A recursive construction for simple $t$-designs using resolutions

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Abstract This work presents a recursive construction for simple $t$-designs using resolutions of the ingredient designs. The result extends a construction of $t$-designs in our recent paper van Trung (Des Codes Cryptogr 83:493–502, 2017). Essentially, the method in van Trung (Des Codes Cryptogr 83:493–502, 2017) describes the blocks of a constructed design as a collection of block unions from a number of appropriate pairs of disjoint ingredient designs. Now, if some pairs of these ingredient $t$-designs have both suitable $s$-resolutions, then we can define a distance mapping on their resolution classes. Using this mapping enables us to have more possibilities for forming blocks from those pairs. The method makes it possible for constructing many new simple $t$-designs. We give some application results of the new construction.

Keywords Recursive construction · Resolution · Large set · Simple $t$-design

Mathematics Subject Classification 05B05

1 Introduction

In a recent paper [41] we have presented a recursive method for constructing simple $t$-designs for arbitrary $t$. The method is of combinatorial nature since it requires finding solutions for the indices of the ingredient designs that satisfy a certain set of equalities. In essence, the core of the construction is that the blocks of a constructed design are built as a collection of block unions from a number of appropriate pairs of disjoint ingredient designs. In particular, when a pair of ingredient designs is used, we take as new blocks the unions of all the pairs of
blocks in the two ingredient designs. For the sake of simplicity we refer to this construction method as the basic method or the basic construction.

In the present paper we describe an extension of the basic construction by assuming that a subset of pairs of ingredient designs have suitable resolutions. For those given pairs we may define a distance mapping on their resolution classes. By using this mapping we have more possibilities for forming blocks from those pairs other than taking the unions of all possible pairs of blocks in the ingredient designs. This construction actually extends the basic construction since many new simple \( t \)-designs can only be constructed with the new method. The crucial point of this extension is the use of \( s \)-resolutions for \( t \)-designs. The concept of \( s \)-resolutions may be viewed as a generalization of the notion of parallelisms, which may be termed as \((1, 1)\)-resolutions, i.e. the blocks of the \( t \)-design can be partitioned into classes of mutually disjoint blocks such that every point is in exactly one block of each class. To date very little is known about \( s \)-resolutions for \( t \)-designs when \( s \geq 2 \), except for the trivial \( t \)-designs. In this case, an \( s \)-resolution of the trivial \( t \)-design turns out to be a large set of \( s \)-designs. A great deal of results about large sets of \( s \)-designs have been achieved by many researchers, see the references below. We will describe our construction in terms of \( s \)-resolutions for \( t \)-designs in general. However we will restrict its applications just for the case where pairs of trivial designs are used and each has a suitable large set. Even with this limitation we find that the construction using resolutions still possesses its strength since it produces many simple \( t \)-designs.

It is worthwhile to emphasize that constructing simple \( t \)-designs for large \( t \) is a challenging problem in design theory. There are several major approaches to the problem. These include constructing \( t \)-designs from large sets of \( t \)-designs, for instance [1,13,16,18,19,21,22,24,25,28,32–34]; constructing \( t \)-designs by using prescribed automorphism groups, for example [2,3,6–10,14,20,23,26,29]; or constructing \( t \)-designs via recursive construction methods, see for instance [15,17,27,30,36–41].

### 2 Preliminaries

We recall some basic definitions. A \( t \)-design, denoted by \((v, k, \lambda)\), is a pair \((X, \mathcal{B})\), where \( X \) is a \( v \)-set of points and \( \mathcal{B} \) is a collection of \( k \)-subsets, called blocks, of \( X \) having the property that every \( t \)-set of \( X \) is a subset of exactly \( \lambda \) blocks in \( \mathcal{B} \). The parameter \( \lambda \) is called the index of the design. A \( t \)-design is called simple if no two blocks are identical i.e. no block of \( \mathcal{B} \) is repeated; otherwise, it is called non-simple (i.e. \( \mathcal{B} \) is a multiset). It can be shown by simple counting that a \((t, v, k, \lambda)\) design is an \( s-(v, k, \lambda_s)\) design for \( 0 \leq s \leq t \), where \( \lambda_s = \lambda (v-s)^t / (t-s)^t \). Since \( \lambda_s \) is an integer, necessary conditions for the parameters of a \( t \)-design are \((v-t)^s (t-s)^v (s^t)\), for \( 0 \leq s \leq t \). For given \( t, v \) and \( k \), we denote by \( \lambda_{\min}(t, v, k) \), or \( \lambda_{\min} \) for short, the smallest positive integer such that these conditions are satisfied for all \( 0 \leq s \leq t \). By complementing each block in \( X \) of a \((t, v, k, \lambda)\) design, we obtain a \((t, v, v-k, \lambda^*)\) design with \( \lambda^* = \lambda (v-k)^t / k^t \), hence we shall assume that \( k \leq v/2 \). The largest value for \( \lambda \) for which a simple \((t, v, k, \lambda)\) design exists is denoted by \( \lambda_{\max} \) and we have \( \lambda_{\max} = (v-t)^t / (k-t)^t \). The simple \((t, v, k, \lambda_{\max})\) design is called the complete design or the trivial design. A \((t, v, k, 1)\) design is called a \( t \)-Steiner system.

We refer the reader to [5,12] for more information about designs.

**Definition 2.1** A \((t, v, k, \lambda)\)-design \((X, \mathcal{B})\) is said to be \((s, \tau)\)-resolvable with \( 0 \leq s < t \), if its block set \( \mathcal{B} \) can be partitioned into \( N \) classes \( \mathfrak{A}_1, \ldots, \mathfrak{A}_N \) such that \((X, \mathfrak{A}_t)\) is an \( s\)-
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constructed by using the group PGL

resolution classes in the partition.

Definition 2.2

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2.1 The basic construction

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\text{resolvability) for BIBD in [31]. A definition of (s, \tau)-resolvability for t-designs with

t \geq 3 may be found in [4]. In that paper Baker shows that the Steiner quadruple system

3-

(4^m, 4, 1) constructed from an even dimensional affine space over the field of two elements

has a (2, 1)-resolution. Also, Teirlinck shows for example that there exists a 2-resolvable

3-(2p^n + 2, 4, 1) design with \( p \in \{7, 31, 127\} \), for any positive integer \( n \), [35].

To date, very little is known about \( s \)-resolutions of non-trivial \( t-(v, k, \lambda) \) designs for \( t \geq 3 \)

and \( s \geq 2 \). Here are examples with \( t = 4 \) and \( s = 3 \). In [2], Alltop has shown that there

exists a simple 4-(9, 1, 5, 5) design for every \( q = 2^n \), \( n \geq 5 \), \( n \) odd. This 4-design \((X, \mathcal{B})\)

is constructed by using the group PGL(2, \( q \)), which acts sharply 3-transitively on the projective

design in the Alltop’s family has a (3, 15)-resolution.

When \((X, \mathcal{B})\) is the trivial \( t-(v, k, \binom{v-1}{k-1}) \) design, then an \( (s, \tau) \)-resolution of \((X, \mathcal{B})\)

called a large set. Thus, a large set is a partition of the complete \( t-(v, k, \binom{v-1}{k-1}) \) design into

\( s \)-(v, k, \( \tau \)) designs, and is denoted by LS\{N\}(s, k, v), where \( N = \frac{\binom{v-1}{k-1}}{\tau} \) is the number of

resolution classes in the partition.

We define a distance on the resolution classes of a \( t \)-design as follows.

**Definition 2.2** Let \( D \) be a \( t-(v, k, \lambda) \) design admitting an \((s, \tau)\)-resolution with \( \mathcal{A}_1, \ldots, \mathcal{A}_N \)

as resolution classes. Define a distance between any two classes \( \mathcal{A}_i \) and \( \mathcal{A}_j \) by \( d(\mathcal{A}_i, \mathcal{A}_j) = \min\{|i - j|, N - |i - j|\} \).

