SOBOLEV-LIKE HILBERT SPACES
INDUCED BY ELLIPTIC OPERATORS

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Abstract. We investigate properties of function spaces induced by the inner product Sobolev spaces $H^s(\Omega)$ over a bounded Euclidean domain $\Omega$ and by an elliptic differential operator $A$ on $\overline{\Omega}$. The domain and the coefficients of $A$ are of the class $C^\infty$. These spaces consist of all distributions $u \in H^s(\Omega)$ such that $Au \in H^\lambda(\Omega)$ and are endowed with the corresponding graph norm, with $s, \lambda \in \mathbb{R}$. We prove an interpolation formula for these spaces and discuss their application to elliptic boundary-value problems.

1. Introduction

This paper is devoted to the function spaces induced by the scale \{H^s(\Omega) : s \in \mathbb{R}\} of inner product Sobolev spaces over a bounded Euclidean domain $\Omega$ and by an elliptic partial differential operator $A$ on $\overline{\Omega}$. The boundary of $\Omega$ and the coefficients of $A$ are supposed to be infinitely smooth. We investigate the inner product spaces $H^s_{A,\lambda}(\Omega)$, where $s, \lambda \in \mathbb{R}$, that consist of all distributions $u \in H^s(\Omega)$ subject to the condition $Au \in H^\lambda(\Omega)$ and that are endowed with the corresponding graph norm. We call $H^s_{A,\lambda}(\Omega)$ the Sobolev-like space induced by $A$. Such spaces are used in the theory of elliptic differential operators and elliptic boundary-value problems.

Specifically, the space $H^0_{A,0}(\Omega)$ is the domain of the maximal operator that corresponds to the unbounded operator $A$ defined on $C^\infty(\overline{\Omega})$ and acting in $L_2(\Omega)$ [1, 2, 3]. Lions and Magenes [4, 5] systematically use the space $H^s_{A,0}(\Omega)$ for $s < \text{ord } A := 2q$ in the theory of solvability of regular elliptic boundary-value problems in distribution spaces. In the case of the Dirichlet problem, the space $H^s_{A,-q}(\Omega)$ for $s < q$ is also useful [6]. These spaces serve as domains of Fredholm bounded operators that correspond to elliptic problems. A more general class of the spaces $H^s_{A,\lambda}(\Omega)$ is considered in [7, 8]. Some of their versions are introduced for $L_p$-Sobolev spaces [4, 5, 9, 10], weighted Sobolev spaces [10, 11, 12], and generalized Sobolev spaces [8, 13, 14] and are also used in the theory of elliptic differential equations.

The purpose of this paper is to investigate some properties of the spaces $H^s_{A,\lambda}(\Omega)$ concerning their completeness, separability, and dense subsets. Besides, we will prove an interpolation formula for these spaces. We also give their application to elliptic boundary-value problems.

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Main Results

Let \( 1 \leq n \in \mathbb{Z} \) and \( s \in \mathbb{R} \). We let \( H^s(\mathbb{R}^n) \) denote the complex inner product Sobolev space of order \( s \) over \( \mathbb{R}^n \). By definition, this space consists of all tempered distributions \( w \) in \( \mathbb{R}^n \) that their Fourier transform \( \hat{w} \) is locally Lebesgue integrable over \( \mathbb{R}^n \) and satisfies the condition
\[
\|w\|_{s,\mathbb{R}^n}^2 := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{w}(\xi)|^2 d\xi < \infty.
\]
The space \( H^s(\mathbb{R}^n) \) is Hilbert and separable with respect to the norm \( \| \cdot \|_{s,\mathbb{R}^n} \). If \( \Omega \) is an open subset of \( \mathbb{R}^n \), then the Sobolev space \( H^s(\Omega) \) of order \( s \) over \( \Omega \) is defined to consist of the restrictions \( u := w \upharpoonright \Omega \) of all distributions \( w \in H^s(\mathbb{R}^n) \) to \( \Omega \). This space is Hilbert and separable with respect to the norm
\[
\|u\|_{s,\Omega} := \inf \{ \|w\|_{s,\mathbb{R}^n} : w \in H^s(\mathbb{R}^n), \ u = w \upharpoonright \Omega \}
\]
and is continuously embedded in the linear topological space \( \mathcal{D}'(\Omega) \) of all distributions in \( \Omega \).

