Experimental Implementation of a Concatenated Quantum Error-Correcting Code

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Concentrated coding provides a general strategy to achieve the desired level of noise protection in quantum information storage and transmission. We report the implementation of a concatenated quantum error-correcting code able to correct against phase errors with a strong correlated component. The experiment was performed using liquid-state nuclear magnetic resonance techniques on a four spin subsystem of labeled crotonic acid. Our results show that concatenation between active and passive quantum error correcting codes offers a practical tool to handle realistic noise contributed by both independent and correlated errors.

Scalable quantum information processing (QIP) requires the ability to realize information fault-tolerantly in the presence of both environmental and control errors. A variety of approaches has been developed to meet this challenge, including active quantum error-correcting codes (QECCs), dynamical decoupling techniques, passive QECCs based on decoherence-free subspaces (DFSs) and noiseless subsystems (NSSs), and topological schemes. While the exploration of viable routes to quantum fault-tolerance is witnessing continual advances (see Refs. 4 and 5 for recent threshold analyses of post-selected QIP and concatenated decoupling schemes, respectively), QECCs remain to date the method of choice under a relatively wide range of error and control assumptions. In particular, concatenated QECCs are instrumental to ensure that a final accuracy can be reached without requiring arbitrarily low error rates at intermediate stages. The basic idea is to use multiple levels of encoding to recursively obtain logical qubits with improved robustness. In its standard setting, a concatenated code consists of hierarchically implementing a fixed QECC, provided that the errors for the encoded information satisfy at each level appropriate assumptions. For the procedure to be successful, it is critical that the implementation begins with sufficiently high fidelity, which requires the entry-level physical qubits to be subjected to a sufficiently weak noise.

If small error rates are not available from the start, concatenated QECCs are still valuable if the originating noise process is highly correlated, which makes it possible to exploit the existence of efficient DFS or NS encodings. Because the latter are tied to the occurrence of symmetries in the error process, such infinite-distance QECCs are capable of tolerating arbitrarily high error rates as long as the underlying symmetry is exact. While infinite-distance behavior is not retained for imperfect symmetry, stability results ensure that the residual errors remain small if the symmetry is broken perturbatively, with short-time fidelity solely determined by the perturbing noise strength. Concatenation schemes taking advantage of both finite- and infinite-distance codes were originally developed in Ref. 12 and subsequently in the context of the so-called cluster error model, where a dominant collective symmetry is perturbed by independent errors on individual qubits.

Here, we theoretically expand and experimentally demonstrate the usefulness of concatenating active and passive QECCs. Our approach is tailored to realistic hybrid noise models where errors do not follow a cluster pattern. In particular, while being to a large extent independent, they are dominated by a large error rate which prevents quantum error correction (QEC) from being affordable with feasible control resources. Two guiding principles emerge for error control design: (1) treat errors in order of their importance; (2) at each stage, realize logical qubits with reduced error rates. Unlike standard concatenation schemes, where physical qubits are uniformly replaced by logical ones at the first level of encoding, this leads in general to effect such a replacement only partially at a given stage, with physical and logical qubits being treated alike as needed.

Concatenated active and passive QECCs.- Let $S$ be the quantum system of interest, and imagine that noise on $S$ is to a good approximation Markovian. Then the state $\rho$ evolves as $\rho_t = e^{\mathcal{L}t}\rho_0$, where the infinitesimal noise super-operator $\mathcal{L}$ takes the standard Lindblad form:

$$\mathcal{L} = \sum_{\mu} D_{L_{\mu}}^{\dagger}D_{L_{\mu}}$$

(1)

Given the set of error generators $\{L_{\mu}\}$, a measure of the overall noise strength is given by $\lambda = \sum_{\mu} |L_{\mu}|^2 = |\sum_{\mu} L_{\mu}^\dagger L_{\mu}|$, where $|X| = \text{Max} \text{Re} \{\text{eig}(X^\dagger X)\}$ Ref. For independent noise on qubits, each $L_{\mu}$ involves a single-qubit Pauli operator, and $\lambda$ can be thought of as resulting from the sum of the partial noise strengths $\lambda_{\mu} = 2|L_{\mu}|^2$ associated to each error generator. In general, the error probability for information stored in $S$ is a complicated...
function of time. By letting $F_c(t)$ denote entanglement fidelity \(16\), one may write an error expansion

\[
F_c(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{t}{\tau_k} \right)^k,
\]

where the $k$th order error rate $1/\tau_k^c$ is upper-bounded by $\lambda^k$. If information is protected in a $w$-error-correcting code, error rates up to order $w$ are effectively canceled, improving the fidelity to $F_c(t) = 1 - O[(t/\tau_{w+1})^{w+1}] \geq 1 - O((\lambda t)^{w+1})$ \(11\). The larger the window where $F_c(t)$ remains approximately flat, the longer the time interval after which a one-time use of the code still succeeds at retaining the information with high fidelity. Since such an interval is determined by $\lambda$, the condition that noise is sufficiently weak is critical for QEC.

