Quantum spin chains and integrable many-body systems of classical mechanics

A. Zabrodin *

September 2014

Abstract

This note is a review of the recently revealed intriguing connection between integrable quantum spin chains and integrable many-body systems of classical mechanics. The essence of this connection lies in the fact that the spectral problem for quantum Hamiltonians of the former models is closely related to a sort of inverse spectral problem for Lax matrices of the latter ones. For simplicity, we focus on the most transparent and familiar case of spin chains on $N$ sites constructed by means of the $GL(2)$-invariant $R$-matrix. They are related to the classical Ruijsenaars-Schneider system of $N$ particles, which is known to be an integrable deformation of the Calogero-Moser system. As an explicit example the case $N = 2$ is considered in detail.

1 Introduction

In this paper we present some results of [1]-[4] in a short compressed form and in the simplest possible setting. First of all let us explain what we mean by “quantum spin chains” and “integrable many-body systems of classical mechanics”.

The best known example of integrable quantum spin chain is the isotropic (XXX) homogeneous Heisenberg model with spin $\frac{1}{2}$ on a 1D lattice with coupling between nearest neighbours. Throughout the paper, we use the words “spin chain” in a broader sense, not implying existence of any local Hamiltonian of the Heisenberg type. In fact integrable local Hamiltonians in general do not exist for inhomogeneous spin chains which are closely involved in our story. However, such models still make sense as generalized spin chains with long-range interaction and a family of commuting (non-local) Hamiltonians. We call them inhomogeneous XXX spin

*Institute of Biochemical Physics, 4 Kosygina, 119334, Moscow, Russia; ITEP, 25 B. Cheremushkinskaya, 117218, Moscow, Russia; National Research University Higher School of Economics, International Laboratory of Representation Theory and Mathematical Physics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia
chains. Alternatively, one may prefer to keep in mind inhomogeneous integrable lattice models of statistical mechanics rather than spin chains as such. In either case, the final goal of the theory is diagonalization of transfer matrices which are generating functions of commuting conserved quantities. This is usually achieved by one or another version of the Bethe ansatz method.

The integrable model of classical mechanics we are mainly interested in is the $N$-body system of particles on the line called the Ruijsenaars-Schneider (RS) model [5]. It is often referred to as an integrable relativistic deformation of the famous Calogero-Moser (CM) model with inversely quadratic pair potential [6, 7].

As is common for integrable models, the classical dynamics can be represented in the Lax form, i.e., as an isospectral deformation of a $N \times N$ matrix called the Lax matrix. Matrix elements of this matrix are simple functions of coordinates and momenta of the particles while the eigenvalues are integrals of motion. In a nutshell, the essence of the quantum-classical (QC) duality

$$\text{Quantum integrable models} \leftrightarrow \text{Classical many-body systems}. \quad (1)$$

lies in the fact that spectra of quantum Hamiltonians of a model from the left hand side appear to be encoded in the algebraic properties of the Lax matrix for a classical system from the right hand side.

In the case of the inhomogeneous XXX spin chain, a refined version of (1) is

$$\text{Quantum XXX spin-}\frac{1}{2}\text{ chain on }N\text{ sites} \leftrightarrow \text{Classical }N\text{-body RS model}. \quad (2)$$

More precisely, the spectral problem for the quantum Hamiltonians of the inhomogeneous XXX spin chain on $N$ sites is reduced to a sort of an inverse spectral problem for the $N \times N$ Lax matrix for the classical RS system. Given its spectrum and the coordinates of the particles, the problem is to find possible values of their momenta compatible with these data. In general this problem has many solutions which just yield different eigenvalues of the quantum Hamiltonians. In a special scaling limit, the XXX spin chain turns into the Gaudin spin model [8]. On the right hand side of (2), this corresponds to the non-relativistic limit of the RS system:

$$\text{Quantum Gaudin model} \leftrightarrow \text{Classical CM model}. \quad (3)$$

The QC duality is traced back to [9], where joint spectra of some finite-dimensional operators were linked to the classical Toda chain. The existence of an unexpected link between the quantum Gaudin and the classical CM models was first pointed out in [10], see also [11]. In a more general set-up, the correspondence between quantum and classical integrable systems was independently derived [11, 12, 13] as a corollary of an embedding of the commutative algebra of spin chain Hamiltonians into an infinite integrable hierarchy of soliton equations known as the modified Kadomtsev-Petviashvili (mKP) hierarchy. Namely, the most general generating function of commuting integrals of motion of the spin chain (the “master $T$-operator”) was shown to satisfy the bilinear identity and the Hirota bilinear equations for the tau-function of the mKP hierarchy [14].

Although only a limited number of examples are available at the moment, the very phenomenon of the existence of hidden non-standard connections between quantum and classical integrable systems seems to be rather general. Presumably, it can be thought of as a new kind of a correspondence (or duality) principle in the realm of integrable systems. In [11], the QC duality (2), (3) was checked directly using the Bethe ansatz solution of integrable spin chains. The role of this duality in the context of supersymmetric gauge theories and branes was discussed in [15, 16, 4].
It is worthwhile to stress that both sides of the correspondence, i.e., quantum and classical integrable systems, participate in the game as two faces of one entity on an equal-rights basis. In the theory of quantum models, there are some fundamental relations, exact for any $\hbar \neq 0$, which assume the form of classical equations of motion for some other system. (One of such examples is the classical integrable dynamics naturally realized in the space of conserved quantities of quantum integrable models, see [1] and earlier works [17, 18].) At the same time, given a many-body problem of classical mechanics, one may extract from it, by addressing some non-traditional questions about the system, the spectral properties of a quantum model. This picture becomes valid and meaningful if the systems from both sides are integrable. It might be interesting to combine the hypothetical “correspondence principle” based on the QC duality with the standard correspondence principle of quantum mechanics.

Let us outline the contents of the paper.

In section 2, we start with the most familiar example of integrable spin chain: the Heisenberg model with spin $\frac{1}{2}$ and periodic boundary conditions (the XXX magnet) solved by H.Bethe in 1931 [19]. The “spin variables” are vectors from the spaces $\mathbb{C}^2$ at each site. However, this model itself is too degenerate to be directly linked to a classical many-body system. To this end, we need an inhomogeneous version of the model with twisted boundary conditions. Such a generalized XXX model has $N + 2$ free parameters which are $N$ “inhomogeneity parameters” on each site and 2 eigenvalues of the twist matrix which is assumed to be diagonal. The generalized XXX model can be naturally constructed in the framework of the Quantum Inverse Scattering Method (QISM) developed by the former Leningrad school [20, 21]. In the inhomogeneous model, the locality of spin interactions does not take place. Instead, there are $N$ non-local commuting Hamiltonians (which are cousins of the Gaudin ones). They can be simultaneously diagonalized using the algebraic Bethe ansatz.

In section 3, the necessary formulae related to the classical RS model are presented, including the Lax matrix. The rules of the quantum-classical correspondence between the integrable models are explained in section 4. As an example we consider the case $N = 2$, where all calculations can be done directly by hands (section 5). Finally, in section 6 we give some remarks on the scaling limit to the Gaudin model which corresponds, on the classical side, to the non-relativistic limit of the RS system. Some generalizations and perspectives are briefly discussed in the concluding section 7.

