EQUIDIMENSIONALITY OF THE BASIC AFFINE 
DELIGNE-LUSZTIG VARIETY IN MIXED CHARACTERISTIC

YUTA TAKAYA

Abstract. We prove equidimensionality of the basic affine Deligne-Lusztig variety in 
mixed characteristic. This verifies a conjecture made by Rapoport. The method is a 
mixed characteristic analogue of the work of Hartl and Viehmann.

Contents

Introduction 1
Acknowledgements 2
1. Preliminaries 2
1.1. Notations 2
1.2. Formal local rings 2
1.3. \( F \)-standard ideals 3
1.4. Dimensions of local bounded \( f \)-adic rings 3
2. Moduli spaces of interest 4
2.1. Affine Grassmannians 5
2.2. Affine Deligne-Lusztig varieties 5
2.3. A link between two moduli spaces 5
References 8

Introduction

The affine Deligne-Lusztig variety was introduced by Rapoport in [Rap05]. It arises as 
the underlying space of a moduli space of local shtukas with a parahoric level structure. 
In this paper, we only deal with the hyperspecial case. Let \( F \) be a non-archimedean 
local field with a uniformizer \( \pi \) and let \( L \) be the completion of the maximal unramified 
extension of \( F \) with a residue field \( k \). Let \( O_F \) and \( O_L \) be their ring of integers. Let \( G \) be 
an unramified connected reductive group over \( F \). We fix a Borel pair \( T \subset B \subset G \) over \( F \) 
and a reductive model \( \mathcal{G} \) over \( O_F \). Let \( b \) be an element of \( G(L) \) and let \( \mu \in X^*(T)_+ \) be a 
dominant cocharacter. The affine Deligne-Lusztig variety \( X_\mu(b) \) associated to the datum 
(\( \mathcal{G}, b, \mu \)) is a locally closed subscheme of the affine Grassmannian having the following 
description:

\[
X_\mu(b)(k) = \left\{ g \in G(L)/\mathcal{G}(O_L) \mid g^{-1}b\sigma(g) \in \mathcal{G}(O_L)\pi^\mu\mathcal{G}(O_L) \right\}.
\]

It is known to be “locally of finite type” over \( k \): locally of finite type over \( k \) in the 
equal characteristic case, and locally perfectly of finite type in the mixed characteristic 
case. A lot of research has been done regarding its geometric properties, such as the 
nonemptiness criterion, known as Kottwitz-Rapoport conjecture, proved in [RR96] and 
[Gas10], and the dimension formula proved in [Ham15] and [Zhu17]. One of the recent 
interesting problems, conjectured by Chen and Zhu and settled in [ZZ20] and [Nie22], 
is the determination of the \( J_b(F) \)-orbits of top-dimensional irreducible components of
$X_\mu(b)$. It relates the $J_b(F)$-orbits to the Mirkovic-Vilonen basis for some weight space of the representation of the dual group $\hat{G}$ corresponding to $\mu$. It gives an elementary way of computing the number of the $J_b(F)$-orbits of top-dimensional irreducible components.

Though it only concerns top-dimensional irreducible components, it was conjectured in [Rap05, Conjecture 5.10] (at least when $b$ is basic) that $X_\mu(b)$ is equidimensional. In the equal characteristic case, it was already proved in [HV11] (the basic case) and [HV18] (the general case). However, according to [HV18, Theorem 3.4], equidimensionality in mixed characteristic is available only when we can reduce to the underlying space of a Rapoport-Zink space. Avoiding such a reduction, we prove it in the basic case (see Theorem 2.11):

**Theorem.** $X_\mu(b)$ is equidimensional if $b$ is basic.

The strategy of the proof is to adapt the arguments in [HV11] to the mixed characteristic case: we relate the completion of $X_\mu(b)$ at a closed point to the closed Newton stratum of the completion of the affine Grassmannian, whose dimension can be estimated. Since $X_\mu(b)$ is a perfect scheme in the mixed characteristic case, we do not have a meaningful completion in the usual sense. Instead, we use the perfection of the completion of some deperfection. Its Noetherian property is important in the proof of Lemma 2.9, which is the reason why we do not use the completion of it. The main difference in the proof is the behavior of a deformation on the completion of the affine Grassmannian. Unlike the result of [HV11, Theorem 5.6], we do not know if the deformation is universal: in the first place, it is not clear to us which category we should work on to give moduli interpretations. A careful choice made in Lemma 2.7 enables us to compare the dimensions in Theorem 2.10. However, we hope for the existence of the universal deformation, which makes the morphism in Theorem 2.10 an isomorphism. It would help us to give a mixed characteristic analogue of [HV18] in the general case.

