Funnel control of nonlinear systems

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Abstract Tracking of reference signals is addressed in the context of a class of nonlinear controlled systems modelled by $r$-th order functional differential equations, encompassing inter alia systems with unknown “control direction” and dead-zone input effects. A control structure is developed which ensures that, for every member of the underlying system class and every admissible reference signal, the tracking error evolves in a prescribed funnel chosen to reflect transient and asymptotic performance objectives. Two fundamental properties underpin the system class: bounded-input bounded-output stable internal dynamics, and a weak high-gain property (an antecedent of which is the concept of sign-definite high-frequency gain in the context of linear systems).

Keywords nonlinear systems · adaptive control · asymptotic tracking · funnel control · relative degree · functional differential equations

Mathematics Subject Classification (2010) 93C10 · 93C23 · 93C40

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Nomenclature

| N, N₀ | the set of positive, non-negative integers, respectively |
| R≥₀, C≥₀ | the sets [0, ∞), \{λ ∈ C| \text{Re}(λ) ≥ 0\}, respectively |
| ⟨v, w⟩ | the Euclidean inner product of vectors v, w ∈ Rⁿ |
| ∥x∥ | \sqrt{⟨x, x⟩}, the Euclidean norm of x ∈ Rⁿ |
| L∞(I, Rⁿ) | the Lebesgue space of measurable, essentially bounded functions \(f : I → Rⁿ\), where \(I ⊆ R\) is some interval |
| L∞₂loc(I, Rⁿ) | the set of measurable, locally essentially bounded functions \(f : I → Rⁿ\), where \(I ⊆ R\) is some interval |
| Wk,∞(I, Rⁿ) | the Sobolev space of all functions \(f : I → Rⁿ\) with \(k\)-th order weak derivative \(f^{(k)}\) and \(f, f^{(1)}, \ldots, f^{(k)} ∈ L∞(I, Rⁿ)\), where \(I ⊆ R\) is some interval and \(k ∈ N\) |
| \(C^{k}(V, Rⁿ)\) | the set of \(k\)-times continuously differentiable functions \(f : V → Rⁿ\), where \(V ⊆ Rᵐ\) and \(k ∈ N₀\); \(C(V, Rⁿ) := C^{0}(V, Rⁿ)\) |

1 Introduction

Since its inception in 2002, the concept of funnel control has been widely investigated. In its essence, the approach considers the following basic question: for a given class of dynamical systems, with input \(u\) and output \(y\), and a given class of reference signals \(y_{\text{ref}}\), does there exist a single control strategy (generating \(u\)) which ensures that, for every member of the system class and every admissible reference signal, the output \(y\) approaches the reference \(y_{\text{ref}}\) with prescribed transient behaviour and prescribed asymptotic accuracy? The twofold objective of “prescribed transient behaviour and asymptotic accuracy” is encompassed by the adoption of a so-called “performance funnel” in which the error function \(t ↦ e(t) := y(t) = y_{\text{ref}(t)}\) is required to evolve; see Fig.1. The present paper considers a large class of systems, described by \(r\)-th order functional differential equations with a parameter \(r ∈ N\) related to the control-theoretic concept of relative degree. Loosely speaking, one may think of the relative degree of a system as the order of differentiation of its output required to cause its input to appear explicitly. The information available for feedback to the controller is comprised of the instantaneous values of the output and its first \(r−1\) derivatives, together with the instantaneous values of the reference signal and its first \(\hat{r}−1\) derivatives, where \(1 ≤ \hat{r} ≤ r\). A feedback strategy is developed which assures attainment of the above twofold performance objective. First, we make explicit the underlying class of systems.
1.1 System class

We consider a class \( \mathcal{N}^{m,r} \) of systems, modelled by nonlinear functional differential equations of the form

\[
y'(t) = f(d(t), T(y, y', \ldots, y^{(r-1)})(t), u(t)) \quad y|_{[-h,0]} = y^0 \in C^{r-1}([-h,0], \mathbb{R}^m),
\]

where \( h \geq 0 \) quantifies the “memory” in the system, \( r \in \mathbb{N} \) is related to the concept of relative degree, \( m \in \mathbb{N} \) is the dimension of both the input \( u(t) \) and output \( y(t) \) at time \( t \geq 0 \), \( d \in L^m(\mathbb{R}_{\geq 0}, \mathbb{R}^p) \) is a “disturbance”, and \( f \in C(\mathbb{R}^p \times \mathbb{R}^s \times \mathbb{R}^m, \mathbb{R}^m) \) belongs to a set of nonlinear functions characterized by a particular high-gain property (made precise in Definition 1.2). The operator \( T \) belongs to a set of nonlinear functions characterized by a particular a high-gain property (made precise in Definition 1.1). The most simple, but non-trivial, prototype of the system local Lipschitz condition, and map bounded functions to bounded functions (made precise in Definition 1.1). The most simple, but non-trivial, prototype of the system local Lipschitz condition, and map bounded functions to bounded functions (made precise in Definition 1.1). The most simple, but non-trivial, prototype of the system local Lipschitz condition, and map bounded functions to bounded functions (made precise in Definition 1.1). The most simple, but non-trivial, prototype of the system local Lipschitz condition, and map bounded functions to bounded functions (made precise in Definition 1.1). The most simple, but non-trivial, prototype of the system local Lipschitz condition, and map bounded functions to bounded functions (made precise in Definition 1.1).

**Definition 1.1 (Operator class)**. For \( n, q \in \mathbb{N} \) and \( h \geq 0 \), the set \( \mathcal{N}^{n,q}_h \) denotes the class of operators

\[
\mathcal{N}^{n,q}_h := \{ T : C([-h, \infty), \mathbb{R}^n) \rightarrow L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^q) \mid (\text{TP1}) - (\text{TP3}) \text{ hold} \},
\]

where (TP1) – (TP3) denote the following properties.

**(TP1) Causality**: for all \( \zeta, \theta \in C([-h, \infty), \mathbb{R}^n) \) and all \( t \geq 0 \),

\[
\zeta|_{[-h,t]} = \theta|_{[-h,t]} \implies T(\zeta)|_{[0,t]} = T(\theta)|_{[0,t]}.
\]

**(TP2) Local Lipschitz property**: for each \( t \geq 0 \) and all \( \zeta \in C([-h,t], \mathbb{R}^n) \), there exist positive constants \( c_0, \delta, \tau > 0 \) such that, for all \( \zeta_1, \zeta_2 \in C([-h, \infty), \mathbb{R}^n) \) with \( \zeta_1|_{[-h,t]} = \zeta_2|_{[-h,t]} \) and \( \| \zeta_i(s) - \zeta_j(s) \| < \delta \) for all \( s \in [t, t+\tau] \) and \( i = 1, 2 \), we have

\[
\sup_{s \in [t,t+\tau]} \| T(\zeta_1)(s) - T(\zeta_2)(s) \| \leq c_0 \sup_{s \in [t,t+\tau]} \| \zeta_1(s) - \zeta_2(s) \|.
\]

**(TP3) Bounded-input bounded-output (BIBO) property**: for each \( c_1 > 0 \) there exists \( c_2 > 0 \) such that, for all \( \zeta \in C([-h, \infty), \mathbb{R}^n) \),

\[
\sup_{t \in [-h,\infty)} \| \zeta(t) \| < c_1 \implies \sup_{t \geq 0} \| T(\zeta)(t) \| < c_2.
\]

Property (TP1) is entirely natural in the context of physically-motivated controlled systems. Property (TP2) is a technical condition which (in conjunction with continuity of \( f \)) plays a role in ensuring well-posedness of the initial-value problem (1) under feedback control. Property (TP3) is, loosely speaking, a stability condition on the “internal dynamics” of (1). For linear systems with strict relative degree, the first two conditions are trivially satisfied, whilst the third is equivalent to a minimum-phase assumption: this is shown in Section 2.1.2.

The formulation also embraces nonlinear delay elements and hysteretic effects, as we shall briefly illustrate.
Nonlinear delay elements. For $i = 0, \ldots, k$, let $\Psi_i : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^q$ be measurable in its first argument and locally Lipschitz in its second argument, uniformly with respect to its first argument. Precisely, for each $\xi \in \mathbb{R}^m$, $\Psi_i(\cdot, \xi)$ is measurable, and for every compact $C \subset \mathbb{R}^m$, there exists a constant $c > 0$ such that

$$\text{for a.a. } t \in \mathbb{R} \ \forall \xi_1, \xi_2 \in C: \|\Psi_i(t, \xi_1) - \Psi_i(t, \xi_2)\| \leq c\|\xi_1 - \xi_2\|.$$ 

Let $h_i > 0$, $i = 0, \ldots, k$, and set $h := \max_i h_i$. For $y \in C([-h, \infty), \mathbb{R}^m)$, let

$$T(y)(t) := \int_{-h}^0 \Psi_0(s, y(t + s)) \, ds + \sum_{i=1}^k \Psi_i(t, y(t - h_i)), \ t \geq 0.$$ 

The operator $T$, so defined (which models distributed and point delays), is of class $T_h$; for details, see [50].

Hysteresis. A large class of nonlinear operators $T : C(\mathbb{R}_{\geq 0}, \mathbb{R}) \to C(\mathbb{R}_{\geq 0}, \mathbb{R})$, which includes many physically-motivated hysteretic effects, is defined in [40]. These operators are contained in the class $T_h$ of the present paper. Specific examples include relay hysteresis, backlash hysteresis, elastic-plastic hysteresis, and Preisach operators. For further details, see [30].

Next, we introduce a high-gain property which, in effect, characterizes the class of admissible nonlinearities $f$.

Definition 1.2 (High-gain property). For $p, q, m \in \mathbb{N}$, a function $f \in C(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^m)$ is said to have the high-gain property, if there exists $\nu^* \in (0, 1)$ such that, for every compact $K_p \subset \mathbb{R}^p$ and compact $K_q \subset \mathbb{R}^q$ the (continuous) function

$$\chi : \mathbb{R} \to \mathbb{R}, \ s \mapsto \min \{ \langle v, f(\delta, z, -sv) \rangle : (\delta, z) \in K_p \times K_q, v \in \mathbb{R}^m, \nu^* \leq \|v\| \leq 1 \}$$

is such that $\sup_{s \in \mathbb{R}} \chi(s) = \infty$.

We elucidate the high-gain property – which at first sight might seem somewhat arcane – in the following remark.

Remark 1.3.

(a) The high-gain property holds if, and only if, at least one of the following is true

$$\text{(i) } \sup_{s \geq 0} \chi(s) = \infty \quad \text{or} \quad \text{(ii) } \sup_{s \leq 0} \chi(s) = \infty. \ \quad (2)$$

However, the controller needs not know which of these (possibly both) is valid. That properties (i) and (ii) may hold simultaneously for a function $f$ with the high-gain property is illustrated by following example. Let $m = 1$ and let $f$ be given by

$$f(\delta, z, u) = u \sin \left( \ln(1 + |u|) \right), \quad (\delta, z, u) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R},$$

which has the set of zeros $\{ u_k, -u_k \}$ with

$$u_k = e^{k\pi} - 1, \quad k \in \mathbb{N}_0.$$
Define the sequence \( \{s_k\} \) by
\[
s_k := \frac{1}{2}(u_{k+1} - u_k) = \frac{1}{2}e^{k\pi}(e^{\pi} - 1) > 0, \quad k \in \mathbb{N}.
\]
Noting that \( 4e^{\pi/2} < e^{\pi} - 1 \), we have
\[
\ln \left( 1 + \frac{1}{2}s_k \right) = \ln \left( e^{k\pi} \left( e^{-k\pi} + \frac{1}{2}(e^{\pi} - 1) \right) \right) > k\pi + \frac{\pi}{2}.
\]
Also,
\[
\ln \left( 1 + s_k \right) = \ln \left( e^{k\pi} \left( e^{-k\pi} + \frac{1}{2}(e^{\pi} - 1) \right) \right) < \ln \left( e^{k\pi} (e^{\pi}/2) \right) = (k+1)\pi - \ln 2.
\]
Therefore, for all \( v \in \mathbb{R} \) with \( \frac{1}{4} \leq |v| \leq 1 \) we have
\[
k\pi + \frac{\pi}{2} < \ln(1 + s_k|v|) < (k+1)\pi - \ln 2.
\]
It follows that
\[
0 < \sin(\pi - 2) < \left\{ \begin{array}{ll}
+ \sin(\ln(1 + s_k|v|)), & k \text{ even} \\
- \sin(\ln(1 + s_k|v|)), & k \text{ odd}
\end{array} \right\} < 1.
\]
Set \( v^* = \frac{1}{2} \). Then we find that
\[
\chi(-s_{2k}) = \min_{\frac{1}{2} \leq |v| \leq 1} s_{2k} v^2 \sin(\ln(1 + s_{2k}|v|)) > \frac{1}{2} s_{2k} \sin(\pi - 2)
\]
and
\[
\chi(s_{2k+1}) = \min_{\frac{1}{2} \leq |v| \leq 1} -s_{2k+1} v^2 \sin(\ln(1 + s_{2k+1}|v|)) > \frac{1}{2} s_{2k+1} \sin(\pi - 2).
\]
Since \( \sin(\pi - 2) > 0 \), it follows that \( \sup_{s > 0} \chi(s) = \infty = \sup_{s < 0} \chi(s) \). Therefore, the high-gain property holds and both properties (i) and (ii) also hold.

(b) If (i) (respectively, (ii)) holds, then we say that the system has the **negative-definite high-gain property** (respectively, the **positive-definite high-gain property**).

(c) For linear systems with strict relative degree, we will show in Section 2.1.3 that (i) (respectively, (ii)) is equivalent to the high-frequency gain matrix being negative definite (respectively, positive definite).

(d) If it is known in advance that the negative-definite (respectively, positive-definite) high-gain property holds, then the controller structure can be simplified considerably as we will discuss in Remark 1.6.

Now we are in a position to define the general system class to be considered.

**Definition 1.4 (System class).** For \( m, r \in \mathbb{N} \) we say that system \( \{1\} \) belongs to the system class \( \mathcal{N}^{m,r} \), written \( (d, f, T) \in \mathcal{N}^{m,r} \), if, for some \( p, q \in \mathbb{N} \) and \( h \geq 0 \) the following hold: \( d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \), \( f \in C(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m, \mathbb{R}^m) \) has the high-gain property, and the operator \( T \) is of class \( T_{h,0}^{1/r} \).

We emphasize that the system class \( \mathcal{N}^{m,r} \) is parameterized only by two integers, namely, \( m \) (which denotes the common dimension of the input and output spaces) and \( r \) (which is related to the concept of relative degree). In particular, the class \( \mathcal{N}^{m,r} \) encompasses systems with arbitrary state space dimension, including systems with infinite-dimensional internal dynamics, see e.g. \( \{1\} \); we will elaborate further on this in Section 4.


1.2 Control objectives

The control problem to be addressed is to determine an output derivative feedback strategy which ensures that, for every system of class $\mathcal{N}^{m,r}$ and any reference signal $y_{\text{ref}} \in W^{r,m}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, the output $y$ approaches the reference $y_{\text{ref}}$ with prescribed transient behaviour and asymptotic accuracy. This objective is reflected in the adoption of a so-called “performance funnel”, defined by

$$\mathcal{F}_\varphi := \{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t) \| e \| < 1 \},$$

in which the error function $t \mapsto e(t) := y(t) - y_{\text{ref}}(t)$ is required to evolve; see Fig. 1.

The funnel is shaped – through the choice of its boundary (determined by the reciprocal of $\varphi$) – in accordance with the specified transient behaviour and asymptotic accuracy; $\varphi$ is assumed to belong to the set

$$\Phi := \left\{ \varphi \in \text{AC}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \mid \forall t > 0 : \varphi(t) > 0, \liminf_{t \to \infty} \varphi(t) > 0, \exists c > 0 : |\varphi(t)| \leq c(1 + \varphi(t)) \text{ for a.a. } t \geq 0 \right\}, \quad (4)$$

where $\text{AC}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ denotes the set of locally absolutely continuous functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Note that, for $t > 0$, the funnel $t$-section $\mathcal{F}_\varphi \cap \{t\} \times \mathbb{R}^m$ is the open ball in $\mathbb{R}^m$ of radius $1/\varphi(t)$.

While it is often convenient to adopt a monotonically shrinking funnel (through the choice of a monotonically increasing function $\varphi$), it might be advantageous to widen the funnel over some later time intervals to accommodate, for instance, periodic disturbances or strongly varying reference signals.

1.3 Funnel control structure

We outline the design of funnel control for any system $\mathcal{N}^{m,r}$ of class $\mathcal{N}^{m,r}$. 

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Fig. 1: Performance funnel $\mathcal{F}_\varphi$. 

