PROBLEMS OF EXISTING UNCONDITIONAL SECURITY PROOFS IN QUANTUM KEY DISTRIBUTION

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Abstract

It is repeatedly and persistently claimed in the literature that a specific trace criterion $d$ would guarantee universal composition security in quantum cryptography. Currently that is the sole basis of unconditional security claim in quantum key distribution. In this paper, it is shown that just security against known-plaintext attacks when the generated key is used in direct encryption is not provided by $d$. The problem is directly connected with several general problems in the existing unconditional security proofs in quantum key distribution. A number of issues will be clarified concerning the nature of true security, privacy amplification, key generation rate and the mathematical approach needed for their determination in concrete protocols.

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I. INTRODUCTION AND SUMMARY

In quantum key distribution (QKD) there have been many security proofs offered on the "unconditional security" of various protocols of the BB84 variety. (For a recent review see ref [1].) Until 2004-2005 and in many papers till the present day, the security criterion adopted is the attacker Eve’s quantum accessible information ($I_{ac}$) on the generated key $K$, which is the maximum mutual information Eve has on $K$ from a measurement result on her probe she may set during the key generation process. Security of $K$ before it is actually used is called "raw security" [2], to distinguish it with composition security when $K$ is actually used in an application for which Eve may possess additional information related to $K$. In particular, when $K$ is used for encryption, part of $K$ may be known to Eve in a known-plaintext attack (KPA) to help her get at the rest of $K$. KPA takes a particularly simple form when $K$ is used in the often suggested one-time pad format.

While "universal" composition security is a complicated matter which is perhaps not needed in its full generality, KPA security is necessary because that is one main weakness of conventional symmetric key ciphers QKD purports to overcome. Indeed, there is otherwise no need for QKD since its raw security is worse than that of conventional ciphers in which the key is also typically totally hidden by uniformly random data [2]. In this paper all security under discussion is information-theoretic (IT), and symmetric key cipher is the proper comparison with QKD, not purely complexity-based cipher such as RSA. This is because a shared message authentication key is necessary in QKD during key generation, and in any case a short shared secret key can always be employed.

It was claimed in [3] that an exponentially small $I_{ac}$ (for an $n$-bit key $K$) would guarantee universal composition security and that is applicable to most previous security proofs. The claim was established through an inequality between $I_{ac}$ and a trace distance criterion that has been given by the notation "$d$" in many papers since [4], an abbreviation we adopt in this paper. This $d$ is supposed to give the trace distance between the states of an "ideal" protocol and the "real" protocol under Eve’s attack. It was shown in [4] through an explicit construction involving quantum information locking that exponentially small $I_{ac}$ does not imply KPA security, specifically the last bit of $K$ may be leaked deterministically when $n - 1$ bits of $K$ are known to Eve from a KPA. This small leak has been enlarged to a "spectacular
failure" of the $I_{ac}$ guarantee, under which it is possible that "leakage of even a logarithmic number of key bits compromises the secrecy of all the others". [5]

The remedy, according to [4-7], is to use the criterion $d$ directly. Indeed, $d$ is the only basis of QKD unconditional security claim at present including any use of privacy amplification [1, footnote 20], [6-7]. There are three different "interpretations" on what $d \leq \epsilon$ asserts, each of which is claimed to imply universal composition security. We would concentrate on KPA security in this paper, which is much simpler and can be treated directly. The three interpretations are:

(i) "The real and the ideal setting can be considered to be identical with probability at least $1 - \epsilon$". [6]

(ii) The parameter $\epsilon$ can be understood as the "maximum failure probability" of the real protocol, i.e., the maximum probability that the real protocol "deviates from the behavior of the ideal protocol". [8]

(iii) "Distinguishability advantage" between the real and the ideal protocols is bounded by $\epsilon$. [3]

In this paper all three interpretations will be analyzed, only briefly on (i) because [2] already shows that (i) cannot be true. With (ii) interpreted with respect to a specific scenario so that it is different from the more general (i), it is refuted by a specific KPA counter-example. In fact, $d$ could be interpreted as the difference between two probabilities but it does not have a probability interpretation itself. We will explain why (iii) does not lead to KPA security in general. A different criterion $d'$ is needed for such interpretation.

