Manifestly covariant current matrix elements in the Point Form Relativistic Hamiltonian Dynamics

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Abstract

A manifestly covariant expression for the current matrix elements of three quark bound systems is derived in the framework of the Point Form Relativistic Hamiltonian Dynamics. The relativistic impulse approximation is assumed in the model. A critical comparison is made with other expressions usually given in the literature.

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1. Introduction

Aim of the present work is to show that, in the context of the Point Form Relativistic Hamiltonian Dynamics (PF RHD), it is possible to write, for hadronic bound systems, manifestly covariant matrix elements of the current operators.

For clarity, we define as manifest covariance the property of an equation of being written in terms of quantities that (i) transform in a well-known way under Lorentz transformations and (ii) are not related to a specific reference frame.

We use a relativistic impulse approximation [1,2] (RIA), that generalizes at a relativistic level the widely known model used for the study of the electromagnetic interactions of nuclear systems.

We shall examine in more detail the case of the four-vector electromagnetic current, but the method has been developed to study any kind of interaction,
in particular the axial-vector case [3,4] that is relevant for the weak structure of the hadrons.

In this article we shall specifically refer to the nucleon as a system composed by \( N = 3 \) (identical), \( s = 1/2 \), constituent quarks. The same procedure can be immediately generalized to the study of the \( N \neq 3 \) composite systems. The cases of nonidentical constituents and \( s \neq 1/2 \) will be studied, within the same theoretical framework, in subsequent works.

We also show that our model, that is the result of an independent investigation [5], is completely equivalent to the standard formalism (SF) of PF RHD, developed in refs.[1-3] and implemented with great clarity and precision in ref.[6]. In this concern we anticipate that the main differences between our model and SF are the following:

(a) we use, in our work, the formalism of the Dirac equation boosting covariantly the Dirac spinors, while in SF Wigner rotation matrices (not manifestly covariant) are employed;

(b) the spatial part of the bound system four-velocity and the (independent) three-momenta of two quarks are used in our work as spatial variables for the representation (or projection) states, while the velocity states are considered in SF; the use of these states, that are related to the rest frame of the bound system does not fulfill the requirement (ii) given above to have manifestly covariant equations;

(c) furthermore, we perform manifestly covariant integrations over that spatial variables to calculate the matrix elements of the current operators.

We highlight that the electroweak matrix elements calculated by means of PF RHD allow to reproduce with good accuracy the experimental nucleon form factors [2-7].

The model studied in the present work allows for two further developments (to be studied in different works): the definition of a dynamically conserved electromagnetic current by means of a suitable minimal coupling procedure and the introduction of (virtual) negative energy states in the current matrix elements.

The paper is organized as follows. In sect.2 we revise the construction of the Poincaré algebra generators, in-
Introducing, at the same time, the operators that are used to describe the dynamics of the bound system. In sect.3, by introducing the projection states of our model, we define the wave functions for the bound system in the framework of PF RHD, also discussing their boost properties. In sect.4 we explicitly construct the Dirac wave functions of our formalism, showing the equivalence of their boost properties with those of the SF. Finally, in sect.5, the matrix elements of the current operators are studied by means of the RIA. The main result of this work is our covariant expression given in eq.(5.3c). An accurate comparison with SF is performed transforming eq.(5.3c) into the standard form of eq.(5.20).

2. The Poincaré Algebra

In the present work, considering particles of mass $m$, always on-shell, we transform the four-momentum $p^\mu = (\epsilon(p), p)$ by means of a canonical boost written in the following standard form

$$\begin{align*}
\epsilon_b(p; v) &= \epsilon(p_b(p; v)) = v^0\epsilon(p) + vp \\
p_b(p; v) &= p + v(\frac{1}{v^0 + 1} + \epsilon(p))
\end{align*}
$$

The two previous equations are usually resumed in the form

$$p_b^\mu = L^\mu_\nu(v)p^\nu$$

In eqs.(2.1a,b) we have introduced the time component of the four-momentum of the particle, i.e. the energy, as $\epsilon(p) = [(p^2 + m^2)^{1/2}]$ and $v^\mu = ([v^2 + 1]^{1/2}, v)$, that is the four-velocity boost parameter. We recall that the physical velocity of the initial frame measured from the boosted one is $u = v/v^0$. The independent transformation equation, that is used to define the boost in the Hilbert space, is eq.(2.1b), while eq.(2.1a) can be obtained from that one by calculating the on-shell energy of the particle with the boosted momentum.

As anticipated, we follow the scheme of the PF RHD when defining the generators of the Poincaré algebra [1]. In more detail, for a system of three quarks, the total angular momentum $J$ and the total boost $K$, being free of
the interaction, are written as the sum of the single particle generators, in the form:

\[ J = \sum_{i=1}^{3} (r_i \times p_i + s_i) \]  

(2.2a)

\[ K = \sum_{i=1}^{3} \left[ \frac{1}{2} (r_i \epsilon(p_i) + \epsilon(p_i) r_i) + \frac{p_i \times s_i}{\epsilon(p_i) + m} \right] \]  

(2.2b)

where \( p_i, r_i, s_i = \frac{1}{2} \vec{\sigma}_i \), \( m \) and \( \epsilon(p_i) \) respectively represent the three-momentum, the conjugated (position) variable, the spin, the mass and the energy of the \( i \)-th quark.

