Properties of the Strong-Field Approximation

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Abstract

The SFA (Strong-Field Approximation) is routinely cited as the standard analytical approximation method for treatment of strong-field laser-induced processes. The difficulty with SFA is that it is not well-defined. Some authors equate it with an alternative terminology – KFR (Keldysh, Faisal, Reiss) – which is ambiguous, since the three source papers employ inequivalent approximations. A rational system for naming strong-field approximation methods is developed here, beginning with the Maxwell equations. When the dipole approximation is employed \textit{ab initio}, the four source-free Maxwell equations that apply to laser fields (i.e. propagating fields) are replaced by a single Maxwell equation for an oscillatory electric field dependent on a virtual source current. The sole approximation based on propagating fields is labeled “SPFA” (Strong Propagating-Field Approximation). Dipole-approximation methods are labeled “SEFA” (Strong Electric-Field Approximation). Numerical solution of the time-dependent Schrödinger equation (TDSE) is in the SEFA category. The SPFA is found to have remarkably broad applicability. This is because it connects continuously into relativistic results at both high and low frequencies, and because effects of a laser field are distinct from those of an oscillatory electric field of the same frequency and amplitude even when the dipole approximation is nominally valid. SPFA and SEFA methods approach equivalency at high frequencies as long as conditions are nonrelativistic, but the SEFA becomes increasingly deficient as frequencies decline into the mid-infrared and beyond. The major differences between SPFA and SEFA methods at low frequencies are explained in physical terms. This elucidates why the failure of the dipole approximation at low frequencies is so much more consequential than at high frequencies.

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I. INTRODUCTION

Nonperturbative effects arising from the use of strong-field lasers were first identified in laboratory experiments in 1979 [1], and came to be examined with nonperturbative analytical methods of earlier origin [2, 3]. The capabilities and applications of strong-field lasers have mushroomed in the years following those early beginnings. It is thus surprising that the foundations of the theoretical methods employed to describe strong-laser interactions with matter possess an ambiguity that is unacceptable in such an important and still-expanding field of physics. The primary purpose of this article is to clarify the major differences that distinguish each analytical method from its alternatives. Another purpose is to present further evidence [4] that the electric component of a laser field differs fundamentally from an oscillatory electric field of the same frequency and amplitude.

Theoretical approaches to the description of nonperturbative phenomena can be categorized as analytical and numerical, with analytical methods being the principal focus of this article. Basic tools employed in nonperturbative analytical approaches are the Volkov solution [5, 6], the tunneling model [7, 8], and the KH (Kramers-Henneberger) transformation [9, 10].

The Volkov solution [6] is an exact solution of the Dirac equation for a charged, spin-1/2 particle in a plane-wave field. Since most nonrelativistic applications of the Volkov solution to date do not consider the spin of the particle, it is actually the earlier Gordon solution [5] that is being employed. Gordon found the solution to the Klein-Gordon equation for a spinless charged particle in a plane-wave field. The convention to label both the Gordon and Volkov solutions as the “Volkov solution” is now so ingrained in the strong-field community, that it is a convention that will be followed here.

The tunneling model has been used since the early days of quantum mechanics [7, 8]. As employed in strong-field physics, tunneling represents the interaction of two scalar fields, one of which is the potential that binds an electron in an atom or ion, and the other is the laser field as approximated by an oscillatory electric field. The dipole approximation (DA) is inherent in the selection of a scalar field to represent the laser field. The physical meaning of representing the laser field by a DA potential is that it replaces the vector field of a laser, propagating with the speed of light, with an oscillatory electric field that does not propagate at all [11]; it simply oscillates with time.
The KH transformation, as employed in strong-field physics, is a transformation applied to an electron bound in an atom, considered in the context of the DA. The transformation corresponds to shifting the origin of coordinates from the center of the atom to an orbit corresponding to the motion of a free electron responding to an applied oscillatory electric field. When the frequency of the applied field is sufficiently high, the electron’s motion is so rapid that an accepted approximation is to employ the averaged location of the electron as the center of motion, which is just the nucleus of the atom. This high-frequency approximation has been employed by Faisal \[12\] and by Gavrila \[13, 14\]. The limitation inherent in using the DA at high frequencies is that high frequency violates the underlying assumption involved in using the DA for laser fields. High-frequency employment of the DA means that it is applicable only to quasistatic electric fields, since the field frequencies necessary for validity of the KH transformation violate the DA for propagating fields.

The DA is widely used in strong-field applications. As employed in the AMO (Atomic, Molecular, Optical) physics community, the DA has a simple definition: the electric field is a function only of time, and the magnetic field is neglected entirely. In the usual AMO and condensed-matter treatments of the effects of low- and moderate-intensity laser fields, this approximation has been so useful that its employment has become conventional. The view expounded in this article is that the strong-field environment severely limits the DA \[15–18\], and requires care in using the physical intuition that follows from it.

Most strong-field analytical approximation methods apply the DA from the outset, the sole exception being Ref. \[19\], which employs the nonrelativistic limit of a relativistic formalism \[20, 21\]. This property imparts significant capabilities to this method that are not otherwise available. The explication of these advantages is a major focus of this article.

The \textit{ab initio} use of the DA extends also to direct numerical solution of the Schrödinger equation, referred to as TDSE (Time-Dependent Schrödinger Equation). The widespread assumption that TDSE is exact overlooks the limitations in strong-field applications that follow from the DA.

Another aspect of strong-field approximations that has received considerable attention is the matter of the choice of electromagnetic gauge in the formulation of the theory. There are two gauges in widespread use in the AMO community. One is the so-called \textit{velocity gauge}, in which the interaction Hamiltonian contains a term of the form of $A \cdot p$, where $A$ is the vector potential of the applied field. The other, called \textit{length gauge}, was introduced...
by Göppert-Mayer [22], and has been accorded special status by some researchers [23–25].
The interaction Hamiltonian in the length gauge is proportional to $r \cdot E$, where $E$ is the
electric field vector. Some authors refer to a third choice, called the acceleration gauge, that
follows from the KH transformation. This is not a true gauge transformation. The KH
transformation requires the use of an operator-valued function, which excludes it from the
rules that apply to gauge transformations. The KH transformation is a change of coordinate
system, not a change in the way that electromagnetic fields are represented by potential
functions.

The length gauge and the velocity gauge are equivalent within the DA, as has been ver-
ified by TDSE. There is another gauge that has been shown to be universal [4] for laser
fields when the DA is eschewed. This is the radiation gauge or Coulomb gauge, in which
spatial dependence is retained in the statement of the field potentials for propagating fields.
A useful feature of this gauge is that longitudinal fields like the Coulomb binding potential
are described by a scalar potential $\phi(r)$, and transverse fields like laser fields are described
by a 3-vector potential $A(t, r)$. A basic matter, largely unnoticed within the strong-field
community, is that approximations formulated relativistically and then reduced to a non-
relativistic long-wavelength limit, produce potentials that have the general appearance of
the potentials of the velocity gauge, but are nevertheless importantly different [20, 21, 26].
There is a simple distinguishing feature: the term $A^2(t)$ in the velocity gauge has no physi-
cal consequences. It has no more significance than a change in the origin of energy measure,
and it does not appear in dynamical equations of motion. However, it is vitally important
to retain this term in the radiation gauge [20, 21, 27, 28], where the $A^2$ term is of basic
importance in dynamics [29]. It represents the ponderomotive potential, that can be very
large in strong fields and must be retained even in nonrelativistic conditions. In a DA theory,
the ponderomotive energy appears as a kinetic quiver energy.

The following Section in this paper focuses on an essential matter, already introduced
in a more limited context in Ref. [11]. The ab initio use of the DA is equivalent to the
introduction of an altered single Maxwell equation replacing the four Maxwell equations
descriptive of actual laser fields. This reduction to a single Maxwell equation requires the
introduction of a postulated virtual source current that does not exist in the laboratory, but
is hidden from scrutiny within the DA. However, this virtual source current cannot provide
a universally valid compensation for the reduction of four sourceless Maxwell equations to a
single equation. An examination of the limitations imposed by this virtual source is essential to the determination of the true limits of the DA. These limits are most profound as the laser frequency declines [16, 18], in contradiction to the DA view that decreasing laser frequency points to a simple adiabatic limit.

