HERMITE AND BERNSTEIN STYLE BASIS FUNCTIONS FOR CUBIC SERENDIPITY SPACES ON SQUARES AND CUBES

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ABSTRACT. We introduce new Hermite-style and Bernstein-style geometric decompositions of the cubic order serendipity finite element spaces $S_3(I^2)$ and $S_3(I^3)$, as defined in the recent work of Arnold and Awanou [Found. Comput. Math. 11 (2011), 337–344]. The serendipity spaces are substantially smaller in dimension than the more commonly used bicubic and tricubic Hermite tensor product spaces - 12 instead of 20 for the square and 32 instead of 64 for the cube - yet are still guaranteed to obtain cubic order a priori error estimates when used in finite element methods. The basis functions we define have a canonical relationship both to the finite element degrees of freedom as well as to the geometry of their graphs; this makes the bases ideal for applications employing isogeometric analysis where domain geometry and functions supported on the domain are described by the same basis functions. Moreover, the basis functions are linear combinations of the commonly used bicubic and tricubic polynomial Bernstein or Hermite basis functions, allowing their rapid incorporation into existing finite element codes.

1. INTRODUCTION

Serendipity spaces offer a rigorous means to reduce the degrees of freedom associated to a finite element method while still ensuring optimal order convergence. The ‘serendipity’ moniker came from the observation of this phenomenon among finite element practitioners before its mathematical justification was fully understood; see e.g. [14, 10, 11, 7]. Recent work by Arnold and Awanou [1, 2] classifies serendipity spaces on cubical meshes in $n \geq 2$ dimensions by giving a simple and precise definition of the space of polynomials that must be spanned, as well as a unisolvent set of degrees of freedom for them. For degree $r$ convergence, the serendipity space is defined as the span of $m$ monomials in $n$ variables and is denoted $S_r(I^n)$.

Since these $m$ monomials bear no obvious correspondence to the $m$ domain points of the $n$-dimensional unit cube $I^n$ nor to the $m$ degrees of freedom described in the papers, alternative local bases for the serendipity spaces must be derived.

In this paper, we provide two coordinate-independent geometric decompositions for both $S_3(I^2)$ and $S_3(I^3)$, the cubic serendipity spaces in two and three dimensions, respectively. We present sets of polynomial basis functions associated in a natural and symmetric fashion to the cubic serendipity domain points of the square or cube and prove that they provide a basis for the corresponding cubic serendipity space. Each basis is designated as either Bernstein or Hermite style, as each function restricts to one of these common basis function types on each edge of the square or cube.

The Hermite style bases have the desirable property of being Lagrange-like with respect to vertex values and coordinate directional derivative values at vertices. More precisely, if a Hermite style function is associated to a vertex, it has value 1 at that vertex,

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1The definition of $S_r(I^n)$ and its dimension $m$ is reviewed in Section 2.1.
value 0 at other vertices, and coordinate directional derivatives equal to 0 at all vertices. If a Hermite style function is associated to an edge domain point, it has directional derivative equal to 1 at the nearest vertex (in the direction of that domain point) with all other coordinate directional derivative values equal to 0 (at vertices) and all vertex function values equal to zero. The Bernstein style basis functions have the more simply-stated property of obtaining a unique maximal value at the associated domain point. Graphs of both types of functions over $I^2$ are shown in Figures 4 and 5.

\begin{align*}
\frac{1}{4}(x - 1)(y - 1) & \quad -\frac{1}{4}\sqrt{\frac{2}{5}}(x^2 - 1)(y - 1) & \quad -\frac{1}{4}\sqrt{\frac{5}{2}}x(x^2 - 1)(y - 1)
\end{align*}

**Figure 1.** Cubic serendipity functions on $I^2$ from [15]. The left function is associated to the vertex below the peak. The middle and right functions are associated to the edge $y = -1$ but do not correspond to the domain points $(\pm \frac{1}{3}, -1)$ in any canonical or symmetric fashion, making them less useful for geometric modeling or isogeometric analysis.

To the author’s knowledge, the only basis functions previously available for cubic order serendipity finite element purposes employ Legendre polynomials, which lack a clear relationship to the domain points. Definitions of these basis functions can be found in Szabó and Babuška [15, Sections 6.1 and 13.3]; the two functions from [15] associated to the edge $y = -1$ of $I^2$, are shown in Figure 1 (middle and right). The restriction of these functions to the edge gives an even polynomial in one case and an odd polynomial in the other, forcing an *ad hoc* choice of how to associate the functions to the corresponding domain points $(\pm \frac{1}{3}, -1)$. On the other hand, as was just described, the functions presented in this paper do have a natural correspondence to the domain points of the geometry.

Maintaining a concrete and canonical relationship between domain points and basis functions is an essential component of the growing field of *isogeometric analysis* (IGA). One of the main goals of IGA is to employ basis functions that can be used both for geometry modeling and finite element analysis, exactly as we provide here for cubic serendipity spaces. Each function is a linear combination of bicubic or tricubic Bernstein or Hermite polynomials; the specific coefficients of the combination are given in the proofs of the theorems. This makes the incorporation of the functions into a variety of existing application contexts relatively easy. Note that tensor product bases in two and three dimensions are commonly available in finite element software packages (e.g. deal.II [4]) and cubic order tensor products in particular are commonly used both in modern theory (e.g. isogeometric analysis [9]) and applications (e.g. cardiac electrophysiology models [16]). Hence, a variety of areas of computational science could directly employ the new cubic order serendipity basis functions presented here.

The benefit of serendipity finite element methods is a significant reduction in the computational effort required for optimal order (in this case, cubic) convergence. Cubic
serendipity methods on meshes of squares requires 12 functions per element, an improvement over the 16 functions per element required for bicubic tensor product methods. On meshes of cubes, the cubic serendipity method requires 32 functions per element instead of the 64 functions per element required for tricubic tensor product methods. Using fewer basis functions per element reduces the size of the overall linear system that must be solved, thereby saving computational time and effort. An additional computational advantage occurs when the functions presented here are used in an isogeometric fashion. The process of converting between computational geometry bases and finite element bases is a well-known computational bottleneck in engineering applications [8] but is easily avoided when basis functions suited to both purposes are employed.

The outline of the paper is as follows. In Section 2, we fix notation and summarize relevant background on Hermite and barycentric basis functions as well as serendipity spaces. In Section 3, we present polynomial Bernstein and Hermite style basis functions for $S_3(I^2)$ that agree with the standard bicubics on edges of $I^2$ and provide a novel geometric decomposition of the space. In Section 4, we present polynomial Bernstein and Hermite style basis functions for $S_3(I^3)$ that agree with the standard tricubics on edges of $I^3$, reduce to our bases for $I^2$ on faces of $I^3$, and provide a novel geometric decomposition of the space. Finally, we state our conclusions and discuss future directions in Section 5.

