IMPROVEMENTS TO THE MONTEL–CARATHÉODORY THEOREM FOR FAMILIES OF $\mathbb{P}^n$-VALUED HOLOMORPHIC CURVES

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Abstract. In this paper, we establish various sufficient conditions for a family of holomorphic mappings on a domain $D \subseteq \mathbb{C}$ into $\mathbb{P}^n$ to be normal. Our results are improvements to the Montel–Carathéodory Theorem for a family of $\mathbb{P}^n$-valued holomorphic curves.

1. Introduction and main results

The work in this paper is influenced, mainly, by the Montel–Carathéodory Theorem and its extensions in higher dimensions. The classical Montel–Carathéodory Theorem states that a family of holomorphic $\mathbb{P}^1$-valued mappings on a planar domain is normal if this family omits three fixed, distinct points in $\mathbb{P}^1$. This result has gone through various generalizations in one and higher dimensions. Xu [8] proved, among the other things, the following result that improves the classical Montel–Carathéodory Theorem.

Result 1.1 ([8, Theorem 1]). Let $F$ be a family of meromorphic functions on a planar domain $D \subseteq \mathbb{C}$. Suppose that
   (a) for each pair of functions $f, g \in F$;
   (i) $f^{-1}(\{0\}) = g^{-1}(\{0\})$, and
   (ii) $f^{-1}(\{\infty\}) = g^{-1}(\{\infty\})$, i.e., $f$ and $g$ have the same set of poles;
   (b) all zeros of $f - 1$ are of multiplicity at least 2 in $D$.

Then $F$ is normal on $D$.

We asserted that Result 1.1 is an improvement of the classical Montel–Carathéodory Theorem because the latter follows as a simple corollary of Result 1.1. To see this, suppose a family $F$ of meromorphic functions omits three distinct values — say, $\alpha, \beta, \gamma$ in $\mathbb{C} \cup \{\infty\}$. Then, consider the following family

$$G := \left\{ g(z) = \frac{\alpha - \gamma}{\beta - \gamma} \cdot f(z) - \beta : f \in F \right\}$$

(with the understanding that if $\infty \in \{\alpha, \beta, \gamma\}$ then we label it as $\gamma$, and $(\alpha - \gamma)/(\beta - \gamma)$ is understood to be 1 in that case). It is straightforward to see that $g^{-1}(\{0\}) = g^{-1}(\{1\}) = g^{-1}(\{\infty\}) = \emptyset$ for all $g \in G$. Thus, $G$ satisfies, obviously, all the conditions of Result 1.1. Therefore, $G$ is normal and the normality of $F$ then follows.

The Montel–Carathéodory Theorem was generalized to higher dimensions by Dufresnoy [2]. To state the Montel–Carathéodory Theorem in higher dimensions, we need to introduce some essential notions. To this end, we fix a system of homogeneous coordinates $w = [w_0 : w_1 : \cdots : w_n]$ on the $n$-dimensional complex projective space $\mathbb{P}^n$, $n \geq 1$. A hyperplane $H$ in $\mathbb{P}^n$ can be given by

$$H := \left\{ [w_0 : w_1 : \cdots : w_n] \in \mathbb{P}^n \mid \sum_{l=0}^{n} a_l w_l = 0 \right\},$$

where $(a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$ is a non-zero vector. Recall that for any collection of $q(\geq n+1)$ hyperplanes $H_1, \ldots, H_q$, we say that these hyperplanes are in general position if for any arbitrary subset $R \subset \{1, \ldots, q\}$ — with cardinality $|R| = n+1$ — the intersection $\bigcap_{k \in R} H_k = \emptyset$.
We are now in a position to state the Montel–Carathéodory Theorem in higher dimensions: A family of holomorphic \( \mathbb{P}^n \)-valued mappings on a domain \( D \subseteq \mathbb{C}^m \) is normal if this family omits \( 2n + 1 \) hyperplanes in general position in \( \mathbb{P}^n \).

A natural question arises immediately in connection with the above result:

1. Does a version of Result 1.1 — that improves the Montel–Carathéodory Theorem — holds true for any family of holomorphic mappings from a planar domain into \( \mathbb{P}^n \)?

Concerning Result 1.1, note that the family \( \mathcal{F} \) in the statement of the Result 1.1 is precisely a family of holomorphic mappings from \( D \) into \( \mathbb{P}^1 \). We recall, at this point, that a holomorphic mapping from a Riemann surface into a complex manifold is called a holomorphic curve.

We call the following the coordinate hyperplanes of \( \mathbb{P}^n \). Note that these \( n + 1 \) coordinate hyperplanes are in general position in \( \mathbb{P}^n \). In the set up of Result 1.1 the values 0 and \( \infty \) can be viewed as the coordinate hyperplanes of the projective space \( \mathbb{P}^1 \). With these observations, we give an answer to (1) by introducing our first theorem. Once we examine the Condition (b) in our theorem in the context of Result 1.1 it will be clear that the latter is a special case of the following:

**Theorem 1.2.** Let \( \mathcal{F} \) be a family of holomorphic curves on a planar domain \( D \) into \( \mathbb{P}^n \). Let \( H_1, \ldots, H_{2n+1} \) be \( 2n + 1 \) hyperplanes in general position in \( \mathbb{P}^n \). Assume that the first \( n + 1 \) of these hyperplanes, i.e., \( H_1, \ldots, H_{n+1} \), are the coordinate hyperplanes of \( \mathbb{P}^n \). Suppose that

(a) for each pair \( f, g \in \mathcal{F} \); \( f^{-1}(H_k) = g^{-1}(H_k) \) for each \( k \in \{1, \ldots, n + 1\} \);

(b) for each compact set \( K \subset D \) there exists a finite number \( M = M(K) > 0 \) such that for each \( f \in \mathcal{F} \) and each \( l \in \{0, \ldots, n\} \) for which \( U_{f, l} := \{ z \in D : f(z) \in \{w_0 : \ldots : w_n | w_l \neq 0\} \} \neq \emptyset \), the inequality

\[
\left| (\tilde{t}^f H_k)^{(q)}(z) \right|_{1 \leq q \leq n} \leq M \quad \forall z \in f^{-1}(H_k) \cap K
\]

holds for each \( k \in \{n + 2, \ldots, 2n + 1\} \), where \( \tilde{t}^f = (\tilde{t}^{f_0}, \ldots, \tilde{t}^{f_n}) \) is the reduced representation of \( f \) on the subset \( U_{f, l} \subset D \) such that \( \tilde{t}^f \equiv 1 \) on \( U_{f, l} \).

Then the family \( \mathcal{F} \) is normal.

**Remark 1.3.** The Montel–Carathéodory Theorem for \( \mathbb{P}^n \)-valued holomorphic curves is not deducible from Theorem 1.2 when \( n \geq 2 \). Nevertheless, there are examples of families of \( \mathbb{P}^n \)-valued holomorphic curves, \( n \geq 2 \), for which no information about their normality can be deduced from the Montel–Carathéodory Theorem, but we know these families to be normal owing to Theorem 1.2—see Example 3.3 for instance. We must also remark that while the need to control the specific reduced representations \( \tilde{t}^f \) in Theorem 1.2 may appear to be technical, Example 3.1 below reveals that this is essential.

The object \( \tilde{t}^{\ast} H_k \) refers to a holomorphic function on \( U_{f, l} \) whose precise definition is given in Section 2 (see (2.1)). But to get a sense of these functions (and what Condition (b) says) consider the case \( n = 1 \). Then a hyperplane \( H \) is just a point \( \alpha \in \mathbb{C} \cup \{\infty\} \). Also, for a non-constant holomorphic curve \( f : D \to \mathbb{P}^1 \), we classically understand \( f \) to be equal to a ratio \( f = f_1/f_0 \), where \( f_0, f_1 \neq 0 \) are holomorphic functions on \( D \) having no common zeros, and where the points in \( f^{-1}\{\infty\} \) (if any) are the poles of \( f_1/f_0 \) or, equivalently, the zeros of \( f_0 \). In this case \( U_{f, 0} = D \setminus f_0^{-1}\{0\} \), the reduced representation \( \tilde{t}^f \) referenced by Condition (b) is simply \( (1, f_1/f_0) \). Furthermore (see (2.1) below),

\[
\tilde{t}^f \ast H := f_1/f_0 - \alpha \quad \text{(on } U_{f, 0} = D \setminus f_0^{-1}\{0\} \text{) if } \alpha \neq \infty.
\]

