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ON THE HOCHSCHILD HOMOLOGY OF SINGULARITY CATEGORIES

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ABSTRACT. Let $k$ be an algebraically closed field and $A$ a finite-dimensional $k$-algebra. In this note, we determine complexes which compute the Hochschild homology of the canonical dg enhancement of the bounded derived category of $A$ and of the canonical dg enhancement of the singularity category of $A$. As an application, we obtain a new approach to the computation of Hochschild homology of Leavitt path algebras.

1. Reminder on Hochschild homology of algebras and categories

Let $k$ be a field. We write $\otimes$ for $\otimes_k$. Let $A$ be a $k$-algebra (associative, with 1). We write $\text{Mod}_A$ for the category of all (right) $A$-modules and $D_A = D(\text{Mod} A)$ for its unbounded derived category. Let $A^e = A \otimes A^{\text{op}}$ be the enveloping algebra of $A$ so that $A^e$-modules identify with $A$-bimodules. The Hochschild homology of $A$ is defined by $HH_p(A) = \text{Tor}_{A^e}^p(A,A)$, $p \in \mathbb{Z}$.

Alternatively, we may define it as the $p$th homology group of the Hochschild chain complex $HH(A)$ of $A$, i.e. the complex $C_*A$ concentrated in homological degrees $\geq 0$

$$A \leftarrow A \otimes A \leftarrow \ldots \leftarrow A^{\otimes p} \leftarrow A^{\otimes (p+1)} \leftarrow \ldots$$

with $C_pA = A^{\otimes (p+1)}$, $p \geq 0$, and differential given by

$$(1.0.1) \quad d(a_0, \ldots, a_p) = \sum_{i=0}^{p-1} (-1)^i(a_0, \ldots, a_ia_{i+1}, \ldots, a_p) + (-1)^p(a_pa_0, \ldots, a_{p-1}),$$

where we write $(a_0, \ldots, a_p)$ for $a_0 \otimes \cdots \otimes a_p$. Notice that the first differential takes $a \otimes b$ to the commutator $ab - ba$.

We see that $HH_0(A)$ is the quotient $A/[A,A]$ of the vector space $A$ by its subspace generated by all commutators and that $HH_p(A)$ and $HH(A) \in Dk$ are functorial in the algebra $A$. The definitions extend from $k$-algebras to small $k$-categories. For example, the Hochschild complex then becomes the complex

$$\bigoplus \mathcal{A}(X_0, X_0) \leftarrow \bigoplus \mathcal{A}(X_1, X_0) \otimes \mathcal{A}(X_0, X_1) \leftarrow \ldots$$

whose $p$th term $(p \geq 0)$ is the sum

$$\bigoplus \mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_{p-1}, X_p) \otimes \cdots \otimes \mathcal{A}(X_0, X_1)$$

taken over all sequences of objects $X_0, X_1, \ldots, X_p$ of $\mathcal{A}$ and whose horizontal differential is given by formula (1.0.1). One then shows that the inclusion $A \rightarrow \text{proj}(A)$ of the one-object category given by $A$ into the category $\text{proj}(A)$ of finitely generated projective right

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A-modules induces a quasi-isomorphism
\[ HH(A) \xrightarrow{\sim} HH(\text{proj } A). \]
In particular, this yields Morita invariance of Hochschild homology. The definitions further extend to small differential graded (dg) categories \( \mathcal{A} \), for example the dg category \( C_{dg}^b(\text{proj } A) \) of bounded complexes over \( \text{proj } (A) \). We refer the reader to [9] for more information on this example and dg categories in general. The inclusion \( \text{proj } (A) \to C_{dg}^b(\text{proj } A) \) yields an isomorphism
\[ HH(\text{proj } A) \xrightarrow{\sim} HH(C_{dg}^b(\text{proj } A)) \]
and this yields the invariance of Hochschild homology under derived equivalences. We will need the following localization theorem.

**Theorem 1.1** ([8]). Let
\[ \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \]
be a sequence of dg categories such that the induced sequence of derived categories
\[ 0 \to \mathcal{D}_A \xrightarrow{F^*} \mathcal{D}_B \xrightarrow{G^*} \mathcal{D}_C \to 0 \]
is exact. Then there is a canonical triangle
\[ HH(\mathcal{A}) \xrightarrow{HH(F)} HH(\mathcal{B}) \xrightarrow{HH(G)} HH(\mathcal{C}) \xrightarrow{} \Sigma HH(\mathcal{A}) \]
in \( D_k \) and hence long exact sequences in Hochschild (and cyclic) homology.

