Decoherence of a particle across a medium:

a microscopic derivation

Alexandre Domínguez-Clarimon *
Departament d’Estructura i Constituents de la Matèria,
Facultat de Física, Universitat de Barcelona
Diagonal, 647, 08028 Barcelona, Spain

Abstract

We deduce from a microscopic point of view the equation that describes how the state of a particle crossing a medium decoheres. We apply our results to the example of a particle crossing a gas, computing explicitly the Lindblad operators in terms of the interaction potential between the particle and a target of the medium. We interpret the imaginary part of the refraction index as a loss of quantum coherence, that is reflected in the disappearance of interference patterns in a Young experiment.
1 Introduction

When a particle crosses a medium the loss of quantum coherence in its state is unavoidable. This means that the statistical mixture of the particle state increases as it moves on. There are two ways of increasing such mixture. One of them is purely classical and comes from a probabilistic knowledge of which the state of the medium is. The other one is specific of Quantum Mechanics (QM) and it requires a quantum description for the medium too. Here, decoherence appears because the particle gets entangled with the medium through the interaction with it.

Some of these effects that the medium exerts on the particle have long been understood and well described by means of an effective hamiltonian, often expressed as an index of refraction of the medium for the particle propagation. This refraction index was introduced by Fermi \[1\] in the context of neutron dispersion by matter, and is of the form:

\[
\frac{k'}{k} = 1 + 2\pi nf(k, k)\frac{k^2}{k^2}
\]  

(1)

where \(f(k, k)\) is the forward scattering amplitude of the particle by a single dispersor center and \(n\) is the density of targets. It closely follows the calculation of the refraction index for light in Classical Electrodynamics when it propagates through a dielectric medium, first derived by H.A.Lorentz (see \[2\]).

These corrections induced by the medium to the free evolution are crucial in the effects that matter has on neutrino oscillations \([3, 4]\) (relevant for the MSW effect), and on kaons \([5, 6]\).

One feature of the refraction index type of correction is that the evolution of the particle state is still hamiltonian and, thus, its coherence is strictly preserved in time. On the contrary, this article is focused on the decoherence effects that are not included in a refraction index.

A detailed study of the decoherence process would, of course, require a tracking of the whole system, including the environment besides the particle, which would follow a standard QM evolution. It is when one disregards the environment and focuses just on the particle state that its dynamics departs from the one described by the usual Schrödinger equation. It becomes non-local in time, i.e., it depends on the entire history of its evolution. Under certain conditions, this non-locality is very weak and decoherence can be effectively treated by adding new, non-hamiltonian, terms to the differential equation for the density matrix. (Notice that here, as decoherence is at work, the state of the system is naturally described in terms of a density matrix rather than a wave function). Such differential equation is known as the Lindblad equation and is of the form:

\[
\dot{\rho} = -i[H_{eff}, \rho] + \sum_j \left(2A_j\rho A_j^\dagger - A_j^\dagger A_j \rho - \rho A_j^\dagger A_j\right)
\]  

(2)
The presence of an environment has two effects: it modifies in part the Hamiltonian and it moreover generates the operators $A_i$, which make quantum coherence of the particle itself decrease irreversibly. Lindblad proved equation (2) with the assumption that the evolution of the density matrix in a decoherent such process fulfills quantum dynamical, completely positive, semigroup sort of composition law, which increases entropy [7]. This is a very general treatment and has been widely used to describe many different physical situations. Examples of these are Quantum Optics [8, 9] (where it appeared for the first time); the time dependence of the optical activity in chiral isomer molecules [10]; the emergence of a classical description for the macroscopic objects as well as the attempts to explain the mechanism of some issues of foundation in QM [11, 12].

Recently, it has also been the framework to address the question if decoherence could blur the oscillations in the experiments of neutral kaons [13, 14, 15], or the evidences of neutrino oscillations ([16] and references therein).

In this article we explore the loss of coherence for a particle as it crosses a diluted medium. We assume the interaction of the particle with each target is short range. For this reason memory effects are washed out on time-scales larger than the interaction time. We use the scattering $S$ matrix to describe the interaction, since our coarsed grained time is infinite with respect to the interaction scale, for all purposes. A Lindblad equation thus naturally emerges.

We discern two different sources of decoherence: footprint and mixture, each one contributing to distinct Lindblad operators. We give explicit expressions for these operators in terms of the particle-target potential, their masses and the target wave functions. In particular we compute the corrections to Fermi’s equation [11] in terms particle and target mass ratio. We also see how the unitarity of this equation is restored once all pieces of decoherence in the Lindblad equation are properly included. Finally, we argue that the presence of a medium washes out the interference pattern of a double slit Young experiment.

2 The decoherence mechanisms: the footprint and the mixture

We introduce, in this section, the basic ideas necessary to understand which are the microscopic mechanisms that produce decoherence in the case of a particle crossing a medium. We classify the sources of decoherence in two types: the footprint and the mixture. Let us clarify this distinction with the help of a toy model that consists of a particle and a box. The particle will pass through the box and will interact with it. For simplicity, suppose they both have two-dimensional Hilbert spaces. The orthonormal basis for the particle is $\{|1\rangle, |2\rangle\}$, and for the box is $\{|a\rangle, |b\rangle\}$. After the particle crosses the box, we do a partial trace over
the box degrees of freedom. The two distinct mechanisms that we envisage are the following:

**The footprint.** Suppose that the initial state is $|1\rangle|a\rangle$. Suppose also that after they have interacted the final state is $\frac{1}{\sqrt{2}} (|1\rangle|a\rangle - |2\rangle|b\rangle)$, which is entangled. The corresponding reduced density matrix for the particle is the identity, which is not a pure state anymore. After the interaction occurs, the box "knows" what the out state for the particle is. The particle has left a footprint: if the box is in state $|a\rangle$, the particle is in $|1\rangle$ and if the box is in $|b\rangle$, the particle is in $|2\rangle$. With the partial trace we overlook this information, that remains in the box as a footprint.

**The mixture.** Now we start with the particle in $|1\rangle$, but the box in the mixed state $\frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|$. Choose an interaction as follows:

\[
\begin{align*}
|1\rangle |a\rangle & \rightarrow |1\rangle |a\rangle & |2\rangle |a\rangle & \rightarrow |2\rangle |a\rangle \\
|1\rangle |b\rangle & \rightarrow |2\rangle |b\rangle & |2\rangle |b\rangle & \rightarrow |1\rangle |b\rangle.
\end{align*}
\]

(3) (4)

Notice that with this interaction and this initial state, the box does not have any "knowledge" about what the out state for the particle is: regardless of whether the box starts in $|a\rangle$ or $|b\rangle$, it remains unchanged. Yet, decoherence also appears: the reduced density matrix for the particle is the identity again. The source for decoherence is, in this case, the initial mixed state for the box.

3 A model for the medium

In this section we put forward the approximations that we make, as well as the procedures that will eventually lead to a differential equation for the particle reduced density matrix as it crosses the medium.

Our medium is made of just one kind of targets. There is neither interaction between them nor overlapping of their wave functions. We also consider that the interaction between the particle and the medium is weak; therefore, we neglect terms higher than second order in the potential.

Our procedure consists in dividing the medium in thin slabs of matter that will be crossed one by one by the particle. Each time the particle crosses a slab, we do a partial trace over the slab degrees of freedom and obtain a step by step equation for the particle reduced density matrix. Eventually, we will get a differential equation in time for it.

The targets of each slab are in a mixed state: target $j$ is in state $|m_j\rangle$ with probability
Since the medium is homogeneous, the states of the different targets are related simply by translations, and the corresponding weights are the same.

If the initial state is pure $|\text{in}\rangle = |\text{particle}\rangle |\text{slab}\rangle$, after the particle crosses the slab the system is in state

$$|\text{out}\rangle \simeq (1 + i \sum_{j=1}^{N} T^{(j)}) |\text{particle}\rangle |\text{slab}\rangle$$

(5)

were $T^{(j)}$ is the scattering operator of the particle with target $j$ only, and $N$ is the number of targets in the slab.

