Maximal estimates for averages over space curves

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Abstract Let $M$ be the maximal operator associated to a smooth curve in $\mathbb{R}^3$ which has nonvanishing curvature and torsion. We prove that $M$ is bounded on $L^p$ if and only if $p > 3$.

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1 Introduction

Let $\gamma$ be a smooth curve defined from the interval $\mathbb{J} := [-1, 1]$ to $\mathbb{R}^3$. We consider the average $Af$ over the dilations of $\gamma$ which is given by

$$Af(x, t) = \int f(x - t\gamma(s))\psi(s)\, ds, \quad t > 0.$$ 

Here $\psi$ is a smooth function with $\text{supp}\psi \subset (-1, 1)$. We assume that the curve $\gamma$ has nonvanishing curvature and torsion, equivalently,
\[ \det(\gamma'(s), \gamma''(s), \gamma'''(s)) \neq 0 \quad (1.1) \]

for \( s \in \mathbb{J} \). The condition is the natural nondegeneracy condition which is commonly used in studies related to space curves and the most typical examples are the helix and the moment curve \((s, s^2, s^3)\). In this paper we are concerned with \( L^p \) boundedness of the maximal operator

\[ Mf(x) = \sup_{0 < t} |Af(x, t)|. \]

The study of the maximal average over dilated submanifolds has a long history and there is a lot of literature (for example, see \([13, 31]\) and references therein). The celebrated Stein's spherical maximal theorem \([32]\) tells that the spherical maximal function is bounded on \( L^p \) if and only if \( p > d/(d - 1) \) for \( d \geq 3 \). The case \( d = 2 \) was later proved by Bourgain \([5]\). As it turned out, the problem became more difficult for the circle or the curves with nonvanishing curvature in \( \mathbb{R}^2 \) since the typical interpolation argument relying on \( L^2 \) estimate no longer works. In such cases the maximal estimates were obtained via the methods of continuum incidence geometry \([5, 27, 28, 30]\) or by utilizing the local smoothing property of the averaging operator \([14, 18, 29]\).

Concerning the maximal average over the curve in three or higher dimensional spaces, the \( L^p \) boundedness is naturally expected to be even harder to prove since Fourier transform of the measure supported on a space curve has slower decay. \( L^p \) boundedness of such maximal operators has been of interest for a long time (see \([23]\) for a historical comment) but no positive result was known until recently. It was Pramanik and Seeger \([23]\) who proved for the first time that \( M \) is \( L^p \) bounded for \( p > 38 \). (Also see \([20, 21, 23, 24]\) for the developments related to \( L^p \) Sobolev estimate for the operator \( f \to Af(\cdot, t) \).) Their result was obtained by relying on Wolff’s sharp \( \ell^p \) decoupling inequality for the cone in \( \mathbb{R}^3 \) \([33]\). More precisely, it was shown in \([23]\) that the maximal operator \( M \) is bounded on \( L^p \) for \( p > (p_o + 2)/2 \) if the sharp \( \ell^p \) decoupling inequality holds for \( p > p_o \). Combined with the recent \( \ell^p \) decoupling inequality on the optimal range \( p \geq 6 \) which is due to Bourgain and Demeter \([6]\), this establishes the \( L^p \) boundedness for \( p > 4 \). However, a modification of Stein’s example in \([32]\) shows that \( M \) can not be bounded on \( L^p \) for \( p \leq 3 \) (see Sect. 4.2).

In this paper we fill the gap and settle the problem of \( L^p \) boundedness of \( M \).

**Theorem 1.1** Suppose that \( \gamma : \mathbb{J} \to \mathbb{R}^3 \) is a smooth curve which has nonvanishing curvature and torsion, and \( \psi \) is a nontrivial, nonnegative, smooth function supported in \((-1, 1)\). Then, there is a constant \( C \) such that

\[ \|Mf\|_{L^p(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)} \quad (1.2) \]
for all $f \in L^p(\mathbb{R}^3)$ if and only if $p > 3$.

The assumption that $\psi$ is smooth is not necessary and it is clear that the theorem holds true for a continuous $\psi$. Even though $\gamma$ is assumed to be smooth, there is a positive integer $D$ such that (1.2) holds for $\gamma \in C^D(J)$ (see Remark 1 at the end of Sect. 3).

The maximal estimate in [23] was shown by exploiting $L^p$ local smoothing phenomena of the averaging operator. However, compared with the average over hypersurfaces or curves in $\mathbb{R}^2$, the $L^p$ local smoothing property of $A$ is not well understood. We instead try to make use of an $L^p$-$L^q$ type smoothing estimate which has a close connection to the adjoint restriction estimate. Usefulness of such estimates has been manifested in the study of $L^p$ improving property of the localized circular and spherical maximal functions [14,29] (also see [1,2,25]).

Our argument in this paper is closely related to the induction strategy developed by Ham and one of the authors [11]. They obtained the sharp adjoint restriction estimate for the space curve in $L^p(\mu)$ when $\mu$ is an $\alpha$-dimensional measure (see Sect. 2.1 for the definition). The work was in turn inspired by the multilinear approach due to Bourgain and Guth [7]. Main novelty of the current paper lies in devising an induction argument which directly works for the maximal operator. In contrast to the adjoint restriction operator a suitable form of multilinear estimate is not so obvious for the averaging operator $A$. In order to prove a multilinear estimate for $A$ which enjoys a better boundedness property under a certain additional assumption, we first express the operator $A$ as a sum of adjoint restriction operators and then relate them to geometry of the curves so that the transversality condition can be reformulated in terms of the relative positions between the associated curves. Unfortunately, some the consequent adjoint restriction operators are associated to $C^{1,1/2}$ surfaces but not to $C^2$ surfaces, so we cannot directly apply the multilinear restriction estimate which is due to Bennett, Carbery, and Tao [4]. However, it is not difficult to see that the argument in [4] continues to work for the $C^{1,1/2}$ surfaces (see Theorem 3.6 below). We also make use of some of the results from [23] to strengthen the multilinear estimate and also to deal with the nondegenerate part, whereas the difficult degenerate part is to be handled by the multilinear estimate which we prove in Sect. 3.

The argument here can be further developed to prove not only $L^p$ improving property of the maximal operator $\sup_{1 \leq t \leq 2} |Af(x,t)|$ but also maximal estimates with respect to $\alpha$-dimensional measures (see Remark 2 at the end of Sect. 4.1). Nonetheless, we do not attempt to pursue the matter in this paper.

Structure of the paper. In Sect. 2 we show that the maximal estimate can be deduced from a form of weighted estimate, and we formalize the induction setup to prove the weighted estimate. In Sect. 3 we obtain a weighted mul-
tilinear estimate for \( A \) under a certain separation condition. In Sect. 4 we establish the maximal bound putting the previous estimates together and show the optimality of the range of \( p \).

### 2 Reductions and preliminaries

In this section we reduce the proof of maximal estimate to showing a form of weighted estimates for the averaging operators which are given by the curves close to a specific curve. We also obtain some preparatory results which are to be used to prove the estimates in Sects. 3 and 4.

By the argument in [5] (also see [26]), which relies on Littlewood-Paley decomposition and scaling, one can obtain the maximal estimate (1.2) from that for \( \sup_{1 \leq t \leq 2} |Af(x, t)| \). More precisely, it is sufficient to show that there is an \( \varepsilon_p > 0 \) such that

\[
\|Af\|_{L^p_t L^\infty_x(\mathbb{R}^3 \times [1, 2])} \leq C \lambda^{-\varepsilon_p} \|f\|_{L^p(\mathbb{R}^3)}
\]

holds for all \( f \in \mathcal{S}(\mathbb{R}^3) \) whenever

\[
\text{supp} \hat{f} \subset A_\lambda := \{ \xi \in \mathbb{R}^3 : 3\lambda/4 \leq |\xi| \leq 7\lambda/4 \}, \quad \lambda \geq 1.
\]  

For the rest of the paper, we assume (2.2) unless it is mentioned otherwise.

