ASYMPTOTICS OF THE AVERAGE HEIGHT OF 2–WATERMELONS WITH A WALL.

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Abstract. We generalize the classical work of de Bruijn, Knuth and Rice (giving the asymptotics of the average height of Dyck paths of length \( n \)) to the case of \( p \)–watermelons with a wall (i.e., to a certain family of \( p \) nonintersecting Dyck paths; simple Dyck paths being the special case \( p = 1 \).) We work out this asymptotics for the case \( p = 2 \) only, since the computations involved are already quite complicated (but might be of some interest in their own right).

1. Introduction

The model of vicious walkers was originally introduced by Fisher [9] and received much interest, since it leads to challenging enumerative questions. Here, we consider special configurations of vicious walkers called \( p \)–watermelons with a wall.

Briefly stated, a \( p \)–watermelon of length \( n \) is a family \( P_1, \ldots P_p \) of \( p \) nonintersecting lattice paths in \( \mathbb{Z}^2 \), where

- \( P_i \) starts at \((0, 2i - 2)\) and ends at \((2n, 2i - 2)\), for \( i = 1, \ldots, p \),
- all the steps are directed north–east or south–east, i.e., lead from lattice point \((i, j)\) to \((i + 1, j + 1)\) or to \((i + 1, j - 1)\),
- no two paths \( P_i, P_j \) have a point in common (this is the meaning of “nonintersecting”).

The height of a \( p \)–watermelon is the \( y \)–coordinate of the highest lattice point contained in any of its paths (since the paths are nonintersecting, it suffices to consider the lattice points contained in the highest path \( P_p \); see Figure 1 for an illustration.)

A \( p \)–watermelon of length \( n \) with a wall has the additional property that none of the paths ever goes below the line \( y = 0 \) (since the paths are nonintersecting, it suffices to impose this condition on the lowest path \( P_1 \); see Figure 2 for an illustration.).

In [2], Bonichon and Mosbah considered (amongst other things) the average height \( H(n, p) \) of \( p \)–watermelons of length \( n \) with a wall,

\[
H(n, p) = \frac{1}{\#(\text{all } p\text{-watermelons of length } n)} \times \sum_h h \cdot \#(\text{all } p\text{-watermelons of length } n \text{ and height } h),
\]

and derived by computer experiments the following conjectural asymptotics [2 4.1]:

\[
H(n, p) \sim \sqrt{(1.67p - 0.06)2n + o(\sqrt{n})}.
\]
The purpose of this paper is to work out the exact asymptotics for the simple special case $p = 2$. This will be done by imitating the classical reasoning of de Bruijn, Knuth and Rice [5] for the case $p = 1$ (i.e., for the average height of Dyck paths). However, even the case $p = 2$ involves rather complicated computations. In particular, we shall need informations about residues and evaluations of a double Dirichlet series, which we shall (partly) obtain by imitating Riemann’s representation of the zeta function [6, section 1.12, (16)].

1.1. Notational conventions. For $k, n \in \mathbb{Z}$, we shall use the notation introduced in [12] for the rising and falling factorial powers, i.e.

$$(n)^k := 0 \text{ if } k < 0, \quad (n)_k := 1, \quad (n)^k := n \cdot (n + 1) \cdots (n + k - 1) \text{ if } k > 0$$

and

$$(n)^k := 0 \text{ if } k < 0, \quad (n)_k := 1, \quad (n)^k := n \cdot (n - 1) \cdots (n - k + 1) \text{ if } k > 0.$$ 

For the binomial coefficient we adopt the convention

$$\binom{n}{k} := \begin{cases} \frac{(n)^k}{k!} & \text{if } 0 \leq k \leq n, \\ 0 & \text{else.} \end{cases}$$
Moreover, we shall use Iverson’s notation:

\[ [\text{some assertion}] = \begin{cases} 
1 & \text{if “some assertion” is true,} \\
0 & \text{else.} 
\end{cases} \]

1.2. Organization of the material presented. This paper is organized as follows:

- In Section 2 we present exact enumeration formulas for the average height of \( p \)-watermelons with a wall in terms of certain determinants. Moreover, we make these formulas more explicit (in terms of sums of binomial coefficients) for the simple cases \( p = 1 \) and \( p = 2 \).
- In Section 3 we first review the classical reasoning for the asymptotics of the average height of 1–watermelons with a wall, which was given by de Bruijn, Knuth and Rice [5]. Then we show how this reasoning can be modified for the case of 2–watermelons with a wall.
- In appendix A we summarize background information on
  - Stirling’s approximation,
  - certain residues and values of the gamma and zeta function,
  - a certain double Dirichlet series and Jacobi’s theta function which are needed in our presentation.

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2. Exact enumeration

For a start, we gather some exact enumeration results.

2.1. The number of \( p \)-watermelons with a wall. We have the following generalization of the enumeration of 1–watermelons with a wall (i.e., Dyck paths) of length \( n \), which is given by the Catalan numbers

\[ C(n) = \frac{1}{n+1} \binom{2n}{n}. \]

**Proposition 1.** The number \( C(n, p) \) of all \( p \)-watermelons with a wall of length \( n \) is given as

\[ C(n, p) = \prod_{j=0}^{p-1} \frac{\binom{2n+2j}{n}}{\binom{n+2j+1}{n}}. \]  

**Proof.** This is a special case of Theorem 6 in [13]. \( \square \)

2.2. 1–watermelons with a wall and height restrictions. In order to obtain the average height, we count \( p \)-watermelons with a wall of length \( n \) which do not exceed height \( h \).

