_{i \geq 1}$ be a family of i.i.d. random variables in a measurable space $(S, \mathcal{B}(S))$ and suppose that $(X_1^j, \ldots, X_n^j)_{j=1}^k$ are independent copies of $(X_1, \ldots, X_n)$. Let $\Psi$ be any convex increasing function on $[0, +\infty)$. Let $\Phi : S^k \to \mathbb{R}$ be a symmetric function of $k$ variables such that

$$\mathbb{E}[\Phi(X_1, \ldots, X_k)] < +\infty,$$

then

$$\mathbb{E}[\Phi(X_1, \ldots, X_k) \leq \mathbb{E}[\sum_{(i_1, i_2, \ldots, i_k) \in I^k_n} \Phi(X_{i_1}, \ldots, X_{i_k})] \leq \mathbb{E}[\sum_{(i_1, i_2, \ldots, i_k) \in I^k_n} \Phi(X_{i_1}^1, \ldots, X_{i_k}^k)],$$

where $C_2 = 8$ and $C_k = 2^k(k^k - 1)((k - 1)^{k-1} - 1) \times \cdots \times 3$ for $k > 2$.

The following lemma will be used for proving our key lemma.

**Lemma 3.4.** Let $1 \leq k \leq n$, and $\{X_i^j \mid 1 \leq i \leq n, 1 \leq j \leq k\}$ be independent random variables. For any $(i_1, \ldots, i_k) \in I^k_n$, let $\Phi_{i_1, \ldots, i_k} : S^k \to \mathbb{R}$ be a function of $k$ variables, then

$$\log \mathbb{E}\exp\left(\frac{(n-k)!}{n!} \sum_{(i_1, \ldots, i_k) \in I^k_n} \Phi_{i_1, \ldots, i_k}(X_{i_1}^1, \ldots, X_{i_k}^k) \right) \leq \frac{(n-k+1)!}{(n-k+1)!} \sum_{(i_1, \ldots, i_k) \in I^k_n} \log \mathbb{E}\exp\left(\frac{1}{n-k+1} \Phi_{i_1, \ldots, i_k}(X_{i_1}^1, \ldots, X_{i_k}^k) \right).$$

**Proof.** It is obvious that (3.10) becomes equality for $k = 1$. Next we prove this lemma by induction. Assume that (3.10) is valid for $1, \ldots, k-1$. Denote the left hand side of (3.10) by $B_k$ and write $\sum_{(i_1, \ldots, i_{k-1}) \in I^{k-1}_n} X_{i_1}^1, \ldots, X_{i_{k-1}}^k$ for simplicity. We have

$$B_k = \log \mathbb{E}X^k \left\{ \mathbb{E}\left[ \exp\left(\frac{(n-k+1)!}{n!} \sum_{I^{k-1}_n \backslash i_k} \sum_{i_1, \ldots, i_{k-1}} \frac{1}{n-k+1} \Phi_{i_1, \ldots, i_k}(X_{i_1}^1, \ldots, X_{i_k}^k) \right) \right] \right\}. $$

Given $X^k = (X_1^1, \ldots, X_n^k)$, let

$$\tilde{\Phi}_{i_1, \ldots, i_{k-1}} := \sum_{i_k : i_k \notin \{i_1, \ldots, i_{k-1}\}} \frac{1}{n-k+1} \Phi_{i_1, \ldots, i_k}(X_{i_1}^1, \ldots, X_{i_k}^k).$$

By the assumption of $(k-1)^{th}$ step, we get
\[ B_k \leq \log E^x \left\{ \exp \left( \frac{(n-k+2)!}{n!} \sum_{I_n^{k-1}} \log E[\exp(\frac{1}{n-k+2} \Phi_{i_1,\ldots,i_{k-1}}) | X^k] \right) \right\} \]

\[ = \log E^x \left\{ \exp \left( \frac{(n-k+1)!}{n!} \sum_{I_n^{k-1}} \log \left\{ E[\exp(\frac{1}{n-k+2} \Phi_{i_1,\ldots,i_{k-1}}) | X^k] \right\}^{n-k+2} \right) \right\} \]

\[ \leq \frac{(n-k+1)!}{n!} \sum_{I_n^{k-1}} \log E^x \left\{ \left[ E \exp \left( \frac{1}{n-k+2} \Phi_{i_1,\ldots,i_{k-1}} \right) | X^k \right]^{n-k+2} \right\} \]

where the last inequality follows by the convexity of \( X \rightarrow \log E e^X \). Now we deal with the logarithmic term in the last inequality above. Given \((i_1, \ldots, i_{k-1})\),

\[ E^x \left\{ [E \exp(\frac{1}{n-k+2} \Phi_{i_1,\ldots,i_{k-1}}) | X^k]^{n-k+2} \right\} \]

\[ = E^x \left\{ \left[ E \exp \left( \frac{1}{n-k+2} \Phi_{i_1,\ldots,i_{k-1}} (X_{i_1}^1, \ldots, X_{i_k}^{k-1}) | X^k \right) \right]^{n-k+2} \right\} \]

\[ \leq E^x \left\{ \left[ \prod_{i_k : i_k \not\in \{i_1, \ldots, i_{k-1}\}} E[\exp(\frac{1}{n-k+2} \Phi_{i_1,\ldots,i_{k-1}} (X_{i_1}^1, \ldots, X_{i_k}^{k-1}, X_{i_k}^k) | X^k) \right]^{n-k+2} \right\} \]

