Non–perturbative definition of five–dimensional
gauge theories on the $\mathbb{R}^4 \times S^1/\mathbb{Z}_2$ orbifold

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**Abstract**

We construct a $\mathbb{Z}_2$ orbifold projection of SU($N$) gauge theories formulated in five dimensions with a compact fifth dimension. We show through a non–perturbative argument that no boundary mass term for the Higgs field, identified with some of the fifth dimensional components of the gauge field, is generated, which would be quadratically divergent in the five–dimensional ultraviolet cutoff. This opens the possibility of studying these theories non–perturbatively in order to establish if they can be used as effective weakly interacting theories at low energies. We make preparations for a study on the lattice. In particular we show that only Dirichlet boundary conditions are needed, which specify the breaking pattern of the gauge group at the orbifold fixpoints.

**1 Introduction**

Gauge theories with extra space–like dimensions have attracted interest during the last few years. Even though departing from four dimensions leads into the “wild” domain of non–renormalizable theories, there are perhaps reasons they should not be discarded immediately: an ultraviolet (UV) cutoff $\Lambda$ (like the inverse lattice spacing) can be introduced and the theory can be treated as an effective low–energy theory. One is however not guaranteed that this is a consistent program unless there exists a range of the cutoff $\Lambda$ where the low–energy physical properties depend only weakly on $\Lambda$ (this is called the scaling region) and the theory is weakly interacting. If this is the case then these theories could provide a solid starting point basis for constructing non–supersymmetric extensions of known and well tested physical theories.
We discuss more in detail this point for SU($N$) gauge theories. Gauge invariance guides the construction of the theory. We assume that the effective gauge theory in $d$ Euclidean dimensions can be written as

$$S = \frac{1}{4} \int d^d x \left[ b F_{MN}^A F_{MN}^A + c D_L F_{MN}^A D_L F_{MN}^A \right]. \quad (1.1)$$

All quantities appearing in the above are dimensionless, for example the action $S$ can be discretized on a lattice and all quantities are pure numbers in units of the lattice spacing. We neglect in Eq. (1.1) other terms allowed by gauge invariance for reasons which will become clear later. The index $A$ is the adjoint gauge group index, the indices $M, N, L$ are the Euclidean indices and $D_L$ is the gauge covariant derivative. The field strength components $F_{MN}^A$ contain the bare gauge coupling $g_0$. The theory is defined in terms of the parameters $b$ and $c$. We rescale coordinates, gauge field $A$ and coupling as

$$x = \Lambda x', \quad (1.2)$$

$$A^A_M(x) = \zeta A^A_M(x'), \quad (1.3)$$

$$g_0 = \xi g_0'. \quad (1.4)$$

$\Lambda$ can be thought of a (large) momentum cutoff giving a dimension to the physical primed (rescaled) quantities. The action Eq. (1.1) can be interpreted as a low–energy effective theory for energies $E \ll \Lambda$. The coupling $g_0'$ is the effective coupling at the scale $\Lambda$, renormalized up to slowly varying renormalization factors. Requiring that the kinetic term for $A'$ has the standard coefficient $1/4$ fixes

$$\zeta = b^{-1/2} \Lambda^{(2-d)/2} \quad \text{and} \quad \xi = b^{1/2} \Lambda^{(d-4)/2}. \quad (1.5)$$

In terms of the rescaled quantities the action Eq. (1.1) becomes

$$S = \frac{1}{4} \int d^d x' \left[ F'^A_{MN} F'^A_{MN} + c \frac{1}{b \Lambda^2} D'_L F'^A_{MN} D'_L F'^A_{MN} \right]. \quad (1.6)$$

We observe that the term with two covariant derivatives gives contributions suppressed by a factor $(E/\Lambda)^2$. Even more suppressed are further terms containing more fields and their derivatives. The action Eq. (1.6) is reminiscent of the Symanzik effective action for renormalizable lattice theories [1–4]. The low–energy effective theory, for energies $E \ll \Lambda = 1/a$ where $a$ is the lattice spacing, is a continuum theory where the lattice spacing is the expansion parameter appearing in positive powers.

In $d = 5$ dimensions pure SU($N$) gauge theories have at least two phases in infinite volume. On a lattice when the bare gauge coupling $g_0$ is very large the theory is confining, while at very small $g_0$ the theory is in a Coulomb phase with massless gluons [5,6]. There must therefore be at least one phase–transition point between these phases. An attempt to approach the continuum limit in the Coulomb phase is by keeping the renormalized gauge coupling $g_0'$ fixed and increasing the UV cutoff $\Lambda$. From Eq. (1.1), $g_0' = (b \Lambda)^{-1/2} g_0$ which requires the bare gauge coupling $g_0$ to

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1\ We would like to thank Jean Zinn–Justin for pointing out this to us.
be scaled to larger values roughly proportionally to the square root of $\Lambda$. Eventually the phase transition point is reached which implies that at fixed renormalized coupling there is a maximal value of the cutoff and thus the theory is trivial.

Triviality is a property shared also by some renormalizable theories, like $\phi^4$ theory in four dimensions. Studies of this theory formulated on a lattice provided strong evidence of the triviality of the continuum limit [7, 8], i.e. removing the ultraviolet cutoff $\Lambda$ leads to a zero renormalized gauge coupling. Nevertheless appreciable interactions are observed in a scaling region for finite values of the lattice spacing. An important difference is that the renormalized gauge coupling in the $\phi^4$ theory approaches zero as the inverse logarithm of $a$. In non-renormalizable theories the renormalized coupling approaches zero with a power of $a$.

There exist alternative approaches to construct an effective theory for five-dimensional gauge theories. In the $D$–theory regularization of field theory, a five-dimensional gauge theory on the lattice arises as low–energy effective theory of quantum link models [9, 10]. It has been shown that dimensional reduction to a four-dimensional SU($N$) gauge theory can occur [11]. The gauge coupling constant $g$ of the dimensionally reduced theory is given by $g^2 = g_0^2/\beta$, where $g_0$ is the dimensionful gauge coupling of the five-dimensional quantum link model and $\beta$ is the extension of the fifth dimension. Yet another approach to five-dimensional gauge theories is the investigation of a non-perturbative UV fixed point. In case such a fixed point exists, the limit of infinite UV cutoff $\Lambda$ could be taken. Its existence is suggested by the epsilon expansion [12, 13] but it has been elusive so far in lattice simulations.

