Particles, Generalized Statistics and Categories

Władysław Marcinek
Institute of Theoretical Physics
University of Wrocław
Poland

Abstract

Interacting systems of particles with generalized statistics are considered on both classical and quantum level. It is shown that all possible quantum states and corresponding processes can be represented in terms of certain specific categories. The corresponding Fock space representation is discussed. The problem of existence of well-defined scalar product is considered. It is shown that commutation relations corresponding to a system with generalized statistics can be constructed from relations corresponding to Boltzmann statistics.

1. Introduction

Let us assume that we have a system of \( n \)–identical and spinless particles moving on a smooth manifold \( \mathcal{M} \) without boundary, of dimension \( d \geq 2 \). All possible classical trajectories of the particle system are path in the \( n \)-th Cartesian product \( \mathcal{M}^{\times n} \). It is natural to assume that two or more particles can not occupy the same position. We also assume that our particles are identical. This means that the configuration space for the system of particles is

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Q_n(\mathcal{M}) = \left(\mathcal{M}^{\times n}\setminus D_n\right)/S_n,
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1. Introduction

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\[
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\]

\[\text{(1)}\]

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where $D$ is the subset of the Cartesian product $\mathcal{M}^n$ on which two or more particles occupy the same position and $S_n$ is the symmetric group. Let us denote by $P_m \mathcal{M}$ the space of all homotopy classes of trajectories which starts at certain point $m_0 \in \mathcal{M}$ and end at arbitrary point $m \in \mathcal{M}$. The homotopy classes $P_{m_0} \mathcal{M}$ of all paths which start at $m_0$ and end at the same point $m_0$, (i.e. a loop space) can be naturally endowed with a group structure. This group is known as the fundamental group of $\mathcal{M}$ at $m_0$ and is denoted by $\pi_1(\mathcal{M}, m_0)$. One can prove that under some assumptions all fundamental groups $\pi_1(\mathcal{M}, m_0)$ corresponding to different base point $m_0$ are isomorphic one to other. Hence we can define the fundamental group $\pi_1(\mathcal{M})$ of $\mathcal{M}$ as $\pi_1(\mathcal{M}, m_0)$ for arbitrary $m_0$. We denote by $\pi_* : \pi_1(\tilde{\mathcal{M}}) \to \pi_1(\mathcal{M})$ the homomorphism induced by $\pi$. The fundamental group $\pi_1(\mathcal{Q}_n(\mathcal{M}))$ of the space $\mathcal{Q}_n(\mathcal{M})$ is said to be the $n$–string braid group on $\mathcal{M}$ and is denoted by $B_n(\mathcal{M})$, i.e.

$$B_n(\mathcal{M}) := \pi_1(\mathcal{Q}_n(\mathcal{M})).$$

Note that elements of $B_n(\mathcal{M})$ are loops in $\mathcal{Q}_n$. It is easy to see that all loops in $B_n(\mathcal{M})$ can be naturally divided into two classes. The first class corresponds to noncontractible loops with any interchanges of particles. This class describe the topology of the base space $\mathcal{M}$. The second class contain loops which corresponds to all mutual interchanges of particles one by other. It describes the statistics of a given system of particles. Let us consider this class in more details. Let $\Sigma_i = (i, i + 1) \in S_n$ be the transposition which describes the interchange of the $i$-th particle with the $i+1$-th one. Let us denote by $\xi_i$ the path in $\mathcal{M}^n \setminus D_n$ which realize this interchange. We assume that for the path $\xi_i$ we have the following parametrization

$$\xi_i : t \in [0, 1] \to \xi_i(t) = [(\xi_i)_1, \ldots, (\xi_i)_n](t).$$

We assume in addition that all $(\xi_i)_k$ for $k$ not in $\{i, i + 1\}$ are fixed, the loop $[(\xi_i)_i, (\xi_i)_{i+1}](t) \subset \mathcal{M}$ is counterclockwise and all particles $k$ not in $\{i, i + 1\}$ are outside of this loop. The projection of $\xi_i$ on $\mathcal{Q}_n$ is a loop in $\mathcal{Q}_n$. The homotopy class of this loop is denoted by $\Sigma_i$. One can see that elements $\{s_i : i = 1, \ldots, n - 1\}$ satisfy two following conditions

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad \text{for} \quad j = 1, \ldots, n - 2,$$

$$s_is_j = s_js_i \quad \text{for} \quad |i - j| > 2.$$
This means that $s_i$ for $i = 1, \ldots, n - 1$ generate a group which is denoted by $\Sigma_n(M)$.