Let \( Q \) be a finite quiver and \( I \) an admissible ideal in \( kQ \), i.e. a two-sided ideal contained in the ideal generated by the arrows and such that the quotient \( kQ/I \) is finite-dimensional. Let \( R \) be the quotient of \( A \) by its radical. Thus, as an \( A \)-module, the algebra \( R \) is the direct sum of the simple \( A \)-modules. Following [7], we define the Koszul dual of \( A \) to be the dg algebra
\[ A^! = \text{RHom}_A(R, R). \]
Thus, if \( P \) is a projective resolution of the \( A \)-module \( R \), then the Koszul dual is quasi-isomorphic to the dg endomorphism algebra \( \text{Hom}_A(P, P) \) of \( P \). The following theorem is a special case of Corollary D.2 of Van den Bergh’s [12]. We write \( D \) for the dual \( \text{Hom}_k(?, k) \) over the ground field.

**Theorem 1.2** (Van den Bergh). We have a canonical isomorphism
\[ HH(A^!) \xrightarrow{\sim} DHH(A). \]
We refer to [6] for a comparison taking into account much more structure.

2. **Hochschild homology of derived categories and singularity categories**

Let \( Q \) be a finite quiver and \( I \) an admissible ideal in \( kQ \). Let \( \text{mod } A \) be the category of \( k \)-finite-dimensional right \( A \)-modules. Denote by \( \mathcal{D}^b(A) = \mathcal{D}^b(\text{mod } A) \) the bounded derived category of \( A \) and by \( \text{per } (A) \) the perfect derived category, i.e. the thick subcategory generated by the free \( A \)-module of rank 1. Following Buchweitz [2] and Orlov [10], one defines the singularity category of \( A \) as the Verdier quotient
\[ \text{sg}(A) = \mathcal{D}^b(A)/\text{per } (A). \]
Using the canonical dg enhancements of \( \mathcal{D}^b(A) \) and \( \text{per } (A) \), cf. [9], we obtain a canonical exact sequence of dg categories
\[ 0 \to \text{per }_{dg}(A) \to \mathcal{D}^b_{dg}(A) \to \text{sg}_{dg}(A) \to 0. \]
where the dg quotient $sg_{dg}(A)$ yields a canonical dg enhancement for $sg(A)$. It is not hard to see that, in the homotopy category of dg categories, it is functorial with respect to bimodule complexes $X \in D(A^{op} \otimes B)$ such that $X_B$ is perfect over $B$ and $A X$ is perfect over $A$. From the localization theorem 1.1, we deduce a triangle

$$
(2.0.1) \quad H H(\text{per}_{dg}(A)) \rightarrow H H(D^b_{dg}(A)) \rightarrow H H(sg_{dg}(A)) \rightarrow \Sigma H H(\text{per}_{dg}(A))
$$

in the derived category of vector spaces.

**Theorem 2.1.** We have a canonical isomorphism $H H(D^b_{dg}(A)) \cong D H H(A)$.

**Proof.** Recall that we have defined $R$ to be the quotient of $A$ by its radical and the Koszul dual $A^!$ as $R \text{Hom}_A(R, R)$. Since the module $R$ is a classical generator of the bounded derived category $D^b(A)$, we deduce from the results of [7] that we have a triangle equivalence

$$R \text{Hom}_A(R, ?) : D^b(A) \rightarrow \text{per}(A^!).$$

This lifts to a quasi-equivalence

$$D^b_{dg}(A) \rightarrow \text{per}_{dg}(A^!).$$

By Morita invariance of Hochschild homology, we have

$$H H(A!) \rightarrow H H(\text{per}_{dg}(A^!)).$$

By Van den Bergh’s theorem 1.2, we have

$$H H(A!) \rightarrow D H H(A).$$

The claim follows if we combine these isomorphisms. √

Define a linear map $\tau : A \rightarrow D A$ by sending an element $a \in A$ to the linear form which takes $b \in A$ to the trace of the linear map