We illustrate our error control methodology by focusing on the following example. Consider three physical qubits with independent phase errors. Strong noise on one of the qubits, say qubit 3, causes the overall strength to be too high for QEC to produce a significant improvement. However, noise on qubit 3 is dominated by correlated dephasing involving an additional qubit, say qubit 4. The error model may be specified in terms of the following generators:

\[
L_1 = \sqrt{\lambda_3/2} \sigma_3^1, \quad L_2 = \sqrt{\lambda_3/2} \sigma_3^2, \quad L_3 = \sqrt{\lambda_3/2} (\sigma_3^1 + \sigma_3^2), \quad L_r = \sqrt{\lambda_3/2} \sigma_3^3;
\]

for positive parameters $\lambda_3$, $\mu = 1, 2, 3, r$. Here, $\lambda_3$ and $\lambda_r$ are the strengths of the phase errors on qubits 1, 2, whereas $\lambda_r$ and $\lambda_r$ characterize the dominant (collective) and residual (independent) dephasing on qubit 3. For simplicity, we imagine a situation where $|L_r|/|L_3| = \epsilon$, with $\epsilon < 1$, and $\lambda_1 \approx \lambda_2 \approx \lambda_3 \approx \lambda_0 \approx \epsilon^2 \lambda_3$. Physically, the collective and independent error processes affecting qubit 3 may or (b) may not have the same origin. In the latter case (case b), qubits 3 and 4 are identically coupled to some environment, and qubit 3 is additionally weakly interacting with a second environment. $L_c$ and $L_r$ are then separate error generators, with an overall noise strength on qubit 3 given by $\lambda_3 = \lambda_3 (1 + \epsilon^2)$. If the interaction involves a single environment instead (case a), then the symmetry between qubits 3 and 4 is perturbatively broken by independent errors on qubit 3. Accordingly, $L_c$ and $L_r$ should be combined into a single error generator $L_3 = L_c + L_r$, resulting in a noise strength $\lambda_3 = \lambda_3 (1 + \epsilon^2)^2$. In both cases, $\lambda \approx \lambda_3 / \epsilon^2$ small enough.

The presence of strong correlated noise naturally suggests the use of passive QECCs as the first step toward reducing the noise. Let $|0_L\rangle = |01\rangle_{34}$, $|1_L\rangle = |10\rangle_{34}$ define logical DFS basis states for collective dephasing on qubits 3, 4 \(17\), and work with the new system $S'$ composed of the physical qubits 1, 2, and the logical DFS qubit. Noise for $S'$ may be analyzed by examining the action of the error generators on a state $\hat{\rho}$ that is properly initialized to $S'$. The basic observation is that, thanks to the degenerate action of $L_c$ on DFS states \(17\), $L_c|i_L\rangle\langle j_L| = (\ell \mathbb{1}) |i_L\rangle\langle j_L|$ for some $\ell$ ($\ell = 0$ in our case), the errors caused by $L_c$ and $L_r$ on information encoded in $S'$ can be described, in both cases a and b, as

\[
\mathcal{D}_{L_c}[\hat{\rho}] + \mathcal{D}_{L_r}[\hat{\rho}] = \mathcal{D}_{L_r}[\hat{\rho}].
\]

Thus, the strong collective noise disappears and, since $\sigma_z^3$ acts as an encoded $\sigma_z^L$ observable, the residual noise from $L_r$ corresponds to logical phase errors with a reduced strength. The overall noise strength for $S'$ becomes $\lambda' = 3\lambda_0$, suitable for compensation by an additional level of QEC. In particular, a standard three-bit QECC is able to improve the one-qubit memory fidelity from $F_c(t) = 1 - O(\lambda_0 t)$ to $F_c^{\text{QEC}}(t) = 1 - O((\lambda_0 t)^2)$.

