Hybrid Spectral Difference/Embedded Finite Volume Method for Conservation Laws

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Abstract
Recently, interest has been increasing towards applying high-order methods to engineering applications with complex geometries [29]. As a result, a family of discontinuous high-order methods, such as Discontinuous Galerkin (DG), Spectral Volume (SV) and Spectral Difference (SD) methods, are under active development. These methods provide spectral-like solutions and are highly parallelizable due to local solution reconstruction within each element. But, these methods suffer from Gibbs phenomenon when discontinuities present in the flow fields. Various type of limiters [40, 41, 42] and artificial viscosity [43, 45] have been employed to overcome this problem.

A novel hybrid spectral difference/embedded finite volume method is introduced in order to apply a discontinuous high-order method for large scale engineering applications involving discontinuities in flows with complex geometries. In the proposed hybrid approach, structured finite volume (FV) cells are embedded in hexahedral SD elements containing discontinuities, and FV based high-order shock-capturing scheme is employed to overcome Gibbs phenomenon. Thus, discontinuities are captured at the resolution of embedded FV cells within an SD element. In smooth flow regions, the SD method is chosen for its low numerical dissipation and computational efficiency preserving spectral-like solutions. The coupling between the SD elements and the elements with embedded FV cells are achieved by the mortar method. In this paper, the 5th-order WENO scheme with characteristic decomposition is employed as the shock-capturing scheme in the embedded FV cells, and the 5th-order SD method is used in the smooth flow field.

The order of accuracy study and various 1D and 2D test cases are carried out, which involve the discontinuities and vortex flows. Overall, it is shown that the proposed hybrid method results in comparable or better simulation results compared to the standalone WENO scheme with the same number of solution DOF.

Keywords: Shock-turbulence interaction, hybrid method, spectral difference method, finite volume method, shock-capturing scheme

1. Introduction
The interaction of turbulence and discontinuities, e.g. shock, detonation and contact surface, in high speed flows is commonly encountered in many engineering applications. Turbulence interacting with shock formed by high speed aircraft generates noise downstream of the shock, which travels to ground and is called sonic boom. In such case, large scale turbulent motion interacting with shock can intensify or thicken shock, which in turn affects the level of noise downstream of the shock. Fundamental study of shock/turbulence interaction to understand the physics of sonic boom and its reduction have been given a great attention theoretically and numerically [11, 2]. The shock/shear layer interactions in the high speed turbulent jet from the aircraft engine also produces screech noise, and the reduction of such noise without compensating engine performance is of great interests [3]. For high speed propulsion systems, such as ramjet/scramjet, shock interacting with turbulent flows at high speed can enhance fuel/air mixing for stable combustion with the aid of flame holding apparatus [4]. The interaction of shock and turbulent flame plays a crucial role in deflagration to detonation transition [5].

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Turbulence is a vortex dominated chaotic process with a wide range of spatial and temporal scales while shock is in the length scale of molecular mean free path and is approximated by a mathematical discontinuity. These physical processes occurring at different length scales pose a conflicting numerical requirements for successful simulations. In order to capture a wide range of length scales in turbulent flows, low numerical dissipation is required especially to capture high wave number spectrum. On the other hand, introduction of some numerical dissipation is needed for numerical stability for capturing discontinuities in the flow field on a computational grid appropriate for resolving turbulent length scale. Therefore, devising a stable and accurate numerical method for simulating turbulent flows with discontinuities in the flow field is a challenging task.

Various low-dissipation high-order methods have been extensively developed to reduce numerical dissipation in smooth or turbulent flow fields yet provide the discontinuity capturing capability with some numerical dissipation introduced in the vicinity of discontinuity. Among various high-order schemes, weighted essentially non-oscillatory (WENO) type schemes [6,7] and spectral-like compact scheme [8] are widely employed. The WENO family schemes are mainly shock-capturing scheme where the order of accuracy is reduced at the location of discontinuities while the order of accuracy is preserved in smooth flow fields. However, it has been noted that the WENO scheme becomes dissipative in the smooth or turbulent flows. The compact scheme is designed to have very low dissipation resulting in spectral-like accuracy in the smooth or turbulent flows. But the compact scheme suffers from the Gibbs phenomena when discontinuities are present in the flow field. Various efforts have been made to address these issues associated with WENO and compact schemes since they are introduced, especially in applying these schemes for large eddy simulation (LES) and direct numerical simulation (DNS).

With the success of early WENO scheme, a lot of efforts have been made in order to lower the numerical dissipation in smooth or turbulent flows. Balsara and Shu [9] noted that WENO schemes are not monotonicity preserving and, thus, introduced a monotonicity preserving weighted essentially non-oscillatory (MPWENO) scheme. They employed the monotonicity preserving bounds of Suresh and Hyunh [10] and showed that the 9th or higher order MP-WENO scheme has high phase accuracy, thus suitable for compressible turbulent flows. Martin [12] et al. introduced a bandwidth-optimized WENO scheme for the direct numerical simulation of turbulent compressible flows. They proposed a set of candidate stencils to be symmetric with an additional candidate stencil and computed the bandwidth optimized weights for the optimal stencil by minimizing the truncation error on a given grid to maintain a small dissipation error at high wavenumbers. Simulations of incompressible and high turbulent Mach number isotropic turbulence show good agreement compared with the simulations by the 6th-order central Pade scheme and show good high wavenumber characteristics of turbulent compressible flows. For reviews of ENO and WENO schemes and other variants, see references [7,13,14].

The compact scheme provides spectral-like accuracy with very low numerical dissipation, but it is shown that the compact scheme suffers from Gibbs phenomenon when discontinuity are present. In order to remedy this problem, artificial viscosity/diffusivity, filtering schemes and TVD limiter are utilized in compact scheme for discontinuities in the flow fields. Cook and Cabot [15,16] introduced a high wavenumber biased spectral-like artificial shear and bulk viscosity based on strain rate for discontinuity capturing in compact scheme. Considering supersonic reacting flows with discontinuities, Fiorina and Lele [17] added artificial diffusivities for energy and species equations in addition to the artificial viscosity introduced in Cook and Cabot [16], and Kawai and Lele [18] extended this approach for curvilinear meshes. Mani et al. [19] noted that the artificial bulk viscosity in Cook and Cabot [16] significantly damps out the sound field while turbulent fields are not affected. Therefore, Mani et al. proposed to replace the strain rate by the dilatation and multiply it by the Heaviside function to localize artificial bulk viscosity near the shock and showed the improvement in sound field prediction. Yee et al. [20] employed the artificial compression method (ACM) switch [21] as a characteristic filter to stabilize numerical solution and minimize numerical dissipation near discontinuities for compact schemes. The alternative approach was proposed by Cockburn and Shu [22] by using TVD limiter and was modified by Yee [23] reducing spurious oscillations near the discontinuities which originates from TVD limiter.

