Features of Quantum Mechanics on a Ring

Bernhard K. Meister

Department of Physics, Renmin University of China, Beijing, China 100872

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Aspects of quantum mechanics on a ring are studied. Either one or two impenetrable barriers are inserted at nodal and non-nodal points to turn the ring into either one or two infinite square wells. In the process, the wave function of a particle can change its energy, as it gets entangled with the barriers and the insertion points become nodes. Two seemingly innocent assumptions representing locality and linearity are investigated. Namely, a barrier insertion at a fixed node needs no energy, and barrier insertions can be described by linear maps. It will be shown that the two assumptions are incompatible.

I. INTRODUCTION

Quantum mechanics of a particle trapped on a ring of radius one with mass $M$ is described by a Hamiltonian of the form

$$H = -\frac{\hbar^2}{2M} \frac{d^2}{d\theta^2},$$

defined for $\theta \in [0, 2\pi]$. The energy eigenvalues are $E_n = \frac{\hbar^2}{2M} n^2$ for $n \in \mathbb{N}$ and possess a two-fold degeneracy with the associated eigenstates $\sin(n\theta)$ and $\cos(n\theta)$. Another basis combination is $\sin(n\theta - \alpha) = \sin(n\theta) \cos(\alpha) - \cos(n\theta) \sin(\alpha)$ and $\cos(n\theta - \alpha) = \cos(n\theta) \cos(\alpha) + \sin(n\theta) \sin(\alpha)$.

The purpose of the paper is to investigate the effect of inserting impenetrable barriers at either one point, the origin, or simultaneously at both the origin and $\alpha$. Related investigations were carried out for a barrier insertion in the one-dimensional infinite well by Bender et al. [1] and the author [2, 3]. The problem of one and two barrier insertions on a ring was investigated by the author [4].

Inserting one impenetrable barrier turns the ring into a one-dimensional infinite well, while two barriers turn the ring into two separate chambers. For a given initial state $\psi(\theta, 0)$ of the system, there are three possibilities for the insertion point $\theta_0$ of the barrier. It can be a fixed or transitory node of the wave function or a general non-zero amplitude point. By a ‘fixed node’ we mean a point $\theta_0$ for which the time-dependent wave function satisfies $\psi(\theta_0, t) = 0$ for all $t \geq 0$. Wave functions that are superposition of eigenfunctions of $H$ can have zero amplitude points that change with time called ‘transitory nodes’. We confine ourselves to ‘fixed nodes’ to deliver a more striking result. The third group of ‘non-nodal’ points have a non-zero amplitude throughout time.

Cases considered in subsequent sections:

The first case involves one impenetrable barrier at a node. Since the wave function vanishes at the insertion point, and therefore there is no interaction with the barrier, the energy of the wave function is unchanged. After the insertion, the configuration is turned into one-dimensional infinite well.

Inserting a barrier at a point that is not a node provides another example. In this case inserting a barrier changes the energy by an amount that depends on the rate at which the barrier is inserted. For conceptual simplicity, we restrict ourselves to the instantaneous insertion.

Finally, two barriers are inserted at a fixed node and at a non-nodal point dividing the ring into two independent infinite wells. The energy of the wave function is changed and the energy of the barriers is transformed in a complementary way to guarantee energy conservation. The exact nature of the entanglement of the particle with the barriers during an insertion is dependent on the insertion speed and not obvious. A variety of answers are possible. We take the different energy eigenstates of the particle in the post-insertion basis to be individually entangled with different barrier states. Other choices lead to other transfer energies and are experimentally distinguishable.