Another aspect of the difference in the four Maxwell equations describing laser fields and the single Maxwell equation for oscillatory electric fields is that it reinforces the basis for the conclusion in Ref. [4] that potentials are more fundamental than fields.

Section II also reviews the S-matrix methods that provide an exact formalism that can be regarded as underlying all of the diverse elements of the KFR terminology. The basic structure of the S matrix suggests the most fruitful approximation to employ for strong-field applications.

Section III in this paper explicates the important distinctions that make the elements comprising KFR so different from each other. Each of the KFR components can be identified with a source publication [3, 12, 19], making it possible to be explicit about the differences. The current understanding of the SFA is examined in Section IV. Although the SFA terminology has now largely supplanted KFR, it is even more vague since authors that employ the SFA terminology typically do not identify a source publication. Many authors equate SFA with KFR, although an alternate view is expressed in a review article [30] that claims to explain the difference between KFR and SFA.

The SFA terminology was introduced [20] to distinguish the R [19] of KFR from the other methods. It is the sole analytical approximation for strong fields that is a reduction from a relativistic formulation that treats a laser field as a propagating field [20, 21]. That identifying feature of the SFA terminology was not recognized by other researchers, so it is here renamed SPFA (Strong Propagating-Field Approximation) to establish its unique character. Approximations that assume the DA from the outset describe only electric-field effects, so they are labeled SEFA (Strong Electric-Field Approximation).

The performance of the SPFA is examined over a wide variety of physical conditions in Section IV, first using comparisons with experimental results and then with alternative theories. At high frequencies, SPFA and SEFA results converge, but increasing divergences are revealed as the frequency declines, with the SPFA showing much better agreement with laboratory observations.

Section V contains commentary on the applicability of the DA to high-frequency fields
(\omega > 1 \ a.u.), where the SPFA and SEFA methods become equivalent. However, failure of the DA at very high frequencies means that this common SPFA/SEFA result applies only to oscillatory electric fields, a fact whose ramifications are explored. The known relativistic extension of the SPFA becomes necessary at high frequencies.

Section VI examines the important distinctions between the role of the ponderomotive potential in the SPFA and the SEFA.

A final Section reviews how SPFA, SEFA, and gauge choice are related. This is a matter that continues to cause confusion almost forty years after the inception of laboratory strong-field physics.

II. PRELIMINARIES

The well-established usefulness of the DA in AMO and condensed-matter physics masks its limitations when employed in the physics of strong-laser effects. These limitations are evident when the single underlying Maxwell equation descriptive of the DA is confronted with the four Maxwell equations governing laser fields. This aspect of differing Maxwell equations has been presented earlier in the context of a critique of the length gauge [11], but it applies to any DA theory. It also has direct bearing on the inadequately examined assumption that an oscillatory electric field accurately models the effect of a laser field whose electric-field component matches that of the oscillatory pure-electric field.

A second element of this preliminary Section is a review of the S-matrix method for the expression of exact quantum transition amplitudes, emphasizing the qualities that make it so useful for strong-field applications.
A. Maxwell equations

The vacuum Maxwell equations in Gaussian units for the electric field \( E \) and the magnetic field \( B \) are

\[
\nabla \cdot E = 4\pi \rho , \tag{1}
\]
\[
\nabla \cdot B = 0 , \tag{2}
\]
\[
\nabla \times B - \frac{1}{c} \partial_t E = \frac{4\pi}{c} J , \tag{3}
\]
\[
\nabla \times E + \frac{1}{c} \partial_t B = 0 . \tag{4}
\]

where \( \rho \) and \( J \) are the source charge and current densities. For laser fields, these equations apply with no source terms, so that laser fields are governed by the equations

\[
\nabla \cdot E = 0 , \tag{5}
\]
\[
\nabla \cdot B = 0 , \tag{6}
\]
\[
\nabla \times B - \frac{1}{c} \partial_t E = 0 , \tag{7}
\]
\[
\nabla \times E + \frac{1}{c} \partial_t B = 0 . \tag{8}
\]

The almost complete symmetry between electric and magnetic fields in Eqs. (5) - (8) is a hallmark of transverse fields. (The descriptions transverse, propagating, plane-wave and laser are used interchangeably here.)

In any laser experiment, the only fields that can actually reach the target are propagating fields. Any extraneous fields cannot persist beyond a few wavelengths from the disturbing influence. The beam reaching the target can consist only of fields that obey the plane-wave equations (5) to (8).

The DA, as conventionally employed in the AMO community, is defined by the substitutions

\[
E (t, r) \rightarrow E^{DA} (t) , \quad B (t, r) \rightarrow B^{DA} = 0 . \tag{9}
\]

The applicable Maxwell equations follow from zero values for all expressions in (1) - (4) containing the \( \nabla \) operator, and the complete absence of \( B \). The only surviving equation is (3), which can be written as

\[
\partial_t E^{DA} (t) = -4\pi J^{DA} (t) , \tag{10}
\]
where $J^{DA}(t)$ is a virtual source current that must be posited to avoid a null result. The fidelity of DA mimicry to actual laser fields is determined by how well the effects of the proxy field governed by Eq. (10) replicate the effects of laser fields in Eqs. (5) - (8). This is a basic matter that has not been adequately scrutinized.

An example of effective mimicry (with important limitations) is the matter of rescattering. The classical path of a free electron in the laser field \cite{31} in the nonrelativistic case, closely resembles that of the electron driven by the virtual source $J^{DA}$ \cite{32,34}, but only if the comparison is limited to not much more than a wavelength in space or a wave period in time. Beyond those narrow limits, the virtual source current can inject unphysical energy into the problem. The rescattering case is instructive because it illustrates that effective mimicry evokes models that differ from those for laser fields. The virtual source forces an oscillatory electron motion, called a *quiver motion*, with a kinetic energy identified as the ponderomotive energy. In the plane-wave case, the same magnitude of ponderomotive energy is involved, but it is a potential energy that arises from a *mass shift* of a charged particle in a strong plane-wave field \cite{29}. The oscillatory motion in the plane-wave case is not forced by an external source, but is an inherent motion associated with the laser field parameters. See Ref. \cite{31} for a description of that motion.

An example of rescattering failure is provided by the path of a photoelectron after ionization by a circularly polarized field. Using the photoelectron trajectory according to a tunneling theory \cite{34}, one can calculate the resulting angular momentum $l$ of the photoelectron which, after the first few cycles, is \cite{11}

$$l = \omega^2 x_0^2 t \sin \omega t,$$

(11)

which oscillates in sign and increases in amplitude with time. This is unphysical. A circularly polarized laser field, since it lacks an external source, can transfer angular momentum to an electron only through the intrinsic angular momentum of absorbed photons. Absorbed circularly polarized photons can impart only one sign of angular momentum to the photoelectron; oscillation in sign is impossible. Each absorbed photon contributes one quantum unit of angular momentum. An angular momentum of the photoelectron increasing indefinitely with time is impossible. Both of these anomalies arise from the virtual current $J^{DA}$, which continues to transfer energy and momentum to the photoelectron after the brief time allowed by accurate mimicry.
B. S-Matrix formalism

The S matrix is an exact transition amplitude that can be employed in two fundamental forms: direct-time (or post), and time-reversed (or prior). Derivations of S matrices for strong-field problems have been presented in many places. A formally complete version is in Section IV of Ref. [35], with other simple derivations shown in Refs. [15, 20]. The general idea is that a quantum transition starts with a target free of the influence that produces the transitions, and ends with the results of the process analyzed by measuring instruments that are also free of the transition-causing influence. The effect of the interaction is evaluated by expressing an interacting state as a superposition of noninteracting states. The result is that the S-matrix transition amplitude always contains one field-free state $\Phi$ and one completely interacting state $\Psi$ satisfying the Schrödinger equations

$$i\hbar \partial_t \Phi = H_0 \Phi,$$

$$i\hbar \partial_t \Psi = (H_0 + H_I) \Psi.$$  \hspace{1cm} (12) \hspace{1cm} (13)

The direct-time and time-reversed transition amplitudes are, respectively,

$$ (S - 1)_{fi} = -\frac{i}{\hbar} \int dt \langle \Phi_f, H_I \Psi_i \rangle, \hspace{1cm} (14)$$

$$ (S - 1)_{fi} = -\frac{i}{\hbar} \int dt \langle \Psi_f, H_I \Phi_i \rangle, \hspace{1cm} (15)$$

where the subscripts $i$ and $f$ refer to initial and final states, and $H_I$ is the interaction Hamiltonian. The $\Phi$ states can always be described accurately, so it is the accurate rendition of $\Psi$ states that is the challenge. The direct-time amplitude (14) possesses the difficulty with strong-field ionization that $\Psi_i$ combines the binding potential and the strong field influence on a bound electron in an intimately coupled form where neither influence can safely be assumed to be negligible. In contrast to this difficulty, the reversed-time amplitude (15) contains $\Psi_f$ for a detached electron, where a strong laser field can be regarded as dominant over residual binding-potential effects. This feature of the time-reversed S matrix is the motivation for the SPFA [19]. The more limited direct-time S matrix is the basis for the SEFA of Faisal [12].

