STRINGS NEAR A RINDLER OR BLACK HOLE HORIZON

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Orbifold techniques are used to study bosonic, type II and heterotic strings in Rindler space at integer multiples $N$ of the Rindler temperature, and near a black hole horizon at integer multiples of the Hawking temperature, extending earlier results of Dabholkar. It is argued that a Hagedorn transition occurs nears the horizon for all $N > 1$. 

October, 1994

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1. Introduction

The usual perturbative description of string theory as a theory of quantum strings propagating in a fixed spacetime background fails miserably at the Planck scale, just when string theory is supposed to be telling us something really interesting. It appears that the proper description must be fundamentally different and does not involve either strings or spacetime. Evidence for this, arising from calculations involving strings in various kinds of extreme circumstances, is given by the existence of the Hagedorn transition \[1\], \( R \rightarrow 1/R \) duality in spacetime \[2\], lattice formulations \[3\] and symmetries in high energy scattering \[4\]. These few concrete calculations have served as precious guideposts in our attempts to understand planckian string theory.

It is important to develop additional guideposts. One such potential guidepost is the behavior of strings near a black hole horizon. This may also bear on the information problem \[5,6\]. In the limit that the black hole becomes very large, its horizon approaches that of the Rindler wedge, and one is led to the simpler problem of strings near the Rindler horizon.

A complete understanding of strings in Rindler space has been elusive. This subject has previously been considered in \[7\]. Ordinary quantum field theory in Rindler space is defined by constructing annihilation and creation operators which are positive and negative frequency with respect to Rindler time. Rindler time is proportional to the proper time along a family of uniformly accelerating trajectories which fill the Rindler wedge, with the proportionality constant equal to the acceleration. Of course the thermal Rindler state at Rindler temperature \( T = 1/2\pi \) is equivalent to the Minkowski vacuum state at \( T = 0 \) (restricted to the Rindler wedge), so many of its properties can be trivially deduced, even for the case of string theory\[3\]. For example the energy density in this state vanishes everywhere. The uniformly accelerated Rindler observer explains this as due to a miraculous cancellation between the Casimir energy of the Rindler vacuum and the thermal energy of the gas of excited strings. These both increase in equal but opposite manners near the Rindler horizon, the former due to the proximity of the boundary, the latter due to the blue-shifting of the local temperature.

It is much more non-trivial to understand Rindler strings at \( T \neq 1/2\pi \) (including \( T = 0 \), the Rindler vacuum). In this case the cancellation will not occur, because the thermal energy will change with the temperature but the vacuum energy will not. Since the local temperature diverges as the inverse distance to the horizon, one may expect a Hagedorn transition near the horizon \[8\]. However, this increasing temperature does not in itself necessarily imply a Hagedorn transition because the size of the region in which the local temperature exceeds the Hagedorn temperature is itself of order the string scale. Thus a more careful analysis is required.

The general problem of Rindler strings at \( T \neq 1/2\pi \) has not been solved. Indeed it is not clear that there is any consistent description. However for the special cases \( T = N/2\pi \), where \( N \) is an integer greater than zero, the partition function can be computed as a \( Z_N \) orbifold. This orbifold corresponds to a cone with deficit angle \( \epsilon = 2\pi(N - 1)/N \). We shall see that tachyons appear (even for the superstring) with masses \( M^2 = -2\epsilon/\pi \). This

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3 Although questions involving uniformly accelerated observers in this state remain puzzling.
indeed signals the existence of a super-Hagedorn region near the horizon.

This same partition function also approximates the near-horizon behavior of a black hole of arbitrary mass $M$ immersed in a heat bath of temperature $N$ times the Hawking temperature $T_H = 1/8\pi M$.

The appearance of tachyons and a super-Hagedorn region of course implies a breakdown of string perturbation theory. What we have learned is that ill-understood planckian string physics – perhaps a genus-zero condensate\[9\]– is needed to describe physics near the horizon.

It has been suggested \[4\] that the entanglement entropy of the quantum state outside a black hole horizon can be computed in string theory as the derivative of the partition function with respect to the deficit angle $\epsilon$ evaluated at $\epsilon = 0$. It might seem that this suggestion would be foiled by the genus-zero condensate which occurs for non-zero $\epsilon$. In fact we shall argue – using an analytic continuation of our results at discrete values of $\epsilon$ – that this is not the case: the uncalculable genus-zero corrections to the partition function come in at order $\epsilon^2$ and thus do not affect the entropy. This is perhaps to be expected since other perturbative methods of computing the entropy would not require consideration of strings on singular spaces.

