THE DERIVATION OF THE COMPRESSIBLE EULER EQUATION FROM QUANTUM MANY-BODY DYNAMICS

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Abstract. We study the three dimensional many-particle quantum dynamics in mean-field setting. We forge together the hierarchy method and the modulated energy method. We prove rigorously that the compressible Euler equation is the limit as the particle number tends to infinity and the Planck’s constant tends to zero. We establish strong and quantitative microscopic to macroscopic convergence of mass and momentum densities up to the 1st blow up time of the limiting Euler equation. We justify that the macroscopic pressure emerges from the space-time averages of microscopic interactions, which are in fact, Strichartz-type bounds. We have hence found a physical meaning for Strichartz type bounds which were first raised by Klainerman and Machedon in this context.

Contents

1. Introduction 1
1.1. Statement of the Main Theorem 3
1.2. Outline of the Proof 7
2. BBGKY Hierarchy v.s. H-NLS: Long-time Uniform in $\hbar$ Estimates 9
2.1. A Tool Box of Space-time Estimates 15
2.2. A Klainerman-Machedon Bound 1st 21
2.3. Feeding the Strichartz Bound into the $H^1$ Estimate 24
2.4. Convergence Rate for Every Finite Time 27
3. H-NLS v.s. the Compressible Euler Equation: a Modulated Energy Approach 30
3.1. The Evolution of the Modulated Energy 32
3.2. Modulated Energy Estimate 35
Appendix A. Miscellaneous Lemmas 39
A.1. Collapsing Estimate and Strichartz Estimates 39
A.2. Convolution and Commutator Estimates 42
Appendix B. Energy Estimate 43
References 45

1. INTRODUCTION

The analysis of the nonlinear fluid equations like the Euler equations and the Navier-Stokes equations, is an important (if not vital) part of many areas of pure and applied mathematics, science, and engineering. On the one hand, their validity has certainly been checked countless times against the experiments. On the other hand, the rigorous derivation

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of these macroscopic continuum equations from basic microscopic Newtonian / Maxwell / quantum particle models has largely remained open. It is certainly of fundamental interest in mathematics to establish such derivations and prove that macroscopic quantities like pressure and viscosity emerge from the averaging of microscopic quantities. In this paper, we prove the derivation of the compressible Euler equation from the quantum $N$-body dynamic in the mean-field setting. We choose to start from the quantum theory as it is, at the moment, the most accurate microscopic model and such a derivation would also establish (again) that there is no obvious gap between the basic models in quantum and classical scales.

In the setting of classical mechanics, a strategy of the derivation of fluid equations from particle systems is to 1st pass to a mesoscopic Boltzmann equation, then derive the desired fluid equation from the Boltzmann equation. (See, for example, the standard monographs [8, 32, 53] and references within.) However, such a route may not suit our purpose here. On the one hand, the validity of the classical Boltzmann equations is only justified up to a sufficiently small time and is not clear if it covers the 1st blow up time of the Euler equation. On the other hand, the derivation of the quantum Boltzmann equation is at a rudimentary stage. (See, for example, [10, 15, 27] and the references within.) Not to mention the possibility that one might need to pass to another classical Boltzmann equation if one takes such a route. Moreover, we would like to understand the fine interplay between $\hbar$ and $N$, the two fundamental constants, which differ by $10^{57}$ in SI units. In fact, starting from 2019, the mass unit is defined via the Planck’s constant. Thus, we choose to derive the compressible Euler equation directly from quantum many-body dynamics.

We consider Bosons in this paper as it is more directly related to the Newton-Maxwell particles due to the assumption that particles are indistinguishable. (Fermions are also interesting, see for example, the survey [51].) We consider the 3D linear $N$-body bosonic Schrödinger equation:

$$i\hbar \partial_t \psi_{N,\hbar} = H_{N,\hbar} \psi_{N,\hbar}$$

with Hamiltonian $H_{N,\hbar}$ given by

$$H_{N,\hbar} = \sum_{j=1}^{N} -\frac{1}{2} \hbar^2 \Delta x_j + \frac{1}{N} \sum_{1 \leq j < k \leq N} V_N(x_j - x_k)$$

where

$$V_N(x) = N^{3\beta} V(N^{\beta} x),$$

and the factor $1/N$ is to make sure the interactions grow like $N$ instead $N^2$, a mean-field like scaling. The marginal densities $\gamma_{N,\hbar}^{(k)}$ associated with $\psi_{N,\hbar}$ in kernel form are given by

$$\gamma_{N,\hbar}^{(k)}(t, x_k, x_k') = \int \psi_{N,\hbar}(t, x_k, x_{N-k}) \overline{\psi_{N,\hbar}(t, x'_k, x_{N-k})} dx_{N-k}$$

where $x_k = (x_1, ..., x_k) \in \mathbb{R}^{3k}$ and $x_{N-k} = (x_{k+1}, ..., x_N) \in \mathbb{R}^{3(N-k)}$. Notably, one can derive cubic nonlinear Schrödinger equation (NLS) as the $N \to \infty$ limit of (1.1) with $\hbar$ fixed, then the well-known Madelung transform [49] relates Schrödinger type equation and the macroscopic Euler equations in a formal limit process as $\hbar$ tends to zero. That is, the macroscopic equations could formally emerge from (1.1) as an iterated limit: $\lim_{\hbar \to 0} \lim_{N \to \infty}$. Such an iterated limit is far from satisfactory in either mathematics or physics. Not only an iterated limit could lose information in any one limit, it kills the fine interplay between
\( \hbar \) and \( N \) and hence cannot show the \((N, \hbar)\) threshold at which classical behavior starts to dominate. Therefore, for a more complete and deeper understanding, we deal with the \((N, \hbar)\) double limit which is also a more challenging problem.

Our limiting macroscopic equation is the 3D compressible Euler equation, which is,

\[
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
\partial_t u + (u \cdot \nabla) u + b_0 \nabla \rho = 0,
\end{cases}
\]

if written in velocity form, or

\[
\begin{cases}
\partial_t \rho + \text{div} J = 0, \\
\partial_t J + \text{div} \left( \frac{J \otimes J}{\rho} \right) + \frac{1}{2} \nabla (b_0 \rho^2) = 0,
\end{cases}
\]

if written in momentum form. Here, as usual, \( \rho(t, x) : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R} \) is the mass density, \( u(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x)) : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \) denotes the velocity of the fluid, \( J(t, x) = (p(u))(t, x) : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \) denotes the momentum of the fluid as the coupling constant\(^1\) \( b_0 = \int V \) which is the macroscopic effect of the microscopic interaction \( V \) and hints that pressure \( b_0 \rho^2 \) should originate from the microscopic interaction between particles.

1.1. Statement of the Main Theorem.

**Theorem 1.1.** Let \( d = 3, \beta < \frac{2}{5} \), the marginal densities \( \Gamma_{N, \hbar} = \{ \gamma_{N, \hbar}^{(k)} \} \) associated with \( \psi_{N, \hbar} \) be the solution to the \( N \)-body dynamics with a Schwarz even pair interaction \( V \geq 0 \). The \( N \)-body initial data satisfies the following condition:

(a) \( \psi_{N, \hbar}(0) \) is normalized, that is, \( \| \psi_{N, \hbar}(0) \|_{L^2} = 1 \).

(b) The \( N \)-body energy bounds hold:

\[
(\psi_{N, \hbar}(0), (H_{N, \hbar}/N + 1)^k \psi_{N, \hbar}(0)) \leq (E_{0, \hbar})^k
\]

for \( k \leq (\ln N)^{100} \).

(c) \( \Gamma_{N, \hbar}(0) \) is asymptotically factorized in the sense that

\[
\left\| \prod_{j=1}^{k} \langle \hbar \nabla_{x_j} \rangle \langle \hbar \nabla_{x_j'} \rangle \left[ \gamma_{N, \hbar}^{(k)}(0) - |\phi_{N, \hbar}^{in}| \langle \phi_{N, \hbar}^{in} \rangle \otimes^k \right] \right\|_{L^2_{x, x'}} \leq (E_{0, \hbar})^k N^{\frac{2}{5} \beta - 1}
\]

for \( k \leq (\ln N)^{100} \), where \( \phi_{N, \hbar}^{in} \) is normalized that \( \| \phi_{N, \hbar}^{in} \|_{L^2} = 1 \) and has finite energy\(^2\), that is

\[
\frac{1}{2} \| \phi_{N, \hbar}^{in} \|_{L^2}^2 + \frac{1}{2} \| \hbar \nabla \phi_{N, \hbar}^{in} \|_{L^2}^2 + \frac{1}{2} V_N \ast |\phi_{N, \hbar}^{in}|^2, |\phi_{N, \hbar}^{in}|^2 \| \leq E_0.
\]

(d) The initial datum \((\rho^{in}, u^{in})\) to \((1.5)\) satisfy

\[
\rho^{in} \geq 0, \quad \int \rho^{in}(x) dx = 1,
\]

\(^1\)The equations \((1.5)\) and \((1.0)\) are not hyperbolic if the microscopic potential \( V \) is focusing or \( b_0 < 0 \).

\(^2\)It is expected that \( E_0 \leq E_{0, \hbar} \) due to the correction structure.
and is such that the Euler system (1.3) has a solution \((\rho, u)\) satisfying

\[
\begin{align*}
\rho &\geq 0, \quad \int_{\mathbb{R}^d} \rho(t, x)dx = 1, \\
&\quad (\rho, u) \in C([0, T_0]; H^s) \cap C^1([0, T_0]; H^{s-1}),
\end{align*}
\]

where \(s > \frac{d}{2} + 3\). The modulated / renormalized energy at initial time tends to zero:

\[
\int_{\mathbb{R}^d} |(ih\nabla - u^{in})\phi_{N,h}^{in}|^2 dx + b_0 \int_{\mathbb{R}^d} (|\phi_{N,h}^{in}|^2 - \rho^{in})^2 dx \leq C\hbar^2.
\]

Then under the restriction that\(^3\)

\[
N \geq e^{(2)} \left( \frac{C^2 V E_{0,h}^2 T_0}{\hbar^7} \right)^2,
\]

for \(N \geq N_0(\beta)\) and \((\rho, u)\) satisfying (1.3), we have the quantitative estimates on the convergence of the mass density

\[
\|\gamma_{N,h}^{(1)}(t, x; x) - \rho(t, x)\|_{L^\infty_T[0,T_0]L^2_x(\mathbb{R}^d)} \leq C(T_0) \left( \frac{1}{\ln N} + \hbar \right),
\]

on the convergence of the momentum density for \(r \in (1, 4/3)\)

\[
\left\| \text{Im} \left( \hbar \nabla x_1 \gamma_{N,h}^{(1)}(t, x; x) - (\rho u)(t, x) \right) \right\|_{L^\infty_T[0,T_0]L^r_x(\mathbb{R}^d)} \leq C(T_0) \left( \frac{1}{(\ln N)^{5(1 - \frac{1}{r})}} + \hbar^{\frac{4-3r}{r}} \right),
\]

and on the emergence of pressure

\[
\left\| \int V_N(x - x_2)\gamma_{N,h}^{(2)}(t, x, x_2; x, x_2)dx_2 - b_0 \rho(t, x)^2 \right\|_{L^1_T[0,T_0]L^1_x(B_0^\delta)} \leq C(T_0) \left( \frac{R^{d/2}}{\ln N} + \hbar \right).
\]

where coupling constant is \(b_0 = \int V\).

Theorem 1.4 is the first of its type and involves the up-to-date techniques in the hierarchy method as well as well-developed modulated energy approach and we can in fact see it from its assumptions. The \(N\)-body energy condition in (b) is inspired by purely factorized or statistically independent datum, and has been used since the 1st wave of work \[1, 26, 28, 29, 30, 31\] on deriving NLS using hierarchy methods. It is usually cashed in as the \(H^1\) bound on the marginals\(^4\)

\[
\left\| \prod_{j=1}^k \langle \hbar \nabla x_j \rangle \langle \hbar \nabla x'_j \rangle \gamma_{N,h}^{(k)}(t) \right\|_{L^2_{x,x'}} \leq (2E_{0,h})^k
\]

for \(k \leq (\ln N)^{100},\ N \geq N_0(\beta)\) which is independent of \(k\) and \(\hbar\), and all \(t \in (-\infty, +\infty)\). Here, we allow the \(k \geq 2\) energy bound \(E_{0,h}\) to depend on \(\hbar\) (the \(k = 1\) case can be the same \(E_0\) as in (1.9)) as long as it is finite for every nonzero \(\hbar\), so that a larger variety of initial datum are included at the cost of the restriction (1.13) with an unspecific factor \(E_{0,h}\). This is a natural requirement as the \(k > 2\) energy includes higher derivatives which do not play well with \(\hbar\). Though the initially asymptotic statistically independent assumption (1.8) in (c) is like

\(^3\)The composite function \(e^{(n)}(x) := e^{(n-1)(x)}\) and \(C_V\) is a constant which only depends on some Sobolev norms of \(V\) as needed in the proof.

\(^4\)We include a proof as Proposition 3.1 for completeness.
usual in this line of work, the optimal decay rate is believed (and proved in some cases, see for example, [3, 6]) to be $1/N$ for every given $\hbar$. We assume $N^{\frac{\alpha}{2}-1}$ here so that the paper is self-contained as we will prove this rate at the first step of bootstrapping argument. Indeed, for $\hbar = 1$, the convergence rate has been achieved in [23]. On the other hand, compared to the $N$-body energy bounds (1.7), the energy bound $E_0$ for $\phi_{N,h}^{\text{in}}$ is independent of $\hbar$ to be compatible with the modulated energy bound.

As for the assumptions regarding the initial datum of (1.5), the local well-posedness of compressible Euler equations has been studied by many authors, for example, see the monograph [50]. But we remark that, there are many variants / choices / constructions of the modulated energy (1.12) which look seemingly different but are intuitively and closely related up to an error term as the initial quantities like $|\phi_{N,h}^{\text{in}}|$ and $\rho^{\text{in}}$ are supposed to be close. In fact, the full modulated energy which we will use and is going to be controlled by (1.12) takes the form

$$
\mathcal{M} [\phi_{N,h}, \rho, u] (t) = \frac{1}{2} \int_{\mathbb{R}^d} |(i\hbar \nabla - u) \phi_{N,h}(t)|^2 dx + \frac{1}{2} \langle V_N * |\phi_{N,h}|^2, |\phi_{N,h}|^2 \rangle + \frac{b_0}{2} \int_{\mathbb{R}^d} \rho^2 dx - b_0 \int_{\mathbb{R}^d} \rho |\phi_{N,h}|^2 dx.
$$

We assume the convergence rate (1.12) to be $\hbar^2$ which should also be optimal, since the smallness factor in the modulated kinetic part is at most $\hbar^2$. Besides, the $\hbar^2$ rate can be achieved with WKB type initial datum.

Theorem 1.1 rigorously establishes the derivation of the macroscopic equation (1.5) in classical mechanics from the quantum many-body systems as a regional double limit and provides convergent rate estimates in the strong norm sense. It also justifies the emergence of the macroscopic pressure from the space-time averages of microscopic interactions, which are in fact, Strichartz-type bounds. Notice that, the microscopic quantity converging to the pressure $\rho^2$ is basically $\gamma_{N,h}^{(2)}(x, x, x, x)$. It is not necessarily finite or defined a.e. if we are below $H^{9/8}$ in 3D by the Sobolev embeddings, and we only have $H^1$ here. The Strichartz bound, 1st raised by Klanerman-Machedon (KM) [45] in this context, makes this quantity well-defined and have unexpectedly verified the theory that pressure is the space-time averaging of the microscopic interactions under the physical $H^1$ assumption. We have hence found the 1st physical meaning for Strichartz type bounds since its original discovery in [60]. Such a discovery is part of the main novelty of this paper. On the other hand, the limit in Theorem 1.1 is taken within the region (1.13) which proves the dominance of classical behaviors when $N >> \hbar$. Such a requirement is physical as they indeed differ by $10^{57}$ in reality but we believe (1.13) is not optimal and searching for the sharp threshold (may not exist, some mesoscopic behaviors might happen) between classical and quantum behaviors is certainly of interest. However, it would not be surprising to have totally independent $N$ and $h$ in weak / weak* limits as the classical-now-elementary Riemann-Lebesgue lemma shows that a weak convergent sequence can be uniformly bounded away from its weak limit. To work with the 3D $N$-body equation smoothly in the physical $H^1$ energy space, we improvise and extend the up-to-date hierarchy method in KM format.

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5Such an averaging effect certainly cannot be observed if one assumes higher than $H^{9/8}$ regularity at the $N$-body level, but we remark that it cannot be observed either if one passes through the NLS in the $H^1$ setting as $|\phi|^4$ is already defined a.e. without any need to appeal to Strichartz.
The hierarchy method in general was 1st suggested by M. Kacs and proved to be successful in Lanford’s work [46] regarding the Boltzmann equation. The hierarchy method we use in the paper is actually more originated from the 1st wave of work [1, 29, 30, 31] by Adami-Golse-Teta and Erdös-Schlein-Yau on deriving NLS from quantum many-body dynamics around 2005 as suggested by Spohn [59]. At that time, the main difficulty lies in the uniqueness of the infinite Gross-Pitaevskii (GP) hierarchy. With a sophisticated Feynman graph analysis in the fundamental papers [29, 30, 31] which derived the 3D cubic defocusing NLS, Erdös, Schlein, and Yau proved the $H^1$-type unconditional uniqueness of the $\mathbb{R}^3$ cubic GP hierarchy. The first series of ground breaking papers have motivated a large amount of work.

Subsequently in 2007, by imposing an additional a-prior condition on space-time norm, Klainerman and Machedon [45], inspired by [29, 44], gave another uniqueness criterion of the GP hierarchy in a different space of density matrices defined by Strichartz type norms. They provided a different combinatorial argument, the now so-called Klainerman-Machedon board game, to combine the inhomogeneous terms effectively reducing their numbers and then derived a space-time estimate to control these terms. At that time, it was open on how to prove that the limits coming from the $N$-body dynamics satisfy the now so called KM space-time bound required for uniqueness. Nonetheless, [45] has made the delicate analysis of the GP hierarchy approachable from the perspective of PDE. Klainerman and Machedon also did not know the KM bound required for uniqueness, which is an usual product of Strichartz type well-posedness theory, actually has a physical meaning.

Later, Kirkpatrick, Schlein, and Staffilani [43] obtained the KM space-time bound via a simple trace theorem in both $\mathbb{R}^2$ and $\mathbb{T}^2$ and derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic. Such a scheme also motivated many works [11, 13, 18, 20, 35, 38, 57, 58, 61] for the uniqueness of GP hierarchies and enables the hierarchy method on the derivation 1D or 2D NLS directly from 3D [16, 20, 55], which is quite different but has some similar flavor with our Theorem 1 here. However, how to verify the KM bound in the 3D cubic case remained fully open at that time.

Then in 2011, T. Chen and Pavlović proved that the 3D cubic KM space-time bound held for the defocusing $\beta < 1/4$ case in [12]. The result was quickly improved to $\beta < 2/7$ by X. Chen in [14] and then extended to the almost optimal case, $\beta < 1$, by X. Chen and Holmer in [17, 19], by lifting the $X^{1,b}_{1,\theta}$ space techniques from NLS theory into the field. Away from being the first work to prove the 3D KM bound, the work [12] hinted two unforeseen directions of the hierarchy method: one direction is to prove new NLS results via the more complicated hierarchies, while the other is that it is possible to derive NLS without a compactness or uniqueness argument as in the 1st wave of papers.

In 2013, by introducing the quantum de Finetti theorem from [47] to the field, T. Chen, Hainzl, Pavlović and Seiringer [9] provided a simplified proof of the $L^\infty_t H^1_x$-type 3D cubic uniqueness theorem as stated in [29]. This method motivated many work [25, 40, 41, 56] and has climbed to a climax recently as the previously open $\mathbb{T}^d$ energy-critical and supercritical NLS unconditional uniqueness problems progressed in [39] were completely and unifiedly resolved via the analysis of the supposedly more complicated GP hierarchy in [21, 22, 24] which used, the $l^2$ decoupling theorem [5] and has helped in the derivation of the energy-critical NLS [21, 22]. With these new exciting developments, it seems that KM bound method is obsolete though the KM board game stays useful. Such an impression or conclusion is apparently wrong.