What we learn from these cases is that quantum mechanics on the ring cannot simultaneously satisfy two seemingly harmless assumptions

1) a barrier insertion at a ‘fixed node’ needs no energy, and
2) barrier insertions can be described by linear maps,

abbreviated LOC & LIN, respectively. LOC is linked to the principle of locality. The paper provides a novel view on the clash between locality and linearity in quantum mechanics. Initially, we conjecture both LOC and LIN to

*Electronic address: bernhard.k.meister@gmail.com*
be correct, and then point out a contradiction, i.e. there is no map consistent with LIN that can account for the simultaneous insertion of two barriers (at the origin and $\alpha$) such that a barrier inserted at a nodal point requires no energy. To exploit linearity we consider both $\sin(\theta - \alpha)$ with its node at $\alpha$ as well its decomposition in the $\sin(\theta)$ and $\cos(\theta)$ basis, i.e. $\sin(\theta - \alpha) = \sin(\theta) \cos(\alpha) - \cos(\theta) \sin(\alpha)$, whose $\sin(\theta)$ component has a node at the origin, but not at $\alpha$. Due to linearity we can map each component separately observing the LOC condition and sum the results. It will be shown that the combination does not coincide with the mapping of $\sin(\theta - \alpha)$ itself.

As a consequence, one has to choose between LIN and LOC. In an earlier paper\cite{4}, we presupposed LOC and this led to the breaking of the Helstrom bound. If instead one assumes LIN, then one can distinguish between inserting one or two barriers, since simultaneous insertions can require energy even at nodal points, whereas individual insertions at fixed nodes are energy-free.

The structure of the rest of paper is as follows. In section II the impact of one instantaneous insertion of one barrier on a ring is studied. In section III the simultaneous insertion of two barriers is considered. In section IV the inconsistency of LOC & LIN is proven and some implications are discussed. In the conclusion the result is briefly reviewed and some general comments added.

II. INSTANTANEOUS INSERTION OF ONE BARRIER

In this section the insertion, considered to be instantaneous, of one barrier at both nodal and non-nodal points is reviewed. The nodal point insertion is dealt with first. This is easier, since the particle wave function and energy is left unchanged - for background material see section II of Bender et al.\cite{1}, where a series of results for a particle in a one dimensional box, directly applicable to quantum mechanics on a ring, were established. In the penultimate section, the energy-free nature of individual insertions at fixed nodes will be employed.

The situation is more intricate for an insertion at a non-nodal point. Energy is needed to modify the wave function. In the idealised setting considered here the required energy is infinite. The energy localised in the barrier point inserted at $t = 0$ propagates through the system at $t > 0$ and increases the energy on the ring. The result is a fractal wave function. The details of the calculation can be found in sections IV & VI of Bender et al.\cite{1} or in section II of \cite{2,3}. The question how fast changes in the wave function and energy propagate through the system is of interest as one moves beyond non-relativistic quantum mechanics.

An example is provided next. The insertion at zero into the wave function $\sin(x)$, which can also be represented as $\cos(a) \sin(x-a) + \sin(a) \cos(x-a)$, can be expanded into the unique post insertion basis, which has lost its degeneracy due to the new boundary conditions. The expansion has the following coefficients

$$f_n = \frac{1}{\pi} \int_0^{2\pi} d\theta \sin(n\theta/2)(\cos(a) \sin(\theta-a) + \sin(a) \cos(\theta-a)) = 4\frac{\sin(\pi n)}{n^2-4}$$

which are zero for all $n$, except for $n = 2$ with $f_2$ equal to $\pi$ and coincides with the expected result of $\sin(\theta)/\pi$. In the coming section we move from one to two barriers.

III. THE INSTANTANEOUS INSERTION OF TWO IMPENETRABLE BARRIERS

The case of two simultaneous insertions changing the ring into two separate infinite square wells is considered in this section. We rely in this section on the LOC assumption to restrict the transfer of energy to barriers at non-nodal points. In the following paragraphs the wave functions of a particle

$$\phi(\theta) := \frac{1}{\sqrt{\pi}} \sin(\theta),$$
$$\psi(\theta) := \frac{1}{\sqrt{\pi}} \sin(\theta-\alpha),$$

are evaluated before and after the insertion of two barriers, at $t = 0$, where $\theta \in [0, 2\pi]$, and $\alpha \in (0, \pi/2)$.

Due to the pre-insertion symmetry both candidate wave functions are eigenfunctions of the Hamiltonian of a particle on a ring. The symmetry is only broken by the barriers. The barriers are inserted at the point 0 and $\alpha$ at time $t = 0$. The barrier inserted at point 0 hits a fixed node of $\phi$, but the second barrier at point $\alpha$ hits a non-nodal point. The situation is the reversed for $\psi$. At this stage we assume that at nodal points no energy transfer occurs (consistent with LOC), but a barrier at a non-nodal point is associated with a change of energy.