In sum, there is no QKD unconditional security proof at all against attacks with quantum memory. The ramification of this security failure will be elaborated. The actual QKD security situation will be discussed in regard to the secure key generation rate, privacy amplification, and the necessity of using $M$-ary quantum detection theory in quantifying fundamental security performance unless $d'$ is bound.
II. CLASSICAL VARIATIONAL DISTANCE AND QUANTUM TRACE DISTANCE

The classical variational distance \( v(P, Q) \) between two probability distributions \( P = \{p_i\} \) and \( Q = \{q_i\} \) on the same sample space is given by [9]

\[
v(P, Q) = \frac{1}{2} \sum_i |p_i - q_i|
\]

with \( 0 \leq v \leq 1 \). The guarantee \( v \leq \varepsilon \) is equivalent to, for any event \( \mathcal{E} \), the probabilities of \( \mathcal{E} \) from \( P \) and \( Q \) satisfy

\[
| p(\mathcal{E}) - q(\mathcal{E}) | \leq 2\varepsilon
\]

Indeed we have [9, p.299]

\[
2v(P, Q) = \max \varepsilon \left| p(\mathcal{E}) - q(\mathcal{E}) \right|
\]

An important case for our purpose is when \( Q \) equals the uniform distribution \( U, u_i = 1/N \) for sample space of size \( N = 2^{|K|} \) while Eve has distribution \( P \) for \( K \). Then (2) shows

\[
| p(\tilde{K}) - \frac{1}{2^{|K|}} | \leq 2\varepsilon
\]

for any subset \( \tilde{K} \) of \( K \). Thus, (4) shows that any \( m \)-bit subsequence \( \tilde{K} \) of \( K \) also has a probability different from that of a uniform distribution by at most \( 2\varepsilon \). In particular, if \( \varepsilon \leq 2^{-n} \), it shows \( P \) and \( U \) are not much different quantitatively at all. However, when \( \frac{1}{2^{-n}} \gg 1 \), \( P \) may be very different from \( U \) in regards to the possible \( p(\mathcal{E}) \) even when \( \varepsilon \) is exponentially small, say \( \varepsilon = 2^{-n/2} \). Whether something is small in a cryptographic context has to be judged with respect to the key length or data length with exponentiation if appropriate.

The quantum trace distance between two density operators \( \rho \) and \( \sigma \) on the same state space is

\[
D(\rho, \sigma) = \frac{1}{2} \| \rho - \sigma \|_1
\]

with \( 0 \leq D \leq 1 \). It can be readily shown that \( D(\rho, \sigma) \leq \varepsilon \) implies \( v(P, Q) \leq \varepsilon \) for any quantum measurement which gives \( P \) and \( Q \) from \( \rho \) and \( \sigma \) [10]. By using the basis that
diagonalizes $\rho - \sigma$, $D(\rho, \sigma)$ itself can be achieved by a measurement in the form $\nu(P, Q)$. Thus, we have the equivalence of variational distance with trace distance as a criterion.

It is important to stress that $D(\rho, \sigma) \leq \varepsilon$ does not imply that $\rho$ and $\sigma$ are close, similar to $\nu(P, Q) \leq \varepsilon$ does not imply $P$ and $Q$ are close, unless $\varepsilon$ is small enough. Incorrect understanding of the security situation would result if the quantitative level of $\varepsilon$ relative to $2^{-n}$ is not attended to for an $n$-bit or $n$-qubit sequence. This is due to the large freedom of $P$, in particular $p_1$, that is possible under such a constraint for fixed $\sigma$ or $Q$. This has been emphasized in [2,11-13].

III. PROBLEM FORMULATION AND RAW SECURITY GUARANTEE

During the key generation process, Eve sets her probe and the protocol goes ahead after intrusion level estimation. We assume that everything goes well on the user’s end. At whatever time when Eve measures on her probe, she would obtain a whole probability distribution on correctly estimating the different possible values of $K$ [13]. Classically the quantitative raw security problem can be formulated as follows. We will use upper case letter for a random variable (vector) with its specific value denoted by the corresponding lower case letter.

