STRONG SOLUTIONS TO STOCHASTIC HYDRODYNAMICAL SYSTEMS WITH MULTIPLICATIVE NOISE OF JUMP TYPE

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Abstract. In this paper we prove the existence and uniqueness of maximal strong (in PDE sense) solution to several stochastic hydrodynamical systems on unbounded and bounded domains of $\mathbb{R}^n$, $n = 2, 3$. This maximal solution turns out to be a global one in the case of 2D stochastic hydrodynamical systems. Our framework is general in the sense that it allows us to solve the Navier-Stokes equations, MHD equations, Magnetic Bénard problems, Boussinesq model of the Bénard convection, Shell models of turbulence and the Leray-$\alpha$ model with jump type perturbation. Our goal is achieved by proving general results about the existence of maximal and global solution to an abstract stochastic partial differential equations with locally Lipschitz continuous coefficients. The method of the proofs are based on some truncation and fixed point methods.

1. Introduction

Stochastic Partial Differential Equations (SPDEs) are a powerful tool for understanding and investigating mathematically hydrodynamic and turbulence theory. To model turbulent fluids, mathematicians often use stochastic equations obtained from adding a noise term in the dynamical equations of the fluids. This approach is basically motivated by Reynolds’ work which stipulates that turbulent flows are composed of slow (deterministic) and fast (stochastic) components. Recently by following the statistical approach of turbulence theory, Flandoli et al [23], Kupiainen [32] confirm the importance of studying the stochastic version of fluids dynamics. Indeed, the authors of [23] pointed out that some rigorous information on questions of turbulence theory might be obtained from these stochastic versions. It is worth emphasizing that the presence of the stochastic term (noise) in these models often leads to qualitatively new types of behavior for the processes. Since the pioneering work of Bensoussan and Temam [4], there has been an extensive literature on stochastic Navier-Stokes equations with Wiener noise and related equations, we refer to [1], [2], [5], [6], [16], [19], [24], [42] amongst other.

In the last five years, there has been an extensive effort to tackle SPDEs with Lévy noise. There are several examples where the Gaussian noise is not well suited to represent realistically external forces. For example, if the ratio between the time scale of the deterministic part and that of the stochastic noise is large, then the temporal structure of the forcing in the course of each event has no influence on the overall dynamics, and - at the time scale of the deterministic process - the external forcing can be modeled as a sequence of episodic instantaneous impulses. This happens for example in Climatology (see, for instance, [29]). Often the noise observed by time series is typically asymmetric, heavy-tailed and has non trivial kurtosis. These are all features which cannot be captured by a Gaussian noise, but rather by a Lévy noise with appropriate parameters. Lévy randomness requires different techniques from the ones used for Brownian motion and are less amenable to mathematical analysis. We refer to [9], [11],[20], [28] and [36] that deal with stochastic hydrodynamical systems driven by Lévy type noise. Most of these articles are about the existence of solution which are weak in the PDEs sense.

2000 Mathematics Subject Classification. 60H15, 35Q35, 60H30, 35R15.

Key words and phrases. Strong solution, Hydrodynamical systems, Navier-Stokes, MHD, Bénard convection, Boussinesq equations, Shell models, Leray-$\alpha$, Levy noise, Poisson random measure.
In this paper, we are interested in proving the existence and uniqueness of maximal and global strong solution of Lévy driven hydrodynamical systems such as the Navier-Stokes equations (NS), Magnetohydrodynamics equations (MHD), Magnetic Bénard problem (MB), Boussinesq model for Bénard convection (BBC), Shell models of turbulence, and 3-D Leray-α for Navier-Stokes equations. Here, strong solutions should be understood in both the Probability and PDEs senses. Our objectives are achieved by adopting the unified approach initiated and developed in [16] and used later in [9]. This approach is based on rewriting the various equations above into an abstract stochastic evolution equations in a Hilbert space $V$ of the following form

$$u(t) = u_0 - \int_0^t [Au(s) + F(u(s))] \, ds + \int_0^t \int Z G(z, u(s)) \tilde{\eta}(dz, ds), \quad (1.1)$$

where $\int Z G(z, u(s)) \tilde{\eta}(dz, ds)$ represents a global Lipschitz continuous multiplicative noise of jump type. In Theorem 3.5 we give sufficient conditions (on $A$ and $F$) for the existence and uniqueness of a maximal solution to (1.1). Sufficient conditions for non-explosion of the maximal solution in finite time is given in Theorem 3.7. These two theorems are our main results and their assumptions are carefully chosen so that they are verified by the NS, MHD, MB, BBC, Shell models and the Leray-α models. We give a detailed account of this discussion in Section 4.

The book [40] contains several results about existence of solution to abstract SPDEs driven by Lévy noise in Hilbert space setting, but the hypotheses in this book do not cover the various hydrodynamical systems that we enumerated above. We also note that while there are several results about the existence of solution which are strong in PDEs sense for stochastic hydrodynamical systems perturbed by Wiener noise (see, for instance, [3], [8], [37], [31], [35], [42] and references therein), it seems that this is the first paper treating the existence of strong (in PDE sense) solution for stochastic hydrodynamical systems with Lévy noise. To prove our results we closely follow [10] (see also [8]) which in turn followed methods elaborated in two papers by De Bouard and Debussche [17, 18].

The layout of the present paper is as follows. In Section 2, we introduce the abstract stochastic evolution equation that our result will be based on. In the very section we give the notations and standing assumptions, and prove some preliminary results that are very important for our analysis. Section 3 is devoted to the statements and the proofs of our main results. We will mainly show that under the assumptions introduced in Section 2 the Eq. (1.1) admits a unique maximal local solution, and with additional conditions on $F$ and $G$ we prove that this maximal local solution turns out to be a global one. The results are obtained by use of cut-off and fixed point methods introduced in [10]. In Section 4 we give a detailed discussion on how our abstract results are used to solve the stochastic NS, MHD, MB, BBC, Shell models and Leray-α models driven by multiplicative noise of jump type. In appendix we prove the well-posedness of a linear stochastic evolution equations driven by compensated Poisson random measure which is very important for our analysis.

2. Description of an abstract stochastic evolution equation

In this paper we give the necessary notations and standing assumptions used throughout the paper. We also prove some preliminary results that are very important for our analysis.

2.1. Notations and Preliminary results. In this section we start with some notations, then introduce the assumptions used throughout the paper and our abstract stochastic equation.

Let $(V, \|\cdot\|)$, $(H, \|\cdot\|)$ and $(E, \|\cdot\|_*)$ be three separable and reflexive Hilbert spaces. The scaler product in $H$ is denoted by $\langle u, v \rangle$ for any $u, v \in H$. The same symbol $\langle \phi, v \rangle$ will also be used to denoted duality pairing of $\phi \in V^*$ and $v \in V$. We denote by $L(Y_1, Y_2)$ be the space of bounded linear maps from a Banach space $Y_1$ into another Banach space $Y_2$. 

For $T_2 > T_1 \geq 0$ we set

$$X_{T_1, T_2} = L^\infty(T_1, T_2; V) \cap L^2(T_1, T_2; E),$$

(2.1)

with the norm $\|u\|_{X_{T_1, T_2}}$ defined by

$$\|u\|_{X_{T_1, T_2}}^2 = \sup_{s \in [T_1, T_2]} \|u_n(s)\|^2 + \int_{T_1}^{T_2} \|u_n(s)\|^2_{\ast} ds.$$  

(2.2)

For $T_1 = 0$ and $T_2 = T > 0$ we simply write $X_T := X_{0, T}$.

Let $Y$ be a separable and complete metric space and $T > 0$. The space $D([0, T]; Y)$ denotes the space of all right continuous functions $x : [0, 1] \rightarrow Y$ with left limits. We equip $D([0, T]; Y)$ with the Skorohod topology in which $D([0, T]; Y)$ is both separable and complete. For more information about Skorohod space and topology we refer to Ethier and Kurtz [22].

Let $(Z, \mathcal{Z})$ be a separable metric space and let $\nu$ be a $\sigma$-finite positive measure on it. Suppose that $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space, where $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration, and $\eta : \Omega \times B(\mathbb{R}_+) \times Z \rightarrow \mathbb{N}$ is a time homogeneous Poisson random measure with the intensity measure $\nu$ defined over the filtered probability space $\mathcal{P}$. We will denote by $\tilde{\eta} = \eta - \gamma$ the compensated Poisson random measure associated to $\eta$ where the compensator $\gamma$ is given by

$$B(\mathbb{R}_+) \times Z \ni (I, A) \mapsto \gamma(I, A) = \nu(A)\lambda(I) \in \mathbb{R}_+.$$  

For each Banach space $B$ we denote by $M^2(0, T; B)$ the space of all progressively measurable $B$-valued processes such that

$$\|u\|^2_{M^2(0, T; B)} = \mathbb{E} \int_0^T \|u(s)\|^2_B ds < \infty.$$  

Throughout the paper, let us denote by $M^2(X_T)$, the space of all progressively measurable $V \cap H$-valued processes whose trajectories belong to $X_T$ almost surely, endowed with a norm

$$\|u\|^2_{M^2(X_T)} = \mathbb{E} \left[ \sup_{s \in [0, T]} \|u(s)\|^2 + \int_0^T \|u(s)\|^2_{\ast} ds \right].$$  

(2.3)

Let $H$ be a separable Hilbert space. Following the notation of [7], let $\mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ be the class of all progressively measurable processes $\xi : \mathbb{R}_+ \times Z \times \Omega \rightarrow H$ satisfying the condition

$$\mathbb{E} \int_0^T \int_Z |\xi(r,z)|^2_{\ast} \nu(dz) dr < \infty, \quad \forall T > 0.$$  

(2.4)

If $T > 0$, the class of all progressively measurable processes $\xi : [0, T] \times Z \times \Omega \rightarrow H$ satisfying the condition (2.4) just for this one $T$, will be denoted by $\mathcal{M}^2(0, T, L^2(Z, \nu, H))$. Also, let $\mathcal{M}^2_{\text{step}}(\mathbb{R}_+, L^2(Z, \nu, H))$ be the space of all processes $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ such that

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1}, t_j]}(r)\xi_j, \quad 0 \leq r,$$

where $\{0 = t_0 < t_1 < \ldots < t_n < \infty\}$ is a partition of $[0, \infty)$, and for all $j$, $\xi_j$ is an $\mathcal{F}_{t_{j-1}}$ measurable random variable. For any $\xi \in \mathcal{M}^2_{\text{step}}(\mathbb{R}_+, L^2(Z, \nu, H))$ we set

$$\bar{I}(\xi) = \sum_{j=1}^n \int_Z \xi_j(z)\tilde{\eta} (dz, (t_{j-1}, t_j]).$$  

(2.5)

Basically, this is the definition of stochastic integral of a random step process $\xi$ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ is possible thanks to the following result which is taken from [7, Theorem C.1].
Theorem 2.1. There exists a unique bounded linear operator

\[ I : \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu; H)) \to L^2(\Omega, F; H) \]

such that for \( \xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu; H)) \) we have \( I(\xi) = \tilde{I}(\xi) \). In particular, there exists a constant \( C = C(H) \) such that for any \( \xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu; H)) \),

\[ \mathbb{E}\left| \int_0^t \int_Z \xi(r, z)\hat{\eta}(dz, dr) \right|^2 \leq C \mathbb{E} \int_0^t \int_Z |\xi(r, z)|^2 H_\nu(dz) \, dr, \quad t \geq 0. \]  

(2.6)

Moreover, for each \( \xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu; H)) \), the process \( I(1_{[0,t]}\xi), t \geq 0 \), is an \( H \)-valued càdlàg martingale. The process \( 1_{[0,t]}\xi \) is defined by \( 1_{[0,t]}\xi(r, z, \omega) := 1_{[0,t]}(r)\xi(r, z, \omega), \quad t \geq 0, \quad r \in \mathbb{R}_+, \) \( z \in Z \) and \( \omega \in \Omega \).

As usual we will write

\[ \int_0^t \int_Z \xi(r, z)\hat{\eta}(dz, dr) := I(\xi)(t), \quad t \geq 0. \]

Now we introduce the following standing assumptions.

Assumption 2.1. We will identify \( H \) with its dual \( H^* \), and we assume that the embeddings

\[ E \subset V \subset H \subset V^* \subset E^* \]

are continuous and dense.

Let \( N \) be a self-adjoint operator on \( H \) such that \( N \in \mathcal{L}(E, H) \cap \mathcal{L}(V, V^*) \). Also let \( A \) be a bounded linear map from \( E \) into \( H \). We assume that there exist \( C_N, C_A > 0 \) such that

\[ \langle Au, Nu \rangle \geq C_A\|u\|^2 \quad \text{and} \quad \langle Nu, u \rangle \geq C_N\|u\|^2, \]

for any \( u \in V \). The norm of \( N \in \mathcal{L}(E, H) \) and \( N \in \mathcal{L}(V, V^*) \) will be denoted respectively by \( \|N\|_{\mathcal{L}(E,H)} \) and \( \|N\|_{\mathcal{L}(V,V^*)} \) throughout.

Let \( F \) and \( G \) be two nonlinear mappings satisfying the following sets of conditions.

Assumption 2.2. Suppose that \( F : E \to H \) is such that \( F(0) = 0 \) and there exists \( p \geq 1, \alpha \in [0,1) \) and \( C > 0 \) such that

\[ |F(y) - F(x)| \leq C \left[ \|y - x\|_p \|y\|^{p-\alpha} \|y\|_V^{\alpha} + \|y - x\|_V \|y - x\|_V^{1-\alpha} \|x\|_p \right], \]

(2.7)

for any \( x, y \in E \).

Assumption 2.3. (i) Assume that \( G : V \to L^{2p}(Z, \nu, V) \) and there exists a constant \( \ell_p > 0 \) such that

\[ \|G(x) - G(y)\|_{L^{2p}(Z,\nu,V)} \leq \ell_p\|x - y\|_V^{2p}, \]

(2.8)

for any \( x, y \in V \) and \( p = 1,2 \).

Note that this implies in particular that there exists a constant \( \ell_p > 0 \) such that

\[ \|G(x)\|_{L^{2p}(Z,\nu,V)} \leq \ell_p(1+\|x\|_V^{2p}), \]

(2.9)

for any \( x \in V \) and \( p = 1,2 \).

(ii) We also assume that \( G \) satisfies the inequality (2.7) with the norm of \( V \) replaced by the norm of \( H \). More precisely, there exists \( \ell_p > 0 \) such that

\[ \|G(x) - G(y)\|_{L^{2p}(Z,\nu,H)} \leq \ell_p\|x - y\|_H^{2p}, \]

(10.2)

for any \( x, y \in V \) and \( p = 1,2 \).
Throughout this work we fix a positive number $T$. One of our objectives is to prove the existence and uniqueness of maximal/local solution of the following stochastic evolution equation

$$u(t) = u_0 - \int_0^t \left[Au(s) + F(u(s))\right] ds + \int_0^t \int_Z G(z, u(s)) \tilde{\eta}(dz, ds).$$  \tag{2.11}

The above identity is the shorthand of the following identity

$$\langle u(t), v \rangle = \langle u_0, v \rangle - \int_0^t \langle \left[Au(s) + F(u(s))\right], v \rangle ds + \int_0^t \int_Z \langle G(u(s)), v \rangle \tilde{\eta}(dz, ds),$$  \tag{2.12}

for any $t \in [0, T]$ and $v \in H$.

Now, let us introduce the concept of local and maximal local solution.

