THE $\theta$-BUMP THEOREM FOR PRODUCT FRACTIONAL INTEGRALS

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Abstract. We extend the one parameter $\theta$-bump theorem for fractional integrals of Sawyer and Wheeden to the setting of two parameters, as well as improving the multiparameter result of Tanaka and Yabuta for doubling weights to classical reverse doubling weights.

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1. Introduction

In [SaWh, Theorem 1(A)], Sawyer and Wheeden proved that the fractional integral $I_\alpha f(x) = \int |x-u|^{\alpha-n} f(u) \, du$, $x \in \mathbb{R}^N$, is bounded from one weighted space $L^p(v^p)$ to another $L^q(w^q)$ provided there is $\theta > 1$ such that

$$A_{p,q;\theta}^{\alpha,m}(v,w) \equiv \sup_{I \in \mathcal{D}^N} |I|^{\frac{m-\frac{\alpha-m}{p}}{\frac{1}{p}+\frac{1}{q}} + \frac{1}{\theta}} \left( \frac{1}{|I|} \int_I v^\theta \right)^{\frac{1}{p\theta}} \left( \frac{1}{|I|} \int_I w^\theta \right)^{\frac{1}{q\theta}} < \infty.$$ 

Here $1 < p \leq q < \infty$, $0 < \alpha < N$ and $v,w$ are nonnegative measurable functions on $\mathbb{R}^N$, $N \geq 1$. The finiteness of $A_{p,q;\theta}^{\alpha,m}(v,w)$ when $\theta = 1$ is a well-known necessary condition for the boundedness of $I_\alpha$, and the above strengthening of that condition is usually referred to as a $\theta$-bump condition. In the same paper [SaWh, the second assertion of Theorem 1(B)], it was shown that in the case $p < q$, if $v^\theta$ and $w^\theta$ are both reverse doubling weights, then the necessary condition $A_{p,q;1}^{\alpha,m}(v,w) < \infty$ is also sufficient for the boundedness of $I_\alpha$ from $L^p(v^p)$ to $L^q(w^q)$. Here a measure $\mu$ is reverse doubling in $\mathbb{R}^N$ if there are $C, \varepsilon > 0$ such that

$$|2^{-s}I|_\mu \leq C 2^{-\varepsilon s} |I|_\mu,$$

for all $s > 0$ and cubes $I \subset \mathbb{R}^N$, where $2^{-s}I$ denotes the cube concentric with $I$ and having side length $\ell(2^{-s}I)$ equal to $2^{-s}\ell(I)$.

Recently, H. Tanaka and K. Yabuta [TaYa] used a clever iteration\footnote{In a nutshell, they use $p < r < q$ and Hölder’s inequality with $r$ and $r'$ to separate the measures $\sigma$ and $\omega$ early on, and then use iteration on the resulting ‘one weight’ Carleson embeddings, the point being that iteration works better with one weight than with two weights.} to obtain an $n$-linear embedding theorem for rectangles that has as a corollary the following result for certain product fractional integrals $\tilde{I}_\alpha^N$ on $\mathbb{R}^N$ given by

$$\tilde{I}_\alpha^N f(x) \equiv \int_{\mathbb{R}^N} \prod_{j=1}^N |x_j - u_j|^{\alpha-1} f(u) \, du, \quad x \in \mathbb{R}^N, \quad 0 < \alpha < 1.$$ 

Let $\mathcal{R}^N$ denote the partial grid of all rectangles in $\mathbb{R}^N$ with sides parallel to the coordinate axes (which is not a grid). A weight $\mu$ is a rectangle doubling weight on $\mathbb{R}^N$ if there is $C > 0$ such that

$$|2R|_\mu \leq C |R|_\mu,$$

for all rectangles $R \in \mathcal{R}^N$. 

References
Theorem 1 (H. Tanaka and K. Yabuta [TaYa Proposition 5.1]). Suppose $1 < p < q < \infty$ and that both $v^{-\nu}$ and $w^q$ are rectangle doubling weights on $\mathbb{R}^N$. Then $I^N_\alpha$ is bounded from $L^p(v^\nu)$ to $L^q(w^q)$ if and only if
\[
\sup_{R \in \mathbb{R}^N} \left( \frac{1}{|R|} \int_R v^{-\nu} \right)^\frac{1}{\nu} \left( \frac{1}{|R|} \int_R w^q \right)^\frac{1}{q} < \infty.
\]
Thus the theorem of Tanaka and Yabuta extends the second assertion in Theorem 1(B) of [SaWh] to product fractional integrals with rectangle doubling weights. The purpose of this paper is to extend both Theorem 1(A) and the second assertion in Theorem 1(B) of [SaWh] to product fractional integrals of the form (more than two factors in the kernel are handled similarly)
\[
I^{m,n}_{\alpha,\beta} f(x,y) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |x-u|^{-\tau-1} |y-t|^{-\tau-1} f(u,t) \, dt du, \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n,
\]
more precisely, to show that a product $\theta$-bump condition is always sufficient for the norm inequality, and that the same condition without a bump is sufficient provided the weights $v^{-\nu}$ and $w^q$ are product reverse doubling on $\mathbb{R}^m \times \mathbb{R}^n$ in this sense: a weight $\mu$ is product reverse doubling on $\mathbb{R}^m \times \mathbb{R}^n$ if there are $C, \varepsilon_1, \varepsilon_2 > 0$ such that
\[
(1.1) \quad \left| (2^{-s} I) \times (2^{-t} J) \right|_\mu \leq C 2^{-s \varepsilon_1 - \varepsilon_2 t} |I \times J|_\mu,
\]
for all $s, t > 0$ and cubes $I \subset \mathbb{R}^m$ and $J \subset \mathbb{R}^n$.

Our proof of the first result adapts the Tanaka-Yabuta argument to the $\theta$-bump functional used in [SaWh], while the second result regarding reverse doubling weights adapts the Tanaka-Yabuta argument to the use of NTV good/bad grids in place of the Str"{o}mberg $\frac{1}{2}$-trick that was used in [TaYa]. Additional results for the product situation can be found in our paper [SaWa]. See the appendix below for a discussion of the doubling and various reverse doubling conditions.

Acknowledgement 1. We are grateful to Hitoshi Tanaka for bringing our attention to his beautiful paper [TaYa] with K. Yabuta.