To this end, we employ the following formula (see [15] p. 6, Theorem 2):
**Theorem 1.** Let $u, d$ be nonnegative integers, and let $b, t$ be positive integers, such that $-b < u - d < t$. The number of lattice paths from $(0,0)$ to $(u+d,u-d)$, which do not touch neither line $y = -b$ nor line $y = t$, equals

$$
\sum_{k \in \mathbb{Z}} \left( \binom{u+d}{u-k(b+t)} - \binom{u+d}{u-k(b+t)+b} \right).
$$

(3)

**Corollary 1.** Let $i, j$ and $h$ be integers such that $0 \leq 2i, 2j \leq h$. The number of all lattice paths from $(0,0)$ to $(2n,2j)$ which do not exceed height $m$ and lie between the lines $y = 0$ and $y = h \geq 0$ is

$$
m(n,i,j,h) := \sum_{k \in \mathbb{Z}} \left( \binom{2n}{n-i+j-k(h+2)} - \binom{2n}{n+i-j-k(h+2)+1} \right).
$$

(4)

The special case $i = j = 0$ can be written as

$$
m(n,h) := m(n,0,0,h) = \frac{1}{n+1} \binom{2n}{n} - \sum_{k \geq 1} \left( \binom{2n}{n-k(h+2)-1} - 2 \binom{2n}{n-k(h+2)} + \binom{2n}{n-k(h+2)+1} \right).
$$

(5)

Proof. Set $u = n - i + j, d = n - i - j, b = 2i + 1$ and $t = h + 1 - 2i$ in [3].

**Corollary 2.** For $n > 0$, the number of all $p$–watermelons with a wall of length $2n$, which do not exceed height $h$, is given by the following determinant:

$$
C(n,p,h) = \left| m(n,i,j,h) \right|_{i,j=0}^{p-1}.
$$

(6)

Proof. For $h > 2p - 2$ this follows by a direct application of the Lindström–Gessel–Viennot method [4].

For $0 \leq h \leq 2p - 2$, the determinant equals 0 (as it should), since $m(n,i,j,2i) = 0$ and $m(n,i,j,2i+1) + m(n,i+1,j,2i+1) = 0$ for all $j$.

\[ \square \]

2.3. The average height of 1–watermelons with a wall. The following is a condensed version of the reasoning given in [5]. Denote by $\overline{m}(n,h)$ the number of all Dyck paths, starting at $(0,0)$ and ending at $(2n,0)$, which reach at least height $h$. By [5], we obtain

$$
\overline{m}(n,h) = m(n,n) - m(n,h-1) = C(n,1) - C(n,1,h-1)
$$

$$
= \sum_{k \geq 1} \left( \binom{2n}{n-k(h+1)-1} - 2 \binom{2n}{n-k(h+1)} + \binom{2n}{n-k(h+1)+1} \right).
$$

So, the average height of 1–watermelons with a wall (i.e., Dyck paths) of length $n$ is

$$
H(n,1) = \frac{1}{C_n} \sum_{h=1}^{n} \overline{m}(n,h) = \frac{1}{C_n} \left( -C_n + \sum_{h=0}^{n} \overline{m}(n,h) \right)
$$

$$
= -1 + (n + 1) \left( \sum_{k \geq 1} d(k) \left( \frac{2^n}{n} \binom{n-k-1}{k} - 2 \binom{2n}{n-k} + \binom{2n}{n-k+1} \right) \right),
$$

(7)
where \( d(k) \) denotes the number of positive divisors of \( k \). Introducing the notation
\[
S(n, a) := \sum_{k \geq 1} d(k) \frac{\binom{2n}{n-k+a}}{\binom{2n}{n}},
\]
we arrive at
\[
H(n, 1) = (n + 1) \left( S(n, 1) - 2 S(n, 0) + S(n, -1) \right) - 1,
\]
which is equivalent to equation (23) in [3].

2.4. The average height of \( p \)-watermelons with a wall. In generalization of the above notation, denote by \( m(n, p, h) \) the number of all \( p \)-watermelons of length \( n \), which reach at least height \( h \), i.e.,
\[
m(n, p, h) = C(n, p) - C(n, p, h - 1).
\]
Clearly, we have the following exact formula for the average height of \( p \)-watermelons of length \( n \):
\[
H(n, p) = \frac{1}{C(n, p)} \sum_{h=1}^{n+2p-2} m(n, p, h).
\]

2.4.1. The average height of 2–watermelons with a wall. Let
\[
S(n, a, b) = \sum_{j \geq 1} \sum_{k \geq 1} d(\gcd(j, k)) \frac{\binom{2n}{n-j+a}}{\binom{2n}{n}} \frac{\binom{2n}{n-k+b}}{\binom{2n}{n}}.
\]
From (3), straightforward (but rather tedious) computations lead to the following formula:
\[
H(n, 2) = \frac{(n + 1)^3}{12 (2n + 1)} \left( (n + 1)^3 S_2(n) + S_1(n) \right) - 1,
\]
where
\[
S_1(n) = -20(n - 1)(n + 2)S(n, 0) + 15n(n + 1)(S(n, -1) + S(n, 1)) + (n + 3)(6S(n, -1) - 16S(n, 0) + 6S(n, 1)) + (n - 2)(6S(n, -1) + 8S(n, 0) + 6S(n, 1)) - 6n(n + 3)(S(n, -2) + S(n, 2)) + (n + 2)(n + 3)(S(n, -3) + S(n, 3)),
\]
\[
S_2(n) = S(n, -2, -2) - S(n, -1, -3) - 2S(n, -1, -2) + S(n, -1, -1) + 2S(n, -1, 0) - S(n, -1, 3) + 2S(n, 0, -3) - 4S(n, 0, 0) + 2S(n, 0, 3) - S(n, 1, -3) + 2S(n, 1, -2) + 2S(n, 1, -1) + 2S(n, 1, 0) + S(n, 1, 1) - S(n, 1, 3) + 2S(n, 2, -2) - 2S(n, 2, -1) - 2S(n, 2, 1) + S(n, 2, 2).
\]

3. Asymptotic enumeration

3.1. Asymptotics of the average height of 1–watermelons. In the case of 1–watermelons, the asymptotic of the average height [7] is well-known, see [10] Proposition 7.7] or [5] equation (34):
\[
H(n, 1) \simeq \sqrt{\pi n} - \frac{3}{2} + O\left( n^{-\frac{3}{2}+\epsilon} \right).
\]
We repeat the classical reasoning of de Bruijn, Knuth and Rice \[5\], in order to make clear the basic idea, which we shall also employ for the case $$p = 2$$ later.

**Proof of (15):** From (9) it is clear that we need to investigate the asymptotic behaviour of $$S(n, a) = \sum_{k=1}^{n} d(k) \left(\binom{2n}{n^2} \right)^{2n-a-k}$$. Note that the sums $$S(n, a)$$ in (9) are multiplied with a factor of order 1. So if we are interested in the asymptotics of $$H(n, 1)$$ up to some $$O(n^{-\alpha})$$, we need the asymptotics for $$S(n, a)$$ up to $$O(n^{-\alpha-1})$$; for our case, $$\alpha = 1 - \epsilon$$ is sufficient.

