Two Groups in a Curie-Weiss Model

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1 Introduction

The Curie-Weiss model is probably the easiest model of magnetism which shows a phase transition between a diamagnetic and a ferromagnetic phase. In this model the spins can take values in \{-1, 1\} (or up/down), each spin interacts with all the others in the same way. More precisely, for finitely many spins \((X_1, X_2, \ldots, X_N) \in \{-1, 1\}\) the energy of the spins is given by

\[
H = H(X_1, \ldots, X_N) := \frac{-J}{2N} \left( \sum_{j=1}^{N} X_j \right)^2, \tag{1}
\]

where \(J\) is a positive real number.

Consequently, in the ‘canonical ensemble’ with inverse temperature \(\beta \geq 0\) the probability of a spin configuration is given by

\[
P(X_1 = x_1, \ldots, X_N = x_N) := Z^{-1} e^{-\beta H(x_1, \ldots, x_N)} \tag{2}
\]

where \(x_i \in \{-1, 1\}\) and \(Z\) is a normalization constant which depends on \(N\), \(J\) and \(\beta\). Since only the product of \(\beta\) and \(J\) occurs in (2) we may set \(J = 1\) without loss of generality.

The quantity

\[
S_N = \sum_{j=1}^{N} X_j \tag{3}
\]

is called the (total) magnetization. It is well known (see e.g. Ellis [2] or [3]) that the Curie-Weiss model has a phase transition at \(\beta = 1\) in the following sense

\[
\frac{1}{N}S_N \Rightarrow \frac{1}{2}(\delta_{-m(\beta)} + \delta_{m(\beta)}) \tag{4}
\]

where \(\Rightarrow\) denotes convergence in distribution, \(\delta_x\) the Dirac measure in \(x\).
For $\beta \leq 1$ we have $m(\beta) = 0$ which is the unique solution of
\begin{equation}
\tanh(\beta x) = x
\end{equation}
for this case.

If $\beta > 1$ equation (5) has exactly three solutions and $m(\beta)$ is the unique positive one.

Equation (4) is a substitute for the law of large numbers for i.i.d. random variables.

Moreover, for $\beta < 1$ there is a central limit theorem, i.e.
\begin{equation}
\frac{1}{\sqrt{N}} S_N \implies \mathcal{N}(0, 1/(1-\beta))
\end{equation}
for $\beta = 1$ there is no such central limit theorem. In fact, the random variables
\begin{equation}
\frac{1}{N^{3/4}} S_N
\end{equation}
converge in distribution to a limit which is not a normal distribution.

In this paper we form out of $N$ Curie-Weiss spins two disjoint groups $X_1, \ldots, X_N$, and $Y_1, \ldots, Y_N$, with $N_1 + N_2 \leq N$. We let $N_1$ and $N_2$ depend on $N$ in such a way that both $N_1$ and $N_2$ go to infinity as $N$ does. We consider the asymptotic behaviour of the two-dimensional random variables
\begin{equation}
\left(\sum_{i=1}^{N_1} X_i, \sum_{j=1}^{N_2} Y_j\right)
\end{equation}
as $N$ goes to infinity.

We prove

**Theorem 1** (Law of Large Numbers). *If $N_1, N_2 \to \infty$ as $N \to \infty$, then we have for all $\beta$
\begin{equation}
\left(\frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j\right) \implies \frac{1}{2}(\delta(-m(\beta), -m(\beta)) + \delta(m(\beta), m(\beta))) \quad (\text{as } N \to \infty)
\end{equation}

Above `$\implies$' denotes convergence in distribution of the 2-dimensional random variable on the left hand side.

**Remark 2.** If we consider a model without interaction between the groups $X_i$ and $Y_j$ then the limit in (9) is
\begin{equation}
\frac{1}{4}(\delta(-m(\beta), -m(\beta)) + \delta(-m(\beta), m(\beta)) + \delta(m(\beta), -m(\beta)) + \delta(m(\beta), m(\beta)))
\end{equation}
For $\beta < 1$ we also have a central limit theorem. The covariance of the limiting normal distribution depends on the growth rate of $N_1$ and $N_2$. We set
\begin{equation}
\alpha_1 = \lim_{N \to \infty} \frac{N_1}{N} \quad \alpha_2 = \lim_{N \to \infty} \frac{N_2}{N}
\end{equation}
and assume that these limits exist.
Theorem 3 (Central Limit Theorem). If $\beta < 1$, then

\[
\left( \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right) \xrightarrow{N \to \infty} N((0,0), C),
\]

where the covariance matrix $C$ is given by

\[
C = \begin{bmatrix}
1 + \alpha_1 \frac{\beta}{1-\beta} & \sqrt{\alpha_1 \alpha_2} \frac{\beta}{1-\beta} \\
\sqrt{\alpha_1 \alpha_2} \frac{\beta}{1-\beta} & 1 + \alpha_2 \frac{\beta}{1-\beta}
\end{bmatrix}
\]

In particular, for sublinear growth of either $N_1$ or $N_2$, i.e. if $\alpha_1 = 0$ or $\alpha_2 = 0$, the standardized sums in (11) are asymptotically independent.

We mention that the Curie-Weiss model is also used to model the behaviour of voters who have the choice to vote ‘Yea’ (spin=1, say) or ‘Nay’ (spin=-1) (see [6]).