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Tracking via funnel control. Throughout, it is assumed that the instantaneous value of the output $y(t)$ and its first $r-1$ derivatives $\dot{y}(t), \ldots, \ddot{y}(r-1)(t)$ are available for feedback. Admissible reference signals are functions $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. The instantaneous reference value $y_{\text{ref}}(t)$ is assumed to be accessible to the controller and, if $r \geq 2$, then, for some $\hat{r} \in \{1, \ldots, r\}$, the derivatives $\dot{y}_{\text{ref}}(t), \ldots, \ddot{y}_{\text{ref}}^{(\hat{r}-1)}(t)$ are also accessible for feedback. In summary, for some $\hat{r} \in \{1, \ldots, r\}$, the following instantaneous vector is available for feedback purposes:

$$
e(t) = (e^{(0)}(t), \ldots, e^{(\hat{r}-1)}(t), \ldots, y^{(r-1)}(t)) \in \mathbb{R}^m, \quad e(t) := y(t) - y_{\text{ref}}(t), \quad (5)$$

with the notational convention that $e^{(0)} \equiv e$.

Feedback strategy. Preliminary ingredients in the feedback construction, called funnel control design parameters, are:

$$\begin{align*}
\Phi \subseteq \Phi, & \quad \text{bounded if } \hat{r} < r, \\
N \subseteq C(\mathbb{R}_{\geq 0}; \mathbb{R}), & \quad \text{a surjection,} \\
\alpha \subseteq C^1([0, 1), [1, \infty)), & \quad \text{a bijection.} \\
\end{align*} \quad \text{(6)}$$

These functions are open to choice. For notational convenience, we define

$$\gamma: \mathcal{B} \to \mathbb{R}^m, \quad w \mapsto \alpha(\|w\|^2) w, \quad \text{where } \mathcal{B} := \{ w \in \mathbb{R}^m \mid \|w\| < 1 \}. \quad (7)$$

Next, we introduce continuous maps $\rho_k: \mathcal{B}_k \to \mathcal{B}$, $k = 1, \ldots, r$, recursively as follows:

$$\begin{align*}
\mathcal{B}_1 & := \mathcal{B}, \\
\rho_1: \mathcal{B}_1 & \to \mathcal{B}, \quad \eta_1 \mapsto \eta_1, \\
\mathcal{B}_k & := \left\{ (\eta_1, \ldots, \eta_k) \in [\mathbb{R}^m \times \mathcal{B}_{k-1} \mid \eta_1, \ldots, \eta_k \in \mathcal{B}_{k-1}, \eta_k + \gamma(\rho_{k-1}(\eta_1, \ldots, \eta_{k-1})) \in \mathcal{B} \right\}, \\
\rho_k: \mathcal{B}_k & \to \mathcal{B}, \quad (\eta_1, \ldots, \eta_k) \mapsto \eta_k + \gamma(\rho_{k-1}(\eta_1, \ldots, \eta_{k-1})). \\
\end{align*} \quad (8)$$

Note that each of the sets $\mathcal{B}_k$ is non-empty and open. With reference to Fig. 2 and with $e$ and $\rho$, defined by (5) and (8), the funnel controller is given by

$$u(t) = (N \circ \alpha)(\|w(t)\|^2) w(t), \quad w(t) := \rho_k(\varphi(t) e(t)). \quad (9)$$

Example 1.5. Choosing the design parameter triple

$$\varphi \in \Phi \cap L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0}), \quad N: s \mapsto s \sin(s), \quad \alpha: s \mapsto 1/(1-s),$$

the feedback becomes

$$u(t) = \left(1 - \|w(t)\|^2\right)^{-1} \sin \left(\left(1 - \|w(t)\|^2\right)^{-1}\right) \cdot w(t),$$

where the signal $w(t)$ is, for example,

$$w(t) = \begin{cases} 
\varphi(t) e(t), & \text{if } r = 1 = \hat{r}, \\
\varphi(t) \dot{y}(t) + \gamma(\varphi(t) e(t)), & \text{if } r = 2, \hat{r} = 1, \\
\varphi(t) \ddot{y}(t) + \gamma(\varphi(t) e(t)), & \text{if } r = 2, \hat{r} = 2, \\
\varphi(t) \dddot{y}(t) + \gamma(\varphi(t) \ddot{y}(t) + \gamma(\varphi(t) e(t))), & \text{if } r = 3, \hat{r} = 1, 
\end{cases}$$

with $\gamma$ given by (7).
Remark 1.6. Some comments are warranted.

(a) The intermediate signal \( w(t) \) in (9) is a feedback – via the function \( \gamma \) – of the available information, given by (5), “weighted” by \( \varphi(t) \).

(b) Note the striking simplicity of the control (9): proportional feedback of the information vector \( w(t) \), with scalar gain. We point out that the complexity of the controller is much lower than in previous approaches such as [8], where successive derivatives of auxiliary error variables need to be calculated before implementation. This complicates the feedback structure for larger values of the parameter \( r \). In (9) all required signals are explicitly given by the recursion in (8) and can be implemented directly.

(c) The parameter \( \hat{r} \in \{1, \ldots, r\} \) specifies the number of derivatives of \( y_{ref} \) available for feedback. With increasing \( \hat{r} \), more information becomes accessible and so, not unreasonably, it might be expected that, loosely speaking, controller performance improves: this expectation is borne out by numerical simulations in Section 3.

(d) Note that, if \( \hat{r} = r \), then polynomial or exponentially increasing funnel functions \( \varphi \) are admissible. For example, the choices \( \varphi: t \mapsto at^\ell \) or \( \varphi: t \mapsto e^{at} - 1 \), \( a > 0, \ell \in \mathbb{N} \), ensure polynomial/exponential decay (to zero) of the tracking error \( t \mapsto e(t) = y(t) - y_{ref}(t) \). If \( \hat{r} < r \), then boundedness of \( \varphi \) is required. As an example in this case, the choice \( \varphi: t \mapsto \min\{e^{at} - 1, b\} \), \( a, b > 0 \), ensures that the tracking error approaches the ball of (arbitrarily small) radius \( b^{-1} \) exponentially fast and resides in that ball for all \( t \geq a^{-1} \ln(1 + b) \).

(e) Funnel control presents an anomaly: its performance might seem to contradict the internal model principle which asserts that “a regulator is structurally stable only if the controller […] incorporates […] a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process” [63, p. 210]. Diverse sources echo this principle – one such source is noted in [27]: a young Mark Twain, when apprenticed to a Mississippi river pilot, recorded the latter’s advice on navigating the river in the words “you can always steer by the shape that’s in your head, and never mind the one that’s before.

Footnote 1: The auxiliary error variables are given by \( e_i(t) \) in equation (5) of [8] for \( i = 0, \ldots, r - 1 \).
your eyes" [60, Ch.VIII]. But the funnel controller has no “shape” in its “head”,
it operates only on what is before its eyes. It does not incorporate “a suitably
reduplicated model […] of the exogenous signals”. How is this anomaly to be
resolved? The internal model principle applies in the context of exact asymptotic
tracking of reference signals. In the case of a bounded funnel function \( \phi \), only
approximate tracking, with non-zero prescribed asymptotic accuracy, is assured
and thus the anomaly is circumvented.

(f) But what of the case of unbounded funnel function \( \phi \), which is permissi-
ble whenever \( \hat{r} = r \)? In this case, exact asymptotic tracking is achieved. Return-
ning to the control-theoretic origins of the internal model principle, summarised
in [63, p. 210] as “every good regulator must incorporate a model of the outside
world”, we regard the term “good regulator” as most pertinent. A fundamental
ingredient of the funnel controller is the quantity

\[ \phi(t) e(t) \]

which, in the case of unbounded \( \phi \), inevitably leads to an ill-conditioned computation of the product
of “infinitely large” and “infinitesimally small” terms. Such a controller cannot
be deemed “good”. Whilst of theoretical interest, the case of unbounded \( \phi \) is of
limited practical utility.

Remark 1.7. We comment on the function \( N \in C(\mathbb{R}_{\geq 0}, \mathbb{R}) \) in (6). Note that \( N \) is a
surjection if, and only if,

\[ \limsup_{s \to \infty} N(s) = +\infty \quad \text{and} \quad \liminf_{s \to \infty} N(s) = -\infty. \]  \hspace{1cm} (10)

These two conditions are a generalization of so called Nussbaum functions discussed
in Section 2.2. Reiterating Remark 1.3, the high-gain property implies that at least
one of the conditions in (2) must hold. In the absence of knowledge of which of these
two possibilities is valid, the role of the function \( N \) is to provide the controller with the
capability of “probing” or implicitly accommodating each possibility. However, if it
is known that (i) in (2) holds, then the choice \( N : s \mapsto s \) suffices and the feedback takes the form

\[ u(t) = \alpha(|w(t)|^2) w(t). \]

Similarly, if (ii) holds, then \( N : s \mapsto -s \) suffices and the feedback takes the form

\[ u(t) = -\alpha(|w(t)|^2) w(t). \]

As the example in Remark 1.3(a) shows, it is also possible that (i) and (ii) hold
simultaneously, in which case both feedback laws are feasible. To illustrate this, we
consider the system

\[ \dot{x}(t) = u(t) \sin\ln(1 + |u(t)|), \quad x(0) = 1, \]

with \( x(t), u(t) \in \mathbb{R} \) under the control (9) with \( \alpha : s \mapsto 1/(1 - s) \) and \( N : s \mapsto \sigma s \) for
\( \sigma \in \{-1, 1\} \), that is

\[ u(t) = \frac{\sigma w(t)}{1 - w(t)^2}, \quad w(t) = \varphi(t) \left(y(t) - y_{\text{ref}}(t)\right). \]

We choose \( \varphi(t) = t^2 \), \( y_{\text{ref}}(t) = \sin t \) for \( t \geq 0 \) and perform the simulation over the
time interval [0, 10].

The results are shown in Fig. 3 where the tracking error and input function for \( \sigma =
\]

\footnote{All simulations in the paper are MATLAB generated (solver: ode45, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}).}
-1 are depicted in Figs. 3a and 3c and for $\sigma = 1$ in Figs. 3b and 3d, resp. In the latter case, the input exhibits a sharp increase when the tracking error approaches the funnel boundary, and it stays within the interval $[20, 25]$ thereafter, while for $\sigma = -1$ the input stays within the interval $[-1.5, 1.5]$. This suggests that the system structure allows the input to “probe” for an appropriate interval of control values, independent of the sign of $\sigma$.

### 1.4 Funnel control – main result

If the funnel controller (9) is applied to a system (1), then the first issue is to prove the existence of solutions of the closed-loop initial-value problem and to establish the efficacy of the control. We stress that the proof is quite delicate – even in the case of linear systems of the form (13). The reason is that the function $\alpha$ used in the feedback (9) introduces a potential singularity on the right hand side of the closed-loop differential equation.
By a solution of (1), (2) on \([-h, \omega]\) we mean a function \(y \in C_r^{-1}(\[-h, \omega\], \mathbb{R}^m)\), \(\omega \in (0, \infty)\), with \(y|_{[-h,0]} = y^0\) such that \(y^{r-1}|_{[0,\omega]}\) is locally absolutely continuous and satisfies the differential equation in (1) with \(u\) defined in (2) for almost all \(t \in [0,\omega)\); \(y\) is said to be maximal, if it has no right extension that is also a solution.

We are now in the position to present the main result for systems belonging to the system class \(\mathcal{N}^m_r\).

**Theorem 1.8.** Consider system (1) with \((d, f, T) \in \mathcal{N}^m_r, m, r \in \mathbb{N}\), and initial data \(y^0 \in C_r^{-1}(\[-h,0\], \mathbb{R}^m)\). Choose the triple \((\alpha, N, \varphi)\) of funnel control design parameters as in (6) and let \(y_{\text{ref}} \in W^{r, \infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)\) be arbitrary. Assume that, for some \(\hat{r} \in \{1, \ldots, r\}\), the instantaneous vector \(\varepsilon(t)\), given by (5), is available for feedback and the following holds:

\[
\varphi(0)e(0) \in \mathcal{P}_r,
\]

(trivially satisfied if \(\varphi(0) = 0\)). Then the funnel control (2) applied to (1) yields an initial-value problem which has a solution, every solution can be maximally extended and every maximal solution \(y : [-h, \omega) \to \mathbb{R}^m\) has the properties:

(i) \(\omega = \infty\) (global existence);
(ii) \(u \in L^r(\mathbb{R}_{\geq 0}, \mathbb{R}^n)\), \(y \in W^{r, \infty}(\[-h, \infty), \mathbb{R}^m)\);
(iii) the tracking error \(e : \mathbb{R}_{\geq 0} \to \mathbb{R}^m\) as in (5) evolves in the funnel \(\mathcal{F}_\varphi\) and is uniformly bounded away from the funnel boundary

\[
\partial \mathcal{F}_\varphi = \{(t, \zeta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t) \parallel \zeta \parallel = 1\}
\]

in the sense that there exists \(\varepsilon \in (0, 1)\) such that \(\varphi(t) \parallel e(t) \parallel \leq \varepsilon\) for all \(t \geq 0\).
(iv) If \(\hat{r} > 1\) and \(\varphi\) is unbounded, then \(e^{(k)}(t) \to 0\) as \(t \to \infty, k = 0, \ldots, \hat{r} - 1\).
(v) If the system is known to satisfy the negative-definite (respectively, positive-definite) high-gain property (see Remark [7,3]), then the feedback (3) may be simplified by choosing \(N\): \(s \mapsto s\) (respectively, \(N\): \(s \mapsto -s\)) and Assertions (i)–(iv) remain valid.

The proof is relegated to Appendix A.

When interpreted in specific cases, the initial condition constraint (11) becomes more transparent. For example, in the relative-degree-one case \(r = \hat{r} = 1\), it is simply the requirement that \(\varphi(0) \parallel e(0) \parallel < 1\), where \(e(0) = y^0(0) - y_{\text{ref}}(0)\) and, in the case \(r = 2\), it is equivalent to the same requirement augmented by

\[
\varphi(0)e(0) + \gamma(\varphi(0)e(0)) < 1,
\]

with \(z = \begin{cases} y^0(0) - y_{\text{ref}}(0), & \text{if } \hat{r} = 2, \\ y^0(0), & \text{if } \hat{r} = 1. \end{cases}\)

In some specific circumstances, computation of a priori bounds on the evolution of the tracking error \(e\) and (some of) its derivatives is possible. We highlight one such circumstance. Assume that \(\hat{r} \geq 2\) and \(\varphi \in \Phi\) is such that \(\varphi(0) > 0\). Define

\[
\mu_0 := \text{ess sup}_{t \geq 0} \left(\|\varphi(t)\|/\|\varphi(t)\|\right).
\]
Let $\alpha^\dagger \in C^1(\mathbb{R}_{\geq 0}, [0, 1))$ denote the inverse of the continuously differentiable bijection $[0, 1) \to \mathbb{R}_{\geq 0}, s \mapsto s\alpha(s)$ and, for notational convenience, introduce the continuous function

$$\tilde{\alpha}: [0, 1) \to \mathbb{R}_{\geq 0}, s \mapsto 2s\alpha'(s) + \alpha(s).$$

Define $(\mu_k, e_0^k, c_k), k = 1, \ldots, \hat{r} - 1$, recursively as follows:

$$
\begin{align*}
\mu_k &:= 1 + \mu_0(1 + c_{k-1}\alpha(c_k^{2-1})) + \tilde{\alpha}(c_k^{2-1}),(\mu_{k-1} + c_{k-1}\alpha(c_{k-1}^{2-1})), \\
e_0^k &:= \varphi(0)e_0(0), \quad c_1 := \max\{|e_0^2|, (1 + \mu_0))\}^{1/2} < 1, \quad \mu_1 := 1 + \mu_0c_1, \\
\mu_k &:= 1 + \mu_0(1 + c_{k-1}\alpha(c_k^{2-1})) + \tilde{\alpha}(c_k^{2-1}),(\mu_{k-1} + c_{k-1}\alpha(c_{k-1}^{2-1})), \\
e_k^0 &:= \varphi(0)e_k^{(k-1)}(0) + (\alpha(e_k^{0})^2)e_k^{0}, \\
c_k &:= \max\{|e_k^2/\alpha(\mu_k))\}^{1/2} < 1.
\end{align*}
$$

(12)

We emphasize that the constants $c_k$ are determined by the design parameters $\varphi$ and $\alpha$, together with the known initial data: $y(0), \ldots, y^{(\hat{r}-1)}(0)$ and $y_{\text{ref}}(0), \ldots, y_{\text{ref}}^{(\hat{r}-1)}(0)$.

**Corollary 1.9.** Let all hypotheses of Theorem 1.8 hold. Assume, in addition, that $\hat{r} \geq 2, \quad \varphi(0) > 0$ and $\alpha^\dagger$ is monotonically non-decreasing.

Then, for every maximal solution $y: [-h, \infty) \to \mathbb{R}^m$ of the feedback system (1) & (2), the tracking error $e = y - y_{\text{ref}}$ and its first $\hat{r} - 2$ derivatives satisfy, for all $k = 1, \ldots, \hat{r} - 2$ and all $t \geq 0$,

$$
\|e(t)\| \leq \varphi(t)^{-1}c_1, \quad \|e^{(k)}(t)\| \leq \varphi(t)^{-1}(c_{k+1} + c_k\alpha(c_k^{2}))
$$

where the constants $c_k$ are given by (12).

The proof is relegated to Appendix A.

Note that these findings are much simpler than the complicated bounds derived in [3] Prop. 3.2.

