Stability and steady-state analysis of distributed cooperative droop controlled DC microgrids

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Abstract: Distributed cooperative droop control consisting of the primary decentralised droop control and the secondary distributed control is an exact current sharing between generators, worked in the voltage control mode, of DC microgrids. For the DC microgrids with the distributed cooperative droop control, the dynamic stability has not been well investigated although its steady performance has been widely reported. This study focuses on the stability problem of DC microgrids with fixed topology and shows it is equivalent to the semistability problem of a class of second-order matrix systems. Some further sufficient conditions as well followed. The steady state is analysed deeply for some special cases. A DC microgrid of four nodes is simulated on the Matlab/Simulink platform to illustrate the efficacy of analytic results.

1 Introduction

DC microgrids have been attracting considerable attention in more recent years [1–3], since most renewable energy sources and storage systems, as well as many loads like vehicles, data centres and telecom systems, have a nature of direct current and thereby it is preferable to aggregate them by DC microgrids. Particularly, there are no reactive power and frequency synchronisation in DC microgrid, both of them being the main challenging problems of AC systems. Similar with AC microgrids, an important control objective of DC microgrids is to share the power demanded by loads among different sources. The sharing control for microgrids including both AC and DC can be roughly categorised into three types: centralised, decentralised and distributed control [4]. In contrast to the centralised control requiring high-bandwidth communications and to the decentralised control with a biased sharing, the distributed control as a strategy between the centralised control and the decentralised control is more robust and expandable. It can make an exact sharing among all sources in the cost of low-bandwidth communications between neighbouring sources.

DC decentralised droop control can be traced back to about twenty years ago for a superconductive DC system [5], where the voltage is uniform for all terminals and thereby is utilised as the same parameter like the frequency of AC systems, to coordinate terminal currents. In the presence of line resistances, the DC voltage is no longer the uniform measure such that the power sharing has a large deviation from that in a lossless DC microgrid [6]. In [7], the steady state of decentralised droop control is addressed. The influences of droop gains on the power sharing and voltage deviations are illustrated and an optimal droop gain setting problem is issued.

To improve the current sharing accuracy, a hierarchical controller is presented in [8] where the large droop gains are adopted in the primary control for a small deviation current sharing and the large deviation of voltage is compensated by the secondary control. In [9], a distributed droop controller based on the average current of all terminals is proposed to make an exact current sharing. However, an extra wire laying along with power lines is required to connect the measured values of currents such that the information of average current is available in real time. In [10], a distributed droop control based on a low-bandwidth communication is proposed, in which the converter’s voltage and current are exchanged between neighbouring converters. The stability analysis has been made only on a two-node DC microgrid. A distributed droop controller including two modules, voltage regulator and current regulator for a meshed DC microgrid has been proposed in [11]. An extension to adaptive droop gains is reported in [12]. Although the analysis of steady states and performances has been made in the frequency domain, the stability of overall closed-loop system is not addressed.

It is difficult to analyse the overall stability of microgrids. The conventional frequency domain method is difficult to tackle the whole dynamics with high dimensions. A few works on this topic often are based on the small-signal model. In [13], the stability of battery converter with adaptive droop gains has been addressed. In [14], it was shown that even if each converter is stable by itself the stability of overall DC microgrid is not ensured because of the coupling between converter regulators. In [15], a linearised model including sources, lines and loads is presented for DC microgrids. The eigenvalues of the system matrix determine the stability of DC microgrid. The relationship between the eigenvalue locations and the line impedance is discussed.

This paper addresses the distributed cooperative droop controller with only one module of current sharing regulator. The controller has two control levels. The primary control is a decentralised voltage droop control to regulate the converter output voltage according to its output current. The secondary control is a distributed current sharing control in the sense that the neighbouring converters exchange their p.u. currents via a low-bandwidth communication. Our focus is on the stability problem owing to the secondary distributed control.

As for as the current sharing is concerned, the control goal is to force the currents of generator to reach the same ratio with respect to their maximal/nominal currents. The desired sharing currents cannot be assigned a prior because of the fluctuation of loads in a power system. Such a scenario can be characterised by the notation of semistability [16–18], which means that the steady state is not completely determined by the system dynamics, but depends on the system initial conditions as well. Semistability is an appropriate notation for the analysis of self-organised behaviours of networked systems which rely on the initial configuration, and has been applied for the consensus problem of linear [19] and non-linear networked systems [20]. This paper will also use the tool of semistability to analysis the stability of the closed-loop system under the secondary distributed current sharing control.

