Explicit Round Fold Maps on Some Fundamental Manifolds

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Abstract. (Stable) fold maps are fundamental tools in a generalization of the theory of Morse functions on smooth manifolds and its application to studies of geometry of smooth manifolds. Round fold maps were introduced by the author 2013–2014 as stable fold maps with singular value sets, defined as the set consisting of all the singular values, of concentric spheres; for example, some special generic maps on homotopy spheres are regarded as round fold maps whose singular value sets are connected. The author studied algebraic topological invariants such as homology and homotopy groups of manifolds admitting round fold maps and more geometric information of manifolds, the homeomorphism and diffeomorphism types of manifolds admitting such maps having appropriate differential topological structures.

Moreover, explicit round fold maps into the Euclidean space of dimension larger than 1 have been constructed on some fundamental manifolds such as manifolds having the structures of bundles over the standard sphere of dimension equal to the Euclidean space whose fibers are closed smooth manifolds and manifolds of dimension not smaller than twice the dimension of the Euclidean space represented as the connected sum of manifolds having the structures of bundles over the standard sphere of dimension equal to the Euclidean space whose fibers are diffeomorphic to another standard sphere by the author. The author have constructed these maps by decomposing the manifolds into submanifolds of dimension equal to that of the original manifolds, constructing local maps on these submanifolds and gluing them together compatibly, which is based on fundamental technique of differential topology of manifolds but had not been used except in some simple cases such as special generic maps and generic maps including stable fold maps between low dimensional manifolds.

Succeeding in constructions of explicit fold maps will help us to study geometry of manifolds by using the theory of fold maps with good geometric properties. For this purpose, in this paper, we construct more new explicit fold maps on some fundamental manifolds including the manifolds before by using extended methods of ones used in the constructions before.

1. Introduction and Terminologies

Fold maps are fundamental tools in a generalization of the theory of Morse functions and its application to studies of geometry of manifolds, which is defined as a smooth map with singular points being of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)$$

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for two positive integers $m \geq n$ and an integer $0 \leq i \leq \frac{m-n+1}{2}$; note that $i$ is determined uniquely for each singular point (we call $i$ the index of the singular point). Studies of such maps were started by Whitney ([28]) and Thom ([27]) in the 1950’s. A Morse function is regarded as a fold map in the case where $n = 1$ holds and for a fold map from a closed smooth manifold of dimension $m$ into a smooth manifold of dimension $n$ without boundary with $m \geq n \geq 1$, the followings hold.

1. The singular set, which is defined as the set of all the singular points, and the set of all the singular points of index $i$ are closed smooth submanifolds of dimension $n - 1$ of the source manifold.
2. The restriction map to the singular set is a smooth immersion of codimension 1.

Although Morse functions exist densely on all smooth manifolds, there exist families of (closed) smooth manifolds admitting no fold maps into the Euclidean space of a dimension. For example, a closed smooth manifold of dimension $k \geq 2$ admits a fold map into the plane if and only if the Euler number of the manifold is even. In [4], [5] and other proceeding papers, more general existence problems for fold maps were studied. As a simplest example, it has been known that smooth homotopy spheres of dimension $k$ (not necessarily diffeomorphic to the standard sphere $S^k$) admits a fold map into the $k'$-dimensional Euclidean space $\mathbb{R}^{k'}$ for any integer $1 \leq k' \leq k$.

Since around the 1990s, fold maps with additional conditions have been actively studied. For example, in [1], [6], [21], [22], [23] and [25], special generic maps, which are defined as fold maps whose singular points are of the form

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{n-1}, \sum_{k=n}^{m} x_k^2)$$

for two positive integers $m \geq n$, were studied. In [25], Sakuma studied simple fold maps, which are defined as fold maps such that fibers of singular values do not have any connected component with more than one singular points (see also [20]). For example, special generic maps are simple. In [16], Kobayashi and Saeki investigated topological properties of stable maps into the plane including fold maps which are stable (for stable maps, see [7] for example). In [24], Saeki and Suzuoka found good topological properties of manifolds admitting stable maps whose regular fibers, which are defined as the fibers of regular values, are disjoint unions of spheres.

Later, in [13], round fold maps, which will be mainly studied in this paper, were introduced. A round fold map is defined as a fold map satisfying the followings.

1. The singular set is a disjoint union of standard spheres.
2. The restriction map to the singular set is an embedding.
3. The singular value set, which is defined as the image of the singular set, is a disjoint union of spheres embedded concentrically.

For example, some special generic maps on spheres are round fold maps whose singular sets are connected. Any standard sphere whose dimension is $m > 1$ admits such a map into $\mathbb{R}^n$ with $m \geq n \geq 2$ and any smooth homotopy sphere whose dimension is larger than 1 and not 4 admits such a map into the plane. See also [21] and Example 1 of the present paper.

In [13] and [14], homology groups and homotopy groups of manifolds admitting round fold maps are studied. In [12] and [14], under appropriate conditions, the
homeomorphism and diffeomorphism types of manifolds admitting round fold maps were studied under appropriate conditions. Moreover, explicit examples of (round) fold maps on these manifolds have been constructed in the proofs of the results. By the way, it is a fundamental and difficult problem in the theory of fold maps to construct explicit fold maps on explicit manifolds, although existence problems for fold maps into Euclidean spaces on these manifolds have been solved in the studies of general existence problems explained before. In the present paper, we obtain more new examples of round fold maps with the homeomorphism and diffeomorphism types of the source manifolds in new manners.

This paper is organized as follows.

In section 2, we recall round fold maps and some terminologies on round fold maps such as axes and proper cores. We also recall a $C^\infty$ trivial round fold map, which is defined as a round fold map whose differential topological structure satisfies a kind of triviality.

In section 3, we study the homeomorphism and diffeomorphism types of manifolds admitting round fold maps. Mainly, we give new examples of round fold maps with the diffeomorphism types of their source manifolds.

First, under appropriate conditions, we construct a new round fold map on each connected sum of two closed and connected $m$-dimensional manifolds admitting round fold maps into $\mathbb{R}^n$ $(m \geq n \geq 2)$ with $m > n$ in Proposition 1, which has been shown in [12], [13] and [15] under the assumptions that $m \geq 2n$ holds and that the manifolds are oriented. As an application, we construct round fold maps into $\mathbb{R}^4$ on 7-dimensional smooth homotopy spheres by applying this construction with known facts on 7-dimensional homotopy spheres of [3], [11] and [17] (Theorem 1). Second, we introduce another easy application of Proposition 1 as Theorem 2. Third, by applying our Proposition 1, on a manifold represented as a connected sum of manifolds having the structures of bundles over the standard $n$-dimensional sphere $S^n$ with $m - n \geq 1$ and $n \geq 2$ and a manifold admitting a round fold map satisfying appropriate differential topological conditions, we construct a new round fold map into $\mathbb{R}^n$ (Theorem 3). As an application of this theorem, for example, on a manifold represented as a connected sum of a finite number of $m$-dimensional smooth manifolds having the structures of bundles over the standard $n$-dimensional sphere $S^n$ whose fibers are diffeomorphic to the $(m-n)$-dimensional standard sphere $S^{m-n}$ with $m - n \geq 1$ and $n \geq 2$ assumed, we construct a round fold map into $\mathbb{R}^n$ (Theorem 5 (1)), which has been obtained also in the three listed papers by the author under the assumption that $m \geq 2n$ holds. Last, under appropriate conditions a bit different from the previous conditions, we construct a new round fold map on a connected sum of two closed and connected manifolds admitting round fold maps again in Proposition 2 and show Theorem 6, which states that a manifold represented as a connected sum of a smooth homotopy sphere of dimension 7 and a 7-dimensional smooth manifold admitting a round fold map into $\mathbb{R}^4$ admits another round fold map into $\mathbb{R}^4$.

