Abstract
In this paper, we first prove an interpolation inequality of Ehring-type, which is an improvement of a special case to the well known Gagliardo-Nirenberg inequality. Then we apply it to study the classical Keller-Segel system
\[
\begin{aligned}
  u_t &= \Delta u - \nabla \cdot (u \nabla v), \\
  v_t &= \Delta v - v + u,
\end{aligned}
\]
in a bounded domain \( \Omega \subset \mathbb{R}^N \) \((N \geq 2)\) with smooth boundary. It is known that for any \( \delta > 0 \), if \( \int_\Omega u^{N/2 + \delta} \) is bounded, then the solution is global and bounded. Here we show that the same conclusion holds for a weaker assumption: the equi-integrability of \( \left\{ \int_\Omega u^{N/2} t \mid t \in (0, T_{\text{max}}) \right\} \) can prevent blow up.

Keywords: Interpolation inequality, chemotaxis, global existence, boundedness

Math Subject Classification (2010): 35K55, 35B40 35Q35, 92C17, 35B35

1 Introduction
The following system called Keller-Segel model is proposed in [14] to model chemotactic migration
\[
\begin{aligned}
  u_t &= \Delta u - \nabla \cdot (u \nabla v), \\
  v_t &= \Delta v - v + u, \\
  \partial_\nu u &= \partial_\nu v = 0, \\
  u(x, 0) &= u_0(x), \\
  v(x, 0) &= v_0(x), \\
  x &\in \Omega.
\end{aligned}
\] (1.1)
Here \( \Omega \subset \mathbb{R}^N \) \((N \geq 2)\) is a bounded smooth domain, \( T \in (0, \infty] \), and \( \nu \) denotes the outer normal vector on \( \partial \Omega \). Let \((u_0, v_0)\) be a nonnegative function pair, \( u \) and \( v \) denote the density of cells and chemical concentration, respectively. The system [14] describes an interesting interaction between the cells and the chemical signal. This chemical substance is released by the cells themselves, and on the other hand, it also attracts cells; meaning that the movement of cells is oriented to the higher density of chemical signal. The latter mechanism is known as chemotaxis, which is represented by the cross-diffusion term \(-\nabla \cdot (u \nabla v)\) in the first equation. This biological model plays an important role in numerous biological processes such as wound healing, cancer invasion. It also draws interests from many mathematicians, for surveys in this area we refer to [1, 11, 10] and the references therein.

A striking feature of this model is the occurrence of a blow up phenomenon caused by the aggregation of cells, related research can be found in [9, 12, 19, 18, 23, 16]. The spatial dimension seems crucial in the mathematical analysis of detecting blow up. In the one dimensional setting, blow up never happens.

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However, considering the two-dimensional case, one can prove the existence of radial blow up solutions if the initial data \((u_0, v_0)\) exceed the critical mass: \(\int_\Omega u_0 > 8\pi \)\(^2\); otherwise, the solution always remains bounded \(17\). In higher dimensions, whether a solution blows up does not depend on the total mass any more; blow up solutions are constructed with any small mass \(23\). On the other hand, looking for a sufficient condition which can prevent blow up may be of some interest, especially in two or higher dimensions.

Throughout the paper, we consider the classical solution \((u, v)\) of (1.1) on \(\Omega \times [0, T_{\text{max}}]\) emanating from the nonnegative initial pair \((u_0, v_0)\) where \(T_{\text{max}} \in (0, \infty)\) denotes the maximal existence time of the solution. The local existence theory concerning this issue is presented in Lemma 3.1. Beyond this, a well known sufficient condition for global solutions is the following [11 Lemma 3.2]:

**Proposition 1.1.** Let \(N \geq 1\) and \(p > \frac{N}{2}\). Assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary and \((u, v)\) is a nonnegative classical solution of (1.1) in \(\Omega \times (0, T_{\text{max}})\) with maximal existence time \(T_{\text{max}} \in (0, \infty)\). If

\[
\sup_{t \in (0, T_{\text{max}})} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty, \tag{1.2}
\]

then

\[
\sup_{t \in (0, T_{\text{max}})} \left( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right) < \infty.
\]

The proof is carried out either by using Neumann heat semigroup estimates or by studying a coupled energy evolution of \(\int_\Omega u^p\) and \(\int_\Omega |\nabla v|^{2q}\) with \(p, q\) sufficiently large \(21, 4\). Generally, the condition in the above proposition can not reach the borderline value \(p = \frac{N}{2}\). In the special case when \(N = 2\) and thus \(\frac{N}{2} = 1\), we already mentioned that blow up can happen even though \(\int_\Omega u(\cdot, t) = \int_\Omega u_0\) is bounded \(16\). Therefore, we cannot expect that boundedness of \(\|u(\cdot, t)\|_{L^\infty(\Omega)}\) can prevent blow up. However, if we require a little more, namely that \(\{u^\frac{N}{N-2}(\cdot, t)\}_{t \in (0, T_{\text{max}})}\) is not only bounded with respect to the spatial \(L^1\)-norm, but also enjoys an additional equi-integrability property, we will be able to show global existence and boundedness for the system. Accordingly, the main result in the paper reads as follows:

**Theorem 1.2.** Assume that \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\) is a bounded domain with smooth boundary, and that the nonnegative initial data \((u_0, v_0)\) satisfy \(u_0 \in C^0(\Omega)\) and \(v_0 \in W^{1,\infty}(\Omega)\). Let \((u, v)\) be a nonnegative classical solution of (1.1) on \(\Omega \times (0, T_{\text{max}})\) with maximal existence time \(T_{\text{max}} \in (0, \infty)\). If

\[
\sup_{t \in (0, T_{\text{max}})} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty, \tag{1.3}
\]

and \(\{u^\frac{N}{N-2}(\cdot, t)\}_{t \in (0, T_{\text{max}})}\) is equi-integrable, \(\tag{1.4}\)

then \((u, v)\) is global and bounded.

Recalling De la Vallée-Poussin Theorem, we obtain the following equivalent extension criterion:

**Corollary 1.3.** Assume that \((u, v)\) be a nonnegative classical solution of (1.1) on \(\Omega \times (0, T_{\text{max}})\) with \(T_{\text{max}} \in (0, \infty)\). Let \(f : [0, \infty) \to [0, \infty)\) be continuous and such that

\[
\lim_{s \to \infty} \frac{f(s)}{s^\frac{N}{N-2}} = \infty.
\]

If we have

\[
\sup_{t \in (0, T_{\text{max}})} \int_\Omega f(u(\cdot, t)) < \infty, \tag{1.5}
\]

then \((u, v)\) is global and bounded.
The above corollary inter alia shows that the boundedness of \( \int_\Omega u \frac{\nabla |u|^2}{|u|^2} \log u \) is sufficient for our conclusion, which is obviously not covered by Proposition 1.1.