**Acknowledgements**

I would like to thank my advisor Yoichi Mieda for his constant support and encouragement. I am also grateful to Ryosuke Shimada for introducing this topic to me. This work was supported by the WINGS-FMSP program at the Graduate School of Mathematical Sciences, the University of Tokyo.

1. Preliminaries

1.1. **Notations.** We fix a prime number $p$. All rings are assumed to be commutative. A bounded $\mathfrak{f}$-adic ring is a ring with a linear topology for a finitely generated ideal. The completion of an object $A$ is denoted by $A^\wedge$, or $A^\wedge_I$ when the completion is done along an object $I$. The category of affine perfect schemes over a perfect field $k$ is denoted by $\text{AffPSch}_k$.

1.2. **Formal local rings.** Here, $k$ is a perfect field of characteristic $p$.

**Definition 1.1.** Let $X$ be a perfect scheme, locally perfectly of finite type over $k$, and let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighborhood of $x$ and let $U_0 = \text{Spec}(A_0)$ be a deperfection of $U$ with $A_0$ a finite type $k$-algebra. We call the perfection of the completion of $A_0$ at $x$, which is independent of the choice of $U$ and $U_0$, the formal local ring of $X$ at $x$. We call the completion of $A_0$ at $x$ for some $A_0$ a deperfection of the formal local ring.
A formal local ring is naturally equipped with a structure of a bounded $f$-adic ring. When $x$ is a $k'$-valued point where $k'/k$ is a field extension, we define (a deperfection of) the formal local ring of $X$ at $x$ as that of $X_{k'}$ at $x$. This notion is well-behaved even if $x$ is a non-$k$-valued closed point of $X$.

**Lemma 1.2.** Let $X$ be a perfect scheme, locally perfectly of finite type over $k$, and let $x \in X$ be a closed point. The formal local ring of $X$ at $x$ is henselian. If $x$ is a $k'$-valued point where $k'/k$ is an algebraically closed field extension, the formal local ring is strictly henselian.

**Proof.** The formal local ring can be written as $R_{\text{perf}}$ with $R$ a complete local Noetherian ring over $k$. Since $R$ is henselian, $R_{\text{perf}}$, written as the colimit of a sequence consisting of $R$, is also henselian. If $x$ is a geometric point, the residue field of $R$ is algebraically closed, so $R_{\text{perf}}$ is strictly henselian. □

### 1.3. $F$-standard ideals

**Definition 1.3.** Let $A$ be a perfect ring. An ideal $J$ of $A$ is $F$-standard if $J^p = \text{Frob}_A(J)$, where $\text{Frob}_A$ is the Frobenius map on $A$.

**Definition 1.4.** Let $A$ be a perfect ring and let $I$ be an ideal of $A$. We call the minimum $F$-standard ideal containing $I$, which is $\bigcup_{n \in \mathbb{Z}_{\geq 0}} (\text{Frob}^{-n}_A(I))^p$, the $F$-standardization of $I$.

**Lemma 1.5.** Let $A$ be a perfect bounded $f$-adic ring and let $I$ be a finitely generated ideal of definition. Then, its $F$-standardization $J$ is an ideal of definition of $A$.

**Proof.** Let $f_1, f_2, \ldots, f_m$ be a set of generators of $I$. Then, $J$ is the union of the ideals $(f_1^{1/p^n}, f_2^{1/p^n}, \ldots, f_m^{1/p^n})^p$. We have $(f_1^{1/p^n}, f_2^{1/p^n}, \ldots, f_m^{1/p^n})^{mp^n} \subset (f_1, f_2, \ldots, f_m)$, so $J^{mp^n} \subset I \subset J$ and $J$ is an ideal of definition of $A$. □

**Corollary 1.6.** The completion of a perfect bounded $f$-adic ring is perfect.

**Proof.** Let $A$ be a perfect bounded $f$-adic ring and let $J$ be an $F$-standard ideal of definition. The Frobenius map induces an isomorphism $A/J^n \cong A/J^{mp^n}$. Thus, the Frobenius map on $A$ is an isomorphism. □

**Definition 1.7.** Let $A$ be a perfect ring and let $J$ be an $F$-standard ideal. We denote the ideal $\left\{ \sum_{n=0}^{\infty} [c_n]p^n \mid c_n \in J \right\}$ of $W_0(A)$ by $[J]$.

