Nonlinear analysis of PLL by the harmonic balance method.

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Abstract: In this paper we discuss the application of the harmonic balance method for the global analysis of the classical phase-locked loop (PLL) circuit. The harmonic balance is a non-rigorous method, which is widely used for the computation of periodic solutions and the checking of global stability. The proof of the absence of periodic solutions is a key step to establish the global stability of PLL and estimate the pull-in range (which is an interval of the frequency deviations such that any solution tends to one of the equilibria). The advantages and limitations of the study of the classical PLL with lead-lag filter using the harmonic balance method is discussed.

1. INTRODUCTION

Phase-locked loop (PLL) is a nonlinear control system, which various modifications are widely used in telecommunication and computer architecture for the master-slave synchronization of oscillators and data demodulations. Rigorous analysis of the mathematical models of PLLs is a challenging task and, thus, the simulation and non-rigorous methods are often used in engineering literature for their analysis.

In this paper we discuss the application of the harmonic balance (HB) method for the global analysis of the classical PLL. The harmonic balance is a non-rigorous analytical method, which allows to study periodic solutions in control systems. It is widely applied for the study of PLL (see, e.g. Margaris (2004); Suárez et al. (2012); Homayoun and Razavi (2016)). The proof of the absence of periodic solutions is a key step to establish the global stability of the PLL model and estimate the pull-in range (which is an interval of the frequency deviations such that any solution tends to one of the equilibria). It is known that the harmonic balance method may lead to wrong conclusion on the global stability, e.g. it states that well-known Aizerman’s and Kalman’s conjectures on the global stability of nonlinear control systems are valid, while there are known counterexamples with hidden oscillations (see, e.g. Pliss (1958); Fitts (1966); Barabanov (1988); Bernat and Libre (1996); Leonov et al. (2010); Bragin et al. (2011); Leonov and Kuznetsov (2011, 2013); the corresponding discrete examples are considered in Alli-Oke et al. (2012); Heath et al. (2015)). Below we consider advantages and limitations of the study of classical PLL with lead-lag filter using the harmonic balance method. Section 2 introduces the mathematical model of PLL in a signal’s phase space (Leonov et al., 2012, 2015b). In Section 3 the harmonic balance equations are derived, in Section 4 the harmonic balance equations are solved numerically and the obtained results are compared with the result of the direct simulation of the model.

2. CLASSICAL NONLINEAR MATHEMATICAL MODELS OF PLL-BASED CIRCUITS IN A SIGNAL’S PHASE SPACE

In classical engineering works (see (Viterbi, 1966; Gardner, 1966; Best, 2007)) various analog PLL-based circuits are represented in a signal’s phase space (Leonov et al., 2015b) (also named frequency-domain (Davis, 2011, p.338)) by a block diagram shown in Fig. 1.

Here the Phase Detector (PD) is a nonlinear block; the phases \( \theta_{\text{ref}, \text{vco}}(t) \) of the input (reference) and voltage controlled oscillator (VCO) signals are the PD block inputs, and the output is the function \( \varphi(\theta_e(t)) = \varphi(\theta_{\text{ref}}(t) - \theta_{\text{vco}}(t)) \) called a phase detector characteristic, where

\[
\theta_e(t) = \theta_{\text{ref}}(t) - \theta_{\text{vco}}(t), \quad \text{(1)}
\]

is called the phase error. For the classical PLL-based circuits with sinusoidal signal’s waveforms the phase-detector characteristics is sinusoidal:

\[
\varphi(\theta_e) = \frac{1}{2} \sin(\theta_e). \quad \text{(2)}
\]
The relationship between the input \( \varphi(\theta_c(t)) \) and the output \( g(t) \) of the linear filter (Loop filter) is as follows:
\[
\dot{x} = Ax + b\varphi(\theta_c(t)), \quad g(t) = c^*x + h\varphi(\theta_c(t)),
\]
where \( A \) is a constant \( n \times n \) matrix, \( x(t) \in \mathbb{R}^n \) is the filter state, \( x(0) \) is the initial state of filter, \( b \) and \( c \) are constant vectors, and \( h \) is a number. The filter transfer function is:
\[
H(s) = c^*(A - sI)^{-1}b + h.
\]

