Buckling Analysis of Symmetrically Laminated Rectangular Thin Plates under Biaxial Compression

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ABSTRACT
In this parametric study, the buckling analysis of symmetrically laminated rectangular thin plates assumed to be subjected to biaxial compression was presented. The simply supported boundary condition was considered at the edges of the symmetrically laminated quasi-isotropic, cross-ply and angle-ply plates. The Rayleigh-Ritz Method was used to specify the critical buckling load of the plates based on the Classical Laminated Plate Theory (CLPT). A convergence study was achieved by increasing the number of parameters of assumed shape function. Validation of isotropic case was verified. The effects of the lamination types, plate aspect ratios (a/b, b/a) and thickness on the critical buckling load of the laminated plates under bi-axial compression were then investigated. The results were compared with Finite Element Method (FEM) solutions performed by ANSYS software package and fairly good agreement was obtained. Non-dimensional results were tabulated and presented for practical use for designers.

Keywords: Bi-axial buckling, symmetrically laminated thin plate, Rayleigh-Ritz Method, Finite Element Method, parametric study.

1. INTRODUCTION
Laminated composite thin plates have been extensively used in a diverse field of application in engineering structures such as civil, wind, aerospace, automotive and ship hull and superstructures etc., due to their excellent high strength-to-weight ratio and modulus-to-weight ratio. Being a structural element, buckling is a significant problem for these plates. Buckling of composite plates, which is often encountered in such structures, commonly occurs at a low applied stress levels and generates large deformations. Therefore, buckling

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of composite plates is a critical problem and focusing on the buckling phenomenon is of importance.

Early studies about uniaxial and biaxial buckling analysis of laminated rectangular composite plates were carried out by several researchers [1-19] before this century. Many researchers have investigated mechanical buckling of composite rectangular plates for the last two decades. Veres and Kollar [20] carried out buckling of orthotropic plates subjected to biaxial load based on the Ritz method. Biaxial buckling behavior of anisotropic rectangular plates under simply supported, clamped and mixed boundary conditions was investigated analytically, experimentally and numerically by Romeo and Ferrero [21] and results demonstrated good correlation. Narita and Turvey [22] studied determining of the optimum lay-ups and maximum buckling loads of symmetrically laminated rectangular plates by new layerwise optimization (LO) iterative procedure. Ni et al. [23] presented buckling behavior for rectangular laminated composite plates subjected to biaxial loading by higher-order shear deformation theory and the pb-2 Ritz method for arbitrary boundary conditions. Shukla et al. [24] performed critical buckling analysis of cross-ply and angle-ply plates under uniaxial and biaxial loading based on the first-order shear deformation theory and von-Karman-type nonlinearity for different boundary conditions. Buckling of cross-ply square plates under uniaxial and biaxial loading on the basis of a unified five-degree-of-freedom shear deformable plate theory was presented by Timarci and Aydogdu [25].

Bert and Malik [26] analyzed buckling of cross-ply plates subject to uniaxial and biaxial compression using classical laminated plate theory, third order shear deformable theory and differential quadrature method for simply supported boundary condition. Qiao and Shan [27] studied buckling analysis of rotationally restrained plates subjected to biaxial load using the Ritz method. Aktas [28] studied buckling of carbon/epoxy laminated composite plates under biaxial loading using the Veres-Kollar approach [20] and Finite Element Method software ANSYS. Good agreements were obtained between analytical and numerical results. Latalski [29] dealt with the plies thicknesses on optimal design of multi-layered laminated plates under uniaxial and biaxial compression. Sayad and Ghugal [30] developed a trigonometric shear and normal deformation theory for buckling of isotropic, transversely isotropic, orthotropic composite rectangular plates subject to in-plane compressive forces. Bourada et al. [31] analyzed buckling of isotropic and orthotropic plates subject to uniaxial and biaxial compression by proposing a new four variable refined plate theory. Becheri et al. [32] presented exact analytical solution of buckling analysis of symmetrically cross-ply laminated plates subject to biaxial in-plane loads. Rajanna et al. [33] examined the effect of tension and compression buckling of cross-ply and angle-ply plates with circular and square cutouts subject to biaxial in-plane varying edge loads by Finite Element Method. Belkacem et al. [34] studied buckling behavior of hybrid (carbon/glass) laminated cross-ply plates under different boundary conditions, taking account the shear effect. Topal et al. [35] focused on the maximization of the critical buckling load of angle-ply plates resting on elastic foundation subjected to compressive loads using teaching learning based optimization method (TLBO) based on the governing equations of the first order shear deformation theory. Bourada et al. [36] have investigated buckling behavior of rectangular isotropic plates under uniaxial and biaxial compression by analyzing by the first order shear deformation theory. Fellah et al. [37] have presented a novel refined shear deformation theory for the buckling analysis of thick isotropic plates. Altekin [38, 39] have investigated bending, free vibration and buckling of super-elliptical plates.
In view of the literature, the majority of the articles are concerned with critical buckling loads of mainly orthotropic rectangular plates (such as cross-ply laminates) with different theories and methods. Recently, Altunsaray and Bayer [40] investigated buckling analysis of symmetrically laminated quasi-isotropic thin rectangular plates subject to uniaxial compressive loading by Galerkin Method and Finite Difference Method based on Classical Laminated Plate Theory. The authors also used Finite Element Method software package ANSYS to compare the results. The importance of using the symmetrically laminated quasi-isotropic plates which are constructed with -45°, +45°, 0° and 90° orientations used in engineering applications was indicated in the study of Altunsaray and Bayer [40]. An advantage of the symmetric laminate is that the bending-extension coupling matrix (Bij) is zero. Thus, symmetrically laminated plates are preferred in production because such plates remain flat after curing due to thermal strains encountered during the curing process. To the best knowledge of the authors, no comparative parametric study has been done on the biaxial buckling analysis of symmetrically laminated quasi-isotropic, cross-ply and angle-ply thin plates by using Rayleigh-Ritz method and FEM in the literature. The motivation of this paper is to study the buckling analysis of symmetrically laminated quasi-isotropic, cross-ply and angle-ply thin rectangular plates under biaxial compressive load and to estimate the influence of lamination types, aspect ratio and plate thickness on these types of plates. The plates are analyzed when subject to simply supported boundary condition at the edge. Rayleigh Ritz method is used for the solution of integral equations based on the Classical Laminated Plate Theory. Finite Element Method software package ANSYS is used to compare the results.

2. ANALYSIS

2.1. Geometry of plates, material properties and lamination types

Positive rotation of principle material with local and global axes is given by Figure 1.

![Figure 1 - Positive Rotation of Principal Material Axes from 1'-2' Axes (1-2 local axes, 1'-2' global axes)](image-url)
Material properties of carbon/epoxy composite, selected aspect ratios, lamination types and bending stiffness matrix are given in Table 1, 2, 3 and 4 respectively. All laminated plates are symmetric, Quasi-isotropic plates have four different sequences (−45°, 0°, 45° and 90°), Cross-ply laminated plates consist of two different sequences (0° and 90°) and Angle-ply laminates have two different sequences (−45° and 45°). Thickness of each lamina (t) is equal to 0.2 mm thus the total thickness of a laminated plate is equal to 3.2 mm.

