A NEW EVEN ENTIRE FUNCTION AND THE EXTENDED Riemann Hypothesis

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Abstract. The extended Riemann hypothesis states that all of the nontrivial zeros of the Dedekind zeta function are on the critical line $\Re(s) = 1/2$. In this paper we study an even entire function by Li’s integral representation of the Dedekind xi function. We obtain the inequality for its coefficients by the result of Coffey. We set up its product given by the Hadamard’s factorization theorem. We discover all of the zeros of this function are on the critical line $\Re(s) = 0$. With use of the obtained result, we prove that the extended Riemann hypothesis is true.

1. Introduction

In 1877 Julius Wilhelm Richard Dedekind [1], who is a German mathematician, proposed the well-known Dedekind zeta function of an algebraic number field. This function is considered as a generalization of the Riemann zeta function from integers to number fields. We now suppose that this algebraic number field $F$ with $\alpha_1$ real places and $\alpha_2$ complex places. Let $M = \alpha_1 + 2\alpha_2 > 1$. Assume that $C$, $R$ and $N$ are the sets of the complex, real and natural numbers, respectively, and that $\Re(s)$ and $\Im(s)$ are the real and imaginary parts for $s \in C$. The Dedekind zeta function $\zeta_F(s)$ of the algebraic number field $F$ is defined by [2, 3]

$$
\zeta_F(s) = \sum_{m=1}^{\infty} \varphi_m m^{-s},
$$

where $\varphi_m$ denotes the number of ideals with the norm equal to $m$ and $\Re(s) > 1$.

The Dedekind xi function $\xi_F(s)$ of the algebraic number field $F$ is expressed as [4]

$$
\xi_F(s) = s(s-1) \zeta_F(s) d_F^{-\frac{s}{2}} \pi^{-\frac{\alpha_1 s}{2}} \Gamma^{\alpha_1 \left(\frac{s}{2}\right)} (2\pi)^{-\alpha_2} \Gamma^{\alpha_2}(s),
$$

where $d_F$ is the absolute value of the discriminant of $F$ and $\Gamma(s)$ is the Gamma function [6].

For $s \in C$, $\xi_F(s)$ has the followings:

- (A1) There exists the functional equation [5]

$$
\xi_F(s) = \xi_F(1-s)
$$
for \( s \in \mathbb{C} \).

- (A2) \( \xi_F(s) \) is an entire function of order \( \lambda = 1 \) [6, 7, 8].
- (A3) The extended Riemann hypothesis is equivalent to the statement that all of the zeros of the Dedekind xi function \( \xi_F(s) \) are on the critical line \( \Re(s) = 1/2 \) (see Extension 6.6 in [9], p.58).

We assume that \( w_F \) is the number of roots of unity of \( F \) and that \( D_F \) is the discriminant of \( F \). Following Li [4], we let the quadratic form

\[
\Theta (y, z) = \phi_1 y^2 + \phi_2 yz + \phi_3 z^2
\]

satisfy the conditions

\[
\phi_2^2 - 4\phi_1 \phi_3 = d_F = |D_F|, \quad \begin{cases} -\phi_1 < \phi_2 \leq \phi_1 < \phi_3, \\ \text{or} \\ 0 < \phi_2 \leq \phi_1 = \phi_3. \end{cases}
\]

The Li’s function \( \Lambda(v) \) is defined as [6]

\[
\Lambda(h) = \sum_{\Theta} \sum_{p, q = -\infty}^{\infty} \frac{\pi \Theta(p, q)}{\sqrt{|D_F|}} \left( \frac{\pi h \Theta(p, q)}{\sqrt{|D_F|}} - 1 \right) \exp \left( -\frac{2\pi h \Theta(p, q)}{\sqrt{|D_F|}} \right),
\]

where the first sum takes over the inequivalent classes of the positive definite integral quadratic forms of \( D_F \) for all integers \( p, q \) with \( q \neq 0 \).

The integral representation of the Dedekind xi function \( \xi_F(s) \) is given by Li (see Theorem 1 in [4]) as

\[
\xi_F(s) = \frac{4}{w_F} \int_1^{\infty} \Lambda(h) (h^s + h^{1-s}) \, dh
\]

where \( s \in \mathbb{C} \).

