DETERMINING THE SHAPE OF A SOLID OF REVOLUTION

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Abstract. We show how to reconstruct the shape of a solid of revolution by measuring its temperature on the boundary. This inverse problem reduces to finding a coefficient of a parabolic equation from values of the trace of its solution on the boundary. This is achieved by using the inverse spectral theory of the string, as developed by M.G. Krein, which provides uniqueness and also a reconstruction algorithm.

1. Introduction. We are concerned with the reconstruction of the coefficient \( s \) appearing in the one dimensional heat equation

\[
\begin{cases}
\partial_t u(x, t) = \partial_{xx} u(x, t) + s(x) \partial_x u(x, t) & 0 \leq x \leq \pi \\
\partial_x u(0, t) = 0 \quad \text{and} \quad \partial_x u(\pi, t) + \beta u(\pi, t) = 0 \\
u(x, 0) = a(x)
\end{cases}
\]

from a single reading of the trace of the solution on the boundary, that is from the knowledge of the traces

\[ G(T) := \{(u(0, t), u(\pi, t)) : 0 < t < T \leq \infty\}. \]

The above equation models the heat propagation in a solid of revolution with \( s(x) \) being its cross section and with rotation along the x-axis. For example \( s \) is constant for a cylinder and a linear function for a cone. Usually one uses optimal control to reconstruct the coefficient \( s \) from the knowledge of the final time overdetermination of the solution \( u(x, T) \) for all \( x \in [0, \pi] \), see [12]. Their method is to minimize a regularized functional so to approximate the function \( s \). The purpose of this note is to provide a simpler alternative and a direct treatment of the same problem which avoids inside measurements, as they may not be possible in practice. Also the minimization of a functional is sometimes difficult to achieve as the minimizer may not even exist. In [3, 4], inverse parabolic equations of the type \( u_t = Au \), where \( A := D^2 - q(x) \) is self-adjoint in \( L^2(0, \pi) \), have been worked out by using the Gelfand-Levitan inverse spectral theory, where it was shown that two measurements only were enough to find \( q \). In the same spirit, the authors in [1] extended the idea to the Schrodinger equation \( iu_t = Au \). However in (1), the underlying operator \( A = D^2 + sD \) is non symmetric in \( L^2(0, \pi) \) and so it is not clear how to extend the previous method to the nonself-adjoint case as the Gelfand-Levitan theory is no more applicable.