Let $\tilde{Q}_n$ be the space which cover the configuration space $Q_n$. The above definition means that we have the relation

$$\Sigma_n(M) = \pi_\ast[\pi_1(\tilde{Q}_n(M))].$$

(5)

Note that the group $\Sigma_n(M)$ is a subgroup of $B_n(M)$ and is an extension of the symmetric group $S_n$ describing the interchange process of two arbitrary indistinguishable particles. It is obvious that the statistics of the given system of particles is determined by this group, see [8, 9]. A system of $n$–identical and spinless particles moving on a manifold $M$ and equipped with statistics described by the group $\Sigma_i(M)$ is said to be a particle system with the braid-group statistics.

We must indicate that the presented above braid group approach to quantum arbitrary statistics is in our opinion not adequate for the study of charged particles interacting with an external quantum field. First of all the statistics is determined by the interchange of two different particles. In order to do such interchange we need two particles and a place, a suitable space. The statistics of a single particle has no sense. Hence we need a new and more general approach to the concept of statistics. In this paper we are going to study a system of charged particles moving under influence of a quantum field. A generalized statistics is considered as a result of interaction of charged particles with quantum field. Our discussion is based on the algebraic formalism previously considered in [22, 23, 24, 25, 39, 41]. The short summary of this paper has been published in the Proceedings of the XII-th Max Born Symposium [42]. The fundamental assumption is that every charged particle is transform under interaction into a system consisting a charge and quanta of the field. Such system behaves like free particles moving in certain effective space. The so–called cross statistics is discussed. This statistics is described by an operator $T$. The existence of exchange braid statistic is related to the existence of additional operator $B$ and corresponding consistency conditions. For the description of our system we need a solution of these conditions. It is shown that these solutions are related to a construction of a category of states.
2. Interaction as coaction

Let us consider for example a charged particle moving on a space $\mathcal{M}$ under influence of singular magnetic field [24]. We assume that $\mathcal{M}$ is a path-connected topological space with a base point $m_0$. All possible classical trajectories of the particle are path in $\mathcal{M}$. We denote by $P_m\mathcal{M}$ the space of all homotopy classes of paths which starts at $m_0$ and end at arbitrary point $m \in \mathcal{M}$. It is obvious that the union $\cup_m P_m\mathcal{M} = P\mathcal{M}$ is a covering space $P = (P\mathcal{M}, \pi, \mathcal{M})$. The projection $\pi : P\mathcal{M} \to \mathcal{M}$ is given by

$$\pi(\xi) = m \quad \text{iff} \quad \xi \in P_m\mathcal{M}. \quad (6)$$

The homotopy class $P_{m_0}\mathcal{M}$ of all paths which start at $m_0$ and end at the same point $m_0$, (i.e. a loop space) forms the fundamental group $\pi_1(\mathcal{M}, m_0)$ of $\mathcal{M}$ at $m_0$. Generators of the fundamental group are denoted by $\Sigma_i$. In our case we have $\mathcal{M} = S^1 \times \ldots \times S^1$ and the fundamental group is

$$\pi_1(\mathcal{M}, m_0) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \quad (N\text{-sumands}). \quad (7)$$

Let $G$ be a subgroup of the fundamental group $\pi_1(\mathcal{M}, m_0)$. We assume that quantum states of magnetic field can be represented as linear combinations of elements (over a field $\mathbb{C}$ of complex numbers) of the group $G$. It is known such that linear combinations of elements of certain group $G$ form a group algebra $H := \mathbb{C}G$. It is also known that there is a Hopf algebra structure defined on the group algebra $H$. Let us denote by $E$ a Hilbert space of quantum states of particle which is not coupled to magnetic field. Quantum states of particle coupled to our singular magnetic field is described by the tensor product $E \otimes H$. Every attaching of magnetic flux to the particle moving in singular magnetic field can be represented by certain coaction $\rho_E$ of the Hopf algebra $H$ on the space $E$, i.e. by a linear mapping

$$\rho_E : E \to E \otimes H, \quad (8)$$

which define a (right-) $H$-comodule structure on $E$. Note that if $E$ is a $H$-comodule, where $H = \mathbb{C}G$, then $E$ is also a $G$-graded vector space, i.e

$$E = \bigoplus_{\alpha \in G} E_{\alpha}. \quad (9)$$

Note that the family of all $H$-comodules for a given Hopf algebra $H$ forms a monoidal category $\mathcal{C} = \mathcal{M}^H$. In this way we obtain a category of comodules
for the description of particles in singular magnetic field. This means that there is a closed relation between interacting particle system and categories. Obviously we can generalize the above particular case and study interactions and quantum processes in terms of categories in an general way.

