Extended pseudo-fermions from non commutative bosons

S. Twareque Ali

F. Bagarello

Jean Pierre Gazeau

Abstract

We consider some modifications of the two dimensional canonical commutation relations, leading to non commutative bosons and we show how biorthogonal bases of the Hilbert space of the system can be obtained out of them. Our construction extends those recently introduced by one of us (FB), modifying the canonical anticommutation relations. We also briefly discuss how bicoherent states, producing a resolution of the identity, can be defined.

1Department of Mathematics and Statistics, Concordia University, Montréal, Québec, CANADA H3G 1M8
e-mail: twareque.ali@concordia.ca
2Dipartimento di Energia, ingegneria dell’Informazione e modelli Matematici, Facoltà di Ingegneria, Università di Palermo, I-90128 Palermo, ITALY
e-mail: fabio.bagarello@unipa.it Home page: www.unipa.it/fabio.bagarello
3Laboratoire APC, Université Paris Diderot, Sorbonne Paris-Cité, 10, rue A. Domon et L. Duquet, 75205 Paris Cedex 13, France, and Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, 22290-180 Rio de Janeiro, Brasil
e-mail: gazeau@apc.univ-paris7.fr
I Introduction

In a recent series of papers the possibility of modifying the canonical (anti)commutation relations in order to get biorthogonal (Riesz) bases has been considered in some detail, see [1], [2] and references therein. The articles [3]-[8] are more concerned with physical applications. The functional structure arising from these modifications turns out to be rather rich and, moreover, it appears to be closely related to the so-called non hermitian quantum mechanics (or some variations of it). This particular aspect is discussed, for instance, in [5, 6].

On the different side, a rather fashionable topic in the recent literature is the so called noncommutative quantum mechanics which, in the form that is of relevance to us, is essentially a two dimensional version of the position and momentum operators, \( \hat{x}_j \) and \( \hat{p}_j \), which satisfy, other than \([\hat{x}_j, \hat{p}_k] = i \delta_{j,k}, \ j = 1, 2\), also the commutation rule \([x_1, x_2] = i \theta\), for some real parameter \( \theta \) (see [9, 10]). These were originally adopted to describe a possible quantized space at very small length scales. For recent coherent state and group theoretical approaches to non commutative quantum mechanics see, for example, [11, 12, 13]. In [13] it has been shown that the commutation relations of non commutative quantum mechanics are essentially those satisfied by the generators of the Galilei group in (2+1) space-time dimensions with two central extensions, a fact which had earlier been also noted in [14]. Additionally, using the coherent states introduced in [13], these same commutation relations have been derived by the method of coherent state quantization.

In this paper we will show how these two topics could be indeed related, and how a non commutative version of the annihilation and creation operators acting on a Hilbert space \( \mathcal{H} \) naturally gives rise to two biorthonormal bases of \( \mathcal{H} \). More in details, after a first no-go result, showing that, not surprisingly, not all the non commutative extensions of two-dimensional quantum mechanics gives rise to these kind of bases, we discuss a second possibility for which several interesting facts can be established. In particular, two biorthogonal sets are explicitly found, together with several extended versions of non self-adjoint number-like operators of the same kind as those introduced in [2] (In a related paper [15] such biorthogonal bases have been realized as biorthogonal sets of polynomials in a complex variable). These non self-adjoint operators are related to their adjoints by certain intertwining operators, which can be easily identified in our setting. This is the content of Section II, while in Section III we give an explicit realization of the construction. In Section IV we return to the general construction for the infinite-dimensional Hilbert space, and in Section V we construct related coherent states. Section VI contains our conclusions.
II Generalized pseudo-fermions

We will discuss in this section how non commutative quantum mechanics can be used to generate biorthogonal bases of an infinite dimensional Hilbert space which arise from a numerable union of other, finite-dimensional, pairs of biorthogonal bases, each pair in some finite dimensional Hilbert space. We will also describe in detail the intertwining operators relating, see below, the number-like operators which naturally arise in our framework. Moreover, since these latter are not self-adjoint, we will briefly discuss the natural connections between our results and non hermitian quantum mechanics.

We begin our analysis with a simple but useful exercise, showing that not all the non commutative deformations of ordinary two-dimensional quantum mechanics give rise to an interesting structure, when analyzed from the point of view of [1] and [2].

II.1 A no-go result

Consider the following transformation \((x, y, p_x, p_y) \rightarrow (X, Y, P_X, P_Y)\), where

\[
\begin{align*}
X &= x - \frac{\theta}{2} p_y, \quad P_X = p_x \\
Y &= y + \frac{\theta}{2} p_x, \quad P_Y = p_y,
\end{align*}
\]

which maps the canonical operators \((x, y, p_x, p_y)\), satisfying the usual \([x, p_x] = [y, p_y] = i\), with all the other commutators being zero, to the capital operators, for which, again, \([X, P_X] = [Y, P_Y] = i\). However, \([X, Y] = i\theta\), while \([P_X, P_Y] = 0\). These are the non commutative operators we are going to consider here. They can be obtained from the original ones using, in a non-standard fashion, the unitary operator \(V_\theta := e^{-i\frac{\theta}{2}p_x p_y}:
\]

\[
\begin{align*}
X &= V_\theta x V_\theta^{-1}, \quad P_X = V_\theta p_x V_\theta^{-1}, \\
Y &= V_\theta^{-1} y V_\theta, \quad P_Y = V_\theta^{-1} p_y V_\theta.
\end{align*}
\]

These show that each pair of variables transforms unitarily, but the two pairs transform in different ways. Thus each single pair \((X, P_X)\) and \((Y, P_Y)\) is canonical, but the two pairs together are not.

One could try to introduce creation and annihilation operators in the usual way: \(a_x = \frac{1}{\sqrt{2}} (x + ip_x)\) and \(a_y = \frac{1}{\sqrt{2}} (y + ip_y)\), and the related capital operators: \(A_X = \frac{1}{\sqrt{2}} (X + iP_X) = V_\theta a_x V_\theta^{-1}\) and \(A_Y = \frac{1}{\sqrt{2}} (Y + iP_Y) = V_\theta^{-1} a_y V_\theta\). Since \([A_X, A_Y^\dagger] = [A_Y, A_X^\dagger] = I\), one could also think to construct eigenstates of \(N_X = A_X^\dagger A_X^\dagger\) and \(N_Y = A_Y^\dagger A_Y\) in a standard fashion. This is possible, of course: one starts with, say, the vacuum of \(A_X, \Phi_0, A_X \Phi_0 = 0\), and acts on this vector with powers of \(A_X^\dagger\). Then \(N_X(A_X^\dagger n \Phi_0) = n(A_X^\dagger n \Phi_0)\). The same procedure can be
repeated for $A_Y$. However, when we try to extend this construction to the two dimensional system, problems arise already at a very essential level: in fact, whenever the non commutativity parameter $\theta$ is not zero, it is not possible to find any non zero, square-integrable, function $f(x, y)$ satisfying simultaneously $A_X f = 0$ and $A_Y f = 0$. In fact, using the definition of $A_X$ and $A_Y$, and the definitions in (2.1), these conditions read as
\[
\begin{aligned}
(x + \partial_x + i^{\theta} \partial_y) f(x, y) &= 0 \\
(y + \partial_y - i^{\theta} \partial_x) f(x, y) &= 0.
\end{aligned}
\]
It is not hard to check that these equations cannot be solved together, except when $\theta = 0$. In this case we recover the standard gaussian, $f(x, y) = N e^{-(x^2+y^2)/2}$, which coincides with the common vacuum $\Phi_{0,0}(x, y)$ for $a_x$ and $a_y$, with the choice of the normalization $N = \frac{1}{\sqrt{2\pi}}$. The conclusion is that, because of (2.1), we lose the possibility of having a common vacuum for the two annihilation operators $A_X$ and $A_Y$. Other more interesting choices of non commuting operators will be considered below.

