Extended Cubic B-spline Collocation Method for Pricing European Call Option

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Abstract. A numerical method is proposed based on the extended cubic B-spline functions for solving the model of European call option. The Black-Scholes equation is fully-discretized by using the extended cubic B-spline collocation for spatial discretization and the finite difference method for the time discretization. The stability of difference scheme is analysed and the stability condition is given. The accuracy of this method can be improved by properly choosing the values of the parameter $\lambda$. A numerical experiment is performed to show the accuracy and stability of the proposed method.

1. Introduction
Options pricing in financial application fields is mathematically one of the most complicated problems. The Black-Scholes equation[1] is often chosen in option pricing. Various forms of numerical methods, such as the lattice method[2], the binomial method[3], the finite element method[4], the finite difference method[5,6], and the finite volume method[7,8], have been presented to compute the Black-Scholes equation.

It is useful for the B-spline collocation method with low computational cost to solve partial differential equations. B-spline functions, which are piecewise polynomial functions, have compact support and can easily be integrated and differentiated. Some numerical methods have been developed by using B-spline functions to solve option pricing problems. For example, Kadlabajoo et al.[9] and Rashidinia et al.[10,11] used cubic B-spline collocation methods for the Black-Scholes equation. Mohammadi[12] proposed a quintic B-spline collocation method for the European option pricing problems.

As a generalization, the cubic B-spline basis functions are extended by introducing the parameter $\lambda$. Recently, the extended cubic B-spline collocation method has been used for solving linear two-point boundary value problems[13], the advection-diffusion equation[14], the modified regularized long wave equation[15], etc. The use of the extended cubic B-spline collocation method is expanded to price European call option by terms of the $\theta$-scheme in this paper.

2. Black-Scholes equation
Consider the following Black-Scholes equation

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf = 0, \quad (S,t) \in \mathbb{R}^+ \times (0,T),$$ (1)

with a terminal condition
\[ f(S,T) = \max\{S - E, 0\}. \] (2)

Here \( f(S,t) \) is the European call option price, \( S \) is the asset price, \( t \) is the time, \( \sigma \) is the volatility of the underlying asset, \( r \) is the risk-free interest rate, \( T \) is the maturity and \( E \) is the exercise price.

For computational purposes, the infinite domain can be truncated \( R^+ \times (0, T) \) into \((a,b) \times (0, T)\), where \( a \) and \( b \) are two positive numbers. The generalized Black-Scholes equation is considered as follows

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - r f = 0, \quad (S,t) \in (a,b) \times (0,T),
\] (3)

with the terminal condition given as (2) and the boundary conditions

\[
f(a,t) = \alpha(t), \quad f(b,t) = \beta(t).
\] (4)

By using the transformation \( S = e^x \), we transform (3) into a non-degenerate partial differential equation

\[
\frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2 e^{2x} \frac{\partial^2 U}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial U}{\partial x} - r U = 0, \quad (x,t) \in (\tilde{a}, \tilde{b}) \times (0,T),
\] (5)

with \( \tilde{a} = \ln a, \tilde{b} = \ln b \), the terminal condition

\[
U(x,T) = g(x) = \max\{e^x - E, 0\},
\] (6)

and the boundary conditions

\[
U(\tilde{a},t) = \alpha(t), \quad U(\tilde{b},t) = \beta(t).
\] (7)

3. Extended Cubic B-spline Collocation Method

The region \([\tilde{a}, \tilde{b}] \times [0, T]\) is discretized by the grid points \((x_j, t_i)\) with \( x_j = \tilde{a} + jh, \)

\[ h = (\tilde{b} - \tilde{a})/M, \quad j = 0, 1, \ldots, M, \text{ and } t_i = l\Delta t, \quad \Delta t = T / N, \quad l = 0, 1, \cdots, N. \]

The extended cubic B-spline functions are defined by[13-15]

\[
Q_j(x) = \frac{1}{24h^4}
\]

\[
\begin{cases}
4h(1-\lambda)(x-x_{j-2})^3 + 3\lambda(x-x_{j-2})^4, & x \in [x_{j-2}, x_{j-1}], \\
(4-\lambda)h^4 + 12h^4(x-x_{j-1}) + 6h^2(2+\lambda)(x-x_{j-1})^2 & - 12h(x-x_{j-1})^3 - 3\lambda(x-x_{j-1})^4, & x \in [x_{j-1}, x_{j}], \\
(4-\lambda)h^4 + 12h^4(x_{j+1}-x) + 6h^2(2+\lambda)(x_{j+1}-x)^2 & - 12h(x_{j+1}-x)^3 - 3\lambda(x_{j+1}-x)^4, & x \in [x_{j}, x_{j+1}], \\
4h(1-\lambda)(x_{j+2}-x)^3 + 3\lambda(x_{j+2}-x)^4, & x \in [x_{j+1}, x_{j+2}], \\
0, & \text{otherwise}.
\end{cases}
\] (8)

The exact solution \( U(x,t) \) is approximated by \( u(x,t) \) in the following form

\[
u(x,t) = \sum_{j=1}^{M+1} c_j(t)Q_j(x),
\] (9)
Here \( c_j(t) \) are time-dependent unknowns to be determined. The basis functions \( Q_j(x) \) will degenerate into the cubic B-spline basis functions if \( \lambda = 0 \).

