CDG DECOMPOSITION OF QCD IN THE CONSTRAINTLESS CLAIRAUT-TYPE FORMALISM

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ABSTRACT. We apply the recently-derived constraintless Clairaut-type formalism to the Cho-Duan-Ge decomposition in $SU(2)$ QCD. We find non-trivial corrections to the physical equations of motion, and that the contribution of the topological degrees of freedom is qualitatively different from that found by treating the monopole potential as though it were dynamic. We also find alterations to the field commutation relations that undermine the particle interpretation in the presence of the chromomonopole condensate.

CONTENTS

1. Introduction 1
2. Representing the Gluon Field 2
3. The $q^\alpha$ gauge fields of the monopole field 3
4. The $q^\alpha$-curvature 4
5. Altered equations of motion 5
6. The fundamental representation 6
7. Monopole corrections to the quantum commutation relations 6
8. Discussion 7
Appendix A. Appendix. The Clairaut type formalism 8
References 9

1. Introduction

The occurrence of “redundant” degrees of freedom not determined by equations of motion (EOMs) is a characteristic property of any physical system having symmetry [1, 2]. In gauge theories the covariance of EOM under symmetry transformations leads to gauge ambiguity, i.e. the appearance of undetermined functions. In this situation some dynamical variables obey first order differential equations [4]. One then employs a suitably modified Hamiltonian formalism, such as the Dirac theory of constraints [5].

A constraintless generalization of the Hamiltonian formalism based on a Clairaut-type formulation was recently put forward by one of the authors [6, 7]. It generalises the standard Hamiltonian formalism to include Hessians with zero determinant, providing a rigorous treatment of the non-physical degrees of freedom in the derivation of EOMs and the quantum commutation relations. An outline is given in appendix A.

The Cho-Duan-Ge (CDG) decomposition of the gluon field in Quantum Chromodynamics (QCD), published by Duan and Ge [8] and also by Cho [9], specifies the Abelian components of the background field in a gauge covariant manner. In so doing it identifies the monopole degrees of freedom (DOFs) of the gluon field naturally, making it preferable to the conventional maximal Abelian gauge [10]. It can also generate a gauge invariant canonical momentum, which makes it of interest to studies of nucleon spin decomposition [11, 12, 13, 14].
Up until now, the monopole DOFs have not been rigorously handled. Indeed, merely accounting for the physical and gauge DOFs proved to be a long and difficult task [13, 16, 17, 18, 19]. An important observation of the monopole DOFs by Cho et al. is that the Euler-Lagrange equation for the Abelian direction does not yield a new EOM. Their interpretation is that the monopole is the “slow-changing background part” of the gauge field while the physical gluons constituted the “fast-changing quantum part”.

In this paper we apply the Clairaut formalism to the monopole DOFs in two-colour QCD. We consider both the gluon field and scalar “quarks” in the fundamental field. We find that the interaction between monopole and physical DOFs vanishes from the EOMs, but that the canonical commutation relations are altered in a manner that leaves particle number undefined.

Section 2 describes the CDG decomposition and establishes notation. In section 3 we identify the field theory equivalent of \( q^a \) and go on to find the \( q^a \) curvature in section 4. The curvature’s non-zero value leads to alterations in the EOMs elucidated in section 5, while corresponding results are found in section 6 for colour-charged scalars in the fundamental representation. Our most important results, alterations to the commutation relations and their implications for the particle interpretation, are discussed in section 7. We give a final discussion in section 8 and a detailed summary of the Clairaut formalism in appendix A.

2. Representing the Gluon Field

The Cho-Duan-Ge (CDG) decomposition [8, 9], and another like it [20], was (re-)discovered [15] at about the turn of the century when several groups were readdressing the stability of the chromomonomopole condensate [16, 17, 18, 21, 22, 23]. Some authors [17, 18, 22], including one of the current ones [23], have overlooked the differences between the CDG decomposition and that of Faddeev and Niemi, referring to the former as either the Cho-Faddeev-Niemi (CFN) or the Cho-Faddeev-Niemi-Shabanov (CFNS) decomposition. In this paper we label it the CDG decomposition, as per the convention of Cho et al. [14].

