COVARIANT QUANTIZATION OF
GREEN SCHWARZ SUPERSTRING

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Abstract

We describe a canonical covariant approach to the quantization of the Green-Schwarz superstring. The approach is first applied to the canonical covariant quantization of the Brink and Schwarz superparticle. The Kallosh action is obtained in this case, with the correct BRST cohomology.
1. Introduction

The construction of the second quantized, Lorentz covariant Superstring action is one of the main problems to be solved in String Theory. The Lorentz covariant, BRST quantization of the Green-Schwarz Superstring (GSSS) [1] is the first step we have to take towards that goal.

A non-perturbative, second quantized action for superstrings was formulated by Green-Schwarz [2] in the Light Cone Gauge (LCG). The action is cubic in terms of the functions of the transverse modes of the superstrings with a non trivial insertion factor. It was shown in [3] that the action requires in addition complicated contact terms in order to allow the closure of the Super-Poincare algebra. Recently a new approach to Superstring in the LCG has been proposed [4]. It is obtained by patching together the local Green-Schwarz LCG action, using the unique complex analytic structure of Teichmüller space. It may solve completely the problem of contact terms at the second quantized level for closed type IIB and Heterotic strings. However, in order to understand the geometric principles of the Superstring Theory a Lorentz covariant formulation is required. As usually we are interested in a covariant, canonical, BRST quantization of the Green-Schwarz Superstring [1], in order to have simultaneously a manifest covariant formulation and unitary theory. Given a gauge theory whose evolution is determined by a Hamiltonian $H$ on a phase space $\mathcal{M}$ of local coordinates $(q,p)$ with first class, irreducible, regular constraints $\varphi_i$, then the construction of the effective action follows by considering the BRST charge $Q$,

\[
\{Q,Q\}_{\text{Poisson}} = 0,
Q|_{\mu=0} = c^i \varphi_i,
\]

where $(c^i, \mu_i)$ are the ghost fields and its conjugate momenta. The BRST invariant Hamiltonian $\hat{H}$ is obtained from

\[
\{\hat{H},Q\} = 0,
\hat{H}|_{\mu=0} = H.
\]

The effective action is given by

\[
S_{\text{eff}} = \langle p\dot{q} + \mu\dot{c} - \hat{H} + \delta(\lambda^i \mu_i) + \delta(\tau_i \chi^i) \rangle,
\]

where $\delta$ denotes BRST transformed [5], and $\chi^i$ are the gauge fixing conditions.
The functional integral

\[ I(\chi) = \int \mathcal{D}_\mu \exp \frac{i}{\hbar} S_{\text{eff}}, \]

with \( \mathcal{D}_\mu \) the Liouville measure, is then independent of \( \chi \), that is,

\[ \frac{\delta I}{\delta \chi} = 0 \quad (1.5) \]

within the admissible set of gauge fixing conditions, provided initial and final (in time) conditions on the ghost fields are satisfied [5].

Property (1.5) allows to shown the equivalence between the quantization of the gauge theory in a physical gauge, like the LCG where only physical modes are present and where physical unitary is directly proven, and the manifest Lorentz covariant quantization obtained in a covariant gauge, for example \( \lambda_0 = 1 \lambda_1 = 0 \) in the Bosonic String Theory. \( \lambda_0 \) is the Lagrange multiplier (L.m.) associated to the Hamiltonian constraint while \( \lambda_1 \) is the L.m. associated to the generator for \( \sigma \)-reparametrizations.

The construction of \( S_{\text{eff}} \) for the GSSS and property (1.5) for a non-trivial topology are the ingredients for a proof of the equivalence between the perturbative multiloop amplitudes associated of the theory in the LCG and in a Lorentz Covariant gauge. The corresponding equivalence for the bosonic string theory following this line was done in [6]. The construction of the covariant, BRST effective action for the GSSS in thus relevant in several aspects. The main difficulty in this construction has been the covariant gauge fixing of the local \( K \) supersymmetry. The first class constraints associated with the gauge symmetries appear mixed with second class ones, and so far no local, Lorentz covariant, and finite reducible approach to disentangle them has been found.

The same difficulties, but at less sophisticated level, are present in the covariant quantization of the Brink-Schwarz superparticle (BSSP) which correspond to the zero mode of the GSSS. For this reason all approaches to the covariant quantization of superstrings have been first tested with the BSSP.

The approach we are going to follow in order to obtain the covariant, BRST invariant, effective action for the GSSS was developed in Ref. [7]. It is based on the idea [8] of extending a theory with second class constraints to another one with only first class constraints in such a way that the functional integral of the latest reduces by partial gauge fixing, within the admissible set, to the original one with the correct Fradkin-Senjanovic measure. It is an off-shell approach allowing the systematic construction of
the off-shell nilpotent BRST charge and of the BRST invariant effective action. In the
extended phase space where all constraints are first class the operatorial quantization
construction of Batalin and Fradkin [5] may be directly implemented. However for the
GSSS as well as for the BSSP due to the infinite reducibility structure of the first class
constraints, generating functions should be introduced in order to handle the infinite set
of auxiliary fields. An analogous situation occurs in the construction of the covariant
unconstrained formulation of Super Yang-Mills and Supergravity [9].

In section 2 we present the GSSS action and perform the canonical analysis, which
was first done in [10]. In section 3 discuss the structure of the constraints in relation
with the previous approaches to deal with the problem, which involved the introduction
of irregular constraints [11]. In section 4 we present our method applied to the covariant
quantization of the superparticle. We obtain in a canonical way the action proposed
by Kallosh action [12]. Finally in section 5 we show the construction for the GSSS and
discuss related problems.

