Vortex Dynamics and the Problem of the Transverse Force in Clean Superconductors and Fermi Superfluids

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Abstract
We review the basic ideas and results on the vortex dynamics in clean superfluid Fermi systems. The forces acting on moving vortices are discussed including the problem of the transverse force which was a matter of confusion for quite a time. Finally, we formulate the equations of the vortex dynamics which include all the forces and the inertial term associated with excitations bound to the moving vortex.

1 Introduction
Study of the vortex dynamics opens promising perspectives for understanding the most fundamental properties of condensed matter especially for superconductors and superfluids. Among them, clean systems, i.e., the systems where the mean free path of quasiparticles is much longer than the characteristic coherence length, offer more intriguing physics than dirty superconductors. For example, one of the fundamental problems can be formulated as follows: Speaking of clean superconductors one can, in particular, think of such a system where no relaxation processes are available, i.e., the mean free path of excitations is infinite. In particular, vortices in such systems should move without dissipation since there is no mechanism to absorb the energy. The vortex velocity should then be parallel to the transport current, which makes the induced electric field perpendicular to the current; the dissipation thus vanishes $j \cdot E = 0$. This contrasts with what we know of dirty superconductors: the vortex motion there is dissipative, and each moving vortex experiences a large friction; it generates an electric field parallel to the transport current which produces energy dissipation.
Clearly, a crossover should occur from dissipative to non–dissipative vortex motion as the quasiparticle mean free path increases. The question is what is the condition which controls the crossover?

This question is of a fundamental importance for our understanding of the dynamics of superconductors and, in a more broad sense, for the understanding of dynamic properties of quantum condensed matter in general. To illustrate the problem one can consider a simple example as follows. One can argue that a time dependent non–dissipative superconducting state, similarly to any other quantum state, can be described by a Hamiltonian dynamics based on a time dependent Schrödinger equation. Such a description for a weakly interacting Bose gas has been suggested by Pitaevskii and Gross [1, 2]. It is widely used for superfluid helium II as well. The Gross–Pitaevskii equation is essentially a nonlinear Schrödinger equation, it has the imaginary factor \(-i\hbar\) in front of the time derivative of the condensate wave function \(\partial\Psi/\partial t\). On the other hand, the time dependent Ginzburg–Landau model which is a particular case of a more general Model F dynamics [3] is believed to describe a relaxation dynamics of superconductors near the transition temperature. In contrast to the Gross–Pitaevskii equation, it has the time derivative \(\partial\Psi/\partial t\) with a real factor in front of it. The question which we are interested in can be formulated as follows: What is the condition when the imaginary prefactor transforms into a real one?

It seems that there is no universal answer to this simple question in general. However, the problem of crossover from non–dissipative to dissipative behavior of a condensed matter state can be solved for the particular example of vortex dynamics. It is known that the relaxation constant in the time dependent Ginzburg–Landau model has in fact a small imaginary part [4, 5, 6] which results in appearance of a small transverse component of the electric field with respect to the current. We shall see later that the transverse component of the electric field increases at the expense of the longitudinal component as the mean free path of excitations grows. The crossover condition, however, does not coincide simply with the condition which divides superconductors between dirty and clean ones. The criterion rather involves the spectrum of excitations localized in the vortex cores; the distance between their levels takes the part of the energy gap. The condition for a non–dissipative vortex motion requires that the relaxation rate of localized excitations is smaller than the distance between the levels. This implies a much longer mean free path of excitations than the condition for a superconductor to be just in a clean limit.

In the present paper we review the basic ideas and results on the vortex dynamics in clean Fermi systems. We concentrate largely on superconductors, the case of superfluid $^3$He can be easily incorporated if one takes the limit of zero charge of carriers in the final results (to be discussed in more detail later). We consider forces acting on moving vortices and clarify the conditions for the crossover from dissipative to non–dissipative vortex dynamics. A great deal of attention is given to the problem of the so called transverse force, i.e., the force perpendicular to the vortex velocity, which was a matter of confusion.
for quite a time since its discovery. Finally, we formulate the equations of the vortex dynamics which include all the forces and the inertial term associated with excitations bound to the moving vortex.

2 Boltzmann kinetic equation approach

The forces on a vortex come from several different sources including the hydrodynamic Magnus force, the force produced by excitations scattered from the vortex, and the force associated with the momentum flow from the heat bath to the vortex through the localized excitations. The whole rich and exciting physics involved in the vortex dynamics can be successfully described by the general microscopic time dependent theory. Here we rather discuss a general picture using a simple semi–classical approach based on the Boltzmann kinetic equation. The semi–classical approach assumes that the wavelength of excitations is much shorter than the superfluid coherence length, $p_F\xi \gg 1$ (from now on we put $\hbar = 1$). This condition is quite safely fulfilled in almost all superconductors and in superfluid $^3$He. Only in some high–temperature superconductors, its accuracy may be not very good. For simplicity, we consider $s$–wave superconductors. We concentrate on isolated vortices such that their cores do not overlap, i.e., on the region of magnetic fields $H \ll H_c^2$.

One can distinguish two types of excitations: Excitations localized in the vortex core, and those which are not localized but move in the vortex potential under the action of magnetic filed. We start our discussion with the localized excitations.

2.1 Localized excitations

We remind that the profile of the order parameter $\Delta (r)$ near the vortex core produces a potential well where localized states with a discrete spectrum exist. The localized states correspond to energies $|\epsilon| < \Delta_\infty$. The spectrum has the so called anomalous branch with the radial quantum number $n = 0$ whose energy varies from $-\Delta_\infty$ to $+\Delta_\infty$ as the particle impact parameter $b$ changes from $-\infty$ to $+\infty$ and crosses $\epsilon = 0$, being an odd function of $b$. For low $\epsilon \ll \Delta_\infty$, the anomalous branch is $\epsilon_0 = -\omega_0\mu$ where $\mu = -bp_\perp$ is the angular momentum, and $p_\perp$ is the momentum in the plane perpendicular to the vortex axis (see also Fig. 1). In a $s$–wave superconductor with an axisymmetric vortex, the angular momentum $\mu$ is quantized and so is the spectrum, $\omega_0$ being the distance between the discrete levels in the vortex core. The spectrum also has branches with $n \neq 0$ which are separated from the one with $n = 0$ by energies of the order of $\Delta$. They are even in $b$ and symmetric in energy with respect to $\epsilon = 0$. We denote the separation between the levels with neighboring angular momenta through

$$\omega_n = p_\perp^{-1} \frac{\partial \epsilon_n}{\partial b} = -\frac{\partial \epsilon_n}{\partial \mu}. \quad (1)$$
Figure 1: Spectrum of excitations localized in the vortex core.

