Confinement of monopole field lines in a superconductor at $T \neq 0$

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We apply the Bogoliubov-de Gennes equations to the confinement of a monopole-antimonopole pair in a superconductor. This is related to the problem of a quark-antiquark pair bound by a confining string, consisting of a colour-electric flux tube, dual to the magnetic vortex of type-II superconductors. We study the confinement of the field lines due to the superconducting state and calculate the effective potential between the two monopoles. At short distances the potential is Coulombic and at large distances the potential is linear, as previously determined solving the Ginzburg-Landau equations. The magnetic field lines and the string tension are also studied as a function of the temperature $T$. Because we take into account the explicit fermionic degrees of freedom, this work may open new perspectives to the breaking of chiral symmetry or to colour superconductivity.

I. INTRODUCTION

The confining string is an important candidate to solve the confinement problem of quantum chromodynamics (QCD). Right after QCD was proposed as the theory of strong interactions [1, 2], different ideas for confinement were developed. Nielsen and Olesen [3] suggested that the magnetic flux tube vortex of type-II superconductors could be applied, after a dual transformation, to colour-electric flux tube strings [4] in QCD. Soon afterwards Nambu [5], 't Hooft [6], and Mandelstam [7] developed the thesis that the open colour-electric string was able to confine quark-antiquark systems, where the quark-antiquark pair is dual to a Dirac [8] monopole-antimonopole pair. This idea was further developed by Baker, Ball and Zachariasen [9] and co-authors, who developed dual QCD, and also studied systems similar to a monopole-antimonopole pair in a superconductor in the Ginzburg-Landau framework. At the onset of QCD, lattice QCD was also developed by Wilson [11], who showed analytically that the strong coupling limit of QCD is equivalent to a string theory. The study of strings at the realistic transition between the strong and weak coupling of Lattice QCD was further explored by several authors. Bali [12, 13] and Polikarkov [14, 15] and co-authors studied in detail the string formation between static quark-antiquark sources in Lattice QCD. Importantly, color superconductivity was also proposed to exist at finite baryon density by Alford, Rajagopal and Wilczek [16].

It is clear that QCD differs from type-II superconductors in many details. Baker, Ball and Zachariasen found that dual QCD is at the frontier between type-I and type-II superconductivity. The spin dependence of the quark-antiquark interaction, unparallel to monopole-antimonopole interaction, was also addressed by De Rujula, Georgi and Glashow, and [17] Henriques, Kellett and Moorhouse [18] identified the role of the confining string in the spin dependent interactions. Moreover Takahashi, Suganuma and co-authors [19, 20] also studied the three-legged string formation between three static quark sources, proposed to exist in Baryons. It is clear that this needs a SU(3) symmetry, unparallel to type-I or type-II superconductors. Okiharu, Suganuma and Takahashi [21, 22], further studied the tetraquark and pentaquark potentials. The idea of bag model, separating the interior of the flux tube from the remaining of the universe, was developed by Chodos, Jaffe, Johnson, Thorn and Weisskopf [23], and this lead Jaffe and Johnson to propose the existence of exotic hadrons, including glueballs, needing also a SU(3) symmetry, [24]. Moreover, many problems in QCD confinement with strings remain to be solved. For instance in the limit of infinitely thin relativistic strings [25], the world sheet swept out by a string in space time can only be quantized in 26 dimensions. For example light quarks, which condense the vacuum with chiral symmetry spontaneously breaking $^3P_0$ scalar quark-antiquark pairs described by Bicudo and Ribeiro [26], are not yet fully compatible with string confinement. These problems, and other open problems of QCD confinement, motivate very interesting and active investigations.

Nevertheless the main inspiration of QCD confinement, the monopole-antimonopole pair in a superconductor, remains to be fully explored. Ball and Caticha [27], already studied the monopole-antimonopole pair in Ginzburg-Landau type-II superconductor. The confinement of magnetic field lines in superconductors occurs through an Anderson-Higgs mechanism such that the photons of the electromagnetic field acquire a mass and are therefore exponentially damped in the superconductor. This constitutes the Meissner effect discovered experimentally in 1933. Indeed at the boundary between the exterior and the superconductor, the parallel component of the

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In this paper we apply the Bogoliubov-de Gennes (BdG) equations, enabling a microscopic understanding of the role of electronic states in the confinement of magnetic and electric fields in superconductors. This is an inspiring model for QCD, where colour superconductivity and chiral symmetry breaking can both be addressed at the microscopic level of fermion (quark or antiquark) pairing. We will be interested in the situation where two solenoids or magnetic rods are inserted in a superconductor (perfect diamagnet) over a distance defined by the penetration length. If $d >> \lambda$, the superconductor is of type-II and the magnetic field penetrates in the material through quantized vortices, whose density increases until their cores (with size given by the coherence length) overlap and the system becomes normal.

