Metastability for Non-Linear Random Perturbations of Dynamical Systems

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Abstract

In this paper we describe the long time behavior of solutions to quasi-linear parabolic equations with a small parameter at the second order term and the long time behavior of corresponding diffusion processes.

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1 Introduction

Consider a dynamical system
\[ \dot{X}^x_t = b(X^x_t), \quad X^x_0 = x \in \mathbb{R}^d, \]  \hspace{1cm} (1)

together with its stochastic perturbations
\[ dX^{x, \varepsilon}_t = b(X^{x, \varepsilon}_t)dt + \varepsilon \sigma(X^{x, \varepsilon}_t) dW_t, \quad X^{x, \varepsilon}_0 = x \in \mathbb{R}^d. \]  \hspace{1cm} (2)

Here \( \varepsilon > 0 \) is a small parameter, \( W_t \) is a Wiener process in \( \mathbb{R}^d \), and the coefficients \( \sigma \) and \( b \) are assumed to be Lipschitz continuous. The diffusion matrix \( a(x) = (a_{ij}(x)) = \sigma(x)\sigma^*(x) \) is assumed to be uniformly positive definite.

Together with (2), we can consider the corresponding Cauchy problem
\[ \frac{\partial u^\varepsilon(t,x)}{\partial t} = L^\varepsilon u^\varepsilon := \frac{\varepsilon^2}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u^\varepsilon(t,x)}{\partial x_i \partial x_j} + b(x) \cdot \nabla_x u^\varepsilon(t,x), \quad x \in \mathbb{R}^d, \quad t > 0, \]  \hspace{1cm} (3)

\[ u^\varepsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \]  \hspace{1cm} (4)

where \( g \) is a bounded continuous function.

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Suppose for a moment that the vector field \( b \) has just one asymptotically stable equilibrium point \( O \) such that all the points get attracted to \( O \) and \( (b(x), x - O) \leq -c|x - O| \) for some positive constant \( c \) and all sufficiently large \( |x| \). Then it is easy to check that

\[
\lim_{(\varepsilon, t) \to (0, \infty)} P(X^x_{t,\varepsilon} \in U) = 1
\]

for any neighborhood \( U \) of the equilibrium \( O \). Taking into account that the solution \( u^\varepsilon \) of (3)-(4) can be written in the form \( u^\varepsilon(t, x) = E_g(X^x_{t,\varepsilon}) \) and the continuity of \( g \), we conclude that

\[
\lim_{(\varepsilon, t) \to (0, \infty)} u^\varepsilon(t, x) = g(O).
\]

A similar result holds in the case of a unique compact global attractor if the system (1) has a unique normalized invariant measure on the attractor. This is the case, for example, if system (1) in \( \mathbb{R}^2 \) has a unique limit cycle attracting all the trajectories except the unstable equilibrium inside the cycle.

The situation becomes more complicated if the dynamical system has more than one asymptotically stable attractor. Assume, for brevity, that all the attractors are equilibriums \( O_1, \ldots, O_n \). Let \( D_i \) be the basin of \( O_i \), \( 1 \leq i \leq n \), and assume that the set \( \mathbb{R}^d \setminus (D_1 \cup \ldots \cup D_n) \) belongs to a finite union of surfaces of dimension \( d - 1 \). The long time behavior of \( X^x_{t,\varepsilon} \) and \( u^\varepsilon(t, x) \) is now determined by the transitions between the attractors \( O_1, \ldots, O_n \). These transitions are described by the large deviation theory for stochastic perturbations of dynamical systems developed in the late 1960-s (see [7] and references there). In particular, the weak limit \( \mu \) of the invariant measure \( \mu^\varepsilon \) of the family of processes (2) was found. In the generic case, the limiting measure \( \mu \) is concentrated on one of the attractors, which will be denoted by \( O^* \). Then

\[
\lim_{\varepsilon \downarrow 0} \lim_{t \to \infty} u^\varepsilon(t, x) = g(O^*).
\]

However, in the case of many attractors, the limiting behavior of \( X^x_{t,\varepsilon} \) and \( u^\varepsilon(t, x) \) as \( \varepsilon \downarrow 0 \) and \( t \to \infty \) depends on the way in which \( (\varepsilon, t) \) approaches \( (0, \infty) \). Roughly speaking, under natural additional assumptions, there exist a finite number of regions in the neighborhood of \( (0, \infty) \) such that the limiting distribution of \( X^x_{t,\varepsilon} \) and the limit of \( u^\varepsilon(t, x) \) exist if \( (\varepsilon, t) \) approaches \( (0, \infty) \) while staying inside one region. For different regions, these limits are, in general, different.

The corresponding theory of metastability (of sublimiting distributions) was developed in [3] (see also [3], [7], [9]). The notion of a hierarchy of cycles, which is discussed below, was introduced there. Let \( S_{0,T}(\varphi) \) be the action functional for the family \( X^x_{t,\varepsilon} \) in \( C([0, T], \mathbb{R}^d) \) as \( \varepsilon \downarrow 0 \) ([7]):

\[
S_{0,T}(\varphi) = \frac{1}{2} \int_0^T \sum_{i,j=1}^d a^{ij}(\varphi_t)(\dot{\varphi}_t^i - b_i(\varphi_t))(\dot{\varphi}_t^j - b_j(\varphi_t))dt, \quad T \geq 0, \quad \varphi \in C([0, T], \mathbb{R}^d),
\]
where \(a^{ij}\) are the elements of the inverse matrix, that is \(a^{ij} = (a^{-1})_{ij}\), and \(S_{0,T}(\varphi) = +\infty\) if \(\varphi\) is not absolutely continuous. The quasi-potential is defined as

\[
V(x, y) = \inf_{t, \varphi} \{S_{0,T}(\varphi) : \varphi \in C([0, T], \mathbb{R}^d), \varphi(0) = x, \varphi(T) = y\}, \quad x, y \in \mathbb{R}^d.
\]

Note that while the term “quasi-potential” is normally applied to the function \(V\) of the variable \(y\) with \(x\) being a fixed equilibrium point, we use the same term for the function of two variables. The hierarchy of cycles is determined by the numbers

\[
V_{ij} = V(O_i, O_j), \quad 1 \leq i, j \leq n.
\]

The equilibriums \(O_1, ..., O_n\) are the cycles of rank zero. In the generic case, for each \(O_i\) there exists a unique “next” equilibrium \(O_i = \mathcal{N}(O_i)\) defined by \(V_{ii} = \min_{k \neq i} V_{ik}\). For all sufficiently small \(\delta > 0\), with probability close to one as \(\varepsilon \downarrow 0\), the process \(X^{\varepsilon, \delta}_{t}\) that starts in a \(\delta\)-neighborhood of \(O_i\) will enter a \(\delta\)-neighborhood of \(\mathcal{N}(O_i)\) before visiting the basins of any of the equilibriums other than \(O_i\) and \(\mathcal{N}(O_i)\). The time before the process enters the neighborhood of \(O_i = \mathcal{N}(O_i)\) is logarithmically equivalent to \(\exp(V_{ii}/\varepsilon^2)\). If the sequence \(O_i, \mathcal{N}(O_i), \mathcal{N}^2(O_i) = \mathcal{N}(\mathcal{N}(O_i)), ..., \mathcal{N}^n(O_i), ...,\) is periodic, that is \(\mathcal{N}^n(O_i) = O_i\) for some \(n\), then a cycle of rank one appears. It contains the cycles or rank zero \(O_i, \mathcal{N}(O_i), ..., \mathcal{N}^{n-1}(O_i)\). If \(\mathcal{N}^n(O_i) \neq O_i\) for any \(n \geq 1\), we say that \(O_i\) forms a cycle of rank one. The entire set of equilibriums is decomposed into cycles of rank one, which will be denoted by \(C^1_1, ..., C^r_{m_r}\). Note that some of the cycles of rank one may consist of one cycle of rank zero.

Next, the transitions between cycles of rank one can be considered. Namely, in the generic case, for each cycle \(C^1_i\) there is a different cycle \(\mathcal{N}(C^1_i)\) of rank one determined by \(V_{ij}\), \(1 \leq i, j \leq n\), with the following property: if the process starts at one of the equilibrium points in \(C^1_i\), then, with probability close to one as \(\varepsilon \downarrow 0\), it will enter a \(\delta\)-neighborhood of one of the equilibrium points inside the cycle \(\mathcal{N}(C^1_i)\) before visiting basins of any of the equilibriums outside \(C^1_i\) and \(\mathcal{N}(C^1_i)\). This leads to the decomposition of the set of cycles or rank one into cycles of rank two.

