THE PROJECTIVE SYMPLECTIC GEOMETRY OF HIGHER ORDER VARIATIONAL PROBLEMS: MINIMALITY CONDITIONS

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(Communicated by Manuel de León)

Abstract. We associate curves of isotropic, Lagrangian and coisotropic subspaces to higher order, one parameter variational problems. Minimality and conjugacy properties of extremals are described in terms self-intersections of these curves.

1. Introduction. We introduce the subject by briefly describing the classical case of Riemannian manifolds: Let \((M, g)\) be a Riemannian manifold, and \(E = \frac{1}{2}g(v, v)\) its associated energy function. On the space \(\Omega_{p,q}\) of smooth curves \(\gamma\) joining \(p = \gamma(a)\) with \(q = \gamma(b)\), we consider the variational problem with fixed endpoints associated to the action functional

\[ A(\gamma) = \int_a^b E(\dot{\gamma}(s)) \, ds. \]

The local minimality of a geodesic \(\gamma\) is assured, using the second variation formula, by the strong positive definiteness of the Hessian of the functional, which in turn is given by the non-existence of conjugate points on the interval \((a, b)\).

It is well-known that the constructions mentioned in the previous paragraph can be studied via the local, global and self-intersection properties of certain curves of Lagrangian planes in a symplectic vector space. This viewpoint, in addition to be of great value in understanding purely Riemannian geometry [20, 26], is particularly suited to the study of more general variational problems, e.g., Finsler geometry [14], Lorentzian and semi-Riemannian geometry [19] and sub-Riemannian geometry [1].

The aim of this paper is to develop this projective symplectic viewpoint for higher order one-parameter variational problems, that is, functionals associated to Lagrangians depending on higher derivatives of curves, using a purely variational approach: directly from the Hessian of the given functional and its associated Euler-Lagrange equation, without detouring through non-holonomic mechanics. Higher order variational problems appear, for example in control theory (e.g., [13]) and applications to physics (e.g., higher-order mechanics considered in the book [18] and in the papers [6, 25]).

We use, extend and interpret in the symplectic-projective setting the line of classical identities of Cimino, Picone, Eastham, Easwaran, among others [4, 9, 10, 15, 17, 24], which allow the writing of higher order Lagrangians as a perfect square.

2010 Mathematics Subject Classification. Primary: 49K05; Secondary: 53D12.

Key words and phrases. Calculus of variations, isotropic subspaces, Lagrangian Grassmannian.

Diego Otero was financially supported by CNPq, grant number 140837/2012-4.
plus a total differential term. For the higher dimensional case, non-commutativity produces combinatorial difficulties in the establishment of the aforementioned identities. In this case we extend to arbitrary dimensions the work of Coppel [5], which uses the Legendre transformation to produce a Hamiltonian version that is much more suitable for this computation. As a by-product, we get a version of Easwaran’s identity for the higher dimensional case.

We prove that for higher order variational problems there exists a curve of isotropic subspaces whose successive prolongations determine the conjugacy, positive definiteness of the Hessian and, therefore, the local minimality of the extremals.

As in the first order problem, the main tool for determining minimality and conjugacy is the self-intersection properties of a (prolonged) curve of Lagrangian subspaces which we call the Jacobi curve. However, in contrast with the first-order case, this curve is always degenerate, even for positive definite Lagrangians. Another purpose of this work is to precisely state the degree of degeneracy of the Jacobi curve for higher-order problems. The present paper focuses on projectivization of the sufficient conditions for minimality; more precise information concerning the index of the Hessian (that in particular furnishes necessary conditions) will be studied in a forthcoming work [8].

The paper is organized as follows: a brief review of the preliminary set up is done in Section 2. The core of the paper is then given in Sections 3 and 4, where the Jacobi curve is constructed and studied respectively for one-dimensional and n-dimensional, higher order variational problems. The reason for the separation is that the one-dimensional case can be treated using the one-dimensional classical identities, which makes it a good starting point to understand the structure of the Jacobi curve, whereas in the higher dimensional case we use the Legendre transformation in order to apply the work of Coppel [5], which we extend to the n-dimensional case.

2. Preliminaries.

2.1. Higher order jet bundles. We consider Lagrangians that depend on higher order derivatives of curves in $M$: the natural setting of our work is then the space $J^k(\mathbb{R}, M)$ of k-jets of curves in a n-dimensional manifold $M$. Let us briefly describe some basic language on jet spaces of curves that is needed for this work. We refer to [18] and [12] for details and the general theory.

Let $M$ be a n-dimensional smooth manifold, $p \in M$ and $k \in \mathbb{N}$. We say that two curves in $M$, $\sigma, \gamma : (-\epsilon; \epsilon) \to M$ with $\sigma(0) = \gamma(0) = p$ have the same jet of order $k$ at $p$ if

$$\left. \frac{d^i}{dt^i} (f \circ \sigma)(t) \right|_{t=0} = \left. \frac{d^j}{dt^j} (f \circ \gamma)(t) \right|_{t=0}, \quad j = 1, 2, \ldots, k,$$

for every real smooth function $f$ defined in a neighborhood of $p$. This is an equivalence relation in the set of curves passing at $p$. The equivalence class of a curve $\sigma$ as above is denoted by $j^k\sigma(0)$ and the family of the equivalence classes of this relation is denoted by $J^k_p(\mathbb{R}, M)$. The collection of these families $J^k(\mathbb{R}, M) = \cup_{p \in M} J^k_p(\mathbb{R}, M)$ is called the space of k-jets of curves.

We have canonical projections between the spaces of jets of curves given by

$$\rho^k_r : J^k(\mathbb{R}, M) \to J^r(\mathbb{R}, M), \quad \rho^k_r(j^k\sigma(0)) = j^r\sigma(0),$$
with \( k \geq r \) and where \( J^0(\mathbb{R}, M) = M \) and \( j^0\sigma(0) = \sigma(0) \). It is possible to show that \( J^k(\mathbb{R}, M) \) has a structure of smooth manifold induced by local charts of \( M \).

**Proposition 1.** The space of \( k \)-jets of curves, \( J^k(\mathbb{R}, M) \), admits a structure of smooth manifold when endowed with the following charts: for every point \( j^k\sigma(0) \) of \( J^k(\mathbb{R}, M) \), if we denote by \((U, \phi)\) a local chart at \( \sigma(0) \) in \( M \), we have a local chart \((\tilde{U}, \tilde{\phi})\) at \( j^k\sigma(0) \) in \( J^k(\mathbb{R}, M) \), where \( U = (\rho^{k_0})^{-1}(U) \) and

\[
\tilde{\phi}(j^k\sigma(0)) = (q(0), q(1), \ldots, q(k)) = (q(0), q(1), \ldots, q(k)) \text{,}
\]

for \( j^k\sigma(0) \in \tilde{U} \), with \( q(s) = (q^A(s)) \), \( A = 1, \ldots, n \), \( q(s) = \frac{d^s}{dt^s}(\phi \circ \tilde{\sigma}(t))|_{t=0} \) and \( s = 0, 1, 2, \ldots, k \).