For completeness we also give the expression of the finite boost operator, that in the PF RHD is not modified by the interaction:

\[ B(v) = \exp(iK \cdot U) \simeq 1 + i\delta u \cdot K \]  

(2.2c)

with

\[ U = \frac{v}{|v|} \tanh^{-1}(\frac{|v|}{v^0}) = \frac{u}{|u|} \tanh^{-1}(|u|) \]  

(2.2d)

On the other hand, the total four-momentum operator of the system, that is \( P^\mu = (P^0 = H, \, P) \) depends on the interaction among the constituent quarks. We shall define the operator \( P^\mu \) in eq.(2.19). To this aim, we have, previously, to introduce:

(a) the quantum-mechanical operator \( V^\mu \), that represents the four-velocity of the bound system measured from a generic reference frame (GF);

(b) the other dynamical variables of the quantum mechanical model.

We first consider point (a), that is the construction of the quantum mechanical operator \( V^\mu \). In order to help the reader to understand the physical meaning of the following procedure, we note that the four-momentum \( P^\mu \) of a system, as a classical quantity, can be written in terms of \( V^\mu \) as

\[ P^\mu = MV^\mu \]  

(2.3)

where the physical mass \( M \) of the (bound) system has been introduced. The corresponding quantum-mechanical expression will be given in eq.(2.19).
To derive this expression, we have to write $V^\mu$ as a function of the momenta of the constituents. As first step, we introduce the rest frame (RF) four-momentum of the i-th quark

$$p_i^{\ast \mu} = (\epsilon(p_i^\ast), p_i^\ast)$$

(2.4)

Here and in the following, the asterisk denotes the quantities observed in the RF. The sum of the four-momenta of the three constituent quarks, is, by definition of the RF

$$\sum_{i=1}^{3} p_i^{\ast \mu} = (\sum_{i=1}^{3} \epsilon(p_i^\ast) = M_f, 0)$$

(2.5)

where we have also introduced $M_f$ that represents the free mass operator of the system. By applying the Lorentz transformation of eqs.(2.1a,b) (as a function of the parameter $V$) to the $p_i^{\ast \mu}$ and also using eq.(2.5), one can write the sum of the four-momenta of the particles in a GF as

$$\sum_{i=1}^{3} p_i^\mu = V^\mu M_f$$

(2.6)

with

$$p_i^\mu = (\epsilon(p_i), p_i)$$

We highlight that $M_f$, as defined in eq.(2.5), is a nonvanishing and Lorentz invariant quantity. The nonvanishing character of $M_f$ allows to solve the previous equation with respect to $V^\mu$. Lorentz invariance allows to write $M_f$ in terms of the $p_i^\mu$ observed in a GF. In this way one can express $V^\mu$ as a function the $p_i^\mu$, or, more precisely, of the three-momenta $p_i$. (Note that, the $p_i$, with $i = 1, 2, 3$ represent, in the first step of the construction, the spatial dynamical variables of the relativistic model. The final choice of the spatial variables will be given in the following.)

In more detail, $M_f$ is expressed as a function of the momenta in a GF in the form

$$M_f = M_f(p_1, p_2, p_3) = \left[ \sum_{ij=1}^{3} p_i^\mu p_j^\nu g_{\mu\nu} \right]^{1/2}$$

(2.7)

that will be taken as the definition of the operator $M_f$. In consequence, we can also write
\[ V^\mu(p_1, p_2, p_3) = [M_f(p_1, p_2, p_3)]^{-1} \sum_{i=1}^{3} p_i^\mu \] (2.8a)

and, obviously
\[ V^\mu V_\mu = 1 \] (2.8b)
\[ V^0(V) = [1 + V^2]^{1/2} \] (2.8c)

Let us note that the observable four-vector \( V^\mu \), as given in eq.(2.8a), transforms in the same way as a standard four-momentum, that is replacing \( p \) with \( V \), \( \epsilon(p) \) with \( V^0(V) \) in eq.(2.1b). In this way we introduce
\[ V^0_b = V^0_b(V; v) = V^0(V_b(V; v)) \] (2.9a)
\[ V_b = V_b(V; v) \] (2.9b)

This result, that is also consistent with eq.(2.3), can be easily derived by transforming, with the help of eqs.(2.1a,b), the \( p_i^\mu \) that appear in eq.(2.8a).

As for point (b), we can now introduce the final choice for the complete set of commuting operators that will be used for the quantum mechanical description of the system. To this aim we note that, due to its definition in eq.(2.8a), the operator \( V^\mu \) commutes with the momenta of all the particles. In consequence, it is possible to choose the following operators:
(i) as spatial variables, the three-momenta of 2 quarks, say \( p_2, p_3 \), and the spatial components of the four-velocity \( V \); those variables replace the first step choice of \( p_1, p_2, p_3 \);
(ii) the spin operators of the three quarks; the eigenvalues of their projections on the z axis will be denoted as \( \sigma_1, \sigma_2, \sigma_3 \).

For further developments, it is necessary to express \( p_i^\mu \) and \( M_f \) as functions of \( p_2, p_3 \) and \( V \).