This procedure can be continued inductively until we arrive at a single cycle of finite rank \(R\) which contains all the equilibrium points. The cycles of rank \(r \leq R\) will be denoted by \(C^1_1, ..., C^r_{m_r}\).

Let \(T^\varepsilon(\lambda) = \exp(\lambda/\varepsilon^2)\). (The results stated in the paper also hold for \(T^\varepsilon(\lambda) \asymp \exp(\lambda/\varepsilon^2)\), that is if \(\ln T^\varepsilon(\lambda) \sim \lambda/\varepsilon^2\) as \(\varepsilon \downarrow 0\).) In the generic case, there is a finite set \(\Lambda \subset (0, \infty)\) such that for each \(x \in D_1 \cup ... \cup D_n\) and each \(\lambda \in (0, \infty) \setminus \Lambda\), one equilibrium \(O_{M(x, \lambda)}\) is defined such that the measures \(\lambda^\varepsilon(\Gamma) = \mu^\varepsilon(\Gamma) = \mu^\varepsilon(\Gamma)\) converge weakly to the \(\delta\)-measure concentrated at \(O_{M(x, \lambda)}\). The state \(O_{M(x, \lambda)}\) is called the metastable state for the initial point \(x\) and the time scale \(T^\varepsilon(\lambda)\).

In this paper, instead of the linear problem \((3)-(4)\), we will consider the Cauchy problem for the quasi-linear equation with a small parameter

\[
\frac{\partial u^\varepsilon(t, x)}{\partial t} = L^\varepsilon u^\varepsilon := \frac{\varepsilon^2}{2} \sum_{i,j=1}^{d} a_{ij}(x, u) \frac{\partial^2 u^\varepsilon(t, x)}{\partial x_i \partial x_j} + b(x) \cdot \nabla_x u^\varepsilon(t, x), \quad x \in \mathbb{R}^d, \quad t > 0,
\]

(5)
We assume that the coefficients of this equation are Lipschitz continuous and bounded; the matrix \((a_{ij}(x,u))\) is assumed to be uniformly positive definite. Under these conditions, problem (5)-(6) has a unique solution for any continuous bounded \(g(x)\) (see, for instance, \([8]\)).

For \(t > 0\) and \(x \in \mathbb{R}^d\), we can define \(X_{t,x,\varepsilon}^s\), \(s \in [0,t]\), as the process which starts at \(x\) and solves

\[
dX_{t,x,\varepsilon}^s = b(X_{t,x,\varepsilon}^s)ds + \varepsilon \sigma(X_{t,x,\varepsilon}^s, u_{\varepsilon}(t-s, X_{t,x,\varepsilon}^s))dW_s, \quad s \leq t,\]

where the entries \(\sigma_{ij}\), \(1 \leq i, j \leq d\), of the matrix \(\sigma(x, u)\) are Lipschitz continuous and such that \(\sigma \sigma^* = a\). The process \(X_{t,x,\varepsilon}^t\) can be viewed as a nonlinear stochastic perturbation of (1). In the linear case, we have the following relation between \(u_{\varepsilon}\) and the process \(X_{t,x,\varepsilon}^t\):

\[
u_{\varepsilon}(t, x) = E g(X_{t,x,\varepsilon}^t).\]

Conversely, (7)-(8) can be viewed as a system of equations with unknown function \(u_{\varepsilon}\) and process \(X_{t,x,\varepsilon}^t\). Under the above assumptions on the coefficients and the function \(g\), the solution of this system is unique and the function \(u_{\varepsilon}\) is the solution of the problem (5)-(6). This approach of examining systems of the type (7)-(8) was employed in \([4]\) in order to study linear and non-linear degenerate equations. The first initial-boundary value problem for quasi-linear parabolic equation with a small diffusion and the exit problem for the corresponding processes were studied in \([6]\). The results of the latter paper will be used here.

In the nonlinear case, the role of \(V_{ij}\), \(1 \leq i, j \leq n\), will be played by the functions

\[
V_{ij}(c) = \inf_{T, \varphi} \{ S_{0,T}^c(\varphi) : \varphi \in C([0, T], \mathbb{R}^d), \varphi(0) = O_i, \varphi(T) = O_j \}, \quad c \in \mathbb{R},
\]

where \(S_{0,T}^c(\varphi)\) is defined using the function \(a(x,c)\) instead of \(a(x)\) used in the linear case. If \(\lambda\) is sufficiently large, then the distribution of \(X_{T,\varepsilon}^{\lambda}x,\varepsilon\), even in a generic case, converges not necessarily to a \(\delta\)-measure concentrated at an equilibrium point, but to a distribution on the set of equilibrium points. This happens, for example, in the case of two equilibrium points if \(V_{12}(c) = V_{21}(c)\) for some value of \(c\). Therefore, in the case of nonlinear perturbations, the notion of a metastable state should be replaced by the notion of a metastable distribution. Note that metastable distributions arise also when perturbations of nearly-Hamiltonian systems are considered (see \([1]\), \([2]\)), but because of different reasons.

An additional difficulty in analyzing the asymptotics of the process and the solution to the parabolic equation is due to the fact that the entire hierarchy of cycles may change with time in the non-linear case. An example of this phenomenon is provided in Section 6.

In Section 2, we introduce some of the definitions and discuss the notion of the hierarchy of cycles in more detail. In Sections 3 and 4, we consider a system with two equilibriums and a system with three equilibriums on the real line in the case when the hierarchy of
cycles is preserved. In Section 5 we formulate a general result for the case when the hierarchy of cycles is preserved. In Section 6 we study the asymptotics of the solution to the parabolic equation for a system in which a bifurcation in the hierarchy of cycles occurs.

2 Preliminaries and notations

Let \( \alpha(x) \) be a symmetric \( d \times d \) matrix whose elements \( \alpha_{ij}(x) \) are Lipschitz continuous and satisfy \( k|\xi|^2 \leq \sum_{i,j=1}^d \alpha_{ij}(x)\xi_i\xi_j, \ x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d \). Let \( \alpha^{ij} \) be the elements of the inverse matrix, that is \( \alpha^{ij} = (\alpha^{-1})_{ij} \), and \( \sigma \) be a square matrix such that \( \alpha = \sigma \sigma^* \). We choose \( \sigma \) in such a way that \( \sigma_{ij} \) are also Lipschitz continuous.

Let \( S_{0,T}^\alpha \) be the normalized action functional for the family of processes \( X_{t}^{x,\varepsilon} \) satisfying

\[
dX_{t}^{x,\varepsilon} = b(X_{t}^{x,\varepsilon})dt + \varepsilon\sigma(X_{t}^{x,\varepsilon})dW_t, \quad X_{0}^{x,\varepsilon} = x,
\]

where \( b \) is a Lipschitz continuous vector field on \( \mathbb{R}^d \). Thus

\[
S_{0,T}^\alpha(\varphi) = \frac{1}{2} \int_0^T \sum_{i,j=1}^d \alpha^{ij}(\varphi_t)(\dot{\varphi}_i^t - b_i(\varphi_t))\dot{\varphi}_j^t - b_j(\varphi_t))dt
\]

for absolutely continuous \( \varphi \) defined on \([0,T]\), \( \varphi_0 = x \), and \( S_{0,T}^\alpha(\varphi) = \infty \) if \( \varphi \) is not absolutely continuous or if \( \varphi_0 \neq x \) (see [3]). Let \( V^\alpha(x,y) \) be the quasi-potential for the family \( X_{t}^{x,\varepsilon} \) in \( \mathbb{R}^d \), that is

\[
V^\alpha(x,y) = \inf_{T,\varphi} \{ S_{0,T}^\alpha(\varphi) : \varphi \in C([0,T],\mathbb{R}^d), \varphi(0) = x, \varphi(T) = y \}, \quad x, y \in \mathbb{R}^d.
\]

Let \( V_0^\alpha = V^\alpha(O_1, O_2) \). For a given function \( \alpha \), we define inductively the following objects (see [3], [7] for a detailed exposition).

(a) The hierarchy of cycles \( C_1^r, ..., C_{m_r}^r \), \( r \leq R \).

(b) The notion of the “next” equilibrium \( \nu(C_i^r) \) and the “next” cycle \( N(C_i^r) \) of the same rank for a cycle \( C_i^r \) of rank less than \( R \).

