Isospectral Hamiltonians and $W_{1+\infty}$ algebra

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Abstract

We discuss a spectrum generating algebra in the supersymmetric quantum mechanical system which is defined as a series of solutions to a specific differential equation. All Hamiltonians have equally spaced eigenvalues, and we realize both positive and negative mode generators of a subalgebra of $W_{1+\infty}$ without use of negative power of raising/lowering operators of the system. All features in the supersymmetric case are generalized to the parasupersymmetric systems of order 2.

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1 Introduction

We study spectrum generating algebras in para-/supersymmetric quantum mechanical (PSQM/SQM) systems with equally spaced energy eigenvalues (For a review on supersymmetric quantum mechanics, see [1]). In this article, we adopt the formalism [5] that we can define a series of Hamiltonians as well as raising/lowering operators to every solution of a specific nonlinear differential equation [2]-[7]. Supersymmetry is an important concept nowadays, however it is generally difficult to explicitly construct raising/lowering operators for eigenvalues separated by inhomogeneous intervals. To open a more general and complicated energy spectrum, it may be useful to investigate various properties of the spectrum generating algebra of our systems. Our algebra, generated by Hamiltonian and raising/lowering operators, is not a finite dimensional one like the harmonic oscillator (HO).

The importance of infinite dimensional Lie algebras in theoretical physics has increased since the discovery of Kac-Moody and Virasoro algebras. Among other things, \( W_{1+\infty} \) algebra and its subalgebras have become popular in recent years in various subjects [8] as they describe symmetries of systems. As an interesting connection to this stream, we report that subalgebras of \( W_{1+\infty} \) take part in a spectrum generating algebra, not in a symmetry algebra.

First, we discuss the similar algebraic structure to the HO system in the SQM Hamiltonian with equally spaced eigenvalues. Second, a different structure from HO algebra is discussed in relevance to \( W_{\infty} \) algebra. Finally, our scheme is generalized to the PSQM system.

2 Definition of \( W_{1+\infty} \) and \( W_{\infty} \)

Let us here note the definitions of \( W_{1+\infty} \) and \( W_{\infty} \) algebras. \( W_{1+\infty} \) algebra is defined as a central extension of Lie algebra of (higher order) differential operators on a circle. Let \( z \) be a point on a circle and \( D = z \frac{d}{dz} \), then the commutation relations of the algebra generated by \( \{ z^n D^m \mid n, m \in \mathbb{Z}, m \geq 0 \} \) are easily obtained. For the arbitrary polynomials of \( D \), \( f \) and \( g \) (which may include a constant term), the commutation relation of \( W_{1+\infty} \) is [1]

\[
[W(z^n f(D)), W(z^m g(D))] = W(z^{n+m} f(D + m) g(D)) - W(z^{n+m} f(D) g(D + n)) + c \Psi(z^n f(D), z^m g(D)) \tag{2.1}
\]

where \( W(z^n D^m) \) is a generator corresponding to \( z^n D^m \) and \( c \) is called a central charge. \( \Psi \) is the 2-cocycle given by

\[
\Psi(z^n f(D), z^m g(D)) = \delta_{n+m,0} \begin{cases} 
\sum_{j=1}^{n} f(-j) g(n-j) & \text{for } n \geq 1 \\
-\sum_{j=1}^{m} f(m-j) g(-j) & \text{for } m \geq 1 
\end{cases} \tag{2.2}
\]
$W_\infty$ algebra is a subalgebra of $W_{1+\infty}$ and is generated by $W(z^n f(D)D)\), $(n \in \mathbb{Z})$. In other words, $W_\infty$ algebra is obtained by subtracting an infinite number of generators $W(z^n)$ $(n = \pm 1, \pm 2, \cdots)$ from $W_{1+\infty}$. The unitary representations of these algebras have been discussed in [9] and [10].

