EXPLORING MODULAR FORMS AND THE
COHOMOLOGY OF LOCAL SYSTEMS ON MODULI
SPACES BY COUNTING POINTS

GERARD VAN DER GEER

ABSTRACT. We report on a joint project in experimental mathematics
with Jonas Bergström and Carel Faber where we obtain information
about modular forms by counting curves over finite fields.

1. Introduction

According to Arnold mathematics is the part of physics where experi-
ments are cheap. I take it that he meant ‘cheap’ in the literal sense, that
is, not requiring large investments of goods and money. But experiments in
mathematics may require a considerable investment of time, mental energy
and computational power. On the other hand they can be extremely reward-
ing. Here we report on a project of experimental mathematics concerning
moduli spaces and modular forms that has been going on for quite some
time. We used counts of points over finite fields to explore the cohomology
of local systems on moduli spaces of low genus curves ([24, 7, 8, 11]). As a
side result it produces a lot of information about Siegel modular forms of
low degree. This comes about since on the one hand modular forms admit
a cohomological description as part of the cohomology of local systems on
moduli spaces of abelian varieties and on the other hand abelian varieties of
low dimension allow a description in terms of curves. By using comparison
theorems for the cohomology we can gauge this cohomology by looking at
the situation over a finite field and listing all the isomorphism classes of
curves of this genus over the finite field and the order of their automorphism
groups, and by calculating the action of Frobenius on the cohomology of the
corresponding curve.

Birch made such counts for elliptic curves in the 1960s in [12] and from
his formulas one sees that one gets information about the traces of the Hecke
operators on the spaces of cusp forms. In a joint project with Carel Faber
we started counting points of the moduli spaces $\mathcal{M}_{2,n}$ of $n$-pointed curves of
genus 2 over finite fields in order to probe the cohomology of these moduli
spaces. We realized that knowledge of the cohomology of the moduli spaces
$\mathcal{M}_{2,n}$ is equivalent to information about the cohomology of local systems on
$\mathcal{M}_2$ and also realized that this way we could get information about vector-
valued Siegel modular forms. It turned out to be wonderfully effective and
our heuristic results have since then been turned into theorems. In joint
work with Bergström we extended this to local systems on the moduli space \( \mathcal{M}_2 \) of curves of genus 2 with a level 2 structure and then later also to the case of \( \mathcal{M}_3 \), the moduli of curves of genus 3, where it makes predictions about Siegel modular forms of degree 3. It was a surprise that this was feasible and it thus created a new window on Siegel modular forms. Apart from these cases we also looked at the case of Picard modular forms. Another chapter is the case of modular forms on \( \mathcal{M}_3 \), called Teichmüller modular forms. We will not treat this last aspect here and have to refer to future papers.

One of the nice aspects of the theory of elliptic modular forms is the availability of concrete examples and accessibility of the topic for doing experiments. Concrete examples for higher degree Siegel modular forms were much harder to get by, especially for vector-valued modular forms. But recently things have changed quite a lot. For example the work of Chenevier-Renard and Taibi (see [15, 59]) where applying Arthur’s results on the trace formula (even if all has not yet been proved) provided a wealth of new data on Siegel modular forms. No doubt we can await many new surprises and discoveries in this beautiful corner of mathematical nature.

Finally I would like to thank my collaborators Jonas Bergström and Carel Faber for their comments and the continued pleasant cooperation in this project. I also thank Fabien Cléry with whom I worked on the purely modular forms side of the project.

2. Modular Forms

Every mathematician knows or should know the definition of (elliptic) modular form: a holomorphic function \( f : \mathfrak{H} \to \mathbb{C} \) on the upper half plane \( \mathfrak{H} \) of \( \mathbb{C} \) satisfying (for a fixed \( k \in \mathbb{Z} \))
\[
f((a\tau + b)(c\tau + d)^{-1}) = (c\tau + d)^k f(\tau)
\]
for all \( \tau \in \mathfrak{H} \) and \( (a \ b \\ c \ d) \in \text{SL}(2, \mathbb{Z}) \), in particular \( f(\tau + 1) = f(\tau) \), and thus admitting a Fourier series
\[
f = \sum_{n \in \mathbb{Z}} a(n) q^n \quad \text{with } q = e^{2\pi i \tau},
\]
where we require additionally that \( a(n) = 0 \) for \( n < 0 \). The exponent \( k \in \mathbb{Z} \) is called the weight of the modular form. Examples of modular forms can be found everywhere in mathematics, and in fact they appeared early on in our history, for example in the papers of Jacobi, but the notion of a modular form was formalized only at the end of the nineteenth century, apparently by Klein who introduced the word ‘Modulform.’

Modular forms of given weight \( k \) form a finite-dimensional complex vector space denoted by \( M_k \) (or \( M_k(\text{SL}(2, \mathbb{Z})) \)) and since the product of two modular forms of weight \( k \) and \( l \) is a modular form of weight \( k + l \) they form a graded ring and \( \mathbb{C} \)-algebra
\[
M = \bigoplus_k M_k.
\]
A modular form vanishing at infinity, that is, with \( a(0) = 0 \), is called a cusp form. The subspace of cusp forms is denoted by \( S_k \). The cusp form of smallest non-zero weight is \( \Delta = \sum_{n=1}^{\infty} \tau(n) q^n \) which can be defined by an infinite product

\[
q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots
\]

and this turns out to be a cusp form of weight 12 generating the ideal of cusp forms in the algebra \( M \).

The structure of this \( \mathbb{C} \)-algebra \( M \) is known: it is a polynomial algebra

\[ M = \mathbb{C}[E_4, E_6], \]

where the modular forms \( E_4 \) and \( E_6 \) of weight 4 and 6 are Eisenstein series defined for even \( k \geq 4 \) via

\[
E_k = \frac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-k}
\]

where \( (c,d) \) runs over coprime pairs of integers and we assume \( k \geq 4 \) to get convergence; it has the Fourier series

\[
1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n
\]

where \( \sigma_r(n) = \sum_{1 \leq d \mid n} d^r \) and \( B_k \) is the \( k \)th Bernoulli number. The cusp form \( \Delta \) equals the expression \((E_4^3 - E_6^2)/1728\).

We can replace \( \text{SL}(2, \mathbb{Z}) \) by a subgroup of finite index, like for example the subgroup \( \Gamma_0(N) \) where we require for a matrix \((a,b;c,d)\) that \( c \) is divisible by \( N \). Then we find more modular forms. However, for the most part of this report we restrict to the case of level \( N = 1 \).

Modular forms show up unexpectedly in many parts of mathematics, for example in the theory of lattices as generating series for the number of vectors of given norm in an even definite unimodular lattice. They show up ubiquitously if one studies the moduli of elliptic curves or other algebro-geometric objects. But they also show up in mathematical physics. As a rule the Fourier coefficients of these modular forms show interesting arithmetical properties and seem to contain magical information.

It was Hecke who saw around 1937 how to extract this information from the Fourier series. He introduced the operators \( T(n) \) for \( n \in \mathbb{Z}_{\geq 1} \) named after him. These operators form a commutative algebra of operators. Hecke’s student Petersson introduced a positive definite hermitian product on the space of cusp forms and the Hecke operators are hermitian for this product. We thus can find a basis of eigenvectors, called eigenforms which are eigenvectors for all the Hecke operators. Then one shows that for such a nonzero eigenform \( f = \sum a(n) q^n \) with eigenvalue \( \lambda(m) \) under \( T(m) \) we have

\[
\lambda(m)a(1) = a(m)
\]
so that \( a(1) \neq 0 \) and when we divide \( f \) by \( a(1) \) to get \( a(1) = 1 \) we see that then the Fourier coefficients are the eigenvalues.

It was also Hecke who noticed that the fact that a modular form of weight \( k \) is an eigenform for the Hecke algebra implies that the formal \( L \)-series of \( f \) has an Euler product:

\[
L(f, s) = \sum_{n \geq 1} \frac{a(n)}{n^s} = \prod_p \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}}.
\]

For example, for the Eisenstein series one finds

\[
L(E_k, s) = \zeta(s)\zeta(s+k-1).
\]

with \( \zeta(s) \) the Riemann zeta function. Hecke also showed that the \( L \)-series of a cusp form can be extended analytically to a holomorphic function of \( s \) for all \( s \in \mathbb{C} \) and satisfies a functional equation, namely

\[
\Lambda(f, s) = (-1)^{k/2}\Lambda(f, k-s)
\]

with \( \Lambda(f, s) = \Gamma(s)(2\pi)^{-s}L(f, s) \).

Let \( f \) be a (normalized with \( a(1) = 1 \)) cusp form of weight \( k \) that is an eigenform of the Hecke algebra. We denote the totally real number field obtained by adjoining the eigenvalues of \( f \) to \( \mathbb{Q} \) by \( K_f \). Deligne showed [21] that to such a cusp form one can associate an \( \ell \)-adic representation

\[
\rho_{\ell, f} : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{GL}(2, K_f \otimes \mathbb{Q}_\ell)
\]

of the Galois group of \( \mathbb{Q} \) with the property that for a prime \( p \) different from \( \ell \) the image \( \rho_{\ell, f}(F_p) \) of a geometric Frobenius element \( F_p \) has trace \( \lambda(p) \), the eigenvalue of \( f \) under the operator \( T(p) \), and determinant \( p^{k-1} \). This was conjectured by Serre [54]. This result exhibited the number theoretical meaning of modular forms. The origin of these representations (namely the cohomology of modular varieties) also enabled Deligne to apply his (later) result on the eigenvalues of Frobenius on the cohomology of algebraic varieties to these eigenvalues, see [21]. We shall see in Section 3 where this representation lives.

3. Siegel Modular Forms

The notion of modular form (on \( \text{SL}(2, \mathbb{Z}) \)) generalizes. Consider a symplectic lattice \( \mathbb{Z}^{2g} \), say with basis \( e_1, \ldots, e_g, f_1, \ldots, f_g \), which pair by \( \langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle \) and \( \langle e_i, f_j \rangle = \delta_{ij} \), the Kronecker delta. Then we let

\[
\Gamma_g = \text{Sp}(2g, \mathbb{Z}) = \text{Aut}(\mathbb{Z}^{2g}, \langle , \rangle)
\]

be the automorphism group of this lattice, the so-called Siegel modular group. For \( g = 1 \) one finds back \( \text{SL}(2, \mathbb{Z}) \). The action of \( \text{SL}(2, \mathbb{Z}) \) on the upper half plane generalizes to an action of \( \Gamma_g \) on the Siegel upper half space

\[
\mathfrak{H}_g = \{ \tau \in \text{Mat}(g \times g, \mathbb{C}) : \tau^t = \tau, \text{Im}(\tau) > 0 \}
\]

via

\[
\tau \mapsto (a \tau + b)(c \tau + d)^{-1}
\]
for an element
\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \]
where the decomposition of the \(2g \times 2g\) matrix \(\gamma\) in four \(g \times g\) blocks corresponds to the decomposition of \(\mathbb{Z}^{2g}\) as a direct sum of two isotropic sublattices generated by the \(e_i\) and the \(f_i\). Often we shall write \(\gamma = (a, b; c, d)\) for such a matrix.

