Non-abelian Electrons

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We analyze critical and massive $SO(5)$ superspin regimes for correlated electrons on a two-chain ladder. We identify fundamental low energy excitations, which carry the quantum numbers of a free electron, and can be probed in (inverse) photo-emission experiments. These excitations do not obey the usual Pauli Principle, but are governed by specific forms of so-called non-abelian exclusion statistics.

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It is well-known that the quantum statistics of particles in one or two spatial dimensions are not necessarily bosonic or fermionic. In two spatial dimensions all that is needed is a representation of the braid group, and this allows for various alternative possibilities. A particularly intriguing possibility is that of so-called non-abelian braiding statistics, in which case the braiding of particles is represented by non-trivial matrices acting on multi-component wave functions. A prototype example for this is the so-called quasi-hole excitations over the pfaffian quantum Hall state [2].

In a recent paper [3], one of us investigated the exclusion statistics properties of the edge quasi-holes over the pfaffian quantum Hall state [4]. A key quantity is the thermodynamic distribution function $n_{qh}(\epsilon)$, which describes the expected occupation of specific one-quasi-hole states and which generalizes the Fermi-Dirac distribution for fermions. The edge quasi-hole distribution, which was computed in [3], has the following characteristic features:

$$n_{qh}(\epsilon) \sim 8 \quad \text{for } \epsilon \ll \mu$$

$$\sim \sqrt{2} e^{-\beta \epsilon} \quad \text{for } \epsilon \gg \mu$$

(1)

with $\beta = (k_B T)^{-1}$ and $\mu$ the chemical potential. The nontrivial prefactor $\sqrt{2}$ in the high-energy Boltzmann tail of the distribution $n_{qh}(\epsilon)$ is a direct manifestation of the non-abelian nature of the braiding statistics of the quasiholes over the pfaffian quantum Hall state [3].

One may demonstrate that, in general, non-abelian braiding statistics in a Conformal Field Theory (CFT) lead to non-trivial prefactors in the Boltzmann tails of the corresponding thermodynamic distribution functions [5]. We therefore propose the term 'non-abelian exclusion statistics' for all forms of exclusion statistics that exhibit such non-trivial prefactors. In the theoretical literature, a number of examples, other than the pfaffian, have been discussed. These include spinons in $S \geq 1$ critical spin chains [6], and generalized fermions in minimal models of CFT [2].

In this Letter we focus on situations where excitations with the quantum numbers of free electrons obey specific forms of non-abelian exclusion statistics. The concrete setting for this is the so-called $SO(5)$ superspin regime for correlated electrons on a two-chain ladder. The excitations, which we call 'non-abelian electrons', can be probed via (inverse) photo-emission and, in principle, the non-abelian nature of their exclusion statistics can be tested in experiments.

Before discussing the $SO(5)$ ladder models and the associated non-abelian electrons, we shall briefly discuss the exclusion statistics properties of the spinons and holons of free or weakly interacting electrons in one dimension. In a continuum description, free electrons in one dimension are described by a Conformal Field Theory (CFT), with the electron degrees of freedom represented by four real (Majorana) fermion fields

$$\left(\psi_{\uparrow,\epsilon}, \psi_{\downarrow,\epsilon}, \psi_{\uparrow,-\epsilon}, \psi_{\downarrow,-\epsilon}\right).$$

(2)

These fields transform as a vector under the $SO(4)$ symmetry. Through non-abelian bosonization, the free fermion CFT is equivalent to an $SO(4)$ Wess-Zumino-Witten theory with level $k = 1$ ($SO(4)_1$ WZW).

As soon as (weak, repulsive) interactions are introduced in a one-dimensional electron theory, a separation of spin and charge sets in and the fundamental excitations are found to be spinons (carrying spin $\frac{1}{2}$ and no charge) and holons (carrying charge $\pm e$ and no spin). An explicit example is the $U > 0$ Hubbard model, where one may use the exact Bethe Ansatz solution to demonstrate the fundamental role of the spinon and holon excitations [6]. The free electron CFT may equally well be described in terms of spinons and holons, which we write as

$$\left(\phi_{\uparrow}, \phi_{\downarrow}\right), \quad \left(\phi_\epsilon, \phi_{-\epsilon}\right).$$

(3)

Together, the spinon and holon fields form a spinor representation of the $SO(4)$ symmetry. In figure 1 we show in an $SO(4)$ weight diagram the quantum numbers of the electron, spinon and holon fields.