Let $X$ be an $m$-bit data sequence random variable picked by user $A$ and $Y$ Eve’s observation random variable of any possible length and alphabet size. The transition probability $p(y|x)$ and a priori distribution $p(x)$ are fixed by the cryptosystem and chosen attack. The user $B$ observes the random variable $Z$ specified by the cryptosystem, applies an openly known known error-correcting code (ECC) to get a data estimate $\hat{X}(Z)$ which is presumably error free, and then an openly known privacy amplification code (PAC) to yield a final generated key $K$. The ECC and PAC can be combined to yield directly $K(Z)$. From $Y$ Eve forms her estimate $K(Y)$. The timing of Eve’s knowledge of various openly known codes is implicit in the possible $p(y|x)$ she could obtain.

With Bayes rule and the known ECC+PAC, Eve forms from $y$ the conditional probability distribution (CPD) on $K$, $p(k|y)$, which gives Eve’s success probability of getting the entire $k$ for each possible value of $K$. We will use $P = \{p_i\}, i \in \{1, \ldots, N\}$, for this CPD, suppressing the dependence on $y$. Any single-number criterion on $K$, be it mutual information or
variational distance, merely expresses a constraint on $P$. The Markov Inequality \cite{9} can be used to convert an average constraint to an individual one for a nonnegative random variable, here it is $p(k|Y)$ for each $k$ and random $Y$. We order $p_i$ so that $p_1 \geq p_2 \geq \ldots \geq p_N$. Thus, $p_1$ is Eve’s optimal probability of estimating $K$ correctly given $y$. It is a most significant number concerning the security of $K$, as we will see.

With $I(K;Y)$ denoting the mutual information between any two random variables $K$ and $Y$, we use the following notations

$$\delta_E \equiv v(P,U), \quad I_E \equiv I(P;U)$$

For simplicity, we take the data $X$ to be uniformly distributed and the same for $K$ obtained from it via ECC+PAC, the ideal situation. Thus $\delta_E$ and $I_E$ in (6) are indeed single-number measures of Eve’s "information" on $K$.

Note that it is not sufficient to employ a criterion that would give perfect IT security when it has its limiting value, say $\delta_E = 0$ or $I_E = 0$, but using it for a relatively large nonzero value. The issue is a quantitative one and whether the security guarantee is adequate depends on the exact value that can be obtained in a concrete protocol, as we will see.

We have shown in \cite{11-13} that for $I_E = 2^{-l'}, l' > 0$, Eve’s maximum probability of getting the whole $K$ can be as big as

$$p_1 \sim 2^{-l}, \quad l = l' + \log n$$

Unless $l \sim n$, the raw security guarantee of $I_E \leq 2^{-l'}$ is very far from that of a uniform key. The subsets of $K$ suffer similarly \cite{13}. When $l'$ approaches $n - \log n$, more exact estimate of $p_1$ \cite{16} needs to be used in lieu of $2^{-l}$ since $l$ cannot exceed $n$. The practical experimental value of $l' \sim 21$ for $n \sim 4000$ \cite{14, 15} is quite an inadequate guarantee, especially after the application of Markov Inequality \cite{2,13}. Generally, “exponentially small in $n$” can be very misleading because the rate $\lambda$ in $l = \lambda n$ is the real crux of the security situation. We will see this repeatedly in the following.

The $\delta_E$ guarantee suffers a similar problem \cite{2,13} because for $\delta_E \leq 2^{-l}$, the averaged (over
Y) $\overline{p}_1$ can be as big as

$$\overline{p}_1 = 2^{-l} - \frac{1}{N} \quad \text{with} \quad \delta_E = 2^{-l} \quad (8)$$

Thus, unless $l \sim n$ as indicated after (4) above, a $\delta_E \leq \epsilon$ raw security guarantee is not really better than that of $I_E \leq \epsilon$.

IV. INCORRECTNESS OF INTERPRETATIONS (i) AND (ii)

Let $\rho^k_E$ be Eve’s probe state when $K$ has value $k$ with probability $p_0(k)$ before B measures, $p_0(k) = U$ in the ideal case. The possible $\rho^k_E$ are limited by the users’ intrusion level estimation. Let

$$\rho_K \equiv \sum_k p_0(k) \left| k \right\rangle \left\langle k \right| \quad (9)$$

be the $p_0(k)$-mixed state on $N$ orthonormal $| k \rangle$’s. Let $\rho_E$ be the $K$-averaged state, and $\rho_{KE}$ the joint state

$$\rho_E \equiv \sum_k p_0(k) \rho^k_E \quad (10)$$

$$\rho_{KE} \equiv \sum_k p_0(k) \left| k \right\rangle \left\langle k \right| \otimes \rho^k_E \quad (11)$$

The criterion $d$ is defined to be,

$$d \equiv \frac{1}{2} \| \rho_{KE} - \rho_K \otimes \rho_E \|_1 \quad (12)$$

A key satisfying $d \leq \epsilon$ is called "$\epsilon$-secure" by definition [6] in the case $p_0(k) = U$.