First, we recall that the rest frame quark energies are invariant quantities [8], that can be written as
\[ \epsilon^*_i = \epsilon(p^*_i) = V_\mu p^\mu_i \] (2.10)

Second, we write eq.(2.6) in the form
\[ p_2^\mu + p_3^\mu = -p_1^\mu + V_\mu \cdot [\epsilon(p^*_1) + \epsilon(p^*_2) + \epsilon(p^*_3)] \] (2.11)
Then, squaring both sides, with the help of eq.(2.10), one obtains the RF energy of the quark #1 as a function of $p_1$, $p_2$ and $V$:

$$\epsilon(p_1^*) = \epsilon_1^*(p_2, p_3, V) =$$

$$m^2 - (p_2^\mu + p_3^\mu)(p_2^\nu + p_3^\nu)g_{\mu\nu} + [(p_2^\mu + p_3^\mu)V_{\mu}]^2 \right)^{1/2}$$

(2.12)

where $p_2^\mu$, $p_3^\mu$ and $V^\mu$ are functions of $p_2$, $p_3$, and $V$, respectively. Finally, we find

$$M_f(p_2, p_3, V) = (p_2^\mu + p_3^\mu)V_{\mu} + \epsilon_1^*(p_2, p_3, V)$$

(2.13)

and, by means of eq.(6)

$$p_1^\mu(p_2, p_3, V) = -(p_2^\mu + p_3^\mu) + V^\mu \cdot M_f(p_2, p_3, V)$$

(2.14)

By definition, $M_f$ is a Lorentz invariant operator, that is

$$[K, M_f] = 0$$

(2.15)

We now introduce the interaction among the quarks by means of the total mass operator $M$ that, according to the Bakamjian-Thomas construction [1,9], is defined as

$$M = M_f + W$$

(2.16)

where $W$ represents a Lorentz invariant interaction operator, that means

$$[K, W] = 0$$

(2.17a)

and, in consequence,

$$[K, M] = 0$$

(2.17b)

In this work, we do not enter into the details of the definition of $W$. We only point out that rotationally scalar operators, defined in the RF (as the phenomenological potentials generally used for the relativized constituent quark models, in particular the hypercentral potentials [10]), are formally Lorentz invariant and can be also written in an explicit invariant form by means of the dynamical variables of the model. Note that, if the interaction operator $W$ represents a quasi-potential derived from an underlying field theory, its expression is, in general, highly momentum dependent.

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In any case, being defined in the RF, the interaction operator \( W \) has nonvanishing matrix elements only between states with the same \( V^\mu \), that means
\[
[V^\mu, W] = 0 \quad (2.18)
\]
We can now introduce the generators of the time and space translation of the system, that is the four-momentum operator, as
\[
P^\mu = M \cdot V^\mu = (M_f + W) \cdot V^\mu \quad (2.19)
\]
that is the same expression of eq.(2.3), but considered as a definition of a quantum mechanical operator.

Standard calculations \[1\] show that the total generators defined in eqs.(2.2a,b) and (2.19) fulfill the Poincaré group commutation rules \[11,12\].

3. The wave functions of the model

We now turn to introduce the representation states that will be used to write down explicitly the wave functions of the model. Following the definitions of the dynamical variables given in the previous section, one has
\[
|\psi_r> = |p_2, p_3; V; \sigma_1, \sigma_2, \sigma_3>
\]
with the normalization
\[
<\psi_r|\psi'_r> = <p_2, p_3; V; \sigma_1, \sigma_2, \sigma_3|p'_2, p'_3, V'; \sigma'_1, \sigma'_2, \sigma'_3> =
\]
\[
\delta(p'_2 - p_2)\delta(p'_3 - p_3)\delta(V' - V)\delta_{\sigma'_1 \sigma_1}\delta_{\sigma'_2 \sigma_2}\delta_{\sigma'_3 \sigma_3} \quad (3.2)
\]
The choice of eq.(3.1), as it will be shown in sect.4, helps to introduce in a very clear way the relativistic impulse approximation for the current matrix elements. On the other hand, in SF a different type of representation states, currently denoted as *velocity states*, is generally used to study the relativistic bound state wave functions. In the velocity states the spatial variables are represented by \( V \) and by the three (not independent) rest frame momenta \( p_1^*, p_2^*, p_3^* \) or better by the two (independent) Jacobi momenta \( p_\rho, p_\lambda \). As shown in ref.[2], the Lorentz transformation of these states is given by the
standard boost of $V$ and by a Wigner rotation of the rest frame momenta or of the Jacobi momenta. If also the spin projections are referred to the RF, the same Wigner rotation acts on the spin variables.

By applying the boost operator of eq.\((2.2c)\) to the representation states, one obtains:

$$B(v)|p_2, p_3, V; \sigma_1, \sigma_2, \sigma_3 >=$$

$$R(p_1; v)R(p_2; v)R(p_3; v)|p_{2b}, p_{3b}, V_b; \sigma_1, \sigma_2, \sigma_3 > G(p_2, p_3, V; v)$$ \hspace{1cm} (3.3a)

with

$$G(p_2, p_3, V; v) = \left[ \frac{\epsilon_b(p_2; V)}{\epsilon(p_2)} \frac{\epsilon_b(p_3; V)}{\epsilon(p_3)} \frac{V_0(V; v)}{V_0(V)} \right]^{1/2}$$ \hspace{1cm} (3.3b)

The previous equations show that the action of the boost operator on the representation states can be divided into a spatial (a) and a spin (b) part.