(c) The transition rates \( V_{C_i^r, O_j}^\alpha, 1 \leq i \leq m_r, 1 \leq j \leq n, \ O_j \notin C_i^r \), from a cycle to equilibriums outside this cycle.

For \( r = 0 \), we define \( C_1^0 = \{ O_1 \} \), \( V_{C_1^0, O_j}^\alpha = V_{ij}^\alpha \). Assume that the cycles of rank \( r \) and the transition rates from those cycles to equilibrium points have been defined. We define \( O_j \) to be the next equilibrium after \( C_i^r \) if \( \min_{j:O_j \notin C_i^r} V_{C_i^r, O_j}^\alpha \) is achieved at \( j \).

**Assumption A.** The minimum \( \min_{j:O_j \notin C_i^r} V_{C_i^r, O_j}^\alpha \) is achieved for a single value of \( j \).

We will write \( O_j = \nu(C_i^r) \) to express that \( O_j \) is the next equilibrium after \( C_i^r \). We say that the cycle \( C_i^r \) of rank \( r \) is the next after \( C_i^r \) if \( C_i^r \) contains \( \nu(C_i^r) \). We will express this relation by writing \( C_i^r = N(C_i^r) \). Starting from a cycle \( C_i^r \) of rank \( r \), we can form
the sequence $C^r_t, N(C^r_t), N^2(C^r_t), \ldots$ by using the operation “next”. If this sequence is periodic, that is $C^r_t = N^n(C^r_t)$ for some $n$, then the cycles $C^r_t, \ldots, N^{n-1}(C^r_t)$ form a cycle of rank $r + 1$. If $C^r_t \neq N^n(C^r_t)$ for any $n \geq 1$, then $C^r_t$ is said to form a cycle of rank $r + 1$. This way, the collection of all the cycles of rank $r$ is decomposed in a non-intersecting union of cycles of rank $r + 1$.

If $C^r_t, \ldots, C^r_s$ form a cycle of rank $r + 1$, which will be denoted by $\Gamma$, we define $V_{\Gamma, O_j}^{\alpha}$ as

$$V_{\Gamma, O_j}^{\alpha} = \max_{1 \leq m \leq s} V_{C^m_n, \nu(C^m_n)}^{\alpha} + \min_{1 \leq m \leq s} (V_{C^m_n, O_j}^{\alpha} - V_{C^m_n, \nu(C^m_n)})$$

We can continue this procedure until we arrive at a single cycle of rank $R$.

If $\Gamma$ is a cycle, we define $D_{\Gamma} = \cup_{i:O_i \in \Gamma} D_i$. As follows from [3], [7], if the process (9) starts in $D_{\Gamma}$, where $\Gamma$ is a cycle of rank $r < R$, then with probability which tends to one as $\varepsilon \downarrow 0$ it will leave $D_{\Gamma}$ and enter a small neighborhood of $\nu(\Gamma)$ in time $T(\varepsilon) \sim \exp(V_{\Gamma, \nu(\Gamma)}^{\alpha}/\varepsilon^2)$.

Note that the quasi-potential can be defined by (10) even if $\alpha$ has some discontinuities. We shall be particularly interested in the structure of the hierarchy of cycles and the exponential transition times for functions $\alpha$ which are of the form $\alpha = \alpha(x, f(x))$, where $f$ is constant on each $D_i$. The reason for that is that the solution of (5)-(6) is nearly constant inside each of the domains $D_i = \{x \in D_i : \text{dist}(x, \partial D_i) \geq \delta, |x| \leq 1/\delta\}$, $\delta > 0$, $1 \leq i \leq n$, as follows from the following lemma.

**Lemma 2.1.** For every positive $\lambda_0$ and $\delta$ there is positive $\varepsilon_0$ such that

$$|u(\varepsilon(T(\varepsilon)(\lambda), x) - u(\varepsilon(T(\varepsilon)(\lambda), O_i))| \leq \delta$$

whenever $x \in D_i^\delta$, $\varepsilon \leq \varepsilon_0$ and $\lambda \geq \lambda_0$.

**Proof.** A similar lemma was proved in [3], so we only recall the main steps here. The solution at time $T(\varepsilon)(\lambda)$ can be viewed as the solution of the same equation at time $T(\lambda)$ with initial data $g$ replaced by the solution of (5)-(6) at time $T(\varepsilon)(\lambda) - T(\varepsilon)(\lambda)$ (semigroup property). We use a priori estimates to show that the solution at time $T(\varepsilon)(\lambda) - T(\varepsilon)(\lambda)$ is almost constant in a small neighborhood of $O_i$. Instead of (8), we can use the representation

$$u(\varepsilon(T(\varepsilon)(\lambda), x) = \mathbb{E}u(\varepsilon(T(\varepsilon)(\lambda), T(\varepsilon)(\lambda), x, \varepsilon)).$$

Finally, we note that $\mathbb{X}_{T(\varepsilon)(\lambda), 0, \lambda}$ is in a small neighborhood of $O_i$ with high probability if $\lambda$ is sufficiently small and $x \in D_i^\delta$.

\[\square\]

### 3 The case of two equilibrium points

In this section we assume that there are two asymptotically stable equilibrium points $O_1, O_2 \in \mathbb{R}^d$. Let $D_1 \subset \mathbb{R}^d$ be the set of points in $\mathbb{R}^d$ which are attracted to $O_1$ and $D_2 \subset \mathbb{R}^d$ the set of points attracted to $O_2$. Note that in the case of two equilibrium
points, the hierarchy of cycles is always the same: \( O_1 \) and \( O_2 \) are cycles of rank zero, and there is one cycle of rank one which contains both \( O_1 \) and \( O_2 \).

Let \( g_{\min} = \inf_{x \in \mathbb{R}^d} g(x) \) and \( g_{\max} = \sup_{x \in \mathbb{R}^d} g(x) \). Define the functions \( M_{12}, M_{21} : [g_{\min}, g_{\max}] \to \mathbb{R} \) via

\[
M_{12}(c) = V^{a(\cdot, c)}_{O_1, O_2}, \quad M_{21}(c) = V^{a(\cdot, c)}_{O_2, O_1}.
\]

These functions are shown on Figure 1. It is not difficult to check that the constant \( c \) in the second argument of \( a \) can be replaced by any function which equal to \( c \) on \( D_1 \) in the definition of \( M_{12} \) and equal to \( c \) on \( D_2 \) in the definition of \( M_{21} \).

![Figure 1: The case of two equilibrium points](image)

Without loss of generality we may assume that \( g(O_1) \leq g(O_2) \). Let \( \lambda_1 = M_{12}(g(O_1)) \) and \( \lambda_2 = M_{21}(g(O_2)) \). In order to formulate the results on the asymptotics of \( u^ε(T^ε(\lambda), x) \), we need the functions \( c^1(\lambda) \) and \( c^2(\lambda) \) defined as follows:

\[
c^1(\lambda) = \begin{cases} g(O_1), & \lambda < \lambda_1, \\ \min\{g(O_2), \min\{c : c \in [g(O_1), g(O_2)], M_{12}(c) = \lambda\}\}, & 0 < \lambda < \lambda_1, \\ \lambda, & \lambda \geq \lambda_1. \end{cases}
\]

\[
c^2(\lambda) = \begin{cases} g(O_2), & \lambda < \lambda_2, \\ \max\{g(O_1), \max\{c : c \in [g(O_1), g(O_2)], M_{21}(c) = \lambda\}\}, & 0 < \lambda < \lambda_2, \\ \lambda, & \lambda \geq \lambda_2. \end{cases}
\]

Let \( \lambda_3 = \inf\{\lambda : c^1(\lambda) \geq c^2(\lambda)\} \). Assume that at least one of the functions \( c^1 \) and \( c^2 \) is continuous at \( \lambda_3 \). Let \( c^* = c^1(\lambda_3) \) if \( c^1 \) is continuous at \( \lambda_3 \) and \( c^* = c^2(\lambda_3) \) otherwise.
Let \( \overline{c}^1(\lambda) = \min(c^1(\lambda), c^*) \) and \( \overline{c}^2(\lambda) = \max(c^2(\lambda), c^*) \). On Figure 2, the graphs of \( \overline{c}^1 \) and \( \overline{c}^2 \) are denoted by the thick and the dotted lines, respectively.

The asymptotics of \( u^\varepsilon(T^\varepsilon(\lambda), x) \) is described by the following theorem.