2. The Green-Schwarz covariant Superstring

The Green-Schwarz covariant action for Superstring Theory in 10 dim may be
written as

\[ S = \frac{1}{\pi} \int d\sigma d\tau (L_1 + L_2 + \lambda (\det e + 1)), \]  

(2.1)

where

\[ L_1 = -\frac{1}{2} \epsilon^{\alpha\beta} \pi^\mu_{\alpha} \pi^\mu_{\beta} \]
\[ L_2 = -\epsilon^{\alpha\beta} \partial_{\alpha} X^\mu \{K^{1\mu}_{\beta} - K^{2\mu}_{\beta}\} - \epsilon^{\alpha\beta} K^{1\mu}_{\alpha} K^{2\mu}_{\beta} \]
\[ \pi^\mu_{\alpha} = \partial_{\alpha} X^\mu - \sum_A i \bar{\theta}^A_{\gamma} \gamma^\mu \partial_{\alpha} \theta^A \]
\[ K^{A\mu}_{\alpha} = i \bar{\theta}^A_{\gamma} \gamma^\mu \partial_{\alpha} \theta^A \]  

(2.2)

\[ \theta^A, A = 1, 2, \] are Majorana Weyl spinors in ten dimensions, \( \alpha, \beta \) denote the world sheet
indices.

(2.1) is the functional action in terms of the independent variables \( X^\mu, \theta^A \), and
\( e^{\alpha\beta} \).

The action has manifest Poincare invariance in 10-dim and local reparametrization
invariance on the world sheet. Additionally it has global and local Supersymmetries that
we discuss later on using a canonical approach. Here we are going to discuss explicitly
the closed Superstring Theory and then comment on the results for Heterotic String Theory in 10-dim.

We impose periodic boundary conditions on $X^\mu$, $\theta^A$, $e^{\alpha\beta}$ at $\sigma = 0$ and $\sigma = \pi$ and denote $p_\mu$, $p_{\alpha\beta}$ and $\xi^A$ the canonical conjugate momenta associated to $X^\mu$, $e^{\alpha\beta}$ and $\theta^A$ respectively.

The primary constraints are

$$p_{\alpha\beta} = 0, \quad (2.3a)$$

$$F^A \equiv \xi^A + i \bar{\theta}^A \gamma^\mu \{ p_\mu + (-1)^A \frac{1}{\pi} (\partial_\sigma X^\mu - K^A_{\mu \sigma}) \} = 0 \quad (2.3b)$$

By solving $\det e + 1 = 0$ and using the conservation of (2.3) one may get rid of $e^{\alpha\beta}$ and $p_{\alpha\beta}$ as canonical variables. The Hamiltonian is the given by

$$H = \int d\sigma \{ \frac{1}{e_{\sigma\sigma}} \phi_1 + \frac{e_{\sigma\sigma}}{e_{\sigma\sigma}} \phi_2 + \bar{F}^1 \Lambda_1 + \bar{F}^2 \Lambda_2 \}, \quad (2.4)$$

where

$$\phi_1 \equiv \frac{1}{2\pi} (\pi^2 \tilde{p}^2 + \pi^2 - 2\pi F^1 \partial_\sigma \omega^1 + 2\pi F^2 \partial_\sigma \theta^2)$$

$$= \frac{1}{2} (\pi p^2 + x^2) - \xi^1 \theta^{1'} + \xi^2 \theta^{2'}, \quad (2.5a)$$

$$\phi_2 \equiv \bar{p} \pi_\sigma + F^1 \partial_\sigma \theta^1 + F^2 \partial_\sigma \theta^2 =$$

$$= px' + \xi^1 \theta^{1'} + \xi^2 \theta^{2'}, \quad (2.5b)$$

$$\bar{F}^A \equiv F^A \Gamma_A. \quad (2.5c)$$

We have denoted

$$\tilde{p}_\mu \equiv p_\mu + \frac{1}{\pi} (K^1_{\mu \sigma} - K^2_{\mu \sigma}),$$

$$\Gamma_A \equiv \gamma^\mu (\pi \tilde{p}_\mu + (-1)^A \pi_{\sigma \mu}),$$

and repetition of $A$ indices does not denote summation.

Eq.(2.4) is subject to the second class fermionic constraints coming from (2.3b)

$$G^1 \equiv F^1 \Gamma_2 = 0,$$

$$G^2 \equiv F^2 \Gamma_1 = 0. \quad (2.6)$$

The Lagrange multipliers $\Lambda_A$ are restricted to the subspace

$$\Gamma_2 \Lambda_1 = 0$$

$$\Gamma_1 \Lambda_2 = 0. \quad (2.7)$$
The constraints (2.3b) have been decoupled into a first class part given by $\tilde{F}^A = 0$ and a second class part given by $G^A = 0$. The Lagrange multipliers $\frac{1}{\epsilon_{\sigma \sigma}}, \frac{e_{\sigma \sigma}}{\epsilon_{\sigma \sigma}}$ and $\Lambda_A$ cannot be determined by Dirac approach. They are thus associated to first class constraints.

Eq.(2.6) were first obtained by Hori and Kamimura using the projectors

\[
P_1 = \frac{\Gamma_1 \Gamma_2}{2(\tilde{p}^2 - \pi_Z^2)},
\]

\[
P_2 = \frac{\Gamma_2 \Gamma_1}{2(\tilde{p}^2 - \pi_Z^2)}.
\]

We notice however that $P_1$ and $P_2$ are projectors only over the submanifold of constraints. That is not the situation on an effective action where a covariant gauge fixing condition, through restrictions on the Lagrange multipliers, has been imposed. After all we are just interested in that case. We thus conclude that the approach of Hori and Kamimura does not solve the problem of covariant decoupling of the first and second class constraints in Superstring Theory.

Expressions(2.5a,b) are the generators of local reparametrization on the world sheet, they satisfy the same algebra as the Virasoro constraints of the Nambu-Goto String. $X^\mu$ and $\theta^A$ transform as scalars under reparametrizations. This transformation law for $\theta^A$ is essential in the Green-Schwarz construction of the interacting theory. Since $\theta^A$ behave as scalars under reparametrization there is no summation on spin structures involved in the evaluation of Multiloop Amplitudes.