## 2 The Heisenberg spin chain and its generalizations

The Hamiltonian of the isotropic Heisenberg spin chain (also called the XXX-magnet) with periodic boundary condition is

$$H_{\text{xxx}} = 2 \sum_{j=1}^{N} \left( s_x^{(j)} s_x^{(j+1)} + s_y^{(j)} s_y^{(j+1)} + s_z^{(j)} s_z^{(j+1)} - I \right), \quad N + 1 \equiv 1,$$

where the spin operators $(s_x, s_y, s_z) = \vec{s}$ are expressed through the Pauli matrices as

$$s_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $I = 1^\otimes N$ is the identity operator. (Hereafter $1$ stands for the identity matrix in $\mathbb{C}^2$). We will also use $s_+ = s_x + is_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $s_- = s_x - is_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $s_1 = \frac{1}{2} I + s_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
and $s_2 = \frac{1}{2} \mathbf{1} - s_z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The operator $\mathbf{s}^{(j)} = \mathbf{1} \otimes (j-1) \otimes \mathbf{s} \otimes \mathbf{1} \otimes (N-j)$ acts non-trivially at the $j$th site of the chain. Clearly, they commute for any $j \neq j'$. The Hamiltonian acts in the $2^N$-dimensional linear space $\mathcal{V} = \otimes_{j=1}^N V_j$, $V_j \cong \mathbb{C}^2$. Basis vectors in this space can be constructed as tensor products of local vectors with definite $z$-projection of spin, i.e., eigenvectors of $s_z$.

Note that $P_{ij} = \frac{1}{2} (\mathbf{1} + 4s^{(i)}s^{(j)})$ is the permutation operator of the $i$th and $j$th spaces, and so the Heisenberg Hamiltonian can be written in the form $H_{xxx} = \sum_j P_{j,j+1} - N\mathbf{1}$. The Hamiltonian commutes with the operator $M = \frac{1}{2} \sum_{j=1}^N (\mathbf{1} - 2s_z^{(j)}) = \sum_{j=1}^N s_z^{(j)}$, which counts the total number of spins in the chain with negative $z$-projection. Namely, the states in which $M$ spins look down (and so the rest $N - M$ spins look up) are eigenstates for the operator $M$ with the eigenvalue $M$. The space of states $\mathcal{V}$ is decomposed in the direct sum of eigenspaces for the operator $M$: $\mathcal{V} = \bigoplus_{M=0}^N \mathcal{V}(M)$, $M\mathcal{V}(M) = M\mathcal{V}(M)$. It is clear that

$$\dim \mathcal{V}(M) = \binom{N}{M} = \frac{N!}{M!(N-M)!}.$$ 

In particular, $\mathcal{V}(0)$ and $\mathcal{V}(N)$ are one-dimensional spaces generated by the states in which all spins look up or down respectively.

The common spectral problem for the operators $H_{xxx}$ and $M$, $H_{xxx} \Psi = E \Psi$, $M \Psi = M \Psi$, has the famous Bethe ansatz solution [19]. The eigenvalues $E$ for $0 \leq M \leq [N/2]$ are given by the formula

$$E = \sum_{\alpha=1}^M \varepsilon(v_\alpha), \quad \varepsilon(v) = -\frac{4}{1 + 4v^2},$$

where the auxiliary quantities $v_\alpha$ (the Bethe roots) are to be found from the system of algebraic equations

$$\left(\frac{v_\alpha + i}{v_\alpha - i}\right)^N = \prod_{\beta=1,\beta \neq \alpha}^M \frac{v_\alpha - v_\beta + i}{v_\alpha - v_\beta - i}$$

(the Bethe equations). Different solutions to this system give energies of different eigenstates.

The exact solution of the Heisenberg spin chain is possible due to the fact that the model is integrable. This means that there is a sufficiently large family of independent commuting operators, one of which is the Heisenberg Hamiltonian. The other operators of this family are higher integrals of motion. A general prescription how to construct models possessing higher integrals of motion is provided by the Quantum Inverse Scattering Method (QISM) [20].

We start by reformulating the XXX spin chain in the framework of the QISM, following [21]. Such a reformulation makes integrability of the model explicit and, what is even more important, it suggests natural integrable generalizations of the XXX chain.

Let $V_0 \cong \mathbb{C}^2$ be another copy of the complex linear space $\mathbb{C}^2$ (the auxiliary space). The quantum Lax operator at the $j$th site acts non-trivially in $V_0 \otimes V_j$. It is

$$L_j(x) = x\mathbf{1} \otimes \mathbf{1} + \eta P_{0j} = \left(x + \frac{\eta}{2}\right) \mathbf{1} \otimes \mathbf{1} + 2\eta \mathbf{s} \otimes \mathbf{s},$$
or, in the block-matrix form,

\[ L_j(x) = \begin{pmatrix} xI + \eta s_1^{(j)} & \eta s_2^{(j)} \\ \eta s_2^{(j)} & xI + \eta s_1^{(j)} \end{pmatrix}. \tag{8} \]

The variable \( x \in \mathbb{C} \) is called the (quantum) spectral parameter. The extra parameter \( \eta \) introduced here for the reason clarified below is not actually essential because it can be eliminated by a rescaling of the spectral parameter (unless one tends it to 0 as in the limit to the Gaudin model \cite{SS}). The Heisenberg Hamiltonian does not depend on \( \eta \) which is usually put equal to \( i = \sqrt{-1} \) in this context. The \( L \)-operator satisfies the “\( RLL = LLR \)” intertwining relation

\[ R(x - x') L_j(x) \otimes L_j(x') = L_j(x') \otimes L_j(x) R(x - x'), \]

where the quantum \( R \)-matrix \( R(x) \) acts in the tensor product of two auxiliary spaces \( V_0 \cong V_0' \cong \mathbb{C}^2 \). In the natural basis in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) it is

\[ R(x) = \begin{pmatrix} x + \eta & 0 & 0 & 0 \\ 0 & \eta & x & 0 \\ 0 & x & \eta & 0 \\ 0 & 0 & 0 & x + \eta \end{pmatrix} = \eta I \otimes 1 + xP_{00'}. \tag{9} \]

Note that in this particular case the \( R \)-matrix is almost the same object as the quantum \( L \)-operator: they differ only by a permutation operator of the two spaces, so that the intertwining relation is equivalent to the Yang-Baxter equation for the \( R \)-matrix. The quantum transfer matrix is defined as

\[ T(x) = \text{tr}_0 \left[ L_1(x) L_2(x) \ldots L_N(x) \right] = 2I x^N + J_{N-1} x^{N-1} + \ldots + J_1 x + J_0. \tag{10} \]

The intertwining relation implies that the transfer matrices with different spectral parameters (and the same \( \eta \)) commute: \([T(x), T(x')]=0\) for any \( x, x' \). In its turn, this implies that the operators \( J_k \) in (11) all commute with each other. At the same time, the operator \( J_0 \) is proportional to the cyclic permutation of the chain:

\[ J_0 = T(0) = \eta^N P_{12} P_{23} P_{34} \ldots P_{N-1} N P_{N1} \]

while the Hamiltonian of the spin chain is given by

\[ H^{xxx} = \eta \frac{d}{dx} \log T(x) \bigg|_{x=0} - NI = \eta J_0^{-1} J_1 - NI. \]

The operators \( J_0^{-1} J_k \) are then the higher integrals of motion. They are local due to the special property of the quantum Lax operator \( L_j(0) = \eta P_{0j} \) and the homogeneity of the chain. The operator \( M \) (see (11)) commutes not only with \( H^{xxx} \) but with the whole one-parametric family \( T(x) \), and the Bethe states are common eigenstates for the \( T(x) \) and \( M \): \( T(x) \Psi = T(x) \Psi, M \Psi = M \Psi \).