In extra dimensional theories typically one assumes that the extra dimensions are compactified on some manifold, a torus in the simplest case. The minimal of the clearly large number of possibilities is a gauge theory with a single compact extra dimension. The advantage of such a simple model is that one can make considerable progress in understanding its quantum properties which becomes increasingly hard as the number of the extra dimensions grows or when the extra dimensional theory is coupled to gravity. The UV behavior of the compact theory is the same as that of the uncompactified theory so all the above comments and questions apply to it. Nevertheless, compactification is well motivated by phenomenology. The first phenomenological applications of large ($\text{TeV}^{-1}$) extra dimensions to the Standard Model were proposed in [14, 15]. A striking example is that if the compact dimension is as large as 1 TeV$^{-1}$ then the electroweak symmetry breaking could proceed by the Hosotani mechanism [16, 17] without supersymmetry and avoiding the hierarchy problem. The idea is to identify the Standard Model Higgs field with an extra dimensional component of the gauge field for which a non-trivial effective potential is conjectured. Results at 1-loop [18–21] support the viability of this scenario but only a non-perturbative computation can prove its true validity. Extra dimensions in connection with further alternatives to the Higgs mechanism have received attention from recent lattice studies [22]. Also the dimensional reduction and localization of gauge fields have been studied on the lattice in a three-dimensional model [23].

Compactification introduces a new scale in the theory, in the case of circle compactifications the radius $R$ of the circle $S^1$. Like in the case of field theories at nonzero temperature [24]
dimensional reduction to four dimensions can be investigated. The regime where the effective four–dimensional theory can behave like a weakly interacting field theory at low energies $E$ is now

$$E \ll \frac{1}{R} \ll \Lambda.$$  \hspace{1cm} (1.7)

The four–dimensional theory is effectively a theory of the Kaluza–Klein zero modes of the five–dimensional fields. To understand this better recall that in the perturbative approach one typically fixes the gauge by imposing that the gauge field is periodic in the compact coordinate. Then a Kaluza-Klein expansion of the fields is possible and perturbative calculations can be performed in a four–dimensional language, where one has four–dimensional massless fields together with infinite towers of massive fields. The breaking of the five–dimensional gauge invariance can be interpreted as a Higgs mechanism where the non–zero modes of the four–dimensional part of the gauge field absorb the non–zero modes of the fifth dimensional component of the gauge field [25]. In perturbative calculations, after summing over the massive states using certain infinite sum regularization methods, one can arrive at interesting results, such as the 1–loop mass of the adjoint Kaluza–Klein scalar (the Higgs field in this approach), which is found to be finite. It would be interesting to verify that this result is not just a gauge artifact, result of the specific gauge fixing method.

An obvious practical problem with a five–dimensional gauge theory (intended to be used for four–dimensional physics) is how to take a four–dimensional slice of it in such a way that this slice resembles the physics that we observe. A possible solution to this problem turns out to be the same as the solution to the problem of the non–existence of chiral fermions in five dimensions. By changing the compact space from a circle $S^1$ of radius $R$ parametrized by the coordinate $x_5$ into an interval $S^1/Z_2$ by the identification $x_5 \rightarrow -x_5$, one obtains naturally four dimensional boundaries at the two ends of the interval (which are just the fixed points of the projection) with chiral fermions localized on them. The new space obtained in this way is called an orbifold. One can embed the orbifold projection in a gauge theory by imposing certain boundary conditions on the gauge fields. The orbifold projection can thus reduce the gauge symmetry at the four–dimensional fixed points of the orbifold. As a result, the Higgs field does not transform in the adjoint representation of the gauge group as in the $S^1$ compactification but in some lower dimensional representation, a property shared also by the Standard Model Higgs field. For recent promising phenomenological applications where such theories are used to construct models for extensions of the Standard Model, see Refs. [26–28].

The introduction of a fifth dimension in connection with chiral fermions on the lattice is at the basis of the domain wall fermion formulation [29]. It is also known that in the domain wall construction of chiral fermions the domain wall can be replaced by a boundary through Dirichlet boundary conditions [30] which is precisely what one has in the orbifold construction. The derivation of light four–dimensional fermions from a five–dimensional theory with boundaries may be a concrete hint of the physical reality of compact extra dimensions [31]. Stimulating progress related to the fermions comes from a recent work where the orbifold construction has
been used to formulate in four dimensions lattice chiral fermions with Schrödinger functional boundary conditions [32].

In the orbifold compactification of the gauge theory there is a new problem that appears due to the presence of the boundaries: fields acquire Dirichlet or Neumann boundary conditions at the fixpoints of the orbifold. The formulation of a field theory with prescribed boundary values for some of the field components requires in general additional renormalization. This has been first studied for renormalizable theories by Symanzik [33, 34]. There it was found that the presence of boundaries introduces additional divergences and these induce boundary counterterms with renormalization factors calculable in perturbation theory. The important lesson therefore is that renormalization requires counterterms localized on the boundaries of the theory and this applies also to non–renormalizable theories in the parameter range Eq. (1.7). The renormalization pattern of a five–dimensional Yukawa theory formulated on the $M^4 \times S^1/Z_2$ orbifold has been first discussed in [35]. There, counterterms localized on the boundaries and logarithmically divergent in the cutoff have been computed in perturbation theory at 1–loop order. Five–dimensional gauge theories formulated on $M^4 \times S^1/Z_2$ have been considered in [36–40]. The main result of [38] was that at 1–loop level a boundary mass counterterm for the Higgs field (identified with some of the five–dimensional components of the gauge field) is absent. This term would represent a correction to the Higgs mass squared proportional to $g_0^2 \Lambda^2 / R$ for the zero modes of the four–dimensional low energy theory defined at the boundaries, introducing a hierarchy problem. It was not clear though if a boundary counterterm is absent also at higher orders in perturbation theory. A strong indication for this is the shift symmetry argument given in Refs. [41, 42].

The scenario of dimensional reduction of five–dimensional orbifold gauge theories to four–dimensional theories of gauge and Higgs fields at the orbifold fixpoints is supported by perturbative calculations. The mass of the Higgs field is generated through radiative corrections. At 1–loop it is found independent of the five–dimensional UV cutoff $\Lambda$. However, since the theory is non–renormalizable, a sensitivity to $\Lambda$ is expected at higher orders in perturbation theory. The results of Ref. [40] show that at 2–loop order the Higgs mass receives a contribution logarithmic in $\Lambda$, generated by insertions of boundary terms in finite 1–loop bulk graphs. It is not clear to us whether radiative corrections will generate power divergent contributions at even higher orders in perturbation theory.

In this paper we make preparations to study on the lattice $SU(N)$ pure gauge theories on the orbifold $R^4 \times S^1/Z_2$. The idea we would like to investigate is if in principle one could have a four–dimensional non–supersymmetric effective theory coupled to a Higgs field without a hierarchy problem. In section 2 and 3 we present a proof that a boundary mass counterterm for the Higgs field is absent. Regarding boundary counterterms, in a non–perturbative formulation the main problem turns out to be to develop a gauge invariant method for their classification, despite the fact that gauge invariance may be broken at the boundaries. The basic tool for developing such a method is the introduction of a spurion field which restores the gauge invariance of the theory broken by the orbifold boundaries. In section 4 we construct the lattice orbifold theory.
In section 5 we make a short summary of this work.