$\tilde{F}^A$ generates local supersymmetric transformation. $X^\mu$ and $\theta^A$ transform as

\[
\delta X^\mu(\sigma) = \{X^\mu(\sigma), \sum_B \int d\tilde{\sigma} \tilde{F}^B(\tilde{\sigma}) \xi^B(\tilde{\sigma})\} = \sum_B i\tilde{\theta}^B(\sigma) \gamma^\mu \delta \theta^B(\sigma)
\]

(2.8a)

\[
\delta \theta^A(\sigma) = \{\theta^A(\sigma), \sum_B \int d\tilde{\sigma} \tilde{F}^B(\tilde{\sigma}) \xi^B(\tilde{\sigma})\} = \Gamma_A(\sigma) \xi^A(\sigma).
\]

(2.8b)

3. Regular and Irregular Constraints.

In order to circumvent the problem presented by mixing, in a covariant treatment, of the first and second class constraints other actions for description of the superparticles
and GSSS, following original ideas of Siegel, have been proposed [13]. They allow
a formulation in terms of first class constraints only. The formulations however are
given in terms of irregular constraints. The presence of irregular constraints [11] does
not allow a straightforward application of the Batalin-Fradkin or Batalin-Vilkovisky
approach. As a consequence the BRST invariant effective action for these models has
not been constructed.

The distinction between regular and irregular constraints is very revelant since all
the quantization procedures break-down in the presence of irregular constraints.

Let $f$ be a $C^m$ map between Banach manifolds

$$f : M \rightarrow N$$

$n \in N$ is a regular value of $f$ if for each $m(\in f^{-1}(n)$ the map between tangent spaces at
$m$, $T_m f$ is surjective. If $T_m f$ is not surjective, $m \in M$ is a critical point and $n = f(m)$
is a critical value of $f$.

The constraints on the evolution of dynamical system may be defined from a map

$$\phi : M \rightarrow N$$

between Banach manifolds, as the set of restrictions

$$\phi(m) = 0 \quad , m \in M \quad , 0 \in N.$$ When $0 \in N$ is a critical value of $\phi$ we say that $\phi = 0$ an irregular constraints. That is,
there exist at least one critical point $m \in M$.

If $\phi$ is a smooth submersion then the constraint $\phi = 0$ is a regular constraint.

The regularity of the constraints is a requirement for the application of the Lagrange
Multiplier theorem. Let $\phi$ be a smooth submersion (3.1), denote

$$L = \phi^{-1}(0)$$

and $f$ a functional on $M$

$$f : M \rightarrow N.$$ Then, $m \in M$ is a critical point of $f|_L$ if and only if there exists $\lambda \in N^*$, the dual to $N$,
such that $m$ is a critical point of $f - \lambda \cdot \phi$.It is not difficult to construct examples with
irregular constraints where a critical point of $f|_L$ is not a critical point of $f - \lambda \cdot \phi$. 
In the presence of irregular constraints the Batalin-Fradkin-Vilkovisky [5] theorem is not valid. In fact, since in general the Lagrange Multiplier theorem does not apply in the presence of irregular constraints one may suspect that the BFV approach which explicitly introduces the constraints into the effective action through Lagrange multipliers may have some obstruction. The problem arises when writing the Fradkin-Senjanovic measure on the functional integral. This is so, since the $\delta$ Dirac delta is ill defined on an irregular constraint. In fact, since

$$\delta(\phi(x)) = \sum \frac{\delta^{-1}(\phi(0))}{det \frac{\delta \phi}{\delta x}|_{\phi^{-1}(0)}},$$

if $\phi$ is irregular the determinant in the denominator is zero. Consequently one is not allowed to impose a canonical gauge in the effective action associated to a system with irregular constraints, since integration on the ghost and antighost fields one ends up with a Fradkin-Senjanovic like measure

$$D_\mu \delta(\phi_{\text{irregular}}) \delta(\chi) det\{\chi, \phi\}.$$

It is still possible, however, to consider covariant gauges in the effective action. In fact a covariant gauge is implemented by imposing restrictions on the Lagrange multipliers. One is not working then on the submanifold of constraints and there is no inconsistency in the procedure. Nevertheless, the equivalence between a manifest covariant quantization and the quantization on a physical gauge, where physical unitary is manifest, breaks down.

Finally we would like to remark that the Dirac algorithm to obtain the constraints on a canonical formulation of a gauge theory has to be supplemented with the following procedure in order to detect the complete set of constraints. At each step of the conservation procedure, from a set of constraints $\phi$ one considers the conservation condition

$$\{H, \phi\} = 0 \quad (3.2)$$

a) If $\phi$ is regular, the algorithm stops when $\{H, \phi\}$ is weakly zero or when only determines Lagrange multipliers, which must then be associated to second class constraints.

b) If $\phi$ is irregular, even when the above conditions are satisfied the algorithm does not stop, one has also to evaluate

$$\{H, \{H, \phi\}\} = 0, \quad (3.3)$$
if \{H, \phi\} is regular then a) applies if it is irregular one has to continue the conservation procedure.

The above remark is based in the following point. If \(\phi = 0\) is a regular constraint
\[\{H, \phi\} \approx 0\]
implies
\[\{H, \ldots \{H, \phi\} \ldots \} = 0\]
for all \(n\).

This remark can be proven as follows; if \(F\) is weakly zero \(F \approx 0\) and \(\phi\) are regular then
\[F = \Sigma a\phi\]
This property is not valid in general when \(\phi\) are irregular constraints.

