ON THE LARGEST BELL VIOLATION ATTAINABLE BY A QUANTUM STATE

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Abstract. We study the projective tensor norm as a measure of the largest Bell violation of a quantum state. In order to do this, we consider a truncated version of a well-known SDP relaxation for the quantum value of a two-prover one-round game, one which has extra restrictions on the dimension of the SDP solutions. Our main result provides a quite accurate upper bound for the distance between the classical value of a Bell inequality and the corresponding value of the relaxation. Along the way, we give a simple proof that the best complementation constant of $\ell_2^n$ in $\ell_1(\ell_\infty)$ is of order $\sqrt{\ln n}$. As a direct consequence, we show that we cannot remove a logarithmic factor when we are computing the largest Bell violation attainable by the maximally entangled state.

1. Introduction

A standard scenario to study quantum non-locality consists of two spatially separated and non-communicating parties, usually called Alice and Bob. Each of them can choose among different measurements, labeled by $x = 1, \ldots, N$ in the case of Alice and $y = 1, \ldots, N$ in the case of Bob. The possible outcomes of these measurements are labeled by $a = 1, \ldots, K$ in the case of Alice and $b = 1, \ldots, K$ in the case of Bob. Following the standard notation, we will refer the observables $x$ and $y$ as inputs and call $a$ and $b$ outputs. For fixed $x, y$, we will consider the probability distribution $(P(a, b| x, y))_{a,b=1}^K$ of positive real number satisfying

$$\sum_{a,b=1}^K P(ab| xy) = 1.$$ 

The collection $P = (P(a, b| x, y))_{x,y,a,b=1}^{N,K}$ will be also referred as a probability distribution. Given a probability distribution $P = (P(a, b| x, y))_{x,y,a,b=1}^{N,K}$, we will say that $P$ is

a) Classical if 

$$P(a, b| x, y) = \int_{\Omega} P_\omega(a| x)Q_\omega(b| y)dP(\omega)$$

for every $x, y, a, b$, where $(\Omega, \Sigma, P)$ is a probability space, $P_\omega(a| x) \geq 0$ for all $a, x, \omega$, $\sum_a P_\omega(a| x) = 1$ for all $x, \omega$ and analogous conditions for the $Q_\omega(b| y)$'s. We denote the set of classical probability distributions by $\mathcal{L}$.

b) Quantum if there exist two Hilbert spaces $H_1, H_2$ such that 

$$P(a, b| x, y) = tr(E_x^a \otimes F_y^b \rho)$$

for every $x, y, a, b$, where $E_x^a$ and $F_y^b$ are operators acting on $H_1$ and $H_2$, respectively, and $\rho$ is a density matrix on $H_1 \otimes H_2$. The quantum value of the game is the maximum value of $P$ over all quantum strategies. The classical value is the maximum value of $P$ over all classical strategies. The Bell value is the maximum value of $P$ over all strategies.
for every \( x, y, a, b \), where \( \rho \in B(H_1 \otimes H_2) \) is a density operator and \( (E^x_a)_{x,a} \subset B(H_1) \), \( (F^y_b)_{y,b} \subset B(H_2) \) are two sets of operators representing POVM measurements on Alice and Bob systems. That is, \( E^x_a \geq 0 \) for every \( x, a \), \( \sum_a E^x_a = 1 \) for every \( x \), and analogous conditions for the \( F^y_b \)’s. We denote the set of quantum probability distributions by \( Q \).

It is not difficult to see that both \( L \) and \( Q \) are convex sets and, furthermore, \( L \) is a polytope. Note that in order to talk about \( L \) and \( Q \) we must fixed the number of inputs \( N \) and outputs \( K \) in the Alice-Bob scenario. However, we will just write \( P \in L \) (resp. \( Q \in Q \)), where \( N \) and \( K \) will be clear from the context.

Given an element \( M = (M^a_{x,y})_{x,y}^{N,K} \), we denote
\[
\langle M, P \rangle = \sum_{x,y; a,b=1}^{N,K} M^a_{x,y} P(a, b|x, y)
\]
for every probability distribution \( P = (P(a, b|x, y))_{x,y; a,b=1}^{N,K} \). Then, we define the largest Bell violation of \( M \in \mathbb{R}^{N^2K^2} \) by
\[
LV(M) = \frac{\omega^*(M)}{\omega(M)},
\]
where \( \omega^*(M) := \sup \{ |\langle M, Q \rangle| : Q \in Q \} \) and \( \omega(M) := \sup \{ |\langle M, P \rangle| : P \in L \} \) (see\cite{19, 20, 21}). Actually, we must restrict this definition to those elements \( M \) which do not vanish on all \( L \). In the following we will assume this fact and we will write \( M \in M^{N,K} \). Any \( M \in M^{N,K} \) will be referred as a Bell inequality.\footnote{Formally, Bell inequalities are those inequalities which describe the facets of the set \( L \). However, it is very convenient considering this more general definition to quantify quantum non-locality.} There is a particularly interesting kind of Bell inequalities called two-prover one-round (2P1R) games. These are Bell inequalities of the form \( M^a_{x,y} = \pi(x,y)V(a, b|x, y) \) for every \( x, y = 1, \cdots, N \), \( a, b = 1, \cdots, K \); where \( \pi : [N] \times [N] \to [0, 1] \) is a probability distribution and \( V : [K] \times [K] \times [N] \times [N] \to \{0, 1\} \) is a boolean function, usually called predicate function. Note that, in particular, 2P1R-games have positive coefficients. These kinds of Bell inequalities are very relevant in computer science because most of the important problems in complexity theory can be stated in terms of these games. We will keep notation \( G = (G^a_{x,y})_{x,y; a,b=1}^{N,K} \) to denote these kinds of Bell inequalities. We talk about a Bell inequality violation when we have \( LV(M) > 1 \) for some \( M \in M^{N,K} \) (see\cite{37}). Note that this fact is equivalent to say that \( L \) is strictly contained in \( Q \). This is also referred as quantum non-locality.

Quantum non-locality is a crucial point in many different areas of quantum information and quantum computation. Some examples can be found in quantum cryptography\cite{11, 12}, in testing random numbers\cite{30}, in complexity theory\cite{13, 12, 22} and in communication complexity\cite{8}. This has motivated an increased interest in the study of the value \( LV(M) \) for some fixed \( M \)’s and also in the study of \( \sup_M LV(M) \), as a way...
of quantifying quantum non-locality (see [13], [20], [21], [9] for some recent works on the topic). In this work we will be also concerned with quantifying quantum non-locality but we will change our perspective. In this case, we will focus on the quantum states. Our main question is:

*Given an $n$-dimensional bipartite state $\rho$, how large can its Bell violations be?*

This question requires some extra notation. We will denote by $Q_\rho$ the set of all quantum probability distributions constructed with the state $\rho$ and, given $M \in \mathcal{M}^{N,K}$, we will denote

$$\omega_\rho^*(M) = \sup \{ |\langle M, Q \rangle| : Q \in Q_\rho \}$$

and

$$LV_\rho(M) = \frac{\omega_\rho^*(M)}{\omega(M)}.$$

Finally, we will define our key object:

$$LV_\rho := \sup_{N,K} \sup_{M \in \mathcal{M}^{N,K}} LV_\rho(M).$$

When we are dealing with pure states $\rho = |\psi\rangle\langle\psi|$ we will just write $LV_{|\psi\rangle}$. Then, the previous question can be written as: *How large can $LV_\rho$ be?*

As we will see in Theorem 2.1 though the supremum above runs over all $N, K \in \mathbb{N}$ and $M \in \mathcal{M}^{N,K}$, we have that $LV_\rho < \infty$ for every finite dimensional state $\rho$. The quantity $LV_\rho$ was first considered in [19] as a natural measure of how non-local a quantum state $\rho$ is. Indeed, since non-locality refers to probability distributions, it is natural to quantify the non-locality of a state $\rho$ by measuring how non-local the quantum probability distributions constructed with $\rho$ can be. $LV_\rho$ measures exactly this. Actually, [21, Proposition 3] allows us to write $LV_\rho$ in the following alternative way, which emphasizes its non-locality nature:

$$LV_\rho = \frac{2}{\pi_\rho} - 1,$$

such that

$$\pi_\rho = \sup_{P \in Q_\rho} \sup_{\lambda \in [0, 1]} \{ \lambda P + (1 - \lambda)P' \in \mathcal{L} : \text{ for some } P' \in \mathcal{L} \},$$

where the first supremum must be understood as a supremum in $N$ and $K$ too.