3. Categories and interactions

We are going here to study systems of charged particles with certain dynamical interaction with a quantum field. It is natural to expect that some new and specific quantum states of the system have appear as a result of interaction. We would like to describe all such states. It is natural to assume that there is a specific category $C$ which represent all possible quantum states of interacting systems of particles and corresponding processes. The category $C$ is said to be a category of states.

Our fundamental assumption is that every charged particle is transform under interaction into a system consisting a charge and $N$–species of quanta of the field. A system which contains a charge and certain number of quanta as a result of interaction with the quantum field is said to be a dressed particle. Next we assume that every dressed particle is a composite object equipped with an internal structure. Obviously the structure of dressed particles is determined by the interaction with the quantum field. We describe a dressed particle as a non-local system which contains $n$ centers (vertexes). Two systems with $n$ and $m$ centers, respectively, can be "composed" into one system with $n + m$ centers. All centers as members of a given system behave like free particles moving on certain effective space. Every center is also equipped with ability for absorption and emission of quanta of the intermediate field. A centre dressed with a single quantum of the field is said to be a quasiparticle. In our approach a center equipped with two quanta forms a system of two quasiparticles.

A center with an empty place for a single quantum is said to be a quasihole. A centre which contains any quantum is said to be neutral. A neutral center can be transform into a quasiparticle or a quasihole by an absorption or emission process of single quantum, respectively. In this way the process of absorption of quanta of quantum field by a charged particle is equivalent to a creation of quasiparticles and emission – to annihilation of quasiparticles. Note that there is also the process of mutual annihilation of quasiparticles and quasiholes.
Quasiparticles and quasiholes as components of certain dressed particle have also their own statistics. It is interesting that there is a statistics of new kind, namely a cross statistics. This statistics is determined by an exchange process of quasiholes and quasiparticles. Note that the exchange is not a real process but an effect of interaction. Such exchange means annihilation of a quasiparticle on certain place and simultaneous creation of quasihole on an another place.

Summarizing we assume that for a system of quasiparticles and quasiholes we have the following possible processes: (i) composition, (ii) creation and annihilation, (iii) mutual exchanges of quasiholes and quasiparticles, (iv) exchange of quasiparticles or quasiholes itself. We are going to use the concept of monoidal categories in order to describe these processes. Let $\mathcal{C} \equiv \mathcal{C}(\otimes, k)$ be a monoidal category. This means that there are a collection $\text{Ob}(\mathcal{C})$ of objects, a collection $\text{hom}(\mathcal{C})$ of arrows (morphisms), an identity object $k$ (a field) and a monoidal operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying some known axioms, see [43] for details. The collection $\text{hom}(\mathcal{C})$ is the union of mutually disjoint sets $\text{hom}(U, V)$ of arrows $f : U \to V$ from $U$ to $V$ defined for every pair of objects $U, V \in \text{Ob}(\mathcal{C})$. It may happen that for a pair $U, V \in \text{Ob}(\mathcal{C})$ the set $\text{hom}(U, V)$ is empty. The associative composition of morphisms is also defined. The monoidal operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor which has a two-sided identity object $k$. The category $\mathcal{C}$ can contain some special objects like algebras, coalgebras, modules or comodules, etc...

Our fundamental assumption is that all possible quantum processes are represented as arrows of certain monoidal category $\mathcal{C} = \mathcal{C}(\otimes, k)$. In our case $k \equiv \mathbb{C}$ is the field of complex numbers. If $f : U \to V$ is an arrow from $U$ to $V$, then the object $U$ represents physical objects before interactions and $V$ represents possible results of interactions. We assume that different objects of the monoidal category $\mathcal{C}$ describe physical objects of different nature, charged particles, quasiparticles or different species of quanta of an external field, etc... Let $U$ be an object of the category $\mathcal{C}$, then the object $U^*$ corresponds for antiparticles, holes or quasiholes or dual field, respectively. If $U$ and $V$ are two different objects of the category $\mathcal{C}$, then $U \otimes V$ is also an object of the category, it represents a composite quantum system composed from object of different nature.

Let $U$ and $V$ be two different objects of $\mathcal{M}$. If for example $U$ represents charged particles and $V$ – a quantum field, then the product $U \otimes V$ describes the composite system containing both particles and the quantum field. Observe that the arrow $U \to U \otimes V$ describes the process of absorption and the
arrow $U \otimes V \rightarrow U$ describes the process of emission. If $B$ is a coalgebra in $C$, then the result of comultiplication $\triangle : B \rightarrow B \otimes B$ represents a composite object composed from copies of objects of the same nature. In other words the arrow $\triangle : C \rightarrow C \otimes C$ represents a process of making a copy of an object.