Arguing further: in the final analysis we deduce that the zeros of the holomorphic function

\[
\tilde{t}^f \ast H(z) \text{ are the zeros of (the meromorphic function) } f - \alpha \text{ in } D \text{ when } \alpha \neq \infty; \text{ and}
\]

\[
\tilde{t}^f \ast H(z) \text{ are the zeros of } f \text{ in } D \text{ when } (f^{-1}\{\infty\} \text{ if any}) \text{ are the poles of } f \text{ and } \tilde{t}^f \text{ is the reduced representation of } f \text{ on } D \text{ that is a holomorphic function on } U_{f, l}.
\]
The notation \( (g_f^* H_k)\{(q) \} \) — denoting any of the relevant reduced representations — represents the \( q \)-th derivative of \( g_f^* H_k \). It is now not hard to notice that Condition (b) in the statement of Theorem \( \ref{thm:1.2} \) is weaker than the Condition (b) in the statement of Result \( \ref{cor:1.1} \). In fact, if the zeros of \( g_f^* H_k \) are of multiplicity at least \((n+1)\) (with \( g_f \) as above and \( n + 2 \leq k \leq 2n + 1 \) then Condition (b), in the statement of Theorem \( \ref{thm:1.2} \) automatically holds true. With this observation, we have a corollary with a more attractive statement:

**Corollary 1.4.** Let \( \mathcal{F} \) be a family of holomorphic curves on a planar domain \( D \) into \( \mathbb{P}^n \). Let \( H_k, k = 1, \ldots, 2n + 1 \), be hyperplanes as in the statement of Theorem \( \ref{thm:1.2} \). Suppose that

(a) for each pair \( f, g \in \mathcal{F}, f^{-1}(H_k) = g^{-1}(H_k) \) for each \( k \in \{1, \ldots, n + 1\} \);

(b) for each \( f \in \mathcal{F} \) and each \( k \in \{n + 2, \ldots, 2n + 1\} \), \( f \) intersects \( H_k \) with multiplicity at least \( n + 1 \).

Then the family \( \mathcal{F} \) is normal.

We now ask a related question: whether one can replace Condition (b) in the statement of Corollary \( \ref{cor:1.4} \) by the condition (\( \ast \)), given below, and yet yields the same conclusion.

(\( \ast \)) the volumes of \( f^{-1}(H_k) \), viewing \( f^{-1}(H_k) \) as divisors, are locally uniformly bounded. Example \( \ref{ex:3.3} \) below will show that this replacement is not the right replacement. We therefore ask: how close a condition to Condition (\( \ast \)) would give the same conclusion as in the Corollary \( \ref{cor:1.4} \). In order to answer this question we state our next result which is a generalization of the Montel–Carathéodory Theorem (we shall presently see the relevance, in the following theorem, of considering volumes):

**Theorem 1.5.** Let \( \mathcal{F} \) be a family of holomorphic curves on a planar domain \( D \) into \( \mathbb{P}^n \), \( n \geq 2 \), and \( H_1, \ldots, H_{2n+1} \) be hyperplanes in general position in \( \mathbb{P}^n \). Suppose that

(a) for each \( f \in \mathcal{F}, f(D) \cap H_k = \emptyset \) for all \( k = 1, \ldots, n + 1 \);

and that there exists an integer \( t, n + 2 \leq t < 2n + 1 \), such that

(b) for each pair of holomorphic curves \( f, g \in \mathcal{F}, f^{-1}(H_k) = g^{-1}(H_k) \) for each \( n + 2 \leq k \leq t \);

(c) for each \( f \in \mathcal{F} \) and each \( k \in \{t + 1, \ldots, 2n + 1\} \), \( f(D) \not\subset H_k \) and, for each compact set \( K \subset D \), the volumes of \( f^{-1}(H_k) \cap K \), viewing \( f^{-1}(H_k) \) as divisors, are uniformly bounded (i.e., for \( k = t + 1, \ldots, 2n + 1, and all f \in \mathcal{F} \)).

Then the family \( \mathcal{F} \) is normal.

We clarify here that the volume of \( f^{-1}(H_k) \cap K \) is identified with the cardinality of the set of zeros — counting with multiplicity — of \( f^* H_k \) in \( K \), for some reduced representation \( f \) of \( f \). We end this section with a brief explanation for the above-mentioned assertion that Theorem \( \ref{thm:1.5} \) is a generalization of the Montel–Carathéodory Theorem (for holomorphic curves). To this end, suppose a family \( \mathcal{F} \) of \( \mathbb{P}^n \)-valued holomorphic curves on a planar domain \( D \) omits \( 2n + 1 \) hyperplanes in general position — say \( H_1, \ldots, H_{2n+1} \) in \( \mathbb{P}^n \). This means that

\[
\text{Vol}(f^* H_k) = 0 \text{ for all } f \in \mathcal{F} \text{ and } k = 1, \ldots, 2n + 1.
\]

We now fix \( t = 2n \) (\( \geq n + 2 \) and notice, easily, that \( \mathcal{F} \) satisfies Conditions (a) and (b) of Theorem \( \ref{thm:1.5} \). We next infer, from \( \ref{eq:1.2} \), that \( f(D) \not\subset H_{2n+1} \) and the volume of \( f^{-1}(H_{2n+1}) \cap K \) is 0 for each \( f \in \mathcal{F} \) and each compact subset \( K \) of \( D \). Thus, Condition (c) of Theorem \( \ref{thm:1.5} \) is also satisfied and hence \( \mathcal{F} \) is normal. Finally, to make the case that Theorem \( \ref{thm:1.5} \) is a generalization of the Montel–Carathéodory Theorem, we must produce an example of a planar domain \( D \), a family \( \mathcal{F} \), and hyperplanes \( H_1, \ldots, H_{2n+1} \) as in Theorem \( \ref{thm:1.5} \) (and satisfy the conditions thereof) that do not satisfy the hypothesis of Montel–Carathéodory Theorem. Interestingly, the example referred to in Remark \( \ref{rem:1.3} \) — see Section \( \ref{sec:3} \) — for details — serves the latter purpose as well.
2. Basic notions

This section is devoted to elaborating upon concepts and terms that made an appearance in Section 1 and to introducing certain basic notions needed in our proofs. In this section, \( D \subseteq \mathbb{C} \) will always denote a planar domain.

Let \( f : D \to \mathbb{P}^n \) be a holomorphic mapping. Fixing a system of homogeneous coordinates on the \( n \)-dimensional complex projective space \( \mathbb{P}^n \), for each \( a \in D \), we have a holomorphic map \( \tilde{f}(z) := (f_0(z), f_1(z), \ldots, f_n(z)) \) on some neighborhood \( U \) of \( a \) such that \( \{ z \in U \mid f_0(z) = f_1(z) = \cdots = f_n(z) = 0 \} = \emptyset \) and \( f(z) = [f_0(z) : f_1(z) : \cdots : f_n(z)] \) for each \( z \in U \). We shall call any such holomorphic map \( \tilde{f} : U \to \mathbb{C}^{n+1} \) a reduced representation (or an admissible representation) of \( f \) on \( U \). A holomorphic mapping \( f : D \to \mathbb{P}^n \) is called a holomorphic curve of \( D \) into \( \mathbb{P}^n \). Note that, as \( \dim_{\mathbb{C}} D = 1 \), every holomorphic curve on \( D \) into \( \mathbb{P}^n \) has a reduced representation globally on \( D \). However, we will find it useful to work with local reduced representations as well.

For a fixed system of homogeneous coordinates on \( \mathbb{P}^n \) we set
\[
V_l := \{ [w_0 : \cdots : w_n] \mid w_l \neq 0 \}, \quad \text{for } l = 0, \ldots, n.
\]
Then every point \( a \in D \) has a neighborhood \( U \ni a \) such that \( f(U) \subset V_l \) for some \( l \), and \( f \) has a reduced representation \( \tilde{f} = (f_0, \ldots, f_{l-1}, 1, f_{l+1}, \ldots, f_n) \) on \( U \), where \( f_0, \ldots, f_{l-1}, f_{l+1,}, \ldots, f_n \) are holomorphic functions on \( U \).

