EMBEDDING TOPOLOGICAL SEMIGROUPS INTO THE HYPERSPACES OVER TOPOLOGICAL GROUPS

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Abstract. We study algebraic and topological properties of subsemigroups of the hyperspace exp(G) of non-empty compact subsets of a topological group G endowed with the Vietoris topology and the natural semigroup operation. On this base we prove that a compact Clifford topological semigroup S is topologically isomorphic to a subsemigroup of exp(G) for a suitable topological group G if and only if S is a topological inverse semigroup with zero-dimensional idempotent semilattice.

1. Introduction

According to [2] (and [13]) each (commutative) semigroup S embeds into the global semigroup Γ(G) over a suitable (Abelian) group G. The global semigroup Γ(G) over G is the set of all non-empty subsets of G endowed with the semigroup operation (A, B) → AB = \{ab : a ∈ A, b ∈ B\}. If G is a topological group, then the global semigroup Γ(G) contains a subsemigroup exp(G) consisting of all non-empty compact subsets of G and carrying a natural topology turning it into a topological semigroup. This is the Vietoris topology generated by the sub-base consisting of the sets

\[ U^+ = \{K ∈ \exp(S) : K ⊂ U\} \text{ and } U^- = \{K ∈ \exp(S) : K ∩ U \neq ∅\} \]

where U runs over open subsets of S. Endowed with the Vietoris topology the semigroup exp(G) will be referred to as the hypersemigroup over G (because its underlying topological space is the hyperspace exp(G) of G, see [12]). Since each topological group G is Tychonov, so is the hypersemigroup exp(G). The group G can be identified with the subgroup \{K ∈ \exp(G) : |K| = 1\} of exp(G) consisting of singletons.

The main object of our study in this paper is the class \(\mathcal{H}\) of topological semigroups S that embed into the hypersemigroups exp(G) over topological groups G. We shall say that a topological semigroup \(S_1\) embeds into another topological semigroup \(S_2\) if there is a semigroup homomorphism \(h : S_1 → S_2\) that is a topological embedding. In is clear that the class \(\mathcal{H}\) contains all topological groups. On the other hand, the compact topological semigroup ([0, 1], min) does not belong to \(\mathcal{H}\), see [3]. In this paper we establish some inheritance properties of the class \(\mathcal{H}\) and on this base detect compact Clifford semigroups belonging to \(\mathcal{H}\): those are precisely compact Clifford inverse semigroups with zero-dimensional idempotent semilattice.

Let us recall that a semigroup S is

- **Clifford** if each element \(x \in S\) lies in a subgroup of S;

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\[ \alpha \in (s, g) \text{ given by the formula } (s, g) \cdot (s', g') = (sg(s'), gg') \text{.} \]

- **inverse** if each element \( x \in S \) is *uniquely invertible* in the sense that there is a unique element \( x^{-1} \in S \) called the inverse of \( x \) such that \( xx^{-1}x = x \) and \( x^{-1}xx^{-1} = x^{-1} \);
- **algebraically regular** if each element \( x \in S \) is regular in the sense that \( xyx = x \) for some \( y \in S \);
- a **semilattice** if \( xx = x \) and \( xy = yx \) for all \( x, y \in S \).

It is known [7, 1.17], [11, II.1.2] that a semigroup \( S \) is inverse if and only if \( S \) is algebraically regular and the set \( E = \{ x \in S : xx = x \} \) of idempotents is a commutative subsemigroup of \( S \). The subsemigroup \( E \) will be called the *idempotent semilattice* of \( S \). An inverse semigroup \( S \) is Clifford if and only if \( xx^{-1} = x^{-1}x \) for all \( x \in S \). In this case \( S = \bigcup_{e \in E} H_e \) where \( H_e = \{ x \in S : xx^{-1} = e = x^{-1}x \} \) are the maximal subgroups of \( S \) corresponding to the idempotents \( e \) of \( S \).

The above classes of semigroups relate as follows:

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group -> Clifford inverse -> inverse
semilattice -> Clifford inverse
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These classes form varieties of semigroups, which means that they are closed under taking subdirect products and homomorphic images. As we shall see later, the class \( \mathcal{H} \) is not closed under homomorphic images and thus does not form a variety but is invariant with respect to many operations over topological semigroups.

By a *topological semigroup* we understand a topological space \( S \) endowed with a continuous semigroup operation. A topological semigroup \( S \) is called a *topological inverse semigroup* if \( S \) is an inverse semigroup and the inversion map \( ( \cdot )^{-1} : S \to S \), \( ( \cdot )^{-1} : x \mapsto x^{-1} \) is continuous.

Now we define three operations over topological semigroups that do not lead out the class \( \mathcal{H} \).

We say that a topological semigroup \( S \) is a *subdirect product* of a family \( \{ S_\alpha : \alpha \in A \} \) of topological semigroups if \( S \) embeds into the Tychonov product \( \prod_{\alpha \in A} S_\alpha \) endowed the coordinatewise semigroup operation.

Another operation is the *semidirect product* \( S \ltimes^\sigma G \) of a topological semigroup \( S \) and a topological group \( G \) acting on \( S \) by automorphisms. More precisely, let \( Aut(S) \) denote the group of topological auto-isomorphisms of the semigroup \( S \) and \( \sigma : G \to Aut(S) \) be a group homomorphism defined on a topological group \( G \) and such that the induced map

\[ \tilde{\sigma} : G \times S \to S, \tilde{\sigma} : (g, s) \mapsto \sigma(g)(s) \]

is continuous. By \( S \ltimes^\sigma G \) we denote the topological semigroup whose underlying topological space is the Tychonov product \( S \times G \) and the semigroup operation is given by the formula \( (s, g) * (s', g') = (sg(s'), gg') \). The semidirect product \( S \ltimes^{id} Aut(S) \) of a semigroup \( S \) with its automorphism group is called the *holomorph* of \( S \) and is denoted by Hol(S).

