General Properties of Classical $W$ Algebras

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Abstract

After some definitions, we review in the first part of this talk the construction and classification of classical $W$ (super)algebras symmetries of Toda theories. The second part deals with more recently obtained properties. At first, we show that chains of $W$ algebras can be obtained by imposing constraints on some $W$ generators: we call secondary reduction such a gauge procedure on $W$ algebras. Then we emphasize the role of the Kac-Moody part, when it exists, in a $W$ (super) algebra. Factorizing out this spin 1 subalgebra gives rise to a new $W$ structure which we interpret either as a rational finitely generated $W$ algebra, or as a polynomial non linear $W_\infty$ realization.
1 Introduction

$W$ algebras constitute today a rather broad subject: on the one hand they play a role in different parts of 2 dimensional Conformal Field Theories (CFT), on the other hand much has still to be done for a complete knowledge of these algebras and their algebraic properties. First it was thought that they can be used to facilitate the analysis of rational CFT (i.e. theories in which the main parameters, namely central charge $c$ and conformal dimensions $h_i$ are all rational numbers): this extra symmetry, bigger than the conformal one, could help to characterize degeneracies, and to classify in a simpler way the physical states. After that it was realized that they show up in several places. We currently talk nowadays about $W$ gravity. $W$ algebras appear in the quantum Hall effect, black holes models, in lattice models of statistical mechanics at criticality, and in Toda models\cite{1} as symmetry algebras \cite{2}.

After some definitions (Section 2), we will concentrate on classical $W$ algebras and superalgebras which are finitely generated -we generically denote them $W_n$-. Two remarkable facts can then be mentioned (Section 3):

-i) The constants of motion of a Toda theory form a $W_n$ algebra, and such a Toda theory can be seen as a gauged WZW model, on which constraints have been imposed \cite{2}.

-ii) As a consequence, one can explicitly construct such $W_n$ algebras, and give a group theoretical classification of them \cite{3}.

Two comments:
- this classification is based on the $Sl(2)$ embeddings in a simple Lie (super)algebra $G$ and on the $OSp(1|2)$ embeddings in a simple superalgebra $SG$. We will try to insist on the property of $Sl(2)$ to be intimately linked to a $W_n$ algebra from its definition: this is important for our construction, but also allows to think that the classification of $W_n$ algebras symmetries of Toda models hereafter given is ”not far” from exhausting the set of $W_n$ algebras.

- there are two main types of $W_n$ algebras: those that we will call the Abelian ones because they are related to Abelian Toda models: for example, if the underlying group of the Toda model is $Sl(n)$, one gets the algebra generated by $W_2, W_3, ... W_n$.

There is a second type of $W_n$ algebra, less well-known: they are associated to non Abelian Toda models\cite{1}, and we call them non Abelian $W_n$ algebras, and we will come back to this class of algebras.

The above classification can be simplified using two interesting features, directly suggested by properties of simple Lie algebras and superalgebras, namely:
- deduction of $W_n$ algebras related to non simply laced algebras $B_n, C_n, ...$ from $W_n$ algebras related to $A_n$ series by ”foldings” \cite{4} analogous to the folding technics which produce $B_n, C_n, ...$ algebras from $A_n$ ones (Section 4).
existence of chains of $W_n$ algebras mimicking chains of embeddings of sub-algebras in a simple Lie Algebra \([5]\). Imposing constraints, when possible, on a the $W$ algebra itself, one can reduce $W$ into another algebra $W$: we will call this technics a secondary reduction (Section 5).

Finally coming back to the non Abelian $W_n$ algebras, one can remark that most of them contain a Kac Moody part. Such a Kac Moody subalgebra should play a particular role. In particular, we will see that factorizing out this ”spin one” part in the $W_n$ algebra gives rise to an algebra which can be seen either as an $W_\infty$ algebra, that is an infinitely generated $W$ algebra, or as a finitely generated $W$ algebra but of a new type; we will call it ”rational” $W_n$ algebras \([6]\). This problem as well as its supersymmetric generalisation is the subject of Section 6 which ends up by a comparative study of the factorizations of spin 1/2 fermions and spin 1 bosons in a $W$ algebra.

We have chosen to illustrate each property which is introduced on an example instead of presenting general proofs. We hope that this approach will make the reading as easy for the non experts as for those familiar with $W$ algebras, these last ones being invited to directly go to the three last sections.

2 Definitions

We know from $d = 2$ CFT that the stress energy tensor has a short-distance O.P.E. of the form, with $z, w$ complex variables:

$$T(z),T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} + \ldots \quad (2.1)$$

Expressing $T(z)$ into Laurent modes

$$T(z) = \sum_{m \in \mathbb{Z}} z^{-m-2}L_m \quad L_m = \oint \frac{dz}{2\pi i z} z^{m+2}T(z) \quad (2.2)$$

the integral being understood around the origin clockwise, we have the C.R. of the Virasoro algebra:

$$[L_m,L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \quad (2.3)$$

Note that $\{L_{+1},L_{-1},L_0\}$ generate an $Sl(2,R)$ algebra, while $c$ is the central charge.

In a CFT, primary fields are those which transform as tensors of weight $(h,\bar{h})$ under conformal transformations:

$$z \rightarrow w(z), \quad \bar{z} \rightarrow \bar{w}(\bar{z})$$

$$\phi_{h,\bar{h}}(z,\bar{z}) = \phi_{h,\bar{h}}(w(z),\bar{w}(\bar{z})) \left( \frac{dw}{dz} \right)^h \left( \frac{d\bar{w}}{d\bar{z}} \right)^{\bar{h}} \quad (2.4)$$
$T(z)$ being the generator of local scale transformations, one gets the O.P.E., after restricting to the $z$-part:

$$T(z) \phi_h(w) = \frac{h \phi_h(w)}{(z-w)^2} + \frac{\partial \phi_h(w)}{(z-w)} + ...$$  \hspace{1cm} (2.5)

$h$ is called the conformal spin of the primary field $\phi_h(z)$. One can deduce from eq. (2.5) the CR:

$$[L_m, \phi_h(z)] = (m+1)hz^m \phi_h(z) + z^{m+1} \partial \phi_h(z)$$  \hspace{1cm} (2.6)

Now let us add to the Virasoro algebra some primary fields. With some precautions, we can obtain a $W$ algebra.