In order to justify this approximation, let us look at the case where there is just one particle and two targets, and consider the amplitude of the process of a particle that goes from $x$ to $y$, and targets go from $x_a, x_b$ to $y_a, y_b$, in a time interval $t$. The terms to second order in the potential are those shown in Fig 1. We must integrate over the coordinates and the instants in which the interactions take place. Recall that the free propagator is proportional to the phase factor $\exp(im \frac{(y-x)^2}{2t})$, $m$ being the mass of the particle. When we integrate over the internal coordinates, $z$ for instance, the oscillation of the exponential is much faster in picture 1.b than in picture 1.a. This is so because the size of the region where a particle and a target do interact is much smaller than the distance among targets. Therefore, we can neglect those terms of 1.b in front of those in 1.a. Ultimately, this is the same as (5).

It is worth pointing out that unitary relations hold both for each $1 + iT^{(j)}$ and for $1 + i \sum_{j=1}^{N} T^{(j)}$ (within the approximations above explained). Thus, terms like the crossed product $T^{(i)} T^{(j)}$ are neglected by consistency.
4 The one step equation

In this section we obtain the particle density matrix after it has crossed one slab. In the next section we will iterate this one step evolution \( r \) times.

If the initial state for the particle is \( \ket{\phi^n} \) and for the slab is \( \ket{m} \equiv \prod_{j=1}^{N} \ket{m_j} \), the outgoing state for both together is:

\[
\ket{\text{out}^{n,m}} \simeq (1 + i \sum_{j=1}^{N} T^{(j)} \ket{\phi^n} \ket{m})
\]  

(6)

The reduced density matrix for the particle is obtained by performing a partial trace over the slab:

\[
\rho_{n,m}^{\text{red}} = \text{TR}_{\text{slab}} \left[ \ket{\text{out}^{n,m}} \bra{\text{out}^{n,m}} \right]
\]  

(7)

In general, the initial state will be mixed, and will have \( \ket{\phi^n} \) with probability \( p_n \) and \( \prod_{j=1}^{N} \ket{m_j} \) with joint probability \( \prod_{j=1}^{N} q_{m_j} \). Then, an average over all possible initial states is due in (7).

4.1 The footprint in the slab and the reduced density matrix for the particle

Let us separate from \( \ket{\text{out}^{n,m}} \) the part that contains the footprint left by the particle. This part can be understood as the change that the particle leaves in the slab. It is thus natural to define it with the help of the following projector:

\[
P = I \otimes \ket{m} \bra{m}
\]  

(8)

In order to separate the footprint term, we apply the projector \((1-P)\) to \( \ket{\text{out}^{n,m}} \), which only keeps the part of the slab that has changed:

\[
(1 - P) \ket{\text{out}^{n,m}} \equiv \ket{\text{footprint}^{n,m}}
\]  

(9)

The part of \( \ket{\text{out}^{n,m}} \) that leaves \( \ket{m} \) intact is:

\[
P \ket{\text{out}^{n,m}} \equiv \ket{\phi'^{n,m}} \ket{m}
\]  

(10)

One can thus write (6) as a sum of two orthogonal terms:

\[
\ket{\text{out}^{n,m}} = \ket{\phi'^{n,m}} \ket{m} + \ket{\text{footprint}^{n,m}}
\]  

(11)

and with it (7) reads:

\[
\rho_{n,m}^{\text{red}} = \ket{\phi'^{n,m}} \bra{\phi'^{n,m}} + \text{TR} \left[ \ket{\text{footprint}^{n,m}} \bra{\text{footprint}^{n,m}} \right]
\]  

(12)
In this way we separate the part in which the particle changes and the slab does not (the first term), from the one which contains any change produced in the slab (the second term).

It is worth pointing out that the second term in (12) encodes the entanglement between the slab and the particle. From (6) and (10):

\[ |\phi^{',m}_n\rangle = |\phi^n\rangle + i \sum_{j=1}^{N} \langle m_j | T^{(j)} | m_j \rangle |\phi^n\rangle. \quad (13) \]

In terms of the operators \( D_{M}^{(m_j)} \) the first term in (12):

\[ |\phi^{',m}_n\rangle \langle \phi^{',m}_n| = \left( 1 + i \sum_{j=1}^{N} D_{M}^{(m_j)} \right) |\phi^n\rangle \langle \phi^n| \left( 1 - i \sum_{j=1}^{N} D_{M}^{\dagger(m_j)} \right) \quad (14) \]

As for the footprint term, from (6) and (9):

\[ |\text{footprint}^{n,m}\rangle = i \sum_{j=1}^{N} \left( T^{(j)} - \langle m_j | T^{(j)} | m_j \rangle \otimes I^{(\otimes N)} \right) |\phi^n\rangle \prod_{i=1}^{N} |m_i\rangle \quad (15) \]

Each term ”\( j \)” of this sum is orthogonal to any state which has target \( j \) in state \( |m_j\rangle \). For this reason, in the second term of (12):

\[ \sum_{j,k} \text{TR} \left[ \left( T^{(j)} - \langle m_j | T^{(j)} | m_j \rangle \otimes I^{(\otimes N)} \right) |\phi^n\rangle \prod_{i=1}^{N} |m_i\rangle \langle \phi^n| \prod_{k=1}^{N} \left( T^{\dagger(k)} - \langle m_k | T^{\dagger(k)} | m_k \rangle \otimes I^{(\otimes N)} \right) \right] \quad (16) \]

only the \( j = k \) terms do contribute. In order to evaluate the partial trace we introduce as a basis for the Hilbert space of the particle the \( \{|l_j\rangle\} \); (16) then reads:

\[ \sum_{j} \sum_{l_j} \left( \langle l_j | T^{(j)} | m_j \rangle - \langle m_j | T^{(j)} | m_j \rangle \langle l_j | m_j \rangle \right) |\phi^n\rangle \langle \phi^n| \left( \langle m_j | T^{(j)} | l_j \rangle - \langle m_j | T^{(j)} | m_j \rangle \langle m_j | l_j \rangle \right) \quad (17) \]

which in terms of the operators \( A_{E}^{(l_j,m_j)} \equiv \langle l_j | T^{(j)} | m_j \rangle - \langle m_j | T^{(j)} | l_j \rangle \) is:

\[ \sum_{j,l_j} A_{E}^{(l_j,m_j)} |\phi^n\rangle \langle \phi^n| A_{E}^{\dagger(l_j,m_j)} \quad (18) \]

Finally, we average in (12) over all possible initial states, for both the particle and the slab, with their corresponding probabilities\(^1\).

\[ \rho_{\text{red}}^{\text{out}} = \sum_{(m_j)} \prod_{j} q_{m_j} \left( 1 + i \sum_{i} D_{M}^{(m_i)} \right) \rho_{\text{red}}^{\text{in}} \left( 1 - i \sum_{i} D_{M}^{\dagger(m_i)} \right) + \sum_{(m_j)} \prod_{j} q_{m_j} \left( \sum_{i} A_{E}^{(l_i,m_i)} \rho_{\text{red}}^{\text{in}} A_{E}^{\dagger(l_i,m_i)} \right) \quad (19) \]

\(^1\)The unitary relations of the next section guarantee the correct normalization of \( \rho_{\text{red}}^{\text{out}} \)
where $\rho_{\text{in}}^\text{red} = \sum_n p_n |\phi^n\rangle\langle \phi^n|$. We call this the one step equation. Notice that the footprint operators $A_E^{(l_i,m_i)}$ are independent of the probabilities $q_{m_j}$.