Let $A$ be an algebra in the category $M$. The multiplication $m : A \otimes A \rightarrow A$ is a morphism in this category. In our interpretation it represents the creation process of a single object from a composite system of objects of the same species.

Let $A$ be an algebra and $B$ be a coalgebra in $C$. If we have a bilinear pairing $< -, - > : B \otimes A \rightarrow k$ such that

$$< \triangle f, s \otimes t > = < f, s \cdot t >, \quad (10)$$

where $f \in B$ and $s, t \in A$. then we say that $B$ and $A$ are in duality. In our physical interpretation this duality means that the annihilation process of quasiparticles as components of two independent systems is equivalent to their annihilation as components of one bigger system!

The cross symmetry $\Psi_{U^*, V} : U^* \otimes V \rightarrow V \otimes U^*$ corresponds for exchanging process of quasiparticles and quasiholes. If $V \equiv U$, then the pairing $g_{U} : U^* \otimes U \rightarrow C$ describes the process of annihilation of a pair, quasiparticle and quasi-hole. In this way we can characterize in general a category $C$ representing all possible quantum processes of certain nature.

Let $U$ represents physical objects before interactions and $V$ represents possible results of interactions. If results of a given interaction can be obtained in a few different ways, then we assume that the only difference is in the phase. We identify all objects with different phase factor. Such identification is a natural isomorphism in the category. In this case we say that the process of interaction is consistent. If every possible process which can be represented as certain sequence arrows and objects in the category $C$ is consistent, then this category is said to be consistent. The construction and classification of categories for the description of quantum possible processes in a consistent way is a problem.

4. Quantum states and generalized statistics

In this section we are going to formulate conditions which form a starting point for the construction of category representing particles dressed with
some quanta. Let us consider a single center of interaction which can be transform into system which contains a charged particle dressed in one quantum in \( N \) different way, i.e. a system of \( N \) quasiparticles. In our approach such system is represented by as a finite set of elements

\[ Q := \{ x^i : i = 1, \ldots, N < \infty \} \quad (11) \]

which form a basis for a finite-dimensional Hilbert space \( E \) over a field of complex numbers \( \mathbb{C} \). A center with empty places (quasiholes) correspond to the basis

\[ Q^* := \{ x^{*i} : i = N, N - 1, \ldots, 1 \}. \quad (12) \]

for the complex conjugate space \( E^* \).

The process of "composition" of quasiparticles and quasiholes is described by a tensor product of spaces \( E \) and \( E^* \), respectively. In this way the product \( E \otimes \cdots \otimes E \) represents a system of \( n \) centers which can be equipped with several quanta in \( nN \) different ways. The product \( E^* \otimes \cdots \otimes E^* \) represents a system of \( n \) centers with empty places and an arbitrary mixed product of spaces \( E \) and \( E^* \) represents centers equipped with quanta and empty places.

The field \( \mathbb{C} \) of complex numbers represent the state without centers of interaction, i.e. a free and undressing particle in a vacuum state. The pairing \( g_E : E^* \otimes E \to \mathbb{C} \) and the corresponding scalar product is given by

\[ g_E(x^{*i} \otimes x^j) = \langle x^i | x^j \rangle := \delta^{ij}. \quad (13) \]

It represent the annihilation of empty places by corresponding quanta, i.e. annihilation of pairs, quasiholes by quasiparticles.

The cross statistics is described by an operator \( T \) called an elementary cross or twist \([10, 14]\). This operator is linear, invertible and Hermitian. It is given by its matrix elements

\[ T(x^{*i} \otimes x^j) = \sum T^{ij}_{kl} x^k \otimes x^{*l}. \quad (14) \]

We assume here that both spaces \( E \) and \( E^* \), the \(*\)–operation, the pairing \( g_E \) and operators \( T \) are the starting point for a construction of our category \( \mathcal{C} \). The most simple example is provided by the category \( \mathcal{C}(E, E^*, g_E, T) \) which consists of all possible tensor product of spaces \( E \) and \( E^* \) and their direct
sums. The $\ast$–operation, the pairing $g_E$ and the cross $T$ can be extended to the whole category [39].