We can consider for \(g > 1\) holomorphic functions \(f : \mathfrak{H}_g \to \mathbb{C}\) satisfying
\[ f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^k f(\tau) \]
for all \(\tau \in \mathfrak{H}_g\) and \(\gamma \in \Gamma_g\). This is the notion of a Siegel modular form of degree \(g\) (or genus \(g\)) and weight \(k\). But there is a wider generalization. Let \(\rho : \text{GL}(g, \mathbb{C}) \to \text{Aut}(W)\)
be a finite dimensional irreducible complex representation of \(\text{GL}(g, \mathbb{C})\) on the vector space \(W\). Then we can consider holomorphic maps
\[ f : \mathfrak{H}_g \to W \]
satisfying
\[ f((a\tau + b)(c\tau + d)^{-1}) = \rho(c\tau + d) f(\tau) \]
for all \(\tau \in \mathfrak{H}_g\) and \((a, b; c, d) \in \Gamma_g\). Such a function \(f\) is called a Siegel modular form of degree \(g\) and weight \(\rho\). The space of all such modular forms is denoted by \(M_\rho(\Gamma_g)\). If \(\rho = \det k\) we retrieve the notion of scalar-valued Siegel modular form of weight \(k\). It is a reflection of the Hartogs extension theorem that for \(g > 1\) we need not demand a holomorphicity condition like we did for \(g = 1\) (when we demanded \(a(n) = 0\) for \(n < 0\)). In fact, we can write the Fourier series of a Siegel modular form \(f\) as
\[ f(\tau) = \sum_n a(n) q^n \]
where the index \(n\) runs over all \(g \times g\) symmetric matrices whose entries are half-integral and with integral diagonal entries and where we employ the shorthand
\[ q^n = e^{2\pi i \text{Tr}(n \tau)} \]
and \(a(n)\) is a vector in the vector space \(W\); the Koecher principle says that if \(a(n) \neq 0\) then \(n\) is positive semi-definite.

There is a way to obtain a Siegel modular form of degree \(g - 1\) from a form \(f\) of degree \(g\) by taking a limit
\[ \Phi(f)(\tau') = \lim_{t \to \infty} f\left(\frac{\tau'}{0 \ 1} \ 0 \ i t \right) \quad \text{with } t \in \mathbb{R}_{>0} \text{ and } \tau' \in \mathfrak{H}_{g-1}. \]
This operator is called the Siegel operator. Forms for which \(\Phi(f)\) is zero are called cusp forms.

Like for \(g = 1\) we can form a graded \(\mathbb{C}\)-algebra of scalar-valued Siegel modular forms on \(\Gamma_g\)
\[ M(\Gamma_g) = \bigoplus_k M_k(\Gamma_g) \]
where \( M_k(\Gamma_2) = M_{\det k}(\Gamma_2) \). The structure of this algebra is known for \( g = 2 \) and \( g = 3 \) only. For \( g = 2 \) Igusa proved that the subalgebra of forms of even weight is a polynomial algebra given by

\[
M^e(\Gamma_2) = \bigoplus_{k \equiv 0 \pmod{2}} M_k(\Gamma_2) = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}].
\]

Here \( E_4 \) and \( E_6 \) are Eisenstein series, now for \( g = 2 \), defined by series

\[
E_k = \sum_{(c,d)} \det(c\tau + d)^{-k}
\]

with a sum over non-associated pairs of co-prime symmetric integral matrices (where non-associated is taken with respect to left multiplication by elements of \( \text{GL}(2, \mathbb{Z}) \)), and \( \chi_{10} \) and \( \chi_{12} \) are cusp forms; up to a normalization they can be given by \( E_{10} - E_4 E_6 \) and \( E_{12} - E_6^2 \). The algebra of modular forms of all weights is an extension of \( M^e(\Gamma_2) \) generated by a form \( \chi_{35} \) of weight 35 that satisfies a quadratic relation \( \chi_{35}^2 = P(E_4, E_6, \chi_{10}, \chi_{12}) \) expressing it as a polynomial in the even weight modular forms. For \( g = 3 \) the structure of \( M(\Gamma_3) \) was determined by Tsuyumine \[62\]; it has 34 generators and thus becomes rather complicated to handle.

Vector-valued Siegel modular forms have attracted much less attention than the scalar-valued ones, but as we shall see they are the natural generalization of the elliptic modular forms. Much less is known about vector-valued Siegel modular forms than about scalar-valued forms.

The quotient space \( \text{SL}(2, \mathbb{Z})/\mathcal{H}_1 \) is the moduli space of complex elliptic curves; similarly, \( \text{Sp}(2g, \mathbb{Z})/\mathcal{H}_g \) is the moduli space of principally polarized complex abelian varieties of dimension \( g \):

\[
\mathcal{A}_g(\mathbb{C}) := \text{Sp}(2g, \mathbb{Z})/\mathcal{H}_g.
\]

It can be viewed as the fiber over the infinite place of a moduli stack \( \mathcal{A}_g \) over \( \mathbb{Z} \) parametrizing principally polarized abelian varieties and this comes equipped with a universal abelian variety

\[
\pi : \mathcal{X}_g \longrightarrow \mathcal{A}_g
\]

with fibre \( \mathcal{X}_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau \mathbb{Z}^g \) over the orbit of \( \tau \) in \( \mathcal{A}_g(\mathbb{C}) \). We should view these moduli spaces (or quotients) as stacks or orbifolds.

Modular forms can be interpreted as sections of a vector bundle: \( \Gamma_2 \) acts on \( \mathcal{H}_g \times \mathbb{C}^g \) by

\[
(\tau, z) \mapsto (\gamma \tau, (c\tau + d)z)
\]

where \( \gamma = (a, b; c, d) \). This defines a vector bundle (in the orbifold sense). More intrinsically, we have

\[
E_g = \pi_* \Omega^1_{\mathcal{X}_g/\mathcal{A}_g},
\]

the Hodge bundle, and its restriction to \( \mathcal{A}_g(\mathbb{C}) \) corresponds to the standard representation of \( \text{GL}(g, \mathbb{C}) \). For every irreducible representation \( \rho \) of \( \text{GL}(g, \mathbb{C}) \) we have a corresponding bundle \( U_\rho \) on \( \mathcal{A}_g(\mathbb{C}) \) produced by applying an appropriate Schur functor to \( E \); or more concretely, it can be described as the quotient of the action of \( \Gamma_2 \) on \( \mathcal{H}_g \times W \) with \( W \) the representation
space of ρ by \((τ, w) \mapsto (γτ, ρ(ct + d)w)\). Then we can interpret modular forms as sections: for \(g ≥ 2\) we have
\[
M_ρ(Γ_g) = \text{space of sections of } U_ρ \text{ on } Α_ρ(ℂ).
\]
Our moduli space can be compactified by a toroidal compactification over which the Hodge bundle extends. Then the bundles \(U_ρ\) extend too.

By the Koecher principle these sections extend over a toroidal compactification and we find that the space of cusp forms \(S_ρ(Γ_g)\) consists of the sections that vanish at infinity, that is, on the divisor that is added to \(Α_ρ(ℂ)\) to compactify it.

Like for \(g = 1\) one can define Hecke operators. They are induced by correspondences and give a commutative algebra of operators. In particular for every prime \(p\) we have an operator \(T(p)\), but we have also operators \(T_i(p^2)\) for \(i = 1, \ldots, g\). We refer to [28, 1]. Also the Petersson product has its analogue and we thus can find bases of eigenforms for the action of the Hecke algebra on the spaces of cusp forms.

4. Cohomological interpretation

As alluded to in Section 2 modular forms admit a cohomological interpretation and this is a powerful tool as we shall see. Let \(π : X_1 → A_1\) be the universal elliptic curve. Then we find a local system \(V = R^1π_*Q\) on \(A_1(ℂ)\) of \(Q\)-vector spaces with fibre \(H^1(E, Q)\) over \([E]\). This is a local system of rank 2 associated to the standard representation of \(Γ_1\).

By taking \(\text{Sym}^a(V)\), the \(a\)th symmetric power of \(V\), we obtain a local system \(V_a\) of rank \(a + 1\).

This local system enters in the cohomological interpretation of elliptic modular cusp forms that Eichler and Shimura gave in the 1950s.

**Theorem 4.1. (Eichler-Shimura)** For even \(a ≥ 2\) we have
\[
H^1_c(A_1 ⊗ ℂ, V_a ⊗ ℂ) \cong S_{a+2}(Γ_1) ⊕ \overline{S}_{a+2}(Γ_1) ⊕ ℂ
\]
where \(H^1_c\) stands for compactly supported cohomology and the right hand side displays the mixed Hodge structure of the left hand side. The space \(S_{a+2}(Γ_1)\) has Hodge weight \((a + 1, 0)\) and \(\overline{S}_{a+2}\) is the complex conjugate of \(S_{a+2}\).

The case \(a = 0\) corresponds to the compactly supported cohomology of the moduli space \(A_1(ℂ)\). In this case \(S_2 = (0)\); but if we replace \(SL(2, ℤ)\) by a congruence subgroup \(Γ_0(N)\) then cusp forms of weight 2 might show up and these would give differential forms via \(f(τ) → f(τ)dτ\) and these would contribute to the cohomology of the quotient space \(Γ_0(N)\backslash \overline{H}\) and its compactification \(Γ_0(N)\backslash \overline{H}\). This is how Eichler and Shimura were led to their result. Then the Galois representation associated to \(f\) can be found on the Tate module of the Jacobian of the compactification \(Γ_0(N)\backslash \overline{H}\).