The basis of the following considerations is the asymptotic expansion of the quotient of binomial coefficients \[29\] (see appendix A.1).

### 3.1.1. The asymptotics of $$S(n, a)$$ for a fixed, $$n \to \infty$$.

First we observe that

$$\left(\frac{2n}{(2n)^{n^2}}\right)^{2n-a-k} = O\left(\exp\left(-n^{2\epsilon}\right)\right)$$

if $$\left|\frac{k-a}{n} \right| \geq n^{\epsilon-\frac{1}{2}}$$, i.e., if $$k \geq n^{1/2+\epsilon} + a$$ (see \[29\] in appendix A.1). Therefore, the sum of all terms with $$k \geq n^{1/2+\epsilon} + a$$ is negligible in (8), being $$O(n^{-m})$$ for all $$m > 0$$, and we may take $$\frac{k-a}{n} = O\left(n^{\epsilon-\frac{1}{2}}\right)$$ in (29).

Next, we take (29) up to order $$n^{-1}$$ and substitute $$x = \frac{k-a}{n}$$: Pulling out the leading term $$e^{-k^2}$$ and expanding the rest with respect to $$k$$ gives

$$\left(\frac{2n}{(2n)^{n^2}}\right)^{2n-a-k} = e^{-k^2/n} \left(1 - \frac{a^2}{n} + k \left(2 \frac{a}{n} - \frac{a + 2a^3}{n^2}\right) + \frac{(1 + 4a^2)k^2}{2n^2} + \frac{(5a + 4a^3)k^3}{3n^3} - \frac{k^4}{6n^4} + O(n^{-2\epsilon}) \right). \quad (16)$$

Now we consider the following function

$$g(n, b) := \sum_{k \geq 1} k^b d(k) e^{-k^2/n}, \quad (17)$$

and observe that here the terms for $$k \geq n^{1/2+\epsilon}$$ are again negligible:

$$\sum_{k \geq n^{1/2+\epsilon}} k^b d(k) e^{-k^2/n} = O\left(n^{-m}\right) \quad \text{for all } m > 0.$$ 

Hence we directly obtain from (16):

$$S(n, a) = \left(1 - \frac{a^2}{n}\right) g(n, 0) + \left(\frac{2a}{n} - \frac{2a^3 + a}{n^2}\right) g(n, 1) + \left(\frac{4a^2 + 1}{2n^2}\right) g(n, 2)$$

$$+ \left(\frac{4a^3 + 5a}{3n^3}\right) g(n, 3) - \left(\frac{1}{6n^3}\right) g(n, 4) - \frac{a}{3n^4} g(n, 5) + O\left(n^{-2+\epsilon} g(n, 0)\right). \quad (18)$$

(This is equation (27) in \[5\] ) Note that the coefficients for $$g(n, k)$$ are odd functions of $$a$$ for odd $$k$$ and obtain:

$$S(n, 1) - 2S(n, 0) + S(n, -1) = -\frac{2}{n} g(n, 0) + \frac{4}{n^2} g(n, 2) + O\left(n^{-2+\epsilon} g(n, 0)\right). \quad (19)$$ 

So we reduced our problem to that of obtaining an asymptotic expansion for $$g(n, b)$$. Note that we need this information only for $$b$$ even. It follows from the computations
presented in appendix A.2.1 that \( g(n, 2b) = O\left(n^{b+1/2} \log(n)\right) \), and that we have for all \( m \geq 0 \):

\[
\begin{align*}
    g(n, 0) &= \frac{1}{4} \sqrt{\pi n \log(n)} + \left(\frac{3}{4} \gamma - \frac{1}{2} \log(2)\right) \sqrt{\pi n} + \frac{1}{4} + O(n^{-m}), \\
    g(n, 2) &= \frac{n}{8} \sqrt{\pi n \log(n)} + \left(\frac{1}{4} + \frac{3}{8} \gamma - \frac{1}{4} \log(2)\right) n \sqrt{\pi n} + O(n^{-m}).
\end{align*}
\]

(20)

Inserting this information in (19) we immediately obtain the desired result (15). \( \square \)

3.2. Asymptotics of the average height of 2–watermelons. We shall modify the reasoning from section 3.1 appropriately. In doing so, it turns out that we have to deal with the double Dirichlet series \( \sum_{k,l \geq 1} \frac{k^{a}l^{b}}{(k^{2}+l^{2})^{s}} \) for integers \( a, b \geq 0 \). Proposition 3 (see section A.3) states that this series is convergent in the half–plane \( \Re(z) > a + b + 1 \) and defines a meromorphic function \( Z(a, b; z) : \mathbb{C} \to \mathbb{C} \) which has a simple pole at \( z = a + b + 1 \), and an additional simple pole at \( z = a + b + \frac{1}{2} \) only if \( a = 0 \) or \( b = 0 \). Hence, we can write

\[
Z(a, b; z) = \frac{r_{a,b}}{z - a - b - \frac{1}{2}} + c_{a,b} + O\left(z - a - b - \frac{1}{2}\right).
\]

(21)

Given this “implicit” definition of the numbers \( c_{a,b} \) we will show:

\[
H(n, 2) \simeq \sqrt{\pi n} \left(-2c_{0,0} + 8c_{1,0} - 9c_{1,1} - 9c_{2,0} + 15c_{2,1} + 35c_{2,2} + 5c_{3,0} - 35c_{3,1}\right) - 2 + O\left(n^{-\frac{1}{2}+\epsilon}\right).
\]

(22)

Using the representations of the constants \( c_{a,b} \) by certain integrals (see (49) in appendix A.3), we obtain the following approximative asymptotics by numerical integration (carried out with Mathemtica)

\[
H(n, 2) \simeq 2.57758 \sqrt{n} - 2 + O\left(n^{-\frac{1}{2}+\epsilon}\right),
\]

which conforms well to Bonichon’s and Mosbah’s conjecture (1) for the case \( p = 2 \), which yields approximately 2.56125 \( \sqrt{n} \). Figure 3 shows the quotient \( q(n) = \frac{H(n, 2)}{2.57758 \sqrt{n} - 2} \) for small \( n \). For example, \( q(1000) = 1.00734 \).