The entanglement created during the insertion between the particle and the barriers could take different forms. One possible answer is to assume that the wave function in each compartment in its totality is entangled with the non-nodal barrier. Another answer, which we favour, is to have different energy eigenstates of the particle in the new basis individually entangled with the non-nodal barrier.
An instantaneous insertion requires, due to the change of the configuration space, an expansion of the original wave functions into the energy basis of the two separate one dimensional infinite wells. This will be carried out next. The first expansion is in the interval \((0, \alpha)\) with the discrete energy levels \(E_n^\alpha = \frac{n^2 \pi^2 \hbar^2}{2M \alpha^2}\) and the second expansion is for the interval \((\alpha, 2\pi)\) with the discrete energy levels \(E_n^{2\pi-\alpha} = \frac{n^2 \pi^2 \hbar^2}{2M(2\pi-\alpha)^2}\) such that the first candidate wave function has directly after the insertion the following form

\[
\phi_{after}(\theta) := \begin{cases} 
\sqrt{\frac{1}{\alpha}} \sum_{n=1}^{\infty} a_n \sin \left( \frac{n \pi \theta}{\alpha} \right) & 0 < \theta < \alpha, \\
\sqrt{\frac{1}{\alpha}} \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi (\theta-\alpha)}{2\pi-\alpha} \right) & \alpha < \theta < 2\pi,
\end{cases}
\]

where

\[
a_n := \frac{1}{\pi} \int_0^\alpha d\theta \sin(\theta) \sin \left( \frac{n \pi \theta}{\alpha} \right) = (-1)^n \frac{n \pi}{2} \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \sin(\alpha),
\]

\[
A_n := \frac{1}{\pi} \int_\alpha^{2\pi} d\theta \sin(\theta) \sin \left( \frac{n \pi (\theta-\alpha)}{2\pi-\alpha} \right) = -\frac{2\pi \alpha n}{(\alpha - (n + 2)\pi)(\alpha + (n - 2)\pi)} \sin(\alpha).
\]

For ease of notation we replace \(\sin \left( \frac{n \pi \theta}{\alpha} \right)\) by \(|\eta_n\) in the interval \(0 < \theta < \alpha\), and \(\sin \left( \frac{n \pi (\theta-\alpha)}{2\pi-\alpha} \right)\) by \(|\chi_n\) in the interval \(\alpha < \theta < 2\pi\). The transition of \(\phi\) to \(|\eta_n\rangle + |\chi_n\rangle\) is accompanied by the energy transfer to the barrier inserted at the non-nodal point \(\alpha\) of

\[
\Delta E_{nm}^\phi := \frac{\pi^2 \hbar^2}{2M} \left( \frac{|a_n|^2}{|a_n|^2 + |A_M|^2} n^2 \pi^2 \alpha^2 + \frac{|A_M|^2}{|a_n|^2 + |A_M|^2} \frac{m^2}{(2\pi - \alpha)^2} - \frac{1}{4\pi^2} \right).
\]

For an appropriate choice of \(\alpha\), e.g. \(\alpha = \pi/4\), the energy change \(\Delta E_{nm}^\phi\) is always non-zero.

The second candidate wave function can be expanded into

\[
\psi_{after}(\theta) := \begin{cases} 
\sqrt{\frac{1}{\alpha}} \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi \theta}{\alpha} \right) & 0 < \theta < \alpha, \\
\sqrt{\frac{1}{\alpha}} \sum_{n=1}^{\infty} C_n \sin \left( \frac{n \pi (\theta-\alpha)}{2\pi-\alpha} \right) & \alpha < \theta < 2\pi,
\end{cases}
\]

where

\[
c_n := \frac{1}{\pi} \int_0^\alpha d\theta \sin(\theta - \alpha) \sin \left( \frac{n \pi \theta}{\alpha} \right) = \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \sin(\alpha),
\]

\[
C_n := \frac{1}{\pi} \int_\alpha^{2\pi} d\theta \sin(\theta - \alpha) \sin \left( \frac{n \pi (\theta-\alpha)}{2\pi-\alpha} \right) = (-1)^{n+1} \frac{(2\pi - \alpha)n}{(\alpha - (n + 2)\pi)(\alpha + (n - 2)\pi)} \sin(\alpha).
\]