But the moduli space \(A_1\) is defined over \(ℤ\) and the Eichler-Shimura isomorphism has an \(ℓ\)-adic counterpart, due to Deligne [21]. It is embodied in the identity
\[
\text{Tr}(T(p), S_{a+2}) = \text{Tr}(F_p, H^1_c(A_1 ⊗ ℍ_p, V^{(ℓ)}_a)) - 1,
\]
where $\mathcal{V}^{(i)}$ denotes the $\ell$-adic sheaf $R^1\pi_*\mathbb{Q}_\ell$ and $\mathcal{V}_a^{(i)}$ its $a$th symmetric power, and $F_p$ represents a geometric Frobenius element in the Galois group. A convenient way to see where this identity comes from uses the Euler characteristic

$$e_c(A_1, \mathcal{V}) = \sum (-1)^i [H^i_c(A, \mathcal{V}_a)],$$

where we take the class $[H^i]$ of the cohomology group in an appropriate Grothendieck group of mixed Hodge modules or of Galois representations depending on whether we take complex cohomology or $\ell$-adic étale cohomology; then the statement is that

$$e_c(A_1, \mathcal{V}_a) = -S[a + 2] - 1,$$  \hspace{1cm} (2)

where $S[a + 2]$ is the Chow motive defined by Scholl associated to the space of cusp forms of weight $a + 2$ on $\text{SL}(2, \mathbb{Z})$. The rank of this motive is $2s_{a+2}$ with $s_a = \dim S_{a+2}(\Gamma_1)$. It is cut out by projectors on the cohomology of a symmetric power of the universal elliptic curve over the moduli space and its compactification. Then the $\ell$-adic Galois representation associated to $S_{a+2}(\text{SL}(2, \mathbb{Z}))$ is to be found in $H^1_c(A_1 \otimes \overline{\mathbb{F}}_p, \mathcal{V}_a^{(i)}); \text{ here we use the comparison isomorphisms } H^i_c(A_1 \otimes \overline{\mathbb{F}}_p, \mathcal{V}_a^{(i)}) \cong H^i_c(A \otimes \overline{\mathbb{Q}}_p, \mathcal{V}_a^{(i)}) \cong H^i_c(A \otimes \overline{\mathbb{Q}}, \mathcal{V}_a^{(i)})$ for $\ell \neq p$ and the surjection of Galois groups $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

The Hecke operators $T(n)$ are defined by correspondences on $A_1$ and act on the cohomology. Moreover, in characteristic $p$ there is the congruence relation that relates the Hecke correspondence and the correspondence defined by Frobenius and its transpose. The Hecke correspondence $T(p)$ is given by all pairs $(E, E')$ of elliptic curves linked by a cyclic isogeny $E \rightarrow E'$ of degree $p$. But in characteristic $p$ if $j$ and $j'$ are the $j$-invariants of such $E$ and $E'$ we have $j^p = j'$ or $j'^p = j$.

Then the identity (2) and the congruence relation imply that we can calculate the trace of the Hecke operator $T(n)$ on $S_{a+2}$ from the trace of Frobenius on the cohomology $H^1_c(A_1 \otimes \overline{\mathbb{F}}_p, \mathcal{V}_a^{(i)})$ via Deligne’s identity (1).

From a calculational perspective this might seem pointless since we know generators for the ring of modular forms and we can calculate the action of the Hecke operators explicitly. Moreover, we have an explicit formula for the trace of the Hecke operator $T(p)$, the Eichler-Selberg trace formula, see for example Zagier’s appendix in [44]. Nevertheless, on the contrary, our approach is a practical one. Indeed, by the philosophy of Weil we can calculate cohomology by counting points over finite fields. And it turns out that we can use this to obtain information about modular forms. In practice we can do this by the following strategy:

1. make a list of all elliptic curves over $\mathbb{F}_p$ up to isomorphism over $\mathbb{F}_p$;
2. for each elliptic curve $E$ in this list determine $\#E(\mathbb{F}_p)$ and $\#\text{Aut}_{\mathbb{F}_p}(E)$.

We know by Hasse that the number of rational points of an elliptic curve over $\mathbb{F}_p$ satisfies $\#E(\mathbb{F}_p) = p + 1 - \alpha - \overline{\alpha}$ with $\alpha$ an algebraic integer with
\[ |\alpha| = \sqrt{p} \text{ and } \bar{\alpha} \text{ its complex conjugate. Then we can calculate the trace of } T(p) \text{ on } S_{a+2} \text{ by} \]
\[ 1 + \text{Tr}(T(p), S_{a+2}) = -\sum_E \frac{\alpha^a + \alpha^{a-1}\bar{\alpha} + \ldots + \bar{\alpha}^a}{\# \text{Aut}_{F_p}(E)}, \tag{3} \]

where the sum is over the elliptic curves in our list. This formula is the concrete embodiment of Deligne's formula (1). For example for the Fourier coefficients of \( \Delta \) we have
\[ \tau(p) = -1 - \sum_E \frac{\alpha^{10} + \alpha^9\bar{\alpha} + \ldots + \bar{\alpha}^{10}}{\# \text{Aut}_{F_p}(E)}. \]

Given a prime \( p \) we can take the \( E \) with the same trace \( t \) together and then have to list how often a certain trace \( t = \alpha + \bar{\alpha} \) occurs in our list, where we count the frequency as
\[ w(t) = \sum_{E: \#(F_p) = p+1-t} \frac{1}{\# \text{Aut}_{F_p}(E)}, \]

where each \( E \) with given trace counts with weight \( 1/\# \text{Aut}_{F_p}(E) \). For example for \( p = 17 \) we thus get the list
\[
\begin{array}{c|ccccccccc}
 t & \pm 8 & \pm 7 & \pm 6 & \pm 5 & \pm 4 & \pm 3 & \pm 2 & \pm 1 & 0 \\
 w & 1/4 & 1/2 & 3/2 & 1/2 & 1 & 3/2 & 7/4 & 1/2 & 2 \\
\end{array}
\]

With this list we can calculate the trace of \( T(17) \) on the space \( S_k \) of cusp forms for all weights \( k \geq 4 \) by formula (3). Moreover, by having the list for a certain prime, computing the trace for an arbitrary weight is (almost) immediate. For prime powers \( q \) we have a slightly modified formula.

As a final remark we note that the 1 in formula (3) comes from the Eisenstein series \( E_{a+2} \). Indeed, for even \( a \geq 2 \) we have
\[ H_i^1(A_1, V_a) = S[a + 2] \]
where \( H_i^1 \) is the image of \( H_c^i \) in \( H^i \). The \(-1\) appearing in (1) and (2) comes from part of the eigenvalue \( 1 + p^{a+1} \) of \( E_{a+2} \) and it appears in the kernel of the map \( H_c^1 \rightarrow H^1 \); the other part \( p^{a+1} \) appears in the cokernel.

5. Degree Two

Since our knowledge about Siegel modular forms of degree \( g \geq 2 \) is much more limited there is every reason to try to generalize this approach to higher \( g \). We consider then
\[ \pi: \mathcal{X}_g \rightarrow A_g \]
the universal abelian variety over our moduli space. This yields a local system \( V = \mathbb{R}^1 \pi_* \mathbb{Q} \) of rank \( 2g \) on \( A_g(\mathbb{C}) \) and its \( \ell \)-adic variant \( V^{(\ell)} \) on \( A_g \otimes \mathbb{C} \) and on \( A_g \otimes F_p \). The fibre over the point \([X]\) of a principally polarized abelian variety \( X \) is \( H^1(X, \mathbb{Q}) \) or \( H^1_{et}(X, \mathbb{Q}_\ell) \). We shall simply write \( V \)
for \( \mathcal{V}(\ell) \). From this local system we can construct other local systems as follows. For every irreducible representation of \( \text{Sp}(2g, \mathbb{Q}) \) of highest weight \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g) \) we get an associated local system \( \mathcal{V}_\lambda \); so with our conventions we have

\[
\mathcal{V}_{1,0,...,0} = \mathcal{V}.
\]

We consider then the cohomology of this local system. As before, a convenient way to deal with it is by using the so-called motivic Euler characteristic

\[
e_c(A_g(\mathbb{C}), \mathcal{V}_\lambda \otimes \mathbb{C}) = \sum_i (-1)^i [H^i_c(A_g(\mathbb{C}), \mathcal{V}_\lambda \otimes \mathbb{C})],
\]

where \( H^i_c \) is the compactly supported cohomology and the brackets refer to the class in a Grothendieck group of mixed Hodge structures. But we can also consider \( \ell \)-adic étale cohomology and in this case we consider \( H^i_c \) as a Galois representation and then \( e_c \) lives in a Grothendieck group of Galois representations.

It is a result of Faltings [26, 25], but see also [46], that \( H^i \) and \( H^i_c \) have a mixed Hodge filtration; if we define the interior cohomology by

\[
H^i \! = \text{image of } H^i_c \text{ in } H^i,
\]

then \( H^i \! \) has a pure Hodge structure. Moreover, it follows from [25] that if \( \lambda \) is regular (that is, \( \lambda_1 > \lambda_2 > \cdots > \lambda_g \)) then if \( H^i \! \neq 0 \) we must have \( i = g(g + 1)/2 \), the dimension of \( A_g \).

Let us specialize to \( g = 2 \). Then \( \lambda \) is given as a pair \( \lambda = (a, b) \) of integers with \( a \geq b \geq 0 \) and according to Faltings we have a Hodge filtration

\[
F^{a+b+3} \subset F^{a+2} \subset F^{b+1} \subset F^0 = H^3_c(A_2(\mathbb{C}), \mathcal{V}_{a,b} \otimes \mathbb{C}).
\]

A main point is now that we have an interpretation of the first step:

\[
F^{a+b+3} \cong S_{a-b,b+3},
\]

with \( S_{a-b,b+3} \) the space of cusp forms on \( \Gamma_2 \) of weight \((a - b, b + 3)\); that is, these modular forms are the sections on \( \tilde{A}_2 \) of

\[
\text{Sym}^{a-b}(E) \otimes \text{det}(E)^{b+3} \otimes \mathcal{O}(-D),
\]

where \( D \) is the divisor added to \( A_2 \) to obtain an appropriate toroidal compactification \( \tilde{A}_2 \) of \( A_2 \).

Based on ample numerical evidence Carel Faber and I formulated a conjecture [24] in 2004 that has now been confirmed completely. For regular local systems \((a > b > 0)\) this was done by Weissauer [64] in 2009 and the irregular cases by Petersen [49] in 2013.

