GEOMETRIC DESCRIPTION OF $(2)$-POLARISED HILBERT SQUARES OF GENERIC K3 SURFACES

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Abstract. A generic K3 surface of degree $2t$ is a general complex projective K3 surface $S_{2t}$ whose Picard group is generated by the class of an ample divisor $H \in \text{Div}(S_{2t})$ such that $H^2 = 2t$ with respect to the intersection form. We show that if $X$ is the Hilbert square of a generic K3 surface of degree $2t$ with $t \neq 2$ which admits an ample divisor $D \in \text{Div}(X)$ with $q_X(D) = 2$, where $q_X$ is the Beauville–Bogomolov–Fujiki form, then $X$ is a double EPW sextic.

1. Introduction

A classical result in the theory of complex projective K3 surfaces says that if a K3 surface admits an ample divisor $D$ with $D^2 = 2$ with respect to the intersection form, then the complete linear system $|D|$ is basepoint free and the morphism that it induces is a double cover of the plane $\mathbb{P}^2$ ramified on a sextic curve. This was proved by Saint–Donat in [SD74].

It is natural to study a similar problem for a projective IHS (Irreducible Holomorphic Symplectic) manifold of dimension $2n$ with $n \geq 2$, a sort of higher dimensional generalization of K3 surfaces. The importance of these varieties is linked to the Beauville–Bogomolov decomposition theorem, see [Bea83b]: up to a finite étale cover, any compact Kähler manifold with trivial first Chern class is the product of a complex torus, irreducible Calabi–Yau manifolds and IHS manifolds.

Let $S$ be a K3 surface. We denote by $T(S)$ its transcendental lattice, which coincides with $\text{NS}(S)^\perp$, where $\text{NS}(S)$ is the Néron–Severi group of $S$ and the orthogonal is taken with respect to the intersection form. Then $T(S)_Q := T(S) \otimes \mathbb{Q}$ is a rational Hodge structure of weight 2 and we denote by

$$E_S := \text{Hom}_{\mathbb{Q}}(T(S)_Q, T(S)_Q)$$

the algebra of endomorphisms of weight 0 on $T(S)_Q$. See [Huy16, §3] for details on Hodge structures. By a result of Zarhin, see [Zar83], $E_S$ is either $\mathbb{Q}$, or a totally real field or a CM field. We say that $S$ is a K3 surface general in its rank, or simply general, if $E_S \cong \mathbb{Q}$. If $E_S$ is a totally real field then $S$ has real multiplication (RM), while if $E_S$ is a CM field then $S$ has complex multiplication (CM). Then K3 surfaces with RM or CM describe a locus of positive codimension in the analytic moduli space of K3 surfaces of fixed Picard rank, except for K3 surfaces of Picard rank 20, which have all CM. See [vG08] and [EJ16] for details.

A generic K3 surface is a general projective K3 surface whose Picard group has rank 1. We say that a generic K3 surface $S_{2t}$ has degree $2t$ if $\text{Pic}(S_{2t}) = \mathbb{Z}H$ with $H^2 = 2t$. In this paper we focus on Hilbert schemes of 2 points on a generic K3 surface, also known as Hilbert squares of generic K3 surfaces: the Hilbert square of a K3 surface $S$ will be denoted by $S^{[2]}$. The main problem of this paper is to determine the geometric description of a $(2)$-polarised Hilbert square $X$ of a generic K3 surface, i.e., $X$ admits an ample divisor $D \in \text{Div}(X)$ with $q_X(D) = 2$, where $q_X$ is the Beauville–Bogomolov–Fujiki quadratic form on $H^2(X, \mathbb{Z})$. This can be seen as a generalization of the problem studied by Saint–Donat presented in the first lines. We denote by $|D|$ the complete linear system associated to $D \in \text{Div}(X)$, i.e., the set of effective divisors linearly equivalent to $D$, and by $\text{Bs}|D|$ the base locus of $|D|$, which is the set of points on $X$ where all the global sections in $H^0(X, \mathcal{O}_X(D))$ vanish. Our aim is to study the rational map $\varphi|D|$ induced by the complete linear system $|D|$, i.e.,

$$\varphi|D| : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee),$$

where by definition $x \in X \setminus \text{Bs}|D|$ is mapped to the hyperplane in $H^0(X, \mathcal{O}_X(D))$ consisting of sections vanishing at $x$. If $\text{Bs}|D|$ is empty, we say that $|D|$ is basepoint free: in this case $\varphi|D|$ is a morphism.

Let $X$ and $D$ be as above. Then there exists an anti-symplectic involution $\iota$ which generates the group $\text{Aut}(X)$ of biregular automorphisms on $X$ by a result obtained by Boissière, Cattaneo, Nierop-Wijfikirchen and Sarti in [BCNWS16], and $\iota$ is such that $\iota^*|D| = |D|$ in the Néron–Severi group $\text{NS}(X)$. Here anti-symplectic means that $\iota^*\sigma_X = -\sigma_X$, where $\sigma_X \in H^0(X, \Omega_X^2)$ is a symplectic form of $X$. The main theorem of this paper, which we now
present, gives a geometrical description of the map induced by the complete linear system \( |D| \), cf. Theorem 6.4 and Theorem 8.5.

**Theorem.** Let \( X \) be the Hilbert square of a generic K3 surface \( S_{2t} \) of degree \( 2t \) such that \( X \) admits an ample divisor \( D \in \text{Div}(X) \) with \( q_X(D) = 2 \). Suppose that \( t \neq 2 \), and denote by \( \iota \) the anti-symplectic involution which generates the group \( \text{Aut}(X) \). Then the complete linear system \( |D| \) is basepoint free, the morphism

\[
\varphi_{|D|} : X \to Y \subset \mathbb{P}^5
\]

is a double cover whose ramification locus is the surface \( F \) of points fixed by \( \iota \), and \( Y \cong X/\langle \iota \rangle \) is an EPW sextic, in particular \( X \) is a double EPW sextic. Moreover, if \( H^{2,2}(X,\mathbb{Z}) := H^4(X,\mathbb{Z}) \cap H^{2,2}(X) \), then

\[
[F] = 5D^2 - \frac{1}{3}c_2(X) \in H^{2,2}(X,\mathbb{Z}),
\]

where \( c_2(X) \in H^4(X,\mathbb{Z}) \) is the second Chern class of \( X \).

Here a double EPW sextic is a double cover of an EPW sextic ramified in its singular locus: see [EPW01] for the definition of EPW sextic and [O’G06] for details on double EPW sextics. The paper [O’G08], where O’Grady gives a classification, up to deformation equivalence, of numerical K3\(^{[2]} \), will play an important role. A numerical K3\(^{[2]} \) is by definition an IHS manifold \( M \) which admits an isomorphism of abelian groups \( \psi : H^2(M,\mathbb{Z}) \to H^2(S^{[2]},\mathbb{Z}) \) for some K3 surface \( S \) such that \( \int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \) for every \( \alpha \in H^2(M,\mathbb{Z}) \). In particular he showed that a numerical K3\(^{[2]} \) is deformation equivalent either to a double EPW sextic or to an IHS manifold \( Z \) admitting a rational map \( f : Z \to \mathbb{P}^5 \) which is birational onto its image \( Y \), with \( 6 \leq \deg(Y) \leq 12 \). The link between our problem and the one studied in [O’G08] is given by the fact that O’Grady proved the following: a numerical K3\(^{[2]} \) is deformation equivalent to an IHS manifold \( Z \) of K3\(^{[2]} \)-type such that \( \text{Pic}(Z) \) is generated by the class of an ample divisor \( H \in \text{Pic}(Z) \) with \( q_Z(H) = 2 \). Let \( X \) be the Hilbert square of a general K3 surface of degree \( 2t \) and suppose that there is an ample divisor \( D \in \text{Div}(X) \) with \( q_X(D) = 2 \). In this specific case the result obtained by O’Grady implies that \( X \) is deformation equivalent to a double EPW sextic, while our main theorem is stronger, since it shows that \( X = S_{2t} \), if \( t \neq 2 \), is exactly a double EPW sextic, giving a more precise geometric description of these varieties. The case \( t = 2 \) was already studied by Welters and Beauville in [Wei81] and [Bea83a]: the map \( \varphi_{|D|} \) is a finite morphism of degree 6 with image \( \mathbb{G}(1,\mathbb{P}^3) \), the Grassmannian of lines in \( \mathbb{P}^3 \).

Our strategy is to follow [O’G08], using the anti-symplectic involution which generates \( \text{Aut}(X) \) given by [BCNWS16] in order to get as much information as possible on the geometry of the complete linear system.

The paper is organised as follows. In Section 2 we give basics on Pell equations and Pell-type equations. In Section 3 we recall the definition of IHS manifold, together with main properties of this family of varieties. In Section 4 we present some useful results on IHS manifolds of K3\(^{[2]} \)-type, in particular we recall the description of the lattice \( H^{2,2}(S^{[2]},\mathbb{Z}) := H^4(S^{[2]},\mathbb{Z}) \cap H^{2,2}(S^{[2]}) \) of integral Hodge classes of type \((2,2)\) on \( S^{[2]} \) obtained in [Nov21] for a general projective K3 surface \( S \): this will be extremely useful in several steps of this paper. In Section 5 we recall the most important results on the Hilbert square \( X \) of a generic K3 surface, in particular the description of the group of biregular automorphisms obtained in [BCNWS16]: \( \text{Aut}(X) \) is not trivial and generated by a non-natural anti-symplectic involution \( \iota \) when \( X \) is the Hilbert square of a generic K3 surface of degree \( 2t \) admitting an ample divisor \( D \in \text{Div}(X) \) with \( q_X(D) = 2 \). We also briefly describe the case \( t = 2 \), already studied by Welters and Beauville in [Wei81] and [Bea83a]. From now on \( X \) is the Hilbert square of a generic K3 surface admitting an ample divisor \( D \) with \( q_X(D) = 2 \). In Section 6 we study the surface \( F \) of points on \( X \) fixed by the anti-symplectic involution \( \iota \): we compute the class of \( F \) in \( H^{2,2}(X,\mathbb{Z}) \) and we show that the rational map \( \varphi_{|D|} \) factors through the quotient with respect to the involution \( \iota \). In Section 7 we prove the irreducibility property: if \( D_1, D_2 \in |D| \) are distinct and \( t \neq 2 \), where \( 2t \) is the degree of the underlying generic K3 surface, then \( D_1 \cap D_2 \) is a reduced and irreducible surface. This is a fundamental step in the geometrical analysis of the rational map \( \varphi_{|D|} \). When \( t = 2 \), the surface \( D_1 \cap D_2 \) can be reducible: we give a geometrical interpretation of the two irreducible components. We conclude with Section 8, where we show the main result of this paper: keeping notation as above, if \( t \neq 2 \) then \( X \) is a double EPW sextic and \( \varphi_{|D|} \) is the double cover associated.

The topic of this paper is related to [DM19, §7]. The approach and the techniques that we use are different: main tools are results on integral Hodge classes of type \( (2,2) \) on the Hilbert square of a generic projective K3 surface obtained in [Nov21], which are exploited in particular in Theorem 6.4 and in Theorem 7.3, and the papers [O’G08] and [BCNWS16]. Moreover, the irreducibility property of Theorem 7.3 explains geometrically why the case \( t \neq 2 \) is special, compare with [DM19, Remark 7.7].
This paper is based on Chapter 4 and Chapter 5 of the author’s PhD thesis, see [Nov].

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2. Pell equations and Pell-type equations

We give a brief overview of Pell equations and Pell-type equations.

Definition 2.1. A Pell-type equation is a diophantine equation of the form

\[ x^2 - dy^2 = n, \]

where \( d \in \mathbb{Z}_{>0} \) is a positive integer, \( n \in \mathbb{Z} \setminus \{0\} \) is a non-zero integer and \( x, y \) are variables. We denote an equation of this form by \( P_d(n) \). We call \( P_d(1) \) a Pell equation and \( P_d(-1) \) a negative Pell equation.

We are interested in integral solutions of Pell equations and Pell-type equations. If \( d = c^2 \) is a square, the only solutions of the Pell equation \( P_d(1) \) are \((x, y) = (\pm 1, 0)\), and the Pell-type equation \( P_d(n) \) can be written as \((x + cy)(x - cy) = n\), so it can be easily solved. From now on, we will assume that \( d \) is not a square. In this case the Pell-type equation \( P_d(n) \) can be written in \( \mathbb{Z}[\sqrt{d}] := \mathbb{Z}[x]/(x^2 - d) \) as

\[ (x + y\sqrt{d})(x - y\sqrt{d}) = n. \]

Let \( z = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}] \); we define its conjugate as \( \bar{z} := x - y\sqrt{d} \in \mathbb{Z}[\sqrt{d}] \), and its norm as \( N(z) := z\bar{z} = x^2 - dy^2 \in \mathbb{Z} \). With this notation, \( P_d(1) \) and \( P_d(n) \) can be written respectively as \( N(z) = 1 \) and \( N(z) = n \), where \( z = x + y\sqrt{d} \).

Definition 2.2. Given a Pell equation \( P_d(1) \) or a Pell-type equation \( P_d(n) \), two solutions \((X, Y) \) and \((X', Y') \) are said to be equivalent if

\[ \frac{XX' - dYY'}{n} \in \mathbb{Z}, \quad \frac{XY' - X'Y}{n} \in \mathbb{Z}. \]

Note that two solutions \((X, Y) \) and \((X', Y') \) of a Pell-type equation \( P_d(n) \) are equivalent if \( \frac{XX' - dYY'}{n} \in \mathbb{Z}[\sqrt{d}] \), where \( z_1 = X + Y\sqrt{d} \) and \( z_2 = X' + Y'\sqrt{d} \).

Definition 2.3. Consider a Pell equation or a Pell-type equation. The fundamental solution \((X, Y) \) in an equivalence class of solutions is the one with smallest non-negative \( Y \) if such a solution is unique in its class. Otherwise, there are two solutions \((X, Y) \), \((-X, Y) \), which are said to be conjugated, with smallest non-negative \( Y \): the fundamental solution is the one of the form \((X, Y) \) with \( X > 0 \).

One can show that a Pell equation \( P_d(1) \) is always solvable and all its solutions are equivalent. If \( z_0 = a + b\sqrt{d} \) is the fundamental solution, then all the other solutions are of the form

\[ z = \pm z_0^m, \quad m \in \mathbb{Z}_{>0}. \]

Similarly, if \( z_0 \) is a fundamental solution of \( P_d(n) \) and \( \bar{z}_0 \) is the fundamental solution of \( P_d(1) \), all the other solutions equivalent to \( z_0 \) are of the form

\[ z = \pm z_0^m \cdot z_0, \quad m \in \mathbb{Z}_{>0}. \]

We now recall the definition of positive and minimal solution of a Pell-type equation.

Definition 2.4. Let \( P_d(n) \) be a Pell-type equation. A solution \((X, Y) \) is positive if \( X, Y > 0 \). The minimal solution of \( P_d(n) \) is the positive solution with smallest \( X \).