The transfer matrix \( T(x) \) can be diagonalized by means of the algebraic Bethe ansatz method. The eigenvalues \( T(x) \) are given by the formula

\[ T(x) = (x + \eta)^N \prod_{\alpha=1}^{M} \frac{x - u_\alpha - \eta}{x - u_\alpha} + x^N \prod_{\alpha=1}^{M} \frac{x - u_\alpha + \eta}{x - u_\alpha}. \tag{11} \]
The Bethe roots $u_\alpha$ are to be found from the system of Bethe equations
\begin{equation}
\left(\frac{u_\alpha + \eta}{u_\alpha}\right)^N = \prod_{\beta=1, \beta \neq \alpha}^M \frac{u_\alpha - u_\beta + \eta}{u_\alpha - u_\beta - \eta},
\end{equation}
where is implied that $0 \leq M \leq [N/2]$. The eigenvalues of the Heisenberg Hamiltonian, in terms of the Bethe roots, are given by the formula
\begin{equation}
E = \sum_{\alpha=1}^M \frac{\eta^2}{u_\alpha(u_\alpha + \eta)},
\end{equation}
which is equivalent to (5) under the substitution $v_\alpha = \frac{iu_\alpha}{\eta} + \frac{i}{2}$.

The XXX model can be generalized, preserving integrability, in two ways: a) by making it inhomogeneous and b) by imposing twisted boundary conditions. The former is based on the possibility to introduce an inhomogeneity parameter at each site which does not spoil the intertwining relation:
\begin{equation}
R(x - x') L_j(x - x) \otimes L_j(x' - x_j) = L_j(x' - x_j) \otimes L_j(x - x_j) R(x - x').
\end{equation}
The latter is due to the $GL(2)$-invariance of the $R$-matrix [9]: $g \otimes g R(x) = R(x) g \otimes g$ for any $g \in GL(2)$. This property implies that commutativity of the transfer matrices still holds if one inserts a matrix $g \in GL(2)$ in the auxiliary space before taking trace. For simplicity, we assume that $g$ is diagonal:
\begin{equation}
g = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}.
\end{equation}
The generalizations a) and b) can be applied simultaneously, which leads to the most general one-parametric family of commuting operator-valued polynomials in $x$:
\begin{equation}
T(x) = T(x; g, \eta, \{x_j\}) = \text{tr}_0 \left[ g L_1(x - x) L_2(x - x_2) \ldots L_N(x - x_N) \right].
\end{equation}
These operators commute for different $x$’s and the same $\eta, g$ and $x_j$:
\begin{equation}
[T(x; g, \eta, \{x_j\}), T(x'; g, \eta, \{x_j\})] = 0.
\end{equation}
Similarly to (15), one can expand
\begin{equation}
T(x) = I \text{tr} g x^N + J_{N-1} x^{N-1} + \ldots + J_1 x + J_0,
\end{equation}
the $J_k$’s being commuting integrals of motion. Note, in particular, that $J_{N-1} = \eta \sum_i g^{(i)}$, where $g^{(i)}$ is the operator acting as the matrix $g$ at the $i$th site: $g^{(i)} := 1^{\otimes (i-1)} \otimes g \otimes 1^{\otimes (N-i)}$. In general there is no way to construct local Hamiltonians from the $J_k$’s. Instead, assuming that all the $x_j$’s are distinct and in general position (meaning that $x_i - x_j \neq \pm \eta$ for all $i, j$), one can define non-local Hamiltonians as residues of $T(x) / \prod_{j=1}^N (x - x_j)$ (cf. [22]):
\begin{equation}
\frac{T(x)}{\prod_{j=1}^N (x - x_j)} = \text{tr} g \cdot I + \sum_{j=1}^N \frac{\eta H_j}{x - x_j}
\end{equation}
In general, the Hamiltonians $H_j = H_j(\eta, g, \{x_i\})$ imply a long-range interaction involving all spins in the chain. Their explicit form is
\begin{equation}
H_i = \prod_{j=i+1}^N \left( I + \frac{\eta P_{ij}}{x_i - x_j} \right) g^{(i)} \prod_{j=1}^{i-1} \left( I + \frac{\eta P_{ij}}{x_i - x_j} \right),
\end{equation}
where we use the notation \( \prod_{j=1}^m A_j = A_1 A_2 \ldots A_m \) for the ordered product. It follows from the definition that \( \sum_{j=1}^N H_j = \sum_{j=1}^N g^{(j)} \).

The operator \( M(4) \) still commutes with \( T(x) \) and all the \( H_j \)'s, so, again, all these operators are diagonalized simultaneously: \( T(x) \Psi = T(x) \Psi, H_j \Psi = H_j \Psi, M \Psi = M \Psi \). The algebraic Bethe ansatz gives the following result. The eigenvalues \( T(x) \) and \( H_j \) are given by the formulae

\[
T(x) = w_1 \prod_{k=1}^N (x - x_k + \eta) \prod_{\alpha=1}^M \frac{x - u_\alpha - \eta}{x - u_\alpha},
\]

\[
H_j = w_1 \prod_{k=1, k \neq j}^N \frac{x_j - x_k + \eta}{x_j - x_k} \prod_{\alpha=1}^M \frac{x_j - u_\alpha - \eta}{x_j - u_\alpha}.
\]

The Bethe roots \( u_\alpha \) are to be found from the system of Bethe equations

\[
\frac{w_1}{w_2} \prod_{k=1}^N \frac{u_\alpha - x_k + \eta}{u_\alpha - x_k} = \prod_{\beta=1, \beta \neq \alpha}^M \frac{u_\alpha - u_\beta + \eta}{u_\alpha - u_\beta - \eta},
\]

where it is implied that \( 0 \leq M \leq [N/2] \).