So the approximant \( u_j' \) at the point \((x_j, t_j)\) over interval \([x_j, x_{j+1}]\) can be defined as

\[
u_j' = \sum_{i=j+1}^{j+2} c_i'(x)Q_i(x).
\]

By terms of (9) and (10), we have the following approximants

\[
u_j' = p_1c_i'(x) + p_2c_j'(x) + p_3c_{j+1}'(x),
\]

\[
u_{xxj}' = p_4c_i'(x) - p_3c_{j+1}'(x),
\]

\[
u_{xxxj}' = p_4c_{j-1}'(x) + p_3c_j'(x) + p_4c_{j+1}'(x),
\]

where

\[p_1 = \frac{4 - \lambda}{2h}, \quad p_2 = \frac{8 + \lambda}{12}, \quad p_3 = -\frac{1}{2h}, \quad p_4 = \frac{2 + \lambda}{2h}, \quad p_5 = -\frac{2 + \lambda}{h}.
\]

We discretize time variables of the unknown solution \( u \) in the Black-Scholes equation by using the \( \theta \)-scheme. Then the following equation is obtained

\[
u_j'^{i+1} - \nu_j'^{i} = \frac{1}{\Delta t}(1 - \theta)(\frac{1}{2} \sigma^2 \frac{\partial^2 \nu_j'^{i+1}}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial \nu_j'^{i+1}}{\partial x} - \nu_j'^{i+1}) + \theta(\frac{1}{2} \sigma^2 \frac{\partial^2 \nu_j'^{i}}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial \nu_j'^{i}}{\partial x} - \nu_j'^{i}) = 0, \quad 0 \leq \theta \leq 1.
\]

It will become an explicit scheme if \( \theta = 0 \), a Crank-Nicolson scheme if \( \theta = 1/2 \), and a fully implicit scheme if \( \theta = 1 \). Substituting (11) into (12) at the knots \( x_j \) (\( j = 0, 1, \cdots, M \)), the system of equations is obtained

\[
u_j'^{i+1} + \nu_j'^{j+1} + \nu_j'^{j-1} = \nu_j'^{i+1} + \nu_j'^{i+1} + \nu_j'^{i+1}
\]

with

\[\nu' = (1 + r(\theta \Delta t))p_1 - \theta(r - \frac{1}{2} \sigma^2)\Delta t p_3 - \frac{1}{2} \sigma^2 \theta \Delta t p_4, \quad \nu'' = (1 + r(\theta \Delta t))p_2 - \frac{1}{2} \sigma^2 \theta \Delta t p_5,
\]

\[\nu' = (1 + r(\theta \Delta t))p_1 + \theta(r - \frac{1}{2} \sigma^2)\Delta t p_3 - \frac{1}{2} \sigma^2 \theta \Delta t p_4, \quad \mu = (1 - r(1 - \theta)\Delta t)p_2 + \frac{1}{2} \sigma^2 (1 - \theta) \Delta t p_5,
\]

\[\gamma = (1 - r(1 - \theta)\Delta t)p_1 + (1 - \theta)(r - \frac{1}{2} \sigma^2)\Delta t p_3 + \frac{1}{2} \sigma^2 (1 - \theta) \Delta t p_4, \quad
\]

\[\nu = (1 - r(1 - \theta)\Delta t)p_1 - (1 - \theta)(r - \frac{1}{2} \sigma^2)\Delta t p_3 + \frac{1}{2} \sigma^2 (1 - \theta) \Delta t p_4.
\]

The system (13) consists of \( M + 1 \) equations in \( M + 3 \) unknowns. To obtain a unique solution of the system, we need two additional equations. Thus, the boundary conditions (7) can be written as

\[
u(\alpha, t_i) = p_1c_0^{i+1} + p_2c_0^{i+1} + p_3c_1^{i+1} = \alpha(t_i),
\]

\[
u(\beta, t_i) = p_1c_M^{i+1} + p_2c_M^{i+1} + p_3c_{M+1}^{i+1} = \beta(t_i).
\]

To start iterations of the matrix system (13), the final parameters \( c_{i+1}^N, c_0^N, \cdots, c_M^N, c_{M+1}^N \) can be obtained from the terminal condition as follows:
\[(u_{s})_{0}^{N} = 1, \quad u_{j}^{N} = g(x_{j}), \quad j = 0, 1, \ldots, M, \quad (u_{s})_{M}^{N} = 1.\]

4. Stability Analysis

By terms of the Von Neumann theory, the stability of the proposed method can be investigated. Consider the trial solutions (one Fourier mode out of the full solution) at a given point \(x_{j}\),

\[c_{j}^{i} = \delta^{i} \exp(i\varphi),\]  

where \(i = \sqrt{-1}\) and \(\varphi\) ranges from \(-\pi\) to \(\pi\).