Using this, we can consider the congruence of $W_0(A)$-valued points modulo $J$: we say that $x, x' \in X(W_0(A))$ are congruent modulo $J$ if their reductions to $(W_0(A)/[J])$-valued points are equal. Here, $X$ is a presheaf on a suitable category.

### 1.4. Dimensions of local bounded $f$-adic rings

**Definition 1.8.** A bounded $f$-adic ring $A$ is said to be local if $A$ is a local ring with the maximal ideal $A^\circ$.

**Definition 1.9.** Let $A$ be a local bounded $f$-adic ring. A sequence of elements $f_1, \ldots, f_n$ is said to be a system of parameters for $A$ if the ideal $(f_1, \ldots, f_n)$ is an ideal of definition. We define the dimension of $A$ as the minimum length of a system of parameters. It is denoted by $\dim A$.

**Lemma 1.10.** Let $(A, m)$ be a local Noetherian ring. Then, the dimension of $A$ as a local bounded $f$-adic ring with an ideal of definition $m$ is equal to a classical one.
Proof. This follows from the fact that a system of parameters in the above sense is equal to a classical one for a local Noetherian ring. \qed

**Lemma 1.11.** Let $A$ be a local bounded $f$-adic ring and let $I$ be an ideal of definition of $A$ generated by $f_1, \ldots, f_n$. Then, for any elements $a_1, \ldots, a_n$ of $I^2$, the elements $f_1 + a_1, \ldots, f_n + a_n$ generate $I$.

**Proof.** We have $I = (f_1 + a_1, \ldots, f_n + a_n) + I \cdot A^\circ$. Since $I$ is a finite $A$-module, we may apply Nakayama’s lemma to obtain $I = (f_1 + a_1, \ldots, f_n + a_n)$. \qed

**Corollary 1.12.** Let $A$ be a local bounded $f$-adic ring and let $I$ be an ideal of $A$. If the closure of $I$ is open, then $I$ itself is open.

**Proof.** Let $J$ be a finitely generated ideal of definition of $A$ contained in the closure of $I$. For any element $f \in J$, there exists an element $a \in J^2$ such that $f + a \in I$. By Lemma 1.11, we have a set of generators of $J$ contained in $I$. Thus, $J \subset I$. \qed

**Lemma 1.13.** The dimension of a local bounded $f$-adic ring is invariant under completion.

**Proof.** Let $A$ be a local bounded $f$-adic ring. Since a system of parameters for $A$ is also that for $A^\wedge$, we have $\dim A \geq \dim A^\wedge$. We prove the converse inequality. Let $\hat{f}_1, \ldots, \hat{f}_n$ be a set of generators of an ideal of definition $J$ of $A^\wedge$. By Lemma 1.11, we may assume that each $\hat{f}_i$ comes from an element $f_i$ of $A$. Let $g_1, \ldots, g_m$ be a set of generators of an ideal of definition $I$ of $A$. We may assume that $I \cdot A^\wedge \subset J$. Then, there exist elements $a_1, \ldots, a_m$ of $I^2$ such that $g_1 + a_1, \ldots, g_m + a_m$ is contained in the ideal $(f_1, \ldots, f_n)$ of $A$. They generate $I$ by Lemma 1.11, thus $f_1, \ldots, f_n$ is also a system of parameters for $A$. \qed

**Lemma 1.14.** The dimension of a local bounded $f$-adic ring of characteristic $p$ is invariant under perfection.

**Proof.** Let $A$ be a local bounded $f$-adic ring of characteristic $p$. Since a system of parameters for $A$ is also that for $A^\perf$, we have $\dim A \geq \dim A^\perf$. We prove the converse inequality. Let $f_1, \ldots, f_n$ be a set of generators of an ideal of definition $J$ of $A^\perf$ which may be assumed to be in $A$ by taking the $p^N$-th power for sufficiently large $N$. Let $g_1, \ldots, g_m$ be a set of generators of an ideal of definition $I$ of $A$. We may assume that $I \cdot A^\perf \subset J$. Then, we can write $g_i = \sum a_{ij} f_j$ where $a_{ij} \in A^\perf$. Again by taking the $p^N$-th power for sufficiently large $N$, we may assume that $a_{ij} \in A$. Then, $f_1, \ldots, f_n$ generate an ideal of definition of $A$. \qed

2. Moduli spaces of interest

Let $F$ be a local field over $\mathbb{Q}_p$ with a finite residue field $k$ consisting of $q$ elements and let $O_F$ be its ring of integers. Let $\pi$ be a uniformizer of $F$. For a $k$-algebra $R$, let $W_O(R) = W(R) \otimes_{W(k)} O_F$ and $W_{O,n}(R) = W_O(R) \otimes_{O_F} O_F/\pi^n$. Here, $O$ stands for $O_F$. The Frobenius map on $W_O(R)$ is denoted by $\sigma_R$ or simply by $\sigma$. We fix an algebraic closure $\overline{k}$ of $k$ and let $L = W_O(\overline{k})[\frac{1}{p}]$, $O_L = W_O(\overline{k})$.