2 The orbifold

The orbifold projection identifies field components under the transformations of a discrete symmetry group $K$. Here we consider five–dimensional gauge theories with gauge group $G = SU(N)$ and $K = \mathbb{Z}_2$ formulated in Euclidean space. We use capital Latin letters $M, N, \ldots = 0, 1, 2, 3, 5$ to denote the five–dimensional Euclidean index and small Greek letters $\mu, \nu, \ldots = 0, 1, 2, 3$ to denote the four–dimensional part. For the coordinates we will use the shorthand notation $z = (x_\mu, x_5)$ and $\bar{z} = (x_\mu, -x_5)$. In the following we will suppress $x_\mu$ whenever its explicit appearance is not necessary.

We introduce the $\mathbb{Z}_2$ reflection $R$ in the fifth dimension

$$Rz = \bar{z}. \tag{2.1}$$

Next, we define the $\mathbb{Z}_2$ reflection on a rank–$r$ tensor field $C(z)$ as

$$(RC_{M_1 M_2 \cdots M_r})(z) = \alpha_{M_1} \alpha_{M_2} \cdots \alpha_{M_r} C_{M_1 M_2 \cdots M_r}(Rz), \tag{2.2}$$

where no sum on the $M_i$ is implied on the right hand side. The intrinsic parities $\alpha_M$ are defined by $\alpha_\mu = 1$ and $\alpha_5 = -1$. Since tensor fields can be obtained through derivatives of fields, the relation

$$(R \partial_M C_{M_1 M_2 \cdots M_r})(z) = \alpha_M \alpha_{M_1} \alpha_{M_2} \cdots \alpha_{M_r} (\partial_M C_{M_1 M_2 \cdots M_r})(Rz), \tag{2.3}$$

holds. Incidentally this implies that $R$ and the derivative operator $\partial_M$ commute

$$[R, \partial_M] = 0. \tag{2.4}$$

Also Eq. (2.2) is consistent with the property

$$(RC \cdot D)(z) = (RC)(z) \cdot (RD)(z) \tag{2.5}$$

for the product of any two tensor fields $C(z)$ and $D(z)$. In the following for $R$ and similarly for all the other operators we will write $R C(z)$ as a shorthand for $(RC)(z)$.

Inspired by the geometric description of gauge fields on $R^4 \times S^1/\mathbb{Z}_2$ (see Appendix B) one can formulate the orbifold theory on the strip

$$I_0 = \{ x_\mu, 0 \leq x_5 \leq \pi R \} \tag{2.6}$$

without reference to the circle. The following construction yields the proper boundary conditions on the boundary planes at $x_5 = 0$ and $x_5 = \pi R$.

One starts with an $SU(N)$ gauge theory defined on the open set $I_e = \{ x_\mu, x_5 \in (-\epsilon, \pi R + \epsilon) \}$ with a gauge field $A_M(z)$ defined everywhere on $I_e$ and a spurion field$^2$ $\mathcal{G}(z) \in SU(N)$ defined in the neighborhoods $O_1 = \{ x_\mu, x_5 \in (-\epsilon, \epsilon) \}$ and $O_2 = \{ x_\mu, x_5 \in (\pi R - \epsilon, \pi R + \epsilon) \}$ that satisfies

$$(R \mathcal{G}) \mathcal{G} = \pm 1, \tag{2.7}$$

$^2$We would like to thank Martin Lüscher for suggesting this to us.
with \( \mathcal{R} \) the reflection operator. At the fixpoints \( x_5 = 0 \) and \( x_5 = \pi R \), Eq. (2.7) states that \( \mathcal{G}^2 = 1 \). The gauge field on \( O_i \) is constrained by

\[
\mathcal{R} A_M = \mathcal{G} A_M \mathcal{G}^{-1} + \mathcal{G} \partial_M \mathcal{G}^{-1},
\]

which implies \( \mathcal{R} F_{MN} = \mathcal{G} F_{MN} \mathcal{G}^{-1} \). In Appendix B it is shown that the spurion field can be identified with a transition function that is required when defining gauge fields on the circle using two charts. The property Eq. (2.7) expresses the gluing condition of the two charts. The transformation property of the spurion field under a gauge transformation is such that the constraint Eq. (2.8) is covariant under gauge transformations \( \Omega \in \text{SU}(N) \). This is the case for

\[
\mathcal{G} \xrightarrow{\Omega} (\mathcal{R} \Omega) \mathcal{G} \Omega^{-1}.
\]

The covariant derivative of \( \mathcal{G} \) can be defined on the neighborhoods \( O_i \) by requiring that it transforms like \( \mathcal{G} \). Such a covariant derivative is

\[
D_M \mathcal{G} = \partial_M \mathcal{G} + (\mathcal{R} A_M) \mathcal{G} - \mathcal{G} A_M
\]

and in fact, Eq. (2.8) implies that

\[
D_M \mathcal{G} \equiv 0.
\]

By means of the constraints Eq. (2.7), Eq. (2.8) and

\[
\mathcal{R} \Omega = \Omega
\]

which ensures that all gauge transformations on \( I_\epsilon \) are local, the orbifold theory can be consistently defined on the strip \( I_0 \) respecting the SU\((N)\) gauge symmetry.

For any \( \epsilon \neq 0 \) the theories are gauge invariant and equivalent. The breaking of the gauge symmetry is realized by taking the limit \( \epsilon \to 0 \). In this limit the neighborhoods \( O_i \) shrink to single points and one is left with boundaries at \( x_5 = 0 \) and \( x_5 = \pi R \). We approach the limit \( \epsilon \to 0 \) so that (in the limit), the spurion field and its derivatives take the value

\[
\mathcal{G}(0) = \mathcal{G}(\pi R) = g,
\]

\[
\partial_5^p \mathcal{G}(0) = \partial_5^p \mathcal{G}(\pi R) = 0, \quad p \in \mathbb{N}, \; p > 0
\]

for a constant matrix\(^3\) \( g \) obeying \( g^2 = \pm 1 \) by virtue of Eq. (2.7). We will specify the matrix \( g \) below. Since \( g \) is constant all derivatives \( \partial_\mu \) of \( \mathcal{G} \) vanish as \( \epsilon \to 0 \). From Eq. (2.9) it is immediately clear that only gauge transformations for which

\[
\Omega = g \Omega g \quad \text{at } x_5 = 0 \quad \text{and } x_5 = \pi R
\]

are still a symmetry of the theory. Taking the limit \( \epsilon \to 0 \) in Eq. (2.8) yields the Dirichlet boundary conditions

\[
\alpha_M A_M = g A_M g \quad \text{at } x_5 = 0 \quad \text{and } x_5 = \pi R,
\]

\(^3\)One could in principle take a different matrix \( g \) for \( x_5 = 0 \) and \( x_5 = \pi R \).
where no sum on $M$ is implied on the left hand side.

