Vertex Operator Superalgebras and Their Representations \*†

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0 Introduction

Vertex operator algebras (VOA) were introduced in physics by Belavin, Polyakov and Zamolodchikov [BPZ] and in mathematics by Borcherds [B]. For a detailed exposition of the theory of VOAs see [FLM] and [FHL]. In a remarkable development of the theory, Zhu [Z] constructed an associative algebra $A(V)$ corresponding to a VOA $V$ and established a 1-1 correspondence between the irreducible representations of $V$ and those of $A(V)$. Furthermore, Frenkel and Zhu [FZ] defined an $A(V)$-module $A(M)$ for any $V$-module $M$ and then described the fusion rules in terms of the modules $A(M)$. An important feature of these constructions is that $A(V)$ and $A(M)$ can usually be computed explicitly. For example, they enabled Frenkel and Zhu to prove the rationality and compute the fusion rules of

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VOAs associated to the representations of affine Kac-Moody algebras with a positive integral level. They also allowed one of the authors [W] to prove the rationality and compute the fusion rules of VOAs associated to the minimal series representations of the Virasoro algebra. (Independently, Dong, Mason and Zhu [DMZ] proved the rationality for the unitary minimal series of the Virasoro algebra and calculated the fusion rules in the case of central charge \( c = \frac{1}{2} \)).

In this paper we generalize Frenkel-Zhu’s construction to vertex operator superalgebras (SVOA) and then discuss in detail several interesting classes of SVOAs. We present explicit formulas for the “top” singular vectors and defining relations for the integrable representations of the affine Kac-Moody superalgebras. These formulas are not only crucial for the theory of the associated SVOAs and their modules, but also of independent interest.

We organize this paper in the following way. In Subsec.1.1 we present definitions of vertex operator superalgebras and their modules, emphasizing the existence of the Neveu-Schwarz element in the so-called \( N = 1 \) (NS-type) SVOAs. We define in Subsec.1.2 an associative algebra \( A(V) \) corresponding to a SVOA \( V \) and establish a bijective correspondence between the irreducible representations of \( V \) and the irreducible representations of \( A(V) \). In Subsec.1.3 we define an \( A(V) \)-module \( A(M) \) for every \( V \)-module \( M \) and then describe the fusion rules in terms of modules \( A(M) \). Needless to say that, if we view a VOA as a SVOA with zero odd part, then our construction reduces to Frenkel-Zhu’s original one.

In Subsec.2.1 we construct \( N = 1 \) SVOAs \( M_{k,0} \) and \( L_{k,0} \) corresponding to the representations of an affine Kac-Moody superalgebra \( \hat{g} \). In [KT], the minimal representation \( L(h^\vee \Lambda_0) \) of \( \hat{g} \) was realized in
a Fock space $F$ of a certain infinite-dimensional Clifford algebra contained in $\hat{\mathfrak{g}}$. Kac and Todorov [KT] proved that any unitary highest weight representation of $\hat{\mathfrak{g}}$ is of the form $L(\Lambda + h^\vee \Lambda_0) = F \otimes \bar{L}(\Lambda)$, where $\bar{L}(\Lambda)$ is the irreducible unitary highest weight representation of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$. Explicit formulas for the “top” singular vectors of the Verma module $M(\Lambda + h^\vee \Lambda_0)$ of $\hat{\mathfrak{g}}$ and the defining relations of $L(\Lambda + h^\vee \Lambda_0)$ are presented in detail in the Appendix. With the help of the theory developed in Sec.1, we prove in Subsec.2.2 that the SVOA $L_{k,0}$ is rational for positive integral $k$ and that the representations and fusion rules for the SVOA $L_{k,0}$ are in 1-1 correspondence with those for the VOA $\bar{L}_{k,0}$.

In Subsec.3.1 we construct $N = 1$ SVOAs $M_c$ and $V_c$ corresponding to the representations of the Neveu-Schwarz algebra. We then discuss the rationality and the fusion rules of $V_c$.

In Sec.4 we construct the SVOAs generated by charged and neutral free fermionic fields. We prove that such an SVOA is rational and has a unique irreducible representation, namely itself.

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1 General constructions and theorems

1.1 Definitions

For a rational function $f(z, w)$, with possible poles only at $z = w, z = 0$ and $w = 0$, we denote by $\iota_{z,w} f(z, w)$ the power series expansion of $f(z, w)$ in the domain $|z| > |w|$. Set $Z_+ = \{0, 1, 2, \cdots\}, \mathbb{N} = \{1, 2, 3, \cdots\}$. 
A superalgebra is an algebra $V$ with a $\mathbb{Z}_2$-gradation $V = V_0 \oplus V_1$. Elements in $V_0$ (resp. $V_1$) are called even (resp. odd). Let $\tilde{a}$ be 0 if $a \in V_0$, and 1 if $a \in V_1$. The general principle to extend identities in VOAs to SVOAs is the usual one: if in certain formulas of VOAs there are some monomials of vertex operators with interchanged terms, then in the corresponding formulas in SVOAs every interchange of neighboring terms, say $a$ and $b$, is accompanied by multiplication of the monomial by the factor $(-1)^{\tilde{a}\tilde{b}}$.

**Definition 1.1** A vertex operator superalgebra is a $\frac{1}{2}\mathbb{Z}_+\text{-graded vector space}$ $V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} V_n$ with a sequence of linear operators $\{a(n) \mid n \in \mathbb{Z}\} \subset \text{End } V$ associated to every $a \in V$, whose generating series $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]$, called the vertex operators associated to $a$, satisfy the following axioms:

(A1) $Y(a, z) = 0$ iff $a = 0$.

(A2) There is a vacuum vector, which we denote by 1, such that

$$Y(1, z) = I_V \quad (I_V \text{ is the identity of } \text{End } V).$$

(A3) There is a special element $\omega \in V$ (called the Virasoro element), whose vertex operator we write in the form

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega(n)z^{-n-1} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

such that

$$L_0 \mid_{V_n} = nI \mid_{V_n},$$

$$Y(L_{-1}a, z) = \frac{d}{dz} Y(a, z) \quad \text{for every } a \in V,$$

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c,$$

where $c$ is some constant in $\mathbb{C}$, which is called the rank of $V$. 
The Jacobi identity holds, i.e.

\[
\text{Res}_{z-w}(Y(Y(a, z-w)b, w)t_{w, z-w}((z-w)^m z^n))
\]

\[
= \text{Res}_z(Y(a, z)Y(b, w)t_{z, w}(z-w)^m z^n)
\]

\[
-(-1)^{\tilde{a}\tilde{b}}\text{Res}_z(Y(b, w)Y(a, z)t_{w, z}(z-w)^m z^n)
\]

for any \(m, n \in \mathbb{Z}\).

An element \(a \in V\) is called homogeneous of degree \(n\) if \(a\) is in \(V_n\).

In this case we write \(\deg a = n\).

Define a natural \(\mathbb{Z}_2\)-gradation of \(V\) by letting

\[
V = V_0 + V_1, \quad V_0 = \bigoplus_{n \in \mathbb{Z}_+} V_n, \quad V_1 = \bigoplus_{n \in \frac{1}{2} + \mathbb{Z}_+} V_n.
\]

\(V = V_0 + V_1\). \(V_0\) (resp. \(V_1\)) is called the even (resp. odd) part of \(V\).

Elements in \(V_0\) (resp. \(V_1\)) are called even (resp. odd).

We now introduce the notion of an \(N = 1\) SVOA.

**Definition 1.2** \(V\) is called an \(N = 1\) (NS-type) SVOA if axiom \((A3)\) is replaced by the following stronger axiom:

\((A3')\) There is a special element \(\tau \in V\) (called the Neveu-Schwarz element), whose corresponding vertex operator we write in the form

\[
Y(\tau, z) = \sum_{n \in \mathbb{Z}} \tau(n)z^{-n-1} = \sum_{n \in \mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2},
\]

such that the element \(\omega := \frac{1}{2}G_{-\frac{1}{2}}\tau\) satisfies \((A3)\), and the commutation relations

\[
\left[G_{m+\frac{1}{2}}, L_n\right] = (m + \frac{1}{2} - \frac{n}{2})G_{m+n+\frac{1}{2}},
\]

\[
\left[G_{m+\frac{1}{2}}, G_{n+\frac{1}{2}}\right]_+ = 2L_{m+n} + \frac{1}{3}m(m+1)\delta_{m+n,0}c, \quad m, n \in \mathbb{Z}
\]
also hold.

We list some properties of SVOAs which are analogous to those in the VOA case. For more detail see [FLM].

\begin{equation}
[a(n), Y(b, z)]_\pm = \sum_{i \geq 0} \binom{n}{i} z^{n-i} Y(a(i)b, z),
\end{equation}

\begin{equation}
[L_0, Y(a, z)] = (z \frac{d}{dz} + \text{deg } a)Y(a, z),
\end{equation}

\begin{equation}
[L_{-1}, Y(a, z)] = \frac{d}{dz} Y(a, z),
\end{equation}

\begin{equation}
a(n)V_m \subset V_{m+\text{deg } a-n-1},
\end{equation}

\begin{equation}
Y(a, z)1 = e^z L_{-1} a,
\end{equation}

\begin{equation}
Y(a, z)b = (-1)^{\tilde{a} \tilde{b}} e^z L_{-1} Y(b, -z) a,
\end{equation}

\begin{equation}
a(n)1 = 0, \text{ for } n \geq 0,
\end{equation}

\begin{equation}
a(-n-1)1 = \frac{1}{n!} L_{-1}^n a \text{ for } n \geq 0.
\end{equation}

Moreover, \( N = 1 \) SVOAs have the extra property that:

\begin{equation}
[G_{-\frac{1}{2}}, Y(a, z)]_\pm = Y(G_{-\frac{1}{2}} a, z).
\end{equation}

**Definition 1.3** Given an SVOA \( V \), a representation of \( V \) (or \( V \)-module) is a \( \frac{1}{2} \mathbb{Z}_+ \)-graded vector space \( M = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}_+} M_n \) and a linear map

\begin{equation}
V \rightarrow (\text{End } M)[[z, z^{-1}]],
\end{equation}

\begin{equation}
a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1},
\end{equation}

satisfying

\begin{equation}
(R1) \ a(n)M_m \subset M_{m+\text{deg } a-n-1} \text{ for every homogeneous element } a.
\end{equation}
(R2) \( Y_M(1, z) = I_M \), and setting \( Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \), we have
\[
[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} c,
\]
\( Y_M(L_{-1} a, z) = \frac{d}{dz} Y_M(a, z) \) for every \( a \in V \).

(R3) The Jacobi identity holds, i.e.
\[
\text{Res}_{z-w}(Y_M(Y(a, z-w)b, w)\iota_{w,z-w}((z-w)^m z^n)) = \text{Res}_z(Y_M(a, z)Y_M(b, w)\iota_{z,w}(z-w)^m z^n)
\]
\[= (-1)^{\tilde{a} \tilde{b}} \text{Res}_z(Y_M(b, w)Y_M(a, z)\iota_{w,z}(z-w)^m z^n) \]
for any \( m, n \in \mathbb{Z} \).

**Definition 1.4** Given an \( N = 1 \) SVOA \( V \), \( M \) is called a representation of \( V \) if axiom (R2) is replaced by the following stronger axiom:

(R2') Set \( Y_M(\tau, z) = \sum_{n \in \mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2} \) and \( \omega := \frac{1}{2} G_{-\frac{1}{2}} \tau \). Then \( \omega \) satisfies (R2), and the commutation relations
\[
[G_{m+\frac{1}{2}}, L_n] = (m + \frac{1}{2} - \frac{n}{2}) G_{m+n+\frac{1}{2}},
\]
\[
[G_{m+\frac{1}{2}}, G_{n+\frac{1}{2}}] = 2L_{m+n} + \frac{1}{3} m(m+1) \delta_{m+n,0} c, m, n \in \mathbb{Z}
\]
also hold.

The notions of submodules, quotient modules, submodules generated by a subset, direct sums, irreducible modules, completely reducible modules, etc., can be introduced in the usual way. As a module over itself, \( V \) is called the adjoint module. A submodule of the adjoint module is called an ideal of \( V \). Given an ideal \( I \) in \( V \) such that \( 1 \notin I, \omega \notin I \), the quotient \( V/I \) admits a natural SVOA structure.
Definition 1.5 A SVOA is called rational if it has finitely many irreducible modules and every module is a direct sum of irreducibles.

We will now extend the definition of intertwining operators and fusion rules of representations of VOAs ([FHL]) to SVOAs.

For simplicity, we will only define an intertwining operator for $V$-modules $M^i = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}} M^i(n)$, $(i = 1, 2, 3)$, satisfying $L_0 |_{M^i(n)} = (h_i + n) I |_{M^i(n)}$, for some complex numbers $h_1, h_2, h_3$. We define a $\mathbb{Z}_2$-gradation of $M^i$ by letting $\tilde{v} = 0$ if $v \in M^i(n), n \in \mathbb{Z}$; $\tilde{v} = 1$ if $v \in M^i(n)$, $n \in \frac{1}{2} + \mathbb{Z}$.