**Definition 2.2** (Local solution). By a local solution of (2.11) we mean a pair $(u, \tau_\infty)$ such that

1. the symbol $\tau_\infty$ is a stopping time such that $\tau_\infty \leq T$ a.s. and there exists a nondecreasing sequence $\{\tau_n, n \geq 1\}$ stopping times with $\tau_n \uparrow \tau_\infty$ a.s. as $n \uparrow \infty$,
2. the symbol $u$ denotes a progressively measurable stochastic process such that $u \in X_t$ a.s. for any $t \in [0, \tau_\infty)$ and

$$u(t \wedge \tau_n) = u_0 - \int_0^{t \wedge \tau_n} \left[Au(s) + F(u(s))\right] ds + \int_0^{t \wedge \tau_n} \int_Z G(z, u(s)) \tilde{\eta}(dz, ds),$$  \tag{2.13}

holds for any $t \in [0, T]$ and $n \geq 1$ with probability 1.

The identity (2.13) is the shorthand of the following

$$\langle u(t \wedge \tau_n), v \rangle = \langle u_0, v \rangle - \int_0^{t \wedge \tau_n} \langle \left[Au(s) + F(u(s))\right], v \rangle ds + \int_0^{t \wedge \tau_n} \int_Z \langle G(u(s)), v \rangle \tilde{\eta}(dz, ds),$$  \tag{2.14}

holds for any $t \in [0, T], v \in H$, and $n \geq 1$ with probability 1.

We also define the maximal local solution to (2.11).

**Definition 2.3** (Maximal local solution). (1) Let $(u, \tau_\infty)$ be a local solution to (2.11) such that$

\lim_{\tau \nearrow \tau_\infty} \|u\|_{X_t} = \infty$ on $\{\omega, \tau_\infty < T\}$, then the local process $(u, \tau_\infty)$ is called a maximal local solution. If $\tau_\infty < T$, then the stopping time $\tau_\infty$ is called the explosion time of the stochastic process $u$.

2. A maximal local solution $(u, \tau_\infty)$ is said unique if for any other maximal local solution $(v, \sigma_\infty)$ we have $\sigma_\infty = \tau_\infty$ and $u(t) = v(t)$ for any $0 \leq t < \tau_\infty$ with probability one.

3. If the explosion time of the stochastic process $u$ is equal to $T$ with probability 1, then the stochastic process $\{u(t), t \in [0, T]\}$ is called a global solution.

As in [8], we let $\theta : \mathbb{R}_+ \rightarrow [0, 1]$ be a $C_0^\infty$ non increasing function such that

$$\inf_{x \in \mathbb{R}_+} \theta'(x) \geq -1, \quad \theta(x) = 1 \text{ iff } x \in [0, 1] \quad \text{and} \quad \theta(x) = 0 \text{ iff } x \in [2, \infty).$$  \tag{2.15}

and for $n \geq 1$ set $\theta_n(x) = \theta(\frac{x}{n})$. Note that if $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non decreasing function, then for every $x, y \in \mathbb{R}$,

$$\theta_n(x)h(x) \leq h(2n), \quad |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x - y|.$$  \tag{2.16}

**Proposition 2.4.** Let $F$ be a nonlinear mapping satisfying Assumption 2.2. Let us consider a map $B_n^T : X_T \rightarrow L^2(0, T; H)$ defined by

$$B_n^T(u)(t) := \theta_n(\|u_0\|_{X_s})F(u(t)), \quad u \in X_T, \quad t \in [0, T].$$

Then $B_n^T$ is globally Lipschitz and moreover, for any $u_1, u_2 \in X_T$,

$$\|B_n^T(u_1) - B_n^T(u_2)\|_{L^2(0, T; H)} \leq C(2n)^p \left[(2n)C + 1\right] T^{1 - \alpha} \|u_1 - u_2\|_{X_T}.$$  \tag{2.17}
Note that by definition, if the set on the RHS above is empty, then by Assumption 2.2 $B^T_n(0) = 0$. Assume that $u_1, u_2 \in X_T$. Denote, for $i = 1, 2$,

$$
\tau_i = \inf\{t \in [0, T): \|u_i\|_{X_\tau} \geq 2n\}.
$$

Note that by definition, if the set on the RHS above is empty, then $\tau_i = T$. Without loss of generality we may assume that $\tau_1 \leq \tau_2$.

We have the following chain of inequalities/equalities

$$
\|B^T_n(u_1) - B^T_n(u_2)\|_{L^2(0, T; \mathbf{H})} = \left[ \int_0^T |\theta_n(\|u_1\|_{X_\tau}) - \theta_n(\|u_2\|_{X_\tau})| F(u_1(t)) - F(u_2(t))|^2 dt \right]^{1/2}
$$

because for $i = 1, 2$, $\theta_n(\|u_i\|_{X_\tau}) = 0$ for $t \geq \tau_2$

$$
= \left[ \int_0^{\tau_2} |\theta_n(\|u_1\|_{X_\tau}) - \theta_n(\|u_2\|_{X_\tau})| F(u_1(t)) - F(u_2(t))|^2 dt \right]^{1/2}
$$

$$
\leq \left[ \int_0^{\tau_2} |\theta_n(\|u_1\|_{X_\tau}) - \theta_n(\|u_2\|_{X_\tau})| F(u_1(t))|^2 dt \right]^{1/2}
$$

$$
+ \left[ \int_0^{\tau_2} |\theta_n(\|u_1\|_{X_\tau}) - \theta_n(\|u_2\|_{X_\tau})| F(u_2(t))|^2 dt \right]^{1/2}
$$

because for $i = 1, 2$, $\theta_n(\|u_i\|_{X_\tau}) = 0$ for $t \geq \tau_2$

$$
=: I_1 + I_2.
$$

Next, since $\theta_n$ is Lipschitz with Lipschitz constant $n^{-1}$ we have

$$
I_1^2 = \int_0^{\tau_2} |\theta_n(\|u_1\|_{X_\tau}) - \theta_n(\|u_2\|_{X_\tau})| F(u_1(t))|^2 dt
$$

$$
\leq n^{-2} C^2 \int_0^{\tau_2} \|u_1 - u_2\|^2_{X_\tau} |F(u_1(t))|^2 dt
$$

by Minkowski inequality

$$
\leq n^{-2} C^2 \int_0^{\tau_2} \|u_1 - u_2\|^2_{X_\tau} |F(u_2(t))|^2 dt \leq 4n^2 C^2 \int_0^{\tau_2} \|u_1 - u_2\|^2_{X_\tau} |F(u_2(t))|^2 dt
$$

$$
\leq n^{-2} C^2 \|u_1 - u_2\|^2_{X_\tau} \int_0^{\tau_2} |F(u_2(t))|^2 dt.
$$

Next, by assumptions

$$
\int_0^{\tau_2} |F(u_2(t))|^2 dt \leq C^2 \int_0^{\tau_2} \|u(t)\|^{2p+2-2\alpha} \|u(t)\|^{2\alpha} dt
$$

$$
\leq C^2 \sup_{t \in [0, \tau_2]} \|u(t)\|^{2p+2-2\alpha} \left( \int_0^{\tau_2} \|u(t)\|^{2\alpha} dt \right)^{\alpha - 1/2}
$$

$$
\leq C^2 \tau_2^{1-\alpha} \|u\|^{2p+2}_{X_{\tau_2}} \leq C^2 \tau_2^{1-\alpha} (2n)^{2p+2}.
$$

Therefore,

$$
I_1 \leq C^2 \tau_2^{(1-\alpha)/2} (2n)^p \|u_1 - u_2\|_{X_\tau}.
$$
Also, because \( \theta_n(\|u_1\|_{X_T}) = 0 \) for \( t \geq \tau_1 \), and \( \tau_1 \leq \tau_2 \), we have

\[
I_2 = \left[ \int_0^{\tau_2} \theta_n(\|u_1\|_{X_T}) \left[ F(u_1(t)) - F(u_2(t)) \right]^2 dt \right]^{1/2}
\]

\[
= \left[ \int_0^{\tau_1} \theta_n(\|u_1\|_{X_T}) \left[ F(u_1(t)) - F(u_2(t)) \right]^2 dt \right]^{1/2}
\]

because \( \theta_n(\|u_1\|_{X_T}) \leq 1 \) for \( t \in [0, \tau_1] \)

\[
\leq \left[ \int_0^{\tau_1} \|F(u_1(t)) - F(u_2(t))\|^2 dt \right]^{1/2}
\]

\[
\leq C \left[ \int_0^{\tau_1} \|u_1(t) - u_2(t)\|^2 \|u_1(t)\|^{2p-2\alpha} \|u_1(t)\|^{2\alpha} dt \right]^{1/2}
\]

\[
+ C \left[ \int_0^{\tau_1} \|u_1(t) - u_2(t)\|^{2\alpha} \|u_1(t) - u_2(t)\|^{2-2\alpha} \|u_2(t)\|^{2p} dt \right]^{1/2}
\]

\[
\leq C \sup_{t \in [0, \tau_1]} \|u_1(t) - u_2(t)\|^{1-\alpha} \|u_2(t)\|^p \left[ \int_0^{\tau_1} \|u_1(t) - u_2(t)\|^{2\alpha} dt \right]^{1/2}
\]

\[
+ C \sup_{t \in [0, T]} \|u_1(t) - u_2(t)\| \|u_1(t)\|^{p-\alpha} \left[ \int_0^{\tau_1} \|u_1(t)\|^{2\alpha} dt \right]^{1/2}
\]

\[
\leq C \sup_{t \in [0, T]} \|u_1(t) - u_2(t)\| \sup_{t \in [0, \tau_1]} \|u_1(t)\|^{p-\alpha} \left[ \int_0^{\tau_1} \|u_1(t)\|^{2\alpha} dt \right]^{1/2} \tau_1^{(1-\alpha)/2}
\]

\[
+ C \sup_{t \in [0, T]} \|u_1(t) - u_2(t)\|^{1-\alpha} \sup_{t \in [0, \tau_1]} \|u_2(t)\|^p \left[ \int_0^{\tau_1} \|u_1(t) - u_2(t)\|^{2\alpha} dt \right]^{1/2} \tau_1^{(1-\alpha)/2}
\]

\[
\leq C\|u_1 - u_2\|_{X_T} \|u_1\|_{X_{\tau_1}}^{\tau_1^{(1-\alpha)/2}} + C\|u_1 - u_2\|_{X_T} \|u_2\|_{X_{\tau_1}}^{\tau_1^{(1-\alpha)/2}}
\]

because \( \|u_1\|_{X_{\tau_1}} \leq 2n \) and \( \|u_2\|_{X_{\tau_1}} \leq 2n \)

\[
\leq C\tau_1^{(1-\alpha)/2} \|u_1 - u_2\|_{X_T} \left[ \|u_1\|_{X_{\tau_1}}^{\tau_1^{(1-\alpha)/2}} + \|u_2\|_{X_{\tau_1}}^{\tau_1^{(1-\alpha)/2}} \right] \leq C(2n)^{p+1} \tau_1^{(1-\alpha)/2} \|u_1 - u_2\|_{X_T}
\]

Summing up, we proved

\[
\|B_n^T(u_1) - B_n^T(u_2)\|_{L^2(0,T;H)} \leq \left[ C^2 \tau_2^{(1-\alpha)/2} (2n)^p + C(2n)^{p+1} \tau_1^{(1-\alpha)/2} \right] \|u_1 - u_2\|_{X_T}
\]

\[
= C(2n)^p [2nC + 1] \tau_2^{(1-\alpha)/2} \|u_1 - u_2\|_{X_T}
\]

The proof is complete.

\[ \Box \]

3. Existence of Maximal local and Global solution of Eq. (2.11)

This section is devoted to the solvability of (2.11). We will mainly show that under Assumption 2.1-2.3, Eq. (2.11) admits a unique maximal local solution. Under additional conditions on \( F \) and \( G \) we prove that this maximal local solution turns out to be a global solution. The results are obtained by use of cut-off and fixed point arguments.

3.1. Global solution of a truncated equation. For simplicity we set \( B_n^T(u)(s) = B_n^T(u(s)) \) for any \( u \in X_T \) and \( s \geq 0 \). Let

\[
u_n(t) + \int_0^t [Au_n(s) + B_n^T(u_n(s))] ds = u_0 + \int_0^t \int_Z G(z, u_n(s)) \eta(dz, ds), \quad t \in [0, T],
\]

(3.1)
which is understood as
\[
\langle u_n(t), v \rangle + \int_0^t \langle A u_n(s) + B_n^T(u_n(s)), v \rangle \, ds = \langle u_0, v \rangle + \int_0^t \int_Z \langle G(z, u_n(s)), v \rangle \tilde{\eta}(dz, ds), \quad t \in [0, T],
\]
for any \( v \in \mathbf{H} \). Here, we previously set
\[
B_n^T(u(t)) = \theta_n(\|u\|_{X_1}) F(u(t)),
\]
for any \( u \in X_t \) and \( t \geq 0 \). For \( n \in \mathbb{N} \) we also set
\[
\phi(n) = C^2(2n)^2 \left[ 2nC + 1 \right]^2.
\]
Now, let \( v \in M^2(X_T) \), \( n > 0 \) and let us consider the linear stochastic evolution equation
\[
\begin{cases}
du_n(t) + Au_n(t) dt = -B_n^t(v(t)) dt + \int_Z G(z, v(t)) \tilde{\eta}(dz, dt), \\
u_n(0) = u_0.
\end{cases}
\]
(3.4)

Thanks to Theorem A.1 for each \( v \in M^2(X_T) \) and \( n \geq 1 \), there exists a unique \( \nu \)-valued progressively measurable process \( u_n \) solving (3.4). Moreover, \( u^n \in D(0, T; \nu) \cap L^2(0, T; \nu) \) with probability 1.

**Lemma 3.1.** For each \( n \geq 1 \) let \( \Lambda_n \) be the mapping defined by
\[
\Lambda_n : M^2(X_T) \ni v \mapsto u_n = \Lambda_n(v),
\]
where \( u_n \) is the unique solution to (3.4). For any \( v \in M^2(X_T) \), the stochastic process \( u_n \) belongs to \( M^2(X_T) \).

**Proof.** Let \( \Psi : \mathbf{H} \to \mathbb{R} \) be the mapping defined by
\[
\Psi(u) = \langle u, Nu \rangle,
\]
for any \( u \in \mathbf{H} \). This mapping is Fréchet differentiable with first derivative defined by
\[
\Psi'(u)[h] = \langle h, Nu \rangle + \langle u, Nh \rangle.
\]
Since \( N \) is self-adjoint we have
\[
\Psi'(u)[h] = 2 \langle h, Nu \rangle.
\]
Applying Itô’s formula (see, for instance, [40, Appendix D]) to \( \Psi(u) \) with (3.4) we obtain
\[
\begin{align*}
\Psi(u_n(t)) - \Psi(u_0) + 2 \int_0^t \langle Au_n(s) + B_n^T(v(s)), Nu_n(s) \rangle \, ds \\
= \int_0^t \int_Z \left[ \Psi(u_n(s-)) + G(z, v(s)) \right] - \Psi(u_n(s-)) - \Psi'(u_n(s-)) [G(z, v(s))] \, \nu(dz) \, ds \\
+ \int_0^t \int_Z \Psi'(u_n(s-)) [G(z, v(s))] \, \tilde{\eta}(dz, ds).
\end{align*}
\]
(3.5)

From the Cauchy-Schwarz inequality we derive that
\[
\left| \int_0^t \langle B_n^T(v(s)), Nu_n(s) \rangle \, ds \right| \leq \int_0^t \| B_n^T(v(s)) \| \| Nu_n(s) \| \, ds,
\]
\[
\leq \| N \|_{L(E, H)} \int_0^t \| B_n^T(v(s)) \| \| u_n(s) \| \, ds.
\]
From the last line along with Cauchy’s inequality with \( \epsilon \) we deduce that
\[
\mathbb{E} \left| \int_0^t \langle B_n^T(v(s)), Nu_n(s) \rangle \, ds \right| \leq \epsilon \mathbb{E} \int_0^t \| u_n(s) \|^2 \, ds + \frac{\| N \|^2_{L(E, H)}}{4\epsilon} \mathbb{E} \int_0^t \| B_n^T(v(s)) \|^2 \, ds.
\]
Owing to the BDG inequality (see, for instance, [41, Theorem 4.8]) we infer that
\[
\mathbb{E} \left| \int_0^t \langle B_n^T(v(s)), Nu_n(s) \rangle ds \right| \leq \varepsilon \mathbb{E} \int_0^t \|u_n(s)\|^2 ds + \frac{\|N\|_{L(E,H)}^2 \phi(n)}{4\varepsilon} t^{\alpha-1} \|\nu\|^2_{M^2(X_T)}, \quad (3.6)
\]
where \( \phi(n) \) is defined in (3.3).

