The entangling power of passive optical elements

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We investigate the entangling capability of passive optical elements, both qualitatively and quantitatively. We present a general necessary and sufficient condition for the possibility of creating distillable entanglement in an arbitrary multi-mode Gaussian state with the help of passive optical elements, thereby establishing a general connection between squeezing and the entanglement that is attainable by non-squeezing operations. Special attention is devoted to general two-mode Gaussian states, for which we provide the optimal entangling procedure, present an explicit formula for the attainable degree of entanglement measured in terms of the logarithmic negativity, and discuss several practically important special cases.

Entangled states of light field modes may be generated by transmitting two squeezed states through a beam splitter [1]. This is one of the experimentally accessible procedures for generating continuous-variable entanglement in optical systems [2]. Moreover, it is a particular example of a situation where passive optical elements exhibit their entangling power when applied to Gaussian input states. It is well known that the presence of squeezing is necessary for obtaining entanglement in this manner [1]. However, the degree of the attained entanglement is by no means the same for all input states: it depends to a large extent on the degree and direction of squeezing of the incoming modes and on the specific properties of the beam splitter. This raises the question under what circumstances such an entangling procedure is optimal in the sense of generating states which have the maximal attainable amount of entanglement. And in general, by means of arbitrary passive optical elements, what are the requirements such that entanglement can be generated between any bi-partite split of a system in a multi-mode Gaussian state?

In this letter we address the question of the entangling power of passive optical elements acting on any number of modes in an arbitrary Gaussian state, qualitatively as well as quantitatively. Passive optical operations can be implemented by using beam splitters and phase shifters [3]. These are cheap operations and easy to implement in contrast to squeezing operations. Therefore we will consider squeezing as a potential resource for entanglement and ask for the requirements and the optimal way of entangling a squeezed state by means of passive operations, which we assume to be available in arbitrary quantities. The main result and starting point is a necessary and sufficient condition for the possibility of creating distillable entanglement on general Gaussian initial states – pure or mixed – between any bi-partite split of an \( n \)-mode system with the help of passive optical elements. We then introduce a lower bound for the attainable degree of entanglement measured in terms of the logarithmic negativity [4] for \( n \)-mode systems. Moreover, we derive a general formula for the largest degree of entanglement of an arbitrary subsystem consisting of two modes. The operations that can be implemented with passive optical elements can be identified with the non-squeezing operations. In this sense we establish a quantitative connection between the degree of squeezing of a Gaussian state and the degree of entanglement that is attainable with the application of non-squeezing operations. Of particular interest is the case where only two modes are present. We will discuss this situation in more detail by explicitly constructing the optimal entangling procedure and discussing several meaningful special cases.

We start by introducing the formalism that we will use extensively. Gaussian states are completely characterized by their first and second moments, where only the latter, given in terms of a covariance matrix \( \Gamma \), carry information about entanglement and squeezing. For this reason we will set the first moments to zero, which can always be achieved by unitary operations on individual modes. The covariance matrix is then given by \( \Gamma_{kl} = 2 \langle R_k R_l \rangle - i \sigma_{kl}, k,l = 1, \ldots, 2n \), where the vector \( R = (Q_1, \ldots, Q_n, P_1, \ldots, P_n) \) consists of the canonical coordinates for \( n \) modes and the symplectic matrix

\[
\sigma = \begin{pmatrix}
0 & \mathbb{1}_n \\
-\mathbb{1}_n & 0
\end{pmatrix}
\]

(1)
governs the canonical commutation relations (CCR) \([\hat{R}_k, \hat{R}_l] = i \sigma_{kl} \). A matrix represents an admissible covariance matrix if it satisfies the Heisenberg uncertainty relations \( \Gamma + i \sigma \geq 0 \). Symplectic transformations \( \Gamma \mapsto S^T \Gamma S \) preserve the CCR and therefore satisfy \( S^T \sigma S = \sigma \). All symplectic transformations correspond to unitary Gaussian
operations [6] on the level of states, in the sense that the Gaussian character of arbitrary input states is preserved under such unitary operations. They can be decomposed [5,7] into active/nonlinear and passive/linear operations [8]. The latter can be implemented by using passive optical elements such as beam splitters and phase plates only [3], and are of the form \( \Gamma \mapsto K^T \Gamma K \),

\[
K = \Omega \left( \begin{array}{cc} U & 0 \\ 0 & U \end{array} \right) \Omega = \left( \begin{array}{cc} X & Y \\ -Y & X \end{array} \right).
\]