**Notation.** Throughout the paper \( C, C_1, \ldots \) and \( c \) are supposed to be independent positive constants, and \( C_\varepsilon, C_\delta \) are constants depending on \( \varepsilon, \delta \) but all of these constants may vary at each appearance. In addition to the conventional notation \( \hat{\cdot} \) we use \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) to denote the Fourier and inverse Fourier transforms, respectively. By \( Q_1 = \mathcal{O}(Q_2) \) we denote \( |Q_1| \leq C Q_2 \) for a constant \( C \) and we also use the notation \( Q_1 = \mathcal{O}_3(Q_2) \) if \( |Q_1| \leq Q_2 \).

### 2.1 Estimate with \( \alpha \)-dimensional measure

Let \( \mathbb{B}^d(z, r) \) denote the ball of radius \( r \) which is centered at \( z \in \mathbb{R}^d \). Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^4 \). For \( 0 < \alpha \leq 4 \) we say \( \mu \) is \( \alpha \)-dimensional if there is a constant \( C \) such that

\[
\mu(\mathbb{B}^4(z, r)) \leq C r^\alpha
\]

for all \( r > 0 \) and \( z \in \mathbb{R}^4 \). For an \( \alpha \)-dimensional measure \( \mu \) we define

\[
\langle \mu \rangle_\alpha = \sup_{z \in \mathbb{R}^4, r > 0} r^{-\alpha} \mu(\mathbb{B}^4(z, r)).
\]
Instead of directly proving the maximal estimate (2.1) we obtain estimates for $Af$ with $\alpha$-dimensional measures. From those estimates we can deduce the estimate (2.1). As far as the authors are aware, it seems that this type of argument deducing the maximal estimate from the estimates with $\alpha$-dimensional measures first appeared in [19]. (See also [33, p. 1283] for a related discussion.)

**Theorem 2.1** Let $\mu$ be 3-dimensional. Suppose that $\gamma : \mathbb{J} \to \mathbb{R}^3$ is a smooth curve satisfying (1.1). Then, for $p > 3$ there is an $\varepsilon_p > 0$ such that

$$\|Af\|_{L^p(\mathbb{R}^3 \times [1,2], d\mu)} \leq C(\mu)^{\frac{1}{3}} \lambda^{-\varepsilon_p} \|f\|_{L^p(\mathbb{R}^3)}$$

(2.3)

holds whenever $\hat{f}$ is supported on $\mathbb{A}_\lambda$.

We shall work only with 3-dimensional measures even though it is possible to prove such estimates with $\alpha$-dimensional measure, $\alpha \neq 3$ on a certain range of $p$ (see Remark 2). The following shows the estimate (2.3) implies (2.1).

**Lemma 2.2** Suppose (2.3) holds true for all 3-dimensional measures $\mu$. Then the estimate (2.1) holds.

To prove this, we start with an elementary lemma.

**Lemma 2.3** Let $\eta \in C_0^\infty([2^{-3}, 2^3])$ and $\psi \in C_0^\infty(\mathbb{J})$. Set $r_0 = 1 + 4 \max\{|\gamma(s)| : s \in \text{supp}\psi\}$ and

$$K_\eta(x, t) = (2\pi)^{-3} \int\int e^{i(x \cdot \xi - t\gamma(s) \cdot \xi)} \psi(s) \, ds \, \eta(\lambda^{-1}|\xi|) \, d\xi.$$ 

If $|x| \geq r_0$ and $|t| \leq 2$, then $|K_\eta(x, t)| \leq C\|\eta\|_{C^{2N+3}} E_N(x)$ for any $N \geq 1$ where $E_N(x) := \lambda^{-N}(1 + |x|)^{-N}$.

**Proof** We see $K_\eta(x, t) = \frac{2^3}{(2\pi)^3} \int\int e^{i\lambda(x \cdot \xi - t\gamma(s) \cdot \xi)} \psi(s) \, ds \, \eta(|\xi|) \, d\xi$ by changing variables $\xi \to \lambda \xi$. Then repeated integration by parts in $\xi$ gives the desired estimate since $|\nabla_\xi (x \cdot \xi - t\gamma(s) \cdot \xi)| \geq 2^{-1}|x|$ if $|x| \geq r_0$ and $|t| \leq 2$. \hfill \Box

**Proof of Lemma 2.2** To obtain (2.1) it suffices to show the local estimate

$$\|Af\|_{L^p(\mathbb{R}^3 \times (0,1) \times [1,2])} \leq C\lambda^{-\varepsilon_p} \|f\|_{L^p(\mathbb{R}^3)}.$$ 

(2.4)

This is obvious if $\hat{f}$ is not assumed to be supported in $\mathbb{A}_\lambda$. However, we may handle $f$ as if it were supported on a ball of radius $r_0$. Since $\text{supp} \hat{f} \subset \mathbb{A}_\lambda$, $Af(\cdot, t) = K_\eta(\cdot, t) * f$ for an $\eta$ such that $\eta \in C_0^\infty((2^{-1}, 2))$ and $\eta = 1$ on $[3/4, 7/4]$. So, Lemma 2.3 gives $|K_\eta(x, t)| \leq C E_N(x)$ if $|x| \geq r_0$ and $|t| \leq 2$. Thus, by the typical localization argument (e.g., see the proof of Lemma 3.10) one can easily see that (2.4) implies (2.1).
In order to prove (2.4), using the Kolmogorov-Seliverstov-Plessner linearization, it is enough to show
\[ \| Af(\cdot, t(\cdot)) \|_{L^p(B^3(0,1))} \leq C \lambda^{-\varepsilon_p} \| f \|_{L^p(R^3)} \] (2.5)
for all measurable function \( t : B^3(0,1) \rightarrow [1, 2] \) with \( C \) independent of \( t \). Since \( \hat{f} \) is supported in \( A_\lambda \), \( Af \) is uniformly continuous on every compact subset. So, for (2.4) it is sufficient to show (2.5) while assuming \( t \) is continuous.

With a continuous function \( t \), the positive linear functional \( C_c(R^4) \ni F \mapsto \int_{B^3(0,1)} F(x, t(x)) \, dx \) defines a measure \( \mu \) by the relation
\[ \int F(x, t) \, d\mu(x, t) = \int_{B^3(0,1)} F(x, t(x)) \, dx, \quad F \in C_c(R^4). \]

We now notice that \( \mu \) is a 3-dimensional measure. Since \( B^4((x_o, t_o), r) \subset \{(x, t) \in R^3 \times R : |x - x_o| \leq r \} \),
\[ \mu(B^4((x_o, t_o), r)) = \int_{B^3(0,1)} \chi_{B^4((x_o, t_o), r)}(x, t(x)) \, dx \]
\[ \leq \int \chi_{B^3(x_o, r)}(x) \, dx = \frac{4}{3} \pi r^3 \]
for any \( r > 0 \) and \( (x_o, t_o) \in R^3 \times R \). Thus we have \( \langle \mu \rangle_3 \leq 4\pi/3 \). Noting \( \| Af(\cdot, t(\cdot)) \|_{L^p(B^3(0,1))} = \| Af \|_{L^p(d\mu)} \), we apply Theorem 2.1 and get (2.5) with \( C \) independent of \( t \).

\[ \Box \]

### 2.2 Weighted estimate

For \( 0 < \alpha \leq 4 \), let us denote by \( \Omega^\alpha \) the collection of nonnegative measurable functions \( \omega \) on \( R^4 \) such that the measure \( \omega \, dx \, dt \) is \( \alpha \)-dimensional. For a simpler notation we denote
\[ [\omega]_\alpha = \langle \omega \, dx \, dt \rangle_\alpha. \]

Even though \( \Omega^\alpha \) is properly contained in the set of \( \alpha \)-dimensional measures, the fact that \( \text{supp} \hat{f} \subset A_\lambda \) allows us to recover the estimate (2.3) from the estimates against \( \omega \in \Omega^\alpha \).

\[ ^1 \text{In fact, we see } \mu \text{ is a regular Borel measure by the Riesz-Markov-Kakutani representation theorem.} \]
Lemma 2.4  Let $I = [2^{-1}, 2^2]$. Suppose that

$$\|Af\|_{L^p(\mathbb{R}^3 \times I, \omega)} \leq C[\omega]^{\frac{1}{p}} \lambda^{-\beta_p} \|f\|_{L^p(\mathbb{R}^3)}$$

(2.6)

holds whenever $\omega \in \Omega^3$ and $\hat{f}$ is supported on $A_{\lambda}$. Then (2.3) holds for any 3-dimensional measure $\mu$.

