Groups having 11 cyclic subgroups

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Abstract

Let \( c(G) \) denotes the number of cyclic subgroups of a finite group \( G \). A group \( G \) is \( n \)-cyclic if \( c(G) = n \). In this paper, we show that \( c(G) = 11 \) if and only if \( G \cong H \), where \( H \in \{ \mathbb{Z}_{p^{10}}, \mathbb{Z}_{27} \times \mathbb{Z}_3, \mathbb{Z}_{27} \times \mathbb{Z}_3, \text{Dic}_7, \mathbb{Z}_7 \times \mathbb{Z}_3, \mathbb{Z}_3 \times S_3, \mathbb{Z}_5 \times \mathbb{Z}_8, \mathbb{Z}_3 \times \mathbb{Z}_{16} \} \) and \( p \) is a prime number.

Keywords: \( n \)-cyclic group, Sylow theorem, Maximal subgroup

Mathematics Subject Classification: 20D20, 20D25

1 Introduction

A group \( G \) is \( n \)-cyclic, if it has \( n \) cyclic subgroups. It is easy to verify that \( G \) is 1-cyclic or 2-cyclic if and only if \( G \cong \{ e \}, G \cong \mathbb{Z}_p \), where \( p \) is a prime number respectively. Zohu [9] found all the groups which are \( n \)-cyclic for \( n = 3, 4 \) and 5. Later, Kalra [3] classified all the \( n \)-cyclic groups for \( n \in \{ 6, 7, 8 \} \). Recently, Ashrafi and Haghi [1] have found all 9, 10-cyclic groups. In the present work, we find all 11-cyclic groups by using the same approach as that of Ashrafi and Haghi [1].

To elucidate further on this literature, we examined the incremental work done over time in counting these \( n \)-cyclic groups, beginning with the work by Tóth [8] which helped to count the number of cyclic subgroups of a finite abelian group. Later, Tărnăuceanu [7] classified all finite groups \( G \) having \( |G| - 1 \) number of cyclic subgroups. It is well known that \( G \) is \( |G| \)-cyclic if and only if \( G \) is an elementary abelian 2-group [1].

The notations \( \mathbb{Z}_n, D_{2n}, Q_{2^n}, M(p^a) \) and \( \text{Dic}_n \) denote the cyclic groups of order \( n \), dihedral group of order \( 2n \), generalized quaternion group of order \( 2^n \), modular group of order \( p^a \) and dicyclic group of order \( 4n \) respectively. Here, a group \( G \) is CLT, if it has subgroups corresponding to every divisor of \( |G| \). Throughout this paper \( p, q \) and \( r \) are distinct prime numbers. We refer to the website [2] to count the number of cyclic subgroups of any group of order 1 to 500, using their Hasse diagram.
Let $G$ be a group and $c(G)$ denotes the number of cyclic subgroups of $G$. If $c(G) = n$, we can also say that $G$ is $n$-cyclic. Let $d(n), \omega(n)$ denote the number of positive divisors and number of distinct prime divisors of $n$ respectively. It is easy see that $c(G) \leq |G|$. Richard’s Theorem [4] provides a lower bound on $c(G)$, in particular, we have $c(G) \geq d(|G|)$. Suppose $G$ is an 11-cyclic group of order $n$. An immediate consequence from Richard’s Theorem [4] is that $\omega(n) \leq 3$ in particular, $n$ is one of the form $p^k, pq, p^2q, pqr, p^3q$ or $p^4q$, where $k \leq 10$. Let $c_G(m)$ (simply $c(m)$, when $G$ is clear from the context) denotes the number of cyclic subgroups of order $m$ in $G$ and define $T(G) = |G| - \sum_{m|\mid G} c(m)\phi(m)$. We can easily see that $T(G) = 0$. In most of the cases to find all 11-cyclic groups we take different possibilities of $c(m)$, for all the divisors $m$ of $n$ and then examine all the solutions of $T(G) = 0$. In the next section we will give the proof to find all 11-cyclic groups.

2 Proof

Let $G$ be an 11-cyclic group. We will prove the result casewise by taking different possibilities on $|G|$.