**Table 1 - Material properties of carbon/epoxy (T300-934) [41]**

| Property                        | Value                      |
|---------------------------------|----------------------------|
| Longitudinal Young Modulus (E_{11}) | 148x10^9 (N/m^2)          |
| Transversal Young Modulus (E_{22}) | 9.65x10^9 (N/m^2)         |
| Longitudinal Shear Modulus (G_{12}) | 4.55x10^9 (N/m^2)         |
| Longitudinal Poisson ratio (ν_{12}) | 0.3                       |
| Lamina thickness (t)            | 0.185x10^{-3} – 0.213x10^{-3} (m) |

**Table 2 - Aspect ratios**

| a/b | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2  |
|-----|---|-----|-----|-----|-----|----|
| b/a | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2  |

**Table 3 - Symmetrically laminated composite plate types**

| LT1  | [-45°/0°/45°/90°]_s | LT15 | [45°/0°/45°/90°]_s |
|----------------|---------------------|-----|-------------------|
| LT2  | [-45°/0°/90°/45°]_s | LT16 | [45°/0°/90°/-45°]_s |
| LT3  | [-45°/45°/0°/90°]_s | LT17 | [45°/90°/45°/0°]_s |
| LT4  | [-45°/45°/90°/0°]_s | LT18 | [45°/90°/-45°/0°]_s |
| LT5  | [-45°/90°/0°/45°]_s | LT19 | [90°/-45°/0°/45°]_s |
| LT6  | [-45°/90°/45°/0°]_s | LT20 | [90°/-45°/45°/0°]_s |
| LT7  | [0°/-45°/45°/90°]_s | LT21 | [90°/-45°/45°/90°]_s |
| LT8  | [0°/-45°/90°/45°]_s | LT22 | [90°/-45°/45°/45°]_s |
| LT9  | [0°/45°/90°/0°]_s  | LT23 | [90°/45°/-45°/0°]_s |
| LT10 | [0°/45°/90°/-45°]_s | LT24 | [90°/45°/0°/-45°]_s |
| LT11 | [0°/90°/-45°/45°]_s | LT25 | [0°/90°/0°/90°]_s |
| LT12 | [0°/90°/45°/-45°]_s | LT26 | [90°/90°/0°/90°]_s |
| LT13 | [45°/-45°/0°/90°]_s | LT27 | [-45°/45°/-45°/45°]_s |
| LT14 | [45°/-45°/90°/0°]_s | LT28 | [45°/-45°/45°/-45°]_s |
When the laminate is symmetrical with respect to the midplane, it is referred to be a symmetrical laminate. Notation of the layup in LT1 \([-45/0/45/90]\) plate is given by Figure 2.

![Figure 2 - Notation of the layup in LT1 [-45/0/45/90] plate](image)

**Table 4 - Bending stiffness matrix of isotropic and symmetrically laminated plate types**

| Plate Types                  | Bending Stiffness Matrix | Explanations                        |
|------------------------------|--------------------------|-------------------------------------|
| Isotropic (single isotropic layer) | $D \begin{bmatrix} D & vD & 0 & 0 \\ vD & D & 0 & 0 \\ 0 & 0 & (1 - v)D & 0 \\ 0 & 0 & 0 & 2D \end{bmatrix}$ | ($D_{11} = D_{22} = D$) |
| Symmetrical Orthotropic (Cross-Ply) Example: LT25 = $[0/90/0/90]$, LT26 = $[90/0/90/0]$ | $D_{11} \begin{bmatrix} D_{11} & D_{12} & 0 & 0 \\ D_{12} & D_{22} & 0 & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$ | ($D_{16} = D_{26} = 0$) |
| Symmetrical Angle-ply Example: LT27 = $[-45/45/45/45]$, LT28 = $[45/-45/45/-45]$ | $D_{11} \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$ | ($D_{11} = D_{22}, D_{16} = D_{26}$) |
| Symmetrical Quasi-isotropic Example: LT1 = $[-45/0/45/90]$ | $D_{11} \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$ | ($D_{16} = D_{26}$) |
Bending stiffness matrix of isotropic and symmetrically laminated plate types were given below (Table 4.) It can be seen from the Table 4, bend-twist coupling terms are equal zero ($D_{16}=D_{26}=0$) for Isotropic and Cross-Ply plates, while these terms are different from zero for Angle-ply and Quasi-isotropic plates. Bending stiffness matrix elements ($D_{11}=D_{22}$) of Angle-ply plates are similar to isotropic plates. Explanations are given in Section 3.1 in detail.

2.2. Approximate solution methods in stability analysis of plates

Exact analytical solutions for certain geometries and boundary conditions are possible with methods such as Navier or Levy. Approximate solution methods such as Galerkin Method, which is one of the weighted residual methods, Rayleigh-Ritz Method which is one of the variational methods and Finite Element Method is one of the powerful numerical solution techniques can be used for different situations.

The mathematical model in the differential equation form can be solved by the Galerkin method, while the model in the form of integral equation can be solved by the Rayleigh-Ritz method. When the same trial function is used, the results obtained by Rayleigh-Ritz and Galerkin Method are identical.

The Rayleigh-Ritz method is based on the principle of minimum potential energy. An approximate trial function that satisfies the geometric boundary conditions of the system is selected and placed in the total potential energy equation. Then, the total potential energy is minimized with respect to the unknown coefficients of the approximate trial function, which gives a linear homogeneous equation system. The determinant of the coefficient matrix should be equal to zero, for a non-trivial solution, which leads to a characteristic equation involving a polynomial. Finally, the lowest critical buckling load may be found by the smallest root of this equation.

Galerkin method is another form of the Ritz Method. For the Galerkin method, the governing differential equation for the problem is needed. First, an approximate deflection function including unknown coefficients and shape functions is chosen. When the selected approximate deflection function is placed in to the governing differential equation, there will be a remaining part different from zero which is called ‘residual’. The Galerkin method minimizes the sum of the product of this residual by the shape functions over the entire region of the problem. The rest of the problem will be similar to R-R method mentioned above [42].

In the Finite Element Method, the system is divided into a finite number of elements (meshing). Each of the elements that make up the system is called a finite element and the corner points where they join are called nodal points. The deformation of the finite element surface is expressed depending on the displacement parameters (displacement components, displacement vectors such as displacement components, rotations and torsional curves). Thousands of nodes are often needed to achieve a reasonably accurate solution, so using a computer is inevitable. In general, the accuracy of the solution increases as the number of elements (and nodes) increases at the expense of calculation time [43,44].