A lead-lag filter (Best, 2007) (with \( H(0) = 1 \)) or a PI filter (\( H(0) \) is infinite) is usually used as the loop filter. The control signal \( g(t) \) adjusts the VCO frequency to the frequency of the input signal:
\[
\dot{\theta}_{vco} = \omega_{vco} = \omega_{vco}^{\text{free}} + K_{vco}g(t),
\]
where \( \omega_{vco}^{\text{free}} \) is the VCO free-running frequency (i.e. for \( g(t) \equiv 0 \)) and \( K_{vco} \) is the VCO gain. Nonlinear VCO models can be considered similarly, see, e.g. (Margaris, 2004; Bianchi et al., 2016a). The frequency of the input signal (reference frequency) is usually assumed to be constant:
\[
\dot{\theta}_{\text{ref}}(t) = \omega_{\text{ref}}(t) \equiv \omega_{\text{ref}}.
\]

By using the first two elements of the following equations (1), (3), and (5)–(7), we get:
\[
\dot{\theta}_e = \omega_e^{\text{free}} - K_{vco}g(t).
\]

The difference between the reference frequency and the VCO free-running frequency is denoted as \( \omega_e^{\text{free}} \):
\[
\omega_e^{\text{free}} \equiv \omega_{\text{ref}} - \omega_{vco}^{\text{free}}.
\]

Combining equations (1), (3), and (5)–(7), we get:
\[
\dot{\theta}_e = \omega_{\text{free}}^e - K_{vco}g(t).
\]

In the case of PD characteristic (2), system (9) is not changed under the transformation
\[
(\omega_e^{\text{free}}, x(t), \theta_c(t)) \rightarrow \left(-\omega_e^{\text{free}}, -x(t), -\theta_c(t)\right)
\]
and, thus, we can analyze system (9) only with \( \omega_e^{\text{free}} > 0 \) and introduce the concept of frequency deviation
\[
|\omega_e^{\text{free}}| \equiv |\omega_{\text{ref}} - \omega_{vco}^{\text{free}}|.
\]

The pull-in range is a widely used engineering concept (see, e.g. (Gardner, 1966, p.40), (Best, 2007, p.61)). The following rigorous definition is suggested (Kuznetsov et al., 2015; Leonov et al., 2015b; Best et al., 2016). The largest interval of frequency deviations \( 0 \leq |\omega_e^{\text{free}}| < \omega_{\text{pull-in}} \) such that the nonlinear mathematical model of PLL in the signal’s phase space acquires lock for arbitrary initial phase difference and filter state (i.e. any trajectory tends to an equilibrium point) is called a pull-in range, \( \omega_{\text{pull-in}} \) is called a pull-in frequency.

This definition implies that for any frequency deviation from pull-in range the mathematical model of PLL does not contain periodic solutions. This property can be used to obtain necessary conditions of pull-in range (see, e.g. (Homayoun and Razavi, 2016; Bianchi et al., 2016b,a)). In the next section the application of harmonic balance method to the PLL with lead-lag filter is discussed.