We call the charts \((\tilde{U}, \tilde{\phi})\) given in the proposition above the *natural coordinates* of \( J^k(\mathbb{R}, M) \). In the rest of the paper, we adopt the convention that the sub-index \( s \) of the coordinates of \( J^k(\mathbb{R}, M) \) \((q(s) = (q^A(s)))\) is the order of differentiation and the super-index \( A = 1, \ldots, n \) ranges through the coordinates on \( M \). Notice that \( \dim J^k(\mathbb{R}, M) = (k + 1)n \) and the canonical projections \( \rho^k \) are smooth with respect to the structure described above. For a proof of the previous proposition see for example chapter 2 of [12] or section 2 of [6].

Given a smooth curve \( \sigma : [a; b] \to M \) there is a natural lift \( j^k\sigma : [a; b] \to J^k(\mathbb{R}, M) \) such that the following diagram commutes

\[
\begin{array}{ccc}
J^k(\mathbb{R}, M) & \xrightarrow{j^k\sigma} & J^k(\mathbb{R}, M) \\
\downarrow \rho^k & & \downarrow \rho^k \\
[a; b] & \xrightarrow{\sigma} & M
\end{array}
\]

This lift is also called the kth *prolongation* of the curve \( \sigma \), and can be constructed as follows: consider a family of curves \( \sigma_t : [a - t; b - t] \to M \) given by \( \sigma_t(s) = \sigma(t + s) \) for each \( t \in [a; b] \). Then, for each \( t \in [a; b] \), let \( j^k\sigma_t(0) \) be the k-jet equivalence class at \( \sigma(t) \). Define the lift \( j^k\sigma : [a; b] \to J^k(\mathbb{R}, M) \) by the map \( t \mapsto j^k\sigma(t) := j^k\sigma_t(0) \).

If \( q(t) \) is the expression of \( \sigma(t) \) in a local chart \((U, \phi)\) of \( M \), then in the natural coordinates of \( J^k(\mathbb{R}, M) \) induced by \((U, \phi)\), the curve \( j^k\sigma \) is given by

\[
j^k\sigma(t) = \left( q(t), \frac{d}{dt}q(t), \ldots, \frac{d^k}{dt^k}q(t) \right).
\]

### 2.2. Higher order variational problem and the Hessian

Let us consider now a *k*-th order Lagrangian, that is, a function \( L : J^k(\mathbb{R}, M) \to \mathbb{R} \).

We assume the *strong Legendre condition*, which is that the restriction of the Lagrangian \( L \) to the inverse image of a point by the projection \( \rho^k_{k-1} : J^k(\mathbb{R}, M) \to J^{k-1}(\mathbb{R}, M) \) is strictly convex; this makes intrinsic sense since this fiber is an affine space (see for example Theorem 6.2.9 in [27]). In the local coordinates described above, the strong Legendre condition takes the form

\[
\frac{\partial^2}{\partial q_i \partial q_j} L \geq 0
\]

is positive definite.
From this data, we associate the functional
\[ \mathcal{L}[\sigma] = \int_{a}^{b} L(j^{k}\sigma(s))ds. \]
Take \( p, q \in J^{k-1}(\mathbb{R}, M) \), and let us, as usual, restrict \( \mathcal{L} \) to the set of curves \( \Omega_{p,q}[a,b] \) given by
\[ \Omega_{p,q}[a,b] = \{ \sigma : [a,b] \to M \mid j^{k-1}\sigma(a) = p \text{ and } j^{k-1}\sigma(b) = q \}. \]

Consider now an extremal \( \gamma(t) \) of the functional \( \mathcal{L} \). This curve satisfies the Euler-Lagrange equation, which might be understood in local coordinates (giving an elementary approach as in [11], 2.11), a symplectic approach as in [18, 28] or a more abstract one as in [3]. The approach taken makes no difference in what follows.

The crucial point is that if \( \gamma \) is a critical point of \( \mathcal{L} \), that is, \( d\mathcal{L}_{\gamma} = 0 \) for some reasonable notion of differentiability (see, e.g., [16, 23] for the global analysis and infinite dimensional manifold theory needed for these notions), then the Hessian of \( \mathcal{L} \) at \( \gamma \) is intrinsically defined, for example as in [21, 22]. This Hessian is a quadratic form on the space \( T_{\gamma}\Omega_{p,q}[a,b] \) of vector fields along \( \gamma(t) \) which vanish up to order \( k-1 \) at the endpoints.

Since we know \textit{a priori} that the Hessian is well defined, we can compute it in a convenient way: let us choose a frame \( v_1(t), \ldots, v_n(t) \) of vector fields along \( \gamma(t) \) that span \( T_{\gamma(t)}M \) at each point. In terms of this frame, each element \( X(t) \in T_{\gamma(t)}M \) can be written as \( X(t) = \sum h_i(t)v_i(t) \), and then \( T_{\gamma}\Omega_{p,q}[a,b] \) can be identified (with respect to the frame \( v_1, \ldots, v_n \)) with the space
\[ C^{\infty}_{k}([a,b], \mathbb{R}^n) = \{ h \in C^{\infty}([a,b], \mathbb{R}^n) \mid h^{(s)}(a) = h^{(s)}(b) = 0, \text{ for } 0 \leq s \leq k-1 \} \]
then we can work in local coordinates to show that the Hessian of \( \mathcal{L} \) at an extremum is given by a quadratic form as follows:
\[ \mathcal{Q}[h] = \int_{a}^{b} \sum_{1 \leq i \leq j \leq k} h^{(j)}(t)^{\top}Q_{ij}(t)h^{(i)}(t), \]
where each \( Q_{ij}(t) \) is a smooth \((n \times n)\)-matrix valued function. The matrices \( Q_{ij} \) depend on the Lagrangian and its derivatives, but the crucial information is that the top level \( Q_{kk}(t) \) is symmetric and positive definite. These two conditions are independent of the chosen frame; if the frame comes from prolonged local coordinates, the term \( Q_{kk}(t) \) is essentially the Hessian of the Lagrangian restricted to the coordinates \( q^4_k \).

The aim of this paper is, starting from the classical identities of higher order calculus of variations, to construct and study the projective curves whose lack of self-intersection imply that \( \mathcal{Q} \) is positive definite. Let us remark that the absence of self-intersection of our Jacobi curves on \( (a,b) \) implies positive definiteness, but a simple step similar to Sections 28 and 29.4 of [11] allows the passage from positive definite to \textit{strongly} positive definite (that is, \( \mathcal{Q}(h) \geq C|h| \) for some constant \( C > 0 \) and an appropriate \( C^k \)-norm for \( h \)). Thus the projective topology of the Jacobi curves (plus the approximation up to order 2 of the functional \( \mathcal{L} \)) does indeed provide sufficient conditions for local minimality.
2.3. Curves in the half- and divisible - Grassmannians and their rank.

As mentioned in the introduction, Jacobi curves in the context of higher-order variational problems are always degenerate. In this section we furnish the tools to quantify this degeneracy.

Let us denote by $Gr(k, V)$ the Grassmann manifold of $k$-dimensional subspaces of a real vector space $V$, which we consider to be finite dimensional. The local geometry (as opposed to the global topology) of curves in Grassmann manifolds can be studied in terms of frames spanning the given curve. This is the approach used in [2] and [7], where the reader can find details about the construction of local invariants. In this section we use this viewpoint to define the rank of a curve in the Grassmannian, which measures its degeneracy.

