SIMPLICIAL APPROXIMATION AND COMPLEXITY GROWTH

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Abstract: This work is motivated by two problems: 1) The approach of manifolds and spaces by triangulations. 2) The complexity growth in sequences of polyhedra. Considering both problems as related, new criteria and methods for approximating smooth manifolds are deduced. When the sequences of polyhedra are obtained by the action of a discrete group or semigroup, further control is given by geometric, topologic and complexity observables. We give a set of relevant examples to illustrate the results, both in infinite and finite dimensions.

1 Introduction

Analysis situs, an ancestor of modern topology, arose as a clandestine area of mathematics in the nineteenth century. Gradually it became more accepted, thanks to the work of H. Poincaré, P. Alexandrov, O. Veblen, H. Hopf, J. Alexander, A. Kolmogorov, H. Weyl, L. Brouwer, H. Whitney, W. Hodge and S. Lefschetz, among others.

One of its driving forces, the approximation of shapes (and spaces) through the juxtaposition of prisms or polyhedra, permeated to science and art, becoming essential in our view of the world. From P. Picasso’s cubism to quantum gravity, human perception seemed to accept simplices as elementary blocks to approach forms and space.

It is standard, from a mathematical perspective, to infer estimates of error, complexity, and changes in both complexity and error in a process of approximation, also to establish quantitative and qualitative criteria for convergence, and infer bounds for the speed at which such a convergence (if any) occurs.

This paper is motivated by those problems; we obtain, using suitable tools, results of this kind for evolving polyhedra on manifolds. We describe sequences of complexes associated to coverings of spaces by open sets. Sequences of this type, considered by P. Alexandrov (see [Ale-Pon]) under the name of projective spectra, yield, under suitable convergence assumptions, approximations of a paracompact Hausdorff space up to homeomorphism.

We regard the number of simplices and the dimension of each complex in the sequence as a measure of its complexity, and control its growth not only in the limit, but also at every stage. This delivers sequences of irreducible complexes, those for which the excess of complexity is eliminated, say. If the space in question is a differentiable
manifold endowed with a Riemannian metric, those irreducible complexes, together
with available tools from geometric measure theory, yield a quantitative approximation
as well.

To perform those constructions in a systematic way, we consider actions of discrete
groups and semigroups, say $\Gamma$, on complexes associated to coverings by open sets.
We describe $\Gamma$-representations/actions that yield convergent sequences of complexes,
to make a connection with expansive systems, or $e$-systems; in those systems the
convergent sequence of complexes is obtained by iteration of a suitable initial simplicial
complex, a generator, say.

If the space where $\Gamma$ acts expansively is a closed Riemannian manifold, estimates
for the minimal complexity of the generating complex are achieved. This is possible
thanks to comparison results in differential geometry.

We briefly mention the contents of this work.
Sections 2.1, 2.2, and 2.3 provide some notation and framework.

In Section 2.4 complexity functions for simplicial complexes are proposed, and we
mention their main and useful properties.

Section 2.5 deals with concrete realizations of complexes in Euclidean space; this
is needed, together with the functions introduced in Section 2.4 to obtain better
approximations of spaces when compared with those achieved by arbitrary convergent
sequences (Section 2.6); this is developed in Section 2.8 both from a qualitative and
quantitative perspective.

In Section 2.7 increasing sequences of numbers control the complexity growth in
sequences of complexes constructed from finer and finer coverings, as measured by the
functions introduced in Section 2.4 yielding a quantitative description of the process
in the limit. Those growths are measured by what we call the simplicial growth up to
dimension $k$, denoted by $\text{ent}_k$, and by the dimension growth, denoted by $\text{Dim}$. In fact
$\text{ent}_k$ is a generalization of what is known as topological entropy (see [Wal]), meanwhile
$\text{Dim}$ is a relative of mean dimension (see [Gro3]).

Section 3 begins with a natural framework for groups and semigroups actions on
spaces, usually known as $\Gamma$-spaces. We mention the natural morphisms between objects
of this type, some advantages of this perspective, to define the evolution of simplicial
complexes in $\Gamma$-spaces, where the growths of complexity can be measured.

In Section 3.1 the exponential growth of the 0-simplices is studied under assump-
tions on $\Gamma$, to infer some quantitative control at every stage.

In Section 3.2 we describe a particular type of $\Gamma$-spaces, namely $\Gamma$-spaces with
property-e. The first remarkable issue of the expansive property, or property-e, is that
it can be characterized using either topological (set theoretic) or geometric tools. The
set theoretic characterization leads to the concept of a generator, an open cover that
has a good response to the action of $\Gamma$, say. It could be seen as a complex that under
the action of $\Gamma$ evolves towards an acceptable approximation of the space. We describe
in which sense the evolving nerves of generating covers approximate the space, and
recall fundamental results in geometry and topology that suit our developments. All
the results from previous Sections can be used in this scenario, and the adaptation of
them is left to the reader.
In Section 3.3, assuming that the space is of Riemannian type, we provide estimates to have a better control of the generating process. Those estimates find concrete applications in Section 5.

Finally, in Section 4 and Section 5 we present some examples. Section 4 deals with infinite dimensional examples where estimates for the simplicial growth, as measured by the family \( \{ \text{ent}_k \mid k \in \mathbb{N} \} \), and the dimension growth, as measured by \( \text{Dim} \), appear. Section 5 describes finite dimensional closed manifolds for which an expansive action can be constructed. Some of the examples in finite dimension are not new, and the list of examples is far from being exhaustive nor definitive; their (not so detailed) description is included for many purposes:

1. To ensure that the results of Sections 3.2 and 3.3 are non-void, enabling concrete constructions and estimates.
2. To have an idea of the methods used to construct them.
3. To allow the construction of new examples from known ones.

Sections 4 and 5 are not entirely independent: all the examples in Section 5 can be used in Section 4.1 to construct infinite dimensional closed manifolds with property-e.

2 Simplicial complexes, complexity and convergence

We state properties of the canonical simplicial complex associated to the covering of a space by open sets, known as the nerve of the covering. Some statements can be found in [Ale-Pon], [Hur-Wall], [Lef], and the references therein. Other properties are new (at least for the author), and all of them will be used in this article.

2.1 The nerve of open covers

If \( V \) is a compact Hausdorff space\(^1\), we denote by \( C_V \) the set of covers of \( V \) by open sets: one calls the members of \( C_V \) open covers.

**Remark 2.1.** Since \( V \) is compact, it suffices to identify \( C_V \) with the totality of all finite covers by open sets of \( V \) to simplify.

If \( \alpha \) and \( \beta \) belong to \( C_V \), one says that \( \alpha \) is finer than \( \beta \) if whenever \( A \) is an element in \( \alpha \) there exists some \( B \) in \( \beta \) such that \( A \subseteq B \), and writes \( \alpha \succ \beta \) if that is the case. This notion induces a partial order on \( C_V \).

If \( \{ \alpha, \beta \} \subset C_V \), one denotes by \( \alpha \cap \beta \) the refinement of \( \alpha \) by \( \beta \) (or equivalently the refinement of \( \beta \) by \( \alpha \)): its elements are intersections of one element from \( \alpha \) and another from \( \beta \). One can write

\[
\alpha \cap \beta := \text{l.u.b.}\{ \gamma \mid \gamma \succ \alpha, \gamma \succ \beta \},
\]

where l.u.b. denotes the supremum (or least upper bound) in \( C_V \) induced by \( \succ \).

\(^1\)Some of the constructions and results are valid in more general spaces, but our intention is to provide examples in compact manifolds, usually without a boundary.
Remark 2.2. One can play further with those notions and use the language of lattices, something that we give for granted.

Let $\alpha$ be given as $\{ A_i \mid i \in I \}$, where $I$ is an indexing set (finite since $V$ is compact). Associated to $\alpha$ is a simplicial complex, known as the nerve of $\alpha$, that we denote by $K(\alpha)$, uniquely defined up to homotopy, and whose simplices are constructed as follows: for every $k$ in $\mathbb{N}$ the set of $k$-dimensional simplices of $K(\alpha)$, denoted by $\triangle_k(\alpha)$, is given by

$$\{ [a_{i(0)}, \ldots, a_{i(k)}] \mid \bigcap_{r=0}^{k} A_{i(r)} \neq \emptyset \},$$

where for each $i$ in $I$ we identify the open set $A_i$ with the 0-simplex $[a_i]$.

2.2 Dimension, simplicial mappings and irreducibility

Given $\alpha$ in $C_V$, for every $k$ we denote by $|\triangle_k(\alpha)|$ the cardinality of $\triangle_k(\alpha)$, i.e. the number of $k$-simplices in $K(\alpha)$. By those means one introduces the dimension of $K(\alpha)$, denoted by $\dim K(\alpha)$, as the maximal $k$ for which $|\triangle_k(\alpha)|$ is different from zero.

For $\alpha$ and $\beta$ in $C_V$ with $\alpha \succ \beta$ there exists a simplicial map from $K(\alpha)$ to $K(\beta)$, say $T^\alpha_\beta : K(\alpha) \to K(\beta)$, defined up to homotopy, satisfying the following properties:

1. If $A_i \subseteq B_j$, then $T^\alpha_\beta[a_i] = [b_j]$.

2. Whenever $k > 0$ and $\sigma$ is in $\triangle_k(\alpha)$, then the image of $\sigma$ under $T^\alpha_\beta$ is completely determined by the image of the 0-simplices making up $\sigma$: this allows the possibility that $T^\alpha_\beta \sigma$ is in $\triangle_l(\beta)$ for some $l \leq k$ (for example when different vertices of $\sigma$ are mapped to the same 0-simplex in $K(\beta)$).

One says that $T^\alpha_\beta$ is compatible with $\succ$. It is important to note:

1. Such a map need not be unique.

2. If $\alpha \succ \beta \succ \gamma$ and we have constructed two simplicial maps $T^\alpha_\beta : K(\alpha) \to K(\beta)$ and $T^\beta_\gamma : K(\beta) \to K(\gamma)$ compatible with $\succ$, then we have a simplicial map $T^\alpha_\gamma : K(\alpha) \to K(\gamma)$ given by $T^\alpha_\gamma = T^\beta_\gamma \cdot T^\alpha_\beta$ that is also compatible with $\succ$.

There are open covers we distinguish for later purposes.

Definition 2.3. One says that $\alpha$ in $C_V$ is irreducible if no open refinement of $\alpha$ has a nerve isomorphic with a proper sub-complex of $K(\alpha)$, i.e. if there is no $\beta$ finer than $\alpha$ that admits a strict simplicial embedding from its nerve to the nerve of $\alpha$.

Lemma 2.4. Irreducible covers have the following properties:

1. If $\alpha$ is irreducible then every member of it contains a point in $V$ that is not contained in other member.
Remark 2.5.

To handle a better notation, given an open cover \( \sigma \) modulo permutations. By those means we identify the \( \alpha \) sequence of complexes, with \( 2.3 \) Chain complexes, homology

Proof. 1. If \( \alpha = \{ A_i \mid i \in I \} \) has some element, say \( A_j \), that contains no point that is not contained in the rest of the \( A_i \)'s, then \( \alpha' = \{ A_i \mid i \in I \setminus \{ j \} \} \) is a refinement of \( \alpha \) and \( K(\alpha') \) is a proper subcomplex of \( K(\alpha) \).

2. This is clear from the definition.

3. Let \( \alpha \) in \( C_V \) be given, and consider a sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) in \( C_V \) so that \( \alpha_0 \prec \alpha_1 \prec \alpha_2 \prec \ldots \), where all the \( \alpha_n \)'s are reducible, and such that the corresponding nerves form a sequence of complexes, with \( K(\alpha_{n+1}) \) being a proper subcomplex of \( K(\alpha_n) \), where \( \alpha_0 = \alpha \). By compacity of \( V \) the sequence must stop, and the last term is an irreducible refinement of \( \alpha \).

4. Being \( V \) a manifold of dimension \( n \), it suffices to prove the result on some open set homeomorphic to \( \mathbb{R}^n \), where the statement is obviously true.

2.3 Chain complexes, homology

To handle a better notation, given an open cover \( \alpha = \{ A_i \mid i \in I \} \), we denote by \( \Lambda_k(\alpha) \) the set of injective mappings \( \tilde{r} : \{0, \ldots, k\} \rightarrow I^{k+1} \) such that \( \cap_{r=0}^k A_{\tilde{r}(r)} \neq \emptyset \), modulo permutations. By those means we identify the \( k \)-simplex \([a_{\tilde{r}(0)}, \ldots, a_{\tilde{r}(k)}]\) with \( \sigma_\tilde{r}^k \) whenever \( \tilde{r} \) is in \( \Lambda_k(\alpha) \). Therefore we have a bijection between \( \Delta_k(\alpha) \) and \( \Lambda_k(\alpha) \).