Henceforth we suppose that \( n \geq 2 \) and that \( \Omega \) is a bounded domain whose boundary \( \Gamma := \partial \Omega \) is an infinitely smooth closed manifold of dimension \( n - 1 \), the \( C^\infty \)-structure on \( \Gamma \) being induced by \( \mathbb{R}^n \). In the closed domain \( \overline{\Omega} := \Omega \cup \Gamma \), we consider an arbitrary elliptic linear partial differential expression
\[
A := \sum_{|\mu| \leq 2q} a_\mu(x) \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}
\]
of an even order \( 2q \geq 2 \) with complex-valued coefficients \( a_\mu \in C^\infty(\overline{\Omega}) \). Here, \( \mu := (\mu_1, \ldots, \mu_n) \) is a multi-index with nonnegative integer-valued components, \( |\mu| := \mu_1 + \cdots + \mu_n \), and \( x = (x_1, \ldots, x_n) \) is an arbitrary point in \( \mathbb{R}^n \). The ellipticity condition for \( A \) means that
\[
\sum_{|\mu| = 2q} a_\mu(x) \xi_1^{\mu_1} \cdots \xi_n^{\mu_n} \neq 0
\]
for every point \( x \in \overline{\Omega} \) and each vector \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{0\} \). In the \( n = 2 \) case, we suppose in addition that \( A \) is properly elliptic on \( \overline{\Omega} \) (see, e.g., [15, Section 5.2.1]).

We associate some spaces with the elliptic formal differential operator \( A \). Choosing \( s, \lambda \in \mathbb{R} \) arbitrarily, we let \( H^s_{A,\lambda}(\Omega) \) denote the linear space
\[
\{ u \in H^s(\Omega) : Au \in H^\lambda(\Omega) \}
\]
endowed with the graph norm
\[
\|u\|_{s, A, \lambda} := \left( \|u\|_{s, \Omega}^2 + \|Au\|_{\lambda, \Omega}^2 \right)^{1/2}.
\]
Here and below, \( Au \) is understood in the sense of the theory of distributions in \( \Omega \).

Theorem 1. Let \( s, \lambda \in \mathbb{R} \). The following assertions are true:

(i) The space \( H^s_{A,\lambda}(\Omega) \) is Hilbert and separable.
(ii) The equality of linear spaces $H^s_{A,\lambda}(\Omega)$ and $H^s(\Omega)$ holds if and only if $\lambda \leq s - 2q$. If $\lambda \leq s - 2q$, the norms in these spaces are equivalent.

(iii) The set $C^\infty(\Omega)$ is dense in $H^s_{A,\lambda}(\Omega)$.

(iv) If $s \leq 1/2$ and $\lambda \leq 1/2 - 2q$, then the set $C^\infty_0(\Omega)$ is dense in $H^s_{A,\lambda}(\Omega)$.

Here and below, $C^\infty_0(\Omega)$ denotes the set of all functions from $C^\infty(\Omega)$ that vanish near the boundary of $\Omega$.

Note that, if $\lambda > s - 2q$, the space $H^s_{A,\lambda}(\Omega)$ depends on the coefficients of $A$, even when all of them are constant. For example, if $A_1$ and $A_2$ are elliptic constant-coefficient linear differential expressions, the embedding $H^0_{A_1,0}(\Omega) \subseteq H^0_{A_2,0}(\Omega)$ implies that $A_2 = \alpha A_1 + \beta$ for certain $\alpha, \beta \in \mathbb{C}$. This follows from Hörmander’s result [2, Theorem 3.1].