(by Hölder’s inequality)

\[ \leq E^x \left\{ \prod_{i_k : i_k \not\in \{i_1, \ldots, i_{k-1}\}} E[\exp(\frac{1}{n-k+1} \Phi_{i_1,\ldots,i_{k-1}} (X_{i_1}^1, \ldots, X_{i_k}^{k-1}, X_{i_k}^k) | X^k) \right\} \]

(by Jensen’s inequality)

\[ = \prod_{i_k : i_k \not\in \{i_1, \ldots, i_{k-1}\}} \exp(\frac{1}{n-k+1} \Phi_{i_1,\ldots,i_{k-1}} (X_{i_1}^1, \ldots, X_{i_k}^{k-1}, X_{i_k}^k) \right) \]

(by the independence of \( X_1^1, \ldots, X_n^k \))

(3.14)

Plugging (3.14) into (3.13), we get the desired inequality (3.10).

Let \( \{X_i\}_{i \geq 1} \) be a family of i.i.d. random variables of law \( \alpha \). Denote by \( \Lambda_n(\cdot; W^k) \) the logarithmic moment generating function associated with the \( U \)-statistics of order \( k \), i.e., for any \( n \geq k \geq 2 \) and \( \lambda > 0 \),

\[ \Lambda_n(\lambda; W^k) := \frac{1}{n} \log E[\exp(\lambda n U_n^k(W^k))] \]

(3.15)

where

\[ U_n^k(W^k) = \frac{(n-k)!}{n!} \sum_{(i_1, \ldots, i_k) \in I_n^k} W^k(X_{i_1}, \ldots, X_{i_k}) \]

Now we present our key lemma.
**Lemma 3.5.** If $\mathbb{E}|W^k(X_1, \cdots, X_k)| < +\infty$, then for any $n \geq k \geq 2$ and $\lambda > 0$,
\[
\Lambda_n(\lambda; W^k) \leq \frac{1}{k} \log \mathbb{E}[\exp(kC_k \lambda |W^k(X_1, \cdots, X_k)|)],
\] (3.16)
where $C_k$ is defined as in Proposition 3.3.

**Proof.** Let $(X_i^j, \cdots, X_n^j)_{j=1}^k$ be independent copies of $(X_1, \cdots, X_n)$. By Lemma 3.3 (the decoupling inequality of de la Pêna) and Lemma 3.4, taking $\Phi$ as $\Phi_{i_1, \cdots, i_k} \equiv W^k$ for any $(i_1, \cdots, i_k) \in I_n^{k}$, we get for any $\lambda > 0$,
\[
\Lambda_n(\lambda; W^k) = \frac{1}{n} \log \mathbb{E}[\exp(\lambda n (n-k)! \sum_{(i_1, \cdots, i_k) \in I_n^{k}} W^k(X_{i_1}, \cdots, X_{i_k}))]
\leq \frac{1}{n} \log \mathbb{E}[\exp\left(\frac{(n-k)!}{n!} \sum_{(i_1, \cdots, i_k) \in I_n^{k}} \lambda n C_k |W^k| (X_{i_1}^1, \cdots, X_{i_k}^k)\right)] \quad \text{(by (3.9))}
\leq \frac{1}{n} \log \mathbb{E}\left[\exp\left(\frac{\lambda n C_k}{n-k+1} |W^k| (X_{i_1}^1, \cdots, X_{i_k}^k)\right)\right] \quad \text{(by (3.10))}
= \frac{n-k+1}{n} \log \mathbb{E}\left[\exp\left(\frac{\lambda n C_k}{n-k+1} W^k\right)\right]
\leq \frac{1}{k} \log \mathbb{E}[\exp(kC_k \lambda |W^k|(X_1, \cdots, X_k))],
\] (3.17)
where the last inequality follows by Jensen’s inequality, for $\frac{n}{n-k+1} \leq k$ for all $n \geq k$. □

We have the following exponential approximation of the $U$-statistics.