The interlevel spacing $\omega_n(b)$ is an even function of $b$ for $n = 0$, and it is an odd function for $n \neq 0$.

We choose the direction of the $z$ axis in such a way that the vortex has a positive circulation. The $z$ axis is thus parallel to the magnetic field for positive charge of carriers, and it is antiparallel to it for negative charge: $\hat{z} = \hbar \text{sign}(e)$.

Since the particle velocity $\mathbf{v}_\perp$ in the plane perpendicular to the vortex axis makes an angle $\alpha$ with the $x$ axis, the cylindrical coordinates of the position point $(\rho, \phi)$ are connected with the impact parameter and the coordinate along the trajectory through $\rho^2 = b^2 + s^2$ where

$$b = \rho \sin(\phi - \alpha) ; \quad s = \rho \cos(\phi - \alpha).$$  \hspace{1cm} (2)

The coordinates are shown in Fig. 2.

The first step is as follows. We assume that the quasiclassical spectrum $\epsilon_n(b)$ of a particle plays the role of its effective Hamiltonian. We can thus invoke the Boltzmann equation in the canonical form

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \alpha} \frac{\partial \epsilon_n}{\partial \alpha} - \frac{\partial \epsilon_n}{\partial \alpha} \frac{\partial f}{\partial \mu} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}},$$  \hspace{1cm} (3)

to describe the quasiparticle distribution $f$. Equation (3) has been derived in Ref. [9] from the set of microscopic kinetic equations.

In the time derivative, the energy $\epsilon_n$ contains a time dependence through $\mu(t) = [(r - v_L^t) \times \mathbf{p}] \cdot \hat{z}$ such that

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial \epsilon} \frac{\partial \epsilon_n}{\partial \mu} \frac{\partial \epsilon_n}{\partial t} = \frac{\partial f^{(0)}}{\partial \epsilon} \frac{\partial \epsilon_n}{\partial \mu} \left( [\mathbf{p}_\perp \times \mathbf{v}_L] \cdot \hat{z} \right).$$  \hspace{1cm} (4)
Kinetic equation takes the form

\[
\frac{\partial f_1}{\partial t} + \frac{\partial f^{(0)}}{\partial \epsilon} \frac{\partial \epsilon_n}{\partial \mu} ([p_\perp \times v_L] \cdot \hat{z}) + \frac{\partial f_1}{\partial \alpha} \frac{\partial \epsilon_n}{\partial \mu} - \frac{\partial \epsilon_n}{\partial \alpha} \frac{\partial f_1}{\partial \mu} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}},
\]

where the distribution function is separated into an equilibrium and a non-equilibrium parts, \( f = f^{(0)} + f_1 \), respectively. Here \( f^{(0)} = 1 - 2n_\epsilon = \tanh(\epsilon/2T) \) with \( n_\epsilon \) being the Fermi function.

We shall simplify the collision integral in Eq. (3) using the relaxation–time approximation

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = - \frac{f_1}{\tau_n}
\]

where \( \tau_n \sim \tau \). With this approximation, the mean free time can be of any origin. For definiteness, we assume that the most effective relaxation is brought about by impurities, as is the case in almost all practical superconducting compounds. However, the relaxation time \( \tau \) can be due to the electron–phonon interaction, as well, or due to quasiparticle–quasiparticle scattering. The latter is, in fact, the only relaxation mechanism available in \( ^3\text{He} \). Therefore, the term “clean” does not necessarily mean a low concentration of impurities but simply refers to a situation when the mean free time is long.

Equation (4) with the collision integral in the form of Eq. (3) is easy to solve. For an axi–symmetric s-wave vortex the energies \( \epsilon_n \) do not depend on \( \alpha \) and the term \( \partial \epsilon_n/\partial \alpha \) vanishes. Let us take the distribution function in the
form
\[ f_1 = -\frac{\partial f(0)}{\partial \epsilon} \left[ \gamma_O (v_L \times p_{\perp}) \hat{z} \right] + \gamma_H (v_L \cdot p_{\perp}) \] (7)

where the factors \( \gamma_{O,H} \) are to be found from the Boltzmann equation (8). The result for a steady vortex motion is

\[ \gamma_O = \frac{\omega_n \tau_n}{\omega_n \tau_n^2 + 1}, \quad \gamma_H = \frac{\omega_n^2 \tau_n^2}{\omega_n \tau_n^2 + 1}. \] (8)

This generalizes the result first obtained in Ref. [1].

2.2 Delocalized excitations

A delocalized particle moves mostly far from the vortex core where the order parameter is constant and the superfluid velocity potential \( pv_s \) is small compared to \( \Delta \). Kinetic equation for delocalized excitations can thus be written as for a particle in a magnetic field with a semi–classical spectrum

\[ \epsilon_p = \sqrt{\xi^2_p + \Delta^2_{\infty}} \] (9)

where \( \xi_p = p^2/2m - E_F \). As shown in Ref. [12] the kinetic equation for a particle moving in a vortex array has the conventional form

\[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p} \cdot v_g + \frac{\partial f}{\partial r} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}. \]

The force is the elementary Lorentz force

\[ f = \frac{\partial p}{\partial t} = \frac{e}{c} v_g \times H = \frac{\omega_c}{g} [p_F \times \hat{z}] \] (10)

where \( \omega_c = |e| H/mc \) is the cyclotron frequency,

\[ v_g = \frac{\partial \epsilon_p}{\partial p} = \frac{v_F}{g} \] (11)

is the group velocity, and

\[ g = \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2_{\infty}}}. \]

The driving term can be written as

\[ \frac{\partial f}{\partial t} = \frac{\partial f(0)}{\partial \epsilon} \frac{\partial \epsilon}{\partial t} \]

where

\[ \frac{\partial \epsilon}{\partial t} = e v_g \cdot \mathbf{E} = \frac{e}{c} v_g \cdot [H \times v_L] = \frac{\omega_c}{g} p_{\perp} \cdot [\hat{z} \times v_L]. \]
The kinetic equation becomes
\[ \frac{\partial f_1}{\partial t} + \frac{\partial f^{(0)}}{\partial \epsilon} \frac{\omega_r}{g} p_\perp \cdot [\hat{z} \times \mathbf{v}_L] + \frac{\partial f_1}{\partial p} \cdot \frac{\omega_r}{g} [p_F \times \hat{z}] = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}. \] (12)

We omit the spatial derivative of the distribution function since it is constant in space.

