Explicit Maximum Likelihood Loss Estimator in Multicast Tomography

Weiping Zhu, member, IEEE,

Abstract—For the tree topology, previous studies show the maximum likelihood estimate (MLE) of a link/path takes a polynomial form with a degree that is one less than the number of descendants connected to the link/path. Since then, the main concern is focused on searching for methods to solve the high degree polynomial without using iterative approximation. An explicit estimator based on the Law of Large Numbers has been proposed to speed up the estimation. However, the estimate obtained from the estimator is not a MLE. When \( n \to \infty \), the estimate may be noticeable different from the MLE. To overcome this, an explicit MLE estimator is presented in this paper and a comparison between the MLE estimator and the explicit estimator proposed previously is presented to unveil the insight of the MLE estimator and point out the pitfall of the previous one.

Index Terms—Loss tomography, Tree topology, Transformation, Explicit Estimator.

I. INTRODUCTION

Network tomography is proposed in [1] to obtain network characteristics without modifying network infrastructure, where the author suggests the use of end-to-end measurement and statistical inference together to estimate the characteristics instead of direct measurement. The end-to-end measurement can be divided into two classes: passive and active, depending on whether probe packets are sent from sources to receivers. Without probing, the passive methods depend on the data collected from log files to estimate network characteristics. However, the data collected in the log files are either unrelated or poorly related that makes inference hard if not impossible. In contrast, the active approach attaches a number of sources to some end nodes of a network that send probe packets to the receivers attached to the other side of the network, where the paths from the sources to the receivers cover the links of interest. Since the probes are multicast to the receivers, the observations obtained by the receivers are strongly correlated. Then, statistical inference is applied on the data collected by the receivers to estimate the network characteristics, such as link-level loss rates [2], delay distribution [3], [4], [5], [6], [7], and loss pattern [8]. In this paper, our focus is on using active approach to estimate the loss rate of a path/link.

Loss rate estimation is also called loss tomography in literature, where the main focus is on searching for efficient maximum likelihood estimators that can avoid the use of iterative procedures to approximate the MLE. To achieve this, a deep analysis of the likelihood function and a comprehensive study of the likelihood equations obtained from the likelihood function are essential. Unfortunately, there are only a few analytical results presented in the literature, [9] and [10] are two of the few. Both papers show that when the Bernoulli loss model is assumed for the loss process of a link and independent identical distributed (i.i.d) probing is used in end-to-end measurement, the maximum likelihood equation of the pass/loss rate of a path/link takes a polynomial form. The difference between them is that [9] is for the pass rate of a path connecting the source to an internal node, [10] is for the loss rate of a path connecting two nodes that form a parent and child pair. We call them path-based estimator and link-based estimator, respectively. Both estimators target the tree topology, and their advantages and disadvantages are presented in [11]. Apart from agreeing on the polynomial form, both report that the degree of the polynomial is one less than the number of descendants connected to the path/link being estimated. Then, how to solve a high degree polynomial becomes the critical issue since there is no analytical solution for a polynomial that is 5 degree or greater from Galois theory. Unfortunately, there has been little progress in this regard until [12], where a connection between observations and the degree of the polynomial is established that provides the theoretical foundation to reduce the degree of the polynomial obtained from the likelihood equation. Prior to [12], the authors of [13] introduced an explicit estimator built on the law of large numbers. The estimator has been proved to be a consistent estimator and has the same asymptotic variances as that of an MLE to first order. When \( n \to \infty \), the estimate obtained by the estimator can be very different from the MLE. Considering the cost of probing and dynamic nature of network traffic, we argue here that despite the importance of the large sample theory in statistics, it is unwise to use an estimator that is purely based on the law of large numbers in practice because the accuracy of the estimator depends on a large number of samples that can take a long time to collect and cost a lot of resources.

The question then becomes whether there is an explicit maximum likelihood estimator for multicast loss tomography. If so, what is that and how different between the estimates obtained by the explicit MLE estimator and the estimator presented in [13] when \( n \to \infty \). The two issues will be addressed in this paper. Firstly, we present an explicit MLE estimator for the tree topology under the Bernoulli model that has a similar computation complexity as the one presented in [13]. Secondly, a comparison between the two estimators is presented that shows the newly proposed estimator is better than the previous one when \( n \ll \infty \). The new estimator is also better than the previous one in terms of the rate of convergence when \( n \to \infty \) since the MLE is asymptotic efficient and the
best asymptotically normal estimate.