More recently, Andreasson et al., regarded the terminals of HVDC transmission systems as the controlled current sources and
presented three kinds of distributed controllers to regulate the terminal voltages, as well as the related sufficient conditions for the stability of the closed-loop system in [21]. Moreover, the stability analysis here goes along the line of the semistability which is as well different from the line of the characteristic equation used in [21].

2 Decentralised droop control

Consider a DC microgrid of $n$ generator nodes connected by $m$ lines. Fig. 1 shows an illustration with two energy storages and two PV generators. Denote the node sets by $\mathcal{N} = \{1, 2, \ldots, n\}$. A line connecting with nodes $k$ and $j$ is associated with a branch conductance $G_{kj} \geq 0$. $G_{kj} = 0$ if and only if there is no connection between nodes $k$ and $j$. A node $k$ is associated with a current injection $i_k$, an output voltage $u_k$ and a shunt conductance $G_{kk} \geq 0$. If $G_{kk} = 0$, then node $k$ has no local load. The conductance matrix $Y = (Y_{kj})_{n \times n}$ is given by

$$Y_{kj} = \begin{cases} \sum_{i=1}^{n} G_{ki} & k = j \\ -G_{kj} & k \neq j \end{cases}. \quad (1)$$

The conductance matrix can be decomposed by $Y = Y_s + Y_e$, where $Y_s = \text{diag}(G_{11}, G_{22}, \ldots, G_{nn})$ denotes the shunt conductance matrix and $Y_e$ the branch conductance matrix, defined by

$$Y_e = \begin{bmatrix} G_{e11} & -G_{e12} & \cdots & -G_{e1n} \\ -G_{e21} & G_{e22} & \cdots & -G_{e2n} \\ \vdots & \vdots & \ddots & \vdots \\ -G_{en1} & -G_{en2} & \cdots & G_{enn} \end{bmatrix}. \quad (2)$$

Remark 1: The model above is general in that load nodes can be cancelled by a network reduction [22, 23].

Due to the ultra fast responses of converters, the generator with DC/DC converters can be simplified as a DC voltage source whose voltage is regulated instantaneously, namely

$$u_k = u_k^{\text{ref}}, \quad k \in \mathcal{N}, \quad (3)$$

where $u_k$ and $u_k^{\text{ref}}$ are the output voltage and the output voltage reference of node $k$. A DC microgrid makes a load current sharing by a voltage droop controller with which the voltage reference of generator will reduce when its output current increases, being implemented as

$$u_k^{\text{ref}} = u_k - R_k i_k^{\text{ref}}, \quad (4)$$

where $u_k^\text{ref}$ is the rated output voltage, $R_k$ is the internal resistance that might be a virtual one to be designed and $i_k^{\text{ref}}$ is the output current measured.

Since of interest is the DC current, a low-pass filter on the output current $i_k$ is used to get $i_k^m$,

$$\tau_k i_k^m = -i_k + i_k^m, \quad (5)$$

where $\tau_k$ is the time constant of low-pass filter. Using the Kirchhoff’s current law, one has

$$i_k = \sum_{j=1}^{n} Y_{kj} i_j, \quad k \in \mathcal{N}. \quad (6)$$

Combining (3) and (4) and replacing (6) into (5) yield the following dynamic equation of the DC microgrid:

$$\tau_k i_k^m = -i_k + \sum_{j=1}^{n} Y_{kj} (u_j - R_k i_j^{\text{ref}}), \quad k \in \mathcal{N}. \quad (7)$$

Define $I^m = \text{col}(i_{1}^m, \ldots, i_{n}^m)$ and $u^d = \text{col}(u_1^d, \ldots, u_n^d)$, then the compact form of (7) is given by

$$D I^m = -(E + Y R) I^m + Y u^d, \quad (8)$$

where $D = \text{diag}(\tau_1, \tau_2, \ldots, \tau_n)$, $R = \text{diag}(R_1, R_2, \ldots, R_n)$ and $E$ denotes the identity matrix.

Since $\tau_i > 0$ for all $i \in \mathcal{N}$, the steady state can be solved from setting the right side of (8) to zero

$$I_s^m = I^m = (E + Y R)^{-1} Y u^d, \quad (9)$$

where the superscript $ss$ denotes the steady state of variable and the first equality comes from (5). Subsequently, the steady output voltage of each node is given by

$$u_s^m = U^d - R I_s^m = (E + Y R)^{-1} Y u^d. \quad (10)$$

3 Distributed current sharing control

It can be seen from (9) that the same voltage reference $u^d = u_1^d 1_n$ in general cannot lead to the same current by the primary decentralised control. Throughout of this paper, $1$ denotes the vector with all elements being 1, and $1_n$ denotes such a vector with order being $n$. This section address a secondary distributed control to realise an exact current sharing. Only the neighbouring nodes
exchange information each other and each node adjusts the rated voltage according to the current bias from their neighbouring nodes.