One can easily check that for an algebraically regular (inverse) topological semigroup \( S \) the semidirect product \( S \ltimes^\sigma G \) with any topological group \( G \) acting on \( S \) is an algebraically regular (inverse) topological semigroup. The situation is different for Clifford topological semigroups: the semidirect product \( S \ltimes^\sigma G \) is Clifford...
inverse if and only if \( S \) is Clifford inverse and the group \( G \) acts trivially on the idempotents of \( S \), see Proposition 3.

The third operation that does not lead out the class \( \mathcal{H} \) is attaching zero to a compact semigroup from \( \mathcal{H} \). Given a topological semigroup \( S \) let \( S^0 = S \cup \{0\} \) denote the extension of \( S \) by an isolated point \( 0 \not\in S \) such that \( s0 = 0s = 0 \) for all \( s \in S^0 \).

**Theorem 1.** The class \( \mathcal{H} \) is closed under the following three operations:

1. subdirect products;
2. semidirect products with Abelian topological groups;
3. attaching zero to compact semigroups from \( \mathcal{H} \).

**Proof.**

1. The first item follows from the fact that for any family \( \{H_\alpha\}_{\alpha \in A} \) of topological groups the map 
   \[ E : \prod_{\alpha \in A} \exp(H_\alpha) \to \exp(\prod_{\alpha \in A} H_\alpha), \quad E : (K_\alpha)_{\alpha \in A} \mapsto \prod_{\alpha \in A} K_\alpha \]
   is an embedding of topological semigroups.

2. The second item is less trivial and will be proved in Section 2.

3. If \( S \in \mathcal{H} \) is a compact semigroup, then there is an embedding \( f : S \to \exp(H) \) of \( S \) into the hypersemigroup \( \exp(H) \) of some compact topological group \( H \), see Proposition 1 below. Take any compact topological group \( G \) containing \( H \) so that \( H \neq G \) and define the map \( f^0 : S^0 \to \exp(G) \) letting \( f^0|S = f \) and \( f^0(0) = G \). It can be shown that \( f^0 \) is a topological embedding and thus \( S^0 \in \mathcal{H} \). \( \square \)

**Problem 1.** Is the class \( \mathcal{H} \) closed under taking semidirect products with arbitrary (not necessarily abelian) topological groups?

In light of Theorem 1 it is natural to consider the smallest class \( \mathcal{H}_0 \) of topological semigroups, closed under subdirect products, semidirect products with Abelian topological groups and attaching zero to compact semigroups from \( \mathcal{H} \). Since the class of topological inverse semigroups is closed under those three operations, we conclude that \( \mathcal{H}_0 \) is a subclass of the class of topological inverse semigroups. Consequently, \( \mathcal{H}_0 \) is strictly smaller than the class \( \mathcal{H} \) (because for a topological group \( G \) the semigroup \( \exp(G) \) is inverse if and only if \( |G| \leq 2 \)).

Nonetheless we can ask the following

**Question 1.** Does each (compact) topological inverse semigroup \( S \in \mathcal{H} \) belong to the class \( \mathcal{H}_0 \)?

In this respect let us note the following property of compact semigroups from the class \( \mathcal{H} \).

**Proposition 1.** A compact topological semigroup \( S \) belongs to the class \( \mathcal{H} \) if and only if \( S \) embeds into the hypersemigroup \( \exp(G) \) over a compact topological group \( G \).

**Proof.** Given a compact topological semigroup \( S \in \mathcal{H} \) find an embedding \( h : S \to \exp(G) \) of \( S \) into the hypersemigroup \( \exp(G) \) over a topological group \( G \). It follows from [4, 2.1.2] that the union \( H = \bigcup_{s \in S} h(s) \subset G \) is compact. Moreover, \( H \) is a subsemigroup of \( G \). Indeed, given arbitrary points \( y, y' \in H \) find points \( x, x' \in S \) with \( y \in h(x) \) and \( y' \in h(x') \). Then \( yy' \in h(x)h(x') = h(xx') \subset H \). Being a compact cancellative semigroup, \( H \) is a topological group by [5, Th.1.10]. Since
We shall affirmatively answer the “compact” part of Question 1 under an additional assumption that $S \in \mathcal{H}$ is Clifford. For this we first establish some specific algebraic and topological properties of algebraically regular semigroups $S \in \mathcal{H}$.

Let us call two elements $x, y$ of an inverse semigroup $S$ conjugated if $x = zyz^{-1}$ and $y = z^{-1}xz$ for some element $z \in S$. For an element $e \in E$ of a semilattice $E$ let $\uparrow e = \{ f \in E : ef = e \}$ denote the principal filter of $e$. We say that two elements $e, f \in E$ are incomparable if their product $ef$ differs from $e$ and $f$ (this is equivalent to $e \notin \uparrow f$ and $f \notin \uparrow e$).

A topological space $X$ is called
- totally disconnected if for any distinct points $x, y \in X$ there is a closed-and-open subset $U \subset X$ containing $x$ but not $y$;
- zero-dimensional if the family of closed-and-open sets forms a base of the topology of $X$.

It is known that a compact Hausdorff space is zero-dimensional if and only if it is totally disconnected.

**Theorem 2.** If a topological semigroup $S \in \mathcal{H}$ is algebraically regular, then

1. $S$ is a topological inverse semigroup;
2. the idempotent semilattice $E$ of $S$ has totally disconnected principal filters $\uparrow e$, $e \in E$;
3. an element $x \in S$ is an idempotent if and only if $x^2x^{-1}$ is an idempotent;
4. any distinct conjugated idempotents of $S$ are incomparable.

This theorem will be proved in Section 3.

**Remark 1.** Theorem 2 allows us to construct many examples of algebraically regular topological semigroups non-embeddable into the hypersemigroups over topological groups. The first two items of this proposition imply the result of [3] that non-trivial rectangular semigroups and connected topological semilattices do not belong to the class $\mathcal{H}$. The last two items imply that the class $\mathcal{H}$ does not contain neither Brandt nor bicyclic semigroups. A bicyclic semigroup is a semigroup generated by two elements $p, q$ connected by the relation $qp = 1$.

