Stochastic Quantization of the Hořava Gravity

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The stochastic quantization method is applied to the recent proposal by Hořava for gravity. We show that in contrast to General Relativity, the Hořava’s action, satisfying the detailed balance condition, has a stable, non-perturbative quantum vacuum when the DeWitt parameter $\lambda$ is not greater than 1/3, providing a possible candidate for consistent quantum gravity.

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\textbf{Introduction.} The goal of formulating a consistent and renormalizable quantum theory of gravity has been pursued for more than half century. Attempts of quantizing General Relativity (Einstein’s theory of gravitation) have met tremendous difficulties. On one hand, the canonical quantization is shown to be perturbatively non-renormalizable in four dimensions\textsuperscript{[1,2]} and, therefore, loses predictability, because the Einstein-Hilbert action is non-polynomial. On the other hand, the Euclidean path integral approach suffers the indefiniteness problem\textsuperscript{[3]}: Namely the Einstein-Hilbert action is not positive-definite, because conformal transformations can make the action arbitrarily negative. In this letter we apply stochastic quantization to the Hořava gravity, where the gauge symmetry is spatial diffeomorphisms. We will show that the quantized theory to the Hořava gravity, where the gauge symmetry is spatial diffeomorphisms. We will show that the quantized theory does not lead to a stable vacuum (ground state). See below.)

A recent effort attempting to overcome these difficulties is the proposal made by Hořava\textsuperscript{[4]}. (For the ideas that led to this proposal, see also refs. \textsuperscript{[5,6]}. This proposal is a non-Lorentz invariant theory of gravity in 3+1 dimensions, inspired by the Lifshitz model\textsuperscript{[7]} studied in condensed matter physics. At the microscopic (ultraviolet) level this model exhibits anisotropic scaling between space and time, with the dynamical critical exponent $z$ set equal to 3. (Namely, the action is invariant under the scaling $x^i \rightarrow bx^i (i=1,2,3), t \rightarrow b^z t$, where $z \neq 1$ violates the Lorentz symmetry.) The action is assumed to satisfy the so-called detailed balance condition, and is renormalizable by power counting. It is argued that the renormalization group flow in the model approaches an infrared (IR) fixed point theory with $z=1$, thus Einstein’s General Relativity (with local Lorentz symmetry) is naturally emergent or recovered at the macroscopic level. It is this perspective that has enabled the proposal to attract a lot of interests in recent literature. Many papers have appeared to study the classical solutions or consequences of the Hořava’s proposal (e.g. see refs. \textsuperscript{[8,9,10]}). A number of fundamental questions remained unanswered. In this letter we report a study of the most fundamental questions on Hořava gravity: whether the action can really be quantized in a consistent and non-perturbative manner? If yes, whether this will put any constraint(s) on the parameters appearing in the action or not? (A recent paper\textsuperscript{[11]} on the renormalizability of the model did not address these issues, assuming no problem with quantization.)

Among the three existing – canonical, path integral and stochastic – quantization approaches, only the last (stochastic quantization) is constructive through stochastic differential equation, so that the question of whether a stable vacuum (ground state) really exists or not can be easily investigated and answered. Also it has the great advantage\textsuperscript{[12]} of no need for gauge-fixing when applied to theories with gauge symmetry. In this letter we apply stochastic quantization to the Hořava gravity, where the gauge symmetry is spatial diffeomorphisms. We will show that the quantized theory with a stable vacuum indeed exists only when the parameter $\lambda$ in the DeWitt metric in the space of three-metrics is not greater than a critical value 1/3: (i.e. $\lambda \leq \lambda_c = 1/3$). This is the range of the values of $\lambda$ for which Hořava’s action may make sense for a consistent quantum theory of gravity. (In contrast, stochastic quantization of General Relativity does not lead to a stable vacuum (ground state). See below.)