On the other hand, Corollary 1.3 also improves the previous knowledge in the two-dimensional Keller-Segel model; it is known that the boundedness of \( \int_\Omega u \log u \) and \( \int_\Omega |\nabla u|^2 \) can exclude blow up [1, Lemma 3.3]. Now we can immediately remove the requirement on \( \int_\Omega u \log u \) is bounded without applying the current result. However, since a corresponding estimate for \( \int_\Omega |\nabla u|^2 \) in a parabolic equation appears to be lacking, the outcome of the above corollary seems not trivial in the fully parabolic model. Moreover, the condition can be weakened to the boundedness of the \( L^1 \)-norm of essentially any superlinear functional of \( u \), e.g. \( \int_\Omega u \log (u + \epsilon) \).

Additionally, by virtue of an equivalent definition of equi-integrability, Theorem 1.2 can be rephrased in the following way:

**Corollary 1.4.** Let \( (u, v) \) be a classical solution of \((1.1)\) on \( \Omega \times (0, T_{\text{max}}) \) with \( T_{\text{max}} \in (0, \infty) \). For all \( \epsilon > 0 \) there is \( \delta > 0 \) such that for any measurable set \( E \subset \Omega \) with \( |E| < \delta \), if we have

\[
\sup_{t \in (0, T_{\text{max}})} \int_E u \frac{\nabla |u|^2}{|u|^2} \log u < \epsilon,
\]

then

\[
\sup_{t \in (0, T_{\text{max}})} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.
\]

We note that this property resembles the feature of \( \varepsilon \)-regularity derived in [20] for a porous medium type parabolic-elliptic Keller-Segel model in the whole space or for a corresponding degenerate fully parabolic system in a bounded domain [13]. Since our result in the above corollary is independent of time, this analogy is further underlined in the following consequence describing the behavior of unbounded solutions, which also applies infinite time blow-up.

**Theorem 1.5.** Assume that \( \Omega \subset \mathbb{R}^N \) \( (N \geq 2) \) is a bounded domain with smooth boundary. Let \( (u, v) \) be a classical solution of \((1.1)\) on \( \Omega \times (0, T_{\text{max}}) \) with \( T_{\text{max}} \in (0, \infty) \). Suppose that

\[
\sup_{t \in (0, T_{\text{max}})} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]

Then \( \{u \frac{\nabla |u|^2}{|u|^2}(\cdot, t)\}_{t \in (0, T_{\text{max}})} \) is not equi-integrable. In other words, there are \( \varepsilon_0 > 0 \), and \( x_0 \in \Omega \) such that for all \( \rho > 0 \),

\[
\sup_{t \in (0, T_{\text{max}})} \int_{B_\rho(x_0) \cap \Omega} u \frac{\nabla |u|^2}{|u|^2}(\cdot, t) > \varepsilon_0.
\]

### 2 An interpolation inequality

In the analysis of chemotaxis models, the Gagliardo-Nirenberg inequality is frequently used, especially in the style of the following form

\[
\|\varphi\|_{L^p(\Omega)} \leq C_1 \|\nabla \varphi\|_{L^q(\Omega)}^{a} \|\varphi\|_{L^r(\Omega)}^{1-a} + C_2 \|\varphi\|_{L^r(\Omega)} \quad \text{for all } \varphi \in W^{1,r}(\Omega),
\]  

(2.1)

where \( a = \frac{\frac{N}{r} - \frac{N}{p}}{1 - \frac{N}{p} + \frac{N}{r}} \in (0, 1) \) [6, Theorem 10.1]. Here the constant \( C_1 > 0 \) depends on \( p, q, r \) and \( \Omega \). When applying the Gagliardo-Nirenberg inequality, we usually require the exponent \( a \) to be strictly less than a given power in order to control a target term. One can imagine that if \( C_1 > 0 \) could be chosen arbitrarily small, we would be able to deal with more subtle critical cases [2].

The purpose of this section is to investigate a kind of interpolation inequality with the aforementioned ambition that the constant \( C_1 \) can be arbitrarily small. However, this is not generally true. Following the
Lemma 2.1. Let \( \Omega \subset \mathbb{R}^N \) be bounded with smooth boundary. Let \( r \geq 1, 0 < q < \frac{Nr}{(N-r)r} \). For any \( 0 < \theta < q \), we define
\[
p := \begin{cases} N(\frac{2}{q} - 1), & \text{if } q > r, \\ \theta, & \text{if } q \leq r, \end{cases}
\]
\[
q_0 := \begin{cases} q, & \text{if } q > r, \\ r(1 + \frac{N}{Nr}), & \text{if } q \leq r. \end{cases}
\]
Let \( \delta : (0, 1) \rightarrow (0, \infty) \) be nondecreasing. Then for each \( \varepsilon > 0 \), we can find \( C_\varepsilon > 0 \) such that
\[
\| \varphi \|_{L^q(\Omega)} \leq \varepsilon \| \nabla \varphi \|_{L^r(\Omega)}^{\frac{1}{N} - b} + C_\varepsilon \| \varphi \|_{L^r(\Omega)}^{(1 - \frac{N}{Nr} + \frac{q}{Nq})b + (1 - b)} + C_\varepsilon \| \varphi \|_{L^r(\Omega)} \cdot \| \varphi \|_{L^p(\Omega)} \cdot \| \varphi \|_{L^p(\Omega)}^{\frac{1}{N} - b},
\]
is valid for any
\[
\varphi \in \mathcal{F}_\delta := \left\{ \psi \in W^{1,r}(\Omega) \Big| \text{ For all } \varepsilon' \in (0, 1), \text{ we have } \int_E \psi^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega \text{ with } |E| < \delta(\varepsilon') \right\}.
\]
Proof. We first consider the case \( q > r \), hence \( \frac{2}{q} - 1 > 0 \). We abbreviate \( s := \frac{Nr}{N + r} < \min\{N, r\} \). Then according to the Sobolev embedding: \( W^{1,s}_0(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N) \), there is a constant \( c_1 > 0 \) such that
\[
\| \psi \|_{L^r(\mathbb{R}^N)} \leq c_1 \| \nabla \psi \|_{L^s(\mathbb{R}^N)} \tag{2.5}
\]
for all \( \psi \in W^{1,s}_0(\mathbb{R}^N) \). Let \( \Omega' \) be a bounded open set such that \( \Omega \subseteq \Omega' \). In light of Theorem A.1, we can find \( c_2 > 0 \) and extend \( \varphi \in W^{1,r}(\Omega) \) to \( \tilde{\varphi} \in W^{1,r}_0(\mathbb{R}^N) \) in such a way that
\[
\tilde{\varphi} = \varphi \text{ a.e. in } \Omega, \quad \text{supp } \tilde{\varphi} \subset \Omega', \quad \| \tilde{\varphi} \|_{L^s(\Omega')} \leq c_2 \| \varphi \|_{L^s(\Omega)}, \quad \| \nabla \tilde{\varphi} \|_{L^r(\Omega')} \leq c_2 \| \nabla \varphi \|_{L^r(\Omega)}, \tag{2.6}
\]
and that there is a nondecreasing function \( \tilde{\delta} : (0, 1) \rightarrow (0, \infty) \) such that
\[
\tilde{\varphi} \in \mathcal{F}_{\tilde{\delta}} := \left\{ \psi \in W^{1,r}(\Omega) \Big| \text{ For all } \varepsilon' \in (0, 1), \text{ we have } \int_E \psi^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega' \text{ with } |E| < \tilde{\delta}(\varepsilon') \right\}.
\]
Given \( \varepsilon > 0 \), let \( \varepsilon' := \left( \frac{\varepsilon^p}{\overline{\varepsilon}^p} \right)^{\frac{1}{p} - \frac{1}{q}} \) and let \( \delta := \tilde{\delta}(\varepsilon') > 0 \). We have
\[
\int_B |\tilde{\varphi}|^p < \varepsilon' \tag{2.8}
\]
for any ball \( B \subset \Omega' \) and with radius no bigger than \( \eta := \left( \frac{\delta}{\overline{\varepsilon}} \right)^{\frac{1}{p} - \frac{1}{q}} \), where \( \overline{\varepsilon} \) denotes the volume of the unit ball in \( \mathbb{R}^N \).

