Partitions of trees and $\text{ACA}'_0$

Bernard A. Anderson
Jeffry L. Hirst
Appalachian State University
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Abstract

We show that a version of Ramsey’s theorem for trees for arbitrary exponents is equivalent to the subsystem $\text{ACA}'_0$ of reverse mathematics.

In [1], a version of Ramsey’s theorem for trees is analyzed using techniques from computability theory and reverse mathematics. In particular, it is shown that for each standard integer $n \geq 3$, the usual Ramsey’s theorem for $n$-tuples is equivalent to the tree version for $n$-tuples. The main result of this note shows that the universally quantified versions of these forms of Ramsey’s theorem are also equivalent. Because there are so few examples of proofs involving $\text{ACA}'_0$ in the literature, we have included a somewhat detailed exposition of the proof.

The main subsystems of second order arithmetic used in this paper are $\text{RCA}_0$, which includes a comprehension axiom for computable sets, and $\text{ACA}_0$, which appends a comprehension axiom for sets definable by arithmetical formulas. For details on the axiomatization of these subsystems, see [4]. More about the subsystem $\text{ACA}'_0$ appears below.

If $2^{< N}$ is the full binary tree of height $\omega$, we may identify each node with a finite sequence of zeros and ones. We refer to any subset of the nodes as a subtree, and say that a subtree $S$ is isomorphic to $2^{< N}$ if every node of $S$ has exactly two immediate successors in $S$. Formally, $S \subseteq 2^{< N}$ is isomorphic to $2^{< N}$ if and only if there is a bijection $b : 2^{< N} \to S$ such that for all $\sigma, \tau \in 2^{< N}$, we have $\sigma \subseteq \tau$ if and only if $b(\sigma) \subseteq b(\tau)$. (For sequences, $\sigma \subseteq \tau$ means $\sigma$ is an initial segment of $\tau$, and $\sigma \subset \tau$ means $\sigma$ is a proper initial segment of $\tau$.)
For any subtree $S$, we write $[S]^n$ for the set of linearly ordered $n$-tuples of nodes in $S$. All the nodes in any such $n$-tuple are pairwise comparable in the tree ordering. In [1], the following version of Ramsey’s theorem is presented.

**TT**($n$): Fix $k \in \mathbb{N}$. Suppose that $[2^{<\mathbb{N}}]^n$ is colored with $k$ colors. Then there is a subtree $S$ isomorphic to $2^{<\mathbb{N}}$ such that $[S]^n$ is monochromatic.

In applying **TT**($n$), we often think of the coloring as a function $f : [2^{<\mathbb{N}}]^n \to k$, in which case $S$ is monochromatic precisely when $f$ is constant on $[S]^n$.

Let $\Phi_{X,e,t}^X(m) \downarrow$ denote a fixed formalization of the assertion that the Turing machine with code number $e$, using an oracle for the set $X$, halts on input $m$ with the entire computation bounded by $t$. We will assume that $t$ is a bound on all aspects of the computation, including codes for inputs from the oracle. This formalization can be based on Kleene’s $T$-predicate or any similar arithmetization of computation. In **RCA$_0$**, we use the notation $Y \leq_T X$ to denote the existence of two codes $e$ and $e'$ such that

$$\forall m (m \in Y \iff \exists t \Phi_{X,e,t}^X(m) \downarrow)$$

and

$$\forall m (m \notin Y \iff \exists t \Phi_{X,e',t}^X(m) \downarrow).$$

The preceding formalizes the notion that $Y$ is Turing reducible to $X$ if and only if both $Y$ and its complement are computably enumerable in $X$.

As in [2], we can also use this notation to formalize **ACA’$_0$**. Given any set $X$, let $Y = X'$ denote the statement

$$\forall \langle m,e \rangle (\langle m,e \rangle \in Y \iff \exists t \Phi_{X,e,t}^X(m) \downarrow),$$

where $\langle m,e \rangle$ denotes an integer code for the ordered pair $(m,e)$. To formalize the $n$th jump for $n \geq 1$, we write $Y = X^{(n)}$ if there is a finite sequence $X_0, \ldots, X_n$ such that $X_0 = X$, $X_n = Y$, and for every $i < n$, $X_{i+1} = X'_i$. In this notation, $Y = X'$ if and only if $Y = X^{(1)}$, and we will often write $X''$ for $X^{(2)}$. The subsystem **ACA’$_0$** consists of **ACA$_0$** plus the assertion that for every $X$ and every $n$, there is a set $Y$ such that $Y = X^{(n)}$.