Here \( U = X + iY \) (\( X, Y \) real) is any unitary matrix and

\[
\Omega = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} \mathbb{I}_n & i\mathbb{I}_n \\ \mathbb{I}_n & -i\mathbb{I}_n \end{array} \right)
\]

relates real and complex representations by mapping creation/annihilation operators to position/momentum operators via \( \Omega(R_1, \ldots, R_{2n})^T = (a_1, \ldots, a_n, a_1^*, \ldots, a_n^*)^T \). Transformations of the type \( \Gamma \mapsto K^T \Gamma K \) with \( K \) as above will from now on be denoted as **passive transformations**. Any such \( K \) is both symplectic and orthogonal, i.e., \( K^T K = \mathbb{I} \), and the set of all symplectic transformations that can be implemented with passive optical elements form a group, the maximal compact subgroup \( K(n) \) of the group of symplectic transformations \( Sp(2n, \mathbb{R}) \) [5].

A Gaussian state is said to be **squeezed** if there exists a basis in phase space such that at least one diagonal element of the covariance matrix is smaller than one. From now on we order the eigenvalues of \( \Gamma \) in non-increasing order, so that this implies that the smallest eigenvalue \( \lambda_1 \) of \( \Gamma \) is smaller than one [5]. Since every passive transformation \( K \) is orthogonal, it does not affect the squeezing of a state.

Let us now turn to entanglement properties. A Gaussian state of a bi-partite system consisting of parts \( A \) and \( B \) with \( n_A + n_B = n \) modes is **separable**, i.e., unentangled between \( A \) and \( B \), iff there exist covariance matrices \( \Gamma_A, \Gamma_B \) for \( n_A \) resp. \( n_B \) modes such that \( \Gamma \geq \Gamma_A \oplus \Gamma_B \) [9,10]. A necessary and for \( 1 \times n_B \) modes also sufficient condition for separability [9,11] is that the partial transpose of the state is positive semi-definite. This, in turn is equivalent to \( \Gamma \geq i\sigma \), with the partially transposed symplectic matrix \( \sigma = (\mathbb{I}_n \oplus E)\sigma(\mathbb{I}_n \oplus E) \) and \( E = \mathbb{I}_{n_A} \oplus (-\mathbb{I}_{n_B}) \) being the partial transposition operator that reverses all momenta on one side. It has been shown that a Gaussian state is distillable, i.e., that its entanglement can be revealed using local operations and classical communication, iff its partial transpose is non-positive [12].

Obviously, every entangled Gaussian state is squeezed since \( \lambda_1 \geq 1 \) would mean that \( \Gamma \geq \mathbb{I} \oplus \mathbb{I} \) which in turn implies separability. Hence, a state can only be entangled by means of passive operations if it is squeezed initially. The following proposition gives a necessary and sufficient condition for the possibility of transforming a general Gaussian state into a distillable one by means of passive transformations (see Fig. 1):

**Proposition 1** Let \( \Gamma \) be a covariance matrix corresponding to a Gaussian state of \( n \) modes. A passive transformation \( \Gamma \mapsto \Gamma' = K^T \Gamma K \leading to an entangled state having a non-positive partial transpose with respect to a partition into \( n_A + n_B = n \) modes exists iff

\[
\lambda_1\lambda_2 < 1,
\]

where \( \lambda_1, \lambda_2 \) are the two smallest eigenvalues of \( \Gamma \).

**Proof:** A Gaussian state of an \( n \)-mode system with covariance matrix \( \Gamma' \) has a positive partial transpose iff all symplectic eigenvalues of the respective partially transposed covariance matrix \( \Gamma'' = (\mathbb{I}_n \oplus E)\Gamma'(\mathbb{I}_n \oplus E) \) are larger than or equal to one. The symplectic eigenvalues of \( \Gamma'' \) are in turn equal to the square roots of the ordinary eigenvalues of \(- (\Gamma' \sigma)^2 \). The square of the smallest symplectic eigenvalue additionally minimized over all passive transformations is thus given by

\[
\nu := \inf_{K} \inf_{||\xi||=1} \langle \xi|\Gamma^1/2 M^T \Gamma M \Gamma^1/2 |\xi \rangle,
\]

where \( M := K \sigma K^T \) and \( ||\cdot|| \) denotes the standard vector norm. Hence, we have to show that inequality (4) is equivalent to \( \nu < 1 \).