We now have a prescription to obtain the correct boundary conditions for any field derived from $A_M$. One starts in the gauge invariant theory ($\epsilon \neq 0$) where the field $A_M$ is constrained by Eq. (2.8). Then the limit $\epsilon \rightarrow 0$ is taken using the properties of $G$ in Eq. (2.13) and Eq. (2.14). For example we obtain the following Neumann boundary conditions

$$-\alpha_M \partial_5 A_M = g \partial_5 A_M g \quad \text{at } x_5 = 0 \text{ and } x_5 = \pi R,$$

where no sum on $M$ is implied on the left hand side. From Eq. (2.8) Dirichlet boundary conditions follow for the field strength tensor

$$\alpha_M \alpha_N F_{MN} = g F_{MN} g \quad \text{at } x_5 = 0 \text{ and } x_5 = \pi R,$$

where no sum on $M$ and $N$ is implied on the left hand side. The point we would like to emphasize here is that our construction provides all the necessary boundary conditions that define the orbifold theory.

The gauge symmetry at the boundaries is broken to a subgroup $\mathcal{H}$ of $SU(N)$ by the group conjugation in Eq. (2.15). The latter is an inner automorphism of the Lie algebra and for $g$ one can take

$$g = e^{-2\pi i V \cdot H},$$

with $H = \{H_i\}$, $i = 1, \ldots, \text{rank}(SU(N)) = N - 1$ the hermitian generators of the Cartan subalgebra of $SU(N)$. $V = \{V_i\}$ is a constant $(N - 1)$–dimensional twist vector. In general, an inner automorphism breaks the gauge group as

$$G = SU(p+q) \rightarrow \mathcal{H} = SU(p) \times SU(q) \times U(1).$$

As shown in Appendix A under a group conjugation by $g$ the hermitian $SU(N)$ generators $T^A$, $A = 1, \ldots, N^2 - 1$ transform as $g T^A g = \eta^A T^A$, where $\eta^A = \pm 1$ is their parity. The generators are divided into unbroken generators $T^a$ with $\eta^a = 1$ and broken generators $\hat{T}^\alpha$ with $\eta^{\hat{\alpha}} = -1$. The above imply that in the adjoint representation the matrix elements of the conjugation matrix are simply $g^{AA'} = \eta^A \delta^{AA'}$. In terms of the gauge field components $A_M = -i g_0 A^A_M T^A$ the boundary conditions Eq. (2.16) and Eq. (2.17) read

$$A^\alpha_\mu = 0 \quad \text{and} \quad A_5^\alpha = 0 \quad \text{at } x_5 = 0 \text{ and } x_5 = \pi R,$$

$$\partial_5 A^\alpha_\mu = 0 \quad \text{and} \quad \partial_5 A_5^\alpha = 0 \quad \text{at } x_5 = 0 \text{ and } x_5 = \pi R.$$  

In the Kaluza-Klein decomposition, the zero modes of $A_5^\alpha$ are the Higgs fields of the four–dimensional low–energy effective theory defined at the orbifold boundaries. The zero modes of $A^\alpha_\mu$ are the gauge bosons, which generate the residual gauge group $\mathcal{H}$. The Higgs fields transform in some representation of $\mathcal{H}$.

The simplest possibility in Eq. (2.20) is the breaking pattern $SU(2) \rightarrow U(1)$ which can be achieved with the twist vector $V = 1/2$. This twist vector results in $\{\eta^A\} = \{-1, -1, +1\}$ (which
specifies $g$ in the adjoint representation) and the gauge boson branching $3 = 1_1 + 1_{-1} + 1_0$, where the subscripts are the $U(1)$ charges. There are two charged Higgs scalars. In the fundamental representation one can use $g = -i\sigma_3$.

The next simplest case is $SU(3) \rightarrow SU(2) \times U(1)$. The twist vector is $V = (0, \sqrt{3})$ and $\{\eta^A\} = \{+1, +1, +1, -1, -1, -1, -1, +1\}$. The gauge bosons branch as $8 = 3_0 + 2_1 + 2_{-1} + 1_0$. There are two Higgs fields in the fundamental representation of the unbroken $SU(2)$ gauge group with $U(1)$ charges $\pm 1$. The matrix $g$ in the fundamental representation in this case is $g = \text{diag}(-1, -1, +1)$.

3 The boundary terms

In general the presence of boundaries in a field theory leads to new divergences. Symanzik studied the Schrödinger functional for renormalizable theories [33, 34]. The Schrödinger functional is a formulation of field theories with prescribed boundary values for some of the field components. The expectation is that to make these theories finite all the possible (i.e. consistent with the symmetries of the theory) counterterms localized on the boundaries have to be added. This expectation has been confirmed for the massless scalar $\phi^4$ theory [33] and for QCD [43–46].

The aim of our work is to define the orbifold theory on a Euclidean lattice. For renormalizable quantum field theories the dependence on the lattice spacing can be described at low energies in terms of a continuum local effective theory. The associated Symanzik effective action [1–4] includes bulk and boundary terms. As explained in the Introduction the non–renormalizable five–dimensional orbifold theory makes sense only as an effective theory for energies much below a finite cutoff, in our case given by the inverse lattice spacing $1/a$. In this regime the theory behaves effectively like a continuum theory and renormalized perturbation theory applies. The difference with respect to renormalizable theories is that renormalization requires at each order the subtraction in the effective action of new divergent terms of increasing dimension. We expect that the orbifold theory defined on the strip $I_0$ and put on a lattice can be described by a continuum Symanzik effective action.

In Section 2 we have constructed the orbifold theory on the strip $I_0$ as a limit $\epsilon \rightarrow 0$ of a gauge invariant theory defined on the open set $I_\epsilon$. The latter theory has the full $SU(N)$ gauge invariance. In particular we showed that our construction provides the orbifold boundary conditions on $I_0$ for any field. The only dangerous terms at tree level in the Symanzik effective action on $I_0$ are boundary terms described by local composite fields $\mathcal{O}_i(x)$ of dimension less than or equal to four contributing with a boundary action

$$\delta S_b[A] = \int d^4x \sum_i Z_i \{\mathcal{O}_i(x)|_{x_5=0} + \mathcal{O}_i(x)|_{x_5=\pi R}\}. \quad (3.1)$$

The canonical dimensions of the operators $\mathcal{O}_i$ determine the superficial degree of divergence with the cutoff of the renormalization constants $Z_i$. The boundary counterterms are generated by terms in the Symanzik effective action for the gauge invariant theory on $I_\epsilon$ containing $\mathcal{G}$. To
make a list of all possible gauge invariant terms involving \( G \), we first note that \( \text{tr}\{G\}, \text{tr}\{G^2\}, \ldots \) contribute only an irrelevant constant and there is no kinetic term for \( G \). The lowest dimensional gauge invariant terms are therefore the dimension five terms

\[
\frac{1}{g_0'} \text{Re} \text{tr}\{GF_{MN}F_{MN}\} = \frac{1}{2g_0'} \left( \text{tr}\{GF_{MN}F_{MN}\} + \text{tr}\{G^{-1}F_{MN}F_{MN}\} \right),
\]

\[
\frac{1}{g_0'} \text{Re} \text{tr} \{GF_{MN}G_{MN}\} = \frac{1}{2g_0'} \left( \text{tr}\{GF_{MN}G_{MN}\} + \text{tr}\{G^{-1}F_{MN}G^{-1}F_{MN}\} \right). \tag{3.2}
\]