3 Hamiltonians with equally spaced eigenvalues

It is known that a SQM system is described by a pair of Hamiltonians written in the following factorized form [11]

$$A^\dagger = \frac{1}{\sqrt{2}}(-ip + w(x)), \quad A = \frac{1}{\sqrt{2}}(ip + w(x)), \quad p = -i\frac{d}{dx},$$

$$H_- \equiv A^\dagger A = \frac{1}{2}(p^2 + w(x)^2 - w'(x)),$$

$$H_+ \equiv AA^\dagger = \frac{1}{2}(p^2 + w(x)^2 + w'(x)),$$

and possesses the following three important properties: (i) the eigenvalues of $H_\pm$ are positive or zero; (ii) the ground state of $H_-$ given by $\langle G |$ $\propto \exp(-\int_{-\infty}^{x} w(t)dt)$ has zero energy because it is defined as the state annihilated by the operator $A$; (iii) $(n+1)$-th eigenstate of $H_-$ and $n$-th one of $H_+$ are transformed into each other by $A$ and $A^\dagger$, and hence $H_\pm$ have the same energy spectra except for the ground state $\langle G |$. All these properties stem from the factorized forms of $H_\pm$.

We suppose the following condition throughout this paper to discuss Hamiltonians with equally spaced eigenvalues

$$w(x)^2 + w'(x) = x^2 + k, \quad (3.2)$$

where $H_+$ becomes a harmonic oscillator and the constant $k$ yields the lowest energy of $H_+$ (the energy gap between $\langle G |$ and $\langle 0 |$)

$$H_+ \langle 0 | = E_0 \langle 0 |, \quad E_0 = (k + 1)/2. \quad (3.3)$$

The case $k = 1$ is discussed in [3, 4, 5]. For $k = 1$, it is known that equation (3.2) has a series of non-trivial solutions which have been found analytically (see Appendix). For $k \neq 1$, solutions could be found numerically, at least. We generalize their arguments independently of the value of $k$. Our prescription will clarify the role of $k$ as central extensions of a spectrum generating algebra of $H_-$. 

First, we mention general features of our spectrum generating algebra. Eq. (3.2) is a nonlinear differential equation with respect to $w(x)$. Owing to property (iii), we can consider $H_-$ as a Hamiltonian with equally spaced eigenvalues for each solution of (3.2). In fact, we obtain the raising and lowering operators for excited states of $H_-$

$$O^\dagger = A^\dagger b^\dagger A, \quad O = A^\dagger b A, \quad (3.4)$$

which satisfy

$$[H_-, O^\dagger] = O^\dagger, \quad [H_-, O] = -O. \quad (3.5)$$

3
where \( b \) and \( b^\dagger \) are harmonic oscillator’s raising/lowering operators in \( H_+ \)
\[
b^\dagger = \frac{1}{\sqrt{2}}(-ip + x), \quad b = \frac{1}{\sqrt{2}}(ip + x).
\]
(3.6)

Note that the ground state of \( H_- \) is annihilated by these
\[
O^\dagger |G\rangle = O |G\rangle = H_- |G\rangle = 0.
\]
(3.7)

These relations are summarized in Figure 1.

Similar to the case of usual harmonic oscillator, we can construct the Fock representation and coherent states (eigenstates of the lowering operator). The \((n + 1)\)-th excited state \( |\psi_n\rangle \) of \( H_- \) starting from its first excited state is
\[
|\psi_n\rangle = \left( \prod_{t=0}^{n-1} \xi_t \right)^{-1/2} (O^\dagger)^n |\psi_0\rangle, \quad \xi_t = \sum_m \left\{ 3(E_0 + m)^2 - k(E_0 + m) \right\},
\]
which is orthonormal and satisfies
\[
O^\dagger |\psi_n\rangle = \sqrt{\xi_n} |\psi_{n+1}\rangle, \quad O |\psi_n\rangle = \sqrt{\xi_{n-1}} |\psi_{n-1}\rangle.
\]
(3.8)