Let $G$ be a connected reductive group over $F$. We assume that $G$ is unramified, and fix a Borel pair $T \subset B \subset G$ over $F$ and a reductive model $\mathcal{G}$ over $O_F$. Let $\mu \in X^*(T)_+$ be a dominant cocharacter. In this paper, the trivial $\mathcal{G}$-torsor (resp. $\mathcal{G}$-torsor) is simply denoted by $\mathcal{G}$ (resp. $G$).
2.1. Affine Grassmannians.

**Definition 2.1.** (cf. [Zhu17, Lemma 1.3], [SW20, Definition 20.3.3]) The Witt vector affine Grassmannian $\text{Gr}_W^G$ is the functor on AffPSch taking $\text{Spec}(R)$ to the set of pairs $(\mathcal{E}, \beta)$ where $\mathcal{E}$ is a $G$-torsor on $\text{Spec}W_O(R)$ and $\beta$ is a trivialization of $\mathcal{E}$ over $\text{Spec}W_O(R)[\frac{1}{p}]$; it is also the étale sheafification of $R \mapsto G(W_O(R)[\frac{1}{p}])/\mathcal{G}(W_O(R))$.

**Definition 2.2.** (cf. [Zhu17, p.428], [SW20, p.186]) The Schubert variety $\text{Gr}_W^G \leq \mu$ is the subfunctor of $\text{Gr}_W^G$ taking $\text{Spec}(R)$ to the set of pairs $(\mathcal{E}, \beta)$ such that the relative position of $\beta$ at every geometric point of $\text{Spec}(R)$ is bounded by $\mu$.

**Proposition 2.3.** (cf. [Zhu17, Proposition 1.23], [BS17, Corollary 9.6]) The Schubert variety $\text{Gr}_W^G \leq \mu$ is represented by the perfection of a projective variety over $k$.

2.2. Affine Deligne-Lusztig varieties.

**Definition 2.4.** An $F$-crystal with $\mathcal{G}$-structure on a perfect $k$-algebra $R$ is a pair $(\mathcal{P}, \varphi)$ where $\mathcal{P}$ is a $\mathcal{G}$-torsor on $W_O(R)$, $\varphi$ is an isomorphism $\sigma_R \mathcal{P} \rightarrow \mathcal{P}$ over $W_O(R)[\frac{1}{p}]$. We say that $(\mathcal{P}, \varphi)$ is bounded by $\mu$ if its Hodge polygon at each geometric point of $\text{Spec}(R)$ is bounded by $\mu$.

We simply say that $\mathcal{P}$ is an $F$-crystal with $\mathcal{G}$-structure without mentioning the Frobenius action. We use $\varphi_\mathcal{P}$ to denote its Frobenius action and use $\mathcal{P}_\eta$ to denote the $F$-isocrystal with $G$-structure obtained by restricting to $W_O(R)[\frac{1}{p}]$.

**Definition 2.5.** (cf. [Zhu17, 3.1.2]) Let $b$ be an element of $G(L)$ and let $\mathcal{E}^b$ be the $F$-crystal with $\mathcal{G}$-structure $(\mathcal{G}, b\sigma)$ on $\bar{k}$. The closed affine Deligne-Lusztig variety $X_{\leq \mu}(b)$ is the functor on AffPSch taking $\text{Spec}(R)$ to the set of pairs $(\mathcal{P}, \beta)$ where $\mathcal{P}$ is an $F$-crystal with $G$-structure on $R$ bounded by $\mu$, and $\beta$ is an isomorphism of $F$-isocrystals with $G$-structure between $\mathcal{P}_\eta$ and $\mathcal{E}_{\eta,R}^b$.

**Proposition 2.6.** (cf. [HV18, Lemma 1.1]) $X_{\leq \mu}(b)$ is represented by a perfect scheme locally perfectly of finite type over $\bar{k}$.

