Multi-gap discrete vector solitons

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We analyze nonlinear collective effects in periodic systems with multi-gap transmission spectra such as light in waveguide arrays or Bose-Einstein condensates in optical lattices. We demonstrate that the inter-band interactions in nonlinear periodic gratings can be efficiently managed by controlling their geometry, and predict novel types of discrete vector solitons supported by nonlinear coupling between different bandgaps and study their stability.

Periodic structures are common in nature, with the crystalline lattice being the most familiar example. One of the important common features of such systems is the existence of frequency gaps in the transmission spectra which can dramatically affect both propagation and localization of waves. Moreover, the modern technology allows creating different structures with an artificial periodicity, and the recent examples are photonic crystals, which can control propagation and emission of electromagnetic waves, and optical lattices, which are used to trap and manipulate atomic Bose-Einstein condensates (BECs). An unprecedented level of control over such engineered structures can be realized by tailoring both location and width of multiple band-gaps with additional modulation of the structure parameters. For example, it has been shown that atomic BEC can demonstrate a rich variety of phase transitions in optical superlattices, and the reduced-symmetry photonic crystals allow self-localization of waves in mini-gaps.

The response of many systems becomes nonlinear at higher energy densities. This phenomenon may have various physical origins, such as the charge recombination in biased photorefractive crystals, excitation of higher energy levels in semiconductors, or atomic interaction in BEC. In periodic media, nonlinearity produces a shift of the band-gap spectrum, and this physical mechanism is responsible for a number of remarkable effects, including the formation of gap solitons. Such nonlinear localized modes can be excited within multiple spectral gaps of a periodic structure, as was first demonstrated experimentally for temporal optical pulses in fiber Bragg gratings. Since the pulses extend over hundreds of grating periods, the gap-soliton dynamics is usually described by averaged coupled-mode equations with constant coefficients. In contrast, spatial optical beams in waveguide arrays and matter waves in BEC can span over a few periods of the structure. Under such conditions the wave profiles are essentially discretized by the underlying periodic structure, strongly affecting the properties of discrete solitons.

Spatial optical solitons associated with the first spectral band of the multi-band transmission spectrum have been extensively studied both theoretically and experimentally in arrays of coupled optical waveguides. Very recently, spatial gap solitons localized in higher-order bands have been observed as well. Similar observation of matter-wave gap solitons in BEC loaded into an optical lattice is expected soon as well. However, the interaction properties of localized modes which belong to different gaps are not known. The existence of gap solitons is directly related to the band-gap spectrum, and the latter can be fine-tuned in superlattices. In this Letter, we reveal, for the first time to our knowledge, the fundamental links between periodic modulation of the medium parameters and nonlinear wave coupling between different gaps, and also predict the existence of multi-gap discrete vector solitons with nontrivial symmetry and stability properties. We believe that our results can stimulate and guide the future experiments on discrete gap solitons in optics and matter-wave physics.

Self-action and interaction of optical beams in a one-dimensional periodic structure of coupled optical waveguides with the normalized index profile \( V(x) = V(x + h) \) can be described by a set of coupled nonlinear Schrödinger equations,

\[
\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + G(\psi)\psi = 0,
\]

where, in the case of optical waveguides, \( x \) and \( z \) are the transverse and longitudinal coordinates, respectively, \( h \) is the spatial period, \( \psi = (\psi^{(1)}, \psi^{(2)}, ...) \) are the normalized electric field envelopes of several co-propagating beams having different polarizations or detuned frequencies. It is assumed that the beams interact incoherently through the Kerr-type nonlinear change of the refractive index, \( G_m = \sum_j \Gamma_{mj}|\psi^{(j)}|^2 \), where \( \Gamma_{m=j} \) and \( \Gamma_{m\neq j} \) are the self- and cross-phase modulation coefficients, respectively.

We note that the model is equivalent to a system of the coupled Gross-Pitaevskii equations describing the dynamics of multi-component BEC in 1D optical lattice, where \( \psi^{(m)} \) is the mean-field wave function for atoms in the \( m \)-th quantum state, \( z \) stands for time, \( V(x) \) is the periodic potential of an optical lattice, and \( G \) is the effective mean-field nonlinearity which appears due to the s-wave atom interaction. Although below we use the terminology from guided wave optics where the rapid progress in the experimental study of spatial optical solitons is observed, our results are equally applicable to the nonlinear dynamics of BEC in optical lattices.
Linear wave propagation through a periodic structure can be entirely described by the Floquet-Bloch spectrum of the eigenmode solutions of Eqs. (1) in the form
$$\psi = \psi_b \exp(i\beta z + iK_b x/h),$$
where $K_b$ is the normalized Bloch-wave number and $\beta$ is the propagation constant. The Bloch wave amplitudes decay exponentially when $\text{Im}K_b(\beta) \neq 0$, and this condition defines the location of gaps in the transmission spectrum. At the gap edges, $K_b = 0, \pi$.