\[\text{1}\] In the control theory the transfer function is often defined with the opposite sign (see, e.g. (Leonov et al., 2015b)): \( H(s) = c^*(A - sI)^{-1}b + h \).
Substituting (16) and (17) in PLL equation (8), we have
\[ \psi_c - \omega_c \beta_c \sin(\psi_c) = \frac{K_{\text{vco}}}{2} \left[ J_0(\beta_c) - J_2(\beta_c) \right] \cos(\theta_c) \cos(\omega_c t - \psi_c), \]
where \( \psi_c \) and \( |H(i\omega_c)| \) are the filter phase shift and gain for the frequency \( \omega_c \). Taking derivative of (11), we get
\[ \dot{\theta}_c(t) = \omega_c - \beta_c \omega_c \cos(\omega_c t). \]  
Substituting (16) and (17) in PLL equation (8), we have
\[ \omega_c - \omega_c \beta_c \left( \cos(\omega_c t - \psi_c) \cos(\psi_c) - \sin(\omega_c t - \psi_c) \sin(\psi_c) \right) + \frac{1}{2} H(i\omega_c) [J_0(\beta_c) - J_2(\beta_c)] \sin(\omega_c t - \psi_c) = \frac{K_{\text{vco}}}{2} \left( J_1(\beta_c) \cos(\theta_c) \right) \]
where \( \omega_c \beta_c \cos(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) + J_2(\beta_c)] \cos(\theta_c), \]
\[ \omega_c \beta_c \sin(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) - J_2(\beta_c)] \sin(\theta_c). \]
By equations (18) we get the following harmonic balance equations
\[ \omega_c + \frac{K_{\text{vco}}}{2} J_1(\beta_c) \cos(\theta_c) = \omega_c, \]
\[ \omega_c \beta_c \cos(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) + J_2(\beta_c)] \cos(\theta_c), \]
\[ \omega_c \beta_c \sin(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) - J_2(\beta_c)] \sin(\theta_c). \]
Using the property of Bessel functions:
\[ J_0(\beta_c) + J_2(\beta_c) = \frac{2 J_1(\beta_c)}{\beta_c}, \]
we have
\[ \cos(\theta_c) = \frac{\omega_c - \omega_c}{\frac{K_{\text{vco}}}{2}} J_1(\beta_c), \]
\[ \omega_c \beta_c \cos(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) - J_2(\beta_c)] \cos(\theta_c), \]
\[ \omega_c \beta_c \sin(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) - J_2(\beta_c)] \sin(\theta_c). \]
(21)
(22)
(23)

Finally,
\[ \sin(\theta_c) = \frac{\omega_c - \omega_c}{\frac{K_{\text{vco}}}{2}} J_1(\beta_c), \]
\[ \omega_c = \frac{2 |H(i\omega_c)|}{\beta_c^2 \cos(\psi_c)} \left( \omega_c^\text{free} - \omega_c \right), \]
\[ \omega_c \beta_c \sin(\psi_c) = \frac{K_{\text{vco}}}{2} H(i\omega_c) [J_0(\beta_c) - J_2(\beta_c)] \sin(\theta_c). \]

Here \( J_{0,1,2} \) are Bessel functions; \( \omega_c, \beta_c, \) and \( \theta_c \) are unknown parameters of the solution.

In the next section we consider numerical solution of (23).

4. NUMERICAL SOLUTIONS OF HARMONIC-BALANCE EQUATIONS FOR LEAD-LAG FILTER

For lead-lag filter we have
\[ |H(i\omega_c)| = \left| \frac{1 + i \tau_2 \omega_c}{1 + i \tau_1 \omega_c} \right|, \]
\[ \cos(\psi_c) = \cos \left( \arg \left( \frac{1 + i \tau_2 \omega_c}{1 + i \tau_1 \omega_c} \right) \right), \]
\[ \sin(\psi_c) = \sin \left( \arg \left( \frac{1 + i \tau_2 \omega_c}{1 + i \tau_1 \omega_c} \right) \right), \]
(24)
Let us find numerically the solutions of (23). To solve nonlinear equations (23), it is possible to apply MATLAB function “vpasolve”, but the result depends on the initial guess that is not convenient for the checking of the absence of solutions of (23). Thus, we consider the difference between the right-hand side and left-hand side of equations (23). Since we cannot find exact solution, we plot the points on \((\omega_c, \beta_c)\)-plane for which the absolute value of the differences between the right-hand side and left-hand-side of (23) is less than \(\Delta\), i.e.

\[
\begin{align*}
\left| \omega_c - \omega_c^{\text{free}} \right| &< \Delta, \\
\left| \omega_c \beta_c \sin(\psi_c) - K_{\text{vco}} |H(\omega_c)| \left( J_0(\beta_c) - J_2(\beta_c) \right) \right| &< \Delta.
\end{align*}
\]

(25)

The values \((\omega_c, \beta_c)\) satisfying conditions (25) with \(\Delta = 1\) are shown in Fig. 2. In the left subfigure there are two areas, which corresponds to the first and the second equations of (25). The intersection of these areas gives an approximation of solution of harmonic-balance equations, e.g. the intersection contains the following point \(\omega_c \approx 76\), \(\beta_c \approx 0.98\), \(\theta_c \approx 0.88\). For the parameters \(\omega_c \approx 76\), \(\beta_c \approx 0.98\), \(\theta_c \approx 0.88\) we plot (11) that is the results of numerical simulation of system (9) with zero initial conditions. As shown in Fig. 3 the solution \(\theta_e(t)\) tends to infinity and the approximation given by the harmonic balance method is correct and contains periodic part (cycle).