As an example, let us consider the $N = 1$ superconformal algebra: it is made from the (conformal spin 2) stress energy tensor $T(z)$ and a conformal spin 3/2 fermionic field $G(z)$. Developing $T(z)$ and $G(z)$ in Laurent modes:

$$G(z) = \sum z^{-3/2-r} G_r$$  \hspace{1cm} (2.7)

with $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + \frac{1}{2}$ following we are in the Ramond or Neveu-Schwarz sector, we get the (anti) C.R.:

$$[L_m, G_r] = \left( \frac{1}{2}m - r \right) G_{m+r}$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - 1/4)\delta_{r+s,0}$$  \hspace{1cm} (2.8)

We have a $W$ (super)algebra. It is specially simple since it closes linearly on the generators $L_m$ and $G_r$. Let us add two remarks which will be relevant for the future.

First $\{L_{\pm 1}, L_0, G_{\pm 1/2}, G_{-1/2}\}$ constitute the $OSp(1|2)$ superalgebra, that is the "supersymmetric" $Sl(2)$ extension. In the following $OSp(1|2)$ will play for $W_n$ superalgebras the role of $Sl(2, \mathbb{R})$ for $W_n$ algebras.

Secondly $\{G_{\pm 1/2}\}$ constitutes a spin 1/2 representation of the algebra $\{L_{\pm 1}, L_0\}$. More generally [7] if $W_h(z)$ is a $h$ primary field under $T(z)$ the modes $W_n$ with $-h + 1 \leq n \leq h - 1$ will form a spin $(h-1)$ representation of $\{L_{\pm 1}, L_0\}$.

The above definitions and properties stand for the above OPE to be radially ordered. We will relax this last feature in the following and restrict ourselves to the classical case.

Then a classical finitely generated $W_n$ algebra will be defined as a Lie algebra with a Poisson bracket $\{,\}_{P.B.}$, and a set of generators involving a stress-energy tensor $T$ as well as a finite number of primary fields $W_{h_i}(i = 1, \ldots n-1)$ under $T$ satisfying:

$$\{T(z), T(w)\}_{P.B.} = -2T(w)\delta'(z-w) + \partial T(w)\delta(z-w) +$$
\[ + \frac{\alpha}{2} \delta''(z - w) \]
\[ \{ T(z), W_{h_i}(w) \}_\text{P.B.} = -h_i W_{h_i}(w) \delta'(z - w) + \partial W_{h_i}(w) \delta(z - w) \]

and
\[ \{ W_{h_i}(z), W_{h_j}(w) \} = \sum_{\alpha} P_{i,j;\alpha}(w) \delta^{(\alpha)}(z - w) \]

where \( P_{i,j;\alpha}(w) \) are polynomials in the primary fields \( W_{h_i}, T \) and their derivatives.

Let us remark that the property of a primary field \( W_h \) of conformal spin \( h \) to be connected to the representation \( D_{h-1} \) of the \( \text{Sl}(2, \mathbb{R}) \) algebra \( \{ L_\pm, L_0 \} \) limitates through the tensorial product \( D_{h_1-1} \times D_{h_2-1} \) the allowed conformal spin of the \( P_{i,j;\alpha} \) polynomials.

3 From a WZW model to a Toda theory

3.1 The method

It has been elegantly shown that, starting from a WZW model, the action of which is \( S(g) \) and the fields \( g(x) \) belong to the group \( G \), and imposing some of the components of the conserved currents to be constant or zero leads to a Toda model [4].

Let us denote \( S_{WZW}(g) \) the action of the WZW model based on a real connected Lie group \( G \), and \( g \in G \). Then from the Kac-Moody invariance \( G_1 \times G_2 \) with \( G_1 \cong G_2 \cong G \) of the model
\[ g(x) \rightarrow g_1(x^-)g(x)g_2(x^+) \]

with \( x = (x^+, x^-) \) denoting the two-dimensional variable, we get the currents:
\[ J_+ = g^{-1} \partial_+ g \quad \text{and} \quad J_- = \partial_- gg^{-1} \]

which, due to the equations of motion, are conserved:
\[ \partial_\pm J_\mp = 0 \]

In order to perform the gauge theory approach which will be relevant, we need \( G \) to be non compact: let us consider as an example the \( \text{Sl}(n, R) \) group. We decompose its Lie algebra \( \mathcal{G} \) as follows:
\[ \mathcal{G} = \mathcal{G}_- \oplus \mathcal{H} \oplus \mathcal{G}_+ \]

where \( \mathcal{G}_+(\mathcal{G}_-) \) is the subalgebra of positive (negative) root generators and \( \mathcal{H} \) the Cartan part, i.e.:
\[
\begin{pmatrix}
* & \cdots & \mathcal{G}_+ \\
\cdots & \cdots & \cdots \\
\mathcal{G}_- & \cdots & * \\
\end{pmatrix}
\]
Note that the generators $E_{\alpha_i}(i = 1...n - 1)$ associated to the (positive) simple roots are in the positions $E_{12}, E_{23}, ...E_{n-1,n}$ in the above matrix, while $E_{-\alpha_i}$ occupy the position $E_{21}, ..., E_{n,n-1}$ ($E_{ij}$ being the $n \times n$ matrix with 1 in position $(i,j)$ only).