Proof of (22): Note that in (12), the “single sums” \( S(n, a) \) are multiplied with a rational function in \( n \) of order at most 3, while the “double sums” \( S(n, a, b) \) are multiplied with a factor of order 4: So if we are interested in the asymptotics of \( H(n, 2) \) up to some \( O(n^{-\alpha}) \), we need the asymptotics for \( S(n, a) \) up to \( O(n^{-\alpha-3}) \) and for \( S(n, a, b) \) up to \( O(n^{-\alpha-4}) \); for our case, \( \alpha = 1 - \epsilon \) is sufficient.

3.2.1. The asymptotics of \( S(n, a) \) for a fixed, \( n \to \infty \). Basically, we repeat the computations from section 3.1.1. The only difference is that we need higher orders now. After
some calculations, we obtain:

\[
S_1(n) = -\frac{24}{n^3} \left(4n^2 + 20n + 89\right) g(n, 0) + \frac{4}{n^4} \left(96n^2 + 1065n + 3656\right) g(n, 2) \\
- \frac{1}{n^5} \left(288n^2 + 4060n + 12213\right) g(n, 4) + \frac{8}{n^6} \left(8n^2 + 107n + 335\right) g(n, 6) \\
- \frac{96n + 521}{3n^7} g(n, 8) + \frac{10g(n, 10)}{3n^8} + O\left(n^{-4+\epsilon} g(n, 0)\right).
\] (23)

Recall that in appendix A.2.1 it is proved that \( g(n, 2b) = O\left(n^{b+1/2} \log(n)\right) \). Moreover, the arguments in appendix A.2.1 show that we have (in addition to (20)) for all \( m \geq 0 \):

\[
\begin{align*}
&g(n, 4) = \frac{3n^2}{16} \sqrt{\pi n \log(n)} + \left(\frac{1}{2} + \frac{9}{16} \gamma - \frac{3}{8} \log(2)\right) n^2 \sqrt{\pi n} + O\left(n^{-m}\right), \\
&g(n, 6) = \frac{15n^3}{32} \sqrt{\pi n \log(n)} + \left(\frac{23}{16} + \frac{45}{32} \gamma - \frac{15}{16} \log(2)\right) n^3 \sqrt{\pi n} + O\left(n^{-m}\right), \\
&g(n, 8) = \frac{105n^4}{64} \sqrt{\pi n \log(n)} + \left(\frac{11}{2} + \frac{315}{64} \gamma - \frac{365}{16} \log(2)\right) n^4 \sqrt{\pi n} + O\left(n^{-m}\right), \\
&g(n, 10) = \frac{945n^5}{128} \sqrt{\pi n \log(n)} + \left(\frac{945}{64} \gamma - \frac{2835}{128} \log(2)\right) n^5 \sqrt{\pi n} + O\left(n^{-m}\right).
\end{align*}
\]

Inserting this information in (23) we immediately obtain the first part of the desired result:

\[
\frac{(n+1)^{\gamma}}{12 (2n+1)} S_1(n) = \frac{11 \sqrt{\pi n}}{6} - 1 + O\left(n^{-1/2+\epsilon}\right).
\] (24)

3.2.2. The asymptotics of \( S(n, a, b) \) for \( a, b \) fixed, \( n \to \infty \). Basically, we mimic the computations from section 3.1.1. In doing so, we are led to consider the following
function

\[ g(n, a, b) := \sum_{k \geq 1} \sum_{l \geq 1} k^a l^b \phi(\gcd(k, l)) e^{-(k^2+l^2)/n}. \]

Observe again that the terms for \( k, l \geq n^{1/2+\varepsilon} \) are negligible in this sum. For obtaining the following formula, we made use of the fact that \( g(n, 2a, 2b) = O(n^{a+b+1}) \) (which is shown in appendix A.2.2):

\[
S_2(n) = \frac{1}{2} \left( -\frac{192}{n^4} + \frac{1632}{n^5} - \frac{8736}{n^6} \right) g(n, 0, 0) + \left( \frac{768}{n^5} - \frac{8928}{n^6} + \frac{61744}{n^7} \right) g(n, 2, 0) \\
+ \frac{1}{2} \left( -\frac{1152}{n^6} + \frac{18432}{n^7} - \frac{161488}{n^8} \right) g(n, 2, 2) + \left( -\frac{576}{n^6} + \frac{10336}{n^7} - \frac{99336}{n^8} \right) g(n, 4, 0) \\
+ \left( \frac{384}{n^7} - \frac{10784}{n^8} + \frac{138128}{n^9} \right) g(n, 4, 2) + \frac{1}{2} \left( \frac{512}{n^8} - \frac{7872}{n^9} + \frac{58368}{n^{10}} \right) g(n, 4, 4) \\
+ \left( \frac{128}{n^7} - \frac{4192}{n^8} + \frac{300624}{n^9} \right) g(n, 6, 0) + \left( -\frac{256}{n^8} + \frac{6848}{n^9} - \frac{1517888}{n^{10}} \right) g(n, 6, 2) \\
+ \left( \frac{2432}{3n^{10}} - \frac{62368}{5n^{11}} \right) g(n, 6, 4) + \frac{1}{2} \left( \frac{256}{3n^{11}} - \frac{416}{15n^{12}} \right) g(n, 6, 6) + \left( \frac{544}{n^9} - \frac{225488}{15n^{10}} \right) g(n, 8, 0) \\
+ \left( \frac{398912}{15n^{11}} - \frac{960}{n^{10}} \right) g(n, 8, 2) + \left( \frac{31736}{15n^{12}} - \frac{256}{3n^{11}} \right) g(n, 8, 4) + \frac{1328g(n, 8, 6)}{45n^{13}} + \frac{64g(n, 8, 8)}{9n^{14}} \\
+ \left( \frac{8672}{5n^{11}} - \frac{64}{3n^{10}} \right) g(n, 10, 0) + \left( \frac{128}{3n^{11}} - \frac{47504}{15n^{12}} \right) g(n, 10, 2) - \frac{7856g(n, 10, 4)}{45n^{13}} - \frac{32g(n, 10, 6)}{3n^{14}} \\
- \frac{456g(n, 12, 0)}{5n^{12}} + \frac{2576g(n, 12, 2)}{15n^{13}} + \frac{64g(n, 12, 4)}{9n^{14}} + \frac{16g(n, 14, 0)}{9n^{13}} \\
- \frac{32g(n, 14, 2)}{9n^{14}} + O(n^{-5+\varepsilon}g(n, 0, 0)). \quad (25)
\]