II.2 A different choice

We have shown that the choice of the non commuting variables in (2.1) is not appropriate if we are interested, for instance, in extending the construction of the eigenstates of the hamiltonian of the Landau levels, since this construction is mainly based on the existence of a common vacuum for two different annihilation operators. For this reason we discuss here a slightly different point of view, in which the existence of such a vector is ensured by the construction itself. For this reason, rather than introducing non commutative coordinates, it is more convenient to introduce directly *non commutative bosonic operators*.

Let $a_x$ and $a_y$ be as above, and let us introduce two linear combinations of these,
\[
A_1 := \alpha_x a_x + \alpha_y a_y, \quad A_2 := \beta_x a_x + \beta_y a_y,
\]
for some complex $\alpha$’s and $\beta$’s. This is the simplest non trivial choice we can consider for our task. In fact, since $\Phi_{0,0}(x, y)$ is annihilated by both $a_x$ and $a_y$, it is also annihilated by $A_1$ and $A_2$, for all choices of the coefficients: $A_1 \Phi_{0,0} = A_2 \Phi_{0,0} = 0$. If we assume the following
\[
|\alpha_x|^2 + |\alpha_y|^2 = |\beta_x|^2 + |\beta_y|^2 = 1, \quad \alpha_x \beta_x + \alpha_y \beta_y = \gamma,
\]
\footnote{We want to remark that this is not an automatic consequence of the fact that $[A_X, A_Y] \neq 0$, since we are not trying to find a common set of eigenstates for these two operators, but just a single common vacuum.}
for some \( \gamma \in \mathbb{C} \), the following commutator rules follow:

\[
[A_1, A_2] = 0, \quad [A_1, A_1^\dagger] = [A_2, A_2^\dagger] = 1, \quad [A_1, A_2^\dagger] = \gamma 1
\]

(2.5)

which also imply that \([A_1^\dagger, A_2^\dagger] = 0\) and that \([A_2, A_1^\dagger] = \gamma 1\). In order to be able to invert (2.3), we also impose that

\[
\alpha_x \beta_y - \alpha_y \beta_x \neq 0.
\]

(2.6)

This will be useful in a moment². Let us now proceed as usual: we define

\[
\Phi_{n_1,n_2} := \frac{1}{\sqrt{n_1!n_2!}} (A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\Phi_{0,0},
\]

(2.7)

for \( n_j \geq 0 \). Because of (2.6) it is possible to check that these vectors are linearly independent. This follows from the following fact: let us construct the vectors \( \varphi_{n_1,n_2} := \frac{1}{\sqrt{n_1!n_2!}} (a_x^\dagger)^{n_1}(a_y^\dagger)^{n_2}\Phi_{0,0} \).

The set \( \mathcal{F}^\varphi := \{ \varphi_{n_1,n_2}, n_j \geq 0 \} \) is an orthonormal (o.n.) basis of \( \mathcal{H} \). It is obvious that the linear span of \( \mathcal{F}_M^\Phi := \{ \Phi_{n_1,n_2}, n_1 + n_2 = M \} \) coincides with that of \( \{ \varphi_{n_1,n_2}, n_1 + n_2 = M \} \), for each \( M = 0, 1, 2, \ldots \). Therefore both these sets span the same finite dimensional Hilbert space, \( \mathcal{H}_M \), whose dimension is \( M + 1 \). Let us now introduce the operators \( N_j = A_j^\dagger A_j, j = 1, 2 \). Due to the unusual commutation rules, \( N_j \) is not a number operator. Indeed, using (2.5), we find

\[
N_1 \Phi_{n_1,n_2} = n_1 \Phi_{n_1,n_2} + \gamma \sqrt{(n_1 + 1)n_2}\Phi_{n_1+1,n_2-1}, \quad N_2 \Phi_{n_1,n_2} = n_2 \Phi_{n_1,n_2} + \gamma \sqrt{n_1(n_2 + 1)}\Phi_{n_1-1,n_2+1}.
\]

We see that two extra terms appear. However, two manifestly non self-adjoint number operators can be still introduced into the game. Indeed, let us assume that \( |\gamma| \neq 1 \). Then, if we introduce

\[
M_1 = \frac{1}{1 - |\gamma|^2} \left( N_1 - \gamma A_1^\dagger A_2 \right), \quad M_2 = \frac{1}{1 - |\gamma|^2} \left( N_2 - \gamma A_2^\dagger A_1 \right),
\]

(2.8)

we get

\[
M_1 \Phi_{n_1,n_2} = n_1 \Phi_{n_1,n_2}, \quad M_2 \Phi_{n_1,n_2} = n_2 \Phi_{n_1,n_2}.
\]

(2.9)

These two operators commute: \([M_1, M_2] = 0\), so that it is not surprising they admit a common set of eigenstates. However, since \( M_j^\dagger \neq M_j \), we do not expect, in principle, that the \( \Phi_{n_1,n_2} \) are mutually orthogonal. Nevertheless, it is easy to see that some orthogonality can be established. For that we introduce the following operator:

\[
H = M_1 + M_2 = \frac{1}{1 - |\gamma|^2} \left( N_1 + N_2 - \gamma A_1^\dagger A_2 - \gamma A_2^\dagger A_1 \right),
\]

(2.10)

²It has been shown in [15] that it is in fact possible to work with more general \( GL(n, \mathbb{C}) \) transformations to define the operators \( A_i \), at least for the aspects concerning construction of the biorthogonal bases.
which is clearly self-adjoint. Moreover $H\Phi_{n_1,n_2} = (n_1 + n_2)\Phi_{n_1,n_2}$. This means the following: whenever $n_1 + n_2 \neq k_1 + k_2$,
$$\langle \Phi_{n_1,n_2}, \Phi_{k_1,k_2} \rangle = 0.$$ 
In other words, taking $f \in \mathcal{H}_M$ and $g \in \mathcal{H}_L$, with $M \neq L$, then $\langle f, g \rangle = 0$. This is in agreement with what we have stated before, i.e. that the linear span of $\{\Phi_{n_1,n_2}, n_1 + n_2 = M\}$ coincides with that of $\{\varphi_{n_1,n_2}, n_1 + n_2 = M\}$, and with the fact that the different $\varphi_{n_1,n_2}$'s are mutually orthogonal.

Moreover, vectors $\Phi_{n_1,n_2}$ with the same value of $n_1 + n_2$ are not, in general, mutually orthogonal. For instance, assuming that $\|\Phi_{0,0}\| = 1$, we have
$$\langle \Phi_{1,0}, \Phi_{0,1} \rangle = \langle \Phi_{0,0}, \Phi_{1,0} \rangle = \langle \left[A_1^\dagger, A_2^\dagger + A_2^\dagger A_1\right] \Phi_{0,0} \rangle = \gamma. \quad (2.11)$$

Analogously, we easily deduce that $\langle \Phi_{2,0}, \Phi_{0,2} \rangle = \gamma^2$, while $\langle \Phi_{2,0}, \Phi_{1,1} \rangle = \langle \Phi_{1,1}, \Phi_{0,2} \rangle = \sqrt{2} \gamma$, and so on.