The coherent state, which has been obtained for \( k = 1 \), is generalized to any value of \( k \);
\[
O |\alpha\rangle = \alpha |\alpha\rangle, \quad |\alpha\rangle = \sum_{n=0}^{\infty} \left( \prod_{t=0}^{n-1} \xi_t \right)^{-1/2} \alpha^n |\psi_n\rangle.
\]
(3.9)

The norm of this state converges for any value of \(|\alpha|^2\)
\[
\langle \alpha |\alpha\rangle = \sum_{n=0}^{\infty} |\alpha|^2 \left( \prod_{t=0}^{n-1} \xi_t \right)^{-1}
\]
regarding the power series of \(|\alpha|^2\) where the radius of convergence \( R \) is infinity
\[
R^{-1} = \lim_{n \to \infty} \left| \left( \prod_{t=0}^{n} \xi_t \right)^{-1} \left( \prod_{t=0}^{n-1} \xi_t \right) \right| = 0.
\]
(3.10)

4 Relationship to Subalgebras of \( W_{1+\infty} \)

Next, we focus our attention on the infinite dimensionality of the algebraic relation of (3.4) and (3.5) for \( H_- \). This property is different from the finite dimensionality of the harmonic oscillator \( H_+ \). The operators \( O^\dagger \) and \( O \) satisfy the following relations
\[
OO^\dagger = H_-^3 + \frac{1}{2}(3 - k)H_-^2 + \frac{1}{2}(1 - k)H_,
\]
\[
O^\dagger O = H_-^3 - \frac{1}{2}(3 + k)H_-^2 + \frac{1}{2}(1 + k)H_-, \quad (4.1)
\]
and
\[ [O, O^\dagger] = 3H_0^2 - kH_-\].

(4.2)

RHS of (4.2) is not linear but quadratic. In order to treat it as linear algebra (Lie algebra), we must regard \( H_0^2 \) as a new element and further consider additional commutation relations. The commutators among \( H_0^2, O \) and \( O^\dagger \) yield new elements \( O^\dagger H_- \) and \( OH_- \). \( [O^\dagger H_-, O^\dagger] \) creates \((O^\dagger)^2\), and in general \((O^\dagger)^{n+1}\) follows from \([O^\dagger H_-, (O^\dagger)^n]\). An infinite number of commutation relations is thus brought about. The fundamental elements of our algebra are represented as \( \{ (O^\dagger)^m H_n^-, O^m H_n^-, \ m, n = 0, 1, 2, \cdots \}\), to which elements such as \( O^n(O^\dagger)^m \) are reduced because of (4.1). We refer to this algebra as \( S \). Note that \( k \) appears as a structure constant in (4.2).

The purpose of this section is to discuss the relationship between algebras \( S \) and \( W_{\infty} \). It is convenient to first note that algebra \( S \) can be realized in terms of linear combinations of parts of \( W_{1+\infty} \) generators;

\[
(O^\dagger)^m H_n^- \rightarrow W(z^m(D + a_1 + m - 1)(D + a_1 + m - 2) \cdots (D + a_1)(D + a_0)^n)
\]
\[
O^m H_n^- \rightarrow W(z^{-m}(D + a_{-1} - m)(D + a_{-1} - m + 1) \cdots (D + a_{-1} - 1))
\]
\[
	imes (D - m + 1)(D - m + 2) \cdots D(D + a_0)^n),
\]

(4.3)

where the possible values of \( a_{\pm 1} \) and \( a_0 \) are listed on Table 1. It is worth while noticing that these linear combinations are particular combinations which do not produce any central extensions whether or not the original \( W_{1+\infty} \) generators participate in central extensions.

