ON ORIENTED SUPERSINGULAR ELLIPTIC CURVES

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Abstract. We revisit theoretical background on OSIDH, that is an isogeny-based key-exchange protocol proposed by Colò and Kohel at NutMIC 2019. We give a proof of a fundamental theorem for OSIDH. The theorem was stated by Colò and Kohel without proof. Furthermore, we consider parameters of OSIDH, give a sufficient condition on the parameters that the protocol works, and estimate the size of the parameters for a certain security level.

1. Introduction

Isogeny-based cryptography is based on hardness of the isogeny problem, that is a problem to find an isogeny between given two elliptic curves. The isogeny problem is considered to be hard even if one uses a quantum computer. Therefore, Isogeny-based cryptography is one of the candidates for post quantum cryptography. The first isogeny-based cryptosystem was proposed by Couveignes [4] in 1997. But his work was not published and posted on ePrint in 2006. The same result was rediscovered by Rostovsev and Stolbunov [15,18]. Their cryptosystem is a key-exchange protocol using isogenies between ordinary elliptic curves. In 2011, Jao and De Feo [9] proposed an isogeny-based exchange protocol using supersingular elliptic curves, named SIDH (Supersingular Isogeny Diffie-Hellman). Castryck, Lange, Martindale, Panny, and Renes [1] proposed another isogeny-based key-exchange protocol CSIDH (Commutative SIDH). CSIDH uses an action of an ideal class group on a set of classes of supersingular elliptic curves.

Currently, many researches focus on the protocols using supersingular elliptic curves due to their efficiency. Even after optimizations by De Feo, Kieffer, and Smith [6], the protocol using ordinary elliptic curves is much slower than SIDH and CSIDH. For a supersingular elliptic curve, its endomorphism rings is isomorphic to a maximal order of a quaternion algebra. The isogeny problem is closely related to the structure of the endomorphism ring. Therefore, it is important for cryptanalysis to study the endomorphism ring. Indeed, there are several researches on this topic. For example, see [10,11,8,2,13].

In 2019, Colò and Kohel [3] proposed a new isogeny-based key-exchange protocol using isogenies between supersingular elliptic curves, named OSIDH. OSIDH uses an inclusion

\[ \mathcal{O} \hookrightarrow \text{End}(E), \]

where \( \mathcal{O} \) is an order of an imaginary quadratic field, and \( E \) is a supersingular elliptic curve over a finite field. Colò and Kohel stated that the ideal class group \( \mathcal{O}(\mathcal{O}) \) acts freely and transitively on a set of equivalence classes of supersingular elliptic

2010 Mathematics Subject Classification. Primary: 11G07, 11Z05.
Key words and phrases. Supersingular elliptic curves, isogeny graphs.
curves which have the above inclusion. It can be seen as a generalization of the result by Waterhouse [20]. However, they did not give any proof of it.

In this paper, we show that their claim has to be slightly modified and give a proof of a modified theorem. Furthermore, we show that a method to calculate the group action proposed by Colò and Kohel does not work in some case, and give a sufficient condition that the method works. Under this condition, we estimate the size of parameters of OSIDH for a certain security level.

2. Notation

Throughout this paper, we use the following notation.

We fix an algebraic closure of a finite field of characteristic \( p \) and denote it by \( k \).

We let \( K \) be a quadratic imaginary field, \( \mathcal{O}_K \) the ring of integer of \( K \), and \( \mathcal{O} \) an order in \( K \). There exists a unique positive integer \( c \) such that \( \mathcal{O} = \mathbb{Z} + c\mathcal{O}_K \). We call the number \( c \) the conductor of \( \mathcal{O} \).

We denote the class group of \( \mathcal{O} \) by \( C_\ell(\mathcal{O}) \) and the class number of \( \mathcal{O} \) by \( h(\mathcal{O}) \). For \( \alpha \in K \), we denote the complex conjugate of \( \alpha \) by \( \overline{\alpha} \).

For an elliptic curve \( E \), we denote the identity element of \( E \) by \( 0_E \), the \( j \)-invariant of \( E \) by \( j(E) \), and the endomorphism ring of \( E \) by \( \text{End}(E) \). We define \( \text{End}^0(E) := \text{End}(E) \otimes \mathbb{Q} \). For an isogeny \( \varphi \), we denote the dual isogeny of \( \varphi \) by \( \hat{\varphi} \).

For a set \( S \), we denote the cardinality of \( S \) by \( \#S \). For a group \( G \) and \( g \in G \), we denote the subgroup of \( G \) generated by \( g \) by \( \langle g \rangle \).

3. Oriented supersingular elliptic curves

In this section, we define orientations on elliptic curves over \( k \) and prove that an ideal class group acts freely and transitively on a set of equivalence classes of oriented supersingular elliptic curves.

3.1. Orientations. We recall definitions about orientations on elliptic curves, which are given in [3].

**Definition 3.1.** A \( K \)-Orientation on an elliptic curve \( E/k \) is a ring homomorphism \( \iota : K \hookrightarrow \text{End}^0(E) \).

A \( K \)-Orientation on \( E \) is a \( \mathcal{O} \)-orientation if \( \iota(\mathcal{O}) \subseteq \text{End}(E) \cap \iota(\mathbb{Z}) \). A \( \mathcal{O} \)-orientation is primitive if \( \iota(\mathcal{O}) = \text{End}(E) \cap \iota(\mathbb{Z}) \). If \( \iota \) is a \( K \)-orientation on \( E \) (resp. (primitive) \( \mathcal{O} \)-orientation), a pair \( (E, \iota) \) is called a \( K \)-oriented (resp. (primitive) \( \mathcal{O} \)-oriented) elliptic curve.

Let \( (E, \iota) \) be a \( K \)-oriented elliptic curve and \( \alpha \in K \) such that \( \iota(\alpha) \in \text{End}(E) \). We have \( \iota(\overline{\alpha}) = \iota(\overline{\alpha}) \). The degree and the trace of \( \iota(\alpha) \) is equal to the norm and the trace of \( \alpha \) respectively. In particular, \( \alpha \) is integral over \( \mathbb{Z} \).

**Definition 3.2.** Let \( (E, \iota) \) be a \( K \)-oriented elliptic curve and \( \varphi : E \to F \) an isogeny of degree \( \ell \). We define a \( K \)-orientation \( \varphi_*(\iota) \) on \( F \) by

\[
\varphi_*(\iota)(\alpha) = \frac{1}{\ell} \varphi \circ \iota(\alpha) \circ \hat{\varphi} \quad \text{for} \quad \alpha \in K.
\]

Given two \( K \)-orientations \( (E, \iota_E) \) and \( (F, \iota_F) \), an isogeny \( \varphi : E \to F \) is \( K \)-oriented if \( \varphi_*(\iota_E) = \iota_F \). We denote this by \( \varphi : (E, \iota_E) \to (F, \iota_F) \).
For $K$-oriented isogeny $\varphi : (E, i_E) \to (F, i_F)$, let $O = \text{End}(E) \cap i_E(K)$ and $O' = \text{End}(F) \cap i_F(K)$ so that $i_E$ is a primitive $O$-orientation and $i_F$ is a primitive $O'$-orientation. We say that $\varphi$ is horizontal if $O = O'$, ascending if $O \subseteq O'$, and descending if $O \supseteq O'$.

