Correlation function of the velocity field of a thin suspended liquid film

Doriano Brogioli

January 18, 2013

Abstract

In this paper, I consider a thin suspended liquid film, surrounded by a different fluid. Examples of such a system are soap films and liquid crystal films, surrounded by air. They are considered good models for two dimensional fluid dynamics, although air drag is known to introduce strong deviations. I exactly solve the hydrodynamic equations of the system composed by the liquid layer and the surrounding fluid in order to evaluate the correlation function of the thermally excited velocity fluctuations. Both the temporal and the spatial statistical properties are the ones of a two dimensional, hypothetical fluid, only for temporal and spatial frequencies higher than critical values; at lower frequencies they show significative deviations due to the interactions of the liquid layer with the surrounding fluid.

1 Introduction

A thin, freely suspended liquid layer is generally considered a system that follows two dimensional hydrodynamics, on length scales longer than the layer thickness. Many experiments have been performed on soap films, that consist of two monolayers of amphiphilic molecules around a layer of water. The layer thickness ranges from the 4nm of the Newton black film to many micrometers; features down to a fraction of a millimeter should be considered two dimensional [6].

Also smectic A liquid crystals can form films, by a different effect. It spontaneously form a layered structure, with the elongated molecules aligned to the surface normal. When a film is drawn from this kind of liquid crystal, it is formed by a set of monolayers, ranging from two to many hundreds [1]; the thickness can be selected with extreme accuracy. Because of the absence of translational order in the monolayers, the film behaves as a two dimensional system.

Many papers [6, 7, 11, 4, 5] describe experiments in which turbulent flows have been detected, and discuss the agreement with two dimensional hydrodynamics. One result is that the role of the air surrounding the film cannot be
neglected. In turbulent flows, it damps the turbulent vortices. From a theoretical point of view, an exact evaluation of this effect is extremely difficult; for example, Rivera et al. [9] modeled this damping as a linear drag term in the two dimensional Navier-Stokes equations, in order to describe velocity fluctuations observed in a turbulent flow. They evaluated the amplitude of the drag term from experimental data; from this value, they found the dissipation due to air friction to be a significant energy dissipation mechanism in their experimental system.

Another well known effect of air friction on thin liquid films can be found in diffusion of particles, whose diameter is about the thickness of the layer. Neglecting air friction, the mobility of a particle in a two dimensional system should be infinite, that is, a particle subjected to a steady force should accelerate indefinitely. This would result in an infinite diffusion coefficient. Saffman [11] found that air friction reduces the mobility to a finite value: he evaluated the diffusion coefficient and found that it strongly depends on the viscosity of the surrounding fluid, and diverges as the viscosity vanishes. Saffman formula has been tested both in soap films [3] and in liquid crystal films [1].

The diffusion coefficient can be evaluated once we know the correlation function of the thermally excited velocity fluctuations; this means that the velocity correlation function must be strongly affected by air friction. In this paper, I evaluate the correlation function of the velocity field of a thin suspended liquid film, by solving the hydrodynamic equation of the whole system, including the liquid film and the fluid surrounding it.

Then I discuss the result, showing that both the temporal and the spatial statistical properties are the ones of a two dimensional, hypothetical fluid, only for temporal and spatial frequencies higher than critical values; at lower frequencies they show significant deviations due to the interactions of the liquid layer with the surrounding fluid.

Last, I derive approximately the value of diffusion coefficient, and I compare it with Saffman formula.

## 2 Derivation of the power spectrum of the velocity fluctuations

I describe the film and the fluid surrounding it by using Landau’s fluctuating hydrodynamics equations:

\[
\begin{align*}
\rho_0 \frac{\partial \vec{u}_0 (\vec{x}, t)}{\partial t} &= \eta_0 \nabla^2 \vec{u}_0 (\vec{x}, t) - \nabla p_0 (\vec{x}, t) + \frac{\partial}{\partial z} [\frac{\partial}{\partial z} \vec{u}_1 (\vec{x}, z = 0, t) - \frac{\partial}{\partial z} \vec{u}_2 (\vec{x}, z = 0, t)] \\
\rho_1 \frac{\partial \vec{u}_{1,2} (\vec{x}, z, t)}{\partial t} &= \eta_1 \nabla^2 \vec{u}_{1,2} (\vec{x}, z, t) - \nabla p_{1,2} (\vec{x}, z, t)
\end{align*}
\]