The "lemma 1" of [6] and [16] was given the following interpretation on $v(P, Q) \leq \epsilon$: the two random variables $K$ and $Y$ described by $P$ and $Q$ take on the same value with probability $\geq 1 - \epsilon$. (The lemma says there exists a joint distribution of $K$ and $Y$ which gives this result. That joint distribution is in fact the optimal one for this interpretation.) Given this incorrect interpretation as premise, it can be validly deduced [6, p.414],[2] that under $d \leq \epsilon$, "the real and the ideal setting can be considered to be identical with probability at least $1 - \epsilon$". As discussed in [2,13,17], this interpretation of $d$ is not a consequence of "lemma 1" in [6] or
[16] but an incorrect interpretation of that lemma 1. We may note here that there is no physically meaningful joint distribution that gives $P$ and $Q$ as marginals other than the product distribution $PQ$ which applies in this situation. Thus, the two random variables $K$ and $Y$ would take the same value only as a result of random collision with probability $1/N$, $N$ the size of the sample space, even when $P$ and $Q$ are the same distribution. As concluded in [2], interpretation (i) is simply false and not just unproven.

Going onto interpretation (ii), observe that its wording in [8] is very ambiguous. It can mean either interpretation (i), or (iii) with $\varepsilon$ as the probability difference between the real and the ideal cases. We would give this "failure probability" a distinct literal interpretation from the words of [8], since it is the sole basis of the QKD unconditional security claim in the recent review [1]. In lieu of random variable identity or coincidence of (i), we restrict (ii) to apply just to specific KPA scenarios in which performance can be readily quantified. The following simple counter-example shows such interpretation (ii) cannot be expected to hold, not just unproven.

Consider the following simplest information locking example, for a two-bit $K$ with $\rho^k_E = \rho^{k_1} \otimes \rho^{k_2}$ for the two bits $k_1$ and $k_2$. Let $|i\rangle$, $i \in \{1, 2, 3, 4\}$, be the four BB84 states on a qubit, with $\langle 1|3 \rangle = \langle 2|4 \rangle = 0$. Let $P_i$ be the projectors into $|i\rangle$, and

$$
\begin{align*}
\rho^{11}_E &= \frac{1}{2}(P_1 \otimes P_1 + P_3 \otimes P_2) \\
\rho^{10}_E &= \frac{1}{2}(P_1 \otimes P_3 + P_3 \otimes P_4) \\
\rho^{01}_E &= \frac{1}{2}(P_2 \otimes P_1 + P_4 \otimes P_2) \\
\rho^{00}_E &= \frac{1}{2}(P_4 \otimes P_3 + P_4 \otimes P_4)
\end{align*}
$$

Thus, $k_2$ is locked into the second qubit through $k_1$, and is unlocked by measuring on the 1-3 or 2-4 basis given the knowledge of $k_1$. This $\rho^k_E$ does not yield a $\rho_E = I/4$, but since $d$ is equal to [6, lemma 2]

$$
d = \frac{1}{2} E_K[\| \rho^k_E - \rho_E \|_1],
$$

let us evaluate "ideal" comparison $\| \rho^k_E - I/4 \|_1$ which is easily computed to be $1/2$. However, knowing $k_1$ implies $k_2$ is compromised for sure, not with a maximum failure probability $1/2$, contradicting the interpretation (ii) in this specific situation. (Note that the $d \leq \varepsilon$ guarantee is supposed to apply to any $\rho^k_E$ in (12)).
Indeed, any locking information scenario provides a counter-example to (ii) similar to the example of (13). Let \( I_{ac} = 2^{-t'} \) and with a \( \rho_E^k \) that leaks the rest of \( K \) from its \( l \) bits according to (7). The corresponding \( d \) must be less than 1 since \( D(\rho, \sigma) = 1 \) if and only if \( \rho \) and \( \sigma \) have orthogonal ranges.