The set of the closed points of $X_{\leq \mu}(b)$ has the following description:

$$X_{\leq \mu}(b)(\bar{k}) = \left\{ g \in G(L)/\mathcal{G}(O_L) \mid g^{-1} b \sigma(g) \in \bigcup_{\mu' \leq \mu} \mathcal{G}(O_L) / \mathcal{G}(O_L) \right\}.$$

Here, $\preceq$ denotes the Bruhat order on $X^*(T)_\pm$: we write $\mu' \preceq \mu$ if $\mu - \mu'$ can be written as a sum of positive coroots with nonnegative integral coefficients. The affine Deligne-Lusztig variety $X_{\mu}(b)$ is the open subfunctor of $X_{\leq \mu}(b)$ consisting of those pairs $(\mathcal{P}, \beta)$ whose Hodge polygons at each geometric point are equal to $\mu$.

2.3. A link between two moduli spaces. We fix an $F$-crystal with $\mathcal{G}$-structure $\mathcal{P}_0$ bounded by $\mu$ on $\bar{k}$ corresponding to an element $g \in G(L)$. That is, $\mathcal{P}_0 = (\mathcal{G}, b'\sigma)$ where $b' = g^{-1} b \sigma(g)$, and the isomorphism $\beta_0 : (\mathcal{P}_0)_\eta \cong \mathcal{E}_{\eta,R}^b$ is given by $g$. Here, the element $b \in G(L)$ satisfies $[b] \in B(G, \mu)$, which is the condition for the affine Deligne-Lusztig variety to be non-empty (cf. [Gas10, Theorem 5.1]). Let $\mu^* = w_0(-\mu)$ where $w_0$ is the longest element of the Weyl group. We first construct a “universal deformation” of $\mathcal{P}_0$.

By Proposition 2.3, we can take the formal local ring of $\text{Gr}_W^G \leq \mu^*$ at the $\bar{k}$-valued point $[b^{-1}]$. Let $A$ be the ring. There exists a natural map $\text{Spec}(A) \rightarrow \text{Gr}_W^G \leq \mu^*$. Let $(\mathcal{E}, \beta)$ be the pair associated to this map. Since $A$ is strictly henselian, $\mathcal{E}$ is trivial as a $\mathcal{G}$-torsor. We can trivialize it “continuously” in the following sense.
Lemma 2.7. There exists a trivialization of $\mathcal{E}$, giving rise to an element $t \in G(W_0(A)[[t]])$ as the difference to $\beta$, satisfying the following “continuity” conditions:

- There exists an $F$-standard ideal of definition $I$ of $A$ such that $t$ is congruent to $b^{-1}$ modulo $I$ as an element of $G(W_0(A)[[t]])$.
- Let $B$ be a perfect $A$-algebra and let $J$ be an $F$-standard ideal of $B$. Let $t_B$ be the pull back of $t$ to $G(W_0(B)[[t]])$. If $[t_B]$ and $[b^{-1}]$ are congruent modulo $J$ as an element of $G(W_0(B)[[t]])/G(W_0(B))$, then $t_B$ and $b^{-1}$ are congruent modulo $J$ as an element of $G(W_0(B)[[t]])$.

Proof. Let $[t]$ be the element of $G(W_0(A)[[t]])/G(W_0(A))$ corresponding to $(\mathcal{E}, \beta)$. The difference between $b^{-1}$ and $t_{\mathbb{F}_p}$, where $t$ is a representative of $[t]$, is given by $G(O_L)$. Since there exists a canonical section of $G(W_0(A)) \to G(O_L)$, we can take a representative $t$ so that $t_{\mathbb{F}_p} = b^{-1}$.

Let $f_1, \ldots, f_n$ be a set of generators of $\Gamma(G, O_G)$ as an $O_F$-algebra such that $f_i(id_G) = 0$ for all $i$. Let $B$ be a perfect $A$-algebra and let $J$ be an $F$-standard ideal of $B$. The congruence of $[t_B]$ and $[b^{-1}]$ modulo $J$ is equivalent to the condition that the pull back of $b^it_B$ to $G(W_0(B)[[t]])$ lies in $G(W_0(B)/[J])$, that is, the coefficients of $f_i(b^it_B)$ in negative degrees vanish modulo $J$ for all $i$. Let $I$ be the $F$-standardization of the ideal of $A$ generated by the coefficients of $f_i(b^it)$ in negative degrees. Then, by the congruence of $t_B$ and $b^{-1}$ modulo $J$ implies the congruence of $t_B$ and $b^{-1}$ modulo $J$. Let $B' = B \cdot I$ and $b' = b \cdot I$. If we consider the case $B = A$, we see that $I$ is a proper ideal independent of the choice of $f_1, \ldots, f_n$ and $n$. Thus, every coefficient of $f_i(b^it)$ vanishes modulo $I$.