Before we state it we remark that it follows from the action of \(-1_X\) on an abelian surface \( X \) that if \( a \not\equiv b \pmod{2} \) then \( e_c = 0 \). We thus may restrict ourselves to the case \( a \equiv b \pmod{2} \).
Theorem 5.1. We have
\[ \text{Tr}(T(p), S_{a-b,b+3}) = -\text{Tr}(F_p, e_c(A \otimes \overline{\mathbb{F}}_p, V_{a,b})) + \text{Tr}(F_p, e_{2,\text{extra}}(a,b)) \]
with \( e_{2,\text{extra}}(a,b) \) a correction term given by
\[
s_{a-b+2} - s_{a+b+4}(S[a-b+2] + 1)L^{b+1} + \begin{cases} 
S[b+2] + 1 & \text{a even} \\
-S[a+3] & \text{a odd}
\end{cases}
\]

Here \( s_n = \dim S_n(\Gamma_1) \) and \( L \) stands for the Lefschetz motive \( (= h^2(\mathbb{P}^1)) \).

And in order to make it work for \( a = 0 \) we should define
\[ S[2] = -L - 1 \quad \text{and} \quad s_2 = -1. \]

We arrived at this conjecture by applying the strategy mentioned for the case \( g = 1 \): first note that the moduli space of stable curves of genus 2 of compact type coincides with the moduli space \( \mathcal{A}_2 \) of principally polarized abelian surfaces. We then make a list of all stable curves of compact type of genus 2 over our finite field \( \mathbb{F}_q \) up to isomorphism over that finite field and for each Jacobian \( X \) of such a curve we compute the order of the group of automorphisms defined over \( \mathbb{F}_q \) and the eigenvalues \( \alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2 \) of Frobenius acting on its \( \ell \)-adic Tate module (or on \( H^1(X \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell) \)) for a prime \( \ell \) different from the characteristic. Then the trace of Frobenius on the cohomology of the local system \( V^{(\ell)}_{a,b} \) is given by summing over the \( X \) in our list the values of a Schur function in the eigenvalues \( \alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2 \), that is, the \( g = 2 \) analogue of (2). For each \( q \) this will give a number and we then tried to interpret this number as a function of \( q \) as a polynomial plus eigenvalue of an elliptic modular form for small values of \( a \) and \( b \). This gave us an idea about the extra term \( e_{2,\text{extra}}(a,b) \).

In order to make the list and in order to avoid the calculational problem of determining the order of the automorphism group what we do is the following. We take a suitable family of curves lying finitely over the moduli space and then we count in this family and divide the numbers by the degree of the map to the moduli space.

The way we arrived at the conjecture is a good example of experimental mathematics. The fact that we had a dimensional formula for the spaces of Siegel modular forms of degree 2, due to Tsushima [61], was very helpful. We also knew the numerical Euler characterics of the local systems.

If for given \( q \) we have made such a list then this result enables us to compute \( \text{Tr}(T(q), S_{a-b,b+3}) \) for all values of \( (a,b) \) ! We did this for all \( q < 37 \) (and have extended this now to \( q < 200 \)).

Even for classical Siegel modular forms this gave information without reach before. For example, the space \( S_{0,35} \) of scalar-valued Siegel modular forms of degree 2 and weight 35 is 1-dimensional. It is generated by a cusp form \( \chi_{35} \) constructed by Igusa. It occurs in the cohomology of the local system \( V_{32,32} \) where \( e_{2,\text{extra}} \) has the form
\[ e_{2,\text{extra}}(32, 32) = 5L^{34} + S[34] \]
and one finds as eigenvalue $\lambda(p)$ of $T(p)$ for $p \leq 37$: 

| $p$ | $\lambda(p)$ on $S_{0,35}$ |
|-----|----------------------------|
| 2   | $-29073418240$             |
| 3   | $-11824551571578840$       |
| 5   | $94700816423199930937500$  |
| 7   | $-10370198954152041951342796400$ |
| 11  | $-801507168963203485836418146947656$ |
| 13  | $-2023213625610765093833898249808244330$ |
| 17  | $11864313906984767985086867381297555266980$ |
| 19  | $2995917272720633250746754589685425572441160$ |
| 23  | $-19113726221407800133722231270080156034989320$ |
| 29  | $-21293272738730115477693459164181205732243180385460$ |
| 31  | $-15734859849821844552162028787565695196444874180822976$ |
| 37  | $-4778585615454984035267859493926208327050656971703460$ |

Or consider the form $E_8 \chi_{35}$ that generates the 1-dimensional space $S_{0,43}$. We have 

$$e_{2,\text{extra}}(40, 40) = S[42] + 7 L^{42}.$$ 

Here we find the following eigenvalues $\lambda(p)$ of $T(p)$: 

| $p$ | $\lambda(p)$ on $S_{0,43}$ |
|-----|----------------------------|
| 2   | $-4069732515840$           |
| 3   | $-65782425978552959640$    |
| 5   | $-44890110453444530263489062500$ |
| 7   | $-1986958479133933968101320293932400$ |
| 11  | $425721965935227369149493869974304329235064$ |
| 13  | $1189605571437888391664528082353560596010166220$ |
| 17  | $-13929961324386673984950242621374936127528473925020$ |
| 19  | $-155890765104968381621459579332178224814423111191589240$ |
| 23  | $-128837520803382146891405898440571781690554910722934311120$ |
| 29  | $47168509025563817366328057558079480585601761063857570397101740$ |
| 31  | $351859132076831108347355005851115474157273721509187497250584$ |
| 37  | $-809124574416380620433562441711130529360039653719132889553889096260$ |

For vector-valued modular forms it is just as powerful. We illustrate this by giving the Hecke eigenvalues for two vector-valued modular cusp forms, namely of weight $(14, 7)$ and $(4, 17)$. In both cases the dimension of the space of Siegel modular cusp forms is 1. We have 

$$e_{2,\text{extra}}(18, 4) = -L^5(S[16] + 1) + 2,$$

and 

$$e_{2,\text{extra}}(18, 14) = -3L^{15} + S[16] + 1.$$ 

The graphs formed by the digits illustrate the growth of the eigenvalues as roughly $p^{(j+2k-3)/2}$. They also illustrate the fact that these eigenvalues tend to be ‘smooth’ (highly divisible) numbers for small values of $(j, k)$. 

Here we go only till $p \leq 37$, but in fact, we have extended the calculations to all $q < 200$. The data will be made available on a website.

As an application of the availability of all these data we can mention the paper [6] where all these data (traces for $q < 150$) are used to approximate critical $L$-values used in checking congruences.

In the case of degree 2 the analogue of the result of Deligne on the existence of a semi-simple $\ell$-adic Galois representation for a Hecke eigenform can be deduced from work of Taylor, Weissauer and Laumon [60, 63, 45]. The dimension of the representation is 4. But unlike the situation for $g = 1$ here it is not necessarily true that the four eigenvalues of a Frobenius element at $p$ have a fixed absolute value $p^w$ for some $w$. When this was discovered this fact came as a surprise.

In fact, at the end of the 1970s Kurokawa and Saito discovered (see [42]) that there are Siegel modular forms of degree 2 that are lifts from elliptic modular forms. In fact, if $f = \sum_n a(n)q^n$ is an elliptic cusp form of weight $2k - 2$ with $k$ even and a normalized (i.e. $a(1) = 1$) eigenform for the Hecke algebra, then there is a scalar-valued Siegel modular form of even weight $k$ on $\Gamma_2$, also an eigenform for the Hecke algebra, with Hecke eigenvalues

$$\lambda(p) = p^{k-2} + a(p) + p^{k-1}.$$  \hspace{1cm} (4)

Note that $a(p) = \alpha + \bar{\alpha}$ with $|\alpha| = p^{k-3/2}$. So there is a 4-dimensional Galois representation corresponding to (4) which we can realize as

$$\mathbb{Q}(-k + 2) \oplus R_f \oplus \mathbb{Q}(-k + 1),$$

where $R_f$ is the 2-dimensional representation $\rho_{\ell,f}$ associated to $f$ by Deligne. But for reasons of cohomological weight only the 2-dimensional part $R_f$ defined by $f$ can occur in the interior cohomology of the local system $\mathcal{V}_{k-3,k-3}$ on $A_2$. So the full 4-dimensional Galois representation is not to be found in the cohomology of the local system, only a part.
These lifts found by Kurokawa and Saito lie in a subspace of $S_k(\Gamma_2)$ called the Maass subspace consisting of Siegel modular forms whose Fourier series
\[ \sum_{N \geq 0} a(N)e^{2\pi i \text{Tr}(N\tau)} \]
has the property that for $N = (n, r/2; r/2, m)$ the coefficient depends only on the discriminant $d(N) = 4mn - r^2$ and the content $\text{g.c.d}(n, r, m)$. The $1-1$ correspondence between the Hecke eigenforms in $S_{2k-2}(\Gamma_1)$ and Hecke eigenforms in $S_k(\Gamma_2)$ was proved to a large extent by Maass, and completed by Andrianov and Zagier, see [65].

In joint work with Bergström and Faber [7] we extended all this to the case of degree 2 and level 2. Here the group $\Gamma_2$ is replaced by the kernel $\Gamma_2[2] = \ker(\text{Sp}(4, \mathbb{Z}) \to \text{Sp}(4, \mathbb{Z}/2\mathbb{Z}))$ of the natural reduction map. Since $\text{Sp}(4, \mathbb{Z}/2\mathbb{Z}) \cong S_6$, the symmetric group on six letters, everything comes with an action of this group. In [7] we formulated the conjectural analogue of the above theorem. The cohomology and spaces of modular forms appear as representation spaces of $S_6$. Although it is conjectural it is based on very firm evidence. In joint work with Fabien Cléry and Sam Grushevsky [18] we used these conjectural results to determine the structure of a number of $M(\Gamma_2[2])$-modules of vector-valued modular forms. By work of Igusa we know that the ring $M^{\text{ev}}(\Gamma_2[2])$ of even-weight scalar-valued modular forms on $\Gamma_2[2]$ is generated by a 5-dimensional vector space of modular forms of weight 2 that form a representation of type $[2,1,1,1,1]$ for the group $S_6$ satisfying a quartic relation. The ring of modular forms of all weights is a quadratic extension generated by a modular form of weight 5. Then the direct sums
\[ \bigoplus_k M_{j,k}(\Gamma_2[2]) \]
for fixed value of $j$ are modules over $M(\Gamma_2[2])$. We determined the structure of such modules for small values of $j$. The conjectural results suggested where to find generators of these modules and by working with explicit generators and using the dimension formulas the structure could be determined. We showed that the $M(\Gamma_2[2])$-module
\[ \Sigma_2 = \bigoplus_{k:\text{odd}} S_{2,k}(\Gamma_2[2]) \]
is generated by ten modular forms $\Phi_i \in S_{2,5}$ ($i = 1, \ldots, 10$) with $\sum \Phi_i = 0$ that span a 9-dimensional $S_6$-representation of type $[2,2,1,1]$. Since we know the decomposition of $S_{2,k}$ as a $S_6$-representation we can read off the relations. Similarly, the module
\[ \bigoplus_{k:\text{even}} M_{2,k}(\Gamma_2[2]) \]
is generated by 15 modular forms $G_{ij} \in S_{2,4}(\Gamma_2[2])$ with $1 \leq i < j \leq 6$ that generate a $S_6$-representation of type $[3,1,1,1] \oplus [2,1,1,1,1]$. 
Before we move to degree 3 we remark that we can formulate the result for genus 2 differently, namely in the form

\[ e_c(A_2, V_{\lambda}) = - S[a - b, b + 3] + e_{2, \text{extra}}(a, b) \]  

in analogy with the case \( g = 1 \), by assuming that there is a motive \( S[a - b, b + 3] \) of rank \( 4 \dim S_{a-b,b+3} \) with the property that

\[ \text{Tr}(T(p), S_{a-b,b+3}) = \text{Tr}(F_p, S[a - b, b + 3]) = - \text{Tr}(F_p, e_c(A_2 \otimes F_p, V_{a,b}) + e_{2, \text{extra}}(a, b)) \]