\textbf{Preliminaries.} Assume the spacetime allows a $(3+1)$-decomposition:
\begin{equation}
\text{d}s_5^2 = -N^2 \text{d}t^2 + g_{ij}(\text{d}x^i - N^i \text{d}t)(\text{d}x^j - N^j \text{d}t),
\end{equation}
where $g_{ij}(i,j=1,2,3)$ is the 3-metric, $N$ and $N_i$ are the lapse function and shift vector, respectively. The Hořava action with $z=3$ is given by\textsuperscript{[4]}
\begin{equation}
S = \int \text{d}t \text{d}^3 x \sqrt{g} N \left[ \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl} + \frac{\kappa^2}{8} E_{ij} G_{ijkl} E^{kl} \right],
\end{equation}
where $g$ denotes the determinant of the 3-metric $g_{ij}$ and $\kappa^2$ is the coupling constant, to be identified with $32\pi G c$ in the IR regime with $z=1$ ($G$ and $c$ the Newton’s gravitational constant and the speed of light, respectively). The
extrinsic curvature $K_{ij}$ and the DeWitt metric $g^{ijkl}$ in (2) are defined by

$$K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i),$$

$$g^{ijkl} = \frac{1}{2} \left( g^{ik}g^{jl} + g^{il}g^{jk} \right) - \lambda g^{ij} g^{kl}$$

with $\lambda$ a free parameter. The potential term in (2), when $E^{ij}$ is given by $\sqrt{g}E^{ij} = \frac{\delta W}{\delta g^{ij}}$, is said to satisfy the so-called detailed balance condition. Hořava took $W$ to be

$$W = \frac{1}{2} \int \omega_3(\Gamma) + \mu \int d^3x \sqrt{g}(R - 2\Lambda_W).$$

Here $\mu, w$ and $\Lambda_W$ are coupling constants, and $\omega_3$ is the gravitational Chern-Simons term:

$$\omega_3 \equiv \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right),$$

with $\Gamma$ the Christoffel symbols. Simple dimensional analysis for the coupling constants shows that the theory is ultraviolet (UV) renormalizable. The renormalizability beyond the power counting of this theory has recently been confirmed in [11], assuming no problem with quantization. Here we will examine the more fundamental question of the non-perturbative existence of quantum vacuum.

**Stochastic Quantization.** Stochastic quantization [12] has been proved to be an effective tool for quantizing a field theory, in particular a gauge theory [13, 14]. Stochastic quantization is based on the principle that quantum dynamics of a $d$-dimensional system is equivalent to classical equilibrium statistical mechanics of a $(d + 1)$-dimensional system. The essence of stochastic quantization is to use a stochastic evolution – the Langevin equation – in fictitious time, driven by white noises, to construct the equilibrium state corresponding to the quantum ground state. The existence of an equilibrium state can be proved or disproved by studying the corresponding Fokker-Planck equation associated with the Langevin equation. In this spirit, we start with the Langevin equation of the Hořava gravity:

$$\begin{cases}
\dot{N} = -\frac{1}{\sqrt{g}} \frac{\delta S_E}{\delta N} + \eta^I,
\dot{N}_i = -\frac{1}{\sqrt{g}} \frac{\delta S_E}{\delta N_i} + \zeta_i,
\dot{g}^{ij} = -G^{Ij} \partial_I S_E + \xi^I,
\end{cases}$$

where the dot represents derivative with respect to the fictitious time $\tau$ and following notations have been introduced:

$$g_{ij} \equiv g^{ij}, \quad G^{Ij} \equiv G^{ijkl}, \quad \partial_I S_E \equiv \frac{1}{\sqrt{g}} \frac{\delta S_E}{\delta g^{ij}}.$$ 

In eq. (7), $\eta^I$, $\zeta_i$, and $\xi^I$ are noises, and $S_E$ is the Euclidean version of the action (2).

Note that the indices $I, J (=1, 2, \ldots, 6)$ are raised and lowered by $G^{Ij}$ and its inverse $G_{Ij}$. The stochastic correlation of a gauge invariant functional $F(N, N_i, g_I)$ is defined as the expectation value of the functional with respect to the noises

$$\langle F(N, N_i, g_I) \rangle \sim \int D[\eta] D[\zeta] D[\xi] F(N, N_i, g_I) \exp \left[ -\frac{1}{4} \int dt d^3x d\tau \sqrt{g}N \left( \eta^2 + g^{ij} \zeta_i \zeta_j + G^{Ij} \xi_I \xi_J \right) \right]$$

where $g^{ij}$ and $G^{Ij}$ are solutions of the Langevin equation (7) and hence are functions of $\zeta^I$ and $\xi^I$, respectively. The Wick rotation to imaginary time $\tau$ has been applied and $\tau$ is the fictitious time. Eq. (8) indicates that the noises $\zeta_i$ and $\xi_I$ are not Gaussian. As suggested in [11], one can overcome this difficulty by introducing a set of new noises via vielbein. That is $\zeta^a \equiv e^a_i \zeta_i$, $\xi^A \equiv E^A_I \xi_I$, and its inverse $\zeta^i = e_i^a \zeta^a$, $\xi_I^a = E_I^A \xi_A$, where $e^a_i$ and $E_A^I$ are the vielbeins. The following relations hold

$$e_a^i e_b^j g_{ij} = \delta_{ab}, \quad E^A_I E_B^J G^{Ij} = \delta_{AB}, \quad e_a^i e_b^j \delta^{ab} = g^{ij}, \quad E^A_I E_B^J \delta^{AB} = G^{Ij}. $$