Now, note that
\[
\Psi(u + h) - \Psi(u) = \langle Nh, h \rangle.
\]
Hence
\[
I_1 := \mathbb{E} \left| \int_0^t \int Z \left[ (\Psi(u_n(s)) + G(z, v(s))) - (\Psi(u_n(s)) - \Psi'(u_n(s)) [G(z, v(s))] \nu (dz) ds \right] \right| 
\leq \|N\|_{L(\mathbb{V}, \mathbb{V}^\ast)} \mathbb{E} \int_0^t \int Z |G(z, v(s))|^2 \nu (dz) ds. \quad (3.7)
\]
By making use of (2.9) we easily derive from the last inequality that
\[
I_1 \leq t \|N\|_{L(\mathbb{V}, \mathbb{V}^\ast)} \tilde{E}_1 (1 + \|\nu\|^2_{M^2(X_T)}). \quad (3.8)
\]
Notice also that
\[
\Psi(u + h) - \Psi(u) = 2 \langle Nu, h \rangle + \langle Nh, h \rangle,
\]
thus
\[
I_2 := \mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s \int Z \left[ (\Psi(u_n(r)) + G(z, v(r))) - (\Psi(u_n(r)) - \Psi'(u_n(r)) [G(z, v(r))] \nu (dz, dr) \right] \right| 
\leq \mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s \int Z \langle Nu_n(s), G(z, v(s)) \rangle \tilde{\eta}(dz, ds) \right| + \mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s \int Z \langle NG(z, v(s)), G(z, v(s)) \rangle \tilde{\eta}(dz, ds) \right| 
\leq I_{2,1} + I_{2,2}.
\]
Owing to the BDG inequality (see, for instance, [41, Theorem 4.8]) we infer that
\[
I_{2,1} := \mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s \int Z \langle Nu_n(s), G(z, v(s)) \rangle \tilde{\eta}(dz, ds) \right| 
\leq C \mathbb{E} \left[ \int_0^t \int Z \langle Nu_n(s), Tu_n(s) \rangle ^2 ds \right]^{\frac{1}{2}},
\leq C \|N\|_{L(\mathbb{V}, \mathbb{V}^\ast)} \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_n(s)\| \left( \int_0^t \int Z \|G(z, v(s))\|^2 \nu (dz) ds \right)^{\frac{1}{2}} \right]
\]
(by the Young inequality with \( \delta > 0 \) arbitrary)
\[
\leq \delta \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_n(s)\|^2 \right] + \frac{C^2 \|N\|_{L(\mathbb{V}, \mathbb{V}^\ast)}^2}{4\delta} \mathbb{E} \int_0^t \int Z \|G(z, v(s))\|^2 \nu (dz) ds
\]
(by the inequality (2.9))
\[
I_{2,1} \leq \delta \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_n(s)\|^2 \right] + \frac{C^2 \|N\|_{L(\mathbb{V}, \mathbb{V}^\ast)}^2}{4\delta} t (1 + \|\nu\|^2_{M^2(X_T)}). \quad (3.9)
\]
Using again the BDG inequality yields

\[ I_{2.2} := \mathbb{E} \sup_{s \in [0,t]} \int_0^s \int_Z (NG(z,v(s)), G(z, v(s))) \tilde{m}(dz, ds) \]

\[ \leq C \mathbb{E} \left[ \int_0^t \int_Z [NG(z,v(s)), G(z, v(s))]^2 \nu(dz)ds \right]^{\frac{1}{2}} \]

\[ \leq C \|N\|_{\mathcal{L}(V)} \mathbb{E} \left[ \int_0^t \|G(z, v(s))\|^4 \nu(dz)ds \right]^{\frac{1}{2}} \]

(by the inequality (2.9))

\[ \leq C \|N\|_{\mathcal{L}(V)} \tilde{t}^2 (1 + \mathbb{E} \sup_{s \in [0,t]} \|v(s)\|^2), \]

\[ I_{2.2} \leq C \|N\|_{\mathcal{L}(V)} \tilde{t}^2 (1 + \|v\|^2_{M^2(X_T)}). \] (3.10)

Now it follows from Eqs. (3.5), (3.6), (3.8), (3.9) and (3.10) that

\[ \mathbb{E} \sup_{s \in [0,t]} \Psi(u_n(s)) - \Psi(u_0) + 2 \mathbb{E} \int_0^t (Au_n(s), Nu_n(s))ds \leq 2C(\|N\|, \varepsilon, \delta, n, t) (1 + \|v\|^2_{M^2(X_T)}) \]

\[ + \varepsilon \mathbb{E} \int_0^t \|u_n(s)\|^2 ds + \delta \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_n(s)\|^2 \right], \]

where

\[ \|N\| := \max (\|N\|_{\mathcal{L}(E,H)}, \|N\|_{\mathcal{L}(V,V^*)}), \]

\[ C(\|N\|, \varepsilon, \delta, n, t) := \left( \frac{\|N\| \phi(n) t^\alpha - 1}{4\varepsilon} + t \left[ \frac{C^2 \|N\| \tilde{E}_1}{4\delta} + \tilde{E}_1 + C\tilde{E}_2 \right] \right) \|N\|. \]

Since \((u, Nu) \geq C_N \|u\|^2\) and \((Au, Nu) \geq C_A \|u\|^2\), it follows that

\[ (C_N - \varepsilon) \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_n(s)\|^2 \right] + (2C_A - \varepsilon) \mathbb{E} \int_0^t \|u_n(s)\|^2 ds \leq C(\|N\|, \varepsilon, \delta, n, t) (1 + \|v\|^2_{M^2(X_T)}) \]

\[ + \Psi(u_0). \]

Choosing \(\varepsilon = C_A\) and \(\delta = C_N/2\), we derive from the last inequality that

\[ \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_n(s)\|^2 \right] + \mathbb{E} \int_0^t \|u_n(s)\|^2 ds \leq \frac{\Psi(u_0)}{\min(C_N/2, C_A)} + \frac{C(\|N\|, C_A, C_N, n, t)}{\min(C_N/2, C_A)} (1 + \|v\|^2_{M^2(X_T)}). \]

With this last inequality we easily conclude the proof of the claim. \(\square\)

**Lemma 3.2.** Let \(\Lambda_n\) be the mapping defined in Lemma 3.1 and

\[ \|N\| := \max (\|N\|_{\mathcal{L}(E,H)}, \|N\|_{\mathcal{L}(V,V^*)}). \]

Then, there exists a constant \(\kappa > 0\) depending only on \(\|N\|, n\) and the constants in Assumptions 2.1-2.3 such that

\[ \|\Lambda_n(v_1) - \Lambda_n(v_2)\|^2_{M^2(X_T)} \leq \kappa [T^{\alpha - 1} \vee T] \|v_1 - v_2\|^2_{M^2(X_T)}, \]

for any \(v_1, v_2 \in M^2(X_T)\).

**Proof.** Let \(v_i, i = 1, 2\), be two elements of \(M^2(X_T)\). To each \(v_i\) one can associate a unique element \(u_i \in M^2(X_T)\) which is a solution to Eq. (3.4) with the stochastic perturbation \(B_i^0(v_i(t))dt + \)
\[ f_z G(z, v_1(t)) \eta(dz, dt) \] and initial condition \( u_0 \). In this proof we suppress the dependence on \( n \) of the solution to (3.4). The difference \( u = u_1 - u_2 \) solves the linear equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
    d u(t) + A u(t) dt = [B_n^T(v_2(t)) - B_n^T(v_1(t))] dt + \int_Z [G(z, v_1(t)) - G(z, v_2(t))] \eta(dz, dt), \\
    u(0) = 0.
\end{array} \right.
\end{aligned}
\]  

To simplify our notation we also set \( v = v_1 - v_2 \).

As before we apply Itô’s formula (see, for instance, [40, Appendix D]) to \( \Psi(u) = \langle Nu, u \rangle \) with (3.11). We then obtain

\[
\Psi(u(t)) + 2 \int_0^t \langle A u(s), N u(s) \rangle ds \leq 2 \int_0^t |B_n^T(v_1(s)) - B_n^T(v_2(s))||Nu(s)| ds
\]

\[ + \int_0^t \int_Z f(z, s, v_1, v_2) \nu(dz) ds \]  

\[ + \int_0^t \int_Z g(z, s, v_1, v_2) \eta(dz, ds), \]

with

\[ g(z, s, v_1, v_2) := \langle N[G(z, v_1(s)) - G(z, v_2(s))], G(z, v_1(s)) - G(z, v_2(s)) \rangle \]

\[ + 2 \langle [G(z, v_1(s)) - G(z, v_2(s))], Nu(s) \rangle, \]

and

\[ f(z, s, v_1, v_2) := \langle N[G(z, v_1(s)) - G(z, v_2(s))], G(z, v_1(s)) - G(z, v_2(s)) \rangle. \]

Arguing as in the proofs of Eq. (3.6), (3.8), (3.9) and (3.10), respectively, we obtain the following inequalities

\[
\mathbb{E} \int_0^t |B_n^T(v_1(s)) - B_n^T(v_2(s))||Nu(s)| ds \leq \frac{\|N\|^2\phi(n)}{4\varepsilon} t^{\alpha - 1} \|v\|^2_{L^2(X_T)}
\]

\[ + \varepsilon \mathbb{E} \int_0^t \|u(s)\|^2_2 ds, \]

\[
\mathbb{E} \sup_{s \in [0, t]} \left| \int_0^t \int_Z g(z, s, v_1, v_2) \eta(dz, ds) \right| \leq \left[ \frac{C^2\|N\|^2\ell_1}{4\delta} + \|N\|\ell_2 \right] t \|v\|^2_{L^2(X_T)}
\]

\[ + \delta \mathbb{E} \left( \sup_{s \in [0, t]} \|u(s)\|^2_{2} \right), \]

\[
\mathbb{E} \int_0^t \int_Z f(z, s, v_1, v_2) \nu(dz) ds \leq \|N\|_2^2 \|\ell_1\|_2 \|v\|^2_{L^2(X_T)},
\]

where \( \varepsilon, \delta \) are arbitrary positive numbers. By setting \( T^* = T \vee T^\alpha - 1 \) and

\[ \bar{\kappa} := \left( \|N\|_2^2 \left[ \frac{\phi(n)}{4\varepsilon} + \frac{C^2\ell_1}{4\delta} \right] + \ell_1 + C\ell_2 \right) \|N\|, \]

it follows from these inequalities and Eq. (3.12) that

\[
(C_N - \delta) \mathbb{E} \left( \sup_{s \in [0, t]} \|u(s)\|^2_{2} \right) + (2C_A - \varepsilon) \mathbb{E} \int_0^t \|u(s)\|^2_{2} ds \leq \bar{\kappa}T^* \|v\|^2_{L^2(X_T)},
\]

where we have used the fact that \( \langle u, Nu \rangle \geq C_N \|u\|^2 \) and \( \langle A u, Nu \rangle \geq C_A \|u\|^2_{2} \). By choosing \( \delta = C_N/2 \) and \( \varepsilon = C_A \) we get from the last estimate that

\[
\mathbb{E} \left( \sup_{s \in [0, t]} \|u(s)\|^2_{2} \right) + \mathbb{E} \int_0^t \|u(s)\|^2_{2} ds \leq \kappa T^* \|v\|^2_{L^2(X_T)},
\]
where \( \kappa := \kappa / \min(C_N/2, C_A) \). The last estimate means that
\[
\| \Lambda_n(v_1) - \Lambda_n(v_2) \|^2_{M^2(X_T)} \leq \kappa T^* \| v_1 - v_2 \|^2_{M^2(X_T)}.
\]
This completes the proof of our lemma. \( \square \)

Let \( n \) be a fixed positive integer. It follows from Lemma 3.1 that \( \Lambda_{n,1}^n \) maps \( M^2(X_T) \) into itself. From the proof of Lemma 3.2 we deduce that \( \Lambda_{n,1}^n \) is globally Lipschitz. Moreover it is a strict contraction for small \( T \). Therefore we can find a time \( \delta_n > 0 \) that is independent of the initial condition \( u_0 \) such that \( \Lambda_{\delta_n,0}^n \) is \( \frac{1}{2} \)-contraction. Hence it admits a unique fixed point \( u_{n,\delta_n} \in M^2(X_{\delta_n}) \) which solves on the small interval \([0, \delta_n]\) the nonlinear stochastic evolution equation
\[
u(t) + \int_0^t [Au(s) + B_n^T(u(s))]ds = u_0 + \int_0^t \int_Z G(z, u(s))h(dz, ds), \quad t \in [0, \delta_n].
\]

**Lemma 3.3.** Let \( u_{n,\delta_n} \) be a solution of \((3.13)\). Then \( \mathbb{P} \)-almost surely \( u_{n,\delta_n} : [0, \delta_n] \to V \) is càdlàg.

**Proof.** For sake of simplicity we just write \( \delta := \delta_n \). Since the solution \( u_{n,\delta} \) to the truncated equation \((3.1)\) belongs to \( M^2(X_\delta) \), from Proposition 2.4 and the fact that \( A \in \mathcal{L}(E, H) \) we infer that \( Au_{n,\delta}(\cdot) + B_n^T(u_{n,\delta}(\cdot)) \) is an element of \( M^2(0, \delta; H) \). From Theorem 2.1 we derive that the process \( \int_0^t \int_Z G(u_{n,\delta}(s), \tilde{\eta}(dz, ds) \) belongs to \( L^2(\Omega, D(0, \delta; V)) \) and define an \( \mathbb{P} \)-martingale. Since \( \mathbb{P} \)-a.s.
\[
u_{n,\delta}(t) + \int_0^t [Au_{n,\delta}(s) + B_{n,\delta}^T(u_{n,\delta}(s))]ds = u_0 + \int_0^t \int_Z G(z, u_{n,\delta}(s))\tilde{\eta}(dz, ds), \quad t \in [0, \delta],
\]

it follows from the above remarks and [27, Theorem 2] that \( \mathbb{P} \)-a.s. \( u_{n,\delta} \in D(0, \delta; V) \). \( \square \)

Now, we are able to formulate the result about the global existence of solution to the truncated equation \((3.1)\).

**Theorem 3.4.** Let Assumption 2.1, Assumption 2.2 and Assumption 2.3 hold. Then, for each \( n \geq 1 \) the truncated equation \((3.1)\) admits a unique global solution \( u^n \in M^2(X_T) \) for any \( T \in (0, \infty) \).