They generate the boundary terms

\[
\frac{1}{g_0'} \text{tr}\{gF_{MN}(z)F_{MN}(z)\}, \quad \frac{1}{g_0'} \text{tr}\{gF_{MN}(z)gF_{MN}(z)\}, \tag{3.3}
\]

invariant under the residual gauge transformations \( \Omega \) satisfying Eq. (2.15). Being of dimension five these terms give at tree level a contribution to the Symanzik action proportional to the lattice spacing. We have hence proven that no boundary terms proportional at tree level to inverse powers of the lattice spacing exist for the orbifold theory.

The first term in Eq. (3.3) can be evaluated taking the generators in the adjoint representation

\[
(T^A)_{BC} = -i f^{ABC}.
\]

Evaluating the trace, one has

\[
\text{tr}\{gT^AT'_A\} = f^{ABC} f^{A'B'C} \eta^C = \left( C_2(\mathcal{H}) - \frac{1}{2} C_2(G) \right) (\eta^A + 1) \delta^{AA'}, \tag{3.4}
\]

where \( \mathcal{H} \) is the unbroken gauge subgroup at the boundaries and thus we get the boundary term

\[
- \left( C_2(\mathcal{H}) - \frac{1}{2} C_2(G) \right) F^{a}_{\mu\nu} F^{a}_{\mu\nu}. \tag{3.5}
\]

In the above \( C_2(G) \) and \( C_2(\mathcal{H}) \) is the quadratic Casimir invariant of the unbroken group \( G \) and its subgroup \( \mathcal{H} \) respectively. In the fundamental representation \( C_2(\text{SU}(N)) = 1/2 \). Counterterms of the type Eq. (3.5) were indeed encountered at 1–loop in perturbation theory \([38, 39]\) with logarithmically divergent \( Z \)–factors.

The second term in Eq. (3.3) can be evaluated using \( \text{tr}\{gT^A gT^A'\} = C_2(G) \eta^A \delta^{AA'} \) yielding the boundary term

\[
- \frac{1}{2} C_2(G) \{ F^{a}_{\mu\nu} F^{a}_{\mu\nu} - F^{a}_{\mu\nu} F^{a}_{\mu\nu} \}. \tag{3.6}
\]

Terms of this type do not appear at 1–loop, however they are expected to arise at 2–loops even though such perturbative computation has not been performed yet.

Finally, notice that the term

\[
\text{tr}\{[A_M(z),g][A_M(z),g]\}, \tag{3.7}
\]

is invariant under Eq. (2.15). Using the boundary conditions Eq. (2.21) this term is equal to

\[
2g_0'^2 \delta^{AA'} A^a_{\mu} A^a_{\mu}, \tag{3.8}
\]

a would be quadratically divergent boundary mass term for the Higgs. An operator of the \( \epsilon \neq 0 \) effective action that could give rise to such a term is

\[
\text{tr}\{D_M G D_M G\}, \tag{3.8}
\]

10
which is however identically zero, by Eq. (2.11). In fact, it is not hard to check that none of the operators of the $\epsilon \neq 0$ effective action containing the spurion field (such as those in Eq. (3.3)) can induce a boundary Higgs mass term.

It would be tempting at this point to conclude that the Higgs mass in the orbifold theory is non—perturbatively finite. Indeed, if dimensional reduction occurs due to the compactification of the fifth dimension, as it happens for SU($N$) gauge theories at nonzero temperature in four dimensions [24], the Kaluza–Klein zero modes of the fields $A^{\hat{5}}$ play the role of Higgs fields in the four–dimensional low–energy effective theory. In the dimensionally reduced theory a bulk mass term for the Higgs is allowed. The 1–loop perturbative prediction is [38, 39]

$$m_h^2 = \frac{3}{32\pi^4 R^2} \frac{g_0'^2}{\pi R} \zeta(3) 3C_2(G),$$

a manifestly finite result ($g_0'$ is the renormalized gauge coupling). At higher orders of perturbation theory there can be mixing of bulk and boundary radiative effects. For example, one can have a finite bulk 1–loop correction to the Higgs mass infected at 2–loop order by divergences due to insertion of boundary counterterms like Eq. (3.3) [38]. The explicit 2–loop computation of Ref. [40] has indeed found these effects. At 2–loop order the Higgs mass is logarithmically sensitive to the cutoff. At this point only a non–perturbative computation can establish if there is a scaling region where these higher order corrections to the 1–loop prediction Eq. (3.9) are negligible.

4 Lattice formulation

We consider now a Euclidean five–dimensional hypercubic lattice with lattice spacing $a$. The points have coordinates $z = a (n_0, n_1, n_2, n_3, n_5)$ with $n_\mu = 0, 1, \ldots, N_\mu - 1$, $\mu = 0, 1, 2, 3$ and $n_5 = -N_5, -N_5 + 1, \ldots, N_5 - 1$. The physical extensions of the lattice are $L_\mu = N_\mu a$ and $2\pi R = 2N_5 a$. The gauge variables on the lattice consist of the links $U(z, M) \in \text{SU}(N)$, which are the parallel transporters for SU($N$) vectors from $z + a\hat{M}$ to $z$ along the straight line connecting these two points ($\hat{M}$ is the unit vector in direction $M$). Under gauge transformations

$$U(z, M) \xrightarrow{\Omega} \Omega(z) U(z, M) \Omega^\dagger(z + a\hat{M}).$$

We impose periodic boundary conditions on the gauge field and on the gauge transformations in all five directions. As the gauge action we take the Wilson action

$$S_W[U] = \frac{\beta}{2N} \sum_p \text{tr}\{1 - U(p)\},$$

where the sum runs over all oriented plaquettes $p$ on the lattice. We set

$$\beta = \frac{2N}{g_0'^2 a}$$

and identify $g_0'$ as the bare dimensionful gauge coupling on the lattice.
Unlike in perturbation theory in the continuum, on the lattice the periodicity of the gauge links does not break gauge invariance. There is no need to introduce transition functions like we had to do in the continuum formulation in Appendix B. The orbifold theory can be therefore defined in a more straightforward way. Given a continuum gauge field $A_M$ the gauge links can be reconstructed as

$$U(z, M) = \mathcal{P} \exp \left\{ a \int_0^1 dt A_M(z + a \hat{M} - t a \hat{M}) \right\}, \quad (4.4)$$

where the symbol $\mathcal{P}$ implies a path ordered exponential such that the fields at larger values of the integration variable $t$ stand to the left of those with smaller $t$. From the reflection $\mathcal{R}$, defined in Eq. (2.1) and Eq. (2.2), and the group conjugation

$$\mathcal{T}_g A_M = g A_M g, \quad (4.5)$$

with $g$ specified in Eq. (2.19), it is easy to derive the corresponding $\mathbb{Z}_2$ transformations acting on the gauge links. Under the reflection $\mathcal{R}$ the gauge links transform as

$$\mathcal{R} U(z, \mu) = U(\bar{z}, \mu), \quad \mathcal{R} U(z, 5) = U^\dagger(\bar{z} - \hat{a} 5, 5). \quad (4.6)$$