1.1. Preliminaries. Let $D^m$ denote the grid of dyadic cubes in $\mathbb{R}^m$, and let $\mathcal{R}^{m,n} = D^m \times D^n$ denote the partial grid of dyadic rectangles in $\mathbb{R}^m \times \mathbb{R}^n$ (which is not actually a grid since it fails the nested property). For $d\mu(x) = u(x) \, dx$ absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^N$, we will use the following $\theta$-bump functional for a cube $Q$ and $\theta > 1$ ([SaWh] see page 830):
\[
|Q|_{\mu,\theta} = \left( \int_Q u^\theta \right)^{\frac{1}{\theta}}.
\]
We have $|Q|_\mu \leq |Q|_{\mu,\theta}$, and if $P = \bigcup_{i=1}^\infty Q_i$ is a pairwise disjoint union of the cubes $Q_i$, then we have
\[
\sum_{i=1}^\infty |Q_i|_{\mu,\theta} = \sum_{i=1}^\infty \left( \int_{Q_i} u^\theta \right)^{\frac{1}{\theta}} \leq \left( \sum_{i=1}^\infty |Q_i| \right)^{\frac{1}{\theta}} \left( \sum_{i=1}^\infty \int_{Q_i} u^\theta \right)^{\frac{1}{\theta}} = |P|^{\frac{1}{\theta}} \left( \int_P u^\theta \right)^{\frac{1}{\theta}} = |P|_{\mu,\theta}.
\]

The important property of the $\theta$-bump functional on cubes for us is that, when taken to a power larger than 1, it automatically satisfies a Carleson condition taken over all dyadic subcubes. More precisely, if $\rho > 1$, then
\[
(1.2) \quad \sum_{Q \in D^N: Q \subset P} |Q|_{\mu,\theta}^\rho = \sum_{k=-\infty}^\infty \sum_{Q \in D^N: \ell(Q) = 2^{-k} \ell(P)} |Q|_{\mu,\theta} \left( \int_Q u^\theta \right)^{\frac{\rho-1}{\rho}} |Q|_{\mu,\theta}^\rho \leq \sum_{k=-\infty}^\infty (C 2^{-kN\varepsilon} |P|)^{\frac{\rho-1}{\rho}} \left( \int_P u^\theta \right)^{\frac{\rho-1}{\rho}} |Q|_{\mu,\theta}^\rho \leq \sum_{k=-\infty}^\infty (C 2^{-kN\varepsilon} |P|)^{\frac{\rho-1}{\rho}} \left( \int_P u^\theta \right)^{\frac{\rho-1}{\rho}} |P|_{\mu,\theta}^\rho = C N \varepsilon^{\frac{\rho-1}{\rho}} |P|_{\mu,\theta}^\rho.
\]

This automatic Carleson condition leads to a corresponding automatic Carleson embedding lemma.

\(^2\)In [TaYa] the authors use a strong form of reverse doubling on rectangles, which is equivalent to rectangle doubling. See the appendix below.
Lemma 1. Suppose that $1 < s < r < \infty$, $\theta > 1$, and that $d\mu(x) = u(x)\,dx$ is a locally $L^\theta$ absolutely continuous measure on $\mathbb{R}^N$. Then we have

$$\left\{ \sum_{Q \in \mathcal{D}^N} |Q|^{\frac{\theta}{\theta-1}} \left( \frac{1}{|Q|^{\theta}} \int_Q f\,d\mu \right)^r \right\}^{\frac{1}{r}} \leq C_{r,s,\theta} \|f\|_{L^r(\mu)}, \quad f \geq 0.$$  

Proof. The cubes in $\mathcal{D}^N$ form a grid, and so for each integer $k \in \mathbb{Z}$, we can consider the maximal dyadic cubes $\{M_i^k\}_{i=1}^\infty$ from $\mathcal{D}^N$ such that

$$\frac{1}{|M_i^k|^{\mu,\theta}} \int_{M_i^k} f\,d\mu > 2^k.$$  

Then we can estimate using (1.2) that

$$\sum_{Q \in \mathcal{D}^N} |Q|^{\frac{\theta}{\theta-1}} \left( \frac{1}{|Q|^{\theta}} \int_Q f\,d\mu \right)^r \leq \sum_{k=-\infty}^\infty \sum_{Q \in \mathcal{D}^N: Q \subseteq M_i^k} |Q|^{\frac{\theta}{\theta-1}} \left( 2^{k+1} \right)^r$$

$$\leq 2^r \sum_{k=-\infty}^\infty \sum_{i=1}^\infty \sum_{Q \in \mathcal{D}^N: Q \subseteq M_i^k} |Q|^{\frac{\theta}{\theta-1}} \left( 2^{k+1} \right)^r$$

$$= 2^r \sum_{k=-\infty}^\infty \sum_{i=1}^\infty \left( \sum_{Q \in \mathcal{D}^N: Q \subseteq M_i^k} |Q|^{\frac{\theta}{\theta-1}} \right)^{2kr} \leq 2^r C_{N,s} \sum_{k=-\infty}^\infty \sum_{i=1}^\infty |M_i^k|^{\frac{\theta}{\theta-1}} 2^{kr}.$$  

Now we use the fact that

$$\frac{1}{|M_i^k|^{\mu,\theta}} \int_{M_i^k \cap \{f > 2^k-1\}} f\,d\mu = \frac{1}{|M_i^k|^{\mu,\theta}} \int_{M_i^k} f\,d\mu - \frac{1}{|M_i^k|^{\mu,\theta}} \int_{M_i^k \cap \{f \leq 2^k-1\}} f\,d\mu$$

$$\geq \frac{1}{|M_i^k|^{\mu,\theta}} \int_{M_i^k} f\,d\mu - \frac{1}{|M_i^k|^{\mu,\theta}} \int_{M_i^k} 2^{k-1} d\mu$$

$$> 2^k - 2^{k-1} \frac{|M_i^k|^{\mu,\theta}}{|M_i^k|^{\mu,\theta}} \geq 2^{k-1},$$  

to obtain

$$\sum_{Q \in \mathcal{D}^N} |Q|^{\frac{\theta}{\theta-1}} \left( \frac{1}{|Q|^{\theta}} \int_Q f\,d\mu \right)^r \leq 2^r C_{N,s} \sum_{k=-\infty}^\infty \sum_{i=1}^\infty |M_i^k|^{\frac{\theta}{\theta-1}} 2^{kr}$$

$$\leq C_{r,s,\theta} \sum_{k=-\infty}^\infty \sum_{i=1}^\infty \left( 2^{-k} \int_{M_i^k \cap \{f > 2^k-1\}} f\,d\mu \right)^{\frac{\theta}{\theta-1}} 2^{kr}$$

$$\leq C_{r,s,\theta} \left( \sum_{k=-\infty}^\infty \sum_{i=1}^\infty 2^{k(s-1)} \int_{M_i^k \cap \{f > 2^k-1\}} f\,d\mu \right)^{\frac{\theta}{\theta-1}}.$$  

We now use that the cubes \( \{ M^k_i \}_{i=1}^\infty \) are pairwise disjoint in \( i \) to continue with the estimate

\[
\left( \sum_{k=-\infty}^\infty \sum_{i=1}^{2^k} \int_{M^k_i \cap \{ f > 2^{k-1} \}} f \, d\mu \right)^{\frac{p}{r}} \leq \left( \sum_{k=-\infty}^\infty \left( 2^{k(s-1)} \int_{\{ f > 2^{k-1} \}} |f| \, d\mu \right) \right)^{\frac{r}{s}}
\]

\[
= \left( \left\{ \sum_{k \in \mathbb{Z}} 2^{k(s-1)} f(x) \, d\mu(x) \right\} \right)^{\frac{r}{s}}
\]

\[
\leq C_s \left( \int f(x)^s \, d\mu(x) \right)^{\frac{r}{s}} = C_s \|f\|_{L^s(\mu)}^r.
\]

\[\square\]

2. The 2-parameter theory

Here we state and prove our extensions of Theorem 1(A) and the second assertion of Theorem 1(B) in [SaWh]. We begin with the \( \theta \)-bump condition.