Since \( \Omega \) is bounded, we can find a family of finite balls \( \{ B_j \}_{1 \leq j \leq M} \) with radius larger than \( \eta \) to cover \( \Omega \) with \( \Omega \subset \bigcup_{1 \leq j \leq M} B_j \subset \Omega' \). Moreover, there exists \( c_3 > 0 \) and a smooth partition of unity for \( \bigcup_{1 \leq j \leq M} B_j \) is given by a family of nonnegative functions \( \{ \zeta_j \}_{1 \leq j \leq M} \) satisfying
\[
\text{supp } \zeta_j \subset B_j, \quad |\nabla \zeta_j|^p < \frac{c_3}{\eta} \text{ for all } 1 \leq j \leq M, \quad \text{and } \sum_{j=1}^M \zeta_j = 1. \tag{2.9}
\]
It can be easily checked that \( \tilde{\varphi}^\frac{1}{q} \zeta_j \) belongs to \( W_0^{1,q}(B_j) \) since \( \varphi \in W_0^{1,q}(\mathbb{R}^N) \) and \( q < \frac{N_r}{(N-r_+)} \), therefore we can invoke (2.5) and the elementary inequality
\[
(a + b)^s \leq 2^{s-1}a^s + 2^{s-1}b^s
\]
for all \( s > 1 \) and \( a, b > 0 \), to obtain that
\[
\int_{\Omega'} \tilde{\varphi}^q \zeta_j = \| \tilde{\varphi}^\frac{1}{q} \zeta_j \|^q_{L^r(B_j)}
\]
\[
\leq c_1 \| \nabla (\tilde{\varphi}^\frac{1}{q} \zeta_j) \|_{L^r(B_j)}^r
\]
\[
\leq c_1 \| \frac{q}{r} \tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi} + \tilde{\varphi}^\frac{1}{q} \zeta_j \|_{L^r(B_j)}^r
\]
\[
\leq c_1 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}} + c_1 2^{r-1} \left( \frac{q}{\eta} \right)^r \left( \int_{B_j} \tilde{\varphi}^\frac{1}{r} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}.
\]
This implies the existence of \( \tilde{\varphi}^\frac{1}{q} \zeta_j \) in \( L^r(B_j) \) for all \( r < q < \frac{N_r}{(N-r_+)} \), there are constants \( c_2, c_4, c_5 > 0 \) such that
\[
c_1 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}} \leq c_1 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}
\]
\[
= c_1 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}
\]
\[
= c_1 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}
\]
\[
\leq c_1 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}} \leq c_1 \varepsilon^q 2^{r-1} \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}.
\]
Now we claim that for all \( r < q < \frac{N_r}{(N-r_+)} \), there are constants \( c_2, c_4, c_5 > 0 \) such that
\[
2^{r-1} c_1 \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}} \leq 2^{r-1} c_1 \left( \frac{q}{\eta} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}
\]
\[
\leq c_4 \| \nabla \tilde{\varphi} \|_{L^r(\Omega')}^q \| \tilde{\varphi} \|_{L^N(\Omega')}^{1-q} + c_4 \| \tilde{\varphi} \|_{L^N(\Omega')}^q
\]
\[
= c_4 \left( \int_{\Omega'} |\nabla \tilde{\varphi}|^r \right)^{\frac{q}{r}} \left( \int_{\Omega'} \tilde{\varphi}^{N(\frac{r}{r} - 1)} \right)^{\frac{1-q}{r}} + c_4 \| \tilde{\varphi} \|_{L^N(\Omega')}^q
\]
\[
\leq \varepsilon^q 2^{r-1} c_4 \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}} \leq c_5 \varepsilon^q 2^{r-1} c_4 \left( \frac{q}{r} \right)^r \left( \int_{B_j} |\tilde{\varphi}^{\frac{q}{r} - 1} \zeta_j \nabla \tilde{\varphi}|^r \right)^{\frac{1}{r}}.
\]
Hence (2.12) holds for all \( r < q < \frac{N_{r}}{(N_{r})} \). Combining (2.10, 2.12), we see that for each \( 1 \leq j \leq M \),
\[
\int_{\Omega'} |\varphi|^q \zeta_j \leq \frac{\varepsilon^q}{2c_2} \int_{\Omega'} \zeta_j \nabla |\varphi|^r + \frac{\varepsilon^q}{2c_2 M} \int_{\Omega'} |\nabla \varphi|^r
+ c_\varepsilon \|\varphi\|_L^{q + \frac{N_{r} + N + r}{N_{r} - 1} (\Omega')} + c_4 \|\varphi\|_L^{q + \frac{N_{r}}{N_{r} - 1} (\Omega')} + c_5 \varepsilon^\frac{q^2}{q - r}.
\]
Finally, we obtain from (2.14) and (2.9) that
\[
\|\varphi\|_{L^q(\Omega')} \leq \|\varphi\|_{L^q(\Omega')} \left( \sum_{j=1}^{\frac{M^q}{M}} \right) = \sum_{j=1}^{\frac{M^q}{M}} \int_{\Omega'} |\varphi|^q \zeta_j \leq \sum_{j=1}^{\frac{M^q}{M}} \left( \frac{\varepsilon^q}{2c_2} \int_{\Omega'} |\nabla |\varphi|^r + \frac{\varepsilon^q}{2c_2 M} \int_{\Omega'} |\nabla \varphi|^r + c_\varepsilon \|\varphi\|_L^{q + \frac{N_{r} + N + r}{N_{r} - 1} (\Omega')} + c_4 \|\varphi\|_L^{q + \frac{N_{r}}{N_{r} - 1} (\Omega')} + c_5 \varepsilon^\frac{q^2}{q - r} \right)
\leq \frac{\varepsilon^q}{2c_2} \int_{\Omega'} |\nabla |\varphi|^r + \frac{\varepsilon^q}{2c_2 M} \int_{\Omega'} |\nabla \varphi|^r + c_\varepsilon \|\varphi\|_L^{q + \frac{N_{r} + N + r}{N_{r} - 1} (\Omega')} + c_4 \|\varphi\|_L^{q + \frac{N_{r}}{N_{r} - 1} (\Omega')} + c_5 \varepsilon^\frac{q^2}{q - r}
\leq \frac{\varepsilon^q}{2c_2} \int_{\Omega} |\nabla \varphi|^r + c_\varepsilon \|\varphi\|_{L^{q + \frac{N_{r} + N + r}{N_{r} - 1} (\Omega')} + c_4 \|\varphi\|_{L^{q + \frac{N_{r}}{N_{r} - 1} (\Omega')} + c_5 \varepsilon^\frac{q^2}{q - r}
\]
with some constant \( C_\varepsilon > 0 \). Note that \( b = 1 \) if \( q > r \), taking the \( q \)-th root on both sides leads to (2.3) for the case \( q > r \).