Some generalizations are worth mentioning. For hybrid error models where strong correlated dephasing coexists with weak arbitrary single-qubit errors, concatenation between an “inner” DFS coding on the appropriate pairs and an “outer” five-bit QECC is applicable. If the strong noise involves arbitrary collective errors, then three-bit NSs \(12\) \(13\) offer the most efficient code to be used at the lowest level. In this case, the analysis is easier under the assumption that the residual noise is in the commutant of the primary collective generators. This assumption, which parallels the no-leakage assumption of standard concatenated coding \(10\), ensures that the residual errors for the logical subsystem can be, as above, described in terms of encoded observables. Together with the identity action of the collective generators on the NS \(5\), this implies (similar to Eq. \(3\)) that the strong noise is fully compensated for after the first stage of encoding. Concatenation with active QEC can then further suppress errors. The no-leakage assumption may be relaxed at the expense of complicating the error control strategy. The consequences of leakage appear at first more serious in the perturbative scenario $a$, as the action of $\mathcal{D}_L$ would include, besides $\mathcal{D}_F$, additional terms mixing $L_c$ and $L_r$ or order $\epsilon^2$. However, general stability results \(11\) \(12\) ensure that the contribution from these terms to all the error rates $1/\tau_k^c$ remains of order $\epsilon^2$, as they are already in the independent scenario $b$. This makes concatenation with QEC still advantageous, provided that the procedure is modified to detect and handle leakage appropriately \(13\) \(14\) \(15\).

Experimental implementation.- The experiment implemented the above-mentioned DFS-QEC code for hybrid phase errors on four qubits using liquid-state NMR techniques \(20\). A 400 MHz Bruker AVANCE spectrometer was used with a sample of $^{13}$C labeled crotonic acid in a deuterated acetone solvent \(21\). The experiment combined basic steps used in the implementation of both active \(22\) \(23\) and passive QECCs \(17\) \(18\). The quantum network is shown in Fig. 4. The required independent and collective errors were engineered by combining the action of unitary radio-frequency pulses with the non-unitary dynamics induced by magnetic field gradients integrated over the three independent directions of the spatially distributed sample \(24\). If $n_i$, $i = 1, 2, 3,$
is the wavenumber of the gradient ramp along the \(i\)th axis \([24]\), the phase coherence of the corresponding qubit averaged along that direction is attenuated by a factor 

\[
\sin(k_i(t)L_i/2),
\]

where \(L_i\), \(\gamma\), and \(G_i\) denote the length of the sample in the \(i\)th direction, the gyromagnetic ratio of the nuclear species, the gradient strength and the duration of the gradient pulse, respectively. For independent errors, the gradients were calibrated to yield equal noise strengths on the different qubits and 4 so that \(k_3 = k_c + k_0\), corresponding to a single environment situation (case a) with \(k_0/k_c \sim 0.5\). In order to minimize the impact of natural decoherence, error models corresponding to a different noise strength were engineered by varying the value of \(G_i\) while keeping the total length of the experiment fixed.

Four different scenarios were investigated. First, the regular three-qubit QECC with independent \(z\) noise on qubits 1, 2, and 3, was implemented as a reference for the concatenated DFS-QEC code. Second, strong \(z\) noise on qubit 3 was added. Because, due to the incoherent nature of the applied noise, the attenuation of the phase coherence still yields an initial zero slope even in the absence of QEC, we also applied the error model to different input states with no QEC. In the fourth scenario, the four-qubit concatenated DFS-QEC code was realized, starting from the conditional pseudo-pure states \([20, 21]\)

\[
|0\rangle^3|0\rangle \otimes \sigma^z_u \otimes |0\rangle^3|0\rangle \otimes |0\rangle^4|0\rangle, \quad u = x, y, z. \]

Ideally, the data qubit should be recovered with the same accuracy as in the original QEC setting. Four- and one-qubit state tomography \([20]\) was performed to verify both input and output states, as well as their correlations. A total of 18 readout pulses was used to reconstruct the four-qubit state while two pulses sufficed for the state of the data qubit 2 alone. Strongly modulating control pulses designed to be robust against radio-frequency power inhomogeneity were employed \([26]\). A feedback loop was implemented to correct for systematic errors arising from the response of the electronics chain \([28]\). The protons were decoupled during both the experiment and the acquisition to avoid additional incoherence.