2.1 The basic construction

In this section, we summarize the basic construction as described in [41]. This preparation

is necessary for the description of the construction using resolutions in the next section.

We first give notation and definitions. Let \( t, \ v, \ k \) be non-negative integers such that

\( v \geq k \geq t \geq 0 \). Let \( X \) be a v-set and let \( X = X_1 \cup X_2 \) be a partition of \( X \) (i.e \( X_1 \cap X_2 = \emptyset \))

with \(|X_1| = v_1\) and \(|X_2| = v_2\).

The parameter set \( t-(v_2, j, \lambda^{(j)}) \) for a design indicates that the point set of the design is

\( X_2 \). Also, a design defined on the point set \( X_2 \) is denoted by \( \bar{D} = (X_2, \mathcal{B}) \).

(i) For \( i = 0, \ldots, t \), let \( D_i = (X_1, \mathcal{B}^{(i)}) \) be the complete \( i-(v_1, i, 1) \) design. For \( i = t + 1, \ldots, k \), let \( D_i = (X_1, \mathcal{B}^{(i)}) \) be a simple \( t-(v_1, i, \lambda^{(i)}) \) design.

(ii) Similarly, for \( i = 0, \ldots, t \), let \( \bar{D}_i = (X_2, \mathcal{B}^{(i)}) \) be the complete \( i-(v_2, i, 1) \) design.

And for \( i = t + 1, \ldots, k \), let \( \bar{D}_i = (X_2, \mathcal{B}^{(i)}) \) be a simple \( t-(v_2, i, \lambda^{(i)}) \) design.

(iii) Two degenerate cases for designs occur when either \( k = t = 0 \) or \( v = k \). The first case

\( k = t = 0 \) gives an “empty” design, denoted by \( \emptyset \), however we use the convention that

the number of blocks of the empty design is 1 (i.e. the unique block is the empty block).

The second case \( v = k \) gives a degenerate \( k \)-design having just 1 block consisting of all \( v \) points. Thus, in these two extreme cases the number of blocks of the designs is always 1.

(iv) We denote by \( T_{(r,t-r)} \) a \( t \)-subset \( T \) of \( X \) with \(|T \cap X_1| = r \) and hence \(|T \cap X_2| = t - r \), for \( r = 0, \ldots, t \). It is clear that any \( t \)-subset of \( X \) is a \( T_{(r,t-r)} \) set for some \( r \in \{0, \ldots, t\} \).
Theorem 2.1 (Basic construction) Let \( v, k, t \) be integers with \( v > k > t \geq 2 \). Let \( X \) be a \( v \)-set and let \( X = X_1 \cup X_2 \) be a partition of \( X \) with \( |X_1| = v_1 \) and \( |X_2| = v_2 \). Let \( D_1 = (X_1, \mathcal{B}(i)) \) be the complete \( i \)-(v_1, i, 1) design for \( i = 0, \ldots, t \) and let \( D_2 = (X_2, \mathcal{B}(i)) \) be a simple \( t \)-(v_2, k \( - \) \( i \), \( \lambda(i) \)) design for \( i = t + 1, \ldots, k \). Define \( \mathcal{B}(i) = \mathcal{B}(0, k) \times [u_0] \cup \mathcal{B}(1, k - 1) \times [u_1] \cup \cdots \cup \mathcal{B}(k - 1, 1) \times [u_{k - 1}] \cup \mathcal{B}(k, 0) \times [u_k] \), where \( u_i \in \{0, 1\} \), for \( i = 0, \ldots, k \).

Assume that

\[
L_{0,t} = L_{1,t-1} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda,
\]

for a positive integer \( \Lambda \), where

\[
L_{r,t-r} = \sum_{i=0}^{k} u_i \lambda_r(i) \lambda_{t-r}^{(k-i)},
\]

\( r = 0, \ldots, t \), and \( u_i \in \{0, 1\} \), for \( i = 0, \ldots, k \). Then \( (X, \mathcal{B}) \) is a simple \( t \)-(v, k, \( \Lambda \)) design.
3 The construction using resolutions

In this section we describe a recursive construction of simple $t$-designs using resolutions. Note that in the basic construction, if a pair $(D_i, \bar{D}_{k-i})$ is used in the construction (i.e. $u_i = 1$), then the new blocks formed by this pair consist of taking the union of each block of $D_i$ with each block of $\bar{D}_{k-i}$. The crucial idea of the construction using resolutions is that if $D_i$ and $\bar{D}_{k-i}$ have appropriate $s_1$- and $s_2$-resolutions with the same number of resolution classes, then the new blocks are formed according to the distance mapping defined on the resolution classes of $D_i$ and $\bar{D}_{k-i}$ rather than taking the unions of each block of $D_i$ with each block of $\bar{D}_{k-i}$.

In the following we go into detail of the construction. We make use of the notation and definitions for the basic construction in the previous section. When for a certain $i \in \{0, \ldots, k\}$ the $t$-$(v_1, i, \lambda_i^{(i)})$ design $D_i = (X_1, \mathcal{B}^{(i)})$ has an $s_1$-resolution, i.e. $D_i$ can be partitioned into $N_i$ disjoint $(X_1, \mathcal{A}_h^{(i)})$ designs with parameters $s_1$-$(v_1, i, \lambda_{s_1}^{(i)})$, $s_i < t$, then we write

$$\mathcal{B}^{(i)} = \bigcup_{h=1}^{N_i} \mathcal{A}_h^{(i)},$$

where

$$N_i = \lambda_i^{(i)} \left( \frac{v_1 - s_i}{t - s_i} \right) / \lambda_{s_1}^{(i)} \left( \frac{t - s_i}{t - s_i} \right).$$

Similarly, we write

$$\bar{\mathcal{B}}^{(k-i)} = \bigcup_{h=1}^{\bar{N}_{k-i}} \bar{\mathcal{A}}_h^{(i)},$$

when the blocks of a $t$-$(v_2, k - i, \bar{\lambda}_i^{(k-i)})$ design $\bar{D}_{k-i} = (X_2, \bar{\mathcal{B}}^{(k-i)})$ can be partitioned into $\bar{N}_{k-i}$ disjoint $(X_2, \bar{\mathcal{A}}_h^{(k-i)})$ designs with parameters $s_{k-i}$-$(v_2, k - i, \bar{\lambda}_{s_{k-i}}^{(k-i)})$, where

$$\bar{N}_{k-i} = \bar{\lambda}_i^{(k-i)} \left( \frac{v_2 - s_{k-i}}{t - s_{k-i}} \right) / \bar{\lambda}_{s_{k-i}}^{(k-i)} \left( \frac{k - i - s_{k-i}}{t - s_{k-i}} \right)$$

is the number of $s_{k-i}$-resolution classes.

Let $K = \{(0, k), (1, k - 1), \ldots, (k - 1, 1), (k, 0)\}$. Assume there exists a subset $R \subseteq K$ such that if $(i, k - i) \in R$, then $D_i$ and $\bar{D}_{k-i}$ have an $s_i$-resolution of size $N_i$ and an $s_{k-i}$-resolution of size $\bar{N}_{k-i}$, respectively, satisfying the following conditions.

(i) $N_i = \bar{N}_{k-i}$,

(ii) $s_i + s_{k-i} \geq 2 \left\lceil \frac{t}{2} \right\rceil$.

The construction consists of building two types of blocks.