Another class of favored approach is a hybrid method [24]. The hybrid method combines a high-order low-dissipative method in smooth or turbulent flow field and a high-order shock-capturing scheme localizing numerical dissipation in the proximity of discontinuity. In order to switch between two schemes, a discontinuity detector is devised. Adams and Shariff [25] introduced a hybrid method combining nonconservative compact scheme and ENO scheme. Following Adams and Shariff, Pirozzoli [26] used WENO scheme in the discontinuity region while conservative compact scheme is used in smooth region. Hill and Pullin [27] proposed tuned hybrid centered-difference/WENO method by developing tuned centered difference scheme for bandwidth optimization and coupling it with WENO
scheme for large eddy simulation with strong shocks. In these hybrid methods, it is noted that computing convective flux is switched between low dissipative scheme and shock-capturing scheme depending on the structure of flow fields. It has been shown that a hybrid scheme produces more favorable results compared to the approaches employing artificial viscosity or characteristic-based filter method [24, 28].

With significant developments in high-order methods, there have been increased interests in applying high-order method to engineering applications with complex geometries [29]. In this regard, a family of discontinuous high-order methods, such as Discontinuous Galerkin (DG) [30,31], Spectral Volume (SV) [32,33] and Spectral Difference (SD) methods [34, 35], has been given a great attention. Recently, Huynh [36] proposed the flux reconstruction scheme, and it has been shown that the flux reconstruction scheme is stable and efficient higher order method [37, 38, 39]. The advantage of discontinuous high-order methods is that the solution reconstruction is highly localized by using a polynomial basis function within each element. These methods result in spectral-like solutions and are highly parallelizable. Due to these properties of discontinuous high-order method, it is well suited for simulations on unstructured grids while applying WENO and compact schemes may become cumbersome for unstructured grids. However, it has been shown that the family of discontinuous high-order methods also suffers from Gibbs phenomenon as in the compact scheme when discontinuities are present. Limiters and artificial viscosity are employed to remedy this problem. Du et al. [40] and Zhong et al. [41] used WENO limiter for the Runge-Kutta discontinuous galerkin (RKDG) and the correction procedure via reconstruction (CPR) methods. Yang and Wang [42] introduced TVD limiter approach for the SD method based on Cockburn and Shu’s TVD limiter. Premasuthan et al. [43] employed artificial viscosity by Cook and Cabot [16] for shock capturing in the SD method for unstructured grid and further extended it to adaptive mesh refinement [44]. Persson and Peraire [45] proposed sub-cell shock capturing in DG method by introducing polynomial order dependent artificial viscosity.

In an effort to applying discontinuous high-order methods for large scale engineering applications involving the interaction of discontinuities (such as shock and detonation) and turbulent flows in complex geometries, a novel hybrid spectral difference (SD)/embedded finite volume (FV) method is introduced for hexahedral elements in this paper. The choice of SD method is made based on the computational efficiency of the method compared to DG or SV method and its simple standard form of numerical formulations without resorting to its variants, e.g. flux reconstruction method. In the proposed approach, the SD method is used in the smooth flow region. Where discontinuities are present in the flow, the structured finite volume cells are embedded in the hexahedral SD elements. High-order FV (or finite difference) discontinuity capturing scheme is employed for those embedded FV cells to capture discontinuities. The flux coupling at the common interface between standard SD elements and the elements with embedded FV cells is achieved by the mortar method [46]. The advantages of the proposed hybrid method are that (a) the discontinuities are captured at the resolution of embedded FV cells, resulting in sharp capturing of discontinuities within one or two SD elements depending on the nature of discontinuities (e.g. shock, detonation or contact surface) and the location of discontinuity in SD elements, (b) the number of embedded FV cells can be varied for high resolution discontinuity capturing, (c) without having to devise new schemes, current FV based state-of-the-art high-order shock-capturing schemes can be readily used in the embedded FV cells and (d) the proposed hybrid method still keeps the spatial compactness of the overall scheme for efficient parallel implementation since any information needed to compute fluxes depends only on the immediate neighbors of the SD element. The proposed approach can be applied directly to unstructured hexahedral grids as presented. However, in order to focus on delivering the main idea of the proposed hybrid method, the approach is described for curvilinear structured hexahedral grids in this paper.

The paper is organized as follows. First, the system of governing equations for the conservation laws is introduced in the physical space in Section 2. Then, the proposed hybrid method is introduced in Section 3 with brief descriptions of SD and WENO method and the flux coupling between standard SD elements and the elements with embedded FV cells. In Section 4 the overall order of accuracy of the hybrid method is examined. Standard test cases in 1D and 2D are presented in Section 5 and results are discussed. Section 6 summarizes the results of the current study and discusses the future works for further improvement of the proposed hybrid method.

2. Governing Equations

A system of 3D unsteady compressible Euler equation is considered and written in conservation form as,
\[
\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0,
\]
where
\[
Q = [\rho, \rho u, \rho v, \rho w, \rho E]^T.
\]
Inviscid flux vectors are
\[
F = \begin{pmatrix}
\rho u \\
\rho uu + p \\
\rho uv \\
\rho uw \\
\rho u(E + p/\rho)
\end{pmatrix},
G = \begin{pmatrix}
\rho v \\
\rho vv + p \\
\rho vw \\
\rho vw \\
\rho v(E + p/\rho)
\end{pmatrix},
H = \begin{pmatrix}
\rho w \\
\rho pw \\
\rho uv \\
\rho pw + p \\
\rho w(E + p/\rho)
\end{pmatrix},
\]
where \( \rho \) is the density, \( u, v \) and \( w \) are velocities, \( p \) is the pressure. The total energy is denoted by \( E \) and defined as
\[
E = e + \frac{1}{2}(u^2 + v^2 + w^2),
\]
where \( e \) is the internal energy, and \( e = C_v T \) under the calorically perfect gas assumption. The system of governing equations is closed with the ideal gas equation, \( p = \rho RT \), where \( R \) is the specific gas constant and \( T \) is the temperature.

3. Hybrid Method

The main idea of the proposed hybrid approach is to utilize the finite volume based high-order shock-capturing scheme locally in the region where the discontinuities are present by dynamically embedding finite volume cells within the SD elements, as depicted in Fig. 1(a). The SD method is employed in the region where the flow fields are smooth. As mentioned in Section 1, the SD method is chosen for computational efficiency and simple numerical formulations. For the finite volume based high-order shock-capturing method, the standard WENO scheme is employed while other high-order shock-capturing schemes can also be used. For the purpose of presenting main idea of the proposed hybrid method, the 5\textsuperscript{th}-order SD method and the 5\textsuperscript{th}-order WENO scheme with characteristic decomposition are employed in the present study. The same order of accuracy is chosen such that the overall order of the hybrid method is to be
the 5th order. The number of FV cells embedded is $5^3$ in order to have the same DOF in terms of the solution points as in the $5^{\text{th}}$-order SD element. The location of solution points is shown in Fig. 1(b) for SD element and embedded FV cells. The essential part of the current hybrid method lies in computing common interface flux coupling between standard SD elements and the elements with embedded FV cells, which is achieved by the mortar method [46].