Let \( G = (\mathcal{N}, \mathcal{E}) \) express the information flow between nodes, where \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \) denotes the edge set. Let \( L = (l_{ij}) \in \mathbb{R}^{n \times n} \) be the associated Laplacian matrix of \( G \). If \( Y_{ij} \neq 0 \), then \( l_{ij} = -1 \); or else \( l_{ij} = 0 \). The information graph has no self-loop, so \( l_{ii} = \sum_{j=1}^{n} l_{ij} = 1 \).

In contrast to the same current, the same current ratio is a more rational index for current sharing. Denote by \( i_{i}^{\max} \) the maximum current of node \( i \), then our goal is to realise

\[
\frac{i_{i}^{\max}}{l_{i}} = \frac{i_{1}^{\max}}{l_{1}} = \cdots = \frac{i_{n}^{\max}}{l_{n}}. \tag{11}
\]

Let current ratio \( \mathcal{I}_k = \frac{i_{k}^{m}}{i_{k}^{\max}} \) be the information exchanged between neighbouring nodes. The following PI controller is proposed for the rated output voltage:

\[
u_{k}^{d} = \left(-\alpha_{k} - \frac{\beta_{k}}{s} \right) \sum_{j} l_{kj} \mathcal{I}_{j}, \quad k \in \mathcal{N}, \tag{12}
\]

where \( \alpha > 0 \) and \( \beta > 0 \) are proportional and integrator gains of node \( k \), respectively. The block diagram of the closed-loop system of node \( k \) is shown in Fig. 2. Its closed-loop dynamics is described by

\[
\begin{align*}
\dot{i}_{k} &= -\frac{1}{\tau_{k}} i_{k} + \frac{1}{\tau_{k}} \sum_{j=1}^{n} Y_{kj} \mathcal{I}_{j} - \frac{1}{\tau_{k}} \sum_{j=1}^{n} Y_{kj} R \mathcal{I}_{j}, \\
\dot{\mathcal{I}}_{k} &= -\alpha_{k} \sum_{j=1}^{n} l_{kj} \mathcal{I}_{j} - \beta_{k} \sum_{j=1}^{n} l_{kj} \mathcal{I}_{j},
\end{align*}
\]

Define \( Y = \text{diag}(\mathcal{I}_{1}^{\max}, \ldots, \mathcal{I}_{n}^{\max}) \), \( \Phi = \text{diag}(\alpha_{1}, \ldots, \alpha_{n}) \) and \( \Psi = \text{diag}(\beta_{1}, \ldots, \beta_{n}) \). Then the above closed-loop system can be rewritten as the following compact form:

\[
\begin{bmatrix}
\dot{i}_{k} \\
\dot{\mathcal{I}}_{k}
\end{bmatrix} =
\begin{bmatrix}
-D^{-1}(E + YR) & D^{-1}Y \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
i_{k} \\
\mathcal{I}_{k}
\end{bmatrix}, \tag{13}
\]

where \( M_{21} = -\Psi L(Y)^{-1} + \Phi L(Y)^{-1} D^{-1}(E + YR) \) and \( M_{22} = -\Phi L(Y)^{-1} D^{-1}Y \).

Remark 2: We would like to point out that the secondary control (12) aims to make the current sharing between nodes and is distributed because of the use of the information of neighbouring nodes. While the secondary control in [8] aims to compensate the voltage drifts due to the primary droop control and is decentralised because only the local voltage and reference voltage are used.

4 Stability analysis

As a linear autonomous system, system (13) is required to be neither unstable nor asymptotically stable. An asymptotical stability means output voltages and currents converging to zero, which certainly is not what we want. Actually, we hope that the nodes have the same output current ratio with their output voltage in the desired region. Such a case corresponds to the term of semistable that is recalled in the Appendix.

**Lemma 1:** System (13) is semistable if and only if the following matrix second-order system:

\[
DY\dot{x}(t) + (Y + YRY + Y\Phi L)\dot{x} + Y\Psi L = 0, \tag{14}
\]

is semistable.