If \((G, +)\) is an Abelian group one identifies \( C_k(\alpha, G) \) with the (Abelian) group of \( k \)-chains in \( K(\alpha) \) with coefficients in \( G \), so that

\[
C_k(\alpha, G) := \{ \sum_{\tilde{r} \in \Lambda_k(\alpha)} g_{\tilde{r}} \sigma_{\tilde{r}}^k \mid g_{\tilde{r}} \in G \}.
\]

Remark 2.5. Given a permutation of \((k+1)\) letters, say \( \xi \), we are identifying \( \sigma_\tilde{r}^k \) with \( \sigma_{\tilde{r}}^k = [a_{\tilde{r}(0)}, \ldots, a_{\tilde{r}(k)}] \) in \( \Delta_k(\alpha) \), although in \( C_k(\alpha, G) \) we have \( \sigma_{\tilde{r}}^k = \text{sgn}(\xi) \sigma_{\tilde{r}}^k \), where \( \text{sgn}(\xi) \) denotes the sign of \( \xi \).

Introduce the boundary operator, denoted by \( \partial \), as the map that sends \( k \)-chains to \((k-1)\)-chains in a \( G \)-linear way. Since \( C_k(\alpha, G) \) is generated by the elements in \( \Delta_k(\alpha) \), it suffices to define the action of \( \partial \) on the elements of \( \Delta_k(\alpha) \).

Thus given \( \tilde{r} \) in \( \Lambda_k(\alpha) \) we set

\[
\partial \sigma_{\tilde{r}}^k := \sum_{r=0}^k (-1)^r \sigma_{\tilde{r} \setminus \{r\}}^{k-1},
\]
Lemma 2.7. \[ \sigma^k_i \in c \chi \] provided that \( \sigma^k_i = [a_{i(0)}, ..., a_{i(k)}] \), where \( \widehat{a} \) means that \( a \) is deleted.

One verifies that for every \( c \in C_k(\alpha, G) \) one has \( \partial^2 c = \partial \partial c = 0 \) in \( C_{k-2}(\alpha, G) \), i.e. the boundary of the boundary of every \( k \)-chain is equal to zero.

Using the boundary operator one defines two subgroups of \( C_k(\alpha, G) \):

1. The subgroup of \( k \)-cycles, denoted by \( Z_k(\alpha, G) \), and defined through
   \[ Z_k(\alpha, G) := C_k(\alpha, G) \cap \{ c | \partial c = 0 \} . \]

2. The subgroup of \( k \)-boundaries, denoted by \( B_k(\alpha, G) \), and defined through
   \[ B_k(\alpha, G) := C_k(\alpha, G) \cap \partial C_{k+1}(\alpha, G) . \]

By those means the \( k \)-th homology group of \( K(\alpha) \) with coefficients in \( G \) is defined, namely
   \[ H_k(\alpha, G) := \frac{Z_k(\alpha, G)}{B_k(\alpha, G)} . \]

**Remark 2.6.** An algebraist would say that \( H_k \) measures the inexactness of the sequence
   \[ \ldots \to C_{k+1} \overset{\partial}{\to} C_k \overset{\partial}{\to} C_{k-1} \to \ldots \]

A geometer/topologist would say that \( H_k \) measures the amount of closed \( k \)-chains that are not filled in the space in question (i.e. that are not boundaries) up to bordism.

Let \( H_*(\alpha, G) = \bigoplus_{i=0}^{\dim K(\alpha)} H_i(\alpha, G) \) be the graded \( G \)-module associated to the homology of \( K(\alpha) \) with coefficients in \( G \). In particular if \( G \) is taken as \( \mathbb{R} \), one denotes by \( B_i(\alpha) := \dim_{\mathbb{R}} H_i(\alpha, \mathbb{R}) \) the \( i \)-th Betti number of \( K(\alpha) \). Regarding the structure of the complex \((C_*(\alpha, \mathbb{R}), \partial)\), one has the isomorphism \( C_i(\alpha, \mathbb{R}) = Z_i(\alpha, \mathbb{R}) \bigoplus B_{i-1}(\alpha, \mathbb{R}) \).

If no confusion arises we identify \( \mathbf{c}_i(\alpha) \), \( \mathbf{z}_i(\alpha) \) and \( \mathbf{b}_i(\alpha) \) with the real dimension of \( C_i(\alpha, \mathbb{R}), Z_i(\alpha, \mathbb{R}) \) and \( B_i(\alpha, \mathbb{R}) \) respectively, whence in particular \( \mathbf{B}_i(\alpha) = \mathbf{z}_i(\alpha) - \mathbf{b}_i(\alpha) \) and \( \mathbf{c}_i(\alpha) = \mathbf{z}_i(\alpha) + \mathbf{b}_{i-1}(\alpha) \) follow.

Using the previous nomenclature one defines \( \chi_i(\alpha) := \sum_{i=0}^{\dim K(\alpha)} t^i \mathbf{B}_i(\alpha) \), so that \( \chi_{-1}(\alpha) \) is the Euler-Poincaré characteristic of \( K(\alpha) \). The equalities for \( \mathbf{B}_i(\alpha) \) and \( \mathbf{c}_i(\alpha) \) entail that \( \chi_{-1}(\alpha) \) is equal to the sum \( \sum_{i=0}^{\dim K(\alpha)} (-1)^i \mathbf{c}_i(\alpha) \).

From the definitions/constructions one has the equality \( \mathbf{c}_i(\alpha) = |\Delta_i(\alpha)| \) for every \( i \) in \( \mathbb{N} \), therefore:

**Lemma 2.7.** Whenever \( \alpha \) is in \( C_V \) one has the identity
\[ \chi_{-1}(\alpha) = \sum_{i=0}^{\dim K(\alpha)} (-1)^i |\Delta_i(\alpha)| . \]
2.4 Complexity functions

In this Section we define complexity functions for the simplices of an open cover on $V$. We infer some properties of their minimizers and some estimates for them. The next observation is fundamental.

**Lemma 2.8.** For every $k$ in $\mathbb{N}$ and $\alpha$ in $C_V$ the minimum of $|\triangle_k(\beta)|$ among those $\beta$’s finer than $\alpha$ is obtained for irreducible $\beta$’s. In particular, if $\alpha$ is irreducible, then the minimum mentioned above is obtained for $\alpha$ itself. The same is true for the sum $\sum_{i=0}^{k} |\triangle_i(\beta)|$ and for the dimension $\dim K(\beta)$.

**Proof.** Follows from 3 and 2 in Lemma 2.4, namely that irreducible covers are cofinal in the directed set $(C_V, \succ)$: hence if $\alpha$ is irreducible then for every $\beta$ finer than $\alpha$ all the simplicial maps from the nerve of $\beta$ to the nerve of $\alpha$ compatible with $\succ$ are surjective. \qed

To quantify the complexity of $K(\alpha)$, that we measure in terms of its dimension and its number of simplices, also by similar quantities in $K(\beta)$ whenever $\beta$ is finer than $\alpha$, we introduce the functions $\text{Dim} K(\cdot)$, $G_k(\cdot)$ and $S_k(\cdot)$ from $C_V$ to $\mathbb{N}$ through:

\[
\text{Dim} K(\alpha) := \min_{\beta \succ \alpha} \dim K(\beta),
\]

\[
G_k(\alpha) := \sum_{i=0}^{k} |\triangle_i(\alpha)|
\]

and

\[
S_k(\alpha) := \min_{\beta \succ \alpha} G_k(\beta).
\]

From the definitions, Lemma 2.8 and the identity in Lemma 2.7 we observe:

**Lemma 2.9.** For every $\alpha$ in $C_V$ we have

1. If $k$ is larger than zero

\[
G_0(\alpha) \leq G_{k-1}(\alpha) \leq G_k(\alpha),
\]

\[
\max_{I \subseteq \{0, \ldots, k\}} |\triangle_I(\alpha)| \leq G_k(\alpha) \leq (k+1) \max_{I \subseteq \{0, \ldots, k\}} |\triangle_I(\alpha)|,
\]

with

\[
|\triangle_I(\alpha)| \leq \frac{(G_0(\alpha))^{I+1}}{(I+1)!}.
\]

2. $\text{Dim} K(\alpha)$ is equal to $\dim K(\beta)$ for some irreducible $\beta$ finer than $\alpha$.

3. $S_k(\alpha)$ is equal to $G_k(\beta)$ for some irreducible $\beta$ finer than $\alpha$.

4. The identity

\[
\chi_{-1}(\alpha) = 2 \sum_{i=0}^{\text{dim} K(\alpha)-1} (-1)^i G_i(\alpha) + (-1)^{\text{dim} K(\alpha)} G_{\text{dim} K(\alpha)}(\alpha).
\]
2.5 Euclidean realization of nerves

Let \( \alpha = \{ A_i \mid i \in I \} \) be an open cover for \( V \). We say that a partition of unity for \( V \) is compatible with \( \alpha \) if it satisfies the following conditions:

1. \( \sum_{i \in I} x_i(v) = 1 \) for every \( v \) in \( V \).
2. For every \( i \) in \( I \) we have that \( x_i(v) = 0 \) whenever \( v \) is not in \( A_i \).

Identify the 0-simplex \([a_i]\) of \( K(\alpha) \) corresponding to \( A_i \) with the unit vector in \( \mathbb{R}^{|I|} \) along the \( i \)-th direction, to denote the image of the map

\[
x : V \rightarrow \mathbb{R}^{|I|}
\]

\[
v \mapsto x(v) = \sum_{i \in I} x_i(v)[a_i]
\]

by \( |K(\alpha)| \), and call it an Euclidean realization of \( K(\alpha) \). Observe that different partitions of unit on \( V \) compatible with \( \alpha \) induce maps from \( V \) to \( \mathbb{R}^{|I|} \) that are homotopic.

Sometimes we identify \(|K(\alpha)|\) with a polyhedral current in \( \mathbb{R}^{|I|} \). We do this as follows: for every \( i \) in \( I \) we have the 0-current \([a_i]\) that corresponds to the pure point measure supported at distance one from the origin along the \( i \)-th axis. Using the convention of Section 2.3, for \( \vec{i} \) in \( \wedge_k(\alpha) \) we identify \( \sigma^k_{\vec{i}} \) with the polyhedral \( k \)-current

\[
\|\sigma^k_{\vec{i}}\| \wedge \sigma^l_{\vec{i}}, \quad \text{where} \quad \|\sigma^k_{\vec{i}}\| = \mathcal{H}^k_{\text{spt}\sigma^k_{\vec{i}}}
\]

whose support is the convex hull of \( \{[a_i(0)],...,[a_i(k)]\} \), meanwhile \( \sigma^k_{\vec{i}} \) is a \( k \)-vectorfield of unit length tangent to such a plane (see [Fed] for all the details).

Observe that the chain complex associated to \(|K(\alpha)|\) is isomorphic with that defined in Section 2.3 for \( K(\alpha) \).

2.6 Sequences of nerves: convergence

The results in this Section are a simplified version, suitable for the applications in this work, of general results attributed to P. Alexandrov, S. Lefschetz and V. Ponomarev (see [Ale-Pon]-[Lef]). In the literature the nomenclature is not uniform: we try to unify some notions as well.

Consider a sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) in \( C_V \) with \( \alpha_{n+1} \succ \alpha_n \). If \( K(\alpha) \) is identified with the simplicial complex that corresponds to the nerve of \( \alpha \), then for every \( n \) we have a simplicial map \( T_n : K(\alpha_{n+1}) \rightarrow K(\alpha_n) \) compatible with \( \succ \) and defined up to homotopy (see Section 2.2). Those simplicial maps can be composed inductively to get a map \( T_m \) from \( K(\alpha_m) \) to \( K(\alpha_n) \) whenever \( m \succ n \) in the usual way, where \( T_m := T_m \circ T_{m+1} \circ \cdots \circ T_{m-1} \).

We have an infinite sequence of simplicial complexes and mappings making up a directed set \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\).

One says that the sequence \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) is convergent if every member of \( \alpha_n \) consists at most of a point when \( n \) goes to infinity.
As \( m \) tends to infinity we have a surjective simplicial map from \( K(\alpha_m) \) to \( K(\alpha_n) \) for every \( n \). We naturally identify the inverse or projective limit of the directed set \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) with the nerve of \( \alpha_n \) when \( n \) tends to infinity, that we denote by

\[
\lim_{\leftarrow}(K(\alpha_n), T_n),
\]

to state:

**Proposition 2.10.** (Alexandrov-Ponomarev [Ale-Pon]) Assume that the sequence \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) is convergent. Then when \( n \) goes to infinity the nerve of \( \alpha_n \) and \( V \) are homeomorphic.

**Proof.** We describe the projective limit of the directed set \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\), to see that there is a homeomorphism between such a limit and \( V \).

Let \( \sigma := \{\sigma_n\}_{n \in \mathbb{N}} \) be a sequence of simplices, with \( \sigma_n \) in \( K(\alpha_n) \) for every \( n \). We say that \( \sigma \) is an **admissible sequence** or a **thread** for \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) if \( \sigma_n = T_m^m \sigma_m \) whenever \( m \) is larger than \( n \), and say that an admissible sequence \( \sigma' \) is an **extension** of \( \sigma \) if for every \( n \) the simplex \( \sigma_n \) is a face (not necessarily a proper one) of \( \sigma'_n \). If the admissible sequence \( \sigma \) has no extensions other than itself, we say that it is a **maximal admissible sequence** (or a **maximal thread**).