**Example 1.10.** Assume $\hat{r} = 3, \varphi: t \mapsto a + bt, a, b > 0$, and $\alpha: s \mapsto 1/(1 - s)$. In this case, we have $\mu_0 = b/a$ and $\alpha^\dagger: s \mapsto s/(1 + s)$. Therefore, for all $t \geq 0$,

$$
e_0^1 = ae(0), \quad c_1 = \max\{|e_0^2|, (1 + \mu_0)/2 + \mu_0\}^{1/2} \quad \text{and} \quad \|e(t)\| \leq \frac{c_1}{a + bt}.
$$

Furthermore, $\mu_1 = 1 + \mu_0c_1$,

$$
\tilde{\alpha}: s \mapsto (1 + s)/(1 - s)^2, \quad \mu_2 = 1 + \mu_0(1 + c_1\alpha(c_1^2)) + \tilde{\alpha}(c_1^2)(\mu_1 + c_1\alpha(c_1^2)), \\
e_0^2 = ae(0) + (1 - |e_0^2|)^{-1}e_0^0, \quad c_2 = \max\{|e_0^2|, \mu_2/(1 + \mu_2\}^{1/2}
$$

and $\forall t \geq 0: \|e(t)\| \leq \frac{c_2 + c_1/(1 - c_1^2)}{a + bt}$.\)
1.5 Novelties and literature

Predecessors and relative degree: The parameter $r$ in (1) coincides with the concept of relative degree for many nonlinear examples belonging to the class (1). However, Theorem 1.8 is more general and holds for systems which do not necessarily have a relative degree as defined in, for example, (36). Adaptive control for systems with relative degree $r > 1$ has been an issue since the early days of high-gain adaptive control; see the early contribution (31) from 1984. An early approach which takes transient behaviour into account is (41) in 1991, using a feedback strategy that differs intrinsically from the funnel control methodology. Funnel control was introduced in 2002 by (31) for nonlinear functional systems of the form (1) with relative degree one, using a variant of the high-gain property from Definition 1.2. The efficacy of funnel control for systems (1) with arbitrary $r \in \mathbb{N}$ was demonstrated in (33) in 2007. However, the control structure in that paper is based on backstepping with attendant (but unavoidable) escalating controller complexity vis à vis the striking simplicity of the funnel controller for relative-degree-one systems. An alternative controller was developed in (39) for a special class of systems with $m = 1$ and arbitrary $r \in \mathbb{N}$, which is called the bang-bang funnel controller. Since the control input switches only between two values, it is able to respect input constraints; however, it requires various feasibility assumptions and involves a complicated switching logic. A simpler control strategy for nonlinear systems has been introduced by (26) for $r = 2$ in 2013 and by (8) for $r \in \mathbb{N}$ in 2018.

Prescribed Performance Control: A relative of funnel control is the approach of Prescribed Performance Control developed by Bechlioulis and Rovithakis (1) in 2008. Using so-called performance functions, which are special funnel boundaries, and a transformation that incorporates these performance functions, the original controlled system is expressed in a form for which boundedness of the states, via the prescribed performance control input, can be proved – achieving evolution of the tracking error within the funnel defined by the performance functions. The controller presented in (1) is not of high-gain type. Instead, neural networks are used to approximate the unknown nonlinearities of the system, but results in a complicated controller structure. After some developments, the complexity issue has been addressed in (2) in 2014, where Prescribed Performance Control is shown to be feasible for so-called pure feedback systems. We show in Section 2.3 that these systems are included in the class $\mathcal{A}^{m,r}$.

Controller complexity: Although implicitly explained in the two paragraphs above, we like to explicitly mention the issue of controller complexity, which was a guiding principle since the early days of adaptive control. For implementation purposes it is crucial to keep the controller complexity at a minimum. The first approaches to funnel control for systems with arbitrary relative degree in (32), (33) showed a controller complexity significantly increasing with the relative degree. Although these contributions have the advantage that only the output – and not its derivatives – need to be known, the intrinsic backstepping procedure yields high powers of the gain function (which typically takes very large values) for large relative degree and thus becomes impractical.

Avoiding the backstepping procedure, a low-complexity funnel controller has been developed in [26] for relative degree two systems and in [8] for arbitrary relative degree. Nevertheless, as pointed out in Remark 1.6, the control design developed in [8] involves successive derivatives of auxiliary error variables, which exhibit an increasing complexity for higher relative degree. The simple funnel control design (9) helps to resolve these issues.

**Unknown control direction and feedback gain:** In the early days of high-gain adaptive control without system identification, linear systems with relative degree one and positive high-frequency gain were studied, cf. Section 2.1. In 1983, Morse [43] conjectured the non-existence of a smooth adaptive controller which stabilizes every linear system under the assumption that the high-frequency gain is not zero but its sign is unknown. Nussbaum [45] showed that Morse’s conjecture is false and introduced a sign-sensing or probing “switching function” in the feedback law, see Section 2.2. In the present work, we allow for a much larger class of switching functions satisfying condition (10), and this might be advantageous in applications. We also emphasize that, as discussed in Remarks 1.3 and 1.7, no switching may be necessary, but there are systems where both feedback laws (resting on either a positive or negative high-frequency gain assumption) are feasible.

**Dead-zone input:** A dead-zone input is a special case of input nonlinearity where the value of the nonlinearity is zero when the input is between some prescribed deadband parameters, see Sections 2.4 and 2.5. A dead-zone input may appear in practical applications such as hydraulic servo valves and electronic motors, and it may severely affect the performance of a control system, see e.g. [57, 58]. Several approaches have been undertaken to treat these problems, see [44, 57, 58] and the references therein. We show that the system class \( \mathcal{N}^{m,r} \) encompasses a much larger class of dead-zone inputs than previously considered in the literature.

**Initial data constraints:** Consider the funnel controller (9). If \( \phi(0) = 0 \), then the funnel 0-section is the whole space \( \mathbb{R}^m \) and so there is no restriction required on the initial tracking error signals composed of \( y_0^{(k)}(0) \) and \( y_{\text{ref}}^{(k)}(0) \). Otherwise, the initial condition (11) has to hold. This is “standard” in previous and related works such as [2, 8, 31, 39]. However, for the higher relative degree cases considered in [2, 8, 39] a set of \( r \) funnel functions is used, each of which needs to satisfy an initial condition.

**Non-asymptotic tracking:** If \( \phi \in \Phi \) is bounded, then the funnel boundary is uniformly bounded away from zero and asymptotic tracking is not achieved. However, the design of the funnel boundary is at the designer’s discretion and may be chosen close to zero, in which practical tracking is met. This assumption is widespread in papers concerned with funnel control.

**Asymptotic tracking:** In the present paper we consider funnel control with possibly unbounded funnel function \( \phi \in \Phi \). This means asymptotic tracking – the funnel boundary tends asymptotically to zero – is achieved. This result has already been shown in [51] for a class of nonlinear relative degree one systems: in [28] a predecessor for linear relative degree one systems was developed utilizing the internal model principle. Recently (and unaware of the latter results) it was observed in [38]
that asymptotic funnel control is possible for a class of nonlinear single-input single-output systems, more restricted than the class \( \mathcal{N}^{m,r} \) of the present paper. Note also that asymptotic tracking via funnel control for systems with relative degree two has been shown by \([61],[62]\). However, the funnel boundaries used in these works are bounded away from zero and the additional property of exact asymptotic tracking is achieved via a discontinuous control scheme. In Prescribed Performance Control the performance functions are required to have a positive distance to zero, i.e., asymptotic tracking cannot be achieved, see e.g. \([2]\).

We emphasize that in case of unbounded \( \varphi \in \Phi \) the funnel controller \( (9) \) additionally achieves convergence of the tracking error and its derivatives \( e^{(k)}(t) \to 0 \), \( k = 0, \ldots, \hat{r} - 1 \); this is an interesting novel feature.

**Parameter \( \hat{r} \leq r \):** In many applications the derivatives of the reference trajectory for the output are not known. In these situations it is desirable that the feedback does not require the instantaneous values \( \dot{y}_{\text{ref}}(t), \ldots, y_{\text{ref}}^{(r-1)}(t) \). To take this into account, the parameter \( \hat{r} \leq r \) has been introduced in \([5]\). With \( \hat{r} \) it is possible to prescribe the number of derivatives of the error \( e^{(k)}(t) \), \( k = 0, \ldots, \hat{r} - 1 \), which need to be available for feedback; the higher derivatives need only be known for the output \( y^{(k)}(t) \), \( k = \hat{r}, \ldots, r - 1 \). This might be advantageous for applications: the more information that is available and implemented in the feedback law \( (9) \), the better the controller performance is expected to be.

**Applications:** The new funnel control strategy has a potential impact on various applications. Since its development in \([31]\) the funnel controller proved an appropriate tool for tracking problems in various applications such as temperature control of chemical reactor models \([34]\), control of industrial servo-systems \([25]\) and underactuated multibody systems \([9]\), speed control of wind turbine systems \([22],[24]\), current control for synchronous machines \([23]\), DC-link power flow control \([53]\), voltage and current control of electrical circuits \([14]\), oxygenation control during artificial ventilation therapy \([46]\), control of peak inspiratory pressure \([47]\), and adaptive cruise control \([12],[13]\).

**Summary of novelties:** The results of the present paper are also new for linear minimum phase systems with strict relative degree and sign-definite high-frequency gain matrix. The recursive design of the feedback signal leads to a further simplification and a better intuitive understanding of the controller as well as more flexibility in the control objectives. Moreover, the high-gain property allow to encompass a considerably larger class of systems and we are able to achieve exact asymptotic tracking.

2 **Subclasses and examples**

In this section, we show that the class of systems \( \mathcal{N}^{m,r} \) encompasses the prototype of linear multi-input multi-output systems with strict relative degree \( r \in \mathbb{N} \) and asymptotically stable zero dynamics, see Subsection \([2.1]\) and also a nonlinear generalization of it in pure feedback form, see Subsection \([2.3]\). Furthermore, the issues of control directions are discussed in Subsection \([2.2]\) and input nonlinearities in Subsection \([2.4]\).
2.1 The prototypical linear system class

As a concrete example we consider linear, finite-dimensional systems of the form
\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\
y(t) &= Cx(t)
\end{align*} \] (13)
where \((A,B,C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times n}, m \leq n\), and discuss its relationships to Properties (TP1)–(TP3) and the high-gain property.

2.1.1 Strict relative degree

We show that system (13) can be equivalently written in the form (1), if system (13) has (strict) relative degree \(r \in \mathbb{N}\), that is

\[ CA^kB = 0, \quad k = 0, \ldots, r - 2 \quad \text{and} \quad \Gamma := CA^{-1}B \quad \text{is invertible.} \]

It is shown in [33] that under this assumption there exists a state space transformation

\[ z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = Ux, \quad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}, \quad U \in \mathbb{R}^{n \times n} \text{ invertible,} \]

which transforms (13) into Byrnes-Isidori form

\[ \begin{align*}
\dot{z}(t) &= \tilde{A}z(t) + \tilde{B}u(t), \\
y(t) &= \tilde{C}z(t),
\end{align*} \]

where

\[ (\tilde{A}, \tilde{B}, \tilde{C}) = (UA^{-1}U, UB, CU^{-1}) \]

with

\[ \tilde{A} = \begin{bmatrix}
0 & I_m & 0 & \cdots & 0 & 0 \\
0 & 0 & I_m & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & I_m & 0 \\
R_1 & R_2 & \cdots & R_{r-1} & R_r & S \\
P & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
0_{m \times m} \\
\vdots \\
0_{m \times m} \\
\Gamma \\
0_{(n-rm) \times m}
\end{bmatrix}, \quad \tilde{C} = \begin{bmatrix}
I_m & 0_{m \times m} & \cdots & 0_{m \times m} & 0_{m \times (n-rm)}
\end{bmatrix}. \]

In the new coordinates, the system representation of (13) becomes

\[ \begin{align*}
\dot{\xi}_k(t) &= \xi_{k+1}(t), \quad k = 1, \ldots, r - 1, \\
\dot{\xi}_r(t) &= \sum_{k=1}^{r-1} R_k \xi_k(t) + S\eta(t) + \Gamma u(t), \\
\eta(t) &= P\xi_1(t) + Q\eta(t)
\end{align*} \]

with output \( y(t) = \xi_1(t) \). (14)
With the last equation in (14), the so-called internal dynamics, we may associate a linear operator
\[
L: y(\cdot) \mapsto \left( t \mapsto \int_0^t S_\tau y(t) d\tau \right).
\]
(15)

With initial data \( \eta(0) = [0, I_{n-m}] U x^0 \) and \( d(\cdot) := S \eta^0 \), we find that
\[
S \eta(t) = d(t) + L(y)(t).
\]

Introducing the (linear) operator
\[
T: C(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \rightarrow L_{\text{loc}}^m(\mathbb{R}_{\geq 0}, \mathbb{R}^m),
\]
\[
\zeta = (\zeta_1, \ldots, \zeta_r) \mapsto \left( t \mapsto \sum_{k=1}^r R_k \zeta_k(t) + L(\zeta_1)(t) \right),
\]
(16)

it follows from (14) that (13) is equivalent to the functional differential system
\[
\begin{aligned}
y^{(r)}(t) &= d(t) + T(y, \ldots, y^{(r-1)})(t) + \Gamma u(t) \\
y(0) &= C x^0, \ldots, y^{(r-1)}(0) = C A^{r-1} x^0.
\end{aligned}
\]
(17)

It is easy to see that the operator \( T \) satisfies properties (TP1) and (TP2) from Definition 1.1. The following section is devoted to (TP3).

2.1.2 Minimum phase

Suppose that system (13) has strict relative degree \( r \in \mathbb{N} \). Then the BIBO property (TP3) of the operator \( T \) in (17) is closely related to system (13) having asymptotically stable zero dynamics, i.e.,
\[
\forall \lambda \in \mathbb{C}_{\geq 0} : \det \left[ \begin{array}{cc}
\lambda I - A & B \\
C & 0
\end{array} \right] \neq 0.
\]
(18)

This concept (also closely related to the minimum phase property in the literature, cf. [17, 41]) is extensively studied since its relevance has been revealed in classical works such as [17, 41]. To be precise, assume that the transfer function \( C(sI - A)^{-1} B \in \mathbb{R}(s)^{m \times m} \) of \( (A, B, C) \) is invertible over \( \mathbb{R}(s) \), then we have the following:

\[(A, B, C) \text{ satisfies (18)} \quad \Leftrightarrow \quad (A, B, C) \text{ stabilizable & detectable,} \quad C(sI - A)^{-1} B \text{ has no zeros in } \mathbb{C}_{\geq 0}
\]
\[
\quad \Downarrow \quad (A, B, C) \text{ stab. & det.,} \quad S(sI - Q)^{-1} P \text{ has no poles in } \mathbb{C}_{\geq 0}
\]
(3, Cor. 2.8)

\[(A, B, C) \text{ satisfies (18)} \quad \Leftrightarrow \quad (A, B, C) \text{ stabilizable & detectable,}
\]
\[(A, B, C) \text{ stab. & det.,}\]
\[T \text{ satisfies (TP3)} \quad \Leftrightarrow \quad (A, B, C) \text{ stabilizable & detectable,} \]
\[
S(sI - Q)^{-1} P \text{ has no poles in } \mathbb{C}_{\geq 0}
\]

(3, Cor. 3.3)

For the last equivalence above we note that by [59, Thm. 3.21] it is straightforward that \( S(sI - Q)^{-1} P \) having no poles in \( \mathbb{C}_{\geq 0} \) is equivalent to \( (Q, P, S) \) being externally stable or, in other words, the operator \( L \) from (15) satisfies (TP3). It is easily seen that this is the same as \( T \) satisfying (TP3).
2.1.3 Sign-definite high-frequency gain matrix

We show that system (13) satisfies the high-gain property (recall Definition 1.2) if, and only if, the high-frequency gain matrix \( \Gamma = CA^{-1}B \) is sign-definite\(^3\). Otherwise stated, we seek to establish the following equivalence:

(a) \( (13) \) has the high-gain property \( \iff \forall v \in \mathbb{R}^m \setminus \{0\} : v^\top \Gamma v \neq 0 \). 

(a) \( \implies \) (b): Assume (a). Let \( v' \in (0, 1) \) be given and choose \( K_p = \{0\}, K_q = \{0\} \). Write \( A_m := \{v \in \mathbb{R}^m \mid v^* \leq \|v\| \leq 1\} \). Suppose (b) is false. Then there exists \( \hat{v} \in A_m \) such that \( \hat{v}^\top \Gamma \hat{v} = 0 \), thus

\[ \forall s \in \mathbb{R} : \chi(s) = \min_{v \in \mathbb{R}^m} (-sv^\top \Gamma v) \leq -s\hat{v}^\top \Gamma \hat{v} = 0, \]

which contradicts (a).

