Bose-Einstein Condensate strings

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We consider the possible existence of gravitationally bound general relativistic strings consisting of Bose-Einstein Condensate (BEC) matter which is described, in the Newtonian limit, by the zero temperature time-dependent nonlinear Schrödinger equation (the Gross-Pitaevskii equation), with repulsive inter-particle interactions. In the Madelung representation of the wave function, the quantum dynamics of the condensate can be formulated in terms of the classical continuity equation and the hydrodynamic Euler equations. In the case of a condensate with quartic nonlinearity, the condensates can be described as a gas with two pressure terms, the interaction pressure, which is proportional to the square of the matter density, and the quantum pressure, which is without any classical analogue though, when the number of particles in the system is high enough, the latter may be neglected. By assuming cylindrical symmetry, we analyze the physical properties of the BEC strings in both the interaction pressure and quantum pressure dominated limits, by numerically integrating the gravitational field equations. In this way we obtain a large class of stable string-like astrophysical objects, whose basic parameters (mass density and radius) depend sensitively on the mass and scattering length of the condensate particles, as well as on the quantum pressure of the Bose-Einstein gas.

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I. INTRODUCTION

The observation, in 1995, of Bose-Einstein condensation in dilute alkali gases, such as vapors of rubidium and sodium, confined in a magnetic trap and cooled to very low temperatures\textsuperscript{1}, represented a major breakthrough in experimental condensed matter physics, as well as a major confirmation of an important prediction in theoretical statistical physics. At very low temperatures, all particles in a dilute Bose gas condense to the same quantum ground state, forming a Bose-Einstein Condensate (BEC), which appears as a sharp peak over a broader distribution in both coordinate and momentum space. Particles become correlated with each other when their wavelengths overlap, that is, when the thermal wavelength $\lambda_T$ is greater than the mean inter-particle distance $l$. This happens at a critical temperature $T_c < 2\pi\hbar^2 n^{2/3}/mk_B$, where $m$ is the mass of an individual condensate particle, $n$ is the number density, and $k_B$ is Boltzmann’s constant\textsuperscript{2,3}. A coherent state develops when the particle density is high enough, or the temperature is sufficiently low. From an experimental point of view, the condensation is indicated by the generation of a sharp peak in the velocity distribution, which is observed below the critical temperature\textsuperscript{1}. More recently, quantum degenerate gases have been created by a combination of laser and evaporative cooling techniques, opening several new lines of research at the border of atomic, statistical and condensed matter physics\textsuperscript{2,4}.

Since Bose-Einstein condensation is a phenomenon that has been observed and well studied in the laboratory, the possibility that it may occur on astrophysical or cosmic scales cannot be rejected \textit{a priori}. Thus, dark matter, which is required to explain the dynamics of the neutral hydrogen clouds at large distances from the galactic center, and which is a cold bosonic system, could also exist as a Bose-Einstein condensate\textsuperscript{5}. In fact, since there exists a formal analogy between classical scalar fields and BECs, any theory of scalar field dark matter may also be viewed as condensate system\textsuperscript{6}. In early studies, such as those given in\textsuperscript{8}, either a phenomenological approach was adopted, or the solutions of the Gross-Pitaevskii equation, which describes the condensate in the non-relativistic limit, were investigated numerically. A systematic study of the properties of condensed galactic dark matter halos was performed in\textsuperscript{10}, and these systems have been further investigated by numerous authors\textsuperscript{11}.

By introducing the Madelung representation of the wave function, the dynamics of the dark matter halo can be formulated in terms of the continuity equation and the hydrodynamic Euler equations. Hence, condensed dark matter can be described as a Newtonian gas, whose density and pressure are related by a barotropic equation of state. However, in the case of a condensate with quartic nonlinearity, the equation of state is polytropic with index $n = 1$\textsuperscript{10}.

Furthermore, though superfluids, such as liquid $^4$He, are far from being dilute, there is, nevertheless, good
reason to believe that the phenomenon of superfluidity is related to that of Bose-Einstein condensation. Both experimental observations and theoretical calculations estimate the condensate fraction at $T = 0$ (denoted $n_0$) for superfluid helium to be around $n_0 \approx 0.10$ and, since a strongly correlated pair of fermions can be treated approximately like a boson, the arising superfluidity can be interpreted as the condensation of coupled fermions. Similarly, the transition to a superconducting state in a solid material may be described as the condensation of electrons or holes into Cooper pairs, which drastically reduces the friction caused by the flow of current $\text{[1]}$.

The possibility of Bose-Einstein condensation in nuclear and quark matter has also been considered in the framework of the so-called Bardeen-Cooper-Schrieffer to Bose-Einstein condensate (BCS-BEC) crossover $\text{[12]}$. At ultra-high density, matter is expected to form a degenerate Fermi gas of quarks in which Cooper pairs of quarks form a condensate near the Fermi surface (a color superconductor $\text{[13]}$). If the attractive interaction is strong enough, at some critical temperature the fermions may condense into the bosonic zero mode, forming a quark BEC $\text{[14]}$. In general, fermions exist in a BCS state when the attractive interaction between particles is weak. The system then exhibits superfluidity, characterized by the energy gap for single particle excitations which is created by the formation of the Cooper pairs. Conversely, a BEC exists when the attractive interaction between fermions is strong, causing them to first form bound “molecules” (i.e. bosons), before starting to condense into the bosonic zero mode at some critical temperature. An important point is that the BCS and BEC states are smoothly connected, without a phase transition between the two $\text{[15]}$.

Remarkably, the critical temperature in the BEC region is, in fact, independent of the precise strength of the coupling for the attractive interaction between fermions. This is because an increase in the coupling strength affects only the internal structure of the bosons, whereas the critical temperature is determined by their kinetic energy. Thus, the critical temperature reaches an upper limit for strong coupling, as long as the effect of the binding energy on the total mass of the boson is small, and can be neglected. In this limit, we are able to use a non-relativistic framework to describe the BCS-BEC crossover $\text{[12]}$.

However, in relativistic systems, the binding energy makes a significant contribution to the total mass of the boson and cannot be neglected. In this case, two crossovers are possible. Firstly, an ordinary BCS-BEC crossover may occur, though the critical temperature in the BEC region no longer tends towards an upper bound, due to relativistic effects. Secondly, the non-relativistic BEC state undergoes a transition to a relativistic BEC (RBEC) state, in which the critical temperature increases to the order of the Fermi energy $\text{[12]}$.

The possibility of Bose condensates existing in neutron stars has been considered (see Glendenning $\text{[16]}$ for a detailed discussion), as the condensation of negatively charged mesons in neutron star matter is favored, since these mesons would replace electrons with very high Fermi momenta. Recently, Bose-Einstein condensates of kaons/anti-kaons in compact objects were also discussed $\text{[17, 18]}$. Pion as well as kaon condensates would have two important effects on neutron stars. Firstly, condensates soften the equation of state above the critical density for the onset of condensation, which reduces the maximal possible neutron star mass. At the same time, however, the central stellar density increases, due to the softening. Secondly, meson condensates would lead to neutrino luminosities which are considerably enhanced over those of normal neutron star matter. This would speed up neutron star cooling considerably $\text{[16]}$. Another particle which may form a condensate is the H-dibaryon, a doubly strange six-quark composite with zero spin and isospin, and baryon number $B = 2$. In neutron star matter, which may contain a significant fraction of $\Lambda$ hyperons (i.e. neutral subatomic hadrons consisting of one up, one down and one strange quark, labelled $\Lambda^0$), these particles could combine to form H-dibaryons $\text{[19]}$. Thus, H-matter condensates may thus exist at the center of neutron stars $\text{[16]}$. Neutrino superfluidity, as suggested by Kapusta $\text{[20]}$, may also lead to Bose-Einstein condensation $\text{[21]}$.

These results show that the possibility of the existence of Bose-Einstein condensed matter inside compact astrophysical objects, or even the existence of stars formed entirely from a BEC, cannot be excluded a priori. The properties of BEC stars have been considered in $\text{[22]}$, and it was shown that these hypothetical astrophysical objects have mass and radii ranges that are compatibles with the observed physical parameters of some neutron stars. Bearing this in mind, it is the purpose of the current paper to consider another possible astrophysical BEC system, with potentially important cosmological implications, namely, the Bose-Einstein condensate string. By this, we mean a cylindrically-symmetric system consisting of bosonic matter in a Bose-Einstein condensed phase.