(a) The spatial part, denoted in the following as $\hat{B}(v)$, produces an eigenstate of the boosted momenta $p_{2b}, p_{3b}, V_b$ that are taken as functions of the corresponding unboosted variables by means of eq.\((2.1b)\); the numerical factor $G(p_2, p_3, V)$ is due to the nonlinearity, with respect to the momenta, of the boost generator of eq.\((2.2b)\) and provides for the correct normalization of the state, being $\hat{B}(v)$ a unitary operator. To simplify further developments we introduce the following spatial matrix element

$$< p_2, p_3, V|\hat{B}(v)|p_2', p_3', V' >= G(p_2, p_3, V; v)$$

$$\delta(p_2 - p_{2b}(p_2'; v))\delta(p_3 - p_{3b}(p_3'; v))\delta(V - V_b(V'; v))$$ \hspace{1cm} (3.4)

and recall the following property of the delta functions

$$\delta(p_i - p_{ib}(p_i'; v)) = \delta(p_i' - p_{ib}(p_i; -v)) \frac{\epsilon(p_i')}{\epsilon_b(p_i'; v)}$$ \hspace{1cm} (3.5)

with $i = 2, 3$. Note that $p_{ib}(p_i; -v)$ represents the inverse Lorentz transformation on $p_i$ that is obtained using in eq.\((1.b)\) the boost parameter $-v$.

(b) In eq.\((3.3a)\) the spin part is given by the product of the $R(p_i; v)$ that represent the Wigner spin rotation operators (due to the second term in the generator of eq.\((2.2b)\)) that depend on the numerical values of the $p_i$. The (not independent) momentum $p_1$ is obtained by means of eq.\((2.14)\).
considering the Pauli spinor representation for the spin states, for further developments we introduce the following matrix elements

$$w_{\sigma_i}^+ R(p_i; v) w_{\sigma_i} = R_{\sigma_i' \sigma_i}(p_i; v)$$  \hspace{1cm} (3.6)

In SF the matrix elements of the spin rotation operators have been denoted as

$$R_{\sigma_i' \sigma_i}(p_i; v) = D_{\sigma_i}^{1/2} [R_W(p_i, B(v))]$$  \hspace{1cm} (3.7)

Such notation is used to represent the spin 1/2 rotation matrices considered as functions of the Wigner rotation related to the momentum $p_i$ and to the boost $B(v)$.

The wave function of our model is determined in the RF, as a function of the Jacobi momenta $p_\rho, p_\lambda$ solving the mass eigenvalue equation for the mass operator introduced in eq.(2.16). This solution is a velocity state solution with $V = 0$, as indicated in the next equation by a Dirac $\delta$ function. It is written as

$$\psi_{RF}(p_\rho, p_\lambda, V) = \psi^{J \Sigma}(p_\rho, p_\lambda) \delta(V)$$  \hspace{1cm} (3.8a)

with

$$\psi^{J \Sigma}(p_\rho, p_\lambda) = \sum_{\sigma_1, \sigma_2, \sigma_3} \psi^{J \Sigma}_{\sigma_1 \sigma_2 \sigma_3}(p_\rho, p_\lambda) w_{\sigma_1} w_{\sigma_2} w_{\sigma_3}$$  \hspace{1cm} (3.8b)

In the previous expression $J, \Sigma$ respectively represent the total angular momentum (absolute value) and its projection on the z axis. This state is constructed by coupling the angular momenta with Clebsch-Gordan coefficients, for example according to the standard scheme [13]

$$[[l_\rho \otimes l_\lambda]^L \otimes S]^{J \Sigma}$$

with

$$[[s_1 \otimes s_2]^{S_{12}} \otimes s_3]^{S_{MS}}$$

Note that, in eq.(3.8b) the dependence on the quark Pauli spinors has been highlighted in order to make a comparison with SF.

For the following developments, it is convenient to introduce, as spatial variables, instead of the Jacobi momenta, the RF momenta $p_\rho^*, p_\lambda^*$. The former and the latter momenta are connected by a standard linear relation. We have
\[ \psi^{J\Sigma}(\mathbf{p}_2^*, \mathbf{p}_3^*) = j^{1/2} \psi^{J\Sigma}(\mathbf{p}_\rho(\mathbf{p}_2^*, \mathbf{p}_3^*), \mathbf{p}_\lambda(\mathbf{p}_2^*, \mathbf{p}_3^*)) \]  

(3.9)

where \( j^{1/2} \) represents the (numerical) constant factor that is used to keep the normalization to unity for the wave function when using the new variables \( \mathbf{p}_2^*, \mathbf{p}_3^* \). The wave function of the previous equation can be decomposed with respect to the Pauli spinors in the same way as the wave function given in eq.(3.8b). As before,

\[ \psi_{RF}(\mathbf{p}_2^*, \mathbf{p}_3^*, \mathbf{V}) = \psi^{J\Sigma}(\mathbf{p}_2^*, \mathbf{p}_3^*) \delta(\mathbf{V}) \]  

(3.10)

We can now determine the wave function of the system in a GF boosting the RF wave function given in the previous equation. We use the boost parameter \( \mathbf{v}_G \), that, as usual, represents the spatial part of the four-velocity of the bound system observed from the GF. One has

\[ \psi_G(\mathbf{p}_2, \mathbf{p}_3, \mathbf{V}) = \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{V} | B(\mathbf{v}_G) | \psi_{RF} \rangle = \int d^3p_2' d^3p_3' d^3\mathbf{V}' \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{V} | B(\mathbf{v}_G) | \mathbf{p}_2', \mathbf{p}_3', \mathbf{V}' \rangle \langle \mathbf{p}_2', \mathbf{p}_3', \mathbf{V}' | \psi_{RF} \rangle \]  

(3.11)

By using the explicit expression of the RF wave function of eq.(3.10), the property of the spatial part of the boost and of the \( \delta \) functions, respectively given in eqs.(3.4) and (3.5) and, finally, the spin rotation operators of eq.(3.6), one obtains

\[ \psi_G(\mathbf{p}_2, \mathbf{p}_3, \mathbf{V}) = R(\mathbf{p}_G^*; \mathbf{v}_G) R(\mathbf{p}_{2G}^*; \mathbf{v}_G) R(\mathbf{p}_{3G}^*; \mathbf{v}_G) \]

\[ \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{V} | \hat{B}(\mathbf{v}_G) | \psi_{RF} \rangle \]  

