ON THE PERIPHERAL SUBGROUPS OF IRREDUCIBLE 3-MANIFOLD GROUPS AND ACYLINDRICAL SPLITTINGS

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Abstract. We discuss the problem of separation under conjugacy and malnormality of the abelian peripheral subgroups of an orientable, irreducible 3-manifold $X$. We shall focus on the relation between this problem and the existence of acylindrical splittings of $\pi_1(X)$ as an amalgamated product or HNN-extension along the abelian subgroups corresponding to the JSJ-tori, providing new proofs of results in [WZ10] and [dlHW14b].

1. Introduction

A classical result proved in the late seventies by Jaco-Shalen and independently by Johannson (see [JS79], [Joh79]) says that an orientable, irreducible 3-manifold $X$ with (possibly empty) boundary can be splitted along a minimal collection of embedded, incompressible tori which are not boundary parallel and which decompose the manifold $X$ into connected components which are either atoroidal (i.e. any embedded, incompressible torus is boundary parallel) or Seifert fibered (a well understood class of 3-manifolds, see subsection §2.1). Such a collection is unique up to isotopy, and is called the collection of the JSJ-tori of $X$. The decomposition along this collection of tori is called JSJ-decomposition and the connected components obtained via the previous procedure are the JSJ-components of $X$. The set of atoroidal irreducible 3-manifolds and the set of Seifert fibered 3-manifolds have a non empty intersection: actually, atoroidal, Seifert fibered 3-manifolds are classified (see [JS79], §IV.2.5 and §IV.2.6). By Thurston’s Hyperbolization Theorem and by the work of Perelman, the interior of an atoroidal, non-Seifert fibered, irreducible 3-manifold can be endowed with a complete hyperbolic metric which has finite volume if and only if one of the following holds: either the manifold is closed or all of its boundary components are homeomorphic to tori. We shall refer to the atoroidal, non-Seifert fibered, irreducible 3-manifolds as to the manifolds of hyperbolic type. We refer to subsection §2.2 for further details and references.

We are interested in the study of the peripheral subgroups corresponding to the boundary tori of an orientable, irreducible 3-manifold $X$, i.e. the abelian peripheral subgroups of $\pi_1(X)$. The peripheral subgroups of the fundamental group of a given

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We say that a compact surface $S$ properly embedded in a compact 3-manifold $X$ is incompressible if for any embedded disk $D$ such that $\partial D \subset S$, there exists an embedded disk $D' \subset S$ such that $\partial D' = \partial D$.

A properly embedded compact submanifold $Y$ of a compact manifold $X$ is said to be boundary parallel if it can be isotoped into $\partial X$ by an isotopy fixing $\partial Y \subset \partial X$. 
compact 3-manifold $X$ are the subgroups of $\pi_1(X)$ which are images via the natural inclusions of the fundamental groups of the connected components of $\partial X$. In particular we shall focus on the following question: how do the abelian peripheral subgroups of $\pi_1(X)$ behave under conjugacy in $\pi_1(X)$?

In [dlHW14], de la Harpe and Weber proved the following:

**Theorem (dlHW14, Theorem 3).** Let $X$ be an orientable, irreducible 3-manifold $X$ and let $T \subset \partial X$ be a boundary torus. The peripheral subgroup $\pi_1(T)$ is malnormal in $\pi_1(X)$ if and only if $T$ bounds a JSJ-component of hyperbolic type.

The malnormality of a suitable family of subgroups may have interesting consequences. As an example, the previous result can be used to characterize those 3-manifold groups which are CA or CSA (see Corollary 2.5.11 in [AFW15]).

Given an orientable, irreducible 3-manifold $X$, we shall focus instead on the boundary tori of $X$ lying into Seifert fibered JSJ-components of $X$. The next result, together with Theorem 3 in [dlHW14], gives a picture of the cases that can occur (except for $D^2 \times S^1$, $T^2 \times I$, $K \times I$ — the twisted interval bundle over the Klein bottle — where the answer is trivial):

**Theorem 1.1.** Let $X$ be an orientable, irreducible 3-manifold with non-empty boundary and assume that $X$ is not homeomorphic to $D^2 \times S^1$, $T^2 \times I$, $K \times I$. Let $T^0_{(k_1,i_1)}$, $T^0_{(k_2,i_2)}$ denote two (possibly equal) boundary tori of $X$ contained into two (possibly equal) Seifert fibered JSJ-components $X_{k_1}$, $X_{k_2}$ of $X$.

(i) Assume that $(k_1,i_1) = (k_2,i_2) = (k,i)$ and let $g \in \pi_1(X) \setminus \pi_1(T^0_{(k,i)})$. Then

$$g\pi_1(T^0_{(k_1,i_1)}) g^{-1} \cap \pi_1(T^0_{(k_1,i_1)}) \neq \{1\}$$

if and only if $g \in \pi_1(X_k)$. In this case $g\pi_1(T^0_{(k,i)}) g^{-1} \cap \pi_1(T^0_{(k,i)}) = \langle f_k \rangle$, where $f_k$ is a regular fiber of $\pi_1(X_k)$.

(ii) Assume that $(k_1,i_1) \neq (k_2,i_2)$, and let $g \in \pi_1(X)$. Then

$$g\pi_1(T^0_{(k_1,i_1)}) g^{-1} \cap \pi_1(T^0_{(k_2,i_2)}) \neq \{1\}$$

if and only if $k_1 = k_2 = k$, and $g \in \pi_1(X_k)$. In this case we have $g\pi_1(T^0_{(k,i)}) g^{-1} \cap \pi_1(T^0_{(k,i)}) = \langle f_k \rangle$, where $f_k$ is a regular fiber of $\pi_1(X_k)$.

By a regular fiber of $\pi_1(X_k)$ we mean an element representing the homotopy class of the regular fiber of a Seifert fibration on $X_k$. It turns out (see Remark 2.6) that, in restriction to the cases which can appear in Theorem 1.1, the elements representing the homotopy classes of the regular fibers of the different Seifert fibrations of $X_k$ coincide (up to take inverses). This provides a univoquely determined regular fiber of $\pi_1(X_k)$, which depends only on $X_k$.

**Remark 1.2.** Theorem 1.1 and Theorem 3 in [dlHW14] will appear as consequences of Proposition 3.6 in this paper, where we study the intersection of the leaves stabilizers of the Bass-Serre tree of a suitable graph of groups associated to the manifold $X$ (see section §3).

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3Notice that the peripheral subgroups are defined up to the choice of basepoints $x_0$ in $X$, $x_i$ in $\partial_i X$ and of a path connecting $x_0$ to $x_i$.

4A subgroup $H < G$ is said to be malnormal in $G$ if $gHg^{-1} \cap H = \{1\}$ for any $g \in G \setminus H$.

5A CA group is a group $G$ such that for each element $g \in G$ the centralizer $C(g)$ is abelian; a CSA group is a group whose maximal abelian subgroups are malnormal.
Remark 1.3. Notice that the 3-manifolds excluded by the previous statements are precisely the only three orientable, Seifert fibered 3-manifolds with non-empty boundary having virtually abelian fundamental group.

We shall use Theorem 1.1 to give a proof of the existence of a particular class of splittings of the fundamental group of an orientable, irreducible 3-manifold \( X \). In order to state the result we shall briefly recall some facts about \( k \)-step malnormal amalgamated products, \( k \)-step malnormal HNN-extensions and \( k \)-acylindrical splittings.

In 1971 Karrass-Solitar (see [KS71]) introduced the notion of \( k \)-step malnormal amalgamated product. Roughly speaking, a \( k \)-step malnormal amalgamated product \( A \ast_C \mathbb{B} \) is an amalgamated product where the amalgam \( C \) is malnormal with respect to the elements having syllable length greater or equal to \( k + 1 \). In analogy we can define a \( k \)-step malnormal HNN-extension as a HNN-extension \( A \ast_{C, \varphi} \), with respect to an isomorphism of subgroups \( \varphi : C_1 \rightarrow C_1 \), such that the associated subgroups, \( C_{\pm 1} \), are malnormal and conjugately separated in \( A \ast_{\varphi} \) with respect to the set of elements having a reduced form containing at least \( k + 1 \) times the stable letter \( t \) to the power \( \pm 1 \) (see subsection §4.1.1 for a precise definition).

These notions have a natural generalization in the context of Bass-Serre theory: the notion of \( k \)-acylindrical splitting introduced by Sela in [Sel97]. We recall that a finitely generated group \( G \) admits a \( k \)-acylindrical splitting if there exists a graph of groups \( (G, \Gamma) \) such that \( G = \pi_1(G, \Gamma) \) and the action of \( G \) on the corresponding Bass-Serre tree \( T_{(G, \Gamma)} \) is \( k \)-acylindrical, i.e. the diameter of the fixed points set of any element of \( G \) is smaller or equal to \( k \). It turns out that a splitting of \( G \) as a \( k \)-step malnormal amalgamated product gives a \((k + 1)\)-acylindrical splitting and a splitting of \( G \) as a \( k \)-step malnormal HNN-extension is a \( k \)-acylindrical splitting. Conversely, a \( k \)-acylindrical splitting whose underlying graph has only one edge with two distinct vertices corresponds to a splitting as a \((k - 1)\)-step malnormal amalgamated product and a \( k \)-acylindrical splitting having a loop as underlying graph corresponds to a \( k \)-step malnormal HNN-extension. We shall explain this correspondence in subsection §4.1.2.

The existence of acylindrical splittings for the fundamental groups of closed, orientable, irreducible 3-manifolds with non-trivial JSJ-decomposition, which are not \( Sol \)-manifolds, has been proved by Wilton-Zalesskii (see [WZ10], section §2). The main ingredients of their proof are explained in full detail in section §2, and their approach inspired the proof of Proposition 3.6. We shall work with compact 3-manifolds with boundary, but no claim of originality is made as Wilton-Zalesskii’s argument extends to this case without effort. Given an orientable, irreducible 3-manifold \( X \) we shall look to “more elementary” splittings of \( \pi_1(X) \) than the one considered in [WZ10]. Namely, given the collection \( \{T_i\} \) of the JSJ-tori of \( X \), for each \( i \) we shall study the splitting of \( \pi_1(X) \) along the subgroup \( \pi_1(T_i) \) as an amalgamated product or HNN-extension, rather than the splitting provided by the whole collection. This gives acylindrical splittings of \( \pi_1(X) \) as an amalgamated product or HNN-extension, whose constant \( k \) is often smaller than the one of the JSJ-splitting. The interest in finding acylindrical splittings comes for instance from Riemannian geometry and geometric group theory: we refer to [CS17a] and [CS17b] where acylindrical splittings are exploited to prove some finiteness results for various classes of Riemannian manifolds and groups, and to [CS17c] where a
systolic estimate and some topological rigidity results are proven for non-geometric 3-manifolds. As we shall consider splittings of $\pi_1(X)$ as an amalgamated product or HNN-extension, we prefer to speak of malnormal splittings, since this does not require any reference to Bass-Serre theory.

**Proposition 1.4** (Malnormal splittings). Let $X$ be an orientable, irreducible 3-manifold having a non-trivial JSJ-decomposition and which is not finitely covered by a torus bundle over the circle. Let $T$ be a JSJ-torus of $X$. We denote by $T^\pm$ the boundary tori of $X$ obtained by cutting $X$ along $T$, and by $X_{k\pm 1}$ the JSJ-components of $X$ adjacent to $T$. Let $\varphi : \pi_1(T^{-1}) \to \pi_1(T^{+1})$ be the isomorphism given by the gluing.

**D1** Assume that $T$ separates $X$. Let $X_{k\pm 1}$ be hyperbolic JSJ-components. Then $\pi_1(X)$ splits as a 0-step malnormal product.

**D2** Assume that $T$ separates $X$. Let $X_{k\pm 1}$ be a hyperbolic JSJ-component and $X_{k+1}$ be a Seifert fibered JSJ-component. Then $\pi_1(X)$ splits as a 1-step malnormal product.

**D3** Assume that $T$ separates $X$. Let $X_{k\pm 1}$ be Seifert fibered JSJ-components not homeomorphic to $K\tilde{\times}I$. Then $\pi_1(X)$ splits as a 1-step malnormal product.

**D4** Assume that $T$ separates $X$. Let $X_{k\pm 1}$ be a Seifert fibered JSJ-component with hyperbolic base orbifold and $X_{k+1} \simeq K\tilde{\times}I$. Then $\pi_1(X)$ splits as a 3-step malnormal product.

**ND1** Assume that $T$ does not separate $X$. Let $X_{k\pm 1}$ be two (possibly equal) hyperbolic JSJ-components. Then $\pi_1(X)$ splits as a 1-step malnormal HNN-extension.

**ND2** Assume that $T$ does not separate $X$. If we are not in case (ND1) then $\pi_1(X)$ splits as a 2-step malnormal HNN-extension.

**Remark 1.5.** The assumption “not finitely covered by a torus bundle” is necessary to avoid the case of $\text{Sol}$-manifolds, which do not admit this kind of splittings (see [WZ10], §2). Actually, as a byproduct of Theorem 3.2 and of Example 5.1 in [CS17c], $\text{Sol}$-manifolds do not admit any kind of acylindrical splitting.

**Remark 1.6.** Proposition 1.4 is to compare with Lemma 2.2 and Lemma 2.4 in [WZ10]. Notice that it can be obtained from Wilton-Zalesskii’s result by collapsing subgraphs of the JSJ-splitting as explained in [Kap01a], Proposition 3.6. We shall provide a proof which does not make use of Bass-Serre theory. It is worth to say here that analogous results have been proved for some classes of high dimensional graph manifolds in [FLS15].