Definition 1.6 Under the above assumptions, an intertwining operator of type $\left( M^1^M^3 \ M^2 \right)$ is a linear map

$I(\cdot, z) : v \mapsto \sum_{k \in I} v(n) z^{-n-1+(h_3-h_1-h_2)}$, $v \in M^1$, $v(n) \in Hom_C(M^2, M^3)$

satisfying

(I1) For homogeneous $v \in M^1$,

$$ I(L_{-1}v, z) = \frac{d}{dz} I(v, z) \text{ for every } v \in M^1, $$

(I2) For any $a \in V$, $v \in M^1$, and $m, n \in \mathbb{Z}$,

$$ Res_{z-w}(I(Y(a, z-w)v, w) \iota_{w,z-w}((z-w)^m z^n)) $$

$$ = Res_z(Y(a, z)I(v, w) \iota_{z,w}(z-w)^m z^n) $$

$$ - (-1)^{\tilde{v}} Res_z(I(v, w)Y(a, z) \iota_{w,z}(z-w)^m z^n). $$

We denote by $I\left( M^1^M^3 \ M^2 \right)$ the vector space of intertwining operators of type $\left( M^1^M^3 \ M^2 \right)$.
An immediate consequence of this definition is that for homogeneous \( v \in M^1 \),
\[
v(n)M^2_m \subset M^3_{m+\deg v-n-1},
\]
where \( \deg v = k \) means that \( v \in M^1_k \).

We now assume that \( V \) is a rational SVOA and \( \{M^i, \ i \in J\} \) is the complete set of the irreducible modules of \( V \). Denote by \( N^{ij}_k \) the dimension of the vector space \( I(M^i_{M^j_{M^k}}) \). We define the fusion rules as the formal product rules
\[
M^i \times M^j = \sum_{k \in J} N^{ij}_k M^k.
\]

### 1.2 The associative algebra \( A(V) \) and related theorems

**Definition 1.7** We define bilinear maps \( \ast : V \times V \to V \), \( \circ : V \times V \to V \) as follows. For homogeneous \( a, b \), let
\[
a \ast b = \begin{cases} 
\text{Res}_z \left(Y(a,z)\frac{(z+1)^{\deg a}}{z}b\right), & \text{if } a, b \in V_0, \\
0, & \text{if } a \text{ or } b \in V_1.
\end{cases}
\]
\[
a \circ b = \begin{cases} 
\text{Res}_z \left(Y(a,z)\frac{(z+1)^{\deg a}}{z}b\right), & \text{for } a \in V_0 \\
\text{Res}_z \left(Y(a,z)\frac{(z+1)^{\deg a - \frac{1}{2}}}{z}b\right), & \text{for } a \in V_1.
\end{cases}
\]

Extend to \( V \times V \) by linearity, denote by \( O(V) \subset V \) the linear span of elements of the form \( a \circ b \), and by \( A(V) \) the quotient space \( V/O(V) \).

**Remark 1.1**

1) \( O(V) \) is a \( \mathbb{Z}_2 \)-graded subspace of \( V \).

2) If \( a \in V_1 \), then
\[
a \circ 1 = \text{Res}_z \left(Y(a,z)\frac{(z+1)^{\deg a - \frac{1}{2}}}{z}1\right) = a.
\]
Hence $O(V) = O_0(V) + V_1$, where $O_0(V) = O(V) \cap V_0$. Thus $A(V) = V_0/O_0(V)$. Denote by $O_e(V)$ (resp. $O_d(V)$) the linear span of the elements $a \circ b$ for $a, b \in V_0$ (resp. $V_1$). The intersection $O_e(V) \cap O_d(V)$ need not be empty.

It is convenient to introduce an equivalence relation $\sim$ as follows. For $a, b \in V$, $a \sim b$ means $a - b \equiv 0 \mod O(V)$. For $f, g \in \text{End} V$, $f \sim g$ means $f \cdot c \sim g \cdot c$ for any $c \in V$. Let $[a]$ denote the image of $a$ in $V$ under the projection of $V$ onto $A(V)$.

**Lemma 1.1**

1) $L_{-1}a + L_0a \sim 0$ if $a \in V_0$.

2) For every homogeneous element $a \in V$, and $m \geq n \geq 0$, one has
   $$\text{Res}_z \left( (Y(a, z) \frac{(z + 1)^{\text{deg} a + n}}{z^{2+m}}) \right) \sim 0, \text{ if } a \in V_0.$$  $$\text{Res}_z \left( (Y(a, z) \frac{(z + 1)^{\text{deg} a + n - \frac{1}{2}}}{z^{1+m}}) \right) \sim 0, \text{ if } a \in V_1.$$  

3) For any homogeneous element $a, b \in V_0$, one has
   $$a \ast b \sim \text{Res}_z \left( Y(b, z) \frac{(z + 1)^{\text{deg} b - 1}}{z} a \right).$$

**Proof.** Noting that $V_0$ is a vertex operator algebra, we see that 1), the first part of 2) and 3) are the same as Lemma 2.1.1, 2.1.2 and 2.1.3 in [Z]. The proof of the second part of 2) is similar to that of the first part. 

The following theorem is an analog of Theorem 2.1.1 in [Z].

**Theorem 1.1**

1) $O(V)$ is a two-sided ideal of $V$ under the multiplication $\ast$. Moreover, the quotient algebra $(A(V), \ast)$ is associative.
2) $[1]$ is the unit element of the algebra $A(V)$.

3) $[\omega]$ is in the center of $A(V)$.

4) $A(V)$ has a filtration $A_0(V) \subset A_1(V) \subset \cdots$, where $A_n(V)$ is the image of $\bigoplus_{i \in \frac{1}{2}\mathbb{Z}_+, i \leq n} V_i$.

**Sketch of a proof.** To prove 1), it is enough to prove the following relations:

$$O_0(V) \ast V \subset O(V),$$

$$V_0 \ast O_0(V) \subset O(V),$$

$$(a \ast b) \ast c - a \ast (b \ast c) \in O(V).$$

By the definition of the operation $\ast$ and Remark 1.1, it suffices to prove that for homogeneous $a, b, c$ one has

$$a \circ b \ast c \subset O(V) \text{ for } a, b, c \in V_0,$$

$$a \ast (b \circ c) \subset O(V) \text{ for } a, b, c \in V_0,$$

$$a \ast (b \circ c) \subset O(V) \text{ for } a, b \in V_1,$$

$$a \ast (b \circ c) \subset O(V) \text{ for } a \in V_0, b, c \in V_1,$$

$$a \ast (b \circ c) \subset O(V) \text{ for } a, b \in V_1, c \subset V_0,$$

The proofs of (1.6), (1.7) and (1.10) are the same as in the VOA cases (see the proof of Theorem 2.1.1 in [Z]).

To prove (1.8), for $a, b \in V_1, c \in V$ homogeneous, we have

$$(a \circ b) \ast c$$

$$= \text{Res}_z (Y(a, z)^{(z+1)\deg a - \frac{1}{2}} b) \ast c$$
\[ \deg a^{-\frac{1}{2}} \sum_{i=0}^{\deg a^{-\frac{1}{2}}} (\deg a^{-\frac{1}{2}}) (a(i - 1)b) \ast c \]

\[ = \sum_{i=0}^{\deg a^{-\frac{1}{2}}} \left( \deg a^{-\frac{1}{2}} \right) \text{Res}_w \left( Y(a(i - 1)b, w) \frac{(w + 1)^{\deg a + \deg b - i}}{w} c \right) \]

\[ = \sum_{i=0}^{\deg a^{-\frac{1}{2}}} \left( \deg a^{-\frac{1}{2}} \right) \times \text{Res}_w, \text{Res}_z - w \left( Y(a(i - 1)b, w) (z - w)^{i - 1} \frac{(w + 1)^{\deg a + \deg b - i}}{w} c \right) \]

By Lemma 1.1 the right hand side of the last identity is in \( O(V) \).

To prove (1.9), for \( a \in V_0, b, c \in V_1 \) homogeneous, we have

\[ a \ast (b \circ c) - b \circ (a \ast c) = \text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\deg a}}{z} \right) \text{Res}_w \left( Y(b, w) \frac{(w + 1)^{\deg b - \frac{1}{2}}}{w} c \right) \]
\[- \text{Res}_w \left( Y(b, w) \frac{(w + 1)^{\deg b - \frac{1}{2}}}{w} \right) \text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\deg a}}{z} \right) c \]

\[= \text{Res}_w \text{Res}_{z-w} \left( Y(Y(a, z-w)b, w) \frac{(z + 1)^{\deg a} (w + 1)^{\deg b - \frac{1}{2}}}{w} \right) \]

\[= \sum_{i=0}^{\deg a} \sum_{j \in \mathbb{Z}_+} \binom{\deg a}{i} \text{Res}_w \left( Y(a(i + j)b, w) (-1)^j \frac{(w + 1)^{\deg a + \deg b - i - \frac{1}{2}}}{w^{j+2}} \right). \]

Since $\deg(a(i+j)b) = \deg a + \deg b - i - j - 1$, and $a(i+j)b \in V_1$, by Lemma 1.1, the right-hand side of the last identity is in $O(V)$. The second term of the left-hand side is also in $O(V)$ by definition. Then so is the first term.

The proof of statements 2), 3) and 4) is the same as in the VOA case. (For details see the proof of Theorem 2.1.1 in [Z]). \(\square\)

The following proposition follows from the definition of $A(V)$.

**Proposition 1.1** Let $I$ be an ideal of $V$ with the $\mathbb{Z}_2$-gradation $I_0 \oplus I_1$ consistent with that of $V$. Assume $1 \notin I$, $\omega \notin I$. Then the associative algebra $A(V/I)$ is isomorphic to $A(V)/[I]$, where $[I]$ is the image of $I$ in $A(V)$.

For any homogeneous $a \in V_0$ we define $o(a) = a(\deg a - 1)$ and extend this map linearly to $V_0$. It follows from (1.3) that $o(a)M_n \subset M_n$. In particular, $o(a)$ maps $M_0$ into itself. We may assume that $M_0 \neq 0$ without loss of generality.

**Theorem 1.2** Let $M = \bigoplus_{n \in \mathbb{Z}_+} M_n$ be a $V$-module. Then $M_0$ is an $A(V)$-module defined as follows: for $[a] \in A(V)$, let $a \in V_0$ be a preimage of $[a]$. Then $[a]$ acts on $M_0$ as $o(a)$.

**Proof.** An equivalent way to state this theorem is that for $a, b \in V_0$, $o(a)o(b) \mid M_0 = o(a \ast b) \mid M_0$, and for $c \in O(V) = O_d(V) + O_e(V)$,
\( o(c) |_{M_0} = 0 \). We only need to prove that \( o(c) |_{M_0} = 0 \) for \( c \in O_d(V) \) since \( V_0 \) is a vertex operator algebra and so the rest of the statements above holds (For details see Theorem 2.1.2 and its proof in [Z]).

Given \( a, b \in V_1 \) homogeneous, we have

\[
o(a \circ b) = o(Res_z(Y(a, z) \frac{(z+1)^{\deg a - \frac{1}{2}}}{z} b))
\]

\[
= \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} o(a(i-1)b)
\]

\[
= \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} (a(i-1)b) (\deg a + \deg b - i - 1)
\]

\[
= Res_w Res_{z-w} \sum_{i=0}^{\deg a - \frac{1}{2}} \binom{\deg a - \frac{1}{2}}{i} \times
\]

\[
\times (Y((a, z-w)b, w)(z-w)^{i-1} w^{\deg a + \deg b - i - 1})
\]

\[
= Res_w Res_{z-w} (Y((a, z-w)b, w) \frac{z^{\deg a - \frac{1}{2}} w^{\deg b - \frac{1}{2}}}{z-w})
\]

\[
= Res_z Res_w \left( Y(a, z) Y(b, w) \frac{z^{\deg a - \frac{1}{2}} w^{\deg b - \frac{1}{2}}}{z-w} \right)
\]

\[
+ Res_w Res_z \left( Y(b, w) Y(a, z) \frac{z^{\deg a - \frac{1}{2}} w^{\deg b - \frac{1}{2}}}{z-w} \right)
\]

\[
= \sum_{i \in \mathbb{Z}_+} Res_z Res_w \left( Y(a, z) Y(b, w) z^{\deg a - i - \frac{3}{2}} w^{\deg b - i - \frac{3}{2}} \right)
\]

\[
- \sum_{i \in \mathbb{Z}_+} Res_w Res_z \left( Y(b, w) Y(a, z) z^{\deg a + i - \frac{3}{2}} w^{\deg b - i - \frac{3}{2}} \right)
\]

\[
= \sum_{i \in \mathbb{Z}_+} a(\deg a - i - \frac{3}{2}) b(\deg b + i - \frac{1}{2})
\]

\[
- \sum_{i \in \mathbb{Z}_+} b(\deg b - i - \frac{3}{2}) a(\deg a + i - \frac{1}{2}).
\]
The right-hand side of the above identities acting on $M_0$ is 0 since
$$a(\deg a + i - \frac{1}{2}) |_{M_0} = b(\deg b + i - \frac{1}{2}) |_{M_0} = 0.$$  

\[\square\]

**Theorem 1.3** Given an $A(V)$-module $(W, \pi)$, there exists a $V$-module $M = \bigoplus_{n \in \frac{1}{2}Z_+} M_n$ such that the $A(V)$-modules $M_0$ and $W$ are isomorphic. Moreover, this gives a bijective correspondence between the set of irreducible $A(V)$-modules and the set of irreducible $V$-modules.