The proof of the maximal estimate (2.1) is now reduced to showing (2.6). Lemma 2.4 of course remains valid for any $\alpha \in (0, 4]$.

To show Lemma 2.4 we make use of the next two lemmas: Lemma 2.5 and 2.6. The former can be shown following the standard argument (for example, see [17, pp. 47–49]), so we omit the proof.

Lemma 2.5  Let $0 < \alpha < 4$ and $\varphi \in \mathcal{S}(\mathbb{R}^4)$. Set $\varphi_\lambda = \lambda^4 \varphi(\lambda \cdot)$. If $\mu$ is an $\alpha$-dimensional measure, then $|\varphi| \lambda \ast \mu \in \Omega^\alpha$ and $||\varphi| \lambda \ast \mu||_\alpha \leq C \varphi(\mu)_\alpha$.

In what follows, $\tilde{\varphi}$ denotes a function in $C^\infty_0(I)$ which satisfies $\tilde{\varphi} = 1$ on $[1, 2]$, and $\beta, \beta_0$ respectively denote the functions such that $\beta \in C^\infty_0([2^{-1}, 2])$, $\beta = 1$ on $[3/4, 7/4]; \beta_0 \in C^\infty_0([-2, 2])$, $\beta_0 = 1$ on $[-1, 1]$.

Lemma 2.6  Let $r_0 = 1 + 4 \max\{|\gamma(s)| : s \in \text{supp} \psi\}$ and let

$$m(\xi, \tau) = \int \int \tilde{\varphi}(t) e^{-it(\tau + \gamma(s) \cdot \xi)} \psi(s) \, ds \, dt \beta(\lambda^{-1}|\xi|), \quad (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R}.$$  

Then, we have $|\mathcal{F}^{-1}(m(\xi, \tau)(1 - \beta_0((\lambda r_0)^{-1}\tau)))| \leq C_N \|\psi\|_\infty \tilde{E}_I^N$ for any $N > 0$ where $\tilde{E}_I^N := (1 + |\tau|)^{-N} E_N$.

Proof  Let $\rho_\ell(t) = (-it)^{k+l} \tilde{\varphi}(t)$ and note that $\partial_{\xi}^\alpha \partial_{\tau}^\alpha m(\xi, \tau)$ is a sum of the intergrals $\int \rho_{|\alpha|}(\tau + \gamma(s) \cdot \xi)(\gamma(s))^{\alpha_1} \psi(s) \, ds \times O(\lambda^{-|\alpha_2|})$ with $\alpha_1 + \alpha_2 = \alpha$. Thus it follows that $|\partial_{\xi}^\alpha \partial_{\tau}^\alpha m(\xi, \tau)| \leq C_N \|\psi\|_\infty \int_0^{r_0 \lambda} (r_0 \lambda)^{-N} (1 + |\tau|)^{-N}$ for any $N$ if $|\tau| \geq r_0 \lambda$. We then get the desired estimate by routine integration by parts. \hfill $\square$

Proof of Lemma 2.4  We define an auxiliary operator $\tilde{A}$ by

$$\mathcal{F}(\tilde{A}h)(\xi, \tau) = \beta_0((\lambda r_0)^{-1}\tau) \mathcal{F}(\tilde{\varphi}(t) Ah)(\xi, \tau).$$

Since $\hat{f}$ is supported in $A_{\lambda}$, we have $|((\tilde{\varphi}(t) A - \tilde{A}) f)| \leq C \tilde{E}_I^N \ast |f|$ by Lemma 2.6. We then note that $\int \tilde{E}_I^N(x - y) d\mu(x, t) \leq C \lambda^{-N} \langle \mu \rangle_3$ and $\int \tilde{E}_I^N(x - y) dy \leq C \lambda^{-N}$. Thus by Schur’s test we get

$$\|\tilde{E}_I^N \ast h\|_{L^p(\mathbb{R}^3 \times \mathbb{R}, d\mu)} \leq C \langle \mu \rangle_3^{\frac{1}{p}} \lambda^{-N} \|h\|_{L^p(\mathbb{R}^3)}$$

(2.7)
for $1 \leq p \leq \infty$ and a large $N$. So, in order to show (2.3), it suffices to prove

$$
\| \tilde{A} f \|_{L^p([\mathbb{R}^3 \times [1,2], d\mu])} \leq C \langle \mu \rangle \frac{1}{2} \lambda^{-\varepsilon p} \| f \|_{L^p(\mathbb{R}^3)}.
$$

(2.8)

Since the space time Fourier transform of $\tilde{A} f$ is supported in $B^4_4(0, 2^2 r_0 \lambda)$, $\tilde{A} f = \tilde{A} f * \psi_{r_0 \lambda}$ for some $\psi \in S(\mathbb{R}^4)$. This gives $|\tilde{A} f|^p \leq C |\tilde{A} f|^p * |\psi_{r_0 \lambda}|$ via Hölder’s inequality. Thus we have

$$
\| \tilde{A} f \|_{L^p([\mathbb{R}^3 \times [1,2], d\mu])} \leq C \| \tilde{A} f \|_{L^p(\mathbb{R}^3 \times \mathbb{R}, \omega)},
$$

where $\omega = |\psi_{r_0 \lambda}| * \mu$. Therefore, using $|(\tilde{\chi}(t) A - \tilde{A}) f| \leq C \tilde{E}_i^N * |f|$ again, we have only to obtain the estimate for $\tilde{\chi}(t) A f$ in $L^p(\mathbb{R}^3 \times \mathbb{R}, \omega)$ since the minor part can be handled as before. Since $[\omega]_3 \leq C \langle \mu \rangle_3$ by Lemma 2.5, the estimate (2.8) follows from (2.6) because supp $\tilde{\chi} \subset I$. □

### 2.3 Normalization of curves and weights

In order to prove the estimate (2.6), as mentioned before, we use an induction type argument over a class of curves. For the purpose we need to normalize the curves properly so that the induction assumption applies. This step is especially important for defining the induction quantity and proving uniform estimates (cf. \[11,15\]).