The energy transfer to \(\psi\) from the barrier inserted at the origin has the form

\[
\Delta E_{nm}^\psi := \frac{\pi^2 \hbar^2}{2M} \left( \frac{|c_n|^2}{|c_n|^2 + |C_M|^2} n^2 \pi^2 \alpha^2 + \frac{|C_M|^2}{|c_n|^2 + |C_M|^2} \frac{m^2}{(2\pi - \alpha)^2} - \frac{1}{4\pi^2} \right),
\]

and is identical to \(\Delta E_{nm}^\phi\).

Due to energy conservation, there has to be a source for the energy increase of the particle. A laser beam could be a possible realisation for the barrier. The photons of the laser beam would have an energy dependent entanglement with the expanded wave function on the ring, where each combination of energy levels in the two chambers is linked to a complementary state for the barriers to achieve energy conservation. The energy transfer is to \(\phi\) from the barrier at \(\alpha\) and to \(\psi\) from the barrier at \(0\), since each candidate wave function has its energy only modified through one specific barrier corresponding to the initial non-nodal point.

The extended wave functions including the barrier can be written before the insertion as either

\[
\Phi_{before} = \phi \otimes \omega_0^{before}(0) \otimes \omega_0^{before}(\alpha),
\]

or

\[
\Psi_{before} = \psi \otimes \omega_0^{before}(0) \otimes \omega_0^{before}(\alpha),
\]
where \( \omega_0 \) and \( \omega_\alpha \) correspond to the wave functions associated with the pre-insertion barriers at the points 0 and \( \alpha \) respectively. Directly after the insertion the extended wave functions are transformed into

\[
\Phi_{\text{after}} = \left( \sum_{n=1}^{\infty} a_n |\eta_n\rangle + \sum_{m=1}^{\infty} A_m |\chi_m\rangle \right) \otimes \omega_0^{\text{after}}(0) \otimes \omega_{\Delta E_{nm}}^{\text{after}}(\alpha)
\]

and

\[
\Psi_{\text{after}} = \left( \sum_{n=1}^{\infty} c_n |\eta_n\rangle + \sum_{m=1}^{\infty} C_m |\chi_m\rangle \right) \otimes \omega_{\Delta E_{nm}}^{\text{after}}(0) \otimes \omega_0^{\text{after}}(\alpha),
\]

\( \omega_0^{\text{after}}(0) \) & \( \omega_\alpha^{\text{after}}(\alpha) \) are the barrier wave functions, if inserted at either a nodal point at 0 or \( \alpha \). \( \omega_{\Delta E_{nm}}^{\text{after}}(\alpha) \) and \( \omega_{\Delta E_{nm}}^{\text{after}}(0) \) correspond to barriers that transfer \( \Delta E_{nm} \) to the candidate wave functions \( \phi_{\text{after}} \) and \( \psi_{\text{after}} \) respectively. The double insertion case is further studied in the next section.

**IV. INCONSISTENCY OF LINEARITY AND ENERGY-FREE BARRIER INSERTION AT A NODE AND ITS IMPLICATIONS**

In this section, we point out the impossibility to be able to construct a certain two barrier insertion map to satisfy the following two assumptions:

- LOC: a barrier insertion at a ‘fixed node’ needs no energy, and
- LIN: barrier insertions can be described by linear maps.

This is done by first assuming that there exist a map for which both assumptions are true, and then pointing out a contradiction, i.e. comparing the LOC&LIN calculation for \( \sin(\theta - \alpha) \) with the one for \( \sin(\theta) \cos(\alpha) - \cos(\theta) \sin(\alpha) \) and being unable to match them.