Remark 2.5. The minimal solution of the Pell equation \( P_1(1) \) coincides with the square of the minimal solution of the negative Pell equation \( P_1(-1) \), if this exists, i.e., if \( a + b\sqrt{t} \in \mathbb{Z}[\sqrt{t}] \) is the minimal solution of \( N(z) = -1 \) and \( c + d\sqrt{t} \in \mathbb{Z}[\sqrt{t}] \) is the minimal solution of \( N(z) = 1 \), then \( c + d\sqrt{t} = (a + b\sqrt{t})^2 = a^2 + b^2 + 2ab\sqrt{t} \), hence

\[ a^2 + tb^2 = c, \quad d = 2ab. \quad (2.1) \]

We conclude this section with the following elementary result, which will be useful in Section 6 and in Section 7.

Proposition 2.6. Let \((a, b) \) be the minimal solution of the Pell-type equation \( P_d(-1) \). Then \( b \) is odd.

Proof. Suppose that \( b \) is even. Then \( a^2 + 1 \) is divisible by 4, since \( a^2 - db^2 = -1 \), i.e., \( a^2 + 1 = 4X \) for some \( X \in \mathbb{Z} \).
be even, i.e., \( a = 2Y \) for some \( Y \in \mathbb{Z} \), then \( 4Y^2 + 1 = 4X \), which is not possible.

- If \( a \) is odd, i.e., \( a = 2Y + 1 \) for some \( Y \in \mathbb{Z} \), then \( 4Y^2 + 1 + 4Y + 1 = 4X \), which gives \( 4(X - Y^2 - Y) = 2 \), which is not possible.

We conclude that \( b \) is odd.

\[ \square \]

3. Generalities on IHS manifolds

In this section we recall basics on irreducible holomorphic symplectic manifolds.

**Definition 3.1.** An **irreducible holomorphic symplectic** (IHS) **manifold** is a simply connected compact complex Kähler manifold \( X \) such that \( H^0(X, \Omega^2_X) \) is generated by a non-degenerate holomorphic 2-form, called **symplectic form**.

The definition of IHS manifold generalises to higher dimensions the one of K3 surface, the only example of IHS manifold of dimension 2, as shown by the Enriques–Kodaira classification of compact complex surfaces. The existence of a symplectic form implies that the dimension of an IHS manifold is necessarily even. If \( X \) is an IHS manifold, then the \( \mathbb{C} \)-vector space \( H^0(X, \Omega^2_X) \) is zero if \( p \) is odd, and it is generated by \( \sigma^2 \) if \( 0 \leq p \leq \dim(X) \) is even, where \( \sigma \) is a symplectic form, see [Bea83b, Proposition 3]. The Picard group \( \text{Pic}(X) \) and the Néron–Severi group, defined as \( \text{NS}(X) := H^{1,1}(X)_{\mathbb{R}} \cap H^2(X, \mathbb{Z}) \), are isomorphic: \( \text{Pic}(X) \) then embeds in the second cohomology group \( H^2(X, \mathbb{Z}) \).

We recall that a **lattice** is a free \( \mathbb{Z} \)-module \( L \) of finite rank with a symmetric bilinear form \( b : L \times L \to \mathbb{Z} \): we denote by \( q : L \to \mathbb{Z} \) the quadratic form \( q(x) := b(x, x) \) for every \( x \in L \). If \( B := \{e_1, \ldots, e_n\} \) is a \( \mathbb{Z} \)-basis of \( L \), the **Gram matrix** of \( L \) associated to \( B \) is the \( n \times n \) symmetric matrix

\[
\begin{pmatrix}
    b(e_1, e_1) & \cdots & b(e_1, e_n) \\
    \vdots & \ddots & \vdots \\
    b(e_n, e_1) & \cdots & b(e_n, e_n)
\end{pmatrix}
\]

We say that a lattice \( L \) of rank \( n \) is **non-degenerate** if for any non-zero \( l \in L \) there exists \( l' \in L \) such that \( b(l, l') \neq 0 \), equivalently, \( \det(G) \neq 0 \) if \( G \) is a Gram matrix of \( L \). A lattice \( L \) is **even** if \( b(l, l) \in 2\mathbb{Z} \) for every \( l \in L \), and **odd** if it is not even. The determinant of a lattice \( L \) is the determinant of a Gram matrix \( G \) of the lattice, and the **discriminant** of \( L \) is \( \text{disc}(L) := |\det(G)| \). A lattice \( L \) is **unimodular** if \( \text{disc}(L) = 1 \). If \( L \) is a unimodular lattice, for every \( x \in L \) there exists \( y \in L \) such that \( b(x, y) = 1 \). A **sublattice** of a lattice \( L \) is a free submodule \( L' \subseteq L \) with symmetric bilinear form \( b' := b|_{L' \times L'} \). A sublattice \( L' \subseteq L \) is **primitive** if \( L/L' \) is a free module. We define the direct sum of two lattices \( L_1 \) and \( L_2 \) as the lattice \( L_1 \oplus L_2 \) whose bilinear form is \( b(v_1 + v_2, w_1 + w_2) := b_1(v_1, w_1) + b_2(v_2, w_2) \) for every \( v_1, w_1 \in L_1 \) and \( v_2, w_2 \in L_2 \), where \( b_1 \) and \( b_2 \) are the bilinear forms of \( L_1 \) and \( L_2 \) respectively. If \( L \) and \( L' \) are two lattices with bilinear forms \( b \) and \( b' \) respectively, we call **morphism of lattices** \( \varphi : L \to L' \) a morphism of \( \mathbb{Z} \)-modules such that for every \( l_1, l_2 \in L \) we have \( b(l_1, l_2) = b'(\varphi(l_1), \varphi(l_2)) \). Note that morphisms between two non-degenerate lattices are injective. We say that a lattice **embeds primitively** in a lattice \( L' \) if there is a morphism \( \varphi : L \to L' \) such that \( \varphi(L) \) is a primitive sublattice of \( L' \). An **isometry** is a bijective morphism of lattices. The **divisibility** of an element \( l \in L \) in a lattice \( L \) is the positive generator of the ideal \( \{b(l, m) \mid m \in L\} \subseteq \mathbb{Z} \).

For a lattice \( L \) of rank \( n \) we write \( L_\mathbb{R} := L \otimes_{\mathbb{Z}} \mathbb{R} \) and we extend \( \mathbb{R} \)-bilinearly the bilinear form \( b \) to \( L_\mathbb{R} \), similarly we extend \( q \) to \( L_\mathbb{R} \). If the lattice is non-degenerate, the **signature** of \( L \) is the signature \( (l_{(+)}, l_{(-)}) \) of the quadratic form on \( L_\mathbb{R} \). A non-degenerate lattice is **positive definite** if \( l_{(-)} = 0 \), similarly it is **negative definite** if \( l_{(+)} = 0 \), while it is **indefinite** if \( l_{(+)}, l_{(-)} \neq 0 \).

**Example 3.2.** If \( k \) is a non-zero integer, let \( \langle k \rangle \) be the rank one lattice \( L = \mathbb{Z} e \) with bilinear form \( b(e, e) = k \).

**Example 3.3.** Let \( U \) be the **hyperbolic lattice**, i.e., the unique unimodular lattice of rank 2 and signature \((1,1)\). Its Gram matrix is the following:

\[
\begin{pmatrix}
    0 & 1 \\
    1 & 0
\end{pmatrix}
\]
Example 3.4. Let $E_8(-1)$ be the even unimodular lattice of signature $(0,8)$ whose Gram matrix is the following:

\[
\begin{pmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & 1 & -2 & 1 \\
& & & 1 & -2 \\
& & & & -2 \\
\end{pmatrix}
\]

Let $X$ be an IHS manifold. By the universal coefficient theorem, the second singular cohomology group $H^2(X, \mathbb{Z})$ is torsion free, and this can be equipped with a non-degenerate integral quadratic form by the following result due to Beauville, Bogomolov and Fujiki, see [Bea83b] and [Fuj87].

Theorem 3.5 (Beauville–Bogomolov–Fujiki form). Let $X$ be an IHS manifold of dimension $2n$. Then there exists an integral quadratic form $q_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ and a constant $c_X \in \mathbb{Q}_{>0}$ such that

\[
\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X(\alpha)^n \quad \text{for all } \alpha \in H^2(X, \mathbb{Z}).
\]

The quadratic form $q_X$ is called Beauville–Bogomolov–Fujiki (BBF) form, and $c_X$ is called Fujiki constant of $X$. Moreover, $(H^2(X, \mathbb{Z}), q_X)$ is a lattice of signature $(3, b_2(X) - 3)$, where $b_2(X)$ is the second Betti number of $X$.

An example of IHS manifold of dimension $2n$, where $n \geq 2$, is given by the Hilbert scheme of $n$ points on a K3 surface $S$, the scheme which parametrises zero-dimensional closed subschemes of length $n$ on a K3 surface: we denote it by $S^{[n]}$. Let $S^{(n)}$ be the quotient of $S^n = S \times \cdots \times S$ by the symmetric group of $n$ elements, so $S^{(n)}$ is the variety of 0-cycles of degree $n$. Then the Hilbert–Chow morphism $\rho : S^{[n]} \to S^{(n)}$ is defined as follows: a point $[x] \in S^{[n]}$ is mapped to the cycle $\sum_x i(\mathcal{O}_{\mathbb{P}^1}) \cdot x$, see for instance [Iv69]. The singular locus of $S^{(n)}$ is the so-called diagonal, i.e., the set of cycles $p_1 + \cdots + p_n$ such that there exist distinct $i$ and $j$ with $p_i = p_j$. The Hilbert–Chow morphism is a desingularization of $S^{(n)}$, and the pre-image of the diagonal is an irreducible divisor $E$ on $S^{[1]}$. The Hilbert scheme of $n$ points on a K3 surface is an IHS manifold by [Bea83b, Théorème 3]. There exists a primitive class $\delta \in \text{Pic}(S^{[n]})$ such that $2\delta = [E]$. Moreover, there is a primitive embedding of lattices

\[
i : H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[1]}, \mathbb{Z})
\]

such that $H^2(S^{[1]}, \mathbb{Z}) = i(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta$, and $q_{S^{[1]}}(\delta) = -2(n - 1)$. Recall that $H^2(S, \mathbb{Z}) \cong U \oplus \mathbb{Z}$, in particular $H^2(S, \mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$, see for instance [BHPV15, §VII.3]. Then there is an isometry of lattices

\[
H^2(S^{[1]}, \mathbb{Z}) \cong U \oplus \mathbb{Z} \oplus \mathbb{Z}\delta \cong U \oplus \mathbb{Z} \oplus E_8(-1) \oplus \langle -2(n - 1) \rangle,
\]

Similarly $\text{Pic}(S^{[n]}) = i(\text{Pic}(S)) \oplus \mathbb{Z}\delta$: see [Bea83b, §6] for details. The Fujiki constant of the Hilbert scheme of $n$ points on a K3 surface $S$ is $c_{S^{[n]}} = 1$, see [Bea83b, §9]. In particular for K3 surfaces the BBF form coincides with the intersection form. Moreover, the singular cohomology ring $H^*(S^{[1]}, \mathbb{Z})$ for a K3 surface $S$ and $n \geq 1$ is torsion free by [Mar07, Theorem 1]. When $n = 2$, the variety $S^{[2]}$ is usually called Hilbert square of a K3 surface. An IHS manifold which is deformation equivalent to the Hilbert square of a K3 surface is said to be of $K3^{[2]}$-type.

The other known examples of IHS manifolds up to deformation equivalence are generalised Kummer varieties, see [Bea83b, §7], an isolated example of dimension 10 and second Betti number $b_2 = 24$, see [OG99], and an isolated example of dimension 6 and second Betti number $b_2 = 8$, see [OG03]. We do not discuss details on these examples in this paper we deal only with Hilbert squares of K3 surfaces.

We conclude this section with the following useful correspondence between primitive elements in $H^2(X, \mathbb{Z})$ and primitive elements in $H_2(X, \mathbb{Z})_f$, where $X$ is an IHS manifold and $H_2(X, \mathbb{Z})_f$ is the torsion free quotient group of the second singular homology $H_2(X, \mathbb{Z})$. See [HT01] for details.

Proposition 3.6 (Hassett–Tschinkel). Let $X$ be an IHS manifold and denote by $H_2(X, \mathbb{Z})_f$ the torsion free quotient group of $H_2(X, \mathbb{Z})$. Let $(\cdot, \cdot)$ be the BBF bilinear form. Then there is a correspondence between primitive elements in $H^2(X, \mathbb{Z})$ and primitive elements in $H_2(X, \mathbb{Z})_f$. In particular:
(i) For every primitive element \( R \in H_2(X, \mathbb{Z}) \) there exists a unique class \( \omega \in H^2(X, \mathbb{Q}) \) such that
\[
\epsilon(R \cap v) = (\omega, v) \quad \text{for every } v \in H^2(X, \mathbb{Z}),
\]
where \( \epsilon : H_2(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \) is the isomorphism in [Bre13, Theorem IV.2.1] and \( \cap \) is the cup product. The primitive \( \rho \in H^2(X, \mathbb{Z}) \) associated to \( R \) is the primitive element such that \( c \rho = \omega \) for some \( c \in \mathbb{Q}_{>0}. \)

(ii) For every primitive element \( \rho \in H^2(X, \mathbb{Z}) \) of divisibility \( \text{div}(\rho) = d \) in \( (H^2(X, \mathbb{Z}), q_X) \), there exists a unique primitive \( R \in H_2(X, \mathbb{Z}) \) such that
\[
d \cdot \epsilon(R \cap v) = (\rho, v) \quad \text{for every } v \in H^2(X, \mathbb{Z}).
\]

4. Generalities on IHS manifolds of \( K^{[2]} \)-type

Let \( X \) be an IHS manifold of dimension 4 of \( K^{[2]} \)-type. In this section we recall the link between the intersection pairing on \( H^4(X, \mathbb{Q}) \) and the \( \mathbb{Q} \)-bilinear extension on \( H^2(X, \mathbb{Q}) \) of the BBF form, and we introduce the dual \( q^\vee_X \) of the BBF quadratic form. We refer to [O’G08, §2].

First of all, we state the following corollary of Verbitsky’s results in [Ver96], obtained by Guan in [Gua01], see also [O’G10, Corollary 2.5]. We denote by \( b_i(X) \) the \( i \)-th Betti number of \( X \).

**Proposition 4.1.** Let \( X \) be an IHS manifold of dimension 4. Then \( b_2(X) \leq 23 \). If equality holds then \( b_3(X) = 0 \) and the map
\[
\text{Sym}^2 H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q})
\]
induced by the cup product is an isomorphism. In particular this happens when \( X \) is an IHS fourfold of \( K^{[2]} \)-type.

Since \( X \) is a compact complex manifold of dimension \( \dim_{\mathbb{C}}(X) = 4 \), the singular cohomology group \( H^4(X, \mathbb{Z}) \) has an intersection pairing induced by the cup product:
\[
\langle \cdot, \cdot \rangle : H^4(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}) \to \mathbb{Z}, \quad \langle \alpha, \beta \rangle := \int_X \alpha \beta.
\]
We write \( \langle \cdot, \cdot \rangle \) also for the \( \mathbb{Q} \)-bilinear extension of the intersection pairing above to \( H^4(X, \mathbb{Q}) \times H^4(X, \mathbb{Q}) \), obtaining a \( \mathbb{Q} \)-valued intersection pairing on \( \text{Sym}^2 H^2(X, \mathbb{Q}) \). Let \( X \) be an IHS manifold of \( K^{[2]} \)-type: the following relation between \( \langle \cdot, \cdot \rangle \) and the \( \mathbb{Q} \)-extension of the BBF form on \( H^2(X, \mathbb{Q}) \) holds, see [O’G08, Remark 2.1].