Let us show a brief sketch of how to determine \( a_i \) and \( \Psi = 0 \) for RHS of \((4.3)\). Consider (3.3) and (4.2) in this realization;

\[
H_- \rightarrow W(D + a_0), \quad O^\dagger \rightarrow W(z(D + a_1)), \quad O \rightarrow W(z^{-1}(D + a_{-1} - 1)D).
\]

(4.4)

We see that (3.3) holds without any restriction on \( a_i \) as usual.

\[
[H_-, O^\dagger] \rightarrow [W(D + a_0), W(z(D + a_1))] = W(z(D + a_1)),
\]
\[
[H_-, O] \rightarrow [W(D + a_0), W(z^{-1}(D + a_{-1} - 1)D)] = -W(z^{-1}(D + a_{-1} - 1)D).
\]

Comparing the image of (4.2) on both sides;

\[
[O, O^\dagger] \rightarrow [W(z^{-1}(D + a_{-1} - 1)D), W(z(D + a_1))]
\]
\[
= 3W(D^2) + (2a_1 + 2a_{-1} - 1)W(D) + a_1a_{-1},
\]

(4.5)

and

\[
3H_0^2 - kH_- \rightarrow 3W((D + a_0)^2) - kW(D + a_0)
\]
\[
= 3W(D^2) + (6a_0 - k)W(D) + 3a_0^2 - ka_0,
\]

(4.6)

we obtain the following two equations

\[
2a_1 + 2a_{-1} = 6a_0 + 1 - k,
\]
\[
a_1a_{-1} = 3a_0^2 - ka_0.
\]

(4.7)
Recalling that (4.1) is imposed on the definition of $S$, these equations are not enough to conclude that the realization (4.3) is consistent with (4.1). We in fact obtain one more relation for $a_i$ considering

$$[O, O^\dagger H_{-}] = [O, O^\dagger]H_{-} + O^\dagger O = (3H^2_{-} - kH_{-})H_{-} + \sigma(H_{-}), \quad (4.8)$$

where $\sigma(H_{-}) = O^\dagger O$ is given by RHS of (4.1). Comparing the coefficients of $W(D^2)$ and $W(D)$ on LHS of (4.8)

$$[O, O^\dagger H_{-}] \rightarrow [W(z^{-1}(D + a_{-1} - 1)D), W(z(D + a_1)(D + a_0))];$$

which gives

$$4W(D^3) + 3(a_1 + a_{-1} + a_0 - 1)W(D^2) + (2a_1a_{-1} + (2a_0 - 1)(a_1 + a_{-1} - a_0 + 1)W(D) + a_0a_1a_{-1},$$

with those on RHS

$$(3H^2_{-} - kH_{-})H_{-} + \sigma(H_{-})$$

$$\rightarrow 4W(D^3) + \frac{1}{2}(24a_0 - 3 - 3k)W(D^2) + (\frac{1}{2} - 3a_0 + 12a_0^2 + \frac{k}{2} - 3a_0k)W(D) + \frac{1}{2}(a_0 - 3a_0^2 + a_0k - 3a_0^2k) + 4a_0^3, \quad (4.10)$$

we obtain exactly the same relations as (4.7). A comparison between constant terms gives another relation

$$2a_0^3 - (k + 3)a_0^2 + (k + 1)a_0 = 0. \quad (4.11)$$

No further constraint is produced from $[O^mH_{-}, (O^\dagger)^mH^l_{-}]$, and we can determine $a_i$ as solutions of (4.7) and (4.11) as a result.