As we noticed before, \( \sin(\theta) \) and \( \sin(\theta - \alpha) \) have permanent nodes at zero and \( \alpha \) respectively. Expansion coefficients on \((0, \alpha)\) and \((\alpha, 2\pi)\) for \( \cos(\theta) \) are

\[
b_n := \frac{1}{\pi} \int_{0}^{\alpha} d\theta \sin \left( \frac{n\pi \theta}{\alpha} \right) \cos(\theta) = \frac{\eta_n}{\alpha^2 - \pi^2 n^2} \left((-1)^n \cos(\alpha) - 1\right),
\]

\[
B_n := \frac{1}{\pi} \int_{0}^{\alpha} d\theta \sin \left( \frac{n\pi \theta}{2\pi - \alpha} \right) \cos(\theta) = \frac{(2\pi - \alpha)n}{(\alpha + \pi(n-2))(\pi(n+2)-\alpha)} \left(\cos(\alpha) - (-1)^n\right).
\]

As before, the extended wave function directly after the insertions is, due to the LOC assumption,

\[
\Psi_{\text{after}} = \left( \sum_{n=1}^{\infty} c_n |\eta_n\rangle + \sum_{m=1}^{\infty} C_m |\chi_m\rangle \right) \otimes \omega_{\Delta E_{nm}}^{\text{after}}(0) \otimes \omega_0^{\text{after}}(\alpha).
\]

Replacing \( \sin(\theta - \alpha) \) by \( \sin(\theta) \cos(\alpha) - \cos(\theta) \sin(\alpha) \), and assuming both LIN as well as LOC, one gets

\[
\Psi_{\text{after}} = \cos(\alpha) \left( \sum_{n=1}^{\infty} a_n |\eta_n\rangle + \sum_{m=1}^{\infty} A_m |\chi_m\rangle \right) \otimes \omega_0^{\text{after}}(0) \otimes \omega_{\Delta E_{nm}}^{\text{after}}(\alpha)
\]

\[
- \sin(\alpha) \left( \sum_{n=1}^{\infty} b_n |\eta_n\rangle + \sum_{m=1}^{\infty} B_m |\chi_m\rangle \right) \otimes \sum_{k,k'} R_{m,n,k,k'} |\Delta E_{nmk}^{\text{after}}(0) \otimes \omega_{\Delta E_{nmk}}^{\text{after}}(\alpha),
\]

where the two barriers \( \omega_{\Delta E_{nmk}}^{\text{after}}(0) \) and \( \omega_{\Delta E_{nmk}}^{\text{after}}(\alpha) \), with insertion energy transfer of \( \Delta E_{nmk}(0) \) and \( \Delta E_{nmk}(\alpha) \) respectively, split the transfer energy \( \Delta E_{nm} \) between themselves, i.e. \( \Delta E_{nm} = \Delta E_{nmk}(0) + \Delta E_{nmk}(\alpha) \). For reasons of convenience, \( \alpha \) will be restricted to values with exclusively non-zero \( \Delta E_{nm} \). For different \( k \) the \( \omega_{\Delta E_{nmk}}^{\text{after}}(0) \) are mutually orthogonal. The same applies to \( \omega_{\Delta E_{nmk}}^{\text{after}}(\alpha) \) for different \( k' \). \( R_{m,n,k,k'} \) are the weights attached to different combinations of \( \omega_{\Delta E_{nmk}}^{\text{after}}(0) \otimes \omega_{\Delta E_{nmk}}^{\text{after}}(\alpha) \langle b_n |\eta_n\rangle + B_m |\chi_m\rangle \), which have to satisfy the formal sum \( \sum_{k,k'} |R_{m,n,k,k'}|^2 = 1 \).

To satisfy for this type of map both LIN & LOC, the total weight of each unique orthogonal product state for both expansions, i.e. equations (1) and (2), has to coincide. Can this be done by choosing appropriate \( R_{m,n,k,k'} \)?