For the slab we obtain an equation similar to (19):

\[
\rho_{\text{slab}}^\text{out} = \sum_n p_n (1 + iD_M^n) \rho_{\text{slab}}^\text{in} (1 - iD_M^n)
\]

\[+
\sum_n p_n \left( \sum_l A_E^{(l,n)} p_{\text{red}}^\text{in} A_E^{(l,n)} \right),
\]

(20)

It is interesting to note that if $\rho_{\text{slab}}^\text{in}$ is the identity, and $[A_E^{(l,n)}, A_E^{(l,n)}] = 0$, it remains unchanged. This may seem contradictory, because footprint effects still appear in (19), given that $A_E^{(l_i,m_i)} \neq 0$, whereas the medium has not changed. This is fixed by realizing that the initial state for the slab is a mixture of pure states, and the footprint is left in each one of them. By averaging over these pure states one erases the footprint that the particle leaves on the slab. Yet, the footprint effects on the particle are still there.

### 4.2 Unitarity relations

Before we finish this section, let us put forward some unitarity relations among the operators $A$’s and $D$’s defined above. In order to simplify the notation we omit the subindex $j$, and we write $D_M^{(m)}$ and $q_m$ instead of $\sum_i D_M^{(m,i)}$ and $\prod_j q_{m_j}$. Unitarity of $S$ matrix reads:

\[ T - T^\dagger = iT^\dagger T \]

(21)

Let us define the operator $D_E^{(m)}$ such that: $T = D_M^{(m)} \otimes I^{(\otimes N)} + D_E^{(m)}$. Notice that then $\langle l_j | D_E | m_j \rangle$ is nothing but $A_E^{(l_j,m_j)}$. We can rewrite (21) with the help of this decomposition:

\[
\left( D_M^{(m)} - D_M^{\dagger (m)} - iD_M^{\dagger (m)} D_M^{(m)} \right) \otimes I^{(\otimes N)} + \left( D_E^{(m)} - D_E^{\dagger (m)} - iD_E^{\dagger (m)} D_E^{(m)} \right) = i \left( D_M^{\dagger (m)} \otimes I^{(\otimes N)} \right) D_M^{(m)} + iD_E^{\dagger (m)} \left( D_M^{(m)} \otimes I^{(\otimes N)} \right)
\]

(22)

Now let us sandwich each term of (22) with $|m\rangle$. Since $\langle m | D_E^{(m)} | m \rangle = 0$, then:

\[
\sum_{j,l_j} A_E^{(l_j,m_j)} A_E^{(l_j,m_j)} = - \left( iD_M^{(m)} - iD_M^{\dagger (m)} + D_M^{\dagger (m)} D_M^{(m)} \right)
\]

(23)

This relation has a clear intuitive interpretation. With the approximation for the $T$ matrix, the particle cannot change the state of more than one target at a time. Thus, for a given initial configuration $(n, \{m\})$, the probability to change target $j$ from state $m_j$ to state $l_j$ is:

\[ \| \langle l_j | T^{(j)} | \phi^n \rangle | m_j \| \|^2 \]

(24)

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2 There is similar relations between $D_M^{(m)}$ and $A_E$ that comes from unitary relations like in (28).
The probability that any target suffers any kind of change for some initial particle state is, thus, written in terms of the $A_E$ operators:

$$\sum_n p_n \sum_j \sum_{\{m_j\}} q_{m_j} \sum_{l_j \neq m_j} \langle \phi^n | A^\dagger_{E}(l,m_j) A_{E}(l,m_j) | \phi^n \rangle$$  \hspace{1cm} (25)$$

The opposite situation, in which there is no change in the slab, has a probability:

$$\sum_{\{m\}} q_{m} \sum_n p_n \| \prod_j \langle m_j | (1 + i \sum_j T^{(j)}) | \phi^n \rangle \prod_j | m_j \rangle \|^2$$  \hspace{1cm} (26)$$

that, written in terms of the $D_M$'s:

$$1 - \sum_n p_n \sum_m q_m \langle \phi^n | i(D^{\dagger m}_M - D^{m}_M) - D^{\dagger m}_M D^{m}_M | \phi^n \rangle$$  \hspace{1cm} (27)$$

These two probabilities must add up to one, which expresses the unitarity relation (23).

## 5 The equation of motion

In the previous section we have found an equation that relates the density matrices for the particle before and after it crosses a slab of matter, which we called the one-step equation. From it we now derive the differential equation for the evolution of this density matrix.

### 5.1 From the one-step to a time equation

Let us first justify the procedure we follow with a simple example which also applies to the general case.

Suppose the particle is in a pure state and the medium is made out of static scattering centers. After the particle crosses the $r$th slab, the state for the particle is:

$$|\phi(r)\rangle = (1 + iT) |\phi(r-1)\rangle$$  \hspace{1cm} (28)$$

The particle spends a time $\Delta t = \frac{\delta}{v}$ crossing the slab, where $\delta$ is its thickness and $v$ is the mean velocity of the particle. Then, one could be naively tempted to write an equation of the sort:

$$\frac{|\phi(r)\rangle - |\phi(r-1)\rangle}{\Delta t} = i\frac{v}{\delta} T |\phi(r-1)\rangle \xrightarrow{\text{continuum}} i \frac{d}{dt} |\phi\rangle = -\frac{v}{\delta} T |\phi\rangle,$$  \hspace{1cm} (29)$$

But this cannot be a Schrödinger equation because $T$ is not an hermitian operator.Demanding that $(1 + iT)$ should be of the form $exp(-iH_{eff}\Delta t)$, the correct differential equation is:

$$\frac{d}{dt} |\phi\rangle = \frac{v}{\delta} \ln(1 + iT) |\phi\rangle$$  \hspace{1cm} (30)$$
Unfortunately, these arguments cannot be immediately generalized to the case of equation (19), because in that case the particle gets entangled with the medium, and the one-step equation is much more complicated than (28); moreover, we are aiming at an evolution equation of the Lindblad type, which is not a mere Schrödinger evolution. For that purpose let’s go back to our example in (28) and re derive (30) in such a way that now it does generalize for the one step equation (19). 

Consider that the particle crosses \( r \) slabs in a time \( \Delta t = \frac{\delta}{v} \). We can iterate equation (28) \( r \) times and expand \((1 + iT)^r\):

\[
|\phi(r)\rangle = (1 + r(iT) + \frac{r(r-1)}{2}(iT)^2 + ...)\phi(0))
= (1 + r(iT - \frac{(iT)^2}{2} + ...) + O(r^2))\phi(0)) \tag{31}
\]

Compare (31) with the Taylor expansion of \(|\phi(t + \Delta t)\rangle\):

\[
|\phi(t + \Delta t)\rangle = |\phi(t)\rangle + \Delta t \frac{d}{dt}|\phi(t)\rangle + ...
\tag{32}
\]

where \(|\phi(t + \Delta t)\rangle = |\phi(r)\rangle\) and \(|\phi(t)\rangle = |\phi(0)\rangle\). Now, identify the terms of (31) linear in \(\Delta t\) with those of (32) through the relation \(\Delta t = \frac{\delta}{v}\), and write:

\[
\frac{d}{dt}|\phi(t)\rangle = \frac{v}{\delta}(iT - \frac{(iT)^2}{2} + ...)\phi(t)) \tag{33}
\]

which is nothing but the Taylor expansion of \(\ln(1 + iT)\). Thus, we retrieve (30).