(1)

where \( \vec{u}_1 (\vec{x}, z, t) \) and \( \vec{u}_2 (\vec{x}, z, t) \) are three dimensional vector fields, describing the velocities in the space above and below the film; \( \vec{u}_0 (\vec{x}, t) \) is a two dimensional vector field, describing the velocity in the film; \( \eta_0 \) and \( \eta_1 \) are the shear viscosity of the film and the surrounding fluid; \( \rho_0 \) and \( \rho_1 \) are the densities of the film and
the surrounding fluid, \( h \) is the thickness of the film; \( p_{1,2}(\vec{x}, z, t) \) and \( p_0(\vec{x}, t) \) are the pressures.

I suppose that both the fluids are incompressible, the film is placed at \( z = 0 \), and the fluid does not slip at the surface of the sheet:

\[
\nabla \cdot \vec{u}_0(\vec{x}, t) = 0 \tag{2}
\]
\[
\nabla \cdot \vec{u}_{1,2}(\vec{x}, z, t) = 0 \tag{3}
\]
\[
\vec{u}_{1,2}(\vec{x}, z = 0, t) = [\vec{u}_0(\vec{x}, t), 0] \tag{4}
\]

The pressures \( p_{1,2}(\vec{x}, z, t) \) and \( p_0(\vec{x}, t) \) must be determined in order to fulfill the incompressibility equations Eqs (2, 3).

In Fourier space:

\[
\begin{aligned}
\left\{ \begin{array}{l}
-i \rho_0 \omega + \eta_0 q^2 \; \vec{u}_0(\vec{q}, \omega) &= i \vec{q} \vec{p}_0(\vec{q}, \omega) - i \frac{\eta_0}{h^2} \int q_z \left[ \vec{u}_1(\vec{q}, q_z, \omega) - \vec{u}_2(\vec{q}, q_z, \omega) \right] dq_z \\
-i \rho_1 \omega + \eta_1 \left( q^2 + q_z^2 \right) \; \vec{u}_{1,2}(\vec{q}, q_z, \omega) &= i \frac{\vec{q}}{\vec{q}^2} p_{1,2}(\vec{q}, q_z, \omega)
\end{array} \right. \tag{5}
\end{aligned}
\]

with the conditions:

\[
\vec{q} \cdot \vec{u}_0(\vec{q}, \omega) = 0 \tag{6}
\]
\[
[\vec{q}, q_z] \cdot \vec{u}_{1,2}(\vec{q}, q_z, \omega) = 0 \tag{7}
\]
\[
\int \vec{u}_{1,2}(\vec{q}, q_z, \omega) dq_z = [\vec{u}_0(\vec{q}, \omega), 0] \tag{8}
\]

where the Fourier transform is defined as follows:

\[
f(\vec{q}) = \frac{1}{(2\pi)^D} \int e^{i\vec{q} \cdot \vec{x}} f(\vec{x}) d\vec{x} \tag{9}
\]

Now I use the incompressibility equations Eqs (2, 3) in order to evaluate the pressures:

\[
\begin{aligned}
\left\{ \begin{array}{l}
-i \rho_0 \omega + \eta_0 q^2 \; \vec{u}_0(\vec{q}, \omega) &= -i \frac{\eta_0}{h} \left( \vec{1} - \frac{\vec{q} \cdot \vec{x}}{\vec{q}^2} \right) \int q_z \left[ \vec{u}_1(\vec{q}, q_z, \omega) - \vec{u}_2(\vec{q}, q_z, \omega) \right] dq_z \\
-i \rho_1 \omega + \eta_1 \left( q^2 + q_z^2 \right) \; \vec{u}_{1,2}(\vec{q}, q_z, \omega) &= 0
\end{array} \right. \tag{10}
\end{aligned}
\]

with the conditions:

\[
\vec{q} \cdot \vec{u}_0(\vec{q}, \omega) = 0 \tag{11}
\]
\[
[\vec{q}, q_z] \cdot \vec{u}_{1,2}(\vec{q}, q_z, \omega) = 0 \tag{12}
\]
\[
\int \vec{u}_{1,2}(\vec{q}, q_z, \omega) dq_z = [\vec{u}_0(\vec{q}, \omega), 0] \tag{13}
\]