**Proposition 4.2** (O’Grady). Let \( X \) be an IHS fourfold of \( K^{[2]} \)-type. The intersection pairing \( \langle \cdot, \cdot \rangle \) defined above is the bilinear form on \( \text{Sym}^2 H^2(X, \mathbb{Q}) \) given by
\[
\langle \alpha_1 \alpha_2, \alpha_3 \alpha_4 \rangle = (\alpha_1, \alpha_2)(\alpha_3, \alpha_4) + (\alpha_1, \alpha_3)(\alpha_2, \alpha_4) + (\alpha_1, \alpha_4)(\alpha_2, \alpha_3)
\]
for every \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^2(X, \mathbb{Q}) \), where \( \langle \cdot, \cdot \rangle \) denotes the BBF bilinear form on \( H^2(X, \mathbb{Q}) \).

Let \( q_X \) be the BBF quadratic form on \( X \). Let \( \{e_1, \ldots, e_{23}\} \) be a basis of \( H^2(X, \mathbb{Q}) \) and \( \{e^\vee_1, \ldots, e^\vee_{23}\} \) be the dual basis in \( H^2(X, \mathbb{Q})^\vee \), i.e., \( e^\vee_I (e_J) = \delta_{I,J} \). Then we have
\[
q_X = \sum_{i,j} g_{i,j} e_i^\vee \otimes e_j^\vee, \quad q^\vee_X = \sum_{i,j} m_{i,j} e_i e_j,
\]
where \( g_{i,j} := (e_i, e_j) \), the matrix \( (g_{i,j}) \) is symmetric and \( (m_{i,j}) = (g_{i,j})^{-1} \). The values of the products \( \langle q^\vee_X, \alpha \rangle \) for every \( \alpha \in H^2(X, \mathbb{Q}) \) are given by the following proposition, see [O’G08, Proposition 2.2].

**Proposition 4.3** (O’Grady). Let \( X \) be an IHS fourfold of \( K^{[2]} \)-type. Let \( \langle \cdot, \cdot \rangle \) be the bilinear form described in Proposition 4.2. Then \( \langle \cdot, \cdot \rangle \) is non-degenerate and
\[
\langle q^\vee_X, \alpha \beta \rangle = 25 \langle \alpha, \beta \rangle \quad \text{for all } \alpha, \beta \in H^2(X, \mathbb{Q}),
\]
\[
\langle q^\vee_X, q^\vee_X \rangle = 23 \cdot 25.
\]

Recall that rational Hodge classes and integral Hodge classes of type \( (k,k) \) on a projective manifold \( Y \) are elements belonging respectively to
\[
H^{k,k}(Y, \mathbb{Q}) := H^{2k}(Y, \mathbb{Q}) \cap H^{k,k}(Y), \quad H^{k,k}(Y, \mathbb{Z}) := H^{2k}(Y, \mathbb{Z}) \cap H^{k,k}(Y).
\]
O’Grady has shown in [O’G08, §3] that \( q^\vee_X \) is a rational multiple of \( c_2(X) \), the second Chern class of the tangent bundle of \( X \), in particular it is an element of \( H^{2,2}(X, \mathbb{Q}) \), i.e., it is a rational Hodge class of type \( (2,2) \).
Proposition 4.4 (O’Grady). Let $X$ be an IHS fourfold of $K3^{[2]}$-type. Then $q_X^* \in H^{2,2}(X, \mathbb{Q})$, i.e., $q_X^*$ is a rational Hodge class of $X$ of type $(2,2)$, and

$$\frac{6}{5}q_X^* = c_2(X) \in H^{2,2}(X, \mathbb{Z}).$$

Moreover, $\frac{2}{5}q_X^* \in H^{2,2}(X, \mathbb{Z})$ is an integral Hodge class of $X$ of type $(2,2)$.

If $S$ is a projective K3 surface, then $H^{2,2}(S^{[2]}, \mathbb{Z})$ is a lattice, where the bilinear form considered is the cup product. We recall [Nov21, Theorem 8.3], which gives a basis of the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$ when $S$ is general and the Picard group of $S$ is known.

Theorem 4.5 (N.). Let $S$ be a general projective K3 surface, $\{b_1, \ldots, b_r\}$ be a basis of $\text{Pic}(S)$ and $X := S^{[2]}$. Then:

(i) $\text{rk}(H^{2,2}(X, \mathbb{Z})) = \frac{r(r+1)r}{2} + r + 2$.

(ii) The following is a basis of the lattice $H^{2,2}(X, \mathbb{Z})$, which is odd:

$$\left\{ b_ib_j, \frac{b_i^2 - b_i \delta}{2}, \frac{1}{8} \left( \delta^2 + \frac{2}{5} q_X^* \right), \delta^2 \right\}_{1 \leq i \leq j \leq r}.$$

In particular, if $S = S_{2t}$ is a generic K3 surface of degree $2t$, and $h \in \text{Pic}(S^{[2]}_{2t})$ is the class induced by the ample generator of $\text{Pic}(S_{2t})$, then

$$H^{2,2}(S^{[2]}_{2t}, \mathbb{Z}) = Z\delta^2 \oplus Z\frac{h^2 - h \delta}{2} \oplus Z\frac{1}{8} \left( \delta^2 + \frac{2}{5} q_X^* \right) \oplus Z\frac{2}{5} q_X^*.$$

Moreover, $\text{disc}(H^{2,2}(S^{[2]}_{2t}, \mathbb{Z})) = 84t^3$ and the Gram matrix in the basis given above is the following:

$$
\begin{pmatrix}
12t^2 & 6t^2 & 2t & 20t \\
6t^2 & t(3t - 1) & t & 10t \\
2t & t & 1 & 9 \\
20t & 10t & 9 & 92
\end{pmatrix}.
$$

Note that in the case of the Hilbert square of a generic K3 surface $S_{2t}$ we have substituted the element $\delta^2$ in a basis of $H^{2,2}(X, \mathbb{Z})$ with $\frac{2}{5} q_X^*$: this choice slightly simplifies computations in Section 7.

5. Generalities on Hilbert squares of generic K3 surfaces

In this section we recall some important results on Hilbert squares of generic K3 surfaces of degree $2t$ and we present the main problem of this paper. Recall that the nef cone of a projective variety $Y$ is the convex cone $\text{Nef}(Y) \subseteq \text{NS}(Y)_\mathbb{R}$ generated by classes of nef divisors in $\text{NS}(Y)_\mathbb{R} = \text{NS}(Y) \otimes \mathbb{R}$, where $\text{NS}(Y)$ is the Néron–Severi group of $Y$, the moving cone is the convex cone $\text{Mov}(Y) \subseteq \text{NS}(Y)_\mathbb{R}$ generated by classes of divisors whose complete linear system has base locus of codimension greater than 1, and the pseudoeffective cone is the convex cone $\text{Eff}(Y) \subseteq \text{NS}(Y)_\mathbb{R}$ generated by classes of pseudoeffective divisors. We present the following version of the Bayer–Macrì theorem, see [BM14, Proposition 13.1], describes the closure of the moving cone, the nef cone and the pseudoeffective cone of the Hilbert square of a generic K3 surface.

Theorem 5.1 (Bayer–Macrì). Let $S_{2t}$ be a projective K3 surface with $\text{Pic}(S_{2t}) = Z\delta$ and $H^2 = 2t$. Let $X := S^{[2]}_{2t}$ be its Hilbert square. Denote by $h \in \text{Pic}(X)$ the line bundle induced by the ample generator of $\text{Pic}(S_{2t})$. Then the cones of classes of divisors can be described as follows.

1. The extremal rays of the moving cone $\text{Mov}(X)$ are spanned by $h$ and $h - \mu_t \delta$, where:
   - if $t$ is a perfect square, $\mu_t = \sqrt{t}$;
   - $t$ is not a perfect square and $(c, d)$ is the minimal solution of the Pell equation $P_t(1)$, then $\mu_t = t \cdot \frac{4}{c}$.

2. The extremal rays of the nef cone $\text{Nef}(X)$ are spanned by $h$ and $h - \nu_t \delta$, where:
   - if the equation $P_{4t}(5)$ is not solvable, $\nu_t = \mu_t$;
   - if the equation $P_{4t}(5)$ is solvable and $(a_4, b_4)$ is its minimal solution, $\nu_t = 2t \cdot \frac{b_4}{a_4}$.

3. The extremal rays of the pseudoeffective cone $\text{Eff}(X)$ are spanned by $\delta$ and $h - \omega_t \delta$, where:
   - if $t$ is a perfect square, $\omega_t = \sqrt{t}$;
   - $t$ is not a perfect square, $\omega_t = \frac{\omega_t}{\sqrt{t}}$, where $(c, d)$ is the minimal solution of $P_t(1)$.

Let $X$ be as in Theorem 5.1. We recall the description of the group of biregular automorphisms $\text{Aut}(X)$ given by Boissière, Cattaneo, Nieper-Wißkirchen and Sarti in [BCNWS16].
Theorem 5.2 (Proposition 4.3, Proposition 5.1, Lemma 5.3, Theorem 5.5 in [BCNWS16]). Let $S_{2t}$ be a projective $K3$ surface with $\text{Pic}(S_{2t}) = ZH$ and $H^2 = 2t$. Let $h \in \text{Pic}(S_{2t}^{[2]})(\mathbb{Z})$ be the line bundle induced by the ample generator of the Picard group $\text{Pic}(S_{2t})$.

1. If $t = 1$, then $S_{2}^{[2]}$ is the double cover of $\mathbb{P}^2$ branched along a smooth sextic curve and if $\iota$ is the covering involution then
   $$\text{Aut}(S_{2}^{[2]}) = \{ \text{id}_{S_{2}^{[2]}}, \iota \} \cong \mathbb{Z}/2\mathbb{Z}.$$  

2. If $t \geq 2$, then $X := S_{2t}^{[2]}$ admits a non-trivial automorphism if and only if one of the following equivalent conditions is satisfied.
   
   • The integer $t$ is not a square, the Pell-type equation $P_t(5)$ has no solution and the negative Pell equation $P_t(-1)$ has a solution.
   
   • There exists an ample class $D \in \text{NS}(S_{2t}^{[2]})$ such that $q_X(D) = 2$.

   In this case the class $D$ is unique, the automorphism $\iota$ is unique and it is a non-natural anti-symplectic involution. Its action on $\text{NS}(S_{2t}^{[2]})$ is the reflection in the span of the class $D$ of square 2 represented in the basis $\{ h, -\delta \}$ by the matrix
   $$\left( \begin{array}{cc} c & -d \\ td & -c \end{array} \right),$$
   i.e., $\iota^* (xh - y\delta) = (cx - dy)h - (tdx - cy)\delta$, where $(c, d)$ is the minimal solution of the Pell equation $P_t(1)$.

Let $X$ be the Hilbert square of a generic $K3$ surface of degree $2t$, and suppose that $X$ admits an ample class $D \in \text{Pic}(X)$ with $q_X(D) = 2$. By Theorem 5.1 the nef cone $\text{Nef}(X)$ and the closure of the moving cone $\text{Mov}(X)$ coincide, and by Theorem 5.2 the action on $\text{NS}(S_{2t}^{[2]}) \otimes \mathbb{R}$ of the anti-symplectic involution $\iota$ preserves the nef cone of $S_{2t}^{[2]}$, exchanging the two extremal rays. This is true also for a non-natural non-trivial birational automorphism $\iota \in \text{Bir}(S_{2t}^{[2]})$ on the Hilbert square of a generic $K3$ surface $S_{2t}$ of degree $2t$, whose action on $\text{NS}(S_{2t}^{[2]}) \otimes \mathbb{R}$ preserves the closure of the moving cone of the two extremal rays, see [Mar11, Lemma 6.22]. For a description of $\text{Bir}(S_{2t}^{[2]})$ see [DM19, Proposition B.3], for a description of $\text{Aut}(S_{2t}^{[2]})$ for $n \geq 3$ see [Cat19] and for a description of $\text{Bir}(S_{2t}^{[2]})$ for $n \geq 3$ see [BC20].

Let $X = S_{2t}^{[2]}$ be as above: the ample divisor $D$ with $q_X(D) = 2$ is unique by Theorem 5.2 and its class in $\text{Pic}(X)$ is $bh - a\delta$, where $h \in \text{Pic}(X)$ is the line bundle induced by the ample generator of $\text{Pic}(S_{2t})$, and $(a, b)$ is the minimal solution of the negative Pell equation $P_t(-1)$. By the Kodaira vanishing theorem and a Riemann–Roch formula for IHS fourfolds of $K3^{[2]}$-type, see [O’G10, Formula (2.2.7)], we have $\dim(H^0(X, \mathcal{O}_X(D))) = 6$. Hence the rational map $\varphi_{|D|}$ induced by the complete linear system $|D|$ has $\mathbb{P}^5$ as codomain, i.e.,

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5.$$ 

We can now state the main problem of this paper.

Problem. Let $X$ be the Hilbert square of a generic $K3$ surface of degree $2t$ such that $X$ admits an ample divisor $D$ with $q_X(D) = 2$. Describe the base locus of the complete linear system $|D|$ and describe the rational map

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5.$$ 

As observed in [BCNWS16, §1], the first values of $t$ such that $X = S_{2t}^{[2]}$ as in Theorem 5.2 has an ample divisor $D$ with $q_X(D) = 2$ are $t = 2, 10, 13$, etc. The case $t = 2$ was studied by Welters in [Wel81] and Beauville in [Bea83a]: in this case $S_2 \subset \mathbb{P}^3$ is a smooth quartic surface of $\mathbb{P}^3$ with Picard rank 1, in particular $S_2$ does not contain any line, the map $\varphi_{|D|}$ is a finite morphism of degree 6 with image $G(1, \mathbb{P}^3) \subset \mathbb{P}^5$, the Grassmannian of lines in $\mathbb{P}^3$, and $\varphi_{|D|}(x) := l_x$ is the unique line in $\mathbb{P}^3$ which contains the support of the subscheme $x$ of $S_4$. Moreover, if $l_x \cap S_4 = x \cup x'$, where $x, x' \in S_2^{[2]}$, then the anti-symplectic involution $\iota \in \text{Aut}(X)$ is the Beauville involution, defined as $\iota(x) := x'$. Since $S_4$ does not contain any line, $\iota$ is everywhere well defined. Note that $\varphi_{|D|}$ factors through the quotient $\pi : S_4^{[2]} \rightarrow S_4^{[2]}/\langle \iota \rangle$ with respect to the Beauville involution.

A first step in the analysis of the base locus of $|D|$ in the Problem above is given by the following result, which is a corollary of [Rie18, Theorem 4.7].

Proposition 5.3. Let $X$ be the Hilbert square of a projective $K3$ surface of Picard rank 1. Consider a big and nef divisor $D \in \text{Div}(X)$. Then $|D|$ has no fixed part, i.e., the base locus of $|D|$ has codimension greater than 1.
Proof. By [Rie18, Theorem 4.7] the complete linear system has a fixed part if and only if $D = mL + F$, where $m \geq 2$, the class of $L$ is movable with $q_X(L) = 0$ and $F$ is a reduced and irreducible divisor with $q_X(F) < 0$ and $(L, F) = 1$. Since $\operatorname{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}\delta$, where $h$ is the line bundle induced by the ample generator of the Picard group of the underlying K3 surface, and $(h, \delta) = 0$, there are no elements $L$ and $F$ in $\operatorname{Pic}(X)$ such that $(L, F) = 1$. We conclude that $|D|$ has no fixed part. \hfill \qed

We now describe the set $H^{3,3}(S_2^{[2]}, \mathcal{Z})$ of integral Hodge classes of type $(3, 3)$ on the Hilbert square of a K3 surface $S_2$ with $\operatorname{Pic}(S_2) = \mathbb{Z}H$ and $H^2 = 2t$, this will be useful in the proof of Proposition 8.3.