$$\lambda_a \rho_b : A \rightarrow A, \ x \mapsto axb,$$

where $\lambda_a$ is left multiplication by $a$ and $\rho_b$ right multiplication by $b$. Notice that since $A$ is finite-dimensional, this is well-defined. Moreover, the value of $\langle a, b \rangle = (\tau(a))(b)$ only depends on the classes of $a$ and $b$ in $H H_0(A)$, which is canonically isomorphic to $R$. It is not hard to check that in the basis formed by the $e_i$, the matrix of the induced bilinear form

$$H H_0(A) \times H H_0(A) \rightarrow k$$

is the Cartan matrix of $A$, whose $(i, j)$-entry is the dimension of $e_i A e_j$. Define the double Hochschild complex of $A$ to be the complex

$$\cdots \rightarrow A \otimes A \xrightarrow{b} A \xrightarrow{\tau} D A \xrightarrow{D h} D(A \otimes A) \xrightarrow{D h} \cdots ,$$

where $DA$ sits in degree 0, the differentials $b$ are those of the Hochschild complex and the $Dh$ their duals.

Let us abbreviate $S = sg_{dg}(A)$.

**Theorem 2.2.** In $D k$, we have a canonical isomorphism between $H H(S)$ and the double Hochschild complex of $A$.

Notice that this implies in particular that $H H_n(S)$ is finite-dimensional for all $n$. This is surprising since the singularity category $sg(A)$ is usually not Hom-finite (except if $A$ is Gorenstein), cf. for example [3].
Proof. We use the triangle
\[ HH(\text{per}_{dg}(A)) \longrightarrow HH(D^b_{dg}(A)) \longrightarrow HH(S) \longrightarrow \Sigma HH(\text{per}_{dg}(A)) \]
obtained from the localization theorem 1.1. We have already seen that it is isomorphic to a triangle
\[ HH(A) \rightarrow HH(A^!_1) \rightarrow HH(S) \rightarrow \Sigma HH(A), \]
where the first morphism is induced by the inclusion per\(_{dg}(A) \rightarrow D^b_{dg}(A)\). Thus, the complex \( HH(S) \) identifies with the mapping cone over the morphism \( HH(A) \rightarrow HH(A^1) \).

Let us determine this morphism explicitly. Recall that the functor \( HH \), considered as a functor on the homotopy category of small dg categories with values in the derived category \( Dk \), commutes with tensor products. We have the following commutative square
\[
\begin{array}{ccc}
\text{per}_{dg}(A^{op}) \otimes \text{per}_{dg}(A) & \longrightarrow & \text{per}_{dg}(k) \\
\downarrow & & \downarrow \\
\text{per}_{dg}(A)^{op} \otimes D^b_{dg}(A) & \longrightarrow & \text{per}_{dg}(k)
\end{array}
\]
Here, a pair \((P_1, P_2)\), \(P_1 \in \text{proj}\(A^{op}\))\), \(P_2 \in \text{proj}\(A\)) is taken to \(P_2 \otimes_A P_1\) by the top arrow and to \((\text{Hom}_A(P_1, A), P_2)\) by the left vertical arrow. It follows from Appendix D in [12] that the lower horizontal arrow induces a non degenerate pairing
\[ HH(A) \otimes HH(D^b_{dg}(A)) \rightarrow HH(k) = k. \]

A direct computation now shows that the morphism
\[ HH(A) \rightarrow DHH(A) \]
is the composition
\[ HH(A) \rightarrow HH_0(A) \rightarrow DHH_0(A) \rightarrow DHH(A) \]
where the middle morphism is induced by the map \( \tau \).

Corollary 2.3. For \( n \geq 2 \), we have canonical isomorphisms
\[ HH_n(S) \cong HH_{n-1}(A) \cong DHH_{1-n}(S). \]
Moreover, we have
\[ HH_1(S) \cong \ker(HH_0(A) \xrightarrow{\tau} DHH_0(A)) \cong DHH_0(S). \]