In this parametric study, Rayleigh-Ritz Method, an energy method which is one of the approximate solution methods, and ANSYS [45] software based on Finite Element Method developed since 1969 were used.
2.2.1. Isotropic Plate Case and Applying of the Rayleigh-Ritz Method

According to energy approach the strain energy of isotropic plate is given below [46]

\[
U = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} D \left( \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2 \left( 1 - \nu \right) \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \, dx \, dy
\]  

(1)

Potential energy of the plate due to \( N_x \) and \( N_y \)

\[
V = -\frac{1}{2} \int_{0}^{a} \int_{0}^{b} \left( N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 \right) \, dx \, dy
\]  

(2)

\( N_x = \gamma N_y \)  

(3)

where \( \gamma = 0 \) for uniaxial loading and \( \gamma = 1 \) for bi-axial compressive loading (\( N_x = N_y \)). For this study \( \gamma = 1 \) is assumed and hence:

\[
V = -\frac{1}{2} \int_{0}^{a} \int_{0}^{b} N \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) \, dx \, dy
\]  

(4)

The potential energy functional is given below

\[
F = U + V
\]  

(5)

Substituting Eq. 1 and Eq. 4 into Eq. 5, the total potential energy is

\[
F = \frac{1}{2} \int_{0}^{a} \int_{0}^{b} D \left( \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2 \left( 1 - \nu \right) \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \, dx \, dy
\]  

\[
-\frac{1}{2} \int_{0}^{a} \int_{0}^{b} N \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) \, dx \, dy
\]  

(6)

Boundary conditions at edges of the plate;

(i) Simply supported; as the edges are free to rotate, the moment \( M_x \) or \( M_y \) must be zero,

\[
w = M_x = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a
\]  

(7)

\[
w = M_y = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b
\]  

(8)

(ii) Clamped edges; as the edges cannot rotate, the first derivative of \( w \) with respect to \( x \) and \( y \) must be zero,

\[
w = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a
\]  

(9)
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\[ w = \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b \quad (10) \]

Deflection function which satisfies the boundary conditions is given below;

\[ \phi_{mn} = X_m. Y_n = \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \quad \text{(for all edges simply supported)} \quad (11) \]

\[ \phi_{mn} = X_m. Y_n = x^{2m} (a - x)^{2m} y^{2n} (b - y)^{2n} \quad \text{(for all edges clamped)} \quad (12) \]

\[ w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \phi_{mn} \quad (13) \]

In order to find the lowest set of critical buckling loads, equation (6) is minimized with respect to the coefficients \( C_{mn} \)

\[ \frac{\partial F}{\partial C_{mn}} = 0 \quad (14) \]

Then, the following equation is obtained:

\[ [K - \lambda_b M_b] [C_{mn}] = 0 \quad (15) \]

where \( \lambda_b \) is the buckling load parameter including material properties, characteristic dimensions and in-plane uniform load of the plate. \( K \) is the stiffness matrix related with the strain energy and \( M_b \) is the mass matrix related to potential energy. This is a generalized eigenvalue problem. For a non-trivial solution, the determinant of the coefficient matrix should be equal to zero:

\[ |K - \lambda_b M_b| = 0 \quad (16) \]

Solution of equation (16) leads to a characteristic equation involving a polynomial, whose degree depends on the number of the terms of the deflection function, in \( \lambda_b \), from which the lowest critical buckling loads \( (N_{cr}) \) may be found.

\[ 2.2.2. \text{Symmetrically Laminated Composite Plate Cases and Applying of the Rayleigh-Ritz Method} \]

In this study buckling of symmetrically laminated Cross-Ply, Angle-Ply and Quasi-Isotropic thin plates were investigated based on the Classical Laminated Plate Theory (CLPT).

The strain energy \( (U) \) of the symmetrically laminated plate is given by the following [47]:

\[ U = \frac{1}{2} \int_0^a \int_0^b \left[ D_{11} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12} \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) + D_{22} \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{16} \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{26} \left( \frac{\partial^2 w}{\partial y^2} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) + 4D_{66} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \ dx \ dy \quad (17) \]
Where $D_{11}$, $D_{12}$, $D_{22}$, $D_{16}$, $D_{26}$ and $D_{66}$ indicate the elements of bending stiffness matrix $D_{ij}$ which are found by the following [47]:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} Q_{ij}^{k} (z_{k+1}^{2} - z_{k}^{2})$$

(18)

Where, $Q_{ij}$, $n$, $z_{k}$ and $z_{k-1}$ indicate the transformed reduced stiffness matrix, total number of plies and distance from the reference plane respectively [47]. The components of transformed reduced stiffness matrix, $Q_{ij}$, calculated for each lamina is:

$$Q_{11} = Q_{11} c^{4} + 2(Q_{12} + 2Q_{66}) s^{2} c^{2} + Q_{22} s^{4}$$

$$Q_{12} = (Q_{11} + Q_{22} - 4Q_{66}) s^{2} c^{2} + Q_{12}(s^{4} + c^{4})$$

$$Q_{22} = Q_{11} s^{4} + 2(Q_{12} + 2Q_{66}) s^{2} c^{2} + Q_{22} c^{4}$$

(19)

$$Q_{16} = (Q_{11} - Q_{12} - 2Q_{66}) s c^{3} + (Q_{12} - Q_{22} + 2Q_{66}) c s^{3}$$

$$Q_{26} = (Q_{11} - Q_{12} - 2Q_{66}) s^{3} c + (Q_{12} - Q_{22} + 2Q_{66}) s^{3} c$$

$$Q_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) s^{2} c^{2} + Q_{66}(s^{4} + c^{4})$$

Where $c=\cos(\theta)$ and $s=\sin(\theta)$ respectively. The reduced stiffness matrix elements, $Q_{ij}$, are given below:

$$Q_{11} = \frac{E_{11}}{1-v_{12}v_{21}}$$

$$Q_{12} = \frac{v_{12}E_{11}}{1-v_{12}v_{21}}$$

$$Q_{22} = \frac{E_{22}}{1-v_{12}v_{21}}$$

$$Q_{66} = G_{12}$$

(20)

For symmetrically laminated composite plate cases, only the simply supported boundary condition at the all four edges of plates was considered. Then the lowest critical buckling loads ($N_{cr}$) can be found as applying the same procedure as the isotropic case in Section 2.2.1.

2.2.3. Finite Element Method (FEM) software package ANSYS

In this study, in order to compare the results obtained by Rayleigh-Ritz Method, Finite Element Method software ANSYS was used and the numerical results were given in Table 11, 12 and 13. It can be seen that the results of the two methods are correlated. Then, the non-dimensionial results calculated by Rayleigh-Ritz Method and are presented in Table 14 and 15 to give practical data for designers.

A four nodal point shell element (SHELL 181) with six degrees of freedom at each node (see Figure 3) was used in finite element software package ANSYS [45]. SHELL181 element,
which is capable of modeling up to 250 plies, was selected for layered applications. The accuracy in modeling composite shells is governed by the first-order shear-deformation theory [45].

For meshing geometry, the rectangular element size was taken as 0.01 m. x 0.01 m. (Small edge of plate / Length of SHELL181 finite element = 20). There are 400 elements in square plates (a/b = b/a = 1) and 800 elements in rectangular plates (a/b = b/a = 2). Convergence study with the number of finite elements is given in Table 5. The ratio of “Small edge of plate / Length of SHELL181 finite element” was taken to be 20 in order to obtain good convergence and high accuracy with low computational time.

Table 5 - Convergence study with increasing number of finite elements for LT1 [-45\degree/0\degree/45\degree/90\degree], plate

| a/b | Small edge of plate / Length of SHELL181 finite element | 2 | 4 | 8 | 10 | 20 | 40 | 50 |
|-----|------------------------------------------------------|---|---|---|----|----|----|----|
|     | Ncr (N/m)                                            | Ncr (N/m) | Ncr (N/m) | Ncr (N/m) | Ncr (N/m) | Ncr (N/m) | Ncr (N/m) | Ncr (N/m) |
| 1   |                                                      | 150604    | 93301      | 84726      | 83781      | 82537      | 82224      | 82186      |
| 2   |                                                      | 58495     | 42770      | 40016      | 39706      | 39297      | 39194      | 39181      |

3. RESULTS

3.1. Isotropic plate case

A convergence study was done for the all edges simply supported case of isotropic plates. The results obtained by R-R method were compared with the results given in [48]. It can be seen from Table 6 that a convergence is observed after the 2nd terms (Table 5).