Provide \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) with the following topology: given a simplex \( \sigma_n \) in \( K(\alpha_n) \) for some \( n \), a basic open set around \( \sigma_n \) consists of all maximal admissible sequences \( \sigma' \) such that \( \sigma'_n \) is a face of \( \sigma_n \). In such a way one generates a topology for the limit space, namely the set of all maximal admissible sequences.

Whenever \( v \) is a point in \( V \) we have a simplex \( \sigma_n(v) \) in \( K(\alpha_n) \) that corresponds to all the open sets in \( \alpha_n \) to which \( v \) belongs; due to the convergence assumption we note that \( \sigma(v) = \{\sigma_n(v)\}_{n \in \mathbb{N}} \) is a maximal admissible sequence, and conversely, every maximal admissible sequence in \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) is of the form \( \sigma(v) \) for some \( v \) in \( V \).

Therefore \( V \) is isomorphic to the inverse limit of \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\), and at this stage it is easy to see that they are homeomorphic. \( \square \)

**Remark 2.11.** Neither a metric nor a differentiable structure on \( V \) are required in Proposition 2.10.

**Remark 2.12.** Proposition 2.10 can be refined sometimes: it might happen that for some finite \( n \) all the elements in \( \alpha_n \) together with their intersections are contractible (see Figure 1 in Section 5). Then \( \alpha_n \) is said to be a ‘good cover’, and it is known that in such a case \( K(\alpha_n) \) is homotopically equivalent to \( V \) (see [Hat] for example).

One is led to consider convergent sequences of coverings to reconstruct and/or approximate a given space up to homeomorphism in the limit. On every paracompact Hausdorff space a convergent sequence can be constructed in an arbitrary way. It is of interest, however, to create them under some quantitative and qualitative control. We will see in Section 2.7 that the family of complexity functions introduced in Section 2.4 are of much use for those purposes. Moreover, if we endow \( V \) with a Riemannian metric, one can consider subsequences of complexes associated to those complexity functions, and have a better approximation of \( V \) in the limit (Section 2.8).
2.7 Controlling sequences

Let \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\) be a sequence of nerves and simplicial mappings built up from a sequence \(\{\alpha_n\}_{n \in \mathbb{N}}\) of open covers for \(V\), with \(\alpha_{n+1} \succ \alpha_n\). From Lemma 2.9 we know that if we consider the sequence \(\{S_k(\alpha_n)\}_{n \in \mathbb{N}}\) of positive integers there exists, for every \(n\) in \(\mathbb{N}\), at least one irreducible \(\beta_{k,n}\) finer than \(\alpha_n\) so that \(S_k(\alpha_n)\) is equal to \(G_k(\beta_{k,n})\).

Fix \(k\) and let \(\{\beta_{k,n}\}_{n \in \mathbb{N}}\) be a sequence of irreducible covers that achieve, for each \(n\) in \(\mathbb{N}\), the minimum of \(S_k(\alpha_n)\). Since \(\beta_{k,n}\) is finer than \(\alpha_n\), then when \(n\) goes to infinity we have, under the hypothesis of Proposition 2.10, that \(K(\beta_{k,n})\) is homeomorphic to \(V\); since \(\beta_{k,n}\) is irreducible the dimension of \(K(\beta_{k,n})\) is equal to the dimension of \(V\).

For every \(i \in \{0, \ldots, \dim V\}\) consider the increasing sequence of positive integers \(\{G_i(\beta_{k,n})\}_{n \in \mathbb{N}}\); each sequence goes to infinity as \(n\) increases. The next Proposition provides a correlation between those sequences thanks to Lemma 2.9.

**Proposition 2.13.** Let \(V\) be a compact Hausdorff space, without a boundary, whose topological dimension is uniform and finite. For a fixed \(k\) let \(\{\beta_{k,n}\}_{n \in \mathbb{N}}\) be a sequence of irreducible open covers associated to a convergent sequence \((K(\alpha_n), T_n)_{n \in \mathbb{N}}\). Then as \(n\) goes to infinity we have the equality

\[
\chi_{-1}(V) = 2 \sum_{i=0}^{\dim V - 1} (-1)^i G_i(\beta_{k,n}) + (-1)^{\dim V} \sum_{i=0}^{\dim V} G_{\dim V}(\beta_{k,n}).
\]

To have more control on a sequence \(\{K(\alpha_n)\}_{n \in \mathbb{N}}\) we consider strictly increasing sequences of positive real numbers, say \(\{c(n)\}_{n \in \mathbb{N}}\), going to infinity and such that

\[
0 < \liminf_n \frac{\log S_k(\alpha_n)}{c(n)} =: \text{ent}_k^+(\alpha_n, c(n)) \leq \text{ent}_k^+(\alpha_n, c(n)) := \limsup_n \frac{\log S_k(\alpha_n)}{c(n)} < \infty.
\]

If the sequence \(\{c(n)\}_{n \in \mathbb{N}}\) satisfies those estimates, we say that it **controls the simplicial growth** of \(\{K(\alpha_n)\}_{n \in \mathbb{N}}\) up to dimension \(k\). If \(\lim_n \log S_k(\alpha_n)/c(n)\) exists, then

\[
\text{ent}_k^+(\alpha_n, c(n)) = \text{ent}_k^+(\alpha_n, c(n)) =: \text{ent}_k(\alpha_n, c(n)).
\]

Similarly, if

\[
0 < \liminf_n \frac{\dim K(\alpha_n)}{c(n)} =: \dim^+(\alpha_n, c(n)) \leq \dim^+(\alpha_n, c(n)) := \limsup_n \frac{\dim K(\alpha_n)}{c(n)} < \infty,
\]

we say that \(\{c(n)\}_{n \in \mathbb{N}}\) **controls the dimension growth** of \(\{K(\alpha_n)\}_{n \in \mathbb{N}}\).

Of course if \(\lim_n \dim K(\alpha_n)/c(n)\) exists, then

\[
\dim^+(\alpha_n, c(n)) = \dim^+(\alpha_n, c(n)) =: \dim(\alpha_n, c(n)).
\]

Using Lemma 2.9 we deduce:

**Theorem 1.** Assume that \(\{c(n)\}_{n \in \mathbb{N}}\) controls the simplicial growth of \(\{K(\alpha_n)\}_{n \in \mathbb{N}}\) up to dimension \(k\) for some finite \(k\). Then \(\{c(n)\}_{n \in \mathbb{N}}\) is a controlling sequence for the growth of simplices of \(\{K(\alpha_n)\}_{n \in \mathbb{N}}\) up to dimension \(k\) for every finite \(k\).
Proof. From Lemma 2.9 we see that
\[ G_0(\alpha_n) \leq G_k(\alpha_n) \leq (k + 1) \max_{l \in \{0, \ldots, k\}} \frac{G_0(\alpha_n)^{l+1}}{(l + 1)!}, \]
therefore we have
\[ \text{ent}^\downarrow_0(\alpha_n, c(n)) \leq \text{ent}^\downarrow_k(\alpha_n, c(n)) \leq (k + 1)\text{ent}^\downarrow_0(\alpha_n, c(n)), \]
and similarly
\[ \text{ent}^\uparrow_0(\alpha_n, c(n)) \leq \text{ent}^\uparrow_k(\alpha_n, c(n)) \leq (k + 1)\text{ent}^\uparrow_0(\alpha_n, c(n)). \]

By a simple interpolation we deduce that \( \text{ent}^\downarrow_l(\alpha_n, c(n)) \) and \( \text{ent}^\downarrow_k(\alpha_n, c(n)) \) are comparable if both \( k \) and \( l \) are finite, and similarly for \( \text{ent}^\uparrow_l(\alpha_n, c(n)) \) and \( \text{ent}^\uparrow_k(\alpha_n, c(n)) \).

Natural choices for controlling sequences are:
1. \( c(n) = n \), and then (in the strict sense)
   (a) The simplicial growth is of exponential type.
   (b) The dimension growth is of linear type.
2. \( c(n) = \log n \), and then
   (a) The simplicial growth is of polynomial type.
   (b) The dimension growth is of logarithmic type.

Of course there are other possibilities.

Theorem 1 says that the order of the simplicial growth up to dimension \( k \) in a sequence \( (K(\alpha_n), T_n)_{n \in \mathbb{N}} \) is comparable to the order of growth of 0-simplices if \( k \) is finite or \( V \) is finite dimensional.

On the other hand, observe that a necessary condition for \( \{K(\alpha_n)\}_{n \in \mathbb{N}} \) to have a sequence controlling its dimension growth is that the underlying space \( V \) must have infinite topological dimension, i.e. the supremum of \( \dim K(\alpha) \) as \( \alpha \) varies in \( C_V \) must be unbounded; this condition is also sufficient if \( \{K(\alpha_n)\}_{n \in \mathbb{N}} \) is convergent.

Remark 2.14. We understand that \( \{n\}_{n \in \mathbb{N}} \) is the standard sequence; due to that we will omit \( c(n) \) from the expressions whenever such a sequence is used.

2.8 Life with a Riemannian metric

Now \( V \) is a smooth closed manifold, provided with a distance function arising from some Riemannian metric \( g \), say \( d = d_g \). Then the convergence of \( (K(\alpha_n), T_n)_{n \in \mathbb{N}} \) is equivalent to the statement that all the members of \( \alpha_n \) have a diameter that goes to zero as \( n \) goes to infinity, where we assume that \( \alpha_{n+1} \succ \alpha_n \) for every \( n \).
As in Section 2.7 consider, for every $k$ and $n$, an irreducible cover $\beta_{k,n}$ finer than $\alpha_n$ so that $S_k(\alpha_k)$ is equal to $G_k(\beta_{k,n})$. Then for each $n$ we have a simplicial embedding from the nerve of $\beta_{k,n}$ to the nerve of $\alpha_n$, but there is no guarantee that $\beta_{k,n+1} \succ \beta_{k,n}$, nor that the members of $\beta_{k,n}$ are contractible.

But we can do better; since the diameter of the members of $\alpha_n$ are decreasing as $n$ increases, then for every $n$ there exists some $m$ large enough such that every member of the irreducible cover $\beta_{k,n+m}$ has a diameter smaller than some Lebesgue number of $\beta_{k,n}$, yielding a surjective simplicial map $T(k)^{n+m}_n : K(\beta_{k,n+m}) \to K(\beta_{k,n})$.

Assume that $k$ is fixed: we have a subsequence $\{\beta_{k,\phi(n)}\}_{n \in \mathbb{N}} \equiv \{\beta_{\phi(n)}\}_{n \in \mathbb{N}}$ of $\{\beta_{k,n}\}_{n \in \mathbb{N}} \equiv \{\beta_n\}_{n \in \mathbb{N}}$ made up of irreducible covers and endowed with surjective simplicial maps $T_{\phi(n)} : K(\beta_{\phi(n+1)}) \to K(\beta_{\phi(n)})$ making up a directed set $\{K(\beta_{\phi(n)}), T_{\phi(n)}\}_{n \in \mathbb{N}}$.

We recover Proposition 2.10 for the projective limit

$$\lim_{\leftarrow}(K(\beta_{\phi(n)}), T_{\phi(n)}),$$

although with better quantitative control. This is due to the results in Section 2.7, and because the dimension of $K(\beta_{\phi(n)})$ is bounded by the dimension of $V$ at every stage (Lemma 2.4).

If $\epsilon(\phi(n))$ is the largest diameter of a member in $\beta_{\phi(n)} = \{ B_i \mid i \in I(\phi(n)) \}$, we assume that $n$ is large enough so that $\epsilon(\phi(n))$ is smaller than the injectivity radius of $(V,g)$. Then for every $i$ in $I(\phi(n))$ we can choose some $b_i$ in $B_i$ such that $d(b_i, b_j) \leq \epsilon(\phi(n))$ whenever $B_i \cap B_j \neq \emptyset$, and identify $b_i$ with the 0-simplex $[b_i]$ that corresponds to $B_i$.

Let $x_{\phi(n)} : V \to |K(\beta_{\phi(n)})|$ be an Euclidean realization of $K(\beta_{\phi(n)})$ (Section 2.5). Embed the 1-simplices of $|K(\beta_{\phi(n)})|$ in $V$ using a Lipschitz map $y^1_{\phi(n)}$ so that the image of the 1-simplex $[b_i, b_j]$ corresponds to the distance minimizing path or geodesic between the points $b_i$ and $b_j$, that we regard as a rectifiable path (or current) in $V$.

We observe (see [Gro2]):

**Lemma 2.15.** Endow the set of 0-simplices in $|K(\beta_{\phi(n)})|$, namely $\Delta_0(|K(\beta_{\phi(n)})|)$, with the distance induced by the embedding $y^1_{\phi(n)}$ of the 1-simplices in $(V,d)$, extending it to all $\Delta_0(|K(\beta_{\phi(n)})|)$ in the natural way; denote such a distance by $d^0_{\phi(n)}$. Then, as $n$ goes to infinity, the metric space $(\Delta_0(|K(\beta_{\phi(n)})|), d^0_{\phi(n)})$ converges to $(V,d)$ in the Gromov-Hausdorff sense.

