ON MODEL THEORY, ZILBER AND PHYSICS

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Abstract

We study B. Zilber’s method of calculating the Feynman propagator for the free particle and harmonic oscillator.

In Jouko Väänänen’s 60th birthday meeting B. Zilber gave a talk on the use of model theory in quantum physics. We got interested and tried to read [Zi] in which Zilber calculates the Feynman propagator for the free particle and the harmonic oscillator. But we failed to follow the calculations. However, the paper suggests various ways of constructing models that resemble the Hilbert space operator model used in quantum physics for a single particle (this kind of hyperfinite approach is not completely new, see e.g. [Ya]). In addition, we noticed that by using number theory, at least in the case of the free particle, the Feynman propagator is easy to calculate in these models. We did several straightforward such constructions and each time the model calculated the propagator incorrectly. So we asked, is there a way of constructing a model that calculates the propagator correctly and how would the physics look like in such a model? In this paper we will give one such construction, calculate the Feynman propagator for the free particle and discuss a bit about the properties of the model.

In quantum mechanics physical observables are described by self adjoint operators on an infinite-dimensional complex Hilbert space. The unit sphere of the space corresponds to the possible states of the system and the possible outcomes of measurements are the eigenvalues of the operators. If a self adjoint operator has non-degenerate eigenvalues, the corresponding eigenvectors will be orthogonal. Physicists work under the assumption that there not only are eigenvalues, but that the eigenvectors of any of the operators considered span the whole space. This is called ’inserting a complete set of states’. So they work in a Hilbert space spanned by the eigenvectors corresponding to the possible outcomes of measurements. Linear combinations of these are considered ’superpositions’, states in which the observable is not yet determined until it is measured (upon which the state ’collapses’ onto one of the eigenstates). The coefficients in the linear combinations determine the probability that the state will collapse onto the corresponding eigenstate when the observable is measured.

We start by looking at the Hilbert space operator model from quantum physics for a single particle in one dimensional space (from the point of view of e.g. the
Feynman propagator the dimension assumption is w.l.o.g.). This is governed by two theorems. The first one is Stone’s theorem which states that if for all \( t \in \mathbb{R} \), \( U^t \) is a unitary operator in a (complex) Hilbert space \( H \), \( t \mapsto U^t \) is continuous (i.e. \( t \mapsto U^t(x) \) is continuous for all \( x \in H \)) and \( U^{t+t'} = U^t U^{t'} \), then there is a unique selfadjoint operator \( Q \) such that \( U^t = e^{itQ} \) for all \( t \in \mathbb{R} \) (typically the selfadjoint operator is not bounded and so by \( e^{itQ} \) we mean the continuation to the whole space). We will call the maps \( t \mapsto U^t \) as above continuous unitary representations of \((\mathbb{R},+)\). The other theorem is the Stone-von Neumann theorem. It states that the class of Hilbert spaces \( H \) with continuous unitary representations \( t \mapsto U^t \) and \( t \mapsto V^t \) of \((\mathbb{R},+)\) with the property \( V^w U^t = e^{itw} U^t V^w \) is essentially categorical (in the sense of model theory i.e. 'unitarily’) in every cardinality (here \( \hbar = h/2\pi \) is the reduced Plank constant). However, we want to point out that if \( t \mapsto U^t \) and \( t \mapsto V^t \) are as above and \( a \) is e.g. a positive real then so are \( t \mapsto U^t_\ast \) and \( t \mapsto V^t_\ast \) where \( U^t_\ast = U^{at} \) and \( V^t_\ast = V^{a^{-1}t} \) and that this kind of scaling may have an effect in calculations. We want to point out also that although, following Zilber, we use \( t \) as the parameter in these representations, we do not think of it as time, it is just a parameter. But we also use \( t \) to denote the time, it should be clear from the context which we mean.

Now following roughly Zilber we start to build a model. Since there is room for scaling, we pick two positive reals \( a \) and \( b \) for the scaling and in the end we will study the question of the effects of this scaling. Also in our model the space (in which the particle lives - not the Hilbert space) will be of finite length. There does not seem to be a way of avoiding this in this construction (if one wants a model that calculates the Feynman propagator correctly). In the end we will study the size of the space.