Since \( M \) is an antisymmetric orthogonal matrix, it maps any real unit vector onto the two-dimensional unit sphere of its orthogonal complement. The vector

\[
||\xi'(K, \xi) := M \Gamma^1/2 |\xi \rangle \cdot \langle \xi|\Gamma|\xi \rangle^{-1/2},
\]

therefore satisfies \( \langle \xi|\Gamma^1/2 |\xi'(K, \xi) \rangle = 0 \). Inserting Eq. (6) in Eq. (5) we get

\[
\nu = \inf_{K} \inf_{||\xi||=1} \langle \xi|\Gamma^1/2 \xi'(K, \xi) \rangle \langle \xi'(K, \xi)|\Gamma'|\xi'(K, \xi) \rangle
\]

\[
\geq \inf_{||\xi||=1} \langle \xi|\Gamma|\xi \rangle \langle \xi'|\Gamma' \xi' \rangle,
\]

where the infimum in Eq. (8) is taken over all real unit vectors satisfying \( \langle \xi|\Gamma^1/2 \xi' \rangle = 0 \). This relaxes the requirement that \( |\xi' \rangle \) has to be of the form in Eq. (6) and therefore leads to the lower bound. The minimum in Eq. (8) is now attained for vectors lying in the two-dimensional space corresponding to the two smallest eigenvalues \( \lambda_1, \lambda_2 \) of \( \Gamma \). Hence, \( |\xi \rangle = \cos \phi |\lambda_1 \rangle + \sin \phi |\lambda_2 \rangle \) for some \( \phi \) and \( |\xi' \rangle \propto \sqrt{\lambda_2 \sin \phi |\lambda_1 \rangle - \sqrt{\lambda_1 \cos \phi |\lambda_2 \rangle}}. \) However, every \( \phi \) leads to the same value and we have

\[
\nu \geq \lambda_1 \lambda_2,
\]

showing that \( \lambda_1 \lambda_2 < 1 \) is indeed necessary for \( \nu < 1 \).

In order to prove sufficiency, we have to show that there always exists a passive transformation \( K \) such that \( |\xi' \rangle \) is of the form (6) and the inequalities (8,9) thus become equalities. Note that this is in turn equivalent to the statement that for every pair of orthogonal real unit vectors \( |\xi \rangle \downarrow |\eta \rangle \) there is a passive transformation \( K \) such that \( |\eta \rangle K \sigma K^T |\xi \rangle = 1 \). We first show that the problem can be reduced to a two-mode problem. Let \( |\lambda_1 \rangle \) and \( |\lambda_2 \rangle \) be the eigenvectors associated with \( \lambda_1 \) and \( \lambda_2 \). Decomposing \( |\lambda_1 \rangle, |\lambda_2 \rangle \) into position and momentum
components, one may define the complex form of $|\lambda_1\rangle$ and $|\lambda_2\rangle$ according to

$$|\Psi_i\rangle := |\lambda_1^{(Q)}\rangle + i|\lambda_1^{(P)}\rangle, \ i = 1, 2.$$ (10)

Then there always exists a unitary $U$ such that the vectors $U|\Psi_i\rangle$ have only two non-zero components in the first two entries of the vector. In turn, according to Eqs. (2,3) this implies that there exists a passive transformation $S$ such that $\Gamma' := S^{T}\Gamma S$ has the property that the leading principle submatrix of $\Gamma'$ associated with the first two modes has the same two smallest eigenvalues $\lambda_1, \lambda_2$.

Similarly, $|\xi\rangle, |\eta\rangle$ can be decomposed into position and momentum components, and define $|\Psi\rangle := |\xi^{(Q)}\rangle + i|\xi^{(P)}\rangle$ and $|\Phi\rangle := |\eta^{(Q)}\rangle + i|\eta^{(P)}\rangle$ such that $\Omega|\xi\rangle = \Omega|\xi^{(Q)}\oplus\xi^{(P)}\rangle = \sqrt{2}|\Psi\rangle \otimes|\overline{\Psi}\rangle$ and analogous for $|\eta\rangle$ and $|\Phi\rangle$. Then $|\Psi\rangle|\langle\Psi| = |\Phi\rangle|\langle\Phi|$ and

$$\langle\eta|K\tilde{\sigma}K^T\xi\rangle = |\langle\Psi|U\rho U^\dagger|\Psi\rangle|,$$ (11)

$$\langle\eta|\xi\rangle = \text{Re}[\langle\Phi|\Psi]\rangle \equiv 0.$$ (12)

Without loss of generality we fix $|\Psi\rangle = (1, 0)^T$, which can always be achieved by applying an additional unitary. Then, every two-dimensional unit vector $|\Phi\rangle$ for which $\text{Re}[\langle\Phi|\Psi]\rangle = 0$ is of the form

$$|\Phi\rangle = -i(\cos(2\gamma), e^{2i\alpha}\sin(2\gamma))^T.$$ (13)

Choosing $U = \begin{pmatrix} e^{-i\alpha}\cos(\gamma) & -e^{-i\alpha}\sin(\gamma) \\ e^{i\alpha}\sin(\gamma) & e^{i\alpha}\cos(\gamma) \end{pmatrix},$ (14)

we obtain with $E = \text{diag}(1, -1)$

$$1 = \text{Im}[\langle\Phi|U\rho U^\dagger|\Psi\rangle] = \langle\eta|K\tilde{\sigma}K^T\xi\rangle,$$

which completes the proof.