Fig. 1 schematically represents the reflection $\mathcal{R}$ on the lattice. Under the group conjugation $\mathcal{T}_g$ the gauge links transform as

$$\mathcal{T}_g U(z, M) = g U(z, M) g. \quad (4.7)$$

The action Eq. (4.2) is invariant under the combined $\mathbb{Z}_2$ transformation $\Gamma = \mathcal{R} \mathcal{T}_g$. Consequently we embed the orbifold projection in the lattice theory through

$$\frac{1 - \Gamma}{2} U(z, M) = 0, \quad (4.8)$$

**Figure 1**: Representation of the $S^1/\mathbb{Z}_2$ orbifold projection in the fifth dimension on the lattice.
where \((1 - \Gamma)/2\) is a projector. For the gauge links in the four-dimensional planes defined by the fixpoints \(z = \bar{z}\) of \(R\), Eq. (4.8) implies

\[
U(z, \mu) = g U(z, \mu) g \quad \text{at} \quad n_5 = 0 \quad \text{and} \quad n_5 = N_5 = \pi R/a.
\]

These constraints break the gauge group \(SU(N)\) down to the subgroup \(H\) Eq. (2.20) depending on the choice of \(g\). As discussed in Section 2, the generators \(T^a\) of \(H\) satisfy \([T^a, g] = 0\).

The lattice orbifold theory can now be defined on the strip \(I_0 = \{n_\mu, 0 \leq n_5 \leq N_5\}\). The action is

\[
S_{\text{orb}}^W[U] = \frac{\beta}{2N} \sum_p w(p) \tr\{1 - U(p)\}
\]

where the sum runs now over all oriented plaquettes in the strip. The weight \(w(p)\) is

\[
w(p) = \begin{cases} \frac{1}{2} & \text{if } p \text{ is a plaquette in the } (\mu \nu)\text{-planes at } n_5 = 0 \text{ and } n_5 = N_5, \\ 1 & \text{in all other cases.} \end{cases}
\]

The Dirichlet boundary conditions are specified by Eq. (4.9). The normalization \(\beta/(2N) = a/g_0^2\) in Eq. (4.10) is such that the continuum action on the strip is reproduced in the naive continuum limit. The theory is invariant under gauge transformations

\[
\Omega(z) \in \begin{cases} H & \text{at the boundary planes } n_5 = 0 \text{ and } n_5 = N_5, \\ SU(N) & \text{otherwise.} \end{cases}
\]

We are left to prove that in the continuum limit the lattice orbifold projection Eq. (4.8) reproduces the boundary conditions Eq. (2.16) and Eq. (2.17). To this end, a gauge field on the lattice can be introduced through \(U(z, M) = \exp\{aA_M(z)\}\).

In the classical continuum limit, i.e. expanding Eq. (4.8) in powers of the lattice spacing \(a\), we get at the fixpoints

\[
A_\mu(z) = g A_\mu(\bar{z}) g + \mathcal{O}(a),
\]

\[
A_5(z) = -g A_5(\bar{z}) g + \mathcal{O}(a).
\]

The leading term at \(z = \bar{z}\) gives the Dirichlet boundary conditions Eq. (2.16).

At the quantum level the gluon propagator (not the vertices) carries the information about the boundaries. The propagator on the lattice can be constructed extending a trick used in Refs. [35,47]. We observe that the orbifold constraint Eq. (4.8) is satisfied automatically by the gauge links

\[
U_\Gamma(z, M) = \exp\left\{a \frac{1 + \Gamma}{2} B_M(z)\right\},
\]

where \(B_M(z)\) is an unconstrained gauge field on the full periodic lattice. It is easy to check that \(U_\Gamma(z, M) \in SU(N)\) and in particular that at the fixpoints \(z = \bar{z}\) of \(R\) the links \(U_\Gamma\) are elements of its subgroup \(H\), as expected. We use the hermitian basis of generators \(T^A\) and under group
conjugation $g T^A g = \eta^A T^A$ (no sum on $A$), see Appendix A. The gauge field components are $A_M = -ig_0 A^C_M T^C$. The building block is the propagator on a five–dimensional periodic lattice

$$\Delta_{MM'}^{CC'}(z-z') = \left( \prod_\mu N_\mu N_5 \right)^{-1} \sum_p e^{ip(z-z')} e^{i(p_M-p_{M'})/2} \tilde{\Delta}_{MM'}^{CC'}(p),$$

(4.16)

$$\tilde{\Delta}_{MM'}^{CC'}(p) = \delta_{MM'} \left\{ \frac{\delta_{MM'} \eta_{CC'}}{p^2} - \left( 1 - \xi \right) \hat{p}_M \hat{p}_{M'} \right\},$$

(4.17)

where $\hat{p}_M = (2/a) \sin(ap_M/2)$ and the sum in Eq. (4.16) runs over the momenta in the Brillouin zones $p_\mu = 2\pi n_\mu / L_\mu$ and $p_5 = n_5 / R$ with $n_\mu = 0,1,\ldots,N_\mu -1$, $\mu = 0,1,2,3$ and $n_5 = -N_5, -N_5+1, \ldots, N_5 -1$. Note that the gauge field on the lattice is naturally associated with the midpoint of the link. To check that the correct Neumann boundary conditions are obtained in the continuum limit of the lattice theory we use for example the identities

$$(\Delta_{\text{orb}})^{CC'}_{55'}(\bar{z}, z') = -\eta^C (\Delta_{\text{orb}})^{CC'}_{55}(z, z'),$$

(4.19)

$$(\Delta_{\text{orb}})^{CC'}_{55}(\bar{z}, z') = -\eta^C (\Delta_{\text{orb}})^{CC'}_{55}(z + a\hat{5}, z').$$

(4.20)

Setting $C = \hat{a}$ and using the lattice forward $\partial$ and backward $\partial^*$ derivatives, yields

$$\partial_5^*(\Delta_{\text{orb}})^{CC'}_{55}(z, z')|_{z=\bar{z}} = 0,$$

(4.21)

$$\partial_5^*(\Delta_{\text{orb}})^{CC'}_{55}(z, z')|_{z=\bar{z}} = 0,$$

(4.22)

which give in the continuum limit $\partial_5 A^a_5 = 0$ at $z = \bar{z}$. Similarly we obtain

$$\partial_5^*(\Delta_{\text{orb}})^{CC'}_{55}(z, z')|_{z=\bar{z}} = 0,$$

(4.23)

$$\partial_5^*(\Delta_{\text{orb}})^{CC'}_{55}(z, z')|_{z=\bar{z}} = 0,$$

(4.24)

which give in the continuum limit $\partial_5 A^a_5 = 0$ at $z = \bar{z}$. We have therefore proven that the lattice orbifold propagator carries in the continuum limit the Neumann boundary conditions Eq. (4.22).