Let \( f \neq 0 \) be a holomorphic function on \( D \). For a point \( a \in D \), the number
\[
\nu_f(a) := \begin{cases} 0, & \text{if } f(a) \neq 0 \\ m, & \text{if } f \text{ has a zero of multiplicity } m \text{ at } a \end{cases}
\]
is said to be the zero-multiplicity of \( f \) at \( a \). An integer-valued function \( \nu : D \to \mathbb{Z} \) is called a divisor on \( D \) if for each point \( a \in D \) there exist holomorphic functions \( g \neq 0 \) and \( h \neq 0 \) in a neighborhood \( U \) of \( a \) such that \( \nu(z) = \nu_g(z) - \nu_h(z) \) for all \( z \in U \). A divisor \( \nu \) on \( D \) is said to be non-negative if \( \nu(z) \geq 0 \) for all \( z \in D \). We define the support \( \text{supp} \nu \) of a non-negative divisor \( \nu \) on \( D \) by
\[
\text{supp} \nu := \{ z \in D \mid \nu(z) \neq 0 \}.
\]

Let \( H \) be a hyperplane in \( \mathbb{P}^n \) as defined in 1.4. Let \( f : D \to \mathbb{P}^n \) be a holomorphic curve such that \( f(D) \not\subseteq H \). Under this condition it is possible to define a divisor on \( D \) that is canonically associated with the pair \((f, H)\), which we shall denote by \( \nu(f, H) \). (We have not defined divisors on general complex manifolds, but we observe that \( H \) determines a divisor on \( \mathbb{P}^n \) and the divisor we shall define is the pullback of the latter.) To do so, consider any \( a \in D \), take a reduced representation \( \tilde{f} := (f_0, f_1, \ldots, f_n) \) of \( f \) on a neighborhood \( U \) of \( a \), and consider the following holomorphic function
\[
\tilde{f}^*H := a_0f_0 + a_1f_1 + \cdots + a_nf_n. \tag{2.1}
\]
It follows from the definition of a reduced representation that the values of \( \nu_{\tilde{f}^*H} \) agree, on an appropriate neighborhood of \( a \), for any two reduced representations of \( f \) around \( a \). It is now easy to check that if one defines \( \nu(f, H) \) by
\[
\nu(f, H)|_U(z) := \nu_{\tilde{f}^*H}(z), \quad z \in U,
\]
then \( \nu(f, H) \) is well defined globally to give a divisor on \( D \).

We say that the holomorphic mapping \( f \) intersects \( H \) with multiplicity at least \( m \) on \( D \), \( m \in \mathbb{Z}_+ \), if \( \nu(f, H) \geq m \) for all \( z \in \text{supp} \nu(f, H) \). Furthermore, we say that \( f \) intersects \( H \) with multiplicity \( \infty \) on \( D \) if either \( f(D) \subset H \) or \( f(D) \cap H = \emptyset \). Here we remark that one can identify the support \( \text{supp} \nu(f, H) \) of \( \nu(f, H) \) by \( f^{-1}(H_k) \) which is again identified locally with the set of zeros of the function \( f^*H \) ignoring multiplicities.
We now give the definition that is central to the discussion in Section 1. Let \( M \) be a compact connected Hermitian manifold. The space \( \text{Hol}(D, M) \) of holomorphic mappings from \( D \) into \( M \) is endowed with the compact-open topology.

**Definition 2.1.** A family \( \mathcal{F} \subset \text{Hol}(D, M) \) is said to be normal if \( \mathcal{F} \) is relatively compact in \( \text{Hol}(D, M) \).

### 3. Some Examples

We now provide the examples alluded to in Section 1.

**Example 3.1.** The “technical restriction” imposed on the reduced representations of holomorphic curves of the family \( \mathcal{F} \) in the statement of Corollary 1.2 is essential. More precisely: the conclusion of Theorem 1.2 need not follow if the bounds in Condition (b) apply to some reduced representation of \( f \in \mathcal{F} \) but which are not the reduced representations \( \overline{f}_j \) of Condition (b).

Let \( D = \mathbb{C} \) and \( \mathcal{F}_1 = \{f_j(z) \mid j \in \mathbb{Z}_+\} \), where \( f_j : D \to \mathbb{P}^2 \) is defined by

\[
f_j(z) := [jz^2 : 1 : 1].
\]

Let \( H_1, H_2 \) and \( H_5 \) be the coordinate hyperplanes of \( \mathbb{P}^2 \) given by

\[
H_1 := \{[w_0 : w_1 : w_2] \mid w_{l-1} = 0\}, \quad l = 1, 2, 3.
\]

Let

\[
H_4 := \{[w_0 : w_1 : w_2] \mid w_0 + w_1 + w_3 = 0\}; \quad \text{and} \quad H_5 := \{[w_0 : w_1 : w_2] \mid w_0 + 2w_1 + 3w_3 = 0\}.
\]

Clearly, hyperplanes \( H_1, \ldots, H_5 \) are in general position. It is easy to see that \( f_j^{-1}(\{H_1\}) = \{0\} \) and \( f_j^{-1}(\{H_2\}) = \emptyset = f_j^{-1}(\{H_3\}) \) for all \( j \). Suppose we consider the following reduced representation of \( f \in \mathcal{F}_1 \) — where \( f_l \neq 1 \) for all \( 0 \leq l \leq 2 \):

\[
\overline{f}_j(z) := (z^2, 1/j, 1/j); \quad z \in \mathbb{C} \text{ and } \mathbb{Z}_+ \ni j \geq 2.
\]

Then \( \overline{f}_j^*H_4(z) = z^2 + 2/j \), and \( \overline{f}_j^*H_5(z) = z^2 + 5/j \) for all \( z \in D, j \geq 2 \). It is easy to see that for \( k = 4, 5 \), we have

\[
|\left(\overline{f}_j^*H_k\right)^{(q)}(z)| \leq 2 \text{ for } z \in f_j^{-1}(H_k), \quad \text{and } q = 1, 2.
\]

Thus, all the conditions in the statement of Theorem 1.2 hold true — with the only exception that the estimate stated in Condition (b) applies a reduced representation of \( f \in \mathcal{F}_1 \) such that \( f_l \neq 1 \) for all \( 0 \leq l \leq n \) (as opposed to the reduced representations identified in Theorem 1.2). However, the family \( \mathcal{F}_1 \) is not normal.

Let us look back at the family \( \mathcal{F}_1 \). Clearly, from the discussion in Example 3.1, we expect Condition (b) of Theorem 1.2 to fail for \( \mathcal{F}_1 \). Let us examine precisely how it fails. To this end, observe that

\[
\{z \in D : f_j(z) \in \{[w_0 : w_1 : w_2] \mid w_l \neq 0\} \} = \mathbb{C} \text{ for } l = 1, 2 \text{ and for all } j \in \mathbb{Z}_+.
\]

Consider the reduced representations \( \overline{f}_j(z) = \overline{f}_{1j}(z) := (jz^2, 1, 1) \) of \( f_j \) in \( \mathbb{C} \), \( j \in \mathbb{Z}_+ \). Let us use the notation \( \overline{f}_j \) for \( \overline{f}_{1j} \) and \( \overline{2f}_j \). Then for \( k = 4, 5 \), we have

\[
|\left(\overline{f}_j^*H_k\right)^{(q)}(z)| \to \infty, \quad \text{as } j \to \infty \text{ for } z \in f_j^{-1}(H_k) \text{ and } q = 1, 2.
\]

This implies the (anticipated) failure of Condition (b) in the statement of Theorem 1.2 for the family \( \mathcal{F}_1 \).

**Example 3.2.** The condition that holomorphic curves \( f \in \mathcal{F} \) intersect \( H_k \) with multiplicity at least \( n + 1 \) in Condition (b) of Corollary 1.4 is essential, when \( n = 1 \).
Let $D = \mathbb{C}$, and $\mathcal{F}_2 = \{f_j(z) \mid j \in \mathbb{Z}_+\}$, where $f_j : D \to \mathbb{P}^1$ is defined by
\[ f_j(z) := [jz : 1]. \]

Let $H_1, H_2$ be the coordinate hyperplanes of $\mathbb{P}^1$ such that
\[ H_l := \{[w_0 : w_1] \mid w_{l-1} = 0\}, \quad l = 1, 2. \]

Let $H_3 := \{[w_0 : w_1] \mid w_0 - w_1 = 0\}$. The hyperplanes $H_1, H_2, H_3$ are in general position. Clearly, $f_j^{-1}(\{H_1\}) = \emptyset$ and $f_j^{-1}(\{H_2\}) = \emptyset$ for all $j$. Whereas $f_j^*H_3$ has a zero of multiplicity 1 for each $j$, where $f_j(z) := (z, 1/j)$, $z \in \mathbb{C}$, for all $j \in \mathbb{Z}_+$.

Thus, all the conditions in the statement of Corollary 1.4 hold true except for (b). However, the family $\mathcal{F}_2$ fails to be normal.

**Example 3.3.** The number $n + 1$ is sharp in the statement of Condition (a) of Theorem 1.6. Specifically: The conclusion of Theorem 1.6 need not hold true if at most $n$ hyperplanes are omitted by the family $\mathcal{F}$.

