On the evaluation of the norm of an integral operator associated with the stability of one-electron atoms

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Abstract

The norm of an integral operator occurring in the partial wave decomposition of an operator $B$ introduced by Brown and Ravenhall in a model for relativistic one-electron atoms is determined. The result implies that $B$ is non-negative and has no eigenvalue at 0 when the nuclear charge does not exceed a specified critical value.

1 Introduction

The operator referred to in the title is defined on $L^2(0, \infty)$ by

$$(T\phi)(x) := \int_0^\infty t(x,y)\phi(y)dy, \quad 0 < x < \infty,$$  \hfill (1.1)

where,

$$t(x,y) = \frac{1}{2} \left\{ \sqrt{\frac{x^2 + 1 + 1}{x^2 + 1}} g_0(x/y) \sqrt{\frac{y^2 + 1 + 1}{y^2 + 1}} ight.$$  \hfill (1.2)

$$+ \sqrt{\frac{x^2 + 1 - 1}{x^2 + 1}} g_1(x/y) \sqrt{\frac{y^2 + 1 - 1}{y^2 + 1}} \right\}$$

with

$$g_0(u) = \log \left| \frac{u + 1}{u - 1} \right|, \quad g_1(u) = \frac{1}{2} \left( u + \frac{1}{u} \right) \log \left| \frac{u + 1}{u - 1} \right| - 1, \quad u > 0.$$  \hfill (1.2)

To describe its role in relativistic stability, we require some background information. It is well-known that the Dirac operator describing relativistic one-particle
systems is unbounded below, and that problems occur when it is extended as a model for multi-particle systems. The root of the problem is that the Dirac operator describes two different particles, namely electrons and positrons. In the paper [2] Brown and Ravenhall overcame this difficulty by projecting onto the electron subspaces only. Specifically, for a relativistic electron in the field of its nucleus, their operator is

$$B := \Lambda_+(D_0 - \frac{e^2 Z}{|\cdot|})\Lambda_+. \quad (1.3)$$

The notation in (1.3) is as follows:

- $D_0$ is the free Dirac operator

$$D_0 = c\alpha \cdot \frac{\hbar}{i} \nabla + mc^2\beta \equiv \sum_{j=1}^{3} c\frac{\hbar}{i} \alpha_j \frac{\partial}{\partial x_j} + mc^2\beta$$

where $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ are the Dirac matrices given by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}$$

with $0_2, 1_2$ the zero and unit $2 \times 2$ matrices respectively, and $\sigma_j$ the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

- $\Lambda_+$ denotes the projection of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ onto the positive spectral subspace of $D_0$, that is $\chi_{(0,\infty)}D_0$. If we set

$$\hat{f}(p) \equiv [\mathcal{F}(f)](p) := \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int_{\mathbb{R}^3} e^{-ip\cdot x/\hbar} f(x) dx$$

for the Fourier transform of $f$, then it follows that

$$(\Lambda_+ f)(p) = \Lambda_+(p)\hat{f}(p)$$

where

$$\Lambda_+(p) = \frac{1}{2} + \frac{c\alpha \cdot p + mc^2\beta}{2e(p)} e(p) = \sqrt{c^2p^2 + m^2c^4}, \quad (1.4)$$

with $p = |p|$;

- $2\pi\hbar$ is Planck’s constant, $c$ the velocity of light, $m$ the electron mass, $-e$ the electron charge and $Z$ the nuclear charge.
The underlying Hilbert space in which $B$ acts is

$$\mathcal{H} = \Lambda_+(L^2(R^3) \otimes C^4)$$

and when it is bounded below, $B$ generates a self-adjoint operator (also denoted by $B$) which is the Friedrichs extension of the restriction of $B$ to $\Lambda_+(C^\infty(R^3) \otimes C^4)$.