Finally, a brief but important comment about the Higgs mass. It will be certainly very interesting to compare the perturbative result Eq. (3.9) with the corresponding mass extracted from lattice simulations. For this we have to construct within the five–dimensional orbifold lattice theory a gauge invariant operator which has the proper symmetries. This has been discussed in Ref. [48] for pure SU($N$) gauge theories at nonzero temperature. The Debye mass, which in our context is the Higgs mass, can be extracted from the exponential fall–off of correlation functions of gauge invariant operators which are odd under the reflection $\mathcal{R}$. The operators proposed in [48] can be easily extended to the orbifold theory and will be studied in forthcoming simulations.
5 Conclusion

In this work we constructed non–perturbatively five–dimensional gauge theories in Euclidean space with the fifth dimension compactified on the $S^1/Z_2$ orbifold.

We discussed the possibility of studying these theories on the lattice at a finite value of the cutoff $\Lambda = 1/a$ given by the inverse lattice spacing and for energies in the range specified by Eq. (1.7). The five–dimensional (four–dimensional) components of the gauge field with positive “parity” under the orbifold projection play the role of the Higgs (gluon) field in the dimensionally reduced theory, defined at the orbifold fixpoints in terms of the Kaluza–Klein zero modes of these fields. The ultimate goal of our work is to provide a non–perturbative proof whether this is a viable field–theoretic scenario, in other words if a scaling region at finite cutoff exists where the interactions are appreciable.

We discussed the possible boundary terms localized at the fixpoints of the orbifold and give a prescription how to derive them. In particular a non–perturbative proof is given that no boundary term for the Higgs mass can occur, which would be quadratically divergent in the cutoff $\Lambda$. We showed that the theories can be formulated in a straightforward way on the lattice. Boundary conditions are imposed only for the links in the four–dimensional boundary planes which belong to the broken gauge group. In the naive continuum limit the gauge field propagator implements the correct Neumann boundary conditions.

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A Notational conventions

The Euclidean gauge action for gauge group SU($N$) on the manifold $\mathbb{R}^5$ is given by

$$S_5[A] = -\frac{1}{2g_0^2} \int d^5z \text{tr}\{F_{MN}(z)F_{MN}(z)\}. \quad (A.1)$$

The gauge field $A_M(z)$ belongs to the Lie algebra su($N$) of SU($N$)

$$A_M^\dagger(z) = -A_M(z), \quad \text{tr}\{A_M\} = 0. \quad (A.2)$$

The field strength tensor $F_{MN}(z)$ is defined through

$$F_{MN} = [D_M, D_N] = \partial_M A_N - \partial_N A_M + [A_M, A_N], \quad (A.3)$$
where we introduced the gauge covariant derivative
\[ D_M = \partial_M + A_M. \]  
(A.4)

We denote by \( \Omega(z) \in SU(N) \) a gauge transformation in five dimensions. The gauge field transforms as
\[ A_M(z) \xrightarrow{\Omega} \Omega(z)A_M(z)\Omega(z)^{-1} + \Omega\partial_M\Omega(z)^{-1}. \]  
(A.5)

The gauge transformations of the covariant derivative Eq. (A.4) and the field strength tensor Eq. (A.3) are easily derived
\[ D_M \xrightarrow{\Omega} \Omega(z)D_M\Omega(z)^{-1}, \]  
(A.6)
\[ F_{MN}(z) \xrightarrow{\Omega} \Omega(z)F_{MN}(z)\Omega(z)^{-1}. \]  
(A.7)

The covariant derivative of the field strength tensor is defined through
\[ D_L F_{MN} = \partial_L F_{MN} + [A_L, F_{MN}], \]  
(A.8)

and under gauge transformation it transforms (as its name suggests) like
\[ D_L F_{MN}(z) \xrightarrow{\Omega} \Omega(z)D_L F_{MN}(z)\Omega(z)^{-1}. \]  
(A.9)

The generators \( T_A, A = 1, \ldots, N^2-1 \) of \( SU(N) \) are typically taken to be hermitian and traceless. This basis is spanned for \( SU(2) \) by the Pauli \( \sigma \)-matrices, for \( SU(3) \) by the Gell-Mann \( l \)-matrices and so forth. The generators have the properties
\[ [T^A, T^B] = if^{ABC}T^C, \quad \text{tr} \{T^A T^B\} = \frac{1}{2} \delta^{AB}. \]  
(A.10)

The connection of this basis with the Cartan–Weyl basis is simply to take the Cartan sub-algebra, i.e. the commuting generators \( H = \{H_i\}, \ i = 1, \ldots, N-1 \) to be the same. The remaining generators are combined in pairs of ladder operators (a raising and a lowering operator) \( E_{\pm \alpha}, \alpha = 1, \ldots, N(N-1)/2 \), which are defined through (with normalization by \( 1/\sqrt{2} \))
\[ E_\alpha = \frac{1}{\sqrt{2}}(T^\alpha(S) + iT^\alpha(A)) \quad \text{and} \quad E_{-\alpha} = \frac{1}{\sqrt{2}}(T^\alpha(S) - iT^\alpha(A)). \]  
(A.11)

Here by \( T^\alpha(S) \) we mean the symmetric \( SU(N) \) generator with a 1 in the \( mn \)-th position and by \( T^\alpha(A) \) the anti-symmetric \( SU(N) \) generator with a \(-i \) in the \( mn \)-th position (\( \alpha \) labels all possible pairs \( mn, m \neq n \), with \( m, n = 1, \ldots, N-1 \)). Each operator \( E_\alpha \) has associated an \( (N-1) \)-dimensional vector \( \alpha = \{\alpha_i\} \) (called root of the operator) such that
\[ [H_i, E_{\pm \alpha}] = \pm \alpha_i E_{\pm \alpha}. \]  
(A.12)

In the orbifold boundary conditions, the breaking of the gauge symmetry is realized by a group conjugation with the matrix \( g \) defined in Eq. (2.19). Using the properties Eq. (A.12) it follows [49]
\[ g H_i g = H_i, \]  
(A.13)
\[ g E_{\pm \alpha} g = e^{-2\pi i \alpha \cdot V} E_{\pm \alpha} \quad \text{with} \quad e^{-2\pi i \alpha \cdot V} = \pm 1. \]  
(A.14)
The parity of the generators $\exp(-2\pi i \alpha \cdot V)$ is determined by the twist vector $V$. From the relations Eq. (A.11) it follows immediately that also the hermitian generators $T^A$ have a definite parity $\eta^A$ under group conjugation $g T^A g = \eta^A T^A$. We label by $T^a$ the unbroken generators with $\eta^a = 1$ and by $T^\hat{a}$ the broken generators with $\eta^\hat{a} = -1$. In the adjoint representation for the generators $T^A$, the matrix $g$ takes the form $g = \text{diag}\{\{\eta^a\}\}$.

