Quantized lobe level theoretical estimates for phased array antennas applicability

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Abstract. This paper raises the question of the applicability of some theoretical estimates of the quantization lobe level. Existing assessment methods and their features are considered. Based on numerical simulated method, statistical characteristics are constructed for the magnitude of the level of quantization lobes and, according to the proposed criterion, a conclusion is given on the consistency of theoretical estimates.

1. Introduction
To date, one of the areas attracting the attention of researchers and developers of airborne radar systems is the use of phased antenna arrays. Due to the ability of the antenna array to change the phase relationships between the currents in the emitters, you can control the direction of the beam in a given sector of angles, which contributes to solving the tasks set for the onboard radar system, namely: target detection, angular coordinates determining, velocities determining, moving objects tracking, and objects classifying.

The need to place such antennas on the aircraft board imposes a number of restrictions, one of which is the overall mass characteristics. The desire to minimize the size of the antenna array at the same time, while maintaining the ability to scan, has led to widespread use of discrete-switching phase shifters as phase control devices. These phase shifters are a set of \( n \) fixed delay lines with values \( 2\pi/2, 2\pi/2^2 \ldots \ 2\pi/n \), \( n \) is called the digitizer phase capacity. The ability to implement phases only multiples of the discrete \( \Delta \varphi \) leads to errors in the phase distribution. This adversely affects the parameters of the pattern. There is a decrease in the directional coefficient, the increase in the level of side lobes, as well as the inaccuracy of the installation of the beam in space.

Studies on the effect of phase distribution errors on the characteristics of the antenna array are presented in [1–6]. One may notice that most of the previous works treated the effect of random errors on the average sidelobe level. In a modern radar system, the peak side lobe level (SLL), however, is more important. Operation requirements are usually specified in terms of peak sidelobe level, in particular for a low sidelobe system. The reason is that for jamming rejection, it is most important that a jammer noise comes from a spatial location at the peak of a sidelobe and this noise must not interfere with the radar detection. Average sidelobe level in this case does not help. An array may have a very low average sidelobe level, but at the same time, it may have a few high peak sidelobes located at the wrong spot, which renders the radar system vulnerable.

2. The theoretical estimates of quantization lobe level
A linear array far field and a normalized far field can be written as follows:
AF(θ) = \sum_{n=1}^{N} A_n e^{ikd(n-1)\cos(θ)} e^{i\Delta \varphi_n}

(1)

where \( i \) is the imaginary unit, \( \theta \) the angle measured from the direction of the array, \( k \) is the wave number \( (k = 2\pi/\lambda) \), \( N \) is the elements number, \( A_n \) is \( n^{th} \) element excitation current, \( \Delta \varphi_n \) is the phase error at the \( n^{th} \) element and \( d \) is the interelement spacing.

In order to steer the beam to a given direction, we must set up the phases computed according to the steering direction in the phase shifters. Although the array is required to produce a smooth phase taper, an \( N \)-bit phase shifters has phase states separated by the least significant bit: \( \Delta \varphi = 2\pi / 2^n \). These can only approximate to the desired phase distribution across the array, and if a straightforward rounding off is used, a periodic phase error arises which causes a pointing deviation of the main beam and introduces a quantization lobe.

Figure 1 shows that this quantization allows only a staircase approximation of the continuous progressive shift required for the array. The staircase phase front results in a periodic triangular phase error that produces the pattern with grating lobe-like sidelobes.

![Figure 1. Phase error due to phase quantization: 1-actual phase shift 2-desired phase shift 3-phase error.](image)

The phase errors are

\[ \Delta \varphi(x) = \begin{cases} 
\Delta \varphi(1 - \frac{(x-x_m)d}{l}) & \text{with probability } p(x) = \frac{1}{1+p(x)} \\
\Delta \varphi(x-x_m) & \text{with probability } q(x) = 1 - p(x) 
\end{cases} \]

where \( x_m = (m-1)l/d \) is the coordinate of the start point of the \( m^{th} \) period; \( l = Md \) is the length of a period; \( M \) is number of element in subarrays. As the variation of the current amplitude in a period is not large, we can approximately obtain

\[ AF(u) = f_1(u) \cdot f_2(u) \]

where

\[ u = kd(\cos \theta - \cos \theta_0) \]

\[ f_1(u) = \frac{1}{d} \sum_{n=1}^{N} A_n e^{i(kd-1)u} \]

\[ f_2(u) = \frac{d}{l} \int_0^l \left( p(x)e^{i\frac{\pi dx}{d}} + q(x)e^{i\frac{\pi dx}{d}} \right) e^{iu} dx \]