One can also add some quotients and a braid symmetry [39, 45, 41]. The usual exchange statistics of quasiparticles is described a linear $B$ satisfying the standard braid relations

$$B_{(1)}B_{(2)}B_{(1)} = B_{(2)}B_{(1)}B_{(2)},$$

(15)

where $B_{(1)} := B \otimes id$ and $B_{(2)} := id \otimes B$. The exchange process determined by the operator $B$ is a real process. Such exchange process is possible if the dimension of the effective space is equal or great than two. Hence in this case we need two operators $T$ and $B$ for the description of our system with generalized statistics. These operators are not arbitrary. They must satisfy the following consistency conditions

$$B^{(1)}T^{(2)}T^{(1)} = T^{(2)}T^{(1)}B^{(2)},$$

$$id_{E \otimes E} + \tilde{T}(id_{E \otimes E} - B) = 0,$$

(16)

where the operator $\tilde{T} : E \otimes E \to E \otimes E$ is given by its matrix elements

$$(\tilde{T})_{ij}^{kl} = T_{ij}^{kl}.$$

(17)

We need a solution of these conditions for the construction of an example of corresponding category.

5. Hermitian Wick algebras and Fock space representation

Let us assume that a category of states $\mathcal{C}$ is given. We consider here a pair of unital and associative algebras $\mathcal{A}$ and $\mathcal{A}^*$ in $\mathcal{C}$. We assume that they are conjugated. This means that there is an anti-linear and involutive anti-isomorphism $(-)^*: \mathcal{A} \to \mathcal{A}^*$ and we have the following relations

$$b^* \cdot a^* = (a \cdot b)^*, \quad (a^*)^* = a,$$

(18)

where $a, b \in \mathcal{A}$ and $a^*, b^*$ are their images under the ant-isomorphisms $(-)^*$. Both algebras $\mathcal{A}$ and $\mathcal{A}^*$ are graded

$$\mathcal{A} := \bigoplus_n \mathcal{A}^n, \quad \mathcal{A}^* := \bigoplus_n \mathcal{A}^{*n}.$$
A linear mapping $\Psi : A^* \otimes A \rightarrow A \otimes A^*$ such that we have the following relations

\[
\begin{align*}
\Psi \circ (id_A^* \otimes m_A) &= (m_A \otimes id_A^*) \circ (id_A \otimes \Psi) \circ (\Psi \otimes id_A), \\
\Psi \circ (m_A^* \otimes id_A) &= (id_A \otimes m_A^*) \circ (\Psi \otimes id_A^*) \circ (id_A^* \otimes \Psi) \\
(\Psi(b^* \otimes a))^* &= \Psi(a^* \otimes b)
\end{align*}
\]

is said to be a cross symmetry or $*$-twist [44]. We use here the notation

\[
\Psi(b^* \otimes a) = \Sigma_{a(1)} \otimes b_{(2)}^*
\]

for $a \in A, b^* \in A^*$.

The tensor product $A \otimes A^*$ of algebras $A$ and $A^*$ equipped with the multiplication

\[
m_\Psi := (m_A \otimes m_A^*) \circ (id_A \otimes \Psi \otimes id_A^*)
\]

is an associative algebra called a Hermitian Wick algebra [27, 44] and it is denoted by $W = W_\Psi(A) = A \otimes_\Psi A^*$. This means that the Hermitian Wick algebra $W$ is the tensor cross product of algebras $A$ and $A^*$ with respect to the cross symmetry $\Psi$ [44]. Let $H$ be a linear space. We denote by $L(H)$ the algebra of linear operators acting on $H$. One can prove [44] that we have the theorem: Let $W \equiv A \otimes_\Psi A^*$ be a Hermitian Wick algebra. If $\pi_A : A \rightarrow L(H)$ is a representation of the algebra $A$, such that we have the relation

\[
(\pi_A(b))^* \pi_A(a) = \Sigma \pi_A(a(1))(\pi_A(b(2)))^*,
\]

then there is a representation $\pi_W : W \rightarrow L(H)$ of the algebra $W$. We use the following notation

\[
\pi_A(x^i) \equiv a^+_{x^i}, \quad \pi_A(x^{*i}) \equiv a_{x^{*i}}.
\]

The relations (23) are said to be commutation relations if there is a positive definite scalar product on $H$ such that operators $a^+_{x^i}$ are adjoint to $a_{x^{*i}}$ and vice versa. Let us consider a Hermitian Wick algebra $W$ corresponding for a system with generalized statistics. For the construction of such algebra we need a pair of algebras $A, A^*$ and a cross symmetry $\Psi$. It is natural to assume that these algebras have $E$ and $E^*$ as generating spaces, respectively, and there is the following condition for the cross symmetry

\[
\Psi|_{E^* \otimes E} = T + g_E.
\]
Let us consider the Fock space representation of the algebra \( \mathcal{W} \) corresponding for a system with generalized statistics. For the ground state and annihilation operators we assume that

\[
\langle 0 | 0 \rangle = 0, \quad a_s^* | 0 \rangle = 0 \quad \text{for} \quad s^* \in \mathcal{A}^*.
\]

In this case the representation act on the algebra \( \mathcal{A} \). Creation operators are defined as the multiplication in the algebra \( \mathcal{A} \)

\[
a_s^+ t := m_{\mathcal{A}}(s \otimes t), \quad \text{for} \quad s, t \in \mathcal{A}.
\]

The proper definition of the action of annihilation operators on the whole algebra \( \mathcal{A} \) is a problem.