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[6] Private communication with M. Machedon.
Recently, on the basis of \cite{12, 14, 17, 19}, X. Chen and Holmer in \cite{23} reformatted the hierarchy method with KM space-time estimates and proved a bi-scattering theorem for the NLS to obtain almost optimal local in time convergence rate estimates under $H^1$ regularity. They integrate the idea from the Fock space approach (see, for example, \cite{2, 4, 6, 36, 37} and references within\cite{7}), that, using H-NLS as an intermediate dynamic, into the hierarchy method. Most notably, the work \cite{23}, though it did not use the KM bound, sheds light on our principal part in which we prove strong, quantitative, uniform in $\hbar$, estimates regarding the BBGKY hierarchy and the H-NLS hierarchy.

On the other hand, the behavior of the wave function of cubic defocusing NLS as the Planck’s constant goes to zero is studied by many authors using various approaches. In \cite{34}, Grenier derived compressible Euler equations for small time from cubic NLS by WKB. Jin, Levermore and McLaughlin in \cite{42} established the semiclassical limit of the 1D defocusing cubic NLS for all time by using the complete integrability. In \cite{48}, F. Lin and P. Zhang investigated Gross-Pitaevskii equation (a cubic Schrödinger equation nonzero at infinity) in 2D exterior domains by adopting the modulated energy method. For a more detailed survey related to semiclassical limits of NLS, see \cite{7, 62} and references within.

As seen from above, it is highly nontrivial to derive Euler equations from NLS, let alone from quantum $N$-body dynamics. As the first breakthrough, Golse and Paul \cite{33}, with the help of Serfaty’s inequality \cite{54} Corollary 3.4, used the modulated energy method in the quantum $N$-body setting to justify the validity of the joint mean-field and classical limit of the quantum $N$-body dynamics leading to the pressureless Euler-Poisson with repulsive Coulomb potential. Subsequently, Rosenzweig complemented \cite{33} in \cite{52} by combining mean-field, semiclassical and quasi-neutral limits to reach a derivation of an incompressible Euler equation on $\mathbb{T}^d$ with binary Coulomb interactions.

Though both singular, the $\delta$-interaction, which results in a compressible Euler equation, is substantially different from the Coulomb potential and calls for new ideas. The strong convergence and quantitative estimates are much more demanding as well. Our proof combines improvisation and extension of up-to-date techniques in the hierarchy method and the well-developed modulated energy method.

1.2. Outline of the Proof. \eqref{1.1} is very different from our goal \eqref{1.5} or \eqref{1.6}, at least by the look of them. Key quantities of $\gamma^{(k)}_{N,\hbar}$ in \eqref{1.14} – \eqref{1.16} are all traces and thus as usual, are regularity thirsty and does not react well as $\hbar \to 0$, while solutions to \eqref{1.5} will blow up in finite time. Thus we insert H-NLS \eqref{2.1} as an intermediate dynamic. We hence divide the proof of Theorem \ref{1.1} into two parts in Sections 2 and 3 respectively. The first part is the quantitative estimate between the BBGKY hierarchy and the H-NLS using an improvised and extended version of cutting edge hierarchy methods, while the second part is comparing the H-NLS equation with the compressible Euler equation \eqref{1.5} by means of modulated energy approach. Here, we are using the BBGKY hierarchy directly satisfied by $\gamma^{(k)}_{N,\hbar}$. We are not using any Wigner transforms in this paper. Theorem \ref{1.1} then follows from summing the concluding estimates in Sections 2 and 3 respectively.

There are two main difficulties in Section 2. One is to make sure all the differences estimates are uniform in $\hbar$. The other one is to make sure the estimates hold for every finite
time despite that the method [23] only works local in time. How to circumvent these two difficulties is also the main technical novelty of this paper. The key is to implement the Klainerman-Machedon space-time bound, which was thought of only as a part of uniqueness, to strengthen our local in time quantitative estimate. The whole process is still very technical, we illustrate the principle logic of the proof of Section 2 by the following diagram.

Global $H^1$ bound on the difference $w_{N,h}^{(k)}$

↓

KM bound on $w_{N,h}^{(k)}$

↓ Feedback

Summable, decay in $N$, $H^1$ estimate on $w_{N,h}^{(k)}$

↓ Feedback

Summable, decay in $N$, KM bound on $w_{N,h}^{(k)}$

Sum up (iteration argument)

Convergence rate for every finite time

The logic above looks quite like proving global well-posedness for a $H^1$ subcritical NLS. However, this is the 1st time such a diagram is carried out for the hierarchy analysis. The technical reason is exactly as mention before (and in almost all paper in this field), though the $N$-body equations and hierarchies are linear, we are dealing with traces instead of powers.

In Section 2.1, we first provide some preliminary or crude estimates for the difference between BBGKY hierarchy and H-NLS hierarchy. We then prove in Section 2.2 that $w_{N,h}^{(k)}$ satisfies the Klainerman-Machedon bound by gathering information from the $(\ln N)^{10}$ coupling level. Subsequently in Section 2.3, we feed the KM bound / a Strichartz bound back, to strengthen the $H^1$ estimate for $k < (\ln N)^2$ to obtain summable and decay in $N$ estimates. We can further feed the $H^1$ estimate of $w_{N,h}^{(k)}$ back into the KM bound proof and deduce that the KM bound actually decays in $N$. Notice the difference between the given $k$-th marginal and the selectable coupling level. For a given $k$-th marginal, how to select a suitable coupling level to yield desired information is a fine technical point. Section 2.2 to 2.3 addresses this issue. Finally, in Section 2.4, with the conclusion in Section 2.3, we can sacrifice some decays in $N$ to bootstrap the quantitative estimates to every finite time by a clever but elementary manipulation.

As the $N$-body estimates have been set ready in Section 2, in Section 3 we adopt modulated energy method to compare directly the H-NLS equation with compressible Euler equations before the blowup time. The idea of proving convergence is via a Gronwall argument on modulated energies assuming and using the regularity of the limiting solution. Therefore, in Section 3.1, we compute the evolution of modulated energy. Subsequently in Section 3.2, we control the error term originating from the evolution of modulated energy to obtain a Gronwall type estimate. Due to the work in Section 2, we are able to have a close match inside the modulated energy, and hence the error term is very tractable.

The main novelty of the paper is Theorem 1.1 which establishes a strong microscopic to macroscopic derivation up to the 1st blow up time of the limiting Euler equation from the fundamental quantum $N$-body dynamics. The proof also combines the hierarchy method
and the modulated energy method for the 1st time. We indeed anticipated more fusion of these two methods in the future. During the course of proof, we have implemented the Klainerman-Machedon Strichartz type bound and hence verified the emergence of pressure as the space-time averagings of microscopic interaction. This argument thus discovers a physical meaning for Strichartz type bounds for PDE and harmonic analysis.

2. BBGKY Hierarchy v.s. H-NLS: Long-time Uniform in \( h \) Estimates

The main goal in this section is to establish long-time uniform in \( h \) estimate for the difference \( \gamma_{N,h}^{(k)} - |\phi_{N,h}\rangle \langle \phi_{N,h}|^{\otimes k} \) where \( \phi_{N,h} \) is the solution to H-NLS equation as below

\[
(2.1) \quad \begin{cases}
  i\hbar \partial_t \phi_{N,h} = -\frac{1}{2} \hbar^2 \Delta \phi_{N,h} + (V_N * |\phi_{N,h}|^2) \phi_{N,h}, \\
  \phi_{N,h}(0) = \phi_{in}.
\end{cases}
\]

Our strategy is to use the hierarchy approach. It is well-known that \( \Gamma_{N,h}(t) = \{ \gamma_{N,h}^{(k)} \} \) satisfies the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy

\[
(2.2) \quad i\hbar \partial_t \gamma_{N,h}^{(k)} = \sum_{j=1}^k \left[ -\frac{\hbar^2}{2} \Delta x_j, \gamma_{N,h}^{(k)} \right] + \frac{1}{N} \sum_{1 \leq i < j \leq k} \left[ V_N(x_i-x_j), \gamma_{N,h}^{(k)} \right] + \frac{N-k}{N} \sum_{j=1}^k \text{Tr}_{k+1} \left[ V_N(x_j-x_{k+1}), \gamma_{N,h}^{(k+1)} \right].
\]

In addition to \((2.2)\), we will use the so-called H-NLS hierarchy which takes the form

\[
(2.3) \quad i\hbar \partial_t \gamma_{H,h}^{(k)} = \sum_{j=1}^k \left[ -\frac{\hbar^2}{2} \Delta x_j, \gamma_{H,h}^{(k)} \right] + \sum_{j=1}^k \text{Tr}_{k+1} \left[ V_N(x_j-x_{k+1}), \gamma_{H,h}^{(k+1)} \right],
\]

generated by

\[
\{ \gamma_{H,h}^{(k)}(t, x_k; x'_k) = |\phi_{N,h}\rangle \langle \phi_{N,h}|^{\otimes k} \},
\]

the tensor product of solutions to H-NLS equation \((2.1)\).

Denote the difference between the BBGKY hierarchy and the H-NLS hierarchy by

\[
(2.4) \quad w_{N,h}^{(k)} = \gamma_{N,h}^{(k)} - \gamma_{H,h}^{(k)}.
\]

For convenience, we first set up some notations. Define

\[
(2.5) \quad S_{(1,k)}^h = \prod_{j=1}^k \langle h \nabla x_j \rangle \langle h \nabla x'_j \rangle,
\]

the collision operator

\[
(2.6) \quad B_{N,j,k+1}f^{(k+1)} = B_{N,j,k+1}^+ f^{(k+1)} - B_{N,j,k+1}^- f^{(k+1)}
\]

\[
= \int V_N(x_j - x_{k+1}) f^{(k+1)}(x_k, x_{k+1}; x'_k, x_{k+1}) dx_{k+1}
\]

\[
- \int V_N(x'_j - x_{k+1}) f^{(k+1)}(x_k, x_{k+1}; x'_k, x_{k+1}) dx_{k+1},
\]

\footnote{As it is indeed a tensor product, the energy bound \((1.17)\) also holds for \( \gamma_{H,h}^{(k)} \) with \( E_{0,h} \) replaced by \( E_0 \).}
and

\begin{equation}
B_{N,j,k+1} = \frac{1}{\hbar} B_{N,j,k+1}, \quad B_{N,j,k+1}^\pm = \frac{1}{\hbar} B_{N,j,k+1}^\pm.
\end{equation}

Define the quantum mass density and momentum density in the quantum \(N\)-body setting

\begin{equation}
\gamma_{N,j}^{(1)}(t; x; x), \quad J_{N,j}^{(1)}(t; x; x) = \text{Im} \left( \hbar \nabla_x \gamma_{N,j}^{(1)}(t; x; x) \right)
\end{equation}

and

\begin{equation}
\rho_{N,j}(t; x) = |\phi_{N,j}(t; x)|^2, \quad J_{N,j}(t; x) = \hbar \text{Im} \left( \phi_{N,j}(t; x) \nabla_x \phi_{N,j}(t; x) \right)
\end{equation}

with respect to H-NLS equation.

Our main theorem of this section is the following.

**Theorem 2.1.** Let \(\phi_{N,j}(t)\) be the solution to H-NLS equation with the initial data \(\phi_{N,j}^{in}\). Under the same conditions (a), (b) and (c) of Theorem 1 and the restriction that

\begin{equation}
N \geq \epsilon^2 \left( \left[ C_N^2 E_{0,j}^2 T_0 / \hbar \right]^2 \right),
\end{equation}

then for \(N \geq N_0(\beta)\) we have the quantitative estimates

\begin{align}
\sup_{t \in [0,T_0]} \| S^{(1,1)}_N \|_{L^2} \leq \left( \frac{1}{\ln N} \right)^{100}, \\
\int_{[0,T_0]} \| S^{(1,1)}_N B_{N,1,2}^{\pm} \|_{L^2} dt \leq \left( \frac{1}{\ln N} \right)^{100},
\end{align}

which implies that

\begin{align}
\| \gamma_{N,j}^{(1)}(t; x; x) - \rho_{N,j}(t; x) \|_{L^\infty} \leq \frac{C}{\ln N}, \\
\| J_{N,j}^{(1)}(t; x; x) - J_{N,j}(t; x) \|_{L^\infty} \leq \frac{C}{\ln N^{5 \min \{1, \beta - \frac{2}{r+1} \}}}, \\
\left\| \left( B_{N,1,2}^{\pm} \gamma_{N,j}^{(2)}(t; x; x) \right) - \left( \rho_{N,j} V_N \ast \rho_{N,j} \right)(t; x) \right\|_{L^1_{t,L^2_x} \left(B_R \right)} \leq \frac{C R^{\beta/2} + T_0}{\ln N},
\end{align}

where \(r \in \left(1, \frac{r}{2} \right)\). Here \(\pm\) does not matter as \(B_{N,1,2}^{\pm} \gamma_{N,j}^{(2)}(t; x; x) = B_{N,1,2}^{-} \gamma_{N,j}^{(2)}(t; x; x)\).

**Proof of Theorem 2.1.** We prove (2.11) and (2.12) in Proposition 2.8. Here, we prove (2.13) - (2.15) using (2.11) and (2.12). For the mass density estimate (2.13), we split

\begin{equation}
w^{(1)}_{N,j} = \left( P^{\prime}_{\leq M} + P^{\prime}_{> M} \right) w^{(1)}_{N,j},
\end{equation}

where \(P_{\leq M}\) denotes the Littlewood-Paley projection with \(M\) to be determined.

For the low frequency part, by Bernstein inequality and estimate (2.11), we have

\begin{equation}
\left\| \left( P_{\leq M}^{\prime} w^{(1)}_{N,j} \right)(t; x; x) \right\|_{L^2_x} \leq \left\| \left( P_{\leq M}^{\prime} w^{(1)}_{N,j} \right)(t; x; x') \right\|_{L^2_x L^\infty_x} \lesssim M^4 \| w^{(1)}_{N,j} \|_{L^2_{t,x; x'}} \lesssim \frac{M^4}{(\ln N)^{100}}.
\end{equation}

For the high frequency part, by triangle inequality we have

\begin{equation}
\left\| \left( P_{> M}^{\prime} w^{(1)}_{N,j} \right)(t; x; x) \right\|_{L^2_x} \leq \left\| \left( P_{> M}^{\prime} \gamma^{(1)}_{N,j} \right)(t; x; x) \right\|_{L^2_x} + \left\| \left( P_{> M}^{\prime} \gamma^{(1)}_{N,j} \right)(t; x; x) \right\|_{L^2_x}.
\end{equation}
It suffices to deal with $\gamma^{(1)}_{N,h}$ as we can estimate $\gamma^{(1)}_{H,h}$ in the same way. We use interpolation between $L^1$ and $L^3$

\begin{equation}
(2.17) \quad \left\| \left( P^1 \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^2_x} \leq \left\| \left( P^{1'} \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^2_x} \left( \left\| \left( P^{1'} \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^4_x} \right)^{\frac{1}{2}}.
\end{equation}

For the $L^1_x$ norm, we have, by definition of $\gamma_{N,h}^{(1)}$

\begin{equation}
\left\| \left( P^{1'} \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^1_x} = \int_{\mathbb{R}^d} \int \psi_{N,h}(t, x, x_{2, N}) P^{1'}_{>M} \psi_{N,h}(t, x, x_{2, N}) dx_{2, N} dx
\end{equation}

where we have used $x_{2, N} = (x_2, \ldots, x_N)$ for short. By Cauchy-Schwarz and Bernstein,

\begin{align*}
&\leq \| \psi_{N,h} \|_{L^2} \| P^{1'}_{>M} \psi_{N,h} \|_{L^2} \\
&\leq \| \psi_{N,h} \|_{L^2} \frac{1}{\hbar M} \| \langle \hbar \nabla_{x_1} \rangle \psi_{N,h} \|_{L^2}
\end{align*}

By the $N$-body energy bound (1.17), we reach

\begin{equation}
(2.18) \quad \left\| \left( P^{1'} \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^1_x} \lesssim \frac{E_{0,h}^{1/2}}{\hbar M}.
\end{equation}

Similarly, for the $L^3_x$ norm, we have

\begin{equation}
(2.19) \quad \left\| \left( P^{1'} \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^3_x} = \left[ \int_{\mathbb{R}^d} \int \psi_{N,h}(t, x, x_{2, N}) P^{1'}_{>M} \psi_{N,h}(t, x, x_{2, N}) dx_{2, N} \right]^{\frac{1}{2}}
\end{equation}

By Hölder, Minkowski, Sobolev, and the $N$-body energy bound (1.17), we get

\begin{align*}
&\leq \| \psi_{N,h} \|_{L^2_{x_{2, N}}} \| L_{x_{2, N}} \|_{L^6_x} \| P^{1'}_{>M} \psi_{N,h} \|_{L^2_{x_{2, N}}} \| L_{x_{2, N}} \|_{L^6_x} \\
&\lesssim \| \langle \nabla_{x_1} \rangle \psi_{N,h} \|_{L^2} \| \langle \nabla_{x_1} \rangle P^{1'}_{>M} \psi_{N,h} \|_{L^2} \lesssim \frac{E_{0,h}}{\hbar^2}.
\end{align*}

Combining (2.18) and (2.19), we obtain

\begin{equation}
(2.20) \quad \left\| \left( P^{1'} \gamma^{(1)}_{N,h} \right)^{\varepsilon} (t, x; x) \right\|_{L^2_x} \lesssim \frac{1}{M^{1/4}} \left( \frac{E_{0,h}}{\hbar^2} \right)^{\frac{7}{8}}.
\end{equation}

By taking $M = (\ln N)^{10}$ and adopting the restriction (2.10), we obtain (2.13).

For the momentum estimate (2.14), we set

\begin{equation}
\gamma^{(1)}_{N,h}(t, x_1; x'_1) = h\nabla_{x_1} \gamma^{(1)}_{N,h}(t, x_1; x'_1) - h\nabla_{x_1} \gamma^{(1)}_{H,h}(t, x_1; x'_1).
\end{equation}

We split

\begin{equation}
\gamma^{(1)}_{N,h} = (P^{1'}_{\leq M} + P^{1'}_{>M}) \gamma^{(1)}_{N,h}
\end{equation}
with $M$ to be determined. By Interpolation, (2.23)
\[ \| (P'_{\leq M} g_{N,h} ) (t, x; x) \|_{L^2_t} \leq \| (P'_{\leq M} g_{N,h} )(t, x; x) \|_{L^2_t}^{\frac{1}{12}} \| (P'_{\leq M} g_{N,h} )(t, x; x) \|_{L^2_t}^{\frac{7}{12}} \leq I \cdot II, \]
(2.24)
\[ \| (P'_{> M} g_{N,h} ) (t, x; x) \|_{L^2_t} \leq \| (P'_{> M} g_{N,h} )(t, x; x) \|_{L^2_t}^{\frac{1}{12}} \| (P'_{> M} g_{N,h} )(t, x; x) \|_{L^2_t}^{\frac{7}{12}} \leq III \cdot IV. \]

Next, we separately estimate the above terms on the right hand side of (2.23) and (2.24).