**Proof:** Let \( A_{c} \) denote the system matrix of (13), which is equivalent to

\[
A_{c} = \begin{bmatrix}
E & 0 \\
-\Phi L(Y)^{-1}E & D^{-1}Y
\end{bmatrix}. \tag{15}
\]

With the transform matrix

\[
\begin{bmatrix}
Y^{-1} & 0 \\
-\Phi LY^{-1} & E
\end{bmatrix}
\]

\( A_{c} \) is similar to

\[
\tilde{A}_{c} = \begin{bmatrix}
-(DY)^{-1}(Y + YRY + Y\Phi L) & (DY)^{-1}Y \\
-\Psi L & 0
\end{bmatrix}. \tag{16}
\]

Below the proof will be completed by showing that the dynamic system \( \tilde{x} = \tilde{A}_{c}\tilde{x} \) has the characteristic equation (14). Partition \( \tilde{x} \) by \( \tilde{x} = [\tilde{x}_{1}^{T}, \tilde{x}_{2}^{T}]^{T} \) with \( \tilde{x} \in \mathbb{R}^{\mathcal{N}} \). It can be verified that

\[
\dot{\tilde{x}}_{1} = -(DY)^{-1}(Y + YRY + Y\Phi L)\dot{\tilde{x}}_{1} - (DY)^{-1}Y\Psi L\tilde{x}_{2}. \tag{17}
\]

Left-multiplying the above by \( DY \) yields nothing, but (14), which completes the proof.

**Lemma 1** converts the stability of (13) into a stability problem of a matrix second-order system that by itself owns fundamental importance in many fields, such as vibration and structure analysis, spacecraft control and robotics control. There are many results for the stability analysis of (14) with symmetric matrix coefficients, which however are not directly applied here due to asymmetric matrix coefficients arising from the heterogeneity between nodes. Below we further present some sufficient condition for the stability of (13) for some special cases. Before to proceed, the following results are recalled [17].

Given a second-order dynamic system \((M_{1} + M_{2})\dot{x}(t) + (D_{1} + D_{2})x + (K_{1} + K_{2}) = 0\), its eigensolution can be written as

\[
(\lambda_{i}^{2}(M_{1} + M_{2}) + \lambda_{i}(D_{1} + D_{2}) + (K_{1} + K_{2}))x_{i} = 0, \quad i = 1, \ldots, 2n,
\]

where \( \lambda_{i} \) and \( x_{i} \) are the \( i \)-th eigenvalue and the corresponding complex eigenvector, respectively. The subscripts \( s \) and \( k \) denote the symmetric part and skew symmetric part. For any matrix \( M \), the associated \( M_{s} = (M + M^{T})/2 \) and \( M_{k} = (M - M^{T})/2 \). Write
\( x_i = x_{di} + jx_{ri} \) with \( x_{di} \) and \( x_{ri} \) being real and imaginary parts, respectively. Multiplying the above equation by \( x_i^* \) on both sides yields the following second-order scalar equation:

\[
(a_{mi} + jb_{mi})x_i^2 + (a_{di} + jb_{di})x_i + (a_{ki} + jb_{ki}) = 0, \quad i = 1, \ldots, 2n_x
\]  

(19)

where \( x_i^* \) is the conjugated transpose of \( x_i \), \( a_{mi} = x_i^T M_x x_i, b_{mi} = 2x_i^T M_x x_i, a_{di}, b_{di}, a_{ki}, b_{ki} \) are similarly expressed. The solutions of (19) satisfy the following lemma 2 [(18)]:
The solution \( \lambda_i \) of (19) has negative real parts if and only if

\[
b_{mi}b_{di} + a_{mi}a_{di} > 0, \quad \text{for} \quad i = 1, \ldots, 2n_x
\]  

(20)

and

\[
(a_{di}a_{ki} + b_{di}b_{ki}) (a_{mi}a_{di} + b_{mi}b_{di}) > (a_{mi}b_{di} - a_{di}b_{mi})^2.
\]  

(21)

The above lemma is closely related to the symmetric degree of matrices. If all the involved matrices are symmetric positive definite, then (20) (and 21) hold and the corresponding dynamic system is asymptotically stable.

To describe in which degree a matrix \( M \) is symmetric positive definite, we define a measurement variable as follows:

\[
\theta(M) = \max \left\{ \theta \geq 0 : \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix} \geq 0 \right\}
\]  

(22)

The above matrix inequality implies that \( M \geq 0 \) and so does \( M \). If \( M \) is negative definite, then the matrix inequality in (22) is infeasible and therefore \( \theta(M) \) does not exist.

The physical meaning of the above definition is that complex matrix \( M \) is non-negative. It can be seen that \( \theta(M) = 0 \) if and only if \( M = 0 \), and \( \theta(M) = \infty \) if and only if \( M = 0 \).