The operator $B$ was later used by Bethe-Salpeter (see [1]) and is referred to with their name in [3]. In [3] it is proved that $B$ is bounded below if and only if the nuclear charge $Z$ does not exceed the critical value

$$Z_c = \frac{2}{(\frac{\pi}{2} + \frac{2}{\pi})\alpha}, \quad \alpha = \frac{e^2}{\hbar c},$$

where $\alpha$ is Sommerfeld’s fine structure constant; this range of $Z$ covers all natural elements. When $Z = Z_c$, it is proved in [3] that

$$B \geq -\left(\frac{\pi^2}{\pi^2 + 4}\right)mc^2.$$  

However, in [4] Hardekopf and Sucher had investigated $B$ numerically and predicted that $B$ is in fact non-negative, and that, as for the Dirac operator, the ground state energy vanishes for $Z = Z_c$, i.e. $0$ is an eigenvalue of $B$. The first part of this prediction of Hardekopf and Sucher has recently been confirmed, but the second part contradicted, by Tix in [5]. Following the basic strategy in [3], but with a better choice of trial functions, Tix obtains a lower bound for $B$ which is shown to be positive for $Z \leq Z_c$, specifically

$$B \geq mc^2(1 - \alpha Z - 0.002 \frac{Z}{Z_c}) > 0;$$

the numerical factor is roughly $0.09$ for $Z = Z_c$.

From the partial wave analysis of $B$, it is shown in [3] that for all $\psi \in \Lambda_+(C^\infty_0(R^3) \otimes C^4)$ and with $(\cdot, \cdot)$ the standard inner product on $L^2(R^3) \otimes C^4$,

$$\begin{align*}
(B\psi, \psi) &= \sum_{(l,m,s) \in I} \left\{ \int_0^\infty e(p)|a_{l,m,s}(p)|^2 dp \\
&\quad - \frac{\alpha c Z}{\pi} \int_0^\infty \int_0^\infty \sigma_{l,m,s}(p')k_{l,s}(p', p)a_{l,m,s}(p)dpdp' \right\}
\end{align*}$$

where $I$ is the index set

$$I = \{(l, m, s) : l \in N_0, \quad m = -l - 1/2, \ldots, l + 1/2, s = 1/2, -1/2, \quad |m| \neq l + 1/2 \text{ when } s = -1/2\},$$
the kernels \( k_{l,s}(p', p) \) are given by

\[
k_{l,s}(p', p) = \left[ e(p') + e(0) \right] Q_l \left( \frac{1}{2} \left[ \frac{p'}{p} + \frac{p}{p'} \right] \right) + c^2 p' Q_{l+2s} \left( \frac{1}{2} \left[ \frac{p}{p'} + \frac{p'}{p} \right] \right) \]

\[
\frac{2e(p)|e(p) + e(0)|^{1/2}(2e(p')|e(p') + e(0)|)^{1/2}}{(2e(p)|e(p) + e(0)|)^{1/2}(2e(p')|e(p') + e(0)|)^{1/2}}
\]

(1.10)

and

\[
\sum_{(l,m,s) \in I} \int_0^\infty |a_{l,m,s}(p)|^2 dp = \| \psi \|_2^2 := \sum_{j=1}^4 \int_{R^3} |\psi_j|^2 dx.
\]

(1.11)

In (1.10) the \( Q_l \) are the Legendre functions of the second kind. The strategy in [3] was based on this decomposition of \( B \) and the observation that

\[
0 \leq k_{l,s}(p', p) \leq k_{0,1/2}(p', p), \quad l \in N_0, s = 1/2, -1/2.
\]

(1.12)

It would follow that \( B \geq 0 \) for \( Z \leq Z_c \) if and only if

\[
\int_0^\infty \int_0^\infty a(p') k_{0,1/2}(p', p) a(p) dp dp' \leq \frac{\pi}{\alpha c Z_c} \int_0^\infty e(p)a(p)^2 dp
\]

(1.13)

for all non-negative measurable functions \( a \). On setting

\[
g_l(u) = Q_l \left( \frac{1}{2} \left[ u + \frac{1}{u} \right] \right), \quad l \in N_0
\]

(1.14)

\[
p = mc x, p' = mcy, \phi(x) = \sqrt{e(mc x)} a(mc x),
\]

(1.15)

where \( t(\cdot, \cdot) \) is defined in (1.2). What we prove in this paper is that the constant \( \frac{\pi^2}{4} + 1 \) in the inequality (1.15) is sharp, and there are no extremal functions. Furthermore, we show that these results imply that \( B \geq (1 - \frac{Z}{Z_c})mc^2 \) for \( Z \leq Z_c \) and 0 is not an eigenvalue of \( B \) when \( Z = Z_c \). Much of the analysis continues to be valid for analogous inequalities defined by general kernel functions \( t_{l,s} \) derived from the \( k_{l,s} \).