Following Chandrasekharan and Narasimhan [10], we substitute

\[
s = \frac{1}{2} + ig
\]

into \( \xi_F(s) \) (see (2)) for \( g \in \mathbb{C} \) such that [10]

\[
\Xi_F(g) = \xi_F \left( \frac{1}{2} + ig \right).
\]

Adopting (3) and (8), the function \( \Xi_F(g) \) of the algebraic number field \( F \) has:

- (B1) \( \Xi_F(g) \) has the functional equation [10]

\[
\Xi_F(g) = \Xi_F(-g).
\]

- (B2) The extended Riemann hypothesis is equivalent to the statement that all of the zeros of the function \( \Xi_F(g) \) are on the critical line \( \Im(g) = 0 \) [10].
Motivated by the work of Chandrasekharan and Narasimhan [10], we introduce the function $\mathcal{N}_F(t)$ of the algebraic number field $F$ by

\begin{equation}
\mathcal{N}_F(t) = \xi_F \left( \frac{1}{2} + t \right),
\end{equation}

with the fact

\begin{equation}
\Xi_F(g) = \mathcal{N}_F(ig).
\end{equation}

Applying (B2) and (11), we discover that the extended Riemann hypothesis is equivalent to the statement that all of the zeros of the function $\mathcal{N}_F(t)$ are on the critical line $\Re(s) = 0$.

The main aim of this paper is to prove:

**Theorem 1.** All of the zeros of the function $\mathcal{N}_F(t)$ are on the critical line $\Re(t) = 0$.

The outline of the present paper is suggested as follows. In Section 2, we investigate the series representation for $\xi_F(s)$. In Section 3 we obtain the integral, series and product representations and order of $\mathcal{N}_F(t)$. In Section 4 we give the proof of Theorem 1. In Section 5 we suggest the representations for $\xi_F(s)$ and $\Xi_F(g)$.

### 2. The series representation for $\xi_F(s)$

With the work of Li [4], we have the followings:

**Theorem 2.**

- There is

\begin{equation}
\xi_F(s) = \frac{8}{w_F} \int_1^\infty \Lambda(h) h^{\frac{1}{2}} \cosh \left[ \left( s - \frac{1}{2} \right) \log h \right] dh.
\end{equation}

- There is

\begin{equation}
\xi_F(s) = \sum_{a=0}^{\infty} \Omega_{2a} \left( s - \frac{1}{2} \right)^{2a},
\end{equation}

where

\begin{equation}
\Omega_{2a} = \frac{8}{w_F} \int_1^\infty \Lambda(h) h^{\frac{1}{2}} \left( \log h \right)^{2a} \frac{1}{(2a)!} dh.
\end{equation}

**Proof.** By (6), we present

\begin{equation}
\xi_F(s) = \frac{4}{w_F} \int_1^\infty \Lambda(h) \left( h^s + h^{1-s} \right) dh = \frac{4}{w_F} \int_1^\infty \Lambda(h) h^{\frac{1}{2}} \left( h^{s-\frac{1}{2}} + h^{1-s-\frac{1}{2}} \right) dh
\end{equation}

\begin{equation}
= \frac{8}{w_F} \int_1^\infty \Lambda(h) h^{\frac{1}{2}} \cosh \left[ \left( s - \frac{1}{2} \right) \log h \right] dh.
\end{equation}
Taking
\[
\cosh \left[ \left( s - \frac{1}{2} \right) \log h \right] = \sum_{a=0}^{\infty} \left[ \frac{(\log h)^{2a}}{(2a)!} \left( s - \frac{1}{2} \right)^{2a} \right]
\]
into (15), we obtain
\[
\xi_F \left( s \right) = \sum_{a=0}^{\infty} \Omega_{2a} \left( s - \frac{1}{2} \right)^{2a},
\]
where
\[
\Omega_{2a} = \frac{8}{w_F} \int_{1}^{\infty} \Lambda \left( h \right) h^{\frac{1}{2}} \frac{(\log h)^{2a}}{(2a)!} \, dh.
\]
We therefore complete the proof of Theorem 2. \(\Box\)

3. Results for \(N_F \left( t \right) \)

By (6) and (10), we obtain the followings:

**Theorem 3.**

- There exists
  \(\tilde{N}_F \left( t \right) = \frac{8}{w_F} \int_{1}^{\infty} \Lambda \left( h \right) h^{\frac{1}{2}} \cosh \left( t \log h \right) \, dh.\)
  \((16)\)

- There exists
  \(N_F \left( t \right) = N_F \left( -t \right).\)
  \((17)\)

*Proof.* Substituting \(s = 1/2 + t\) into (12), we obtain
\[
N_F \left( t \right) = \xi_F \left( \frac{1}{2} + t \right) = \frac{8}{w_F} \int_{1}^{\infty} \Lambda \left( h \right) h^{\frac{1}{2}} \cosh \left\{ \left[ \left( \frac{1}{2} + t \right) - \frac{1}{2} \right] \log h \right\} \, dh
\]
\[
= \frac{8}{w_F} \int_{1}^{\infty} \Lambda \left( h \right) h^{\frac{1}{2}} \cosh \left( t \log h \right) \, dh.
\]
This is the desired result.

We thus complete the proof of Theorem 3. \(\Box\)

**Theorem 4.** If \(\Omega_{2a}\) is defined as in Theorem 2, then there is
\[
N_F \left( t \right) = \sum_{a=0}^{\infty} \Omega_{2a} t^{2a}.
\]
\((18)\)

*Proof.* Putting \(s = 1/2 + t\) into (13), we may give
\[
N_F \left( t \right) = \xi_F \left( \frac{1}{2} + t \right) = \sum_{a=0}^{\infty} \Omega_{2a} \left[ \left( \frac{1}{2} + t \right) - \frac{1}{2} \right]^{2a} = \sum_{a=0}^{\infty} \Omega_{2a} t^{2a},
\]
which is the required result.
We hence complete the proof of Theorem 4.

□

**Theorem 5.** \( \mathcal{N}_F(t) \) is an even entire function of order \( \varpi = 1 \).

**Proof.** Since \( \xi_F(s) \) is an entire function of order \( \lambda = 1 \) [6, 8], there exist (see [12], p.4)
\[
\lim_{a \to \infty} \sqrt[\varpi]{|\Omega_{2a}|} = 0,
\]
and (see Theorem 2.2.2 in Boas’ book [9], p.4)
\[
\lambda = \limsup_{a \to \infty} \frac{a \ln a}{\ln (1/\ln |\Omega_{2a}|)} = 1.
\]
By (13), (18) and (21), the same coefficient \( \Omega_{2a} \) implies that \( \mathcal{N}_F(t) \) is an entire function of order \( \lambda = 1 \).

From (17) we know that
\[
\mathcal{N}_F(t) = \mathcal{N}_F(-t).
\]
We thus complete the proof of Theorem 5.

□

**Theorem 6.** Let \( \ell \in \mathbb{N} \). We have
\[
\mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t_{\ell}) > 0} \left( 1 - \frac{t_{\ell}^2}{t_{\ell}^2} \right),
\]
where the product runs all of the zeros of \( t_{\ell} \) of \( \mathcal{N}_F(t) \).

**Proof.** Following the result of Coffey (see Proposition in [11], p.250), we know that
\[
\xi_F(s) > 0
\]
for all \( s \in \mathbb{R} \).

This implies
\[
\xi_F\left( \frac{1}{2} \right) > 0.
\]
With (19), we have
\[
\mathcal{N}_F(t) = \xi_F\left( \frac{1}{2} + t \right),
\]
such that
\[
\mathcal{N}_F(0) = \xi_F\left( \frac{1}{2} + 0 \right) = \xi_F\left( \frac{1}{2} \right) > 0.
\]
By (16) and (27), we see
\[
\mathcal{N}_F(0) = \frac{8}{\nu_F} \int_1^\infty \Lambda(h) h^{\frac{1}{2}} dh > 0.
\]
Because $\aleph_F(t)$ is an even entire function of order $\varpi = 1$ and (28) is true, the Hadamard's factorization theorem (see [13], p.22) implies that

$$\aleph_F(t) = \aleph_F(0) e^{\mu t} \prod_{t_\ell} \left(1 - \frac{t}{t_\ell}\right) \exp\left(\frac{t}{t_\ell}\right),$$

where the product run all of the zeros $t_\ell$ of $\aleph_F(t)$ and $\mu$ is a constant.