**Definition 3.3.** A $K$-orientated isogeny $\varphi : (E, i_E) \to (F, i_F)$ is a $K$-orientated isomorphism if there exists a $K$-oriented isogeny $\psi : (F, i_F) \to (E, i_E)$ such that $\psi \circ \varphi = \text{id}_E$ and $\varphi \circ \psi = \text{id}_F$ as maps. If this happens, we say that $(E, i_E)$ and $(F, i_F)$ are $K$-isomorphic and write $(E, i_E) \cong (F, i_F)$.

Note that a $K$-isomorphism $\varphi$ and its inverse $\varphi^{-1}$ are horizontal.

Let $(E, i)$ be a $K$-oriented elliptic curve over $k$. There is the $p$-th power Frobenius map $\phi_p : E \to E^{(p)}$, where $E^{(p)}$ is the elliptic curve obtained from $E$ by raising each coefficients of $E$ to the $p$-th power. Then we denote $(\phi_p)_*(i)$ by $i^{(p)}$. It can be easily checked that $K$-oriented isogeny $\phi_p : (E, i) \to (E^{(p)}, i^{(p)})$ is horizontal. Furthermore, if $E$ is supersingular then $(E, i)$ is $K$-isomorphic to $((E^{(p)})^{(p)}, (i^{(p)})^{(p)})$, since $E$ is isomorphic to an elliptic curve defined over $\mathbb{F}_{p^2}$ and its endomorphism ring is also defined over $\mathbb{F}_{p^2}$.

We denote the set of primitive $O$-oriented supersingular elliptic curves up to $K$-isomorphism by $\text{SS}_O^{pr}(p)$. We write a $K$-isomorphism class by the same symbol as one of its representatives for brevity. Note that we can always take a representative of a class in $\text{SS}_O^{pr}(p)$ defined over $\mathbb{F}_{p^2}$.

In [3], it is claimed that $O(O)$ acts freely and transitively on $\text{SS}_O^{pr}(p)$. However, rigorously it is not correct. We should slightly modify their claim. We explain by showing a counter example.

Let $E$ be an elliptic curve over $k$ defined by $y^2 = x^3 + x$. As is well know, $\text{End}(E)$ contains a subring isomorphic to $\mathbb{Z}[i]$, where $i$ is a square root of $-1$ in $\mathbb{C}$. We assume $p \equiv 3 \pmod{4}$. Then $E$ is supersingular. Let $a$ be a square root of $-1$ in $\mathbb{F}_{p^2}$. Then there are two orientations $\iota : \mathbb{Q}(i) \to \text{End}^0(E)$, $i \mapsto ((x, y) \mapsto (-x, ay)),$ and two primitive $\mathbb{Z}[i]$-oriented elliptic curves $(E, \iota)$ and $(E, \iota')$. It is easy to show that $(E, \iota)$ and $(E, \iota')$ are not $K$-isomorphic by checking all the automorphisms of $E$ (there are exactly four automorphisms). Therefore, there exists at least two $\mathbb{Q}(i)$-isomorphism classes of primitive $\mathbb{Z}[i]$-oriented supersingular elliptic curves. On the other hand, the class number of $\mathbb{Z}[i]$ is one, so the class group of $\mathbb{Z}[i]$ never acts transitively on the set of these classes.

To fix the claim in [3], we consider reductions of elliptic curves over number fields in the next subsection.

3.2. Reductions. Let $L$ be a number field containing $K$ and $E$ an elliptic curve over $L$ with $\text{End}(E) \cong O$. Let $\left[ \cdot \right]_E : O \to \text{End}(E)$ be an isomorphism such that $(E, \left[ \cdot \right]_E)$ is normalized, i.e., for any invariant differential $\omega$ on $E$, $$(\left[ \alpha \right]_E)^* \omega = \alpha \omega, \quad \text{for all } \alpha \in O.$$ (See II.1 in [16].)

Let $p$ be a prime ideal of $L$ lying above $p$ at which $E$ has a good reduction. A pair $(E, \left[ \cdot \right]_E)$ determines a $K$-oriented elliptic curve $(\tilde{E}, \left[ \cdot \right]_{\tilde{E}})$ by the reduction
modular p, where \([ \cdot ]_{\tilde{E}} : K \to \End^0(\tilde{E})\) is defined by
\[ [\alpha]_{\tilde{E}} = [\alpha]_E \mod p \text{ for all } \alpha \in \mathcal{O}. \]

For two isomorphic elliptic curves \(E\) and \(E'\) over \(L\) and an isomorphism \(\lambda : E \to E'\), it holds that \([ \cdot ]_{E'} = \lambda \circ [ \cdot ]_E \circ \lambda^{-1}\). Therefore, the \(K\)-isomorphism class of the reduction is determined by the isomorphism class of an elliptic curve over a number field. The following lemma states properties of these reductions.

**Lemma 3.1.** Let \(E\) be an elliptic curve over a number field \(L\) containing \(K\) with \(\End(E) \cong \mathcal{O}\), and \(p\) a prime ideal of \(L\) lying above \(p\). Then the reduction curve \(E\) is supersingular if and only if \(p\) does not split in \(K\). Furthermore, let \(c\) be the conductor of \(\mathcal{O}\) and write \(c = p^r c_0\), where \(p \nmid c_0\). Then
\[ \End(\tilde{E}) \cap [K]_{\tilde{E}} = [\mathbb{Z} + c\mathcal{O}_K]_p. \]

**Proof.** See Theorem 12 in chapter 13 in [12]. The statement of about the endomorphism ring can be proved in a similar way to that of ordinary elliptic curves. \(\square\)

This lemma shows that if \(p\) does not split in \(K\) and does not divide the conductor of \(\mathcal{O}\), the reduction \((\tilde{E}, [ \cdot ]_{\tilde{E}})\) is a primitive \(\mathcal{O}\)-oriented supersingular elliptic curve. From this lemma, we obtain the following proposition.

**Proposition 3.2.** The set \(\text{SS}^\text{pr}_0(p)\) is not empty if and only if \(p\) does not split in \(K\) and does not divide the conductor of \(\mathcal{O}\).

**Proof.** First, we assume that \(p\) does not split in \(K\) and does not divide the conductor of \(\mathcal{O}\). There exists an elliptic curve over \(E\) over a number field \(L\) with \(\End(E) \cong \mathcal{O}\). Since \(j\)-invariant of \(E\) is an algebraic integer (Theorem II.6 in [16]), \(E\) has a potential good reduction at every prime ideal (Proposition VII.5.5 in [17]). Therefore, we may assume that \(E\) has a good reduction at a prime ideal of \(L\) lying above \(p\). By Lemma 3.1, the reduction \((\tilde{E}, [ \cdot ]_{\tilde{E}})\) is a primitive \(\mathcal{O}\)-oriented supersingular elliptic curve. Therefore, \(\text{SS}^\text{pr}_0(p)\) is not empty.