In the last section 4, under appropriate assumptions, we show that for two closed and connected $m$-dimensional manifolds admitting $C^\infty$ trivial round fold map into $\mathbb{R}^n$ with $m > n \geq 2$ assumed, we can construct a $C^\infty$ trivial round fold map from a manifold represented as a connected sum of the manifolds into $\mathbb{R}^n$.

We note about terminologies on spaces and maps in this paper.
On a topological space $X$, we denote the identity map on $X$ by $\text{id}_X$. If the space $X$ is a topological manifold, then we denote the interior of $X$ by $\text{Int}X$, the closure of $X$ by $\overline{X}$ and the boundary of $X$ by $\partial X$. For two topological spaces $X_1$ and $X_2$, we denote their disjoint union by $X_1 \sqcup X_2$. For a map $c : X_1 \to X_2$ and subspaces $Y_1 \subset X_1$ and $Y_2 \subset X_2$ such that $c(Y_1) \subset Y_2$ holds, $c|_{Y_1} : Y_1 \to Y_2$ is the restriction map of $c$ to $Y_1$. For a homeomorphism $\phi : Y_2 \to Y_1$ in the same situation, by gluing $X_1$ and $X_2$ together by $\phi$, we obtain a new topological space and denote the space by $X_1 \sqcup_{\phi} X_2$. We often omit $\phi$ of $X_1 \sqcup_{\phi} X_2$ and denote it by $X_1 \sqcup X_2$ in case we consider a natural identification. As before, for a smooth map $c$, we define the singular set of $c$ by the set consisting of all the singular points of $c$ and denote this set by $S(c)$. In addition, as before, for the map $c$, we call the set $c(S(c))$ the singular value set of $c$ and a regular fiber of $c$ means the fiber of a point of a regular value of the map.

Throughout this paper, we assume that $M$ is a closed smooth manifold of dimension $m$, that $N$ is a smooth manifold of dimension $n$ without boundary, that $f : M \to N$ is a smooth map and that $m \geq n \geq 1$ holds. In the proceeding sections, manifolds, maps between manifolds and (closed) tubular neighborhoods of submanifolds of manifolds are of class $C^\infty$ and in addition, for bundles whose fibers are $(C^\infty)$ manifolds, the structure groups consist of $(C^\infty)$ diffeomorphisms on the fibers unless otherwise stated. Moreover, for a manifold $X$, an $X$-bundle means a bundle whose fiber is diffeomorphic to $X$.

This paper is partially based on the doctoral dissertation of the author [13]. For example, Theorems 1, 3, 4 and 5 are also in the doctoral dissertation.

2. Preliminaries on round fold maps

In this section, we review round fold maps. See also [13] and [14] for example. First we recall $C^\infty$ equivalence. For two $C^\infty$ maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, we say that they are $C^\infty$ equivalent if there exist diffeomorphisms $\phi_X : X_1 \to X_2$ and $\phi_Y : Y_1 \to Y_2$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\phi_X} & X_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
Y_1 & \xrightarrow{\phi_Y} & Y_2
\end{array}
$$

For $C^\infty$ equivalence, see also [7] for example.

**Definition 1** (round fold map([13])). $f : M \to \mathbb{R}^n$ ($n \geq 2$) is said to be a round fold map if $f$ is $C^\infty$ equivalent to a fold map $f_0 : M_0 \to \mathbb{R}^n$ on a closed manifold $M_0$ such that the followings hold.

1. The singular set $S(f_0)$ is a disjoint union of $(n-1)$-dimensional standard spheres and consists of $l \in \mathbb{N}$ connected components.
2. The restriction map $f_0|_{S(f_0)}$ is an embedding.
3. Let $D^n_r := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^{n} x_k^2 \leq r\}$. Then, $f_0(S(f_0)) = \bigcup_{k=1}^{l} \partial D^n_k$ holds.

We call $f_0$ a normal form of $f$. We call a ray $L$ from $0 \in \mathbb{R}^n$ an axis of $f_0$ and $D^n_{\frac{1}{2}}$ the proper core of $f_0$. Suppose that for a round fold map $f$, its normal form $f_0$ and diffeomorphisms $\Phi : M \to M_0$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, the relation $\phi \circ f = f_0 \circ \Phi$
holds. Then, for an axis $L$ of $f_0$, we also call $\phi^{-1}(L)$ an axis of $f$ and for the proper core $D^n_{\frac{1}{2}}$ of $f_0$, we also call $\phi^{-1}(D^n_{\frac{1}{2}})$ a proper core of $f$.

For a round fold map $f : M \to \mathbb{R}^n$ and for any connected component $C$ of the singular value set of $f(S(f))$, there exists a small smooth closed tubular neighborhood $N(C)$ regarded as a product bundle $C \times [-1, 1]$ over $C$ such that the composition of the restriction map to the set $f^{-1}(N(C))$ of $f$ and the projection onto $C$ is a submersion and gives $f^{-1}(N(C))$ the structure of a bundle over $C$ whose fiber is a compact manifold. Especially, if $C$ is the image of a connected component of the singular set consisting of points of index 0, then the resulting bundle is a bundle whose fiber is the $(m - n + 1)$-dimensional standard closed disc $D^{m-n+1}$ and whose structure group consists of linear transformations. We call a bundle whose fiber is a standard disc (sphere) and whose structure group consists of linear transformations a linear bundle.

Let $f$ be a normal form of a round fold map and $P_1 := D^n_{\frac{1}{2}}$. We set $E := f^{-1}(P_1)$ and $E' := M - f^{-1}(\text{Int } P_1)$. We set $F := f^{-1}(p)$ for $p \in \partial P_1$. We put $P_2 := \mathbb{R}^n - \text{Int } P_1$. Let $f_1 := f|_{E} : E \to P_1$ if $F$ is non-empty and let $f_2 := f|_{E'} : E' \to P_2$.