In section 2 we compute the bosonic string partition function for Rindler strings at $T = N/2\pi$. Section 3 contains the type II superstring in the RNS formalism and a discussion of the genus-zero condensate and its effect on the calculation of the horizon entropy. Section 4 describes the heterotic string.

Many of the results in sections 2 and 3 are contained in a previous paper of Dabholkar \[11\]. This paper may be viewed as an extension of his work with some differences in the discussion and perspective.

2. The Bosonic String

We are interested in describing strings propagating on Rindler space at finite temperature. The thermodynamic partition function is obtained by performing the path integral on the Euclidean continuation of this space. The metric on this space may be written in the following form

$$ds^2 = r^2d\theta^2 + dr^2 + d\vec{x}^2,$$

where the imaginary time coordinate $\theta$ is periodically identified, with period $\beta$, the inverse temperature.

This metric is the analytic continuation of a cosmic string metric

$$ds^2 = -dt^2 + r^2d\theta^2 + dr^2 + d\vec{x}^2,$$

in which the $x_2$ coordinate is continued to $t = -ix_2$. The mass of the cosmic string introduces a conical singularity at $r = 0$ with non-trivial deficit angle $2\pi - \beta$. Quantum field theory on cones has been studied in \[12\]. Strings propagating on spaces with cosmic string type singularities have been previously considered in \[11,13\]. The cosmic string continuation of \eqref{eq:2.1} allows us to fix light-cone gauge, which will be helpful when we consider the spectrum of the string theory.
In the following we will be considering critical string theories obtained by orbifolding flat space [14]. These models satisfy on-shell conformal invariance (unlike the off-shell prescription proposed in [6] to continue the the partition function to non-trivial \( \beta \)). A consequence of this will be that the genus-one partition functions we obtain may be written as integrals of modular invariant expressions over the fundamental domain. Because the UV region is excluded from these integrals, all our expressions will be UV finite. Unfortunately, the only known modular invariant theories are \( \mathbb{Z}_N \) orbifolds of flat space, which restricts the deficit angle to a discrete set of values \( 2\pi(1 - 1/N) \).

Consider the critical closed bosonic string on the Euclidean orbifold \( \mathbb{R}^{24} \times \mathbb{R}^2/\mathbb{Z}_N \). It is convenient to represent \( X_1 \) and \( X_2 \) as a complex coordinate, \( X = (X_1 + iX_2)/\sqrt{2} \) and \( \tilde{X} = (X_1 - iX_2)/\sqrt{2} \). The Hilbert space of the string states splits up into an untwisted sector and a number of twisted sectors labeled by the integer \( k = 0, \cdots, N-1 \). The boundary condition in the \( k \)'th twisted sector is then

\[
X(\sigma + 1) = e^{2\pi i k/N} X(\sigma).
\]

This leads to the following mode expansion for the \( X \) field

\[
X = \delta_{k,0} x + \frac{i}{2} \sum_n \frac{\alpha_{n+k/N}}{n+k/N} e^{2i\pi(n+k/N)\sigma} + \frac{i}{2} \sum_n \frac{\tilde{\alpha}_{n-k/N}}{n-k/N} e^{-2i\pi(n-k/N)\sigma},
\]

in units where \( \alpha' = 1/2 \). The oscillators satisfy the commutation relations

\[
[\alpha_{m+k/N}, \tilde{\alpha}_{n-k/N}] = (m + k/N)\delta_{m+n}
\]

\[
[\tilde{\alpha}_{m-k/N}, \tilde{\alpha}_{n+k/N}] = (m - k/N)\delta_{m+n}.
\]

The twist in the boundary condition of the \( X \) field introduces a shift in the vacuum energy. The contribution to the vacuum energy of \( X \) and \( \tilde{X} \) is given by \(-1/2(\eta^2 - \eta + 1/6)\), where \( \eta = k/N \). The other coordinates contribute a \(-11/12\) to the vacuum energy, so the total vacuum energy in the \( k \)'th twisted sector is \(-1/2(\eta^2 - \eta + 2)\). Each twisted sector therefore contains one set of tachyon states with \( M^2 = 4(-\eta^2 + \eta - 2) \).