For $I$, by triangle inequality we have
\[ \| (P'_{\leq M} g_{N,h} )(t, x; x) \|_{L^2_t} \leq \| (P'_{\leq M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t} + \| (P'_{\leq M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t}. \]
By Cauchy-Schwarz and the $N$-body energy bound (1.17), we have
(2.25)
\[ \| (P'_{\leq M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t} \leq \| h\nabla \psi_{N,h} \|_{L^2} \| P_{\leq M} \psi_{N,h} \|_{L^2} \leq E_{0,h}^{1/2}. \]
Similarly, by Cauchy-Schwarz and the energy bound for $\phi_{N,h}$, we have
(2.26)
\[ \| (P'_{\leq M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t} = \| (h\nabla_x \phi_{N,h})(t, x; x) \|_{L^2_t} \leq \| h\nabla_x \phi_{N,h} \|_{L^2} \| P_{\leq M} \phi_{N,h} \|_{L^2} \leq E_{0,h}^{1/2}. \]

With $E_0 \leq E_{0,h}$, we combine (2.25) and (2.26) to obtain
(2.27)
\[ I = \| (P'_{\leq M} g_{N,h} )(t, x; x) \|_{L^2_t}^{\frac{1}{12}} \| (P'_{\leq M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t} \leq E_{0,h}^{1/2}. \]

For $II$, we use Bernstein inequality and estimate (2.11) to get
(2.28)
\[ \| (P'_{\leq M} g_{N,h} )(t, x; x) \|_{L^2_t} \leq \| (P'_{\leq M} g_{N,h} )(t, x; x') \|_{L^2_t L^2_x} \leq M_1^{\frac{1}{2}} \| g_{N,h}(t, x; x') \|_{L^2_t L^2_x} \leq \frac{M_1^{1/2}}{\ln N^{100}}, \]
and hence
(2.29)
\[ II \leq \left( \frac{M_1^{1/2}}{\ln N^{100}} \right)^{2-\frac{2}{2}} . \]

For $III$, by triangle inequality we have
(3.30)
\[ \| (P'_{> M} g_{N,h} )(t, x; x) \|_{L^2_t} \leq \| (P'_{> M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t} + \| (P'_{> M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t}. \]
We use Cauchy-Schwarz, Bernstein, and the $N$-body energy bound (1.17) to obtain
(3.31)
\[ \| (P'_{> M} h\nabla_x \gamma_{N,h}^{(1)})(t, x; x) \|_{L^2_t} = \int \| \int h\nabla_x \psi_{N,h}(t, x, x_2, h) \|_{L^2} \| P_{> M} \psi_{N,h} \|_{L^2} dx_2, h \right) dx \]
\[ \leq \| h\nabla_x \psi_{N,h} \|_{L^2} \| P_{> M} \psi_{N,h} \|_{L^2} \leq M^{-1} \| h\nabla_x \psi_{N,h} \|_{L^2} \| (\nabla_x) P_{> M} \psi_{N,h} \|_{L^2} \]
\[ \leq \frac{E_{0,h}}{hM} \]
In the same method, we use the energy bound for \( \phi_{N,h} \) to get

\[
\left\| \left( P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{H,h} \right) (t, x; x) \right\|_{L^1_t L^2_x} \lesssim \frac{E_0}{\hbar M}.
\]

Combining (2.31) with (2.32), we have

\[
III \lesssim \left( \frac{E_{0,h}}{\hbar M} \right)^{\frac{3}{r} - 2}.
\]

For \( IV \), we use Hölder, Minkowski, Sobolev, and the \( N \)-body energy bound (1.17) to obtain

\[
\left\| \left( P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{H,h} \right) (t, x; x) \right\|_{L^{3/2}_t L^6_x} \lesssim \frac{E_0}{\hbar^{3/2}}
\]

In the same method, we use the energy bound for \( \phi_{N,h} \) to get

\[
\left\| \left( P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{H,h} \right) (t, x; x) \right\|_{L^{3/2}_t L^6_x} \lesssim \frac{E_0}{\hbar}.
\]

Combining (2.34) with (2.35), we have

\[
IV \lesssim \left( \frac{E_{0,h}}{\hbar} \right)^{3 - \frac{3}{r}}.
\]

Putting together with estimates (2.27), (2.29), (2.33) and (2.36), we arrive at

\[
\left\| J^{(1)}_{N,h}(t, x; x) - J_{N,h}(t, x) \right\|_{L^r_x} \lesssim E_{0,h}^{\frac{1}{r} - \frac{1}{2}} \left( \frac{M^{\frac{r}{2}}}{(\ln N)^{100}} \right)^{\frac{2 - \frac{4}{r}}{r}} + \left( \frac{E_{0,h}}{\hbar M} \right)^{\frac{3}{r} - 2} \left( \frac{E_{0,h}}{\hbar} \right)^{3 - \frac{3}{r}}
\]

Setting \( M = (\ln N)^{20} \),

\[
\lesssim E_{0,h}^{\frac{1}{r} - \frac{1}{2}} \left( \frac{1}{(\ln N)^{10}} \right)^{\min\{1 - \frac{1}{r}, \frac{3}{2} - 2\}}.
\]

For fixed \( r \in (1, 3/2) \), we make use of the restriction (2.10) to obtain

\[
\left\| J^{(1)}_{N,h}(t, x; x) - J_{N,h}(t, x) \right\|_{L^r_x} \lesssim \frac{1}{(\ln N)^{5\min\{1 - \frac{1}{r}, \frac{3}{2} - 2\}}},
\]

which completes the proof of (2.14).
For the pressure estimate (2.15), we set
\[
(2.39) \quad p_{N,h}^\pm(t, x_1, x_1') = \left[ B_{N,1,2}^\pm \left( \gamma_{N,h}^{(2)} - \gamma_{H,h}^{(2)} \right) \right](t, x_1, x_1').
\]
Again we split
\[
(2.40) \quad p_{N,h}^\pm = \left( P_{\leq M}^\prime + P_{> M}^\prime \right) p_{N,h}^\pm
\]
with $M$ to be determined. We use Hölder and Bernstein inequalities to obtain
\[
(2.41) \quad \left\| \left( P_{\leq M}^\prime p_{N,h}^\pm \right) (t, x; x) \right\|_{L^1([0,T];L^1_\text{loc}(B_R))} \leq R^{\frac{d}{2}} \left\| \left( P_{\leq M}^\prime p_{N,h}^\pm \right)(t, x; x) \right\|_{L^1([0,T];L^2(B_R))} \leq M^{\frac{d}{2}} R^{\frac{d}{2}} \left\| \left( P_{\leq M}^\prime p_{N,h}^\pm \right)(t, x; x') \right\|_{L^1([0,T];L^2 L^\infty_{x'}(\mathbb{R}^d))}
\]
By estimate (2.12), we arrive at
\[
(2.42) \quad \left\| \left( P_{\leq M}^\prime p_{N,h}^\pm \right)(t, x; x) \right\|_{L^1([0,T];L^1_\text{loc}(B_R))} \leq \frac{\hbar M^{\frac{d}{2}} R^{\frac{d}{2}}}{(\ln N)^{100}}.
\]
On the other hand, we note that
\[
P_{> M}^\prime B_{N,1,2}^+ \gamma_{N,h}^{(2)}(t, x_1; x_1') = \int V_N(x_1 - x_2) \psi_{N,h}(t, x_1, x_2) \psi_{N,h}(t, x_1, x_2, x_1') \, dx_2.
\]
Hence, by Cauchy–Schwarz we have
\[
(2.43) \quad \int |P_{> M}^\prime B_{N,1,2}^+ \gamma_{N,h}^{(2)}(t, x_1; x_1)| \, dx_1 \leq \langle \psi_{N,h}, V_N(x_1 - x_2) \psi_{N,h} \rangle^{1/2} \langle P_{> M}^1 \psi_{N,h}, V_N(x_1 - x_2) P_{> M}^1 \psi_{N,h} \rangle^{1/2}.
\]
By estimate (A.17) in Lemma A.8, Bernstein inequality, and the $N$-body energy bound (1.17),
\[
\lesssim \langle \psi_{N,h}, (1 - \Delta_{x_1})(1 - \Delta_{x_2}) \psi_{N,h} \rangle^{1/2} \langle P_{> M}^1 \psi_{N,h}, [(1 - \Delta_{x_1})(1 - \Delta_{x_2})]^{\frac{d}{4} +} P_{> M}^1 \psi_{N,h} \rangle^{1/2} \lesssim \frac{1}{M^{(1 - \frac{d}{4})}} \langle \nabla x_1 \rangle \langle \nabla x_2 \rangle \psi_{N,h} \|_{L^2} \langle \nabla x_1 \rangle \langle \nabla x_2 \rangle P_{> M}^1 \psi_{N,h} \|_{L^2} \lesssim \frac{E_0^2}{\hbar^4 M^{(1 - \frac{d}{4})}}.
\]
In the same method, we use the energy bound for $\phi_{N,h}$ to get
\[
(2.44) \quad \left\| P_{> M}^\prime B_{N,1,2}^+ \gamma_{H,h}^{(2)}(t, x; x) \right\|_{L^1_\text{loc}} \lesssim \frac{E_0^2}{\hbar^4 M^{(1 - \frac{d}{4})}}.
\]
Estimates (2.42), (2.43), and (2.44) together give

\[
(2.45) \quad \left\| \left( B_{N,1,2}^{\pm} \gamma_{N,h}^{(2)} \right) (t, x, x) - \left( \rho_{N,h} \rho_{N,h} \right) (t, x) \right\|_{L^1([0,T_0]; L^1(B_R))} = \left\| \left[ B_{N,1,2}^{\pm} \left( \gamma_{N,h}^{(2)} - \gamma_{R,h}^{(2)} \right) \right] (t, x; x) \right\|_{L^1([0,T_0]; L^1(B_R))} \leq \frac{\hbar M + R^2}{(\ln N)^{10}} + \frac{T_0 E_{0,h}^2}{\hbar^4 M^{1-\frac{d}{2}}},
\]

By taking \( M = (\ln N)^{10} \),

\[
\leq \frac{\hbar R^2}{(\ln N)^{10}} + \frac{T_0 E_{0,h}^4}{\hbar^4 (\ln N)^{10}}.
\]

For fixed \( T_0 \), we utilize the restriction (2.10) to get

\[
\left\| \left( B_{N,1,2}^{\pm} \gamma_{N,h}^{(2)} \right) (t, x; x) - \left( \rho_{N,h} \rho_{N,h} \right) (t, x) \right\|_{L^1([0,T_0]; L^1(B_R))} \leq \frac{R^{d/2} + T_0}{\ln N},
\]

which completes the proof of (2.15).

The proof of Theorem 2.1 is hence concluded assuming (2.11) and (2.12) included in Proposition 2.8. The rest of Section 2 is to prove Proposition 2.8.

2.1. A Tool Box of Space-time Estimates. We reproduce and rewrite [23, Section 2] with \( \hbar \) for our purpose here and provide some preliminary estimates for \( w_{N,h}^{(k)} \). We start by rewriting the 3D cubic BBGKY hierarchy (2.2) in integral form

\[
(2.46) \quad \gamma_{N,h}^{(k)} = U_h^{(k)} \gamma_{N,h}^{(0)} + \int_0^{T_0} U_h^{(k)} (t_k - t_{k+1}) V_{N,h}^{(k)} \gamma_{N,h}^{(k)} (t_{k+1}) dt_{k+1} + \frac{N - k}{N} \int_0^{T_0} U_h^{(k)} (t_k - t_{k+1}) B_{N,h}^{(k+1)} \gamma_{N,h}^{(k+1)} (t_{k+1}) dt_{k+1}
\]

where we have adopted the shorthands\[1^{10}\]

\[
(2.47) \quad U_h^{(k)} = \prod_{j=1}^{k} e^{\langle \hbar \Delta_{x_j} / 2 \rangle} e^{\langle -\hbar \Delta_{x_j} / 2 \rangle},
\]

\[
(2.48) \quad V_{N,h}^{(k)} \gamma_{N,h}^{(k)} = \frac{1}{N} \sum_{1 \leq i < j \leq k} [V_{N,h} (x_i - x_j), \gamma_{N,h}^{(k)}],
\]

\[
(2.49) \quad V_{N,h} (x) = \frac{1}{\hbar} N^{d/2} \tilde{V} (N x),
\]

\[
(2.50) \quad B_{N,h}^{(k+1)} \gamma_{N,h}^{(k+1)} = \sum_{j=1}^{k} B_{N,h,j,k+1} \gamma_{N,h}^{(k+1)} = \sum_{j=1}^{k} \text{Tr}_{k+1} \left[ V_{N,h} (x_j - x_{k+1}), \gamma_{N,h}^{(k+1)} \right],
\]

\[1^{10}\text{Please notice that we have divided by } \hbar \text{ to use (2.47).}
and we have omitted the \((-i\) in front of the second and third terms in the right hand side of (2.46) as it serves as 1 in our estimates. In addition to (2.46), we write (2.3) in integral form

\begin{equation}
(2.51) \quad \gamma^{(k)}_{11}(t_k) = U^{(k)}_h(t_k)\gamma^{(k)}_{11}(0) + \int_0^{t_k} U^{(k)}_h(t_k - t_{k+1})B^{(k+1)}_{N,h}\gamma^{(k+1)}_{11}(t_{k+1})dt_{k+1}.
\end{equation}

The difference \(w^{(k)}_{N,h} = \gamma^{(k)}_{N,h} - \gamma^{(k)}_{11}\) solves the hierarchy

\begin{equation}
(2.52) \quad w^{(k)}_{N,h}(t_k) = U^{(k)}_h(t_k)w^{(k)}_{N,h}(0) + \int_0^{t_k} U^{(k)}_h(t_k - t_{k+1})V^{(k)}_{N,h}\gamma^{(k)}_{N,h}(t_{k+1})dt_{k+1}
- \frac{k}{N}\int_0^{t_k} U^{(k)}_h(t_k - t_{k+1})B^{(k+1)}_{N,h}\gamma^{(k+1)}_{N,h}(t_{k+1})dt_{k+1}
+ \int_0^{t_k} U^{(k)}_h(t_k - t_{k+1})B^{(k+1)}_{N,h}w^{(k+1)}_{N,h}(t_{k+1})dt_{k+1}.
\end{equation}

Iterating hierarchy (2.52) \(l_c\) times\(^{11}\) at the last term of (2.52), we have

\begin{equation}
(2.53) \quad w^{(k)}_{N,h}(t_k) = FP^{(k,l_c)}(t_k) + DP^{(k,l_c)}(t_k) + EP^{(k,l_c)}(t_k) + IP^{(k,l_c)}(t_k)
\end{equation}

where we have grouped the terms in \(w^{(k)}_{N,h}(t_k)\) into four parts: the free/ driving/ error/ interaction parts. We remark that (2.53) holds for all \(l_c \geq 1\) and we will select \(l_c\) depending on what aspect of \(w^{(k)}_{N,h}\) we need in Section 2.7.2–2.7.4. To write out the four parts of \(w^{(k)}_{N,h}\), we define the notation that, for \(j \geq 1\),

\begin{equation}
(2.54) \quad J^{(k,j)}_{N,h}(t_k, t_{(k,j)}(f^{(k+1)}(t_{k,j})),
= \left(U^{(k)}_h(t_k - t_{k+1})B^{(k+1)}_{N,h}\right) \cdots \left(U^{(k+j-1)}_h(t_{k+j-1} - t_{k+1})B^{(k+1)}_{N,h}\right) f^{(k+1)}(t_{k+1}),
\end{equation}

and \(J^{(k,0)}_{N,h}(t_k, t_{(k,0)}(f^{(k)}(t_k)) = f^{(k)}(t_k)\), where \(t_{(k,j)} = (t_{k+1}, \ldots, t_{k+j})\) for \(j \geq 1\). In this notation, the free part of \(w^{(k)}_{N,h}\) at \(l_c\) coupling level is

\begin{equation}
(2.55) \quad FP^{(k,l_c)}(t_k) = U^{(k)}_h(t_k)w^{(k)}_{N,h}(0) + \sum_{j=1}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U^{(k)}_h(t_k - t_{k+1})B^{(k+1)}_{N,h} \cdots \times U^{(k+j-1)}_h(t_{k+j-1} - t_{k+1})B^{(k+1)}_{N,h} f^{(k+j)}(t_{k+j}) dt_{(k,j)}
\end{equation}

where in the \(j = 0\) case, it is meant that there are no time integrals and \(J^{(k,0)}_{N,h}\) is the identity operator, and

\begin{equation}
(2.56) \quad f^{(k,j)}_{FP}(t_{k+1}) = U^{(k+j)}_h(t_{k+1})w^{(k+j)}_{N,h}(0).
\end{equation}

\(^{11}\)\(l_c\) means “coupling level”.
The driving part is given by
\begin{equation}
DP^{(k,l_c)}(t_k) = \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V_{N,h}^{(k)} \gamma_{N,h}^{(k)} (t_{k+1}) dt_{k+1} + \sum_{j=1}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U_h^{(k)}(t_k - t_{k+1}) B_{N,h}^{(k+1)} \cdots U_h^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,h}^{(k+j)} \nonumber \\
\times \left( \int_0^{t_{k+j}} U_h^{(k+j)}(t_{k+j} - t_{k+j+1}) V_{N,h}^{(k+j)} \gamma_{N,h}^{(k+j)} (t_{k+j+1}) dt_{k+j+1} \right) dt_{k,j} \nonumber \\
= \sum_{j=0}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_{N,h}^{(k,j)}(t_{k,j}) \left( f_{DP}^{(k,j)}(t_{k+j}) \right) dt_{k,j}
\end{equation}

where in the $j = 0$ case, it is meant that there are no time integrals and $J_{N,h}^{(k,0)}$ is the identity operator, and
\begin{equation}
f_{DP}^{(k,j)}(t_{k+j}) = \int_0^{t_{k+j}} U_h^{(k+j)}(t_{k+j} - t_{k+j+1}) V_{N,h}^{(k+j)} \gamma_{N,h}^{(k+j)} (t_{k+j+1}) dt_{k+j+1}.
\end{equation}

The error part is given by
\begin{equation}
EP^{(k,l_c)}(t_k) 
= - \frac{k}{N} \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) B_{N,h}^{(k+1)} \gamma_{N,h}^{(k+1)} (t_{k+1}) dt_{k+1} 
\nonumber \\
- \sum_{j=1}^{l_c} \frac{k+j}{N} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U_h^{(k)}(t_k - t_{k+1}) B_{N,h}^{(k+1)} \cdots U_h^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,h}^{(k+j)} 
\nonumber \\
\times \left( \int_0^{t_{k+j}} U_h^{(k+j)}(t_{k+j} - t_{k+j+1}) B_{N,h}^{(k+j+1)} \gamma_{N,h}^{(k+j+1)} (t_{k+j+1}) dt_{k+j+1} \right) dt_{k,j} 
\nonumber \\
= \sum_{j=0}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_{N,h}^{(k,j)}(t_{k,j}) \left( f_{EP}^{(k,j)}(t_{k+j}) \right) dt_{k,j}
\end{equation}

where in the $j = 0$ case, it is meant that there are no time integrals and $J_{N,h}^{(k,0)}$ is the identity operator, and
\begin{equation}
f_{EP}^{(k,j)}(t_{k+j}) = - \frac{k+j-1}{N} \gamma_{N,h}^{(k+j)}.
\end{equation}

The interaction part is given by
\begin{equation}
IP^{(k,l_c)}(t_k) = \int_0^{t_k} \cdots \int_0^{t_{k+l_c}} U_h^{(k)}(t_k - t_{k+l_c}) B_{N,h}^{(k+1)} 
\nonumber \\
\cdots U_h^{(k+l_c)}(t_{k+l_c} - t_{k+l_c+1}) B_{N,h}^{(k+l_c+1)} \left( w_{N,h}^{(k+l_c+1)}(t_{k+l_c+1}) \right) dt_{k+1} \cdots dt_{k+l_c+1} 
\nonumber \\
= \int_0^{t_k} \cdots \int_0^{t_{k+l_c}} J_{N,h}^{(k,l_c+1)}(t_k, t_{k+l_c+1}) \left( w_{N,h}^{(k+l_c+1)}(t_{k+l_c+1}) \right) dt_{k(l_c+1)}
\end{equation}
where
\[
 f_{IP}^{(k,l+1)} = w_{N,h}^{(k+l+1)}(t_{k+l+1}).
\]

There are around \( \frac{(k+l)!}{k!} \) many summands in each part. They can be grouped together by using the KM board game argument \cite{15}, which is below.