(1) For each pair $(i, k - i) \in K \setminus R$ form a subset of new blocks $\mathcal{B}_{(i,k-i)}$ from the pair $(D_i, \bar{D}_{k-i})$ as

$$\mathcal{B}_{(i,k-i)} := \{ B = B_i \cup \bar{B}_{k-i} \mid B_i \in \mathcal{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathcal{B}}^{(k-i)} \}.$$  

(2) For each pair $(i, k - i) \in R$ form a subset of new blocks $\mathcal{B}_{(i,k-i)}^*$ from $(D_i, \bar{D}_{k-i})$ by using an $s_i$-resolution of $D_i$ and an $s_{k-i}$-resolution of $\bar{D}_{k-i}$ as follows.

$$\mathcal{B}_{(i,k-i)}^* := \left\{ B_i \cup \bar{B}_{k-i} \mid B_i \in \mathcal{A}_h^{(i)}, \bar{B}_{k-i} \in \bar{\mathcal{A}}_j^{(k-i)}, \varepsilon_i \leq d \left( \mathcal{A}_h^{(i)}, \mathcal{A}_j^{(i)} \right) \leq w_i, \right\}$$
\[ \varepsilon_i = 0, 1; \quad w_i \leq \left\lfloor \frac{N_i}{2} \right\rfloor. \]

Further, define
\[ z_i := (2w_i + 1 - \varepsilon_i), \quad \text{if} \quad w_i < \frac{N_i}{2}, \quad \text{and} \quad z_i := (2w_i - \varepsilon_i), \quad \text{if} \quad w_i = \frac{N_i}{2}. \]

Note that \( w_i \) and \( z_i \) are considered as variables.

Now, let \( T_{(r,t-r)} \) be a \( t \)-set of \( X \) for \( r = 0, \ldots, t \). According to the property of \( s_i \) and \( s_{k-i} \), one of the following cases has to occur.

\( (a) \quad r \leq s_i \) and \( t - r \leq s_{k-i} \). Then \( T_{(r,t-r)} \) is contained in
\[ \Lambda^{*(i,k-i)}_{r,t-r} = \lambda_r^{*}(i) \lambda_{t-r}^{*(k-i)} . N_i . z_i \]
blocks of \( \mathfrak{B}^{*}_{(i,k-i)} \).

\( (b) \quad r \leq s_i \) and \( t - r > s_{k-i} \). Then \( T_{(r,t-r)} \) is contained in
\[ \Lambda^{*(i,k-i)}_{r,t-r} = \lambda_r^{*}(i) \lambda_{t-r} \cdot z_i \]
blocks of \( \mathfrak{B}^{*}_{(i,k-i)} \).

\( (c) \quad r > s_i \) and \( t - r \leq s_{k-i} \). Then \( T_{(r,t-r)} \) is contained in
\[ \Lambda^{*(i,k-i)}_{r,t-r} = \lambda_r^{*}(i) \lambda_{t-r}^{*(k-i)} . z_i \]
blocks of \( \mathfrak{B}^{*}_{(i,k-i)} \).

It is straightforward to verify the values of \( \Lambda^{*(i,k-i)}_{r,t-r} \) for the cases (a), (b) and (c) above. In case (a) each \( r \)-subset of \( X_1 \) is contained in \( \lambda_r^{*}(i) \) blocks of \( \mathfrak{A}_h^{(i)} \) and each \( (t - r) \)-subset of \( X_2 \) in \( \lambda_{t-r}^{*(k-i)} \) blocks of \( \mathfrak{A}_j^{(k-i)} \). Thus each pair \( (\mathfrak{A}_h^{(i)}, \mathfrak{A}_j^{(k-i)}) \) contributes \( \lambda_r^{*}(i) \lambda_{t-r}^{*(k-i)} \) blocks to \( \mathfrak{B}^{*}_{(i,k-i)} \). Now each of the \( N_i \) resolution classes \( \mathfrak{A}_h^{(i)} \), \( \ldots, \mathfrak{A}_N^{(i)} \) is combined with \( z_i \) resolution classes of \( \mathfrak{A}_1^{(k-i)}, \ldots, \mathfrak{A}_N^{(k-i)} \), therefore we have \( \Lambda^{*(i,k-i)}_{r,t-r} = \lambda_r^{*}(i) \lambda_{t-r}^{*(k-i)} . N_i . z_i \).

In case (b) each \( r \)-subset of \( X_1 \) is contained in \( \lambda_r^{*}(i) \) blocks of \( \mathfrak{A}_h^{(i)} \) and each \( (t - r) \)-subset of \( X_2 \) in \( \lambda_{t-r}^{*(k-i)} \) blocks of \( \mathfrak{B}^{(k-i)} \). These blocks are distributed in the \( N_i \) resolution classes \( \mathfrak{A}_1^{(k-i)}, \ldots, \mathfrak{A}_N^{(k-i)} \). Each class \( \lambda_{t-r}^{*(k-i)} \) is combined \( z_i \) times with \( \mathfrak{A}_h^{(i)} \). Hence, in this case, the contribution of the blocks to \( \mathfrak{B}^{*}_{(i,k-i)} \) is \( \Lambda^{*(i,k-i)}_{r,t-r} = \lambda_r^{*}(i) \lambda_{t-r} \cdot z_i \).

The case (c) is similar to case (b).

Define
\[ \mathfrak{B} := \bigcup_{(i,k-i) \in R} \mathfrak{B}^{*}_{(i,k-i)} \times [u_i] \cup \bigcup_{(i,k-i) \in K \setminus R} \mathfrak{B}^{*}_{(i,k-i)} \times [u_i], \]
with \( u_i \in \{0, 1\} \), \( i = 0, \ldots, k \).

The above presentation can be summarized as follows. Let \( T_{(r,t-r)} \) be a \( t \)-subset of \( X \) for \( r = 0, \ldots, t \). The number of blocks in \( \mathfrak{B}^{*}_{(i,k-i)} \) containing \( T_{(r,t-r)} \), for all \( (i, k-i) \in K \setminus R \), is then
\[ \sum_{(i,k-i) \in K \setminus R} u_i . \Lambda^{*(i,k-i)}_{r,t-r} = \sum_{(i,k-i) \in K \setminus R} u_i . \lambda_r^{*}(i) \lambda_{t-r}^{*(k-i)}. \]

The number of blocks in \( \mathfrak{B}^{*}_{(i,k-i)} \) containing \( T_{(r,t-r)} \), for all \( (i, k-i) \in R \), is then
\[ \sum_{(i,k-i) \in R} u_i . \Lambda^{*(i,k-i)}_{r,t-r}. \]
where

\[
\Lambda_{(r,t-r)}^{*(i,k-i)} = \begin{cases} 
\lambda_r^{(i)} \tilde{\lambda}_{t-r}^{(k-i)} , N_i, z_i & \text{if } r \leq s_i, \ t - r \leq s_{k-i}, \\
\lambda_r^{(i)} \tilde{\lambda}_{t-r}^{(k-i)} & \text{if } r \leq s_i, \ t - r > s_{k-i}, \\
\lambda_r^{(i)} \tilde{\lambda}_{t-r}^{(k-i)} , z_i & \text{if } r > s_i, \ t - r \leq s_{k-i}.
\end{cases}
\]

It follows that the number of blocks in \( \mathcal{B} \) containing \( T_{r,t-r} \) is equal to

\[
L_{r,t-r} := \sum_{(i,k-i) \in R} u_i \Lambda_{(r,t-r)}^{*(i,k-i)} + \sum_{(i,k-i) \in K \setminus R} u_i \Lambda_{(r,t-r)}^{*(i,k-i)}.
\]

Since any \( t \)-subset of \( X \) is of form \( T_{r,t-r} \) for some \( r \in \{0, \ldots, t\} \), we see that if

\[
L_{0,t} = L_{1,t-1} = \cdots = L_{t,0} := \Lambda
\]

for a positive integer \( \Lambda \), then \( (X, \mathcal{B}) \) forms a simple \( t \)-design with parameters \( t-(v, k, \Lambda) \).

We record the construction above in the following theorem.