2. Background and Notation

2.1. Serendipity Elements. We first review the definition of serendipity spaces and their accompanying notation from the work of Arnold and Awanou [1, 2].

**Definition 2.1.** The superlinear degree of a monomial in $n$ variables, denoted $\text{sldeg}(\cdot)$, is given by

$$\text{sldeg}(x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}) := \left(\sum_{i=1}^{n} e_i\right) - |\{e_i : e_i = 1\}| \quad (2.1)$$

In words, $\text{sldeg}(q)$ is the ordinary degree of $q$, ignoring variables that enter linearly. For instance, the superlinear degree of $xy^2z^3$ is 5.

**Definition 2.2.** Define the following spaces of polynomials, each of which is restricted to the domain $I^n = [-1, 1]^n \subset \mathbb{R}^n$:

- $\mathcal{P}_r(I^n) := \text{span}_\mathbb{R} \{\text{monomials in } n \text{ variables with degree } \leq r\}$
- $\mathcal{S}_r(I^n) := \text{span}_\mathbb{R} \{\text{monomials in } n \text{ variables with superlinear degree } \leq r\}$
- $\mathcal{Q}_r(I^n) := \text{span}_\mathbb{R} \{\text{monomials in } n \text{ variables with each variable degree } \leq r\}$

Note that $\mathcal{P}_r(I^n) \subset \mathcal{S}_r(I^n) \subset \mathcal{Q}_r(I^n)$, with proper containments when $r, n > 1$. The space $\mathcal{S}_r(I^n)$ is called the degree $r$ serendipity space on the $n$-dimensional unit cube $I^n$. In the notation of the more recent paper by Arnold and Awanou [2], the serendipity spaces discussed in this work would be denoted $\mathcal{S}_r^0(I^n)$, indicating that they are differential 0-form spaces. The spaces have dimension given by the following formulas (cf. [1]).

$$\dim \mathcal{P}_r(I^n) = \binom{n + r}{n}$$
$$\dim \mathcal{S}_r(I^n) = \min(n, \lfloor r/2 \rfloor) \sum_{d=0}^{\min(n, \lfloor r/2 \rfloor)} 2^{n-d} \binom{n}{d} \binom{r-d}{d}$$
$$\dim \mathcal{Q}_r(I^n) = (r + 1)^n$$
We write out standard bases for these spaces more precisely in the cubic order cases of concern here.

\[
\mathcal{P}_3(I^2) = \text{span}\{1, \frac{1}{2} x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^2y^2, x^3y^2, x^3y^3\}
\] (2.2)

\[
\mathcal{S}_3(I^2) = \mathcal{P}_3(I^2) \cup \text{span}\{x^3y, xy^3\}
\] (2.3)

\[
\mathcal{Q}_3(I^2) = \mathcal{S}_3(I^2) \cup \text{span}\{x^2y^2, x^3y^2, x^2y^3, x^3y^3\}
\] (2.4)

Observe that the dimensions of the three spaces are 10, 12, and 16, respectively.

\[
\mathcal{P}_3(I^3) = \text{span}\{1, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^3, y^3, z^3, x^2y, x^2z, y^2z, x^3z, y^3z, xyz\}
\] (2.5)

\[
\mathcal{S}_3(I^3) = \mathcal{P}_3(I^3) \cup \text{span}\{x^3y, x^3z, y^3z, xy^3, xz^3, yz^3, x^2y^2, x^2y^3, x^2z^2, y^2z^2, x^2yz, x^2y^2z\}
\] (2.6)

\[
\mathcal{Q}_3(I^3) = \mathcal{S}_3(I^3) \cup \text{span}\{x^3y^2, \ldots, x^3y^3z^3\}
\] (2.7)

Observe that the dimensions of the three spaces are 20, 32, and 64, respectively.

The serendipity spaces are associated to specific degrees of freedom in the classical finite element sense. For a face \( f \) of \( I^n \) of dimension \( d \geq 0 \), the degrees of freedom associated to \( f \) for \( \mathcal{S}_r(I^n) \) are (cf. [1])

\[
\mathbf{u} \mapsto \int_f \mathbf{u} \mathbf{q}, \quad \mathbf{q} \in \mathcal{P}_{r-2d}(f).
\]

For the cases considered in this work, \( n = 2 \) or 3 and \( r = 3 \) so the only non-zero degrees of freedom are when \( f \) is a vertex (\( d = 0 \)) or an edge (\( d = 1 \)). Thus, the degrees of freedom for our cases are the values

\[
\mathbf{u}(v), \quad \int_e \mathbf{u} \ dt, \quad \text{and} \quad \int_e \mathbf{u} t \ dt,
\]

for each vertex \( v \) and each edge \( e \) of the square or cube.

\[\text{Figure 2. The Bernstein basis } [\beta] \text{ and the Hermite basis } [\psi] \text{ on } [0,1].\]

2.2. Cubic Bernstein and Hermite Bases. For cubic order approximation on square or cubical grids, tensor product bases are typically built from one of two alternative bases
for $\mathcal{P}_3([0, 1])$:

$$[\beta] = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix} := \begin{bmatrix} 1 - 3x + 3x^2 - x^3 \\
x - 2x^2 + x^3 \\
x^2 - x^3 \\
x^3 
\end{bmatrix} = \begin{bmatrix} (1 - x)^3 \\
(1 - x)^2x \\
(1 - x)x^2 \\
x^3 
\end{bmatrix}$$

$$[\psi] = \begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{bmatrix} := \begin{bmatrix} 1 - 3x^2 + 2x^3 \\
x - 2x^2 + x^3 \\
3x^2 - 2x^3 \\
0 
\end{bmatrix} = \begin{bmatrix} \psi_0^3 \\
\psi_0^4 \\
\psi_1^4 \\
\psi_1^3 
\end{bmatrix}$$

The set $[\beta]$ is the cubic Bernstein basis and the set $[\psi]$ is the cubic Hermite basis. Both bases are shown in Figure 2. Bernstein functions have been used recently to provide a geometric decomposition of finite element spaces over simplices [3]. Hermite functions, while more common in geometric modeling contexts [12] have also been studied in finite element contexts for some time [6].