The S matrices in Eqs. (14) and (15) are for the Schrödinger case. Relativistic S matrices for the Klein-Gordon case are given in Ref. [20], and for the Dirac case in Ref. [21].
III. KFR

The first laboratory observation of strong-laser effects [1] noted the existence of a spectral feature not previously observed in atomic ionization, which is the simultaneous presence of the contributions of more than just the lowest order of interaction. This was named ATI (Above-Threshold Ionization), a term that remains in use. It was surprising because the usual prediction of perturbation theory is that the lowest allowable photon order should be completely dominant. The original paper ascribed the effect to post-ionization perturbative interaction of the photoelectron with the laser field. Other interpretations assumed the effect to be a manifestation of higher orders of perturbation in the ionization process itself. These perturbative interpretations were accepted [36] by much of the strong-laser community for almost a decade after the initial ATI observation, but the likelihood of a nonperturbative alternative became increasingly evident [19, 37–39].

The term “KFR” was introduced in Ref. [40] to refer generically to nonperturbative analytical approximations. The continued use of the KFR acronym is unfortunate in view of the disparate approximations employed in the three papers that gave rise to the terminology. These differences will be explained after first establishing conventions about domains for frequency and wavelength measures.

A. Frequencies and wavelengths

Frequencies will be referred to as high if \( \omega > 1 \text{ a.u.} \) and low if \( \omega < 1 \text{ a.u.} \). Frequency \( \omega \) and wavelength \( \lambda \) are related as

\[
\omega = \frac{2\pi c}{\lambda},
\]

an expression that applies in atomic units as well as full units. When \( \omega = 1 \), marking the change from low to high frequency, \( \lambda \) has the value

\[
\lambda = 2\pi c \approx 861 \text{ a.u.}
\]

This means that it is possible to have rather long wavelengths be associated with high frequencies despite their inverse relationship, because of this value that is near \( 10^3 \). The rationale for employing the dipole approximation on the basis of long wavelength follows from the fact that the phase of a propagating field, \( \omega t - k \cdot r \), will have only slight variation.
over a distance of the size of the atom (i.e. $|r| = a_0 = 1 \text{ a.u.}$) if

$$|k| = \frac{\omega}{c} \ll \pi, \text{ or } \lambda \gg 2 \text{ a.u.}$$

(18)

using the relation (16). The disparity between the numerical values in Eqs. (17) and (18) means that there is a considerable range where high frequency need not infer short wavelength.

**B. The Keldysh approximation**

Keldysh entered the field of strong-laser effects with an established reputation in solid-state physics. The concept of tunneling through a potential barrier is widely employed in a solid-state context, and his physical reasoning is based on that concept. The formal basis Keldysh employed produces a tunneling result [3]. Since $F$ and $R$ are defined in terms of $S$ matrices, it is useful to analyze the 1964 K result in those terms as well, even though the applicability of $S$-matrix techniques had been confined to scattering problems until its universal extension for nonperturbative problems was demonstrated in 1970 [35, 41].

The essence of the Keldysh approximation can be stated in simple terms. Everything is treated within the DA; the interaction Hamiltonian is stated in the length gauge; the time-reversed transition amplitude of Eq. (15) is employed; and the final state $\Psi_f$ is replaced by a Volkov state with a gauge-transformation factor introduced to transform the Volkov solution expressed in the Coulomb gauge to the length gauge instead.

The limitations imposed by the approximations made by Keldysh as applied to laser fields can be stated concisely. The DA fails at high field frequencies (as is well-known) and also at low frequencies, since strong fields lead to increasing importance of the magnetic component of a laser field as $\omega \to 0$ [16, 18]. The $r \cdot E$ potential of the length gauge restricts the field to being purely an oscillatory electric field [11]. The “dipole-approximated Volkov solution” is not a Volkov solution in the usual sense of describing the behavior of a free electron in a transverse field. As Keldysh shows, it is actually a solution for a free electron in an oscillatory electric field. A more suitable terminology would be to call it a “Keldysh solution” rather than a Volkov solution. The Keldysh approximation is therefore an approximation for ionization or detachment of an electron from a bound state by a strong, low-frequency, oscillatory electric field.
Keldysh, understandably unaware in 1964 of the existence of the exact transition amplitude of Eq. (13), regards the non-interacting nature of the $\Phi_i$ as an additional approximation. Less understandable is the continued belief by many authors in that inappropriate assessment.

The Keldysh approach was recast in Ref. [42] as ionization by a low-frequency electric field using an adiabatic approximation, rather than with Volkov-solution concepts.

C. The Faisal approximation

The essence of the Faisal approximation [12] can also be stated in simple terms. Everything is treated within the DA; the interaction Hamiltonian is expressed in the velocity gauge; and the direct-time transition amplitude of Eq. (14) is employed.

Since no statement about field interaction of the electron in the final state is involved in Eq. (14), the F approximation uses neither the Volkov solution nor the Keldysh solution. The difficult problem of finding a suitable approximation for the initial state $\Psi_i$ in the direct-time S-matrix formalism is approached by using the KH transformation to a moving frame of reference, and then replacing that oscillatory orbit with the assumption that spatial coordinates remain centered at the atomic nucleus. As stated above, this is a high-frequency approximation. The additional limitations of the DA are also incurred.

The Faisal approximation can be summarized as an approximation for ionization or photodetachment of an electron from a bound state by a high-frequency oscillatory electric field.

D. The Reiss approximation

The genesis of the R method [19] is to be found in the long history [2, 43–46] of the author with the properties of the relativistic Volkov solution [6]. The intended application to nonrelativistic problems is approached by using the full Volkov solution (or Gordon solution [5]) in a nonrelativistic, long-wavelength limit. The Coulomb-gauge interaction Hamiltonian is also employed in its nonrelativistic form. This nonrelativistic long-wavelength version of the Coulomb gauge is similar in appearance to the velocity-gauge expression of the interaction Hamiltonian in the DA; the essential difference being in the $A^2$ term. In the DA,
A^2 (t) plays no dynamical role; whereas A^2 (t) has a central role to play in the nonrelativistic R theory. (See also Ref. [29].) This distinction is made clear in two 1990 papers. In the first of these papers [20], the 1980 R formalism is shown to be the nonrelativistic limit of a Klein-Gordon (i.e. spinless relativistic) treatment of atomic ionization. The second 1990 paper [21] is a completely Dirac-relativistic version of the SPFA, where initial- and final-state wave functions as well as the interaction term are Dirac-relativistic. The nonrelativistic, long-wavelength limit of the final outcome of the Dirac-relativistic atomic ionization calculation replicates exactly the 1980 R paper [19].

The insistence in Ref. [20] of the need to retain A^2 as a dynamical part of the formalism drew an understandable reaction [27] from a community accustomed to the DA, but the response [28] was direct, and supported in detail by the strong-field Dirac theory [21] that followed. See also Ref. [15].

The R approximation is therefore an approximation for ionization or detachment of an electron from a bound state by a strong, propagating field; i.e. by the field produced by a laser.

Hereafter, the R approximation will be referred to as the SPFA.

Frequency limitations in the SPFA are much milder than with other methods. This stands in contrast to the low-frequency restriction inherent in tunneling methods and the high-frequency-only domain of the KH-based F approximation. An interesting development is that the SPFA produces results at high frequencies equivalent to the KH-based methods of Faisal and Gavrila, and to TDSE. That is, the SPFA, fundamentally different from DA methods at low frequencies, becomes equivalent to DA theories at high frequencies. Strictly speaking, all of these high-frequency results apply only to quasistatic electric fields.