**Proof.** Let \( n \) be a positive integer and \( \delta_n > 0 \) such that \( \Lambda_{\delta_n,0}^n \) is a \( \frac{1}{2} \)-contraction. To keep the notation simple we just write \( \delta := \delta_n \). For \( k \in \mathbb{N} \) let \( (t_k)_{k \in \mathbb{N}} \) be a sequence of times defined by \( t_k = k\delta \). By the \( \frac{1}{2} \)-contraction property of \( \Lambda_{\delta_n,0}^n \) we can find \( u^{[n,i]} \in M^2(X_{\delta}) \) such that \( u^{[n,i]} = \Lambda_{\delta_n,0}^n(u^{[n,i-1]}) \). Since \( u^{[n,1]} \in M^2(X_\delta) \) it follows from Lemma 3.3 that \( u^{[n,1]} \) is \( \mathcal{F}_\tau \)-measurable and \( u^{[n,1]}(t) \in L^2(\Omega, \mathbb{P}; V) \) for any \( t \in [0, \delta] \). Thus replacing \( u_0 \) with \( u^{[n,1]}(\delta) \) where
\[
u^{[n,1]}(\delta) := u^{[n,1]}(\delta-) + \int_0^\delta G(z, u^{[n,1]}(\delta-))\tilde{\eta}(dz, \{\delta\}),
\]

and using the same argument as above we can find \( u^{[n,2]} \in M^2(X_{t_{k-1}, t_k}) \) such that \( u^{[n,2]} = \Lambda_{\delta_n,0}^{u^{[n,1]}(\delta)}(u^{[n,2]}) \). By induction we can construct a sequence \( u^{[n,k]} \) such that \( u^{[n,k]} = \Lambda_{\delta_n,0}^{u^{[n,k-1]}(\delta)}(u^{[n,k]}) \). Now let \( u^n \) be the process defined by \( u^n(t) = u^{[n,i]}(t), t \in [0, \delta] \), and for \( k \in \mathbb{N} \) and \( 0 \leq t < \delta \), let \( u^n(t + k\delta) = u^{[n,k]}(t) \). By construction \( u^n \in M^2(X_T) \) and \( u^n \) is a global solution to the truncated equation \((3.1)\).

Now let \((v, \tau)\) be another local solution of Eq. \((3.1)\), we shall show that \( u^n(t) = v(t) \), for all \( t \in [0, \tau] \) almost surely. For this purpose let \( t_1 = \tau \land \delta \) and \( t_k = \tau \land (k\delta) \) where \( k \) and \( \delta \) are as above; note that as \( k \to \infty \) we have \( t_k \uparrow \tau \) almost surely. With the same contraction principle used above we infer that \( 1_{(0, \tau \land \delta)}u^n(\cdot) = 1_{(0, \tau \land \delta)}v(\cdot) \) and \( 1_{(0, t_k)}u^n(\cdot) = 1_{(0, t_k)}v(\cdot) \) almost surely. By letting \( k \to \infty \) we infer that \( u^n(t) = v(t) \), for all \( t \in [0, \tau] \) almost surely. \( \square \)
3.2. Existence and uniqueness of maximal/global solution to Eq. (2.11). In this subsection we will use what we have learnt from the solvability of the truncated equation (3.1) to construct a unique maximal local and global solution to the original problem (2.11).

We start with the existence and uniqueness of maximal local solution.

**Theorem 3.5.** Let Assumption 2.1-2.3 be satisfied, then there exists a unique pair \((u, \tau_\infty)\) which is a maximal local solution to (2.11).

**Proof.** We have seen that for each \(n \in \mathbb{N}\) Eq. (3.1) has an unique global strong solution \(u^n\). Let us construct a sequence of stopping times \(\{\tau_n, n \in \mathbb{N}\}\) as follows

\[
\tau_n = \inf \{t \geq 0, \|u^n\|_{X_t} \geq n\} \land T, n \in \mathbb{N}.
\]

Now let \(k > n\) and \(\tau_{n,k} = \inf \{t \geq 0, \|u^k\|_{X_t} \geq n\} \land T\). Since \(\tau_{n,k} \leq \tau_k\) a.s., \((u^k, \tau_{n,k})\) is a local solution to Eq. (3.1) and \((u^n, \tau_n)\) is also a local solution to Eq. (3.1). Hence by the uniqueness we proved in Theorem 3.4 we infer that \(u^n(t) = u^k(t)\) a.s. for all \(t \in [0, \tau_n \land \tau_{n,k})\) which implies that

\[
u^n(t) = u^k(t)\text{ a.s. for } t \in [0, \tau_n).
\]

This also proves that \(\tau_n < \tau_k\) a.s. for all \(n < k\), and the sequence \(\{\tau_n, n \in \mathbb{N}\}\) has a limit \(\tau_\infty := \lim_{n \uparrow \infty} \tau_n\) a.s.

Now let \(\{u(t), 0 \leq t < \tau_\infty\}\) be the stochastic process defined by

\[
u(t) = u^n(t), \quad t \in [\tau_{n-1}, \tau_n), \quad n \geq 1,
\]

where \(\tau_0 = 0\). Since by definition \(\theta_n(\|u^n(s)\|_{X_s}) = 1\) for any \(s \in [0, t \land \tau_n]\), it follows that \(B^T_n(u^n(s)) = F(u^n(s))\) for any \(s \in [0, t \land \tau_n]\). By (3.14) we have \(u(t \land \tau_n) = u^n(t \land \tau_n)\), thus we can derive that \(P\text{-a.s.}
\]

\[
u(t \land \tau_n) = u_0 - \int_0^{t \land \tau_n} \left[Au^n(s) + B^T_n(u^n(s))\right] ds + \int_0^{t \land \tau_n} \int_Z G(z, u^n(s))\tilde{\eta}(dz, s),
\]

\[
u(t \land \tau_n) = u_0 - \int_0^{t \land \tau_n} \left[Au(s) + F(u(s))\right] ds + \int_0^{t \land \tau_n} \int_Z G(z, u(s))\tilde{\eta}(dz, s),
\]

for any \(t \in [0, T]\). This proves that \((u, \tau_n)\) is a local solution to (2.11). On \(\{\tau_\infty(\omega) < T\}\) we have

\[
\lim_{t \uparrow \tau_\infty} \|u\|_{X_t} \geq \lim_{n \uparrow \infty} \|u^n\|_{X_{\tau_n}},
\]

\[
\geq \lim_{n \uparrow \infty} \|u^n\|_{X_{\tau_n}} = \infty.
\]

Therefore \((u, \tau_\infty)\) is a maximal local solution to Eq. (2.11).

We will prove that this maximal solution is unique. For this let \((v, \sigma_\infty)\) be another maximal local solution and \(\{\sigma_n, n \geq 0\}\) a sequence of stopping times converging to \(\sigma_\infty\) defined by

\[
\sigma_n = \inf \{t \geq 0, \|v\|_{X_t} \geq n\} \land \sigma_\infty \land T.
\]

Arguing as above we can prove that \(u(t) = v(t)\) for all \(t \in [0, \tau_n \land \sigma_n]\) a.s. which, upon letting \(n \uparrow \infty\), implies that

\[
u(t) = v(t)\text{ for all } t \in [0, \tau_\infty \land \sigma_\infty]\text{ a.s.}
\]

From this last identity we can conclude that \(\tau_\infty = \sigma_\infty\) almost surely. Indeed if the last conclusion were not true then we either have

\[
\lim_{t \uparrow \tau_\infty} \|1_{\{\sigma_\infty > \tau_\infty\}}v\|_{X_{\sigma_\infty}} = \lim_{n \uparrow} \|1_{\{\sigma_\infty > \tau_\infty\}}v\|_{X_{\tau_n}} = \infty,
\]

(3.16)
or

\[
\lim_{t \uparrow \tau_\infty} \|1_{\{\sigma_\infty < \tau_\infty\}} u\|_X = \lim_{n \uparrow} \|1_{\{\sigma_\infty < \tau_\infty\}} u\|_X = \lim_{n \uparrow} \|1_{\{\sigma_\infty < \tau_\infty\}} v\|_X = \infty.
\] (3.17)

The identity (3.16) (resp. Eq. (3.17)) contradicts the fact that \( v \) (resp. \( u \)) does not explode before time \( \sigma_\infty \) (resp. \( \tau_\infty \)). Therefore one must have \( \tau_\infty = \sigma_\infty \) almost surely, which yields the uniqueness of the maximal local solution \((u, \tau_\infty)\). □

**Proposition 3.6.** In addition to the assumptions of Theorem 3.7 we assume that \( \mathbb{E}|u_0|^4 < \infty \) and there exists \( \tilde{C}_A > 0 \) such that \( \langle Au, u \rangle \geq \tilde{C}_A \|u\|^2 \) for any \( u \in V \). We also suppose that \( F \)

\[
\langle F(u), u \rangle = 0,
\]

(3.18)

for all \( u \in V \). Let \( u \) be the stochastic process we constructed in Theorem 3.5. Let \( (\tau_n)_{n \geq 1} \) be a sequence of stopping times defined by

\[
\tau_n = \inf\{t \geq 0 : \|u\|^2_{X_t} \geq n^2\} \land T.
\]

Then for \( r = 1, 2 \), for any \( t \geq 0 \) there exists constant \( \tilde{C} > 0 \) such that the local solution \((u, \tau_n)\) to (2.11) satisfies

\[
\mathbb{E} \sup_{s \in [0, t \land \tau_n]} |u(s)|^{2r} + \mathbb{E} \int_0^{t \land \tau_n} |u(s)|^{2r-2} \|u(s)\|^2 ds \leq \tilde{C},
\]

(3.19)

and

\[
\mathbb{E} \left[ \int_0^{t \land \tau_n} \|u(s)\|^2 ds \right]^2 \leq \tilde{C},
\]

(3.20)

for any \( n \geq 1 \).

**Proof.** Let \( \Psi(u) := |u|^2 \) and

\[
g(s, z, u) := \langle G(z, u(s)), G(z, u(s)) \rangle,
\]

\[
f(z, s, u) := g(s, z, u) + 2 \langle G(z, u(s)), u(s- \rangle).
\]

Note that \((u, \tau_\infty)\), where \( \tau_\infty = \lim_{n \uparrow} \tau_n \) a.s., is the unique maximal solution to (2.11). Throughout let \( n \) be a fixed positive integer. To shorten notation we define \( t_n = t \land \tau_n \) for any \( t \in [0, T] \).

Estimates (3.19) can be proved by using the Itô’ formula to \([\Psi(u(t_n))]^r, r = 1, 2\), with \( t_n = t \land \tau_n \) for every \( t \in [0, T] \). For \( r = 1 \) the same calculations with \( N = Id \) as in proof of Theorem 3.7 yields

\[
\mathbb{E} \sup_{s \in [0, t_n]} \Psi(u(s)) + 2\mathbb{E} \int_0^{t_n} \langle A(u(s), u(s)) ds \leq \mathbb{E} \Psi(u_0) + \bar{\ell}_1 \mathbb{E} \int_0^{t_n} |u(s)|^2 ds + \bar{\ell}_1 T + \mathbb{E} \sup_{s \in [0, t_n]} \left[ \int_0^s \left\langle f(z, s, u, \tilde{\eta}(dz, ds) \right\rangle \right],
\]

(3.21)

for any \( t \in [0, T] \) and \( n \geq 1 \). Arguing as in the proofs of Eq. (3.9) and (3.10) we obtain the following inequality

\[
\mathbb{E} \sup_{s \in [0, t_n]} \left[ \int_0^s \left\langle f(z, s, u, \tilde{\eta}(dz, ds) \right\rangle \leq \left\{ C^2 \bar{\ell}_1 + \bar{\ell}_2 \right\} \mathbb{E} \int_0^{t_n} |u(s)|^2 ds + \epsilon \mathbb{E} \left( \sup_{s \in [0, t_n]} |u(s)|^2 \right),
\]
which along with (3.21) implies that
\[
\mathbb{E} \sup_{s \in [0, t_n]} \Psi(u(s)) + 2\mathbb{E} \int_0^{t_n} \langle Au(s), u(s) \rangle ds \leq \mathbb{E} \Psi(u_0) + \varepsilon \mathbb{E} \left[ \sup_{s \in [0, t_n]} |u(s)|^2 \right] + \left[ \bar{\ell}_1 + \bar{\ell}_2 \right] \mathbb{E} \int_0^{t_n} |u(s)|^2 ds.
\]
Using Assumption (2.1), choosing \( \varepsilon = 1/2 \) and invoking the Gronwall lemma yield
\[
\mathbb{E} \sup_{s \in [0, t_n]} |u(s)|^2 + \mathbb{E} \int_0^{t_n} |u(s)|^2 ds \leq \frac{\mathbb{E} \Psi(u_0)}{\min\left(\frac{1}{2}, 2C_A\right)} [\ell T + 1],
\]
where
\[
\ell = \frac{1}{\min\left(\frac{1}{2}, 2C_A\right)} \left[ \bar{\ell}_1 + \bar{\ell}_2 + 1 \right].
\]
This completes the proof of the theorem for \( r = 1 \).

Now, for \( t \geq 0 \) let \( y(t) := \langle u(t), u(t) \rangle \) and \( \Psi'(u(t))[h] = 2\langle u(t), h \rangle \) for any \( h \in H \).
First we should notice that by Itô’s formula and the assumption about \( F \) in Proposition 3.6 we have
\[
y(t_n) + \int_0^{t_n} \Psi'(u(s))[Au(s)] ds = y(0) + \int_0^{t_n} \int_Z g(s, z, u) \nu(dz) ds + \int_0^{t_n} \int_Z f(s, z, u) \bar{\eta}(dz, ds).
\]
By applying Itô’s formula to \( |y(t)|^2 := z(t) \) we obtain
\[
\mathbb{E} \sup_{r \in [0, t_n]} \left[ z(r) + 2 \int_0^r y(s) \Psi'(u(s))[Au(s)] ds - 2 \int_0^r \int_Z y(s) g(s, z, u) \nu(dz) ds \right] = \mathbb{E} \sup_{r \in [0, t_n]} \left[ z(0) + \int_0^r \int_Z [f(s, z, u)]^2 \nu(dz) ds \right] + \mathbb{E} \sup_{r \in [0, t_n]} \left[ \int_0^r \int_Z \left( [f(s, z, u)]^2 + 2y(s-)[f(s, z, u)] \bar{\eta}(dz, ds) \right) \right].
\]
By performing elementary calculation and using part (ii) of Assumption 2.3 one can show that
\[
\mathbb{E} \int_0^{t_n} \int_Z [f(s, z, u)]^2 \nu(dz) ds \leq 2\mathbb{E} \int_0^{t_n} \int_Z \left( [G(z, u(s)), G(z, u(s))] \right)^2 \nu(dz) ds
\]
\[
+ 2\mathbb{E} \int_0^{t_n} \int_Z \left( [u(s-), G(z, u(s))] \right)^2 \nu(dz) ds,
\]
\[
\leq 2[\bar{\ell}_1 + \bar{\ell}_2] \left( T + \mathbb{E} \int_0^{t_n} |u(s)|^4 ds \right) + \mathbb{E} \int_0^{t_n} \int_Z \left( [G(z, u(s)), G(z, u(s))] \right)^2 \nu(dz) ds.
\]
Similarly,
\[
2\mathbb{E} \int_0^{t_n} \int_Z y(s) g(s, z, u) \nu(dz) ds \leq \bar{\ell}_1 \left( T + \mathbb{E} \int_0^{t_n} |u(s)|^4 ds \right)
\]
Since \( [\psi(s, z, u)]^2 > 0 \), by using [43, Theorem 3.10, Eq. (3.10)] we derive that
\[
\mathbb{E} \sup_{r \in [0, t_n]} \left| \int_0^r \int_Z [f(s, z, u)]^2 \bar{\eta}(dz, ds) \right| \leq \mathbb{E} \int_0^{t_n} \int_Z [f(s, z, u)]^2 \nu(dz) ds
\]
and by arguing as above we infer that
\[
\mathbb{E} \int_0^{t_n} \int_Z [f(s, z, u)]^2 \bar{\eta}(dz, ds) \leq 2^2[\bar{\ell}_1 + \bar{\ell}_2^2] \left( T + \mathbb{E} \int_0^{t_n} |u(s)|^4 ds \right).
\]
By using the BDG inequality and Cauchy inequality with epsilon we obtain
\[
\mathbb{E} \sup_{r \in [0, t_n]} \left| \int_0^r \int_Z 2y(s-) f(s, z, u) \tilde{\eta}(dz, ds) \right| \leq 4K \mathbb{E} \left[ \int_0^{t_n} \int_Z |y(s)|^2 |f(s, z, u)|^2 \nu(dz) ds \right]^{\frac{1}{2}},
\]
\[
\leq \frac{16K^2}{4\varepsilon} \mathbb{E} \int_0^{t_n} \int_Z |f(s, z, u)|^2 \nu(dz) ds + \varepsilon \mathbb{E} \sup_{s \in [0, t_n]} |u(s)|^4. \tag{3.28}
\]

And arguing as in (3.27) we derive that
\[
\mathbb{E} \sup_{r \in [0, t_n]} \left| \int_0^r \int_Z 2y(s-) f(s, z, u) \tilde{\eta}(dz, ds) \right| - \varepsilon \mathbb{E} \sup_{s \in [0, t_n]} |u(s)|^4 \leq \frac{32K^2(1 + \ell_1 + \ell_2^2)}{4\varepsilon} (T + \mathbb{E} \int_0^{t_n} |u(s)|^4 ds)
\]
(3.29)

Plugging (3.25), (3.26), (3.27) and (3.29) in (3.24), using Assumption 2.1 and choosing \( \varepsilon = 1/2 \) yield the existence of positive constants \( \bar{L}, \bar{\ell} \) such that
\[
\mathbb{E} \sup_{s \in [0, t_n]} |u(s)|^4 + \int_0^{t_n} |u(s)|^2 \|u(s)\|^2 ds \leq \bar{L}T + \bar{\ell} \mathbb{E} \int_0^{t_n} |u(s)|^4 ds + \bar{\ell} \mathbb{E} |\Psi(u_0)|^2.
\]

Thanks to the Gronwall lemma we infer that
\[
\mathbb{E} \sup_{s \in [0, t_n]} |u(s)|^4 + \int_0^{t_n} |u(s)|^2 \|u(s)\|^2 ds \leq \left( \bar{L}T + \bar{\ell} \mathbb{E} |\Psi(u_0)|^2 \right) [\varepsilon \bar{L}T + 1]. \tag{3.30}
\]

The above inequality completes the proof of (3.19) for \( r = 2 \), and hence the first part of our theorem.

