Chiral phase transition in relativistic heavy-ion collisions with weak magnetic fields:
ring diagrams in the linear sigma model

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Working in the linear sigma model with quarks, we compute the finite-temperature effective potential in the presence of a weak magnetic field, including the contribution of the pion ring diagrams and considering the sigma as a classical field. In the approximation where the pion self-energy is computed perturbatively, we show that there is a region of the parameter space where the effect of the ring diagrams is to preclude the phase transition from happening. Inclusion of the magnetic field has small effects that however become more important as the system evolves to the lowest temperatures allowed in the analysis.

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I. INTRODUCTION

In recent years it has been possible to produce and study hadronic matter at high densities and temperatures by means of collisions of heavy nuclei at high energies [1]. There are convincing signals that reveal the production of deconfined matter where the degrees of freedom involved are the quarks and gluons of QCD, forming the so-called quark-gluon plasma (QGP). A common feature of these signals is their strengthening as the centrality of the collision increases.

At the same time it has been realized that a host of new phenomena can also happen for not so central collisions. Among these, it has been pointed out that for peripheral collisions, a magnetic field of a non-negligible strength is generated [2]. The origin of this field is two-fold: On one hand, in non-central collisions, there is a local imbalance in the momentum carried by the colliding nucleons in the target and projectile that generates a non-vanishing local angular momentum [3,4] which in turn produces a magnetic field, given the net positive charge present in the collision. On the other hand, the spectator nucleons can be thought of as currents of net positive charge moving in opposite, off center, directions which in turn produce a magnetic field that adds up in the interaction region.

An interesting question that emerges from this scenario is whether a magnetic field can influence the phase transitions that may occur during the reaction, in particular the chiral phase transition. There are known examples where magnetic fields are able to change the nature of a phase transition. Most notably is the Meissner effect where the phase transition of type I superconductors changes from second to first order when an external magnetic field is applied. Magnetic catalysis is another phenomenon whereby a magnetic field is able to dynamically generate masses in QED, regardless of the strength of the field [5]. More recently, it has been shown that in the presence of primordial magnetic fields, the electroweak phase transition, that took place in the early universe for temperatures of order 100 GeV, gets also strengthened [6]. In a similar connection, the dynamical generation of anomalous magnetic moment of the electron has also been unveiled in Ref. [7].

Calculations of the intensity of the field produced in this kind of collisions show that for very early proper times after the reaction (\(\tau \lesssim 0.1\) fm) the field reaches values \(eB \simeq 6(m_{\pi}^{\text{vac}})^2\), where \(m_{\pi}^{\text{vac}}\) is the vacuum pion mass, even for mid-peripheral collisions. The intensity decreases with the proper time \(\tau\) as \(eB \propto 1/\tau^3\) in such a way that for \(\tau \simeq 1\) fm, namely, for times when the standard picture of a heavy-ion reaction places the existence of the equilibrated QGP, \(eB \lesssim 0.1(m_{\pi}^{\text{vac}})^2\) [2], that is, already two orders of magnitude smaller than at the very early stages of the collision.

In a recent work, the chiral phase transition in relativistic heavy-ion collisions has been examined in the presence of strong magnetic fields using the linear sigma model [8]. Working with the hierarchy of scales where \((m_{\pi}^{\text{vac}})^2 \ll T^2 \ll eB\), with \(T\) being the temperature around the phase transition, the authors conclude that the effect is to turn a crossover into a weak first order transition, a result that might be relevant even for the physics of the primordial QCD transition. Nevertheless, as mentioned above, a more realistic scenario in heavy-ion collisions should be to consider that for the times when the initial chromoelectric fields decohere in the aftermath of the collision and give rise to partons – which in turn are the appropriate degrees of freedom to describe the chiral phase transition – the magnetic field in the interaction region might not be that strong. At these times the hierarchy of scales is such that the magnetic field is the smallest of all and the deconfinement/chiral phase transition temperature is the largest one, while the pion vacuum mass, occupies an intermediate place. Furthermore, the analysis of Ref. [8] neglects the contri-
duction from the so called ring diagrams which are known to be important at high temperatures to account for the infrared properties of the plasma [8, 10]. Similar considerations, in QED for instance, have already been reported [11]. In the context of the QGP and hadron matter, the effects of magnetic fields have also been recently looked at. Such studies include their influence on making evident possible topological-charge transitions in heavy-ion collisions [2, 12], on the confinement/deconfinement phase transition—in the Abelian approximation of the chromomagnetic field—[13] and on the hadron structure [14].