## 2 The main results

The operator \( T \) defined on \( L^2(0, \infty) \) by (1.1) is readily seen to be a bounded symmetric operator and so

\[
\sup \left\{ \frac{|(T \phi, \phi)|}{\| \phi \|^2} : \phi \in L^2(0, \infty), \phi \neq 0 \right\} = \| T \|_{L^2(0, \infty) \rightarrow L^2(0, \infty)}.
\]

(2.1)

Our main result is
Theorem 2.1 Let $T$ be defined by (1.1). Then

1. $\|T\|_{L^2(0, \infty) \to L^2(0, \infty)} = \frac{\pi^2}{4} + 1$;

2. the operator $T$ has no extremal functions.

Remark 2.2 We recall that $\phi$ is an extremal function of a bounded symmetric operator $T$ if $\phi \in L^2(0, \infty), \phi \neq 0$ a.e. and $\|T\|_{L^2(0, \infty) \to L^2(0, \infty)} = |(T\phi, \phi)|/\|\phi\|^2$. Hence Theorem 2.1 means the following: for all non-negative measurable functions $\phi \in L^2(0, \infty)$ with $\phi \neq 0$ a.e.

$$\int_0^\infty \int_0^\infty t(x,y)\phi(x)\phi(y)dx dy < \left(\frac{\pi^2}{4} + 1\right) \int_0^\infty \phi^2(x)dx$$

(2.2)

and the constant $\frac{\pi^2}{4} + 1$ is sharp. In turn, this implies that the inequality (1.13) is valid, the constant $\pi/\alpha c Z_c$ is sharp and there is no function $a \in L^2(0, \infty; e(p)dp)$ which is not null and for which there is equality in (1.13). A consequence of Theorem 2.1 is

Theorem 2.3 Let $B$ be the self-adjoint operator generated in $H$ by (1.3) and let $Z_c$ be given by (1.6). Then

1. if $Z \leq Z_c$, $B \geq (1 - Z/Z_c)mc^2$;

2. if $Z = Z_c$, 0 is not an eigenvalue of $B$;

3. if $Z > Z_c$, $B$ is unbounded below.

Proof. Part 3 is proved in [3]. From (1.8),(1.10),(1.11) and (1.12) it follows that

$$(B\psi, \psi) \geq \sum_{(l,m,s) \in I} \left\{ \int_0^\infty e(p)|a_{l,m,s}(p)|^2 dp ight.$$

$$- \frac{\alpha c Z_c}{\pi} \int_0^\infty \int_0^\infty |a_{l,m,s}(p')|k_{0.1/2}(p',p)|a_{l,m,s}(p)|dp dp' \right\}$$

$$\geq \sum_{(l,m,s) \in I} \int_0^\infty \left(1 - \frac{Z}{Z_c}\right) e(p)|a_{l,m,s}(p)|^2 dp$$

$$\geq \left(1 - \frac{Z}{Z_c}\right) mc^2 \|\psi\|^2$$

which establishes part 1. To prove part 2, suppose 0 is an eigenvalue of $B$ with corresponding eigenfunction $\psi$. By (1.12) and (1.13), all the summands on the right-hand side of (1.8) are non-negative and consequently are zero as now $B\psi = 0$. Also (1.11) implies that at least one of the functions $a_{l_0,m_0,s_0}$ say, is not null. But this would imply that there is equality in (1.13) with the function $a = |a_{l_0,m_0,s_0}|$, contrary to Remark 2.2. Hence the proof is complete.
Remark 2.4 When the mass \( m = 0 \), a proof of Theorem 2.3 is given in [3].
On setting \( p = x, p' = y, \phi(x) = \sqrt{\text{det}}(x) \) in (1.13) when \( m = 0 \) we obtain
\[
\int_0^\infty \int_0^\infty t_0(x,y)\phi(x)\phi(y)dx\,dy \leq \left( \frac{x^2}{4} + 1 \right) \int_0^\infty \phi(x)^2 \, dx \tag{2. 3}
\]
where
\[
t_0(x,y) := \frac{1}{2\sqrt{xy}} \left\{ g_0 \left( \frac{x}{y} \right) + g_1 \left( \frac{x}{y} \right) \right\}. \tag{2. 4}
\]
We shall prove in Section 3 that the integral operator \( T_0 \) with kernel \( t_0 \) satisfies
Theorem 2.1, and thus yields the analogue of Theorem 2.3 in the case \( m = 0 \).