**Theorem 3.1.** Let the above assumptions be satisfied. Suppose that the function \( \overline{c}^1(\lambda) \) is continuous at a point \( \lambda \in (0, \infty) \). Then for every \( \delta > 0 \) the following limit

\[
\lim_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), x) = \overline{c}^1(\lambda)
\]

is uniform in \( x \in D_1^\delta \). Suppose that the function \( \overline{c}^2(\lambda) \) is continuous at a point \( \lambda \in (0, \infty) \). Then for every \( \delta > 0 \) the following limit

\[
\lim_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), x) = \overline{c}^2(\lambda)
\]

is uniform in \( x \in D_2^\delta \).

**Proof.** The proof essentially relies on the results of [6], where a similar statement was proved for the case of the initial-boundary value problem with one equilibrium point inside the domain. We only sketch one of the main steps since it illustrates the ideas involved in the entire proof. Namely, we will show that if \( 0 < \lambda < \lambda_3 \) and \( \overline{c}^1, \overline{c}^2 \) are continuous at \( \lambda \), then

\[
\limsup_{\varepsilon \downarrow 0} \sup_{x \in D_1^\delta} u^\varepsilon(T^\varepsilon(\lambda), x) \leq \overline{c}^1(\lambda)
\]

and

\[
\liminf_{\varepsilon \downarrow 0} \inf_{x \in D_2^\delta} u^\varepsilon(T^\varepsilon(\lambda), x) \geq \overline{c}^2(\lambda).
\]

Due to Lemma 2.1, it is sufficient to show that

\[
\limsup_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), O_1(\lambda)) \leq \overline{c}^1(\lambda), \quad \liminf_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), O_2(\lambda)) \geq \overline{c}^2(\lambda). \tag{15}
\]

Note that there is a positive \( v_0 \) such that for every \( 0 < \delta < v_0 \) there is \( \varepsilon_0 > 0 \) such that

\[
|u^\varepsilon(T^\varepsilon(\lambda), x) - g(O_i)| \leq \delta \tag{16}
\]

whenever \( x \in D^\delta, \ 0 < \varepsilon \leq \varepsilon_0 \) and \( \delta \leq \lambda \leq v_0 \). Indeed, (16) follows from the representation (8) of the solution in terms of the process \( X^\varepsilon_T(\lambda, x, \varepsilon) \) and the fact that \( X^\varepsilon_T(\lambda, x, \varepsilon) \) belongs to a small neighborhood of \( O_i \) with high probability if \( v_0 \) is sufficiently small, as follows from the basic properties of the action functional (see [7]).

If (15) fails for a certain value of \( \lambda \), then due to continuity of the functions \( u^\varepsilon(t, O_i) \) in \( t \), it follows from (16) that for an arbitrarily small \( \delta' > 0 \) there are sequences \( \varepsilon_n \downarrow 0 \) and \( \lambda_n \in [\delta', \lambda] \) such that

\[
u^\varepsilon(t, O_1) \leq \overline{c}^1(\lambda) + \delta', \quad u^\varepsilon(t, O_2) \geq \overline{c}^2(\lambda) - \delta', \quad T^\varepsilon_{\varepsilon_n}(\delta') \leq t \leq T^\varepsilon_{\varepsilon_n}(\lambda_n),\]

where

\[
T^\varepsilon_T(\lambda, x, \varepsilon) = \inf\{t : X^\varepsilon_T(\lambda, x, \varepsilon) = g(O_i), t \geq 0\}\]
and either
\[ u_\varepsilon^n(T_\varepsilon^n(\lambda_n), O_1) = \overline{c}(\lambda) + \delta' \]

or
\[ u_\varepsilon^n(T_\varepsilon^n(\lambda_n), O_2) = \overline{c}(\lambda) - \delta'. \]
Assume, for example, that the former is the case. Take \( \delta'' \in (0, \delta') \) which will be specified later. Due to continuity of \( u_\varepsilon^n(t, O_1) \) in \( t \), we can find a sequence \( \mu_n \in [\delta', \lambda_n) \) such that
\[ u_\varepsilon^n(T_\varepsilon^n(\mu_n), O_1) = \overline{c}(\lambda) + \delta'' \]
and
\[ u_\varepsilon^n(t, O_1) \in [\overline{c}(\lambda) + \delta'', \overline{c}(\lambda) + \delta'] \quad \text{for} \quad t \in [T_\varepsilon^n(\mu_n), T_\varepsilon^n(\lambda_n)]. \]
(17)
We can express \( u_\varepsilon^n(T_\varepsilon^n(\lambda_n), O_1) \) in terms of the process \( X_{T_\varepsilon^n(\lambda_n)}^{T_\varepsilon^n(\lambda_n), O_1, \varepsilon} \) and the solution at the earlier time \( T_\varepsilon^n(\mu_n) \) as follows
\[ u_\varepsilon^n(T_\varepsilon^n(\lambda_n), O_1) = \mathbb{E} u_\varepsilon^n \left( T_\varepsilon^n(\mu_n), X_{T_\varepsilon^n(\lambda_n) - T_\varepsilon^n(\mu_n)}^{T_\varepsilon^n(\lambda_n), O_1, \varepsilon} \right). \]
(18)
Since \( \overline{c} \) is continuous at \( \lambda \), there are arbitrarily small \( \delta' > 0 \) such that \( M_{12}(\overline{c}(\lambda) + \delta') > M_{12}(\overline{c}(\lambda)) = \lambda \). Since \( \lambda_n \leq \lambda \), a process starting at \( O_1 \) and satisfying (9) with
\[ \sigma \sigma^s(x) = a(x, u_\varepsilon^n(T_\varepsilon^n(\lambda_n), O_1)) = a(x, \overline{c}(\lambda) + \delta') \]
will be in an arbitrarily small neighborhood of \( O_1 \) at time \( T_\varepsilon^n(\lambda_n) - T_\varepsilon^n(\mu_n) \) with probability which tends to one when \( \varepsilon_n \downarrow 0 \). This remains true if the constant \( u_\varepsilon^n(T_\varepsilon^n(\lambda_n), O_1) \) is replaced by a function which is sufficiently close to this constant in \( D_\delta^0 \), where \( \delta \) is sufficiently small. Therefore, due to (17) and Lemma 2.1 we can choose \( \delta'' \) sufficiently close to \( \delta' \) so that \( X_{T_\varepsilon^n(\lambda_n) - T_\varepsilon^n(\mu_n)}^{T_\varepsilon^n(\lambda_n), O_1, \varepsilon} \) will be in a small neighborhood of \( O_1 \) with probability which tends to one when \( \varepsilon_n \downarrow 0 \). With \( \delta' \) and \( \delta'' \) thus fixed, we let \( \varepsilon_n \downarrow 0 \) in (18). The left hand side is equal to \( \overline{c}(\lambda) + \delta' \), while the right hand side tends to \( \overline{c}(\lambda) + \delta'' \). This leads to a contradiction which proves that (15) holds. \( \square \)

**Remark.** If \( \lambda > \lambda_3 \), then \( \overline{c} = \overline{c}(\lambda) = c^* \). It is possible to show that the limit
\[ \lim_{\varepsilon \downarrow 0} u_\varepsilon(T_\varepsilon(\lambda), x) = c^* \]
is uniform in \( (x, \lambda) \in B_{1/\delta} \times [\overline{\lambda}, \infty) \) for each \( \overline{\lambda} > \lambda_3 \). Therefore, for each \( \delta > 0 \) and \( \overline{\lambda} > \lambda_3 \) there is \( \varepsilon_0 > 0 \) such that
\[ |u_\varepsilon(t, x) - c^*| \leq \delta \]
whenever \( \varepsilon \in (0, \varepsilon_0) \), \( x \in B_{1/\delta} \) and \( t \geq T_\varepsilon(\overline{\lambda}) \).

Let \( X_{T_\varepsilon(\lambda)}^{T_\varepsilon(\lambda), x, \varepsilon} \), \( s \in [0, T_\varepsilon(\lambda)] \), be the process defined in (7). As follows from the large deviation theory (see [7]), the distribution of the random variable \( X_{T_\varepsilon(\lambda)}^{T_\varepsilon(\lambda), x, \varepsilon} \) will be concentrated near the points \( O_1 \) and \( O_2 \). From Theorem 3.1 and the representation (8) for the solution, we obtain the following theorem.
Theorem 3.2. Suppose that \( g(O_1) < g(O_2) \). If the function \( \overline{c}^1(\lambda) \) is continuous at a point \( \lambda \in (0, \infty) \) and \( x \in D_1 \), then the distribution of the random variable \( X_{T^\lambda(\Gamma_3, x, \varepsilon)} \) converges to the measure \( \mu_1^\lambda = a_1\delta_{x_1} + a_2\delta_{x_2} \), where the coefficients \( a_1 \) and \( a_2 \) can be found from the equations \( \overline{c}^1(\lambda) = a_1g(O_1) + a_2g(O_2) \), \( a_1 + a_2 = 1 \).