The Hermite functions have the following important property relating them to the geometry of the graph of their associated interpolant:

$$u = u(0)\psi_1 + u'(0)\psi_2 - u'(1)\psi_3 + u(1)\psi_4, \quad \forall u \in \mathcal{P}_3([0, 1]). \quad (2.9)$$

We have chosen these sign and basis ordering conventions so that both bases have the same symmetry property:

$$\beta_k(1 - x) = \beta_{5-k}(x), \quad \psi_k(1 - x) = \psi_{5-k}(x). \quad (2.10)$$

The bases $[\beta]$ and $[\psi]$ are related by $[\beta] = \mathbb{V}[\psi]$ and $[\psi] = \mathbb{V}^{-1}[\beta]$ where

$$\mathbb{V} = \begin{bmatrix}
1 & -3 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -3 & 1
\end{bmatrix}, \quad \mathbb{V}^{-1} = \begin{bmatrix}
1 & 3 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 3 & 1
\end{bmatrix}. \quad (2.11)$$

Bases for $\mathcal{Q}_3([0, 1]^n)$ are easily constructed by taking tensor products of bases for $\mathcal{P}_r([0, 1])$. For instance, note that $[\beta] \otimes [\beta]$ (denoted $[\beta^2]$) is a basis for $\mathcal{Q}_3([0, 1]^2)$ where

$$[\beta] \otimes [\beta] = [\beta^2] = \{ \beta_i(x)\beta_j(y) : i, j \in \{1, 2, 3, 4\} \}$$

The basis function $\beta_i(x)\beta_j(y)$ will be denoted $\beta_{ij}$. Similarly, $[\beta] \otimes [\beta] \otimes [\beta]$ (denoted $[\beta^3]$) is a basis for $\mathcal{Q}_3([0, 1]^3)$ with basis functions denoted $\beta_{ijk}$.

We now rephrase the fact that $[\beta]$ is a basis for $\mathcal{P}_3([0, 1])$ as a ‘precision property,’ i.e. that linear combinations of the basis functions recover certain monomials exactly. The extension of this property to tensor product bases will be used to prove that the sets of functions presented in Sections 3 and 4 are indeed bases for $\mathcal{S}_3(I^2)$ and $\mathcal{S}_3(I^3)$. 


Proposition 2.3. For $0 \leq r, s, t \leq 3$, the precision properties of $[\beta]$, $[\beta^2]$, and $[\beta^3]$ take on the respective forms

$$x^r = \sum_{i=1}^{4} \left( \frac{3 - r}{4 - i} \right) \beta_i,$$  \hspace{1cm} (2.12)

$$x^r y^s = \sum_{i=1}^{4} \sum_{j=1}^{4} \left( \frac{3 - r}{4 - i} \right) \left( \frac{3 - s}{4 - j} \right) \beta_{ij},$$ \hspace{1cm} (2.13)

$$x^r y^s z^t = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \left( \frac{3 - r}{4 - i} \right) \left( \frac{3 - s}{4 - j} \right) \left( \frac{3 - t}{4 - k} \right) \beta_{ijk}.$$ \hspace{1cm} (2.14)

Proof. One easily confirms that $x^3 = \beta_4$, $x^2 = \beta_4 + \beta_3$, $x^1 = \beta_4 + 2\beta_3 + \beta_2$, and $x^0 = \beta_4 + 3\beta_3 + 3\beta_2 + \beta_1$, which is the statement of (2.12). The other two statements follow immediately from the first. \hfill \Box

We have a similar proposition for the Hermite basis $[\psi]$ and its tensor products $[\psi^2] := [\psi] \otimes [\psi]$ and $[\psi^3] := [\psi] \otimes [\psi] \otimes [\psi]$. As before, $\psi_{ij}$ means $\psi_i(x)\psi_j(y)$ and $\psi_{ijk}$ means $\psi_i(x)\psi_j(y)\psi_k(z)$.

Proposition 2.4. Let

$$\varepsilon_{r,i} := \sum_{a=1}^{4} \left( \frac{3 - r}{4 - a} \right) v_{ai}$$ \hspace{1cm} (2.15)

where $v_{ai}$ denotes the $(a, i)$ entry (row, column) of $\nabla$ from (2.11). For $0 \leq r, s, t \leq 3$, the precision properties of $[\psi]$, $[\psi^2]$, and $[\psi^3]$ take on the respective forms

$$x^r = \sum_{i=1}^{4} \varepsilon_{r,i} \psi_i,$$ \hspace{1cm} (2.16)

$$x^r y^s = \sum_{i=1}^{4} \sum_{j=1}^{4} \varepsilon_{r,i} \varepsilon_{s,j} \psi_{ij},$$ \hspace{1cm} (2.17)

$$x^r y^s z^t = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} \psi_{ijk}.$$ \hspace{1cm} (2.18)

Proof. By swapping the order of summation, we see that

$$\sum_{i=1}^{4} \varepsilon_{r,i} \psi_i = \sum_{a=1}^{4} \left( \frac{3 - r}{4 - a} \right) \left( \sum_{i=1}^{4} v_{ai} \psi_i \right) = \sum_{a=1}^{4} \left( \frac{3 - r}{4 - a} \right) \beta_a = x^r,$$

by (2.12), proving (2.16). The other two statements follow immediately from the first. \hfill \Box

Transforming the bases $[\beta]$ and $[\psi]$ to domains other than $[0, 1]$ is straightforward. If $T : [a, b] \to [0, 1]$ is linear, then replacing $x$ with $T(x)$ in each basis function expression for $[\beta]$ and $[\psi]$ gives bases for $P_3([a, b])$. Note, however, that the derivative interpolation property for $[\psi]$ must be adjusted to account for the scaling:

$$u = u(1)\psi_1(T(x)) + |b - a|u'(0)\psi_2(T(x))$$

$$- |b - a|u'(1)\psi_3(T(x)) + u(1)\psi_4(T(x)), \hspace{1cm} \forall u \in P_3([a, b]).$$ \hspace{1cm} (2.19)
In geometric modeling applications, the coefficient $|b - a|$ is sometimes left as an adjustable parameter, usually denoted $s$ for scale factor. [5, 16]. For all the Hermite and Hermite style functions, we will use derivative-preserving scaling which will include scale factors on those functions related to derivatives; this will be made explicit in the various contexts where it is relevant.

Remark 2.5. Both $[\beta]$ and $[\psi]$ are Lagrange like at the endpoints of $[0, 1]$, i.e. at an endpoint, the only basis function with non-zero value is the function associated to that endpoint ($\beta_1$ or $\psi_1$ for 0, $\beta_4$ or $\psi_4$ for 1). This means the two remaining basis functions of each type ($\beta_2, \beta_3$ or $\psi_2, \psi_3$) are naturally associated to the two edge degrees of freedom (2.8). We will refer to these associations between basis functions and geometrical objects as the standard geometrical decompositions of $[\beta]$ and $[\psi]$.