2.1. The \( \theta \)-bump condition for bilinear embeddings. Here is a variation on the Tanaka-Yabuta theorem [TaYa] Theorem 1.1 involving general weights that satisfy a \( \theta \)-bump analogue of the ‘rectangle testing’ condition in [TaYa]. We extend the definition of the \( \theta \)-bump functional to rectangles in the obvious way,

\[|R|_{\mu, \theta} \equiv |R|^{\frac{1}{p}} \left( \int_R u^\theta \right)^{\frac{1}{p}},\]

for \( d\mu(x, y) = u(x, y) \, dx \, dy \) absolutely continuous and \( R \) a rectangle in \( \mathbb{R}^m \times \mathbb{R}^n \).

**Theorem 2.** Suppose \( 1 < p < q < \infty \). Let \( d\sigma = v^{-p'} \, dx \) and \( d\omega = u^q \, dx \) be locally finite absolutely continuous weights on \( \mathbb{R}^m \times \mathbb{R}^n \), let \( \theta > 1 \), and let \( K : \mathcal{R}^{m,n} \to [0, \infty) \). Then the norm \( \mathcal{N}_K(\sigma, \omega) \) of the positive bilinear inequality,

\[\sum_{R \in \mathcal{R}^{m,n}} K(R) \left( \int_R f \, d\sigma \right) \left( \int_R g \, d\omega \right) \leq \mathcal{N}_K(\sigma, \omega) \|f\|_{L^p(\sigma)} \|g\|_{L^q(\omega)} , \quad f, g \geq 0 ,\]

is finite independent of all partial grids \( \mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n \) if the \( \theta \)-bump product characteristic \( \mathcal{A}_{K, \theta}(\sigma, \omega) \) is finite, where

\[\mathcal{A}_{K, \theta}(\sigma, \omega) \equiv \sup_{R \in \mathcal{R}^{m,n}} K(R) \left| R \right|^{\frac{1}{p}} \left| \int_R v^{-p'} \, d\sigma \right|^{\frac{1}{p}} \left| R \right|^{\frac{1}{q}} \left| \int_R u^q \, d\omega \right|^{\frac{1}{q}} \left[ \left| R \right|^{\frac{1}{p}} \left( \int_R u^q \, d\omega \right)^{\frac{1}{q}} \right].
\]

**Proof.** As in [TaYa], we choose \( p < r < q \). Then the definition of the \( \theta \)-bump characteristic, followed by Hölder’s inequality with exponents \( r \) and \( r' \), gives

\[
\sum_{R \in \mathcal{R}^{m,n}} K(R) \left( \int_R f \, d\sigma \right) \left( \int_R g \, d\omega \right) = \sum_{R \in \mathcal{R}^{m,n}} \left\{ K(R) \left| R \right|^{\frac{1}{p}} \left| R \right|^{\frac{1}{q}} \left| \int_R f \, d\sigma \right| \left( \int_R \frac{1}{\left| R \right|^{\frac{1}{p}} \left| R \right|^{\frac{1}{q}}} \, d\sigma \right)^{\frac{1}{p}} \left( \int_R \frac{1}{\left| R \right|^{\frac{1}{q}}} \, d\omega \right)^{\frac{1}{q}} \right\}.
\]

\[
\leq \mathcal{A}_{K, \theta}(\sigma, \omega) \left\{ \sum_{R \in \mathcal{R}^{m,n}} \left| R \right|^{\frac{1}{p}} \left( \int_R \frac{1}{\left| R \right|^{\frac{1}{p}}} \, d\sigma \right)^{\frac{r}{p}} \right\} \left\{ \sum_{R \in \mathcal{R}^{m,n}} \left| R \right|^{\frac{1}{q}} \left( \int_R \frac{1}{\left| R \right|^{\frac{1}{q}}} \, d\omega \right)^{\frac{r'}{q}} \right\},
\]

and the theorem now follows from the following proposition. \[\square\]
Proposition 1. Suppose that $1 < s < r < \infty$, $\theta > 1$, and that $\mu$ is a locally finite absolutely continuous measure on $\mathbb{R}^m \times \mathbb{R}^n$. Then we have

$$
\left\{ \sum_{R \in \mathbb{R}^{m,n}} |R|^{\frac{\theta}{r}} \left( \frac{1}{|R|^{\mu,\theta}} \int_R f \, d\mu \right)^r \right\}^{\frac{1}{r}} \leq C_{s,r,\theta} \| f \|_{L^r(\mu)}, \quad f \geq 0.
$$

Proof. We follow the outline of the iteration argument in H. Tanaka and K. Yabuta [TaYa], but adapted to $\theta$-bump functionals. Let $d\mu(x,y) = u(x,y) \, dx \, dy$ and define

$$
u^\theta(x) = u(x,y), \quad d\mu^\theta(x) = u^\theta(x) \, dx \quad \text{and} \quad d\mu_x(y) = u_x(y) \, dy
$$

for a.e. $x \in \mathbb{R}^m$, a.e. $y \in \mathbb{R}^n$.

and note that

$$|J|_{\mu,\theta} = |J|^\theta \left( \int_I u_x(y)^\theta \, dy \right)^{\frac{1}{\theta}} \quad \text{and} \quad |I|_{\mu,\theta} = |I|^\theta \left( \int_I u_y(y)^\theta \, dx \right)^{\frac{1}{\theta}}
$$

for a.e. $x \in \mathbb{R}^m$, a.e. $y \in \mathbb{R}^n$.