If the function \( \overline{c}^2(\lambda) \) is continuous at a point \( \lambda \in (0, \infty) \) and \( x \in D_2 \), then the distribution of the random variable \( X_{T^\lambda(\Gamma_3, x, \varepsilon)} \) converges to the measure \( \mu_2^\lambda = a_1\delta_{x_1} + a_2\delta_{x_2} \), where the coefficients \( a_1 \) and \( a_2 \) can be found from the equations \( \overline{c}^2(\lambda) = a_1g(O_1) + a_2g(O_2) \), \( a_1 + a_2 = 1 \).

If \( \lambda \in (\lambda_3, \infty) \) and \( x \in D \), then the distribution of the random variable \( X_{T^\lambda(\Gamma_3, x, \varepsilon)} \) converges to the measure \( \mu^* = a_1\delta_{x_1} + a_2\delta_{x_2} \), where the coefficients \( a_1 \) and \( a_2 \) can be found from the equations \( c^* = a_1g(O_1) + a_2g(O_2) \), \( a_1 + a_2 = 1 \).

4 Three equilibrium points without changes in the hierarchy of cycles

In this section we assume that there are three asymptotically stable equilibrium points \( O_1, O_2, O_3 \) such that \( g(O_1) \leq g(O_2) \leq g(O_3) \). For \( c_1, c_2, c_3 \in [g_{\text{min}}, g_{\text{max}}] \), let

\[
f_{c_1,c_2,c_3}(x) = c_1\chi_{D_1}(x) + c_2\chi_{D_2}(x) + c_3\chi_{D_3}(x), \quad x \in \mathbb{R}^d.
\]  

Recall the definition of the hierarchy of cycles from Section 2. We will assume that, for each choice of constants \( c_i \in [g_{\text{min}}, g_{\text{max}}] \) in the function \( \alpha = \alpha(x, f_{c_1,c_2,c_3}(x)) \), Assumption A holds and \( O_1 \) and \( O_2 \) form a cycle of rank one. Consequently \( O_1, O_2 \) and \( O_3 \) form a cycle of rank two for each choice of the constants. Define

\[
M_{12}(c) = V_{O_1,O_2}^{a(c)}, \quad M_{21}(c) = V_{O_2,O_1}^{a(c)},
\]

\[
M_{31}(c) = V_{T_{\lambda_3},O_1}^{a(c)}, \quad M_{32}(c) = V_{O_3,U(\lambda)}^{a(c)}.
\]

Let \( \lambda_1 = M_{12}(g(O_1)) \) and \( \lambda_2 = M_{21}(g(O_2)) \). Define functions \( c^1 \) and \( c^2 \) by (13) and (14), respectively. Let \( \lambda_3 = \inf\{\lambda : c^1(\lambda) \geq c^2(\lambda)\} \). Assume that at least one of the functions \( c^1 \) and \( c^2 \) is continuous at \( \lambda_3 \). Let \( c^* = c^1(\lambda_3) \) if \( c^1 \) is continuous at \( \lambda_3 \) and \( c^* = c^2(\lambda_3) \) otherwise. Let \( \overline{c}^1(\lambda) = \min\{c^1(\lambda), c^*\} \) and \( \overline{c}^2(\lambda) = \max\{c^2(\lambda), c^*\} \), \( \lambda < \lambda_3 \). Let \( \lambda_4 = M_{31}(c^*) \) and \( \lambda_5 = M_{32}(g(O_3)) \).

Let us assume that \( \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 \) (see Figure 2). For \( \lambda < \lambda_3 \), the behavior of the solution in \( D_1 \) and \( D_2 \) is still governed by Theorem 3.1. For each \( \lambda > \lambda_3 \), the value of \( u^\varepsilon(T^\varepsilon(\lambda), x) \) will be nearly constant on \( D_1 \cup D_3 \), and we can treat the cycle \( \Gamma = \{O_1, O_2\} \) in the same way as a single equilibrium was treated in Section 3. Namely, let

\[
c^\Gamma(\lambda) = \begin{cases} c^*, \\
\min\{g(O_3), \min\{c : c \in [c^*, g(O_3)] \}, M_{13}(c) = \lambda\}\end{cases}, \quad \lambda_3 \leq \lambda < \lambda_4,
\]

\[
c^\Gamma(\lambda) = \begin{cases} c^*, \\
\min\{g(O_3), \min\{c : c \in [c^*, g(O_3)] \}, M_{23}(c) = \lambda\}\end{cases}, \quad \lambda \geq \lambda_4,
\]
Figure 2: A case of three equilibrium points without changes in the hierarchy of cycles

\[ c^3(\lambda) = \begin{cases} 
g(O_3), & 0 < \lambda < \lambda_5, \\
\max\{c^*, \max\{c : c \in [c^*, g(O_3)], M_{3\Gamma}(c) = \lambda\}\}, & \lambda \geq \lambda_5,
\end{cases} \]

Define \( \lambda_6 = \inf\{\lambda > \lambda_3 : c^\Gamma(\lambda) \geq c^3(\lambda)\} \). Assume that \( \lambda_5 < \lambda_6 \) and that at least one of the functions \( c^\Gamma \) and \( c^3 \) is continuous at \( \lambda_6 \). Let \( c^{**} = c^\Gamma(\lambda_6) \) if \( c^\Gamma \) is continuous at \( \lambda_6 \) and \( c^{**} = c^3(\lambda_6) \) otherwise. Define \( \overline{c}^1(\lambda) = \overline{c}^2(\lambda) = \min(c^\Gamma(\lambda), c^{**}), \lambda \geq \lambda_3, \) and \( \overline{c}^3(\lambda) = \max(c^3(\lambda), c^{**}), \lambda > 0. \)

Having thus defined the functions \( \overline{c}^i(\lambda), i = 1, 2, 3, \) for all \( \lambda > 0 \), we can now state that for each \( \lambda > 0 \) such that \( \overline{c}^i \) is continuous at \( \lambda \) and every \( \delta > 0 \), the limit

\[ \lim_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), x) = \overline{c}(\lambda) \]

is uniform in \( x \in D^i_\delta \).

On Figure 2, the limits \( \lim_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), x) \), as functions of \( \lambda \), for \( x \in D^i_1, D^i_2 \) and \( D^i_3 \) are depicted by thick, dotted and dashed lines, respectively.

5 A general result for the case when the hierarchy of cycles does not change

In this section we will suppose that, in addition to Assumption A, the hierarchy of cycles and the equilibrium points \( \nu(\Gamma) \) for each cycle \( \Gamma \) of rank less than \( R \) do not depend on
the choice of constants $c_i \in [g_{\min}, g_{\max}]$ in the function $\alpha = a(x; \sum_{i=1}^{n} c_i \chi_{D_i}(x))$.

We will say that a cycle $\Gamma$ is active for a given value of $\lambda > 0$ if $V_{\Gamma, \nu(\Gamma)}^{\alpha} = \lambda$. We will say that it is engaged if $V_{\Gamma, \nu(\Gamma)}^{\alpha} = \lambda$ and passive if $V_{\Gamma, \nu(\Gamma)}^{\alpha} > \lambda$. We will say that a cycle $\Gamma_0$ is connected to a cycle $\Gamma$ by a chain if there is a sequence of cycles $\Gamma_1, \ldots, \Gamma_k$ and equilibriums $O_1 \in \Gamma_1, \ldots, O_k \in \Gamma_k$, $O_{k+1} \in \Gamma$ such that $\Gamma_i$ are engaged or active and $O_{i+1} = \nu(\Gamma_i)$ for $0 \leq i \leq k$. The collection of all the cycles that do not belong to $\Gamma$ and are connected to $\Gamma$ by a chain will be called the cluster connected to $\Gamma$. For each cycle $\Gamma$ of less than maximal rank and $c \in [g_{\min}, g_{\max}]$, we define

$$M_\Gamma(c) = V_{\Gamma, \nu(\Gamma)}^{\alpha(c)}$$

and, for $\lambda > 0$ and $c_2 \geq c_1$, define $C(c_1, c_2, \lambda, \Gamma) = \min(c_2, \inf(c > c_1 : M_\Gamma(c) \geq \lambda))$. Similarly, if $c_2 \leq c_1$, define $C(c_1, c_2, \lambda, \Gamma) = \max(c_2, \sup(c < c_1 : M_\Gamma(c) \geq \lambda))$.