We can give $E'$ the structure of a bundle over $\partial P_2$ as follows. Since for $\pi_P(x) := \frac{1}{2}\frac{1}{|p|}(x \in P_2)$, $\pi_P \circ f|_{E'}$ is a proper submersion, this map gives $E'$ the structure of a $f^{-1}(L)$-bundle over $\partial P_2$ (apply Ehresmann’s fibration theorem [2]). We call this bundle the surrounding bundle of $f$. Note that the structure group of this bundle is regarded as the group of diffeomorphisms on $f^{-1}(L)$ preserving the function $f|_{f^{-1}(L)} : f^{-1}(L) \to L(\subset \mathbb{R})$, which is naturally regarded as a Morse function.

For a round fold map $f$ which is not a normal form, we can consider similar objects. We call a bundle naturally corresponding to the surrounding bundle of a normal form of $f$ a surrounding bundle of $f$.

We can define the following condition for a round fold map.

**Definition 2.** Let $f : M \to \mathbb{R}^n$ ($n \geq 2$) be a round fold map. If a surrounding bundle of $f$ as above is a trivial bundle, then $f$ is said to be $C^\infty$ trivial.

We introduce a fundamental example of round fold maps.

**Example 1.** Let $m, n$ be integers such that $m \geq n \geq 2$ holds. Then, by a fundamental discussion of [21], a round fold map $f : S^m \to \mathbb{R}^n$ whose singular set is connected exists. The map is special generic. Furthermore, any homotopy sphere of dimension $m > 1$ admits a map into the plane as above unless $m = 4$ according to a discussion in section 5 of [21]. Round fold maps here are $C^\infty$ trivial (see also Example 3 (1) of [13]).

Let $m \geq 4$ and let $n$ be an integer satisfying $m - n = 1, 2, 3$. In section 4 of [21] and [22], it is shown that if a homotopy sphere of dimension $m$ admits a special generic map into $\mathbb{R}^n$, then the sphere is diffeomorphic to $S^m$. Thus, on a homotopy sphere of dimension $m$, a round fold map into $\mathbb{R}^n$ whose singular set is connected exists, then the homotopy sphere is diffeomorphic to $S^m$. 


We easily obtain a lot of round fold maps which are $C^\infty$ trivial by using the following method.
Let $\bar{M}$ be a compact manifold with non-empty boundary $\partial \bar{M}$. Let $a \in \mathbb{R}$. Then, there exists a Morse function $\bar{f} : \bar{M} \to [a, +\infty)$ satisfying the followings.

1. $a$ is the minimum of $\bar{f}$ and $\bar{f}^{-1}(a) = \partial \bar{M}$ holds.
2. All the singular points of $\bar{f} : \bar{M} \to [a, +\infty)$ are in $\bar{M} - \partial \bar{M}$ and at distinct singular points, the values are always distinct.

Let $\Phi : \partial(\bar{M} \times \partial(\mathbb{R}^n - \text{Int}D^n)) \to \partial(\partial \bar{M} \times D^n)$ and $\phi : \partial(\mathbb{R}^n - \text{Int}D^n) \to \partial D^n$ be diffeomorphisms. Let $p_1 : \partial \bar{M} \times \partial(\mathbb{R}^n - \text{Int}D^n) \to \partial(\mathbb{R}^n - \text{Int}D^n)$ and $p_2 : \partial \bar{M} \times D^n \to \partial D^n$ be the canonical projections. Suppose that the following diagram commutes.

$$
\begin{array}{ccc}
\partial \bar{M} \times \partial(\mathbb{R}^n - \text{Int}D^n) & \xrightarrow{\Phi} & \partial \bar{M} \times D^n \\
\downarrow{p_1} & & \downarrow{p_2} \\
\partial(\mathbb{R}^n - \text{Int}D^n) & \xrightarrow{\phi} & \partial D^n
\end{array}
$$

By using the diffeomorphism $\Phi$, we construct $M := (\partial \bar{M} \times D^n) \cup_{\phi}(\bar{M} \times \partial(\mathbb{R}^n - \text{Int}D^n))$. Let $p : \partial M \times D^n \to D^n$ be the canonical projection. Then gluing the two maps $p$ and $\bar{f} \times \text{id}_{S^{n-1}}$ together by using the two diffeomorphisms $\Phi$ and $\phi$, we obtain a round fold map $f : M \to \mathbb{R}^n$.

If $M$ is a compact manifold without boundary, then there exists a Morse function $\bar{f} : \bar{M} \to [a, +\infty)$ such that $\bar{f}(\bar{M}) \subset (a, +\infty)$ and that at distinct singular points, the values are always distinct. We are enough to consider $\bar{f} \times \text{id}_{S^{n-1}}$ and embed $(a, +\infty) \times S^{n-1}$ into $\mathbb{R}^n$ to construct a round fold map whose source manifold is $\bar{M} \times S^{n-1}$.

We call this construction of a round fold map a **trivial spinning construction**.

### 3. New examples of round fold maps

In this section, we give new examples of round fold maps with their source manifolds.

In this section and the next section, we define a **trivial embedding** of the standard sphere $S^p$ ($p \geq 1$) into a smooth manifold $X$ of dimension $q > p$ without boundary as a smooth embedding smoothly isotopic to an embedding into a smoothly embedded open disc $\text{Int}D^q \subset X$ which is unknot in the $C^\infty$ category.

**Proposition 1.** Let $M_1$ and $M_2$ be closed and connected $m$-dimensional manifolds. Assume that two round fold maps $f_1 : M_1 \to \mathbb{R}^n$ and $f_2 : M_2 \to \mathbb{R}^n$ ($m > n \geq 2$) exist. We also assume the followings.

1. The fiber of a point in a proper core of $f_1$ has a connected component diffeomorphic to $S^{m-n}$.

2. Let $C$ be the connected component of $\partial f_2(M_2)$ bounding the unbounded connected component of $\mathbb{R}^n - \text{Int}f_2(M_2)$. Then, the embedding of $f_2^{-1}(C)$ into $M_2$ is a trivial embedding into $M_2$.

Then, on each connected sum $M$ of the manifolds $M_1$ and $M_2$, there exists a round fold map $f : M \to \mathbb{R}^n$ such that the fiber of a point in a proper core of $f$ consist of all the connected components of the fiber of point in a proper core of $f_1$ except...
one diffeomorphic to $S^{m-n}$ mentioned in the first condition and all the connected components of the fiber of a point in a proper core of $f_2$.