Next, we consider the converse. Assume \(\text{SS}^\text{pr}_0(p)\) is not empty and let \((F, \iota) \in \text{SS}^\text{pr}_0(p)\). Let \(\theta \in \mathcal{O}\) such that \(\mathcal{O} = \mathbb{Z}[\theta]\). The Deuring lifting theorem (Theorem 14 in Chapter 13 in [12]) says that there exist an elliptic curve \(\tilde{E}\) over a number field \(L\), an endomorphism \(\alpha \in \End(E)\), and a prime ideal \(p\) lying above \(p\) such that \(\tilde{E}\) isomorphic to \(F\) and \(\alpha \mod p\) corresponds to \(\iota(\theta)\) under the isomorphism. Since \(\alpha\) has the same degree and trace as \(\iota(\theta)\), \(\End(E)\) contains a subset isomorphic to \(\mathcal{O}\). The reduction map \(\End(E) \to \End(F)\) is injective. Therefore, \(\End(E) \cong \mathcal{O}\). By Lemma 3.1, \(p\) does not split in \(K\) and does not divide the conductor of \(\mathcal{O}\). \(\square\)

Let \(p\) does not split in \(K\) and does not divide the conductor of \(\mathcal{O}\). We denote the set of \(j\)-invariants of elliptic curves \(E\) over \(\mathbb{C}\) with \(\End(E) \cong \mathcal{O}\) by \(\mathcal{J}_\mathcal{O}\). Since all elements in \(\mathcal{J}_\mathcal{O}\) are algebraic integer, an elliptic curve whose \(j\)-invariant is in \(\mathcal{J}_\mathcal{O}\) has a potential good reduction at any prime ideal. Since \(\mathcal{J}_\mathcal{O}\) is finite, we can take a number field \(L\) and a prime ideal \(p\) of \(L\) lying above \(p\) such that for all \(j \in \mathcal{J}_\mathcal{O}\), there exists an elliptic curve over \(L\) whose \(j\)-invariant is \(j\) and that has a good reduction at \(p\). Hereafter, we fix a number field \(L\) and a prime ideal \(p\) which have these properties, and an inclusion from the residue field of \(L\) modulo \(p\) to \(k\). We denote the set of isomorphism classes of elliptic curves \(E\) over \(L\) such that
We show that \( \rho : \mathcal{E}(\mathcal{O}) \to \operatorname{SS}^\text{pr}_O(p) \), \( E \to (\tilde{E}, [\cdot]_{\tilde{E}}) \).

We prove that \( \rho \) is surjective up to the \( p \)-th power Frobenius map.

**Proposition 3.3.** For all \((F, \iota) \in \operatorname{SS}^\text{pr}_O(p)\), we have

\[ (F, \iota) \text{ or } (F^{(p)}, \iota^{(p)}) \in \rho(\mathcal{E}(\mathcal{O})). \]

**Proof.** Let \((F, \iota) \in \operatorname{SS}^\text{pr}_O(p)\). As in the proof of Proposition 3.2, there exist a number field \( L' \), a prime ideal \( p' \) of \( L' \), and an elliptic curve over \( E \) with \( \operatorname{End}(E) \cong \mathcal{O} \) such that \( E \) has a good reduction at \( p' \) and the reduction \( \tilde{E} \) is isomorphism to \( F \). We can assume that \( L' \) is a Galois extension over \( \mathbb{Q} \). Then the induced inclusion \([\cdot]_{\tilde{E}}\) is equal to \( \iota \) or it holds that \( [\alpha]_{\tilde{E}} = \iota(\tilde{\alpha}) \), for all \( \alpha \in \mathcal{O} \).

Assume the later holds. Let \( G_{p'} \) be the decomposition group of \( p' \), i.e.,

\[ G_{p'} = \{ \sigma \in \operatorname{Gal}(L'/\mathbb{Q}) \mid p'^{\sigma} = p' \}. \]

Since \( p \) does not split in \( K \), there exists \( \sigma \in G_{p'} \) such that the restriction of \( \sigma \) on \( K \) is not trivial. Then, for \( \alpha \in \mathcal{O} \), we have \([\alpha]_{E^\sigma} = ([\alpha]_E)^{\sigma}\). Therefore, the reduction \((E^\sigma, [\cdot]_{E^\sigma})\) is \( K \)-isomorphism to \((F, \iota)\) or \((F^{(p)}, \iota^{(p)})\) (it is determined by the reduction of \( \sigma \) modulo \( p' \)).

Consequently, we obtain a number field \( L' \), a prime ideal \( p' \) and an elliptic curve \( E \) over \( L' \) such that the reduction \((\tilde{E}, [\cdot]_{\tilde{E}})\) is \( K \)-isomorphic to \((F, \iota)\) or \((F^{(p)}, \iota^{(p)})\).

Let \( M \) be a finite Galois extension over \( K \) containing \( L \) and \( L' \). Let \( \mathfrak{P} \) and \( \mathfrak{P}' \) be prime ideals of \( M \) lying above \( p \) and \( p' \), respectively. Since \( K \) has the only one prime ideal lying above \( p \), there exists \( \sigma \in \operatorname{Gal}(M/K) \) such that \( \mathfrak{P} = \mathfrak{P}^{\sigma} \). Then we have \([\alpha]_{E^{\sigma}} = ([\alpha]_E)^{\sigma}\), for all \( \alpha \in \mathcal{O} \). Therefore, the reduction \((E^{\sigma}, [\cdot]_{E^{\sigma}})\) modulo \( \mathfrak{P} \) is \( K \)-isomorphic to the reduction \((\tilde{E}, [\cdot]_{\tilde{E}})\) modulo \( \mathfrak{P}' \) or its image \((\tilde{E}, [\cdot]_{\tilde{E}}) \) under the \( p \)-th power Frobenius map. Since \( j(E^{\sigma}) \in J_\mathcal{O} \), there exists an elliptic curve \( E' \) over \( L \) such that \( E' \cong E^{\sigma} \) and \( E' \) has a good reduction at \( p \). Then we have \( \rho(E') = (F, \iota) \) or \((F^{(p)}, \iota^{(p)})\).

3.3. **Group action.** The first object in this paper is to prove the following theorem. This is a slightly modified version of Proposition 3.1 in [3], that is stated without proof.

**Theorem 3.4.** Let \( K \) be an imaginary quadratic field such that \( p \) does not split in \( K \), and \( \mathcal{O} \) an order in \( K \) such that \( p \) does not divide the conductor of \( \mathcal{O} \). Then the ideal class group \( \mathcal{O}(\mathcal{O}) \) acts freely and transitively on \( \rho(\mathcal{E}(\mathcal{O})). \)

Before we prove this theorem, we define an action of \( \mathcal{O}(\mathcal{O}) \) on \( \rho(\mathcal{E}(\mathcal{O})). \) For \((E, \iota) \in \operatorname{SS}^\text{pr}_O(p) \) and an integral ideal \( \alpha \) of \( \mathcal{O} \) prime to \( p \), we define the \( \alpha \)-torsion subgroup \( E[\iota(\alpha)] \) by

\[ E[\alpha] = \bigcap_{\alpha \in \mathfrak{a}} \ker \iota(\alpha). \]

Then there are an elliptic curve \( F \) and a separable isogeny \( \varphi : E \to F \) with \( \ker \varphi = E[\alpha] \) (Proposition III.4.12 in [17]). Furthermore, if there are an elliptic curve \( F' \) and a separable isogeny \( \varphi' : E \to F' \) with \( \ker \varphi' = E[\alpha] \), there exists an isomorphism \( \lambda : F \to F' \) such that \( \varphi' = \lambda \circ \varphi \) (Corollary III.4.11 in [17]). Therefore, we have \((F, \varphi_*(\iota)) \) is \( K \)-isomorphic to \((F', \varphi'_*(\iota)) \). We denote this \( K \)-isomorphism class by \( \alpha * (E, \iota) \). Then we have the following proposition.
Proposition 3.5. Let $(E, \iota)$ be a primitive $\mathcal{O}$-oriented elliptic curve, $a$ an integral ideal of $\mathcal{O}$ prime to $p$. Then $K$-oriented isogeny with kernel $E[a]$

$$\varphi : (E, \iota) \rightarrow a \ast (E, \iota)$$

is horizontal or ascending. Furthermore, if $a$ is invertible then $\varphi$ is horizontal.

