FLAG VARIETIES AS EQUIVARIANT COMPACTIFICATIONS OF $G^n_a$

IVAN V. ARZHANTSEV

Abstract. Let $G$ be a semisimple affine algebraic group and $P$ a parabolic subgroup of $G$. We classify all flag varieties $G/P$ which admit an action of the commutative unipotent group $G^n_u$ with an open orbit.

Introduction

Let $G$ be a connected semisimple affine algebraic group of adjoint type over an algebraically closed field of characteristic zero, and $P$ be a parabolic subgroup of $G$. The homogeneous space $G/P$ is called a (generalized) flag variety. Recall that $G/P$ is complete and the action of the unipotent radical $P_u$ of the opposite parabolic subgroup $P^-$ on $G/P$ by left multiplication is generically transitive. The open orbit $O$ of this action is called the big Schubert cell on $G/P$. Since $O$ is isomorphic to the affine space $A^n$, where $n = \dim G/P$, every flag variety may be regarded as a compactification of an affine space.

Notice that the affine space $A^n$ has a structure of the vector group, or, equivalently, of the commutative unipotent affine algebraic group $G^n_u$. We say that a complete variety $X$ of dimension $n$ is an equivariant compactification of the group $G^n_u$, if there exists a regular action $G^n_u \times X \to X$ with a dense open orbit. A systematic study of equivariant compactifications of the group $G^n_u$ was initiated by B. Hassett and Yu. Tschinkel in [4], see also [10] and [1].

In this note we address the question whether a flag variety $G/P$ may be realized as an equivariant compactification of $G^n_u$. Clearly, this is the case when the group $P_u^-$, or, equivalently, the group $P_u$ is commutative. It is a classical result that the connected component $\tilde{G}$ of the automorphism group of the variety $G/P$ is a semisimple group of adjoint type, and $G/P = \tilde{G}/Q$ for some parabolic subgroup $Q \subset \tilde{G}$. In most cases the group $\tilde{G}$ coincides with $G$, and all exceptions are well known, see [6], [7, Theorem 7.1], [12, page 118], [8, Section 2]. If $\tilde{G} \neq G$, we say that $(\tilde{G}, Q)$ is the covering pair of the exceptional pair $(G, P)$. For a simple group $G$, the exceptional pairs are $(\text{PSp}(2r), P_1)$, $(\text{SO}(2r + 1), P_r)$ and $(G_2, P_1)$ with the covering pairs $(\text{PSL}(2r), P_1)$, $(\text{PSO}(2r + 2), P_{r+1})$ and $(\text{SO}(7), P_1)$ respectively, where $PH$ denotes the quotient of the group $H$ by its center, and $P_i$ is the maximal parabolic subgroup associated with the $i$th simple root. It turns out that for a simple group $G$ the condition $\tilde{G} \neq G$ implies that the unipotent radical $Q_u$ is commutative and $P_u$ is not. In particular, in this case $G/P$ is an equivariant compactification of $G^n_u$. Our main result states that these are the only possible cases.

\textit{Date}: March 19, 2010.
\textit{2010 Mathematics Subject Classification}. Primary 14M15; Secondary 14L30.
\textit{Key words and phrases}. Semisimple group, parabolic subgroup, flag variety, automorphism.
\textit{Supported by RFBR grants 09-01-00648-a, 09-01-90416-Ukr-f-a, and the Deligne 2004 Balzan prize in mathematics.
Theorem 1. Let $G$ be a connected semisimple group of adjoint type and $P$ a parabolic subgroup of $G$. Then the flag variety $G/P$ is an equivariant compactification of $\mathbb{G}_a^n$ if and only if for every pair $(G^{(i)}, P^{(i)})$, where $G^{(i)}$ is a simple component of $G$ and $P^{(i)} = G^{(i)} \cap P$, one of the following conditions holds:

1. The unipotent radical $P_u^{(i)}$ is commutative;
2. The pair $(G^{(i)}, P^{(i)})$ is exceptional.

For convenience of the reader, we list all pairs $(G, P)$, where $G$ is a simple group (up to local isomorphism) and $P$ is a parabolic subgroup with a commutative unipotent radical:

- $(\text{SL}(r + 1), P_i)$, $i = 1, \ldots, r$; $(\text{SO}(2r + 1), P_i)$; $(\text{Sp}(2r), P_i)$;
- $(\text{SO}(2r), P_i)$, $i = 1, r - 1$, $(E_6, P_i)$, $i = 1, 6$; $(E_7, P_i)$,

see [9 Section 2]. The simple roots $\{\alpha_1, \ldots, \alpha_r\}$ are indexed as in [2] Planches I-IX. Note that the unipotent radical of $P_i$ is commutative if and only if the simple root $\alpha_i$ occurs in the highest root $\rho$ with coefficient 1, see [11 Lemma 2.2]. Another equivalent condition is that the fundamental weight $\omega_i$ of the dual group $G^v$ is minuscule, i.e., the weight system of the simple $G^v$-module $V(\omega_i)$ with the highest weight $\omega_i$ coincides with the orbit $W\omega_i$ of the Weyl group $W$.

1. Proof of Theorem

If the unipotent radical $P_u$ is commutative, then the action of $P_u$ on $G/P$ by left multiplication is the desired generically transitive $\mathbb{G}_a^n$-action, see, for example, [5 pp. 22-24]. The same arguments work when for the connected component $\tilde{G}$ of the automorphism group $\text{Aut}(G/P)$, one has $G/P = \tilde{G}/Q$ and the unipotent radical $Q_u$ is commutative. Since

$G/P \cong G^{(1)}/P^{(1)} \times \cdots \times G^{(k)}/P^{(k)},$

where $G^{(1)}, \ldots, G^{(k)}$ are the simple components of the group $G$, the group $\tilde{G}$ is isomorphic to the direct product $G^{(1)} \times \cdots \times G^{(k)}$, cf. [5 Chapter 4]. Moreover, $Q_u \cong Q_u^{(1)} \times \cdots \times Q_u^{(k)}$ with $Q^{(i)} = \tilde{G}^{(i)} \cap Q$, Thus the group $Q_u$ is commutative if and only if for each pair $(G^{(i)}, P^{(i)})$ either $P_u^{(i)}$ is commutative or the pair $(G^{(i)}, P^{(i)})$ is exceptional.

Conversely, assume that $G/P$ admits a generically transitive $\mathbb{G}_a^n$-action. One may identify $\mathbb{G}_a^n$ with a commutative unipotent subgroup $H$ of $\tilde{G}$, and the flag variety $G/P$ with $\tilde{G}/Q$, where $Q$ is a parabolic subgroup of $\tilde{G}$.

Let $T \subset B$ be a maximal torus and a Borel subgroup of the group $\tilde{G}$ such that $B \subseteq Q$. Consider the root system $\Phi$ of the tangent algebra $\mathfrak{g} = \text{Lie}(\tilde{G})$ defined by the torus $T$, its decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots associated with $B$, the set of simple roots $\Delta \subseteq \Phi^+$, $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, and the root decomposition

$\mathfrak{g} = \bigoplus_{\beta \in \Phi^-} \mathfrak{g}_\beta \oplus \mathfrak{t} \oplus \bigoplus_{\beta \in \Phi^+} \mathfrak{g}_\beta,$

where $\mathfrak{t} = \text{Lie}(T)$ is a Cartan subalgebra in $\mathfrak{g}$ and $\mathfrak{g}_\beta = \{x \in \mathfrak{g} : [y, x] = \beta(y)x$ for all $y \in \mathfrak{t}\}$ is the root subspace. Set $\mathfrak{q} = \text{Lie}(Q)$ and $\Delta_Q = \{\alpha \in \Delta : \mathfrak{g}_-\alpha \not\subseteq \mathfrak{q}\}$. For every root
\[ \beta = a_1 \alpha_1 + \ldots + a_r \alpha_r \] define \( \deg(\beta) = \sum_{\alpha_i \in \Delta^+} a_i \). This gives a \( \mathbb{Z} \)-grading on the Lie algebra \( \mathfrak{g} \):

\[ \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \quad \text{where} \quad \mathfrak{t} \subseteq \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{g}_\beta \subseteq \mathfrak{g}_k \quad \text{with} \quad k = \deg(\beta). \]