It is well known that the Sobolev scale $\{H^s(\Omega) : s \in \mathbb{R}\}$ has the following interpolation property: if $s_0, s_1 \in \mathbb{R}$ and $0 < \theta < 1$, then

$$\left[H^{s_0}(\Omega), H^{s_1}(\Omega)\right]_\theta = H^{(1-\theta)s_0 + \theta s_1}(\Omega)$$

up to equivalence of norms (see, e.g., [15, Section 4.3.1, Theorem 1]). Here and below, $[H_0, H_1]_\theta$ denotes the result of the complex interpolation with the parameter $\theta$ of a compatible pair of Hilbert (or, more generally, Banach) spaces $H_0$ and $H_1$ (see, e.g., [15, Section 1.9]). The pair $[H_0, H_1]$ of these spaces is called compatible if they are continuously embedded in a certain linear Hausdorff space.

Let us formulate a version of the interpolation property (1) for the spaces $H^s_{A,\lambda_j}(\Omega)$, where $j \in \{0, 1\}$. In view of assertion (ii) of Theorem 1, it is worthwhile to restrict ourselves to the case when $\lambda_j \geq s_j - 2q$.

**Theorem 2.** Suppose that $s_0, s_1, \lambda_0, \lambda_1$, and $\theta$ are real numbers satisfying the inequalities $\lambda_0 \geq s_0 - 2q$, $\lambda_1 \geq s_1 - 2q$, and $0 < \theta < 1$. Put $s := (1 - \theta)s_0 + \theta s_1$ and $\lambda := (1 - \theta)\lambda_0 + \theta \lambda_1$. Then

$$\left[H^{s_0}_{A,\lambda_0}(\Omega), H^{s_1}_{A,\lambda_1}(\Omega)\right]_\theta = H^s_{A,\lambda}(\Omega)$$

up to equivalence of norms.

3. **Proof of the main results**

We will give a joint proof of Theorems 1 and 2.

Let us first prove that the space $H^s_{A,\lambda}(\Omega)$ is Hilbert for arbitrary $s, \lambda \in \mathbb{R}$. Evidently, the norm in it is induced by a graph inner product. Besides, the space $H^s_{A,\lambda}(\Omega)$ is complete. Indeed, if $(u_k)$ is a Cauchy sequence in this space, then there exist the limits $u := \lim u_k$ in $H^s(\Omega)$ and $f := \lim Au_k$ in $H^\lambda(\Omega)$ because the last two spaces are complete. The differential operator $A$ is continuous on $\mathcal{D}'(\Omega)$; therefore, $Au = \lim Au_k = f$ in $\mathcal{D}'(\Omega)$. Here, $u \in H^s(\Omega)$ and $f \in H^\lambda(\Omega)$. Hence, $u \in H^s_{A,\lambda}(\Omega)$, and $\lim u_k = u$ in the space $H^s_{A,\lambda}(\Omega)$. Thus, this space is complete.

Let us now prove assertion (ii) of Theorem 1. Since $\text{ord} A = 2q$, the differential operator $A$ acts continuously from $H^s(\Omega)$ to $H^{s-2q}(\Omega)$ for every $s \in \mathbb{R}$. Hence, if
\( \lambda \leq s - 2q, \) then \( H^s(\Omega) = H^s_{A,\lambda}(\Omega) \) up to equivalence of norms due to the continuous embedding \( H^{s-2q}(\Omega) \subseteq H^s(\Omega). \)

Conversely, assume that \( H^s(\Omega) = H^s_{A,\lambda}(\Omega) \) for certain \( s, \lambda \in \mathbb{R}. \) Choose an arbitrary distribution \( w \in H^s(\mathbb{R}^n) \) such that \( \text{supp} w \subseteq \Omega, \) and put \( u := w | \Omega. \) Since \( u \in H^s(\Omega), \) we have the inclusion \( A u \in H^\Psi(\Omega) \) by our assumption. Hence, \( \chi u \in H^{s+2q}(\Omega) \) for every function \( \chi \in C_0^\infty(\Omega) \) by virtue of the ellipticity of \( A \) (see, e.g., \([16, \text{Theorem 7.4.1}]\)). We can choose the function \( \chi \) so that \( \chi = 1 \) on \( \text{supp} w. \) Therefore \( u \in H^{s+2q}(\Omega), \) which implies the inclusion \( w \in H^{s+2q}(\mathbb{R}^n). \) Thus,

\[
\{ w \in H^s(\mathbb{R}^n) : \text{supp} w \subseteq \Omega \} \subseteq H^{s+2q}(\mathbb{R}^n).
\]