Proof. Let $(E', \iota') = a \ast (E, \iota)$ and $\mathcal{O}' = \text{End}(E') \cap \iota'(K)$.

If $a$ is divisible by some integer $n$, we have $\varphi = \varphi' \circ \iota(n)$, where $\varphi'$ is an isogeny with kernel $E[a/n]$. Since the multiplication by $n$ in $\text{End}(E)$ is horizontal, we can assume that $a$ is not divisible by any integer greater than 1, i.e., $E[a]$ is cyclic.

Let $a$ be the absolute norm of $a$ and $x \in \mathcal{O}$. By definition,

$$\iota'(x) = \frac{1}{a} \varphi \circ \iota(x) \circ \hat{\varphi}.$$ 

Therefore, $\mathcal{O} \subseteq \mathcal{O}'$ if and only if $\varphi \circ \iota(x) \circ \hat{\varphi} \in a \text{End}(E')$ for all $x \in \mathcal{O}$. Let $\{P, Q\}$ be a pair of points in $E$ generating $E[a]$ such that $P$ generates $E[a]$, and $P'$ a point in $E$ such that $aP' = P$. Then it can be easily checked that $\varphi(P')$ and $\varphi(Q)$ generate $E'[a]$. We have

$$\varphi \circ \iota(x) \circ \hat{\varphi}(P) = \varphi \circ \iota(x)(P) = 0_E, \quad (\because \iota(x)(P) \in E[a],)$$

$$\varphi \circ \iota(x) \circ \hat{\varphi}(Q) = \varphi \circ \iota(x)(0_E) = 0_E.$$ 

Therefore, $E'[a] \subseteq \ker(\varphi \circ \iota(x) \circ \hat{\varphi})$. This means $\varphi \circ \iota(x) \circ \hat{\varphi} \in a \text{End}(E')$.

Assume that $a$ is an invertible ideal of $\mathcal{O}$. Then $aa^{-1}$ is an integral ideal of $\mathcal{O}$. We show $\ker(\hat{\varphi}) = E'[aa^{-1}\mathcal{O}']$. From this and the first assertion, we have $\mathcal{O}' = \mathcal{O}$.

We note that $\ker(\hat{\varphi})$ is generated by $\varphi(Q)$. Let $x \in a^{-1}$. For $\alpha \in a$, we have

$$\iota(\alpha x) Q = \iota(\alpha x) a Q = 0_E.$$ 

Therefore, $\iota(\alpha x) Q \in E[a] = \ker(\varphi)$. So we have $\iota'(ax) \varphi(Q) = \varphi(\iota(ax) Q) = 0_E$. This means $\ker(\hat{\varphi}) \subseteq E'[aa^{-1}\mathcal{O}']$.

Conversely, let $R \in E'[aa^{-1}\mathcal{O}']$. Since a non-constant isogeny is surjective, there exists $S \in E$ such that $R = \varphi(S)$. For $x \in a^{-1}$, we have

$$\varphi(\iota(ax) S) = \iota'(ax) \varphi(S) = \iota'(ax) R = 0_E.$$ 

Let $\alpha_1, \ldots, \alpha_n \in a$ and $x_1, \ldots, x_n \in a^{-1}$ such that $\sum_i \alpha_i x_i = 1$. Then

$$a S = \iota(\sum_i \alpha_i x_i) a S = \sum_i \iota(\alpha_i) \iota(ax_i) S = 0_E.$$ 

Therefore, $S \in E[a]$. So, $R = \varphi(S) \in \varphi(E[a]) = \langle \varphi(Q) \rangle$. \hfill $\square$

Now, we can define an action of $\mathcal{O}(\mathcal{O})$ on $\rho(\mathcal{E}\ell(\mathcal{O}))$.

Proposition 3.6. For $(E, \iota) \in \rho(\mathcal{E}\ell(\mathcal{O}))$ and an invertible integral ideal $a$ of $\mathcal{O}$ prime to $p$, the map $(a, (E, \iota)) \mapsto a \ast (E, \iota)$ defines an action of $\mathcal{O}(\mathcal{O})$ on $\rho(\mathcal{E}\ell(\mathcal{O}))$.

Proof. Let $(E, \iota) \in S_{\mathcal{O}}(p)$ and $F \in \mathcal{E}\ell(\mathcal{O})$ such that $(\widetilde{F}, [\cdot]_p) = (E, \iota)$. Then $F[a] := \cap_{a \in a} \ker[a]_F$ corresponds to $E[a]$ by the reduction modulo $p$, since $a$ is prime to $p$. By the complex multiplication theory for elliptic curves over number fields, there exist an elliptic curve $F' \in \mathcal{E}\ell(\mathcal{O})$ and an isogeny $\varphi : F \rightarrow F'$ with $\ker \varphi = F'[a]$. Let $\tilde{\varphi} : \widetilde{F} \rightarrow \widetilde{F}'$ be the reduction of $\varphi$ modulo $p$. Then we have

$$a \ast (E, \iota) = (\widetilde{F}, \tilde{\varphi}(\iota)) = \rho(F) \in \rho(\mathcal{E}\ell(\mathcal{O})).$$
Let \((E, \iota) \in \rho(\mathcal{Ell}(O))\) and \(a, b\) invertible integral ideals of \(O\) prime to \(p\). Write \((E', \iota') = a \ast (E, \iota)\) and \((E'', \iota'') = b \ast (E', \iota')\), and let \(\varphi_a : E \rightarrow E'\) be a separable isogeny with \(\ker \varphi_a = E[a]\) and \(\varphi_b : E' \rightarrow E''\) a separable isogeny with \(\ker \varphi_b = E[b]\). Then Proposition 3.12 in [20] says \(\varphi_b \circ \varphi_a\) has kernel \(E[ba]\). We have
\[
\iota'' = \frac{1}{\deg \varphi_b} \varphi_b \circ \iota' \circ \varphi_b
= \frac{1}{\deg \varphi_b \deg \varphi_a} \varphi_b \circ \varphi_a \circ \iota \circ \varphi_a \circ \hat{\varphi}_b
= (\varphi_b \circ \varphi_a) \ast (\iota).
\]
Therefore, we have
\[
b \ast (a \ast (E, \iota)) = (ba) \ast (E, \iota).
\]
It is easy to show that any principal ideal of \(O\) acts trivially on \(\rho(\mathcal{Ell}(O))\). Therefore, the map \((a, (E, \iota)) \rightarrow a \ast (E, \iota)\) defines an action of \(\mathcal{O}(O)\) on \(\rho(\mathcal{Ell}(O))\).