Since $\aleph_F(t)$ is an even entire function, form (17) we have $\aleph_F(t) = \aleph_F(-t)$ such that

$$\aleph_F(0) e^{\mu t} \prod_{t_\ell \in \mathbb{N}} \left(1 - \frac{t}{t_\ell}\right) \left(1 + \frac{t}{t_\ell}\right) \exp\left(\frac{t}{t_\ell} - \frac{t}{t_\ell}\right)$$

(30)

and

$$\aleph_F(-t) = \aleph_F(0) e^{-\mu t} \prod_{t_\ell \in \mathbb{N}} \left(1 - \frac{t^2}{t_\ell^2}\right).$$

(31)

By (30) and (31), there is

$$\aleph_F(0) e^{\mu t} \prod_{t_\ell \in \mathbb{N}} \left(1 - \frac{t^2}{t_\ell^2}\right) = \aleph_F(0) e^{-\mu t} \prod_{t_\ell \in \mathbb{N}} \left(1 - \frac{t^2}{t_\ell^2}\right),$$

such that $\mu = 0$

This implies that (30) can be written as

$$\aleph_F(t) = \aleph_F(0) \prod_{t_\ell \in \mathbb{N}} \left(1 - \frac{t^2}{t_\ell^2}\right).$$

(33)

Hence, the proof of Theorem 6 is completed.

\begin{proof}

**Theorem 7.** The series

$$\sum_{t_\ell = 0}^{\infty} \frac{1}{|t_\ell|^{1+v}}$$

is convergent for any $v > 0$.

**Proof.** Assume that $\{t_\ell\}_{1}^{\infty}$ is a sequence of complex numbers, numbered in order of modulus $|t_\ell| < |t_{\ell+1}|$ for $\ell \in \mathbb{N}$, with $t_\ell \neq 0$ (see [13], p.18). Since $\aleph_F(t)$ is an entire function of order

...
\(\omega = 1\), and \(\{t_\ell\}^\infty_1\) is a sequence of zeros of \(\mathcal{N}_F(t)\), the series \(\text{see, for example, [12], p.9}\)

\[
\sum_{\ell} \frac{1}{|t_\ell|^{1+v}}
\]

converges for any \(v > 0\).

We hence complete the proof of Theorem 7. \(\square\)

\[\text{4. The proof of Theorem 1}\]

By using Theorem 4 and Theorem 6, we have

\[
\mathcal{N}_F(t) = \sum_{a=0}^{\infty} \Omega_{2a} t^{2a}
\]

and

\[
\mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t_\ell) > 0} \left(1 - \frac{t^2}{t_\ell^2}\right).
\]

such that there exists

\[
\mathcal{N}_F(t) = \sum_{a=0}^{\infty} \Omega_{2a} t^{2a} = \mathcal{N}_F(0) \prod_{\Im(t_\ell) > 0} \left(1 - \frac{t^2}{t_\ell^2}\right).
\]

With use of the results of Li [4] and Coffey ([11], p.250), (5) implies that there exists

\[
\Lambda(h) = \sum_\Theta \sum_{p,q=-\infty}^{\infty} \frac{\pi \Theta(p,q)}{\sqrt{|D_F|}} \left(\frac{\pi h \Theta(p,q)}{\sqrt{|D_F|}} - 1\right) \exp\left(-\frac{2\pi h \Theta(p,q)}{\sqrt{|D_F|}}\right),
\]

such that for \(h > 1\),

\[
\Lambda(h) > 0
\]

since ([11], p.250)

\[
\Theta(p,q) \geq \frac{\sqrt{|D_F|}}{2}
\]

for all integers \(p, q\) with \(q \neq 0\), ([11], p.250)

\[
\frac{\pi \Theta(p,q)}{\sqrt{|D_F|}} \geq \frac{\pi}{2},
\]

\[
\frac{\pi h \Theta(p,q)}{\sqrt{|D_F|}} - 1 \geq \frac{\pi h}{2} - 1 > 0
\]

for \(h > 1\), and

\[
\exp\left(-\frac{2\pi h \Theta(p,q)}{\sqrt{|D_F|}}\right) > 1.
\]
Thus, by (14) and (40), we show that
\begin{equation}
\Omega_{2a} = \frac{8}{w_F} \int_1^\infty \Lambda(h) \frac{1}{h^2} \frac{(\log h)^{2a}}{(2a)!} dh > 0.
\end{equation}