One notices, \( R_{m,n,k,k'} \) has for fixed \( n \) & \( m \) to be zero for all \( k \) & \( k' \), except for two terms corresponding to energy transfers exclusively to either the barrier at 0 or \( \alpha \). First, it is the pair \( \Delta E_{nm,k,k'}(0) = \Delta E_{nm} \) and \( \Delta E_{nm,k,k'}(\alpha) = 0 \), and second it is the pair \( \Delta E_{nmk}(0) = 0 \) and \( \Delta E_{nmk}(\alpha) = \Delta E_{nm} \). The corresponding weights we call \( R_0^{\alpha} \) and \( R_\alpha \).
respectively. All other $R_{m,n,k,k'}$ must be zero, since the relevant $\omega_{\Delta E_{nm,k}}^\alpha(0) \otimes \omega_{\Delta E_{nm,k'}}^\alpha(0)$ terms, where there is a non-zero energy transfer to both barriers, only appear once in the expansion of equation (2), but not at all in equation (1). In addition, if $n \neq n'$ and $m \neq m'$, then $\langle \eta_n | b_n^* + (\chi_m | B_m^* | \eta_{m'}) + B_{m'} | \chi_m' \rangle = 0$. As a consequence,

$$|R_{m,n}^0|^2 + |R_{m,n}^\alpha|^2 = 1.$$

The coefficients can be split into four sets of terms from equation (1) and (2). The first set of terms corresponding to $\omega_{\Delta E_{nm}}^\alpha(0) \otimes \omega_{\Delta E_{nm}}^\alpha(0)$ in the first interval, i.e. $(0, \alpha)$, is

$$c_n = \frac{\sin(\alpha)}{\alpha^2 - \pi^2 n^2} \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \sin(\alpha) = -\sin(\alpha) R_{m,n}^0 \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \left( (-1)^n \cos(\alpha) - 1 \right),$$

which simplifies to

$$1 = R_{m,n}^0 \left( 1 - (-1)^n \cos(\alpha) \right).$$

(3)

For the second set of terms corresponding to $\omega_{\Delta E_{nm}}^\alpha(0) \otimes \omega_{\Delta E_{nm}}^\alpha(0)$ in the second interval, i.e. $(\alpha, 2\pi)$, one gets

$$C_m = (-1)^m \frac{(2\pi - \alpha)^m}{(\alpha + \pi(m - 2))(\pi(m + 2) - \alpha)} \sin^2(\alpha)$$

$$= -\sin(\alpha) R_{m,n}^0 R_{m,n}^\alpha \left( \frac{(2\pi - \alpha)^m}{(\alpha + \pi(m - 2))(\pi(m + 2) - \alpha)} \cos(\alpha) - (-1)^m \left( \frac{(2\pi - \alpha)^m}{(\alpha + \pi(m - 2))(\pi(m + 2) - \alpha)} \cos(\alpha) - (-1)^m \right) \right),$$

to be rewritten as

$$(-1)^m \sin(\alpha) = R_{m,n}^0 \left( (-1)^m \cos(\alpha) \right).$$

(4)

Equation (3) and (4) combine to form

$$\left( (-1)^m \cos(\alpha) \right) = (-1)^m \sin(\alpha) \left( 1 - (-1)^n \cos(\alpha) \right).$$

(5)

The third set of terms corresponding to $\omega_{\Delta E_{nm}}^\alpha(0) \otimes \omega_{\Delta E_{nm}}^\alpha(0)$ in the first interval leads to

$$\cos(\alpha) a_n = \cos(\alpha) (-1)^n \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \sin(\alpha)$$

$$= \sin(\alpha) b_n R_{m,n}^\alpha \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \left( (-1)^n \cos(\alpha) - 1 \right) R_{m,n}^\alpha,$$

and the fourth set of terms corresponding to $\omega_{\Delta E_{nm}}^\alpha(0) \otimes \omega_{\Delta E_{nm}}^\alpha(0)$ in the second interval is

$$\cos(\alpha) A_m = \cos(\alpha) \left( \frac{(2\pi - \alpha)^m}{(\alpha + \pi(m - 2))(\pi(m + 2) - \alpha)} \sin(\alpha) \right)$$

$$= \sin(\alpha) B_m R_{m,n}^\alpha \left( \frac{(2\pi - \alpha)^m}{(\alpha + \pi(m - 2))(\pi(m + 2) - \alpha)} \cos(\alpha) - (-1)^m \right) R_{m,n}^\alpha.$$