In many cases one is interested in studying the value $LV_\rho(M)$ for fixed $M$ and $\rho$ and also in $\sup_\rho LV_\rho(M) = LV(M)$ for a fixed Bell inequality $M \in \mathcal{M}^{N,K}$. In this context one can find very interesting works that mainly deal with particular Bell inequalities like CHSH ([11]), CGLMP ([14]) or $I_{3322}$ ([16]). Recent results have treated this problem from a more general point of view by studying the asymptotic behavior of $\sup_{M \in \mathcal{M}^{N,K}} LV(M)$ for fixed $N$ and $K$. Note that in these problems we fix the number of inputs $N$ and outputs $K$, whereas the dimension $n$ of the state (and measurements) is a free parameter in the optimization. In contrast, $n$ is the fixed parameter in the problem considered in this work (since we fix our state $\rho$), whereas we must consider $N$ and $K$ as free parameters.
in order to optimize over all Bell inequalities and all quantum measurements. This means that the problem considered here is, somehow, dual of those considered before.

The paper is organized as follows. In Section 2 we show that the projective tensor norm can be seen as a good measure for the largest Bell violation of a quantum state $L_V$. In particular, we provide upper and lower bounds for this largest Bell violation. This motivates the question of whether the projective tensor norm can actually measure the largest Bell violation of a quantum state up to, maybe, a constant factor. In order to study this question, in Section 3 we introduce a modified version of a SDP relaxation already used in computer sciences to approximate the classical and quantum value of a $2P1R$-game. The modification considered here consists of restricting the dimension of the SDP solutions. Then, we present a tight upper bound for the distance between the classical value of a Bell inequality and the corresponding value of the relaxation. We believe that these results can be of independent interest for computer scientists and they are the main point of this work. We will postpone the proof of this upper bound to Section 4. In the last part of Section 3 we use the previous estimates to show that there is a logarithmic factor (in the dimension) that cannot be removed when we compute the largest Bell violation of a quantum state by means of its projective tensor norm. Finally, in Section 4 we present the proof of our main theorem and we comment some points about its optimality.

2. The projective tensor norm as a measure of the largest Bell violation

2.1. An upper bound for the largest Bell violation of a quantum state. In order to study the value $L_V$ we will start with an alternative (somehow dual) statement of [21, Proposition 2] (see also [26, Theorem 3] for a related result). For the sake of completeness we will present a very simple new proof of this result avoiding operator space terminology and showing that no constant is required in the inequality. Before, we need to recall the definition of the projective and injective tensor norms, already used in several contexts of quantum information theory (see for instance [35], [17], [34]).

Given a finite dimensional normed space $X$, we denote by $B_X = \{x \in X : \|x\| \leq 1\}$ its (closed) unit ball. Also, we consider its dual space, $X^* = \{x^* : X \to \mathbb{C} : x^* \text{ is linear}\}$, with the norm $\|x^*\| = \sup_{x \in B_X} |x^*(x)|$. If $X, Y$ are finite dimensional normed spaces, we will denote the algebraic tensor product by $X \otimes Y$. Then, for a given $u \in X \otimes Y$ we define its projective tensor norm as

$$\pi(u) = \inf \left\{ \sum_{i=1}^{N} \|x_i\| \|y_i\| : N \in \mathbb{N}, u = \sum_{i=1}^{N} x_i \otimes y_i \right\}.$$

We will denote $X \otimes_p Y$ the space $X \otimes Y$ endowed with the projective tensor norm. It is very well known that $\ell_2^p \otimes \pi \ell_2^p = S_n^p$, where $\ell_2^p$ is the $n$-dimensional Hilbert complex space.
and \( S_1^n \) is the space of trace class operators from \( \ell_2^n \) to \( \ell_2^n \). On the other hand, for any \( v = \sum_{i=1}^{N} x_i \otimes y_i \in X \otimes Y \) we define its injective tensor norm as

\[
\epsilon(v) = \sup \left\{ \left| \sum_{i=1}^{N} x^* (x_i) y^* (y_i) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}.
\]

We will denote \( X \otimes \epsilon Y \) the space \( X \otimes Y \) endowed with the injective tensor norm. One can check that the projective and injective tensor norms are dual of each other. Specifically, for any pair of finite dimensional normed spaces \( X, Y \) we have

\[
(X \otimes \pi Y)^* = X^* \otimes \epsilon Y^* \text{ isometrically}.
\]

In particular, we recover the duality relation

\[
(S_1^n)^* = (\ell_2^n \otimes \pi \ell_2^n)^* = \ell_2^n \otimes \epsilon \ell_2^n = M_n,
\]

where \( M_n \) denotes the space of maps from \( \ell_2^n \) to \( \ell_2^n \) with the operator norm.

Finally, we will just mention that both tensor norms, projective and injective, can be defined in the tensor product of \( N \) spaces exactly in the same way. One can see that Equation (2.1) still holds in this general context and, furthermore, both norms are commutative and associative (respect to the spaces in the tensor products).

Note that, given an \( n \)-dimensional bipartite state \( \rho \), it can be realized as an element of the algebraic tensor product \( S_1^n \otimes S_1^n = S_1^{n^2} \).

**Theorem 2.1.** Given an \( n \)-dimensional bipartite state \( \rho \in S_1^n \otimes S_1^n \), we have

\[
LV_\rho \leq \| \rho \|_{S_1^n \otimes \pi S_1^n}.
\]

In particular,

\[
\sup_{\rho} LV_\rho \leq n,
\]

where the sup runs over all \( n \)-dimensional bipartite states.

**Proof.** Let’s consider a quantum strategy constructed with the state \( \rho \):

\[
Q = Q(a, b|x, y) = tr(E^a_x \otimes F^b_y \rho)
\]

for every \( x, y, a, b \); where \( \{ E^a_x \}_{x,a} \) and \( \{ F^b_y \}_{y,b} \) denote POVMs. We do not specify the number of inputs nor outputs because the result will not depend on that. Then, for every \( M \) verifying \( \omega(M) \leq 1 \) we have

\[
|\langle M, Q \rangle| \leq \left| tr\left( \sum_{x,y,a,b} M^{a,b}_{x,y} E^a_x \otimes F^b_y \rho \right) \right| \leq \| \rho \|_{S_1^n \otimes \pi S_1^n} \left\| \sum_{x,y,a,b} M^{a,b}_{x,y} E^a_x \otimes F^b_y \right\|_{M_n \otimes M_n}.
\]
Here, we have used that \((M_n \otimes \pi M_n)^* = S_1^n \otimes_n S_1^n\) and the dual action is given by the trace. On the other hand, if \(S_n\) denotes the set of states on \(M_n\) we have
\[
\left\| \sum_{x,y,a,b} M^{a,b}_{x,y} E^a_x \otimes E^b_y \right\|_{M_n \otimes M_n} = \sup_{\varrho_1, \varrho_2 \in S_n} \left| \sum_{x,y,a,b} M^{a,b}_{x,y} \text{tr}(E^a_x \varrho_1) \text{tr}(E^b_y \varrho_2) \right| \leq \omega(M) \leq 1.
\]
Here we have used that the operator \(\sum_{x,y,a,b} M^{a,b}_{x,y} E^a_x \otimes E^b_y\) is self adjoint and, therefore, we can take \(\varrho_1, \varrho_2 \in S_n \cup (-S_n)\). The absolute value allows us to restrict to \(S_n\).

In order to see the second assertion note that, by convexity, it suffices to show it for pure states. On the other hand, note that \(S_1^n \otimes \pi S_1^n = \ell_2^n \otimes \pi \ell_2^n \otimes \pi \ell_2^n\). Therefore, using that the projective tensor norm does not change if we apply a unitary on each space in the tensor product, one can even assume that our state is diagonal \(|\psi\rangle = \sum_{i=1}^n \alpha_i |ii\rangle\) and it is defined with positive coefficients. Furthermore, using the commutativity property of the projective tensor norm we have
\[
\|\langle \psi | \psi \rangle\|_{S_1^n \otimes S_1^n} = \left\| \sum_{i,j=1}^n \alpha_i \alpha_j |i jj\rangle \right\|_{\ell_2^n} = \left\| \sum_{i=1}^n \alpha_i |ii\rangle \otimes \sum_{j=1}^n \alpha_j |jj\rangle \right\|_{\ell_2^n} = \left\| \sum_{i=1}^n \alpha_i |ii\rangle \right\|_{\ell_2^n}^2 = \left( \sum_{i=1}^n \alpha_i \right)^2 = \|\psi\|^2.
\]
Since \(\sum_{i=1}^n \alpha_i^2 = 1\), the statement follows from the inequality \(\sum_{i=1}^n \alpha_i \leq \sqrt{n} (\sum_{i=1}^n \alpha_i^2)^{\frac{1}{2}}\). \(\square\)

In this paper we will restrict to pure states. As it was explained in the previous proof, given a diagonal unit element with positive coefficients\(^2\) \(|\varphi\rangle = \sum_{i=1}^n \alpha_i |ii\rangle \in \ell_2^n \otimes \ell_2^n\) and denoting \(\rho = |\varphi\rangle \langle \varphi|\in S_1^n \otimes S_1^n\), we have that
\[
(2.2) \quad \|\rho\|_{S_1^n \otimes S_1^n} = \|\varphi\|_1^2 = \left( \sum_{i=1}^n \alpha_i \right)^2.
\]
Note that Theorem \(2.1\) states that for every pure state \(|\varphi\rangle\) we have
\[
LV_{|\varphi\rangle} \leq \|\varphi\|_1^2.
\]

2.2. Lower bounds for every pure state. In the remarkable paper \([9]\) the authors showed that the upper bound \(O(n)\) given in Theorem \(2.1\) is almost tight. Specifically,

**Theorem 2.2** \([9]\). Let \(n\) be a natural number. There exists a game \(G_{KV}\) such that
\[
(2.3) \quad LV_{|\psi_n\rangle}(G_{KV}) \geq C \frac{n}{(\ln n)^2},
\]
where \(|\psi_n\rangle\) is the maximally entangled state in dimension \(n\). Here \(C\) is a universal constant which does not depend on the dimension.