Theorem 1: System (13) is semistable if the network is connected and one of the following conditions is satisfied:

1. \( Y \neq 0 \), the physical meaning of the above definition is that complex matrix \( M \) is non-negative. It can be seen that \( \theta(M) = 0 \) if and only if \( M = 0 \), and \( \theta(M) = \infty \) if and only if \( M = 0 \).

Proof: We firstly prove that system (13) has only one zero eigenvalue when the network is connected. Rewrite \( A_c \) by

\[
A_c = A_{ce} A_{ci} = A_{cc} A_{cr}
\]  

(27)

with

\[
A_{ci} = \begin{bmatrix} D^{-1} & 0 \\ -P(Y)^{-1} D^{-1} & E \end{bmatrix}, \quad A_{cc} = \begin{bmatrix} E & 0 \\ -P(Y)^{-1} Y \end{bmatrix}, \quad A_{cr} = \begin{bmatrix} -D^{-1} (E + YR) & D^{-1} Y \\ E & 0 \end{bmatrix}
\]

The rank of system matrix \( A_c \) equals to that of \( A_{ce} \) as well as to that of \( A_{ci} \). For a connected network, \( \text{rank}(L) = \text{rank}(Y) = n - 1 \) and both have \( \begin{bmatrix} 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \end{bmatrix} \) as the associated left and right eigenvectors, respectively, to the trivial eigenvalue 0. It is known that if \( Y \neq 0 \), namely at least one shunt conductance exists, then \( \text{rank}(L) = 2n - 1 \). Below we proceed by two cases.

i. \( Y \neq 0 \). In this case \( Y \) is a \( M \)-matrix [24], whose inverse matrix \( Y^{-1} \) is non-negative matrix. It can be seen that \( \text{rank}(A_c) = n + \text{rank}(L) = 2n - 1 \) due to \( Y \) being non-singular. Therefore 0 is one eigenvalue of \( A_c \). We further show that the 0 is semisimple by showing \( \text{rank}(A_c) = 2n - 1 \) (according to Proposition 1 in the Appendix). Noting that \( \text{rank}(A_c) = \text{rank}(A_{ci}) \), we consider the null space of \( A_{ci} \). Suppose there is a vector \( x = [x_1, x_2]^T \in \mathbb{R}^{2n} \) such that

\[
-A_{ci} A_c x = \begin{bmatrix} -E + YR - Y \Psi L(Y)^{-1} \Psi L(Y)^{-1} \end{bmatrix} x = 0
\]

(28)

The solution of \( x \) has \( x_i = \sigma_i Y_1 \) for some \( \sigma_i \in \mathbb{R} \), which as well should satisfy

\[
\sigma_i (Y^{-1} + R) Y_1 = \Psi L(Y)^{-1} x_i,
\]

(29)

which implies \( \sigma_i = 0 \) because positive definite symmetric matrix \( Y^{-1} + R \) is non-negative. Thus, the solution of (29) has the form \( \sigma_i [0, Y_1 \mathbb{R}]^T \) for some \( \sigma_i \in \mathbb{R} \). This means that \( \text{rank}(A_{ci}) = 2n - 1 \). Below we further show \( \text{rank}(A_{ci}) = 2n - 1 \). Suppose there is a vector \( x = [x_1, x_2]^T \in \mathbb{R}^{2n} \) such that \( A_c x = 0 \), then \( A_c x = \sigma_i [0, Y_1 \mathbb{R}]^T \) for some scalar \( \sigma_i \), which implies

\[
A_c x = \begin{bmatrix} -E + YR & \Psi L(Y)^{-1} \\ -\Psi L(Y)^{-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sigma_i A_c x \]

(30)
Due to the existence of $\sigma_i$ which implies $\sigma_1 = 0$. Thus, it can be concluded that if $A_0x = 0$ then $A_1x = 0$. That means $\text{rank}(A_0) = \text{rank}(A_1) = 2n - 1$ and subsequently $A_1$ has one simple zero eigenvalue as well.

The reminder is to prove that all non-zero eigenvalues are of negative real parts for every condition.

(c1) With the condition (c1), recasting system (13) into the form of (18) yields the following matrix equations:

$$
M_x = DY, \quad D_x = Y + uY + rK_x, \quad D_k = rK_k
$$

(31)

$$
K_x = \frac{\Psi L + LY}{2}, \quad K_k = \frac{\Psi L - LY}{2}.
$$

(32)

Due to the existence of $\theta(\Psi L)$, $K_x \geq 0$ and subsequently $D_x > 0$. Therefore $a_{mi} > 0$ and $a_{di} > 0$ for all $i$. If $a_{kl} = 0$, then $b_{ki} = 0$ due to $a_{kl} \geq \theta^2(\Psi L)b_{mi}$ and the non-negative solution of (19) is $-a_{di}/b_{di} > a_{mi}/b_{mi}$ being of negative real parts.