(3.12a)

where we have introduced the spatial part of the boosted wave function

\[ \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{V} | \hat{B}(\mathbf{v}_G) | \psi_{RF} \rangle = \left[ \frac{\epsilon(\mathbf{p}_{2G}^*) \epsilon(\mathbf{p}_{3G}^*)}{\epsilon(\mathbf{p}_2) \epsilon(\mathbf{p}_3)} \right]^{1/2} \left( 1 + v_{2G}^2 \right)^{-1/4} \]

\[ \psi^{J\Sigma}(\mathbf{p}_{2G}^*, \mathbf{p}_{3G}^*) \delta(\mathbf{V} - \mathbf{v}_G) \]  

(3.12b)

also, by means eq.(2.1b), we have used

\[ \mathbf{p}_{iG}^* = \mathbf{p}_0(\mathbf{p}_i; -\mathbf{v}_G) \]  

(3.12c)
that represent the rest frame three-momenta considered as functions of the three-momenta of the GF, transformed by means of the parameter $v_G$.

4. The Dirac equation formalism

In order to construct operators that manifestly transform as Lorentz tensors, it is very useful to make use of the Dirac equation formalism. First, we define the RF Dirac wave function in the form

$$\psi_{RF}(\mathbf{p}_2^*, \mathbf{p}_3^*, \mathbf{V}) = u(\mathbf{p}_1)u(\mathbf{p}_2^*)u(\mathbf{p}_3^*)\psi_{RF}(\mathbf{p}_2^*, \mathbf{p}_3^*, \mathbf{V}) \quad (4.1a)$$

with the Dirac spinors,

$$u(p_i) = \frac{1}{\sqrt{2m}} \left[ \sqrt{\epsilon(p_i) + m} \frac{(p_i, \vec{\sigma})}{\sqrt{\epsilon(p_i) + m}} \right] \quad (4.1b)$$

For brevity, we denote these quantities, here and in the following, as (positive energy) Dirac spinors, taking into account that they represent $4 \times 2$ matrices acting onto the Pauli spinors $w_{\sigma_i}$ contained in $\psi_{RF}(\mathbf{p}_2^*, \mathbf{p}_3^*, \mathbf{V})$. They are covariantly normalized as $\bar{u}(p_i)u(p_i) = 1$.

We recall that the Dirac spinors are boosted by means of the nonunitary Dirac boost operator

$$B_i^D(v) = [B_i^D(v)]^+ = \left[ \frac{1}{2} (v^0 + 1) \right]^{1/2} + \left[ \frac{1}{2} (v^0 - 1) \right]^{1/2} \frac{(v \gamma_i^0, \vec{\gamma}_i)}{|v|} \quad (4.2)$$

where we have introduced the Dirac the gamma matrices $\gamma_i^\mu = (\gamma_i^0, \gamma_i^\mu)$ for the $i$-th particle; also, $v^0$ is the time component of the four-velocity boost parameter. Standard calculations show the following very important property of the Dirac boost when applied to the Dirac spinors

$$B_i^D(v)u(p_i) = u(p_0(p_i; v))R(p_i; v) \quad (4.3)$$

It shows that the Dirac boost produces a Dirac spinor of the boosted momentum applied to the spin rotation operator, given in eq.(3.7), that acts onto the Pauli spinor.

Introducing

$$B^D(v_G) = B_1^D(v_G) \otimes B_2^D(v_G) \otimes B_3^D(v_G) \quad (4.4)$$
we now construct the GF Dirac wave function for the three quark system by means of the following boost

$$\psi_D^G(p_2, p_3; V) = B_D^G(v_G)u(p^*_1G)u(p^*_2G)u(p^*_3G)$$

$$< p_2, p_3, V | \hat{B}(v_G) | \psi_{RF} > = u(p_1)u(p_2)u(p_3)\psi_G(p_2, p_3, V)$$

where eqs.(3.12a,b) and (4.3) have been taken into account. Also, equivalently, making explicit use of eq.(3.12b), one can write

$$\psi_D^G(p_2, p_3, V) = \left[ \frac{\epsilon(p^*_2G)\epsilon(p^*_3G)}{\epsilon(p^*_2)\epsilon(p^*_3)} \right]^{1/2} \varphi_D^G(p_2, p_3; v_G)(1 + v_G^2)^{-1/4} \delta(V - v_G)$$

with

$$\varphi_D^G(p_2, p_3; v_G) = B_D^G(v_G)u(p^*_1G)u(p^*_2G)u(p^*_3G)\psi_D^G(p_2, p_3; v_G)$$

The expression $$\psi_D^G(p_2, p_3, V)$$ of eq.(4.5c) is the boosted Dirac wave function of the model. Also, $$\varphi_D^G(p_2, p_3; v_G)$$ of eq.(4.5d) can be defined as the boosted intrinsic Dirac wave function. This expression will be used in the next section for writing the manifestly covariant current operators.

Finally, we recall that, in all the previous eqs.(4.5a-d), the expression of the $$p_iG$$ given in eq.(3.12c) must be used.

The previous discussion has been focussed on the boost transformation from the RF to a GF. However, recalling the general property of eq.(4.3), one can immediately verify the equivalence of our model to SF in the case of a transformation from a GF to another GF.