For all natural numbers \( N > 1 \), let \( H_N \) be a vector space over the complex numbers \( \mathbb{C} \) with basis \( \{ u(x) \mid x \in \omega, x < N \} \). In the end, we will take an ultraproduct (and a metric ultraproduct) of these and so when we claim that something is true of \( H_N \), we mean that it is true assuming \( N \) is a sufficiently divisible natural number. Similarly, when we define something, it is enough if it makes sense in those \( H_N \) in which \( N \) is a sufficiently divisible natural number.

We make Hilbert spaces out of the vector spaces \( H_N \) by defining an inner product \( \langle \cdot , \cdot \rangle \) so that \( \langle u(x)|u(x') \rangle = 0 \) if \( x \neq x' \) and otherwise \( \langle u(x)|u(x') \rangle = 1 \) (and for complex numbers \( c \) and \( d \), \( \langle cu(x)|du(x) \rangle = \overline{cd} \)).

By \( q = q_N \) we denote the complex number \( e^{i2\pi/N} \) and denote

\[
v(x) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{xy} u(y).
\]

Notice that for \( x \neq x' \),

\[
\langle v(x)|v(x') \rangle = (1/N) \sum_{y=0}^{N-1} e^{i2\pi z y/N}
\]

for some integer \( z \neq 0 \) with \( |z| < N \) and thus \( \langle v(x)|v(x') \rangle = 0 \). Similarly \( \langle v(x)|v(x) \rangle = 1 \) and so \( \{v(x) \mid x \in \omega, x < N \} \) is another orthonormal basis of
Furthermore, similarly, one can see that

$$u(x) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} v(y)$$

since

$$u(x) \mapsto (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} v(y)$$

and

$$v(x) \mapsto (1/N)^{1/2} \sum_{y=0}^{N-1} q^{xy} u(y)$$

are inverses of each other.

For all real numbers $t$, let $t^u = at/2\pi$ and $t^v = bt/2\pi$. Also for all real numbers $t$, we define operators $U^t$ and $V^t$ as follows: For all $0 \leq x < N$, $U^t(u(x)) = q^{xt^u} u(x)$ and $V^t(v(x)) = q^{xt^v} v(x)$. Notice that if $t^v$ is a natural number, then

$$(1) \quad V^t(u(x)) = u(x - t^v)$$

where the sum is taken 'modulo $N$' i.e. $-y = N - y$ if $N \geq y > 0$ (since

$$u(x - t^v) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-(x-t^v)y} v(y)$$

and

$$V^t(u(x)) = V^t((1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} v(y)) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} q^{t^vy} v(y))$$

Let us recall what we have: $U^t$ and $V^w$ are unitary onto operators, $u(x)$ is an eigenvector of $U^t$ with eigenvalue $e^{ixat/N}$ and $v(x)$ is an eigenvector of $V^w$ with eigenvalue $e^{ixbw/N}$ and assuming that $w^w$ is a natural number, $V^w U^t(u(x)) = q^{xt^u} u(x - w^w)$ and $U^t V^w(u(x)) = q^{(x-w^w)t^w} u(x - w^w)$ i.e. $V^w U^t = q^{w^w t^w} U^t V^w = e^{iabtw/2\pi N U^t V^w}$.

Now it is time to look at the commutator law. We want that

$$V^w U^t = e^{ihtw} U^t V^w$$

(Zilber uses here $V^w U^t = e^{2\pi ihtw} U^t V^w$. ) So should we require that $ab = Nh$? We could do this (making $a$ or $b$ or both depend on $N$) and to a point things would go much as how we actually do. However we follow, in a sense, Zilber i.e. we require that $ab = h$, $a$ and $b$ do not depend on $N$ and consider $h_N = h/N$ in the model $H_N$. How is this possible? We scale units, i.e. units in $H_N$ are not the same as
in the ‘real world’. So what unit(s) do we scale? We scale time or mass or both, it will turn out that this choice makes no difference. This is because everywhere in our calculations these units cancel out i.e. after introducing \( V_t \) we can forget this scaling (e.g. we let 1 unit of time be \( N \) units of time in \( H_N \), keep in mind that the unit of \( h \) is that of \( mass \times length^2/time \)). This is not the case if we scale length. In fact this would have small unwanted consequences beyond the trouble of keeping this scaling in mind all the time.