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2. Preliminaries. The main idea in [3, 4] is to say if \( A \) is a self-adjoint operator acting in a certain Hilbert space and has a simple discrete spectrum then its eigenfunctions form an orthonormal basis in that space. Thus if \( Ay_n = \lambda_n y_n \), where \( \|y_n\| = 1 \), then the evolution equation \( u' = Au \) has the solution
\[
\sum_{n \geq 1} c_n e^{\lambda_n t} y_n.
\]
Therefore if we can observe the solution \( \sum c_n e^{\lambda_n t} y_n \), and can extract the complete spectral data, [3], by spectral estimation methods [3, 4, 2], then we can recover the operator \( A \). The main question now is how to extend the method in [3, 4] to case of a non self-adjoint operator such as in (1). We first show that although the Gelfand-Levitan theory is not applicable, Krein’s inverse spectral theory is. We first observe that the operator
\[
A(y)(x) = y''(x) + s(x)y'(x) \quad 0 \leq x \leq \pi
\]
with \( s \in L^1(0, \pi) \), which at first sight, is not symmetric in \( L^2(0, \pi) \) happens to be in fact similar to a Krein self-adjoint string \( \frac{dy^+}{dM} \) in a certain weighted \( L^2_{\text{dM}} \) space. Here \( y^+ \) denotes the right-derivative and \( dM \) is a Lebesgue-Stieltjes measure generated by a non decreasing function \( M(x) \) which represents the mass of the string over the interval \((0, x)\), \( M(x) = 0 \) if \( x < 0 \), and
\[
L^2_{\text{dM}} = \{ f \text{ is measurable : } \int_R |f(x)|^2 dM(x) < \infty \}. 
\]
Therefore the operator \( A \) would enjoy many of the spectral properties of a string and especially its inverse spectral theory which can be found in [7, 8]. For a numerical solution for the recovery of a string, based on continued fraction, see [6, 11]. All we need is to combine both derivatives in \( A \) by using the function \( p \) defined as a solution to
\[
p'(x) = s(x)p(x) \quad \text{and} \quad p(0) = 1. \quad (4)
\]
Note that (4) establishes a one-to-one correspondence between \( p \) and \( s \), that is given by \( p(x) = e^{\int_0^x s(\eta)d\eta} \). We deduce that for any \( s \in L(0, \pi) \), \( p \) is a positive absolutely continuous function, \( p \in AC(0, \pi) \), that is bounded away from 0, i.e. \( \inf_{0 \leq x \leq \pi} p(x) \geq \delta > 0 \). Using \( p \) we can rewrite the operator \( A \) in impedance form
\[
\begin{align*}
A(y)(x) &= \frac{1}{p(x)} (p(x)y'(x))' \quad 0 \leq x \leq \pi \\
y'(0) &= 0 \quad \text{and} \quad y'(\pi) + \beta y(\pi) = 0
\end{align*}
\]
which is readily seen to be a self-adjoint operator acting in \( L^2_{pdx}(0, \pi) \), and has a discrete spectrum since we have
\[
p \in AC(0, \pi), \ p > 0 \quad \text{and} \quad p, \frac{1}{p} \in L(0, \pi). \quad (6)
\]
To see that \( A \) in (5) is similar to a string, we simply make use of the change of variables
\[
t(x) = \int_0^x \frac{1}{p(\eta)}d\eta, \quad \text{and denote by} \quad \tau = t(\pi)
\]
which leads to \( \frac{d}{d\tau} = p(x)\frac{d}{dx} \). It follows at once that the operator \( A \) given by (5) is equivalent to
\[
\begin{align*}
B(\phi)(t) &= \frac{1}{w(t)} \phi''(t) \quad 0 \leq t \leq \tau \\
\phi'(0) &= 0 \quad \text{and} \quad \phi'(\tau) + \beta p(\pi)\phi(\tau) = 0
\end{align*}
\]
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where

\[ w(t(x)) = p(x)^2 \geq 0. \]  

(9)

The operator \( B \) is a particular case of Krein string, [7], as the mass is absolutely continuous, \( dM(t) = w(t)dt \) and is a self-adjoint operator acting in \( L^2_{w(t)}(0, \tau) \). To see the similarity between the operator \( B \) and \( A \), define the transformation operator \( T \) first from \( C(0, \tau) \) into \( C(0, \pi) \) by

\[ T(\phi)(x) := \phi(t(x)). \]  

(10)

We have the following:

**Lemma 2.1.** The transformation operator in (10) is an isometry \( L^2_{w(t)}(0, \tau) \rightarrow L^2_{p(x)}(0, \pi) \) and \( AT = TB \), i.e. \( A \) and \( B \) are similar operators.

**Proof.** We have

\[ \|Tf(x)\|^2_p = \|f\|^2_w \]

which follows from

\[ \int_0^\pi |T(f)(x)|^2 p(x) dx = \int_0^\pi |f(t(x))|^2 p^2(x) t'(x) dx = \int_0^\tau |f(t)|^2 w(t) dt \]

since \( t'(x) = \frac{1}{p(x)} \), \( w(t(x)) = p^2(x) \) and \( t(\pi) = \tau \). It also follows that from (8), (10) and (5), that if

\[ B(\phi_n)(t) = \lambda_n \phi_n(t) \]

then

\[ y_n(x) := T(\phi_n)(x) \]  

(11)

is an eigenfunction of \( A \),

\[ Ay_n(x) = \lambda_n y_n(x). \]

The boundary conditions are also easily verified. This implies that

\[ AT \phi_n = Ay_n = \lambda_n y_n = \lambda_n T \phi_n = T(\lambda_n \phi_n) = TB \phi_n \]

and since \( \{ \phi_n \} \) is a basis in \( L^2_{w(t)}(0, \tau) \) it follows that the operators \( A \) and \( B \) are similar

\[ AT = TB, \]

their spectra are identical and by (11), and Lemma 1 we have

\[ \|\phi_n\|_w = \|y_n\|_p. \]  

(12)

Thus we have proved the following

**Proposition 1.** The recovery of the operator \( A \) is equivalent to recovering the string operator \( B \), which requires the complete spectral data \( \{ \lambda_n, \|\phi_n\|_w \} \) or equivalently \( \{ \lambda_n, \|y_n\|_p \} \).