4. Canonical Covariant Quantization of the Brink-Schwarz Superparticle.

The first order action for the ten dimensional BS superparticle is
\[S = < P_\mu \partial_\tau \chi^\mu + \overline{\eta} P \partial_\tau \xi + eP^2 >\]  
(4.1)
where \(e\) is a Lagrange multiplier associated to the constraint
\[P^2 = 0 \quad .\]  
(4.2)

Let \(\eta\) be the momenta canonically conjugate to \(\xi\). Since the action (4.1) is first order in \(\partial_\tau \xi\) its dynamics is restricted by,
\[\phi = \eta - P\xi = 0 \quad .\]  
(4.3)

The canonical Hamiltonian action of the system is
\[S = < P_\mu \partial_\tau \chi^\mu + \overline{\eta} \partial_\tau \xi + eP^2 + \overline{\psi}(\eta - P\xi) >\]  
(4.4)
where \(\psi\) are Lagrange multipliers associated to the constraints (4.3).

Constraints (4.3) are a combination of first and second class ones. This is best observed computing the Poisson algebra of the constraints which yields
\[\{\phi, \phi\} = 2 P \quad .\]  
(4.5)
Over the manifold defined by (4.2), $\mathcal{P}$ is non-invertible and in fact conservation of (4.3) fixes only half of the multipliers $\bar{\psi}$. The constraints associated to the other half are first class.

One can covariantly project from (4.3) the first class constraints by application of $\mathcal{P}$

$$\varphi \equiv \mathcal{P}\eta = 0 .$$

(4.6)

$$\{\varphi, \varphi\} = 0 = \{\varphi, \phi\}$$

(4.7)

The price one has to pay is that (4.6) is a set of infinite reducible constraints since

$$\mathcal{P}\varphi \equiv 0 .$$

(4.8)

Over the manifold defined in (4.2) and (4.6) constraints (4.3) become reducible since

$$\mathcal{P}\phi = 0$$

(4.9)

holds identically. The manifold defined by (4.2) and (4.3) may thus be equivalently described by the first class infinite reducible constraint (4.2), (4.6) and the reducible second class constraints (4.3). In order to write the effective action of the system one has to include the ghost fields adequate for the first class reducible constraints $\varphi$ and devise a method to handle the second class reducible constraints (4.3).

This decomposition of the constraints into first class and reducible second class ones, over the manifold of first class constraints, is best understood by introducing tranverse + longitudinal ($T+L$) decomposition of the geometrical object. Although $\mathcal{P}$ is not an invertible matrix it serves to define such decomposition of spinors. Due to the identity

$$\mathcal{P}\gamma^+ + \gamma^+ \mathcal{P} = \mathcal{P}-\Lambda 1$$

(4.10a)

for any spinors $\xi$ one has the ($T+L$) decomposition

$$\xi = \xi_T + \gamma^+\xi_L$$

(4.10b)

with

$$\mathcal{P}\xi_T = 0 .$$

(4.10c)

$\xi_L$ is not uniquely defined but $\gamma^+\xi_L$ and $\xi_T$ are uniquely determined.
Equation (4.9) imposes that the longitudinal part of $\phi$, more precisely $\gamma^+ \phi_L$, be identically zero over the manifold of first class constraints. The true content of the reducible constraints (4.3) is then only

$$\phi_T = 0$$  \hspace{1cm} (4.11)

Let us translate the above situation to a general notation. We have a constrained system with Hamiltonian $H_0$ subject to a set of reducible constraints $\phi_{a_1}$ ($a_1 = 1, \cdots, n$) and a set of first class constraints $\varphi_i$ ($i = 1, \cdots, k$) which we omit in the explicit construction that follows. We limit ourselves to remark on the modifications to be done when included. So we have,

$$\phi_{a_1} = 0 \hspace{1cm} (4.12)$$

$$a_{a_2}^{a_1} \phi_{a_1} = 0 \hspace{1cm} a_1 = 1 \cdots n, \hspace{0.1cm} a_2 = 1 \cdots m \hspace{1cm} (4.12a)$$

We will not suppose $a_{a_2}^{a_1}$ to be of maximal rank. Instead we will impose that a $(T+L)$ decomposition similar to (4.10) is allowed.

We have then for any object $V_{a_1}$

$$V_{a_1} = V_{a_1}^T + A_{a_1}^{a_2} V_{a_2}^L$$

$$a_{a_2}^{a_1} V_{a_1}^T = 0, \hspace{1cm} V_{a_2}^L = a_{a_2}^{a_1} V_{a_1}$$

and for any object $W^{a_1}$

$$W^{a_1} = W^{a_1}_T + a_{a_2}^{a_1} W^{a_2}_L$$

$$A^{a_2}_{a_1} W^{a_1}_T = 0, \hspace{1cm} W^{a_2}_L = A^{a_2}_{a_1} W^{a_1}_T$$

It follows that

$$V_{a_2}^L = a_{a_2}^{a_1} A^{b_2}_{a_1} V_{b_2}^L, \hspace{1cm} W^{a_2}_L = A^{a_2}_{a_1} a_{a_2}^{a_1} W^{b_2}_L$$

and

$$W^{a_1} V_{a_1} = W^{a_1}_T V_{a_1}^T + W^{a_2}_L V_{a_2}^L$$

In the irreducible case ($a_{a_2}^{a_1}$ of a maximum rank) $A^{a_2}_{a_1}$ is the inverse of $a_{a_2}^{a_1}$. In the finite reducible case this decomposition may always be done in a unique way for a given pair $A, a$. For infinite reducible system, we will assume that there exists such a
decomposition. The constraints (4.12) are second class in the sense that they have an invertible Poisson Bracket matrix in the transverse sub-space.