Recall that if $V$ is a real vector space, the tangent space of a Grassmann manifold $Gr(k, V)$ can be canonically identified with the quotient vector space $T_tGr(k, V) \cong \text{Hom}(\ell(t), V/\ell(t))$.

**Definition 2.1.** Let $I$ be an interval and $\ell : I \rightarrow Gr(k, V)$ be a curve in a Grassmann manifold. The rank of $\ell$ at $t \in I$ is the rank of $\ell(t)$ considered as an element of $\text{Hom}(\ell(t), V/\ell(t))$.

We now fix a basis on $V$ so that we may consider $V \cong \mathbb{R}^n$ and denote the Grassmannian by $Gr(k, n)$. Given a curve $\ell(t) \in Gr(k, n)$, we can lift it to a curve of $k$ linearly independent vectors $a_1(t), \ldots, a_k(t)$ in $\mathbb{R}^n$ that span $\ell(t)$. Let us consider the $n \times k$ matrix $\mathcal{A}(t)$ whose columns are the vectors $a_i(t)$, $\mathcal{A}(t) = (a_1(t) | \ldots | a_k(t))$ (the vertical bars denote juxtaposition). We have

**Proposition 2.** In the situation above, the rank of $\ell(t)$ is given by $\text{rank}(\mathcal{A}(t) | \dot{\mathcal{A}}(t)) - k$.

**Proof.** Let us recall concretely how the identification $T_tGr(k, V) \cong \text{Hom}(\ell(t), V/\ell(t))$ is established, in a way that is useful for computations involving the frame once a basis of $V$ is fixed: given a curve $\ell(t)$, choose a curve of idempotent matrices $\rho(t)$ representing projections (not necessarily orthogonal, since we do not assume any Euclidean structure in $V$) such that the image of $\rho(t)$ is $\ell(t)$. Then we have:

1. The derivative of a curve of projections $\rho(t)$ is a curve of endomorphisms that maps $\text{Im} \rho(t)$ into $\ker \rho(t)$ and vice-versa.
2. The quotient $V/\ell(0)$ can be identified with $\ker \rho(0)$.

Then, the derivative $\dot{\rho}(t_0)$ provides a map form $\text{Im} \rho(t_0) = \ell(t_0)$ into $\ker \rho(t_0) \cong V/\ell(t_0)$. It is straightforward, after unraveling the identifications, that this map is independent of the curve of projections chosen to represent $\ell(t)$.

Fix one such adapted curve of projection matrices $\rho(t)$. Then, since $\rho(t) \mathcal{A}(t) = \dot{\mathcal{A}}(t)$, we have

$$\dot{\mathcal{A}}(t) = \dot{\rho}(t) \mathcal{A}(t) + \rho(t) \dot{\mathcal{A}}(t).$$

Juxtaposing and computing the rank, we get

$$\text{rank}(\mathcal{A}(t) | \dot{\mathcal{A}}(t)) = \text{rank}(\mathcal{A}(t) | \dot{\rho}(t) \mathcal{A}(t) + \rho(t) \dot{\mathcal{A}}(t))$$

$$= \text{rank}(\mathcal{A}(t) | \dot{\rho}(t) \mathcal{A}(t)),$$

where the last equality follows from $\text{Im} \rho(t) = \ell(t)$ and therefore the columns $\rho(t) \dot{\mathcal{A}}(t)$ are in the span of the columns before the vertical bar. Now since the columns of $\mathcal{A}(t)$ are in the image of $\rho(t)$ and the columns $\dot{\rho}(t) \mathcal{A}(t)$ are in the kernel of $\rho(t)$, it follows that the rank of the right-hand side of the equation above is $k + \text{rank} \dot{\rho}(t) \mathcal{A}(t) = k + \text{rank} \ell(t)$. \qed
In the case of the half-Grassmannian $Gr(k, 2k)$ a curve $\ell(t)$ is called fanning if it has maximal rank, that is the rank of $\ell(t)$ is $k$ and $\dot{\ell}(t) : \ell(t) \to \mathbb{R}^{2k}/\ell(t)$ is invertible. This (generic) condition is the starting point of the invariants defined in [2]: the invariantly defined nilpotent endomorphism $F(t)$ of $\mathbb{R}^{2k}$, whose derivatives provide the invariants, is given by the composition $\mathbb{R}^{2k} \to \mathbb{R}^{2k}/\ell(t) \to \mathbb{R}^{2k}$ and therefore uses the fanning condition in an essential way.

All of the invariants studied in [2] for the Grassmannian can be used for the local study of curves in the Lagrangian Grassmannian; there is a single extra discrete invariant, the signature, that solves the equivalence problem for curves in the Lagrangian Grassmannian.

It is important to note that, in the case of higher order variational problems, we construct a curve of Lagrangian subspaces whose self-intersections (or, rather, lack thereof) control the positivity of the second variation. However, in the context of higher order variational problems, this curve is never fanning in the sense of [2], and thus a different approach to the the invariants must be taken. One possibility is using the theory developed in [30]. An alternative approach could be the use of curves $\ell(t)$ in a divisible Grassmannian $Gr(k, nk)$ and the invariant theory that has been developed in [7]. This is particularly adapted to the projective geometry of higher order linear differential equations [29], such as the Euler-Lagrange equation of a quadratic Lagrangian of higher order. The concept of fanning curve in this case requires the $(n-1)$-jet of the curve: if $\ell(t)$ is a curve in a divisible Grassmannian $Gr(k, nk)$ spanned as above by the columns of a $nk \times k$ matrix $A(t)$, we say that $\ell$ is fanning if the matrix $(A(t)|\dot{A}(t)|\ldots|A^{(n-3)}(t))$ is invertible. This is, again, a generic condition on smooth curves in a divisible Grassmannian, and that is satisfied in our case of isotropic curves whose prolongation gives the curve in the Lagrangian Grassmannian.

An important construction for these curves is the canonical linear flag

$$
\text{Span}\{A(t)\} \subset \text{Span}\{A(t), \dot{A}(t)\} \subset \ldots,
$$

which only depends on the curve $\ell(t)$ and its successive jets. We refer to each step of this sequence of inclusions as a prolongation of the curve $\ell$.

Fanning curves in the divisible Grassmannian satisfy that the canonical linear flag jumps dimension by $k$ at each stage up to the forced stabilization at maximal dimension $nk$; compare [30] where the general case, that is, jumps of arbitrary size on each prolongation, is considered.

3. The symplectic projective geometry of higher order quadratic functionals: The one dimensional case. Let $a < b \in \mathbb{R}$. Consider the general quadratic $k$th order functional

$$
Q[h] = \int_a^b \sum_{1 \leq i \leq j \leq k} Q_{ij}(t) h^{(i)}(t) h^{(j)}(t) \, dt,
$$

defined on the subspace of $C^\infty([a, b])$ consisting of functions vanishing up to order $k - 1$ at $a$ and $b$. By repeated integration by parts, $Q$ can be written as

$$
Q[h] = \int_a^b P_0(t) h(t)^2 + \ldots + P_1(t) (\dot{h}(t))^2 + \ldots + P_k(t) (h^{(k)})^2 \, dt. \quad (1)
$$
The Euler-Lagrange equation of \( Q \) is given by

\[ P_0 h - \frac{d}{dt} (P_1 \dot{h}) + \ldots + (-1)^k \frac{d^k}{dt^k} (P_k h^{(k)}) = 0. \tag{2} \]

We call this equation the Jacobi equation of the functional \( Q \), or, more generally, of a functional whose Hessian is given by \( Q \). The Jacobi equation is an actual differential equation of order 2\( k \) if \( P_k(t) \neq 0 \) for all \( t \in [a, b] \); we assume the strict Legendre condition: \( P_k(t) \) is actually positive on the interval \([a, b]\).