Now consider the case of $a_{kl} > 0$. Since $b_{mi} > 0$ (20) holds always and (21) reduces to

$$
(a_d a_{kl} + b_{di} b_{ki}) a_{di} > a_{di} b_{ki}.
$$

(33)

If $b_{ki} = 0$ and $a_{di} = 0$, then the above is obvious. Matrix inequality (23) means $a_{di} > (1/T_{\text{max}}) a_{mi}$ because of $a_{di} > 0$. Noticing $b_{di} = r b_{ki}$ and $a_{kl} \geq \theta^2(\Psi L)b_{ki}$, one has

$$
(a_d a_{kl} + b_{di} b_{ki}) a_{di} > (v + r) \theta^2(\Psi L) + r b_{di} a_{mi}/T_{\text{max}}
$$

(34)

which together with (24) and $b_{ki} \neq 0$ implies (34) and therefore (21). By Lemma 2, all eigenvalues associated with $a_{kl} > 0$ have negative real parts.

Now it can be concluded that system (21) has only one simple 0 eigenvalue, and all other eigenvalues be of negative real parts. Thus, it is semistable.

(c2) With condition (c2), $Y > 0$. Multiplying by $Y^{-1}$ on both sides of (14) and then casting it into the form of (18) obtain

$$
M_x = \frac{Y^{-1}Y + YY^{-1}}{2}, \quad D_x = RY + rK_x + M_x / \tau,
$$

(35)

$$
K_x = \frac{\Psi L + LY}{2}, \quad D_k = \frac{\Psi L - LY}{2}.
$$

(36)

Since $\theta(Y^{-1}Y) = \theta(Y)$ exists, $a_{mi} > 0$ and $a_{di}^{\frac{1}{2}} \geq \theta^2(Y)b_{mi}$. Also $a_{di} > a_{di}/2$ and $a_{di} > (v + r) a_{di}$ due to (25).

Firstly consider $b_{mi} = 0$. If $a_{di} = 0$ as well, then (19) has the solution

$$
\lambda_i = -a_{di}/b_{di} > 0
$$

(37)

which is either 0 or of negative real parts when $a_{di} = 0$ or $a_{di} > 0$, respectively. If $a_{di} > 0$, but $a_{di} = 0$, then similarly it can be seen that the non-zero solution of (19) has negative real parts.

Now consider $b_{mi} \neq 0$. Noting that $b_{di} = b_{mi}/\tau$, (20) holds always, which as well ensures that the non-zero solution of (19) with $a_{kl} > 0$ has negative real parts. For $a_{kl} 
eq 0$, (21) reduces to

$$
(a_d a_{kl} + b_{di} b_{mi}) > \frac{a_{mi}^2 a_{kl}}{b_{mi} a_{kl}}.
$$

Noticing that $a_d a_{kl} \geq (v + r) a_{kl}$ and $a_{mi} a_{di} + b_{di} b_{mi} > (\theta^2(Y) + 1)b_{mi}/\tau$, inequality (20) holds by (26). This by Lemma 2 shows that all the eigenvalues associated with $b_{mi} \neq 0$ and $a_{di} > 0$ have negative real parts. Therefore system (21) has only one simple 0 eigenvalue and all other eigenvalues be of negative real parts and is semistable.

Notice that $r$ can be set a value larger than $\tau_{\text{max}}$ that in general is a small value less than 0.1. Therefore conditions from (23) to (26) are easy to satisfy. However, more critical are the implied conditions in Theorem 1, the existence of $\theta(\Psi L)$ and $\theta(Y)$ for cases (c1) and (c2), respectively. If $1/T_{\text{max}} = \cdots = 1/T_{\text{max}}$, then $\theta(Y)$ always exists. This means the heterogeneous extent between nodes influences the stability of DC microgrids with distributed control (12).

It should be pointed out that the information network and physical network are not required to have the same topology. For a special case that $L = Y$, and moreover $Y_{\text{c}} = gE$, $\Psi = \beta E$, then $\theta(\Psi L)$ always exists as well.

### 5 Steady state

The steady state of closed-loop system (13) is the mode determined by the 0 eigenvalue. This section issues not only what is the steady state, but also the relationship with the initial condition and network topology. Due to the space limitation, all proofs of this section are omitted. Interested readers can refer to [23].