By a Brandt semigroup we understand a semigroup of the form

$$B(H, \kappa) = (\kappa \times H \times \kappa) \cup \{0\}$$

where $H$ is a group, $\kappa$ is a non-empty set, and the product $(\alpha, h, \beta) \ast (\alpha', h', \beta')$ of two non-zero elements of $B(H, \kappa)$ is equal to $(\alpha, hh', \beta')$ if $\beta = \alpha'$ and 0 otherwise. Brandt and bicyclic semigroups play an important role in the structure theory of inverse semigroups, see [11].

The following theorem answers affirmatively the “compact” part of Question 1.

**Theorem 3.** For a compact topological Clifford semigroup $S$ the following conditions are equivalent:

1. $S$ belongs to the class $\mathcal{H}$;
2. $S$ belongs to the class $\mathcal{H}_0$;
3. $S$ is a topological inverse semigroup with zero-dimensional idempotent semilattice $E$;
We claim that
\[ \text{elements of } \mathcal{E} \text{ being compact, is zero-dimensional.} \]

\[ \exists \text{ smallest element } e \iff e \text{ be the retraction of } \mathcal{S}, \]

\[ \pi \exp(\mathcal{E}) \text{ into the latter homomorphism defined by} \]

\[ \prod_{e \in \mathcal{E}} \exp(\mathcal{H}_e) \to \exp(\prod_{e \in \mathcal{E}} \mathcal{H}_e). \]

\[ (1) \Rightarrow (3) \text{ Assume that } \mathcal{S} \in \mathcal{H}. \text{ Then } \mathcal{S} \text{ is a compact topological inverse semigroup according to Theorem 2(1).} \]

\[ \text{The semigroup } \mathcal{E} \text{ of idempotents of } \mathcal{S} \text{ is compact and thus contains the smallest idempotent } e \in \mathcal{E} \text{ (in the sense that } ee' = e \text{ for all } e' \in \mathcal{E}). \]

\[ \text{By Theorem 2 the principal filter } \uparrow e = \mathcal{E} \text{ is totally disconnected and being compact, is zero-dimensional.} \]

\[ (3) \Rightarrow (4) \text{ Assume that } \mathcal{S} \text{ is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice } \mathcal{E}. \]

\[ \pi : \mathcal{S} \to \mathcal{E}, \pi : x \mapsto xx^{-1} = x^{-1}x \text{ be the retraction of } \mathcal{S} \text{ onto } \mathcal{E}. \]

\[ \text{The set } \mathcal{E} \text{ carries a natural partial order } \leq: e \leq e' \text{ iff } ee' = e. \]

\[ \mathcal{E}_0 = \{ e \in \mathcal{E} : \uparrow e \text{ is open} \} \text{ stands for the set of locally minimal elements of } \mathcal{E}. \]

\[ \text{For every } e \in \mathcal{E} \setminus \mathcal{E}_0 \text{ let } h_e : \mathcal{S} \to \mathcal{E}_0 \text{ be the trivial homomorphism mapping } \mathcal{S} \text{ into the zero of } \mathcal{E}_0. \]

\[ \text{Next, for every } e \in \mathcal{E}_0 \text{ consider the homomorphism } h_e : \mathcal{S} \to \mathcal{E}_0 \text{ defined by} \]

\[ h_e(s) = \begin{cases} es, & \text{if } s \in \pi^{-1}(\uparrow e); \\ 0, & \text{otherwise} \end{cases} \]

\[ \text{Taking the diagonal product of the homomorphisms } h_e, e \in \mathcal{E}, \text{ we obtain a homomorphism} \]

\[ h = (h_e)_{e \in \mathcal{E}} : \mathcal{S} \to \prod_{e \in \mathcal{E}} \mathcal{E}_0, \quad h : s \mapsto (h_e(s))_{e \in \mathcal{E}}. \]

\[ \text{We claim that } h \text{ is injective and thus an embedding of the compact semigroup } \mathcal{S} \text{ into } \prod_{e \in \mathcal{E}} \mathcal{E}_0. \]

\[ \text{Let } x, y \in \mathcal{S} \text{ be two distinct points. If } \pi(x) \neq \pi(y), \text{ then either } \pi(x) \notin \uparrow \pi(y) \text{ or } \pi(y) \notin \uparrow \pi(x). \]

\[ \text{We lose no generality assuming the first case. Consider the set } \mathcal{U} = \{ u \in \mathcal{E} : \pi(x) \notin \uparrow u \} \text{ and note that it is open and } \mathcal{U} = \uparrow \mathcal{U} \text{ where} \]

\[ \uparrow \mathcal{U} = \{ v \in \mathcal{E} : 3u \in \mathcal{U} \text{ with } u \leq v \}. \]

\[ \text{Also } \pi(y) \in \mathcal{U}. \text{ By Proposition 1 of [8] there is a continuous semilattice homomorphism } h : \mathcal{E} \to \{0,1\} \text{ such that} \]

\[ \pi(y) \in h^{-1}(1) \subset \uparrow \mathcal{U}. \]

\[ \text{The preimage } h^{-1}(1), \text{ being a compact subsemilattice of } \mathcal{E}, \]

\[ \text{has the smallest element } e, \text{ that belongs to } \mathcal{E}_0 \text{ because } h^{-1}(1) = \uparrow e. \]
Now the definition of the homomorphism \( h_e \) and the non-inclusion \( \pi(x) \notin \uparrow e \) imply that \( h_e(x) = 0 \) while \( h_e(y) \in H_e \). Hence \( h_e(x) \neq h_e(y) \) and \( h(x) \neq h(y) \).

Finally consider the case \( \pi(x) = \pi(y) \). Observe that the set \( U = \{ e \in E : xe \neq ye \} \) contains the idempotent \( \pi(x) = \pi(y) \) and coincides with \( \uparrow U \). Again applying Proposition 1 of \([9]\) we can find a continuous semilattice homomorphism \( h : E \rightarrow \{0,1\} \) such that \( \pi(x) = \pi(y) \in h^{-1}(1) \subset \uparrow U \). The preimage \( h^{-1}(1) \), being a compact subsemilattice of \( E \), has the smallest element \( e \). Since \( h^{-1}(1) = \uparrow e \) is open in \( E \), \( e \in E_0 \). It follows from \( e \in U \) that \( h_e(x) = ex \neq ey = h_e(y) \) and hence \( h(x) \neq h(y) \).