**Sketch of a proof.** First we have the following recurrent formula for $n$-correlation functions on $\langle M_0, (M_0)^* \rangle$ for a given $V$-module $M = \bigoplus_{i \in \frac{1}{2}Z_+} M_i$, where $M_0^*$ is the dual space of $M_0$. (The proof is similar to that of Lemma 2.2.1 in [Z].) Given $v \in M_0$, $v' \in M_0^*$, and homogeneous $a_1 \in V$, we have

$$\langle v', Y(a_1, z_1)Y(a_2, z_2) \cdots Y(a_m, z_m) v \rangle = \begin{cases} \sum_{k=2}^{m} \sum_{i \in Z_+} (-1)^{\tilde{a}_2 + \cdots + \tilde{a}_{k-1}} F_{\deg a_1 - \frac{1}{2}i}(z_1, z_k) \times \times \langle v', Y(a_2, z_2) \cdots Y(a_1(i)a_k, z_k) \cdots Y(a_m, z_m) v \rangle & \text{if } a_1 \in V_1, \\ \sum_{k=2}^{m} \sum_{i \in Z_+} F_{\deg a_1, i}(z_1, z_k) \times \times \langle v', Y(a_2, z_2) \cdots Y(a_1(i)a_k, z_k) \cdots Y(a_m, z_m) v \rangle + z_1^{-\deg a_1} \langle a_1(\deg a_1 - 1)^* v', Y(a_2, z_2) \cdots Y(a_m, z_m) v \rangle & \text{if } a_1 \in V_0, \end{cases}$$

where $F_{\deg a_1, i}$ is defined by

$$F_{n, i}(z, w) = \sum_{j \in Z_+} \binom{n+j}{i} z^{-n-j} w^{n+j-i} = t_{z, w} \left( z^{-n} \frac{1}{i!} \left( \frac{d^n}{dw^n} \right) \frac{w^n}{z-w} \right).$$

This recurrent formula means that the $n$-correlation functions on $\langle M_0, (M_0)^* \rangle$ are determined by the $A(V)$-module structure on $M_0$. The completion of the proof of this theorem is similar to that in Theorem 2.2.1 in [Z].  

\[\square\]
Remark 1.2  Thus we have a functor from the category of $V$-modules to the category of $A(V)$-modules which is bijective on the sets of irreducibles.

1.3  Fusion rules

In this subsection, to generalize the construction of [FZ], we define a bimodule $A(M)$ of $A(V)$ for every $V$-module $M$. We then give a description of the fusion rules in terms of $A(M)$. The proofs are only sketched.

Definition 1.8  For a $V$-module $M$, we define bilinear operations $a \ast v$ and $v \ast a$, for $a \in V$ homogeneous and $v \in M$, as follows

\[(1.11) \quad a \ast v = \text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\deg a}}{z^2} v \right), \quad \text{for } a \in V_0, \]

\[(1.12) \quad v \ast a = \text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\deg a - 1}}{z} v \right), \quad \text{for } a \in V_0, \]

\[a \ast v = 0, \quad v \ast a = 0, \quad \text{for } a \in V_1, \]

and extend linearly to $V$. We also define $O(M) \subset M$ to be the linear span of elements of the forms

\[\text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\deg a}}{z^2} v \right), \quad \text{for } a \in V_0 \text{ and} \]

\[\text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\deg a - 1}}{z} v \right), \quad \text{for } a \in V_1. \]

Let $A(M)$ be the quotient space $M/O(M)$.

We have the following theorem which is an analogue of Theorem 1.5.1 in [Z].
Theorem 1.4 $A(M)$ is an $A(V)$-bimodule with the left action of $A(V)$ defined by (1.11) and the right action by (1.12). Moreover the left and right action of $A(V)$ commute with each other.

Sketch of a proof. By a similar argument to Lemma 1.1, we see that
\[ \text{Res}_z(Y(a, z) \frac{(z + 1)^{\deg a + n}}{z^{2+m}} v) \in O(M), \text{ for } a \in V_0, \]
\[ \text{Res}_z(Y(a, z) \frac{(z + 1)^{\deg a + n - \frac{1}{2}}}{z^{1+m}} v) \in O(M), \text{ for } a \in V_1, \]
for $m \geq n \geq 0$, $v \in M$.

Recall that $O(V) = O_d(V) + O_e(V)$. To prove the theorem, we need to check that
\[ O_d(V) \ast v \subset O(M), \quad v \ast O_d(V) \subset O(M), \tag{1.13} \]
\[ O_e(V) \ast v \subset O(M), \quad v \ast O_e(V) \subset O(M), \tag{1.14} \]
\[ a \ast O(M) \subset O(M), \quad O(M) \ast a \subset O(M), \tag{1.15} \]
\[ (a \ast b) \ast v - a \ast (b \ast v) \in O(M), \tag{1.16} \]
\[ (v \ast a) \ast b - v \ast (a \ast b) \in O(M), \tag{1.17} \]
\[ (a \ast v) \ast b - a \ast (v \ast b) \in O(M). \tag{1.18} \]

The proof of (1.13) is similar to that of Theorem 1.1. The proofs of (1.14), (1.15), (1.16), (1.17) and (1.18) are similar to those in [Z]. □

Consider left $V$-modules $M^i = \bigoplus_{n \in \mathbb{Z}_+} M^i(n)$, $i = 1, 2, 3$. Note that $M^2(0)$ is a left module over $A(V)$, $(M^3(0))^*$ is a right module over $A(V)$, and $A(M^1)$ is a bimodule over $A(V)$. Hence we can consider the tensor product $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$ of $A(V)$-modules.
The following theorem is an analogue of Theorems 1.5.2 and 1.5.3 in [FZ].

**Theorem 1.5** Let \( M^i = \sum_{n \in \frac{1}{2} \mathbb{Z}_+} M^i(n)(i = 1, 2, 3) \) be \( V \)-modules, satisfying \( L_0 \mid_{M^i(n)} = (h_i + n)I \mid_{M^i(n)} \), for some complex numbers \( h_1, h_2, h_3 \).

1) Let \( I (\cdot, z) \) be an intertwining operator of type \( \begin{pmatrix} M^3 & M^1 \\ M^2 & \end{pmatrix} \). Then \( \langle v'_3, o(v_1)v_2 \rangle \) defines a linear functional \( f_I \) on \( M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0) \), where \( v'_3 \in M^3(0)^*, v_1 \in M^1, v_2 \in M^2 \).

2) The map \( I \mapsto f_I \) given in 1) defines an isomorphism of vector spaces \( I(\begin{pmatrix} M^3 & M^1 \\ M^2 & \end{pmatrix}) \) and \( (M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^* \) if \( M^i (i = 1, 2, 3) \) are irreducible.

**Proof.** The argument is similar to that in Theorem [FZ]. \( \square \)

As a consequence, we obtain the following proposition, which is an analogue of Proposition 1.5.4 in [FZ].

**Proposition 1.2**

1) Given a \( V \)-module \( M \) and a submodule \( M^1 \) of \( M \), then the image \( A(M^1) \) of \( M^1 \) in \( A(M) \), is a submodule of \( A(V) \)-bimodule \( A(M) \), and the quotient \( A(M)/A(M^1) \) is isomorphic to the bimodule \( A(M/M^1) \) corresponding to the quotient \( V \)-module \( M/M^1 \).

2) If \( I \) is an ideal of \( V \), \( 1 \notin I, \omega \notin I \), and \( I \cdot M \subset M^1 \), then \( A(V/I)\)-bimodule \( A(M)/A(M^1) \) is isomorphic to the \( A(M/M^1) \).

**Remark 1.3** One can also consider the pre-SVOA (i.e. the SVOA which may not admit a Virasoro element). Similarly to the VOA...
case, one can still define the associative algebra $A(V)$ and the $A(V)$-module $A(M)$ for any $V$-module $M$ [L]. Theorems 1.3 and 1.5 are valid for the pre-SVOAs.

2 SVOA associated to representations of affine Kac-Moody superalgebras

2.1 SVOA structures on $M_{k,0}$ and $L_{k,0}$

In this subsection, we construct the SVOAs associated to representations of affine Kac-Moody superalgebras which are analogous to the construction of VOAs associated to representations of affine algebras [FZ]. First let us recall some basic notions of affine Kac-Moody (super)algebras. Given a simple finite-dimensional Lie algebra $\mathfrak{g}$ of rank $l$ over $\mathbb{C}$, we fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Delta \subset \mathfrak{h}^*$ and a set of positive roots $\Delta_+ \subset \Delta$. Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$. Let $e_i, f_i, h_i (i = 1, \cdots, l)$ be the corresponding Chevalley generators. Denote by $\theta$ the highest root and normalize the Killing form

$$ (, ) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C} $$

by the condition $(\theta, \theta) = 2$. Let $\sigma$ be the antilinear anti-involution of $\mathfrak{g}$. We choose $f_\theta \in \mathfrak{g}_{-\theta}$ so that $(f_\theta, \sigma(f_\theta)) = 1$, and set $e_\theta = \sigma(f_\theta)$. We denote by $r_\alpha$ the reflection with respect to $\alpha \in \Delta$ in the Weyl group $W \in GL(\mathfrak{h})$ of $\mathfrak{g}$.

The affine Kac-Moody superalgebra (of NS type) is then defined by

$$ \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}, \xi] \oplus \mathbb{C}k \oplus \mathbb{C}d $$
with the following commutation relations

(2.1) \[[a(m), b(n)] = [a, b](m + n) + m\delta_{m+n,0}(a, b)k,\]

(2.2) \[[\bar{a}(m), \bar{b}(n)]_+ = \delta_{m+n+1,0}(a, b)k,\]

(2.3) \[[a(m), \bar{b}(n)] = \bar{a}(m + n),\]

(2.4) \[[k, a(m)] = 0,\]

(2.5) \[[d, a(m)] = ma(m),\]

(2.6) \[[d, \bar{a}(m)] = (m + \frac{1}{2})\bar{a}(m),\]

where \(a, b \in \mathfrak{g}\), \(m, n \in \mathbb{Z}\), \(a(m) := a \otimes t^m\), \(\bar{a}(m) := a \otimes \xi t^m\).

Let

\[\hat{\mathfrak{g}} = \mathfrak{g} \otimes \xi\]

\[\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\mathbb{C}[t] \bigoplus \bar{\mathfrak{g}} \otimes \mathbb{C}[t]\]

\[\hat{\mathfrak{g}}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \bigoplus \bar{\mathfrak{g}} \otimes t^{-1}\mathbb{C}[t^{-1}].\]

Then \(\hat{\mathfrak{g}}_+\) and \(\hat{\mathfrak{g}}_-\) are subalgebras of \(\hat{\mathfrak{g}}\) and

\[\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \bigoplus \hat{\mathfrak{g}}_- \bigoplus \mathfrak{g} \bigoplus \mathbb{C}k \bigoplus \mathbb{C}d.\]

Here we identify \(\mathfrak{g} \otimes 1\) with \(\mathfrak{g}\). We let

\[\mathfrak{h} = \mathfrak{h} \bigoplus \mathbb{C}k \bigoplus \mathbb{C}d,\]

and extend the Killing form on \(\mathfrak{h}\) to \(\hat{\mathfrak{h}}\) by letting \((k, d) = 1, (k, k) = 0, (d, d) = 0, (\mathbb{C}k + \mathbb{C}d, \mathfrak{h}) = 0\). We identify \(\hat{\mathfrak{h}}^*\) with \(\hat{\mathfrak{h}}\) using this bilinear form on \(\hat{\mathfrak{h}}\).

Given a \(\mathfrak{g}\)-module \(V\) and a complex number \(k\), we can define the induced module \(\tilde{V}_k\) over \(\hat{\mathfrak{g}}\) as follows: \(V\) can be viewed as a module
over $\hat{\mathfrak{g}}_+ + \mathfrak{g} + \mathbb{C}k + \mathbb{C}d$ by letting $(\hat{\mathfrak{g}}_+ \oplus \mathbb{C}d)V = 0$ and $k = (k + h^\vee)I|_V$. Then we let

$$\tilde{V}_k = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{g}}_+ + \mathfrak{g} + \mathbb{C}k + \mathbb{C}d)} V.$$ 

Here and further $\mathcal{U}(\mathfrak{a})$ denotes the universal enveloping algebra of a Lie (super)algebra $\mathfrak{a}$. In particular for any $\lambda \in \mathfrak{h}^*$, we let $L(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda$, and denote the $\hat{\mathfrak{g}}$-module $\tilde{L}(\lambda)_k$ by $M_{k,\lambda}$. Let $J_{k,\lambda}$ be the maximal proper submodule of the $\hat{\mathfrak{g}}$-module $M_{k,\lambda}$. Denote $M_{k,\lambda}/J_{k,\lambda}$ by $L_{k,\lambda}$. Note that if $\lambda = 0$, $L(0)$ is the trivial $\mathfrak{g}$-module $\mathbb{C}$ and $M_{k,0} \cong \mathcal{U}(\hat{\mathfrak{g}}_-)$ as $\hat{\mathfrak{g}}_-$-modules.

Define a $\frac{1}{2}\mathbb{Z}$-gradation of $\hat{\mathfrak{g}}$ by the eigenvalues of $-d$:

$$\deg k = 0, \quad \deg a(n) = -n, \quad \deg \bar{a}(n) = -n - \frac{1}{2}, \quad a \in \mathfrak{g}.$$ 

This induces $\frac{1}{2}\mathbb{Z}$-gradations of $\mathcal{U}(\hat{\mathfrak{g}}), \mathcal{U}(\hat{\mathfrak{g}}_-)$ and $\frac{1}{2}\mathbb{Z}_+$-gradations of $M_{k,\lambda}$ if we let the degree of the highest weight of $L(\lambda)$ to be zero. We denote the gradation decompositions by

$$\mathcal{U}(\hat{\mathfrak{g}}) = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} \mathcal{U}(\hat{\mathfrak{g}})(n), \quad \mathcal{U}(\hat{\mathfrak{g}}_-) = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} \mathcal{U}(\hat{\mathfrak{g}}_-)(n),$$ 

$$M_{k,\lambda} = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} M_{k,\lambda}(n),$$

where $M_{k,\lambda}(n) = \mathcal{U}(\hat{\mathfrak{g}}_-)(n)L(\lambda)$.