**Lemma 3.6.** Assume that for any $\lambda > 0$,
\[
\mathbb{E}[\exp(\lambda |W^k|(X_1, \cdots, X_k))] < +\infty.
\] (3.18)
Then there exists a sequence of bounded continuous functions $\{W_m\}_{m \geq 1}$ such that for any $\delta > 0$,
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \frac{1}{n} \log \mathbb{P}\{|U^k_n(W^k) - U^k_n(W_m)| > \delta\} = -\infty.
\] (3.19)

**Proof.** For any function $W^k$ satisfying (3.18), there exists a sequence of bounded continuous functions $\{W_m\}_{m \in \mathbb{N}}$ such that for any $\lambda > 0$,
\[
\varepsilon(\lambda, m) := \log \mathbb{E}[\exp(\lambda |W^k - W_m|(X_1, \cdots, X_k))] \to 0
\] (3.20)
as $m \to +\infty$.

For any $\delta, \lambda > 0$, by Chebyshev’s inequality we have
\[
\mathbb{P}\{|U^k_n(W^k) - U^k_n(W_m)| > \delta\} \leq e^{-n\lambda \delta} \mathbb{E}[\exp(\lambda n U^k_n(|W^k - W_m|))].
\] (3.21)
Applying Lemma 3.5, we get
\[
\frac{1}{n} \log \mathbb{P}\{U_n(W) - U_n(W_m) > \delta\} \leq -\lambda \delta + \frac{1}{k} \log \mathbb{E}[\exp(kC_k \lambda |W - W_m|(X_1, \cdots, X_k))].
\] (3.22)
Let $m \to +\infty$ and by (3.20), we get the desired result (3.19) since $\lambda > 0$ is arbitrary. □
3.3. LDP upper bound. We deal with the large deviation upper bound in the case of $k = 2$ in this subsection. The proof is divided into three steps in terms of $W$: bounded, lower bounded and then unbounded.

3.3.1. Bounded case.

**Lemma 3.7.** Assume that $W$ is bounded and measurable, then $\{(L_n, U_n(W))\}_{n \geq 2}$ satisfies the LDP under $\alpha \otimes \nu$ in the product space $\mathcal{M}_1(S) \times \mathbb{R}$, with good rate function $I$ defined by

$$I(\nu, z) := \begin{cases} H(\nu|\alpha), & \text{if } z = W(\nu); \\ +\infty, & \text{otherwise} \end{cases} \quad (3.23)$$

for any $(\nu, z) \in \mathcal{M}_1(S) \times \mathbb{R}$.

**Proof.** If $W$ is bounded and continuous, then $\nu \mapsto \iint_{S^2} W(x, y) d\nu(x) d\nu(y)$ is continuous in the weak convergence topology. In fact let $\nu_n \to \nu$ in $(\mathcal{M}_1(S), d_w)$. By Skorokhod’s lemma, one can construct a sequence of $S$-valued random variables $X_n$ of law $\nu$, converging a.s. to $X$ of law $\nu$. Let $(Y_n, n \geq 0; Y)$ be an independent copy of $(X_n, n \geq 0; X)$. Then $Y_n \to Y$, a.s. too. Thus $(X_n, Y_n) \to (X, Y)$, a.s., which shows that $\nu_n \otimes \nu_n \to \nu \otimes \nu$ weakly on $S^2$.

Furthermore,

$$|U_n(W) - \iint_{S^2} W(x, y) L_n(dx) L_n(dy)| \leq \frac{1}{n^2} \sum_{i=1}^n W(X_i, X_i) \leq \frac{1}{n} \|W\|_{\infty} \to 0,$$

as $n \to +\infty$, where $L_n(dx) = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}(dx)$. By the Sanov theorem and the contraction principle (see [10]), we obtain the desired LDP with good rate function $I$.

When $W$ is only bounded and measurable, it is unknown whether $\nu \mapsto \iint_{S^2} W(x, y) d\nu(x) d\nu(y)$ is continuous in the $\tau$-topology. That is why we do the approximation (3.19). Let $\{W_m\}_{m \geq 1}$ be the sequence of bounded continuous functions as in Lemma 3.6 such that for any $\lambda > 0$,

$$\varepsilon(\lambda, m) := \log \iint_{S^2} e^{\lambda |W_m - W|} d\alpha \otimes 2 \to 0, \text{ as } m \to +\infty. \quad (3.24)$$

Let

$$f_m(\nu) := \left(\nu, \iint_{S^2} W_m(x, y) d\nu \otimes 2\right), \quad f(\nu) := \left(\nu, \iint_{S^2} W(x, y) d\nu \otimes 2\right),$$

and $d(f_m(\nu), f(\nu)) = \left|\iint_{S^2} (W_m - W) d\nu \otimes 2\right|$ where

$$d((\nu_1, z_1), (\nu_2, z_2)) := d_w(\nu_1, \nu_2) + |z_1 - z_2|$$
is the metric on the product space $\mathcal{M}_1((S)) \times \mathbb{R}$. For any $\lambda > 0$, $L > 0$ and $\nu$ with $H(\nu|\alpha) \leq L$, we have by Donsker-Varadhan’s variational formula,

$$
\left| \int_{S \times S} (W_m - W) d\nu^{\otimes 2} \right| \leq \int_{S \times S} |W_m - W| d\nu^{\otimes 2}
\leq \frac{1}{\lambda} \left( H(\nu^{\otimes 2}|\alpha^{\otimes 2}) + \log \int_{S \times S} e^{\lambda |W_m - W|} d\alpha^{\otimes 2} \right) \quad (3.25)
\leq \frac{1}{\lambda}(2L + \varepsilon(\lambda, m)).
$$