$|G| = p^a$ : We will prove that

$$G \cong \begin{cases} 
\mathbb{Z}_{p^{10}} \text{ or } \mathbb{Z}_{27} \times \mathbb{Z}_3 & \text{if } G \text{ is abelian,} \\
\mathbb{Z}_{27} \times \mathbb{Z}_3 & \text{otherwise.}
\end{cases}$$

Let $M$ be a maximal subgroup of $G$. Then we have the following two cases:

**M is a cyclic subgroup of $G$.** We first assume that $G$ is abelian. Since $|M| = p^{a-1}$, then either $G$ is isomorphic to $\mathbb{Z}_{p^a}$ or $\mathbb{Z}_p \times \mathbb{Z}_{p^{a-1}}$. If $G \cong \mathbb{Z}_{p^a}$, then we can easily prove that $a = 10$. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{a-1}}$, then by Theorem 1.1 of [1], $c(G) = (n - 1)p + 2$, consequently, $p = 3, a = 4$ and $|G| = 81$. Further from [2] we have $G \cong \mathbb{Z}_{27} \times \mathbb{Z}_3$.

Now if $G$ is non-abelian, then depending on $p$, we have the following two situations: If $p$ is odd, then $G \cong M(p^a)$ and by Theorem 1.1 [1], $c(G) = (n - 1)p + 2$. This shows that $p = 3, a = 4$. Thus $G \cong \mathbb{Z}_{27} \times \mathbb{Z}_3$ by [2]. When $p = 2$ by classification theorem of finite non-abelian 2-groups containing cyclic maximal subgroup, $G \cong D_{2^2}, Q_{2^2}, M(2^a)$ or $S_{2^2}$. Using Theorem 1.1 [1], we have $c(D_{2^2}) = a + 2^{a-1}, c(Q_{2^2}) = a + 2^{a-2}, c(M(2^a)) = 2a$ and $c(S_{2^2}) = a + 3.2^{a-3}$. Now by simple calculation, one can check that none of these groups are 11-cyclic.

**M is a non-cyclic subgroup of $G$.** Then $c(M) \leq 10$. Let us consider the set $S = \{ \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_7 \times \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{Z}_{16}, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times Q_{16}, SD_{16}, \mathbb{Z}_{16} \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_9 \times \mathbb{Z}_3, \mathbb{Z}_8 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, Q_{16}, D_8, Q_8, S_3 \}$ of all non-cyclic groups, which are $n$-cyclic and $n \leq 10$. 


Then $M$ is isomorphic to a member of $S$ by Theorem 1.2 and Lemma 2.1 \cite{1}. Therefore, $p = 2$ and $a = 3, 5$ or $6$, $p = 3$ and $a = 3, 5$, $p = 5, 7$ and $a = 3$. Moreover, $|G| \in \{8, 16, 27, 32, 64, 81, 125, 343\}$. We can easily verify that no group of these orders is 11-cyclic by checking their Hasse diagram. Hence, the only non-abelian 11-cyclic $p$-group is $\mathbb{Z}_{27} \rtimes \mathbb{Z}_3$.

The following conclusion can be made by using the above case. A group $G$ is cyclic and 11-cyclic if and only if $G \cong \mathbb{Z}_{p^{10}}$. If $G$ is an abelian group of order $n$, where $n \in \{pq, p^2q, pqr, p^3q, p^4q\}$, then by using Theorem 1 \cite{8}, one can see that $G$ is not 11-cyclic. Hence, the only non-cyclic abelian 11-cyclic group is $\mathbb{Z}_{27} \times \mathbb{Z}_3$.

From now onwards, all the groups are supposed to be non-abelian.

$|G| = pq$: If $p < q$, then by Lemma 3.1 of \cite{3} $c(G) = q + 2$. Therefore $q = 9$, which is a contradiction. Hence no group of order $pq$ is 11-cyclic.

$|G| = p^2q$: If $p < q$, then according to Proposition 3.2 of \cite{3}, $c(G) \in \{6, 2p+4, pq+4, q+4, 2q+2\}$. Hence $G$ is 11-cyclic only if $p = 2, 3$ and $q = 7$. Moreover, from \cite{2} one can notice that $G \cong Dic_7$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_9$.

If $p > q$, then from Proposition 3.2 of \cite{3}, $c(G) \in \{6, 2p+4, p^2+3, p^2+p+2, 2p+3, 3p+2\}$. Thus $c(G) = 11$ is possible only if $p = 3$ and $q = 2$. Again from \cite{2}, we have $G \cong \mathbb{Z}_3 \times S_3$.