**Theorem 3.1** Let \( v, k, t \) be integers with \( v > k > t \geq 2 \). Let \( X \) be a \( v \)-set and let \( X = X_1 \cup X_2 \) be a partition of \( X \) with \( |X_1| = v_1 \) and \( |X_2| = v_2 \). Let \( D_i = (X_1, \mathcal{B}^{(i)}) \) be the complete \( i-(v_1, 1, 1) \) design for \( i = 0, \ldots, t \) and let \( D_i = (X_2, \mathcal{B}^{(i)}) \) be a simple \( t-(v_1, i, \lambda_i^{(i)}) \) design for \( i = t + 1, \ldots, k \). Similarly, let \( \tilde{D}_i = (X_2, \tilde{\mathcal{B}}^{(i)}) \) be the complete \( i-(v_2, 1, 1) \) design for \( i = 0, \ldots, t \), and let \( \tilde{D}_i = (X_2, \tilde{\mathcal{B}}^{(i)}) \) be a simple \( t-(v_2, i, \tilde{\lambda}_i^{(i)}) \) design for \( i = t + 1, \ldots, k \). Let \( K = \{(0, k), (1, k-1), \ldots, (k-1, 1), (k, 0)\} \). Suppose there exists a subset \( R \subseteq K \) such that for each \( (i, k-i) \in R \), the designs \( D_i \) and \( \tilde{D}_{k-i} \) have an \( s_i \)-resolution with \( N_i \) classes and an \( s_{k-i} \)-resolution with \( \tilde{N}_{k-i} \) classes, respectively, satisfying the following conditions.

(i) \( N_i = \tilde{N}_{k-i} \),

(ii) \( s_i + s_{k-i} \geq 2 \lfloor \frac{t}{2} \rfloor \).

Define

\[
\mathcal{B} = \bigcup_{(i,k-i) \in R} \mathcal{B}_{(i,k-i)}^* \times [u_i] \cup \bigcup_{(i,k-i) \in K \setminus R} \mathcal{B}_{(i,k-i)} \times [u_i],
\]

for \( u_i \in \{0, 1\}, i = 0, \ldots, k \),

\[
\mathcal{B}_{(i,k-i)}^* := \left\{ B_i \cup \tilde{B}_{k-i} \mid B_i \in \mathcal{A}_{h}^{(i)}, \tilde{B}_{k-i} \in \tilde{\mathcal{A}}_{j}^{(k-i)}, \varepsilon_i \leq d(\mathcal{A}_{h}^{(i)}, \mathcal{A}_{j}^{(i)}) \leq w_i, \right. \\
\varepsilon_i = 0, 1; w_i \leq \lfloor \frac{N_i}{2} \rfloor \right\},
\]

with \( w_i \) as variable, where \( \mathcal{A}_{h}^{(i)}, \ldots, \mathcal{A}_{N_i}^{(i)} \) are \( s_i \)-resolution classes of \( D_i \), with \( (X_1, \mathcal{A}_{h}^{(i)}) \) as an \( s_i-(v_1, i, \lambda_{s_i}^{(i)}) \) design; and \( \tilde{\mathcal{A}}_{j}^{(k-i)}, \ldots, \tilde{\mathcal{A}}_{N_j}^{(k-i)} \) are \( s_{k-i} \)-resolution classes of \( \tilde{D}_{k-i} \), with \( (X_2, \tilde{\mathcal{A}}_{j}^{(k-i)}) \) as an \( s_{k-i}-(v_2, k-i, \tilde{\lambda}_{s_{k-i}}^{(k-i)}) \) design; and

\[
\mathcal{B}_{(i,k-i)} := \{ B = B_i \cup \tilde{B}_{k-i} \mid B_i \in \mathcal{B}^{(i)}, \tilde{B}_{k-i} \in \tilde{\mathcal{B}}^{(k-i)} \}.
\]

Define

\[
L_{r,t-r} := \sum_{(i,k-i) \in R} u_i \Lambda_{(r,t-r)}^{*(i,k-i)} + \sum_{(i,k-i) \in K \setminus R} u_i \Lambda_{(r,t-r)}^{*(i,k-i)}.
\]
for $r = 0, \ldots, t$, where

$$
\Lambda^{*}(i, k-i)_{(r, t-r)} = \begin{cases} 
\lambda^{*}(i) \cdot \lambda^{*}(k-i) . N_i . z_i & \text{if } r \leq s_i, \ t - r \leq s_{k-i}, \\
\lambda^{*}(i) \cdot \lambda^{*}(k-i) . z_i & \text{if } r \leq s_i, \ t - r > s_{k-i}, \\
\lambda^{*}(i) \cdot \lambda^{*}(k-i) . z_i & \text{if } r > s_i, \ t - r \leq s_{k-i}, 
\end{cases}
$$

with $z_i = (2w_i + 1 - \varepsilon_i)$, if $w_i < \frac{N_i}{2}$, and $z_i = (2w_i - \varepsilon_i)$, if $w_i = \frac{N_i}{2}$; and

$$
\Lambda^{*}(i, k-i)_{(r, t-r)} = \lambda^{*}(i) \cdot \lambda^{*}(k-i)
$$

Assume that

$$L_{0,t} = L_{1,t-1} = \cdots = L_{t,0} := \Lambda$$

for a positive integer $\Lambda$, then $(X, \mathcal{B})$ is a simple $t$-$(v, k, \Lambda)$ design.

Remark 3.1

1. In the basic construction the set $\mathcal{B}_{(i, k-i)}$ of the new blocks is uniquely determined as the unions of all the pairs of blocks in $D_i$ and $\overline{D}_{k-i}$. Whereas in the construction using resolutions in Theorem 3.1 the set $\mathcal{B}^{*}_{(i, k-i)}$ is no longer unique. Its size varies according to the variable $z_i$.

2. Theorem 3.1 does not restrict to constructing simple $t$-designs. Obviously, if any of the ingredient designs is non-simple, then the construction will yield non-simple designs.

4 Applications

In this section we illustrate the construction in Theorem 3.1 through a number of examples which show the strength of the method.

In the following we will employ the notation from Chapter II.4 : $t$-Designs with $t \geq 3$ of the Handbook of Combinatorial Designs [12]. The parameter set $t$-$(v, k, \lambda)$ of a design will be written as $t$-$(v, k, m\lambda_{\min})$. Since the supplement of a simple $t$-$(v, k, \lambda)$ design is a $t$-$(v, k, \lambda_{\max} - \lambda)$ design, we usually consider simple $t$-$(v, k, \lambda)$ designs with $\lambda \leq \lambda_{\max}/2$. Thus, the upper limit of $m$ of a constructed design will be $\text{LIM} = \left\lfloor \frac{\lambda_{\max}}{2\lambda_{\min}} \right\rfloor$. But, it should be remarked that, when an ingredient design with index $\lambda$ is used, then $\lambda$ can take on all possible values, i.e. $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$.

4.1 Simple 5-$(38, k, \Lambda)$ designs with $k = 8, 9, 10$

We apply the construction in Theorem 3.1 to the cases $t = 5, v_1 = v_2 = 19$ and $k = 8, 9, 10$.

4.1.1 Simple 5-$(38, 8, \Lambda)$ designs

Here we show a detailed example to illustrate the construction.

Let $X = X_1 \cup X_2$ be a partition of the point set $X$ with $|X| = 38$ into two subsets $X_1$ and $X_2$ with $|X_1| = |X_2| = 19$. For $i = 0, 1, 2, 3, 4, 5$ let $D_i = (X_1, \mathcal{B}^{(i)})$ be the complete $i$-$(19, i, \lambda^{(i)}) := i$-$(19, i, 1)$ design. For $i = 6, 7, 8$ let $D_i = (X_1, \mathcal{B}^{(i)})$ be a simple 5-$(19, i, \lambda^{(i)})$ design. These designs have the following parameters.

- $5$-$(19, 6, \lambda^{(6)}) = 5$-$(19, 6, m2), m = 1, 2, \ldots, 7$
- $5$-$(19, 7, \lambda^{(7)}) = 5$-$(19, 7, m7), m = 1, 2, \ldots, 13$
5-\((19, 8, \lambda_5^{(8)}) = 5-\((19, 8, m28), m = 1, 2, \ldots, 13\)

Correspondingly, let \(\tilde{D}_i = (X_2, \tilde{G}^{(i)})\) be simple designs defined on \(X_2\). Here \(K = \{0, 8, (1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1), (8, 0)\}.