Non-uniform hexahedral elements in physical space is transformed to the standard unit cubic elements in the computational domain by the geometric mapping. Accordingly, the system of governing equations, Eq. (1), in the physical domain is transformed to the computational domain using the Jacobian transformation matrix,

$$\frac{\partial \tilde{\mathbf{Q}}}{\partial t} + \frac{\partial \tilde{\mathbf{F}}}{\partial x} + \frac{\partial \tilde{\mathbf{G}}}{\partial y} + \frac{\partial \tilde{\mathbf{H}}}{\partial z} = 0,$$

where $\tilde{\mathbf{Q}} = |\mathbf{J}|\mathbf{Q}$ with the Jacobian matrix, $\mathbf{J}$, and

$$\begin{pmatrix} \tilde{\mathbf{F}} \\ \tilde{\mathbf{G}} \\ \tilde{\mathbf{H}} \end{pmatrix} = |\mathbf{J}|\mathbf{J}^{-1} \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix}.$$  \hspace{1cm} (5)

The time integrations in both SD and WENO methods are carried out by the $3^{\text{rd}}$-order strong stability preserving (SSP) Runge-Kutta (RK) scheme with low storage method [47], and it is given in general form as

$$\begin{align*}
\tilde{\mathbf{Q}}^{(0)} &= \tilde{\mathbf{Q}}^0, & d\tilde{\mathbf{Q}}^{(0)} &= 0, \\
d\tilde{\mathbf{Q}}^{(i)} &= A_i d\tilde{\mathbf{Q}}^{(i-1)} + \Delta t L_i(\tilde{\mathbf{Q}}^{(i-1)}), & i &= 1, \ldots, m, \\
\tilde{\mathbf{Q}}^{(i+1)} &= \tilde{\mathbf{Q}}^{(i)} + B_i d\tilde{\mathbf{Q}}^{(i-1)}, & i &= 1, \ldots, m, \\
\tilde{\mathbf{Q}}^{(n+1)} &= \tilde{\mathbf{Q}}^{(n)},
\end{align*}$$

with $A_1 = 0$, and $m = 3$ for 3 stage. The coefficients $A_i$ and $B_i$ are given in [47].

In the following, brief descriptions of standard SD and $5^{\text{th}}$-order WENO scheme with characteristic decomposition is given followed by the reconstruction of common interface flux between the standard SD elements and the elements with embedded FV cells by the mortar method.

3.1. Spectral difference method

In the spectral difference method, two sets of points are required inside the element, namely solution points and flux points. The unknown solutions are located at the solution points while flux values are located at the flux points. The solution points are defined by the Gauss points given by

$$X_s = \frac{1}{2} \left[ 1 - \cos \left( \frac{2s - 1}{2n} \pi \right) \right], \quad s = 1, 2, \ldots, n.$$  \hspace{1cm} (7)

For the flux points, the Legendre-Gauss points with additional two end points, 0 and 1, are used. The Legendre-Gauss polynomial is given by

$$P_s(\xi) = \frac{2s - 1}{s} (2\xi - 1) P_{s-1}(\xi) - \frac{s - 1}{s} P_{s-2}(\xi), \quad s = 1, \ldots, n - 1,$$

with $P_{-1}(\xi) = 0$ and $P_0(\xi) = 1$. The flux points are then defined with the roots of the Legendre-Gauss polynomial $P_{n-1}$ with two end points for $n^{\text{th}}$-order SD element.

The solution points are used to construct $(n - 1)$ degree Lagrange polynomials given by

$$h_i(X) = \prod_{s=1, s\neq i}^{n} \frac{X - X_s}{X_i - X_s},$$

for solution reconstruction. Likewise, $n$ degree Lagrange polynomials is constructed for flux reconstruction given by

$$l_{i+1/2}(X) = \prod_{s=0, s\neq i}^{n} \frac{X - X_{s+1/2}}{X_{i+1/2} - X_{s+1/2}}.$$  \hspace{1cm} (10)
Then, the solution is reconstructed by the tensor products of one-dimensional Lagrange polynomials.

\[ Q(\xi, \eta, \zeta) = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \hat{Q}_{i,j,k} h_{i}(\xi) \cdot h_{j}(\eta) \cdot h_{k}(\zeta). \]  

(11)

And the flux reconstruction are given by

\[ \tilde{F}(\xi, \eta, \zeta) = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{n} \hat{F}_{i+1/2, j, k} l_{i+1/2}(\xi) \cdot h_{j}(\eta) \cdot h_{k}(\zeta), \]  

(12)

\[ \tilde{G}(\xi, \eta, \zeta) = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{n} \hat{G}_{i+1/2, j, k} h_{i}(\xi) \cdot l_{i+1/2}(\eta) \cdot h_{k}(\zeta), \]  

(13)

\[ \tilde{H}(\xi, \eta, \zeta) = \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{n} \hat{H}_{i+1/2, j, k} h_{i}(\xi) \cdot h_{j}(\eta) \cdot l_{i+1/2}(\zeta). \]  

(14)

In advancing the solution in time, the conservative variable, \( \tilde{Q} \), are interpolated to the flux points using Eq. (11). The fluxes are constructed at the flux points using Eqs. (12)-(14) with \( Q \) at flux points. Note that the fluxes reconstructed are only element-wise continuous, but discontinuous between element interface. Thus, it requires a Riemann solver, such as Rusanov solver [48] or AUSM+ [49], to compute common fluxes at the element interface. Then, the derivatives of fluxes at the solution points are computed by

\[ \left( \frac{\partial \tilde{F}}{\partial \xi} \right)_{i,j,k} = \sum_{m=0}^{n} \hat{F}_{i+1/2, j, k} \cdot l'_{m+1/2}(\xi), \]  

(15)

\[ \left( \frac{\partial \tilde{G}}{\partial \xi} \right)_{i,j,k} = \sum_{m=0}^{n} \hat{G}_{i, j+1/2, k} \cdot l'_{m+1/2}(\eta), \]  

(16)

\[ \left( \frac{\partial \tilde{H}}{\partial \xi} \right)_{i,j,k} = \sum_{m=0}^{n} \hat{H}_{i, j, k+1/2} \cdot l'_{m+1/2}(\zeta), \]  

(17)

where \( l' \) is the spatial derivative of the Lagrange polynomial. And finally, the solution is updated by

\[ \frac{\partial \tilde{Q}_{i,j,k}}{\partial t} = - \left( \frac{\partial \tilde{F}}{\partial \xi} + \frac{\partial \tilde{G}}{\partial \eta} + \frac{\partial \tilde{H}}{\partial \zeta} \right)_{i,j,k} \]  

(18)

at each solution point.