In section §2 we introduce some background material. In particular, we shall survey topological results concerning Seifert fibered manifold, focusing on the properties of their fundamental groups.

Section §3 is devoted to the proof of Theorem 1.1. We shall give a simultaneous proof of Theorem 1.1 and of Theorem 3 in [dlHW14b] by looking at the Bass-Serre tree associated to a particular extension of the JSJ-splitting of the fundamental group (see Proposition 3.6). Key ingredients in the proof are three lemmata: Lemma 2.7 and Lemma 2.10 which describes the algebraic properties of the abelian peripheral subgroups when $X$ has an empty JSJ-decomposition, and Lemma 2.9 which provides a necessary and sufficient condition for a map between two boundary tori of Seifert fibered manifolds to give rise to a graph manifold.
2. Basic facts about $3$-manifolds and $3$-manifold groups

This section is devoted to survey some classical facts of $3$-manifold geometry and topology regarding Seifert fibered manifolds, atoroidal manifolds and the JSJ-decomposition. We are mainly interested in their consequences on the algebraic structure of $3$-manifold groups. All the $3$-manifolds considered in this section and in the rest of the paper are assumed to be orientable. General references for the results stated in this section are the following: [Sco83], [FM97], [Thu97], [Hat00], [Bon02], [AFW15] (Chapters 1 and 2), [Mar16].

2.1. Seifert fibered manifolds

We shall briefly review some topological result concerning Seifert fibered $3$-manifolds.

2.1.1. Topological construction of Seifert fibrations and their $π_1$. We recall that any Seifert fibration on a $3$-manifold can be constructed as follows. Let $Σ_{g,h}$ be a surface of genus $g ∈ ℤ$, having $h$ boundary components. Excise from $Σ_{g,h}$ a set of $k ≥ 0$ disks $D = \{D_1,..,D_k\}$. This gives rise to $k$ new boundary components, represented by the loops $c_1,..,c_k$. Let us consider $(Σ_{g,h} \setminus D)^{\sim}$ the (unique) orientable $S^1$-bundle over $Σ_{g,h} \setminus D$ and let $f$ denote the homotopy class of the fiber of the $S^1$-bundle. Glue $k$ copies of the solid torus $D^2 × S^1$ to the boundary tori of $(Σ_{g,h} \setminus D)^{\sim}$ which project on the loops $c_i$ in the following way: choose a slope $p_i,c_i + q_if$, with $(p_i,q_i)$ a pair of coprime integers with $p_i > 0$, and send it in the meridian loop of the $i$th solid torus $D^2 × S^1$. This procedure is known as Dehn filling along the curve $p_i,c_i + q_if$. We shall denote $S(g,h; (p_1,q_1),..,(p_k,q_k))$ the Seifert fibration obtained via the previous construction.

Let $X$ be a Seifert fibered manifold and let $S = S(g,h; (p_1,q_1),..,(p_k,q_k))$ be a Seifert fibration on $X$. There is a presentation $ℙ(S)$ of $π_1(X)$ canonically associated to $S$ which can be obtained from the topological construction using Van-Kampen’s Theorem:

(i) $g ≥ 0$

\[
ℙ(S) = \left\langle a_1, b_1,..,b_y,c_1,..,c_k,d_1,..,d_h, f \mid \prod_{i=1}^{g} [a_i,b_i] \cdot \prod_{j=1}^{h} c_j \cdot \prod_{\ell=1}^{h} d_\ell = 1, \quad \prod_{i=1}^{y} b_i f b_i^{-1} = 1, \quad f \leftrightarrow a_i, b_i, c_j, d_\ell \right\rangle
\]

(ii) $g < 0$

\[
ℙ(S) = \left\langle a_1,..,a_{|g|},c_1,..,c_k,d_1,..,d_h, f \mid \prod_{i=1}^{|g|} a_i^2 \cdot \prod_{j=1}^{h} c_j \cdot \prod_{\ell=1}^{h} d_\ell = 1, \quad a_i f a_i^{-1} f^{-1} = 1, \quad f \leftrightarrow c_j, d_\ell \right\rangle
\]
2.1.2. **Classification of Seifert fibrations up to (oriented) isomorphisms.** It is natural to ask when two Seifert fibrations $S$ and $S'$ are oriented isomorphic, i.e. when there exists an orientation preserving homeomorphism which sends fibers into fibers. It turns out that given $S = S(g, h; (p_1, q_1), ..., (p_k, q_k))$ the following moves do not change its oriented isomorphism type (see [JN83] Theorem 1.5 or [Mar16] Proposition 10.3.11):

1. $\{(p_i, q_i), (p_j, q_j)\} \mapsto \{(p_i, q_i - p_i), (p_j, q_j + p_j)\}$;
2. $\{(p_1, q_1), ..., (p_k, q_k)\} \leftrightarrow \{(p_1, q_1), ..., (p_k, q_k), (1, 0)\}$;
3. $(p_i, q_i) \mapsto (p_i, q_i \pm p_i)$, if $h > 0$;
4. permutations of the indices of the collection of pairs $\{(p_i, q_i)\}$.

Moreover, two Seifert fibrations are oriented isomorphic if and only if they are related via a finite sequence of these moves (see [Mar16], Proposition 10.3.11).

**Remark 2.1.** The Seifert fibrations

$$S(g, h; (p_1, q_1), ..., (p_k, q_k)), \quad S(g, h; (p_1, -q_1), ..., (p_k, -q_k))$$

are isomorphic via the homeomorphism which sends fibers into fibers with the opposite orientation, but they are not in general oriented isomorphic.

These moves allow to classify Seifert fibered manifolds up to (oriented) isomorphism, but before to give the precise statement we need to introduce a useful invariant: the Euler number of a Seifert fibration. It is a well known fact that the oriented $S^1$-bundles over a closed surface $\Sigma_g$ can be realized as Seifert fibrations $\{S(g, 0; (1, q))\}_{q \in \mathbb{Z}}$ (see for instance [Mon87], Chapter 1). A classical result of algebraic topology says that oriented $S^1$-bundles over closed surfaces are classified by their Euler number, i.e. the integral over the base surface of a suitable characteristic class, the Euler class. The Euler number can be seen as an obstruction for the $S^1$-bundle to admit a global section. Given the $S^1$-bundle $S(g, 0; (1, q))$, we know that, by construction, it has a section $s$ in $\Sigma_g \setminus \{x\}$ where $x$ is the projection of the central fiber of the fibered solid torus. It turns out that the Euler number coincides ([BT82], Theorem 11.16) with the local degree of $s$ at $x$, i.e. with $q$. Given a general Seifert fibration $S = S(g, h; (p_1, q_1), ..., (p_k, q_k))$, we define its Euler number as $e(S) = \sum_{i}^{k} \frac{q_i}{p_i}$, where the previous sum is defined only mod $\mathbb{Z}$ when $h \neq 0$.

Notice that, by definition, the Euler number (possibly mod $\mathbb{Z}$) is invariant under the moves (1)-(4). This definition of the Euler number coincides for $S^1$-bundles over closed surfaces with the classical one. The following proposition holds (see [JN83] Theorem 1.5 or [Mar16], Corollary 10.3.13):

**Proposition 2.2** (Classification of Seifert fibration up to isomorphism). **Consider** $S = S(g, h; (p_1, q_1), ..., (p_k, q_k))$ and $S' = S(g', h'; (p'_1, q'_1), ..., (p'_k, q'_k))$, two Seifert fibrations with $p_i, p'_i \geq 2$. They are (orientation preservingly) isomorphic if and only if $g = g'$, $h = h'$, $k = k'$, $e(S) = e(S')$, and, up to reordering the indices, $p_i = p'_i$, $q_i = q'_i$ mod $p_i$.

**Remark 2.3.** Let $S' = S(g, h; (p'_1, q'_1), ..., (p'_k, q'_k))$ with $h > 0$ be a Seifert fibration. A consequence of Proposition 2.2 is that either $S'$ is isomorphic to $S = S(g, h; (p_1, q_1), ..., (p_k, q_k))$ with $0 < q_i < p_i$ or $S'$ is isomorphic to $S = S(g, h; \cdot)$, and the two possibilities are mutually exclusive. These Seifert fibrations are canonical in their isomorphism class. In fact, they are the unique enjoying these properties.
2.1.3. Classification of Seifert manifolds up to diffeomorphism. When does a Seifert manifold \( X \) admit non-isomorphic Seifert fibrations? The next result ([Jac80] Theorems VI.17 and VI.18) answer the question:

**Proposition 2.4** (Classification of Seifert fibered manifolds up to diffeomorphism). Every Seifert fibered manifold with non-empty boundary admits a unique fibration up to isomorphism, except in the following cases:

- \( D^2 \times S^1 \), which admits the fibrations \( S = S(0,1;(p,q)) \) for any pair of coprime integers \( (p,q) \);
- \( K \times I \), which admits two non-isomorphic Seifert fibrations:
  \[ S_1 = S(-1,1; ), S_2 = S(1,1;(2,1),(2,1)). \]

Every closed Seifert fibered manifold \( X \) which is not covered by \( S^3 \) or \( S^2 \times S^1 \) admits a single Seifert fibration up to isomorphism, except for \( K \times S^1 \) which admits the non-isomorphic fibrations \( S_3 = S(0,0;(2,1),(2,1),(2,-1),(2,-1)) \) and \( S_4 = S(-2,0; ). \)

2.1.4. Seifert fibrations as \( S^1 \)-bundles over 2-dimensional orbifolds. There is a second way to look at (oriented) Seifert fibered manifolds: they can be seen as \( S^1 \)-bundles over 2-dimensional orbifolds having only conical singular points. We refer to the classical references [Sco83] and [Thu97] for an account on 2-dimensional orbifolds. It is worth to stress that this point of view is particularly helpful in order to show the existence of complete, locally homogeneous metrics on Seifert fibered manifolds ([Sco83], [Ohs87], [Bon02]).

Given \( S = S(g,h;(p_1,q_1),...,(p_k,q_k)) \), we associate to \( S \) its base orbifold \( O_S \). The orbifold \( O_S \) has an underlying topological space the surface of genus \( g \) and \( h \) boundary components, and \( k \) conical singular points of order \( p_1,...,p_k \). There is a natural extension of the usual Euler characteristic to 2-dimensional orbifolds, known as the orbifold characteristic (see [Sco83] or [Bon02]), denoted \( \chi_{orb} \). The orbifold characteristic allows in first instance to distinguish between the orbifolds admitting a manifold cover (good orbifolds) and those who do not admit such a cover (bad orbifolds).

A complete list of good 2-dimensional orbifolds can be found in [Sco83]. In particular, the bad 2-dimensional orbifolds which may appear as base orbifolds of Seifert fibered manifolds are either of type \( S^2(p) \) (underlying surface \( S^2 \) and a single conical singular point of order \( p \)) or of type \( S^2(p,q) \) (underlying surface \( S^2 \) and two conical singular point of order \( p \) and \( q \) with \( \gcd(p,q) = 1 \)). The Seifert fibered manifolds which fibers over bad 2-dimensional orbifolds belong to a specific and well understood class of quotients of the 3-sphere: the class of lens spaces (see [LN83], section I.4) which in particular have fundamental groups isomorphic to finite cyclic groups. Secondly, good orbifolds carry geometric structures and the orbifold characteristic detects which kind of geometry a good orbifold carries: namely, good orbifolds are divided into elliptic, Euclidean and hyperbolic orbifolds depending whether \( \chi_{orb} > 0 \), \( \chi_{orb} = 0 \) or \( \chi_{orb} < 0 \) (see [Thu97], Theorem 13.3.6).

Our focus is on Seifert fibered manifolds with non-empty boundary, which clearly fibers over 2-orbifolds with boundary. The set of 2-orbifolds with boundary consists of the collection of disks having a unique singular point, three Euclidean orbifolds and hyperbolic orbifolds. The Euclidean orbifolds are: \( D^2(2,2) \), the disk with two singular points of order 2, \( Mb \), the Möbius strip, and \( C \), the cylinder. Notice that there are only two Seifert fibered manifolds fibering over these orbifolds: \( K \times I \), fibering over \( D^2(2,2) \) or over \( Mb \), and \( T^2 \times I \), fibering over \( C \). Finally, there is a
unique Seifert fibered manifold fibered over the collection of disks with a unique singular point: the solid torus $D^2 \times S^1$. Hence, except for $K \times I$, $T^2 \times I$, $D^2 \times S^1$, all Seifert fibered 3-manifolds with boundary are fibered over hyperbolic base orbifolds.