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Consider a fermionic system with an effective attractive interaction between the particles defined by an Hamiltonian

$$H = \int dr \sum_{\sigma} c^\dagger(r, \sigma) \left( \frac{p^2}{2m} + \left( \frac{eA}{2m} \right)^2 - E_F \right) c(r, \sigma)$$

where $c^\dagger(r, \sigma)$ creates a fermion at site $r$ with spin $\sigma$. $A$ is the vector potential. In BCS theory, this leads to fermion-fermion and antifermion-antifermion pairing. Defining the field of a pair as $\Delta(r)$, and considering a conventional s-wave superconductor, the Hamiltonian is decoupled like

$$H_{BCS} = \int dr \sum_{\sigma} c^\dagger(r, \sigma) \left[ \left( \frac{p^2}{2m} + \left( \frac{eA}{2m} \right)^2 - E_F \right) - \Delta^2(r) \right] c(r, \sigma)$$

Here $E_F$ is the Fermi energy. The Hamiltonian is diagonalized through a canonical transformation

$$c(r, \uparrow) = \sum_i \left( u_i(r) \gamma_{i,\uparrow} - v_i^*(r) \gamma^\dagger_{i,\downarrow} \right)$$

$$c(r, \downarrow) = \sum_i \left( u_i(r) \gamma_{i,\downarrow} + v_i^*(r) \gamma^\dagger_{i,\uparrow} \right)$$

Here the fermionic operators $\gamma_{i,\sigma}$ annihilate a quasiparticle in the level $i$ and with spin $\sigma$. Defining the diagonalized Hamiltonian as

$$H = E_g + \sum_{i,\sigma} E_i \gamma^\dagger_{i,\sigma} \gamma_{i,\sigma}$$
where $E_g$ is the groundstate energy, the physical meaning of the new operators is that they create the excitations of the system, and therefore we must restrict ourselves to the positive energies. Using the commutation relations of the operators with the Hamiltonian leads to the Bogoliubov-de Gennes equations (BdG) \[28\] for the amplitudes $u(r)$ and $v(r)$:

\[
\begin{align*}
\left( \frac{1}{2m}(p - \frac{e}{c}A)^2 - E_F \right) u_i(r) + \Delta(r) v_i(r) &= E_i u_i(r) \quad (5) \\
- \left( \frac{1}{2m}(p + \frac{e}{c}A)^2 - E_F \right) v_i(r) + \Delta^*(r) u_i(r) &= E_i v_i(r) \quad (6)
\end{align*}
\]

$A(r)$ is the vector potential, $\Delta(r)$ is the pairing function given by

\[
\Delta(r) = g \sum_{0 < E_i \leq \hbar \omega_D} u_i(r) v_i^*(r) (1 - 2 f(E_i)). \quad (7)
\]

Here $f(E_i)$ is the Fermi-Dirac distribution given by

\[
f(E_i) = \frac{1}{e^{\frac{E_i}{k_B T}} + 1} \quad (8)
\]

where $T$ is the temperature, assumed smaller than the critical temperature $T_c$, below which superconductivity arises. The pairing between the electrons occurs on an energy scale $\hbar \omega_D$, called the Debye energy. In conventional superconductors this is the cutoff energy provided by the phonon exchange between electrons. An example of electronic wavefunctions $u$ and $v$ obtained in this paper is depicted in Fig. 2. The vector potential is given by Maxwell’s equations

\[
\nabla \times B = \nabla \times \nabla \times A = \frac{4\pi}{c} J_{total} \quad (9)
\]

which, in the Coulomb gauge ($\nabla \cdot A = 0$), is given by

\[
\nabla^2 A = -\frac{4\pi}{c} J_{total} \quad (10)
\]