For the energy spectrum of Eq. (9) the collision integral is
\[ \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{1}{g \tau} f_1. \]

Kinetic equation (12) takes the final form
\[ p_\perp \cdot [\hat{z} \times \mathbf{v}_L] \frac{\partial f^{(0)}}{\partial \epsilon} - \frac{\partial f_1}{\partial \alpha} = -\frac{1}{\omega_c \tau} f_1. \]

Its solution is Eq. (7) with
\[ \gamma'_O = \frac{\omega_c \tau}{\omega_c^2 \tau^2 + 1}, \quad \gamma'_H = \frac{\omega_c^2 \tau^2}{\omega_c^2 \tau^2 + 1}. \] (13)

### 3 Forces

At the second step we calculate the force acting on a vortex from the environment. This force is exerted via excitations which travel near the vortex through transfer of their momenta to the vortex. Consider first localized excitations. The transferred momentum is
\[ \mathbf{F}_{\text{env}}^{(\text{loc})} = \frac{1}{2} \sum_n \int \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{d\mu}{2\pi} \frac{\partial p_n}{\partial t} f_1 = -\frac{1}{2} \sum_n \int \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{d\mu}{2\pi} \frac{\partial \epsilon_n}{\partial b} [\hat{z} \times \hat{v}_\perp] f_1. \] (14)

Here we make use of the Hamilton equation
\[ \frac{\partial p_n}{\partial t} = -\nabla \epsilon_n = -\frac{\partial \epsilon_n}{\partial b} [\hat{z} \times \hat{v}_\perp]. \] (15)

The second equality follows from the fact that, in the coordinate frame \((s, b)\) of Fig. 3, the energy only depends on the particle impact parameter \(b\).

The transfer of the momentum from delocalized excitations in the semi-classical approximation has the form
\[ \mathbf{F}_{\text{env}}^{(\text{del})} = \frac{1}{2} \sum_n \int \frac{dp_z}{2\pi} \int \frac{d\alpha}{2\pi} \frac{\partial p_n}{\partial t} f_1 \]

where \(\partial p_n \partial t\) is given by Eq. (15). The force becomes
\[ \mathbf{F}_{\text{env}}^{(\text{del})} = \frac{1}{2} \sum_n \int \frac{dp_z}{2\pi} \int \frac{d\alpha}{2\pi} [p_F \times \hat{z}] \frac{\omega_c}{g} f_1. \]
The sum is over quantized states of a semi–classical quasiparticle in a magnetic field. Indeed, the energy spectrum Eq. (9) becomes quantized \[12\] in a magnetic field and turns into a modified Landau spectrum $\epsilon_n$ such that

$$\sqrt{\epsilon_n^2 - \Delta_\infty^2} = \omega_c n - p_\perp^2 / 2m.$$ 

The quantity

$$\frac{g}{\omega_c} = \frac{\partial n}{\partial \epsilon}$$

is thus the density of states. As a result,

$$\sum_n \rightarrow \int \frac{\partial n}{\partial \epsilon} \mathrm{d}\epsilon$$

and the force becomes \[14\]

$$F_{\text{env}}^{(\text{del})} = \int_{\epsilon > \Delta_\infty} \mathrm{d}\epsilon \int \frac{dp_\perp}{2\pi} \int \frac{d\alpha}{2\pi} \left[ p_\perp \times \hat{z} \right] f_1. \quad (16)$$

The total force is

$$F_{\text{env}} = F_{\text{env}}^{(\text{loc})} + F_{\text{env}}^{(\text{del})}. \quad (17)$$

With the Ansatz \[7\], the total force Eq. (17) splits into two terms $F_{\text{env}} = F_\parallel + F_\perp$, with the friction $F_\parallel$ and transverse $F_\perp$ forces given by

$$F_\parallel = -\pi N \left[ \left\langle \left\langle \sum_n \int \omega_n \gamma_0 \frac{\partial f^{(0)}}{\partial \epsilon} \mathrm{d}\mu / 2 \right\rangle + \left( 1 - \tanh \frac{\Delta_\infty}{2T} \right) \gamma_0' \right\rangle \right] v_L, \quad (18)$$

$$F_\perp = \pi N \left[ \left\langle \left\langle \int \omega_0 \gamma_H \frac{\partial f^{(0)}}{\partial \epsilon} \mathrm{d}\mu / 2 \right\rangle + \left( 1 - \tanh \frac{\Delta_\infty}{2T} \right) \gamma_H' \right\rangle \left[ \hat{z} \times v_L \right] \right], \quad (19)$$

where $N$ is the quasiparticle (electron) density, $\left\langle \left\langle \ldots \right\rangle \right\rangle$ is the average over the Fermi surface with the weight $\pi p_\perp^2$,

$$\left\langle \left\langle \ldots \right\rangle \right\rangle = V_F^{-1} \int \pi p_\perp^2 \mathrm{d}p_z \ldots,$$ \quad (20)

and $V_F$ is the volume encompassed by the Fermi surface. For an isotropic Fermi surface,

$$\left\langle \left\langle \ldots \right\rangle \right\rangle = \frac{3}{4} \int \sin^3 \theta \, d\theta \ldots,$$

where $p_z = p_F \cos \theta$. Only the spectral branch with $n = 0$ contributes to the transverse force $F_\perp$ as all $\omega_n$ with $n \neq 0$ are odd functions of $\mu$ and thus drop out of the sum over $n$ in Eq. \[14\].