Now, $b^it$ modulo $I$ lies in $G(W_0(A)/[I])$, thus if we can take its lift to $G(W_0(A))$, then by modifying $t$ by the lift, we can take $t$ so that all of the coefficients of $f_i(b^it)$ vanish modulo $I$. Then, $t$ satisfies the desired continuity conditions (in the first condition, we can take any ideal of definition containing $I$; actually, $I$ itself is an ideal of definition.) Thus, it is enough to show the surjectivity of the map $G(W_0(A)) \to G(W_0(A)/[I])$.

Let $a$ be an element of $G(W_0(A)/[I])$. We successively construct its $\pi^n$-approximation $a_n \in G(W_{0,n}(A))$. First, we consider the case $n = 1$. In this case, the claim is that $G(A) \to G(A/I)$ is surjective. Since $G$ is finitely presented, an $(A/I)$-valued point of $G$ can be lifted to an $(A_0/(I \cap A_0))$-valued point for some deperfection $A_0$ of $A$. Since $A_0$ is a complete local Noetherian ring, and thus $(I \cap A_0)$-adically complete, such a lift can also be lifted to an $A_0$-valued point, then to an $A$-valued point. Next, suppose that we have a lift $a_n$ for some $n$. Take a lift $a'$ of $a_n$ to a $W_{0,n+1}(A)$-valued point. The difference between $a'$ and $a$ in $G(W_{0,n+1}(A)/[I])$ is trivial modulo $\pi^n$, so it corresponds to an element of $\text{Lie}_{A/I} G$. Since $\text{Lie}_A G \to \text{Lie}_{A/I} G$ is surjective, the difference can be lifted to a $W_{0,n+1}(A)$-valued point trivial modulo $\pi^n$. By modifying $a'$ by the lift, we have a lift $a_{n+1}$ compatible with $a_n$. Finally, we can take a lift of $a$ as $\lim_{n \to \infty} a_n$. □

We fix a trivialization $t$ of $\mathcal{E}$ constructed in Lemma 2.7. Let $\mathcal{P} = (G, t^{-1} \sigma)$ be an $F$-crystal with $G$-structure on $A$. By [RR96, Theorem 3.6], we have the Newton stratification of $\mathcal{P}_n$ on $\text{Spec}(A)$. This satisfies the purity property by [Vas06, Main Theorem B], [HV11, Theorem 7.4]. As the perfection preserves the purity by [Zhu17, Lemma A.7(6)], the stratification, even if regarded as a stratification on $\text{Spec}(A_0)$ for some deperfection $A_0$ of $A$, satisfies the purity property. Thus, we may apply the arguments in [HV11, Corollary 7.7]. Let $\nu$ be the Newton polygon of $b$ and let $N_\nu$ be the closed Newton stratum of $\mathcal{P}_\nu$ on $\text{Spec}(A)$. We have the following:
Proposition 2.8. \( \dim N_\nu \geq (\rho, \mu + \nu) - \defi(b)/2 \).

Proof. By [Zhu17, Proposition 1.23], we have \( \dim A_0 = (2\rho, \mu^*) = (2\rho, \mu) \). Thus, the same argument as in [HV11, Proposition 7.8] can be applied.

Since the Kottwitz map is locally constant by [Zhu17, Proposition 1.21], \( \mathcal{P}_\eta \) is isomorphic to \( \mathcal{E}_{\nu}^b \) at each geometric point of \( N_\nu \).

From now on, we assume that \( b \) is basic. Let \( X_{\leq \mu}(b)^{\wedge}/g \) be the spectrum of the formal local ring of \( X_{\leq \mu}(b) \) at \( g \) and let \( N_\nu \) be the completion of \( N_\nu \) along an ideal of definition. We construct a morphism from \( N_\nu \to X_{\leq \mu}(b)^{\wedge}/g \), by giving a quasi-isogeny from the “universal” \( F \)-crystal with \( G \)-structure to a constant one.

Lemma 2.9. (cf. [HV11, Proposition 8.1]) Let \( R \) be the perfection of a complete local Noetherian ring over \( \mathbb{R} \) and let \( P = (G, x) \sigma \) be an \( F \)-isocrystal with \( G \)-structure on \( R \). Suppose that \( x \) is congruent to \( b' \) modulo an \( F \)-standard ideal of definition \( J \) of \( R \) and \( P \) is isomorphic to \( \mathcal{E}_{\nu}^b \) at each geometric point of \( \text{Spec}(R) \). Then, on the completion \( R^\wedge \) of \( R \), there exists an isomorphism \( P \cong \mathcal{E}_{\nu_0}^b \), which is trivial modulo \( J \).