**Theorem 5.4.** Let $S$ be a K3 surface and suppose that $\{b_1, \ldots, b_r\}$ is a basis of $\operatorname{Pic}(S^{[2]})$. Then a basis of the $\mathbb{Z}$-module $H^{3,3}(S^{[2]}, \mathcal{Z})$ is given by $\{c(b_1^\vee), \ldots, c(b_r^\vee)\}$, where $b_i^\vee \in H_2(S^{[2]}, \mathcal{Z})$ is the primitive element associated to $b_i \in H^2(S^{[2]}, \mathcal{Z})$ by Proposition 3.6, and $c(b_i^\vee)$ is the inverse of the Poincaré dual of $b_i^\vee$. In particular, let $X = S_2^{[2]}$ be the Hilbert square of a projective K3 surface with $\operatorname{Pic}(S_2) = \mathbb{Z}H$ and $H^2 = 2t$, and let $h \in \operatorname{Pic}(X)$ be the class induced by the ample generator of $\operatorname{Pic}(S_2)$. Consider $h^\vee_2 := c(h^\vee)$ and $\delta^\vee_2 := c(\delta^\vee) \in H^6(X, \mathcal{Z})$. Then

\[ H^{3,3}(X, \mathcal{Z}) \cong \mathbb{Z}h^\vee_2 \oplus \mathbb{Z}\delta^\vee_2. \]

Moreover, $h^\vee_2 = \frac{1}{6t} h^3$ and $\delta^\vee_2 = \frac{1}{2t} h^2 \delta$, and the following equalities hold in $H^{3,3}(X, \mathcal{Z})$:

\[
\delta^3 = -\frac{3}{t} h^2 \delta, \quad h^3 \delta = -\frac{1}{3t} h^3, \quad q_X^h h = \frac{25}{6t} h^3, \quad q_X^h \delta = \frac{25}{2t} h^2 \delta.
\]

**Proof.** The torsion free quotient group $H_2(S_2^{[2]}, \mathcal{Z})_f$ is a Hodge structure of weight $-2$. Moreover the cap product

\[ H^2(S_2^{[2]}, \mathcal{Z}) \otimes H_2(S_2^{[2]}, \mathcal{Z})_f \to H_0(S_2^{[2]}, \mathcal{Z}) \]

is a morphism of Hodge structures of weight 0, see for instance [PS08], hence if $R \in H_2(S_2^{[2]}, \mathcal{Z})_f$ is primitive of type $(-1, -1)$, the corresponding $\rho \in H^2(S_2^{[2]}, \mathcal{Z})$ is of type $(1, 1)$, $\{b_i, \ldots, b_r\}$ is a basis of $H_{-1, -1}(S_2^{[2]}, \mathcal{Z}) := H_2(S_2^{[2]}, \mathcal{Z}) \cap H_{-1, -1}(S_2^{[2]}),$

where $H_{-1, -1}(S_2^{[2]})$ is the component of type $(-1, -1)$ of $H_2(S_2^{[2]}, \mathcal{C})$. The Poincaré duality

\[ PD : H^6(S_2^{[2]}, \mathcal{Z}) \cong H_2(S_2^{[2]}, \mathcal{Z})_f \]

is an isomorphism of Hodge structures of weight $-4$, hence $H^{3,3}(S_2^{[2]}, \mathcal{Z}) \cong H_{-1, -1}(S_2^{[2]}, \mathcal{Z})$, so $\{c(b_i^\vee), \ldots, c(b_r^\vee)\}$ is a basis of $H^{3,3}(S^{[2]}, \mathcal{Z})$.

Let now $X = S_2^{[2]}$ be the Hilbert square of a projective K3 surface with $\operatorname{Pic}(S_2) = \mathbb{Z}H$ and $H^2 = 2t$, and let $h \in \operatorname{Pic}(X)$ be the line bundle induced by the ample generator of $\operatorname{Pic}(S_2)$. The cup product with $h^2$ gives an isomorphism between $H^{1,1}(X, \mathbb{Q})$ and $H^{3,3}(X, \mathbb{Q})$. Then $H^{3,3}(X, \mathbb{Q})$ is 2-dimensional and it is generated by $h^2$ and $h^2 \delta$ since $H^{{1,1}}(X, \mathbb{Q}) \cong \operatorname{Pic}(X) \otimes \mathbb{Q}$. We now represent $\delta^3$, $h^3 \delta$, $q_X^h h$ and $q_X^h \delta$ in terms of this basis. If $\delta^3 = x h^3 + y h^2 \delta$ with $x, y \in \mathbb{Q}$, by Proposition 4.2 we have

\[ h^3 \delta = 0 = x(h^2, h^2) + y(h^2, h \delta) = 12t^2 x, \quad \delta^4 = 3 \cdot (2)^2 = x(h^2, h \delta) + y(h^2, \delta^2) = -4ty, \]

which give $x = 0$ and $y = -\frac{3}{4} t$, hence

\[ \delta^3 = -\frac{3}{t} h^2 \delta. \]

Similarly we obtain

\[ h^3 \delta^2 = -\frac{1}{3t} h^3, \quad q_X^h h = \frac{25}{6t} h^3, \quad q_X^h \delta = \frac{25}{2t} h^2 \delta. \]

Consider now $H^{3,3}(X, \mathcal{Z})$. The cohomology group $H^6(X, \mathcal{Z})$ is torsion free, and $H^{3,3}(X, \mathbb{Q})$ has dimension 2, hence $\operatorname{rk}(H^{3,3}(X, \mathcal{Z})) = 2$. Let $h^\vee, \delta^\vee \in H_2(X, \mathcal{Z})$ be the primitive classes which correspond, by Proposition 3.6, respectively to the primitive classes $h, \delta \in H^2(X, \mathcal{Z})$. Using Proposition 3.6 one gets $h^\vee = h$ and $\delta^\vee = \frac{\delta}{t}$ seen as elements in $H^2(X, \mathcal{Z})$. Let $PD$ be the Poincaré duality: then $h^\vee_2 := PD^{-1}(h^\vee)$ and $\delta^\vee_2 := PD^{-1}(\delta^\vee)$ give a basis of $H^{3,3}(X, \mathcal{Z})$, so we have

\[ H^{3,3}(X, \mathcal{Z}) \cong \mathbb{Z}h^\vee_2 \oplus \mathbb{Z}\delta^\vee_2. \]

Moreover, if $\cdot$ denotes the cup product and $\cap$ the cap product, we get

\[
\int_X h^\vee_2 \cdot x = \epsilon(h^\vee \cap x), \quad \int_X \delta^\vee_2 \cdot x = \epsilon(\delta^\vee \cap x)
\]

(5.2)
for every \( x \in H^2(X, \mathbb{Z}) \), see [Hat05, p.249], and \( \epsilon : H^0(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \). Again by Proposition 3.6 we have

\[
\epsilon(h^\vee \cap h) = (h, h) = 2t, \quad \epsilon(h^\vee \cap \delta) = (h, \delta) = 0.
\] (5.3)

If we write \( h_0^\vee = \alpha h^3 + \beta h^2 \delta \) for some \( \alpha, \beta \in \mathbb{Q} \), by Proposition 4.2 we obtain

\[
\int_X h_0^\vee \cdot h = \alpha (h^2, h^2) + \beta (h^2, h\delta) = 12\alpha t^2, \quad \int_X h_0^\vee \cdot \delta = \alpha (h^2, h\delta) + \beta (h^2, \delta^2) = -4\beta t.
\] (5.4)

Then (5.2), (5.3) and (5.4) give \( \alpha = \frac{1}{12t} \) and \( \beta = 0 \), hence \( h_0^\vee = \frac{1}{12t} h^3 \). Similarly we obtain \( \delta_0^\vee = \frac{1}{12t} h^2 \delta \). \( \square \)

6. Fixed locus of the anti-symplectic involution

Let \( X \) be the Hilbert square of a generic K3 surface \( S_{2t} \) of degree \( 2t \) such that \( X \) admits an ample divisor \( D \) with \( q_X(D) = 2 \). We recall some properties of \( F := \text{Fix}(\iota) \), the locus of points of \( X \) fixed by the anti-symplectic involution \( \iota \in \text{Aut}(X) \), then we compute its integral Hodge class in \( H^{2,2}(X, \mathbb{Z}) \) and we show that the rational map \( \varphi_{|D|} \) induced by the complete linear system \( |D| \) factors through the quotient \( \pi : X \to X/\langle \iota \rangle \) with respect to \( \iota \).

First of all, we prove that \( F \) is a connected Lagrangian surface in our case.

**Proposition 6.1.** Keep notation as above. Then \( F = \text{Fix}(\iota) \) is a connected Lagrangian surface in \( X \).

**Proof.** By [Beal11, Lemma 1] the fixed locus \( F \) is a Lagrangian surface. We now show that \( F \) is connected. Let \( \mathcal{M}_0^{2} \) be the moduli space which parametrizes triples \((X, \iota_X, i_X)\), where \( X \) is an IHS manifold of \( K3^{[2]} \)-type, \( \iota_X \in \text{Aut}(X) \) is an anti-symplectic involution whose action on \( H^2(X, \mathbb{Z}) \) is the reflection in the class of an ample divisor \( D \) with \( q_X(D) = 2 \) and \( i_X : (2) \to \text{NS}(X) \) is a primitive embedding such that \( i_X(2) \) contains the class of \( D \). Any two points in the moduli space \( \mathcal{M}_0^{2} \) are deformation equivalent by [BCMS19, Corollary 4.1, Theorem 5.1]. Since deformation equivalence preserves topological properties of \( \text{Fix}(\iota) \), it suffices to find a point \((X, \iota_X, i_X) \in \mathcal{M}_0^{2} \) such that \( \text{Fix}(\iota_X) \) is connected. Let \( X = S_{4}^{[2]} \) be the Hilbert square of a smooth quartic surface of \( \mathbb{P}^3 \) with Picard rank 1, and let \( \iota_X \in \text{Aut}(X) \) be the Beauville involution: the surface \( \text{Fix}(\iota_X) \) is connected by [Weil81, Corollary 3.4.4]. Alternatively, if we show that the class of \( D \), which is \( bb - a\delta \), has divisibility 1 in the lattice \( (H^2(X, \mathbb{Z}), q_X) \), then the surface \( \text{Fix}(\iota) \) is connected by [FMOS20, Main Theorem]. Since \((a, b)\) is the minimal solution of the negative Pell equation \( P_t(-1) \), the integers \( a \) and \( b \) are coprime and \( b \) is odd by Proposition 2.6. Since \( H^2(S_{2t}, \mathbb{Z}) \) is unimodular, there exists \( x \in H^2(X, \mathbb{Z}) \) such that \( (h, x) = 1 \), hence there exist \( \alpha, \beta \in \mathbb{Z} \) such that \( (D, \alpha \delta + \beta x) = 1 \). We conclude that \( \text{div}(bb - a\delta) = 1 \), and \( F \) is connected. \( \square \)

Keep notation as above. By [CGM19, §3] the quotient variety \( X/\langle \iota \rangle \) is singular along \( \pi'(F) \), where \( \pi' : X \to X/\langle \iota \rangle \) is the quotient map. The desingularization of \( X/\langle \iota \rangle \), which we call \( W \), is the blow-up of \( X/\langle \iota \rangle \) along its singular locus, and it is a Calabi–Yau variety. Equivalently, consider the blow-up \( \text{Bl}_F(X) \) of \( X \) in the locus \( F = \text{Fix}(\iota) \) of points fixed by \( \iota \). The involution \( \iota \) gives rise to an involution \( \tilde{\iota} \) on \( \text{Bl}_F(X) \) which fixes the exceptional divisor \( E \subset \text{Bl}_F(X) \). One can show that the quotient \( \text{Bl}_F(X)/\langle \tilde{\iota} \rangle \) is isomorphic to \( W \), obtaining the following commutative diagram, see [CGM19, Theorem 3.6] for more details:

\[
\begin{array}{ccc}
\text{Bl}_F(X) & \xrightarrow{\pi} & W \\
\downarrow{\beta} & & \downarrow{\beta'} \\
X & \xrightarrow{\pi'} & X/\langle \iota \rangle.
\end{array}
\] (6.1)

We now state the following useful technical lemma. See [DHH+15, Lemma 4.3] for a more general statement and for the proof.

**Lemma 6.2.** Let \( \pi : X \to Y \) be a double cover of a smooth projective variety such that the branch locus is a smooth prime divisor \( B \in \text{Div}(Y) \), and denote by \( \iota \in \text{Aut}(X) \) the involution associated to the double cover. If \( \text{Pic}(X)^\iota \) is the subgroup of \( \iota \)-invariant line bundles on \( X \), then \( \pi^*\text{Pic}(Y) \cong \text{Pic}(X)^\iota \).

Similarly to the case in [vGS07, §2.6], since \( \iota^*D \cong D \) there is an involution induced by \( \iota \) on \( \mathbb{P}(H^0(X, \mathcal{O}_X(D)))^\vee \) which has two fixed spaces \( \mathbb{P}^a \) and \( \mathbb{P}^b \), where \( a + 1 + b + 1 = 6 \) and \( a = -1 \) if the corresponding eigenspace of \( \iota^* \) on \( H^0(X, \mathcal{O}_X(D)) \) is zero, similarly for \( b \). See [MFK94, p.32] for details on the action of \( \iota \) on \( H^0(X, \mathcal{O}_X(D)) \). In particular the rational map \( \varphi_{|D|} \) induced by the complete linear system \( |D| \) factors through the quotient \( X \to X/\langle \iota \rangle \) if and only if the action induced by \( \iota \) on \( \mathbb{P}(H^0(X, \mathcal{O}_X(D)))^\vee \) is trivial.
Let $B \in \text{Div}(W)$ be the branch divisor of $\pi : \text{Bl}_F(X) \to W$. By [BHPVdV15, Lemma I.17.1] there exists a divisor $N \in \text{Div}(W)$ such that $O_W(2N) = O_W(B)$ in $\text{Pic}(W)$. Let $D$ be the ample divisor on $X$ with $q_X(D) = 2$. We show the following result, similar to [vGS07, Proposition 2.7, Item (2)] obtained in the case of K3 surfaces admitting a symplectic involution.

**Proposition 6.3.** Keep notation as above. Consider diagram (6.1). Let $D := O_X(D)$ and $N := O_W(N)$. There exists a line bundle $D_W \in \text{Pic}(W)$ such that $\pi^*D_W \cong \beta^*D$. Moreover, the vector space $H^0(X, D)$ decomposes as

$$H^0(X, D) \cong H^0(W, D_W) \oplus H^0(W, D_W - N),$$

(6.2)

which is the decomposition of $H^0(X, D)$ into $\nu^*$-eigenspaces.