Since $\lambda > 0$ is arbitrary, we get by (3.24)

$$
\sup_{\nu:H(\nu|\alpha) \leq L} d(f_m(\nu), f(\nu)) \to 0, \text{ as } m \to +\infty, \forall L > 0. \quad (3.26)
$$

Combining (3.19) and (3.26), we complete the proof by using the exponential approximation result of LDP (see [19, Proposition 4.1, Chapter 1]).

**Remark 3.8.** As seen from the proof above, the LDP in Lemma 3.7 still holds when $W$ satisfies the strong exponential integrability condition

$$
\mathbb{E} \exp(\lambda|W(X_1, X_2)|) < +\infty, \forall \lambda > 0.
$$

3.3.2. Lower bounded case.

**Lemma 3.9.** Assume that $W$ is lower bounded and measurable. $\{P_n\{L_n \in \cdot\}\}_{n \geq 2}$ satisfies the following large deviation upper bound: for any closed subset $C \in \mathcal{B}(\mathcal{M}_1(S))$,

$$
\limsup_{n \to +\infty} \frac{1}{n} \log P_n\{L_n \in C\} \leq - \inf_{\nu \in C} H_{W}(\nu), \quad (3.27)
$$

where $H_{W}(\nu) := H(\nu|\alpha) + \mathcal{W}(\nu)$ is inf-compact.

**Proof.** If $W$ is bounded, then by Lemma 3.7 and Gibbs principle, $\{P_n\{L_n \in \cdot\}\}_{n \geq 2}$ satisfies the LDP with good rate function $I_W$.

If $W$ is lower bounded, letting $W^L := W \wedge L$ for any given positive constant $L$, then $H_{W^L}$ is inf-compact. Therefore $H_W$ is also inf-compact since $H_{W^L} \uparrow H_W$ as $L \to +\infty$.

For any closed subset $C \in \mathcal{B}(\mathcal{M}_1(S))$, we have

$$
P_n\{L_n \in C\} = \int_{S^n} 1\{L_n \in C\} e^{-nU_n(W)} d\alpha^{\otimes n}
\leq \int_{S^n} 1\{L_n \in C\} e^{-nU_n(W^L)} d\alpha^{\otimes n}
\leq \exp \left( -n \inf_{\nu \in C} \left\{ \int_{S \times S} W^L(x, y) d\nu(x) d\nu(y) + H(\nu|\alpha) \right\} + o(n) \right),
$$

where the second inequality follows from Lemma 3.7 and Varadhan’s Laplace principle. Hence we get

$$
\limsup_{n \to +\infty} \frac{1}{n} \log P_n\{L_n \in C\} \leq - \inf_{\nu \in C} H_{W^L}(\nu). \quad (3.29)
$$
Since $H_{W\cdot}$ is inf-compact, we have
\[
\inf_{\nu \in C} H_{W\cdot}(\nu) \uparrow \inf_{\nu \in C} H_W(\nu), \text{ as } L \to +\infty.
\]

Letting $L \to +\infty$ in (3.29), we get the large deviation upper bound (3.27). \hfill \square

### 3.3.3. General unbounded case.

**Lemma 3.10.** Assume the assumption (A1). Then for any closed subset $C \in \mathcal{B}(M_1(S))$,
\[
\limsup_{n \to +\infty} \frac{1}{n} \log P_n^* \{ L_n \in C \} \leq -\inf_{\nu \in C} H_W(\nu),
\]
where $H_W$ is inf-compact. Furthermore, we have
\[
\limsup_{n \to +\infty} \frac{1}{n} \log \tilde{Z}_n \leq -\inf_{\nu \in M_1(S)} H_W(\nu).
\]

**Proof.** We first prove that $H_W$ is inf-compact. Let $W_L := W \vee (-L)$ for any given positive constant $L$, then as in Lemma 3.10, under the assumption (A1), we have for any $\lambda > 0$,
\[
\varepsilon(\lambda, L) := \log \mathbb{E}[\exp(\lambda(W_L - W)(X, Y))] = 0, \text{ as } L \to +\infty.
\]