From the results obtained in appendix A.2.2 and A.3.1, we easily derive the following asymptotic expansions (using the “implicit” definition of the numbers \( c_{a,b} \) given in (21)):

\[
g(n, 0, 0) = \frac{\pi^3 n}{24} + \frac{\sqrt{\pi n}}{4} \left( 2c_{0,0} - \log(n) - \psi\left( \frac{1}{2} \right) - 2\gamma \right) + O(n^{-m}) , \\
g(n, 2a, 0) = \frac{2^{-2a-3}n^a(2a)!}{a!} \left( \frac{\pi^3 n}{3} + \sqrt{\pi n} \left( 4c_{a,0} - \log(n) - \psi\left( a + \frac{1}{2} \right) - 2\gamma \right) \right) + O(n^{-m}) , \\
g(n, 2a, 2b) = \frac{2^{-2a-2b-3}n^{a+b}(2a)!(2b)!}{ab!} \left( \frac{\pi^3 n}{3} + 4\sqrt{\pi n} \frac{c_{a,b}(2a+2b)!}{(a+b)!} \right) + O(n^{-m}) \quad (26)
\]

for all \( m \geq 0 \). Inserting the information from (26) in (25) shows that all the \( \log(n) \)–terms cancel, as well as all evaluations of the digamma function \( \psi \) (see appendix A.2.1). So we obtain the second part of the desired result:

\[
\frac{(n + 1)^3}{12(2n + 1)} (n + 1)^5 S_2(n) = \\
\sqrt{\pi n}\left( -\frac{11}{6} - 2c_{0,0} + 8c_{1,0} - 9c_{1,1} - 9c_{2,0} \\
+ 15c_{2,1} + 35c_{2,2} + 5c_{3,0} - 35c_{3,1} \right) + O(n^{-1/2+\varepsilon}) . \quad (27)
\]
A.2. Integral representations of the exponential function and applications.

Inserting the expressions (24) and (27) in (12) gives the desired result (22). □

Appendix A. Background information and relevant results

A.1. Stirling’s approximation applied to quotients of binomial coefficients.

We have the following asymptotic series for \( \log(\Gamma(z)) \), valid for \(|\arg z| < \pi - \delta, 0 < \delta < \pi, |z| \to \infty \) (see [4, equation (3.10.7)]):

\[
\log(\Gamma(z)) \approx \left( z - \frac{1}{2} \right) \log(z) - z + \frac{\log(2\pi)}{2} + \sum_{j=1}^{\infty} z^{1-2j} \frac{B_{2j}}{(2j)(2j-1)},
\]

where \( B_j \) denotes the \( j \)-th Bernoulli number.

Setting \( x = \frac{k-a}{n} \) we thus obtain

\[
\left( \frac{2^n}{\binom{n}{a-k}} \right) = \exp \left( -2n \left( \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{30} + \frac{x^8}{56} + \ldots \right) + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \ldots \right)
\]

\[
- \frac{1}{6n} \left( x^2 + x^4 + x^6 + x^8 + \ldots \right) + \frac{1}{n^3} \left( \frac{x^2}{30} + \frac{x^4}{12} + \frac{7x^6}{45} + \frac{x^8}{4} + \ldots \right)
\]

\[
- \frac{1}{n^5} \left( \frac{x^2}{42} + \frac{x^4}{9} + \frac{x^6}{14} + \ldots \right) + \frac{1}{n^7} \left( \frac{x^2}{30} + \frac{x^4}{4} + \frac{11x^6}{10} + \frac{143x^8}{40} + \ldots \right)
\]

\[
- \frac{1}{n^9} \left( \frac{5x^2}{66} + \frac{5x^4}{6} + \frac{91x^6}{18} + \frac{65x^8}{3} + \ldots \right) + O(x^2n^{-11})
\]

(29)

Note that \( \left( \frac{2^n}{\binom{n}{a-k}} \right) \) is zero for \( |x| > 1 \). The approximation given by (24) is very good if, say, \( |x| \leq \frac{1}{2} \).

A.2. Integral representations of the exponential function and applications.

A.2.1. The asymptotics of \( g(n, b) \) for \( b \) fixed, \( n \to \infty \). Starting with the formula

\[
e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z) x^{-z} dz \text{ for } c > 0, x > 1,
\]

(see [4] (2.4.1)) and using

\[
\zeta(z)^2 = \sum_{k \geq 1} d(k) k^{-z},
\]

we obtain

\[
g(n, b) = \sum_{k \geq 1} \frac{d(k)}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) k^{b-2z} dz
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) \zeta(2z - b)^2 dz,
\]

where \( c > \frac{b+1}{2} \). Denote the integrand in the above formula by \( G_1(b; z) \).

For any fixed positive number \( q \) and \( \Re(s) \geq -q \), we have \( \zeta(s) = O(|s|^{q+1/2}) \) as \( s \to \infty \). Since \( n^z \Gamma(z) \) becomes small on vertical lines, we can shift the line of integration to the left as far as we want to, if we take into account the residues of our integrand \( G_1(b; z) \).

There is a double pole at \( z = \frac{b+1}{2} \), and possibly some simple poles at \( z = 0, -1, -2, \ldots \).
For obtaining the residues, we use the power series expansion
\[
\zeta(s) - \frac{1}{s - 1} = \gamma + \sum_{n=1}^{\infty} \gamma_n (s - 1)^n,
\]
where \( \gamma \) is Euler’s constant and \( \gamma_n = \lim_{m \to \infty} (\sum_{l=1}^{m} l^{-1} (\log l)^n - (n + 1)^{-1} (\log l)^{n+1}) \) (see \textbf{6} 1.12, (17)), which gives the Laurent expansion at \( \frac{b+1}{2} \)
\[
\zeta(2z - b)^2 = \frac{\gamma}{(z - \frac{b+1}{2})^2} + \frac{1}{4 (z - \frac{b+1}{2})} + \cdots
\]
Combining this with the series expansions
\[
n^z = n^{z_0} (1 + \log(n) (z - z_0) + \ldots),
\]
\[
\Gamma(z) = \Gamma(z_0) (1 + \psi(z_0) (z - z_0) + \ldots),
\]
for \( z_0 = \frac{b+1}{2} \), where \( \psi(z) \) is the digamma function (i.e., the logarithmic derivative of the gamma function \( \Gamma'(z)/\Gamma(z) \), see \textbf{6} 1.7), we can easily express the residue of our integrand \( G_1(b; z) \) at \( z = \frac{b+1}{2} \):
\[
n^{\frac{b+1}{2}} \Gamma\left(\frac{b+1}{2}\right) \left(\frac{1}{4} \log(n) + \frac{1}{4} \psi\left(\frac{b+1}{2}\right) + \gamma\right).
\]
(32)