Proof. This proposition is also shown in [12], [13] and [15] under the assumptions that $m \geq 2n$ holds, that the two manifolds $M_1$ and $M_2$ are oriented and that the embedding $f_2^{-1}(C) \subset M_2$ is null-homotopic (and trivial as a result). We prove this proposition as the review of these proofs. Let $P_1$ be a proper core of $f_1$ and $P_2$ be a small closed tubular neighborhood of the connected component $C$ of $f_2(S(f_2))$. Let $V_1$ be a connected component of $f_1^{-1}(P_1)$ such that $f_1|_{V_1} : V_1 \to P_1$ gives the structure of a trivial $S^{m-n}$-bundle over $P_1$ and $V_2 := f_2^{-1}(P_2)$. $V_2$ is a closed tubular neighborhood of $f_2^{-1}(C) \subset M_2$ and $V_2$ has the structure of a trivial linear $D^{m-n+1}$-bundle over $C$ by the assumptions that $C$ is the connected component of $\partial f_2(M_2)$ bounding the unbounded connected component of $\mathbb{R}^n - \text{Int} f_2(M_2)$ and the image of a connected component of the singular set consisting of singular points of index 0 and that the embedding $f_2^{-1}(C) \subset M_2$ is a trivial embedding into $M_2$. Note also that $f_2|_{\partial V_2}$ gives the structure of a subbundle of the bundle.

For any diffeomorphism $\Psi : \partial D^m \to \partial D^m$ extending to a diffeomorphism on $D^m$ or from $M_2 - (M_2 - D^m)$ onto $M_1 - (M_1 - D^m)$, we may regard that there exists a diffeomorphism $\Phi : \partial V_2 \to \partial V_1$ regarded as a bundle isomorphism between the two trivial $S^{m-n+1}$-bundles over the $(n-1)$-dimensional standard spheres inducing a diffeomorphism between the base spaces and that the following relation holds, where for two smooth manifolds $X_1$ and $X_2$, $X_1 \cong X_2$ means that $X_1$ and $X_2$ are diffeomorphic.

\[
(M_1 - \text{Int} V_1) \bigcup_{\Phi} (M_2 - \text{Int} V_2) \\
\cong (M_1 - \text{Int} V_1) \bigcup_{\Phi} ([D^m - \text{Int} V_2] \bigcup (M_2 - \text{Int} D^m)) \\
\cong (M_1 - \text{Int} V_1) \bigcup_{\Phi} ([S^m - (\text{Int} V_2 \cup \text{Int} D^m)] \bigcup \Phi(M_2 - \text{Int} D^m)) \\
\cong (M_1 - \text{Int} D^m) \bigcup_{\Phi} (M_2 - \text{Int} D^m)
\]

This means that the resulting manifold is a connected sum $M$ of $M_1$ and $M_2$ and that $M$ admits a round fold map $f : M \to \mathbb{R}^n$. More precisely, $f$ is obtained by gluing the two maps $f_1|_{M_1 - \text{Int} V_1}$ and $f_2|_{M_2 - \text{Int} V_2}$. We also note that we can realize each connected sum of the manifolds $M_1$ and $M_2$ as the resulting manifold $M$ and that we obtain a round fold map $f : M \to \mathbb{R}^n$ satisfying the assumption. □

Corollary 1. Let $M_1$ be a closed and connected manifold of dimension $m$ and $M_2$ be a homotopy sphere of dimension $m$. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ ($n \geq 2$) such that the fiber of a point in a proper core of $f_1$ has a connected component diffeomorphic to $S^{m-n}$ and a round fold map $f_2 : M_2 \to \mathbb{R}^n$. We also assume that $3n < 2m$ holds.

Then, on each connected sum $M$ of the manifolds $M_1$ and $M_2$, there exists a round fold map $f : M \to \mathbb{R}^n$ such that the fiber of a point in a proper core of $f$ consists of all the connected components of the fiber of point in a proper core of $f_1$ except one diffeomorphic to $S^{m-n}$ mentioned in the first condition and all the connected components of the fiber of a point in a proper core of $f_2$. 
Since the inequality $3n = 3\{(n - 1) + 1\} < 2m$ holds, from the theory of [8] or [9], it follows that two embeddings of $S^{m-1}$ into $M_2$ are always smoothly isotopic. We may apply Proposition 1 to complete the proof. □

We have the following corollary.

**Corollary 2.** Let $m, n \in \mathbb{N}$, $n \geq 2$ and $3n < 2m$. If a homotopy sphere $\Sigma$ of dimension $m$ is represented as a connected sum of a finite number of homotopy spheres having the structures of $S^{m-n}$-bundles over $S^n$, then there exists a round fold map $f : \Sigma \to \mathbb{R}^n$ such that regular fibers are disjoint unions of finite copies of $S^{m-n}$ and that the number of connected components of the singular set and the number of connected components of the fiber of a point in a proper core agree.

**Proof.** In the situation of Corollary 1, we consider round fold maps from $m$-dimensional homotopy spheres having the structures of $S^{m-n}$-bundles over $S^n$ as presented in Example 2 later. The singular set of each map has 2 connected components and the fiber of a point in a proper core of each map is a disjoint union of two copies of $S^{m-n}$. From Proposition 1, we easily obtain a desired round fold map $f : \Sigma \to \mathbb{R}^n$. □

It is well-known that if an $m$-dimensional sphere has the structure of a linear bundle over an $n$-dimensional sphere whose fiber is an $(m - n)$-dimensional sphere, then $(m, n) = (3, 2), (7, 4), (15, 8)$ must hold. We note that $3n < 2m$ holds for $(m, n) = (7, 4)$ and $(m, n) = (15, 8)$.

It is also known that $S^3$, $S^7$ and $S^{15}$ have the structures of linear bundles over $S^2$, $S^4$ and $S^8$ whose fibers are $S^1$, $S^3$ and $S^7$, respectively. In [3] and [17], there are some examples of 7-dimensional homotopy spheres not diffeomorphic to $S^7$ having the structures of linear $S^3$-bundles over $S^4$.

We have the following theorem.

**Theorem 1.** Every homotopy sphere of dimension 7 admits a round fold map into $\mathbb{R}^4$ such that regular fibers are disjoint unions of finite copies of $S^3$ and that the number of connected components of the singular set and the number of connected components of the fiber of a point in a proper core agree.

**Proof.** By virtue of the theory of [11] and [17], every homotopy sphere of dimension 7 is represented as a connected sum of a finite number of oriented 7-dimensional homotopy spheres admitting the structures of linear $S^3$-bundles over $S^4$. In fact, we can choose an oriented homotopy sphere of dimension 7 so that it is a generator of the oriented h-cobordism group of 7-dimensional smooth homotopy spheres. From Corollary 2, the result follows. □

Theorem 1 states that every homotopy sphere of dimension 7 admits a round fold map into $\mathbb{R}^4$, although a homotopy sphere of dimension 7 not diffeomorphic to $S^7$ does not admit a round fold map into $\mathbb{R}^4$ whose singular set is connected as in Example 1.

We also note that all the homotopy spheres admit fold maps into Euclidean spaces whose dimensions are not larger than those of the source manifolds ([4], [5]) and that it has been difficult to construct explicit examples of such fold maps as mentioned in the introduction. Theorem 1 gives explicit examples.