Let $D = \mathbb{C}$, and $\mathcal{F}_3 = \{f_j(z) \mid j \in \mathbb{Z}_+\}$, where $f_j : D \to \mathbb{P}^n$ is defined by
\[ f_j(z) := [1 : 2 : \cdots : n : jz]. \]

Let $H_1, \ldots, H_{n+1}$ be coordinate hyperplanes of $\mathbb{P}^n$ such that
\[ H_l := \{[w_0 : w_1 : \ldots : w_n] \mid w_{l-1} = 0\}, \quad l = 1, \ldots, n+1. \]

Let $H_{n+2}, \ldots, H_{2n+1}$ be any hyperplanes such that $H_1, \ldots, H_{2n+1}$ are in general position. Clearly, for each $j$,
\begin{itemize}
  \item $f_j^{-1}(\{H_1\}) = \emptyset$ for $l = 1, \ldots, n$; and
  \item $f_j^{-1}(\{H_{n+1}\}) = \emptyset$.
\end{itemize}

It is easy to see that for each $j$ and each $k \in \{n+2, \ldots, 2n+1\}$, $f_j^{-1}(H_k)$ consists of only one point in $D$. Thus, $\mathcal{F}_3$ satisfies all the conditions—with the understanding that $k$ is limited to $k = 1, \ldots, n$ in the condition (a)—in the statement of Theorem 1.5. However, the family $\mathcal{F}_3$ is not normal.

Our final example is the one alluded to in Remark 1.3. Note that the same example also serves as an example that does not satisfy the hypothesis of the Montel–Carathéodory Theorem but does satisfy the conditions of Theorem 1.5.

**Example 3.4.** There exist a family of $\mathbb{P}^n$-valued holomorphic curves, $n = 2$, for which the Montel–Carathéodory Theorem does not give any information but whose normality is deducible either from Theorem 1.2 or Theorem 1.5.

Let $D = \mathbb{C}$, $n = 2$, and $\mathcal{F}_4 = \{f_j : j \in \mathbb{Z}_+\}$, where $f_j : \mathbb{C} \to \mathbb{P}^2$ is defined by
\[ f_j(z) := [je^z : 1 : -1]. \]

Let $H_1, H_2, H_3$ be the coordinate hyperplanes in $\mathbb{P}^2$ given by
\[ H_l := \{[w_0 : w_1 : w_2] \in \mathbb{P}^2 \mid w_{l-1} = 0\}, \quad l = 1, 2, 3. \]

Let
\[ H_4 := \{[w_0 : w_1 : w_2] \mid w_0 + w_1 + w_2 = 0\}; \] and
\[ H_5 := \{[w_0 : w_1 : w_2] \mid w_0 + 2w_1 + 3w_2 = 0\}. \]

Clearly, hyperplanes $H_1, \ldots, H_5$ are in general position. Since $\mathcal{F}_4$ consists of non-constant holomorphic curves $f_j : \mathbb{C} \to \mathbb{P}^2$, $\mathcal{F}_4$ cannot omit $5(=2n+1)$ hyperplanes in general position in $\mathbb{P}^2$. Therefore, the Montel–Carathéodory theorem cannot give any information about the normality of $\mathcal{F}_4$.

We now show, by using Theorem 1.2 and Theorem 1.5, that the family $\mathcal{F}_4$ is normal. Clearly, $f_j^{-1}(H_k) = \emptyset$ for all $j \in \mathbb{Z}_+$ and $k = 1, 2, 3, 4$. Thus,
• Condition (a) of Theorem \ref{thm:nf1} is satisfied.
• Conditions (a) & (b) of Theorem \ref{thm:nf2} are satisfied after fixing $t = 4$.

Since $f_j^{-1}(H_4) = \emptyset$, the condition (b) of Theorem \ref{thm:nf2} hold true for $H_4$. We now show that Condition (b) of Theorem \ref{thm:nf2} holds true for $H_5$:

We first consider the following reduced representation of $f_j$

$$\tilde{0}_j^*(z) := (1, e^{-z}/j, -e^{-z}/j), \quad z \in \mathbb{C}. $$

Then, we have

$$\tilde{0}_j^* H_5(z) = 1 - e^{-z}/j; \quad (\tilde{0}_j^* H_5)'(z) = e^{-z}/j; \quad \text{and} \quad (\tilde{0}_j^* H_5)''(z) = -e^{-z}/j. $$

For any compact set $K \subset \mathbb{C}$, there exists an $R = R(K) > 0$ such that $K \subset B(0, R)$, where $B(0, R)$ is the open disc centered at the origin with radius $R$. Therefore, we have

$$|f_j^*(H_5)^{(q)}(z)| = e^{-z}/j |z| \leq e^R \quad \text{for all } j \in \mathbb{Z}_+, \ z \in B(0, R) \text{ and } q = 1, 2.$$  

Choose $M \geq e^R$, whence Condition (b) of Theorem \ref{thm:nf2} is satisfied in this case. We now consider the following reduced representation of $f_j$

$$\tilde{1}_j(z) := (je^z, 1, -1), \quad z \in \mathbb{C}. $$

Then, we have

$$\tilde{1}_j^* H_5(z) = je^z - 1 \quad \text{and} \quad (\tilde{1}_j^* H_5)'(z) = je^z = (\tilde{1}_j^* H_5)''(z).$$

Clearly, for each $z \in f_j^{-1}(H_5)$, we have $e^z = 1/j$. This implies that

$$|(\tilde{1}_j^* H_5)^{(q)}(z)| = 1 \quad \text{for all } z \in f_j^{-1}(H_5) \text{ and } q = 1, 2, $$

Similar calculation holds for the reduced representation $\tilde{2}_j(z) := (-je^z, -1, 1)$. Thus, Condition (b) of Theorem \ref{thm:nf2} is satisfied for $H_5$ and hence $\mathcal{F}_4$ is normal.

Now we show that Condition (c) of Theorem \ref{thm:nf2} is also satisfied. Consider the reduced representations $\tilde{f}_j(z) := (e^z, 1/j, -1/j)$ of $f_j$ on $\mathbb{C}$, $j \in \mathbb{Z}_+$. Clearly, $f_j(\mathbb{C}) \not\subset H_5$ and for any compact subset $K \subset \mathbb{C}$ the volumes of $f_j^{-1}(H_5) \cap K$ are uniformly bounded, which is evident from the following observation: For a closed ball $\overline{B}(0, R)$ of radius $R > 0$ centered at $0$, the cardinality of the set of zeros of $\tilde{f}_j^* H_5$ is bounded by $[R/\pi] + 1$ for all $j \in \mathbb{Z}_+$. Where, $[\bullet]$ is the ceiling function. Thus, Condition (c) of Theorem \ref{thm:nf2} is also satisfied, therefore $\mathcal{F}_4$ is normal. Hence we conclude that $\mathcal{F}_4$ is normal by Theorem \ref{thm:nf2} as well as by Theorem \ref{thm:nf4} but we cannot conclude the normality of $\mathcal{F}_4$ from the Montel–Carathéodory theorem.

4. Essential lemmas

In order to prove our theorems, we need to state certain known results and prove some essential lemmas.

One of the well-known tools in the theory of normal families in one complex variable is Zalcman’s lemma. Roughly speaking, it says that the failure of normality implies that a certain kind of infinitesimal convergence must take place. The higher dimensional analogue of Zalcman’s rescaling lemma is as follows:

**Lemma 4.1** ([1] Theorem 3.1, [7] Corollary 2.8). Let $M$ be a compact complex space, and $\mathcal{F}$ a family of holomorphic mappings from a domain $D \subset \mathbb{C}^m$ into $M$. The family $\mathcal{F}$ is not normal if and only if there exist

(a) a point $\xi_0 \in D$ and a sequence $\{\xi_j\} \subset D$ such that $\xi_j \to \xi_0$;
(b) a sequence $\{f_j\} \subset \mathcal{F}$;
(c) a sequence $\{r_j\} \subset \mathbb{R}$ with $r_j > 0$ and $r_j \to 0$. 


such that $h_j(\zeta) := f_j(\xi_j + r_j \zeta)$, where $\zeta \in \mathbb{C}^m$ satisfies $\xi_j + r_j \zeta \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant holomorphic mapping $h : \mathbb{C} \to M$.

Remark 4.2. We remark that the result of Aladro–Krantz in [1] has a weaker hypothesis than in Lemma 4.1. In the case where $M$ is non-compact, there is a case missing from their analysis. The arguments needed in this case were provided by [7, Theorem 2.5]. At any rate, Lemma 4.1 is the version of the Aladro–Krantz theorem that we need.

4.1. Results on Nevanlinna theory. Let $\nu$ be a non-negative divisor on $\mathbb{C}$. Given a $p \in \mathbb{Z}_+ \cup \{\infty\}$, we define the truncated counting function of $\nu$—multiplicities are truncated by $p$—by

$$N^p(r, \nu) := \int_{p}^{r} \frac{n^p(t)}{t} dt \quad (1 < r < +\infty),$$

where $n^p(t) := \sum_{|z| \leq t} \min\{\nu(z), p\}$.

Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve, and $H$ be a hyperplane in $\mathbb{P}^n$. If $f(\mathbb{C}) \not\subset H$ then we define

$$N_f^p(r) := N^p(r, \nu(f, H)); \quad \text{and} \quad N_f(r, H) := N_f^{[+\infty]}(r, H).$$

Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve. Let $\tilde{f} = (f_0, \ldots, f_n)$—for arbitrarily fixed homogeneous coordinates on $\mathbb{P}^n$—be a reduced representation of $f$. Then we define the characteristic function $T_f(r)$ of $f$ by

$$T_f(r) := \frac{1}{2\pi} \int_{0}^{2\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|,$$

where $\|f\| := (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$. We now state the following version of the First Main Theorem in Nevanlinna Theory which relates the characteristic function and the counting function.

First Main Theorem ([5, Theorem 2.3.31]). Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve and $H$ be a hyperplane such that $f(\mathbb{C}) \not\subset H$. Then

$$N_f(r, H) \leq T_f(r) + O(1) \quad \text{for all} \quad r > 1.$$

Generalizing H. Cartan’s work, Nochka established the following Second Main Theorem of Nevanlinna theory for $s$-nondegenerate holomorphic curves into $\mathbb{P}^n$ (see [4]). Following Nochka, a holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n$ is said to be $s$-nondegenerate, $0 \leq s \leq n$, if the dimension of the smallest linear subspace containing the image $f(\mathbb{C})$ is $s$.

Second Main Theorem ([4,5 Theorem 4.2.11]). Let $f : \mathbb{C} \to \mathbb{P}^n$ be an $s$-nondegenerate holomorphic curve, where $0 < s \leq n$. Let $H_k \not\subset f(\mathbb{C})$, $1 \leq k \leq q$, be hyperplanes in general position in $\mathbb{P}^n$. Then the following estimate

$$(q - 2n + s - 1)T_f(r) \leq \sum_{k=1}^{q} N_f^{[s]}(r, H_k) + O(\log(rT_f(r)))$$

holds for all $r$ excluding a subset of $(1, +\infty)$ of finite Lebesgue measure.

Nochka established the following Picard’s type results for holomorphic curves into $\mathbb{P}^n$.

Lemma 4.3 ([4,5 Corollary 4.2.15]). Let $H_1, \ldots, H_q$ be $q(\geq 2n + 1)$ hyperplanes in general position in $\mathbb{P}^n$. Suppose that $m_1, \ldots, m_q \subset \mathbb{Z}_+ \cup \{+\infty\}$. If

$$\sum_{k=1}^{q} \left(1 - \frac{n}{m_k}\right) > n + 1,$$

then there does not exist a non-constant holomorphic mapping $f : \mathbb{C} \to \mathbb{P}^n$ such that $f$ intersects $H_k$ with multiplicity at least $m_k$, $k = 1, \ldots, q$. 


The following lemma asserts that the logarithmic growth of the characteristic function is attained for rational mappings and only for them. Recall that a mapping \( f : \mathbb{C} \to \mathbb{P}^n \) is rational if \( f \) can be represented, in homogeneous coordinates, as \( f = [f_0 : \cdots : f_n] \) with polynomial coordinates \( f_l, l = 0, \ldots , n \).

**Lemma 4.4** ([3, Page 42, (7.3)]). Let \( f : \mathbb{C} \to \mathbb{P}^n \) be a holomorphic curve. Then \( f \) is rational if and only if
\[
\lim_{r \to \infty} \frac{T_f(r)}{\log r} < \infty.
\]

Yang–Fang–Pang [9] — by using the above lemma — established the following result wherein a holomorphic curve that intersects with \( 2n + 1 \) hyperplanes reduces to a rational map into \( \mathbb{P}^n \).

**Lemma 4.5** ([3, Lemma 3.7]). Let \( f : \mathbb{C} \to \mathbb{P}^n \) be a holomorphic curve, and let \( H_1, \ldots , H_{2n+1} \) be hyperplanes in general position in \( \mathbb{P}^n \). Suppose that for each \( H_k \in \{H_1, \ldots , H_{2n+1}\} \) either
(a) \( f(\mathbb{C}) \subset H_k \); or
(b) \( f(\mathbb{C}) \not\subset H_k \) and \( \supp \nu(f, H_k) \) is of finite cardinality in \( \mathbb{C} \).

Then the map \( f \) is rational.

We prove the following proposition in a similar spirit as Lemma 4.5. We shall use this proposition in order to prove Theorem 1.2.

**Proposition 4.6.** Let \( f : \mathbb{C} \to \mathbb{P}^n \) be a holomorphic curve, and \( H_1, \ldots , H_{2n+1} \) be hyperplanes in general position in \( \mathbb{P}^n \). Suppose that for each \( H_k \), either
(a) \( f(\mathbb{C}) \subset H_k \); or
(b) \( f(\mathbb{C}) \not\subset H_k \) and one of the following holds:
(i) \( f \) intersects \( H_k \) with multiplicity at least \( m_k \), where \( m_k \in \mathbb{Z}_+ \cup \{+\infty\} \); or
(ii) \( \supp \nu(f, H_k) \) is of finite cardinality in \( \mathbb{C} \).

Write \( \mathcal{I} := \{k \in \{1, \ldots , 2n+1\} : \supp \nu(f, H_k) \text{ is of finite cardinality in } \mathbb{C} \} \). Suppose
\[
\sum_{k \in \{1, \ldots , 2n+1\} \setminus \mathcal{I}} \frac{1}{m_k} < 1.
\]

Then \( f \) is rational.

**Proof.** Let \( \tilde{f} \) be a reduced representation of \( f \) on \( \mathbb{C} \). We may, without loss of generality, assume that \( f \) is an \( s \)-nondegenerate holomorphic curve, \( 0 \leq s \leq n \). If \( s = 0 \), then \( f \) is constant and hence rational. We therefore assume that \( s > 0 \). We divide the set \( \{1, \ldots , 2n+1\} \) into three disjoint subsets
\[
I_1 := \{k \in \{1, \ldots , 2n+1\} : f(\mathbb{C}) \subset H_k \};
I_2 := \mathcal{I};
I_3 := \{1, \ldots , 2n+1\} \setminus (I_1 \cup I_2).
\]
If \( I_3 = \emptyset \), then the conclusion follows from Lemma 4.5. And if \( I_2 = \emptyset \), then it follows from Lemma 4.3 — owing to the discussions in Section 2 — that \( f \) intersects \( H_k \) with the multiplicity \(+\infty\) if \( f(\mathbb{C}) \subset H_k \) or \( f(\mathbb{C}) \cap H_k = \emptyset \), and with the understanding that \( 1/(+\infty) = 0 \) — that \( f \) is a constant map. If \( I_2 \cup I_3 = \emptyset \), then again \( f \) is constant because \( H_1, \ldots , H_{2n+1} \) are in general position. Therefore, we assume that \( I_2 \neq \emptyset \neq I_3 \) and consider the following:

**Case 1.** \( I_1 = \emptyset \), i.e., \( f(\mathbb{C}) \not\subset H_k \) for each \( k \in \{1, \ldots , 2n+1\} \). It follows from the Second Main Theorem for \( s \)-nondegenerate holomorphic curve that the following inequality holds for all \( r \) outside a set of finite Lebesgue measure:
\[
sT_f(r) \leq \sum_{k=1}^{2n+1} N_f^{[s]}(r, H_k) + O(\log(rT_f(r)))
\]
There exists a neighborhood \( U \) of \( f \) initially made for the chosen \( H \), and we consider \( \tilde{f}^{s} H_{k} \) for \( k \in I_{2} \). Since \( s > 0 \) and \( \sum_{k \in I_{2}} (1/m_{k}) < 1 \), we have

\[
\lim_{r \to \infty} \frac{\log r}{\log T_{f}(r)} < \infty.
\]

Hence, by Lemma 4.4, \( f \) is rational.

**Case 2.** \( I_{1} \neq \emptyset \). Put

\[
X_{I_{1}} := \bigcap_{k \in I_{1}} H_{k}.
\]

Clearly, \( f(\mathbb{C}) \subset X_{I_{1}} \), and we can identify \( X_{I_{1}} \) with a projective space of dimension \((n-t)\), where \( t = |I_{1}| \). Again, for \( k \not\in I_{1} \), the restrictions \( H_{k} \cap X_{I_{1}} =: H_{k}^{s} \) are in general positions in \( X_{I_{1}} = \mathbb{P}^{n-t} \). At this stage, we appeal to the Second Main Theorem which yields the following inequality

\[
(s + t)T_{f}(r) = (2n + 1 - t - 2(n-t) + s - 1)T_{f}(r)
\leq \sum_{k \not\in I_{1}} N_{f}^{s}(r, H_{k}^{s}) + O(\log(T_{f}(r)))).
\]

Combining this inequality with conditions \((b(i)), (b(ii))\) and the First Main Theorem for holomorphic curves, we get the following inequality

\[
\left( s + t - \sum_{k \in I_{3}} (s/m_{k}) \right) T_{f}(r) \leq O(\log r) + O(\log(T_{f}(r)))).
\]

Since \( s + t - \sum_{k \in I_{3}} (s/m_{k}) > 0 \), the above inequality, together with Lemma 4.4, implies that \( f \) is rational.