For our purposes, we need \( \psi(z) \) at positive integral or half–integral values \( z \), which can be derived from the following information:
\[
\psi(1) = -\gamma \quad \text{(see \textbf{6} section 1.7, equation (4))},
\]
\[
\psi\left(\frac{1}{2}\right) = -\gamma - 2 \log(2) \quad \text{(see \textbf{16} p. 104)},
\]
\[
\psi(z + n) = \psi(z) + \sum_{j=0}^{n-1} \frac{1}{z + j} \quad \text{(see \textbf{6} section 1.7, equation (10))}.
\]

The residue of \( G_1(b; z) \) at \( z = -m \) is
\[
n^{-m}(\frac{-1)^m}{m!} \zeta(-2m - b)^2 = n^{-m}(\frac{-1)^m}{m!} \left(\frac{B_{2m+b+1}}{(2m + b + 1)}\right)^2,
\]
(33)

where \( B_k \) denotes the \( k \)-th Bernoulli number (see \textbf{6} 1.12, (20)). Note that this number is non–zero only if \( b \) is odd or \( m = b = 0 \).

The sum of (32) and (33) for all \( m \geq 0 \) gives an asymptotic series for \( g(n, b) \).

A.2.2. The asymptotics of \( g(n, a, b) \) for \( a, b \) fixed, \( n \to \infty \). In the same manner as in section A.2.1 we obtain
\[
g(n, a, b) = \sum_{k,l \geq 1} \frac{d(\gcd(k, l))}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) k^a l^b (k^2 + l^2)^{-z} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) \sum_{k,l \geq 1} d(\gcd(k, l)) k^a l^b (k^2 + l^2)^{-z} \, dz,
\]
where \( c > \frac{a+b+1}{2} \).
For \( k \) and \( l \) fixed, set \( j = \gcd(k, l) \). Then we may write \( k = k_1 j \) and \( l = l_1 j \) with \( \gcd(k_1, l_1) = 1 \). This leads to

\[
g(n, a, b) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) \sum_{j, k_1, l_1 \geq 1 \atop \gcd(k_1, l_1) = 1} \vartheta(j) j^{a+b-2z} k_1^{a} l_1^{b} (k_1^2 + l_1^2)^{-z} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) \zeta(2z-a-b)^2 \sum_{k_1, l_1 \geq 1 \atop \gcd(k_1, l_1) = 1} k_1^{a} l_1^{b} (k_1^2 + l_1^2)^{-z} \, dz.
\]

Now we get rid of the constraint \( \gcd(k_1, l_1) = 1 \):

**Proposition 2.** We have the following identity:

\[
\sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z} = \frac{1}{\zeta(2z-a-b)} \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z}.
\]

**Proof.** By inclusion–exclusion, the left–hand side equals the sum over all pairs \((k, l)\) minus the sum over all pairs \((k, l)\), where some prime number \( p \) divides \( \gcd(k, l) \), plus the sum over all pairs \((k, l)\), where the product of two different primes \( p_1 p_2 \), divides \( \gcd(k, l) \), and so on:

\[
\sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z} = \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z} - \sum_{p \text{ prime}} \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{(kp)^a (lp)^b}{((kp)^2 + (lp)^2)^z}
\]

\[
+ \sum_{p_1 \neq p_2 \text{ prime}} \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{(kp_1 p_2)^a (lp_1 p_2)^b}{((kp_1 p_2)^2 + (lp_1 p_2)^2)^z} - + \cdots
\]

\[
= \left(1 - \sum_{p \text{ prime}} \frac{1}{p^{2z-a-b}} + \sum_{p_1 \neq p_2 \text{ prime}} \frac{1}{(p_1 p_2)^{2z-a-b}} - + \cdots \right) \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z}
\]

\[
= \left( \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{2z-a-b}}\right) \right) \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z}.
\]

The product in the last line is the reciprocal of the Euler product for \( \zeta(2z-a-b) \) ([8, p. 225]), which proves the assertion. \( \square \)

Thus, we arrive at

\[
g(n, a, b) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^z \Gamma(z) \zeta(2z-a-b) \sum_{k, l \geq 1 \atop \gcd(k, l) = 1} \frac{k^a l^b}{(k^2 + l^2)^z} \, dz.
\]

Denote the integrand in the above formula by \( G_2(a, b; z) \).

Again, we may shift the line of integration to the left as far as we want to, if we take into account the residues of our integrand \( G_2(b; z) \). Computing the poles and residues clearly depends on some information about the double Dirichlet series involved. This information will be provided in the next (and final) subsection.
A.3. The Dirichlet series $\sum_{k,l \geq 1} \frac{k^{2a}l^{2b}}{(k^2 + l^2)^s}$. Note that for our purposes, we only need $g(n, 2a, 2b)$, so the series we are interested in is

$$Z(a, b; s) := \sum_{k,l \geq 1} \frac{k^{2a}l^{2b}}{(k^2 + l^2)^s}.$$  \hspace{1cm} (36)

Clearly, this is closely related to the following function:

$$Z_{*}(a, b; s) := \sum_{(k,l) \in \mathbb{Z}^2 \setminus \{0\}} \frac{k^{2a}l^{2b}}{(k^2 + l^2)^s} = 4 \cdot Z(a, b; s) + 2 \cdot [b = 0] \zeta(2s - 2a) + 2 \cdot [a = 0] \zeta(2s - 2b).$$  \hspace{1cm} (37)

Informations on the poles and residues of $Z(a, b; s)$ could be directly derived from the work of Pierrette Cassou–Noguès [3, p. 41ff], but there is a simpler way by using the reciprocity law for Jacobi’s theta function. This reasoning is a generalization of Riemann’s representation (see [6, section 1.12, (16)]) of $\zeta(s)$,

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s - 1} + \int_{1}^{\infty} \left( t^{1/2(1-s)} + t^{s/2} \right) t^{-1} \omega(t) \, dt,$$  \hspace{1cm} (38)

where

$$\omega(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t} = 1/2 (\theta_{3}(0, it) - 1).$$  \hspace{1cm} (39)

Here, $\theta_{3}$ denotes (one variant of) Jacobi’s theta function (see [7, section 13.19, (8)])

$$\theta_{3}(z, t) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz},$$  \hspace{1cm} (40)

where $q = e^{i \pi t}$.