In this work we undertake the calculation of the effective potential at finite temperature, in the presence of weak magnetic fields. We use the linear sigma model as the working tool to describe the chiral phase transition in relativistic heavy-ion collisions. We work explicitly with the hierarchy of energy scales where $eB \ll m^2 \ll T^2$, with $m$ being a generic mass appearing in the calculation and considering that the interaction region is subject to an external magnetic field directed along the positive $z$-axis. We work up to the contributions of the ring diagrams since for the aforementioned hierarchy of scales, the effects of the magnetic fields appear only at this level. Computation of the ring diagrams calls for the calculation of the pion finite temperature self-energy in the presence of the field. We approximate this self-energy by its perturbative value although, for the large couplings involved, the calculation most likely overshoots the true result. We parameterize the lack of an accurate non-perturbative treatment introducing a parameter to control the strength of the self-energy and analyze the consequences in this parameter space. To incorporate the weak magnetic field we use the Schwinger proper time method to write the charged particle propagators in a weak field expansion.

The work is organized as follows: After a brief summary in Sec. II of the linear sigma model Lagrangian, we perform the calculation of the finite temperature effective potential in Sec. III considering that the sigma field is classical and going up to the contribution of the ring diagrams in the presence of a weak magnetic field. In Sec. IV we present the numerical results of the work and show that in a certain region of the parameter space, even in the absence of magnetic fields, the ring diagrams preclude the development of the phase transition. Inclusion of the weak magnetic field has little impact on the phase transition. Finally in Sec. V we discuss our results and present the conclusions.

II. THE LINEAR SIGMA MODEL WITH QUARKS

The Lagrangian for the linear sigma model is given by

$$
\mathcal{L} = \frac{1}{2} (\partial_{\mu} \sigma)^2 + \frac{1}{2} (\partial_{\mu} \pi)^2 + \frac{\mu^2}{2} (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 + \frac{i}{2} \bar{\psi} \gamma_{\mu} \partial_{\mu} \psi + \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 \quad (1)
$$

where $\psi$ is a SU(2) isospin doublet of massless quarks, $\pi = (\pi_1, \pi_2, \pi_3)$ is an isospin triplet representing the pions and $\sigma$ is an isospin singlet.

When the mass parameter $\mu^2$ is positive, the Lagrangian admits a broken symmetry vacuum solution given by the minimum of the classical potential

$$
V^{(cl)} = -\frac{\mu^2}{2} (\sigma^2 + \pi^2) + \frac{\lambda}{4} (\sigma^2 + \pi^2)^2. \quad (2)
$$

Choosing this minimum along the $\sigma$ direction, the vacuum expectation values for the sigma and pion fields are given by

$$
\langle \sigma \rangle = \nu_0 \equiv \mu / \sqrt{\lambda}, \quad \langle \pi \rangle = 0. \quad (3)
$$

$\nu_0$ is also called the classical vacuum which minimizes the action for uniform field configurations. To study the quantum properties of the system, we define the shifted field $\sigma'$ by

$$
\sigma = \nu + \sigma', \quad (4)
$$

where $\nu$ is taken as a variable. When $\nu = \nu_0$, $\sigma'$ represents the field configuration around the classical vacuum.

As a function of the shifted field, after symmetry breaking, the Lagrangian of Eq. (1) becomes a theory describing a $\sigma'$ field, three pion fields and a quark-doublet field with masses given by

$$
\begin{align*}
    m_{\sigma}^2 (\nu) &= 3 \lambda \nu^2 - \mu^2 \\
    m_{\pi}^2 (\nu) &= \lambda \nu^2 - \mu^2 \\
    m_q (\nu) &= g \nu.
\end{align*} \quad (5)
$$

Armed with the fundamentals of the linear sigma model, we proceed to compute the effective potential.

III. THE EFFECTIVE POTENTIAL

A. Tree and one-loop

The tree level potential is given by

$$
V^{(tree)} = -\frac{\mu^2}{2} \nu^2 + \frac{\lambda}{4} \nu^4. \quad (6)
$$
We consider that the \( \sigma' \) field is very heavy and thus treat it only classically. To one-loop order, the contribution to the effective potential in the imaginary-time formulation of thermal field theory is given, for bosons, by

\[
V_b^{(1)} = s_b T \sum_n \int \frac{d^3k}{(2\pi)^3} \ln(D^{-1})^{1/2},
\]

(7)

whereas for fermions, by

\[
V_f^{(1)} = s_f T \sum_n \int \frac{d^3k}{(2\pi)^3} \text{Tr} \ln(S^{-1}),
\]

(8)

where \( s_{b,f} \) are the degeneracy factors accounting for the internal degrees of freedom for bosons (isospin) and fermions (isospin and color), respectively, \( n \) is the index for the Matsubara frequency and \( D \) and \( S \) represent the boson and fermion Matsubara propagators. For charged particles, these propagators should include the effect of the external magnetic field. Nevertheless, it has been shown [6] that for the hierarchy of energy scales considered, the terms containing the effects of the magnetic field are subdominant.