In the proof of both results we employ the moment method (see e.g. [1] or [5]). Thus, to show the convergence in distribution of a sequence $(X_n, Y_n)$ of two-dimensional random variables to a measure $\mu$ on $\mathbb{R}^2$ we prove that

\[
\mathbb{E}\left( X_n^K \cdot Y_n^L \right) \longrightarrow \int x^K y^L \mu(dx, dy)
\]

for all $K, L \in \mathbb{N}$.

Equation (13) implies convergence in distribution if the moments of $\mu$ grow only moderately, namely if for some constant $A$ and $C$ and all $K, L$

\[
\int |x|^K |y|^L \mu(dx, dy) \leq A C^{K+L} (K + L)!
\]

holds.

Some time after publishing the first version of this paper on arXiv, we became aware of the articles [3] and [4] which contain the above results as special cases. The methods used by those authors is very different from ours. We are grateful to Francesca Collet for drawing our attention to the papers [3] and [4].

2 Preparation

To use the method of moments we have to evaluate sums of the form

\[
\mathbb{E} \left[ \left( \sum_{i=1}^{N_1} X_i \right)^K \left( \sum_{j=1}^{N_2} Y_j \right)^L \right]
\]

\[
= \sum_{i_1, \ldots, i_K} \sum_{j_1, \ldots, j_L} \mathbb{E} \left( X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_K} \cdot Y_{j_1} \cdot Y_{j_2} \cdot \ldots \cdot Y_{j_L} \right). \tag{15}
\]

To do the book-keeping for these huge sums we introduce a few combinatorial concepts taken from [5].
Definition 4. We define a multiindex \( \underline{i} = (i_1, i_2, \ldots, i_L) \in \{1, 2, \ldots, N\}^L \).

1. For \( j \in \{1, 2, \ldots, N\} \) we set

\[
\nu_j(\underline{i}) = |\{k \in \{1, 2, \ldots, L\} | i_k = j\}|
\]

where \(|M|\) denotes the number of elements in the set \( M \).

2. For \( l = 0, 1, \ldots, L \) we define

\[
\rho_l(\underline{i}) = |\{j | \nu_j(\underline{i}) = l\}|
\]

and

\[
\underline{\rho}(\underline{i}) = (\rho_1(\underline{i}), \ldots, \rho_L(\underline{i})).
\]

The numbers \( \nu_j(\underline{i}) \) represent the multiplicity of each index \( j \in \{1, 2, \ldots, N\} \) in the multiindex \( \underline{i} \), and \( \rho_l(\underline{i}) \) represents the number of indices in \( \underline{i} \) that occur exactly \( l \) times. We shall call such \( \underline{\rho}(\underline{i}) \) profile vectors.

Lemma 5 (Lemma 3.8). For all \( \underline{i} = (i_1, i_2, \ldots, i_L) \in \{1, 2, \ldots, N\}^L \) we have

\[
\sum_{l=1}^L l \rho_l(\underline{i}) = L.
\]

Definition 6. Let \( \underline{r} = (r_1, \ldots, r_L) \), \( \sum_{l=1}^L lr_l = L \), be a profile vector. We define

\[
w_L(\underline{r}) = |\{\underline{i} \in \{1, \ldots, N\}^L | \underline{\rho}(\underline{i}) = \underline{r}\}|
\]

to represent the number of multiindices \( \underline{i} \) that have a given profile vector \( \underline{r} \).

We now define the set of all profile vectors for a given \( L \in \mathbb{N} \).

Definition 7. Let \( \Pi^L \) = \( \{\underline{r} \in \{0, 1, \ldots, L\}^L | \sum_{l=1}^L lr_l = L\} \). Some important subsets of \( \Pi^L \) are \( \Pi^L_1 = \{\underline{r} \in \Pi^L | r_1 = k\} \), \( \Pi^L_0 = \{\underline{r} \in \Pi^L | r_l = 0 \text{ for all } l \geq 3\} \) and \( \Pi^L_+ = \{\underline{r} \in \Pi^L | r_l > 0 \text{ for some } l \geq 3\} \).

Proposition 8. For \( \underline{r} \in \Pi^L \) set \( r_0 = N - \sum_{l=1}^L r_l \), then

\[
\frac{N!}{r_1!r_2! \ldots r_L!r_0!} \frac{L!}{1!r_1!2!r_2! \ldots L!r_L!}.
\]
3 Proofs

3.1 Law of Large Numbers

We are interested in the behaviour of the partial sums

\[ \frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j. \]

Suppose \( K, L \in \mathbb{N} \). We want to calculate the moment

\[ \mathbb{E} \left[ \left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i \right)^K \left( \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right)^L \right]. \tag{16} \]

We distinguish between multiindices that have a repeated index and those that do not. The following proposition shows that only the multiindices \( \mathbf{i} \in \Pi^{(K)} \), \( \mathbf{j} \in \Pi^{(L)} \) in which each index occurs exactly once contribute asymptotically to the moments

\[ \frac{1}{N_1 N_2} \sum_{\mathbf{i} \in \Pi^{(K)}} \sum_{\mathbf{j} \in \Pi^{(L)}} w_K(\mathbf{i}) w_L(\mathbf{j}) \mathbb{E}(X_\mathbf{i} Y_\mathbf{j}). \tag{17} \]

**Proposition 9.** If \( \mathbf{i} \in \Pi^{(K)} \) or \( \mathbf{j} \in \Pi^{(L)} \) has an index that occurs more than once, then \( \frac{1}{N_1 N_2} w_K(\mathbf{i}) w_L(\mathbf{j}) \mathbb{E}(X_\mathbf{i} Y_\mathbf{j}) \) converges to 0 as \( N \to \infty \).