### 3.1. Easwaran identities

Let \( \{\sigma_i, i = 1, \ldots, k\} \) be a set of linearly independent solutions of (2) (in this work, unless otherwise specified, linear independence is taken with respect to the \( \mathbb{R} \)-vector space structure of the space of functions), which satisfy

\[ \sigma_i^{(j)}(a) = 0, \text{ for } i = 1, 2, \ldots, k, \text{ and } j = 0, 1, 2, \ldots, k - 1, \tag{3} \]

and consider the “sub-Wronskian”

\[ W[\sigma_1, \sigma_2, \ldots, \sigma_k](t) = \det \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_k \\ \sigma'_1 & \sigma'_2 & \cdots & \sigma'_k \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{(k-1)} & \sigma_{(k-1)} & \cdots & \sigma_{(k-1)} \end{pmatrix}. \tag{4} \]

**Definition 3.1.** A point \( t^* \in (a, b) \) is said to be conjugate to \( a \) if \( W[\sigma_1, \sigma_2, \ldots, \sigma_k](t^*) = 0 \).

We have have then the following theorem:

**Theorem 3.2.** (Easwaran [10]) If there are no conjugate points to \( a \) on \( (a, b) \), then the functional \( Q \) is positive definite, that is, \( Q(h) \geq 0 \) and \( Q(h) = 0 \) only when \( h \equiv 0 \).

The main idea behind Easwaran’s result is the following identity which enables us, under the disconjugacy hypothesis, to write the quadratic functional \( Q \) as a perfect square plus a total differential:

**Theorem 3.3.** (Easwaran [10], following Eastham [9]) Let \( \sigma_1, \ldots, \sigma_k \) be a linearly independent set of solutions of the Euler-Lagrange equations associated to a \( k \)-th order Lagrangian and satisfying satisfying (3), such that the sub-Wronskian satisfies \( W[\sigma_1, \sigma_2, \ldots, \sigma_k] \neq 0 \) in the interval \((a; b)\). Then for any \( h \in C^k([a, b]) \), we have the identity

\[ \sum_{i=0}^{k} P_i(t)(h^{(i)})^2 = P_k \left( \frac{W[h, \sigma_1, \sigma_2, \ldots, \sigma_k]}{W[\sigma_1, \sigma_2, \ldots, \sigma_k]} \right)^2 + \frac{dR}{dt}, \tag{5} \]

where \( W[h, \sigma_1, \sigma_2, \ldots, \sigma_k] \) is defined as

\[ W[h, \sigma_1, \sigma_2, \ldots, \sigma_k] = \det \begin{pmatrix} h & \sigma_1 & \sigma_2 & \cdots & \sigma_k \\ h & \sigma'_1 & \sigma'_2 & \cdots & \sigma'_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h^{(k-1)} & \sigma_{(k-1)} & \sigma_{(k-1)} & \cdots & \sigma_{(k-1)} \\ h^{(k)} & \sigma^{(k)} & \sigma^{(k)} & \cdots & \sigma^{(k)} \end{pmatrix}, \]

and \( R \) is a rational expression in \( \sigma_i, h^{(i)} \) and \( P_i \), such that \( R(t^*) = 0 \) if \( h^{(i)}(t^*) = 0 \) for all \( i = 0, 1, 2, \ldots k - 1 \).
Thus, under the hypothesis of Theorem 3.3, we can rewrite $Q$ as

$$Q(h) = \int_a^b \sum_{l=0}^{k-1} P_l(t) h^{(l)}(t) \, dt = \int_a^b \left( P_k \left( \frac{W[h,\sigma_1,\sigma_2,\ldots,\sigma_k]}{W[\sigma_1,\sigma_2,\ldots,\sigma_k]} \right)^2 + \frac{dR}{dt} \right) \, dt,$$

from which Theorem 3.2 immediately follows.

These kinds of identities were studied by Cimmino and Picone in [4, 24] for the first order case, by Leighton and Kreith in [17, 15] for the second order case, and by Eastham [9] for the general case in one dimensional problems (recall that the Euler-Lagrange equations generally have twice the order of the Lagrangian).

Easwaran [10] uses the identity developed in [9] to attain the minimality conditions in the one dimensional case. Coppel [5] also develops the minimality conditions for extremals of higher order one-dimensional variational problems assisted by the Legendre transformation, in order to be in the context of linear Hamiltonian systems. In Section 4, we observe that being careful about the order of multiplication of certain matrices, Coppel’s approach can be extended to arbitrary finite dimensional problems.

3.2. Jacobi curves. Let us note that Theorem 3.2 is projective in two senses. First, if we choose a different set of linearly independent solutions, say $\eta_i(t) = \sum a_{ij} \sigma_j(t)$ for some constant invertible $k \times k$ matrix, then the conjugacy condition is the same. This means that the conjugacy condition depends only on the subspace of the space of all solutions defined by the vanishing of the first $k$ derivatives, which we call the vertical subspace.

Second, and more importantly, we do not need to have actual solutions but only their (simultaneous) projective class: if $\phi: [a, b] \to \mathbb{R}$ is a never-vanishing differentiable function and we replace every $\sigma_i(t)$ by $\eta_i(t) = \phi(t) \sigma_i(t)$, then $W[\eta_1, \eta_2, \ldots, \eta_k](t) = \phi^k(t) W[\sigma_1, \sigma_2, \ldots, \sigma_k](t)$ and therefore their zeros coincide.

This motivates us to consider the following moving frame in $\mathbb{R}^{2k}$: let

$$\sigma_1(t), \ldots, \sigma_k(t), \sigma_{k+1}(t), \ldots, \sigma_{2k}(t)$$

be linearly independent solutions of the Euler-Lagrange equations, where the first $k$ solutions vanish up to order $k - 1$ at $t = a$, and

$$C(t) = \begin{pmatrix}
\sigma_1(t) \\
\vdots \\
\sigma_k(t) \\
\sigma_{k+1}(t) \\
\vdots \\
\sigma_{2k}(t)
\end{pmatrix}.$$

Let now $p(t)$ denote the class of $C$ under the projection to $\mathbb{R} P^{2k-1}$. We have

**Proposition 3.** The curve $p(t)$ is fanning as a curve in the divisible Grassmannian.

**Proof.** This follows directly from the fact that the determinant of the juxtaposed matrix $(C(t)|\dot{C}(t)|\ldots|C^{2k-1}(t))$ is the Wronskian of the $2k$ linearly independent solutions of the Jacobi equation. \qed
We now consider the frame obtained by the \((k-1)\) prolongation of \(C\),

\[
\mathcal{A}(t) = \begin{pmatrix}
\sigma_1(t) & \dot{\sigma}_1(t) & \ldots & \sigma_1^{(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_k(t) & \dot{\sigma}_k(t) & \ldots & \sigma_k^{(k-1)} \\
\sigma_{k+1}(t) & \dot{\sigma}_{k+1}(t) & \ldots & \sigma_{k+1}^{(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{2k}(t) & \dot{\sigma}_{2k}(t) & \ldots & \sigma_{2k}^{(k-1)}
\end{pmatrix},
\]