The basic idea is to impose constraints on some components of these $J_\pm$ currents. Let us impose the restriction of $J_-$ to its $\mathcal{G}_-$ components to be:

$$J_-|_{\mathcal{G}_-} = M_- = \sum_{i=1}^{n-1} \mu_i E_{-\alpha_i}, \quad J_+|_{\mathcal{G}_+} = \sum_{i=1}^n \nu_i E_{\alpha_i}$$

(3.6)

with $\mu_i$ and $\nu_i$ real positive constants.

Such constraints can be obtained as a part of the equations of motion of a new model resulting from a Lagrange multiplier treatment on the WZW action. More precisely, it is a gauge theoretical approach involving as gauge group the (non compact) part $G_+$ in $G_1$ and $G_-$ in $G_2$, associated to the Lie $\mathcal{G}$ subalgebra $\mathcal{G}_+$ and $\mathcal{G}_-$ respectively with elements $g_+(x) \in G_+$ and $g_-(x) \in G_-$ which will lead to the Euler equations (3.3) and (3.6). The use of the local Gauss decomposition

$$g = g_+ \cdot h \cdot g_-$$

(3.7)

with

$$h(x) = \exp \sum_{i=1}^r \phi_i(x) H_i$$

(3.8)

provides in the Euler equations the differential equations of the Toda theory based on the group $G$, the $\phi_i$'s being the corresponding fields.

$$\partial_+ \partial_- \phi_i = \mu_i \nu_i \exp \sum_j K_{ij} \phi_j$$

(3.9)

where $K_{ij}$ is the Cartan matrix associated to the Lie algebra $\mathcal{G}$ of $G$.

Two remarks can be made at this point.

i) The above $G$ Toda theory involves $r = \text{rank } \mathcal{G}$ fields in one-to-one correspondence with the Cartan part $\mathcal{H}$ of $\mathcal{G}$, and it is usually called the ”Abelian” Toda theory on $\mathcal{G}$.

ii) The above construction actually involves the principal $SL(2)$ subalgebra of $\mathcal{G}$ with generators:

$$H = \sum_{i,j=1}^r K^{ij} H_j \quad E_- = \sum_{i=1}^r E_{-\alpha_i} \quad E_+ = \sum_{i,j=1}^r K^{ij} E_{\alpha_i}$$

(3.10)

(note that a rescaling in Eq.(3.6) allows to take all the $\mu_i = 1$; $K^{ij}$ is the inverse Cartan matrix).
Moreover the currents $J_-$ (resp. $J_+$) are not invariant under the gauge transformations generated by the constraints (3.6). Focussing on $J_-$, these transformations read:

$$ J_-(x_-) \to J_0^g(x_-) = g_+(x_-)J_-(x_-)g_+(x_-)^{-1} + \partial_-g_+(x_-) \cdot g_+(x_-)^{-1} \quad (3.11) $$

where $g_+(x_-) \in G_+$. This will allow to bring the currents to the gauge-fixed form:

$$ J^g_\cdot = M_- + \sum_{j \geq 0} W_{j+1}(J)M_j \quad (3.12) $$

where the $W_{j+1}$ are polynomials in the currents $J_-$ and their derivatives $\partial_-^n J_-$. In the so-called "Drinfeld-Sokolov highest weight gauge" each generator $M_j$ is the highest weight in the $Sl(2)_{ppal}$ representation $\mathcal{G}_j$ space obtained by reducing with respect to $Sl(2)_{ppal}$ the Lie algebra $\mathcal{G}$: considered as a vector space, $\mathcal{G}$ writes

$$ \mathcal{G} = \bigoplus_{j=1}^k D_j \quad (3.13) $$

with $D_j$ of dimension $(2j + 1)$. The Poisson brackets among the $W_j$’s can be obtained from the Poisson-Lie algebra satisfied by the current components:

$$ \{ J_\cdot_\cdot_\cdot_{a}(x_-), J_\cdot_\cdot_\cdot_{b}(x'_-) \}_PB = if_{c}^{ab}J'_c(x'_-)\delta(x_- - x'_-) + k\delta_{c}^b\delta'(x_- - x'_-) \quad (3.14) $$

where $f_{c}^{ab}$ are the structure constants for a given basis of $\mathcal{G}$.

Then each $W_{j+1}$ is associated to a $D_j$ and its conformal spin is $(j + 1)$ with respect to the stress energy tensor itself relative to the $D_1$ representation spanned by the generators of $Sl(2)_{ppal}$:

$$ T = T_0 + trH.\partial J \quad (3.15) $$

with

$$ T_0 = \frac{1}{2k} tr(J.J). \quad (3.16) $$

Note also that each $W_{j+1}$ can always be seen as a primary field with respect to $T$, after adjunction of an extra term in the $J'$s and derivatives.

Before going to examples, let us remark that, in this approach, a classical $W$-algebra is a subalgebra of the enveloping algebra of (3.14), itself symmetry of a WZW model: the constraints reduce the symmetry in such a way that only some polynomials in the $J^a$’s and their derivatives generate the residual symmetry.

### 3.2 Examples

Let us take for $\mathcal{G}$ the $Sl(3)$ algebra.