$|G| = pqr$: We will prove that there is no 11-cyclic group of order $pqr$. Since every group of square free order is solvable, so $G$ has Hall subgroups of order $pq, pr$ and $qr$. These Hall subgroups are either cyclic or they are isomorphic to $S_3, D_{14}, D_{10}$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ by Theorem 1.2 and Lemma 2.1 of \cite{1}. As a consequence of Lemma 19 \cite{5}, at least one of them is non-abelian. Let $M$ be a non-abelian, maximal Hall subgroup of $G$. Then we have the following sub-cases:

$M \cong \mathbb{Z}_7 \times \mathbb{Z}_3$. Table 1 lists all the cases for the potential number of cyclic subgroups of $G$.

\begin{table}[h]
\begin{center}
\begin{tabular}{cccccccc}
\hline
$c(1)$ & $c(7)$ & $c(3)$ & $c(p)$ & $c(7p)$ & $c(3p)$ & $c(21)$ & $T(G)$ \\
\hline
1 & 1 & 7 & 1 & 1 & 0 & 0 & 14$p$ – 14 \\
1 & 2 & 7 & 1 & 0 & 0 & 0 & 20$p$ – 26 \\
1 & 1 & 7 & 1 & 0 & 1 & 0 & 18$p$ – 18 \\
1 & 1 & 7 & 1 & 0 & 0 & 1 & 20$p$ – 32 \\
\end{tabular}
\end{center}
\end{table}

Table 1

It is clear that $T(G) \neq 0$ in every case.

$M \cong D_{14}$. In this case we have $|G| = 14p$ and $c(M) = 9$. Since $c(G) = 11$ by Sylow theorem $G$ has unique subgroups of order 7 and $p$. Therefore Table 2 includes

\begin{table}[h]
\begin{center}
\begin{tabular}{cccccccc}
\hline
$c(1)$ & $c(7)$ & $c(3)$ & $c(p)$ & $c(7p)$ & $c(3p)$ & $c(21)$ & $T(G)$ \\
\hline
1 & 1 & 7 & 1 & 1 & 0 & 0 & 14$p$ – 14 \\
1 & 2 & 7 & 1 & 0 & 0 & 0 & 20$p$ – 26 \\
1 & 1 & 7 & 1 & 0 & 1 & 0 & 18$p$ – 18 \\
1 & 1 & 7 & 1 & 0 & 0 & 1 & 20$p$ – 32 \\
\end{tabular}
\end{center}
\end{table}

Table 2
all possible cases for the number of cyclic subgroups of $G$.

| 1 | 7 | 1 | 1 | 0 | 0 | $13p - 19$ |
|---|---|---|---|---|---|------------|
| 1 | 7 | 1 | 1 | 0 | 1 | $12p - 12$ |
| 1 | 7 | 1 | 1 | 0 | 1 | $7p - 7$ |

Table 2

In every case we can easily see that $T(G) = 0$ has no solution.

$M \cong S_3$. Then from $c(G) = 11$ and Sylow theorem, we get the following possibilities for the number of cyclic subgroups of $G$ recorded in Table 3.

| 1 | 3 | 1 | 1 | 3 | 2 | 0 | $2p - 2$ |
|---|---|---|---|---|---|---|------------|
| 1 | 3 | 1 | 1 | 2 | 3 | 0 | $3p - 3$ |
| 1 | 3 | 1 | 1 | 3 | 0 | 2 | $p - 3$ |
| 1 | 3 | 1 | 1 | 2 | 0 | 3 | $3p - 9$ |
| 1 | 3 | 1 | 1 | 0 | 3 | 2 | $p + 3$ |
| 1 | 3 | 1 | 1 | 0 | 2 | 3 | $p - 7$ |
| 1 | 3 | 1 | 1 | 5 | 0 | 0 | $0$ |
| 1 | 3 | 1 | 1 | 0 | 5 | 0 | $5p - 5$ |
| 1 | 3 | 1 | 1 | 0 | 0 | 5 | $5p - 15$ |
| 1 | 3 | 5 | 1 | 1 | 0 | 0 | $4p - 12$ |
| 1 | 3 | 5 | 1 | 0 | 1 | 0 | $6p - 11$ |
| 1 | 3 | 5 | 1 | 0 | 0 | 1 | $5p - 15$ |
| 1 | 3 | 1 | 6 | 0 | 0 | 0 | $0$ |
| 1 | 6 | 1 | 1 | 2 | 0 | 0 | $3p - 6$ |
| 1 | 6 | 1 | 1 | 0 | 2 | 0 | $p - 4$ |
| 1 | 6 | 1 | 1 | 0 | 0 | 2 | $5p - 12$ |
| 1 | 6 | 1 | 1 | 1 | 1 | 0 | $2p - 5$ |
| 1 | 6 | 1 | 1 | 0 | 1 | 1 | $3p - 8$ |
| 1 | 6 | 1 | 1 | 1 | 0 | 1 | $4p - 9$ |