(b) \( \implies \) (a): Assume (b). Then there exists \( \sigma \in \{-1, 1\} \) such that \( \sigma \Gamma \) is positive definite. Let \( G := (\sigma/2)(\Gamma + \Gamma^\top) \) denote the symmetric part of \( \sigma \Gamma \) and let \( \lambda_1 > 0 \) be the smallest eigenvalue of \( G \). Set \( v^* = \frac{1}{2} \), choose compact \( K_p \subset \mathbb{R}^p \) and \( K_q \subset \mathbb{R}^q \) and define

\[ c_1 := \min \left\{ v^\top (\delta + z) \mid (\delta, z, v) \in K_p \times K_q \times A_m \right\} . \]

Then,

\[ \forall s \in \mathbb{R} : \chi(s) - c_1 \geq \min_{v \in A_m} (-sv^\top \Gamma v) = \min_{v \in \mathbb{A}_m} (-sv^\top Gv). \]

Let \( (s_n) \) be a real sequence with \( \sigma s_n < 0 \) for all \( n \in \mathbb{N} \) and \( \sigma s_n \to -\infty \) as \( n \to \infty \). It follows that

\[ \forall n \in \mathbb{N} \forall v \in A_m : -\sigma s_n v^\top Gv \geq -\sigma s_n \|v\|^2 \geq -\frac{\sigma s_n \lambda_1}{4} \]

and so we have

\[ \forall n \in \mathbb{N} : \chi(s_n) \geq c_1 - \frac{\|s_n\|}{4} . \]

Therefore, \( \chi(s_n) \to \infty \) as \( n \to \infty \) and so (a) holds.

2.2 Known and unknown control directions

For linear systems (13) with relative degree \( r \in \mathbb{N} \) the notion of “control direction” is captured by the sign of the high-frequency gain matrix \( \Gamma = CA^{-1}B \) as discussed in Section 2.1.3. More precisely, if \( \sigma \Gamma \) is positive definite for some \( \sigma \in \{-1, 1\} \), then \( \sigma \) is called the control direction. If \( \sigma \) is known and the system (13) has asymptotically stable zero dynamics, see (13), then it can be shown that the “classical high-gain adaptive feedback”

\[ u(t) = -\sigma k(t)y(t) , \quad \dot{k}(t) = \|y(t)\|^2 , \quad (19) \]

\(^3\) Recall that \( \Gamma \in \mathbb{R}^{m \times m} \) is positive definite, if \( v^\top \Gamma v > 0 \) for all \( v \in \mathbb{R}^m \setminus \{0\} \). \( \Gamma \) is negative definite, if \( -\Gamma \) is positive definite. \( \Gamma \) is sign definite, if it is either positive or negative definite: equivalently, if \( v^\top \Gamma v = 0 \iff v = 0 \).
with \( k(0) = k^0 \geq 0 \), applied to \((13)\) yields a closed-loop system, where for any solution \((x,k)\) we have that \( x(t) \to 0 \) as \( t \to \infty \) and \( k(\cdot) \) is bounded; see \([17,41,43]\).

For the case of unknown control direction \( \sigma \), the adaptive stabilization was an obstacle over many years. Morse \([43]\) conjectured the non-existence of a smooth adaptive controller which stabilizes every linear single-input single-output system \((13)\), i.e. \( m = 1 \), under the assumption that \( \Gamma \neq 0 \). It was shown by Nussbaum in \([45]\) that this conjecture is false: One has to incorporate a “sign-sensing function” in the feedback law \((19)\) so that it becomes

\[
    u(t) = -N(k(t))y(t), \quad \dot{k}(t) = \|y(t)\|^2, \tag{20}
\]

where the smooth function \( N: \mathbb{R}_\geq 0 \to \mathbb{R} \) satisfies the so-called Nussbaum property

\[
    \forall k^0 \geq 0: \sup_{k^0 \geq k} \int_{k^0}^k N(\kappa) d\kappa = \infty \quad \text{and} \quad \inf_{k^0 \geq k} \int_{k^0}^k N(\kappa) d\kappa = -\infty, \tag{21}
\]

see, for example, \([18–20,37,64]\). Loosely speaking, when incorporated in the control design, “Nussbaum” functions provide a mechanism that can “probe” in both control directions.

The present paper utilizes a larger class of “probing” functions: in particular, the proposed control design permits the adoption of any continuous function \( N: \mathbb{R}_\geq 0 \to \mathbb{R} \) which is surjective or, equivalently, satisfies \((10)\). Properties \((21)\) imply properties \((10)\), but the reverse implication is false: for example, the function \( s \mapsto N(s) = s \sin s \) exhibits properties \((10)\), but fails to exhibit the Nussbaum properties \((21)\).

### 2.3 A nonlinear generalization of the linear prototype

Consider the system

\[
    \begin{align*}
    \dot{\xi}_k(t) &= f_k(\xi_1(t), \ldots, \xi_{k+1}(t)), \quad k = 1, \ldots, r-1, \\
    \dot{\xi}_r(t) &= f_r(d(t), \xi_1(t), \ldots, \xi_r(t), \eta(t), u(t)), \\
    \eta(t) &= g(d(t), \xi_1(t), \ldots, \xi_r(t), \eta(t)),
    \end{align*}
\]

with output \( y(t) = \xi_1(t) \), \( (22) \)

where \( d \in L^\infty(\mathbb{R}_\geq 0, \mathbb{R}^p) \) is a disturbance or perturbation. We remark there is no loss of generality in assuming that both the second and third of the above differential equations are subject to the same disturbance \( d \). If these equations are subject to separate disturbances \( d_1 \) and \( d_2 \), then, on writing \( d = (d_1, d_2) \) and re-defining the functions \( f_k \) and \( g \) by the inclusion of projections \( \pi_1: d \mapsto d_1 \) and \( \pi_2: d \mapsto d_2 \), we recover \((22)\). The functions \( f_k: \mathbb{R}^{(k+1)m} \to \mathbb{R}^m \), \( k = 1, \ldots, r-1 \), \( f_r: \mathbb{R}^{p+rm+q+m} \to \mathbb{R}^m \) and \( g: \mathbb{R}^{p+rm+q} \to \mathbb{R}^q \) are assumed sufficiently regular in a sense that will be made precise.

System \((22)\) is a nonlinear generalization of the linear prototype \((14)\) in Byrnes-Isidori form. If the variable \( \eta \) and its generating differential equation (a counterpart of the internal dynamics of \((14)\)) are excised from \((22)\), the resulting reduced system is of so-called pure feedback form, as studied e.g. in \([2]\) (see also references therein). Pure-feedback systems are fundamental to the “backstepping” control methodology.
The ability to encompass the additional internal dynamics is a distinguishing feature of the approach of the present paper. We proceed to identify conditions under which system (22) is of class (1).

First, we introduce regularity assumptions on the functions \( f_k \). This necessitates some notation: for \( f \in C^1(\mathbb{R}^{nm}, \mathbb{R}^m) \) we denote by \( \partial f(x) = \frac{\partial f}{\partial x}(x) \) the \((m \times m)\)-matrix-valued function of partial derivatives of the components of \( f \) with respect to components of its \( k \)-th argument evaluated at \( x \). Specifically, for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^{nm} \), the \( ij \)-th entry of \( \partial f \) at \( x \) is

\[
[\partial f(x)]_{ij} = \frac{\partial [f(x)]_i}{\partial [x]_j} = \left( \frac{\partial f}{\partial x} \right)_{ij}, \quad i, j = 1, \ldots, m,
\]

Returning to the context of (22), we assume that, for each \( k = 1, \ldots, r - 1 \), \( f_k \) is \( r - k \) times continuously differentiable and its derivative with respect to its final argument \( \partial f_k \) is everywhere invertible; here we use the notation \( GL_m(\mathbb{R}) \) for set of all invertible matrices in \( \mathbb{R}^{m \times m} \). Specifically, we assume:

(P1) \( f_k \in C^{r-k}(\mathbb{R}^{(k+1)m}, \mathbb{R}^m) \), \( \partial f_{k+1} \in C^{r-k-1}(\mathbb{R}^{(k+1)m}, GL_m(\mathbb{R})) \), \( k = 1, \ldots, r - 1 \), \( f_r \in C^1(\mathbb{R}^{p+q}, \mathbb{R}^m) \) and \( g \in C^1(\mathbb{R}^{p+q}, \mathbb{R}^q) \), where \( q = rm + q \).

(P2) For each \( k = 1, \ldots, r - 1 \), there exists a nondecreasing function \( \zeta_k \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{> 0}) \) with

\[
\int_0^\infty \zeta_k(t)^{-1} \, dt = \infty,
\]

such that, for all \( x = (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{(k+1)m} \),

\[
\left\| (\partial f_{k+1}(x))^{-1} \cdot \left\| \left[ \partial f_k(x), \ldots, \partial f_k(x), -L_m \right] \right\| \leq \zeta_k(\|x_{k+1}\|) \right. \]

(P3) \( \forall c_0 > 0 \exists c_1 > 0 \forall \eta^0 \in \mathbb{R}^q \forall w \in L^m_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{p+rm}) : \)

\[
\|\eta^0\| + \sup_{t \geq 0} \|w(t)\| \leq c_0 \quad \Rightarrow \quad \sup_{t \geq 0} \|z(t; w, \eta^0)\| \leq c_1,
\]

where \( z(\cdot; w, \eta^0) \) denotes the unique maximal solution to the initial-value problem

\[
\eta(t) = g \left( w(t), \eta(t) \right), \quad \eta(0) = \eta^0.
\]

(P4) For all compact \( K \subset \mathbb{R}^{p+rm+q} \) there exist \( c_0, c_1 > 0 \) and \( \sigma \in \{-1, 1\} \) such that

\[
\forall (\delta, x_r, z) \in K \forall v \in \mathbb{R}^m \text{ with } \|v\| \geq c_0 : \min_{\|\theta\|=1} \sigma \theta^\top G(\delta, x_r, z, v) \theta \geq c_1,
\]

where we write \( x_k := (x_1, \ldots, x_k) \) for \( k = 1, \ldots, r \) and \( G \) is defined by

\[
G : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^{m \times m} \quad \delta, x_r, z, v \mapsto \partial f_1(x_1) \cdot \partial f_2(x_2) \cdot \cdots \cdot \partial f_{r-1}(x_{r-1}) \cdot \partial f_r(x_r, \delta, x_r, z, v).
\]
Some comments are warranted. Property (P1) ensures sufficient smoothness of the right hand side of (22). (P2) allows a later application of a global implicit function theorem. (P3) is a BIBO stability property of the internal dynamics required to obtain (TP3) for a suitable operator T. (P4) will ensure the high-gain property of the right hand side of (22). Essentially it means that, for every compact \( K \subset \mathbb{R}^{p+\gamma+q} \), there exist \( \sigma \in \{-1, 1\} \) (defining the control direction – whether known or unknown) and an open ball \( B \) centred at 0 such that \( \sigma G \) is positive definite and uniformly bounded away from zero on \( K \times (\mathbb{R}^m \setminus B) \).

We are now in a position to show that the class of systems (22) is encompassed by the class of systems (1), provided the functions on the right hand side of (22) satisfy properties (P1)–(P4).

**Proposition 2.1.** Any system in “pure feedback” form (22) satisfying the properties (P1)–(P4) is equivalent to a system (1) with \( (d, f, T) \in \mathcal{A}^{m'} \), where \( d, m, r \) are the same as in (22).

The proof is relegated to Appendix A.

**Example 2.2.** As a concrete illustrative example with \( r = 2 \) and \( m = p = q = 1 \), consider the system

\[
\begin{align*}
\dot{z}_1(t) &= (1 + z_1(t)^2) \dot{z}_2(t) \\
\dot{z}_2(t) &= f_0(d(t), z_1(t), \eta(t)) + \beta(u(t)) \\
\dot{\eta}(t) &= -\eta(t)^2 (a_1 \dot{z}_1(t) + a_2 \dot{z}_2(t) + \eta(t))
\end{align*}
\]

with output \( y(t) = \xi_1(t) \),

where, \( a_1, a_2 \in \mathbb{R} \), the functions \( f_0 \) and \( \beta \) are continuously differentiable and the disturbance is bounded, i.e., \( d \in L^m(\mathbb{R}_\geq 0, \mathbb{R}) \). In terms of (22), we have

\[
\begin{align*}
f_1(x_1, x_2) &= (1 + x_1^2)x_2, \\
f_2(\delta, x_1, x_2, z, v) &= f_0(\delta, x_1, x_2, z) + \beta(v), \\
g(\delta, x_1, x_2, z) &= -z^2(a_1 x_1 + a_2 z + z) =: g(x_1, x_2, z).
\end{align*}
\]

Therefore, \( \partial_2 f_1(x_1, x_2) = 1 + x_1^2 \) and property (P1) is evident. To show that property (P2) holds, consider the increasing function \( \zeta_1 \in C(\mathbb{R}_\geq 0, \mathbb{R}_> 0) \) given by \( \zeta_1(t) := \sqrt{2} (1 + t) \) for which \( \lim_{t \to \infty} \frac{\partial_1 \zeta_1}{\zeta_1} = \infty \). Moreover, for all \( x = (x_1, x_2) \in \mathbb{R}^2 \), we have

\[
\left\| (\partial_2 f_1(x))^{-1} \right\| \cdot \left\| \partial_1 f_1(x), -1 \right\| = (1 + x_1^2)^{-1} ((2x_1^2 + 1)^{1/2}) \leq \sqrt{2} (1 + |x_2|) = \zeta_1(|x_2|),
\]

whence property (P2). It remains to establish properties (P3) and (P4). For \( V(z) := \frac{1}{2} z^2, z \in \mathbb{R} \), and writing \( a_0 := \sqrt{a_1^2 + a_2^2} \), we have

\[
\forall (x, z) \in \mathbb{R}^2 \times \mathbb{R} : \, V'(z) \hat{g}(x, z) \leq -z^4 + a_0 |z|^3 \|x\| \leq -\frac{1}{4} z^4 + \frac{1}{2} (a_0 \|x\|)^4,
\]

wherein Young’s inequality has been used. Therefore, \( V \) is an iss-Lyapunov function for the system \( \dot{\eta} = \hat{g}(\xi_1, \xi_2, \eta) \) which, in consequence is input-to-state stable, see [55] Rem. 2.4 & Lem. 2.14. Therefore, (P3) holds *a fortiori*. Finally, we have

\[
G(\delta, x_1, x_2, z, v) = \partial_2 f_1(x_1, x_2) \cdot \partial_1 f_2(\delta, x_1, x_2, z, v) = (1 + x_1^2) \cdot B'(v)
\]
and so, if we assume the existence of \( c_0, c_1 > 0 \) and \( \sigma \in \{-1, +1\} \) such that

\[
\forall v \in \mathbb{R} \text{ with } |v| \geq c_0 : \; \sigma \beta'(v) \geq c_1,
\]

then (P4) is valid. We remark that the above assumption on the derivative \( \beta' \) implies that \( \beta \) is either strictly increasing or strictly decreasing – a property that is a special case of the subsequent considerations.

### 2.4 Input nonlinearities

In addition to accommodating the issue of (unknown) control direction (cf. Section 2.2), the generic formulation (1) with associated high-gain property encompasses a wide variety of input nonlinearities. Consideration of a scalar system of the simple form

\[
y'(t) = f_1(y(t)) + f_2(y(t)) \beta(u(t))
\]

(23)

with \( f_1 \in C(\mathbb{R}, \mathbb{R}) \), \( f_2 \in C(\mathbb{R}, \mathbb{R}\setminus\{0\}) \) and \( \beta \in C(\mathbb{R}, \mathbb{R}) \), will serve to illustrate this variety. The assumption that \( f_2 \) is a non-zero-valued continuous function ensures a well-defined control direction (unknown to the controller). Without loss of generality, we may assume that \( f_2 \in C(\mathbb{R}, \mathbb{R}_{>0}) \); if \( f_2 \) is negative-valued, then, in (23), simply replace \( f_2 \) by \(-f_2\) and \( \beta \) by \(-\beta\). We impose the following conditions on \( \beta \in C(\mathbb{R}, \mathbb{R}) \):

\[
\beta \text{ is surjective, with } |\beta(\tau)| \to \infty \text{ as } |\tau| \to \infty,
\]

(24)

which is equivalent to the requirement that one of the following conditions hold:

\[
\lim_{\tau \to \pm \infty} \beta(\tau) = \pm \infty \quad \text{or} \quad \lim_{\tau \to \pm \infty} \beta(\tau) = \mp \infty.
\]

We proceed to show that system (23) has the high-gain property. Set \( v^* = \frac{1}{2} \), let \( K_1 \subset \mathbb{R} \) be compact and define

\[
A_1 := [-1, -\frac{1}{2}] \cup \left[\frac{1}{2}, 1\right], \quad c_1 := \min \{ \, |vf_1(z)| \mid (z, v) \in K_1 \times A_1 \} \in \mathbb{R}.
\]

Consider the function

\[
\chi : \mathbb{R} \to \mathbb{R}, \; s \mapsto \min \{ \, \nu f_1(z) + f_2(z)\beta(-sv) \mid (z, v) \in K_1 \times A_1 \},
\]

Then

\[
\forall s \in \mathbb{R} : \; \chi(s) \geq c_1 + \min \{ \, vf_2(z)\beta(-sv) \mid (z, v) \in K_1 \times A_1 \}.
\]

(25)

Let \( M > 0 \) be arbitrary. To conclude that the high-gain property holds, it suffices to show that there exists \( s \in \mathbb{R} \) such that

\[
\forall (z, v) \in K_1 \times A_1 : \; vf_2(z)\beta(-sv) > M.
\]

Define

\[
c_2 := \min_{z \in K_1} f_2(z) > 0 \quad \text{and} \quad c_3 := 2M/c_2.
\]

By properties of \( \beta \), there exist \( \sigma \in \{-1, 1\} \) and \( c_4 > 0 \) such that

\[
\forall \tau > c_4 : \; \beta(\sigma \tau) > c_3 \quad \text{and} \quad \beta(-\sigma \tau) > c_3.
\]
Let \((z, v) \in K_1 \times A_1\) be arbitrary. Fix \(s \in \mathbb{R}\) such that \(\sigma s < -2c_4\) and so \(|sv| > c_4\). Then
\[
v f_2(z) \beta(-sv) = \begin{cases} 
|v|f_2(z)\beta(\sigma|sv|), & \text{if } v > 0 \\
|v|f_2(z)(-\beta(-\sigma|sv|)), & \text{if } v < 0 
\end{cases} > \frac{c_2c_3}{2} = M.
\]
Therefore, the high-gain property holds.