**Theorem 2:** A semistable dynamic system (13) will reach the current sharing in the sense that all nodes have the same current ratio $i_k = r_{ci}$ for all $k \in N$, where $r_{ci}$ is given by

$$
r_{ci} = \begin{cases} 
\frac{1}{2} Y_{\text{c}}^{-1} (U(0) + \Phi Y^{\bar{Y}_{\text{c}}}) & \text{if } Y_{\text{c}} \neq 0, \\
0 & \text{if } Y_{\text{c}} = 0.
\end{cases}
$$

(38)

Moreover, the output voltage of nodes is given by

$$
U = \begin{cases} 
\frac{1}{2} Y_{\text{c}}^{-1} \Psi L x_{\text{c}} & \text{if } Y_{\text{c}} \neq 0, \\
0 & \text{if } Y_{\text{c}} = 0.
\end{cases}
$$

(39)

where $r_{ci}$ is defined by

$$
r_{ci} = \frac{1}{2} Y_{\text{c}}^{-1} \left(U(0) + \Phi Y^{\bar{Y}_{\text{c}}}(0)\right) = \frac{1}{2} Y_{\text{c}}^{-1} Y_{\text{c}}^{\bar{Y}_{\text{c}}}(0).
$$

(40)

In general the initial values $U(0)$ of the low-pass filter for output currents are set to zeros; or else they will influence the steady states according to (38), which is not what we want. In the following, we always assume that $U(0) = 0$. With this, the proportional gain $a_i$ will influence the stability, but not the steady state.

It is rational to assume that the steady currents in the decentralised droop control, stated in (9), do not exceed the maximum permissible currents for all nodes. That is

$$
(E + Y R)^{-1} Y U(0) \leq U_{\text{max}},
$$

(41)

where $U_{\text{max}} \in \mathbb{R}^n = \left[U_{\text{max,1}}, \ldots, U_{\text{max,n}}\right]^T$. Define the component ratio vector by $R_{\text{c}} = [i_{\text{cl}}, \ldots, i_{\text{cn}}]^T$. Denote the steady-current ratio in the decentralised droop control by $R_{\text{Dec}}^{\text{c}}$, which satisfies $R_{\text{Dec}}^{\text{c}} = Y(E + Y R)^{-1} Y U_{\text{Dec}}^{\text{c}}(0)$. Denote by $R_{\text{Dec}}$ and $R_{\text{Dec}}^{\text{c}}$ the maximum and minimum elements of $R_{\text{Dec}}$, respectively.

With $U(0) = 0$, several further discussions on the steady states for some special cases are made.
Corollary 1: Given \( I^m(0) = 0 \) and \( Y_s \neq 0 \), the steady states of system (13) with the cooperative droop control satisfy the following properties:

(P1) With (41), the common current ratio satisfies \( r_{ci} < 1 \) and \( r_{ci} \in [R^\text{dec}_{cr}, R^\text{dec}] \).

(P2) With (41), the output voltage satisfies

\[
\frac{r_{ci}^\text{ss} U^\text{ss}}{r_{ci}} \leq U \leq \frac{r_{ci}^\text{ss} U^\text{ss}}{r_{ci}}.
\]

(P3) If \( \Psi = \beta E \) and \( R = rE \), then \( r_{ci} < \frac{\sum U_{s,ci}(0)}{\sum \lambda_k} \frac{\lambda_n}{1 + \lambda_n r} \), where \( \lambda_n \) is the maximal eigenvalue of \( Y \).

(P4) If \( Y_s = gE \), \( \Psi = \beta E \), and \( R = rE \), then

\[
r_{ci} = \frac{\sum U_{s,ci}(0)}{\sum \lambda_k} \frac{g}{1 + gr}.
\]

(P5) All the nodes have the same output voltage if and only if there is a positive scalar \( \varepsilon \) such that \( Y = \varepsilon Y_s \).

6 Simulation example

Consider a microgrid with four nodes connected by four lines. Each node is a DC voltage interfaced to the DC bus via a buck converter. As illustrated in Fig. 3, the simulation is made on the Matlab and Simulink. The electrical parameters are listed in Table 1.