\(^2\) To compute Bell violations we can always assume that our state is of this form. Indeed, this can be done by composing the corresponding POVMs with certain unitaries.
The game $G_{KV}$ is usually called *Khot-Visnoi game* (or KV game) because it was first defined by Khot and Visnoi to show a large integrality gap for a semidefinite programming (SDP) relaxation of certain complexity problems (see [23] for details). Since the KV game will play an important role in this work we will give a brief description of it (see [9] for a much more complete explanation). For any $n = 2^l$ with $l \in \mathbb{N}$ and every $\eta \in [0, \frac{1}{2}]$ we consider the group $\{0, 1\}^n$ and the Hadamard subgroup $H$. Then, we consider the quotient group $G = \{0, 1\}^n/H$ which is formed by $\frac{2^n}{n}$ cosets $[x]$ each with $n$ elements. The questions of the games $(x, y)$ are associated to the cosets whereas the answers $a$ and $b$ are indexed in $[n]$. The game works as follows: The referee chooses a uniformly random coset $[x]$ and one element $z \in \{0, 1\}^n$ according to the probability distribution $pr(z(i) = 1) = \eta$, $pr(z(i) = 0) = 1 - \eta$ independently of $i$. Then, the referee asks question $[x]$ to Alice and question $[x \oplus z]$ to Bob. Alice and Bob must answer one element of their corresponding cosets and they win the game if and only if $a \oplus b = z$. Given a probability distribution $P = \left( P([x], [y]|a, b) \right)_{[x],[y]=1}^{2^n,n}$ it is easy to see that

$$
\langle G_{KV}, P \rangle = \mathbb{E}_z \frac{n}{2^n} \sum_{[x]} \sum_{a \in [x]} P(a, a \oplus z|[x], [x+z]).
$$

Now, as a consequence of a clever use of hypercontractive inequality one can see that $\omega(G_{KV}) \leq n^{-\frac{n}{4}}$ (see [9] Theorem 7)). Furthermore, one can define, for any $a \in \{0, 1\}^n$, the vector $|u_a\rangle \in \mathbb{C}^n$ by $u_a(i) = \frac{(-1)^{a(i)}}{\sqrt{n}}$ for every $i = 1, \cdots, n$. It is easy from the properties of the Hadamard group that $(P_a = |u_a\rangle \langle u_a|)_{a \in [x]}$ defines a von Neumann measurement (vNm) for every $[x]$. These measurements will define Alice and Bob’s quantum strategies.

A careful study of the KV game shows that for every pure state $|\varphi\rangle$ in dimension $n$ we have

$$(2.4) \quad LV_{\varphi}(G_{KV}) \geq C \left( 1 + 4 \frac{\|\varphi\|_1^2 - 1}{(\ln n)^2} \right),$$

where $C$ is a universal constant (which can be taken $C = e^{-4}$). Indeed, as we have explained before, we can assume that our state is diagonal with non negative coefficients $|\varphi\rangle = \sum_{i=1}^n \alpha_i |ii\rangle$. Therefore, considering the same vNms as above (with respect to the basis $(|i\rangle)_{i=1}^n$) one can check that the quantum winning probability is greater or equal than

$$(2.5) \quad \mathbb{E}_z \left[ \frac{1}{n^2} \sum_{[x]} \sum_{a \in [x]} \sum_{i,j=1}^n \alpha_i \alpha_j (-1)^{a(i)}(-1)^{a(j)}(-1)^{a(i)+z(i)}(-1)^{a(j)+z(j)} \right]$$

$$= \frac{1}{n} \mathbb{E}_z \left[ \sum_{i,j=1}^n \alpha_i \alpha_j (-1)^{z(i)}(-1)^{z(j)} \right] = \frac{1}{n} \sum_{i=1}^n \alpha_i^2 + \frac{1}{n} \sum_{i \neq j} \alpha_i \alpha_j \mathbb{E}_z \left[ (-1)^{z(i)+z(j)} \right]$$

$$= \frac{1}{n} \sum_{i=1}^n \alpha_i^2 + \frac{1}{n} \sum_{i \neq j} \alpha_i \alpha_j (1-2\eta)^2 = \frac{1}{n} + \frac{1}{n} \left( \sum_{i=1}^n \alpha_i \right)^2 (1 - 2\eta)^2,$$
where we have used that $\mathbb{E}_2 \left[ (-1)^{z(i)+z(j)} \right] = (1 - 2\eta)^2$ is independent of $i, j$ with $i \neq j$.

On the other hand, as we have said before the classical value of $G_{KV}$ is upper bounded by $n^{-\frac{1}{m n}}$. If we consider $\eta = \frac{1}{2} - \frac{1}{m n} \in [0, \frac{1}{2}]$, as in [9], we have $n^{-\frac{1}{m n}} \leq C \frac{1}{n}$ and the last term in Expression (2.5) becomes $\frac{1}{n} + \frac{1}{n} \left( \left( \sum_{i=1}^{n} \alpha_i \right)^2 - 1 \right) \left( \frac{1}{mn} \right)^2$. Thus, we obtain Equation (2.4). Since we are thinking about states $|\varphi\rangle$ which typically have very large 1-norm, it is helpful to understand the previous result as

$$LV_{|\varphi\rangle} \geq C \left( \frac{||\varphi||_1}{\ln n} \right)^2$$

for every $n$-dimensional pure state $|\varphi\rangle$.

**Remark 2.1.** Actually, the KV game is defined for $n = 2^l$ with $l$ any natural number. However, an easy modification of the game allows us to state Equation (2.4) (so Equation (2.6) too) for a general $n$ with a slightly different constant. Indeed, for a given state $|\varphi\rangle$ in dimension $n$ we define $l_0 = \max \{ l : 2^l \leq n \}$. Then, we can consider the KV game in dimension $m = 2^{l_0}$ and artificially add an extra $m + 1$ output for Alice and Bob so that the predicate function of the game is always zero for these new values. Then, the only difference in the classical value of the game is that we must optimize over all families of non negative numbers $(P(a|x))_{x,a}$, $(Q(b|y))_{y,b}$ such that $\sum_a P(a|x) \leq 1$ for every $x$ and $\sum_b Q(b|y) \leq 1$ for every $y$. However, since all coefficients of the game are positive it is trivial to deduce that the optimum families will verify equality in the previous expressions. Therefore, the classical value of the new game is exactly the classical value of the KV game in dimension $m$. On the other hand, for every $a \in \{0, 1\}^m$ we can define the vector $|u_a\rangle \in \mathbb{C}^n$, by $u(i) = \frac{(-1)^{n(i)}}{\sqrt{m}}$ if $1 \leq i \leq m$ and $u(i) = 0$ otherwise. Then, the same calculation as above shows that if we consider the quantum probability distribution $Q$ constructed with the state $|\varphi\rangle$ and the von Neumann measurements

$$\left\{ \left( P^a_{[x]} = |u_a\rangle \langle u_a| \right)_{a[x]}, P^{m+1}_{[x]} = 1_{M_n} - \sum_{a[x]} |u_a\rangle \langle u_a| \right\}$$

we obtain that

$$\langle G_{KV}, Q \rangle = \frac{1}{m} \left( \sum_{i=1}^{m} \alpha_i^2 \right) \left( 1 - (1 - 2\eta)^2 \right) + \frac{1}{m} \left( \sum_{i=1}^{m} \alpha_i \right)^2 \left( 1 - 2\eta \right)^2.$$

Then, considering $\eta = \frac{1}{2} - \frac{1}{\log m}$ and using that $n \geq m = 2^{l_0} \geq \frac{n}{2}$ we recover the same estimates as in (2.4) with a slight modification in the constant.

In fact, since we are looking for a good measure of $LV_{|\varphi\rangle}$ for a general pure state $|\varphi\rangle$, we must be careful about giving lower bounds depending on the rank (or dimension) of the state. Indeed, in many case this can distort the essence of a state. With the computations above and the same ideas as in Remark 2.1 it is easy to see that one can give the following better lower bound for the largest Bell violation of a pure state $|\varphi\rangle = \sum_{i=1}^{n} \alpha_i |ii\rangle$. 