**Definition 3.4.** The Jacobi curve \(\ell(t)\) is the space spanned by the columns of \(\mathcal{A}(t)\).

It follows from Theorem 3 that \(\ell(t)\) is \(k\)-dimensional, that is, \(\ell\) is a curve in the half-Grassmannian \(\text{Gr}(k, 2k)\). Now, if we define the vertical space \(\mathcal{V} \subset \mathbb{R}^{2k}\) as the vectors such that their first \(k\)-coordinates vanish, we have that \(\mathcal{V} = \ell(a)\) and Theorem 3.2 translates to

**Theorem 3.5.** If \(\ell(t^*) \cap \mathcal{V} = \{0\}\) for all \(t^* \in (a, b]\), then the functional \(Q(t)\) is positive definite.

In contrast to the first order variational problems, we have that the curve \(\ell(t)\) is not fanning. Indeed,

**Proposition 4.** The rank of the curve \(\ell(t)\) is one.

**Proof.** It follows immediately by computing the rank of the extended matrix \((\mathcal{A}(t) | \dot{\mathcal{A}}(t))\); the columns \(k, k+1, \ldots, 2k-1\) are repeats of the columns \(2, \ldots, k\) but the first \(k\) columns and the last column are linearly independent by proposition 3.

An important feature of the curve \(p(t)\) is its symplectic behavior. This is better observed after applying the Legendre transformation which is done, in general dimension, in the following section, where we show that the spaces given by the canonical flag given by the successive prolongations of \(p(t)\) are isotropic, Lagrangian or coisotropic, according to the dimension.

4. The symplectic projective geometry of higher order quadratic functionals: The general case. We now consider a quadratic functional

\[
Q = \int_a^b L(t, \dot{h}(t), \ddot{h}(t), \ldots, h^{(k)}(t)) \, dt
\]

where

\[
L(t, h, \dot{h}, \ldots, h^{(k)}) = \sum_{1 \leq i \leq j \leq k} h^{(i)\top} M_{ij}(t) h^{(j)}
\]

and now \(h : [a, b] \to \mathbb{R}^n\) is a vector-valued smooth function, vanishing up to order \(k - 1\) at the ends of the interval, as in Section 2.2. In contrast with the one dimensional case of the previous section, here non-commutativity prevents us from transforming \(Q\) into a functional of the form (1). However some simplifications can be done; first, without loss of generality we can assume that \(M_{ij}(t)\) is symmetric, since \(v^\top A v\) vanishes if \(A\) is antisymmetric; we also assume the strong Legendre condition: the top term \(M_{kk}(t)\), in addition to being symmetric, is positive definite for all \(t \in [a, b]\). Also, it is easy to see that by integrating by parts, we can make \(M_{ij}(t) = 0\) if unless \(j = i\) or \(j = i + 1\). Then \(L\) can be written as
\[
L(t, h, \dot{h}, \ldots, h^{(k)}) = \frac{1}{2} \sum_{i=0}^{k} h^{(i)\top} M_{ii} h^{(i)} + \sum_{i=0}^{k-1} h^{(i)\top} M_{i(i+1)} h^{(i+1)},
\]
where \(M_{ii} = M_{ij}^\top\); these reductions greatly simplify the computations. From now on we drop the independent variable \(t\) from the notation.

The Euler-Lagrange equation (which we again call Jacobi equation, if the quadratic functional is the Hessian of a functional along an extremal) is then

\[
\frac{\partial L}{\partial q_0} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_0} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{q}_2} \right) + \ldots + (-1)^k \frac{d^k}{dt^k} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = 0. \tag{6}
\]

Where the partial derivatives above can be easily obtained from the form of \(L\):

\[
\frac{\partial L}{\partial q_0} = M_{00}h + M_{01}\dot{h},
\]

\[
\frac{\partial L}{\partial \dot{q}_j} = M_{jj}h^{(j)} + M_{(j-1)j}^\top h^{(j-1)} + M_{(j+1)j} h^{(j+1)}, \quad \text{for } j = 1, \ldots, k-1,
\]

\[
\frac{\partial L}{\partial \dot{q}_k} = M_{kk}h^{(k)} + M_{(k-1)k}^\top h^{(k-1)}.
\]

These relations can be written in the following form

\[
\begin{pmatrix}
\frac{\partial L}{\partial \dot{q}_0} \\
\vdots \\
\frac{\partial L}{\partial \dot{q}_{k-1}}
\end{pmatrix} = C(t) \begin{pmatrix} h \\ \vdots \\ h^{(k-1)} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial \dot{q}_k} \end{pmatrix},
\]

where \(C\) is a curve of \(kn \times kn\) matrices that have the \(M_{ij}\) blocks arranged as

\[
C(t) = \begin{pmatrix}
M_{00} & M_{01} & 0 & \ldots & 0 \\
M_{01}^\top & M_{11} & M_{12} & \ldots & 0 \\
0 & M_{12}^\top & M_{22} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & M_{(k-3)(k-2)}^\top & M_{(k-2)(k-2)} \\
0 & \ldots & 0 & 0 & M_{(k-2)(k-1)}^\top \\
\end{pmatrix},
\]

and \(M_{(k-1)(k-1)} = M_{(k-1)(k-1)} - M_{(k-1)k} M_{kk}^{-1} M_{(k-1)k}^\top\).

### 4.1. The Legendre transformation and Hamiltonian presentation.

As remarked at the end of Section 3.1, a Hamiltonian version of the Jacobi equation, using a higher dimensional extension of the methods of [5], greatly simplifies the computations needed to establish the Jacobi curve and its relationship with minimality.

Consider then the Legendre transformation \(\text{Leg} : \mathbb{R}^{2kn} \to \mathbb{R}^{2kn}\) associated with \(L\). The Legendre transformation applied to the \((2k-1)\)-jet of a curve \(h\) is given by:

\[
(h, \dot{h}, \ldots, h^{(k-1)}, h^{(k)}, \ldots, h^{(2k-1)}) \equiv (h, \dot{h}, \ldots, h^{(k-1)}, z_1 \ldots z_k) \equiv (y, z)
\]

where

\[
z_i = \sum_{j=i}^{k} (-1)^{j-i} \left( \frac{d}{dt} \right)^{j-i} \left( \frac{\partial L}{\partial \dot{q}_j} \right).
\]

Computation shows that the Legendre transformation takes solutions of (6) into solutions of a Hamiltonian system in the variables \((y, z)\) given by
where $C(t)$ is the same as above, $A(t)$ is given by the following block matrices

$$A(t) = \begin{pmatrix}
0_n & \Id_n & 0_n & \cdots & 0_n & 0_n \\
0_n & 0_n & \Id_n & \cdots & 0_n & 0_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0_n & 0_n & 0_n & \cdots & \Id_n & 0_n \\
0_n & 0_n & 0_n & \cdots & 0_n & \Id_n \\
0_n & 0_n & 0_n & \cdots & 0_n & -M^{-1}_{kk}M^T_{(k-1)k}
\end{pmatrix},$$

and $B(t)$ is given by

$$B(t) = \begin{pmatrix}
0_n & 0_n & 0_n & \cdots & 0_n & 0_n \\
0_n & 0_n & 0_n & \cdots & 0_n & 0_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0_n & 0_n & 0_n & \cdots & 0_n & 0_n \\
0_n & 0_n & 0_n & \cdots & 0_n & (M_{kk})^{-1}
\end{pmatrix}.$$

Note that $B(t)^T = B(t)$ and $C(t)^T = C(t)$, and then the matrix with the $A$, $B$ and $C$ blocks in the expression (7) is in the Lie algebra of the symplectic group (with respect to the canonical symplectic form on $\mathbb{R}^{2kn}$ that makes $(y, 0)$ and $(0, z)$ Lagrangian subspaces). Thus, equation (7) is indeed a Hamiltonian system.