3.2. 5th-order WENO scheme

In WENO scheme, the numerical flux at the mid point, \( \tilde{F}_{i+1/2, j, k} \), is constructed from the left and right states by the weighted averaging procedure and using Riemann solver as following,

\[ \tilde{F}_{i+1/2, j, k} = R \left( \tilde{Q}_{i+1/2, j, k}^{L}, \tilde{Q}_{i+1/2, j, k}^{R} \right), \]  

(19)

where \( \tilde{Q}_{i+1/2, j, k}^{L} \) and \( \tilde{Q}_{i+1/2, j, k}^{R} \) are obtained from WENO interpolation, and \( R \) is an operator denoting a Riemann solver. For brevity, only WENO interpolation on the left state in \( i \) direction is described. The right state can be computed in the same way using symmetric arrangement of stencils. The \( \tilde{Q} \) variable is transformed into characteristic forms as follows

\[ \tilde{Q}_{i,j,m} \equiv l_{j+1/2, m} \tilde{Q}_{i,j,m}^{L} \quad (k = j - 2, j - 1, j, j + 1, j + 2), \]  

(20)

where \( \tilde{Q}_{i,j,m} \) denotes the \( m \)th characteristic variable, and \( l_{j+1/2, m} \) denotes the \( m \)th left eigenvector of the matrix \( \partial \tilde{F}/\partial \tilde{Q} \) at \( j + 1/2 \), which is computed with the Roe average of values at \( j \) and \( j + 1 \). Then, the characteristic form of \( \tilde{Q}_{j+1/2,m} \) is constructed as follows

\[ \tilde{Q}_{j+1/2,m}^{L} = w_{j,m}^{1} \tilde{Q}_{j+1/2,m}^{L,1} + w_{j,m}^{2} \tilde{Q}_{j+1/2,m}^{L,2} + w_{j,m}^{3} \tilde{Q}_{j+1/2,m}^{L,3}, \]  

(21)
where

\[
Q_{j+1/2,m}^{t,1} = \frac{1}{3} Q_{j-2,m}^{t} - \frac{7}{6} Q_{j-2,m}^{t} + \frac{11}{6} Q_{j,m}^{t},
\]

(22)

\[
Q_{j+1/2,m}^{t,2} = -\frac{1}{6} Q_{j-1,m}^{t} + \frac{5}{6} Q_{j,m}^{t} + \frac{1}{3} Q_{j+1,m}^{t},
\]

(23)

\[
Q_{j+1/2,m}^{t,3} = \frac{1}{3} Q_{j,m}^{t} + \frac{5}{6} Q_{j+1,m}^{t} - \frac{1}{6} Q_{j+2,m}^{t},
\]

(24)

and \( w^1, w^2 \) and \( w^3 \) are nonlinear weights, which are determined as

\[
w_{j,m}^k = \frac{\alpha_{j,m}^k}{\alpha_{j,m}^1 + \alpha_{j,m}^2 + \alpha_{j,m}^3}, \quad k = 1, 2, 3,
\]

(25)

where

\[
\alpha_{j,m}^k = \frac{C^k}{(IS_{j,m}^k + \epsilon)^p} \quad k = 1, 2, 3,
\]

(26)

with

\[
C^1 = \frac{1}{10}, \quad C^2 = \frac{6}{10}, \quad C^3 = \frac{3}{10},
\]

(27)

and

\[
IS_{j,m}^1 = \frac{1}{4} \left( Q_{j-2,m}^{t} - 4Q_{j-1,m}^{t} + 4Q_{j-1,m}^{t} - Q_{j,m}^{t} \right)^2 + \frac{13}{12} \left( Q_{j-2,m}^{t} - 2Q_{j-1,m}^{t} + Q_{j,m}^{t} \right)^2,
\]

(28)

\[
IS_{j,m}^2 = \frac{1}{4} \left( -Q_{j-1,m}^{t} + Q_{j+1,m}^{t} \right)^2 + \frac{13}{12} \left( Q_{j-1,m}^{t} - 2Q_{j,m}^{t} + Q_{j+1,m}^{t} \right)^2,
\]

(29)

\[
IS_{j,m}^3 = \frac{1}{4} \left( -3Q_{j,m}^{t} + 4Q_{j+1,m}^{t} - Q_{j+2,m}^{t} \right)^2 + \frac{13}{12} \left( Q_{j,m}^{t} - 2Q_{j+1,m}^{t} + Q_{j+2,m}^{t} \right)^2.
\]

(30)
Here, $\epsilon = 10^{-6}$ and $p = 2$. Then the characteristic form of the $\tilde{Q}$ variable is transformed into conservative form as follows

$$Q_{j+1/2}^L = \sum_m Q_{j+1/2,m}^L r_{j+1/2,m},$$

(31)

where $r_{j+1/2,m}$ is the $m^{th}$ right eigenvector of the matrix $\partial \tilde{F}/\partial \tilde{Q}$ at $j + 1/2$.

Then, the derivative of flux is evaluated as,

$$\left( \frac{\partial \tilde{F}}{\partial \xi} \right)_{i,j,k} = \frac{1}{\Delta \xi} \left( \tilde{F}_{i+1/2,j,k} - \tilde{F}_{i-1/2,j,k} \right).$$

(32)

Likewise, $\left( \frac{\partial \tilde{G}}{\partial \eta} \right)_{i,j,k}$ and $\left( \frac{\partial \tilde{H}}{\partial \zeta} \right)_{i,j,k}$ are computed. Then, the solution is advanced using Eq. (18).

In the case that the neighboring element is the SD element as shown in Fig. 2, the solution in the neighboring SD element is interpolated to the solution points in FV cells using Eq. (11) with the same order of Lagrange polynomial. Then, those interpolated values at FV solution points in the neighboring SD element is used to construct fluxes in the embedded FV cells.

In some cases, such as presence of strong discontinuities, a limiter is required in WENO scheme for the numerical stability [9]. The following form of total variation diminishing (TVD) limiter is employed as described in [50].

$$\phi = \max \left[ 0, \min \left( \alpha, \frac{\tilde{Q}_{i+1} - \tilde{Q}_i}{\tilde{Q}_i - \tilde{Q}_{i-1}}, 2 \frac{\tilde{Q}_{i+1/2} - \tilde{Q}_i}{\tilde{Q}_i - \tilde{Q}_{i-1}} \right) \right],$$

(33)

where $\tilde{Q}_{i+1/2}$ is the interpolated value using $5^{th}$-order WENO interpolation, and $\alpha$ is a constant which is set to 2. Then, the slope limited interpolation is defined by

$$\tilde{Q}_{i+1/2} = \tilde{Q}_i + 0.5 (\tilde{Q}_i - \tilde{Q}_{i-1}) \phi.$$

(34)

It is noted when the TVD limiter is used for the test cases presented in this study.

3.3. SD/Embedded FV interface flux reconstruction

When the neighboring elements are the same type at the common element interface, the standard SD or WENO interpolation procedure is used to compute the common flux at the element interface. However, as the FV cells are embedded in the discontinuity region, there are element interfaces that have different types of elements in the neighbor with different solution locations in each element. For these interfaces, consistent and conservative common flux should be evaluated. In this study, reconstruction of common flux at this type of interface is achieved by employing the mortar method [46]. The schematic diagram of mortar procedure is shown in Fig. 3.