Following Ref. [7] and [8] let us enlarge the phase space using a set of auxiliary variables $\xi^{a_1}$ and $\eta^{b_1}$ conjugate to each other. We also introduce the combinations

$$
\Phi^{a_1} = \eta^{a_1} - \frac{1}{2} \omega^{a_1 b_1} (p, q) \xi^{b_1} \\
\bar{\Phi}^{a_1} = \eta^{a_1} + \frac{1}{2} \omega^{a_1 b_1} (p, q) \xi^{b_1} .
$$

(4.14)

Here $\omega^{ab}$ is an antisymmetric matrix with vanishing Poisson Bracket with itself to be fixed by the procedure. $\Phi$ and $\bar{\Phi}$ satisfy

$$
\{ \Phi^{a_1}, \Phi^{b_1} \} = -\omega^{a_1 b_1} \\
\{ \Phi^{a_1}, \bar{\Phi}^{b_1} \} = \omega^{a_1 b_1} \\
\{ \bar{\Phi}^{a_1}, \bar{\Phi}^{b_1} \} = 0 .
$$

(4.15)

In order to introduce only the complications necessary to deal with the case of the BS superparticle we will suppose in the following that $\omega_{a_1 b_1}$ is transverse

$$
a^{a_1}_{a_2} \omega_{a_1 b_1} = 0 \quad a_1 = 1 \cdots n \quad a_2 = 1 \cdots m
$$

(4.16)

and invertible in the transverse space. Now we extend the constraints in the enlarged space to

$$
\tilde{\phi}^{a_1} = \phi^{a_1} + V^{c_1}_{a_1 b_1} \Phi^{c_1} = 0
$$

(4.17)

where $V^{c_1}_{a_1} (q, p)$ is also to be fixed. In general the first class constraints $\phi$ may also have to be extended in order the complete set of extended constraints be first class. In this case of the superparticle, however, the extension is not necessary. We assume $V^{b_1}_{a_1}$ to be invertible. In this case we impose the constraints (4.17) to be irreducible, first class and with structure functions at most linear in $\Phi^{a_1}$. We then have

$$
\{ \tilde{\phi}^{a_1}, \tilde{\phi}^{b_1} \} = U^{c_1}_{a_1 b_1} \tilde{\phi}^{c_1} = -2(u^{c_1}_{a_1 b_1} + v^{d_1}_{a_1 b_1} \Phi^{d_1}) \tilde{\phi}^{c_1} .
$$

(4.18)

The structure functions $U^{c_1}_{a_1 b_1}$ may depend on the phase space variables $p$ and $q$. Substitution of (4.17) in (4.18) yields.

$$
\{ \phi^{a_1}, \phi^{b_1} \} - V^{c_1}_{a_1 b_1} V^{d_1}_{b_1} \omega_{c_1 d_1} + 2u^{c_1}_{a_1 b_1} \phi^{c_1} = 0 \\
\{ \phi^{a_1}, V^{c_1}_{b_1} \} + \{ V^{c_1}_{a_1}, \phi^{b_1} \} + 2v^{d_1}_{a_1 b_1} \phi^{d_1} + 2u^{d_1}_{a_1 b_1} V^{c_1}_{d_1} = 0 \\
\{ V^{c_1}_{a_1}, V^{d_1}_{b_1} \} + \{ V^{d_1}_{a_1}, V^{c_1}_{b_1} \} + 2v^{e_1}_{a_1 b_1} v^{d_1}_{e_1 b_1} + 2u^{e_1}_{a_1 b_1} V^{e_1 c_1}_{a_1 b_1} = 0 .
$$

(4.19)
We suppose here that

\[ \{ \phi_{a_1}, \Phi_{1a_1} \} = 0, \quad \{ V_{1a_1}^b, \Phi_{1c_1} \} = 0. \quad (4.20) \]

Let us suppose that we are able to find a solution to (4.20) with all the required conditions. In order to demonstrate the equivalence of our system in the enlarged phase space to the original system we have to impose additional restriction besides (4.17). A counting of the degrees of freedom suggests which ones should be chosen. The original model has 2\(N\) phase space variables \(p, q\) restricted by \((n - m_L)\) transverse constraints with \(m_L\) the rank of \(a_{a_1}^{a_2}\). The enlarged model has 2\(N\) variables \(p, q\) and 2\(n\) variables \(\xi, \eta\) restricted by \(n\) constraints \(\tilde{\phi}_{a_1}\), and \(n\) gauge fixing conditions \(\tilde{\chi}_{a_1}\). To match we need \((n - m_L)\) additional constraints. We take them to be

\[ \Phi^{a_1 \top} = 0. \quad (4.21) \]

Since

\[ [\Phi^{a_1 \top}, \Phi^{a_2 \top}] = \omega^{a_1 a_2} \quad (4.22) \]

the constraints (4.21) are in our hypothesis second class. The advantage of this formulation is that the field dependence in \(\omega^{a_1 a_2}\) (whose determinant will appear in functional measure) may be simpler than in \(\{ \phi_{a_1}, \phi_{b_1} \}\) since \(V_{a_1}^{a_2}\) may be also a field dependent object. This justifies the enlarging of the phase space and the modification of the constraints. Moreover iterating the process one can hope to obtain a field independent functional measure and a pure gauge model. For the BS superparticle, as we will show below infinitely many iterations are needed to this end, but in other cases only finite steps may be necessary.

A gauge invariant extension of the hamiltonian \(H_0\) may be written in the form [11]

\[ \tilde{H} = H_0 + h^{a_1} \Phi_{a_1}. \quad (4.23) \]

\(h^{a_1}\) is fixed imposing

\[ \{ \tilde{H}, \tilde{\phi}_{a_1} \} = W_{a_1}^{b_1} \tilde{\phi}_{b_1}. \quad (4.24) \]

Introducing the ghost variables \(C^{a_1}\) and \(\mu_{a_1}\) the BRST operator is obtained by solving [6]

\[ \{ \Omega, \Omega \} = 0 \quad (4.25a) \]

\[ \frac{\partial \Omega}{\partial C^{a_1}} \bigg|_{\mu=0} = \tilde{\phi}_{a_1}. \quad (4.25b) \]
It takes the form
\[ \Omega = C^{a_1} \tilde{\phi}_a + C^{a_1} U_{a_1 b_1}^{c_1} C^{b_1} \mu_{c_1} + \cdots \] (4.25c)
with a non-trivial tail, when the algebra of first class constraints has structure functions of a higher order. In the general case when first class constraints \( \varphi \) are also present, one has to include, of course, associated ghost fields and condition (4.25b) must also be satisfied for the extended first class constraints \( \tilde{\varphi} \).