As we consider \( V_t = e^{itP} \), where \( P \) is the momentum operator, and the eigenvalue of \( V_t \) at \( v(x) \) is \( e^{ibtx/N} \), we consider \( v(x) \) an eigenvector of \( P \) with eigenvalue \( bx/N \) which is interpreted as momentum. Similarly, we consider \( U_t = e^{itQ} \), where \( Q \) is the position operator. Now keeping in mind the scaling, in the units of the real world, the \( P \)-eigenvalue of \( v(x) \) is \( bx \) (the unit of momentum is that of \( mass \times length/time \)). Thus we also look at the operators that directly give the eigenvalue in the right units, i.e. the operators \( V'_* = (V_t)^N \) for which \( V'_*(v(x)) = e^{ibtx}v(x) \). And then if \( Nw^n \) is a natural number, then \( V'_wU_t = e^{iabtw/2\pi}UTV'_w = e^{ibtw}UTV'_w \).

Now we take an ultraproduct of these structures. There are two kinds of ultraproducts we may consider. First the ordinary one with equivalence classes defined by the ultrafilter. The second is the metric ultraproduct, where one first takes the product, then defines norms as ultralimits of the coordinate wise norms, then consider only the elements with finite norms and finally mods out the infinitesimals.

We would like to take a metric ultraproduct but unfortunately it does not work. First of all, the operators \( V'_* \) do not have uniform moduli of uniform continuity. In fact it is easy to see that infinitesimal changes in the argument may result in a non-infinitesimal change in the value. So the operators are far too turbulent. This same problem arises when we introduce the time evolution operators. And if we just forget the operators \( V'_* \) and take the metric ultraproduct we lose the commutator law information, the operators \( U_t \) and \( V_t \) commute in the metric ultraproduct (but not in the usual ultraproduct where the unit of time is infinitesimal). So what we do is that we take the usual ultraproduct and then inside that we find the metric ultraproduct and just live with the fact that some of our operators are well-defined only in the usual ultraproduct.

Let \( X = \{x \in \omega| x > 1\} \) and \( D \) be an ultrafilter on \( X \) such that for all \( n > 0, X_n \in D \), where the set \( X_n \) consists of those \( N \in X \) which are divisible by \( n \). We think of our structures \( H_N \) as two-sorted structures: in one sort we have the elements of the Hilbert space and in the other one the complex numbers \( C \). The vocabulary consists of the addition in the Hilbert space, the inner product from \((H_N)^2 \) to \( C \), the scalar multiplication \( C \times H_N \rightarrow H_N \), the field structure on \( C \), the norm \( |.| \) in \( C \) (or the complex conjugation), the complex exponentiation and constants for \( i, h \) and \( \pi \) and, of course, the operators \( U_t, V_t \) and \( V'_* \), \( t \in \mathbb{R} \). When we introduce the time evolution operators, we add them to this list. Then we let \((H_*, C_*) \) be the ultraproduct \( \Pi_{N \in X}(H_N, C)/D \). Notice that \( C_* \) with the field structure is isomorphic to the field of complex numbers but \( \mathbb{R}_* \) i.e. the image of the norm closed under subtraction is not isomorphic to the reals. It is a real closed
subfield of \( \mathbb{C}_* \) but contains e.g. infinitesimals. However \( \mathbb{C}_* = \mathbb{R}_*[i] \). \( H_* \) looks a lot like a Hilbert space: it is a vector space over \( \mathbb{C}_* \) with something that looks like an inner product, i.e. the inner product satisfies everything one requires from an inner product if as complex conjugation one uses the one that arises from the fact that \( \mathbb{C}_* = \mathbb{R}_*[i] \). From this we get a 'norm' on \( H_* \) the usual way, which now can get infinitesimal values. When we talk about the norm of an element of \( H_* \), it is this that we mean. Similarly, the norm of elements of \( \mathbb{C}_* \) is the one that is in our vocabulary which is the same as the one one gets from our complex conjugation. Notice that now the completeness of \( H_N \) loses its meaning.