According to \([16, \text{Theorem 2.2.2}]\), this embedding implies that

\[
(1 + |\xi|)^{s+2q} \leq c (1 + |\xi|)^s
\]

for every \( \xi \in \mathbb{R}^n \) with some number \( c > 0 \) not depending on \( \xi. \) Hence, \( \lambda + 2q \leq s. \) We have proved assertion (ii) of Theorem 1.

To prove Theorem 2 and the remaining part of Theorem 1, we use a theorem on interpolation of some Hilbert spaces induced by linear continuous operators. Before we formulate this result, let us make some notation. If \( H, \Phi \) and \( \Psi \) are Hilbert spaces subject to the continuous embedding \( \Phi \subseteq \Psi \) and if \( T : H \to \Psi \) is a continuous linear operator, we put

\[
(H)_{T,\Phi} := \{ u \in H : Tu \in \Phi \}
\]

and endow the linear space \((H)_{T,\Phi}\) with the graph norm

\[
\|u\|_{(H)_{T,\Phi}} := \left( \|u\|_H^2 + \|Tu\|^2_\Phi \right)^{1/2}.
\]

This norm does not depend on \( \Psi, \) and the space \((H)_{T,\Phi}\) is Hilbert. The latter is proved in a quite similar way as the proof of the completeness of \( H^s_{A,\lambda}(\Omega). \)

**Proposition 1.** Assume that six separable Hilbert spaces \( X_0, Y_0, Z_0, X_1, Y_1, \) and \( Z_1 \) and three linear mappings \( T, R, \) and \( S \) are given and satisfy the following conditions:

(i) The pairs \([X_0, X_1]\) and \([Y_0, Y_1]\) are compatible.

(ii) The spaces \( Z_0 \) and \( Z_1 \) are linear manifolds in a certain locally convex topological space \( E. \)

(iii) The continuous embeddings \( Y_0 \subseteq Z_0 \) and \( Y_1 \subseteq Z_1 \) hold.

(iv) The mapping \( T \) is given on \( X_0 + X_1 \) and defines the bounded operators \( T : X_0 \to Z_0 \) and \( T : X_1 \to Z_1. \)

(v) The mapping \( R \) is given on \( E \) and defines the bounded operators \( R : Z_0 \to X_0 \) and \( R : Z_1 \to X_1. \)

(vi) The mapping \( S \) is given on \( E \) and defines the bounded operators \( S : Z_0 \to Y_0 \) and \( S : Z_1 \to Y_1. \)

(vii) The equality \( TRu = u + Su \) holds for every \( u \in E. \)

Then

\[
[(X_0)_{T,Y_0}, (X_1)_{T,Y_1}]_\theta = ([X_0, X_1]_\theta)_{T,[Y_0,Y_1]_\theta}
\]
up to equivalence of norms for every \( \theta \in (0, 1) \).

This result is proved by Lions and Magenes [12, Chapter 1, Theorem 14.3] in a more general case of Banach spaces.

Let us now prove Theorem 2. Our reasoning is motivated by [12, Chapter 2, Proof of Theorem 4.2] and [8, Section 3.3.4]. We choose an integer \( r \geq 1 \) arbitrarily and consider the linear differential expression \( A^r A^{r^+} + 1 \) of order \( 4qr \). Here, as usual, \( A^r \) denotes the differential expression which is formally adjoint to the \( r \)-th iteration \( A^r \) of \( A \). Given an integer \( \sigma \geq 2qr \), we let \( H^\sigma_D(\Omega) \) denote the linear manifold of all distributions \( u \in H^\sigma(\Omega) \) such that \( \partial^j_\nu u = 0 \) on \( \Gamma \) for each \( j \in \{0, \ldots, 2qr - 1\} \). Here, \( \partial_\nu \) is the operator of the differentiation with respect to the inward normal to the boundary \( \Gamma \) of \( \Omega \). This definition is reasonable in view of the theorem on traces of distributions from \( H^\sigma(\Omega) \) (see, e.g., [15, Section 4.7.1]). According to this theorem, \( H^\sigma_D(\Omega) \) is a (closed) subspace of \( H^\sigma(\Omega) \). We have the isomorphism