Define a topological completion $\hat{\mathcal{U}}(\hat{\mathfrak{g}})$ of $\mathcal{U}(\hat{\mathfrak{g}})$ as follows. Let

$$\mathcal{U}(\hat{\mathfrak{g}})_n^m = \sum_{i \leq m, j \in \frac{1}{2}\mathbb{Z}} \mathcal{U}(\hat{\mathfrak{g}})_{n-i} \mathcal{U}(\hat{\mathfrak{g}})_i, \quad \text{for } m \in \frac{1}{2}\mathbb{Z}.$$

It is easy to see that

$$\mathcal{U}(\hat{\mathfrak{g}})^{m + \frac{1}{2}}_n \subset \mathcal{U}(\hat{\mathfrak{g}})^m_n, \quad \bigcap_{m \in \frac{1}{2}\mathbb{Z}} \mathcal{U}(\hat{\mathfrak{g}})_n^m = 0, \quad \bigcup_{m \in \frac{1}{2}\mathbb{Z}} \mathcal{U}(\hat{\mathfrak{g}})_n^m = \mathcal{U}(\hat{\mathfrak{g}})_n.$$
We take \( \{ \mathfrak{u}(\hat{g})_m^n, m \in \frac{1}{2} \mathbb{Z} \} \) for a fundamental neighborhood system of \( \mathfrak{u}(\hat{g})_n \), and denote the corresponding completion by \( \hat{\mathfrak{u}}(\hat{g})_n \). We let

\[
\hat{\mathfrak{u}}(\hat{g}) = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}} \hat{\mathfrak{u}}(\hat{g})_n.
\]

Let \( \langle \mathfrak{k} - (k + h^\vee) \rangle \) be the two-sided ideal of the associative superalgebra \( \hat{\mathfrak{u}}(\hat{g}) \) generated by the element \( \mathfrak{k} - (k + h^\vee) \). Denote \( \hat{\mathfrak{u}}(\hat{g})/\langle \mathfrak{k} - (k + h^\vee) \rangle \) by \( \hat{\mathfrak{u}}(\hat{g}, k) \).

A \( \hat{\mathfrak{g}} \)-module \( \mathfrak{M} \) is called restricted if for any fixed \( v \in M, x(n)v = 0 \) for \( n \gg 0 \). For example, \( \hat{\mathcal{V}}_k \) is a restricted module. The action of \( \hat{\mathfrak{g}} \) on any restricted module can be extended to \( \hat{\mathfrak{u}}(\hat{g}) \) naturally.

Let \( \hat{\mathfrak{u}}(\hat{g}, k)[[z, z^{-1}]] \) be the space of power series of \( z, z^{-1} \) with coefficients in \( \hat{\mathfrak{u}}(\hat{g}, k) \). An element \( b(z) = \sum_{n \in \mathbb{Z}} b(n)z^{-n-1} \) in \( \hat{\mathfrak{u}}(\hat{g}, k)[[z, z^{-1}]] \) is called regular if every \( b(n) \) is homogeneous in \( \hat{\mathfrak{u}}(\hat{g}, k) \) and \( \deg(b(n)) = -n + N_b \), where \( N_b \in \frac{1}{2} \mathbb{Z} \) is a constant independent of \( n \). We say that the regular element is odd if \( N_b \in \frac{1}{2} + \mathbb{Z} \), even if \( N_b \in \mathbb{Z} \). We define \( \tilde{b} \) to be 1, if \( b(z) \) is odd and 0 if \( b(z) \) even. We denote by \( \hat{\mathfrak{u}}(\hat{g}, k)\langle z \rangle \) the subspace linearly spanned by the regular elements in \( \hat{\mathfrak{u}}(\hat{g}, k)[[z, z^{-1}]] \).

Recall the \( \frac{1}{2} \mathbb{Z}_+ \)-gradation \( M_{k,0} = \bigoplus_{n \in \frac{1}{2} \mathbb{Z}_+} M_{k,0}(n) \). Let \( 1 \in M_{k,0}(0) \) be the vacuum element of \( M_{k,0} \). Define \( \hat{Y}(1, z) = I \big|_{M_{k,0}} \). We have

\[
M_{k,0}(0) = \mathbb{C} \cdot 1, \quad M_{k,0}(\frac{1}{2}) = \hat{\mathfrak{g}}(-1) \cdot 1 \cong \hat{\mathfrak{g}}, \quad M_{k,0}(1) = \mathfrak{g}(-1) \cdot 1 \cong \mathfrak{g}.
\]

For \( a \in \mathfrak{g} \subset M_{k,0}, \bar{a} \in \hat{\mathfrak{g}} \subset M_{k,0} \), we define

\[
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \quad \bar{a}(z) = \sum_{n \in \mathbb{Z}} \bar{a}(n)z^{-n-1}.
\]

It is clear that \( a(z), \bar{a}(z) \in \hat{\mathfrak{u}}(\hat{g}, k)\langle z \rangle \). \( a(z) \) is even while \( \bar{a}(z) \) is odd.

**Definition 2.1** For \( b(z) = \sum_{n \in \mathbb{Z}} b(n)z^{-n-1} \), and \( a, \bar{a} \in M_{k,0} \), we define

\[
a(n) \cdot b(z) = \text{Res}_w(a(w)b(z)\nu_w(z)(w - z)^n - b(z)a(w)\nu_{z,w}(w - z)^n)
\]
\[ \bar{a}(n) \cdot b(z) = \text{Res}_w(\bar{a}(w)b(z)\omega_{w,z}(w-z)^n - (-1)^b(z)\bar{a}(w)\omega_{z,w}(w-z)^n) \]

By direct calculation, the following proposition which is an analogue of Proposition 2.2.1 in [FZ] follows from Definition 2.1.

**Proposition 2.1** The definition above gives \( \tilde{U}(\hat{\mathfrak{g}},k)\langle z \rangle \) the structure of a \( \hat{\mathfrak{g}} \)-module, where \( k \in \hat{\mathfrak{g}} \) acts as \((k + h^\vee)I\).

As a consequence of Definition 2.1, we have the following.

**Corollary 2.1** For \( a \in \mathfrak{g}, \bar{a} \in \bar{\mathfrak{g}}, \) we have

\[
\begin{align*}
a(n) \cdot 1 &= \begin{cases} 0, & n \geq 0 \\ \frac{1}{(-n-1)!}(\frac{d}{dz})^{n-1}a(z), & n < 0, \end{cases} \\
\bar{a}(n) \cdot 1 &= \begin{cases} 0, & n \geq 0 \\ \frac{1}{(-n-1)!}(\frac{d}{dz})^{n-1}\bar{a}(z), & n < 0. \end{cases}
\end{align*}
\]

By Propositions 2.1 and Corollary 2.1, we have a well-defined homomorphism of \( \hat{\mathfrak{g}} \)-modules from \( M_{k,0} \) to \( \tilde{U}(\hat{\mathfrak{g}},k)\langle z \rangle \):

\[
Y(\ , \ z) : a_1(-i_1) \cdots a_n(-i_n)\bar{b}_1(-j_1) \cdots \bar{b}_m(-j_m)1 \\
\mapsto a_1(-i_1) \cdots \bullet a_n(-i_n) \bullet \bar{b}_1(-j_1) \cdots \bullet \bar{b}_m(-j_m) \bullet 1.
\]

Moreover, \( Y(a(-1)1, z) = a(z), Y(\bar{a}(-1)1, z) = \bar{a}(z) \) for \( a \in \mathfrak{g}, \bar{a} \in \bar{\mathfrak{g}}. \)

Since \( M_{k,0} \) is a \( \tilde{U}(\hat{\mathfrak{g}},k) \)-module, we have a map from \( \tilde{U}(\hat{\mathfrak{g}},k) \) to \( \text{End}(M_{k,0}) \). Now we have a series of maps

\[
M_{k,0} \rightarrow \tilde{U}(\hat{\mathfrak{g}},k)\langle z \rangle \subset \tilde{U}(\hat{\mathfrak{g}},k)[[z, z^{-1}]] \rightarrow \text{End}(M_{k,0})[[z, z^{-1}]].
\]
We still denote the composition of these maps by

\[ Y(,z) : M_{k,0} \to \text{End} (M_{k,0})[[z, z^{-1}]]. \]

For \( b \in M_{k,0} \), we call \( Y(b,z) \) the vertex operator of \( b \).

We use small letters \( a, b, c \cdot \cdot \cdot \) to denote the index among \( 1, 2, \cdot \cdot \cdot, \dim \mathfrak{g} \).

Choose a basis \( \{ u_a \} \) of \( \mathfrak{g} \) satisfying

\[ (u_a, u_b) = \frac{1}{2} \delta_{ab}, \quad [u_a, u_b] = i f_{abc} u_c, \]

where \( f_{abc} \) is anti-symmetric in \( a, b, c \) and real valued (These notations agree with those in [KS]). Here and below we assume, as usual, summation over repeated indices.

**Theorem 2.1** \((M_{k,0}, 1, \omega, \tau, Y(,z))\) is an \( N = 1 \) SVOA of rank \( c_k \) provided that \( k \neq -h^\vee \), where

\[
c_k = \frac{\dim \mathfrak{g}}{2} + \frac{k \dim \mathfrak{g}}{k + h^\vee},
\]

\[
\tau = \frac{2}{k + h^\vee} u_a(-1) \bar{u}_a(-1)1 + \frac{4i}{3(k + h^\vee)^2} f_{abc} \bar{u}_a(-1) \bar{u}_b(-1) \bar{u}_c(-1)1,
\]

\[
\omega = \frac{1}{k + h^\vee} \{ u_a(-1) u_a(-1)1 \\
+ \bar{u}_a(-2) \bar{u}_a(-1)1 \} + \frac{2i}{3(k + h^\vee)^2} f_{abc} \bar{u}_a(-1) \bar{u}_b(-1) u_c(-1)1.
\]

The fact that the components of the fields

\[
Y(\tau, z) = \sum_{n \in \mathbb{Z}} G_{n+\frac{1}{2}} z^{-n-2}, \quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]

satisfy the commutation relations of the Neveu-Schwarz algebra with the central charge \( c_k \) is ensured by Theorem 4 in [KT]. The rest of the proof of the above theorem is similar to Theorem 2.4.1 in [FZ].
It follows from [KT] that \([L_0, a(m)] = -ma(m), [L_0, \bar{a}(m)] = -(m+\frac{1}{2})\bar{a}(m)\). Thus \(L_0+d\), which commutes with all \(a(m), \bar{a}(m) \in \hat{\mathfrak{g}}\).

We call the generalized Casimir operator the element \(\Omega = 2(k + h^\vee)(L_0 + d)\). It follows from [KT] that
\[
\Omega(v) = (\lambda + 2\rho, \lambda)v \tag{2.7}
\]
if \(v\) is a singular vector of weight \(\lambda\).

Let \(J_{k,0}\) be the maximal proper submodule of \(M_{k,0}\). It is easy to see that if \(k \neq -h^\vee\) then \(1 \not\in J_{k,0}, \tau \not\in J_{k,0}\) and hence the quotient \(L_{k,0} = M_{k,0}/J_{k,0}\) is also a SVOA. To understand what \(J_{k,0}\) is, we need to find the formulas for singular vectors of \(M_{k,0}\). This is done in the Appendix (Sec.5).

**Remark 2.1** One may construct the \(N = 2\) SVOA (i.e. the SVOA which admits vertex operators whose Fourier components satisfy the \(N = 2\) superconformal algebra) from the \((N = 1)\) affine Kac-Moody superalgebra [KS].

### 2.2 Rationality and fusion rules of the SVOA \(L_{k,0}\)

**Lemma 2.1** The associative algebra \(A(M_{k,0})\) is canonically isomorphic to \(\mathfrak{u}(\mathfrak{g})\).

**Proof:** By the definition of \(A(M_{k,0})\) and Lemma 1.1 we have
\[
[c] \ast [a(-1)1] = [a(-1)c],
\]
where \(a \in \mathfrak{g}, c \in M_{k,0}\). Hence
\[
[a_m(-1)1] \ast \cdots \ast [a_1(-1)1] = [a_1(-1)\cdots a_m(-1)1].
\]
Therefore we have a homomorphism of associative algebras

\begin{equation}
F : \mathfrak{g} \longrightarrow A(M_{k,0})
\end{equation}

given by

\[ a_m \cdots a_1 \mapsto [a_1(-1) \cdots a_n(-1)1]. \]

It is clear that

\begin{align*}
(a(-n - 2) + a(-n - 1))c &= \text{Res}_z(Y(a(-1)1, z)\frac{z + 1}{z^{n+2}}c), \\
\bar{b}(-n - 1)c &= \text{Res}_z(Y(\bar{b}(-1)1, z)\frac{1}{z^{n+1}}c).
\end{align*}

By Lemma 1.1, we have

\[ O'(M_{k,0}) \subset O(M_{k,0}), \]

where

\[ O'(M_{k,0}) = \{(a(-n - 2) + a(-n - 1))c, \ \bar{b}(-n - 1)c \text{ for } n \geq 0\}. \]

Then it follows that

\[ [a_1(-i_1 - 1) \cdots a_m(-i_m - 1)] = (-1)^{i_1 + \cdots + i_m}[a_1(-1) \cdots a_n(-1)1] \]

for \(i_1, \cdots i_m \geq 0\). So F is an epimorphism. To show that F is indeed an isomorphism, we still need to show that

\begin{equation}
O'(M_{k,0}) = O(M_{k,0}).
\end{equation}

However this is standard (see the proof of a similar fact in Appendix of [W]). \(\square\)

**Lemma 2.2** If \(k\) is a positive integer, then the map (2.8) induces an isomorphism from \(\mathfrak{g}/\langle e_g^{k+1}\rangle\) onto \(A(L_{k,0})\), where \(\langle e_g^{k+1}\rangle\) is the two-sided ideal of \(\mathfrak{g}\) generated by \(e_g^{k+1}\).
Proof. It follows from Theorem 5.3 in the Appendix that the SVOA $M_{k,0}$ is isomorphic to

$$M(\Lambda + h^\vee \Lambda_0)/\langle f_i 1, i = 1, \cdots, l \rangle,$$

with $\Lambda$ given by $\lambda_i = \Lambda(h_i) = 0, i = 1, \cdots, l,$ $\lambda_0 = \Lambda(k) = k.$ Then the SVOA $L_{k,0}$ is isomorphic to $M_{k,0}/\langle v_k \rangle,$ where $v_k$ is defined by Theorem 5.2. By Remark 5.3 and the identity (2.9), we see that under the isomorphism (2.8), $v_{\lambda_0}$ corresponds to $e_\theta^{k+1} \in \mathfrak{u}(g).$ Hence the lemma follows from Proposition 1.1. □

Lemma 2.3 If $x \in \mathfrak{g},$ and $N \in \mathbb{N},$ then the algebra $\mathfrak{u}(g)/\langle x^N \rangle$ is finite dimensional and semisimple.