By expanding both the statistical model used in the likelihood equation and the observations obtained from receivers, we found that the accuracy of an estimator is related to the consideration of the correlated observations; while the efficiency of an estimator is inversely related to the degree of the likelihood equation that is proportional to the number of correlations. Then, how to keep the accuracy without losing efficiency becomes the key issue that has been under investigation for some time. As a result, the connection between the degree of the polynomial and the observations obtained by receivers is established that sets up the foundation to have an explicit maximum likelihood estimator. Meanwhile, the exactly cause of the larger variance created by the explicit estimator presented in [13] is identified in the paper.

The rest of the paper is organized as follows. In Section 2, we introduce the notations used in the paper. In addition to the notation, the set of sufficient statistics used in this paper is introduced in this section. We then present the explicit MLE in Section 3 that unveils the connection between sufficient statistics and the likelihood model used to describe the loss process of a path/link. Section 4 is devoted to compare and analyze the explicit MLE with the estimator presented in [13]. The last section is devoted to concluding remark.

II. Problem Formulation

A. Notation

In order to make correlated observations at the receivers, multicast is used to send probes to receivers, where the multicast tree or subtree used to connect the source to receivers is slightly different from an ordinary one at its root, that has only a single child. Let \( T = (V, E) \) donate the multicast tree, where \( V = \{v_0, v_1, \ldots, v_m\} \) is a set of nodes representing routers and switches of a network; \( E = \{e_1, \ldots, e_m\} \) is a set of directed links connecting node \( f(i) \) to node \( i \), where \( f(i) \) is the parent of node \( i \). To distinguish a link from another, each link is assigned a unique number from 1 to \( m \); similarly, each node also has a unique number from 0 to \( m \), where link \( i \) is used to connect node \( f(i) \) to node \( i \). The numbers are assigned to the nodes from small to big along the tree structure from top to bottom and left to right. The source is attached to node 0 to send probes to the receivers attached to the leaf nodes of \( T \). \( R \) is used to denote all receivers. Let \( A = \{A_1, \ldots, A_m\} \) be an \( m \)-element vector, where \( A_i, i \in \{1, \ldots, m\} \), is the pass rate of the path connecting node 0 to node \( i \). In addition, except leaf nodes each node has a number of children, where \( d_i \) denotes the children of node \( i \) and \( |d_i| \) denotes the number of children of node \( i \). Note that a multicast subtree is different from an ordinary subtree, where multicast subtree \( i, T(i), \) is rooted at node \( f(i) \) that has link \( i \) as its root link. The group of receivers attached to \( T(i) \) is denoted by \( R(i) \). If \( n \) probes are dispatched from the source, each probe \( i = 1, \ldots, n \) gives rise of an independent realization \( X^{(i)} \) of the probe process \( X, X_k^i = 1, k \in R \) if probe \( i \) reaches receiver \( k \); otherwise \( X_k^i = 0 \). The observations of \( \Omega = \{X^{(i)}\}_{i \in \{1, \ldots, n\}} \) comprise the data for inference.

Given observation \( X^{(j)} \), let

\[
Y_j^j = \bigvee_{k \in R(i)} X_k^j, \quad j \in \{1, \ldots, n\}. \tag{1}
\]

be the observation obtained by \( R(i) \) for probe \( j \). If \( Y_j^j = 1 \) probe \( j \) reaches at least one receiver attached to \( T(i) \), that also implies the probe reaches node \( i \). Then,

\[
n_i(1) = \sum_{j=1}^n Y_j^j,
\]

is the number of probes confirmed from observations that reach node \( i \). \( n_i(1), i \in V \setminus 0 \) have been proved to be a set of minimal sufficient statistics in [12].

In addition to \( n_i(1), i \in V \setminus 0 \), we also introduced another set of numbers for each node, \( k \), where \( n_{ij}(1) = \sum_{u=1}^n (Y_u^i \land Y_u^j), i, j \in d_k \) is for the number of probes confirmed from observations that reach at least one receiver of \( R(i) \) and one of \( R(j) \) simultaneously; and \( n_{ijk}(1) = \sum_{u=1}^n (Y_u^i \land Y_u^j \land Y_u^k), i, j, k \in d_k \) is for the number of probes confirmed from observations that reach simultaneously to at least one receiver in each of \( R(i), R(j) \) and \( R(k) \); \( \cdots \); and \( n_G(1) = \sum_{u=1}^n (\bigwedge_{j \in G} Y_u^j) \) is for the probes observed by at least one receiver in each subtree rooted at node \( k \). We did not realize that this set of numbers is a set of sufficient statistics until recently, we then name them as the set of alternative sufficient statistics. The following theorem confirms this:

**Theorem 1.** The alternative set of statistics defined above is a set of sufficient statistics.

**Proof:** As stated, \( n_i(1), i \in V \setminus 0 \) has been proved to be a set of minimal sufficient statistics. Then, if there is a function \( \Gamma \) that can map the alternative set to \( n_i(1), i \in V \setminus 0 \), the alternative set is a set of sufficient statistics. The function \( \Gamma \) is as follows

\[
n_i(1) = \sum_{j \in d_i} n_j(1) - \sum_{j < k \in d_i} n_{jk}(1) \cdots + (-1)^{|d_i| - 1} n_{dk}(1), \quad i \in (V \setminus (0 \cup R(i)))
\]

\[
n_i(1) = \sum_{j=1}^n Y_j^j, \quad i \in R \tag{2}
\]

The function is a recursive function from bottom up along the tree topology.

III. The explicit maximum likelihood estimator

**A. Explicit Estimator**

Among the few studies providing analytical results, Multicast Inference of Network Characters (MINC) is the most influence one that covers almost all of the areas in network tomography, including link-level loss, link delay and topology tomographies. In loss tomography, it uses a Bernoulli model to describe the losses occurred on a link. Using this model, the authors of [9] derive an MLE for the pass rate of a path connecting the source to an internal node. The MLE is expressed in a set of polynomials, one for a path [9], [14], [15]. Once knowing the pass rates to two nodes that form a
parent and child pair, the loss rate of the link connecting the two nodes can be calculated by $1 - A_i / A_f(i)$. Considering the complexity of using numeric method to solve higher degree polynomials (> 5), the authors of [13] propose an explicit estimator on the basis of the law of large numbers, where the authors define $Z_k^{(i)} = \min_{j \in d_i} Y_j^{(i)}$ and $B_k = P(Z_k = 1)$. The key of [13] is based on the following theorem.

Theorem 2. 1) For $k \in V \setminus R$, 
$$A_k = \Phi(B_k, \gamma) := \left( \prod_{j \in d_k} \gamma_j / B_k \right)^{1/(|d_k|-1)} \quad (3)$$

2) Define $\hat{A}_k = \hat{\gamma}_k$ for $k \in R$, and $\hat{A}_k = \Phi(\hat{B}_k, \hat{\gamma})$ otherwise. Then $\hat{A}_k$ is a consistent estimator of $A_k$, and hence $\hat{A}_k = \hat{A}_k/\hat{A}(\hat{k})$ is a consistent estimator of $A_k$.

where $\hat{B}_k$, the empirical probability of $B_k$, is equal to $n^{-1} \sum_{i=1}^{n} Z_i^{(i)}$. Note that the consistent property proved only if $n \to \infty$. $\hat{A}_k$ is almost surely approach to $A_k$, the true pass rate of the path from the source to node $k$.

This property is the basic requirement for an estimator. If an estimator cannot ensure consistency, it should not be called an estimator. Then, the main concern with the explicit estimator is its accuracy in comparison with the MLE when $n < \infty$ since it only uses a part of all available information.

B. Insight of MLE

A minimum variance and unbiased estimator (MVUE) is normally regarded as a good estimator in statistics. A maximum likelihood estimator is a MVUE if it meets some simple regularity conditions. Unfortunately, the estimator proposed in [13] is not a MLE. Apart from that, there are a number of concerns with the applicability of the estimator in practice because the accuracy of the estimate requires $n \to \infty$. Then, the scalability of the estimator must be addressed, where the time and resources spent on measurement, time spent on processing the data collected from measurement, and the stationary period of network traffic must be considered in practice.

To remedy the scalability of the explicit estimator, we start to search for an explicit maximum likelihood estimator and have a close look at the maximum likelihood estimator proposed in [9], which is as follows:

$$H(A_k, k) = 1 - \frac{\gamma_k}{A_k} - \prod_{j \in d_k} \left( 1 - \frac{\gamma_j}{A_k} \right) = 0 \quad (4)$$

where $\gamma_i, i \in V \setminus 0$ is the pass rate of the multicast tree with its root link connecting node 0 to node $i$. Rewriting (4) as

$$1 - \frac{\gamma_k}{A_k} = \prod_{j \in d_k} \left( 1 - \frac{\gamma_j}{A_k} \right) \quad (5)$$