Theorem 2.3. For every $n$-dimensional pure state $|\varphi\rangle = \sum_{i=1}^{n} \alpha_i |ii\rangle$ with $(\alpha_i)_{i=1}^{n}$ a non-increasing sequence of positive numbers we have

$$LV_{|\varphi\rangle} \geq C \sup_{k=1,\ldots,n} \left( \frac{\sum_{i=1}^{k} \alpha_i}{\log k} \right)^{2},$$

where $C$ is a universal constant.

Note that for the maximally entangled state we obtain $LV_{|\psi_n\rangle} \geq C \frac{n}{(\ln n)^2}$ as it is stated in Theorem 2.2.

The previous study shows the projective tensor norm as a good candidate to measure the largest Bell violation attainable by a (pure) state. This reminds us Rudolph’s characterization of entangled states:

Theorem 2.4 [34]. A quantum state $\rho \in S_1^n \otimes S_1^n$ is entangled if and only if $\|\rho\|_{S_1^n \otimes \pi S_1^n} > 1$.

In this sense the previous estimates show a link between quantum entanglement and quantum non-locality, contrary to the spirit of the most recent results on the topic (see for instance [27], [5], [19]). The results above encourage us to ask whether the projective tensor norm of a state $\|\rho\|_{S_1^n \otimes S_1^n}$ could measure its largest Bell violation $LV_{\rho}$ up to, maybe, a constant factor. Unfortunately, the following theorem shows that this is not the case. That is, we cannot completely remove the logarithmic factor in Equation (2.3) (so we cannot in (2.6) either). Specifically,

Theorem 2.5. Let $|\psi_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |ii\rangle$ be the maximally entangled state in dimension $n$, then

$$LV_{|\psi_n\rangle} \leq D\frac{n}{\sqrt{\ln n}}$$

for certain universal constant $D$.

Theorem 2.5 will be obtained as a consequence of our main result presented in Section 3.

Note that Theorem 2.2 and Theorem 2.5 clarify the asymptotic behavior of the largest Bell violation of the maximally entangled state up to the order of the logarithmic factor:

$$C \frac{n}{(\ln n)^2} \leq LV_{|\psi_n\rangle} \leq D \frac{n}{\sqrt{\ln n}}.$$
It is very interesting to say that, beyond their own interest, these logarithmic-like estimates are very useful to obtain non-multiplicative results. Indeed, in the very recent paper [28] the previous estimates have been used to show that the largest Bell violation of a state $LV_\rho$ is a highly non-multiplicative measure. Actually, similar techniques have been used to show super-activation of quantum non-locality.

We must also mention that when we restrict to the easier case of von Neumann measurements (vNms) rather than general POVMs one can improve the upper bound in Theorem 2.5 to obtain $O(\frac{n}{\ln n})$. Indeed, following Werner's construction ([39]), in [3] the authors showed that for certain $p \geq K' \ln n$ (with $K' \geq 0.8$) the state $\xi_p = p|\psi_n\rangle\langle\psi_n| + (1-p)\frac{1}{n^2}$ is vNm-local; that is, one can construct a local hidden variable model to describe any quantum probability distribution $(\text{tr}(P_a \otimes Q_y^b \xi_p))_{x,y}$ constructed with vNms $\{P_a\}, \{Q^b_y\}$. Here, we denote by $\frac{1}{n^2}$ the maximally mixed state. Since $\frac{1}{n^2}$ is a separable state, one immediately deduces that

$$LV_{|\psi_n\rangle}^v \leq K' \frac{n}{\ln n},$$

where $LV_{|\psi\rangle}^v$ denotes the measure $LV_{|\psi\rangle}$ restricted to vNms in the construction of the quantum probability distributions and $K' \leq \frac{n}{2}$. We must mention, however, that restricting to vNms, though very natural from a physical point of view, simplifies very much the geometry of the problem. Actually, the best estimate in [3] for $p$ to verify that $\xi_p$ is local (with general POVMs) is $\Omega(\frac{1}{n})$, which leads to an estimate $LV_{|\psi_n\rangle} \leq Dn$. It is also worth mentioning that the KV game can be used to improve the upper bound estimates in [3]. Indeed, since the quantum strategy used in Theorem 2.2 is constructed with vNms acting on the maximally entangled state in dimension $n$, we immediately conclude that $\frac{(\ln n)^2}{n}$ is an upper bound for the value $p^L_n$ considered in [3].

### 3. A RELAXATION OF THE PROBLEM

Let’s consider the following semi-definite programming (SDP) relaxation for the quantum value of a $2P1R$-game $G$ with $N$ questions and $K$ answers, which optimizes over families of real vectors $\{u^a_x\}_{x,a=1}^{N,K}$, $\{v^b_y\}_{y,b=1}^{N,K}$.

(3.1)

$$SDP(G) := \text{Maximize: } \left| \sum_{x,y,a,b=1}^{N,K} G_{x,y}^a b (u^a_x, v^b_y) \right|$$

Subject to: $\forall x, \forall a \neq a', \langle u^a_x, u^{a'}_x \rangle = 0$ and $\forall y, \forall b \neq b', \langle v^b_y, v^{b'}_y \rangle = 0$, $\forall x, \| \sum_a u^a_x \| = 1$ and $\forall y, \| \sum_b v^b_y \| = 1$.

Note that the orthogonality restriction in (3.1) when seen as a relaxation of the quantum value of a $2P1R$-game $G$ is not natural when one is interested in studying the dimension of the considered quantum states; as we are in this work. Indeed, such an orthogonality
condition comes from the fact that any quantum probability distribution \( Q \in \mathcal{Q} \) can be written by using von Neumann measurements. However, that process involves an increase in the dimension of the Hilbert spaces. Furthermore, since we are interested here in fixing the dimension of our quantum states \( \rho \), we would like to truncate the previous SDP relaxation by requiring the families of vectors \( \{ u_x^a \}_{x,a=1}^{N,K}, \{ v_y^b \}_{y,b=1}^{N,K} \) to have a fixed dimension \( n \). Note, however, that this is not possible in general since the orthogonality condition implies that \( K \) must be smaller or equal than \( n \) (while we are typically interested in the opposite case). In order to save this problem we will consider the following optimization problem, which optimizes over families of real vectors \( \{ u_x^a \}_{x,a=1}^{N,K}, \{ v_y^b \}_{y,b=1}^{N,K} \):

\[
\omega_{OP\infty}(G) := \text{Maximize: } \left| \sum_{x,y,a,b=1}^{N,K} C_{x,y}^{a,b}(u_x^a, v_y^b) \right|
\]

Subject to:
\[
\forall x, \sup_{\alpha_x^a=\pm 1} \left\| \sum_{a=1}^{K} \alpha_x^a u_x^a \right\| \leq 1, \\
\forall y, \sup_{\alpha_y^b=\pm 1} \left\| \sum_{b=1}^{K} \alpha_y^b v_y^b \right\| \leq 1.
\]

Then, it is very easy to see that \( \omega_{OP\infty}(G) \) is a relaxation for the problem of computing the quantum value of a 2P1R-game \( G \) and it verifies that \( \text{SDP}(G) \leq \omega_{OP\infty}(G) \) for every \( G \). The value \( \omega_{OP\infty}(G) \) is a natural generalization of \( \text{SDP}(G) \) which removes the orthogonality conditions and so admits restrictions in the dimension of the vectors. We will call \( \omega_{OP\infty}(G) \) the value of the previous optimization problem with the extra restriction: \( u_x^a, v_y^b \in \mathbb{R}^n \) for every \( x, y, a, b \).

Then, we will show:

**Theorem 3.1.**

\[
\omega_{OP_n}(G) \leq D \frac{n}{\sqrt{\ln n}} \omega(G)
\]

for every 2P1R-game \( G \), where \( D \) is a universal constant.

We think that Theorem 3.1 and Theorem 3.2 (see below) can be of independent interest for computer scientists. \( \omega_{OP_n} \) is the natural generalization of \( \text{SDP} \) when we want to impose “low dimensional solutions” (where orthogonality restrictions no longer make sense since we will have \( K > n \)). As far as we know the question of rounding low-dimensional solutions of these kinds of optimization problems has not received much attention. Some interesting papers in this direction are [4], [6], [7].