To prove the second part we will use (3.22). In fact, from (3.22) we derive that
\[
\mathbb{E} \left[ \int_0^{t_n} \Psi'(u(s)) [Au(s)] ds \right] \leq C \mathbb{E} |y(0)|^2 + C \mathbb{E} \left[ \int_0^{t_n} \int_Z g(s, z, u) \nu(dz) ds \right]^2 + C \mathbb{E} \left[ \int_0^{t_n} \int_Z f(s, z, u) \tilde{\eta}(dz, ds) \right]^2. \tag{3.31}
\]

Note that the stochastic integral in the last term of the RHS of the above estimate is real-valued, so from Itô’s isometry we infer that
\[
\mathbb{E} \left[ \int_0^{t_n} \int_Z f(s, z, u) \tilde{\eta}(dz, ds) \right]^2 = \mathbb{E} \int_0^{t_n} \int_Z |f(s, z, u)|^2 \nu(dz) ds,
\]
from which altogether with (3.27) and (3.19) we derive that for any \( t \geq 0 \) there exists a constant \( \bar{C} > 0 \) such that
\[
\mathbb{E} \left[ \int_0^{t_n} \int_Z f(s, z, u) \tilde{\eta}(dz, ds) \right]^2 \leq \bar{C} \tag{3.32}
\]
for any \( n \geq 1 \). By imitating the proof of (3.25) we infer that there exists \( \bar{C} > 0 \) such that
\[
\mathbb{E} \left[ \int_0^{t_n} \int_Z g(s, z, u) \nu(dz) ds \right]^2 \leq \bar{C} \left( t + \mathbb{E} \int_0^{t_n} |u(s)|^4 ds \right), \tag{3.33}
\]
from which and (3.19) we deduce that for any \( t \geq 0 \) there exists \( \bar{C} > 0 \) such that
\[
\mathbb{E} \left[ \int_0^{t_n} \int_Z g(s, z, u) \nu(dz) ds \right]^2 \leq \bar{C}, \tag{3.34}
\]
for any \( n \geq 1 \). Taking (3.32) and (3.34) into (3.31) implies that

\[
\mathbb{E}
\left[
\int_0^{t_n} \Psi'(u(s))[Au(s)]ds
\right]
\leq C\mathbb{E}[y(0)]^2 + 2\tilde{C}.
\]

(3.35)

Thanks to this last estimate and the fact that \( \langle Au, u \rangle \geq \tilde{C}_A \|u\|^2 \) we easily derive that for any \( t \geq 0 \) there exists \( \tilde{C} > 0 \) such that for any \( n \geq 1 \)

\[
\mathbb{E}
\left[
\int_0^{t_n} \|u(s)\|^2 ds
\right]
\leq \tilde{C}.
\]

This completes the proof of (3.20), and hence the whole Proposition.

Now we turn our attention to the existence and uniqueness of global solution.

**Theorem 3.7.** Assume that \( F \) satisfies the assumptions of Proposition 3.6 with \( p = 1 \) and \( \alpha \in [0, \frac{1}{2}] \). Moreover, we suppose that there exists \( \tilde{c} > 0 \) such that

\[
|F(u) - F(v)| \leq \tilde{c} \left[ |u|^{1-\alpha} \|u\|^\alpha |u - v|^1 |u - v|^\alpha + |u - v|^{1-\alpha} \|u - v\|^\alpha |v|^{1-\alpha} \|v\|^\alpha \right]
\]

(3.36)

for any \( u, v \in \mathbf{E} \). Then Problem (2.11) has a unique global solution.

**Proof.** Let \( u \) be the stochastic process we constructed in Theorem 3.5 and

\[
\|N\| := \max \left( \|N\|_{L(E,H)}, \|N\|_{L(V,V^*)} \right).
\]

Let \( (\tau_n)_{n \geq 1} \) be a sequence of stopping times defined by

\[
\tau_n = \inf \{ t \in [0,T] : \|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds \geq n^2 \}.
\]

Note that \( (u, \tau_\infty) \), where \( \tau_\infty = \lim_{n \to \infty} \tau_n \) a.s., is the unique maximal solution to (2.11). To deal with the structure of the nonlinearity \( F \) (see Eq. (3.36)) we introduce another sequence of stopping times \( (\sigma_m)_{m \geq 1} \) defined by

\[
\sigma_m = \inf \left\{ t \in [0,T] : \int_0^t \|u(s)\|^2 \|u(s)\|^\frac{2\alpha}{1-\alpha} ds \geq m \right\}, \text{ for any } m \geq 1.
\]

To shorten notation we define \( t_{m,n} = t \wedge (\sigma_m \wedge \tau_n) \) for any \( t \in [0,T] \), \( n \geq 1 \) and \( m \geq 1 \). Let

\[
\begin{align*}
  f(z,s,u) &:= \langle NG(z,u(s)), G(z,u(s)) \rangle \\
  &\quad + 2 \langle G(z,u(s)), Nu(s) \rangle,
\end{align*}
\]

and

\[
g(z,s,u) := \langle NG(z,u(s)), G(z,u(s)) \rangle.
\]

Applying Itô’s formula to \( \Psi(u) = \langle u, Nu \rangle \) we obtain

\[
\Psi(u(t_{m,n})) = \Psi(u_0) - 2 \int_0^{t_{m,n}} \left[ \langle Au(s) + F(u(s)), Nu(s) \rangle \right] ds + \int_0^{t_{m,n}} \int \langle g(z,s,u)\nu(dz)ds \right. \\
\left. \quad + \int_0^{t_{m,n}} \int f(z,s,u)\tilde{n}(dz,ds),
\]

for any \( t \in [0,T] \). For any \( \delta > 0 \) and \( p, q \geq 1 \) with \( p^{-1} + q^{-1} = 1 \) let \( C(\delta, p, q) \) be the constant from the Young inequality

\[
ab \leq C(\delta, p, q)a^p + \delta b^q.
\]

From Eq. (3.36) and the above Young inequality with \( p = \frac{2}{1+\alpha} \), \( q = \frac{2}{1-\alpha} \), and \( \delta = C_A \) we obtain

\[
|2\langle F(u(s)), Nu(s) \rangle| \leq C(C_A, p, q)[2\tilde{c}\|N\|] \|u(s)\|^2 \|u(s)\|^\frac{2\alpha}{1-\alpha} + C_A \|u(s)\|^2.
\]
By making use of the definition of $\sigma_m$ we get that
\[
2\int_0^{t_{m,n}} \langle F(u(s)), N\dot{u}(s) \rangle ds \leq C(C_A, p, q) |2e\|N\|^q mT + C_A \int_0^{t_{m,n}} \|u(s)\|^2 ds. \tag{3.37}
\]
From the assumption on $G$ we derive that
\[
\int_0^{t_{m,n}} \int_Z g(z, s, u) \nu(dz) ds \leq \|N\| \bar{\ell}_1 T + \|N\| \bar{\ell}_1 \int_0^{t_{m,n}} \|u(s)\|^2 ds. \tag{3.38}
\]

By taking the mathematical expectation to both sides of this estimate and by using Assumption 2.1 altogether with Eqs. (3.37), (3.38) we infer that
\[
E[\|u(t_{m,n})\|^2] + E \int_0^{t_{m,n}} \|u(s)\|^2 ds \leq \tilde{L}^{-1} \|N\| \bar{\ell}_1 E \int_0^{t_{m,n}} \|u(s)\|^2 ds
\]
\[
+ \tilde{L}^{-1} [E\Psi(u_0) + \|N\| \bar{\ell}_1 T + C_{mA} T],
\]
where $\tilde{L} = \min(C_N, C_A)$ and $C_{mA} := C(C_A, p, q) |2e\|N\|^q m$. From the Gronwall’s lemma we infer that
\[
E[\|u(t_{m,n})\|^2] + E \int_0^{t_{m,n}} \|u(s)\|^2 ds \leq \tilde{L}^{-1} [E\Psi(u_0) + \|N\| \bar{\ell}_1 T + C_{mA} T] e^{\tilde{L}^{-1} \|N\| \bar{\ell}_1 t_{m,n}} [1 + \|N\| \bar{\ell}_1 T]. \tag{3.39}
\]

Next, note that
\[
P(\tau_n < t) = P(\{\tau < t\} \cap (\Omega_m \cup \Omega_m')) \\
= P(\{\tau < t\} \cap \Omega_m) + P(\{\tau < t\} \cap \Omega_m'),
\]
where $\Omega_m = \{\sigma \geq T\}$, $m \geq 1$. Now, by arguing as in [12, pp. 123] we have
\[
P(\tau_n < t) \leq \frac{1}{n^2} E \left\{ 1_{\{\tau_n < t\}} \cap \Omega_m \left[ \|u(t_{m,n})\|^2 + \int_0^{t_{m,n}} \|u(s)\|^2 ds \right] \right\} + P[\Omega_m'],
\]
\[
\leq \frac{1}{n^2} E \left[ \|u(t_{m,n})\|^2 + \int_0^{t_{m,n}} \|u(s)\|^2 ds \right] + \frac{1}{m} E \int_0^{t_{m,n}} |u(s)|^2 \|u(s)\|^{\frac{2\alpha}{1-\alpha}} ds.
\]

Thanks to Eq. (3.39)
\[
P(\tau_n < t) \leq \frac{1}{n^2} \tilde{L}^{-1} [E\Psi(u_0) + \|N\| \bar{\ell}_1 T + C_{mA} T] e^{\tilde{L}^{-1} \|N\| \bar{\ell}_1 T} + \frac{1}{m} E \int_0^{t_{m,n}} |u(s)|^2 \|u(s)\|^{\frac{2\alpha}{1-\alpha}} ds,
\]
from which we derive that
\[
\lim_{n \to \infty} P(\tau_n < t) \leq \frac{1}{m} \left\{ E \left[ \sup_{s \in [0, t_{m,n}]} |u(s)|^4 \right] + \left( E \left[ \int_0^{t_{m,n}} \|u(s)\|^2 ds \right]^2 \right)^{\frac{2\alpha}{1-\alpha}} \right\}.
\]

Since $\alpha \in [0, \frac{1}{2}]$ it follows from Proposition 3.6 (see (3.19)-(3.20)) that the solution $u$ satisfies
\[
E[ \sup_{s \in [0, t_{m,n}]} |u(s)|^4 ] + \left( E \left[ \int_0^{t_{m,n}} \|u(s)\|^2 ds \right] \right)^2 \leq \tilde{C}.
\]

Hence, combining this latter equation with the former one yields that
\[
\lim_{n \to \infty} P(\tau_n < t) = 0,
\]
from which we derive that $P(\tau_\infty < T) = 0$ for any $T > 0$. This implies that $u$ is a global solution. \hfill \Box

**Remark 3.8.** All of our results in this section remain true if we replace $F(u)$ by $B(u) + R(u)$ with $R \in L(H, H)$ and $B$ satisfying the assumptions of Theorem 3.5 and Theorem 3.7.
4. Examples

4.1. Notations. Let $n \in \{2, 3\}$ and assume that $\mathcal{O} \subset \mathbb{R}^n$ is a Poincaré’s domain (its definition is given below) with boundary $\partial \mathcal{O}$ of class $C^\infty$. For any $p \in [1, \infty)$ and $k \in \mathbb{N}$, $L^p(\mathcal{O})$ and $W^{k,p}(\mathcal{O})$ are the well-known Lebesgue and Sobolev spaces, respectively, of $\mathbb{R}^n$-valued functions. The corresponding spaces of scalar functions we will denote by standard letter, e.g. $W^{k,p}(\mathcal{O})$.

A domain $\mathcal{O} \subset \mathbb{R}^d$ is called a Poincaré’s domains if following Poincaré’s inequality holds

$$|u| \leq c|\nabla u|, \text{ for all } u \in H^1(\mathcal{O}).$$

(4.1)

For $p = 2$ we denote $W^{k,2}(\mathcal{O}) = \mathbb{H}^k$ and its norm are denoted by $|u|_k$. By $\mathbb{H}^1$ we mean the space of functions in $\mathbb{H}^1$ that vanish on the boundary on $\mathcal{O}$; $\mathbb{H}^1$ is a Hilbert space when endowed with the scalar product induced by that of $\mathbb{H}^1$. The usual scalar product on $L^2$ is denoted by $(u,v)$ for $u, v \in L^2$. Its associated norm is $|u|_2, u \in L^2$. We also introduce the following spaces

$$\mathcal{V}_1 = \{u \in [C^\infty_c(\mathcal{O}, \mathbb{R}^n)] \text{ such that } \nabla \cdot u = 0\}$$

$$\mathcal{V}_1 = \text{closure of } \mathcal{V} \text{ in } \mathbb{H}^1(\mathcal{O})$$

$$\mathcal{H}_1 = \text{closure of } \mathcal{V} \text{ in } L^2(\mathcal{O}).$$

We also consider the Hilbert spaces $\mathcal{H}_2 = \mathcal{H}_1$ and $\mathcal{V}_2 = \mathbb{H}^1 \cap \mathcal{H}_2$.