This brings us to the statistics of the spinons and holons in a free electron theory. That these statistics are unusual can be demonstrated by the following argument. We consider the capacitance of the free electron...
gas, i.e., the amount of charge $\Delta Q$ pulled from the Fermi sea by an applied voltage $V$. In the electron-operator picture, this response is generated by the 4 fermionic fields, eq. (3), of charge $\pm e$. In the spinon/holon picture, the response is generated by the holons alone, i.e., by 2 fields of charge $\pm e$. Assuming fermionic statistics for the holons thus leads to a mismatch by a factor of 2 between the two approaches. As it turns out, the actual, fractional, statistics of the holons are such that the response to a voltage indeed exhibits an enhancement by a factor of 2 w.r.t. free fermions (see below).

The appropriate framework for discussing the statistics of spinons and holons is the notion of ‘fractional exclusion statistics’, which was introduced by Haldane in [6] and elaborated on in [7]. The early applications of this new paradigm in many-body theory have hinged on the specifics of exactly solvable models of quantum mechanics with inverse square exchange. In an important step forward [8], it was demonstrated that a fractional statistics assignment can be done as soon as an effective CFT for low-energy behavior has been identified. The method introduced in [8] is based on recursion relations for truncated conformal spectra. In the mean time, a large number of examples of this ‘statistics from field theory’ approach have been worked out [9][10][11].

To obtain the exclusion statistics of spinons and holons, we rely on our recent results, presented in [3], for quasi-particle formulations of level-1 WZW models. For the case $SO(4)_1$, we obtained explicit results for the exclusion statistics of quasi-particles transforming in the spinor of $SO(4)$, which consists of the spinon and holon fields combined. The result may be phrased in terms of a quantity $\lambda(x^+_{s,c}, x^\pm_{c,c})$, which is the approximate partition sum for a single one-particle level in the spectrum. This partition sum depends on the one-particle energy $\epsilon$ and on chemical potentials $\mu^\pm_s, \mu^\pm_c$ for the various spinor components through the quantities $x^\pm_{s,c} = e^{\beta(\mu^\pm_{s,c} - \epsilon)}$. In [3] we showed that

$$\lambda(x^+_{s,c}, x^\pm_{c,c}) = \tilde{\lambda}(x^+_{s,c}, x^-_{c,c}) \tilde{\lambda}(x^+_c, x^-_c) ,$$  \hspace{1cm} (4)

with the $\tilde{\lambda}(x^+, x^-)$ given by the single-level partition sum $\mu(x, z)$ for spinons in the $SU(2)_1$ WZW model [8]:

$$\tilde{\lambda}(x, z) = 1 + (z^2 + z^{-2}) \frac{x^2}{2} + (z + z^{-1}) \frac{x}{2} \sqrt{(z - z^{-1})^2 x^2 + 4} .$$  \hspace{1cm} (5)

The parameter $x$ is related to a chemical potential for spinons and $z$ keeps track of their $SU(2)$ quantum number $S^z$. The factorization (4) is a direct consequence of the identity $SO(4)_1 = SU(2)_1 \times SU(2)_1$ at the CFT level [10]. The exclusion statistics encoded in $\mu(x, z)$ agree with Haldane’s fractional exclusion statistics, with the statistics matrix given by $G = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

From eq. (3), (3), the spinon and holon thermodynamics are easily derived. The thermal response follows from

$$\mu(x, z) = (1 + x)^2 ,$$

leading to a CFT central charge $c_{CFT} = 1$ for both spinons and holons. The response to a field follows from

$$z\partial_x \log[\mu(x, z)] = \frac{2x(z - z^{-1})}{(z - z^{-1})^2 x^2 + 4} .$$  \hspace{1cm} (6)

The above-mentioned factor of 2 in the holon response to a voltage is established by comparing the integral expressions (with $\mu^\pm_c = \pm eV$ and $\rho$ the density of states)

$$\Delta Q_{\text{fermion}}(\beta, V) =$$

$$\rho e \int_0^\infty \frac{e^{-\beta(\epsilon - eV)} - e^{-\beta(\epsilon + eV)}}{(1 + e^{-\beta(\epsilon - eV)})(1 + e^{-\beta(\epsilon + eV)})} \, d\epsilon = \rho e^2 V ,$$

$$\Delta Q_{\text{holon}}(\beta, V) =$$

$$\rho e \int_0^\infty \frac{2(e^{-\beta(\epsilon - eV)} - e^{-\beta(\epsilon + eV)})}{(e^{\beta eV} - e^{-\beta eV})^2 e^{-2\beta e} + 4} = 2 \rho e^2 V$$  \hspace{1cm} (7)

for a single charged fermion and a holon, respectively.