Since in this paper we want to work in the general context of Bell inequalities (rather than restricting to the specific case of 2P1R-games) we have to consider a modification of the definition of \( \omega_{OP\infty} \) (resp. \( \omega_{OP_n} \)). Indeed, the non-signaling condition verified by the classical and quantum probability distributions plays an important role in this case and one has to impose an extra restriction to avoid trivial cases where \( \omega(M) = 0 \) and \( \omega_{OP_n}(M) > 0 \), which makes not possible any result like Theorem 3.1 (see [19] Section 5) for a complete study on the geometry of the problem). Then, for a given Bell inequality \( M \in \mathcal{M}^{N,K} \), we consider the following optimization problem, which optimizes over families
of real $n$ dimensional vectors $\{u^a_{x, a, b = 1}^N, \{v^b_{y, b = 1}^K, z:\$

$$\mathcal{w}_{OP_n}(M) := \text{Maximize: } \left| \sum_{x, y, a, b = 1}^{N, K} M_{x, y}^{a, b} \langle u^a_x, v^b_y \rangle \right|$$

Subject to:

$$\forall x, y, \sum_{a = 1}^{K} u^a_x = \sum_{b = 1}^{K} v^b_y = z \, (*) \,$$

$$\forall x, \sup_{a = 1}^{\alpha_x^a} \| \sum_{a = 1}^{K} \alpha_x^a u^a_x \| \leq 1,$$

$$\forall y, \sup_{b = 1}^{\alpha_y^b} \| \sum_{b = 1}^{K} \alpha_y^b v^b_y \| \leq 1.$$  

Our main theorem states

**Theorem 3.2.** For all natural numbers $n$, $N$, $K$ and every $M \in \mathcal{M}^{N,K}$ we have

$$\mathcal{w}_{OP_n}(M) \leq D \frac{n}{\sqrt{\ln n}} \omega(M),$$

where $D$ is a universal constant.

We will postpone the proof of Theorem 3.2 to Section 4. The proofs of Theorem 3.1 and Theorem 3.2 are the same, but in the second case we have the extra difficulty of restricting to a certain affine subspace described by condition $(*)$. In particular, Theorem 3.1 can be obtained by following exactly the same proof as the one we will present in Section 4 for Theorem 3.2 with obvious modifications.

It can be deduced from [19] that for some $M \in \mathcal{M}^{n,n}$ we have

$$\mathcal{w}_{OP_n}(M) \geq C \frac{n}{\ln n} \omega(M)$$

for some universal constant $C$. We do not know whether one can get the upper bound $O(\frac{n}{\ln n})$ in Theorem 3.2. However, in order to obtain this estimate a different approach from the one followed in this work would be required. Indeed, as we will explain in Section 4, our result is optimal in some sense. We must also mention that $\mathcal{w}_{OP_n}(M)$ can be much larger than $LV_{\omega_{\psi}}(M)$ for some $M \in \mathcal{M}^{N,K}$. A particularly extreme case can be found for $N = 1$, where one can find some elements $M$ such that $\omega(M) = \omega^*(M) = 1$ and also $\mathcal{w}_{OP_n}(M) \geq \sqrt{n}$. Thus, another approach more focused on the specific properties of the maximally entangled state could give a better upper bound in Theorem 2.5 without improving Theorem 3.2. On the other hand, note that Theorem 2.2 says that Theorem 2.5 is already very tight.

**Remark 3.1.** Theorem 3.1 and Theorem 3.2 can be stated in terms of the so called $\gamma_2^*$ tensor norm. Given two Banach spaces $X$, $Y$ and their algebraic tensor product $X \otimes Y$, for a given $z \in X \otimes Y$ we define

$$\|z\|_{\gamma_2^*} = \sup \{ \|u \otimes v\|_{\ell_2^2 \otimes \ell_2^2} : \|u\|, \|v\| \leq 1 \},$$

where the supremum runs over all linear maps $u : X \to \ell_2^2$, $v : Y \to \ell_2^2$ verifying $\|u\|, \|v\| \leq 1$. Then, it is very easy to see that for every $M \in \mathcal{R}^{N^2K^2}$ we have

$$\omega_{OP_n}(M) = \|M\|_{\ell_2^N(\ell_2^K) \otimes \gamma_2^*(\ell_2^N(\ell_2^K))}.$$
Thus, Theorem 3.1 is equivalent to prove
\[
(3.4) \quad \left\| id \otimes id : \ell_1^n(\ell_\infty^K) \otimes_\epsilon \ell_1^n(\ell_\infty^K) \rightarrow \ell_1^n(\ell_\infty^K) \otimes_\gamma_{\tau_2,n} \ell_1^n(\ell_\infty^K) \right\| \leq D \frac{n}{\sqrt{\ln n}}.
\]
To deal with general Bell inequalities one must replace the space \(\ell_1^n(\ell_\infty^K)\) with the space \(NSG^*(N, K)\) introduced in [19, Section 5]. Then, Theorem 3.2 is equivalent to prove
\[
(3.5) \quad \left\| id \otimes id : NSG^*(N, K) \otimes_\epsilon NSG^*(N, K) \rightarrow NSG^*(N, K) \otimes_\gamma_{\tau_2,n} NSG^*(N, K) \right\| \leq D \frac{n}{\sqrt{\ln n}}.
\]
It was proven in [19, Section 5] that \(NSG^*(N, K)\) is a twisted version of the space \(\ell_1^n(\ell_\infty^K)\) and it can be seen that proving (3.4) is equivalent to prove (3.5). However, in order to present a work accessible for a general audience, we will not make use of extra results and we will provide a completely self-contained proof of Theorem 3.2.

We finish this section by showing how to obtain Theorem 2.5 from Theorem 3.2.

Proof of Theorem 2.5. Let’s consider a Bell inequality \(M \in M^{N,K}\) such that \(\omega(M) \leq 1\). We must show that
\[
\left| \sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} Q_{x,y}^{a,b} \right| \leq D \frac{n}{\sqrt{\ln n}}
\]
for every \(Q \in Q^{N,K}_{\{|\psi_n\rangle\}}\). By definition \(Q_{x,y}^{a,b} = tr(E_x^a \otimes F_y^b |\psi_n\rangle \langle \psi_n|)\) for every \(x, y, a, b\), where \(\{E_x^a\}_{x,a}\) and \(\{F_y^b\}_{y,b}\) are POVMs in dimension \(n\) and \(|\psi_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle\). Then, we can write
\[
tr(E_x^a \otimes F_y^b |\psi_n\rangle \langle \psi_n|) = \frac{1}{n} \sum_{i,j=1}^n E_x^a(i, j) F_y^b(i, j) = \frac{1}{n} \sum_{i,j=1}^n (E_x^a)^t(i, i) F_y^b(i, j) = \frac{1}{n} tr((E_x^a)^t F_y^b),
\]
where \(^t\) denotes the transpose. Therefore,
\[
\left| \sum_{x,y,a,b} M_{x,y}^{a,b} Q_{x,y}^{a,b} \right| = \left| \frac{1}{n} \sum_{i=1}^n \sum_{x,y,a,b} M_{x,y}^{a,b} ((E_x^a)^t F_y^b)(i, i) \right| \leq \sup_{i=1,\ldots,n} \left| \sum_{x,y,a,b} M_{x,y}^{a,b} (u_x^{a,i}, v_y^{b,i}) \right|
\]
(3.6)
where \(|u_x^{a,i}| = |E_x^a|\) for every \(x, a, i\) and \(|v_y^{b,j}| = |F_y^b|\) for every \(y, b, j\).

Note that for a fixed \(i = 1, \ldots, n\), we trivially have
\[
(3.7) \quad \sum_{a=1}^K u_x^{a,i} = \sum_{b=1}^K v_y^{b,i} = |i|
\]
for every \(x, y\). Furthermore, for every \(x\) and every \((\alpha_a)_{a=1}^K \in \{-1, 1\}^K\), we have that
\[
(3.8) \quad \left\| \sum_{a=1}^K \alpha_a u_x^{a,i} \right\| = \left\| \sum_{a=1}^K \alpha_a E_x^a i \right\| \leq \left\| \sum_{a=1}^K \alpha_a E_x^a \right\|_{M_n} \leq 1;
\]
and analogously for the \( v_{y,i}'s \), where the last inequality follows from the fact that \( \{ E_z^2 \}_a \) is a POVM. Therefore, for every \( i = 1, \cdots, n \) the families of \( n \)-dimensional (possibly complex) vectors \( \{ u_{x,i}^a \}_{x,a} \) and \( \{ v_{y,i}^b \}_{y,b} \) verify the conditions in (3.3). The only thing left to do is to show that these vectors can be assumed to be real. Indeed, if this is true, we can apply Theorem 3.2 to conclude
\[
\left| \sum_{x,y,a,b} M_{x,y}^{a,b} Q_{x,y}^{a,b} \right| \leq \sup_{i=1,\cdots,n} \left| \sum_{x,y,a,b} M_{x,y}^{a,b} \langle u_{x,i}^a, v_{y,i}^b \rangle \right| \leq D \frac{n}{\sqrt{\ln n}}
\]
We can assume the families \( \{ u_{x,i}^a \}_{x,a} \) and \( \{ v_{y,i}^b \}_{y,b} \) to be formed by real vectors replacing \( n \) with \( 2n \) (which means just a slight modification in the constant \( D \)). To see this we note that Equation (3.6) can be read as
\[
\left| \sum_{x,y,a,b} M_{x,y}^{a,b} Q_{x,y}^{a,b} \right| = \left| Re \left( \sum_{x,y,a,b} M_{x,y}^{a,b} Q_{x,y}^{a,b} \right) \right| \leq \sup_{i=1,\cdots,n} \left| \sum_{x,y,a,b} M_{x,y}^{a,b} Re(\langle u_{x,i}^a, v_{y,i}^b \rangle) \right|
\]
where \( Re(z) \) denote the real part of \( z \). On the other hand, if we define the vectors \( \overline{u}_{x,i}^a = Re(u_{x,i}^a) \oplus Im(u_{x,i}^a) \in \mathbb{R}^{2n} \) and \( \overline{v}_{y,i}^b = Re(v_{y,i}^b) \oplus Im(v_{y,i}^b) \in \mathbb{R}^{2n} \), we obtain new real vectors verifying \( \langle \overline{u}_{x,i}^a, \overline{v}_{y,i}^b \rangle = Re(\langle u_{x,i}^a, v_{y,i}^b \rangle) \) for every \( x, a \) and also conditions (3.7) and (3.8). \( \square \)