Next we find the metric ultraproduct. We let \( \mathbb{C}'_* \) be the set of those elements \( q \in \mathbb{C}_* \) whose norm is less than some natural number \( n = 1 + 1 + ... + 1 \). \( \mathbb{H}'_* \) is defined similarly. Notice that \( \mathbb{H}'_* \) and \( \mathbb{C}'_* \) are closed under all functions of our vocabulary. Then we define an equivalence relation \( E \) so that for \( q, q' \in \mathbb{C}'_* \), \( qE q' \) if \( |q - q'| < 1/n \) for all natural numbers \( n \) and on \( \mathbb{H}'_* \), \( E \) is defined similarly. We denote \( H = \mathbb{H}'_/E \) and \( \mathbb{C} = \mathbb{C}'_/E \) (we will justify this notation soon). As mentioned above, excluding \( V^t_* \), we can define all functions from our vocabulary by using representatives from the equivalence classes (e.g. \( U^t(x/E) = U^t(x)/E \)). Let \( e \) be the canonical embedding of complex numbers to \( \mathbb{C}_* \) and then it is easy to see that \( \mathbb{C} \) is the set of all \( e(q)/E \), where \( q \) is a complex number (bounded closed subsets of complex numbers are compact) i.e. \( q \mapsto e(q)/E \) is an isomorphism from the real complex numbers to (our) \( \mathbb{C} \) (in the vocabulary that consist of all the structure we have put on \( \mathbb{C} \) alone). Thus we think of our \( \mathbb{C} \) as the complex numbers. In the context of \( \mathbb{C}_* \), by a complex number \( q \) we mean \( e(q) \). And now \( H \) is a complex Hilbert space. From now on we work in \( H \) keeping in mind that \( V^t_* \) and later \( K^t \) are well-defined only in \( H_* \) i.e. we move to \( H_* \) if necessary. It will be clear from the context in which structure we work.

Now the functions \( t \mapsto U^t \) and \( t \mapsto V^t \) are continuous unitary representations of \( (\mathbb{R}, +) \) and for all \( t \) and such \( w \) that \( w^w \) is a rational number, \( V^w_*U^t = e^{ihtw}U^tV^w \) (assuming \( ab = h \) and notice that the latter claim talks about \( H_* \)). Also assuming \( a^{-1}x \) is a rational number < 1, (e.g.) \( u(x) = (u(a^{-1}N x)| N \in X)/D/E \) is an eigenvector of each \( U^t \) with eigenvalue \( e^{itx} \). Similarly, (e.g.) \( v(x) = (v(x)| N \in X)/D \) is an eigenvector of each \( V^t_* \) with eigenvalue \( e^{itbx} \).

So let us consider the time evolution operator. For a time independent Hamiltonian \( H \) it is

\[
K^t = e^{-iHt/h}.
\]

We wish to calculate the Feynman propagator \( < x_1|K^t|x_0 > \) for the free particle in our model (where \( < x_1|K^t|x_0 > \) is the inner product between \( |x_1 > \) i.e. \( Q \)-eigenvector with eigenvalue \( x_1 \) and \( K^t|x_0 > = K^t(|x_0 >) \)).