The structure of this paper is then as follows. In Section $\text{II}$ we briefly review the general treatment of gravitationally bound Bose-Einstein condensates in the non-relativistic limit, including their description by the generalized Gross-Pitkaevskii equation (Section $\text{IIA}$) and the hydrodynamical representation (Section $\text{IIB}$) in which a quantum potential term, which is significant close to the boundary of the condensate, arises. Using the non-relativistic analysis as a guide, we determine that the thermodynamic (“interaction”) pressure of BEC dark matter is governed by a polytropic equation of state which, together with the assumption of cylindrical-symmetry, allows us to fix the general form of the metric, and of the components of the energy-momentum tensor, which are expressed in terms of the energy density and pressure of the fluid. The latter is, in turn, decomposed into “interaction” and “quantum” pressure terms, denoted $p$ and $p_Q$, respectively, and the two limiting cases $p \gg p_Q$ and $p_Q \gg p$ are considered separately. Thus, the
string field geometry and the corresponding form of the Einstein field equations are determined in Section III and, by appropriately redefining the relevant dynamical variables, these are then recast (in each limiting case) as an autonomous system of differential equations, which may be solved numerically. Specifically, Section III A reviews the “semi-classical” approximation for the relativistic treatment of the quantum pressure term and Section III B deals with the string geometry and components of Einstein tensor, while the numerical solutions of the field equations for interaction dominated and quantum pressure dominated strings are given in Sects. III C and III D respectively. The meaning and significance of the variation, with respect to the radial coordinate $r$, of the physical parameters of the BEC string (i.e. the three-dimensional energy density and pressure and the mass per unit length) and of the components of metric tensor that determine the resulting spacetime, is discussed, in each limiting case, at the end of the relevant subsection. Finally, Section IV contains a brief treatment of BEC strings in the Newtonian approximation of the gravitational field, from which it is seen that constraints on the order of magnitude values of physically important quantities can be easily obtained. A summary of the results for both the thermodynamic and quantum pressure dominated regimes, together with some brief remarks regarding their possible cosmological and astrophysical significance, and suggestions for future work, are given in Section V.

II. BOSE-EINSTEIN CONDENSATION

In a quantum system of $N$ interacting condensed bosons most of the bosons lie in the same single-particle quantum state. For a system consisting of a large number of particles, the calculation of the ground state of the system with the direct use of the Hamiltonian is impracticable, due to the high computational cost. However, the use of some approximate methods can lead to a significant simplification of the formalism. One such approach is the mean field description of the condensate, which is based on the idea of separating out the condensate contribution to the bosonic field operator. We also assume that, in a medium composed of scalar particles with nonzero mass, when the transition to a Bose-Einstein condensed phase occurs, the range of Van der Waals-type scalar mediated interactions among particles becomes infinite.

A. The generalized Gross-Pitaevskii equation

The many-body Hamiltonian describing the interacting bosons confined by an external potential $V_{ext}$ is given, in the second quantization, by

$$\hat{H} = \int d^3\vec{r} \hat{\Phi}^+ (\vec{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{rot} (\vec{r}) + V_{ext} (\vec{r}) \right] \hat{\Phi} (\vec{r}) + \frac{1}{2} \int d^3d^3\vec{r} \hat{\Phi}^+ (\vec{r}) \hat{\Phi}^+ (\vec{r'}) V_{int} (\vec{r} - \vec{r'}) \hat{\Phi} (\vec{r}) \hat{\Phi} (\vec{r'}) , \quad (1)$$

where $\nabla$ is the three-dimensional Laplacian, $\hat{\Phi} (\vec{r})$ and $\hat{\Phi}^+ (\vec{r})$ are the boson field operators that annihilate and create a particle at the position $\vec{r}$, respectively, and $V_{int} (\vec{r} - \vec{r'})$ is the two-body interatomic potential. By treating the operator $\hat{\Phi}^+ (\vec{r})$ as a small perturbation, one can develop the first order theory for the excitations of the interacting Bose gases [2, 22].

In the general case of a non-uniform and time-dependent configuration the field operator in the Heisenberg representation is given by $\hat{\Phi} (\vec{r}, t) = \psi (\vec{r}, t) + \hat{\Phi}' (\vec{r}, t)$, where $\psi (\vec{r}, t)$, also called the condensate wave function, is the expectation value of the field operator, $\psi (\vec{r}, t) = \langle \hat{\Phi} (\vec{r}, t) \rangle$. It is a classical field, and its absolute value fixes the number density of the condensate particles according to $\rho_N (\vec{r}, t) = |\psi (\vec{r}, t)|^2$. The normalization condition is $N = \int d^3\vec{r} |\psi (\vec{r}, t)|^2$, where $N$ is the total number of particles in the condensate.

The equation of motion for the condensate wave function is the Heisenberg equation corresponding to the many-body Hamiltonian given by Eq. (1).

$$i\hbar \frac{\partial}{\partial t} \hat{\Phi} (\vec{r}, t) = \left[ \hat{\Phi}, \hat{H} \right] =$$

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{rot} (\vec{r}) + V_{ext} (\vec{r}) + \int d^3\vec{r} \hat{\Phi}^+ (\vec{r}', t) V_{int} (\vec{r} - \vec{r}') \hat{\Phi} (\vec{r}', t) \right] \hat{\Phi} (\vec{r}, t) . \quad (2)$$

The zeroth-order approximation to the Heisenberg equation is obtained by replacing $\hat{\Phi} (\vec{r}, t)$ with the condensate wave function $\psi (\vec{r}, t)$. In the integral containing the particle-particle interaction, $V_{int} (\vec{r} - \vec{r'})$, this replacement is, in general, a poor approximation for short distances. However, in a dilute and cold gas, only binary collisions at low energy are relevant, and these collisions are characterized by a single parameter, the s-wave scattering length $l_s$, independently of the details of the two-body potential. Therefore, one can replace $V_{int} (\vec{r} - \vec{r'})$
with an effective interaction $V_{\text{int}}(\vec{r}' - \vec{r}) = \lambda \delta(\vec{r}' - \vec{r})$, where the coupling constant $\lambda$ is related to the scattering length $l_s$ via $\lambda = 4\pi\hbar^2l_s/m$, where $m$ is the mass of an individual condensate particle. With the use of the effective potential the integral in the bracket of Eq. (2) gives $\lambda |\psi(\vec{r}', t)|^2$, and the resulting equation is the Schrödinger equation with a quartic nonlinear term $\frac{\hbar}{2m} i \partial(\vec{r}, t) = V_{\text{rot}}(\vec{r}) + V_{\text{ext}}(\vec{r}) + g\left(|\psi(\vec{r}, t)|^2\right)^2 \psi(\vec{r}, t)$, where we have denoted $g' = dg/d\rho_N$. As for $V_{\text{ext}}(\vec{r})$, we assume that it is the Newtonian gravitational potential, which we denote as $V_{\text{grav}}$, which satisfies the Poisson equation

$$\nabla^2 V_{\text{grav}} = 4\pi G\rho,$$

where $\rho = m\rho_N = m|\psi(\vec{r}, t)|^2$ is the mass density inside the condensate.

B. The hydrodynamical representation

The physical properties of a Bose-Einstein condensate described by the generalized Gross-Pitaevskii equation, Eq. (10), can be understood much more easily using the so-called Madelung representation of the wave function \[2\], which here consists of writing $\psi(\vec{r}, t)$ in the form

$$\psi(\vec{r}, t) = \sqrt{\frac{\rho(\vec{r}, t)}{m}} \exp\left[i\frac{\hbar}{m} S(\vec{r}, t)\right],$$

where the function $S(\vec{r}, t)$ has the dimensions of an action. Substituting the above expression for $\psi(\vec{r}, t)$ into Eq. (10), the latter decouples into a system of two differential equations for the real functions $\rho$ and $\vec{v}$, given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0,$$

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla P \left(\frac{\rho}{m}\right) - \rho \nabla \left(\frac{V_{\text{rot}}}{m}\right) - \nabla V_Q,$$

where we have introduced the quantum potential

$$V_Q = \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}},$$

and the velocity of the quantum fluid, defined as

$$\vec{v} = \frac{\nabla S}{m},$$

and have denoted

$$P \left(\frac{\rho}{m}\right) = g' \left(\frac{\rho}{m}\right) \frac{\rho}{m} - g \left(\frac{\rho}{m}\right).$$

From this definition it follows that the velocity field is irrotational, satisfying the condition $\nabla \times \vec{v} = 0$. Therefore the equations of motion for the gravitationally bound, ideal Bose-Einstein condensate (in the non-relativistic limit), take the form of the continuity equation plus the hydrodynamic Euler equation. Since the gravitational Bose-Einstein condensate can be described as a gas whose density and pressure are related by a barotropic equation of state \[2, 4, 6\], the explicit form of this equation therefore depends on the form of the nonlinear term $g(\rho)$.

When the number of particles becomes large enough, the quantum pressure term makes a significant contribution only near the boundary of the condensate. Hence it is much smaller than the nonlinear interaction term. Thus, the quantum stress term in the equation of motion of the condensate can be neglected. This is the Thomas-Fermi approximation, which has been extensively used for the study of the Bose-Einstein condensates \[2, 4, 6\]. As the number of particles in the condensate becomes infinite, the Thomas-Fermi approximation becomes exact. This approximation also corresponds to the classical limit of the theory, i.e. to neglecting all terms in nonzero powers of $\hbar$, or, equivalently, to the regime of strong repulsive interactions among particles. In the hydrodynamical representation, the Thomas-Fermi approximation corresponds to neglecting all terms containing $\nabla \rho$ (or $\nabla \sqrt{\rho}$) and $\nabla S$ in the equations of motion.