Now take $f \in L^p(\mu)$ and let

$$F^J(x) = \frac{1}{|J|_{\mu,\theta}} \int_J f(x,y) u(x,y) \, dy \quad \text{for a.e.} \quad x \in \mathbb{R}^m.
$$

Note that

$$|I \times J|_{\mu,\theta} = |I| \cdot |J|_{\mu,\theta}^{\frac{\theta}{r}} \left( \int_I \left\{ \int_J u(x,y)^\theta \, dy \right\} \, dx \right)^{\frac{1}{\theta}}
$$

where we can interpret the term in braces as

$$\left| J \right|^{\frac{1}{\theta}} \left( \int_J u_x(y)^\theta \, dy \right)^{\frac{1}{\theta}} = |J|_{\mu,\theta}
$$

so that we have

$$|I \times J|_{\mu,\theta} = |I|^{\frac{\theta}{r}} \left( \int_I |J|_{\mu,\theta}^\theta \, dx \right)^{\frac{1}{\theta}} = |I|^{\frac{\theta}{r}} \left( \int_I (J_{\mu,\theta}(x))^\theta \, dx \right)^{\frac{1}{\theta}} = |I|_{J_{\mu,\theta}}(x)
$$

where we have defined the absolutely continuous measure $J_{\mu,\theta}$ by $dJ_{\mu,\theta}(x) = J_{\mu,\theta}(x) \, dx$ and where its density function, which with a small abuse of notation we also denote by $J_{\mu,\theta}$, is given by

$$J_{\mu,\theta}(x) = |J|_{\mu,\theta}, \quad x \in \mathbb{R}^m.
$$

We then estimate

$$\sum_{R \in \mathbb{R}^{m,n}} |R|^{\frac{\theta}{r}} \left( \frac{1}{|R|^{\mu,\theta}} \int_R f(x,y) \, u(x,y) \, dx \, dy \right)^r
$$

$$= \sum_{I \times J \in \mathbb{R}^{m,n}} |I \times J|^{\frac{\theta}{r}} \left( \frac{1}{|I \times J|^{\mu,\theta}} \int_{I \times J} f(x,y) \, u(x,y) \, dx \, dy \right)^r
$$

$$= \sum_{J \in D^m} \sum_{I \in D^m} |I|^{\frac{\theta}{r}} \left( \frac{1}{|I|^{\mu,\theta}} \int_I \left( \int_J f(x,y) \, u(x,y) \, dy \right)^{\frac{1}{r}} \, J_{\mu,\theta}(x) \, dx \right)^r
$$

$$= \sum_{J \in D^m} \left\{ \sum_{I \in D^m} |I|^{\frac{\theta}{r}} \left( \frac{1}{|I|^{\mu,\theta}} \int_I F^J(x) \, J_{\mu,\theta}(x) \, dx \right)^{\frac{1}{r}} \right\} \leq \sum_{J \in D^m} \left( \int_{\mathbb{R}^m} F^J(x)^{\theta} \, J_{\mu,\theta}(x) \, dx \right)^{\frac{1}{r}},
$$
by Lemma II above applied with the locally finite absolutely continuous measures \( J_{\mu, \theta} \) on \( \mathbb{R}^m \), \( J \in \mathcal{D}^n \). Now we continue to estimate the latter sum raised to the power \( \frac{r}{s} \) by Minkowski's inequality,

\[
\left\{ \sum_{J \in \mathcal{D}^n} \left( \int_{\mathbb{R}^m} F^J (x)^s J_{\mu, \theta} (x) \, dx \right)^{\frac{r}{s}} \right\}^{\frac{s}{r}} \leq \int_{\mathbb{R}^m} \left\{ \sum_{J \in \mathcal{D}^n} \left( F^J (x)^s \right)^{\frac{r}{s}} J_{\mu, \theta} (x) \, dx \right\}^{\frac{s}{r}} = \int_{\mathbb{R}^m} \left\{ \sum_{J \in \mathcal{D}^n} J_{\mu, \theta} (x)^{\frac{r}{s}} F^J (x)^r \right\}^{\frac{s}{r}} \, dx.
\]

Now apply Lemma II above with the locally finite absolutely continuous measures \( \mu_x \) on \( \mathbb{R}^n \) for a.e. \( x \in \mathbb{R}^m \) to obtain

\[
\sum_{J \in \mathcal{D}^n} J_{\mu, \theta} (x)^{\frac{r}{s}} F^J (x)^r = \sum_{J \in \mathcal{D}^n} J_{\mu, \theta} (x)^{\frac{r}{s}} \left( \frac{1}{|J|^{\mu, \theta}} \int_J f (x, y) u_x (y) \, dy \right)^r
\]

\[
= \sum_{J \in \mathcal{D}^n} |J|^{\mu, \theta} \left( \frac{1}{|J|^{\mu, \theta}} \int_J f (x, y) u_x (y) \, dy \right)^r
\]

\[
\lesssim \left( \int_{\mathbb{R}^m} f (x, y)^s u_x (y) \, dy \right)^{\frac{r}{s}} = \left( \int_{\mathbb{R}^m} f (x, y)^s u (x, y) \, dy \right)^{\frac{r}{s}},
\]

uniformly for a.e. \( x \in \mathbb{R}^m \). Plugging this into the previous display gives

\[
\left\{ \sum_{J \in \mathcal{D}^n} \left( \int_{\mathbb{R}^m} F^J (x)^s J_{\mu, \theta} (x) \, dx \right)^{\frac{r}{s}} \right\}^{\frac{s}{r}} \lesssim \int_{\mathbb{R}^m} \left\{ \left( \int_{\mathbb{R}^m} f (x, y)^s u (x, y) \, dy \right)^{\frac{r}{s}} \right\}^{\frac{s}{r}} \, dx
\]

\[
= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f (x, y)^s u (x, y) \, dy \, dx = \|f\|_{L^r (\mu)}^r.
\]

Altogether then we have

\[
\sum_{R \in \mathcal{R}^{m,n}} |R|^{\mu, \theta} \frac{1}{|R|^{\mu, \theta}} \int_R f (x, y) u (x, y) \, dxdy
\]

\[
\lesssim \sum_{J \in \mathcal{D}^n} \left( \int_{\mathbb{R}^m} F^J (x)^s J_{\mu, \theta} (x) \, dx \right)^{\frac{r}{s}} \lesssim \|f\|_{L^r (\mu)}^r.
\]