we found two interesting features of the polynomial; one is the expandable feature, the other is merge-able feature. The former allows us to expand both sides of (5) to have a number of terms on each side that are corresponded to each other. The latter, on the other hand, allows us to merge a number of terms located on the right hand side (RHS) and in the product into a single term as the one located on left hand side (LHS) of (5). The advantage of the merge-able feature will be detailed in the next subsection. We now put our attention on the expanding feature to unveil the internal correlation embedded in (5). Expanding the RHS of (5) and dividing all terms by $A_k$, we have

$$\gamma_k = \sum_{j \in d_k} \gamma_j - \sum_{j < k, j, k \in d_k} \frac{\gamma_j \gamma_k}{A_k} \cdots + (-1)^{|d_k|-1} \prod_{j \in d_k} \gamma_j \left. \frac{A_k}{A_k^{d_{d_k}}} \right|_{A_k} \quad (6)$$

Using the empirical probability $\hat{\gamma}_j = n_j(1)$ to replace $\gamma_j$ in (6), we have a $|d_k|-1$ degree polynomial of $A_k$. Solving the polynomial, the MLE of path $i$, $\hat{A}_k$, is obtained. Note that $\hat{\gamma}_k$ can be replaced by the alternative sufficient statistics, then the LHS of (6) becomes

$$\frac{1}{n} \left( \sum_{j \in d_k} n_j(1) - \sum_{j < k, j, k \in d_k} n_{jk}(1) \cdots + (-1)^{|d_k|-1} n_{d_k}(1) \right) \quad (7)$$

Comparing the RHS of (6) with (7), one is able to find the correspondences between the terms. Each term of (6) represents a type of correlation in the model among/between the members of the term, while each term of (7) is the statistics or evidence obtained from an experiment for the corresponding term of (6). Except the first term of (6) and the first term of (7) that are exactly equal to each other, other pairs between the two can be different from each other if $n < \infty$. Taking the first terms of (6) and (7) out, we have

$$\sum_{j < k, j, k \in d_k} \frac{\gamma_j \gamma_k}{A_k} \cdots + (-1)^{|d_k|-1} \prod_{j \in d_k} \gamma_j \left. \frac{A_k}{A_k^{d_{d_k}}} \right|_{A_k} = \frac{1}{n} \left( \sum_{j < k, j, k \in d_k} n_{jk}(1) \cdots + (-1)^{|d_k|-1} n_{d_k}(1) \right) \quad (8)$$

Statistical inference aims to estimate $\hat{A}_k$ from (8), i.e. matching the model presented on the LHS to the statistics presented on the RHS. This equation also shows that in order to have the MLE of a path, one must consider all available information embedded in the observations of $R(i)$, in particular the correlations between the descendants. Without correlations and/or the corresponding statistics, inference is impossible. This corresponds to the data consistent problem raised in [9] and [11]. If we only consider matching a part of (6) to the corresponding part of (7) when $n < \infty$, the estimate obtained would not be the MLE unless the ignored correlations are negligible. Then, the explicit estimator proposed in [13] is not an MLE since it only pairs the last term of (6) to the last term of (7).

C. The Explicit Maximum Likelihood Estimator

As stated, the degree of the polynomial is proportional to the number of descendants connected to the path being
estimated and the estimation relies on the observed correlations between the descendants to estimate the unknown characteristic. Under the i.i.d. assumption, the likelihood function takes a product form as \( \hat{d}_k \). Unfortunately, the previously stated merge-able feature has not been given enough attention although \( \hat{d} \) clearly expresses the loss rate of subtree \( k \) is equal to the product of the loss rates of those sub-multicast trees rooted at node \( k \). In probability, \( \hat{d} \) states such a fact that the loss rate of subtree \( k \) depends on a number of independent events, one for a sub-multicast tree rooted at node \( k \). More importantly, this implies that those independent events can be merged into a single event, i.e. the LHS of \( \hat{d} \). With this in mind, whether the degree of \( \hat{d}_k \) can be reduced depends on whether we are able to obtain the empirical pass rate of the tree that has a path from the source to node \( k \) plus some of the multicast subtrees rooted at node \( k \). Let \( \gamma_k \) denote the pass rate, where \( k \) is for the end node of the path being estimated, \( g \) denotes the group of subtrees being merged. Based on the alternative sufficient statistics of \( d_k, \hat{\gamma}_k \), the empirical probability of \( \gamma_k \), can be computed \([12]\). Then, the degree of \( \hat{d} \) can be reduced to 1 that can be solved easily. Further, we have the following theorem to calculate the pass rate of a path explicitly for the tree topology.