2.1.5. The short exact sequence. The description of Seifert fibered manifolds as $S^1$-bundles over 2-dimensional orbifolds provides a short exact sequence (see [Sco83]):

$$1 \to \langle f \rangle \to \pi_1(S) \to \pi_1^{orb}(O_S) \to 1$$

where $\pi_1^{orb}(O_S)$ is the orbifold fundamental group (for the definition see for instance [BMP03], Chapter 2) and the surjective morphism corresponds to the quotient by the normal subgroup of the fiber of $S$. We thus obtain the following presentations for the orbifold fundamental groups:

$$\pi_1^{orb}(O_S) = \left< \bar{a}_1, \bar{b}_1, ..., \bar{b}_g, \bar{c}_1, ..., \bar{c}_k, \bar{d}_1, ..., \bar{d}_h \mid \prod_{i=1}^g \bar{a}_i \cdot \prod_{j=1}^k \bar{c}_j \cdot \prod_{\ell=1}^h \bar{d}_\ell = 1 \right>$$

$$\pi_1^{orb}(O_S) = \left< \bar{a}_1, ..., \bar{a}_g, \bar{c}_1, ..., \bar{c}_k, \bar{d}_1, ..., \bar{d}_h \mid \prod_{i=1}^g \bar{a}_i^2 \cdot \prod_{j=1}^k \bar{c}_j \cdot \prod_{\ell=1}^h \bar{d}_\ell = 1 \right>$$

depending on the sign of $g$, i.e. depending on the orientability of the topological space underlying the base orbifold. In the previous presentations is understood that $p(a_i) = \bar{a}_i$, $p(b_i) = \bar{b}_i$, $p(c_j) = \bar{c}_j$, $p(d_\ell) = \bar{d}_\ell$, where $p : \pi_1(S) \to \pi_1^{orb}(O_S)$ is the surjective morphism of the exact sequence.

Lemma 2.5. If $h > 0$ and $O_S$ is a hyperbolic orbifold, then the collection of subgroups $\{\langle \bar{d}_i \rangle\}_{i=1}^h$ is a collection of malnormal, conjugately separated subgroups in $\pi_1^{orb}(O_S)$

Proof. Since $O_S$ is a compact 2-orbifold with non empty boundary we can express $\bar{d}_h$ in terms of the other generators. We see that $\pi_1^{orb}(O_S) \cong \mathbb{F}_{2g+h-1} \ast \mathbb{Z}_{p_1} \ast \cdots \ast \mathbb{Z}_{p_k}$ freely generated by $\{\bar{a}_i, \bar{b}_i\}_{i=1}^g \cup \{\bar{d}_i\}_{\ell=1}^h \cup \{\bar{c}_j\}_{j=1}^k$ if the genus $g$ of $|O|$ is positive and $\pi_1^{orb}(O_S) \cong \mathbb{F}_{g+h-1} \ast \mathbb{Z}_{p_1} \ast \cdots \ast \mathbb{Z}_{p_k}$ freely generated by $\{\bar{a}_i\}_{i=1}^g \cup \{\bar{d}_i\}_{\ell=1}^h \cup \{\bar{c}_j\}_{j=1}^k$ if the genus $g$ of $|O_S|$ is negative. Moreover, as the orbifold $O_S$ has negative orbifold Euler characteristic we see that the previous free products are always different from $\mathbb{Z}_2 \ast \mathbb{Z}_2$. Observe that $\{\langle \bar{d}_i \rangle\}_{i=1}^h$ is a collection of malnormal, conjugately separated infinite cyclic subgroups. In fact, they represent by definition $h - 1$ infinite cyclic free factors of the free product which determines $\pi_1^{orb}(O_S)$, because $g \bar{d}_h g^{-1}$ has syllable length greater than 1, unless $g = \bar{d}_k$ for some $k \in \mathbb{Z}$. On the other hand, writing down the expression for $\bar{d}_h$ in terms of the other generators we see that it is a primitive element of infinite order in a free product of cyclic groups, which can be represented as a cyclically reduced word of length strictly greater than one. It follows that $\bar{d}_h$ is not conjugate to any other $\bar{d}_i$ (because their cyclically reduced length is equal to 1). On the other hand, if $g \bar{d}_h g^{-1} = \bar{d}_h$ then we would have $g^2 \bar{d}_h g^{-2} = \bar{d}_h$. Since the centralizer of a primitive element of cyclically reduced length greater than 1 in a free product is infinite cyclic, generated by the primitive element we see that $g^2 = d_h$. But in a free product of cyclic groups any element of infinite order possesses a unique root, and since $\bar{d}_h$ is primitive we conclude that $k = 2m$ and $g = \bar{d}_m$, proving that $\langle \bar{d}_m \rangle$ is malnormal in $\pi_1^{orb}(O_S)$.

Remark 2.6 (The regular fiber). Let $X$ be a Seifert fibered 3-manifold with non-empty boundary, not homeomorphic to $T^2 \times I$ and $D^2 \times S^1$. We recall (subsection §2.1.1) that any Seifert fibration of $X$ determines a presentation of $\pi_1(X)$ and a maximal, infinite cyclic, normal subgroup: the subgroup corresponding to the
Further remarks about binomial group theory arguments using Lemma 2.5 and the short exact sequence. It will play an important role in the proof of Theorem 1.1. It can be proved via general subgroups and the regular fibers of manifolds resuming in the next lemma few straightforward facts about the peripheral subgroups of the Seifert fibration Lemma 2.8.

Let JSJ-decomposition Theorem. Let embedded, non boundary parallel, incompressible tori ([JS79], [Joh79]): On the particular fibration. The previous lemma is implicit in Wilton-Zalesskii’s proof of the existence of acylindrical splittings for irreducible 3-manifold groups ([WZ10], section §2) and will play an important role in the proof of Theorem 1.1. It can be proved via combinatorial group theory arguments using Lemma 2.3 and the short exact sequence.

2.1.6. Further remarks about \( \pi_1(K \times I) \). We conclude this section on Seifert fibered manifolds resuming in the next lemma few straightforward facts about the peripheral subgroups and the regular fibers of \( \pi_1(K \times I) \).

Lemma 2.7. Let \( X \) be a Seifert fibered manifold with non-empty boundary which fibers over a hyperbolic base orbifold and let \( f \in \pi_1(X) \) be the regular fiber:

(i) If \( T^0 \subset \partial X \) is a boundary torus and \( g \in \pi_1(X) \setminus \pi_1(T^0) \), then
\[
g \pi_1(T^0) g^{-1} \cap \pi_1(T^0) = \langle f \rangle
\]
(ii) If \( T^0_1, T^0_2 \subset \partial X \) are two boundary tori and \( g \in \pi_1(X) \), then
\[
g \pi_1(T^0_1) g^{-1} \cap \pi_1(T^0_2) = \langle f \rangle
\]

The previous lemma is implicit in Wilton-Zalesskii’s proof of the existence of acylindrical splittings for irreducible 3-manifold groups ([WZ10], section §2) and will play an important role in the proof of Theorem 1.1. It can be proved via combinatorial group theory arguments using Lemma 2.3 and the short exact sequence.

The homotopy class of the regular fibers of the Seifert fibration. From Proposition 2.4 we know that if \( X \) has non-empty boundary and it is not homeomorphic to \( T^2 \times I \), \( D^2 \times I \) or \( K \times I \), all Seifert fibrations are isomorphic. On the other hand, by the explicit description of the fundamental group of a Seifert fibered manifold and arguments similar to the ones used in Lemma 2.5 it is not difficult to check that \( \pi_1(X) \) possesses a unique maximal, infinite cyclic normal subgroup. Hence, despite the fact that distinct Seifert fibrations of \( X \) in the same isomorphism class give rise to distinct presentations of \( \pi_1(X) \), they all determine the same maximal, infinite cyclic, normal subgroup. In view of this fact we shall call the generator — uniquely defined up to take inverses — of this unique maximal, infinite cyclic, normal subgroup the regular fiber of \( \pi_1(X) \), since this element does no longer depend on the particular fibration.

Similarly, if we look at \( K \times I \) there are only two isomorphism classes of Seifert fibrations, which are identified by the only two distinct maximal, infinite cyclic, normal subgroups in \( \pi_1(K \times I) = \langle a, f \mid a f a^{-1} = f^{-1} \rangle \), namely the subgroups \( \langle a^2 \rangle \) and \( \langle f \rangle \). We shall call \( a^2 \) and \( f \) the regular fibers of \( K \times I \).

The next lemma states that the peripheral subgroups \( \langle d_i, f \rangle = p^{-1}(\langle d_i \rangle) \) of \( S \) partially inherit the properties of the peripheral subgroups of \( \partial S \):

Lemma 2.8. Let \( \pi_1(K \times I) = \langle a, f \mid a f a^{-1} = f^{-1} \rangle \) be the presentation associated to the Seifert fibration \( S(-1,1) \) of \( K \times I \). Then:

(i) \( \pi_1(\partial(K \times I)) = \langle a^2, f \rangle \) is a normal subgroup of \( \pi_1(K \times I) \);
(ii) the two regular fibers of \( \pi_1(K \times I) \) are \( a^2 \) and \( f \);
(iii) if \( g = a^{2p} f^q \) (with \( \gcd(p,q) = 1 \)) is a primitive element in \( \pi_1(K \times I) \) which is not equal to \( a^{\pm 2} \) or \( f^{\pm 1} \), then \( a^{\pm 1} (a^{2p} f^q) a^\pm \cap \langle a^{2p} f^q \rangle = \{1\} \).

2.2. The JSJ-decomposition Theorem. One of the cornerstones of 3-manifolds theory is the existence of a decomposition for compact, irreducible 3-manifolds along embedded, non boundary parallel, incompressible tori ([JS79], [Joh79]):

JSJ-decomposition Theorem. Let \( X \) be a compact, irreducible 3-manifold. In the interior of \( X \), there exists a family \( C = \{T_1, ..., T_r\} \) of disjoint tori that are incompressible and not boundary parallel, with the following properties:
(i) each connected component of $X \setminus C$ is either a Seifert manifold or is atoroidal;

(ii) the family $C$ is minimal among those satisfying (i).

Moreover, such a family $C$ is unique up to ambient isotopy.

Following Thurston [Thu82], we say that an irreducible 3-manifold $X$ is homotopically atoroidal if every $\pi_1$-injective map from the torus to the irreducible manifold is homotopic to a map into the boundary. Being homotopically atoroidal is a stronger property than being atoroidal, where we consider embeddings instead of $\pi_1$-injective maps. Nevertheless, the two notions agree except for some Seifert fibered manifolds. In fact, the atoroidal, Seifert fibered manifolds are the 3-manifolds which admit one of the Seifert fibrations from the list IV.2.5 in [JS79] (see [JS79] Lemma IV.2.6). Among the manifold in the list only a few of them are homotopically atoroidal. To our purposes it is sufficient to say that the list of compact, homotopically atoroidal, Seifert fibered manifold with non-empty boundary is reduced to the following list: $K\tilde{\times}I, T^2 \times I, D^2 \times S^1$.

In 1982 ([Thu82]) Thurston proposed the Geometrization conjecture and announced a series of papers proving the Hyperbolization Theorem for Haken 3-manifolds ([Thu86], [Thu98a], [Thu98b] —the last two are unpublished—):

**Hyperbolization Theorem** (Thurston). The interior of a compact, irreducible, Haken 3-manifold admits a complete hyperbolic metric if and only if the manifold is homotopically atoroidal and not homeomorphic to $K\tilde{\times}I$.

A complete proof of Thurston’s Hyperbolization Theorem can be found in [Ota96], [Ota98] and [Kap01b]. Since compact, irreducible 3-manifolds with non-empty boundary and not homeomorphic to $D^2 \times S^1$ are Haken manifolds, Thurston’s result implies that such a 3-manifold admits a complete hyperbolic metric if and only if is homotopically atoroidal and not homeomorphic to $K\tilde{\times}I$. Moreover, if the 3-manifold has only toroidal boundary and in addition is not homeomorphic to $T^2 \times I$, the complete hyperbolic metric has finite volume. Thanks to the work of Perelman ([Per02], [Per03a], [Per03b]) we are allowed to replace the assumption “Haken” with the assumption “with infinite fundamental group”. Coherently with the picture provided by the Hyperbolization Theorem it is customary to call manifolds of hyperbolic type the homotopically atoroidal, irreducible 3-manifolds (possibly with non empty boundary) not homeomorphic to $D^2 \times S^1, T^2 \times I$ or $K\tilde{\times}I$.

Assume that $X \neq T^2 \times I, D^2 \times S^1$ and $\partial X \neq \emptyset$. Then $T^2 \times I, D^2 \times S^1$ cannot appear as a JSJ-component: the first by minimality of the collection $C$; the second because a JSJ-torus bounding $D^2 \times S^1$ would not be incompressible. Hence if $X_k$ is a JSJ-component of $X$ one of the following mutually exclusive conditions holds:

- $X_k$ is of hyperbolic type;
- $X_k$ is Seifert fibered with hyperbolic base orbifold;
- $X_k$ is homeomorphic to $K\tilde{\times}I$.

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6In order to check which closed Seifert fibered manifold among the atoroidal are homotopically atoroidal a way to proceed is to compute the fundamental groups and consider the ones which do not admit a subgroup isomorphic to $\mathbb{Z}^2$.

7We recall that an Haken 3-manifold is an irreducible 3-manifolds which contains a properly embedded, 2-sided, incompressible surface.
The next lemma is well known (see for example [AFW15, Lemma 1.5.3) and it gives a necessary and sufficient condition for the gluing of two Seifert fibered 3-manifolds not to be Seifert fibered.

**Lemma 2.9.** Let $X, Y \neq D^2 \times S^1$ be two Seifert fibered manifolds (possibly $X = Y$) with non-empty boundary. Let $g : T^0_X = T^0_Y$ be a diffeomorphism between two boundary tori of $X$. Let $Z$ be the resulting manifold. The following are equivalent:

(i) the manifold $Z$ is not Seifert fibered;
(ii) for any choice of regular fibers $f_X$ in $\pi_1(X)$ and $f_Y$ in $\pi_1(Y)$ the following condition holds: $g_*(\langle f_X \rangle) \cap \langle f_Y \rangle = \{1\}$.