Let $D \geq 2^5$ be a positive integer which is taken to be large. Let $\gamma \in C^D([J])$ which satisfies (1.1). Then, for $s_0$ and $0 < \delta \ll 1$ such that $[s_0 - \delta, s_0 + \delta] \subset J$, we define

$$
M^\delta_{\gamma}(s_0) = (\delta \gamma'(s_0), \delta^2 \gamma''(s_0), \delta^3 \gamma'''(s_0))
$$

and

$$
\gamma^\delta_{s_0}(s) = (M^\delta_{\gamma}(s_0))^{-1}(\gamma(\delta s + s_0) - \gamma(s_0)).
$$

(2.9)

Let $\gamma_0(s) = (s, s^2/2!, s^3/3!)$. We consider a class of curves which are small perturbations of the curve $\gamma_0$ in $C^D([J])$. For $\varepsilon_0 > 0$, we set

$$
C^D(\varepsilon_0) = \{ \gamma \in C^D([J]) : \| \gamma - \gamma_0 \|_{C^D([J])} \leq \varepsilon_0 \}.
$$

Using an affine map, one can transform a small enough sub-curve of any $\gamma \in C^D([J])$ satisfying (1.1) so as to be contained in $C^D(\varepsilon_0)$. The following lemma is a slight modification of [11, Lemma 2.1].
Lemma 2.7 Let \( s_0 \in (-1, 1) \) and \( \gamma \in C^D(J) \) satisfy (1.1) on \( J \). Then, for any \( \varepsilon_0 > 0 \), there exists \( \delta_* = \delta_* (\varepsilon_0, \gamma) > 0 \) such that \( \gamma_\delta^{s_0} \in C^D(\varepsilon_0) \) whenever \([s_0 - \delta, s_0 + \delta] \subset J \) and \( |\delta| \leq \delta_* \). Additionally, if \( \gamma \in C^D(\varepsilon_0) \) and \( \varepsilon_0 < 2^{-5} \), then there is a uniform \( \delta > 0 \) such that \( \gamma_\delta^{s_0} \in C^D(\varepsilon_0) \) whenever \([s_0 - \delta, s_0 + \delta] \subset J \) with \( |\delta| \leq \delta_0 \).

For a matrix \( M \) we denote \( \|M\| = \sup_{|z|=1} |Mz| \).

Proof By Taylor expansion of \( \gamma(s + \delta) \) about \( s = 0 \), we have

\[
\gamma(s + \delta) = \gamma(s) + \delta \gamma'(s) + \frac{\delta^2}{2!} \gamma''(s) + \frac{\delta^3}{3!} \gamma'''(s) + \cdots \]

and \( \|\gamma(s + \delta) - \gamma(s)\| \leq C \delta^4 \).

For \( \delta > 0 \) we denote by \( D_\delta \) the diagonal matrix \((\delta e_1, \delta^2 e_2, \delta^3 e_3)\). To normalize the weights we need the next lemma, which one can show by following the argument in [11].

Lemma 2.8 Let \( 0 < \alpha \leq 4 \), \( 0 < \delta \ll 1 \) and \( \omega \in \Omega^\alpha \), and let \( M \) be a \( 4 \times 4 \) nonsingular matrix. Set \( \omega^\delta(x, t) = \omega(D_\delta x, t) \) and \( \omega_M(x, t) = \omega(M(x, t)) \).

Then, for a constant \( C \) independent of \( \omega \) and \( \delta \), we have

\[
[\omega^\delta]_\alpha = C \delta^{3\alpha - 12} [\omega]_\alpha, \tag{2.10}
\]

\[
[\omega_M]_\alpha \leq |\det M|^{-1} \|M\|^{\alpha} [\omega]_\alpha. \tag{2.11}
\]

Proof The inequality (2.10) is equivalent to

\[
\int_{B^4(y, r)} \omega(D_\delta x, t) \, dx \, dt \leq C \delta^{3\alpha - 12} [\omega]_\alpha \alpha
\]

for \( y \in \mathbb{R}^4 \) and \( r > 0 \). Changing variables \( x \to D_\delta^{-1} x \), we see the left hand side is equal to \( \delta^{-6} \int \chi_{B^4(y, r)}(D_\delta^{-1} x, t) \omega(x, t) \, dx \, dt \). Then we note that the set \( \{(x, t) : (D_\delta^{-1} x, t) \in B^4(y, r)\} \) is contained in a rectangle \( \mathcal{R}_\delta \) of dimensions about \( \delta r \times \delta^2 r \times \delta^3 r \times r \). Since \( \mathcal{R}_\delta \) is covered by at most \( C \delta^{-6} \) many balls of radius \( \delta^3 r \), (2.10) follows.
For (2.11) we only have to show
\[ \int_{\mathbb{B}^4(y, r)} \omega(M(x, t)) \, dx \, dt \leq |\det M|^{-1} \|M\|^\alpha [\omega]_\alpha r^\alpha \]
for \( y \in \mathbb{R}^4 \) and \( r > 0 \). Changing variables, we see that the left hand side equals
\[ |\det M|^{-1} \int_{\chi_{\mathbb{B}^4(y, r)}(\mathbb{M}^{-1}(x, t))} \omega(x, t) \, dx \, dt. \]
So, we get the inequality (2.11) since \((x, t) \in \mathbb{B}^4(My, \|M\|r)\) if \( M^{-1}(x, t) \in \mathbb{B}^4(y, r) \).

\[ \square \]

2.4 Reduction and the induction quantity

Throughout the paper we fix a small positive constant \( c_\circ \). To show (2.6) for a smooth curve satisfying (1.1), it is sufficient to handle \( \gamma \in \mathcal{C}^D(\varepsilon_\circ) \) with a small \( \varepsilon_\circ > 0 \) while \( \psi \in \mathcal{C}^D \) and \( \text{supp } \psi \subset [-c_\circ, c_\circ] \). As we shall see later, this can be shown by a finite decomposition and changing variables via affine transformations.

Definition 2.1 Let \( c_\circ, \varepsilon_\circ \) and \( \delta \) be the numbers such that \( 0 < c_\circ \leq 2^{-10} \), \( 0 < \varepsilon_\circ \leq c_\circ^2 \), and
\[ 0 < \delta \leq \min(c_\circ, \delta_\circ) \quad (2.12) \]
where \( \delta_\circ \) is given in Lemma 2.7. The number \( \delta \) is to be chosen later (see Sect. 4.1). We denote \( J_\circ = [-c_\circ, c_\circ] \), and we set
\[ \mathfrak{J}(\delta) = \{ J : J = [c_\circ \delta(k-1), c_\circ \delta(k+1)], k \in \mathbb{Z}, |k| \leq (c_\circ \delta)^{-1} + 1 \} \]
so that the intervals in \( \mathfrak{J}(\delta) \) cover \( \mathbb{J} \). For each \( J \in \mathfrak{J}(\delta) \) we define \( \mathcal{M}^D(J) \) to be the set of functions such that \( \psi \in \mathcal{C}^D(J) \) and \( \|\psi(|J| \cdot )\|_{\mathcal{C}^D(\mathbb{R})} \leq 1 \). For a given interval \( J \) we denote by \( \psi_J \) a function in \( \mathcal{M}^D(J) \).

For a smooth function \( a \) on \( \mathbb{J} \times I \times \mathbb{A}_\lambda \), following [23], we define an integral operator by setting
\[ A^\gamma[a]f(x, t) = (2\pi)^{-3} \int \int e^{i(x-ty(s))} a(s, t, \xi) \, ds \, \hat{f}(\xi) \, d\xi. \quad (2.13) \]
In particular, we note \( Af = A^\gamma[\psi]f \) as is clear by Fourier inversion.

Let us take \( \zeta \in \mathcal{C}_0^\infty([-1, 1]) \) such that \( \zeta \geq 0 \) and \( \sum_{k \in \mathbb{Z}} \zeta(s - k) = 1 \). For an interval \( J \) we denote by \( c_J \) the center of \( J \) and set \( \zeta_J(s) = \zeta(2(s - c_J)/|J|) \). Consequently, \( \zeta_J \in \mathcal{C}_0^\infty(J) \) and \( \sum_{J \in \mathfrak{J}(\delta)} \zeta_J(s) = 1 \) for \( s \in \mathbb{J} \). As a result,
we have

$$A^\gamma [\psi] f(x, t) = \sum_{J \in \mathbb{J}(\delta)} A^\gamma [\psi \xi_J] f(x, t)$$  \hspace{1cm} (2.14)

if supp$\psi \subset \mathbb{J}$. The following is one of the key lemmas which relates the estimate for the average over a short curve to that over a larger one.