3. Local Bases for $S_3(I^2)$

Before defining local bases on the square, we fix notation for the domain points to which they are associated. For $[0, 1]^2$, define the ordered paired index set

$$X := \{\{i, j\} | i, j \in \{1, \ldots, 4\}\}.$$  

Then $X$ is the disjoint union $V \cup E \cup D$ where

$$V := \{\{i, j\} | i, j \in \{1, 4\}\}; \quad (3.1)$$

$$E := \{\{i, j\} | \text{exactly one of } \{i, j\} \text{ is in } \{1, 4\}\}; \quad (3.2)$$

$$D := \{\{i, j\} | i, j \in \{2, 3\}\} \quad (3.3)$$

The $V$ indices are associated with vertices of $[0, 1]^2$, the $E$ indices to edges of $[0, 1]^2$, and the $D$ vertices to the domain interior to $[0, 1]^2$. The relation between indices and domain points of the square is shown in Figure 3.

3.1. A local Bernstein style basis for $S_3(I^2)$. Under the notation and conventions established in Section 2, we establish a local Bernstein style basis for $S_3(I^3)$ where $I := [-1, 1]$. Define the following set of 12 functions, indexed by $V \cup E$; note that each
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function is to be scaled by a factor of \(1/16\).

\[
[\xi^2] = \begin{bmatrix}
\xi_{11} \\
\xi_{14} \\
\xi_{41} \\
\xi_{44} \\
\xi_{12} \\
\xi_{13} \\
\xi_{42} \\
\xi_{43} \\
\xi_{21} \\
\xi_{31} \\
\xi_{24} \\
\xi_{34}
\end{bmatrix} = \begin{bmatrix}
(x - 1)(y - 1)(-2 - 2x + x^2 - 2y + y^2) \\
-(x - 1)(y + 1)(-2 - 2x + x^2 + 2y + y^2) \\
-(x + 1)(y - 1)(-2 + 2x + x^2 - 2y + y^2) \\
(x + 1)(y + 1)(-2 + 2x + x^2 + 2y + y^2)
\end{bmatrix} \cdot \frac{1}{16} \quad (3.4)
\]

Fix the basis orderings

\[
[\xi^2] := \begin{bmatrix}
\xi_{11}, \xi_{14}, \xi_{41}, \xi_{44}, \\
\xi_{12}, \xi_{13}, \xi_{42}, \xi_{43}, \xi_{21}, \xi_{31}, \xi_{24}, \xi_{34}
\end{bmatrix},
\]

indices in \(V\), \(\xi^2\)

\[
[\beta^2] := \begin{bmatrix}
\beta_{11}, \beta_{14}, \beta_{41}, \beta_{44}, \\
\beta_{12}, \beta_{13}, \beta_{42}, \beta_{43}, \beta_{21}, \beta_{24}, \beta_{31}, \beta_{22}, \beta_{23}, \beta_{32}, \beta_{33}
\end{bmatrix}
\]

indices in \(V\), \(\beta^2\)

\[
\beta_{I,11} := \beta_{11}((x + 1)/2), \beta_{I,21} := \beta_{21}((y + 1)/2).
\]

The set \([\xi^2]\) has the following properties:

\(\xi_{11}\), \(\xi_{21}\), \(\xi_{31}\)

\(\beta_{I,11}\), \(\beta_{I,21}\), \(\beta_{I,31}\)

\(\xi_{21}\), \(\xi_{31}\)

\(\beta_{I,21}\), \(\beta_{I,31}\)

\(\xi_{31}\)

\(\beta_{I,31}\)

**Figure 4.** The top row shows 3 of the 16 bicubic Bernstein functions on \(I^2\) while the bottom row shows 3 of the 12 cubic Bernstein style serendipity functions. The visual differences are subtle, although some changes in concavity can be observed. Note that functions in the same column have the same values on the edges of \(I^2\).

**Theorem 3.1.** Let \(\beta_{I,m}\) denote the scaling of \(\beta_{I,m}\) to \(I^2\), i.e.

\[
\beta_{I,m} := \beta_{I,m}((x + 1)/2)\beta_{m}((y + 1)/2).
\]

The set \([\xi^2]\) has the following properties:
(i) \([\xi^2]\) is a basis for \(S_3(I^2)\).
(ii) For any \(\ell m \in V \cup E\), \(\xi_{\ell m}\) is identical to \(\beta_{\ell m}^I\) on the edges of \(I^2\).
(iii) \([\xi^2]\) is a geometric decomposition of \(S_3(I^2)\).

**Proof.** For (i), we scale \([\xi^2]\) to \([0, 1]^2\) to take advantage of a simple characterization of the precision properties. Let \([\xi^2]^{[0,1]}\) denote the set of scaled basis functions \(\xi_{\ell m}^{[0,1]}(x, y) := \xi_{\ell m}(2x - 1, 2y - 1)\). Given the basis orderings in (3.5)-(3.6), it can be confirmed directly that \([\xi^2]^{[0,1]}\) is related to \([\beta^2]\) by

\[
[\xi^2]^{[0,1]} = B[\beta^2]
\]  

where \(B\) is the \(12 \times 16\) matrix with the structure

\[
B := \begin{bmatrix} I & B' \end{bmatrix}
\]  

where \(I\) is the \(12 \times 12\) identity matrix and \(B'\) is the \(12 \times 4\) matrix

\[
B' = \begin{bmatrix}
-4 & -2 & -2 & -1 \\
-2 & -4 & -1 & -2 \\
-2 & -1 & -4 & -2 \\
-1 & -2 & -2 & -4 \\
2 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 \\
2 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 \\
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]  

Using \(ij \in X\) to denote an index for \(\beta_{ij}\) and \(\ell m \in V \cup E\) to denote an index for \(\xi_{\ell m}^{[0,1]}\), the entries of \(B\) can be denoted by \(b_{ij}^{\ell m}\) so that

\[
B := \begin{bmatrix}
b_{11}^{11} & \ldots & b_{ij}^{11} & \ldots & b_{33}^{11} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
b_{11}^{\ell m} & \ldots & b_{ij}^{\ell m} & \ldots & b_{33}^{\ell m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
b_{11}^{34} & \ldots & b_{ij}^{34} & \ldots & b_{33}^{34}
\end{bmatrix}
\]  