The critical buckling load equation obtained for the bi-axial buckling condition is given below [48]:

\[
N_{cr} = ... \]
The critical buckling load for isotropic plates may be found by equation (22). It can be noticed that when the aspect ratios \( a/b = b/a \), the results will be the same as each other. However, the situation is different in symmetrically laminated composite plates, which can be seen from the critical buckling load equation presented in Table 10.

\[
N_{cr} = \frac{\pi^2 D}{b^2} \left[ 1 + \left( \frac{b}{a} \right)^2 \right]
\] (21)

In this equation, while the coefficient of \( a^4 \) is \( D_{22} \) that of \( b^4 \) is \( D_{11} \). Elements of bending stiffness matrix \( D_{11} \) and \( D_{22} \) are not equal for Cross-Ply laminated plates (LT25 and LT26) and Quasi-isotropic laminated plates (LT1-LT24) except Angle-Ply laminated plates (LT27 and LT28) in Table 8. Hence, if only \( a=b \), the \( N_{cr} \) (Equation 22) gives the same result, and the results for the different edge ratios of \( a \) and \( b \) are different for symmetrically laminated composite plate cases (Cross-Ply and Quasi-isotropic plates).

A comparison for the clamped case was not achieved for isotropic plates, because no results were found for this particular case in the literature, no results were obtained by ANSYS software, either. However, a convergence is observed in this present study for the all edges clamped case, which can be observed after the 3rd term (Table 7).

For the deflection function, a trigonometric trial function for the simply supported condition was selected, while an algebraic polynomial trial function given in Section 2.2.1 was selected for the clamped support condition.

\[
\phi_{mn} = X_m \cdot Y_n = \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)
\] (11)

\[
\phi_{mn} = X_m \cdot Y_n = x^{2m}(a-x)^{2m}y^{2n}(b-y)^{2n}
\] (12)

| Table 6 - Convergence study of isotropic plates with all edges are simply supported |
|-----------------------------------------------|------------------|------------------|------------------|------------------|
| Critical buckling load \( N_{cr} \)           | Ventsel and Krauthammer, 2001 [48] | Present (Rayleigh-Ritz) |
| \( a/b \) | | 1 term | 2 terms | 3 terms | 4 terms |
| 1 | 19,739 D | 20,800 D | 19,739 D | 19,739 D | 19,739 D |
| 1,2 | 16,723 D | 17,930 D | 16,723 D | 16,723 D | 16,723 D |
| 1,4 | 14,905 D | 16,595 D | 14,905 D | 14,905 D | 14,905 D |
| 1,6 | 13,724 D | 15,957 D | 13,724 D | 13,724 D | 13,724 D |
| 1,8 | 12,915 D | 15,655 D | 12,915 D | 12,915 D | 12,915 D |
| 2 | 12,337 D | 15,520 D | 12,337 D | 12,337 D | 12,337 D |
For the simply supported case of isotropic plates, even though m and n values increase, the results remain the same after 2nd term. It is thought that it may be as a result of the trigonometric shape function which is widely used in the literature. The same situation is observed in the convergence analysis results given in Table 9 for symmetrically laminated composite plates.

Table 7 - Convergence study of isotropic plates with all edges are clamped

| a/b | Present (Rayleigh-Ritz) | 1 term | 2 terms | 3 terms | 4 terms |
|-----|-------------------------|--------|---------|---------|---------|
| 1   | 54,0000 D               | 53,2226 D | 52,5145 D | 52,5145 D |
| 1,2 | 46,5765 D               | 46,0815 D | 45,2341 D | 45,2341 D |
| 1,4 | 43,1583 D               | 42,7891 D | 41,7908 D | 41,7908 D |
| 1,6 | 41,5523 D               | 41,2406 D | 40,0979 D | 40,0979 D |
| 1,8 | 40,8120 D               | 40,5229 D | 39,2512 D | 39,2512 D |
| 2   | 40,5000 D               | 40,2112 D | 38,8304 D | 38,8304 D |

3.2. Symmetrically laminated composite plates cases

3.2.1. Elements of bending stiffness matrix of lamination types

Elements of bending stiffness matrix of 28 different lamination types calculated by CLPT given are in Table 8.

Table 8 – Elements of bending stiffness matrix of 28 different lamination types

| LT1 | D_{11} (N.m) | D_{12} (N.m) | D_{16} (N.m) | D_{22} (N.m) | D_{26} (N.m) | D_{66} (N.m) |
|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| LT2 | 206.80       | 71.10        | -44.53       | 99.92        | -44.53       | 75.58        |
| LT3 | 197.60       | 62.49        | -53.44       | 126.35       | -53.44       | 66.96        |
| LT4 | 153.95       | 88.33        | -26.72       | 118.33       | -26.72       | 92.80        |
| LT5 | 118.33       | 88.33        | -26.72       | 153.95       | -26.72       | 92.80        |
| LT6 | 126.35       | 62.49        | -53.44       | 197.60       | -53.44       | 66.96        |
| LT7 | 99.92        | 71.10        | -44.53       | 206.80       | -44.53       | 75.58        |
| LT8 | 286.08       | 45.27        | -17.81       | 72.32        | -17.81       | 49.74        |
| LT9 | 276.88       | 36.66        | -26.72       | 98.74        | -26.72       | 41.13        |

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Table 8 – Elements of bending stiffness matrix of 28 different lamination types (continue)

|     | D_{11} (N.m) | D_{12} (N.m) | D_{16} (N.m) | D_{22} (N.m) | D_{26} (N.m) | D_{66} (N.m) |
|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| LT10| 276.88       | 36.66        | 26.72        | 98.74        | 26.72        | 41.13        |
| LT11| 258.47       | 19.43        | -8.91        | 151.59       | -8.91        | 23.91        |
| LT12| 258.47       | 19.43        | 8.91         | 151.59       | 8.91         | 23.91        |
| LT13| 153.95       | 88.33        | 26.72        | 118.33       | 26.72        | 92.80        |
| LT14| 118.33       | 88.33        | 26.72        | 153.95       | 26.72        | 92.80        |
| LT15| 206.80       | 71.10        | 44.53        | 99.92        | 44.53        | 75.58        |
| LT16| 197.60       | 62.49        | 53.44        | 126.35       | 53.44        | 66.96        |
| LT17| 99.92        | 71.10        | 44.53        | 206.80       | 44.53        | 75.58        |
| LT18| 126.35       | 62.49        | 53.44        | 197.60       | 53.44        | 66.96        |
| LT19| 98.74        | 36.66        | -26.72       | 276.88       | -26.72       | 41.13        |
| LT20| 72.32        | 45.27        | -17.81       | 286.08       | -17.81       | 49.74        |
| LT21| 151.59       | 19.43        | -8.91        | 258.47       | -8.91        | 23.91        |
| LT22| 151.59       | 19.43        | 8.91         | 258.47       | 8.91         | 23.91        |
| LT23| 72.32        | 45.27        | 17.81        | 286.08       | 17.81        | 49.74        |
| LT24| 98.74        | 36.66        | 26.72        | 276.88       | 26.72        | 41.13        |
| LT25| 287.7687     | 7.9519       | 0            | 145.2620     | 0            | 12.4245      |
| LT26| 145.2620     | 7.9519       | 0            | 287.7687     | 0            | 12.4245      |
| LT27| LT28         | LT29         | LT30         | LT31         | LT32         | LT33         |