Consider $D_*(V)$, the graded $\mathbb{Z}$-module of general currents on $V$, and for every $\alpha$ in $\mathcal{C}_V$ let $\mathcal{P}_*([K(\alpha)])$ denote the graded $\mathbb{Z}$-module of polyhedral currents on the Euclidean realization of $K(\alpha)$. Hence if $y : |K(\alpha)| \to V$ is of Lipschitz type, then we get a linear map $y_* : \mathcal{P}_*([K(\alpha)]) \to D_*(V)$.

Let $M \equiv M_g$ be the mass norm on $D_*(V)$ induced by the Riemannian metric $g$. A fundamental fact, to be found in [Fed], asserts that the closure in $D_*(V)$ with respect to $M$ of pushforwarded polyhedral currents by Lipschitz maps into $V$ is $R_*(V)$, the $\mathbb{Z}$-module of rectifiable currents on $V$. An important sub-module of $R_*(V)$, denoted by $I_*(V)$, is the $\mathbb{Z}$-module of integral currents; it consists of rectifiable currents whose

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2See [Fed] for more details.
boundary is also rectifiable. The choice of an atlas on $V$ and elementary constructions from geometric measure theory give:

**Proposition 2.16.** $\mathcal{R}_*(V)$ and $\mathcal{I}_*(V)$ depend on the Lipschitz structure on $V$ chosen.

Embed now the 2-simplices of $|K(\beta_{(\phi(n))})|$ in $V$ by means of a Lipschitz map $y^2_{\phi(n)}$ using the geodesics that correspond to the 1-simplices as their boundary, straightening them as much as possible, so that in the process their mass (with respect to $g$) tends to minimize.

Then proceed inductively to get, for every $n$ and each $j$ not bigger than the dimension of $V$, a Lipschitz embedding $y^j_{\phi(n)} : \Delta_j (|K(\beta_{(\phi(n))})|) \rightarrow V$ of the $j$-simplices in the Euclidean realization of $K(\beta_{(\phi(n))})$ inside $V$, all whose images have a diameter not larger than $\epsilon (\phi(n))$; we obtain a map at the level of currents

$$y_{\phi(n)} : \mathcal{P}_* (|K(\beta_{(\phi(n))})|) \rightarrow \mathcal{I}_* (V) \subset \mathcal{R}_* (V).$$

More precisely, if $\vec{i}$ is in $\Lambda_j (\beta_{(\phi(n))})$, then $y_{\phi(n)}|\sigma^j_{\vec{i}}$ is almost minimal among those rectifiable currents whose boundary is $\partial y_{\phi(n)}|\sigma^j_{\vec{i}} = y_{\phi(n)}|\partial \sigma^j_{\vec{i}}$, for every $j \leq \text{dim}V$. By almost minimal we mean that the minimum of mass might not occur, however there is a sequence of Lipschitz maps leading to an infimum.

This process of approximation gives:

**Theorem 2.** Let $(V,g)$ be a smooth closed Riemannian manifold. Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}_V$, with $\alpha_{n+1} \triangleright \alpha_n$, and such that the diameter of each member of $\alpha_n$ goes to zero as $n$ goes to infinity. Then for every $k$ and every positive $\epsilon$ there exists some $m \equiv m(\epsilon)$, a cover $\beta_{k,m}$ minimizing $S_k(\alpha_m)$, and a Lipschitz map $y_m : |K(\beta_{k,m})| \rightarrow V$ such that

$$M_g (y_m |K(\beta_{k,m})| - V) < \epsilon,$$

where $V = \|\nabla\| \wedge \nabla$ is the current representing $V$, and $|K(\beta_{k,m})|$ is the polyhedral current that corresponds to an Euclidean realization of $K(\beta_{k,m})$. If the dimension of $V$ is not 4, such a map is independent of the Lipschitz structure on $V$.

**Remark 2.17.** We know, thanks to the work of E. Moise, S. Donaldson and D. Sullivan, that only in dimension 4 there are smooth manifolds that are homeomorphic but not Lipschitz equivalent.

### 3 The category of $\Gamma$-spaces

We denote by $\Gamma$ a countable or discrete group or semigroup whose cardinality is $\aleph_0$. Let $\rho : \Gamma \rightarrow \text{Map}(V,V)$ be a representation of $\Gamma$ on the set of mappings of $V$, where we understand, if $\Gamma$ is a group, that $\rho(\Gamma)$ is a sub-group of $\text{Homeo}(V)$, the group of homeomorphisms on $V$. If $\Gamma$ is a semigroup, then $\rho(\Gamma)$ is a sub-semigroup of $\text{End}(V)$, the semigroup of endomorphisms on $V$. We denote a structure of this type by a tuple $(V,\Gamma,\rho)$, and speak of a system, a representation of $\Gamma$, or a $\Gamma$-space.

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3Visit Section 2.5 if needed, for the nomenclature used.
Remark 3.1. We restrict to discrete groups and semigroups since:

1. Some of the results are not valid for arbitrary $\Gamma$’s.

2. All the examples in this article belong to this class.

We identify the totality of those structures with the objects in the category of $\Gamma$-spaces. The standard morphisms between objects in this category are:

1- Conjugations: Two systems $(V, \Gamma, \rho)$ and $(W, \Gamma, \rho')$ are said to be conjugated if there exists a homeomorphism $x : V \to W$ that interwinds the action of $\Gamma$, i.e. a homeomorphism between $V$ and $W$ that is $\Gamma$-equivariant. The notion of being conjugated is a strong equivalence relation; not only the underlying (topological-geometric) spaces in question are homeomorphic, moreover the dynamics induced by the maps are, up to a continuous change of coordinates say, equivalent.

2- Factors/Extensions: $(V, \Gamma, \rho)$ is said to be an extension of $(W, \Gamma, \rho')$, or $(W, \Gamma, \rho')$ is said to be a factor of $(V, \Gamma, \rho)$, if there is a continuous surjection $x : V \to W$, so that whenever $\gamma$ is in $\Gamma$ we have $x \cdot \rho(\gamma) = \rho'(\gamma) \cdot x$.

Remark 3.2. Of course if $(V, \Gamma, \rho)$ is both a factor and an extension of $(W, \Gamma, \rho')$ then both systems are conjugated.

Remark 3.3. The advantage of using the language of categories in this context is that some operations of algebraic topology (for example the loop functor, the suspension functor and the smash product) can be used as self-functors. By those means one obtains new systems from known ones (see Section 5.3.3).

Consider the action of $\rho(\Gamma)$’s inverses on elements of $C_V$. If $\gamma$ is in $\Gamma$ we have a map $\rho(\gamma) : C_V \to C_V$ and also an induced map $\rho(\gamma)^{-1} : C_V \to C_V$, where in the case of semigroups we understand that $\rho(\gamma)^{-1}A$ is given by $V \cap \{ v \mid \rho(\gamma)v \in A \}$ for every subset $A$ of $V$. Whenever $F$ is a finite subset of $\Gamma$ and $\alpha$ is in $C_V$ we set

$$\alpha_F := \bigcap_{\gamma \in F} \rho(\gamma)^{-1}\alpha.$$

In what follows we describe $K(\alpha_F)$ as $F$ increases both from a quantitative and a qualitative perspective.

3.1 Exponential simplicial growth: topological entropy

Assume that $\Gamma$ is generated by a finite subset of elements, say $H$, where we assume, if $\Gamma$ is a group, that $H$ contains all its inverses, i.e. that $H = H^{-1}$. Then whenever $F$ is a finite subset of $\Gamma$ we define its boundary with respect to $H$, that we denote by $\partial_H F$, as the subset of $F$ made up of those $\gamma$’s such that $h\gamma$ is not in $F$ for some $h$ in $H$. Consider an increasing sequence of subsets exhausting $\Gamma$, say $\{F(n)\}_{n \in \mathbb{N}}$. Such a sequence is said to be of Følner type if the quotient between $|\partial_H F(n)|$ and $|F(n)|$ goes to zero as $n$ goes to infinity. If such a sequence exists, then $\Gamma$ is said to be amenable (see [Gro2] for more about this).
Remark 3.4. Since all the results in Section 2.7 are valid in this context, we will not repeat analogous statements unless this is relevant; one should replace \(c(n)\) by \(c(|F(n)|)\), and \(\alpha_n\) by

\[
\bigcap_{\gamma \in F(n)} \rho(\gamma)^{-1} \alpha,
\]

for example. We will write \(\text{ent}_k(\alpha, \Gamma, \rho, c)\) instead of \(\text{ent}_k(\alpha_n, c(n))\) in what follows, and similarly for \(\text{Dim}(\alpha, \Gamma, \rho, c)\).

Consider the standard sequence \(\{c(n)\}_{n \in \mathbb{N}} = \{n\}_{n \in \mathbb{N}}\) (see Remarks 2.14 and 3.4). The next Lemma asserts that the upper and lower limits in Theorem 1 for \(\text{ent}_0\) coincide.

**Lemma 3.5.** Let \(\{F(n)\}_{n \in \mathbb{N}}\) be a Følner sequence for \(\Gamma\). Then the following limit exists, and is independent of the Følner sequence:

For every open cover \(\alpha\)

\[
\lim_n \frac{\log S_0(\alpha F(n))}{|F(n)|} =: \text{ent}_0(\alpha, \Gamma, \rho).
\]

**Proof.** The proof follows from a general convergence result for subadditive functions, known as the Orstein-Weiss Lemma; a proof can be found in [Gro3]. To use such a result it suffices to note that (see [Wal], for example)

\[
|\Delta_0(\alpha \cap \beta)| = \mathcal{G}_0(\alpha \cap \beta) \leq \mathcal{G}_0(\alpha) \mathcal{G}_0(\beta),
\]

hence

\[
S_0(\alpha \cap \beta) \leq S_0(\alpha) S_0(\beta).
\]

Remark 3.6. If the growth of the number of simplices in \(\{K(\alpha_{F(n)})\}_{n \in \mathbb{N}}\) is strictly exponential, then \((V, \Gamma, \rho)\) is said to have non zero finite topological entropy with respect to the cover \(\alpha\). The supremum of \(\text{ent}_0(\alpha, \Gamma, \rho)\) among all \(\alpha\)'s in \(C_V\), that might not be finite, is known as the topological entropy for \((V, \Gamma, \rho)\), and denoted by \(\text{ent}_0(V, \Gamma, \rho)\).

Recall some constructions/results in Section 2.7. For a fixed \(k\) there exists a sequence \(\{\beta_{k,F(n)}\}_{n \in \mathbb{N}}\) of irreducible covers that achieve, for each \(n\), the minimum of \(S_k(\alpha_{F(n)})\). In particular, by Lemma 3.5 we have that \(\text{ent}_0(\alpha, \Gamma, \rho)\) is given by

\[
\lim_n \frac{\log S_0(\alpha_{F(n)})}{|F(n)|},
\]

hence

\[
\text{ent}_0(\alpha, \Gamma, \rho) = \lim_n \frac{\log \mathcal{G}_0(\beta_{0,F(n)})}{|F(n)|}.
\]

On the other hand, since

\[
S_0(\alpha_{F(n)}) = S_0\left( \alpha_{F(n-1)} \cap \bigcap_{\gamma \in F(n) \setminus F(n-1)} \rho(\gamma)^{-1} \alpha \right)
\]
we observe that
\[ G_0(\beta_{0,F(n)}) = S_0(\alpha_{F(n)}) \leq G_0(\beta_{0,F(n-1)}) S_0(\alpha)^{|F(n)\setminus F(n-1)|}, \]
where we recall that \( S_0(\alpha) \) is the best lower bound for \( G_0(\beta) \) among those \( \beta \)'s finer than \( \alpha \), to infer by recursion:

**Proposition 3.7.** For every \( \alpha \) the number of 0-simplices in the sequence \( \{ K(\beta_{0,F(n)}) \}_{n \in \mathbb{N}} \) is bounded at every stage by
\[ G_0(\beta_{0,F(n)}) \leq S_0(\alpha)^{|F(n)|}. \]

In particular, Theorem 1 ensures that for each \( k \) we have the bound
\[ \mathrm{ent}_k^\ell(\alpha, \Gamma, \rho) \leq (k + 1) \log S_0(\alpha). \]

**Remark 3.8.** The sensibility of the simplicial growth with respect to the initial condition \( \alpha \) can be grasped thanks to Proposition 3.7. Indeed, for every \( n \) and \( k \) we have
\[ S_k(\alpha_{F(n)}) \leq S_0(\alpha)^{(k+1)|F(n)|}. \]

### 3.2 Convergence of nerves: e-systems

Let \( \rho : \Gamma \to \text{Map}(V, V) \) be a representation of the group (or semigroup) \( \Gamma \) acting on \( V \), and choose some metric \( d_V \) on \( V \). We say \( (V, \Gamma, \rho) \) is expansive, or has property-e, if there exists a constant \( \epsilon \) strictly larger than zero, such that for every \( u \) different from \( v \) there exists some \( \gamma \) in \( \Gamma \) with \( \rho(\gamma)^*d_V(u, v) := d_V(\rho(\gamma)u, \rho(\gamma)v) \) larger than \( \epsilon \).