Let us define another transformation on the $k$-jets of functions $h$, which we call the zeroing transformation since it maps the functional $Q$ to a functional that hides the derivatives.

Under the zeroing transformation, the $k$-jet of $h$ is mapped to $(y, \hat{z})$, where $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_k)$ is defined by

$$\hat{z}_j = \begin{cases} 
0 & \text{if } 0 \leq j < k \\
M_{kk}h^{(k)} + M^T_{(k-1)k}h^{(k-1)} & \text{if } j = k
\end{cases} \quad (8)$$

There are two important properties of the zeroing transformation. First, even though the pair $(y, \hat{z})$ usually does not satisfy equation (7), the equation $\dot{y} = A(t)y + B(t)\hat{z}(t)$ is an identity implied by the form of the matrices $A$ and $B$. This identity is needed in the generalized Picone identity 9 that appears in theorem 4.5.

The zeroing transform also enables us to simplify the expression of the functional $Q$. Let $h \in C^k([a, b], \mathbb{R}^n)$ be a function, and $(y, \hat{z})$ the pair defined above. Then we have:

$$\hat{z}^T B \hat{z} = \left(\hat{z}_1^T \cdots \hat{z}_k^T\right) \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & (M_{kk})^{-1}
\end{pmatrix} \begin{pmatrix}
\hat{z}_1 \\
\vdots \\
\hat{z}_k
\end{pmatrix}$$

$$= \hat{z}_k^T M^{-1}_{kk} \hat{z}_k = \left(h^{(k)T}M_{kk} + h^{(k-1)T}M_{(k-1)k}\right) M^{-1}_{kk} \left(M_{kk}h^{(k)} + M^T_{(k-1)k}h^{(k-1)}\right)$$

$$= h^{(k)T}M_{kk}h^{(k)} + 2h^{(k-1)T}M_{(k-1)k}h^{(k)} + h^{(k-1)T}M_{(k-1)k}M^{-1}_{kk}M^T_{(k-1)k}h^{(k-1)},$$
and

\[
y^\top Cy = y^\top \begin{pmatrix} \frac{\partial L}{\partial q_0} \\ \vdots \\ \frac{\partial L}{\partial q_{k-1}} \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k} \end{pmatrix} \\
= y^\top \begin{pmatrix} \frac{\partial L}{\partial q_0} \\ \vdots \\ \frac{\partial L}{\partial q_{k-1}} \end{pmatrix} - y^\top \begin{pmatrix} 0 \\ \vdots \\ M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k} \end{pmatrix} \\
= \sum_{i=0}^{k-1} h^{(i)} \top M_{ii} h^{(i)} + 2 \sum_{i=0}^{k-2} h^{(i)} \top M_{i(i+1)} h^{(i+1)} + h^{(k-1)} \top M_{(k-1)k} h^{(k)} \\
- h^{(k-1)} \top M_{(k-1)k} M_{kk}^{-1} \frac{\partial L}{\partial q_k}
\]

Adding \( \dot{z}^\top B \dot{z} \) and \( y^\top Cy \) leads to

\[
\dot{z}^\top B \dot{z} + y^\top Cy = \sum_{i=0}^{k} h^{(i)} \top M_{ii} h^{(i)} + 2 \sum_{i=0}^{k-1} h^{(i)} \top M_{i(i+1)} h^{(i+1)} = 2L.
\]

Then, the Legendre transformation maps solutions of (6) to solutions of (7), and the zeroing transformation gives the equality of the quadratic functionals (up to a factor 2):

\[
\dot{Q} = \int_a^b \dot{z}^\top B \dot{z} + y^\top Cy \, dt = 2Q = 2 \int_a^b L(t, \tilde{h}(t), \tilde{h}(t), \ldots, \tilde{h}^{(k)}(t)) \, dt.
\]

Let \( h_1, \ldots, h_{kn} \) be a linearly independent set of solutions of (6) such that all the derivatives up to the \( k-1 \) order vanish at \( t = a \). Consider the following sub-Wronskian

\[
W[h_1, \ldots, h_{kn}](t) = \det \begin{pmatrix} h_1(t) & h_2(t) & \cdots & h_{kn}(t) \\ h_1^{(1)}(t) & h_2^{(1)}(t) & \cdots & h_{kn}^{(1)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(k-1)}(t) & h_2^{(k-1)}(t) & \cdots & h_{kn}^{(k-1)}(t) \end{pmatrix}
\]

**Definition 4.1.** A point \( t^* \in (a, b] \) is said to be **conjugate** to \( a \) if \( W[h_1, \ldots, h_{kn}](t^*) = 0 \).

Then we have the following straightforward result:

**Lemma 4.2.** A point \( t^* \) is conjugate to \( a \) if, and only if, there exists a non-trivial solution \( h \) of (6) such that \( h^{(i)}(a) = h^{(i)}(t^*) = 0 \), for \( i = 0, 1, \ldots, k-1 \).
As for the one-dimensional case, we have the following result:

**Theorem 4.3.** If there are no conjugate points to a on \((a, b]\), then the functional \(Q\) is positive definite, that is, \(Q(h) \geq 0\) and \(Q(h) = 0\) only when \(h \equiv 0\).

The proof of this theorem involves a generalized Picone identity of the Hamiltonian system (7). Let us fix a set \(\{h_1, \ldots, h_{kn}\}\) of linearly independent solutions of (6) such that all the derivatives up to the \(k - 1\) order vanish at \(t = a\), and consider the image of the Legendre transformation of the \((2k - 1)\)-jet of each \(h_j\),

\[
\text{Leg} \left( h_j, h_j', \ldots, h_j^{(2k-1)} \right) = \begin{pmatrix} \mu_j \\ \zeta_j \end{pmatrix}.
\]

We now put them together by constructing the \(2kn \times kn\) matrix whose columns are the image of the Legendre transformation for each \(h_j\):

\[
\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} \mu_1 & \cdots & \mu_j \\ \zeta_1 & \cdots & \zeta_j \end{pmatrix}.
\]

We have the following lemma, where the proof follows by the fact that each image of the Legendre transformation of \(h_j\) satisfies equation (7) and using the initial condition at \(t = a\):

**Lemma 4.4.** In the conditions above we have

\[
Y(t)^\top Z(t) - Z(t)^\top Y(t) = 0, \ \forall t \in [a; b].
\]

**Corollary 1.** In the conditions above, and supposing that \(Y(t)\) is invertible for all \(t\), we have that \(Z(t)Y(t)^{-1}\) is symmetric for all \(t \in [a; b]\).