Theorem 6 will be applied to characterize Clifford compact topological semigroups embeddable into the hypersemigroups of topological groups \( G \) belonging to certain varieties of compact topological groups. A class \( \mathcal{G} \) of topological groups is called a variety if it is closed under taking arbitrary Tychonov products, taking closed subgroups, and quotient groups by closed normal subgroups.

**Theorem 4.** Let \( \mathcal{G} \) be a non-trivial variety of compact topological groups. A Clifford compact topological semigroup \( S \) embeds into the hypersemigroup \( \exp(G) \) of a topological group \( G \in \mathcal{G} \) if and only if \( S \) is a topological inverse semigroup whose idempotent semilattice \( E \) is zero-dimensional and all maximal groups \( H_e, e \in E \), belong to the class \( \mathcal{G} \).

This theorem will be proved in Section 4 after establishing the nature of group elements in the hypersemigroups.

The classes \( \mathcal{H} \) and \( \mathcal{H}_0 \) are closed under subdirect products but are very far from being closed under homomorphic images. We shall show that the class of continuous homomorphic images of compact Clifford semigroups \( S \in \mathcal{H}_0 \) coincides with the class of all compact Clifford inverse semigroups with Lawson idempotent semilattices. We recall that a topological semilattice \( E \) is called **Lawson** if open subsemilattices form a base of the topology of \( E \). By the fundamental Lawson Theorem \([6\, Th. 2.13]\) a compact topological semilattice is Lawson if and only if the continuous homomorphisms to the min-interval \([0,1]\) separate points of \( S \). It is known \([6\, Th. 2.6]\) that each zero-dimensional compact topological semilattice is Lawson.

**Proposition 2.** A topological semigroup \( S \) is a continuous homomorphic image of a compact Clifford semigroup \( S_0 \in \mathcal{H}_0 \) if and only if \( S \) is a compact Clifford topological inverse semigroup with Lawson idempotent semilattice.

**Proof.** To prove the “only if” part, assume that a topological semigroup \( S \) is the image of a compact Clifford semigroup \( S_0 \in \mathcal{H}_0 \) under a continuous homomorphism \( h : S_0 \rightarrow S \). By Theorem 3(3), \( S_0 \) is a topological inverse Clifford semigroup with zero-dimensional idempotent semilattice \( E_0 \). Then \( S \) is an inverse Clifford semigroup, being the homomorphic image of \( S_0 \), see \([11\, L.II.1.10]\). Moreover, being compact topological semigroup, \( S \) is a topological inverse semigroup, see \([9,\, 10]\) or \([1]\). It follows that the semigroup \( E \) of idempotents of \( S \) is the homomorphic image of the semilattice \( E_0 \). Being zero-dimensional and compact, the semilattice \( E_0 \) is Lawson \([6\, Th.2.6]\). Then \( E \) is Lawson as the compact homomorphic image of a Lawson semilattice \([6\, Th.2.4]\).

To prove the “if” part, assume that \( S \) is a compact topological inverse Clifford semigroup \( S \) with Lawson semilattice \( E \) of idempotents. By Corollary 1 of \([8]\), \( S \)
embeds into a product \( \prod_{\alpha \in A} \hat{H}_\alpha \) of the cones over compact topological groups \( H_\alpha \). By definition, for a compact topological group \( G \) the semigroup
\[
\hat{H} = H \times [0, 1]/H \times \{0\}
\]
that is the quotient semigroup of the product \( H \times [0, 1] \) of \( H \) with the min-interval \([0, 1]\) by the ideal \( H \times \{0\} \) of \( H \times [0, 1] \).

Observe that the unit interval \([0, 1]\) is the image of the standard Cantor set \( C \subset [0, 1] \) under a continuous monotone map \( h : C \to [0, 1] \) well-known under the name “Cantor ladder”. The map \( h \) can be thought as a continuous semilattice homomorphism \( h : C \to [0, 1] \), where both \( C \) and \([0, 1]\) are endowed with the operation of minimum. Then \( \hat{H} \) is the image of the semigroup \( H \times C \), which is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice \( C \).

Thus for each index \( \alpha \in A \) we can construct a continuous surjective homomorphism \( h_\alpha : S_\alpha \to \hat{H}_\alpha \) of a compact topological inverse Clifford semigroup \( S_\alpha \) with zero-dimensional idempotent semilattice onto the semigroups \( \hat{H}_\alpha \). Taking the product of those homomorphisms we obtains a continuous surjective homomorphism
\[
h : \prod_{\alpha \in A} S_\alpha \to \prod_{\alpha \in A} \hat{H}_\alpha.
\]
It is clear that \( \prod_{\alpha \in A} S_\alpha \) is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice. By Theorem \( \text{3}(2) \), this semigroup belongs to the class \( \mathcal{H}_0 \) and so does its subsemigroup \( S_0 = h^{-1}(S) \). It remains to observe that \( S \) is the continuous homomorphic image of the semigroup \( S_0 \in \mathcal{H}_0 \).

This proposition yields many examples of compact Clifford semigroups \( S \notin \mathcal{H} \) that are continuous homomorphic images of compact Clifford semigroups \( S_0 \in \mathcal{H}_0 \subset \mathcal{H} \). We have also a non-Clifford example.

**Example 1.** The holomorph \( \text{Hol}(E_3) = E_3 \times \text{Aut}(E_3) \) of the 3-element semilattice \( E_3 = \{e, f, ef\} \) belongs to the class \( \mathcal{H}_0 \) but contains the 2-element ideal \( I = \{ef\} \times \text{Aut}(E_3) \) such that quotient semigroup \( \text{Hol}(E_3)/I \) is isomorphic to the 5-element Brandt semigroup \( B(\mathbb{Z}_1, 2) \) and thus does not belong to the class \( \mathcal{H} \).

The remaining part of the paper is devoted to the proofs of the results announced in the Introduction.