Being linearly independent, however, the $M + 1$ vectors of $\mathcal{F}_M^\Phi$ form a basis of the $M + 1$-dimensional space $\mathcal{H}_M$. In what follows we will discuss how, extending what was done in [2], it is possible to construct a second family of vectors which is biorthogonal to $\mathcal{F}_M^\Phi$. These vectors are expected, among other things, to be eigenvectors of $M_1^\dagger$ and $M_2^\dagger$. We will return on this aspect later on.

Since the role of finite-dimensional Hilbert spaces will be quite important in what follows, we want to stress that finite-dimensional systems are, quite often, very important in quantum mechanics, and in non hermitian quantum mechanics as well, since they usually allow a reasonably simple comprehension of some aspects of the theory which, otherwise, would be hidden by the many (mathematical) technicalities which are intrinsic to infinite-dimensional systems. The recent literature on this subject is rather rich and a complete list would be too long. We just want to cite here some of those papers which are more relevant for us, [16].

**II.3 Relation with extended pseudo-fermions**

It is now convenient to rename the vectors of $\mathcal{H}_M$ as follows
$$h_0^{(M)} = \Phi_{M,0}, \quad h_1^{(M)} = \Phi_{M-1,1}, \ldots, \quad h_M^{(M)} = \Phi_{0,M},$$
and let $\mathcal{F}_M^{(h)}$ be the set of these $M + 1$ vectors: $\mathcal{F}_M^{(h)} = \{h_0^{(M)}, h_1^{(M)}, \ldots, h_M^{(M)}\}$. Of course, $\mathcal{F}_M^{(h)}$ coincides with the set $\mathcal{F}_M^\Phi$ introduced previously. For each fixed $M$ these vectors are linearly
independent for those choice of $\alpha$’s and $\beta$’s which satisfy (2.4) and (2.6). Moreover, vectors $h_i^{(M)}$ and $h_j^{(M')}$ are automatically orthogonal if $M \neq M'$.

Following and extending the original idea introduced in [2] we will show now how a rather general algebraic procedure, again related to some suitable deformation of the anticommutation relations and in particular to some generalized raising and lowering operators, can be introduced into the game in order to produce, in each finite-dimensional Hilbert space $\mathcal{H}_M$, a new family of vectors, $\mathcal{F}_M^{(e)} = \{e_0^{(M)}, e_1^{(M)}, \ldots, e_M^{(M)}\}$, which is biorthogonal to the vectors in $\mathcal{F}_M^{(h)}$ (see also, [15]) and how these can be used to construct some families of intertwining operators. For that, after considering briefly what happens in $\mathcal{H}_0$, we will show in detail how the procedure works in $\mathcal{H}_1$ and $\mathcal{H}_2$, and then we will discuss how to extend this procedure to higher values of $M$.

Needless to say, we are not claiming here that ours is the only procedure which produces two biorthogonal families in a finite-dimensional Hilbert space, or that this is crucially related to noncommutative quantum mechanics: what we are saying is that the procedure we are going to describe is natural and interesting both from a mathematical side and for possible physical applications.

First of all, it is obvious that the set $\mathcal{F}_0^{(e)}$ simply coincides with $\mathcal{F}_0^{(h)}$, except possibly for a normalization factor: indeed, if we define $e_0^{(0)} := \frac{1}{\|h_0^{(0)}\|^2} h_0^{(0)}$, then $\langle e_0^{(0)}, h_0^{(0)} \rangle = 1$. Both sets are bases in the 1-dimensional Hilbert space $\mathcal{H}_0$.

More interesting is the situation for $M = 1$. In this case we introduce the bounded operators $a_1$ and $b_1$ via their action on the basis $\mathcal{F}_1^{(h)}$:

\[ a_1 h_0^{(1)} := 0, \quad a_1 h_1^{(1)} := h_0^{(1)}, \quad \text{and} \quad b_1 h_0^{(1)} := h_1^{(1)}, \quad b_1 h_1^{(1)} := 0. \tag{2.12} \]

We see that $a_1$ and $b_1$ act as lowering and raising operators on $\mathcal{F}_1^{(h)}$. From this definition we deduce that

\[ a_1^2 = 0 \quad b_1^2 = 0, \quad \{a_1, b_1\} = \mathbb{I}_1, \tag{2.13} \]

where $\mathbb{I}_1$ is the identity operator on $\mathcal{H}_1$. These are exactly the pseudo-fermionic anti-commutation rules considered in [2], so that the same construction proposed there can be repeated here. The starting point is a non-zero vector, $e_0^{(1)}$, orthogonal to $h_1^{(1)}$. Such a vector surely exists, since $\dim(\mathcal{H}_1) = 2$. Moreover, it is always possible to choose its normalization in such a way $\langle e_0^{(1)}, h_0^{(1)} \rangle = 1$. It is easy to check that $e_0^{(1)}$ is the vacuum for $b_1^\dagger$: $b_1^\dagger e_0^{(1)} = 0$. In fact, taken a generic vector $f \in \mathcal{H}_1$ and recalling that $f$ can be written as $f = c_0 h_0^{(1)} + c_1 h_1^{(1)}$, for some complex $c_0$ and $c_1$, using (2.12) we deduce that $\langle f, b_1^\dagger e_0^{(1)} \rangle = c_0 \langle h_1^{(1)}, e_0^{(1)} \rangle = 0$. Then our claim follows from the arbitrariness of $f$. 

7
Let us now define the vector $e_1^{(1)} := a_1^* e_0^{(1)}$. Since $\langle e_1^{(1)}, h_0^{(1)} \rangle = \langle e_0^{(1)}, a_1 h_0^{(1)} \rangle = 0$ and $\langle e_1^{(1)}, h_1^{(1)} \rangle = \langle e_0^{(1)}, a_1 h_1^{(1)} \rangle = \langle e_0^{(1)}, h_0^{(1)} \rangle = 1$, we conclude that $F_1^{(e)} = \{e_0^{(1)}, e_1^{(1)}\}$ is biorthonormal to $F_1^{(h)}$. These two sets are respectively eigenstates of $N_1^\dagger = a_1^* b_1^\dagger$ and $N_1 = b_1 a_1$:

$$N_1 h_k^{(1)} = k h_k^{(1)}, \quad N_1^\dagger e_k^{(1)} = k e_k^{(1)},$$

for $k = 0, 1$. Moreover, they resolve the identity in $\mathcal{H}_1$:

$$\sum_{k=0}^1 |e_k^{(1)}\rangle \langle h_k^{(1)}| = \sum_{k=0}^1 |h_k^{(1)}\rangle \langle e_k^{(1)}| = \mathbb{1}_1.$$

We can also introduce two self-adjoint, positive and invertible operators

$$S_1^{(h)} = \sum_{k=0}^1 |h_k^{(1)}\rangle \langle h_k^{(1)}|, \quad S_1^{(e)} = \sum_{k=0}^1 |e_k^{(1)}\rangle \langle e_k^{(1)}|,$$

or, more explicitly,

$$S_1^{(h)} f = \sum_{k=0}^1 \langle h_k^{(1)}, f \rangle h_k^{(1)}, \quad S_1^{(e)} f = \sum_{k=0}^1 \langle e_k^{(1)}, f \rangle e_k^{(1)},$$

for each $f \in \mathcal{H}_1$. These operators are inverses of one another: $S_1^{(h)} = (S_1^{(e)})^{-1}$. Moreover, they map $F_1^{(e)}$ into $F_1^{(h)}$ and viceversa:

$$S_1^{(h)} e_k^{(1)} = h_k^{(1)}, \quad S_1^{(e)} h_k^{(1)} = e_k^{(1)},$$

$k = 0, 1$, and they satisfy the following intertwining relations:

$$S_1^{(e)} N_1 = N_1^\dagger S_1^{(e)}, \quad N_1 S_1^{(h)} = S_1^{(h)} N_1^\dagger.$$

There is something more: since they are positive operators, the square roots of $S_1^{(h)}$ and $S_1^{(e)}$ surely exist. Therefore, we can define

$$n_1 := \left( S_1^{(e)} \right)^{1/2} N_1 \left( S_1^{(h)} \right)^{1/2}, \quad c_k^{(1)} := \left( S_1^{(e)} \right)^{1/2} h_k^{(1)},$$

$k = 0, 1$. It is easy to check that $n_1$ is a self-adjoint operator on $\mathcal{H}_1$, and that $F_1^{(e)} = \{c_0^{(1)}, c_1^{(1)}\}$ is an o.n. basis of $\mathcal{H}_1$.