We can now state the generalized Picone identity:

**Theorem 4.5** (Generalized Picone Identity). Let \((Y; Z)\) satisfy the conclusion of Lemma 4.4 above and suppose that \(Y(t)\) is invertible for all \(t \in (a; b]\). Also consider a pair \((y; z)\) satisfying the relation \(\dot{y} = A(t)y + B(t)z\). In these conditions we have

\[
\frac{d}{dt}(y^\top Z Y^{-1} y) = z^\top B z + y^\top C y - (z - Z Y^{-1} y)^\top B (z - Z Y^{-1} y). \tag{9}
\]

**Proof.** Calculating the derivative above, using \(\dot{y} = A(t)y + B(t)z\) and

\[
\dot{Y} = AY + BZ, \quad \dot{Z} = CY - A^\top Z,
\]

we have

\[
\frac{d}{dt}(y^\top Z Y^{-1} y) = y^\top Z Y^{-1} y + y^\top Z Y^{-1} y + y^\top Z(-Y^{-1} \dot{Y} Y^{-1}) y + y^\top Z Y^{-1} \dot{y} =
\]

\[
= (Ay + Bz)^\top Z Y^{-1} y + y^\top (CY - A^\top Z) Y^{-1} y - y^\top Z(Y^{-1}(AY + BZ) Y^{-1}) y +
\]

\[
+ y^\top Z Y^{-1} (Ay + Bz) =
\]

\[
= (y^\top A + z^\top B) Z Y^{-1} y + y^\top (C - A^\top Z Y^{-1}) y - y^\top Z(Y^{-1} A^\top + Y^{-1} B Z Y^{-1}) y +
\]

\[
+ y^\top Z Y^{-1} (A^\top B + B) =
\]

\[
= z^\top B Z Y^{-1} y + y^\top C y - y^\top Z Y^{-1} B Z Y^{-1} y + y^\top Z Y^{-1} B z + z^\top B z - z^\top B z =
\]

\[
= z^\top B + y^\top C y + \left( z^\top B Z Y^{-1} y - y^\top Z Y^{-1} B Z Y^{-1} y + y^\top Z Y^{-1} B z - z^\top B z \right). \Delta
\]

\[
\]
Expanding the last term in (9) and using that $ZY^{-1}$ is symmetric we have

$$- (z - ZY^{-1} y)^\top B(z - ZY^{-1} y) = (y^\top (ZY^{-1})^\top - z^\top) B(z - ZY^{-1} y) =$$

$$= (y^\top ZY^{-1} - z^\top) B(z - ZY^{-1} y) = y^\top ZY^{-1} Bz - y^\top ZY^{-1} BZY^{-1} y - z^\top Bz +$$

$$+ z^\top BZY^{-1} y = \Delta,$$

and then, the identity follows.

With the hypothesis above, let $h \in C^k([a, b], \mathbb{R}^n)$ satisfying $h^{(i)}(a) = h^{(i)}(b) = 0$, $0 \leq i \leq k - 1$, and consider the image $(y, \hat{z})$ of the zeroing transformation of the $k$-jet of $h$ (as in (8)). We have

$$2Q[h] = \tilde{Q}[y, \hat{z}] = \int_a^b \hat{z}^\top B\hat{z} + y^\top Cy \, dt =$$

$$= \int_a^b \frac{d}{dt}(y^\top ZY^{-1} y) + (\hat{z} - ZY^{-1} y)^\top B(\hat{z} - ZY^{-1} y) \, dt =$$

$$= \int_a^b (\hat{z} - ZY^{-1} y)^\top B(\hat{z} - ZY^{-1} y) \, dt \geq 0$$

and

$$2Q = \tilde{Q} = 0 \iff \int_a^b (\hat{z} - ZY^{-1} y)^\top B(\hat{z} - ZY^{-1} y) \, dt = 0 \iff$$

$$\iff B(\hat{z} - ZY^{-1} y) = 0 \iff$$

$$\iff \hat{y} - Ay - BZY^{-1} y = 0 \iff$$

$$\iff \hat{y} = (A + BZY^{-1}) y \iff$$

$$\iff y \equiv 0.$$

Then Theorem 4.3 follows, since $y \equiv 0$ implies that $h \equiv 0$.

4.2. Jacobi curves. Let us projectivize as in the 1-dimensional case. Define the $2kn \times n$ frame

$$\mathcal{A}(t) = \begin{pmatrix} h_1^\top \\ \vdots \\ h_{2kn}^\top \end{pmatrix},$$

where $\{h_1, \ldots, h_{2kn}\}$ is a fundamental set of solutions of (6) such that for $i = 1, \ldots, kn$, each $h_i$ has all of its derivatives vanishing up to order $k - 1$ at $t = a$.

Again considering the space $p(t)$ spanned by the columns of $\mathcal{A}$ at each $t$ we have

**Theorem 4.6.** The curve $p(t)$ is a fanning curve in the Grassmannian $Gr(n, 2kn)$

**Proof.** Considering the prolongation to $(2k - 1)$-jet of $p$ written in terms of the frame $\mathcal{A}$, the rank of this prolongation is maximal, that is, for each $t$ the $2kn \times 2kn$ matrix bellow is non-degenerate:

$$\left( \mathcal{A}(t) \left| \dot{\mathcal{A}}(t) \right| \cdots \left| \mathcal{A}^{(2k-1)}(t) \right. \right).$$
The non-degeneracy comes from the fact that the determinant of the matrix above is the Wronskian of a set of fundamental solutions of (6) (which is non-zero for all \( t \)), and then the assertion follows.

**Definition 4.7.** The Jacobi curve \( \ell : [a, b] \to \text{Gr}(kn, 2kn) \) is the \((k - 1)\)-jet prolongation of \( p \), that is, the curve of subspaces spanned by the columns of the matrix

\[
\left( \begin{array}{c|c|c|c}
A(t) & \dot{A}(t) & \cdots & A^{(k-1)}(t) \\
\hline
\end{array} \right) .
\]

(11)

Now if we define the vertical space \( \mathcal{V}^{2kn}_{kn} \subset \mathbb{R}^{2kn} \) as the vectors having vanishing first \( kn \)-coordinates, we have that \( \mathcal{V}^{2kn}_{kn} = \ell(a) \) and then Theorem 4.3 can be formulated as

**Theorem 4.8.** If \( \ell(t^*) \cap \mathcal{V}^{2kn}_{kn} = \{0\} \) for all \( t^* \in (a, b] \), then the functional \( Q(t) \) is positive definite.

Therefore the lack of self-intersections of the Jacobi curve implies the positivity of the quadratic functional \( Q \).

In contrast with first order variational problems, this Jacobi curve is highly degenerate. Indeed, we have:

**Theorem 4.9.** The rank of the Jacobi curve \( \ell(t) \) is \( n \).

**Proof.** Proposition 2 can applied to the frame (11) of \( \ell(t) \). The rank of \( \ell \) is then given by

\[
\text{rank} \left( \begin{array}{c|c|c|c}
A(t) & \dot{A}(t) & \cdots & A^{(k-1)}(t) \\
\hline
\end{array} \right) - kn,
\]

which, by Theorem 4.6, is \( n(k + 1) - kn = n \). \( \square \)

Observe that the rank of a fanning curve in the half-Grassmannian \( \text{Gr}(kn, 2kn) \) is \( kn \); thus the Jacobi curve of higher order variational problems \((k > 1)\) is never fanning.