Nonlinearity manifests itself through an effective change of the optical refractive index which results in a local shift of the bands and gaps. This physical mechanism is responsible for the formation of gap solitons. When the band shifts are small, we can seek solutions of Eqs. (2) near the gap edges ($\beta = \beta_m$) in the form of modulated Bloch waves
$$\psi^{(m)} = \varphi^{(m)}(x) \exp(i\beta_m z + iK_b^{(m)} x/h),$$
and obtain a system of coupled nonlinear Schrödinger equations for the slowly varying envelopes,
$$i \frac{\partial \varphi^{(m)}}{\partial z} + \frac{D^{(m)}}{2} \frac{\partial^2 \varphi^{(m)}}{\partial x^2} + \sum_j \gamma_{mj} \Gamma_{mj} |\varphi^{(j)}|^2 \varphi^{(m)} = 0.$$  

Here $D^{(m)} = -\hbar^2 \beta^2/\partial \beta K_b^{(m)}$ are the effective diffraction coefficients, and $\gamma_{mj} = \int_0^h |\psi_b^{(m)}(x)\psi^{(j)}(x)|^2dx$ are the nonlinear coupling coefficients, where we assume the normalization $\int_0^h |\psi_b^{(m)}|^2dx = 1$. It can be demonstrated that the diffraction coefficients $D^{(m)}$ are positive near the lower gap edges, and negative at the upper edges and, therefore, both bright and dark solitons can co-exist in the nonlinear media with either self-focusing or self-defocusing nonlinearities. Moreover, it immediately follows from Eqs. (2) that each soliton can support multiple guided modes in other band gaps, all such modes can be coupled together to form multi-gap vector solitons.

The simplified model (2) predicts stability of bright vector solitons [14] when all $\Gamma_{mj}$ have the same sign. However, the applicability of Eqs. (2) is limited to small-amplitude solitons in the vicinity of the gap edges. Even in this regime, the important effect of the inter-band resonances, which can lead to the soliton instabilities [17], is not taken into account by the envelope approximation (2). As the input power grows, the soliton width decreases and becomes comparable to the spatial period of the structure. This suggests that nonlinear properties should depend on the actual profile of the periodic potential $V(x)$. Modulated periodic structures or superlattices can then become an important tool to engineer both linear and nonlinear properties of the Bloch waves and gap solitons.

As an example, we consider a binary superlattice where the effective periodic potential is composed of two types (A and B) of separated individual potential wells, $V(x) = \sum_n [V_A(x + nh) + V_B(x + nh)]$. Such superlattices can be produced by etching waveguides on top of a AlGaAs substrate [18], or induced dynamically by two overlapping mutually incoherent interference patterns in a photorefractive medium [19], see Figs. 1a,b. In order to analyze the properties of nonlinear waves of such superlattices, we employ the tight-binding approximation. This approach allows us to describe correctly the first two spectral bands, and the Rowland gap ghost [20] which is defined by a difference between the A- and B-type lattice cites. We present the total field as a superposition of the guided modes supported by individual potential wells, $\psi(x, z) = \sum_n [A_n(z)\psi_A(x) + B_n(z)\psi_B(x)]$, where $A_n$ and $B_n$ are (yet unknown) mode amplitudes. Finally, we derive a system of coupled discrete equations for the

![FIG. 1: Examples of a binary superlattice which can be created by (a) an array of two types of coupled waveguides, or (b) an optical superlattice induced by two overlapping mutually incoherent interference patterns.](image)

![FIG. 2: (a) Characteristic dependence of the Bloch wave number ($K_b$) on the propagation constant $\beta$. Gray shadings mark the transmission bands. (b) Dependence of the normalized self- and cross-phase modulation nonlinear coefficients ($\gamma_{mj} = |A^{(m)}A^{(j)}|^2 + |B^{(m)}B^{(j)}|^2$) between the gap edges $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_4$ vs. the parameter $\rho$. The values of $\beta_2$ correspond to the plot (a) by a proper choice of $\kappa$. Insets show possible symmetries of superlattices corresponding to different parameter values, but the same linear dispersion.](image)
normalized amplitudes \( a_n \) and \( b_n \),

\[
\frac{da_n}{dz} + \rho \alpha_n + \kappa^{-1}b_{n-1} + \kappa b_n + \chi_\alpha |a_n|^2 a_n = 0,
\]

\[
\frac{db_n}{dz} - \rho b_n + \kappa a_n + \kappa^{-1}a_{n+1} + \chi_\beta |b_n|^2 b_n = 0.
\]

Here, the key characteristics of the binary superlattice are defined by free parameters: \( \rho \) is proportional to the detuning between the propagation constants of the A and B-type guided modes, \( \kappa \) characterizes the relative coupling strength between the neighboring wells on the right- and left-hand sides, and \( \chi_{a,b} \) are the normalized nonlinear coefficients.