IV. SPFA EVALUATION

A variety of tests will be applied to evaluate the scope and accuracy of the SPFA.

The first comparisons to be displayed relate to experiments that illustrate aspects of the SPFA that are beyond the scope of tunneling theories or display a precision unmatched by tunneling theories or even TDSE. That set of comparisons with experiments is then followed by a survey of analytical matters, where theories alternative to the SPFA are applicable.
A. Comparisons with experimental results

The first comparison is with an experiment in the “multiphoton” regime, in DA parlance. This is followed by an example in the “tunneling” regime, again using DA terminology, where very high intensity was employed, and where especially precise measurement of a spectrum was accomplished. The next example shows that low-frequency limitations of the DA become detectable while still in the near-infrared, that can be regarded as a “transition” regime. The last case is one where radiation pressure (a relativistic phenomenon) was detected in experiments that were otherwise nonrelativistic.

1. The “multiphoton” regime

An early ATI experiment with circularly polarized light was carried out by Bucksbaum et al. in 1986 [37], which exhibited a spectrum that the authors considered to be revolutionary. Contributions from many photon orders were revealed, and the most probable order was significantly larger than the lowest detectable order. Although this effect had already been predicted (see Fig. 9 in Ref. [19]), this was not known to the experimental group, and they regarded their results to be strongly counter-intuitive.

Clearly distinguishable peaks in the spectrum, separated by $\hbar \omega$ in energy, placed the experiment in the so-called “multiphoton” regime in DA terminology, where the Keldysh parameter

$$\gamma_K = \sqrt{E_B/2U_p}$$  \hspace{1cm} (19)

is employed to distinguish between a multiphoton regime ($\gamma_K > 1$) and a tunneling regime ($\gamma_K < 1$). $E_B$ is the energy by which the atomic electron is bound (also known as the ionization potential $IP$ or $I_p$), and $U_p$ is the ponderomotive energy

$$U_p = I/(4\omega^2)$$  \hspace{1cm} (20)

in atomic units, where $I$ is the intensity of the field of frequency $\omega$. For this experiment, $\gamma_K \approx 1.7$.

The first attempt at a theoretical fit to ATI data employed the data of Ref. [37] and the method of Ref. [19], and yielded the satisfactory result reported in Ref. [38]. A more refined calculation was published in Ref. [39]. The bar graph spectra displayed in Refs.
can now be replaced by a much more sophisticated modern calculation shown in Fig. 1. The target atom is xenon, with the ground state wave function treated by an analytical Hartree-Fock method. The peak laser intensity employed represents the best post-experiment estimate by one of the experimenters. The only fitting parameter employed to produce Fig. 1 is the fraction of the ponderomotive energy returned to the photoelectrons in the very long pulse length of the experiment. This was used to set the absolute location of the ATI peaks.

Figure 1 is notable for several reasons. Not only was it the first successful application of nonperturbative theory to an ATI spectrum, but the parameters are such that a tunneling method is inapplicable, and the combination of circular polarization and long pulse length present formidable challenges for a TDSE approach.

A further remark about this 1986 experiment is that it is unequivocally in a nonperturbative domain because the peak $U_p$ of 2.12 eV exceeded the single-photon energy of 1.17 eV, a sufficient condition for failure of perturbation theory. Many authors continue to interpret the existence of distinguishable ATI peaks as an indication of perturbative behavior. This is an important misunderstanding.

2. The “tunneling” regime

The next example is in the “tunneling” domain, using DA nomenclature. It is selected because the experimenters achieved exceptionally small error bars for their circular polarization data, thus presenting an especially demanding challenge for an analytical theory. The experimental paper by Mohideen, et al. included an attempt to fit the data with the 1980 theory, but they neglected to include focal averaging, which accounts for the temporal and spatial variation of laser intensity in the focal region. When focal averaging is done, the result is shown in Fig. 2.

The spectral peak covers about 70 eV of energy, and the laser wavelength of 815 nm corresponds to a single-photon energy of 1.52 eV. There are thus about 50 ATI peaks included in the spectrum. These ATI peaks are so closely spaced on the scale of the total spectrum that both the laboratory measurement and the analytical calculation show a smooth energy distribution with no hint remaining of ATI structure. Nevertheless, the actual SPFA calculation requires a sum over photon orders. That is, from the point of
FIG. 1: This figure shows the ability of the transverse-field theory of Ref. [19] to replicate the experimental results presented in Ref. [37]. In DA terminology, this is in the multiphoton domain. The black curve (with wide peaks) is the measured photoelectron spectrum and the red curve (with narrow peaks) is the theoretical fit. Laser parameters: 1064 nm, peak intensity $2 \times 10^{13}$ W/cm$^2$, pulse duration 100 ps on a xenon target ($\gamma_K = 1.69$). The calculation includes focal averaging in a Gaussian beam with Gaussian time distribution, and with partial ponderomotive energy ($U_p$) recovery in the very long pulse. The only fitting parameter employed was the relative fraction of recovered $U_p$ (about 80%) selected to fix the absolute energy locations of the peaks. The theory used was in existence prior to the first observation of ATI [1].

view of a transverse-field theory, this experiment can be regarded as “multiphoton” and not “tunneling”, as demanded by longitudinal-field concepts. The Keldysh parameter has the value $\gamma_K = 0.40$, which places this experiment in the putative tunneling domain.

3. The “transition” regime

The parameter used as an index for the significance of magnetic effects is [16, 52]

$$I_{mag} = 8c\omega^3$$  \hspace{1cm} (21)

16
FIG. 2: This figure, based on Ref. [50], shows the extreme accuracy possible with the SPFA of 1980 [19] applied to the description of an experiment [49] at the relatively high intensity of $1.27 \times 10^{15}$ W/cm$^2$ at 815 nm wavelength and a pulse length of 180 fs in the ionization of helium. By SEFA terminology, this is in the tunneling domain ($\gamma K = 0.40$). The SPFA fit is within the very small experimental error bars. Irregularities in the low energy part of the spectrum are experimental artifacts [51].

in atomic units, using as a criterion that the displacement in the propagation direction of a photoelectron due to the magnetic component of a laser field reaches one atomic unit. That is a large displacement, and so it is to be expected that magnetic effects would become manifest at a much smaller intensity than that given by Eq. (21). The strong dependence on the frequency suggests that magnetic effects should be most evident at low frequencies.

Other factors, not particularly associated with magnetic fields, begin to manifest themselves as field frequency declines. For example, ponderomotive energy, so different in meaning in laser fields as opposed to oscillatory electric fields, depends on frequency as $1/\omega^2$. Non-
perturbative effects in general become more important at low frequencies. Collectively, low frequencies are more likely to degrade the effectiveness of the mimicry of the transverse-field Eqs. (5) to (8) by the longitudinal-field Eq. (10), a matter that will be revisited below.

The experiment to be examined here was done at Freiburg on photodetachment of the negative fluorine ion by a circularly polarized laser beam of wavelength $1.510 \, \mu m$ [53]. This is a useful experiment because it was designed to test the putative “gauge noninvariance of the SFA”, in addition to the present purpose of exploring a transition region.

A central issue in discussion of the Freiburg experiment is measurement of the peak laser intensity. Taking special care to be accurate about this difficult-to-measure quantity, the peak intensity was found to be $2.6 \times 10^{13} \, W/cm^2$, accurate to within 15% [53]. The published momentum distribution of the detached electrons was shown to provide an independent measure of the peak intensity [26] that confirmed the value $2.6 \times 10^{13} \, W/cm^2$. The prediction of the SPFA is compared with the experimental spectrum in Fig. 3 using only measured quantities, with no fitting done.

The authors of Ref. [53] compared their results with an SEFA calculation using the length gauge, and found that they had to assume that the peak laser intensity was 145% of the measured value to obtain a fit to the experiment. A TDSE calculation was also done, and matched the SEFA analytical approximation well, meaning that it also required an upward adjustment of the peak intensity by 145%. In view of the agreement between TDSE and the SEFA analytical approximation, and assuming TDSE to be exact, the authors concluded that they must have erred in their measurement of peak intensity. They pronounced the length gauge to be the “proper gauge” to use for analytical approximations.