A similar procedure can be repeated also for $\mathcal{H}_2$. In this case, however, we lose the relations in (2.13), but we still maintain the main aspects of the functional structure. The starting point
is, as before, the basis $\mathcal{F}^{(h)}_2 = \{h_0^{(2)}, h_1^{(2)}, h_2^{(2)}\}$. In this case the raising and lowering operators, $b_2$ and $a_2$, are defined by an extended version of (2.12):

$$a_2 h_0^{(2)} := 0, \quad a_2 h_1^{(2)} := h_0^{(2)}, \quad a_2 h_2^{(2)} := \sqrt{2} h_1^{(2)}, \quad (2.14)$$

and

$$b_2 h_0^{(2)} := h_1^{(2)}, \quad b_2 h_1^{(2)} := \sqrt{2} h_2^{(2)}, \quad b_2 h_2^{(2)} := 0. \quad (2.15)$$

In this case $a_2^3 = b_2^3 = 0$, but $\{a_2, b_2\} \neq \mathbb{I}_2$. Nevertheless, if we define $N_2 = b_2 a_2$, we still get $N_2 h_k^{(2)} = k h_k^{(2)}$, $k = 0, 1, 2$, so that the vectors $h_k^{(2)}$ are eigenstates of a number-like operator.

The biorthogonal set $\mathcal{F}^{(e)}_2$ is now constructed extending the previous procedure: we begin considering a vector, $e_0^{(2)}$, which is orthogonal to both $h_1^{(2)}$ and $h_2^{(2)}$. This vector is unique up to a normalization, which we choose in such a way that $\langle e_0^{(2)}, h_0^{(2)} \rangle = 1$. We find that $b_2^\dagger e_0^{(2)} = 0$. Defining further $e_1^{(2)} := a_2^\dagger e_0^{(2)}$ and $e_2^{(2)} := \frac{1}{\sqrt{2}} a_2^\dagger e_1^{(2)}$, we get

$$\langle e_j^{(2)}, h_k^{(2)} \rangle = \delta_{j,k},$$

$j, k = 0, 1, 2$. Hence $\mathcal{F}^{(e)}_2$ into $\mathcal{F}^{(h)}_2$ are biorthogonal bases of $\mathcal{H}_2$. The vector $e_k^{(2)}$ is eigenstate of $N_2^\dagger$: $N_2^\dagger e_k^{(2)} = k e_k^{(2)}$, $k = 0, 1, 2$. This can be proved by using the lowering nature of $b_2^\dagger$ on $\mathcal{F}^{(e)}_2$.

In fact, other than $b_2^\dagger e_0^{(2)} = 0$, we can also check that $b_2^\dagger e_1^{(2)} = e_0^{(2)}$, and that $b_2^\dagger e_2^{(2)} = \sqrt{2} e_1^{(2)}$.

Two operators, $S_2^{(h)}$ and $S_2^{(e)}$, can be defined as before, and for these we can prove exactly analogous results as those for $S_1^{(h)}$ and $S_1^{(e)}$. Hence, what appears to be really relevant in this construction, is not really the anticommutation rule $\{a, b\} = \mathbb{I}$, but the definition of the raising and lowering operators. For this reason, we call these particles, generalized pseudo-fermions.

Other interesting generalizations of fermions could be found, for instance, in [17, 18] and in [19].

For generic $M$ we could repeat the same construction, starting from $\mathcal{F}^{(h)}_M$. The two operators $a_M$ and $b_M$, defined extending formulas (2.14) and (2.15), satisfy the following property: $a_M^{M+1} = b_M^{M+1} = 0$. As for the anti-commutator rule, we can write $\{a_M, b_M\} = \sum_{k=0}^{M} \alpha_k^{(M)} |e_k^{(M)}\rangle \langle h_k^{(M)}|$, where the coefficients can be easily found. For instance we have $\alpha_0^{(1)} = \alpha_1^{(1)} = 1$, $\alpha_0^{(2)} = 1$, $\alpha_1^{(2)} = 3$ and $\alpha_2^{(2)} = 2$, and yet $\alpha_0^{(3)} = 1$, $\alpha_1^{(3)} = 3$, $\alpha_2^{(3)} = 5$ and $\alpha_3^{(3)} = 3$. In matrix form we
have:

\[
\{a_M, b_M\} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 3 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 5 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 7 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 2M-1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & M
\end{pmatrix},
\]

which, for instance, taking \(M = 1\), gives back \(\{a_M, b_M\} = 1\): \(M = 1\) is the only choice which furnishes the identity in the right-hand side of \(\{a_M, b_M\}\). The vectors of \(F^h_M\) are eigenstates of \(N_M = b_M a_M\), while those of the biorthogonal set \(F^e_M = \{e^{(M)}_l, l = 0, 1, \ldots, M\}\), are eigenstates of its adjoint \(N^\dagger_M\), and we have \(\langle e^{(M)}_l, h^{(N)}_k \rangle = \delta_{l,k} \delta_{M,N}\).

### III An explicit realization

It is interesting now to see how the above operators and bases can be explicitly constructed. In particular, we will show that the main ingredient of the construction is provided by the overlaps between the different vectors in \(F^h_M\): these are, in turn, fixed by the commutation rules in (2.5) and by the definition of the vectors \(\Phi_{n_1,n_2}\), (2.7), as briefly shown, for instance, by formula (2.11): this is the way in which the non commutativity of the bosons came into the game. Once these coefficients are found, there is not a single way to chose the \(M + 1\)-dimensional vectors \(h^{(M)}_k\) which reproduce these overlaps. Different realizations, i.e. different choices of \(F^h_M\), are possible. However, once this first set is chosen, the biorthogonal set \(F^e_M\) is uniquely determined. Stated differently, we are changing here our point of view, focusing on the values of \(\langle \Phi_{n_1,n_2}, \Phi_{k_1,k_2} \rangle\), for \(n_1 + n_2 = k_1 + k_2\), rather than on the vectors \(\Phi_{n_1,n_2}\) themselves. Once these values are found, we will look for those finite-dimensional vectors \(h^{(M)}_k\) which produce, inside each \(H_M\), these particular overlaps. In practice, we will now represent each infinite-dimensional vector \(\Phi_{m_1,m_2}\) as a finite dimensional vector in \(H_{m_1+m_2}\).