We emphasize that the force $F_{\text{env}}$ is defined as the response of the whole environment to the vortex displacement. It is therefore the total force acting on
the vortex from the ambient system, including all partial forces such as the longitudinal friction force and the non-dissipative transverse force. The transverse force, in turn, includes various parts which can be identified historically as the Iordanskii force, the spectral flow force, and the Magnus force. We shall discuss this later in more detail in Section 4.

3.1 Flux flow conductivity

The force from the environment is balanced by the Lorentz force:

\[
F_L = \frac{\Phi_0}{c} j_t \times \hat{z} \text{sign}(e),
\]

with the flux quantum \(\Phi_0 = \pi c/|e|\). The force balance equation

\[
F_L + F_{\text{env}} = 0 \tag{22}
\]
determines the transport current in terms of the vortex velocity and thus allows us to find the flux flow conductivity tensor. The longitudinal force defines the friction coefficient in the vortex equation of motion and determines the Ohmic component of the conductivity \(\sigma_O\). Expressing the vortex velocity \(v_L\) through the average electric field \(E\), as

\[
v_L = \frac{c}{\mu_0} \left( E \times \hat{z} \right) \frac{\omega_c}{B \text{sign}(e)},
\]

we find

\[
\sigma_O = \frac{N|e|c}{B} \left[ \left\langle \sum_n \int \frac{\omega_n \tau_n}{\omega_n^2 \tau_n^2 + 1} \frac{\partial f(0)}{\partial \epsilon} \, d\epsilon \right\rangle + \left( 1 - \tanh \frac{\Delta_{\infty}}{2T} \right) \frac{\omega_c \tau}{\omega_c^2 \tau^2 + 1} \right].
\]

(23)

The transverse force determines the Hall conductivity

\[
\sigma_H = \frac{Ne_c}{B} \left[ \left\langle \int \frac{\omega_n^2 \tau_n^2}{\omega_n^2 \tau_n^2 + 1} \frac{\partial f(0)}{\partial \epsilon} \, d\epsilon \right\rangle + \left( 1 - \tanh \frac{\Delta_{\infty}}{2T} \right) \frac{\omega_c^2 \tau^2}{\omega_c^2 \tau^2 + 1} \right].
\]

(24)

The main conclusion is that the Ohmic and Hall conductivities depend on the purity of the sample through the parameters \(\omega_0 \tau\) (remind that \(\omega_n \sim \omega_0\)) and \(\omega_c \tau\). Note that \(\omega_c \sim (H/H_c) \omega_0\) thus \(\omega_c \ll \omega_0\). One can distinguish two regimes: moderately clean \(\omega_0 \tau \ll 1\) and superclean \(\omega_0 \tau \gg 1\). Note that the moderately clean regime still requires that the superconductor is clean in the usual sense \(\Delta_{\infty} \tau \gg 1\).

In the moderately clean limit where \(\omega_0 \tau \ll 1\), the conductivity roughly follows the Bardeen and Stephen expression [20] at low temperatures though it exhibits an extra temperature-dependent factor \(\Delta_{\infty}/T_c\) on approaching \(T_c\) [14]

\[
\sigma_O \sim \sigma_n \frac{H_c^2 \Delta_{\infty}}{H_c T_c}.
\]

The factor \(\Delta_{\infty}/T_c\) appears because the number of delocalized quasiparticles contributing to the vortex dynamics decreases near \(T_c\). This extra factor has
been recently identified experimentally \[21\]. The Hall conductivity and the Hall angle are small \(\sigma_H \sim (\omega_0 \tau) \sigma_0\) and \(\tan \Theta_H \sim (\omega_0 \tau)\), respectively. The contribution from delocalized states are not important since \(\omega_c \ll \omega_0\).

In the superclean limit \(\omega_0 \tau \gg 1\), on the contrary, the Ohmic conductivity is small. The vortex dynamics becomes non–dissipative. In particular, if \(\omega_c \tau \ll 1\), the corresponding Hall conductivity is

\[
\sigma_H = \frac{Nec}{B} \tanh \left( \frac{\Delta \infty}{2T} \right).
\]

If \(\omega_c \tau \gg 1\), the hyperbolic tangent should be replaced with unity.

## 4 Transverse force

Let us now discuss the forces acting on a moving vortex in more detail. The friction force is determined by the Eq. (18). It is proportional to the mean free path of excitations in the moderately clean regime and vanishes in the superclean limit. The transverse force Eq. (19) deserves a more careful discussion because it has been a matter of controversy for a long time since first calculated for vortices in helium II by Lifshitz and Pitaevskii \[22\] and then by Iordanskii \[16\]. The review \[15\] tells about old disputes (see also \[23\]). Recently, the presence of the transverse force has been questioned in Refs. \[24, 25\].

As we see, however, the microscopic picture gives a finite transverse force in a full accordance with the symmetry arguments which allow a transverse force in a chiral system such as a moving vortex in presence of a superflow. Of course, the magnitude of the transverse force is not an universal quantity. It appears to depend on parameters of the system. Indeed, in the superclean limit, when both \(\omega_c \tau\) and \(\omega_0 \tau\) are much larger than unity, the factors \(\gamma_H = \gamma'_H = 1\), and the transverse force Eq. (19) becomes

\[
F_\perp = \pi N[\hat{z} \times \mathbf{v}_L].
\] (25)

The balance Eq. (22) gives the transport current in the form

\[
\mathbf{j}_{tr} = Ne v_L.
\] (26)

This equation is consistent with the Helmholtz theorem of conservation of circulation in an ideal fluid: vortices move together with the flow. The correction to the distribution function of excitations is simply

\[
f_1 = -\frac{\partial f^{(0)}}{\partial \epsilon}(\mathbf{v}_L \cdot \mathbf{p}_\perp)
\] (27)

such that the full distribution function is \(f = f^{(0)} (\epsilon - \mathbf{v}_L \cdot \mathbf{p}):\) excitations move together with the vortex being in equilibrium with the vortex array. To understand this let us consider first the mean free path of delocalized excitations with...
respect to their collisions with vortices. If the vortex cross section is $\sigma_v$, the mean free path is $\ell_v = 1/\sigma_v n_v$ where $n_v = B/\Phi_0$ is the density of vortices. We shall see in a moment that the vortex cross section is $\sigma_v \sim p^{−1}F$ so that

$$\ell_v \sim pF/n_v \sim v_F/\omega_c$$

(28)
i.e., $\ell \sim r_L$ where $r_L$ is the Larmor radius. In the limit $\omega_c \tau \gg 1$, the vortex mean free path $\ell_v$ becomes shorter than the impurity mean free path $\ell_{imp} = v_F \tau$ so that the delocalized excitations scatter on vortices more frequently than on impurities and thus come to equilibrium with moving vortices. A similar consideration also applies to localized excitations: In the limit $\omega_0 \tau \gg 1$ interaction with a vortex is more effective than relaxation on impurities, and the excitations come to equilibrium with the moving vortex.