In the standard approach to the Bose-Einstein condensates, the nonlinear term $g$ is given by

$$g(\rho) = \frac{u_0}{2} |\psi|^4 = \frac{u_0}{2} \vec{v}^2,$$

where $u_0 = 4\pi\hbar^2 l_s/m$ \[2, 4, 6\] and the corresponding equation of state is

$$P(\rho) = U_0 \rho^2,$$

where

$$U_0 = \frac{2\pi\hbar^2 l_s}{m^3} = 1.232 \times 10^{50} \left(\frac{m}{1 \text{meV}}\right)^{-3} \left(\frac{l_s}{10^9 \text{fm}}\right) \text{cm}^5/\text{g s}^2,$$

or

$$U_0 = \frac{2\pi\hbar^2 l_s}{m^3} = 0.1856 \times 10^5 \left(\frac{l_s}{1 \text{fm}}\right) \left(\frac{m}{2m_n}\right)^{-3},$$

and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0,$$

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla P \left(\frac{\rho}{m}\right) - \rho \nabla \left(\frac{V_{\text{rot}}}{m}\right) - \nabla V_Q.$$
where $m_n = 1.6749 \times 10^{-24}$ g is the mass of the neutron.

Therefore, the equation of state of the Bose-Einstein condensate with quartic nonlinearity is a polytropic with index $n = 1$. However, in the case of low dimensional systems it has been shown that, in many experimentally interesting cases, the nonlinearity will be cubic, or even logarithmic in $\rho$.

For cylindrical distributions, such as those considered (relativistically) in the remainder of this study, the corresponding equation of motion in the non-relativistic limit is the “cylindrical Lane-Emden equation”, 

$$(1/\xi) \frac{d}{d\xi} \left( \xi (d\phi/d\xi) \tau^{n-1} \right) / d\xi + \tau = 0,$$

where $\tau = \rho/\rho_c$, for finite $n$, where $\rho_c$ is the central density at $r = 0$. For a barotropic fluid, the limit $n \to \infty$ must be taken before substituting the equation of state into the Poisson equation, leading to the separate (critical) case, for which $n \to 2$, $\alpha \to U_0$ and $\xi = [U_0/(4\pi G \rho_c)]^{1/2}$. This equation can also be generalized to include rotating fluids by adding a term of the form $\Omega^2 \xi$, where $\Omega = \Omega(\xi)$ is a function of the rescaled radial coordinate $\xi$ only, to the right-hand-side.

Hence, all Bose-Einstein condensate forms of matter can generally be described as fluids satisfying a polytropic equation of state of index $n$ and, importantly, this remains true relativistically (at least within the “semi-classical” approximation), as well as in the non-relativistic limit, which, in effect, has been thoroughly studied in Refs. [30, 32], at least in the limit of the Fermi-Thomas approximation. The remainder of this study is therefore dedicated to determining the astrophysical and cosmological significance of string-like (i.e. cylindrically-symmetric) BEC dark matter structures, by studying them in a general relativistic context. For simplicity, we consider only the case of the condensates with quartic nonlinearity since, in this case, the physical properties of the condensate are relatively well known from laboratory experiments and can be described in terms of only two free parameters, the mass $m$ of the condensate particle, and the scattering length $l_s$.

III. STATIC BOSE-EINSTEIN CONDENSATE STRINGS IN CYLINDRICALLY-SYMMETRIC GEOMETRIES

In the present Section we consider the properties of a string consisting of matter in a Bose-Einstein condensed state. In the hydrodynamical description of the condensate the equilibrium properties of this system are determined by two physical parameters, the interaction pressure $p$ and the quantum pressure $p_Q$, respectively. In the following we will investigate two classes of BEC strings, corresponding to the conditions $p >> p_Q$ (interaction energy dominated strings), and $p Q >> p$ (quantum pressure dominated strings), respectively.

A. General relativistic Bose-Einstein Condensates - the semiclassical approximation

In formulating a general relativistic model of gravitationally bound Bose-Einstein condensates we consider that bosonic matter at temperatures below the critical temperature $T_c$ represents a hybrid system, in which the gravitational field remains classical, while the bosonic condensate is described by quantum fields in which gravitational effects induced by the non-Euclidean spacetime geometry can be effectively neglected. In the standard approach used for coupling quantum fields to a classical gravitational field (i.e. semiclassical gravity), the energy momentum tensor that serves as the source in the Einstein equations is replaced by the expectation value of the energy momentum operator $\hat{T}_{\mu \nu}$, with respect to some quantum state $\Psi$.

$$R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu \nu} | \Psi \rangle,$$  \hspace{1cm} (15)

where $R_{\mu \nu}$ is the Ricci tensor, $R$ is the curvature scalar, and $g_{\mu \nu}$ is the metric tensor of the space-time. In the non-relativistic limit, the state function $\Psi$ evolves according to the Gross-Pitaevskii equation, so that, for a quartic nonlinear term, the evolution of the condensate wave function $\psi$ is determined by the equation:

$$i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + m V_{ext}(\vec{r}, t) + u_0 |\psi(\vec{r}, t)|^2 \right] \psi(\vec{r}, t),$$  \hspace{1cm} (16)

where $u_0 = 4\pi \hbar^2 l_s /m$, $l_s$ is the coherent scattering length (defined as the zero-energy limit of the scattering amplitude, $l_s = \text{lim}_{T \to 0} f_{\text{scat}}$, $m$ is the mass of the condensate particle, and $V_{ext}$ is the gravitational potential $V_{\text{grav}}$, given by the Newtonian limit of Eq. [14] for a cylindrically-symmetric system. Hence, in the semi-classical approach, we obtain

$$\langle \Psi | \hat{T}_{\mu \nu} | \Psi \rangle = \langle \psi | \hat{T}_{\mu \nu} | \psi \rangle = T_{\mu \nu},$$  \hspace{1cm} (17)

as the source term in the Einstein field equations, where $T_{\mu \nu}$ is the effective energy-momentum tensor of the condensate system obtained from the Gross-Pitaevskii equation. In a comoving frame $T_{\mu \nu}$ is diagonal with components

$$T_{\mu \nu} = (\rho c^2, -P, -P, -P),$$  \hspace{1cm} (18)

where $\rho c^2$ denotes the three-dimensional energy density and $P$ denotes the total effective thermodynamic pressure of the system, obtained from the hydrodynamic representation. That is, $P \approx p + p_Q$, where $p$ is the genuine classical (interaction) thermodynamic pressure and $p_Q$ is the quantum pressure term, as discussed previously.
Therefore, in the following, we assume that the effective thermodynamic properties (energy density and pressure) of the Bose-Einstein condensed matter are given by the relations derived from the quantum Gross-Pitaevskii equation in the Newtonian approximation of the gravitational field and in a standard Euclidian quantum geometry.

B. Geometry and gravitational field equations of Bose-Einstein condensate strings

For the geometrical description of Bose-Einstein condensate strings we adopt cylindrical polar coordinates \((x^0 = t, x^1 = r, x^2 = \phi, x^3 = z)\) and assume a cylindrically-symmetric metric, which gives rise to a line element of the form \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(r)dt^2 - dr^2 - L^2(r)d\phi^2 - K^2(r)dz^2\), \((19)\)

where \(N(r)\), \(L(r)\) and \(K(r)\) are arbitrary functions of the radial coordinate \(r\). The nonzero Christoffel symbols associated to the metric \([27]\) are given by

\[
\Gamma^r_{\phi\phi} = -L(r)L'(r), \quad \Gamma^z_{\phi\phi} = \frac{K'(r)}{K(r)}, \quad \Gamma^r_{zz} = -K(r)K'(r),
\]

\((20)\)

where a prime denotes the derivative with respect to \(r\), and the nonzero components of the Ricci tensor are \([27]\)

\[
R^r_t = \frac{(LN'K)'N}{NLK}, \quad R^r_r = \frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K},
\]

\[
R^z_z = \frac{(NLK')'NLK}{NLK}.
\]

\((21)\)

C. Interaction energy dominated Bose-Einstein condensate strings

In this section, we assume that the quantum pressure term inside the BEC string is negligible. This corresponds to the Thomas-Fermi approximation, which is valid when the number of particles is very large. However, in the context of a cylindrically-symmetric distribution, we must remember that the term “large” refers to the number of particles present in a thin, effectively two-dimensional slice of the string, i.e. a perpendicular cross-section, rather than the total number within the string as a whole. As such, it is unlikely that this assumption will hold with any degree of accuracy for very narrow strings and, in general, we would expect the ratio of the string surface area to its internal volume to play a significant role in its dynamics, especially for narrow strings in which the “boundary region” occupies a significant proportion of the overall volume. In other words, we would expect the internal quantum dynamics of the string to play a significant role in determining its macroscopic (essentially classical) dynamics, through the generation of an effective surface tension.