Using all this terminology, we can prove a formalized version of the implication from **TT**($n$) to **TT**($n + 1$), including a formalized computability theoretic upper bound.

**Lemma 1.** (**RCA$_0$**) Suppose $R$ is a tree isomorphic to $2^{<\mathbb{N}}$, $f : [R]^{n+1} \to k$ is a finite coloring of the $(n+1)$-tuples of comparable nodes of $R$, and both
\( R \leq_T A \) and \( f \leq_T A \). Suppose that \( A'' \) exists. Then we can find a tree \( S \) and a coloring \( g : [S]^n \rightarrow k \) such that \( S \leq_T A'' \), \( g \leq_T A'' \), \( S \) is a subtree of \( R \) isomorphic to \( 2^{<\mathbb{N}} \), and every monochromatic subtree of \( S \) for \( g \) is also monochromatic for \( f \).

**Proof.** Working in \( \text{RCA}_0 \), suppose \( R \), \( f \), and \( A \) are as in the statement of the lemma. We will essentially carry out the proof of Theorem 1.4 of [1], replacing uses of arithmetical comprehension by recursive comprehension relative to \( A'' \). Toward this end, given a sequence \( P = \{\rho_\tau \mid \tau \subseteq \sigma\} \) of comparable nodes of \( R \) such that the sequence terminates in \( \rho_\sigma \), define an induced coloring of single nodes \( \tau \supset \rho_\sigma \) by setting

\[
f_{\rho_\sigma}(\tau) = \langle\langle \bar{m}, f(\bar{m}, \tau) \rangle \mid \bar{m} \in [P]^n \rangle\rangle
\]

where the angle brackets denote an integer code for the finite set. Since \( f \leq_T A \), for any finite set \( P \) we have \( f_{\rho_\sigma} \leq_T A \).

For each \( \sigma \in 2^{<\mathbb{N}} \), define \( \rho_\sigma, T_\sigma \), and \( c_\sigma \) as follows. Let \( \rho_\emptyset \) be the root of \( R \) and \( T_\emptyset = R \). Given \( \rho_\sigma \) and \( T_\sigma \) computable from \( A \), use \( A'' \) to compute a \( c_\sigma \) which is the greatest integer in the range of \( f_{\rho_\sigma} \) such that

\[
\exists \rho \in T_\sigma (\rho \supset \rho_\sigma \land \forall \tau \in T_\sigma (\tau \supset \rho \rightarrow c_\sigma \leq f_{\rho_\sigma}(\tau))).
\]

Using \( A'' \), compute the least such \( \rho \). Let \( T \) denote the subtree of \( T_\sigma \) isomorphic to \( 2^{<\mathbb{N}} \) defined by taking \( \rho \) as the root and letting the immediate successors of each node be the least pair of incomparable extensions in \( T_\sigma \) that are assigned \( c_\sigma \) by \( f_{\rho_\sigma} \). Because of the choice of \( c_\sigma \), \( T \) is isomorphic to \( 2^{<\mathbb{N}} \), and its nodes can be located in an effective manner. (In [1], this \( T \) is called the standard \( c_\sigma \)-colored subtree of \( T_\sigma \) for \( \rho \) using \( f_{\rho_\sigma} \).) Let \( \rho_{\sigma^0} \) and \( \rho_{\sigma^{-1}} \) be the two level one elements of \( T \) and let \( T_{\sigma^0} \) be the subtree of \( T \) with root \( \rho_{\sigma^0} \) for each \( \varepsilon \in \{0,1\} \). Note that given any finite chain of elements and colors \( \langle\langle \rho_\tau, c_\tau \rangle \mid \tau \subseteq \sigma\rangle \), sufficiently large initial segments of each \( T_\tau \) can be computed to determine \( \rho_{\sigma^0}, \rho_{\sigma^{-1}}, c_{\sigma^0}, \) and \( c_{\sigma^{-1}} \), using only \( A'' \). Consequently, the subtree \( S = \{\rho_\sigma \mid \sigma \in 2^{<\mathbb{N}}\} \) is computable from \( A'' \).