**Proof.** We shall assume without loss of generality that \( \mathbf{i} \) contains a repeated index. This implies that the profile vector \( \mathbf{r} = (r_1, \ldots, r_K) = \rho(\mathbf{i}) \) has some \( r_j > 0 \) where \( j > 1 \). For multiindex \( \mathbf{j} \), define the profile vector \( \mathbf{s} = (s_1, \ldots, s_L) \). There are

\[
\begin{align*}
w_K(\mathbf{r}) &= \frac{1}{r_1! \cdots r_K!} \frac{K!}{1!^r \cdots K!^r} N_1^{\sum_{k=1}^K r_k} \\
&= C_K N_1^{\sum_{k=1}^K r_k}
\end{align*}
\]

and

\[
\begin{align*}
w_L(\mathbf{s}) &= \frac{1}{s_1! \cdots s_L!} \frac{L!}{1!^{s_1} \cdots L!^{s_L}} N_1^{\sum_{l=1}^L s_l} \\
&= C_L N_1^{\sum_{l=1}^L s_l}
\end{align*}
\]
such multiindices. Note that $\sum_{k=1}^K r_k < K$ due to the existence of $r_j > 0$ for some $j > 1$, as well as $\sum_{k=1}^K kr_k = K$, and $\sum_{l=1}^L sl \leq L$. Hence

$$\frac{1}{N_1^K N_2^L} w_K(j) w_L(j) = \frac{1}{N_1^K N_2^L} C_K N_1^{\sum_{k=1}^K r_k} C_L N_1^{\sum_{l=1}^L s_l} \leq \frac{1}{N_1^K N_2^L} C_K C_L N_1^{K-1} N_2^L = C_K C_L N_1^{-1}\to 0$$

and the assertion holds. \(\square\)

Since multiindices with repeated indices do not contribute asymptotically to the moment in (16), that leaves us with all those multiindices that do not contain the same index more than once. Of these there are asymptotically $N_1^K$ in $\Pi^K$ and $N_2^L$ in $\Pi^L$. Hence the moment is asymptotically given by

$$\frac{1}{N_1^K N_2^L} N_1^K N_2^L E(X_1 Y_2) = E(X_1 Y_2).$$

So the moment is asymptotically equal to the correlations $E(X_1 Y_2)$. As shown in [5], this expression depends on the value of $\beta$. We have for $\beta \leq 1$

$$E(X_1 Y_2) \to 0,$$

as $N \to \infty$. For $\beta > 1$, however,

$$E(X_1 Y_2) \approx m(\beta)^{K+L}.$$

We can conclude, that as in the case of a single group in the Curie-Weiss model, the law of large numbers (theorem 1) holds and the random vectors

$$\left(\frac{1}{N_1^K} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2^L} \sum_{j=1}^{N_2} Y_j\right)_N$$

converge to the random vector $\frac{1}{2}(\delta - (m(\beta), m(\beta))) + \delta(m(\beta), m(\beta)))$.

### 3.2 Central Limit Theorem

Now, we are interested in the behaviour of the partial sums

$$\frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j.$$

Suppose $K, L \in \mathbb{N}$. We want to calculate the moment
\[
\mathbb{E} \left[ \left( \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i \right)^K \left( \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right)^L \right].
\]  \tag{18}

Since within each group the \(X_i, Y_j\) are exchangeable, only the number of indices that occur 0, 1, \ldots times matters. Hence we write for all \((i_1, \ldots, i_K)\) and all \((j_1, \ldots, j_L)\) with \(X(\underline{i}) = X_{i_1} X_{i_2} \cdots X_{i_K}\) and \(Y(j) = Y_{j_1} Y_{j_2} \cdots Y_{j_L}\), provided that the multiindices \((i_1, \ldots, i_K)\) and \((j_1, \ldots, j_L)\) have the profile vectors \(\underline{n}_i\) and \(\underline{n}_j\) respectively.

The moments in (18) are thus given by

\[
\frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K} \sum_{\underline{i} \in \Pi_k^{(K)}} w_K(\underline{i}) \sum_{l=0}^{L} \sum_{\underline{j} \in \Pi_l^{(L)}} w_L(\underline{j}) \mathbb{E}(X_{\underline{i}} Y_{\underline{j}}).
\]  \tag{19}

The value of the above depends on the value of the inverse temperature parameter \(\beta\).

We separate the above sum into four summands:

\[
A_1 = \frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K} \sum_{\underline{i} \in \Pi_k^{(K)}} w_K(\underline{i}) \sum_{l=0}^{L} \sum_{\underline{j} \in \Pi_l^{(L)}} w_L(\underline{j}) \mathbb{E}(X_{\underline{i}} Y_{\underline{j}}),
\]

\[
A_2 = \frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K} \sum_{\underline{i} \in \Pi_k^{(K)}} w_K(\underline{i}) \sum_{l=0}^{L} \sum_{\underline{j} \in \Pi_l^{+}(K)} w_L(\underline{j}) \mathbb{E}(X_{\underline{i}} Y_{\underline{j}}),
\]

\[
A_3 = \frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K} \sum_{\underline{i} \in \Pi_k^{(K)}} w_K(\underline{i}) \sum_{l=0}^{L} \sum_{\underline{j} \in \Pi_l^{0}(L)} w_L(\underline{j}) \mathbb{E}(X_{\underline{i}} Y_{\underline{j}}),
\]

\[
A_4 = \frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K} \sum_{\underline{i} \in \Pi_k^{(K)}} w_K(\underline{i}) \sum_{l=0}^{L} \sum_{\underline{j} \in \Pi_l^{+}(L)} w_L(\underline{j}) \mathbb{E}(X_{\underline{i}} Y_{\underline{j}}).
\]

We will show that, asymptotically, only \(A_1\) contributes to the sum (19).