The Abelian Toda theory is obtained by imposing on the $J$ currents the constraints:

$$ J_- = \begin{bmatrix} \phi_1 & \phi_3 & \phi_4 \\ 1 & \phi_2 & \phi_5 \\ 0 & 1 & -\phi_1 - \phi_2 \end{bmatrix} \quad \text{leading by the} \quad J^g_\cdot = \begin{bmatrix} 0 & T & W_3 \\ 1 & 0 & T \\ 0 & 1 & 0 \end{bmatrix} $$

$$ g_+(x_-) \in G_+ \quad \text{to} $$

$\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$
Involving $SL(2)_{ppal}$ generated by:

\[
E_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad E_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\] (3.18)

$G$ decomposes under the (adjoint) action of $SL(2)_{ppal}$ as:

\[
G/SL(2) = D_1 \oplus D_2
\] (3.19)

to which are associated resp. with the spin 2 and 3 quantities $T$ and $W_3$ generating the well known Zamolodchikov $\{T, W_3\}$ algebra.

But still with $SL(3)$ there exists another kind of constraints which allows for a similar treatment of the WZW model. It reads

\[
J_-= \begin{bmatrix} \varphi_1 & \varphi_3 & \varphi_4 \\ 1 & \varphi_2 & \varphi_5 \\ 0 & -\varphi_1 - \varphi_2 \end{bmatrix}
\] (3.20)

Now the $SL(2)$ subalgebra which is involved is the following:

\[
E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_{+\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\] (3.21)

with respect to this $SL(2)$, $G$ decomposes as:

\[
G = D_1 \oplus D_{1/2} \oplus D_{1/2} \oplus D_0
\] (3.22)

and the gauge invariant matrix current takes the form:

\[
J_{\ell}^g = \begin{bmatrix} W_1 & W_2 & W_{3/2}^+ \\ 1 & W_1 & 0 \\ 0 & W_{3/2}^- & -2W_1 \end{bmatrix}
\] (3.23)

The algebra $\{W_2, W_{3/2}^+, W_{3/2}^-, W_1\}$ is usually called the classical Bershadsky algebra $\mathcal{B}$. It is the symmetry algebra of the "non Abelian" Toda model constructed from the $SL(2)$ algebra defined in (3.21).

There are only two different $SL(2)$ subalgebras in $SL(3)$; therefore we have exhausted the different Toda models and the associated $W$-algebras relative to $SL(3)$. More generally, starting from a simple algebra $G$, each admissible choice of $J$ components which can be set to constant (i.e. first class constraints in Dirac terminology) will correspond to an $SL(2)$ in $G$ and vice-versa. Then to determine all the different $W$-algebras symmetries of Toda theories associated to $G$, one has first to consider all the different $SL(2)$ in $G$. (This mathematical problem has
been solved by Dynkin). In each case, the decomposition of $\mathcal{G}$ with respect to $Sl(2)$ representations will provide the conformal spin of the associated $W$ algebra $[3]$. 

Supersymmetric Toda theories can also be considered. A supersymmetric treatment of the WZW models, based on simple superalgebras $\mathcal{SG}$ has to be done, constraints being written in a superspace formulation $[10]$. Then $Sl(2)$ is replaced by its supersymmetric extension $OSp(1|2)$. The classification of $OSp(1|2)$ subsuperalgebras in simple superalgebras followed by the reduction for each $\mathcal{SG}$ of its adjoint representation with respect to each $OSp(1|2)$ subpart provide the conformal superspin content of the $W$ superalgebras symmetries of Super Toda theories $[3]$. 

From such a classification, general properties of the $W$ (super)algebras, allowing a simplified and synthetic overview, can be deduced: this will be the object of the two next sections.

4 Folding the $W$ (super)algebras

Using the properties of a non simply laced simple algebra to appear as a subalgebra of $Sl(n)$ after a suitable identification of $Sl(n)$ simple roots, one can obtain $W$ algebras related to B-C-D series from $W$ algebras related to unitary ones $[4]$. Let us give an example, based again on the $Sl(3)$ group. Its Dynkin diagram (DD) is:

$$
\alpha_1 \quad \alpha_2
\bigcirc \quad \bigcirc
$$

(4.1)

$\alpha_1$ and $\alpha_2$ representing the simple roots, to which are associated the generators $E_{\alpha_1}$ and $E_{\alpha_2}$. It is known that the transformation $\tau$ such that: $\tau(\alpha_i) = \alpha_j \ i \neq j = 1, 2$ which is a symmetry of DD can be lifted up to an (outer) automorphism on the Lie algebra of $Sl(3)$ by defining:

$$
\hat{\tau}(E_{\pm\alpha_i}) = E_{\pm\tau(\alpha_i)} \ i = 1, 2
$$

(4.2)

with

$$
\hat{\tau}[E_{\alpha_i}, E_{-\alpha_i}] = \tau(\alpha_i)H
$$

(4.3)

The $Sl(3)$ subalgebra $\mathcal{G}$ invariant under $\hat{\tau}$ is then generated from:

$$
E_{\pm\alpha_1} + E_{\pm\alpha_2}
$$

(4.4)

That is, by "folding" the root $\alpha_1$ onto $\alpha_2$, $Sl(3)$ reduces to the Lie algebra $\mathcal{G}^F$ of
the (non compact) 3 dimensional orthogonal group:

\[
\begin{array}{ccc}
\alpha_1 & \alpha_2 & \alpha_1 + \alpha_2 \\
E_{\alpha_1} & E_{\alpha_2} & \rightarrow \\
\end{array}
\]

(4.5)

On the $3 \times 3$ matrix representation, where $E_{\alpha_1}$ is identified with $E_{12}$ and $E_{\alpha_2}$ with $E_{23}$, it will result that from the $G$ matrices $M = m^{ij} E_{ij}$, $m^{ij}$ being real numbers satisfying the traceless condition $\sum_{i=1}^{3} m^{ii} = 0$, one obtains a representation of $G^F$ by imposing the conditions:

\[
m^{ij} = (-1)^{i+j+1} m^{4-j,4-i}
\]

(4.6)

Identifying in the Abelian Toda theory on $SL(3)$ the $J^a$ current components as in (4.6), it is not a surprise to get, by Hamiltonian reduction:

\[
J^g_{SL(3)} = \begin{pmatrix} 0 & T & W_3 \\ 1 & 0 & T \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow J^g_{SO(3)} = \begin{pmatrix} 0 & T' & 0 \\ 1 & 0 & T' \\ 0 & 1 & 0 \end{pmatrix}
\]

(4.7)

as can be expected in a rank 1 algebra.