First, the conservative variable, $\tilde{Q}$, is computed at own flux points at the participating face in each element. Referring to Fig. 3, the values of $\tilde{Q}$ in the SD element are computed at the flux points at the face on the right side of the SD element, shown as blue cross mark in Fig. 3. In the FV cells, the $\tilde{Q}$ at flux points at the face, denoted by blue
cross mark in Fig. 3 on the left side of the FV cells are computed following WENO procedure. A mortar interface is defined with the Gauss points with the maximum number of solution points in each direction of 2D face between SD and embedded FV cells on the participating faces. Then, the Lagrange polynomial is constructed with these points on the mortar interface with the polynomial order, \( J - 1 \), where \( J = \max(M_{sd}, M_{fv}) \). Here, \( M_{sd} - 1 \) and \( M_{fv} - 1 \) are the order of polynomials in SD element and the element with embedded FV cells. The \( \tilde{Q} \) values from SD element and the element with embedded FV cells are projected onto the mortar interface by the least square projection given by

\[
\sum_{i=0}^{J-1} \sum_{j=0}^{J-1} \Phi_{i,j} M_{ij,mn} = \sum_{i=0}^{M_{sd} - 1} \sum_{j=0}^{M_{fv} - 1} U_{i,j} S_{ij,mn}, \quad m, n = 0, \ldots, J - 1,
\]

where \( \Phi_{i,j} \) is the interpolated values on the mortar interface, and \( U_{i,j} \) is the \( \tilde{Q} \) variables on the participating faces of either SD element or the element with embedded FV cells. The projection matrices, \( M_{ij,mn} \) and \( S_{ij,mn} \) are defined as

\[
M_{ij,mn} = \int_0^1 h_i^M(x) h_j^M(y) h_m(x) h_n(y) dxdy, \tag{36}
\]

\[
S_{ij,mn} = \int_0^1 h_i(x) h_j(y) h_m^M(x) h_n^M(y) dxdy, \tag{37}
\]

where \( h_i \) and \( h_j \) are the Lagrange polynomials defined on the face of SD elements or the elements with embedded FV cells, and \( h_i^M \) and \( h_j^M \) are the Lagrange polynomials defined on the mortar interface. The integrals in projection matrices, \( M_{ij,mn} \) and \( S_{ij,mn} \), can be evaluated by using a Clenshaw–Curtis quadrature [51].

These projected \( \tilde{Q} \) values to the mortar interface from both elements then serve as left and right states for the Riemann solver. The common interface fluxes are computed by the Riemann solver at the mortar flux points. Finally, these common fluxes at the mortar interface are projected back to each element’s participating face by

\[
\sum_{i=0}^{M_{sd} - 1} \sum_{j=0}^{M_{fv} - 1} \Phi_{i,j} M_{ij,mn} = \sum_{i=0}^{M_{sd} - 1} \sum_{j=0}^{M_{fv} - 1} U_{i,j} S_{ij,mn}, \quad m, n = 0, \ldots, M_{sd/fv} - 1,
\]

where \( \Phi_{i,j} \) is the interpolated flux on participating element faces, and \( U_{i,j} \) is the flux values on the mortar interface. The projection matrices, \( M_{ij,mn} \) and \( S_{ij,mn} \) are defined as

\[
M_{ij,mn} = \int_0^1 h_i(x) h_j(y) h_m(x) h_n(y) dxdy, \tag{39}
\]
Table 1: Translating density sine wave: error and order of accuracy.

| Mesh       | $L_1$ error | Order | $L_2$ error | Order | $L_\infty$ error | Order |
|------------|-------------|-------|-------------|-------|------------------|-------|
| **SD_RUS** |             |       |             |       |                  |       |
| 20x1x1     | 4.3287e-08  | -     | 4.9659e-08  | -     | 1.0150e-07      | -     |
| 40x1x1     | 1.6219e-09  | 4.74  | 1.8590e-09  | 4.74  | 3.7739e-09      | 4.75  |
| 80x1x1     | 5.1751e-11  | 4.97  | 5.9458e-11  | 4.97  | 1.0771e-10      | 5.13  |
| **SD_AUSM** |            |       |             |       |                  |       |
| 20x1x1     | 1.2598e-08  | -     | 1.8073e-08  | -     | 4.6771e-08      | -     |
| 40x1x1     | 3.9029e-10  | 5.01  | 5.6199e-10  | 5.01  | 1.4511e-09      | 5.01  |
| 80x1x1     | 1.5775e-11  | 4.63  | 1.9871e-11  | 4.82  | 4.6229e-11      | 4.97  |
| **WENO_AUSM** |          |       |             |       |                  |       |
| 100x1x1    | 2.1569e-07  | -     | 2.4392e-07  | -     | 4.1328e-07      | -     |
| 200x1x1    | 6.7398e-09  | 5.00  | 7.5514e-09  | 5.01  | 1.2911e-08      | 5.00  |
| 400x1x1    | 2.1319e-10  | 4.98  | 2.3769e-10  | 4.99  | 3.9799e-10      | 5.02  |
| **WENO_AUSM_TVD** | | | | | | |
| 100x1x1    | 6.5065e-04  | -     | 1.2092e-03  | -     | 3.5669e-03      | -     |
| 200x1x1    | 1.2804e-04  | 2.35  | 3.1984e-04  | 1.92  | 1.2412e-03      | 1.52  |
| 400x1x1    | 2.5361e-05  | 2.34  | 8.2726e-05  | 1.95  | 4.2338e-04      | 1.55  |
| **HYBRID** |             |       |             |       |                  |       |
| 20x1x1     | 4.2916e-07  | -     | 8.0529e-07  | -     | 3.5788e-06      | -     |
| 40x1x1     | 5.4189e-08  | 5.00  | 1.2892e-07  | 2.64  | 7.7709e-07      | 2.20  |
| 80x1x1     | 7.4116e-09  | 2.87  | 2.1448e-08  | 2.59  | 1.6716e-07      | 2.22  |

$$S_{ij,mn} = \int_0^1 h_i^M(\xi) h_j^M(\eta) h_m(x_i) h_n(\eta) d\xi d\eta.$$  \hspace{1cm} (40)

The projected fluxes at the face of element are used in computing the derivatives of fluxes.

3.4. Discontinuity detector

In order to identify the SD elements that contains the discontinuities in the flow fields, the detector described in \[52\] is employed and is given by

$$\left| \phi_{i+1} - \frac{2\phi_i + \phi_{i+1}}{3} \right| > \epsilon_r,$$ \hspace{1cm} (41)

where $\phi$ is pressure or density, and $\epsilon_r$ is the user defined constant. The value of $\epsilon_r$ is varied to keep the embedded FV regions close to discontinuities.

4. Order of Accuracy

In this section, the order of accuracy of the 5th-order SD method, 5th-order WENO scheme and the hybrid method with 5th-order SD and 5th-order WENO scheme is examined. For the SD method, two Riemann solvers are tested, namely Rusanov solver, which is more popular with SD method, and AUSM$^+$-up. Each of these combinations is denoted by SD_RUS and SD_AUSM, respectively. Based on the accuracy study with SD_RUS and SD_AUSM, AUSM$^+$-up method is chosen for the Riemann solver in WENO scheme, denoted by WENO_AUSM. Since WENO scheme is primarily used as a shock-capturing scheme, a limiter is needed in some cases with strong shock. Therefore, the TVD limiter, Eq. (33), is employed, denoted by WENO_AUSM_TVD, and its effect on WENO solution is examined. For the hybrid method, AUSM$^+$-up scheme is used for both SD and WENO scheme and is denoted by HYBRID. The standard tests of translating density sine wave and translating vortex are used for accuracy study.