2. **Semidirect products of topological semigroups**

In this section we shall prove that the class \( \mathcal{H} \) is closed under semidirect products with Abelian topological groups.

Let \( G \) be a topological group. By a topological \( G \)-semigroup we understand a topological semigroup \( S \) endowed with a homomorphism \( \sigma : G \to \text{Aut}(S) \) of \( G \) to the group of topological automorphisms of \( S \) such that the induced action \( \tilde{\sigma} : G \times S \to S, \tilde{\sigma} : (g, s) \to \sigma(g)(s) \), is continuous. It will be convenient to denote the element \( \sigma(g)(s) \) by the symbol \( gs \).

The *semidirect product* \( S \ltimes^\sigma G \) of a topological \( G \)-semigroup \( S \) with \( G \) is the topological semigroup whose underlying topological space is \( S \times G \) and the semigroup operation is defined by \((s, g) \cdot (s', g') = (s \cdot gs'g, gg') \). If the action \( \sigma \) of the group \( G \) on \( S \) is clear from the context, then we shall omit the symbol \( \sigma \) and will write \( S \ltimes G \) instead of \( S \ltimes^\sigma G \).
The following proposition describes some algebraic properties of semidirect products.

**Proposition 3.** Let $S \rtimes G$ be the semidirect product of a topological $G$-semigroup $S$ and a topological group $G$.

1. $S \rtimes G$ is a (topological) inverse semigroup if and only if $S$ is a (topological) inverse semigroup;
2. $S \rtimes G$ is a topological group if and only if $S$ is a topological group;
3. $S \rtimes G$ is an inverse Clifford semigroup if and only if $S$ is an inverse Clifford semigroup and $ge = e$ for any $g \in G$ and any idempotent $e$ of $S$.

**Proof.** First observe that $S$ can be identified with the subsemigroup $S \times \{e\}$ of $S \rtimes G$ where $e$ is the unique idempotent of $G$.

1. Assume that $S$ is an inverse semigroup. To show that $S \rtimes G$ is an inverse semigroup we should check that the idempotents of $S \rtimes G$ commute and each element $(s, g) \in S \rtimes G$ has an inverse. For this observe that $(g^{-1}s^{-1}, g^{-1})$ is an inverse element to $(s, g)$. Indeed,

$$(s, g) * (g^{-1}s^{-1}, g^{-1}) = (ss^{-1}, e)(s, g) = (ss^{-1}s, g) = (s, g).$$

By analogy we can check that

$$(g^{-1}s^{-1}, g^{-1})(s, g)(g^{-1}s^{-1}, g^{-1}) = (g^{-1}s^{-1}, g^{-1}).$$

Observe that an element $(s, g)$ is an idempotent of the semigroup $S \rtimes G$ if and only if $s$ and $g$ are idempotents. This observation easily implies that the idempotents of the semigroup $S \rtimes G$ commute (because the idempotents of $S$ commute).

If $S$ is a topological inverse semigroup, then the map $(\cdot)^{-1} : S \to S$, $(\cdot)^{-1} : s \mapsto s^{-1}$ is continuous. The continuity of this map can be used to show that the map

$$(\cdot)^{-1} : S \rtimes G \to S \rtimes G, \quad (\cdot)^{-1} : (s, g) \mapsto (g^{-1}s^{-1}, g^{-1})$$

is continuous too.

Next, assume that $S \rtimes G$ is an inverse semigroup. Given any element $s$ consider the element $x = (s, e) \in S \rtimes G$ and find its inverse $x^{-1} = (s', g)$ in $S \rtimes G$. It follows from $(s, e)(s', g)(s, e) = xx^{-1}x = x = (s, e)$ that $g = e$ and then $ss'ss's = s$ and $s's's'ss = s$, which means that $s'$ is the inverse element to $s$ in the semigroup $S$. Since the idempotents of $S \rtimes G$ commute and lie in the subsemigroup $S \times \{e\}$, the idempotents of $S$ commute too, which yields that $S$ is an inverse semigroup.

If $S \rtimes G$ is a topological inverse semigroups, then $S$ is a topological inverse semigroup, being a subsemigroup of $S \rtimes G$.

2. The second item follows from the first one and the fact that a topological semigroup is a topological group if and only if it is a topological inverse semigroup with a unique idempotent.

3. Assume that the semigroup $S$ is inverse and Clifford, and $G$ acts trivially on the idempotents of $S$. By the first item, $S \rtimes G$ is an inverse semigroup. So it remains to prove that $xx^{-1} = x^{-1}x$ for all $x = (s, g) \in S \rtimes G$. Observe that

$$x^{-1} = (g^{-1}s^{-1}, g^{-1})$$

and thus

$$x^{-1}x = (g^{-1}s^{-1}, g^{-1})(s, g) = (g^{-1}s^{-1}g^{-1}s, e) = (g^{-1}(s^{-1}s), e) = (s^{-1}ss^{-1}, e) = (s, g)(g^{-1}s^{-1}, g^{-1}) = xx^{-1}.$$

Here we used that $G$ acts trivially on the idempotents of $S$ and hence $g^{-1}(s^{-1}s) = s^{-1}s$. We also used that fact that $g^{-1} : s \mapsto g^{-1}s$ is an automorphism of the
semigroup $S$ and thus $g^{-1}(s^{-1}s) = (g^{-1}s^{-1})(g^{-1}s)$. Now assume that the semigroup $S \times G$ is Clifford and inverse. Then $S$ is Clifford, being a subsemigroup of $S \times G$. It remains to show that $G$ acts trivially on the idempotents of $S$. Take any idempotent $s \in S$, any $g \in G$, and consider the element $x = (s, g)$ and its inverse $x^{-1} = (g^{-1}s^{-1}, g^{-1})$. Since $S \times G$ is Clifford, $xx^{-1} = x^{-1}x$, which implies that $x^{-1}x = (g^{-1}s^{-1}, g^{-1})(s, g) = (g^{-1}s^{-1}g^{-1}s, e) = (g^{-1}s^{-1}s, s) = (s, e)$ and thus $gs = s$. □

If $S$ is a topological $G$-semigroup, then $\exp(S)$ has a structure of a topological $G$-semigroup with respect to the induced action

$G \times \exp(S) \to \exp(S), \quad (g, K) \mapsto gK = \{gs : s \in K\}$.