Table 3

By using the fact that $p$ is prime and $p \geq 5$ the only possible solution of $T(G) = 0$ is $p = 7$. Therefore the order of $G$ is 42. After checking the Hasse diagram of all groups of order 42 (see [2]), we can say that no such group is 11-cyclic.
In this case $|G| = 10p$ and $G$ has a unique subgroup of order $p$ as it has at least 5 subgroups of order 2. Thus, by using Sylow theorem all the possibilities for the number of cyclic subgroups of $G$ are recorded in the Table 4.

Since $p \notin \{2, 5\}$ so it is easy to check that $T(G) = 0$ has no solution. Hence the conclusion is no group of order $pqr$, is 11-cyclic.

$|G| = p^2q^2$: We will show that none of the groups of order $p^2q^2$ are 11-cyclic. We will prove the same by examining the following cases.

G has unique subgroup of order $p$ and unique subgroup of order $q$. In this case Sylow $p$ and $q$ subgroups of $G$ are cyclic. Note that $n_p(G) = n_q(G) = 1$ is not possible as $G$ is not cyclic.

If $n_p(G), n_q(G) > 1$ and $p < q$, then by Sylow theorem, $n_p(G) \geq q, n_q(G) = p^2$, so $c(G) \geq p^2 + q + 3$. Thus, if we take $q > 3$, then $c(G) > 11$. Therefore, $p = 2, q = 3$ and $|G| = 36$. By [2], we can check that no group of order 36 is 11-cyclic.

Now, assume that $n_p(G) > 1$ and $n_q(G) = 1$. Then by using Sylow theorem, $n_p(G) \geq 1 + p$ and $n_p(G) \in \{q, q^2\}$. Also, $G$ has unique cyclic subgroup of order $pq$. If $n_p(G) = q$, then $p < q$ and $c(G) > q + 5$. Hence $p \in \{2, 3\}$ and $q \in \{2, 3, 5\}$. If $p = 2$, then $q \in \{3, 5\}$ and $|G| = 36, 100$. From [2], we can conclude that no group of order 36 and 100 is 11-cyclic. Let $p = 3$, then all the possible cases for the number of cyclic subgroups of $G$ are listed in Table 5.

By simple calculation we can observe that $T(G) = 0$ has no solution. If $n_p(G) = q^2$, then either $|G| = 36$ or $c(G) > 11$. In this case $G$ is not 11-cyclic.

G has unique subgroup of order $q$ and at least $p + 1$ subgroups of order $p$.

This implies that Sylow $q$-subgroup of $G$ is cyclic. Also $G$ has unique cyclic subgroups of order 1, $q$ and $q^2$. Therefore, $c(G) \geq p + 4$, which shows that
Table 5

\[
p \in \{2, 3, 5, 7\}. \text{ If } p = 2, 3, 5 \text{ or } 7, \text{ then all the possible number of cyclic subgroups of } G \text{ are recorded in the below Table 6.}
\]

It is easy to check that \( T(G) = 0 \) has no solution in all the cases.

Table 6

Subgroups of order \( p \) and \( q \) are not unique. By Sylow theorem, \( G \) has at least \( p + 1 \) and \( q + 1 \) subgroups of order \( p \) and \( q \) respectively. Therefore, \( c(G) \geq \)
\[ p + q + 3. \] Also, as a consequence we have \( p + q \in \{2, 3, 4, 5, 6, 7, 8\} \), equivalently \( |G| \in \{36, 100, 225\} \). From [2], after analyzing the Hasse diagram of all the groups of order 36, 100 and 225 we can say that none of them is 11-cyclic.