**Proposition 10.** The limit of \(A_2\) as \(N_2\) goes to infinity is 0.

**Proof.** For fixed \(k, l\) let \(\underline{i} \in \Pi_k^{0(K)}\) and \(\underline{j} \in \Pi_l^{+}(L)\). Then
The constant $c$ in the fourth line above represents the product of all those factors that do not depend on $N_2$ and $N$. In the inequality above we used

$$\sum_{j=1}^L s_j \leq \frac{L}{2} \sum_{j=1}^L s_j - \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \sum_{j=1}^L j s_j - \frac{1}{2} = \frac{L^2 - 1}{2}$$

and

$$N_2 \approx N$$ for large $N_2$ and fixed $N_1$.

Since each summand goes to zero, and there are only finitely many summands, $A_2$ goes to zero as $N_2$ goes to infinity.

Note that although we assumed a fixed $N_1$ in both the statement and the proof, the assertion would also hold if we assumed $N_1 = \alpha \nu N$ for both groups and let $N$ go to infinity.

**Proposition 11.** The following limits hold:

1. The limit of $A_3$ as $N_1$ goes to infinity is 0.
2. The limit of $A_4$ as $N_1$ goes to infinity is 0.
3. The limit of $A_4$ as $N_2$ goes to infinity is 0.

The proof of this proposition is very similar to the previous one, so we shall omit it.

Using these propositions, we obtain

**Corollary 12.** Asymptotically, i.e. if both $N_1$ and $N_2$ go to infinity, we have

$$\frac{1}{N_1^{K/2} N_2^L/2} \sum_{r=0}^K \sum_{k \in \Pi^{(K)}} \sum_{s=0}^L \sum_{j \in \Pi^{(L)}} w_K(k) w_L(j) E(X_k Y_j) \approx \lim_{N_1 \to \infty, N_2 \to \infty} A_1.$$
which states that for $r + s$ odd $E(X_i Y_j) = E(X_{11}X_{12} \cdots X_{1,r+s}) = 0$. (We used the fact, that in $A_1$ both $i$ and $j$ are such multiindices that each index occurs once or twice. Hence we can ignore all indices that occur more than once.)

Now note that $k + 2r_2 = K$ and $l + 2s_2 = L$. This implies that $K$ must have the same parity as $k$ and $L$ the same as $l$. Hence $K + L$ must be even as well. We have to distinguish two cases here:

1. $K, L$ even,

2. $K, L$ odd.

If both $K$ and $L$ are even, we can express $A_1$ as

$$A_1 = \frac{1}{N_1^{K/2}N_2^{L/2}} \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} w_K(2k)w_L(2l)E(X(r)Y(s))$$

$$\approx \frac{1}{N_1^{K/2}N_2^{L/2}} \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} N_1^{K/2+k} \frac{K!}{(2k)!(K/2-k)!2^{K/2-k}} N_2^{L/2+l} \cdot \frac{L!}{(2l)!(L/2-l)!2^{L/2-l}} (2(k + l) - 1)!! \left( \frac{\beta}{1 - \beta} \right)^{k+l} N^{-(k+l)}$$

$$\approx \frac{K!L!}{(\frac{K}{2})!(\frac{L}{2})!2^{K/2}2^{L/2}} \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \alpha_k \alpha_l \frac{(\frac{K}{2})!}{(\frac{K}{2} - k)!(2k)2^{-k}} \cdot \frac{(\frac{L}{2})!}{(\frac{L}{2} - l)!(2l)2^{-l}} (2(k + l) - 1)!! \left( \frac{\beta}{1 - \beta} \right)^{k+l}$$

$$= (K - 1)!!(L - 1)!! \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \alpha_k \alpha_l \frac{(\frac{K}{2})!}{(\frac{K}{2} - k)!2^k} \frac{(\frac{L}{2})!}{(\frac{L}{2} - l)!2^l} (2(k + l) - 1)!! \left( \frac{\beta}{1 - \beta} \right)^{k+l}$$

$$= (K - 1)!!(L - 1)!! \left( \frac{2(k + l) - 1}{(2k - 1)(2l - 1)} \right) \left( \frac{\beta}{1 - \beta} \right)^{k+l}$$

$$= (K - 1)!!(L - 1)!! \left( \frac{(2k + l) - 1}{(2k - 1)(2l - 1)} \right) \left( \frac{\beta}{1 - \beta} \right)^{k+l}$$

$$= (K - 1)!!(L - 1)!! \left( \frac{2k + l}{2k} \right) \left( \frac{\beta}{1 - \beta} \right)^{k+l}.$$
The case where $K, L$ are odd is similar to the above.