Then, I express the vectors on a suitable basis:

\[
\vec{u}_0(\vec{q}, \omega) = u_{0R}(\vec{q}, \omega) \frac{[q_x, q_y]}{\sqrt{q_x^2 + q_y^2}} + u_{0T}(\vec{q}, \omega) \frac{[q_y, -q_x]}{\sqrt{q_x^2 + q_y^2}} \tag{14}
\]
\[ \vec{u}_{1,2} (\vec{q}, z, \omega) = u_{1,2R} (\vec{q}, z, \omega) \left[ \frac{q_x q_y q_z}{\sqrt{q_x^2 + q_y^2 + q_z^2}} \right] + u_{1,2T} (\vec{q}, z, \omega) \left[ \frac{q_x q_y - q_x, 0}{\sqrt{q_x^2 + q_y^2}} \right] \]

The equations become:

\[
\begin{align*}
\{ & -i\rho_0 \omega + \eta_0 q^2 \} u_{0T} (\vec{q}, \omega) = -i \frac{\eta_0}{k} \int q_x \left[ u_{1T} (\vec{q}, z, \omega) - u_{2T} (\vec{q}, z, \omega) \right] dq_x \\
- i\rho_1 \omega + \eta_1 (q^2 + q_z^2) u_{1,2T} (\vec{q}, z, \omega) = 0 \\
- i\rho_1 \omega + \eta_1 (q^2 + q_z^2) u_{1,2R} (\vec{q}, z, \omega) = 0 \\
\int u_{1,2T} (\vec{q}, z, \omega) dq_x = u_{0T} (\vec{q}, \omega) \\
\int u_{1,2R} (\vec{q}, z, \omega) dq_x = 0 \\
\int u_{1,2R} (\vec{q}, z, \omega) dq_x = 0
\end{align*}
\]

Only the components \( u_{0T} \) and \( u_{1,2T} \) are needed in order to evaluate \( u_0 \); in the following, I will drop the subscript \( T \).

\[
\begin{align*}
\{ & -i\rho_0 \omega + \eta_0 q^2 \} u_0 (\vec{q}, \omega) = -i \frac{\eta_0}{k} \int q_x \left[ u_1 (\vec{q}, z, \omega) - u_2 (\vec{q}, z, \omega) \right] dq_x \\
- i\rho_1 \omega + \eta_1 (q^2 + q_z^2) u_{1,2} (\vec{q}, z, \omega) = 0 \\
\int u_{1,2} (\vec{q}, z, \omega) dq_x = u_0 (\vec{q}, \omega)
\end{align*}
\]

In real space:

\[
\begin{align*}
\frac{\partial}{\partial \vec{q}} u_0 (\vec{q}, t) &= -\nu_0 q^2 u_0 (\vec{q}, t) + \frac{\eta_0}{k_0} \left[ \frac{\partial}{\partial z} u_1 (\vec{q}, z = 0, t) - \frac{\partial}{\partial z} u_2 (\vec{q}, z = 0, t) \right] \\
\frac{\partial}{\partial \vec{q}} u_{1,2} (\vec{q}, z, t) &= -\nu_1 q^2 u_{1,2} (\vec{q}, z, t) + \frac{\eta_1}{k_0} \left[ \frac{\partial^2}{\partial z^2} u_{1,2} (\vec{q}, z, t) \right] \\
u_{1,2} (\vec{q}, z = 0, t) &= u_0 (\vec{q}, t)
\end{align*}
\]

where \( \nu = \eta/\rho \).