**Theorem 3.** For the tree topology that uses the Bernoulli model to describe the loss process of a link, there is an explicit MLE estimator to estimate the pass rate of the path connecting the source to node \( k, k \in V \setminus (0 \cup R) \), which is as follows:

\[
\hat{A}_k = \frac{\hat{\gamma}_k \hat{\gamma}_2}{\hat{\gamma}_1 + \hat{\gamma}_2 - \hat{\gamma}_k}
\]

where \( \hat{\gamma}_k = \frac{n_k(1)}{n}, \hat{\gamma}_1 = \frac{n_1(1)}{n}, \) and \( \hat{\gamma}_2 = \frac{n_2(1)}{n} \). \( n_k(1) \) and \( n_1(1) \) are the number of probes confirmed from observations reaching at least one receiver attached to the merged subtree 1 and 2, respectively.

**Proof:** Since node \( k \) is not a leaf node, the subtrees rooted at the node can be divided into two exclusive groups: \( d_{k_1} \) and \( d_{k_2} \) where \( d_{k_1} \cup d_{k_2} = d_k \) and \( d_{k_1} \cap d_{k_2} = \emptyset \). The statistics of the merged subtrees can be computed by using \( d_{k_1} \) or \( d_{k_2} \) to replace \( d_k \) in \( \hat{A}_k \). Then, we have

\[
1 - \frac{\hat{\gamma}_k}{\hat{A}_k} = \prod_{j \in d_k} (1 - \frac{\hat{\gamma}_j}{A_k})
\]

\[
= \prod_{j \in d_{k_1}} (1 - \frac{\hat{\gamma}_j}{A_k}) \prod_{k \in d_{k_2}} (1 - \frac{\hat{\gamma}_k}{A_k})
\]

\[
= (1 - \frac{\hat{\gamma}_{k_1}}{A_k})(1 - \frac{\hat{\gamma}_{k_2}}{A_k})
\]

Solving \( \hat{A}_k \), we have

\[
\hat{A}_k = \frac{\hat{\gamma}_{k_1} \hat{\gamma}_{k_2}}{\hat{\gamma}_{k_1} + \hat{\gamma}_{k_2} - \hat{\gamma}_k}
\]

The theorem shows that using \( \hat{d} \) to merge the alternative statistics of multiple multicast subtrees rooted at the same node would not affect the estimation of the pass rate of the path that ends at the node.

---

**Fig. 1. A Binary Tree**

**IV. COMPARISON OF ESTIMATORS**

In this section, we tackle the second task set at the beginning of the paper, i.e. compare the explicit MLE estimator against the explicit one presented in \([13]\) for \( n < \infty \). Two scenarios: 2 descendants and 3 descendants are connected to the path being estimated, are considered to illustrate the estimates received from the explicit estimator proposed in \([13]\) drifts away from the MLE as the number of descendants increases. To make the variance approximate to the first order of the MLE, the explicit estimator needs to send more probes to receivers. We will use \( H(A_i, i) \) as a reference in the following comparison to measure the accuracy between an estimator and its MLE counterpart.

**A. Binary Tree**

For a tree with binary descendants as Figure 1, the pass rate of link/path \( i \) estimated by \( H(A_i, i) \) is equal to

\[
\hat{A}_i = \frac{\hat{\gamma}_2 \hat{\gamma}_1}{\hat{\gamma}_2 + \hat{\gamma}_1 - \hat{\gamma}_i}
\]

Using the explicit estimator of \([13]\), we have

\[
\hat{A}_i = \frac{\hat{\gamma}_2 \hat{\gamma}_1}{n}
\]

\[
= \frac{\hat{\gamma}_2 \hat{\gamma}_1}{n_2(1) + n_1(1) - n_i(1)}
\]

\[
= \frac{\hat{\gamma}_2 \hat{\gamma}_1}{\frac{n_2(1) + n_1(1) - n_i(1)}{n}}
\]

\[
\frac{n_2(1) + n_1(1) - n_i(1)}{n}
\]

It is the same as the MLE. This is because there is only one type of correlation between the model and the observations, which are considered by the estimator. Thus, \( \hat{A}_i = \hat{A}_i \).