### 3.2.1 Linear Population Growth

We now show the central limit theorem for two groups in a Curie-Weiss model (theorem 3). Let $\beta < 1$ be the inverse temperature parameter, and define for convenience $\bar{\beta} = \beta - 1$. Let $N$ be the overall size of the population and assume that for $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 \leq 1$, $N_1 \approx \alpha_1 N$ and $N_2 \approx \alpha_2 N$ are two groups within this population. We shall use the symbols $X_i$ and $Y_j$ to refer to individual votes within groups 1 and 2, respectively. Define the normed sums $S_1 = \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i$ and $S_2 = \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j$. Using Isserlis’s Theorem (see [7]) and the recursive structure it implies for the moments of a bivariate normal distribution, we show that asymptotically the moments of the infinite sequence the random vectors $(S_1, S_2)_N$ converge to those of a bivariate normal distribution with zero mean and covariance matrix

$$
\begin{bmatrix}
E(S_1^2) & E(S_1 S_2) \\
E(S_1 S_2) & E(S_2^2)
\end{bmatrix} = \begin{bmatrix}
1 + \alpha_1 \bar{\beta} & \sqrt{\alpha_1 \alpha_2 \bar{\beta}} \\
\sqrt{\alpha_1 \alpha_2 \bar{\beta}} & 1 + \alpha_2 \bar{\beta}
\end{bmatrix}.
$$

Let $(Z_1, Z_2)$ be such a bivariate normal random vector. Isserlis’s Theorem states that higher moments of the bivariate normal distribution can be calculated

$$
E(Z_1^K Z_2^L) = \sum_{\pi \in \mathcal{P}} \prod_{i=1}^{K+L} E(\xi_{i1} \xi_{i2}),
$$

where $\mathcal{P}$ is the set of all pair partitions on the set $\{1, 2, \ldots, K + L\}$ and $\xi_{i1} \xi_{i2}$ are two of the $K + L$ variables grouped together by a particular pair partition $\pi$.

We start by using Isserlis’s Theorem to express higher moments of the bivariate normal distribution recursively.

**Lemma 13.** For all $K, L \in \mathbb{N}_0$, the moments $m_{K,L} = E(Z_1^K Z_2^L)$ satisfy the following equalities:

1. $m_{K+2,L} = (K + 1) m_{2,0} m_{K,L} + L m_{1,1} m_{K+1,L-1}$.
2. $m_{K+2,L} = (K + 1) m_{2,0} m_{K,L} + L m_{1,1} m_{K+1,L-1}$.

If $K = 0$ or $L = 0$, then the formulas still hold, setting any moments with negative indices equal to 0. Note that these two formulas suffice to calculate any higher moment as a function of $K, L$ and $m_{2,0}, m_{1,1}, m_{0,2}$.

**Proof.** We shall prove the first equation. By Isserlis’s Theorem, we can calculate $m_{K+2,L}$ as the sum over all permutations of product of expectations. If we take away the two additional $Z_2$ variables, what remains is a set of $K Z_1$s and $L Z_2$s. We have three possibilities here:

...
1. join one of the two additional $Z_2$s with one of the $K$ $Z_1$s.
2. join one of the two additional $Z_2$s with one of the $L$ $Z_2$s.
3. join the two additional $Z_2$s.

In the first case, we have $K$ times

$$E(Z_1Z_2) \sum_{\pi \in P_{K-1,L+1}} \prod_{i=1}^{K+L} E(\xi_i) = m_{1,1}m_{K-1,L+1}.$$ 

In the second case, we have $L$ times

$$E(Z_2^2) \sum_{\pi \in P_{K,L}} \prod_{i=1}^{K+L} E(\xi_i) = m_{0,2}m_{K,L}.$$ 

In the third case, we have $E(Z_2^2) \sum_{\pi \in P_{K,L}} \prod_{i=1}^{K+L} E(\xi_i) = m_{0,2}m_{K,L}$. The result follows by summing over the three possible cases.

Let $M_{K,L}$ stand for the limit of the moments of the random vector $(S_1, S_2)_N$. Our goal is to show that $M_{K,L} = m_{K,L}$ for all $K$ and $L$ and therefore the central limit theorem holds. We accomplish this by showing that the two formulas in lemma 1 hold for $M_{K,L}$ instead of $m_{K,L}$. Then, since by definition $M_{2,0} = m_{2,0}, M_{1,1} = m_{1,1}$ and $M_{0,2} = m_{0,2}$, all higher moments must be equal, too.

In the two dimensional CW model, only $K + L$ has to be even for the moment $M_{K,L}$ to be positive, allowing for two different cases we need to consider: $K$ and $L$ being both even and $K$ and $L$ being both odd. Asymptotically, the moments are given by the formulas

$$M_{K,L} = \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \frac{K!}{(2k)!(K/2-k)!2^{K/2-k}} \frac{L!}{(2l)!(L/2-l)!2^{L/2-l}} \cdot (2(k + l) - 1)!! \beta^k \alpha^l,$$

when $K, L$ are even.