This simplifies to

$$(-1)^n \cos(\alpha) = R_{m,n}^\alpha \left( (-1)^n \cos(\alpha) - 1 \right)$$

(6)

and

$$\cos \alpha = R_{m,n}^\alpha \left( \cos(\alpha) - (-1)^m \right).$$

(7)

By combining the equations (6) and (7) one gets

$$(-1)^n \left( \cos(\alpha) - (-1)^m \right) = \left( (-1)^n \cos(\alpha) - 1 \right).$$

(8)
Equations (5) and (8) together produce
\[ \sin(\alpha) = (-1)^n(-1)^m, \]
which cannot be satisfied simultaneously for all \( n \) and \( m \) for any \( \alpha \) under consideration, and therefore there is no choice of \( R_{m,n,k,k'} \) satisfying the requirements.

As a consequence, the two statements LOC & LIN cannot both be unequivocally true for quantum mechanics on the ring with barrier insertions. Therefore, one has to restrict oneself to either LOC or LIN. There is the possibility of both statements not to be universally true, but this will not be explored. Each of the two statements illuminates an aspect of quantum mechanics. This will be described in the upcoming conclusion.

V. CONCLUSION

The aim of the paper was to point out a paradoxical feature of quantum mechanics on a ring. A class of linear maps was constructed such that energy is even required for an insertion at a fixed node, and implying assumptions LOC & LIN cannot both be true. The tension between locality (LOC) and linearity (LIN) is central to quantum mechanics, and it is brought to the fore on a rotational symmetric configuration space, like a ring, and the respective Hamiltonian with a degeneracy. Experimental implementation, for example with Bose-Einstein condensate and a laser beam as a barrier, should be of interest.

Based on the LOC condition, the barrier insertion at a fixed node is always energy-free, and implies, as discussed in an earlier paper [1], the ability to break the Helstrom bound, since pre- and post-insertion transition probabilities of the extended wave functions including the barriers were suitably modified. It is maybe not surprising that forfeiting linearity has such ramifications. The next paragraph, where we discard LOC, but keep linearity, is maybe more startling.

If barrier insertions can be described by linear maps (condition: LIN), then there is a difference in the energy required at the fixed node depending on the total number of simultaneous insertions. To amplify the effect, we can work with a particle on a scaled up ring of radius \( R \), where all the results derived for a ring of radius one still hold. We notice that for appropriately chosen insertion points there is a difference in the energy needed to insert one or two simultaneously placed barriers. One barrier at a fixed node never requires energy (section I), but two simultaneous insertions at widely separated points can even demand energy at a fixed node. This energy can be measured and, if these operations are carried out in parallel on many rings, to reduce the error probability, can be used for communication. The requirement of non-zero energy at a fixed node forces one to conclude that another simultaneous (or earlier) barrier insertion has occurred somewhere along the ring, while zero energy suggests an isolated insertion.

The paper emphasises the highly non-local nature of linear quantum mechanics. The energy needed to insert a barrier at a fixed node depends in a non-local way on the amplitude of the wave function in the whole configuration space and the actions (like other barrier insertions) undertaken at arbitrarily separated places. The Schrödinger equation, a diffusion equation without an upper propagation speed, has limitations.

Naturally, one can criticise the failure to provide a realistic time evolution; only the infinitely fast insertion was examined. In defence, one can point to the idealised nature of the proposal and the statement that more realistic examples can be viewed as an extrapolation of the procedure under consideration. The exact nature of the entanglement between the particle wave function and the barriers is of interest and could be checked experimentally.

The following Gedankenexperiment might be instructive, because it shows that it is possible to construct an example were no information is transferred to the experimenter, when the potential representing the barrier is altered. Imagine an experimenter doing a fixed amount of work per unit of time to insert the barrier, i.e. pushes in the barrier with constant power. Dependent on the test wave function the change of the potential is either larger or smaller. The potential takes on different shapes and affects the states in different ways without direct, energy based, leakage of information to the outside.

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