Completely anticanonical form of Sp(2)-symmetric Lagrangian quantization

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Abstract

The Sp(2)-symmetric Lagrangian quantization scheme is represented in a completely anticanonical form. Antifields are assigned to all field variables including former "parametric" ones $\pi^{Aa}$. The antibrackets $(F,G)^a$ as well as the operators $\Delta^a$ and $V^a$ are extended to include the new anticanonical pairs $\pi^{Aa}$, $\phi_A$. A new version of the gauge fixing mechanism in the Lagrangian effective action is proposed. The corresponding functional integral is shown to be gauge independent.

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1 Introduction

In papers [1, 2, 3] an Sp(2)-symmetric scheme of the Lagrangian (field-antifield) quantization was developed. (The global Sp(2) symmetry allows one to consider ghost and antighost field variables in a uniform way.) Geometric aspects as well as various possibilities of interpretation of this formalism have been considered in papers [4, 5, 6, 7, 8].

In the Sp(2)-symmetric scheme one deals with a complete set of field variables $\phi^A$, which includes the original fields, the ghosts, the antighosts and the Lagrange multipliers. In addition one assigns to each $\phi^A$ three kinds of antifields: $\phi^*_{A1}$, $\phi^*_{A2}$ and $\overline{\phi}^A$. From the Hamiltonian point of view $\phi^*_{A1}$ and $\phi^*_{A2}$ correspond to the sources of BRST and antiBRST field transformations, $[\phi^A, \Omega^1]$ and $[\phi^A, \Omega^2]$, while $\overline{\phi}^A$ corresponds to the source of their combined transformation, $\epsilon_{ab}[\phi^A, \Omega^b]$. Unfortunately, these three kinds of antifields enter the present Sp(2) symmetric formalism in a nonsymmetric way. While $\phi^*_{Aa}$ are anticanonically conjugate to $\phi^A$ in the usual sense, $\overline{\phi}^A$ do not possess their own conjugate fields from the very beginning. On the other hand, when a gauge fixing procedure was introduced into the Sp(2) symmetric theory one had to make use of some auxiliary field variables $\pi^{Aa}$ to parametrize the differential operator containing the gauge fixing function [1, 2, 3].

The main idea of the present paper is to identify the previous parametric variable $\pi^{Aa}$ with an auxiliary field variable which is conjugated to the antifield $\overline{\phi}^A$ in the usual sense. Then we define an extended antibracket $(F, G)^a$ and an operator $\Delta^a$ into which all fields-antifields enter on equal footing as anticanonical pairs. An extended version of the master equation whose solution generally depends on the complete set of anticanonical pairs is then set up. In this way we are able to generalize the description of abelian Lagrangian surface to cover the Sp(2)-symmetric case. We formulate a completely anticanonical version of generalized BRST-antiBRST transformations and claim that the functional integral is gauge independent. This is explicitly verified for a simple consistent set of gauge fixing conditions which are explicitly solved with respect to $\phi^*_{Aa}$.

2 Main definitions

Let $\phi^A$, $\epsilon(\phi^A) \equiv \epsilon_A$, be a complete set of field variables including original fields, ghosts, antighosts and Lagrange multipliers. Let us assign to each of them a pair of antifields $\phi^*_{Aa}$ $(a = 1, 2)$, $\epsilon(\phi^*_{Aa}) \equiv \epsilon_A + 1$. Next let us introduce a set of pairs of auxiliary field variables $\pi^{Aa}$, $\epsilon(\pi^{Aa}) \equiv \epsilon_A + 1$, whose antifields are $\overline{\phi}^A$, $\epsilon(\overline{\phi}^A) \equiv \epsilon_A$. In terms of these variables we define an extended antibracket by

$$(F, G)^a \equiv F \left( \frac{\partial}{\partial \phi^A} - \epsilon_{ab} \frac{\partial}{\partial \pi^{Ab}} \frac{\partial}{\partial \phi^A} \right) G - (F \leftrightarrow G)(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}$$

and the corresponding operator $\Delta^a$ given by

$$\Delta^a \equiv (-1)^{\epsilon_A} \frac{\partial}{\partial \phi^A} - \epsilon_{ab} (-1)^{\epsilon_A+1} \frac{\partial}{\partial \pi^{Ab}} \frac{\partial}{\partial \phi^A}$$