That conclusion reached from the SEFA cannot be correct, since the measured peak intensity in the experiment is verified by the momentum distribution. Therefore, both the analytical SEFA and TDSE calculations are substantially in error, and the SPFA is accurate. Since TDSE is based on \emph{a priori} use of the DA, as is the SEFA analytical calculation, the underlying problem is the DA. It is known that SPFA and SEFA calculations come into agreement at high frequencies (see below), so the Freiburg experiment should provide an indication of where the primacy of potentials over fields becomes manifest [4]. This is not simply a matter of the onset of magnetic field effects. Designating the measured peak intensity as $I_{lab}$, the result is

$$I_{lab} = I_{mag}/40.$$ (22)
Magnetic effects are still rather small in the Freiburg experiment, and the comparison shown in Fig. 3 is done with a theory where the fields are described without spatial dependence. The experiment therefore provides a direct demonstration that the electric field component of a laser field is not equivalent to an oscillatory electric field of the same frequency. This is a fundamentally important matter that is examined in depth in an article in preparation.

The parameters for the Freiburg experiment also place it in a transition region between tunneling and multiphoton domains, with $\gamma_K = 0.55$. As seen in Fig. 3 both the measured and the SPFA spectra show evidences of multiphoton behavior superposed on a mostly smooth, continuous spectrum. The SEFA spectra published in Ref. [53] show no evidence of multiphoton behavior, further evidence that the 145% increased peak intensity is inappropriate. (A caution is that Ref. [53] displays a putative SPFA spectrum that is not accurate. Figure 3 here is a proper rendition of the SPFA spectrum.)

Briefly, the SPFA fits the experiment accurately using a peak laser intensity that matches that of the experiment as established by two independent measuring techniques. The SEFA approach, both analytical and numerical, fails because it requires an assumed peak laser intensity substantially in excess of the measured value, and because it shows only a featureless spectrum, lacking the traces of ATI peaks found in the experiment and in the SPFA.

A further comment is that the language of Ref. [53] in describing the experiment as critical in the choice of gauge was (unfortunately) adopted in Ref. [26]. That is, both papers discuss the matter as a choice of gauge, which is not the case at all. The experiment was a test of SPFA vs. SEFA. SPFA is always expressed in radiation gauge, as required by the propagation property [4]. DA theory, both analytical and numerical, describes phenomena as caused by an oscillatory electric field, where length gauge and velocity gauge are equivalent.

4. **SPFA vs. SEFA: radiation pressure**

The lowest frequency range is beyond the scope of the present nonrelativistic examination, since it implies a relativistic environment. The reason for this is simple: the ponderomotive potential $U_p$ experienced by an electron in a laser field is given in atomic units as proportional to $1/\omega^2$ as shown in Eq. (20). As $\omega \to 0$, the ponderomotive potential infers relativistic behavior [16, 18, 52].

Experimental information about ionization properties in the relativistic regime is lack-
FIG. 3: Experimental data [53] are shown as dots and the SPFA calculation [26] as the continuous curve, for the spectrum of electrons from the photodetachment of the negative fluorine ion ($F^-$) by a laser operating at 1.510 $\mu m$. The peak laser intensity in the calculation is taken to be the same as that measured in the laboratory. The agreement can be regarded as satisfactory, since residual ATI effects are evident in both the theory and the experiment, and the intensity is not high enough to expect the precise agreement shown in Fig. [2]. TDSE and SEFA analytical approximations could fit the data only if there was an assumed 45% increase in peak laser intensity from that found in the laboratory. The confluence of TDSE and length-gauge analytical calculations occurs because both use the DA. The assumption of faulty peak intensity measurement is not sustainable because the momentum distribution supported the originally reported peak intensity. It is the DA that has to be the cause of the conflict in theoretical approaches [26].

It was found possible to measure the effects of radiation pressure using a sensitive technique to detect the displacement in the propagation direction of the plane of symmetry of photoelectrons created by a circularly polarized laser [55]. A relativistic analysis of the findings is elementary, as shown in the following paragraph, using simple properties of propagating fields. Analysis with the SEFA is impossible since radia-
tion pressure is a consequence of the momentum carried by photons, and there is no photon momentum in a DA context.

Photoelectrons produced by a strong circularly polarized laser will enter into a circular orbit around the remnant ion with a kinetic energy $U_p$ \[11, 26, 56\]. The momentum component perpendicular to the propagation direction of the laser beam is found from $p^2_\perp/2 = U_p$, or $p_\perp = \sqrt{2U_p}$. The number of photons required to acquire this kinetic energy is $n = U_p/\hbar \omega$, and each of these photons contributes a momentum $\hbar \omega/c$ in the direction of propagation of the laser field. The component of momentum parallel to the propagation direction is thus $p_n = U_p/c$. The angle $\theta$ by which the plane of circulation of the photoelectrons is tilted forward is then \[21, 57\]

$$\tan \theta = \frac{p_n}{p_\perp} = \frac{1}{c} \sqrt{\frac{U_p}{2}}. \quad (23)$$

The expression (23) is independent of the identity of the atom being ionized, and it is in complete accord with the observations of Ref. \[55\].

Intensive efforts to explain the experiments with SEFA methods \[55, 58, 59\], produced no meaningful results.

The experimental results can, of course, be explained by completely relativistic dynamics. Equation (5.5) in Ref. \[21\] is identical to Eq. (23) here, albeit with different terminology.

**B. Comparisons with other computational methods**

Several lines of reasoning suggest that SPFA and SEFA methods coalesce as the frequency increases. This was shown in clear fashion by Bondar, et al. \[60\] with TDSE results at $\omega = 1 \text{ a.u.}$ and $\omega = 3 \text{ a.u.}$ Analytically, the most convincing comparison comes from the fact that the Faisal approximation \[12\], derived from a high-frequency ansatz, produces a result analytically identical to the SPFA \[19\]. A numerical comparison with another KH-related high-frequency method \[13, 14\] also shows good agreement \[62\] at $\omega = 2 \text{ a.u.}$ and $\omega = 8 \text{ a.u.}$

These areas of agreement between SPFA and SEFA methods at high frequencies come with the caution that the DA high-frequency limitation means that agreement is being shown in a frequency domain that has limited significance for laser fields because of the high-frequency limitation on the DA. These high-frequency comparisons are carried out in a frequency range discussed in Section III.A, where $\omega > 1$, but $\lambda \gg 1$ as well. That is SPFA and SEFA correspondence is in the wavelength range bounded by Eqs. (17) and (18). At
FIG. 4: This is a reproduction of Fig. 1 in Ref. [60]. A superposition of two linearly polarized fields is analyzed, with $\omega_1 = 1 \text{ a.u.}$ and $\omega_2 = 3 \text{ a.u.}$, and the intensity of each field is $3.5 \times 10^{14} \text{ W/cm}^2$. For the significance of other labeling in the figure, see Ref. [60]. For present purposes, the important features are the remarkably close correspondences between the panels labeled “SFA” (SPFA in this article) and those labeled “Numerical” (TDSE).

very high frequencies, as at very low frequencies, the way to surpass the DA inadequacy is to employ the relativistic SPFA [20, 21, 61].

1. High-frequency TDSE

As part of a project designed to test the capabilities of a proposed analytical technique called SCVA, a three-way comparison was done with TDSE and SPFA (labeled SFA in Ref. [60]). For the specifics of what was calculated, Ref. [60] should be consulted, but the qualitative results are very clear, as seen in Figs. 4 and 5. As commented upon in Ref. [60], the agreement between TDSE and SFA (actually SPFA) is remarkably close.
The stabilization phenomenon is a property of strong-field ionization where the increase of ionization probability with laser intensity reaches a maximum and then declines as the intensity is increased further. It can occur both with laser fields and with oscillatory electric fields. TDSE calculations are difficult because of the required high intensity, but Popov, et al. found a set of parameters for which it was possible to calculate the intensity at which the maximum ionization rate occurred, using a one-dimensional model. Their results are shown in Fig. 6, along with SPFA results for the same parameters. The mechanism by which stabilization occurs in the SPFA is importantly different for low frequencies and for high frequencies, and there is a gap between those two possibilities where the SPFA validity conditions fail. Results appear quite chaotic in that intermediate regime. The
FIG. 6: This figure is reproduced from Ref. [64], showing SPFA results (labeled SFA in the figure) compared with stabilization calculations reported in Ref. [63]. The $I_{stab}$ referred to on the vertical axis is the intensity at which a maximum in the transition rate occurs for the frequency $\omega$ indicated on the $x$ axis. The agreement is very good, especially in view of the fact that the high-frequency and low-frequency domains for the SPFA arise from very different stabilization mechanisms [65, 66]. The gap in the SPFA results follows from a failure of validity conditions for the SPFA within the gap in the neighborhood of the onset of stabilization for those frequencies.

correspondence between the SEFA calculation of Ref. [63] is surprisingly good since some of the frequencies involved are quite low, and the SPFA results are done in three dimensions, rather than one.