The motive \( S[a - b, b + 3] \) is still hypothetical, but would be the analogue of the motive \( S[a + 2] \) constructed by Scholl. But even if we do not know whether such a motive exists we can use \( S[a - b, b + 3] \) in the formulation above just as a bookkeeping device with the property that

\[ \text{Tr}(T(p), S_{a-b,b+3}) = \text{Tr}(F_p, S[a - b, b + 3]). \]

6. DEGREE THREE

Now we move to degree \( g = 3 \). In this case the local systems are indexed by \( \lambda = (a, b, c) \) with integers \( a \geq b \geq c \geq 0 \). The goal is now to find an analogue of the formula (5); in other words the problem is to come up with a formula for \( e_{3, \text{extra}}(a, b, c) \). In [8] we formulated a conjecture.

**Conjecture 6.1.** We have

\[ e_c(A_3 \otimes \overline{F}_p, V_{\lambda}) = S[a - b, b - c, c + 4] + e_{3, \text{extra}}(a, b, c) \]

with the correction term \( e_{3, \text{extra}}(a, b, c) \) defined by

\[ e_{3, \text{extra}}(a, b, c) = -e_c(A_2 \otimes \overline{F}_p, V_{a+1,b+1}) - e_{2, \text{extra}}(a + 1, b + 1) \otimes S[c + 2] + e_c(A_2 \otimes \overline{F}_p, V_{a+1,c}) + e_{2, \text{extra}}(a + 1, c) \otimes S[b + 3] - e_c(A_2 \otimes \overline{F}_p, V_{b,c}) - e_{2, \text{extra}}(b, c) \otimes S[a + 4] \]

and where \( S[a - b, b - c, c + 4] \) denotes a hypothetical motive or bookkeeping device such that

\[ \text{Tr}(T(p), S_{a-b,b-c,c+4}) = \text{Tr}(F_p, S[a - b, b - c, c + 4]). \]

Again we arrived at this conjecture by using the fact that in genus 3 the moduli of principally polarized abelian varieties are close to the moduli of genus three curves. In fact, the map \( \mathcal{M}_3 \to A_3 \) is of degree 2 as a map of stacks. Then we can calculate the trace of a power of Frobenius on the cohomology of \( V_{\lambda} \) by making a list of all principally polarized abelian varieties of dimension 3 over \( \overline{F}_q \) up to isomorphism over \( F_q \) that are Jacobians of curves and calculating the order of the automorphism group and the eigenvalues of Frobenius. By summing a Schur function in these eigenvalues over this list (weighted by the inverse orders of the automorphism groups) we find the trace. Again we then tried to interpret this number as a function of \( q \) in terms of ‘known’ functions, like polynomials in \( q \), traces of Hecke operators on spaces of cusp forms of degree 1 and 2. That is how we arrived at the
expressions for $e_{3,\text{extra}}(a, b, c)$. In fact, the form of part of the Eisenstein cohomology found in [30] guided our guesses and suggested a first approximation. Surprisingly the formula for degree 3 looks simpler than the formula for degree 2. We did not know the dimension of the spaces of cusp forms, but we knew the numerical Euler characteristics $E_c$ (obtained by replacing the terms in the conjecture by their dimensions) of the local systems by [13].

The evidence for this conjecture is considerable:

1. While we are summing rational numbers, since we divide by the order of the automorphism groups, the procedure produces integral values for the traces of the Hecke operators;
2. What we know about the dimensions of the spaces of modular forms agrees with the conjecture; for example, if we know that the space of cusp forms vanishes we always find that $\text{Tr}(e_c) = \text{Tr}(e_{\text{extra}})$;
3. We know the numerical Euler characteristics $E_c$ by [13]; as it turns out $E_c - E_{\text{extra}}$ is always divisible by 8.
4. In all vector-valued cases when $E_c = E_{\text{extra}}$ (i.e. when we conjecture that the space of cusp forms vanishes), we have $\text{Tr}(e_c) = \text{Tr}(e_{\text{extra}})$.
5. Recently Taïbi [59] has been able to calculate dimensions of Siegel modular forms assuming the validity of Arthur’s trace formula. His values fit ours.
6. There is consistency with the results of Chenevier and Renard [15] on the number of automorphic level one polarized algebraic regular automorphic representation of $\text{GL}_n$ over $\mathbb{Q}$ for $n \leq 8$.
7. Harder type congruences between elliptic modular forms and Siegel modular forms, see below.
8. Mégarbané has computed a number of eigenvalues (up to $q = 13$) for several Siegel modular forms of degree 3 as they appear among the automorphic forms of $\text{SO}(8)$. A number of consistency checks have been made.
9. There is agreement with the eigenvalues computed in [17].

Let us give two examples. We have

$$e_c(A_3, V_{11,5,2}) = S[6, 3, 6] - S[12]L^3 + L^7 - L^3 + 1,$$

where $S[6, 3, 6]$ stands as above for a bookkeeping device. In this case we should have $\dim S_{6,3,6} = 1$. The other example is for weight $(4, 2, 8)$. Again we should have $\dim S_{4,2,8} = 1$. We have

$$e_c(10, 6, 4) = L^8 + S[4, 10] + S[4, 2, 8] - 1.$$

The corresponding modular form is of type $G_2$. We have the following table of eigenvalues (where for powers of a prime we use the convention of [8]):
Our results for $g = 3$ also tell what kind of lifts one finds. We even could identify lifts that come from the Lie group $G_2$ as suggested by Gross and Savin. We refer to [8, 9.1].

7. **Harder Type Congruences**

Congruences between modular forms have a considerable history. There are well-known congruences between cusp forms and Eisenstein series like the congruence between the Hecke eigenvalues

$$\tau(p) \equiv p^{11} + 1 \pmod{691}$$

for $\Delta$ and the Eisenstein series of weight 12. Such congruences occur for primes dividing the numerator of $B_k/2k$ with $B_k$ the $k$th Bernoulli number, see [22]. Swinnerton-Dyer determined all congruences modulo a prime for modular forms on $\text{SL}(2, \mathbb{Z})$, see [58, 55].

Kurokawa found several congruences between Siegel modular forms in 1979 in [43]. In particular, he conjectured the following. Suppose that a prime $\ell$ divides the critical value $2k - 2$ of the symmetric square $L$-series of an eigenform $f$ of weight $k$ on $\text{SL}(2, \mathbb{Z})$. Then there should exist a cusp form $F$ of degree 2, an eigenform of the Hecke operators, whose eigenvalues for $T(p)$ are congruent to those of the weight $k$ Klingen Eisenstein series modulo $\ell$ for every $p$. Some of these congruences were proven by Mizumoto in 1986, see [48] and reconsidered by Katsurada and Mizumoto [41, 23].

In general if there is a Hecke invariant splitting of the vector space of Siegel modular forms, one might expect congruences between the factors of this splitting. Harder suspected congruences between cusp forms and Saito-Kurokawa lifts. He was motivated by his work on the Eisenstein cohomology of $\Gamma_2$.

The moduli space $A_g$ admits a 'minimal' compactification $A_g^\ast$ obtained by mapping the quotient $\Gamma_g \backslash \mathcal{H}_g$ into projective space by using a basis of the

| $p$ | $\lambda(q)$ on $S_{6,3,6}$ | $\lambda(q)$ on $S_{4,2,8}$ |
|-----|-----------------|-----------------|
| 2   | 0               | 9504            |
| 3   | $-453600$       | 970272          |
| 4   | 10649600        | 89719808        |
| 5   | $-119410200$    | $-106051896$    |
| 7   | 12572892800     | 112911962240    |
| 8   | 0               | 1156260593664   |
| 9   | $-29108532600$  | 5756589166536   |
| 11  | $-57063064032$  | 44411629220640  |
| 13  | $-25198577349400$ | 209295820896008 |
| 16  | 341411782197248 | $-369164249202688$ |
| 17  | $-107529004510200$ | 123094220187664 |
| 19  | 1091588958605600 | 51084504993278240 |

| $p$ | $\lambda(q)$ on $S_{6,3,6}$ | $\lambda(q)$ on $S_{4,2,8}$ |
|-----|-----------------|-----------------|
| 2   | 0               | 9504            |
| 3   | $-453600$       | 970272          |
| 4   | 10649600        | 89719808        |
| 5   | $-119410200$    | $-106051896$    |
| 7   | 12572892800     | 112911962240    |
| 8   | 0               | 1156260593664   |
| 9   | $-29108532600$  | 5756589166536   |
| 11  | $-57063064032$  | 44411629220640  |
| 13  | $-25198577349400$ | 209295820896008 |
| 16  | 341411782197248 | $-369164249202688$ |
| 17  | $-107529004510200$ | 123094220187664 |
| 19  | 1091588958605600 | 51084504993278240 |
space $M_k(\Gamma_g)$ of scalar-valued modular forms of sufficiently high weight $k$ and then taking the closure. In other words, by taking $\text{Proj}$ of the graded ring of scalar-valued Siegel modular forms. Set-theoretically we have

$$A_g^* = A_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_0.$$  

It is usually called the Satake or Baily-Borel compactification. This decomposition is mirrored in the Siegel operator that associates to a Siegel modular form of degree $g$ a form of degree $g-1$ by the limiting procedure

$$\Phi_g f(\tau') = \lim_{t \to \infty} f \left( \begin{array}{cc} \tau' & 0 \\ 0 & it \end{array} \right)$$  

with $\tau' \in \mathfrak{H}_{g-1}$, $t \in \mathbb{R}_{>0}$.