**Lemma 2.9** Let $I' \subset I$ be an interval, and let $\omega \in \Omega^3$, $J = [s_o - c_o \delta, s_o + c_o \delta] \in \mathbb{J}(\delta)$ and $\psi_J \in \mathcal{H}^D(J)$. Suppose that $\gamma \in C^3(\mathbb{J})$ satisfies (1.1) and supp$\tilde{f} \subset A_\lambda$. Then, there are $\tilde{\omega} \in \Omega^3$, $\tilde{f}$ with $\|\tilde{f}\|_p = \|f\|_p$, and $\psi_{I_o} \in \mathcal{H}^D(J_o)$ which satisfy the following:

$$\|A^\gamma [\psi_J] f\|_{L^p(\mathbb{R}^3 \times I', \omega)} = \delta^{1-\frac{3}{p}} \|A^\gamma [\psi_{I_o}] \tilde{f}\|_{L^p(\mathbb{R}^3 \times I', \tilde{\omega})},$$  \hspace{1cm} (2.15)

$$[\tilde{\omega}]_3 \leq C (1 + |\gamma(s_o)|)^3 \text{det} M^1_\gamma(s_o) \|f\|_p$$  \hspace{1cm} (2.16)

and

$$\text{supp}\mathcal{F}(\tilde{f}) \subset \left\{ \xi : \frac{3}{4} d* \delta^3 \lambda \leq |\xi| \leq \frac{7}{4} d^* \delta \lambda \right\}.$$  \hspace{1cm} (2.17)

where $1/d* = \|(M^1_\gamma(s_o))^{-1}\|$ and $1/d^* = \text{inf}|z|=1 |(M^1_\gamma(s_o))^{-1}z|$.

**Proof** We denote $\psi_{I_o}(s) = \psi_J(\delta s + s_o)$. It is clear that $\psi_{I_o} \in \mathcal{H}^D(J_o)$. We set

$$\tilde{f}(x) = |\text{det}(M^\delta_\gamma(s_o))|^{\frac{1}{p}} f(M^\delta_\gamma(s_o)x).$$

Then, we see $\|\tilde{f}\|_p = \|f\|_p$ and $\mathcal{F}(\tilde{f})$ is supported in the set $S_\lambda = \{\xi : 3\lambda/4 \leq |(M^\delta_\gamma(s_o))^{-1}\xi| \leq 7\lambda/4\}$ because supp$\tilde{f} \subset A_\lambda$. Since $M^\gamma(s_o) = M^1_\gamma(s_o) D_\delta$, it is easy to see that $S_\lambda \subset \{\xi : 3\lambda d*/4 \leq |D_\delta^{-1}\xi| \leq 7\lambda d^*/4\}$, thus we get (2.17).

We now define $\tilde{\omega}$ and $\tilde{\omega}$ by setting $\tilde{\omega}(x, t) = \omega(x + t\gamma(s_o), t)$ and

$$\tilde{\omega}(x, t) = \delta^3 \tilde{\omega}(M^\delta_\gamma(s_o)x, t),$$

respectively. Denoting by $M$ the matrix such that $M(x, t) = (x + t\gamma(s_o), t)$, we note that $\tilde{\omega} = \omega_M$, det $M = 1$, and $\|M\| \leq 1 + |\gamma(s_o)|$. Thus using (2.11) we have $[\tilde{\omega}]_3 \leq (1 + |\gamma(s_o)|)^3[\omega]_3$. We also denote $M'(x, t) = (M^1_\gamma(s_o)x, t)$. Since $M^\gamma(s_o) = M^1_\gamma(s_o) D_\delta$, we have $\tilde{\omega} = \delta^3 (\omega_M)^\delta$ (see Lemma 2.8 for its definition). Using (2.10) and (2.11), we get $[\tilde{\omega}]_3 \leq C|\text{det} M^1_\gamma(s_o)|^{-1}(1 + \|M^1_\gamma(s_o)\|)^3[\omega]_3$ because $\text{det} M' = \text{det} M^1_\gamma(s_o)$ and $\|M'\| \leq 1 + \|M^1_\gamma(s_o)\|$. Combining these two inequalities gives (2.16).
To complete the proof it remains to show (2.15). Changing variables $s \to \delta s + s_0$ and using (2.9), we see $A^\gamma[\psi J]f(x, t) = \delta \int f(x - t\gamma(s_0) - tM_\gamma^\delta(s_0)\gamma(s))\psi J(\delta s + s_0)ds$. We thus have

$$A^\gamma[\psi J]f(x, t) = \delta |\det M_\gamma^\delta(s_0)|^{-\frac{1}{p}} \times \int \tilde{f}((M_\gamma^\delta(s_0))^{-1}(x - t\gamma(s_0)) - t\gamma(s_0))\psi J_\circ(s)ds.$$ 

Therefore the change of variables $x \to M_\gamma^\delta(s_0)x + t\gamma(s_0)$ yields (2.15). $\square$

**Reduction**

Let $\gamma \in \mathcal{C}^D(\mathbb{I})$ be a curve satisfying (1.1). For a given $\varepsilon_0 > 0$ we take $\delta = \delta_*$ where $\delta_*$ is the number given in Lemma 2.7. Applying (2.14) to $\psi \in \mathcal{C}_0^D(\mathbb{I})$ and then Lemma 2.9 to each interval $J$, we have

$$\|A^\gamma[\psi J]f\|_{L^p(\mathbb{R}^3 \times I, \omega)} \leq \delta^{1 - \frac{3}{p}} \sum_{J \in \mathcal{D}(\delta)} \|A^\gamma_{cJ}[\psi J]\tilde{f}^J\|_{L^p(\mathbb{R}^3 \times I, \tilde{\omega}^J)},$$

where $\gamma_{cJ}^\delta \in \mathcal{C}^D(\varepsilon_0)$ (by Lemma 2.7), $[\tilde{\omega}^J]_3 \leq C_J[\omega]_3$, $C^{-1}\psi J \in \mathcal{M}^D(J_\circ)$ for some constants $C_J$, $C > 0$, and $\tilde{f}^J$ satisfies that $\|\tilde{f}^J\|_p \leq \|f\|_p$ and supp$\mathcal{F}(\tilde{f}^J) \subset \{\xi : (B^J)^{-1}\lambda \leq |\xi| \leq B^J \lambda\}$ for a constant $B^J$. Since there are at most $C\delta_*^{-1}$ many intervals, for the estimate (2.6) it is enough to obtain estimate for each $A^\gamma_{cJ}[\psi J]\tilde{f}^J$ against the weight $\tilde{\omega}^J$. Hence, in order to show (2.6), after decomposing $\tilde{f}^J$ via Littlewood-Paley projection and replacing $\tilde{\omega}^J$ with $(C_J[\omega]_3)^{-1}\tilde{\omega}^J$, we need only to consider the curve $\gamma \in \mathcal{C}^D(\varepsilon_0)$ and the weight $\omega$ with $[\omega]_3 \leq 1$.

Furthermore, since $A^\gamma[\psi J]f(x, t) = A^\gamma[\psi J](r x, r t)$, by scaling after splitting $J$ into three intervals $[2^{-1}, 1], [1, 2]$ and $[2, 4]$, the proof of (2.6) now reduces to showing

$$\|A^\gamma[\psi J]f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C\lambda^{-\varepsilon_p} \|f\|_{L^p(\mathbb{R}^3)}$$

for $[\omega]_3 \leq 1$, $\gamma \in \mathcal{C}^D(\varepsilon_0)$, and $\psi \in \mathcal{M}^D(J_\circ)$ for some $D$.