The new noises turn out to be Gaussian and we have

$$\langle \eta(x, \tau) \rangle = 0, \quad < \zeta^a(x, \tau) > = 0, \quad < \xi^A(x, \tau) > = 0,$$

$$\langle \eta(x, \tau) \eta(y, \tau') \rangle = 2 \delta(x - y) \delta(\tau - \tau'),$$

$$\langle \zeta^a(x, \tau) \zeta^b(y, \tau') \rangle = 2 \delta^{ab} \delta(x - y) \delta(\tau - \tau'),$$

$$\langle \xi^A(x, \tau) \xi^B(y, \tau') \rangle = 2 \delta^{AB} \delta(x - y) \delta(\tau - \tau').$$
negative. At the critical value $\lambda$ in this case: It follows from eq. (22) that
\[
\frac{\delta S_E}{\delta N_i} + \eta, \\
\frac{\delta S_E}{\delta N^i} + \zeta a^i, \\
g^I = -G^{IJ} \partial J S_E + \xi^A E_A^I,
\]
and the correlation functional is redefined with respect to $\eta, \zeta^a$ and $\xi^A$ by
\[
<F(N, N^i, g_I) > \sim \int D[\eta] D[\xi] D[\zeta] F(N, N^i, g_I) \exp \left[ -\frac{1}{2} \int dtd\delta x N(\eta^2 + \zeta^a \zeta_a + \xi^A \xi_A) \right],
\]
which is obviously Gaussian as desired.
To study whether the Langevin process (15) really converges to a stationary equilibrium distribution, we examine the probability density functional associated with it:
\[
P(N, N^i, g_I, \tau) = \frac{\exp \left[ -\frac{1}{2} \int dtd\delta x N(\eta^2 + \zeta^a \zeta_a + \xi^A \xi_A) \right]}{\int D[\eta] D[\xi] D[\zeta] \exp \left[ -\frac{1}{2} \int dtd\delta x N(\eta^2 + \zeta^a \zeta_a + \xi^A \xi_A) \right]}.
\]
We introduce
\[
Q(N, N^i, g_I, \tau) \equiv P(N, N^i, g_I, \tau)e^{S_E/2}.
\]
and the Fokker-Planck equation for the probability distribution is
\[
\frac{\partial Q(N, N^i, g_I, \tau)}{\partial \tau} = -\mathcal{H}_{FP} Q(N, N^i, g_I, \tau),
\]
where the Fokker-Planck Hamiltonian $\mathcal{H}_{FP}$ is of the form
\[
\mathcal{H}_{FP} = a^\dagger a + g^{ij} a^i a^j + G^{IJ} A_I^\dagger A_J.
\]
Here
\[
a = i\pi + \frac{1}{2} \frac{\delta S_E}{\delta N}, \quad a^i = i\pi^i + \frac{1}{2} \frac{\delta S_E}{\delta N^i}, \quad A_I^\dagger = i\pi^I + \frac{1}{2} g^{IJ} S_E,
\]
with $\pi, \pi^i$ and $\pi^I$, respectively, the conjugate momenta of $N, N^i$ and $g^{ij}$.

The time independent eigenvalue equation associated with Eq. (19) is
\[
\mathcal{H}_{FP} Q_n(N, N^i, g_I, \tau) = E_n Q_n(N, N^i, g_I, \tau).
\]
The solutions of Eq. (19) lead to the general solution
\[
P(N, N^i, g_I, \tau) = \sum_{n=0}^{\infty} a_n Q_n(N, N^i, g_I, \tau)e^{-S_E/2 - E_n \tau}.
\]
The stationary candidate for the equilibrium state is given by $Q_0(N, N^i, g_I) = e^{-S_E/2}$ with $E_0 = 0$. From the above formula we see that the theory will approach an equilibrium state for large $\tau$ if and only if all other $E_n > 0$. To find the condition(s) under which the Fokker-Planck Hamiltonian (20) is non-negative definite, we note that the sum of the first two terms $(a^\dagger a + g^{ij} a^i a^j)$ always respects this property. So we only need to find condition(s) under which the eigenvalues of the DeWitt metric $G^{IJ}$ are all non-negative. By a straightforward computation, the desired condition is found to be $\lambda \leq 1/3$: When $\lambda < 1/3$, $G^{IJ}$ is positive definite; if $\lambda > 1/3$, one and only one eigenvalue of $G^{IJ}$ becomes negative. At the critical value $\lambda = 1/3$, one eigenvalue of $G^{IJ}$ becomes zero, while all others remain positive.