If \( q \leq r \), we see that \( q_0 > r > q > \theta \). The Hölder inequality with \( b = \frac{q_0 - q}{q_0} \) shows that
\[
\|\varphi\|_{L^q(\Omega')} \leq \|\varphi\|_{L^{q_0}(\Omega')} \|\varphi\|_{L^\frac{q_0}{q_0}(\Omega')},
\]
(2.15) Since \( q_0 > r \), we have already proven that for all \( \varepsilon > 0 \), there is \( C_\varepsilon > 0 \) such that
\[
\|\varphi\|_{L^{q_0}(\Omega)} \leq \left( \varepsilon^\frac{q_0}{q} \|\nabla \varphi\|_{L^{q_0}(\Omega')} + c_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} \right)^\frac{1}{q_0}
\leq \varepsilon^\frac{q_0}{q} \|\nabla \varphi\|_{L^{q_0}(\Omega')} + c_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')},
\]
which combined with the previous interpolation inequality (2.14) yields that
\[
\|\varphi\|_{L^{q_0}(\Omega)} \leq \left( \varepsilon^\frac{q_0}{q} \|\nabla \varphi\|_{L^{q_0}(\Omega')} + c_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} \right)^\frac{1}{q_0}
\leq \varepsilon^\frac{b q_0}{q} \|\nabla \varphi\|_{L^{q_0}(\Omega')} + c_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')} + C_\varepsilon \|\varphi\|_{L^{q_0}(\Omega')}
\]
We easily check that \( b \cdot \frac{q_0}{q_0} = \frac{q_0 - q}{q_0} + \frac{q_0}{q_0} = 1 \), thus (2.3) is valid for \( q \leq r \) as well.

\( \square \)

Remark 2.2. The exponent \( a \) in (2.3) is exactly the one from the Gagliardo-Nirenberg inequality
\[
\|\varphi\|_{L^q(\Omega')} \leq C \|\nabla \varphi\|_{L^{q_0}(\Omega')} \|\varphi\|_{L^{q_0}(\Omega')} + C \|\varphi\|_{L^{q_0}(\Omega')} \quad \text{for all } \varphi \in W^{1, r}(\Omega).
\]
However \( 1 - b \neq 1 - a \). In fact, following the proof we can find \( a + 1 - b < 1 \).

Remark 2.3. Given a family of functions \( \{f_j\}_{j \in \mathbb{N}} \) such that \( \{f_j\}_{j \in \mathbb{N}} \) is equi-integrable, there exists \( \delta : (0, 1) \to (0, \infty) \) nondecreasing such that \( f_j \in F_\delta \), where \( F_\delta \) is defined in (2.4). Therefore, we can apply Lemma 2.1 to a family of functions enjoying equi-integrability.
3 Preliminaries for the Keller-Segel model

In this section, some basic knowledge on the Keller-Segel system is prepared. We first introduce the well-established local existence theory for (1.1). The proof can be found in many previous work, e.g. [1, Lemma 3.1].

**Lemma 3.1.** Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary that the initial data $(u_0,v_0)$ are nonnegative and satisfy $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$. There exists $T_{\max} \in (0,\infty]$ with the property such that the problem possesses a unique nonnegative classical solution $(u,v)$ satisfying

\[
\begin{align*}
    u &\in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\
    v &\in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap L^\infty_{\text{loc}}([0,T_{\max}); W^{1,\infty}(\Omega)).
\end{align*}
\]

Moreover, if $T_{\max} < \infty$, then

\[
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty, \text{ as } t \to T_{\max}.
\]

The following properties can be easily checked.

**Lemma 3.2.** We have

\[
\begin{align*}
    \int_{\Omega} u(\cdot,t) &= \int_{\Omega} u_0, \quad (3.1) \\
    \int_{\Omega} v(\cdot,t) &\leq \max \left\{ \int_{\Omega} v_0, \int_{\Omega} u_0 \right\}, \text{ for all } t \in (0,T_{\max}). \quad (3.2)
\end{align*}
\]

In order to deal with a kind of spatial derivative estimate involving a time potential function, we introduce the following version of maximal Sobolev regularity, which has been used in [3, Lemma 2.5] and [24].

**Lemma 3.3.** Let $r,q \in (1,\infty)$, and $T \in (0,\infty]$, $f \in L^r((0,T); L^q(\Omega))$. Let $v$ be the unique strong solution to the following evolution equation

\[
\begin{align*}
    v_t &= \Delta v - v + f, \quad (x,t) \in \Omega \times (0,T) \\
    \partial_r v &= 0, \quad (x,t) \in \partial\Omega \times (0,T) \\
    v(x,0) &= v_0(x), \quad x \in \Omega.
\end{align*}
\]

There exists $C > 0$, such that if $t_0 \in [0,T)$, $v(\cdot,t_0)$ satisfies $v(\cdot,t_0) \in W^{2,r}(\Omega)$ with $\partial_r v(\cdot,t_0) = 0$, we have

\[
\int_{t_0}^{T} e^{\frac{r}{2} s} \|\Delta v(\cdot,t)\|_{L^r(\Omega)} ds \leq C \int_{t_0}^{T} e^{\frac{r}{2} s} \|f(\cdot,t)\|_{L^r(\Omega)} ds + C e^{\frac{r}{2} t_0} \|v(\cdot,t_0)\|_{W^{2,r}(\Omega)},
\]

where $C$ depends on $q,r,\Omega$.