Proof. We may assume that \( b' \) is decent (cf. [RZ96, Definition 1.8]). Let \( s \) be a positive integer satisfying \( sv \in X^*(T) \) and \( (b' \sigma)^s = sv(\pi)\sigma^s \). We construct an element \( \epsilon \in G(W_O(R^\wedge)(\overline{\eta})) \) which is trivial modulo \( J \) and satisfies \((x\sigma)^s(\epsilon) = sv(\pi)\epsilon \). We define its approximate solution \( \epsilon_0, \epsilon_1, \ldots \) as follows: \( \epsilon_0 = 1, \epsilon_{n+1} = sv(\pi)(-\epsilon_n)^s(\epsilon_n) \). Since \( \epsilon_{n+1} = sv(\pi)^{-ns}(x\sigma)^{ns}(\epsilon_1) \) and \( \epsilon_1 \) is congruent to 1 modulo \( J \), \( \epsilon_{n+1} \) is congruent to \( \epsilon_n \) modulo \( J^{n+1} \). We also see that \((x\sigma)^s(\epsilon_n) \) is congruent to \( sv(\pi)\epsilon_n \) modulo \( J^{n+1} \). We prove that the limit of the sequence \( \epsilon_0, \epsilon_1, \ldots \), which is a priori an element of \( G(W_O(R)(\overline{\eta}); \mathcal{E}_{\nu}^b) \), lies in \( G(W_O(R^\wedge)(\overline{\eta})) \). It is enough to check the boundedness of the sequence with respect to the \( \pi \)-adic topology. Since the number of minimal points of \( \text{Spec}(R) \) is finite, it is enough to check at each minimal point of \( \text{Spec}(R) \). Let \( \xi \) be a minimal geometric point of \( \text{Spec}(R) \) with an algebraically closed residue field \( \kappa(\xi) \). Let \( \bar{x} \) be the image of \( x \) in \( G(W_O(\kappa(\xi))(\overline{\eta})) \). By the constancy of \( P \), we may write \( \bar{x} = y^{-1}b'\sigma(y) \) with \( y \) an element of \( G(W_O(\kappa(\xi))(\overline{\eta})) \). Then, we have \( \bar{\epsilon}_n = y^{-1}\sigma^{ns}(y) \). Thus, the sequence \( \epsilon_1, \epsilon_2, \ldots \) is \( \pi \)-adically bounded.

Let \( \bar{x}' = \epsilon^{-1}x\sigma(\epsilon) \). We see that \( \bar{x}' \) is congruent to \( b' \) modulo \( J \) and satisfies \((x'\sigma)^s = sv(\pi)\sigma^s \). We inductively prove that \( \bar{x}' \) is congruent to \( b' \) modulo \( J^{n+1} \) for every integer \( n \). Suppose that \( \bar{x}' \) is congruent to \( b' \) modulo \( J^{n+1} \). By \((x'\sigma)^s(x'\sigma) = (x'\sigma)(x'\sigma)^s \), we have \( \sigma^s\bar{x}' = \bar{x}' \). Similarly, we have \( \sigma^sb' = b' \). Since \( \sigma^s\bar{x}' \) is congruent to \( \sigma^sb' \) modulo \( J^{n+1} \), \( \bar{x}' \) is congruent to \( b' \) modulo \( J^{n+1} \). Thus, we have \( \bar{x}' = b' \), so \( \epsilon \) gives the desired isomorphism.

Theorem 2.10. There exists an adic morphism from \( N_\nu^{\wedge} \to X_{\leq \mu}(b)^{\wedge}/g \). We have \( \dim N_\nu \leq \dim X_{\leq \mu}(b)^{\wedge}/g \).

Proof. Let \( i: \mathcal{P}_\eta \to \mathcal{E}_{\nu}^b \) be an isomorphism constructed in Lemma 2.9. There is a morphism \( \varphi: \mathcal{N}_\nu \to X_{\leq \mu}(b) \) associated to the object \((\mathcal{G}, t_{\nu_{\varphi}}^{-1} \sigma, g \circ i) \). Since the closed point of \( \mathcal{N}_\nu^{\wedge} \) maps to \( g \), \( \varphi \) factors through \( X_{\leq \mu}(b)^{\wedge}/g \). We prove that \( \varphi \) is adic.