3 The Ruijsenaars-Schneider model

The RS model [5] is an integrable model of classical mechanics. It is an \( N \)-body system of interacting particles on the line with the Hamiltonian

\[
H_{1}^{RS} = \eta^{-1} \sum_{i=1}^N e^{-\eta p_i} \prod_{k=1, k \neq i}^N \frac{x_i - x_k + \eta}{x_i - x_k}.
\]

For some reason it is often called the relativistic deformation of the Calogero-Moser model, the parameter \( \eta \) being the inverse "velocity of light". The Hamiltonian equations of motion

\[
\left( \begin{array}{c} \dot{x}_i \\ \dot{p}_i \end{array} \right) = \left( \begin{array}{c} \frac{\partial_{p_i} H_1^{RS}}{\partial_{x_i} H_1^{RS}} \\ -\partial_{x_i} H_1^{RS} \end{array} \right)
\]

give the following connection between velocity and momentum

\[
\dot{x}_i = -e^{-\eta p_i} \prod_{k=1, k \neq i}^N \frac{x_i - x_k + \eta}{x_i - x_k}
\]

and the equations of motion

\[
\ddot{x}_i = -\sum_{k \neq i} \frac{2 \eta^2 \dot{x}_i \dot{x}_k}{(x_i - x_k)((x_i - x_k)^2 - \eta^2)}, \quad i = 1, \ldots, N.
\]

The RS model is known to be integrable, with the higher integrals of motion in involution being given by \( H_k^{RS} = \eta^{-1} \text{tr} (Y^{RS})^k \), where \( Y^{RS} = Y^{RS} (\{x_i\}; \{\dot{x}_i\}) \) is the Lax matrix of the
model. Its matrix elements are $Y_{ij}^\text{RS} = \frac{\eta \dot{x}_i}{x_i - x_j - \eta}$, i.e.,

$$Y^\text{RS} = \begin{pmatrix} -\dot{x}_1 & \frac{\eta \dot{x}_1}{x_1 - x_2 - \eta} & \frac{\eta \dot{x}_1}{x_1 - x_3 - \eta} & \cdots & \frac{\eta \dot{x}_1}{x_1 - x_N - \eta} \\ \frac{\eta \dot{x}_2}{x_2 - x_1 - \eta} & -\dot{x}_2 & \frac{\eta \dot{x}_2}{x_2 - x_3 - \eta} & \cdots & \frac{\eta \dot{x}_2}{x_2 - x_N - \eta} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta \dot{x}_N}{x_N - x_1 - \eta} & \frac{\eta \dot{x}_N}{x_N - x_2 - \eta} & \frac{\eta \dot{x}_N}{x_N - x_3 - \eta} & \cdots & -\dot{x}_N \end{pmatrix}. \quad (23)$$

Equations of motion (22) are equivalent to the Lax equation $\dot{Y}^\text{RS} = [B, Y^\text{RS}]$, where

$$B_{ij} = \left( \sum_{k \neq i} \frac{\eta \dot{x}_k}{x_i - x_k} - \sum_k \frac{\eta \dot{x}_k}{x_i - x_k + \eta} \right) \delta_{ij} + \frac{\eta \dot{x}_i}{x_i - x_j} (1 - \delta_{ij}).$$

The Lax equation implies that all eigenvalues of the Lax matrix are integrals of motion.

Let $X = \text{diag}(x_1, x_2, \ldots, x_N)$ be the diagonal matrix with the diagonal entries being coordinates of the particles. It is easy to check that the matrices $X$, $Y^\text{RS}$ satisfy the commutation relation

$$[X, Y^\text{RS}] = \eta Y^\text{RS} + \eta \dot{X} E,$$

where $E$ is the $N \times N$ matrix of rank 1 with all entries equal to 1. Note also that the Lax matrix $Y^\text{RS}$ can be represented in the form

$$Y^\text{RS} = \dot{X} C,$$

where $C$ is the Cauchy matrix $C_{ij} = \frac{\eta}{x_i - x_j - \eta}$.

### 4 The quantum-classical duality

Consider the Lax matrix (23) of the $N$-particle RS model, where the $x_i$'s are identified with the inhomogeneity parameters $x_i$ at the sites of the spin chain and the inverse “velocity of light”, $\eta$, is identified with the parameter $\eta$ introduced in the quantum $L$-operator (8). Let us also substitute $\dot{x}_i = -H_i$:

$$Y^\text{RS} = \begin{pmatrix} H_1 & \frac{\eta H_1}{x_1 - x_2 + \eta} & \frac{\eta H_1}{x_1 - x_3 + \eta} & \cdots & \frac{\eta H_1}{x_1 - x_N + \eta} \\ \frac{\eta H_2}{x_1 - x_2 + \eta} & H_2 & \frac{\eta H_2}{x_2 - x_3 + \eta} & \cdots & \frac{\eta H_2}{x_2 - x_N + \eta} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta H_N}{x_1 - x_N + \eta} & \frac{\eta H_N}{x_2 - x_N + \eta} & \frac{\eta H_N}{x_3 - x_N + \eta} & \cdots & H_N \end{pmatrix}. \quad (26)$$

The decomposition (25) for the matrix (26) acquires the form

$$Y^\text{RS} = -HC,$$
where \( H = \text{diag}(H_1, H_2, \ldots, H_N) \).

The claim is that if the \( H_i \)'s are eigenvalues of the Hamiltonians of the spin chain in the invariant subspace \( \mathcal{V}(M) \), then the first \( N - M \) eigenvalues of this matrix coincide with eigenvalues of the twist matrix \( w_1 \) while the rest \( M \) eigenvalues coincide with \( w_2 \):

\[
\text{Spec}(Y^{RS}) = (w_1, \ldots, w_1, w_2, \ldots, w_2).
\]  

(28)

This means that the values of the higher RS Hamiltonians are

\[
\eta H^{RS}_k = (N - M)w_1^k + Mw_2^k.
\]  

(29)

In general, the matrix \( Y^{RS} \) with multiple eigenvalues is not diagonalizable and contains Jordan cells.

To put it somewhat differently, one can say that the eigenstates of the quantum spin chain Hamiltonians correspond to the intersection points of two Lagrangian submanifolds in the phase space of the RS model. One of them is the hyperplane defined by fixing all the coordinates \( x_i \) while the other one is the Lagrangian submanifold obtained by fixing values \( H^{RS}_k \) of the \( N \) integrals of motion in involution. In general, there are many such intersection points numbered by a finite set \( I \), with coordinates, say \( (x_1, \ldots, x_N, p_1^{(\alpha)}, \ldots, p_N^{(\alpha)}) \), \( \alpha \in I \). The values of \( p_j^{(\alpha)} \) give, through equation (21), the spectrum of \( H_j \):

\[
H_j^{(\alpha)} = e^{-\eta p_j^{(\alpha)}} \prod_{k=1, k \neq j}^{N} \frac{x_j - x_k + \eta}{x_j - x_k}.
\]

However, we can not claim that all the intersection points correspond to the energy levels of the spin chain Hamiltonians. The example of \( N = 2 \) considered below in detail suggests that some intersection points do not correspond to the energy levels of a given spin chain. Their meaning is to be clarified.

Anyway, the spectral problem for the non-local inhomogeneous spin chain Hamiltonians \( H_j \) in the subspace \( \mathcal{V}(M) \) appears to be closely linked to the following inverse spectral problem for the RS Lax matrix \( Y^{RS} \) of the form (26). Let us fix the spectrum of the matrix \( Y^{RS} \) to be (28), where \( w_1, w_2 \) are eigenvalues of the (diagonal) twist matrix \( g \). Then we ask what is the set of possible values of the \( H_j \)'s allowed by these constraints. The eigenvalues \( H_j \) of the quantum Hamiltonians are contained in this set.

A similar correspondence between quantum and classical integrable systems was suggested in [10], see also [11]. In a more general set-up, this assertion was derived [1, 3, 12, 13] as a corollary of the embedding of the spin chain into an infinite integrable hierarchy of non-linear PDE's. In [14], it was checked directly using the Bethe ansatz solution.