For this we need to define the time evolution operators \( K^t \) in the spaces \( H_N \): We let

\[
K^t(v(x)) = e^{-it(bx)^2/2hm}v(x),
\]

for all \( 0 \leq x < N \) (keep in mind that \( bx \) is the \( P \)-eigenvalue of \( v(x) \) and as Hamiltonian we use \( P^2/2m \), where \( m \) is the mass of the particle, see [Ze]; in [Zi]...
there is probably a typo here). The right answer according to physicists is

\[(m/2\pi iht)^{1/2}e^{im(x_0-x_1)^2/2ht}\]

but in our calculations the answer depends on our scaling and there is only one choice that gives the right answer, namely that \(a = \frac{ht}{m}\) and \(b = \frac{m}{t}\) (keep in mind that \(ab\) should be \(h\)) and we do the calculations only for these, in the general case the formulas become ugly. Notice that then \(Q\) and \(P\) depend on time (cf. Heisenberg picture), so to get the right answer for the time-independent Hamiltonian we actually make it depend on time. We discuss this after the calculations.

So now

\[K^t(v(x)) = e^{-i\pi a^{-1}x^2}v(x).\]

Warning: So \(a\) and \(b\) are real numbers i.e. they come without units and we do the calculation without trying to track the units. Even in the calculations physicists do, the tracking of the units appears difficult.

In order to be able to use number theory in our calculations, we assume that in the real world the units are chosen so that \(a\) is an even natural number. We also assume that \(x_0\) and \(x_1\) are non-negative rational numbers \(< a\) such that \(x_0 - x_1\) is an integer (in a sense this assumption is w.l.o.g. since we can make the unit of length as small as we want, see our considerations on the size of our space below).

It is enough to show that

\[<x_1|K^t|x_0> = (m/2\pi iht)^{1/2}e^{im(x_0-x_1)^2/2ht}\]

holds in every \(H_N\) for \(N\) divisible enough.

Notice that \(|x_0>\) is \(u(a^{-1}N x_0)\) and so

\[|x_0> = (1/N)^{1/2} \sum_{n=0}^{N-1} q^{-a^{-1}N x_0 n} v(n)\]

and thus

\[K^t|x_0> = (1/N)^{1/2} \sum_{n=0}^{N-1} q^{-a^{-1}N x_0 n} e^{-i\pi a^{-1}n^2} v(n)\]

i.e.

\[K^t|x_0> = 1/N \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} q^{-a^{-1}N x_0 n} e^{-i\pi a^{-1}n^2} q^n k u(k).\]

And so

\[<x_1|K^t|x_0> = 1/N \sum_{n=0}^{N-1} q^{-a^{-1}N x_0 n} e^{-i\pi a^{-1}n^2} q^{a^{-1}N x_1 n} = \]

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$$1/N \sum_{n=0}^{N-1} e^{i\pi(2a^{-1}(x_1-x_0)n-a^{-1}n^2)} =$$

$$1/N \sum_{n=0}^{N-1} e^{i\pi(2(x_1-x_0)n-n^2)/a}.$$ 

Since $a$ is even and $x_0 - x_1$ is an integer,

$$e^{i\pi(a+n)/a} = e^{i\pi n^2/a},$$

and also $N = Na^{-1}a$ and so

$$<x_1|K^t|x_0> = (1/N)Na^{-1} \sum_{n=0}^{a-1} e^{i\pi(2(x_1-x_0)n-n^2)/a} = a^{-1} \sum_{n=0}^{a-1} e^{i\pi(2(x_1-x_0)n-n^2)/a}.$$ 

Now we apply number theory: For integers $c, d, g$, if $cg \neq 0$ and $cg - d$ is even, then

$$\sum_{n=0}^{|g|-1} e^{\pi i(cn^2+dn)/g} = |g/c|^{1/2} e^{\pi i(|cg|-d^2)/4cg} \sum_{n=0}^{|c|-1} e^{-\pi i(gn^2+dn)/c},$$

see [BE]. In our case, $g = a$, $d = 2(x_1-x_0)$ and $c = -1$. In particular, since $c = -1$,

$$\sum_{n=0}^{|c|-1} e^{-\pi i(gn^2+dn)/c} = 1.$$

So

$$<x_1|K^t|x_0> = a^{-1} a^{1/2} e^{i\pi(-(1/4)+(x_0-x_1)^2/a))} =$$

$$e^{-i\pi/4} (m/\hbar t)^{1/2} e^{im(x_0-x_1)^2/2\hbar t} =$$

$$(m/2\pi i\hbar t)^{1/2} e^{im(x_0-x_1)^2/2\hbar t}.$$ 

In the last equation we used the fact that $(e^{-i\pi/4})^2 = -i = 1/i$.