The current density originated in the supercurrents is obtained self-consistently by

\[
J(r) = \frac{e\hbar}{im} \sum_i f(E_i) u_i^*(r) (\nabla - \frac{ie}{\hbar c} A(r)) u_i(r) + (1 - f(E_i)) v_i(r) (\nabla - \frac{ie}{\hbar c} A(r)) v_i^*(r) - c.c. \quad (11)
\]

Let us take the order parameter in the form

\[
\Delta(r) = \Delta(\rho, z) e^{-in\varphi} \quad (12)
\]

This form describes a magnetic flux equal to $n$ flux quanta ($\Phi = n\phi_0 = n\frac{\hbar e}{2\pi}$). The wave functions $u_i$ and $v_i$ are expanded in a way similar to ref. \[28\]

\[
\begin{align*}
u_i(r) &= \sum_{\mu, j, k} c_{\mu, j, k} \phi_{\mu, j, \rho + n/2, k} e^{i(\mu + n/2)\varphi} \quad (13)
\end{align*}
\]

**FIG. 2:** The $\rho$ dependence, in the equatorial plane ($z = 0$), of the lowest electronic wavefunction. Notice that this is a localized wavefunction. We show in a) wavefunction $u$, and in b) the wavefunction $v$

\[
v_i(r) = \sum_{\mu, j, k} d_{\mu, j, k} \phi_{\mu, j, \rho + n/2, k} e^{i(\mu + n/2)\varphi} \quad (14)
\]

where

\[
\phi_{jmk} = \frac{\sqrt{2}}{RJ_{m+1}(\alpha_m) J_m (\alpha_m R)} \frac{e^{i \frac{2\pi j}{2\rho h}}}{\sqrt{\hbar}} \quad (15)
\]

Here $\mu$ is an half-odd integer if $n$ is odd and an integer if $n$ is even. $J_m$ is a Bessel function. The system is placed in a cylinder of radius $R$ and height $h$. Given the azimuthal symmetry of the system $A$ does not depend on $\varphi$, therefore the Hamiltonian of the BdG equations may be simultaneously diagonalized for each value of $\mu$. It is therefore enough to diagonalize the matrix

\[
\begin{pmatrix}
T^- & \Delta \\
\Delta^T & T^+
\end{pmatrix} \begin{pmatrix}
c_i \\
d_i
\end{pmatrix} = E_i \begin{pmatrix}
c_i \\
d_i
\end{pmatrix} \quad (16)
\]

where

\[
T_{jj'kk'}^{\pm} = \pm \frac{\hbar^2}{2m} (\alpha_{j\rho + n/2} + \frac{k^2 \pi^2}{h^2}) \delta_{jj'} \delta_{kk'} - (\mu \pm n/2) \frac{e}{\hbar c} I_1 \pm \frac{e^2}{h^2 c^2} I_2 - E_F \quad (17)
\]

with

\[
I_1 = \int_0^h dz \int_0^R \rho dp \quad (18)
\]
\[
\phi_{j, \mu \pm n/2, k} (\rho, z) \frac{\Delta^*(\rho, z)}{\rho} \phi_{j', \mu \pm n/2, k'} (\rho, z)
\]
the monopole, is given by

\[
\phi_{j,\mu \pm n/2, k}(\rho, z) A_{\varphi}(\rho, z)^2 \phi_{j',\mu \pm n/2, k'}(\rho, z)
\]

(19)

We also obtain that

\[
\Delta_{j,j',k,k'} = \int_0^h dz \int_0^R dp \rho \rho \phi_{j,\mu - n/2, k}(\rho, z) \Delta(\rho, z) \phi_{j',\mu + n/2, k'}(\rho, z)
\]

(20)

B. Monopolar gauge field

The vector potential has two contributions one due to the magnetic monopoles, \( A^{\text{ext}} \) and the other due to the currents that are formed in the superconductor, \( \mathbf{a} \). Due to the linearity of the Maxwell equations we have that

\[
\mathbf{A} = \mathbf{A}^{\text{ext}} + \mathbf{a}
\]

(21)

The contribution from the magnetic monopoles corresponding to a magnetic field

\[
\mathbf{B} = \frac{g_M}{4\pi} \frac{\mathbf{r}}{r^3}
\]