Now, we prove Theorem 3.4.

Proof of Theorem 3.4. It remains to show the action in Proposition 3.6 is free and transitive.

Let \(a\) be an invertible integral ideal of \(O\) prime to \(p\) such that \(a \ast (E, \iota) = (E, \iota)\). This means that there exists a separable endomorphism \(\varphi\) of \(E\) with \(\ker \varphi = E[a]\) such that \(\iota = \varphi \ast (\iota)\). Then \(\varphi\) commutes with the endomorphisms in the image of \(\iota\). Therefore, \(\varphi \in \iota(O)\). Let \(\alpha \in O\) such that \(\varphi = \iota(\alpha)\). Since \(\varphi\) is separable and \(a\) is prime to \(p\), it holds that \(aO = a\). Therefore, the action of \(\mathcal{O}(O)\) on \(\rho(\mathcal{Ell}(O))\) is free.

By Proposition 3.2 and the assumption on \(K\) and \(O\), \(\rho(\mathcal{Ell}(O))\) is not empty. By the definition, \(\rho(\mathcal{Ell}(O)) \leq \# \mathcal{Ell}(O) = h(O)\). On the other hand, since \(\mathcal{O}(O)\) acts freely on \(\rho(\mathcal{Ell}(O))\), we have \(\# \rho(\mathcal{Ell}(O)) \geq h(O)\). Therefore, \(\# \rho(\mathcal{Ell}(O)) = h(O)\). This shows that the action is transitive. \(\square\)

4. ORIENTING SUPERSINGULAR ISOGENY GRAPHS

In this section, we consider a graph related to oriented supersingular elliptic curves.

First, we define an equivalent relation on isogenies.

Definition 4.1. Two \(K\)-oriented isogenies
\[
\varphi : (E, \iota_E) \rightarrow (F, \iota_F) \quad \text{and} \quad \psi : (E', \iota_E') \rightarrow (F', \iota_{F'})
\]
are \(K\)-equivalent if there exist \(K\)-oriented isomorphisms \(\lambda : E \rightarrow E'\) and \(\lambda' : F' \rightarrow F\) such that \(\lambda' \circ \psi \circ \lambda = \varphi\).

Let \(\ell \neq p\) be a prime number. An \(\ell\)-isogeny is an isogeny of degree \(\ell\). We define
\[
\mathcal{O}_\ell^{(n)} := \mathbb{Z} + \ell^nO_K, \quad n \in \mathbb{Z}_{\geq 0}.
\]

We define an \(\ell\)-orienting supersingular \(\ell\)-isogeny graph \(G_\ell(K, p)\) as follows: The vertex set of \(G_\ell(K, p)\) is \(\bigcap_{n \geq 0} \rho(\mathcal{Ell}(O_\ell^{(n)}))\), and the edges of \(G_\ell(K, p)\) are \(K\)-oriented \(\ell\)-isogenies up to \(K\)-equivalence.

Since the reduction map \(\rho\) is bijective, and an \(\ell\)-isogeny between \(K\)-oriented supersingular elliptic curves corresponds to an \(\ell\)-isogeny between elliptic curves over a
number field, $G_\ell(K, p)$ has the same structure as that of the $\ell$-isogeny graph of elliptic curves over a number filed with complex multiplication by orders of $K$. In particular, every $\ell$-isogeny from $\bigcap_{n \geq 0} \rho(\ell\ell(O^{(n)}_\ell))$ has codomain in $\bigcap_{n \geq 0} \rho(\ell\ell(O^{(n)}_\ell))$. Moreover, the following an analogue of Proposition 23 in [10] can be obtained by the graph structure of elliptic curves over a number field with complex multiplication.

**Proposition 4.1.** Let $(E, \iota) \in \text{SS}_O^p(p)$, $D$ be the discriminant of $K$ and $(\frac{D}{\ell})$ the Legendre symbol. If $\ell$ does not divide the conductor of $O$, $(E, \iota)$ has no ascending $\ell$-isogeny, $(\frac{D}{\ell}) + 1$ horizontal $\ell$-isogenies, and $\ell - (\frac{D}{\ell})$ descending $\ell$-isogenies. If $\ell$ divides the conductor of $O$, $(E, \iota)$ has exactly one ascending $\ell$-isogeny, no horizontal $\ell$-isogeny, and $\ell$ descending $\ell$-isogenies. Furthermore, every codomain of a descending $\ell$-isogeny has exactly $[O^\times : (\mathbb{Z} + \ell O)^\times]$ $\ell$-isogenies from $(E, \iota)$.

By this proposition, the number of primitive $O^{(n)}_\ell$-oriented supersingular elliptic curves connected to $\rho(\ell\ell(O_K))$ is

$$\frac{h(O_K)}{|O^\times : (O^{(n)}_\ell)^\times|} \ell^{c - 1} \left( \ell - \left( \frac{D}{\ell} \right) \right).$$

The formula for the class number of an order (see §7 in [5]) says this number is equal to $h(O^{(n)}_\ell) = \#\rho(\ell\ell(O^{(n)}_\ell))$. Therefore, all elements in $\rho(\ell\ell(O^{(n)}_\ell))$ have exactly one descending path from an element in $\rho(\ell\ell(O_K))$.

5. OSIDH

Colò and Kohel [3] proposed a Diffie-Hellman type key-exchange protocol using oriented supersingular elliptic curves, named OSIDH. OSIDH uses the action of an ideal class group described in [5,3]. A method to calculate the group action was proposed in [3]. However, in some cases, it does not work.

In this section, we recall the method to calculate the group action in [3] and the protocol of OSIDH.

5.1. **Group action.** We recall the method to calculate the group action proposed in [3].

Let $(E_0, t_0) \in \rho(\ell\ell(O_K))$. By Proposition 4.1, there is a chain of descending $K$-oriented $\ell$-isogenies:

$$(1) \quad (E_0, t_0) \xrightarrow{\varphi_0} (E_1, t_1) \xrightarrow{\varphi_1} (E_2, t_2) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} (E_n, t_n),$$

where $(E_i, t_i) \in \rho(\ell\ell(O^{(n)}_\ell))$ for $i = 0, 1, \ldots, n$. We denote this chain by $((E_i, t_i), \varphi_i)$. Let $q \neq \ell$ be a prime splitting in $K$ and $\mathfrak{q}$ a prime ideal in $O_K$ lying above $q$. For brevity, we use the same symbol $\mathfrak{q}$ for the prime ideal $\mathfrak{q} \cap O^{(i)}_\ell$ for $i = 0, 1, \ldots, n$. Let $(F_i, t'_i) = q \ast (E_i, t_i)$ and $\psi_i : (E_i, t_i) \rightarrow (F_i, t'_i)$ be a $K$-oriented isogeny with $\ker \psi_i = E_i[\mathfrak{q}]$. Then there exists a descending $K$-oriented $\ell$-isogeny $\varphi'_i : (F_i, t'_i) \rightarrow (F_{i+1}, t_{i+1})$ with $\ker \varphi'_i = \psi_i(\ker \varphi_i)$. Therefore, we obtain the following commutative diagram of $K$-oriented isogenies:

$$\begin{array}{ccc}
(E_0, t_0) & \xrightarrow{\varphi_0} & (E_1, t_1) \\
\downarrow \psi_0 & & \downarrow \psi_1 \\
(F_0, t'_0) & \xrightarrow{\varphi'_0} & (F_1, t'_1)
\end{array} \quad \begin{array}{ccc}
(E_1, t_1) & \xrightarrow{\varphi_1} & (E_2, t_2) \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
(F_1, t'_1) & \xrightarrow{\varphi'_1} & (F_2, t'_2)
\end{array} \quad \begin{array}{ccc}
(E_2, t_2) & \xrightarrow{\varphi_2} & \cdots & \xrightarrow{\varphi_{n-1}} & (E_n, t_n) \\
\downarrow \psi_2 & & \cdots & & \downarrow \psi_{n-1} \\
(F_2, t'_2) & \xrightarrow{\varphi'_2} & \cdots & \xrightarrow{\varphi'_{n-1}} & (F_n, t'_n)
\end{array}$$

We denote the chain $((F_i, t'_i), \varphi'_i)$ by $q \ast ((E_i, t_i), \varphi_i)$. 

The method to calculate the group action in [3] is based on the following assumption, though it is not explicitly stated.

**Assumption 5.1.** Let \( \ell \neq p \) be a prime number, \( q \neq \ell \) a prime number splitting in \( K \), and \( q \) a prime ideal in \( K \) lying above \( q \), and \( (E, t) \rightarrow (E', t') \) a descending \( K \)-oriented \( \ell \)-isogeny. We denote \( q \ast ((E, t), \varphi) \) by \( (E', t') \). Let \( F \) be an elliptic curve such that there exist \( q \)-isogeny \( \psi : E' \rightarrow F \) and \( \ell \)-isogeny \( \varphi : E'' \rightarrow F \). Then \( \psi \) and \( \varphi \) induce the same orientation on \( F \), i.e.,

\[
(F, \psi_\ast(t')) \cong (F, \varphi_\ast(t'')).
\]

Now we explain the method to the group action in [3]. The idea in [3] is to compute \( j \)-invariants of the chain \( q \ast ((E_i, t_i), \varphi_i) \) in the commutative diagram (2) by using the greatest common divisor of modular polynomials.

For a prime \( r \), we denote the \( r \)-th modular polynomial by \( \Phi_r(X, Y) \). We assume that the class number of \( \ell \) is one. Then, in the commutative diagram (2), it holds that \((F_0, t'_0) = (E_0, t_0)\). For simplicity, we assume that \( q^2 \) is not principal in \( \mathcal{O}_\ell^{(1)} \).

First, we need to determine the “direction” of \( q \)-isogeny. By Assumption 5.1, an elliptic curve that has an \( \ell \)-isogeny from \( E_0 \) and an \( q \)-isogeny from \( E_1 \) is isomorphic to \( q \ast (E_1, t_1) \) or \( q \ast (E_1, t_1) \), where \( \bar{q} \) is the complex conjugate of \( q \). To distinguish these curves, we compute \( j \)-invariant of \( (F_1, t'_0) = q \ast (E_1, t_1) \) by, for example, Vélu’s formula [19]. Then, by Assumption 5.1 an elliptic curve that has an \( \ell \)-isogeny from \( F_1 \) and a \( q \)-isogeny from \( E_2 \) is isomorphic to \( q \ast (E_2, t_2) \). Therefore, the \( j \)-invariant of \( q \ast (E_2, t_2) \) is the unique solution of

\[
\gcd(\Phi_e(X, j(F_1)), \Phi_q(X, j(E_2))) = 0.
\]

We obtain the \( j \)-invariant of \( q \ast (E_2, t_2) \) by calculating the g.c.d. of these polynomials. In the same way, we obtain all the \( j \)-invariants of the chain \( q \ast ((E_i, t_i), \varphi_i) \). By repeating this process, we can obtain the \( j \)-invariants of the chain \( q^{e} \ast ((E_i, t_i), \varphi_i) \) for an integer \( e \). Note that for \( e \geq 2 \), the calculation of \( q^{e} \ast ((E_i, t_i), \varphi_i) \) does not need to determine the direction. Because the chain \( \bar{q} \ast (q^{e-1} \ast ((E_i, t_i), \varphi_i)) \) and we have already know the \( j \)-invariants of this chain.

Let \( q_1, \ldots, q_t \neq \ell \) be prime numbers splitting in \( K \) and \( q_1 \) a prime ideal of \( \mathcal{O}_K \) lying above \( q_1 \) for \( i = 1, \ldots, t \). We further assume that \( q_i \neq q_i^{\pm 2} \) in \( \mathcal{O}(\mathcal{O}_\ell^{(n)}) \) for \( i \neq i' \). As the above, we use the same symbol \( q_i \) for \( q_i \cap \mathcal{O}_\ell^{(k)} \) for \( k = 0, 1, \ldots, n \). By the method explained in the previous paragraph, we can calculate the \( j \)-invariants of chains of \( K \)-oriented \( q_i \)-isogenies

\[
(E_n, t_n) \rightarrow q_1 \ast (E_n, t_n) \rightarrow \cdots \rightarrow q_i^{e_i} \ast (E_n, t_n)
\]

for \( i = 1, \ldots, t \), where \( e_i \) is a positive integer. As in the previous paragraph, we can obtain the \( j \)-invariant of \( (\prod_{i=1}^{t} q_i^{e_i}) \ast (E_n, t_n) \) by calculating the g.c.d.s of modular polynomials.

### 5.2. Protocol

The protocol of OSIDH is as follows:

**Public data:** At the first, Alice and Bob publicly share the following system information.

- The \( j \)-invariants of a chain of descending \( K \)-oriented \( \ell \)-isogenies

\[
(E_0, t_0) \rightarrow (E_1, t_1) \rightarrow \cdots \rightarrow (E_n, t_n),
\]

where \((E_0, t_0)\) is a primitive \( \mathcal{O}_K \)-oriented supersingular elliptic curve with \( h(\mathcal{O}_K) = 1 \).
• Prime numbers \( q_1, \ldots, q_t \) splitting in \( K \), and prime ideals \( q_1, \ldots, q_t \) of \( \mathcal{O}_K \) above \( q_1, \ldots, q_t \), respectively. We assume that \( q_i^2 \neq q_i^{\pm2} \) in \( \mathcal{O}(\mathcal{O}_K^{(n)}) \) for \( i \neq i' \).

• The \( j \) invariant of \( q_i \ast (E_{k_i}, t_{k_i}) \) for \( i = 1, \ldots, t \), where \( k_i \) is the smallest integer such that \( q_i^2 \) is not principal in \( \mathcal{O}_K^{(k_i)} \).

**Secret key:** Secret keys are an integer vector in \([-B, B]^t\). We let Alice’s secret key be \((e_1, \ldots, e_t)\) and Bob’s secret key \((d_1, \ldots, d_t)\).