If the action of annihilation operators are given in such a way that there is unique, nondegenerate, positive definite scalar product on \( \mathcal{A} \), creation operators are adjoint to annihilation ones and vice versa, then we say that we have the well-defined system with generalized statistics in the Fock representation \( \mathcal{W} \).

Let us consider some examples for such systems. Assume that quasiparticles and quasiholes are moving on one dimensional effective space. In this case we can construct the category \( \mathcal{C}(E, E^*, g_E) \) which consists of all possible tensor product of spaces \( E \) and \( E^* \) and their direct sums.

The algebra of states \( \mathcal{A} \) is the full tensor algebra \( TE \) over the space \( E \), and the conjugate algebra \( \mathcal{A}^* \) is identical with the tensor algebra \( TE^* \). If \( T \equiv 0 \) then we obtain the most simple example of well-defined system with generalized statistics. The corresponding statistics is the so-called infinite Bolzman) statistics \( \mathcal{W} \). The action of annihilation operators is given by the formula

\[
a_{x^*i_1 \otimes \cdots \otimes x^*i_1}(x^{j_1} \otimes \cdots \otimes x^{j_n}) := \delta_{i_1}^{j_1} \cdots \delta_{i_k}^{j_k} x^{j_{n-k+1}} \otimes \cdots \otimes x^{j_n}.
\]

For the scalar product we have the equation

\[
\langle i_n \cdots i_1 | j_1 \cdots j_n \rangle^n := \delta_{i_1}^{j_1} \cdots \delta_{i_n}^{j_n},
\]

where \( i_n \cdots i_1 := x^{i_n} \cdots x^{i_1} \). It is easy to see that we have the relation and

\[
a_{x^*i} a_{x^*j}^+ := \delta_{i}^{j} 1.
\]
given, then the corresponding cross symmetry $\Psi^T : TE^* \otimes TE \to TE \otimes TE^*$ is defined by a set of mappings $\Psi_{k,l} : E^* \otimes E \otimes E^* \otimes E \to E \otimes E^* \otimes E \otimes E^*$, where $\Psi_{1,1} \equiv R := T + g_E$, and

$$
\Psi_{1,l} := R_l^{(l)} \circ \ldots \circ R_l^{(1)}, \\
\Psi_{k,l} := (\Psi_{1,l})^{(l)} \circ \ldots \circ (\Psi_{1,l})^{(k)},
$$

here $R_l^{(i)} : E_l^{(i)} \to E_l^{(i+1)}$, $E_l^{(i)} := E \otimes \ldots \otimes E^* \otimes E \otimes \ldots \otimes E$ ($l + 1$-factors, $E^*$ on the $i$-th place, $i \leq l$) is given by the relation

$$
R_l^{(i)} := \underbrace{id_E \otimes \ldots \otimes R \otimes \ldots \otimes id_E}_{l \text{ times}},
$$

where $R$ is on the $i$-th place, $(\Psi_{1,l})^{(i)}$ is defined in similar way like $R^{(i)}$. The commutation relations (23) can be given here in the following form

$$
a_{x^*}a_{x^+}^+ - T^{ij}_{ki} a_{x^+}^+ a_{x^+} = \delta^{ij} 1. \quad (32)
$$

If the operator $\tilde{T}$ is a bounded operator acting on some Hilbert space such that we have the following Yang-Baxter equation on $E \otimes E \otimes E$

$$
(\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) = (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}), \quad (33)
$$

and $||\tilde{T}|| \leq 1$, then according to Bożejko and Speicher $[3]$ there is a positive definite scalar product

$$
\langle s|t \rangle^n_T := \langle s|P_n t \rangle^n_0, \quad (34)
$$

where $s, t \in TE$, $P_1 \equiv id$, $P_2 \equiv R_2 \equiv id_{E \otimes E} + \tilde{T}$, and

$$
P_{n+1} := (id \otimes P_n) \circ R_{n+1}, \quad (35)
$$

the operator $R_n$ is given by the formula

$$
R_n := id + \tilde{T}^{(1)} + \tilde{T}^{(1)}\tilde{T}^{(2)} + \ldots + \tilde{T}^{(1)}\ldots\tilde{T}^{(n-1)}. \quad (36)
$$

Note that the existence of nontrivial kernel of operator $P_2 \equiv id_{E \otimes E} + \tilde{T}$ is essential for the nondegeneracy of the scalar product $[27]$. One can see that
if this kernel is trivial, then we obtain well-defined system with generalized statistics \[28, 30\].