The Lie group \( SU(N) \) has \( N^2 - 1 \) generators \( \lambda^{(a)} \) \( (a = 1, \ldots, N^2 - 1) \), of which \( N - 1 \) are Abelian generators \( \lambda^{(i)} \) \( (i = 1, \ldots, N - 1) \). The gauge transformed Abelian directions (Cartan generators) are denoted as

\[
\hat{n}_i(x) = U(x)\lambda^{(i)} U(x). \tag{1}
\]

Gluon fluctuations in the \( \hat{n}_i(x) \) directions are described by \( c^{(i)}_\mu(x) \), where \( \mu \) is the Minkowski index. There is a covariant derivative which leaves the \( \hat{n}_i(x) \) invariant,

\[
\hat{D}_\mu \hat{n}_i(x) \equiv (\partial_\mu + g\vec{V}_\mu(x)\times) \hat{n}_i(x) = 0, \tag{2}
\]

where \( \vec{V}_\mu(x) \) is of the form

\[
\vec{V}_\mu(x) = c^{(i)}_\mu(x) \hat{n}_i(x) + \vec{C}_\mu(x), \quad \vec{C}_\mu(x) = g^{-1}\partial_\mu \hat{n}_i(x) \times \hat{n}_i(x). \tag{3}
\]

The vector notation refers to the internal space, and summation is implied over \( i = 1, \ldots, N - 1 \). For later convenience we define

\[
F^{(i)}_{\mu\nu}(x) = \partial_\mu c^{(i)}_\nu(x) - \partial_\nu c^{(i)}_\mu(x) \tag{4}
\]

\[
\vec{H}_{\mu\nu}(x) = \partial_\mu \vec{C}_\nu(x) - \partial_\nu \vec{C}_\mu(x) + g\vec{C}_\mu(x) \times \vec{C}_\nu(x) = H^{(i)}_{\mu\nu}(x) \hat{n}_i(x), \tag{5}
\]

\[
H^{(i)}_{\mu\nu}(x) = \vec{H}_{\mu\nu}(x) \cdot \hat{n}_i(x). \tag{6}
\]

The vectors \( \vec{X}_\mu(x) \) denote the dynamical components of the gluon field in the off-diagonal directions of the internal space, so if \( \vec{A}_\mu(x) \) is the gluon field then
\( \vec{A}_\mu(x) = \vec{V}_\mu(x) + \vec{X}_\mu(x) = c^{(i)}_\mu(x)\hat{n}_i(x) + \vec{C}_\mu(x) + \vec{X}_\mu(x), \) \hspace{1cm} (7)

where

\( \vec{X}_\mu(x) \perp \hat{n}_i(x), \forall 1 \leq i < N, \quad \vec{D}_\mu = \partial_\mu + gA_\mu(x). \) \hspace{1cm} (8)

The Lagrangian density is still

\[ \mathcal{L}_{\text{gauge}}(x) = -\frac{1}{4} F^{\mu\nu}(x) \cdot \tilde{F}^{\mu\nu}(x) \]

where the field strength tensor of QCD expressed in terms of the CDG decomposition is

\[ F^{\mu\nu}(x) = (F^{(i)}_{\mu\nu}(x) + H^{(i)}_{\mu\nu}(x))\hat{n}_i(x) \]
\[ + (\vec{D}_\mu \vec{X}_\nu(x) - \vec{D}_\nu \vec{X}_\mu(x)) + g\vec{X}_\mu(x) \times \vec{X}_\nu(x). \] \hspace{1cm} (9)

Henceforth we restrict ourselves to the \( SU(2) \) theory, for which there is only one \( \hat{n}(x) \) lying in a three dimensional internal space, and neglect the \( (i) \) indices. The results can be extended to larger \( SU(N) \) gauge groups \([23]\).