**Proof.** For given $t_0 \in (0,T)$, we know that $\partial_r v(\cdot,t_0) = 0$ on $\partial\Omega$. Let $d := \min\{\frac{T-t_0}{4},1\}$ and let $\chi \in C_0^\infty((0,\infty))$ be a cut-off function satisfying

\[
\begin{align*}
    \chi(s) &= 1, \quad s = 0, \\
    \chi(s) &\leq 1, \quad 0 < s < d, \\
    \chi(s) &= 0, \quad s \geq d.
\end{align*}
\]

Moreover, $|\chi'(s)| \leq \frac{2}{d}$ for all $s \in [0,\infty)$. Let $w(x,s) := e^{\frac{r}{2} s} v(x,s+t_0) - \chi(s)v(x,t_0)$ for $(x,s) \in \Omega \times [0,T-t_0)$. We see that $w$ solves the following equation

\[
\begin{align*}
    w_t &= (\Delta - \frac{r}{2})w(x,s) + e^{\frac{r}{2} s} f(x,s+t_0) + g(x,s), \quad (x,s) \in \Omega \times (0,T-t_0), \\
    \nabla w \cdot \nu &= 0, \quad (x,s) \in \partial\Omega \times [0,T-t_0), \\
    w(x,0) &= 0, \quad x \in \Omega,
\end{align*}
\]

where $g(x,s)$ is given by

\[
\begin{align*}
    g(x,s) &= \frac{r}{2} v(x,s+t_0) + \frac{r}{2} |\chi'(s)v(x,t_0)|^2 + e^{\frac{r}{2} s} g(x,s+t_0), \\
    g(x,s) &= \frac{r}{2} v(x,s+t_0) + e^{\frac{r}{2} s} g(x,s+t_0) + \frac{r}{2} \chi'(s)v(x,t_0)^2, \\
    g(x,s) &= \frac{r}{2} v(x,s+t_0) + e^{\frac{r}{2} s} g(x,s+t_0) + \frac{r}{2} \chi'(s)v(x,t_0)^2, \\
    g(x,s) &= \frac{r}{2} v(x,s+t_0) + e^{\frac{r}{2} s} g(x,s+t_0) + \frac{r}{2} \chi'(s)v(x,t_0)^2.
\end{align*}
\]
where \( g(x, s) := \chi(s)\Delta v(x, t_0) - \chi'(s)v(x, t_0) - \frac{1}{2}\chi(s)v(x, t_0) \) in \( \Omega \times [0, T - t_0] \).

An application of the maximal Sobolev regularity result from [8] implies the existence of \( C_{q, r} > 0 \) such that
\[
\int_0^{T-t_0} \| \Delta w(\cdot, s) \|_{L^q(\Omega)} ds \\
\leq C_{q, r} \int_0^{T-t_0} \| e^{\frac{2r}{d}} f(x, s + t_0) \|_{L^q(\Omega)} ds \\
+ C_{q, r} \int_0^{T-t_0} \| \chi(s)\Delta v(x, t_0) - \chi'(s)v(x, t_0) - \frac{1}{2}\chi(s)v(x, t_0) \|_{L^q(\Omega)} ds \\
\leq C_{q, r} \int_0^{T-t_0} \| e^{\frac{2r}{d}} f(x, s + t_0) \|_{L^q(\Omega)} ds + 3^{r-1} C_{q, r} d(\frac{2}{d} + \frac{3}{2}) \| v(\cdot, t_0) \|_{W^{2, q}(\Omega)} \\
\leq C_{q, r} \int_0^{T-t_0} \| e^{\frac{2r}{d}} f(x, s + t_0) \|_{L^q(\Omega)} ds + 4^r C_{q, r} \| v(\cdot, t_0) \|_{W^{2, q}(\Omega)}.
\]

Since \( e^{\frac{2r}{d}} \Delta v(x, s + t_0) = \Delta w(x, s) + \chi(s)\Delta v(x, t_0) \), we have
\[
\int_0^{T-t_0} e^{\frac{2r}{d}} \| \Delta v(\cdot, s + t_0) \|_{L^q(\Omega)} ds \\
\leq 2^{r-1} \int_0^{T-t_0} \| \Delta w(\cdot, s) \|_{L^q(\Omega)} ds + 2^{r-1} \int_0^{T-t_0} \| \chi(s)\Delta v(\cdot, t_0) \|_{L^q(\Omega)} ds
\]
\[
\leq 2^{r-1} C_{q, r} \int_0^{T-t_0} \| e^{\frac{2r}{d}} f(x, s + t_0) \|_{L^q(\Omega)} ds + 2^{r-1}(4^r C_{q, r} + 1) \| v(\cdot, t_0) \|_{W^{2, q}(\Omega)}.
\]

Upon changing variables, we obtain that
\[
\int_0^{T} e^{\frac{2r}{d}(t-t_0)} \| \Delta v(\cdot, t) \|_{L^q(\Omega)} dt \\
\leq 2^{r-1} C_{q, r} \int_0^{T} e^{\frac{2r}{d}(t-t_0)} \| f(\cdot, t) \|_{L^q(\Omega)} dt + (8^r C_{q, r} + 2^{r-1}) \| v(\cdot, t_0) \|_{W^{2, q}(\Omega)},
\]
(3.7)
where (3.4) follows by multiplying (3.7) by \( e^{\frac{2r}{d}t_0} \) and choosing \( C := 8^r C_{q, r} + 2^{r-1} \).

\section{Proof of Theorem \ref{thm:1.2}}

Having in hand Proposition \ref{prop:1.1} we see that it is sufficient to show that (1.2) holds for some \( p > \frac{N}{2} \). Before going into details, let us first prepare the following embedding lemma.

\begin{lemma}
Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary, and let \( \alpha \in (1, N) \). For all \( s \in (0, \infty) \), there is \( C > 0 \) such that
\[
\| \nabla \varphi \|_{L^{\frac{N\alpha}{N-\alpha}(\Omega)}} \leq C \| \Delta \varphi \|_{L^\alpha(\Omega)} + C \| \varphi \|_{L^\alpha(\Omega)}, \quad \text{for all } \varphi \in W^{2, \alpha}(\Omega) \text{ with } \partial_\nu \varphi = 0 \text{ on } \partial \Omega.
\]
(4.1)
\end{lemma}