3. The High-Frequency Approximation (HFA) of Gavrila

The primary focus of the work of Gavrila [13, 14] was the stabilization phenomenon, which he believed occurred only at high frequencies. He employed the KH transformation with some embellishments that tended to smooth the effects of channel closings. His results
FIG. 7: The curve labeled “HFA” comes from the high-frequency approximation of Gavrila [13, 14]. It gives the photoelectron yield from ground-state hydrogen by a DA field at $\omega = 2$ a.u. The curve is adapted from Ref. [13]. The curve labeled “SFA” follows from what is called “SPFA” in this article. The intensity labeled “$z_1 = 1$” corresponds to $2U_p = E_B$, which is at the value of the Keldysh parameter $\gamma_K = 1$. The peak in the curve corresponds to the closing of the single-photon channel due to the increasing ponderomotive potential $U_p$.

for $\omega = 2$ a.u. are shown in Fig. 7 and for $\omega = 8$ a.u. in Fig. 8. SPFA calculations [62] are superimposed on the HFA curves. The correspondence is quite good except near the peaks. The decline in ionization probability occurs when the first-order channel closes because of the increasing demands of the ponderomotive energy. The SPFA curves show a slight increase just before the first-order channel closes because the contributions of higher orders become evident before the lowest-order channel closes.
FIG. 8: The comments in the caption of Fig. 7 apply here as well, except that the frequency is now $\omega = 8 \text{ a.u.}$ The range of intensity shown in both figures covers 8 orders of magnitude. The increase in the ionization rate shortly before the dominant single-photon channel closes is due to contributions from higher-order terms becoming significant.

V. ROLE OF THE PONDEROMOTIVE ENERGY

The aim of this Section is to achieve qualitative insights into differing behaviors of SPFA and SEFA methods at both high and low frequencies. Detailed inspection of specific problems have established that there are important differences as frequencies decline, but comparisons at high frequencies have revealed broad correspondences. The goal now is to provide a qualitative understanding of the causes of these frequency differences.

A. High frequencies

A starting point is given by Fig. 9 reproduced from Ref. [64]. This figure compares relativistic and nonrelativistic single-frequency (i.e. with no focal averaging) total ioniza-
tion rates from ground-state hydrogen by circularly polarized light. Low frequencies are represented by \( \omega = 1/16 \text{ a.u.} \) and \( 1/8 \text{ a.u.} \), and high frequencies by \( \omega = 4 \text{ a.u.} \) and \( \omega = 8 \text{ a.u.} \). Differences are visible at high intensity for all cases. Differences are inevitable because the magnetic field eventually becomes important at high intensities and because relativistic kinematics differs from nonrelativistic kinematics. None of the high-frequency comparisons examined earlier in this article went to sufficiently high intensity to observe the onset of relativistic kinematical behavior, even when they exceeded substantially the usual DA limit that requires \( \lambda > 1 \).

An important qualitative observation is that total rates constitute a relatively “blunt instrument” for examining physical distinctions. For example, although slight distinctions are seen in Fig. 9 at \( \omega = 1/8 \text{ a.u.} \) between Schrödinger SPFA and Dirac SPFA total rates, differences are actually quite clear if spectra or angular distributions are examined. This is shown in Figs. 10 and 11.

The ponderomotive energy \( U_p \) has an important role in strong-field problems. As Eq. (20) shows, the ponderomotive energy depends on frequency as \( 1/\omega^2 \), establishing the decreasing importance of ponderomotive considerations as the frequency increases.

Magnetic fields were mentioned as being of considerable significance at low frequencies. The onset of magnetic field effects is proportional to \( 1/\omega^3 \), as shown in Eq. (21). Explicitly relativistic effects (as distinct from magnetic effects \[16, 52\]) depend on the ratio of \( U_p \) to \( mc^2 \), and so frequency dependence is proportional to \( 1/\omega^2 \). Fully relativistic effects will occur before explicit magnetic-field phenomena become separately visible at high frequencies – opposite to the low-frequency case.

An important consideration, not often mentioned in strong-field studies, is the field intensity at which perturbation theory fails. The radius of convergence of perturbation theory was examined formally in Refs. \[2, 19, 43, 44\]. The results of these studies can be expressed concisely. An upper bound on the field intensity at which perturbative considerations fail is that intensity at which the first channel closing occurs due to the demands of the ponderomotive energy. Channel closing depends on the system being examined as well as the characteristics of the laser field. The energy threshold for ionization is given by the sum of the binding energy \( E_B \) and the ponderomotive energy \( U_p \). Suppose that the minimum photon order for ionization is \( n_0 \) without any allowance for ponderomotive energy requirements. When the ponderomotive energy increases to the point where \( E_B + U_p \geq n_0\hbar\omega \),
FIG. 9: Rates for ionization by circularly polarized light for ground-state hydrogen as calculated
nonrelativistically (Ref. [19]) and relativistically (Ref. [21]). The calculation is for a fixed intensity.
There is neither focal averaging nor saturation effects. This figure is reproduced from Ref. [64].
(The character “w” in the legend refers to the frequency $\omega$.) For both low frequencies: $\omega = 1/16$
a.u. and $\omega = 1/8$ a.u.; and for high frequencies: $\omega = 4$ a.u. and $\omega = 8$ a.u., nonrelativistic and
nonrelativistic ionization rates remain almost equal for a large range of intensities. Deviations
become evident at much higher intensity when frequencies are high then when they are low.

then the lowest-order channel closes and $n_0 + 1$ photons becomes the required minimum.
In formal terms, this represents an essential singularity in the complex coupling-constant
plane, and is thus an upper bound on perturbation theory. For example, using atomic units,
the field-free binding energy of ground-state hydrogen is $1/2$. In the presence of a laser field,
the lowest-order channel will close when $U_p = 1/2$. From Eq. (20), this will occur when
$I = 2\omega^2$. Therefore, perturbation theory can remain valid to much higher intensities at
FIG. 10: The physical conditions that pertain for this comparison of nonrelativistic and relativistic spectra relate to Fig. 9 for the case of $\omega = 1/8 \ a.u.$ at an intensity $I = 10^3 \ a.u.$ The intent is to show that spectra can be significantly different for the two cases even when the total ionization rates are very similar. This figure is for illustrative purposes only. Ground-state hydrogen in an actual laser pulse would be completely ionized long before the intensity shown could be reached.

high frequency than at low frequency. Currently, strong-field lasers that operate in the IR to UV range generally exceed the bounds on perturbation theory, whereas high-frequency free-electron lasers (FELs) can operate at very high intensities and still produce effects that are explicable perturbatively.

B. Low frequencies

It has been noted previously [16, 52] that, at low frequencies, magnetic-field effects can occur in a broad frequency domain before completely relativistic properties become evident. In itself, this is not a full explanation for the significant differences that begin to appear
FIG. 11: The remarks in the caption of Fig. 10 apply also to this demonstration of the extent to which radiation pressure causes photoelectrons to be shifted in the propagation direction even when the conditions of $\omega = 1/8 \ a.u.$ and $I = 10^3 \ a.u.$ have little consequence for total rates. Radiation pressure is a phenomenon caused by momentum transferred from the laser field to the photoelectron. This can only be observable when magnetic effects are included. Oscillatory electric fields do not lead to radiation pressure.

between SPFA and SEFA predictions when the $\beta_0$ parameter measuring magnetic field influences has a magnitude of only 1/40, as indicated by Eqs. (21) and (22). A further puzzle is that the electromagnetic field that is employed in the SPFA calculation that gives rise to Fig. 3 is employed without the presence of $\mathbf{r}$ dependence in the potentials employed, signifying an absence of a magnetic field.