5. The matrix elements of the current operators. Comparison with SF.

In this section we first examine the construction of transition matrix elements introducing the RIA; then, we critically discuss the equivalence of our formalism with SF. We recall that, in order to compare the theoretical model with the experimental data, the electromagnetic and weak form factors can be easily extracted from the corresponding current matrix elements [1-3].
The main hypothesis of the RIA, as in the nonrelativistic case, consists in assuming that, formally, only one constituent quark interacts with the external probe while the others act as spectators. Considering the choice of the independent momenta performed in the previous sections, we conveniently take the quark #1 as the interacting one and the quarks #2 and #3 as spectators. The matrix element calculated according to this hypothesis, is then multiplied by a factor 3 to obtain the total amplitude (when considering three identical particles).

In order to construct current transition matrix elements with explicit relativistic tensor properties, we shall use the boosted Dirac wave functions of eqs.(4.5a-d) and make, in a GF, the integrations over $p_2$ and $p_3$, that are the spatial variables of the spectator quarks. According to the impulse approximation, these momenta remain unchanged in the initial and final state of the scattering process.

In more detail, we shall denote the four-momentum of the bound system, observed in the GF, as $P^\mu_G$. The index $G$ will be set to $I$ and $F$ for the initial and final state, respectively. The same notation will be used extensively in the following of this section.

The numerical parameters $v^\mu_G$ (introduced in Sect.3) for boosting the wave function from the initial or final RF to the GF, are determined by means of eq.(2.3) in the form

$$v^\mu_G = P^\mu_G / M_G$$

with

$$M_G = \sqrt{P^\mu_G P^\nu_G g_{\mu\nu}}$$

As before, the independent components are the spatial ones, i.e. $v_G$. As shown in eq.(3.12b), the bound system is in an eigenstate with $V = v_G$.

In this work we consider, for definiteness, elastic transition amplitudes, that is with $M_G = M$, but the method can be generalized to the case of inelastic processes.

For the whole bound system, we introduce the total (measured) four-momentum transfer $q^\mu$, that is $P_F^\mu - P_I^\mu = q^\mu = (q^0, \mathbf{q})$, and $Q^2 = -q_\mu q^\mu > 0$.

We observe that, on the other hand, the four-momentum (denoted as $\bar{q}^\mu$) acquired by the interacting quark #1, that remains on shell in the scattering
process, can be easily calculated from eq. (2.14) and depends on the dynamical state of the system. Explicitly, it has the form
\[
\bar{q}^\mu = p_1^\mu_F - p_1^\mu_I = v_F^\mu \cdot M_f(p_2, p_3, v_F) - v_I^\mu \cdot M_f(p_2, p_3, v_I)
\] (5.2)

At variance with the nonrelativistic impulse approximation, \(\bar{q}^\mu\) is not equal to the measured momentum transfer \(q^\mu\) [2,3].

According to the previous considerations, the current matrix element can be written in the following general form
\[
\hat{I}_{FI} = 3 \int d^3p_2 d^3p_3 d^3V d^3V' \bar{\psi}_F(p_2, p_3, V) N_F \epsilon_1 \hat{\chi}_1 N_I \psi_I^D(p_2, p_3, V')
\]
\[
\frac{1}{M} (1 + V^2)^{1/4} \delta(V - V' - Mq)(1 + V'^2)^{1/4}
\]
\[
= \hat{J}_{FI} \delta(P_F - P_I - q)
\] (5.3a)

with
\[
\hat{J}_{FI} = 3 \int \frac{d^3p_2}{\epsilon(p_2)} \frac{d^3p_3}{\epsilon(p_3)} \varphi_F^D(p_2, p_3; v_F)[\epsilon(p_2^*) \epsilon(p_3^*)]^{1/2}
\]
\[
N_F \epsilon_1 \hat{\chi}_1 N_I
\]
\[
[\epsilon(p_{2I}^*) \epsilon(p_{3I}^*)]^{1/2} \varphi_I^D(p_2, p_3; v_I)
\] (5.3c)

Let us now comment the previous expressions.

As anticipated, the factor 3 that multiplies the matrix element, by means of the antisymmetry of the wave function, takes into account the contributions of the quarks #2 and #3, when these quarks are interacting with the virtual photon field.

The factors in the last line of eq.(5.3a) represent the matrix element, in the \(V\) representation, of the operator that changes the total momentum of the system.

Eq.(4.5c) has been used to transform eq.(5.3a) into eqs.(5.3b,c). The Dirac adjoint wave functions have been introduced multiplying the Hermitic conjugate by \(\gamma_1^0 \otimes \gamma_2^0 \otimes \gamma_3^0\).

The factors \(N_G\) represent invariant but, in some extent, arbitrary normalization functions [6] that will be briefly discussed in the following for the electromagnetic form factors.
The generalized charge operator $e_1$ for the interacting quark has been introduced. The specific form of this operator, in the isospin space, will be given in eqs.(5.5) and (5.11) for the electromagnetic and axial current, respectively. The symbol $\hat{\Gamma}_1$ denotes the covariant quark interaction vertex. It is given by a subset of the 16 Dirac covariant matrices for the quark #1 multiplied by spatial functions with definite Lorentz tensor properties. We recall that one has the following Dirac matrices: $\hat{\Gamma}_1 = 1, \gamma^\mu_1, \gamma^5 \gamma^\mu_1, ...$ for scalar, vector, axial-vector, ... matrix elements, respectively.