4. PROOF OF THE MAIN RESULT

4.1. Proof of Theorem 3.2. The proof of Theorem 3.2 will follow the same lines as [19, Theorem 18]. However, we will present here a simpler approach to the problem avoiding, in particular, the use of [19] (via [19, Theorem 19]). Actually, our proof relies on Lemma 4.1 proven below which will lead to an improvement of [19, Theorem 19]. Furthermore, as we will show in Theorem 4.3 we will give the optimal complementation constant of \( \ell_2^n \) in \( \ell_1(\ell_\infty) \), so the optimal estimate in [19, Theorem 19]. In order to make the proof of Theorem 3.2 completely understandable for every reader, we will start by introducing a few definitions and basic results. In the following we will denote by \( \ell_2^n \) the space \( \mathbb{R}^n \) with the Euclidean norm \( \| (a_i)_i \|_2 = \left( \sum_i |a_i|^2 \right)^{\frac{1}{2}} \) and by \( \ell_\infty^n \) the space \( \mathbb{R}^n \) with the norm \( \| (a_i)_i \|_\infty = \sup_{i=1,\cdots,n} |a_i| \). We will denote by \( \ell_2 \) and \( \ell_\infty \) the corresponding infinite dimensional spaces. On the other hand, given a linear map \( T : X \to Y \) between two finite dimensional normed spaces, we will denote the norm of \( T \) by
\[
\| T \| := \sup_{x \in B_X} \| T(x) \|_Y,
\]
\footnote{Note that we used this notation in Theorem 2.1 to denote the complex \( n \)-dimensional Hilbert space. Here, we will restrict to real spaces.}
where $B_X$ is the unit ball of $X$. Note that for a linear map $T : \ell_\infty^K \to \ell_2^n$ with $T(|a_i\rangle) = |u_a\rangle$ for every $a$, where $(|a\rangle)_{a=1}^K$ denotes the standard basis in $\mathbb{R}^K$, we have
\begin{equation}
\|T\| = \sup_{(a_n)_n \in (-1,1)^K} \left\| \sum_{a=1}^K \alpha_a |u_a\rangle \right\|_{\ell_2^n} = \sup_{\sum_{a=1}^K |\alpha_a|^2 = 1} \sum_{a=1}^K \left| \sum_{i=1}^n \beta_i \langle a_i | i \rangle \right|.
\end{equation}
In the particular case where $T : \ell_2^n \to \mathcal{Y}$, we will be also interested in the following norm of $T$
\begin{equation}
\ell(T) := \mathbb{E}\left( \left\| \sum_{i=1}^n g_i T(|i\rangle) \right\|_{\mathcal{Y}}^2 \right)^{\frac{1}{2}},
\end{equation}
where $\{|i\rangle\}_{i=1}^n$ denotes the standard basis of $\ell_2^n$ and $(g_i)_{i=1}^n$ is a sequence of independent normalized real random Gaussian variables. Note that a trivial computation shows
\begin{equation}
\ell(T) = \mathbb{E}\left( \left\| T\left( \sum_{i=1}^n g_i |i\rangle \right) \right\|_{\mathcal{Y}}^2 \right)^{\frac{1}{2}} \leq \|T\| \mathbb{E}\left( \left\| \sum_{i=1}^n g_i |i\rangle \right\|_{\ell_2^n}^2 \right)^{\frac{1}{2}} \leq \sqrt{\mathbb{E}\|T\|}
\end{equation}
for every operator $T : \ell_2^n \to \mathcal{Y}$. According to Kahane-Khinchin inequality (see for instance \cite{3}, pp 16) we know that
\begin{equation}
\ell(T) \leq K_{1,2} \mathbb{E}\left\| \sum_{i=1}^n g_i T(|i\rangle) \right\|_{\mathcal{Y}}
\end{equation}
for every $T : \ell_2^n \to \mathcal{Y}$, where $K_{1,2}$ is a universal constant.

Finally we will introduce a third norm for a given linear map $T : X \to Y$. We say that $T$ is 2-summing if there exists a constant $C \geq 0$ such that for every $N \in \mathbb{N}$ and every sequence $x_1, \cdots, x_N$ in $X$ the following inequality holds:
\begin{equation}
\left( \sum_{i=1}^N \| T(x_i) \|_Y^2 \right)^{\frac{1}{2}} \leq C \sup_{x^* \in B_{X^*}} \left( \sum_{i=1}^N |x^*(x_i)|^2 \right)^{\frac{1}{2}}.
\end{equation}
In this case we define the 2-summing norm of $T$ as $\pi_2(T) := \inf \{ C : C$ verifies (1.4) $\}$. A particularly simple case is when $T : \ell_\infty \to \ell_\infty$ is a diagonal map defined by a sequence $(\lambda_i)_{i=1}^\infty$ (that is, $T(|i\rangle) = \lambda_i |i\rangle$ for every $i$). In this case, one can see that $\pi_2(T) = \| (\lambda_i)_{i=1}^\infty \|_2$. It is also easy to verify from its definition that the 2-summing operators form an operator ideal. In particular, for all linear maps between Banach spaces $T : X \to Y$, $S : Y \to Z$ and $Q : Z \to W$ we have that $\pi_2(Q \circ S \circ T) \leq \|Q\| \pi_2(S) \|T\|$. Grothendieck inequality has been already used in several problems of quantum information theory (see \cite{32} for a complete survey of the topic). As an immediate consequence of Grothendieck inequality we deduce that for every linear map $T : \ell_\infty \to \ell_2$ we have
\begin{equation}
\pi_2(T) \leq K_G \|T\|,
\end{equation}
where $K_G$ is the Grothendeick constant, which is known to verify $K_G < 1.78$. Finally, the following inequality will be very helpful in the proof of Lemma 4.1. Let $a : \ell_2 \to \ell_\infty$ and $b : \ell_\infty \to \ell_2$ be two linear maps, then
\begin{equation}
|\text{tr}(b \circ a)| \leq \pi_2(b)\pi_2(a) \leq K_G\|b\|\pi_2(a).
\end{equation}
Here, the first inequality is a consequence of trace duality (see for instance [15]) and the second one follows from Equation (4.5).

The following lemma will be crucial in the proof of Theorem 3.2.

Lemma 4.1. For all natural numbers $n, N \in \mathbb{N}$ and all linear maps $S : \ell_n^2 \to \ell_N^\infty$ and $T : \ell_\infty^N \to \ell_2^n$ we have
\begin{equation}
|\text{tr}(T \circ S)| \leq C\sqrt{\frac{n}{\ln n}}\|T\|\ell(S),
\end{equation}
where $C$ is a universal constant.

The key point in the proof of Lemma 4.1 is a nice consequence of the concentration of measure phenomenon given by Ledoux and Talagrand. It has already been used in the study of cotype constants in Banach space theory. In particular, we develop here some ideas from [18].

