Comments on the Aharonov–Cashier Effect

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Abstract

We study the basic requirements for neutron confinement in the framework of some 3-D Aharonov–Cashier configurations.

Aharonov and Cashier (AC) pointed out the existence of a quantum mechanical process [1–3] wherein the behavior of an uncharged dipole is affected by the presence of an electric field. Let us imagine an electrically charged object with axial symmetry centered around the z axis. The nearly point dipoles, e.g., neutrons, are completely polarized along, say, the positive y direction. It is straightforward to see that this system can be recast in a supersymmetric form [4–6]. To study supersymmetry breaking, one solves the corresponding eigenvalue problem for the ground state of the given geometrical configuration.

In this report we shall treat three cases: (1) a finite spherical charge distribution, (2) an infinite plane with uniform charge density, and (3) the standard Aharonov–Cashier configuration. We conclude that, although in this circumstance there is apparently no force on the particles, slow neutrons will tend to move toward regions where the gradient of the field increases.

To start with we assume connectivity in the configuration space in order to define a normalizable ground state. The problem turns out to have exact supersymmetry only under the fulfillment of a condition for the magnitude of the charge density for the cases of infinite charge distribution. We also discuss the possibility of supersymmetry breaking by examining the requirements for the existence of lower energy bound states.

To be specific, let us consider a spin 1/2 charged particle with an anomalous magnetic moment $\kappa_n$. The Dirac equation can be written [7] in a covariant form ($h = c = 1$) as

$$\left(\gamma_{\mu}p^{\mu} - \frac{e\kappa_n}{2M_n} F^{\mu\nu} \gamma_{\mu} = M_n \right) \Phi(x) = 0,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor.

The Aharonov–Cashier effective wave equation is obtained by making $A^\mu(x) \neq 0$, $B(x) = 0$, with $\nabla \cdot E(x) = 4\pi \rho(x)$. Equation (1) can be recast in the form

$$\left(\gamma \cdot \left( p + \frac{i e \kappa_n}{M_n} \gamma_0 E(x) \right) + \gamma_0 M_n \right) \Phi(x,t) = i \frac{\partial}{\partial t} \Phi(x,t).$$

For stationary states of energy $E$ we write

$$\Phi_E(x,t) = \Phi_E(x)e^{-iEt} = \left( \phi_E(x) \right) e^{-iEt}.$$  \hspace{1cm} (3)

Thus from (2) and (3) we get

$$\alpha_k \otimes \left( p^k - \tau_3 \frac{i e \kappa_n}{M_n} E^k \right) \Phi_E(x) = e \Phi_E(x),$$

where $\tau_3$ is a $z$-Pauli matrix, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_i$ Pauli matrices, $\eta = e\kappa_n/M_n$, and $e = E^2 - M_n^2$. A $\mathcal{N} = 1$ supersymmetry algebra can be constructed in the form

$$\mathcal{H}_{S} = \left[ Q, Q^\dagger \right], \quad [\mathcal{H}_{S}, Q] = [\mathcal{H}_{S}, Q^\dagger] = 0,$$

with

$$\mathcal{H}_{S} \Phi_E(x) = \frac{e}{2M_n} \Phi_E(x).$$

Here

$$Q(x, p) \equiv \frac{1}{\sqrt{2M_n}} \tau^- \otimes \sigma \cdot (p - i eE(x))$$

is the supersymmetric charge and $\tau^- = (1/2)(\tau_1 - i\tau_2)$, where the $\tau_1, \tau_2$ are Pauli matrices. Thus $\mathcal{H}_{S}$ is invariant under $Q$ and $Q^\dagger$. From (6) we find that the equations for $\phi_E$ and $\chi_E$ are decoupled. In particular, for thermal neutrons we consider the upper components of $\Phi_E$ which satisfy

$$\left\{ p^2 - \eta \nabla \cdot E(x) - \frac{2\eta E(x)}{r} \sigma \cdot L + 3\eta^2 \gamma_0^2(x) \right\} \phi_E(x) = e \phi_E(x),$$

where $r = |x|$, with $L$ the orbital angular momentum operator. The supersymmetric generators annihilate the vacuum state in order to have unbroken symmetry:

$$Q \phi_{(0)}(x) = 0, \quad Q^\dagger \phi_{(0)}(x) = 0,$$

where $\phi_{(0)}$ is the ground state of the system.