Let \( \mathcal{N}_\nu = \text{Spf}(A') \), \( X_{\leq \mu}(b)^{\wedge}/g = \text{Spec}(B) \) and let \( J \) be an ideal of definition of \( B \). The quotient \( A'' \) of \( A' \) by the closure of \((J \cdot A')^{\text{perf}} \) is a complete, perfect \( A \)-algebra. By the universality of \( X_{\leq \mu}(b) \), there exists an isomorphism \((\mathcal{G}, t_{\nu_{\varphi}}^{-1} \sigma, g \circ i_{A''}) \cong (\mathcal{G}, b' \sigma, g) \). Thus, \( i_{A''} \) is given by an element \( \epsilon \in \mathcal{G}(W_O(A'')) \). We see that \( t_{\nu_{\varphi}} = \sigma(\epsilon)b'^{-1} \) and \( \epsilon \) is trivial.
modulo an $F$-standard ideal of definition $I$ of $A''$. Now, if $\epsilon$ is trivial modulo $I^{n''}$, then $[t_{A''}]$ and $[b''^{-1}]$ are congruent modulo $I^{n''+1}$, and by the continuity of $t$ in Lemma 2.7, $t_{A''}$ and $b''^{-1}$ are congruent modulo $I^{n''+1}$, and thus $\epsilon$ is trivial modulo $I^{n''+1}$. Since $A''$ is $I$-adically separated, we have $\epsilon = 1$. In particular, $[t_{A''}] = [b''^{-1}]$, and by the universality of $\text{Gr}^W_{\mu''}$, the canonical map $\text{Spec}(A'') \to \text{Gr}^W_{\mu''}$ factors through $[b''^{-1}]$. Thus, the natural map $\mathcal{A} \to A''$ factors through the residue field of $A$, and so is the quotient map $A' \to A''$. In particular, $J \cdot A'$ is open, thus $\varphi$ is adic.

By the definition of dimensions in Definition 1.9, we have $\dim N^\wedge_\varphi \leq \dim X_{\leq \mu}(b)^\wedge$. By Lemma 1.13 and Lemma 1.14, $\dim N_\varphi = \dim N^\wedge_\varphi$ holds, so we have the claim.

**Theorem 2.11.** When $b$ is basic, the closed affine Deligne-Lusztig variety $X_{\leq \mu}(b)$ is equidimensional, and the affine Deligne-Lusztig variety $X_\mu(b)$ is a dense open subscheme of $X_{\leq \mu}(b)$.

**Proof.** The dimension of $X_{\leq \mu}(b)$ is $\langle \rho, \mu - \nu \rangle - \text{def}(b) / 2$ by [Zhu17, Theorem 3.1]. When $b$ is basic, we have $\langle \rho, \nu \rangle = 0$. Thus, the dimension of $X_{\leq \mu}(b)$ at each closed point is equal to the global dimension by Proposition 2.8 and Theorem 2.10. Since the dimension of the complement of $X_{\mu}(b)$ is smaller than the dimension of $X_{\leq \mu}(b)$, $X_\mu(b)$ must be dense in $X_{\leq \mu}(b)$.

**References**

[BSS17] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423.

[Gas10] Qëndrim R. Gashi, *On a conjecture of Kottwitz and Rapoport*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 6, 1017–1038.

[Ham15] Paul Hamacher, *The dimension of affine Deligne-Lusztig varieties in the affine Grassmannian*, Int. Math. Res. Not. IMRN (2015), no. 23, 12804–12839.

[HV11] Urs Hartl and Eva Viehmann, *The Newton stratification on deformations of local $G$-shtukas*, J. Reine Angew. Math. **656** (2011), 87–129.

[HV18] Paul Hamacher and Eva Viehmann, *Irreducible components of minuscule affine Deligne-Lusztig varieties*, Algebra Number Theory **12** (2018), no. 7, 1611–1634.

[Nie22] Sian Nie, *Irreducible components of affine Deligne-Lusztig varieties*, Camb. J. Math. **10** (2022), no. 2, 433–510.

[Rap05] Michael Rapoport, *A guide to the reduction modulo $p$ of Shimura varieties*, no. 298, 2005, Automorphic forms, 1, pp. 271–318.

[RR96] Michael Rapoport and Melanie Richartz, *On the classification and specialization of $F$-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181.

[RZ96] Michael Rapoport and Thomas Zink, *Period spaces for $p$-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on $p$-adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020.

[Vas06] Adrian Vasiu, *Crystalline boundedness principle*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 2, 245–300.

[Zhu17] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Ann. of Math. (2) **185** (2017), no. 2, 403–492.

[ZZ20] Rong Zhou and Yihang Zhu, *Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties*, Camb. J. Math. **8** (2020), no. 1, 149–241.