Thus it is legal to consider the semidirect product $\exp(S) \rtimes G$.

The proof of the following proposition is easy and is left to the reader.

**Lemma 1.** The map

$E : \exp(S) \times G \to \exp(S \times G), \quad E : (K, g) \mapsto K \times \{g\}$

is a topological embedding of the topological semigroups.

For a topological semigroup $S$ consider the Tychonov power $S^G$ as a topological $G$-semigroup with the following action of $G$:

$(g, (s_\alpha)_{\alpha \in G}) \mapsto (s_\alpha g_\alpha)_{\alpha \in G}$.

A homomorphism $h : S \to S'$ between two $G$-semigroups is called $G$-equivariant if $h(gs) = gh(s)$ for every $g \in G$ and $s \in S$. The proof of the following lemma also is left to the reader.

**Lemma 2.** For any topological semigroup $H$ the map

$E : \exp(H)^G \to \exp(H^G), \quad E : (K_\alpha)_{\alpha \in G} \mapsto \prod_{\alpha \in G} K_\alpha$

is a $G$-equivariant embedding of the corresponding $G$-semigroups.

The following immediate lemma helps to transform semigroup embedding into $G$-equivariant embedding.

**Lemma 3.** Let $G$ be an Abelian topological group. If $f : S \to H$ is an embedding of a topological $G$-semigroup $H$ into a topological semigroup $A$, then the map

$F : S \to H^G, \quad F : s \mapsto (f(gs))_{g \in G}$

is a $G$-equivariant embedding of the $G$-semigroup $S$ into the $G$-semigroup $H^G$.

Finally we are able to prove the second item of Theorem 1.

**Theorem 5.** Let $G$ be an Abelian topological group. If a topological $G$-semigroup $S$ embeds into the hypersemigroup $\exp(H)$ of a topological group $H$, then the semidirect product $S \rtimes G$ embeds into the hypersemigroup $\exp(H^G \rtimes G)$ of the topological group $H^G \rtimes G$.\[ ]
Proof. Let \( f : S \to \exp(H) \) be an embedding. By Lemmas 3 and 2 the map
\[
F : S \to \exp(H^G), \quad F : s \mapsto \prod_{\alpha \in G} f(\alpha s)
\]
is a \( G \)-equivariant embedding. The \( G \)-equivariantness of \( F \) guarantees that the map
\[
E : S \times G \to \exp(H^G) \times G, \quad E : (s, g) \mapsto (F(s), g)
\]
is an embedding of the corresponding topological semigroups. Finally, applying Lemma 1 we see that the semigroup \( S \times G \) admits an embedding into the hypersemigroup \( \exp(H^G \times G) \) of the topological group \( H^G \times G \).

\[ \square \]

3. Idempotents and invertible elements of the hypersemigroups

In this section given a topological group \( G \) we characterize idempotent and related special elements of the hypersemigroup \( \exp(G) \). We recall that an element \( x \) of a semigroup \( S \) is called

- an idempotent if \( xx = x \);
- regular if there is an element \( y \in S \) such that \( xyx = x \);
- (uniquely) invertible if there is a (unique) element \( x^{-1} \in S \) such that \( xx^{-1}x = x \) and \( x^{-1}xx^{-1} = x^{-1} \);
- a group element if \( x \) lies in some subgroup of \( S \).

It is possible to prove our results in a more general setting of cancellative topological semigroups. We recall that a semigroup \( S \) is cancellative if for any \( x, y, z \in S \) the equality \( xz = yz \) implies \( x = y \) and the equality \( zx = zy \) implies \( x = y \). It is easy to check that the invertible elements of a cancellative semigroup form a subgroup.

Proposition 4. Let \( X \) be a cancellative topological semigroup. A non-empty compact subset \( K \subset X \) is

1. an idempotent of the semigroup \( \exp(X) \) if and only if \( K \) is a compact subgroup of \( X \);
2. a regular element of the semigroup \( \exp(X) \) if and only if \( K \) uniquely invertible in \( \exp(X) \) if and only if \( K = Hx \) for some compact subgroup \( H \subset X \) and some invertible element \( x \in X \);
3. a group element in \( \exp(X) \) if and only if \( K = Hx = xH \) for some compact subgroup \( H \subset X \) and some invertible element \( x \in X \).

Proof. 1. If a compact subset \( K \subset X \) is an idempotent of the semigroup \( \exp(X) \) that is \( KK = K \), then \( K \) is a compact cancellative semigroup. It is known [3, Th. 1.10] that a compact cancellative semigroup is a group. If \( K \) is subgroup of \( X \) then \( KK = K \).

2. Assume that \( K \in \exp(X) \) is a regular element of the semigroup \( \exp(X) \) which means that \( KAK = K \) for some non-empty compact subset \( A \subset X \). Fix any elements \( x \in K \) and \( a \in A \). The set \( KA \), being an idempotent of the semigroup \( \exp(X) \), coincides with some compact subgroup \( H \) of \( X \). We claim that \( K = Hx \) and the element \( x \) is invertible in \( X \). Observe that \( Hx \subset HK = KAK = K \) and thus \( Hxa \subset KA = H \), which implies that \( xa \in H \) is invertible. Consequently, \( xa(xa)^{-1} = e = (xa)^{-1}xa \) which means that \( x \) and \( a \) are invertible. It follows from \( Ka \subset KA = H \) that
\[
K \subset Ha^{-1} = Ha^{-1}x^{-1}x = H(xa)^{-1}x \subset HHx = Hx \subset K
\]
and thus \( K = Hx \).

To show that \( K \) is uniquely invertible, assume additionally that \( AKA = A \). In this case \( A = AKA \supset aKa = aHxa = aH = x^{-1}xaH = x^{-1}H \). On the other hand, the equality \( KAK = K \) implies \( xAx \subset Hx \) and \( A \subset x^{-1}H \). Therefore \( A = x^{-1}H \) is a unique inverse element to \( K \).