For any $K > 0$, $\{ \nu|H_{W_L}(\nu) \leq K \}$ is compact by Lemma 3.9. Since
\[
\int \int W^-(x, y) d\nu(x) d\nu(y) \leq \frac{2}{\lambda} H(\nu|\alpha) + \frac{1}{\lambda} \log \mathbb{E} e^{\lambda W^-(x, y)}
\]
as noted in Remark 2.4, we have for any $\lambda > 2$,
\[
H_W(\nu) \geq H(\nu|\alpha) - \int \int W^-(x, y) d\nu(x) d\nu(y) \geq \left( 1 - \frac{2}{\lambda} \right) H(\nu|\alpha) - C_\lambda
\]
where $C_\lambda := \frac{1}{\lambda} \log \mathbb{E} e^{\lambda W^-(x, y)}$. Consequently
\[
\{ \nu|H_{W_L}(\nu) \leq K \} \subset \{ \nu|H_W(\nu) \leq K \} \subset \{ \nu|H(\nu|\alpha) \leq \frac{\lambda}{\lambda - 2}(K + C_\lambda) \}.
\]

For $\nu$ with $H(\nu|\alpha) \leq \frac{\lambda}{\lambda - 2}(K + C_\lambda)$, we have by (3.25)
\[
|H_{W_L}(\nu) - H_W(\nu)| \leq \int \int_{S \times S} |W_L - W| d\nu^\otimes 2 \leq \frac{1}{\lambda} \left( \frac{2\lambda}{\lambda - 2}(K + C_\lambda) + \varepsilon(\lambda, L) \right).
\]

Therefore $H_{W_L}$ converges to $H_W$ uniformly on the compact set $\{ \nu|H(\nu|\alpha) \leq \frac{\lambda}{\lambda - 2}(K + C_\lambda) \}$ as $L \to +\infty$, which implies that $\{ \nu|H_W(\nu) \leq K \}$ is compact.
For any closed subset $C \in \mathcal{B}(\mathcal{M}_1(S))$, we have by Hölder’s inequality

$$P^*_n \{ L_n \in C \}$$

$$= \int_{S^n} 1_{\{ L_n \in C \}} \exp\{-nU_n(W_L) + nU_n(W_L - W)\} d\alpha^n$$

$$\leq \left( \int_{S^n} 1_{\{ L_n \in C \}} \exp(nU_n(W_L - W)) d\alpha^n \right)^{\frac{1}{p}} \left( \int_{S^n} 1_{\{ L_n \in C \}} \exp(-nqU_n(W_L)) d\alpha^n \right)^{\frac{1}{q}}$$

$$= R_{1,n} \times R_{2,n}$$

where $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\limsup_{n \to +\infty} \frac{1}{n} P^*_n \{ L_n \in C \} \leq \limsup_{n \to +\infty} \frac{1}{n} \log R_{1,n} + \limsup_{n \to +\infty} \frac{1}{n} \log R_{2,n}. \quad (3.34)$$

Now we estimate these two items in the right hand of (3.34) separately.

By Lemma 3.9, we get for any $n \geq 2$ and $p > 1$,

$$\frac{1}{n} \log R_{1,n} \leq \frac{1}{pn} \log \int_{S^n} \exp(nU_n(W_L - W)) d\alpha^n$$

$$\leq \frac{1}{2p} \log E[\exp(16p(W_L - W)(X, Y))] \to 0, \quad \text{as } L \to +\infty. \quad (3.35)$$

By Lemma 3.9 we have

$$\limsup_{n \to +\infty} \frac{1}{n} \log R_{2,n} \leq -\frac{1}{q} \inf_{\nu \in C} \left\{ \int_{S \times S} qW_L(x, y) d\nu(x) d\nu(y) + H(\nu|\alpha) \right\}$$

$$= -\inf_{\nu \in C} \left\{ \int_{S \times S} W_L(x, y) d\nu(x) d\nu(y) + \frac{1}{q} H(\nu|\alpha) \right\}. \quad (3.36)$$

Since $W = \lim_{L \to +\infty} W_L = \inf_{L \geq 0} W_L$, then

$$\lim_{L \to +\infty} \limsup_{n \to +\infty} \frac{1}{n} \log R_{2,n} \leq -\inf_{\nu \in C} \left\{ \int_{S \times S} W(x, y) d\nu(x) d\nu(y) + \frac{1}{q} H(\nu|\alpha) \right\}$$

$$:= -\inf_{\nu \in C} H_{W,q}(\nu). \quad (3.37)$$

It can be proved as before that $H_{W,q}$ is inf-compact for any $q > 1$. Since $H_{W,q}$ increases to $H_W$ as $q \downarrow 1$, we get

$$\lim_{q \to 1} \lim_{L \to +\infty} \limsup_{n \to +\infty} \frac{1}{n} \log R_{2,n} \leq -\inf_{\nu \in C} H_W(\nu). \quad (3.38)$$

Combining this with (3.34) and (3.33), we obtain (3.30). Finally (3.31) follows from (3.30) by taking $C = \mathcal{M}_1(S)$. □
3.4. Proof of Theorem 2.5

Proof of Theorem 2.5. (1) At first $H_W : M_1(S) \to (-\infty, +\infty]$ is inf-compact by Lemma 3.10. Hence its infimum is attained, then $\inf_{M_1(S)} H_W > -\infty$, and it is $< +\infty$ by condition (2.10) in (A1).