In the untwisted sector, states are labeled by the continuous momenta \( p \) and \( \bar{p} \), together with momenta in the directions transverse to the plane. Note that there is a projection onto \( \mathbb{Z}_N \) invariant states, so the allowed combinations of \( p \) and \( \bar{p} \) are reduced by a factor \( N \). In the twisted sectors, the \( X \) field has no zero-mode, so states are built on the vacuum state which has zero momentum in the \( X_1 \) and \( X_2 \) directions. This means that all states in the twisted sectors may be interpreted as states localized near the tip of the cone, which may only propagate in the transverse directions.

The orbifold partition function involves a sum over all the twisted sectors, together with a projection onto \( \mathbb{Z}_N \) invariant states. This is achieved by summing over the \( N^2 \) contributions \( \mathcal{Z}_{k,l} \) where \( \mathcal{Z}_{k,l} \) is the partition function for a single complex boson on a torus with twisted boundary conditions

\[
X(\sigma_1 + 1,\sigma_2) = e^{2\pi i k/N} X(\sigma_1,\sigma_2), \quad X(\sigma_1,\sigma_2 + 1) = e^{-2\pi i l/N} X(\sigma_1,\sigma_2).
\]
Here $\sigma_1, \sigma_2$ are coordinates with period 1 which parametrize the torus. The partition functions $Z_{k,l}$ are then simply determinants of Laplacians on the torus, subject to the above boundary conditions. These partition functions may be easily computed using the oscillators defined in (2.4),

$$Z_{k,l} = \left| \frac{\eta}{\theta \left[ \frac{1}{2} + \frac{k}{N} \right]} \right|^2,$$

for $k$ and $l$ not both zero. Here $\theta[\alpha, \beta]$ is the theta function with characteristics $\alpha$ and $\beta$ [16]. The $Z_{0,0}$ term involves zero modes which yield a factor proportional to the area of the plane $A_p$. One then obtains

$$Z_{0,0} = \frac{A_p}{\text{Im} \tau |\eta^2|^2}.$$

This term is modular invariant on its own. The other $Z_{k,l}$ mix under modular transformations.

The final partition function is then

$$Z_N = \frac{1}{N} \frac{1}{|\eta^2|^2 \text{Im} \tau |\eta^2|^2} \sum_{k,l=0}^{N-1} Z_{k,l},$$

and the genus one amplitude is

$$A_N = V_T \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \tau)^2} Z_N,$$

where $\mathcal{F}$ is the fundamental domain of the modular group and $V_T$ is the 24-dimensional transverse volume. The free energy of strings in Rindler space at finite temperature is related to this via

$$\beta F = -A_N.$$

Since each twisted sector contains a tachyon, the free energy (2.11) will diverge in the infrared. The limit $\text{Im} \tau \to \infty$ part of the integral in the $k$'th twisted sector looks like

$$- \int_{-\infty}^\infty \frac{d \tau_2}{\tau_2^2} e^{-\pi M_k^2 \tau_2/2 \tau_2^2},$$

where $M_k^2$ is the mass squared of the tachyon in the $k$'th twisted sector. Thus the free energy in these models is negative infinity.

The presence of tachyons in the twisted sectors is reminiscent of the appearance of tachyonic winding states in string theory in flat space at finite temperature as the Hagedorn temperature is approached [17]. Indeed the twisted sector states wind around the cone in the euclidean time direction. They also lead to divergence of the thermodynamic partition function and a breakdown of the canonical ensemble. In [14] it was argued that as the
temperature approaches the Hagedorn temperature, a first order phase transition should occur with a large latent heat. At this phase transition a condensate of the winding modes will form. This in turn leads to a genus-zero contribution to the free energy.

Exactly the same kind of argument will apply here, with the role of the winding modes replaced by the tachyonic fields appearing in the twisted sectors. The infrared divergence of (2.10) is a consequence of expanding around an unstable vacuum in the computation of the free energy. The correct procedure is to give the winding modes expectation values which lead to stable solutions, minimize the free energy. In general one might also expect the dilaton and possibly other massless fields to acquire expectation values. The problem of finding this non-trivial minimum, and describing the physics beyond this phase transition, appears to require non-perturbative string physics and is at present intractable.