As an immediate corollary, we see that, were we to attempt to derive an effective action for a BEC string, it would not be possible to simply take the “wire approximation” (i.e. the zero thickness limit \([33, 34]\)), as used, for example, to derive the Nambu-Goto action \([35]\) as the effective action for Nielsen-Olesen strings \([36]\). Rather, we would need to develop a modified Nambu-type action (such as those corresponding to species of chiral, superconducting, or current-carrying string, e.g. in \([37, 39]\)) incorporating surface tension effects, possibly through the existence of an additional rigidity and/or elasticity terms \((c.f. [40-42])\). Though such an analysis remains beyond the scope of the current paper, it would form the next logical step in the study of BEC strings \([33]\).

In principle, for strings in which the thermodynamic pressure \(p\) and quantum pressure \(p_Q\) are both significant (which are not considered in the present study), both finite-thickness corrections \((c.f. [44])\) and surface tension effects may also need to be incorporated.

However, within the limit of the Thomas-Fermi approximation, the Bose-Einstein condensate can be described as a quantum gas satisfying a polytropic equation of state with index \(n = 1\). Therefore, the source term in the field equations is given by the energy-momentum tensor of a BEC, with the following components

\[
T^t_t = \rho(r)c^2, \quad T^\phi^\phi = T^z_z = -p(r),
\]

\((22)\)

We then have

\[
T = T^\mu_\mu = \rho(r)c^2 - 3p(r), \quad p(r) = U_0\rho^2(r),
\]

\((23)\)

and the field equations describing cylindrically-symmetric string type solutions in general relativity,

\[
R_{\mu\nu} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right),
\]

\((24)\)

can be written as

\[
\frac{(LN'K)'N}{NLK} = \frac{4\pi G}{c^4} \left( \rho c^2 + 3p \right),
\]

\((25)\)

\[
\frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K} = \frac{4\pi G}{c^4} \left( p - \rho c^2 \right),
\]

\((26)\)

\[
\frac{(NLK')'NLK}{NLK} = \frac{4\pi G}{c^4} \left( p - \rho c^2 \right),
\]

\((27)\)

\[
\frac{(NLK')'NLK}{NLK} = \frac{4\pi G}{c^4} \left( p - \rho c^2 \right).
\]

\((28)\)

Generally, the regularity of the geometry on the symmetry axis is imposed via the initial conditions,

\[
L(0) = 0, L'(0) = 1, N(0) = 1, N'(0) = 0.
\]

\((29)\)
Next, we consider the relations that follow from the conservation of the energy-momentum tensor, given by Eq. (22). Since all the components of the energy-momentum tensor are independent of the coordinates $t$, $\phi$ and $z$, the relations
\[ \nabla_{\mu} T^{\mu}_{\nu} = 0, \quad \nabla_{\mu} T^{\mu}_{\phi} = 0, \quad \nabla_{\mu} T^{\mu}_{z} = 0. \] (30)
hold automatically. Furthermore, the divergence of the $T^\mu$ can be obtained in a general form as
\[ \nabla_{\mu} T^{\mu}_{\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} T^{\mu}_{\nu}) - \frac{1}{2} \frac{\partial g_{\alpha \beta}}{\partial x^\nu} T_{\alpha \beta}. \] (31)

It follows that Eqs. (30) are identically satisfied, with the only non-trivial component of the divergence of the energy-momentum tensor being given by
\[ \nabla_{\mu} T^{\mu}_{\nu} = \frac{1}{\sqrt{-g}} \frac{d}{dr} (\sqrt{-g} T^{\nu}_r) - \frac{1}{2} \frac{\partial g_{\alpha \beta}}{\partial r} T_{\alpha \beta} = 0, \] (32)
where $\sqrt{-g} = NKL$ and $\alpha = \beta \in \{r, \phi, z\}$ for our choice of metric in Eq. (19), and energy-momentum tensor, Eq. (22).

Substituting from Eq. (22) into Eq. (32), the conservation equation for the BEC string takes the form
\[ \frac{dp}{dr} + (\rho c^2 + p) \frac{N'}{N} = 0. \] (33)

An upper bound for the density of the string arises from imposing the trace energy condition $T = \rho c^2 - 3p \geq 0$, which restricts the range of allowed densities to
\[ \rho \leq \frac{c^2}{3U_0}. \] (34)

To describe the physical characteristics of the Bose Einstein Condensate string we introduce the Tolman mass per unit length, $M$, defined as
\[ M = \frac{1}{c^2} \int \left( \rho c^2 - 3U_0 \rho^2 \right) \sqrt{-g} d^2r = 2\pi \int_{r=0}^{\infty} \left( \rho - 3U_0 \frac{\rho^2}{c^2} \right) NKL dr. \] (35)

We also introduce the parameter $W$, which can be related to the physical deficit angle in the spacetime of the string, and which is defined as
\[ W = -\frac{2\pi}{c^2} \int_{r=0}^{\infty} \left( \rho c^2 - p \right) \sqrt{-g} d^2r = -2\pi \int_{r=0}^{\infty} \rho \left( 1 - \frac{U_0}{c^2} \rho \right) NKL dr. \] (36)

From the field equations, Eqs. (25)–(28), together with the definitions for $M$ and $W$, it follows that
\[ [N'LK]_{r=0}^{\infty} = 2GM, \] (37)
\[ [NL'K]_{r=0}^{\infty} = 2GW, \] (38)
\[ [NLK']_{r=0}^{\infty} = 2GW, \] (39)
and
\[ \frac{L'}{L}_{r=0}^{\infty} = \frac{K'}{K}_{r=0}^{\infty}. \] (40)

We then have
\[ L'(\infty) = \frac{2G\rho c^2 + K(0)}{N(\infty)K(\infty)}, \] (41)
for $L'(0) = N(0) = 1$, and the angular deficit in the cylindrical geometry, due to the presence of the string, is given by
\[ \Delta \phi = 2\pi(1 - L'(\infty)), \] (42)
which can be numerically estimated by substituting values of $N(r)$ and $K(r)$, for large $r$. However, in order to do this, we would require a precise knowledge of the vacuum solution surrounding the string core. The general form of this solution is well known: it is the Kasner solution, which is the unique, cylindrically-symmetric solution in general relativity but, for the sake brevity, we treat only the interior solution for the BEC string in this paper. In order to estimate the angular deficit using the formula given in Eq. (42), we would therefore need to match the Kasner spacetime onto the interior solution at the string boundary, $r = R_s$ (defined as the point beyond which the energy and pressure density vanish, or become negative, so that $\rho c^2(R_s) = p(R_s) = 0$), thereby fixing the numerical values of the Kasner parameters, before taking the limit $r \to \infty$ to determine $L'(\infty)$. As a first approximation, however, we may substitute $L'(R_s)$ into Eq. (42), in place of $L'(\infty)$, and use the formula
\[ \Delta \phi \approx 2\pi(1 - L'(R_s)), \] (43)
from which we can obtain an order of magnitude estimate of the deficit angle in the exterior BEC string geometry, using the numerical solutions of the field equations for the string interior.

For the sake of notational simplicity, we now introduce the variables
\[ \sqrt{-g} = \Sigma = NKL, \quad H_t = \frac{N'}{N}, \quad H_\phi = \frac{L'}{L}, \quad H_z = \frac{K'}{K}, \] (44)
and
\[ H = \frac{1}{3} (H_t + H_\phi + H_z) = \frac{1}{3} \Sigma'. \] (45)
The field equations describing an interaction energy dominated BEC string then take the form
\[ \frac{1}{\Sigma} \frac{d}{dr} (\Sigma H_t) = \frac{4\pi G}{c^4} \left( \rho c^2 + 3U_0 \rho^2 \right), \] (46)
\[ 3\frac{dH}{dr} + H_t^2 + H_\phi^2 + H_z^2 = \frac{4\pi G}{c^4} \left( U_0 \rho^2 - \rho c^2 \right), \] (47)
\[
\frac{1}{\Sigma} \frac{d}{dr} (\Sigma H_i) = \frac{4\pi G}{c^4} \left( U_0 \rho^2 - \rho c^2 \right), \quad i = \phi, z.
\] (48)