(2) By the lower bound (3.5) in Proposition 3.2 and the upper bound (3.31) in Lemma 3.10 and the fact that $H_W(\nu) = +\infty$ once if $W \notin L^1(\nu \otimes \nu)$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \tilde{Z}_n = - \inf_{\nu \in M_1(S)} H_W(\nu).$$

(3.39)

So the LDP of $P_n(L_n \in \cdot)$ follows from the lower bound in Proposition 3.2 and the upper bound in Lemma 3.10.

(3) Finally for any $\nu$ such that $\nu \ll \alpha$ and $W^- \in L^1(\nu \otimes \nu)$, we have

$$\frac{1}{n} H(\nu \otimes \nu | P_n) = \frac{1}{n} E_{\nu \otimes \nu} \left( \log \frac{d\nu \otimes \nu}{d\alpha \otimes \alpha} + \frac{1}{n-1} \sum_{1 \leq i,j \leq n; i \neq j} W(x_i, x_j) + \log \tilde{Z}_n \right)$$

$$= H(\nu | \alpha) + \int_{S \times S} W(x, y) d\nu(x) d\nu(y) + \frac{1}{n} \log \tilde{Z}_n$$

which yields (2.14) by (3.39). \qed

3.5. Proof of Theorem 2.9. We first present the result of Sanov’s theorem in the Wasserstein distance by the second author et. al. [18].

Proposition 3.11. Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a Polish space $(S, \rho)$, of common law $\alpha$, then $\{P(L_n \in \cdot)\}_{n \geq 1}$ satisfies the LDP on $(M^p_1(S), W_p)$ with speed $n$ and the good rate function $H(\cdot | \alpha)$, if and only if

$$\int_S \exp\{\lambda \rho^p(x, x_0)\} \alpha(dx) < +\infty, \quad \forall \lambda > 0$$

for some (hence for any) $x_0 \in S$.

Proof of Theorem 2.9. Since we have established the LDP of the empirical measure $L_n$ under $P_n$ on $M_1(S)$ equipped with the weak convergence topology, it is sufficient to prove the exponential tightness of $\{P_n(L_n \in \cdot)\}_{n \geq 2}$ in $(M^p_1(S), W_p)$. Under the exponential integrability condition (2.18), by Proposition 3.11, the LDP holds for $L_n$ under $\alpha \otimes \alpha$ with respect to the Wasserstein topology. Thus we have for any $L > 0$, there exists a compact subset $K_L \subset M_1^p(S)$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} \log \alpha \otimes \alpha \{L_n \notin K_L\} \leq -L.$$ (3.40)
For any fixed \( a, b \in (1, +\infty) \) with \( \frac{1}{a} + \frac{1}{b} = 1 \), we have by Hölder’s inequality

\[
P_n\{L_n \notin K_L\} = \frac{1}{Z_n} \int 1_{\{L_n \notin K_L\}} \exp(-nU_n(W)) d\alpha^n \\
\leq \frac{1}{Z_n} \left[ \alpha^n \{L_n \notin K_L\} \right]^{1/a} \times \left( \int \exp(-bnU_n(W)) d\alpha^n \right)^{1/b},
\]

(3.41)

Hence by (3.39) and (3.40), we get

\[
\limsup_{n \to +\infty} \frac{1}{n} \log P_n\{L_n \notin K_L\} \\
\leq \frac{1}{a} \limsup_{n \to +\infty} \frac{1}{n} \log \alpha^n \{L_n \notin K_L\} - \limsup_{n \to +\infty} \frac{1}{n} \log Z_n + \frac{1}{b} \limsup_{n \to +\infty} \frac{1}{n} \log \int \exp(-nU_n(bW)) d\alpha^n \\
\leq -\frac{1}{a} L + \inf_{\nu \in M_1(S)} H_W(\nu) - \frac{1}{b} \inf_{\nu \in M_1(S)} H_{bW}(\nu),
\]

(3.42)

which implies the exponential tightness since \( \inf_{\nu \in M_1(S)} H_W(\nu) \) and \( \inf_{\nu \in M_1(S)} H_{bW}(\nu) \) are finite constants by (2.12) in Theorem 2.5.