**Definition 2.2** Fixing $p, \varepsilon_0, D$, for $\lambda \geq 1$ we define the quantity $Q(\lambda)$ by

$$Q(\lambda) = \sup \left\{\|A^\gamma[\psi J]f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} : \gamma \in \mathcal{C}^D(\varepsilon_0), \psi \in \mathcal{M}^D(J_\circ), [\omega]_3 \leq 1, \text{supp } \hat{f} \subset A_\lambda, \|f\|_{L^p(\mathbb{R}^3)} \leq 1 \right\}.$$

An elementary estimate gives $Q(\lambda) \leq C\lambda^2$ for $1 \leq p \leq \infty$. 

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Thanks to the discussion in the above and Lemma 2.4, Theorem 2.1 now follows from the next proposition, which we prove in Sect. 4.1.

**Proposition 2.10** For $p \in (3, \infty)$, there are positive constants $\varepsilon_0$, $D$, $\varepsilon_p$, and $C$ such that

$$Q(\lambda) \leq C\lambda^{-\varepsilon_p}. \tag{2.18}$$

In order to show (2.18) we need only to handle $A^\gamma[\psi]$ with $\psi \in \mathcal{N}^D(J_0)$, which we decompose in the fashion of (2.14). Thus it suffices work with the intervals $J \cap J_0 \neq \emptyset$. We set

$$J_0(\delta) = \{ J \in J(\delta) : J \subset (1 + 2c_0)J_0 \}.$$  

What follows next is a consequence of Lemma 2.9, which plays an important role in proving (2.18).

**Lemma 2.11** Let $J \in J_0(\delta)$ and $\psi_J \in \mathcal{N}^D(J)$. Suppose $\gamma \in \mathcal{C}^D(\varepsilon_0)$, $|\omega|_3 \leq 1$, and $\text{supp} \tilde{f} \subset A_{\lambda}^\gamma$. If $\delta^3 \lambda \geq 2^2$ and $\varepsilon_0 > 0$ is sufficiently small, there is a constant $C$, independent of $\gamma$, $\omega$, and $\psi_J$, such that

$$\left\| A^\gamma[\psi_J]f \right\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C\delta^{1 - \frac{3}{p}} K_\delta(\lambda) \left\| f \right\|_{L^p(\mathbb{R}^3)}, \tag{2.19}$$

where

$$K_\delta(\lambda) = \sum_{2^{-2} \delta^3 \lambda \leq 2^j \leq 2^2 \delta \lambda} Q(2^j).$$

**Proof** We denote $J = [s_0 - c_0 \delta, s_0 + c_0 \delta]$. Since $\gamma \in \mathcal{C}^D(\varepsilon_0)$, $\gamma_{s_0}^\delta \in \mathcal{C}^D(\varepsilon_0)$ by Lemma 2.7 and our choice of $\delta$, i.e., (2.12). Noting that $s_0 \in 2J_0$, $\gamma \in \mathcal{C}^D(\varepsilon_0)$ and $\varepsilon_0 \leq c_0^2$, we see that $|\gamma(s_0)| \leq 3c_0$ and $\|M_{\gamma}^1(s_0) - I_3\| \leq 5c_0$. If we use $\sum_{\ell=0}^{\infty}(I_3 - M_{\gamma}^1(s_0))^{2^\ell} = (M_{\gamma}^1(s_0))^{-1}$, it follows $\|M_{\gamma}^1(s_0))^{-1} - I_3\| \leq \frac{5c_0}{1 - 5c_0}$. Since $\|M\| = \|M^t\|$ for any matrix $M$, $\|(M_{\gamma}^1(s_0))^{-1} - I_3\| < 1/100$. So, we have

$$\frac{99}{100} \leq \inf_{|z|=1} |(M_{\gamma}^1(s_0))^{-1}z|, \quad \|M_{\gamma}^1(s_0))^{-1}\|, \quad |\det M_{\gamma}^1(s_0)| \leq \frac{101}{100}.$$  

Therefore, by (2.16) and (2.17) we see, respectively, that $[\tilde{\omega}]_3 \leq C$ with a constant $C$ independent of $\gamma$ and that $\text{supp} \mathcal{F}(\tilde{f}) \subset \{ \xi : 2^{-1} \delta^3 \lambda \leq |\xi| \leq 2\delta \lambda \}$. Let $\beta_* \in C^\infty_0([3/4, 7/4])$ be such that $\sum_j \beta_*(2^{-j} \cdot) = 1$. We decompose

$$\tilde{f} = \sum_{2^{-2} \delta^3 \lambda \leq 2^j \leq 2^2 \delta \lambda} \tilde{f}_j,$$
where \( \tilde{f}_j = \mathcal{F}^{-1}(\beta \cdot |\cdot| \mathcal{F}(\tilde{f})) \). By (2.15) it follows that

\[
\|A^\gamma [\psi] f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq \delta^{1 - \frac{3}{p}} \sum_{2^{-2} \delta \lambda \leq \frac{1}{2} \lambda \leq 2^2 \delta \lambda} \|A^{\gamma^0} [\psi] \tilde{f}_j\|_{L^p(\mathbb{R}^3 \times [1, 2], \tilde{\omega})}.
\]

Since \( \text{supp} \mathcal{F}(\tilde{f}_j) \subset A_{2j} \) and \( \| \tilde{f}_j \|_p \leq C_{\beta} \| f \|_p \) and since \( \gamma^0 \in C^D(\varepsilon_0) \), \( \psi_{J_0} \in \mathcal{M}^D(J_0) \) and \( [\tilde{\omega}]_3 \leq C \), we have \( \|A^{\gamma^0} [\psi] \tilde{f}_j\|_{L^p(\mathbb{R}^3 \times [1, 2], \tilde{\omega})} \leq CQ(2^j) \| f \|_p \) while \( C \) is independent of \( \gamma, \omega \), and \( \psi_{J_0} \). Therefore we get (2.19).

\[\square\]

### 2.5 Decomposition on the Fourier side

To show the inequality (2.18) we need only to deal with \( \gamma \in C^D(\varepsilon_0) \) and \( \psi \in \mathcal{M}^D(J_0) \), therefore it suffices to consider the curve \( \gamma \) over the interval \( (1 + 2c_0)J_0 \). This additional localization helps to simplify the argument, which comes after.

Since \( \varepsilon_0 \leq c_0^2 \), it is clear that

\[
|\gamma'(s) - e_1| \leq 2c_0, \quad |\gamma''(s) - e_2| \leq 2c_0, \quad |\gamma'''(s) - e_3| \leq 2c_0 \quad (2.20)
\]

for \( s \in (1 + 2c_0)J_0 \) and \( \gamma \in C^D(\varepsilon_0) \). Thus we have \( |\gamma'(s) \cdot \xi| + |\gamma''(s) \cdot \xi| \geq c_0 |\xi| \) if \( |\xi_1| \geq 3c_0 |\xi| \) or \( |\xi_2| \geq 3c_0 |\xi| \). Using Proposition 2.20 below we can handle the contribution from the part of frequency \( |\xi_1| \geq 3c_0 |\xi| \) or \( |\xi_2| \geq 3c_0 |\xi| \) since the condition (2.26) is satisfied. We shall mainly concentrate on the case where \( \xi \) is included in the set

\[A^*_{\lambda} := \{ \xi : 2^{-1} \lambda \leq |\xi| \leq 2\lambda, \quad |\xi_1| \leq 2^2 c_0 |\xi|, \quad |\xi_2| \leq 2^2 c_0 |\xi| \}.\]

The following is easy to see.