It is known that the complete designs \(D_i\) and \(\tilde{D}_i\) for \(i = 3, 4, 5\) have each a 2-resolution with the number of resolution classes \(N_i = 17\), i.e. the large sets \(LS[17](2, i, 19)\), see for instance Chapter II.4 [12]. We choose

\[ R = \{(3, 5), (4, 4), (5, 3)\}. \]

Thus we have

- \(\mathcal{B}^{(3)} = \bigcup_{j=1}^{17} A_j^{(3)}\), where \((X_1, A_j^{(3)})\) is a 2-(19, 3, \(\lambda_2^{(3)} = 2\)) design, and \(\lambda_2^{(3)} = 1, \lambda_4^{(3)} = 1, \lambda_5^{(3)} = 9, \lambda_0^{(3)} = 57\);
- \(\mathcal{B}^{(4)} = \bigcup_{j=1}^{17} A_j^{(4)}\), where \((X_1, A_j^{(4)})\) is a 2-(19, 4, \(\lambda_2^{(4)} = 2\)) design and \(\lambda_4^{(4)} = 1, \lambda_5^{(4)} = 16, \lambda_0^{(4)} = 8, \lambda_1^{(4)} = 48, \lambda_3^{(4)} = 228\);
- \(\mathcal{B}^{(5)} = \bigcup_{j=1}^{17} A_j^{(5)}\), where \((X_1, A_j^{(5)})\) is a 2-(19, 5, \(\lambda_2^{(5)} = 2\)) design and \(\lambda_5^{(5)} = 1, \lambda_4^{(5)} = 15, \lambda_3^{(5)} = 120, \lambda_2^{(5)} = 40, \lambda_1^{(5)} = 180, \lambda_0^{(5)} = 684\).

Similarly, the complete designs \(\tilde{D}_i\) have the same 2-resolutions as \(D_i\), each having \(\tilde{N}_i = 17\) resolution classes, for \(i = 3, 4, 5\). Thus \(\mathcal{B}^{(i)} = \bigcup_{j=1}^{17} \tilde{A}_j^{(i)}\), and each \((X_2, \tilde{A}_j^{(i)})\) is a 2-(19, \(i, \lambda_2^{(i)})\) design with \(\tilde{\lambda}_2^{(i)} = \lambda_2^{(i)}\).

We compute

\[ L_{r,5-r} = \sum_{(i,8-i) \in R} u_i \cdot \Lambda^{(i,8-i)}_{r,5-r} + \sum_{(i,8-i) \in K \setminus R} u_i \cdot \Lambda^{(i,8-i)}_{r,5-r}, \]

for \(r = 0, \ldots, 5\), and \(u_i = 0, 1\). If \((i, 8-i) \in K \setminus R\), then

\[ \Lambda^{(i,8-i)}_{r,5-r} = \lambda_r^{(i)} \cdot \Lambda_{5-r}^{(8-i)}. \]

If \((i, 8-i) \in R\), then the values of \(\Lambda^{(i,8-i)}_{r,5-r}\) are computed by using the formula

\[ \Lambda^{(i,k-i)}_{r,t-r} = \begin{cases} \lambda_r^{(i)} \cdot \lambda_r^{(k-i)} \cdot \tilde{\lambda}_{i} \cdot z_i & \text{if } r \leq s_i, t \leq s_{k-i}, \\ \lambda_r^{(i)} \cdot \tilde{\lambda}_{r-t} \cdot z_i & \text{if } r > s_i, t \leq s_{k-i}, \\ \lambda_r^{(k-i)} \cdot \tilde{\lambda}_{r-t} \cdot z_i & \text{if } r \leq s_i, t > s_{k-i}. \end{cases} \]

Here we have

\[ \Lambda^{(3,5)}_{0,5} = \lambda_4^{(3)} \cdot \tilde{\lambda}_5^{(5)} \cdot z_3 = 57z_3, \quad \Lambda^{(3,5)}_{1,4} = \lambda_1^{(3)} \cdot \lambda_4^{(5)} \cdot z_3 = 9 \times 15z_3, \]
\[ \Lambda^{(3,5)}_{2,3} = \lambda_2^{(3)} \cdot \tilde{\lambda}_3^{(5)} \cdot z_3 = 120z_3, \quad \Lambda^{(3,5)}_{3,2} = \lambda_3^{(3)} \cdot \tilde{\lambda}_2^{(5)} \cdot z_3 = 40z_3, \]
\[ \Lambda^{(3,5)}_{4,1} = \lambda_5^{(5)} = 0, \quad \Lambda^{(3,5)}_{5,0} = \lambda_0^{(3)} \cdot \tilde{\lambda}_5^{(5)} \cdot z_3 = 15z_3, \]
\[ \Lambda^{(5,3)}_{0,5} = \lambda_4^{(5)} \cdot \tilde{\lambda}_5^{(3)} \cdot z_5 = 57z_5, \quad \Lambda^{(5,3)}_{1,4} = \lambda_1^{(5)} \cdot \tilde{\lambda}_4^{(3)} \cdot z_5 = 9 \times 15z_5, \]
\[ \Lambda^{(5,3)}_{2,3} = \lambda_2^{(5)} \cdot \tilde{\lambda}_3^{(3)} \cdot z_5 = 120z_5, \quad \Lambda^{(5,3)}_{3,2} = \lambda_3^{(5)} \cdot \tilde{\lambda}_2^{(3)} \cdot z_5 = 15 \times 9z_5, \]
\[ \Lambda^{(5,3)}_{4,1} = \lambda_5^{(3)} \cdot \tilde{\lambda}_0^{(3)} \cdot z_5 = 57z_5, \quad \Lambda^{(5,3)}_{5,0} = \lambda_0^{(3)} \cdot \tilde{\lambda}_5^{(3)} \cdot z_5 = 15z_5, \]
\[ \Lambda^{(4,4)}_{0,5} = \lambda_4^{(4)} \cdot \tilde{\lambda}_5^{(4)} \cdot z_4 = 8 \times 16z_4, \quad \Lambda^{(4,4)}_{1,4} = \lambda_1^{(4)} \cdot \tilde{\lambda}_4^{(4)} \cdot z_4 = 48z_4, \]
\[ \Lambda^{(4,4)}_{2,3} = \lambda_2^{(4)} \cdot \tilde{\lambda}_3^{(4)} \cdot z_4 = 8 \times 16z_4, \quad \Lambda^{(4,4)}_{3,2} = \lambda_3^{(4)} \cdot \tilde{\lambda}_2^{(4)} \cdot z_4 = 16 \times 8z_4. \]
we have two distinct non isomorphic solutions for this value of $m = \lfloor 123/\Lambda_1 \rfloor$. Again we assume that $\Lambda_1 = 5$.  

**Table 1** Constructed simple $5$-$(38, 8, \Lambda)$ designs

| $m$  | $z_3$ | $z_4$ | $z_5$ | $\lambda^*_5(6)$ | $\lambda^*_5(7)$ | $\lambda^*_5(8)$ |
|------|-------|-------|-------|------------------|------------------|------------------|
| 280  | 7     | 0     | 7     | 0               | 35               | 56               |
| 488  | 8     | 4     | 8     | 4               | 28               | 280              |
| 524  | 6     | 7     | 6     | 6               | 28               | 196              |
| 560  | 4     | 10    | 4     | 8               | 28               | 112              |
| 560  | 9     | 5     | 9     | 4               | 49               | 112              |

$\Lambda^*_4(4,4) = \lambda^*_4(4) \Lambda^*_1(4) \cdot z_4 = 48z_4$.

It follows that

$L_{0,5} = u_0 \lambda^*_5(8) + u_1 19\lambda^*_5(7) + u_2 171\lambda^*_5(6) + u_3 57z_3$,

$L_{1,4} = u_1 5\lambda^*_5(7) + u_2 29 \times 15\lambda^*_5(6) + u_3 9 \times 15z_5 + u_4 48z_4$,

$L_{2,3} = u_2 40\lambda^*_5(6) + u_3 120z_3 + u_4 48 \times 16z_4 + u_5 40z_5$,

$L_{3,2} = u_6 40\lambda^*_5(6) + u_5 120z_5 + u_4 16 \times 8z_4 + u_3 40z_3$,

$L_{4,1} = u_7 5\lambda^*_5(7) + u_6 15 \times 9\lambda^*_5(6) + u_5 15 \times 9z_5 + u_4 48z_4$,

$L_{5,0} = u_8 \lambda^*_5(8) + u_7 19\lambda^*_5(7) + u_6 171\lambda^*_5(6) + u_5 57z_5$.