2.5 Dead-zone input

An important example of a nonlinearity \(\beta = D\) with properties (24) is a so-called dead-zone input of the form
\[
D : \mathbb{R} \to \mathbb{R}, \quad v \mapsto \begin{cases} 
D_r(v), & v \geq b_r, \\
0, & b_l < v < b_r, \\
D_l(v), & v \leq b_l
\end{cases}
\]
with unknown deadband parameters \(b_l < 0 < b_r\) and unknown functions \(D_l, D_r \in C(\mathbb{R}, \mathbb{R})\) which satisfy, for unknown \(\sigma \in \{-1, 1\}\),
\[
D_l(b_l) = D_r(b_r) = 0 \quad \text{and} \quad \lim_{\sigma \to -\infty} \sigma D_r(s) = \infty, \quad \lim_{\sigma \to -\infty} \sigma D_l(s) = -\infty.
\]
Note that the above assumptions allow for a much larger class of functions \(D_l, D_r\) compared to e.g. [44], where assumptions on their derivatives are used. In particular, in the present context, \(D_l\) and \(D_r\) need not be differentiable or monotone.

3 Simulations

We compare the controller (9) to the controller presented in [8] and, to this end, consider the simulation examples presented therein.

3.1 Mass-on-car system

To illustrate the controller (9), we consider a mass-spring system mounted on a car from [52], see Fig. 4. The mass \(m_2\) (in kg) moves on a ramp inclined by the angle \(\vartheta\) (in rad) and mounted on a car with mass \(m_1\) (in kg), for which it is possible to control the force with \(u = F\) (in N) acting on it. The equations of motion for the system are given by
\[
\begin{bmatrix} m_1 + m_2 & m_2 \cos \vartheta \\ m_2 \cos \vartheta & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}(t) \\ \ddot{s}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ 0 \end{bmatrix},
\]
where \(t\) is current time (in s), \(x\) (in m) is the horizontal car position and \(s\) (in m) the relative position of the mass on the ramp. The constants \(k\) (in N/m), \(d\) (in Ns/m) are the coefficients of the spring and damper, respectively. The output \(y\) (in m) of the system is given by the horizontal position of the mass on the ramp,
\[
y(t) = x(t) + s(t) \cos \vartheta.
\]
As shown in [52], system (26) can be reformulated in the form (22) and hence belongs to $\mathcal{N}^{(1)}_1$ by Proposition 2.1, with a relative degree $r$ depending on the angle $\vartheta$ and the damping $d$. Furthermore, it satisfies the positive-definite high-gain property since the function $G$ in (P4) is a positive constant. Invoking Assertion (v) of Theorem 1.8, the function $N$ in (9) may be taken as $N(s) = -s$.

For the simulation, we choose the parameters $m_1 = 4$, $m_2 = 1$, $k = 2$, $d = 1$, the initial values $x(0) = s(0) = 0$, $\dot{x}(0) = \dot{s} = 0$ and the reference trajectory $y_{\text{ref}}: t \mapsto \cos t$.

We emphasize that the function $y_{\text{ref}}(\cdot)$ is not available a priori to the controller: all that is available is the function value at the current time $t$ together with the values of its first $\hat{r} - 1$ derivatives, $y_{\text{ref}}^{(i)}(t)$, $i = 0, \ldots, \hat{r} - 1$. We consider two cases.

Case 1: If $0 < \vartheta < \frac{\pi}{2}$, then system (26) has relative degree $r = 2$, and the funnel controller (9) with $\hat{r} = r = 2$ is

$$u(t) = -\alpha(w(t)^2)w(t), \quad \text{with} \quad w(t) = \phi(t)\dot{e}(t) + \alpha(\varphi(t)^2 e(t)^2) \varphi(t)e(t),$$

where $\alpha(s) = 1/(1 - s)$ for $s \in [0, 1)$. The controller presented in [8] takes the form

$$u(t) = -\alpha(\varphi_1(t)^2 w_1(t)^2)w_1(t), \quad \text{with} \quad w_1(t) = \dot{e}(t) + \alpha(\varphi(t)^2 e(t)^2) e(t),$$

(27)

where $\varphi_1$ is a second funnel function, chosen appropriately, cf. [8]. Note that $w(t) = \varphi(t)w_1(t)$. As simulations show, the performance of the controller (27) can be improved compared to the simulations in [8], by choosing $\varphi_1 = \varphi$. As in [8], we set $\varphi(t) = (5e^{-2t} + 0.1)^{-1}$ for $t \geq 0$.

The performance of the controllers (9) and (27) applied to (26) is depicted in Fig. 5. Fig. 5a shows the tracking errors generated by the two different controllers, while Fig. 5b shows the respective input functions. Comparable performance is evident, suggesting broadly similar efficacy in cases wherein both controllers are feasible. However, (9) is feasible in certain situations which are outside the scope of (27).

For example, (9) is able to achieve asymptotic tracking, to address the issue of an unknown control direction and is applicable when the instantaneous value $y_{\text{ref}}(t)$ is not available to the controller: these features form the basis of our re-visiting Example 2.2 below.
Case 2: If $\vartheta = 0$ and $d \neq 0$, then system (26) has relative degree $r = 3$. Then the
funnel controller (9), with $\hat{r} = r = 3$, takes the form

$$w(t) = \varphi(t)\dot{e}(t) + \gamma(\varphi(t)\dot{e}(t) + \varphi(t)\dot{e}(t)),
$$

$$u(t) = -\gamma(w(t)),$$

where $\gamma(s) = s\alpha(s^2)$ for $s \in (-1, 1)$. The controller presented in [8] reads

$$w_1(t) = \dot{e}(t) + \alpha(\varphi(t)^2\dot{e}(t)^2) e(t),
$$

$$w_2(t) = w_1(t) + \alpha(\varphi_1(t)^2 w_1(t)^2) w_1(t)
$$

$$= \dot{e}(t) + 2\alpha(\varphi(t)^2\dot{e}(t)^2) (\varphi(t)\varphi(t)\dot{e}(t)^2) + \varphi(t)^2\dot{e}(t)\ddot{e}(t) e(t)
$$

$$+ \alpha(\varphi_1(t)^2 e(t)^2) \dot{e}(t) + \alpha(\varphi_1(t)^2 w_1(t)^2) w_1(t),
$$

$$u(t) = -\alpha(\varphi_2(t)^2 w_2(t)^2) w_2(t),$$

(28)

where $\varphi_1, \varphi_2$ are appropriate additional funnel functions, cf. [8]. Here, we choose

$$\varphi_1 = \varphi_2 = \varphi, \text{ with } \varphi(t) = (3e^{-t} + 0.1)^{-1} \text{ for } t \geq 0$$

and compare the controller (9) with (28).

The simulation suggests that the controllers are broadly similar in performance. While
controller (9) requires more input action than controller (28), the latter exhibits a
significantly higher level of complexity, which makes it more difficult to implement (this
issue becomes even more severe for relative degrees higher than three).

### 3.2 Nonlinear MIMO system

As a nonlinear multi-input, multi-output example we consider the robotic manipulator
from [25, Ch. 13] as depicted in Fig. 7. It is planar, rigid, with revolute joints and has
two degrees of freedom.

The two joints are actuated by $u_1$ and $u_2$ (in Nm). The links are assumed to be
massless and have lengths $l_1$ and $l_2$ (in m), resp., with point masses $m_1$ and $m_2$ (in kg)
attached to their ends. The two outputs are the joint angles $y_1$ and $y_2$ (in rad) and the equations of motion are given by (see also [56, p. 259])

$$M(y(t))\ddot{y}(t) + C(y(t), \dot{y}(t))\dot{y}(t) + G(y(t)) = u(t)$$  \hfill (29)

with initial value $(y(0),\dot{y}(0)) = (0,0)$, inertia matrix

$$M : \mathbb{R}^2 \rightarrow \mathbb{R}^{2\times2}, \quad (y_1, y_2) \mapsto \begin{bmatrix}
  m_1 l_1^2 + m_2 (l_1^2 + l_2^2 + 2l_1l_2 \cos(y_2)) & m_2 (l_2^2 + l_1l_2 \cos(y_2)) \\
  m_2 (l_2^2 + l_1l_2 \cos(y_2)) & m_2 l_2^2
\end{bmatrix}$$

centrifugal and Coriolis force matrix

$$C : \mathbb{R}^4 \rightarrow \mathbb{R}^{2\times2}, \quad (y_1, y_2, v_1, v_2) \mapsto \begin{bmatrix}
  -2m_2l_2l_1 \sin(y_2) v_1 - m_2l_2 \sin(y_2) v_2 \\
  -m_2l_1l_2 \sin(y_2) v_1
\end{bmatrix},$$

and gravity vector

$$G : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (y_1, y_2) \mapsto g \begin{bmatrix}
  m_1 \cos(y_1) + m_2 (l_1 \cos(y_1) + l_2 \cos(y_1 + y_2)) \\
  m_2l_2 \cos(y_1 + y_2)
\end{bmatrix},$$

where $g = 9.81 \text{ m/s}^2$ is the acceleration of gravity. Multiplying $(29)$ with $M(y(t))^{-1}$, which is pointwise positive definite, from the left we see that the resulting system is
of the form (1) and satisfies the positive-definite high-gain property, hence it belongs to $\mathcal{N}^{2,2}$.

For the simulation, we choose the parameters $m_1 = m_2 = 1, l_1 = l_2 = 1$ and the reference signal $y_{\text{ref}}: t \mapsto (\sin t, \sin 2t)$. We compare the controller (9) to the multivariate version of (27) from [8], that is

$$u(t) = -\alpha(\varphi_1(t)^2 \|w_1(t)\|^2) w_1(t),$$

with

$$w_1(t) = e(t) + \alpha(\varphi(t)^2 \|e(t)\|^2) e(t),$$

where $\alpha(s) = 1/(1-s)$ for $s \in [0, 1)$. We choose $\varphi(t) = (4e^{-2t} + 0.1)^{-1} = \varphi_1(t)$ for $t \geq 0$.

![Funnel and first tracking error components](image1)

![Funnel and second tracking error components](image2)

![First input components](image3)

![Second input components](image4)

**Fig. 8:** Simulation of the controllers (9) and (30) applied to (29).

The simulation of the controllers (9) and (30) applied to (29) over the time interval $[0, 10]$ is depicted in Fig. 8. It can be seen that for this example both controllers exhibit a nearly identical performance.

**Remark 3.1.** A closer look at the simulations reveals that the controller performance of (9) differs from that of the controller presented in [8] for the example in Subsection 3.1, while it is practically identical for the example in Subsection 3.2. Since the different dimensions of input/output spaces ($m = 1$ compared to $m = 2$)
is probably not the reason here, the presumable cause seems to be the internal dynamics. System (26) has two-dimensional internal dynamics in Case 1 \((r = 2)\) and one-dimensional internal dynamics in Case 2 \((r = 3)\), while system (29) has trivial internal dynamics. This seems to suggest that the controllers exhibit a different behaviour in the presence of non-trivial internal dynamics.

### 3.3 Example 2.2 revisited

We consider the system from Example 2.2 to demonstrate that the controller (9) can achieve asymptotic tracking and is feasible when the control direction is unknown, a dead-zone input is present and \(\dot{y}_{\text{ref}}(t)\) is not available for feedback. To this end, we consider

\[
\begin{align*}
\dot{\xi}_1(t) &= (1 + \xi_1(t)^2)\xi_2(t), \\
\dot{\xi}_2(t) &= \xi_1(t) - 2\xi_2(t) + \eta(t) + \beta(u(t)), \\
\eta(t) &= -\eta(t)^2(2\xi_1(t) + \xi_2(t) + \eta(t)), \\
y(t) &= \dot{\xi}_1(t)
\end{align*}
\]

with the dead-zone input

\[
\beta: \mathbb{R} \to \mathbb{R}, \quad v \mapsto \begin{cases} 
  v - 1, & v \geq 1, \\
  0, & -1 < v < 1, \\
  v + 1, & v \leq -1.
\end{cases}
\]

By Proposition 2.1, system (31) belongs to the class of systems \(\mathcal{A}^{1,2}\). The initial values are chosen as \(\dot{\xi}_1(0) = \xi_2(0) = \eta(0) = 0\) and the reference signal is \(y_{\text{ref}}: t \mapsto \cos t\). For the funnel controller (9), we choose the design parameters \(\alpha: s \mapsto 1/(1 - s)\) and \(N: s \mapsto s \sin s\); the latter choice is based on the assumption that the exact shape of \(\beta\) (and in particular the control direction) is unknown to the controller.

We consider two different cases: If information of the instantaneous signals \(\dot{y}_{\text{ref}}(t)\) are available to the controller, then we choose \(\hat{r} = 2 = r\) and an unbounded funnel function \(\varphi: t \mapsto t^2\). If information of \(\dot{y}_{\text{ref}}(t)\) is not available, then we choose \(\hat{r} = 1 < 2\) and a bounded funnel function \(\varphi: t \mapsto (2e^{-t} + 0.01)^{-1}\).

The simulation of the controller (9) applied to (31) in the cases \(\hat{r} = 1\) and \(\hat{r} = 2\) is depicted in Fig. 9. The “jumps” in the input \(u\) are due to the dead-zone induced by the function \(\beta\). Comparing Figs. 9a and 9b, a degradation in performance may be observed. However, this is not surprising in view of the enhanced information available for feedback in case \(\hat{r} = 2\). We may also observe, that in the latter case asymptotic tracking is achieved.

### 4 Conclusion

We have solved the asymptotic and non-asymptotic tracking control objective for a fairly large class of nonlinear systems with “higher relative degree” described by functional differential equations that satisfy a weak high-gain property. We have designed a feedback strategy – simple in the sense of funnel control and as “simple” as
one may expect for higher relative degree. We believe that the present paper is somehow the “definitive paper” on funnel control for nonlinear systems, whose internal dynamics satisfy a BIBO property. First results on funnel control for systems which are not minimum phase are given in [5] for uncertain linear systems and in [7] for a nonlinear robotic manipulator.

In the present paper we did not treat funnel control for systems described by partial differential equations. This is however, a very important field and in fact very different. On the one hand, there are systems which have a well-defined relative degree and exhibit infinite-dimensional internal dynamics, see e.g. [11]. Such systems are susceptible to funnel control with the control laws presented in the present paper; for instance, a linearized model of a moving water tank, where sloshing effects appear, is discussed in [10]. On the other hand, not even every linear infinite-dimensional system has a well-defined relative degree, in which case the results presented here cannot be applied. For such systems, the feasibility of funnel control has to be investigated directly for the (nonlinear) closed-loop system, see e.g. [49] for a boundary controlled heat equation, [48] for a general class of boundary control systems, [6] for the monodomain equations (which represents defibrillation processes of the human heart) and [4] for the Fokker-Planck equation corresponding to the Ornstein-Uhlenbeck process.
One important problem remains: non-derivative funnel control, that is, when only the output $y$ is available for feedback, but not its first $r-1$ derivatives $y, \ldots, y^{(r-1)}$. First results on this have been obtained in [32, 33] using a backstepping approach. However, these results necessitate a level of controller complexity which, on the evidence of numerical simulation, can lead to practical performance drawbacks. An attempt to overcome these backstepping-induced drawbacks through the adoption of pre-compensators can be found in [15, 16] but only for systems with relative degree at most three: the higher relative degree case remains open, even in the context of single-input, single-output linear systems with positive high-frequency gain and asymptotically stable zero dynamics.

Appendix A Proofs

**Proof of Theorem 1.8.** For $k = 1, \ldots, r$, we define

$$
\pi_k : \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \to \mathbb{R}^{km},
$$

$$(t, \xi) \mapsto \begin{cases} 
\varphi(t) \left( \xi_1 - y_{\text{ref}}(t), \ldots, \xi_k - y_{\text{ref}}^{(k-1)}(t) \right), & k = 1, \ldots, r, \\
\varphi(t) \left( \xi_1 - y_{\text{ref}}(t), \ldots, \xi_{\hat{r}} - y_{\text{ref}}^{(\hat{r}-1)}(t), \xi_{\hat{r}+1}, \ldots, \xi_k \right), & k = \hat{r} + 1, \ldots, r.
\end{cases}
$$

The proof now proceeds in several steps.