It does not appear there is any probability meaning one can sensibly give to \( \varepsilon \) in \( d \leq \varepsilon \), which is just a numerical measure of the difference between two density operators similar to \( \varepsilon \) between \( P \) and \( Q \) in \( v(P, Q) \leq \varepsilon \). There is simply no basis to assign probability distribution to the security situation after all the parameters are fixed, i.e., there is no more random system parameter that could give rise to such probability distribution. The incorrect probability interpretation of \( \varepsilon \) in \( v(P, Q) \leq \varepsilon \) is responsible for the incorrectness of interpretation (i). Here we see that any probability interpretation of \( \varepsilon \) itself, whatever the "failure probability" may be, would fail similarly.

V. SECURITY FAILURE UNDER INTERPRETATION (iii)

Going on to interpretation (iii), note the huge difference between it and the other two interpretations. According to (i) and (ii), \( d = 1/2 \) in the example of (13) is the probability Eve can succeed, which is not true. According to interpretation (iii), Eve may succeed at \( 1/2 + d = 1 \), which is true (we actually use \( d' \) to get the ideal situation.)

It may be observed that \( \varepsilon \) in \( D(\rho, \sigma) \leq \varepsilon \) is not the success probability of distinguishing \( \rho \) from \( \sigma \) by a measurement. That is given by the well known [18] probability \( P_c \) of correct decision,

\[
P_c = \frac{1}{2} + \frac{1}{2} D(\rho, \sigma)
\]  

(15)

Note that Eve is not usually trying to make a binary decision with her attack. A major source of confusion may have arisen from calling \( \rho \) and \( \sigma \) "\( \varepsilon \)-indistinguishable" when \( D(\rho, \sigma) \leq \varepsilon \), as if \( \rho \) and \( \sigma \) can only be distinguished with probability \( \sim \varepsilon \). Actually just (15), or (2) for any measurement, is the mathematical statement of "\( \varepsilon \)-indistinguishable". Any other claimed consequence needs to be mathematically expressed and derived from this mathematical given. Such development has not been provided for universal composition security (which is different from "composability" of the criterion), not just for KPA. Before going into the KPA issue we
will show that (iii) is not generally true even for raw security, because the "ideal" situation is not captured by $\rho_U \otimes \rho_E$, as follows.

Consider a term $\| \rho_E^k - \rho_E \|_1$ in (14) apart from averaging over K. It implies

$$v(p(y|k), p(y)) \leq 2\varepsilon$$

for Eve’s observation $Y$ on her probe. This $v$ does not compare the real situation to the ideal one unless $p(y) = U(y)$. Generally, this means $\rho_E$ should be a completely random state proportional to the identity operator $I$ for finite-dimensional state spaces. However, there is no reason to expect that to be the case. Indeed, there is no way the users can guarantee that since $\rho_E$ depends on Eve’s chosen attack in contrast to a classical scenario with a single complete Y. Thus $\rho_E$ cannot be $\propto I$ for all attacks and probe measurements, and there is no justification to consider $\rho_U \otimes \rho_E$ a general representation of the "ideal" situation when it is inside a trace, although it does as a whole state (or probability distribution) since $\rho_E$ is independent of $k$. This is why interpretation (i) does lead to universal composition as well as perfect raw security with a high probability, and why (i) or (ii) is crucial for an unconditional security claim.

Notice that under (iii), $\varepsilon$ is merely a single-number quantitative measure of difference, and thus has much weaker meaning than the equality of whole state or distribution. Indeed, it is clear that the level of $\varepsilon$ becomes crucial, as we saw in section II, even if it is measured with respect to the "ideal".

The following different criteria $d'$ should be used for interpretation (iii),

$$d' \equiv \frac{1}{2} E_K[\| \rho_E^k - \frac{I}{N'} \|_1],$$

where $N'$ is the dimension of the range of $\rho_E$. Note that there is representation problem with infinite-dimensional qumodes.

