On the asymptotic behavior of Bessel-like diffusions

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Abstract
We derive the asymptotic behavior of the transition probability density of the Bessel-like diffusions for “dimension” $\rho = 0$.

1 Introduction

1.1 Background
Let $\rho > 0$. A Bessel process of dimension $\rho$ is a diffusion process on $[0, \infty)$ with generator

$$L_\rho := \frac{1}{2} \left( \frac{d^2}{dx^2} + \frac{\rho - 1}{x} \frac{d}{dx} \right), \quad x > 0.$$ 

If the origin is a regular boundary (i.e., $0 < \rho < 2$), we impose the reflecting boundary condition. Then the transition probability density with respect to the speed measure $m_\rho(dx) = 2x^{\rho-1}dx$ is

$$P_\rho(t, x, y) := \frac{1}{2t} (xy)^{-\nu} \exp \left( -\frac{x^2 + y^2}{2t} \right) I_\nu \left( \frac{xy}{t} \right)$$ 

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where $I_\nu$ is the modified Bessel function and $\nu := \frac{\rho}{2} - 1$. We thus have

$$P_\rho(t; x, y) \sim \frac{1}{2^{\rho/2} \Gamma(\rho/2)} \cdot \frac{1}{t^{\rho/2}}, \quad t \to \infty.$$  

Here and henceforth we denote by $f \sim g$ if $\lim \frac{f}{g} = 1$. In this paper we consider a diffusion process on $[0, \infty)$ with generator:

$$\mathcal{L} := \frac{1}{2} \left( \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \right), \quad x > 0,$$

where $b \in L^1_{\text{loc}}[0, \infty)$ so that the left boundary 0 is regular where the reflecting boundary condition is imposed. We assume that $\mathcal{L}$ is asymptotically equal to the generator of the Bessel process:

**Assumption**

$$b(x) = \frac{\rho - 1 + \epsilon(x)}{x} + \eta(x), \quad x \geq 1$$

where $\rho \in \mathbb{R}$, $\lim_{x \to \infty} \epsilon(x) = 0$, $\eta \in L^1_{\text{loc}}[0, \infty)$ such that the following limit exists: $A := \lim_{x \to \infty} \int_1^x \eta(u)du \in \mathbb{R}$.

Assumption implies that the function

$$W(x) := \exp \left( \int_1^x b(u)du \right), \quad x > 0$$

varies regularly at $\infty$ with index $\rho - 1$; that is

$$\lim_{x \to \infty} \frac{W(\lambda x)}{W(x)} = \lambda^{\rho - 1},$$

for any $\lambda > 0$. We denote by $R_\alpha(\infty)$ (resp. $R_\alpha(0)$) the totality of regularly varying functions at infinity (resp. zero) with index $\alpha$. Our aim is to study the asymptotic behavior of the transition probability density of this diffusion as $t \to \infty$. The answer is known for $\rho \neq 0$ [2,3] which we recall in Subsection 1.2.
1.2 Known Results

Set

\[ W(x) = \exp \left( \int_1^x b(u)du \right), \quad x > 0 \]

\[ s(x; c) := \int_c^x \frac{du}{W(u)}, \quad m(x) = 2\int_0^x W(u)du \]

which leads to the canonical form \( \mathcal{L} = \frac{d}{dm(x)} \frac{d}{ds(x)} \). Let \( p(t; x, y) \) be the transition probability density with respect to \( m(dx) \) which is equal to the Laplace transform of the spectral function \( \sigma \).

\[ p(t; 0, 0) = \int_{[0, \infty)} e^{-\lambda t}d\sigma(\lambda), \quad t > 0. \]

Let

\[ G_s(x, y) = \int_0^\infty e^{-st}p(t; x, y)dt, \quad s > 0 \]

be Green’s function. Then \( h(s) := G_s(0, 0) \) satisfies

\[ h(s) = \int_{[0, \infty)} \frac{d\sigma(\xi)}{s + \xi}, \quad s > 0, \]

and \( h \) is the characteristic function associated to \( \tilde{m}(x) := m(s^{-1}(x)) \) by Krein’s correspondence \[4\]. When \( \rho \neq 0 \), the answer to our question is:

**Theorem 1.1**

(1) ([2] Theorem 4.2) If \( \rho > 0 \),

\[ p(t; x, y) \sim \frac{1}{2^{\rho/2}\Gamma(\rho/2)} \cdot \frac{1}{\sqrt{t}W(\sqrt{t})}, \quad t \to \infty. \]

(2) ([3] Theorem 5.1) If \( \rho < 0 \),

\[ p(t; 0, 0) - \frac{1}{m(\infty)} \sim \frac{1}{m(\infty)^2} \frac{2^{\rho+1}}{|\rho|}\Gamma((2 - \rho)/2)\sqrt{t}W(\sqrt{t}), \quad t \to \infty. \]