Substituting (15) into (13) and simplifying the equation, we obtain the amplification factor

\[\kappa(\varphi) = \frac{x_{1} + i y_{1}}{x_{2} + i y_{2}},\]  

where

\[x_{1} = \mu + (\gamma + \nu) \cos \varphi, \quad y_{1} = (\nu - \gamma) \sin \varphi,\]  

\[x_{2} = \mu^{*} + (\gamma^{*} + \nu^{*}) \cos \varphi, \quad y_{2} = (\nu^{*} - \gamma^{*}) \sin \varphi.\]  

(17)

The proposed scheme will be stable if the amplification factor

\[|\kappa(\varphi)| = \sqrt{\frac{\gamma^{2} + \mu^{2} + \nu^{2} + 2 \mu^{2} \gamma^{2} + 2 \nu^{2} \nu^{2} \cos 2\varphi}{\gamma^{*2} + \mu^{*2} + \nu^{*2} + 2 \mu^{*2} \gamma^{*2} + 2 \nu^{*2} \nu^{*2} \cos 2\varphi}} \leq 1,\]  

(18)

It is easy to verify that the extreme value of \(\kappa(\varphi)\) can be obtained at \(\varphi = 0\). Setting \(\varphi = 0\) in (18), we have

\[\left|\frac{\gamma + \mu + \nu}{\gamma^{*} + \mu^{*} + \nu^{*}}\right| = \frac{1 - (1 - \theta)\Delta t}{1 + \theta\Delta t} \leq 1.\]

From the inequality, we obtain \(\theta \geq 1/2 - 1/\Delta t\). Thus the proposed scheme is unconditionally stable for \(1/2 \leq \theta \leq 1\).

5. Numerical Experiment

A European call option is given with a domain \(S \in [1,31]\), and \(\sigma = 0.2, \quad r = 0.05, \quad T = 0.5, \quad E = 10\). The boundary conditions are

\[\alpha(t) = 0, \quad \beta(t) = S - E e^{-r(T-t)}.\]

The extended cubic B-spline (ECBS) method is employed for solving this problem numerically. To measure the accuracy of the proposed method, the relative error \((L_{2})\) norm is calculated by

\[L_{2} = \sqrt{\frac{\sum_{j} |U_{j} - u_{j}|^{2}}{\sum_{j} |U_{j}|^{2}}},\]

where \(u_{j}\) and \(U_{j}\) are respectively approximate solutions and analytical solutions. The value of \(\lambda\) with the lowest norm is identified.

Figure 1 and Figure 2 depict the approximate solutions and relative errors with \(N=M=300, \lambda=0.000025\) for \(\theta=1/2\) and \(\theta=1\), respectively.
Figure 1. Approximate solutions with N=M=300, λ=0.000025: (a) θ=1/2; (b) θ=1.

Figure 2. Relative errors with N=M=300, λ=0.000025: (a) θ=1/2; (b) θ=1.

Table 1. Comparison of two methods on L₂ with θ=1/2.

| (N,M)    | ECBS          | CBS           |
|----------|---------------|---------------|
| (10,100) | 5.62e-005(λ=0.004) | 7.06e-005     |
| (20,100) | 5.28e-005(λ=0.004) | 6.84e-005     |
| (20,200) | 1.23e-005(λ=0.00048) | 1.34e-005     |
| (40,200) | 1.10e-005(λ=0.0005)  | 1.23e-005     |
| (40,400) | 3.48e-006(λ=0.000088) | 3.61e-006     |
| (80,400) | 3.15e-006(λ=0.000083) | 3.28e-006     |

For the purpose of comparison, the values of the relative error L₂ are given in Table 1 by the proposed method and the cubic B-spline (CBS) method [10] for θ=1/2.

The numerical experiment is performed to illustrate the validity of the proposed method. It is shown that the extended cubic B-spline collocation method is superior to the cubic B-spline collocation method, which provides a better approximation to the Black-Scholes model by altering the values of the parameter λ.

### 6. Conclusion

The proposed method has been used to solve the Black-Scholes model for European call option pricing. A numerical experiment is presented to demonstrate the efficiency of the extended cubic B-spline collocation method. It is more flexible for this method to give slightly better results than the cubic B-spline collocation method by choosing an appropriate parameter λ. The proposed method in
this paper provides a valid method for pricing European call option.

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