The low-temperature behavior of a one-dimensional electron gas (away from half-filling) in the presence of weak repulsive interactions is described by the Luttinger Liquid, which is parametrized by a single interaction parameter $K_c$ and two velocities $u_s$ and $u_d$ for gap-less excitations. The above discussion of the spinon and holon statistics is easily generalized to the general Luttinger Liquid. The absence of a non-trivial parameter $K_s$ makes clear that the spinon statistics in a Luttinger Liquid are identical to those in the free electron theory, while a non-trivial value $K_c \neq 1$ will lead to a modification of the holon statistics.

We now turn to a specific systems of strongly correlated electrons, in which the fundamental degrees of freedom belong to the spinor representation of $SO(5)$, and are radically different from the spinons and holons in the weakly interacting case. $SO(5)$ symmetry has recently been proposed as a structure that unifies antiferromagnetic (AF) ordering and $d$-wave superconductivity (dSC). As such, it may be invoked in attempts to sort out the competition between AF and dSC ordering in the cuprate superconductors [11]. As a laboratory for studying the implications of $SO(5)$ symmetry, Scalapino, Zhang and Hanke (SZH) [12] have introduced a class of two-chain ladder models with exact $SO(5)$ symmetry [13]. In their paper, SZH give a strong-coupling phase diagram for these models, as a function of the couplings $U, V$ within a single rung. Among others, they identify a so-called $SO(5)$ superspin phase (called $E_1$), where the electron degrees of freedom on a single rung reduce to an $SO(5)$ vector $n_a(x), a = 1, \ldots, 5$, composed of electron-bilinear terms that represent the AF and dSC order parameters,
In the superspin phase, inter-rung interactions lead to an effective $SO(5)$ spin-chain, with the fundamental 'spins' in the vector-representation of $SO(5)$. This may be compared to an $SO(3)$ spin-chain with spins in the vector representation, which is an $SU(2)$, $S = 1$ spin-chain. On the basis of this analogy, one expects that generic inter-rung couplings in an $SO(5)$ ladder lead to a 'Haldane' gap. A particular example of a gapped phase (the so-called AKLT point) was studied in \cite{2}. One also expects a single critical point, which avoids the Haldane gap, and which is analogous to the integrable $S = 1$ chain. It will be represented by inter-rung interactions

$$n_a = \left( \frac{\Delta^t + \Delta}{2}, N_x, N_y, N_z, \frac{\Delta^t - \Delta}{2t} \right). \quad (8)$$

with $L^{ab}$ the 10 generators of $SO(5)$ and the ratio $\alpha_2/\alpha_1$ tuned to a critical value. Deviations from the critical $\alpha_2/\alpha_1$ will be relevant and will destroy the critical behavior.

These qualitative expectations are in agreement with the perturbative RG analysis of \cite{3}, which has identified two $SO(5)$-invariant fixed points: a massless point, described by the $SO(5)_1$ WZW theory and a massive point corresponding to the so-called $SO(5)$ Gross-Neveu (GN) field theory.

In what follows, we shall first focus on the critical $SO(5)_1$ WZW theory, where we establish the existence of non-abelian electrons and give full quantitative results for their exclusion statistics. After that, we consider the massive $SO(5)$ GN theory, and argue that also in this massive phase non-abelian electrons are present. We shall also explain how these excitations can be probed in photo-emission type experiments.

Through non-abelian bosonization, the critical $SO(5)_1$ WZW theory is equivalent to a theory of 5 free fermions $\psi_a$. These fermions carry the quantum numbers of the $SO(5)$ order parameter $n_a$. If we consider a 4-\phi state, and insist that it forms an $SO(5)$ singlet, we have contributions from two distinct 'fusion channels' depending on the nature of the intermediate state with the first two \phi-quanta in place. When building the full spectrum in terms of \phi-quanta, one has to specify a 'fusion channel' for each multi-\phi state, and this gives a degeneracy factor of 2 for every two \phi-quanta that are added. The geometric average of these degeneracy factors equals $\sqrt{2}$ per \phi-quantum and this factor ends up in the Boltzmann tail of the generalized distribution for these quasi-particles.