We now observe that for each \(ij \in X\),

\[
\begin{bmatrix} 3 - r \\ 4 - i \end{bmatrix} \begin{bmatrix} 3 - s \\ 4 - j \end{bmatrix} = \sum_{\ell m \in V \cup E} \begin{bmatrix} 3 - r \\ 4 - \ell \end{bmatrix} \begin{bmatrix} 3 - s \\ 4 - m \end{bmatrix} b_{ij}^{\ell m},
\]  

for all \((r, s)\) pairs such that \(\text{sldeg}(x^r y^s) \leq 3\) (recall Definition 2.1). Note that this claim holds trivially for the first 12 columns of \(B\), i.e. for those \(ij \in V \cup E \subset X\). For \(ij \in D \subset X\), (3.11) defines an invertible linear system of 12 equations with 12 unknowns whose solution is the \(ij\) column of \(B'\); the 12 \((r, s)\) pairs correspond to the exponents of \(x\) and \(y\) in the basis ordering of \(S_3(I^2)\) given in (2.2)-(2.3). Substituting (3.11) into (2.13) yields:

\[
x^r y^s = \sum_{ij \in X} \left( \sum_{\ell m \in V \cup E} \begin{bmatrix} 3 - r \\ 4 - \ell \end{bmatrix} \begin{bmatrix} 3 - s \\ 4 - m \end{bmatrix} b_{ij}^{\ell m} \right) \beta_{ij}
\]
Swapping the order of summation and regrouping yields
\[
x^r y^s = \sum_{\ell m \in V \cup E} \left( \begin{array}{c} 3 - r \\ 4 - \ell \\ 4 - m \end{array} \right) \left( \sum_{ij \in X} b_{ij}^{\ell m} \beta_{ij} \right).
\]
The inner summation is exactly \( \xi_{\ell m}^{[0,1]} \) by (3.7), implying that
\[
x^r y^s = \sum_{\ell m \in V \cup E} \left( \begin{array}{c} 3 - r \\ 4 - \ell \\ 4 - m \end{array} \right) \xi_{\ell m}^{[0,1]},
\]
for all \((r, s)\) pairs with \( \text{sddeg}(x^r y^s) \leq 3 \). Since \([\xi^2]^{[0,1]}\) has 12 elements which span the 12 dimensional space \( \mathcal{S}_3([0, 1]^2) \), it is a basis for \( \mathcal{S}_3([0, 1]^2) \). By scaling, \([\xi^2]\) is a basis for \( \mathcal{S}_3(I^2) \).

For (ii), note that an edge of \([0, 1]^2\) is described by an equation of the form \( \{ x \text{ or } y \} = \{0 \text{ or } 1\} \). Since \( \beta_2(t) \) and \( \beta_3(t) \) are equal to 0 at \( t = 0 \) and \( t = 1 \), \( \beta_{ij} \equiv 0 \) on the edges of \([0, 1]^2\) for any \( ij \in D \). By the structure of \( \mathcal{B} \) from (3.8), we see that for any \( \ell m \in V \cup E \),
\[
\xi_{\ell m}^{[0,1]} = \beta_{\ell m} + \sum_{ij \in D} b_{ij}^{\ell m} \beta_{ij}.
\]
Thus, on the edges of \([0, 1]^2\), \( \xi_{\ell m}^{[0,1]} \) and \( \beta_{\ell m} \) are identical. After scaling back, we have \( \xi_{\ell m} \) and \( \beta_{\ell m} \) identical on the edges of \( I^2 \), as desired.

For (iii), the geometric decomposition is given by the indices of the basis functions, i.e. the function \( \xi_{\ell m} \) is associated to the domain point for \( \ell m \in V \cup E \). This follows immediately from (ii), the fact that \([\beta^2]\) is a tensor product basis, and Remark 2.5.

\[ \square \]

**Remark 3.2.** It is worth noting that the basis \([\xi^2]\) was derived by essentially the reverse order of the proof of part (i) of the theorem. More precisely, the twelve coefficients in each column of \( \mathcal{B} \) define an invertible linear system given by (3.11). After solving for the coefficients, we can immediately derive the basis functions via (3.7). By the nature of this approach, the edge agreement property (ii) is guaranteed by the symmetry properties of the basis \([\beta]\). This technique was inspired by a previous work for Lagrange-like quadratic serendipity elements on convex polygons [13].

### 3.2. A local Hermite style basis for \( \mathcal{S}_3(I^2) \)

We now establish a local Hermite style basis \([\vartheta^2]\) for \( \mathcal{S}_3(I^2) \) using the bicubic Hermite basis \([\psi^2]\) for \( \mathcal{Q}_3([0, 1]^2) \). Define the following set of 12 functions, indexed by \( V \cup E \); note that each function is to be scaled by a factor of \( 1/8 \).

\[
[\vartheta^2] = \begin{bmatrix}
\vartheta_{11} & \vartheta_{14} & \vartheta_{11} & \vartheta_{14} & \vartheta_{12} & \vartheta_{13} & \vartheta_{12} & \vartheta_{13} & \vartheta_{21} & \vartheta_{24} & \vartheta_{21} & \vartheta_{24} \\
\vartheta_{41} & \vartheta_{44} & \vartheta_{41} & \vartheta_{44} & \vartheta_{42} & \vartheta_{43} & \vartheta_{42} & \vartheta_{43} & \vartheta_{21} & \vartheta_{24} & \vartheta_{21} & \vartheta_{24}
\end{bmatrix}
\begin{bmatrix}
-(x-1)(y-1)(-2+x+x^2+y+y^2) \\
(x-1)(y+1)(-2+x+x^2-y+y^2) \\
(x+1)(y-1)(-2-x+x^2+y+y^2) \\
-(x+1)(y+1)(-2-x+x^2-y+y^2) \\
-(x-1)(y-1)^2(y+1) \\
(x-1)(y+1)(y+1)^2 \\
(x+1)(y-1)^2(y+1) \\
-(x+1)(y-1)(y+1)^2 \\
-(x-1)^2(x+1)(y-1) \\
(x-1)(x+1)^2(y-1) \\
-(x-1)^2(x+1)(y+1)
\end{bmatrix}
\cdot \frac{1}{8} \tag{3.14}
\]
Fix the basis orderings
\[ [\vartheta^2] := \left[ \vartheta_{11}, \vartheta_{14}, \vartheta_{41}, \vartheta_{44}, \vartheta_{12}, \vartheta_{13}, \vartheta_{42}, \vartheta_{43}, \vartheta_{21}, \vartheta_{31}, \vartheta_{24}, \vartheta_{34} \right], \tag{3.15} \]
\[ [\psi^2] := \left[ \psi_{11}, \psi_{14}, \psi_{41}, \psi_{44}, \psi_{12}, \psi_{13}, \psi_{42}, \psi_{43}, \psi_{21}, \psi_{31}, \psi_{24}, \psi_{34}, \psi_{22}, \psi_{23}, \psi_{32}, \psi_{33} \right], \tag{3.16} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{hermite_functions.png}
\caption{The top row shows 3 of the 16 bicubic Hermite functions on \( I^2 \) while the bottom row shows 3 of the 12 cubic Hermite style serendipity functions. The visual differences are subtle, although some changes in concavity can be observed. Note that functions in the same column have the same values on the edges of \( I^2 \).}
\end{figure}

**Theorem 3.3.** Let \( \psi_{\ell m}^I \) denote the derivative-preserving scaling of \( \psi_{\ell m} \) to \( I^2 \), i.e.
\[ \psi_{\ell m}^I := \psi_{\ell}((x+1)/2)\psi_{m}((y+1)/2), \quad \ell m \in V, \]
\[ \psi_{\ell m}^I := 2\psi_{\ell}((x+1)/2)\psi_{m}((y+1)/2), \quad \ell m \in E. \]

The set \([\vartheta^2]\) has the following properties:

(i) \([\vartheta^2]\) is a basis for \( S_3(I^2) \).