Hence no group of order \( p^2q^2 \) is 11-cyclic.

\[ |G| = p^3q : \] Specifically, we will prove that \( G \cong \mathbb{Z}_5 \times \mathbb{Z}_8 \). If \( G \) is a non-CLT group of order \( p^3q \), then by Theorem 1.1 of [6], either \( G \) is isomorphic to \( SL(2, 3) \) or \( E(p^3) \times \mathbb{Z}_q \), where, \( E(p^3) \) is the elementary abelian \( p \)-group of order \( p^3 \). We can check \( c(SL(2, 3)) = 13 \), from [2]. Since every non-identity element of \( E(p^3) \) has order \( p \), then \( c(E(p^3)) = p^2 + p + 2 \). If \( p > 2 \), then \( c(E(p^3)) \geq 14 \). If \( p = 2 \), then \( c(E(p^3)) = 8 \) also \( E(p^3) \times \mathbb{Z}_q \) has at least \( q + 1 \) subgroups of order \( q \), where \( q \geq 3 \). Therefore, \( c(E(p^3) \times \mathbb{Z}_q) > 11 \), which is a contradiction. Thus \( G \) is a CLT group, so it has a subgroup of order \( p^2q \), say \( M \). By using the fact that \( c(M) < 11 \) together with Theorem 1.2 and Lemma 2.1 [1], we have \( M \in \{ \mathbb{Z}_{p^2q}, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_{2p}, A_4, \mathbb{Z}_5 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_{2q}, D_{12}, \mathbb{Z}_7 \times \mathbb{Z}_9, \mathbb{Z}_9 \times S_3 \} \). If \( M \in \{ A_4, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_5 \times \mathbb{Z}_4, D_{12}, \mathbb{Z}_7 \times \mathbb{Z}_9, \mathbb{Z}_3 \times S_3 \} \), then \( |G| \in \{24, 40, 54, 189\} \). Hence by [2], we can check that no such group is 11-cyclic. If \( M \cong \mathbb{Z}_{p^2q} \) then there are following sub-cases:

**G has at least \( p + 1 \) subgroups of order \( p \).** Since \( c(M) = 6 \) and \( M \) has unique subgroup of order \( p \), \( G \) has \( p \) subgroups of order \( p \) different from those, which are contained in \( M \). Thus \( c(G) \geq 6 + p \) and \( p \leq 5 \). All the possibilities of numbers of cyclic subgroups of \( G \) are given in Table [7]. We are getting \( q = 3 \) for \( p = 2 \) after solving the equation \( T(G) = 0 \), from Table [7]. Also \( |G| = 24 \), by [2], we can check that no group of order 24 is 11-cyclic.

**G has unique subgroup of order \( p \).** In this case Sylow \( p \)-subgroup of \( G \) is either cyclic or generalized quaternion. Suppose \( G \) has unique cyclic Sylow \( p \)-subgroup, then \( n_q(G) \geq 1 + q \). This shows that, \( c(G) \geq 7 + q \) and \( q = 2 \) or 3. We have the following possibilities for the number of cyclic subgroups of \( G \) recorded in Table [8].

We can not find values of \( p \) after solving the equation \( T(G) = 0 \) as \( p > 2 \). Therefore, if Sylow \( p \)-subgroup of \( G \) is cyclic then \( n_q(G) > 1 \). Now, by Sylow theorem, \( G \) has at least \( 1 + p \) Sylow \( p \)-subgroups. Moreover, \( c(G) \geq 7 + p \) and \( p = 2 \) or 3. All the possible number of cyclic subgroups of \( G \) are listed in Table [9]. From, Table [9] after solving the equation \( T(G) = 0 \), we are getting \( q = 5 \) for \( p = 2 \) and \( |G| = 40 \). By [2], after checking the structure of all the groups of order 40, \( G \cong \mathbb{Z}_5 \times \mathbb{Z}_8 \).

If Sylow \( p \)-subgroup of \( G \) is generalized quaternion then the possible number of cyclic subgroups of \( G \) are given in Table [10].