4.1. Translating density sine wave

First, the error and the order of accuracy of the 5th-order SD, the 5th-order WENO and the hybrid method are examined in 1D by a translating density sine wave. The physical domain in $x$ is [0, 1]. For SD_AUSM and SD_RUS, the physical domain is discretized with 20, 40 and 80 hexahedral SD elements in $x$ direction. For WENO_AUSM and WENO_AUSM_TVD, $5^3$ uniform FV cells are embedded in all SD elements, resulting in the same solution DOF as in 5th-order SD element and the resolution of 100, 200 and 400 FV cells in $x$ direction. For HYBRID case, the physical
domain is discretized with 20, 40 and 80 SD elements, and $5^3$ FV cells are statically embedded in SD elements for $x > 0.5$. The density profile is initialized with a sine function by

$$\rho = \rho_0 (1 + 0.2 \sin(2\pi x)), \quad (42)$$

where $\rho_0$ is reference density, $\rho_0 = 1.179$ kg/m$^3$. The pressure is set constant at 1 atm, the temperature is computed from equation of state, $T = \rho / \rho R$, with $R = 288.18$ J/kg K. The density sine wave is translated with velocity, $u = 100$ m/s. The time step is set to $\Delta t = 5 \times 10^{-7}$. Periodic boundary conditions are used in all directions. The initial density profile in the mesh configuration for HYBRID case is depicted in Fig. 4(a).

The density sine wave is translated for two periods of a cycle. Then, the density profile is compared with the exact solution. The error norm and the order of accuracy of various cases are listed in Table [1]. It shows that the designed order of accuracy is nearly reached with SD_AUSM, SD_RUS and WENO_AUSM. It should be noted that the error norm is smallest in SD_AUSM among these three cases. In case of WENO_AUSM_TVD, the order of accuracy is much reduced to around 2nd order. This is due to the cutoff behavior of TVD limiter clipping local extrema, degrading the scheme to lower order. The HYBRID case also shows the degradation of order to around 3rd order. The origin of error stems from the interpolation error in the mortar projection process, primarily interpolating conservative variables from embedded FV cells to the mortar interface and interpolation of computed common fluxes back to the embedded FV cells. However, it should be pointed out that the magnitude of the error norm is much smaller than WENO_AUSM_TVD and only an order of magnitude larger than WENO_AUSM as the mesh is refined. The density profiles of various cases are shown in Fig. [5]. It shows that the density profiles of various cases are in good agreement with the exact solution except the WENO_AUSM_TVD case. The WENO_AUSM_TVD case is depicted by dashed line with diamond mark in Fig. [5(b)], and it shows the cutoff behavior of TVD limiter at local extrema.

4.2. Translating vortex

Based on the error and the order of accuracy study in 1D translating density wave, SD_AUSM, WENO_AUSM and HYBRID cases are chosen for the order of accuracy test in 2D translating vortex. The physical domain in x and y is $[0, 10] \times [0, 10]$. The physical domain is discretized by $20 \times 20$, $40 \times 40$ and $80 \times 80$ hexahedral SD elements for SD_AUSM. For WENO_AUSM, $5^3$ FV cells are embedded in all SD elements, resulting in the resolution of $100 \times 100$, $200 \times 200$ and $400 \times 400$ in x and y directions constructing the same solution DOF as in SD_AUSM. For HYBRID
Table 2: Translating vortex: error and order of accuracy.

| Mesh      | $L_1$ error   | Order | $L_2$ error   | Order | $L_{\infty}$ error | Order |
|-----------|---------------|-------|---------------|-------|---------------------|-------|
| SD_AUSM   |               |       |               |       |                     |       |
| 20\times20\times1 | 2.5311e-06   | -     | 5.1440e-06   | -     | 6.2432e-05         | -     |
| 40\times40\times1 | 9.1350e-08   | 4.79  | 1.6210e-07   | 4.99  | 2.4500e-06         | 4.67  |
| 80\times80\times1 | 2.5259e-09   | 5.18  | 5.3222e-09   | 4.93  | 8.9683e-08         | 4.77  |
| WENO_AUSM |               |       |               |       |                     |       |
| 100\times100\times1 | 1.1542e-04   | -     | 3.1883e-04   | -     | 2.5294e-03         | -     |
| 200\times200\times1 | 2.7858e-05   | 2.05  | 7.7676e-05   | 2.04  | 5.4659e-04         | 2.21  |
| 400\times400\times1 | 7.9988e-06   | 1.80  | 1.9391e-05   | 2.00  | 1.3792e-04         | 1.99  |
| HYBRID    |               |       |               |       |                     |       |
| 20\times20\times1 | 1.2735e-04   | -     | 2.8108e-04   | -     | 2.4218e-03         | -     |
| 40\times40\times1 | 3.0947e-05   | 2.04  | 7.7679e-05   | 2.05  | 5.0281e-04         | 2.27  |
| 80\times80\times1 | 7.7032e-06   | 2.01  | 1.7005e-05   | 2.00  | 1.2438e-04         | 2.02  |

case, the physical domain is discretized in the same way as in SD_AUSM except that $5^3$ FV cells are embedded in SD elements for $x > 5$. The mean flow conditions are $(\rho, u, v, w, p) = (1, 1, 0, 0, 1)$. The isotropic vortex is superimposed on the mean flow with the following perturbations.

$$
\Delta u = \frac{\epsilon}{2\pi} e^{(1-r^2)/2}(5-y),
\Delta v = \frac{\epsilon}{2\pi} e^{(1-r^2)/2}(x-5),
\Delta T = -\frac{(\gamma - 1)\epsilon^2}{8\gamma\pi^2} e^{(1-r^2)},
\Delta S = 0,
$$

where $r^2 = (x-5)^2 + (y-5)^2$, the vortex strength, $\epsilon = 5.0$, and $\gamma = 1.4$. Initially, the center of isotropic vortex is located at $x = 5$ and $y = 5$. The time step is set to $\Delta t = 2 \times 10^{-3}$. Periodic boundary conditions are used in all directions. The initial density profile in the mesh configuration for HYBRID case is depicted in Fig. 4(b).

The vortex is translated for two periods of a cycle. Then, the density profile is compared with the exact solution. The error norm and the order of accuracy of three cases are listed in Table 2. It shows that the designed order of accuracy is achieved only in SD_AUSM while the order of accuracy is much reduced in WENO_AUSM, indicating the dissipative nature of WENO scheme for smooth but rapidly varying flow fields. However, the results by WENO_AUSM can be improved to achieve the designed order of accuracy in this test case by employing more advanced variants of WENO scheme, e.g. [12]. Examining the magnitude of the error norm and the order of accuracy in HYBRID case, it can be seen that they are in about the same range as in WENO_AUSM. This shows that the error is more dominated by the WENO scheme than by the interpolation error in the mortar projection step in this test case.

While the WENO scheme is used in statically embedded FV cells in the domain where $x > 5$ in this test case, it should be noted that the WENO scheme is primarily employed in the proposed hybrid method as a shock-capturing scheme dynamically applied only in the vicinity of the region where discontinuities are present, thus the order of accuracy of WENO scheme itself degrades only in these regions. Therefore, it is expected that the degradation of the order of accuracy of the WENO scheme does not affect the overall performance of the proposed hybrid method.