A.3.1. Jacobi’s theta function. We rewrite (40) by setting $t = y$ and $z = \pi x$ for $x, y \in \mathbb{C}$; i.e.:

$$\vartheta(x, y) = \sum_{n=-\infty}^{\infty} e^{2\pi i(xn + \frac{y}{2} n^2)}$$  \hspace{1cm} (41)

This series is absolute convergent for all $x$ and all $y$ with $\Im(y) > 0$. Therefore, for fixed $y$ the function $f : z \mapsto \vartheta(z, y)$ is an entire function. We have the properties

$$\vartheta(x+1, y) = \vartheta(x, y),$$

$$e^{2\pi i(x + \frac{y}{2})} \vartheta(x + y, y) = \vartheta(x, y),$$

which in fact determine the theta function up to a multiplicative factor $c(y)$ (see [14, section 2.3]). Moreover, we have the following functional equation (see [14, Theorem 2.12, equation (2.29)]):

$$\vartheta(x, y) = \vartheta \left( \frac{x}{y}, - \frac{1}{y} \right) e^{-\pi i x^2 / y} \sqrt{\frac{i}{y}}$$

Setting $\bar{\vartheta}(y) := \vartheta(0, iy)$, we obtain as a special case the following reciprocity law, valid for all $y$ with $\Re(y) > 0$:

$$\bar{\vartheta}(y) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} = \sqrt{\frac{1}{y}} \cdot \bar{\vartheta} \left( \frac{1}{y} \right).$$  \hspace{1cm} (42)
\( \tilde{\vartheta}(y) \) is a holomorphic function in the half plane \( \Re(y) > 0 \), with \( \Re(y) = 0 \) as essential singular line, see [14, Satz 2.13].

Interchanging summation, differentiation and integration in the appropriate places, we obtain

\[
\frac{(-\pi)^{a+b}}{\pi^s} \Gamma(s) Z_s(a, b; s) = \sum_{(k, l) \in \mathbb{Z}^2 \setminus \{0\}} \Gamma(s) \frac{(-\pi)^{a+b}}{\pi^s} \frac{k^{2a} l^{2b}}{(k^2 + l^2)^s}
\]

\[
= \sum_{(k, l) \in \mathbb{Z}^2 \setminus \{0\}} (\pi (k^2 + l^2))^{-s} \int_0^\infty t^{s-1} e^{-t} (-\pi)^{a+b} k^{2a} l^{2b} dt
\]

\[
= \sum_{(k, l) \in \mathbb{Z}^2 \setminus \{0\}} \int_0^\infty u^{s-1} (-\pi k^2)^a (-\pi l^2)^b e^{-\pi (k^2+l^2) u} du \tag{43}
\]

\[
= \sum_{(k, l) \in \mathbb{Z}^2 \setminus \{0\}} \int_0^\infty u^{s-1} \left( \frac{d^a}{du^a} e^{-\pi k^2 u} \right) \left( \frac{d^b}{du^b} e^{-\pi l^2 u} \right) du.
\]

For (43), we used \( \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = \alpha^s \int_0^\infty u^{s-1} e^{-\alpha u} du \) with \( \alpha = (k^2 + l^2) \pi \).

Clearly, \( Z_s(a, b; s) = Z_s(b, a; s) \). So w.l.o.g. we may assume \( a \geq b \). We have to distinguish the following two cases, where we assume \( a > 0 \) and \( b \geq 0 \):

\[
\frac{1}{\pi^s} \Gamma(s) Z_s(0, 0; s) = \int_0^\infty u^{s-1} (\tilde{\vartheta}(u)^2 - 1) du, \tag{44}
\]

\[
\frac{(-\pi)^{a+b}}{\pi^s} \Gamma(s) Z_s(a, b; s) = \int_0^\infty u^{s-1} \left( \frac{d^a}{du^a} \tilde{\vartheta}(u) \right) \left( \frac{d^b}{du^b} \tilde{\vartheta}(u) \right) du. \tag{45}
\]

A.3.1.1. Case 1. Considering (44), we basically repeat the reasoning in [14, p. 203]. From (42), we get \( \frac{1}{u} \tilde{\vartheta}\left(\frac{1}{u}\right)^2 = \tilde{\vartheta}(u)^2 \). Assuming \( \Re(s) > 1 \), we compute:

\[
\int_0^\infty u^{s-1} (\tilde{\vartheta}(u)^2 - 1) du = \int_0^1 u^{s-1} \left( \frac{1}{u} \tilde{\vartheta}\left(\frac{1}{u}\right)^2 - 1 \right) du + \int_1^\infty u^{s-1} (\tilde{\vartheta}(u)^2 - 1) du
\]

\[
= \int_0^1 u^{s-1} \left( \frac{1}{u} \left( \tilde{\vartheta}\left(\frac{1}{u}\right)^2 - 1 \right) + 1 - 1 \right) du + \int_1^\infty u^{s-1} (\tilde{\vartheta}(u)^2 - 1) du
\]

\[
= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty t^{-s} (\tilde{\vartheta}(t)^2 - 1) dt + \int_1^\infty u^{s-1} (\tilde{\vartheta}(u)^2 - 1) du. \tag{46}
\]

The integrals in the last line converge for all \( s \in \mathbb{C} \) and constitute holomorphic functions, so \( Z_s(0, 0; s) \) only has a simple pole for \( s = 1 \).