2.1.1. **Product fractional integrals.** The Tanaka-Yabuta theorem [TakYa Theorem 1.1], as well as the variant in Theorem 2 above, uses an arbitrary nonnegative function \( K (R) \) defined on dyadic rectangles \( R \in \mathcal{R}^{m,n} \). If for \( 0 < \frac{\alpha}{m}, \frac{\beta}{n} < 1 \), we define

\[
K_{n, \alpha, \beta}^{m,n} (R) = K (I \times J) \equiv |I|^{\frac{\alpha}{m} - 1} |J|^{\frac{\beta}{n} - 1},
\]

for \( R = I \times J \in \mathcal{R}^{m,n} \), then in the special case \( K = K_{n, \alpha, \beta}^{m,n} \) we have the following pointwise estimate,

\[
\sum_{R \in \mathcal{R}^{m,n}} K_{n, \alpha, \beta}^{m,n} (R) 1_R (x, y) 1_R (u, v) = \sum_{I \times J \in \mathcal{R}^{m,n}} \{ K (I \times J) : x, u \in I \text{ and } y, v \in J \}
\]

\[
= \sum_{I \times J \in \mathcal{R}^{m,n}} \left| I \right|^{\frac{\alpha}{m} - 1} \left| J \right|^{\frac{\beta}{n} - 1} : x, u \in I \text{ and } y, v \in J \}
\]

\[
= \sum_{I \in \mathcal{D}^m} \left| I \right|^{\frac{\alpha}{m} - 1} : x, u \in I \} \times \sum_{J \in \mathcal{D}^n} \left| J \right|^{\frac{\beta}{n} - 1} : y, v \in J \}
\]

\[
\approx d(x, u)^{\frac{\alpha}{m} - 1} d(y, v)^{\frac{\beta}{n} - 1} \lesssim |x - u|^{\frac{\alpha}{m} - 1} |y - v|^{\frac{\beta}{n} - 1},
\]

where \( d_{dy} (x, u) \) denotes the *dyadic* distance between \( x \) and \( u \) in \( \mathbb{R}^m \), and \( d_{dy} (y, v) \) denotes the *dyadic* distance between \( y \) and \( v \) in \( \mathbb{R}^n \). Here the dyadic distance between two points \( p \) and \( q \) in \( \mathbb{R}^k \) is defined to be the side length of the smallest dyadic cube containing \( p \) and \( q \). Note that the dyadic distance is at least \( \frac{1}{\sqrt{k}} \) times the Euclidean distance since any dyadic cube \( Q \) containing \( x \) and \( y \) must satisfy

\[
\ell (Q) \geq \max_{1 \leq i \leq k} |x_i - y_i| \geq \frac{1}{\sqrt{k}} \sum_{i=1}^k |x_i - y_i|^2 = \frac{1}{\sqrt{k}} |x - y|.
\]
So in order to apply the next theorem to the product fractional integral operator with kernel $|x-u|^\frac{-\alpha}{m-1}|y-v|^\frac{-\beta}{n-1}$, it suffices to appeal to Stromberg's well-known $\frac{1}{3}$-trick for the dyadic grids $\{D_i^{(m)}\}_{i=1}^3$ and $\{D_j^{(n)}\}_{j=1}^3$, to obtain

$$\sum_{i=1}^{3^m} \sum_{j=1}^{3^n} \left[ \sum_{R \in I \times J \in D_i^{(m)} \times D_j^{(n)}} K(R) 1_R(x,y) 1_R(u,v) \right] \approx |x-u|^{\frac{-\alpha}{m-1}} |y-v|^{\frac{-\beta}{n-1}}. \tag{2.2}$$

Variants of the following lemma can be found many times over in the literature, too numerous to mention here. Let $\mathcal{P}^N$ denote the collection of all cubes in $\mathbb{R}^N$ with sides parallel to the coordinate axes.

**Lemma 2.** For $K(R)$ defined as in (2.1) we have (2.2).

**Proof.** For convenience we recall a variation on the $\frac{1}{3}$-trick given in Lemma 2.5 of [HyLaPe]. For a given dyadic grid $D \subset \mathcal{P}^N$ with side lengths in $\{2^m\}_{m \in \mathbb{Z}}$, partition the collection of tripled cubes $\{3I\}_{I \in D}$ into $3^N$ subcollections $\{S_u\}_{u=1}^{3^N}$, with the property that for each subcollection $S_u$ there exists a dyadic grid $D_u$ with side lengths in $\{2^m\}_{m \in \mathbb{Z}}$, such that $S_u \subset D_u$. With these grids $\{D_u\}_{u=1}^{3^N}$ fixed, we have the following sandwiching property. For each cube $P \in \mathcal{P}^N$ and each integer $j \in \mathbb{N}$, there is a choice of $u = u(P,j)$ with $1 \leq u \leq 3^n$ and a cube $I = I_{u(P,j)} \in D_u$ such that

$$\ell(I) \leq 18 \ell(P), \quad 3P \subset I, \quad 2^j P \subset \pi^{(j)}D_u I,$$

where $\pi^{(j)}D_u I$ denotes the $j^{th}$ grandparent of $I$ in the grid $D_u$.

Now fix $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$. For $x \in \mathbb{R}^m$, let $P(x,\ell)$ denote the cube centered at $x$ with side length $\ell \in \{2^k\}_{k \in \mathbb{Z}}$. Then with $R_{a,b} \equiv P(x,2^a) \times Q(y,2^b)$ for $a,b \in \mathbb{Z}$, we note that the right hand side of (2.2) is equivalent to

$$\sum_{a,b \in \mathbb{Z}} K(R_{a,b}(x,y)) 1_{R_{a,b}(x,y)}(x,y) 1_{R_{a,b}(x,y)}(u,v), \quad (u,v) \in \mathbb{R}^m \times \mathbb{R}^n. \tag{2.3}$$

The first two lines in (2.3) now prove (2.2), since for each rectangle $R_{a,b}(x,y) \equiv P(x,2^a) \times Q(y,2^b)$ there is $I \times J \in \bigcup_{i=1}^{3^m} \bigcup_{j=1}^{3^n} (D_i^{(m)} \times D_j^{(n)})$ such that

$$3R_{a,b}(x,y) \subset I \times J \subset 18R_{a,b}(x,y),$$

and moreover, by the definition of $K$ in (2.1), we then have $K(R_{a,b}(x,y)) \approx K(I \times J)$. We do not need the third line in (2.3) here. \hfill \square

**Corollary 1.** Let $1 < p < q < \infty$, $0 < \alpha < m$, $0 < \beta < n$, $\theta > 1$, and let $v$ and $w$ be absolutely continuous weights on $\mathbb{R}^m \times \mathbb{R}^n$. Then the product fractional integral $I_{\alpha,\beta}^{m,n}$ is bounded from $L^p(v^p)$ to $L^q(w^q)$ if the $\theta$-bump rectangular characteristic $K_{p,q,\theta}^{(\alpha,\beta)(m,n)}(v,w)$ is finite, where

$$K_{p,q,\theta}^{(\alpha,\beta)(m,n)}(v,w) \equiv \sup_{I \times J \in \mathbb{R}_m \times \mathbb{R}_n} \left| I \right|^{\frac{-\alpha}{m-1} + \frac{1}{p} + \frac{1}{\theta}} \left| J \right|^{\frac{-\beta}{n-1} + \frac{1}{q} + \frac{1}{\theta}} \left( \frac{1}{|I \times J|} \int_{I \times J} \int_{I \times J} v^{-\alpha'} w^{-\beta'} \right)^{\frac{1}{\theta}} \left( \frac{1}{|I \times J|} \int_{I \times J} \int_{I \times J} \right)^{\frac{1}{\theta}}.