**Proof.** Consider $\pi : \text{Bl}_F(X) \to W$ appearing in diagram (6.1): by Lemma 6.2 we obtain a divisor $D_W \in \text{Div}(W)$ whose class $D_W$ in $\text{Pic}(W)$ is such that $\pi^*D_W \cong \beta^*D$ in $\text{Pic}(\text{Bl}_F(X))$. Note that Lemma 6.2 can be applied: $F \subset X$ is smooth and connected by Proposition 6.1, so the exceptional divisor of $\beta$ and the branch divisor of $\pi$ are smooth prime divisors. Since $\beta$ is a birational morphism, we have $H^0(X, D) \cong H^0(\text{Bl}_F(X), \beta^*D)$, which is isomorphic to $H^0(\text{Bl}_F(X), \pi^*D_W)$, being $\beta^*D = \pi^*D_W$. Moreover,

$$\pi_*(\pi^*D_W) \cong \pi_*(\pi^*D_W \otimes O_{\text{Bl}_F(X)}) \cong D_W \otimes \pi_*O_{\text{Bl}_F(X)},$$

where in the second isomorphism we use the projection formula. By [BHPVdV15, Lemma I.17.2] we have

$$\pi_*O_{\text{Bl}_F(X)} \cong O_W \oplus O_W(-N),$$

where $B \in \text{Div}(W)$ is the branch divisor and $N \in \text{Div}(W)$ is the divisor such that $O_W(2N) = O_W(B)$ in $\text{Pic}(W)$. Thus we obtain the isomorphism

$$\pi_*(\pi^*D_W) \cong D_W \otimes (O_W \oplus O_W(-N)) \cong D_W \oplus (D_W - N).$$

Hence decomposition (6.2) holds. Moreover, this is the decomposition of $H^0(X, D)$ in $\nu^*$-eigenspaces. Indeed, two global sections $s, t \in H^0(X, D)$ are in the same eigenspace if and only if the rational function $f = s/t$ is $\nu$-invariant, and this is true when the sections belong both to $H^0(W, D_W)$ or $H^0(W, D_W - N)$ in the decomposition, hence each of these two spaces is contained in an eigenspace of $H^0(X, D)$. We conclude that $H^0(W, D_W)$ and $H^0(W, D_W - N)$ are isomorphic to the two eigenspaces by decomposition (6.2).

We then see from Proposition 6.3 that the action induced by $\nu$ on $\mathbb{P}(H^0(X, O_X(D))^\vee)$ is trivial, i.e., $\varphi_{|D|}$ factors through the quotient $X \to X/\langle \nu \rangle$, if and only if either $H^0(X, D_W)$ or $H^0(W, D_W - N)$ is zero. We can now prove the main theorem of this section, which is the first step of the geometrical description of the rational map $\varphi_{|D|}$ induced by the complete linear system $|D|$.

**Theorem 6.4.** Let $X = S_{2t}^2$ be the Hilbert square of a generic K3 surface of degree $2t$. Suppose that $X$ admits an ample divisor $D$ with $q_X(D) = 2$. Let $\iota : X \to X$ be the anti-symplectic involution which generates $\text{Aut}(X)$ and $F := \text{Fix}(\iota)$ be the locus of points of $X$ fixed by $\iota$. Then

$$[F] = 5D^2 - \frac{2}{5}q_X^\vee \in H^{2,2}(X, \mathbb{Z}),$$

(6.3)

where $[F]$ denotes the fundamental cohomological class of $F$ in $H^{2,2}(X, \mathbb{Z})$. Moreover, let $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$ be the rational map induced by the complete linear system $|D|$. Then $\varphi_{|D|}$ factors through the quotient $X \to X/\langle \iota \rangle$, i.e., the following diagram is commutative:

$$\begin{array}{ccc}
X & \xrightarrow{\varphi_{|D|}} & \mathbb{P}^5 \\
& \downarrow & \downarrow \\
& X/\langle \iota \rangle & \\
\end{array}$$

(6.4)

**Proof.** By Proposition 6.3 there exists $D_W \in \text{Pic}(W)$ such that $\pi^*D_W \cong \beta^*D$, using the notation of diagram (6.1). Applying [CGM19, Proposition 7.3] we have

$$\dim(H^0(W, D_W)) = \frac{7}{2} + \frac{1}{16}(D|F|^2),$$

where $[F] = 5D^2 - \frac{2}{5}q_X^\vee$. Theorem 6.4 follows.
hence by decomposition (6.2) and \( \dim(H^0(X, D)) = 6 \) we get \( \dim(H^0(W, D_W)) \in \{0, 1, \ldots, 6\} \). Thus we obtain the following possible values for \((D|_F)^2\):

\[
\begin{align*}
\dim(H^0(W, D_W)) = 0 & \iff (D|_F)^2 = -56, & \dim(H^0(W, D_W)) = 4 & \iff (D|_F)^2 = 8, \\
\dim(H^0(W, D_W)) = 1 & \iff (D|_F)^2 = -40, & \dim(H^0(W, D_W)) = 5 & \iff (D|_F)^2 = 24, \\
\dim(H^0(W, D_W)) = 2 & \iff (D|_F)^2 = -24, & \dim(H^0(W, D_W)) = 6 & \iff (D|_F)^2 = 40. \\
\end{align*}
\]

(6.5)

Since \( D \) is ample, \((D|_F)^2 > 0\) by the Nakai–Moishezon criterion. This implies that \( \dim(H^0(W, D_W)) \in \{4, 5, 6\} \). We show that \( \dim(H^0(W, D_W)) = 6 \) by computing \( [F] \in H^{2,2}(X, \mathbb{Z}) \), where \( [F] \) is the fundamental cohomological class of the fixed locus \( F = \text{Fix}(\iota) \) of the involution \( \iota \). Let \( h \in \text{Pic}(X) \) be the line bundle induced by the ample generator of \( \text{Pic}(S_{24}) \). We can write, see for instance [Nov21, Proposition 7.1],

\[
[F] = xh^2 + yh\delta + z\delta^2 + w\frac{2}{5}q_X^\gamma \in H^{2,2}(X, \mathbb{Z}),
\]

(6.6) with \( x, y, z, w \in \mathbb{Q} \) to determine. We write \( D \) also for its class in \( \text{Pic}(X) \): then \( D = bh - a\delta \), with \((a, b)\) minimal solution of the Pell-type equation \( P_t(1) \). We denote by \( \langle \cdot, \cdot \rangle \) the bilinear form of \( H^*(X, \mathbb{Z}) \) of Proposition 4.2. We have the following four conditions.

1. \( \langle [F], (\sigma + \bar{\sigma})^2 \rangle = 0 \), where \( \sigma \) is the symplectic form \( \sigma \in H^0(X, \Omega^2_X) \), since by [Bea11, Lemma 1] the surface \( F \) is Lagrangian. If \( \eta := (\sigma + \bar{\sigma}, \sigma + \bar{\sigma}) \), we have:

\[
\langle h^2, (\sigma + \bar{\sigma})^2 \rangle = 2t_\eta, \quad \langle \delta^2, (\sigma + \bar{\sigma})^2 \rangle = -2\eta, \quad \langle h\delta, (\sigma + \bar{\sigma})^2 \rangle = 0, \quad \langle \frac{2}{5}q_X^\gamma, (\sigma + \bar{\sigma})^2 \rangle = 10\eta.
\]

(6.7)

Note that \( \eta \neq 0 \), see [GHJ12, Definition 22.10, Theorem 23.14], so we obtain from \( \langle [F], (\sigma + \bar{\sigma})^2 \rangle = 0 \) and (6.6) the following condition:

\[
tx - z + 5w = 0.
\]

(6.8)

2. Applying [Bea11, Theorem 2] in our case we have \( c_2(F) = 192 \), i.e., \( \langle [F], [F] \rangle = 192 \). This gives, together with (6.6), the following condition:

\[
3t^2x^2 - ty^2 + 3z^2 + 23w^2 - 2txz + 10txw - 10zw = 48.
\]

(6.9)

3. Recall that by Theorem 5.2 the action of \( \iota \) on \( \text{Pic}(X) \otimes \mathbb{Q} \cong H^{1,1}(X, \mathbb{Q}) \) is described in the basis \( \{h, -\delta\} \) by the matrix (5.1). Consider the action induced by \( \iota \) on \( H^{2,2}(X, \mathbb{Q}) \), described in the basis \( \{h^2, h\delta, \delta^2, q_X^\gamma\} \) by

\[
\iota^*(h^2) = \iota^* h \cdot \iota^* h, \quad \iota^*(h\delta) = \iota^* h \cdot \iota^* \delta, \quad \iota^*(\delta^2) = \iota^* \delta \cdot \iota^* \delta, \quad \iota^*(q_X^\gamma) = q_X^\gamma,
\]

(6.10)

where \( \cdot \) denotes the cup product: the last equation comes from the equality \( c_2(X) = \frac{2}{5}q_X^\gamma \) of Proposition 4.4 and the fact that \( \iota \) is an automorphism. Since \( F \) is the locus of points fixed by \( \iota \), we have \( \iota^*([F]) = [F] \), i.e., if

\[
\iota^*([F]) = \tilde{x}h^2 + \tilde{y}h\delta + \tilde{z}\delta^2 + \frac{2}{5}\tilde{w}q_X^\gamma,
\]

with \( \tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} \in \mathbb{Q} \), then \( x = \tilde{x}, y = \tilde{y}, z = \tilde{z}, w = \tilde{w} \). Imposing \( x = \tilde{x} \) we obtain the following condition:

\[
x - c^2x - cdy - d^2z = 0,
\]

(6.11)

where \( (c, d) \) is the minimal solution of the Pell equation \( P_t(1) \). One remarks a posteriori that \( y = \tilde{y}, z = \tilde{z} \) and \( w = \tilde{w} \) give the same condition (6.11).

4. By \((D|_F)^2 \in \{8, 24, 40\}\) we have \( \langle [F], D^2 \rangle \in \{8, 24, 40\} \), since the bilinear form \( \langle \cdot, \cdot \rangle \) represents the intersection form by Proposition 4.2. This gives, together with (6.6), the following possibilities:

\[
\begin{align*}
(t + 2t^2b^2)x + 2abty + (2a^2 - 1)z + 5w = 2 & \iff \langle [F], D^2 \rangle = 8, \\
(t + 2t^2b^2)x + 2abty + (2a^2 - 1)z + 5w = 6 & \iff \langle [F], D^2 \rangle = 24, \\
(t + 2t^2b^2)x + 2abty + (2a^2 - 1)z + 5w = 10 & \iff \langle [F], D^2 \rangle = 40.
\end{align*}
\]

(6.12)

If \( \langle [F], D^2 \rangle = 8 \), which is equivalent to \( \dim(H^0(W, D_W)) = 4 \) by (6.5), the system given by (6.8), (6.9), (6.11) and the first condition in (6.12) has solutions with \( x, y, z, w \in \mathbb{Q} \), which is impossible. Similarly we cannot have \( \langle [F], D^2 \rangle = 24 \), i.e., \( \dim(H^0(W, D_W)) = 5 \). We conclude that \( \langle [F], D^2 \rangle = 40 \), which is equivalent to \( \dim(H^0(W, D_W)) = 6 \). With the help of a computer, the system given by (6.8), (6.9), (6.11) and the third condition in (6.12) implies \( w \in \{-13/12, -1\} \).
By Theorem 4.5 we cannot have \( w = -\frac{13}{12} \), hence \( w = -1 \). Imposing \( w = -1 \), we necessarily obtain only one admissible solution, which is the following:

\[
\begin{align*}
  x &= 5b^2, \\
  y &= -10ab, \\
  z &= 5a^2, \\
  w &= -1.
\end{align*}
\]

We conclude that

\[
[F] = 5D^2 - 2q_X^2 \in H^{2,2}(X, \mathbb{Z}).
\]

Moreover, we have obtained \( H^0(X, D) \cong H^0(W, D_W) \), so \( H^0(W, D_W - \mathcal{N}) = \{0\} \), which shows that the action induced by \( \iota \) on \( \mathbb{P}(H^0(X, D)^\vee) \) is trivial, i.e., \( \varphi_{|D} \) factors through the quotient \( X \to X/(\iota) \).

**Remark 6.5.** Ferretti in [Fer12, Lemma 4.1] obtained the same relation of (6.3) in the Chow ring of a smooth double EPW sextic \( X \): in that case \( F \) is the branch locus of the double cover \( f : X \to Y \subset \mathbb{P}^5 \), where \( Y \) is an EPW sextic.

## 7. Irreducibility Property

Let \( X \) be the Hilbert square of a generic K3 surface of degree \( 2t \) such that \( X \) admits an ample divisor \( D \) with \( q_X(D) = 2 \). In this section we prove the irreducibility property when \( t \neq 2 \), which will be, together with Theorem 6.4, fundamental to obtain the main result of this paper. This can be seen as an analogue of the irreducibility property in [O’G08, Proposition 4.1.(1)]. First of all we show that every divisor in the complete linear system \(|D|\) is a prime divisor.

**Proposition 7.1.** Keep notation as above. Then every divisor \( D' \in |D| \) is reduced and irreducible.

**Proof.** By abuse of notation, we write \( D \) for an effective divisor \( D' \in |D| \). Since \( q_X(D) = 2 \), the divisor \( D \) is reduced, i.e., it is not of the form \( D = \alpha E \) with \( \alpha \in \mathbb{Z} \), \( \alpha \neq \pm 1 \), and \( E \in \text{Div}(X) \). Suppose that

\[
D = D_1 + D_2
\]

(7.1)

where \( D_1 = \sum_j n_j D_{1,j} \) and \( D_2 = \sum_j m_j D_{2,j} \) are effective divisors with \( D_{1,j} , D_{2,j} \) prime divisors which are pairwise distinct, and \( n_j , m_j \in \mathbb{Z}_{\geq 0} \). Without loss of generality we can assume that \( D_1 \) has only one component. By abuse of notation, we write \( D_i \) and \( D_{i,j} \) also for their classes in \( H^2(X, \mathbb{Z}) \): we show that (7.1) seen in \( H^2(X, \mathbb{Z}) \) gives a contradiction. We have

\[
q_X(D) = 2 = q_X(D_1) + q_X(D_2) + 2(D_1, D_2).
\]

(7.2)

Since \( D_1 \) and \( D_2 \) are effective divisors with no common components, we can apply [Bou04, Proposition 4.2.(i)], obtaining \( (D_1, D_2) \geq 0 \). Note that we cannot have neither \( q_X(D_1) = 0 \) nor \( q_X(D_2) = 0 \). Indeed, the equation \( q_x(xh - y\delta) = 0 \) has a non zero solution \((x, y) \in \mathbb{Z}^2 \) only when \( t \) is a perfect square. If \( t \) was a perfect square, the Pell-type equation \( P_t(-1) \) would be solvable only for \( t = 1 \), which is a case that we are not considering. Hence \( t \) is not a perfect square and \( q_X(D_1) \neq 0 \), \( q_X(D_2) \neq 0 \).