We will later have need of the conjugate momenta. These are only defined up to a gauge transformation, so to avoid complications we take the Lorentz gauge. The conjugate momentum for the Abelian component is then

\[ \Pi^\mu(x) = \frac{\delta H_{\text{phys}}}{\delta \partial_0 c_\mu(x)} = -\frac{1}{2} \tilde{F}^{0\mu}(x) \cdot \hat{n}(x), \] \hspace{1cm} (10)

while the conjugate momentum of \( \vec{X}_\mu(x) \) is

\[ \vec{\Pi}^\mu(x) = \frac{\delta H_{\text{phys}}}{\delta \vec{D}_0 \vec{X}_\mu(x)} = -\frac{1}{2} \vec{D}^0 \vec{X}_\mu(x) - \vec{D}^\mu \vec{X}^0(x). \] \hspace{1cm} (11)

The above outline neglects various mathematical subtleties involved in a fully consistent application of the CDG decomposition. In fact, its proper interpretation and gauge-fixing took considerable effort by several independent groups. The interested reader is referred to \([15, 16, 17, 18, 19]\) for further details.

3. The \( q^a \) Gauge Fields of the Monopole Field

Now we adapt the Clairaut approach (see appendix \[A\]) \([21, 22, 23]\) to quantum field theory and apply it to the CDG decomposition of the QCD gauge field, leaving the fundamental representation until section \[3\]. Substituting the polar angles,

\[ \hat{n}(x) = \cos \theta(x) \sin \phi(x) \hat{e}_1 + \sin \theta(x) \sin \phi(x) \hat{e}_2 + \cos \phi(x) \hat{e}_3. \] \hspace{1cm} (12)

and defining

\[ \sin \phi(x) \hat{n}_\theta(x) \equiv \int dy^4 \frac{d\hat{n}(y)}{d\theta(y)} = \sin \phi(x) (-\sin \theta(x) \hat{e}_1 + \cos \theta(x) \hat{e}_2) \]
\[ \hat{n}_\phi(x) \equiv \int dy^4 \frac{d\hat{n}(y)}{d\phi(y)} = \cos \theta(x) \cos \phi(x) \hat{e}_1 + \sin \theta(x) \cos \phi(x) \hat{e}_2 - \sin \phi(x) \hat{e}_3, \] \hspace{1cm} (13)

for later convenience, we note that the vectors \( \hat{n}(x) = \hat{n}_\phi(x) \times \hat{n}_\theta(x) \) form an orthonormal basis of the internal space.

Substituting the above into the Cho connection in eq. \([3]\) gives

\[ g\vec{C}_\mu(x) = (\cos \theta(x) \cos \phi(x) \sin \phi(x) \partial_\mu \theta(x) + \sin \theta(x) \partial_\phi(x)) \hat{e}_1 \]
\[ + (\sin \theta(x) \cos \phi(x) \sin \phi(x) \partial_\mu \theta(x) - \cos \theta(x) \partial_\phi(x)) \hat{e}_2 - \sin^2 \phi(x) \partial_\mu \theta(x) \hat{e}_3 \]
\[ = \sin \phi(x) \partial_\mu \theta(x) \hat{n}_\phi(x) - \partial_\mu \phi(x) \hat{n}_\theta(x) \] \hspace{1cm} (14)
from which it follows that
\[ g^2 \tilde{C}_\mu(x) \times \tilde{C}_\nu(x) = \sin \phi(x) (\partial_\mu \phi(x) \partial_\nu \theta(x) - \partial_\nu \phi(x) \partial_\mu \theta(x)) \hat{n}(x), \] (15)

A Hessian of determinant zero is equivalent to
\[ \left\| \frac{\delta^2 \mathcal{L}}{\delta q A \delta q B} \right\| = 0, \] (16)

which follows from Cho and Pak’s [16], and Bae et al.’s [19] finding that \( \hat{n}(x) \) (and by extension \( \theta(x), \phi(x) \)) does not generate an independent EOM.