The extended hamiltonian is then obtained by solving [5]
\[ \{ \tilde{H}, \Omega \} = 0 \]
\[ \tilde{H} \bigg|_{\mu=0} = \tilde{H} \] . (4.26)

The BRST invariant effective action in a phase space representation is given by [5]
\[ S_{eff} = \langle p \dot{q} + \mu_{a_1} \dot{C}^{a_1} + \eta_{a_1} \dot{\xi}^{a_1} - \hat{H} + \delta(\lambda^{a_1} \mu_{a_1}) + \delta(C_{a_1} \chi^{a_1}) \rangle \] (4.27)
where \( \chi^{a_1} \) are the gauge fixing conditions and \( \hat{\delta} \) is defined by
\[ \hat{\delta}F = [\Omega, F] \] (4.28)
for any function \( F \) of the canonical variables of the enlarged superphase-space. For the non-canonical sector we have
\[ \hat{\delta}\lambda^{a_1} = \theta^{a_1} , \quad \hat{\delta}\theta^{a_1} = 0 \] (4.29a)
\[ \hat{\delta}C_{a_1} = B_{a_1} , \quad \hat{\delta}B_{a_1} = 0 \] (4.29b)
\[ \delta\mu_{a_1} = \tilde{\varphi}_{a_1} \] . (4.29c)

We claim that the gauge invariant system defined by (4.27) and constrained by (4.21) is canonically equivalent to the original system. To prove this, we will show that with an adequate gauge fixing condition one can reduce the path integral corresponding to the enlarged system to the Senjanovic-Fradkin expression for the original system.

The classical gauge transformation law for \( \xi \) is
\[ \delta\xi^{a_1} = \{ \xi^{a_1} , \epsilon^{b_1} \tilde{\phi}_{b_1} \} = V^{a_1}_{b_1} \epsilon^{b_1} \] (4.30)
where \( \epsilon^{b_1} \) are the infinitesimal parameters of the transformation.
We may then choose the gauge conditions

\[ \chi^{a_1} = \xi^{a_1} \quad . \] (4.31)

Using (4.25), (4.28) and (4.29) we have

\[
\hat{\delta}(\overline{C}_{a_1} \chi^{a_1}) = B_{a_1} \chi^{a_1} - \overline{C}_{a_1} \delta \chi^{a_1} C^{b_1} + O(\mu)
\]

\[
= B_{a_1} \xi^{a_1} - \overline{C}_{a_1} V_{b_1}^{a_1} C^{b_1} + O(\mu) \] (4.32)

where \( O(\mu) \) may appear if the structure functions depend explicitly on the phase space coordinates. We also have

\[
\hat{\delta}(\lambda^{a_1} \mu_{a_1}) = \lambda^{a_1} \tilde{\phi}_{a_1} + \theta^{a_1} \mu_{a_1}
\]

\[
= \lambda^{a_1} _T \phi_{a_1} + \lambda^{a_2} _L a_{a_2} \phi_{a_1} + \theta^{a_1} \mu_{a_1}
\]

\[
= \lambda^{a_1} _T \tilde{\phi}_{a_1} + \lambda^{a_2} _L a_{a_2} V_{a_1}^{b_1} \Phi_{b_1} + \theta^{a_1} \mu_{a_1} \] (4.33)

The functional integral is

\[
\langle \chi \rangle = \int \mathcal{D}z \delta(\Phi^T)(\det \omega^T)^{1/2} e^{-S_{eff}}
\] (4.34)

where \( \mathcal{D}z \) is the Liouville measure

\[
\mathcal{D}z = \mathcal{D}p \mathcal{D}q \mathcal{D}C \mathcal{D}C \mathcal{D}B \mathcal{D}B \mathcal{D}\theta \mathcal{D}\eta \mathcal{D}\xi
\] (4.35)

Integrating in \( \theta \) one gets \( \delta(\mu) \) so that in particular \( O(\mu) \) in (4.32) does not contribute. Integrating in \( B_{a_1}, \overline{C}_{a_1}, \) and \( \lambda^{a_2}_L \) and using Eq.(4.19) the factor in the measure of (4.34) becomes

\[
(det \omega^T)^{1/2} \delta(\Phi^T) \delta(\xi) \delta(V_{Tb_1}^{a_1} C_{T}^{b_1}) \delta(a_{b_2}^{b_1} V_{b_1}^{a_1} A_{a_1}^{a_2} C_{b_2}^{b_1}) \delta(a_{a_2}^{a_1} V_{a_1}^{b_1} \Phi_{b_1}) =
\]

\[
(det \omega^T)^{1/2} \delta(\eta^T) \delta(\xi) \delta(V_{Tb_1}^{a_1} C_{T}^{b_1}) \delta(a_{b_2}^{b_1} V_{b_1}^{a_1} A_{a_1}^{a_2} C_{L}^{b_2}) \delta(a_{a_2}^{a_1} V_{a_1}^{b_1} A_{b_1}^{a_2} \eta_{b_2}^{L})
\] (4.36)

In (4.36) the arguments of the last two factors have opposite statistics. Hence

\[
\delta(a_{b_2}^{b_1} V_{b_1}^{a_1} A_{a_1}^{a_2} C_{L}^{b_2}) \delta(a_{a_2}^{a_1} V_{a_1}^{b_1} A_{b_1}^{a_2} \eta_{b_2}^{L}) = \delta(C_{L}^{b_2}) \delta(\eta_{a_2}^{L})
\] . (4.37)