Since \( H^2(X, \mathbb{Z}) \) is an even lattice, we have only two possibilities: either one between \( D_1 \) and \( D_2 \) is zero, or (at least) one of the two has negative square with respect to the BBF form. We show that \( q_X(D_i) > 0 \) for \( i = 1, 2 \). Assume by contradiction that at least one between \( D_1 \) and \( D_2 \) has negative square. This can happen only if there exists a component \( D_{1,i} \) or \( D_{2,j} \), whose square with respect to the BBF form is negative. Without loss of generality, we can suppose that \( q_X(D_{1,1}) < 0 \). Recall that \( H^2(X, \mathbb{Z}) \cong H^2(S_{2t}, \mathbb{Z}) \oplus (-2) \), where \( S_{2t} \) is a K3 surface, and \( H^2(S_{2t}, \mathbb{Z}) \) is a unimodular lattice. Then the divisibility in \( H^2(X, \mathbb{Z}) \) of a primitive class can be only 1 or 2. Hence we have either \( q_X(D_{1,1}) = -2 \) or \( q_X(D_{1,1}) = -4 \) by [Mar13, Lemma 3.7], see also [Ric18, Lemma 3.5]. We show that \( q_X(D_{1,1}) = -4 \) is not possible. Indeed, if \( t = 2 \), the class of \( D_{1,1} \) in \( NS(X) \) is necessarily \( h - 2\delta \), which is outside the pseudoeffective cone, whose extremal rays are generated by \( \delta \) and \( 2h - 3\delta \) by Theorem 5.1, obtaining a contradiction. If \( t \neq 2 \), there exists a \((-4)\)-class if and only if the Pell-type equation \( P_t(2) \) is solvable: since by assumption \( P_t(-1) \) has solutions, by [Per13, p.106-109], see also [Yok94, Proposition 1], we have that \( P_t(2) \) has no solution, hence there are no \((-4)\)-classes. We conclude that \( D_{1,1} \) must be a \((-2)\)-class, hence it is either \( \delta \) or \( \iota^*\delta \), where \( \iota \) is the anti-symplectic involution on \( X \) which generates \( \text{Aut}(X) \), see Theorem 5.2. Since \( \iota^* D = D \), it is enough to show that \( D_{1,1} = \delta \) is not possible. If \( D_{1,1} = \delta \), from (7.1) we get \( D = n_1 \delta + D_2 \), where \( n_1 \geq 1 \) since \( D_1 \) is effective by assumption, hence

\[
D_2 = D - n_1 \delta = bh - (a + n_1)\delta.
\]
where as usual \((a, b)\) is the minimal solution of the Pell equation \(P_l(-1)\). We show that \(D_2\) is outside the pseudoeffective cone. By Theorem 5.1 the extremal rays of the pseudoeffective cone are generated by the classes \(\delta\) and \(\tau^*\delta = dh - c\delta\), where \((c, d)\) is the minimal solution of the Pell equation \(P_l(1)\). Then \(\tau^*\delta = 2abh - (a^2 + tb^2)\delta\) by (2.1), hence we need to check that \(a + n_1 > \frac{a^2 + tb^2}{2a}\) in order to show that \(D_2\) is outside the pseudoeffective cone. This is true since \(a^2 - tb^2 = -1\). We obtain a contradiction, so \(q_X(D_i) \geq 0\) for \(i = 1, 2\). Moreover, we have already remarked that \(q_X(D_1) \neq 0\) for \(i = 1, 2\), thus \(q_X(D_i) > 0\). We get a contradiction with (7.2), so one between \(D_1\) and \(D_2\) is zero. If \(D_2 = 0\), then \(n_1 = 1\) since \(q_X(D) = n_1^2 q_X(D_{1,1}) = 2\), so \(D\) is reduced and irreducible. If \(D_1 = 0\), we repeat the argument for \(D = D_2\) until we obtain a \(D\) which is reduced and irreducible.

Suppose now that \(D_1, D_2 \in |D|\) are two distinct divisors. We want to study the surface \(D_1 \cap D_2\) and see if it is reduced and irreducible. We first show what happens when \(t = 2\), i.e., when \(X = S_4^{[2]}\) is the Hilbert square of a smooth quartic surface of \(\mathbb{P}^3\) with Picard rank 1, a case already studied in [Wel81] and [Bea83a]. We briefly recall some notion from [GH78, §1.5, §6.2] on Schubert varieties of \(G(1, \mathbb{P}^3)\). Fix a complete flag in \(\mathbb{P}^3\), i.e., let \(v_0\) be a point in \(\mathbb{P}^3\), let \(L_0\) be a line in \(\mathbb{P}^3\) and let \(H_0\) be a plane in \(\mathbb{P}^3\) such that \(v_0 \in L_0 \subset H_0\). Consider the following two Schubert varieties of dimension 2 of the Grassmannian \(G(1, \mathbb{P}^3)\),

\[
\Sigma_{1,1} := \{L \in G(1, \mathbb{P}^3) | L \subset H_0\}, \quad \Sigma_2 := \{L \in G(1, \mathbb{P}^3) | v_0 \in L\},
\]

and denote by \(\sigma_{1,1}\) and \(\sigma_2\) the corresponding classes in \(H^4(G(1, \mathbb{P}^3), \mathbb{Z})\). Recall that \(G(1, \mathbb{P}^3)\) can be embedded in \(\mathbb{P}^5\) as a quadric. Then \(\sigma_{1,1} + \sigma_2 = \sigma_1^2\), where \(\sigma_1\) is the hyperplane class of \(\mathbb{P}^5\) restricted to \(G(1, \mathbb{P}^3)\), moreover \(\int_{G(1, \mathbb{P}^3)} \sigma_{1,1} \cdot \sigma_2 = 0\) and \(\int_{G(1, \mathbb{P}^3)} \sigma_2^2 = \int_{G(1, \mathbb{P}^3)} \sigma_1^2 = 1\). Recall that \(S_4\) does not contain any line, since it has Picard rank 1. Let \(f := \varphi_D\) be the map seen in Section 5, i.e.,

\[
f : S_4^{[2]} \to G(1, \mathbb{P}^3) \subset \mathbb{P}^5, \quad x \mapsto l_x,
\]

where \(l_x\) is the unique line in \(\mathbb{P}^3\) which contains the support of the subscheme \(x\). Moreover, the class of \(D\) is \(h - \delta\), where as usual \(h \in \text{Pic}(S_4^{[2]})\) is the class induced by the ample generator of \(\text{Pic}(S_4)\). Since \(f\) is the map induced by the complete linear system \(|D|\), we have \(f^*\sigma_1 = c_1(O_X(D))\), hence \(f^*(\sigma_{1,1} + \sigma_2) = c_1(O_X(D)^2)\). This shows that, given two distinct divisors \(D_1, D_2 \in |D|\), the surface \(D_1 \cap D_2\) can be reducible, and if so, we have \(|D_1 \cap D_2| = A + B \in H^{2,2}(X, \mathbb{Z})\), where \(A, B \in H^{2,2}(X, \mathbb{Z})\) are the effective classes of the two irreducible components of \(D_1 \cap D_2\). Moreover, \(A = f^*\sigma_{1,1}\) and \(B = f^*\sigma_2\). If \(S_4\) is generic it is possible to compute the classes of \(f^*\sigma_{1,1}\) and \(f^*\sigma_2\), obtaining the following equalities:

\[
f^*\sigma_{1,1} = \frac{1}{2} h^2 - \frac{1}{4} \delta^2 - \frac{1}{2} h\delta - \frac{1}{10} q^\vee, \quad f^*\sigma_2 = \frac{1}{2} h^2 + \frac{5}{4} \delta^2 - \frac{3}{2} h\delta + \frac{1}{10} q^\vee.
\]

See [Nov, Proposition 4.4.4] for details. It is also possible to geometrically describe the surface \(F \subset S_4^{[2]}\) of points fixed by the Beauville involution. We call bitangent of \(S_4 \subset \mathbb{P}^3\) a line of \(\mathbb{P}^3\) which intersects \(S_4\) in two points. We define the bitangent variety \(\text{Bit}(S_4)\) of \(S_4 \subset \mathbb{P}^3\). Then we have \(f|_{F} : F \to \text{Bit}(S_4)\). If \(S_4\) is generic, starting from \(\text{Bit}(S_4) = 12S_2 + 28S_{1,1} \subset H^{2,2}(G(1, \mathbb{P}^3), \mathbb{Z})\), see for instance [ABT01, Proposition 3.3], it is possible to show that \(f^*\text{Bit}(S_4) = 20h^2 + 8\delta^2 - 32h\delta - \frac{1}{5} q^\vee \in H^{2,2}(S_4^{[2]}, \mathbb{Z})\), see [Nov, Proposition 4.4.4].

We come back to the general problem of the Hilbert square \(X\) of a generic K3 surface admitting an ample divisor \(D \in \text{Div}(X)\) with \(q_X(D) = 2\). Before stating the irreducibility property, we show the following technical lemma.

**Lemma 7.2.** Let \(X\) and \(D\) be as in Theorem 6.4, and let \(D_1, D_2 \in |D|\) be two distinct divisors. Denote by \(\iota\) the anti-symplectic involution which generates the automorphism group \(\text{Aut}(X)\). Suppose that there is no decomposition of the form

\[
|D_1 \cap D_2| = A + B \in H^{2,2}(X, \mathbb{Z}),
\]

where \(|D_1 \cap D_2|\) is the fundamental cohomological class of the surface \(D_1 \cap D_2\), and \(A, B \in H^{2,2}(X, \mathbb{Z})\) are effective classes such that \(\iota^*(A) = A\) and \(\iota^*(B) = B\) with \(\iota^*\) described in (6.10). Then the surface \(D_1 \cap D_2\) is reduced and irreducible.

**Proof.** Suppose that

\[
|D_1 \cap D_2| = A_1 + A_2 + \cdots + A_n \in H^{2,2}(X, \mathbb{Z}),
\]

where \(A_i \in H^{2,2}(X, \mathbb{Z})\) are effective classes, not necessarily pairwise distinct. By Theorem 5.2 we have \(\iota^*|D| = |D|\), hence \(\iota^*|D_1 \cap D_2| = |D_1 \cap D_2|\). If \(n > 1\) is odd, then there exists \(i\) such that \(\iota^*A_i = A_i\), since \(\iota\) is an involution, hence we take \(A := A_i\) and \(B := A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n\). Thus \(\iota^*A = A\) and \(\iota^*B = B\). We obtain a contradiction.
with the assumption that there is no decomposition of this form. Suppose now that $n = 2$ and $\iota^* A_1 = A_2$. We show that this is not possible. Recall that the class of $D$ in $\text{Pic}(X)$ is $bh - a\delta$, with $(a, b)$ minimal solution of the Pell-type equation $P_t(-1)$. By Theorem 4.5 the classes $A_1$ and $A_2$ in $H^{2,2}(X, \mathbb{Z})$ are of the form

\[ A_1 = \left( x + \frac{y}{2} \right) h^2 + \frac{z}{8} d^2 - \frac{w}{2} h \delta + \left( \frac{1}{16} z + w \right) \frac{2}{5} q_Y, \]

\[ A_2 = \left( b^2 - x - \frac{y}{2} \right) h^2 + \left( a^2 - \frac{z}{8} \right) \delta^2 + \left( -2ab + \frac{2}{5} \right) h \delta - \left( \frac{1}{2} z + w \right) \frac{2}{5} q_Y, \]

for some $x, y, z, w \in \mathbb{Z}$. By Theorem 5.2 and (6.10), we obtain

\[ \iota^* A_1 = \left( c^2 \left( x + \frac{y}{2} \right) + cd \left( \frac{y}{2} - 2ab \right) + d^2 \left( a^2 - \frac{z}{8} \right) \right) h^2 \]

\[ + \left( t^2 d^2 \left( x + \frac{1}{2} \right) - cdt \left( \frac{y}{2} + c^2 \frac{z}{8} \right) \right) \delta^2 \]

\[ + \left( -2cdt \left( x + \frac{y}{2} \right) + c^2 \frac{y}{8} + td^2 \frac{y}{2} - 2cd \frac{z}{8} \right) \frac{2}{5} h \delta \]

\[ + \left( \frac{1}{2} z + w \right) \frac{2}{5} q_Y. \]

and similarly

\[ \iota^* A_2 = \left( c^2 \left( b^2 - x - \frac{y}{2} \right) + cd \left( \frac{y}{2} - 2ab \right) + d^2 \left( a^2 - \frac{z}{8} \right) \right) h^2 \]

\[ + \left( t^2 d^2 \left( b^2 - x - \frac{1}{2} \right) + cdt \left( a^2 - \frac{z}{8} \right) \right) \delta^2 \]

\[ + \left( -2cdt \left( b^2 - x - \frac{y}{2} \right) + c^2 + td^2 \left( 2ab - \frac{2}{5} \right) - 2cd \left( a^2 - \frac{4}{5} \right) \right) \frac{2}{5} h \delta \]

\[ - \left( \frac{1}{2} z + w \right) \frac{2}{5} q_Y. \]

Imposing $\iota^* A_1 = A_2$, we obtain a system whose solution is

\[
\begin{cases}
 x = \frac{1}{2} b^2 - ab, \\
 y = 2ab, \\
 z = 4a^2, \\
 w = -\frac{2}{5}.
\end{cases}
\]

Recall that $b$ is odd by Proposition 2.6, so $x \not\in \mathbb{Z}$, which is not possible. Since by assumption we cannot have $\iota^* A_1 = A_1$, one between $A_1$ and $A_2$ is zero, so $D_1 \cap D_2$ is reduced and irreducible. If $n > 1$ is even, if there exists an $i$ such that $\iota^* A_i = A_i$, we proceed as in the case of $n$ odd, otherwise without loss of generality we can assume that $\iota^* A_1 = A_2$. Then, taking $A := A_1 + A_2$ and $B := A_3 + \cdots + A_n$, we have $\iota^* A = A$ and $\iota^* B = B$, which contradicts the assumption. We conclude that $D_1 \cap D_2$ is reduced and irreducible.

We can now state the main theorem of this section, the irreducibility property.

**Theorem 7.3.** Let $X$ be the Hilbert square of a projective K3 surface $S_{2t}$ with $\text{Pic}(S_{2t}) = \mathbb{Z}H$ and $H^2 = 2t$ and such that $X$ admits an ample divisor $D \in \text{Div}(X)$ with $q_X(D) = 2$. Let $D_1, D_2 \in |D|$ be two distinct divisors.

(i) If $t = 2$, then the surface $D_1 \cap D_2$ can be reducible. If $[D_1 \cap D_2] = A + B$, where $[D_1 \cap D_2]$ is the fundamental cohomological class of $D_1 \cap D_2$ in $H^{2,2}(X, \mathbb{Z})$, then $A = f^* \sigma_{1,1}$ and $B = f^* \sigma_2$, where $f : X \to \mathbb{G}(1, \mathbb{P}^3)$ is the map given in (7.4) and $\sigma_{1,1}$ and $\sigma_2$ are the classes in $H^4(\mathbb{G}(1, \mathbb{P}^3), \mathbb{Z})$ of the Schubert varieties in (7.3). Moreover, if $S_4$ is generic, then the effective classes $A$ and $B$ in $H^{2,2}(X, \mathbb{Z})$ are

\[ A = \frac{1}{2} h^2 - \frac{1}{4} \delta^2 - \frac{1}{2} h \delta - \frac{1}{10} q_Y, \]

\[ B = \frac{1}{2} h^2 + \frac{5}{4} \delta^2 - \frac{3}{2} h \delta + \frac{1}{10} q_Y. \]

(ii) If $t \neq 2$ and $S_{2t}$ is generic, then $D_1 \cap D_2$ is a reduced and irreducible surface.