The second equation (9) is satisfied identically in the nonrelativistic limit the lower components $\Phi_{E=0}$ vanish. The first one yields

$$\sigma \cdot (p - i eE(x)) \phi_{(0)}(x) = 0.$$  \hspace{1cm} (10)

Without loss of generality we can set

$$\phi_{(0)}(x) = \left( \begin{array}{c} \phi(x) \\ 0 \end{array} \right), \quad \chi_{(0)}(x) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).$$

Furthermore, in a system with axial symmetry we have also the condition $L_3 \phi_{(0)}(x) = 0$, i.e. $\phi_{(0)}(x) = \phi_{(0)}(r)$. Here then we are concerned with states for which $E^2 = M_n^2$ ($e = 0$).

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We begin by considering a solid sphere with uniform charge per unit volume \( \rho_0 \) centered in the origin of the laboratory frame, so that there exists an electric field

\[
E_\varphi (x) = 4\pi \rho_0 x / 3, \quad 0 \leq r \leq r_0; \quad E_\varphi (x) = 4\pi \rho_0 r^3 / 3r^3, \quad r_0 \leq r < \infty,
\]

where \( r_0 \) is the radius of the sphere. In this circumstance there is apparently no force on the neutrons but there exists a kind of Aharonov–Bohm effect [1–3,7]. Nevertheless, if we allow the neutrons to penetrate the sphere we can consider the problem of the possible bound states of the neutron in this new AC configuration.

Then from (10) we find the first order differential equations

\[
\left( \frac{d}{dr} - \beta r \right) \phi_\varphi (r) = 0, \quad 0 \leq r \leq r_0; \quad \left( \frac{d}{dr} - \beta \frac{r^3}{r^2} \right) \psi_\varphi (r) = 0, \quad r_0 \leq r < \infty,
\]

where \( \beta = 4\pi \rho_0 / 3 \). Thus

\[
\phi_\varphi (r) = A_\varphi e^{-\beta r^2 / 2}, \quad 0 \leq r \leq r_0;
\]

\[
\psi_\varphi (r) = A_\psi \exp \left( -\frac{\beta r^2}{r} \right), \quad r_0 \leq r < \infty,
\]

with \( A \) complex constants.

Next we demand continuity of the wavefunction and its derivative at \( r = r_0 \). Both conditions yield the same information:

\[
\frac{A_\varphi}{A_\psi} = \exp \left( -\frac{1}{2} \beta r_0^2 \right).
\]

Moreover, if \( \Psi_{E=M} \) belongs to the Hilbert space, \( \phi \) must be normalizable in \( R^3 \):

\[
4\pi \lim_{r \to \infty} \int_0^r \left| \phi (r) \right|^2 r^2 dr = 1.
\]

However, as \( r \to \infty \) this integral diverges since \( \exp (-\beta r^2 / r) \to 1 \). Therefore supersymmetry is broken in the case.

Next we solve the general problem (8) by separation of variables: \( \phi (x) = \phi (r) \chi_{l,m}(\theta, \varphi) \), where in terms of the spherical harmonics \( Y_{l,m} \),

\[
Y_{l,\pm 1/2,m_0}(\theta, \varphi) = \frac{1}{\sqrt{2l + 1}} \left\{ \pm \sqrt{l \pm m_0 + 1} \left( Y_{l-m_0-1/2}(\theta, \varphi) \right) \right. \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right. \\
+ \left. \left( l \mp m_0 + 1 \right) Y_{l+m_0+1/2}(\theta, \varphi) \right\} \left( \begin{array}{c} 0 \\ 1 \end{array} \right).
\]