In order to find the characteristic polynomial of the matrix (26) explicitly, we use the well known fact that the coefficient in front of \( \lambda^{N-k} \) in the polynomial \( \det_{N \times N} (\lambda + A) \) equals the sum of all diagonal \( k \times k \) minors of the matrix \( A \). All such minors can be found using decomposition (27) and the explicit expression for the determinant of the Cauchy matrix:

\[
\det_{1 \leq i, j \leq n} \frac{\eta}{x_i - x_j - \eta} = (-1)^n \prod_{1 \leq i < j \leq n} \left( 1 - \frac{\eta^2}{(x_i - x_j)^2} \right)^{-1}.
\]

The result is:

\[
\det_{N \times N} (\lambda - Y) = \det_{N \times N} (\lambda - H \mathcal{C}) = \sum_{n=0}^{N} \mathcal{J}_n \lambda^{N-n},
\]  

(30)
where
\[
J_n = (-1)^n \sum_{1 \leq i_1 < \ldots < i_n \leq N} H_{i_1} \ldots H_{i_n} \prod_{1 \leq \alpha < \beta \leq n} \left(1 - \frac{\eta^2}{(x_{i_\alpha} - x_{i_\beta})^2}\right)^{-1}. \tag{31}
\]
In particular, the highest coefficient is given by the following simple formula:
\[
J_N = (-1)^N H_1 H_2 \ldots H_N \prod_{i < j} \left(1 - \frac{\eta^2}{(x_i - x_j)^2}\right)^{-1}.
\]
For completeness, we point out that the integrals \(H_k\) introduced in the previous section are connected with the integrals \(J_k\) by the Newton’s formula \[23\] \[
N \sum_{k=0}^{N} J_{N-k} H_k = 0 \quad (\text{we have set} \quad H_0 = \eta^{-1} \text{tr}(Y^{RS})^0 = N/\eta).
\]
Another way to write expressions (30), (31) is through a sum over \(\epsilon_1, \ldots, \epsilon_N\), with \(\epsilon_i \in \{0, 1\}\):
\[
\det_{N \times N}(\lambda I - Y) = \lambda^N \sum_{\{\epsilon_1, \ldots, \epsilon_N\} \in \mathbb{Z}_2^N} N \prod_{i=1}^{N} (-H_i/\lambda_i)^{\epsilon_i} \prod_{1 \leq j < k \leq N} \left(1 - \frac{\eta^2}{(x_j - x_k)^2}\right)^{-\epsilon_j \epsilon_k}. \tag{32}
\]
The similarity of these expressions with tau-functions for \(N\)-soliton solutions to the KP hierarchy is not accidental. This point will be discussed elsewhere.

We conclude this section by writing down the system of algebraic equations for spectra of the operators \(H_i\). Combining (28) and (31), we obtain \(N\) polynomial equations for \(N\) unknown quantities \(H_1, \ldots, H_N:\)
\[
\sum_{1 \leq i_1 < \ldots < i_n \leq N} H_{i_1} \ldots H_{i_n} \prod_{1 \leq \alpha < \beta \leq n} \left(1 - \frac{\eta^2}{(x_{i_\alpha} - x_{i_\beta})^2}\right)^{-1} = C_n(N, M), \tag{33}
\]
where \(C_n(N, M) = \frac{1}{2\pi i} \oint_{|z|=1} (1 + zw_1)^{N-M}(1 + zw_2)^{M} z^{-n-1} dz, \ n = 1, 2, \ldots, N. \) Let us emphasize that in contrast to the Bethe ansatz solution, the algebraic equations are written here not for some auxiliary quantities like Bethe roots but for the spectrum itself.

The state where all spins look up \((M = 0)\) is an obvious eigenvector of the operators \(H_i\) with the eigenvalues
\[
H_i = w_1 \prod_{j=1, j \neq i}^{N} \left(1 + \frac{\eta}{x_i - x_j}\right). \tag{34}
\]
One can check that these \(H_i\)'s indeed solve the system (33) with \(C_n(N, 0) = \frac{N!w_1^n}{n!(N-n)!}. \)

5 Examples: \(N = 1\) and \(N = 2\)

The case \(N = 1\) is trivial. The only quantum Hamiltonian \(H_1\) is diagonal in the standard basis of \(\mathbb{C}^2\) and coincides with the twist matrix, so we have two eigenvalues: \(H_1 = w_1\) or \(H_1 = w_2\). The one-particle RS model is the model of a free particle on the line, the Lax “matrix” is just the number \(-\dot{x}_1\). Fixing it to be \(w_1\) or \(w_2\), as required by the QC duality, we obtain the two eigenvalues of \(H_1\) by the identification \(H_i = -\dot{x}_i\), see (25).
The case $N = 2$ is meaningful and instructive. First, let us find the spectrum of the quantum Hamiltonians directly. The transfer matrix is:

$$
T(x) = \text{tr} \left[ \begin{pmatrix}
    w_1 & 0 \\
    0 & w_2
\end{pmatrix} \begin{pmatrix}
    (x-x_1)I + \eta s_1^{(1)} & \eta s_1^{(-)} \\
    \eta s_1^{(1)} & (x-x_1)I + \eta s_1^{(2)}
\end{pmatrix} \begin{pmatrix}
    (x-x_2)I + \eta s_2^{(1)} & \eta s_2^{(-)} \\
    \eta s_2^{(1)} & (x-x_2)I + \eta s_2^{(2)}
\end{pmatrix} \right]
$$

A simple calculation gives the following explicit form of the Hamiltonians:

$$
H_1 = w_1 s_1^{(1)} + w_2 s_2^{(1)} + \frac{\eta w_1}{x_1 - x_2} (s_1^{(1)} s_1^{(2)} + s_1^{(-)} s_1^{(-)}) + \frac{\eta w_2}{x_2 - x_1} (s_2^{(1)} s_2^{(2)} + s_2^{(-)} s_2^{(-)}),
$$

$$
H_2 = w_1 s_1^{(2)} + w_2 s_2^{(2)} + \frac{\eta w_1}{x_2 - x_1} (s_1^{(1)} s_1^{(2)} + s_1^{(-)} s_1^{(-)}) + \frac{\eta w_2}{x_2 - x_1} (s_2^{(1)} s_2^{(2)} + s_2^{(-)} s_2^{(-)}).
$$

We see that $H_1 + H_2 = g^{(1)} + g^{(2)}$, as it should be. The space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is decomposed into the direct sum of the one-dimensional space $V(0)$ generated by the vector $|++\rangle$ ($M = 0$), two-dimensional space $V(1)$ generated by the vectors $|+-\rangle, |--\rangle$ ($M = 1$) and one-dimensional space $V(2)$ generated by the vector $|--\rangle$ ($M = 2$). We have:

$$
H_1 |++\rangle = w_1 \left(1 + \frac{\eta}{x_1 - x_2}\right) |++\rangle, \quad H_1 |--\rangle = w_2 \left(1 + \frac{\eta}{x_2 - x_1}\right) |--\rangle,
$$

$$
H_1 |+-\rangle = w_1 |+-\rangle + \frac{\eta w_1}{x_1 - x_2} |--\rangle,
$$

$$
H_1 |--\rangle = w_2 |--\rangle + \frac{\eta w_2}{x_2 - x_1} |+-\rangle.
$$

Here we use the usual notation for the basis vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$
|++\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |--\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and so on.}
$$