5. Test Cases

In this section, the standard 1D and 2D test cases are presented using the proposed hybrid method and the 5th-order WENO scheme, denoted by HYBRID and WENO respectively, for comparison. In HYBRID, structured $5^3$ FV cells are embedded dynamically only in the SD elements where the discontinuities are present. For WENO, structured $5^3$ FV cells are embedded in all SD elements, resulting in 5 times fine grid in all directions to have the same solution DOF as in the 5th-order SD element. For all test cases presented, the AUSM+ up [49] is employed as the Riemann solver. For the test cases with a strong shock, the TVD limiter is used.
5.1. SOD shock tube problem

The SOD shock tube problem \cite{53} is considered to examine the performance of shock capturing capability by dynamically embedded FV cells in a SD element following the contact surface and the shock. The computed solutions between HYBRID and WENO are compared. The physical domain in \( x \) is \([0, 1]\), is discretized by 20, 40, 80 and 160 hexahedral SD elements, which results in 100, 200, 400 and 800 FV cells in \( x \) direction for WENO. The initial condition is set by
\[
\begin{align}
(\rho, u, v, w, p) &= (10\rho_0, 0, 0, 10p_0) \quad \text{for} \quad x < 0.5, \quad (47) \\
(\rho, u, v, w, p) &= (\rho_0, 0, 0, p_0) \quad \text{for} \quad x \geq 0.5, \quad (48)
\end{align}
\]
where \( \rho_0 = 1.179 \text{ kg/m}^3 \) and \( p_0 = 1 \text{ atm} \). The specific gas constant, \( R \), is 288.18 J/kg K. The density and pressure discontinuities in the SD elements are initially detected, and the 5\( ^3 \) FV cells are embedded in those SD elements. Then, the flow fields are initialized at the solution points in the embedded FV cells containing the discontinuities. As the solution is advanced in time with constant \( \Delta t = 1 \times 10^{-6} \), the contact surface and the shock are dynamically detected. The discontinuity detector is used on both density and pressure in order to capture the contact surface and the shock with \( \epsilon_s = 0.01 \).

The computed density profiles are shown in Fig. \ref{fig:fig6} for various grid sizes with \( N \) denoting the number of SD elements. Figure \ref{fig:fig7} shows the close-up view near the contact surface and the shock. The exact solution is also shown in these figures for comparison. The horizontal dots in Fig. \ref{fig:fig7} indicates the locations of embedded FV solution points. It can be seen that the region where FV cells are embedded become smaller close to the discontinuities and that the discontinuities are captured within a few embedded FV cells as the mesh is refined with the hybrid method. The density profiles computed by both method compare well with the exact solution for the given spatial resolution. The density profiles show that HYBRID with coarser mesh produces comparable results as WENO with 5 times finer grid for the same solution DOF. For example, the hybrid method with 160 SD elements produces very similar results as the 5\( ^{th} \)-order WENO scheme on 800 FV cells. Overall, by using the current hybrid method, the contact surface and shocks are captured in sub-SD element resolution compared to other methods, such as using artificial viscosity or limiter, where the discontinuities are usually captured in a few SD elements.

5.2. Shu-Osher problem

In this example, the interaction between propagating shock and perturbed entropy waves is simulated \cite{54}. The physical domain in \( x \) is \([-5, 5]\). The domain is discretized with 20, 40, 80 and 160 hexahedral SD elements, which
results in 100, 200, 400 and 800 FV cells in x direction for WENO. The nondimensional initial condition is given by

\[(\rho, u, v, w, p) = (3.857143, 2.269369, 0, 0, 10.3333) \quad \text{for} \quad x < -4, \quad (49)\]

\[(\rho, u, v, w, p) = (1 + 0.2 \sin(5x), 0, 0, 0, 1) \quad \text{for} \quad x \geq -4. \quad (50)\]

with \(\gamma = 1.4\).

The initial shock is detected in SD elements, and FV cells are embedded in these elements. The solution is advanced with constant \(\Delta t = 1 \times 10^{-4}\). The FV cells are dynamically embedded in the SD elements where the steep density gradient or the shock is detected with \(\epsilon_s = 0.02\). The numerical "exact" solution is obtained by the 5th-order WENO scheme employed in this study on a 3000 FV cells.

The simulation is run until \(t = 1.8\). The computed density profiles as well as the numerical "exact" solution are shown in Fig. 7 for various grid sizes with \(N\) denoting the number of SD elements. The close-up view is shown in Fig. 10. In Fig. 9 the horizontal dots shows the location of embedded FV solution points. The computed results
show that the compressed entropy waves behind the shock are not well capture by both HYBRID and WENO at low resolutions, e.g. \( N = 20 \) and \( N = 40 \). But, it can be seen that HYBRID gives a slightly better results with \( N = 40 \) than WENO. When the grid is refined further to \( N = 80 \) and \( N = 160 \), both HYBRID and WENO, having the same solution DOF, are able to capture the compressed entropy waves behind the shock showing good agreement with the numerical "exact" solution.

5.3. Weak shock/vortex interaction

This test simulates the interaction of a moving vortex with a stationary weak shock \([6] \). The physical domain in \( x \) and \( y \) is \([0, 2] \times [0, 1] \), and the domain is discretized by uniform \( 80 \times 40 \) hexahedral SD elements, which gives \( 400 \times 200 \) FV cells for WENO. At \( x = 0.5 \), a stationary shock with Mach 1.1 positioned, where the left state condition is \((\rho, u, v, w, p) = (1, 1.1 \sqrt{\gamma}, 0, 0, 1) \) with \( \gamma = 1.4 \). A small vortex is superposed to the flow left of the shock at \((x_c, y_c) = (0.25, 0.5) \). The vortex is defined by perturbations to the mean flow defined by the left state of the shock, and
Figure 9: Shu-Osher problem. Density profiles on different grid sizes with \( N \) denoting the number of SD elements. HYBRID, \( \circ \); WENO, \( \times \) and the numerical "exact" solution, solid line. Horizontal dots indicate the locations of FV solution points.

The perturbations are given by

\[
\begin{align*}
\Delta u &= \epsilon \tau e^{\alpha(1-\tau^2)} \sin \theta, \\
\Delta v &= -\epsilon \tau e^{\alpha(1-\tau^2)} \cos \theta, \\
\Delta T &= \frac{(\gamma - 1)\epsilon^2 e^{2\alpha(1-\tau^2)}}{4\alpha \gamma}, \\
\Delta S &= 0,
\end{align*}
\]

where \( \tau = r/r_c \) and \( r = \sqrt{(x-x_c)^2 + (y-y_c)^2} \). \( \epsilon \) is the strength of the vortex, \( \alpha \) is the decay rate of vortex, \( r_c \) is the critical radius for which the vortex has the maximum strength. In this study, \( \epsilon = 0.3 \), \( r_c = 0.05 \) and \( \alpha = 0.24 \). The reflective boundary conditions are used on the upper and lower boundary. The simulation is carried out with constant \( \Delta t = 1 \times 10^{-4} \). The discontinuity detector is used only on the pressure with \( \epsilon_s = 0.01 \).
The density contours at $t = 0.05$, $t = 0.2$ and $t = 0.35$ are shown in Fig. 11(a) for WENO and Fig. 11(b) for HYBRID, and Fig. 12 (a) and (b) shows the pressure contour at $t = 0.6$ and $t = 0.8$ for WENO and HYBRID, respectively. It can be seen that the shock is sharply captured in both HYBRID and WENO with the 5th-order WENO scheme at the same resolution. The moving vortex is captured by the SD method in HYBRID. It is shown that the shape of vortex after interacting with shock is still well defined in simulations with both methods. At $t = 0.8$, one of the shock bifurcations is reached the top boundary and reflected. Overall, the simulations by both method shows very similar results with sharp shock capturing and vortex preservation after interaction with shock. Figure 13 shows the SD elements where FV cells are embedded at $t = 0.35$ and $t = 0.8$. It can be seen that the shock is captured within one or two SD elements, of which it depends on the location of shock within the SD element. The weak reflected shock in HYBRID is captured by the SD method without severe oscillations as shown in Fig. 13(b).
5.4. Double Mach reflection