**Proof.** We begin with $t = 2$, so $X = S_{2t}^{[2]}$ is the Hilbert square of a smooth quartic surface of $\mathbb{P}^3$ with Picard rank 1 and the class of $D$ is $h - \delta \in \text{Pic}(X)$, where $h$ is the class induced by the ample generator of $\text{Pic}(S_4)$. For the reader’s convenience, we work under the assumption that $S_4$ is generic: the procedure that we will develop in this way is exactly the same that we will use for Item (ii), where $S_{2t}$ is generic by assumption, but easier to follow for the case $t = 2$. By Lemma 7.2, the surface $D_1 \cap D_2$ can be reducible only if there exist effective classes $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $[D_1 \cap D_2] = A + B$ and $\iota^*(A) = A, \iota^*(B) = B$, where $\iota$ is the Beauville involution. Since $(h - \delta)^2 = h^2 + \delta^2 - 2h \delta$, by Theorem 4.5 we can write, for some $x, y, z, w \in \mathbb{Z}$,

\[ A = \left( x + \frac{y}{2} \right) h^2 + \frac{z}{8} d^2 - \frac{w}{2} h \delta + \left( \frac{1}{16} z + w \right) \frac{2}{5} q_Y \in H^{2,2}(X, \mathbb{Z}), \]

\[ B = \left( 1 - x - \frac{y}{2} \right) h^2 + \left( 1 - \frac{z}{8} \right) \delta^2 + \left( \frac{y}{2} - 2 \right) h \delta - \left( \frac{1}{2} z + w \right) \frac{2}{5} q_Y \in H^{2,2}(X, \mathbb{Z}). \]
• By assumption $A$ and $B$ are effective. Moreover, $h \in \text{Pic}(X)$ is nef by Theorem 5.1. By Kleiman’s theorem we have $\langle A, h^2 \rangle \geq 0$ and $\langle B, h^2 \rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ is the bilinear form of $H^4(X, \mathbb{Z})$ given in Proposition 4.2, which coincides with the intersection pairing. We obtain the following condition:

$$0 \leq 12x + 6y + z + 10w \leq 10.$$  \hfill (7.6)

• Let $\sigma \in H^0(X, \Omega^2_X)$ be the symplectic form. Then, since $A$ and $B$ are effective classes in $H^{2,2}(X, \mathbb{Z})$ by assumption, we have

$$\int_A (\sigma + \delta)^2 = 2 \int_A \sigma \wedge \delta \geq 0, \quad \int_B (\sigma + \delta)^2 = 2 \int_B \sigma \wedge \delta \geq 0,$$

since $\sigma \wedge \delta \in H^{2,2}(X)$ is a volume form on $A$ and on $B$. Note that $\sigma \wedge \delta$ can be zero on $A$ or $B$, for instance when $A$ or $B$ are Lagrangian. Hence $\langle A, (\sigma + \delta)^2 \rangle \geq 0$ and $\langle B, (\sigma + \delta)^2 \rangle \geq 0$. Using (6.7) we obtain the following condition:

$$0 \leq 4x + 2y + z + 10w \leq 2.$$  \hfill (7.7)

• By abuse of notation we write $D$ also for its class in Pic$(X)$. Since $D$ is ample and by assumption $A$ and $B$ are effective, by the Nakai–Moishezon criterion we have $\langle A, D^2 \rangle > 0$ and $\langle B, D^2 \rangle > 0$, and we obtain the following condition:

$$0 < 40x + 12y + 3z + 20w < 12.$$  \hfill (7.8)

• By Theorem 5.2 and (6.10) we have

$$\iota^*A = \left(9x + \frac{3}{2}y + \frac{1}{2}z\right)h^2 + \left(16x + 2y + \frac{9}{8}\right)\delta^2 - \left(24x + \frac{7}{2}y + \frac{3}{2}z\right)h\delta + \left(\frac{1}{8}z + w\right)\frac{2}{5}\psi^\vee. $$

Since $\iota^*A = A$, we obtain the system

$$\begin{cases}
9x + \frac{3}{2}y + \frac{1}{2}z = x + \frac{q}{r}, \\
16x + 2y + \frac{9}{8}z = \frac{q}{r}, \\
24x + \frac{7}{2}y + \frac{3}{2}z = \frac{q}{r},
\end{cases}$$

which gives the following condition:

$$16x + 2y + z = 0.$$  \hfill (7.9)

We look for $x, y, z, w \in \mathbb{Z}$ which satisfy (7.6), (7.7), (7.8) and (7.9). Since $-2 \leq 8x + 4y \leq 10$ by (7.6) and (7.7), and $x, y \in \mathbb{Z}$, we have

$$2x + y \in \{0, 1, 2\}.$$

• Suppose that $2x + y = 0$. By (7.9) we have $z = -12x$, and (7.8) becomes $0 < w - x < \frac{2}{3}$. Since $w - x \in \mathbb{Z}$, this condition is never satisfied.

• Suppose that $2x + y = 1$. By (7.9) we have $z = -12x - 2$, and (7.8) becomes $-\frac{4}{15} < w - x < \frac{4}{15}$. Since $w - x \in \mathbb{Z}$, we get $x = w$, and (7.7) gives $x \in \{-1, 0\}$. We obtain the following two solutions:

$$\begin{cases}
x = 0, \\
y = 1,
\end{cases} \quad \text{and} \quad \begin{cases}
x = -1, \\
y = 3,
\end{cases}$$

which coincide with the effective classes $f^*\sigma_{1,1}$ and $f^*\sigma_2$ respectively, see (7.5) and [Nov, Proposition 4.4.4]. Moreover, with the same technique it is possible to show that $f^*\sigma_{1,1}$ and $f^*\sigma_2$ are reduced and irreducible.

• The case $2x + y = 2$ is symmetric to the case $2x + y = 0$, i.e., if $A$ is a class obtained in this case, then $A$ coincides with a class $B$ obtained in the case $2x + y = 0$. Since there are no classes in the case $2x + y = 0$, there are no classes in the cases $2x + y = 2$.

We conclude that if $D_1 \cap D_2$ is a reducible surface, then it is reduced with two irreducible components whose classes in $H^{2,2}(X, \mathbb{Z})$ are $f^*\sigma_{1,1}$ and $f^*\sigma_2$.

Suppose now that $t \neq 2$ and $S_{2t}$ is generic. We want to show that the surface $D_1 \cap D_2$ is reduced and irreducible. By Lemma 7.2 it is enough to show that we cannot have $|D_1 \cap D_2| = A + B$ for effective $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $\iota^*A = A$ and $\iota^*B = B$. The technique is the same seen before. By abuse of notation we write $D$ also for its class in Pic$(X)$. Recall that $D = bh - a\delta$, where $(a, b)$ is the minimal solution of the Pell-type equation $P_t(1)$. We have already remarked that the first value of $t$ different from 2 which satisfies our assumptions is $t = 10$, and in this
case $a = 3$, hence for other values of $t$ that we consider we have $a > 3$. Thus we can suppose that $t \geq 10$ and $a \geq 3$. Since $S_{2t}$ is generic by assumption, by Theorem 4.5 we can write

$$A = (x + \frac{b}{2})h^2 + \frac{1}{2}h z + w \quad \text{and} \quad B = (b^2 - x - \frac{b}{2})h^2 + (a^2 - \frac{b}{2})h d + (-2ab + \frac{b}{2})h^2 + (b z + w)\eta_Y,$$

for some $x, y, z, w \in \mathbb{Z}$. Proceeding as for the case $t = 2$ we obtain the following conditions:

$$0 \leq 6tx + 3ty + z + 10w \leq 6b^2 - 2a^2, \quad (7.10)$$

$$0 \leq 2tx + ty + z + 10w \leq 2, \quad (7.11)$$

$$0 < (4t + 8t^2b^2)x + (2t + 4t^2b^2 - 4ab)\eta y + (1 + tb^2)z + 20w < 12, \quad (7.12)$$

$$8tdx + 4(td - c)y + dz = 0. \quad (7.13)$$

where condition (7.10) is given by $\langle A, h^2 \rangle \geq 0$ and $\langle B, h^2 \rangle \geq 0$, condition (7.11) is given by $(\sigma + \bar{\sigma})^2 \geq 0$ and $(B, (\sigma + \bar{\sigma})^2) \geq 0$, condition (7.12) is given by $\langle A, D^2 \rangle > 0$ and $(B, D^2) > 0$ and condition (7.13) is given by $\mathfrak{r}^*A = A$. Note that (7.10) and (7.11) implies $-\frac{1}{4} < 2x + y < \frac{1}{4}$. Since $2x + y \in \mathbb{Z}$ and $t \geq 10$ we have $2x + y \in \{0, 1, \ldots, 2b^2\}$.

Note that, similarly to the case $t = 2$ seen above, a class $A$ obtained by imposing $2x + y \in \{b^2 + 1, \ldots, 2b^2\}$ coincide with a class $B$ obtained for $2x + y \in \{0, 1, \ldots, b^2\}$. Hence it suffices to study $2x + y \in \{0, 1, \ldots, b^2\}$. Suppose that $2x + y = k$, where $k \in \{0, 1, \ldots, b^2\}$. By (7.11) we have $-tk \leq z + 10w \leq -2 - tk$, and since $z + 10w \in \mathbb{Z}$ we have $z + 10w \in \{-tk, -tk + 1, -tk + 2\}$.

Suppose that $z + 10w = -tk$. Then (7.12) gives, after some computations, $-4t^2b^2k - 4abty + a^2z < -4t^2b^2k + 12$. Since $-4abty + a^2z \in \mathbb{Z}$, we have $-4abty + a^2z = -4t^2b^2k + h$, where $h \in \{1, 2, \ldots, 11\}$. With the help of a computer we obtain

$$w = -5a^2t k + 2ha^2 - 4tk + h.$$

Then $w$ is an integer only if $4tk - h \equiv 0 \pmod{a^2}$. If $k = 0$, then $-h \equiv a^2 0$ only if $h = 9$ and $a = 3$, being $a \geq 3$. This happens only when $t = 10$, and we get $w = 10a^2$, which is not an integer. Suppose now that $k \neq 0$. Since $a \leq b^2$, we have $4tk - h \leq 4a^2 + 4 - h$, hence in order to get $4tk - h \equiv a^2 0$ we must have $4tk - h \in \{a^2, 2a^2, 3a^2, 4a^2\}$. If $4tk - h = a^2$, then $k = \frac{b^2 + h - 1}{4t}$, which is not an integer by Proposition 2.6. If $h = 11$, then $h - 1 = 10$ is divisible by $t \geq 10$ and $k$ is not an integer. If $h = 1$, then $k = \frac{b^2}{4t}$, which is not an integer by Proposition 2.6. In a similar way it is possible to show that all the other remaining cases are not possible: we omit the details, which can be found in [Nov, Appendix B]. We conclude that there are no effective classes $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $[D_1 \cap D_2] = A + B$, hence $D_1 \cap D_2$ is a reduced and irreducible surface.

8. Geometric description

Let $X$ be the Hilbert square of a generic K3 surface of degree $2t$. Suppose that $X$ admits an ample divisor $D$ with $q_X(D) = 2$ and $t \neq 2$. In this section, using Theorem 6.4 and Theorem 7.3, we show that $\varphi_D : X \to Y \subset \mathbb{P}^5$ is a double cover of an EPW sextic $Y$, hence $X$ is a double EPW sextic. Our idea is to follow the strategy developed by O’Grady in [O’G08] with remarkable simplifications obtained thanks to the existence of the anti-symplectic involution given by Theorem 5.2. We will omit proofs which are identical to the ones in [O’G08].

Let $X$ and $D$ be as above. We introduce the following notation from [O’G08, §4]. We choose an isomorphism $|D|^\vee \cong \mathbb{P}^5$ and we denote by $f : X \to \mathbb{P}^5$ the composition $X \dasharrow |D|^\vee \cong \mathbb{P}^5$; this is basically the rational map $\varphi_D$. Let $B$ be the base locus of $|D|$, and $\beta_B : \tilde{X} \to X$ be the blow-up of $X$ in $B$. We denote by $f \circ \beta_B : \tilde{X} \to \mathbb{P}^5$ the regular map which resolves the indeterminacies of $f$. Let $Y := \text{Im}(f)$, which is a closed subset of $\mathbb{P}^5$. We obtain a dominant map, which we call $\beta$ by abuse of notation:

$$\beta : X \dasharrow Y.$$

Note that $\text{deg}(f)$ is even by Theorem 6.4. Let $Y_0$ be the interior of $\text{Im}(X \setminus B)$, thus $Y_0 \subseteq Y$ is open and dense. Let $X_0 := (X \setminus B) \cap \text{Im}^{-1}(Y_0)$, so $X_0 \subseteq X$ is open and dense. We call $f_0$ the restriction of $f$ to $X_0$, which is a regular surjective map:

$$f_0 : X_0 \to Y_0.$$

We now summarize the main steps which will lead us to the main theorem of this section:
(i) \( \dim(Y) = 4 \).
(ii) \( \deg(f) \cdot \deg(Y) \leq 12 \) and the equality holds if and only if \( \text{Bs}|D| = \emptyset \).
(iii) Either \( \deg(f) = 2 \) and \( \deg(Y) = 6 \) or \( \deg(f) = 4 \) and \( \deg(Y) = 3 \). In particular \( \text{Bs}|D| = \emptyset \) and \( f \) is a morphism.
(iv) The case \( \deg(f) = 4 \) and \( \deg(Y) = 3 \) never holds.
(v) The variety \( X \) is a double EPW sextic.

The following corollary of Theorem 7.3 will be fundamental to prove the other results of this section.

**Corollary 8.1.** Let \( X \) and \( D \) be as in Theorem 7.3.(ii), and keep notation as above.

(i) If \( L \subset \mathbb{P}^5 \) is a linear subset of codimension at most 2, then \( L \cap Y_o \) is reduced and irreducible and, if non-empty, it has pure codimension equal to \( \text{cod}(L, \mathbb{P}^5) \).

(ii) The base locus \( B = \text{Bs}|D| \) has dimension at most 1. Let \( \text{Bs}_{\text{red}} \) be the reduced scheme associated to \( B \) and \( B_{\text{red}}^{1} \) be the union of the irreducible components of \( \text{Bs}_{\text{red}} \) of dimension 1. If \( D_1, D_2, D_3 \in |D| \) are linearly independent, then \( D_1 \cap D_2 \cap D_3 \) has pure dimension 1 and there exists a unique decomposition

\[
[D_1 \cap D_2 \cap D_3] = \Gamma + \Sigma,
\]

where \( \Gamma, \Sigma \) are (classes of) effective 1-cycles such that

- \( \text{Supp}(\Gamma) \cap \text{Bs}_{\text{red}} \) is either 0-dimensional or empty.
- \( \text{Supp}(\Sigma) = B_{\text{red}}^{1} \).

We only prove Item (i), even if it is almost the same as [O’G08, Corollary 4.2.(i)], since in our setting it is a direct consequence of Proposition 7.1 and of Theorem 7.3.(ii).