Remark 2.5 Tix’s lower bound (1.7) for \( B \) is an improvement on that in Theorem 2.3(1). If 0 is not in the essential spectrum \( \sigma_{\text{ess}}(B) \) of \( B \) when \( Z = Z_c \), as is the case when \( Z < Z_c \) for \( \sigma_{\text{ess}}(B) = [mc^2, \infty) \) is established in [3, Theorem 2], then Parts 1 and 2 of Theorem 2.3 imply that \( B \) is strictly positive. However, no specific positive lower bound can be deduced from Theorem 2.1 alone.

3 Proof of Theorem 2.1

The starting point is the following simple result (cf[3, Section 2.3]). We shall denote by \( (\cdot, \cdot) \) and \( \| \cdot \| \) the standard inner-product and norm respectively in \( L^2(0, \infty) \). It is sufficient to consider only real-valued functions in \( L^2(0, \infty) \) throughout this section.

Lemma 3.1 Let \( f, g, h \) be real-valued, measurable functions on \( (0, \infty) \). Moreover, let \( g \) and \( h \) be positive and \( g(1/u) = g(u), \ 0 < u < \infty \). (3. 1)

Then
\[
\int_0^\infty \int_0^\infty f(x)g \left( \frac{x}{y} \right) f(y)dx\,dy \leq \int_0^\infty f(x)^2 \left\{ \int_0^\infty \frac{h(y)}{h(x)} g \left( \frac{y}{x} \right) dy \right\} dx. \tag{3. 2}
\]
Equality holds if and only if \( f(x) = Ah(x) \) a.e. on \( (0, \infty) \), where \( A \) is a constant.

Proof. By the Cauchy-Schwarz inequality,
\[
\int_0^\infty \int_0^\infty f(x)g(x/y)f(y)dx\,dy
\leq \left( \int_0^\infty \int_0^\infty f(x)^2 g(x/y)h(y)h(x)dx\,dy \right)^{1/2} \left( \int_0^\infty \int_0^\infty f(y)^2 g(y/x)h(x)h(y)dx\,dy \right)^{1/2}
= \int_0^\infty f(x)^2 \left( \int_0^\infty g(y/x)h(y)h(x)dy \right) dx
\]
Equality holds if and only if, for some constants \( \mu \) and \( \lambda \)
\[
\mu f(x) \sqrt{\frac{g(x/y)}{h(x)}} = \lambda f(y) \sqrt{\frac{g(y/x)}{h(y)}}
\]
a.e. on \((0, \infty) \times (0, \infty)\). This is equivalent to \( f(x) = Ah(x) \) a.e. on \((0, \infty)\), where \( A \) is a constant.

**Lemma 3.2** Let \( G \) be the symmetric operator defined on \( L^2(0, \infty) \) by
\[
Gf(x) := \int_0^\infty \frac{g(x/y)}{\sqrt{xy}} f(y) dy, \quad 0 < x < \infty,
\]
where \( g \) is a positive measurable function satisfying (3.1). Then
\[
\|G|L^2(0, \infty) \rightarrow L^2(0, \infty)\| = \int_0^{\infty} g(u) \frac{du}{u}.
\]
Moreover, there are no extremal functions.

**Proof.** By Lemma 3.1 with \( h(u) = 1/u \) we get
\[
|(Gf, f)| = \left| \int_0^\infty \int_0^\infty \frac{f(x) g(x/y)}{\sqrt{xy}} \frac{f(y)}{\sqrt{y}} dx dy \right|
\leq \int_0^\infty \left( \frac{f(x)}{\sqrt{x}} \right)^2 \left( \int_0^\infty \frac{x g(x/y)}{y} dy \right) dx
= \int_0^\infty g(u) \frac{du}{u} \int_0^\infty f(x)^2 dx.
\]
Hence,
\[
\|G|L^2(0, \infty) \rightarrow L^2(0, \infty)\| \leq \int_0^\infty g(u) \frac{du}{u}.
\]
Furthermore, equality in (3.5) can hold if and only if \( f(x) = A/\sqrt{x} \) a.e. on \((0, \infty)\). Since \( A/\sqrt{x} \notin L^2(0, \infty) \) unless \( A = 0 \), it follows that for all \( f \in L^2(0, \infty) \), \( f \neq 0 \) a.e.,
\[
|(Gf, f)| < \left( \int_0^\infty g(u) \frac{du}{u} \right) \|f\|^2.
\]
In order to establish the inequality converse to (3.5) we take \( f_\delta(x) = \frac{\chi(1, \delta)(x)}{\sqrt{x}} \) as a test function, where \( \chi(1, \delta) \) denotes the characteristic function of \((1, \delta), 1 < \delta < \infty\). By l’Hospital’s Rule, we have as \( \delta \rightarrow \infty \),
\[
\|G|L^2(0, \infty) \rightarrow L^2(0, \infty)\| \geq \lim_{\delta \rightarrow \infty} \frac{(Gf_\delta, f_\delta)}{\|f_\delta\|^2}
\]
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\[
\begin{align*}
\lim_{\delta \to \infty} \left\{ \left( \ln \delta \right)^{-1} \int_1^\delta \left( \int_1^\delta \frac{g(y/x) \, dy}{y} \right) \frac{dx}{x} \right\} \\
= \lim_{\delta \to \infty} \left\{ \int_1^\delta \frac{g(y/\delta) \, dy}{y} + \int_1^\delta \frac{g(\delta/x) \, dx}{x} \right\} \\
= \lim_{\delta \to \infty} \int_{1/\delta}^\delta g(u) \frac{du}{u} \\
= \int_0^\infty g(u) \frac{du}{u}.
\end{align*}
\]