If the dimension of the effective space is greater than one and the kernel of \(P_2\) is nontrivial, then the scalar product is degenerate. Hence we must remove this degeneracy by factoring the mentioned above scalar product by the kernel. In this case our category contains some quotients and we have \(A := TE / I, \quad A^* := TE^*/I^*\), where \(I := \text{gen}\{id_{E \otimes E} - B\}\) is an ideal in \(TE\) and \(B : E \otimes E \to E \otimes E\) is a linear and invertible operator satisfying the braid relation \[13\] and the consistency conditions \[16\], \(I^*\) is the corresponding conjugated ideal in \(TE^*\). One can see that there is the cross symmetry and the action of annihilation operators can be defined in such a way that we obtain the well-defined system with the usual braid statistics \[28, 30, 41\].

We have here the following commutation relations

\[
\begin{align*}
a_{x^i} a^+_{x^j} - T^{ij}_{kl} a^+_{x^k} a_{x^l} &= \delta^{ij} 1, \\
a_{x^i} a^+_{x^j} - B^{ij}_{kl} a_{x^k} a^+_{x^l} &= 0, \\
a^+_{x^i} a^+_{x^j} - B^{ij}_{kl} a^+_{x^k} a^+_{x^l} &= 0.
\end{align*}
\]

Observe that for \(T \equiv B \equiv \tau\), where \(\tau\) represents the transposition \(\tau(x^i \otimes x^j) := x^j \otimes x^i\) we obtain the usual canonical commutation relations of bosons or fermions. Note that similar relations are described by Fiore \[46\].

We can see that the commutation relations \[32\] corresponding for the system with cross symmetry can be understood as a deformation of relations \[30\] of Boltzmann statistics. The commutation relations \[37\] for the system with a braid group statistics are in fact certain degenerated system. In this way the infinite statistics seems to be the most general statistics and others statistics are its deformed or degenerated version.

6. Category of comodules

Let us consider a category of \(H\)-comodules, where \(H\) is a finite Hopf algebra \(H = H(m, u, \Delta, \eta, S)\), equipped with the multiplication \(m\), the unit \(u\), the comultiplication \(\Delta\), the counit \(\eta\) and the antipode \(S\). According to our physical interpretation the algebra represents an external quantum field and comodules – a charged particle dressed with quanta of this field. We use the following notation for the coproduct in \(H\): if \(h \in H\), then \(\Delta(h) := \sum h_{(1)} \otimes h_{(2)} \in H \otimes H\). We assume that \(H\) is coquasitrivial Hopf algebra.

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This means that $H$ equipped with a bilinear form $b : H \otimes H \to \mathbb{C}$ such that

$$
\sum b(\xi_{(1)}, \eta_{(2)}) \eta_{(2)} = \sum \eta_{(1)} \xi_{(1)} b(\eta_{(2)}, \eta_{(2)}),
$$

$$
b(h, kl) = \sum b(h_{(1)}, k) b(h_{(2)}, l),
$$

$$
b(hk, l) = \sum b(h, l_{(2)}) b(k, l_{(1)}),
$$

(38)

for every $h, k, l \in H$. If such bilinear form $b$ exists for a given Hopf algebra $H$, then we say that there is a coquasitriangular structure on $H$. Let us assume for simplicity that $H \equiv \mathbb{C}G$ is a group algebra, where $G$ is an Abelian group. The group algebra $H := \mathbb{C}G$ is a Hopf algebra for which the comultiplication, the counit, and the antipode are given by the formulae

$$
\Delta(g) := g \otimes g, \quad \eta(g) := 1, \quad S(g) := g^{-1} \quad \text{for } g \in G.
$$

respectively. If $G$ is an Abelian group, then the coquasitriangular structure on $H = \mathbb{C}G$ is given by a commutation factor $\epsilon : G \otimes G \to \mathbb{C} \setminus \{0\}$ on $G$.