In Figure 3 we have an example of a hierarchy of cycles with the thick arrows between the actively connected cycles and the corresponding equilibrium points. The dashed arrows are used for the engaged cycles and the dotted arrows for the passively connected cycles.

![Figure 3: The hierarchy of cycles](image)

In order to describe the asymptotics of $u^\varepsilon(T^\varepsilon(\lambda), x)$, we will define a finite number of "special" points $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_m = \infty$. We claim that there are functions $\overline{c}_i(\lambda)$, $1 \leq i \leq n$, which are continuous on each of the intervals $(\lambda_k, \lambda_{k+1})$, $0 \leq k \leq m - 1$, have
one-sided limits as \( \lambda \) approaches the end points of the intervals, and are such that the limits

\[
\lim_{\varepsilon \downarrow 0} u^\varepsilon(T^\varepsilon(\lambda), x) = \overline{\sigma}(\lambda)
\]

are uniform in \( x \in D^\varepsilon_i \) for each \( \delta > 0, \lambda \in \mathbb{R}^+ \setminus \Lambda \). Moreover, neither of the cycles changes its type (between passive, engaged and active) for \( \lambda \in (\lambda_k, \lambda_{k+1}) \) and \( \alpha(x) = \lim_{\varepsilon \downarrow 0} a(x, u^\varepsilon(T^\varepsilon(\lambda), x)) \). We will use induction on \( k \) in order to define the functions \( \overline{\sigma}(\lambda) \) and describe for each cycle whether it is passive, engaged or active for \( \lambda \in (\lambda_k, \lambda_{k+1}) \) with \( \alpha(x) = \lim_{\varepsilon \downarrow 0} a(x, u^\varepsilon(T^\varepsilon(\lambda), x)) \). In the process, we will make several assumptions about the functions \( M_\Gamma \).

Assuming that we have defined \( \overline{\sigma}(\lambda) \), let

\[
\lambda_\Gamma = \inf\{ \lambda > 0 : \overline{\sigma}(\lambda') \text{ does not depend on } i \text{ for } \lambda' \geq \lambda \text{ and } O_i \in \Gamma \}.
\]

From the inductive construction of the functions \( \overline{\sigma}(\lambda) \) it will follow that \( \lambda_\Gamma < \infty \). Let \( a_\Gamma = \lim_{\lambda \downarrow \lambda_\Gamma} \overline{\sigma}(\lambda), O_i \in \Gamma \), and \( A_\Gamma = M_\Gamma(a_\Gamma) \). We assume that all \( A_\Gamma \) are distinct and define

\[
\Lambda^1 = \{ A_\Gamma, \text{ rank}(\Gamma) < R \}.
\]

We assume that \( M_\Gamma \) has a finite number of critical points on \([g_{\min}, g_{\max}]\) for each \( \Gamma \) with \( \text{rank}(\Gamma) < R \). Let \( c^\Gamma_1, ..., c^\Gamma_{k_\Gamma} \) be all the local maxima of \( M_\Gamma \). We assume that \( M_\Gamma(c^\Gamma_i) \) are distinct for all \( \Gamma \) with \( \text{rank}(\Gamma) < R \) and \( i \). Define

\[
\Lambda^2 = \{ M_\Gamma(c^\Gamma_i), \text{ rank}(\Gamma) < R, 1 \leq i \leq k_\Gamma \}.
\]

Let \( \Gamma \) be a cycle of rank \( r < R, \Gamma \) the cycle of rank \( r + 1 \) that contains \( \Gamma \), and \( \Upsilon \) a cycle that is contained in \( \overline{\Gamma} \setminus \Gamma \). Let \( I_{\Gamma, \Upsilon} = \{ c : M_\Gamma(c) = M_\Upsilon(c) \} \). We assume that the sets \( I_{\Gamma, \Upsilon} \) are finite and \( I_{\Gamma_1, \Upsilon_1} \cap I_{\Gamma_2, \Upsilon_2} = \emptyset \) unless \( (\Gamma_1, \Upsilon_1) = (\Gamma_2, \Upsilon_2) \) or \( (\Gamma_1, \Upsilon_1) = (\Upsilon_2, \Gamma_2) \). Define

\[
\Lambda^3 = \{ M_\Gamma(c), c \in I_{\Gamma, \Upsilon}, \text{ rank}(\Gamma) < R, \Upsilon \subseteq \overline{\Gamma} \setminus \Gamma \}.
\]

We assume that the numbers \( M_\Gamma(a_\Upsilon) \) are distinct for all choices of cycles \( \Gamma \) and \( \Upsilon \) such that \( \text{rank}(\Gamma) < R, \text{rank}(\Upsilon) \leq \text{rank}(\Gamma) \) and \( \nu(\Gamma) \in \Upsilon \). Define

\[
\Lambda^4 = \{ M_\Gamma(a_\Upsilon), \text{ rank}(\Gamma) < R, \text{rank}(\Upsilon) \leq \text{rank}(\Gamma), \nu(\Gamma) \in \Upsilon \}.
\]

Finally, we assume that the sets \( \Lambda^1, \Lambda^2, \Lambda^3 \) and \( \Lambda^4 \) do not intersect and define

\[
\Lambda = \{ \lambda_0, \lambda_1, ..., \lambda_m \} := \{ 0 \} \cup \Lambda^1 \cup \Lambda^2 \cup \Lambda^3 \cup \Lambda^4 \cup \{ \infty \},
\]

where we arrange \( \lambda_k \) in the increasing order.

Below we will define \( \overline{\sigma}(\lambda) \) on the successive intervals \((\lambda_k, \lambda_{k+1})\) using induction on \( k \) while assuming that \( \lambda_k \) are known. The above definition of \( \Lambda^1 \) and \( \Lambda^4 \) in terms of \( \overline{\sigma}(\lambda) \) does not constitute a circular argument, since we could instead define the pairs \((\lambda_{k+1}, \overline{\sigma}(\lambda)) \) for \( \lambda \in (\lambda_k, \lambda_{k+1}) \) inductively. Such an approach would lead to more complicated notations, though, so we avoid it.
Let us proceed with the inductive definition of $\mathfrak{s}(\lambda)$. For $\lambda \in (\lambda_0, \lambda_1)$ all cycles are passive and $\mathfrak{s}(\lambda) = q(O_i)$ for all $i$. Assuming that the types of the cycles and the limits $q(O_i) = \lim_{\lambda \uparrow \lambda_k} \mathfrak{s}(\lambda)$ are known for $\lambda \in (\lambda_{k-1}, \lambda_k)$ with some $0 < k < m$, we will describe the types of the cycles for $\lambda \in (\lambda_k, \lambda_{k+1})$ and specify the limits $s(O_i) = \lim_{\lambda \uparrow \lambda_k} \mathfrak{s}(\lambda)$. Then, assuming that the types of the cycles are specified for $\lambda \in (\lambda_k, \lambda_{k+1})$ and the values of $s(O_i)$ are known, we will define the functions $\mathfrak{s}(\lambda)$ for $\lambda \in (\lambda_k, \lambda_{k+1})$. We distinguish a number of cases depending on whether $\lambda_k$ belongs to $\Lambda^1$, $\Lambda^2$, $\Lambda^3$ or $\Lambda^4$.

First, however, we describe the procedure for determining the values of $s(O_i)$ for $O_i$ which belong to a cluster.

**Determining the values of $s(O_i)$ and the types of cycles within a cluster.** Suppose that we have defined $s(O_i) = \lim_{\lambda \uparrow \lambda_k} \mathfrak{s}(\lambda)$ for all $O_i$ that belong to a cycle $\Gamma$. Consider the cluster of cycles that are connected to $\Gamma$ for $\lambda \in (\lambda_{k-1}, \lambda_k)$. For each cycle $\Gamma'$ in the cluster, we will define the values of $s(O)$ for $O \in \Gamma'$ and specify its type for $\lambda \in (\lambda_k, \lambda_{k+1})$.

First assume that $\nu(\Gamma') = O_i \in \Gamma$ for $\lambda \in (\lambda_{k-1}, \lambda_k)$. It will follow from the inductive construction that $q(\Gamma') = q(O_i)$ if $O_i, O'' \in \Gamma'$. Let $q(\Gamma') = q(O_i)$. For $O \in \Gamma'$, we define $s(O) = C(q(\Gamma'), s(O_i), \lambda_k, \Gamma')$.