(22)

where \( g_M \) is the magnetic charge and \( r \) the distance to the monopole, is given by

\[
A^{\text{ext}} = -\frac{g_M}{4\pi} \frac{\rho}{r(r - z)} \mathbf{e}_\varphi
\]

(23)

to replacing the magnetic monopole by a semi-infinite infinitely thin solenoid corresponding to the magnetic charge and the Dirac string that carries the magnetic flux from the infinity to the charge. The string guarantees that \( \nabla \cdot \mathbf{B} = 0 \). We are interested in a situation where the two solenoids (or magnetic whiskers) inserted in the superconductor originate from one monopole flux lines that penetrate the superconductor and are recovered in the other monopole. In order for the magnetic field lines to diverge from one monopole and converge in the other the magnetic field created by the two monopoles has the form

\[
A^{\text{ext}} = -\frac{g_M}{4\pi} \left\{ \frac{\rho}{r_1 r_2 (z - z_1) (z - z_2)} \right\} \mathbf{e}_\varphi
\]

(24)

with \( r_i = \sqrt{\rho^2 + (z - z_i)^2} \).

Close to each Dirac string and far from the monopoles the external vector potential is approximately given by a term like

\[
A^{\text{ext}} = n^{\text{ext}} \frac{\hbar c}{2|e|\rho}
\]

(25)

since we can relate \( g_M \) with the total flux \( \Phi \) coming out from a monopole where \( \Phi = n^{\text{ext}} \Phi_0 \). Then it can be shown [30] that the BdG equations can be recast in a form such that they only depend on the difference \( n_a = n - n^{\text{ext}} \) between the vorticity of the gap function \( n \) and the vorticity of the external line \( n^{\text{ext}} \). In the BdG equations we just have to replace \( n \to n_a \) and the effect of the external infinitely thin solenoid is taken into account exactly if \( n^{\text{ext}} \) is an integer. In our case the vector potential is more complex.

The choice of form for the vector potential implies that the Dirac strings go from each monopole to infinity aligned with the \( z \)-axis in such a way that the direction of flux goes from \(-\infty\) to \(+\infty\). Assuming that \( g_M = \Phi_0 \) we have an unit of flux along the \( z \)-axis, as shown in Fig. 3. The flux lines diverge (converge) from (to) the south (north) monopoles. The flux due to the field lines coming out (or in) of each monopole is equally divided 1/2 (−1/2) between the north and the south. In addition we have to take into account the flux carried by the strings. Therefore considering any plane perpendicular to the \( z \)-axis, the flux traversing it will contain a flux quantum. In the south part the contributions add like: string (+1 flux), south monopole (−1/2) and north monopole (+1/2). In the region between the monopoles where a flux tube confined by the superconductor will appear we have: south monopole (+1/2) and north monopole (−1/2). Notice that with this choice there is no string traversing an horizontal plane when \( z_1 < z < z_2 \). In the north part the contributions are such: string (+1 flux), south monopole (+1/2) and north monopole (−1/2). Therefore it always adds to a unit of flux.

We can as well consider that the Dirac string has a finite extension and goes from the north monopole to the

\[
\int_0^h dz \int_0^R d\rho \rho \phi_{j,\mu \pm n/2, k}(\rho, z) A_{\varphi}(\rho, z)^2 \phi_{j',\mu \pm n/2, k'}(\rho, z)
\]

(21)

and therefore we have

\[
\Delta(\rho, z) \phi_{j',\mu \pm n/2, k'}(\rho, z)
\]

(22)

Since we can relate \( g_M \) with the total flux \( \Phi \) coming out from a monopole where \( \Phi = n^{\text{ext}} \Phi_0 \). Then it can be shown [30] that the BdG equations can be recast in a form such that they only depend on the difference \( n_a = n - n^{\text{ext}} \) between the vorticity of the gap function \( n \) and the vorticity of the external line \( n^{\text{ext}} \). In the BdG equations we just have to replace \( n \to n_a \) and the effect of the external infinitely thin solenoid is taken into account exactly if \( n^{\text{ext}} \) is an integer. In our case the vector potential is more complex.
south monopole, as shown as well in Fig. 3. In this case it is easy to see that the total flux is zero. It seems therefore natural to look for solutions where in the first case \( n = 1 \) and in the second case \( n = 0 \), reflecting the different total magnetic fluxes.