From Eqs. (48) we immediately obtain
\[
H_\phi = H_z + \frac{C}{\Sigma},
\] (49)

where \( C \) is an arbitrary constant of integration. By adding Eqs. (46) and (48) we have
\[
\frac{3}{\Sigma} \frac{d}{dr} (\Sigma H) = \frac{\Sigma''}{\Sigma} + \frac{d}{dr} \left( \frac{\Sigma'}{\Sigma} \right)^2 = \frac{4\pi G}{c^4} \left( 5U_0 \rho^2 - \rho c^2 \right).
\] (50)

and from the conservation equation Eq. (33) it follows that
\[
H_t = -2 \frac{(U_0/c^2) \rho'}{1 + (U_0/c^2) \rho},
\] (51)

giving
\[
N(r) = \frac{N_0}{\left[ 1 + (U_0/c^2) \rho(r) \right]^2},
\] (52)

where \( N_0 \) is an arbitrary constant of integration. Combining Eqs. (46) and (51) gives
\[
\frac{\Sigma'}{\Sigma} H_t + H'_t = \frac{4\pi G}{c^4} (\rho c^2 + 3U_0 \rho^2),
\] (53)

and, with the help of Eq. (51), Eq. (53) allows to express \( \Sigma'/\Sigma \) as
\[
\Sigma' = \frac{\rho''}{\rho'} + \frac{\left( U_0/c^2 \right) \rho'}{1 + (U_0/c^2) \rho} - \frac{4\pi G \rho \left[ 1 + (U_0/c^2) \rho \right] \left[ 1 + (3U_0/c^2) \rho \right]}{2 \left( U_0/c^2 \right) \rho'}. \] (54)

At this point we rescale the energy density, by introducing a new dimensionless variable \( \theta \), so that
\[
\theta(r) = \frac{U_0}{c^2} \rho(r),
\] (55)

and, from here on, denote
\[
\lambda = \frac{4\pi G}{U_0}. \] (56)

In the new variable \( \theta \) the pressure distribution inside the string is obtained as
\[
p(r) = \frac{\rho'_2}{U_0} \theta^2(r),
\] (57)

while the Tolman mass (hereafter, we will often use the phrase “Tolman mass” to refer to the Tolman mass per unit length), is given by
\[
M = \frac{2\pi c^2}{U_0} \int_{r_0}^{\infty} \theta(1-\theta) \Sigma dr.
\] (58)

The parameter \( W \) is given by
\[
W = -\frac{2\pi c^2}{U_0} \int_{r_0}^{\infty} \theta(1-\theta) \Sigma dr.
\] (59)

Then Eq. (54) can be written as
\[
\frac{\Sigma'}{\Sigma} = \frac{\theta''}{1 + \theta} - \lambda \frac{\theta(1+\theta)(1+3\theta)}{2\theta'}.
\] (60)

Thus, by combining Eq. (50), written as
\[
\frac{d}{dr} \left( \frac{\Sigma'}{\Sigma} \right) + \left( \frac{\Sigma'}{\Sigma} \right)^2 = \lambda \theta(5\theta - 1),
\] (61)

with Eq. (60), we obtain the following third order ordinary nonlinear differential equation for the energy density distribution inside the BEC string,
\[
\frac{\theta'''}{\theta'} - \frac{2\theta''^2}{\theta'^2} = \left[ 3\lambda \theta(1+\theta)(3\theta + 1) \right] - \frac{1}{\theta + 1} \theta'' - \frac{\lambda^2 \theta^2 (1+\theta)^2 (3\theta + 1)^2}{4\theta'^2} + \frac{1}{2} \lambda (25\theta^2 + 8\theta + 1) = 0.
\] (62)

However, instead of studying Eq. (62) numerically, it is more advantageous to study the following equivalent autonomous system of differential equations,
\[
\Sigma' = u,
\] (63)
\[
u' = \lambda \theta(5\theta - 1) \Sigma,
\] (64)
\[
\theta' = v,
\] (65)
\[
\nu' = -\frac{u}{\Sigma} v + \frac{v^2}{1 + \theta} - \frac{\theta(1+\theta)(1+3\theta)}{2},
\] (66)
\[
K' = k,
\] (67)
\[
k' = \frac{k^2}{K} - \frac{u}{\Sigma} k + \lambda \theta(\theta - 1) K.
\] (68)

where the expression for \( k' \) follows directly from Eq. (18), and a prime represents differentiation with respect to \( r \).

The system of Eqs. (63–68) must be considered with the initial conditions \( \Sigma(0) = \Sigma_0, u(0) = \Sigma'(0) = 3H(0)\Sigma(0), \theta(0) = (U_0/c^2)\rho_0, K(0) = K_0, v(0) = \theta'(0) = -[(1 + \theta_0)/2] (N'/N)|_{r=0} \) and \( k(0) = K'(0) = k'_0 \), respectively. Once the solution of the system given in Eqs. (63–68) is obtained, the variation of metric tensor component \( L \) can be obtained from the equation \( L = \Sigma/NK \). The initial condition \( N(0) = 1 \), imposed on the metric function \( N \), determines the integration constant \( N_0 \) as \( N_0 = \left[ 1 + (U_0/c^2) \rho_0 \right]^2 \).

The variation of the functions \( \Sigma(r) = \sqrt{-g}, \theta(r) \propto \rho(r), \theta'(r) \propto \rho(r), N(r) = \sqrt{g_{tt}}, L(r) = \sqrt{-g_{xx}} \) and \( K(r) = \sqrt{-g_{zz}} \), obtained from the numerical solution of Eqs. (63–68) together with appropriate (example) numerical values for the initial conditions, are represented in Figs. 1–4 respectively. The variation of the Tolman mass is given in Fig. 4 while the variation of the parameter \( W \) is presented in Fig. 5. In all Figs. 1–5 the initial conditions used to numerically integrate the system,
Eqs. \(63-68\), are: \(\Sigma(0) = 10^{-8}\), \(\rho(0) = 10^{14} \text{g/cm}^2\), \(H(0) = 0.75 \text{cm}^{-1}\), \(N'/N|_{r=0} = 10^{-5} \text{cm}^{-1}\), \(K(0) = 0.1\), \(K'(0) = 0.001\), \(M(0) = 0\) and \(W(0) = 0\), respectively.

As one can see from Fig. 1, the function \(\Sigma(r) = \sqrt{-g}\) appears to be (approximately) proportional to \(r\), but with constant of proportionality much less than one (compared to \(\Sigma(r) = \sqrt{-g} = r\) for a flat conical geometry, such as that obtained for a vacuum string). The dimensionless density \(\theta(r) = U_0 \rho(r)/c^2\), presented in Fig. 2, appears, roughly, to be a Bell-shaped curve. It monotonically decreases from a maximum central value at \(r = 0\) (as expected intuitively), and reaches the value \(\theta = 0\) at some finite value of \(r\), which defines the radius of the string \(R_s\), \(\theta(R_s) = 0\). In the examples considered, the radius of the string is of the order of \(R_s \approx 3 \times 10^6 \text{cm} \approx 30 \text{km}\). The physical pressure, \(p(r) \propto \theta^2(r)\), of the Bose Einstein condensate that forms the string interior, which is shown in Fig. 3, also becomes zero for \(r = R_s\), that is, at the vacuum boundary of the string. The behavior of the functions that determine the metric tensor components, depicted in Figs. 4-6, show very different variation with respect to the radial coordinate: \(N(r)/N_0 \propto \sqrt{g_{tt}}\) is a monotonically increasing function of \(r\), roughly proportional to \(r\) for large values of \(r\). \(L(r) = \sqrt{-g_{zz}}\) is also roughly proportional to \(r\), but with a constant of proportionality much less than one. For large \(r\), \(L(r)\) also varies according to some power of \(r\), \(r^\delta\), where \(0 < \delta < 1\), but the effect appears more pronounced than for \(\Sigma(r)\) (i.e. \(\delta < \gamma\)), which exhibits a similar dependence as \(r \to \infty\). This is in sharp contrast with the flat conical geometry of a vacuum string, for which \(L(r) = \sqrt{g_{zz}} = r\). The metric function \(K(r) = \sqrt{-g_{zz}}\) rises sharply from it’s
initial value, and peaks rapidly, before decreasing monotonically for large $r$. This behavior is again very different with respect to $K(r) = \sqrt{-g_{zz}} = 1$, corresponding to a flat conical Minkowski space. The Tolman mass function, presented in Fig. 6, is monotonically increasing, and reaches its maximum value at $r = R_s$, giving a total mass per unit length for the BEC string of the order $M \approx 4 - 6 \times 10^3 \times (2\pi c^2) / U_0 \approx 2 - 3 \times 10^{19} \mathrm{g}$. This is extremely small (for reasonable values of $U_0$), as compared to the masses of possible BEC matter neutron stars, for example, which are of the order of $M \approx 2.8 \times 10^{33} \mathrm{g}$. The angular deficit parameter $W$, plotted in Fig. 8, has negative values inside the string, and it monotonically decreases while approaching the string boundary.