From Table [10] after solving the equation \( T(G) = 0 \) we are getting \( q = 3 \) and \( |G| = 24 \). By [2], no group of order 24 in which Sylow \( p \)-subgroup is
generalized quaternion is 11-cyclic. If \( M \cong \mathbb{Z}_3 \times \mathbb{Z}_{3q} \), then \( |G| = 27q \) and \( G \) has no cyclic subgroup of order 27. Also, by Sylow theorem, \( n_3(G) \in \{1, q\} \) and \( n_q(G) \in \{1, 3, 9, 27\} \). If \( n_q(G) \in \{3, 9, 27\} \), then we can easily check that \( G \) is not 11-cyclic. Thus \( n_q(G) = 1 \), all the possibilities for number of cyclic subgroups of \( G \) are listed in Table 11.

### Table 7

| \( p \) | \( c(1) \) | \( c(p) \) | \( c(p^2) \) | \( c(p^3) \) | \( c(q) \) | \( c(pq) \) | \( c(p^2q) \) | \( T(G) \) |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 3 | 1 | 3 | 1 | 1 | 1 | \( 2q - 7 \) |
| 1 | 3 | 1 | 2 | 1 | 2 | 1 | \( q - 3 \) |
| 1 | 3 | 1 | 2 | 1 | 1 | 2 | \( q + 2 \) |
| 1 | 3 | 2 | 2 | 1 | 1 | 1 | \( q - 3 \) |
| 1 | 3 | 3 | 1 | 1 | 1 | 1 | \( 2q - 5 \) |
| 1 | 3 | 2 | 1 | 1 | 1 | 2 | \( q - 3 \) |
| 1 | 3 | 2 | 1 | 1 | 2 | 1 | \( 3q - 7 \) |
| 1 | 3 | 2 | 0 | 1 | 3 | 1 | \( q - 1 \) |
| 1 | 3 | 2 | 0 | 1 | 1 | 3 | 0 |
| 1 | 3 | 1 | 0 | 1 | 2 | 2 | \( q - 1 \) |
| 1 | 3 | 1 | 0 | 1 | 2 | 3 | \( q - 3 \) |
| 1 | 3 | 1 | 0 | 1 | 3 | 2 | \( q \) |
| 1 | 3 | 1 | 0 | 1 | 4 | 1 | 1 | \( q + 1 \) |
| 3 | 1 | 4 | 2 | 1 | 1 | 1 | 1 | \( 3q - 5 \) |
| 1 | 4 | 1 | 1 | 2 | 1 | 1 | \( 17q - 23 \) |
| 1 | 4 | 1 | 0 | 2 | 2 | 1 | \( 5q - 1 \) |
| 1 | 4 | 1 | 0 | 1 | 2 | 2 | \( 5q + 1 \) |
| 1 | 4 | 2 | 0 | 1 | 1 | 2 | \( 2q - 1 \) |
| 1 | 4 | 3 | 0 | 1 | 1 | 1 | \( q - 1 \) |
| 1 | 4 | 1 | 2 | 1 | 1 | 1 | \( 3q - 7 \) |
| 1 | 4 | 1 | 0 | 3 | 1 | 1 | \( 4q - 1 \) |
| 1 | 4 | 1 | 0 | 1 | 3 | 1 | \( 7q - 1 \) |
| 1 | 4 | 1 | 0 | 1 | 1 | 3 | \( q + 1 \) |
| 5 | 1 | 6 | 1 | 0 | 1 | 1 | 1 | \( 10q - 1 \) |

### Table 8

| \( c(1) \) | \( c(p) \) | \( c(p^2) \) | \( c(p^3) \) | \( c(q) \) | \( c(pq) \) | \( c(p^2q) \) | \( T(G) \) |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 3 | 1 | 3 | \( p^3 - 3p^2 + 2p - p \) |
| 1 | 1 | 1 | 1 | 3 | 3 | 1 | \( p^3 - p^2 - 2p \) |
| 1 | 1 | 1 | 1 | 3 | 2 | 2 | \( p^3 - 2p^2 - 1 \) |
| 1 | 1 | 1 | 1 | 5 | 1 | 1 | \( p^3 - p^2 - 4 \) |
| 1 | 1 | 1 | 1 | 4 | 2 | 1 | \( p^3 - p^2 - p - 4 \) |
| 1 | 1 | 1 | 1 | 4 | 1 | 2 | \( p^3 - 2p^2 + p - 3 \) |
No Such group is 11-cyclic as from Table 11, there is no solution for the equation $T(G) = 0$. Now assume that $M \cong \mathbb{Z}_2 \times \mathbb{Z}_2 q$. By Sylow theorem $n_2(G) \in \{1, q\}$ and $n_q(G) \in \{1, 2, 4, 8\}$. If $n_q(G) = 8$ then $G$ is not 11-cyclic. Now, we have the following possibilities for the number cyclic subgroups of $G$ given in the below Table 12.