Thus from (8) and (17) we get

\[
\left( \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + \frac{\sigma}{2r} - 2\beta \left( \frac{\sigma - L - \frac{3}{2}}{2} \right) - \beta^2 r^2 \right) \psi_\varphi (r) = 0, \quad r \leq r_0,
\]

\[
\left( \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + \frac{\sigma}{2r} - 2\beta \left( \frac{\sigma - L - \frac{3}{2}}{2} \right) - \beta^2 r^2 \right) \psi_\psi (r) = 0, \quad r > r_0,
\]

where \( \psi (r) \equiv r \phi (r) \). The radial solutions must be normalizable in the range \( 0 \leq r < \infty \), and we also demand continuity at \( r_0 \) on the corresponding solutions. For \( \psi_\psi (r) \) we find

\[
\left( \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + \varepsilon - 2\beta \frac{\sigma - L - \frac{3}{2}}{2} - \beta^2 r^2 \right) \psi_\psi (r) = 0,
\]

with \( \varepsilon = \varepsilon + \beta(3 + 2j - 1/2) \). Thus

\[
\psi_\psi (r) = C_1 \hat{F}_1 \left( \frac{l + 3/2 - \varepsilon}{2}; l + 3/2 + 1; \beta^2 r^2 \right)^{j+1} e^{-\beta r^2 / 2},
\]

where \( C \) is a complex constant and \( \hat{F}_1 \) is the confluent hypergeometric function.

For \( r \geq r_0 \) we obtain

\[
\left( \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + \varepsilon + 2\beta \frac{\sigma - L - \frac{3}{2}}{2} - \beta^2 \left( \frac{\sigma - L - \frac{3}{2}}{2} \right)^2 \right) \psi_\varphi (r) = 0.
\]

Eq. (22) has only one kind of solution: non-normalizable scattering-like states for \( \varepsilon > 0 \) \((E > M^2)\) since the potential decreases as \( V \sim 1 / r^2 \) \((1 / r^2 \sim dV / dr)\), and also because there is a term proportional to \( 1 / r^4 \) induced by the electric moment of the particle [1].

The second case considers an infinite plane of thickness \( L \) with uniform charge density \( \rho \), situated symmetrically on the \( xy \) plane. This configuration resembles a potential well in one-dimensional quantum mechanics. The electric field is given by

\[
E_\varphi (z) = 4\pi \rho_0 \hat{z}, \quad |z| \leq \frac{L}{2};
\]

\[
E_\varphi (z) = 4\pi \rho_0 L \frac{z}{|z|} \hat{z}, \quad |z| > \frac{L}{2}.
\]

We assume that the neutrons are completely polarized along the positive \( z \) direction. Thus

\[
\phi (x) = R(r) \phi (z) \exp (\pm \imath \varphi) \phi (z).
\]

In order that the wavefunction be single valued when the full azimuth is allowed, \( \varphi \) must be an integer. Therefore from (8) and (24) we obtain the differential equations

\[
\frac{1}{R(r)} \frac{d}{dr} \left( \frac{dR(r)}{dr} \right) - \frac{\varepsilon}{r^2} + \frac{1}{\phi(z)} \frac{d^2 \phi(z)}{dz^2} + 4\pi \rho_0
\]

\[
- 16\pi^2 \rho_0^2 \frac{z^2}{r^2} + \varepsilon = 0, \quad |z| \leq \frac{L}{2},
\]

\[
\frac{1}{R(r)} \frac{d}{dr} \left( \frac{dR(r)}{dr} \right) - \frac{\varepsilon}{r^2} + \frac{1}{\phi(z)} \frac{d^2 \phi(z)}{dz^2} + \varepsilon = 0, \quad |z| > \frac{L}{2}.
\]

Thus

\[
\left( \frac{1}{r} \frac{d}{dr} \left( \frac{d}{dr} + \frac{k^2 - \frac{\varepsilon}{r^2}}{r^2} \right) \right) R_{zk}(r) = 0
\]

is the radial equation for a Bessel function, with \( R_{zk}(r) = C J_l(kr) \), where \( k^2 = k^2 - \varepsilon \), with \( k \) a real positive parameter.
Hence for the ground state ($\varepsilon = 0$) we have

$$\left( \frac{d^2}{dz^2} + 4\pi\eta \rho_0 - k^2 - 16\pi^2 \eta^2 z^2 \right) \phi(z) = 0, \quad L/2 \geq |z|,$$

$$\left( \frac{d^2}{dz^2} - k^2 \right) \phi(z) = 0, \quad L/2 < |z|.$$  \hspace{1cm} (28)