**Remark 3.9.** Every \( \epsilon \) that satisfies the condition given before is called an e-constant for \( (V, \Gamma, \rho) \). The e-constants depend on the given metric \( d_V \), but the existence of those constants does not (see Lemma 3.10), and therefore one can omit the metric and say that \( (\Gamma, \rho) \) acts expansively on \( V \), or that \( (V, \Gamma, \rho) \) has property-e.

Given \( \alpha = \{ A_i \mid i \in I \} \in C_V \) we say that it is a generator for \( (V, \Gamma, \rho) \), or a generating cover, if for every array \( \{ i(\gamma) \mid \gamma \in \Gamma \} \) in \( I \) the intersection
\[ \bigcap_{\gamma \in \Gamma} \rho(\gamma)^{-1} A_{i(\gamma)} \]
contains at most one point. Thus if \( \alpha \) is a generator for \( (V, \Gamma, \rho) \) then the maximum of the diameter of all the open sets making up \( \alpha_F \) decreases as \( F \) increases, and it goes to zero as \( F \) exhausts \( \Gamma \).

The next Lemma provides a rough relation between the e-constant and the diameter of a generator (see [Wal]).

**Lemma 3.10.** Assume that \( (V, \Gamma, \rho) \) is expansive. Let \( \epsilon \) be some e-constant for some metric on \( V \), say \( d_V \). If \( \alpha = \{ A_i \mid i \in I \} \in C_V \) is such that the diameter of each \( A_i \) is at most \( \epsilon \), then \( \alpha \) is a generator for \( (V, \Gamma, \rho) \).
Proof. Assume that we have a pair of points \((u, v)\) in \(V\) that belong to \(\bigcap_{\gamma \in F} \rho(\gamma)^{-1} A_i\) for some \(i\) in \(I\), this for every subset \(F\) of \(\Gamma\). Then for every \(\gamma\) in \(F\) the distance between \(\rho(\gamma)u\) and \(\rho(\gamma)v\) is at most \(\epsilon\), the diameter of \(A_i\); since \(\epsilon\) is an e-constant then \(u\) and \(v\) coincide, and \(\alpha\) is a generator. 

Remark 3.11. In Theorem \([4]\) we will see that it is natural to estimate, for a fixed metric \(d_V\) on \(V\), the largest \(e\)-constant. In Corollary \([3.17]\) an upper bound will be provided if \(d = d_g\) for a Riemannian metric \(g\) that is regular enough.

Our next aim is to observe that a generator should be considered as being a good initial condition to reconstruct the skeleton of \(V\) using \((\Gamma, \rho)\); in other words, a generator generates a simplicial complex that is an acceptable approximation of the space.

For those purposes let \(\{F(n)\}_{n \in \mathbb{N}}\) be an increasing sequence exhausting \(\Gamma\), not necessarily amenable. If \(\alpha\) is a generator for \((V, \Gamma, \rho)\) then we have an infinite sequence of simplicial complexes and mappings making up a directed set \((K(\alpha_F(n)), T_n)_{n \in \mathbb{N}}\) that is convergent in the sense of Section \([2.6]\). From Proposition \([2.10]\) and Theorem \([2]\) we infer:

**Theorem 3.** Assume that \((V, \Gamma, \rho)\) has property-e, that \(\alpha\) is a generator for \((V, \Gamma, \rho)\). Then

1. When \(F\) exhausts \(\Gamma\) the complex \(K(\alpha_F)\) and \(V\) are homeomorphic.

2. If \((V, g)\) is Riemannian, closed and smooth, then for every positive \(\epsilon\) and every integer \(k\) we can find a subset \(F = F(\epsilon)\) of \(\Gamma\) such that the minimizer \(\beta_k(F)\) of \(S_k(\alpha_F)\) has the following property: there exists a Lipschitz map sending the polyhedral current corresponding to an Euclidean realization of \(K(\beta_k(F))\) into \(D_\epsilon(\gamma)\), leaving it at a distance not bigger than \(\epsilon\), with respect to \(M_g\), from the current \(V\) associated to \(V\). If the dimension of \(V\) is not 4, then the map is independent of the Lipschitz structure on \(V\).

As mentioned in Section \([3]\) the notion of conjugacy is an equivalence relation in the category of \(\Gamma\)-spaces. It is natural to expect that property-e, having both a metric and a set-theoretic characterization, will be invariant under conjugation. The next Lemma confirms this is the case:

**Lemma 3.12.** Assume that \((V, \Gamma, \rho)\) has property-e, and let \(x : V \to W\) be a homeomorphism between \(V\) and \(W\). Then \((W, \Gamma, \rho')\) is also expansive, where \(\rho'\), the induced representation of \(\Gamma\) in \(W\), is given by \(\rho'(\gamma) = x \cdot \rho(\gamma) \cdot x^{-1}\) for every \(\gamma\) in \(\Gamma\).

**Proof.** We use the set theoretic characterization of property-e. Assume that \(\alpha = \{A_i \mid i \in I\}\) is a generator for \((V, \Gamma, \rho)\), and let \(\alpha' = \{A'_i = x(A_i) \mid i \in I\}\) be its image in \(C_W\). If \(\{i(\gamma) \mid \gamma \in \Gamma\}\) is an array in \(I\) indexed by \(\Gamma\), then \(\bigcap_{\gamma \in \Gamma} \rho(\gamma)^{-1} A_i(\gamma)\) consists at most of one point, hence so does its image under \(x\). Since \(x\) is a homeomorphism

\[
x(\bigcap_{\gamma \in \Gamma} \rho(\gamma)^{-1} A_i(\gamma)) = \bigcap_{\gamma \in \Gamma} x \cdot \rho(\gamma)^{-1} \cdot x^{-1} \cdot x(A_i(\gamma)) = \bigcap_{\gamma \in \Gamma} \rho'(\gamma)^{-1} A'_i(\gamma),
\]

and we conclude that \(\alpha'\) is a generator for \((W, \Gamma, \rho')\). 

\(\square\)
Remark 3.13. From Theorem 3 and Lemma 3.12 we infer that if $(V, \Gamma, \rho)$ and $(W, \Gamma, \rho')$ are conjugated and if we know that one of them is expansive, then the simplicial complexes associated to the nerve of any of their generators evolve to complexes that are homeomorphic. This cannot be strengthened to differentiable mappings in full generality (see also Remark 2.17).

One counterexample is provided by expansive actions of $\mathbb{N}^d$ in $S^d$ (see Corollary 5.3). If $d \geq 7$ the work of M. Kervaire and J. Milnor yields a finite number of homeomorphic but not diffeomorphic $S^d$’s.

Other counterexamples are obtained if one glues these spheres, by connected sum, on manifolds that admit an expansive action (see Section 5); to the author’s knowledge, this was first done by T. Farrel and L. Jones for Anosov diffeomorphisms on $T^d$ (see [Fa-Jo]).

3.3 Choosing good generators, estimates for the $\epsilon$-constant

In what follows we consider a Riemannian manifold, compact and without a boundary, whose dimension is $d$, and whose Riemannian metric $g$ is at least of type $C^2$. If $Rc(g)$ denotes the Ricci tensor of $g$, we denote by $\lambda$ the biggest real number so that

$$Rc(g) \geq \lambda(d - 1)g$$

all over $V$. Let $d_g$ be the distance function induced by $g$; we denote by $\|B_R(v)\|_g$ the mass (or volume) of the ball of radius $R$ centered at $v$ whenever $R$ is a positive real number and $v$ is some point in $V$.

Let $S^d(\lambda)$ be the simply connected space of dimension $d$ whose sectional curvature is everywhere equal to $\lambda$; then $\|\widetilde{B}_R\|_\lambda$ denotes the mass of any ball of radius $R$ in $S^d(\lambda)$. If $D$ denotes the diameter of $(V, g)$ then by Bishop’s comparison (see [Be] or [Gro2] for example) whenever $R \leq D$ and $t \leq 1$ one has, for every point $v$ in $V$, the estimate

$$\|B_R(v)\|_g / \|B_{tR}(v)\|_g \leq \|\widetilde{B}_R\|_\lambda / \|\widetilde{B}_{tR}\|_\lambda.$$

Let $C^V_R$ denote the minimal number of balls of radius $R$ needed to cover $(V, g)$; it is not difficult to see that such a number of balls is not bigger than the largest value of $\|B_D(v)\|_g / \|B_{R/2}(v)\|_g$ as $v$ varies in $V$. Recalling Lemma 3.10 we conclude:

Theorem 4. Assume that $(V, \Gamma, \rho)$ has property-$\epsilon$, and let $\epsilon = \epsilon(g)$ be an expansivity constant for the distance function $d_g$ induced by $g$. If $Rc(g) \geq \lambda(d - 1)g$, where $d$ is the dimension of $V$, and the diameter of $(V, g)$ is $D$, then there exists a generator for $(V, \Gamma, \rho)$ whose cardinality is at most $\|\widetilde{B}_D\|_\lambda / \|\widetilde{B}_{\epsilon/4}\|_\lambda$.

Abbreviate $\|\widetilde{B}_D\|_\lambda / \|\widetilde{B}_{\epsilon/4}\|_\lambda$ by $\Theta(\lambda, D, \epsilon)(g) \equiv \Theta(\lambda(g), D(g), \epsilon(g))$, and observe that $\Theta(\lambda, D, \epsilon)(g)$ is invariant under scalings of $g$. In particular, if $\lambda$ is larger than zero, then $\Theta(\lambda, D, \epsilon)$ is not bigger than $\Theta(\lambda, \sqrt{\frac{\pi^2}{(d-1)^2}}, \epsilon)$ by Bonnet-Myers’ comparison (see [Be] or [Gro2] again).

It becomes clear that, for a fixed $g$, better estimates for the expansivity constant $\epsilon(g)$ will improve the upper estimate for the minimal number of zero simplices that the
nerve associated to a generator for \((V, \Gamma, \rho)\) can have; we denote that minimal number by \(|\Delta_0(V, \Gamma, \rho)|\). The next definition provides an upper bound for \(|\Delta_0(V, \Gamma, \rho)|\).

**Definition 3.14.** The real number \(\Theta(V, \Gamma, \rho)\) is defined as the best lower bound for \(\Theta(\lambda, D, \epsilon)(g)\) as \(g\) varies within the Riemannian metrics on \(V\) of type \(C^2\).

To obtain a lower bound for \(|\Delta_0(V, \Gamma, \rho)|\), consider the Følner sequence \(\{F(n)\}_{n \in \mathbb{N}}\) exhausting \(\Gamma\). Let \(\zeta \in C_V\) have \(\delta = \delta(g)\) as a Lebesgue number (for some metric \(d_g\) on \(V\)); if \(\alpha\) is a generator for \((V, \Gamma, \rho)\) then for \(t\) large enough the open cover \(\alpha_{F(t)}\) has a diameter not bigger than \(\delta\), hence \(\alpha_{F(t)}\) is finer than \(\zeta\), therefore by the definition of \(S_0\) (see Section 2.4) the estimate

\[
\text{ent}_0(\zeta, \Gamma, \rho) \leq \text{ent}_0(\alpha_{F(t)}, \Gamma, \rho) = \text{ent}_0(\alpha, \Gamma, \rho)
\]

follows, hence:

**Lemma 3.15.** If \((V, \Gamma, \rho)\) has property-e, then the supremum of \(\text{ent}_0(\zeta, \Gamma, \rho)\) as \(\zeta\) varies in \(C_V\) is a maximum, denoted by \(\text{ent}_0(V, \Gamma, \rho)\), and is attained when \(\zeta\) is a generating cover.

From Theorem 4, Definition 3.14, Proposition 3.7 and Lemma 3.5 we get a relation between \(\Theta(V, \Gamma, \rho)\), \(|\Delta_0(V, \Gamma, \rho)|\) and \(\text{ent}_0(V, \Gamma, \rho)\).

**Theorem 5.** Assume that \((V, \Gamma, \rho)\) has property-e. Then

\[
e^{\text{ent}_0(V, \Gamma, \rho)} \leq |\Delta_0(V, \Gamma, \rho)| \leq \Theta(V, \Gamma, \rho).
\]

**Remark 3.16.** The sense and value of the Theorems for e-systems depend on the taste of the reader:

1. If she/he is interested in constructing \(V\) starting from a simplicial complex of lower complexity, it has been proved in Theorem 3 that a method to achieve such a task is to find a group or semigroup \(\Gamma\) together with a representation \(\rho\) such that \((V, \Gamma, \rho)\) has the expansive property.

2. Once \(\Gamma\) and \(\rho\) have been found, bounds on the 0-simplices of the initial complex are provided by Theorem 3, assuming that \(\text{ent}_0(V, \Gamma, \rho)\) and/or \(\Theta(V, \Gamma, \rho)\) have been computed.