From, Table 12 after solving the equation $T(G) = 0$ we are getting $q = 3$ or 5, that is $|G| = 24$ or 40. By [2], no such group is 11-cyclic.
Table 12

| G | \( p^4q \) : We will prove that, \( G \cong \mathbb{Z}_3 \times \mathbb{Z}_{16} \). If \( G \) is a CLT group, then it has a subgroup of order \( p^3q \), call it \( M \) such that \( c(M) \leq 10 \). By Theorem 1.2 and Lemma 2.1 of [1], \( M \in \{ \mathbb{Z}_{p^3q}, \mathbb{Z}_3 \times \mathbb{Z}_8, \mathbb{Z}_9 \times \mathbb{Z}_8 \} \). If \( M \cong \mathbb{Z}_3 \times \mathbb{Z}_8 \) then \( |G| = 48 \). By [2], no such group of order 48 is 11-cyclic. If \( M \cong \mathbb{Z}_9 \times \mathbb{Z}_8 \), then \( |G| = 16q \) and \( c(M) = 10 \) by Theorem 1.2 of [1]. All the possibilities for the number of cyclic subgroups of \( G \) are recorded in the Table 13.

We cannot find the value of \( q \) by solving the equation \( T(G) = 0 \). Thus, no such group is 11-cyclic. Now, suppose that \( G \) has a subgroup \( M \cong \mathbb{Z}_{p^3q} \), then we have the following sub-cases.

**G has a unique subgroup of order \( p \).** In this case, Sylow \( p \)-subgroup of \( G \) is either cyclic or generalized quaternion. If \( G \) has unique cyclic Sylow \( p \)-subgroup, then \( n_q(G) \geq 1 + q \). Therefore \( q = 2 \). All the possibilities of the number of cyclic subgroups of \( G \) are given in the Table 14.

By simple calculation we cannot find value of \( p \) after solving the equation \( T(G) = 0 \), from Table 14 Therefore, if Sylow \( p \)-subgroup of \( G \) is cyclic, then

\[
c(1) \quad c(2) \quad c(2^2) \quad c(2^3) \quad c(q) \quad c(2q) \quad c(4q) \quad T(G)
\begin{array}{cccccccc}
1 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \\
1 & 3 & 2 & 0 & 1 & 3 & 1 & q - 1 \\
1 & 3 & 0 & 0 & 4 & 3 & 0 & q + 3 \\
1 & 4 & 1 & 0 & 1 & 3 & 1 & 2q - 11 \\
1 & 3 & 1 & 0 & 1 & 4 & 1 & q + 1 \\
1 & 4 & 2 & 0 & 1 & 3 & 0 & 4q - 5 \\
1 & 3 & 2 & 0 & 1 & 4 & 0 & q - 1 \\
1 & 3 & 3 & 0 & 1 & 3 & 0 & 2q - 3 \\
1 & 3 & 0 & 0 & 1 & 3 & 3 & q - 3 \\
1 & 3 & 0 & 0 & 1 & 4 & 2 & q - 5 \\
1 & 4 & 0 & 0 & 1 & 3 & 2 & 3 \\
1 & 5 & 0 & 0 & 1 & 4 & 0 & 3p - 1 \\
1 & 4 & 0 & 0 & 1 & 5 & 0 & 2p + 1
\end{array}
\]

Table 12

\[
c(1) \quad c(2) \quad c(2^2) \quad c(2^3) \quad c(q) \quad c(2q) \quad c(4q) \quad c(8q) \quad T(G)
\begin{array}{cccccccc}
1 & 1 & 3 & 1 & 0 & 1 & 1 & 3 & 0 & 2q - 1 \\
1 & 2 & 3 & 0 & 0 & 1 & 1 & 3 & 0 & 8q - 1 \\
1 & 1 & 4 & 0 & 0 & 1 & 1 & 3 & 0 & 4q - 1 \\
1 & 1 & 3 & 0 & 0 & 2 & 1 & 3 & 0 & 7q + 1 \\
1 & 1 & 3 & 0 & 0 & 1 & 2 & 3 & 0 & 7q + 1 \\
1 & 1 & 3 & 0 & 0 & 1 & 1 & 4 & 0 & 3q + 1
\end{array}
\]