We are now ready to apply this procedure to the one-step equation (19). To derive the desired relation let us Taylor expand \(\rho\) about the instant \(t\), as in (32):

\[
\rho(t + \Delta t) \simeq \rho(t) + \Delta t \dot{\rho}(t) + ...
\tag{34}
\]

On the other hand we know \(\rho\) after \(r\) steps by iterating equation (19) \(r\) times. Expanding the result in powers of \(r\):

\[
\rho(r) \simeq \rho(0) + r \Lambda[\rho] + ...
\tag{35}
\]

where \(\Lambda\) is a linear operator acting on \(\rho\). By comparing equal powers of \(\Delta t\) in the two expressions, one finally gets:

\[
\dot{\rho}(t) = \frac{v}{\delta} \Lambda[\rho] \tag{36}
\]

This is the equation of motion, that, as we will see in the following section, has a Lindblad form.
5.2 The equation of motion for the reduced density matrix

Here we compute $\Lambda[\rho]$ by iterating eq. (19), as explained in the previous section. If we only keep terms linear in the potential, the iteration only gives a linear term in $r$:

$$r \sum_m q_m \left( i D^{(m)}_M \rho - i \rho D^{\dagger (m)}_M + D^{(m)}_M \rho D^{\dagger (m)}_M \right), \quad (37)$$

and (36) is an equation of Schrödinger type. It is not until we retain terms up to quadratic in the potential that we find any sign of decoherence and, hence, corrections to this standard evolution. These terms are:

$$r \sum_m q_m \left( i D^{(m)}_M \rho - i \rho D^{\dagger (m)}_M + D^{(m)}_M \rho D^{\dagger (m)}_M \right)$$

$$+ \frac{r(r-1)}{2} \sum_{m,n} q_m q_n \left( -D^{(m)}_M D^{(n)}_M \rho - \rho D^{\dagger (m)}_M D^{\dagger (n)}_M \right)$$

$$+ \sum_{m,j,l} q_{m,j} A^{(l,j,m)}_E \rho A^{(l,j,m)}_E \quad (38)$$

from which we retain only the pieces linear in $r$. Defining the operators $A^{n,m}_M \equiv D^{(n)}_M - D^{(m)}_M$, we can write the equation of motion (36) as:

$$\dot{\rho}(t) = -i H \rho + i \rho H^{\dagger} + \frac{\nu}{2} \sum_{m,n} q_m q_n A^{n,m}_M \rho A^{\dagger n,m}_M + \frac{\nu}{2} \sum_{l,j,m} q_{m,j} A^{(l,j,m)}_E \rho A^{(l,j,m)}_E \quad (39)$$

Where $H \equiv -\frac{\nu}{2} \left( \sum_{\{m\}} q_m D^{(m)}_M - i \frac{1}{2} \left( \sum_{\{m\}} q_m D^{(m)}_M \right)^2 \right)$

This equation is not in the Lindblad form yet. In order to do that, we use the unitarity relations of section 4.2.3 and rewrite it as:

$$\dot{\rho} = -i \left[ H_{eff}, \rho \right] + \frac{1}{4} \frac{\nu}{\delta} \sum_{m,n} q_m q_n L^{(n,m)}_M [\rho] + \frac{1}{2} \frac{\nu}{\delta} \sum_{l,j,m} q_{m,j} L^{(l,j,m)}_E [\rho] \quad (40)$$

This is the final expression. We have defined the effective Hamiltonian and the Lindblad terms as:

$$H_{eff} = -\frac{1}{2} \frac{\nu}{\delta} \left( \sum_{\{m\}} q_m D^{(m)}_M + \sum_{\{m\}} q_m D^{\dagger (m)}_M \right) \quad (41)$$

$$L^{(n,m)}_M [\rho] = 2 A^{n,m}_M \rho A^{\dagger n,m}_M - A^{\dagger n,m}_M A^{n,m}_M \rho - \rho A^{\dagger n,m}_M A^{n,m}_M \quad (42)$$

$$L^{(l,j,m)}_E [\rho] = 2 A^{(l,j,m)}_E \rho A^{(l,j,m)}_E - A^{(l,j,m)}_E A^{(l,j,m)}_E \rho - \rho A^{(l,j,m)}_E A^{(l,j,m)}_E$$
As we can see, the Hamiltonian is Hermitian, and it contains the "mean" effects that the medium does on the particle. To first order in the potential it is nothing but the mean energy felt by the particle while it propagates inside the medium.

An important remark concerning the Lindblad operators $A_M$ and $A_E$ is in order. Mixture effects only depend upon how much mixed the medium is: if it is pure $A_M = 0$ and they are not present. However, footprint effects are unavoidable once the particle has modified the state of the medium. This always carries $A_E \neq 0$, regardless of whether the state is mixed or not.

5.3 The independence on $\delta$ of the equation of motion

Physics cannot depend on $\delta$, the width of the elementary slab of our partition; any dependence on $\delta$ in equation (39) is an artifact of the approximations that have been used and it will be automatically removed whenever the approximations hold. In order to qualify this statement, let us regard again, for simplicity, the case in section 5.1 where the particle always evolves as a pure state: we expect that the effective Hamiltonian $H = \frac{v}{\delta} \ln(1 + iT^\delta)$ does not depend on the width of the slab, while $T^\delta$ does.

This follows from the relation between the $S$ matrix for a slab of width $\delta$ and the $S$ matrix for a slab twice as thick:

$$(1 + iT^\delta)(1 + iT^{2\delta}) = (1 + iT^{2\delta})$$

from which we obtain

$$\frac{v}{2\delta} \ln(1 + iT^{2\delta}) = \frac{v}{\delta} \ln(1 + iT^\delta)$$

This shows the independence of $H$ on the $\delta$ parameter. If we make some approximation in the computation of $T^\delta$, the relation (43) will fail to hold and it will lead us to a spurious dependence on $\delta$. Nevertheless, this is not substantial in the domain where the approximation for $T$ is valid.

In our case, the approximation consists of taking as $T$ matrix the sum of the individual $T^j$ matrices of scattering of just one target $j$. In order for equation (43) to be satisfied:

$$(1 + i \sum_{j \in \text{slab1}} T^j)(1 + i \sum_{j \in \text{slab2}} T^j) = (1 + i \sum_{j \in \text{slab1+slab2}} T^j)$$

the crossed terms $T^jT^i$ should be negligible. These are similar conditions to those imposed by unitarity, put forward in section 3. Recall that this approximation amounts to neglecting re-scattering, which implies that the density of targets must be small.
6 The Lindblad equation for a gas

In this section we treat the example of a gas. We expect in this case some simplifications to occur in the general equation \( [\mathbf{39}] \), because the gas is an homogeneous medium. In particular, the simplifications apply to the matrix elements \( \langle k' | H | k \rangle \), that are related to the refraction index of the gas, as we shall see in section 7.

6.1 The hamiltonian part

Let us consider the hamiltonian piece from \( [\mathbf{39}] \) in momentum representation:

\[
\langle k | H | q \rangle \equiv -\frac{\gamma}{\delta} \langle k | \sum_m q_m D_{M}^{(m)} | q \rangle + i \frac{1}{2} \frac{\gamma}{\delta} \langle k | \left( \frac{\gamma}{\delta} \sum_m q_m D_{M}^{(m)} \right)^2 | q \rangle \tag{46}
\]

In order to perform the computation of these sums, we divide the slab in boxes of volume \( 1/n \) so as to have one target inside each box (\( n \) is the density of targets). We also write \( |m_j, x_j \rangle \) to label explicitly the centers \( x_j \) of the target wave functions, and similarly for the weights \( q_{m_j, x_j} \). The sum over all possible locations \( x_j \) inside box \( j \) becomes an integral:

\[
\langle k | \sum_{\{m\}} q_m D_M^{(m)} | q \rangle = \sum_j \sum_{\{m_j, x_j\}} q_{m_j, x_j} \langle k | D_M^{(m_j, x_j)} | q \rangle
\]

\[
= \sum_j \sum_{\{m_j\}} q_{m_j} \int_{\text{box}_j} dx_j e^{i(q-k)x_j} \langle k | \langle m_j, 0 | T | m_j, 0 \rangle | q \rangle \tag{47}
\]

We have used the fact that the matrix element \( \langle k | \langle m_j, x_j | T | m_j, x_j \rangle | q \rangle = e^{i(q-k)x_j} \langle k | \langle m_j | T | m_j \rangle | q \rangle \), where it is understood that the states \( |m_j\rangle \) are centered at the origin now. Since the weights \( q_{m_j} \) are the same for all targets, the sum over \( j \), together with the integral over each box, becomes an integral over the slab. Then,

\[
\langle k | \sum_{\{m\}} q_m D_M^{(m)} | q \rangle = n \sum_{\{m\}} q_m \int_{\text{slab}} dxe^{i(q-k)x} \langle k | \langle m | T | m \rangle | q \rangle \tag{48}
\]