**Lemma 2.2** (\cite{15}, Lemma 2.1). \footnote{More advanced version of this combinatoric is now available, see \cite{22,24}.} For \( j \geq 1 \), one can express
\[
\begin{aligned}
 (2.62) \quad & \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} j^{(k,j)}_{N,h}(t_k, L_{(k,j)})(f^{(k+j)}) dt_{(k,j)} \\
 = & \sum_m \int_D j^{(k,j)}_{N,h}(t_k, L_{(k,j)}, \mu_m)(f^{(k+j)}) dt_{(k,j)}.
\end{aligned}
\]

Here \( D \subset [0, t_k]^j \), \( \mu_m \) are a set of maps from \( \{k+1, \ldots, k+j\} \) to \( \{1, \ldots, k+j-1\} \) and \( \mu_m(l) < l \) for all \( l \), and
\[
\begin{aligned}
(2.63) \quad & j^{(k,j)}_{N,h}(t_k, L_{(k,j)}, \mu_m)(f^{(k+j)}) \\
= & \left( u^{(k)}(t_k - t_k)B_{N,h, \mu_m(k+1),k+1} \right) \cdots \left( u^{(k+j-1)}(t_{k+j-1} - t_k)B_{N,h, \mu_m(k+j),k+j} \right) f^{(k+j)}(t_k).
\end{aligned}
\]

The summing number can be controlled by \( 2^{k+2j-2} \).

Then we are able to estimate \( j^{(k,j)}_{N,h}(t_k, L_{(k,j)})(f^{(k+j)}) \) via collapsing estimates in Lemma A.2.

**Lemma 2.3.** Let \( d = 3 \) and \( \alpha = d + 1/2 \). For \( j \geq 1 \),
\[
\begin{aligned}
(2.65) \quad & \| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S_{h}^{(1,k)} j^{(k,j)}_{N,h}(t_k, L_{(k,j)})(f^{(k+j)}) dt_{(k,j)} \|_{L^\infty([0,T]) L^2_{x,x'}} \\
\leq & 2^{k+4j} \| (C \cdot h^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} S_{h}^{(1,k+j-1)} B_{N,h,1,k+j}(f^{(k+j)}(t_k)) \|_{L^2_{x,x'}} dt_{k+j}
\end{aligned}
\]
\[
\begin{aligned}
(2.66) \quad & \| S_{h}^{(1,k)} B_{N,h,1,k+1} \int_0^{t_{k+1}} \cdots \int_0^{t_{k+j}} j^{(k+1,j)}_{N,h}(t_{k+1}, L_{(k+1,j)})(f^{(k+j+1)}) dt_{(k+1,j)} \|_{L^2_{x,x'}} \\
\leq & 2^{k+4j} \| (C \cdot h^{-\alpha} T^{1/2})^{j} \int_{[0,T]} S_{h}^{(1,k+j)} B_{N,h,1,k+j+1}(f^{(k+j+1)}(t_{k+j+1})) \|_{L^2_{x,x'}} dt_{k+j+1}.
\end{aligned}
\]

**Proof.** This is well-known for \( h = 1 \). We include a proof for completeness. For (2.65), we start by using Lemma 2.2.
\[
\begin{aligned}
(2.67) \quad & \| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S_{h}^{(1,k)} j^{(k,j)}_{N,h}(t_k, L_{(k,j)})(f^{(k+j)}) dt_{(k,j)} \|_{L^\infty([0,T]) L^2_{x,x'}} \\
\leq & 2^{k+4j} \int_D S_{h}^{(1,k)} j^{(k,j)}_{N,h}(t_k, L_{(k,j)}, \mu_m)(f^{(k+j)}) dt_{(k,j)} \|_{L^\infty([0,T]) L^2_{x,x'}} \\
\leq & 2^{k+4j} \int_{[0,T]} \| S_{h}^{(1,k)} j^{(k,j)}_{N,h}(t_k, L_{(k,j)}, \mu_m)(f^{(k+j)}) \|_{L^2_{x,x'}} dt_{(k,j)}
\end{aligned}
\]
Cauchy-Schwarz at $dt_{k+1}$,

\[(2.68)\]

$$
\leq 2^k 4^j T^{1/2} \int_{[0,T]} \left\| S_h^{(1,k)} N_{h,1,1} U_h^{(k+1)}(t_{k+1} - t_{k+2}) \right\|_{L_{k+1}^2[0,T]L_{x,x'}} dt_{k+1,j-1}
$$

By Lemma [A.2]

\[(2.69)\]

$$
\leq 2^k 4^j C_v h^{-\alpha} T^{1/2} \int_{[0,T]} \left\| S_h^{(1,k+1)} B_{N,h,1,m_0(m+1)+1} U_h^{(k+1)}(t_{k+2} - t_{k+3}) \right\|_{L_{k+1}^2[0,T]L_{x,x'}} dt_{k+1,j-1}
$$

Repeating such a process gives

\[(2.70)\]

$$
\leq 2^k 4^j (C_v h^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,m_0(m+1)+1} f^{(k+j)}(t_{k+j}) \right\|_{L_{k+1,j}^2[0,T]L_{x,x'}} dt_{k+j}
$$

By symmetry,

\[(2.71)\]

$$
= 2^k 4^j (C_v h^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,m_0(m+1)+1} f^{(k+j)}(t_{k+j}) \right\|_{L_{k+1,j}^2[0,T]L_{x,x'}} dt_{k+j}
$$

For (2.66), we apply Lemma [2.2] again to obtain

\[(2.72)\]

$$
\int_{[0,T]} \left\| S_h^{(1,k)} B_{N,h,1,1} f^{(k+1)}(t_{k+1} + t_{k+3}) dt_{k+1,j} \right\|_{L_{k+1,j}^2[0,T]L_{x,x'}} dt_{k+1,j}
$$

\[(2.73)\]

$$
\leq 2^k 4^j T^{1/2} \int_{[0,T]} \left\| S_h^{(1,k)} B_{N,h,1,1} U_h^{(k+1)}(t_{k+1} - t_{k+2}) \right\|_{L_{k+1,j}^2[0,T]L_{x,x'}} dt_{k+1,j}
$$

Iterating the same process as (2.68), we obtain

\[(2.73)\]

$$
\leq 2^k 4^j (C_v h^{-\alpha} T^{1/2})^j \int_{[0,T]} \left\| S_h^{(1,k+j)} B_{N,h,1,m_0(m+1)+1} f^{(k+j+1)}(t_{k+j+1}) \right\|_{L_{k+1,j}^2[0,T]L_{x,x'}} dt_{k+j+1}
$$

Away from Lemma [2.3], we obtain below crude estimates of the driving part, error part and the interaction part.

**Lemma 2.4.** Let $k \leq (\ln N)^{10}$ and $j \leq (\ln N)^{10}$. For the driving part, we have

\[(2.74)\]

$$
\left\| S_h^{(1,k)} f^{(k+1)}(t_k) \right\|_{L_{k}^\infty[0,T]L_{x,x'}} \leq N^{2\beta-1} (C_v h^{-\alpha} T^{1/2}) k^2 (2E_{0,h})^k
$$

and

\[(2.75)\]

$$
\int_{[0,T]} \left\| S_h^{(1,k+j)} B_{N,h,1,m_0(m+1)+1} f^{(k+j)}(t_{k+j}) \right\|_{L_{k+1,j}^2[0,T]L_{x,x'}} dt_{k+j}
$$

\[\leq N^{2\beta-1} (C_v h^{-\alpha} T^{1/2})^2 (k+j)^2 (2E_{0,h})^{k+j}.
\]
For the error part, we have

\[
\int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (f^{(k,j)}_{EP} (t_{k+j})) \right\|_{L^2_{x,x'}} \ dt_{k+j} \\
\leq N^{\frac{\beta}{2}} \alpha^{-1} (C_V h^{-\alpha} T^{1/2}) (k + j) (2E_{0,h})^{k+j}.
\]

For the interaction part, we have

\[
\int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (f^{(k,j)}_{IP} (t_{k+j})) \right\|_{L^2_{x,x'}} \ dt_{k+j} \\
\leq N^{\frac{\beta}{2}} (C_V h^{-\alpha} T^{1/2}) (4E_{0,h})^{k+j}.
\]

**Proof.** For (2.74), plugging in \( f^{(k,0)}_{DP} \), we need to estimate

\[
\left\| S_h^{(1,k)} \int_0^{t_k} U_h^{(k)} (t_k - t_{k+1}) V_{N,h}^{(k)} \gamma_{N,h} (t_{k+1}) dt_{k+1} \right\|_{L^2_{t_k,0,T} L^2_{x,x'}}
\]

By (A.7) in Lemma A.4,

\[
\leq N^{\frac{\beta}{2}} h (C_V h^{-\alpha} T^{1/2}) k^2 \left\| S_h^{(1,k)} \gamma_{N,h} (t_{k+1}) \right\|_{L^2_{t_k,0,T} L^2_{x,x'}}.
\]

Using the \( N \)-body energy bound (1.17) and discarding the unimportant factor \( h^2 \), we arrive at

\[
\left\| S_h^{(1,k)} \int_0^{t_k} U_h^{(k)} (t_k - t_{k+1}) V_{N,h}^{(k)} \gamma_{N,h} (t_{k+1}) dt_{k+1} \right\|_{L^2_{t_k,0,T} L^2_{x,x'}} \\
\leq N^{\frac{\beta}{2}} h (C_V h^{-\alpha} T^{1/2}) k^2 (2E_{0,h})^k,
\]

which completes the proof of (2.74).

For (2.75), we insert \( f^{(k,j)}_{DP} \) defined in (2.58) to obtain

\[
\int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (f^{(k,j)}_{DP} (t_{k+j})) \right\|_{L^2_{x,x'}} \ dt_{k+j} \\
= \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} \int_0^{t_{k+j}} U_h^{(k+j)} (t_{k+j} - t_{k+j+1}) V_{N,h}^{(k+j)} \gamma_{N,h}^{(k+j)} (t_{k+j+1}) dt_{k+j+1} \right\|_{L^2_{t_{k+j},0,T} L^2_{x,x'}}
\]

Utilizing (A.8) in Lemma A.4

\[
\leq N^{\frac{\beta}{2}} h (C_V h^{-\alpha} T^{1/2})^2 (k + j)^2 \left\| S_h^{(1,k+j)} \gamma_{N,h}^{(k+j)} (t_{k+j+1}) \right\|_{L^2_{t_{k+j+1},0,T} L^2_{x,x'}}.
\]

Making use of the \( N \)-body energy bound (1.17) and discarding the unimportant small factor \( h \), (2.75) is then proved.
For the error part (2.76), inserting \( f^{(k,j)}_{IP} \) we have

\[
\int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (f^{(k,j)}_{IP} (t_{k+j})) \right\|_{L^2_{x,x'}} \, dt_{k+j} = \frac{k+j-1}{N} \int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (\gamma_{N,h}^{(k+j)} (t_{k+j})) \right\|_{L^2_{x,x'}} \, dt_{k+j}.
\]

By (A.11) in Lemma A.5 and the \( N \)-body energy bound (1.17),

\[
\leq N^{\frac{2}{3} - 1} h^2 T^{1/2} \left( C_V h^{-\alpha} T^{1/2} \right) (k + j) (2E_{0,h})^{k+j}.
\]

Discarding the unimportant small factor \( h^2 T^{1/2} \), we complete the proof of (2.76).

For the interaction part (2.77), inserting \( f^{(k,j)}_{IP} \) we have

\[
\int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (f^{(k,j)}_{IP} (t_{k+j})) \right\|_{L^2_{x,x'}} \, dt_{k+j} \leq \int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (\gamma_{N,h}^{(k+j)} (t_{k+j})) \right\|_{L^2_{x,x'}} \, dt_{k+j} + \int_{[0,T]} \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} (\gamma_{H,h}^{(k+j)} (t_{k+j})) \right\|_{L^2_{x,x'}} \, dt_{k+j}.
\]

By (A.11) in Lemma A.5 and the \( N \)-body energy bound (1.17),

\[
\leq N^{\frac{2}{3} - 1} h^2 T^{1/2} \left( C_V h^{-\alpha} T^{1/2} \right) (4E_{0,h})^{k+j}.
\]

By discarding the unimportant small factor \( h^2 T^{1/2} \), we complete the proof of (2.77). \( \square \)

2.2. A Klainerman-Machedon Bound 1st. Via the preliminary estimates in Section 2.1 we are able to provide a “preliminary” Klainerman-Machedon bound for \( w^{(k)}_{N,h} \). Here, “preliminary” certainly means, “not final” as we will improve it once we have used it to prove (2.11).

**Lemma 2.5.** Let \( t_0 \in [0, \infty) \), \( T \leq \frac{k^{2\alpha}}{(64E_{0,h}C_V e)} \), and \( \alpha = d + \frac{1}{2} \). For \( k \leq (\ln N)^{10} \), we have

\[
\int_{[t_0,t_0+T]} \left\| S_h^{(1,k)} B_{N,h,1,k+1} w^{(k+1)}_{N,h} (t_{k+1}) \right\|_{L^2_{x,x'}} \, dt_{k+1} \leq (16E_{0,h})^k.
\]

It holds for sufficiently small \( T \) but independent of the initial time.

**Proof.** We give a proof following the method in [14, 17, 19] which was inspired by [12]. We might as well take \( t_0 = 0 \) for convenience, as the general case also holds from time translation.
Decompose $w_{N,h}^{(k)}$ as in (2.53), it suffices to prove that

\[(2.84) \quad \int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} F P^{(k+1,l_{c})}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \leq (8E_{0,h})^{k}, \]

\[(2.85) \quad \int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} D P^{(k+1,l_{c})}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \leq (8E_{0,h})^{k}, \]

\[(2.86) \quad \int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} E P^{(k+1,l_{c})}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \leq (8E_{0,h})^{k}, \]

\[(2.87) \quad \int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} I P^{(k+1,l_{c})}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \leq (8E_{0,h})^{k}. \]

For the FP part (2.84), we start by using estimate (2.66) in Lemma 2.3 to obtain

\[
\int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} F P^{(k+1,l_{c})}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \\
\leq \int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} f_{FP}^{(k+1,0)}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \\
+ \sum_{j=1}^{l_{c}} 2^{k+1} 4^{j} (C_{V}h^{-\alpha}T^{1/2})^{j} \int_{[0,T]} \left\| S_{h}^{(1,k+j)} B_{N,h,1,k+j+1}(f_{FP}^{(k+1,j)}(t_{k+j+1})) \right\|_{L_{x,x'}}^2 \, dt_{k+j+1}
\]

Plugging in $f_{FP}^{(k+j)}$, applying Cauchy-Schwarz at $dt_{k+j+1}$ and then Lemma A.2

\[
\leq 2^{k+1} \sum_{j=0}^{l_{c}} \left( 4C_{V}h^{-\alpha}T^{1/2} \right)^{j+1} \left\| S_{h}^{(1,k+j+1)} w_{N,h}^{(k+j+1)}(0) \right\|_{L_{x,x'}}^2 \]

We have required that $l_{c} \leq \ln N$ thus we can use the $N$-body energy bound (1.17) to obtain

\[
\leq (8E_{0,h})^{k} \sum_{j=0}^{l_{c}} (16E_{0,h}C_{V}h^{-\alpha}T^{1/2})^{j+1} \leq (8E_{0,h})^{k}
\]

if we plug in $T \leq \frac{h^{2\alpha}}{(64E_{0,h}C_{V}e)}$.

For the DP Part (2.85), the above process gives

\[
\int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} D P^{(k+1,l_{c})}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \\
\leq \int_{[0,T]} \left\| S_{h}^{(1,k)} B_{N,h,1,k+1} f_{DP}^{(k+1,0)}(t_{k+1}) \right\|_{L_{x,x'}}^2 \, dt_{k+1} \\
+ 2^{k+1} \sum_{j=1}^{l_{c}} 4^{j} (C_{V}h^{-\alpha}T^{1/2})^{j} \int_{[0,T]} \left\| S_{h}^{(1,k+j)} B_{N,h,1,k+j+1}(f_{DP}^{(k+1,j)}(t_{k+j+1})) \right\|_{L_{x,x'}}^2 \, dt_{k+j+1}
\]
As $k \leq (\ln N)^{10}$ and $j \leq l_c \leq \ln N$, we can use estimate (2.73) in Lemma 2.4 to get

\begin{align*}
&\leq N^{\frac{\alpha}{2} - 1} 2^{k+1} \sum_{j=0}^{l_c} (4C_V h^{-\alpha} T^{1/2})^{j+2} (k + j + 1)^2 (2E_{0,h})^{k+j+1} \\
&\leq N^{\frac{\alpha}{2} - 1} (8E_{0,h})^k \sum_{j=0}^{l_c} (16E_{0,h} C_V h^{-\alpha} T^{1/2})^{j+2} \\
&\leq (8E_{0,h})^k
\end{align*}

if we plug in $T \leq \frac{\text{const}}{(64E_{0,h} C_V e)^{\alpha}}$.

Similarly, for the error part (2.86), we have

\begin{align*}
&\int_{[0,T]} \left\| S_h^{(1,k)} B_{N,h,1,k+1} E_P^{(k+1,l_c)} (t_{k+1}) \right\|_{L_2}^2 dt_{k+1} \\
&\leq 2^{k+1} \sum_{j=1}^{l_c} 2^j (C_V h^{-\alpha} T^{1/2})^j \int_{[0,T]} \left\| S_h^{(1,k+j)} B_{N,h,1,k+j+1} (f_{E_P}^{(k+j+1)} (t_{k+j+1})) \right\|_{L_2}^2 dt_{k+j+1}
\end{align*}

Plugging in $f_{E_P}^{(k,j)}$ and using estimate (2.76) in Lemma 2.4

\begin{align*}
&\leq N^{\frac{\alpha}{2} - 1} 2^{k+1} \sum_{j=1}^{l_c} (4C_V h^{-\alpha} T^{1/2})^{j+1} (k + j + 1) (2E_{0,h})^{k+j+1} \\
&\leq N^{\frac{\alpha}{2} - 1} (8E_{0,h})^k \sum_{j=1}^{l_c} (16E_{0,h} C_V h^{-\alpha} T^{1/2})^{j+1} \\
&\leq (8E_{0,h})^k
\end{align*}

if we plug in $T \leq \frac{\text{const}}{(64E_{0,h} C_V e)^{\alpha}}$.

Finally, for the interaction part (2.87), we have

\begin{align*}
&\int_{[0,T]} \left\| S_h^{(1,k)} B_{N,h,1,k+1} I P^{(k+1,l_c)} (t_{k+1}) \right\|_{L_2}^2 dt_{k+1} \\
&\leq 2^{k+1} 4^{l_c+1} (C_V h^{-\alpha} T^{1/2})^{l_c+1} \int_{[0,T]} \left\| S_h^{(1,k+l_c+1)} B_{N,h,1,k+l_c+2} w_{N,h}^{(k+l_c+2)} (t_{k+l_c+2}) \right\|_{L_2}^2 dt_{k+l_c+2}
\end{align*}

By estimate (2.77) in Lemma 2.4

\begin{align*}
&\leq N^{\frac{\alpha}{2} - 2} 2^{k+1} (4C_V h^{-\alpha} T^{1/2})^{l_c+2} (4E_{0,h})^{k+l_c+2} \\
&\leq 2 N^{\frac{\alpha}{2} - 1} (8E_{0,h})^k
\end{align*}

Plugging in $T \leq \frac{\text{const}}{(64E_{0,h} C_V e)^{\alpha}}$ and taking $l_c + 1 = \ln N$,

\begin{align*}
&\leq 2 N^{\frac{\alpha}{2} - 1} (8E_{0,h})^k
\end{align*}

and we have completed the proof of Lemma 2.5.
2.3. Feeding the Strichartz Bound into the $H^1$ Estimate. In the section, we first provide estimates for the four parts in the expansion of $w^{(k)}_{N,h}$ via the preliminary crude estimates established in Section 2.1. Then with the help of the KM bound we prove in Section 2.2, we can establish a strong stepping estimate for $w^{(k)}_{N,h}$ which is Proposition 2.7.

Lemma 2.6. Let $\alpha = d + 1/2$. For $k \leq (\ln N)^2$ and $l_c \leq \ln N$, we have the following estimates.