We now show a formula that helps compute the norming constants from the values of \( y_n(0) \) and \( y_n(\pi) \). To this end denote by \( y(x, \lambda) \) the eigensolutions of (3) of the initial value problem

\[
\begin{align*}
(p(x)y'(x, \lambda))' &= \lambda p(x)y(x, \lambda) \quad 0 \leq x \leq \pi \\
y(0, \lambda) &= 1 & \text{and} & y'(0, \lambda) &= 0.
\end{align*}
\]

(13)

Clearly \( y(x, \lambda) \) is an entire function of \( \lambda \) and the eigenvalues \( \lambda_n \) of \( A \), are the roots of

\[ \Delta(\lambda) := y'(\pi, \lambda) + \beta y(\pi, \lambda) = 0, \]

(14)
and so connecting with the previous notation in (5) we have
\[ y_n(x) = y_n(0)y(x, \lambda_n). \] (15)

If we denote by \( y_\lambda = \partial_\lambda y(x, \lambda) \) then \((py_\lambda)' = \lambda py_\lambda + py\) leads to
\[
(ppy_\lambda - py'y_\lambda)' = y(ppy_\lambda)' + y'(ppy_\lambda) - (py'y_\lambda)' = \lambda(ppy_\lambda + py) - (py'y_\lambda) = py^2 + y\lambda py_\lambda - \lambda py_y \lambda \\
= py^2
\]
and, since \( y'(0, \lambda) = 0 \) for all \( \lambda \) means \( y'_\lambda(0, \lambda) = 0 \), we have
\[
\|y\|^2_p = \int_0^\pi (ppy_\lambda - py'y_\lambda)' dx = ypy_\lambda' - py'y_\lambda|_0^\pi
\]
\[
= y(\pi, \lambda) p(\pi) y'_\lambda(\pi, \lambda) - p(\pi) y'(\pi, \lambda) y_\lambda(\pi, \lambda) - y(0, \lambda) p(0) y'_\lambda(0, \lambda) + p(0) y'(0, \lambda) y_\lambda(0, \lambda) \\
= y(\pi, \lambda) p(\pi) y'_\lambda(\pi, \lambda) - p(\pi) y'(\pi, \lambda) y_\lambda(\pi, \lambda) \\
= y(\pi, \lambda) p(\pi) y'_\lambda(\pi, \lambda) + p(\pi) \beta y(\pi, \lambda) y_\lambda(\pi, \lambda) \\
= y(\pi, \lambda) p(\pi) \partial_\lambda (y'(\pi, \lambda) + \beta y(\pi, \lambda)) \\
= y(\pi, \lambda) p(\pi) \Delta'(\lambda).
\]
Thus we have obtained
\[
\|y_n\|^2_p = p(\pi) y(\pi, \lambda_n) \Delta'(\lambda_n). \quad (16)
\]

Then to compute the norming constants \( \|y_n\|^2_p \) we need the values \( y(\pi, \lambda_n) = \frac{y_n(\pi)}{y_n(0)} \) by (15), which can be extracted from \( G(T) \), and also \( \Delta'(\lambda_n) \) which is obtained by interpolating \( \Delta \) as follows. Since the eigenvalues \( \lambda_n \) are its zeros by (14), and \( y(\pi, \lambda) \) is an entire function of order 1/2, we can write
\[
\Delta(\lambda) = c \prod_{n \geq 1} \left(1 - \frac{\lambda}{\lambda_n}\right)
\]
where \( c \) is a constant that can be determined by evaluating \( \Delta(0) \) in (14) which yields
\[
\Delta(0) = c = y'(\pi, 0) + \beta y(0, 0) = \beta.
\]
Therefore we explicitly know the function
\[
\Delta(\lambda) = \beta \prod_{n \geq 1} \left(1 - \frac{\lambda}{\lambda_n}\right)
\]
and \( \Delta'(\lambda_n) \) can be computed explicitly since we now know the eigenvalues \( \{\lambda_n\} \). Furthermore since the eigenvalues are known to be simple, \( \Delta'(\lambda_n) \neq 0 \) and so all \( \|y_n\|^2_p \) can be evaluated by (16). Thus we have proved