Of course, \(H_0\) being a one-dimensional space, there is not much to say. In this case \(h^{(0)}_0 = e^{(0)}_0\), with a normalization chosen in such a way \(\langle h^{(0)}_0, e^{(0)}_0 \rangle = 1\), and \(a_0, b_0\) both annihilate these states.
Here the situation is more interesting. To avoid useless complications, from now on we consider $\gamma > 0$. As we have discussed above, in order to produce our biorthogonal set and the related operators, we just need to find two two-dimensional vectors $h^{(1)}_j$ which represent $\Phi_{k,l}$, $j = 0, 1$ and $k + l = 1$: we simply write $h^{(1)}_0 = \Phi_{1,0}$ and $h^{(1)}_1 = \Phi_{0,1}$, and we ask for $\langle h^{(1)}_0, h^{(1)}_1 \rangle = \gamma$, see (2.11). A possible choice (clearly highly non-unique!) of these vectors is the following:

$$h^{(1)}_0 = \sqrt{\gamma} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h^{(1)}_1 = \sqrt{\gamma} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Then, the two two-by-two matrices $a_1$ and $b_1$ which satisfy the raising and lowering identities given in (2.12) are the following:

$$a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$  

It is clear that $a_1^2 = b_1^2 = 0$, and that $\{a_1, b_1\} = \mathbb{1}_1$. Moreover $N_1 = b_1 a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, which is not self-adjoint. But $N_1$ is still a number-like operator, since $N_1 h^{(1)}_k = k h^{(1)}_k$, $k = 0, 1$. Let us now construct the biorthonormal basis $F^{(e)}_1$. The first step consists in finding the kernel of $b_1^\dagger$. This is one dimensional and it is proportional to the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence, in order to have $\langle h^{(1)}_0, e^{(1)}_0 \rangle = 1$, we take $e^{(1)}_0 = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $e^{(1)}_1 = a_1^\dagger e^{(1)}_0 = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is now an easy exercise to check the resolutions of the identity $\sum_{k=0}^1 |h^{(1)}_k\rangle \langle h^{(1)}_k| = \sum_{k=0}^1 |e^{(1)}_k\rangle \langle e^{(1)}_k| h^{(1)}_k| = \mathbb{1}_1$, and to deduce the matrix form of the intertwining operators

$$S^{(h)}_1 = \sum_{k=0}^1 |h^{(1)}_k\rangle \langle h^{(1)}_k| = \gamma \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$S^{(e)}_1 = \sum_{k=0}^1 |e^{(1)}_k\rangle \langle e^{(1)}_k| = \frac{1}{\gamma} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$  

For these matrices our formulas can be easily checked. For instance they are both self-adjoint, one is the inverse of the other, and $S^{(h)}_1 e^{(1)}_k = h^{(1)}_k$, $k = 0, 1$. Also, $S^{(e)}_1 N_1 = N_1^\dagger S^{(e)}_1$. Taking
one of the square roots of \( S_1^{(e)} \) as,

\[
\left( S_1^{(e)} \right)^{1/2} = \frac{1}{\sqrt{5} \gamma} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix},
\]

and its inverse, it is possible to define a new self-adjoint number operator,

\[
n_1 = \left( S_1^{(e)} \right)^{1/2} N_1 \left( S_1^{(e)} \right)^{-1/2} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},
\]

and its eigenvectors \( c_0^{(1)} = \left( S_1^{(e)} \right)^{1/2} h_0^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \), and \( c_1^{(1)} = \left( S_1^{(e)} \right)^{1/2} h_1^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \),

which, as we can see, is an o.n. basis in \( \mathcal{H}_1 \). Incidentally we observe that these vectors are different from the canonical basis in \( \mathbb{C}^2 \).

### III.2 \( M = 2 \)

Similar computations can be performed also for \( \mathcal{H}_2 \). Again, the starting point is a fixed choice of three linearly independent vectors \( h_j^{(2)} \) which represent \( \Phi_{k,l}, j = 0, 1, 2 \) and \( k + l = 2 \): again we simply write \( h_0^{(2)} = \Phi_{2,0}, h_1^{(2)} = \Phi_{1,1} \) and \( h_2^{(2)} = \Phi_{0,2} \), and we ask that \( \langle h_0^{(2)}, h_1^{(2)} \rangle = \gamma \sqrt{2} \) and \( \langle h_0^{(2)}, h_2^{(2)} \rangle = \gamma^2 \), see Section II. Again, this choice is not unique. Here we consider the following:

\[
h_0^{(2)} = \gamma \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad h_1^{(2)} = \begin{pmatrix} 1 \\ \gamma/\sqrt{2} \\ 0 \end{pmatrix}, \quad h_2^{(2)} = \begin{pmatrix} \gamma/\sqrt{2} \\ 1 \\ 1 \end{pmatrix}.
\]

Hence the matrices \( a_2 \) and \( b_2 \) are uniquely found to be

\[
a_2 = \begin{pmatrix} 0 & 2 & \sqrt{2} - 2 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} \frac{1}{\sqrt{2} \gamma} & \frac{\sqrt{2}}{\gamma} (\gamma - \frac{1}{\sqrt{2} \gamma}) & -\frac{1}{2} - \sqrt{2} + \frac{1}{\gamma^2} \\ \frac{1}{\sqrt{2} \gamma} & (\sqrt{2} - \frac{1}{2}) & -\frac{\gamma}{2 \sqrt{2}} - \frac{\sqrt{2}}{\gamma} (\sqrt{2} - \frac{1}{2}) \end{pmatrix}.
\]

They satisfy, in particular, the condition \( a_2^3 = b_2^3 = 0 \). The form of the (non self-adjoint) number operator is now

\[
N_2 = b_2 a_2 = \begin{pmatrix} 0 & \frac{\sqrt{2}}{\gamma} & \frac{\sqrt{2} (\gamma^2 - 1)}{\gamma} \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.
\]
Then $N_2 h_k^{(2)} = k h_k^{(2)}$. In order to fix the vectors $e_k^{(2)}$ we start again looking for the kernel of $b_2^\dagger$. The normalization is fixed by requiring that $\langle e_0^{(2)}, h_0^{(2)} \rangle = 1$. Then we define $e_1^{(2)} = a_2^\dagger e_0^{(2)}$ and $e_2^{(2)} = \frac{1}{\sqrt{2}} a_2^\dagger e_1^{(2)}$:

$$e_0^{(2)} = \frac{1}{\gamma^2} \begin{pmatrix} \gamma^2 \\ -1 \\ 1 - \gamma^2 \end{pmatrix}, \quad e_1^{(2)} = \frac{\sqrt{2}}{\gamma} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad e_2^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Hence $\langle e_k^{(2)}, h_n^{(2)} \rangle = \delta_{k,n}$. It is easy to recover the resolution of the identity in $H_2$, and to deduce the expressions for the operators $S_2^{(h)}$ and $S_2^{(e)}$:

$$S_2^{(h)} = \frac{1}{\gamma^2} \begin{pmatrix} 1 + \frac{5}{2} \gamma^2 & \sqrt{2} \gamma & \gamma \\ \sqrt{2} \gamma & 1 + \gamma^2 & 1 \\ \gamma & 1 & 1 \end{pmatrix},$$

and

$$S_2^{(e)} = \begin{pmatrix} \frac{1}{\gamma^2} & -\frac{1}{\sqrt{2} \gamma^3} & \frac{1}{\sqrt{2} \gamma^3} \left( 1 - \frac{\gamma^2}{2} \right) \\ -\frac{1}{\sqrt{2} \gamma^3} & \frac{1}{\gamma} \left( 2 + \frac{1}{\gamma^2} \right) & -\frac{1}{\gamma} \left( \frac{3}{2} + \frac{1}{\gamma^2} \right) \\ \frac{1}{\sqrt{2} \gamma^3} \left( 1 - \frac{\gamma^2}{2} \right) & -\frac{1}{\gamma} \left( \frac{3}{2} + \frac{1}{\gamma^2} \right) & \frac{1}{\gamma} + \frac{1}{\gamma^2} + \frac{5}{4} \end{pmatrix}.$$ 

Even for these matrices it is possible to check all the properties found previously. For instance, they are one the inverse of the other, and they intertwine between $N_2$ and $N_2^\dagger$. The analytic expression for (one of) their square roots is more complicated, and will not be given here.