Define \( g : [S]^n \rightarrow k \) by \( g(\rho_{\sigma_1}, \ldots, \rho_{\sigma_n}) = f(\rho_{\sigma_1}, \ldots, \rho_{\sigma_n}, \rho_{\sigma^0}) \). Since \( S \leq_T A'' \), we also have \( g \leq_T A'' \). By the construction of \( S \), given any increasing sequence of elements of \( S \) of the form \( \rho_1 \prec \rho_2 \prec \cdots \prec \rho_n \), and extensions \( \rho_n \subset \rho_{n+1} \) and \( \rho_n \subset \rho_{n+2} \), we have \( f_{\rho_n}(\rho_{n+1}) = f_{\rho_n}(\rho_{n+2}) \), so \( f(\rho_1, \ldots, \rho_n, \rho_{n+1}) = f(\rho_1, \ldots, \rho_n, \rho_{n+2}) \). Thus any monochromatic subtree for \( g \) is also monochromatic for \( f \), and the proof is complete. \( \square \)
Extracting the computability theoretic content of the previous argument, given a computable coloring of $n$-tuples we can find a monochromatic set computable from $0^{(2n-2)}$. This is not an optimal bound, since applying the Strong Hierarchy Theorem to Theorem 2.7 of [1] yields a monochromatic set computable from $0^{(n)}$. However, the preceding result does enable us to complete the proof of the next theorem, and avoids formalization of the long proof of Theorem 2.7 of [1].

**Theorem 2.** (RCA$_0$) The following are equivalent:

1. $\text{ACA}'_0$
2. $\forall n \text{TT}(n)$

*Proof.* To prove that (1) implies (2), assume $\text{ACA}'_0$ and let $f : [2^{<\mathbb{N}}]^n \to k$ be a coloring. By $\text{ACA}'_0$, the jump $f^{(2n-2)}$ exists, so by discarding the odd jumps we can find a sequence of sets $X_0, X_1, \ldots, X_{n-1}$ such that $X_0 = f$ and for each $i$, $X_{i+1} = X_i''$. Note that $f \leq_T X_0$ and $2^{<\mathbb{N}} \leq_T X_0$. By Lemma 1, for any $X_i$, given indices witnessing that a subtree isomorphic to $2^{<\mathbb{N}}$ and a coloring of the $(n-i)$-tuples of that subtree are each computable from $X_i$, we can find indices for computing an infinite subtree and a coloring of $(n-i-1)$-tuples from $X_{i+1}$ satisfying the conclusion of Lemma 1. Thus, by induction on arithmetical formulas (which is a consequence of $\text{ACA}'_0$), we can prove the existence of a sequence of indices, the last of which can be used to compute a subtree $T_{n-1}$ and a function $f_{n-1} : [T_{n-1}]^1 \to k$ such that $T_{n-1}$ is isomorphic to $2^{<\mathbb{N}}$ and any monochromatic subtree for $f_{n-1}$ is also monochromatic for $f$. Since $\text{ACA}'_0$ includes RCA$_0$ plus induction for $\Sigma^0_2$ formulas, by Theorem 1.2 of [1], $T_{n-1}$ contains a subtree which is monochromatic for $f_{n-1}$ and isomorphic to $2^{<\mathbb{N}}$. This subtree is also monochromatic for $f$, so $\text{TT}(n)$ holds for $f$.

To prove that (2) implies (1), assume RCA$_0$ and (2). Given any coloring of $n$-tuples of integers, $f : [\mathbb{N}]^n \to k$, we may define a coloring $g : [2^{<\mathbb{N}}]^n \to k$ on $n$-tuples of elements of $2^{<\mathbb{N}}$ by

$$g(\sigma_1, \ldots, \sigma_n) = f(\text{lh}(\sigma_1), \ldots, \text{lh}(\sigma_n))$$

where $\text{lh}(\sigma)$ denotes the length of the sequence $\sigma$. Any monochromatic tree for $g$ contains an infinite path which encodes an infinite monochromatic set for $f$. Thus, as noted in the proof of Theorem 1.5 of [1], $\forall n \text{TT}(n)$ implies the usual full Ramsey’s theorem, denoted by $\forall n \text{RT}(n)$. $\text{ACA}'_0$ can be deduced from $\forall n \text{RT}(n)$ by Theorem 8.4 of [3], or by applying Proposition 4.4 of [2]. □
A typical proof of $\forall n \text{TT}(n)$ would proceed by induction on $n$ and require the use of induction on $\Pi^1_2$ formulas. In the preceding argument, the existence of the $n$th jump is used to push the application of induction down to arithmetical formulas. The proof of Theorem 2 together with Proposition 4.4 of [2] provide a detailed exposition of a proof and reversal in $\text{ACA}'_0$ and show that the full versions of the usual Ramsey's theorem, the polarized version of Ramsey's theorem, and Ramsey's theorem for trees are all equivalent to $\text{ACA}'_0$ over $\text{RCA}_0$.

Bibliography

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