For the free part,

(2.88) $\sup_{t_k \in [t_0, t_0 + T]} \|S^{(1,k)}_h F P^{(k,l_c)}(t_k)\|_{L^2_{x,t}} \leq 2^k \sum_{j=0}^{l_c} \left(4C_V h^{-\alpha} T^{1/2}\right)^j \|S^{(1,k+j)}_h w^{(k+j)}_{N,h}(t_0)\|_{L^2_{x,t}}$

For the driving part,

(2.89) $\sup_{t_k \in [t_0, t_0 + T]} \|S^{(1,k)}_h D P^{(k,l_c)}(t_k)\|_{L^2_{x,t}} \leq (8E_{0,h})^k N^{2\beta-1} \sum_{j=0}^{l_c} (16E_{0,h}C_V h^{-\alpha} T^{1/2})^{j+1}$

For the error part,

(2.90) $\sup_{t_k \in [t_0, t_0 + T]} \|S^{(1,k)}_h E P^{(k,l_c)}(t_k)\|_{L^2_{x,t}} \leq (8E_{0,h})^k N^{2\beta-1} \sum_{j=0}^{l_c} (16E_{0,h}C_V h^{-\alpha} T^{1/2})^{j+1}$

For the interaction part,

(2.91) $\sup_{t_k \in [t_0, t_0 + T]} \|S^{(1,k)}_h I P^{(k,l_c)}(t_k)\|_{L^2_{x,t}} \leq 2^k 4^{l_c+1} (C_V h^{-\alpha} T^{1/2})^{l_c} \int_{[t_0, t_0 + T]} \left\|S^{(1,k+l_c)}_h B_{N,h,1,k+l_c+1} w^{(k+l_c+1)}_{N,h}(t_{k+l_c+1})\right\|_{L^2_{x,t}} dt_{k+l_c+1}$

Proof. For convenience, we might as well take $t_0 = 0$ as the proof works the same for general case by time translation.

For the free part, applying estimate (2.65) in Lemma 2.3 we arrive at

$$\|S^{(1,k)}_h F P^{(k,l_c)}\|_{L^\infty_t[0,T]L^2_{x,t}} \leq \|S^{(1,k)}_h f^{(k,0)}_{FP}(t_k)\|_{L^\infty_t[0,T]L^2_{x,t}} + \sum_{j=1}^{l_c} \left(4C_V h^{-\alpha} T^{1/2}\right)^{j-1} \int_{[0,T]} \left\|S^{(1,k+j-1)}_h B_{N,h,1,k+j} f^{(k+j)}_{FP}(t)\right\|_{L^2_{x,t}} dt_{k+j}$$

Plugging in $f^{(k+j)}_{FP}$ and applying Cauchy-Schwarz at $dt_{k+j}$,

$$\leq \|S^{(1,k)}_h U^{(k)}_{N,h}(t_k) w^{(k)}_{N,h}(0)\|_{L^\infty_t[0,T]L^2_{x,t}} + \sum_{j=1}^{l_c} 2^k 4^j (C_V h^{-\alpha} T^{1/2})^{j-1} T^{1/2} \left\|S^{(1,k+j-1)}_h B_{N,h,1,k+j} U^{(k+j)}_{N,h}(t_{k+j}) w^{(k+j)}_{N,h}(0)\right\|_{L^2_{x,t}[0,T]}$$

Applying the KM collapsing estimate (Lemma A.2) for $j \geq 1$,

(2.92) $\leq \sum_{j=0}^{l_c} 2^k (4C_V h^{-\alpha} T^{1/2})^j \left\|S^{(1,k+j)}_h w^{(k+j)}_{N,h}(0)\right\|_{L^2_{x,t}}$.
We have (2.88) as claimed.

For the driving part, the same process yields

\[
\| S^{(1,k)}_h P^{(k,l_c)} \|_{L^\infty_k[0,T] L^2_{x,x'}} \\
\leq \| S^{(1,k)}_h f^{(k,0)}_D (t_k) \|_{L^\infty_k[0,T] L^2_{x,x'}} \\
+ 2^k \sum_{j=1}^{l_c} 4^j \left( C_V h^{-\alpha} T^{1/2} \right)^{j-1} \int_{[0,T]} \| S^{(1,k,j-1)}_h B_{N,h,1,k+j} (f^{(k,j)}_D (t_{k+j})) \|_{L^2_{x,x'}} dt_{k+j}
\]

Plugging in \( f^{(k,j)}_D \) and using estimates (2.74) and (2.75) gives

\[
\leq N^\frac{2}{\beta} 2^{k-1} \sum_{j=0}^{l_c} (k+j)^2 \left( 4C_V h^{-\alpha} T^{1/2} \right)^{j+1} (2E_{0,h})^{k+j}
\]

\[
\leq N^\frac{2}{\beta} 2^{k-1}(8E_{0,h})^k \sum_{j=0}^{l_c} \left( 16E_{0,h} C_V h^{-\alpha} T^{1/2} \right)^{j+1}
\]

which completes the proof for the driving part.

For the error part, it reads

\[
\| S^{(1,k)}_h E P^{(k,l_c)} \|_{L^\infty_k[0,T] L^2_{x,x'}} \\
\leq 2^k \sum_{j=1}^{l_c+1} 4^j \left( C_V h^{-\alpha} T^{1/2} \right)^{j-1} \int_{[0,T]} \| S^{(1,k,j-1)}_h B_{N,h,1,k+j} (f^{(k,j)}_E (t_{k+j})) \|_{L^2_{x,x'}} dt_{k+j}
\]

Plugging in \( f^{(k,j)}_E \) and using estimate (2.76) provides

\[
\leq N^\frac{2}{\beta} 2^{k-1} \sum_{j=1}^{l_c+1} (k+j) \left( 4C_V h^{-\alpha} T^{1/2} \right)^{j} (2E_{0,h})^{k+j}
\]

\[
\leq N^\frac{2}{\beta} 2^{k-1}(8E_{0,h})^k \sum_{j=1}^{l_c+1} \left( 16E_{0,h} C_V h^{-\alpha} T^{1/2} \right)^{j}
\]

which completes the proof for the error part.

For the interaction part, we have similarly

\[
\| S^{(1,k)}_h I P^{(k,l_c)} \|_{L^\infty_k[0,T] L^2_{x,x'}} \\
= \left\| \int_{t_k}^{t_{k+l_c}} \ldots \int_{t_{k+l_c}}^{t_{k+l_c+1}} S^{(1,k)}_h J^{(k,l_c+1)} (t_k, t_{k+l_c+1}) \left( w^{(k+l_c+1)}_{N,h} (t_{k+l_c+1}) \right) dt_{k+l_c+1} \right\|_{L^\infty_k[0,T] L^2_{x,x'}}
\]

\[
\leq 2^k 4^{l_c+1} \left( C_V h^{-\alpha} T^{1/2} \right)^{l_c} \int_{[0,T]} \| S^{(1,k+l_c)}_h B_{N,h,1,k+l_c+1} w^{(k+l_c+1)}_{N,h} (t_{k+l_c+1}) \|_{L^2_{x,x'}} dt_{k+l_c+1}
\]

which is (2.91).

Notice that, we are not using the crude estimates in Lemma 2.4 for (2.91). We will use the KM bound we refined in Lemma 2.5 to strengthen our estimate in Proposition 2.7. Before we start, we recall that (2.53) is true for all \( l_c \geq 1 \), hence properties regarding \( w^{(k)}_{N,h} \) using \( l_c \).
equal to some number $A$ can be fed into the proof of another property of $w^{(k)}_{N,h}$ using $l_c$ equal to some number $B$.

**Proposition 2.7.** Let $T \leq \frac{h^{2\alpha}}{(64E_{0,h}C\epsilon)}$, and $\alpha = d + 1/2$. For $k \leq (\ln N)^2$, $l_c \leq \ln N$, we have

\[
\sup_{t \in [t_0, t_0 + T]} \|S_h^{(1,k)}w_{N,h}^{(k)}(t)\|_{L^2_{x,x'}}^{(2.93)} \leq 2k \sum_{j=0}^{l_c} (4C_V h^{-\alpha}T^{1/2})^j \|S_h^{(1,k+j)}w_{N,h}^{(k+j)}(t_0)\|_{L^2_{x,x'}}^j + (C_{0,h})^k N^{\frac{2\beta}{\alpha} - 1} + (C_{0,h})^k \left(\frac{1}{e}\right)^{l_c + 1},
\]

and

\[
\int_{[t_0, t_0 + T]} \|S_h^{(1,1)}B_{N,h,1,2}w_{N,h}^{(2)}(t)\|_{L^2_{x,x'}} dt \leq 4 \sum_{j=0}^{l_c} (4C_V h^{-\alpha}T^{1/2})^{j+1} \|S_h^{(1,2+j)}w_{N,h}^{(2+j)}(t_0)\|_{L^2_{x,x'}}^{j+1} + (C_{0,h})^2 N^{\frac{2\beta}{\alpha} - 1} + (C_{0,h})^2 \left(\frac{1}{e}\right)^{l_c + 1},
\]

where $C_{0,h} = 64E_{0,h}$. Notice that (2.94) is stronger than (2.83).

**Proof.** The conclusion of Lemma 2.6 reads

\[
\sup_{t \in [t_0, t_0 + T]} \|S_h^{(1,k)}w_{N,h}^{(k)}(t)\|_{L^2_{x,x'}}^{(2.95)} \leq 2k \sum_{j=0}^{l_c} (4C_V h^{-\alpha}T^{1/2})^j \|S_h^{(1,k+j)}w_{N,h}^{(k+j)}(t_0)\|_{L^2_{x,x'}}^j + 2N^{\frac{2\beta}{\alpha} - 1}(8E_{0,h})^k \sum_{j=0}^{l_c} (C_V h^{-\alpha}T^{1/2})^j
\]

\[+ 2k 4^{l_c+1} (C_V h^{-\alpha}T^{1/2})^{l_c} \int_{[t_0, t_0 + T]} \|S_h^{(1,0)}B_{N,h,1,1+k+l_c}w_{N,h}^{(k+l_c+1)}(t_{k+l_c+1})\|_{L^2_{x,x'}} dt_{k+l_c+1}.
\]

Since $k + l_c \leq (\ln N)^2$ and $T \leq \frac{h^{2\alpha}}{(64E_{0,h}C\epsilon)}$, we can employ KM bound in Lemma 2.5 to get

\[
\leq 2k \sum_{j=0}^{l_c} (4C_V h^{-\alpha}T^{1/2})^j \|S_h^{(1,k+j)}w_{N,h}^{(k+j)}(t_0)\|_{L^2_{x,x'}}^j + 2N^{\frac{2\beta}{\alpha} - 1}(8E_{0,h})^k \sum_{j=0}^{l_c} (C_V h^{-\alpha}T^{1/2})^j
\]

\[+ 2k 4^{l_c+1} (C_V h^{-\alpha}T^{1/2})^{l_c} (16E_{0,h})^{k+l_c}.
\]

Plugging in $T \leq \frac{h^{2\alpha}}{(64E_{0,h}C\epsilon)}$ and $C_{0,h} = 64E_{0,h}$, we obtain (2.93).

For (2.94), repeating the proof of KM bound in Lemma 2.5, we have

\[
\int_{[t_0, t_0 + T]} \|S_h^{(1,1)}B_{N,h,1,2}w_{N,h}^{(2)}(t)\|_{L^2_{x,x'}} dt \leq 4 \sum_{j=0}^{l_c} (4C_V h^{-\alpha}T^{1/2})^{j+1} \|S_h^{(1,2+j)}w_{N,h}^{(2+j)}(t_0)\|_{L^2_{x,x'}}^{j+1} + 2N^{\frac{2\beta}{\alpha} - 1}(8E_{0,h})^2
\]

\[+ 4^{l_c+2} (C_V h^{-\alpha}T^{1/2})^{l_c+1} \int_{[t_0, t_0 + T]} \|S_h^{(1,2+l_c)}B_{N,h,1,3+l_c}w_{N,h}^{(3+l_c)}(t_{3+l_c})\|_{L^2_{x,x'}} dt_{3+l_c}.
\]
Since \( 2 + l_c \leq (\ln N)^{10} \) and \( T \leq \frac{\hbar^{2\alpha}}{(64E_{0,h}C_Ve)} \), we can employ KM bound in Lemma \(2.5\) to get

\[
\leq 4 \sum_{j=0}^{l_c} (4 C_V \hbar^{-\alpha} T^{1/2})^{j+1} \| S_h^{(1,2+j)} w_{N,h}^{(2+j)}(t_0) \|_{L^2_{x,a'}} + 2 N^{\frac{5\beta-1}{2}} (8E_{0,h})^2
\]

\[ + 4 l_c^2 (C_V \hbar^{-\alpha} T^{1/2})^{l_c+1} (16E_{0,h})^{3+l_c} \]

Plugging in \( T \leq \frac{\hbar^{2\alpha}}{(64E_{0,h}C_Ve)} \) and \( C_{0,h} = 64E_{0,h} \), we obtain \((2.94)\).

2.4. Convergence Rate for Every Finite Time. In the section, we will iteratively use Proposition \(2.7\) to obtain the convergence rate for every finite time at the price of weakening the convergence rate.

**Proposition 2.8.** Let \( T_0 < +\infty \) and \( \alpha = d + 1/2 \). For \( k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \sum_{j=0}^{n(T_0,h)} \frac{\ln N}{2j} \), we have

\[
(2.97) \quad \sup_{t \in [0,T_0]} \| S_h^{(1,k)} w_{N,h}^{(k)}(t) \|_{L^2_{x,a'}} \leq (e^{n(T_0,h)} C_{0,h})^{k} N^{\frac{5\beta-1}{2}} (2 + l_c) \]

and

\[
(2.98) \quad \int_{[0,T_0]} \| S_h^{(1,1)} B_{N,h,1,2}^{(1)} w_{N,h}^{(2)}(t) \|_{L^2_{x,a'}} dt \leq 8 n(T_0,h) C_{0,h}^{2} N^{\frac{5\beta-1}{2}} (2 + l_c) \]

where \( n(T_0,h) = (8cC_V C_{0,h})^2 T_0/\hbar^{2\alpha} \) and \( C_{0,h} = 64E_{0,h} \) as defined in Proposition \(2.7\). Moreover, under the restriction \((2.10)\) that

\[
(2.99) \quad N \geq e^{(2)} \left( \left[ C_V E_{0,h}^2 T_0/\hbar^{2\alpha} \right]^{2} \right),
\]

for \( N \geq N_0(\beta) \) we have \((2.11)\) and \((2.12)\) which we restate here

\[
(2.100) \quad \sup_{t \in [0,T_0]} \| S_h^{(1,k)} w_{N,h}^{(k)}(t) \|_{L^2_{x,a'}} \leq \left( \frac{1}{\ln N} \right)^{100},
\]

\[
\int_{[0,T_0]} \| S_h^{(1,1)} B_{N,h,1,2}^{(1)} w_{N,h}^{(2)}(t) \|_{L^2_{x,a'}} dt \leq \left( \frac{1}{\ln N} \right)^{100}.
\]

**Proof.** Step 0. Set \( \lambda = \frac{1}{8cC_V C_{0,h}} \). Then for

\[
k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \ln N, \quad l_c \leq (1 - \frac{5}{2}\beta) \ln N,
\]

by estimate \((2.93)\) in Proposition \(2.7\) we have

\[
(2.100) \quad \sup_{t \in [0,\lambda^{2\alpha}]^{2}} \| S_h^{(1,k)} w_{N,h}^{(k)}(t) \|_{L^2_{x,a'}} \leq 2k \sum_{j=0}^{l_c} (4 C_V \lambda)^j \| S_h^{(1,2+j)} w_{N,h}^{(2+j)}(0) \|_{L^2_{x,a'}} + (C_{0,h})^{k} N^{\frac{5\beta-1}{2}} + (C_{0,h})^{k} \left( \frac{1}{\epsilon} \right)^{l_c+1}.
\]
By initial condition (2.8) in condition (c), we plug in $\lambda = \frac{1}{8eC_Vc_{0,h}}$ and take $l_c = (1 - \frac{5}{2}\beta) \ln N$ to get

$$\sup_{t \in [0, \lambda^2h^{2\alpha}] | L_2(x,t) \leq 4(C_{0,h})^k N^{\frac{5}{4}\beta - 1}$$

for every $k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \ln N$.

Step 1. Let $t_1 = \lambda^2h^{2\alpha}$. For

$$k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \left(\ln N + \frac{\ln N}{2}\right), \quad l_c \leq (1 - \frac{5}{2}\beta) \ln N,$$

we make use of estimate (2.93) in Proposition 2.7 again to obtain

$$\sup_{t \in [t_1, t_1 + \lambda^2h^{2\alpha}] | L_2(x,t) \leq 2^k \sum_{j=0}^{l_c} (4C_V\lambda)^j ||S_h^{(1,k)} w_{N,h}(t)||_{L_2(x)} + (C_{0,h})^k N^{\frac{5}{4}\beta - 1} + (C_{0,h})^k \left(\frac{1}{e}\right)^{l_c+1}$$

Since $k + l_c \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \ln N$, one can adopt estimate (2.101) in Step 0 to reach

$$\leq N^{\frac{5}{4}\beta - 1} (C_{0,h})^k \sum_{j=0}^{l_c} (4C_VC_{0,h}\lambda)^j + (C_{0,h})^k N^{\frac{5}{4}\beta - 1} + (C_{0,h})^k \left(\frac{1}{e}\right)^{l_c+1}$$

Recalling $\lambda = \frac{1}{8eC_Vc_{0,h}}$,

$$\leq N^{\frac{5}{4}\beta - 1} 8(C_{0,h})^k + (C_{0,h})^k N^{\frac{5}{4}\beta - 1} + (C_{0,h})^k \left(\frac{1}{e}\right)^{l_c+1}$$

By taking $l_c = (1 - \frac{5}{2}\beta) \ln N/2$, we arrive at

$$\sup_{t \in [t_1, t_1 + \lambda^2h^{2\alpha}] | L_2(x,t) \leq (eC_{0,h})^k N^{\frac{5}{4}\beta - 1}$$

for every $k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \left(\ln N + \frac{\ln N}{2}\right)$.

Step m. Let $t_m = m\lambda^2h^{2\alpha}$. Now we assume (2.102) is true for the case $n = m$, that is,

$$\sup_{t \in [t_m, t_m + \lambda^2h^{2\alpha}] | L_2(x,t) \leq (eC_{0,h})^k N^{\frac{5}{4}\beta - 1}$$

for every $k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \sum_{j=0}^{m} \frac{\ln N}{2j+1}$. Then we will prove it for $n = m + 1$. For

$$k \leq (\ln N)^2 - (1 - \frac{5}{2}\beta) \sum_{j=0}^{m+1} \frac{\ln N}{2j+1}, \quad l_c \leq (1 - \frac{5}{2}\beta) \ln N \frac{1}{2^{m+1}(m+1)!}$$

one can employ estimate (2.93) in Proposition 2.7 to reach

$$\sup_{t \in [t_m, t_m + \lambda^2h^{2\alpha}] | L_2(x,t) \leq 2^k \sum_{j=0}^{l_c} (4C_V\lambda)^j ||S_h^{(1,k+j)} w_{N,h}(t)||_{L_2(x)} + (C_{0,h})^k N^{\frac{5}{4}\beta - 1} + (C_{0,h})^k \left(\frac{1}{e}\right)^{l_c+1}$$

$$\leq 2^k \sum_{j=0}^{l_c} (4C_V\lambda)^j ||S_h^{(1,k+j)} w_{N,h}(t)||_{L_2(x)} + (C_{0,h})^k N^{\frac{5}{4}\beta - 1} + (C_{0,h})^k \left(\frac{1}{e}\right)^{l_c+1}$$
Since \( k + l_c \leq (\ln N)^2 - (1 - \frac{5}{2} \beta) \sum_{j=0}^{m} \frac{\ln N}{2^j} \), one can use estimate (2.103) in the case \( n = m \) to get

\[
\leq N^{\frac{5}{2m+1}} (2e^m C_{0,h})^k \sum_{j=0}^{l_c} (4C_V \lambda)^j (e^m C_{0,h})^j + (C_{0,h})^k N^{\frac{5}{2} \beta - 1} + (C_{0,h})^k \left( \frac{1}{e} \right)^{l_c+1}
\]

Recalling \( \lambda = \frac{1}{8eC_V C_{0,h}} \),

\[
\leq (2e^m C_{0,h})^k N^{\frac{5}{2m+1}} (e^m)^{l_c+1} + (C_{0,h})^k N^{\frac{5}{2} \beta - 1} + (C_{0,h})^k \left( \frac{1}{e} \right)^{l_c+1}
\]

Taking \( l_c + 1 = \frac{(1 - \frac{5}{2} \beta) \ln N}{2m+1(m+1)!} \), we arrive at

\[
\sup_{t \in [t_m, t_{m+1} + \lambda^2 \hbar^{2\alpha}]} \| S_h^{(1,k)} w_{N,h}^{(k)}(t) \|_{L^2_{x,x'}} \leq (2e^m C_{0,h})^k N^{\frac{5}{2m+1}} (e^m)^{l_c+1} + (C_{0,h})^k N^{\frac{5}{2} \beta - 1} + (C_{0,h})^k \left( \frac{1}{e} \right)^{l_c+1}
\]

This proves (2.103) and completes the proof of (2.97) as we can take \( m = n(T_0, \hbar) = (8eC_V C_{0,h})^2 T_0 / \hbar^{2\alpha} \).