For any cycle $\Gamma''$ such that $\nu(\Gamma'') \in \Gamma'$, we can similarly determine the values of $s(O)$ for $O \in \Gamma''$. Continuing this procedure inductively, we define the values of $s(O)$ when $O$ belongs to the cycles from the cluster. A cycle $\Gamma'$ from the cluster will be engaged for $\lambda \in (\lambda_k, \lambda_{k+1})$ if $\lambda_k = M_{\Gamma'}(s(O))$ for $O \in \Gamma'$ and active if $\lambda_k > M_{\Gamma'}(s(O))$ for $O \in \Gamma'$.

**Case 1.** Assume that $\lambda_k \in \Lambda_1$. Let $\Gamma$ be such that $\lambda_k = A_{\Gamma}$. For $O_i \in \Gamma$, we define $s(O_i) = C(q(O_i), q(\nu(\Gamma)), \lambda_k, \Gamma)$. The cycle will be engaged for $\lambda \in (\lambda_k, \lambda_{k+1})$ if $\lambda_k = M_{\Gamma}(s(O))$ for $O \in \Gamma$ and active if $\lambda_k > M_{\Gamma}(s(O))$ for $O \in \Gamma$.

The types of cycles that belong to the cluster connected to $\Gamma$ for $\lambda \in (\lambda_{k-1}, \lambda_k)$, and the values of $s(O_j)$ for the equilibrium points in those cycles are determined according to the procedure described above. For the remaining equilibrium points $O$, we define $s(O) = q(O)$. The remaining cycles don’t change type.

**Case 2.** Assume that $\lambda_k \in \Lambda_2$. Let $c$ be the local maximum of a cycle $\Gamma$ such that $M_{\Gamma}(c) = \lambda_k$. If $\Gamma$ was not engaged for $\lambda \in (\lambda_{k-1}, \lambda_k)$ or if $q(O) \neq c$ for some $O \in \Gamma$, then we define $s(O) = q(O)$ for all the equilibrium points, and all the cycles have the same type on $(\lambda_k, \lambda_{k+1})$ as on $(\lambda_{k-1}, \lambda_k)$.

If $\Gamma$ was engaged and $q(O) = c$ for $O \in \Gamma$, then for $O_i \in \Gamma$, we define $s(O_i) = C(q(O_i), q(\nu(\Gamma)), \lambda_k, \Gamma)$. The cycle will be engaged for $\lambda \in (\lambda_k, \lambda_{k+1})$ if $\lambda_k = M_{\Gamma}(s(O))$ for $O \in \Gamma$ and active if $\lambda_k > M_{\Gamma}(s(O))$ for $O \in \Gamma$.

The types of cycles that belong to the cluster connected to $\Gamma$ for $\lambda \in (\lambda_{k-1}, \lambda_k)$, and the values of $s(O_j)$ for the equilibrium points in those cycles are determined according to the procedure described above. For the remaining equilibrium points $O$, we define $s(O) = q(O)$. The remaining cycles don’t change type.

**Case 3.** Assume that $\lambda_k \in \Lambda_3$. Let $\Gamma$ be a cycle of rank $r < R$, $\bar{\Gamma}$ the cycle of rank $r + 1$ that contains $\Gamma$, and $\Gamma$ a cycle that is contained in $\bar{\Gamma} \setminus \Gamma$. Suppose that $c$ is such
that $M_\Gamma(c) = M_\Gamma(c)$ and $\lambda_k = M_\Gamma(c)$.

We define $s(O) = q(O)$ for all the equilibrium points. All the cycles, other than perhaps $\Gamma$ and $\Upsilon$, have the same type on $(\lambda_k, \lambda_{k+1})$ as on $(\lambda_{k-1}, \lambda_k)$. To determine the type of cycles $\Gamma$ and $\Upsilon$ on $(\lambda_k, \lambda_{k+1})$, we examine several cases.

(a) If $q(O) = c$ for all $O \in \Gamma \cup \Upsilon$, $\Gamma$ and $\Upsilon$ were engaged by a chain that contained only active cycles (other than $\Upsilon$ itself) and $\Gamma$ was connected to $\Upsilon$ by a chain that contained only active cycles (other than $\Gamma$ itself), then $\Gamma$ and $\Upsilon$ becomes active. The type of $\Upsilon$ stays the same. If no such cycle exists, that is if $\lambda_k$, then $\Gamma$ becomes active if it was engaged on $(\lambda_{k-1}, \lambda_k)$ and becomes engaged if it was active.

(b) If $q(O) = c$ for all $O \in \Gamma \cup \Upsilon$, $\Gamma$ was connected to $\Upsilon$ by a chain that contained only active cycles (other than $\Gamma$ itself), but $\Upsilon$ was not connected to $\Gamma$ by a chain that contained only active cycles (other than $\Upsilon$ itself), and $\Upsilon$ was not passive, then $\Gamma$ becomes active on $(\lambda_k, \lambda_{k+1})$ if it was engaged on $(\lambda_{k-1}, \lambda_k)$ and becomes engaged if it was active.

(c) The same as (b) with $\Gamma$ and $\Upsilon$ interchanged.

(d) If none of the cases (a)-(c) applies, then $\Gamma$ and $\Upsilon$ have the same types on $(\lambda_k, \lambda_{k+1})$ as on $(\lambda_{k-1}, \lambda_k)$.

**Case 4.** Assume that $\lambda_k \in \Lambda_4$. Let $\lambda_k = M_\Gamma(a_\Gamma)$, where cycles $\Gamma$ and $\Upsilon$ are such that $\text{rank}(\Gamma) < R$, $\text{rank}(\Upsilon) \leq \text{rank}(\Gamma)$ and $\nu(\Gamma) \in \Upsilon$. We define $s(O) = q(O)$ for all the equilibrium points. All the cycles, other than perhaps $\Gamma$, have the same type on $(\lambda_k, \lambda_{k+1})$ as on $(\lambda_{k-1}, \lambda_k)$.

The cycle $\Gamma$ becomes active if it was engaged on $(\lambda_{k-1}, \lambda_k)$, $q(O) = a_\Upsilon$ for all $O \in \Gamma$ and $M_\Gamma(a_\Upsilon) < A_\Upsilon$. Otherwise, $\Gamma$ has the same type on $(\lambda_k, \lambda_{k+1})$ as on $(\lambda_{k-1}, \lambda_k)$.

Now let us define the functions $\overline{\sigma}(\lambda)$ on $(\lambda_k, \lambda_{k+1})$ assuming that the values of $s(O_i)$ and the cycle types are known. For an equilibrium point $O_i$, we identify the cycle $\Gamma$ with the smallest possible rank $r$ such that $O_i \in \Gamma$ and the values of $s(O_i)$, $O_j \in \Gamma$, are not all the same. If no such cycle exists, that is if $s(O_j)$, $1 \leq j \leq n$, does not depend on $j$, then we define $\overline{\sigma}(\lambda) = s(O_i)$ for $\lambda > \lambda_k$.

Assuming that such a cycle $\Gamma$ exists, let $\Gamma_1, \ldots, \Gamma_l$ be the cycles of rank $r - 1$ which comprise $\Gamma$, and let $O \in \Gamma_1$. Here we number the cycles in such a way that $\mathcal{N}(\Gamma_1) = \Gamma_2, \ldots, \mathcal{N}(\Gamma_l) = \Gamma_1$. Take the least $j$ such that $\Gamma_j$ is is passive or engaged (it can not happen that all the cycles $\Gamma_1, \ldots, \Gamma_l$ are active, since then all the values of $s(O)$, $O \in \Gamma$, would be the same, as follows from the inductive construction above). If $\Gamma_j$ is passive, we define $\overline{\sigma}(\lambda) = s(O_i)$ for $\lambda \in (\lambda_k, \lambda_{k+1})$. If $\Gamma_j$ is engaged, we define $\overline{\sigma}(\lambda) = C(r(O_i), \zeta, \lambda, \Gamma_j)$ for $\lambda \in (\lambda_k, \lambda_{k+1})$, where $\zeta = +\infty$ if $M_{\Gamma_j}$ is locally increasing at $r(O_i)$ and $\zeta = -\infty$ if $M_{\Gamma_j}$ is locally decreasing at $r(O_i)$.

We can now summarize the above discussion.