We also recall the following result which is an important ingredient of the proof of our main theorem. Let \( h^*(s) = (sh(s))^{-1} \) be the dual of \( h \) which is the characteristic function associated to \( \tilde{m}^{-1}(x) \) \[4\]. Let \( \sigma^* \) be the corresponding spectral function.
Theorem 1.2 ([3], Proposition 5.1)
If $\rho < 2$,
\[
\sigma^\ast(\lambda) \sim \frac{2^{\frac{\rho}{2}+1}}{(2 - \rho)\Gamma\left(\frac{2-\rho}{2}\right)^2} \sqrt{\lambda W}\left(\frac{1}{\sqrt{\lambda}}\right), \quad \lambda \to +0.
\]
We note that Theorem 1.2 is valid even for $\rho = 0$.

1.3 Results in this paper
In this paper, we consider the case $\rho = 0$. Then we could have both $m(+\infty) = \infty$ and $m(+\infty) < +\infty$. Let $m_\infty := m(+\infty)$. Since $\sigma(+0) = 1/m_\infty$, $m_\infty < \infty$ implies $\sigma(+0) > 0$ and $p(t;0,0) \to 1/m_\infty$.

Theorem 1.3 If $\rho = 0$ and $m_\infty = \infty$,
\[
p(t; x, y) \sim \frac{1}{m(\sqrt{t})}, \quad t \to \infty.
\]

Theorem 1.4 If $\rho = 0$ and $m_\infty < \infty$,
\[
p(t; 0, 0) - \frac{1}{m_\infty} \sim \frac{1}{2^{\rho/2}\Gamma\left(\frac{\rho}{2} + 1\right)} \cdot \frac{1}{m(\sqrt{t})} \left(m_\infty - m(\sqrt{t})\right), \quad t \to \infty.
\]

Remark 1.1 To summarize the statements in [2] Theorem 4.1, [3] Theorem 5.1 and Theorems 1.3, 1.4, we have
(1) $\rho \geq 0$, $m(+\infty) = \infty$ :
\[
p(t; x, y) \sim \frac{1}{2^{\rho/2}\Gamma\left(\frac{\rho}{2} + 1\right)} \cdot \frac{1}{m(\sqrt{t})}, \quad t \to \infty. \tag{1.1}
\]
(2) $\rho \leq 0$, $m(+\infty) < \infty$ :
\[
p(t; 0, 0) - \frac{1}{m_\infty} \sim \frac{1}{2^{\rho/2}\Gamma\left(\frac{\rho}{2} + 1\right)} \cdot \frac{1}{m_\infty} \left(1 - \frac{m(\sqrt{t})}{m_\infty}\right), \quad t \to \infty. \tag{1.2}
\]
In Section 2, we prove Theorems 1.3, 1.4 and apply them to some concrete examples. A strategy of the proof is to study the behavior of the following quantities in the arranged order, using Theorem 1.2 and Tauberian theorems.
\[
\sigma^\ast(\lambda) \to h^\ast(s) \to h(s) = \frac{1}{sh^\ast(s)} \to \sigma(\lambda)
\]
In Section 3, we shall quote some Tauberian Theorems used frequently in this paper.
2 Proof of Theorems

2.1 Proof of Theorem 1.3

First of all, by a property of the regularly varying functions \[1\] we have

\[m(x) = 2 \int_0^x W(u)du \sim \frac{2}{\rho} xW(x), \quad x \to \infty.\]

Applying it to Theorem 1.1 yields (1.1) in Remark 1.1 for \(\rho > 0\).

Proof of Theorem 1.3

By the argument in \[2\] Corollary 5.3,

\[p(t, x, y) \sim p(t, 0, 0), \quad t \to \infty\]

so that we may suppose \(x = y = 0\). \[3\] Proposition 5.1 (\(\rho = 0\)) implies

\[\sigma^*(\lambda) \sim \sqrt{\lambda}W\left(\frac{1}{\sqrt{\lambda}}\right) \in R_1(0), \quad \lambda \downarrow 0.\]

Thus \[3\] Proposition 5.1 (\(\rho = 0\)) and Theorem 3.2 (\(\alpha = 1, n = 1\)) below yield

\[(-1) \cdot \frac{d}{ds} h^*(s) \sim s^{-2} \sigma^*(s) \sim s^{-\frac{3}{2}}W\left(\frac{1}{\sqrt{s}}\right), \quad s \to +0.\]

On the other hand, by the definition of \(m\),

\[\frac{d}{ds} m\left(\frac{1}{\sqrt{s}}\right) = -s^{-\frac{3}{2}}W\left(\frac{1}{\sqrt{s}}\right).\]