The vectors $|++\rangle$ and $|--\rangle$ are eigenvectors of $H_1$. The rest of the spectrum is found by diagonalizing the $2 \times 2$ matrix

$$
\left( \begin{array}{cc}
    w_1 & \frac{\eta w_2}{x_1 - x_2} \\
    \frac{\eta w_1}{x_2 - x_1} & w_2
\end{array} \right).
$$

The two eigenvalues are

$$
\frac{1}{2} \left( w_1 + w_2 \pm \sqrt{R} \right),
$$

where

$$
R = (w_1 - w_2)^2 + \frac{4\eta^2 w_1 w_2}{(x_1 - x_2)^2}.
$$

The final result for the joint spectrum of the operators $H_i$ is as follows:

$$
(H_1, H_2) = \begin{cases}
    \left( \begin{array}{cc}
        w_1 + \frac{\eta w_1}{x_1 - x_2}, & w_1 - \frac{\eta w_1}{x_1 - x_2} \\
        \frac{w_1 + w_2 + \sqrt{R}}{2}, & \frac{w_1 + w_2 - \sqrt{R}}{2}
    \end{array} \right), & M = 0, \\
    \left( \begin{array}{cc}
        \frac{w_1 + w_2 + \sqrt{R}}{2}, & \frac{w_1 + w_2 - \sqrt{R}}{2} \\
        2 & 2
    \end{array} \right), & M = 1, \\
    \left( \begin{array}{cc}
        \frac{w_1 + w_2 - \sqrt{R}}{2}, & \frac{w_1 + w_2 + \sqrt{R}}{2} \\
        2 & 2
    \end{array} \right), & M = 1, \\
    \left( \begin{array}{cc}
        \frac{w_1 + w_2}{x_1 - x_2}, & \frac{w_1 + w_2}{x_1 - x_2} \\
        \frac{w_2}{x_1 - x_2}, & \frac{w_2}{x_1 - x_2}
    \end{array} \right), & M = 2.
\end{cases}
$$

Note that in the case of the periodic boundary condition $w_1 = w_2 = 1$ the eigenvalue $H_1 = 1 + \frac{\eta}{x_1 - x_2}$ becomes 3-fold degenerate as it should be due to the $GL(2)$-invariance of the $R$-matrix.
Now consider the Lax matrix of the 2-particle RS model, where we substitute \( \dot{x}_i = -H_i \):

\[
Y = \begin{pmatrix} H_1 & \eta H_1 \\ \eta H_2 & H_2 \end{pmatrix},
\]

The characteristic equation \( \det(Y - \lambda I) = 0 \) reads

\[
\lambda^2 - (H_1 + H_2)\lambda + \frac{x_{12}^2 H_1 H_2}{x_{12}^2 - \eta^2} = 0,
\]

where \( x_{12} = x_1 - x_2 \) and the two eigenvalues are

\[
\frac{1}{2} \left( H_1 + H_2 \pm \sqrt{(H_1 + H_2)^2 - \frac{4x_{12}^2 H_1 H_2}{x_{12}^2 - \eta^2}} \right).
\]

In the subspace with \( M = 0 \) the eigenvalue of \( H_1 + H_2 \) is \( 2w_1 \) and the Lax matrix has the double eigenvalue \( w_1 \). This implies that the expression under the square root vanishes, i.e., we arrive at the system

\[
\begin{aligned}
H_1 + H_2 &= 2w_1 \\
H_1 H_2 &= w_1^2 \left( 1 - \frac{\eta^2}{x_{12}^2} \right)
\end{aligned}
\]

which is a particular case \( N = 2 \) of the general system (33). There are two solutions:

\[
(H_1, H_2) = \left( w_1 \pm \frac{\eta w_1}{x_1 - x_2}, w_1 \mp \frac{\eta w_1}{x_1 - x_2} \right), \quad M = 0.
\]

The choice of the upper sign corresponds to the first line in (35). The meaning of the other solution is to be clarified. In a similar way, for \( M = 2 \) we obtain two solutions

\[
(H_1, H_2) = \left( w_2 \pm \frac{\eta w_2}{x_1 - x_2}, w_2 \mp \frac{\eta w_2}{x_1 - x_2} \right), \quad M = 2,
\]

of which the one with the upper sign corresponds to the last line in (35). Finally, at \( M = 1 \) we have the system

\[
\begin{aligned}
H_1 + H_2 &= w_1 + w_2 \\
H_1 H_2 &= w_1 w_2 \left( 1 - \frac{\eta^2}{x_{12}^2} \right)
\end{aligned}
\]

There are two solutions which coincide with the second and the third lines in (35).

6 The limit to the quantum Gaudin model and the classical Calogero-Moser system

In the limit \( \eta \to 0 \) the QC duality discussed above becomes a correspondence (3) between the quantum Gaudin model and the classical Calogero-Moser system with inversely quadratic pair potential. Some details are given below.

The rational \( GL(2) \) Gaudin model [8] is the \( \eta \to 0 \) limit of the inhomogeneous spin chain with the transfer matrix \( T(x; e^{\eta h}, \eta, \{x_j\}) \). The expansion as \( \eta \to 0 \) gives:

\[
T(x; e^{\eta h}, \eta, \{x_j\}) = 2I + \eta \left( \text{tr } h + \sum_{i=1}^N \frac{1}{x - x_i} \right) I + \eta^2 \left( \frac{1}{2} \text{tr } h^2 I + \sum_{i=1}^N \frac{H^G_i}{x - x_i} \right) + O(\eta^3),
\]

12
where \( h = \left( \begin{array}{cc} \omega_1 & 0 \\ 0 & \omega_2 \end{array} \right) \) is the Gaudin analogue of the twist matrix, and

\[
\mathbf{H}^G_i = \lim_{\eta \to 0} \frac{\mathbf{H}_i(\eta, e^{\eta h}, \{x_j\}) - \mathbf{I}}{\eta} = h^{(i)} + \sum_{j \neq i} \frac{\mathbf{P}_{ij}}{x_i - x_j}
\]

\[
= \sum_{j \neq i} \frac{\mathbf{I}}{x_i - x_j} + h^{(i)} + 2 \sum_{j \neq i} \frac{\mathbf{s}^{(i)} \mathbf{s}^{(j)}}{x_i - x_j}
\]