**Remark 1.** The above proof of the Corollary, when restricted to the 1-parameter case, gives a short and elegant proof of Theorem 1.4 in [SaWi] in the special case $p < q$.

**2.2. Reverse doubling weights for bilinear embeddings.** Here is a slight improvement of the theorem of Tanaka and Yabuta [YuYa], valid for the product fractional integral kernel, as well as more general kernels $K$ satisfying property (2.1) below regarding expectations taken over partial grids $\mathcal{R}^{m,n} = D^m \times D^n$. Recall that $\mu$ is a product reverse doubling weight on $\mathbb{R}^m \times \mathbb{R}^n$ if (1.1) holds.

**Theorem 3.** Suppose $1 < p < q < \infty$. Let $\sigma$ and $\omega$ be product reverse doubling weights on $\mathbb{R}^m \times \mathbb{R}^n$, and let $K = K_{\alpha,\beta}^{m,n} : \mathcal{R}^{m,n} \to [0,\infty)$ be as in (2.1), or more generally satisfy the expectation inequality (2.2) below. Then the norm $\|K\|_{\mathcal{N}^{m,n}}$ of the positive bilinear inequality,

$$\sum_{R \in \mathcal{R}^{m,n}} K(R) \left( \int_R f d\sigma \right) \left( \int_R g d\omega \right) \leq \mathcal{N}_K(\sigma,\omega) \|f\|_{L^p(\sigma)} \|g\|_{L^q(\omega)}, \quad f, g \geq 0,$$
is finite for all partial grids \( R^{m,n} = D^m \times D^n \) if and only if
\[
\mathcal{A}_K (\sigma, \omega) \equiv \sup_{R \in \mathcal{P}^m \times \mathcal{P}^n} K(R) \left( |R|_{\sigma}^{\frac{1}{\nu}} |R|_{\omega}^{\frac{1}{\nu}} \right) < \infty, \quad \text{for all rectangles } R \in \mathcal{P}^m \times \mathcal{P}^n.
\]

Proof. We begin the proof with a brief review of the good/bad grid technology of Nazarov, Treil and Volberg. See [NTV2], [NTV4], or [Vol] for more detail. We restrict to dimension \( n = 1 \) for the moment. Let \( 0 < \varepsilon < 1 \) and \( r \in \mathbb{N} \) to be chosen later. Define \( J \) to be \( \varepsilon \)-good in an interval \( K \) if
\[
d(J, \text{skeleton of } K) > 2 |J|^\varepsilon |K|^{1-\varepsilon},
\]
where the skeleton \( \text{skeleton of } K \) consists of its two endpoints and its midpoint. Define \( D_{(r, \varepsilon)} \)-good to consist of those \( J \in D \) such that \( J \) is good in every superinterval \( K \in D \) that lies at least \( r \) levels above \( J \).

As the goodness parameters \( \varepsilon \) and \( r \) will eventually be fixed throughout the proof, we sometimes suppress the parameters, and simply write \( D_{\text{good}} \) in place of \( D_{(r, \varepsilon)} \)-good, and say "\( J \) is good" instead of "\( J \) is good in every superinterval \( K \in D \) that lies at least \( r \) levels above \( J \)." We also define \( D_{\text{bad}} \equiv D \setminus D_{\text{good}} \).

**Parameters of dyadic grids:** Here we recall a construction from [SaShUr10] that was in turn based on that of Hytönen in [Hyt2]. Momentarily fix a large positive integer \( M \in \mathbb{N} \), and consider the tiling of \( \mathbb{R} \) by the family of intervals \( D_M \equiv \{ I_M^\alpha \}_{\alpha \in \mathbb{Z}} \) having side length \( 2^{-M} \) and given by \( I_M^\alpha \equiv I_0^M + 2^{-M} \alpha \) where \( I_0^M = [0, 2^{-M}) \). A dyadic grid \( D \) built on \( D_M \) is to be a family of intervals \( D \) satisfying:

1. Each \( I \in D \) has side length \( 2^{-\ell} \) for some \( \ell \in \mathbb{Z} \) with \( \ell \leq M \), and \( I \) is a union of \( 2^{M-\ell} \) intervals from the tiling \( D_M \).
2. For \( \ell \leq M \), the collection \( D_\ell \) of intervals in \( D \) having side length \( 2^{-\ell} \) forms a pairwise disjoint decomposition of the space \( \mathbb{R} \).
3. Given \( I \in D_\ell \) and \( J \in D_j \) with \( j \leq i \leq M \), it is the case that either \( I \cap J = \emptyset \) or \( I \subset J \).

We now momentarily fix a negative integer \( N \in \mathbb{Z} \), and restrict the above grids to intervals of side length at most \( 2^{-N} \):
\[
D_N \equiv \{ I \in D : \text{side length of } I \text{ is at most } 2^{-N} \}.
\]
We refer to such grids \( D_N \) as a (truncated) dyadic grid \( D \) built on \( D_M \) of size \( 2^{-N} \). There are now two traditional methods of constructing probability measures on collections of such dyadic grids, namely parameterization by choice of parent, and parameterization by translation. We will only need the former parameterization here. For any
\[
\beta = \{ \beta_i \}_{i \in \mathbb{Z}_M} \in \omega_M^N \equiv \{ 0, 1 \}^{\mathbb{Z}_M},
\]
where \( \mathbb{Z}_M \equiv \{ \ell \in \mathbb{Z} : 0 \leq \ell \leq M \} \), define the dyadic grid \( D_\beta \) built on \( D_M \) of size \( 2^{-N} \) by
\[
D_\beta = \left\{ 2^{-\ell} \left( [0, 1) + k + \sum_{i : \ell+i \leq M} 2^{-i} \beta_i \right) \right\}_{\ell \leq N, k \in \mathbb{Z}}.
\]
Place the uniform probability measure \( \rho_M^N \) on the finite index space \( \omega_M^N = \{ 0, 1 \}^{\mathbb{Z}_M} \), namely that which charges each \( \beta \in \omega_M^N \) equally.

This construction may be thought of as being parameterized by scales - each component \( \beta_i \) in \( \beta = \{ \beta_i \}_{i \in \mathbb{Z}_M} \in \omega_M^N \) amounting to a choice of the two possible tilings at level \( i \) that respect the choice of tiling at the level below. For purposes of notation and clarity, we now suppress all reference to \( M \) and \( N \) in our families of grids, and use the notation \( \Omega \) instead of \( \omega_M^N \) for the index or parameter set, and then use \( P_\Omega \) and \( E_\Omega \) to denote probability and expectation with respect to families of grids. We will now instead proceed as if all grids considered are unrestricted. The careful reader can supply the modifications necessary to handle the assumptions made above on the grids \( D \) regarding \( M \) and \( N \).