Extending what we have discussed so far, we can conclude that in any finite dimensional Hilbert space $\hat{H}$, given a non-orthogonal basis, it is possible introduce a pair of generalized pseudo-fermionic operators $a$ and $b$ which give rise, looking at the kernel of $b^\dagger$ and acting on this vector with powers of $a^\dagger$ (with suitable normalization factors), to another basis of $\hat{H}$ which is biorthonormal to the first one. Both bases are eigenstates of two non self-adjoint number-like operators. Moreover, interesting intertwining relations between these two operators can be deduced.

It might be worth noticing that, while the existence and the construction of biorthogonal bases are widely discussed in the mathematical literature, the procedure described here has many aspects which, in our knowledge, have not been considered before. In particular, relations with self-adjoint or crypto-hermitian number-like operators and the existence of some intertwining relations are, in our opinion, peculiar of the present construction.
IV Back to $\mathcal{H}$

In each $\mathcal{H}_M$, due to the fact that finite matrices are always bounded operators, it is clear that each set $\mathcal{F}^{(h)}_M$ or $\mathcal{F}^{(e)}_M$ are Riesz bases. This is true for each fixed value of $M$, but appears not to be necessarily so when we go back to $\mathcal{H}$, since $\mathcal{H} = \bigoplus_{M=0}^{\infty} \mathcal{H}_M$. Therefore, even if it is reasonable to expect that $\mathcal{F}^{(h)} = \{h_k^{(M)}, M \geq 0, k = 0, 1, 2, \ldots, M\}$ and $\mathcal{F}^{(e)} = \{e_k^{(M)}, M \geq 0, k = 0, 1, 2, \ldots, M\}$ are indeed (biorthogonal), but not necessarily Riesz, bases for $\mathcal{H}$, this remains an open problem. However, supposing that this is indeed the case, using the mutual orthogonality between different $\mathcal{H}_M$, we can write

$$\mathbb{I} = \sum_{M=0}^{\infty} \mathbb{I}_M = \sum_{M=0}^{\infty} \sum_{l=0}^{M} |e_l^{(M)}\rangle\langle h_l^{(M)}| = \sum_{M=0}^{\infty} P_M,$$

where $P_M := \sum_{l=0}^{M} |e_l^{(M)}\rangle\langle h_l^{(M)}| = \mathbb{I}_M$ is an orthogonal projection on $\mathcal{H}_M$: $P_M = P_M^2 = P_M^\dagger$.

Moreover, if $M \neq N$, $P_M P_N = 0$. Now we define

$$A = \sum_{M=0}^{\infty} P_M a_M P_M, \quad B = \sum_{M=0}^{\infty} P_M b_M P_M. \quad (4.1)$$

These operators are not defined on the whole $\mathcal{H}$, but on the domains

$$D(A) = \left\{ f \in \mathcal{H} : \left\{ f_N^A := \sum_{M=1}^{N} \sum_{l=1}^{M} \langle e_l^{(M)}, f \rangle \sqrt{l} h_l^{(M)} \right\} \text{ is a Cauchy sequence in } \mathcal{H} \right\},$$

while

$$D(B) = \left\{ f \in \mathcal{H} : \left\{ f_N^B := \sum_{M=1}^{N} \sum_{l=0}^{M-1} \langle e_l^{(M)}, f \rangle \sqrt{l+1} h_l^{(M)} \right\} \text{ is a Cauchy sequence in } \mathcal{H} \right\}.$$

Then, $\forall f \in D(A)$ and $\forall g \in D(B)$, we find

$$Af = \sum_{M=1}^{\infty} \sum_{l=1}^{M} \langle e_l^{(M)}, f \rangle \sqrt{l} h_l^{(M)}^{(M)}, \quad Bg = \sum_{M=1}^{\infty} \sum_{l=0}^{M-1} \langle e_l^{(M)}, g \rangle \sqrt{l+1} h_l^{(M)}^{(M)},$$

which are convergent by construction. The above domains are dense in $\mathcal{H}$, since they both contain $\mathcal{F}^{(h)}$, which, as stated, is a basis for $\mathcal{H}$. In particular, recalling that $k = 0, 1, 2, \ldots, M$, we find that

$$Ah_k^{(M)} = \begin{cases} 0, & \text{if } k = 0, \forall M \\ \sqrt{k} h_k^{(M)}^{(M)}, & \text{if } M \geq 1, \forall k = 1, 2, \ldots, M, \end{cases}$$
while
\[ B h_k^{(M)} = \begin{cases} 0, & \text{if } k = M, \forall M \\ \sqrt{k+1} h_{k+1}^{(M)}, & \text{otherwise.} \end{cases} \]

We can now interpret the operator \( N = BA \) as a (partial) number operator on \( \mathcal{H} \), since \( Nh_k^{(M)} = kh_k^{(M)} \) for all \( k \) and \( M \). Of course, \( D(N) \) is the following proper subspace of \( \mathcal{H} \):
\[ D(N) = \{ f \in D(A) : Af \in D(B) \}. \]

The operators \( A^\dagger \) and \( B^\dagger \) are densely defined, since \( D(A^\dagger) \) and \( D(B^\dagger) \) both contain \( \mathcal{F}^{(e)} \), see below, which is also a basis for \( \mathcal{H} \). In particular, for our present purposes, it is enough to notice that
\[
A^\dagger e_k^{(M)} = \begin{cases} 0, & \text{if } k = M, \forall M \\ \frac{1}{\sqrt{k+1}} e_{k+1}^{(M)}, & \text{if } k = 0, 1, \ldots, M-1, \end{cases}
\]

while
\[
B^\dagger e_k^{(M)} = \begin{cases} 0, & \text{if } k = 0, \forall M \\ \frac{1}{\sqrt{k}} e_{k-1}^{(M)}, & \text{otherwise.} \end{cases}
\]

In fact, let us first introduce \( D(A^\dagger) = \{ g \in \mathcal{H} \text{ such that } \exists g_A \in \mathcal{H} : \langle g_A, f \rangle = \langle g, Af \rangle, \forall f \in D(A) \} \). Then, \( \forall g \in D(A^\dagger) \), \( A^\dagger \) is defined as \( A^\dagger g := g_A \). Now, taken \( f \in D(A) \), it is easy to check that \( e_k^{(M)} \in D(A^\dagger) \) and that the first formula above holds. For that we observe that, because of the above expansion for \( Af \),
\[
\langle A^\dagger e_k^{(M)}, f \rangle = \langle e_k^{(M)}, Af \rangle = \sum_{N=1}^{\infty} \sum_{l=1}^{N} \langle e_l^{(N)}, f \rangle \sqrt{l} \langle e_k^{(M)}, h_{l-1}^{(N)} \rangle = \]
\[
= \sum_{N=1}^{\infty} \sum_{l=1}^{N} \langle e_l^{(N)}, f \rangle \sqrt{l} \delta_{N,M} \delta_{l-1,k} = \langle \sqrt{k+1} e_{k+1}^{(M)}, f \rangle,
\]

where we have considered here \( k \) ranging between 0 and \( M-1 \). Our conclusion follows from the arbitrariness of \( f \). The case \( k = M \) is even simpler. With similar computations we could also check the formula for \( B^\dagger e_k^{(M)} \).