3.2.2. Convergence study for composite plates

For the study of the convergence of results, critical buckling load of LT1 ([−45/0°/45/90°]s) plate with simply supported boundary condition is investigated. The shape functions with increasing terms were employed in order to reach convergence and the results are given in Table 9. It can be noticed from Table 9 that the convergence achieved is sufficient, if a shape function with 4 terms is selected. Four-term solutions have more economical computational time than those of six or nine terms. Additionally, another important reason for why calculation with 4 terms is preferred, as shown in Table 10, is that bending-twisting coupling terms D_{16} and D_{26} are not included in the calculation with 3 terms, while they are included in 4-term calculation. Thus, this shape function with four terms will be used for all calculations for the rest of the study. The effect of bending-twisting coupling terms D_{16} and D_{26} for critical
buckling loads of plates demonstrated in Table 11. From the results it seems that bending-twisting coupling-terms decrease the critical buckling load.

Table 9 - Convergence study of LT1 plate for aspect ratio = a/b = 1 (Xm = sin(m \cdot \pi \cdot x/a), Yn = sin(n \cdot \pi \cdot y/b))

| m/n | Critical buckling load Ncr (N/m) |
|-----|---------------------------------|
|     | 1 term | 2 terms | 3 terms |
| 1   | X1.Y1  | X1.Y1 + X1.Y2 | X1.Y1 + X1.Y2 + X1.Y3 |
|     | 92680.5 | 92680.5 | 92680.5 |
| 2   | X1.Y1 + X2.Y1 | X1.Y1 + X2.Y1 + X2.Y1 + X2.Y2 | X1.Y1 + X2.Y1 + X2.Y1 + X2.Y1 + X2.Y2 |
|     | 92680.5 | 87154.6 | 86848.6 |
| 3   | X1.Y1 + X2.Y1 + X3.Y1 | X1.Y1 + X2.Y1 + X2.Y1 + X3.Y1 + X2.Y1 + X3.Y1 |
|     | 92680.5 | 87003.5 | 86286.9 |

Table 10 - Comparison of three and four terms solution of critical buckling load of LT1 Plate

| Terms | Computations of critical buckling load Ncr (N/m) by Mathematica | a/b = 1 |
|-------|---------------------------------------------------------------|--------|
| 3     | \( \frac{b^4 \cdot 11 \cdot \pi^2 + 2 \cdot a^4 \cdot b^4 \cdot 12 \cdot \pi^2 + 4 \cdot a^4 \cdot b^4 \cdot 66 \cdot \pi^2}{a^2 \cdot b^2 \cdot (a^2 + b^2)} \) | 92680.5 |
| 4     | \( \left\{ \frac{1}{2} \left( \frac{b^4 \cdot 111 \cdot \pi^2 + 405 \cdot a^4 \cdot b^4 \cdot 111 \cdot \pi^2 + 810 \cdot a^8 \cdot b^8 \cdot 111 \cdot \pi^2 + 810 \cdot a^8 \cdot b^8 \cdot 111 \cdot \pi^2 + 405 \cdot a^4 \cdot b^4 \cdot 111 \cdot \pi^2 + 1620 \cdot a^8 \cdot b^8 \cdot \pi^2 + 1620 \cdot a^8 \cdot b^8 \cdot \pi^2 \right) \right\} \) | 87154.6 |
Table 11 - Effect of bending-twisting coupling terms ($D_{16}$, $D_{26}$) for critical buckling load $N_{cr}$

| 4 terms solution (a/b=1) | Critical buckling load $N_{cr}$ (N/m) |  |  |
|--------------------------|---------------------------------------|---|---|
|                          | with $D_{16}$ and $D_{26}$ terms       |   |   |
|                          | neglecting $D_{16}$ and $D_{26}$      |   |   |
|                          | terms ($D_{16}=D_{26}=0$)             |   |   |
| LT25 LT26                | 61516                                 | 61516 |   |
| LT11 LT12 LT21 LT22     | 66872                                 | 67183 |   |
| LT8 LT10 LT19 LT24      | 73224                                 | 75682 |   |
| LT7 LT19 LT20 LT23      | 78891                                 | 79932 |   |
| LT2 LT5 LT16 LT18       | 80182                                 | 88431 |   |
| LT1 LT6 LT15 LT17       | 87155                                 | 92681 |   |
| LT3 LT4 LT13 LT14       | 99333                                 | 101180|   |
| LT27 LT28               | 103747                                | 106846|   |

3.2.3. Effect of thickness

Critical buckling loads of symmetrically laminated rectangular plates for three different thicknesses (3.2, 4.8 and 6.4 mm) and six aspect ratios (a/b and b/a) were investigated and the results are presented in Table 12. It can be seen from the results that the critical buckling loads increase with the increase of the plate thickness. It can be noticed from Table 12 that the critical buckling loads decrease with the increase of the aspect ratio. From the tabulated results, differences between the results of Rayleigh-Ritz and FEM (ANSYS) grow with the increases of the thickness. In this study, because thin plates ($t=3.2$ mm) are studied, Classical Laminated Plate Theory (CLPT) is suitable. For thicker plates, shear deformable plate theories should be considered.

Table 12 - Critical buckling load $N_{cr}$ (N/m) of different thinner or thicker plates

| a/ b | Critical buckling load (N/m) |
|------|-------------------------------|
|      | [-45°/0°/45°/90°]s | [-45°/0°/45°/90°]s | [-45°/0°/45°/90°]s |
| t=3.2 mm | Rayleigh-Ritz | FEM(ANSYS S) | Rayleigh-Ritz | FEM(ANSYS S) | Rayleigh-Ritz | FEM(ANSYS S) |
| t=4.8 mm | Rayleigh-Ritz | FEM(ANSYS S) | Rayleigh-Ritz | FEM(ANSYS S) | Rayleigh-Ritz | FEM(ANSYS S) |
| t=6.4 mm | Rayleigh-Ritz | FEM(ANSYS S) | Rayleigh-Ritz | FEM(ANSYS S) | Rayleigh-Ritz | FEM(ANSYS S) |
| 1.0 | 87155 | 82537 | 291181 | 275262 | 698266 | 641889 |
| 1.2 | 69227 | 65368 | 229920 | 218516 | 552296 | 511192 |
| 1.4 | 57983 | 54732 | 191931 | 183233 | 461300 | 429522 |
| 1.6 | 50452 | 47695 | 166710 | 159836 | 400630 | 375192 |
| 1.8 | 45166 | 42812 | 149117 | 143575 | 358169 | 337350 |
| 2.0 | 41321 | 39297 | 136371 | 131855 | 327330 | 310035 |
Table 12 - Critical buckling load $N_{cr}$ (N/m) of different thinner or thicker plates (continue)