Finally we record a result concerning the abelian peripheral subgroups of manifolds of hyperbolic type. In analogy with the case of hyperbolic 2-orbifolds with boundary (Lemma 2.3), the collection of the abelian peripheral subgroups of an irreducible 3-manifold of hyperbolic type are malnormal and conjugately separated:

**Lemma 2.10.** Let $X$ be an irreducible 3-manifold of hyperbolic type with non-empty boundary and let $T^0_1, T^0_2, T^0_3$ be toroidal boundary components in $\partial X$.

(i) if $g \in \pi_1(X) \setminus \pi_1(T^0_1)$, then $g \pi_1(T^0_1)g^{-1} \cap \pi_1(T^0_1) = \{1\}$;
(ii) if $g \in \pi_1(X)$, then $g \pi_1(T^0_1)g^{-1} \cap \pi_1(T^0_3) = \{1\}$.

A reference for the previous result is [dHW14a, §3, Example 6].

3. Abelian peripheral subgroups of irreducible 3-manifolds

Throughout this section $X$ will be an irreducible 3-manifold with non-empty boundary, not homeomorphic to $K \times I$, $T^2 \times I$, $D^2 \times S^1$. Let $\{X_k\}_{k=1}^N$ be the JSJ-components of $X$, let $\{T^0_{(k,i)}\}_{i=1}^{m_k}$ be the set of the boundary tori of $X$ which belong to the JSJ-component $X_k$ (possibly $m_k = 0$, if the previous collection is empty) and $\{T^i_{(k_1,k_2)}\}_{j=1}^{n_{(k_1,k_2)}} (k_1 \leq k_2$, possibly $n_{(k_1,k_2)} = 0)$ be the collection of the JSJ-tori which bound both the JSJ-components $X_{k_1}, X_{k_2}$.

We shall also assume to have fixed:

- a basepoint $x_0$ in $X$, basepoints $x_k$ in the interior of each JSJ-component $X_k$, and paths from $x_0$ to the basepoints $x_k$;
- a basepoint for each boundary torus of the JSJ-component $X_k$ and paths connecting them to the chosen basepoint $x_k \in X_k$.

With this proviso the fundamental groups of the tori $T^0_{(k,i)}$, $T^i_{(k,j)}$, $T^r_{(k,l)}$ are naturally identified with suitable subgroups of $\pi_1(X_k)$, and the fundamental groups of the JSJ-components are naturally identified with subgroups of $\pi_1(X)$.

3.1. **The peripheral extension of the JSJ-splitting.** We shall first recall the definition of the JSJ-splitting of the fundamental group of the irreducible 3-manifold $X$. For the general theory of graph or groups we refer to [Bas76, Ser77, Ser80].

**Definition 3.1 (JSJ-splitting).** We define the JSJ-splitting of $\pi_1(X)$ as the graph of groups $(\mathcal{G}_X, \Gamma_X)$ of $\pi_1(X)$ (with the assignement of an orientation for the edges) defined in the following way:

- $\Gamma_X$ is the graph defined by the vertices $V(\Gamma_X) = \{v_k\}_{k=1}^N$, and the edges $E(\Gamma_X) = \{e_{(k_1,k_2)} \mid k_1 \leq k_2, k_1 = 1, \ldots, N \text{ and } j = 1, \ldots, n_{(k_1,k_2)}\}$ where $e_{(k_1,k_2)} = \overline{e}_{(k_1,k_2)}$ and where $s(e_{(k_1,k_2)}) = v_{k_1}, t(e_{(k_1,k_2)}) = v_{k_2}$ (given an
oriented edge \( e \) we define by \( s(e) \) and \( t(e) \) respectively the source and the target of the edge \( e \). We choose the following orientation of \((\Gamma_X, \mathcal{G}_X)\): 
\[ E_+ (\Gamma_X) = \{ e_{(k_1, k_2)}^j \mid k_1 \leq k_2 \text{ and } j = 1, \ldots, n(k_1, k_2) \}. \]

- The collection of subgroups \( \mathcal{G}_X \) is given by 
  \[ G_{v_k} = \pi_1(X_k), \quad G_{e_{(k_1, k_2)}} = G_{e_{(k_1, k_2)}} = \pi_1(T^i_{(k_1, k_2)}) \cong \mathbb{Z}^2 \]
  and the monomorphisms 
  \[ \varphi(k_1, k_2): G_{e_{(k_1, k_2)}} \to G_{v_{k_1}}, \quad \varphi(k_1, k_2): G_{e_{(k_1, k_2)}} \to G_{v_{k_2}} \]
  are such that the composition \( \psi(k_1, k_2) = \varphi(k_1, k_2) \circ \varphi^{-1}(k_1, k_2) \) coincides with the isomorphism induced by the gluing map between the two boundary tori of the JSJ-components \( X_{k_1}, X_{k_2} \) giving rise to the JSJ-torus \( T^i_{(k_1, k_2)} \).

We shall now define the announced extension of the graph of groups \((\mathcal{G}_X, \Gamma_X)\):

**Definition 3.2** (Peripheral extension of the JSJ-splitting). We shall denote by \((\mathcal{G}_X, \hat{\Gamma}_X)\) the peripheral extension of the JSJ-splitting of \( X \), i.e. the graph of groups defined as follows:

- \( \hat{\Gamma}_X \) is the graph given by the vertices 
  \[ V(\hat{\Gamma}_X) = V(\Gamma_X) \cup \{ v_{(k, i)} \mid k = 1, \ldots, \ell \text{ and } i = 1, \ldots, m_k \} \]
  and the edges \( E(\hat{\Gamma}_X) = E(\Gamma_X) \cup \{ e_{(k, i), j}^0, e_{(k, i)}^0 \mid k = 1, \ldots, \ell \text{ and } i = 1, \ldots, m_k \} \)
  where \( e_{(k, i), j}^0 = e_{(k, i)}^0 \) and \( s(e_{(k, i)}^0) = v_{(k, i)}, \quad t(e_{(k, i)}^0) = v_k \).
  The orientation of \((\mathcal{G}_X, \hat{\Gamma}_X)\) is: \( E_+ (\hat{\Gamma}_X) = E_+ (\Gamma_X) \cup \{ e_{(k, i)}^0 \mid k = 1, \ldots, \ell \text{ and } i = 1, \ldots, m_k \} \).

- The collection of subgroups \( \mathcal{G}_X \) is the union:
  \[ \mathcal{G}_X = \mathcal{G}_X \cup \{ G_{e_{(k, i)}}^0, G_{e_{(k, i)}}^0 = G_{\varphi(k, i)}^0 \mid k = 1, \ldots, \ell \text{ and } i = 1, \ldots, m_k \} \]
  where \( G_{e_{(k, i)}}^0 \cong \mathbb{Z}^2, \quad G_{e_{(k, i)}}^0 = G_{\varphi(k, i)}^0 \cong \mathbb{Z}^2 \) and the monomorphisms 
  \[ \varphi(k, i): G_{e_{(k, i)}}^0 \to G_{e_{(k, i)}}^0, \quad \varphi(k, i): G_{e_{(k, i)}}^0 \to G_{e_{(k, i)}}^0 \]
  are such that \( \psi(k, i) = \varphi(k, i) \circ \varphi^{-1}(k, i) \) is the isomorphism induced by the inclusion \( T^i_{(k, i)} \to X_k \).

Given \( \hat{T}_X \subseteq \hat{\Gamma}_X \) a maximal subtree of \( \hat{\Gamma}_X \) we recall that is well defined the fundamental group \( \pi_1(\mathcal{G}_X, \hat{\Gamma}_X, \hat{T}_X) \) of the graph of groups \((\mathcal{G}_X, \hat{\Gamma}_X)\) at \( \hat{T}_X \) (see Bas76, Ser77 and Ser80 section §1.5), which is the group generated by the vertex groups in \( \mathcal{G}_X \), with their own relations, together with a set of elements \( \{ g_e \} \) verifying the following relations:

\[ g_e h^e g_e^{-1} = h^e \] for each \( h \in G_e \); 
\[ g_e^{-1} = g_e^{-1} \] for each \( e \in E(\hat{\Gamma}_X) \)

where we denote by \( h^e \) the image of \( h \in G_e \) into \( G_{t(e)} \). It turns out (Ser80) that the isomorphism class of \( \pi_1(\mathcal{G}_X, \hat{\Gamma}_X, \hat{T}_X) \) does not depend from the choice of a maximal tree in \( \hat{\Gamma}_X \), hence we are allowed to speak of fundamental group of \((\mathcal{G}_X, \hat{\Gamma}_X)\), which in this case (by construction) coincides with the fundamental group of the irreducible 3-manifold \( X \).
The Bass-Serre tree of the peripheral extension of the JSJ-splitting. We shall now describe the Bass-Serre tree of the peripheral extension of the JSJ-splitting. The tree can be constructed from the data \((\mathcal{G}_X, \hat{\Gamma}_X, \hat{T}_X)\). In the sequel we implicitly assume to have fixed the isomorphism \(\pi\). We call \(\tilde{\pi}\) (as they are left cosets of subgroups of \(\hat{\Gamma}_X\)) the coset in \((\hat{\Gamma}_X)\) corresponding to the boundary to \(\partial \mathbb{D}^3\), where \(\mathbb{D}^3\) is the 3-dimensional disk. In the previous section. Before stating and proving Proposition 3.6 we need to establish two easy facts.

\begin{itemize}
\item \(\chi\) is the characteristic function of \(\hat{\Gamma}_X\). Then we define: \(\chi(g) = g g_k \chi_{k,i}) = g_k \chi_{k,i})\); \(\chi(g) = g g_k \chi_{k,i}) t((k,i)) = g_k \chi_{k,i}) t((k,i))\); \(\chi(g) = g g_k \chi_{k,i}) t((k,i))\).
\end{itemize}

**Remark 3.6.** By construction, the tree \(T_X\) verifies the following:

1. the vertices \(G_{e_{(i,j)}}\) are the (only) leaves of the Bass-Serre tree \(T_X\);
2. \(v = g G_{e_{(i,j)}}\) is adjacent to \(v' = g' G_{e_{(i,j)}}\) if and only if \(j = k\) and \(g' - 1 g' \in G_{e_{(i,j)}}\).

The group \(\pi_1(X)\) acts on the vertices and the edges of \(T_X\) by left multiplication (as they are left cosets of subgroups of \(\pi_1(X)\)). This action turns out to be an action by automorphisms (without edge inversions) of \(T_X\). By definition of \(T_X\) and of the action of \(\pi_1(X)\) on \(T_X\) the stabilizer of the vertex \(g G_{e_{(i,j)}} \in V(T_X)\) is the subgroup \(\text{Stab}_{\pi_1(X)}(g G_{e_{(i,j)}}) = g G_{e_{(i,j)}} g^{-1}\). The same holds for the stabilizers of the edges.

**Glossary.** We shall distinguish four families of vertices in \(T_X\):

- **peripheral vertices**, i.e. \(h G_{e_{(i,j)}}\);
- **hyperbolic vertices**, i.e. \(h G_{e_{(i,j)}}\) with \(X_k\) a JSJ-component of hyperbolic type;
- **Seifert fibered vertices**, i.e. \(h G_{e_{(i,j)}}\) with \(X_k\) a JSJ-component which is Seifert fibered with hyperbolic base orbifold;
- **vertices of type \(K \times I\)**, i.e. \(h G_{e_{(i,j)}}\) with \(X_k \simeq K \times I\).

3.2. **The intersection of the leaves stabilizers.** We use the notation introduced in the previous section. Before stating and proving Proposition 3.6 we need to establish two easy facts.

**Lemma 3.4.** Let \(g \in \text{Stab}_{\pi_1(X)}(h G_{e_{(i,j)}})\), and assume that \(v_k\) is a hyperbolic vertex. Then \(g\) can stabilize at most a single oriented edge having \(h G_{e_{(i,j)}}\) as source.

**Proof.** Up to act on \(T_X\) by left multiplication with \(h^{-1}\), we may assume that \(h = 1\). By construction of \(T_X\) the (oriented) edges having \(G_{e_{(i,j)}} = \pi_1(X_k)\) as source are either of type \(b G_{e_{(i,j)}}\) or \(b G_{e_{(i,j)}}\), where \(b \in \pi_1(X_k)\), which correspond to the left cosets in \(\pi_1(X_k)\) of the peripheral subgroups corresponding to the boundary tori \(T^1_{(i,j)}, T^2_{(i,j)}, T^3_{(i,j)}, T^4_{(i,j)}\) of \(X_k\). Hence the stabilizers of the edges starting from \(G_{e_{(i,j)}}\) corresponds to suitable conjugates \(b \pi_1(T) b^{-1} < \pi_1(X_k)\) for some \(T\) in \(\{T^1_{(i,j)}, T^2_{(i,j)}, T^3_{(i,j)}, T^4_{(i,j)}\}\) and the conclusion follows applying Lemma 2.11 and conjugating by \(h\).

**Lemma 3.5.** Any geodesic path connecting two leaves of \(T_X\) and containing a vertex \(h G_{e_{(i,j)}}\) of type \(K \times I\) has length greater or equal to 4. Moreover, a vertex of
type $K \bar{\times} I$ is source of precisely two oriented edges, whose targets are two distinct left cosets associated to the JSJ-component adjacent to $X_k$.