3. Type II Superstring

In this section, we consider the case of type II superstrings to verify that a similar condensate forms in this case. In the following, we use the covariant Ramond-Neveu-Schwarz formalism. These theories have previously been considered in [11] using the light–cone Green–Schwarz formalism. In superconformal gauge, the worldsheet action for the RNS string is

\[ S = -\frac{1}{2\pi} \int d^2\sigma \left( \partial_\alpha X^\mu \partial^\alpha X^\nu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \right) \eta_{\mu\nu}, \]  

(3.1)

where \( \psi^\mu(\sigma) \) are Majorana–Weyl worldsheet spinors transforming as a spacetime vector, i.e. in the same way as \( X^\mu(\sigma) \). As before, we combine \( X_0 \) and \( X_1 \) into a complex coordinate \( X \), and likewise combine \( \psi^0 \) and \( \psi^1 \) into a complex spinor \( \psi \). To construct physical states it is necessary to do a GSO projection, which leads to a sum over spin structures on the worldsheet.

3.1. Theories without spacetime fermions

To begin with, we will ignore the level matching conditions which will lead to theories with an infinite number of spacetime fermionic states. In the \( k \)'th twisted sector \( (k = 0, \ldots, N - 1) \), the boundary conditions are

\[ \psi(\sigma_1 + 1, \sigma_2) = e^{2\pi i k/N} \psi(\sigma_1, \sigma_2) \quad (R) \]

\[ \psi(\sigma_1 + 1, \sigma_2) = -e^{2\pi i k/N} \psi(\sigma_1, \sigma_2) \quad (NS) \]

\[ X(\sigma_1 + 1, \sigma_2) = e^{2\pi i k/N} X(\sigma_1, \sigma_2) \]

(3.2)

where R refers to the Ramond sector, and NS refers to the Neveu–Schwarz sector. The partition functions for the twisted fermions are easily computed [15], and lead to the following modular invariant partition function for the orbifold

\[ Z_N = \frac{1}{4N} \frac{1}{|\eta^2 \bar{\eta}^2 \Im \tau|^3} \sum_{k,l=0}^{N-1} \sum_{\alpha,\beta} \left| \frac{\theta^3 \left[ \frac{\alpha}{\beta} \right] \theta \left[ \frac{\alpha + \frac{1}{2}}{\beta + \frac{1}{2}} \right]}{\theta \left[ \frac{\alpha + \frac{1}{2}}{\beta + \frac{1}{2}} \right] \eta^3} \right|^2. \]  

(3.3)
Here we are performing a diagonal sum over spin structures, labeled by $\alpha, \beta = 0, 1/2$. This corresponds to projecting onto states with $(-1)^F = +1$ in the (NS,NS) sector, where $F$ is the sum of the fermion numbers of the left and right movers. In the (R,R) sector the projection is also onto states with $(-1)^F = +1$. The $k = l = 0$ term should be interpreted as in (2.8).

In the usual type II string there is a separate GSO projection for left and right movers which leads to spacetime fermions in the (R,NS) and (NS,R) sectors. However, here these sectors are absent and these models therefore contain only spacetime bosons. The NS ground state is not projected out, so tachyons are present both in the untwisted and twisted sectors, with $M^2 = 4(-1 + k/N)$. Note that in the Ramond sector, states could alternatively be projected onto $(-1)^F = -1$ which would lead to the same partition function, but a different spectrum. The theories with $N = 1$ have been considered previously by Seiberg and Witten [18]. Using the Riemann theta identity [16] it may be shown that (3.3) agrees with [11] (equation (2.21) for $N$ even, and equation (2.22) for $N$ odd).

3.2. Theories with spacetime fermions

To obtain consistent theories with spacetime fermions some care is required. The action of the discrete group $Z_N$ on the worldsheet fields must be chosen in such a way that there is level matching between the right moving NS sector and the left moving R sector (and vice versa), to ensure an infinite number of fermionic states appear in the spectrum. In addition, the GSO projection must be chosen in a manner consistent with modular invariance. It should be noted that any theory with non-trivial deficit angle will necessarily break spacetime supersymmetry.

Let us consider the contributions to the partition function coming from the worldsheet fermion determinants in the (R,NS) sector. Inserting the GSO projection, this corresponds to $\frac{1}{4} \text{Tr} (1 + (-1)^{F_L})(1 - (-1)^{F_R})q^{H_L}q^{H_R}$ where the left-moving states are Ramond fermions twisted by $h \in Z_N$, and the right-moving states are Neveu-Schwarz fermions twisted by $h$. $H_L$ ($H_R$) denotes the left-(right-)moving hamiltonian for the fermions, and $F_L$ ($F_R$) denotes the left-(right-)moving worldsheet fermion number. Because $h^N = 1$, the boundary conditions on the fermions are unchanged under a modular transformation $\tau \rightarrow \tau + N$, so the trace should not pick up a phase. This will be true, provided that for an arbitrary state which is invariant under the GSO projections,

$$N(E_L - E_R) = 0 \text{ mod } 1. \quad (3.4)$$

We take the action of the group element $h$ on the fermions to be

$$\psi(\sigma_1 + 1, \sigma_2) = e^{2\pi i s/N} \psi(\sigma_1, \sigma_2) \quad (R)$$

$$\psi(\sigma_1 + 1, \sigma_2) = -e^{2\pi i s/N} \psi(\sigma_1, \sigma_2) \quad (NS). \quad (3.5)$$