A right $H$-Hopf module is a $k$-linear space $M$ such that

(i) there is a right $H$-module action $\lhd : M \otimes H \to M$,

(ii) there is a right $H$-comodule map $\delta : M \to M \otimes H$,

(iii) $\delta$ is a right $H$-module map, this means that we have the relation

$$
\sum (m \lhd h)_{(0)} \otimes (m \lhd h)_{(1)} = \sum m_{(0)} \lhd h_{(1)} \otimes m_{(1)} h_{(2)},
$$

(39)

where $m \in M, h \in H, \delta(m) = \sum m_{(0)} \otimes m_{(1)}$, and $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

It is very interesting that every right $H$-Hopf module $M$ is a tensor product $E = V \otimes H$, where $V \equiv E^{coH}$ is also a trivial right $H$-module, i. e. $m \lhd h = \eta(h)m$ for every $m \in E$. If $V$ is a finite–dimensional vector space equipped with a basis $\{\xi^i : i = 1, \ldots, n\}$ and $H$ is a finite Hopf algebra equipped with a vector space basis $\{h^j : i = 1, \ldots, N\}$, then we have the following basis in $E$

$$
x^i := \xi^i \otimes h^j \quad \text{(no sum).}
$$

(40)

In this basis the right coaction is given by the relation

$$
\delta(x^i) := \sum x^{i(1)} \otimes h^{i(2)}.
$$

(41)

Let $H$ be a CQTHA with coquasitriangular structure $(-,-)$. The family of all $H$-comodules forms a category $\mathcal{C} = \mathcal{M}^H$. The category $\mathcal{C}$ is braided.
monoidal. The braid symmetry $\Psi \equiv \{\Psi_{U,V} : U \otimes V \rightarrow V \otimes U; U, V \in \text{Ob}\mathcal{C}\}$ in $\mathcal{C}$ is defined by the equation

$$\Psi_{U,V}(u \otimes v) = \Sigma b(v_1, u_1) \ v_0 \otimes u_0, \quad (42)$$

where $\rho(u) = \Sigma u_0 \otimes u_1 \in U \otimes H$, and $\rho(v) = \Sigma v_0 \otimes v_1 \in V \otimes H$ for every $u \in U, v \in V$. Let $H$ be a CQTHA with coquasitriangular structure given by a form $b : H \otimes H \rightarrow k$ and let $\mathcal{A}$ be a (right) $H$-comodule algebra with coaction $\rho$. Then the algebra $\mathcal{A}$ is said to be quantum commutative with respect to the coaction of $(H, b)$ if an only if we have the relation

$$a \ b = \Sigma b(a_1, b_1) \ b_0 \ a_0, \quad (43)$$

where $\rho(a) = \Sigma a_0 \otimes a_1 \in \mathcal{A} \otimes H$, and $\rho(b) = \Sigma b_0 \otimes b_1 \in \mathcal{A} \otimes H$ for every $a, b \in \mathcal{A}$, see [48]. The Hopf algebra $H$ is said to be a quantum symmetry for $\mathcal{A}$.

If $H \equiv kG$, where $G$ is an Abelian group, $k \equiv \mathbb{C}$ is the field of complex numbers, then the coquasitriangular structure on $H$ is given as a bicharacter on $G$ [47], i.e. a mapping $\epsilon : G \times G \rightarrow \mathbb{C} \setminus \{0\}$ such that we have the following relations

$$\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma), \quad \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma) \quad (44)$$

for $\alpha, \beta, \gamma \in G$. If in addition

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = 1, \quad (45)$$

for $\alpha, \beta \in G$. It is interesting that the coaction of $H$ on certain space $\mathcal{V}$, where $H \equiv kG$ is equivalent to the $G$-gradation of $\mathcal{V}$, see [47]. In this case the quantum commutative algebra $\mathcal{A}$ becomes graded commutative

$$x^i x^j = \epsilon(|i|, |j|) x^j x^i \quad (46)$$

for homogeneous elements $x^i$ and $x^j$ of grade $|i|$ and $|j|$, respectively. One can describe the corresponding algebra $\mathcal{A}^*$ in a similar way. For both two operators $T$ and $B$ we have the following relations

$$T(x^i \otimes x^j) := \epsilon(|j|, |i|) x^j \otimes x^i, \quad B(x^i \otimes x^j) := \epsilon(|i|, |j|) x^j \otimes x^i \quad (47)$$

The construction of related Hermitian Wick algebra and Fock space representation is evident, see [41]. Note that in this case the category of comodules $\mathcal{C} = \mathcal{M}^H$ is denoted by $\mathcal{C} \equiv \mathcal{C}(G, \epsilon)$. 

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