In the intermediate case when $\omega_c \tau \to 0$ but $\omega_0 \tau \sim 1$, only the localized excitations are out of equilibrium. The excitations with energies above the energy gap $\Delta_\infty$ have $\ell_v \gg \ell_{imp}$ and are now in equilibrium with the heat bath: both $\gamma'_0 = 0$, and $\gamma'_H = 0$. As a result these excitations do not influence considerably the vortex motion. Delocalized excitations, however, do affect the vortex motion if $\omega_c \tau$ is comparable with unity. Finally, in the limit $\omega_0 \tau, \omega_c \tau \ll 1$ the transverse force Eq. (19) disappears. The longitudinal force dominates: the vortex dynamics is dissipative.

In general, we can identify several contribution to the transverse force. One can present the full transverse force Eq. (19) in the form

$$\mathbf{F}_\perp = -\pi N_s [\mathbf{v}_L \times \hat{z}] - \pi N_n [\mathbf{v}_L \times \hat{z}] + \mathbf{F}_{sf}.$$  

(29)
The force

$$\mathbf{F}_{sf} = \pi N \left\{ \int \frac{d\mu}{2} \frac{\partial f^{(0)}}{\partial \epsilon} \frac{\omega_0}{\omega_0^2 + 1} \right\} [\mathbf{v}_L \times \hat{z}] + \pi N \left( 1 - \tanh \frac{\Delta_\infty}{2T} \right) \frac{1}{\omega_c^2 \tau^2 + 1} [\mathbf{v}_L \times \hat{z}]$$  

(30)
is called the spectral flow force [17].

The spectral flow force $\mathbf{F}_{sf}$ is due to the momentum flow from the Fermi sea of normal excitations to the moving vortex via the gapless spectral branch going through the vortex core from negative to positive energies [18], [19]. Due to the time dependent angular momentum of the excitations $\mu(t) = [(\mathbf{r} - \mathbf{v}_L t) \times \hat{r}] \cdot \hat{z}$, there appears a flow (with the velocity $\partial \mu/\partial t$) of spectral levels characterized by the angular momentum $\mu$. Each particle on a level carries a momentum $p_F$. The momentum transfer to the vortex is effective if the quasiparticle relaxation occurs quickly: the factor $(\omega_0^2 + 1)^{-1}$ in the first line of Eq. (30) accounts for the relaxation on localized levels: the relaxation and hence the momentum transfer is complete for $\omega_0 \tau < 1$, and vanishes in the opposite limit. The first term in Eq. (30) thus describes a disorder–mediated momentum flow along
the anomalous chiral branch $E_0(\mu)$ for energies below the gap. The second line in Eq. (30) accounts for the spectral flow for energies above the gap. The corresponding factor which takes into account the relaxation rate is $(\omega_c^2 \tau^2 + 1)^{-1}$.

Note that $F_{env}$ is calculated in the reference frame where the normal component is at rest $v_n = 0$ for stationary vortices. In superconductors, the normal velocity is always zero in equilibrium, the excitations being at rest in the reference frame associated with the crystal lattice. In superfluid $^3$He, the situation is similar: the normal component has a rather large viscosity so that it is normally at rest in the container frame of reference. For a nonzero $v_n$, we would need to replace $v_L$ with $v_L - v_n$ everywhere in $F_{env}$. Simultaneously, the Lorentz force would also include the contribution due to the normal velocity.

Let us now turn to the force balance equation (22). In presence of an electric field, the transport current is not entirely due to a supercurrent, a part of it being carried by delocalized quasiparticles. Far from the vortex core, the quasiparticle current is

$$j^{(qp)} = -2\nu(0)e\int_{\epsilon > \Delta} v_\perp g f_1 d\epsilon \frac{d\Omega p}{4\pi} \tag{31}$$

where $\nu(0)$ is the single-spin normal density of states, and $d\Omega_p$ is the elementary solid angle in the direction of the momentum $p$. Using Eqs. (7) and (13) we find

$$j^{(qp)} = N_n e\gamma'_H v_L + N_n e\gamma'_O [z \times v_L] \tag{32}$$

where the density of normal quasiparticles is

$$N_n = N \int_{\epsilon > \Delta} \partial f^{(0)} \partial \epsilon d\epsilon. \tag{33}$$

Writing the transport current as $j_{tr} = N_s e v_s + j^{(qp)}$ we get the force balance Eq. (22) in the form

$$F_M + F_L^{(qp)} + F_I + F_{sf} + F_{\parallel} = 0. \tag{34}$$

Here

$$F_M = \pi N_s [(v_s - v_L) \times \hat{z}]$$

is the Magnus force;

$$F_L^{(qp)} = \frac{\Phi_0}{c} [j^{(qp)} \times \hat{z}] \text{sign} (e) \tag{35}$$

is the Lorentz force from the quasiparticle current Eq. (32), and

$$F_I = \pi N_n [(v_n - v_L) \times \hat{z}]$$

is called the Iordanskii force [14]. The Iordanskii force is the counterpart of the Magnus force for normal excitations. We have included the normal velocity $v_n$ to make this similarity more transparent.
The spectral flow force vanishes in the limit $\tau \to \infty$ when both $\omega_0 \tau \gg 1$ and $\omega_c \tau \gg 1$. In this limit the transverse force is given by Eq. (29). The quasiparticle current is $j^{(qp)} = N_n e v_L$ so that the force from the quasiparticle current compensates the Iordanskii force. The force balance Eq. (34) reduces to $F_M = 0$. The vortex thus moves with the superfluid velocity $v_L = v_s$. As follows from Eq. (27), quasiparticles also have a velocity $v_L$ so that all the particles move together which amounts to the total current as in Eq. (26).