3. From another perspective, if upper bounds for \(\Theta(V, \Gamma, \rho)\) are available, she/he has upper bounds for \(\text{ent}_0(V, \Gamma, \rho)\), the rate at which the approximation of \(V\) is achieved, and conversely.

Recall that the e-constant is the minimal distance that any two points in \(V\) become separated under the action of \((\Gamma, \rho)\) for a given distance function (not necessarily arising from a Riemannian metric) on \(V\), and it is of interest to know its order of magnitude (how large it is). To achieve that, observe that if \(\lambda\) and \(d \geq 2\) are fixed, the function \(\|B_R\|_\lambda\) is a strictly increasing function of \(R\).

Using Theorems 4 and 5 we can state:
Corollary 3.17. Let $g$ be a metric of type $C^2$, and let $\lambda$ be the biggest real number satisfying $Rc(g) \geq \lambda(d - 1)g$ all over $V$. Then every $\epsilon$ such that
\[ \|B_{\epsilon/4}\| \leq \frac{\|B_D\|_\lambda}{\exp(\text{ent}_0(V, \Gamma, \rho))} \]
is an $\epsilon$-constant for the system $(V, \Gamma, \rho)$ with respect to the distance $d_g$.

Remark 3.18. If some $\epsilon$-constant $\epsilon$ for a distance function $d = d_g$ has been found, from Lemma 3.10 every cover whose components have a diameter at most $\epsilon$ is a generator. An advantage of choosing the biggest $\epsilon$-constant allowed by Corollary 3.17 is that the intersection pattern of covers with larger diameter becomes simpler, hence the associated simplicial complex is of lower complexity. This gives better upper bounds for the simplicial growth in every dimension at every stage, as predicted by Proposition 3.7.

Remark 3.19. Being property-e invariant under homeomorphisms (or conjugation) of $\Gamma$-systems, then so are the numbers associated to $\text{ent}_0(V, \Gamma, \rho)$ and the minimal complexity that a generator can have. In contrast, if we intersect Remark 3.13 with N. Hitchin’s result asserting that in every dimension bigger than 8 and equal to either $1 \pmod{8}$ or $2 \pmod{8}$ there are exotic spheres not admitting metrics of positive scalar curvature (see [Law-Mi]), then the estimate in Corollary 3.17 becomes more intriguing (after normalization of volume or diameter, say), despite its simplicity.

4 Infinite dimensional examples

4.1 $\Gamma$-shifts

Consider a compact finite dimensional and connected manifold, say $V$, and infinitely many copies of it indexed by a discrete amenable group $\Gamma$. Hence the total space is $V := V^\Gamma$. Endow $V$ with the weakest topology that makes the projection in all the copies of $V$ continuous; then by Tychonov’s Lemma the space $V$ is compact for this topology. Moreover, the set of all open covers for $V$, namely $C_V$, coincides with $(C_V)^\Gamma$.

Elements in $V$ are maps $v : \Gamma \to V$, thus the natural representation of $\Gamma$ on $\text{Aut}(V)$ is given, for each $v$ in $V$ and every pair $\{\gamma, \gamma'\}$ in $\Gamma$, by
\[ (\rho(\gamma')v)(\gamma) = v(\gamma'\gamma). \]

A systematic study of $\Gamma$-spaces of this type, of $\Gamma$-invariant subsets of them (called $\Gamma$-subshifts), was presented in [Gro3]. Estimates on the complexity growth of those systems in the spirit of $\text{Dim}(V, \Gamma, \rho)$ are provided in [Gro3], but with an emphasis on metric and (co)homological observables. Instead of explaining results from [Gro3] (an interesting task indeed !) we obtain, for some $\tilde{\alpha}$ in $C_V$, estimates for $\text{Dim}(\tilde{\alpha}, \Gamma, \rho)$ and for the family $\{ \text{ent}_k(\tilde{\alpha}, \Gamma, \rho) \mid k \in \mathbb{N} \}$.

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4Thanks to Professor T. Friedrich for some key points on the subject.
Consider $\vec{\alpha} = \prod_{\gamma} \alpha(\gamma)$ in $C_{\mathcal{V}} = (C_{\mathcal{V}})^{\Gamma}$. Then $\vec{\alpha}(\gamma)$ denotes the component of $\vec{\alpha}$ indexed by the coordinate $\gamma$, namely $\alpha(\gamma)$. Natural constructions attributed to Künneth for Cartesian products of simplicial complexes (see [Hat] or [Led], for example) ensure that for every $k$ the set $\triangle_k(\vec{\alpha})$ of $k$-simplices in $K(\vec{\alpha})$ is decomposed in products of simplices in each of the $K(\alpha(\gamma))$ as $\gamma$ varies, i.e.

$$\triangle_k(\vec{\alpha}) = \coprod_{\{\vec{k} \mid |\vec{k}| = k\}} \prod_{\gamma \in \Gamma} \triangle_k(\alpha(\gamma)),$$

where the (disjoint) union is over all the vectors $\vec{k}$ in $\mathbb{N}^\Gamma$ such that

$$|\vec{k}| = \sum_{\gamma \in \Gamma} k(\gamma)$$

is equal to $k$. That decomposition extends naturally to the Abelian group $C_k(\vec{\alpha}, G)$ of $k$ chains on $K(\vec{\alpha})$ with coefficients in $(G, +)$ (see Section 2.3), and the boundary operator is compatible with such a decomposition (with the obvious plus or minus signs).

The representation $\rho : \Gamma \to \text{Aut}(\mathcal{V})$ induces (see Section 3) an action on $C_{\mathcal{V}}$. If $\vec{\alpha} = \prod_{\gamma \in \Gamma} \alpha(\gamma)$ then for every $\delta$ in $\Gamma$ we have $\rho(\delta)\vec{\alpha} = \prod_{\gamma \in \Gamma} \alpha(\delta(\gamma))$, i.e. the components of $\vec{\alpha}$ are translated by $\delta$, so that $(\rho(\delta)\vec{\alpha})(\gamma) = \alpha(\delta(\gamma))$.

Thus whenever $F$ is a finite subset of $\Gamma$ the refinement of $\vec{\alpha}$ under the action of the inverse of the elements in $F$ is given by

$$\vec{\alpha}_F := \bigcap_{\delta \in F} \rho(\delta)^{-1} \vec{\alpha} = \prod_{\gamma \in \Gamma} \bigcap_{\delta \in F} \alpha(\delta(\gamma)),$$

i.e. the component of $\vec{\alpha}_F$ in the coordinate $\gamma$ is given by the common refinement of the subset $\{ \alpha(\delta(\gamma)) \mid \delta \in F \}$ of components of $\vec{\alpha}$, so that $\vec{\alpha}_F(\gamma) = \bigcap_{\delta \in F} \alpha(\delta(\gamma))$.

Choose some $\gamma'$ in $\Gamma$ and consider $\vec{\alpha}$ as being:

- $\vec{\alpha}(\gamma) = \alpha_p$ if $\gamma = \gamma'$, where $\alpha_p$ is an irreducible cover whose nerve $K(\alpha_p)$ has dimension $p$, where $1 \leq p \leq d$, and $d$ is the dimension of $V$.

- $\vec{\alpha}(\gamma) = \{V\}$ if $\gamma \neq \gamma'$, where $\{V\}$ is the trivial cover for $V$.

Hence for every subset $F$ of $\Gamma$ the open cover $\vec{\alpha}_F$ is given by:

- $\vec{\alpha}_F(\gamma) = \alpha_p$ if $\gamma = \delta\gamma'$ for some $\delta$ in $F$.

- $\vec{\alpha}_F(\gamma) = \{V\}$ otherwise.

Making use of Künneth’s relations we check that $\text{Dim}K(\vec{\alpha}_F)$ is given by

$$|F| \text{dim}K(\alpha_p) = p|F|$$

for every subset $F$ of $\Gamma$, whence $\text{Dim}(\vec{\alpha}, \Gamma, \rho)$ is equal to $p$. 21
To estimate the simplicial growth we observe that
\[ G_0(\vec{\alpha}_F) = G_0(\alpha_p)^{|F|}, \]
and a little bit of algebra along the lines of Lemma 2.9 ensures, making use of Künneth’s relations, that
\[ \text{ent}_k(\vec{\alpha}, \Gamma, \rho) = \text{ent}_0(\vec{\alpha}, \Gamma, \rho) = \log|\Delta_0(\alpha_p)| \]
for every \( k \).

We conclude that \{c(n)\}_{n \in \mathbb{N}} = \{n\}_{n \in \mathbb{N}} is a controlling sequence both for the simplicial and the dimension growth of \{K(\vec{\alpha}_n)\}_{n \in \mathbb{N}} \equiv \{K(\vec{\alpha}_F(n))\}_{n \in \mathbb{N}}.

### 4.2 Pyramids and prismatic covers

Consider on \( V \) irreducible covers all of whose members have at least one point in common; if \( \alpha \) is an open cover with this property we say that \( \alpha \) is a prismatic cover for \( V \). Prismatic covers satisfy:

1. \( \dim K(\alpha) = G_0(\alpha) - 1 \).

2. If \( k \) is less or equal than \( \dim K(\alpha) \) then
\[ G_k(\alpha) = \left( \frac{G_0(\alpha)}{k + 1} \right) + G_{k-1}(\alpha). \]

Observe that if \( V \) is connected, then a prismatic cover whose nerve has dimension smaller or equal than the dimension of \( V \) always exists (indeed, prismatic covers get rid of all the topology on \( V \), if any).

Here \((\mathbb{R}^\Gamma, \| \cdot \|_{l_1}) \equiv (X, \| \cdot \|_X)\) is the Banach space of arrays of real numbers indexed by elements of a discrete group \( \Gamma \) with the \( l_1 \)-norm. Elements in \( X \) are functions \( x : \Gamma \to \mathbb{R} \) such that \( \|x\|_1 := \sum_{\gamma \in \Gamma} |x(\gamma)| < \infty \), and a basis for this linear space is given by the maps \{ \( e_\gamma : \Gamma \to \{0, 1\} \) \( | \gamma \in \Gamma \) \} satisfying \( e_\gamma(\gamma') = 1 \) if \( \gamma = \gamma' \), and zero otherwise, so every element \( x \in X \) can also be written as a sum \( \sum_{\gamma \in \Gamma} x(\gamma) e_\gamma \).

The predual of \((\mathbb{R}^\Gamma, \| \cdot \|_1)\) is the Banach space \((\mathbb{R}^\Gamma, \| \cdot \|_{\infty}) = (Y, \| \cdot \|_Y)\) of functions \( y : \Gamma \to \mathbb{R} \) such that \( \|y\|_\infty := \sup_{\gamma \in \Gamma} |y(\gamma)| < \infty \). A basis for this linear space is given by the maps \{ \( e_\gamma : \Gamma \to \{0, 1\} \) \( | \gamma \in \Gamma \) \} such that \( \langle e_\gamma, e_{\gamma'} \rangle = 1 \) if \( \gamma = \gamma' \), and zero if \( \gamma \neq \gamma' \), therefore \( \langle e_\gamma, x \rangle = x(\gamma) \) for every \( x \in X \).

On \( X \) one can also consider families of seminorms given by
\[ p_C(x) := \max \{ |\langle y, x \rangle| := \left| \sum_{\gamma \in \Gamma} y(\gamma)x(\gamma) \right| \ y \in C \}, \]
where \( C \) ranges over arbitrary finite subsets on \( Y \). The open sets associated to that family of seminorms generate a Hausdorff topology on \( X \); an application of Tychonov’s Lemma ensures that bounded sets in \( X \) are compact in this topology.

Let \( V := X \cap \{ x \ | \ |x|_1 \leq 1 \ , \ x(\gamma) \geq 0 \ \text{for every} \ \gamma \} \) be the part of the unit ball in \((\mathbb{R}^\Gamma, \| \cdot \|_1)\) all of whose coordinates are non-negative, and consider the topology
induced by the family of seminorms \( \{ p_C \} \). Then \( V \) is closed, convex, Hausdorff and compact, and can be spanned by convex linear combinations of its extremal points, say \( E(V) \), that consists of the set \( \{ e_\gamma \mid \gamma \in \Gamma \} \) together with the origin in \( X \), that we denote by \( e_0 \) if no confusion arises. We regard \( e_0 \) as the apex of the pyramid \( V \), and say that the convex set spanned by the rest of the extreme points is the base of \( V \). Note that the base of \( V \) consists of maps \( v : \Gamma \to [0, 1] \) such that \( \|v\|_1 = \sum_{\gamma \in \Gamma} v(\gamma) = 1 \).