Using the above formulas to deduce the size of the quantization lobe for the rounding-off method, using the "probability" distribution:

\[ p(t) = \begin{cases} 
1, & t \geq 0 \\
0, & t < 0 
\end{cases} \]
With this approximation, the first quantization lobe level [4] is given as
\[ L_1 \approx 20 \log \left( \frac{\delta}{\pi - \delta} \right) = 20 \log \left( \frac{\pi}{2 \pi} \left( \frac{\pi}{2 \pi} \right)^{-1} \right) = -6n \]

To obtain an estimate of the average quantization lobes level due to a single quantization error is decomposed in equation (1) by the factor characterizing the phase error in the Maclaurin series, representing the sum in the form of the scalar product of two N-dimensional vectors.

As a result, the value of the level of switching lobes is written as
\[ L_2 = \frac{1}{\sqrt{2}} \beta_{\text{mean}} \]
where \( \beta_{\text{mean}} \) is the RMS value of phase errors.

The mean square of the systematic phase error can be found by taking the integral of the square of the phase error magnitude over the entire length of the antenna. According to Figure 1 we have
\[ \delta \varphi_{\text{rms}} = \frac{1}{2} \left( \delta \varphi \chi \right) \]

Then
\[ \left( \delta \varphi_{\text{rms}} \right)^2 = \frac{\pi^2}{3.2^{2n}} \]

Let the value of the random phase error introduced by the phase shifter be small compared with the value of the systematic phase error.
\[ \beta_{\text{mean}} = \delta \varphi_{\text{rms}} \]

Then
\[ L_2 = \left( \frac{\pi^2}{3.2^{2n+1}} \right)^{\frac{1}{2}} \]

Miller [7] published analysis of the adverse effects of using phase shifters with discrete phase states for large phase arrays with uniformly illuminated contiguous subarrays. The general expression for power at the peak of the quantization lobe [3] is written as:
\[ L_3 \approx -20 \log(M) + 9.94 - 6.02N \]

Theoretical estimates calculated for the antenna array with 3, 4, 5-bit phase shifters are summarized in table 1.

| Phase shifter bitness | Values of theoretical estimates |
|-----------------------|---------------------------------|
|                       | \( L_1 \) | \( L_2 \) | \( L_3 \) |
| 3                     | -18      | -15.9    | -14.2    |
| 4                     | -24      | -21.92   | -20.1    |
| 5                     | -30      | 27.94    | -26.2    |

3. Numerical simulated method and statistical approach for estimate quantization lobe level
Errors due to the discreteness of the phase shifter are systematic, and the level of quantization lobes for a particular spatial direction of radiation is determined (by the function of the initial phase distribution). We will consider the entire set of the level values of the switching lobe for all possible radiation directions of the main maximum. Then we can talk about the frequency of occurrence of the quantization lobe level in the scanning sector and thereby assigning the probability of its occurrence. Formulating the problem in this way, we use the apparatus of mathematical statistics. Based on the results obtained by the method of numerical simulations of a 32 element array antenna, the phasing of which is carried out by 3, 4, and 5-bit phase shifters, the corresponding statistical series of the realized quantization lobe
levels are obtained. Some radiation patterns obtained by numerical simulation are presented on Figures 2–7. Phase shifts between radiating elements are set in the rounding-off method and are calculated by the following formula:

$$\Delta \phi_n = \frac{\phi_{n, \theta}}{\Delta \phi} + 0.5$$

where $\phi_{n, \theta} = n \cdot \frac{2\pi d}{\lambda} \cdot \cos(\theta_0)$ is required phase shift on the $n^{th}$ element to form a beam in the direction of $\theta_0$; $[a]$ the integer part of $a$.