On the contrary, the spectral flow force has its maximum value for moderately clean limit, $\omega_0 \tau \ll 1$. Equation (30) gives in this limit

$$F_{sf} = \pi N [v_L \times \hat{z}].$$

This completely compensates the first two terms in Eq. (29), i.e., the Iordanskii force and the part of the Magnus force that contains the vortex velocity; the transverse force vanishes. The quasiparticle current vanishes even faster because $\omega_c \ll \omega_0$. The Lorentz force is balanced only by a friction force $F_\parallel$. As a result, the dissipative dynamics is restored.

4.1 Low–field limit and superfluid $^3$He

The low–field limit when $\omega_c \tau \ll 1$ is most practical for superconductors. Moreover, this regime is realized in electrically neutral superfluids such as $^3$He. At the first glance, it is simply because $\omega_c$ vanishes together with the charge of carriers. However, this is not completely correct. In fact, to estimate a deviation from equilibrium of delocalized excitations in this case one has again to compare the mean free path of excitations with their mean free path with respect to scattering by vortices. Keeping in mind that the vortex density is $n_v = 2\Omega/\kappa$ where $\Omega$ is an angular velocity of a rotating container and $\kappa = \pi/m$ is the circulation quantum, Eq. (28) gives $\ell_v \sim v_F/\Omega$. We observe that the cyclotron frequency is replaced with the rotation velocity in a full compliance with the Larmor theorem. The ratio of the particle–particle mean free path $\ell$ to the vortex mean free path is $\ell/\ell_v \sim \Omega \tau$. With the practical rotation velocity $\Omega$ of a few radians per second one always has $\ell_v$ exceedingly larger than $\ell$. Delocalized excitations are thus at rest in the container frame.

Consider this regime in more detail. Since delocalized excitations are in equilibrium the quasiparticle current vanishes. The force balance becomes

$$F_M + F_I + F_{sf} + F_\parallel = 0$$

where the spectral flow force is

$$F_{sf} = \pi N \left\langle \int \frac{d\mu}{2} \frac{\partial f(0)}{\partial \epsilon} \frac{\omega_0}{\omega_0^2 + 1} \right\rangle [v_L \times \hat{z}]$$

$$+ \pi N \left( 1 - \tanh \frac{\Delta_\infty}{2T} \right) [v_L \times \hat{z}].$$

13
It is interesting to observe that the spectral flow force from delocalized states in this case is related to the anomalous contribution to the transverse vortex cross section for scattering of delocalized quasiparticles. The vortex cross sections were calculated in Refs. [26, 27]. The transverse cross section is

$$
\sigma_\perp = \frac{\pi}{p_\perp} \left[ \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta_\infty^2}} - 1 \right]
$$

per unit vortex length. Note that the transport cross section which is responsible for the scattering contribution to the longitudinal force is vanishingly small in the semi-classical limit. Inserting Eq. (38) into the expression for the force exerted on the vortex by scattered excitations,

$$
F^{(sc)}_\perp = \int_{\epsilon > \Delta_\infty} \frac{d\epsilon}{2} \frac{\partial f^{(0)}}{\partial \epsilon} \int \frac{dp_\perp}{2\pi} p_\perp^3 \sigma_\perp \hat{z} \times \mathbf{v}_L, \quad (39)
$$

we recover those contributions to Eq. (29) which are due to the normal excitations, namely the term $-\pi N_n [\mathbf{v}_L \times \hat{z}]$ and the last term in the spectral flow force Eq. (37).

The first term in Eq. (38) corresponds to the cross section of a vortex in a Bose superfluid [15]

$$
\sigma'_\perp = \frac{2\pi}{m_B v_g}
$$

where $v_g$ is the group velocity which, in our case, is defined by Eq. (11), and $m_B$ is the mass of a Bosonic atom. Note that, in our case, $m_B = 2m$. The corresponding part of the transverse cross section can be easily obtained from the semi-classical description. Indeed, the Doppler energy due to the vortex velocity is $p \cdot \mathbf{v}_s = p \cdot \nabla \chi/m_B$. Its contribution to the quasiparticle action is

$$
A = -\frac{1}{m_B} \int p \cdot \nabla \chi \, dt = -\frac{p}{m_B v_g} \int \frac{\partial \chi}{\partial s} \, ds = -\frac{p}{m_B v_g} \delta \chi.
$$

Here $s$ is the coordinate along the particle trajectory as in Fig. 2 and $\delta \chi$ is the variation of the order parameter phase along the trajectory. The change in the transverse momentum of the particle is $\delta p_\perp = \partial A/\partial b$ hence the transverse cross section becomes

$$
\sigma'_\perp = \int \frac{\delta p_\perp}{p} \, db = \frac{A_+ - A_-}{p}
$$

where $A_\pm$ is the action along the trajectory passing on the left (right) side of the vortex. Since $\delta \chi_+ - \delta \chi_- = -2\pi$ we recover Eq. (40). With this expression for the cross section, Eq. (39) gives the Iordanskii force $F_I = -\pi N_n [\mathbf{v}_L \times \hat{z}]$.

The second term in Eq. (38) originates [26] from the fact that here, as distinct from the situation in a Bose superfluid, the phase of the single-particle wave function changes by $\pi$ upon encircling the vortex, while it is the order
parameter phase which changes by $2\pi$. It is this singularity produced by the vortex in the single-particle wave function which results in the anomalous contribution to the cross section in Eq. (38). Inserted into Eq. (39), it exactly reproduces the second term in Eq. (37). We see that the spectral flow force is related to a single-particle anomaly associated with the vortex.