Whereas every entangled state is squeezed, Proposition 1 implies that conversely any squeezed state can be entangled by using passive optical elements supplemented by a single additional vacuum mode (empty port of a beam splitter), because the joint covariance matrix $\Gamma \oplus I_2$ [10] then satisfies inequality (4). The optimal entangling procedure consists then of two steps: (i) One first applies a passive transformation $S$ such that the smallest eigenvalue $\lambda_1$ of $\Gamma$ is also the smallest eigenvalue of the $2 \times 2$ principal submatrix of $S^T\Gamma S$ corresponding to the first mode. (ii) One then applies the optimal entangling procedure on this mode and the vacuum mode, which will be derived in Proposition 3.

The proof of proposition 1 leads to a lower bound for the attainable entanglement measured in terms of the logarithmic negativity [4]. The latter is so far the only calculable entanglement measure for mixed Gaussian states. For an $n$-mode Gaussian state $\rho$ it is given by

$$E_N = \sum_i \min(0, \log_2(s_i)).$$ (15)

where the $s_i, i = 1, \ldots, n$ are the symplectic eigenvalues of the partially transposed covariance matrix. Since $\nu = \lambda_1\lambda_2$ is the square of the smallest symplectic eigenvalue, we obtain

$$E_N \geq \max(0, -\log_2(\lambda_1\lambda_2)/2)$$ (16)

for the attainable entanglement, with equality if there is only one $s_i$ smaller than one. A particularly transparent situation is now the case where we consider only the entanglement present in an arbitrary two-mode subsystem obtained when tracing out the other modes at the end.

**Proposition 2** Let $\Gamma \mapsto \Gamma' = K^T\Gamma K$ be a passive transformation acting on a Gaussian state of $n \geq 2$ modes with covariance matrix $\Gamma$. The maximum attainable amount of entanglement obtained for an arbitrary two-mode subsystem of $\Gamma'$ is then given by

$$E_N = \max(0, -\log_2(\lambda_1\lambda_2)/2),$$ (17)

where $\lambda_1, \lambda_2$ are the two smallest eigenvalues of $\Gamma$.

**Proof:** First note that for the case of a two-mode state only one of the two symplectic eigenvalues $s_1, s_2$ of the partially transposed covariance matrix $\Gamma_{(2)}'$ can be smaller than one, since $(s_1s_2)^2 = \text{det} \Gamma_{(2)}' = \text{det} \Gamma_{(2)}' \geq 1$ [13]. Following the same argument as in Proposition 1, there exists always a partial transformation $S$ such that a two-mode principal submatrix of the covariance matrix $S^T\Gamma S$ has the two smallest eigenvalues $\lambda_1$ and $\lambda_2$, which leads to equality in Eq. (16).

A special instance of Proposition 2 is the case where the input already is a two-mode system, i.e. $n = 2$. For this case we will now explicitly construct the optimal entangling procedure. We will show that it is always sufficient to perform a single phase rotation in one of the two modes, for example in A, succeeded by a beam splitter operation on both modes. Again, it is most convenient to employ the complex version of the problem. In their complex forms, a beam splitter $B(\gamma)$ and a phase shift $L(\alpha)$ in system $A$ are represented by the matrices

$$B(\gamma) = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix}, \quad L(\alpha) = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & 1 \end{pmatrix},$$ (18)

where $\gamma \in [0, 2\pi)$ determines the transmission coefficient of the beam splitter, and $\alpha \in [0, \pi)$ is the phase difference of the incoming and outgoing fields. Without loss of generality the beam splitter itself is assumed to induce no phase difference.