**Figure 2.** Radiation patterns for scan angles $\theta_0 = 1^\circ$ (a) for a 3-bit phase shifter.

**Figure 3.** Radiation patterns for scan angles $\theta_0 = 1^\circ$ for a 4-bit phase shifter.

**Figure 4.** Radiation patterns for scan angles $\theta_0 = 1^\circ$ for a 5-bit phase shifter.

**Figure 5.** Radiation patterns for scan angles $\theta_0 = 5^\circ$ for a 3-bit phase shifter.

**Figure 6.** Radiation patterns for scan angles $\theta_0 = 5^\circ$ for a 4-bit phase shifter.

**Figure 7.** Radiation patterns for scan angles $\theta_0 = 5^\circ$ for a 5-bit phase shifter.

Since the type of statistical distribution of the quantization lobe level is not known in advance, the Pearson curves, based on solving a differential equation expressing the general properties of the distribution function, was used to approximate the probability distribution density. To estimate the distribution parameters, the method of moments was used, the essence of which is to calculate the required number of distribution parameters from experimental data and equate them to theoretical values. As is known, the Pearson curve is uniquely determined by its first four moments given in Table 2. Based on their values, the types of Pearson curves are determined that are suitable for specific distributions obtained by the method of numerical simulations and the approximated probability density functions of the distribution of Figures 8–10 are constructed. Probability distribution density satisfies the chi-square criterion with significance level $\alpha = 0.05$. 

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Figure 8. The normalized probability distribution density for a 3-bit phase shifter.

Figure 9. The normalized probability distribution density for a 4-bit phase shifter.

Figure 10. The normalized probability distribution density for 5-bit phase shifter.

Table 2. Values of statistical moments.

| Phase shifter bitness | $m_1$  | $\mu_2$ | $\mu_3$  | $\mu_4$  |
|-----------------------|--------|---------|----------|----------|
| 3                     | -16.83 | 4.47    | -39.6    | 499.47   |
| 4                     | -21.82 | 3.56    | -21.68   | 192.99   |
| 5                     | -25.59 | 1.96    | -4.29    | 25.17    |

All probability distribution densities were centered relative to the expectation and were normalized according to the condition:

$$P\left(-\infty < L < +\infty \right) = \int_{-\infty}^{+\infty} \rho(t)dt = 1$$

Since the method of numerical simulations allows to obtain the actual realizable values of the quantization lobe level, taking into account the accepted assumptions, it is possible to draw a conclusion on the reliability of the theoretical estimates from the obtained probability densities. To do this, we introduce the validity criterion: we will consider the $L_k$ estimate to be reliable if, with a probability of $\geq 90\%$, the quantization lobe level lies in the interval $(-\infty, L_k)$. Using the obtained probability distribution densities, we find the probability that the quantization lobe level does not exceed the theoretical estimate

$$F(L_k) = P\left(-\infty < L < L_k \right) = \int_{-\infty}^{L_k} \rho(t)dt$$

The probabilities calculated in equation (2) are given in the table 3.
Table 3. Probability quantization lobe level less than theoretical estimate.

| Phase shifter bitness | $F(L_1)$ | $F(L_2)$ | $F(L_3)$ |
|-----------------------|----------|----------|----------|
| 3                     | 0.55     | 0.78     | 0.94     |
| 4                     | 0.35     | 0.62     | 0.85     |
| 5                     | 0.01     | 0.06     | 0.23     |

$L_1$, $L_2$ are not satisfactory for estimating the peak value of the quantization lobe. Evaluation of $L_3$ satisfies the criterion in the case of application in the antenna array of a 3-bit phase shifter, in the later cases, $L_3$ does not give a value close to the peak quantization lobe level. Since the estimate of $L_3$ takes into account a certain general symmetry of errors, its value is closest to the realizable peak value of the quantization lobes (for $n = 3, 4$). In the case $n = 5$ where the influence of systematic errors increases, the consideration of the general symmetry of errors becomes insufficient.

Moreover, with an increase in the digitizer of the phase shifter, the probability tends to zero, which indicates a drop in the quality of the estimate and its insolvency. It is possible to draw conclusions about the insufficient consideration of only average phase errors over the lattice and about the predominant influence of their symmetries.

The proposed approach can be applied to assess the radiation patterns of the onboard radio frequency systems.

4. Conclusion
Various theoretical estimates of the level of quantization lobes are considered and analyzed. Based on statistical series, methods for calculating numerical symmetries are obtained using 3, 4, and 5-bit phase shifters. Based on the proposed reliability criterion, a conclusion was made on the applicability of theoretical estimates for antenna arrays as part of airborne radar systems.

References
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