This can be taken as valid even in the case of \( a_{a_2}^{a_1} \) and \( A_{b_2}^{b_1} \) being non invertible. The factor in the measure reduces to

\[
(det \omega^T)^{1/2} \delta(\eta) \delta(\xi) \delta(C) \det V_{T}^{T}
\] . (4.38)
Now we note from (4.19) that

$$\left(\text{det} \omega^T\right)^{1/2} \text{det} V_T^T = \left(\text{det} \{\phi_T, \phi_T\}\right).$$  \hspace{1cm} (4.39)$$

Doing the trivial integrals in $\eta, \xi$ and $C$, we finally obtain

$$I = \int \mathcal{D}q \mathcal{D}p \mathcal{D}\lambda^T (\text{det} \{\phi_T, \phi_T\}) \exp - \langle p\dot{q} - H + \lambda_T \phi^T \rangle$$  \hspace{1cm} (4.40)

which is the correct Senjanovic-Fradkin expression of the functional integral of this system.

The discussion above only supposes the uniqueness of decomposition (4.13). For the superparticle we identify $a_{a_1}^a$ with $\mathcal{P}$ and $A_{a_1}^{a_2}$ with $\gamma^+/P_-$. The $T + L$ decomposition is given by (4.11). Solving (4.19) by taking $V$ as a Dirac $\delta$ we have

$$\omega = -2 \mathcal{P}. \hspace{1cm} (4.41)$$

In terms of the auxiliary Majorana spinors $\eta_1$ and $\xi_1$ we have

$$\Phi_1 = \eta_1 + \mathcal{P} \xi_1 \hspace{1cm} (4.42a)$$

$$\Phi_1^\dagger = \eta_1 - \mathcal{P} \xi_1. \hspace{1cm} (4.42b)$$

The enlarged constraints (4.17) are in this case

$$\Phi_0 = \eta - \mathcal{P} \xi + \Phi_1 \hspace{1cm} (4.43)$$

The additional restrictions corresponding to (4.21) are

$$\Phi_1^\dagger = 0 \hspace{1cm} (4.44)$$

Since we choose $V_{a_1}^{b_1}$ to be field independent the factor $\text{det} \{\Phi_1^\dagger, \Phi_1^\dagger\}^{1/2}$ in the measure of functional integral appears in principle in this case as problematic as the factor $\text{det} \{\phi^T, \phi^T\}^{1/2}$ in the direct approach. Nevertheless we observe that constraint (4.44) is equivalent to reducible constraint

$$\mathcal{P} \Phi_1 |_{\text{first class}} = \mathcal{P} \eta_1 \equiv 0, \hspace{1cm} (4.45a)$$

$$\Phi_1 = 0. \hspace{1cm} (4.45b)$$
We iterate now the process and introduce $\xi_2$, $\eta_2$ and $\omega_2$. We obtain again

\[
\begin{align*}
\omega_2 &= -2P \\
\Phi_2 &= \eta_2 + P\xi_2 \\
\bar{\Phi}_2 &= \eta_2 - P\xi_2
\end{align*}
\]  
(4.46)

and we have the new constraints

\[
\begin{align*}
\bar{\phi}_1 &= \bar{\Phi}_1 + \Phi_2 \\
\bar{\Phi}_2^\top &= 0
\end{align*}
\]  
(4.47a, 4.47b)

For the same reason as above we take instead of (4.47b) the reducible constraint

\[
\begin{align*}
P\Phi_2 \mid_{\text{first class}} &= P\eta_2 \equiv 0 \\
\bar{\Phi}_2 &= 0.
\end{align*}
\]  
(4.48a, 4.48b)

and continue the process. After $\ell$ steps we have

\[
\begin{align*}
\bar{\phi}_{i-1} &= \bar{\Phi}_{i-1} + \Phi_i, \quad i = 1, \ldots, \ell \\
\bar{\Phi}_\ell^\top &= 0
\end{align*}
\]  
(4.49)

with $\bar{\Phi}_0 \equiv \phi$.

At this level the classical action may be written in terms of the canonical variables in the form

\[
S_\ell = \left< P_\mu \dot{x}^\mu + \sum_{i=0}^{\ell} \eta_i \dot{\xi}_i + \lambda P^2 + \sum_{i=0}^{\ell-1} \bar{\psi}^{\dot{i}} P\eta_i + \sum_{i=0}^{\ell} \lambda^{i} \bar{\phi}_{i-1} \right>.
\]  
(4.50a)

subject to the second class constraints

\[
\bar{\Phi}_\ell^\top = 0.
\]  
(4.50b)

In (4.50a) $\bar{\psi}$ is the Lagrange multiplier associated to the i-esim analog to (4.45a) and (4.48a). The constraints $\bar{\phi}_\ell$ in (4.49) are irreducible, the other constraints being infinite reducible. At each level $\ell$ the formulation (4.50) is not Lorentz covariant due to the transverse projection in (4.50b).

In order to avoid this problem we may introduce infinite auxiliary fields. We then obtain

\[
S_\infty = \left< P_\mu \dot{x}^\mu + \sum_{i=0}^{\infty} \eta_i \dot{\xi}_i + \lambda P^2 + \sum_{i=0}^{\infty} \bar{\psi}^{\dot{i}} P\eta_i + \sum_{i=0}^{\infty} \lambda^{i} \bar{\phi}_{i-1} \right>.
\]  
(4.51)
This is the action proposed by Kallosh in [12] which is associated to the BRST charge with the correct cohomology for the BSSP. The effective action associated to (4.51) may be truncated at any level $\ell$ by imposing the gauge fixing conditions (4.31) for $i = \ell + 1, \cdots, \infty$, and the effective action associated to (4.50) is regained. In the limit case $P\eta_i = 0$ and $\bar{\phi}_i = 0$ $i = 0, \cdots, \infty$ are regular infinite reducible first class constraints. Other approaches for the quantization of the superparticle may be found in [14].