The equality (3.4) follows from (3.5). From (3.6) it follows that there is no extremal function.

**Lemma 3.3** Let \( T_0 \) be the symmetric operator in \( L^2(0, \infty) \) defined by

\[
T_0 f(x) := \int_0^\infty t_0(x, y) f(y) \, dy,
\]

where \( t_0 \) is given by (2.4). Then

1. \( \| T_0 \| \) \( L^2(0, \infty) \to L^2(0, \infty) \) \( = \frac{\pi^2}{4} + 1 \);
2. there are no extremal functions.

**Proof.** The results follow from Lemma 3.2 since

\[
\int_0^\infty g_0(u) \frac{du}{u} = 2 \int_0^1 g_0(u) \frac{du}{u} \\
= 2 \int_0^1 \log \left| \frac{u + 1}{u - 1} \right| \frac{du}{u} \\
= 4 \int_0^1 \left( \sum_{k=0}^\infty \frac{u^{2k}}{2k+1} \right) \frac{du}{u} \\
= 4 \sum_{k=0}^\infty \frac{1}{(2k+1)^2} \\
= \frac{\pi^2}{2} \quad (3.7)
\]

and

\[
\int_0^\infty g_1(u) \frac{du}{u} = 2 \int_0^1 g_1(u) \frac{du}{u} \\
= 2 \int_0^1 \left( \frac{1}{2} \left[ u + \frac{1}{u} \right] \log \left| \frac{u + 1}{u - 1} \right| - 1 \right) \frac{du}{u}
\]

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Lemma 3.4  The operator $T$ defined in (1.1) satisfies

$$\|T\|_{L^2(0, \infty) \to L^2(0, \infty)} \geq \frac{\pi^2}{4} + 1.$$  

Proof. As in Lemma 3.2, we take $f_\delta(x) = \frac{\chi_{(1, \delta)}(x)}{\sqrt{x}}$, $1 < \delta < \infty$, as a test function. By l'Hospital's rule we obtain

$$\|T\|_{L^2(0, \infty) \to L^2(0, \infty)} \geq \lim_{\delta \to \infty} \frac{(Tf_\delta, f_\delta)}{\|f_\delta\|^2}$$

$$= \lim_{\delta \to \infty} \left\{ \frac{1}{(\ln \delta)^{-1}} \int_1^\delta \left( \int_1^\delta t(x, y) \frac{dy}{\sqrt{y}} \right) \frac{dx}{\sqrt{x}} \right\}$$

$$= \lim_{\delta \to \infty} \left\{ \sqrt{\delta} \int_1^\delta t(x, \delta) \frac{dx}{\sqrt{x}} + \sqrt{\delta} \int_1^\delta t(\delta, y) \frac{dy}{\sqrt{y}} \right\}$$

$$= 2 \lim_{\delta \to \infty} \int_{1/\delta}^1 \frac{\delta t(\delta, \delta u)}{\sqrt{u}} \frac{du}{\sqrt{u}}.$$  