Of course, this simple example can be generalized, the foldings of $A_{2n-1} = SL(2n)$ and $A_{2n} = SL(2n+1)$ providing the symplectic $C_n = Sp(2n)$ and $B_n = SO(2n+1)$ algebras respectively. If one notes that $SO(2n)$ can be obtained from $SO(2n+1)$ by a regular embedding, one realizes that the $W$ algebras associated to the $A_n$ series can be ”folded” into the $W$ algebras relative to the other infinite series (note also that for the exceptional cases, the $G_2$ ones can be deduced from $D_4 \equiv SO(8)$ and $F_4 W$-algebras from the $E_6$ ones). The same procedure can be applied to superalgebras (see [4]).

An useful consequence of this technics is to get identities between structure constants of $W$-algebras relative to different simple algebras: denoting by $C_{ij}^k$ the general structure constant of the ”fusion rule”:

\[
[W_i] \cdot [W_j] = \delta_{ij} \frac{c}{2} [I] + C_{ij}^k (G) [W_k]
\]

(4.8)

We have as examples, in the Abelian case:

\[
C_{ij}^k (D_n) = C_{ij}^k (A_{2n}) \quad i, j, k \neq n
\]

(4.9)

\[
C_{ij}^k (C_n) = C_{ij}^k (A_{2n-1}) \quad C_{ij}^k (B_n) = C_{ij}^k (A_{2n}),
\]

(4.10)

such relations being sometimes precious, due to the difficulty to obtain explicit commutation relations.
5 Secondary reductions

Let us consider again \( G = SL(3) \) and the two \( W \)-algebras which can be constructed, via Toda theories, from such an underlying simple algebra; they are the Zamolodchikov algebra \( \{ T, W_3 \} \) and the Bershadsky algebra generated by \( \{ W_2, W_{3/2}^+, W_{3/2}^-, W_1 \} \). The corresponding \( J^g \) matrices read (see Eq. (3.17) and (3.23)):

\[
J^g_{\text{Abel}} = \begin{pmatrix}
0 & T & W_3 \\
1 & 0 & T \\
0 & 1 & 0
\end{pmatrix} \quad J^g_{\text{Non Abel}} = \begin{pmatrix}
W_1 & W_2 & W_{3/2}^+ \\
1 & W_1 & 0 \\
0 & W_{3/2}^- & -2W_1
\end{pmatrix}
\] (5.1)

One remarks that the constraints imposed in the Non Abelian case

\[
\{ trJ_+ \cdot E_{-\alpha_1} = 1 ; \ trJ_+ \cdot E_{-(\alpha_1+\alpha_2)} = 0 \} \quad (5.2)
\]

form a subset of the constraints corresponding to the Abelian case:

\[
\{ trJ_+ \cdot E_{-\alpha_1} = trJ_+ \cdot E_{-\alpha_2} = 1 ; \ trJ_+ \cdot E_{-(\alpha_1+\alpha_2)} = 0 \} \quad (5.3)
\]

It is time to give explicitly the P.B. of the Classical Bershadsky algebra: let us, for convenience, make a little change in the notations and denote \( W_1 \) by \( J \) and \( W_2 + \frac{1}{3c} J \cdot J \) by \( T \).

\[
\begin{align*}
\{ J(z), J(w) \} &= -\frac{3}{2} \alpha \delta'(z - w) \\
\{ J(z), W_{3/2}^{\pm}(w) \} &= \pm \frac{3}{2} W_{3/2}^{\pm} \delta(z - w) \\
\{ T(z), W_{3/2}^{\pm}(w) \} &= \frac{3}{2} W_{3/2}^{\pm}(w) \delta'(z - w) + \partial W_{3/2}^{\pm}(w) \delta(z - w) \\
\{ T(z), J(w) \} &= -J(w) \delta'(z - w) + \partial J(w) \delta(z - w) \\
\{ T(z), T(w) \} &= -2T(w) \delta'(z - w) + \partial T(w) \delta(z - w) + \frac{c}{2} \delta'''(z - w) \\
\{ W_{3/2}^+(z), W_{3/2}^-(w) \} &= 2J(w) \delta'(z - w) - c \delta''(z - w) + \\
&+ (T - \frac{4}{3c} J^2 - \partial J)(w) \delta(z - w) \\
\{ W_{3/2}^+(z), W_{3/2}^-(w) \} &= 0
\end{align*}
\] (5.4)

The last relation, which expresses the nilpotency of \( W_{3/2}^- \) (and \( W_{3/2}^+ \)), allows to consider the constraint

\[
W_{3/2}^- = 1 \quad (5.5)
\]
as a gauge constraint (first class constraint).