For (2.98), we can use estimate (2.94) in Proposition 2.7 to get to

\[
(2.104) \quad \int_{[t_m, t_{m+1} + \lambda^2 \hbar^{2\alpha}]} \| S_h^{(1,1)} B^+_{N,h,1,2} w_{N,h}^{(2)}(t) \|_{L^2_{x,x'}} dt \leq 4 \sum_{j=0}^{l_c} (4C_V \lambda)^{j+1} \left\| S_h^{(1,2+j)} w_{N,h}^{(2+j)}(t_m) \right\|_{L^2_{x,x'}} + C_{0,h}^2 N^{\frac{5}{2} \beta - 1} + C_{0,h}^2 \left( \frac{1}{e} \right)^{l_c+1}
\]

Plugging in estimate (2.103),

\[
\leq 4 \sum_{j=0}^{l_c} (4C_V \lambda)^{j+1} (e^m C_{0,h})^{j+2} N^{\frac{5}{2m+1}} + C_{0,h}^2 N^{\frac{5}{2} \beta - 1} + C_{0,h}^2 \left( \frac{1}{e} \right)^{l_c+1}
\]

Recalling \( \lambda = \frac{1}{8eC_V C_{0,h}} \),

\[
\leq 4 C_{0,h} (e^m)^{l_c+2} N^{\frac{5}{2m+1}} + C_{0,h}^2 N^{\frac{5}{2} \beta - 1} + C_{0,h}^2 \left( \frac{1}{e} \right)^{l_c+1}
\]

Setting \( l_c + 2 = \frac{(1 - \frac{5}{2} \beta) \ln N}{2m+1(m+1)!} \), we arrive at

\[
\leq 4 C_{0,h} N^{\frac{5}{2(m+1)}} + C_{0,h}^2 N^{\frac{5}{2} \beta - 1} + e C_{0,h}^2 N^{\frac{5}{2m+1}} \leq 8 C_{0,h}^2 N^{\frac{5}{2m+1}}.
\]
Then by summing the integration time domain, we obtain

\begin{equation}
\int_{[0,T_0]} \| S_h^{(1,1)} B_{N,h}^{\pm} w_{N,h}(t) \|_{L^2_{t,x}} dt \\
\leq \sum_{m=0}^{n(T_0,h)} \int_{[m,T_0)} \| S_h^{(1,1)} B_{N,h}^{\pm} w_{N,h}(t) \|_{L^2_{t,x}} dt \\
\leq \sum_{m=0}^{n(T_0,h)} 8C_{0,h}^2 N \frac{\hat{d}^{\beta-1}}{2^{m+1}(m+1)!} \\
\leq 8n(T_0,h)C_{0,h}^2 N \frac{\hat{d}^{\beta-1}}{2^{m+1}(m+1)!}.
\end{equation}

This completes the proof of \((2.98)\).

For estimates \((2.11)\) and \((2.12)\), under the restriction \((2.10)\) that

\begin{equation}
N \geq e^{(2)} \left( \left[ C_1^2 E_{0,h}^2 T_0^2 / \hbar^2 \right]^{1/2} \right)
\end{equation}

which implies that \(n(T_0,h) \leq \sqrt{C \ln \ln N}\) with an absolute constant \(C\), we have

\[2^n(T_0,h) n(T_0,h)! \leq n(T_0,h) n(T_0,h) \leq \sqrt{C \ln \ln N} \sqrt{C \ln \ln N} \leq \sqrt{\ln N}.
\]

Also, we have

\[8n(T_0,h)C_{0,h}^2 \leq n(T_0,h) C_{0,h} \leq n(T_0,h) n(T_0,h) \leq \sqrt{\ln N}.
\]

Hence, we obtain

\[\sup_{t \in [0,T_0]} \| S_h^{(1,1)} w_{N,h}(t) \|_{L^2_{t,x}} \leq e^{n(T_0,h)} C_{0,h} N \frac{\hat{d}^{\beta-1}}{2^{m+1}(m+1)!} \leq \frac{\sqrt{\ln N}}{\sqrt{\ln N}} \leq \left( \frac{1}{\ln N} \right)^{100} ,
\]

\[\int_{[0,T_0]} \| S_h^{(1,1)} B_{N,h}^{\pm} w_{N,h}(t) \|_{L^2_{t,x}} dt \leq 8n(T_0,h)C_{0,h}^2 N \frac{\hat{d}^{\beta-1}}{2^{m+1}(m+1)!} \leq \left( \frac{1}{\ln N} \right)^{100} ,
\]

for \(N \geq N_0(\beta)\). This completes the proof of estimates \((2.11)\) and \((2.12)\).

\[\Box\]

3. H-NLS v.s. the Compressible Euler Equation: A Modulated Energy Approach

We will compare the H-NLS equation \((2.1)\) and the compressible Euler equation \((1.5)\) before its blowup time by the method of modulated energy. Recall the H-NLS equation \((2.1)\)

\[
\left\{
\begin{array}{l}
\hbar \phi_{t,N,h} = -\frac{i}{2} \hbar^2 \Delta \phi_{N,h} + (V_N * |\phi_{N,h}|^2) \phi_{N,h}, \\
\phi_{N,h}(0) = \phi_{N,h}^m,
\end{array}
\right.
\]

with the mass density and momentum density defined by \((2.29)\)

\[\rho_{N,h}(t,x) = |\phi_{N,h}(t,x)|^2, \quad J_{N,h}(t,x) = \hbar \Im (\overline{\phi_{N,h}(t,x)} \nabla \phi_{N,h}(t,x)) .
\]
and the compressible Euler equation \((1.5)\)
\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t u + (u \cdot \nabla) u + b_0 \nabla \rho &= 0, \\
(\rho, u)|_{t=0} &= (\rho^m, u^m).
\end{aligned}
\]

Here is the main theorem of the section.

**Theorem 3.1.** Let \(\phi_{N,h}(t)\) be the solution to H-NLS equation with the initial data \(\phi_{N,h}^0\). Under the same conditions of Theorem \((1.1)\), then we have\(^{14}\) \((3.1)\)
\[
\|\rho_{N,h} - \rho\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0) \left( \frac{1}{h^4N^\beta} + h^2 \right)^{\frac{1}{2}},
\]
\[(3.2)\]
\[
\|J_{N,h} - \rho u\|_{L^\infty([0,T_0];L^r(\mathbb{R}^d))} \leq C(T_0) \left( \frac{1}{h^4N^\beta} + h^2 \right)^{\frac{1}{2}(\frac{4}{r} - 3)},
\]
where \(r \in [1, 4/3)\), \(\beta\) can be absorbed into \(h^2\).

**Proof of Theorem 3.1.** By \((3.37)\) and \((3.38)\) in Proposition 3.5 we have
\[
\|\rho_{N,h} - \rho\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0) \left( \frac{1}{h^4N^\beta} + h^2 \right)^{\frac{1}{2}},
\]
\[(3.3)\]
\[
\|(i\hbar \nabla - u)\phi_{N,h}\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0) \left( \frac{1}{h^4N^\beta} + h^2 \right)^{\frac{1}{2}},
\]
which directly completes the proof of \((3.1)\).

For \((3.2)\), by the triangle and Hölder’s inequalities as well as estimates \((3.37)\) and \((3.38)\) we have
\[
\|J_{N,h} - \rho u\|_{L^1(\mathbb{R}^d)} \leq \|J_{N,h} - \rho_{N,h} u\|_{L^1(\mathbb{R}^d)} + \|\rho_{N,h} u - \rho u\|_{L^1(\mathbb{R}^d)}
\]
\[
= \|\text{Im} \left( \frac{\phi_{N,h}}{\hbar \nabla - iu} \phi_{N,h} \right)\|_{L^1(\mathbb{R}^d)} + \|\rho_{N,h} u - \rho u\|_{L^1(\mathbb{R}^d)}
\]
\[
\leq \|\phi_{N,h}\|_{L^2(\mathbb{R}^d)} \| (i\hbar \nabla - u)\phi_{N,h}\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \|\rho_{N,h} - \rho\|_{L^2(\mathbb{R}^d)}
\]
\[
\leq C(T_0) \left( \frac{1}{h^4N^\beta} + h^2 \right)^{\frac{1}{2}}
\]
On the other hand, by the energy bound for \(\phi_{N,h}\) and the uniform bound for \(\|\rho_{N,h}\|_{L^2}\) we have
\[(3.4)\]
\[
\|J_{N,h}\|_{L^{4/3}} \leq \|h \nabla \phi_{N,h}\|_{L^2} \|\phi_{N,h}\|_{L^1} \lesssim E_0.
\]

\(^{14}\)Under the restriction \((1.13)\), the smallness factor \(1/h^4\) can be absorbed into \(h^2\).
where we used energy bound and uniform bound for \( \|\rho_{N,h}\|_{L^2} \) in the last inequality. Hence, by interpolation inequality we obtain
\[
(3.5) \quad \|J_{N,h} - \rho u\|_{L^\infty([0,T_0];L^1(\mathbb{R}^d))} \leq \|J_{N,h} - \rho u\|_{L^\infty([0,T_0];L^1(\mathbb{R}^d))}^{1-\alpha} \|J_{N,h} - \rho u\|_{L^\infty([0,T_0];L^1/3(\mathbb{R}^d))}^\alpha \leq C \left( \frac{1}{\hbar^4N^\beta} + \hbar^2 \right)^{\frac{1}{2}} E_0^{\frac{1}{r}},
\]
where \( \alpha = 4 - 4/r \). This completes the proof of (3.2).

For (3.3), by triangle inequality we have
\[
(3.6) \quad \|\rho_{N,h}V_N * \rho_{N,h} - b_0\rho^2\|_{L^1(\mathbb{R}^d)} \leq \|\rho_{N,h}V_N * \rho_{N,h} - b_0(\rho_{N,h})^2\|_{L^1(\mathbb{R}^d)} + b_0\|\rho_{N,h} - \rho\|^2_{L^1(\mathbb{R}^d)}.
\]
By the approximation of identity estimate (3.13) which reads
\[
(3.7) \quad \|\rho_{N,h}V_N * \rho_{N,h} - b_0(\rho_{N,h})^2\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\hbar^4N^\beta}
\]
and estimate (3.11), we have
\[
\|\rho_{N,h}V_N * \rho_{N,h} - b_0\rho^2\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\hbar^4N^\beta} + \|\rho_{N,h} - \rho\|_{L^2(\mathbb{R}^d)} \left( \|\rho_{N,h}\|_{L^2(\mathbb{R}^d)} + \|\rho\|_{L^2(\mathbb{R}^d)} \right) \leq C(T_0) \left( \frac{1}{\hbar^4N^\beta} + \hbar^2 \right)^{\frac{1}{2}}.
\]
By taking \( L^\infty \) norm at \( dt \), we complete the proof of (3.3). Thus we have proved Theorem 3.1 assuming Proposition 3.5 and (3.13). The rest of this section is to prove them.

3.1. The Evolution of the Modulated Energy. We consider the following modulated energy
\[
(3.8) \quad \mathcal{M}[\phi_{N,h}, \rho, u](t) = \frac{1}{2} \int_{\mathbb{R}^d} |(i\hbar \nabla - u)\phi_{N,h}(t)|^2 \, dx + \frac{1}{2} \left( V_N * \rho_{N,h}, \rho_{N,h} \right) + \frac{b_0}{2} \int_{\mathbb{R}^d} \rho^2 \, dx - b_0 \int_{\mathbb{R}^d} \rho \rho_{N,h} \, dx.
\]
We need to derive a time evolution equation for \( \mathcal{M}[\phi_{N,h}, \rho, u](t) \). The related quantities for \( \phi_{N,h} \) are given as the following.

Lemma 3.2. We have the following estimates regarding \( \phi_{N,h} \):
\[
(3.9) \quad \partial_t \rho_{N,h} + \text{div } J_{N,h} = 0,
\]
\[
(3.10) \quad \partial_t J_{N,h}^j + \sum_{j,k} \partial_k \left[ \hbar^2 \text{Re} \left( \partial_j \overline{\phi_{N,h} \partial_k \phi_{N,h}} - \frac{\hbar^2}{4} \partial_j \rho_{N,h} \right) \right] + (\partial_j (V_N * \rho_{N,h})) \rho_{N,h} = 0,
\]
(3.11) \( E_{N,h}(t) \equiv E_{N,h}(0) \),
where the energy \( E_{N,h}(t) \) is defined by
\[
(3.12) \quad E_{N,h}(t) = \frac{1}{2} \|\hbar \nabla \phi_{N,h}(t)\|_2^2 + \frac{1}{2} \left( V_N * \rho_{N,h}, \rho_{N,h} \right) (t).
\]
We also have the approximation of identity estimate:
\[
(3.13) \quad \|\rho_{N,h}V_N * \rho_{N,h} - b_0(\rho_{N,h})^2\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\hbar^4N^\beta}.
\]
Proof. We omit the proof of (3.9) - (3.11) as this is a direct computation and is well-known in $H^1$ wellposedness theory. For (3.13), we set $W_N = V_N - b_0 \delta$ and rewrite

$$
\|\rho_{N,h}V_N * \rho_{N,h} - b_0 (\rho_{N,h})^2\|_{L^1(\mathbb{R}^d)} = \|\rho_{N,h}W_N * \rho_{N,h}\|_{L^1(\mathbb{R}^d)}
$$

By Hölder,

$$
\leq \|W_N * \rho_{N,h}\|_{L^{3/2}} \|\rho_{N,h}\|_{L^3}
$$

By Lemma A.6

$$
\lesssim N^{-\beta} \|\nabla \rho_{N,h}\|_{L^{3/2}} \|\rho_{N,h}\|_{L^3}
$$

By fractional Leibniz rule in Lemma A.7 and Sobolev inequality,

$$
\lesssim N^{-\beta} \|\phi_{N,h}\|_{H^1}^4
$$

By the energy bound for $\phi_{N,h},$

$$
\lesssim \frac{1}{\hbar^4 N^\beta},
$$

which completes the proof of (3.13). \qed

Next let us derive the time derivative of $\mathcal{M}[\phi_{N,h}, \rho, u](t)$.

**Proposition 3.3.** There holds

\begin{equation}
\frac{d}{dt} \mathcal{M}[\phi_{N,h}, \rho, u](t) = -\int_{\mathbb{R}^d} \partial_k u^j \text{Re} \left((\hbar \partial_k - i u^k)\phi_{N,h}(\hbar \partial_j - i u^j) \overline{\phi_{N,h}}\right) - \frac{b_0}{2} \int_{\mathbb{R}^d} \text{div} \rho_{N,h} - \rho \|^2 \, dx - \frac{\hbar^2}{4} \int_{\mathbb{R}^d} \rho_{N,h} (\Delta \text{div} u) \, dx + E_r
\end{equation}

where the summation convention for repeated indices is used and the error term is given by

\begin{equation}
E_r = \int_{\mathbb{R}^d} u^j (\partial_j (V_N * \rho_{N,h})) \rho_{N,h} \, dx + \frac{b_0}{2} \int_{\mathbb{R}^d} \text{div} \rho_{N,h} (\rho_{N,h})^2 \, dx.
\end{equation}

**Proof.** By energy conservation law (3.11) in Lemma (3.2), we obtain

$$
\frac{d}{dt} \mathcal{M}[\phi_{N,h}, \rho, u](t) = \frac{1}{2} \frac{d}{dt} \|h \nabla \phi_{N,h}(t)\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |u|^2 \rho_{N,h} \, dx - \frac{d}{dt} \int_{\mathbb{R}^d} J_{N,h} u \, dx
$$

$$
+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (V_N * \rho_{N,h}, \rho_{N,h}) + \frac{b_0}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^2 \, dx - b_0 \frac{d}{dt} \int_{\mathbb{R}^d} \rho_{N,h} \, dx
$$

$$
= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |u|^2 \rho_{N,h} \, dx - \frac{d}{dt} \int_{\mathbb{R}^d} J_{N,h} u \, dx
$$

$$
+ \frac{b_0}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^2 \, dx - b_0 \frac{d}{dt} \int_{\mathbb{R}^d} \rho_{N,h} \, dx.
$$
Next, we calculate the above four terms separately. For the first term, by (1.5) and (3.9) we find

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |u|^2 \rho_{N,h} dx = \int_{\mathbb{R}^d} \left[ u \partial_t u \rho_{N,h} + \frac{1}{2} |u|^2 \partial_t \rho_{N,h} \right] dx
= \int_{\mathbb{R}^d} \left[ \partial_t u^j \rho_{N,h} u^j - \frac{1}{2} |u|^2 \text{div } J_{N,h} \right] dx
= \int_{\mathbb{R}^d} \left[ -\rho_{N,h} u^j \partial_k u_j - b_0 \rho_{N,h} u^j \partial_j \rho + J_{N,h}^j u^k \partial_j u^k \right] dx
$$

where we have used integration by parts in the last equality.

For the second term, via (3.10) and (1.5) we have

$$
- \frac{d}{dt} \int_{\mathbb{R}^d} J_{N,h} u dx
= \int_{\mathbb{R}^d} \left( - \partial_t J_{N,h} u - J_{N,h} \partial_t u \right) dx
= \int_{\mathbb{R}^d} \left( \partial_j \left( \frac{\hbar^2}{2} \text{Re}(\partial_j \phi_{N,h} \partial_k \phi_{N,h}) - \frac{\hbar^2}{4} \partial^2_{jk} \rho_{N,h} \right) + (\partial_j (V_N * \rho_{N,h})) \rho_{N,h} \right) u^j dx
+ \int_{\mathbb{R}^d} J_{N,h}^j u^k \partial_k u^j dx + b_0 \int_{\mathbb{R}^d} J_{N,h}^j \partial_j \rho dx
$$

Integrating by parts and using (3.15),

$$
= \int_{\mathbb{R}^d} -\frac{\hbar^2}{2} \partial_k u^j \left[ \text{Re}(\partial_j \phi_{N,h} \partial_k \phi_{N,h}) \right] dx - \int_{\mathbb{R}^d} \frac{\hbar^2}{4} \rho_{N,h} \partial^2_{jk} \rho_{N,h} u^j dx
- \frac{b_0}{2} \int \text{div } u (\rho_{N,h})^2 dx + Er + \int_{\mathbb{R}^d} J_{N,h}^j u^k \partial_k u^j dx + b_0 \int_{\mathbb{R}^d} J_{N,h}^j \partial_j \rho dx.
$$

For the third term, using (1.5) and integration by parts, we obtain

$$
\frac{b_0}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho^2 dx = b_0 \int_{\mathbb{R}^d} \rho \partial_t \rho dx = -b_0 \int_{\mathbb{R}^d} \rho \text{div } (\rho u) dx
= b_0 \int_{\mathbb{R}^d} (\partial_j \rho) \rho u^j dx = -\frac{b_0}{2} \int_{\mathbb{R}^d} \rho^2 \text{div } u dx.
$$

For the forth term, plugging in (1.5) and (3.9), we integrate by parts to get

$$
- \frac{b_0}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho_{N,h} \rho dx = b_0 \int_{\mathbb{R}^d} -\rho \partial_t \rho_{N,h} - \rho_{N,h} \partial_t \rho dx
= b_0 \int_{\mathbb{R}^d} \rho \text{div } J_{N,h} \rho_{N,h} \rho \text{div } (\rho u) dx
= b_0 \int_{\mathbb{R}^d} -\partial_j \rho J_{N,h}^j + \rho_{N,h} \rho \text{div } u + \rho_{N,h} u^j \partial_j \rho dx.
$$
Summing up (3.16) – (3.19), we conclude
\[
\frac{d}{dt} \mathcal{M} [\phi_{N,h}, \rho, u] (t) \\
= \int_{\mathbb{R}^d} \left[ -\rho_{N,h} u^i \partial_i u^j - b_0 \rho_{N,h} u^i \partial_i \rho + J_{N,h}^j u^k \partial_j u^k \right] dx \\
+ \int_{\mathbb{R}^d} -\hbar^2 \partial_k u^j \left[ \text{Re}(\partial_j \overline{\phi_{N,h}} \partial_k \phi_{N,h}) \right] dx - \int_{\mathbb{R}^d} \frac{\hbar^2}{4} \rho_{N,h} \partial^2 \partial_j u^j dx \\
- \frac{b_0}{2} \int \text{div} u (\rho_{N,h})^2 dx + Er + \int_{\mathbb{R}^d} J_{N,h}^j u^k \partial_j u^j dx + \int_{\mathbb{R}^d} b_0 J_{N,h}^j \partial_j \rho dx \\
- \frac{b_0}{2} \int \rho^2 \text{div} u dx + \int_{\mathbb{R}^d} b_0 \rho_{N,h} \rho \text{div} u + b_0 \rho_{N,h} u^i \partial_j \rho - b_0 \partial_j \rho J_{N,h}^j dx \\
= - \int_{\mathbb{R}^d} \partial_k u^j \left\{ \rho_{N,h} u^i u^k + h^2 \left[ \text{Re}(\partial_j \overline{\phi_{N,h}} \partial_k \phi_{N,h}) \right] - J_{N,h}^j u^k - J_{N,h}^k u^j \right\} dx \\
- \frac{b_0}{2} \int \text{div} u (\rho_{N,h} - \rho)^2 dx - \frac{\hbar^2}{4} \int_{\mathbb{R}^d} \rho_{N,h} (\Delta \text{div} u) dx + Er
\]
which is equivalent to (3.14). This completes the proof.

\[\Box\]

3.2. Modulated Energy Estimate. We first estimate for the error term (3.15) and then establish Gronwall’s inequality for the modulated energy \(\mathcal{M} [\phi_{N,h}, \rho, u] (t)\).