The double Mach reflection is commonly employed to test high resolution schemes \cite{55}. The physical domain in \( x \) and \( y \) is \([0, 4] \times [0, 1]\). This test case is carried out on two mesh sizes, \( 80 \times 20 \) and \( 160 \times 40 \) hexahedral SD elements, which gives \( 400 \times 100 \) and \( 800 \times 200 \) FV cells for WENO. A right-moving Mach 10 shock is positioned at \( x_0 = 1/6 \) at an angle of \( \pi/3 \) with \( x \) axis. The post-shock condition is \((\rho, u, v, w, p) = (8, 8.249, -0.07539, 0, 116.5)\) and \( \gamma = 1.4. \)
In the region from $x = 0$ to $x = 1/6$ at the bottom boundary, the exact post-shock condition is imposed. A reflective boundary condition is used for the remaining part at the bottom boundary. On the top boundary, the exact motion of initial Mach 10 shock is imposed with the position of the shock wave at time $t$ given by

$$s(t) = x_0 + \frac{1 + 20t}{\sqrt{3}}. \tag{55}$$

The left boundary condition is set by the initial post shock condition, and the right boundary condition is set by the zero gradient condition. The simulation is run with constant $\Delta t = 1 \times 10^{-5}$ until $t = 2.0$. The TVD limiter is used, and the discontinuity detector with $\epsilon_s = 0.02$ is used on the pressure only.

The density contours by WENO and HYBRID on two mesh sizes are shown in Fig. 14(a) and (b), respectively. The close-up views near Mach stems and jet formed are shown in Fig. 15. At low resolution, both the WENO and HYBRID produce comparable shock and jet structure. On the other hand, the jet structure is better resolved with HYBRID on the grid by $160 \times 40$ SD elements compared to the one by WENO on $800 \times 200$ FV grid for the same solution DOF. It should be mentioned that the results obtained by the currently implemented 5th-order WENO scheme on the $800 \times 200$ grid is comparable to the results shown in [11] and [56], but the results by the 5th-order WENO scheme can certainly be improved by using more advanced variants of WENO scheme. In Fig. 14(c), the location of SD elements where FV cells are embedded is shown, and it can be seen that FV cells are embedded following shocks and Mach stems. From this test case, it is noted that HYBRID performs better capturing dynamically developing smooth but complex flow structures than WENO due to much less numerical dissipation in SD method compared with WENO scheme.

### 5.5. Inviscid supersonic flow past cylinder

The inviscid supersonic flow past cylinder case is employed for testing the hybrid method on a curvilinear grid. The elliptic grid is constructed with $a = 3$ m and $b = 6$ m, where $a$ and $b$ are the minor and major axis of elliptic equation. The domain is discretized with $40 \times 30$ hexahedral SD elements, resulting in $200 \times 150$ FV cells for WENO. The supersonic flow of air at Mach number 2 passes over the cylinder with radius of 1 m. The flow fields are initialized with $\rho = 1.179$ kg/m$^3$, $p = 1$ atm, and $T = 298.15$ K and $u = 693.31$ m/s. The specific gas constant, $R$, is set to 288.18 J/kg K. On the cylinder wall, inviscid wall condition is imposed. The simulation is run with CFL = 0.3 until the solution is converged. The TVD limiter is employed, and the discontinuity detector threshold, $\epsilon_s$, is set to 0.05 in the discontinuity detector on the pressure.

The density and pressure contours are shown in Fig. 16(a) and (b), respectively. It can be seen that the shock is essentially captured at the resolution of embedded FV cells in HYBRID, which is the same as the resolution of WENO. Thus, the simulation results show comparable solution between HYBRID and WENO. Figure 17(a) shows the location of SD elements where FV cells are embedded. It should be noted that the FV cells are statically embedded on the cylinder wall in order to approximate cylinder wall at the resolution of embedded FV cells, as shown in Fig. 17(b). The issue with the curved boundary representation for high-order scheme has been addressed in [57].
Figure 14: Double Mach reflection. Density contours with 30 contour lines using (a) WENO and (b) HYBRID on two different grids with $N_x$ and $N_y$ denoting the number of SD elements in $x$ and $y$.

Figure 15: Double Mach reflection. The close-up view of density contours near the double Mach stems using (a) WENO and (b) HYBRID.
6. Discussion and Conclusion

In order to remedy the Gibbs phenomena in the spectral difference (SD) method associated with discontinuities in the flow fields, a hybrid SD/Embedded FV method for the hexahedral element is introduced. In this proposed hybrid method, structured FV cells are dynamically embedded in the hexahedral SD elements which contains discontinuities in the flow fields. The 5th-order WENO scheme with characteristic decomposition is currently employed as the shock-capturing scheme solving the governing equations on these embedded FV cells. Away from the discontinuities, the 5th-order SD method is used in the smooth region. The coupling between the SD elements and the elements with embedded FV cells are achieved by the mortar method.

The order of accuracy study shows that the error of hybrid method comes mainly from two error sources, which are (a) the interpolation error in coupling fluxes between SD elements and the elements with embedded FV cells and (b) the numerical dissipation error in WENO scheme. Although the order of accuracy is degraded in the hybrid method, it is seen that the error norm is still smaller than the currently implemented 5th-order WENO scheme, e.g. in the case of translating vortex. It should be noted that the purpose of embedding FV cells with WENO scheme is to capture the discontinuities, where the order of accuracy is essentially degraded in WENO scheme while the solution in the smooth region is obtained by the less dissipative SD method. Thus, it is seen that the solution is less affected by the reduction in the order of accuracy. This is demonstrated in the test cases comparing the simulations by the hybrid method and the standalone WENO scheme.

For the 1D and 2D test cases employed in this study, overall the hybrid method gives comparable or better results compared to the standalone WENO scheme considering the same DOF of solution points in both method. Particularly, when the complex flow structure dynamically develops, the hybrid method performs better compared with the standalone WENO scheme due to low numerical dissipation in SD method. This is shown in the double Mach reflection
test, where developing complex jet structure is better resolved with the hybrid method for the same solution DOF as the standalone WENO scheme. In terms of grid size, hybrid method with much coarser grid is able to produce comparable or better results as the ones on finer grid with WENO scheme. This may provide benefits of applying the proposed hybrid method for complex geometry with easing the grid preparation efforts.

The proposed hybrid method is presented in this paper using the structured hexahedral elements in order to deliver the main idea. However, it should be mentioned that the proposed hybrid method is designed to be applied for the unstructured hexahedral elements. Currently, this hybrid approach is being implemented for the unstructured hexahedral elements, and the results will be reported in the follow-up paper. In addition, variants of WENO schemes or other high-order shock-capturing schemes, with varying the order of the scheme and the number of embedded FV cells, will be examined in order to improve the order of accuracy.

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