We pick the family \( \{ p_C \} \) as follows: if \( F \) is a finite subset of \( \Gamma \) we denote by \( p_F \) the seminorm on \( X \) given by

\[
p_F(x) = \max_{\gamma \in F} |x(\gamma)|.
\]

The family \( \{ p_F \mid F \text{ is a finite subset of } \Gamma \} \) of seminorms on \( X \) provide the desired properties on \( V \). Note that given \( T \) bigger than zero:

- On one side \( p_F(x) < T \) if and only if for every \( \gamma \) in \( F \) we have \( |x(\gamma)| < T \), hence

\[
\{ x \mid p_F(x) < T \} = \bigcap_{\gamma \in F} \{ x \mid p_\gamma(x) < T \}.
\]

- On the other side \( p_F(x) > T \) if and only if there exists some \( \gamma \) in \( F \) with \( |x(\gamma)| > T \), therefore

\[
\{ x \mid p_F(x) > T \} = \bigcup_{\gamma \in F} \{ x \mid p_\gamma(x) > T \}.
\]

Choose a positive number \( a \) small enough (in fact smaller than 0.5), and for every \( \gamma \) in \( \Gamma \) define an open cover \( \alpha[\gamma] \) with two members for \( V \) as follows:

- The open set \( A_\gamma \) is given by those \( v \) in \( V \) such that \( p_\gamma(v) > 0.5 - a \). Thus \( A_\gamma \) consists of those elements in \( V \) whose distance from the vertex \( e_\gamma \) is smaller than \( 0.5 + a \), the distance being the one induced by the norm \( \| \|_1 \) on \( X \).

- The open set \( A'_\gamma \) is given by those \( v \) in \( V \) such that \( p_\gamma(v) < 0.5 + a \), i.e. \( A'_\gamma \) contains the apex of \( V \) and those points whose distance from the vertex \( e_\gamma \) is larger than \( 0.5 - a \).

We construct two examples in this setup, one that does not use the group structure of \( \Gamma \) at all, meanwhile the other uses such a structure.

1. In this example \( \Gamma \) could be any denumerable infinite set. Using the covers \( \alpha[\gamma] \) yet mentioned we construct, for every finite subset \( F \) of \( \Gamma \), an open cover \( \alpha[F] \) whose cardinality is \( |F| + 1 \), containing two types of open sets:

- For each \( \gamma \) in \( F \) we have an open set \( A_\gamma \), as before. Observe that the union \( \bigcup_{\gamma \in F} A_\gamma \) consists of those points in \( V \) with \( p_F(v) > 0.5 - a \).

- An open set \( A'_F \) is given by those \( v \) in \( V \) such that \( p_F(v) < 0.5 + a \), therefore \( A'_F \) is the intersection \( \bigcap_{\gamma \in F} A'_\gamma \).
One verifies that $\alpha[F]$ is a prismatic cover for $V$ whenever $F$ is a finite subset of $\Gamma$, also that $\alpha[F']$ is finer than $\alpha[F]$ whenever $F$ is a subset of $F'$.

Let $\{F(n)\}_{n \in \mathbb{N}}$ be an increasing sequence of subsets exhausting $\Gamma$, and consider the sequence of complexes $\{K(\alpha[F(n)])\}_{n \in \mathbb{N}}$. Since $\alpha[F(m)] > \alpha[F(n)]$ whenever $m > n$ we get a directed sequence $\{K(\alpha[F(n)]), T_n\}_{n \in \mathbb{N}}$ of complexes and maps, thus the constructions/results in Section 2.7 can be used.

Being $\alpha[F(n)]$ a prismatic cover for every $n$, the simplex $K(\alpha[F(n)])$ has the maximal number of simplices allowed, say. It is easy to see that $\{ \log |F(n)| \}_{n \in \mathbb{N}}$ controls the simplicial growth of $\{ K(\alpha[F(n)]) \}_{n \in \mathbb{N}}$, and for every $k$ we have

$$\text{ent}_k(\alpha[F(n)], \log |F(n)|) = k + 1,$$

showing that the estimates in Theorem 1 are sharp.

In this example $\{ |F(n)| \}_{n \in \mathbb{N}}$ controls the dimension growth of $\{ K(\alpha[F(n)]) \}_{n \in \mathbb{N}}$, and we see that $\text{Dim}(\alpha[F(n)], |F(n)|)$ is equal to one.

2. We consider a representation $\rho : \Gamma \rightarrow \text{Aut}(V)$ that leaves the apex fixed and translates the coordinates in the base, so that if

$$v = v(0)e_0 + \sum_{\gamma \in \Gamma} v(\gamma)e_{\gamma},$$

then

$$\rho(\delta)v = v(0)e_0 + \sum_{\gamma \in \Gamma} v(\delta\gamma)e_{\gamma}.$$

For every $\gamma$ the open cover $\alpha[\gamma]$ is given by $\{A_\gamma, A'_\gamma\}$, hence $\rho(\delta)\alpha[\gamma]$ is just $\alpha[\delta\gamma]$. Therefore for every finite subset $F$ of $\Gamma$ the cover $\alpha[\gamma]_F$ is given, according to Section 3 by

$$\bigcap_{\delta \in F} \alpha[\delta^{-1}\gamma] = \bigcap_{\delta \in F} \alpha[\delta^{-1}\gamma].$$

The covering $\alpha[\gamma]_F$ is not irreducible if $F$ has at least two elements, however the cover $\alpha[F^{-1}\gamma]$ constructed in 1 is finer than $\alpha[\gamma]_F$ (and prismatic). Hence for every $k$ whenever $F$ is a finite subset of $\Gamma$ we have the equality

$$S_k(\alpha[\gamma]_F) = G_k(\alpha[F^{-1}\gamma]).$$

As in 1 choose an increasing family of subsets $\{F(n)\}_{n \in \mathbb{N}}$ exhausting $\Gamma$, to infer that the simplicial growth up to dimension $k$ for the sequence $\{ K(\alpha[\gamma]_{F(n)}) \}_{n \in \mathbb{N}}$ is polynomial of degree $(k + 1)$, this for every $k$, and the dimension growth is linear, and equal to one.

**Remark 4.1.** Being the simplicial growth of polynomial type the statements in Remark 3, 8 are not relevant.
5 Finite dimensional manifolds and property-e

As explained in Section \[3.2\] if \( V \) admits an expansive action of a group (semigroup) \( \Gamma \) we can, in a precise sense, reconstruct a complex that is homeomorphic to \( V \) if we take as an initial condition the nerve associated to an open cover that is a generator for \((V, \Gamma, \rho)\). Moreover, if \( V \) is endowed with a Riemannian metric and its dimension is bigger than one all the estimates in Section \[3.3\] for the e-constant, the (minimal) complexity of generating covers, and their relation to the (topological) entropy provide interesting information (sometimes without much effort).

If the dimension of \( V \) is either one or two the classification of closed orientable manifolds is complete and extremely simple. In those cases if \( V \) and \( W \) are homotopically equivalent finite and boundaryless simplicial complexes then they are homeomorphic, and even diffeomorphic if they are endowed with a smooth structure.

In the context of algebra, the simplest groups and semigroups are \( \mathbb{Z} \) and \( \mathbb{N} \) respectively. Thus to understand expansive actions of groups and/or semigroups on closed orientable manifolds it is natural to begin with the simplest examples, i.e. with \( \mathbb{Z} \) and/or \( \mathbb{N} \) actions on closed (orientable) manifolds, to then consider Abelian actions of products of those.

5.1 Dimension one

The only closed one dimensional manifold up to homeomorphism is \( S^1 \). If \( \Gamma \) is equal to \( \mathbb{N} \), then \((S^1, \mathbb{N})\) is expansive if one considers the \( \mathbb{N} \)-action \( n : \theta \mapsto k^n \theta \) for some fixed \( k \) in \( \mathbb{Z} \) whose absolute value is bigger than one. If no confusion arises we denote such a representation by \( f \) so that \( f^n(\theta) = kf^{n-1}(\theta) \) whenever \( n \) is a natural number.

Consider for simplicity the case when \( k \) is equal to two. Let \( \alpha \) be the open cover of \( S^1 \) given by \( \{ ]-a, \pi \} ]a, \pi ]-a, a \} \) for some positive \( a \) that is small enough. Then \( \alpha \) is a generator, and it is easy to see that \( \text{ent}_0(\alpha, \mathbb{Z}, f) = \text{ent}_0(S^1, \mathbb{Z}, f) = \log 2 \) (see Figure 1).

![Figure 1](image-url)

**Figure 1:** A schematic evolution of the nerve of \( \bigcap_{n=0}^{T} f^{-n} \alpha \) when \( f : S^1 \to S^1 \) is given by \( f(\theta) = 2\theta \). Here \( \alpha \) consists of two semicircles overlapping in a neighborhood of \( \theta = 0 \) and \( \theta = \pi \), with \( T \) being equal to 0, 1 or 2 (from left to right).
5.2 Dimension two

Closed manifolds of dimension two, also known as compact Riemann surfaces, are the basic test of (almost) every theory that wishes to be extended to higher dimensions. The classification of them up to diffeomorphism is extremely simple, and everyone can distinguish among them by the number of holes (or the intersection form in the first homology group with $\mathbb{Z}_2$ coefficients). Within the orientable ones we will construct expansive actions of either $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{N}^2$, depending on the genus.

5.2.1 Actions of $\mathbb{Z}$

Expansive homeomorphisms (or expansive actions of $\mathbb{Z}$) in compact Riemann surfaces of positive genus were constructed in [Ob-Re]. We briefly explain some ideas.

Consider the standard Anosov homeomorphism on the 2-torus (see Section 5.3.1 for general definitions), namely the one induced by the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

on $\mathbb{R}^2$. Let $h : T^2 \to T^2$ be the induced homeomorphism that turns out to be expansive, and note that if $h : W \to W$ induces an expansive action of $\mathbb{Z}$ then so does $h^k : W \to W$ whenever $k$ is a positive integer.

Let $\Sigma_g$ denote the orientable Riemann surface of genus $g$, and consider a branched cover $x : \Sigma_g \to \Sigma_1 = T^2$ to construct a homeomorphism $f : \Sigma_g \to \Sigma_g$ by lifting $h^k$ through $x$ for some $k$. If the pair $(x, f)$ can be constructed, then $f : \Sigma_g \to \Sigma_g$ provides an expansive action of $\mathbb{Z}$ on $\Sigma_g$ (observe that this is not true in higher dimensions because the branch set could have strictly positive dimension, and the dynamics of the lifted map, namely $f$, need not be expansive therein).

Considering standard relations that the map $x : \Sigma_g \to T^2$ should satisfy at the level of the fundamental groups to achieve a branched cover, lifts of iterates of $h^3 : T^2 \to T^2$ are constructed for every $g$ bigger than one in [Ob-Re], providing the desired expansive systems $(\Sigma_g, \mathbb{Z})$ whenever $g$ is different from zero.

Some years later K. Hiraide and J. Lewowicz (see [Hir] and [Lew]) found a natural relation between expansive actions of $\mathbb{Z}$ on hyperbolic Riemann surfaces and neat constructions/results on Teichmüller theory due to W. Thurston (see [Th]). The result in [Hir-Lew] can be rephrased using the language developed in [Th] as follows:

Theorem 6. (Hiraide-Lewowicz) Let $\Sigma_g$ be a closed and orientable hyperbolic Riemann surface, and assume that $f : \Sigma_g \to \Sigma_g$ induces an expansive action of $\mathbb{Z}$ (those actions are known to exist due to the constructions in [Ob-Re]). Let $\mathcal{T}(\Sigma_g)$ denote the Teichmüller space of $\Sigma_g$. Then for some $f^*$ conjugated to $f$ the induced action of $f^*$ on the closure of $\mathcal{T}(\Sigma_g)$, denoted by

$$\overline{\mathcal{T}}(f^*) : \overline{\mathcal{T}}(\Sigma_g) \to \overline{\mathcal{T}}(\Sigma_g),$$

has exactly two fixed points. Those points are on the boundary of $\overline{\mathcal{T}}(\Sigma_g)$ and correspond to projective classes of mutually transverse measured laminations on $\Sigma_g$. One of those projective classes has a representative that expands under the action of $f^*$, while the other contracts (one says that $f$ is conjugated to a pseudo-Anosov diffeomorphism).
In [Hir] and [Lew] it is stated that $S^2$ does not admit an expansive action of $\mathbb{Z}$.

5.2.2 Actions of $\mathbb{N}$

It is rather easy to see that $V$ admits an expansive action of $\mathbb{N}$ only if there exists a map $f : V \to V$ whose degree it at least two. A necessary condition for the existence of such a map is that the simplicial volume of $V$ is equal to zero (see [Gro2]), and then $V$ must be either the two sphere or the two torus. On $T^2$ an expansive action of $\mathbb{N}$ can be easily constructed, although in $S^2$ it is not possible to achieve that (see Section 5.3.2 for both issues).

Therefore to complete the program of reconstructing every orientable closed manifold whose dimension is two from a simplicial complex that has a simpler structure we are led to consider higher rank actions.

5.2.3 Actions of $\mathbb{N}^2$

An expansive action of $\mathbb{N}^2$ on $T^2 = S^1 \times S^1$ can be achieved using expansive actions of $\mathbb{N}$ on $S^1$ (see Section 5.1) if one considers the remarks in Section 5.3.3 concerning cartesian products of expansive systems.

An expansive action of $\mathbb{N}^2$ on $S^2$ is constructed as a particular case of Theorem 7 in Section 5.3.3 (see Corollary 5.2).

5.3 Higher dimensional examples

There exist (partial) characterizations of expansive actions on closed manifolds of the simplest groups and semigroups, namely $\mathbb{Z}$ and $\mathbb{N}$ respectively.