**Step 1.** We recast the feedback-controlled system in the form of an initial-value problem to which a variant of an extant existence theory applies. Set $n = rm$ and

$$
\mathcal{D} := \{ (t, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n | \pi_r(t, \xi) \in \mathcal{D}_r \},
$$

which is non-empty and relatively open, and define $\rho : \mathcal{D} \to \mathcal{B}$ by $\rho := \rho_r \circ \pi_r$. Introducing the function $F : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^n$ given by

$$(t, \xi, \eta) \mapsto F(t, \xi, \eta) := \begin{pmatrix} 
\xi_2 \\
\vdots \\
\xi_r \\
f \left( d(t) \eta, (N \circ \alpha)(\|\rho(t, \xi)\|^2) \rho(t, \xi) \right)
\end{pmatrix},
$$

and writing

$$
x(t) = \begin{pmatrix} 
y(t) \\
\vdots \\
y^{(r-1)}(t)
\end{pmatrix},
$$

we see that the (formal) control (9) may be expressed as

$$
u(t) = (N \circ \alpha)(\|\rho(t, x(t))\|^2) \rho(t, x(t)).
$$

The feedback-controlled initial-value problem (1) & (9) may now be formulated as

$$
x(t) = F(t, x(t), T(x)(t)), \quad x|_{[-h,0]} = x^0 \in C([-h,0], \mathbb{R}^n),
$$

(32)
where
\[
x^0(t) := \begin{pmatrix} x^0(t) \\ \vdots \\ (y^0)^{(r-1)}(t) \end{pmatrix}, \quad t \in [-h, 0].
\]
A continuous function \( x \in C(I, \mathbb{R}^n) \) on an interval of the form \( I = [-h, \omega], \) \( 0 < \omega < \infty, \)
or of the form \([-h, \omega], \) \( 0 < \omega \leq \infty, \) is a solution of (32), if \( x|_{[-h,0]} = x^0, \ (t, x(t)) \in \mathcal{G} \)
for all \( t \in I \setminus [-h, 0] \) and
\[
\forall t \in I, \ t \geq 0: \ x(t) = x(t_0) + \int_0^t F(s, x(s), T(x)(s)) \, ds.
\]
A solution is maximal, if it has no right extension that is also a solution. Since \( T \) is an operator with domain \( C([-h, \infty), \mathbb{R}^n), \) some care is required in interpreting the above
notion of a solution \( x \in C(I, \mathbb{R}^n) \) when \( I \) is a bounded interval of the form \( I = [-h, \omega] \)
or \( I = [-h, \omega], \) \( 0 < \omega \leq \infty, \) \( \mathcal{G} \) and, for each \( \tau \in J, \) define \( x_{\tau} \in C([-h, \infty), \mathbb{R}^n) \) by
\[
x_{\tau}(t) := \begin{cases} x(t), & t \in [-h, \tau] \\ x(\tau), & t > \tau. \end{cases}
\]
With \( T \in \mathcal{T}_h^{n,d} \) we may associate \( \hat{T} : C(I, \mathbb{R}^n) \rightarrow L_{\text{loc}}^1(J, \mathbb{R}^q) \) defined by the property
\[
\forall \tau \in J: \ T(x)|_{[0, \tau]} = T(x_{\tau})|_{[0, \tau]}.
\]
The causality property (P1) of \( T \in \mathcal{T}_h^{n,d} \) ensures that \( \hat{T} \) is well defined. Replacing \( T \)
by \( \hat{T} \) in (33) we arrive at the correct interpretation of a solution. However, for simplicity, we will not distinguish notationally between an operator \( T \in \mathcal{T}_h^{n,d} \) and its
“localization” \( \hat{T} \).

It is readily verified that \( F \) has the following properties: If \( I \subset \mathbb{R}_{\geq 0} \) is a compact interval and \( K_n \subset \mathbb{R}^n, K_q \subset \mathbb{R}^q \) are compact with \( I \times K_n \subset \mathcal{G}, \) then
(a) \( F(t, \cdot, \cdot) : K_n \times K_q \rightarrow \mathbb{R}^n \) is continuous for all \( t \in I; \)
(b) \( F(\cdot, v, w) : I \rightarrow \mathbb{R}^n \) is measurable for all \( (v, w) \in K_n \times K_q; \)
(c) there exists \( \bar{f} \in (0, \infty) \) such that \( \| F(t, v, w) \| \leq \bar{f} \) for almost all \( t \in I \) and all \( (v, w) \in K_n \times K_q. \)

Invoking (11), we see that \((0, x^0(0)) \in \mathcal{G}. \) An application of a variant (a straightforward modification tailored to the current context) of [29] Thm. B.1 yields the existence of a maximal solution \( x : [-h, \omega) \rightarrow \mathbb{R}^n, \) \( 0 < \omega \leq \infty, \) of (32) and so
\[
\mathcal{G} = \text{graph}(x|_{[0, \omega)}) \subset \mathcal{G}.
\]
Moreover, the closure of \( \mathcal{G} \) is not a compact subset of \( \mathcal{G}. \)

Step 2. Before embarking on the proof proper, we record some preliminary observations and definitions. Since \((t, x(t)) \in \mathcal{G} \) for all \( t \in [0, \omega), \) we have \( \pi_k(t, x(t)) \in \mathbb{R}_k = \text{dom}(p_k), \ k = 1, \ldots, r. \) Introduce continuous functions
\[
e_k : [0, \omega) \rightarrow \mathcal{B}, \ \alpha_k : [0, \omega) \rightarrow [1, \infty), \ \beta_k : [0, \omega) \rightarrow \mathbb{R}^m, \ k = 1, \ldots, r,
\]
given by
\[ e_k(t) := (p_k \circ \pi_k)(t, x(t)), \quad \alpha_k(t) := \alpha(\|e_k(t)\|^2), \quad \gamma_k(t) := \gamma(e_k(t)) = \alpha_k(t) e_k(t), \]
where \( \gamma \) is given by (7), and, for later notational consistency, we also write \( \gamma_k(\cdot) := 0. \) Clearly,
\[ \forall k = 1, \ldots, r \forall t \in [0, \omega) : \|e_k(t)\| < 1. \tag{34} \]
In particular, for \( k = 1 \) we have \( \|e_1(t)\| = \varphi(t)\|e(t)\| < 1 \) for all \( t \in [0, \omega) \) and so the tracking error \( e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot) \) evolves in the funnel \( \mathcal{F}_\varphi. \)

Observe that, for almost all \( t \in [0, \omega) \) and \( k = 1, \ldots, r, \) we have by definition of \( p_k \) in (8)
\[ e_k(t) - \gamma_{k-1}(t) = \begin{cases} \varphi(t) e^{(k-1)}(t), & \text{if } k \leq \hat{r} \\ \varphi(t) y^{(k-1)}(t), & \text{otherwise.} \end{cases} \tag{36} \]
We also record that
\[ \alpha_k(t) = -2\alpha'(\|e_k(t)\|^2) \langle e_k(t), \dot{e}_k(t) \rangle \quad \text{for a.a. } t \in [0, \omega), \quad k = 1, \ldots, r. \tag{37} \]
Define functions \( \psi_k : [0, \infty) \to \mathbb{R}^n, k = 1, \ldots, r, \) as follows
\[ \hat{r} = r \implies \psi_k(\cdot) := 0, \quad k = 1, \ldots, r \]
\[ \hat{r} < r \implies \psi_k(t) := \begin{cases} 0, & \text{if } k < \hat{r} \\ -\varphi(t) y^{(\hat{r})}_{\text{ref}}(t), & \text{if } k = \hat{r} \\ \dot{\varphi}(t) y^{(\hat{r}-1)}_{\text{ref}}(t) + \varphi(t) y^{(\hat{r})}_{\text{ref}}(t), & \text{if } \hat{r} < k \leq r - 1 \\ \dot{\varphi}(t) y^{(r-1)}_{\text{ref}}(t) + \varphi(t) y^{(r)}_{\text{ref}}(t), & \text{if } k = r. \end{cases} \]
By choice of the design parameters as in (9), \( \varphi \) is bounded (and so \( \dot{\varphi} \) is essentially bounded) if \( \hat{r} < r. \) Therefore, we may infer the existence of \( \psi^* \in \mathbb{R} \) (with \( \psi^* = 0 \) if \( \hat{r} = r \)) such that
\[ \|\psi_k(t)\| \leq \psi^* \text{ for a.a. } t \in [0, \infty), \quad k = 1, \ldots, r. \tag{38} \]
Observe that, for almost all \( t \in [0, \omega), \)
\[ \dot{e}_k(t) = \dot{\varphi}(t) e^{(k-1)}(t) + e_{k+1}(t) - \gamma_k(t) + \gamma_{k-1}(t) + \psi_k(t), \quad k = 1, \ldots, r - 1 \]
\[ \dot{e}_r(t) = \dot{\varphi}(t) e^{(r-1)}(t) + \varphi(t) e^{(r)}(t) + \gamma_{r-1}(t) + \psi_r(t) \tag{39} \]
which, if \( r = 1 = \hat{r}, \) collapses to the tautology: \( \dot{e}_1(t) = (\varphi e^{(1)})(t) \) for a.a. \( t \in [0, \omega). \)
Arbitrarily fix \( \tau \in (0, \omega). \) By continuity, there exists \( \theta \in (0, \infty) \) such that
\[ \forall t \in [0, \tau]: \left( 1 + \varphi(t) \right) \sum_{k=1}^{r} \|e^{(k-1)}(t)\| \leq \theta \tag{40} \]
and so, by properties of $\Phi$, there exists $c > 0$ such that
\[
\| \dot{\phi}(t) e^{k-1}(t) \| \leq c (1 + \phi(t)) \| e^{k-1}(t) \| \leq c \theta \text{ for a.a. } t \in [0, \tau], \ k = 1, \ldots, r. \tag{41}
\]
Again by properties of $\Phi$, the following are well defined:
\[
\sup_{t \in [\tau, \omega)} \left( \frac{1}{\varphi(t)} \right) =: \lambda > 0 \quad \text{and} \quad \esssup_{t \in [\tau, \omega)} \left( \frac{\| \phi(t) \|}{\varphi(t)} \right) =: \mu \geq 0.
\]
For $k \in \{1, \ldots, r\}$ and invoking (34), (36) and (38), we find
\[
\begin{align*}
& (a) \quad \| \dot{\phi}(t) e^{k-1}(t) \| \leq \mu \| \phi(t) e^{k-1}(t) \| \leq \mu (1 + \| y_{k-1}(t) \|) \\
& \quad \text{for a.a. } t \in [\tau, \omega), \text{ if } k \leq \hat{r} \\
& (b) \quad \| \dot{\phi}(t) e^{k-1}(t) \| \leq \mu \| \phi(t) e^{k-1}(t) \| + \| \phi(t) y_{k-1}^{(k-1)} \| \\
& \quad \leq \mu (1 + \| y_{k-1}(t) \|) + \psi^* \\
& \quad \text{for a.a. } t \in [\tau, \omega), \text{ if } k > \hat{r}
\end{align*}
\]
and so, a fortiori, we have
\[
\| \dot{\phi}(t) e^{k-1}(t) \| \leq \mu (1 + \| y_{k-1}(t) \|) + \psi^* \text{ for a.a. } t \in [\tau, \omega), \ k = 1, \ldots, r. \tag{42}
\]
We complete the preliminaries by writing
\[
\hat{e}_k := \max_{t \in [0, \tau]} \| e_k(t) \|^2 < 1, \ k = 1, \ldots, r.
\]
Step 3. Assume that $r \geq 2$, otherwise proceed to Step 5. Let $e_1$ be the unique point of $(0,1)$ such that $e_1 \alpha(e_1) = 1 + \mu + 2\psi^*$ and $e_1^* := \max \{ \hat{e}_1, e_1 \} < 1$. We will show that
\[
\forall t \in [0, \omega) : \| e_1(t) \|^2 \leq e_1^*. \tag{43}
\]
Suppose that this claim is false. Then $\| e_1(s) \|^2 > e_1^*$ for some $s \in (0, \omega)$. Since $\| e_1(t) \|^2 \leq \hat{e}_1 \leq e_1^*$ for all $t \in [0, \tau]$, we have $\tau < s$ and so we may define
\[
\sigma := \max \{ t \in [\tau, s] \mid \| e_1(t) \|^2 = e_1^* \}.
\]
Clearly,
\[
\forall t \in [\sigma, s] : e_1 \leq e_1^* \leq \| e_1(t) \|^2,
\]
whence, by monotonicity of $\alpha$,
\[
\forall t \in [\sigma, s] : \alpha(e_1) \leq \alpha(\| e_1(t) \|^2) = \alpha_1(t).
\]
Therefore,
\[
\forall t \in [\sigma, s] : \alpha_1(t) \| e_1(t) \|^2 \geq e_1 \alpha(e_1) = 1 + \mu + 2\psi^* \tag{44}
\]
which, by the first of relations (39) in conjunction with (33) and (42) (and recalling $\rho_0(\cdot) = 0$), gives
\[
\frac{1}{2} \| e_1(t) \|^2 = \langle e_1(t), \dot{e}_1(t) \rangle \\
= \langle \dot{\phi}(t) e_1(t), e(t) \rangle + \langle e_1(t), e_2(t) \rangle - \alpha_1(t) \| e_1(t) \|^2 + \langle e_1(t), \psi_1(t) \rangle \\
< 1 + \mu + 2\psi^* - \alpha_1(t) \| e_1(t) \|^2 \leq 0
\]
for almost all $t \in [\sigma, s]$ and so $\|e_1(s)\|^2 < \|e_1(\sigma)\|^2$, whence the contradiction

$$e_1^* < \|e_1(s)\|^2 < \|e_1(\sigma)\|^2 = e_1^*.$$  

Therefore (43) holds.

Step 4. For notational convenience, write

$$W_1 := W^{1,m}([0, \omega), \mathbb{R}) \quad \text{and} \quad W_m := W^{1,m}([0, \omega), \mathbb{R}^m).$$

We show by induction that

$$(\alpha_k, e_k, \gamma_k) \in W_1 \times W_m \times W_m \quad \text{for} \quad k = 1, \ldots, r - 1. \tag{45}$$

This step is vacuous in the case $r = 1$. Let $k = 1$. By (43), we see that $e_1$ is bounded by $\sqrt{E_1}$. $\alpha_1$ is bounded by $\alpha(e_1^*)$ and that $\gamma_1$ is bounded by $\sqrt{E_1}\alpha(e_1^*)$. Recalling that $\gamma_0(\cdot) = 0$, essential boundedness of $\dot{\gamma}_1$ follows by the first of relations (39) together with (34), (38), (41), (42). Invoking (37), we may conclude essential boundedness of $\dot{\gamma}_1$. Essential boundedness of $\gamma_1 = \alpha_1 e_1 + \alpha_1 e_1$ then follows. Therefore, $(\alpha_1, e_1, \gamma_1) \in W_1 \times W_m \times W_m$.

Now assume that $k \in \{2, \ldots, r - 1\}$ and

$$(\alpha_j, e_j, \gamma_j) \in W_1 \times W_m \times W_m, \quad j = 1, \ldots, k - 1.$$  

Set

$$\beta := \max \left\{ \psi^*, \sup_{t \in [0, \omega)} \|\gamma_{k-1}(t)\|, \text{ess}\sup_{t \in [0, \omega)} \|\gamma_{k-1}(t)\| \right\} < \infty.$$  

By (34), (39) and (42), we have

$$\langle e_k(t), \dot{e}_k(t) \rangle = \phi(t)\langle e_k(t), e^{(k-1)}(t) \rangle + \langle e_k(t), e_{k+1}(t) \rangle$$

$$+ \langle e_k(t), (\dot{\gamma}_{k-1}(t) + \psi_k(t)) \rangle - \alpha_k(t)\|e_k(t)\|^2$$

$$< 1 + 3\beta + \mu(1 + \beta) - \alpha_k(t)\|e_k(t)\|^2 \tag{46}$$

for almost all $t \in [\tau, \omega)$. Let $\hat{e}_k$ be the unique point of $(0, 1)$ such that $e_k\alpha(\hat{e}_k) = 1 + 3\beta + \mu(1 + \beta)$ and define

$$e_k^* := \max \{ \hat{e}_k, e_k \} > 0.$$  

We first show that

$$\forall t \in [0, \omega) : \|e_k(t)\|^2 \leq e_k^* \tag{47}$$

by the contradiction argument of Step 3 (mutatis mutandis). Suppose that (47) is false. Then $\|e_k(s)\|^2 > e_k^*$ for some $s \in (0, \omega)$. Since $\|e_k(t)\|^2 \leq \hat{e}_k \leq e_k^*$ for all $t \in [0, \tau]$, we have $\tau < s$ and so we may define $\sigma := \max \{ t \in [\tau, s) \mid \|e_k(t)\|^2 = e_k^* \}$. The counterpart of (44) now follows:

$$\forall t \in [\sigma, s) : \alpha_k(t)\|e_k(t)\|^2 \geq e_k\alpha(\hat{e}_k) = 1 + 3\beta + \mu(1 + \beta)$$

which, in conjunction with (45), gives $\frac{d}{dt}\|e_k(t)\|^2 < 0$ for almost all $t \in [\sigma, s]$, whence the contradiction

$$e_k^* < \|e_k(s)\|^2 < \|e_k(\sigma)\|^2 = e_k^*.$$
Therefore, (47) holds which, in turn, implies that $\alpha_k$ is bounded (by $\alpha(e_k^*)$) and that $\gamma_k = \alpha_k e_k$ is bounded (by $\sqrt{e_k^*} \alpha(e_k^*)$). By boundedness of $e_{k+1}$, $\gamma_k$ and essential boundedness of $\gamma_{k-1}$, it follows from (39), together with (41) and (42), that $\dot{\gamma}_k$ is essentially bounded and so $e_k \in W_m$. Invoking (37), we may now infer essential boundedness of $\alpha_k$. Therefore, $\alpha_k \in W_1$. Finally, since $\gamma_k = \alpha_k \dot{e}_k + \alpha_k e_k$, we have essential boundedness of $\dot{e}_k$ and so $\dot{e}_k \in W_m$. In summary, we have shown that, for $k \in \{2, ..., r - 1\}$,

$$(\alpha_j, e_j, \gamma_j) \in W_1 \times W_m \times W_m, \quad j = 1, ..., k - 1 \implies (\alpha_k, e_k, \gamma_k) \in W_1 \times W_m \times W_m,$$

and so, by induction, we conclude (45).