**Lemma 2.12** There exists a function \( \sigma \in C^{D-2}(A^*_{\lambda}) \), homogeneous of degree 0, such that, for \( \xi \in A^*_{\lambda} \), \( |\sigma(\xi)| \leq 5c_0 \) and

\[
\gamma''(\sigma(\xi)) \cdot \xi = 0.
\]

Indeed, we need to solve the equation \( \gamma''(s) \cdot \xi = 0 \) for a given \( \xi \), equivalently, \( \xi_3^{-1} \xi_2 + s + e(\xi, s) = 0 \) where \( e(\xi, s) \) is a function of homogeneous of degree zero and \( \|e(\xi, \cdot)\|_{C^{p-2}} \leq 2\varepsilon_0 \). An elementary argument shows existence of \( \sigma(\xi) \) and the implicit function theorem guarantees that \( \sigma \in C^{D-2}(A^*_{\lambda}) \) since \( \gamma \in C^D(\varepsilon_0) \). It is clear that \( |\sigma(\xi)| \leq 5c_0 \) because \( \xi_3^{-1} \xi_2 + \sigma(\xi) + e(\xi, \sigma(\xi)) = 0 \).
For $\xi \in A^*_\lambda$, we denote
\[
\Lambda_\gamma(\xi) = \gamma'''(\sigma(\xi)) \cdot \xi,
\]
\[
R_\gamma(\xi) = -\frac{\gamma'(\sigma(\xi)) \cdot \xi}{\Lambda_\gamma(\xi)}.
\]

If $\xi \in A^*_\lambda$ and $\sigma(\xi) \in (1 + 2c_o)J_0$, by (2.20) we have $2^{-2}\lambda \leq |\Lambda_\gamma(\xi)| \leq 2^2\lambda$, $|\gamma'(\sigma(\xi)) \cdot \xi - \xi| \leq 2^3 c_o \lambda$, and $|\Lambda_\gamma(\xi) - \xi_3| \leq 2^3 c_o \lambda$, so $|R_\gamma(\xi)| \leq 2^6 c_o$.

**Decomposition of the operator $A^\gamma[\psi_J]$**

By a Taylor expansion we have
\[
\gamma'(s) \cdot \xi = -\Lambda_\gamma(\xi) R_\gamma(\xi) + 2^{-1} \Lambda_\gamma(\xi) (s - \sigma(\xi))^2 + O(e_o \lambda |s - \sigma(\xi)|^3),
\]
and
\[
\gamma''(s) \cdot \xi = \Lambda_\gamma(\xi) (s - \sigma(\xi)) + O(e_o \lambda |s - \sigma(\xi)|^2)
\]
for $s \in J$ and $\xi \in A^*_\lambda$. Thus $\gamma'(s) \cdot \xi$ and $\gamma''(s) \cdot \xi$ have lower bounds if $\sigma(\xi)$ is distanced from $J$, so it is not difficult to have control over the contribution from the associated frequency. However, if $\sigma(\xi)$ is close to $J$ for $\xi \in \text{supp} \hat{f}$, the behavior of $A^\gamma[\psi_J]f$ becomes less favorable. This leads us to define, for $K \geq 1$ and $J \in \mathcal{J}_{\varepsilon}(\delta)$,
\[
\mathcal{R}_J(K) = \{ \xi : |\gamma'(c_J) \cdot \xi| \leq Kc_o^2 \delta^2 \lambda, |\gamma''(c_J) \cdot \xi| \leq Kc_o \delta \lambda, 2^{-2}\lambda \leq |\xi_3| \leq 2^2 \lambda \},
\]
which contains the unfavorable frequency part of $A^\gamma[\psi_J]f$. Concerning the sets $\mathcal{R}_J(K)$ we have the next lemma, which we use later.

**Lemma 2.13** Let $\gamma \in C^D(\varepsilon_o)$. If $\varepsilon_o > 0$ is sufficiently small, we have the following with $C$ independent of $\gamma$ and $\delta$:
\[
\sum_{J \in \mathcal{J}_{\varepsilon}(\delta)} \chi_{\mathcal{R}_J(2^6)} \leq C.
\]

**Proof** In order to show (2.23) it is sufficient to verify that the sets $\mathcal{R}_J := \{ \xi : \lambda \xi \in \mathcal{R}_J(2^6) \}$ overlap each other at most $C$ many times. Note that $\mathcal{R}_J$ is contained in $2^8 c_o \delta$ neighborhood of the line $L_J$ passing through the origin with its direction parallel to $\gamma'(c_J) \times \gamma''(c_J)$. Since $\mathcal{R}_J \subset \{ \xi : 2^{-4} \leq |\xi| \leq 2^4 \}$, it is sufficient to show that the directions of the lines $L_J$ are separated from each other by a distance at least $2^{-1} c_o \delta$. This in turn follows if we show
\[
\frac{d}{ds} \left( \gamma'(s) \times \gamma''(s) \right) = -e_2 + O_\varepsilon(5c_o)
\]
for $\gamma \in C^D(\varepsilon_0)$ because the distance between the centers $c_J$ of $J$ is at least $c_0\delta$. Since $(d/ds)(y'(s) \times y''(s)) = y'(s) \times y'''(s)$, it is enough to show $y'(s) \times y'''(s) = -e_2 + O_h(5c_0)$. Since $s \in [-2c_0, 2c_0]$ and $\gamma \in C^D(\varepsilon_0)$, $|y'(s) - e_1| \leq 2c_0(1 + 2c_0)$ and $|y'''(s) - e_3| \leq c_0^2$. Thus, we have $y'(s) \times y'''(s) = -e_2 + O_h(5c_0)$.

Let $\tilde{\beta} \in C_0^\infty([2^{-2}, 2^2])$ be such that $\tilde{\beta} = 1$ on $[2^{-1}, 2]$. Then we set

$$\tilde{\chi}_{R_J}(\xi) = \beta_0 \left( \frac{|y'(c_J) \cdot \xi|}{2^5 c_0^2 \delta^2 \lambda} \right) \beta_0 \left( \frac{|y''(c_J) \cdot \xi|}{2^5 c_0 \delta \lambda} \right) \frac{\tilde{\beta}(\xi/\lambda)}{\tilde{\beta}(\xi_3)},$$

so that $\tilde{\chi}_{R_J}$ is supported in $R_J(2^6)$ and $\tilde{\chi}_{R_J}(\xi) = 1$ if $\xi \in R_J(2^5) \cap \Lambda^*_\chi$. We also set

$$P_J f = \mathcal{F}^{-1}(\tilde{\chi}_{R_J} \hat{f}).$$

The following is a consequence of (2.23).

**Lemma 2.14** If $\varepsilon_0$ is small enough, we have \( \left( \sum_{J \in \mathfrak{H}(\delta)} \| P_J f \|_p^p \right)^{1/p} \leq C \| f \|_p \) for $2 \leq p \leq \infty$ whenever $\gamma \in C^D(\varepsilon_0)$.

The inequality follows by interpolation between the cases $p = 2$ and $p = \infty$. Plancherel's theorem and (2.23) give $\left( \sum_{J} \| P_J f \|_2^2 \right)^{1/2} \leq C \| f \|_2$ and the estimate $\max_J \| P_J f \|_\infty \leq C \| f \|_\infty$ is obvious.