5.3.1 Actions of $\mathbb{Z}$

Let $V$ be a closed manifold admitting two foliations of complementary dimension that are transversal all over $V$, and let $f : V \to V$ be a diffeomorphism preserving those foliations. Assume furthermore that $f$ strictly expands the current corresponding to one of those foliations and strictly contracts the other one (see [Ru-Su]). One says that $(V, \mathbb{Z}, f)$ is Anosov, and it is easy to see that Anosov systems provide examples of expansive $\mathbb{Z}$-actions.

A good introduction to Anosov systems can be found in [Sm], and modulo examples unknown to the author in all the systems of this type the underlying space is, up to conjugation, an infra-nilmanifold, i.e. up to a finite cover and homeomorphism, a co-compact quotient of a connected simply connected nilpotent Lie group, say $G$, the quotient being induced by the action of a discrete subgroup of $G$, say $\Upsilon$, that is finitely generated, nilpotent, and has no elements of finite order (see [Sm] again), generalizing linear automorphisms on tori. If $f : G/\Upsilon \to G/\Upsilon$ is Anosov, then the linear map induced at the level of Lie algebras has no eigenvalues in the unit circle, and an important part of the structure of these systems can be decoded by algebraic means (see [Lau-Will]).
One interesting feature of infra-nil-automorphisms with the Anosov property is that the observables involved in the estimates in Section 3.3 can be easily found. For instance, if $\lambda$ is the largest eigenvalue of the linearization of $f$, then $\text{ent}_0(V, N, f)$ is equal to $\log \lambda$ (see [Ru-Su]), extending the pseudo-Anosov behavior, where $\lambda$ corresponds to the expansion/contraction coefficient for the transverse measured laminations.

5.3.2 Actions of $N$

In [Co-Re] is shown that on closed manifolds a map $f : V \to V$ represents an expansive action of $N$ if and only if such a map is expanding in the sense of [Gro1], namely if for some metric $d$ on $V$ and every point $v$ in $V$ there exists a neighbourhood of that point such that $f^*d > d$ outside the diagonal therein.

The following discussion is based on [Gro1]. It is proved that a necessary condition for the existence of a map of this type on a closed manifold is that their universal cover is homeomorphic to $\mathbb{R}^n$. To achieve that M. Gromov notes that the lift of those maps to the universal cover are globally expanding for some metric invariant under deck transformations, a condition that is easy to verify.

The simplest examples of this kind are induced by linear maps on $\mathbb{R}^n$ whose eigenvalues are greater than one and that are compatible with the free action of discrete groups on $\mathbb{R}^n$, say $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$, so that $V = \mathbb{R}^n / \Upsilon$ is compact. An invariant metric in those examples is of course the very-flat canonical one (see [Be]): every flat manifold of this type admits an expanding action of $N$, a result that Gromov attributes to D. Epstein and M. Shub.

Assuming an upper bound on the Jacobian of the map one sees that a necessary condition for the existence of an expanding map on a closed manifold, say $V$, is that the fundamental group must have polynomial growth. The analogous result without using the assumption that the map is differentiable and obtained using techniques from geometric group theory is due to J. Franks.

Hence the candidates are closed aspherical manifolds that do no admit metrics of negative sectional curvature (see [Be] or [Gro2], for example). Needless to say, those are necessary conditions.

Posterior work of Shub (see [Gro1]) enables to assert that an expanding system $(V, N)$ is conjugated to an infra-nil-endomorphism if and only if the fundamental group of $V$ contains a nilpotent subgroup of finite index.

Since the main result in [Gro1] claims that every finitely generated group with polynomial growth is virtually nilpotent, one concludes that every expansive $N$-action on a closed manifold is conjugated to an infra-nil-endomorphism.

It is worth mentioning the result of D. Epstein and M. Shub: it provides the only known examples of closed manifolds, that are not products, with special holonomy (see [Be], and the foundational [Har-Law]) and of dimension larger than two, allowing an expansive action. Indeed, all complex tori belong to this class, and the estimates of Section 3.3 enriched with the Monge–Ampère-Aubin-Calabi-Yau developments provide a play-ground.
5.3.3 Higher rank actions

Let \( \{ V_\omega \}_{\omega \in \Omega} \) be a finite collection of closed manifolds so that for each \( \omega \) in \( \Omega \) the space \( V_\omega \) admits an expansive action of a group or semigroup \( (\Gamma_\omega, \rho_\omega) \). By means of the set-theoretic characterization of property-e (Section 3.2), one readily sees that the Cartesian product of them, say \( V := \prod_{\omega \in \Omega} V_\omega \), also admits an expansive action of \( (\Gamma, \rho) := \prod_{\omega \in \Omega} (\Gamma_\omega, \rho_\omega) \).

Consider now the wedge sum of the finite collection of spaces \( \{ V_\omega \}_{\omega \in \Omega} \), denoted by \( \vee_{\omega \in \Omega} V_\omega \), where in each of the \( V_\omega \)'s a base point \( v_{\omega,0} \) is understood. Inside the Cartesian product of the \( V_\omega \)'s collapse the wedge (sum) of the spaces to a point, to get the smash of \( \{ V_\omega \}_{\omega \in \Omega} \), usually written as \( \wedge_{\omega \in \Omega} V_\omega \).

In the category of topological spaces (with base points) the smash product is a commutative and associative self-functor. Hence if \( (\Gamma_\omega, \rho_\omega) \) is a group or semigroup acting on \( V_\omega \) having the base point \( v_{\omega,0} \) as a fixed point for every \( \omega \), then there is a natural action of \( \{ (\Gamma_\omega, \rho_\omega) \}_{\omega \in \Omega} \) on \( \wedge_{\omega \in \Omega} V_\omega \), denoted by

\[
\wedge_{\omega \in \Omega} (\Gamma_\omega, \rho_\omega) : \wedge_{\omega \in \Omega} V_\omega \to \wedge_{\omega \in \Omega} V_\omega,
\]

that is commutative and associative with respect to the different \( \omega \) coordinates (in the same way as \( \prod_{\omega \in \Omega} (\Gamma_\omega, \rho_\omega) : \prod_{\omega \in \Omega} V_\omega \to \prod_{\omega \in \Omega} V_\omega \)).

The next result asserts that the property-e is preserved under the smash product.

**Theorem 7.** Let \( (V_\omega, \Gamma_\omega, \rho_\omega)_{\omega \in \Omega} \) be a finite family of systems with property-e, each of them having at least one fixed point, where \( \Gamma_\omega \) is a given group or semigroup, and \( V_\omega \) is a closed manifold. Then the system \( (\wedge_{\omega \in \Omega} V_\omega, \wedge_{\omega \in \Omega} \Gamma_\omega, \wedge_{\omega \in \Omega} \rho_\omega) \) is expansive as well provided the base points are taken as invariant ones for the representation \( (\Gamma_\omega, \rho_\omega) \), for every \( \omega \) in \( \Omega \).

**Proof.** For simplicity consider the case when \( \Omega \) has two elements, and \( \Gamma_\omega \) coincides with \( \mathbb{N} \) for both \( \omega \)'s. So assume that \( (V, \mathbb{N}, f) \) and \( (W, \mathbb{N}, h) \) correspond to expansive actions of \( \mathbb{N} \) on \( V \) and \( W \) respectively, with \( v_0 \) and \( w_0 \) being fixed points for \( f \) and \( h \), respectively. Then the system \( (V \wedge W, \mathbb{N}^2, f \wedge h) \) corresponds to an action of \( \mathbb{N}^2 \) on \( V \wedge W \).

Let \( v_0 \) and \( w_0 \) denote the base points of \( V \) and \( W \), to construct a family \( \{ g(t) \}_{t \in [0,1]} \) of Riemannian metrics on \( V \times W \) as follows. Let \( \kappa : V \to [0,1] \) and \( \rho : W \to [0,1] \) be functions different from zero outside the base points, smooth enough, but such that \( \lim_{v \to v_0} \kappa(v) = 0 \) and \( \lim_{w \to w_0} \rho(w) = 0 \). Define for each \( t \) in \( ]0,1] \) the Riemannian metric \( g(t) \) on \( V \times W \) by

\[
g(t) := ( t + (1-t)\rho ) \ g_V \oplus ( t + (1-t)\kappa ) \ g_W,
\]

where \( g_V \) and \( g_W \) are metrics on \( V \) and \( W \), both of finite diameter and of a suitable regularity.

Denote by \( d_{g(t)} \) the distance on \( V \times W \) induced by \( g(t) \), and consider the family of metric spaces \( \{ (V \times W, d_{g(t)}) \}_{t \in [0,1]} \). As \( t \) goes to zero the couple \( (V \times W, d_{g(t)}) \) ceases
to be a metric space because all the elements in \( V \lor W \) (recall that base points are understood) are at zero distance.

After those remarks it is interesting to note:

**Lemma 5.1.** One has the convergence

\[
(V \times W, d_{g(t)}) \leadsto (V \land W, d_{g(0)})
\]

in the Gromov-Hausdorff sense as \( t \) goes to zero (see \[Gro2\]).

In the Gromov-Hausdorff metric space identify \((V \times W, d_{g(t)})\) with \((V \times W)_t\) for every \( t \) in \([0, 1]\), to denote by

\[
\{( (V \times W)_t, \mathbb{N}^2, (f \times h)_t ) \}_{t \in [0,1]}
\]

the collection of systems obtained, where

\[
( (V \times W)_0, \mathbb{N}^2, (f \times h)_0 ) = ( (V \land W, d_{g(0)}), \mathbb{N}^2, f \land h ).
\]

Since by assumption both \((V, \mathbb{N}, f)\) and \((W, \mathbb{N}, h)\) are expansive systems, we conclude that \(( (V \times W)_t, \mathbb{N}^2, (f \times h)_t )\) is also expansive for every \( t \) different from zero; indeed, the property of being expansive is a conjugacy invariant that does not depend on the metric chosen (see Section 3.2).

To conclude the proof we add further conditions to the functions \( \kappa \) and \( \rho \) to ensure the expansive property on \((V \land W, \mathbb{N}^2, f \land h)\) thanks to the metric \( d_{g(0)} \).

Let \( c_V \) and \( c_W \) be expansivity constants for \((V, d_V, \mathbb{N}, f)\) and \((W, d_W, \mathbb{N}, h)\), where \( d_V \) and \( d_W \) are the distance functions induced by the Riemannian metrics \( g_V \) and \( g_W \), respectively. If \( d_V(v, v_0) \) is larger than \( c_V \) we require that \( \kappa(v) = 1 \), and if \( d_W(w, w_0) \) is bigger than \( c_W \) we demand that \( \rho(w) = 1 \).

Denote by \([v, w]\) the point in \( V \land W \) that is the image of \((v, w)\) under the map from \( V \times W \) to \( V \land W \). Observe that \([v, w_0] = [v_0, w] = [v_0, w_0]\) for every \((v, w)\) in \( V \times W \), where \([v_0, w_0]\) is a fixed point for

\[
f^{n_1} \land h^{n_2} = (f^{n_1} \land 1_W) \cdot (1_V \land h^{n_2}) = (1_V \land h^{n_2}) \cdot (f^{n_1} \land 1_W)
\]

whenever \((n_1, n_2)\) is in \( \mathbb{N}^2 \).

Choose different points \([v, w]\) and \([v', w']\) in \( V \land W \), and exhaust all the possibilities to infer the expansiveness of \(((V \land W, d_{g(0)}), \mathbb{N}^2, f \land h)\) with e-constant \( \min\{c_V, c_W\} \).

The extension to the general case is direct.

Consider the case when \( \Omega \) has \( d \) elements, and for every \( \omega \) in \( \Omega \) choose \((V_\omega, \Gamma_\omega, \rho_\omega)\) as being conjugated to \((S^1, \mathbb{N}, f)\) with the \( \mathbb{N} \)-action on \( S^1 \) given by \( f(\theta) = 2\theta \). Take \( \theta = 0 \) as the base point in \( S^1 \) to construct the bouquet of \( d \) circles \( \cup_d S^1 \), and note that \( \cup_d S^1 = S^d \) is endowed with an action of \( \mathbb{N}^d \) induced by \( \land_d f^{n(i)} \).

Theorems \[3\] and \[7\] together with Lemma \[3.12\] give, thanks to (the proof of) the generalized Poincaré conjecture\[3\],

**Corollary 5.2.** Let \( V \) be an homotopy \( S^d \). Then there exists an expansive action of \( \mathbb{N}^d \) on \( V \). If \( \alpha \) is a generator for \((V, \mathbb{N}^d, \land^d f)\) and \( \{F(n)\}_{n \in \mathbb{N}} \) is an increasing sequence exhausting \( \mathbb{N}^d \), then when \( n \) goes to infinity the nerve of \( \alpha_{F(n)} \) is homeomorphic to \( S^d \).

\[3\]Finished in the work of W. Thurston, R. Hamilton and G. Perelman in dimension 3, M. Freedman in dimension 4, and S. Smale in higher dimensions.
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