According to Eqs. (3), the linear Bloch-wave dispersion is defined as \( K_b = \cos^{-1}(-\eta/2) \), where \( \eta = \kappa^2 + \kappa^{-2} + \rho^2 - \beta \). The transmission bands correspond to real \( K_b \), and they appear when \( \beta_- \leq |\beta| \leq \beta_+ \), where \( \beta_{\pm} = (\kappa^2+\kappa^{-2}+\rho^2+2)^{1/2} \). A characteristic dispersion relation and the corresponding band-gap structure are presented in Fig. 2(a). The upper gap at \( \beta > \beta_+ \) is due to the effect of total internal reflection (IR), whereas additional gaps appear due to the resonant Bragg reflection (BR).

It is well known that the material dispersion can be completely compensated by the geometrical dispersion in optical fibers. More recently, diffraction management was realized in a periodic waveguide array structure. The natural open question is whether it is possible to control nonlinear coupling between the gaps by appropriate design of periodic structures. In order to answer this fundamental question, we study the dependence of the nonlinear coupling coefficients on the superlattice parameter \( \rho \), while preserving exactly the same linear dispersion of Bloch-waves by a proper choice of \( \kappa \). Our results are presented in Fig. 2(b), and they uncover the remarkable feature: nonlinear inter-band interaction coefficients strongly depend on the symmetry of the periodic structure, and this relation cannot be fully characterized just by the linear Bloch-wave dispersion. Thus, by changing the superlattice parameters it is possible to selectively enhance or suppress inter-band interaction.

To be specific, we consider the superlattice with symmetric inter-site coupling (\( \kappa = 1 \)) created in a self-focusing medium. Such a lattice can support two fundamental types of one-component bright solitons centered at either A or B cites, and these solitons exist in both the IR and BR gaps described by the model. We find that the powers of A- and B-type solitons become significantly different away from the band edges, see Fig. 3(a). The solitons of type-B in the IR gap are always unstable with respect to a translational shift (symmetry breaking), however the stability is reversed for discrete gap solitons in the first BR gap [left part of Fig. 3(a)] where type-A solitons become unstable. Additionally, the discrete gap solitons become oscillatory unstable above a critical power due to (i) internal resonance within the gap, first discovered for the fiber Bragg solitons, and (ii) inter-band resonances, first found for nonlinear defect modes in a layered medium.

The mutual trapping of the modes localized in different gaps and the physics of multi-gap vector solitons can be understood in terms of the soliton-induced waveguides. Therefore, the effect of discreteness on the inter-gap coupling can be captured by studying the guided modes supported by a scalar soliton in other gaps: the larger is the eigenvalue shift from the band edge, the stronger is the interaction. In Figs. 3(b,c), we plot the eigenvalues of the guided modes supported by the BR (gap) and IR solitons and observe two remarkable effects which cannot be captured by the simplified envelope approximation. First, the strength of the inter-gap coupling depends strongly on the soliton symmetry. Indeed, the type B soliton always creates a stronger effective waveguide, despite the fact that the soliton power in the BR gap is smaller compared to the type-A solitons. Second, the nonlinear inter-gap coupling decreases for strongly localized discrete soliton in the IR regime, as follows from the non-monotonic dependence of the gap-mode eigenvalues shown in Fig. 3(c).

The eigenvalues of the linear guided modes define the point where a multi-gap vector soliton bifurcate from their scalar counterparts. Initially the amplitude of the guided mode is very small, but it increases away from the bifurcation point, and the mode interacts with the soliton waveguide creating a coupled inter-gap state, see Fig. 4. In the vicinity of the bifurcation point, the soliton symmetry and stability are defined by the large-amplitude
soliton component. For example, the AA-type discrete vector soliton shown in Fig. 4(III) is stable because the powerful A-type mode in the IR gap suppresses instability of the second component. However, as the power in the second component grows, the soliton properties change dramatically: (i) AA state becomes unstable, and at the same time (ii) a stable AB-type asymmetric vector soliton emerges, see the modes II and I in Figs. 4(bottom), respectively. These complex existence emerges, see the modes II and I in Fig. 4(bottom) trends and their stability. Using the example of a binary waveguide array, we have demonstrated the basic concepts of the engineering of nonlinear inter-band interaction in such structures, which in turn determine the key soliton properties.

In conclusion, we have studied nonlinear coupling and wave localization in periodic systems with multi-gap transmission spectra. We have predicted the existence of novel types of multi-gap vector solitons and studied their stability. Using the example of a binary waveguide array, we have demonstrated the basic concepts of the engineering of nonlinear inter-band interaction in such structures, which in turn determine the key soliton properties.

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**FIG. 4: (a,b) Powers in the IR (a) and BR (b) components of the multi-gap discrete vector solitons vs. propagation constant $\beta_1$ with $\beta_2 = 0.5 - \beta_1/3$ for the families with symmetric (AA) and asymmetric (AB) profiles. Dashed lines mark interesting effects will be discussed elsewhere.**

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