Let $(e_1, e_2)$ be the standard basis in $\mathbb{R}^2$ and $x = (x^1, x^2)$ an element of $\mathbb{R}^2$. When $\mathcal{O} = (0, l) \times (0, 1)$ is a rectangular domain in the vertical plane we consider the following spaces

$$\mathcal{H}_3 = \{u \in L^2, \text{ div } u = 0, u^2|_{x^2=0} = u^2|_{x^2=1} = 0, u^1|_{x^1=0} = u^1|_{x^1=l}\}$$

and $\mathcal{H}_4 = L^2(\mathcal{O})$. We also denote

$$\mathcal{V}_3 = \{u \in \mathcal{H}_3 \cap \mathbb{H}^1, \text{ div } u = 0, \text{ is } l\text{-periodic in } x^1\},$$

$$\mathcal{V}_4 = \{\theta \in H^1(\mathcal{O}), \text{ div } \theta = 0, \text{ is } l\text{-periodic in } x^1\},$$

$$\mathcal{H}_5 = \mathcal{H}_3, \mathcal{V}_5 = \mathcal{H}_5 \cap \mathbb{H}^1.$$

Let $\Pi_i : \mathbb{L}^2 \to \mathcal{H}_i$ be the projection from $L^2$ onto $\mathcal{H}_i$, $i = 1, 2, 3, 4, 5$. We denote by $A_i$ the Stokes operator defined by

$$\begin{cases}
D(A_i) = \{u \in \mathcal{H}_i, \Delta u \in \mathcal{H}_i\},
A_i u = -\Pi_i \Delta u, \text{ } u \in D(A_i),
\end{cases}$$

(4.2)

$i = 1, \ldots, 5$. In all cases the $A_i$-s are self-adjoint, positive linear operators on $\mathcal{H}_i$. Finally we set $E_i = D(A_i), i \in \{1, 2, 3, 4, 5\}$. Note that $E_i \subset \mathbb{H}^2 \cap \mathcal{V}_i, i = 1, 2, 3, 5$ and $E_4 \subset H^2 \cap \mathcal{V}_4$.

We endow the spaces $\mathcal{H}_i, i \in \{1, 2, 3, 4, 5\}$, with the scalar product and norm of $L^2$. We equip the space $\mathcal{V}_i, i \in \{1, 2, 3, 4, 5\}$, with the scalar product $(A_i^* u, A_i^* v)$ which is equivalent to the $\mathbb{H}^1(\mathcal{O})$-scalar product on $\mathcal{V}_i$. The spaces $E_i, i \in \{1, 2, 3, 4, 5\}$ are equipped with the norm $|A_i u|$ which is equivalent to the $\mathbb{H}^2$-norm on $E_i$.

Remark 4.1. In the case of an general unbounded domain we equip the space $\mathcal{V}_i, i \in \{1, 2, 3, 4, 5\}$, with the scalar product $((\text{Id} + A_i)^* u, (\text{Id} + A_i)^* v)$. The spaces $E_i, i \in \{1, 2, 3, 4, 5\}$ are equipped with the norm $|((\text{Id} + A_i) u|$ which is equivalent to the $\mathbb{H}^2$-norm on $E_i$. 


Next we define two trilinear forms $b_1(\cdot, \cdot, \cdot)$ and $b_2(\cdot, \cdot, \cdot)$ by setting

$$b_1(u, v, w) = \sum_{i,j=1}^{n} \int_{\Omega} u^i(x) \frac{\partial}{\partial x_1} v^j(x) w^j(x) dx, \quad \text{for any } (u, v, w) \in L^4 \times W^{1,4} \times L^2, \quad (4.3)$$

$$b_2(u, \theta_2, \theta_3) = \sum_{i=1}^{n} \int_{\Omega} u^i(x) \frac{\partial}{\partial x_1} \theta_2(x \theta_3(x) dx, \quad \text{for any } (u, \theta_2, \theta_3) \in L^4 \times W^{1,4} \times L^2. \quad (4.4)$$

Recall that for $\alpha = \frac{\gamma}{2}$, the following estimate, valid for all $u \in H^1_0$ (or $u \in H^1$), is a special case of Gagliardo-Nirenberg’s inequalities:

$$\|u\|_{H^1} \leq \|u\|_{L^4}^{1-\alpha} |\nabla u|^\alpha. \quad (4.5)$$

The inequality (4.5) can be written in the spirit of the continuous embedding

$$H^1 \subset L^4. \quad (4.6)$$

Using Cauchy-Schwarz inequality, (4.5) and (4.6) in (4.3)-(4.4) we derive that for any $(u, v, w) \in H^1 \times H^2 \times L^2$

$$|b_1(u, v, w)| \leq c\|u\|_{H^1} |\nabla v|^{1-\alpha} |D^2 v|^\alpha |w| \quad \text{for any } (u, v, w) \in H^1 \times H^2 \times L^2, \quad (4.7)$$

$$|b_2(u, \theta_2, \theta_3)| \leq c\|u\|_{H^1} |\nabla \theta_2|^{1-\alpha} |D^2 \theta_2|^\alpha |\theta_3| \quad \text{for any } (u, \theta_2, \theta_3) \in H^1 \times H^2 \times L^2. \quad (4.8)$$

From Eq. (4.7) (resp., Eq. (4.8)) we infer that there exists a bilinear map $B_1(\cdot, \cdot)$ (resp., $B_2(\cdot, \cdot)$) defined on $V_i \times E_i$ and taking values in $H_i$, for appropriate values of $i$. Moreover, there exist $c > 0$ such that

$$|B_1(u, v)| \leq c |u|_{H^1} ||v||_{H^1}^{1-\alpha} |v|^\alpha, \quad \text{for any } (u, v) \in V_i \times E_i, \quad (4.9)$$

$$|B_2(u, \theta)| \leq c |u|_{H^1} ||\theta||_{H^1}^{1-\alpha} |\theta|^\alpha, \quad \text{for any } (u, \theta) \in V_i \times E_i, \quad (4.10)$$

for appropriate values of $i$. Note that using Cauchy-Schwarz inequality, (4.5) and (4.6) in (4.3)-(4.4) we also derive that

$$|B_1(u, v)| \leq c |u|^{1-\alpha} |u|_{H^1} ||v||_{H^1}^{1-\alpha} |v|^\alpha, \quad \text{for any } (u, v) \in V_i \times E_i, \quad (4.11)$$

$$|B_2(u, \theta)| \leq c |u|^{1-\alpha} |\theta|_{H^1} ||\theta||_{H^1}^{1-\alpha} |\theta|^\alpha, \quad \text{for any } (u, \theta) \in V_i \times E_i, \quad (4.12)$$

for appropriate values of $i$.

### 4.2. Stochastic hydrodynamical systems

The first examples that we have in mind are the models studied by Chueshov & Millet in [16].

#### 4.2.1. Stochastic Navier-Stokes Equations

Let $\Omega$ be a bounded, open and simply connected domain of $\mathbb{R}^n$, $n = 2, 3$. The boundary $\partial \Omega$ of $\Omega$ is assumed to be smooth. Let $(Z, \mathcal{Z}, \nu)$ be a measure space where the $\nu$ is a $\sigma$-finite, positive measure and $\tilde{\eta}$ be a compensated Poisson random measure having intensity measure $\nu$ defined on filtered complete probability space $\mathcal{F} = (\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$, where the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. We consider the Navier-Stokes equation with the Dirichlet (no-slip) boundary conditions:

$$du + \left[ - \kappa \Delta u + u \nabla u + \nu \right] dt = \int_{\Omega} G(t, u(t), z) \tilde{\eta}(dz, dt), \quad \text{div } u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial \Omega,$$

where $u = (u^1(x, t), u^2(x, t))$ is the velocity of a fluid, $p(x, t)$ is the pressure, $\kappa$ the kinematic viscosity. Here $\int_{\Omega} G(t, u(t), z) \tilde{\eta}(dz, dt)$ represents a state-dependent random external forcing of jump type.

Let $H = H_1$, $V = V_1$ and $E = E_1$ where the hilbert spaces $H_i$, $V_i$ and $E_i$ are defined as in Eq. (4.2) of Subsection 4.1. The norms of $H$, $V$ and $E$ are denoted by $|\cdot|$, $||\cdot||$, $||\cdot||_*$, respectively.
Let \( A = A_1 \) and \( B = B_1 \) be the linear and bilinear maps defined in Subsection 4.1. We also set \( N = A. \) Note that in this case \( N \) is self-adjoint and \( N \in \mathcal{L}(E, H) \cap \mathcal{L}(V, V^*) \). We suppose that \( \tilde{G} \) satisfies the following sets of conditions.

**Assumption 4.1.** We assume that \( \tilde{G} \) maps \( V \) into \( L^{2p}(Z, \nu, V) \) and there exists a constant \( \tilde{\ell}_p > 0 \) such that

\[
\| \tilde{G}(x) - \tilde{G}(y) \|^{2p}_{L^{2p}(Z, \nu, V)} \leq \tilde{\ell}_p \| x - y \|^{2p},
\]  

(4.14)

for any \( x, y \in V \) and \( p = 1, 2 \).

Note that this implies in particular that there exists a constant \( \tilde{\ell}_p > 0 \) such that

\[
\| \tilde{G}(x) \|^{2p}_{L^{2p}(Z, \nu, H)} \leq \tilde{\ell}_p (1 + \| x \|^{2p}),
\]

(4.15)

for any \( x \in V \) and \( p = 1, 2. \)

By setting \( R \equiv 0 \) and projecting on the space of divergence free vector fields the system (4.13) can be rewritten in the following abstract form

\[
du + [Au + B(u, u)] dt = \int Z \tilde{G}(t, u(t), z) \tilde{\eta}(dz, dt),
\]

(4.16)

\[
u(0) = \xi,
\]

**Theorem 4.2.** The stochastic Navier-Stokes problem (4.16) admits a local maximal strong solution which is global if \( n = 2. \)

**Remark 4.3.** This theorem remains true in the case \( O \) being a general unbounded domain. For the proof it is sufficient to take \( A = A_1 + \text{Id}, \) \( R(u) := -u \) and argue as in the case of bounded domain.

**Proof.** The existence and uniqueness of a maximal local solution will follow from Theorem 3.5 if we are able to prove that \( F(u) = B(u, u) \) satisfies (2.7). But from (4.9) we deduce that there exists \( C > 0 \) such that for

\[
|B(y) - B(x)| \leq C \left[ \| y - x \|^{1 - \frac{\alpha}{4}} \| y \|^{\frac{\alpha}{4}} + \| y - x \|^{\frac{\alpha}{4}} \| y - x \|^{1 - \frac{\alpha}{4}} \| x \| \right],
\]

for any \( x, y \in E. \) This means that \( B \) satisfies (2.7) with \( p = 1 \) and \( \alpha = \frac{2}{3}. \) Since \( \alpha = \frac{2}{3} \notin [0, \frac{1}{2}] \) for \( n = 3, \) the solution is only maximal. For \( n = 2 \) we have \( \alpha = \frac{1}{2} \) and \( \langle B(u, u), u \rangle = 0. \) So thanks to Remark 3.8, we only need to check that (3.36) is verified by \( B. \) But this will follow from (4.11). \( \square \)

### 4.2.2. Magnetohydrodynamic equations

Let \( O \subset \mathbb{R}^n, \) \( n = 2, 3 \) be a simply connected, possibly unbounded domain. As above we assume that \( O \) has a smooth boundary \( \partial O. \) Let \( (Z_i, \mathcal{Z}_i, \nu_i), \) \( i = 1, 2 \) be two measure spaces where the measures \( \nu_i \) are \( \sigma \)-finite and positive. We consider two mutually independent compensated Poisson random measures \( \tilde{\eta}_i \) with intensity measure \( \nu_i \) defined on a complete filtered probability space \( \mathcal{F} = (\Omega, \mathcal{F}, \mathbb{P}). \) We consider the magnetohydrodynamic (MHD) equations

\[
du + [-\Delta u + u \nabla u] dt = [-\nabla \left( p + \frac{1}{2} |b|^2 \right) + b \nabla b] dt + \int_{Z_1} \tilde{f}(t, u(t), b(t), z_1) \tilde{\eta}_1(dz_1, dt),
\]

(4.17)

\[
db + [-\nu_2 \Delta b + u \nabla b] dt = [b \nabla u] dt + \int_{Z_2} \tilde{g}(t, u(t), b(t), z_2) \tilde{\eta}_2(dz_2, dt),
\]

(4.18)

\[
\text{div } u = 0, \quad \text{div } b = 0
\]

(4.19)

where \( u = (u^{(1)}(x, t), u^{(2)}(x, t), u^{(3)}(x, t)) \) and \( b = (b^{(1)}(x, t), b^{(2)}(x, t), b^{(3)}(x, t)) \) denote velocity and magnetic fields, \( p(x, t) \) is a scalar pressure. We consider the following boundary conditions

\[
u = 0, \quad b \cdot n = 0, \quad \text{curl } b \times n = 0 \quad \text{on } \partial O
\]

(4.20)
The terms \( \int_{Z_1} \tilde{f}(t, u(t), b(t), z_1) \tilde{\eta}_1(dz_1, dt) \) and \( \int_{Z_2} \tilde{g}(t, u(t), b(t), z_2) \tilde{\eta}_2(dz_2, dt) \), represent random external volume forces and the curl of random external current applied to the fluid. We refer to [33], [21] and [44] for the mathematical theory for the MHD equations.

Let \( H = H_1 \times H_2, V = V_1 \times V_2 \) and \( E = E_1 \times E_2 \). We define a bilinear map \( B(\cdot, \cdot) \) on \( V \times E \) by
\[
\langle B(z_1, z_2), z_3 \rangle = \langle B_1(u_1, u_2), u_3 \rangle - \langle B_1(b_1, b_2), u_3 \rangle \\
+ \langle B_1(u_1, b_2), b_3 \rangle - \langle B_1(b_1, u_2), b_3 \rangle + \langle B_2(u_1, \theta_2), \theta_3 \rangle,
\]
for \( z_1 = (u_1, b_1) \in V, z_2 = (u_2, b_2) \in E \) and \( z_3 = (u_3, b_3) \in H \). We also set
\[
A_z = \left( \begin{array}{cc} Id + A_1 & 0 \\
0 & Id + A_2 \end{array} \right) \begin{pmatrix} u \\ b \end{pmatrix}
\]
for \( z = (u, b) \in E \).

We set \( u := (u, b) \) and
\[
\int_{Z} \tilde{G}(t, u(t), z) \tilde{\eta}(dz, dt) := \left( \int_{Z_1} \tilde{f}(t, u(t), z_1) \tilde{\eta}_1(dz_1, dt) \right) \left( \int_{Z_2} \tilde{g}(t, u(t), z_2) \tilde{\eta}_2(dz_2, dt) \right).
\]
We assume that \( \tilde{f}, \tilde{g} \) are chosen in such a way that \( \tilde{G} \) maps \( V \) into \( L^{2p}(Z, \nu, V) \) and satisfies Assumption 4.1.

By setting \( R = -Id \) and projecting on \( H \) we can see that (4.17)-(4.18) can be rewritten in the form (4.16). Now, by choosing \( N = A \) we can show by arguing as in Theorem 4.2 that the stochastic Magnetohydrodynamical equations (4.17)-(4.18) has a local maximal solution which is global if the dimension \( n = 2 \).

4.2.3. Magnetic Bénard problem. Let \( O = (0,l) \times (0,1) \) be a rectangular domain in the vertical plane, \( (e_1, e_2) \) the standard basis in \( \mathbb{R}^2 \). Let \( (Z_i, \mathcal{Z}_i, \nu_i), i = 1, 2, 3 \) be two measure spaces where the measures \( \nu_i \) are \( \sigma \)-finite and positive. We consider three mutually independent compensated Poisson random measures \( \tilde{\eta}_i \) with intensity measure \( \nu_i \) defined on a complete filtered probability space \( \mathfrak{F} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \).