Equations (7) and (8) contain both a vacuum and a finite temperature piece. The vacuum piece exhibits the usual ultraviolet divergence that needs to be removed. In the present context, this means that, up to additive constants, the effective potential should contain only finite \( v \)-dependent terms. This is accomplished, for instance, by introducing counter-terms to absorb the infinities. The usual physical conditions implemented to fix the counter-terms require that the position of the minimum of the effective potential, as well as the mass of the \( \sigma' \) field, maintain their classical values [15]. However, for theories with massless modes –such as the pions in the present case, whose mass vanishes at \( \mu = 0 \)—this procedure breaks down. The problem is that in order to get the mass of the \( \sigma' \) field from the effective potential one requires computing the second derivative, since this mass is the inertia along the \( \sigma' \)-axis. This derivative turns out to not be defined at such value. Thus, the second condition needs to be replaced by another appropriate one in a manner that we proceed to explain.

First, let us introduce the general expression for the one-loop renormalized effective potential, where we add the counterterms to absorb the \( v \)-dependent infinites

\[
V_{\text{ren}}^{(1)} = \frac{\mu^2}{2} v^2 + \frac{\lambda}{4} v^4 + \left( \frac{a(\Lambda) - \delta \mu^2}{2} \right) v^2
+ \left( \frac{b(\Lambda) + \delta \lambda}{4} \right) v^4
- 24 I(m_f, \Lambda).
\]

The counter-terms in Eq. (9), \( a(\Lambda) \) and \( b(\Lambda) \), are introduced to take care of the \( v \)-dependent infinites, whereas the counter-terms \( \delta \mu^2 \) and \( \delta \lambda \), account for finite terms that might shift the coefficients of the \( v^2 \) and \( v^4 \) terms, respectively.

After some straightforward algebra where the \( v \)-dependent infinites are absorbed, the one-loop renormalized effective potential becomes

\[
V_{\text{ren}}^{(1)} = - \left( \frac{1}{2} \mu^2 + 3 \frac{\lambda}{64 \pi^2} \mu^2 + \frac{1}{2} \delta \mu^2 \right) v^2
+ \left( \frac{1}{4} \lambda + 3 \frac{\lambda^2}{128 \pi^2} + 6 \frac{g^4}{32 \pi^2} + \frac{1}{4} \delta \lambda \right) v^4
- 24 m_f^4 \ln \left( \frac{m_q^2}{4} \right) + 3 \frac{m^2}{64 \pi^2} \ln \left( \frac{m^2}{4} \right) ,
\]

(11)

where, hereafter, when not explicitly indicated, the pion and quark masses are the \( v \)-dependent ones, given in Eqs. [9].

To fix one of the remaining counter-terms, either \( \delta \mu^2 \) or \( \delta \lambda \), we impose the condition that the minimum of the renormalized effective potential remains at its classical value, namely,

\[
\frac{1}{2v} \frac{\partial}{\partial v} V_{\text{ren}}^{(1)} \bigg|_{v=v_0} = 0,
\]

(12)

which implies that

\[
\delta \lambda = 6 \frac{g^4}{4 \pi^2} \left[ 1 + \ln \left( \frac{g^2 \mu^2}{4 \lambda} \right) \right] + \frac{\lambda}{\mu^2} \delta \mu^2.
\]

(13)

Inserting Eq. (13) into Eq. (11), we get

\[
V_{\text{ren}}^{(1)} = - \left( \frac{1}{2} \mu^2 + 3 \frac{\lambda}{64 \pi^2} \mu^2 + \frac{1}{2} \delta \mu^2 \right) v^2
+ \left( \frac{1}{4} \lambda + 3 \frac{\lambda^2}{128 \pi^2} + 6 \frac{g^4}{32 \pi^2} + \frac{3}{4} \mu^2 \delta \mu^2 \right) v^4
- 6 \frac{m_f^4}{16 \pi^2} \ln \left( \frac{m_q^2}{m_f^2(v_0)} \right) + 3 \frac{m^2}{64 \pi^2} \ln \left( \frac{m^2}{4} \right).
\]

(14)

To fix the second counter-term, \( \delta \mu^2 \), notice that the argument of the last logarithmic function is dimensionfull, though the whole term is well defined as \( m_q \) goes to zero. However, if we choose

\[
\delta \mu^2 = -3 \frac{\lambda \mu^2}{16 \pi^2} \ln \left( \frac{\mu^2}{4} \right),
\]

(15)

we obtain

\[
V_{\text{ren}}^{(1)} = - \left( \frac{1}{2} \mu^2 + 3 \frac{\lambda}{64 \pi^2} \mu^2 \right) v^2
+ \left( \frac{1}{4} \lambda + 3 \frac{\lambda^2}{128 \pi^2} + 6 \frac{g^4}{32 \pi^2} \right) v^4
- 6 \frac{m_f^4}{16 \pi^2} \ln \left( \frac{m_q^2}{m_f^2(v_0)} \right) + 3 \frac{m^2}{64 \pi^2} \ln \left( \frac{m^2}{4} \right),
\]

(16)
which no longer contains dimension-full arguments of the logarithmic functions. The choice of Eq. (15) although seemingly arbitrary, has the advantage of producing a well defined renormalized effective potential which preserves the properties of the tree level one, namely, that the pions are massless at $v_0$ which in turn keeps being the minimum of the potential. Recall that the effective potential is not in itself an observable; only physical properties extracted from it, such as the position of the minimum and the critical temperature are. The choice that produces Eq. (15) is also extensively used in Standard Model calculations [9].