The resolution of this apparent dilemma is to be found in the ponderomotive energy $U_p$, which has importantly different meaning when it arises from a propagating field as opposed to an oscillatory electric field. The concept of ponderomotive energy occurs in electrodynamics with three distinct meanings. The earliest sense in which it arose in electromagnetism is as a force potential. When a charged particle is immersed in a nonuniform electromagnetic field, it experiences a force directed towards lower field intensity. That meaning retains
its importance in modern applications where focused laser beams are involved. A second meaning occurs when a charged particle is subjected to an oscillatory electric field. This produces an oscillation called a “quiver” motion, describable by the Maxwell equation (11), that relates to oscillatory electric fields. A third meaning is associated with laser fields. Laser fields obey the Maxwell equations (5) to (8) that do not have any source terms. However, a charged particle in a plane-wave field possesses a ponderomotive energy that is a potential energy that depends on the local field intensity $U_p$.

To appreciate the importance of the distinctions between the different forms of ponderomotive energy, consider the case of an atom being ionized by a field that just achieves the threshold energy for ionization. When that ionizing field is an oscillatory electric field, the threshold condition is that the field must supply enough energy to overcome the binding potential as well as to provide the kinetic quiver energy that the electron must possess in order to be a free electron in the field. In the case of ionization of an atom by a laser field that just meets threshold condition for ionization, the electron appears as a free electron with zero kinetic energy, but with a potential energy that is required by its presence in the laser field. If that detached electron is free to accelerate under the influence of the laser field until it reaches the boundary of the laser beam, it will then appear outside the beam with a kinetic energy that is just $U_p$. That is, its initial potential energy is converted to kinetic energy through the mechanism of acceleration in the intensity gradient of the field.

This description is the manifestation of the significance of retaining the $A^2$ term even under long-wavelength conditions where position dependence of the $A^2$ appears to be unimportant, as described above in Section III.C. In an oscillatory electric field, the $A^2(t)$ term has no dynamical consequences. The electron detached under threshold conditions possesses an initial kinetic energy from the field in the amount of $U_p$, but not a potential energy. In a plane-wave field, the electron detached under threshold conditions has no kinetic energy, but it possesses instead a potential energy in the amount of $U_p$. Threshold energy conditions have the same magnitude, but the physical consequences are fundamentally different.

The ultimate distinction between oscillatory electric fields and plane-wave fields becomes evident as the field frequency approaches zero. Oscillatory electric fields then approach constant fields. Plane-wave fields, by contrast, propagate at the velocity of light for all frequencies, no matter how low.
VI. SPFA, SEFA, AND GAUGE COMPARISONS

An essential theme in this paper is that the SPFA analytical approximation refers to propagating fields; that is, to laser fields. The SEFA analytical approximations treats the behavior of oscillatory electric fields; that is, there is no magnetic field and no propagation. TDSE maintains the same DA assumptions as analytical SEFA methods.

As field frequency increases, SPFA and SEFA become almost identical, as long as the field wavelength is in excess of the size of the atom. As field frequency declines, the basic ponderomotive distinctions between propagating electromagnetic fields and oscillatory electric fields become manifest, the magnetic component of laser fields becomes increasingly important, and the SEFA becomes increasingly unreliable. Qualitative and quantitative failure of the SEFA becomes extreme at low frequencies, so that the so-called adiabatic limit and/or tunneling limit, once regarded as safe territory for a dipole-approximate theory, is actually where the DA becomes completely meaningless \[16,18\] for intense-field laser applications. It is unfortunate that, for many years, analytical approximation methods for laser-induced processes were rejected \[67\] unless they exhibited the erroneous adiabatic limit as \(\omega \to 0\).

Another unfortunate practice has been to validate analytical approximations by their agreement with TDSE, overlooking the fact that TDSE shares the same DA as SEFA analytical theories. When SPFA and SEFA methods diverge as the frequency declines, it is the SPFA that is related to laser-induced processes, and not the SEFA. This defect has been shown to become significant in experiments where the laboratory intensity reaches only 2.5\% of the magnetic-field intensity indicator \(I_{\text{mag}} = 8c\omega^3\) \[26,53\].

The debate about the relative validity of SPFA and SEFA methods has been cast improperly as a debate about a choice between the velocity gauge and the length gauge. Both of these gauges relate exclusively to DA matters. The SPFA offers no gauge alternatives; only the radiation gauge (also known as Coulomb gauge) is appropriate for propagating fields \[4\]. It is regrettable that much of the literature on the subject was expressed in terms of gauge comparisons (including by the present author in the title of Ref. \[26\]). However, Ref. \[26\] is definitive. The peak laser intensity is properly stated in the paper containing the experimental results \[53\]; the SPFA is in accord with those results, and the SEFA is not, neither in its analytical form nor in its TDSE form.

The predilection of the laser physics community to use the length gauge has led to the
relative neglect of the SPFA. Exact correction terms to the basic SPFA are stated in Appendix A of Ref. [19]. These correction terms contain both the high energy plateau in linear polarization spectra [68] and the low energy structure (LES) found at long wavelengths [54], but major efforts have been expended in finding DA alternatives. Failure of the DA at both low and high frequencies can be treated by the relativistic form of the SPFA, but results have been published only for the ground state of hydrogen [21, 61], whereas it can be formulated for any hydrogenic initial state. Through Hartree-Fock analysis, arbitrary initial bound states can be constructed from hydrogenic states. The SPFA approach is more efficient and more comprehensive than seeking nondipole correction terms to TDSE.

[1] P. Agostini, F. Fabre, G. Mainfray, G. Petite, and N. K. Rahman, Free-free transitions following six-photon ionization of xenon atoms, Phys. Rev. Lett. 42, 1127-1130 (1979).

[2] H. R. Reiss, Absorption of light by light, J. Math. Phys. 3, 59-67 (1962).

[3] L. V. Keldysh, Ionization in the field of a strong electromagnetic wave, Sov. Phys. JETP 20, 1307-1314 (1965) [Zh. Eksp. Teor. Fiz. 47, 1945-1957 (1964)].

[4] H. R. Reiss, Physical restrictions on the choice of electromagnetic gauge and their practical consequences, J. Phys. B 50, 075003 (2017).

[5] W. Gordon, Der Comptoneffekt nach der Schrödingerschen Theorie, Zeit. f. Phys. 40, 117-133 (1926).

[6] D. M. Volkov, Über eine Klasse von Lösungen der Diracschen Gleichung, Zeit. f. Physik 94, 250-260 (1935).

[7] G. Gamow, Zur Quantentheorie des Atomkernes, Z. Physik 51, 204-212 (1928).

[8] J. R. Oppenheimer, Three notes on the quantum theory of aperiodic effects, Phys. Rev. 31, 66-81 (1928).

[9] H. A. Kramers, Collected Scientific Papers (North Holland, Amsterdam, 1956) 262.

[10] W. C. Henneberger, Perturbation method for atoms in intense light beams, Phys. Rev. Lett. 21, 838-841 (1968).

[11] H. R. Reiss, Altered Maxwell equations in the length gauge, J. Phys. B 46, 175601 (2013).

[12] F. H. M. Faisal, Multiple absorption of laser photons by atoms, J. Phys. B 6, L89-L92 (1973).
[13] M. Pont and M. Gavrila, *Stabilization in atomic hydrogen in superintense high-frequency laser fields of circular polarization*, Phys. Rev. Lett. **65**, 2362-2365 (1990).

[14] M. Gavrila, *Atoms in Intense Laser Fields*, M. Gavrila, ed. (Academic Press, San Diego, 1992) 435-510.

[15] H. R. Reiss, *Theoretical methods in quantum optics: S-matrix and Keldysh techniques for strong-field processes*, Prog. Quant. Electronics **16**, 1-71 (1992).

[16] H. R. Reiss, *Limits on tunneling theories of strong-field ionization*, Phys. Rev. Lett. **101**, 043002 (2008); *Erratum*, Phys. Rev. Lett. **101**, 155901(E) (2008).

[17] H. R. Reiss, *Unsuitability of the Keldysh parameter for laser fields*, Phys. Rev. A **82**, 023418 (2010).

[18] H. R. Reiss, *The tunneling model of laser-induced ionization and its failure at low frequencies*, J. Phys. B **47**, 204006 (2014).

[19] H. R. Reiss, *Effect of an intense electromagnetic field on a weakly bound system*, Phys. Rev. A **22**, 1786-1813 (1980).