We shall now show that the first formula in lemma 1 holds for $M_{K,L}$, assuming that $K, L$ are even. Then the left hand side of the formula reads $M_{K,L+2} =$

$$M_{K,L+2} = \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \frac{K!}{(2k)!(K/2-k)!2^{K/2-k}} \frac{(L + 2)!}{(2l)!(L/2+1-l)!2^{L/2+1-l}} \cdot (2(k + l) - 1)!! \beta^k \alpha^l.$$

On the right hand side, we have $K M_{1,1} M_{K-1,L+1} =$
\[
K \sqrt{\alpha_1 \alpha_2 \beta} \sum_{k=0}^{K/2-1} \sum_{l=0}^{L/2} \frac{(K-1)!}{(2k+1)!(\frac{K}{2}-1-k)!2^{\frac{K}{2}-1-k}} \frac{(L+1)!}{(2l+1)!(\frac{L}{2}-l)!2^{\frac{L}{2}-l}} \cdot (2(k+l)+1)!! \bar{\beta}^{k+l+1} \alpha_1^{k+1} \alpha_2^{l+1/2},
\]
and \((L+1)M_{0,2}M_{K,L} = \)
\[
(L+1)(1 + \alpha_2 \bar{\beta}) \sum_{k=0}^{K/2-1} \sum_{l=0}^{L/2} \frac{K!}{(2k)!(K/2-k)!2^{K/2-k}} \frac{L!}{(2l)!(L/2-l)!2^{L/2-l}} \cdot (2(k+l)+1)!! \bar{\beta}^{k+l+1} \alpha_1^{k+1} \alpha_2^{l+1},
\]
(21)
\[
\sum_{k=0}^{K/2-1} \sum_{l=0}^{L/2} \frac{K!}{(2k)!(K/2-k)!2^{K/2-k}} \frac{L!}{(2l)!(L/2-l)!2^{L/2-l}} \cdot (2(k+l)+1)!! \bar{\beta}^{k+l+1} \alpha_1^{k+1} \alpha_2^{l+1},
\]
(22)
\[
\sum_{k=0}^{K/2-1} \sum_{l=0}^{L/2} \frac{K!}{(2k)!(K/2-k)!2^{K/2-k}} \frac{L!}{(2l)!(L/2-l)!2^{L/2-l}} \cdot (2(k+l)+1)!! \bar{\beta}^{k+l+1} \alpha_1^{k} \alpha_2^{l+1},
\]
(23)
Note that the powers of \(\alpha_1\) and \(\alpha_2\) in (20) run through the sets \(\{0, 1, \ldots, K/2\}\) and \(\{0, 1, \ldots, L/2 + 1\}\), respectively. Once both powers are chosen, the power of \(\bar{\beta}\) is given by their sum.

We prove the theorem by showing that for each possible value of said powers \(k_1 \in \{0, 1, \ldots, K/2\}\) and \(k_2 \in \{0, 1, \ldots, L/2 + 1\}\) the coefficient multiplying the term \(\bar{\beta}^{k_1+k_2} \alpha_1^{k_1} \alpha_2^{k_2}\) is equal to the coefficient of the corresponding term on the right hand side, given by the sum of (21), (22) and (23).

Depending on the values of \(k_1\) and \(k_2\), not all of the three sums on the right hand side contribute to the coefficient of \(\bar{\beta}^{k_1+k_2} \alpha_1^{k_1} \alpha_2^{k_2}\). It is only if \(k_1 \in \{1, \ldots, K/2\}\)
and \( k_2 \in \{1, \ldots, L/2\} \) that all three sums contribute. That is the first case we want to inspect.

The coefficient on the left hand side given by (20) is

\[
\frac{K!}{(2k_1)!(K/2 - k_1)!2^{K/2-k_1}} \frac{(L + 2)!}{(2k_2)!((L/2 + 1 - k_2)!2^{L/2+1-k_2})(2(k_1 + k_2) - 1)!!}. 
\]

On the right hand side, we have three summands. The sum in (21) contributes when \( k = k_1 - 1 \) and \( l = k_2 - 1 \):

\[
\frac{K!}{(2k_1 - 1)!(\frac{K}{2} - k_1)!2^{\frac{K}{2}-k_1}} \frac{(L + 1)!}{(2k_2 - 1)!((\frac{L}{2} + 1 - k_2)!2^{\frac{L}{2}+1-k_2})(2(k_1 + k_2) - 3)!!}. 
\]

The sum in (22) contributes when \( k = k_1 \) and \( l = k_2 - 1 \):

\[
\frac{K!}{(2k_1)!(\frac{L}{2} - k_1)!2^{\frac{L}{2}-k_1}} \frac{(L + 1)!}{(2k_2 - 2)!((\frac{L}{2} + 1 - k_2)!2^{\frac{L}{2}+1-k_2})(2(k_1 + k_2) - 3)!!}. 
\]

The sum in (23) contributes when \( k = k_1 \) and \( l = k_2 \):

\[
\frac{K!}{(2k_1)!(\frac{L}{2} - k_1)!2^{\frac{L}{2}-k_1}} \frac{(L + 1)!}{(2k_2)!((\frac{L}{2} - k_2)!2^{\frac{L}{2}-k_2})(2(k_1 + k_2) - 1)!!}. 
\]

The common factor among the three terms on the right hand side is

\[
\frac{K!}{(2k_1 - 1)!(\frac{K}{2} - k_1)!2^{\frac{K}{2}-k_1}} \frac{(L + 1)!}{(2k_2 - 2)!((\frac{L}{2} - k_2)!2^{\frac{L}{2}-k_2})(2(k_1 + k_2) - 3)!!}. 
\]

Since this factor is contained in the term on the left hand side, as well, we can divide both sides by it. The following term remains on the left:

\[
\frac{1}{2k_1 2k_2 (2k_2 - 1)(L/2 + 1 - k_2)2^{(2(k_1 + k_2) - 1)}}. 
\]