The energies of the states are then

$$E = \frac{1}{2} \left( \frac{s}{N} \right)^2 - \frac{1}{2} \left( \frac{s}{N} \right) + \frac{1}{3} + m_1 + m_2 \frac{s}{N} \quad (R) \quad (3.6)$$

$$E = \frac{1}{2} \left( \frac{s}{N} \right)^2 - \frac{1}{6} + n_1 \left( \frac{s}{N} \right) - n_2 \frac{s}{N} + n_3 \quad (NS) \quad (3.6)$$

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where \( m_i \) and \( n_i \) are integers. The GSO projections impose additional restrictions on these integers. It is sufficient to impose the condition (3.4) on the lowest energy states of the left and right sectors, which correspond to \( m_1 = m_2 = 0 \) in the R sector and \( n_1 = 1, n_2 = n_3 = 0 \) in the NS sector. This leads to the condition that \( s \) is even. We take
\[
s = 2. \tag{3.7}
\]
Other non-zero values lead to equivalent theories.

The necessity of taking \( s = 2 \) (or even) arises because of the presence of spacetime fermions. In a theory with spacetime fermions, a \( 2\pi \) rotation is not trivial, so one cannot construct a \( Z_N \) orbifold with a group action whose \( N \)’th power is a \( 2\pi \) rotation. Rather one needs a group action whose \( N \)’th power is a \( 4\pi \) rotation. This is accomplished by setting \( s = 2 \) in (3.5).

The boundary conditions in the \( k \)’th twisted sector are then
\[
\psi(\sigma_1 + 1, \sigma_2) = e^{4\pi ik/N} \psi(\sigma_1, \sigma_2) \quad \text{(R)}
\]
\[
\psi(\sigma_1 + 1, \sigma_2) = -e^{4\pi ik/N} \psi(\sigma_1, \sigma_2) \quad \text{(NS)} \tag{3.8}
\]
\[
X(\sigma_1 + 1, \sigma_2) = e^{4\pi ik/N} X(\sigma_1, \sigma_2).
\]

The usual GSO projection gives the following modular invariant partition function
\[
Z_N = \frac{1}{4N} \frac{1}{\eta^2 \bar{\eta}^2 \text{Im} \tau} \sum_{k,l=0}^{N-1} \sum_{\alpha,\beta,\gamma,\delta} \omega_{\alpha,\beta}(k,l) \bar{\omega}_{\gamma,\delta}(k,l) \frac{\theta^3[\alpha, \beta] \theta^{\gamma + \frac{2}{N}} \bar{\theta}^{\delta + \frac{2}{N}}}{\theta^3\bar{\theta}^{\gamma + \frac{2}{N}} \bar{\eta}^2} \theta^{\frac{1}{2} + \frac{2k}{N}}. \tag{3.9}
\]

The sum over spin structures corresponds to the sum over \( \alpha, \cdots, \delta = 0, \frac{1}{2} \). The coefficients \( \omega_{\alpha,\beta}(k,l) \) are \( \omega_{1/2,1/2} = \pm e^{-2\pi ik/N}, \omega_{1/2,0} = -1, \omega_{0,1/2} = 1 \) and \( \omega_{0,1/2} = -e^{-2\pi ik/N} \). The \( e^{-2\pi ik/N} \) factors arise from the action of \((-1)^F\) on the vacua in the Neveu-Schwarz and Ramond sectors.