Proof. A vertex of type $K \bar{\times} I$ cannot be adjacent to a leaf of $T_X$, because no boundary component of $X$ belong to a JSJ-component homeomorphic to $K \bar{\times} I$. This is sufficient to show that a geodesic path connecting two leaves of $T_X$ and containing a vertex of type $K \bar{\times} I$ must have at least 5 vertices, and thus must have length greater or equal to 4. For what concerns the second claim observe first that the edge in $(\tilde{\Gamma}_X, \tilde{G}_X)$ connecting $X_k$ to the adjacent JSJ-component, say $X_j$, is always contained in any maximal tree $\tilde{T}_X$. On the other hand, by Lemma 2.8 the peripheral subgroup $\pi_1(\partial X_k)$ has index two in $\pi_1(X_k)$ and that the two left cosets of $\pi_1(\partial X_k)$ in $\pi_1(X_k)$, using the notation of Lemma 2.8 are $\pi_1(\partial X_k)$ and $a \pi_1(\partial X_k)$. Combining the previous informations we deduce that we have precisely two oriented edges starting from $h \pi_1(X_k)$ — namely $h \pi_1(\partial X_k) = h G_{v_1}$ and $ha \pi_1(\partial X_k) = ha G_{v_2}$ — whose targets are $h G_{v_j}$ and $ha G_{v_j}$ respectively. $\square$

We are now ready to prove:

Proposition 3.6 (Intersection of leaves stabilizers). Let $X$ be an irreducible 3-manifold with non-empty boundary, not homeomorphic to $K \bar{\times} I$, $T^2 \times I$ or $D^2 \times S^1$. Consider two distinct leaves $h_1G_{v_{(k_1,i_1)}}, h_2G_{v_{(k_2,i_2)}}$ of $T_X$. Then

$$\text{Stab}_{\pi_1(X)}(h_1G_{v_{(k_1,i_1)}}) \cap \text{Stab}_{\pi_1(X)}(h_2G_{v_{(k_2,i_2)}}) \neq \{1\}$$

if and only if the following conditions hold:

1. $k_1 = k_2$;
2. $X_k$ is a Seifert fibered JSJ-component not homeomorphic to $K \bar{\times} I$;
3. $h_1^{-1}h_2 \in G_{v_k}$.

Moreover, if $hG_{v_k}$ is the vertex of $T_X$ adjacent to $h_jG_{v_{(k,j)}}$ ($j = 1, 2$) the previous intersection coincides with the infinite cyclic subgroup $h(f_k)h^{-1}$, where $f_k$ is the regular fiber of $G_{v_k} \cong \pi_1(X_k)$.

Proof. We may assume that the JSJ-decomposition of $X$ is non-trivial, otherwise the result follows trivially from the description of the Bass-Serre tree, Lemma 2.7 and Lemma 2.11. Let $g \in \pi_1(X)$ be a non-trivial element and assume that

$$g \in \text{Stab}_{\pi_1(X)}(h_1G_{v_{(k_1,i_1)}}) \cap \text{Stab}_{\pi_1(X)}(h_2G_{v_{(k_2,i_2)}})$$

Since the fixed point set of an element of a group acting on a tree is a connected subtree, we deduce that $\text{Fix}_{\pi_1}(g)$ contains the geodesic segment $\gamma : [0, M] \to T_X$ joining the vertex $h_1G_{v_{(k_1,i_1)}}$ and the vertex $h_2G_{v_{(k_2,i_2)}}$. Here we have chosen the simplicial distance on $T_X$ — where the edges of $T_X$ are all isometric to $[0, 1] \subset \mathbb{R}$ — so that $\gamma(i)$ is a vertex of $T_X$ for each integer $i = 0, \ldots, M$. By construction of $T_X$ the path between two distinct leaves necessarily has length greater or equal to 2, hence $M \geq 2$. We recall that the vertices $hG_{v_{(k,i)}}$ are leaves of the Bass-Serre tree $T_X$. It follows that the vertices contained into $\gamma([0, M])$ are necessarily of type $hG_{v_k}$ for a suitable $h \in \pi_1(X)$, because a geodesic segment cannot contain backtrackings. We know that the interior of the path $\gamma$ does contain neither hyperbolic vertices (by Lemma 2.9), nor peripheral vertices. Now we shall prove that the length of the geodesic $\gamma$ is necessarily equal to 2.
Lemma 3.7. \( M = 2 \)

Proof. Let \( \gamma(i) = g_iG_{v_{k_i}} \) be the vertices in \( \gamma((0, M)) \). We know that the vertices \( \gamma(i), i \in \{1, \ldots, M - 1\} \) are either Seifert fibered vertices or vertices of type \( K \times I \).

Consider \( \gamma(1) = g_1G_{v_{k_1}} \). By Lemma 3.6 we know that this vertex is a Seifert fibered vertex. Since \( g \) stabilizes \( \gamma \), it should stabilize two distinct (oriented) edges having \( \gamma(1) \) as source and \( \gamma(0), \gamma(2) \) as target. By Lemma 2.7 we know that in this case \( g \in g_1 \langle f_{k_1} \rangle g_1^{-1} \subset \text{Stab}_{\pi_1(X)}(g_1G_{v_{k_1}}) \) where \( f_{k_1} \) is the regular fiber of \( \pi_1(X_{k_1}) \). We deduce that either \( \gamma(2) \) is a peripheral vertex or by Lemma 2.9 \( g \in \text{Stab}_{\pi_1(X)}(g_2G_{v_{k_2}}) \smallsetminus \langle g_2f_{k_2}, g_2^{-1} \rangle \), for any choice of a regular fiber \( f_{k_2} \) in \( \pi_1(X_{k_2}) \).

Our aim is to show that \( \gamma(2) \) is a peripheral vertex. Thus we need to prove that \( \gamma(2) \) is neither a Seifert fibered vertex nor a vertex of type \( K \times I \).

Assume first that \( \gamma(2) \) is a Seifert fibered vertex. Since \( g \notin g_2\langle f_{k_2} \rangle g_2^{-1} \) it follows from Lemma 2.7 that \( g \) does not stabilize edges other than the one connecting \( g_1G_{v_{k_1}} \) to \( g_2G_{v_{k_2}} \). In particular \( g \) would not stabilize any geodesic path of length greater or equal to 3. Since we are assuming the length of the geodesic path greater or equal to 3, we need to exclude that \( \gamma(2) \) is a Seifert fibered vertex.

Now consider the case where \( \gamma(2) \) is a vertex of type \( K \times I \). By Lemma 3.6 in this case \( M \geq 4 \) and there are only two oriented edges starting from \( g_2G_{v_{k_2}} \), which are those linking \( \gamma(2) \) to the vertices \( \gamma(1) = g_1G_{v_{k_1}} \) and \( \gamma(3) = g_3G_{v_{k_3}} = g_3G_{v_{k_4}} \) and from Lemma 3.3 we deduce that \( g_3G_{v_{k_3}} = g_2aG_{v_{k_4}} \). Since \( g \in g_1 \langle f_{k_1} \rangle g_1^{-1} \) we have \( g \in g_2\langle \psi^\varepsilon_{k_1,k_2;1}(f_{k_1}) \rangle g_2^{-1} \), where we have to choose \( \varepsilon \in \{ \pm 1 \} \) depending on the orientation of the edge. By Lemma 2.4 we know that \( \psi^\varepsilon_{k_1,k_2;1}(f_{k_1}) \notin \langle a^2 \rangle \cup \langle f \rangle \).

We can write \( g \) as:

\[
g = g_2\psi^\varepsilon_{k_1,k_2;1}(f_{k_1}) g_2^{-1} = (g_2a)(a^{-1}\psi^\varepsilon_{k_1,k_2;1}(f_{k_1})a)(g_2a)^{-1}
\]

Since \( \psi^\varepsilon_{k_1,k_2;1}(f_{k_1}) \notin \langle a^2 \rangle \cup \langle f \rangle \), it follows from Lemma 2.8 that

\[
\langle a^{-1}\psi^\varepsilon_{k_1,k_2;1}(f_{k_1})a \rangle \cap \langle \psi^\varepsilon_{k_1,k_2;1}(f_{k_1}) \rangle = \{1\}
\]

Thus, looking at \( g \) as an element of \( \text{Stab}_{\pi_1(X)}(g_3G_{v_{k_1}}) = \text{Stab}_{\pi_1(X)}(g_2aG_{v_{k_4}}) \), we obtain the expression \( g = (g_2a)(\psi^\varepsilon_{k_1,k_2;1}(a^{-1}\psi^\varepsilon_{k_1,k_2;1}(f_{k_1})a))(g_2a)^{-1} \). From the previous discussion it follows that \( g \notin (g_2a)(f_{k_1})(g_2a)^{-1} \) and hence \( g \notin g_1(f_{k_1})g_1^{-1} \) (as \( g_3^{-1}g_2a \in G_{v_{k_4}} \) and \( (f_{k_1}) \) is normal in \( G_{v_{k_4}} \)). Using Lemma 2.4 we conclude that \( g \) does not stabilize the edges starting from \( \gamma(3) \) except for the one connecting \( \gamma(3) \) to \( \gamma(2) \). But this is a contradiction, since by Lemma 3.6 \( g \) should stabilize a segment \( \gamma \) of length greater or equal to 4.

Since \( \gamma(2) \) is neither a hyperbolic vertex, nor a Seifert fibered vertex, nor a vertex of type \( K \times I \) we conclude that \( \gamma(2) \) is a peripheral vertex, \( i.e. \) \( M = 2 \) and \( \gamma(2) = h_2G_{v_{k_2}} \).

End of the Proof of Proposition 3.6 By Lemma 3.7 we know that if

\[
\text{Stab}_{\pi_1(X)}(h_1G_{v_{k_1}}) \cap \text{Stab}_{\pi_1(X)}(h_2G_{v_{k_2}}) \neq \{1\}
\]

then the path connecting \( h_1G_{v_{k_1}} \) to \( h_2G_{v_{k_2}} \) has length equal to 2. This means that both the leaves are adjacent to a single vertex \( hG_{v_k} \), which is necessarily a Seifert fibered vertex by Lemma 3.3 and Lemma 3.5. By construction of \( T_X \) we deduce that \( k_1 = k_2 = k \), and \( h^{-1}h_i \in G_{v_i} \) for \( i = 1, 2 \), and thus \( h^{-1}h_2 \in G_{v_k} \).
Now assume that \( i_1 \neq i_2 \). The stabilizers of the leaves are precisely
\[
h_j G_{\varphi, j} h_j^{-1} = h_j \pi_1(T_{(k, i_j)}) h_j^{-1} \quad j = 1, 2
\]
Conjugating both subgroups by \( h_1 \), the problem is reduced to find the intersection of the subgroups \( \pi_1(T_{(k, i_1)}) \) and \( (h_1^{-1} h_2) \pi_1(T_{(k, i_2)}) (h_2^{-1} h_1) \) in \( \pi_1(X_k) \). It follows from Lemma \( \ref{lem:intersection} \) that this intersection is equal to the subgroup \( \langle f_k \rangle \). The same conclusion holds (again in view of Lemma \( \ref{lem:intersection} \)) if \( i = i_1 = i_2 \) and \( h_2^{-1} h_1 \notin \pi_1(T_{(k, i)}) \).

Proof of Theorem \( \ref{thm:main} \) and of \( \ref{thm:main-2} \) Theorem 3. Since \( h_j G_{\varphi, j} h_j \) are two distinct leaves of \( T_X \), then either \( (k_1, i_1) \neq (k_2, i_2) \), or \( (k, i) = (k_1, i_1) = (k_2, i_2) \) but \( h_1^{-1} h_2 \notin G_{\varphi, j} \). Theorem \( \ref{thm:main} \) (i) is equivalent to Proposition \ref{prop:main} case \( (k, i) = (k_1, i_1) = (k_2, i_2) \) and \( h_1^{-1} h_2 \notin \pi_1(T_{(k, i_1)}) \), whereas Theorem \( \ref{thm:main} \) (ii) corresponds to Proposition \ref{prop:main} case \( (k, i_1) \neq (k_2, i_2) \). Proposition \ref{prop:main} (ii) is precisely the content of \( \ref{thm:main-2} \) Theorem 3. Finally, we observe that Proposition \ref{prop:main} implies that abelian peripheral subgroups corresponding to the boundary tori which belong to distinct JSJ-components are conjugately separated.

4. Malnormal splittings of 3-manifold groups

4.1. Malnormal amalgamated products and HNN-extensions.

4.1.1. Definitions. In this section we shall first recall the definition of \( k \)-step malnormal amalgamated product (see \[KS71\]) and we shall give the natural analogous, the notion of \( k \)-step malnormal HNN-extension. We shall recall their relation with the notion of \( k \)-acylindrical splitting (\[Sc97\]) in the setting of Bass-Serre theory. For a general introduction to combinatorial group theory we refer to the textbooks \[LS77\], \[MKS04\]; see also section \( \S \)I in \[SW79\].

Let \( G \) be a group. Let \( H \leq G \) be a subgroup of \( G \). We recall that \( H \) is malnormal in \( G \) if \( gHg^{-1} \cap H \neq \{1\} \) implies \( g \in H \).