On the right, we get

\[
\frac{1}{(2k_2 - 1)(\frac{L}{2} + 1 - k_2)!2^{L/2+1-k_2}} \cdot \frac{1}{2k_1(\frac{L}{2} + 1 - k_2)!2^{L/2+1-k_2}} \cdot \frac{2(k_1 + k_2) - 1}{2k_1 2k_2 (2k_2 - 1)}. 
\]
We need to show that both sides are equal. We start by multiplying both sides by $2k_1 2k_2 (2k_2 - 1)(L/2 + 1 - k_2)^2$ and calculate

$$
(L + 2)(2(k_1 + k_2) - 1) \div 2k_1 2k_2 + 2k_2(2k_2 - 1) + (2(k_1 + k_2) - 1)(L/2 + 1 - k_2)^2
$$

$$
(L + 2)(2(k_1 + k_2) - 1) \div 2k_1 2k_2 + 2k_2(2k_2 - 1) + (2(k_1 + k_2) - 1)(L + 2 - 2k_2)
$$

$$
0 \div 2k_1 2k_2 + 2k_2(2k_2 - 1) + (2(k_1 + k_2) - 1)(L/2 + 1 - k_2) - 2k_2
$$

$$
0 \div 2k_1 2k_2 + 4k_2^2 - 2k_2 - 4k_1 k_2 - 4k_2^2 + 2k_2 = 0.
$$

This concludes the proof that all terms $\tilde{\beta}^{k_1+k_2} \alpha_1^{k_1} \alpha_2^{k_2}$ with $k_1 \in \{1, \ldots, K/2\}$ and $k_2 \in \{1, \ldots, L/2\}$ have equal coefficients on both sides of the formula. We still need to show the same for the marginal cases where $k_1 = 0$ or $k_2 \in \{0, L/2 + 1\}$. In these five cases, on the right hand side, only one or two of the sums in (21), (22) and (23) contribute to the coefficient of $\tilde{\beta}^{k_1+k_2} \alpha_1^{k_1} \alpha_2^{k_2}$. The proofs are very similar, we shall therefore limit ourselves to the case where $k_1 = 0$ and $k_2 = 0$.

On the left hand side, we obtain the coefficient

$$
\frac{K!}{(K/2)!^2} \frac{(L + 2)!}{(L/2 + 1)!^2}
$$

On the right hand side, only (23) contributes to the coefficient, as in (21) and (22) the powers of $\alpha_1$ and $\alpha_2$ can never both be 0. Hence the coefficient on the right is

$$
\frac{K!}{(K/2)!^2} \frac{(L + 1)!}{(L/2)!^2}
$$

Dividing both coefficients by $\frac{K!}{(K/2)!^2} \frac{(L + 1)!}{(L/2)!^2}$, we obtain 1 on the right and $\frac{L + 2}{(L/2 + 1)^2} = 1$ on the left.

Since the coefficients on both sides are equal for all possible powers of $\alpha_1$ and $\alpha_2$, we conclude that the recursive formula holds for $M_{K,L}$. This concludes the proof of theorem 3.

### 3.2.2 Sublinear Population Growth

In this section we shall analyse the limiting distribution of the sums $(S_1, S_2)_N$ if one or both groups grow at lower rates than the overall population $N$. As in the previous section, we allow for the presence of a remainder population, i.e. $N_1 + N_2 \leq N$, where equality need not hold.

Assume again that $\alpha_1 = \lim \frac{N_1}{N}$ and $\alpha_2 = \lim \frac{N_2}{N}$. Now we allow one or both of these limits to be 0: let $\alpha_1 = 0$ and $0 \leq \alpha_2 \leq 1$. 

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We shall show the following results: \( S_1 \) is asymptotically standard normal, even though for finite \( N \) the variance is of course strictly greater than 1. In the large \( N \) limit, \( S_1 \) and \( S_2 \) become independent. Note that it suffices for this that one of the two groups grow more slowly than at linear speed. Hence, contrary to the last section, where we had positive moments for \( K, L \) odd (such as the covariance, e.g.), here only moments for \( K, L \) even are positive.

We already know that only multiindices \( \mathbf{i} = (i_1, \ldots, i_K) \) with the property that each of the indices occurs only once or twice contribute to the asymptotic moments. The moments are given by

\[
E(S_1^K) = \frac{N_1!K!}{N_1^{K/2}} \sum_{k=0}^{K/2} \frac{(2k)!K!}{(K/2 - k)!(N_1 - K/2 + k)!2^{K/2-k}}(2k - 1)!\bar{\beta}^k N^{-k}
\]

The last approximate equality is due to \( \alpha_1 = 0 \); only the summand with \( k = 0 \) contributes asymptotically to the value of the moment \( M_K \). This shows that \( S_1 \) tends to a standard normal distribution.

The bivariate moment \( M_{K,L} = E(S_1^K S_2^L) \) is similar:

\[
M_{K,L} \approx \frac{K!}{(K/2)!2^{K/2}} = (K - 1)!!
\]
\[
\frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \frac{N_1!K!}{(2k)!(K/2 - k)!(N_1 - K/2 + k)!2^{K/2-k}} \cdot \frac{N_2!L!}{(2l)!(L/2 - l)!(N_2 - L/2 + l)!2^{L/2-l}} \cdot (2(k + l) - 1)!!\beta^{k+l}N^{-(k+l)}
\]

\[
\approx \frac{1}{N_1^{K/2} N_2^{L/2}} \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \frac{N_1^{K/2+k} K!}{(2k)!(K/2 - k)!2^{K/2-k}} \cdot \frac{N_2^{L/2+l} L!}{(2l)!(L/2 - l)!2^{L/2-l}} \cdot (2(k + l) - 1)!!\beta^{k+l}N^{-(k+l)}
\]

\[
\approx \sum_{k=0}^{K/2} \sum_{l=0}^{L/2} \frac{K!}{(2k)!(K/2 - k)!2^{K/2-k}} \cdot \frac{L!}{(2l)!(L/2 - l)!2^{L/2-l}} \cdot (2(k + l) - 1)!!\beta^{k+l} \alpha_k \alpha_l
\]

\[
\approx \frac{K!}{(K/2)!2^{K/2}} \sum_{l=0}^{L/2} \frac{L!}{(2l)!(L/2 - l)!2^{L/2-l}} \cdot (2l - 1)!!\beta^l \alpha_l
\]