We will use the following easily provable lemma—we will not discuss the proof of this lemma—in order to prove Theorem 1.2.

**Lemma 4.7.** Let \( p \) be a non-constant polynomial in \( \mathbb{C} \), and \( n \in \mathbb{Z}_{+} \). If \( p \) has a non-zero root of multiplicity \( n \) then \( p \) consists of at least \( n + 1 \) terms in its expression.

5. **The proof of Theorem 1.2**

**Proof of Theorem 1.2.** We begin the proof by setting

\[ H_{a} := \{H_{1}, \ldots, H_{n+1}\} \quad \text{and} \quad H_{b} := \{H_{n+2}, \ldots, H_{2n+1}\}. \]

We shall prove that \( F \) is normal at each point \( z \in D \). Fix an arbitrary point \( z_{0} \in D \) and a holomorphic curve \( f \in F \). The collection \( H_{a} \) is partitioned into two subsets

\[ H_{1}^{1}(f) := \{H \in H_{a} : f(z_{0}) \notin H\} \quad \text{and} \quad H_{2}^{1}(f) := \{H \in H_{a} : f(z_{0}) \in H\}. \]

There exists a neighborhood \( U_{z_{0}} \subset D \) such that

- if \( H \in H_{2}^{1}(f) \) then \( f(U_{z_{0}}) \cap H = \emptyset \); and
- if \( H \in H_{1}^{1}(f) \) and \( f(D) \not\subseteq H \) then \( f(U_{z_{0}}) \cap H = \{f(z_{0})\} \).

This allows us to divide \( H_{a} \) into three disjoint subsets \( H_{1}, H_{2}, H_{3} \) as follows:

\[
\begin{align*}
H_{1} &:= \{H \in H_{a} : f(U_{z_{0}}) \subseteq H\}; \\
H_{2} &:= \{H \in H_{a} : f(U_{z_{0}}) \cap H = \emptyset\}; \\
H_{3} &:= \{H \in H_{a} : f(U_{z_{0}}) \subseteq H \quad \text{and} \quad f(U_{z_{0}}) \cap H = f(z_{0})\}.
\end{align*}
\]

By the condition \((a)\) of Theorem 1.2, we see that the above choice of \( U_{z_{0}} \)—which we had initially made for the chosen \( f \)—is independent of the choice of \( f \in F \). Hence, the sets \( H_{i}, \ i = 1, 2, 3, \) are independent of \( f \in F \) as well.
We now fix an arbitrary sequence \( \{ f_j \} \). We shall prove that the sequence \( \{ f_j|_{U_{z_0}} \} \) has a subsequence that converges uniformly on compact subsets of \( U_{z_0} \). Let us assume that this is not true, and aim for a contradiction. Then, by Lemma 11 there exist a point \( z_0' \in U_{z_0} \) and

(i) a subsequence of \( \{ f_j|_{U_{z_0}} \} \), which we may label without causing confusion as \( \{ f_j|_{U_{z_0}} \} \);

(ii) a sequence \( \{ z_j \} \subset U_{z_0} \) such that \( z_j \to z_0' \);

(iii) a sequence \( \{ r_j \} \subset \mathbb{R} \) with \( r_j > 0 \) such that \( r_j \to 0 \)

such that — defining the maps \( h_j : \zeta \mapsto f_j(z_j + r_j \zeta) \) on suitable neighborhoods of \( 0 \in \mathbb{C} — \{ h_j \} \) converges uniformly on compact subsets of \( \mathbb{C} \) to a non-constant holomorphic mapping \( h : \mathbb{C} \to \mathbb{P}^n \). Then, there exist reduced representations

\[
\tilde{f}_j = (f_{j,0}, f_{j,1}, \ldots, f_{j,n}); \quad \tilde{h}_j = (h_{j,0}, h_{j,1}, \ldots, h_{j,n}); \quad \text{and} \quad \tilde{h} = (h_0, h_1, \ldots, h_n)
\]

(5.1)
of \( f_j, h_j, \) and \( h \) respectively (that are defined globally on their respective domains of definition) such that \( \tilde{h}_j(\zeta) := \tilde{f}_j(z_j + r_j \zeta) \to \tilde{h}(\zeta) \) uniformly on compact subsets of \( \mathbb{C} \)—see [2, §5-6]. This implies that \( \{ \tilde{h}_j^* H_k \} \) converges uniformly on compact subsets of \( \mathbb{C} \) to \( \tilde{h}^* H_k \), \( 1 \leq k \leq 2n + 1 \). Clearly, \( \tilde{h}^* H_k \) is holomorphic on the entire complex plane \( \mathbb{C} \) for all \( k = 1, \ldots, 2n + 1 \). Recall that we defined \( f^* H \), where \( H \) is any hyperplane, in Section 2.

We, now, aim for the following **claim 1:** For each hyperplane \( H \in \mathcal{H}_a \), either \( h(\mathbb{C}) \subset H \) or \( \text{supp } \nu(h, H) \) consists of at most one point in \( \mathbb{C} \).

Clearly, by definitions of \( \mathcal{H}_1, \mathcal{H}_2 \) and Hurwitz’s Theorem, for each \( H \in \mathcal{H}_1 \cup \mathcal{H}_2 \), one of the following holds:

\[
h(\mathbb{C}) \subset H \quad \text{or} \quad h(\mathbb{C}) \cap H = \emptyset. \tag{5.2}
\]

If \( \mathcal{H}_3 = \emptyset \) then our claim is established. We may thus assume that \( \mathcal{H}_3 \neq \emptyset \) and fix an arbitrary hyperplane \( H \in \mathcal{H}_3 \). We may, without loss of generality, assume that \( h^* H \neq c \), where \( c \in \mathbb{C} \). We shall prove that \( \tilde{h}^* H \) has a unique zero. Suppose on the contrary that \( \tilde{h}^* H \) has at least two distinct zeros in \( \mathbb{C} \). Let \( \zeta_1 \) and \( \zeta_2 \) be any two distinct zeros of \( \tilde{h}^* H \) in \( \mathbb{C} \), then there exist two disjoint neighborhoods \( U_{\zeta_1} \ni \zeta_1 \) and \( U_{\zeta_2} \ni \zeta_2 \) in \( \mathbb{C} \) such that \( \tilde{h}^* H \) vanishes only at \( \zeta_1 \) and \( \zeta_2 \) in \( U_{\zeta_1} \cup U_{\zeta_2} \). By Hurwitz’s Theorem, there exist sequences \( \{ \zeta_j^1 \} \) and \( \{ \zeta_j^2 \} \) converging to \( \zeta_1 \) and \( \zeta_2 \) respectively such that for \( j \) sufficiently large we have

\[
\tilde{h}_j^* H(\zeta_j^1) = \tilde{f}_j^* H(z_j + r_j \zeta_j^1) = 0 \quad \text{and} \quad \tilde{h}_j^* H(\zeta_j^2) = \tilde{f}_j^* H(z_j + r_j \zeta_j^2) = 0.
\]

By Condition (a) of Theorem 1.2 for each \( f_i \) in \( \mathcal{F} \) we have

\[
\tilde{f}_i^* H(z_j + r_j \zeta_j^1) = 0 \quad \text{and} \quad \tilde{f}_i^* H(z_j + r_j \zeta_j^2) = 0.
\]

Now we fix \( i \) and let \( j \to \infty \), and notice that \( z_j + r_j \zeta_j^1 \to z_0' \) and \( z_j + r_j \zeta_j^2 \to z_0' \). Thus we get \( \tilde{f}_i^* H(z_0') = 0 \), whence we deduce, from the definition of \( \mathcal{H}_3 \), that (whenever \( \mathcal{H}_3 \neq \emptyset \) ) \( z_0' = z_0 \). Since the zeros of non-constant univariate holomorphic functions are isolated, we have \( z_j + r_j \zeta_j^1 = z_0 \) and \( z_j + r_j \zeta_j^2 = z_0 \) for sufficiently large \( j \). Hence \( \zeta_j^1 = (z_0 - z_j)/r_j = \zeta_j^2 \), which is not possible as \( \zeta_j^1 \to \zeta_1^1 \) and \( \zeta_j^2 \to \zeta_2^1 \), but \( \zeta_1^1 \neq \zeta_2^1 \). Therefore \( \tilde{h}^* H \) has a unique zero, and hence \( \text{supp } \nu(h, H) \) consists of only one point. This, together with \( [5,2] \) establishes **claim 1.** At this point, we make a further observation that will be important. Set

\[
\xi_0 := \lim_{j \to \infty} \frac{z_0 - z_j}{r_j}.
\]

From the above discussion, we notice that \( \xi_0 \) is the unique zero of \( \tilde{h}^* H \) in \( \mathbb{C} \), where \( H \) is as chosen at the beginning of this paragraph. Now, as the parameters \( z_0, z_j \) and \( r_j \) are independent of \( H \in \mathcal{H}_3 \), we get

\[(*) \quad \text{If } \mathcal{H}_3 \neq \emptyset \text{ then there exists a } \xi_0 \subset \mathbb{C} \text{ that is the unique zero of } \tilde{h}^* H \text{ for each } H \in \mathcal{H}_3. \]
We, next, aim for the following **claim 2**: For each hyperplane \(H \in \mathcal{H}_b\), the map \(h\) intersects \(H\) with multiplicity at least \(n + 1\).