In order to simplify the following calculations, I impose periodic boundary conditions on \( z \), with a period \( L \); later, I will evaluate the limit for \( L \to \infty \). I express the hydrodynamic variables on a complete orthonormal basis:

\[
\begin{align*}
u_{1,2} (\vec{q}, z, t) &= \sum_{n=0}^{\infty} \{ a_n^A (t) f_n^A (z) + b_n^B (t) f_n^B (z) \} + c^\vec{q} (t) f^C (z) \\
u_0 (\vec{q}, t) &= \sum_{n=0}^{\infty} b_n^B (t) f_n^B (0) + c^\vec{q} (t) f^C (0)
\end{align*}
\]

The eigenfunctions are:

\[
\begin{align*}
f_n^A (z) &= N_n^A \sin \left[ A_n (z - L/2) \right] \\
f_n^B (z) &= N_n^B \cos \left[ B_n (z - L/2) \right] \\
f^C (z) &= N^C \cosh \left[ C (z - L/2) \right]
\end{align*}
\]
where the wavevectors, for \( n > 0 \), are:

\[
A_n = \frac{2\pi}{L} n, \tag{25}
\]

\( B_n \) is implicitly defined by:

\[
- \frac{1}{2} \frac{\rho_0}{\rho_1} \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2 + B_n^2 = \tan \left( \frac{B_n L}{2} \right), \tag{26}
\]

and the only value of \( C \) is implicitly defined by:

\[
\frac{1}{2} \frac{\rho_0}{\rho_1} \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2 - C^2 = \tanh \left( \frac{C L}{2} \right) \tag{27}
\]

The normalization constants are:

\[
N_n^A = \frac{1}{\sqrt{L \rho_1 / 2 h \rho_0}}, \tag{28}
\]

\[
N_n^B = \frac{1}{\sqrt{L \rho_1 / 2 h \rho_0 + \frac{1}{2} \cos^2 \left( B_n L / 2 \right)}} \frac{B_n^2 - \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2}{B_n^2}, \tag{29}
\]

\[
N^C = \frac{1}{\sqrt{L \rho_1 / 2 h \rho_0 + \frac{1}{2} \cosh^2 \left( C L / 2 \right)}} \frac{C^2 + \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2}{C^2}, \tag{30}
\]

The eigenvalues are:

\[
\lambda_n^A = -\nu_1 \left( q^2 + A_n^2 \right) \tag{31}
\]

\[
\lambda_n^B = -\nu_1 \left( q^2 + B_n^2 \right) \tag{32}
\]

\[
\lambda^C = -\nu_1 \left( q^2 - C^2 \right) \tag{33}
\]

The eigenfunctions are orthogonal with respect to the scalar product:

\[
f \cdot g = \frac{\rho_1}{\hbar \rho_0} \int f(z) g(z) \, dz + f(0) g(0) \tag{34}
\]

The evolution equations for the coefficients are:

\[
\begin{align*}
\frac{\partial}{\partial t} a_n^\xi(t) &= \lambda_n^A a_n^\xi(t) \\
\frac{\partial}{\partial t} b_n^\xi(t) &= \lambda_n^B b_n^\xi(t) \\
\frac{\partial}{\partial t} c_n^\xi(t) &= \lambda^C b_n^\xi(t) \tag{35}
\end{align*}
\]

In order to use equipartition theorem, I evaluate the kinetic energy of the fluid:

\[
E = \frac{1}{2} \rho_0 h \int |\vec{u}_0(\vec{x}, t)|^2 \, d\vec{x} + \frac{1}{2} \rho_1 h \int |\vec{u}_1(\vec{x}, z, t)|^2 \, d\vec{x}dz \tag{36}
\]
In Fourier space, the energy can be decomposed as the sum \( E_T + E_Z \), where:

\[
E_T = \frac{(2\pi)^2}{2} \rho_0 h \int |u_0(q, t)|^2 \, dq + \frac{(2\pi)^2}{2} \rho_1 \int |u_1(q, z, t)|^2 \, dq \, dz
\]  

(37)

The integrals can be expressed in terms of the scalar product defined in Eq. (34):

\[
E_T = \frac{(2\pi)^2}{2} \rho_0 h \int u(q, z, t) \cdot u(q, z, t) \, dq
\]  

(38)

Due to the orthonormality of the eigenfunctions I used:

\[
E_T = \frac{(2\pi)^2}{2} \rho_0 h \int \sum_{n=1}^{+\infty} \left[ |a_n|^2 + |b_n|^2 + |c_n|^2 \right] \, dq
\]  

(39)