Definition 4.1 (Extended normalizer). Let \( H \) be a subgroup of a discrete group \( G \) and let \( h \in H \). The extended normalizer \( \mathcal{E}_G^H(h) \) of \( h \) relative to \( H \) in \( G \) is the set:
\[
\mathcal{E}_G^H(h) = \{ g \in G \mid ghg^{-1} \in H \},
\]
when \( h \neq 1 \), and \( \mathcal{E}_G^H(1) = H \). We define the extended normalizer of \( H \) in \( G \) by \( \mathcal{E}_G(H) = \bigcup_{h \in H} \mathcal{E}_G^H(h) \).

In \[KS71\] Karrass and Solitar introduced the notion of \( k \)-step malnormal amalgamated product (or, for short, \( k \)-step malnormal product) combining the notion of malnormality of a subgroup and the existence of a normal form for the elements of an amalgamated product (and thus of a notion of “length”). Let us denote the syllable length of the element \( g \) by \( |g| \). If \( K \) is a subset of \( G \) we shall denote \( \mathcal{L}(K) = \max_{g \in K} |g| \).

Definition 4.2 (\( k \)-step malnormal product). Let \( G \cong A \ast_B \) be an amalgamated product. The subgroup \( C \) is \( k \)-step malnormal in \( G \) if and only if \( \mathcal{L}(\mathcal{E}_G(C)) \leq k \).

We shall say that \( G \) is a \( k \)-step malnormal product. It is easily seen that \( G \) is a \( 0 \)-step malnormal product if and only if \( C \) is malnormal in both \( A \) and \( B \).

The HNN-extension of a group \( A \) via an isomorphism \( \varphi : C_{-1} \to C_1 \) between two of its subgroups, \( C_{-1}, C_1 \), is defined as the group having the following presentation:
\[
A_{\varphi}^* = \langle A, t \mid \text{rel}(A);\; tct^{-1} = \varphi(c), \forall c \in C_{-1} \rangle
\]
where \( \text{rel}(A) \) are the relations of the group \( A \). We remark that any element \( g \) in \( G \cong A^*_φ \) can be written as: \( g = w_0 t^{e_1} w_1 t^{e_2} \cdots t^{e_n} w_n \) where \( e_i = ±1 \) and \( w_i \in A \).

**Britton's Lemma.** Let \( G \cong A^*_φ \) and let \( G \ni g = w_0 t^{e_1} w_1 \cdots t^{e_n} w_n \), as before. If \( g = 1 \) one of the two following conditions holds:

(i) \( n = 0 \) and \( w_0 = 1 \);

(ii) \( g \) contains either \( t w_i t^{-1} \) with \( w_i \in C_{-1} \) or \( t^{-1} w_i t \) with \( w_i \in C_1 \). For future reference we shall say that such a form has pinches.

We shall say that \( g = w_0 t^{e_1} w_1 t^{e_2} \cdots t^{e_n} w_n \) is a reduced form for \( g \) if it contains no pinches. A consequence of Britton's Lemma is that any element of \( G = A^*_φ \) has a reduced form. The number of occurrences of the stable letter \( t \), raised to the power \( ±1 \) in a reduced form for \( g \) does not change if we change the reduced form. Hence we define the length of the element \( g \) as the number \( |g| \) of occurrences of the stable letter raised to the power \( ±1 \) in a reduced writing for \( g \).

**Definition 4.3.** Let us consider \( A^*_φ \) where \( φ : C_{-1} \rightarrow C_1 \) is the isomorphism. We shall say that \( G \cong A^*_φ \) is a \( k \)-step malnormal HNN-extension if \( G \cong e \) for \( e = gC_φ g^{-1} \cap C_{±e} \neq 1 \) for \( e = ±1 \), implies \( |g| \leq k \).

### 4.1.2. Malnormality and acylindricity.

Let \( G \) be the fundamental group of the graph of groups \((\mathcal{G}, Γ)\) defined as follows: the underlying graph has two vertices \( \{v_A, v_B\} \) and two (oriented) edges \( \{e_C, e_C^*\} \), with \( e_C = e_C^* \). The collection of vertex groups is given by \( A \cong G_{v_A} \), \( B \cong G_{v_B} \) and the edge group is \( C \cong G_{e_C} \cong G_{e_C^*} \) with monomorphisms given by \( t_A : C \hookrightarrow A \) and \( t_B : C \hookrightarrow B \). Hence \( G \cong \pi_1(\mathcal{G}, Γ) \cong A^*_φ \).

**Proposition 4.4.** The splitting \((\mathcal{G}, Γ)\) is \( k \)-acylindrical if and only if \( \pi_1(\mathcal{G}, Γ) \cong A^*_φ \) is a \((k − 1)\)-step malnormal product.

**Proof.** Consider the Bass-Serre tree \( T_{(\mathcal{G}, Γ)} \) associated to the graph of groups previously described and to the choice of \( \{e_C\} \) as orientation (no choice of a maximal tree is needed in this case, since \( Γ \) is a tree). We are assuming here to have chosen the representative of the left coset \( hA \) (resp. \( hB \)) so that \( |ha| > |h| \) for any \( a \in A \setminus C \) (resp. \( |hb| > |h| \) for any \( b \in B \setminus C \)). By construction two vertices \( gA \) and \( hB \) are adjacent if and only if either there exists \( b \in B \setminus C \) such that \( g = hb \) or there exists \( a \in A \setminus C \) such that \( h = ga \) or \( g = h = 1 \).

Consider the action of \( G \cong A^*_φ \) on \( T_{(\mathcal{G}, Γ)} \). The stabilizers of the vertices are given by \( \text{Stab}_G(gA) \cong Ag^{-1} \) and \( \text{Stab}_G(hB) \cong Bh^{-1} \). Moreover the edge \( e \) between \( gA \) and \( hB \) is equal to \( e = δ(A, C) \) where \( δ \) is the longest element between \( g \) and \( h \). The stabilizer is \( \text{Stab}_G(e) \cong δ \). Let \( g \in G \) be an element which stabilizes a set in \( T_{(\mathcal{G}, Γ)} \) having diameter \( k' \). Consider a geodesic of length \( k' \) between the two vertices \( v_1, v_2 \) realizing the diameter of \( \text{Fix}_{T_{(\mathcal{G}, Γ)}}(g) \), the fixed point set of \( g \).

We shall assume that \( v_1 = h_1A \) and \( v_2 = h_2B \) (the other cases are similar). Since the element \( g \) stabilizes both \( v_1 \) and \( v_2 \) we have that \( g \in \text{Stab}_G(v_1) \cap \text{Stab}_G(v_2) \) from the structure of \( T_{(\mathcal{G}, Γ)} \) and the previous choice of a representative system for the left cosets of \( A \) and \( B \), it is straightforward to check that \( k' \) is either equal to \( |h_1^{-1}h_2| + 1 \) or to \( |h_1^{-1}h_2| \) (depending whether the path which leads from \( 1A \) to \( (h_1^{-1}h_2)B \) pass through the edge \( 1C \) or no). Observe that

\[
g \in \text{Stab}_G(v_1) \cap \text{Stab}_G(v_2) \Leftrightarrow g \in h_1Ah_1^{-1} \cap h_2Bh_2^{-1} \Leftrightarrow g \in h_1Ch_1^{-1} \cap h_2Ch_2^{-1}
\]

Assume that \( G \cong A^*_φ \) is a \( k \)-step malnormal product; the latter intersection is non-trivial only if \( k' - 1 \leq |h_1^{-1}h_2| \leq k \). A similar conclusion holds if we replace
Proof. As in the previous case we consider the Bass-Serre tree \( \mathcal{T}_{(\mathcal{G}, \Gamma)} \) of groups \((\mathcal{G}, \Gamma)\) is a \((k+1)\)-acylindrical splitting of \( G \). Vice versa, assume that the graph of groups \((\mathcal{G}, \Gamma)\) is a \((k+1)\)-acylindrical splitting for \( G \cong A^*_\varphi B \). The intersection of the two stabilizers is non-trivial only if \(|h^{-1}_1 h_2| + 1 \leq k+1 \text{ i.e. only if } C \text{ is } k\text{-step malnormal in } G \cong A^*_\varphi B. \)

Now let us consider an amalgamated product \( G = A^*_\varphi B \) where \( \varphi : C_{-1} \to C_1 \) is an isomorphism between two subgroups of \( A \). We can construct the following graph of groups \((\mathcal{G}, \Gamma)\): consider a loop formed by a vertex \( v \) and one edge \( e \) and define \( G_v = A \) and \( C = G_e = G_b \) with isomorphisms \( G_e \to C_{-1} < A, G_b \to C_1 < A. \)

**Proposition 4.5.** The splitting \((\mathcal{G}, \Gamma)\) is \( k\)-acylindrical if and only if \( \pi_1(\mathcal{G}, \Gamma) = A^*_\varphi \) is a \( k\)-step malnormal HNN-extension.

**Proof.** As in the previous case we consider the Bass-Serre tree \( \mathcal{T}_{(\mathcal{G}, \Gamma)} \) associated to the graph of groups previously described and to the choice of \( \{e_C\} \) as orientation (no choice of a maximal tree is needed in this case, since \( \Gamma \) is a tree). The set of vertices of this tree is given by the left cosets \( \{g A\} \); we choose representatives of the left cosets so that their reduced form ends with the stable letter to the power \( \pm 1 \). The set of edges similarly is given by the left cosets \( \{gC_{-1}\} \). Let \( g_1, g_2 \) be two representatives of left cosets of \( A \) and let \(|g_2| > |g_1| \) there exists an edge between \( g_1 A \) and \( g_2 A \) if and only if there exists an \( a \in A \) such that either \( g_2 = g_1 a t \) or \( g_2 = g_1 a t^{-1} \) and in that case the corresponding edge will be respectively either \( g_1 a C_1 \) or \( g_1 a C_{-1} \). Now, assume that \( A^*_\varphi \) is a \( k\)-step malnormal HNN-extension and take \( g \in G \). Assume that \( g \) fixes a path between \( g_1 A \) and \( g_2 A \). We assume that the length of this path is \(|g_1^{-1} g_2| > k \). This means that \( g \in \text{Stab}_C(g_1 A g_1^{-1}) \cap \text{Stab}_C(g_2 A g_2^{-1}) \). By construction of the Bass-Serre tree this is possible if and only if \( g \in g_1 a C_\varepsilon a^{-1} g_1^{-1} \cap g_2 a' C_\delta a'^{-1} g_2^{-1} \) for suitable \( a, a' \in A \), where \( \varepsilon, \delta \in \{\pm 1\} \) and should be chosen depending on the first and the last edge of the path. But this intersection is empty because the intersection \( C_\varepsilon \cap (g_1^{-1} g_2) C_\delta (g_2^{-1} g_1) \) is empty by the \( k\)-step malnormality of the HNN-extension. Conversely, assume that the action of \( G \) on \( \mathcal{T}_{(\mathcal{G}, \Gamma)} \) is \( k\)-acylindrical, this means that if \( g_1 A, g_2 A \) are at distance greater than \( k \) then the respective stabilizers are disjoint. Since the following equality holds \( \text{Stab}_C(g_1 A) \cap \text{Stab}_C(g_2 A) = g_1 a C_\varepsilon a^{-1} g_1^{-1} \cap g_2 a' C_\delta a'^{-1} g_2^{-1} \) we see that the intersection on the right hand side is trivial, which implies that, whenever \(|g| > k \) then \( g C_\varepsilon g^{-1} \cap C_{\pm 1} = \{1\} \), which proves that the HNN-extension is \( k\)-step malnormal.

\[ \square \]

### 4.2. Malnormal splittings

This subsection is devoted to prove Proposition 4.3.

Let us establish first the following preparatory Lemma:

**Lemma 4.6.** Let \( X \neq T^2 \times I \) be an irreducible 3-manifold and let \( \{T^{\pm 1}\} \subseteq \partial X \) be two distinct boundary tori. Let \( f : T^1 \to T^1 \) be a gluing giving rise to a JSJ-torus for \( X' \), the resulting irreducible 3-manifold, and by \( \varphi = f_* : \pi_1(T^{-1}) \to \pi_1(T^1) \) the induced isomorphism. Then for any pair of elements

\[ (g_-, g_+) \in \{\pi_1(X) \setminus \pi_1(T^{-1})\} \times \{\pi_1(X) \setminus \pi_1(T^1)\} \]

we have

\[ (g_+) \varphi ( (g_-) \pi_1(T^{-1})(g_-)^{-1} \cap \pi_1(T^{-1}) ) \quad (g_+)^{-1} \pi_1(T^1) = \{1\} \]

\[ (g_-) \varphi^{-1} ( (g_+) \pi_1(T^1)(g_+)^{-1} \cap \pi_1(T^1) ) \quad (g_-)^{-1} \pi_1(T^{-1}) = \{1\} \]
Proof. We shall denote by $X_{k \pm 1}$ the JSJ-components of $X$ which contain respectively $T^{\pm 1}$ in their boundary (possibly $k = k + 1$). We point out that $T^{-1}$ and $T^{+1}$ belong either to hyperbolic JSJ-components or to Seifert fibered JSJ-components with hyperbolic base orbifolds. We shall consider two different cases:

1. at least one between $X_{k \pm 1}$ is a hyperbolic JSJ-component;
2. both $X_{k \pm 1}$ are Seifert fibered JSJ-components.