(36)

are the Hamiltonians of the \( GL(2)-\)invariant Gaudin model. Here \( h^{(i)} = \frac{\omega_1 + \omega_2}{2} \mathbf{I} + (\omega_1 - \omega_2) \mathbf{s}^{(i)} \) is the twist matrix acting in the space \( V_i \cong \mathbb{C}^2 \) at the \( i \)th site. In the context of the Gaudin model, the parameters \( x_i \) (in general, complex numbers) are often called marked points of the Riemann sphere. Since the first two terms in the \( \eta \to 0 \) expansion of the \( \mathbf{T}(x; e^{\eta h}, \eta, \{x_j\}) \) are proportional to the identity operator and thus commute with everything, commutativity of the transfer matrices implies commutativity of the Gaudin Hamiltonians: \( [\mathbf{H}^G_i, \mathbf{H}^G_j] = 0 \). The Gaudin spectral problem consists in the simultaneous diagonalization of these operators and the operator \( \mathbf{M} \) which has the same form as above: \( \mathbf{H}^G \mathbf{\Psi} = \mathbf{H}^G \mathbf{\Psi}, \mathbf{M} \mathbf{\Psi} = \mathbf{M} \mathbf{\Psi} \). The Bethe ansatz solution is the \( \eta \to 0 \) limit of (18), (19):

\[
H^G_j = \omega_1 + \sum_{k \neq j} \frac{1}{x_j - x_k} + \sum_{\alpha=1}^{M} \frac{1}{u_\alpha - x_j},
\]

(37)

where the Bethe roots \( u_\alpha \) satisfy the system of equations

\[
\omega_1 - \omega_2 + \sum_{k=1}^{N} \frac{1}{u_k - x_k} = 2 \sum_{\beta=1, \neq \alpha}^{M} \frac{1}{u_\alpha - u_\beta}.
\]

(38)

An alternative solution is achieved via the QC duality with the classical CM model with the Hamiltonian \( \mathcal{H}^{CM} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{1}{(x_i - x_j)^2} \). The equations of motion are

\[
\ddot{x}_i = \sum_{k \neq i} \frac{2}{(x_i - x_k)^3}, \quad i = 1, \ldots, N.
\]

(39)

The CM model is known to be integrable, with the higher integrals of motion in involution being given by \( \mathcal{H}^{CM}_k = \frac{1}{k} \text{tr} (\mathcal{Y}^{CM})^k \) (\( \mathcal{H}^{CM}_1 \) being the total momentum \( \mathcal{P}^{CM} = \sum_j p_j \) and \( \mathcal{H}^{CM}_2 = \mathcal{H}^{CM} \)), where

\[
\mathcal{Y}^{CM}(\{x_i\}; \{\dot{x}_i\}) = \begin{pmatrix}
-\dot{x}_1 & \frac{1}{x_2 - x_1} & \frac{1}{x_3 - x_1} & \cdots & \frac{1}{x_N - x_1} \\
\frac{1}{x_1 - x_2} & -\dot{x}_2 & \frac{1}{x_3 - x_2} & \cdots & \frac{1}{x_N - x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{x_1 - x_N} & \frac{1}{x_2 - x_N} & \frac{1}{x_3 - x_N} & \cdots & -\dot{x}_N
\end{pmatrix}
\]

(40)

is the Lax matrix of the model. Its matrix elements are \( \mathcal{Y}^{CM}_{ij} = -\dot{x}_i \delta_{ij} - \frac{1 - \delta_{ij}}{x_i - x_j} \).
Note that the CM model can be treated as a $\eta \to 0$ limit of the RS model meaning that

$$Y^{\text{RS}} = I + \eta Y^{\text{CM}} + O(\eta^2), \quad H_1^{\text{RS}} = \frac{N}{\eta} + p^{\text{CM}} + \eta \tilde{H}^{\text{CM}} + O(\eta^2),$$

where $\tilde{H}^{\text{CM}} = \frac{1}{2} \sum_i \left( p_i + \sum_{k \neq i} \frac{1}{x_i - x_k} \right)^2 - \sum_{i < j} \frac{1}{(x_i - x_j)^2}$ differs from the $H^{\text{CM}}$ by a simple canonical transformation and leads to the same equations of motion.

The rules of the QC duality in this case are as follows [2, 4]. Consider the Lax matrix (40) of the $N$-particle CM model, where the $x_i$’s are identified with the $N$ marked points of the Gaudin model. Let us also substitute $\dot{x}_i = -H_i^G$:

$$Y^{\text{CM}}(\{x_1\}; \{-H_i\}) = \begin{pmatrix}
H_1^G & \frac{1}{x_2-x_1} & \frac{1}{x_3-x_1} & \cdots & \frac{1}{x_N-x_1} \\
\frac{1}{x_1-x_2} & H_2^G & \frac{1}{x_3-x_2} & \cdots & \frac{1}{x_N-x_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_1-x_N} & \frac{1}{x_2-x_N} & \frac{1}{x_3-x_N} & \cdots & H_N^G
\end{pmatrix}. \quad (41)$$

The claim is that if the $H_i^G$’s are eigenvalues of the Gaudin Hamiltonians in the invariant subspace $\mathcal{V}(M)$, then the first $N - M$ eigenvalues of this matrix coincide with eigenvalues of the twist matrix $\omega_1$ while the rest $M$ eigenvalues coincide with $\omega_2$:

$$\text{Spec} (Y^{\text{CM}}) = (\omega_1, \ldots, \omega_{N-M}, \omega_2, \ldots, \omega_2). \quad (42)$$

As it follows from the results of [24, 25], the characteristic polynomial of the matrix $Y^{\text{CM}}$ can be represented in the form

$$\det_{N \times N} (\lambda - Y^{\text{CM}}) = \exp \left( \sum_{i < j} \frac{\partial_y \partial_{y_j}}{(x_i - x_j)^2} \right) \prod_{k=1}^{N} (\lambda - y_k) \bigg|_{y_i = H_i^G} \quad (43)$$

Therefore, the spectrum consists of the values $(H_1, H_2, \ldots, H_N)$ such that the equality

$$\exp \left( \sum_{i < j} \frac{\partial_y \partial_{y_j}}{(x_i - x_j)^2} \right) \prod_{k=1}^{N} (\lambda - y_k) \bigg|_{y_i = H_i^G} = (\lambda - w_1)^{N-M} (\lambda - w_2)^M \quad (44)$$

is satisfied identically in $\lambda$. As in the case of the XXX model, this is equivalent to $N$ algebraic equations for $N$ quantities $H_i^G$.

### 7 Concluding remarks

The QC duality can be more or less straightforwardly extended to quantum inhomogeneous spin chains associated with $GL(n)$-invariant $R$-matrices. These models are solved by the nested
Bethe ansatz (see [26]). On the classical side, the correspondence is with the same rational RS model, with eigenvalues of the Lax matrix being chosen (with some multiplicities) from the elements of the $n \times n$ diagonal twist matrix. The corresponding results can be found in [1, 3, 4]. In the present paper, we have restricted ourselves by the $GL(2)$ case only because of the notational simplicity.

An interesting possible generalization is the $q$-deformation of the QC duality which implies the anisotropic spin chains with trigonometric $R$-matrices (associated with $U_q(gl_n)$) on the quantum side. As is shown in [12], the classical side in this case is represented by the trigonometric RS model. However, some interesting details, including an accurate limit to the trigonometric Gaudin model, are still to be elaborated.

Among future perspectives we mention an extension to the supersymmetric $GL(n|m)$-invariant spin chains and to the spin chains with elliptic $R$-matrices. The latter case seems to be especially non-trivial because integrable magnets constructed with the help of elliptic $R$-matrices do not allow twisted boundary conditions with continuous parameters. That is why it is not clear how to fix values of the classical integrals of motion in the elliptic RS model which would be the most natural candidate for the classical part of the QC duality. Another difficulty is that the Lax matrix for the elliptic RS model contains a spectral parameter. The role of this parameter in the context of the quantum-classical correspondence is not clear at the moment.