From equipartition theorem:

\[
\langle b_n^* b_m^* \rangle = \delta_{n,m} \delta(q + q') \frac{K_B T}{(2\pi)^2 \rho_0 h}
\]  

(40)

\[
\langle c_n^* c_m^* \rangle = \delta(q + q') \frac{K_B T}{(2\pi)^2 \rho_0 h}
\]  

(41)

From Eq. (33), I can derive the time spectrum:

\[
\langle b_n^* (\omega) b_m^* (\omega') \rangle = \delta_{n,m} \delta(q + q') \delta(\omega + \omega') \frac{K_B T}{4\pi^3 \rho_0 h} \frac{-\lambda_n^B}{\omega^2 + \lambda_n^B}
\]  

(42)

\[
\langle c_n^* (\omega) c_m^* (\omega') \rangle = \delta(q + q') \delta(\omega + \omega') \frac{K_B T}{4\pi^3 \rho_0 h} \frac{-\lambda_n^C}{\omega^2 + \lambda_n^C}
\]  

(43)

Using Eq. (21), I evaluate the power dynamic spectrum of the velocity fluctuations of \( \tilde{u}_0 \):

\[
S_u(q, \omega) = \frac{K_B T}{4\pi^3 \rho_0 h} \sum_{n=1}^{+\infty} \frac{-\lambda_n^B}{\omega^2 + \lambda_n^B} N_n^B \cos^2 \left( B_n L \right)
\]  

(44)

\[
+ \frac{K_B T}{4\pi^3 \rho_0 h} \frac{-\lambda_n^C}{\omega^2 + \lambda_n^C} N_n^C \cosh^2 \left( C L \right)
\]  

(45)

where the dynamic power spectrum is defined as follows:

\[
\langle u_0(q, \omega) u_0(q', \omega') \rangle = \delta(q + q') \delta(\omega + \omega') S_u(q, \omega)
\]  

(46)

Now I use Eq. (26) in order to express any trigonometric function of \( B_n \) and \( C_n \):

\[
S_u(q, \omega) = \frac{K_B T}{4\pi^3 \rho_0 h} \sum_{n=1}^{+\infty} 2 \nu_1 \left( q^2 + B_n^2 \right) \frac{B_n^2 \vert}{h\nu_0} \frac{B_n^2 \nu_1}{h\nu_0} \left[ B_n^2 + \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2 \right]^2 + \frac{C^2}{L^2} \left[ B_n^2 - \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2 \right]
\]  

(47)

\[
K_B T \frac{2 \nu_1 \left( q^2 + C_n^2 \right)}{4\pi^3 \rho_0 h} \omega^2 + \frac{\nu_1^2 \left( q^2 + C_n^2 \right)^2}{L} \left[ \frac{h\nu_0}{C_n^2 - 1 - \frac{\nu_0}{\nu_1} \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2} \right] + C^2 + \left( 1 - \frac{\nu_0}{\nu_1} \right) q^2
\]  

(48)
From Eq. (26):

\[ B_{n+1} - B_n = \frac{2\pi}{L} + o\left(\frac{1}{L}\right) \]  

(50)

\[ C = \sqrt{\left(\frac{\rho_1}{h\rho_0}\right)^2 + \left(1 - \frac{\nu_0}{\nu_1}\right) q^2 - \frac{\rho_1}{h\rho_0} + o\left(\frac{1}{L}\right)} \]  

(51)

For \( L \to \infty \):

\[ S_u(\vec{q}, \omega) = 4\pi^3\rho_0 h \int_0^{+\infty} \frac{\nu_1 (q^2 + B^2)}{\omega^2 + \nu_1^2 (q^2 + B^2)^2} \frac{B^2}{B_0^2 + \frac{1}{\nu_1} \left[ 1 - \frac{1}{\nu_0} \right] q^2 + B^2} dB + \]  

(52)

\[ \frac{K_B T}{4\pi^3\rho_0 h} \int_0^{+\infty} \frac{\nu_1 (q^2 - C^2)}{\omega^2 + \nu_1^2 (q^2 - C^2)^2} \frac{C^2}{C_0^2 + \left(1 - \frac{1}{\nu_0} \right) q^2} dB \]  