If $f \in M_k(\Gamma_g)$ then $\Phi_g(f) \in M_k(\Gamma_{g-1})$; if $f$ is a vector-valued Siegel modular form of weight $\rho$ then $\Phi_g f$ is a form of weight $\rho'$ with $\rho'$ the irreducible representation of highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_{g-1})$ if $\rho$ has highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_g)$. The cusp forms are by definition the forms with $\Phi_g f = 0$.

So the boundary of our moduli space is the place where modular forms of different degrees interact. Using the interpretation with cohomology one sees the interaction in the long exact sequences connecting cohomology on the interior and on the boundary. Harder studied this in an extensive way using the Borel-Serre compactification (a real manifold with corners) and Betti cohomology. He considers the cohomology in the local system of $\mathbb{Z}$-modules $V^\mathbb{Z}_\lambda$ and over a ring $R = \mathbb{Z}[1/N]$ with an appropriate integer $N$ inverted, one considers the long exact cohomology sequence

$$H^3_c(\Gamma_2 \setminus \mathfrak{H}_2, V^\mathbb{Z}_\lambda \otimes R) \to H^3(\Gamma_2 \setminus \mathfrak{H}_2, V^\mathbb{Z}_\lambda \otimes R) \to H^3(\partial(\Gamma_2 \setminus \mathfrak{H}_2), V^\mathbb{Z}_\lambda \otimes R).$$

Using the eigenvalues of the Hecke operators that have different absolute values on $H^3$ and on the Eisenstein part, here taken with $\mathbb{Q}$-coefficients, one sees that there is a direct sum decomposition of $H^3(\Gamma_2 \setminus \mathfrak{H}_2, \mathcal{V}_\lambda)$ as $H^3(\Gamma_2 \setminus \mathfrak{H}_2, \mathcal{V}_\lambda)$ plus its image in $H^3(\partial(\Gamma_2 \setminus \mathfrak{H}_2), \mathcal{V}_\lambda)$. This latter part contributes to the Eisenstein cohomology and the classes there can be described by certain Eisenstein series. For an Eisenstein series generating an eigen space of the cohomology the constant term is a product of critical values of $L$-functions associated to a modular form on $\text{SL}(2, \mathbb{Z})$. But if one works with integral cohomology (Harder uses Betti cohomology) and intersects this with the direct sum of the interior integral cohomology and the integral Eisenstein cohomology it might be that this direct sum has a non-trivial index in the integral cohomology $H^3(\Gamma_2 \setminus \mathfrak{H}_2, \mathcal{V}_\lambda^\mathbb{Z})$. This index is related to primes dividing this critical values of the $L$ functions occuring in the denominator of the constant term of this Eisenstein series. This will lead to a congruence between a part of the interior cohomology and part of the Eisenstein cohomology. In concrete terms it leads to a congruence between the eigenvalues of an honest Siegel modular form of genus 2 (that is, not a Saito-Kurokawa lift) and the eigenvalues of an elliptic modular form.

Harder had this idea already quite early, see his Lecture notes [36], but it seemed difficult to check it. We quote "Ich halte es für sehr interessant,
numerische Rechnungen durchzuführen. .... Ich glaube, daß die Implementierung hiervon auf einem Computer, einige schwierige Aufgabe darstellt. Sie wird viel Zeit und Rechenaufwand kosten...” (36, p. 101)]. But more than ten years later when it became feasible to calculate the eigenvalues of examples of Siegel modular forms, Harder set himself the task of making such conjectures explicit. The first case he arrived at was the conjectured congruence for an eigenform in \( F \) in \( S_{4,10}(\Gamma_2) \) and the normalized elliptic cusp form \( f = \sum a(n)q^n \) generating \( S_{22}(\Gamma_1) \): if \( \lambda(p) \) denotes the eigenvalue of \( F \) for the Hecke operator \( T(p) \) we should have
\[
\lambda(p) \equiv p^8 + a(p) + p^{13} \pmod{41} \tag{6}
\]
for all primes \( p \). The eigenvalues we could calculate fitted perfectly, see Harder’s report on this in [37]. Harder formulated a rather precise conjecture and in many examples the congruences he predicted could be confirmed numerically, see [29]. For other numerical checks see [31]. Some of the congruences have now been proved, for example the original congruence (6) modulo 41 for weight \((4,10)\) by Chenevier and Lannes [14] by their work on unimodular lattices of rank 24 and the modular forms related to this.

Harder’s conjectures admit extensions to higher genus. The underlying idea is that direct sum decompositions of rational cohomology that are stable under the action of the Hecke algebra do not necessarily hold for integral cohomology and this leads to congruences. These congruences are predicted by divisibility properties of critical values of \( L \) functions associated to modular forms.

Since we have various Hecke invariant subspaces of the cohomology, like the interior cohomology, Eisenstein cohomology, spaces of lifted forms, we can expect in higher degree many different types of congruences.

In our paper [8] we formulated a number of extensions to degree three Siegel modular forms. Let us give just one example. The conjecture says in this case that if we have eigenforms \( f \in S_{c+2}(\Gamma_1) \) and \( g \in S_{a+b+6}(\Gamma_1) \) and an ordinary prime \( \ell \) (in the field of eigenvalues of \( f \) and \( g \)) dividing the critical value at \( s = a + c + 5 \) of the \( L \)-series belonging to \( \text{Sym}^2(f) \otimes g \), then there should be a genuine Siegel eigenform \( F \in S_{a-b,b-c,c+4}(\Gamma_3) \) such that
\[
\lambda_F(p) \equiv \lambda_f(p)(p^{b+2} + \lambda_g(p) + p^{a+3}) \mod \ell
\]
for all primes \( p \). An example is the case \( (a,b,c) = (13,11,10) \) with \( F \in S_{2,1,14}(\Gamma_3) \). The Euler characteristic is
\[
2(L^{13}S[12] + 1) - 2L^{11} + S[2,1,14]
\]
and \( \dim S_{2,1,14} = 1 \). The forms are \( f = \Delta = q - 24q^2 + \ldots \) in \( S_{12} \) and \( g \in S_{30} \) with Fourier series \( g = 1 + (4320 + 96\sqrt{51349})q + \ldots \) and the prime dividing critical value of the \( L \) series turns out to be \( \ell = 199 \). The heuristic traces of \( T(p) \) for \( F \in S_{2,1,14} \) are in the table below. One checks that the congruence above is indeed satisfied for all \( q \leq 17 \), for example for \( p = 2 \) the norm of
\[
-2073600 + 24(2^{13} + 4320 + 96\sqrt{51349} + 2^{16})
\]
equals \(-232402452480\) and is divisible by \(\ell = 199\).

\[
\begin{array}{|c|c|}
\hline
p & \lambda(g) \text{ on } S_{2,1,14} \\
\hline
2 & -2073600 \\
3 & -1885952160 \\
4 & 1080940298240 \\
5 & -26851408810200 \\
7 & -3490907533294720 \\
8 & -589089814263899200 \\
9 & 2339767572242234760 \\
11 & -660044916805998490272 \\
13 & -10848446812874015943640 \\
16 & 46169746591651767451648 \\
17 & -200932545573210270768120 \\
19 & 17632053727783741943750240 \\
\hline
\end{array}
\]

For further discussion of congruences between modular forms we refer to the paper by Bergström and Dummi
gan [5] where a vast generalization of Harder’s conjecture is made. For finding this generalization the data provided by our calculations were of great help.

8. Picard Modular Forms

The approach that we are advocating in this paper can be applied in other cases as well. Shimura gave in 1964 a list of rational ball quotients [57]; these ball quotients turn out to be moduli spaces of covers of the projective line and therefore are amenable to our approach. Bergström and I have pursued this for two cases of 2-dimensional ball quotients. These ball quotients are Picard modular surfaces.

If \(F\) is an imaginary quadratic field with ring of integers \(O_F\) and discriminant \(D\) we consider the vector space \(V = F^3\) with hermitian form \(h\)

\[
z_1\bar{z}_2 + z_2\bar{z}_1 + z_3\bar{z}_3
\]

and we let

\[
G = \{g \in \text{GL}(3, F) : h(gz, gw) = \eta(g)h(z, w)\}
\]

with \(\eta(g) \in \mathbb{Q}\) the so-called multiplier. Then \(G\) is an algebraic group defined over \(\mathbb{Q}\). If \(G^0 = \ker \eta\) we set \(\Gamma = G^0(\mathbb{Z})\) and \(\Gamma_1 = G^0(\mathbb{Z}) \cap \ker \text{det}\). An element \(g \in G(\mathbb{R})\) with \(\eta(g) > 0\) acts on

\[
B = \{l \in V_\mathbb{C} = V \otimes_{\mathbb{Q}} \mathbb{R} : l \text{ a line with } h|_l < 0\},
\]

the set of lines on which \(h\) is negative definite. It is a complex 2-ball

\[
B = \{(u, v) \in \mathbb{C}^2 : v + \bar{v} + w\bar{u} < 0\},
\]

where \(u = z_3/z_2\) and \(v = z_1/z_2\). The space \(\Gamma \backslash B\) is an orbifold. It is not compact but can be compactified by adding finitely points, called the cusps.
These are singular and can be resolved by elliptic curves. These quotients are moduli spaces of 3-dimensional abelian varieties with endomorphisms from $O_F$.

Picard studied these quotients and the corresponding modular forms already at the end of the nineteenth century, see [50, 51]. Shimura took up their study again in the 1960s and in the 1970s Shintani wrote an unpublished paper about the (vector-valued) modular forms associated to these groups. Later Shiga, Holzapfel, Feustel, Finis and others studied these 2-dimensional spaces and the associated scalar-valued modular forms, see [39, 27].

We looked at two cases in Shimura's list of ball quotients; they have $F = \mathbb{Q}(\sqrt{-1})$ or $F = \mathbb{Q}(\sqrt{-3})$.

Let us consider the case $F = \mathbb{Q}(\sqrt{-3})$ and

$$\Gamma(\sqrt{-3}) = \{ g \in \Gamma : g \equiv 1 (\text{mod } \sqrt{-3}) \}$$

and

$$\Gamma_1(\sqrt{-3}) = \{ g \in \Gamma(\sqrt{-3}) , \det(g) = 1 \} .$$

Then in this case the moduli space $\Gamma_1(\sqrt{-3})\backslash B$ is a moduli space of curves: Galois covers of genus 3 and degree 3 with a marking of the ramification points. These curves can be written as

$$y^3 = f(x)$$

with $f$ a polynomial of degree 4 with non-vanishing discriminant.