Each set of values of $u_i \in \{0, 1\}, i = 0, \ldots, 8$; $z_3, z_4, z_5 = 1, \ldots, 17$; $\lambda^*_5(j)$ and $\bar{\lambda}^*_5(j)$, $j = 6, 7, 8$ for which the equalities

$L_{0,5} = L_{1,4} = L_{2,3} = L_{4,1} = L_{5,0} := \Lambda$

is satisfied for a positive integer $\Lambda$ will yield a simple $5$-$(38, 8, \Lambda)$ design. Recall that a $5$-$(38, 8, \Lambda)$ can be written as $5$-$(38, 8, m4)$ with $\lambda_{\min} = 4$ and $\lambda_{\max} = 5456$. Thus LIM = \{5456/2 \* 4\} = 682. By solving the equalities above we obtain all solutions for $m \leq 1364$. Altogether 33 values for $m$ have been found, of which 16 values of $m \leq \text{LIM}$. Since not all simple $5$-$(19, i, \lambda^*_5(j))$ designs are known to exist, for example, $5$-$(19, 7, m7)$ designs are known for $m = 4, 5, 6, 7, 8, 9, 13$ only, we just obtain the following 5 new simple $5$-$(38, 8, m4)$ designs for $m = 280, 488, 524, 560, 560$ (the number 560 repeats twice, as we have two distinct non isomorphic solutions for this value of $m$). The details of these 5 constructed designs are given in Table 1.

An entry 0 in a column of the table implies that $u_i = 0$, otherwise $u_i = 1$. Here we have $\lambda^*_5(j) = \bar{\lambda}^*_5(j)$, $j = 6, 7, 8$ for all these solutions.

4.1.2 Simple $5$-$(38, k, \Lambda)$ designs with $k = 9, 10$

Again we assume that $v_1 = v_2 = 19$ for the construction of simple $5$-$(38, k, \Lambda)$ designs with $k = 9, 10$.

- For construction of $5$-$(38, 9, \Lambda) = 5$-$(38, 9, m30)$ designs with LIM = 682, we make use of the large sets LS[17](2, i, 19), $i = 3, 4, 5, 6$, i.e. the 2-resolutions of the complete designs $i$-(19, $i$, 1) with resolution class number $N_i = 17$. Thus, we have $R = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$. And the equalities $L_{i,t-i}$ are the following.
A recursive construction for simple $t$-designs using resolutions

Table 2  Constructed simple 5-(38, 9, $\Lambda$) designs

| $m$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ | $\lambda_5^{(7)}$ | $\lambda_5^{(8)}$ | $\lambda_5^{(9)}$ | $\lambda_5^{(7)}$ | $\lambda_5^{(8)}$ | $\lambda_5^{(9)}$ |
|-----|-------|-------|-------|-------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 100 | 2     | 1     | 1     | 2     | 0                | 56                | 56                | 112              | 0                | 112              |
| 200 | 4     | 2     | 2     | 4     | 0                | 112               | 224               | 112              | 0                | 224              |
| 300 | 6     | 3     | 3     | 6     | 0                | 168               | 336               | 168              | 0                | 336              |
| 400 | 8     | 4     | 4     | 8     | 0                | 224               | 448               | 0                | 224              | 448              |
| 402 | 5     | 5     | 5     | 5     | 28               | 84                | 546               | 28               | 84               | 546              |
| 500 | 10    | 5     | 5     | 10    | 0                | 280               | 560               | 0                | 280              | 560              |
| 502 | 7     | 6     | 6     | 7     | 28               | 140               | 658               | 28               | 140              | 658              |
| 504 | 4     | 7     | 7     | 4     | 56               | 0                 | 0                 | 0                | 756              | 756              |
| 582 | 10    | 4     | 11    | 3     | 28               | 168               | 588               | 63               | 84               | 189              |
| 602 | 9     | 7     | 7     | 9     | 28               | 196               | 770               | 28               | 196              | 770              |
| 604 | 6     | 8     | 8     | 6     | 56               | 56                | 868               | 56               | 56               | 868              |
| 660 | 9     | 8     | 8     | 9     | 35               | 252               | 21                | 35               | 252              | 21               |
| 680 | 8     | 11    | 4     | 15    | 0                | 364               | 602               | 35               | 280              | 203              |
| 682 | 5     | 12    | 5     | 12    | 28               | 224               | 700               | 63               | 140              | 301              |

$L_{0,5} = u_0\lambda_5^{(9)} + u_119\lambda_5^{(8)} + u_2171\lambda_5^{(7)} + u_357 \times 14z_3 + u_4228z_4$.

$L_{1,4} = u_115\lambda_5^{(8)}/4 + u_218 \times 5\lambda_5^{(7)} + u_39 \times 105z_3 + u_448 \times 15z_4 + u_5180z_5$.

$L_{2,3} = u_220\lambda_5^{(7)} + u_3560z_3 + u_48 \times 120z_4 + u_540 \times 16z_5 + u_6140z_6$.

$L_{3,2} = u_720\lambda_5^{(7)} + u_6560z_6 + u_5120 \times 8z_5 + u_416 \times 40z_4 + u_3140z_3$.

$L_{4,1} = u_815\lambda_5^{(8)}/4 + u_75 \times 18\lambda_5^{(7)} + u_6105 \times 9z_6 + u_515 \times 48z_5 + u_4180z_4$.

$L_{5,0} = u_9\lambda_5^{(9)} + u_819\lambda_5^{(8)} + u_7171\lambda_5^{(7)} + u_614 \times 57z_6 + u_5228z_5$.

Solving the equalities $L_{0,5} = L_{1,4} = L_{2,3} = L_{3,2} = L_{4,1} = L_{5,0} = \Lambda$ for $\Lambda > 0$ with respect to $z_i = 1, \ldots, 17$ we obtain 20 values for $m$ with $m \leq \text{LIM}$ leading to simple 5-(38, 9, $\Lambda$) = 5-(38, 9, $m30$) designs. Of which 14 designs can be constructed whose details are given in Table 2.

It should be noted that when applying the basic construction for $t = 5$, $v_1 = v_2 = 19$ and $k = 8, 9$ we only obtain the trivial solutions, namely the complete 5-(38, 8, 1364 × 4) and 5-(38, 9, 1364 × 30) designs. This could be explained as follows.

In general, if $k \leq 2t - 1$, then one of the designs in each pair $(D_i, \bar{D}_{k-i})$ is either the empty or the trivial design and at least one pair having both the trivial designs, therefore it leaves little room for the basic construction to produce a non-trivial solution, unless many pairs are unused, i.e. $u_i = 0$. The construction using resolutions indeed makes more room to create non-trivial solutions, as we have seen in the above examples.