The integral over the slab can be split into an integral over its width (z coordinate) times an integral over the plane of the slab. We first perform the integral over the plane, and find that the result is proportional to the delta function of \( (q-k)_\parallel \), i.e., the components parallel to the plane:

\[
\frac{\gamma}{\delta} \langle k | \sum_{\{m\}} q_m D_M^{(m)} | q \rangle = \frac{\gamma}{\delta} (2\pi)^2 n \sum_{m} q_m \int_{0}^{\delta} dz \delta_{\parallel} (q-k)e^{i(q_k-k_k)z} \langle m | T(k,q) | m \rangle \tag{49}
\]

We compute the projection of the \( T \)-matrix in the center of mass coordinates: \( k_{cm} = k_1 + k_2 \) and \( k_r = \frac{m_2}{m_1}k_1 - \frac{m_1}{m_1}k_2 \), where 1 labels the particle and 2, the target. In such
coordinates $T$ reads:

$$\langle k_r | (k_{cm} | T | q_{cm} ) | q_r \rangle = -2\pi \delta(k_{cm} - q_{cm}) \delta(E_{q_r} - E_{k_r}) T_E(k_r, q_r; m_r);$$ (50)

here $m_r$ is the reduced mass. The combination of these two delta functions, together with $\delta(q - k)$ in (49), gives:

$$\delta(q_1 - k_1) \delta(q_2 - k_2) \frac{1}{|k_{2,1} - k_{2,2}|}$$

$$+ \delta(q_1 - k_1) \delta_z \left( k_1 + q_1 - \frac{m_1}{m_2} (k_2 + q_2) \right) \delta(q_2 - k_2) \delta_z \left( k_2 - q_2 - 2q_1 + \frac{m_1}{m_2} (k_2 + q_2) \right) \frac{2m_1}{|k_1 - q_1|}$$

(51)

The delta functions in the two terms provide the energy momentum conservation laws corresponding to one-dimensional elastic scattering. The first term is diagonal in both particle and target momenta; it corresponds to the process where each velocity remains unchanged and gives the dominant contribution to (49), because the imaginary argument of the exponential vanishes. The second term corresponds to the solution where the relative velocity flips its sign, and gives a negligible contribution if the particle wave packet is entirely contained inside the slab of width $\delta$: therefore $k_z \delta \gg 1$ and the oscillations of the exponential factor kill the integral over $z$. Then (49) reads:

$$\frac{v}{\delta} \langle k_1 | \sum_{m_j} q_{m_j} D^{m_j}_M | q_1 \rangle = -(2\pi)^3 n \sum_m q_m \langle m | \frac{1}{|1 - \frac{m_1}{m_2} k_{z,2} |} T_E(k_r, k_r; m_r) | m \rangle \delta(k_1 - q_1)$$ (52)

The hamiltonian (46) contains, also, the square term $\left( \sum_{m_j} q_{m_j} D^{m_j}_M \right)^2$. We discard this piece, because it is quadratic in $n$, which should be small by consistency of the approximation (see (53)). Finally, we get for the $H$ matrix elements:

$$\langle k_1 | H | q_1 \rangle = (2\pi)^3 n \sum_m q_m \langle m | \frac{1}{|1 - \frac{m_1}{m_2} k_{z,2} |} T_E(k_r, k_r; m_r) | m \rangle \delta(k_1 - q_1)$$ (53)

We assume that the typical velocity of a target is much smaller than the particle velocity, and therefore the denominator $|1 - \frac{m_1}{m_2} k_{z,2} | \sim 1$, regardless of whether $\frac{m_1}{m_2}$ is small or not.

This is our final expression for the hamiltonian of the particle. As we can check, there is no-dependence on the width $\delta$ of the slab left over in (53). Further simplifications occur in the limits of both $m_1 \ll m_2$ and $m_1 \gg m_2$, that we now consider.

The limit of a heavy target.

Let us expand (52) in powers of $\frac{m_1}{m_2} \ll 1$. The zeroth order term is:

$$\langle k_1 | H | q_1 \rangle = (2\pi)^3 n \delta(k_1 - q_1) T_E(k_1, k_1; m_1) + O \left( \frac{m_1}{m_2} \right),$$ (54)

\(^3\)Recall that $\frac{k_{z,2}}{m_2} = v$, since we assume that particle state is peacket around velocity $v$ pointing in $z$ direction.
which is independent on the state of the targets.

There are two sources of corrections to this: the denominator of (52), and the $T_E$ matrix elements through their dependence on $k_r$ and $m_r$. In a gas, the target momentum expectation value vanishes, i.e., $\sum_m q_m \langle m | k_2 | m \rangle = 0$, and the corrections coming from the denominator cancel. The first $\frac{m_1}{m_2}$ correction, thus, reads:

$$- \delta(k_1 - q_1) (2\pi)^3 n \frac{m_1}{m_2} k_1 \frac{\partial}{\partial k_1} T_E(k_1, k_1; m_2).$$

(55)

where we have not expanded $m_r$ in $T_E$. Notice that these corrections start quadratic in the potential because the linear term in $V$ does not depend on $k_r$ at all. The final expression in this limit reads:

$$\langle k_1 | H | q_1 \rangle = \delta(k_1 - q_1) (2\pi)^3 n \left( 1 - \frac{m_1}{m_2} k_1 \frac{\partial}{\partial k_1} \right) T_E(k_1, k_1; m_2) + O \left( \frac{m_1}{m_2} \right)^2$$

(56)

The corrections to the infinitely heavy target limit are $O(\frac{m_1}{m_2}, V^2)$.

The limit of a heavy particle.

Let us now consider the opposite limit, and expand (52) in powers of $\frac{m_2}{m_1} \ll 1$. The zeroth order term is:

$$\langle k_1 | H | q_1 \rangle = (2\pi)^3 n \sum_m q_m \langle m | T_E \left( m_2 \frac{1}{m_1} k_1 - k_2, m_2 \frac{1}{m_1} k_1 - k_2, m_2 \right) | m \rangle \delta(k_1 - q_1) + O \left( \frac{m_2}{m_1} \right)^2$$

(57)

$m_2 k_1 = m_2 v_1$ should not be treated as a small quantity as compared to $k_2$ if we assume that the particle velocity $v_1$ is larger than the velocities of the targets in the gas.

The first order corrections contain a term coming from the denominator of (52) which does not cancel as it did before. Finally, the hamiltonian $\langle k_1 | H | q_1 \rangle$ reads:

$$(2\pi)^3 n \sum_m q_m \langle m | (1 + \frac{m_2}{m_1} \frac{k_1 k_2}{k_1^2} + k_2 \frac{\partial}{\partial K}) T_E(K = m_2 \frac{1}{m_1} k_1 - k, K; m_r) | m \rangle \delta(k_1 - q_1) + O \left( \frac{m_2}{m_1} \right)^2$$

(58)

where we have substitute the quotient $k_{z,2}/k_{z,1}$ by $k_1 k_2/k_1^2$: they both coincide in the limit of a particle state pecked in the $z$ direction.