The same control parameters are selected for all the nodes. The desired DC bus voltage is 48 V, the maximum current of all nodes is 30 A, the time constant of LPF is 0.01 s, the virtual resistance \( r = 0.1 \Omega \), the proportional gain \( \alpha = 0 \) and the integrator gain \( \beta = 100 \). The conductance matrix of network and the Laplacian matrix of information graph are given by, respectively,

\[
Y = \begin{bmatrix}
3.25 & -1 & 0 & -2 \\
-1 & 3.7 & -2.5 & 0 \\
0 & -2 & 6.75 & -4 \\
-2 & 0 & -4 & 6.3
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
-1 & 2 & -1 & 0 \\
0 & -1 & -1 & 2 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\]

It can be verified that \( YL \) is not a positive definite matrix, therefore \( \theta(YW) \) does not exist. Since \( Y \) is an unit matrix, the conditions (25) and (26) hold and subsequently the microgrid with distributed controller (12) is semistable. Fig. 4 shows the simulation results. At the beginning, the microgrid runs in the decentralised droop control. At \( t = 1 \text{s} \), the system is closed to the steady state under the decentralised droop control, which are

\[
I^s = \begin{bmatrix}
11.96 \\
10.08 \\
12.06 \\
14.21
\end{bmatrix}
\]

\[
U^s = \begin{bmatrix}
46.80 \\
46.99 \\
46.79 \\
46.58
\end{bmatrix}
\]

In the presence of line resistance and the difference of local load, there is a large bias for current sharing. The minimal and maximal current ratio are \( R^\text{dec}_{cr} = 0.34 \) and \( R^\text{dec} = 0.47 \).

At the time \( t = 1 \text{s} \), the distributed cooperative control applies, with which the reference \( u_{s,ci}^r \) starts varying according to the current sharing errors and the nodal currents then asymptotically converge to the same value, an exact current sharing. All the nodes have the same output current 12 A and current ratio 0.40 that belongs to \( [R^\text{dec}_{cr}, R^\text{dec}] \). The output currents and voltages at \( t = 2 \text{s} \) are

\[
I^m = [12.066, 12.072, 12.053, 12.065]\text{A}
\]

\[
U = [46.744, 47.47, 46.735, 46.18]\text{V}
\]

respectively. They are both very close to the steady states given by (38) and (39), 12.067 and \([46.75, 47.48, 46.75, 46.19]\text{V}\), respectively. The property P2 in Corollary 1 can be verified as well.

There are two transient phases in the simulation result. The first phase is attributed to the primary droop control and the second

![Fig. 3 Four node DC microgrid. The buck converter (left) and network structure (right)](image)

![Fig. 4 Trajectories of currents and voltages of four nodes](image)

| Line | Line 2 | Line 3 | Line 4 | R1 | R2 | R3 | R4 |
|------|--------|--------|--------|----|----|----|----|
| 1 Ω  | 0.5 mH | 0.5 Ω  | 0.1 mH | 0.4 Ω | 0.2 mH | 0.3 Ω | 0.0 mH 21 Ω 5 Ω 4 Ω 3 Ω |
phase to the secondary distributed control. It can be seen that the distributed control has a settling time less than 0.2 s, which is fast enough for secondary control. In fact, the convergence speed of the distributed control can be analytically determined by the eigenvalues of system matrix $A_r$. In the present example, the largest non-zero real part of the eigenvalues is $-24$. The corresponding settling time is $4/24 = 0.16$ that is consistent with the trajectories shown in Fig. 4.

7 Conclusion

The stability of DC microgrids with distributed cooperative control has been investigated and two sufficient semistable conditions are presented. The study on steady state illustrated the current sharing property and its relationship with initial condition and network topology. A DC microgrid with four buck converters was simulated on the Matlab platform to show the developed results. The stability with switching topology is one of our ongoing research, for which the analysis through the locations of eigenvalues as used in this paper is not applicable any more.

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9 Appendix

Given a linear system $x(t) = Ax(t)$ where $t > 0$, $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, the following definitions are made [17].

Definition 1: The system is Lyapunov stable if, for every initial condition $x(0)$, there exists $\epsilon > 0$ such that $\|x(t)\| < \epsilon$ for all $t \geq 0$.

Definition 2: The system is semistable if $\lim_{t \to \infty} x(t)$ exists for all initial conditions $x(0)$.

Definition 3: Given an eigenvalue $\lambda \in \text{spec}(A)$, $\lambda$ is semisimple if every Jordan block of $A$ associated with $\lambda$ is of size one. The following proposition is true.

Proposition 1: An eigenvalue $\lambda$ of $A$ is semisimple if and only if $\text{rank}(\lambda E - A) = \text{rank}(\lambda E - A)^2$, where $E$ is the unit matrix.

Proposition 2: A is semistable if and only if $A$ is Lyapunov stable and $A$ has no non-zero imaginary eigenvalues.