If we consider smaller values of \(\omega_c^{\text{free}}\), then equations (25) still have a solution (see right subfigure in Fig. 2). However in this case the harmonic balance method leads us to a wrong conclusion since we can not reveal corresponding cycle in system (9) by direct simulation (see the comparison of numerical solutions in Fig. 5).

Also it is possible to check that harmonic balance equations (23) have a solution for any parameters. But the solutions with \(\beta_c > 1\) is usually excluded (Shakhgil'dyan and Lyakhovkin, 1966, 1972) because the phase is supposed to be nonnegative. If there exists a solution for \(0 < \beta < 1\), then HB implies the existence of cycle (11). The frequency of the cycle is limited by a cut-off frequency of the filter \(0 < \omega_c < \omega_{\text{cut-off}}\) and \(\theta_c \in [0, 2\pi]\).

Remark that simulation itself may not reveal a complex behavior of PLL: such examples, where the simulation of PLL-based circuits leads to unreliable results, are demonstrated in (Bianchi et al., 2016b; Blagov et al., 2016; Kuznetsov et al., 2017). Consider \(\omega_c^{\text{free}} = 178.545\) and the lead-lag filter with \(\tau_1 = 0.0448\), \(\tau_2 = 0.0185\). This value is close to bifurcation point, where a periodic oscillations appears. Simulation with relatively small precision ('MaxStep', 0.01, 'RelTol', 2e-3, 'AbsTol', 2e-3) shows absence of cycles, while simulation with precision ('MaxStep', 0.001, 'RelTol', 2e-6, 'AbsTol', 2e-6) allows to reveal a cycle (see Fig. 5).

This example demonstrates the difficulties of numerical search of so-called hidden oscillations, whose basin of attraction does not overlap with the neighborhood of the equilibrium point, and thus it may be difficult to find them numerically (Leonov and Kuznetsov, 2013; Leonov et al.,

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**Fig. 3.** Solution obtained by HB vs real solution. Solution of HB equations: \(\omega_c \approx 110\), \(\beta_c \approx 0.785\), \(\theta_c \approx 0.7088\). Parameters: \(\omega_c^{\text{free}} = 178.9\), \(K_{\text{vco}} = 250\), \(\tau_1 = 0.0448\), \(\tau_2 = 0.0185\).

**Fig. 4.** Solution obtained by HB vs real solution. The solution of HB equations: \(\omega_c \approx 76\), \(\beta_c \approx 0.98\), \(\theta_c \approx 0.88\). Parameters: \(\omega_c^{\text{free}} = 145\), \(K_{\text{vco}} = 250\), \(\tau_1 = 0.0448\), \(\tau_2 = 0.0185\).

**Fig. 5.** \(\omega_c^{\text{free}} = 178.545\), \(K_{\text{vco}} = 250\), \(\tau_1 = 0.0448\), \(\tau_2 = 0.0185\). MATLAB ‘odeset’ parameters: black line — odeset(’MaxStep’, 0.01, ’RelTol’, 2e-3, ’AbsTol’, 2e-3), red (grey) line — odeset(’MaxStep’, 0.001, ’RelTol’, 2e-6, ’AbsTol’, 2e-6).
5. CONCLUSIONS

While harmonic balance method is widely used for the estimation of the pull-in range, it may lead to wrong results. Corresponding examples are discussed in the paper. The pull-in range of PLL-based circuits with first-order filters can be estimated using phase plane analysis methods (Tricomi, 1933; Andronov et al., 1937; Shakhtarin, 1969; Belyustina et al., 1970). For a rigorous nonlinear analysis of multidimensional PLL models one may use special modifications of the classical stability criteria developed for the cylindrical phase space in (Leonov et al., 2015b; Leonov and Kuznetsov, 2014).

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