\[
A^r A^{r^+} + 1 : H^\sigma_D(\Omega) \leftrightarrow H^{\sigma - 4qr}(\Omega)
\]

for every integer \( \sigma \geq 2qr \) (see, e.g., [8, Lemma 3.1]). Let \( (A^r A^{r^+} + 1)^{-1} \) denote the inverse of this isomorphism. This inverse defines the continuous linear operator

\[
(A^r A^{r^+} + 1)^{-1} : H^l(\Omega) \to H^{l + 4qr}(\Omega)
\]

for every integer \( l \geq -2qr \). It follows from the interpolation formula (1) that this operator is well defined and continuous for every real \( l \geq -2qr \).

In Proposition 1, we put \( X_j := H^{s_j}(\Omega), Y_j := H^{\lambda_j}(\Omega), Z_j := H^{s_j - 2q}(\Omega), E := H^{\min(s_0,s_1) - 2q}(\Omega), \) and \( T := A \). Evidently, conditions (i)–(iv) of this proposition are fulfilled. Besides, subjecting \( r \) to the restrictions

\[
s_j - 2q \geq -2qr \quad \text{and} \quad s_j - 2q - \lambda_j \geq -4qr
\]

for each \( j \in \{0, 1\} \), we put \( R := A^{r-1}A^r(A^r A^{r^+} + 1)^{-1} \) and \( S := -(A^r A^{r^+} + 1)^{-1} \). Owing to (4), we have the continuous linear operators

\[
R : Z_j = H^{s_j - 2q}(\Omega) \to H^{s_j}(\Omega) = X_j
\]

and

\[
S : Z_j = H^{s_j - 2q}(\Omega) \to H^{s_j - 2q + 4qr}(\Omega) \subseteq H^{\lambda_j}(\Omega) = Y_j
\]

for each \( j \in \{0, 1\} \) (the last embedding is continuous); i.e., conditions (v) and (vi) are fulfilled as well. The last condition (vii) is satisfied because

\[
TRu = (A^r A^{r^+} + 1 - 1)(A^r A^{r^+} + 1)^{-1}u = u + Su
\]

for every \( u \in E \). Thus, owing to Proposition 1 and the interpolation formula (1), we conclude that

\[
[H^{s_0}_{A,\lambda_0}(\Omega), H^{s_1}_{A,\lambda_1}(\Omega)]_\theta = [(X_0)_{T,Y_0}, (X_1)_{T,Y_1}]_\theta
\]

\[
= [(X_0, X_1)_\theta]_{T,[Y_0,Y_1]} = H^{s_0}_{A,\lambda}(\Omega)
\]

up to equivalence of norms, which proves Theorem 2.

Let us prove the remaining part of Theorem 1. We make use of the following supplement to Proposition 1:
Proposition 2. Suppose that the hypothesis of Proposition 1 is fulfilled. If the Hilbert spaces $X_0$, $Y_0$, $X_1$, and $Y_1$ are separable and satisfy the dense continuous embeddings $X_1 \subseteq X_0$ and $Y_1 \subseteq Y_0$, then the Hilbert spaces $(X_0)_{T,Y_0}$ and $(X_1)_{T,Y_1}$ are also separable and satisfy the dense continuous embedding $(X_1)_{T,Y_1} \subseteq (X_0)_{T,Y_0}$.

This result is proved in [17, Theorem 4.1] for a more general method of interpolation between Hilbert spaces than that used in the paper (see also [8, Section 3.3.2]).