The distributions that lead to (7) appear to be of the form suitable for information locking with small $I_E$. Indeed, in [5] there is an $l'$ factor in addition to $O(\log n)$ in the expression for the number of unlocking bits in the key segment, exactly as in (7). In any case, complete raw and composition security would obtain if the required number of bits to increase $p_1$ to
1 goes up to $n$. For $I_{ac} \leq 2^{-l'}$, this would happen at

$$l' \geq n - \log n$$

(18)

For the case $d' \leq 2^{-l}$ or $\delta_E \leq 2^{-l}$, it would happen at

$$l \geq n$$

(19)

It may be observed that KPA may significantly lower $p(\tilde{K})$, Eve’s success probability of getting any $\tilde{K} \subseteq K$, to an unacceptable level without deterministically compromising the whole $K$. Partial information locking of $K$ must, therefore, be dealt with also in a fundamental security analysis. The KPA case alone already shows that there is no universal composition security guarantee from $d$, at least when it is below a certain quantitative level. This is in fact a problem of any single-number criterion, but we will not go into the general issue in this paper.

When $l$ bits unlocks the other $n - l$ bits of $K$, the information locking $\rho^k_E$ that shows interpretation (i) and (ii) are false does not show (iii) is false if $d'$ is used in place of $d$ and $d' \geq 1 - 1/2^{n-l}$. In the following section we give a specific though unrealistic scenario in which interpretation (iii) would fail for $d$.

VI. RELATION BETWEEN HOLEVO QUANTITY $\chi$ AND THE CRITERION $d$

The classical form of $d$, say as obtained from a measurement, is

$$\delta = \frac{1}{2} v(p(y|k)p_0(k); p(y)p_0(k))$$

(20)

The following simple relation between the above $\delta$ and the classical mutual information $I(K; Y)$ is an immediate consequence of the well known [9, p.300] relation between relative entropy and variational distance by considering $p(y, k)$ relative to $p(y)p_0(k)$. We have

Lemma 1:
The $\delta$ of (16) is upper bounded by $I(K; Y)$ in the form

$$2\delta^2 \leq I(K; Y)$$

(21)
From (21), one obtains the weak bound,

**Lemma 2:**
The criterion $d$ is upper bounded by the quantum accessible information $I_{ac}$ that Eve can get from her probe

$$2d^2 \leq 2^{|K|} I_{ac} \tag{22}$$

**Proof:**
From (14) each term, $\| \rho_E^k - \rho_E \|_1$, is bounded by a measurement result $Y^{(k)}$ satisfying (21). Thus,

$$d \leq E_K \left[ \frac{I(K = k; Y^{(k)})}{2} \right]^\frac{1}{2} \tag{23}$$

By Jensen’s Inequality,

$$d \leq \left[ E_K \frac{I(K = k; Y^{(k)})}{2} \right]^\frac{1}{2} \tag{24}$$

which is bounded as (22) by adding many nonnegative terms for each $Y^{(k)}$ inside the $[.]^{1/2}$ of (24) to get $\sum_k I(K; Y) = 2^{|K|} I(K; Y)$.

It is on the basis of equ(16) in [3], which is equivalent to (22), that the incorrect conclusion is drawn in [3] that exponentially small $I_{ac}$ would guarantee composition security in previous security proofs. Our (18) or (22) shows that the exponent needs to be nearly all of $n$ for such conclusion to hold. In the case of [14] with $n \sim 4000$, this means the exponent needs to be as big as $l' \sim 3880$.

The proper quantum generalization of Lemma 1 is not Lemma 2 but the following **Lemma 3:**
The Holevo quantity [10],

$$\chi = S(\rho_E) - E_K [S(\rho_E^k)] \tag{25}$$

bounds $d$ in the form,

$$2d^2 \leq \chi \tag{26}$$
Proof:

Similar to the classical (21), (26) follows from theorem 5.5 of [19] with the quantum relative entropy $S(\rho \parallel \sigma)$ for $\rho = \rho_E^k$ and $\sigma = \rho_K \otimes \rho_E$.

Since the security criterion is supposed to work for each and every $\rho_E^k$, consider the one that leads to $\chi$ insecurity. Let $\chi = 2^{-2m}$, $m > 0$. Thus from (26), $d \leq 2^{-m}$ but it is insecure. However, since $I_{ac} \leq \chi$, an $I_{ac}$ insecurity does not imply a $\chi$ insecurity. Although $I_{ac}/n = \chi/n$ asymptotically [20,21], the total $I_{ac}$ and $\chi$ are not necessarily close for large $n$. One scenario that is the case is when blocks of such $n$ bits are repeated $n'$ times themselves for large $n'$, which is however not realistically applicable to concrete protocols.

The next theorem shows that $\chi$ and $d$ have similar exponential behavior, and thus are actually similar security criteria.