It is readily seen from (1.2) that for $1 < \delta < \infty$,

$$\frac{\delta t(\delta, \delta u)}{\sqrt{u}} \leq \frac{g_0(u) + g_1(u)}{u} \in L(0, 1)$$

and

$$\lim_{\delta \to \infty} \frac{\delta t(\delta, \delta u)}{\sqrt{u}} = \frac{g_0(u) + g_1(u)}{2u}.$$  

Hence, by the Dominated Convergence Theorem, (3.7) and (3.8), we have

$$\|T\|_{L^2(0, \infty) \to L^2(0, \infty)} \geq \int_0^1 \frac{g_0(u) + g_1(u)}{u} \frac{du}{\sqrt{u}} = \frac{\pi^2}{4} + 1.$$  

Lemma 3.5  For all functions $h_0, h_1$ which are positive and measurable on $(0, \infty)$

$$\|T\|_{L^2(0, \infty) \to L^2(0, \infty)} \leq A(h_0, h_1),  
\tag{3.9}$$
where
\[
A(h_0, h_1) = \frac{1}{2} \sup_{0 < x < \infty} \left( \frac{\sqrt{x^2 + 1} + 1}{x^2 + 1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy + \frac{\sqrt{x^2 + 1} - 1}{x^2 + 1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy \right).
\]

The operator $T$ has an extremal function $\phi$ if and only if
\[
\phi(x) = A_0 h_0(x) \sqrt{\frac{x^2 + 1}{x^2 + 1 + 1}} = A_1 h_1(x) \sqrt{\frac{x^2 + 1}{\sqrt{x^2 + 1} - 1}}
\] (3.10)

and
\[
\frac{\sqrt{x^2 + 1} + 1}{x^2 + 1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy + \frac{\sqrt{x^2 + 1} - 1}{x^2 + 1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy = A_3,
\] (3.11)

for some non-zero constants $A_1, A_2, A_3$.

Proof. By Lemma 3.1,
\[
\int_0^\infty \int_0^\infty t(x, y) \phi(x) \phi(y) dx dy = \frac{1}{2} \left\{ \int_0^\infty \int_0^\infty \sqrt{\frac{x^2 + 1 + 1}{x^2 + 1}} \phi(x) g_0(x/y) \sqrt{\frac{y^2 + 1 + 1}{y^2 + 1}} \phi(y) dx dy \\
+ \int_0^\infty \int_0^\infty \sqrt{\frac{x^2 + 1 - 1}{x^2 + 1}} \phi(x) g_1(x/y) \sqrt{\frac{y^2 + 1 - 1}{y^2 + 1}} \phi(y) dx dy \right\}
\]
\[
\leq \frac{1}{2} \left\{ \int_0^\infty \frac{\sqrt{x^2 + 1 + 1}}{x^2 + 1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy \\
+ \frac{\sqrt{x^2 + 1 - 1}}{x^2 + 1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy \right\} \phi(x)^2 dx
\]
\[
\leq A(h_0, h_1) \|\phi\|^2.
\]

Moreover, the first inequality becomes an equality if and only if
\[
\sqrt{\frac{x^2 + 1 + 1}{x^2 + 1}} \phi(x) = A_0 h_0(x), \quad \sqrt{\frac{x^2 + 1 - 1}{x^2 + 1}} \phi(x) = A_1 h_1(x)
\]
a.e. on $(0, \infty)$, for some constants $A_0, A_1$. The second inequality becomes an equality if and only if $\phi(x) = 0$ a.e. on the set of all $x \in (0, \infty)$ for which
\[
\frac{\sqrt{x^2 + 1 + 1}}{x^2 + 1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy + \frac{\sqrt{x^2 + 1 - 1}}{x^2 + 1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy < A(h_0, h_1).
Since an extremal function $\phi$ is not null, this inequality can only be satisfied on a set of zero measure. Consequently, (3.11) holds a.e. for some constant $A_3$.

**Remark 3.6** We note that for all functions $h_0, h_1$ which are positive and measurable on $(0, \infty)$

$$
\liminf_{x \to \infty} \frac{1}{2} \left( \frac{\sqrt{x^2 + 1} + 1}{x^2 + 1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy \\
+ \frac{\sqrt{x^2 + 1} - 1}{x^2 + 1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy \right)
\geq \frac{\pi^2}{4} + 1.
$$