Therefore

\[-\frac{d}{ds} h^*(s) \sim -\frac{d}{ds} m\left(\frac{1}{\sqrt{s}}\right), \quad s \to +0. \quad (2.1)\]

Since \(m(+\infty) = \infty\), we may apply de l'Hospital's theorem to have

\[h^*(s) \sim m\left(\frac{1}{\sqrt{s}}\right), \quad s \to +0.\]

Using \(h^*(s) = (sh(s))^{-1}\),

\[h(s) \sim \frac{1}{sm\left(\frac{1}{\sqrt{s}}\right)}, \quad s \to +0. \quad (2.2)\]
Note that, by Proposition 1.5.9a and by the fact that \( l(x) := xW(x) \) is slowly varying at infinity, \( f(s) := m \left( \frac{1}{\sqrt{s}} \right) \) is slowly varying at 0. By Theorem 3.2 \((\alpha = 0, n = 0)\) below,

\[
\sigma(s) \sim \frac{1}{m \left( \frac{1}{\sqrt{s}} \right)}, \quad s \to +0.
\]

Thus Theorem 3.1 completes the proof.

Remark 2.1 There is another argument starting with (2.2). Using \( h(s) = \int_0^\infty e^{-st}p(t; 0, 0)dt \), Theorem 3.1 implies

\[
\int^t p(s; 0, 0)ds \sim \frac{t}{m(\sqrt{t})}, \quad t \to \infty.
\]

Since \( p(t, 0, 0) \) is monotone as a function of \( t \), monotone density theorem (\[\Pi\], Theorem 1.7.2) yields

\[
p(t; 0, 0) \sim \frac{1}{m(\sqrt{t})}, \quad t \to \infty.
\]

2.2 Proof of Theorem 1.4

We first derive (1.2) in Remark 1.1 for \( \rho < 0 \). Set

\[
m(x) = 2 \int_0^x W(u)du, \quad m_\infty = 2 \int_0^\infty W(u)du, \quad s(x) = \int_0^x \frac{1}{W(u)}du.
\]

Let \( \mathcal{L}^\bullet \) be the dual operator of \( \mathcal{L} \)

\[
\mathcal{L}^\bullet := \frac{1}{2} \left( \frac{d^2}{dx^2} - b(x) \frac{d}{dx} \right)
\]

and let \( m^\bullet, s^\bullet \) be the corresponding speed measure and the scale function, respectively. Then

\[
m^\bullet(x) = 2 \int_0^x \frac{1}{W(u)}du = 2s(x)
\]

\[
s^\bullet(x) = \int_0^x W(u)du = \frac{1}{2}m(x)
\]
so that \( l^\bullet := h^\bullet (+0) = s^\bullet (+\infty) = \frac{1}{2} m_\infty \). Since \( h^\bullet (s) = 2h^\bullet (s) \) \( 3 \), we have
\[
l^\bullet := h^\bullet (+0) = 2h^\bullet (+0) = 2l^\bullet = m_\infty. \tag{2.3}
\]
Thus
\[h(s) = \frac{1}{s h^\bullet (s)} \sim \frac{1}{s m_\infty}, \quad s \to +0.\]
and by Theorem \( 3.2 \) \((\alpha = 0, n = 0, A = m^{-1}_\infty)\) below,
\[\sigma(\lambda) \sim \frac{1}{m_\infty}, \quad \lambda \to +0.\]
Since
\[
\int_\infty^\infty W(u)du \sim \frac{1}{|\rho|} xW(x), \quad x \to \infty
\]
by \([1]\) Proposition 1.5.10, we have
\[m_\infty - m(x) \sim \frac{2}{|\rho|} xW(x) \in R_\rho(0)\]
which, together with \( 3 \) Theorem 5.1, yields \((1.2)\) in Remark 1.1.

**Proof of Theorem 1.4** We note that, by \([1]\) Proposition 1.5.9b, \( g(s) := m_\infty - m\left(\frac{1}{\sqrt{s}}\right)\) is slowly varying at 0. Owing to Theorem \( 3.2 \) it suffices to show the following equation.
\[h(s) - \frac{1}{s m_\infty} \sim \frac{1}{s m^2_\infty} \left( m_\infty - m\left(\frac{1}{\sqrt{s}}\right) \right), \quad s \to +0\]
which is equivalent to
\[
\frac{1}{h^\bullet (s)} - \frac{1}{m_\infty} \sim \frac{1}{m^2_\infty} \left( m_\infty - m\left(\frac{1}{\sqrt{s}}\right) \right).
\]
By de l’Hospital’s theorem,
\[
\frac{1}{m^\infty_\infty} \left( m_\infty - m\left(\frac{1}{\sqrt{s}}\right) \right) \sim \frac{1}{m^\infty_\infty} \left( -\frac{d}{ds} m\left(\frac{1}{\sqrt{s}}\right) \right)
\]
\[
= \frac{1}{m^\infty_\infty} \left( -\frac{d}{ds} m\left(\frac{1}{\sqrt{s}}\right) \right)
\]
\[
\sim \frac{1}{m^\infty_\infty} \frac{d}{ds} m\left(\frac{1}{\sqrt{s}}\right) = 1, \quad s \to +0
\]
which finishes the proof, where we used (2.1), (2.3) in the last line. □