Now, calling \( N^2 = A^\dagger B^\dagger \), we see that \( N^2 e_k^{(M)} = k e_k^{(M)} \). Of course, \( D(N^2) \) is a proper subspace of \( \mathcal{H} \), defined in analogy with \( D(N) \). It is now natural to introduce two operators acting on \( \mathcal{F}^{(e)} \) and \( \mathcal{F}^{(h)} \) in the following way:
\[
S^{(h)} e_k^{(M)} = h_k^{(M)}, \quad S^{(e)} h_k^{(M)} = e_k^{(M)}, \quad (4.2)
\]

\(^3\)We refer to [20] for all the mathematical subtleties which are related to the unbounded nature of the operators introduced in this section, and which are not extremely important for us here, since we are essentially interested in what happens in each \( \mathcal{H}_M \).
and then extend these definitions to finite linear combinations of these vectors. It is clear that, in each subspace $\mathcal{H}_M$, these operators coincide with those considered in the previous sections:

$$S^{(h)}|_{\mathcal{H}_M} = S^{(h)}_M, \quad S^{(e)}|_{\mathcal{H}_M} = S^{(e)}_M.$$ 

Hence they are bounded on each subspace. However, this does not imply that they are also globally bounded, i.e. bounded on $\mathcal{H}$. Exactly for this reason, [6], the intertwining relation should also be considered with a certain care: in fact, condition $S^{(e)}_M N_M = N_M^\dagger S^{(e)}_M$, for each fixed $M \in \mathbb{N}$, does not imply also that $S^{(e)} N = N^\dagger S^{(e)}$. We can only conclude that $(S^{(e)} N - N^\dagger S^{(e)}) h_k^{(M)} = 0$, for all $k$ and $M$. However, in the present situation, this is not enough to recover the operatorial intertwining relation, [21]. This is due to the fact that the operators involved are unbounded. Notice also that they can be represented as diagonal block matrices, the $M$–th block being a matrix acting on $\mathcal{H}_M$. Each block represents a bounded operator, but the infinite matrix is not necessarily bounded.

V Coherent states

In the literature there does not exist full agreement on what a (generalized) coherent state should be: for some authors some properties are more important than others, and this leads to very different generalizations, [22]. Here, we will concentrate on a particular aspect of these states, which is essential for quantization, and thus insist on a resolution of the identity, in an appropriate sense. This is the point of view of, say, [23], where the authors are not so much interested in whether these states are eigenstates of some lowering operator or not.

The idea is quite simple: let $X$ be some subset of $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$, equipped with a measure $\mu$ and let $L^2(X, d\mu)$ the set of all the square-integrable Lebesgue-measurable functions on $X$, with scalar product

$$\langle f, g \rangle_{L^2} := \int_X f(x) g(x) d\mu(x).$$

Let us select, in $L^2(X, d\mu)$, a family of functions $\{\Phi_n(x), x \in X : n = 0, 1, \ldots, N - 1\}$, where $N$ could be finite or not. We require that $\langle \Phi_n, \Phi_m \rangle_{L^2} = \delta_{n,m}$, and that $N(x) := \sum_{n=0}^{N-1} |\Phi_n(x)|^2$ be strictly positive and finite, almost everywhere (a.e.) in $X$: $0 < N(x) < \infty$, a.e. in $X$. Of course, boundedness of $N(x)$ is guaranteed whenever $N < \infty$, otherwise it is not. Let now $\mathcal{H}$ be an $N$-dimensional Hilbert space, with scalar product $\langle ., . \rangle_{\mathcal{H}}$, and with o.n. basis $\mathcal{C} = \{c_n, n = 0, 1, 2, \ldots, N - 1\}$: $\langle c_n, c_m \rangle_{\mathcal{H}} = \delta_{n,m}$. In [23] it is proved that the vector $f_x := \frac{1}{\sqrt{N(x)}} \sum_{n=0}^{N-1} \Phi_n(x) c_n$, $x \in X$, has the following properties: (i) $\langle f_x, f_x \rangle_{\mathcal{H}} = 1$, $\forall x \in X$; (ii) it
satisfies the following resolution of the identity:
\[
\int_X |f_x\rangle\langle f_x| N(x) \, d\mu(x) = \mathbb{1}_N,
\]
where $\mathbb{1}_N$ is the identity operator in $\mathcal{H}_N$. This suggests that the set $\{f_x, x \in X\}$ can be efficiently used to quantize a given classical system, by going from a classical function $h(x)$ to its upper symbol $A_h := \int_X h(x) |f_x\rangle\langle f_x| N(x) \, d\mu(x)$.

In view of what we have seen in the previous sections, it may be interesting to see if a similar strategy can be extended to the case when, rather than an o.n. basis, we have two biorthogonal bases. Notice also that we are mainly interested in considering finite-dimensional Hilbert spaces. As a matter of fact, this extension is quite straightforward: let $\mathcal{L}^2(X, d\mu)$ be as above, and let $\mathcal{F}_N^{(e)} := \{e_n, n = 0, 1, 2, \ldots, N - 1\}$ and $\mathcal{F}_N^{(h)} := \{h_n, n = 0, 1, 2, \ldots, N - 1\}$ be two biorthogonal sets of the $N$-dimensional Hilbert space $\mathcal{H}_N$. Both sets are automatically bases for $\mathcal{H}_N$ and, in so far as $N < \infty$, they are biorthogonal Riesz bases: $\langle e_n, h_m\rangle_{\mathcal{H}} = \delta_{n,m}$ and they are the images, by means of a bounded operator $T$ with bounded inverse, of a certain o.n. basis of $\mathcal{H}_N$, $\mathcal{C} = \{c_n, n = 0, 1, 2, \ldots, N - 1\}$, $\langle c_n, c_m\rangle_{\mathcal{H}} = \delta_{n,m}$: $e_n = T c_n$ and $h_n = T^{-1} c_n$, $\forall n$. For symmetry reasons, it is natural to consider two biorthogonal sets $\{\Phi_n(x), x \in X : n = 0, 1, \ldots, N - 1\}$ and $\{\Psi_n(x), x \in X : n = 0, 1, \ldots, N - 1\}$ in $\mathcal{L}^2(X, d\mu)$, satisfying the following:
\[
0 < \tilde{N}(x) := \sum_{n=0}^{N-1} \Phi_n(x)\Psi_n(x) < \infty,
\]
a.e. in $X$. This apparently strange formula reduces to the previous one if $\Phi_n(x) = \Psi_n(x)$ for all $n = 0, 1, 2, \ldots, N - 1$, or when these functions differ by some exponential term, from an o.n. set $\{\varphi_n(x), x \in X : n = 0, 1, \ldots, N - 1\}$: $\Phi_n(x) = e^{\alpha_n(x)} \varphi_n(x)$ and $\Psi_n(x) = e^{-\alpha_n(x)} \varphi_n(x)$ for all $n = 0, 1, 2, \ldots, N - 1$, for rather general real functions $\alpha_n(x)$.

The next step is to introduce two families of coherent-like states,
\[
e(x) = \frac{1}{\sqrt{\tilde{N}(x)}} \sum_{n=0}^{N-1} \Phi_n(x) e_n, \quad h(x) = \frac{1}{\sqrt{\tilde{N}(x)}} \sum_{n=0}^{N-1} \Psi_n(x) h_n,
\]
extending what we have seen before. It is evident that $\langle e(x), h(x)\rangle_{\mathcal{H}} = 1$ for a.a. $x \in X$, and that
\[
\int_X |e(x)\rangle\langle h(x)| \tilde{N}(x) \, d\mu(x) = \mathbb{1}_N.
\]

\[\text{Of course the main requirement is that these } \alpha_n(x)\text{ must be such that } \Phi_n(x), \Psi_n(x)\text{ and } \varphi_n(x)\text{ all belong to } \mathcal{L}^2(X, d\mu).\]
For this reason, these states are called bi-coherent states and, of course, they could be used to quantize each classical function, producing its upper symbol. Of course, if we were interested in producing eigenstates of some lowering operator, the situation changes drastically, since grassmann or paragrassmann variables are needed, [2] and [23], with many extra difficulties. However, this is not our main interest here, and for this reason we are not considering this problem in this paper.