As another easy application of constructions performed in the proof of Proposition 1, we have the following theorem.
**Theorem 2.** Let $m, n \in \mathbb{N}$ and let $m > n \geq 2$. Let $M_i$ be a closed and connected $m$-dimensional manifold admitting a round fold map $f_i : M_i \to \mathbb{R}^n$ $(i = 1, 2)$. We also assume that the fiber of a point in a proper core of $f_i$ $(i = 1, 2)$ has a connected component diffeomorphic to the $(m-n)$-dimensional standard sphere $S^{m-n}$. Then, on each connected sum $M$ of the manifolds $M_1$ and $M_2$, there exists a round fold map $f : M \to \mathbb{R}^n$.

**Proof.** By a trivial spinning construction, we can construct a $C^\infty$ trivial round fold map as in the following. There exists a Morse function $\tilde{f} : D^{m-n+1} \to [a, +\infty)$ satisfying the followings.

1. $a$ is the minimum of $\tilde{f}$ and $\tilde{f}^{-1}(a) = \partial D^{m-n+1}$ holds.
2. $\tilde{f}$ has at least two singular points of index 0 at which the values of the functions are local maxima.
3. All the singular points of $\tilde{f} : \partial D^{m-n+1} \to [a, +\infty)$ are in the interior $\text{Int}D^{m-n+1}$ of the disc $D^{m-n+1}$ and at distinct singular points, the values are always distinct.

Let $p : \partial D^{m-n+1} \times \partial D^n \to \partial D^n$ be the canonical projection. The following diagram commutes.

$$
\begin{array}{ccc}
\partial D^{m-n+1} \times \partial(\mathbb{R}^n - \text{Int}D^n) & \xrightarrow{\phi \times \text{id}(\mathbb{R}^n - \text{Int}D^n)} & \partial D^{m-n+1} \times \partial D^n \\
\downarrow f|_{\partial D^{m-n+1} \times \text{id}(\mathbb{R}^n - \text{Int}D^n)} & & \downarrow p \\
\{a\} \times \partial(\mathbb{R}^n - \text{Int}D^n) & \xrightarrow{\phi} & \partial D^n
\end{array}
$$

By using the diffeomorphisms $\phi \times \text{id}(\mathbb{R}^n - \text{Int}D^n)$ and $\phi$, we obtain a round fold map $f_0 : S^m \to \mathbb{R}^n$ such that the embedding of any connected component $C$ of the singular set $S(f_0)$ into $S^m$ is a trivial embedding.

As in the proof of Proposition 1, from the two maps $f_1$ and $f_0$, we can construct a new round fold map from $M_1$ into $\mathbb{R}^n$. By deforming the obtained round fold map without changing the singular sets, we obtain a round fold map $f_1' : M_1 \to \mathbb{R}^n$ such that for the connected component $C'$ of the boundary $\partial f_1'(M_1)$ of $f_1'(M_1)$ bounding the unbounded connected component of $\mathbb{R}^n - \text{Int}f_1'(M_1)$, $f_1'^{-1}(C')$ is originally a connected component of $S(f_0) \subset S^m$ consisting of definite fold points. The embedding of $f_1'^{-1}(C')$ into $M_1$ is a trivial embedding. We may apply Proposition 1 to the maps $f_2$ and $f_2'$ to complete the proof. □

**Corollary 3.** Let $m \geq 3$. Let $M_i$ be a closed and connected $m$-dimensional manifold admitting a round fold map $f_i : M_i \to \mathbb{R}^{m-1}$ such that $f_i(M_i)$ is diffeomorphic to $D^{m-1}$ $(i = 1, 2)$. Then, on each connected sum $M$ of $M_1$ and $M_2$, there exists a round fold map $f : M \to \mathbb{R}^{m-1}$.

**Proof.** Regular fibers of the maps $f_1$ and $f_2$ are always disjoint unions of circles. Thus, the statement follows from Theorem 2 immediately. □

We introduce another result.

**Theorem 3.** Let $m$ and $n$ be integers larger than 1 and let $m - n \geq 1$. Then, any $m$-dimensional manifold $M$ represented as a connected sum of two closed and connected manifolds $M_1$ and $M_2$ satisfying the following conditions admits a round fold map $f$ into $\mathbb{R}^n$. 


We consider a map spinning construction as the following. 

As in the presentation of a trivial spinning construction before. We use a trivial map $\Phi$. We may assume that the following diagram commutes.

There exists a Morse function $\tilde{f}$ so that the fiber of a point in a proper core of $\tilde{f}$ consist of two connected components diffeomorphic to $\tilde{M}$ above and all the connected components of the fiber of a point in a proper core of the round fold map $M_1$ admits above but the connected component diffeomorphic to $S^{m-n}$ mentioned above.

To prove Theorem 3, we need the following new result.

**Theorem 4.** Let $F \neq \emptyset$ be a closed and connected manifold. Let $M$ be a closed manifold of dimension $m$ having the structure of an $F$-bundle over $S^n$ $(m \geq n \geq 2)$. Then, $M$ admits a round fold map $f : M \to \mathbb{R}^n$ satisfying the followings.

1. $f$ is $C^\infty$ trivial.
2. For an axis $L$ of $f$, $f^{-1}(L)$ is diffeomorphic to $F \times [-1,1]$.
3. Two connected components of the fiber of a point in a proper core of $f$ is regarded as fibers of the $F$-bundle over $S^n$.
4. $f(M)$ is diffeomorphic to $\mathbb{R}^n$ and for the connected component $C := \partial f(M)$, the embedding of $f^{-1}(C)$ into $M$ is a trivial embedding.

**Proof.** We construct a map satisfying the assumption on a $F$-bundle $M$ over $S^n$. We may represent $S^n$ as $(D^n \sqcup D^n) \cup (S^{n-1} \times [0,1])$, where we identify as $\partial(D^n \sqcup D^n) = S^{n-1} \sqcup S^{n-1}$. For a diffeomorphism $\Phi$ from $(S^{n-1} \sqcup S^{n-1}) \times F$ onto $(\partial D^n \sqcup \partial D^n) \times F$ which is a bundle isomorphism between the trivial $F$-bundles inducing the identification of the base spaces, we may represent $M$ as $((D^n \sqcup D^n) \times F) \cup_{\Phi} (S^{n-1} \times [0,1] \times F) = (D^n \times (F \sqcup F)) \cup_{\Phi} (S^{n-1} \times [0,1] \times F)$ since the base space of the bundle $M$ is a standard sphere. There exists a Morse function $\tilde{f} : F \times [0,1] \to [a, +\infty)$, where $a \in \mathbb{R}$ is the minimum, as in the presentation of a trivial spinning construction before. We use a trivial spinning construction as the following.

We consider a map $\tilde{f} \times \text{id}_{S^{n-1}}$ and the canonical projection $p : D^n \times (F \sqcup F) \to D^n$. For the maps $\Phi$, $\tilde{f} \times \text{id}_{S^{n-1}}$, and $p$ and a diffeomorphism $\phi : \partial(\mathbb{R}^n - \text{Int}D^n) \to \partial D^n$, we may assume that the following diagram commutes.