From (3.10) in [6], we can find a general Hamiltonian \( H_{\text{gen}} = \int dx^3 H_{\text{gen}}(x) \) consistent with conventional representations of QCD taking

\[ p_\theta(x) = \frac{\delta L}{\delta \partial_0 \theta(x)} \equiv B_\theta(x), \] (17)

\[ p_\phi(x) = \frac{\delta L}{\delta \partial_0 \phi(x)} \equiv B_\phi(x). \] (18)

where the definitions of \( B_\phi(x), B_\theta(x) \) are generalised to quantum field theory from those in [6]. It follows from this that \( H_{\text{phys}} = H_{\text{mix}} \) (also defined in [6]). The \( q_\alpha \)-gauge fields are then (see Appendix A)

\[ p_\phi(x) = B_\phi(x) = \int dy^3 \frac{\delta \mathcal{L}}{x \partial_0 \phi(x)} \]

\[ = \int dy^3 \int dy^0 \delta(x^0 - y^0) \left( \sin \phi(y) \partial^\mu \theta(y) \hat{n}(y) + \hat{\theta}(y) \times \vec{X}^\mu(y) \right) \cdot \vec{F}_{0\mu}(y) \delta^3(\vec{x} - \vec{y}) \]

\[ = \left( \sin \phi(x) \partial^\mu \theta \hat{n}(x) + \hat{\theta}(x) \times \vec{X}^\mu(x) \right) \cdot \vec{F}_{0\mu}(x), \] (19)

\[ p_\theta(x) = B_\theta(x) = \int dy^3 \frac{\delta \mathcal{L}}{x \partial_0 \theta(x)} \]

\[ = - \int dy^3 \int dy^0 \delta(x^0 - y^0) \sin \phi(y) \left( \partial^\mu \phi(y) \hat{n}(y) + \sin \phi(y) \hat{\phi}(y) \times \vec{X}^\mu(y) \right) \cdot \vec{F}_{0\mu}(y) \delta^3(\vec{x} - \vec{y}) \]

\[ = - \sin \phi(x) \left( \partial^\mu \phi(x) \hat{n}(x) + \sin \phi(x) \hat{\phi}(x) \times \vec{X}^\mu(x) \right) \cdot \vec{F}_{0\mu}(x). \] (20)

4. The \( q^\alpha \)-curvature

From eqs. [19, 20] we have

\[ \frac{\delta B_\phi(x)}{\delta \theta(y)} = 0, \] (21)

\[ \frac{\delta B_\theta(x)}{\delta \phi(y)} = - \cos^2 \phi(x) \left( \partial^\mu \phi(x) \hat{n}(x) + \hat{\phi}(x) \times \vec{X}^\mu(x) \right) \cdot \vec{F}_{0\mu}(x) \delta^4(x - y) \] (22)

which yields the \( q^\alpha \)-curvature

\[ \mathcal{F}_{\phi \theta}(x) = \int dy^4 \left( \frac{\delta B_\theta(x)}{\delta \phi(y)} - \frac{\delta B_\phi(x)}{\delta \theta(y)} \right) \delta^4(x - y) + \left\{ B_\phi(x), B_\theta(x) \right\}_{\text{phys}} \]

\[ = - \cos \phi(x) \left( \partial^\mu \phi(x) \hat{n}(x) + \hat{\phi}(x) \times \vec{X}^\mu(x) \right) \cdot \vec{F}_{0\mu}(x), \] (23)

where we have used that the bracket \( \left\{ B_\phi(x), B_\theta(x) \right\}_{\text{phys}} \) vanishes because \( B_\phi(x) \) and \( B_\theta(x) \) share the same dependence on the dynamic DOFs and their derivatives.

In earlier work on the Clairaut formalism [24, 6] this was called the \( q^\alpha \)-field strength, but we call it \( q^\alpha \)-curvature in quantum field theory applications to avoid confusion.
This non-zero $\mathcal{F}^{\theta\phi}(x)$ is necessary, and usually sufficient, to indicate a non-dynamic contribution to the conventional Euler-Lagrange EOMs. More significant is a corresponding alteration of the quantum commutators, with repurcussions for canonical quantisation and the particle number.