4. Multiple bodies interaction

In this section, we consider the case that there are more than two particles interacting with each other in the mean-field interacting particle system. Let \( 2 \leq k \leq n \), and for any \( l, 2 \leq l \leq k \), \( W^l : S^l \to \mathbb{R} \) is measurable and symmetric, i.e., \( W^l(x_1, \ldots, x_l) = W^l(x_{i_1}, \ldots, x_{i_l}) \) for any permutation \((i_1, \ldots, i_l)\) of \((1, \ldots, l)\). Recall that

\[
I_n^l := \{(i_1, \ldots, i_l) \in \mathbb{N}^l | i_1, \ldots, i_l \text{ are different, } 1 \leq i_1, \ldots, i_l \leq n\}.
\]

Denote \(|I_n^l|\) by the number of elements in the set \(I_n^l\). The mean field Hamiltonian or energy functional \(H_n^k : S^n \to (-\infty, +\infty)\) corresponding to this configuration is given by

\[
H_n^k(x_1, \ldots, x_n) := \sum_{i=1}^n V(x_i) + n \sum_{l=2}^k U_n^l(W^l) \\
= \sum_{i=1}^n V(x_i) + \sum_{l=2}^k \frac{(n-l)!}{(n-1)!} \sum_{(i_1, \ldots, i_l) \in I_n^l} W^l(x_{i_1}, \ldots, x_{i_l}).
\]

(4.1)

The mean field Gibbs probability measure \(P_n^k\) on \(S^n\) is defined by

\[
dP_n^k(x_1, \ldots, x_n) := \frac{1}{Z_n^k} \exp(-H_n^k(x_1, \ldots, x_n)) m(dx_1) \cdots m(dx_n) \\
= \frac{1}{Z_n^k} \exp(-n \sum_{l=2}^k U_n^l(W^l)) \alpha(dx_1) \cdots \alpha(dx_n),
\]

(4.2)

where \(Z_n^k\) is the normalizing constant

\[
Z_n^k := \int_S \cdots \int_S \exp(-H_n^k(x_1, \ldots, x_n)) m(dx_1) \cdots m(dx_n),
\]

(4.3)
and $\tilde{Z}_n = \frac{Z_n}{C}$, $C = \int_S \exp(-V(x))m(dx)$.

For any measure $\nu \in \mathcal{M}_1(S)$ such that $W^{l,-} := (-W^l) \vee 0 \in L^1(\nu^{\otimes l})$, $\forall 2 \leq l \leq k$, we define

$$W^l(\nu) := \int_{S^l} W^l(x_1, \ldots, x_l) d\nu(x_1) \cdots d\nu(x_l) \in (-\infty, +\infty],$$

and

$$H_k(\nu) := \begin{cases} H(\nu|\alpha) + \sum_{l=2}^k W^l(\nu), & \text{if } H(\nu|\alpha) < +\infty, \text{ and } W^{l,+} \in L^1(\nu^{\otimes 2}), \forall 2 \leq l \leq k; \\ +\infty, & \text{otherwise}. \end{cases}$$

We make the following assumption on the interaction potential $W^k$:

(A2). For each $2 \leq l \leq k$, the function $W^l : S^l \rightarrow (-\infty, +\infty]$ is symmetric, measurable; its positive part $W^{l,+}$ satisfies

$$H(\nu|\alpha) + \int_{S^l} W^{l,+}(x_1, \ldots, x_l) d\nu(x_1) \cdots d\nu(x_l) < +\infty \text{ for some } \nu \in \mathcal{M}_1(S)$$

and its negative part $W^{l,-}$ satisfies the following strong exponential integrability condition

$$\mathbb{E}[\exp(\lambda W^{l,-}(X_1, \ldots, X_l))] < +\infty, \forall \lambda > 0$$

where $X_1, \ldots, X_l$ are i.i.d. random variables of the common law $\alpha$ defined in (2.5).

**Theorem 4.1.** Assume $\int_S \exp(-V(x))m(dx) < +\infty$, and the assumption (A2). Then

$$-\infty < \inf_{\mu \in \mathcal{M}_1(S)} H_k(\mu) < +\infty$$

and the sequence of probability measures $\{P_n^k(L_n \in \cdot)\}_{n \geq k}$ satisfies the LDP on $(\mathcal{M}_1(S), d_w)$ with speed $n$ and the good rate function

$$I_k(\nu) := H_k(\nu) - \inf_{\mu \in \mathcal{M}_1(S)} H_k(\mu), \nu \in \mathcal{M}_1(S).$$

Moreover, for any $\nu$ such that $\nu \ll \alpha$ and $W^{l,-} \in L^1(\nu^{\otimes l})$, $\forall l \leq k$, we have

$$I_k(\nu) = \lim_{n \rightarrow +\infty} \frac{1}{n} H(\nu^{\otimes n} | P_n^k).$$

Furthermore, if

$$\log \int_S \exp(\lambda \rho^k(x,0)) \alpha(dx) < +\infty, \forall \lambda > 0,$$

then the sequence of probability measures $\{P_n^k(L_n \in \cdot)\}_{n \geq k}$ satisfies the LDP on $(\mathcal{M}_1(S), W_p)$ with speed $n$ and the good rate function $I_k$.

Proceeded as in Section 3, the large deviation lower bound holds by the LLN of $U$-statistics of order $k$ (see [11]); for the large deviation upper bound, we use the key Lemma 3.5 of multivariate case and then repeat the proof in Section 3.3. We omit the proof (left to the reader).