This latter observation is perhaps the most intriguing aspect of the BEC string as it allows, in principle, for an “angle excess” in the resulting spacetime, according to Eqs. (41)-(43). The term “angle excess” is used to describe conical geometries in which the deficit angle exceeds $2\pi$. Though these may be considered unphysical, Visser has suggested that, instead, such configurations correspond to negative mass strings (composed of exotic matter), which may be capable of supporting a traversable wormhole [43]. An example of such an “exotic” string, originally proposed in [40], is one with vanishing radial and azimuthal stresses ($T_r = T_\theta = 0, \forall r$), for which the mass per unit length $\mu$ is equal to the longitudinal tension $T_z$, both of which are negative, i.e. $\mu = T_z < 0$, and further work has since been carried out in relation to this idea [47]. The exoticness of such an object may be judged against the corresponding condition for Nambu-Goto or vacuum string, $\mu = -T_z > 0$. Since the BEC string, considered here, clearly has positive mass and positive pressure (i.e. negative tension) in the longitudinal direction, this raises the intriguing question as to whether “exotic” negative mass objects are really
required to support traversable wormholes, or whether other cylindrically-symmetric mass distributions, obeying reasonable sets of energy conditions, may be able to behave “exotically” in this way, given the right initial conditions. If so, BEC strings may have extremely unusual and interesting cosmological and astrophysical consequences which, though beyond the scope of the present investigation, are certainly worthy of further study.

D. Quantum pressure dominated Bose-Einstein Condensate strings

By taking into account the mathematical identity
\[
\tau \nabla \left( \frac{\tau^2 \nabla \tau}{\sqrt{\tau}} \right) = \frac{1}{2} \nabla \left( \tau \nabla^2 \ln \tau \right),
\]
which holds for any dimensionless scalar function \( \tau \) (i.e. including \( \tau = \rho/\rho_c \)), it follows that the quantum potential \( V_Q \) generates a quantum pressure \( \rho \), given in a general form as
\[
p_Q = \frac{\rho \nabla V_Q}{\hbar^2} = -\frac{\hbar^2}{4m^2} \rho \nabla^2 \ln(\rho/\rho_c).
\]
This pressure can play a significant effect for small particle masses and high densities. In this section we assume that the BEC string is supported by its quantum pressure \( \rho_Q \), which satisfies the condition \( \rho_Q \gg \rho = U_0 / \rho^2 \). By adopting the Newtonian approximation for the quantum regime, and assuming that the gravitational field does not affect the fundamental quantum properties of the system, as formulated in the standard Hilbert space approach, the three-dimensional Laplacian operator, \( \nabla^2 \), for cylindrically-symmetric systems, is given by
\[
\Delta = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right).
\]

Therefore, the gravitational field equations describing the quantum pressure supported BEC string take the form
\[
\frac{1}{\Sigma} \frac{d}{dr} (\Sigma H_i) = \frac{4\pi G}{c^2} \rho \left[ 1 - 3 \frac{\hbar^2}{4m^2c^2} \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \ln \frac{\rho}{\rho_c} \right) \right],
\]
\[
3 \frac{dH_i}{dr} + H_i^2 + H_i^2 + H_i^2 =
-4\pi G \rho \left[ 1 + \frac{\hbar^2}{4m^2c^2} \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \ln \frac{\rho}{\rho_c} \right) \right],
\]
\[
\frac{1}{\Sigma} \frac{d}{dr} (\Sigma H_i) = -4\pi G \rho \left[ 1 + \frac{\hbar^2}{4m^2c^2} \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \ln \frac{\rho}{\rho_c} \right) \right],
\]
i = \phi, z.

The corresponding Tolman mass is given by
\[
M = 2\pi \int_0^\infty \rho \left[ 1 + 3 \frac{\hbar}{4m^2c^2} \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \ln \frac{\rho}{\rho_c} \right) \right] \Sigma dr.
\]
As in the case of the interaction energy dominated BEC string, the relation between \( H_\phi \) and \( H_z \) is again given by Eq. (75). By introducing the dimensionless variables \( \xi \) and \( \tau \), defined as
\[
r = \frac{h}{2mc}, \quad \tau = \frac{\rho}{\rho_c},
\]
and by denoting
\[
\lambda_Q = \frac{\pi G \hbar^2}{m^2 c^2 \rho_c},
\]
the field equations take the following dimensionless form,
\[
\frac{1}{\Sigma} \frac{d}{d\xi} (\Sigma H_i) = \lambda_Q \tau \left[ 1 - 3 \frac{1}{\xi} \left( \frac{\xi}{d\xi} \ln \tau \right) \right],
\]
\[
3 \frac{dH_i}{d\xi} + H_i^2 + H_i^2 + H_i^2 = -\lambda_Q \tau \left[ 1 + \frac{1}{\xi} \left( \frac{\xi}{d\xi} \ln \tau \right) \right],
\]
\[
\frac{1}{\Sigma} \frac{d}{d\xi} (\Sigma H_i) = -\lambda_Q \tau \left[ 1 + \frac{1}{\xi} \left( \frac{\xi}{d\xi} \ln \tau \right) \right],
\]
i = \phi, z,

where the \( H_\nu, \nu \in \{ t, \phi, z \} \) and \( H \) are now defined in terms of the derivatives with respect to \( \xi \), i.e. \( H_\nu = (\hbar/2mc)H_\nu \), so that \( H = (1/N)(dN/d\xi) \) etc.

The energy conservation equation for the quantum pressure becomes
\[
\frac{d}{d\xi} \left[ \tau \frac{d}{d\xi} \left( \xi \frac{d}{d\xi} \ln \tau \right) \right] = \tau \left[ 1 - \frac{1}{\xi} \left( \xi \frac{d}{d\xi} \ln \tau \right) \right] H_i,
\]
and its dimensionless equivalent, \( P_Q = \rho_Q/\rho_c \), may be written as
\[
P_Q = -\frac{\tau}{\xi} \frac{d}{d\xi} \left( \frac{\xi}{d\xi} \ln \tau \right).
\]
By adding Eqs. (79) and (80) we obtain
\[
\frac{1}{\Sigma} \frac{d^2\Sigma}{d\xi^2} = -\lambda_Q \tau \left[ 1 + 5 \frac{1}{\xi} \left( \xi \frac{d}{d\xi} \ln \tau \right) \right],
\]
while Eq. (78) and Eq. (80) for \( H_z \) can be written as
\[
\frac{dH_z}{d\xi} + \frac{1}{\Sigma} \frac{d\Sigma}{d\xi} H_i = \lambda_Q \tau \left[ 1 - 3 \frac{1}{\xi} \left( \xi \frac{d}{d\xi} \ln \tau \right) \right],
\]
and
\[
\frac{dH_z}{d\xi} + \frac{1}{\Sigma} \frac{d\Sigma}{d\xi} H_z = -\lambda_Q \tau \left[ 1 + \frac{1}{\xi} \left( \xi \frac{d}{d\xi} \ln \tau \right) \right].
\]
respectively. Equivalently, the system of field equations obtained above, which may be viewed as a relativistic
generalization of the cylindrical Lane-Emden equation discussed in Sect. [113] can then be formulated as a sys-
tem of first order differential equations, i.e.
\[
\frac{d\tau}{d\zeta} = a, \quad \frac{d\Sigma}{d\zeta} = \sigma, \quad \frac{dN}{d\zeta} = n, \quad \frac{dK}{d\zeta} = k, \tag{86}
\]
\[
\frac{da}{d\zeta} = \tau u + \frac{a^2}{\tau} - \frac{a}{\zeta}, \tag{87}
\]
\[
\frac{du}{d\zeta} = (1 - u)\frac{n}{N} - \frac{au}{\tau}, \tag{88}
\]
\[
\frac{dv}{d\zeta} = -\lambda_Q\tau(1 + 5u)\Sigma, \tag{89}
\]
\[
\frac{dn}{d\zeta} = \lambda_Q\tau(1 - 3u)N + \frac{n^2}{N} - \frac{\sigma}{\Sigma}n, \tag{90}
\]
\[
\frac{dk}{d\zeta} = -\lambda_Q\theta(1 + u)K + \frac{k^2}{K} - \frac{\sigma}{\Sigma}k, \tag{91}
\]
which must be integrated subject to the initial conditions
\[
\tau(0) = 1, \quad a(0) = \tau'(0) = \tau_0', \quad u(0) = u_0', \quad N(0) = 1, \quad n(0) = N'(0) = 0, \quad K(0) = \theta(0), \quad K'(0) = \theta'(0), \quad \Sigma(0) = \Sigma_0
\]
and \(\sigma(0) = \Sigma'(0) = \sigma_0',\) where a prime now indicates differentiation with respect to \(\zeta.\) The dimensionless form
of the Tolman mass, \(m = M/(\pi h^2 \rho_c/2m^2 c^2)\), is given by
\[
m = 2\pi \int_{\zeta=0}^{\infty} \left[ 1 + 3 \frac{d}{d\zeta} \left( \zeta \frac{d}{d\zeta} \ln \tau \right) \right] \Sigma d\zeta, \tag{92}
\]
while the dimensionless angular deficit \(w = W/(\pi h^2 \rho_c/2m^2 c^2)\) is obtained as
\[
w = -2\pi \int_{\zeta=0}^{\infty} \tau \left[ 1 + \frac{d}{d\zeta} \left( \zeta \frac{d}{d\zeta} \ln \tau \right) \right] \Sigma d\zeta. \tag{93}
\]