**Proposition 2.** In order to uniquely recover the operator \( B \), i.e. \( w(t) \) for \( t \in (0, \tau) \), we only need the complete spectral data set \( \left\{ \lambda_n, \frac{y_n(\pi)}{y_n(0)} \right\}_{n \geq 1} \).
3. The Fourier coefficients. In all that follows, we assume that the initial condition \( a = 1 \), and \( \beta \neq 0 \) and we can recast equation (1) as

\[
\begin{align*}
\partial_t u(x, t) &= \frac{1}{p(x)} \partial_x (p(x) \partial_x u(x, t)) \quad 0 \leq x \leq \pi \\
\partial_x u(0, t) &= 0 \quad \text{and} \quad \partial_x u(\pi, t) + \beta u(\pi, t) = 0 \\
u(x, 0) &= 1
\end{align*}
\] (17)

Now that we have established that the eigensolutions \( y_n \) form an orthonormal basis in \( L^2_{pdx}(0, \pi) \) then we can express the solution of (1) or equivalently (17), as

\[
u(x, t) = \sum_{n \geq 1} c_n(1) e^{\lambda_n t} y_n(x) \quad \text{for all } x \in (0, \pi) \quad \text{and} \quad t > 0
\] (18)

where

\[c_n(1) = \int_0^\pi y_n(x)p(x)dx.\]

In order for all eigenvalues \( \{\lambda_n\} \) to appear in (18) we need all \( c_n(1) \neq 0 \). Note that since \( p \in AC(0, \pi) \) and \( p \geq \delta > 0 \), the space \( L^2_{pdx}(0, \pi) \) is equivalent to \( L^2_{dx}(0, \pi) \). It is now easy to see that

**Lemma 3.1.** If \( \beta \neq 0 \) then \( c_n(1) \neq 0 \) for all \( n \geq 1 \).

**Proof.** First \( \lambda_n \neq 0 \), otherwise then the corresponding eigenfunction, see (15), satisfies \( (p(x)y'(x, 0))' = 0 \) and with the initial conditions \( p(x)y'(x, 0) = 0 \) we have \( y(x, 0) = 1 \) and also \( y'(\pi, 0) = 0 \). This contradicts the boundary condition at \( x = \pi \) since \( 0 - \beta 1 \neq 0 \). Furthermore we have

\[
\lambda_n c_n(1) = \lambda_n \int_0^\pi y_n(x)p(x)dx = \int_0^\pi (p(x)y'(x))' dx = p(\pi)y_n'(\pi) \neq 0
\]

and so \( c_n(1) \neq 0 \) for all \( n \geq 1 \). \( \square \)

4. The reconstruction of \( s \). Thus we can observe on the boundary points \( x = 0, \pi \) and \( t > 0 \)

\[
u(0, t) = \sum_{n \geq 1} c_n(1) e^{\lambda_n t} y_n(0) \quad \text{and} \quad u(\pi, t) = \sum_{n \geq 1} c_n(1) e^{\lambda_n t} y_n(\pi) \quad \text{for} \quad t \in (0, T).
\] (19)

Using spectral estimation, see [4, 5, 2], we can extract the following complete sets

\[
\{\lambda_n, c_n(1) y_n(0)\}_{n \geq 1} \quad \text{and} \quad \{\lambda_n, c_n(1) y_n(\pi)\}_{n \geq 1}
\]

from (19) for \( T \leq \infty \) since \( c_n(1) \neq 0 \) and also \( y_n(\pi), y_n(0) \neq 0 \) for all \( n \geq 1 \). Thus we recover the complete set \( \{\lambda_n, y_n(\pi) / y_n(0)\} \) which, by proposition 2, yields a unique function \( \nu(t) \) on \( (0, \tau) \) that we now use to find \( s \). From (9) and \( t'(x) = 1/p(x) > 0 \) we deduce that the function \( t(x) \) is invertible while its inverse \( x(t) \) satisfies

\[x'(t) = \sqrt{\nu(t)} \quad \text{with} \quad x(0) = 0\]

and so

\[x(t) = \int_0^t \sqrt{\nu(\eta)}d\eta \quad \text{for} \quad 0 \leq t \leq \tau.\]

Inverting \( x(t) \) we get \( t(x) \) back. The function \( s \) is easily then found from (4) and (7)

\[s(x) = \frac{d}{dx} \ln p(x) = \frac{1}{2} \frac{d}{dx} \ln (\nu(t(x))).\] (20)
We now can state the main result of this note.