For \( N \) odd, this partition function has an interpretation as a string theory living on a cone with deficit angle \( 2\pi(1 - 1/N) \). To see this we use the simple transformation properties of (3.9) under shifts of \( 2k \) and \( 2l \) by \( N \), and reorganize the sums over twisted sectors replacing \( 2k \) and \( 2l \) by \( k \) and \( l \). This yields a partition function whose bosonic contribution is exactly the same as in the bosonic partition function (2.9):
\[
Z_N = \frac{1}{4N} \frac{1}{\eta^2 \bar{\eta}^2 \text{Im} \tau} \sum_{k,l=0}^{N-1} \sum_{\alpha,\beta,\gamma,\delta} \omega'_{\alpha,\beta}(k,l) \bar{\omega}'_{\gamma,\delta}(k,l) \frac{\theta^3[\alpha, \beta] \theta^{\gamma + \frac{2}{N}} \bar{\theta}^{\delta + \frac{2}{N}}}{\theta^3\bar{\theta}^{\gamma + \frac{2}{N}} \bar{\eta}^2} \theta^{\frac{1}{2} + \frac{2k}{N}}. \tag{3.10}
\]

Here we define \( \omega'_{\alpha,\beta}(k,l) = \omega_{\alpha,\beta}(k/2,l)e^{2\pi i(\alpha l - \beta k)} \). Using the Riemann theta identity [16], it may be shown that (3.10) agrees with that calculated in [11] (equation (2.20)).

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4 This condition may be relaxed if we change the GSO projection so that it mixes with the projection onto \( Z_N \) invariant states, as we see later in this section.
To obtain a theory with a thermodynamic interpretation, spacetime fermions should be periodic around the tip of the cone. This is analogous to demanding that spacetime fermions be antiperiodic around the cylinder, as in the usual flat space finite temperature calculations. In that case, care must be taken in defining the GSO projection to ensure that fermions have the correct properties \cite{9}. It turns out that the fermions described by the above orbifold are indeed periodic around the cone, so no additional signs need to be introduced into (3.10) to obtain a thermodynamic partition function.

This model contains fermionic and bosonic states in the left and right-moving sectors. In the twisted sectors, the projection onto $Z_N$ invariant states eliminates all fermions. In the untwisted sector, fermionic states with zero momentum in the $X_1$ and $X_2$ directions are eliminated by the $Z_N$ projection, but non-zero momentum states survive. In each twisted sector, a tachyon is present. When $k$ is odd, these correspond to the ground state of the (NS,NS) sector with $M^2 = 4(-1 + k/N)$, which is not projected out in this case. When $k$ is even, the NS ground state is projected out and the states

$$\psi_{-1/2+k/N} \bar{\psi}_{-1/2+k/N} |0; 0\rangle$$

are tachyonic with $M^2 = -4k/N$. No tachyons are present in the untwisted sector. For $N = 1$, this partition function reproduces that of the usual flat space type II string.

For $N = 2N'$, with $N'$ odd or even, (3.3) similarly has an interpretation as a string theory on a cone – this time with deficit angle $2\pi(1 - 1/N')$. However, it turns out these partition functions are identical to the theories discussed in the previous subsection, and all spacetime fermions are projected out.

Finally, the free energy at genus one is given by

$$\beta F_N = -V_T \int F \frac{d^2 \tau}{(\text{Im} \tau)^2} Z_N ,$$

(3.12)

where $Z_N$ is one of the type II string partition functions discussed above. As for the bosonic string, the free energy will be negative infinity due to the infrared divergence from the tachyons present in the twisted sectors. This should be dealt with by giving these fields expectation values, leading to a genus-zero contribution to the free energy.

Note that in the theory with $N$ odd, (3.10), the leading infrared divergence arises from the sectors with $k = 1$ and $k = N - 1$, which contain tachyonic states with $M^2 = 4(-1 + 1/N)$. It is natural to presume that this leading divergence is the same for superstrings propagating on cones of arbitrary deficit angle. A consistency check on this is that in the limit $N \to 1$ one regains the usual supersymmetric flat space theory. Hence we assume that analogs of these states are present for arbitrary (non-integral) $N$, and that the mass formula may therefore be analytically continued in $N$. As $N \to 1$, these states become massless, consistent with the fact that the $N = 1$ orbifold is just the usual type II superstring.

In the limit $N \to 1$, it is possible to use effective field theory methods as in \cite{4} to make the arguments concerning the formation of the condensate more precise. The effective potential for the two most tachyonic twisted sector fields, denoted $\phi$ and $\phi^*$ is

$$V = 4(\frac{1}{N} - 1)\phi \phi^* + g\mu(\phi \phi^*)^2 + \cdots$$

(3.13)
which should be valid when the fields are sufficiently small. Here $g$ is the closed string coupling constant and $u$ is a constant. This effective potential may then be minimized for $N > 1$ by having $\phi$ jump to non-zero expectation values of order $(N - 1)/g$, leading to a phase transition and a genus-zero contribution to the free energy. Unfortunately the non-trivial minimum can not be found in perturbation theory, and a more detailed description of this phase transition has not been obtained.