Based on Theorem 3 the estimate obtained by the estimator proposed in this paper is also the same as above, which can be written as:

\[
\hat{A}_i = \frac{n_1(1)n_2(1)}{n \cdot (n_1(1) + n_2(1) - n_i(1))}
\]

\[
= \frac{n^2}{n_1(1) + n_2(1) - n_i(1)}
\]

Thus, for the binary tree, the three estimators produce the same result.
B. Tertiary Tree

Let \( i \) have three descendants 1, 2, and 3. Based on \( H(A_i, i) \), we have

\[
\hat{A}_i^2 (\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3 - \hat{\gamma}_i) - \\
A_i (\hat{\gamma}_1 \hat{\gamma}_2 + \hat{\gamma}_1 \hat{\gamma}_3 + \hat{\gamma}_2 \hat{\gamma}_3) + \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 = 0.
\]

(14)

Solving the quadratic function, we have the MLE of \( \hat{A}_i \)

Based on (8), the model and observations are connected by

\[
\sum_{j,k \in \{1,2,3\}} \frac{n_{jk}(1)}{n} - \frac{n_{123}(1)}{n} = \\
\sum_{j,k \in \{1,2,3\}} \frac{\gamma_j \gamma_k}{A_i} - \frac{\gamma_1 \gamma_2 \gamma_3}{A_i^2}.
\]

(15)

It is easy to prove (15) equals (14). Using theorem 3 we have the MLE directly,

\[
\hat{A}_i = \frac{(n_1(1) + n_2(1) - n_{12}(1)) \cdot n_3(1)}{n \cdot (n_{13}(1) + n_{23}(1) - n_{123}(1))}.
\]

(16)

Intuitively, one can notice that (16) considers all correlations between the 3 descendants. To prove this equals (14), we write the RHS of the above as

\[
\frac{(n_1(1) + n_2(1) - n_{12}(1)) \cdot n_3(1)}{n \cdot (n_{13}(1) + n_{23}(1) - n_{123}(1))} = \frac{n}{n}. (17)
\]

The denominator of the above is obtained from

\[
\frac{n_1(1) + n_2(1) - n_{12}(1)}{n} + \frac{n_3(1)}{n} - \gamma_i.
\]

(18)

According to (6) and (7),

\[
\gamma_1 + \gamma_2 - \frac{\gamma_1 \gamma_2}{A_i} = \frac{n_1(1) + n_2(1) - n_{12}(1)}{n}
\]

Using the above in (18), the denominator turns into

\[
(\gamma_1 + \gamma_2 + \gamma_3 - \gamma_i) - \frac{\gamma_1 \gamma_2}{A_i}.
\]

(19)

Similarly, the nominator of (17) is equal to

\[
(\gamma_1 + \gamma_2 - \frac{\gamma_1 \gamma_2}{A_i}) \cdot \gamma_3 = \gamma_1 \gamma_3 + \gamma_2 \gamma_3 - \frac{n_{12}(1)}{n} \gamma_3.
\]

(20)

Using (19) and (20) to replace the denominator and nominator of (16), we have

\[
\hat{A}_i \cdot (\gamma_1 + \gamma_2 + \gamma_3 - \gamma_i) - \frac{\gamma_1 \gamma_2}{A_i} = \\
\gamma_1 \gamma_3 + \gamma_2 \gamma_3 - \frac{\gamma_1 \gamma_2}{A_i} \gamma_3
\]

(21)

Moving every term to the LHS and multiplying \( \hat{A}_i \), we have (14).

Because of the symmetric nature of the 3 descendants, we can also merge descendants 2 and 3 first or merge descendants 1 and 3 first, that lead to

\[
\hat{A}_i = \frac{(n_2(1) + n_3(1) - n_{23}(1)) \cdot n_1(1)}{n \cdot (n_{13}(1) + n_{23}(1) - n_{123}(1))}
\]

and

\[
\hat{A}_i = \frac{(n_1(1) + n_3(1) - n_{13}(1)) \cdot n_2(1)}{n \cdot (n_{12}(1) + n_{23}(1) - n_{123}(1))}
\]

respectively.

In contrast, the explicit estimator presented in [13] has its estimate

\[
\hat{A}_i = \left( \frac{\gamma_1 \gamma_2 \gamma_3}{n_{123}(1)} \right)^{\frac{1}{2}}.
\]

(22)

Comparing (22) with (16), a direct impression is that (22) fails to match the paired correlations, i.e. descendants 1 and 2, descendants 1 and 3, and descendants 2 and 3. If we assume (22) is a solution of a quadratic equation, the equation should be as follows:

\[
\frac{n_{123}(1)}{n} \hat{A}_i^2 - 2 \cdot \left( \frac{n_{123}(1) \gamma_1 \gamma_2 \gamma_3}{n} \right)^{\frac{1}{2}} \hat{A}_i + \gamma_1 \gamma_2 \gamma_3 = 0
\]

(23)

that has a double root. Given the double root assumption, (23) is certainly not the polynomial that leads to MLE since it is contradict to the Lemma 1 introduced in [9] and [12] that states there is one and only one root in (0, 1) for the maximum likelihood equation. Then, the estimator proposed in [13] can be regarded as an estimator based on the method of moments that will be discussed in the next subsection.