\begin{proof}
Using the fact that with some \( c_1 > 0 \), the estimates \( \| \varphi \|_{W^{2, \alpha}(\Omega)} \leq c_1 (\| \varphi \|_{L^\alpha(\Omega)} + \| \Delta \varphi \|_{L^\alpha(\Omega)}) \) holds for all \( \varphi \in W^{2, \alpha}(\Omega) \) with \( \partial_\nu \varphi \big|_{\partial \Omega} = 0 \) [3 Theorem 19.1], we obtain a constant \( c_2 > 0 \) from the embedding \( W^{2, \alpha}(\Omega) \hookrightarrow W^{1, \frac{N\alpha}{N-\alpha}}(\Omega) \) that
\[
\| \nabla \varphi \|_{L^{\frac{N\alpha}{N-\alpha}(\Omega)}} \leq c_2 (\| \Delta \varphi \|_{L^\alpha(\Omega)} + \| \varphi \|_{L^\alpha(\Omega)}).
\]
(4.2)
\end{proof}
If \( s < \alpha \), let \( b = \frac{N}{2} - \frac{N}{2p} \in (0, 1) \). The Gagliardo-Nirenberg inequality together with Poincaré inequality and Young’s inequality implies
\[
\|\varphi\|_{L^\infty(\Omega)} \leq c_3 \|\nabla \varphi\|_{L^{\frac{N}{N-b}}(\Omega)}^{b} \|\varphi\|_{L^p(\Omega)}^{1-b} + c_3 \|\varphi\|_{L^p(\Omega)}
\]
\[
\leq \frac{1}{2c_2} \|\nabla \varphi\|_{L^{\frac{N}{N-b}}(\Omega)} + c_4 \|\varphi\|_{L^p(\Omega)}
\]
with some constant \( c_3, c_4 > 0 \) for all \( \varphi \in W^{2,\alpha}(\Omega) \) with \( \partial_\nu \varphi|_{\partial \Omega} = 0 \). If \( s \geq \alpha \), we use Hölder’s inequality
\[
\|\varphi\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{2} - \frac{s}{2N}} \|\varphi\|_{L^p(\Omega)}
\]
instead of (4.3). Collecting (4.2-4.4) together yields (4.1).

Now we are in a position to proceed the proof of our main ingredient.

**Lemma 4.2.** Assume that \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is a bounded domain with smooth boundary, Let \((u,v)\) be a classical solution of (1.1) on \( \Omega \times (0,T_{\text{max}}) \) with \( T_{\text{max}} \in (0, \infty) \). If
\[
\sup_{t \in (0,T_{\text{max}})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty,
\]
and \( \{u^{\frac{N}{2}}(\cdot,t)\}_{t \in (0,T_{\text{max}})} \) is equi-integrable.

Then there is \( p \in (\frac{N}{2}, N) \) such that
\[
\sup_{t \in (0,T_{\text{max}})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty.
\]

**Proof.** Let \( p \in (\frac{N}{2}, N) \). Let \( \theta \in (1, \infty) \) satisfy \( \frac{1}{\theta} = 1 + \frac{2}{N} - \frac{2}{p} \in (0, 1) \), and \( \theta' \) be such that \( \frac{1}{p'} + \frac{1}{\theta'} = 1 \). We test the first equation in (1.1) with \( pu^{p-1} \) to obtain that
\[
\frac{d}{dt} \int_\Omega u^p + p(p-1) \int_\Omega u^{p-2} |\nabla u|^2 = p(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v
\]
\[
\leq \frac{p(p-1)}{4} \int_\Omega u^{p-2} |\nabla u|^2 + p(p-1) \int_\Omega u^p |\nabla v|^2
\]
for all \( t \in (0,T_{\text{max}}) \). Applying Hölder’s inequality, we get
\[
\frac{d}{dt} \int_\Omega u^p + \frac{3(p-1)}{p} \int_\Omega |\nabla u^{p\theta}|^2 \leq p(p-1) \int_\Omega u^p |\nabla v|^2 \leq p(p-1) \left( \int_\Omega u^{p\theta} \right)^\frac{1}{\theta} \left( \int_\Omega |\nabla v|^{2\theta'} \right)^\frac{1}{\theta'}
\]
for all \( t \in (0,T_{\text{max}}) \). Let \( a := \frac{p - \frac{N}{2}}{1 - \frac{N}{2p}} \in (0, 1) \), and abbreviate \( \frac{1}{1-a} =: \lambda > 1 \). The Gagliardo-Nirenberg inequality implies the existence of \( c_1 > 0 \) such that
\[
p(p-1) \left( \int_\Omega u^{p\theta} \right)^\frac{1}{\theta} = (p-1) \|u^{\frac{N}{2}}\|_{L^{2\theta'}(\Omega)} \leq c_1 \|\nabla u^{\frac{N}{2}}\|_{L^{2}(\Omega)} \|u^{\frac{N}{2}}\|_{L^{\frac{N}{\lambda}}(\Omega)}^{2(1-a)} + c_1 \|u^{\frac{N}{2}}\|_{L^{\frac{N}{\lambda}}(\Omega)}^2.
\]
Using Young’s inequality and the assumption (4.5), we find some constant \( c_2 > 0 \) such that the right-hand side of (4.8) is estimated as
\[
p(p-1) \left( \int_\Omega u^{p\theta} \right)^\frac{1}{\theta} \left( \int_\Omega |\nabla v|^{2\theta'} \right)^\frac{1}{\theta'} \leq \left( c_1 \|\nabla u^{\frac{N}{2}}\|_{L^{2}(\Omega)} + c_1 \|u^{\frac{N}{2}}\|_{L^{\frac{N}{\lambda}}(\Omega)}^{2(1-a)} \right) \|v\|_{L^{2\theta'}(\Omega)}^2
\]
\[
\leq \frac{p-1}{p} \|v\|_{L^{2}(\Omega)}^2 + c_2 \|v\|_{L^{2\theta'}(\Omega)}^2.
\]
Due to the choices of $\theta$ and $\theta'$, we know that $p \in (1, N)$ and $2\theta' = \frac{Np}{N-p}$, hence an application of Lemma 4.3 yields $c_3 > 0$ such that
\[
\|\nabla v\|_{L^{2N/(N-p)}(\Omega)}^{2N/(N-p)} \leq c_3 \|\Delta v\|_{L^p(\Omega)}^{\lambda} + c_3 \|v\|_{L^p(\Omega)}^{2\lambda}.
\] (4.10)
We also recall from the Gagliardo-Nirenberg inequality that there is $c_4 > 0$ fulfilling
\[
\frac{p-1}{p} \int_{\Omega} |\nabla u|^{2} \geq \lambda \int_{\Omega} u^{p} - c_4.
\] (4.11)
Thus we conclude from the previous estimates (4.8-4.11) and Lemma 3.2 that
\[
\frac{d}{dt} \int_{\Omega} u^{p} + \lambda \int_{\Omega} u^{p} + \frac{p-1}{p} \int_{\Omega} |\nabla u|^{2} \leq c_3 \|\Delta v\|_{L^p(\Omega)}^{2\lambda} + c_2 + c_4 + c_3 \|v\|_{L^p(\Omega)}^{2\lambda} \quad \text{for all } t \in (0, T_{\text{max}}).
\] (4.12)
Let $t_0 \in (0, T_{\text{max}})$. Applying the variation-of-constants formula to the above inequality, we find a constant $c_5 > 0$ such that
\[
\int_{\Omega} u^{p}(\cdot, t) \leq e^{-\lambda(t-t_0)} \int_{\Omega} u^{p}(\cdot, t_0) - \frac{p-1}{p} \int_{t_0}^{t} e^{-\lambda(t-s)} \int_{\Omega} |\nabla u|^{2} \, ds + c_5 \int_{t_0}^{t} e^{-\lambda(t-s)} \|\Delta v\|_{L^p(\Omega)}^{2\lambda} \, ds + c_6.
\] (4.13)
for all $t \in (t_0, T_{\text{max}})$. The maximal regularity from Lemma 3.3 provides a constant $c_6 > 0$ satisfying
\[
c_3 \int_{t_0}^{t} e^{-\lambda(t-s)} \|\Delta v\|_{L^p(\Omega)}^{2\lambda} \, ds \leq c_6 \int_{t_0}^{t} e^{-\lambda(t-s)} \|u\|_{L^p(\Omega)}^{2\lambda} \, ds + c_6.
\] (4.14)
Let $d = \frac{p-1}{p}$ and $b = \frac{\frac{N}{p}}{\frac{N}{p} - 1}$. We can easily check that $\frac{\lambda}{p}d = 2$. Since $\{u^{\frac{\lambda}{p}}(\cdot, t)\}_{t \in (0, T_{\text{max}})}$ is uniformly integrable, and therefore belongs to the set $\mathcal{F}_\delta$ defined in (2.4) with some nondecreasing $\delta : (0, 1) \rightarrow (0, \infty)$. Since (4.5), with
\[
\varepsilon := \sup_{t \in (0, T_{\text{max}})} \|u^{\frac{\lambda}{p}}\|_{L^p(\Omega)}^{\frac{\delta}{1-\delta}} > 0,
\]
we can apply Lemma 2.1 (in the case $q = r = 2$, and with $\theta = \frac{N}{p} < q$ by virtue of $p > \frac{N}{2}$) to find $c_\varepsilon > 0$ such that
\[
c_6 \|u\|_{L^p(\Omega)}^{2\lambda} = c_6 \|u^{\frac{\lambda}{p}}\|_{L^p(\Omega)}^{\frac{\lambda}{p}d} \\
\leq c_6 \|\nabla u\|_{L^p(\Omega)}^{\frac{\lambda}{p}d} \|u^{\frac{\lambda}{p}}\|_{L^p(\Omega)}^{\frac{\lambda}{p}(1-\delta)} + c_\varepsilon \leq \frac{p-1}{p} \|\nabla u\|_{L^p(\Omega)}^{\frac{\lambda}{p}d} + c_\varepsilon
\] (4.15)
for all $t \in (0, T_{\text{max}})$, which leads to
\[
c_3 \int_{t_0}^{t} e^{-\lambda(t-s)} \|\Delta v\|_{L^p(\Omega)}^{2\lambda} \, ds \leq \frac{p-1}{p} \int_{t_0}^{t} e^{-\lambda(t-s)} \int_{\Omega} |\nabla u^{\frac{\lambda}{p}}|^{2} \, ds + c_\varepsilon + c_6
\] (4.16)
for all $t \in (t_0, T_{\text{max}})$. Adding this to (4.13) shows that
\[
\int_{\Omega} u^{p}(\cdot, t) \leq e^{-\lambda(t-t_0)} \int_{\Omega} u^{p}(\cdot, t_0) + c_5 + c_6 + c_\varepsilon \leq \int_{\Omega} u^{p}(\cdot, t_0) + c_5 + c_6 + c_\varepsilon
\]
for all $t \in (t_0, T_{\text{max}})$. Since $\sup_{t \in (0, t_0]} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty$ due to the local existence theory, this shows (4.7). □