**Results.** - For an incoherent error dynamics as implemented in the experiment, the behavior of the error-corrected entanglement fidelity for equally distributed, independent phase errors with strength \(k_0\) is

\[
F_{e}^\text{sec.1}(t) = \frac{1}{2} + \frac{1}{4} \left( 3 \sin(k_0(t)L/2) - \sin^3(k_0(t)L/2) \right).
\]

When the strong noise component is added to qubit 3, the above equation is modified as

\[
F_{e}^\text{sec.1}(t) = \frac{1}{2} + \frac{1}{4} \left( 2 \sin(k_0(t)L/2) + \sin(k_3(t)L/2)
\right.

\[ \left. \quad - \sin^2(k_0(t)L/2) \sin(k_3(t)L/2) \right). \]

For irreversible Markovian noise as in Eq. \([11]\), the standard expressions for \(F_{e}^\text{sec.1}(t)\) \([20]\) are recovered upon replacing each sinc function by a corresponding exponential. From the above expressions, it is clear that QEC compensates for errors to first order, irrespective of the collective noise component \(k_c\). However, for longer times the latter decreases \(F_{e}^\text{sec.1}(t)\) much faster than the weaker independent noise, reducing the effectiveness of the code.

For each set of experiments, entanglement fidelities were inferred by calculating \(F_i = (C_x + C_y + C_z + 1)/4\), under the assumption of unital dynamics and with \(C_u = \text{tr}(\rho_{\text{in.u}}\rho_{\text{out.u}})/\text{tr}(\rho^2_{\text{in.u}}), \rho_{\text{in.u}}, \rho_{\text{out.u}} = x, y, z, \) being the measured input and output states, respectively. The results are summarized in Fig. 2(a). The four-qubit input state correlations with the intended states were estimated to be on average 0.93 ± 0.02 after state tomography. As a first remark, QEC with independent noise alone (dots) shows some improvement compared with the uncorrected scenario (squares). The noticeable "hump" in the data may be understood as a consequence of imperfect initialization of the ancillae, combined with the pseudo-pure state nature of the underlying NMR states \([20, 21]\). A similar effect (although less pronounced) is present in the QEC data under additional strong collective noise (asterisks). In any case, the efficiency of QEC in the presence of both independent and collective errors is greatly reduced as expected. The diamonds correspond to the DFS-QEC concatenated code. The initial fidelity drop is primarily explained by coherent errors associated with the longer pulse sequence necessary to realize the code, additional natural decoherence being induced as soon as qubit 4 is brought into the \(xy\) plane. Most of these features were accounted for by extensive simulations including both coherent and incoherent errors, as well as imperfect readouts. In particular, values of \(F_i\) larger than one arise from the joint influence of nuclear Overhauser enhancement, coherent and incoherent errors on the four-qubit system, along with the hump effect mentioned above.

While the entanglement fidelity data of Fig. 2(a) provide a complete representation of the overall implementation accuracy, the significant impact of coherent and
FIG. 2: Experimental results. (a) Entanglement fidelity, (b) Average polarization of the output states as a function of the applied noise strength. Dots: 3-qubit QEC with independent plus collective error bar is ±0.02 for each data point (not displayed for clarity). See text for an explanation of the different effects.

initialization errors evidenced by the simulations makes it difficult to directly assess the performance of different schemes at correcting the intended error model. This suggests to also analyse the data by using a metric which is only sensitive to the length of the output states. A natural choice is provided by the average output polarization, \( P = (P_x + P_y + P_z)/3 \), where \( P_u = \text{tr}(\rho_{\text{out},u}^2)/\text{tr}(\rho_{\text{out},u}^2) \), \( \rho_{\text{out},u} \), \( \rho_{\text{out},\text{w0}} \) being the output corresponding to input \( u \) with and without the applied noise, respectively (the same metric was recently used in [30]). The results are shown in Fig. 2(b). These now clearly demonstrate the advantages of using the DFS-QEC code, if coherent errors and initial polarization losses can be made negligible. As expected, the original QEC behavior with independent noise only (dots) is recovered with good accuracy by the DFS-QEC code under both independent and collective noise (diamonds). Further analysis of the data revealed that the hump visible in Fig. 2(a) is still present but much less important because of the insensitivity of the new metric to the initial coherent errors in \( \rho_{\text{out},\text{w0}} \).

Conclusions.- Our work provides the first experimental instance of a concatenated QEC code, using a combination of both physical and lower-level logical qubits to stabilize the quantum data at a higher logical level. While our results point to the need for improved control capabilities for both unitary and non-unitary dynamics, the implementation convincingly shows the benefits of concatenated active and passive QECCs in handling realistic error models. Notably, collective phase errors are the limiting factor for reliable storage using trapped ions [31]. Thus, our results might allow further advances toward realizing fault-tolerance in scalable device technologies.

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