Acknowledgements

Discussions with A. Alexandrov, A. Gorsky, V. Kazakov, S. Khoroshkin, I. Krichever, S. Leurent, M. Olshanetsky, A. Orlov, T. Takebe, Z. Tsuboi, and A. Zotov are gratefully acknowledged. Some of these results were reported at the International School and Workshop “Nonlinear Mathematical Physics and Natural Hazards” (November 28 - December 2 2013, Sofia, Bulgaria). The author thanks the organizers and especially professors B. Aneva and V. Gerdzhikov for the invitation and support. This work was supported in part by RFBR grant 12-01-00525, by joint RFBR grants 12-02-91052-CNRS, 14-01-90405-Ukr and grant NSh-1500.2014.2 for support of leading scientific schools.

References

[1] A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi and A. Zabrodin, Classical tau-function for quantum spin chains, JHEP 1309 (2013) 064 [arXiv:1112.3310].

[2] A. Alexandrov, S. Leurent, Z. Tsuboi and A. Zabrodin, The master $T$-operator for the Gaudin model and the KP hierarchy, Nucl. Phys. B883 (2014) 173-223 [arXiv:1306.1111].

[3] A. Zabrodin, The master $T$-operator for inhomogeneous XXX spin chain and $mKP$ hierarchy SIGMA 10 (2014) 006 (18 pages), [arXiv:1310.6988].

[4] A. Gorsky, A. Zabrodin and A. Zotov, Spectrum of quantum transfer matrices via classical many-body systems, JHEP 01 (2014) 070, [arXiv:1310.6958].

[5] S.N.M. Ruijsenaars and H. Schneider, A new class of integrable systems and its relation to solitons, Ann. Phys. 170 (1986) 370-405;
S.N.M. Ruijsenaars, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Commun. Math. Phys. 110 (1987) 191-213.
[6] F. Calogero, Solution of the one-dimensional N-body problems with\nquad\nversely \quad quadratc pair potentials, J. Math. Phys. 12 (1971) 419-436;
J. Moser, Three integrable hamiltonian systems connected with isospectrum deformations, Adv. Math. 16 (1976) 354-370.

[7] M. Olshanetsky and A. Perelomov, Classical integrable finite dimensional systems related to Lie algebras, Phys. Reps. 71 (1981) 313-400.

[8] M. Gaudin, Diagonalisation d’une classe d’hamiltoniens de spin, J. de Phys. 37 (1976), no. 10 1087-1098.

[9] A. Givental and B.-S. Kim, Quantum cohomology of flag manifolds and Toda lattices, Commun. Math. Phys. 168 (1995) 609-641 [arXiv:hep-th/9312096].

[10] E. Mukhin, V. Tarasov and A. Varchenko, Gaudin Hamiltonians generate the Bethe algebra of a tensor power of vector representation of gl_N, St. Petersburg Math. J. 22 (2011) 463-472 [arXiv:0904.2131];
E. Mukhin, V. Tarasov and A. Varchenko, Bethe subalgebra of the group algebra of the symmetric group, [arXiv:1004.4248];
E. Mukhin, V. Tarasov and A. Varchenko, Spaces of quasi-exponentials and representations of the Yangian Y(gl_N), [arXiv:1303.1578].

[11] E. Mukhin, V. Tarasov and A. Varchenko, KZ characteristic variety as the zero set of classical Calogero-Moser Hamiltonians, SIGMA 8 (2012) 072 (11 pages) [arXiv:1201.3990];
E. Mukhin, V. Tarasov and A. Varchenko, Bethe subalgebra of the group algebra of the symmetric group, [arXiv:1004.4248];
E. Mukhin, V. Tarasov and A. Varchenko, Spaces of quasi-exponentials and representations of the Yangian Y(gl_N), [arXiv:1303.1578].

[12] A. Zabrodin, The master T-operator for vertex models with trigonometric R-matrices as classical tau-function, Teor. Mat. Fys. 171:1 (2013) 59-76 (Theor. Math. Phys. 174 (2013) 52-67) [arXiv:1205.4152].

[13] A. Zabrodin, Hirota equation and Bethe ansatz in integrable models, Suuri-kagaku Journal (in Japanese), Number 596 (2013) 7-12.

[14] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, in “Nonlinear integrable systems – classical and quantum”, eds. M. Jimbo and T. Miwa, World Scientific, pp. 39-120 (1983);
M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. RIMS, Kyoto Univ. 19 (1983) 943-1001.

[15] N. Nekrasov, A. Rosly and S. Shatashvili, “Darboux coordinates, Yang-Yang functional, and gauge theory”, Nucl. Phys. Proc. Suppl. 216 (2011) 69-93 [arXiv:1103.3919].

[16] D. Gaiotto and P. Koroteev, On three dimensional quiver gauge theories and integrability, JHEP 05 (2013) 126 [arXiv:1304.0779].

[17] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, Quantum Integrable Models and Discrete Classical Hirota Equations, Commun. Math. Phys. 188 (1997) 267-304 [arXiv:hep-th/9604080];
A. Zabrodin, Discrete Hirota’s equation in quantum integrable models, Int. J. Mod. Phys. B11 (1997) 3125-3158;
A. Zabrodin, Hirota equation and Bethe ansatz, Teor. Mat. Fyz., 116 (1998) 54-100 (English translation: Theor. Math. Phys. 116 (1998) 782-819.)
[18] V. Kazakov, A. S. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, Nucl. Phys. B 790 (2008) 345-413 [arXiv:hep-th/0703147];
A. Zabrodin, *Bäcklund transformations for difference Hirota equation and supersymmetric Bethe ansatz*, Teor. Mat. Fyz. 155 (2008) 74-93 (English translation: Theor. Math. Phys. 155 (2008) 567-584) [arXiv:0705.4006].

[19] H. Bethe, *Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette*, Zeitschr. fur Physik 71 (1931) 205-226.

[20] L. Faddeev, E. Sklyanin and L. Takhtajan, *The quantum inverse problem method. I*, Theor. Math. Phys. 40 (1980) 688;
V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge U.K., 1997.

[21] L. Faddeev and L. Takhtajan, *The spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model*, Zap. Nauch. Semin. LOMI 109 (1981) 134-178.

[22] K. Hikami, P. Kulish and M. Wadati, *Construction of integrable spin systems with long-range interaction*, J. Phys. Soc. Japan 61 (1992) 3071-3076.

[23] I. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.

[24] K. Sawada and T. Kotera, *Integrability and a solution for the one-dimensional N-particle system with inversely quadratic pair potential*, J. Phys. Soc. Japan, 39 (1975) 1614-1618.

[25] S. Wojciechowski, *New completely integrable Hamiltonian systems of N-particles on the real line*, Phys. Lett. 59A (1976) 84-86.

[26] P. Kulish and N. Reshetikhin, *Diagonalization of gl_N invariant transfer matrices and quantum N-wave system (Lee model)*, J. Phys. A 16 (1983) L591-L596.