2.3 Example

We apply Theorems 1.3, 1.4 to some examples. In what follows, \( \eta \in L^1_{\text{loc}}[0, \infty) \) such that the limit \( A := \lim_{x \to \infty} \int_1^x \eta(u)du \) exists.

Example 1

\[
b(x) = -\frac{1}{x} 1(x \geq 1) + \eta(x),
\]

Then we have

\[
p(t; x, y) \sim \frac{e^{-A}}{2} \left( \log \sqrt{t} \right)^{-1}, \quad t \to \infty.
\]

Example 2

\[
b(x) = \left( -\frac{1}{x} + \frac{\alpha}{x \log x} \right) 1(x > 1) + \eta(x), \quad \alpha \neq 0, \quad 0 < \beta < 1.
\]

Note that the case \( \beta > 1 \) is reduced to Example 1. Then

\[
p(t; x, y) \sim \frac{\alpha}{2} e^{-A} (\log \sqrt{t})^{-\beta} e^{-\frac{\alpha}{\log \sqrt{t}}^{1-\beta}}, \quad t \to \infty.
\]

Example 3

\[
b(x) = \left( -\frac{1}{x} + \frac{\alpha}{x \log x} \right) 1(x > e) + \eta(x)
\]

Then

(1) \( \alpha > -1, \quad p(t; x, y) \sim \frac{\alpha + 1}{2} e^{-(A+1)} (\log \sqrt{t})^{-(\alpha+1)}, \quad t \to \infty \)

(2) \( \alpha = -1, \quad p(t; x, y) \sim \frac{e^{-(A+1)}}{2} (\log \log \sqrt{t})^{-1}, \quad t \to \infty \)

(3) \( \alpha < -1, \quad p(t; x, y) \sim \frac{1}{m_\infty} \sim \frac{1}{m_\infty^2} \left( -2 \right)^{\alpha+1} e^{A+1} (\log \sqrt{t})^{\alpha+1}, \quad t \to \infty. \)

where \( m_\infty := 2 \int_0^\infty \exp \left( \int_1^u b(v)dv \right) du \).
Example 4
In general, given a function $m : [0, \infty) \to (0, \infty)$, such that $\lim_{t \to \infty} \frac{m''(t)}{m'(t)} = -1$, we can construct a corresponding generator $L$ such that $p(t; x, y) \sim (m(\sqrt{t}))^{-1}$, $t \to \infty$. In fact, we can take
$$b(x) := -\frac{1}{x} + \frac{f''(\log x)}{f'(\log x)} \cdot \frac{1}{x}$$
where $f(x) := m(e^x)$.

3 Appendix

We recall some important facts from the theory of regularly varying functions [1], [3]. For a function $\sigma : [0, \infty) \to \mathbb{R}$ being of locally bounded variation and right-continuous, let
$$\hat{\sigma}(\lambda) = \int_{[0, \infty)} e^{-\lambda x} d\sigma(x)$$
$$H_n(\sigma, \lambda) := \int_{[0, \infty)} \frac{d\sigma(\xi)}{(\lambda + \xi)^{n+1}}, \quad n \geq 0$$
be its Laplace transform, and the generalized Stieltjes transform, respectively.

Theorem 3.1 Let $\rho \geq 0$ and $f \in R_\alpha(0)$. Then
$$\sigma(x) \sim c f(x), \quad x \to \infty \iff \hat{\sigma}(\lambda) \sim c \Gamma(\rho + 1) f \left( \frac{1}{\lambda} \right), \quad \lambda \to +0.$$  

Theorem 3.2 (Theorem 7.1 in [3])
Let $0 \leq \alpha < n + 1$, $A \geq 0$, and $\varphi \in R_\alpha(0)$. Then
$$\sigma(\xi) \sim A \varphi(\xi), \quad \xi \to 0 \iff H_n(\sigma; \lambda) \sim AC_{n,\alpha} \varphi(\lambda) \lambda^{-n-1}, \quad \lambda \to 0$$
where
$$C_{n,\alpha} := \frac{\Gamma(n + 1 - \alpha) \Gamma(\alpha + 1)}{\Gamma(n + 1)}.$$  

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