6.2 The piece of decoherence

Apart from $H$, the evolution equation (39) involves further pieces that contain the $A_M$ and $A_E$ operators, which are the ones that induce decoherence. Let us first consider the terms containing $A_M$ (see section (5.2) for details):

$$A_M \rho A_M^\dagger = \frac{1}{2} \frac{\delta}{\delta} \sum_{m,n} q_m q_n A_M^{n,m} \rho A_M^{n,m} = \frac{\nu}{\delta} \sum_m q_m D_M^m \rho D_M^m - \frac{\nu}{\delta} \sum_m q_m D_M^m \rho \sum_n q_n D_M^n$$

(59)
Making explicit the summation over all targets in (59) we find:

\[ A_M \rho A_M^\dagger = n \frac{V}{\delta} \sum_m \sum_{m,j} (q_{m,j,x_j} - q_0^2) D_{M}^{m_j,x_j} \rho D_M^{\dagger m_j,x_j} \]  

(60)

where we have split label \((m_j)\) into \((m_j, x_j)\) as before, to explicitly display the center of target \(j\) wave function. We proceed as before and convert the sum over \(x_j\) into an integral (note that the squared probability in (60) vanishes). We get:

\[ A_M \rho A_M^\dagger = n \frac{V}{\delta} \sum_m \int_{\text{slab}} d^3 x D_{M}^{m,x} \rho D_M^{\dagger m,x} \]  

(61)

The term coming from the \(A_E\) operator will have a similar expression, yielding:

\[ A_E \rho A_E^\dagger \equiv \frac{V}{\delta} \sum_{m,j,l \neq m_j} q_{m,j} A_{l,j}^{(l,m_j)} \rho A_E^{\dagger (l,j,m_j)} = n \frac{V}{\delta} \sum_m \sum_{l \neq m} \int_{\text{slab}} d^3 x A_{E}^{m,x} \rho A_E^{\dagger m,x} \]  

(62)

The matrix elements \((k', k)\) of these two terms together add up to:

\[ \langle k' | A_M \rho A_M^\dagger + A_E \rho A_E^{\dagger} | k \rangle = n \frac{V}{\delta} \sum_{l,m} \int dq dq' (2\pi)^2 \delta_\parallel (k' + q - k - q') \times \]  

\[ \times \int_0^\delta dz e^{i((k_z'-k_z)-(q_z'-q_z))z} \langle k' | (l|T|m)|q' \rangle \rho (q', q) \langle q | (m|T^{\dagger}|l) | k \rangle \]  

(63)

The T-matrix elements involved in the last expression, to first order in the potential, and in momentum representation, are:

\[ \langle k'_1 | (l|T|m)|q'_1 \rangle = -2\pi \int d^3 k_1' d^3 q_1' \delta_\parallel (k'_1 - q_1' - q') \rho (q_1 - q_1') A_l(q_1) \]  

(64)

where \(A_l, A_m\) are the target wave functions. A resolution of the identity \(\sum_l A_l^{*}(q_2) A_l(q_2) = \delta(k_2 - k_2')\) appears in (64) once expressions (64) are used.

\[ \langle k' | A_M \rho A_M | k \rangle + A_E \rho A_E | k \rangle = n \frac{V}{\delta} (2\pi)^2 \sum_{l,m} \int d^3 k_2 d^3 q_2 d_3 q_2 d^3 k_1 d^3 q_1 \int_0^\delta dz e^{i((k_z'-k_z)-(q_z'-q_z))z} \times \delta(k_2 - k_2') \delta_\parallel (k_1 + q_1 - k_1 - q_1') \delta(k_1 - q_1 - q_1') \rho (q_1 - q_1') A_l(q_1) \times A_m(q_2) A_m^{*}(q_2') \rho (q_1', q_1) \]  

(65)

This is our final expression for Lindblad piece of the equation of motion (39).

Further simplifications occur, again, in the limits of heavy target and heavy particle. The essential difference between the two limits enters through the energy delta functions.

In both cases, the following relation is useful to write a perturbative expansion in the ratio of masses:

\[ \delta(E_{q_1} - E_{k_1}) = \frac{1}{2\pi i} \left( \frac{1}{E_{q_1} - E_{k_1} - i\epsilon} - \frac{1}{E_{q_1} - E_{k_1} + i\epsilon} \right) \]  

(66)
We now study the two different limits separately.

The limit of a heavy target.

In such limit we can expand the terms in (66) as a geometric series:

\[
\frac{1}{E_{q^r} - E_{k^r} + i\epsilon} = \frac{2m_r}{q_1^2 - k_1^2 + i\epsilon} \sum_{s=0}^{\infty} \left( \frac{1}{2m_\ell} \frac{(q_1 + k_2)^2 - (k_1 + q_2)^2}{q_1^2 - k_1^2 + i\epsilon} \right)^s
\]

with the caution that with this expansion the series thus obtained is asymptotic. Here we have used the conservation of the center of mass momentum \(\delta(k_{cm} - q_{cm})\) present in (65). By taking higher derivatives of (66) we have

\[
\frac{1}{2\pi i} \left[ \frac{1}{(E_{q_1} - E_{k_1} - i\epsilon)^{s+1}} - \frac{1}{(E_{q_1} - E_{k_1} + i\epsilon)^{s+1}} \right] = \frac{(-1)^s}{s!} \frac{\partial^s \delta(E_{q_1} - E_{k_1})}{\partial E_{q_1}^s}
\]

that allows us to write (66) as:

\[
\delta(E_{q^r} - E_{k^r}) = \sum_{s=0}^{\infty} \frac{1}{s!} \left( \frac{-1}{4m_2} \right)^s \frac{\partial^s \delta(E_{q_1} - E_{k_1})}{\partial E_{q_1}^s} ((q_1 + k_2)^2 - (q_2 + k_1)^2)^s
\]

suitable for an expansion in inverse powers of the target mass. Let us focus on the zeroth order term of this expansion. The set of delta functions in (65) is now:

\[
\delta(k_2 - k'_2) \delta(k'_1 + q_1 - k_1 - q'_1) \delta(k_{cm} - q_{cm}) \delta(k'_{cm} - q'_{cm}) \delta(E_{q'_1} - E_{k'_1}) \delta(E_{q_1} - E_{k_1})
\]

At this point, we replace such string of deltas by the following expression

\[
\frac{1}{|v_z,1|} \delta(k_2 - k'_2) \delta(k'_1 + q_1 - k_1 - q'_1) \delta(k_{cm} - q_{cm}) \delta(k'_{cm} - q'_{cm}) \delta(E_{q_1} - E_{k_1})
\]

Let us justify this substitution. In first place, the two terms have the same trace in the limit where the state for the particle is peaked around a velocity \(v\) and the size of the wave packet is smaller than \(\delta\), the width of the slab. Also, in such limit (where \(\rho(q,q')\) is different from zero only if \(q \approx q'\)), the regions of momenta \(q,q',k,k'\) for which the arguments of the delta functions vanish are the same in both expressions. In this limit of \(q \approx q'\), from (70) we find in principle two allowed regions:

\[
(k' - k)_\parallel = (q - q')_\parallel \approx 0 ; \quad E_{k'} = E_q' \approx E_q = E_k \quad \rightarrow \quad (k \sim k') \quad \text{or} \quad (k'_\parallel \sim k_\parallel , k_z \sim -k'_z)
\]

but the only one that survives upon \(z\) integration in (65), is the first one, with \(k \sim k'\). This is the same region that makes the arguments of the delta functions in (71) vanish too.
The zeroth order term in $1/m_2$ of (65) is, thus:

$$
\langle k'| A_M \rho A_M + A_E \rho A_E | k \rangle = n(2\pi)^4 \int dq_1 dq_1' \delta (k_1' + q_1 - k_1 - q_1') \delta(E_{q_1} - E_{k_1}) \tilde{V}^*(q_1 - k_1) \tilde{V}(q_1' - k_1') \rho(q_1', q_1)
$$

(73)

Notice that this is totally independent on the specific form of the target states and preserves the mean value of the particle energy.

**The limit of a heavy particle.**

Let us now consider the opposite limit of a heavy particle. The terms in (66) can also be expanded as a geometric series, now in powers of $m_2/m_1$. The relative energy $E_{q_r}$ is:

$$
E_{q_r} = \frac{1}{2m_r}(-q_2 + m_2 q_1 + \frac{m_2}{m_t} q_2)^2
$$

(74)

As in the hamiltonian piece, we cannot treat $\frac{m_2}{m_t} q_1$ as small quantity in front of $q_2$, since we assume that the particle velocity $v_1$ is larger than the velocities of the targets in the gas.