Given a pair of grids \( D^m \) and \( D^n \) in \( \mathbb{R}^m \) and \( \mathbb{R}^n \) respectively, form the corresponding partial grid \( R^{m,n} = D^m \times D^n \) of rectangles. We say that a rectangle \( R = I \times J \in R^{m,n}_{\text{good}} \) (and say \( R \) is good) if both \( I \in D^m \) and \( J \in D^n_{\text{good}} \). Given a positive bilinear form
\[
B_{R^{m,n}} (f, g) \equiv \sum_{R \in R^{m,n}} K(R) \left( \int_R f \, d\sigma \right) \left( \int_R g \, d\omega \right), \quad f \in L^p (\sigma), \, g \in L^q (\omega),
\]
we follow the NTV idea and dominate $B_{\mathcal{D}^m \times \mathcal{D}^n}(f, g) = B_{\mathcal{D}^m \times \mathcal{D}^n}(f, g)$ as follows:

$$B_{\mathcal{D}^m \times \mathcal{D}^n}(f, g) \leq \left\{ \sum_{I \times J \in \mathcal{D}^m \times \mathcal{D}^n_{\text{good}}} + \sum_{I \times J \in \mathcal{D}^m \times \mathcal{D}^n_{\text{bad}}} + \sum_{I \times J \in \mathcal{D}^m_{\text{had}} \times \mathcal{D}^n} \right\} K(I \times J) \left( \int_{I \times J} f d\sigma \right) \left( \int_{I \times J} g d\omega \right) \equiv B_{\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}}(f, g) + B_{\mathcal{D}^m \times \mathcal{D}^n_{\text{bad}}}(f, g) + B_{\mathcal{D}^m_{\text{had}} \times \mathcal{D}^n}(f, g).$$

From the previous subsection we have that the positive bilinear form

$$\mathcal{I}(f, g) \equiv \int_{\mathbb{R}^m \times \mathbb{R}^n} I_{m,n}^{\alpha,\beta}(f \sigma) g \omega$$

satisfies

$$E_{\Omega \times \Omega} E_{\mathcal{D}^m \times \mathcal{D}^n}(f, g) \geq c B_{\mathcal{D}^m \times \mathcal{D}^n}(f, g), \quad \text{for all } \mathcal{D}^m \times \mathcal{D}^n \text{ and some } c > 0.$$ 

It then follows that the norm $\mathcal{N}_\mathcal{I}$ of the bilinear form $\mathcal{I}$ can be estimated using $\|f\|_{L^p(\sigma)} = \|g\|_{L^q(\omega)} = 1$ chosen so that $\mathcal{N}_\mathcal{I} = \mathcal{I}(f, g)$:

$$\mathcal{N}_\mathcal{I} = \mathcal{I}(f, g) = E_{\Omega \times \Omega} B_{\mathcal{D}^m \times \mathcal{D}^n}(f, g) \leq E_{\Omega \times \Omega} B_{\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}}(f, g) + E_{\Omega \times \Omega} B_{\mathcal{D}^m \times \mathcal{D}^n_{\text{bad}}}(f, g) + E_{\Omega \times \Omega} B_{\mathcal{D}^m_{\text{had}} \times \mathcal{D}^n}(f, g).$$

Now the conditional probability that a given cube $I$ is bad in a grid $\mathcal{D}^m$ that contains it is small, in fact (see e.g. [NTV2], [NTV3], [Vol] or [SaShUr] Subsubsection 3.1.1) we have

$$P_{\Omega} \{\mathcal{D}^m : I \text{ is bad in } \mathcal{D}^m | \text{ conditioned on } I \in \mathcal{D}^m \} \leq C 2^{-\varepsilon r}.$$ 

Thus we obtain

$$E_{\Omega \times \Omega} B_{\mathcal{D}^m_{\text{bad}} \times \mathcal{D}^n}(f, g) \leq C 2^{-\varepsilon r} \mathcal{N}_\mathcal{I} \|f\|_{L^p(\sigma)} \|g\|_{L^q(\omega)} = C 2^{-\varepsilon r} \mathcal{N}_\mathcal{I},$$

and hence

$$\mathcal{N}_\mathcal{I} \leq E_{\Omega \times \Omega} B_{\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}}(f, g) + C 2^{-\varepsilon r} \mathcal{N}_\mathcal{I},$$

which gives

$$\mathcal{N}_\mathcal{I} \leq \frac{1}{1 - 2C 2^{-\varepsilon r}} E_{\Omega \times \Omega} B_{\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}}(f, g)$$

if $\varepsilon r$ is chosen sufficiently small.

Thus we see that in order to prove Theorem 3, we need only consider the ‘good’ bilinear form $B_{\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}}(f, g)$ and estimate it independently of the partial grid of good rectangles $\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}$. Then using arguments as in [TaYa] or above, the proof of Theorem 3 is reduced to the following Carleson embedding for ‘good’ rectangles.

**Carleson embedding:** Suppose that $1 < s < r < \infty$ and that $\mu$ is a product reverse doubling measure on $\mathbb{R}^m \times \mathbb{R}^n$. Then we have

$$\left\{ \sum_{R \in \mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}} |R|^s \left( \frac{1}{|R|^s} \int_R f d\mu \right)^r \right\}^{\frac{1}{r}} \leq C_{s, r} \|f\|_{L^r(\mu)}, \quad f \geq 0,$$

where $C_{s, r}$ depends only on $s, r$, the reverse doubling constants for $\mu$, and the goodness parameters $\varepsilon, r$. In particular, $C_{s, r}$ is independent of the partial grid $\mathcal{D}^m_{\text{good}} \times \mathcal{D}^n_{\text{good}}$. Continuing to follow the iteration argument of Tanaka and Yabuta as in [TaYa] or above, further reduces matters to proving the following Carleson condition on cubes for a reverse doubling measure $\mu$ on $\mathbb{R}^N$ with exponent $\eta > 0$, and a power $\rho > 1$:

$$\sum_{Q \in \mathcal{D}^N_{\text{good}} : Q \subset P} |Q|^{\rho} \leq C_{N, r, \varepsilon, \rho} P^{\rho}_{\mu, \theta}.$$ 

Indeed, the reader can easily verify that the arguments work just as well for the subgrids $\mathcal{D}^m_{\text{good}}$ and $\mathcal{D}^n_{\text{good}}$ in place of the grids $\mathcal{D}^m$ and $\mathcal{D}^n$.