Adopting (45), we have
\begin{equation}
\mathcal{N}_F(t) = \sum_{a=0}^{\infty} \Omega_{2a} t^{2a} = \sum_{a=0}^{\infty} (\Omega_{2a} t^{2a}) = \sum_{a=0}^{\infty} (\Omega_{2a} t^{2a}).
\end{equation}

With (40) and (46), we show
\begin{equation}
\mathcal{N}_F(t) = \sum_{a=0}^{\infty} \left( \Omega_{2a} t^{2a} \right).
\end{equation}

By using (36) and (47),
\begin{equation}
\mathcal{N}_F(t) = \mathcal{N}_F(t)
\end{equation}
always holds for \( t \in C \).
From (37) and (48) we obtain
\begin{equation}
\mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left( 1 - \frac{t^2}{t \ell} \right).
\end{equation}
Making use of (37), there exists
\begin{equation}
\mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left( 1 - \frac{t^2}{t \ell} \right) = \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left( 1 - \frac{t^2}{t \ell} \right).
\end{equation}

From (27) we have
\begin{equation}
\mathcal{N}_F(0) > 0
\end{equation}
such that (50) can be rewritten as
\begin{equation}
\mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left( 1 - \frac{t^2}{t \ell} \right) = \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left( 1 - \frac{t^2}{t \ell} \right)
\end{equation}
\begin{equation}
= \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left[ 1 - \left( \frac{t^2}{t \ell} \right) \right].
\end{equation}

To simply (52), we obtain
\begin{equation}
\mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t \ell) > 0} \left( 1 - \frac{t^2}{t \ell} \right).
Combining (49) and (53), we obtain

\[ (54) \quad \mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\tau^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right). \]

By using (48), we have the functional equation

\[ (55) \quad \mathcal{N}_F(t) = \mathcal{N}_F(t) = \mathcal{N}_F(t) \]

such that (54) yields that

\[ (56) \quad \mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\tau^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right). \]

Since

\[ (57) \quad \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\tau^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right) \]

and

\[ (58) \quad \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right) \]

(56) becomes

\[ (59) \quad \mathcal{N}_F(t) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right) = \mathcal{N}_F(0) \prod_{\Im(t) > 0} \left( 1 - \frac{\ell^2}{t^2} \right). \]

By Theorem 7, the series

\[ (60) \quad \sum_{t \in \mathbb{T}_\ell} \frac{1}{|t|^1 + v} \]

is convergent for any \( v > 0 \). This implies that

\[ (61) \quad \sum_{t \in \mathbb{T}_\ell} \frac{1}{|t|^2} \]

is absolutely convergent.
Putting $t = 1$ into (59) implies that

\[(62) \quad \mathcal{N}_F(1) = \mathcal{N}_F(0) \prod_{\Re(t) > 0} \left(1 - \frac{1}{t^2}\right) = \mathcal{N}_F(0) \prod_{\Re(t) > 0} \left(1 - \frac{1}{t^2}\right).\]

From (10), we have

\[(63) \quad \mathcal{N}_F(1) = \xi_F \left(\frac{1}{2} + 1\right) = \xi_F \left(\frac{3}{2}\right) > 0,\]

such that

\[(64) \quad 0 < \mathcal{N}_F(1) < \infty\]

since Coffey (see Proposition in [11], p.250) said that

\[(65) \quad \xi_F(s) > 0\]

for all $s \in \mathbb{R}$.