We consider the equations
\[
du + [u \nabla u - \kappa_1 \Delta u + \nabla \left( p + \frac{s}{2} |b|^2 \right) - sb \nabla b] dt = \theta e_2 dt + \int_{Z_1} \tilde{f}(t, u(t), \theta(t), b(t), z_1) \tilde{\eta}_1(dz_1, dt),
\]
\[
d\theta = u \nabla \theta - \kappa_2 \Delta \theta dt = \int_{Z_2} \tilde{g}(t, u(t), \theta(t), b(t), z_2) \tilde{\eta}_2(dz_2, dt),
\]
\[
db - [-\kappa_3 \Delta b + u \nabla b - b \nabla u] dt = \int_{Z_3} \tilde{h}(t, u(t), \theta(t), b(t), z_3) \tilde{\eta}_3(dz_3, dt),
\]

with boundary conditions
\[
u = 0, \quad \theta = 0, \quad b^{(2)} = 0, \quad \partial_2 b^{(1)} = 0 \quad \text{on} \quad x^{(2)} = 0 \quad \text{and} \quad x^{(2)} = 1,
\]
\[
u, p, \theta, b, u_x, \theta_x, b_x \quad \text{are periodic in} \quad x \quad \text{with period} \ l.
\]
This is the Boussinessq model coupled with magnetic field (see [26]) with stochastic perturbations. Throughout we assume that \( \kappa_1 = \kappa_2 = s = 1 \). Let \( H = H_3 \times H_4 \times H_5, V = V_3 \times V_4 \times V_5 \) and \( E = E_3 \times E_4 \times E_5 \).

We define a bilinear map \( B(\cdot, \cdot) \) on \( V \times E \) by
\[
\langle B(z_1, z_2), z_3 \rangle = \langle B_1(u_1, u_2), u_3 \rangle - \langle B_1(b_1, b_2), u_3 \rangle \\
+ \langle B_1(u_1, b_2), b_3 \rangle - \langle B_1(b_1, u_2), b_3 \rangle + \langle B_2(u_1, \theta_2), \theta_3 \rangle,
\]
for \( z_1 = (u_1, \theta_1, b_1) \in \mathbf{V} \), \( z_2 = (u_2, \theta_2, b_2) \in \mathbf{E} \) and \( z_3 = (u_3, \theta_3, b_3) \in \mathbf{H} \). Using the notations in (4.2), we set

\[
Az = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_5 \end{pmatrix} \begin{pmatrix} u \\ \theta \\ b \end{pmatrix}
\]

for \( z = (u, \theta, b) \in E \).

We also set \( R(u, \theta, b) = -(\theta e_2, u^{(2)}, 0) \) and \( N = A \). Note that in this case \( R \in \mathcal{L}(\mathbf{H}, \mathbf{H}) \) and \( N \in \mathcal{L}(\mathbf{E}, \mathbf{H}) \cap \mathcal{L}(\mathbf{V}, \mathbf{V}^*) \).

We set \( u := (u, \theta, b) \) and

\[
\int_Z \tilde{G}(t, u(t), z) \tilde{\eta}(dz, dt) := \left( \int_Z \tilde{f}(t, u(t), z_1) \tilde{\eta}_1(dz_1, dt) \right) \left( \int_Z \tilde{g}(t, u(t), z_2) \tilde{\eta}_2(dz_2, dt) \right) \left( \int_Z \tilde{h}(t, u(t), z_3) \tilde{\eta}_3(dz_3, dt) \right).
\]

We assume that \( \tilde{f}, \tilde{g}, \tilde{h} \) are chosen such that \( \tilde{G} \) verifies Assumption 4.1. With these notations we can put the stochastic Magnetic Bénard problem into the abstract stochastic evolution equation (4.16).

**Theorem 4.4.** The stochastic Magnetic Bénard problem (4.16) admits a unique global strong solution.

**Proof.** The maximal local solution will follow from Theorem 3.5 if we are able to prove that \( F(u) = B(u, u) + R(u) \) satisfies (2.7). Since \( R \) is a bounded linear map, it follows from Remark 3.8 that it is sufficient to check (2.7) for \( B \). But from (4.9) and (4.10) we deduce that there exists \( C > 0 \) such that for

\[
|B(y) - B(x)| \leq C \left[ \|y - x\|_2 \|y\|_1^{1/2} \|y - x\|_2^{3/2} \|y - x\|_3^{1/2} \right],
\]

for any \( x, y \in \mathbf{E} \). This means that \( B \) satisfies (2.7) with \( p = 2 \) and \( \alpha = 1/3 \). Since \( n = 2 \) we have \( \alpha = 1/2 \) and \( \langle B(u, u), u \rangle = 0 \). So thanks to Remark 3.8, we only need to check that (3.36) is verified by \( B \). But this will follow from (4.11) and (4.12). \( \square \)

**4.2.4. Boussinesq model for the Bénard convection.** Let \( \mathcal{O} \) be a (possibly) domain of \( \mathbb{R}^n \), \( n = 2, 3 \), \( \{e_1, \ldots, e_n\} \) a standard basis in \( \mathbb{R}^n \) and \( x = (x^{(1)}, \ldots, x^{(n)}) \) an element of \( \mathbb{R}^n \). We assume that \( \mathcal{O} \) has a smooth boundary \( \partial \mathcal{O} \). Let \( \mathcal{Z}_i, \mathcal{Z}_i, \nu_i \), \( i = 1, 2 \) be two measure spaces where the measures \( \nu_i \) are \( \sigma \)-finite and positive. We consider two mutually independent compensated Poisson random measures \( \tilde{\eta}_i \) with intensity measure \( \nu_i \) defined on a complete filtered probability space \( \mathbb{Q} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \).

Let us consider the Bénard convection problem (see e.g. [25] and the references therein) given by the following system

\[
du + [u \nabla u - \Delta u + \nabla p]dt = \theta e_n dt + \int_{Z_1} \tilde{f}(t, u(t), b(t), z_1) \tilde{\eta}_1(dz_1, dt), \quad \text{div} \ u = 0, \quad (4.21)
\]

\[
d\theta + [u \nabla \theta - u^{(n)} - \Delta \theta]dt = \int_{Z_2} \tilde{g}(t, u(t), \theta(t), z_2) \tilde{\eta}_2(dz_2, dt), \quad (4.22)
\]

with boundary conditions

\[
u = 0 \quad \text{and} \quad \theta = 0 \quad \text{on} \ \partial \mathcal{O}.
\]

Here \( p(x, t) \) is the pressure field, \( \int_{Z_1} \tilde{f}(t, u(t), b(t), z_1) \tilde{\eta}_1(dz_1, dt) \), \( \int_{Z_2} \tilde{g}(t, u(t), b(t), z_2) \tilde{\eta}_2(dz_2, dt) \) represent two random external forces, \( u = (u^{(1)}(x, t), \ldots, u^{(n)}(x, t)) \) is the velocity field and \( \theta = \theta(x, t) \) is the temperature field.
We set $H = H_3 \times H_4$, $V = V_3 \times V_4$, $E = E_3 \times E_4$. Following the notations given in (4.2) we define

$$A_z = \begin{pmatrix} A_3 & 0 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} u \\ \theta \end{pmatrix}$$

for $z = (u, \theta) \in E$. We define a bilinear map $B(\cdot, \cdot)$ on $V \times E$ by

$$B(z_1, z_2) = (B_1(u_1, u_2), u_3) + (B_2(u_1, \theta_2), \theta_3),$$

for $z_1 = (u_1, \theta_1) \in V$, $z_2 = (u_2, \theta_2) \in E$ and $z_3 = (u_3, \theta_3) \in H$. We also put $R(u, \theta, b) = -(\theta c_2, u^{(n)})$ and $N = A$.

As before we set $u := (u, \theta)$ and

$$\int_E G(t, u(t), z) \eta(dz, dt) := \left( \int_{Z_1} \hat{f}(t, u(t), b(t), z_1) \hat{\eta}_1(dz_1, dt) \right) \left( \int_{Z_2} \hat{g}(t, u(t), b(t), z_2) \hat{\eta}_2(dz_2, dt) \right).$$

We assume that $\tilde{f}$, $\tilde{g}$ are chosen in such a way that $G$ maps $V$ into $L^2(\mathbb{Z}, \nu, \mathbb{V})$ and satisfies Assumption 4.1.

By Arguing as in the case of Navier-Stokes equations, Magnetic Bénard problem and MHD equations we can show that if the random external force satisfies Assumption 4.1, then the Boussinesq model for the Bénard convection admits a unique maximal strong solution which is global is $n = 2$.

### 4.3. Shell models of turbulence

In this section we follows the notation of [16]. Let $H$ be a set of all sequences $u = (u_1, u_2, \ldots)$ of complex numbers such that $\sum_n |u_n|^2 < \infty$. We consider $H$ as a real Hilbert space endowed with the inner product $(\cdot, \cdot)$ and the norm $| \cdot |$ of the form

$$(u, v) = \text{Re} \sum_{n=1}^{\infty} u_n v_n^*,$$

where $v_n^*$ denotes the complex conjugate of $v_n$. In this space $H$ we consider the evolution equation (4.16) with $R = 0$ and with linear operator $A$ and bilinear mapping $B$ defined by the formulas

$$(Au)_n = \nu k_n^2 u_n, \quad n = 1, 2, \ldots, \quad D(A) = \left\{ u \in H : \sum_{n=1}^{\infty} k_n^4 |u_n|^2 < \infty \right\},$$

where $\nu > 0$, $k_n = k_0 \mu^n$ with $k_0 > 0$ and $\mu > 1$, and

$$[B(u, v)]_n = -i \left( ak_n v_{n+1} v_{n+1}^* + bk_n v_{n-1} v_{n-1}^* - ak_{n+1} v_{n+1} v_{n+1}^* - bk_{n+1} v_{n+1} v_{n+1}^* - ak_{n-1} v_{n-1} v_{n-1}^* - bk_{n-1} v_{n-1} v_{n-1}^* \right)$$

for $n = 1, 2, \ldots$, where $a$ and $b$ are real numbers (here above we also assume that $u_{-1} = u_0 = v_{-1} = v_0 = 0$). This choice of $A$ and $B$ corresponds to the so-called GOY-model (see, e.g., [38]).

If we take

$$[B(u, v)]_n = -i \left( ak_{n+1} v_{n+1} v_{n+1}^* + bk_{n+1} v_{n+1} v_{n+1}^* + ak_{n-1} v_{n-1} v_{n-1}^* + bk_{n-1} v_{n-1} v_{n-1}^* \right),$$

then we obtain the Sabra shell model introduced in [34].

One can easily show (see [1] for the GOY model and [15] for the Sabra model) that the trilinear form

$$\langle B(u, v), w \rangle \equiv \text{Re} \sum_{n=1}^{\infty} [B(u, v)]_n w_n^*$$

satisfies the inequality

$$|\langle B(u, v), w \rangle| \leq C |u| A^{1/2} |v| |w|, \quad \forall u, v \in H, \quad \forall v \in D(A^{1/2}).$$

Hence taking $H = H^1(V, \| \cdot \|) = (D(A^2), |A^2 \cdot |)$ and $(E, | \cdot |)_s := (D(A), |A \cdot |)$ we infer that the nonlinear term for the shell models satisfies Assumption 2.2 with $a = 0$ and $p = 1$. By Arguing
as before we can show that if the random external force satisfies Assumption 4.1, then stochastic shell models admits a unique global strong solution.

Let us consider the following dyadic model (see, e.g., [30] and the references therein)

\[
\partial_t u_n + \nu \lambda^{2n} u_n - \lambda^n u_{n-1}^2 + \lambda^{n+1} u_{n+1} = f_n, \quad n = 1, 2, \ldots,
\]

where \( \nu, \alpha > 0 \), \( \lambda > 1 \), \( u_0 = 0 \). By setting \( [B(u, v)]_n = -\lambda^n u_{n-1} v_{n-1} + \lambda^{n+1} u_{n+1} v_{n+1} \) and \( (Au)_n = \nu \lambda^{2n} u_n \), it is not difficult to show that the system (4.23) falls also in the framework of the shell models of turbulence provided that \( \alpha \geq 1/2 \).

4.4. 3D Leray \( \alpha \)-model for Navier-Stokes equations. In a bounded 3D domain \( \Omega \) we consider the following equations:

\[
\begin{align*}
\partial_t u - \Delta u + v \nabla u + \nabla p &= f, \\
(1 - \alpha \Delta)v &= u, \quad \text{div } u = 0, \quad \text{div } v = 0 \quad \text{in } \Omega, \\
v = u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

where \( u = (u(1), u(2), u(3)) \) and \( v = (v(1), v(2), v(3)) \) are unknown fields, \( p(x, t) \) is the pressure. We refer to [13, 14] and references for results related to (4.24)-(4.26).

Let \( H = H_1, \quad V = V_1 \) and \( E = E_1 \) be the Hilbert spaces defined in Subsection 4.1. Set \( A = A_1, \quad G_\alpha = (Id + \alpha A)^{-1} \) and define a bilinear mapping \( B(\cdot, \cdot) \) on \( V \times E \) by setting

\[ B(u, v) = B_1(G_\alpha u, v), \]

for any \( u \in V \) and \( v \in E \).

Arguing as in [16, Subsubsection 2.1.5] we can show that there exists \( C > 0 \) such that

\[ |B(u, v)| \leq C \|u\|_{L^3} \|\nabla v\|_{L^3}, \quad (4.27) \]

for any \( u \in L^3 \) and \( v \in W^{1,3} \). Recall that in three dimensional case we have the following Gagliardo-Nirenberg inequality

\[ \|u\|_{L^3} \leq c \|u\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}}, \quad u \in H^1. \quad (4.28) \]

Now using this inequality and the continuous embedding \( H^1 \subset L^3 \)

we infer from (4.27) that

\[
\begin{align*}
|B(u, v)| &\leq C \|u\|_V \|v\|_V \|v\|_E, \\
|B(u, v)| &\leq C \|u\|_H \|v\|_V \|v\|_E, \quad (4.29)
\end{align*}
\]

for any \( u \in V, \; v \in E \).

Now we set \( R \equiv 0 \) and \( N = A \). Thanks to (4.29)-(4.30) we see that the nonlinear term for the 3D Leray \( \alpha \)-model for Navier-Stokes equations satisfies the assumptions of Theorem 3.7 with \( \alpha = \frac{1}{2} \) and \( p = 1 \). Therefore we can argue as in the case of 2D stochastic Navier-Stokes equations and show that the stochastic 3D Leray \( \alpha \)-model for Navier-Stokes equations admits a global solution if the random external force satisfies Assumption 4.1.

Appendix A. Existence of solution to the linear SPDE (3.4)

Throughout this appendix we assume that the separable Hilbert spaces \( E, V, H \) are defined as before.

Let \( (Z, \mathcal{Z}) \) be a separable metric space and let \( \nu \) be a \( \sigma \)-finite positive measure on it. Let \( \eta : \Omega \times B(\mathbb{R}_+) \times \mathcal{Z} \to \tilde{\mathbb{N}} \) is a time homogeneous Poisson random measure with the intensity
measure $\nu$. We will denote by $\tilde{\eta} = \eta - \gamma$ the compensated Poisson random measure associated to $\eta$ where the compensator $\gamma$ is given by

$$B(\mathbb{R}_+) \times \mathcal{Z} \ni (A, I) \mapsto \gamma(A, I) = \nu(A) \lambda(I) \in \mathbb{R}_+.$$ 

Let $\phi \in M^2(0, T; \mathbf{H})$ and $\psi \in M^2(0, T; L^2(\mathbb{Z}; \nu; \mathbf{V}))$. We will show in the next theorem that the following linear SPDEs (which is (3.4)) has a unique solution

$$\begin{align*}
\left\{ 
\begin{array}{l}
du(t) + [Au(t) + \phi(t)]dt = \int_Z \psi(t, z)\tilde{\eta}(dz, dt), \quad t \in [0, T], \\
u(0) = u_0.
\end{array}
\right.
\end{align*}$$

(A.1)

**Theorem A.1.** Let $A, N$ be as in Assumption 2.1, $\phi \in M^2(0, T; \mathbf{H})$, $\psi \in M^2(0, T; L^2(\mathbb{Z}; \nu; \mathbf{V}))$. Let $u_0$ be a $\mathbf{V}$-valued $\mathcal{F}_0$-measurable random variable satisfying $E|u_0|^2 < \infty$. Then there exists a unique progressively measurable process $u$ taking values in $\mathbf{V}$ such that almost surely

$$\langle u(t), w \rangle + \int_0^t \langle Au(s) + \phi(s), w \rangle ds = \langle u_0, w \rangle + \int_0^t \int_Z \langle \psi(s, z), w \rangle \tilde{\eta}(dz, ds),$$

(A.2)

for all $t \in [0, T]$ and $w \in \mathbf{H}$. Moreover, $u \in L^\infty(0, T; \mathbf{V}) \cap L^2(0, T; \mathbf{E}) \cap D(0, T; \mathbf{V})$ with probability 1.