A.3.1.2. Case 2. Considering (45), we have to adjust the preceding method appropriately. For convenience, set \( \tilde{\vartheta}_a(u) := \frac{d^a}{du^a} \tilde{\vartheta}(u) \). Then we have:

\[
\int_0^\infty u^{s-1} (\tilde{\vartheta}_a(u) \tilde{\vartheta}_b(u)) du = \int_0^1 u^{s-1} (\tilde{\vartheta}_a(u) \tilde{\vartheta}_b(u)) du + \int_1^\infty u^{s-1} (\tilde{\vartheta}_a(u) \tilde{\vartheta}_b(u)) du.
\]

Now use (42) in the form

\[
\tilde{\vartheta}_a(u) \tilde{\vartheta}_b(u) = \left( \frac{d^a}{du^a} \left( \frac{\sqrt{1-\tilde{\vartheta}(1)}}{u} \right) \right) \left( \frac{d^b}{du^b} \left( \frac{\sqrt{1-\tilde{\vartheta}(1)}}{u} \right) \right)
\]
and combine this with the formula

\[
\frac{d^a}{dy^a} \left( \sqrt{\frac{1}{y}} \cdot f \left( \frac{1}{y} \right) \right) = (-1)^a \sum_{k=0}^{a} \frac{(2a)2^k}{4^kk!} \left( \frac{d^{a-k}}{dy^{a-k}} f \right) \left( \frac{1}{y} \right) y^{k-2a-\frac{1}{2}}
\]

to obtain

\[
\bar{\vartheta}_a(u) \bar{\vartheta}_b(u) = \frac{(-1)^{a+b}(2a)!(2b)!}{4^{a+b}a!b!} u^{-a-b-1} + \sum_{k=0}^{a} \sum_{j=0}^{b} \frac{(2a)2^k(2b)2^j}{4^kk!j!} \left( \bar{\vartheta}_{a-k} \left( \frac{1}{u} \right) \bar{\vartheta}_{b-j} \left( \frac{1}{u} \right) - \left[ a = k \land b = j \right] \right) u^{k+j-2a-2b-1}.
\]

Now in the same way as before, this gives

\[
\int_{0}^{\infty} u^{s-1} \left( \bar{\vartheta}_a(u) \bar{\vartheta}_b(u) \right) \, du = \frac{(-1)^{a+b}(2a)!(2b)!}{4^{a+b}a!b! (s-a-b-1)} + \int_{1}^{\infty} u^{s-1} \left( \bar{\vartheta}_a(u) \bar{\vartheta}_b(u) \right) \, du + (-1)^{a+b} \times \sum_{k=0}^{a} \sum_{j=0}^{b} \frac{(2a)2^k(2b)2^j}{4^kk!j!} \int_{1}^{\infty} \left( \bar{\vartheta}_{a-k}(u) \bar{\vartheta}_{b-j}(u) - \left[ a = k \land b = j \right] \right) u^{2a+2b-k-j-s} \, du. \quad (47)
\]

Again, the integrals converge for all \( z \in \mathbb{C} \) and constitute holomorphic functions, whence \( Z_s(a, b; s) \) has only one simple pole at \( s = a + b + 1 \).

We summarize all this information in the following proposition.

**Proposition 3.** For arbitrary nonnegative integers \( a, b \), the series \( \sum_{k,l \geq 1} \frac{k^{2a+l}b^l}{(k^2+l^2)^{a+b+1}} \) is convergent in the half–plane \( \Re(z) > a + b + 1 \) and defines a meromorphic function \( Z(a, b; z) : \mathbb{C} \to \mathbb{C} \) with a simple pole at \( z = a + b + 1 \), where the residue is \( \frac{\pi (2a)! (2b)!}{4^{a+b}a!b!(a+b)!} \).

If \( a > 0 \) and \( b > 0 \), this is the only pole.

If \( a = 0 \) (or \( b = 0 \)), there is another simple pole at \( z = b + \frac{1}{2} \) (or \( z = a + \frac{1}{2} \)), where the residue is \( -\frac{1}{4} (\lfloor b = 0 \rfloor + \lfloor a = 0 \rfloor) \).

Moreover, we have the following information on special evaluations of \( Z(a, b; z) \):

\[
Z(a, b; -n) = \frac{1}{8} \lfloor a = b = n = 0 \rfloor \text{ for } n = 0, 1, 2, \ldots \quad (48)
\]
For the absolute term $c_{a,b}$ in the Laurent series expansion of $Z(a, b; z)$ at $z_0 = a + b + \frac{1}{2}$, we have the following formulas:

$$c_{0,0} = -\gamma - 1 + \frac{1}{2} \int_1^\infty t^{-\frac{1}{2}} \left( \bar{\vartheta}(t)^2 - 1 \right) dt,$$

$$c_{a,0} = -\gamma - 1 + \frac{1}{2} + \frac{4^{-1}a!((-1)^a + 1)}{(2a)!} \int_1^\infty t^{a-\frac{1}{2}} \bar{\vartheta}_a(t) \bar{\vartheta}(t) dt$$

$$+ \sum_{k=1}^a \frac{4^{a-k-1}a!}{k!(2a-2k)!} \int_1^\infty t^{a-k-\frac{1}{2}} \left( \bar{\vartheta}_{a-k}(t) \bar{\vartheta}(t) - [a = k] \right) dt \text{ for } a > 0,$$

$$c_{a,b} = \frac{4^{a+b-1}(a + b)!}{(2a + 2b)!} \left( -2 \frac{(2a)! (2b)!}{4^{a+b}a!b!} + (-1)^{a+b} \int_1^\infty t^{a+b-\frac{1}{2}} \bar{\vartheta}_a(t) \bar{\vartheta}_b(t) dt \right)$$

$$+ \sum_{k=0}^a \sum_{j=0}^b \frac{(2a)^{2k} (2b)^{2j}}{4^{k+j}k!j!} \times \int_1^\infty \left( \bar{\vartheta}_{a-k}(t) \bar{\vartheta}_{b-j}(t) - [a = k \land b = j] \right) t^{a+b-k-j-\frac{1}{2}} dt \right) \text{ for } a \geq b > 0. \quad (49)$$

**Proof.** This information is extracted straightforwardly from [57], together with [14], [10] and [15], [17], respectively.

(The evaluation $\zeta(-2n) = 0$ for nonnegative integers $n$, which is needed for [18], can be found in [8] 1.13, (22).)

---

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