(ii) For any \( \ell m \in V \cup E \), \( \xi_{\ell m} \) is identical to \( \psi_{\ell m}^I \) on the edges of \( I^2 \).

(iii) \([\vartheta^2]\) is a geometric decomposition of \( S_3(I^2) \).

**Proof.** The proof follows that of Theorem 3.1 so we abbreviate proof details that are similar. For (i), let \([\vartheta^2]^{[0,1]}\) denote the derivative-preserving scaling of \([\vartheta^2]\) to \([0,1]^2\); the scale factor is \(1/2\) for functions with indices in \( E \). Given the basis orderings in (3.15)-(3.16), we have
\[ [\vartheta^2]^{[0,1]} = H[\psi^2] \tag{3.17} \]
where \( H \) is the \( 12 \times 16 \) matrix with the structure
\[ H := \left[ \begin{array}{c|c} I & H' \end{array} \right], \tag{3.18} \]
where $I$ is the $12 \times 12$ identity matrix and $ \mathbb{H}' $ is the $12 \times 4$ matrix

$$
\mathbb{H}' = \begin{bmatrix}
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
\end{bmatrix}.
$$

(3.19)

Denote the entries of $H$ by $h_{\ell m}^{ij}$ (cf. (3.10)). Recalling (2.15), observe that for each $ij \in X$,

$$
\varepsilon_{r,i} \varepsilon_{s,j} = \sum_{\ell m \in V \cup E} \varepsilon_{r,i} \varepsilon_{s,j} h_{ij}^{\ell m},
$$

(3.20)

for all $(r, s)$ pairs such that $\text{sldeg}(x^r y^s) \leq 3$. Similar to the Bernstein case, we substitute (3.20) into (2.17), swap the order of summation and regroup, yielding

$$
x^r y^s = \sum_{\ell m \in V \cup E} \varepsilon_{r,\ell} \varepsilon_{s,m} \left( \sum_{ij \in X} h_{ij}^{\ell m} \psi_{ij} \right).
$$

The inner summation is exactly $\vartheta_{\ell m}^{[0,1]}$ by (3.17), implying that

$$
x^r y^s = \sum_{\ell m \in V \cup E} \varepsilon_{r,\ell} \varepsilon_{s,m} \vartheta_{\ell m}^{[0,1]},
$$

(3.21)

for all $(r, s)$ pairs with $\text{sldeg}(x^r y^s) \leq 3$, proving that $[\vartheta^{[2]}_{\ell m}^{[0,1]}]$ is a basis for $S_3([0, 1]^2)$. By derivative-preserving scaling, $[\vartheta^{[2]}_{\ell m}^{[0,1]}]$ is a basis for $S_3(I^2)$.

For (ii), observe that for any $ij \in D$, $\psi_{ij} \equiv 0$ on the edges of $[0, 1]^2$ by virtue of the bicubic Hermite basis functions’ definition. By the structure of $\mathbb{H}$ from (3.18), we see that for any $\ell m \in V \cup E$,

$$
\vartheta_{\ell m}^{[0,1]} = \psi_{\ell m} + \sum_{ij \in D} h_{ij}^{\ell m} \psi_{ij}.
$$

(3.22)

Thus, on the edges of $[0, 1]^2$, $\vartheta_{\ell m}^{[0,1]}$ and $\psi_{\ell m}$ are identical. After scaling back, we have $\vartheta_{\ell m}$ and $\psi_{\ell m}$ identical on the edges of $I^2$, as desired.

For (iii), the geometric decomposition is given by the indices of the basis functions, i.e. the function $\vartheta_{\ell m}$ is associated to the domain point for $\ell m \in V \cup E$. This follows immediately from (ii), the fact that $[\psi^{[2]}_{\ell m}]$ is a tensor product basis, and Remark 2.5. $\square$

4. Local Bases for $S_3(I^3)$

Before defining local bases on the cube, we fix notation for the domain points to which they are associated. For $[0, 1]^3$, define the ordered triple index set

$$
Y := \{ \{i, j, k\} \mid i, j, k \in \{1, \ldots, 4\} \}.
$$
Then $Y$ is the disjoint union $V \cup E \cup F \cup M$ where

- $V := \{\{i, j, k\} | i, j, k \in \{1, 4\}\}$; \hspace{1cm} (4.1)
- $E := \{\{i, j, k\} | \text{exactly two of } \{i, j, k\} \in \{1, 4\}\}$; \hspace{1cm} (4.2)
- $F := \{\{i, j, k\} | \text{exactly one of } \{i, j, k\} \in \{1, 4\}\}$; \hspace{1cm} (4.3)
- $M := \{\{i, j, k\} | i, j, k \in \{2, 3\}\}$ \hspace{1cm} (4.4)

The $V$ indices are associated with vertices of $[0, 1]^3$, the $E$ indices to edges of $[0, 1]^3$, the $F$ indices to face interior points of $[0, 1]^3$, and the $M$ vertices to the domain interior of $[0, 1]^3$. The relation between indices and domain points of the cube is shown in Figure 6.