Since the series

\[(66) \quad \sum_{t} \frac{1}{|t|^2}\]

is absolutely convergent, Knopp said (see [14], p.9) that both

\[(67) \quad \sum_{t} \frac{1}{t^2}\]

and

\[(68) \quad \sum_{t} \frac{1}{t^2}\]

are convergent.
From (62), (67) and (68), we have

\[ (69) \]

\[ \mathbb{F}(0) \prod_{\mathfrak{A}(t_{\ell}) > 0} \left(1 - \frac{1}{t_{\ell}^2}\right) \]
\[ = \prod_{\mathfrak{A}(t_{\ell}) > 0} \left(1 - \frac{1}{t_{\ell}^2}\right) \]
\[ = \prod_{\mathfrak{A}(t_{\ell}) > 0} \left(\frac{t_{\ell}^2 - 1}{t_{\ell}^2 - 1}\right) \]
\[ = \prod_{\mathfrak{A}(t_{\ell}) > 0} \left(\frac{t_{\ell}^2 t_{\ell}^2 - t_{\ell}^2}{t_{\ell}^2 t_{\ell}^2 - t_{\ell}^2}\right) \]
\[ = \prod_{\mathfrak{A}(t_{\ell}) > 0} \left(1 - \frac{t_{\ell}^2 - t_{\ell}^2}{t_{\ell}^2 (t_{\ell}^2 - 1)}\right) = 1 \]

which must require that

\[ \text{(70)} \]

\[ \frac{t_{\ell}^2 - t_{\ell}^2}{t_{\ell}^2 (t_{\ell}^2 - 1)} = 0. \]

Since (27) and (63) implies that \( t_{\ell}^2 \neq 0 \) and \( t_{\ell}^2 - 1 \neq 0 \), (70) gives

\[ \text{(71)} \]

\[ t_{\ell}^2 = t_{\ell}^2 \]

which leads to

\[ \text{(72)} \]

\[ t_{\ell}^2 - t_{\ell}^2 = (t_{\ell} - t_{\ell}) (t_{\ell} + t_{\ell}) = 4\Re(t_{\ell}) \Im(t_{\ell}) = 0. \]

Since \( \Im(t_{\ell}) \neq 0 \), we have from (72) that

\[ \text{(73)} \]

\[ \Re(t_{\ell}) = 0. \]

Let

\[ \text{(74)} \]

\[ |\Im(t_{\ell})| = \theta_{\ell} > 0. \]

Then, substituting (73) into (59) and (69),

\[ \text{(75)} \]

\[ \mathbb{F}_F(t) = \mathbb{F}_F(0) \prod_{\ell=1}^{\infty} \left(1 + \frac{t_{\ell}^2}{\theta_{\ell}^2}\right) \]

and

\[ \text{(76)} \]

\[ \sum_{t_{\ell}} \frac{1}{t_{\ell}^2} = \sum_{t_{\ell}} \frac{1}{t_{\ell}^2} = -\sum_{t_{\ell}} \frac{1}{\theta_{\ell}^2} \]

are convergent.
Thus, we complete the proof of Theorem 1.

**Remark.** From (72) we shown

\[(77) \quad \Im (t_\ell) = 0\]

which implies that all of the zeros of \(F_\ell (t)\) are real.

As is well known, since Coffey (see Proposition in [11], p.250) said that

\[(78) \quad \xi_F (s) > 0\]

for all \(s \in \mathbb{R}\), there exists

\[(79) \quad \Re_F (t) = \sum_{a=0}^{\infty} \Omega_{2a} t^{2a} > 0\]

for \(t \in \mathbb{R}\) since from (45) we obtain the fact \(\Omega_{2a} > 0\). This fact (79) implies that all of the zeros of \(F_\ell (t)\) are not real. Thus, (77) is invalid.

From (75), we directly have:

**Corollary 1.** There exists

\[(80) \quad \Re_F (t) = \Re_F (0) \prod_{\ell=1}^{\infty} \left(1 + \frac{t^2}{\theta_\ell^2}\right).\]

5. **The representations for \(\xi_F (s)\) and \(\Xi_F (g)\)**

Here, we now consider the following:

**Theorem 8.** There is

\[(81) \quad \Re_F (s) = \Re_F \left(\frac{1}{2}\right) \prod_{\ell=1}^{\infty} \left[1 + \left(s - \frac{1}{2}\right)^2\right].\]

**Proof.** By (10), we obtain

\[(82) \quad \Re_F (t) = \xi_F \left(\frac{1}{2} + t\right),\]

which implies that

\[(83) \quad \xi_F (s) = \Re_F \left(s - \frac{1}{2}\right).\]