Since our calculations depend on making $P$ time dependent, we study what changes occur when taking time dependence into account. Now

$$H(v(x)) = (P^2/2m)v(x) = ((bx)^2/2m)v(x) = (mx^2/2t^2)v(x)$$

So the Hamiltonian can be written as $H = t^{-2}H'$ where $H'$ is time independent. Now the Schrödinger equation demands

$$i\hbar \frac{\partial}{\partial t} K^t = H K^t = t^{-2} H' K^t.$$
Solving this yields

\[ K^t = e^{iHt/\hbar} = e^{iH't/\hbar}. \]

So we define \( K^t \) in \( H_N \) by:

\[ K^t(v(x)) = e^{it(bx)^2/2\hbar m}v(x), \]

for all \( 0 \leq x < N \). Using our scaling \( a = ht/m \) and \( b = m/t \) this yields

\[ K^t(v(x)) = e^{i\pi a^{-1} x^2} v(x). \]

With calculations like before we get

\[ \langle x_1 | K^t | x_0 \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{i\pi (2(x_1-x_0)n+n^2)/a} = a^{-1} \sum_{n=0}^{a-1} e^{i\pi (2(x_1-x_0)n+n^2)/a}. \]

Applying the same quadratic Gauss sum formula, with \( g = a \), \( d = 2(x_0 - x_1) \) and \( c = 1 \), we get

\[ \langle x_1 | K^t | x_0 \rangle = a^{-1} a^{1/2} e^{\pi i (1/4 - (x_0-x_1)^2)/2ht}. \]

which is the complex conjugate of the desired result.

We finish by looking at some properties of our model. Clearly, the size of the space (in which the particle lives) depends on time and if \( t = 0 \), the space consists of one point only, see below. In addition to time, the size depends on the mass of the particle. So suppose that the particle is an electron. In our space, we have all the places from the \( Q \)-eigenvalue of

\[ (u(0)| N \in X) / D/E \]

to \( Q \)-eigenvalue of

\[ (u(N-1)| N \in X) / D/E. \]

The first is zero and the latter is \( a \). Now \( a = ht/m \), so at time \( t = 1s \) to calculate \( a \) we compute

\[ \frac{6.626 \cdot 10^{-34} m^2 kg/s \cdot 1s}{9.109 \cdot 10^{-31} kg} \approx 7.27 \cdot 10^{-4} m^2. \]

As the units of time and weight cancel out, scaling these does not affect the result. However, scaling length matters: If we measure length in \( cm \) then \( a \approx 7.27 \). Thus the length of the space at time \( 1s \) is around 7.27cm At the time \( 1h \), \( a \) is around 26200 and thus the length of the space is around 262m. Notice that if the mass of the particle increases, the speed of this expansion decreases.

However, if we scale length and choose the unit for length to be \( 1mm \), then it affects the value of \( a \). At time \( 1s \), \( a \) is around 727 and so the length of our space at time \( 1s \) is around 73cm and at time \( 1h \) it is around 2.6km.

Also there is infinite number of possible finite values for momentum (these are all 'real' real numbers) but the values are not dense in the set of non-negative reals. And the difference between the two consecutive values changes with time. I.e. the laws of physics change with time (in our model that is).
References

[BE] B. Berndt and R. Evans, The determination of Gauss sums, Bull. Amer. Math. Soc. (N.S.) 5 (1981), no. 2, 107-129.

[Ya] H. Yamashita, Nonstandard Methods in Quantum Field Theory I: A Hyperfinite Formalism of Scalar Fields, Internat. J. Theoret. Phys. 41 (2002), no. 3, 511–527.

[Ze] E. Zeidler, Quantum Field Theory I, 2nd ed., Springer, Heidelberg, 2009.

[Zi] B. Zilber, On model theory, non-commutative geometry and physics, manuscript.