Then, by gluing the maps $p$ and $\tilde{f} \times \text{id}_{S^{n-1}}$ by the pair of diffeomorphisms $(\Phi, \phi)$, we obtain a $C^\infty$ trivial round fold map $f : M \to \mathbb{R}^n$.

We may assume that the embedding of $f^{-1}(\partial f(M))$ into $M$ is smoothly isotopic to an embedding $\partial D^n \times \{p\} \subset D^n \times (F \sqcup F) \subset M$. In fact, by considering the bundle structure over $S^n = (D^n \sqcup D^n) \cup (S^{n-1} \times [0,1])$ of $M$, we can easily take $\Phi$ so that this holds. This means that the embedding of $f^{-1}(\partial f(M))$ into $M$ is a trivial embedding into $M_2$. 

We see that $f$ is a round fold map satisfying the given conditions. This completes the proof. □

**Remark 1.** Theorem 4 was also shown in [13] and [15] by the author for a similar map $f$ without the last condition.

**Example 2.** In the situation of the proof of Theorem 4, let $F := S^{m-n} (m > n \geq 2)$ and let $\tilde{f} : F \times [-1, 1] \to [a, +\infty)$ be a Morse function with two singular points, where $a \in \mathbb{R}$ is the minimum, as in the presentation of a trivial spinning construction before (we easily obtain such a Morse function). As a result, on any manifold having the structure of an $S^{m-n}$-bundle over $S^n$, we have a round fold map as in Theorem 4 whose singular set consists of two connected components and the fiber of a proper core of which is a disjoint union of two copies of $S^{m-n}$. The author constructed such a map first in [12] without assuming the last condition mentioned in Theorem 4.

**Proof of Theorem 3.** By applying Proposition 1 to a round fold map from $M_1$ into $\mathbb{R}^n$ appearing in the assumption and a round fold map from $M_2$ into $\mathbb{R}^n$ constructed in Theorem 4, we obtain a round fold map on the manifold $M$. This completes the proof. □

We have the following corollary to Theorem 3.

**Corollary 4.** Let $m$ and $n$ be integers larger than 1 and let $m-n \geq 1$. Then, any $m$-dimensional manifold $M$ represented as a connected sum of two closed and connected manifolds $M_1$ and $M_2$ satisfying the following conditions admits a round fold map $f$ into $\mathbb{R}^n$ the fiber of a point in a proper core of which consist of three connected components.

1. $M_1$ has the structure of an $S^{m-n}$-bundle over $S^n$.
2. For an $(m-n)$-dimensional closed and connected manifold $F_1 \neq \emptyset$, $M_2$ has the structure of an $F_1$-bundle over $S^n$.

Furthermore, one of the connected components of the fiber of a point in a proper core of $f$ is diffeomorphic $S^{m-n}$ and the others are diffeomorphic to $F_1$ and regarded as fibers of the $F_1$-bundle over $S^n$ above.

Let $F_2 \neq \emptyset$ be an $(m-n)$-dimensional closed and connected manifold. Then, any manifold represented as a connected sum of the manifold $M$ and any manifold having the structure of an $F_2$-bundle over $S^n$ admits a round fold map into $\mathbb{R}^n$ such that the fiber of a point in a proper core of $f$ consist of two connected components diffeomorphic to $F_1$ and two connected components diffeomorphic to $F_2$.

We note again that $S^3$, $S^7$ and $S^{15}$ have the structures of a linear $S^1$-bundle over $S^2$, a linear $S^3$-bundle over $S^4$ and a linear $S^7$-bundle over $S^8$, respectively. By applying Corollary 3, we have the following theorem.

**Theorem 5.** Let $m$ and $n$ be integers larger than 1 and let $m-n \geq 1$.

1. Any manifold represented as a connected sum of $l \in \mathbb{N}$ closed manifolds having the structures of $S^{m-n}$-bundles over $S^n$ admits a round fold map $f$ into $\mathbb{R}^n$ satisfying the followings.
   (a) All the regular fibers of $f$ are disjoint unions of finite copies of $S^{m-n}$.
   (b) The number of connected components of $S(f)$ and the number of connected components of the fiber of a point in a proper core of $f$ are $l$.  

(c) All the connected components of the fiber of a point in a proper core of \( f \) are regarded as fibers of the \( S^{m-n} \)-bundles over \( S^n \) and a fiber of any \( S^{m-n} \)-bundle over \( S^n \) appeared in the connected sum is regarded as a connected component of the fiber of a point in a proper core of \( f \).

(d) For any connected component \( C \) of \( f(S(f)) \) and a small closed tubular neighborhood \( N(C) \) of \( C \) such that \( \partial N(C) \) is the disjoint union of two connected components \( C_1 \) and \( C_2 \), \( f^{-1}(N(C)) \) has the structures of trivial bundles over \( C_1 \) and \( C_2 \) and \( f|_{f^{-1}(C_1)} : f^{-1}(C_1) \rightarrow C_1 \) and \( f|_{f^{-1}(C_2)} : f^{-1}(C_2) \rightarrow C_2 \) give the structures of subbundles of the bundles \( f^{-1}(N(C)) \).

(2) Let \((m,n) = (3,2), (7,4), (15,8)\). Then any \( m \)-dimensional manifold \( M \) represented as a connected sum of two closed and connected manifolds \( M_1 \) and \( M_2 \) having the structures of bundles over \( S^n \) admits a round fold map \( f \) into \( \mathbb{R}^n \) such that the fiber of a point in a proper core of \( f \) consist of two connected components diffeomorphic to the fiber of the former bundle and two connected components diffeomorphic to the fiber of the latter bundle.

We also have the following proposition.

**Proposition 2.** Let \( M_1 \) and \( M_2 \) be closed and connected \( m \)-dimensional manifolds. Assume that there exists a round fold map \( f_1 : M_1 \rightarrow \mathbb{R}^n \) (\( m \geq n \geq 2 \)) and that \( m > n \) holds. Assume also that there exists a connected component of the fiber of a point in a proper core of \( f_1 \) diffeomorphic to \( S^{m-n} \) and that the embedding of the connected component into \( M_1 \) is a trivial embedding. Furthermore, we also assume the existence of a round fold map \( f_2 : M_2 \rightarrow \mathbb{R}^n \) satisfying the followings.

1. For a small closed tubular neighborhood \( N(C) \) of the connected component \( C \) of \( \partial f_2(M_2) \) bounding the unbounded connected component of \( \mathbb{R}^n - \text{Int} f_2(M_2) \), \( f_2^{-1}(N(C)) \) has the structure of a trivial \( D^{m-n+1} \)-bundle over the connected component \( C' \) of \( \partial N(C) \) in \( f_2(M_2) \).

2. \( f_2|_{f_2^{-1}(C')} \) gives the structure of a subbundle of the previous bundle \( f_2^{-1}(N(C)) \) over \( C' \).