\section{Gauge fields with one compact extra dimension}

\subsection{Gauge fields on $S^1$}

When compactifying the fifth dimension on the circle one is instructed to define separate gauge fields on overlapping (but not self overlapping) charts that provide an open cover for the compact space. The minimum number of such overlapping open sets for $S^1$ is two, let us call them $O^{(+)}$ and $O^{(-)}$ and their overlaps $O^{(+,-)}_i = (O^{(+)} \cap O^{(-)})_i$, $i = 1, 2$. On each open set there is a gauge field that under a gauge transformation transforms with its own gauge function on $O^{(+)}$:

\begin{align*}
\text{on } O^{(+)}: & \quad A^{(+)}_M \rightarrow \Omega^{(+)} A^{(+)}_M \Omega^{(+)}^{-1} + \Omega^{(+)} \partial_M \Omega^{(+)}^{-1} \\
\text{on } O^{(-)}: & \quad A^{(-)}_M \rightarrow \Omega^{(-)} A^{(-)}_M \Omega^{(-)}^{-1} + \Omega^{(-)} \partial_M \Omega^{(-)}^{-1}.
\end{align*}

(B.1) \quad (B.2)

One requires that the gauge fields on $O^{(+,-)}_i$ (where they are both defined) are related by a gauge transformation:

\begin{align*}
A^{(+)}_M &= G^{(+,-)} A^{(-)}_M G^{(+,-)}^{-1} + G^{(+,-)} \partial_M G^{(+,-)}^{-1} \\
A^{(-)}_M &= G^{(-,+)} A^{(+)}_M G^{(-,+)}^{-1} + G^{(-,+)} \partial_M G^{(-,+)}^{-1}.
\end{align*}

(B.3) \quad (B.4)

The SU($N$)-valued functions $G^{(+,-)}$ and $G^{(-,+)}$ are called transition functions [50] and they are defined on the overlaps of charts $O^{(+,-)}_i$. Eq. (B.3) and Eq. (B.4) are simultaneously satisfied when the gluing condition

\begin{align*}
G^{(+,-)} G^{(-,+)} &= \pm 1 \quad \text{on } O^{(+,-)}_i
\end{align*}

(B.5)

is imposed. Furthermore, their covariance requires that under gauge transformations they must transform as

\begin{align*}
G^{(\pm)} \rightarrow \Omega^{(\pm)} G^{(\pm)} \Omega^{(\mp)}^{-1}.
\end{align*}

(B.6)

Given the above gauge transformations one can define covariant derivatives acting on the transition functions such that

\begin{align*}
D_M G^{(\pm)} \rightarrow \Omega^{(\pm)} D_M G^{(\pm)} \Omega^{(\mp)}^{-1}.
\end{align*}

(B.7)

This fixes the covariant derivatives to be

\begin{align*}
D_M G^{(\pm)} &= \partial_M G^{(\pm)} + A^{(\pm)}_M G^{(\pm)} - G^{(\mp)} A^{(\mp)}_M
\end{align*}

(B.8)

(due to Eq. (B.5) $D_M G^{(\pm)} = (D_M G^{(\pm)})^\dagger$) and one can easily see, using eqs. (B.3) that

\begin{align*}
D_M G^{(+,-)} = D_M G^{(-,+)} = 0.
\end{align*}

(B.9)
B.2 Gauge fields on $S^1/Z_2$

For simplicity we drop in the following the coordinate $x_\mu$ since it is not affected by the transformations considered. For an orbifold construction we define the following charts

$$O^{(+)} = (-\epsilon, \pi R + \epsilon) \quad \text{and} \quad O^{(-)} = (-\pi R - \epsilon, \epsilon) \quad (B.10)$$

with overlaps $O^{(+)}_1 = (-\epsilon, \epsilon)$ and $O^{(+)}_2 = (\pi R - \epsilon, \pi R + \epsilon)$ where $0 \leq \epsilon < (\pi R)/2$. The coordinates are identified modulo $2\pi R$.

Identification under reflection

We introduce the $Z_2$ transformation $R : x_5 \rightarrow -x_5$ which maps $R O^{(\pm)} = O^{(\mp)}$, $R O^{(+)}_i = O^{(+)}_i$. The transformation $R$ can be defined also to act on tensor fields defined on $O^{(\pm)}$ giving as result tensor fields defined on $O^{(\mp)}$.

On the overlaps $O^{(+)}_i$, $i = 1, 2$ we identify the gauge fields under the transformation $R$ through

$$R A^{(+)}_M = A^{(-)}_M. \quad (B.11)$$

This identification is gauge covariant if at the same time the gauge transformations satisfy on the overlaps

$$R \Omega^{(+)} = \Omega^{(-)}. \quad (B.12)$$

Putting together Eq. (B.4) with Eq. (B.11) we obtain the following constraints for $A^{(+)}_M$ on the overlaps $O^{(+)}_i$, $i = 1, 2$

$$R A^{(+)}_M = G^{(-)} A^{(+)}_M G^{(-)}^{-1} + G^{(-)} \partial_M G^{(-)}^{-1}. \quad (B.13)$$

Self-consistency of Eq. (B.13) requires $R G^{(-)} = G^{(-)}$ and using the gluing condition Eq. (B.3) this gives the constraint

$$R G^{(-)} = \pm 1. \quad (B.14)$$

From this it follows that at the fixpoints $x_5 = 0$ and $x_5 = \pi R$ of the $R$ transformation the transition functions satisfy

$$(G^{(-)}(0))^2 = \pm 1 \quad \text{and} \quad (G^{(-)}(\pi R))^2 = \pm 1. \quad (B.15)$$

Outside the overlaps we identify further

$$(R A^{(+)}_M(x_5)) = A^{(-)}_M(-x_5), \quad x_5 \in [\epsilon, \pi R - \epsilon]. \quad (B.16)$$

Therefore we can set up the gauge theory on the one chart $O^{(+)}$ with gauge field $A_M \equiv A^{(+)}_M$ and a spurion field defined on the overlaps $O^{(+)}_i$, $i = 1, 2$ through

$$G = G^{(-)} \quad \text{and} \quad (R G) G = \pm 1 \quad (B.17)$$

which, using Eq. (B.6) and Eq. (B.12) has the gauge transformation under $\Omega \equiv \Omega^{(+)}$

$$G \rightarrow (R \Omega) G \Omega^{-1}. \quad (B.18)$$
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