- For construction of 5-(38, 10, $\Lambda$) = 5-(38, 10, $m6$) designs with LIM = 19778, we again employ the 2-resolutions of the complete designs $i$-(19, i, 1) for $i = 3, 4, 5, 6, 7$ with resolution class number $N_i = 17$. Here,

$$R = \{(3, 7), (4, 6), (5, 5), (6, 4), (7, 3)\}.$$ 

And we have

$L_{0,5} = u_0\lambda_5^{(10)} + u_119\lambda_5^{(9)} + u_2171\lambda_5^{(8)} + u_357 \times 91z_3 + u_4228 \times 14z_4 + u_5684z_5$. 
Remark 4.1

1. It should be noted that when solving the equalities respect to $m$ and $u$.

Solving the equalities $L_{0,5} = L_{1,4} = L_{2,3} = L_{3,2} = L_{4,1} = L_{5,0} = \Lambda$ for $\Lambda > 0$ with respect to $z_i = 1, \ldots, 17$ we obtain an entire number of 479 solutions, of which 239 have $m \leq \text{LIM}$. From these 239 parameters 131 simple 5-(38, 10, m6) designs have been shown to exist. The values of $m$ for these designs are

\[
\begin{align*}
12768 &\quad 17416 &\quad 2604 &\quad 6076 &\quad 7252 &\quad 10724 &\quad 13668 &\quad 15108 \\
15372 &\quad 18580 &\quad 18844 &\quad 3768 &\quad 6976 &\quad 8416 &\quad 8680 &\quad 11624 \\
11888 &\quad 12152 &\quad 16272 &\quad 16536 &\quad 16800 &\quad 19744 &\quad 4932 &\quad 8404 \\
9580 &\quad 9844 &\quad 12788 &\quad 13052 &\quad 13316 &\quad 13580 &\quad 17172 &\quad 17436 \\
17700 &\quad 17964 &\quad 18228 &\quad 6096 &\quad 9040 &\quad 10480 &\quad 10744 &\quad 11008 \\
13952 &\quad 11536 &\quad 14216 &\quad 14480 &\quad 15920 &\quad 18600 &\quad 18864 &\quad 19128 \\
7260 &\quad 11644 &\quad 11908 &\quad 12172 &\quad 14852 &\quad 15116 &\quad 15380 &\quad 15644 \\
16556 &\quad 16820 &\quad 19500 &\quad 17084 &\quad 19764 &\quad 8424 &\quad 11368 &\quad 12544 \\
12808 &\quad 13072 &\quad 13336 &\quad 16016 &\quad 16280 &\quad 16544 &\quad 16808 &\quad 17984 \\
9060 &\quad 9588 &\quad 13972 &\quad 14236 &\quad 16916 &\quad 14500 &\quad 17180 &\quad 17444 \\
17708 &\quad 18884 &\quad 19148 &\quad 10224 &\quad 10752 &\quad 14872 &\quad 15136 &\quad 15400 \\
18080 &\quad 15664 &\quad 18344 &\quad 18608 &\quad 19520 &\quad 11388 &\quad 11916 &\quad 16036 \\
16300 &\quad 16564 &\quad 19244 &\quad 16828 &\quad 19958 &\quad 19772 &\quad 12552 &\quad 13080 \\
17200 &\quad 17464 &\quad 17728 &\quad 13716 &\quad 14244 &\quad 18100 &\quad 18364 &\quad 18628 \\
18892 &\quad 19156 &\quad 14880 &\quad 15408 &\quad 19264 &\quad 19528 &\quad 16044 &\quad 17208 \\
18384 &\quad 18372 &\quad 19536 &\quad 16844 &\quad 11316 &\quad 13908 &\quad 14280 &\quad 14808 \\
19720 &\quad 16872 &\quad 17772
\end{align*}
\]

Here are two examples:

- 5-(38, 10, $2604 \times 6$) with $z_3 = 1$, $z_4 = 2$, $z_6 = 2$, $z_7 = 1$, $\lambda^{(9)}_5 = 147$, $\lambda^{(10)}_5 = 1260$, $u_2 = u_5 = u_8 = 0$ and $\lambda^{(i)}_5 = \lambda^{(i)}_5$ for $i = 9, 10$.

- 5-(38, 10, $11316 \times 6$) with $z_3 = 2$, $z_4 = 8$, $z_5 = 2$, $z_6 = 7$, $z_7 = 4$, $\lambda^{(8)}_5 = 140$, $\lambda^{(9)}_5 = 336$, $\lambda^{(10)}_5 = 294$, $\lambda^{(8)}_5 = 84$, $\lambda^{(9)}_5 = 378$, $\lambda^{(10)}_5 = 1890$.

On the other hand, when the basic construction is applied for this case (i.e. $v_1 = v_2 = 19$ and $k = 10$), we just obtain 5 solutions with $m \leq \text{LIM}$.

Remark 4.1

1. It should be noted that when $v_1 = v_2$, any solution with $\lambda^{(i)}_5 \neq \lambda^{(i)}_5$ will appear twice by reason of symmetry, since $\lambda^{(i)}_5$ and $\lambda^{(i)}_5$ may be interchanged. These two solutions are indeed the same. This fact should be taken into account by counting the number of solutions throughout Sect. 4.

2. Up to now the number of known parameter sets for 5-(38, $k$, $\Lambda$) with $k = 8, 9, 10$ are 8, 14, and 23 respectively, see [12], for instance. For $k = 8, 9$ all the parameters of the
constructed designs differ from the known ones. For \( k = 10 \), only one of the 23 known parameter sets does appear in the list of 131 constructed designs, namely the parameters 5-(38, 10, 11368 \times 6). However, it is not known whether the corresponding designs are isomorphic.

### 4.2 Some further results of applications

We briefly record some further examples of simple \( t \)-designs for \( t = 4, 5, 6 \) by using Theorem 3.1.

#### 4.2.1 \( t = 4 \)

Following are several small parameters for \( t = 4 \).

1. \( 4-(26, 8, m35) \): Take \( v_1 = v_2 = 13 \) and \( R = \{(3, 5), (4, 4), (5, 3)\} \) by using LS[55](2, 4, 13) and LS[11](2, i, 13) for \( i = 3, 5 \). There are 3 non-trivial solutions of Eq. (3) with \( m = 44, 66 \) satisfying \( m \leq \text{LIM}(= 104) \). A design with \( m = 44 \) is known. The two solutions for \( m = 66 \) are non-isomorphic and new. These are

   - \( u_4 = 0, z_3 = z_5 = 7, \lambda_4^{(7)} = 42, \lambda_4^{(8)} = 126, u_2 = u_6 = 0, \bar{\lambda}_4^{(i)} = \lambda_4^{(i)} \) for \( i = 7, 8 \).
   - \( z_4 = 24, z_3 = z_5 = 2, \lambda_4^{(6)} = 18, \lambda_4^{(8)} = 126, u_1 = u_7 = 0, \bar{\lambda}_4^{(i)} = \lambda_4^{(i)} \) for \( i = 6, 8 \).

   The basic construction for \( 4-(26, 8, m35) \) with \( v_1 = v_2 = 13 \) only yields the trivial solution.

2. \( 4-(28, 9, m168) \): Take \( v_1 = v_2 = 14 \) and \( R = \{(4, 5), (5, 4)\} \) by using LS[11](2, i, 14) for \( i = 4, 5 \). There is a unique non-trivial solution of Eq. (3) with \( m = 110 \) satisfying \( m \leq \text{LIM}(= 126) \). This solution with \( z_4 = z_5 = 4, u_2 = u_7 = 0, \lambda_4^{(6)} = 30, \lambda_4^{(8)} = 210, \lambda_4^{(9)} = 252, \) and \( \bar{\lambda}_4^{(i)} = \lambda_4^{(i)} \) for \( i = 6, 8, 9 \) yields a new design.

3. \( 4-(30, 7, m20) \): Take \( v_1 = v_2 = 15 \) and \( R = \{(3, 4), (4, 3)\} \) by using LS[13](2, i, 15) for \( i = 3, 4 \). There are 3 non-trivial solutions of Eq. (3) with \( m = 39, 52, 65 \) satisfying \( m \leq \text{LIM}(= 65) \). The solution for \( m = 52 \) with \( z_3 = z_5 = 5, \lambda_4^{(5)} = 5, \lambda_4^{(6)} = 15, \lambda_4^{(7)} = 115, \) and \( \bar{\lambda}_4^{(i)} = \lambda_4^{(i)} \) for \( i = 5, 6, 7 \) gives a new design.