**Proposition 3.** Using the set of traces in $G(T)$, (2), which is generated with $a = 1$ and $\beta \neq 0$, we can uniquely reconstruct the function $s$ in (1), from a single measurement.

**Proof.** The formula for the function $s$ is given by (20). We need to show its uniqueness. It is clear that $s$ is uniquely defined by $w$. Thus it remains to see that $w$ is also uniquely defined from the set of traces $G(T)$. Let us assume that $u$ in (18) is given by two different expressions with $c_n(1) \neq \tilde{c}_k(1)$ and yet

$$u(x,t) = \sum_{n \geq 1} c_n(1) e^{\lambda_n t} y_n(x) = \sum_{k \geq 1} \tilde{c}_k(1) e^{\tilde{\lambda}_k t} \tilde{y}_k(x) \quad \text{for all} \ x \in (0, \pi) \ \text{and} \ t > 0. \quad (21)$$

The difference can be split into two sums

$$\sum_{\lambda_n = \lambda_k} e^{\lambda_n t} [c_n(1) y_n(x) - \tilde{c}_n(1) \tilde{y}_n(x)]$$

$$+ \sum_{\lambda_n \neq \lambda_k} e^{\lambda_n t} c_n(1) y_n(x) - e^{\tilde{\lambda}_k t} \tilde{c}_k(1) \tilde{y}_k(x) = 0 \quad \text{for} \ 0 < t < T. \quad (22)$$

where all the exponentials are now distinct. Recall that any finite set of distinct exponentials cannot be dependent on any finite interval. This is readily seen that if for $0 < t < T$ we have

$$\sum_{j=m}^{\infty} \alpha_j e^{\mu_j t} = 0 \ \text{where some} \ \alpha_j \neq 0 \ \text{and} \ \mu_j \text{are distinct.}$$

Then looking at the first $m$ derivatives in $t$, we end up with a homogeneous system, with Vandermonde matrix $(\mu_j^t)_{1 \leq i,j \leq m}$, and so the only solution is $e^{\mu_j t} = 0$ for $0 \leq j \leq m$ i.e. $\alpha_j = 0$ and so a contradiction. Thus we deduce that all exponentials involved in (22) are linearly independent over $(0, T)$. Thus from the first sum we have

$$c_n(1) y_n(x) = \tilde{c}_k(1) \tilde{y}_k(x).$$

Using the fact that $\|y_n\| = \|\tilde{y}_k\| = 1$ we deduce that $c_n(1) = \tilde{c}_k(1) = 1$ and so $y_n = \tilde{y}_k$. As for the second sum, we clearly have $c_n(1) y_n(x) = \tilde{c}_k(1) \tilde{y}_k(x) = 0$ whenever $\lambda_n \neq \lambda_k$ and so $y_n = \tilde{y}_k = 0$. Thus we have only the first sum with $\lambda_n = \lambda_k$, $c_n(1) = \tilde{c}_k(1) = 1$ and $y_n = \tilde{y}_k$ and therefore the two sums in (21) have identical terms, which means that we have one complete spectral data set $\{\lambda_n, y_n(\pi)/y_n(0)\}$. It remains to see that the condition for uniqueness of recovery hold. Recall that Krein’s main result says that existence and uniqueness hold for spectral functions $\rho$ such that $\int_0^\infty \frac{1}{1+\lambda} d\rho(\lambda) < \infty$, [7, 5.8 page 202]. In our case this condition is satisfied since $\lambda_n \neq 0$ and we have the bound which follows for short strings, see formula (4) page 192 in [7],

$$\int_0^\infty \frac{1}{1+\lambda} d\rho(\lambda) \leq \int_0^\infty \frac{1}{\lambda} d\rho(\lambda) = \pi \left( \tau + \frac{1}{\beta} \right) < \infty.$$

**Remark.** For the spectral estimation, as we are dealing with a Dirichlet series in $G(T)$, and so all we need is $0 < T \leq \infty$. In fact recently, using interpolation in Hardy spaces, the authors in [5], have shown that all is required is a sequence.
\{u(0, \xi_n), u(\pi, \xi_n)\}_{n \geq 1}, \text{ where } \xi_n \text{ is convergent sequence, in order to extract all the spectral data.}

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