Fix an arbitrary hyperplane \(H \in \mathcal{H}_b\). If \(f_j(U_{z_0}) \cap H = \emptyset\) for sufficiently large \(j\), then by Hurwitz’s Theorem, either \(\tilde{h}^*H \equiv 0\) or \(\tilde{h}^*H\) is non-vanishing on \(\mathbb{C}\). Hence the map \(h\) intersects \(H\) with the multiplicity \(\infty\) and our claim is established. Thus, we may assume that \(\tilde{h}^*H \not\equiv 0\) and \(\tilde{h}^*H\) has zeros in \(\mathbb{C}\). Suppose \(\zeta_0 \in \mathbb{C}\) is such that \(\tilde{h}^*H(\zeta_0) = 0\). As \(h_0, \ldots, h_n\) cannot simultaneously vanish, there exists an index \(l_0 \in \{0, \ldots, n\}\) and an open disc \(B(\zeta_0, r) \subset \mathbb{C}\), centred at \(\zeta_0\) with radius \(r\), such that — fixing homogeneous coordinates on \(\mathbb{P}^n\) — the image \(h(B(\zeta_0, r))\) is a subset of \(V_{l_0} := \{[w_0 : \cdots : w_n] \in \mathbb{P}^n : w_{l_0} \neq 0\}\). Thus, in the reduced representation \((h_0, \ldots, h_n)\) of \(h\) (given in (5.1)) we have \(h_{l_0}(\zeta) \neq 0\) on \(B(\zeta_0, r)\). Hurwitz’s Theorem implies that \(h_{j, l_0}(\zeta) := f_j, l_0(z_j + r_j\zeta) \neq 0\) on \(B(\zeta_0, r)\) when \(j\) is large enough. Now, we consider the following reduced representations \(\tilde{h}_j = (h_{j, 0}/h_{j, l_0}, \ldots, h_{j, n}/h_{j, l_0})\) of \(h_j\) and \(\tilde{h} = (h_0/h_{l_0}, \ldots, h_n/h_{l_0})\) of \(h\) on \(B(\zeta_0, r)\). Thus, for every \(j\) large enough, we get connected open subsets \(U_j \ni z_0'\) of \(D\) and reduced representations \(\tilde{f}_j\) of \(f_j\) defined on \(U_j\) such that the \(l_0\)-th coordinate of \(\tilde{f}_j\) is identically \(1\) on \(U_j\), and \(\tilde{h}_j(\zeta) = \tilde{f}_j(z_j + r_j\zeta)\) on \(B(\zeta_0, r)\). It is clear that \(\tilde{h}_j \to \tilde{h}\) converges uniformly on compact subsets of \(B(\zeta_0, r)\). Hence \(\tilde{h}_j^*H \to \tilde{h}^*H\) converges uniformly on compact subsets of \(B(\zeta_0, r)\), which further implies that

\[
(\tilde{h}_j^*H)^{(q)} \to (\tilde{h}^*H)^{(q)}
\]  

(5.3)

converges uniformly on compact subsets of \(B(\zeta_0, r)\) for all \(q \in \mathbb{Z}_+\).

At this stage, we notice that \(z_j + r_j\zeta\) belongs to a compact subset \(K_0 \ni z_0'\) of \(D\) for all \(\zeta \in B(\zeta_0, r)\) if \(j\) is sufficiently large — this holds true because \(z_j \to z_0'\), \(r_j \to 0\), and \(B(\zeta_0, r) \subset \mathbb{C}\). From Condition (b) of Theorem 2.2 there exists a positive integer \(M\) such that

\[
\left| (\tilde{f}_j^*H)^{(q)}(z) \right|_{1 \leq q \leq n} \leq M, \quad z \in f_j^{-1}(H) \cap K_0,
\]

hold for \(H\) (which we had fixed in the previous paragraph) and \(j \in \mathbb{Z}_+\). Recall that \(\tilde{h}^*H(\zeta_0) = 0\). By Hurwitz’s Theorem, we get a sequence \(\zeta_j \to \zeta_0\) such that \(\tilde{h}_j^*H(\zeta_j) = \tilde{f}_j^*H(z_j + r_j\zeta_j) = 0\) for \(j\) sufficiently large. Which then implies that for sufficiently large \(j\) we have \(z_j + r_j\zeta_j \in f_j^{-1}(H)\). Hence for \(j\) large enough we get that

\[
\left| (\tilde{f}_j^*H)^{(q)}(z_j + r_j\zeta_j) \right| \leq M
\]

for all \(q = 1, \ldots, n\). This further implies that for all \(q = 1, \ldots, n\), we have

\[
\left| (\tilde{h}_j^*H(\zeta_j))^{(q)}(\zeta = \zeta_j) \right| = |r_j^q(\tilde{f}_j^*H)^{(q)}(z_j + r_j\zeta_j)| \leq r_j^qM.
\]  

(5.4)

Combining (5.3) and (5.4), together, we have

\[
(\tilde{h}^*H)^{(q)}(\zeta_0) = \lim_{j \to \infty} (\tilde{h}_j^*H)^{(q)}(\zeta_j) = 0
\]

for all \(q = 1, \ldots, n\). Hence \(\zeta_0\) is a zero of \(\tilde{h}^*H\) of multiplicity at least \(n + 1\). This confirms that \(h\) intersects \(H\) with multiplicity at least \(n + 1\). Since \(H \in \mathcal{H}_b\) was arbitrarily chosen, the latter conclusion holds true for each \(H \in \mathcal{H}_b\), which establishes **claim 2**.

At this stage, we appeal to Proposition 4.6 — in view of **claim 1** and **claim 2** — to conclude that \(h\) is **rational**. We now analyze what consequence this has for \(\tilde{h}^*H\) when \(H \in \mathcal{H}_a\). From (*), (5.2), and the fact that each \(H \in \mathcal{H}_a\) is a coordinate hyperplane, we deduce that there exist \(m_l \in \mathbb{Z}_+, b_l \in \mathbb{C} \setminus \{0\}\), and \(c_l \in \mathbb{C}\) such that either

\[
\tilde{h}^*H_{l+1} = b_l(z - \zeta_0)^{m_l}, \text{ or } \tilde{h}^*H_{l+1} = c_l \text{ for } l = 0, \ldots, n
\]
f ∈ F is normal at each point. A standard diagonal argument gives us a subsequence of \( D \), a collection of open sets.

Proof of Theorem 1.5. Let \( \mathcal{H} \) be a normal family of holomorphic maps. We shall prove that \( \mathcal{H} \) is normal in every neighborhood of a compact subset of \( D \).

Recall now that \( \mathcal{H} \) was chosen arbitrarily. Since we can cover \( D \) by a countable collection of open sets \( U \),

- where \( U \) varies through some countable dense subset of \( D \); and
- \( U \) is such that it has the properties given by the two bullet-points at the beginning of this proof;

a standard diagonal argument gives us a subsequence of \( \{ f_j \} \) that converges uniformly on compact subsets of \( D \). This completes the proof.

\[ \square \]

6. The Proof of Theorem 1.5

Proof of Theorem 1.5. We begin the proof by dividing the set \( \{ H_1, \ldots, H_{2n+1} \} \) of hyperplanes into three disjoint subsets

\[ \mathcal{H}_a := \{ H_1, \ldots, H_{n+1} \}; \quad \mathcal{H}_b := \{ H_{n+2}, \ldots, H_{2n+1} \}; \quad \text{and} \quad \mathcal{H}_c := \{ H_{n+1}, \ldots, H_{2n+1} \}; \]

where \( t \) is a positive integer such that \( n+2 \leq t < 2n+1 \), and \( n \geq 2 \). Existence of \( t \) is ensured by the assumption stated just after condition (a) of the Theorem 1.5. We shall prove that \( \mathcal{F} \) is normal at each point \( z \in D \). Fix an arbitrary point \( z_0 \in D \) and a holomorphic curve \( f \in \mathcal{F} \). Now we repeat the argument, \textit{mutatis mutandis}, in the first paragraph of the proof.
of Theorem 1.2 to get a neighborhood $U_{z_0} \ni z_0$ in $D$ so that $\mathcal{H}_a \cup \mathcal{H}_b = \{H_1, \ldots, H_1\}$ can be divided into the following three disjoint (not necessarily non-empty) subsets

$$
\mathcal{H}_1 := \{H \in \mathcal{H}_a \cup \mathcal{H}_b : f(U_{z_0}) \cap H = \emptyset\};
\mathcal{H}_2 := \{H \in \mathcal{H}_b : f(U_{z_0}) \subseteq H\};
\mathcal{H}_3 := \{H \in \mathcal{H}_b : f(U_{z_0}) \nsubseteq H \text{ and } f(U_{z_0}) \cap H = \{f(z_0)\}\}.
$$

Clearly, $\mathcal{H}_a \subset \mathcal{H}_1$, whence $\mathcal{H}_1 \neq \emptyset$. By the conditions (a) and (b) of Theorem 1.3 we deduce that the open set $U_{z_0}$ and the sets $\mathcal{H}_i$, $i = 1, 2, 3$, are independent of $f \in \mathcal{F}$.