Indeed, let

$$
\hat{h}_j(\xi) := \liminf_{x \to \infty} \frac{h_j(\xi x)}{h_j(x)}, \quad 0 < \xi < \infty, \ j = 0, 1,
$$

where the lim inf can be finite or infinite. Then

$$
\hat{h}_j(1/\xi) = \liminf_{x \to \infty} \frac{h_j(x/\xi)}{h_j(x)} = \liminf_{\xi \to \infty} \frac{h_j(y)}{h_j(\xi y)} = \frac{1}{\hat{h}_j(\xi)}.
$$

By Fatou’s Theorem

$$
\liminf_{x \to \infty} \frac{1}{2} \left( \frac{\sqrt{x^2 + 1} + 1}{x^2 + 1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy \\
+ \frac{\sqrt{x^2 + 1} - 1}{x^2 + 1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy \right)
= \liminf_{x \to \infty} \frac{1}{2} \left( \frac{\sqrt{x^2 + 1} + 1}{x^2 + 1} \int_0^\infty \frac{h_0(ux)}{h_0(x)} g_0(u) du \\
+ \frac{\sqrt{x^2 + 1} - 1}{x^2 + 1} \int_0^\infty \frac{h_1(ux)}{h_1(x)} g_1(u) du \right)
\geq \frac{1}{2} \left( \int_0^\infty \liminf_{x \to \infty} \left[ \frac{h_0(ux)}{h_0(x)} \right] g_0(u) du + \int_0^\infty \liminf_{x \to \infty} \left[ \frac{h_1(ux)}{h_1(x)} \right] g_1(u) du \right)
= \frac{1}{2} \left( \int_0^\infty \hat{h}_0(u) g_0(u) du + \int_0^\infty \hat{h}_1(u) g_1(u) du \right).
$$
Furthermore, on substituting \( u = v - \sqrt{v^2 - 1} \) when \( 0 < u < 1 \) and \( u = v + \sqrt{v^2 - 1} \) when \( u > 1 \) we have

\[
\int_0^\infty \hat{h}_j(u) g_j(u) du = \int_0^\infty \hat{h}_j(u) Q_j \left( \frac{1}{2} \left[ u + \frac{1}{u} \right] \right) du \\
= \int_1^\infty \left\{ \hat{h}_j(v - \sqrt{v^2 - 1})(v - \sqrt{v^2 - 1}) + \hat{h}_j(v + \sqrt{v^2 - 1})(v + \sqrt{v^2 - 1}) \right\} Q_j(v) \frac{dv}{\sqrt{v^2 - 1}} \\
\geq 2 \int_1^\infty Q_j(v) \frac{dv}{\sqrt{v^2 - 1}} \\
= \int_0^\infty Q_j \left( \frac{1}{2} \left[ u + \frac{1}{u} \right] \right) \frac{du}{u} \\
= \int_0^\infty g_j(u) \frac{du}{u}.
\]

This verifies the assertion. We also note that equality holds if and only if \( \hat{h}_j(u) = 1/u \) a.e. on \((0, \infty)\). Thus to prove that \( A(h_0, h_1) \leq \frac{\pi^2}{4} + 1 \), and hence complete the proof of Theorem 2.1, we must choose \( h_0 \) and \( h_1 \) in such a way that \( h_0(u) = h_1(u) = 1/u \) a.e. on \((0, \infty)\).

**Lemma 3.7** For all \( \phi \in L^2(0, \infty) \), \( \phi(x) \neq 0 \) a.e., we have

\[
\int_0^\infty \int_0^\infty t(x, y) \phi(x) \phi(y) dxdy < C \int_0^\infty \phi(x)^2 dx,
\]

where

\[
C = \sup_{0 < x < \infty} F(x)
\]

and

\[
F(x) = \frac{\pi}{2} \left( \sqrt{x^2 + 1} + 1 \right) \frac{\arctan x}{x} + \frac{\left( \sqrt{x^2 + 1} - 1 \right) x}{x^2 + 1}.
\]

**Proof.** We apply Lemma 3.5 with the choice (cf[5])

\[
h_0(x) = \frac{x}{x^2 + 1}, \quad h_1(x) = \frac{1}{x}.
\]