Proof. Let $G$ be the adjoint group of $\mathfrak{g}.$ Since $G$ is generated by $exp(ad y),$ $y \in \mathfrak{g},$ the ideal $\langle x^N \rangle$ is $G$-invariant, hence it contains all elements $g(x)^N, g \in G.$ Since $\mathfrak{g}$ is simple, it coincides with the linear span of the orbit $G(x),$ hence $u_i^N \in \langle x^N \rangle$ for some basis $\{u_i\}$ of $\mathfrak{g}.$ It follows that $dim \mathfrak{u}(g)/\langle x^N \rangle \leq N^{dim \mathfrak{g}}.$

Since any finite-dimensional representation of $\mathfrak{u}(g)$ is semisimple, it follows that any representation of $\mathfrak{u}(g)/\langle x^N \rangle$ is semisimple. Hence the latter algebra is semisimple. □

Theorem 2.2 For any positive integral $k,$ the SVOA $L_{k,0}$ is rational. Moreover, $L_{k,\lambda},$ for $\lambda \in \mathfrak{h}^*$ dominant integrable with $\langle \lambda, \theta \rangle \leq k,$ are precisely all the irreducible $L_{k,0}$-modules.

Proof. The second part of this theorem follows from Theorem 1.3 and Lemma 2.2 because by Lemma 2.3, $L_{k,\lambda},$ for $\lambda \in \mathfrak{h}^*$ dominant integrable with $\langle \lambda, \theta \rangle \leq k,$ are all the irreducible modules of $\mathfrak{u}(g)/\langle e_\theta^{k+1} \rangle.$
Any $L_{k,0}$-module $M$ is a restricted module over $\hat{g}$. Hence any $\hat{g}$-submodule of $M$ is also an $L_{k,0}$-submodule of $M$. To prove the complete reducibility of any $L_{k,0}$-module, we only need to prove the following.

**Lemma 2.4** Given $\lambda, \mu \in P_+$ such that $\langle \lambda, \theta \rangle \leq k, \langle \mu, \theta \rangle \leq k$, any short exact sequence of $\hat{g}$-modules

$$0 \to L_{k,\lambda} \xrightarrow{i} M \xrightarrow{\pi} L_{k,\mu} \to 0$$

splits.

*Proof.* Let $Q_+ = \Sigma_i \mathbb{Z}_+ \alpha_i$. First let us define a partial order in $P_+$ as follows: $\lambda > \mu$ iff $\lambda - \mu \in Q_+$ and $\lambda \neq \mu$. Without loss of generality, we may assume that $\lambda \not> \mu$. Otherwise we can apply the contragredient functor to the short exact sequence to reverse it. Let $v_\mu$ be the vacuum vector of $L_{k,\mu}$. Pick a vector $v'_\mu \in M$ of weight $\mu$ such that $\pi(v'_\mu) = v_\mu$. We claim that $v'_\mu$ is a singular vector of $M$, i.e. $e_i v'_\mu = 0$ for any $i$. Indeed, if $e_i v'_\mu \neq 0$ for some $i$, then

$$\pi(e_i v'_\mu) = e_i \pi(v'_\mu) = e_i v_\mu = 0.$$  

So

$$e_i v'_\mu = i(u)$$

for some nonzero $u \in L_{k,\lambda}$, since the short sequence is exact. Comparing the weights of both sides of equation (2.10), we have $\lambda - \beta = \mu + \alpha_i$ for nonzero $\alpha_i, \beta \in Q_+$. It follows that $\lambda = \mu + \alpha_i + \beta > \mu$, which is a contradiction.

Denote by $M'$ the submodule of $M$ generated by the singular vector $v'_\mu$. It suffices to show that the module $M'$ is irreducible. But
this follows in the same way as in Chapter 11 of [K] by making use of formula (2.7). □

Of course the Lie subalgebra

\[ \hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d \]

of \( \hat{g} \) is the usual affine Kac-Moody algebra. As usual, we let

\[ \hat{g}^+ = g \otimes t\mathbb{C}[t], \]

\[ \hat{g}^- = g \otimes t^{-1}\mathbb{C}[t^{-1}], \]

\[ \hat{h}^\pm = \hat{g}^\pm \oplus \bigoplus_{\alpha \in \Delta_+} g_{\pm \alpha}. \]

Given \( \lambda \in P_+ \) and \( k \in \mathbb{Z}_+ \), consider the irreducible \( g \)-module \( \bar{L}(\lambda) \) as a module over \( \hat{g}^+ \oplus g \oplus \mathbb{C}k \oplus \mathbb{C}d \) by letting \( (\hat{g}^+ \oplus \mathbb{C}d)\bar{L}(\lambda) = 0 \) and \( k = kI \mid_V \). Let

\[ \bar{M}_{k, \lambda} = \mathfrak{u}(\hat{g}) \bigotimes_{\bar{u}(\hat{g}^+ \oplus g \oplus \mathbb{C}k \oplus \mathbb{C}d)} \bar{L}(\lambda), \]

and \( \bar{L}_{k, \lambda} = \bar{M}_{k, \lambda}/\bar{J}_{k, \lambda} \), where \( \bar{J}_{k, \lambda} \) is the unique maximal \( \hat{g} \)-submodule of \( \bar{M}_{k, \lambda} \).

The associative algebra of the VOA \( \bar{L}_{k, 0} \) was computed in [FZ]. Comparing with our results, we see that the associative algebras \( A(L_{k, 0}) \) and \( A(\bar{L}_{k, 0}) \) are the same. And so the irreducible modules of the SVOA \( L_{k, 0} \) are canonically in 1-1 correspondence with those of the VOA \( \bar{L}_{k, 0} \). One can calculate the fusion rules using the \( A(L_{k, 0}) \)-modules similarly to Section 3.2 in [FZ] and find that the fusion rules for the modules of the SVOA \( L_{k, 0} \) are canonically in 1-1 correspondence with those for the VOA \( \bar{L}_{k, 0} \) (see the statements in Theorem 3.2.3 and Corollary 3.2.1 in [FZ]).
3 Neveu-Schwarz SVOAs

3.1 SVOA structure on $M_{c,0}$ and $L_{c,0}$

Let us recall first that the Neveu-Schwarz algebra is the Lie superalgebra

$$\mathfrak{NS} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \bigoplus \bigoplus_{m \in \frac{1}{2} + \mathbb{Z}} \mathbb{C} G_m \bigoplus \mathbb{C} C$$

with commutation relations ($m, n \in \mathbb{Z}$):

$$[L_m, L_n] = (m - n) L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C,$$

$$[G_{m+\frac{1}{2}}, L_n] = (m + \frac{1}{2} - \frac{n}{2}) G_{m+n+\frac{1}{2}},$$

$$[G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}]_+ = 2L_{m+n} + \frac{1}{3} m(m+1) \delta_{m+n,0} C,$$

$$[L_m, C] = 0, \quad [G_{m+\frac{1}{2}}, C] = 0.$$

The $\mathbb{Z}_2$-gradation is given by $\tilde{L}_n = \tilde{C} = 0$, $\tilde{G}_n = \tilde{1}$ (so that the even part is the Virasoro algebra). Set

$$\mathfrak{NS}_\pm = \bigoplus_{n \in \mathbb{N}} \mathbb{C} L_{\pm n} \bigoplus \bigoplus_{m \in \frac{1}{2} + \mathbb{Z}_+} \mathbb{C} G_{\pm m}.$$

Given complex numbers $c$ and $h$, the Verma module $M_{c,h}$ over $\mathfrak{NS}$ is the free $\mathfrak{u}(\mathfrak{NS}_-)$-module generated by 1, such that $\mathfrak{NS}_+ 1 = 0$, $L_0 1 = h \cdot 1$ and $C \cdot 1 = c \cdot 1$. There exists a unique maximal proper submodule $J_{c,h}$ of $M_{c,h}$. Denote the quotient $M_{c,h} / J_{c,h}$ by $L_{c,h}$. Recall that $v \in M_{c,h}$ is called a singular vector if $\mathfrak{NS}_+ 1 = 0$ and $v$ is an eigenvector of $L_0$. For example, $G_{-\frac{1}{2}} 1$ is a singular vector of $M_{c,0}$ for any $c$. 
Denote \( M_{c,0} / \langle G_{-1/2} \rangle \) by \( M_c \), where \( \langle G_{-1/2} \rangle \) is the submodule of \( M_{c,0} \) generated by the singular vector \( G_{-1/2} \). For simplicity we denote \( L_{c,0} \) by \( V_c \).

It is well known that

\[
L_{-i_1} L_{-i_2} \cdots L_{-i_m} G_{-j_1} G_{-j_2} \cdots G_{-j_n},
\]

for \( i_1 \geq \cdots \geq i_m \geq 1, j_1 > \cdots > j_n \geq 1/2, \) \( i_1 \cdots i_m \in \mathbb{N} \), and \( j_1 \cdots j_n \in 1/2 + \mathbb{Z} \) is a basis of \( \mathfrak{U}(\mathfrak{g}_-) \). There is a natural gradation on \( M_{c,0} \), \( M_c \) and \( V_c \) given by the eigenspace decomposition of \( L_0 \):

\[
\text{deg } L_{-i_1} L_{-i_2} \cdots L_{-i_m} G_{-j_1} G_{-j_2} \cdots G_{-j_n} 1 = i_1 + i_2 + \cdots + i_m + j_1 + \cdots + j_n.
\]

We can define \( \check{\mathfrak{U}}(\mathfrak{g}_c, c) \), the completion of \( \mathfrak{U}(\mathfrak{g}_c) \), as in Section 2. The action of \( \mathfrak{g}_c \) on any restricted module of \( \mathfrak{g}_c \) can be extended to \( \check{\mathfrak{U}}(\mathfrak{g}_c) \) naturally. In particular, \( \check{\mathfrak{U}}(\mathfrak{g}_c, c) \) acts on \( M_{c,h} \) and \( M_c \).

We can also define the notions of even and odd regular elements in \( \check{\mathfrak{U}}(\mathfrak{g}_c, c)[[z, z^{-1}]] \). Denote by \( \check{\mathfrak{U}}(\mathfrak{g}_c, c)[z] \) the linear span of regular elements in \( \check{\mathfrak{U}}(\mathfrak{g}_c, c)[[z, z^{-1}]] \). Set

\[
L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]

\[
G(z) = \sum_{n \in \mathbb{Z}} G_{n+1/2} z^{-n-2}.
\]

Clearly \( L(z) \) is even while \( G(z) \) is odd. For a regular element \( b(z) \in \check{\mathfrak{U}}(\mathfrak{g}_c, c)[[z, z^{-1}]] \) we define

\[
L_m \bullet b(z) = \text{Res}_w (L(w)b(z)\iota_{w,z}(w-z)^{n+1} - b(z)L(w)\iota_{z,w}(w-z)^{n+1}),
\]

\[
G_{n+1/2} \bullet b(z) = \text{Res}_w (G(w)b(z)\iota_{w,z}(w-z)^{n+1} - (-1)^{n} b(z)G(w)\iota_{z,w}(w-z)^{n+1}),
\]

\( m \) is a non-negative integer.
where $\bar{b} = 0$ if $b(z)$ is even and $\bar{b} = 1$ if $b(z)$ odd.

Claim 6.1. (3.1) and (3.2) define a $\mathfrak{h}\mathfrak{s}$-module structure on $\tilde{\mathfrak{u}}(\mathfrak{h}\mathfrak{s}, c) \langle z \rangle$ with central charge $c$.

Claim 6.2.

$$L_n \cdot 1 = \begin{cases} 0, & n \geq -1 \\ \frac{1}{(-n-2)!} \left( \frac{d}{dz} \right)^{n-2} L(z), & n < -1, \end{cases}$$

$$G_{n+\frac{1}{2}} \cdot 1 = \begin{cases} 0, & n \geq -1 \\ \frac{1}{(-n-2)!} \left( \frac{d}{dz} \right)^{n-2} G(z), & n < -1, \end{cases}$$

From Claim 6.2 we have a well-defined homomorphism of $\mathfrak{h}\mathfrak{s}$-modules

(3.3) $Y(\cdot, z) : M_{k,0} \rightarrow \tilde{\mathfrak{u}}(\mathfrak{h}\mathfrak{s}, c) \langle z \rangle$

$$\Rightarrow L_{-i_1} L_{-i_2} \cdots L_{-i_m} G_{-j_1} G_{-j_2} \cdots G_{-j_n} 1$$

$$\mapsto L_{-i_1} \cdot L_{-i_2} \cdots \cdot L_{-i_m} \cdot G_{-j_1} \cdot G_{-j_2} \cdots \cdot G_{-j_n} \cdot 1.$$

In particular, if we set $\tau = G_{-3/2} 1$, and $\omega = L_{-2} 1$, we have $Y(\tau, z) = G(z)$, and $Y(\omega, z) = L(z)$.