We now fix an arbitrary sequence $\{f_j\} \subset \mathcal{F}$. We shall prove that the sequence $\{f_j|_{U_{z_0}}\}$ has a subsequence that converges compactly on $U_{z_0}$. Let us assume that the latter is not true, and aim for a contradiction. Then, by Lemma 4.1 there exist a point $z'_0 \in U_{z_0}$ and

(i) a subsequence of $\{f_j|_{U_{z_0}}\}$, which we may label without causing confusion as $\{f_j|_{U_{z_0}}\}$;

(ii) a sequence $\{z_j\} \subset U_{z_0}$ such that $z_j \to z'_0$;

(iii) a sequence $\{r_j\} \subset \mathbb{R}$ with $r_j > 0$ such that $r_j \to 0$

such that — defining the maps $h_j : \zeta \mapsto f_j(z_j + r_j\zeta)$ on suitable neighborhoods of $0 \in \mathbb{C}$ — $\{h_j\}$ converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant holomorphic mapping $h : \mathbb{C} \to \mathbb{C}^n$. Then, there exist reduced representations

$$
\tilde{f}_j = (f_{j,0}, f_{j,1}, \ldots, f_{j,n}); \quad \tilde{h}_j = (h_{j,0}, h_{j,1}, \ldots, h_{j,n}); \quad \text{and} \quad \tilde{h} = (h_0, h_1, \ldots, h_n)
$$

of $f_j$, $h_j$, and $h$ respectively (that are defined globally on their respective domains of definition) such that $\tilde{h}_j(\zeta) := \tilde{f}_j(z_j + r_j\zeta) \to \tilde{h}(\zeta)$ uniformly on compact subsets of $\mathbb{C}$. Clearly, by the definitions of $\mathcal{H}_1$ and $\mathcal{H}_2$, and by Hurwitz’s Theorem, for each $H \in \mathcal{H}_1 \cup \mathcal{H}_2$, one of the following holds:

$$
\begin{align*}
\text{Claim: For each hyperplane } H & \text{ in } \mathcal{H}_3 \cup \mathcal{H}_c, \text{ either } h(\mathbb{C}) \subset H \\
\text{or } \tilde{h}^*H & \text{ has at most finitely many zeros in } \mathbb{C}.
\end{align*}
$$

\textbf{Claim.} For each hyperplane $H$ in $\mathcal{H}_3 \cup \mathcal{H}_c$, either $h(\mathbb{C}) \subset H$ or $\tilde{h}^*H$ has at most finitely many zeros in $\mathbb{C}$. If $\mathcal{H}_3 = \emptyset$, then our claim is established for the case where hyperplanes are in $\mathcal{H}_3$. Thus, we may assume that $\mathcal{H}_3 \neq \emptyset$ and fix an arbitrary hyperplane $H \in \mathcal{H}_3$. Now we repeat verbatim the argument that we used to prove claim 1 in the proof of Theorem 1.2 to establish our claim in this case. We now aim for establishing the claim for the case where hyperplanes are in $\mathcal{H}_c$. Fix an arbitrary hyperplane $H \in \mathcal{H}_c$. If $f_j(U_{z_0}) \cap H = \emptyset$ for sufficiently large $j$, then by Hurwitz’s Theorem, either $h(\mathbb{C}) \subset H$ or $\tilde{h}^*H$ has no zero in $\mathbb{C}$, whence our claim is established. Thus, we may assume that $\tilde{h}^*H \neq 0$ on $\mathbb{C}$. At this stage, we notice that $z_j + r_j\zeta$ belong to a compact subset $K_0 \ni z'_0$ of $D$ for all $\zeta$ contained in a compact subset of $\mathbb{C}$ if $j$ is large enough — this holds true because $\zeta$ lies in a compact subset of $\mathbb{C}$, $z_j \to z'_0$, and $r_j \to 0$. From Condition (c) of Theorem 1.5 there exists a finite integer $M \in \mathbb{Z}_+$ such that the set of zeros, counting with multiplicities, of $f_j^*H$ in $K_0$ is of cardinality less than $M$ for all $j$.

This implies that the cardinality of the set of zeros, counting with multiplicities, of $\tilde{h}^*H$ — in every compact subsets of $\mathbb{C}$, and hence in $\mathbb{C}$ — is less than the positive integer $M$ for all $j$. By the virtue of uniform convergence and the Argument Principle (or, by using Hurwitz’s Theorem), we conclude that $\tilde{h}^*H$ has finitely many (less than $M$ in numbers) zeros in $\mathbb{C}$, which establishes our claim.

From Lemma 4.5—in view of (6.1) and the claim—we conclude that $h$ is rational. This implies that $\tilde{h}^*H_k$ is a polynomial for each $k = 1, \ldots, 2n + 1$. Therefore, we deduce, from (6.1), that there exist constants $c_k \in \mathbb{C}$ such that $\tilde{h}^*H_k \equiv c_k$ for $H_k \in \mathcal{H}_1 \cup \mathcal{H}_2$. Recall that the hyperplanes are in general position and the cardinality of $\mathcal{H}_1 \cup \mathcal{H}_2$ is at least $n + 1$. Thus, for any subset $\mathcal{H}'' \subset \mathcal{H}_1 \cup \mathcal{H}_2$ of cardinality $n + 1$ the following system of equations:

$$
\tilde{h}^*H_k = c_k, \quad H_k \in \mathcal{H}''
$$
confirms that $h_0, \ldots, h_n$ must be constant. Hence, $h$ is a constant map. Which is a contradiction. Thus the sequence $\{ f_j | u_{z_0} \}$ has a subsequence that converges uniformly on compact subsets of $U_{z_0}$. We are now in a position to repeat verbatim the argument in the final paragraph of the proof of Theorem 1.2 to conclude that $\{ f_j \}$ has a subsequence that converges uniformly on compact subsets of $D$. This completes the proof.

\[ \Box \]

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References

[1] G. Aladro and S. G. Krantz, \textit{A criterion for normality in C"{n}}, J. Math. Anal. Appl. \textbf{161} (1991), 1-8.
[2] J. Dufresnoy, \textit{Théorie nouvelle des familles complexes normales: Applications à l'étude des fonctions algébroïdes}, Ann. Sci. École Norm. Sup. (3) \textbf{61} (1944), 1-44.
[3] H. Fujimoto, \textit{On families of meromorphic maps into the complex projective space}, Nagoya Math. J. \textbf{54} (1974), 21-51.
[4] E. I. Nochka, \textit{On the theory of meromorphic functions}, Sov. Math. Dokl. \textbf{27} (1983), 377-381.
[5] J. Noguchi and J. Winkelmann, \textit{Nevanlinna Theory in Several Complex Variables and Diophantine Approximation}, Springer Japan (2014).
[6] W. Stoll, \textit{Die beiden Hauptsätze der Wertverteilungstheorie bei Funktionen mehrerer komplexer Ver"{a}nderlichen (II)}, Acta Math. \textbf{92} (1954), 55-169.
[7] D. D. Thai, P. N. T. Trang, and P. D. Huong, \textit{Families of normal maps in several complex variables and hyperbolicity of complex spaces}, Complex Var. Theory Appl. \textbf{48} (2003), 469-482.
[8] Yan Xu, \textit{Montel’s criterion and shared function}, Publ. Math. Debrecen \textbf{77} (2010), no. 3-4, 471-478.
[9] L. Yang, C. Fang, and X. Pang, \textit{Normal families of holomorphic mappings into complex projective space concerning shared hyperplanes}, Pacific J. Math. \textbf{272} (2014), 245-256.

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