| $b/a$ | Method                  | Plate Types | $N_{cr}$ (N/m) | $N_{cr}$ (N/m) | $N_{cr}$ (N/m) | $N_{cr}$ (N/m) |
|-------|-------------------------|-------------|----------------|----------------|----------------|----------------|
|       |                         | LT1         | LT15           | LT2            | LT16           | LT3            | LT13           | LT4            | LT14           | LT5            | LT18           |
| 1     | Rayleigh-Ritz           | 87155       | 80182          | 99333          | 99333          | 80182          |                 |                 |                 |                 |                 |
|       | FEM (ANSYS)             | 82537       | 73442          | 97414          | 97414          | 73442          |                 |                 |                 |                 |                 |
| 1.2   | Rayleigh-Ritz           | 64265       | 62314          | 220689         | 208453         | 523866         |                 |                 |                 |                 |                 |
|       | FEM (ANSYS)             | 61971       | 60332          | 213835         | 201882         | 506630         |                 |                 |                 |                 |                 |

3.2.4. Effect of lamination types and aspect ratios

Symmetrically laminated composite rectangular thin plates (Quasi-isotropic plates, Cross-Ply plates and Angle-Ply plates) consisted of 28 different types shown in Table 3 are used for the calculations of critical buckling loads $N_{cr}$ (N/m) of plates under simply supported boundary condition and the results are tabulated in Tables 13-14.

It is seen from the results that critical buckling loads depend on lamination types. Critical buckling loads increase with the decrease of the aspect ratios ($a/b$ or $b/a$).

It is seen from the Table 13 (short edge is on the y axis: $a/b$) Angle-ply plates LT27 ([-45/0/45/90]$_s$) and LT28 ([45/0/45/90]$_s$) have the highest value for the lowest critical buckling loads (103747 N/m) for aspect ratio $a/b=1$. For aspect ratio $a/b=2$, both of the Quasi-isotropic plates LT20 ([90/0/45/5/0]$_s$) and LT23 ([0/45/5/0]$_s$) have the highest value for the lowest critical buckling loads (71357 N/m).

It can be noticed from Table 14 (short edge is on the x axis: $b/a$), Angle-ply plates LT27 ([-45/45/45/-45]$_s$) and LT28 ([45/45/45/-45]$_s$) have the highest value for the lowest critical buckling loads (103747 N/m) for aspect ratio $b/a=1$. For aspect ratio $b/a=2$, LT7 ([0/45/5/90]$_s$) and LT9 ([0/45/-5/90]$_s$) have the highest value for the lowest critical buckling loads (71357 N/m).
| a/b | Method          | LT1   | LT15  | LT2   | LT16  | LT3   | LT13  | LT4   | LT14  | LT5   | LT18  |
|-----|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1.4 | Rayleigh-Ritz  | 57983 | 55779 | 70215 | 74596 | 65088 |       |       |       |       |       |
|     | FEM (ANSYS)    | 54232 | 51060 | 69017 | 73390 | 60832 |       |       |       |       |       |
| 1.6 | Rayleigh-Ritz  | 50452 | 49816 | 61978 | 67424 | 61335 |       |       |       |       |       |
|     | FEM (ANSYS)    | 47695 | 45855 | 60998 | 66436 | 57880 |       |       |       |       |       |
| 1.8 | Rayleigh-Ritz  | 45166 | 45745 | 55944 | 62118 | 58738 |       |       |       |       |       |
|     | FEM (ANSYS)    | 42812 | 42401 | 55131 | 61295 | 55889 |       |       |       |       |       |
| 2   | Rayleigh-Ritz  | 41321 | 42852 | 51398 | 58090 | 56866 |       |       |       |       |       |
|     | FEM (ANSYS)    | 39297 | 40007 | 50714 | 57394 | 54482 |       |       |       |       |       |

| a/b | Method          | LT6   | LT17  | LT7   | LT9   | LT8   | LT10  | LT11  | LT12  | LT19  | LT24  |
|-----|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1   | Rayleigh-Ritz  | 87155 | 78891 | 73224 | 66872 | 73224 |       |       |       |       |       |
|     | FEM (ANSYS)    | 82537 | 77900 | 71478 | 66591 | 71478 |       |       |       |       |       |
| 1.2 | Rayleigh-Ritz  | 77739 | 58932 | 55645 | 53557 | 69487 |       |       |       |       |       |
|     | FEM (ANSYS)    | 74110 | 58164 | 54215 | 53356 | 68102 |       |       |       |       |       |
| 1.4 | Rayleigh-Ritz  | 71597 | 47234 | 45751 | 46745 | 67886 |       |       |       |       |       |
|     | FEM (ANSYS)    | 68696 | 46615 | 44561 | 46595 | 66764 |       |       |       |       |       |
| 1.6 | Rayleigh-Ritz  | 67340 | 39844 | 39755 | 43028 | 67214 |       |       |       |       |       |
|     | FEM (ANSYS)    | 64981 | 39335 | 38754 | 42912 | 66287 |       |       |       |       |       |
| 1.8 | Rayleigh-Ritz  | 64265 | 34902 | 35901 | 40895 | 66966 |       |       |       |       |       |
|     | FEM (ANSYS)    | 62314 | 34476 | 35053 | 40802 | 66185 |       |       |       |       |       |
| 2   | Rayleigh-Ritz  | 61971 | 31444 | 33004 | 39618 | 66915 |       |       |       |       |       |
|     | FEM (ANSYS)    | 60332 | 31085 | 32578 | 39542 | 66246 |       |       |       |       |       |

| a/b | Method          | LT20  | LT23  | LT21  | LT22  | LT25  | LT26  | LT27  | LT28  |
|-----|----------------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1   | Rayleigh-Ritz  | 78891 | 66872 | 61516 | 61516 | 103747|       |       |       |
|     | FEM (ANSYS)    | 77900 | 66591 | 61399 | 61399 | 100801|       |       |       |
| 1.2 | Rayleigh-Ritz  | 75251 | 61650 | 58739 | 47995 | 86698 |       |       |       |
|     | FEM (ANSYS)    | 74457 | 61424 | 58629 | 47937 | 84345 |       |       |       |
| 1.4 | Rayleigh-Ritz  | 73373 | 59710 | 58663 | 41440 | 74887 |       |       |       |
|     | FEM (ANSYS)    | 72222 | 59522 | 58557 | 41412 | 72975 |       |       |       |
| 1.6 | Rayleigh-Ritz  | 72335 | 59150 | 59539 | 38112 | 66343 |       |       |       |
|     | FEM (ANSYS)    | 71788 | 58987 | 59434 | 38100 | 64766 |       |       |       |
| 1.8 | Rayleigh-Ritz  | 71728 | 59175 | 60684 | 36375 | 59974 |       |       |       |
|     | FEM (ANSYS)    | 71259 | 59031 | 60578 | 36373 | 58656 |       |       |       |
| 2   | Rayleigh-Ritz  | 71357 | 59440 | 61833 | 35461 | 55115 |       |       |       |
|     | FEM (ANSYS)    | 70946 | 59309 | 61723 | 35466 | 54001 |       |       |       |
Table 14 - Critical buckling load (N/m), short edge is on the x axis