5. Altered equations of motion

Generalizing eqs. (7.1,7.3,7.5) in [6],
\begin{equation}
\partial_0 q(x) = \{q(x), H_{phys}\}_{new} = \frac{\delta H_{phys}}{\delta \partial_0 q(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta \partial_0 q(x)} \partial^0 \alpha(y),
\end{equation}
the derivative of the Abelian component, complete with corrections from the monopole background is
\begin{equation}
\partial_0 c_\sigma(x) = \frac{\delta H}{\delta \partial_0 c^\sigma(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta \partial_0 c^\sigma(x)} \partial^0 \alpha(y).
\end{equation}
Combining this with
\begin{equation}
\frac{\delta B_\phi(y)}{\delta \partial_0 c^\sigma(x)} = -\left( \left( \sin \phi(y) \partial_\phi \phi(y) + \hat{n}_\phi(y) \times \vec{X}_\sigma(y) \right) - \left( \sin \phi(y) \partial_\phi \phi(y) + \hat{n}_\phi(y) \times \vec{X}_0(y) \right) g_\sigma(y) \right) \delta^4(x-y),
\end{equation}
\begin{equation}
\frac{\delta B_\theta(y)}{\delta \partial_0 c^\sigma(x)} = \left( \left( \sin \phi(y) \partial_\theta \theta(y) - \hat{n}_\theta(y) \times \vec{X}_\sigma(y) \right) - \left( \sin \phi(y) \partial_\theta \theta(y) - \hat{n}_\theta(y) \times \vec{X}_0(y) \right) g_\sigma(y) \right) \delta^4(x-y),
\end{equation}
gives
\begin{equation}
\begin{aligned}
\partial_0 c_\sigma(x) &= \frac{\delta H}{\delta \partial_0 c^\sigma(x)} - \frac{1}{2} \left( \sin \phi(x) (\partial_\sigma \phi(x) \partial_0 \theta(x) - \partial_\sigma \theta(x) \partial_0 \phi(x)) \right) \\
&\quad + g \left( \hat{n}_\phi(x) \partial_\theta \theta(x) + \hat{n}_\theta(x) \partial_\phi \phi(x) \right) \times \vec{X}_\sigma(x) \\
&= \frac{\delta H}{\delta \partial_0 c^\sigma(x)} - \frac{1}{2} g^2 \left( 2 \vec{C}_\sigma(x) \times \vec{C}_0(x) + \vec{C}_0(x) \times \vec{X}_\sigma(x) - \vec{C}_\sigma(x) \times \vec{X}_0(x) \right).
\end{aligned}
\end{equation}

The correction exactly cancels the monopole’s electric field contribution to $\{c_\sigma, H_{phys}\}_{phys}$, as required for consistency. Hence the need to treat the monopole as a non-dynamic field is further demonstrated.

We now observe that
\begin{equation}
\frac{\delta B_\theta(x)}{\delta c^\sigma(y)} = \frac{\delta B_\phi(x)}{\delta c^\sigma(y)} = 0,
\end{equation}
from which it follows that the EOM of $c_\sigma$ receives no correction. However its $\{,\}_{phys}$ contribution, corresponding to the terms in the conventional EOM for the Abelian component, already contains a contribution from the monopole field strength.

Repeating the above steps for the valence gluons $\vec{X}_\mu$ finds the converse situation, i.e. its derivative $\partial_0 \vec{X}_\sigma$ is uncorrected while its EOM receives a correction which cancels the monopole’s electric contribution to $\{\partial_0 \vec{X}_\sigma, H_{phys}\}_{phys}$. This is required by the conservation of topological current.
6. The fundamental representation

We consider a complex boson field $a(x), a^\dagger(x)$ in the fundamental representation of the gauge group, and probe the implications of this approach for the quark fields. Although physical quarks are fermions, we study the bosonic case to avoid distracting complications, leaving the fermionic case for a later paper.

The kinetic and interaction terms are given by

$$-(\bar{D}^\mu a)^\dagger(x)\bar{D}_\mu a(x)$$

We do not consider the mass term which makes no contribution to the physics considered here.