Then, on each connected sum \( M \) of the manifolds \( M_1 \) and \( M_2 \), there exists a round fold map \( f : M \rightarrow \mathbb{R}^n \).

**Proof.** We can prove this proposition by the same construction as that of the proof of Proposition 1. However, we present the construction again.

Let \( P_1 \) be a proper core of \( f_1 \) and \( V_1 \) be a connected component of \( f_1^{-1}(P_1) \) such that for \( p \in P_1 \), the embedding of \( f_1^{-1}(p) \cap V_1 \) into \( M_1 \) is a trivial embedding. We note that \( f_1|_{V_1} : V_1 \rightarrow P_1 \) gives the structure of a trivial bundle. Let \( P_2 := N(C) \) and \( V_2 := f_2^{-1}(P_2) \).

Similarly to the proof of Proposition 1, for any diffeomorphism \( \Psi : \partial D^m \rightarrow \partial D^m \) extending to a diffeomorphism on \( D^m \) or from \( M_2 \) (or \( D^m \)) onto \( M_1 \) (or \( D^m \)), we may regard that there exists a diffeomorphism \( \Phi : \partial V_2 \rightarrow \partial V_1 \) regarded as a bundle isomorphism between the two trivial \( S^{m-n+1} \)-bundles over the \((n-1)\)-dimensional standard spheres inducing a diffeomorphism between the base spaces and that the following relation holds, where for two smooth manifolds \( X_1 \) and \( X_2 \), \( X_1 \cong X_2 \) means that \( X_1 \) and \( X_2 \) are diffeomorphic as in the proof of Proposition 1.
Assume that two smooth embeddings of $S$ conditions posed on the map $f$ gives the structure of a subbundle of the bundle $\mathcal{L}$. Let Theorem 7.

Don't know whether a result similar to Theorem 6 holds for round fold maps into $\mathbb{R}$. With the inequality $3 > n > m > 2$, we can realize each connected sum of the manifolds $M_1$ and $M_2$ as the resulting manifold $M$ and that we obtain a round fold map $f : M \to \mathbb{R}^n$. \hfill $\square$

For example, we have the following theorem.

**Theorem 6.** If a closed and connected manifold $M$ of dimension 7 admits a round fold map into $\mathbb{R}^4$, then for any homotopy sphere $\Sigma$ of dimension 7, each connected sum of $M$ and $\Sigma$ admits a round fold map into $\mathbb{R}^4$.

**Proof.** Any linear $D^4$-bundle over $S^3$ is trivial since $\pi_2(SO(4)) \cong \pi_2(S^3) \oplus \pi_2(SO(3)) \cong \{0\}$ holds. Let $f : M \to \mathbb{R}^4$ be a round fold map. For a small closed tubular neighborhood $N(C)$ of the connected component $C$ of $\partial f(M)$ bounding the unbounded connected component of $\mathbb{R}^4 - \text{Int}(f(M))$, $f^{-1}(N(C))$ has the structure of a trivial linear $\mathcal{D}_C$-bundle over the connected component $C'$ of $\partial N(C)$ in $f(M)$ and $f|_{f^{-1}(C')}$ gives the structure of a subbundle of the bundle $f^{-1}(N(C))$. Thus, $f$ satisfies the conditions posed on the map $f_2$ in Proposition 2. Two smooth embeddings of $S^3$ into a 7-dimensional homotopy sphere are always smoothly isotopic from the theory of [8] or [9] with the inequality $3 \times (3 + 1) = 12 < 2 \times 7 = 14$. We may apply Proposition 2 to a round fold map in Theorem 1 and $f : M \to \mathbb{R}^4$ to construct a desired round fold map. This completes the proof. \hfill $\square$

**Remark 2.** If a closed and connected manifold $M$ of dimension 7 admits a round fold map into $\mathbb{R}^2$, then a result similar to Theorem 6 holds. In fact, by [21] or Example 1, every homotopy sphere of dimension 7 admits a round fold map whose singular set is connected into the plane and we only consider a connected sum of this map and the given round fold map (for a connected sum of such maps, for example, see section 5 of [21], in which a connected sum of two special generic maps into the plane was introduced). For the case where $n = 3, 5, 6, 7$ holds, we don’t know whether a result similar to Theorem 6 holds for round fold maps into $\mathbb{R}^n$.

4. **Constructions of $C^\infty$ trivial maps**

We show the following theorem.

**Theorem 7.** Let $M_1$ and $M_2$ be closed and connected $m$-dimensional manifolds. Assume that two $C^\infty$ trivial round fold maps $f_1 : M_1 \to \mathbb{R}^n$ and $f_2 : M_2 \to \mathbb{R}^n$ ($m > n \geq 2$) exist. We also assume the following.
(1) The fiber of a point in a proper core of \( f_1 \) has a connected component diffeomorphic to \( S^{m-n} \).

(2) Isomorphisms on the trivial \( S^{m-n} \)-bundle over \( S^{n-1} \) inducing the identity map on the base space \( S^{n-1} \) are always smoothly isotopic to the identity map on the total space of the trivial bundle if they are orientation preserving diffeomorphisms on the total space.

(3) At least one of the followings holds.

(a) Let \( C \) be the connected component of \( \partial f_2(M_2) \) bounding the unbounded connected component of \( \mathbb{R}^n - \text{Int} f_2(M_2) \). Then, the embedding of \( f_2^{-1}(C) \) into \( M_2 \) is a trivial embedding into \( M_2 \).

(b) The embedding of a connected component of the fiber of a point in a proper core of \( f_1 \) diffeomorphic to \( S^{m-n} \) into \( M_1 \) is a trivial embedding.

Then, on each connected sum \( M \) of the manifolds \( M_1 \) and \( M_2 \), there exists a \( C^\infty \) trivial round fold map \( f : M \to \mathbb{R}^n \).

Proof. If the condition (3a) ((3b)) is assumed, then by the assumption (1), we may apply the proof of Proposition 1 (resp. 2). We abuse notation in the proof of these Propositions such as manifolds \( V_1 \) and \( V_2 \), whose boundaries \( \partial V_1 \) and \( \partial V_2 \) have the structures of trivial \( S^{m-n} \)-bundles over \( S^{n-1} \), and an isomorphism \( \Phi : \partial V_2 \to \partial V_1 \) between the bundles.

We can take an isomorphism \( \Phi \) between the two \( S^{m-n} \)-bundles over \( S^{n-1} \) inducing a diffeomorphism between the base spaces as in these proofs. Furthermore, for any diffeomorphism between the base spaces, we can take \( \Phi \) inducing the diffeomorphism and by the assumption (2), such isomorphisms are always smoothly isotopic to the product of the diffeomorphism between the base spaces and a diffeomorphism between the fibers, which extends to a diffeomorphism between two standard closed discs of dimension \( m-n+1 \) (we regard the fibers as the boundaries of the standard closed discs here).