The surface $\Gamma_1(\sqrt{-3})\backslash B$ can be compactified by adding four cusps. It comes equipped with two vector bundles: we have an exact sequence

$$0 \to L \to V_\mathbb{C} \to Q \to 0$$

where $L$ denotes the tautological line bundle and $Q$ the tautological quotient of rank 2. With $U = Q^\vee$ we have the isomorphisms

$$U \otimes L \cong \Omega^1(\log D), \quad L^3 \cong \Omega^2(\log D) ,$$

where $D$ is the divisor that resolves the four cusps. Correspondingly, we have two factors of automorphy

$$j_1((u,v), g) = g_{21} u + g_{22} v + g_{23} ,$$

corresponding to $L$, and

$$j_2((u,v), g) = \text{Jac}((u,v), g)^{-1} \cdot j_1((u,v), g)^{-1}$$

with Jac the Jacobian of the action, corresponding to $U$. This gives us the notion of a Picard modular form. A scalar-valued modular form of weight $k$ is a holomorphic section of $L^k$ and a scalar-valued cusp form of weight $k$ is a section of $\Omega^2 \otimes L^{k-3}$. The ring of scalar-valued Picard modular forms in this case is known by work of Shiga, Holzapfel and Feustel. We have

$$M(\Gamma(\sqrt{-3})) = \oplus_k M_k(\Gamma(\sqrt{-3})) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2] ,$$
where the $\varphi_i$ are modular forms of weight 3 and
\[ M(\Gamma_1(\sqrt{-3})) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2, \zeta]/(\zeta^3 - P(\varphi_0, \varphi_1, \varphi_2)) \]
with $\zeta$ a cusp form of weight 6. Finis calculated in 1998 (see [27]) some eigenvalues under the Hecke operators of low-weight Picard modular forms for $M(\Gamma_1(\sqrt{-3}))$.

We also have vector-valued modular forms; these correspond to sections of
\[ \text{Sym}^j(U) \otimes L^{k-3} \]
Although Shintani’s unpublished paper treats vector-valued Picard modular forms there were no explicit examples in the literature before we considered this case. Since our quotient parametrizes curves our cohomological approach also works here. We have a universal family
\[ \pi : \mathcal{X} \rightarrow \mathcal{M} \]
of curves of genus 3 with an action of a group of order 3. The local system $V = R^1\pi_*\mathbb{Q}$ is of rank 6 and it or its étale $\ell$-adic variant splits after base change to $F$ as
\[ W \oplus W', \]
where $W$ (resp. $W'$) corresponds to the $\rho$-eigenspace (resp. $\rho^2$-eigenspace) under the action of the group of order 3 and $W$ is of rank 3. Note that after base change we have $G \sim \text{GL}(3, \mathbb{F}) \times \mathbb{G}_m$, where the multiplicative group $\mathbb{G}_m$ corresponds to the multiplier $\eta$. The local systems that we can consider here correspond to representations of this group.

We consider local systems $W_\lambda$ with $\lambda = (a + i, i, -b + i)$
\[ \text{Sym}^aW \otimes \text{Sym}^bW' \otimes \det W^i, \]
where $\det W$ is a local system the 6th power of which is a constant local system. So we may choose $i$ modulo 6 (or even modulo 3). We look at the motivic Euler characteristic
\[ e_c(\mathcal{M}, W_\lambda) = \sum_i (-1)^i[H^i(\mathcal{M}, W_\lambda)], \]
again taken in an appropriate Grothendieck group of mixed Hodge structures or Galois representations. Let us denote as above the interior cohomology, that is, the image of compactly supported cohomology in the usual cohomology, by $H^i$. We know by work of Schwermer-Ragunathan that $H^i \neq (0)$ implies $i = 2$ for regular $\lambda$ and we can show that we have a Hodge filtration on $H^2$ of the form
\[ F^{a+b+2} \subset F^{a+1} \subset F^0 = H^2(\mathcal{M}, W_\lambda). \]
As before, a main point is here that the first step $F^{a+b+2}$ can be interpreted as a space of cusp forms
\[ F^{a+b+2} \cong S_{b,a+3}(\Gamma(\sqrt{-3}), \det^i), \]
namely, cusp forms of weight \((b, a + 3)\) and character \(\det^i\). Such modular forms can be described as functions \(f : B \to \text{Sym}^j(\mathbb{C}^2)\) satisfying
\[
f(\gamma(u, v)) = \det(\gamma)^i (j_1(\gamma, u, v))^{a+3} \text{Sym}^b(j_2(\gamma, u, v)) f(u, v)
\]
for \(\gamma \in \Gamma[\sqrt{-3}]\) and a vanishing condition at the cusps.

We can make counts over finite fields of cardinality \(q \equiv 1(\mod 3)\), and try to find the extra term \(e_{\text{extra}}(a, b)\). It is composed of two parts: the Eisenstein cohomology and the endoscopic part.

The Eisenstein cohomology is given as
\[
e_{\text{Eis}}(M, W_{\lambda}) = \sum_i (-1)^i [\ker(H^i_c \to H^i(M, W_{\lambda})].
\]
This has been determined by Harder [35, 36]. As a Galois representation it is given as an explicit sum of Hecke characters of \(F\). So we subtract it. But then the endoscopy remains. Indeed, there is a plethora of endoscopic terms. In [11] we determined heuristically all those terms. Subtracting these gives us the trace of the Hecke operator on the spaces of honest Picard modular forms. In fact we have a completely explicit but complicated conjectural formula for \(e_c(M, W_{\lambda})\). We refer to [11] and content ourselves by giving some simple examples. We note that the symmetric group \(S_4\) acts on our modular surface (permuting the four cusps) and we describe the cohomology as a representation of the symmetric group \(S_4\). We denote an irreducible representation of \(S_4\) by the corresponding partition, thus writing \(s[4], \ldots, s[1, 1, 1, 1]\).

The Eisenstein cohomology vanishes for \(\lambda = [a + i, i - b + i]\) with \(a \equiv b \equiv 2(\mod 3)\) and \(i = 1, 2\). Take \(\lambda = [6k, 0, 0]\). Then
\[
e_{\text{E}}(M, W_{\lambda}) = k(s[4] + s[3, 1] + s[2, 2])L^{6k+1,1} + S[0, 6k + 3, \det^2]
\]
Here \(L^{a,\beta}\) stands for a 1-dimensional Galois representation of Gal\((F/F)\) or a Hecke character; moreover \(S[0, 6k + 3, \det^2]\) stands for a hypothetical motive or just as a bookkeeping device with the property that
\[
\text{Tr}(F_p, S[0, 6k + 3, \det^2]) = \text{Tr}(T(p), S_{0, 6k+3, \Gamma[\sqrt{-3}], \det^2}).
\]
Using this we can calculated traces of Hecke operators \(T(p)\) for \(p \leq 43\)equivariantly and the results fit for scalar-valued modular forms with the calculations done by Finis. And in fact this has now been extended up to traces for \(q < 1000\).

We add an example involving vector-valued Picard modular forms: we let \(\lambda = (6k + 1, 0, -1)\) and in this case we find the conjectural formula
\[
e_{\text{E}}(M, W_{\lambda}) = (ks[4] + ks[3, 1] + (k + 1)s[2, 2])L^{6k+2,2} + S[1, 6k + 4, \det^2]
\]
We are still in the process of writing up these results. But in the meantime Fabien Cléry and I have used these heuristic results to investigate modules...
of vector-valued Picard modular forms over the ring of scalar-valued Picard modular forms \cite{16}. For example, consider

\[ \Sigma_1^{(i)} = \bigoplus_{k=1}^{\infty} S_{1,3k+1}(\Gamma[\sqrt{-3}], \det^i) \quad \text{for } i = 0, 1, 2. \]

This is a module over the ring of scalar-valued modular forms

\[ M = \bigoplus_k M_{0,3k}(\Gamma[\sqrt{-3}]) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2]. \]

We proved the following result.

**Theorem 8.1.** The module \( \Sigma_1^{0} \) is generated over the ring \( M \) of scalar-valued Picard modular forms by three vector-valued modular forms \( \Phi_0, \Phi_1, \Phi_2 \in S_{1,7}(\Gamma[\sqrt{-3}]) \) with the relation

\[ \varphi_0 \Phi_0 + \varphi_1 \Phi_1 + \varphi_2 \Phi_2 = 0. \]

The modular forms \( \Phi_i \) appearing here are constructed as a sort of Cohen-Rankin bracket, with for example \( \Phi_2 \) given by

\[ \Phi_2 = [\varphi_0, \varphi_1] \sim \varphi_0 \left( \frac{\partial \varphi_1}{\partial u} \right) \frac{\partial}{\partial v} - \varphi_1 \left( \frac{\partial \varphi_0}{\partial u} \right) \frac{\partial}{\partial v}. \]

We can calculate the eigenvalues for these forms and the values fit the heuristic values perfectly. The description of the module makes it possible to construct explicitly eigenforms and calculate eigenvalues and these have been used to confirm the conjectural formulas for \( e_c(M, \mathbb{W}_\lambda) \) of \cite{11}. We just give one table that illustrates the case of the forms \( \Phi_i \).

| \( \alpha \) | \( p \) | \( \lambda_\alpha(\Phi_i) \) |
|----------------|--------|-------------------------------|
| \( 1 + 3\rho \) | 7      | 759 + 261\rho                |
| \( 1 - 3\rho \) | 13     | -4137 + 1633\rho             |
| \( -2 + 3\rho \) | 19     | 24042 + 14733\rho            |
| \( 1 + 6\rho \) | 31     | -145401 - 241830\rho         |
| \( 4 - 3\rho \) | 37     | 12900 - 114849\rho           |
| \( 1 - 6\rho \) | 43     | 246567 - 8946\rho            |
| \( 4 + 9\rho \) | 61     | 1048836 - 173205\rho         |
| \( -2 - 9\rho \) | 67     | -1539510 - 1246887\rho       |
| \( 1 + 9\rho \) | 73     | -1563729 + 1261143\rho       |
| \( 7 - 3\rho \) | 79     | 9921297 + 3294171\rho        |
| \( -8 + 3\rho \) | 97     | 5678616 - 3870891\rho        |
| \( -2 \) | 2      | 72                            |
| \( -5 \) | 5      | 89622                         |

For the details and the structure of \( \Sigma_1^1 \) and \( \Sigma_1^2 \) and various other modules we refer to \cite{16}.
REFERENCES

[1] A.N. Andrianov, V.G. Zhuravlëv: Modular forms and Hecke operators. Translated from the 1990 Russian original by Neal Koblitz. Translations of Mathematical Monographs, 145. American Mathematical Society, Providence, RI, 1995. viii+334 pp.