| b/a | Method       | LT1   | LT15  | LT2   | LT16  | LT3   | LT13  | LT4   | LT14  | LT5   | LT18  |
|-----|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|     |              | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) |
| 1   | Rayleigh-Ritz | 87155 | 80182 | 99333 | 99333 | 80182 |
|     | FEM (ANSYS)  | 82537 | 73442 | 97414 | 97414 | 73442 |
| 1.2 | Rayleigh-Ritz | 77739 | 70807 | 84621 | 81887 | 64974 |
|     | FEM (ANSYS)  | 74110 | 65484 | 83119 | 80392 | 59334 |
| 1.4 | Rayleigh-Ritz | 71597 | 65088 | 74596 | 70215 | 55779 |
|     | FEM (ANSYS)  | 68696 | 60832 | 73390 | 69017 | 51060 |
| 1.6 | Rayleigh-Ritz | 67340 | 61335 | 67424 | 61978 | 49816 |
|     | FEM (ANSYS)  | 64981 | 57880 | 66436 | 60998 | 45855 |
| 1.8 | Rayleigh-Ritz | 64265 | 58738 | 62118 | 55944 | 45745 |
|     | FEM (ANSYS)  | 62314 | 55889 | 61295 | 55131 | 42852 |
| 2   | Rayleigh-Ritz | 61971 | 56866 | 58090 | 51398 | 42852 |
|     | FEM (ANSYS)  | 60332 | 54482 | 57994 | 50714 | 40007 |

| b/a | Method       | LT6   | LT7   | LT9   | LT8   | LT10  | LT11  | LT12  | LT19  | LT24  |
|-----|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|     |              | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) |
| 1   | Rayleigh-Ritz | 87155 | 78891 | 73224 | 66872 | 73224 |
|     | FEM (ANSYS)  | 82537 | 77900 | 71478 | 66591 | 71478 |
| 1.2 | Rayleigh-Ritz | 69227 | 75251 | 69487 | 61650 | 55645 |
|     | FEM (ANSYS)  | 65368 | 74457 | 68102 | 61424 | 54215 |
| 1.4 | Rayleigh-Ritz | 57983 | 73373 | 67886 | 59710 | 45751 |
|     | FEM (ANSYS)  | 54732 | 72722 | 66764 | 59522 | 44561 |
| 1.6 | Rayleigh-Ritz | 50452 | 72335 | 67214 | 59150 | 39755 |
|     | FEM (ANSYS)  | 47695 | 71788 | 66287 | 58987 | 38754 |
| 1.8 | Rayleigh-Ritz | 45166 | 71728 | 66966 | 59175 | 35901 |
|     | FEM (ANSYS)  | 42812 | 71259 | 66185 | 59031 | 35053 |
| 2   | Rayleigh-Ritz | 41321 | 71357 | 66915 | 59440 | 33304 |
|     | FEM (ANSYS)  | 39297 | 70946 | 66246 | 59309 | 32578 |

| b/a | Method       | LT20  | LT21  | LT22  | LT23  | LT25  | LT26  | LT27  | LT28  |
|-----|--------------|-------|-------|-------|-------|-------|-------|-------|-------|
|     |              | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) | N<sub>cr</sub> (N/m) |
| 1   | Rayleigh-Ritz | 78891 | 66872 | 61516 | 61516 | 103747 |
|     | FEM (ANSYS)  | 77900 | 66591 | 61399 | 61399 | 100801 |
| 1.2 | Rayleigh-Ritz | 58932 | 53557 | 47995 | 58739 | 86698 |
|     | FEM (ANSYS)  | 58164 | 53356 | 47937 | 58629 | 84345 |
### Table 14 - Critical buckling load (N/m), short edge is on the x axis (continue)

| b/a | Method           | LT20 $N_{cr}$ (N/m) | LT23 $N_{cr}$ (N/m) | LT21 $N_{cr}$ (N/m) | LT22 $N_{cr}$ (N/m) | LT25 $N_{cr}$ (N/m) | LT26 $N_{cr}$ (N/m) | LT27 $N_{cr}$ (N/m) | LT28 $N_{cr}$ (N/m) |
|------|------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 1.4  | Rayleigh-Ritz    | 47234               | 46745               | 41440               | 58663               | 74887               |                    |                     |                     |
|      | FEM (ANSYS)      | 46615               | 46595               | 41412               | 58557               | 72975               |                    |                     |                     |
| 1.6  | Rayleigh-Ritz    | 39844               | 43028               | 38112               | 59359               | 66343               |                    |                     |                     |
|      | FEM (ANSYS)      | 39335               | 42912               | 38100               | 59434               | 64766               |                    |                     |                     |
| 1.8  | Rayleigh-Ritz    | 34902               | 40895               | 36375               | 60684               | 59974               |                    |                     |                     |
|      | FEM (ANSYS)      | 34476               | 40802               | 36373               | 60578               | 58656               |                    |                     |                     |
| 2    | Rayleigh-Ritz    | 31444               | 39618               | 35461               | 61833               | 55115               |                    |                     |                     |
|      | FEM (ANSYS)      | 31085               | 39542               | 35466               | 61723               | 54001               |                    |                     |                     |

### 3.2.5. Non-dimensional results

Non-dimensional critical buckling loads of symmetrically laminated composite plates are tabulated for practical data for designer and given in Table 15 and 16.

### Table 15 - Non-dimensional critical buckling load, short edge is on the x axis, $N_{cr}' = N_{cr} \frac{b^2}{E_{22}}$

| Plate Type | a/b   | 1      | 1.2    | 1.4    | 1.6    | 1.8    | 2      |
|------------|-------|--------|--------|--------|--------|--------|--------|
| LT1        | 45158 | 35869  | 30043  | 26141  | 23402  | 21410  |
| LT15       | 41545 | 33665  | 28901  | 25811  | 23702  | 22203  |
| LT2        | 51468 | 42428  | 36381  | 32113  | 28987  | 26631  |
| LT16       | 51468 | 43845  | 38651  | 34935  | 32185  | 30098  |
| LT3        | 41545 | 36688  | 33724  | 31780  | 30434  | 29464  |
| LT13       | 45158 | 40279  | 37097  | 34891  | 33298  | 32109  |
| LT4        | 40876 | 30535  | 24474  | 20645  | 18084  | 16292  |
| LT17       | 40876 | 30535  | 24474  | 20645  | 18084  | 16292  |
| LT9        | 40876 | 30535  | 24474  | 20645  | 18084  | 16292  |
Table 15 - Non-dimensional critical buckling load, short edge is on the x axis,
\[ N'_{cr} = N_{cr} \frac{b^2}{t^3 E_{22}} \]

| Plate Type | Aspect Ratio | a/b  | 1   | 1.2 | 1.4 | 1.6 | 1.8 | 2   |
|------------|--------------|------|-----|-----|-----|-----|-----|-----|
| LT8        |              | 37940| 28832| 23705| 20598| 18602| 17256|
| LT10       |              | 34649| 27750| 24220| 22294| 21189| 20527|
| LT11       |              | 37940| 36004| 35174| 34826| 34697| 34671|
| LT12       |              | 40876| 38990| 38017| 37479| 37165| 36973|
| LT19       |              | 31874| 30435| 30395| 30849| 31443| 32038|
| LT24       |              | 34649| 31943| 30938| 30648| 30661| 30798|
| LT20       |              | 53755| 44921| 38802| 34375| 31075| 28557|
| LT23       |              |      |      |      |      |      |      |
| LT21       |              |      |      |      |      |      |      |
| LT22       |              |      |      |      |      |      |      |
| LT25       |              |      |      |      |      |      |      |
| LT26       |              |      |      |      |      |      |      |
| LT27       |              |      |      |      |      |      |      |
| LT28       |              |      |      |      |      |      |      |