In the end of this paper, we give an LDP result for the $U$-statistics.
Theorem 4.2. Assume (A2). Let $m \geq 1$ and $(B, \| \cdot \|)$ be a separable Banach space. Assume that $F : S^m \to (B, \| \cdot \|)$ is a symmetric function of $m$ variables and satisfy the strong exponential integrability condition
\[
\int_{S^m} \exp(\lambda \| F(x_1, \cdots, x_m) \|) \alpha(dx_1) \cdots \alpha(dx_m) < +\infty, \forall \lambda > 0
\] (4.11)
then $\{P^k_n(U^m_n(F) \in \cdot)\}_{n \geq k}^{\infty}$ satisfies the LDP on $(B, \| \cdot \|)$ with speed $n$ and the rate function given by
\[
I_F(z) = \inf \left\{ I_k(\nu)|I_k(\nu) < +\infty, \int_{S^m} F(x_1, \cdots, x_m) \nu(dx_1) \cdots \nu(dx_m) = z \right\}. \quad (4.12)
\]
Proof. Step 1: $F$ is bounded and continuous. Let $\| F \|_\infty = \sup_{(x_1, \cdots, x_m) \in S^m} \| F(x_1, \cdots, x_m) \|$. Since
\[
\| U^m_n(F) - \int_{S^m} F(x_1, \cdots, x_m) dL_n(x_1) \cdots dL_n(x_m) \| \leq 2 \left( 1 - \frac{|I^m_n|}{n^m} \right) \| F \|_\infty \quad (4.13)
\]
and
\[
\nu \to \int_{S^m} F(x_1, \cdots, x_m) \nu(dx_1) \cdots \nu(dx_m)
\]
is continuous from $(\mathcal{M}_1(S), d_w)$ to $(B, \| \cdot \|)$, we have by the contraction principle that $\{P^k_n(U^m_n(F) \in \cdot)\}_{n \geq k}^{\infty}$ satisfies the LDP on $(B, \| \cdot \|)$ with speed $n$ and the rate function $I_F$.

Step 2: General case. By the strong exponential integrability condition (4.11), we can find a sequence of bounded and continuous mappings $F_N : S^m \to B$ such that for every $\lambda > 0$,
\[
\varepsilon(\lambda, N) = \log \int_{S^m} e^{\lambda \| F - F_N \|} \alpha(dx_1) \cdots \alpha(dx_m) \to 0 \text{ as } N \to +\infty. \quad (4.14)
\]
By using Markov inequality and Hölder’s inequality(twice), we have for any $\lambda, \delta > 0$,
\[
P^k_n(\| U^m_n(F) - U^m_n(F_N) \| > \delta) \\
\leq P^k_n(\| F - F_N \| > \delta) \\
\leq \frac{1}{Z^m_k} e^{-\lambda \delta} \int_{S^m} e^{\lambda n U^m_n(\| F - F_N \|)} e^{-\sum_{i=2}^{k} n U^i_{\alpha}(W^j)} d\alpha \otimes^n \\
\leq \frac{1}{Z^m_k} e^{-\lambda \delta} \left( \int_{S^m} e^{a \lambda n U^m_n(\| F - F_N \|)} d\alpha \otimes^n \right)^{\frac{1}{a}} \times \left( \int_{S^m} e^{-\sum_{i=2}^{k} n U^i_{\alpha}(W^j)} d\alpha \otimes^n \right)^{\frac{1}{b}} \quad (4.15)
\]
where $a, b \in (1, +\infty)$ with $\frac{1}{a} + \frac{1}{b} = 1$. 


Hence we get
\[
\frac{1}{n} \log P_n^k (\| U_m^n (F) - U_n^m (F_N) \| > \delta) \\
\leq -\lambda \delta - \frac{1}{n} \log Z_n^k + \frac{1}{an} \int_{S^n} e^{a\lambda n U_m^n (\| F - F_N \|)} d\alpha \otimes^n + \frac{1}{bkn} \sum_{l=2}^{k} \log \int_{S^n} e^{-nbk U_j^n (W_j)} d\alpha \otimes^n
\]

(4.16)

For the third term in the right hand side of the above inequality, we have by Lemma 3.5,
\[
\frac{1}{an} \int_{S^n} e^{a\lambda n U_m^n (\| F - F_N \|)} d\alpha \otimes^n \leq \frac{1}{am} \log \int_{S^m} e^{a\lambda m C_m \| F - F_N \|} d\alpha \otimes^n 
\]

as \(N \to +\infty\) by (4.14).

On the other hand, \(Z_n^k\) and \(\int_{S^n} e^{-nbk U_j^n (W_j)} d\alpha \otimes^n\) are all finite constants by Theorem 4.1. Thus we get
\[
\lim_{n \to +\infty} \lim_{N \to +\infty} \frac{1}{n} \log P_n^k (\| U_m^n (F) - U_n^m (F_N) \| > \delta) = -\infty,
\]

since \(\lambda > 0\) is arbitrary. The proof is then completed by the approximation result of the LDP. \(\square\)

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