3. Application: Hochschild homology of dg Leavitt path algebras

Let \( Q \) be a finite quiver, for example a quiver with one vertex and a unique loop \( \alpha \). Let \( A \) be the associated radical square zero algebra, i.e. the quotient of \( kQ \) by the square of the ideal generated by the arrows. So for the one-loop quiver, we have \( A = k[\varepsilon]/(\varepsilon^2) \). Let \( Q^* \) be the graded quiver obtained from the opposite quiver of \( Q \) by assigning each arrow \( \alpha^* : j \rightarrow i \) corresponding to an arrow \( \alpha : i \rightarrow j \) of \( Q \) the degree +1. For each vertex \( i \) of \( Q \), consider the arrows \( \alpha^*_s : i \rightarrow t(\alpha^*_s) \), \( 1 \leq s \leq t_i \), starting in \( Q^* \) at \( i \). Let
\[ \varphi_i : P_i \rightarrow \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha^*_s)} \]
be the morphism with components $\alpha_s^*$, where $P_i = e_i kQ^*$. For example, for the one-loop quiver, we just have $\varphi(1) = \alpha^*: P_1 \to \Sigma P_1$. Note that if $i$ is a sink of $Q$, then

$$\bigoplus_{s=1}^{t_i} P_{t(\alpha_s^*)} = 0.$$  

For each vertex $i \in Q_0$, let $\varphi_i^{-1} = [\beta_{i1}, \ldots, \beta_{it_i}] : \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha_s^*)} \to P_i$ be the formal inverse of $\varphi(i)$. The graded Leavitt path algebra of $Q$ is obtained from $kQ^*$ by adjoining all coefficients $\beta_{ij}$ of all formal inverses $\varphi(i)^{-1}$, $i \in Q_0$. We endow $L_Q$ with the grading inherited from $Q^*$ and with $d = 0$.

**Theorem 3.1** (Smith [11], Chen–Yang [5]). We have a triangle equivalence $\text{per } (L_Q) \sim \text{sg}(A)$ taking $e_i L_Q$ to the simple $S_i$.  

**Corollary 3.2.** The Hochschild homology $HH_*(L_Q)$ of the Leavitt path algebra is computed by the double Hochschild complex

$$\cdots \longrightarrow A \otimes A \longrightarrow A \longrightarrow DA \longrightarrow D(A \otimes A) \longrightarrow \cdots,$$

(with $DA$ in degree 0). In particular, we have

$$\dim HH_p(L_Q) = 0 < \infty$$

for all $p \in \mathbb{Z}$.

A different description of the Hochschild homology of Leavitt path algebras is due to Ara–Cortiñas [1].

4. Beyond radical square zero

Let $Q$ be a finite quiver and $A = kQ/I$ the quotient of its path algebra by an admissible ideal. Let $J$ be the radical of $A$ and $R = kQ_0$ so that we have $A = R \oplus J$ as $R$-bimodules. Let $A_0 = (T_R J)/(J \otimes R J)$ be the radical square zero algebra associated with $A$. Thus, we have $A_0 = R \oplus J = A$ as $R$-bimodules but we have $xy = 0$ in $A_0$ for any two elements of $J$. We view $A_0$ as a degeneration of $A$ and $A$ as a deformation of $A_0$. As pointed out by Chen–Wang [4], this suggests that the singularity category $\text{sg}(A)$ is a deformation of the singularity category $\text{sg}(A_0)$, which is equivalent to the perfect derived category per $(L_{A_0})$ of the graded Leavitt path algebra $L_{A_0}$. Hence we can hope for the existence of a dg algebra $L_A$ obtained from $L_{A_0}$ by deformation such that per $(L_A)$ is equivalent to $\text{sg}(A)$. We sum up the situation in the following diagram

$$\begin{array}{ccc}
A_0 & \xrightarrow{\text{deformation}} & A \\
\text{sg}(A_0) & \xrightarrow{\text{deformation}} & \text{sg}(A) \\
\text{per } (L_{A_0}) & \xrightarrow{\text{deformation?}} & \text{per } (L_A) \\
L_{A_0} & \xrightarrow{\text{deformation?}} & L_A \\
\end{array}$$

The following theorem confirms this hope.
Theorem 4.1 (Chen–Wang [4]). The graded algebra $L_{A_0}$ admits a canonical differential $d_A$ such that for $L_A = (L_{A_0}, d_A)$, we have a triangle equivalence
\[\text{per}(L_A) \xrightarrow{\sim} \text{sg}(A).\]

Corollary 4.2. The Hochschild homology of the dg Leavitt path algebra $L_A$ is computed by the double Hochschild complex of $A$.

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