The contribution of $a(x), a^\dagger(x)$ to $B_\phi(x), B_\theta(x)$ is

$$B_\phi(x)|_{a,a^\dagger} = (D^\mu a)^\dagger(\partial_\mu \phi(x))a(x) + (\partial_\mu \phi(x))a^\dagger\bar{D}_\mu a(x)$$

$$B_\theta(x)|_{a,a^\dagger} = -(\bar{D}^0 a(x))^\dagger \sin \phi(x)\bar{n}_\phi(x)a(x) - (\sin \phi(x)\bar{n}_\phi(x)a(x))^\dagger\bar{D}_0 a(x)$$

leading to a contribution of

$$F_{\theta\phi}(x)|_{a,a^\dagger} = -(\bar{D}_0 a(x))^\dagger(\cos \phi(x)\bar{n}_\phi(x) - \sin \phi(x)\bar{n}_\phi(x))a(x)$$

$$-(\partial_\phi \theta(x)(\cos \phi(x)\bar{n}_\phi(x) - \sin \phi(x)\bar{n}_\phi(x))a(x))^\dagger \sin \phi(x)\bar{n}_\phi(x)a$$

$$-(\cos \phi(x)\bar{n}_\phi(x) - \sin \phi(x)\bar{n}_\phi(x))a(\partial_\phi \theta(x))a$$

$$+(\bar{n}_\theta(x)a(x)\partial_\phi \theta(x))^\dagger \bar{n}_\theta(x)a(x)$$

$$+(\bar{n}_\theta(x)a(x))^\dagger \bar{n}_\theta(x)a(x)\partial_\phi \theta(x)$$

$$-(\bar{D}_0 a(x))^\dagger \bar{n}_\theta(x)a(x) - (\bar{n}_\theta(x)a(x))^\dagger\bar{D}_0 a(x)$$

(31)

to the $q^a$-curvature. It follows that the complete expression for the $q^a$-curvature in this theory is the sum of eqs. (31). As with the gluon DOFs, the non-zero $F_{\theta\phi}(x)$ leads to the cancellation of the monopole interactions, and generates corrections to the canonical commutation relations.

7. Monopole corrections to the quantum commutation relations

Corrections to the classical Poisson bracket correspond to corrections to the equal-time commutators in the quantum regime. Denoting conventional commutators as $[,]_{phys}$ and the corrected ones as $[,]_{new}$, for $\mu, \nu \neq 0$ we have

$$[c_\mu(x), c_\nu(z)]_{new} = [c_\mu(x), c_\nu(z)]_{phys}$$

$$-\int dy^4 \left( \frac{\delta B_\phi(y)}{\delta \Pi^\mu(x)} \frac{\delta B_\phi(y)}{\delta \Pi^\nu(z)} - \frac{\delta B_\phi(y)}{\delta \Pi^\mu(x)} \frac{\delta B_\phi(y)}{\delta \Pi^\nu(z)} \right) [\bar{F}_{\rho\phi}(y) \cdot \bar{n}(y) \cos \phi(y) \partial^\rho \phi(y)]^{-1} \delta^4(x - z)$$

$$= [c_\mu(x), c_\nu(z)]_{phys}$$

$$- \sin \phi(x) \sin \phi(z)(\partial_\mu \phi(x)\partial_\nu \theta(z) - \partial_\nu \phi(z)\partial_\mu \theta(x))[\bar{F}_{\rho\phi}(z) \cdot \bar{n}(z) \cos \phi(z) \partial^\rho \phi(z)]^{-1} \delta^4(x - z).$$

(33)

The second term on the final line, after integration over $d^4z$, clearly becomes

$$H_{\mu\nu}(x) \tan \phi(x)[\bar{F}_{\rho\phi}(x) \cdot \bar{n}(x) \partial^\rho \phi(x)]^{-1},$$

(34)
indicating the role of the monopole condensate in the correction. By contrast, the commutation relations
\[ [c_\mu(x), \Pi_\nu(z)]_{\text{new}} = [c_\mu(x), \Pi_\nu(z)]_{\text{phys}}, \quad [\Pi_\mu(x), \Pi_\nu(z)]_{\text{new}} = [\Pi_\mu(x), \Pi_\nu(z)]_{\text{phys}}, \]  
are unchanged. Nonetheless, the deviation from the canonical commutation shown in eq. (33) is inconsistent with the particle creation/annihilation operator formalism of conventional second quantization.