We rewrite $E_{q_r}$ as:

$$
E_{q_r} = \frac{1}{2m_r}(-q_2 + m_r v_1 + \frac{m_2}{m_t} (\Delta q_1 + q_2))^2
$$

(75)

where $\Delta q_1 = q_1 - m_1 v_1$. This brings our expressions to a form that can be easily expanded in powers of $\frac{m_2}{m_t}(\Delta q_1 + q_2)$, the fluctuations of the particle and target momenta about their approximate mean values. (66) reads:

$$
\frac{1}{|\langle v_{z,1} \rangle|} \overline{E_{q_r} - E_{k_r}} = \frac{2m_r}{(m_r v_1 - q_2)^2 - (m_r v_1 - k_2)^2} = \frac{2m_2}{m_t} \sum_{s=0}^{\infty} \left( \frac{m_r}{m_t} (\Delta q_1 + q_2) \right)^s
$$

(76)

that is an expansion in inverse powers of the particle mass.

To zeroth order, the delta functions that appear in (65) are now:

$$
\delta(k_2 - k_2')\delta(q_2 - q_2')\delta(k_1' + q_1 - k_1 - q_1')\delta(k_{cm} - q_{cm})\delta(k_{cm}' - q_{cm}')\delta(E_{m_{2v_1} - q_2} - E_{m_{2v_1} - q_2})\delta(E_{m_{2v_1} - k_2} - E_{m_{2v_1} - q_2})
$$

(77)

At this point we use the fact that the velocity of the particle is much bigger than the target velocity, and the last expression reads:

$$
\frac{1}{|\langle v_{z,1} \rangle|} \overline{\delta(k_2 - k_2')\delta(q_2 - q_2')\delta(k_1' + q_1 - k_1 - q_1')\delta(k_2 - q_2)}\delta(E_{m_{2v_1} - k_2} - E_{m_{2v_1} - q_2})
$$

(78)

Finally, equation (65) is, to zeroth order:

$$
\langle k'| A_M \rho A_M + A_E \rho A_E | k \rangle = n(2\pi)^4 \sum_{m} q_m \int d^3k_2 d^3q_2 d^3q_1 d^3q_1' \delta(k_1' + q_1 - k_1 - q_1')\delta(k_1 + k_2 - q_1 - q_2)\delta(E_{m_{2v_1} - k_2} - E_{m_{2v_1} - q_2})\rho(q_1', q_1)|A_m(q_2)|^2|\tilde{V}^*(k_2 - q_2)|^2
$$

(79)
7 The index of refraction

The index of refraction is usually defined as the phase shift that the medium induces on a plane wave as the particle propagates through the medium. In an evolution where coherence is not preserved, this concept, as it stands, does not hold anymore. This is the case of the Lindblad evolution. Yet, a generalization is still possible for the part of $\rho$ that preserves coherence. Let us first identify such part regarding the way a Lindblad equation increases the mixing of the density matrix. A Lindblad evolution is of the form:

$$\dot{\rho} = -iH\rho + i\rho H^\dagger + 2A\rho A^\dagger,$$  \hspace{1cm} (80)

where $H$ is $H_{\text{eff}} - iA^\dagger A$. After an infinitesimal time interval, the density matrix evolves to:

$$\rho(t + \delta t) = \rho(t) + i\delta t(H\rho - \rho H^\dagger) + \delta t(2A\rho A^\dagger)$$ \hspace{1cm} (81)

The first two terms together add up to a density matrix which has trace less than one (since $H$ is non-hermitian) and coherence is preserved on it (since it is a Hamiltonian evolution). There is a further term, the last one in (81), which is also a density matrix on its own. Thus, we see that under such a time evolution a density matrix can always be split as a part in which coherence is preserved, $\rho^{\text{coh}}$, plus the rest, $\rho^{\text{mix}}$, which is also a density matrix (neither one is normalized here). This mechanism reflects the fact that the average of two density matrices is, in general, a density matrix with more mixing. This suggests a definition of $\rho^{\text{coh}}$ so that it satisfies the Hamiltonian equation:

$$\frac{\partial}{\partial t} \rho^{\text{coh}} = -iH\rho^{\text{coh}} + i\rho^{\text{coh}} H^\dagger;$$ \hspace{1cm} (82)

then, for $\rho^{\text{mix}}$:

$$\frac{\partial}{\partial t} \rho^{\text{mix}} = -iH\rho^{\text{mix}} + i\rho^{\text{mix}} H^\dagger + 2A\rho^{\text{mix}} A^\dagger + 2A\rho^{\text{coh}} A^\dagger,$$ \hspace{1cm} (83)

which is the same as (80) plus a term that depends on $\rho^{\text{coh}}$. This non-homogeneous term is the responsible for the increasing of the trace of $\rho^{\text{mix}}$, in order to maintain properly normalized the total density matrix $\rho$. Equation (83) also ensures that $\rho^{\text{mix}}$ is a positive matrix, since its time derivative is a sum of positive matrices.

It is natural, thus, to write $\rho$ in equation (39) as $\rho^{\text{coh}} + \rho^{\text{mix}}$, with $\rho^{\text{coh}}$ and $\rho^{\text{mix}}$ satisfying the equations:

$$\frac{\partial}{\partial t} \rho^{\text{coh}} = -i \left[ H \rho^{\text{coh}} - \rho^{\text{coh}} H^\dagger \right] \hspace{1cm} H \equiv -\frac{v}{\delta} \left( \sum_{\{m\}} q_m D_M^{(m)} - i \frac{1}{2} \left( \sum_{\{m\}} q_m D_M^{(m)} \right)^2 \right)$$ \hspace{1cm} (84)

$$\frac{\partial}{\partial t} \rho^{\text{mix}}(t) = -i \left[ H_{\text{eff}}, \rho^{\text{mix}} \right] + \frac{1}{4v} \sum_{m,n} q_m q_n L_M^{(n,m)} [\rho^{\text{mix}}] + \frac{1}{2v} \sum_{m,j} q_m J_M^{(l_j,m_j)} \rho^{\text{mix}} + \frac{\nu}{\delta} \sum_{m,n} q_m q_n A_M^{(n,m)} \rho^{\text{coh}} A_M^{(n,m)} + \frac{1}{2} \sum_{m,j} q_m J_M^{(l_j,m_j)} \rho^{\text{coh}} A_E^{(l_j,m_j)}$$ \hspace{1cm} (85)
We naturally take $\rho^{coh}(0) = \rho(0)$ and $\rho^{mix}(0) = 0$ as initial conditions.

In spite of the fact that $\rho$ decoheres, an index of refraction can be defined from $\rho^{coh}$, since it still evolves with a Hamiltonian, which has already been computed for the case of an homogeneous gas in terms of the forward scattering amplitude. In order to retrieve the refraction index we should add the free Hamiltonian, and get:

$$\frac{k'}{k} = \sqrt{1 - \frac{\langle k|H|k \rangle}{k^2/2m_1}}$$  \hspace{1cm} (86)

In the limit of heavy targets, $\frac{m_1}{m_2} \ll 1$, we obtain:

$$\frac{k'}{k} \approx 1 + 2\pi n \frac{m_1}{m_r} \frac{(1 - \frac{m_1}{m_2} k \frac{\partial}{\partial k}) f(k, k, m_r)}{k^2} + O \left( \frac{(m_1/m_2)^2}{k^2} \right)$$  \hspace{1cm} (87)

This is the generalization of the result obtained by Fermi in the case of scattering centers [1]. Notice that the imaginary part of (87) has a clear interpretation: it is the depletion of the amplitude modulus due to a loss of coherence by dispersion.