Depending on whether \(\alpha_2 = 0\) or \(\alpha_2 > 0\), we continue in the former case

\[
M_{K,L} = (K - 1)!!(L - 1)!!,
\]

which shows that for \(\alpha_2 = 0\), \((S_1, S_2)\) follows an independent bivariate normal distribution with both variances equal to 1.

In the latter case, we obtain

\[
M_{K,L} = (K - 1)!! \sum_{l=0}^{L/2} \frac{L!}{(2l)!(L/2 - l)!2^{L/2-l}} \cdot (2l - 1)!!\alpha_l \beta^l
\]

\[
= (K - 1)!! \sum_{l=0}^{L/2} \frac{L!}{(2l)!(L/2 - l)!2^{L/2-l}} \cdot (2l - 1)!!(\alpha_2 \beta)^l
\]

\[
= (K - 1)!! \frac{L!}{(L/2)!2^{L/2}} \sum_{l=0}^{L/2} \frac{(L/2)!}{l!(L/2 - l)!}(\alpha_2 \beta)^l
\]

\[
= (K - 1)!!(L - 1)!!(1 + \alpha_2 \beta)^{L/2},
\]

which implies a limiting bivariate normal distribution with zero mean and covariance matrix \(\begin{bmatrix} 1 & 0 \\ 0 & 1 + \alpha_2 \beta \end{bmatrix}\). As mentioned previously, it is enough if one of the two groups grows more slowly for asymptotic independence to occur.

If we inspect the formula for odd \(K, L\), \(M_{K,L} = \)
\[
\sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \frac{K!}{(2k+1)!2^{k+1/2}} \frac{L!}{(2l+1)!2^{l+1/2}} \cdot (2k+1)!(2l+1)! (K-k)! (L-l)! 2^{k-l}.
\]

we notice that even for \( k = 0 \), the power of \( \alpha_1 \) is strictly positive, and therefore the moment disappears.

### 3.3 Remarks on \( \beta = 1 \)

For \( \beta = 1 \), the limiting moments can be calculated as follows:

\[
\frac{1}{N^{3K/4} N_2^{3L/4}} \sum_{k=0}^{K} \sum_{i \in \Pi_k(k)} \sum_{l=0}^{L} \sum_{j \in \Pi_l(l)} w_K(i)w_L(j) \mathbb{E}(X_i Y_j). \tag{24}
\]

**Theorem 14.** Let \( \beta = 1 \). Then the expression in (24) is asymptotically

\[
12 \frac{k+l+1}{4} \frac{\Gamma(K+L+1)}{\Gamma(\frac{1}{4})} \frac{1}{\alpha_1^{K} \alpha_2^{L}}. \tag{25}
\]

**Proof.** We calculate for any \( i \in \Pi_k(k), j \in \Pi_l(l) \):

\[
|\mathbb{E}(X(i, j))| \leq c \frac{1}{N^{k+l+1/4}},
\]

\[
w_K(i)w_L(j) \leq cN_1^{k/2} N_2^{l/2}.
\]

The symbol \( c \) in the above inequalities stands for some constant (not necessarily the same in both lines).

Therefore, each summand is bounded above by

\[
\frac{1}{N^{3K/4} N_2^{3L/4}} w_K(i)w_L(j) |\mathbb{E}(X(i, j))| \leq c \frac{1}{N^{3K/4} N_2^{3L/4}} N_1^{k/2} N_2^{l/2} \frac{1}{N^{k+l+1/4}} = c \alpha_1^{k/4} \alpha_2^{l/4} N_1^{-(K-k)} N_2^{-(L-l)},
\]

which goes to 0 as \( N \to \infty \) if \( K > k \) or \( L > l \).

Just as in the one-dimensional case, the only summand that matters asymptotically is the one where both \( k = K, l = L \) hold. In this case, we have
\[ w_K(i) \approx N_1^K, \]
\[ w_L(j) \approx N_2^L \]

and

\[ \mathbb{E}(X(i,j)) \approx 12 \frac{K+L+1}{\Gamma(\frac{1}{4})} \frac{1}{N^{\frac{K+L+1}{4}}} \]

provided that \( K + L \) is even and 0 otherwise. Multiplying these, we obtain

\[ \frac{1}{N_1^{K/4}N_2^{L/4}} w_K(i)w_L(j)\mathbb{E}(X(i,j)) \approx 12 \frac{K+L+1}{\Gamma(\frac{1}{4})} \frac{1}{N^{\frac{K+L+1}{4}}} \alpha_1^{K/4} \alpha_2^{L/4}. \]

This law of large numbers also implies that the central limit theorem cannot hold for \( \beta \geq 1 \). However, for \( \beta = 1 \), there is a bounded measure \( \mu \) with the moments given by (25).

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