**Public key:** Alice’s public key is the \( j \)-invariant of \((F, \ell') = (\prod_{i=1}^t q_i^{e_i}) \ast (E_n, t_n)\) and the \( j \)-invariants of chains

\[
(F, \ell') \rightarrow q_i \ast (F, \ell') \rightarrow \cdots \rightarrow q_i^B \ast (F, \ell'), \\
(F, \ell') \rightarrow q_i^{-1} \ast (F, \ell') \rightarrow \cdots \rightarrow q_i^{-B} \ast (F, \ell')
\]

for \( i = 1, \ldots, t \). These values are calculated by the method explained in [5.1]. Similarly, Bob’s public key is the \( j \)-invariant of \((G, \ell'') = (\prod_{i=1}^t q_i^{d_i}) \ast (E_n, t_n)\) and the \( j \)-invariants of chains

\[
(G, \ell'') \rightarrow q_i \ast (G, \ell'') \rightarrow \cdots \rightarrow q_i^B \ast (G, \ell''), \\
(G, \ell'') \rightarrow q_i^{-1} \ast (G, \ell'') \rightarrow \cdots \rightarrow q_i^{-B} \ast (G, \ell'')
\]

for \( i = 1, \ldots, t \).

**Shared secret:** By using the method explained in [5.1] Alice calculates the \( j \)-invariant of \((\prod_{i=1}^t q_i^{e_i}) \ast (G, \ell'')\), and Bob calculates the \( j \)-invariant of \((\prod_{i=1}^t q_i^{d_i}) \ast (F, \ell'')\). These values are equal to the \( j \)-invariant of \((\prod_{i=1}^t q_i^{e_i+d_i}) \ast (E_n, t_n)\). Alice and Bob shares this value as a shared secret.

6. Parameter choice on OSIDH

6.1. Counter example. We show that Assumption [5.1] does not hold in general by giving a counter example.

Let \( p = 419 \) and \( E_0 \) be an elliptic curve over \( \mathbb{F}_p \) with \( j(E_0) = 52 \equiv 1728 \) (mod 419). Then \( E_0 \) is supersingular and has a primitive \( \mathbb{Z}[i] \)-orientation, where \( i \) is a square root of -1 in \( \mathbb{C} \). By calculating a chain of 3-isogenies from \( E_0 \), we obtain a sequence of \( j \)-invariants of descending \( \mathbb{Q}(i) \)-oriented isogenies \((52, 367, 288, 308)\). Let \( q \) be a prime ideal of \( \mathbb{Q}(i) \) lying above 5. By applying the method stated in [5.1] we have

\[
q \ast (52, 367, 288) = (52, 356, 333 + 132a) \text{ or } (52, 356, 333 - 132a),
\]

where \( a \) is a square root of -1 in \( \mathbb{F}_{p^2} \). Which equation holds depends on the choice of \( q \). We take \( q \) as the former equation holds. By Assumption [5.1] there is the unique elliptic curve that has a 3-isogeny from the curve with \( j \)-invariant 288 and a 5-isogeny from the curve with \( j \)-invariant 333 + 132a. However, the equation

\[
\gcd(\Phi_3(X, 288), \Phi_5(X, 333 + 132a)) = 0
\]

has two solutions \( 202 + 26a \) and \( 315 + 162a \) in \( \mathbb{F}_p \). I.e., in the following diagram, the most right part of the bottom chain cannot be determined by the modular polynomials.

| 52 | 3 | 288 | 3 | 308 |
|----|---|-----|---|-----|
| 5  |   | 5   |   |     |
| 5  |   | 5   |   |     |
| 52 | 3 | 356 | 3 | 333 + 132a | 3 | 202 + 26a or 315 + 162a |
This contradicts Assumption 5.1.

6.2. Conditions on \( p \). We give a sufficient that condition of Assumption 5.1 holds for the case the starting curve \( E_0 \) has an \( \mathbb{Z}[i] \)-orientation, i.e., \( j(E_0) = 1728 \). We let \( p \equiv 3 \mod{4} \) so that \( E_0 \) is supersingular. Let \( \omega \) be a \( \mathbb{Z}[i] \)-orientation on \( E_0 \). First, we recall the structure of the endomorphism ring of a supersingular elliptic curve. Let \( E \) be a supersingular elliptic curve over \( k \). Then \( \text{End}^0(E) \) is isomorphic to the quaternion algebra \( B_{p,\infty} \) over \( \mathbb{Q} \) ramified at only \( p \) and \( \infty \). In the case \( p \equiv 3 \mod{4} \), Proposition 5.1 in [14] shows that

\[
B_{p,\infty} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k,
\]

where \( i^2 = -1, j^2 = -p, \) and \( ij = -ji = k \). For \( E_0 \), we know the explicit structure of its endomorphism ring.

**Lemma 6.1.** Let \( p \equiv 3 \mod{4} \) and \( E_0 \) be an elliptic curve with \( j(E_0) = 1728 \). Then

\[
\text{End}(E_0) \cong \mathbb{Z} + i\mathbb{Z} + \frac{1+j}{2}\mathbb{Z} + \frac{1+k}{2}\mathbb{Z}.
\]

**Proof.** By Proposition 5.2 in [14], the right hand side on the equation is a maximal order. The Deuring correspondence says that there exists a supersingular elliptic curve \( E \) over \( k \) such that \( \text{End}(E) \) is isomorphic to this maximal order. Since \( \text{End}(E) \) contain an automorphism corresponding to \( i \), the elliptic curve \( E \) has \( \mathbb{Z}[i] \)-orientation, i.e., \( j(E) = 1728 \).

Now we show the following.

**Theorem 6.2.** In Assumption 5.1 we assume that \( K = \mathbb{Q}(i) \), and that \( (E,\epsilon) \) is a primitive \( \mathcal{O}_\ell^{n-1} \)-oriented supersingular elliptic curve. If \( p > 4q\ell^{2n} - 1 \) then Assumption 5.1 holds.

**Proof.** Let \( \varphi : \psi, q \ast (E,\epsilon) \to (E',\epsilon') \) be a descending \( \mathbb{Q}(i) \)-oriented \( \ell \)-isogeny, and \( (E'',\epsilon'') = q \ast (E,\epsilon) \). Assume that there exist an elliptic curve \( F, q \)-isogeny \( \psi : E' \to F \), and \( \ell \)-isogeny \( \varphi' : E'' \to F \) such that \( (F,\psi,\epsilon') \not\cong (F,\varphi',\epsilon'') \). Let \( \eta : (E_0,\iota_0) \to (E,\epsilon) \) and \( \eta' : (E_0,\iota_0) \to (E'',\epsilon'') \) be descending \( \ell^{n-1} \)-isogenies. Then the composition

\[
\alpha : E_0 \xrightarrow{\eta} E \xrightarrow{\varphi,\psi} E' \xrightarrow{\varphi'} E'' \xrightarrow{\eta'} E_0
\]

is in \( \text{End}(E_0) \). Since \( \varphi' \) and \( \psi \) induce distinct orientation on \( F \), the endomorphism \( \alpha \) is not \( \mathbb{Q}(i) \)-oriented on \( (E_0,\iota_0) \). This means that \( \alpha \not\in \iota_0(\mathbb{Z}[i]) \). Therefore, by the isomorphism in Lemma 6.1 \( \alpha \) corresponds to the element of form

\[
a + bi + c\frac{1+j}{2} + d\frac{1+k}{2}, \quad a, b, c, d \in \mathbb{Z}, \ c \text{ or } d \neq 0.
\]

In particular, the degree of \( \alpha \) is greater than or equal to the reduced norm of \( \frac{1+j}{2} \) or \( \frac{i+k}{2} \), i.e., we have \( \deg(\alpha) \geq \frac{\ell+1}{2} \). On the other hand, by definition, \( \deg(\alpha) = q\ell^{2n} \). Therefore, we obtain \( p \leq 4q\ell^{2n} - 1 \). This proves the theorem.