The antibrackets (1) satisfy the generalized Jacobi identities

$$(F, G)^a (H)^b \left( -1 \right)^{(\epsilon(F)+1)(\epsilon(H)+1)} + \text{cycle}(F, G, H) \equiv 0$$

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and $\Delta^a$ satisfies
\begin{equation}
\Delta^a \Delta^b = 0
\end{equation}

\begin{equation}
\Delta^{a}(F,G)^b = (\Delta^{a}F,G)^b - (F,\Delta^{a}G)^b (-1)^{\varepsilon(F)}
\end{equation}

\begin{equation}
\Delta^{a}(FG) = (\Delta^{a}F)G + (F,G)^a (-1)^{\varepsilon(F)} + F(\Delta^{a}G)(-1)^{\varepsilon(F)}
\end{equation}

The curly bracket denotes symmetrization of $a$ and $b$. Notice that formula (6) may be considered to be an alternative definition of the antibracket $\{\}$. Let us also introduce the extended operator $\bar{\Delta}^a$:
\begin{equation}
\bar{\Delta}^a \equiv \Delta^a + \frac{i}{\hbar}V^a
\end{equation}

where
\begin{equation}
V^a \equiv \frac{1}{2} \left( \varepsilon^{ab} \partial_\phi \phi^b \frac{\partial}{\partial \phi} - \pi^a (-1)^{\varepsilon_a} \frac{\partial}{\partial \phi^a} \right)
\end{equation}

One may easily check that the following properties hold
\begin{equation}
V^a(F,G)^b = (V^aF,G)^b - (-1)^{\varepsilon(F)}(F,V^aG)^b
\end{equation}

\begin{equation}
V^{aV^b} = 0
\end{equation}

\begin{equation}
\Delta^aV^b + V^b\Delta^a = 0
\end{equation}

From (10) and (11) it follows then that
\begin{equation}
\bar{\Delta}^{a\bar{\Delta}^b} = 0
\end{equation}

Notice that our definition of $V^a$, (8), differs from the one given in [1, 2, 3]. As a consequence our formulas (9) and (11) are valid without symmetrization in the indices $a$ and $b$.

### 3 Gauge independent functional integral representation

The quantum action $W(\phi, \phi^*, \pi, \bar{\phi}; \hbar)$ is defined to be a solution of the following quantum master equation
\begin{equation}
\bar{\Delta}^a \exp\{\frac{i}{\hbar}W\} = 0
\end{equation}

or equivalently
\begin{equation}
\frac{1}{2}(W,W)^a + V^aW = i\hbar\Delta^aW
\end{equation}

We define then the field-antifield functional integral to be
\begin{equation}
Z = \int [d\phi][d\phi^*][d\pi][d\bar{\phi}][d\lambda] \exp\{\frac{i}{\hbar}[W + X]\}
\end{equation}
where \( X(\phi, \phi^*, \pi, \bar{\phi}; \hbar) \) is a hypergauge fixing action depending on a new variable \( \lambda^A \), \( \varepsilon(\lambda^A) = \varepsilon_A \), corresponding to the hypergauge invariance of \( W \). (If \( X \) is linear in \( \lambda^A \) they are Lagrange multipliers.) \( X \) is required to satisfy the following quantum master equation

\[
\frac{1}{2} (X, X)^a - V^a X = i\hbar \Delta^a X
\]  

which is the same equation as \((11)\) apart from the opposite sign of the \( V \)-term. We expect the classical part of \( X \) to have the structure

\[
X|_{\hbar=0} = G_A \lambda^A + KY
\]  

where \( Y \) and \( G_A \) are functions while \( K \) is the differential operator

\[
K \equiv \varepsilon_{ab} V^a V^b
\]  

One may notice that \( Y \) is only determined up to functions \( A \) satisfying

\[
KA = 0
\]  

\( A \) may \( e.g. \) have the form \( A = V^a R \) any \( R \). When \((11)\) is inserted into \((10)\) we find that the functions \( Y \) and \( G_A \) are subjected to the following conditions

\[
(G_A, G_B)^a = 0
\]  

\[
(KY, G_A)^a = V^a G_A
\]  

\[
(KY, KY)^a = 0
\]  

Let \( \Gamma \) denote the complete set of field-antifield variables. We assert then that \((11)\) is invariant under the following transformation:

\[
\delta \Gamma \equiv (\Gamma, -W + X)^a \mu_a - 2V^a \Gamma (-1)^{\varepsilon(\Gamma)} \mu_a
\]  

where \( \mu_a \) is a constant infinitesimal fermionic parameter. (One has to make use of \((14)\) and \((16)\).)