4.1. **A local Bernstein style basis for $S_3(I^3)$**. Under the notation and conventions established in Section 2, we are ready to establish a local Bernstein style basis for $S_3(I^3)$ where $I := [-1, 1]$. Define the following set of 32 functions, indexed by $V \cup E \subset Y$;
note that each function is to be scaled by a factor of 1/32.

\[
\begin{pmatrix}
\xi_{111} & \xi_{112} & \xi_{113} & \xi_{121} & \xi_{124} & \xi_{134} & \xi_{142} & \xi_{143} & \xi_{211} & \xi_{214} & \xi_{241} & \xi_{244} & \xi_{311} & \xi_{314} & \xi_{341} & \xi_{344} & \xi_{412} & \xi_{413} & \xi_{421} & \xi_{424} & \xi_{431} & \xi_{434} & \xi_{442} & \xi_{443}
\end{pmatrix} = \begin{pmatrix}
-(x - 1)(y - 1)(z - 1)(-5 - 2x + x^2 - 2y + y^2 - 2z + z^2) \\
(x - 1)(y - 1)(z + 1)(-5 - 2x + x^2 - 2y + y^2 + 2z + z^2) \\
(x - 1)(y + 1)(z - 1)(-5 - 2x + x^2 + 2y + y^2 - 2z + z^2) \\
-(x - 1)(y + 1)(z + 1)(-5 - 2x + x^2 + 2y + y^2 + 2z + z^2) \\
(x + 1)(y - 1)(z - 1)(-5 + 2x + x^2 - 2y + y^2 - 2z + z^2) \\
-(x + 1)(y - 1)(z + 1)(-5 + 2x + x^2 - 2y + y^2 + 2z + z^2) \\
(x + 1)(y + 1)(z - 1)(-5 + 2x + x^2 + 2y + y^2 - 2z + z^2) \\
(x + 1)(y + 1)(z + 1)(-5 + 2x + x^2 + 2y + y^2 + 2z + z^2)
\end{pmatrix} \cdot \frac{1}{32}
\]  

(4.5)

We fix the following basis orderings, with omitted basis functions ordered lexicographically by index.

\[
[\xi^3] := \begin{pmatrix} \xi_{111}, \ldots, \xi_{444}, \xi_{112}, \ldots, \xi_{443} \end{pmatrix},
\]

(4.6)

\[
[\beta] := \begin{pmatrix} \beta_{111}, \ldots, \beta_{444}, \beta_{112}, \ldots, \beta_{443}, \beta_{122}, \ldots, \beta_{433}, \beta_{222}, \ldots, \beta_{333} \end{pmatrix}
\]

(4.7)

**Theorem 4.1.** Let \( \beta_{\ell mn}^I \) denote the scaling of \( \beta_{\ell mn} \) to \( I^3 \), i.e.

\[
\beta_{\ell mn}^I := \beta_{\ell}(x + 1)/2)\beta_m((y + 1)/2)\beta_n((z + 1)/2).
\]

The set \([\xi^3]\) has the following properties:

(i) \([\xi^3]\) is a basis for \( S_3(I^3) \).

(ii) \([\xi^3]\) reduces to \([\xi^2]\) on faces of \( I^3 \).

(iii) For any \( \ell mn \in V \cup E \), \( \xi_{\ell mn} \) is identical to \( \beta_{\ell mn}^I \) on edges of \( I^3 \).
(iv) \([\xi^3]\) is a geometric decomposition of \(S_3(I^3)\).

**Proof.** The proof is similar to that of Theorem 3.1 so we abbreviate some details. For (i), let \([\xi^3]^{[0,1]}\) denote the scaling of \([\xi^3]\) to \([0,1]^3\). Given the basis orderings in (4.6)-(4.7), we have

\[
[\xi^3]^{[0,1]} = U[\beta^3]
\]

(4.8)

where \(U\) is the \(32 \times 64\) matrix with the structure

\[
U := \left[ \begin{array}{c|c} I & U' \end{array} \right],
\]

(4.9)

where \(I\) is the \(32 \times 32\) identity matrix and \(U'\) is a specific \(32 \times 32\) matrix whose entries are integers ranging from -16 to 4. Instead of writing out \(U'\) in its entirety, we describe its properties and how it can be constructed (cf. Remark 3.2).

Using \(ijk \in Y\) to denote an index for \(\beta_{ijk}\) and \(\ell mn \in V \cup E \subset Y\) to denote an index for \(\xi_{\ell mn}^{[0,1]}\), the entries of \(U\) will be denoted by \(k_{ijk}^{\ell mn}\) so that

\[
U := \begin{bmatrix}
  u^{111}_{111} & \cdots & u^{111}_{ijk} & \cdots & u^{111}_{333} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  u^{\ell mn}_{111} & \cdots & u^{\ell mn}_{ijk} & \cdots & u^{\ell mn}_{333} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  u^{443}_{111} & \cdots & u^{443}_{ijk} & \cdots & u^{443}_{333}
\end{bmatrix}.
\]

(4.10)

The columns of \(U\) satisfy the relationship

\[
\begin{pmatrix}
  3 - r \\
  4 - i
\end{pmatrix}
\begin{pmatrix}
  3 - s \\
  4 - j
\end{pmatrix}
\begin{pmatrix}
  3 - t \\
  4 - k
\end{pmatrix} = \sum_{\ell mn \in V \cup E} \begin{pmatrix}
  3 - r \\
  4 - \ell
\end{pmatrix}
\begin{pmatrix}
  3 - s \\
  4 - m
\end{pmatrix}
\begin{pmatrix}
  3 - t \\
  4 - n
\end{pmatrix} u^{\ell mn}_{ijk},
\]

(4.11)

for all \((r, s, t)\) tuples such that \(\text{sldeg}(x^ry^sz^t) \leq 3\). This property defines the entries of \(U\) uniquely since for each \(ijk \in Y\) it gives an invertible linear system of 32 equations with 32 unknowns whose solution is the \(ijk\) column of \(U\). The \((r, s, t)\) tuples should be taken in the order given in (2.5)-(2.6). See Remark 3.2 and the text after (3.11) for more on this process.

As in previous proofs, regrouping and recognizing an expression for \(\xi_{\ell mn}^{[0,1]}\) gives

\[
x^ry^sz^t = \sum_{\ell mn \in V \cup E} \begin{pmatrix}
  3 - r \\
  4 - \ell
\end{pmatrix}
\begin{pmatrix}
  3 - s \\
  4 - m
\end{pmatrix}
\begin{pmatrix}
  3 - t \\
  4 - n
\end{pmatrix} \xi_{\ell mn},
\]

(4.12)

proving, after scaling, that \([\xi^3]\) is a basis for \(S_3(I^3)\).

For (ii), the claim can be confirmed directly by calculation, e.g. \(\xi_{111}(x, y, -1) = \xi_{112}(x, 1, z) = \xi_{122}(x, z, z)\), etc.

For (iii), note that an edge of \([0,1]^3\) is described by two equations of the form \(\{ x, y, \text{ or } z \} = \{0 \text{ or } 1\}\) where two distinct variables must be chosen for the two equations. Since \(\beta_2(t)\) and \(\beta_3(t)\) are equal to 0 at \(t = 0\) and \(t = 1\), \(\beta_{ijk} \equiv 0\) on the edges of \([0,1]^2\) for any \(ijk \in M\). Further, for \(ijk \in F\), without loss of generality, assume that \(i \in \{1, 4\}\) so that \(j, k \in \{2, 3\}\). Since every edge is described by at least one equation of the form \(\{ y \text{ or } z \} = \{0 \text{ or } 1\}\), either \(\beta_j(y)\) or \(\beta_k(z)\) is identically zero on every edge. Thus, for \(ijk \in F \cup M\), \(\beta_{ijk} \equiv 0\) on the edges of \([0,1]^3\).