Step 5. Our next goal is to prove boundedness of the solution $x$. Recalling that $y_{ref} \in W^{\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, it suffices to show that the output error $e$ and its derivatives $\dot{e}, ..., e^{(r-1)}$ are bounded on $[0, \omega]$. By (40), we already know that

$$\forall k = 1, ..., r \quad \forall t \in [0, \tau] : \|e^{(k-1)}(t)\| \leq \theta,$$

and so it remains to show that $e^{(k-1)}$ is bounded on $[\tau, \omega)$, $k = 1, ..., r$. Since $\varphi(t)e(t) = e_1(t) \in \mathcal{B}$ for all $t \in [0, \omega)$, we have

$$\forall t \in [\tau, \omega) : \|e(t)\| \leq \frac{1}{\varphi(t)} \leq \lambda.$$ 

By boundedness of the functions $\gamma_k$ (Step 4), there exists $\gamma' > 0$ such that

$$\forall k = 2, ..., r \quad \forall t \in [0, \omega) : \|\gamma_{k-1}(t)\| \leq \gamma'.$$

(48)

Let $k \in \{2, ..., r\}$. By (46), we have

$$\forall t \in [\tau, \omega) : \|e^{(k-1)}(t)\| \leq \lambda (1 + \gamma') + \sup_{t \geq \tau} \|y_{ref}^{(k-1)}(t)\| < \infty.$$ 

This completes Step 5.

Step 6. We prove boundedness of $\alpha_r$. By boundedness of $x$ (Step 5) and property (TP3) of the operator class $\mathcal{F}_h^q$, there exists compact $K_q \subset \mathbb{R}^q$ such that $T(x)(t) \in K_q$ for almost all $t \in [0, \omega)$. Since $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$, there exists compact $K_p \subset \mathbb{R}^p$ such that $d(t) \in K_p$ for almost all $t \in [0, \omega)$. By the high-gain property, there exists $\nu' \in (0, 1)$ such that the continuous function

$$\chi : \mathbb{R} \to \mathbb{R}, \quad s \mapsto \min \left\{ \langle v, f(\delta, z, -sv) \rangle : (\delta, z, v) \in K_p \times K_q \times A_m \right\}$$

is unbounded from above, where, for notational convenience, we have introduced the compact annulus

$$A_m := \left\{ v \in \mathbb{R}^m \mid \nu' \leq \|v\| \leq 1 \right\}.$$ 

Choose a real sequence $(s_j)$ such that the sequence $(\chi(s_j))$ is unbounded, positive, and strictly increasing. By surjectivity and continuity of $N$, for every $a, b \in \mathbb{R}$, the set $\{ \kappa > a \mid N(\kappa) = b \}$ is non-empty. Choose $\kappa_1 > \alpha((1 - \nu')^2) + \alpha, 0$ such that $N(\kappa_1) = s_1$ and define the strictly increasing sequence $(\kappa_j)$ by the recursion

$$\kappa_{j+1} := \inf \left\{ \kappa > \kappa_j \mid N(\kappa) = s_{j+1} \right\}.$$
Observe that
\[
\lim_{j \to \infty} \chi(N(\kappa_j)) = \lim_{j \to \infty} \chi(s_j) = \infty.
\]
Seeking a contradiction, suppose that \(\alpha_r(\cdot)\) is not bounded. Then, since \(\kappa_{j+1} > \kappa_j > \alpha_r(0)\) for all \(j \in \mathbb{N}\), the sequence \((\tau_j)\) in \((0, \omega)\) defined by
\[
\tau_j = \inf \{ t \in [0, \omega) \mid \alpha_r(t) = \kappa_{j+1} \}, \quad j \in \mathbb{N}_0,
\]
is well-defined and strictly increasing with \(N(\alpha_r(\tau_j)) = N(\kappa_{j+1}) = s_{j+1}\) for each \(j \in \mathbb{N}_0\). Now, define the sequence \((\sigma_j)\) in \((0, \omega)\) by
\[
\sigma_j = \sup \{ t \in [\tau_{j-1}, \tau_j) \mid \chi(N(\alpha_r(t))) = \chi(s_j) \}, \quad j \in \mathbb{N}.
\]
Since the sequence \((\chi(s_j))\) is strictly increasing, we have
\[
\forall j \in \mathbb{N}: \quad \chi(N(\alpha_r(\sigma_j))) = \chi(s_j) < \chi(s_{j+1}) = \chi(N(\alpha_r(\tau_j)));
\]
and so
\[
\forall j \in \mathbb{N} \forall t \in ([\sigma_j, \tau_j]: \sigma_j < \tau_j \quad \text{and} \quad \chi(N(\alpha_r(\sigma_j))) = \chi(s_j) < \chi(N(\alpha_r(t))). \quad (49)
\]
Next, suppose that, for some \(j \in \mathbb{N}\), there exists \(t \in [\sigma_j, \tau_j]\) such that \(e_r(t) \not\in A_m\). We first show that \(\alpha_r(t) \geq \kappa_j\). If \(\alpha_r(t) < \kappa_j\), then \(\alpha_r(\tau_j) = \kappa_{j+1} > \kappa_j\) and continuity of \(\alpha_r\) imply that there exists \(\tilde{t} \in (\sigma_j, \tau_j)\) such that \(\alpha_r(\tilde{t}) = \kappa_j\), thus
\[
\chi(N(\alpha_r(\tilde{t}))) = \chi(N(\kappa_j)) = \chi(s_j),
\]
which contradicts the definition of \(\sigma_j\). Therefore, \(\alpha_r(t) \geq \kappa_j\), which, together with the supposition \(\|e_r(t)\| < 1 - \nu^*\), leads to the contradiction:
\[
\alpha((1 - \nu^*)^2) < \kappa_1 \leq \kappa_j \leq \alpha_r(t) = \alpha(\|e_r(t)\|^2) < \alpha((1 - \nu^*)^2).
\]
Therefore,
\[
\forall j \in \mathbb{N} \forall t \in [\sigma_j, \tau_j]: \quad e_r(t) \in A_m, \quad (50)
\]
which, in conjunction with the facts that \(d(t) \in K_p\) and \((T x)(t) \in K_q\) for almost all \(t \in [0, \omega)\) and invoking \((49)\), yields
\[
\langle e_r(t), f(d(t), (T x)(t), u(t)) \rangle = -\langle e_r(t), f(d(t), (T x)(t), -N(\alpha_r(t))(-e_r(t))) \rangle
\leq -\min \{ \langle v, f(\delta, z, -N(\alpha_r(t))v) \rangle \mid (\delta, z, v) \in K_p \times K_q \times A_m \}
= -\chi(N(\alpha_r(t))) \leq -\chi(s_j)
\]
for all \(j \in \mathbb{N}\) and almost all \(t \in [\sigma_j, \tau_j]\). By \((41)\), \((42)\) and \((48)\),
\[
\|
\hat{\phi}(t)e^{(r-1)}(t)\| \leq c \theta + \mu (1 + \gamma^*) + \psi^* = : \Theta^* \quad \text{for a.a.} \ t \in [0, \omega).
\]
Since \(e^{(r)}(t) = f(d(t), T x(t), u(t)) - y^{(r)}_{rel}(t)\) for almost all \(t \in [0, \omega)\) and recalling the last of relations \((49)\), we have
\[
\dot{e}_r(t) = \phi(t)f(d(t), T x(t), u(t)) - y^{(r)}_{rel}(t) + \hat{\phi}(t)e^{(r-1)}(t) + \gamma_{r-1}(t) + \psi_r(t).
\]
for almost all \( t \in [0, \omega) \). By Step 3, \( \gamma_{t-1} \) is essentially bounded and, since \( y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \), we have essential boundedness of \( \gamma_{\text{ref}}^{(r)} \). Write

\[
\dot{c}_1 := \theta^* + \psi^* + \text{ess sup}_{t \in [0, \omega)} \| \gamma_{t-1}(t) \| \quad \text{and} \quad c_2 := \text{ess sup}_{t \geq 0} \| \gamma_{\text{ref}}^{(r)}(t) \|.
\]

Invoking (34), (38) and (51), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \| e_r(t) \|^2 \leq c_1 - \varphi(t)(\chi(s_j) - c_2)
\]

for all \( j \in \mathbb{N} \) and almost all \( t \in [\sigma_j, \tau_j] \). By properties of \( \varphi \in \Phi \) and noting that \( \sigma_1 > 0 \), we have \( \inf_{t \in [\sigma_1, \infty)} \varphi(t) > 0 \). Since \( \chi(s_j) \to \infty \) as \( j \to \infty \), we may choose \( j \) sufficiently large so that \( c_1 - \varphi(t)(\chi(s_j) - c_2) \leq 0 \) for almost all \( t \in [\sigma_j, \tau_j] \), in which case we have \( \| e_r(t) \|^2 < \| e_r(\sigma_j(t)) \|^2 \) and so

\[
\alpha_r(\tau_j) = \alpha(\| e_r(\tau_j) \|^2) < \alpha(\| e_r(\sigma_j) \|^2) = \alpha_r(\sigma_j)
\]

which is impossible since, by definition of \( \tau_j \), we have \( \alpha_r(t) < \alpha_r(\sigma_j) \) for all \( t \in [0, \tau_j) \). Therefore, our original supposition that \( \alpha_r \) is unbounded is false. An immediate consequence is the existence of \( \varepsilon^*_r \in (0, 1) \) such that

\[
\forall t \in [0, \omega) : \| e_r(t) \|^2 \leq \varepsilon_r^*.
\]

This completes Step 6.

Step 7. We prove Assertion [ii] of the theorem. Recalling inequalities (43), (47) and (52) of Steps 3, 4 and 6, we have

\[
\| e_k(t) \| \leq \varepsilon := \sqrt{\max \{ \varepsilon_1^*, \ldots, \varepsilon_r^* \}} < 1
\]

for all \( t \in [0, \omega) \) and all \( k = 1, \ldots, r \). Define

\[
\hat{D}_r := \{ (\eta_1, \ldots, \eta_r) \in \mathbb{R}^{rn} \mid \| \rho_k(\eta_1, \ldots, \eta_k) \| \leq \varepsilon, k = 1, \ldots, r \},
\]

which is evidently a compact subset of \( D_r \) as in (8). Since \( e_k(t) = (\rho_k \circ \pi_k)(t, x(t)) \) for all \( t \in [0, \omega) \), \( k = 1, \ldots, r \), it follows that \( \pi_k(t, x(t)) \in \hat{D}_r \) for all \( t \in [0, \omega) \). Suppose that \( \omega < \infty \). Then

\[
\forall t \in [0, \omega) : (t, x(t)) \in \hat{D} := \left\{ (s, \xi) \in [0, \omega) \times \mathbb{R}^m \mid \pi(s, \xi) \in \hat{D}_r \right\} \subset D.
\]

By compactness of \( \hat{D} \) it follows that the closure of graph \( \{ x \mid [0, \omega) \} \) is a compact subset of \( D \), which contradicts the findings of Step 1. Therefore, \( \omega = \infty \).

Step 8. We complete the proof by establishing Assertions [iii] and [iv] of the theorem. Assertion [iii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Assertion [iv] follows by Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6. Recalling that \( e_1 = \varphi \varepsilon \), we may infer Assertion [iii] from (53), and Assertion [ii] is a direct consequence of Assertion [i] and the results of Steps 5 & 6.
Step 6 is readily modified as follows. By the assumption, there exists a positive (respectively, negative) real sequence \((s_j)\) such that the sequence \(\left\{\chi(s_j)\right\}\) is unbounded, positive, and strictly increasing. Setting \(N : s \mapsto s\) (respectively, \(N : s \mapsto -s\)), the remaining arguments of Step 6 apply \textit{mutatis mutandis} to conclude boundedness of \(\alpha_r\), Steps 7 and 8 then follow as before. This completes the proof of the theorem. 

\textbf{Proof of Corollary 1.9} Let \(y_{\text{ref}} \in W^{\alpha}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)\) and \(y^0 \in W^{\alpha}(\mathbb{R})\). By Theorem 1.8 the feedback-controlled system (1) \& (2) has a solution, every solution can be maximally extended and every maximal solution is global. Let \(y : [\mathbb{R}] \rightarrow \mathbb{R}^m\) be any such global solution. In the following, we adopt the notation introduced in the proof of Theorem 1.8 and recall that, for all \(k = 1, \ldots, \hat{r} - 1, \psi_k(\cdot) = 0\), and for all \(t \geq 0\),

\[
\|e_k(t)\| < 1, \quad \|^\gamma_k(t)\| = \alpha(\|e_k(t)\|^2) \|e_k(t)\|, \\
\|^\gamma_k(t)\| = 2\alpha^\prime(\|e_k(t)\|^2) \langle e_k(t), \dot{e}_k(t) \rangle e_k(t) + \alpha(\|e_k(t)\|^2) \dot{\gamma}_k(t) \leq \alpha(\|e_k(t)\|^2) \|^\gamma_k(t)\|.
\]

Invoking (24), (36) and (39), with the convention that \(\gamma_0(\cdot) = \gamma_0(\cdot)\), we have, for almost all \(t \geq 0\),

\[
\langle \dot{e}_k(t), \dot{\gamma}_k(t) \rangle = \| (\phi(t)/\phi(t)) (e_k(t) - \gamma_{k-1}(t)) + e_{k+1}(t) + \gamma_{k-1}(t) - \gamma_k(t) \| \leq M_k(t) + \| \gamma_k(t) \|, \\
\langle e_k(t), \dot{\gamma}_k(t) \rangle \leq M_k(t) - \alpha_k(t) \| e_k(t) \|^2, \\
M_k(t) := 1 + \mu_0 (\| e_k(t) \| + \| \gamma_{k-1}(t) \|) + \| \gamma_{k-1}(t) \|.
\]

Setting \(k = 1\), we have

\[
\langle \dot{e}_1(t), \dot{\gamma}_1(t) \rangle \leq \mu_0 + 1 - \alpha_1(t) \| e_1(t) \|^2 \quad \text{for a.a.} \ t \geq 0.
\]

With \(e_1^0\) and \(c_1\) as in (12), the argument used in Step 3 of the proof of Theorem 1.8 applies, \textit{mutatis mutandis}, to conclude that \(\|e_1(t)\| \leq c_1\) for all \(t \geq 0\).

With \(\mu_1 = 1 + \mu_0 c_1\) as in (12) we have, for almost all \(t \geq 0\),

\[
\| \gamma_1(t) \| \leq c_1 \alpha(c_1^2) \quad \text{and} \quad \| \gamma_1(t) \| \leq \tilde{\alpha}(\gamma_1^2)(\mu_1 + c_1 \alpha(c_1^2)),
\]

wherein we have used the facts that \(\alpha\) and \(\tilde{\alpha}\) are non-decreasing functions (monotonicity of the latter being assured by the assumption of monotonicity of \(\alpha^\prime\)). Now set \(k = 2\), in which case we have

\[
M_2(t) \leq 1 + \mu_0 (1 + c_1 \alpha(c_1^2)) + \tilde{\alpha}(c_1^2)(\mu_1 + c_1 \alpha(c_1^2)) = \mu_2 \quad \text{for a.a.} \ t \geq 0.
\]

With \(e_2^0\) and \(c_2\) as in (12) the argument used in Step 4 of the proof of Theorem 1.8 applies, \textit{mutatis mutandis}, to conclude that \(\|e_2(t)\| \leq c_2\) for all \(t \geq 0\). Iterating this process, we arrive at

\[
\forall k = 1, \ldots, \hat{r} - 1 \ \forall t \geq 0 : \ |e_k(t)| \leq c_k
\]

To complete the proof, simply note that, for all \(t \geq 0\),

\[
\varphi(t) \| e(t) \| = \| e_1(t) \| \leq c_1 \quad \text{and} \quad \varphi(t) \| e^{(k)}(t) \| = \| e_{k+1}(t) - \gamma_k(t) \| \leq c_{k+1} + c_k \alpha(c_k^2), \ k = 1, \ldots, \hat{r} - 2.
\]

\qed
Proof of Theorem 2.1. The proof proceeds in steps.