The entropy is related to the free energy by

$$S = \beta^2 \frac{\partial F}{\partial \beta} = -2\pi \frac{\partial F}{\partial N}.$$  \hspace{1cm} (3.14)

Since, in general, one expects $\langle \phi \rangle \to 0$ as $N$ approaches 1, the genus-zero condensate contribution to the free energy should approach zero faster than $(N - 1)$ as indicated by (3.13). It follows from (3.14) that the genus-zero condensate contribution to the entropy at $N = 1$ will vanish. Thus the details of the physics of this strong coupling condensate are irrelevant to the calculation of the genus-zero entropy.

There is an important caveat to the foregoing discussion. We have described a second order phase transition at $\epsilon = 0$. However corrections of various kinds could easily perturb this to a first order phase transition, which might bear on the entropy calculation. Indeed in [9] it was suggested that couplings to the string dilaton alter the naively second order phase transition at the Hagedorn temperature to a first order transition at a lower critical temperature. A naive translation of this suggestion to the present context leads to the nonsensical conclusion that a condensate forms at deficit angles greater than some critical value $\epsilon_{\text{crit}} < 0$, and is therefore present even in flat space! While this seems unlikely, a better understanding of the phase transition could certainly alter our conclusion that the condensate does not enter into the genus-zero entropy calculation.

4. Heterotic string

Now we consider $Z_N$ orbifolds of heterotic string theories with single conical singularities. In this case no non-trivial models exist for arbitrary $N$ unless the conical singularity is accompanied by a Wilson line breaking the symmetry of the internal gauge group, or some additional twist of the spacetime modes.

We will work with the $E_8 \times E_8$ version of the heterotic string with a fermionic representation of the gauge degrees of freedom. In superconformal gauge, the worldsheet action is then

$$S = -\frac{1}{2\pi} \int d^2\sigma \left( (\partial_\alpha X^\mu)^2 - i\psi^\mu \rho^- \partial_- \psi_\mu - i\lambda^i \rho^+ \partial_+ \lambda^i - i\tilde{\lambda}^i \rho^+ \partial_+ \tilde{\lambda}^i \right),$$  \hspace{1cm} (4.1)

where $\mu = 0, \ldots, 9$, and $i = 1, \ldots, 16$. The right-moving sector is that of the usual RNS string. The left-moving sector consists of the ten bosonic degrees of freedom plus the two groups of sixteen fermions which generate the $E_8 \times E_8$ current algebra. To remove the tachyon, the usual GSO projection is performed for the right-moving RNS fermions. To correctly generate the $E_8 \times E_8$ current algebra a GSO–like projection must be performed.
on the left-moving fermions, which removes states with an odd number of \( \lambda^i \) and an odd number of \( \tilde{\lambda}^i \) oscillators. The states then divide up into explicit representations of \( SO(16) \times SO(16) \).

We are interested in orbifolding this theory by \( Z_N \). If we work in light-cone gauge, the symmetry group of the theory contains \( K = O(8) \times O(16) \times O(16) \), and an element of \( K \) acts on the worldsheet fields as

\[
h \cdot (X, \psi, \lambda, \tilde{\lambda}) = (\theta X, \theta \psi, \theta_1 \lambda, \theta_2 \tilde{\lambda}) .
\]

In fact, at the genus-one, we should take an eight-fold cover of \( K \) corresponding to the action of \( h \) on the different spin structures of the fermions. If \( h \) lies in a \( Z_N \) subgroup of \( K \), we may write

\[
\begin{align*}
\theta &= (e^{2\pi i r_1/N}, \ldots, e^{2\pi i r_8/N}, \ldots, \text{c.c.}) \\
\theta_1 &= (e^{2\pi i p_1/N}, \ldots, e^{2\pi i p_8/N}, \ldots, \text{c.c.}) \quad (4.3) \\
\theta_2 &= (e^{2\pi i q_1/N}, \ldots, e^{2\pi i q_8/N}, \ldots, \text{c.c.})
\end{align*}
\]

where the \( r_i, p_i \) and \( q_i \) are integers. As shown by Vafa [19], a necessary condition for modular invariance of the resulting orbifold is that

\[
\sum_{1}^{4} (r_i)^2 = \sum_{1}^{8} (p_i)^2 + (q_i)^2 \quad \text{mod } N ,
\]

when \( N \) is odd, and

\[
\sum_{1}^{4} (r_i)^2 = \sum_{1}^{8} (p_i)^2 + (q_i)^2 \quad \text{mod } 2N
\]

\[
\sum_{1}^{4} r_i = \sum_{1}^{8} p_i = \sum_{1}^{8} q_i = 0 \quad \text{mod } 2 ,
\]

when \( N \) is even.