We now proceed to include the finite temperature contribution at one-loop. The finite temperature pieces of Eqs. (7) and (8) are given by

$$V_f^{(1)T\neq 0} = 6 \left[ -\frac{\pi^2 T^4}{180} + \frac{m_\pi^2 T^2}{12} + \frac{m^4}{16\pi^2} \ln \left( \frac{m^2}{T^2} \right) \right]$$

$$V_\pi^{(1)T\neq 0} = 3 \left[ -\frac{\pi^2 T^4}{90} + \frac{m_\pi^2 T^2}{24} - \frac{m^4}{12\pi T} \right] - \frac{m^4}{64\pi^2} \ln \left( \frac{m^2}{(4\pi T)^2} \right). \quad (17)$$

Adding the renormalized effective potential to the finite temperature contributions, we get the full one-loop finite temperature effective potential

$$V_{ren}^{(1)T\neq 0} = -48 \frac{\pi^2 T^4}{180} - \left( 1 + \frac{3 \lambda}{32\pi^2} + 6 \frac{g^4}{8\pi^2} \right) \frac{\mu^2}{2} v^2$$

$$+ \left( \lambda + 3 \frac{\lambda^2}{8\pi^2} + 6 \frac{g^4}{8\pi^2} \right) \frac{v^4}{4}$$

$$+ \frac{12m_q^4 + 3m_\pi^4}{24} \frac{T^2}{2} - 3m_\pi^2 \frac{T}{12\pi}$$

$$- 6 \frac{m_q^4}{16\pi^2} \ln \left( \frac{T^2}{m_q^2(v_0)} \right) \left( 3 \frac{m_\pi^4}{64\pi^2} \ln \left( \frac{(4\pi T)^2}{\mu^2} \right) \right). \quad (18)$$

A few words about the properties of Eq. (18) are in order: First, notice that the dependence on the pion mass in the argument of the logarithmic functions has canceled upon addition of the vacuum and finite-temperature pieces of the renormalized effective potential. This is an important property for especially this function can develop an imaginary part when the pion mass is negative, namely for $v < v_0$. Secondly, notice the appearance of a cubic pion mass term. This is also a dangerous term since it gives rise to an imaginary part when the pion mass is negative. However, as we will show, this term is exactly canceled when considering the contribution from the ring diagrams.

In order to closely examine the behavior of the renormalized finite-temperature effective potential with the temperature, let us re-express Eq. (18) expanding it in powers of $v$

$$V_{ren}^{(1)T\neq 0} = -48 \frac{\pi^2 T^4}{180} - 3 \frac{\mu^2}{24} T^2 + 3 \frac{\mu^4}{64\pi^2} \ln \left( \frac{(4\pi T)^2}{\mu^2} \right)$$

$$- \left[ 1 + 3 \frac{\lambda}{32\pi^2} - 2 \frac{g^2 + \lambda/2}{2\pi^2} \right] \frac{\mu^2}{2} v^2$$

$$+ 3 \frac{\lambda}{16\pi^2} \ln \left( \frac{(4\pi T)^2}{\mu^2} \right) \frac{\mu^2}{2} v^2$$

$$+ \left( \lambda + 3 \frac{\lambda^2}{32\pi^2} + 6 \frac{g^4}{8\pi^2} - 6 \frac{g^4}{4\pi^2} \ln \left( \frac{T^2}{m_q^2(v_0)} \right) \right) \frac{\mu^2}{4} v^4 - 3 \frac{m_q^4}{12\pi} T. \quad (19)$$

Let us ignore, for the time being, the term proportional to $m_q^4$. Notice that the critical temperature $T_c$ for the phase transition is determined by the curvature of the effective potential at $v = 0$. When the curvature changes sign the phase transition starts. At high temperature the effective potential at $v = 0$ is convex and the minimum of the potential happens for $v = 0$, that is the symmetric phase. However, when the effective potential becomes concave at $v = 0$, the minimum of the effective potential is located at a finite value of $v$, that is the broken symmetry phase. The above takes place provided the coefficient of the term $v^4$ is positive for otherwise the effective potential for large $v$ becomes concave and thus unstable. This can happen for very high temperatures larger than a given temperature $T_{max}$. Therefore, the condition for the analysis to be valid is that the critical temperature is smaller than this last temperature, namely, $T_c < T_{max}$.