[20] H. R. Reiss, *Complete Keldysh theory and its limiting cases*, Phys. Rev. A **42**, 1476-1486 (1990).

[21] H. R. Reiss, *Relativistic strong-field photoionization*, J. Opt. Soc. Am. B **7**, 574-586 (1990).

[22] M. Göppert-Mayer, *Über Elementarakte mit zwei Quantensprüngen*, Ann. Phys., Lpz. **9**, 273-294 (1931).

[23] K.-H. Yang, *Gauge transformations and quantum mechanics: I. Gauge invariant interpretation of quantum mechanics*, Ann. Phys. NY **101**, 62-118 (1976).

[24] R. R. Schlicher, W. Becker, J. Bergou, and M. O. Scully, *Interaction Hamiltonian in Quantum Optics* in *Quantum Electrodynamics and Quantum Optics*, A. O. Barut, Ed. (Plenum, New York, 1984).

[25] W. E. Lamb, R. R. Schlicher, and M. O. Scully, *Matter-field interaction in atomic physics and quantum optics*, Phys. Rev. A **36**, 2763-2772 (1987).

[26] H. R. Reiss, *Velocity-gauge theory for the treatment of strong-field photodetachment*, Phys. Rev. A **76**, 033404 (2007).

[27] P. Milonni, *Comment on 'Complete Keldysh theory and its limiting cases'*, Phys. Rev. A **45**, 2138-2139 (1992).

[28] H. R. Reiss, *Reply to "Comment on 'Complete Keldysh theory and its limiting cases'"*, Phys.
Rev. A 45, 2140-2141 (1992).

[29] H. R. Reiss, Mass shell of strong-field quantum electrodynamics, Phys. Rev. A 89, 022116 (2014).

[30] S. V. Popruzhenko, Keldysh theory of strong field ionization: history, applications, difficulties and perspectives, J. Phys. B 47, 204001 (2014).

[31] E. S. Sarachik and G. T. Schappert, Classical theory of the scattering of intense laser radiation by free electrons, Phys. Rev. D 1, 2738-2753 (1970).

[32] G. F. Gribakin and M. Yu. Kuchiev, Multiphoton detachment of electrons from negative ions, Phys. Rev. A 55, 3760-3771 (1997).

[33] K. C. Kulander, K. J. Shafer, and J. L. Krause in Super-Intense Laser-Atom Physics, B. Piraux, A. l’Huiller, and K Rzazewski, Eds. (Plenum, New York, 1973).

[34] P. B. Corkum, Plasma perspective on strong-field multiphoton ionization, Phys. Rev. Lett. 71, 1994-1997 (1993).

[35] H. R. Reiss, Properties of the momentum-translation approximation, Phys. Rev. A 23, 3019-3033 (1981).

[36] S. L. Chin and P. Lambropoulos, eds., Multiphoton Ionization of Atoms (Academic Press, Toronto, 1984).

[37] P. H. Bucksbaum, M. Bashkansky, R. R. Freeman, T. J. McIlrath, and L. F. DiMauro, Suppression of multiphoton ionization with circularly polarized coherent light, Phys. Rev. Lett. 56, 2590-2593 (1986).

[38] H. R. Reiss, Spectrum of atomic electrons ionised by an intense field, J. Phys. B 20, L79-L83 (1987).

[39] H. R. Reiss, Electron spectrum in intense-field photoionization, J. Opt. Soc. Am. B 87, 726-730 (1987).

[40] P. H. Bucksbaum, M. Bashkansky, and D. W. Schumacher, Above-Threshold Ionization in Helium, Phys. Rev. A 37, 3615-3618 (1988).

[41] H. R. Reiss, Semiclassical electrodynamics of bound systems in intense fields, Phys. Rev. A 1, 803-818 (1970).

[42] A. M. Perelomov, V. S. Popov, and M. V. Terent’ev, Ionization of atoms in an alternating electric field, Z. Eksp. Teor. Fiz. 50, 1393-1409 (1966) [Sov. Phys. JETP 23, 924-934 (1966)].

[43] H. R. Reiss, PhD Dissertation, Univ. of Maryland (1958).
[44] H. R. Reiss, *A convergent perturbation expansion in first-quantized electrodynamics*, J. Math. Phys. 3, 387-395 (1962).

[45] J. H. Eberly and H. R. Reiss, *Electron self-energy in intense plane-wave field*, Phys. Rev. 145, 1035-1040 (1966).

[46] H. R. Reiss and J. H. Eberly, *Green’s function in intense-field electrodynamics*, Phys. Rev. 151, 1058-1066 (1966).

[47] A. A. Radzig and B. M. Smirnov, *Reference Data on Atoms, Molecules, and Ions* (Springer, Berlin, 1985).

[48] T. J. McIlrath, Private communication (1987).

[49] U. Mohideen, M. H. Sher, H. W. K. Tom, G. D. Aumiller, O. R. Wood, R. R. Freeman, J. Bokor, and P. H. Bucksbaum, *High intensity above-threshold ionization of He*, Phys. Rev. Lett. 71, 509-512 (1993).

[50] H. R. Reiss, *Energetic electrons in strong-field ionization*, Phys. Rev. A 54, R1765-R1768 (1996).

[51] U. Mohideen, Private communication (1994).

[52] H. R. Reiss, *Dipole-approximation magnetic fields in strong laser beams*, Phys. Rev. A 63, 013409 (2000).

[53] B. Bergues, N. Yongfeng, H. Helm, and I. Yu. Kiyan, *Experimental study of photodetachment in a strong laser field of circular polarization*, Phys. Rev. Lett. 95, 263002 (2005).

[54] C. I. Blaga, F. Catoire, P. Colosimo, G. G. Paulus, H. G. Muller, P. Agostini, and L. F. DiMauro, *Strong-field photoionization revisited*, Nature Phys. 5, 335-338 (2008).

[55] C. T. L. Smeenk, L. Arissian, B. Zhou, A. Mysyrowicz, D. M. Villeneuve, A. Staudte, and P. B. Corkum, *Partitioning of the linear photon momentum in multiphoton ionization*, Phys. Rev. Lett. 106, 193002 (2011).

[56] H. R. Reiss and V. P. Krainov, *Approximation for a Coulomb-Volkov solution in strong fields*, Phys. Rev. A 50, R910 (1994); *Erratum*, Phys. Rev. A 74, 049903(E) (2006).

[57] H. R. Reiss, *Relativistic effects in nonrelativistic ionization*, Phys. Rev. A 87, 033421 (2013).

[58] S. Chelkowski, A. D. Bandrauk, and P. B. Corkum, *Photon momentum sharing between an electron and an ion in photoionization: from one photon (photoelectric effect) to multiphoton absorption*, Phys. Rev. Lett 113, 263005 (2014).

[59] S. Chelkowski, A. D. Bandrauk, and P. B. Corkum, *Photon momentum transfer in multiphoton absorption*
ionization and in time-resolved holography with photoelectrons, Phys. Rev. A 92, 051401(R) (2015).

[60] D. I. Bondar, M. Spanner, W.-K. Liu, and G. L. Yudin, Photoelectron spectrum in strong-field ionization by a high-frequency field, Phys. Rev. A 79, 063404 (2009).

[61] D. P. Crawford, PhD Dissertation, American Univ. (1994).

[62] H. R. Reiss, High-frequency, high-intensity photoionization, J. Opt. Soc. Am. B 13, 355-362 (1996).

[63] A. M. Popov, O. V. Tikhonova, and E. A. Volkova, Applicability of the Kramers-Henneberger approximation in the theory of strong field ionization, J. Phys. B 32, 3331-3345 (1999).

[64] H. R. Reiss, Dependence on frequency of strong-field atomic stabilization, Opt. Exp. 8, 99-105 (2001).

[65] H. R. Reiss, Frequency and polarization effects in stabilization, Phys. Rev. A 46, 391-394 (1992).

[66] H. R. Reiss, Physical basis for strong-field stabilization of atoms, Las. Phys. 7, 543-550 (1997).

[67] C. J. Joachain, N. J. Kylstra, and R. M. Potvliege, Atoms in Intense Laser Fields (Cambridge Univ. Press, Cambridge, 2012).

[68] G. G. Paulus, W. Niklich, H. Xu, P. Lambropoulos, and H. Walther, Plateau in above threshold ionization spectra, Phys. Rev. Lett. 72, 2851-2854 (1994).