It is easy to see that if we wish to construct orbifolds with deficit angles \( 2\pi (1 - 1/N) \), we cannot satisfy the above conditions unless \( h \) also has non-trivial action on either the \( \lambda^i, \tilde{\lambda}^i \), or the transverse \( X^i \) and \( \psi^i \). Here we consider the case \( r_1 = p_1 = 1 \), with all other \( r, p \) and \( q \) zero, which satisfies the above conditions when \( N \) is odd. This corresponds to embedding the spin connection in the first \( E_8 \) gauge group, and will lead to an orbifold with an interpretation as a cosmic string with a Wilson line attached.

The partition function of this theory is

\[
Z_N = \frac{1}{8N} \frac{1}{|\eta^2 \bar{\eta}^2 \Im \tau|^3} \sum_{k,l=0}^{N-1} \sum_{\alpha, \beta, \gamma, \delta, \kappa, \lambda} \omega_{\alpha \beta}(k,l) \bar{p}_{\gamma, \delta}(k,l) \bar{p}_{\kappa, \lambda}(0,0) \\
\times \theta^3 \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \theta \left[ \begin{array}{c} \alpha + \frac{\delta}{2} \\ \beta + \frac{\gamma}{2} \end{array} \right] \bar{\theta}^7 \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] \bar{\theta} \left[ \begin{array}{c} \gamma - \frac{\delta}{2} \\ \delta + \frac{\gamma}{2} \end{array} \right] \bar{\theta}^8 \left[ \begin{array}{c} \kappa \\ \lambda \end{array} \right] \\
\times \theta \left[ \begin{array}{c} \frac{\alpha + \delta}{2} \\ \frac{\beta + \gamma}{2} \end{array} \right] \bar{\eta}^3 \bar{\eta}^{15} .
\]

(4.6)
where $\omega'(k,l)$ is as defined below (3.10), and $\bar{\rho}(k,l)$ are $\bar{\rho}_{1/2,1/2} = \pm e^{i\pi k/N}(-1)^{k+l}$, $\bar{\rho}_{1/2,0} = (-1)^l$, $\bar{\rho}_{0,0} = 1$, and $\bar{\rho}_{0,1/2} = e^{i\pi k/N}(-1)^k$. The sums over $\alpha, \cdots, \lambda = 0, 1/2$ correspond to the sum over spin structures. The genus-one free energy is then

$$\beta F = -V_T \int_F \frac{d^2\tau}{(\text{Im}\tau)^2} Z_N . \quad (4.7)$$

This theory contains tachyons in each of the twisted sectors, localized near the tip of the cone. When $k$ is odd, the ground state of the NS sector is not projected out, and the GSO-like projection on the left-moving fermions removes states with an even number of the $\lambda^i$ or $\bar{\lambda}^i$. The physical states

$$\lambda_{-1/2}^i |0; 0\rangle , \quad \bar{\lambda}_{-1/2}^i |0; 0\rangle , \quad \bar{\lambda}_{-1/2}^i |0; 0\rangle , \quad \bar{\lambda}_{-1/2}^i |0; 0\rangle , \quad (4.8)$$

$(i = 2, \cdots , 8)$ are tachyonic with $M^2 = 4(-1 + k/N)$. When $k$ is even, the tachyonic physical state is

$$\psi_{-1/2+k/N} \bar{\alpha}_{-1+k/N} |0; 0\rangle , \quad (4.9)$$

with $M^2 = -4k/N$. The untwisted sector contains the usual spectrum of the $E_8 \times E_8$ heterotic string in flat space, projected onto $Z_N$ invariant states. The qualitative conclusion that a condensate of these states will form, leading to a genus-zero contribution to the free energy, is the same as in the bosonic and type II superstring cases.

**Acknowledgements**

We would like to thank L. Thorlacius for useful discussions. This work was supported in part by NSF grant PHY-91-16964 and DOE grant 91-ER40618.
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