It is now at this point that the goodness of the cubes $Q$ plays a crucial role in conjunction with the reverse doubling property. To see [215], recall the goodness parameters $0 < \varepsilon < 1$ and $r \in \mathbb{N}$ and observe that if $Q$ is a good cube contained in $P$ then
either $\ell(Q) \geq \ell(P) - r$ and we can use the trivial estimate $|Q|_\mu^p \leq |P|_\mu^p$,
or $\ell(Q) < \ell(P) - r$ in which case $\text{dist}(Q, \partial P) \geq 2\ell(Q)^\varepsilon \ell(P)^{1-\varepsilon}$.
In this latter case we note that if $\ell(Q) = 2^{-k}\ell(P)$ then
\[
2^{k(1-\varepsilon)}Q = \left(\frac{\ell(P)}{\ell(Q)}\right)^{1-\varepsilon}Q \supseteq \frac{2\ell(Q)^\varepsilon \ell(P)^{1-\varepsilon}}{\ell(Q)}Q \subset \text{dist}(Q, \partial P)Q \subset P
\]
and so by reverse doubling we have
\[
|Q|_\mu \leq C 2^{-\eta k(1-\varepsilon)} \left(\frac{\ell(P)}{\ell(Q)}\right)^{1-\varepsilon}Q \leq C 2^{-\eta(1-\varepsilon)k} |P|_\mu^p.
\]
Thus we can estimate
\[
\sum_{Q \in \mathcal{D}_{\text{good}}^N : Q \subset P} |Q|_\mu^p = \sum_{k=0}^r 2^{N r} |P|_\mu^p + \sum_{k=r+1}^{\infty} \sum_{Q \in \mathcal{D}_{\text{good}}^N : \ell(Q) = 2^{-k}\ell(P)} |Q|_\mu^p \leq C_{N,r} |P|_\mu^p + \sum_{k=r+1}^{\infty} \left(C 2^{-\eta(1-\varepsilon)(\rho-1)k}\right) |P|_\mu^p = C_{N,r,\varepsilon,\rho} |P|_\mu^p.
\]
This completes the proof of (2.6), and hence also that of Theorem 3.

2.3. Concluding remarks. In the case of kernels $K = K_{a,b}^{m,n}$ given by (2.1), or more generally that satisfy (2.4), one can assume for each weight separately, either rectangle reverse doubling, or a half $\theta$-bump condition, in order to obtain norm boundedness. For example, the following hybrid theorem holds.

**Theorem 4.** Suppose $1 < p < q < \infty$. Let $\sigma$ be a product reverse doubling weight on $\mathbb{R}^n$, let $d\omega(x) = w(x)^q dx$ be absolutely continuous with respect to Lebesgue measure, and let $K = K_{a,b}^{m,n} : \mathbb{R}^{m,n} \rightarrow [0,\infty)$ be as in (2.7), or more generally satisfy (2.4). Then the norm $\|K(\sigma,\omega)\|$ of the positive bilinear inequality,
\[
\sum_{R \in \mathcal{R}^{m,n}} K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right) \leq N_{K(\sigma,\omega)} \|f\|_{L^p(\sigma)} \|g\|_{L^q(\omega)},
\]
is finite for all products of grids $\mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n$ if the half $\theta$-bump rectangle characteristic $\mathbb{A}^\omega_{K,\theta}(\sigma,\omega)$ is finite, where
\[
\mathbb{A}^\omega_{K,\theta}(\sigma,\omega) = \sup_{R \in \mathcal{R}^{m,n}} K(R) \left(\int_R v^{-\theta} d\sigma\right)^{1/p} \left[|R|_{\sigma}^{1/p} \left(\int_R w^{q\theta} d\omega\right)^{1/q}\right] = \sup_{R \in \mathcal{R}^{m,n}} K(R) |R|_{\omega,\theta}^{1/p} |R|_{\sigma}^{1/p}.
\]
The proof is an easy exercise in combining the proofs of Theorems 2 and 3 above.

3. Appendix

We say that a weight $\mu$ on the real line is **strongly** reverse doubling if there is $\beta < 1$ such that
\[
|I_{\text{left}}|_\mu, |I_{\text{right}}|_\mu \leq \beta |I|_\mu
\]
for all intervals $I$, where if $I = [a,b)$, then $I_{\text{left}} = [a, a + b/2)$ and $I_{\text{right}} = (a + b/2, b)$ are the left and right halves of $I$ respectively. A strongly reverse doubling weight on $\mathbb{R}$ is a doubling weight on $\mathbb{R}$, since if we choose $N$ so large that $\beta^N < 1/4$, then for $I = [a,b)$, we have
\[
\left|\left[a, a + \frac{b-a}{2^N}\right]\right|_\mu, \left|\left[b - \frac{b-a}{2^N}, b\right]\right|_\mu \leq \beta^N |I|_\mu < 1/4 |I|_\mu.
\]
Hence

\[ |a + b - a|/2^N, b - b - a|/2^N \rangle \mu = |(a, b)\rangle_\mu - \left| \left[ a, a + b - a \right] /2^N \rangle_\mu - \left| \left[ b - b - a, b \right] /2^N \rangle_\mu \right. \geq \left. \left( 1 - \frac{1}{4} \right) |I|_\mu = \frac{1}{2} |I|_\mu \right. , \]

where the length of the interval \( \left[ a + b - a, b - b - a \right] /2^N \rangle \) is \( 2^{N-1} - 1 \ell(I) \). Thus with \( \gamma = 2^{N-1} > 1 \), we have for every interval \( K \),

\[ |\gamma K|_\mu \leq 2 |K|_\mu, \]

hence \( |2K|_\mu \leq 2^M |K|_\mu \) if \( \gamma^M \geq 2 \), which shows that \( \mu \) is doubling. Similarly we see that a strongly rectangle reverse doubling weight on \( \mathbb{R}^N \) is a rectangle doubling weight on \( \mathbb{R}^N \). Here \( \mu \) is strongly rectangle reverse doubling if there is \( \beta < 1 \) such that

\[ |I^1 \times \ldots \times I^k_{\text{left}} \times \ldots \times I^N|_\mu, |I^1 \times \ldots \times I^k_{\text{right}} \times \ldots \times I^N|_\mu \leq \beta |I^1 \times \ldots \times I^N|_\mu \]

for all rectangles \( I^1 \times \ldots \times I^N \) and \( 1 \leq k \leq N \), and \( \mu \) is rectangle doubling if there is \( C > 0 \) such that

\[ |(2I^1) \times \ldots \times (2I^N)|_\mu \leq C |I^1 \times \ldots \times I^N|_\mu \]

for all rectangles \( I^1 \times \ldots \times I^N \).

Example 1. Suppose that \( \mu \) is a doubling weight on \( \mathbb{R}^N \). Then \( dv(x) \equiv 1_{[0, \infty)^N}(x) \mu(x) \) is a reverse doubling weight on \( \mathbb{R}^N \) that is not a doubling weight on \( \mathbb{R}^N \).

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