Thus the Fokker-Planck Hamiltonian is non-negative definite if $\lambda \leq 1/3$, and the theory approaches an equilibrium in this case: It follows from eq. (22) that
\[
\lim_{\tau \to \infty} P(N, N^i, g_I, \tau) = a_0 e^{-S_E},
\]
where $a_0$ is determined by the normalization condition. Note that this result is independent of the initial conditions. Any equal-time correlation function (16), if invariant under spatial diffeomorphisms, tends to its equilibrium value
for large time $\tau$. Therefore, though the solution given by (23) is always a stationary state for the Fokker-Planck equation, it represents an equilibrium state (or a stable ground state) reached at large time $\tau$ only when $\lambda \leq 1/3$. In contrast, a similar result would not be obtained with stochastic quantization of Einstein’s gravity, which corresponds to $\lambda = 1 > 1/3$, since the associated Fokker-Planck Hamiltonian is not positive definite and hence does not lead to an equilibrium state at large fictitious times.

In the above derivation, the detailed balance condition is crucial for the Hořava gravity to have a stable vacuum when $\lambda < 1/3$. In fact, with the detailed balance condition satisfied at short distances, $S_E$ is of the form

$$S_E = \int \mathcal{G}^{IJ}(K_I K_J + \alpha E_I E_J),$$

where $\alpha > 0$ and $E_I = \partial_I W$ with $W$ given by (15). $S_E$ has a similar structure to eq. (20), so it is positive definite for $\lambda < 1/3$ and indefinite for $\lambda > 1/3$. As a consequence, the state (23) is a physical ground state for $\lambda < 1/3$ and is unstable for $\lambda > 1/3$.

We have seen that $\lambda_c = 1/3$ is a critical value for the theory: Above it the quantized theory does not make sense, while the opposite is true below it. Exactly at $\lambda = \lambda_c$, extra zero modes develop for the DeWitt metric $\mathcal{G}^{IJ}$ and, hence, for eq. (19) as well. This implies that the gauge symmetry of the theory is enhanced, which now includes local Weyl transformations as already observed in ref. [4]. It would be extremely interesting to understand the fate of the enhanced gauge symmetry in the quantized theory. Anyway, in principle stochastic quantization method should be applicable at $\lambda = 1/3$, and the appearance of extra zero modes does not destroy the stability of the new vacuum, though there are subtle issues to be resolved.

Conclusions and Discussions. In summary, we have applied stochastic quantization to the Hořava gravity. By analyzing the associated Fokker-Planck equation, we have found that with $\lambda < 1/3$ the system will approach to equilibrium as the fictitious time goes to infinity, giving rise to a stable vacuum state for the quantized theory. The key to this property is the detailed balance condition obeyed by the Hořava action. When $\lambda > 1/3$, stochastic quantization does not make sense because of development of a negative mode. The $\lambda = 1/3$ case would be probably alright, but needs more careful examination.

In ref. [4], to make sense of the speed of light in the IR regime with $z = 1$, one needs $\Lambda_W/(1 - 3\lambda)$ to be positive. Our constraint $\lambda < 1/3$ for the stability of gravity vacuum further constrains the cosmological constant to be positive: $\Lambda_W > 0$. This agrees with cosmological observations [15].

Our suggestion opens the door for using stochastic quantization to numerically study the quantized Hořava gravity, in particular to check whether the renormalization group would indeed change the value of $z$ from $z = 3$ in the UV regime to $z = 1$ in the IR regime.

Finally, it should be noted that the stochastic quantization applied in this letter is the standard one that introduces a fictitious time. This is different from the one used in ref. [11], where the time for stochastic evolution is identified with the real time. In this reference, for the purpose of studying the renormalizability of Hořava gravity, they have explored the fact that like any Lifshitz-type models, the Hořava gravity can be viewed as stochastic quantization of a lower dimensional theory [17], which in the present case is three-dimensional topological massive gravity.

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