Proof. According to [24, Theorem 12.10] applied to the Gaussian process $X_t = \sum_{i=1}^n g_i\langle t|S|i\rangle$, $t = 1, \cdots, N$, there exists a Gaussian sequence $(Y_k)_{k \geq 1}$ with $\|Y_k\|_2 \leq C\ell(S)/\sqrt{\ln(k+1)}$ for every $k \geq 1$ and such that for every $t = 1, \cdots, N$ we have
\begin{equation}
X_t = \sum_{k \geq 1} \alpha_k(t)Y_k,
\end{equation}
where $\alpha_k(t) \geq 0$, $\sum_{k \geq 1} \alpha_k \leq 1$ and the series converges almost surely in $L_2$. Then, for every $k \geq 1$ we can define $u_k = \sqrt{\ln(k+1)}\sum_{i=1}^n \langle Y_k, g_i \rangle |i\rangle \in \ell_2^n$, $v_k = \sum_{t=1}^N \alpha_k(t)|y\rangle \in \ell_\infty^N$ and the previous properties guarantee that $\|u_k\| \leq C\ell(S)$ and $\|v_k\| \leq 1$ for every $k$. Let’s consider now the linear maps $A : \ell_2^n \to \ell_\infty$, $D : \ell_\infty \to \ell_\infty$ and $B : \ell_\infty \to \ell_\infty^N$ define by $A(|i\rangle) = \sum_{k \geq 1} \langle u_k|i\rangle |k\rangle$ for every $i = 1, \cdots, n$; $D(|k\rangle) = \frac{1}{\sqrt{\ln(k+1)}}|k\rangle$ for every $k \geq 1$ and $B(|k\rangle) = v_k$ for every $k \geq 1$ respectively. Cauchy-Schwartz inequality shows that $\|A\| \leq C\ell(s)$, whereas it is trivial to check that $\|D\| \leq 1$ and $\|B\| \leq 1$. Furthermore, the following factorization holds:
\begin{equation}
S = B \circ D \circ A.
\end{equation}

Actually, to state inequality (4.18) it is enough to invoke the little Grothendieck theorem which gives us a constant $\sqrt{\pi}$.\footnote{Actually, to state inequality (4.18) it is enough to invoke the little Grothendieck theorem which gives us a constant $\sqrt{\pi}$.}
Following [18, Lemma 3.3] we write $D = D_1 + D_2$ where $D_1$ is the diagonal operator associated to the sequence $\bar{D}_1 = (\frac{1}{\sqrt{\ln 2}}, \ldots, \frac{1}{\sqrt{\ln(n+1)}}, 0, 0, \ldots)$. Then, we have

$$|\text{tr}(T \circ S)| = |\text{tr}(T \circ B \circ D \circ A)| \leq |\text{tr}(T \circ B \circ D_1 \circ A)| + |\text{tr}(T \circ B \circ D_2 \circ A)|.$$  

Now, according to Equation (4.6) and the ideal property of 2-summing operators we have

$$|\text{tr}(T \circ B \circ D_1 \circ A)| \leq K_G \pi_2(D_1) \|T \circ B\| \leq K_G \pi_2(D_1) \|A\| \|T\| \|B\| \leq C'' \sqrt{\frac{n}{\ln n}} \ell(S) \|T\|,$$

where we have used $\pi_2(D_1) = \|\bar{D}_1\| \leq \bar{C} \sqrt{\frac{n}{\ln n}}$. On the other hand, if we denote $id_n : \ell_2^n \to \ell_\infty^n$ the identity map, we have

$$|\text{tr}(T \circ B \circ D_2 \circ A)| = |\text{tr}(id_n^{-1} \circ id_n \circ T \circ B \circ D_2 \circ A)| \leq K_G \pi_2(id_n \circ T \circ B \circ D_2 \circ A) \|id_n^{-1}\| \leq K_G \pi_2(id_n) \|\pi_2(T)\| \|B\| \|D_2\| \|A\| \|id_n^{-1}\| \leq C''' \sqrt{\frac{n}{\ln n}} \|T\| \ell(S),$$

where we have used that $\|D_2\| \leq \frac{1}{\sqrt{\ln(n+1)}}$, $\pi_2(T) \leq K_G \|T\|$, $\|id_n\| = 1$ and $\|id_n^{-1}\| = \sqrt{n}$. Therefore, we obtain that

$$|\text{tr}(T \circ S)| \leq \bar{C} \sqrt{\frac{n}{\ln n}} \|T\| \ell(S),$$

as we wanted. \hfill \square

**Remark 4.1.** We note that Lemma [18] is optimal. Indeed, if we consider the map $id_n : \ell_2^n \to \ell_\infty^n$, it is well known that $\ell(id_n) \leq c \sqrt{\ln n}$ for some universal constant $c$ and we also have $\|id_n^{-1}\| = \sqrt{n}$. On the other hand, we trivially have $\text{tr}(id_n^{-1} \circ id_n) = n$.

The following easy lemma will be very useful in our proof.

**Lemma 4.2.** Let $R = (R(x|a))_{x,a=1}^{N,K}$ be a family of real numbers such that $\sum_{a=1}^K R(x|a) = C$ for every $x = 1, \ldots, N$, where $C$ is a constant. Let’s denote $\Lambda = \sup_{x=1,\ldots,N} \sum_{a=1}^K |R(x|a)|$. Then, we can write $R = \lambda P_1 + \mu P_2$ such that $P_i \in S(N,K)$ for $i = 1, 2$ and $|\lambda| + |\mu| = \Lambda$.

Here, we denote

$$S(N,K) = \left\{(P(x|a))_{x,a=1}^{N,K} : P(x|a) \geq 0 \text{ and } \sum_{a=1,\ldots,K} P(x|a) = 1 \text{ for every } x,a\right\}.$$  

**Proof.** We can assume the constant $C$ to be positive. For every $x$, we denote

$$A^+_x = \{a : R(x|a) > 0\} \text{ and } A^-_x = \{a : R(x|a) \leq 0\}.$$  

Also, we denote

$$M = \sup_x \sum_{a \in A^+_x} R(x|a) \text{ and } m = \inf_x \sum_{a \in A^-_x} R(x|a).$$

Since the case $m = 0$ is trivial we can assume that $m < 0$. The fact that $\sum_a R(x|a) = C$ for every $x$ guarantees that the previous sup and inf are attained in the same $x$. In particular
note that $M + m = C$ and $M - m = \Lambda$. Therefore, we can write $R = MP_1 + mP_2$, where we define, for each $x$: $P_1(a|x) = \frac{R(a|x)}{M}$ for $a \in \{1, \cdots, K - 1\} \cap A_x^+$, $P_1(a|x) = 0$ for $a \in \{1, \cdots, K - 1\} \cap A_x^-$, $P_2(K|x) = 1 - \sum_{a=1}^{K-1} P_1(a|x)$ and $P_2(a|x) = \frac{R(a|x)}{m}$ for $a \in \{1, \cdots, K - 1\} \cap A_x^+$, $P_2(K|x) = 1 - \sum_{a=1}^{K-1} P_2(a|x)$. Since $P_1$ and $P_2$ belong to $S(N, K)$ and $|M| + |m| = \Lambda$ we conclude the proof. □

We are now ready to prove our main result.

Proof of Theorem 3.2. Let’s consider an element $M \in \mathcal{M}^{N,K}$ such that $\omega(M) \leq 1$ and some families of vectors $\{u_x^a\}_{x,a}$ and $\{v_y^b\}_{y,b}$ verifying conditions $3.3$. We must show that

$$\left| \sum_{x,y,a,b} M_{x,y}^{a,b} u_x^a, v_y^b \right| \leq D \frac{\sqrt{n}}{\ln n}$$

for certain universal constant $D$.

In order to fit Lemma 4.1 in our context we must “twist” our Bell inequality $M$ in the spirit of [19, Section 5]. For every fixed $y = 1, \cdots, N$, we consider the linear maps

$$u_y : \ell^n \to \ell^{K-1}_\infty$$

defined by $u_y(|i\rangle) = \sum_{b=1}^{K-1} \sum_{x,a=1}^{N,K} (M_{x,y}^{a,b} - M_{x,y}^{a,K}) u_x^a(i) |b\rangle$ for every $1 \leq i \leq n$ and

$$u_{N+1} : \ell^n \to \ell^{K-1}_\infty$$

defined by $u_{N+1}(|i\rangle) = \sum_{y=1}^{N} \sum_{x,a=1}^{N,K} M_{x,y}^{a,K} u_x^a(i) |1\rangle$ for every $1 \leq i \leq n$.

On the other hand, we will also consider the linear maps

$$v_y : \ell^{K-1}_\infty \to \ell^n_2$$

defined by $v_y(|b\rangle) = v_y^b$ for every $1 \leq b \leq K - 1$,

and

$$v_{N+1} : \ell^{K-1}_\infty \to \ell^n_2$$

defined by $v_{N+1}(|b\rangle) = \begin{cases} 0 & \text{if } |b\rangle \neq |1\rangle \\ \sum_{b=1}^{K} v_y^b & \text{if } |b\rangle = |1\rangle \end{cases}$.