**Theorem 5.1.** Suppose that Assumption A holds and the hierarchy of cycles and the equilibrium points $\nu(\Gamma)$ for each cycle $\Gamma$ of rank less than $R$ do not depend on the choice of constants $c_i \in [g_{\min}, g_{\max}]$ in the function $\alpha = a(x, \sum_{i=1}^n c_i \chi_{D_i}(x))$. Also suppose that the above assumptions on the sets $\Lambda^1$, $\Lambda^2$, $\Lambda^3$ and $\Lambda^4$ hold.
Then the limits
\[
\lim_{\epsilon \to 0} u^\epsilon(T^\epsilon(\lambda), x) = \overline{\gamma}(\lambda)
\]
are uniform in \( x \in D^\delta_i \) for each \( \delta > 0 \), \( \lambda \in \mathbb{R}^+ \setminus \Lambda \), where the functions \( \overline{\gamma}(\lambda) \) were defined via the inductive procedure above.

6 Example of a change in the hierarchy of cycles

As in Section 4 we assume that there are three equilibrium points \( O_1, O_2, O_3 \). For each \( c_1, c_2, c_3 \in [g_{\min}, g_{\max}] \), the function \( f_{c_1,c_2,c_3} \) is defined by (19). We will assume that the hierarchy of cycles for \( \alpha = a(x, f_{c_1,c_2,c_3}(x)) \) depends only on \( c_2 \). This is the case, for example, if \( d = 1 \) and \( O_1 < O_2 < O_3 \). More precisely, suppose that there is \( \overline{c} \in (g_{\min}, g_{\max}) \) such that Assumption A holds for each choice of the constants \( c_i \in [g_{\min}, g_{\max}] \) such that \( c_2 \neq \overline{c} \). We assume that \( O_1 \) and \( O_2 \) form a cycle \( \Gamma' = \{O_1, O_2\} \) of rank one when \( c_2 < \overline{c} \), while \( O_2 \) and \( O_3 \) form a cycle \( \Gamma'' = \{O_2, O_3\} \) of rank one when \( c_2 > \overline{c} \).

As before, we will identify a number of “special” points \( \lambda_k \) and describe the asymptotic behavior of \( u^\epsilon(T^\epsilon(\lambda), x) \) for \( \lambda \in (\lambda_k, \lambda_{k+1}) \) and \( x \in D^\delta_i, i = 1, 2, 3 \). In the process, we will make various assumptions about the quasi-potential that will be specific to the example at hand.

In our example we assume that \( g(O_1) \leq g(O_2) \leq \overline{c} \leq g(O_3) \). Define
\[
M_{12}(c) = V_{O_1,O_2}^{a(c)}, \quad M_{21}(c) = V_{O_2,O_1}^{a(c)}, \quad M_{13}(c) = V_{O_1,O_3}^{a(c)}, \quad M_{32}(c) = V_{O_3,O_2}^{a(c)}, \quad c \in [g_{\min}, \overline{c}];
\]
\[
M_{32}(c) = V_{O_3,O_2}^{a(c)}, \quad M_{23}(c) = V_{O_2,O_3}^{a(c)}, \quad M_{1'3}(c) = V_{O_1,O_3}^{a(c)}, \quad M_{1'2}(c) = V_{O_1,O_2}^{a(c)}; \quad c \in (\overline{c}, g_{\max}].
\]

Let \( \lambda_1 = M_{12}(g(O_1)) \) and \( \lambda_2 = M_{21}(g(O_2)) \). Define functions \( c^1 \) and \( c^2 \) by (13) and (14), respectively. Let \( \lambda_3 = \inf \{ \lambda : c^1(\lambda) \geq c^2(\lambda) \} \). Assume that at least one of the functions \( c^1 \) and \( c^2 \) is continuous at \( \lambda_3 \). Let \( c^* = c^1(\lambda_3) \) if \( c^1 \) is continuous at \( \lambda_3 \) and \( c^* = c^2(\lambda_3) \) otherwise. Let \( \overline{c}^1(\lambda) = \min(c^1(\lambda), c^*) \) and \( \overline{c}^2(\lambda) = \max(c^2(\lambda), c^*) \), \( \lambda < \lambda_3 \). Let \( \lambda_4 = M_{1'3}(c^*), \lambda_5 = \sup_{c \in [\overline{c}, \overline{c}]} M_{1'3}(c) \) and \( \lambda_6 = M_{32}(g(O_3)) \).

Let us assume that \( \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 \) (see Figure 4). Define
\[
\overline{c}^1(\lambda) = \overline{c}^2(\lambda) = \overline{c}^{1'}(\lambda) = \begin{cases} c^*, & \lambda_3 \leq \lambda < \lambda_4; \\ \min\{c : c \in [c^*, \overline{c}], M_{1'3}(c) = \lambda\}, & \lambda_4 \leq \lambda < \lambda_5; \end{cases}
\]
In order to formulate the results on the asymptotics of \( u^\epsilon(T^\epsilon(\lambda), x) \) for \( \lambda > \lambda_5 \), we need the functions \( d^2(\lambda) \) and \( c^3(\lambda) \) defined as follows:
\[
d^2(\lambda) = \min\{g(O_3), \min\{c : c \in [\overline{c}, g(O_3)], M_{23}(c) = \lambda\}\}, \quad \lambda \geq \lambda_5,
\]
\[
c^3(\lambda) = \begin{cases} g(O_3), & 0 < \lambda < \lambda_6; \\ \max\{\overline{c}, \max\{c : c \in [\overline{c}, g(O_3)], M_{32}(c) = \lambda\}\}, & \lambda \geq \lambda_6. \end{cases}
\]
Let $\lambda_7 = \inf \{ \lambda : d^2(\lambda) \geq c^3(\lambda) \}$. Assume that $\lambda_6 < \lambda_7$ and at least one of the functions $d^2$ and $c^3$ is continuous at $\lambda_7$. Let $c^{**} = d^2(\lambda_7)$ if $d^2$ is continuous at $\lambda_7$ and $c^{**} = c^3(\lambda_7)$ otherwise. Let $\lambda_8 = \Gamma_{\lambda_7}(c^{**})$ and assume that $\lambda_7 < \lambda_8$. Define $\bar{c}^2(\lambda) = \min(d^2(\lambda), c^{**})$, $\lambda_5 \leq \lambda < \lambda_8$, and $\bar{c}^3(\lambda) = \max(c^3(\lambda), c^{**})$, $0 < \lambda < \lambda_8$.

Let $d^1(\lambda) = \min\{ c^{**}, \min\{ c : c \in [\bar{c}^2, c^{**}] , M_{\Gamma_{\lambda_7}}(c) = \lambda \} \} \geq \lambda_5$, $\lambda \geq \lambda_5$, $c^{**'}(\lambda) = \max\{ \bar{c}^2, \max\{ c : c \in [\bar{c}^2, c^{**}] , M_{\Gamma_{\lambda_7}}(c) = \lambda \} \}$, $\lambda \geq \lambda_8$.

Let $\lambda_9 = \inf \{ \lambda : d^1(\lambda) \geq c^{**'}(\lambda) \}$. Assume that $\lambda_8 < \lambda_9$ and at least one of the functions $d^1$ and $c^{**'}$ is continuous at $\lambda_9$. Let $c^{***} = d^1(\lambda_9)$ if $d^1$ is continuous at $\lambda_9$ and $c^{***} = c^{**'}(\lambda_9)$ otherwise. Define $\bar{c}^1(\lambda) = \min(d^1(\lambda), c^{***})$, $\lambda_5 \leq \lambda$ and $\bar{c}^2(\lambda) = \bar{c}^3(\lambda) = \max(c^{**'}(\lambda), c^{***})$, $\lambda_8 \leq \lambda$.

Having thus defined the functions $\bar{c}^i(\lambda)$, $i = 1, 2, 3$, for all $\lambda > 0$, we can now state that for each $\lambda > 0$ such that $\bar{c}^i$ is continuous at $\lambda$ and every $\delta > 0$, the limit

$$\lim_{\epsilon \to 0} u^\epsilon(T^\epsilon(\lambda), x) = \bar{c}^i(\lambda)$$

is uniform in $x \in D^\delta_i$.

On Figure 4, the limits $\lim_{\epsilon \to 0} u^\epsilon(T^\epsilon(\lambda), x)$, as functions of $\lambda$, for $x \in D^\delta_1, D^\delta_2$ and $D^\delta_3$ are depicted by thick, dotted and dashed lines, respectively.

Figure 4: A case of three equilibrium when the hierarchy of cycles changes
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