The variation, with respect to \(\zeta,\) of \(\Sigma = \sqrt{-g},\) of the dimensionless energy density of the BEC matter \(\tau,\) the
dimensionless quantum pressure \(pq,\) of the functions \(N, \) \(L\) and \(K\) which determine the components of the metric
tensor, of the dimensionless Tolman mass \(m,\) and of the angular deficit parameter \(w,\) are represented, for differ-
ent values of the parameter \(\lambda_Q,\) in Figs. [9][16]. In each
case, the initial conditions used to numerically integrate
the system of differential equations, Eqs. [86]–[91], were:
\[
\Sigma(0) = 0.01, \quad \sigma(0) = 0.10, \quad \tau(0) = 1, \quad a(0) = 0, \quad N(0) = 1, \quad n(0) = -10^{-8}, \quad u(0) = -1, \quad K(0) = 0.01, \quad k(0) = -0.01, \quad m(0) = 0 \quad \text{and} \quad u(0) = 0.
\]
The function \(\Sigma(\zeta) = \sqrt{-g},\) presented in Fig[9] appears to be approximately proportional to \(\zeta,\) for small \(\zeta,\) but
varies according to a higher power of \(\zeta\) as \(\zeta\) increases. This is in contrast to the interaction pressure dominated case, as well to the \(\Sigma = \sqrt{-g} \propto \zeta\) case for a flat conical
geometry. The dimensionless density \(\tau(\zeta) = \rho(\zeta)/\rho_c(\zeta)\) of the BEC cosmic string, plotted in Fig. [10] monotonically
decreases from a maximum central value at \(\zeta = 0\) and reaches zero for \(\zeta = \zeta_s,\) which defines the vacuum
boundary of the string.

For the parameters used in the numerical simulations it follows that \(R_s \approx 3.56h/2mc = 6.148 \times 10^{-38}/m.\) If the mass of the particle in the condensate is very small, \(m \approx 10^{-44}\) g, then \(R_s \approx 6.16 \times 10^6\) cm. On the other hand, more massive particles can form condensate strings with very small radii. At the vacuum
boundary of the string, \(\zeta = \zeta_s,\) the quantum pressure, depicted in Fig. [11] tends to zero, together with the en-
a sharp immediate drop close to \( \zeta = 0 \), followed by a more gradual decline. By contrast, \( K(r) \) in the interaction pressure dominated case rises sharply close to \( r = 0 \), before peaking suddenly and declining thereafter. Again, the Tolman mass function, plotted in Fig. 15, is monotonically increasing and tends to a constant value \( M \approx 0.8 - 1.4 \times (\pi \hbar^2/2m^2 c^2) \rho_c \approx 2.71 \times 10^{-75} (\rho_c/m^3) \), giving the total mass of the quantum BEC string. For an ultralight particle with \( m = 10^{-44} \) we obtain \( M \approx 2.71 \times 10^{12} \rho_c \). Possible particle species with masses in this range include axions in Quantum Chromodynamics and pseudo Nambu-Goldstone bosons associated with the spontaneous breaking of Peccei-Quinn symmetry, both of which represent important dark matter candidates [48]. By using the most recent cosmological data, including the Planck temperature data, the WMAP E-polarization measurements, the recent BICEP2 observations of B-modes, as well as Baryon Acoustic Oscillation data, in-

energy density of the BEC particles. However, it’s relationship with the energy density is now more complex than in the interaction pressure dominated case. The function \( N(\zeta)/N_0 \propto \sqrt{g_{\mu\nu}} \), presented in Fig. 12, is monotonically increasing with \( \zeta \) and becomes approximately proportional to \( \zeta \) at large radial distances. It’s behavior is qualitatively similar to that found in the spacetime of the interaction pressure dominated string. Likewise, \( L(\zeta) = \sqrt{-g_{\phi\phi}} \), shown in Fig. 13, also behaves in much the same way as in the interaction pressure dominated limit. The major difference in the behavior of the metric components, between the interaction and quantum pressure dominated regimes, comes from \( K = \sqrt{-g_{zz}} \), represented in Fig. 14. In the latter, \( K(\zeta) \) decreases monotonically as a function of the radial distance, with a sharp immediate drop close to \( \zeta = 0 \), followed by

FIG. 11: Variation of the dimensionless quantum pressure \( P_Q \), as a function of \( \zeta \), of the quantum pressure dominated Bose-Einstein condensate string, for different values of \( \lambda_Q \): \( \lambda_Q = 0.10 \) (solid curve), \( \lambda_Q = 0.14 \) (dotted curve), \( \lambda_Q = 0.18 \) (short dashed curve), \( \lambda_Q = 0.20 \) (dashed curve), and \( \lambda_Q = 0.22 \) (long dashed curve), respectively.

FIG. 12: Variation of the \( N(\zeta) = \sqrt{g_{tt}} \) for the spacetime of the quantum pressure dominated Bose-Einstein condensate string, for different values of \( \lambda_Q \): \( \lambda_Q = 0.10 \) (solid curve), \( \lambda_Q = 0.14 \) (dotted curve), \( \lambda_Q = 0.18 \) (short dashed curve), \( \lambda_Q = 0.20 \) (dashed curve), and \( \lambda_Q = 0.22 \) (long dashed curve), respectively.

FIG. 13: Variation of the \( L(\zeta) = \sqrt{-g_{\phi\phi}} \) for the spacetime of the quantum pressure dominated Bose-Einstein condensate string, for different values of \( \lambda_Q \): \( \lambda_Q = 0.10 \) (solid curve), \( \lambda_Q = 0.14 \) (dotted curve), \( \lambda_Q = 0.18 \) (short dashed curve), \( \lambda_Q = 0.20 \) (dashed curve), and \( \lambda_Q = 0.22 \) (long dashed curve), respectively.

FIG. 14: Variation of the \( K(\zeta) = \sqrt{-g_{zz}} \) for the spacetime of the quantum pressure dominated Bose-Einstein condensate string, for different values of \( \lambda_Q \): \( \lambda_Q = 0.10 \) (solid curve), \( \lambda_Q = 0.14 \) (dotted curve), \( \lambda_Q = 0.18 \) (short dashed curve), \( \lambda_Q = 0.20 \) (dashed curve), and \( \lambda_Q = 0.22 \) (long dashed curve), respectively.
including those from the Baryon Oscillation Spectroscopic Survey, in \[49\] it was found that the mass of the dark matter axions is in the range of 70 – 80 µeV. Lighter dark matter particles are also possible.

Thus, if the central density is enough high, quantum pressure dominated BEC strings may have masses of the order of \( M \approx 10^{-7} M_\odot \). The angular deficit \( \omega \), shown in Fig. 16, is again negative inside the string, and it is a monotonically decreasing function of the radial distance. This suggests that the possibility of BEC strings giving rise to spacetimes with “angle excess” cannot be excluded in either the \( p \gg P_Q \) or \( p \ll P_Q \) regimes and, hence, that it may, in principle, be a generic feature of such strings, given appropriate initial conditions. However, further investigation is required in order to determine just how generic these conditions may be, and whether or not they are comparable in each regime.

IV. THE NEWTONIAN APPROXIMATION

Due to the invariance of the action of the BEC system, Eq. (1), under infinitesimal time translations \( t \to t + \delta \) (with \( \delta \bar{t} = \delta \psi = \delta \bar{\psi}^* = 0 \)), the non-relativistic Hamiltonian, and hence the total energy of the condensed particles in the semi-classical approach to quantum BEC strings in general relativity, is conserved. However, in the Newtonian approximation, the total energy \( E \) of a gravitationally bound BEC can, in general, be written in the especially simple way, \( E = E_{\text{kin}} + E_{\text{int}} + E_{\text{grav}} \), where \( E_{\text{kin}}, E_{\text{int}} \) and \( E_{\text{grav}} \) denote the kinetic, interaction, and gravitational energy, respectively. In this, final section, we consider an approximate estimate of the energy only, based on the method used in [3], which assumes the Newtonian limit for the gravitational field. For a cylindrical system, the kinetic energy per particle is of order \( (\hbar^2/2mR^2 + \hbar^2/2m\Delta^2) \), where \( \bar{R} \) gives the radial extension and \( \Delta \) is the length of the cylinder. Therefore, the approximate expression for the total kinetic energy of the system is given by

\[
E_{\text{kin}} \approx \frac{N \hbar^2 (R^2 + \Delta^2)}{2mR^2 \Delta^2},
\]