The covariant matrix element of the model is $\hat{J}_{FI}$. To clarify the meaning of this quantity we recall that, for a single (point-like) spin $1/2$ particle, it would be represented by the standard expression

$$\hat{J}_{FI} = w^+_F \bar{u}(P_F) e \hat{\Gamma} u(P_I) w_{\Sigma_I}.$$ 

We highlight that our model for the current matrix elements of a composite system, represented by eq.(5.3c), is manifestly covariant, according to the definition given in the introduction. In fact, in eq.(5.3c) there appear covariant integrations over the spectator quark momenta and invariant factors. Also, the intrinsic RF wave functions are boosted by means of standard Dirac boosts. Finally, the prove of covariance is completed by using standard boost transformation properties of the Dirac matrices. In particular:

$$B^D(v)\gamma^0B^D(v) = \gamma^0$$

and the corresponding transformations for the other Dirac matrices.

The most relevant case for the study of the hadronic structure is represented by the four-vector electromagnetic interaction. In this case one has the following quark charge operator in the isospin space

$$e_1 = e_1^{em} = \frac{1}{2} \gamma^3_1 + \frac{1}{6}$$

The four-vector vertex can be put in the following phenomenological general form

$$\hat{\Gamma}_1 = \Gamma_1^\mu = \gamma^\mu_1 F_A - \frac{1}{2} \sigma^{\mu\nu}_1 F_B \cdot (K_F v_{\nu F} - K_I v_{\nu I})$$

with the invariant factors

$$F_A = F_A(M; p_2, p_3, v_F, v_I)$$

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In the case of a single but nonpoint-like (on shell) particle they represent
the standard observable form factors

\[ F_A = F_1(Q^2), \quad F_B = F_2(Q^2), \quad K_G = 1 \] \hfill (5.8)

On the other hand, when considering interacting quarks, \( F_A, F_B \) and \( K_G \) can give an effective representation of all the unknown effects that modify the bare quark vertex. Some of this effects can be related to violations of the RIA, others to the strong interactions of the constituent quarks. The latter are usually interpreted in terms of quark substructure and/or exchange of vector mesons between the virtual photon and the quark vertex.

In any case, at zero momentum transfer, vertex charge normalization requires

\[ F_A(M; p_2, p_3, v_F, v_I) = 1 \] \hfill (5.7b)

\[ K_G = K_G(M; v_G, p_2, p_3) \] \hfill (5.7c)

However, the simplest choice is to consider the interaction of the virtual photon with point-like Dirac particles, that is using \( F_A = 1 \) and \( F_B = 0 \) in eq.(5.6). In our opinion a relativistic study of the nucleon form factors should calculate first these quantities with that choice (by using the nucleon wave functions of the quark model), then insert the phenomenological functions \( F_A, F_B \) and \( K_G \) to improve the reproduction of the experimental data.

The study of the counterterms due to dynamical current conservation and the analysis of the contributions due to virtual negative energy states or to quark-antiquark pairs can help to construct a more reliable and consistent model.

As for the invariant normalization factors \( N_G \) of eqs.(5.3a,c), they can be chosen considering the requirement of total charge normalization for the matrix element at zero momentum transfer. It reads

\[ J_{GG}^0 = e_{tot} = +1, \quad 0 \] for the proton and the neutron, respectively. This condition is automatically satisfied (considering the antisymmetry of the wave function and the normalization of the Dirac spinors of eq.(4.1b)) by
\[ N_G = \left[ \frac{m}{\epsilon(P_{1G})} \right]^{1/2} \]  

(5.9)

Within this theoretical framework, various numerical calculations have been performed for the nucleon electromagnetic form factors, by using different constituent quark nucleon wave functions. The results, in good agreement with the new experimental data, show the essential role of relativity in such calculations and the reliability of the RIA \[2,7\] as a starting point for the study of the electromagnetic response of the nucleon.

Similar calculations have been also performed for the study of the axial nucleon form factor \[3,4\]. In this case, the quark interaction vertex is taken as the axial-vector Dirac matrices

\[ \hat{\Gamma}_1 = \gamma_5^\mu \gamma_1^\mu \]  

(5.10)

and the axial charge as an isospin raising operator, that is

\[ e_1 = e_1^{ax} = \tau_1^+ \]  

(5.11)

In this case no charge normalization condition can be found and the form of the vertex spatial functions and of the normalizations factors should be carefully studied \[4,6\].

We now turn to discuss the comparison of our manifestly covariant matrix element of eq.(5.3c) with that of the SF \[2,6\]. For definiteness we refer to a four-vector electromagnetic vertex \( \hat{\Gamma}_1 = \gamma_4^\mu \) with the normalization factors given in eq.(5.9).

To this aim we shall transform our expression of eq.(5.3c) into the SF. We divide this procedure into the following three steps. First (i), we obtain the rotation matrices of the spectator quarks; second (ii), the rotation matrices of the interacting quark; finally (iii), the momentum \( \delta \) functions of the spectator quarks.