**Proof.** If \( L \cong \mathbb{P}^5 \), there is nothing to prove. If \( \text{cod}(L, \mathbb{P}^5) = 1 \), let \( D_1 \in |D| \) be the divisor which corresponds to \( L \) in the isomorphism \( |D|^\vee \cong \mathbb{P}^5 \). Then \( [D_1 \cap X_o] = f_0^*[L] \), where \( [D_1 \cap X_o] \) and \( [L] \) are the fundamental cohomological classes of \( D_1 \cap X_o \) and \( L \) respectively. Since \( X_o \) is open and dense in \( X \) and \( f_0 \) is surjective, Proposition 7.1 implies that \( L \cap Y_o \) is reduced and irreducible of pure codimension 1 if non-empty. If \( \text{cod}(L, \mathbb{P}^5) = 2 \), then \( L = L_1 \cap L_2 \) with \( L_1, L_2 \subset \mathbb{P}^5 \) hyperplanes, hence \( [D_1 \cap D_2 \cap X_o] = f_0^*[L] \), where \( D_1, D_2 \in |D| \) corresponds to \( L_1 \) and \( L_2 \) respectively. Hence \( L \cap Y_o \) is reduced and irreducible of pure codimension 2 if non-empty by Theorem 7.3.(ii). For Item (ii) see [O’G08, Corollary 4.2.(ii)] (one needs Theorem 7.3.(ii) once again).

Note that \( [D_1 \cap D_2 \cap D_3] \) in the statement of Corollary 8.1.(ii) denotes the fundamental cycle of \( D_1 \cap D_2 \cap D_3 \), see [Ful13, §1.5]. The following proposition corresponds to [O’G08, Corollary 4.3].

**Proposition 8.2.** Let \( X \) and \( D \) be as in Theorem 7.3.(ii). Keep notation as above. Then \( \dim(Y) \in \{3, 4\} \).

We want to prove that the case \( \dim(Y) = 3 \) never holds. First of all, similarly to [O’G08, Proposition 4.5], we give boundaries to \( \deg(Y) \). We show the proof, which has little differences with the one by O’Grady: the anti-symplectic involution \( \iota \) of Theorem 5.2 and Theorem 6.4 are strongly used.

**Proposition 8.3.** Keep notation above and suppose that \( \dim(Y) = 3 \). Then \( 3 \leq \deg(Y) \leq 6 \). Moreover, if \( \deg(Y) = 6 \), then the base locus \( \text{Bs}|D| \) is 0-dimensional.

**Proof.** Let \( d_Y := \deg(Y) \). Since \( Y \subset \mathbb{P}^5 \) is a non-degenerate subvariety, by [EH16, Proposition 0] we have \( d_Y \geq 3 \). Let \( L_1, L_2, L_3 \subset \mathbb{P}^5 \) be three generic hyperplanes. Then the intersection \( Y \cap L_1 \cap L_2 \cap L_3 \) is transverse and it is given by \( d_Y \) points, which we call \( p_1, \ldots, p_{d_Y} \). Let \( \Gamma_{i,j} := f_0^{-1}(p_i) \) for \( i = 1, \ldots, d_Y \), and let \( \Gamma_i \) be the closure of \( \Gamma_{0,i} \) in \( X \). Let \( D_1, D_2, D_3 \in |D| \) be the divisors which correspond to \( L_1, L_2, L_3 \) in the isomorphism \( |D|^\vee \cong \mathbb{P}^5 \). Since \( L_1, L_2, L_3 \) are generic, \( D_1, D_2, D_3 \) are linearly independent, and by Corollary 8.1.(ii) we have

\[
[D_1 \cap D_2 \cap D_3] = \sum_{i=1}^{d_Y} \Gamma_i + \Sigma,
\]

where the equality is in \( H^{3,3}(X, \mathbb{Z}) \) and we write by abuse of notation \( \Gamma_i \) and \( \Sigma \) for their classes in \( H^{3,3}(X, \mathbb{Z}) \). We write again \( D \) also for its class in \( \text{Pic}(X) \): recall that \( D = bh - a\delta \) with \( (a, b) \) minimal solution of the Pell-type equation \( P_3(-1) \), and \( b \) is odd by Proposition 2.6. Using Proposition 3.6 one obtains \( D^{\vee} = bh^{\vee} - 2a\delta^{\vee} \) with \( h^{\vee} = h \) and \( \delta^{\vee} = \frac{\delta}{2} \) in \( H^2(X, \mathbb{Q}) \). Moreover, as already remarked, the divisibility of \( D \) in \( H^2(X, \mathbb{Z}) \) is \( \text{div}(D) = 1 \), hence by Proposition 3.6 we have

\[
D \cdot D^{\vee} = \left( bh - a\delta, bh - 2\frac{a}{2} \delta \right) = 2.
\]
Since $D_1, D_2, D_3 \in |D|$ and $\iota^*D \cong D$, we have $\iota^*[D_1 \cap D_2 \cap D_3] = [D_1 \cap D_2 \cap D_3]$, hence
\[
\sum_{i=1}^{dy} \iota^*\Gamma_i + \iota^*\Sigma = \sum_{i=1}^{dy} \Gamma_i + \Sigma.
\]
By Theorem 5.2 and Theorem 5.4 the only class fixed by the action induced by $\iota$ on $H^{3,3}(X, \mathbb{Z})$ is $D^\vee$. Moreover, from Theorem 6.4 we know that $f$ factors through the quotient by $\iota$, so $\iota^*\Gamma_i \cong \Gamma_i$, since $\Gamma_i = \frac{1}{2}(p_1)$. Then the class of $\Gamma_i$ is either some positive multiple of $D^\vee$, or it is of the form $\Gamma_i = \Gamma_i' + \iota^*\Gamma_i'$, where $\Gamma_i'$ is an effective class. Since $q_X(D) = 2$, which gives $\int_X c_1(\mathcal{O}_X(D))^4 = 12$ by Theorem 3.5, we have
\[
12 = \langle D, \sum_{i=1}^{dy} \Gamma_i + \Sigma \rangle.
\]
Note that $\langle D, \alpha D^\vee \rangle = 2\alpha$, where $\alpha \in \mathbb{Z}_{\geq 1}$, or $\langle D, \Gamma_i' + \iota^*\Gamma_i' \rangle \geq 2$, being $D$ ample. Hence we have $12 \geq \langle D, \Sigma \rangle + 2dy$, i.e., $dy \leq 6$. Moreover, if $dy = 6$, then $\Sigma = 0$, otherwise $\langle D, \Sigma \rangle > 0$ by the Nakai–Moishezon criterion, thus the base locus $Bs|D|$ is 0-dimensional, since the 1-dimensional component of $Bs|D|$ is contained in $\Sigma$ by Corollary 8.1.(ii).

Using exactly the same techniques in [O’G08], with important simplifications due to Theorem 6.4, we obtain the following proposition.

**Proposition 8.4.** Let $X$ and $D$ be as in Theorem 7.3.(ii). Consider the map $f : X \dashrightarrow Y \subset \mathbb{P}^5$ induced by the complete linear system $|D|$. Then $\dim(Y) = 4$. Moreover, $|D|$ is basepoint free, i.e., $f$ is a morphism, and one of the following holds.

(i) $\deg(f) = 2$ and $\deg(Y) = 6$.

(ii) $\deg(f) = 4$ and $\deg(Y) = 3$.

**Proof.** After having obtained Proposition 8.3, following [O’G08, §5] one shows that $\dim(Y) = 4$. Then, one obtains results which correspond to [O’G08, Proposition 4.6, Corollary 4.7], and proceeding as in [O’G08, §4] we have the following possibilities:

1. $\deg(Y) = 2$.
2. $\deg(Y) = 3$, $\deg(f) = 3$, $Bs|D| \neq \emptyset$.
3. $\deg(Y) = 3$, $\deg(f) = 4$, $Bs|D| = \emptyset$.
4. $\deg(Y) = 4$, $\deg(f) = 3$, $Bs|D| = \emptyset$.
5. $\deg(Y) = 6$, $\deg(f) = 2$, $Bs|D| = \emptyset$.
6. $\deg(f) = 1$.

As already remarked, $\deg(f)$ is even by Theorem 6.4, hence (2), (4) and (6) are not possible. Case (1) does not hold, otherwise one can find a linear subset $L \subset \mathbb{P}^5$ of dimension 3 such that $L \cap Y_0$ is reducible, which contradicts Corollary 8.1.(i), see [O’G08, §5.2] for details.

Note that in [O’G08] much longer discussions are needed to show that (2) and (4) in the proof of Proposition 8.4 never hold: the existence of the anti-symplectic involution and Theorem 6.4 simplifies the situation a lot. Moreover, recall that Corollary 8.1.(i) is a consequence of the irreducibility property of Theorem 7.3.(ii). When $t = 2$, the irreducibility property does not hold, see Theorem 7.3.(i), hence Corollary 8.1.(i) is not true and case (1) in the proof of Proposition 8.4 is possible, actually it holds: as we have already seen in Section 7, in this case $\deg(f) = 6$ and $\deg(Y) = 2$, in particular $Y$ is the Grassmannian $G(1, \mathbb{P}^3)$ of lines in $\mathbb{P}^3$.

The final step is to show that Item (ii) of Proposition 8.4 is impossible and in Item (i) the variety $Y \subset \mathbb{P}^5$ is an EPW sextic, $X$ is a double EPW sextic and $f$ is the double cover associated. See [EPW01] for the definition of EPW sextic and [O’G06] for details on double EPW sextics. We remark that we cannot proceed following [O’G08, §5.5]: in that case, the fact that the variety is a deformation of the Hilbert square of a K3 surface plays a central role in the proof. If $M$ is an IHS manifold of K3[2]-type and $h \in H^{1,1}(M, \mathbb{Q})$ is an element such that $q_M(h) \neq 0$, O’Grady shows that $H^4(M, \mathbb{C})$ can be decomposed as
\[
H^4(M, \mathbb{C}) = \langle Ch^2 \oplus Cq_M^\vee \rangle \oplus \langle Ch \otimes h^\perp \rangle \oplus W(h),
\]
where the orthogonality is with respect to the BBF form and $W(h) := \langle q_M^\vee \rangle \cap \text{Sym}^2(h^\perp)$, where $\langle q_M^\vee \rangle$ is the orthogonal with respect to the intersection product. Then O’Grady takes an IHS manifold $X$ deformation equivalent
to $M$ such that properties (1)-(6) of [O’G08, Proposition 3.2] hold. In particular property (4) says that if $V \subset H^4(X)$ is a rational sub Hodge structure, then $V_C = V_1 \oplus V_2 \oplus V_3$, where $V_1 \subset (Ch^2 \otimes \mathbb{Q})$, $V_2$ is either 0 or equal to $Ch \otimes h^1$ and $V_3$ is either 0 or equal to $W(h)$. In our case, there is no reason why a similar result holds, without deforming the variety $X = S_{2t}^2$ that we are studying: instead we exploit once again the anti-symplectic involution $\iota$ and Theorem 6.4. We can finally prove the main theorem of this paper.

**Theorem 8.5.** Let $X$ be the Hilbert square of a generic K3 surface $S_{2t}$ of degree $2t$ such that $X$ admits an ample divisor $D$ with $q_X(D) = 2$. Suppose that $t \neq 2$, and denote by $\iota$ the anti-symplectic involution which generates $\text{Aut}(X)$.

Then the complete linear system $|D|$ is basepoint free, the morphism

$$\varphi_D : X \to Y \subset \mathbb{P}^5$$

is a double cover whose ramification locus is the surface $F$ of points fixed by $\iota$, and $Y \cong X/\langle \iota \rangle$ is an EPW sextic, so $X$ is a double EPW sextic.

**Proof.** We show that $\deg(f) = 4$ and $\deg(Y) = 3$ in Proposition 8.4 never holds. Suppose by contradiction that $\deg(f) = 4$ and $\deg(Y) = 3$. First we show that $Y \subset \mathbb{P}^5$ is a normal variety. Since $Y \subset \mathbb{P}^5$ is a hypersurface, by [Har13, Proposition II.8.23] we have that $Y$ is normal if and only if $\text{cody}(\text{Sing}(Y)) \geq 2$. Suppose now that $\dim(\text{Sing}(Y)) = 3$. Note that $Y$ does not contain planes, otherwise we get a contradiction with Corollary 8.1.(i). As observed by O’Grady in [O’G08, Claim 5.10], since in $\mathbb{P}^5$ hyperplanes and quadrics contain planes, a cubic hypersurface of $\mathbb{P}^5$ which is either non-reduced or reducible contains planes. Hence in our case $Y$ is reduced and irreducible. Then, as in [O’G08, Lemma 5.17], the intersection between the variety $Y$ and a generic plane of $\mathbb{P}^5$ is a singular cubic curve which is reduced and irreducible. In particular this has only one singular point, so $\text{Sing}(Y)$ has exactly one irreducible component $\Sigma \cong \mathbb{P}^1$ of degree 1. Thus $Y \supset \Sigma$, and it contains planes. This contradicts Corollary 8.1.(i). We conclude that $\text{cody}(\text{Sing}(Y)) \geq 2$, hence $Y$ is normal. The commutative diagram (6.4) gives the following:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \subset \mathbb{P}^5 \\
\downarrow{\pi'} & & \downarrow{\tilde{f}} \\
X/\langle \iota \rangle.
\end{array}$$

Since $\deg(f) = 4$ by assumption, by the commutativity of the diagram above we have $\deg(\tilde{f}) = 2$. Let $\tilde{Y} := X/\langle \iota \rangle$. Since $X$ is smooth and $\text{Aut}(X) \cong \langle \iota \rangle$ is a finite group, the quotient $\tilde{Y}$ is a normal variety. Both $\tilde{Y}$ and $Y$ are normal varieties, hence by [DK19, Remark 2.4] there is a non trivial involution $\tau : \tilde{Y} \to \tilde{Y}$ such that $Y \cong \tilde{Y}/\langle \tau \rangle$. We show that $\tau$ lifts to an automorphism on $X$. We use a technique from [O’G13, p.179] and [DK18, Proposition B.8]. Let $F = \text{Fix}(\iota)$ be the fixed locus of the anti-symplectic involution $\iota$. Since $\text{Sing}(\tilde{Y}) = \pi'(F)$ and $\tau$ is an automorphism on $\tilde{Y}$, we have $\tau(\text{Sing}(\tilde{Y})) = \text{Sing}(\tilde{Y})$. Let $Y' := \tilde{Y} \backslash \text{Sing}(\tilde{Y})$. Then the restriction of $\tau$ to $Y'$ gives an involution $\tau|_{Y'} : Y' \to Y'$ of $Y'$. We set $\tau' := \tau|_{Y'}$. Since $\text{cody}(F) = 2$ and $X$ is simply connected, we have that $X' := X \backslash F$ is simply connected. Thus if we restrict $\pi'$ to $X'$ we obtain the following universal cover:

$$\pi'|_{X'} : X' \to Y'.$$

By the lifting criterion $\tau'$ lifts to an automorphism on $X'$. Thus we obtain a birational map $\tilde{\tau} : X \dasharrow X$ which is not defined a priori on the fixed locus $F$. By [DM19, Proposition B.3] we have $\text{Bir}(X) \cong \text{Aut}(X) \cong \langle \iota \rangle$, so $\tilde{\tau}$ is either the identity or $\iota$. This implies that the involution $\tau : \tilde{Y} \to \tilde{Y}$ is the identity, which is a contradiction. We conclude that $\deg(\tilde{f})$ cannot be 2, hence we get $\deg(f) = 2$ and $\deg(Y) = 6$. It remains to show that $Y$ is an EPW sextic and $f$ is a double cover ramified over $F$, so that $X$ is a double EPW sextic. This is true by [O’G06, Theorem 1.1], and we are done.

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