**Step 1.** Since the functions in (P1) are sufficiently smooth, by the standard theory of ordinary differential equations, for all initial data

\[(\xi_1(0), \ldots, \xi_r(0), \eta(0)) = (\xi^0_1, \ldots, \xi^0_r, \eta^0) \in \mathbb{R}^d,\]

system \[^{22}\] with input \(u \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)\) and disturbance \(d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n)\), has a unique maximal solution on \((\xi^1_1, \ldots, \xi^1_r, \eta) : [0, \omega) \to \mathbb{R}^d\), where \(0 < \omega \leq \infty\).

**Step 2.** We show that for all \(k = 0, \ldots, r - 1\) there exist functions \(h_k \in C^{r-k}(\mathbb{R}^{(k+1)m}, \mathbb{R}^m)\) such that

\[
\forall t \in [0, \omega) : \xi_{k+1}(t) = h_k(y(t), \ldots, y^{(k)}(t)).
\]

(54)

Setting \(h_0 := \text{id}_{\mathbb{R}^m}\), we trivially have

\[
\xi_1(t) = h_0(y(t)) = y(t), \quad t \in [0, \omega).
\]

(55)

Now consider the case \(k = 1\). In view of (P1), we see that (local) implicit function theory is applicable in the context of the equation

\[
\dot{\xi}_1(t) = f_1(\xi_1(t), \xi_2(t)), \quad t \in [0, \omega).
\]

(56)

In particular, writing

\[F_1 : \mathbb{R}^{2m} \times \mathbb{R}^m \to \mathbb{R}^m, \quad (x_1, x_2, x_3) \mapsto f_1(x_1, x_3) - x_2,\]

then, for each \((x_1, x_2, x_3)\) with \(F_1(x_1, x_2, x_3) = 0\), there exist open neighbourhoods \(U \subset \mathbb{R}^{2m}\) of \((x_1, x_2)\) and \(V \subset \mathbb{R}^m\) of \(x_3\) and a unique function \(h_1 \in C^1(U, V)\) such that

\[
\forall (x_1, x_2) \in U \forall x_3 \in V : \left( F_1(x_1, x_2, x_3) = 0 \iff h_1(x_1, x_2) = x_3 \right).
\]

Moreover, \(h_1\) inherits the smoothness of \(F_1\) (which, in turn, inherits the smoothness of \(f_1\)) and so, by (P1), \(h_1 \in C^{r-1}(U, V)\). Invoking (P2) together with \[^{21}\] Cor. 5.3, this result globalizes to conclude the existence of \(h_1 \in C^{r-1}(\mathbb{R}^{2m}, \mathbb{R}^m)\) such that

\[
\forall (x_1, x_2, x_3) \in \mathbb{R}^{3m} : \left( F_1(x_1, x_2, x_3) = 0 \iff h_1(x_1, x_2) = x_3 \right).
\]

By (56), we have \(F_1(\xi_1(t), \xi_1(t), \xi_2(t)) = 0\) for all \(t \in [0, \omega)\) and so

\[
\forall t \in [0, \omega) : \xi_2(t) = h_1(\xi_1(t), \dot{\xi}_1(t)) = h_1(y(t), \dot{y}(t)).
\]

If \(r \geq 3\), then, repeating the above construction in the context of the equation

\[
\dot{\xi}_2(t) = f_2(\xi_1(t), \xi_2(t), \xi_3(t)), \quad t \in [0, \omega),
\]

we arrive at \(h_2 \in C^{r-2}(\mathbb{R}^{3m}, \mathbb{R}^m)\) such that

\[
\forall t \in [0, \omega) : \xi_3(t) = h_2(\xi_1(t), \xi_2(t), \dot{\xi}_2(t)).
\]

Furthermore,

\[
\dot{\xi}_2(t) = \frac{d}{dt} \left( h_1(y(t), \dot{y}(t)) \right) = \partial_1 h_1(y(t), \dot{y}(t)) \dot{y}(t) + \partial_2 h_1(y(t), \dot{y}(t)) \ddot{y}(t)
\]
Our next goal is to show that iterating the above process gives \((54)\) for some function \(\tilde{\eta}\) we have

\[
\forall t \in [0, \omega) : \quad \tilde{\xi}_1(t) = h_2(y(t), \dot{y}(t), \ddot{y}(t)).
\]

Iterating the above process, gives \((54)\).

**Step 3.** Our next goal is to show that

\[
y^{(r)}(t) = f_r(d(t), y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t), u(t)) \quad \text{for a.a.} \ t \in [0, \omega)
\]

for some function \(f_r \in C^1(\mathbb{R}^{p+rm+q+m}, \mathbb{R}^m)\). If \(r = 1\), then, by (22), we have \(y(t) = f_1(d(t), y(t), \eta(t), u(t))\) for almost all \(t \in [0, \omega)\) and so \((58)\) holds with \(\tilde{f}_1 := f_1\).

Assume now \(r \geq 2\). For notational convenience, set \(\tilde{h}_0 := h_0 = \text{id}_{\mathbb{R}^m}\) and, for \(k = 1, \ldots, r\), write \(x_k := (x_1, \ldots, x_k)\). For \(k = 1, \ldots, r-1\), introduce functions

\[
\tilde{h}_k : \mathbb{R}^{km} \to \mathbb{R}^{km}, \quad \tilde{h}_k(x_{k+1}) = (h_{k-1}(x_k), h_k(x_{k+1})).
\]

For later use, we record that

\[
\tilde{h}_k(\tilde{h}_{k-1}(x_k), x_{k+1}) = (\tilde{h}_{k-1}(x_k), h_k(x_{k+1})). \tag{59}
\]

Set \(\tilde{f}_1 := f_1\) and, for \(k = 2, \ldots, r-1\), recursively define functions

\[
\tilde{f}_k : \mathbb{R}^{(k+1)m} \to \mathbb{R}^m,
\]

\[
(\tilde{x}_k, x_{k+1}) = (x_1, \ldots, x_{k+1}) \mapsto \sum_{j=1}^{k-1} \partial_j \tilde{f}_{k-1}(\tilde{h}_{k-1}(\tilde{x}_k)) \cdot x_{j+1}
\]

\[
+ \partial_k \tilde{f}_{k-1}(\tilde{h}_{k-1}(\tilde{x}_k)) \cdot f_k(\tilde{h}_{k-1}(\tilde{x}_k), x_{k+1}). \tag{60}
\]

A straightforward induction, invoking (22) and (54), shows that for all \(k = 1, \ldots, r-1\) we have

\[
\forall t \in [0, \omega) : \quad y^{(k)}(t) = \tilde{f}_k(y(t), \ldots, y^{(k-1)}(t), \tilde{\xi}_{k+1}(t)).
\]

Differentiating the equation for \(k = r-1\) and again using (54) gives

\[
y^{(r)}(t) = \sum_{k=1}^{r-1} \partial_k \tilde{f}_{r-1} \left( h_{r-1}(y(t), \ldots, y^{(r-1)}(t)) \right) \cdot y^{(k)}(t)
\]

\[
+ \partial_r \tilde{f}_{r-1} \left( h_{r-1}(y(t), \ldots, y^{(r-1)}(t)) \right) \cdot \tilde{\xi}_r(t)
\]

for almost all \(t \in [0, \omega)\). Furthermore, by (22) and (54) we have

\[
\tilde{\xi}_r(t) = f_r(d(t), \tilde{h}_{r-1}(y(t), \ldots, y^{(r-1)}(t)), \eta(t), u(t))
\]
for almost all \( t \in [0, \omega) \). Then, defining
\[
\hat{f}_r : \mathbb{R}^{p + rm + q + m} \rightarrow \mathbb{R}^m,
\]
\[
(\delta, x_r, z, v) = (\delta, x_1, \ldots, x_r, z, v) \mapsto \sum_{k=1}^{r-1} \partial_k \hat{f}_{r-1}(\hat{h}_{r-1}(x_r)) \cdot x_{k+1} \quad (61)
\]
\[
+ \partial_r \hat{f}_{r-1}(\hat{h}_{r-1}(x_r)) \cdot f_r(\delta, \hat{h}_{r-1}(x_r), z, v),
\]
we see that (58) holds.

**Step 4.** Attention is now turned to the internal dynamics, that is, the last of the differential equations in (22). Viewed in isolation as a system with input \( w = (d, \xi_1, \ldots, \xi_r) \), assumption (P1) ensures that it generates a controlled flow, which we denote by \( \Xi \). In particular, for \( \eta^0 \in \mathbb{R}^q \), \( d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p) \), and \( \xi_1, \ldots, \xi_r \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \), the initial-value problem
\[
\dot{\eta}(t) = g(d(t), \xi_1(t), \ldots, \xi_r(t), \eta(t)), \quad \eta(0) = \eta^0
\]
has unique maximal solution \( \eta(\cdot) = \Xi(\cdot; w, \eta^0) : [0, \omega) \rightarrow \mathbb{R}^q \), where \( 0 < \omega \leq \infty \) and with \( w = (d, \xi_1, \ldots, \xi_r) \).

We remark that (P3) ensures that solutions are globally defined: specifically, for each \( \eta^0 \in \mathbb{R}^q \) and \( w = (d, \xi_1, \ldots, \xi_r) \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{p+rm}) \), the unique maximal solution of (62) has interval of existence \([0, \infty)[\). Furthermore, if the internal dynamics are input-to-state stable in the sense of (54), then (P3) holds a fortiori.

Fix \( (\eta^0, d) \in \mathbb{R}^q \times L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p) \) arbitrarily and define the operator
\[
T : C(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm}) \rightarrow L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^q), \quad \zeta = (\xi_1, \ldots, \xi_r) \mapsto \left( \xi, \Xi(\cdot; d, \hat{h}_{r-1}(\xi), \eta^0) \right).
\]
Clearly, \( T \) is causal, i.e., (TP1) holds. Moreover, (P1) and (P3) ensure that properties (TP2) and (TP3) hold. Therefore, \( T \in \mathbb{T}^{0,m,q} \).

Now consider system (22) with \( d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p) \), \( u \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \) and initial data
\[
(\xi(0), \eta(0)) = (\xi_1(0), \ldots, \xi_r(0), \eta(0)) = (\xi_1^0, \ldots, \xi_r^0, \eta^0) = (\xi^0, \eta^0).
\]
Define \( \hat{f}_0 := \text{id}_{\mathbb{R}^m} \) and, for \( k = 1, \ldots, r-1 \), recursively
\[
\hat{f}_k : \mathbb{R}^{(k+1)m} \rightarrow \mathbb{R}^m, \quad (x_1, \ldots, x_{k+1}) \mapsto \sum_{j=1}^{k} \partial_j \hat{f}_{k-1}(x_1, \ldots, x_k) \cdot f_j(x_1, \ldots, x_{k+1}).
\]
Then, setting
\[
y^0 = (y_1^0, \ldots, y_r^0) := (y_1^0, \ldots, \hat{f}_{r-1}(\xi_1^0, \ldots, \xi_r^0)),
\]
we may infer that the above initial-value problem for system (22) is equivalent to
\[
y^{(r)}(t) = \hat{f}_r(d(t), T(y(t), \ldots, y^{(r-1)})(t), u(t)), \quad (y(0), \ldots, y^{(r-1)}(0)) = y^0.
\]
Step 5. To conclude that \((d, f_r, T) \in \mathcal{N}^{m,r}\), we show that the map \(f_r\) satisfies the high-gain property with \(v^* = \frac{1}{2}\). By (61), we have
\[
\forall (\delta, x_r, z, v) \in \mathbb{R}^{p+\hat{q}+m} : \partial_v \tilde{f}_r(\delta, x_r, z, v) = \partial_v \tilde{f}_{r-1}(\tilde{h}_{r-1}(x_r)) \cdot \partial_v f_r(\delta, \tilde{h}_{r-1}(x_r), z, v),
\]
and, by (60) in conjunction with (59),
\[
\partial_v \tilde{f}_{r-1}(\tilde{h}_{r-1}(x_r)) = \partial_v \tilde{f}_{r-2}(\tilde{h}_{r-2}(x_{r-1})) \cdot \partial_v f_r(\delta, \tilde{h}_{r-1}(x_r), z, v).
\]
Therefore,
\[
\partial_v f_r(\delta, x_r, z, v) = \partial_v f_1(\tilde{h}_1(x_2)) \cdot \partial_v f_r(\delta, x_r, z, v).
\]
Iterating this process, we arrive at
\[
\partial_v f_r(\delta, x_r, z, v) = \partial_v f_{r-1}(\tilde{h}_{r-1}(x_r)) \cdot \partial_v f_r(\delta, h_{r-1}(x_r), z, v).
\]

With the function \(G\) defined in (P4) we find that
\[
\forall (\delta, x_r, z, v) \in \mathbb{R}^{p+\hat{q}+m} : \partial_v f_r(\delta, x_r, z, v) = G(\delta, \tilde{h}_{r-1}(x_r), z, v).
\]
(63)

Set \(v^* = \frac{1}{2}\) and let \(K_p \subset \mathbb{R}^p\) and \(K_{\hat{q}} \subset \mathbb{R}^{\hat{q}}\) be compact and define the compact annulus
\[
A_m := \{ v \in \mathbb{R}^m \mid \frac{1}{2} \leq ||v|| \leq 1 \}.
\]
Set
\[
\hat{K}_{\hat{q}} := \{ (\tilde{h}_{r-1}(x_r), z) \mid (x_r, z) \in K_{\hat{q}} \}
\]
which, by continuity of \(\tilde{h}_{r-1}\), is compact, and write \(K = K_p \times \hat{K}_{\hat{q}}\). By (P4), there exist \(c_0, c_1 > 0\) and \(\sigma \in \{-1, 1\}\) such that
\[
\forall (\delta, y_r, z) \in K \forall v \in A_m \forall \lambda \in \mathbb{R} \text{ with } ||\lambda|| \geq 2c_0 : \sigma v^T G(\delta, y_r, z, -\lambda v) v \geq c_1 ||v||^2.
\]
(64)

Fix \(\delta \in K_p\), \((x_r, z) \in \hat{K}_{\hat{q}}\) and \(v \in A_m\) arbitrarily. Invoking (63) and (64), we have
\[
\forall \lambda \in \mathbb{R} \text{ with } ||\lambda|| \geq 2c_0 : \sigma v^T \partial_v f_r(\delta, x_r, z, -\lambda v) v \geq c_1 ||v||^2.
\]
Let \(s \in \mathbb{R}\) be such that \(\sigma s < -2c_0\). By the mean-value theorem applied in the context of the function
\[
\mathbb{R} \rightarrow \mathbb{R}, t \mapsto v^T f_r(\delta, x_r, z, -tv)\text{ with derivative } t \mapsto -v^T \partial_v f_r(\delta, x_r, z, -tv)v,
\]
there exists \(\lambda \in \mathbb{R}\), with \(\lambda \in (s, -2c_0)\) if \(\sigma = 1\) or \(\lambda \in (2c_0, s)\) if \(\sigma = -1\), such that
\[
v^T f_r(\delta, x_r, z, -sv) - v^T f_r(\delta, x_r, z, 2\sigma c_0 v) = -(s + 2\sigma c_0) v^T \partial_v f_r(\delta, x_r, z, -\lambda v) v
\]
\[
\geq |\sigma s + 2c_0| \sigma v^T \partial_v f_r(\delta, x_r, z, -\lambda v) v
\]
\[
\geq |\sigma s + 2c_0| c_1 ||v||^2 \geq \frac{1}{4} c_1 |\sigma s + 2c_0|.
\]
Defining
\[
c_2 := \min \left\{ v^T f_r(\delta, x_r, z, 2c_0 v) \mid \delta \in K_p, (x_r, z) \in K_{\hat{q}}, v \in A_m \right\},
\]
we have
\[ \forall s \in \mathbb{R} : \left( \sigma s < -2c_0 \implies v^\top \tilde{f}_r(\delta, x_r, z, -sv) \geq c_2 + \frac{1}{4} c_1 |\sigma s + 2c_0| \right). \]

Since \( \delta \in K_p, (x_r, z) \in K_q \) and \( v \in A_m \) are arbitrary, it follows that
\[ \chi(s) = \min \left\{ v^\top \tilde{f}_r(\delta, x_r, z, -sv) \mid \delta \in K_p, (x_r, z) \in K_q, v \in A_m \right\} \geq c_2 + \frac{1}{4} c_1 |\sigma s + 2c_0| \]
for all \( s \in \mathbb{R} \) with \( \sigma s < -2c_0 \), whence \( \sup_{s \in \mathbb{R}} \chi(s) = \infty \). Therefore, the high-gain property holds and this finishes the proof.

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