Next, we give a sufficient condition that all the \( j \)-invariant of the elliptic curves in \( \rho(E\ell(\mathcal{O}_\ell^{n})) \) are distinct. The protocol of OSIDH correctly works even if there are two oriented elliptic curves in \( \rho(E\ell(\mathcal{O}_\ell^{n})) \) have the same \( j \)-invariant. But, if the number of \( j \)-invariants in \( \rho(E\ell(\mathcal{O}_\ell^{n})) \) is very small, then the exhaustive search on these \( j \)-invariants could find a secret share.
Fortunately, if \( p \) satisfies the condition in Theorem 6.2 then all the \( j \)-invariant of the elliptic curves in \( \rho(E \ell(\mathcal{O}_E^{(n)})) \) are distinct.

**Theorem 6.3.** Let \( n \) be a positive integer such that \( p > 4\ell^{2n} - 1 \), and \( m_1, m_2 \leq n \) nonnegative integers. Let \( (E_1, \iota_1) \in \rho(E \ell(\mathcal{O}_E^{(m_1)})) \), and \( (E_2, \iota_2) \in \rho(E \ell(\mathcal{O}_E^{(m_2)})) \). Then \( (E_1, \iota_1) \cong (E_2, \iota_2) \) if and only if \( j(E_1) = j(E_2) \).

**Proof.** Obviously, if \( (E_1, \iota_1) \cong (E_2, \iota_2) \) then \( j(E_1) = j(E_2) \).

Assume that \( (E_1, \iota_1) \not\cong (E_2, \iota_2) \) and \( j(E_1) = j(E_2) \). Then there exists an isomorphism \( \lambda : E_1 \to E_2 \) such that \( \lambda(E_2, \iota_2) \not\cong (E_2, \lambda_*(\iota_1)) \). Let \( \eta_1 : (E_0, \iota_0) \to (E_1, \iota_1) \) be a descending \( \ell^{m_1} \)-isogeny, and \( \eta_2 : (E_0, \iota_0) \to (E_2, \iota_2) \) a descending \( \ell^{m_2} \)-isogeny. Then the composition

\[
\beta : E_0 \xrightarrow{\eta_1} E_1 \xrightarrow{\lambda} E_2 \xrightarrow{\eta_2} E_0
\]

is an endomorphism on \( E_0 \) of degree \( \ell^{m_1+m_2} \). By the same discussion in the proof of Theorem 6.2 we have \( p \leq 4\ell^{m_1+m_2} - 1 \). This contradicts our assumption. Therefore, \( j(E_1) \not= j(E_2) \). This completes the proof. \( \square \)

### 6.3. Security

Finally, we discuss parameters of OSIDH for satisfying a certain security level on a classical computer. Let \( \lambda \) be the security level, i.e., we require OSIDH to satisfy that at least \( 2^\lambda \) operations are needed to reveal a shared secret in OSIDH.

As same as CSIDH, Meet-in-the-middle attack (see §7.1 in [1]) can be applied to OSIDH. Therefore, the order of ideal class group \( \mathcal{O}(\mathcal{O}_E^{(n)}) \) has to be greater than \( 2^{2\lambda} \). Unfortunately, this condition is not enough. We need much larger ideal class group. Let \( (e_1, \ldots, e_t) \) be Alice’s secret key and \((E_A, \iota_A) = (\prod_{i=1}^t q^{e_i}_j) \ast (E_n, \iota)\). If one can compute actions of any ideal classes in \( \mathcal{O}(\mathcal{O}_E^{(n)}) \) then s/he can compute the endomorphism \( \iota_A(\ell^n) \) of \( E_A \). The endomorphism \( \iota_A(\ell^n) \) is the composite

\[
E_A \xrightarrow{\varphi} E_0 \xrightarrow{\iota_A(\ell^n)} E_0 \xrightarrow{\varphi} E_A,
\]

where \( \varphi \) is a descending \( \mathbb{Q}(i) \)-oriented \( \ell^n \)-isogeny. Therefore, by computing \( \iota_A(\ell^n) \), one can obtain the descending chain from \((E_0, \iota_0)\) to \((E_A, \iota_A)\). This reveals the secret key. See §5.1 in [3] for the details. For preventing this attack, we have to choose primes \( q_j \) and the range \([-B, B]\) of exponents such that the ideals of form \( \prod_{j=1}^t q^{e_j}_j \), \( e_j \in [-B, B] \) is sufficiently separated from ideals generated by a non-integer element in \( \mathcal{O}_E^{(n)} \). Since the non-integer element in \( \mathcal{O}_E^{(n)} \) that has the smallest norm is \( \ell^n \), if we have

\[
(3) \quad \ell^{2n} \prod_{j=1}^t q_j^B \geq 2^{2\lambda},
\]

Meet-in-the-middle attack needs at least \( 2^\lambda \) operations to calculate the action of ideals generated by an non-integer element.

As a result, we use a subset of \( \mathcal{O}(\mathcal{O}_E^{(n)}) \). To avoid Meet-in-the-middle attack to search the isogeny \( E_n \to E_A \), we have to choose \( t \) and \( B \) so that the cardinality of the subset is greater than \( 2^{3\lambda} \), i.e.,

\[
(4) \quad (2B + 1)^t \geq 2^{3\lambda}.
\]
By Theorem 6.2 if $p$ satisfies

$$p > \max_j \{q_j\} \ell^{2n} - 1,$$

then we can calculate the group action by the method in § 5.1.

Consequently, if we take $p, \ell, n, \{q_j\}$ and $B$ so that these satisfy the conditions § 5.1 and § 6.1, then we can expect that the protocol has a security level of $\lambda$ bit. These constraints make $p$ very large. For example, let $\lambda = 128$, $\ell = 2$, and $\{q_j\}$ be the smallest 100 primes splitting in $\mathbb{Q}(i)$. Then we have to take $p$ greater than 5700 bits. On the other hand, SIDH and CSIDH for the same security level use a finite field of characteristic about 500 bits.

6.4. Future works. The inequality $p > 4q\ell^{2n} - 1$ in Theorem 6.2 is only a sufficient condition but not a necessary condition. Further research on the structure of graphs of oriented supersingular elliptic curves could find a smaller lower bound on $p$. Furthermore, the inequality § 6.2 could also be refined, since our discussion did not take the explicit structure of $\mathcal{O}_\ell^{(n)}$ into account. These could make OSIDH more practical. We leave open these problems for future works.

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