With the help of \( J(z) \), it is possible to redefine the energy momentum tensor \( T \) in such a way that the constraint becomes conformally invariant, that is, shifting \( T \) into

\[
\hat{T} = T - \partial J_+
\] (5.6)
$W_{3/2}^-$ behaves as a spin 0 field:

$$\{ \hat{T}(z), W_{3/2}^-(w) \} = \partial W_{3/2}^-(w) \delta(z - w)$$

$$\simeq 0 \quad \text{using Eq.(5.5)} \quad . \quad (5.7)$$

Then one can look at the reduced $W$ algebra obtained by constructing the polynomials invariant under the gauge transformations associated to $W_{3/2}^-$. Therefore, let us consider the finite gauge transformations on the currents:

$$X(w) \rightarrow \hat{X}(w) = X(w) + \int dz \alpha(z) \{ W_{3/2}^-(z), X(w) \}$$

$$\quad + \frac{1}{2!} \int dz \, dz' \alpha(z) \alpha(z') \left\{ W_{3/2}^-(z), \{ W_{3/2}^-(z'), X(w) \} \right\}$$

$$\quad + \ldots \quad (5.8)$$

where $X = J, T, W_{3/2}^+$, the constraint (5.7) being used on the r.h.s. of the P.B., following Dirac prescriptions on constraints ("weak equations"). Then the $J$ current transforms as:

$$\hat{J}(w) = J(w) + \int dz \alpha(z) \{ W_{3/2}^-(z), W_1^1(w) \} + 0 \quad (5.9)$$

since

$$\{ W_{3/2}^-(z), J(w) \} \simeq \left( \frac{1}{2} \delta(z - w) \right) \quad (5.10)$$

that is:

$$\hat{J}(w) = J(w) + 3 \frac{2}{3} \alpha(w) \quad (5.11)$$

Then, it is clear that a global gauge fixing is given by

$$\hat{J}(w) = 0 \quad (5.12)$$

that is, by taking:

$$\alpha = - \frac{2}{3} J \quad (5.13)$$

It follows for $T$:

$$T(w) \rightarrow \hat{T}(w) = T(w) - \frac{3}{2} \int dz \cdot \alpha(z) \cdot \delta'(z - w) + 0$$

$$\quad = T(w) + \frac{3}{2} \partial \alpha$$

$$\quad = T - \partial J \quad (5.14)$$

as expected from Eq.(5.6) !

In the same way:

$$W_{3/2}^+ \rightarrow \hat{W}_3 = W_{3/2}^+ + \frac{2}{3} J \cdot T + \frac{2}{3} J \cdot \partial J - \frac{8}{27c} J^3 - \frac{2c}{3} \partial^2 J \quad (5.15)$$
the notation $\hat{W}_3$ being justified by the property of $\hat{W}_3$ to behave as a spin 3 field under $\hat{T}$.

At this point, it is not a surprise to realize that the $\hat{T}$ and $\hat{W}_3$ quantities generate a (algebra isomorphic to) Zamolodchikov algebra.

The above illustrated method with $W$ algebras based on $\mathcal{G} = SL(3)$ can be applied to any simple algebra $\mathcal{G}$ up to some obvious technical difficulties. Starting from the weakest constraints and adding new ones on a $W$ algebra relative to some Lie algebra $\mathcal{G}$, one can then obtain chains of $W$ algebras, the "smallest" one being relative to the Abelian Toda case (highest number of constraints). As could be expected by Lie algebra experts, there also exist cases with $\mathcal{G}$ non simply laced, i.e. $B_n$ or $C_n$, for which such a secondary reduction towards the Abelian case cannot be obtained. Finally, in the same way one gets Toda equations by gauging $WZW$ models, a gauging of the Toda action in which a (Non Abelian) $W$ algebra stands as the current algebra of the theory could be performed, leading to a new (more constrained) Toda action. Such an approach for a generalized gauge Toda field theory, as well as a more complete discussion on secondary reductions will soon be available \[5\].

6 Rational $W$ algebras

6.1 Commutant of the spin 1 part

Now let us turn our attention to the particular role of the spin one part, when it is present, in a $W$ algebra. One can easily check, by dimensional arguments, that these fields generate a Kac-Moody algebra $W_1$. Moreover the set of $W$ generators decomposes into irreducible representations under the adjoint action of this Kac Moody algebra. Let us study what happens when factorizing out the spin one part in a $W$ algebra, that is by computing the commutant in $W$ of the $W_1$ Kac-Moody subalgebra \[3\].

Most of $W$ algebras associated to Non Abelian Toda theories contain spin-one fields. Let us perform our calculations on the Bershadsky algebra already considered in the previous sections (see in particular Eq. (5.4)).

First, by the following shift on $T$,

\[ \bar{T} = T - \frac{1}{3c} J^2 \quad (6.1) \]

one gets the P.B.:

\[
\begin{align*}
\{ \bar{T}(z), J(w) \} &= 0 \\
\{ \bar{T}(z), W_\pm(w) \} &= -\frac{3}{2} W_\pm(w) \delta'(z-w) + (D W_\pm)(w) \delta(z-w) \\
\{ W_+(z), W_-(w) \} &= (\bar{T} - cD^2)(w) \delta(z-w)
\end{align*}
\quad (6.2)
\]
while $\bar{T}$ satisfies the usual Virasoro P.B.:

$$\{\bar{T}(z), \bar{T}(w)\} = -2\bar{T}(w)\delta'(z-w) + \partial\bar{T}(w)\delta(z-w) + \frac{c}{2}\delta''(z-w)$$ (6.3)

In the above equations, one has used the covariant derivative $D$ such that

$$DW_\pm = (\partial \mp \frac{1}{c}J)W_\pm$$ (6.4)

while the $D^2$ showing up in the r.h.s. of $\{W_+, W_-\}$ is relative to $w$. The appearance of a covariant derivative may open new perspectives in the field of integrable models. It is here particularly convenient in order to construct the commutant of $J$. Indeed the set of fields commuting with $J$ is generated by the stress energy tensor $\bar{T}$ and the bilinear products:

$$W^{(p,q)} = (D^p W_+)(D^q W_-)$$ (6.5)

with $p, q$ non negative integers.