According to Proposition 2, the spaces $H_{s,A,\lambda_0}(\Omega)$ and $H_{s,A,\lambda_1}(\Omega)$ from Theorem 2 are separable, and the second of them is continuously and densely embedded in the first provided that $s_0 \leq s_1$ and $\lambda_0 \leq \lambda_1$. This specifically implies assertion (i) of Theorem 1 concerning the separability of $H_{s,A,\lambda}(\Omega)$ for all $s \in \mathbb{R}$ and $\lambda > s - 2q$. If $\lambda \leq s - 2q$, this separability follows from assertion (ii).

It remains to prove assertions (iii) and (iv) of Theorem 1. Let $s, \lambda \in \mathbb{R}$; if $\lambda > s - 2q$, we have the dense continuous embedding
\[(5)\quad H^{s_1}(\Omega) = H^{s_1}_{A,s_1-2q}(\Omega) \subseteq H^{s_1}_{A,\lambda}(\Omega)\]
for every number $s_1 \in \mathbb{R}$ such that $s_1 \geq s$ and $s_1 - 2q \geq \lambda$. This has been noted in the previous paragraph. Since the set $C^\infty(\overline{\Omega})$ is dense in $H^{s_1}(\Omega)$, this embedding implies assertion (iii) in the case of $\lambda > s - 2q$. In the opposite case, this assertion is a consequence of assertion (ii).

Finally, assume that $s \leq 1/2$ and $\lambda \leq 1/2 - 2q$. If $\lambda > s - 2q$, assertion (iv) follows from (5) and the density of $C^\infty_0(\Omega)$ in $H^{s_1}(\Omega)$ for $s_1 \leq 1/2$ (see, e.g., [15, Section 4.3.2, Theorem 1 (a)]). If $\lambda \leq s - 2q$, this assertion follows from this density in view of assertion (ii). Our proof is complete.

4. Application

Let us discuss an application of the space $H^{s}_{A,\lambda}(\Omega)$ to elliptic boundary-value problems. We consider a boundary-value problem that consists of the elliptic equation
\[(6)\quad Au = f \quad \text{in} \quad \Omega \]
and the boundary conditions
\[(7)\quad B_j u = g_j \quad \text{on} \quad \Gamma, \quad j = 1, \ldots, q.\]
Here, each
\[B_j := \sum_{|\mu| \leq m_j} b_{j,\mu}(x) \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}}\]
is a linear boundary partial differential expression on $\Gamma$ of order $m_j \leq 2q - 1$ with complex-valued coefficients $b_{j,\mu} \in C^\infty(\Gamma)$. We suppose that the boundary-value problem (6), (7) is elliptic in $\Omega$ (see the definition in, e.g., [19, Section 1.2]). Put $m := \max\{m_1, \ldots, m_q\}$; the case of $m \geq 2q$ is possible.

It is known [18, Theorem 7] that the mapping
\[(8)\quad u \mapsto (Au, Bu) = (Au, B_1 u, \ldots, B_q u),\]
where \( u \in C^\infty(\Omega) \), extends uniquely (by continuity) to a Fredholm continuous linear operator
\[
(A, B) : H^s(\Omega) \rightarrow H^{s-2q}(\Omega) \oplus \bigoplus_{j=1}^q H^{s-m_j-1/2}(\Gamma)
\]
for every real number \( s > m + 1/2 \). Moreover, the kernel of this operator lies in \( C^\infty(\overline{\Omega}) \) and together with the index of the operator does not depend on \( s \). Here and below, \( H^\sigma(\Gamma) \) denotes the inner product Sobolev space over \( \Gamma \) of order \( \sigma \in \mathbb{R} \).

Let \( N \) and \( \varkappa \) respectively denote these kernel and index.

This fundamental property of problem (6), (7) does not remain for \( s \leq m + 1/2 \). Specifically, if \( s \leq m_j + 1/2 \), then the mapping \( u \mapsto B_j u \), where \( u \in C^\infty(\Omega) \), cannot be extended to a continuous linear operator from \( H^s(\Omega) \) to the linear topological space \( D'(\Gamma) \) of all distributions on \( \Gamma \). But it is possible to prove a version of this property for every \( s \leq m + 1/2 \) provided that we use some spaces \( H^{s,\lambda}(\Omega) \) instead of \( H^s(\Omega) \).