In the first case it is easy to see that the effect of the strings is such that, in an approximate way, in the region \( z < z_1 \) close to the string and far from the monopole, the BdG equations are approximately described by the equations where the effect of the external vector potential is equivalent to \( n_a = 0 \), since we choose \( n = 1 \) and the singular contribution from the string is described by \( n_{\text{ext}} = 1 \). A similar analysis for \( z > z_2 \) leads to the same conclusion. Between the monopoles a finite width flux tube forms. As shown in ref. 30 these field lines are not singular and the vorticity of the equations is not affected: there is no string in this region. Therefore \( n_a = n - n_{\text{ext}} = 1 - 0 = 1 \). A similar analysis for the second string configuration leads to a situation where for \( z < z_1 \) and \( z > z_2 \) we have an effective \( n_a = 0 \) and between the monopoles we have \( n_a = 1 \). In this case there is a string between the monopoles (with \( n_{\text{ext}} = -1 \)). Therefore the two configurations are equivalent as expected 8.

C. Numerical method

The considerations above only apply to the singular part of the vector potential due to the strings. The singular part effectively changes the set of basis functions with the replacement \( \mu \pm n/2 \) to \( \mu \pm n_a/2 \). Therefore to take advantage of this basis we just add and subtract to the external vector potential the limiting singular expressions due to the strings. The remaining part of the vector potential is regular. It has been shown by several authors that in this case the contribution of the terms involving the vector potential in the BdG equations is negligible or at least very small. In the following we neglect \( I_1 \) and \( I_2 \). We may therefore divide the system into two regions, using the inversion symmetry around \( z = 0 \).

For \( |z| < |z_1| \) we solve the BdG equations taking a basis with \( n_a = 1 \) (equivalent to the problem of a singly quantized vortex line with no external vector potential) and for \( |z| > |z_1| \) we use a basis with \( n_a = 0 \). We should note that, as shown in ref. 28, we have to consider large systems and consequently a large basis of functions has to be used to properly obtain the exponential decay of the various physical quantities. Otherwise, the exponential regime is not reached. This implies that solving the full 3d problem requires very large matrices to be diagonalized. Therefore we have to resort to this approximation for the \( z \) dependence.

In this approximation the BdG Hamiltonian in each region are plane waves along \( z \). Therefore the potential \( \Delta \) does not depend on \( z \) and the BdG Hamiltonian is independent of \( z \). It may be block diagonalized in terms of the momentum along the \( z \) direction (indexed by \( k \)). In this way the components of the Hamiltonian are reduced to

\[
T^\pm_{jj'} = \pm \frac{\hbar^2}{2m} \left( \frac{\alpha^2}{R^2} + \frac{4k^2\pi^2}{\hbar^2} \right) \delta_{jj'} \mp E_F
\]

\[
\Delta_{jj'} = \int_0^R \phi_{j,\mu-n_a/2}(\rho) \Delta(\rho) \phi_{j',\mu+n_a/2}(\rho) \rho d\rho
\]

It is important to note that the symmetry of the BdG equations \( u_i(\mathbf{r}) \rightarrow v_i^*(\mathbf{r}), v_i(\mathbf{r}) \rightarrow -u_i^*(\mathbf{r}) \) and \( E_i \rightarrow -E_i \) allows to reduce the solution to the positive values of \( \mu \). We obtain the eigenvectors and eigenvalues for negative values of \( \mu \) using the above symmetry.

We may find the internal vector potential solving Poisson’s equation \( \nabla^2 A = -\frac{4\pi}{c} J_\varphi \). Defining \( a_\varphi = F(\rho)/\rho \), Poisson’s equation reduces in cylindrical coordinates to

\[
\frac{\partial^2 F}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial z^2} = -\frac{4\pi}{c} J_\varphi
\]

Decomposing the current as

\[
-\frac{4\pi}{c} J_\varphi \rho = K(\rho, z) + \beta(\rho, z) F(\rho, z)
\]
FIG. 5: Magnetic field profile for the total field (left panels) and the internal field (right panels). We consider the cases of $n = 1$ (upper panels) and $n = 0$ (lower panels). Please note that the magnitude of the field is not visible here when it is smaller than the size of the arrows. This information is contained in Fig. 4. The size of the arrows is chosen for better visualization of the field direction at each point.