The (normalizable) ground state is then

$$\phi_0(z) = \begin{cases} A \exp\left(-\frac{1}{2}k^2z^2\right), & L/2 \geq |z|, \\ A \exp\left(-\frac{1}{2}k^2L^2 + k(L - z)\right), & L/2 < |z|. \end{cases} \hspace{1cm} (30)$$

where the values of $k$ are restricted by the condition $4\pi\eta \rho_0 > k^2$. Note that $k$ denotes an infinite degeneracy in the ground state which proceeds from the unbound motion of the particle on the $xy$ plane. As an example we insert $c$ into the previous inequality and get $4\pi\eta \kappa_0 \rho_0/(M_n c^2) > k^2$. Choosing $\rho_0 = 2.0 \times 10^6$ [esu/cm$^2$] we find the bound 15.28 [cm$^{-1}$] > $k$. As one would expect, this result does not depend on the plane thickness $L$.

Finally let us examine the standard $1 + 2$ AC configuration [5,6]. The problem turns out to have exact super-symmetry only under the fulfillment of a condition for the magnitude of the charge distribution which generates the electric field.

An infinite cylinder with uniform charge per unit volume $\rho$ centered along the $z$ axis, generates the electric field

$$E_z(x) = \rho x/2, \quad 0 \leq r \leq r_0;$$

$$E_z(x) = \rho r_0 x/2r^2, \quad r_0 \leq r < \infty,$$

where $r_0$ is the radius of the cylinder and for simplicity we have chosen $\vec{x} \cdot \vec{z} = 0$. Here $\vec{x}$ and $\vec{z}$ are unit vectors in the $x$ and $z$ directions respectively. The neutrons are completely polarized along the positive $z$ direction. They move on a plane in the presence of $E$.

Then again from (10) we find the differential equations

$$\left( \frac{d}{dr} - \beta r \right) \phi_{+}(r) = 0, \quad 0 \leq r \leq r_0;$$

$$\left( \frac{d}{dr} - \beta^2 r \right) \phi_{-}(r) = 0, \quad r_0 \leq r < \infty,$$

where $\beta = -\varepsilon \kappa_0/4M_n$. Thus

$$\phi_{+}(r) = Ae^{\beta r}, \quad 0 \leq r \leq r_0;$$

$$\phi_{-}(r) = B e^{\beta r}, \quad r_0 \leq r < \infty,$$

with $A, B$ complex constants.

Next we demand continuity of the wavefunction and its derivative at $r = r_0$ yielding the boundary condition

$$A \exp\left[(1/2)\beta r_0^2 \right] = B r_0 e^{\beta r_0}.$$  Furthermore, if $\Psi_{E=M_n}$ belongs to the Hilbert space, $\phi$ must be normalizable on the plane $[0, 2\pi] \times [0, \infty]$ and thus we must require that $\beta r_0^2 < -1$. This inequality constitutes a necessary requirement on the possible values of $\lambda \equiv \rho \kappa_0$ if we want to preserve unbroken supersymmetry. As $\lambda$ depends linearly on $r_0^2$, one can in principle set up a configuration with the required $\lambda [1,5]$. For instance, $c$ into the expression for $\lambda$, we get $|\lambda|_{\min} = 4\pi M_n c^2 / [\kappa_0] = 60, 62 \times 10^6$ [esu/cm]. Naturally, this result is independent of the charged line dimension $2r_0$.

To treat the general eigenvalue problem, we observe that the eigenvalue problem stated by (8) has two kinds of solutions [5]:

(a) non-normalizable scattering-like states for $\varepsilon > 0$ ($E^2 > M_n^2$);

(b) normalizable bound states for $\varepsilon < 0$ ($E^2 < M_n^2$).

The energy levels are obtained by requiring that the radial solutions and their derivatives be continuous at $r = r_0$, i.e., this is the quantization condition for the remaining energy levels.

From the above we can draw the conclusion that electric charge distribution has to be sufficiently spread out in space in order to preserve unbroken supersymmetry. If this be the case, the field is normalizable and thus $\Psi_{E=M_n}$ constitutes a bound state of the system. Furthermore the magnitude of the electric charge distribution has to be sufficiently large in order to generate a bound states.

Confinement is usually achieved by means of diverse magnetic traps. Cold neutrons are extensively used to test quantum theory, and in applied physics [8–10].

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