Actually, the fields $W^{(p,q)}$ and $\bar{T}$ are the building blocks from which one can construct an infinite tower of primary fields of spin 3,4,...

$$W_3 = W_+ W_-$$
$$W_4 = W_+ DW_- - W_- DW_+$$
$$\vdots$$
$$W_{3+n} = W_+ D^n W_- - (D^n W_+)W_- + \cdots \text{ for } n > 2$$ (6.6)

these fields being created by the P.B. of fields of lower conformal spin, for ex.:

$$\{W_3(z), W_3(w)\} = 2W_4(w)\delta'(z-w) - \partial W_4(w)\delta(z-w)$$ (6.7)

and so on.

At this point, one may say that by looking at the commutant of the spin one generator $J$ in the Bershadsky $W$ algebra, one has obtained a polynomial non linear $W_\infty$ realization.

But the primary fields $W_{3+n}$ with $n \geq 2$ are not independent, and can be expressed as rational-and not polynomials- functions of $T, W_3, W_4$: for example $W_5$ can be written in terms of $W_3$ and $W_4$ as follows:

$$W_5 = \frac{1}{4W_3^2} \left[ 7\left(W_4^2 - (\partial W_3)^2\right) + 6W_3(\partial^2 W_3) + \bar{T}W_3 \right]$$ (6.8)

Therefore, the commutant of $J$ exhibits a new structure with respect to the standard $W$ algebras, which can be seen either as a rational finitely generated $W$ algebra or as a polynomial non linear $W_\infty$ realization.

The above example is the simplest one exhibiting such a structure. Of course a general approach with a non Abelian $W_1$ part can be performed (see [4]).
6.2 Supersymmetric extension

The supersymmetric extension of this problem can be considered in an analogous way. Again, let us illustrate the method on an example, the $N = 3$ superconformal algebra $\mathcal{SC}(N = 3)$ generated by a spin 2 generator $T(z)$, 3 spin $\frac{3}{2}$ components $G_{\frac{3}{2}}^a (a = 1, 2, 3)$, 3 spin 1 elements $J^a(z)$, constituting an $\text{Sl}(2)$ Kac-Moody algebra and a spin $\frac{1}{2}$ fermion $\psi(z)$. The C.R. in the classical case can be deduced from the formulas (15) of [11], in which we identify the O.P.E. with the P.B. and the singular terms $\frac{1}{(z-w)^k}$ with $(-1)^{k-1} \frac{1}{(k-1)!} \delta^{(k-1)}(z-w)$. After defining:

$$
G^\pm(z) = \frac{1}{\sqrt{2}}(G^1 \pm iG^2)(z) \quad J^\pm(z) = \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)(z)
$$

we will adopt the superfield formalism (cf. [10]) and define:

$$
T(z) = \frac{1}{2} G^0(z) + \theta T(z) \quad \text{of superspin} \quad \frac{3}{2}
$$

$$
J^\pm(z) = \pm J^\pm(z) + \theta G^\pm(z) \quad \text{of superspin} \quad 1
$$

$$
\Phi(z) = \psi(z) + \theta J^0(z) \quad \text{of superspin} \quad \frac{1}{2}
$$

using the supervariable notations:

$$
Z = (z, \theta), \quad W = (w, \eta) \quad \text{and} \quad Z - W = z - w - \theta \eta
$$

then the P.B. can be “compactly” written as (keeping in mind from above that:

$$
\frac{\theta - \eta}{Z - W} = (\theta - \eta) \delta(Z - W) = \delta(Z - W) \quad \text{and so on for their derivatives, and the O.P.E. being in place of the P.B.}:
$$

$$
\mathcal{T}(z) \cdot \mathcal{O}_s(W) = s \frac{\theta - \eta}{(Z - W)^2} \mathcal{O}_s(W) + \frac{1}{2} \frac{D \mathcal{O}_s(W)}{Z - W} + \frac{\theta - \eta}{Z - W} \partial \mathcal{O}_s(W) + \ldots
$$

if $\mathcal{O}_s(W)$ denotes the superspin $J^\pm(W)$ or $\Phi(W)$ of superspin $s = 1$ or $\frac{1}{2}$, and as usual: $D = \partial_\eta + \eta \partial_w$

$$
T(Z)T(W) = \frac{3}{2} \frac{\theta - \eta}{(Z - W)^2} T(W) + \frac{1}{2} \frac{D T(W)}{Z - W} + \frac{\theta - \eta}{Z - W} \partial T(W) + \frac{c/6}{(Z - W)} + \ldots
$$

$$
\Phi(Z)J^\pm(W) = \pm \frac{\theta - \eta}{Z - W} J^\pm(W) + \ldots
$$

$$
\Phi(Z)\Phi(W) = \frac{c/3}{Z - W} + \ldots
$$

$$
J^+(Z)J^-(W) = - \frac{\theta - \eta}{(Z - W)^2} \Phi(W) - \frac{1}{Z - W} D \Phi(W) - \frac{\theta - \eta}{Z - W} \partial \Phi
$$

$$
- 2 \frac{\theta - \eta}{Z - W} T(W) - \frac{c/3}{(Z - W)^2} + \ldots
$$

(6.13)
We wish to factorize out the superspin \( \frac{1}{2} \) superfield \( \Phi(Z) \). As in the nonsupersymmetric case, we can operate a shift on \( \mathcal{T}(Z) \)

\[
\mathcal{T}_0(Z) = \mathcal{T}(Z) - \frac{3}{2c} \Phi(Z) D\Phi(Z)
\]

(6.14)

such that:

\[
\mathcal{T}_0(Z) \cdot \Phi(W) = 0
\]

(6.15)

We can expect the covariant derivative of Eq.(6.4) to become:

\[
\mathcal{D} = \mathcal{D} - \frac{3q}{c} \Phi
\]