with

$$K(\rho, z) = -\frac{4\pi}{c} \sum_i f(E_i)|u_i|^2(\mu - n_a/2)$$  

$$- (1 - f(E_i))|v_i|^2(\mu + n_a/2) + \beta(\rho) \left( A_{ext} - a \right) \rho, \quad (30)$$

$$\beta(\rho) = -\frac{4\pi |e|}{\hbar c} \sum_i f(E_i)|u_i|^2 + (1 - f(E_i))|v_i|^2, \quad (31)$$

we get

$$\frac{\partial^2 F}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial z^2} = K(\rho, z) + \beta(\rho)F(\rho). \quad (32)$$

These equations may be solved as described in ref. 30.

III. RESULTS FOR $T = 0$

The parameters used are $\epsilon_F = 0.5, \omega_D = 0.25, g/\pi = 1$ in atomic units. We should note that the relevant parameter is the ratio to the critical temperature. For these parameters the critical temperature is of the order of 0.06. We consider the two monopoles in a system with height 960 and radius 480. The distance between the monopoles, $d$, is varied. Let us consider that the strings carry a quantum of flux.

In Fig. 4 we compare the absolute value of the total magnetic field for the cases of no superconductor, and in the superconducting phase considering a $n = 1$ solution and comparing to the case of $n = 0$. The case of no-superconductor is the usual magnetic dipole. Notice that the $n = 1$ and the $n = 0$ cases also correspond to
two different configurations of confinement. The magnetic vortex provides a linear confining potential. In the $n = 1$ configuration the monopole is attracted by the antimonopole, this is the standard picture of confinement, say in a mesonic quark-antiquark system in QCD. In the $n = 0$ configuration the monopole is attracted by the boundary of the superconductor, not by the antimonopole. The $n = 0$ case simulates a situation where the magnetic solenoids are being inserted from the border of the material. When the solenoids are far enough, it is favourable for the flux lines to emerge from the material. As the monopoles get close together it is more favourable to establish a $n = 1$ flux tube between the monopoles.

Taking the solution of $n = 1$ we look for a solution where the flux is quantized and equal to a quantum of flux. Far from either monopole and close to the strings the magnetic field is small. In the case of $n = 0$ there is no flux through the system and the flux lines leave the superconductor from the monopoles to the border of the system. In Fig. 4 we show the total and internal magnetic fields for both $n = 1$ and $n = 0$. This figure provides complementary information with respect to Fig. 5. In the case of $n = 1$ the presence of the flux tube between the monopoles is evident. In the case of $n = 0$ the field lines avoid the central region except along the borders, as expected. The diamagnetic behaviour of the internal field, obtained subtracting the external field from the total field is clearly displayed in the right panels of Fig. 5. In the case of $n = 1$ the internal field close to the monopoles is not strong enough to compensate the external field. Note however that along the flux tube the internal field has the same orientation as the total field. Interestingly the internal field in the vicinity of the $z = 0$ region and far from the axis winds clockwise and counter-clockwise. In the case of $n = 0$ once again along the flux tubes the internal field has the same orientation as the total field. Between the monopoles the internal field opposes the total field, as expected. The diamagnetic behaviour of the internal field, obtained subtracting the external field from the total field is clearly displayed in the right panels of Fig. 5. In the case of $n = 1$ the internal field close to the monopoles is not strong enough to compensate the external field. Note however that along the flux tube the internal field has the same orientation as the total field. Interestingly the internal field in the vicinity of the $z = 0$ region and far from the axis winds clockwise and counter-clockwise. In the case of $n = 0$ once again along the flux tubes the internal field has the same orientation as the total field. Between the monopoles the internal field opposes the total field, as expected.

Clearly it is interesting, in the analogy of the Anderson-Higgs mechanism to the confinement of quarks, to determine the confinement potential between the two monopoles. Taking the analogy of the monopoles with the quarks in QCD the fact that these are linearly confined at large distances should translate in the superconductor to a linear potential energy when the monopoles are moved apart. Indeed in Fig. 6 we show the magnetic energy of the system,

$$ U = \int \frac{|B|^2}{8\pi} dV, \quad (33) $$

as a function of the distance between the monopoles. The behaviour is similar to the one proposed for the confinement problem in QCD. We show the difference in energy between the normal system and the superconductor as a function of $d$.