Since $\tilde{\mathfrak{u}}(\mathfrak{h}\mathfrak{s}, c)$ acts on $M_{c,h}$, we have a map from $\tilde{\mathfrak{u}}(\mathfrak{h}\mathfrak{s}, c)$ to $\text{End}(M_{k,0})$. Then (3.3) induces a linear map

$$Y(\cdot, z) : M_{k,0} \rightarrow \text{End}(M_{k,0})[[z, z^{-1}]].$$

Thus we have

Theorem 3.1 $(M_c, 1, \tau, Y(\cdot, z))$ is an $N = 1$ SVOA.

3.2 Rationality and fusion rules of $V_{c,p,q}$

Lemma 3.1 There exists an isomorphism of associative algebras, $F : A(M_c) \cong \mathbb{C}[x]$, given by $[\omega]^n \mapsto x^n$, where $\mathbb{C}[x]$ is the polynomial algebra on one generator $x$. 
Proof. Set

\[ M_c = M^0_c + M^1_c, \]

where \( M^0_c \) (resp. \( M^1_c \)) is the even (resp. odd) part of \( M_c \). By Lemma 1.1 we have

\begin{equation}
((L_{m-3} + 2L_{m-2} + L_{m-1})b) = \text{Res}_z(Y(\omega, z) \frac{(z + 1)^2}{z^{2+n}} b) \in O(M_c),
\end{equation}

for every \( m \geq 0, b \in M^0_c \).

\begin{equation}
(G_{-n-1} + G_{-n})b = \text{Res}_z(Y(\tau, z) \frac{(z + 1)}{z^{1+n-\frac{1}{2}}} b) \in O(M_c),
\end{equation}

for every \( n \in \frac{1}{2} + \mathbb{Z}_+, b \in M^1_c \). It follows by induction that

\begin{equation}
L_{-m} \sim (-1)^m((m - 1)(L_2 + L_{-1}) + L_0),
\end{equation}

for every \( m \geq 1 \).

\begin{equation}
G_{-n} \sim (-1)^{n-\frac{1}{2}}G_{-\frac{1}{2}}, \text{ for every } n \in \frac{1}{2} + \mathbb{Z}_+.
\end{equation}

By Lemma 1.1 we have

\begin{equation}
[b] * [\omega] = [(L_{-2} + L_{-1})b], b \in M^0_c
\end{equation}

Using (3.4) and (3.5), it is easy to show by induction on \( m + n \) that

\[ [L_{-i_1}L_{-i_2} \cdots L_{-i_m}G_{-j_1}G_{-j_2} \cdots G_{-j_{2n}}1] = P([\omega]) \]

for some \( P(x) \in \mathbb{C}[x] \). Since the elements

\[ L_{-i_1}L_{-i_2} \cdots L_{-i_m}G_{-j_1}G_{-j_2} \cdots G_{-j_{2n}}1 \]

for \( i_1 \geq \cdots \geq i_m \geq 1, j_1 > \cdots > j_{2n} \geq \frac{1}{2}, i_1 \cdots i_m \in \mathbb{N}, j_1 \cdots j_{2n} \in \frac{1}{2} + \mathbb{Z}_+, \) span \( M_c \), the homomorphism of associative algebras

\[ F: \mathbb{C}[x] \to A(M_c) \]
given by $x^n \mapsto [\omega]^n$ is surjective. (This homomorphism is well defined since $[\omega]$ is in the center of $A(M_c).$)

To prove that $F$ is also injective, it suffices to show that $O(M_c)$ is the linear span of the elements of the form (3.3) and (3.4) i.e.

$$O(M_c) = \{ (L_{-n-3}+2L_{-n-2}+L_{-n-1})b, (G_{-n-3/2}+G_{-n-3/2})b n \geq 0 b \in M_c \}.$$ 

This can be proved in a standard way (for a proof of a similar fact see Appendix of [W]). □

Set

$$c_{p,q} = \frac{3}{2} (1 - \frac{2(p-q)^2}{pq}),$$

$$h_{r,s}^{p,q} = \frac{(sp-rq)^2 - (p-q)^2}{8pq}.$$ 

Whenever we mention $c_{p,q}$ again, we always assume that $p, q \in \{2, 3, 4, \cdots \}, p-q \in 2\mathbb{Z}$, and that $(p-q)/2$ and $q$ are relatively prime to each other. The submodule structure of a Verma module over the Neveu-Schwarz algebra [A] is very similar to that for the Virasoro algebras [FF]. From the results of [A], we have the following lemma which is an analogue of the results in [FF] (also see Lemma 4.2 of [W]).

**Lemma 3.2**

1) $J_{c,0}$ is generated by the singular vector $G_{-\frac{1}{2}}$ if $c \neq c_{p,q}$.

2) $J_{c,0}$ is generated by two singular vectors if $c = c_{p,q}$. One of them is $G_{-\frac{1}{2}}$. The other, denoted by $v_{p,q}$ has degree $\frac{1}{2}(p-1)(q-1)$.

From this lemma we immediately derive an analogue of Corollary 4.1 in [W].

**Corollary 3.1** If $c \neq c_{p,q}$, then $V_c$ is not rational.
Proof. See proof of Corollary 4.1 in [W]. $\square$

From now on, we always assume that $c = c_{p,q}$ and that $h^{r,s} = h^{r,s}_{p,q}$. It follows from Lemma 3.2 that $V_c = M_c/\langle v_{p,q} \rangle$, where $\langle v_{p,q} \rangle$ denotes the submodule of $M_c$ generated by $v_{p,q}$. Then we have

**Proposition 3.1** One has:

$$A(V_c) \cong \mathbb{C}[x]/\langle F_{p,q}(x) \rangle,$$

where $\deg F_{p,q} = \frac{1}{4}(p-1)(q-1)$ if $p, q$ are odd; $\deg F = \frac{1}{4}(p-1)(q-1) + \frac{1}{4}$ if $p, q$ are even.

Proof. If $p, q$ are odd, $v_{p,q}$ is an even element of degree $\frac{1}{2}(p-1)(q-1)$ which corresponds to a polynomial $F_{p,q}$ of degree $\frac{1}{4}(p-1)(q-1)$; if $p, q$ are even, $v_{p,q}$ is an odd element of degree $\frac{1}{2}(p-1)(q-1)$. From the definition of the associative algebra $A(V_c)$, it is $G \cdot \frac{1}{2} v_{p,q}$ which corresponds to $F_{p,q}$ of degree $\frac{1}{4}(p-1)(q-1) + \frac{1}{4}$. $\square$

We expect the following conjecture, which is an analogue of Theorem 4.2 in [W], to be true.

**Conjecture 3.1** The vertex operator superalgebra $V_{c_{p,q}}$ is rational. Moreover, the minimal series modules $L_{c, h^{r,s}}$, $0 < r < p, 0 < s < q, r - s \in 2\mathbb{Z}$ are all the irreducible representations of $V_c$.

**Remark 3.1** If $p - q = 2$, $V_{c, h^{r,s}}$ is unitary. In this case we can prove Theorem 3.1 by using the well-known GKO construction [KW] and Theorem 2.2 (see [DMZ] for a similar proof in the Virasoro algebra case). It follows that Conjecture 3.1 holds at least in the cases $p - q = 2$. 
From the argument of Lemma 3.1, we see that
\[ A(V_c) = H_0(\mathfrak{S}, V_c), \]
where \( \mathfrak{S} = \{ L_{-n-2} + 2L_{-n-1} + L_{-n}, G_{-n-1} + G_{-n}, n > 0 \} \) is a locally nilpotent subalgebra of \( \mathfrak{N}\mathfrak{S} \). This conjecture can probably be proved as in [W], by calculating the coinvariants \( H_0(\mathfrak{S}, V_c) \). It is easy to see by Lemma 3.1 that
\[ A(L_{c,h,r,s}) = H_0(\mathfrak{S}, L_{c,h,r,s}). \]

Then by applying Theorem 1.3, and Proposition 1.2, we can obtain the fusion rules for the \( V_{c,p,q} \)-modules \( L_{c,h,r,s}, 0 < r < p, 0 < s < q, r - s \in 2\mathbb{Z} \) if the coinvariants \( H_0(\mathfrak{S}, L_{c,h,r,s}) \) are calculated.

To support our conjecture, we present some examples.

**Example 1.** Consider the case \((p, q) = (5, 3), c_{5,3} = 7/10, h_{1,1} = 0, h_{2,2} = 1/10\). It is easy to check that the singular vector \( v_{5,3} \) is given by
\[ v_{5,3} = 3L_{-4}1 + 10L_{-2}^21 - 15G_{-5/2}G_{-3/2}1. \]

Using (3.6), (3.7) and (3.8), we have
\[ F_{5,3}(x) = 10(x^2 - \frac{1}{10}x) \]
which gives the values of \( h_{1,1} \) and \( h_{2,2} \).

**Example 2.** Let \((p, q) = (8, 2), c_{8,2} = -21/4, h_{1,1} = 0, h_{3,1} = -\frac{1}{4}\). This is a non-unitary case. The singular vector \( v_{8,2} \) is given by
\[ v_{8,2} = 3G_{-7/2}1 - 4L_{-2}G_{-3/2}1. \]
Since \( v_{8,2} \) is an odd element, we consider \( G_{-\frac{7}{2}}v_{8,2} \) in order to get the polynomial \( F_{8,2}(x) \). Using (3.6), (3.7) and (3.8), we get
\[ G_{-\frac{7}{2}}v_{8,2} \sim -8x(x + \frac{1}{4}) \]
which gives the values of \( h_{1,1} \) and \( h_{3,1} \).
4 SVOAs generated by free fermionic fields

The free fermionic fields are

\[
\Phi^a(z) = \sum_{i \in 1/2 + \mathbb{Z}} \phi_i^a z^{-i - \frac{1}{2}}, \quad \text{(neutral)}
\]

\[
\Psi^{a,\pm}(z) = \sum_{i \in 1/2 + \mathbb{Z}} \psi_{i}^{a,\pm} z^{-i - \frac{1}{2}}, \quad \text{(charged)}
\]

with the following nontrivial commutation relations

\[
[\phi_i^a, \phi_j^b]_+ = \delta_{a,b} \delta_{i,j}
\]

\[
[\psi_i^{a,+}, \psi_j^{b,-}]_+ = \delta_{a,b} \delta_{i,j},
\]

where \(a, b = 1, \cdots, l\).

It is easy to see that from a pair of charged free fermionic fields \(\Psi^{\pm}(z)\) one can construct two neutral free fermionic fields \(\Phi(z)\) by letting

\[
\Phi^1(z) = \frac{1}{\sqrt{2}} (\Psi^+(z) + \Psi^-(z)),
\]

\[
\Phi^2(z) = \frac{i}{\sqrt{2}} (\Psi^+(z) - \Psi^-(z)),
\]

and vise versa. Hence we only need to consider the SVOAs generated by neutral free fermionic fields. Let \(F\) be the Fock space defined by \(\phi_{i>0}^a|0\rangle = 0, a = 1, 2, \cdots, l\). \(F\) is a SVOA with the Virasoro element \(\omega = \frac{1}{2} \sum_{a=1}^{l} \phi_{-3/2}^a \phi_{-1/2}^a 1\), and central charge \(c = \frac{l}{2}\). Denote by \(\mathfrak{a}\) the Lie algebra linearly spanned by \(\{\phi_{n+1/2}^a, a = 1, \cdots, l, n \in \mathbb{Z}\}\). \(\mathfrak{u}(\mathfrak{a})\) admits a natural gradation by letting \(\text{deg} \phi_{n+1/2}^a = -n - \frac{1}{2}\). Then, as in Subsec. 2.1, we can define the completion \(\hat{\mathfrak{u}}(\mathfrak{a})\) of \(\mathfrak{u}(\mathfrak{a})\). Similarly, we can define the notion of an (even or odd) regular vector in
\[ \tilde{\mathfrak{u}}(a)[[z, z^{-1}]]. \] Let \( \tilde{\mathfrak{u}}(a)(z) \) be the linear span of all regular vectors. Define \( Y(\phi_{1/2}^a, z) = \Phi^a(z). \) For any regular element \( b(z) \in \tilde{\mathfrak{u}}(a)(z) \) we define an action

\[ \phi_{n+\frac{1}{2}}^a \cdot b(z) = \text{Res}_w(\Phi^a(w)b(z)\tau_{w,z}(w-z)^n - (-1)^n b(z)\Phi^a(w)\tau_{z,w}(w-z)^n). \]

It is easy to see that \( \phi_{n+\frac{1}{2}}^a \cdot b(z) \) is also a regular vector. Then we define the vertex operator associated to any \( v = \phi_{n_1+\frac{1}{2}}^a \cdots \phi_{n_s+\frac{1}{2}}^a \cdot 1 \in F \) by

\[ Y(v, z) = \phi_{n_1+\frac{1}{2}}^a \cdot \cdots \cdot \phi_{n_s+\frac{1}{2}}^a \cdot 1. \]

This definition turns out to be the same as that defined by the normal ordering product \([T]\).