[2] J. Arthur: An introduction to the trace formula. In: Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2005, 1–263. In: Harmonic analysis, the trace formula, and Shimura varieties, 1–263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.

[3] J. Bergström: Cohomology of moduli spaces of curves of genus three via point counts. J. Reine Angew. Math. 622 (2008), 155–187.

[4] J. Bergström: Equivariant counts of points of the moduli spaces of pointed hyperelliptic curves. Doc. Math. 14 (2009), 259–296.

[5] J. Bergström, N. Dummigan: Eisenstein congruences for split reductive groups. arXiv:1502.07827. To appear in Selecta Math.

[6] J. Bergström, N. Dummigan, T. Mégarbané: Eisenstein congruences for SO(4, 3), SO(4, 4), spinor and triple product L-values.

[7] J. Bergström, C. Faber, and G. van der Geer: Siegel modular forms of genus 2 and level 2: conjectures and cohomological computations. IMRN, Art. ID rnn 100 (2008).

[8] J. Bergström, C. Faber, and G. van der Geer: Siegel modular forms of degree three and the cohomology of local systems. Selecta Math. (N.S.) 20 (2014), no. 1, 83-124.

[9] J. Bergström, C. Faber, and G. van der Geer: Teichmüller modular forms and the cohomology of local systems on $M_3$. In preparation.

[10] J. Bergström, G. van der Geer: The Euler characteristic of local systems on the moduli of curves and abelian varieties of genus three. J. Topol. 1 (2008), 651–662.

[11] J. Bergström, G. van der Geer: Picard Modular Forms and Cohomology of Picard Modular Surfaces. In preparation.

[12] B. Birch: How the number of points of an elliptic curve over a fixed prime field varies. J. London Math. Soc. 43 (1968) 57-60.

[13] G. Bini, G. van der Geer: The Euler characteristic of local systems on the moduli of genus 3 hyperelliptic curves. Math. Ann. 332 (2005), 367–379.

[14] G. Chenevier, J. Lannes: Formes automorphes et voisins de Kneser des réseaux de Niemeier. Preprint 2015.

[15] G. Chenevier, D. Renard: Level one algebraic cusp forms of classical groups of small ranks. Memoirs of the A.M.S. 237, 2015.

[16] F. Cléry, G. van der Geer: Generators for modules of vector-valued Picard modular forms. Nagoya Math. J. 212 (2013), 19-57.

[17] F. Cléry, G. van der Geer: Constructing vector-valued Siegel modular forms from scalar-valued Siegel modular forms. Pure Appl. Math. Q. 11 (2015), no. 1, 21-47.

[18] F. Cléry, G. van der Geer, S. Grushevsky: Siegel modular forms of genus 2 and level 2. Internat. J. Math. 26 (2015), no. 5, 1550034, 51 pp.

[19] C. Consani, C. Faber: On the cusp form motives in genus 1 and level 1. In: Moduli spaces and arithmetic geometry, 297–314, Adv. Stud. Pure Math. 45, Math. Soc. Japan, Tokyo, 2006.

[20] H. Darmon, F. Diamond, and R. Taylor: Fermat’s last theorem. In: Current developments in mathematics, 1–154, Int. Press, Cambridge, MA, 1994.

[21] P. Deligne: Formes modulaires et représentations ℓ-adiques, Séminaire Bourbaki 1968-1969, exp. 355, 139–172, Lecture Notes in Math. 179, Springer-Verlag, Berlin, 1971.

[22] P. Deligne, J.-P. Serre: Formes modulaires de poids 1. Ann. Sci. École Norm. Sup. (4) 7 (1974), 507-530.

[23] N. Dummigan: Symmetric square L-functions and Shafarevich-Tate groups. Experiment. Math. 10 (2001), no. 3, 383-400.
26 GERARD VAN DER GEER

[24] C. Faber, G. van der Geer: Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes, I, II. C.R. Acad. Sci. Paris, Sér. I 338 (2004), 381–384, 467–470.

[25] G. Faltings: On the cohomology of locally symmetric Hermitian spaces. Lecture Notes in Math. 1029, Springer-Verlag, Berlin, 1983.

[26] G. Faltings, C-L. Chai: Degeneration of abelian varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22. Springer-Verlag, Berlin, 1990.

[27] T. Finis: Some computational results on Hecke eigenvalues of modular forms on a unitary group. Manuscripta Math. 96 (1998), 149–180.

[28] E. Freitag: Siegelsche Modulfunktionen. Grundlehren der Mathematischen Wissenschaften, 254. Springer-Verlag, Berlin, 1983.

[29] G. van der Geer: Siegel modular forms and their applications. In: J. Bruinier, G. van der Geer, G. Harder, D. Zagier: The 1-2-3 of modular forms. Springer-Verlag, 2008.

[30] G. van der Geer: Rank one Eisenstein cohomology of local systems on the moduli space of abelian varieties. Sci. China Math. 54 (2011), no. 8, 1621–1634.

[31] A. Ghitz, N. Ryan, D. Sulon: Computations of vector-valued Siegel modular forms. J. Number Theory 133 (2013), no. 11, 3921-3940.

[32] B.H. Gross, G. Savin: Motives with Galois group of type $G_2$: an exceptional theta-correspondence. Compositio Math. 114 (1998), no. 2, 153–217.

[33] C. Grundh: Computations of vector valued Siegel modular forms. Ph.D. thesis, KTH, Stockholm, in preparation.

[34] K. Haberland: Perioden von Modulformen einer Variablen und Gruppencohomologie. Math. Nachr. 112 (1983), 245–315.

[35] G. Harder: Eisensteinkohomologie für Gruppen vom Typ $GU(2,1)$. Math. Annalen 278 (1987), 563–592.

[36] G. Harder: Eisensteinkohomologie und die Konstruktion gemischter Motive. Lecture Notes in Math. 1562. Springer Verlag 1993.

[37] G. Harder: A congruence between a Siegel and an elliptic modular form. In: J. Bruinier, G. van der Geer, G. Harder, D. Zagier: The 1-2-3 of modular forms. Springer-Verlag, 2008.

[38] G. Harder: The Eisenstein motive for the cohomology of $GSp_2(Z)$. In: Geometry and Arithmetic, 143–164, EMS Series of Congress Reports, European Mathematical Society, Zürich, 2012.

[39] R.-P. Holzapfel: Geometry and Arithmetic: Around Euler Partial differential equations. Math. Monogr. 20, VEB Deutscher Berlin, 1986.

[40] J. Igusa: On Siegel modular forms genus two. II. Amer. J. Math. 86 (1964), 392–412.

[41] H. Katsurada, S. Mizumoto: Congruences of Hecke eigenvalues of Siegel modular forms. Preprint (2011).

[42] N. Kurokawa: Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two. Invent. Math. 49 (1978), no. 2, 149-165.

[43] N. Kurokawa: Congruences between Siegel modular forms of degree 2. Proc. Japan Acad. Ser. A Math. Sci. 55, Number 10, (1979), 417–422.

[44] S. Lang: Introduction to Modular forms. Grundlehren der mathematischen Wissenschaften 222. Springer Verlag.

[45] G. Laumon: Sur la cohomologie à supports compacts des variétés de Shimura pour $GSp(4, Q)$. Compositio Math. 105 (1997), no. 3, 267-359.

[46] Y. Matsushima, S. Murakami: On certain cohomology groups attached to Hermitian symmetric spaces. Osaka J. Math. 2 (1965) 1-35.

[47] I. Miyawaki: Numerical examples of Siegel cusp forms of degree 3 and their zeta functions. Memoirs of the Faculty of Science, Kyushu University, Ser. A 46 (1992), 307–339.

[48] S. Mizumoto: Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two. Math. Ann. 275 (1986), no. 1, 149–161.
[49] D. Petersen: Cohomology of local systems on the moduli of principally polarized abelian surfaces. Pacific J. Math. 275 (2015), no. 1, 39-61.
[50] E. Picard: Sur une extension aux fonctions de deux variables indépendentes analogues aux fonctions modulaires. Acta Math. 2 (1883), 114–135.
[51] E. Picard: Sur les fonctions hyperfuchsiennes provenant des séries hypergéométriques de deux variables. Ann. École Norm. Sup. 62 (1885), 357–384.
[52] T. Satoh: On certain vector valued Siegel modular forms of degree two. Math. Ann. 274 (1986), no. 2, 335–352.
[53] A.J. Scholl: Motives for modular forms. Invent. Math. 100 (1990), no. 2, 419–430.
[54] J.-P. Serre: Une interprétation des congruences relatives à la fonction τ de Ramanujan. Séminaire Delange-Pisot-Poitou, 1967–1968, ex. 14.
[55] J.-P. Serre: Congruences et formes modulaires. [d'après H.P.F. Swinnerton-Dyer] Séminaire Bourbaki, 24e année, 1971/72, no 416.
[56] J.-P. Serre: Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques. In: Motives (Seattle, WA, 1991), 377–400, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.
[57] G. Shimura: On purely transcendental fields of automorphic functions of several variables. Osaka Math. Journal 1 (1964), 1-14.
[58] P. Swinnerton-Dyer: On ℓ-adic representations and congruences for coefficients of modular forms. In: Modular Functions of One Variable III. Lectures Notes in Math. 350, 1973. Springer Verlag.
[59] O. Taïbi: Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula. arXiv:1406.4247.
[60] R. Taylor: On the ℓ-adic cohomology of Siegel threefolds. Invent. Math. 114 (1993), no. 2, 289–310.
[61] R. Tsushima: An explicit dimension formula for the spaces of generalized automorphic forms with respect to Sp(2, Z). Proc. Japan Acad. Ser. A Math. Sci. 59 (1983), no. 4, 139–142.
[62] S. Tsuyumine: On Siegel Modular Forms of Degree Three. Am. J. Math. 108 (1986), no. 4, 755–862.
[63] R. Weissauer: Endoscopy for GSp(4) and the cohomology of Siegel modular threefolds. Lecture Notes in Math. 1968. Springer-Verlag, Berlin, 2009.
[64] R. Weissauer: The trace of Hecke operators on the space of classical holomorphic Siegel modular forms of genus two. arXiv:0909.1744.
[65] D. Zagier: Sur la conjecture de Saito-Kurokawa (d’après H. Maass). Seminar on Number Theory, Paris 1979–80, 371–394, Progr. Math., 12, Birkhäuser, Boston, MA, 1981.