To test whether the analysis is consistent, we proceed to compute these temperatures. For this purpose, we use the standard values for the parameters

$$\lambda = 20$$

$$\mu = 380 \text{ MeV}$$

$$g = 3.3, \quad (20)$$

which are determined from requiring that the vacuum $\sigma$ mass is, $m_\sigma^{vac} \simeq 600$ MeV, the constituent quark mass $m_q^{vac} \simeq 300$ MeV and by using a value for the pion vacuum decay constant $f_\pi = 93$ MeV. The critical temperature is computed from solving for the temperature where the second derivative of the effective potential with respect to $v$ vanishes, the maximum temperature is computed by finding the temperature for which the coefficient of the quartic term vanishes. This gives

$$T_c = 147.5 \text{ MeV}$$

$$T_{max} = 7, 196.5 \text{ MeV},$$

which confirms that $T_c < T_{max}$. The phase transition becomes a smooth crossover. Figure 1 shows the potential $V_{ren}^{(1)T\neq 0}$, dropping out $v$-independent terms, scaled by $T_c^4$, for different values of the temperature and ignoring the cubic term in the pion mass. Notice how this potential flattens out continuously as the temperature lowers.
down up to the temperature where the phase transition takes place, after which, the potential develops a minimum at a finite value of $v$.

B. Ring diagrams

We now proceed to include in the analysis the contribution from the ring diagrams. As is well known, for theories with massless modes, perturbative calculations can lead to the appearance of infrared divergences which signal the need of a resummation scheme. The leading divergences can be summed up to render an infrared safe quantity and the diagrams corresponding to this divergences are known as the ring diagrams. These are depicted in Fig. 2 and the explicit expression for the resummed series is given by

$$V^{(\text{ring})} = -\frac{T}{2} \sum_{n} \int \frac{d^3k}{(2\pi)^3} \times \sum_{N=1}^{\infty} \frac{1}{N} \left\{ 2[\Pi^* \Delta^B(k)]^N + [\Pi^0 \Delta^0(k)]^N \right\},$$

$$= \frac{T}{2} \sum_{n} \int \frac{d^3k}{(2\pi)^3} \times \left\{ 2 \ln[1 + \Pi^* \Delta^B(k)] + \ln[1 + \Pi^0 \Delta^0(k)] \right\},$$

where $\Pi^*$ and $\Pi^0$ are the charged and neutral pion self-energies in the presence of the magnetic field and $\Delta^B$ and $\Delta^0$ their corresponding propagators. The factor 2 takes into account that there are two charged pions.

At this point we should mention that, as the couplings (expansion parameters) in the expression for the pion self-energy are too large for a perturbative calculation to be strictly valid, it is likely that the one loop expansion overshoots the exact result. For the purposes of this work where we explore whether a weak magnetic field may have some influence on the dynamics of the phase transition, we will content ourselves with a qualitative description of the problem. For this, we parameterize our ignorance on the self-energies computing them perturbatively up to one-loop order but multiplying these times a constant $0 < c < 1$ that we will vary to explore the parameter space. The correct line of action is to calculate the pion self energy non perturbatively. The continuum approach in this connection would be to solve its Schwinger-Dyson equation consistently in the presence of a heat bath and uniform external magnetic field, and then substitute the result into Eqs. (22). However, in the present context, we replace this procedure by introducing the free parameter $c$ as mentioned before.

Equation (22) contains both vacuum as well as $T$-dependent infinities. These last can be canceled by a one-loop vacuum counterterm that renormalizes the pion mass. The procedure is best carried out by explicitly separating the two-loop contribution to Eq. (22) which results in the expression

$$V^{(\text{ring})} = \frac{T}{2} \sum_{n} \int \frac{d^3k}{(2\pi)^3} \times \left\{ 2 \ln[1 + \Pi^* \Delta^B(k)] - \Pi^* \Delta^B(k) \right\}$$

$$+ \left\{ \ln[1 + \Pi^0 \Delta^0(k)] - \Pi^0 \Delta^0(k) \right\}$$

$$+ 2V^{(2)} + V^{(0)},$$

where $V^{(2)}$ and $V^{(0)}$ are the contributions to the two-loop effective potential for a charged and a neutral pion. Once again, the factor 2 takes into account that there are two charged pions.