C. Analysis

The fundamental principle of maximum likelihood estimate is unveiled clearly by (8), where the LHS of (8) is the statistical distribution of the loss process (called model previously) and the RHS is the statistics obtained from observations. There is one to one correspondence between the terms cross the equal sign. The maximum likelihood estimator aims to solve the equation to find the \( A_k \) that fits to the statistic model; while the explicit estimator proposed previously attempts to use the last terms of both sides. When \( n < \infty \), the estimate can be different from MLE. In fact, if only considering asymptotic accuracy, each of the corresponding terms can be connected and considered an explicit estimator as the one presented in [13]. All of the explicit estimators can also be proved to be consistent as their predecessor. Then, the theorem 1 can be extended as

**Theorem 4.** Each of the corresponding pairs of (8) forms an explicit estimator that is consistent as the one proposed in [13], that has the form of

\[
\Phi_i(n_w(1), w, \gamma) = A_i = \frac{1}{C_{|w|}} \sum_{j \in w} \left( \prod_{k \in |w|} \gamma_k \right) |w|^{-1}
\]

(24)

where \( w \) corresponds to one of the pairs cross the equal sign of (8) and \( |w| < |d_i| \) denotes the number of members in the term, and \( n_w(1) \) corresponds to the statistics denoting the number
of probes reaching the receivers attached to the descendants of \( w \).

**Proof:** It can be proved as the proof of Theorem 2 in [13].

Clearly, when \( n < \infty \), the estimates obtained by the explicit estimators are not the MLE. Note that each term on the RHS of (13) is not a sufficient statistic by itself but only a part of the sufficient statistics defined in Theorem 1. Combining a number of the estimators defined above can improve the accuracy of the estimate. When all terms are combined, we have the MLE.

Despite having an explicit maximum likelihood estimator, we still carry out the following analysis to find out why the partial matching could not be an MLE. Firstly, we compare (8) with the explicit one proposed in [13] on the basis of polynomial. The former is a \(|d_k| - 1\) degree of polynomial that has a unique solution in \((0, 1)\), while the latter can be considered a polynomial that has a multiple root in \((0, 1)\). In other words, the explicit one considers the sum of the first \(|d_k| - 2\) terms on both sides of the equal sign are equal to each other, so does the last terms. Instead of using the sum of the first \(|d_k| - 2\) terms to estimate \( 
Ak \), the explicit estimator uses the last one to avoid solving a \(|d_k| - 2\) degree polynomial. Using this approach to estimate \( 
Ak \), an error is inevitable and the amount of the error depends on the number of descendants; and the more the better. This is because if a node has more descendants, we are able to obtain more information about the path connecting the source to the node from observations.

The explicit estimator proposed in [13] can also be viewed as a method of moments. If so, its estimate is normally superseded by the maximum likelihood, because maximum likelihood estimators have higher probability of being close to the quantities to be estimated. Also, the estimates obtained by the method of moments are not necessarily based on sufficient statistics, i.e., they sometimes fail to take into account all or a large part of the relevant information in the sample. Therefore, the accuracy of such an estimator depends on large sample.

As stated, \( n_{d_k}(1) \) itself is not a sufficient statistic for \( Ak \), using \( n_{d_k}(1) \) alone to estimate \( Ak \) fails to consider all other correlations between descendants. Then, as stated, error is inevitable. Let \( n \) be the sample size and \( \delta \) be the error rate, their relation is expressed as

\[
\delta = \frac{\sqrt{n}}{n}
\]

Based on the formula, the explicit estimator relies on sending infinite number of probes to reduce the error. Even though, the effect of ignoring other correlations remains, that makes the variance of the explicit estimator can only approximate to the first order of that of the maximum likelihood estimator.