By the constructions of the manifolds \( M \) and maps \( f \) in these proofs, we can realize any connected sum of the manifolds \( M_1 \) and \( M_2 \) as the resulting source manifold and we can take a diffeomorphism between the base space of the bundles \( \phi \) and an isomorphism \( \Phi : \partial V_2 \to \partial V_1 \) between the bundles inducing the diffeomorphism \( \phi \) so that the resulting map is a \( C^\infty \) trivial round fold map from the connected sum \( M \) into \( \mathbb{R}^n \). \( \square \)

Remark 3. In Theorem 7, we can obtain a \( C^\infty \) trivial round fold map on each connected sum \( M \) of the manifolds \( M_1 \) and \( M_2 \) also by using a method performed in the proof of Theorem 2. In this case, the resulting map is different from the map obtained in the proof above.

As specific cases, we have the following

Theorem 8. Let \( M_1 \) and \( M_2 \) be closed and connected \( m \)-dimensional manifolds. Assume that two \( C^\infty \) trivial round fold maps \( f_1 : M_1 \to \mathbb{R}^n \) and \( f_2 : M_2 \to \mathbb{R}^n \) \((m > n \geq 2)\) exist. We also assume that \( f_2(M_2) \) is diffeomorphic to \( D^n \). One of the followings hold.

(1) \( n \geq 3 \) and \( m-n = 1 \).

(2) \( (m,n) = (5,3) \) or \( (m,n) = (6,3) \) holds and the fibers of points in proper cores of \( f_1 \) and \( f_2 \) have connected components diffeomorphic to the standard sphere \( S^{m-n} \).
Then, on each connected sum $M$ of the manifolds $M_1$ and $M_2$, there exists a $C^\infty$ trivial round fold map $f : M \to \mathbb{R}^n$.

Proof. Let $Q$ be a proper core of $f_2$.

In the first case, regular fibers are always disjoint unions of finite copies of $S^1$. Note that the group of diffeomorphisms consisting of all the orientation preserving diffeomorphisms on $S^1$ has the same homotopy type as $SO(2)$ and $S^1$ and that the natural inclusion of $SO(2)$, which we regard as the group of all the linear transformations on $S^1$, into the group of the diffeomorphism gives a homotopy equivalence. For the connected component $C$ of $\partial f_2(M_2)$ bounding the unbounded connected component of $\mathbb{R}^n - \text{Int} f_2(M_2)$, the embedding of the connected component of $f^{-1}(C)$ of the singular set $S(f_2)$ into $M_2$ is smoothly isotopic to every section of the trivial bundle given by the submersion $f|_{f^{-1}(\partial Q)} : f^{-1}(\partial Q) \to \partial Q$ as a map into $M_2$ by the assumptions that $\pi_{n-1}(S^1)$ is zero ($n \geq 3$ is assumed) and that $f_2$ is $C^\infty$ trivial. Thus, the embedding of $f_2^{-1}(C)$ into $M_2$ is a trivial embedding into $M_2$ since the bundle given by the submersion $f|_{f^{-1}(Q)} : f^{-1}(Q) \to Q$ is a trivial bundle over a standard closed disc whose fiber is diffeomorphic to a disjoint union of finite copies of $S^1$.

In the second case, the fiber of a point in a proper core of $f$ has a connected component diffeomorphic to the standard sphere $S^2$ or $S^3$. Note that the groups of diffeomorphisms consisting of all the orientation preserving diffeomorphisms on $S^2$ and $S^3$ have the same homotopy type as $SO(3)$ and $SO(4)$, respectively, and that the natural inclusions of $SO(3)$ and $SO(4)$, which we regard as the groups of all the linear transformations on $S^2$ and $S^3$, respectively, into the groups of the diffeomorphisms give homotopy equivalences (see [10] and [26]). We also note that the groups $\pi_{n-1}(SO(3)) \cong \pi_2(SO(3))$ and $\pi_{n-1}(SO(4)) \cong \pi_2(SO(4))$ are zero. We obtain a fact similar to one in the first case.

Thus, in both cases, all the assumptions of Theorem 7 are satisfied. This completes the proof.

Theorem 9. Let $M_1$ and $M_2$ be closed and connected $m$-dimensional manifolds. Assume that two $C^\infty$ trivial round fold maps $f_1 : M_1 \to \mathbb{R}^n$ and $f_2 : M_2 \to \mathbb{R}^n$ ($m > n \geq 2$) exist. We also assume the followings.

1. The fibers of points in proper cores of $f_1$ and $f_2$ have connected components diffeomorphic to $S^{m-n}$.
2. Isomorphisms on the trivial $S^{m-n}$-bundle over $S^{n-1}$ inducing the identity map on the base space $S^{n-1}$ are always smoothly isotopic to the identity map on the total space of the trivial bundle if they are orientation preserving diffeomorphisms on the total space.

Then, on each connected sum $M$ of the manifolds $M_1$ and $M_2$, there exists a $C^\infty$ trivial round fold map $f : M \to \mathbb{R}^n$ such that the fiber of a point in a proper core of $f$ has a connected component diffeomorphic to the standard sphere $S^{m-n}$.

Proof. By the assumption (2), for the boundary $C$ of the image $f_2(M)$ of $f_2$, which is diffeomorphic to $D^n$ by the assumption (1) and the assumption that $M_2$ is connected, by using a method similar to that of the proof of Theorem 8, we can show that the embedding of $f_2^{-1}(C)$ into $M_2$ is a trivial embedding into $M_2$. Thus, all the assumptions of Theorem 7 are satisfied. This completes the proof.
Example 3. From Theorems 8 and 9, their proofs and Example 2, a manifold represented as a connected sum of $S^n \times S^1$ and another $(n+1)$-dimensional manifold admitting a $C^\infty$ trivial round fold map into $\mathbb{R}^n$ whose image is diffeomorphic to $D^n$ admits a $C^\infty$ trivial round fold map whose image is diffeomorphic to $D^n$ again where $n \geq 3$ is assumed. In addition, a manifold represented as a connected sum of $S^2 \times S^3$ and another $5$-dimensional (resp. $6$-dimensional) manifold admitting a $C^\infty$ trivial round fold map into $\mathbb{R}^3$ whose image is diffeomorphic to $D^3$ and the fiber of a point in a proper core of which has a connected component diffeomorphic to $S^2$ (resp. $S^3$) admits a $C^\infty$ trivial round fold map whose image is diffeomorphic to $D^3$ again.

For example, manifolds represented as connected sums of finite copies of $S^n \times S^1$ admit $C^\infty$ trivial round fold maps into $\mathbb{R}^n$ satisfying all the conditions of Theorem 5 (1) where $n \geq 3$ is assumed. In addition, manifolds represented as connected sums of finite copies of $S^3 \times S^2$ and ones represented as connected sums of finite copies of $S^3 \times S^3$ admit similar maps into $\mathbb{R}^3$.

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