(53)

I define:

\[ \alpha = 1 - \frac{\nu_0}{\nu_1} \]  

(55)

\[ q_0 = \frac{\rho_1}{h\rho_0} \]  

(56)

\[ h(x, y) = \frac{4}{\pi} \int_0^{+\infty} \frac{1}{\pi} y^2 \left( x^2 + (x^2 + B^2)^2 \right) \frac{B^2}{4B^2 + (ax^2 + B^2)^2} dB \]  

(57)

\[ \Omega(x) = x^2 + 2 \frac{1}{1 - \alpha} \left( \sqrt{1 + \alpha x^2} - 1 \right) \]  

(58)

\[ l(x, y) = \frac{1}{\pi} \frac{\Omega(x)}{y^2 + \Omega^2(x)} \]  

(59)

\[ A(x) = 1 - \frac{1}{\sqrt{1 + \alpha x^2}} \]  

(60)

and obtain:

\[ S_u(\vec{q}, \omega) = \frac{K_B T}{4\pi^2\rho_0 h} \left[ \frac{1}{\nu_1 q_0} h \left( \frac{q}{q_0}, \frac{\omega}{\nu_1 q_0^2} \right) + \frac{1}{\nu_0 q_0^2} A \left( \frac{q}{q_0}, \frac{\omega}{\nu_0 q_0^2} \right) l \left( \frac{q}{q_0}, \frac{\omega}{\nu_0 q_0^2} \right) \right] \]  

(61)

The term with \( l(x, y) \) has a Lorentzian behaviour, and represents the spontaneous velocity fluctuations of the film, damped by the surrounding fluid. The other one describes the velocity fluctuations induced by the fluid on the film. It’s worth noting that the only spatial lengthscale is \( 1/q_0 \), while two different temporal lengthscales are involved in the two processes: \( \nu_0 q_0^2 \) for the fluctuations of the film, and \( \nu_1 q_0^2 \) for the velocity fluctuations of the surrounding fluid.

Some explicit results can be obtained by evaluating the integral:

\[ 4 \int_0^{+\infty} \frac{1}{x^2 + \alpha B^2 \frac{B^2}{4B^2 + (x^2 + B^2)^2}} dB = \frac{1}{x\sqrt{\alpha} + 1} \frac{1}{2\sqrt{\alpha} \sqrt{x^2 + 1} + (\alpha + 1)x} \]  

(62)
With $\alpha = 0$, I obtain, as expected from the equipartition theorem:

$$\int_{-\infty}^{+\infty} S_u(\vec{q}, \omega) d\omega = \frac{K_B T}{4\pi^2 \rho_0 h}$$  \hspace{1cm} (63)

The power spectrum on $\omega = 0$ can be evaluated by means of Eq. (62):

$$S_u(\vec{q}, \omega = 0) = \frac{K_B T}{4\pi^3 \eta_0 h} \frac{1}{q (q_C + q)},$$  \hspace{1cm} (64)

where:

$$q_C = \frac{2\eta_1}{h\eta_0}$$  \hspace{1cm} (65)

$K_B$ is the Boltzmann constant, $T$ the temperature, $\eta_0$ and $\eta_1$ are the shear viscosity of the film and the surrounding fluid, and $h$ is the thickness of the film.

The power spectrum has a $q^{-2}$ dependence for $q \gg q_C$; at smaller wavevectors, the divergence saturates to $q^{-1}$, due to air damping. For a soap film in air, the values of densities are $\rho_0 = 10^3\text{Kg/m}^3$ and $\rho_1 = 1.3\text{Kg/m}^3$. The effective viscosity of the film is $\nu_0 = 1.6 \cdot 10^{-6}\text{m}^2/\text{s}$, evaluated by means of Trapeznikov relation \[12\]. The viscosity of air is $\nu_1 = 1.43 \cdot 10^{-5}\text{m}^2/\text{s}$ \[8\]. From these values, for a 2$\mu$m thick film, $q_C \approx 1.2 \cdot 10^4\text{m}^{-1}$, corresponding to a 0.5mm wavelength.