Combining Corollary 1 and (83), we give

\[(84) \quad \xi_F (s) = \Re_F \left(s - \frac{1}{2}\right) = \Re_F (0) \prod_{\ell=1}^{\infty} \left[1 + \left(s - \frac{1}{2}\right)^2\right].\]

Taking \(s = 1/2\) into (84), we have

\[(85) \quad \xi_F \left(\frac{1}{2}\right) = \Re_F (0).\]
such that (84) becomes
\[
\xi_F(s) = \kappa_F(0) \prod_{\ell=1}^{\infty} \left[ 1 + \left( s - \frac{1}{2} \right)^2 \theta_\ell^2 \right] = \xi_F(0) \prod_{\ell=1}^{\infty} \left[ 1 + \left( s - \frac{1}{2} \right)^2 \theta_\ell^2 \right].
\]
Thus, we complete the proof of Theorem 8.

**Theorem 9.**
- There is
  \[
  \Xi_F(g) = \Xi_F(0) \prod_{\ell=1}^{\infty} \left( 1 - \frac{g^2}{\theta_\ell^2} \right).
  \]
- There is
  \[
  \Xi_F(t) = \frac{8}{w_F} \int_{1}^{\infty} \Lambda(h) h^{\frac{1}{2}} \cos(t \log h) \, dh.
  \]

**Proof.** By using Corollary 1 and (11), we show
\[
\Xi_F(g) = \kappa_F(ig) = \kappa_F(0) \prod_{\ell=1}^{\infty} \left( 1 + \frac{(ig)^2}{\theta_\ell^2} \right) = \kappa_F(0) \prod_{\ell=1}^{\infty} \left( 1 - \frac{g^2}{\theta_\ell^2} \right).
\]
From (11) and Theorem 3 we have
\[
\Xi_F(g) = \kappa_F(ig)
\]
such that
\[
\Xi_F(0) = \kappa_F(0)
\]
and
\[
\Xi_F(g) = \kappa_F(ig) = \frac{8}{w_F} \int_{1}^{\infty} \Lambda(h) h^{\frac{1}{2}} \cosh(ig \log h) \, dh = \frac{8}{w_F} \int_{1}^{\infty} \Lambda(h) h^{\frac{1}{2}} \cos(g \log h) \, dh.
\]
In view of (89) and (91), we obtain
\[
\Xi_F(g) = \kappa_F(0) \prod_{\ell=1}^{\infty} \left( 1 - \frac{g^2}{\theta_\ell^2} \right) = \kappa_F(0) \prod_{\ell=1}^{\infty} \left( 1 - \frac{g^2}{\theta_\ell^2} \right).
\]
Thus, we complete the proof of Theorem 9.

**Remark.** Obviously, Theorem 8 says that all of the zeros of the Dedekind $\xi$ function $\xi_F(s)$ are on the critical line $\Re(s) = 1/2$. And, Theorem 9 implies that all of the zeros of the function $\Xi_F(g)$ are on the critical line $\Im(g) = 0$. The computations of the zeros of the Dedekind zeta functions were investigated by Tollis [15], Grenié and Molteni [16], and Hasanalizade, Shen and Wong [17]. However, (92) can be used to design new technologies for them. It is known that $\{t_\ell\}_{\ell=1}^{\infty}$ is a sequence of complex numbers, numbered in order of modulus $|t_\ell| < |t_{\ell+1}|$ for $\ell \in \mathbb{N}$, with $t_\ell \neq 0$. Since the extended Riemann hypothesis is true, $\{\theta_\ell\}_{\ell=1}^{\infty}$ is a sequence of real positive number, numbered in order of modulus $\theta_\ell < \theta_{\ell+1}$.
for $\ell \in \mathbb{N}$, with $\theta_\ell \neq 0$. This implies that the gaps between zeros of $\alpha$ becomes the gaps between the sequence of real positive number $\{\theta_\ell\}_{\ell=1}^\infty$. The gaps between $\theta_\ell$ and $\theta_{\ell+1}$ remains an open problem in theory of the Dedekind zeta functions [18].

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