For the charged pion contribution to the one-loop pion self-energy, we use the charged scalar propagator which in the weak field limit is given by

$$\Delta^B = \frac{1}{\omega_n^2 + E_k^2} \left\{ 1 - \frac{(eB)^2}{(\omega_n^2 + E_k^2)^2} + \frac{2(eB)^2k_z^2}{(\omega_n^2 + E_k^2)^3} \right\},$$

where $\omega_n = 2n\pi T$, with $n$ an integer, $E_k^2 = k_x^2 + k_y^2 + m_\pi^2$, with $k_z^2 = k_x^2 + k_y^2$ and $m_\pi$ the $v$-dependent pion mass. For the neutral pion contribution to one the loop pion self-energy, we use the same propagator in Eq. (24) setting $eB = 0$. The quark contribution to the pion self-energies should in principle consider that quarks are subject to interact with the external magnetic field. Nevertheless, for the hierarchy of energy scales we work with,
it is easy to see that the $B$-dependent part of this self-energy is subdominant and thus we just consider the $B$-independent quark propagator.

The dominant contribution in Eq. (23) comes from the mode with $n = 0$. Fermions do not contribute to the ring diagrams as their mode with $n = 0$ does not vanish. It is easy to check that, after mass renormalization, the computation of Eq. (23) (for the mode with $n = 0$) reduces to considering only the $T$-dependent terms. As it is outlined in the appendix, carrying out an expansion of the argument of the logarithms and keeping only terms up to $O(eB)^2$, the dominant contribution from the ring diagrams, according to the hierarchy of energy scales we are considering, can be written as

\[
V^{(\text{ring})} = \frac{5\lambda T^4}{192} + \frac{15\lambda}{64\pi^2} T^2 m_\pi^2 + 3\frac{m_\pi^2 T}{12\pi} \\
- \frac{(eB)^2}{192\pi} \left(\frac{m_\pi^2 + \Pi_1}{\Pi_1}\right)^{3/2} \\
- \frac{T}{12\pi} \left(m_\pi^2 + \Pi_0^*\right)^{3/2} - 2\frac{T}{12\pi} \left(m_\pi^2 + \Pi^*\right)^{3/2},
\]

(25)

where in the fourth term on the right-hand side of Eq. (25) we have kept only the leading contribution in $(eB)$. To cope with the lack of a non-perturbative calculation for the self-energies, from now on we set $\Pi_0$, $\Pi^*$, and $\Pi_1$ as given by

\[
\Pi_0 = c \left[ 5\lambda \frac{T^2}{12} - \frac{(eB)^2}{240\pi} \frac{T}{m_\pi^2} + g^2 T^2 \right] \\
\Pi^* = c \left[ 5\lambda \frac{T^2}{12} - \frac{(eB)^2}{120\pi} \frac{T}{m_\pi^2} + g^2 T^2 \right] \\
\Pi_1 = c \left[ 5\lambda \frac{T^2}{12} + g^2 \right] T^2,
\]

(26)

with $0 < c < 1$ and where, as sketched in the appendix, we make use of the same cancellations that effectively substitute $m_\pi \to \tilde{m}_\pi$, with

\[
\tilde{m}_\pi = \sqrt{m_\pi^2 + \Pi_1},
\]

(27)

as in the calculation of the ring diagrams. There are several properties of Eqs. (25) and (26) worth mentioning: First, notice that, as promised, the cubic pion mass term in Eq. (26) exactly cancels the one appearing in Eq. (19). Second, as outlined in the appendix, the leading contribution in $(eB)$, in the first and second of Eqs. (26) need only to consider $\Pi_1$ as a correction to the pion mass.

From Eqs. (19) and (25), the final expression for the effective potential up to order ring can be written as

\[
V_{\text{eff}} = - \left(1 + \frac{3\lambda}{32\pi^2}\right) \frac{\mu^2}{2} v^2 \\
+ \left(\lambda + \frac{3\lambda^2}{8\pi^2} + \left(\frac{4\lambda^3}{36\pi^2} + \frac{6\lambda^3}{8\pi^2} \right)\frac{v^4}{4} \right) \Pi_0 \Pi_1 \\
+ \left(\Pi_0 \Pi_1 \right. \\
+ \left(\Pi_1 \Pi_0 \right) \left(\Pi_1 \Pi_0 + 2\frac{T}{12\pi} \right) \\
- \frac{6}{16\pi^2} \ln \left(\frac{T^2}{\Pi_0 \Pi_1 (v)}\right) + 3 \left(\frac{4\lambda T^2}{\mu^2}\right) \\
- \frac{T}{12\pi} \left( m_\pi^2 + \Pi_0^* \right)^{3/2} - 2\frac{T}{12\pi} \left( m_\pi^2 + \Pi^* \right)^{3/2} \\
- \frac{(eB)^2}{192\pi} \left( m_\pi^2 + \Pi_1 \right)^{3/2},
\]

(28)

where we have dropped out the explicit $v$-independent terms. The potential is real provided that $m_\pi^2 + \Pi_0 > 0$ and $m_\pi^2 + \Pi^* > 0$ (this last condition is enough to make sure that $m_\pi^2 + \Pi_0 > 0$). The first condition holds if for $v = 0$ which yields the requirement that

\[
T > \frac{\mu}{\sqrt{c(5\lambda T^2/12 + g^2)}} \equiv T_1(c).
\]

(29)
The second condition defines a magnetic field dependent value of a temperature \( T_B \) below which the analysis breaks down.