Summarizing the above, we see that the spinor quasi-particles in the $SO(5)_1$ theory, which carry the quantum numbers of a free electron, obey a form of non-abelian exclusion statistics and are properly called 'non-abelian electrons'.

In our paper \cite{3}, we obtained quantitative results for the exclusion statistics of $SO(5)_1$ spinor quasi-particles. These results can be phrased in terms of a single-level partition sum $\lambda(x)$, with $x$ keeping track of the various chemical potentials and energy. For the case where one gives equal weight to all four components, $x = e^{\beta(\mu - \epsilon)}$ with $\mu$ an overall chemical potential, $\lambda(x)$ is a solution of

$$\lambda^2 - (2 + 3x^2)\lambda + (3x^2 - 1)(x^2 - 1)\lambda^2 - x^2(1 - x^2)^2 = 0. \quad (11)$$

The generalized Fermi-Dirac distribution is given by

$$n_{\phi}(\epsilon) = |x\partial_x \log \lambda(x)| \langle x = e^{\beta(\mu - \epsilon)} \rangle. \quad (12)$$

From (11) one immediately derives the properties

$$\lambda(x) \sim x^4 \quad \text{for } x \gg 1, \quad (13)$$

which implies a maximum $n_{\phi}^{\text{max}} = 1$ for the occupation per level of each of the \phi-quanta, and

$$\lambda(x) = 1 + 4\sqrt{2}x + \mathcal{O}(x^2) \quad \text{for } x \ll 1, \quad (14)$$

which implies the prefactor $\sqrt{2}$ in the Boltzmann tail of $n_{\phi}(\epsilon)$.

Comparing the central charge $c_{\text{CFT}}$ and the spin and charge susceptibilities $\chi^s$ and $\chi^c$ to those of standard free electrons (eq. (3)), one finds

$$c_{\phi}^{\text{CFT}}/c_{\text{CFT}} = 5/4, \quad \chi^s_{\phi}/\chi^s = \chi^c_{\phi}/\chi^c = 2. \quad (15)$$

These results are readily checked in a picture with 5 fundamental fermions. Note that the deviations from the free electron results are not fermi-liquid effects, but manifestations of the unusual statistics of the non-abelian electrons.
In the massive $SO(5)$ GN model, the $SO(5)$ vector and spinor excitations exist in the form of massive particles with mass ratio \[ (16) \]

$M_{\psi} = \sqrt{3} M_{\phi}$.

The $\phi$-particles, which are the lightest particles in the theory, may be viewed as ‘kinks’ that carry a topological charge related to the $SO(5)$ group theory \[ (15) \]. In principle, one may derive the thermodynamics of the $\phi$-particles by starting from an exact scattering matrix and applying the Thermodynamic Bethe Ansatz. To our knowledge this procedure has not been carried out, but it is clear that also in this massive theory a Boltzmann tail prefactor equal to $\sqrt{2}$ will arise. [The detailed form of the distribution $n_\phi$ will differ between the critical and the massive cases.] We conclude that also in the massive case, the terminology ‘non-abelian electrons’ is justified.

The physical processes by which non-abelian electrons can be excited inside the $SO(5)$ ladders, both in the critical and in the massive phase, are easily described and can possibly be realized in experiments. The physical set-up is a photo-emission process, where a high energy photon ejects an electron from the ladder. After such a process, or its inverse, one of the rungs in the ladder violates the $SO(5)$ superspin condition, and the ladder finds itself in an excited state with the quantum numbers of a free electron. As we discussed, such excited states correspond to the $\phi$-quanta, i.e., to the non-abelian electrons.

Before concluding, we would like to comment on the the massive $SO(8)$ phase that was found in the perturbative analysis of the $SO(5)$ ladder models \[ (4) \]. This phase features an $SO(8)$ vector of order parameters, bilinear in the bare electron operators, and $SO(8)$ spinors that carry the quantum numbers of free electrons. It is important to realize that the group theory of $SO(8)$ works out in such a way that there are no non-trivial fusion channels in multiple tensor products of the spinor particles, and that as a consequence the phenomenon of non-abelian statistics does not occur for $SO(8)$.

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