Interestingly, the solenoid-antisolonenoid magnetic energy includes a subtle point. To illustrate it let us now perform an analytical computation of the energy of the solenoid-antisolenoid pair in the normal (no superconductor) phase. The magnetic field $\mathbf{B}$ can be decomposed in the sum of the field created by the north antisolenoid 1 and the south solenoid 2. In the limit of infinitely thin solenoids, each with a quantum of magnetic flux, the respective fields are equivalent to a monopole field plus a string field,

$$ \mathbf{B}_1 = \frac{-g_m}{4\pi|\mathbf{r} - \mathbf{r}_0|^2} (\mathbf{r} - \mathbf{r}_0 - g_m \delta (x) \delta (y) \theta (z - z_0) (-\mathbf{e}_z), $$

$$ \mathbf{B}_2 = \frac{+g_m}{4\pi|\mathbf{r} + \mathbf{r}_0|^2} (\mathbf{r} + \mathbf{r}_0 + g_m \delta (x) \delta (y) \theta (-z - z_0) (\mathbf{e}_z). $$

(34)

The $B_1^2 + B_2^2$ contribution is infinite but it is also constant and independent of the distance. The relevant non-constant term comes from the mixed term. Then we get two different contributions for the energy. The energy of the mixed monopole-antimonopole term is similar to the usual electrostatic Coulomb energy,

$$ E_{\text{mono-antimono}} = -\frac{g_m^2}{4\pi|2z_0|.} \quad (35) $$

![FIG. 6: (colour online) a) Comparison of the total magnetic energy in the normal phase and in the superconducting phase. Also, we present the energy difference between the superconducting phase and the normal phase. b) Energy difference between the superconductor and the normal phase as a function of the distance between the monopoles plus the Coulomb term.](image)
However the energy also includes the monopole-string terms $B_{1 \text{monopole}} \cdot B_{2 \text{string}} + B_{1 \text{string}} \cdot B_{2 \text{monopole}}$. An inspection of the directions of the magnetic fields shows that these two terms don’t cancel, in fact they contribute equally. We get,

$$2 E_{\text{string-antimonopole}} = 2 \int_{z_0}^{\infty} dz \frac{+g_m}{4\pi(z + z_0)^2}(\hat{e}_z \cdot \hat{e}_z)g_m$$

$$= 2 \frac{g_m^2}{4\pi} \left( \frac{-1}{z + z_0} \right)_{z_0}^{\infty}$$

$$= 2 \frac{g_m^2}{4\pi |2z_0|} . \quad (36)$$

Summing the two contributions the total energy is,

$$U = + \frac{g_m^2}{4\pi |2z_0|} + \text{const.} . \quad (37)$$

Notice that the final sign in eq. (37) is opposed to the normal coulomb energy of eq. (35). This occurs because any movement of the solenoid in the magnetic field of the other solenoid affects its magnetization and therefore changes its internal energy. In the present case, the magnetization of both solenoids has to be maintained constant, and this requires an extra energy source, say a battery, thus affecting the energy balance of the system. Nevertheless the solenoid-antisolenoide magnetic force is attractive, and quite similar to the charge-anticharge electrostatic force.

Indeed in Fig. 6 a) we depict the magnetic energy of the solenoid-antisolenoide system both in the normal phase (no superconductor) and in the superconductor phase. Although our solenoids are thinner than both the coherence length and the penetration length, they have a finite width, and we are able to account for the inside (string-like) magnetic field created by the solenoids. Notice that the $\frac{1}{r}$ component of the magnetic energy is decreasing. To isolate the linear potential we also show in Fig. 6 a) the difference between the magnetic energy in the two phases. To get the monopole-antimonopole interaction, independently of the string internal energy, we also add to the linear interaction of Fig. 6 a) the monopole-antimonopole Coulomb interaction of eq. (35). This produces the potential depicted in 6 b), similar to the linear+Coulomb potential of QCD.