Theorem 1:
Let $h(\cdot)$ be the binary entropy function. Then

$$2d^2 \leq \chi \leq 8dn + 2h(2d) \quad (27)$$

Proof:

The lower bound is Lemma 3. The upper bound is an immediate consequence of the theorem in [22], again using $S(\rho \parallel \sigma)$ for $\rho = \rho_E^k$ and $\sigma = \rho_K \otimes \rho_E$.

With $d = 2^{-l}$ and $\chi = 2^{-l''}$, the exponents when non-negative are related from (27) for $n \geq l$ as follows,

$$l - \log n - 4 \leq l'' \leq 2l \quad (28)$$

Basically (27)-(28) shows that the exponents of $d$ and $\chi$ for almost any $n$ are within a factor of two.

VII. IT SEMANTIC SECURITY AND PRIVACY AMPLIFICATION

What kind of raw security guarantee on $K$ one should have that is comparable to that of a uniform key? One can introduce the notion of information theoretic semantic security directly as

$$|p(\tilde{K}) - \frac{1}{2^{|K|}}| \leq \epsilon(\tilde{K}) \quad (29)$$
where \( \tilde{K} \) is any subset of \( K \) and \( \epsilon(\tilde{K}) \) is allowed to vary depending on \( \tilde{K} \) in contrast to (4). Such "semantic security" in complexity-based cryptography has been developed extensively [23] and generalized in an IT context [24]. However, in the context of IT physical cryptography in noise we should dispense with any algorithm in the definition and consider the full correlated statistical behavior of the system model. Thus, (29) expresses the direct comparison with the ideal uniform key. In fact, as long as \( \epsilon(\tilde{K}) \) is small enough, such as \( \epsilon(\tilde{K}) = 2^{-|\tilde{K}|} \), we would not need to require that it can be driven to zero. It would be quite adequate, e.g., if \( \epsilon \) is a constant \( = 2^{-n} \) for an \( n \)-bit \( K \) as discussed in section II.

In the case one can guarantee only \( \epsilon = 2^{-m} \) for \( m < n \), it follows immediately that no IT semantically secure key (with arbitrarily small \( \epsilon \)) can be obtained by any further processing on \( K \) which is longer than \( m \). This is an immediate consequence of the fact that \( p_1 \) cannot be improved by any known deterministic transformation on \( K \). Indeed, the original \( p_1 \) that results from Eve's measurement \( Y \) before ECC+PAC also cannot be improved with such codes, i.e., not by the transformation from \( Y \) to her estimate of \( K(Y) \). Thus, the IT semantically secure key rate is reduced from the nominal one by a factor \( m/n \).

Let us consider the security of such an \( m \)-bit key \( K_r \) derived from the \( n \)-bit \( K \). When it is obtained from a \( d' \) guarantee of (17), all the subset probabilities \( p(\tilde{K}) \) Eve may get by any measurement is properly bounded from (2). The users can guarantee \( \frac{\epsilon}{m} \leq 2^{-m} \) for sufficiently large \( m \) under any \( \rho_{E}^{K} \) Eve can launch that passes intrusion level estimation. However, in this case the resulting \( p(\tilde{K}) \) or \( p(\mathcal{E}) \) bound for \( \tilde{K} \) would not be quantum mechanically fundamental because Eve could attack a specific subset \( \tilde{K} \) of \( K \) by an optimal measurement directed toward that subset instead of the whole \( K \). Thus she has a \( 2^{|\tilde{K}|} \)-ary detection problem instead of a \( 2^{|K|} \)-ary one. Specifically, consider \( K \) in two parts \( K_1, K_2 \) (\( K = K_1K_2 \)). In a KPA knowing \( K_1 = k_1 \), the state to Eve is \( \rho_{E}^{k_1K_2} \) and she has a \( 2^{|K_2|} \)-ary quantum detection problem instead of the original \( 2^{|K|} \)-ary one. Her optimum \( 2^{|K_2|} \)-ary quantum detection performance cannot in general be obtained from the \( 2^{|K|} \)-ary performance and subsequent classical reduction to \( 2^{|K_2|} \)-ary case. Quantum mechanically there is no complete measurement which covers all such possibilities while maintaining performance, but there is classically.