3. If \( K = Hx = xH \) for some compact subgroup \( H \subset X \) and some invertible element \( x \in X \), then for the element \( K^{-1} = x^{-1}H = Hx^{-1} \) we get \( K^{-1}K = KK^{-1} = H \), which implies that \( K \) is a group element of \( \exp(X) \). Conversely, if \( K \) is a group element, then \( KK^{-1} = K^{-1}K = H \) for some compact subgroup \( H \subset X \) and \( K = Hx \) for some invertible element \( x \in X \) (because \( K \) is regular). Since \( H = K^{-1}K = x^{-1}HHx = x^{-1}Hx \), we get \( xH = Hx \).

\( \square \)

Theorem 2 is a particular case of the following more general

**Theorem 6.** Let \( X \) be a cancellative topological semigroup and \( G \) be the subgroup of invertible elements of \( X \). Let \( S \) be an algebraically regular subsemigroup of \( \exp(X) \) and \( E \) be the set of idempotents of \( S \).

1. The semigroup \( S \) is inverse and \( S \subset \exp(G) \).
2. If \( G \) is a topological group, then \( S \) is a topological inverse semigroup.
3. An element \( x \in S \) is an idempotent if \( x^2x^{-1} \) is an idempotent.
4. Any distinct conjugate idempotents of \( S \) are incomparable.
5. The set \( E \) is a closed commutative subsemigroup of \( S \) and for every \( e \in E \) the upper cone \( \downarrow e = \{ f \in E : ef = e \} \) is totally disconnected.

**Proof.** 1. Let \( S \) be a regular subsemigroup of \( \exp(X) \). It follows from Proposition 4 that each element \( K \in S \), being regular, is equal to \( Hx \) for some compact subgroup \( H \subset G \) and some invertible element \( x \in X \). Then \( K = Hx \subset G \) and hence \( K \in \exp(G) \subset \exp(X) \). By Proposition 4 \( K \) is uniquely invertible in \( \exp(X) \) and hence in \( S \), which means that \( S \) is an inverse semigroup. Moreover, the inverse \( K^{-1} \) to \( K \) in \( S \) can be found by the natural formula: \( K^{-1} = \{ x^{-1} : x \in K \} \).

2. If the subgroup \( G \) of invertible elements of \( X \) is a topological group, then the inversion

\[
(\cdot)^{-1} : \exp(G) \to \exp(G), \ (\cdot)^{-1} : K \mapsto K^{-1}
\]

is continuous with respect to the Vietoris topology on \( \exp(G) \) and consequently, the inversion map of \( S \) is continuous as well, which yields that \( S \) is a topological inverse semigroup.

3. Let \( K \in S \) be an element such that \( K^2K^{-1} \) is an idempotent in \( S \) and hence is a compact subgroup of \( X \). By Proposition 4 \( K = Hx \) for some compact subgroup \( H \) of \( X \) and some invertible element \( x \in X \). Then \( K^2K^{-1} = HxHx^{-1}H = HxH \). The set \( K^2K^{-1} \), being a subgroup of \( X \), contains the neutral element 1 of \( X \). Then \( 1 \in K^2K^{-1} = HxH \) and hence \( x \in H \), which implies that \( K = Hx = H \) is an idempotent in \( \exp(X) \) and \( S \).

4. Let \( E, F \) be two distinct conjugate idempotents of the semigroup \( S \). Find an element \( K \in S \) such that \( E = KFK^{-1} \) and \( F = K^{-1}EK \). By Proposition 4 find a compact subgroup \( H \) of \( X \) and an invertible element \( x \in X \) such that \( K = Hx \). We claim that \( E = xFx^{-1} \). Indeed, the inclusion

\[
x^{-1}Hx = x^{-1}HHx = K^{-1}K \subset K^{-1}EK = F
\]
implies
\[ E = KFK^{-1} = HxF^{-1}H = xx^{-1}HxFx^{-1}Hxx^{-1} ⊂ xFFx^{-1} = xF^{-1}. \]

On the other hand,
\[ H = Hxx^{-1}H ⊂ HxFx^{-1}H = KFK^{-1} = E \]
implies
\[ F = K^{-1}EK = x^{-1}HEHx ⊂ x^{-1}EEEx = x^{-1}Ex. \]
and hence \( xF^{-1} ⊂ E \).

5. Since \( S \) is an inverse semigroup, the set \( E \) of idempotents of \( S \) is a commutative subsemigroup of \( S \), see [11] II.1.2. To show that \( E \) is closed in \( S \), pick any element \( K \in S \setminus E \). By Proposition [4] \( K = Hx \) for a compact subgroup \( H \subset X \) and an invertible element \( x \in X \). Since \( K \) is not an idempotent, \( Hx \) is not a subgroup, which means that the neutral element \( 1 \) of \( H \) does not belong to \( Hx \). Let \( U = X \setminus \{ x \} \) and observe that \( U^+ = \{ C \in \exp(X) : 1 \notin C \} \) is a neighborhood of \( K \) in \( \exp(X) \) that contains no subgroup of \( X \) and hence does not intersect the set \( E \).

Now given an idempotent \( H \in \mathcal{E} \) we shall prove that the upper cone \( \uparrow H = \{ E \in \mathcal{E} : HE \subset H \} \) of \( H \) is totally disconnected. By Proposition [4] \( H \) is a compact subgroup of \( X \). It follows that \( \uparrow H \subset \exp(H) \). The total disconnectedness of \( \uparrow H \) will be proven as soon as given two distinct elements \( E_0, E_1 \in \uparrow H \) we find a closed-and-open subset \( U \subset \uparrow H \) such that \( E_0 \not\in U \) but \( E_1 \not\in U \). We lose no generality assuming that \( X = H \) and hence \( \mathcal{E} = \uparrow H \subset \exp(H) \).