5 Vortex momentum

The vortex mass in superfluids and superconductors has been a long standing problem in vortex physics and remains to be an issue of controversies. There are different approaches to its definition. In early works on this subject, the vortex mass was determined through an increase in the free energy of a superconductor calculated as an expansion in slow time derivatives of the order parameter. The quasiparticle distribution was assumed to be essentially as in equilibrium. First used by Suhl [28] (see also [29]) this approach yields the mass of the order of one quasiparticle mass (electron, in case of superconductor) per atomic layer. Another approach consists in calculating an electromagnetic energy $E^2/8\pi$ which is proportional to the square of the vortex velocity. This gives rise to the so called electromagnetic mass [30] which, in good metals, is of the same order of magnitude (see Ref. [31] for a review).

A crucial disadvantage of the above definitions of the vortex mass is that they do not take into account the kinetics of excitations disturbed by a moving vortex. We shall see that the inertia of excitations contributes much more to the vortex mass than what the old calculations predict. The kinetic equation approach described here is able to incorporate this effect. To implement this method we find the force necessary to support an unsteady vortex motion. Identifying then the contribution to the force proportional to the vortex acceleration, one defines the vortex mass as a coefficient of proportionality. This method was first applied for vortices in superclean superconductors in Ref. [32] and then was used by other authors (see for example, Refs. [33, 34]). The resulting mass is of the order of the total mass of all electrons within the area occupied by the vortex core. We will refer to this mass as to the dynamic mass. Since the dynamic mass originates from the inertia of excitations localized in the vortex core it can also be calculated through the momentum carried by localized excitations [35]. We shall see that dynamic mass displays a nontrivial feature: it is a tensor whose components depend on the quasiparticle mean free time. In s-wave superconductors, this tensor is diagonal in the superclean limit. The diagonal mass decreases rapidly as a function of the mean free time, and the off-diagonal components dominate in the moderately clean regime. These results were obtained in Ref. [36]. Our results agree with the previous work [32, 33, 35] in the limit $\tau \to \infty$. 
5.1 Equation of vortex dynamics

To introduce the vortex momentum we consider a non–steady motion of a vortex such that its acceleration is small. We again start with the delocalized excitations. Multiplying Eq. (5) by $p_\perp/2$ and summing up over all the quantum numbers, we obtain

$$F^{(\text{loc})}_{\text{env}} = F^{(\text{loc})}_{\text{coll}} - \frac{\partial P^{(\text{loc})}}{\partial t} - \pi N \tanh \left( \frac{\Delta_\infty}{2T} \right) [v_L \times \hat{z}]$$  \hspace{1cm} (41)

where the l.h.s. of Eq. (41) is the force from the environment on a moving vortex Eq. (14).

The first term in the r.h.s. of Eq. (41) is the force exerted on the vortex by the heat bath via excitations localized in the vortex core:

$$F^{(\text{loc})}_{\text{coll}} = -\frac{1}{2} \sum_n \int p_\perp \frac{\partial f}{\partial t}^{(0)} \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{d\mu}{2\pi}.$$ \hspace{1cm} (42)

The second term in the r.h.s. of (41) is the change in the vortex momentum

$$P^{(\text{loc})} = -\frac{1}{2} \sum_n \int p_\perp f_1 \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{d\mu}{2\pi}.$$ \hspace{1cm} (43)

We turn now to delocalized excitations. Multiplying Eq. (12) by $p_\perp/2$ again and summing over all the states we find

$$F^{(\text{del})}_{\text{env}} = F^{(\text{del})}_{\text{coll}} - \frac{\partial P^{(\text{del})}}{\partial t} + \frac{1}{2} \sum_n \int p_\perp \frac{\partial f}{\partial t}^{(0)} \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{d\mu}{2\pi} \frac{p_\perp \cdot (\hat{z} \times v_L)}{g}.$$ \hspace{1cm} (44)

Here

$$F^{(\text{del})}_{\text{coll}} = -\frac{1}{2} \sum_n \int p_\perp \frac{\partial f}{\partial t}^{(0)} \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} = \int_{\epsilon > \Delta_\infty} p_\perp \frac{f_1}{\omega_c} \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{de}{\omega_c}$$

is the force from the heat bath, and the corresponding contribution to the vortex momentum is

$$P^{(\text{del})} = -\frac{1}{2} \sum_n \int p_\perp f_1 \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} = -\int_{\epsilon > \Delta_\infty} p_\perp \frac{g}{\omega_c} f_1 \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} \frac{de}{\omega_c}.$$ \hspace{1cm} (45)

The total momentum is $P = P^{(\text{loc})} + P^{(\text{del})}$.

After a little of algebra, the total force from the heat bath $F_{\text{coll}} = F^{(\text{loc})}_{\text{coll}} + F^{(\text{del})}_{\text{coll}}$ can be written in the form

$$F_{\text{coll}} = F_{\text{sf}} + F_{\parallel}$$
where the friction and spectral flow forces are determined by Eqs. (18) and (30), respectively. The total force from environment Eqs. (41) and (44) takes the form
\[ F_{\text{env}} = F_{\text{sf}} + F_\parallel - \partial P/\partial t - \pi N[v_L \times \hat{z}]. \] (46)

This equation agrees with Eq. (29) for a steady motion of vortices.

The equation of vortex dynamics is obtained by variation of the superconducting free energy plus the external field energy with respect to the vortex displacement. The variation of the superfluid free energy gives the force from the environment, \( F_{\text{env}} \), while the variation of the external field energy produces the external Lorentz force. In the absence of pinning the total energy is translationally invariant. Therefore, the requirement of zero variation of the free energy gives again the condition \( F_L + F_{\text{env}} = 0 \) in the form the force balance.

Using our expression for \( F_{\text{env}} \), the force balance can now be written in the form similar to Eq. (34)
\[ F_M + F_L^{(qp)} + F_I + F_{\text{sf}} + F_\parallel = \partial P/\partial t \] (47)

where the r.h.s. contains the time derivative of the vortex momentum due to a non–steady vortex motion. For a steady vortex motion, Eq. (47) reduces to Eq. (44). The physical meaning of the Eq. (47) is simple. The l.h.s. of this equation accounts for all the forces acting on a moving straight vortex line. The r.h.s. of Eq. (47) comes from the inertia of excitations and is identified as the change in the vortex momentum. The definition of Eq. (43) is similar to that used in Refs. [8, 35]. Note that the delocalized quasiparticles do not contribute to the vortex momentum in the limit \( \omega_c \tau \ll 1 \) because they are in equilibrium with the heat bath.