**IV. RESULTS**

To explore the properties of the effective potential in Eq. (23), we first study the case with \( B = 0 \) and \( c = 0.25 \). Figure 4 shows the effective potential scaled by \( T_1(0.25)^4 \), dropping out \( v \)-independent terms, computed for the same values of the parameters as in Eq. (20). Notice that for the highest temperature the minimum of the effective potential is at 0. As the temperature decreases, the potential flattens at 0 with its curvature remaining positive. However, an interesting phenomenon happens, namely, that for even lower temperatures, the potential becomes steeper at 0. This behavior persists down to the lowest temperature allowed. This effect is caused by the ring diagrams even in the absence of the magnetic field and thus precludes the phase transition to occur.

To see how this behavior is affected by varying the model parameters, we keep \( B = 0 \) but change the parameter \( c \) which we now take as \( c = 0.38 \). Figure 5 shows the effective potential scaled by \( T_1(0.38)^4 \), dropping out \( v \)-independent terms, computed for the same values of the parameters as in Eq. (20). Notice that for the highest temperature the minimum of the effective potential is at 0. As the temperature decreases, the potential flattens and the curvature at 0 becomes negative thus, a minimum at a finite value of 0 develops. This behavior persists down to the lowest temperature allowed. Thus we see that for certain values of \( c \) the phase transition is allowed but below a certain critical value \( c_c \), the phase transition does not occur. This also indicates that, as the temperature decreases after the development of the minimum at \( v \neq 0 \), the system goes back to the symmetry restored phase.

To find the critical value for the parameter \( c \), we can see whether the condition for the curvature at 0 to change sign is satisfied for real values of \( T \). This is shown in Fig. 6 where we plot the second derivative of the effective potential at 0 as a function of \( c \) and \( T \). Notice that this second derivative can change sign from positive to negative as the temperature decreases only above the critical value \( c_c \approx 0.3 \). Notice also from Fig. 6 that for a given temperature, as the value of \( c \) increases the curvature changes sign from positive to negative. This signals that the system starts out in the broken symmetry phase requiring a much larger value of the temperature to restore the symmetry.

We now proceed to analyze the behavior of the effective potential including the effects of the magnetic field. Figure 7 shows the relative difference \( (V_{eff} - V_{eff=0})/V_{eff=0} \) computed for the same values of the parameters as in Eq. (20) and a field intensity \( eB = 0.9 \times (m_{\pi}^{vac})^2 \). For the chosen temperatures, this relative difference amounts only for a 5% – 10% showing that up to this order in the approximation the effects of the magnetic field are small. The difference becomes a bit more significant for the lowest temperatures.

**V. CONCLUSIONS**

In conclusion, we have studied the chiral phase transition in the linear sigma model, including quarks, at finite temperature in the presence of weak magnetic fields. This has been accomplished by looking at the effective potential up to the contribution of the ring diagrams, which in the weak field limit, is the first order where the effects of the magnetic field become evident.

These diagrams are in principle an important ingredient in the calculation, given that the theory contains massless quantum modes which make it necessary to consider a resummation scheme. To accomplish the resummation, one needs to compute the pion self-energy. Nonetheless, given the fact that the theory has large cou-
pling constants, the computation of such self-energy is not perturbatively reliable. In an attempt to qualitatively study the structure of the theory, we have gone ahead and computed this self-energy perturbatively but have parameterized our lack of a non-perturbative calculation introducing a factor $c$ which we allow to vary such that $0 < c < 1$. In these terms, the analysis shows that considering the ring diagrams, there is a critical value for $c$ below which the phase transition is precluded from happening. Above this critical value, the phase transition keeps being second order but as the temperature drops the system comes back to the symmetry restored phase. This means that if the phase transition was completed, then the bounce back does not have any effect. However if the transition is delayed, part of the system could get trapped in the symmetric phase. Finally, the effects of the magnetic field, in the weak field limit, only account for differences up to about $5\% - 10\%$ and become larger as the system cools down to the lowest temperatures for the analysis to be valid.

In order to be able to extract more reliable conclusions, it is clear that one needs to have a better control on the calculation of the self-energy for which a non-perturbative treatment is called for, including the effects of the heat bath as well as those of the magnetic field. Nevertheless, the analysis shows in a qualitative way that the analysis to be valid.