Proof of Theorem 1.2. Employing Lemma 4.2 and Proposition 1.1 proves $\sup_{t \in (0, T_{\text{max}})} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty$, which combined with Lemma 3.1 implies that $T_{\text{max}} = \infty$. Thus the solution is global and bounded. □

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5 Blow up behavior

From another aspect, the extension criterion in Theorem 1.2 also gives the characterization of blow up solutions.

Proof of Theorem 1.2. Suppose on contrary that \( \{ u^\pm(\cdot, t) \}_{t \in (0, T_{\text{max}})} \) is equi-integrable with \( T_{\text{max}} \in (0, \infty] \). We can apply Theorem 1.2 to show that there is a constant \( C > 0 \) such that

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C,
\]

for all \( t \in (0, T_{\text{max}}) \), which is a contradiction. \( \square \)

A Appendix

We claim a basic property of extension functions which we have used in the proof of Lemma 2.1. Namely, the extension function \( \tilde{\varphi} \in W^{1,r}(\Omega') \) is equi-integrable with respect to some power in \( \Omega' \) provided \( \varphi \) has the same property in \( \Omega \). Since we can not find this precise result in any reference, we also give a brief proof here.

Theorem A.1. Assume that \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary and that \( r > 1, 1 \leq q < \frac{N'}{(N-r)'} \). Let \( \Omega' \) be a bounded smooth domain with \( \Omega \subset \Omega' \). Then there is \( C > 0 \) and for any nondecreasing function \( \delta : (0, 1) \to (0, \infty) \), we can find \( \delta_0 : (0, 1) \to (0, \infty) \) nondecreasing such that we can extend any function \( \varphi \in W^{1,r}(\Omega) \) to a function \( \tilde{\varphi} \in W_0^{1,r}(\mathbb{R}^N) \) in such a way that

\[
\tilde{\varphi} = \varphi \text{ a.e. in } \Omega, \quad \text{supp } \tilde{\varphi} \subset \Omega', \tag{A.1}
\]

\[
\| \nabla \tilde{\varphi} \|_{W^{1,r}(\Omega')} \leq C \| \nabla \varphi \|_{W^{1,r}(\Omega)}, \tag{A.2}
\]

\[
\| \tilde{\varphi} \|_{L^q(\Omega')} \leq C \| \varphi \|_{L^q(\Omega)}. \tag{A.3}
\]

Moreover, if \( \varphi \in \mathcal{F}_\delta \) with

\[
\mathcal{F}_\delta := \left\{ \psi \in W^{1,r}(\Omega) \bigg| \text{ For all } \varepsilon' \in (0, 1), \text{ we have } \int_E \psi^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega \text{ with } |E| < \delta(\varepsilon') \right\}, \tag{A.4}
\]

then \( \tilde{\varphi} \in \mathcal{F}_{\delta_0} \) with

\[
\mathcal{F}_{\delta_0} := \left\{ \psi \in W^{1,r}(\Omega') \bigg| \text{ For all } \varepsilon' \in (0, 1), \text{ we have } \int_E \psi^p < \varepsilon' \text{ for all measurable sets } E \subset \Omega' \text{ with } |E| < \tilde{\delta}(\varepsilon') \right\}. \tag{A.5}
\]