**Proof.** We will use the Picard method as presented in [39, Chapitre 3, Section 1]. Throughout this proof we set

$$\|N\| := \max \{\|N\|_{\mathcal{L}(\mathbf{E}, \mathbf{H})}, \|N\|_{\mathcal{L}(\mathbf{V}, \mathbf{V}^*)}\}.$$

For positive integer $n$ we define a sequence $\{u^{[n]}(t), t \in [0, T]\}$ of stochastic processes as follows

$$\begin{align*}
\begin{cases}
u^{[1]}(t) = u_0, \\

u^{[n+1]}(t) = u_0 - \int_0^t [Au^{[n]}(s) + \phi(s)]ds + \int_0^t \int_Z \psi(s, z)\tilde{\eta}(dz, ds), \quad t \in [0, T],
\end{cases}
\end{align*}$$

$n \geq 2$.

Thanks to our assumption and [27, Theorem 2] the stochastic process

$$u^{[2]}(t) = u_0 - \int_0^t [Au^{[1]}(s) + \phi(s)]ds + \int_0^t \int_Z \psi(s, z)\tilde{\eta}(dz, ds)$$

is a well-defined $\mathbf{V}$-valued adapted and càdlàg process. By iterating this definition we see that for each $n \geq 2$ $u^{[n]}$ is also a well-defined $\mathbf{V}$-valued adapted and càdlàg process.

Now we will show that the sequence $u^{[n]}$ is converging in appropriate topology to the solution $u$ of (A.1). In fact we will show that $u^{[n]} \in L^2(\Omega; L^\infty(0, T; \mathbf{V}))$ is a Cauchy sequence. For this aim define $\Phi^n(t) = \mathbb{E}\sup_{\varepsilon \in [0, t]}|u^{[n+1]}(s) - u^{[n]}(s)|^2$ for all $n \geq 1$. We have

$$u^{[n+1]}(t) - u^{[n]}(t) = -\int_0^t A(u^{[n]}(s) - u^{[n-1]}(s))ds,$$

for any $t \in [0, T]$ and $n \geq 2$. Multiplying this equation by $N(u^{[n+1]} - u^{[n]})$, using Assumption 2.1 and the Cauchy inequality with arbitrary $\varepsilon > 0$ we infer that

$$(C_N - \varepsilon) \sup_{s \in [0, t]}\|u^{[n+1]}(s) - u^{[n]}(s)\|^2 \leq \frac{\|N\|^2\|A\|^2}{4\varepsilon} \int_0^t \|u^{[n]}(s) - u^{[n-1]}(s)\|^2 ds.$$ 

Choosing $\varepsilon = C_N/2$ taking the mathematical expectation to both side of the last estimate implies

$$\Phi^n(t) \leq \frac{\|N\|^2\|A\|^2}{2C_N^2} \int_0^t \Phi^{n-1}(s)ds.$$ 

(A.3)

As in [39] we iterate (A.3) and obtain

$$\Phi^n(t) \leq \left(\frac{\|N\|^2\|A\|^2}{2C_N^2}\right)^n \frac{1}{n!}\Phi^1(t),$$
from which we deduce that \((u^{[n]}; n \geq 1)\) forms a Cauchy sequence in \(L^2(\Omega; L^\infty(0, T; \mathbf{V}))\). Therefore, there exists \(u \in L^2(\Omega; L^\infty(0, T; \mathbf{V}))\) such that
\[
\begin{align*}
u^n \to u \text{ strongly in } L^2(\Omega; L^\infty(0, T; \mathbf{V})).\tag{A.4}
\end{align*}
\]

Now, we prove that \(u^{[n]}\) is bounded in \(L^2(\Omega; L^2(0, T; \mathbf{E}))\). For this purpose we apply Itô formula to \(\Psi(u) = (u, Nu)\) and use Assumption 2.3 to infer that
\[
\begin{align*}
C_N\|u^{[n]}(t \wedge \tau)\|^2 + 2C_A \int_0^{t \wedge \tau} \|u^{[n]}(s)\|^2 ds &\leq \int_0^T \left[|\phi(s)|^2 + \int_{\mathbb{Z}} \langle N\psi(s, z), \psi(s, z)\rangle \nu(dz)\right] ds \\
&\quad + \int_0^{t \wedge \tau} \int_{\mathbb{Z}} \left[\langle \psi(s, z), Nu^{[n]}(s)\rangle + \langle \psi(s, z), N\psi(s, z)\rangle\right] \eta(dz, ds) \\
&\quad + \Psi(u_0) + \|N\|^2 \int_0^T \|u^{[n]}(s)\|^2 ds.\tag{A.5}
\end{align*}
\]
where \(\tau\) is an arbitrary stopping time localizing the local martingale
\[
\int_0^t \int_{\mathbb{Z}} \left[\langle \psi(s, z), Nu^{[n]}(s)\rangle + \langle \psi(s, z), N\psi(s, z)\rangle\right] \eta(dz, ds).
\]

We easily derive from (A.5) that
\[
\begin{align*}
C_N\|u^{[n]}(t \wedge \tau)\|^2 + 2C_A \int_0^{t \wedge \tau} \|u^{[n]}(s)\|^2 ds &\leq \int_0^T \left[|\phi(s)|^2 + \|N\|^2 \int_{\mathbb{Z}} \|\psi(s, z)\|^2 \nu(dz)\right] ds \\
&\quad + \int_0^{t \wedge \tau} \int_{\mathbb{Z}} \left[\langle \psi(s, z), Nu^{[n]}(s)\rangle + \langle \psi(s, z), N\psi(s, z)\rangle\right] \eta(dz, ds) \\
&\quad + \Psi(u_0) + \|N\|^2 \int_0^T \|u^{[n]}(s)\|^2 ds.
\end{align*}
\]
Since, by the first part of our proof, \(\int_0^T \mathbb{E}\|u^{[n]}(s)\|^2 ds\) is bounded and \(\tau\) is arbitrary, by taking mathematical expectation to both sides of the last inequality we derive that there exists \(C > 0\) such that
\[
\mathbb{E} \int_0^T \|u^{[n]}(s)\|^2 ds \leq C.
\]
This implies that one can find a subsequence of \(u^{[n]}\), which will be denoted with the same fashion, such that
\[
u^{[n]} \to u \text{ weakly in } L^2(\Omega; L^2(0, T; \mathbf{E})).\tag{A.6}
\]

Since, by assumption, \(A \in \mathcal{L}(\mathbf{E}, \mathbf{H})\) it follows from (A.6) that
\[
A\nu^{[n]} \to Au \text{ weakly in } L^2(\Omega; L^2(0, T; \mathbf{H})).\tag{A.7}
\]
Owing to the convergences (A.4) and (A.7) we easily derive that, with probability 1, \(u\) satisfies (A.2) for all \(t \in [0, T]\) and \(w \in \mathbf{H}\). This means that (A.1) holds for all \(t \in [0, T]\) and all \(w \in \mathbf{H}\) with probability 1. Since \(u\) is the limit in \(L^2(\Omega; L^\infty(0, T; \mathbf{V}))\) of a sequence of adapted processes, we infer that \(u\) is adapted. Thanks to our assumption and [27, Theorem 2] the process \(u\) is càdlàg. Because \(u\) is adapted and càdlàg it admits a progressively measurable version which is still denoted with the same symbol. The proof of our theorem is complete. \(\square\)
ACKNOWLEDGMENTS

E. Hausenblas and P. A. Razafimandimby are funded by the FWF-Austrian Science Fund through the projects P21622 and M1487, respectively. The research on this paper was initiated during the visit of H. Bessaih to the Montanuniversität Leoben in June 2013. She would like to thank the Chair of Applied Mathematics at the Montanuniversität Leoben for hospitality. Part of this paper was written during Razafimandimby’s visit at the University of Wyoming in November 2013. He is very grateful for the warm and kind hospitality of the Department of Mathematics at the University of Wyoming.

REFERENCES

[1] D. Barbato, M. Barsanti, H. Bessaih, & F. Flandoli, Some rigorous results on a stochastic Goy model, Journal of Statistical Physics, 125 (2006) 677–716.
[2] A. Bensoussan, Stochastic Navier-Stokes Equations. Acta Applicandae Mathematicae, 38:267–304, 1995.
[3] A. Bensoussan and J. Frehse. Local solutions for stochastic Navier Stokes equations. M2AN Math. Model. Numer. Anal. 34(2):241-273, 2000.
[4] A. Bensoussan and R. Temam. Equations Stochastiques du Type Navier-Stokes. Journal of Functional Analysis, 13:195–222, 1973.
[5] H. Bessaih and A. Millet. Large deviation principle and inviscid shell models. Electron. J. Probab. 14:2551-2579, 2009.
[6] H. Bessaih, F. Flandoli and E.S. Titi. Stochastic attractors for shell phenomenological models of turbulence. J. Stat. Phys. 140:688–717, 2010.
[7] Z. Brzeźniak, E. Hausenblas. Maximal regularity for stochastic convolutions driven by Lévy processes, Probab. Theory Related Fields. 145(3-4):615–637, 2009.
[8] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby. Stochastic Nonparabolic dissipative systems modeling the flow of Liquid Crystals: Strong solution. Preprint arXiv:1310.8641. To appear in RIMS Kôkyûroku “Proceeding of RIMS Symposium on Mathematical Analysis of Incompressible Flow, February 2013”, 2013.
[9] Z. Brzeźniak,E. Hausenblas and J. Zhu. 2D stochastic Navier-Stokes equations driven by jump noise. Nonlinear Anal. 79:122-139, 2013.
[10] Z. Brzeźniak and A. Millet, On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact riemannian manifold, arXiv:1209.3578, 2012, to appear in Pot. Analysis
[11] Z. Brzeźniak and E. Motyl. Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D domains. J. Differential Equations 254(4): 1627-1685, 2013.
[12] Z. Brzeźniak, B. Maslowski and J. Seidler, Stochastic nonlinear beam equations, Probab. Theory Related Fields. 132(1):119-149 2005.
[13] V. Chepyzhov, E. Titi & M. Vishik, On the convergence of solutions of the Leray-α model to the trajectory attractor of the 3D Navier-Stokes system, Discrete Contin. Dyn. Syst. 17 (2007), 481–500.
[14] A. Cheskidov, D. Holm, E. Olson & E. Titi, On a Leray-α model of turbulence, Proc. R. Soc. Lond. Ser.A 461 (2005), 629–649.
[15] P. Constantin, B. Levant, & E. S. Titi, Analytic study of the shell model of turbulence. Physica D 219 (2006), 120–141.
[16] I. Chueshov and A. Millet. Stochastic 2D hydrodynamical type systems: well posedness and large deviations. Appl. Math. Optim. 61(3): 379-420, 2010.
[17] A. de Bouard and A. Debussche, A stochastic nonlinear Schrödinger equation with multiplicative noise, Comm. Math. Phys. 205(1):161–181 (1999).
[18] A. de Bouard and A. Debussche, The stochastic nonlinear Schrödinger equation in $H^1$, Stochastic Anal. Appl. 21(1):97–126 (2003).
[19] G. Deugoue and M. Sango. On the Strong Solution for the 3D Stochastic Leray-Alpha Model, Boundary Value Problems, vol. 2010, Article ID 723018, 31 pages, 2010. doi:10.1155/2010/723018.
[20] Z. Dong and Z. Jianliang. Martingale solutions and Markov selection of stochastic 3D Navier-Stokes equations with jump, J. Differential Equations, 250:2737–2776, 2011.
[21] G. Duvaut & J.L. Lions, Inéquations en thermoélasticité et magnéto hydrodynamique, Arch. Rational Mech. Anal. 46 (1972), 241–279.
[22] S. Ethier and T. Kurtz. Markov processes, Characterization and convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.
[23] F. Flandoli, M. Gubinelli, M. Hairer, and M. Romito. Rigorous remarks about scaling laws in turbulent fluid. Commun. Math. Phys. 278: 1-29, 2008.
[24] F. Flandoli and D. Gatarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields.* 102(03):367-391, 1995.
[25] C. Foias, O. Manley & R. Temam, Attractors for the Bénard problem: existence and physical bounds on their fractal dimension. *Nonlinear Analysis* 11 (1987), 939–967.
[26] G.P. Galdi & M. Padula, A new approach to energy theory in the stability of fluid motion, *Arch. Rational Mech. Anal.* 110 (1990), 187–286.
[27] I. Gyöngy and N. V. Krylov, On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces. *Stochastics* 6(3-4):153-173, 1981/82.
[28] E. Hausenblas, P. A. Razafimandimby and M. Sango. Martingale solution to differential type fluids of grade two driven by random force of Lévy type. *Potential Anal.* 38(4):1291-1331, 2013.
[29] P. Imkeller and I. Pavlyukevich. First exit times of SDEs driven by stable Lévy processes. *Stochastic Process. Appl.* 116:611–642, 2006.
[30] N. H. Katz & N. Pavlović. Finite time blow-up for a dyadic model of the Euler equations, *Trans. Amer. Math. Soc.* 357 (2005), 695–708.
[31] J. U. Kim. Strong solutions of the stochastic Navier-Stokes equations in $\mathbb{R}^3$. *Indiana Univ. Math. J.* 59(4):1417-1450, 2010.
[32] A. Kupiainen. Statistical theories of turbulence. In *Advances in Mathematical Sciences and Applications*. Gakkotosho, Tokyo, 2003.
[33] O. Ladyzhenskaya & V. Solonnikov, Solution of some nonstationary magnetohydrodynamical problems for incompressible fluid, *Trudy Steklov Math. Inst.* 59 (1960), 115–173; in Russian.
[34] V. S. Lvov, E. Podivilov, A. Pomyalov, I. Procaccia & D. Vandembroucq, Improved shell model of turbulence, *Physical Review E.* 58 (1998), 1811–1822.
[35] R. Mikulevicius. On strong $H^\frac{7}{2}$-solutions of stochastic Navier-Stokes equation in a bounded domain. *SIAM J. Math. Anal.* 41(3):1206-1230, 2009.
[36] E. Motyl. Stochastic Navier-Stokes equations driven by Lévy noise in unbounded 3D domains. *Potential Anal.* 38(3):863-912, 2013.
[37] N. Glatt-Holtz and M. Ziane. Strong pathwise solutions of the stochastic Navier-Stokes system. *Adv. Differential Equations* 14(5-6):567-600, 2009.
[38] K. Ohkitani & M. Yamada, Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully developed model of turbulence, *Prog. Theor. Phys.* 89 (1989), 329–341.
[39] E. Pardoux, *Équations aux dérivées partielles stochastiques non linéaires monotones; Etude de solutions fortes de type Itô*. Thèse (PhD Thesis), Université Paris Sud, 1975.
[40] S. Peszat and J. Zabczyk. *Stochastic partial differential equations with Lévy noise. An evolution equation approach*. Volume 113 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.
[41] P. E. Protter. *Stochastic integration and differential equations*. Second edition. Volume 21 of *Applications of Mathematics* (New York). *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2004.
[42] R. Mikulevicius and B.L. Rozovskii. Stochastic Navier-Stokes Equations and Turbulent Flows. *SIAM J. Math. Anal.*, 35(5):1250-1310, 2004.
[43] B. Rüdiger and G. Ziglio. Itô formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces. *Stochastics.* 78(6):377–410.
[44] M. Sermange & R. Temam, Some mathematical questions related to MHD equations, *Communications in Pure and Applied Mathematics* 36 (1983), 635–664.

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