The opposite situation of heavy particle, when $\frac{m_1}{m_2} \gg 1$, has also been computed in the previous section, with the resulting index of refraction:

$$\frac{k'}{k} \approx 1 + 2\pi n \frac{m_1}{m_r} C$$  \hspace{1cm} (88)

where the function $C$ is:

$$C = \sum m_q m \left( 1 + \frac{m_2}{m_1} (k_1 k_2/k_1^2 + k_2 \frac{\partial}{\partial K}) f(K = \frac{m_2}{m_1} k_1 - k_2, K; m_r) \right)$$  \hspace{1cm} (89)

8 Decoherence effects and interference patterns

The decomposition found in the previous section allows us to see how interference patterns are destroyed by the decoherence induced through the interaction with the medium. Such patterns appear by the interference of two states, $|\phi\rangle$ and $|\psi\rangle$, that travel following different paths and rejoin later. Thus, a superposition of $|\phi\rangle$ and $|\psi\rangle$ must retain enough coherence so that these interference patterns can be observed. For the rest of this section we assume that the initial state $|\Phi\rangle$ is the sum $|\phi\rangle + |\psi\rangle$. We thus start out with $\rho^{coh}(0) = |\Phi\rangle \langle \Phi|$ and $\rho^{mix}(0) = 0$.

The crossed terms $|\phi\rangle \langle \psi|$ of the $\rho$ matrix are the responsible for the interferences: we argue that in the case when $|\phi\rangle$ and $|\psi\rangle$ are well separated and localized they decay exponentially in $\rho(t)$. The proof is in two steps. We first recall that the whole $\rho^{coh}$ decays in time, straightforwardly from its construction, in particular its crossed terms. Then we see that these crossed terms never feed-back into $\rho$ through $\rho^{mix}$ via eq. (83) if the two states are separated enough and localized.
The proof goes by integrating the differential equation for $\rho^{\text{mix}}$ for an infinitesimal time interval $\delta t$:

$$
\rho^{\text{mix}}(\delta t) = \delta t \frac{1}{2} \sum_{m,n} q_m q_n A^{n,m}_M \rho^{\text{coh}}(0) A^\dagger_{n,m} + \delta t \frac{V}{\delta} \sum_{l,j,m_j} q_{m_j} A^{(l_j,m_j)}_E \rho^{\text{coh}}(0) A^\dagger_{(l_j,m_j)}
$$

(90)

All crossed terms $|\phi\rangle\langle\psi|$ appearing on the r.h.s. of this equation vanish. The crossed terms in the second piece are directly zero, because $A^{(l_j,m_j)}_E$ is the amplitude for target $j$ to jump from $|m_j\rangle$ to $|l_j\rangle$ by interacting with the particle: when target $j$ is close to the state $|\phi\rangle$ it is far from the state $|\psi\rangle$, then $A^{(l_j,m_j)}_E|\psi\rangle = 0$, and vice-versa. As for the crossed terms from the first piece, they are proportional to:

$$
\sum_{\{m,n\}} \left( \prod_l q_{m_l} \prod_l q_{n_l} \right) \left( \sum_j (D^{m_j}_M - D^{n_j}_M) \right) |\phi\rangle\langle\psi| \left( \sum_i (D^{i_m}_M - D^{i_n}_M) \right)
$$

(91)

and since states $|\phi\rangle$ and $|\psi\rangle$ are far away, targets that contribute in the summation over the $j$ index are different from those that contribute in the summation over $i$ index. This allows us to write (91) as:

$$
\left( \sum_{\{j \text{ close to } \phi\}} \sum_{m_j,n_j} q_{m_j} q_{n_j} (D^{m_j}_M - D^{n_j}_M) \right) |\phi\rangle\langle\psi| \left( \sum_{\{i \text{ close to } \psi\}} \sum_{m_i,n_i} q_{m_i} q_{n_i} (D^{i_m}_M - D^{i_n}_M) \right) = 0
$$

(92)

Then, after an infinitesimal time $\delta t$, crossed terms stay out from $\rho^{\text{mix}}(t)$. This completes the proof.

We apply this result to the study of the interference patterns in a two slit Young experiment. We shall study separately the probability distribution of hits on the screen that comes out from $\rho^{\text{mix}}$, and from $\rho^{\text{coh}}$, each one being a density matrix on its own (except for an unessential normalization factor). Finally, the observed pattern is retrieved as the sum of both.

The part of $\rho^{\text{coh}}$ is easy: It is essentially the same as if there were no medium, with the wave number corrected $k \rightarrow k'$. This changes the oscillation wave-length to $\frac{2\pi}{\text{Re}(k')} L$ and introduces a global damping factor of $e^{-2\text{Im}(k')L}$ which diminishes the amplitude on the screen by this amount.

Unlike $\rho^{\text{coh}}$, the part of $\rho^{\text{mix}}$ is far more involved to obtain, for it contains all the decoherence effects and it is the one which eventually washes out the interferences when the propagation of the particle between the slits and the screen is through a medium, rather than the vacuum. However, it is easy to see that it produces an approximately constant background in the region close to the central peaks.

The argument is indirect and uses the result of an auxiliary problem, that has as initial state $\tilde{\rho}(0) = \frac{1}{2} |\phi\rangle\langle\phi| + \frac{1}{2} |\psi\rangle\langle\psi|$, for which the solution can be qualitatively described. For
such a totally uncoherent state there are no oscillations and the intensity is the sum of the intensities produced with one of the slits closed. In the presence of matter the curve is rather broad and flat, because each target disperses the particle in all directions, thus increasing the odds that it hits the screen away from the center. Such a broad shape is only possible if the mixing piece itself is even broader, since \( \tilde{\rho}_{\text{coh}} \) only reduces its size by a depletion factor \( e^{-2\text{Im}(k')L} \).

This is already the end of the argument if one realizes that this \( \rho^{\text{mix}} \) has to be the same either with the initial auxiliary state \( \tilde{\rho}(0) \) or with \( |\phi\rangle + |\psi\rangle \), because - and this is crucial - the crossed terms \( |\phi\rangle\langle\psi| \) of the initial state do not intervene at all in the subsequent \( \rho^{\text{mix}} \) evolution, as previously shown.

We thus see that the mixing part adds nothing but a constant background around the central region, that dies off very slowly as one moves away.

Summarizing, the observed interference pattern consists of a constant background, which entails the decoherence due to the medium, superposed to the oscillations of \( \rho^{\text{coh}} \), also damped by the medium (see Fig. 2). The relative strength of the two contributions is dictated by unitarity, and one can roughly estimate the ratio of the oscillation amplitude over the background as:

\[
\frac{2e^{-2\text{Im}(k')L}}{1 - e^{-2\text{Im}(k')L}},
\]

i.e., the smaller this number may get, the more invisible the interference fringes become.

Finally, we would like to stress the observation that from the size of the tiny oscillations of the interference pattern in the central region, one can fit \( e^{-2\text{Im}(k')} \). Precise fits in experiments of this kind thus provide a direct measurement of \( \text{Im}(k') \), related to \( \text{Im}(H) = -A^\dagger A \) in eq. \cite{SU}; i.e., the size of the Lindblad coefficients for the particle crossing a medium.

\section{Conclusions}

In this article we have addressed a microscopic derivation of the Lindblad equation that describes the loss of quantum coherence of a particle that crosses a medium with which it interacts. This is done for a dilute medium made out of targets that interact with a particle through a short range potential. In the case that the medium is a gas, we keep in the evolution equation terms up to second order in the potential. We also use the impulse approximation, i.e., we approximate the scattering amplitude for the particle with the gas by the sum of the scattering amplitudes with each single target. We argue how the imaginary part of the refraction index given by Fermi is related to the Lindblad terms and is, therefore,
due to decoherence. Finally, we show that when we apply our results to a double slit Young experiment, the characteristic interference patterns disappear due to the presence of the medium, a distinctive sign of decoherence.

We stress how the corrections due to the presence of matter split into different terms according to the two different mechanisms that produce decoherence: the *footprint* mechanism, related to the entanglement with the medium; and the *mixture* mechanism, related to the probabilistic knowledge of the medium state.

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