5. On the Covariant Quantization of GSSS

We are now going to extended the approach of section 4 in order to apply it to the GSSS. The constraints (2.5) can be decoupled into left and right sectors

$$\phi_- \equiv \phi_1 - \phi_2$$

$F^1 = 0$ \hspace{1cm} (5.1a)

and

$$\phi_+ \equiv \phi_1 + \phi_2$$

$F^2 = 0$ \hspace{1cm} (5.2a)

We analyse here the problem related to sector (5.1), that is the extension of the phase of section 2 in order to obtain a regular formulation with first class constraints only which reduces off-shell to the left sector of GSSS, with the right canonical functional measure.

We consider the reducible set of constraints

$$H_- = \phi_- + 2F^1\theta^1 = 0$$

$F^1\Gamma_1 = 0$ \hspace{1cm} (5.3a)

$F^1 = 0$ \hspace{1cm} (5.3b)

The $\Gamma_1$ martrices satisfy the property

$$\Gamma_1\Gamma_1 = 2H_- 1,$$ \hspace{1cm} (5.4)

hence the first class constraints (5.3b) are infinite reducible with respect to $\Gamma_1$. The
procedure of section 5 may be generalized as follows, we consider the set of restrictions

\begin{align}
H_- &= 0 \\
\chi^- &= 0 \\
F^1 \Gamma_1 &= 0 \\
\chi_1 &= 0 \\
F^1 &= 0
\end{align} \tag{5.5}

where \(\chi^-\) and \(\chi_1\) are covariant gauge fixing functions associated to \(\phi_-\) and \(F^1 \Gamma_1\) respectively.

(5.5a), (5.5d) are a reducible subset which we shall denote \(\varphi_i\), while \(\chi^-\), \(\chi_1\) are going to be denoted by \(\chi^j\). The constraints (5.5) are seconds class. We now extended (5.5) to obtain a set of first class ones; we enlarge the phase space with auxiliary canonical coordinates \((\mu_j, \rho^j)\) and \((\eta, \xi)\) as follows

\begin{align}
\tilde{\varphi}_i &= \varphi + M^j_i \mu_j + N_{ij} \rho^j \\
\tilde{\chi}^j &= \chi^j + L^j_i \mu_i + \rho^j \\
\tilde{F}^1 &= F^1 + \Phi + P_i \mu_i + Q_j \rho^j
\end{align} \tag{5.6}

where \(\Phi = \eta + \omega \xi\) (see section 4).

It can be shown that associated to system (5.1) or (5.2) there exist \(M, N, P, Q\) such that (5.6) are first class constraints. Moreover, (5.6a) is reducible with respect to \(\Gamma_1\).

It can be also proven that by imposing the partial gauge fixing conditions \(\mu_i = 0\) and \(\chi^j = 0\) associated to (5.6a) and (5.6b) respectively, the functional measure corresponding to (5.6a) and (5.6b) reduces exactly to the Fradkin-Senjanovic measure associated to the constraints \(\phi_-\) and \(F^1 \Gamma_1\).

Constraints (5.6) may be reexpressed as follows

\begin{align}
\hat{\varphi}_i &= \tilde{\varphi}_i + \tilde{M}^j_i \mu_j \\
\hat{\chi}^j &= \tilde{\chi}^j + \rho^j \\
\hat{F}^1 &= \tilde{F}^1 + \Phi
\end{align} \tag{5.7}

where \(\tilde{\varphi}_i, \tilde{\chi}^j\) and \(\tilde{F}^1\) are independent of the auxiliary fields \(\rho, \mu\) and \(\Phi\).
We consider now the set of first class constraints (5.7) together with the second class constraint
\[ \Phi^i + R^i j \mu_j + S_j \rho^j = 0 \] (5.8)
where \( \Phi = \eta - \omega \xi \), and \( R^i, S_j \) can be determined in order that (5.7) are first class constraints. This equation may be reexpressed, by taking linear combinations of (5.7) and (5.8), as
\[ \hat{\Phi} = 0 \] (5.9)
with no dependence on the auxiliary fields.

Eq. (5.7) and (5.9) are the generalization to the superstring case of the superparticle constraints (4.2), (4.17) and (4.21).

The procedure is now exactly as in section 4. We obtain
\[ \hat{\varphi}_i = 0 \]
\[ \hat{\chi}^j = 0 \]
\[ \hat{F}^1 + \Phi = 0 \]
\[ \hat{\Phi} + \Phi_1 = 0 \]
\[ \hat{\Phi}_1 + \Phi_2 = 0 \]
\[ \vdots \]
\[ \hat{\Phi}_n + \Phi_{n+1} = 0 \quad n \to \infty \] (5.10)
which are all first class.

The reduction procedure which we have explicitly shown in section 4 for the superparticle problem can be extended to this case. Moreover it can be shown that it is independent of the \( \chi^j \) function used in (5.10).

It is interesting to compare the formulation of this section with the one in section 4 where no use was made of this constraints \( \tilde{\chi} = 0 \). In those cases the original first class sector commutes with everything else, in particular with \( \omega \) and hence with any extension of the original second class sector. There is no need of extending the original first class sector. However if one introduces the \( \tilde{\chi} = 0 \) restriction as for the more general GSSS case, it can be shown that the \((\rho, \mu)\) auxiliary sector decouples and can be eliminated by functional integration ending up with the original formulation of section 4.

It remains to analyse the admissible set of gauge fixing conditions associated to (5.10) as well as the explicit construction of the BRST charge which nevertheless will
have necessarily the correct cohomology since if is going to be obtained from a regular canonical formulation which reduces correctly to the GSSS. This will be presented with a detailed analysis of our formulation elsewhere [15].

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