**Theorem 3.** Let numbers \( s, \lambda \in \mathbb{R} \) satisfy the conditions \( s \leq m + 1/2 \), \( \lambda > -1/2 \), and \( \lambda > m + 1/2 - 2q \). Then the mapping (8), where \( u \in C^\infty(\Omega) \), extends uniquely (by continuity) to a Fredholm continuous linear operator
\[
(9) \quad (A, B) : H^s_{A,\lambda}(\Omega) \rightarrow H^\lambda(\Omega) \oplus \bigoplus_{j=1}^q H^{s-m_j-1/2}(\Gamma).
\]

The kernel of this operator equals \( N \), and the index equals \( \varkappa \).

If \( s \in \mathbb{Z} \) and \( s \leq 0 \), Theorem 3 is proved in a somewhat similar way as that used in [7] and [8, Sections 4.4.2 and 4.4.3] for regular elliptic boundary-value problems. (The corresponding reasoning is given in [20, Section 5] for \( m \geq 2q \)). In the general situation we prove Theorem 3 with the help of the interpolation formula (2). Choose \( l \in \mathbb{Z} \) such that \( l < s \) and \( l \leq 0 \). We consider the Fredholm continuous operators
\[
(10) \quad (A, B) : H^l_{A,\lambda}(\Omega) \rightarrow H^\lambda(\Omega) \oplus \bigoplus_{j=1}^q H^{l-m_j-1/2}(\Gamma)
\]
and
\[
(11) \quad (A, B) : H^{\lambda+2q}(\Omega) \rightarrow H^\lambda(\Omega) \oplus \bigoplus_{j=1}^q H^{\lambda+2q-m_j-1/2}(\Gamma)
\]
(note that \( \lambda + 2q > 1/2 \) by the hypothesis of the Theorem). They have the common kernel \( N \) and the common index \( \varkappa \). Since \( l < s < \lambda + 2q \), there exists a number \( \theta \in (0, 1) \) such that \( s = (1 - \theta)l + \theta(\lambda + 2q) \). Applying the interpolation with the parameter \( \theta \) to these operators, we conclude by [8, Theorem 1.5] that a restriction of the first operator is a Fredholm continuous operator between the spaces
\[
[H^l_{A,\lambda}(\Omega), H^{\lambda+2q}(\Omega)]_\theta
\]
and
\[ H^\lambda(\Omega) \oplus \bigoplus_{j=1}^{q} H^{l-m_j-1/2}(\Gamma), H^\lambda(\Omega) \oplus \bigoplus_{j=1}^{q} H^{\lambda+2q-m_j-1/2}(\Gamma) \]

Moreover, this operator has the same kernel and index as operators (10) and (11).

Owing to Theorem 2, the first space coincides with \( H^s_{A,\lambda}(\Omega) \) up to equivalence of norms (remark that \( H^{\lambda+2q}(\Omega) = H^{\lambda+2q}_{A,\lambda}(\Omega) \)). Besides, the second space coincides up to equivalence of norms with the target space of operator (9) due to an analog of the interpolation formula (1) for Sobolev spaces over \( \Gamma \) (see, e.g., [12, Chapter 1, Theorem 7.7]). Thus, the latter Fredholm operator acts between the spaces indicated in (9). It is an extension by continuity of the mapping (8), where \( u \in C^\infty(\Omega) \), in view of assertion (iii) of Theorem 1. The proof of Theorem 3 is complete.

Remark that, in the important case of regular elliptic boundary-value problems, Theorem 3 is proved by Lions and Magenes [4, 5] in the framework of \( L_p \)-Sobolev spaces provided that \( \lambda = 0 \) and \( s \notin \{1/p - k : 1 \leq k \in \mathbb{Z}\} \). In the case where \( s \geq 0 \) and \( m \leq 2q - 1 \), this theorem is contained in the result formulated in [19, p. 86].

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