(i) We now reproduce the rotation matrices of the spectator quarks \( i = 2, 3 \). For these quarks, taking into account eqs.(3.8b) and (4.5d), the momenta of eq.(3.12c), in eq.(5.3c) one has the following spinorial bilinear quantities

\[ S_{\sigma_i' \sigma_i} = w_{\sigma_i'}^+ u_+ (p_{1F}^*) B_i^D (v_F) \gamma_5^0 B_i^D (v_I) u(p_{1I}) w_{\sigma_i} \]  

(5.12)
By means of eq.(5.4b) one can write
\[ B^D_i(v_F)\gamma^0_i B^D_i(v_I) = \gamma^0_i [B^D_i(v_F)]^{-1} B^D_i(v_I) = \gamma^0_i B^D_i(-v_F) B^D_i(v_I) \] (5.13)
We now consider the product of the two Dirac boosts in the last equation. We recall that the corresponding boosts on the spectator momenta are
\[ p_b(p_{iI}; v_I) = p_i \] (5.14a)
\[ p_b(p_{iF}; v_F) = p_i \] (5.14b)
The last equation can be rewritten as
\[ p_b(p_i; -v_F) = p_{iF}^* \] (5.14c)
In consequence, applying successively (composing) the boosts of eqs.(5.14a) and (5.14c), one obtains the following total boost
\[ p_b[p_b(p_{iI}^*; v_I); -v_F] = p_{iF}^* \] (5.14d)
Note that for the Dirac spinors the corresponding boost is the product \( B^D_i(-v_F) B^D_i(v_I) \) of eq.(5.13). We use for that product, applied to \( u(p_{iI}^*) \) the property of eq.(4.3). Then, we insert the result in eq.(5.12). Taking the covariant Dirac spinor normalization and the definition of eq.(3.8) for the Wigner rotations, one finally obtains
\[ S_{\sigma_i'/\sigma_i} = D_{\sigma_i'/\sigma_i}^{1/2}[R_W(p_{iI}^*; B^{-1}(v_F)B(v_I))] \]
\[ = \sum_{\lambda_i} D_{\lambda_i',\lambda_i}^{1/2}[R_W(p_{iI}^*, B(v_F))] D_{\lambda_i',\mu_i}^{1/2}[R_W(p_{iF}^*, B(v_I))] \] (5.15)
The second equality is directly obtained without composing the two successive boosts.

(ii) As for the Wigner rotations of the interacting quark, by means of eq.(4.3) and inserting two complete sets of spin states, we introduce the following
(iii) Let us now consider the spatial integrations over the spectator momenta of our eq.(5.3c). We introduce two $\delta$ functions in the following way:

$$
\int d^3p_2 d^3p_3 .... = \int d^3p'_2 d^3p'_3 d^3p_2 d^3p_3 \delta(p'_2 - p_2) \delta(p'_3 - p_3) ....
$$

(5.17)

Furthermore, in eq.(5.3), $p'_2$ and $p'_3$ are then taken as the arguments of the final state wave function. The rest frame final momenta are considered as functions of those momenta. The same holds for the initial state, taking $p_2$ and $p_3$ as arguments.

We now replace the integration variables $p_i$ ($i = 2, 3$) and the primed ones with the corresponding rest frame momenta $p^*_i$. For the initial state momenta, a transformation factor must be introduced according to the following equation

$$
d^3p_i = \frac{\epsilon(p_i)}{\epsilon(p^*_i)} d^3p^*_i
$$

(5.18)

An analogous equation holds for the final state primed momenta. We can identify

$$
p^*_i = p^*_{iI}
$$

(5.19a)

$$
p^*_i = p^*_{iF}
$$

(5.19b)

and use eqs.(5.14a,b), respectively, to express $p_i$ and $p'_i$ as functions of the intrinsic momenta.

Considering eq.(3.9), the rest frame momenta are easily replaced by the Jacobi momenta as integration variables.

Collecting all the previous results, our electromagnetic current matrix element is put in the SF, giving
\[ J_{FI}^\mu = 3 \int d^3p_1 d^3p_2 d^3p_3 \psi_{\sigma_1' \sigma_2' \sigma_3'}^{1/2} (p_1', p_2', p_3') \]

\[
D_{\chi_1' \gamma_1'}^{1/2} \left[ R_W (p_1^*, B(\nu_F)) \right] w_{\chi_1'}^\dagger \bar{u}(p_1') e_1 \gamma_1^\mu u(p_1) w_{\lambda_1} \]

\[
D_{\lambda_1' \gamma_1'}^{1/2} \left[ R_W (p_1^*, B(\nu_I)) \right] D_{\lambda_1' \sigma_1}^{1/2} \left[ R_W (p_1^*, B(\nu_I)) \right] \]

\[
\delta(p_1 - p_2) \delta(p_3 - p_4) \cdot m \epsilon(p_1) \epsilon(p_3) \]

\[
\left[ \epsilon(p_2') \epsilon(p_2) \epsilon(p_3') \epsilon(p_3) \right]^{-1/2} \psi_{\sigma_1' \sigma_2' \sigma_3'}^{1/2} (p_2, p_3) \]

(5.20)

where a sum over the repeated indices is understood.

We note that the previous expression, that has been shown to be equal to eq.(5.3c) for the electromagnetic interaction, is coincident with eqs.(2),(3) and (10) of ref.[6]. Apart from a (probably not relevant) normalization factor, our expression is also equivalent to the result of ref.[2].

After verifying the equivalence of our covariant matrix element with SF, we conclude observing that our expression of eq.(5.3c) presents the following advantages with respect to SF.

(i) As discussed above, it is manifestly covariant.

(ii) It is more compact, in the sense that it contains only two three-dimensional integrations over the spectator momenta with respect to four integrations of the SF.

(iii) Well known Dirac spinors and Dirac boost matrices are used instead of rotation matrices of Wigner rotations.

These features allow for studying the possibility of deriving an expression for a\(^{\text{a}}\) dynamically conserved current\(^{\text{a}}\) by means of a suitable procedure of minimal coupling substitution. The results of this investigation will be presented in subsequent works.

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