Table 13
| \( c(1) \) | \( c(p) \) | \( c(p^2) \) | \( c(p^3) \) | \( c(p^4) \) | \( c(2) \) | \( c(2p) \) | \( c(2p^2) \) | \( c(2p^3) \) | \( T(G) \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | \( p^4 - p^3 - 2 \) |

Table 14

\( n_p(G) \geq 1 + p \) and \( c(G) \geq 8 + 1 + p \). With these observations, we get \( p = 2 \). Now, all the possibilities of the number of cyclic subgroups of \( G \) are listed in Table 15.

| \( c(1) \) | \( c(2) \) | \( c(2^2) \) | \( c(2^3) \) | \( c(2^4) \) | \( c(q) \) | \( c(2q) \) | \( c(4q) \) | \( c(8q) \) | \( T(G) \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | \( q - 3 \) |

Table 15

From Table 15, we get \( q = 3 \) and \( |G| = 48 \). By [2], after seeing the structure of all the groups of order 48, we have \( G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_{16} \).

If Sylow \( p \)-subgroup of \( G \) is generalized quaternion then \( |G| = 16q \). All the possibilities of number of cyclic subgroups of \( G \) are recorded in Table 16.

| \( c(1) \) | \( c(2) \) | \( c(2^2) \) | \( c(2^3) \) | \( c(2^4) \) | \( c(q) \) | \( c(2q) \) | \( c(4q) \) | \( c(8q) \) | \( T(G) \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 4 | 0 | 1 | 1 | 1 | 1 | 1 | \( 4q - 3 \) |
| 1 | 1 | 3 | 0 | 2 | 0 | 1 | 1 | 1 | \( q - 1 \) |
| 1 | 1 | 3 | 1 | 0 | 1 | 2 | 1 | 1 | \( 7q - 3 \) |
| 1 | 1 | 3 | 1 | 0 | 1 | 1 | 2 | 1 | \( 3q + 1 \) |
| 1 | 1 | 3 | 1 | 0 | 1 | 1 | 2 | 1 | \( 2q - 1 \).

Table 16

From the above Table 16, we can not find the value of \( q \) after solving the equation \( T(G) = 0 \). Hence, no such group is 11-cyclic.

\( G \) has at least \( p + 1 \) subgroups of order \( p \). Then \( G \) has at least \( p \) subgroups of order \( p \) different from those, which are contained in \( M \) and so \( p = 2 \) or 3. All the possibilities for the numbers of cyclic subgroups of \( G \) are given in Table 17. Now, it is easy to see we can not find the value of \( q \) after solving the equation \( T(G) = 0 \), from Table 17. Consequently, no such group is 11-cyclic.

| \( p \) | \( c(1) \) | \( c(p) \) | \( c(p^2) \) | \( c(p^3) \) | \( c(p^4) \) | \( c(q) \) | \( c(pq) \) | \( c(p^2q) \) | \( c(p^3q) \) | \( T(G) \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | \( q - 3 \) |
| 3 | 1 | 4 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | \( 3q + 11 \) |

Table 17
Let $G$ be a non-CLT group of order $p^4q$. Then by putting different conditions on the number of subgroups of order $p$ and $q$ and by using Sylow theorem we can check that no such group is 11-cyclic.

### 3 Conclusion

In this work, a classification of 11-cyclic groups is given and they are $\mathbb{Z}_{p^{10}}, \mathbb{Z}_{27} \times \mathbb{Z}_3, \mathbb{Z}_{27} \rtimes \mathbb{Z}_3, Dic_7, \mathbb{Z}_7 \times \mathbb{Z}_9, \mathbb{Z}_3 \times S_3, \mathbb{Z}_5 \rtimes \mathbb{Z}_8$ and $\mathbb{Z}_3 \rtimes \mathbb{Z}_{16}$. Here, we can observe that these groups are supersolvable with abelian Sylow subgroups and centres of these groups are cyclic. Moreover, every $p$-cyclic group, where $p \leq 11$ is supersolvable. We are now working on 12-cyclic groups.

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