IV. RESULTS FOR $T \neq 0$

In Figs. 7 and 8 we study the influence of temperature on the total field. As the temperature increases the flux tube between the two monopoles narrows in the intermediate region and close to the critical temperature, where the superconducting order vanishes, it approaches the normal phase regime where close to the monopoles the field intensity extends away from the axis. As the temperature increases the flux tubes narrow considerably until it breaks down when the superconductor becomes unstable and the system becomes normal. This happens both for $n = 1$ and $n = 0$. Again, the $n = 0$ case simulates a situation where the magnetic solenoids are being
inserted from the border of the material such that it is favourable for the flux lines to emerge from the material. As the monopoles get closer it is more favourable to establish a flux tube between the monopoles.

We may consider different planes perpendicular to the string axis. The flux contained in any plane perpendicular to the strings axis is constant. In the regions far from the monopoles the flux is almost entirely contained in the strings. This is shown in Fig. 9. The flux profile is shown along a plane that intercepts the string and it is clear that the flux reaches its asymptotic value very close to the string. In the region between the monopoles the flux lines emerge and converge in the opposite charge monopole. In Fig. 9 it is clear that the flux grows smoothly from the origin, since in this case there is a considerable value of the magnetic field far from the axis. Also, we present results for $B_z$ and $J_\phi$ also as a function of temperature.

In the central equatorial region ($z = 0$) the magnetic field is maximum on the axis and then decreases rapidly, on the scale of the penetration length. The field is compensated by the supercurrents that also decay rapidly as we move away from the axis.

In a plane such that we are already on the strings, the field is finite and negative at the axis but increases to zero far from the string. There is a cancellation between the field of the string and the field of the monopoles, as explained above. The supercurrents vanish on the string, then go through a maximum at a finite distance from the axis and then decrease far away. Note that the flux is already saturated on the axis, then decreases and the asymptotic value is recovered far from the axis.

As the temperature increases the penetration length increases. As a consequence both the increase and decrease of the field with distance sets on a larger scale as the temperature increases. This is clearly shown on the various quantities plotted in Fig. 9.

Finally, we consider in Fig. 10 the temperature dependence of the string tension. Also, we show the temperature dependence of the bulk value of the pairing function. We see that as the temperature increases both the pairing function and the string tension decrease, particularly as the critical temperature is approached. This critical temperature is defined as the temperature when the pairing function vanishes. Note that Fig. 10 shows that the string tension is already quite small when the pairing function is still finite, even though small and close to the critical temperature.

V. CONCLUSION

Motivated by the superconductor model for the confinement in QCD, we extended the Ball and Caticha work to include both the microscopic fermionic degrees of freedom of the superconductor and finite temperature. We addressed two confining configurations, the $n = 1$ configuration where the monopole is attracted by the antimonopole, and the $n = 0$ configuration where the monopole is attracted by the boundary of the superconductor. Our framework are the Bogoliubov-de Gennes
equations solved in the external field lines of a monopole-antimonopole pair, including the associated semi-infinite magnetic strings.

Notice that this is not yet QCD, here we solve a condensed matter system. The advantage of studying a solenoid-antisolenoid pair in a superconductor is the more detailed nature of our results. For instance, we identify the well known localized fermionic states (unattainable via the Ginzburg-Landau approach [31]), and we analyze the dependence on the temperature of the string tension $\sigma$, of the fermion pairing order parameter $\Delta$ and of the magnetic topological flux $\Phi$.

Nevertheless, this study is inspiring for the understanding of confinement and of temperature and density effects in QCD or in Abelian QED [44, 45]. These effects are usually explored with Lattice QCD, Schwinger-Dyson equations, or the Ginzburg-Landau equations. We note that the present framework can also be extended to model QCD.

Importantly, this extension would be relevant to color superconductivity, where the quark density is large, and to the quark-gluon plasma, where the temperature is finite.

Localized electron states exist in the region where the supercurrents provide a topological charge. As expected in the Bogoliubov-de Gennes framework, we obtain one fermionic localized state per angular momentum $\ell$.

We find that the magnetic field in the vortex, say in the equatorial plane for $n=1$, decreases with increasing temperature. In what concerns the penetration length and the topological magnetic flux, it is seen from the results that the penetration length increases with temperature (as is well known), while the topological magnetic flux decreases. This decreases the string tension of the long distance monopole-antimonopole linear attraction. Notice that the the string tension is already quite small when the pairing function is still finite, even though small and close to the critical temperature.
FIG. 10: a) Order parameter $\Delta$ as a function of temperature, b) String tension $\sigma$ as a function of temperature.