Table 16 - Non-dimensional critical buckling load, short edge is on the y axis,
\[ N'_{cr} = N_{cr} \frac{a^2}{t^3 E_{22}} \]

| Plate Type | Aspect Ratio | b/a  | 1   | 1.2 | 1.4 | 1.6 | 1.8 | 2   |
|------------|--------------|------|-----|-----|-----|-----|-----|-----|
| LT1        |              | 45158| 40279| 37097| 34891| 33298| 32109|
| LT15       |              | 41545| 36688| 33724| 31780| 30434| 29464|
| LT2        |              | 51468| 43845| 38651| 34935| 32185| 30098|
| LT16       |              | 51468| 42428| 36381| 32113| 28987| 26631|
| LT3        |              | 41545| 33665| 28901| 25811| 23702| 22203|
| LT13       |              |      |      |      |      |      |      |
| LT4        |              |      |      |      |      |      |      |
| LT14       |              |      |      |      |      |      |      |
| LT5        |              |      |      |      |      |      |      |
| LT18       |              |      |      |      |      |      |      |
Table 16 - Non-dimensional critical buckling load, short edge is on the y axis,

\[ N'_{cr} = N_{cr} \frac{a^2}{tE_{22}} \] (continue)

| Plate Type | Aspect ratio | b/a |
|------------|--------------|-----|
|            | 1            | 1.2 | 1.4 | 1.6 | 1.8 | 2    |
| LT6        | 45158        | 35869| 30043|26141|23402|21410|
| LT7        | 40876        | 38990|38017|37479|37165|36973|
| LT8        | 37940        | 36004|35174|34826|34697|34671|
| LT9        | 34649        | 31943|30938|30648|30661|30798|
| LT10       | 37940        | 28832|23705|20598|18602|17256|
| LT11       | 34649        | 27750|24220|22294|21189|20527|
| LT12       | 31874        | 24868|21472|19747|18847|18374|
| LT13       | 31874        | 30395|30395|30849|31443|32038|
| LT14       | 31874        | 30435|30395|30849|31443|32038|
| LT15       | 53755        | 44921|38802|34375|31075|28557|

4. CONCLUSIONS

Within this study biaxial buckling analysis of symmetrically laminated quasi-isotropic, cross-ply and angle-ply rectangular thin plates has been examined. Plates are considered simply supported at the edges. Effect of thickness, aspect ratios and lamination types on critical buckling loads has been investigated parametrically by Rayleigh Ritz Method based on the Classical Lamination Plate Theory (CLPT). In addition, Finite Element Method software package ANSYS was used for calculations in order to compare the results and good correlation was obtained.

For the calculation of Rayleigh Ritz Method integral equations were initially solved by using Mathematica [49] then the code prepared by using the MATLAB [50] programming language for different conditions. Results obtained using Rayleigh Ritz Method were reached much faster than those of FEM calculations with ANSYS software package.

The critical buckling load of isotropic plates increases with decreasing of the aspect ratio (a/b or b/a). This situation was observed for the symmetrically composite laminates (Cross-Ply, Angle-Ply and Quasi-isotropic plates) similarly.
The present paper also indicates that the thick plates have a large buckling strength compared to thin plates. However, shear deformable theories should be considered for thick plates.

Symmetrically laminated Cross-ply plates are orthotropic and their bending-twisting coupling terms $D_{16}$ and $D_{26}$ are zero, but these terms are taken into account for quasi-isotropic and angle-ply laminates. Jones [51] mentioned that for laminated plates with bending-twisting coupling decrease buckling loads. The same situation was observed that considering Angle-ply and Quasi-isotropic plates for four-terms solutions in this study. When the bending-twisting coupling terms ($D_{16}$, $D_{26}$) are not taken into account, the critical buckling load is high.

Results show that bending stiffness matrix elements $D_{11}$ and $D_{22}$ are equal for each other for symmetrically laminated Angle-Ply plates ($LT_{27} = [-45^\circ/45^\circ/-45^\circ/45^\circ]$) and $LT_{28} = [45^\circ/-45^\circ/45^\circ/-45^\circ]$) similar to isotropic plates. Thus, the critical buckling load for each aspect ratios ($a/b = b/a$) gives the same result for Angle-Ply plates and isotropic plates. One of the most important results of this study: In terms of highest value for the lowest critical buckling loads ($N_{cr}$), Angle-ply plates are more advantageous than Cross-ply and Quasi-isotropic plates for lowest aspect ratio is ($a/b=1$, 1.2 and 1.4). Symmetrically laminated Quasi-isotropic plates have of highest value for the lowest critical buckling loads for highest aspect ratios ($a/b=1.6$, 1.8 and 2). It is demonstrated that the bending stiffness matrix elements play an important role in the bi-axial buckling of symmetrically laminated plates.

It was aimed to determine the most appropriate stiffest plate types (having highest value for the lowest critical buckling loads) and this aim was accomplished for all conditions (results given in Section 3).

Therefore, it can be concluded that the most suitable plate types may be quickly determined at the design stage of composite engineering structures, with the use of tabulated non-dimensional results obtained by the Rayleigh Ritz method. In addition, the tabulated results should be valuable to engineers as well as researchers working in this field.

Some modes shapes of Quasi-isotropic, Cross-ply and Angle-ply laminates have been obtained and given in Appendix (Figure A1 and Figure A2).

In future studies, stress and strain distributions along the thickness of laminated plates and failure theories can be examined, supported by experimental studies and advantageous lamination types can be investigated by optimization techniques.

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APPENDIX

In this section, critical buckling loads of some plate types for the first three mode shapes are given. It can be seen that the critical buckling load values of some different plate types for some edge ratios are the same as each other. However, it has been observed that this situation may change as the edge ratio changes.

First three modes shapes and buckling loads of some quasi-isotropic, cross-ply and angle-ply laminates are presented in Figure A1 and Figure A2. It can be seen from Figure A1 (a/b=1) the critical buckling loads of LT8 and LT19 are equal but their modes shapes are different. LT27 (Angle-Ply plate) has the highest critical buckling loads for mode-1 and mode-2, while LT25 (Cross-Ply plate) has the highest critical buckling load for mode-3. It may be seen from Figure A2 for a different aspect ratio (a/b=2) the critical buckling loads of LT8 and LT19 are different this time. LT27 (Angle-Ply plate) has the highest critical buckling loads for mode-1, mode-2 and mode-3.
Figure A1 - Some mode shapes of laminated plates (quasi-isotropic, cross-ply, angle-ply) \((a/b=1)\)
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Figure A2 - Some mode shapes of laminated plates (quasi-isotropic, cross-ply, angle-ply) 
(a/b=2)