where \( N \) is the total number of particles. The interaction energy is given by \( E_{\text{int}} \approx (1/2)(\mathcal{V}^2/\mathcal{V})u_0 \), where \( \mathcal{V} \) again denotes the volume of the condensate, while the gravitational potential energy is \( E_{\text{grav}} \approx -GM^2/R \), where \( \mathcal{M} \) is the total mass and, for simplicity, we have neglected the numerical factor in the expression for the gravitational potential energy corresponding to a cylindrically-symmetric mass distribution. Therefore, in the Newtonian approximation, and neglecting numerical factors of order unity, the total energy of the condensate is given by

\[
E \approx \frac{N \hbar^2 (R^2 + \Delta^2)}{mR^2 \Delta^2} + N^2 \frac{\hbar^2 l_s}{mR^2 \Delta} - \frac{GM^2}{R},
\]

where we have used the expressions \( u_0 \sim \lambda \sim \hbar^2 l_s/m \) and \( \mathcal{V} \sim R^2 \Delta \). From Eq. (95) we can obtain a rough estimate of the physical parameters of the BEC string in the interaction energy and quantum potential dominated regimes. The quantum kinetic energy is much bigger than the interaction energy when the parameters of the system satisfy the condition

\[
\Delta \gg Nl_s \left[ 1 + O \left( \frac{R^2}{\Delta} \right)^2 \right],
\]

for \( \Delta \gg R \), or

\[
R \gg Nl_s \left[ 1 + O \left( \frac{\Delta^2}{R^2} \right)^2 \right],
\]

for \( R \gg \Delta \).
for $\Delta \ll R$. Both expressions, Eqs. (96)-(97), hold (approximately), in the limiting case, $\Delta \sim R$, for which the the result for a spherically-symmetric BEC matter distribution is recovered [3]. Assuming the former case $\Delta \gg R$, which is far more probable for astrophysical BEC strings, the approximate total energy is given by

$$E \approx N \frac{\hbar^2}{m^2 R^2} - \frac{G N^2 m^2}{R} \leq 0,$$

where we have used $M \sim mN$. This, in turn, gives a lower bound for the radius of a quantum pressure dominated BEC string, i.e.

$$R_{\text{quant}} \gtrsim \frac{\hbar^2}{G m^3 N},$$

(99)

Equivalently, this may be rewritten as a bound on the (dimensionless) measure of the mass per unit length, which we now label $G \mu_{\text{quant}}$, using $M = \mu_{\text{quant}} \Delta$;

$$G \mu_{\text{quant}} \gtrsim \frac{\hbar^2}{m^2 R \Delta},$$

(100)

which illustrates the problem, already mentioned in Sect. 11C, with taking the wire approximation for BEC strings. Due to the presence of the quantum pressure term, which becomes significant for narrow strings since only a small number of particles inhabit a thin cross-section, the mass per unit length depends sensitively on the product $R \Delta$. Formally, we may consider an infinitely long string of zero width by taking the limits $\Delta \to \infty$, $R \to 0$, such that $R \Delta$ remains finite. Realistically, however, we would like to be able to treat open strings and loops as well as, on cosmological scales, strings whose width is limited simply by causality and the finiteness of the horizon. In such cases, the energy density of the string diverges in the wire approximation because the surface tension becomes focussed on an “area” that shrinks to zero. As stated previously, therefore, we would expect both finite width corrections [44], and effective rigidity terms [40,42] to be important in constructing an effective action for BEC strings. It is interesting to note that, for the quantum pressure dominated BEC, neither the minimum string radius, nor the minimum mass per unit length depend on the scattering length.

By contrast, the interaction energy dominates the internal dynamics of BEC strings when

$$\Delta \ll N l_s \left[ 1 + O \left( \frac{R^2}{\Delta} \right)^2 \right],$$

(101)

assuming $\Delta \gg R$. This condition is obviously satisfied by cylindrical BEC systems with very large numbers of particle number in relation to their length or, in other words, by thicker strings (for fixed $l_s$), though we cannot say a priori whether a string of radius $R$ and length $\Delta$ will be dominated by quantum or interaction pressure. The condition for stability in the latter case is

$$E \approx N \frac{\hbar^2 l_s}{m R^3} - \frac{G m^2 N^2}{R} \leq 0,$$

(102)

which gives the following constraint for the radius, $R_{\text{int}}$, of an interaction pressure dominated string,

$$R_{\text{int}} \gtrsim \sqrt{\frac{\hbar^2 l_s}{G m^3}},$$

(103)

Thus, in this regime, we have that

$$\sqrt{\frac{\hbar^2 l_s}{G m^3}} \lesssim R \ll \Delta \ll N l_s,$$

(104)

from which it follows that interaction pressure dominated strings, with characteristic minimum radius, form when BEC systems with a sufficient number of particles. i.e.

$$N \gg \sqrt{\frac{\hbar^2}{G m^3 l_s}},$$

(105)

adopt cylindrically-symmetric distributions. The corresponding bound on the string tension is

$$G \mu_{\text{int}} \gg \sqrt{\frac{G \hbar^2}{m l_s \Delta^2}}.$$

(106)

Thus, even using the Newtonian approximation, we can obtain rough estimates of the energy and length scales associated with the BEC string, in both the interaction and quantum pressure dominated regimes.

V. DISCUSSIONS AND FINAL REMARKS

We have considered the possible existence, in a cosmological or astrophysical context, of static, cylindrically-symmetric, general relativistic structures consisting of matter in a Bose-Einstein condensed phase or, in other words, of Bose-Einstein condensate strings. By adopting a semi-classical approach, in which we assume that the quantum dynamics of the condensate remain unaffected by the gravitational field, and that the gravitational field remains classical and independent of quantum effects, we identified two limiting regimes corresponding to thermodynamic (i.e. particle interaction) pressure and quantum pressure dominated strings. For both these limiting cases, we solved the gravitational field equations for the string interior numerically and determined the corresponding variation, with respect to the radial coordinate, of the components of the metric, the three-dimensional energy density and pressure, the Tolman mass per unit length, $M$, and the “$W$” parameter which, in conjunction with the initial conditions for the metric components, determines the angular deficit of the cylindrical spacetime.

By defining the boundary of the string as the radius at which the energy density and pressure fall to zero, we also obtained estimates for the order of magnitude values of the string width, which depend sensitively on the mass and scattering length of the BEC particles. However, in general, we are able to conclude that interaction pressure
dominates for “thick” strings, while quantum pressure dominates for “thin” strings in which the ratio of volume to surface area is small. The precise length-scale determining the division between the two is determined by the BEC model parameters. In principle, both interaction and quantum pressure dominated strings may vary from tens of kilometers in diameter to widths comparable to those of more canonical string species.

Though the spacetime of the BEC string exhibits many differences from the flat conical space-time surrounding a vacuum string (or even the spherical(ish) “cap” that regularizes the vacuum string interior), perhaps it’s most interesting feature is that it allows for the existence of an angle excess; that is, for an angular deficit larger than $2\pi$. Though this may considered be unphysical, a more intriguing possibility is that such a string could behave “exotically”, in the sense of being able to support a traversable wormhole, as suggested in [45–47].

The solutions obtained in this study describe the interior of the Bose Einstein Condensate string. The density and pressure both vanish at the string boundary $r = R_s$ and, therefore, for $r \geq R_s$, an exterior vacuum solution of the gravitational field equations describes the physical and geometrical properties of the spacetime. Hence, in order to determine the asymptotic form of the metric, the solutions presented in this paper must be matched with the exterior cylindrically-symmetric vacuum solution. In this case, the appropriate solution of the gravitational field equations is given by the Kasner metric \[50\] \[54\]. Thus, such a matching would determine the parameters $a$, $b$, $c$ of the Kasner metric as a function of the physical parameters of the Bose Einstein Condensate forming the string.

Major particle candidates for the forming of BEC strings are dark matter particles. When the temperature of a cosmological boson gas, which may have existed in the early Universe, became less than the critical temperature, a Bose-Einstein condensation process may have taken place during the early cosmic history. Hence, most of the present day dark matter (DM) may be in the form of a Bose-Einstein Condensate. During the phase transition from normal to condensate dark matter, BEC type topological defects can be generated, and thus condensed dark matter filaments could have been formed. Such dark matter filaments may have some properties in common with the BEC string solutions considered in the present paper. One piece of observational evidence, which may enable us to distinguish between BEC strings and other species of topological defects, is gravitational lensing.

On the other hand, various species of topological defect, especially strings, were originally considered as possible seeds for the formation of large scale structure in the early universe, though this picture has since been overturned in favor of dark matter seeds acting in conjunction with and/or created by perturbations left at the end of the inflationary epoch (c.f. \[34\] \[52\] and reference therein). Thus, it would be extremely interesting to investigate whether Bose-Einstein condensate dark matter strings, which represent, in some sense, a combination of the “standard” topological defect and dark matter seed pictures, may turn out to be a better solution for large scale structure formation than either the dark matter or “string” pictures, individually. In the present paper, we have provided some basic theoretical tools that should enable the further in-depth investigation of the properties of the BEC strings, and of their cosmological implications.

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