By the structure of \(U\) from (4.9), we see that for any \(\ell mn \in V \cup E\),

\[
\xi_{\ell mn}^{[0,1]} = \beta_{\ell mn} + \sum_{ijk \in F \cup M} u^{\ell mn}_{ijk} \beta_{ijk}.
\]

(4.13)
Thus, on the edges of $[0,1]^3$, $\xi_{\ell mn}$ and $\beta_{\ell mn}$ are identical. After scaling back, we have $\xi_{\ell mn}$ and $\beta_{\ell mn}$ identical on the edges of $I^3$, as desired.

For (iv), the geometric decomposition is given by the indices of the basis functions, i.e. the function $\xi_{\ell mn}$ is associated to the domain point for $\ell mn \in V \cup E$. This follows immediately from (ii) and (iii), the fact that $[\varphi^3]$ is a tensor product basis, and Remark 2.5.

4.2. A local Hermite style basis for $S_3(I^3)$. We now establish a local Hermite style basis $[\varphi^3]$ for $S_3(I^3)$ using the tricubic Hermite basis $[\psi^3]$ for $Q_3([0,1]^3)$. Define the following set of 32 functions, indexed by $V \cup E$; note that each function is to be scaled by a factor of $1/16$.

\[
[\varphi^3] = \begin{bmatrix}
\varphi_{111} \\
\varphi_{114} \\
\varphi_{141} \\
\varphi_{144} \\
\varphi_{411} \\
\varphi_{414} \\
\varphi_{441} \\
\varphi_{444}
\end{bmatrix} = \begin{bmatrix}
(x - 1)(y - 1)(z - 1)(-2 + x + x^2 + y + y^2 + z + z^2) \\
-(x - 1)(y - 1)(z - 1)(-2 + x + x^2 + y + y^2 - z + z^2) \\
(x - 1)(y + 1)(z - 1)(-2 + x + x^2 - y + y^2 + z + z^2) \\
-(x - 1)(y - 1)(z - 1)(-2 - x + x^2 + y + y^2 + z + z^2) \\
(x + 1)(y - 1)(z + 1)(-2 + x + x^2 - y + y^2 + z + z^2) \\
-(x + 1)(y + 1)(z - 1)(-2 - x + x^2 - y + y^2 + z + z^2)
\end{bmatrix} \cdot \frac{1}{16}
\]
We fix the following basis orderings, with omitted basis functions ordered lexicographically by index.

\[
[\vartheta^3] := \left\{ \vartheta_{111}, \ldots, \vartheta_{444}, \vartheta_{112}, \ldots, \vartheta_{443} \right\},
\]

\[
\beta := \left\{ \psi_{111}, \ldots, \psi_{444}, \psi_{112}, \ldots, \psi_{443}, \psi_{122}, \ldots, \psi_{433}, \psi_{222}, \ldots, \psi_{333} \right\}
\]

**Theorem 4.2.** Let \( \psi^I_{\ell mn} \) denote the derivative-preserving scaling of \( \psi_{\ell mn} \) to \( I^3 \), i.e.
\[
\psi^I_{\ell mn} := \psi_\ell ((x + 1)/2) \psi_m ((y + 1)/2) \psi_n ((z + 1)/2), \quad \ell mn \in V;
\]
\[
\psi^I_{\ell mn} := 2\psi_\ell ((x + 1)/2) \psi_m ((y + 1)/2) \psi_n ((z + 1)/2), \quad \ell mn \in E.
\]

The set \([\vartheta^3]\) has the following properties:

(i) \([\vartheta^3]\) is a basis for \( S_3(I^3) \).

(ii) \([\vartheta^3]\) reduces to \([\vartheta^2]\) on faces of \( I^3 \).

(iii) For any \( \ell mn \in V \cup E \), \( \vartheta_{\ell mn} \) is identical to \( \psi^I_{\ell mn} \) on edges of \( I^3 \).

(iv) \([\vartheta^3]\) is a geometric decomposition of \( S_3(I^3) \).

**Proof.** The proof is similar to that of Theorem 3.3 so we abbreviate some details. For (i), let \([\vartheta^3]^{0,1}\) denote the scaling of \([\vartheta^3]\) to \([0, 1]^3\). Given the basis orderings in (4.15)-(4.16), we have
\[
[\vartheta^3]^{0,1} = W[\vartheta^3]
\]
where \( W \) is the \( 32 \times 64 \) matrix with the structure
\[
W := \begin{bmatrix} \mathbb{I} & W' \end{bmatrix},
\]
where \( \mathbb{I} \) is the \( 32 \times 32 \) identity matrix and \( W' \) is a specific \( 32 \times 32 \) matrix whose entries are in \( \{-1, 0, 1\} \). The matrix \( W \) is constructed similarly to the matrix \( U \) from the proof of Theorem 4.1; the columns of \( W \) satisfy the relationship
\[
\varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} = \sum_{\ell mn \in V \cup E} \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} W_{ijk}^{\ell mn},
\]
for all \((r, s, t)\) tuples such that \( \text{sldeg}(x^r y^s z^t) \leq 3 \). Similar to previous proofs, this yields
\[
x^r y^s z^t = \sum_{\ell mn \in V \cup E} \varepsilon_{r,\ell} \varepsilon_{s,\ell} \varepsilon_{t,\ell} \vartheta_{\ell mn},
\]
proving, after derivative-preserving scaling, that \([\vartheta^3]\) is a basis for \( S_3(I^3) \).

For (ii)-(iv), the proof is similar to the corresponding parts of the proof of Theorem 4.1.

\[\square\]

5. CONCLUSIONS AND FUTURE DIRECTIONS

The basis functions presented in this work are well-suited for use in both geometric modeling and finite element applications, as discussed in the introduction. Moreover, the proof techniques used for the theorems suggest a number of promising extensions. Similar techniques should be able to produce Bernstein style bases for higher polynomial order serendipity spaces, although the introduction of interior degrees of freedom that occurs when \( r > 3 \) requires some additional care to resolve. Some higher order Hermite style bases may also be available, although the association of directional derivative values to vertices is somewhat unique to the \( r = 3 \) case. Pre-conditioners for finite element methods employing our bases are still needed. The fact that all the functions we defined are fixed linear combinations of standard bicubic or tricubic basis functions suggests that such pre-conditioners will have a straightforward construction.
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