Case (1). Let us assume that $T^{-1}$ is a boundary torus of a hyperbolic JSJ-component. By [dlHW14b] (see also Proposition 3.6) we know that the abelian subgroups associated to the boundary tori of the hyperbolic JSJ-components are malnormal in the whole fundamental group, hence for $g_{-} \in \pi_{1}(X) \setminus \pi_{1}(T^{-1})$ we have $(g_{-}) \pi_{1}(T^{-1}) \cap \pi_{1}(T^{-1}) = \{1\}$, proving the first equality. To prove the second equality we remark that $\varphi^{-1}((g_{+}) \pi_{1}(T^{-1})(g_{+})^{-1} \cap \pi_{1}(T^{-1}))$ is a (possibly trivial) subgroup of $\pi_{1}(T^{+1})$. The latter subgroup is malnormal in $\pi_{1}(X)$, hence a fortiori

$$g_{-} \varphi^{-1}((g_{+}) \pi_{1}(T^{+1})(g_{+})^{-1} \cap \pi_{1}(T^{+1})) g_{-1} \cap \pi_{1}(T^{-1}) = \{1\}$$

On the other hand, it follows from Proposition 3.6 that $\pi_{1}(T^{+1})$ is conjugately separated from $\pi_{1}(T^{-1})$. Hence:

$$g_{-} \varphi^{-1}((g_{+}) \pi_{1}(T^{+1})(g_{+})^{-1} \cap \pi_{1}(T^{+1})) g_{-1} \cap \pi_{1}(T^{+1}) = \{1\}$$

which gives the desired equality.

Case (2). Now suppose that $T^{-1}$ and $T^{+1}$ belong to two Seifert fibered JSJ-components (possibly the same JSJ-component). Let $w \in \pi_{1}(T^{-1})$, by assumption $g_{-} \in \pi_{1}(X) \setminus \pi_{1}(T^{-1})$ hence by Theorem 1.1 (i) we see that $(g_{-}) w (g_{-})^{-1} \in \pi_{1}(T^{-1})$ if and only if $g_{-} \in \pi_{1}(X_{k \pm 1})$ and $w \in \langle f_{k \pm 1} \rangle$. If this is the case then $(g_{-}) w (g_{-})^{-1} = f_{k \pm 1}^{\ell}$ for some $\ell \in \mathbb{Z}$. Thanks to Lemma 2.9 we have that $\varphi((g_{-}) w (g_{-})^{-1} \cap \langle f_{k \pm 1} \rangle) = \{1\}$. Applying Theorem 1.1 (i) to the subgroup $\pi_{1}(T^{+1})$ we deduce that that $(g_{+}) \varphi(f_{k \pm 1}) (g_{+})^{-1} \notin \pi_{1}(T^{+1})$. Also, by Theorem 1.1 (ii), we have that $(g_{+}) \varphi(f_{k \pm 1}) (g_{+})^{-1} \notin \pi_{1}(T^{-1})$ and thus:

$$(g_{+}) \varphi((g_{-}) \pi_{1}(T^{-1})(g_{-})^{-1} \cap \pi_{1}(T^{-1}))(g_{+})^{-1} \cap \pi_{1}(T^{+1}) = \{1\}$$

The proof of the other equality is analogous to this one. □

Proof of Proposition 1.4. Since we excluded $X$ to be finitely covered by a torus bundle over the circle we observe that the different cases listed in the statement of Proposition 1.4 cover any possible scenario. We shall now do a case by case analysis.

Proof of (D1). By assumption $X_{k \pm 1}$ and $X_{k \pm 1}$ are hyperbolic JSJ-components. Let $X'$ and $X''$ be the closures of the two connected components of $X \setminus T$. By [dlHW14b], Theorem 3, $\pi_{1}(T^{-1})$ is malnormal in $\pi_{1}(X')$ and $\pi_{1}(T^{+1})$ is malnormal in $\pi_{1}(X'')$. An amalgamated product $A_{C} \ast B$ where $C$ is malnormal both in $A$ and $B$ it is readily seen to be 0-step malnormal.

Proof of (D2). Assume that $X_{k \pm 1}$ is a hyperbolic JSJ-component. Let $X'$ and $X''$ be the closures of the two connected components of $X \setminus T$. By [dlHW14b], Theorem 3, $\pi_{1}(T^{-1})$ is malnormal in $\pi_{1}(X')$. It is straightforward to check that an amalgamated product $A_{C} \ast B$ where $C$ malnormal in $A$ (or $B$) is a 1-step malnormal
amalgamated product.

Proof of (D3). By assumption $X_{k-1}$ and $X_{k+1}$ are both Seifert fibered manifolds with hyperbolic base orbifolds. Let $X'$ and $X''$ be the closures of the two connected components of $X \setminus T$. Let $g \in \pi_1(X)$ be such that $|g| \geq 2$ and consider a reduced form $g = b_1a_1 \cdots b_ma_m$ (with $b_1,a_m$ possibly equal to 1). Let $c \in \pi_1(T)$ and take $g cg^{-1} = (b_1 \cdots a_m) c (b_1 \cdots a_m)^{-1}$. Notice that, by definition of syllable length, if $|gcg^{-1}| > 0$ then $g cg^{-1} \not\in \pi_1(T)$. Assume that $a_m \neq 1$ (the case $a_m = 1$ is analogous). By Theorem 1.1 (i) either $g cg^{-1}$ is reduced (and $|gcg^{-1}| \geq 3$) or $a_m \in \pi_1(X_{k-1})$ and $c = f^k_{k-1}$ for a suitable $\ell \in \mathbb{Z}$. In this second case by Lemma 2.9 we know that $\varphi(a_m ca_m^{-1}) = \varphi(f^\ell_{k-1}) \not\in \langle f_{k-1} \rangle$. Again by Theorem 1.1 (i) we have that $b_m \varphi(f_{k-1}) b_m^{-1} \not\in \pi_1(T)$ and hence we conclude that either $g cg^{-1} \in \pi_1(X'') \setminus \pi_1(T^1)$ or $|g cg^{-1}| \geq 2$. In both cases $g cg^{-1} \not\in \pi_1(T)$.

Proof of (D4). Without loss of generality we assume that $X'' \simeq K \tilde{X} I$. We recall that $\pi_1(T^1)$ is normal in $\pi_1(X'') \cong \pi_1(K \tilde{X} I)$ (see Lemma 2.8). Let $g \in \pi_1(X)$ be such that $|g| \geq 4$. We shall show that $g cg^{-1} \not\in \pi_1(T)$ for any $c \in \pi_1(T)$; as in the previous case observe that if $|gcg^{-1}| > 0$ then $g cg^{-1} \not\in \pi_1(T)$. Let $g = b_1a_1 \cdots b_ma_m$ be a reduced form for $g$, with $b_1,a_m$ possibly equal to 1. We treat separately the case where $a_m \neq 1$ and the case where $a_m = 1$.

Assume $a_m \neq 1$. By Theorem 1.1 (i) we know that either

\[ g cg^{-1} = (b_1 \cdots b_m)(a_m ca_m^{-1})(b_1 \cdots b_m)^{-1} \]

is a reduced form and $|gcg^{-1}| \geq 7$ or $a_m ca_m^{-1} \in \pi_1(T^1)$, $c = f^k_{k-1}$ for a suitable $\ell \in \mathbb{Z}$ and $a_m \in \pi_1(X_{k-1})$. Thanks to Lemma 2.9 we know that the element $\varphi(a_m ca_m^{-1}) = \varphi(f^\ell_{k-1})$ does not belong to the two infinite cyclic subgroups generated by the regular fibers of $X'' \simeq K \tilde{X} I$. By Lemma 2.8 we have $b_m \varphi(f^\ell_{k-1}) b_m^{-1} \not\in \langle \varphi(f_{k-1}) \rangle$. Hence the element $\varphi^{-1}(b_m \varphi(f^\ell_{k-1}) b_m^{-1})$ does not belong to $\langle f_{k-1} \rangle$, and using Theorem 1.1 (i) we get

\[ a_{m-1} \varphi^{-1}(b_m \varphi(f^\ell_{k-1}) b_m^{-1}) a_{m-1}^{-1} \in \pi_1(X_{k-1}) \setminus \pi_1(T^{-1}) \]

We conclude that $|g cg^{-1}| \geq 3$ and thus $g cg^{-1} \not\in \pi_1(T)$.

Assume now $a_m = 1$. Consider $g cg^{-1} = (b_1 \cdots b_m) c (b_1 \cdots b_m)^{-1}$. The element $b_m c b_m^{-1}$ is still in $\pi_1(T^1)$, since $\pi_1(T^1)$ is normal in $\pi_1(X'')$. By Theorem 1.1 (i) we have $a_{m-1} \varphi^{-1}(b_m c b_m^{-1}) a_{m-1}^{-1} \not\in \pi_1(T^{-1})$, unless $\varphi^{-1}(b_m c b_m^{-1}) = f^\ell_{k-1}$ for some $\ell \in \mathbb{Z}$ and $a_{m-1} \in \pi_1(X_{k-1})$. In the latter case

\[ \varphi(a_{m-1} \varphi^{-1}(b_m c b_m^{-1}) a_{m-1}^{-1}) = \varphi(f^\ell_{k-1}) \]

and by Lemma 2.8 we deduce that $\varphi(f^\ell_{k-1})$ does not belong to one of the two infinite cyclic subgroups generated by the regular fibers of $X'' \simeq K \tilde{X} I$. Using Lemma 2.8 we deduce that $b_{m-1} \varphi(f^\ell_{k-1}) b_{m-1}^{-1} \not\in \langle \varphi(f_{k-1}) \rangle$. Applying $\varphi^{-1}$ we obtain that $\varphi^{-1}(b_{m-1} \varphi(f^\ell_{k-1}) b_{m-1}^{-1}) \not\in \pi_1(T^{-1}) \setminus \langle f_{k-1} \rangle$. By Theorem 1.1 (i) we conclude that $a_{m-2} \varphi^{-1}(b_{m-1} \varphi(f^\ell_{k-1}) b_{m-1}^{-1}) a_{m-2}^{-1} \not\in \pi_1(X') \setminus \pi_1(T^{-1})$. Hence, either $|gcg^{-1}| \geq 3$,\footnote{Here and after the sign \( \pm \) depends on the orientability of the base orbifold of the JSJ-component $X_{k-1}$, and on the element $a_m \in \pi_1(X_{k-1})$. See the presentation of the fundamental group of a Seifert fibered manifold in subsection \$2.1.1$.}
or \(|g^{-1}g| = 1\) and \(g^{-1}g \in \pi_1(X') \setminus \pi_1(T^{-1})\).

Proof of \((\text{ND}1)\). Let \(X'\) be the closure of \(X \setminus T\). Since \(T^{-1}\) and \(T^+\) bound two hyperbolic JSJ-components (possibly the same JSJ-component) we know by Theorem 3 in [11HW14b] (or by Proposition 3.6 that the subgroups \(\pi_1(T^{-1})\) and \(\pi_1(T^+)\) are malnormal in \(\pi_1(X')\)). Moreover, by Proposition 3.6 we know that they are conjugately separated in \(\pi_1(X')\). Consider \(w \in \pi_1(T^{-\varepsilon_s})\) and any \(|g| \geq 2\)
we have that \(|gwg^{-1}| \geq 2\). Let \(g = w_0t^{\varepsilon_1} \cdots t^{\varepsilon_s}w_0\) be a reduced form. We shall distinguish between the case where \(\varepsilon_{s-1} = \varepsilon_s\) and the case where \(\varepsilon_{s-1} = -\varepsilon_s\).
If \(\varepsilon_{s-1} = \varepsilon_s\) it follows from the fact that \(\pi_1(T^{\pm 1})\) are conjugately separated that \(|gwg^{-1}| \geq 2\). Assume that \(\varepsilon_{s-1} = -\varepsilon_s\). Since we have chosen a reduced form for \(g\), we have \(w_{s-1} \in \pi_1(X_{k_{s-1}}) \setminus \pi_1(T^{\varepsilon_s})\). By [11HW14b] Theorem 3 (see also Proposition 3.6) we conclude that \(w_{s-1}t^{\varepsilon_s}w_t^{-1}w_{s-1}^{-1} \notin \pi_1(T^{\pm 1})\) which shows that
\[
gwg^{-1} = (w_0t^{\varepsilon_1} \cdots t^{\varepsilon_s}w_0^{-1})(w_{s-1}t^{\varepsilon_s}w_t w_s^{-1}t^{-\varepsilon_s}w_{s-1}^{-1})(w_0t^{\varepsilon_1} \cdots t^{\varepsilon_s}w_0^{-1})^{-1}
\]
has length \(|gwg^{-1}| \geq 2\), hence \(gwg^{-1} \notin \pi_1(T)\).

Proof of \((\text{ND}2)\). As in the previous case let \(X'\) be the closure of \(X \setminus T\).
Let \(w \in \pi_1(T)\) and let \(g \in \pi_1(X)\) be such that \(|g| \geq 3\). Consider a reduced form
\[
g = w_0t^{\varepsilon_1}w_1 \cdots t^{\varepsilon_s}w_s.
\]
We conjugate \(w\) by \(g\) and we remark that the form \((w_0t^{\varepsilon_1} \cdots t^{\varepsilon_s})w_tw_{s}^{-1}(w_0t^{\varepsilon_1} \cdots t^{\varepsilon_s})^{-1}\) is reduced unless \(w_sw_{s}^{-1} \in \pi_1(T^{-\varepsilon_s})\). If the previous form is reduced then \(gwg^{-1} \in \pi_1(X) \setminus \pi_1(T)\). Otherwise, we need to distinguish several cases, depending whether \(\varepsilon_{s-2}\) and \(\varepsilon_{s-1}\) have the same or the opposite sign with respect to \(\varepsilon_s\).
If \(\varepsilon_{s-1} = -\varepsilon_s\) and \(\varepsilon_{s-2} = \varepsilon_s\) we observe that, since \(|g| \geq 3\) and the previous form is reduced, we have that \(w_{s-1} \notin \pi_1(T^{\varepsilon_s})\) and \(w_{s-2} \notin \pi_1(T^{-\varepsilon_s})\). Hence we can use Lemma 2.1 and conclude that \(gwg^{-1} \in \pi_1(X) \setminus \pi_1(T)\).