In the case when \( \mu_a \) depends on \( \Gamma \) and \( \lambda \) the transformation \((23)\) induces a change in the gauge fixing action \( X \) after the additional transformation

\[
\delta_1 \Gamma = \frac{1}{2} (\Gamma, \delta F_a)^a, \quad \delta F_a(\Gamma) \equiv \frac{2\hbar}{\iota} \mu_a(\Gamma, \lambda)
\]  

is performed. This change is given by

\[
\delta X = (X, \delta F_a)^a - V^a \delta F_a - i\hbar \Delta^a \delta F_a
\]  

One may now show that

\[
(X, \delta X)^a - V^a \delta X = i\hbar \Delta^a \delta X
\]  

provided \( \delta F_a \) is chosen to have the following form

\[
\delta F_a = \varepsilon_{ab} \left\{ (X, \delta \Xi)^b - V^b \delta \Xi - i\hbar \Delta^b \delta \Xi \right\}
\]
The gauge independence of the functional integral is then confirmed.

In order to illustrate the above properties we consider a particular solution of (16). In fact, a simple natural solution is given by

\[ X = G_A \lambda^A + KY \]  

(28)

where

\[ Y = \bar{\phi}_A \phi^A - 2F(\phi) \]  

(29)

\[ G_A = \bar{\phi}_A - F(\phi) \frac{\partial}{\partial \phi^A} \]  

(30)

where \( F(\phi) \) is an arbitrary function of the original fields \( \phi^A \). In this case the transformation (23) is given by

\[\begin{align*}
\delta \phi^A &= - \left( \frac{\partial W}{\partial \phi^*_A} - \frac{3}{2} \pi^{Aa} \right) \mu_a \\
\delta \phi^*_A &= \mu_a \left( \frac{\partial W}{\partial \phi^*_A} + F \frac{\partial}{\partial \phi^A} \frac{\partial}{\partial \phi^B} \lambda^B \right) \\
\delta \pi^{Aa} &= \varepsilon^{ab} \left( \frac{\partial W}{\partial \pi^{Aa}} - \lambda^A \right) \mu_b \\
\delta \bar{\phi}_A &= \mu_a \left( \varepsilon^{ab} \frac{\partial W}{\partial \pi^{Ab}} + \frac{1}{2} \varepsilon^{ab} \phi^*_A \pi_{Aa} + \frac{1}{2} \varepsilon^{ab} \phi^*_A \pi_{Aa} \right).
\end{align*}\]  

(31)-(34)

It is now straightforward to check that the integrand of (15) is invariant under these transformations when \( \mu_a \) is constant. However, if we now let \( \mu_a \) depend on \( \phi^A \) and \( \pi^{Aa} \), then the transformations (31)-(34) lead to the following change of the integrand in (15)

\[ \mu_a \frac{\partial}{\partial \phi^A} \left( \frac{\partial W}{\partial \phi^*_A} - \frac{3}{2} \pi^{Aa} \right) + \varepsilon^{ab} \mu_a \frac{\partial}{\partial \pi^{Aa}} \left( \frac{\partial W}{\partial \phi^*_A} - \lambda^A \right) \]  

(35)

On the other hand, by means of the additional transformations (24) here given by

\[\begin{align*}
\delta_1 \phi^*_A &= - \frac{\hbar}{i} \mu_a \frac{\partial}{\partial \phi^A}, \\
\delta_1 \bar{\phi}_A &= - \frac{\hbar}{i} \varepsilon^{ab} \mu_a \frac{\partial}{\partial \pi^{Ab}}
\end{align*}\]  