We first consider the special case when \( H \) is a Lie group. Without loss of generality \( E_0 \not\in E_1 \) and hence \( E_0 \not\in \downarrow E_1 = \{ E \in \mathcal{E} : E \subset E_1 \} \). So, it remains to prove that the lower cone \( \downarrow E_1 \) is closed-and-open in \( \mathcal{E} \). The closedness of \( \downarrow E_1 \) follows from the continuity of the semigroup operation and the equality \( \downarrow E_1 = \{ E \in \mathcal{E} : EE_1 = E_1 \} \). To prove that \( \downarrow E_1 \) is open in \( \mathcal{E} \), take any \( K \in \downarrow E_1 \). The set \( K \subset \exp(H) \) being an idempotent of the semigroup \( \mathcal{E} \) is a closed subgroup of \( H \).

By Corollary II.5.6 of [4] the subgroup \( K \) of the compact Lie group \( H \) has an open neighborhood \( O(K) \subset H \) such that for each compact subgroup \( C \subset O(K) \) satisfies the inclusion \( xCx^{-1} \subset K \) for a suitable point \( x \in H \). We shall derive from this that \( C = K \) provided \( C \supset K \). Indeed, \( C \supset K \) and \( xCx^{-1} \subset K \) imply \( xKx^{-1} \subset xCx^{-1} \subset K \). Being a homeomorphic copy of the group \( K \), the subgroup \( xKx^{-1} \subset K \) must coincide with \( K \) (it has the same dimension and the same number of connected components). Consequently, \( xCx^{-1} = K \) and hence \( C \), being homeomorphic to its subgroup \( K \), coincides with \( K \) too.

The continuity of the semigroup operation of \( \mathcal{E} \) yields a neighborhood \( O_1(K) \subset \mathcal{E} \) of \( K \) such that \( EK \subset O(K) \) for each \( E \in O_1(K) \). We claim that \( O_1(K) \subset \downarrow E_1 \). Take any element \( E \in O_1(K) \) and observe that the product \( EK \), being an idempotent in the semigroup \( \mathcal{E} \), is a compact subgroup of \( H \) containing the subgroup \( K \). Now the choice of the neighborhood \( O(K) \) guarantees that \( E \subset EK \subset K \subset E_1 \) and hence \( E \in \downarrow E_1 \). This proves that \( O_1(K) \subset \downarrow E_1 \), witnessing that \( \downarrow E_1 \) is open in \( \mathcal{E} \).

Now we are able to finish the proof assuming that \( H \) is an arbitrary compact topological group. Given distinct elements \( E_0, E_1 \in \mathcal{E} \subset \exp(H) \) we should find an closed-and-open subset \( U \subset \mathcal{E} \) containing \( E_0 \) but not \( E_1 \). The topological group \( H \), being compact, is the limit of an inverse spectrum consisting of compact Lie groups. Consequently, we can find a continuous homomorphism \( h : H \to L \) onto a compact Lie group \( L \) such that \( h(E_0) \) and \( h(E_1) \) are distinct subgroups of \( L \). It follows that
$h(\mathcal{E}) = \{ h(E) : E \in \mathcal{E} \}$ is an idempotent semigroup of the hypersemigroup $\exp(L)$. Now the particular case considered above yields a closed-and-open subset $V \subset h(\mathcal{E})$ containing $h(E_0)$ but not $h(E_1)$. By the continuity of the homomorphism $h$ the set $\mathcal{U} = \{ K \in \mathcal{E} : h(K) \in V \}$ is closed-and-open in $\mathcal{E}$. It contains $E_0$ but not $E_1$. This proved the total disconnectedness of the upper cone $\uparrow H$. 

4. Proof of Theorem 4

In this section we will prove the Theorem 4. Given a Clifford compact topological semigroup $S$ and a non-trivial variety $\mathcal{G}$ of compact topological groups we should prove that $S$ embeds into the hypersemigroup $\exp(G)$ of a topological group $G \in \mathcal{G}$ if and only if $S$ is a topological inverse semigroup whose idempotent semilattice $E$ is zero-dimensional and all maximal groups $H_e, e \in E$, belong to the class $\mathcal{G}$.

To prove the “if” part, assume that $S$ is a compact Clifford topological inverse semigroup whose idempotent semilattice $E$ is zero-dimensional and all maximal groups $H_e, e \in E$, belong to the class $\mathcal{G}$. For every $e \in E$ let $\tilde{H}_e = H_e$ if $H_e$ is not trivial and $\tilde{H}_e \in \mathcal{G}$ be any non-trivial compact group if $H_e$ is trivial (such a group $\tilde{H}_e$ exists because the variety $\mathcal{G}$ is not trivial). Since $\mathcal{G}$ is closed under taking Tychonov products, the compact topological group $G = \prod_{e \in E} \tilde{H}_e$ belongs to $\mathcal{G}$. Finally, by Theorem 3(5), the semigroup $S$ embeds into $\exp(G)$.

To prove the “only if” part, assume that $S$ embeds into the hypersemigroup $\exp(G)$ over a topological group $G \in \mathcal{G}$. By Theorem 3(3), $S$ is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice $E$. It remains to show that each maximal group $H_e, e \in E$, of $S$ belongs to $\mathcal{G}$. The embedding of $S$ into $\exp(G)$ induces an embedding $h : H_e \to \exp(G)$. The image $H_0 = h(e)$, being an idempotent in $\exp(G)$, is a compact subsemigroup of $G$ and thus a compact subgroup of $G$ according to Theorem 1.10 [5]. The same is true for the semigroup $H = \bigcup_{x \in H} h(x)$. It is a compact subgroup of $G$. We claim that $H_0$ is a normal subgroup of $H$.

Indeed, for any $x \in H$ we can find a point $z \in H_e$ with $x \in h_e(z)$. It follows from (the proof of) Proposition 4(3) that $h_e(z) = xH_0 = H_0xH_0^{-1} = H_0$, witnessing that the subgroup $H_0$ is normal in $H$.

Let $\pi : H \to H/H_0$ be the quotient homomorphism. It follows from Proposition 4(3) that the composition $\pi \circ h_e : H_e \to H/H_0$ is a bijective continuous homomorphism. Because of the compactness of $H_e$, the group $H_e$ is isomorphic to $H/H_0$, which, being the quotient group of the closed subgroup $H$ of the group $G \in \mathcal{G}$ belongs to the variety $\mathcal{G}$.

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