The following shows that the central extension (2-cocycle) of (4.3) always vanishes. The 2-cocycle that could appear only for $[O^mH_{n}, (O^\dagger)^mH^l_{i}]$, $(m \geq 1)$ is calculated;

$$\Psi(z^{-m}r(D), z^ms(D)) = - \sum_{j=1}^{m} r(m - j)s(-j), \quad (4.12)$$

with

$$r(D) = (D + a_{-1} - m)(D + a_{-1} - m + 1) \cdots (D + a_{-1} - 1) \times (D - m + 1)(D - m + 2) \cdots D(D + a_0)^n,$$

$$s(D) = (D + a_1 + m - 1)(D + a_1 + m - 2) \cdots (D + a_1)(D + a_0)^l,$$

and then

$$\Psi = - \sum_{j=1}^{m} (a_0 + m - j)^n(a_0 - j)^l(a_{-1} - j)(a_{-1} - j + 1) \cdots (a_{-1} + m - j - 1) \times (a_1 + m - j - 1)(a_1 + m - j - 2) \cdots (a_{-1} - j)(1 - j)(2 - j) \cdots (m - j)$$

$$= 0. \quad (4.13)$$
This result is independent of the choice of $a_i$ because the last equality of (4.13) is due to the factor $(1 - j)(2 - j) \cdots (m - j)$. Obviously, this is consistent with the original fact that the generators of $\mathcal{S}$ are combinations of $x$ and $\frac{d}{dx}$. As seen from (4.13), the mapping is one-to-one, however it is not onto because the particular elements $\{W(z^{-m}) \mid m \in \mathbb{Z}_{\geq 1}\}$, which generate central extensions of $W_{1+\infty}$, are missing. This is also a reason that the realization of $\mathcal{S}$ is irrespective of central extensions of original $W_{1+\infty}$ generators.

Now, we give some remarks in the following. We point out there exists a one-to-one and onto relation between $\mathcal{S}$ and a subset of $W_\infty$. Eliminating also the generators associated to positive powers of $z$ $\{W(z^m) \mid m = \pm 1, \pm 2, \pm 3, \cdots\}$, for example, putting $a_0 = a_1 = 0, a_{-1} = (1 - k)/2$, we find

$$(O^\dagger)^m H^n_\alpha \to W(z^m(D + m - 1)(D + m - 2) \cdots (D + 1)D^{n+1}),$$

$$O^m H^n_\alpha \to W(z^{-m}(D + a_{-1} - m)(D + a_{-1} - m + 1) \cdots (D + a_{-1} - 1) \times (D - m + 1)(D - m + 2) \cdots (D - 1)D^{n+1}).$$

Secondly, similar to the Virasoro operators of the harmonic oscillator, $L_n = (b^\dagger)^{n+1}b$ or $b^\dagger b^{n+1}$ for $n \geq 0$, we simply write down the Virasoro operators, which create/annihilate $n$ quanta for an excited state of $H_-$

$$L_n = A^\dagger (b^\dagger)^n A, \quad L^\dagger_n = A^\dagger b^n A. \quad (4.15)$$

These satisfy the centerless Virasoro algebra of positive modes

$$[L_n, L_m] = (m - n)L_{n+m}, \quad [L^\dagger_n, L^\dagger_m] = (n - m)L^\dagger_{n+m}, \quad (4.16)$$

and

$$L_n |G\rangle = L^\dagger_n |G\rangle = 0$$

$$L_n |\psi_m\rangle = \left[ (m + \frac{1}{2}(k + 1))(m + n + \frac{1}{2}(k + 1)) \frac{(m + n)!}{m!} \right]^{1/2} |\psi_{m+n}\rangle \quad (4.17)$$

$$L^\dagger_n |\psi_m\rangle = \begin{cases} \left[ (m + \frac{1}{2}(k + 1))(m - n + \frac{1}{2}(k + 1)) \frac{m!}{(m - n)!} \right]^{1/2} |\psi_{m-n}\rangle & n \leq m \\ 0 & \text{otherwise} \end{cases} \quad (4.18)$$