**Theorem 4.1** \( F \) is a rational SVOA. Moreover, \( F \) has a unique irreducible representation, namely \( F \) itself.

**Proof.** First we calculate the associative algebra \( A(F) \). By Lemma 1.1, we have

\[ \phi_{-\frac{1}{2} - n}^a v = \text{Res}_z(Y(\phi^a, z)\frac{1}{z^{1+n}}) \in O(F), \quad n \geq 0, \quad v \in F. \]

Since

\[ \{ \phi_{-\frac{1}{2} - n}^a v, \quad 1 \leq a \leq \ell, n \geq 0, \quad v \in F \} = \bigoplus_{n \in \frac{1}{2} \mathbb{N}} F_n, \]

we have \( \bigoplus_{n \in \frac{1}{2} \mathbb{N}} F_n \subset O(F). \)

On the other hand, it is easy to check by definition of \( O(F) \) that \( O(F) \subset \bigoplus_{n \in \frac{1}{2} \mathbb{N}} F_n. \) Thus \( O(F) = \bigoplus_{n \in \frac{1}{2} \mathbb{N}} F_n, \) and so \( A(F) = F/O(F) \cong \mathbb{C}. \) Hence there exists a unique representation of the associative algebra \( A(F) \cong \mathbb{C}, \) i.e \( \mathbb{C} \) itself. By Theorem 1.3, there exists a unique representation of \( F, \) i.e. \( F \) itself.

The complete reducibility of modules of \( F \) follows from a similar argument to the proof of Lemma 2.4. So \( F \) is rational. \( \square \)
Remark 4.1 The above SVOA $F$ is not an $N = 1$ SVOA. To construct the $N = 1$ SVOAs one needs to add some bosonic fields. For example, one can see that the Fock space of one free bosonic field and one free neutral fermionic field is an $N = 1$ SVOA of rank $\frac{3}{2}$. This is just the special case of the SVOA associated to the affine Kac-Moody superalgebra corresponding to the 1-dimensional Lie algebra $\mathfrak{g}$.

5 Appendix: Singular vectors and defining relations for the integrable representations of affine Kac-Moody superalgebras

We continue using the notation on affine (super)algebras introduced in Subsec.2.1.

Recall the triangular decomposition $\hat{\mathfrak{g}} = \hat{n}_+ \oplus \hat{h} \oplus \hat{n}_-$, where

$$\hat{n}_\pm = \hat{g}_\pm \bigoplus_{\alpha \in \Delta_+} (\bigoplus_{\pm} \mathfrak{g}_{\pm \alpha}).$$

Recall that for $\Lambda \in \hat{\mathfrak{h}}^*$ we have the Verma module $\bar{M}(\Lambda) = \mathfrak{U}(\hat{\mathfrak{g}}) \otimes_{\mathfrak{U}(\hat{\mathfrak{h}} + \hat{n}_+)} \mathfrak{C}_\Lambda$ over $\hat{\mathfrak{g}}$, where $\mathfrak{C}_\Lambda$ is the 1-dimensional $\mathfrak{U}(\mathfrak{h} + \hat{n}_+)$-module defined by $h \mapsto \Lambda(h)$, $\hat{n}_+ \mapsto 0$. Note that $\bar{M}_{k,\lambda}$, where $k = \Lambda(k)$ and $\lambda = \Lambda|_h$, is a quotient module of $\bar{M}(\Lambda)$ and that $\bar{L}(\lambda)$ is the quotient of $\bar{M}(\Lambda)$ by the maximal submodule.

Similarly we have the triangular decomposition $\hat{\mathfrak{g}} = \hat{n}_- \oplus \hat{h} \oplus \hat{n}_+$, where $\hat{n}_\pm = \hat{g}_\pm \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm \alpha}$, and for a given $\Lambda \in \hat{\mathfrak{h}}^*$, we define the Verma module $M(\Lambda)$ over $\hat{\mathfrak{g}}$, so that $M_{k,\lambda}$ is a quotient of $M(\Lambda)$ and $L(\Lambda)$ is the irreducible quotient of $M(\Lambda)$.

Set $e_0 = e_{-\theta}(1)$, $f_0 = e_{\theta}(-1)$, and $h_0 = \alpha_0 = k - \theta$. Given $\Lambda \in \hat{\mathfrak{h}}^*$, let $\lambda_i = \Lambda(h_i)$. Let $P_+ = \{\Lambda \in \hat{\mathfrak{h}}^* \mid \lambda(h_i) \in \mathbb{Z}_+\}$. 

It is well known [K] that if $\Lambda \in P_+$, then $\{f_i^{\lambda_i+1}, i = 0, 1, \ldots, l\}$ are the singular vectors of $\bar{M}(\Lambda)$ which generate the maximal proper submodule of $\bar{M}(\Lambda)$, denoted by $\langle f_i^{\lambda_i+1}, i = 0, 1, \ldots, l \rangle$, i.e.

$$\bar{L}(\Lambda) \cong \bar{M}(\Lambda) / \langle f_i^{\lambda_i+1}, i = 0, 1, \ldots, l \rangle$$

and that the $\hat{\mathfrak{g}}$-modules $\bar{L}(\Lambda)$, $\Lambda \in P_+$, are all the unitary highest weight modules. For $\hat{\mathfrak{g}}$ the situation is similar. In more detail, there exists a unique hermitian form $H(\cdot, \cdot)$ on the Verma module $M(\Lambda)$ satisfying

$$H(1, 1) = 1,$$

(5.1) $$a(n)^* = \sigma(a)(-n),$$

(5.2) $$\bar{a}(n)^* = \overline{\sigma(a)(-n - 1)},$$

where $^*$ denotes the adjoint operator with respect to the hermitian form $H(\cdot, \cdot)$. Then $L(\Lambda) = M(\Lambda)/\text{Ker } H$. The $\hat{\mathfrak{g}}$-module is called unitary if the form $H$ on $L(\Lambda)$ is positive definite. It is known that there exists a unique unitary highest weight $\hat{\mathfrak{g}}$-module of level $h^\vee$, called the minimal representation $F$ which is given by the Fock space realization of the infinite dimensional Clifford algebra (2.2) and as a $\hat{\mathfrak{g}}$-module is isomorphic to $L(h^\vee d)$ [KT]. Furthermore [KT], any unitary highest weight representation of $\hat{\mathfrak{g}}$ is of the form $L(\Lambda + h^\vee d)$, where $\Lambda \in P_+$, and that one has an isomorphism as $\hat{\mathfrak{g}}$-modules:

$$L(\Lambda + h^\vee d) \cong F \bigotimes \bar{L}(\Lambda).$$

From the construction of the minimal representation, we also see that as $\hat{\mathfrak{g}}$-modules

(5.3) $$\bar{M}(\Lambda + h^\vee d) \cong F \bigotimes \bar{M}(\Lambda)$$
It is clear that \(\{f_i^{\lambda_{i+1}}, i = 1, \ldots, l\}\) are the singular vectors of \(M(\Lambda + h^\vee d)\). By comparing the character formulas of both sides of (5.3), we see that there also exists a unique singular vector of weight \(\Lambda + h^\vee d - (\lambda_0 + 1)\alpha_0\) in \(M(\Lambda + h^\vee d)\). To get an explicit formula for this singular vector, we need to introduce the following notion of special roots.

**Definition 5.1** A root \(\alpha\) in \(\Delta_+\) is called special if \(\theta - \alpha\) is also a root.

Denote by \(\mathcal{S}\) the set of all special roots. The following is an equivalent way to define the set \(\mathcal{S}\):

**Remark 5.1** The set \(\mathcal{S}\) is also characterized by the property: \(r_\theta(\alpha) = \alpha - \theta\), if \(\alpha \in \mathcal{S}\); \(r_\theta(\alpha) = \alpha\), if \(\alpha \in \Delta - (\mathcal{S} \cup \{\theta\})\). Also we have:

\[
\mathcal{S} \cup \{\theta\} = \{\alpha \in \Delta_+ | r_\theta(\alpha) \in -\Delta_+\}. \tag{5.4}
\]

**Lemma 5.1** The number of special roots is \(2(h^\vee - 2)\).

**Proof.** Choose the shortest \(w \in W\) such that \(w(\alpha_i) = \theta\) for some simple root \(\alpha_i\) of \(\mathfrak{g}\). It is not difficult to see that \(l(w) = h^\vee - 2\). Pick a reduced expression \(w = r_{i_1} \cdots r_{i_{h^\vee - 2}}\). We claim that the expression

\[
r_\theta = r_{i_1} \cdots r_{i_{h^\vee - 2}} r_{\theta} r_{i_{h^\vee - 2}} \cdots r_{i_1}, \tag{5.5}
\]

is reduced, i.e. \(l(r_\theta) = 2h^\vee - 3\). Indeed, first from the expression (5.3) of \(r_\theta\) we see that \(l(r_\theta) \leq 2h^\vee - 3\). Let

\[
\beta_1 = \alpha_{i_1}, \ \beta_2 = r_{i_1}(\alpha_{i_2}), \ldots, \ \beta_{h^\vee - 2} = r_{i_1} \cdots r_{i_{h^\vee - 3}}(\alpha_{i_{h^\vee - 2}})
\]
and let
\[ \gamma_s := \theta - \beta_s = w r_i r_{\nu-2} \cdots r_{s+1}(\alpha_s). \]
It is easy to see (using Remark 5.1) that \( \{\beta_1, \cdots, \beta_{h-2}, \gamma_1, \cdots \gamma_{h-2}\} \subset S. \) Hence \( l(r_\theta) \geq 2h^-3. \) Then the lemma follows since \( l(w) = \#\{\alpha \in \Delta_+ | w^{-1}(\alpha) \in -\Delta_+\} \) for any \( w \in W. \) \( \square \)

**Remark 5.2** It follows from the above proof that
\[ S = \{\beta_1, \cdots, \beta_{h-2}, \gamma_1, \cdots \gamma_{h-2}\}. \] (5.6)
The sum of two elements from \( S \cup \{\theta\} \) is a root if and only if one of them is \( \beta_i \) and the other is \( \gamma_i. \)

**Lemma 5.2** Assume that \( \delta_i \in \Delta_+ - \{\theta\}, i = 1, 2, \cdots, p \) satisfy
\[ \sum_{i=1}^{I} \delta_i = p\theta + \sum_{k} \eta_k \quad \text{for some} \quad p \in \mathbb{N} \quad \text{and} \quad \eta_k \in \Delta_. \] (5.7)
Then \( I \geq 2p, \) and at least \( 2p \delta'_i \)s are contained in \( S. \)

Proof. Assume that there are \( q \delta'_i \)s which are contained in \( S. \) By applying \( r_\theta \) to both sides of the equation (5.7), it follows from Remark 5.1 that
\[ \sum_{i=1}^{I} \delta_i - q\theta = -p\theta + \sum_{k} r_\theta(\eta_k). \] (5.8)
By subtracting (5.8) from (5.7), we have
\[ (q - 2p)\theta = \sum_{k} (\eta_k - r_\theta(\eta_k)). \] (5.9)
By Remark 5.1, \( \eta_k - r_\theta(\eta_k) \) is in \( Q_+ = \sum_{\alpha \in \Delta_+} Z_+ \alpha. \) It follows that \( q \geq 2p. \) \( \square \)
Theorem 5.1  The element \( v_{\lambda_0} = \bar{e}_{-\theta}(1)\bar{e}_{-\theta}\Pi_{\alpha \in S}\bar{e}_{-\alpha}\bar{e}_\theta(-1)^{\lambda_0+h^\vee+1}1 \) is a singular vector in \( M(\Lambda + h^\vee d) \) of weight \( \Lambda + h^\vee d - (\lambda_0 + 1)\alpha_0. \) (Here and further, \( \bar{e} \) stands for \( \bar{e}(0) \), where \( e \in g). \)

A different arrangement of order in \( \Pi_{\alpha \in S}\bar{e}_{-\alpha} \) only makes a difference in the sign of \( v_{\lambda_0} \). In the following proof, for convenience we use \( x(m + \frac{1}{2}) \) to denote \( \bar{x}(m) \) in \( \hat{g}, m \in \mathbb{Z}. \)

Proof. It follows from Lemma 5.1 that the weight of \( v_{\lambda_0} \) is \( \mu = \Lambda + h^\vee d - (\lambda_0 + 1)\alpha_0. \) To prove that \( v_{\lambda_0} \) is a singular vector, it suffices to prove that for every homogeneous element \( w \in M^*(\Lambda + h^\vee d)_\mu \), we have \( H(v_{\lambda_0}, w) = 0 \) and that \( v_{\lambda_0} \neq 0. \) The latter statement will follow from another formula for \( v_{\lambda_0} \) given by Theorem 5.2. Here we prove the former one.