Similarly,
\[ [\Pi_\mu(x), \Pi_\nu(z)]_{\text{new}} = [\Pi_\mu(x), \Pi_\nu(z)]_{\text{phys}} \]
\[ = \delta B_\theta(y) \delta B_\phi(y) \delta X^\mu(x) \delta X^\nu(z) \]
\[ - \frac{g^2}{4} \vec{X}^0(z) \cdot \vec{X}^0(z) \sin \phi(x) \sin \phi(z) (\partial_\alpha \phi(x) \partial_\alpha \phi(z) - \partial_\alpha \phi(z) \partial_\alpha \phi(x)) \]
\[ \times [\vec{F}_\rho^\theta(z) \cdot \hat{n} \cos \phi \partial^\rho \phi]^{-1} \delta^4(x - z) \]
\[ \quad \times [\vec{F}_\rho^\phi(z) \cdot \hat{n} \cos \phi \partial^\rho \phi]^{-1} \delta^4(x - z) \]  
while
\[ [\vec{X}_\mu(x), \vec{X}_\nu(z)]_{\text{new}} = [\vec{X}_\mu(x), \vec{X}_\nu(z)]_{\text{phys}}, \quad [\vec{X}_\mu(x), \vec{X}_\nu(z)]_{\text{new}} = [\vec{X}_\mu(x), \vec{X}_\nu(z)]_{\text{phys}}. \]

Indeed, this is not an exhaustive presentation of deviations from canonical quantisation. If a \( q^\alpha \)-gauge field’s derivative with respect to any physical field or its conjugate momentum is non-zero, then that field’s quantisation conditions and particle interpretation are affected unless the \( q^\alpha \)-curvature is exactly zero. Hence any field interacting with the monopole component ceases to have a particle interpretation in the presence of the monopole component. In particular, its particle number becomes ill-defined, which is reminiscent of the parton model.

Arguments that coloured states are ill-defined in the infrared regime, based on either unitarity and/or gauge invariance [25,26,27] date back several decades but, to our knowledge, we are the first to argue that canonical quantisation breaks down.

8. Discussion

We have applied the Clairaut-type formalism to the CDG decomposition. This has shed light on the dynamics of the topologically generated chromomonopole field of QCD. In particular, it addresses the issue of its EOMs, or lack thereof [16,19], and the contribution its DOFs make to the evolution of other fields.

Indeed, the \( q^\alpha \)-curvature was found to be non-zero, leading to corrections to the time derivatives of the gluon’s dynamic DOFs, which cancel all interactions between physical and non-physical fields from the EOMs. This is both necessary for the consistency of eq. (28), and qualitatively consistent with our later finding that the chromomonopole background alters the canonical commutation relations in such a way as to invalidate the particle interpretation of the physical DOFs.

This can be taken to mean that quarks and gluons do not have a well-defined particle number in the monopole condensate, suggestive of both confinement and the parton model, but it remains to repeat this work with a fully quantised, i.e. including ghosts, \( SU(3) \) gauge field, and with fermionic quarks rather than scalar ones. Furthermore, while many papers have found the monopole condensate [28,29,30], especially with the CDG decomposition [31,16,21,32], to be energetically favourable to the perturbative vacuum, this result needs to be repeated within the Clairaut-based quantisation scheme of this paper before strong claims are made.
In summary, this approach offers a rigorous analytic tool for elucidating the role of topological DOFs in the dynamics of quantum field theories, and finds that coloured states have an ill-defined particle number in the presence of a monopole field strength.

Acknowledgments. The author S.D. is thankful to J. Cuntz and R. Wulkenhaar for kind hospitality at the University of Münster, where the work in its final stage was supported by the project “Groups, Geometry and Actions” (SFB 878).