From the power spectrum of the velocity fluctuations, we can derive the diffusion coefficient \[2\]:

$$D \approx \pi \int_{\Lambda} S_u(q, \omega = 0) d^2q$$  \hspace{1cm} (66)

where the cut off wavevector $\Lambda$ is of the order of the inverse of $a$, the radius of the diffusing particle. By integrating the static power spectrum of Eq. (64):

$$D = \frac{K_B T}{2\pi \eta_0 h} \left[ \ln \frac{h\eta_0}{a\eta_1} - \ln (2) \right]$$  \hspace{1cm} (67)

This is the well known Saffman formula for the diffusion coefficient of a particle on a liquid film \[11\], \[3\], \[1\].

This result can be compared with a phenomenological theory \[9\].

### 3 Time correlation function

By Fourier anti-transforming Eq. (61) in $\omega$, I obtain the time velocity time correlation function:

$$S_u(\vec{q}, \tau) = \frac{K_B T}{4\pi^2 \rho_0 h} \left[ e^{-\nu_1 q^2 |\tau|} h (\alpha \nu_1 q_0^2 |\tau|) + A e^{-\nu(\chi) q^2 |\tau|} \right]$$  \hspace{1cm} (68)

where:

$$\alpha = 1 - \frac{\nu_0}{\nu_1}$$  \hspace{1cm} (69)
\[ h(y) = \frac{4}{\pi} \int_{0}^{+\infty} e^{-y B^2} \frac{B^2}{4B^2 + (x^2 + B^2)^2} dB \]  \hspace{1cm} (70)

\[ \nu(x) = \nu_0 + (\nu_1 - \nu_0) \sqrt{1 + \frac{x^2}{1 + x^2}} \]  \hspace{1cm} (71)

\[ x = \frac{q}{q_0} \]  \hspace{1cm} (72)

\[ A(x) = 1 - \frac{1}{\sqrt{1 + x^2}} \]  \hspace{1cm} (73)

\[ q_0 = \frac{\rho_1}{h p_0} \frac{1}{\sqrt{\alpha}} \]  \hspace{1cm} (74)

References

[1] J. Bechhoefer, J. C. Géminard, L. Bocquet, and P. Oswald. Experiments on tracer diffusion in thin free-standing liquid-crystal films. *Phys. Rev. Lett.*, 79(24):4922–4925, 1997.

[2] Doriano Brogioli and Alberto Vailati. Diffusive mass transfer by nonequilibrium fluctuations: Fick’s law revisited. *Phys. Rev. E*, 63, 2001.

[3] C. Cheung, Y. H. Hwang, X. L. Wu, and H. J. Choi. Diffusion of particles in free-standing liquid films. *Phys. Rev. Lett.*, 76(14):2531–2534, 1996.

[4] Y. Couder. Two-dimensional grid turbulence in a thin liquid film. *J. Phys. Lett.*, 45:353–, 1984.

[5] Y. Couder, J. M. Chomaz, and M. Rabaud. On the hydrodynamics of soap films. *Physica D*, 37:384–, 1989.

[6] H. Kellay, X. L. Wu, and W. I. Goldburg. Experiments with turbulent soap films. *Phys. Rev. Lett.*, 74:3875–, 1995.

[7] X. L. Wu, B. K. Martin, H. Kellay, and W. I. Goldburg. Hydrodynamic convection in a two-dimensional couette cell. *Phys. Rev. Lett.*, 75:236–, 1995.

[8] David R. Lide. *Handbook of Chemistry and Physics*. CRC Press, Boston, 1992.

[9] M. Rivera and X. L. Wu. External dissipation in driven two-dimensional turbulence. *Phys. Rev. Lett.*, 85(5):976–979, July 2000.

[10] M. A. Rutgers, X. L. Wu, R. Bhagavatula, A. A. Petersen, and W. I. Goldburg. Two-dimensional velocity profiles and laminar boundary layers in flowing soap film. *Phys. Fluids*, 8(11):2847–2854, November 1996.

[11] P. G. Saffman. Brownian motion in thin sheets of viscous fluid. *J. Fluid Mech.*, 73(4):593–602, 1976.
[12] A. A. Trapeznikov. In Butterworths, editor, Proceedings of the Second International Congress on Surface Activity, page 242, London, 1957.