5 Generalization to Parasupersymmetric Quantum Mechanics

The formalism developed so far can be generalized to PSQM systems. Let us recall the definition of PSQM (of order 2, for example) based on ref. [12]. PSQM is essentially a pair of SQM Hamiltonians (except for vacuum structure)

$$A_\alpha = \frac{1}{\sqrt{2}}(ip + w_\alpha(x)), \quad A^\dagger_\alpha = \frac{1}{\sqrt{2}}(-ip + w_\alpha(x)), \quad \alpha = 1, 2$$
\[ H^{(1)}_{\text{SUSY}} = \begin{pmatrix} A_1 A^\dagger_1 & 0 \\ 0 & A^\dagger_1 A_1 \end{pmatrix}, \]
\[ H^{(2)}_{\text{SUSY}} = \begin{pmatrix} A_2 A^\dagger_2 & 0 \\ 0 & A^\dagger_2 A_2 \end{pmatrix}, \]

equipped with the condition
\[ w^2_2 + w'_2 = w^2_1 - w'_1 + q, \tag{5.2} \]

where \( q \) is a constant. This is called the "shape-invariant condition" \[13\]. Because of condition (5.2), the PSQM of order 2 consists of three distinct Hamiltonians, \textit{i.e.}
\[ H_1 = A_1 A^\dagger_1, \quad H_2 = A^\dagger_1 A_1 = A_2 A^\dagger_2 - \frac{q}{2}, \quad H_3 = A^\dagger_2 A_2. \tag{5.3} \]

The Hamiltonians are isospectral except the states annihilated by \( A_\alpha \), since \( H_1 \) and \( H_2 \), \( H_2 \) and \( H_3 \) form SQM’s respectively. The eigenstates of these Hamiltonians are transformed as follows
\[
\text{eigenstates of } H_1 \xrightarrow{A_1^\dagger} \text{eigenstates of } H_2 \xrightarrow{A_2^\dagger} \text{eigenstates of } H_3
\]

To give the Hamiltonians equally spaced eigenvalues, we require that \( H_1 \) is the harmonic oscillator (similarly to sect.3)
\[ w^2_1 + w'_1 = x^2 + k, \tag{5.4} \]

where \( k \) is a constant. The Hamiltonians \( H_1 \) and \( H_2 \) are identical to \( H_+ \) and \( H_- \) respectively. Hence we have another nonlinear differential equation (5.2) for each solution of the differential equation (5.4). This determines \( H_3 \) as a new Hamiltonian with equally spaced eigenvalues. The previous SQM argument between \( H_1 \) and \( H_2 \) applies another SQM system of \( H_2 \) and \( H_3 \). Instead of \( O \) and \( O^\dagger \), we have
\[ P = A_2^\dagger O A_2, \quad P^\dagger = A_2^\dagger O^\dagger A_2, \tag{5.5} \]

and
\[ [H_3, P^\dagger] = P^\dagger, \quad [H_3, P] = -P, \tag{5.6} \]

where \( O \) and \( O^\dagger \) are the raising and lowering operators for \( H_2 = H_- \) given in (4.4). The counterparts to (4.1) and (4.2) are
\[
PP^\dagger = H^5_3 - \frac{1}{2}(3q + k - 5)H^4_3 + \frac{1}{4}(3q^2 + 2(k - 6)q + 4(2 - k))H^3_3 \\
- \frac{1}{8}(q^3 + (k - 9)q^2 - 2(3k - 7)q + 4(k - 1))H^2_3 \tag{5.7}
\]
\[
\frac{1}{8}(q^3 + (k - 3)q^2 - 2(k - 1)q)H_3,
\]

\[
P^\dagger P = H_3^5 - \frac{1}{2}(3q + k + 5)H_3^4 + \frac{1}{4}(3q^2 + 2(k + 6)q + 4(k + 2))H_3^3
\]

\[- \frac{1}{8}(q^3 + (k - 9)q^2 + 2(3k + 7)q + 4(k + 1))H_3^2
\]

\[+ \frac{1}{8}(q^3 + (k + 3)q^2 + 2(k + 1)q)H_3,
\]

and

\[
[P, P^\dagger] = 5H_3^4 - 2(3q + k)H_3^3 + \left(\frac{9}{4}q^2 + \frac{3}{2}kq + 1\right)H_3^2
\]

\[- \frac{q}{4}(q^2 + kq + 2)H_3.
\]