#### 4.2.2 \( t = 5 \)

1. \( 5-(36, 10, m63) \): Take \( v_1 = v_2 = 18 \) and \( R = \{(5, 5)\} \) by using LS[7](2, 5, 18). There are 164 non-trivial solutions of Eq. (3) with \( m \leq \text{LIM}(= 1348) \). Of which 37 are shown to exist. It is interesting to remark that these 37 designs include the 10 designs constructed using the basic construction [41]. Actually, 27 new designs with parameters 5-(36, 10, m63) have been obtained. These are

\[
\begin{align*}
m &= 611, 818, 921, 945, 969, 1048, 1072, 911, 934, 1094, 1197, 1221, 1245, 1269, \\
&
\quad 1324, 1325, 1348, 1187, 1210, 1234, 1337, 1152, 1176, 1200, 1224, 1303, 1131.
\end{align*}
\]

2. \( 5-(37, 8, m40) \): Take \( v_1 = 13, v_2 = 24 \) and \( R = \{(3, 5), (4, 4), (5, 3)\} \) by using LS[11](2, i, 13), LS[11](2, i, 24) for \( i = 3, 4, 5 \). There are two non-isomorphic non-trivial solutions of Eq. (3) with \( m = 55 \) such that \( m \leq \text{LIM}(= 62) \). Both solutions give
new designs. One of the designs has $z_3 = 2$, $z_5 = 2$, $z_4 = 8$, $\lambda_5^{(6)} = 4$, $\lambda_5^{(7)} = 28$, $\lambda_5^{(8)} = 56$, $\lambda_5^{(6)} = 13$, $\lambda_5^{(7)} = 36$, $\lambda_5^{(8)} = 666$.

3. $5-(37, 9, m_{10})$: Take $v_1 = 13$, $v_2 = 24$ and $R = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$ by using LS[11](2, i, 13), LS[11](2, i, 24) for $i = 3, 4, 5, 6$. There is a unique non-trivial solution of Eq. (3) with $m = 874$ such that $m \leq \text{LIM}(= 1798)$. This solution with $z_3 = 2$, $z_4 = 2$, $z_5 = 4$, $z_6 = 1$, $\lambda_5^{(7)} = 14$, $u_8 = u_9 = 0$, $\lambda_5^{(6)} = 72$, $\lambda_5^{(7)} = 30$, $\lambda_5^{(8)} = 1980$ gives a new design.

4. $5-(44, 8, m)$: Take $v_1 = v_2 = 22$ and $R = \{(4, 4)\}$ by using LS[19](2, 5, 22). There are 9 non-trivial solutions of Eq. (3) with $m \leq \text{LIM}(= 4569)$. Of which one design with $m = 3344$ and $u_3 = u_5 = 0$, $z_4 = 4$, $\lambda_5^{(6)} = 12$, $\lambda_5^{(7)} = 16$, $\lambda_5^{(8)} = 220$, and $\overline{\lambda}_5^{(i)} = \lambda_5^{(i)}$ for $i = 6, 7, 8$, is shown to exist.

5. $5-(46, 10, m_2)$: Take $v_1 = v_2 = 23$ and $R = \{(4, 6), (5, 5), (6, 4)\}$ by using LS[133](2, 5, 23), LS[7](2, i, 23) for $i = 4, 6$. There are 3986 non-trivial solutions of Eq. (3) with $m \leq \text{LIM}(= 187349)$. Of which 176 designs are shown to exist with the following values of $m$.

| $m$   | 65246 | 75487 | 73758 | 83999 | 86526 | 94240 | 96140 | 96767 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 106381| 107008| 116622| 117021| 125134| 123405| 127262| 139004|
| 137503| 142633| 139403| 143887| 149664| 151772| 159885| 162013|
| 164540| 166668| 174781| 185497| 59014  | 79667 | 78166 | 89908 |
| 88179 | 92435 | 98420 | 102676| 108661 | 110561| 120802 | 126160 |
| 122930| 125058| 131043| 136401| 133171 | 139555| 137826 | 145540 |
| 153425| 157054| 158308| 175807| 174705 | 184319| 182818 | 184946 |
| 77064 | 94088 | 99446 | 96216 | 104329 | 109687| 102600 | 119928 |
| 112841| 116698| 121828| 115368| 123082 | 132069| 124982 | 125609 |
| 130967| 142310| 135223| 140581| 135850 | 141208 | 145464 | 150822 |
| 145863| 153976| 156104| 163590| 167846 | 166345| 171475 | 186245 |
| 178486| 183844| 180614| 185972| 86526  | 105355 | 113867 | 112366 |
| 118750| 124108| 126635| 134349| 127262 | 136249 | 134748 | 131518 |
| 136876| 137503| 146490| 139403| 147117 | 140030 | 156731 | 149644 |
| 155002| 151772| 157130| 150271| 153900 | 159885 | 165243 | 162013 |
| 167371| 166668| 177612| 174382| 179512 | 182267 | 185896 | 187150 |
| 121505| 119776| 128288| 132544| 138529 | 142785 | 148770 | 145540 |
| 150670| 160911| 163039| 159809| 171152 | 169423 | 164692 | 170050 |
| 168321| 179664| 174705| 182818 | 184946 | 135850 | 148694 | 152950 |
| 158707| 163191| 165091| 163590| 165718 | 171076 | 175332 | 179189 |
| 175959| 181317| 174230| 185573 | 178486 | 183844 | 180614 | 185972 |
| 184471| 182742| 176985| 164540 | 177612 | 179512 | 187226 | 173052 |

Here is an example with $m = 59014$: $z_4 = z_6 = 1$, $z_5 = 20$, $u_2 = u_8 = 0$, $\lambda_5^{(7)} = 36$, $\lambda_5^{(9)} = 810$, $\lambda_5^{(10)} = 7812$, and $\overline{\lambda}_5^{(i)} = \lambda_5^{(i)}$ for $i = 7, 9, 10$.

4.2.3 $t = 6$

Following are some examples for $t = 6$.

1. $6-(38, 10, m_{10})$: Take $v_1 = v_2 = 19$ and $R = \{(4, 6), (5, 5), (6, 4)\}$ by using LS[4](3, i, 19) for $i = 4, 5, 6$. There are 4 non-trivial solutions of Eq. (3) with $m = 1360$, 892, 1340, 1788 for $m \leq \text{LIM}(= 1798)$. 

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2. 6-(46, 12, m, 420): Take \(v_1 = v_2 = 23\) and \(R = \{(6, 6)\}\) by using LS\(3\)(3, 6, 23). There are 2 non-trivial solutions of Eq. (3) with \(m = 3363, 3819\) for \(m \leq \text{LIM}(= 4569)\). The solution for \(m = 3363\) has \(z_6 = 1, \lambda_6^{(7)} = 7, \lambda_6^{(8)} = 40, \lambda_6^{(9)} = 340, \lambda_6^{(10)} = 350, \lambda_6^{(11)} = 4046, \lambda_6^{(12)} = 5320,\) and \(\lambda_6^{(i)} = \lambda_6^{(i)}\) for \(i = 7, 8, 9, 10, 11, 12\). All the ingredient designs corresponding to \(m = 3363\) exist except that the existence of a 6-(23, 10, 5 \times 70) design is still in doubt. So, we would have a 6-(46, 12, 3364 \times 420) design if a 6-(23, 10, 5 \times 70) design would exist.

3. 6-(50, 12, m, 308): Take \(v_1 = v_2 = 25\) and \(R = \{(6, 6)\}\) by using LS\(7\)(3, 6, 25). There are 195 non-trivial solutions of Eq. (3) for \(m \leq \text{LIM}(= 11459)\).

5 Conclusion

We have presented a recursive construction for simple \(t\)-designs by using the concept of resolutions. This may be viewed as an extension of the basic construction as shown in our previous paper. The \(s\)-resolutions of trivial \(t\)-designs are equivalent to the large sets of \(s\)-designs, which have been extensively studied. Since our construction does not exclude the use of trivial designs as ingredients, we have restricted its applications to resolutions of the trivial ingredient designs only. In spite of this fact, the construction still produces a large number of new simple \(t\)-designs. We strongly believe that the construction would unfold its full impact when we would gain more knowledge about resolutions of non-trivial \(t\)-designs.

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