Then, trivial computations show that

$$\sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} u_x^a, v_y^b = \sum_{y=1}^{N+1} tr(v_y \circ u_y).$$

Now, according to Equation (4.1) and conditions (3.3) we have that $\|v_y\| \leq 1$ for every $y = 1, \cdots, N + 1$. Therefore, according to Lemma 4.1 we have

$$\left| \sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} u_x^a, v_y^b \right| \leq \sum_{y=1}^{N+1} \left| tr(v_y \circ u_y) \right| \leq C \sqrt{\frac{n}{\ln n}} \sum_{y=1}^{N+1} \ell(u_y).$$

Our statement will follow then from the estimate

$$\sum_{y=1}^{N+1} \ell(u_y) \leq 3K_2 \sqrt{n}.$$

First, according to Equation (4.3) we have

\[ \sum_{y=1}^{N} \ell(u_y) \leq K_{1,2} \sum_{y=1}^{N} \mathbb{E} \left| \sum_{i=1}^{n} g_i u_y(i) \right|_{\infty} = K_{1,2} \sum_{y=1}^{N} \mathbb{E} \sup_{x,a=1}^{N,K} \left| \sum_{i=1}^{n} \left( M_{x,y}^{a,b} - M_{x,y}^{a,K} \right) g_i u_x(i) \right|. \]

Now, let’s denote, for every \( y = 1, \ldots, N \), \( b_y \in \{1, \ldots, K - 1\} \) the elements where the previous sup is attained and \( \alpha_y = \text{sign} \left( \sum_{x,a=1}^{N,K} \left( M_{x,y}^{a,b} - M_{x,y}^{a,K} \right) g_i u_x(i) \right) \). Then, defining the element \((R(b|y))_{y=1,\ldots,N}^{N,K}\) by \( R(b|y) = \alpha_y \) if \( b = b_y \), \( R(K|y) = -\alpha_y \) and \( R(b|y) = 0 \) otherwise, it is very easy to check that

\[ \sum_{y=1}^{N} \mathbb{E} \sup_{b=1,\ldots,K-1} \left| \sum_{x,a=1}^{N,K} \left( M_{x,y}^{a,b} - M_{x,y}^{a,K} \right) g_i u_x(i) \right| = \mathbb{E} \sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} \left( \sum_{i=1}^{n} g_i u_x(i) \right) R(b|y). \]

Denoting \( Q(a|x) = \frac{1}{||g||_2^2} \sum_{i=1}^{n} g_i u_x(i) \) for every \( x,a \), conditions (3.3) and Lemma 4.2 tell us that \( Q = \lambda P_1 + \beta P_2 \) and \( R = \gamma P_3 + \delta P_4 \), where \( P_i \in S(N,K) \) for \( i = 1, \ldots, 4 \), \( |\lambda| + |\beta| \leq 1 \) and \( |\gamma| + |\delta| \leq 2 \). The fact that \( \omega(M) \leq 1 \) guarantees that

\[ (4.9) \]

\[ K_{1,2} \mathbb{E} \sum_{x,y,a,b=1}^{N,K} M_{x,y}^{a,b} \left( \sum_{i=1}^{n} g_i u_x(i) \right) R(b|y) = K_{1,2} \mathbb{E} \left| (g_i) \right|_2 \sum_{x,y,a=1}^{N,K} M_{x,y}^{a,b} Q(a|x) R(b|y) \leq 2K_{1,2} \sqrt{n}. \]

On the other hand,

\[ \ell(u_{N+1}) \leq K_{1,2} \mathbb{E} \left| \sum_{x,y,a=1}^{N,K} M_{x,y}^{a,K} \sum_{i=1}^{n} g_i u_x(i) \right| = K_{1,2} \mathbb{E} \left| (g_i) \right|_2 \sum_{x,y,a=1}^{N,K} M_{x,y}^{a,b} Q(a|x) S(b|y), \]

where \( Q \) is defined as above and for every \( y = 1, \ldots, N \) we define \( S(b|y) = 1 \) if \( b = K \) and \( S(b|y) = 0 \) otherwise. Again, \( \omega(M) \leq 1 \) implies that

\[ (4.10) \]

\[ \ell(u_{N+1}) \leq K_{1,2} \mathbb{E} \left| (g_i) \right|_2 \sum_{x,y,a=1}^{N,K} M_{x,y}^{a,b} Q(a|x) S(b|y) \leq K_{1,2} \sqrt{n}. \]

Then, Equation (4.8) follows from Equations (4.9) and (4.10). \( \square \)

4.2. Some comments about the optimality. As we have mentioned before, we do not know if the estimate presented in Theorem 3.2 is optimal or one could actually obtain an order \( O\left(\frac{1}{\sqrt{n}}\right) \). Furthermore, we must emphasize that Theorem 3.2 is much stronger than Theorem 2.5, so a more focused approach on the properties of the maximally entangled state could give a better estimate in Theorem 2.5 without providing an improvement of our main result.

We will finish this work by showing that any possible improvement of Theorem 3.2 must follow a different approach from the one considered here. A careful study of the proof of Theorem 3.2 shows that we are actually reducing the problem to study a picture in which

\footnote{Actually, we should define \( Q_\omega(a|x) \), where \( g_\omega = g_\omega(\omega) \), for every \( \omega \). However, the upper bounds below hold for every \( \omega \), so we avoid that notation for simplicity.}
we have two maps $S : \ell^2_0 \to \ell_1(\ell_\infty)$ and $T : \ell_1(\ell_\infty) \to \ell^2_0$ such that $T \circ S = id_{\ell^2_0}$ and we must study how small $\|T\| \|S\|$ can be. As we show below, Lemma 4.1 is not only optimal in the sense of Remark 4.1, but it can also be used to obtain the best complementation constant of $\ell^2_0$ in $\ell_1(\ell_\infty)$. This gives us the optimal estimate in [19, Theorem 19] and tells us that we cannot get a better upper bound in Theorem 3.2 by reducing the problem in the way we have just explained above. We refer [31] for an introduction to operator space theory.

**Theorem 4.3.** Given linear maps $S : \ell^2_0 \to \ell_1(\ell_\infty)$ and $T : \ell_1(\ell_\infty) \to \ell^2_0$ such that $T \circ S = id_{\ell^2_0}$. We have that $\|T\| \|S\| \geq K\sqrt{\ln n}$, where $K$ is a universal constant.

Furthermore, this estimate is optimal even in the following non-commutative case: There exist linear maps $j : R_n \cap C_n \to \ell_1(\ell_\infty)$ and $P : \ell_1(\ell_\infty) \to R_n \cap C_n$ such that $P \circ j = id_{\ell^2_0}$ and $\|j\|_{cb} \|P\|_{cb} \leq K\sqrt{\ln n}$, where $K$ is a universal constant. Here $R_n \cap C_n$ denotes the complex space $\ell^2_0$ endowed with the $R \cap C$ operator space structure, $\ell_1(\ell_\infty)$ is considered with its natural operator space structure and $\| \cdot \|_{cb}$ denotes the completely bounded norm.

**Proof.** To prove the first part of the statement let’s consider maps $S : \ell^2_0 \to \ell_1(\ell_\infty)$ and $T : \ell_1(\ell_\infty) \to \ell^2_0$ such that $T \circ S = id_{\ell^2_0}$. Then, we can realize $S = (S_x)_{x=1}^{\infty}$ and $T = (T_x)_{x=1}^{\infty}$ such that the linear maps $S_x : \ell^2_0 \to \ell_\infty$ and $T_x : \ell_\infty \to \ell^2_0$ are defined by $S(z) = (S_x(z))_{x=1}^{\infty}$ for every $z \in \ell^2_0$ and $T((y_x)_{x=1}^{\infty}) = \sum_x T_x(y_x)$ for every $(y_x)_{x=1}^{\infty} \in \ell_1(\ell_\infty)$. Note that we trivially have $\|T_x\| \leq \|T\|$ for every $x$. On the other hand, it is very easy to check that

$$tr(T \circ S) = \sum_{x=1}^{\infty} tr(T_x \circ S_x).$$

Furthermore, according to Equation (4.3) we have

$$\sum_x \ell(S_x) \leq K_{1,2} \sum_x E \left\| \sum_{i=1}^{n} g_i S_x(i) \right\|_{\ell_\infty} = K_{1,2} E \sum_x \left\| S_x \left( \sum_{i=1}^{n} g_i i \right) \right\|_{\ell_\infty} = K_{1,2} E \left\| S \left( \sum_{i=1}^{n} g_i i \right) \right\|_{\ell_1(\ell_\infty)} \leq K_{1,2} \ell(S).$$

Therefore, we have

$$n = tr(id_{\ell^2_0}) = tr(T \circ S) = \sum_{x=1}^{\infty} tr(T_x \circ S_x) \leq C \frac{\sqrt{n}}{\ln n} \sum_{x=1}^{\infty} \|T_x\| \|S_x\| \leq C' \frac{\sqrt{n}}{\ln n} \|T\| \|S\| \leq C' \frac{n}{\ln n} \|T\| \|S\|,$$

where for the first inequality we have used Lemma 4.1 and the last inequality follows from Equation (4.2). The first part of the statement follows now trivially.

The second part of the theorem was proven in [19, Theorem 9].

\[\square\]
ACKNOWLEDGMENTS

We would like to thank M. Junge, O. Regev and T. Vidick for many helpful discussions on previous versions.

Author’s research was supported by EU grant QUEVADIS, Spanish projects QUITEMAD, MTM2011-26912 and MINECO: ICMAT Severo Ochoa project SEV-2011-0087 and the “Juan de la Cierva” program.

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