From (3.8),

\[
\int_0^\infty h_1(y) g_1(y/x) dy = 2.
\]
Also, on using Cauchy's Residue Theorem, we obtain
\[
\int_0^\infty h_0(y)g_0(y/x)\,dy = \int_0^\infty \frac{y}{y^2 + 1} \log \left| \frac{x+y}{x-y} \right| \,dy
\]
\[
= \frac{1}{2} \int_{-\infty}^\infty \frac{y}{y^2 + 1} \log \left| \frac{x+y}{x-y} \right| \,dy
\]
\[
= \frac{x^2}{2} \int_{-\infty}^\infty \frac{u}{(xu)^2 + 1} \log \frac{1+u}{1-u} \,du
\]
\[
= \frac{x^2}{2} \Re \left[ \int_{-\infty}^\infty \frac{u}{(xu)^2 + 1} \log \left( \frac{1+u}{1-u} \right) \,du \right]
\]
\[
= \pi \Re \left[ \frac{i}{2} \log \left( \frac{x+i}{x-i} \right) \right]
\]
\[
= \pi \arctan x.
\]
Thus (3.13) is confirmed. Since the equality (3.10) is not satisfied by the choice of \( h_0, h_1 \) in (3.15) for any constants \( A_0, A_1 \), it follows from Lemma 3.5 that there is strict inequality in (3.12).

The final link in the chain of arguments is

**Lemma 3.8** The constant \( C \) in (3.13) is given by
\[
C = \frac{\pi^2}{4} + 1. \tag{3.16}
\]

**Proof.** Since \( \lim_{x\to\infty} F(x) = \frac{\pi^2}{4} + 1 \), we have that \( C \geq \frac{\pi^2}{4} + 1 \). To prove the reverse inequality we start by substituting \( x = \tan 2v \) in \( F(x) \) to obtain
\[
F(\tan 2v) = \frac{\pi v + 4 \sin^4 v}{\tan v}, \quad 0 \leq v \leq \pi/4.
\]
We therefore need to prove that
\[
f(v) := \pi v + 4 \sin^4 v - \left( \frac{\pi^2}{4} + 1 \right) \tan v \leq 0, \quad 0 \leq v \leq \pi/4.
\]
The following identities for the derivatives are easily verified:
\[
f^{(1)}(v) = \pi + 16 \sin^3 v \cos v - \left( \frac{\pi^2}{4} + 1 \right) \sec^2 v,
\]
\[
f^{(2)}(v) = 2 \sin v \sec^3 v g(v),
\]
where
\[
g(v) = 3 \sin 2v + 3 \sin 4v + \sin 6v - \left( \frac{\pi^2}{4} + 1 \right)
\]
and
\[
g^{(1)}(v) = 12 \cos 4v(1 + \cos 2v).
\]
Since $g(0) < 0, g(\pi/8) > 0, g(\pi/4) < 0, g^{(1)}(v) > 0$ on $[0, \pi/8)$ and $g^{(1)}(v) < 0$ on $(\pi/8, \pi/4)$ there exist $v_1, v_2$ such that $0 < v_1 < v_2 < \pi/4, g(v_1) = g(v_2) = 0, g(v) < 0$ on $[0, v_1)$ and $(v_2, \pi/4]$, and $g(v) > 0$ on $(v_1, v_2)$. Thus $f(0) = 0, f^{(1)}(0) < 0, f^{(2)}(0) = 0, f^{(1)}(\pi/4) > 0, f^{(2)}(\pi/4) < 0$. Moreover, $f^{(2)}$ vanishes at $v_1$ and $v_2$, is negative on $(0, v_1)$ and $(v_2, \pi)$, and positive on $(v_1, v_2)$. In particular, it follows that $f^{(1)}$ is negative on $[0, v_1]$ and positive on $[v_2, \pi/4]$.

Suppose that $f(\xi) = 0$ for some $\xi \in (0, \pi/4)$. From $f(0) = f(\xi) = f(\pi/4) = 0$, and the last sentence of the previous paragraph, it follows that there exist $\eta_1, \eta_2$ such that $f^{(1)}(\eta_1) = f^{(1)}(\eta_2) = 0$ and $v_1 < \eta_1 < \eta_2 < v_2$. Consequently, there exists $v_3 \in (v_1, v_2)$ such that $f^{(2)}(v_3) = 0$ which is contrary to what was established in the previous paragraph. Thus $f(v) \neq 0$ on $(0, \pi/4)$. Since $f(0) = f(\pi/4) = 0$ and $f^{(1)}(0) < 0, f^{(1)}(\pi/4) > 0$, it follows that $f(v) \leq 0$ on $[0, \pi/4]$. The proof is therefore complete.

**Proof of Theorem 2.1.** Part 1 follows from Lemmas 3.4,3.7 and 3.8, and part 2 from (3.12).

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