Proof. First, (A.1) and (A.2) are precisely proven in [1; Theorem 5.4.1]. Now we recall the construction of the extension function in the proof to show the remaining properties. Since \( \partial \Omega \) is compact, we can find finitely many points \( \{ x_i \}_{1 \leq i \leq K} \subset \partial \Omega \) and open sets \( \{ W_i \}_{1 \leq i \leq K} \subset \Omega' \) with \( x_i \in W_i \) and \( W_0 \subset \Omega \) such that \( \partial \Omega \subset \bigcup_{1 \leq i \leq K} W_i \) and \( \Omega \subset W_0 \cup \bigcup_{1 \leq i \leq K} W_i \) \( \subset \Omega' \). There exist \( C^1 \) diffeomorphisms \( \Phi_i : W_i \to \mathbb{R}^N \) \( (1 \leq i \leq K) \) which flatten out \( \partial \Omega \) near \( x_i \); namely, if we let \( B_i := \Phi_i(W_i) \) be a ball, it satisfies \( B_i^- = \Phi_i(W_i \cap \Omega) = \{ y = (y_1,...,y_N) \mid y_N < 0 \} \), \( B_i^+ = \Phi_i(W_i \cap \Omega) = \{ y = (y_1,...,y_N) \mid y_N > 0 \} \). Now we define linear transformations

\[
Y_1 : (y_1,...,y_N) \in B_i^- \to (y_1,...,y_{N-1},-y_N) \in B_i^+,
\]

\[
Y_2 : (y_1,...,y_N) \in B_i^- \to (y_1,...,y_{N-1},-\frac{1}{2}y_{N}) \in B_i^+.
\]
Let \( \varphi_i(y) = \varphi(\Phi_i^{-1}(y)) \) \((y \in B_i^+, x = \Phi_i^{-1}(y) \in W_i \cap \Omega)\). A first order reflection of \( \varphi_i(y) \) is given by
\[
\tilde{\varphi}_i(y) := \begin{cases} 
-3\varphi_i(Y_1(\Phi_i(x))) + 4\varphi_i(Y_2(\Phi_i(x))), & y \in B_i^- \\
\varphi_i(y), & y \in B_i^+.
\end{cases}
\] (A.6)

If we let \( \{\zeta_i\}_{0 \leq i \leq K} \) be a partition of unity subordinate to \( \{W_i\}_{0 \leq i \leq K} \), the associated extension \( \tilde{\varphi} : \Omega' \to \mathbb{R}^N \) of \( \varphi \) is defined by converting \( \tilde{\varphi}_i \) back to \( W_i \)
\[
\tilde{\varphi}(x) := \begin{cases} 
\varphi(x), & x \in \Omega = \bigcup_{0 \leq i \leq K} W_i^+, \\
\sum_{i=0}^{i=K} \zeta_i(x) \{-3\varphi_i(Y_1(\Phi_i(x))) + 4\varphi_i(Y_2(\Phi_i(x)))\}, & x \in \Omega \setminus \bigcup_{0 \leq i \leq K} W_i, \\
0, & x \in W_i^-.
\end{cases}
\] (A.7)

where \( W_i^+ := \Phi_i^{-1}(B_i^+) \), \( W_i^- := \Phi_i^{-1}(B_i^-) \). Since the mappings \( \Phi_i, \Phi_i^{-1} \) \((1 \leq i \leq K)\), \( Y_j \) \((j \in \{1, 2\})\) are \( C^1 \), we can find a constant \( c_1 > 0 \) such that \( |\Phi_i^{-1}(Y_j(\Phi_i(U)))| \leq c_1|U| \) for all \( U \subset W_i^- \). For any measurable subset \( \Omega' \subset \Omega \), let \( E_i := \Omega' \cap W_i^- \). We note that \( \Phi_i^{-1}(Y_2(\Phi_i(E_i))) \subset \Phi_i^{-1}(Y_1(\Phi_i(E_i))) \subset \Phi_i^{-1}(B_i^-) \subset \Omega \) and by changing variables, for each \( 1 \leq i \leq K \), we have
\[
\int_{E_i} |\tilde{\varphi}(x)|^p dx = \int_{E_i} |\varphi(x)|^p dx + 3 \int_{\Phi_i(E_i)} |\varphi(Y_1(\Phi_i(x)) - \varphi(Y_2(\Phi_i(x)))|^p dy + \int_{\Phi_i(E_i)} |\varphi(Y_2(\Phi_i(x)))|^p dy
\]
\[
\leq 2^{p-1} \int_{\Phi_i(E_i)} |\varphi(Y_1(\Phi_i(x)))|^{p-1} \varphi(Y_2(\Phi_i(x)))|^{p-1} dy + 2^{p-1} \int_{\Phi_i(E_i)} |\varphi(Y_2(\Phi_i(x)))| dy
\]
\[
\leq 6^p \int_{\Phi_i^{-1}(Y_1(\Phi_i(E_i)))} |\varphi(x)|^p dx + 8^p \int_{\Phi_i^{-1}(Y_2(\Phi_i(E_i)))} |\varphi(x)|^p dx
\]

According to (A.4), given \( \varepsilon' > 0 \), we have that \( \delta(\varepsilon') > 0 \) such that \( \int_E \varphi^p < \varepsilon' \) for all \( E \subset \Omega \) with \( |E| \leq \delta(\varepsilon') \). We let \( \delta := \frac{1}{c_1} \delta \) such that if \( |E| < \min\{\delta, \delta(\varepsilon')\} \), then \( |\Phi_i^{-1}(Y_1(\Phi_i(E_i)))|, |\Phi_i^{-1}(Y_2(\Phi_i(E_i)))| < \delta \) for all \( 1 \leq i \leq K \), hence
\[
\int_{E'} |\tilde{\varphi}(x)|^p dx = \int_{E' \cap \Omega} |\varphi(x)|^p dx + \sum_{i=1}^{i=K} \int_{E_i} |\varphi(x)|^p dx
\]
\[
\leq \int_{E' \cap \Omega} |\varphi(x)|^p dx + \sum_{i=1}^{i=K} \int_{E_i} |\varphi(x)|^p dx
\]
\[
\leq \int_{E' \cap \Omega} |\varphi(x)|^p dx + \sum_{i=1}^{i=K} \left( 6^p \int_{\Phi_i^{-1}(Y_1(\Phi_i(E_i)))} |\varphi(x)|^p dx + 8^p \int_{\Phi_i^{-1}(Y_2(\Phi_i(E_i)))} |\varphi(x)|^p dx \right)
\]
\[
\leq \frac{\varepsilon'}{8^p 3K} + K \left( \frac{6^p \varepsilon'}{8^p 3K} + \frac{8^p \varepsilon'}{8^p 3K} \right) < \varepsilon'.
\]

Therefore, \( \tilde{\varphi} \in \mathcal{F}_{\delta} \) is shown. Using \( \int_{E_i} \tilde{\varphi}^q = \sum_{0 \leq i \leq K} \int_{E_i} \tilde{\varphi}^q \), (A.3) can be proven in a similar way.

\[ \square \]

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