This procedure can be applied in each finite-dimensional Hilbert space of the type considered in Sections III. It turns out that in each $\mathcal{H}_M$ we can naturally introduce two sets of bi-coherent states, related by the operators $S_M^{(h)}$ and $S_M^{(e)}$. We could also extend, at least formally, most of the construction to $\mathcal{H}$ itself. In this case, we very luckily lose the boundedness of the intertwining operators, and a special (mathematical) care would be needed. Again, this is not our main interested here, and the analysis of this problem is postponed to a future paper.

VI Conclusions

We have shown how a certain noncommuting quantum mechanical system produces, in quite a natural way, a sequence of finite dimensional subspaces of $L^2(\mathbb{R}^2)$ in which biorthogonal Riesz bases can be constructed explicitly, which are eigenstates of two families of non self-adjoint operators and of their adjoints. They produce, under suitable conditions, two biorthogonal bases of $L^2(\mathbb{R}^2)$ which are not, in general, Riesz bases. This is done by working in a sequence of mutually orthogonal, finite dimensional, Hilbert spaces, into which $L^2(\mathbb{R}^2)$ can be decomposed. Intertwining operators arising from this structure are also explicitly constructed. Examples in finite dimensional vector spaces are discussed.

We have also shown how bicoherent states can be introduced in this settings, and we have discussed some of their properties.

Acknowledgements

The authors would like to acknowledge financial support from the Università di Palermo through Bando CORI, cap. B.U. 9.3.0002.0001.0001. One of us (STA) would like to acknowledge a grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada. The authors also thank the referee for his very useful remarks.
References

[1] F. Bagarello, *Pseudo-bosons, so far*, Rep. Math. Phys., 68, No. 2, 175-210, 2011

[2] F. Bagarello, *Non linear Pseudo-fermions*, J. Phys. A, 45, 444002, 2012; F. Bagarello, *Damping and Pseudo-fermions*, J. Math. Phys., 54, 023509, (2013)

[3] F. Bagarello, *A note on the Pais-Uhlenbeck model and its coherent states*, Int. J. Theor. Phys., 50, Issue 10, Page 3241-3250, 2011

[4] F. Bagarello, *Dissipation evidence for the quantum damped harmonic oscillator*, Theor. Math. Phys, 171, Page 497-504, 2012

[5] F. Bagarello and M. Znojil, *Non linear pseudo-bosons versus hidden Hermiticity*, J. Phys. A, 44 415305, 2011

[6] F. Bagarello and M. Znojil, *Non linear pseudo-bosons versus hidden Hermiticity. II: The case of unbounded operators*, J. Phys. A, 45, 115311, 2012

[7] F. Bagarello, *Examples of Pseudo-bosons in quantum mechanics*, Phys. Lett. A, 374, 3823-3827, 2010

[8] S.T. Ali, F. Bagarello, and J.-P. Gazeau, *Modified Landau levels, damped harmonic oscillator and two-dimensional pseudo-bosons*, J. Math. Phys., 51, 123502, 2010

[9] S.Doplicher, K.Fredenhagen, and J.E.Roberts, Comm. Math. Phys., *The quantum structure of spacetime at the Planck scale and quantum fields*, 172,187-220, 1995

[10] C. M. Rohwer, K. G. Zloshchastiev, L Gouba, and F. G. Scholtz, *Noncommutative quantum mechanicsa perspective on structure and spatial extent*, J. Phys. A: Math. Theor., 43, 345302, 2010

[11] J. P. Gazeau, M. Baldiotti, and D.M. Gitman, *Semiclassical and quantum description of motion on the noncommutative plane*, Phys. Lett. A 373, 3937-3943, 2009

[12] J. P. Gazeau and F. H. Szafraniec, *Holomorphic Hermite polynomials and non-commutative plane*, J. Phys. A: Math. Theor. 44, 495201-1-13, (2011)

[13] S.H.H. Chowdhury and S.T. Ali, *The symmetry groups of noncommutative quantum mechanics and coherent state quantization*, J. Math. Phys. 54, 032101-1-21, (2013); doi: 10.1063/1.4793992
[14] P.A. Horváthy, L. Martina and P.C. Stichel, *Exotic Galilean symmetry and noncommutative mechanics*, Symmetry, Integrability and Geometry, Methods and Applications (SIGMA), 6, 000 26 pp (2010).

[15] F. Balogh, N.M. Shah and S.T. Ali, *Deformed raising-lowering operators and generalized Hermite polynomials in two variables*, Concordia University preprint, Nov. 2012.

[16] C. Bender, *Making Sense of Non-Hermitian Hamiltonians*, Rep. Progr. Phys., 70, 947-1018, 2007. O. Cherbal, M. Drir, M. Maamache, D. A. Trifonov, *Fermionic coherent states for pseudo-Hermitian two-level systems*, J. Phys. A, 40, 1835-1844, 2007. C. M. Bender, S. P. Klevansky, *PT-symmetric representations of fermionic algebras*, Phys. Rev. A, 84, 024102, 2011. Y. Ben-Aryeh, A. Mann, I. Yaakov, *Rabi oscillations in a two-level atomic system with a pseudo-hermitian hamiltonian*, J. Phys. A, 37, 12059-12066, 2004. A. Mostafazadeh, *Pseudo-Hermitian representation of Quantum Mechanics*, Int. J. Geom. Methods Mod. Phys. 7, 1191-1306, 2010. M. Znojil, *Discrete quantum square well of the first kind*, Phys. Lett. A 375, 2503-2509, 2011. J. Schindler, Z. Lin, J. M. Lee, H. Ramezani, F. M. Ellis and T. Kottos, PT-symmetric electronics, J. Phys. A: Math. Theor., 45, 444029, 2012. H. Ramezani, J. Schindler, F. M. Ellis, Uwe Güther, T. Kottos, *Bypassing the bandwidth theorem with PT symmetry*, Phys. Rev. A, 85, 062122, 2012.

[17] D. A. Trifonov, *Nonlinear fermions and coherent states*, J. Phys. A: Math. Theor., 45, 244037, 2012

[18] D. A. Trifonov, *Nonlinear n-pseudo fermions*, J. Phys. A: Math. Theor., in press

[19] A. K. Mishra, G. Rajasekaran, *Algebra for fermions with a new exclusion principle*, Pramana-J. Phys. 36, 537-555, 1991

[20] F. Bagarello, A. Inoue, C Trapani, *Weak commutation relations of unbounded operators: nonlinear extensions*, J. Math. Phys., 53, 123510, (2012)

[21] P. Halmos, *An Hilbert space problem book*, Springer-Verlag, New York, 1982

[22] J. P. Gazeau, *Coherent States in Quantum Physics*, Wiley-VCH, Berlin 2009

[23] M. El Baz, R. Fresneda, J. P. Gazeau, and Y. Hassouni, *Coherent states quantization of paragrassmann algebras*, J. Phys. A: Math. and Theor., 43, 385202, 2010. and M. El Baz, R. Fresneda, J. P. Gazeau and Y. Hassouni, *Corrigendum: Coherent states quantization of paragrassmann algebras*, J. Phys. A: Math. and Theor., 45, 079501, 2012

20
[24] F. Bagarello, *More mathematics for pseudo-bosons*, Journ. Math. Phys., submitted.