Let \(\varepsilon_{s-1} = \varepsilon_s\) and \(\varepsilon_{s-2} = \varepsilon_s\). Assume that \(w_sw_{s}^{-1} \in \pi_1(T^{-\varepsilon_s})\) (otherwise \(|gwg^{-1}| \geq 6\)) and consider the element \(\varphi^s(w_sw_{s}^{-1}) \in \pi_1(T^{\varepsilon_s})\). Thanks to Theorem 1.1 (ii) the element \(w_{s-1} \varphi^s(w_sw_{s}^{-1})w_{s-1}^{-1}\) belongs to \(\pi_1(T^{-\varepsilon_s})\) if and only if \(k = k_{s-1} = k_{s+1}\), the JSJ-component \(X_k\) is Seifert fibered, the element \(w_{s-1} \in \pi_1(X_k)\) and \(\varphi^s(w_sw_{s}^{-1}) = f_{\ell}^k\) for a suitable \(\ell \in \mathbb{Z}\) (otherwise \(|gwg^{-1}| \geq 4\)). In this case it follows from Lemma 2.3 that
\[
\begin{align*}
(t^{\varepsilon_s-1}w_{s-1} \varphi^s(w_sw_{s}^{-1})w_{s-1}^{-1})t^{\varepsilon_s-1} & = t^{\varepsilon_s}w_{s-1} \varphi^s(w_sw_{s}^{-1})w_{s-1}^{-1}t^{-\varepsilon_s} = \\
 & = t^{\varepsilon_s}f_{\ell}^k t^{-\varepsilon_s} = \varphi^s(f_{\ell}^k) \in \pi_1(T^{\varepsilon_s}) \setminus (f_{\ell}^k)
\end{align*}
\]
where in the first equality we used the assumption \(\varepsilon_{s-1} = \varepsilon_s\). By Theorem 1.1 we see that
\[
|(t^{\varepsilon_s-2}w_{s-2}t^{\varepsilon_s-1}w_{s-1}t^{\varepsilon_s}w_{s})w(t^{\varepsilon_s-2}w_{s-2}t^{\varepsilon_s-1}w_{s-1}t^{\varepsilon_s}w_{s})^{-1}| \geq 2
\]
and we conclude that \(|gwg^{-1}| \geq 2\) and thus \(gwg^{-1} \notin \pi_1(T)\).

If \(\varepsilon_{s-1} = -\varepsilon_s\) and \(\varepsilon_{s-2} = -\varepsilon_{s}\), we observe that \(w_{s-1} \notin \pi_1(T^{\varepsilon_s})\), since we have chosen a reduced form for \(g\). The form for \(gwg^{-1}\) is reduced unless \(w_sw_{s}^{-1} \in \pi_1(T^{-\varepsilon_s})\). In this case, since \(w_{s-1} \notin \pi_1(T^{\varepsilon_s})\) we know by Theorem 1.1 (i) that \(w_{s-1}t^{\varepsilon_s}w_sw_{s}^{-1}t^{-\varepsilon_s}w_{s-1}^{-1} = w_{s-1} \varphi^s(w_sw_{s}^{-1})w_{s-1}^{-1} \in \pi_1(T^{\varepsilon_s})\) if and only if \(X_{k_{s-1}}\) is a Seifert fibered JSJ-component, \(w_{s-1} \in \pi_1(X_{k_{s-1}})\) and \(\varphi^s(w_sw_{s}^{-1}) = f_{\ell}^k\) for a suitable \(\ell \in \mathbb{Z}\). If one of the previous conditions fails \(|gwg^{-1}| \geq 4\), and thus \(gwg^{-1} \notin \pi_1(T)\).

Otherwise we have that
\[
t^{\varepsilon_s-1}(w_{s-1}f_{\ell}^k w_{s-1}^{-1})t^{-\varepsilon_s-1} = t^{-\varepsilon_s}(w_{s-1}f_{\ell}^k w_{s-1}^{-1})t^{\varepsilon_s} = \varphi^{-\varepsilon_s}(f_{\ell}^k) \in \pi_1(T^{-\varepsilon_s})
\]
If $X_{k, \varepsilon_s}$ is not Seifert fibered then it follows from Proposition 3.5 that $\pi_1(T^{\pm 1})$ are conjugately separated. Since $\varepsilon_{s-2} = -\varepsilon_s$, we conclude that $|gw g^{-1}| \geq 2$. Assume that $X_{k, \varepsilon_s}$ is Seifert fibered. By Lemma 2.9 we have that

$$t^{\varepsilon_{s-1}}(w_{s-1} f_{k_s}^t w_{s-1}^{-1}) t^{-\varepsilon_s-1} \in \pi_1(T^{\varepsilon_s}) \setminus \langle f_{k_s} \rangle$$

and by Theorem 1.1 (ii) we conclude that $t^{\varepsilon_{s-1}}(w_{s-1} f_{k_s}^t w_{s}) t^{-\varepsilon_s-1} \notin \pi_1(T^{\varepsilon_s})$. It follows that $|gw g^{-1}| \geq 2$ and thus $gw g^{-1} \notin \pi_1(T)$.

Finally let $\varepsilon_{s-1} = \varepsilon_s$, and $\varepsilon_{s-2} = -\varepsilon_s$. Since the form $g = w_0 t^{\varepsilon_s} w_1 \cdots t^{\varepsilon_s} w_s$ is reduced, we deduce that $w_{s-2} \in \pi_1(X') \setminus \pi(T^{\varepsilon_s})$. Now take $w \in \pi_1(T)$; consider

$$gw g^{-1} = (w_0 t^{\varepsilon_s} \cdots t^{\varepsilon_s} w_s) w (w_0 t^{\varepsilon_s} \cdots t^{\varepsilon_s} w_s)^{-1}$$

The previous form is reduced unless $w_s w w_{s-1} \in \pi_1(T^{\varepsilon_s})$. If this is the case, observe that $w_{s-1} t^{\varepsilon_s} (w_s w w_{s-1}) t^{-\varepsilon_s} w_{s-1} = w_{s-1} \phi(\varepsilon_s) (w_s w w_{s-1}) w_{s-1} \in \pi_1(T^{\varepsilon_s})$ if and only if $T^{\varepsilon_s}$ are both boundary tori of the same Seifert fibered JSJ-component $X_k$, $w_{s-1} \in \pi_1(X_k)$ and $\phi(\varepsilon_s) (w_s w w_{s-1}) = f_k^t$ for a suitable $t \in \mathbb{Z}$. Now observe that

$$t^{\varepsilon_{s-1}} w_{s-1} \phi(\varepsilon_s) (w_s w w_{s-1}) w_{s-2} t^{-\varepsilon_s-1} = t^{\varepsilon_s} f_k^t t^{-\varepsilon_s-1} = \phi(\varepsilon_s) (f_k^t) \in \pi_1(T^{\varepsilon_s}) \setminus \langle f_k \rangle$$

where the last assertion follows from Lemma 2.8. Since $w_{s-2} \in \pi_1(X') \setminus \pi_1(T^{\varepsilon_s})$, we deduce from Theorem 1.1 that $w_{s-2} \phi(\varepsilon_s) (f_k^t) \notin \pi_1(T^{\varepsilon_s})$. Thus $|gw g^{-1}| \geq 2$, which implies that $gw g^{-1} \notin \pi_1(X) \setminus \pi_1(T)$. \hfill $\Box$

References

[AFW15] M. Aschenbrenner, S. Friedl, and H. Wilton, 3-manifold groups, EMS Series of Lectures in Mathematics, European Mathematical Society, 2015.

[Bas76] H. Bass, Some remarks on group actions on trees, Comm. Algebra 4 (1976), no. 12, 1091–1126.

[BMP03] M. Boileau, S. Maillot, and J. Porti, Three-dimensional orbifolds and their geometric structures, Panoramas et Synthèses, vol. 15, Société Mathématique de France, Paris, 2003.

[Bon02] F. Bonahon, Geometric structures on 3-manifolds, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 93–164.

[BT82] R. Bott and L. W. Tu, Differential forms in algebraic topology, Graduate Texts in Mathematics, vol. 82, Springer, Berlin, 1982.

[CS17a] F. Cerocchi and A. Sambusetti, Entropy and convergence of non-geometric 3-manifolds, to appear, 2017.

[CS17b] , Entropy (and finiteness) of groups with acylindrical splittings, to appear, 2017.

[CS17c] , Local topological rigidity of non-geometric 3-manifolds, 2017, arXiv: 1705.06213.

[dHW14a] P. de la Harpe and C. Weber, Malnormal subgroups and Frobenius groups: basics and examples. With an appendix by Denis Osin, Confluentes Math. 6 (2014), no. 1, 65–76.

[dHW14b] , On malnormal peripheral subgroups of the fundamental group of a 3-manifold, Confluentes mathematici 6 (2014), no. 1, 41–64.

[FSL15] R. Frigerio, J.-F. Lafont, and A. Sisto, Rigidity of high dimensional graph manifolds, Astérisque, no. 372, 2015.

[FM97] A. T. Fomenko and S. V. Matveev, Algorithmic and computer methods in three-dimensional topology, Mathematics and its Applications, vol. 425, Kluwer Academic Publishers, Dordrecht, 1997.

[Hat00] A. Hatcher, Notes on basic 3-manifold topology, 2000.

[Jac80] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Mathematics, vol. 43, 1980.

[JD83] M. Jankins and W. D. Neumann, Lectures on Seifert manifolds, Brandeis University, 1983.
[Joh79] K. Johannson, Homotopy equivalences of 3-manifolds with boundaries, Lecture Notes in Mathematics, vol. 761, Springer, Berlin, 1979.

[JS79] W. H. Jaco and P. B. Shalen, Seifert fibered spaces in 3-manifolds, vol. 21, Memoirs of the American Mathematical Society, no. 220, AMS Chelsea Publishing, 1979.

[Kap01a] I. Kapovich, The combination theorem and quasiconvexity, International Journal of Algebra and Computation 11 (2001), no. 02, 185–216.

[Kap01b] M. Kapovich, Hyperbolic manifolds and discrete groups, Progress in Mathematics, vol. 183, Springer, 2001.

[KT71] A. Karrass and D. Solitar, The free product of two groups with a malnormal amalgamated subgroup, Canad. J. Math. 23 (1971), 933–959.

[LS77] R. C. Lyndon and P. E. Schupp, Combinatorial group theory, Springer-Verlag, Berlin-New York, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[Martelli 2016] B. Martelli, An introduction to geometric topology, arxiv : math/1610.02592, 2016.

[EP94] W. Magnus, A. Karrass, and D. Solitar, Combinatorial group theory, second ed., Dover Publications, Inc., Mineola, NY, 2004.

[Mon87] J. M. Montesinos, Classical tessellations and three-manifolds, Universitext, Springer-Verlag, Berlin, 1987.

[Ohs87] K. Ohshika, Teichmüller spaces of seifert fibered manifolds with infinite π1, Topology Appl. 27 (1987), no. 1, 75–93.

[Otal 1996] J.-P. Otal, Théorème d’hyperbolisation pour les 3-variétés fibrées, Astérisque (1996), no. 235.

[Otal 1998] Théorème d’hyperbolisation pour les haken manifolds, Surveys in differential geometry 3 (1998), 77–194.

[Perelman 2002] G. Perelman, The entropy formula for the ricci flow and its geometric applications, arXiv : math/0211159, 2002.

[Perelman 2003a] Finite extinction time for the solutions to the ricci flow on certain three-manifolds, arXiv : math/0307245, 2003.

[Perelman 2003b] Ricci flow with surgery on three-manifolds, arXiv : math/0303109, 2003.

[Scott83] P. Scott, The geometries of 3-manifolds, Bulletin of the London Mathematical Society 15 (1983), no. 5, 401–487.

[Sela97] Z. Sela, Acylindrical accessibility for groups, Inventiones mathematicae 129 (1997), no. 3, 527–565.

[Serre80] J.-P. Serre, Arbres, amalgames, SL2, Astérisque, no. 46, Société Mathématique de France, Paris, 1977.

[Serre80] J. P. Serre, Trees, Springer, 1980.

[SW79] P. Scott and T. Wall, Topological methods in group theory, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 137–203.

[Thu82] W. P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. (N.S.) 6 (1982), no. 3, 357–381.

[Thu86] Hyperbolic structures on 3-manifolds. I. Deformation of acylnindrical manifolds, Ann. of Math. (2) 124 (1986), no. 2, 203–246.

[Thu97] Three-dimensional geometry and topology, vol. 1, Princeton university press, 1997.

[Thu98a] Hyperbolic structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle, arXiv : math/9801045 (1998).

[Thu98b] Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary, arXiv : math/9801058 (1998).

[WZ10] H. Wilton and P. Zalesskii, Profinite properties of graph manifolds, Geometriae Dedicata 147 (2010), no. 1, 29–45.

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