### 5.2 Vortex mass

Having defined the vortex momentum, we calculate the vortex mass. The distribution function is given by Eq. (7). The vortex momentum becomes \( P_i = M_{ik} u_k \); it has both longitudinal and transverse components with respect to the vortex velocity. For a vortex with the symmetry not less than the fourfold, the effective mass tensor per unit length is \( M_{ik} = M_\parallel \delta_{ik} - M_\perp e_{ikj} \hat{z}_j \) where \( M_\parallel = M_{\parallel e} + M_{\parallel h} \) and \( M_\perp = M_{\perp e} - M_{\perp h} \). Each component contains contributions from localized and delocalized states such that \( M_{\parallel,\perp} = M_{\parallel,\perp}^{(\text{loc})} + M_{\parallel,\perp}^{(\text{del})} \) where

\[ M_{\parallel,\perp}^{(\text{loc})} = \frac{1}{4} \sum_n \int \rho^2 \gamma_H \frac{df^{(0)}}{de} \frac{dp_z}{2\pi} \frac{d\mu}{2\pi} = \pi N(e,h) \left( \sum_n \int \gamma_H \frac{df^{(0)}}{de} \frac{d\mu}{2} \right) \] (48)
and

\[ M^{(\text{del})}_{\parallel e,h} = \frac{1}{2} \int_{\epsilon > \Delta_{\infty}} p_{\perp}^2 \gamma'_H \frac{g}{\omega_c} \frac{df^{(0)}}{d\epsilon} \frac{dp_z}{2\pi} \frac{d\alpha}{2\pi} d\epsilon. \]  

The same expression holds for \( M_{\perp e,h} \) where \( \gamma_H \) is replaced with \( \gamma_O \). The indexes \( e, h \) indicate the corresponding momentum integrations over the electron and hole parts of the Fermi surface, respectively. Only the branch with \( n = 0 \) gives the contribution to the transverse mass because \( \gamma_O \) is odd in \( \mu \) for \( n \neq 0 \).

If the vortex acceleration is slow, one can use expressions Eqs. (8), (13) for a steadily moving vortex to calculate the vortex inertia. Consider first the contribution of the states with \( |\epsilon| > \Delta_{\infty} \). Since \( \gamma'_H, \gamma'_O \) do not depend on energy and momentum, Eq. (49) gives

\[ M^{(\text{del})}_{\parallel e,h} = \pi N^{(e,h)}_{\perp} \gamma'_H, \gamma'_O = \frac{\pi N^{(e,h)}_{\perp}}{\omega_c} \gamma'_H, \gamma'_O. \]

The contribution of the delocalized states decreases as \( \omega_c \tau \) gets smaller. In the limit of vanishing \( \omega_c \tau \) the vortex mass is determined by localized excitations. The localized excitations dominate also at low temperatures \( T \ll \Delta_{\infty} \). One has in this case

\[ M_{\parallel e,h}^{(\text{loc})} = \pi N^{(e,h)} \left\langle \frac{\gamma_H}{\omega_0} \right\rangle, \quad M_{\perp e,h}^{(\text{loc})} = \pi N^{(e,h)} \left\langle \frac{\gamma_O}{\omega_0} \right\rangle. \]

For \( \omega_c \tau \gg 1 \) the mass tensor for delocalized states is diagonal \( M_{ik} = M_{i0} \delta_{ik} \) where \( M_{\parallel e,h}^{(\text{del})} = m N_{n} S_0 \); it is equal to the mass of normal particles in the area occupied by the vortex. The mass tensor for localized states becomes diagonal in the superclean limit where \( T_2^2 \tau / E_F \gg 1 \) with \( M_{\parallel e,h}^{(\text{loc})} \sim \pi N \left\langle \frac{\omega_0}{\omega_{n0}} \right\rangle \sim \pi \xi^2 m N \). This is the mass of all electrons in the area occupied by the vortex core. The mass decreases with \( \tau \). In the moderately clean regime \( T_2^2 \tau / E_F \ll 1 \) where \( \omega_{n0} \tau \ll 1 \), the diagonal component vanishes as \( \tau^2 \), and the mass tensor is dominated by the off-diagonal part.

We should emphasize an important point that, in contrast to a conventional physical body, the mass of a vortex is not a constant quantity for a given system: it may depend on the frequency \( \omega \) of the external drive. Indeed, for a nonzero \( \omega \) we find from Eq. (3)

\[ \gamma_H = \frac{\omega_n^2 \tau_n^2}{\omega_n^2 \tau_n^2 + (1 - i \omega \tau_n)^2}, \quad \gamma_O = \frac{\omega_n \tau_n (1 - i \omega \tau_n)}{\omega_n^2 \tau_n^2 + (1 - i \omega \tau_n)^2}. \]

As a result, all the dynamic characteristics of vortices including the conductivity and the effective mass acquire a frequency dispersion. In the limit \( \tau \to \infty \), poles in \( \gamma_{O,H} \) appear at a frequency equal to the energy spacing between the quasiparticle states in the vortex core which gives rise to resonances in absorption of an external electromagnetic field \[ 32, 37 \].
6 Conclusions

All the dynamic characteristics of vortices such as vortex friction, the flux flow conductivity, the Hall effect, and the vortex mass are determined by the mean free path of excitations which interact with vortices. The key parameter is $\omega_0\tau$ where $\omega_0$ is the interlevel spacing for the quasiparticle states in the vortex core. For small $\omega_0\tau$, vortices experience viscous flow; the Lorentz force is opposed by a friction force while the transverse force vanishes. On the contrary, in a superclean regime when $\omega_0\tau \gg 1$, vortices move with a superflow as in an ideal fluid; the Hall angle is $\pi/2$, the friction force is zero, while the transverse force reaches its maximum value. The vortex mass is a tensor. The longitudinal component dominates in the superclean regime: it is the mass of all excitations in the vortex core. On the contrary, the transverse component is the largest one in the limit $\omega_0\tau \ll 1$.

I am pleased to acknowledge helpful discussions with E. Sonin and G. Volovik. This work was supported by the Russian Foundation for Basic Research (Grant No. 99-02-16043).

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