**Lemma 3.4.** Let \(Er\) be defined as in (3.15). We have
\[
|Er| \lesssim \frac{1}{\hbar^4 N^\beta}.
\]

**Proof.** For (3.20), we decompose
\[
Er = \sum_{j=1}^{3} \int_{\mathbb{R}^d} u^j (\partial_j (V_N \ast \rho_{N,h})) \rho_{N,h} dx + \frac{b_0}{2} \int \text{div} u (\rho_{N,h})^2 dx
\]
\[
= I_1 + I_2
\]
where
\[
I_1 = \sum_{j=1}^{3} \int_{\mathbb{R}^d} u^j (\partial_j (V_N \ast \rho_{N,h})) \rho_{N,h} dx
\]
\[
- \sum_{j=1}^{3} \frac{1}{2} \int \partial_j u^j (y) [x^j - y^j] \partial_j [V_N(x - y)] \rho_{N,h}(y) \rho_{N,h}(x) dx dy
\]
and
\[
I_2 = \frac{b_0}{2} \int \text{div} u (\rho_{N,h})^2 dx
\]
\[
+ \sum_{j=1}^{3} \frac{1}{2} \int \partial_j u^j (y) [x^j - y^j] \partial_j [V_N(x - y)] \rho_{N,h}(y) \rho_{N,h}(x) dx dy
\]
with \(x = (x^1, x^2, x^3)\) and \(y = (y^1, y^2, y^3)\).
First, we deal with $I_1$. Note that

$$
(3.24) \quad \int_{\mathbb{R}^d} u^i(\partial_j(V_N \ast \rho_{N,h}))\rho_{N,h}dx
= \int u^i(x) (\partial_j V_N)(x - y)\rho_{N,h}(x)\rho_{N,h}(y)dx\,dy
= \int u^i(y) (\partial_j V_N)(y - x)\rho_{N,h}(y)\rho_{N,h}(x)dx\,dy.
$$

By the anti-symmetry of $\partial_j V_N$,

$$
= - \int u^i(y) (\partial_j V_N)(x - y)\rho_{N,h}(x)\rho_{N,h}(y)dx\,dy.
$$

Hence we obtain

$$
I_1 = \frac{1}{2} \sum_{j=1}^{3} \int (u^i(x) - u^i(y))\partial_j V_N(x - y)\rho_{N,h}(y)\rho_{N,h}(x)dx\,dy
- \frac{1}{2} \sum_{j=1}^{3} \int \partial_j u^j(y) [x^j - y^j] \partial_j V_N(x - y)\rho_{N,h}(y)\rho_{N,h}(x)dx\,dy.
$$

It suffices to estimate the $j = 1$ case. By Taylor’s expansion, we get

$$
(3.25) \quad (u^1(x) - u^1(y)) = \sum_{i=1}^{3} \partial_i u^1(y) [x^i - y^i] + \frac{1}{2} ((x - y) \cdot \nabla)^2 u^1(y + \theta(x - y)),
$$

so we can rewrite

$$
I_1 = A_1 + A_2 + A_3
$$

where

$$
(3.26) \quad A_1 = \frac{1}{2} \int \frac{1}{2} ((x - y) \cdot \nabla)^2 u^1(y + \theta(x - y))\partial_1 V_N(x - y)\rho_{N,h}(y)\rho_{N,h}(x)dx\,dy,
$$

$$
(3.27) \quad A_2 = \frac{1}{2} \int \partial_2 u^1(y) [x^2 - y^2] \partial_1 V_N(x - y)\rho_{N,h}(y)\rho_{N,h}(x)dx\,dy,
$$

$$
(3.28) \quad A_3 = \frac{1}{2} \int \partial_3 u^1(y) [x^3 - y^3] \partial_1 V_N(x - y)\rho_{N,h}(y)\rho_{N,h}(x)dx\,dy.
$$

For $A_1$,

$$
|A_1| \lesssim \frac{||D^2 u||_{L^\infty}}{N^{2\beta}} \int (N^\beta |x - y|^2 |\partial_1 V_N(x - y)|\rho_{N,h}(x)\rho_{N,h}(y)dx\,dy
$$

By Hölder,

$$
\lesssim \frac{||D^2 u||_{L^\infty}}{N^{\beta}} ||(|x|^2 \partial_1 V)_N| \ast \rho_{N,h}||_{L^2} ||\rho_{N,h}||_{L^2}
$$

By Young’s inequality, interpolation inequality, and the energy bound for $\phi_{N,h}$,

$$
\lesssim \frac{||D^2 u||_{L^\infty} \|x\|_1^2 \partial_1 V\|_{L^1}}{h^{4} N^{\beta}}.
$$
For $A_2$,

$$A_2 = \frac{1}{2} \int \partial_2 u_1^1(y) \left[ x^2 - y^2 \right] \partial_1 V_N(x - y) \rho_{N,h}(y) \rho_{N,h}(x) \, dx \, dy \tag{3.29}$$

By integration by parts,

$$= - \frac{1}{2} \int \partial_2 u_1^1(y) \left[ x^2 - y^2 \right] V_N(x - y) \rho_{N,h}(y) \partial_1 \rho_{N,h}(x) \, dx \, dy$$

$$= - \frac{1}{2N^\beta} \int \partial_2 u_1^1(y) \left[ N^\beta (x^2 - y^2) \right] V_N(x - y) \rho_{N,h}(y) \partial_1 \rho_{N,h}(x) \, dx \, dy.$$ 

So we get

$$|A_2| \lesssim \frac{\|Du\|_{L^\infty}}{N^\beta} \|\tilde{V}_N * \rho_{N,h}\|_{L^1} \|\partial_1 \rho_{N,h}\|_{L^{3/2}} \tag{3.30}$$

where we use the notation that $\tilde{V}(x) = x^2 V(x)$. By Young’s inequality and Hölder inequality,

$$\lesssim \frac{\|Du\|_{L^\infty}}{N^\beta} \|\tilde{V}_N\|_{L^1} \|\rho_{N,h}\|_{L^3} \|\phi_{N,h}\|_{L^6} \|\nabla \phi_{N,h}\|_{L^2}$$

By Sobolev,

$$\lesssim \frac{\|Du\|_{L^\infty}}{N^\beta} \|\tilde{V}\|_{L^1} \|\phi_{N,h}\|_{H^1}^4.$$ 

By the energy bound for $\phi_{N,h}$,

$$\lesssim \frac{\|Du\|_{L^\infty}}{h^4 N^\beta} \|\tilde{V}\|_{L^1}.$$ 

For $A_3$, we deal with it in the same way and obtain

$$A_3 \lesssim \frac{\|Du\|_{L^\infty}}{h^4 N^\beta} \|\tilde{V}\|_{L^1} \tag{3.31}$$

For $I_2$, it suffices to treat the case $j = 1$. Let

$$\tilde{V}(x) = -x^1 V(x),$$ 

then we have

$$|I_2| = \frac{1}{2} \left| \langle \partial_1 u_1^1 \tilde{V}_N * \rho_{N,h}, \rho_{N,h} \rangle - b_0 \langle \partial_1 u_1^1 \rho_{N,h}, \rho_{N,h} \rangle \right|$$

$$= \frac{1}{2} \left| \langle \partial_1 u_1^1 \left( \tilde{V}_N - b_0 \delta \right) * \rho_{N,h}, \rho_{N,h} \rangle \right| \tag{3.33}$$

Since $\int \tilde{V} \, dx = \int V \, dx = b_0$, we can repeat the proof of the approximation of identity estimate [3.13] to get to

$$\lesssim \|Du\|_{L^\infty} \|W_N * \rho_{N,h}\|_{L^{3/2}} \|\rho_{N,h}\|_{L^3}$$

$$\lesssim \frac{1}{h^4 N^\beta}.$$ 

Putting together the estimates of $I_1$ and $I_2$ completes the proof. \hfill \Box

We can now provide a closed estimate for the modulated energy.
Proposition 3.5. Let \( \mathcal{M} [\phi_{N,h}, \rho, u] (t) \) be defined as in (3.8), we have the lower bound estimate
\[
(3.34) \quad \mathcal{M} [\phi_{N,h}, \rho, u] (t) + \frac{C}{h^4 N^\beta} \geq 0
\]
and the following Gronwall’s inequality
\[
(3.35) \quad \frac{d}{dt} \mathcal{M} [\phi_{N,h}, \rho, u] (t) \leq \mathcal{M} [\phi_{N,h}, \rho, u] (t) + \frac{1}{h^4 N^\beta} + h^2.
\]
Moreover, we have
\[
(3.36) \quad \mathcal{M} [\phi_{N,h}, \rho, u] (t) + \frac{C}{h^4 N^\beta} \leq \exp (CT_0) \left( \mathcal{M} [\phi_{N,h}, \rho, u] (0) + \frac{C}{h^4 N^\beta} + Ch^2 t \right)
\]
and
\[
(3.37) \quad \| \rho_{N,h} - \rho \|_{L^\infty ([0,T_0]; L^2(\mathbb{R}^d))} \leq C(T_0) \left( \frac{1}{h^4 N^\beta} + h^2 \right)^{1/2},
\]
\[
(3.38) \quad \| (i h \nabla - u) \phi_{N,h} \|_{L^\infty ([0,T_0]; L^2(\mathbb{R}^d))} \leq C(T_0) \left( \frac{1}{h^4 N^\beta} + h^2 \right)^{1/2}.
\]

Proof. For (3.34), we rewrite
\[
(3.39) \quad \mathcal{M} [\phi_{N,h}, \rho, u] (t) = \frac{1}{2} \int_{\mathbb{R}^d} |(i h \nabla - u) \phi_{N,h}(t)|^2 dx + b_0 \int (\rho_{N,h} - \rho)^2 dx + \frac{1}{2} \int L_0 \left( W_N * \rho_{N,h}, \rho_{N,h} \right)
\]
where \( W_N = V_N - b_0 \delta \). By estimate (3.13), we arrive at
\[
(3.40) \quad \mathcal{M} [\phi_{N,h}, \rho, u] (t) \gtrsim \frac{1}{h^4 N^\beta},
\]
which completes the proof of (3.34).

For (3.35), we make use of Proposition 3.3 to obtain
\[
(3.41) \quad \frac{d}{dt} \mathcal{M} [\phi_{N,h}, \rho, u] (t) \lesssim |Du|_{L^\infty} \left( \int_{\mathbb{R}^d} |(i h \nabla - u) \phi_{N,h}(t)|^2 dx + b_0 \int (\rho_{N,h} - \rho)^2 dx \right) + h^2 \| \rho_{N,h} \|_{L^1} \| \Delta \div u \|_{L^\infty} + |E_r|
\]
By the error term estimate (3.20), we reach
\[
(3.42) \quad \frac{d}{dt} \mathcal{M} [\phi_{N,h}, \rho, u] (t) \lesssim \mathcal{M} [\phi_{N,h}, \rho, u] (t) + h^2 + \frac{1}{h^4 N^\beta},
\]
which completes the proof of (3.35).

\[\text{The regularity requirement that } s > \frac{d}{2} + 3 \text{ comes from } \| \Delta \div u \|_{L^\infty}, \text{ the second term on the right side of (3.41). One can reduce one derivative in requirement (d) of Theorem 1.1 by integration by parts at the price of weakening the convergence rate.}\]
Combining (3.34) and (3.35), we have

\[
(3.33) \quad \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (t) + \frac{C}{\hbar^4 N^\beta}
= \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (0) + \frac{C}{\hbar^4 N^\beta} + \int_0^t \frac{d}{d\tau} \left( \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (\tau) + \frac{C}{\hbar^4 N^\beta} \right) d\tau
\leq \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (0) + \frac{C}{\hbar^4 N^\beta} + C \int_0^t \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (\tau) + \frac{C}{\hbar^4 N^\beta} + \hbar^2 d\tau
\]

Then by Gronwall’s inequality, we obtain estimate (3.36).

Finally, we deal with (3.37) and (3.38). By error estimate (3.13), we note that

\[
\int_{\mathbb{R}^d} \left| (i\hbar \nabla - u) \phi_{N,j}(t) \right|^2 dx + b_0 \int (\rho_{N,j} - \rho)^2 dx \leq \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (t) + \frac{1}{\hbar^4 N^\beta},
\]

\[
\mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (0) \leq \int_{\mathbb{R}^d} \left| (i\hbar \nabla - u^\text{in}) \phi_{N,j}^\text{in} \right|^2 dx + b_0 \int_{\mathbb{R}^d} (\rho_{N,j}^\text{in} - \rho^\text{in})^2 dx + \frac{1}{\hbar^4 N^\beta}.
\]

Hence, we can appeal to estimate (3.36) and the initial condition (1.12) to get

\[
(3.34) \quad \int_{\mathbb{R}^d} \left| (i\hbar \nabla - u) \phi_{N,j}(t) \right|^2 dx + b_0 \int (\rho_{N,j} - \rho)^2 dx
\leq C \left( \mathcal{M} \left[ \phi_{N,j}, \rho, u \right] (t) + \frac{1}{\hbar^4 N^\beta} \right)
\leq C(T_0) \left( \hbar^2 + \frac{1}{\hbar^4 N^\beta} \right).
\]

This completes the proof of estimates (3.37) and (3.38). \qed

### Appendix A. Miscellaneous Lemmas

#### A.1. Collapsing Estimate and Strichartz Estimates.

**Lemma A.1.** ([12], [14], [45], KM Collapsing Estimate). There is a $C$ independent of $V$, $j$, $k$ and $N$ such that,

\[
(A.1) \quad \| S^{(1,k)} B_{N,j,k+1} U^{(k+1)} f^{(k+1)} \|_{L^2 L^2_{x,t}} \leq C \| V \|_{L^1} \| S^{(1,k+1)} f^{(k+1)} \|_{L^2_{x,t}},
\]

where $f^{(k+1)}(x_{k+1}; \mathcal{J}_{k+1})$ is independent of $t$.

**Lemma A.2.** Let $d \leq 3$ and $\alpha = d + 1/2$. Then we have

\[
(A.2) \quad \| S_h^{(1,k)} B_{N,h,j,k+1} U_h^{(k+1)} f^{(k+1)} \|_{L^2 L^2_{x,t}} \leq \frac{C \| V \|_{L^1}}{\hbar^\alpha} \| S_h^{(1,k+1)} f^{(k+1)} \|_{L^2_{x,t}}.
\]

**Proof.** Let us define

\[
(A.3) \quad (\delta_x^a f)(x) = f(ax), \quad (\delta_t^a f)(t) = f(at).
\]

\[\text{See also [11], [13], [35], [38], [43], [61] for many different versions of estimates of this type.}\]
By scaling,
\begin{equation}
(A.4) \quad \left\| S_h^{(1,k)} \operatorname{Tr}_{k+1} \left( V_{N,h}(x_j - x_{k+1})U_h^{(k+1)}(t)f^{(k+1)} \right) \right\|_{L^2_t L^2_x}
= h^{kd+\frac{1}{2}} \left\| \delta_x^h \left[ S_h^{(1,k)} \operatorname{Tr}_{k+1} \left( V_{N,h}(x_j - x_{k+1})U_h^{(k+1)}(t)f^{(k+1)} \right) \right] \right\|_{L^2_t L^2_x}
\end{equation}

Noting that $V_{N,h}$ carries $h^{-1}$,
\begin{equation}
= h^{kd-\frac{1}{2}} \left\| S_h^{(1,k)} \operatorname{Tr}_{k+1} \left( h^d V_N(h(x_j - x_{k+1}))U_h^{(k+1)}(t)\delta_x^h [f^{(k+1)}] \right) \right\|_{L^2_t L^2_x}
\end{equation}

By estimate \((A.1)\),
\begin{equation}
\leq h^{kd-\frac{1}{2}} C \left\| h^d V_N(hx) \right\|_{L^1_t} \left\| S_h^{(1,k+1)} \delta_x^h [f^{(k+1)}] \right\|_{L^2_t L^2_x}
= \frac{C \|V\|_{L^1_t}}{h^{d+\frac{1}{2}}} \left\| S_h^{(1,k+1)} f^{(k+1)} \right\|_{L^2_t L^2_x}
\end{equation}

which completes the proof. \qed

**Lemma A.3** \(([19], 	ext{Lemmas 4.1, 4.3, and 4.6})\). \footnote{These are $X_{s,b}$ estimates in disguise. As we are not using the $X_{s,b}$ spaces directly in this paper, we will not go into the details.} Let $\theta$ and $\bar{\theta}$ be cutoff functions supported \([-1,1]\) and $\bar{\theta}_T(t) = \theta(t/T)$. For the case $h = 1$, we have
\begin{equation}
(A.5) \quad \left\| S^{(1,k)} \theta(t_k) \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) V_N(x_1 - x_2) \bar{\theta}_T(t_{k+1}) \gamma_N^{(k)}(t_{k+1}) dt_{k+1} \right\|_{L^1_t L^2_x L^2_{x'}} \leq N^{\frac{5}{2}} C V C \theta \left\| S^{(1,k+1)} \bar{\theta}_T(t_{k+1}) \gamma_N^{(k)}(t_{k+1}) \right\|_{L^2_t L^2_x L^2_{x'}}
\end{equation}

and
\begin{equation}
(A.6) \quad \left\| S^{(1,k+j-1)} B_{N,1,k+j} \theta(t_{k+j}) \int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1}) V_{N,12} \bar{\theta}_T(t_{k+j+1}) \gamma_N^{(k+j)}(t_{k+j+1}) dt_{k+j+1} \right\|_{L^2_t L^2_x L^2_{x'}} \leq N^{\frac{5}{2}} C V C \theta \left\| S^{(1,k+j)} \bar{\theta}_T(t_{k+j+1}) \gamma_N^{(k+j)}(t_{k+j+1}) \right\|_{L^2_t L^2_x L^2_{x'}}
\end{equation}

where $V_{N,12} = N^{d^2} V(N^3(x_1 - x_2))$ and
\begin{equation}
C \theta = |\operatorname{Supp}(\theta)| \left( \|\theta\|_{L^2} + \|\theta'\|_{L^4} \right) + \|\theta\|_{L^4} + \|\nabla \theta\|_{L^2} + \|\theta\|_{L^\infty}
\end{equation}

with $|\operatorname{Supp}(\theta)|$ denoting the Lebesgue measure of the support of $\theta$.