By (5.2), it is enough to show that

\[
(5.10) \quad H(e_\theta(-1)^{\lambda_0+h^\vee+1}1, \bar{w}) = 0,
\]

where

\[
(5.11) \quad \bar{w} = \Pi_{\alpha \in S}e_\alpha(-\frac{1}{2})e_\theta(-\frac{3}{2})e_\theta(-\frac{1}{2}) \cdot w.
\]

Any homogeneous element \( w \) in \( M(\Lambda + h^\vee d)_\mu \) is of the form

\[
\Pi_{i=1}^I e_{\gamma_i}(-m_i)\Pi_{j=1}^J e_\theta(-n_j)\Pi_{k=1}^K e_{-\eta_k}(-l_k) \cdot 1,
\]

where \( \eta_k \in \Delta_+, \gamma_i \in \Delta_+ - \{\theta\} \), \( K, I, J \in \mathbb{Z}_+, l_k \in \frac{1}{2}\mathbb{Z}_+, m_i, n_j \in \frac{1}{2}\mathbb{N}, \) and

\[
(5.12) \quad -\sum_k \eta_k + \sum_i \gamma_i + J\theta = (\lambda_0 + 1)\theta,
\]

\[
(5.13) \quad \sum_k l_k + \sum_i m_i + \sum_j n_j = \lambda_0 + 1.
\]

Case 1) \( n_j = \frac{1}{2} \) for some \( j \).
Note that $e_\theta(-\frac{1}{2})$ commutes up to a sign with elements of the form $e_\theta(-n_j)$ or $e_{n_i}(-m_i)$. Since $e_\theta^2(-\frac{1}{2}) = 0$, we have $\tilde{w} = 0$ by (5.11). (5.10) is satisfied automatically.

Case 2) All $n_j \geq 1$.

On one hand, since all $m_i \geq \frac{1}{2}$, we have the inequality

$$I \leq \frac{1}{2} \leq \lambda_0 + 1 - J \quad \text{(5.14)}$$

since by (5.13) we have

$$I = \sum_i \frac{1}{2} \leq \sum_i m_i = (\lambda_0 + 1) - \sum_j n_j - \sum_k l_k \leq d + 1 - \sum_j 1 = d + 1 - J.$$

And the equality in (5.14) holds iff

$$m_i = \frac{1}{2}, \quad n_j = 1, \quad l_k = 0 \quad \text{(5.15)}$$

for all $i, j, k$.

On the other hand, we rewrite (5.12) as

$$\sum_{i=1}^{I} \gamma_i = (\lambda_0 + 1 - J)\theta + \sum_k \eta_k.$$

By Lemma 5.2

$$I \geq 2(\lambda_0 + 1 - J) \quad \text{(5.16)}.$$

Comparing (5.14) and (5.16), we obtain that $I = 2(\lambda_0 + 1 - J)$ and then (5.13) holds. Furthermore at least $2(\lambda_0 + 1 - J)$ $\gamma_i$’s are in $s$. Now we divide case 2) into two subcases:

Subcase 2.1) $J < \lambda_0 + 1$.

Then at least one $\gamma_{i_0}$ is in $s$, i.e. $w$ can be expressed in the form $e_{\gamma_{i_0}}(-\frac{1}{2})w'$. Then $\tilde{w} = 0$ since $e_{\gamma_{i_0}}^2(-\frac{1}{2}) = 0$.

Subcase 2.2) $J = \lambda_0 + 1$. 

Under this assumption we have $I = K = 0$. $w = e_\theta(-1)^{\lambda_0 + 11}$. Then $w'$ can be expressed in the form $e_\theta(-1)w''$ since $e_\theta(-1)$ commutes with $\Pi_{\alpha \in \mathbb{S}} e_\alpha(-12)e_\theta(-12)$. By (5.2), we see that (5.10) is equivalent to

$$H(e_{-\theta}(1)e_\theta(-1)^{\lambda_0 + h^\vee + 11}, w'') = 0.$$ 

However it is well known that $e_{-\theta}(1)e_\theta(-1)^{\lambda_0 + h^\vee + 11} = 0$. □

We can choose $e_{\pm \alpha} \in g_{\pm \alpha}, \alpha \in \mathbb{S} \cup \{\theta\}$, in such a way that

(5.17) $[e_\alpha, e_{-\alpha}] = -\alpha$

(5.18) $[e_{-\gamma_i}, e_\theta] = e_{\beta_i}$

(5.19) $[e_{-\beta_i}, e_\theta] = -e_{\gamma_i}$

Indeed, we pick $e_{-\gamma_i}$ and $e_\theta$ arbitrarily, and define $e_{\beta_i}$ by the formula (5.18). Then (5.17) fixes $e_{\gamma_i}$ and $e_{-\beta_i}$. The formula (5.19) holds automatically since

$$[[e_{-\beta_i}, e_\theta], e_{-\gamma_i}] = (e_\theta, [e_{-\gamma_i}, e_{-\beta_i}]) = - (e_{\beta_i}, e_{-\beta_i}) = 1.$$

In the above notations, the singular vector $v_{\lambda_0}$ can be written as (cf. (5.6))

$$v_{\lambda_0} = \bar{e}_{-\theta}(1)\bar{e}_{-\theta} \prod_{i=1}^{h^\vee -2} (\bar{e}_{-\beta_i} \bar{e}_{-\gamma_i}) \cdot e_\theta(-1)^{\lambda_0 + h^\vee + 11}.$$

Note that $v_{\lambda_0}$ is independent of the order of $\prod_{i=1}^{h^\vee -2} (\bar{e}_{-\beta_i} \bar{e}_{-\gamma_i})$. Now we rewrite this formula of singular vectors in terms of a PBW basis. We introduce a combinatorial symbol $[m]_n = m(m-1) \cdots (m-n+1)$.

**Theorem 5.2** One has:

$$v_{\lambda_0} = \sum_{s=0}^{h^\vee -2} \sum_{(i_1, \cdots, i_s)} (k + h^\vee)^{h^\vee - 2} [\lambda_0 + h^\vee + 1]_{2(h^\vee - 2) - s} \times$$
\[ \times \left( (k + h^\vee) [\lambda_0 - s + 3] e_\theta (-1)^{\lambda_0 - s + 1} \\
+ [\lambda_0 - s + 3] e_\theta (-1)^{\lambda_0 - s} \bar{e}_\theta (-1) \bar{h}_\theta (-1) \\
+ [\lambda_0 - s + 3] e_\theta (-1)^{\lambda_0 - s - 1} \bar{e}_\theta (-1) \bar{e}_\theta (-2) \right) Q_{i_1} \cdots Q_{i_s} \cdot 1, \]

where \( Q_i = \bar{e}_{\beta_i} (-1) \bar{e}_{\gamma_i} (-1) \), and the sum \( \sum_{(i_1, \ldots, i_s)} \) is taken over all subsets of the set \( \{1, \cdots, h^\vee - 2\} \).

**Proof.** We assume that whenever some negative power of \( e_\theta (-1) \) appears in the following, the corresponding monomial term is zero.

It is not hard to prove by induction that

\[ (5.20) \quad \bar{e}_{-\beta_i} \bar{e}_{-\gamma_i} \cdot e_\theta (-1)^n \]
\[ = n (k + h^\vee) e_\theta (-1)^{n - 1} + n(n - 1) e_\theta (-1)^{n - 2} \bar{e}_{\beta_i} (-1) \bar{e}_{\gamma_i} (-1) \]
\[ + e_\theta (-1)^n \bar{e}_{-\beta_i} \bar{e}_{-\gamma_i} - n e_\theta (-1)^{n - 1} \bar{e}_{\beta_i} (-1) \bar{e}_{-\beta_i} \]
\[ - n e_\theta (-1)^{n - 1} \bar{e}_{\alpha_i} (-1) \bar{e}_{-\alpha_i} \cdot 1. \]

It follows that

\[ (5.21) \quad \bar{e}_{-\beta_i} \bar{e}_{-\gamma_i} \cdot e_\theta (-1)^n \cdot 1 \]
\[ = n (k + h^\vee) e_\theta (-1)^{n - 1} \cdot 1 + n(n - 1) e_\theta (-1)^{n - 2} \bar{e}_{\beta_i} (-1) \bar{e}_{\gamma_i} (-1) \cdot 1. \]

Using (5.20) and (5.21), we get by induction that

\[ v' := \prod_{i=1}^{h^\vee - 2} \bar{e}_{-\beta_i} \bar{e}_{-\gamma_i} \cdot e_\theta (-1)^{\lambda_0 + h^\vee + 1} \]
\[ = \sum_{s=0}^{h^\vee - 2} \sum_{(i_1, \cdots, i_s)} (k + h^\vee)^{h^\vee - s - 2} [\lambda_0 + h^\vee + 1] (2 (h^\vee - 2) - s) \times \]
\[ \times e_\theta (-1)^{\lambda_0 - s + 3} Q_{i_1} \cdots Q_{i_s} \cdot 1. \]

Since

\[ [\bar{e}_{-\theta}, Q_i] = 0 \]
and

\[ [\bar{e}_{-\theta}, e_\theta(-1)^n] = n(n - 1)e_\theta(-1)^{n-2}\bar{e}_{-\theta}(-2) + ne_\theta(-1)^{n-1}\bar{h}_\theta(-1), \]

we have

\[
\bar{e}_{-\theta}v' = \sum_{s=0}^{h^\vee-2} \sum_{(i_1, \ldots, i_s)} (k + h^\vee)(h^\vee-s-2)[\lambda_0 + h^\vee + 1]_{(2(h^\vee-2)-s)} \times \\
\times \left( [\lambda_0 - s + 3]e_\theta(-1)\lambda_0-s+1\bar{e}_{-\theta}(-2) \right. \\
\left. + (\lambda_0 - s + 3)e_\theta(-1)\lambda_0-s+2\bar{h}_\theta(-1) \right) Q_{i_1} \cdots Q_{i_s} \cdot v_0.
\]

Using another identity

\[ [\bar{e}_{-\theta}(1), e_\theta(-1)^n] = n(n - 1)e_\theta(-1)^{n-2}\bar{e}_{-\theta}(-1) + ne_\theta(-1)^{n-1}\bar{h}_\theta, \]

we get the desired formula.

\[ \square \]

**Remark 5.3** It follows from Theorem 5.2 that \( v_{\lambda_0} \neq 0 \). Moreover the only term which does not involve the odd factors is a non-zero multiple of \( e_\theta(-1)\lambda_0+1 \). Therefore the submodule of the Verma module \( M(\Lambda + h^\vee d) \) generated by \( v_{\lambda_0} \) is again a Verma module.

**Theorem 5.3** We have the following isomorphism

\[ L(\Lambda + h^\vee d) \cong M(\Lambda + h^\vee d)/\langle v_{\lambda_0}, f_i^{\lambda_i+1}, i = 1, \ldots, l \rangle, \]

where \( \langle v_{\lambda_0}, f_i^{\lambda_i+1}, i = 1, \ldots, l \rangle \) denotes the submodule of \( M(\Lambda + h^\vee d) \) generated by the singular vectors \( v_{\lambda_0}, f_i^{\lambda_i+1}, i = 1, \ldots, l \).

**Proof:** Since the weights of \( v_{\lambda_0}, f_i^{\lambda_i+1}, i = 1, \ldots, l \) are \( \Lambda + h^\vee d - (\lambda_i + 1)\alpha_i, i = 0, 1, \ldots, l \) respectively, we have the following isomorphism of \( \hat{g} \)-modules:

\[ \langle v_{\lambda_0}, f_i^{\lambda_i+1}, i = 1, \ldots, l \rangle \]
\[ \sum_{i=0}^{l} M(\Lambda + h^\vee d - (\lambda_i + 1)\alpha_i) \]
\[ = \sum_{i=0}^{l} F \otimes \tilde{M}(\Lambda - (\lambda_i + 1)\alpha_i) \]
\[ = F \otimes \{ f_{i+1}^{\lambda_i+1}, i = 0, 1, \ldots, l \}. \]

Therefore we have the following isomorphism of \( \hat{g} \)-modules:

\[ M(\Lambda + h^\vee d)/\langle v_{\lambda_0}, f_{i+1}^{\lambda_i+1}, i = 1, \ldots, l \rangle \]
\[ = F \otimes \tilde{M}(\Lambda) \]
\[ = F \otimes \tilde{L}(\Lambda) \]
\[ = L(\Lambda + h^\vee d). \quad \Box \]

References

[A] A. Astashkevich, *On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras*, preprint (1993)

[BPZ] A. Belavin, A. Polyakov and A. Zamolodchikov, *Infinite conformal symmetries in two-dimensional quantum field theory*, Nucl. Phys. B241(1984) 333-380

[B] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA. 83(1986)3068-3071

[DMZ] C. Dong, G. Mason and Y. Zhu, *Discrete series of the Virasoro algebra and the moonshine module*, preprint

[FF] B.L. Feigin and D.B. Fuchs, *Verma modules over the Virasoro algebra*, Lect. Notes Math. 1060(1984)230-245
[FHL] I. B. Frenkel, Y. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. vol. 104, No. 494 (1993)

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, New York, (1988)

[FZ] I. B. Frenkel and Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebra*, Duke Math. J., vol. 66, No. 1 (1992) 123-168

[K] V. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge Univ. Press (1990)

[KS] Y. Kazama and H. Suzuki, *New $N = 2$ superconformal field theories and superstring compactification*, Nucl. Phys. B321 (1989), 232-268

[KT] V. Kac and I. Todorov, *Superconformal current algebras and their unitary representations*, Comm. Math. Phys. 102 (1985) 337-347

[KW] V. Kac and M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci. USA, vol. 85 (1988) 4956-4960

[KW] V. Kac and M. Wakimoto, *Unitarizable highest weight representations of the Virasoro, Neveu-Schwarz and Ramond algebras*, Lect. notes Phys. 261 (1986) 345-371
[L] B.H. Lian, *On the classification of simple vertex operator algebras*, preprint (1992)

[T] H. Tsukada, *Vertex operator superalgebra*, Comm. Alg. 18(7)(1990)2249-2274

[W] W. Wang, *Rationality of Virasoro vertex operator algebras*, Duke Math. J., IMRN, vol.71, No.1(1993) 197-211

[Z] Y. Zhu, *Vertex operator algebras, elliptic functions and modular forms*, Ph.D. dissertation, Yale Univ. (1990)

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