Note that RHS of the commutator (5.9) is in turn biquadratic compared to (4.2) and that the two energy gap parameters appear in the structure constants. The algebra generated by \(P, P^\dagger\) and \(H_3\) is again infinite dimensional and its elements are \(\{P^nH_3^l, (P^\dagger)^mH_3^l, |n, m, l \in Z_{\geq 0}\}\). We refer to this algebra as \(\mathcal{S}_{ps}\).

Similar to (4.14), the algebra \(\mathcal{S}_{ps}\) can be realized in terms of \(W_\infty\) generators with central extensions. The realization is given by (one-to-one and onto)

\[
P^nH_3^m \rightarrow W(f(D)^nD^m), \quad (P^\dagger)^mH_3^n \rightarrow W(g(D)^nD^m),
\]

where \(f(D)\) and \(g(D)\) are defined by

\[
f(D) = z^{-1}(D^2 + \alpha D + \beta)D, \quad g(D) = z(D + \lambda)D,
\]

and the 2-cocycle vanishes for \([P^nH_3^m, (P^\dagger)^mH_3^n]\). Possible values of \(\alpha, \beta\) and \(\lambda\) (listed on Table 2) are determined through comparing the image on both sides of (5.9)

\[
[P, P^\dagger] \rightarrow [f(D), g(D)]
\]

\[= 5W(D^4) + (4\alpha + 4\lambda + 2)W(D^3) + (3\alpha\lambda + 3\lambda + 3\beta + 1)W(D^2)
\]

\[+ (-\beta + \lambda + \alpha\lambda + 2\beta\lambda)W(D),
\]

\[
\chi(H_3) \rightarrow 5W(D^4) - 2(3q + k)W(D^3) + \left(\frac{9}{4}q^2 + \frac{3}{2}kq + 1\right)W(D^2)
\]

\[- \frac{q}{4}(q^2 + kq + 2)W(D),
\]

where \(\chi(H_3)\) is the RHS of (5.9). We thus obtain the equations

\[
2\alpha + 2\lambda + 1 = -3q - k,
\]

\[4\beta + 4\lambda + 4\alpha\lambda = 3q^2 + 2kq,
\]

\[-4\beta + 4\lambda + 4\alpha\lambda + 8\beta\lambda = -q^3 - kq^2 - 2q.
\]

These equations (5.13) are consistent also with the image of the commutation relation

\[
[P, P^\dagger H_3] = [P, P^\dagger]H_3 + P^\dagger P = \chi(H_3)H_3 + \rho(H_3),
\]

where \(\rho(H_3) = P^\dagger P\) is given by RHS of (5.8).
6 Conclusion

In the present paper, we discussed spectrum generating algebras of the SQM and PSQM systems with equally spaced eigenvalues. (P)SQM Hamiltonians with equally spaced eigenvalues can be obtained for each solution to the nonlinear differential equations (3.2) and (5.2). The pair of Hamiltonians is solvable; namely, we can construct all of their eigenstates using the raising/lowering operators in each system. One satisfies a harmonic oscillator algebra and the other can be realized by \( W_\infty \) with an arbitrary value of \( c \). The commutation relations (3.5) are common with harmonic oscillator algebra, while eq.(4.2) is different. One can discuss different algebraic aspect from ours; for example, Fernández et al modify the definition of \( O \) and \( O^\dagger \) and obtained a boson-like commutation relation \( [4] \). They also constructed a coherent state for their annihilation operator.

Our formalism is dependent on the factorization of Hamiltonians and on the choice of creation/annihilation operators for \( H_+ \), which we have assumed to be a harmonic oscillator. There exist other types of Hamiltonian with equally spaced eigenvalues \( [7] \). It is still an open question whether our formalism is applicable to such systems. As to PSQM, one might get more interesting results as a continuation of this work along the line of \( [14] \).