**Lemma A.4.** For $j \geq 0$ and $k \geq 1$, we have the following estimates
\begin{equation}
(A.7) \quad \left\| S_h^{(1,k)} \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V_N^{(k)}(k) \gamma_N^{(k)}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_t L^2_x} \leq N^{\frac{5}{2}} h C V \left\| S_h^{(1,k)} \gamma_N^{(k)}(t_{k+1}) \right\|_{L^2_t L^2_x L^2_{x'}}
\end{equation}
and

\[ \left\| S_h^{(1,k+j-1)} B_{N,h,1,k+j} \int_0^{t_{k+j}} U_h^{(k+j)}(t_{k+j} - t_{k+j+1}) V^{(k+j)}_{N,h}(t_{k+j+1}) dt_{k+j+1} \right\|_{L^2_{t_{k+j},[0,T] L^2_{x,x'}} \leq N^{2\beta-1} h(C_V h^{-\alpha} T^{1/2})^2 (k+j)^2 \left\| S_h^{(1,k+j)} \gamma^{(k+j)}_{N,h}(t_{k+j+1}) \right\|_{L^\infty_{t_{k+j+1},L^2_{x,x'}}}. \]

Proof. For (A.7), we have

\[ \left\| S_h^{(1,k)} \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V^{(k)}_{N,h} \gamma^{(k)}_{N,h}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_{t_k} L^2_{x,x'}} \leq \left\| S_h^{(1,k)} T_k \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V_{N,h} \gamma^{(k)}_{N,h}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_{t_k} L^2_{x,x'}}. \]

where \( \theta \) and \( \tilde{\theta} \) are cutoff functions supported \([-1, 1]\) and \( \tilde{\theta}(t) = \tilde{\theta}(t/T) \). For simplicity, we set

\[ V_{N,h,12} = (N^\beta h)^d V((N^\beta h(x_1 - x_2)). \]

Then by scaling argument, we arrive at

\[ \left\| S_h^{(1,k)} T_{t_k} \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V_{N,h} \gamma^{(k)}_{N,h}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_{t_k} L^2_{x,x'}} = h^d \left\| \Sigma^{(1,k)} \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V_{N,h} \gamma^{(k)}_{N,h}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_{t_k} L^2_{x,x'}} \]

By using estimate (A.5),

\[ \left\| S_h^{(1,k)} T_{t_k} \int_0^{t_k} U_h^{(k)}(t_k - t_{k+1}) V_{N,h} \gamma^{(k)}_{N,h}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_{t_k} L^2_{x,x'}} \leq \frac{h(N^\beta h)^{2\alpha} C_V C_{\delta^\beta \theta} T^{1/2}}{h^d} \left\| S_h^{(1,k)} \tilde{\theta}(t_{k+1}) \gamma^{(k)}_{N,h}(t_{k+1}) \right\|_{L^2_{t_{k+1},L^2_{x,x'}}}. \]

By taking \( L^\infty \) at \( dt_{k+1} \) and using the estimate that \( h^2 C_{\delta^\beta \theta} \leq C \),

\[ \left\| S_h^{(1,k)} \tilde{\theta}(t_{k+1}) \gamma^{(k)}_{N,h}(t_{k+1}) \right\|_{L^2_{t_{k+1},L^2_{x,x'}}}. \]

By taking \( L^\infty \) at \( dt_{k+1} \) and using the estimate that \( h^2 C_{\delta^\beta \theta} \leq C \),

\[ \left\| S_h^{(1,k)} \tilde{\theta}(t_{k+1}) \gamma^{(k)}_{N,h}(t_{k+1}) \right\|_{L^2_{t_{k+1},L^2_{x,x'}}}. \]

We note that the \( N^{-1}, k^2 \) and \( h^{-1} \) factors come from the expansion of \( V_{N,h}^{(k)} \) and then arrive at (A.7).

Next, we deal with (A.8). With the help of estimate (A.6), we can use scaling argument in the same way as above to arrive at (A.8), where the \( N^{-1}, (k + j)^2 \), and \( h^{-1} \) factors come from the expansion of \( V_{N,h}^{(k+j)} \) and another \( h^{-1} \) factor comes from \( B_{N,h,1,k+j} \).
Lemma A.5. For \( j \geq 0 \) and \( k \geq 1 \), we have
\[
(A.11) \quad \int_{[0,T]} \| S_h^{(1,k+j)} B_{N,h,1,k+j+1} \gamma_{N,h}^{(k+j+1)} (t_{k+j+1}) \|_{L^2_{x,a}} \, dt_{k+j+1} \leq N^{\frac{2\beta}{p}} h^2 T^{1/2} (C_V h^{-\alpha} T^{1/2}) \| S_h^{(1,k+j+1)} \gamma_{N,h}^{(k+j+1)} (t_{k+j+1}) \|_{L^\infty_{k+j+1} L^2_{x,a}}.
\]

Proof. By taking \( L^\infty \) at \( dt_{k+j+1} \), it suffices to prove that
\[
(A.12) \quad \| S_h^{(1,k+j)} B_{N,h,1,k+j+1} \gamma_{N,h}^{(k+j+1)} (t_{k+j+1}) \|_{L^2_{x,a}} \leq N^{\frac{2\beta}{p}} h^2 C_V h^{-\alpha} \| S_h^{(1,k+j+1)} \gamma_{N,h}^{(k+j+1)} \|_{L^2_{x,a}}.
\]

For \( h = 1 \), we have
\[
(A.13) \quad \| S^{(1,k+j)} B_{N,1,k+j+1} \gamma_{N}^{(k+j+1)} (t_{k+j+1}) \|_{L^2_{x,a}} \leq N^{\frac{2\beta}{p}} C_V \| S^{(1,k+j+1)} \gamma_{N}^{(k+j+1)} \|_{L^2_{x,a}}.
\]

By scaling, we arrive at \( (A.12) \).

A.2. Convolution and Commutator Estimates.

Lemma A.6 (\cite{23}, Lemma A.5). Let \( W_N(x) = N^{\beta} V(N^\beta x) - b_0 \delta \), where \( b_0 = \int V(x) \, dx \). For any \( 0 \leq s \leq 1 \),
\[
(A.14) \quad \| W_N * f \|_{L^p} \lesssim N^{-\beta s} \| (\nabla)^s f \|_{L^p}
\]
for any \( 1 < p < \infty \). The implicit constant depends only on \( \| \langle x \rangle V(x) \|_{L^1} \).

Lemma A.7 (Fractional Leibniz Rule).
\[
(A.15) \quad \| (\nabla)^r (fg) \|_{L^r} \lesssim \| (\nabla)^{s_1} f \|_{L^{p_1}} \| g \|_{L^{p_2}} + \| f \|_{L^{q_1}} \| (\nabla)^{s_2} g \|_{L^{q_2}}
\]
where
\[
(A.16) \quad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}
\]
\( r \in [1, \infty) \), \( p_i, q_i \in (1, \infty) \), \( s > 0 \).

Lemma A.8 (\cite{21} \cite{29}). Let \( d = 3 \), \( \eta > d/4 \) and \( V_N(x) = N^{3\beta} V(N^\beta x) \). Then
\[
(A.17) \quad V_N(x_1 - x_2) \leq C(\eta) \| V \|_{L^1} (1 - \Delta x_1)^{\eta} (1 - \Delta x_2)^{\eta},
\]
\[
(A.18) \quad V_N(x_1 - x_2) \leq C N^{\beta} \| V \|_{L^2} (1 - \Delta x_1),
\]
\[
(A.19) \quad V_N(x_1 - x_2) \leq C N^{3\beta} \| V \|_{L^\infty}.
\]

Proof. For \( (A.17) \) with \( \eta = 1 \), \( (A.18) \) and \( (A.19) \), see \cite{29} Lemma A.3]. For \( (A.17) \) with \( \eta > 3/4 \), see \cite{21}. For completeness, we show a proof of \( (A.17) \). By direct calculation,
\[
\langle \psi, V(x_1 - x_2) \psi \rangle = \int dp d\xi_1 d\xi_2 \psi(\xi_1, \xi_2) \overline{\psi(p)} (\xi_1 + p, \xi_2 - p)
\]
\[
= \int dp d\xi_1 d\xi_2 \frac{(\xi_1)^n (\xi_2)^n}{(\xi_1 + p)^n (\xi_2 - p)^n} \psi(\xi_1, \xi_2) \overline{\psi(p)} (\xi_1 + p, \xi_2 - p) + \int dp d\xi_1 d\xi_2 \frac{(\xi_1 + p)^n (\xi_2 - p)^n}{(\xi_1)^n (\xi_2)^n} \psi(\xi_1, \xi_2) \overline{\psi(p)} (\xi_1 + p, \xi_2 - p)
\]
By Hölder,
\[
\leq \|\tilde{V}\|_{L^\infty} \|\langle \xi_1 \rangle^n \langle \xi_2 \rangle^n \tilde{\psi}(\xi_1, \xi_2)\|_{L^\infty_p L^2_{\xi_1, \xi_2}} \|\langle \xi_1 + p \rangle^n \langle \xi_2 - p \rangle^n \tilde{\psi}(\xi_1 + p, \xi_2 - p)\|_{L^\infty_p L^2_{\xi_1, \xi_2}}
\]
\[
\leq C(\eta)\|V\|_{L^1} \|\langle \nabla_{x_1} \rangle \langle \nabla_{x_2} \rangle \tilde{\psi}\|_{L^2}.
\]
where in the last inequality we used
\[
\|\langle \xi_1 \rangle^n \langle \xi_2 \rangle^n \langle \xi_1 + p \rangle^n \langle \xi_2 - p \rangle^n \tilde{\psi}(\xi_1 + p, \xi_2 - p)\|_{L^\infty_p L^2_{\xi_1, \xi_2}} \lesssim \int \frac{1}{\langle p \rangle^{4\eta}} dp \leq C(\eta).
\]

**Appendix B. Energy Estimate**

Recall the Hamiltonian (1.2)
\[
H_{N,\hbar} = \sum_{j=1}^{N} -\frac{1}{2} \hbar^2 \Delta_{x_j} + \frac{1}{N} \sum_{1 \leq j < k \leq N} V_N(x_j - x_k)
\]
and the derivative involving \(\hbar\) in (2.5)
\[
S_{h,j}^2 = 1 - \frac{\hbar^2}{2} \Delta_{x_j}.
\]

**Proposition B.1.** Let \(\beta < \frac{3}{5}\), \(k \leq (\ln N)^{100}\) and \(\hbar^{-1} \leq \ln N\). There exists \(N_0(\beta)\) independent of \(k\) and \(\hbar\), such that
\[
\langle \psi, (H_{N,\hbar} + N)^k \psi \rangle \geq \frac{N^k}{2^k} \langle \psi, S_{h,1}^2 S_{h,2}^2 \cdots S_{h,k}^2 \psi \rangle.
\]
for every \(N \geq N_0(\beta)\).

**Proof.** This proof has been done by many authors in many work. We include one here solely for completeness purposes. For \(k = 0\) and \(k = 1\), the claim is trivial because of the positivity of the potential. Now we assume the proposition is true for all \(k \leq n\), and we prove it for \(k = n + 2\).

\[
\langle \psi, (H_{N,\hbar} + N)^{n+2} \psi \rangle = \langle (H_{N,\hbar} + N)^n (H_{N,\hbar} + N) \psi, (H_{N,\hbar} + N)^{n+1} \psi \rangle
\]
\[
\geq \frac{N^n}{2^n} \langle \psi, (H_{N,\hbar} + N) S_{h,1}^2 \cdots S_{h,n}^2 (H_{N,\hbar} + N) \psi \rangle.
\]

We set
\[
H_{N,\hbar}^{(n)} = \sum_{j=1}^{n} S_{h,j}^2 + \frac{1}{n} \sum_{j<m} V_{jm}
\]

\(\text{The restriction that } \hbar^{-1} \leq \ln N \text{ is not necessary and it can be removed at the price of reducing down the parameter } \beta.\)
Then we have
\[ V_j m = N^{3\beta} V(N^\beta (x_j - x_k)) \]. Then we have
\[ \langle \psi, (H_{N,h} + N)S_{h,1}^2 \cdots S_{h,n}^2 (H_{N,h} + N)\psi \rangle \]
\[ = \sum_{j_1, j_2 \geq n+1} \langle \psi, S_{h,j_1}^2 S_{h,j_2}^2 \cdots S_{h,n}^2 \rangle \]
\[ + \sum_{j \geq n+1} \left( \langle \psi, S_{h,j}^2 S_{h,1}^2 \cdots S_{h,n}^2 H_{N,h}^{(n)} \psi \rangle + c.c. \right) + \langle \psi, H_{N,h}^{(n)} S_{h,1}^2 \cdots S_{h,n}^2 H_{N,h}^{(n)} \psi \rangle. \]

where c.c. denotes the complex conjugate. Since \( H_{N,h}^{(n)} S_{h,1}^2 \cdots S_{h,n}^2 H_{N,h}^{(n)} \geq 0 \), we have, using the symmetry with respect to permutations,
\[ \langle \psi, (H_{N,h} + N)S_{h,1}^2 \cdots S_{h,n}^2 (H_{N,h} + N)\psi \rangle \]
\[ \geq (N - n)(N - n - 1) \langle \psi, S_{h,1}^2 S_{h,2}^2 \cdots S_{h,n+2}^2 \rangle \]
\[ + (2n + 1)(N - n) \langle \psi, S_{h,1}^2 S_{h,2}^2 \cdots S_{h,n+1}^2 \rangle \]
\[ + \frac{n(n + 1)(N - n)}{2N} \langle \psi, V_{12} S_{h,1}^2 S_{h,2}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]
\[ + \frac{(n + 1)(N - n)(N - n - 1)}{N} \langle \psi, V_{1,n+2} S_{h,1}^2 S_{h,2}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]

Here we also used the fact that
\[ \langle \psi, V_j m S_{h,1}^2 \cdots S_{h,n+1}^2 \psi \rangle \geq 0 \]
if \( j, m > n + 1 \), because of the positivity of the potential. Next, we will bound the last two terms on the r.h.s of (B.3) from below, so we might as well set \( S_{h,j}^2 = 1 - h^2 \Delta x_j \) for simplicity. Then we have
\[ \langle \psi, V_1 S_{h,1}^2 S_{h,2}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]
\[ = \langle \psi, V_1 (1 - h^2 \Delta x_1)(1 - h^2 \Delta x_2)S_{h,3}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]
\[ \geq \langle \psi, h \nabla V_{12} h \nabla x_2 S_{h,3}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]
\[ + \langle h \Delta x_1 \psi, h \nabla V_{12} h \nabla x_1 h \nabla x_2 S_{h,3}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]
\[ + \langle \psi, h \nabla V_{12} h^2 \Delta x_1 h \nabla x_2 S_{h,3}^2 \cdots S_{h,n+1}^2 \psi \rangle + c.c. \]
\[ = I + II + III \]

where \( \nabla V_{12} = N^{4\beta} (\nabla V) (N^\beta (x_1 - x_2)) \). Applying Cauchy-Schwarz, we get
\[ I \geq -2 \left\{ \alpha_1 \langle \psi, \langle h \nabla V_{12} | S_{h,3}^2 \cdots S_{h,n+1}^2 \rangle \rangle \right\} \]
\[ + \alpha_1^{-1} \langle h \nabla V_{12} | \psi, \langle h \nabla V_{12} | S_{h,3}^2 \cdots S_{h,n+1}^2 \rangle \rangle \right\}, \]
\[ II \geq -2 \left\{ \alpha_2 \langle h \nabla V_{12} | \psi, \langle h \nabla V_{12} | S_{h,3}^2 \cdots S_{h,n+1}^1 \rangle \rangle \right\} \]
\[ + \alpha_2^{-1} \langle h \nabla V_{12} | \psi, \langle h \nabla V_{12} | S_{h,3}^2 \cdots S_{h,n+1}^1 \rangle \rangle \right\}, \]
\[ III \geq -2 \left\{ \alpha_3 \langle \psi, \langle h \nabla V_{12} | S_{h,3}^2 \cdots S_{h,n+1}^1 \rangle \rangle \right\} \]
\[ + \alpha_3^{-1} \langle \psi, \langle h \nabla V_{12} | S_{h,3}^2 \cdots S_{h,n+1}^1 \rangle \rangle \right\} \].
By Lemma A.8:

\[ I \geq -C \left\{ \alpha_1 N^3 \hbar^{-3} \langle \psi, S^2_{h,1} \cdots S^2_{h,n+1} \rangle + \alpha_1^{-1} N^4 \hbar \langle \psi, S^2_{h,2} \cdots S^2_{h,n+1} \rangle \right\}, \]

\[ II \geq -C \left\{ \alpha_2 N^2 \hbar^{-1} \langle \psi, S^2_{h,1} \cdots S^2_{h,n+1} \rangle + \alpha_2^{-1} N^3 \hbar^{-1} \langle \psi, S^4_{h,2} \cdots S^2_{h,n+1} \rangle \right\}, \]

\[ III \geq -C \left\{ \alpha_3 N^3 \hbar^{-3} \langle \psi, S^2_{h,1} \cdots S^2_{h,n+1} \rangle + \alpha_3^{-1} N^4 \hbar \langle \psi, S^4_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle \right\}. \]

Optimizing the choice of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), we find that

\[ \langle \psi, V_1 S^2_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle + \text{c.c.} \]

\[ \geq -CN^{-3/2} N^\beta \hbar^{-1} \left\{ N^2 \langle \psi, S^2_{h,1} \cdots S^2_{h,n+1} \rangle + N \langle \psi, S^4_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle \right\}. \]

As for the last term on the r.h.s. of (B.3), we have

\[ \langle \psi, V_{1,n+2} S^2_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle + \text{c.c.} \]

\[ \geq \langle \psi, \hbar x_1 S^2_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle + \text{c.c.} \]

\[ \geq \langle \psi, x_1 S^2_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle + \text{c.c.} \]

\[ \geq -\alpha \langle \psi, S^2_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle - \alpha^{-1} \langle \hbar x_1 | \hbar x_1 S^2_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle \]

\[ \geq -C \left( \alpha N^3 \hbar^{-3} + \alpha^{-1} N^2 \hbar^{-1} \right) \langle \psi, S^2_{h,1} \cdots S^2_{h,n+2} \rangle \]

\[ \geq -CN^\beta \hbar^{-2} \]

where we optimized the choice of \( \alpha \). Then we get

\[ \langle \psi, (H_{N,h} + N) S^2_{h,1} \cdots S^2_{h,n} (H_{N,h} + N) \psi \rangle \]

\[ \geq (N-n)(N-n-1) \left( 1 - \frac{CN^\beta \hbar^{-1} n^2}{N^{1/2}(N-n)} - \frac{CN^\beta \hbar^{-2} n}{N} \right) \langle \psi, S^2_{h,1} \cdots S^2_{h,n+2} \rangle \]

\[ + (2n+1)(N-n) \left( 1 - \frac{CN^\beta \hbar^{-1} n}{N^{3/2}} \right) \langle \psi, S^4_{h,1} S^2_{h,2} \cdots S^2_{h,n+1} \rangle \]

Since \( \beta < \frac{3}{2}, n \leq (\ln N)^{100} \) and \( \hbar^{-1} \leq \ln N \), we can find \( N_0(\beta) \) which is independent of \( n \) and \( \hbar \), so that

\[ \langle \psi, (H_{N,h} + N) S^2_{h,1} \cdots S^2_{h,n} (H_{N,h} + N) \psi \rangle \geq \frac{N^2}{4} \langle \psi, S^2_{h,1} \cdots S^2_{h,n+2} \rangle \]

for every \( N \geq N_0(\beta) \). Together with (B.2), this completes the proof. \( \square \)

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