(6.16)

if \( q \) is the super \( U(1) \) charge carried by the primary superfield, i.e.:

\[
\mathcal{D} \mathcal{J}^\pm = (\mathcal{D} \mp \frac{3}{c} \Phi) \mathcal{J}^\pm
\]

(6.17)

Now the spin 2 superfield \( W_2(Z) = \mathcal{J}^+(Z) \cdot \mathcal{J}^-(e) \) is a primary superfield under \( \mathcal{T}_0(Z) \) in the commutant of \( \Phi(Z) \). The properties above obtained with \( W \) algebras generalize here with \( W \) superalgebras. Computing for example the P.B. of \( W_2 \) with itself one gets:

\[
W_2(Z)W_2(W) = - \frac{c}{3} \left( \frac{2W_2(W)}{(Z - W)^2} + \frac{\partial W_2(W)}{Z - W} + \frac{\theta - \eta}{(Z - W)^2} D W_2(W) \right.
\]

\[
+ \frac{3}{5} \left( \frac{\theta - \eta}{Z - W} D \partial W_2(W) \right) - \frac{36}{5} \frac{\theta - \eta}{Z - W} (\mathcal{T}_0 \cdot W_2)(W)
\]

\[
+ \frac{c}{3} \frac{\theta - \eta}{Z - W} W_{7/2}(W) + \ldots
\]

(6.18)

where \( W_{7/2}(W) \) is the (new!) \( 7/2 \) superspin primary superfield defined as:

\[
W_{7/2} = \mathcal{J}^+ \mathcal{D}^3 \mathcal{J}^- + \mathcal{J}^- \mathcal{D}^3 \mathcal{J}^+ - \frac{3}{5} D \partial W_2 - \frac{48}{5c} \mathcal{T}_0 \cdot W_2
\]

(6.19)

### 6.3 Spin 1/2 versus Spin 1 fields

The superalgebra \( \mathcal{SC}(N = 3) \) was the first example considered by the authors of [11] to illustrate their result about the factorization of the spin 1/2 part in a superconformal field theory, more precisely that a meromorphic field theory can be decomposed into the tensor product of a spin 1/2 part and a conformal field theory without spin 1/2 field. We would like to stress that this property can easily be proved, at least at the classical level, by the use of finite gauge transformations already introduced in the previous section (see Eq.(5.8)). Indeed, leaving to the
reader the general proof (which will also be found in [3]) let us stay with the 
SC($N = 3$) algebra and perform on its generators $X(w)$ the transformation:

$$X(w) \rightarrow \hat{X}(w) = X(w) + \int dz \, \alpha(z)\psi(z).X(w) + 0 \quad (6.20)$$

where $\psi(z)$ is the fermion field (we do not use any more the superfield formalism, 
since we wish to only factorize the $\psi(z)$ fermion and not the superspin 1/2 field).

Owing to the OPE relation:

$$\psi(z) \cdot \psi(w) = \frac{c/3}{z - w} \quad (6.21)$$

one directly gets, imposing the ”gauge fixing”:

$$\alpha(w) = -\psi(w) \quad (6.22)$$

the transformed fields:

$$\hat{\psi} = 0 ; \quad \hat{T} = T - \frac{1}{2}\psi\partial\psi ; \quad \hat{G}^a = G^a - T^a\psi \quad \hat{J}^a = J^a \quad \text{with} \quad a = 1, 2, 3 \quad (6.23)$$

In accordance with the results of [11], the O.P.E. among the transformed fields are identical, except for the central charge to the ones relative to the non transformed 
fields, and as expected such that:

$$\hat{T} \cdot \psi = \hat{G}^a \cdot \psi = \hat{J}^a \cdot \psi = 0 \quad (6.24)$$

Note that this gauge transformation can also be done with spin 1/2 bosons, 
and leads to the same conclusion [3]. It has also be shown that the action of such 
a super-Toda model can be rewritten as the sum of two terms, one relative to the 
spin 1/2 part and the other to the factorized $W$ part [12].

It is natural to wonder what happens if, instead of performing a gauge trans-
formation associated with a 1/2 fermion, one involves a spin 1 field. Let us take 
once more as an example the Bershadsky algebra (see Eq.(5.4)): its (simple) Kac 
Moody generator $J(z)$ satisfies:

$$J(z) \cdot J(w) = \frac{3/2c}{(z - w)^2} \quad (6.25)$$

In order to obtain $\hat{J} = 0$ in the transformation:

$$J(w) \rightarrow \hat{J}(w) = J(w) + \int dz \, \alpha(z)J(z) \cdot J(w) + \ldots \quad (6.26)$$

We would have to impose $\alpha$ such that

$$\partial\alpha(w) = J(w) \quad (6.27)$$
The pathology created by this relation appears in different places. In particular, one would get:

$$\alpha(z) \cdot W_{3/2}^\pm(w) = \pm\frac{3}{2} W_{3/2}^\pm(w) \ln(z - w)$$ (6.28)

and some trouble to compute, from:

$$\hat{W}_{3/2}^\pm(w) = e^{\pm3/2a(w)} W_{3/2}^\pm(w)$$ (6.29)

the quantity:

$$\hat{W}_{3/2}^+(z) \cdot \hat{W}_{3/2}^-(w)$$ (6.30)

Thus, gauge transformations relative to spin 1/2 fields allow to recover the result of Ref [11], namely the property that spin 1/2 fermions can be eliminated in a super $W$ algebra, but such a technics does not appear suitable for the factorization of spin 1 fields, as could be expected from the results presented in the first part of this section.

Note that the above discussion has to be compared with the factorization at quantum level, of spin 1/2 and 1 fields considered in [13]: the projection used there appears as a quantum version of our gauge transformation.

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