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Appendix: examples of \( w(x) \)

Roy and Roychoudhury \( [5] \) found an infinite sequence of solutions of (3.2) when \( k = 1 \). That is given by

\[
\begin{align*}
  w_{n+1}(x) &= w_n(x) + u_n(x), \quad w_0 = x \quad n = 0, 1, 2, \cdots \\
  u_n &= \frac{\exp(-2 \int x w_n(t) dt)}{c_n + \int x \exp(-2 \int x w_n(t) dt) dx'}.
\end{align*}
\]

(6.1)

where \( c_n \) is the constant determined so as to make the ground state normalizable. The Hamiltonian corresponding to \( w_n(x) \) is given by

\[
H_{(n)} = \frac{1}{2}(p^2 + x^2 - 1) - \sum_{m=0}^{n-1} \frac{d u_m(x)}{dx}.
\]

(6.2)

Let us look at the case of \( n = 1 \) in detail. The Hamiltonian of this case

\[
H_{(1)} = \frac{1}{2}(p^2 + x^2 - 1) - \frac{du_0(x)}{dx}, \quad u_0(x) = \frac{\exp(-x^2)}{c_0 + \int_{-\infty}^{x} \exp(-t^2) dt}.
\]

(6.3)
is already discussed by several authors [2, 3, 7]. The ground state of this Hamiltonian is given by

$$|G\rangle = \left( \frac{c_0 (c_0 + \sqrt{\pi})}{\sqrt{\pi}} \right)^{1/2} \frac{\exp(-x^2/2)}{c_0 + \int_{-\infty}^{\infty} \exp(-t^2/2) dt}. \quad (6.4)$$

The normalizability of the ground state requires that the constant $c_0$ lies in the range $c_0 > 0$ or $c_0 < -\sqrt{\pi}$. The shape of potential depends on the value of $c_0$, and hence we have obtained uncountable infinite number of solutions of the nonlinear differential equation (3.2). The potentials for some values of $c_0$ are depicted in Figure 2.

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Figure Captions

Figure 1: Energy spectra of $H_{\pm}$ and action of operators

Figure 2: Plots of the potential $V(x) = \frac{x^2}{2} - \frac{du_0(x)}{dx}$ for various values of $c_0$. The thin and thick solid lines correspond to $c_0 = -2, 0.1$, while the dotted and dashed lines to $c_0 = 2, 10$ respectively.
Figure 1: Energy spectra of $H_{\pm}$ and action of operators
| $a_0$    | $a_1$    | $a_{-1}$    |
|---------|---------|-------------|
| 0       | 0       | $\frac{1}{2}(1 - k)$ |
|         | $\frac{k}{2}(1 - k)$ | 0       |
| 1       | 2       | $\frac{1}{2}(3 - k)$ |
|         | $\frac{k}{2}(3 - k)$ | 2       |
| $\frac{1}{2}(k + 1)$ | $\frac{k}{2}(k + 1)$ | $\frac{1}{2}(k + 3)$ |
| $\frac{k}{2}(k + 3)$ | $\frac{1}{2}(k + 1)$ |

**Table 1**: allowed values of constants $a_0$ and $a_{\pm}$
\[
\alpha = \frac{1}{2}(2q + k + 1) \quad \frac{1}{4}(q^2 + (k + 1)q) \quad -\frac{q}{2} \\
-\frac{1}{2}(2q + k + 3) \quad \frac{1}{4}(q^2 + (k + 3)q + 2(k + 1)) \quad \frac{1}{2}(-q + 2) \\
-\frac{1}{2}(-q - 1) \quad \frac{1}{4}(q^2 + 2q) \quad \frac{1}{2}(-q - k + 1)
\]

**Table 2:** allowed values of constants $\alpha, \beta$ and $\lambda$
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