In particular,

\[ \mathfrak{q} = \bigoplus_{k \geq 0} \mathfrak{g}_k \quad \text{and} \quad \mathfrak{q}_- = \bigoplus_{k < 0} \mathfrak{g}_k. \]

Assume that the unipotent radical \( \mathcal{Q}_u^- \) is not commutative, and consider \( \mathfrak{g}_\beta \subseteq [\mathfrak{q}_-, \mathfrak{q}_-] \). For every \( x \in \mathfrak{g}_\beta \setminus \{0\} \) there exist \( z' \in \mathfrak{g}_{\beta'} \subseteq \mathfrak{q}_- \) and \( z'' \in \mathfrak{g}_{\beta''} \subseteq \mathfrak{q}_- \) such that \( x = [z', z''] \). In this case \( \deg(z') > \deg(x) \) and \( \deg(z'') > \deg(x) \).

Since the subgroup \( H \) acts on \( \tilde{G}/Q \) with an open orbit, one may conjugate \( H \) and assume that the \( H \)-orbit of the point \( eQ \) is open in \( \tilde{G}/Q \). This implies \( \mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h} \), where \( \mathfrak{h} = \text{Lie}(H) \). On the other hand, \( \mathfrak{g} = \mathfrak{q}_- \). So every element \( y \in \mathfrak{h} \) may be (uniquely) written as \( y = y_1 + y_2 \), where \( y_1 \in \mathfrak{q} \), \( y_2 \in \mathfrak{q}_- \), and the linear map \( \mathfrak{h} \rightarrow \mathfrak{q}_- \), \( y \mapsto y_2 \), is bijective. Take the elements \( y, y', y'' \in \mathfrak{h} \) with \( y_2 = x, y_2' = z', y_2'' = z'' \). Since the subgroup \( H \) is commutative, one has \([y', y''] = 0\).

Thus

\[ [y'_1 + y'_2, y''_1 + y''_2] = [y'_1, y''_1] + [y'_2, y''_2] + [y'_1, y''_2] + [y'_2, y''_1] = 0. \]

But

\[ [y'_2, y''_2] = x \quad \text{and} \quad [y'_2, y'_1] + [y'_2, y''_2] + [y'_2, y''_2] \in \bigoplus_{k > \deg(x)} \mathfrak{g}_k. \]

This contradiction shows that the group \( \mathcal{Q}_u^- \) is commutative. As we have seen, the latter condition means that for every pair \((G^{(i)}, P^{(i)})\) either the unipotent radical \( P^{(i)}_u \) is commutative or the pair \((G^{(i)}, P^{(i)})\) is exceptional. The proof of Theorem 1 is completed.

2. Concluding Remarks

If a flag variety \( G/P \) is an equivariant compactification of \( G^u_\alpha \), then it is natural to ask for a classification of all generically transitive \( G^u_\alpha \)-actions on \( G/P \) up to equivariant isomorphism. Consider the projective space \( \mathbb{P}^n \cong \text{SL}(n+1)/P_1 \). In [1], a correspondence between equivalence classes of generically transitive \( G^u_\alpha \)-actions on \( \mathbb{P}^n \) and isomorphism classes of local (associative, commutative) algebras of dimension \( n + 1 \) was established. This correspondence together with classification results from [11] yields that for \( n \geq 6 \) the number of equivalence classes of generically transitive \( G^u_\alpha \)-actions on \( \mathbb{P}^n \) is infinite, see [11, Section 3]. On the contrary, a generically transitive \( G^u_\alpha \)-action on a non-degenerate projective quadric \( Q_n \cong \text{SO}(n+2)/P_1 \) is unique [10, Theorem 4]. It would be interesting to study the same problem for the Grassmannians \( \text{Gr}(k, r + 1) \cong \text{SL}(r + 1)/P_k \), where \( 2 \leq k \leq r - 1 \).

Acknowledgement

The author is indebted to N.A. Vavilov for a discussion which results in this note. Thanks are also due to D.A. Timashev and M. Zaidenberg for their interest and valuable comments.
I.V. ARZHANTSEV

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DEPARTMENT OF ALGEBRA, FACULTY OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, LENINSKIE GORY 1, GSP-1, MOSCOW, 119991, RUSSIA
E-mail address: arjantse@mccme.ru