**Proposition 3** Let $\rho$ be a Gaussian $1 \times 1$-mode state with covariance matrix $\Gamma$. Let $|\lambda_1\rangle$ and $|\lambda_2\rangle$ be the eigenvectors of the two smallest eigenvalues of $\Gamma$, with complex versions $|\Psi_1\rangle$ and $|\Psi_2\rangle$. The optimal entangling operation using only passive optical elements is given by a phase rotation $L(\alpha)$ on mode $A$, followed by a beam-splitter $B(\gamma/2)$, such that $\gamma$ and $\alpha$ are the solutions of

$$\cos(\gamma) = \text{Im}[\langle\Psi_2|\sigma_\gamma|\Psi_1\rangle],$$ (19)

$$\sin(\alpha)\sin(\gamma) = \text{Im}[\langle\Psi_2|\sigma_y|\Psi_1\rangle],$$ (20)

$$\cos(\alpha)\sin(\gamma) = \text{Im}[\langle\Psi_2|\sigma_z|\Psi_1\rangle],$$ (21)

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.
Proof: In order to find the optimal entangling procedure one has to identify a unitary $V$ such that

$$\text{Im} \left[ \langle \Psi_2 | V E \Psi_1 \rangle \right] = 1,$$  \hspace{1cm} (22)

and decompose it into a beam splitter and a phase shift. The most general form for $V E \Psi_1 =: F$ is given by

$$F(\gamma, \alpha) = \begin{pmatrix} \cos(\gamma) & e^{-i\alpha} \sin(\gamma) \\ e^{i\alpha} \sin(\gamma) & -\cos(\gamma) \end{pmatrix},$$  \hspace{1cm} (23)

which corresponds to $V = L(\alpha) B(\gamma/2)$. Inserting the decomposition $F(\gamma, \alpha) = \cos(\gamma) \sigma_x + \cos(\alpha) \sin(\gamma) \sigma_y + \sin(\alpha) \sin(\gamma) \sigma_x$ into Eq. (23) one verifies that values $\alpha, \gamma$ that satisfy Eqs. (19-21) provide a solution of Eq. (23). Moreover, the set of equations (19-21) always has a solution, since the vector of the imaginary parts in Eqs. (19-21) can be shown to be a unit vector if $\text{Re} \langle \Psi_2 | \Psi_1 \rangle = 0$.

We will in the following apply this result to some special cases. The covariance matrix $\Gamma$ of the initial state of the $1 \times 1$-mode system will be written in the block form

$$\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$  \hspace{1cm} (24)

where $A$ and $B$ are the reduced covariance matrices corresponding to mode $A$ and $B$ respectively. Depending on the form of the $2 \times 2$-matrices $A$, $B$, and $C$ several optimal entangling protocols can be identified:

(i) A product of arbitrary single mode Gaussian states: If $C = 0$, then a $50 : 50$ beam-splitter is required in the optimal entangling procedure. The phase transformation that is needed will in general depend on the actual form of $A$ and $B$. In particular:

- (ia) A product of two identical single mode states: In this case $A = B$ and $C = 0$, and one finds that $\alpha = \gamma = \pi/2$. The optimal entangling operation is thus a $50 : 50$ beam-splitter, which follows a $\alpha = \pi/2$ phase transformation, as expected. This is the optimal procedure for uncorrelated identical Gaussian input states used in several experiments [2].

- (ib) A product of a Gaussian single mode state and a coherent or thermal state: In this case where $B = b \mathbb{I}$, $b \geq 1$, and $C = 0$ the optimal entangling operation is again the application of a $50 : 50$ beam-splitter. No phase transformation is required.

(ii) States with covariance matrix in Simon normal form [11]: If $A = a \mathbb{I}$, $B = b \mathbb{I}$, $C = \text{diag}(c, d)$, then one eigenvector $\Psi_1$ is real and the other is imaginary. Hence $\alpha \in [0, \pi]$, whereas the optimal beam splitter is in general not balanced.

- (ia) Symmetric states: These are states with identical thermal reductions, meaning that $A = B = a \mathbb{I}$, $a \geq 1$. These states are already optimally entangled, since $E_N(\rho) = \max \left[ 0, -\log(\lambda_1, \lambda_2)/2 \right]$, and the optimal entangling procedure is thus the identity operation.

- (ib) Special cases of symmetric states are two-mode squeezed pure Gaussian states with covariance matrix in Simon normal form, where in addition, $C$ takes the form $C = \text{diag}(c, -c)$ with $c = (1 - a^2)^{1/2}$.

In this letter we have investigated the entangling capabilities of passive optical elements in a general setting. We have presented a necessary and sufficient criterion for the possibility of creating distillable entanglement in a multi-mode system that has been prepared in a Gaussian state. The findings reveal in fact a surprisingly simple close relationship between squeezing and attainable entanglement. We have moreover quantified the maximal degree of entanglement that can be achieved in a two-mode subsystem, and we have identified the optimal entangling procedure for the case of two input modes. In view of recently proposed applications of quantum information science, we hope that the presented results as well as the employed techniques may prove useful tools in the study of feasible sources of continuous-variable entanglement.

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