(36)

eq (35) acquires the form

\[ -2 \mu_a \frac{\partial}{\partial \phi^A} \pi^{Aa} - 2 \varepsilon^{ab} \mu_a \frac{\partial}{\partial \pi^{Aa}} \lambda^A \]  

(37)
Finally choosing the ansatz

$$\mu_a = \frac{i}{2\hbar} \delta F_a(\phi, \pi), \quad \delta F_a(\phi, \pi) = -\frac{1}{2} \epsilon_{ab} \delta F_b \frac{\partial}{\partial \phi} \pi^{Bb}$$

which corresponds to the choice $\delta \Xi = -\frac{1}{2} \delta F$ in (27), one may show that (37) corresponds to the variation

$$F \rightarrow F + \delta F$$

in the integrand of the functional integral (15). Thus, we have proved that the functional integral (15) is gauge independent. One may notice that we in our proof of gauge independence only have made use of superpositions of purely anticanonical transformations and the ones generated by the operator $V^a$ in (8).

4 Conclusion.

Our main results are as follows: First, we have identified the former "parametric" variables $\pi^{4a}$ with fields anticanonically conjugate to the antifields $\bar{\phi}_4$ in the usual sense. The antibrackets $(F,G)^a$ as well as the operators $\Delta^a$, $V^a$ are then extended to include these new anticanonical pairs. The resulting new $V^a$ turns out to act as a derivative on the new antibracket and to anticommute with $\Delta^a$ without symmetrization (eqs (9) and (11)).

We cast then the gauge fixing mechanism into a completely anticanonical form, and give the conditions under which the functional integral is gauge independent. Finally, we verify the formalism for a simple set of allowed hypergauge conditions. One may notice that although we have tripled the number of field variables, the number of hypergauge functions multiplying the Lagrange multipliers are only those of the original fields.

The formulation presented in this paper covers the case when the hypergauge conditons satisfy an abelian algebra in terms of the antibrackets, and when the field-antifield phase space is spanned by anticanonical pairs. The generalization to a geometric description in terms of general phase spaces as well as to arbitrary nonabelian hypergauge conditions is given in [9].

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Appendix

Identities for the proof of eq.(26)

In order to prove that the quantum master eq.(16) is invariant under the variation (25) with \( \delta F_a \) given by (27) one needs the following identities

\[
\varepsilon_{bc}(X, (X, (X, \delta \Xi)^c)^b)^a \equiv \varepsilon_{bc}\left\{(X, (\delta \Xi, (X, X)^b)^c)^a + 
\right\}
\]

\[
\varepsilon_{bc}\left\{(\Delta^a(X, (X, \delta \Xi)^c)^b) + (\delta \Xi, (X, \Delta^c X)^b)^a 
\right\}
\]

\[
\varepsilon_{bc}\left\{(\Delta^a(X, (X, \delta \Xi)^c)^b + (\delta \Xi, (X, \Delta^c X)^b)^a + (\Delta^c X, (X, \delta \Xi)^b)^a - (X, (\Delta^c X, \delta \Xi)^b)^a + 
\right\}
\]

\[
\equiv \varepsilon_{bc}\left\{\Delta^a\left((\delta \Xi, (X, X)^b)^c + (\delta \Xi, \Delta^c (X, X)^b)^a + 2(\Delta^b \delta \Xi, (X, X)^c)^a\right)\right\}
\]

\[
\varepsilon_{bc}\left\{\Delta^a\Delta^b(X, \delta \Xi)^c - \Delta^a(\Delta^c X, \delta \Xi)^b + 2(\Delta^c X, \Delta^b \delta \Xi)^a + 
\right\}
\]

\[
\equiv 0
\]

They may be derived by direct use of the definition (1) of the antibracket, or indirectly from (3) by means of the identities \( \Delta^a \Delta^b \Delta^c (X^3 \delta \Xi) \equiv 0 \), \( \Delta^a \Delta^b \Delta^c (X^2 \delta \Xi) \equiv 0 \) and \( \Delta^a \Delta^b \Delta^c (X \delta \Xi) \equiv 0 \) respectively. One may notice that these identities are even valid when \( \Delta^a \) is replaced by \( \Delta^a + \alpha V^a \) for any constant \( \alpha \).

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