The essential point is that quantum detection theory [18, 25] is the proper approach here for optimal performance analysis, not "information theory" in the narrow sense. Thus, the
raw security guarantee on \( p(\tilde{K}) \) we have discussed is also not ultimate either except for the total \( K \) itself. Note that the effect of PAC on fundamental quantitative security also needs to be ascertained by quantum detection theory. The alternative is to bound \( d' \).

While IT semantic security cannot be improved by any privacy amplification, let us consider the approach of leaving \( m \) in \( I_{ac} \leq \epsilon = 2^{-m} \) much less than \( n \), and keeping the original key generation rate. For example, with \( n \sim 1000 \) we assume \( m \sim 40 \) is sufficiently secure which gives \( p_1 \sim 2^{-50} \) before Markov Inequality is applied and better than \( 10^{-6} \) afterward [2]. In this manner, the above limit on key generation rate may also be extended. An important question is whether one can use PAC to further improve such "relaxed security" without full IT semantic security.

The answer is unknown. The standard reference [26] on general possibility of privacy amplification starts with a Renyi entropy constraint on Eve which is rarely used in security proofs. It is a good measure of "collision" but not "uncertainty". Eve’s total Shannon entropy is always reduced by PAC, though her mutual information may decrease with respect to a shorter key. However, it is not known what PAC could reduce Eve’s mutual information, total or per bit. It appears the answer depends strongly on her CPD on \( K \).

On the other hand, the criterion \( d \) has been used [6-7] for secure privacy amplification that makes \( d \) small, given prior lower bound on Eve’s Shannon entropy or Renyi entropy. The failure of \( d \) for interpretations (i)-(iii) shows that such privacy amplification cannot guarantee security against KPA, and \( d' \) needs to be used instead.

Indeed, let \( p_1 \) be given by (7) before or after PAC, say guaranteed by an \( I_{ac} \leq \epsilon = 2^{-\nu} \). Then knowing \( m = l' + \log(n) \) bits of the \( n \)-bit \( K \) may unlock the rest if Eve knows \( m \) data bits when \( K \) is used in one-time pad encryption. With the above numerical example, that means it has not been ruled out that knowing 50 bits of data may lead to knowledge of the other unknown 950 bits. Thus, the relaxed security scenario would not provide a good enough security guarantee even if PAC is useful. Similarly, the situation is the same under the \( d \leq \epsilon \) guarantee. On the other hand, with full semantic security where only \( \leq m \) bits of \( K \) are generated, such problem would not occur, subject to the above qualification of Eve’s optimal detection under the \( I_{ac} \) guarantee or assuming \( \rho_E \) produces \( U(y) \) upon Eve’s measurement under the \( d \) guarantee. If one ignores the KPA problem for QKD, the question arises as to why not then just use conventional key expansion? The answer lies in the fact
that there is no IT security against KPA at all in conventional ciphers. So QKD is an improvement when Eve has no quantum memory [2]. In such application, one can of course no longer claim "unconditional security".

VIII. PROBLEMS OF UNCONDITIONAL SECURITY PROOFS IN QKD

The criterion $d$ fails to provide KPA security and composition security in general. If $I_{ac} \leq \epsilon = 2^{-l}$ or $d' \leq \epsilon = 2^{-l}$ for $l \geq n$, the KPA security problem does not arise under the qualification described in the last section. In addition, as discussed in section III and reinforced by [27], that is (exponentially) impossible to achieve with the usual key generation rate, and also cannot be achieved by privacy amplification. Furthermore, the key so generated in a concrete protocol is unlikely to be long enough to cover the message authentication key bits spent during key generation, such as in the case of [14]. See ref [2].

There is no "unconditional security" guarantee in QKD. Furthermore, we have now the following broader fundamental QKD security problems.

(i) There is no proof of security against known-plaintext attacks when the generated key $K$ is used in direct encryption and Eve possesses quantum memory, or in other composition security context.

(ii) The fundamental raw security level of Eve’s probability of correctly estimating any proper subset of $K$ is not bounded under either an $I_{ac}$ or $d$ constraint.

(iii) The true secure key rate is far smaller and is determined by quantitative error exponents, the later rarely analyzed in security proofs and for which $M$-ary quantum detection theory would be needed.

(iv) It is not clear what privacy amplification can achieve according to what security measure. It surely cannot improve information theoretic semantic security.

With such difficulties on the foundation of quantum key distribution, it appears radically new approaches are appropriate for fundamental security guarantee.
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