Appendix A. Appendix. The Clairaut type formalism

Here we review the main ideas and formulae of the Clairaut-type formalism for singular theories [24, 6]. Let us consider a singular Lagrangian \( L(q^A, v^A) = L_{\text{deg}}(q^A, v^A) \), \( A = 1, \ldots n \), which is a function of \( 2n \) variables (\( n \) generalized coordinates \( q^A \) and \( n \) velocities \( v^A = \dot{q}^A = dq^A/dt \)) on the configuration space \( TM \), where \( M \) is a smooth manifold, for which the Hessian’s determinant is zero. Therefore, the rank of the Hessian matrix \( W_{AB} = \frac{\partial^2 L(q^A, v^A)}{\partial v_B \partial v_C} \) is \( r < n \), and we suppose that \( r \) is constant. We can rearrange the indices of \( W_{AB} \) in such a way that a nonsingular minor of rank \( r \) appears in the upper left corner. Then, we represent the index \( A \) as follows: if \( A = 1, \ldots, r \), we replace \( A \) with \( i \) (the “regular” index), and, if \( A = r + 1, \ldots, n \) we replace \( A \) with \( \alpha \) (the “degenerate” index). Obviously, \( \det W_{ij} \neq 0 \), and rank \( W_{ij} = r \). Thus any set of variables labelled by a single index splits as a disjoint union of two subsets. We call those subsets regular (having Latin indices) and degenerate (having Greek indices). As was shown in [24, 6], the “physical” Hamiltonian can be presented in the form

\[
H_{\text{phys}} (q^A, p_i) = \sum_{i=1}^{r} p_i V^i (q^A, p_i, v^\alpha) + \sum_{\alpha=r+1}^{n} B_\alpha (q^A, p_i) v^\alpha - L (q^A, V^i (q^A, p_i, v^\alpha), v^\alpha),
\]

where the functions

\[
B_\alpha (q^A, p_i) \overset{\text{def}}{=} \frac{\partial L (q^A, v^A)}{\partial v^\alpha} \bigg|_{v^\beta = V^i (q^A, p_i, v^\alpha)}
\]

are independent of the unresolved velocities \( v^\alpha \) since rank \( W_{AB} = r \). Also, the r.h.s. of (1) does not depend on the degenerate velocities \( v^\alpha \)

\[
\frac{\partial H_{\text{phys}}}{\partial v^\alpha} = 0,
\]

which justifies the term “physical”. The Hamilton-Clairaut system which describes any singular Lagrangian classical system (satisfying the second order Lagrange equations) has the form

\[
\frac{dq^i}{dt} = \{q^i, H_{\text{phys}}\}_{\text{phys}} - \sum_{\beta=r+1}^{n} \{q^i, B_\beta\}_{\text{phys}} \frac{dq^\beta}{dt}, \quad i = 1, \ldots r
\]

\[
\frac{dp_i}{dt} = \{p_i, H_{\text{phys}}\}_{\text{phys}} - \sum_{\beta=r+1}^{n} \{p_i, B_\beta\}_{\text{phys}} \frac{dq^\beta}{dt}, \quad i = 1, \ldots r
\]

\[
\sum_{\beta=r+1}^{n} \left[ \frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \{B_\alpha, B_\beta\}_{\text{phys}} \right] \frac{dq^\beta}{dt}
\]

\[
= \frac{\partial H_{\text{phys}}}{\partial q^\alpha} + \{B_\alpha, H_{\text{phys}}\}_{\text{phys}}, \quad \alpha = r + 1, \ldots, n
\]
where the “physical” Poisson bracket (in regular variables \(q^i, p_i\)) is

\[
\{X, Y\}_{\text{phys}} = \sum_{i=1}^{n-r} \left( \frac{\partial X}{\partial q^i} \frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q^i} \frac{\partial X}{\partial p_i} \right).
\]

Whether the variables \(B_\alpha (q^A, p_i)\) have a nontrivial effect on the time evolution and commutation relations is equivalent to whether or not the so-called “\(q^A\)-field strength”

\[
F_{\alpha\beta} = \frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \{B_\alpha, B_\beta\}_{\text{phys}}
\]

is non-zero. See [6, 7, 24] for more details.

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