**D. Computational Complexity**

The estimator proposed in this paper is the maximum likelihood one with a similar computation complexity as that of the explicit estimator presented in [13]. To determine \( \hat{A}_{k} \) or \( \hat{\lambda}_{k} \), both need to calculate the empirical probabilities \( \hat{\gamma}_{i}, i \in V \setminus 0 \). The two estimators require to compute \( Y_{j}^{2} \) and \( n_{j}(1) \), there are total \( O(n \cdot (|V| - 1)) \) operations. In addition, the estimator proposed here needs to merge descendants into two for those nodes that have more than 2 descendants. That requires to compute \( n_{d_k}(1), \ldots, n_{d_k} / 2(1) \) for node \( k \), and there are \( 2 \times \left( \frac{|d_k|}{2} - 1 \right) \) operations. On the other hand, the previous explicit estimator needs to calculate \( \hat{B}_{k} \) that requires the computation of \( Z_{k} \) for each node that takes a smaller amount of operations than the MLE does. On the other hand, the explicit one needs to perform \( n \)-th root operation for each node that has more than two children, while the MLE one only needs a simple arithmetic operation to estimate \( Ak \). Therefore, in terms of operations, the two are similar to each other.

**V. Conclusion**

In this paper, an explicit MLE is proposed that is built on the unique features of the likelihood equation and the set of alternative sufficient statistics introduced in this paper. The two features of the likelihood equation, i.e. expandable and merge-able, can be considered micro and a macro views of the likelihood equation. The rise of the macro view makes merging possible; and the rise of the micro view unveils the fundamental of the explicit estimator proposed previously and the internal correlations in the model and observations. Based on the macro view, a closed form MLE estimator is proposed and presented in this paper, which is of the simplest one that has even been presented in literature. Applying the micro view on [5], we establish the correlations between descendants, and we also establish the correspondence between the statistical model and the statistics obtained from the leaf nodes of the descendants. This correspondence further unveils the connection between the observations and the degree of the likelihood polynomial. As a result, the explicit MLE is proposed for the tree topology. In addition to the explicit estimator, we in this paper compare the estimator proposed in this paper with the explicit estimator proposed previously, which shows that when \( n < \infty \), the MLE one is substantially better than the explicit one in terms of accuracy.

**References**

[1] Y. Vardi. Network Tomography: Estimating Source-Destination Traffic Intensities from Link Data. *Journal of Amer. Stat. Assoc.*

[2] F. LoPresti & D. Towsley N.G. Duffield, J. Horowitz. Multicast topology inference from measured end-to-end loss. *IEEE Trans. Inform. Theory*, 48, Jan. 2002.

[3] G. Liang and B. Yu. Maximum pseudo likelihood estimation in network tomography. *IEEE trans. on Signal Processing*, 51(8), 2003.

[4] Y. Tsang, M. Coates, and R. Nowak. Network delay tomography. *IEEE Trans on Signal Processing*, (8), 2003.

[5] F.L. Presti, N.G. Duffield, J. Horowitz, and D. Towsley. Multicast-based inference of network-internal delay distribution. *IEEE/ACM Trans on Networking*, (6), 2002.

[6] Meng-Fu Shih and Alfred O. Hero III. Unicast-based inference of network link delay distributions with finite mixture models. *IEEE Trans on Signal Processing*, (8), 2003.

[7] E. Lawrence, G. Michailidis, and V. Nair. Network delay tomography using flexicast experiments. *Journal of Royal Statist. Soc.*, (Part5), 2006.

[8] V. Arya, N.G. Duffield, and D. Veitch. Multicast inference of network-internal loss characteristics. *Performance Evaluation*, (9-12), 2007.

[9] R. Cáceres, N.G. Duffield, J. Horowitz, and D. Towsley. Multicast-based inference of network-internal loss characteristics. *IEEE Trans. on Information Theory*, 45, 1999.

[10] W. Zhu. Loss rate estimation in general topologies. In *Proc. of IEEE Communications Society/CreateNet BROADNETS, San Jose, CA, USA, 2006.*
[11] W. Zhu. Loss rate inference in multi-source and multicast-based general
topologies. \textit{Submitted for Publication}, 2009.

[12] W. Zhu and K. Deng. Loss tomography from tree topology to general
topology. \textit{Submitted for Publication}, 2009.

[13] N. Duffield, J. Horowitz, F. Presti, and D. Towsley. Explicit loss
inference in multicast tomography. \textit{IEEE Trans. on Information Theory},
52(8), Aug., 2006.

[14] R. Cáceres, N.G. Duffield, S.B. Moon, and D. Towsley. Inference of
Internal Loss Rates in the MBone . In \textit{IEEE/ISOC Global Internet'99},
1999.

[15] R. Cáceres, N.G. Duffield, S.B. Moon, and D. Towsley. Inferring link-
level performance from end-to-end multicast measurements. Technical
report, University of Massachusetts, 1999.