4.3. **Isotropic-Lagrangian-coisotropic flag.** We now study the symplectic properties of the fanning curve \( p : [a, b] \to \text{Gr}(n, 2kn) \) and its prolongations. Let us consider in \( \mathbb{R}^{2kn} \) the canonical symplectic form \( \omega_{\text{can}} \) induced by the decomposition \( \mathbb{R}^{2kn} \cong \mathbb{R}^{kn} \oplus \mathbb{R}^{kn} \) in its first and last \( kn \) coordinates. Then we have

**Theorem 4.10.** Considering the symplectic space \((\mathbb{R}^{2kn}, \omega_{\text{can}})\), the curve \( p : [a, b] \to \text{Gr}(n, 2kn) \) defined in the last section satisfies

- \( j^i p : [a, b] \to \text{Gr}(i + 1)n, 2kn) \) is a curve of isotropic subspaces for \( i = 0, \ldots, k - 2 \),
- \( \ell = j^{k-1} p : [a, b] \to \text{Gr}(kn, 2kn) \) is a curve of Lagrangian subspaces,
- \( j^i p : [a, b] \to \text{Gr}(i + 1)n, 2kn) \) is a curve of coisotropic subspaces for \( i = k, \ldots, 2k - 1 \).

**Proof.** The proof relies on the invariance of the isotropic-Lagrangian-coisotropic concepts under certain “tiltings” of the Legendre transformation. Recall from the beginning of this section that the Legendre transformation has the form

\[
\text{Leg}(h, \ldots, h^{(2k-1)}) = \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \text{Id}_{kn} & 0_{kn} \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} h \\ : \\ h^{(2k-1)} \end{pmatrix},
\]
where the blocks $B_1$ and $B_2$ are also $kn \times kn$ matrices. The block $B_1$ is an upper triangular matrix and the block $B_2$ has blocks $M_{kk}$ or $-M_{kk}$ in the anti-diagonal and zeros below the anti-diagonal, that is, $B_2$ is of the form
\[
B_2 = \begin{pmatrix}
* & * & \cdots & * & (\frac{1}{k}k^k) - 1 & M_{kk} \\
* & * & \cdots & (\frac{1}{k}k^k - 2) & M_{kk} & 0_n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
* & -M_{kk} & 0_n & \cdots & 0_n \\
M_{kk} & 0_n & 0_n & \cdots & 0_n
\end{pmatrix}.
\]
From this we get that $z$ is linear combination (as a module considering functions of $t$ as coefficients) of $h^{(i+1)}, h^{(i)}, \ldots, h^{(k)}$, for $i = 1, \ldots, k$.

Now for each $h_i$ given in (10) consider the image $(\mu_i, \zeta_i)$ of the Legendre transformation of the $(2k - 1)$-jet prolongation of $h_i$
\[
\text{Leg}(h_i, \ldots, h_i^{(2k-1)}) = \begin{pmatrix} \mu_i \\ \zeta_i \end{pmatrix},
\]
and construct the following matrix
\[
\begin{pmatrix} \mu_1 & \cdots & \mu_{2kn} \\ \zeta_1 & \cdots & \zeta_{2kn} \end{pmatrix} = \begin{pmatrix} \text{Id}_{kn} & 0_{kn} \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} h_1 & \cdots & h_{2kn} \\ \vdots & \vdots & \vdots \\ h_1^{(2k-1)} & \cdots & h_{2kn}^{(2k-1)} \end{pmatrix}.
\]
Calculating the transpose in the relation above leads to
\[
\begin{pmatrix} \mu_1^T \\ \cdots \\ \mu_{2kn}^T \\ \zeta_1^T \\ \cdots \\ \zeta_{2kn}^T \end{pmatrix} = \begin{pmatrix} \mathcal{A}(t) \mid \mathcal{A}(t) & \cdots & \mathcal{A}^{(2k-1)}(t) \end{pmatrix} \begin{pmatrix} \text{Id}_{kn} & B_1^T \\ 0_{kn} & B_2^T \end{pmatrix} \begin{pmatrix} h_1 & \cdots & h_{2kn} \\ \vdots & \vdots & \vdots \\ h_1^{(2k-1)} & \cdots & h_{2kn}^{(2k-1)} \end{pmatrix}
\]
where the $2kn \times n$ blocks $C^i$ are linear combinations (again as a module considering functions of $t$ as coefficients) of the columns of $A^{(i-1)}, \ldots, A^{(2k-1)}$, for $i = 1, \ldots, k$, that can be written as
\[
C^i = \mathcal{A}^{(i-1)}Q_{i-1}^{(i)} + \cdots + \mathcal{A}^{(2k-1)}Q_{2k-1}^{(i)},
\]
where the $Q_j^i$ are $n \times n$ matrices and also $Q_{2k-1}^{(i)} = \pm M_{kk}$.

Now, the matrix on the left side of the equality (12) lies in the Lie group of symplectic matrices if it satisfies the initial condition
\[
\begin{pmatrix} \mu_1(t) & \zeta_1(t) \\ \vdots & \vdots \\ \mu_{2kn}(t) & \zeta_{2kn}(t) \end{pmatrix} = \begin{pmatrix} 0_{kn} & \text{Id}_{kn} \\ -\text{Id}_{kn} & 0_{kn} \end{pmatrix}.
\]
Then we have that the matrices in (12) are symplectic for each $t$ and the initial condition makes the first columns vanish up to order $k - 1$ at $t = a$, as required.

Denoting by $J$ the matrix of the canonical symplectic form $\omega_{can}$ in $\mathbb{R}^{2kn}$, we will have that
\[
\begin{pmatrix} \mathcal{A}(t) & \cdots & C^k(t) \end{pmatrix}^T J \begin{pmatrix} \mathcal{A}(t) & \cdots & C^k(t) \end{pmatrix} = J \begin{pmatrix} 0_{kn} & \text{Id}_{kn} \\ -\text{Id}_{kn} & 0_{kn} \end{pmatrix},
\]
which in turn implies
\[
\mathcal{A}^{(i-1)}^T J \mathcal{A}^{(j-1)} = 0,
\]
for \(i, j = 1, \ldots, k\), and
\[
C^i \top J A^{j-1} = 0
\]
for \(2 \leq i \leq k\) and \(1 \leq j \leq i - 1\). Developing further the second expression using that \(C^i\) can be written as
\[
C^i = A^{(i-1)} Q^{i-1}_{-1} + \ldots + A^{(2k-i)} Q^{i}_{2k-i},
\]
with \(Q^{i}_{2k-i} = \pm M_{kk}\) (which is non-degenerate), we will have that
\[
A^{(i)} \top J A^{(j)} = 0,
\]
for \(k \leq i \leq 2k - 2\) and \(0 \leq j \leq 2k - i - 2\).

Finally, from the identities (14) and (15) we obtain all items of Theorem 4.10 simultaneously.

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Received October 2015; revised April 2016.

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