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Appendix: Computation of ring diagrams

For the sake of the argument, consider only a single charged scalar field. The generalization to the sigma model is immediate once we consider the isospin factors. The effective potential at ring-order is given by

$$V^{(\text{ring})} = \frac{T}{(2\pi)^2} \int_0^\infty dk k^2 \left\{ \ln \left[ 1 + \frac{\Pi^B}{k^2 + m^2} \right] + V^{(2)} \right\},$$

where $V^{(2)}$ is the two-loop contribution to the effective potential. Notice that the charged scalar Matsubara propagator is given as in Eq. (24) and that we can write

$$V^{(2)} = \sum_{n=0}^{\infty} \int \frac{d^4k}{(2\pi)^4} \left\{ \ln \left[ 1 + \frac{\Pi^B}{k^2 + m^2} \right] - \Pi_1 \left( \frac{(eB)^2}{(k^2 + m^2)^2} - \frac{2(eB)^2k_1^2}{(k^2 + m^2)^4} \right) \right\} + V^{(2)}.$$

The logarithmic term in the above equation can be written as

$$\ln \left[ 1 + \frac{\Pi^B}{k^2 + m^2} - \Pi_1 \left( \frac{(eB)^2}{(k^2 + m^2)^2} - \frac{2(eB)^2k_1^2}{(k^2 + m^2)^4} \right) \right] = \ln \left[ 1 + \frac{\Pi^B}{k^2 + m^2} \right] + \ln \left[ \frac{1}{k^2 + m^2 + \Pi^B} \left( \frac{1}{(k^2 + m^2)^2} - \frac{2k_1^2}{(k^2 + m^2)^3} \right) \right] - \frac{(eB)^2\Pi_1}{k^2 + m^2 + \Pi^B} \left( \frac{1}{(k^2 + m^2)^2} - \frac{2k_1^2}{(k^2 + m^2)^3} \right).$$

With this expansion the ring contribution to the effective potential looks like
\[ V^{(\text{ring})} = \frac{T}{(2\pi)^2} \int_0^\infty dk k^2 \left\{ \ln \left[ 1 + \frac{\Pi^B}{k^2 + m^2} \right] - \frac{\Pi^B}{k^2 + m^2} \right\} - (eB)^2 \Pi_1 \times \left[ \left( \frac{1}{(k^2 + m^2)^2} - \frac{2k^2}{(k^2 + m^2)^3} \right) \frac{1}{(k^2 + m^2 + \Pi^B)^2} - \frac{2k^2}{(k^2 + m^2 + \Pi^B)^3} \right] \right\} + V^{(2)}. \] (33)

In order to render an analytical manageable expression let us make the approximation in the above equation such that

\[ \left( \frac{1}{(k^2 + m^2)^2} - \frac{2k^2}{(k^2 + m^2)^3} \right) \frac{1}{k^2 + m^2 + \Pi^B} \rightarrow \left( \frac{1}{(k^2 + m^2)^2} - \frac{2k^2}{(k^2 + m^2 + \Pi^B)^3} \right) \frac{1}{k^2 + m^2 + \Pi^B}. \] (34)

The error involved is of the order of the leading term in \( \Pi \), that is \( \Pi_1 \), which could be large if this last is proportional to the coupling \( \lambda \), as is the case in this work. In theories with a small coupling this error is small. In the absence of a non-perturbative calculation of \( \Pi_1 \) we will restrict ourselves to this approximation to keep track of the analytical structure of the effective potential.

Thus, under this approximation, we get

\[ V^{(\text{ring})} = \frac{T}{4\pi} \left\{ \frac{m^3}{3} - \frac{(m^2 + \Pi^B)^{3/2}}{3} + \frac{m\Pi^B}{2} \right\} + \frac{(eB)^2\Pi_1}{48} \left\{ \frac{1}{m^3} - \frac{1}{(m^2 + \Pi^B)^{3/2}} \right\} + V^{(2)}. \] (35)

We now recall that the two-loop contribution to the effective potential contains a term that can be written as

\[ V^{(2)} \rightarrow \frac{\lambda}{8} \left[ \frac{2\Pi_1}{\lambda} - \frac{(eB)^2 T}{96\pi m^3} \right]^2 \]
\[ \rightarrow \frac{\lambda}{8} \left[ \frac{4\Pi_1^2}{\lambda^2} - \frac{(eB)^2 T\Pi_1}{24\pi m^4\lambda} \right] \]
\[ = \frac{\lambda}{2} \left[ \frac{T^4}{(24)^2} + \frac{m^2 T^2}{64\pi^2} - \frac{m T^3}{96\pi} \right] - \frac{(eB)^2 T\Pi_1}{192\pi m^3}, \] (36)

Finally, recall that the one-loop contribution to the effective potential contains a term that looks like

\[ \Pi_1 = \frac{T^2}{24} - \frac{mT}{8\pi}. \] (37)

By adding \( V^{(1)} \) and \( V^{(\text{ring})} \) we see that the potentially dangerous terms with odd powers of \( m \) all cancel and that this cancellation effectively amounts for the replacement \( m \rightarrow \sqrt{m^2 + \Pi^B} \). The same discussion applies in the case of the self-energy as can easily be checked.
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