**Decomposition away from the conic surface $C_\lambda$**

We further decompose $A^\gamma[\psi_J] P_J f$ on the Fourier side taking into account how close $\xi$ is to the conic set $C_\lambda := \{ \xi \in \Lambda^*_\chi : R_\gamma(\xi) = 0 \}$. To this end we set

$$\tilde{\chi}_{\Lambda^*_\chi}(\xi) = \beta_0 \left( \frac{\xi_1}{2c_0|\xi|} \right) \beta_0 \left( \frac{\xi_2}{2c_0|\xi|} \right) \beta(\lambda^{-1}|\xi|).$$

For $0 < \nu \ll 1$, we define the cutoff functions $\pi_0, \pi_1, \pi_0^1$, and $\pi_0^0$ by

$$\pi_0(\xi) = \tilde{\chi}_{\Lambda^*_\chi}(\xi) \beta_0(\lambda^{3/2} - 2\nu|R_\gamma(\xi)|),$$

$$\pi_1(\xi) = \beta(\lambda^{-1}|\xi|) - \tilde{\chi}_{\Lambda^*_\chi}(\xi) \beta_0(\delta^{-100}|R_\gamma(\xi)|),$$

and, for $j = 0, 1$,

$$\pi_0^j(\xi) = \tilde{\chi}_{\Lambda^*_\chi}(\xi) \chi_{|\xi|:(-1)^{j+1} R_\gamma(\xi) > 0}(\xi) \times \left( \beta_0(\delta^{-100}|R_\gamma(\xi)|) - \beta_0(\lambda^{3/2} - 2\nu|R_\gamma(\xi)|) \right).$$
The support of \( \tilde{\chi}_{A^*_\lambda} \) is contained in \( \tilde{\mathbb{A}}^*_\lambda \) and \( \pi_c + \pi_o^1 + \pi_o^0 + \pi_e = \beta(\lambda^{-1}| \cdot |) \) almost everywhere. The functions \( \pi_c, \pi_o^1 + \pi_o^0, \) and \( \pi_e \) roughly split the set \( \tilde{\mathbb{A}}^*_\lambda \) into three regions \( \{ \xi : |R_\gamma(\xi)| \leq C\lambda^2\nu^{-2/3} \}, \{ \xi : C\lambda^2\nu^{-2/3} \leq |R_\gamma(\xi)| \leq C_1\delta^{100} \}, \) and \( \{ \xi : C_1\delta^{100} \leq |R_\gamma(\xi)| \} \). The division between the first set and the other two reflects different asymptotic behaviors of the multiplier \( A_\gamma[\psi_J](e^{i(s \cdot \xi)})(0, t) \) as \( |\xi| \to \infty \). The further division of the second and the third sets is necessitated by the transversality condition for the multilinear estimate, which is to be discussed in the next section.

We also define the associated multiplier operators \( \mathcal{P}_c, \mathcal{P}_o^1, \mathcal{P}_o^0, \) and \( \mathcal{P}_e \) by

\[
\hat{\mathcal{P}_c}g(\xi) = \pi_c(\xi)\hat{g}(\xi),
\]

\[
\mathcal{F}(\mathcal{P}_o^j g)(\xi) = \pi_o^j(\xi)\hat{g}(\xi), \quad j = 0, 1,
\]

\[
\hat{\mathcal{P}_e}g(\xi) = \pi_e(\xi)\hat{g}(\xi).
\]

Besides, we set \( \mathcal{P}_n = \mathcal{P}_c + \mathcal{P}_o^1 + \mathcal{P}_o^0 \). Then easy estimates for the kernels of the operators give

\[
\|\mathcal{P}_c\|_{p \to p} \leq C_1\lambda C, \quad \|\mathcal{P}_o^j\|_{p \to p} \leq C_1\lambda C, \quad j = 0, 1,
\]

(2.24)

for \( 1 \leq p \leq \infty \) and some constants \( C, C_1 > 0 \). It is possible to get better bounds if we use the decoupling or the square function estimate for the cone (for example, [10,16]) but we do not attempt to do so since it is irrelevant to our purpose. Similarly, we also have

\[
\|\mathcal{P}_e\|_{p \to p} \leq C_1\delta^{-C}, \quad \|\mathcal{P}_n\|_{p \to p} \leq C_1\delta^{-C}
\]

(2.25)

for \( 1 \leq p \leq \infty \). For the former we need only to note that \( \|\mathcal{F}^{-1}(\pi_e)\|_{L^1(\mathbb{R}^3)} \leq C_1\delta^{-C} \). The latter follows from the former because the multiplier associated to the operator \( \mathcal{P}_n \) is \( \beta(\lambda^{-1}| \cdot |) - \pi_e \).

### 2.6 Nondegenerate part

Decomposition of the operator \( A \) on the Fourier side gives rise to operators of the form of (2.13) such as \( A^\gamma[\psi_J]P_J, A^\gamma[\psi_J](1 - P_J), \ldots, A^\gamma[\psi_J]\mathcal{P}_e \). If \( |\gamma'(s) \cdot \xi| + |\gamma''(s) \cdot \xi| \geq C|\xi| \) on the support of \( a \), we can handle \( A^\gamma[a] \) using the following theorem which is a straightforward consequence of [23, Theorem 4.1].

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2 The subscripts \( c, e \) stand for the (main) conic region, the (minor) error parts, respectively, while \( o \) and \( n \) represent outside of and near the conic region, respectively.
Theorem 2.15 Let $K \geq 1$ and $[s_0 - 2r, s_0 + 2r] \subset \mathbb{J}$ with $K^{-1} \leq r$. Suppose that $a(s, t, \xi)$ is a smooth function supported in $[s_0 - r, s_0 + r] \times I \times \mathbb{A}_\lambda$ and $|\partial_s^{j_1} \partial_t^{j_2} \partial_\xi^\alpha a(s, t, \xi)| \leq B|\xi|^{-|\alpha|}$ for $|\alpha| \leq 5$ and $j_1, j_2 = 0, 1$. Also, assume that

$$|\gamma' (s) \cdot \xi| + |\gamma''(s) \cdot \xi| \geq K^{-1} |\xi|$$

(2.26)

holds whenever $(s, t, \xi) \in \text{supp} a$ for some $t \in I$. Then, if $p \geq 6$ and $\varepsilon_0 > 0$ is small enough, for any $\varepsilon > 0$,

$$\|A^\gamma [a] f\|_{L_p(\mathbb{R}^3 \times I)} \leq C_\varepsilon B K C \lambda^{-\frac{2}{p} + \varepsilon} \|f\|_{L_p(\mathbb{R}^3)}$$

(2.27)

holds whenever $\gamma \in \mathcal{C}^D(\varepsilon_0)$ and $\hat{f}$ is supported in $\mathbb{A}_\lambda$.

The statement of Theorem 2.15 differs from the one in [23] in a couple of aspects. First, the range of $p$ is enlarged to $p \geq 6$ thanks to the $\ell^p$-decoupling inequality for the cone [6]. Secondly, there is an extra factor $K C$ in (2.27). One can easily show the estimate (2.27) by following the argument in [23]. It is also possible to deduce (2.27) from that with $K \in [2^{-1}, 2]$ by finite decomposition and making use of scaling and affine transform. Uniformity of the bound over $\gamma \in \mathcal{C}^D(\varepsilon_0)$ is clear.

The estimate $|\int e^{-it\gamma(s) \cdot \xi} a(s, t, \xi) ds| \leq C_1 B K C |\xi|^{-\frac{1}{2}}$ follows by (2.26) and van der Corput’s Lemma. We thus have $\|A^\gamma [a] f\|_{L^2(\mathbb{R}^3 \times I)} \leq C_1 B K C \lambda^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^3)}$ by Plancherel’s theorem. Interpolation between the estimate and (2.27) with $p = 6$ gives

Corollary 2.16 Under the same assumption as in Theorem 2.15, suppose $2 \leq p \leq 6$ and $\varepsilon_0$ is small enough. Then, for any $\varepsilon > 0$,

$$\|A^\gamma [a] f\|_{L_p(\mathbb{R}^3 \times I)} \leq C_\varepsilon B K C \lambda^{-\frac{1}{2} - \frac{1}{2p} + \varepsilon} \|f\|_{L_p(\mathbb{R}^3)}$$

holds whenever $\gamma \in \mathcal{C}^D(\varepsilon_0)$ and $\hat{f}$ is supported in $\mathbb{A}_\lambda$.

We also make use of the following ([23, Theorem 1.4]).

Theorem 2.17 Let $J \subset \mathbb{J}$ be a compact interval of length $\delta$ and $\psi_J \in \mathcal{N}^D(J)$. Then, if $p \geq 6$ and $\varepsilon_0$ is small enough,

$$\|A^\gamma [\psi_J] f\|_{L_p(\mathbb{R}^3 \times I)} \leq C_\varepsilon \delta^{-C} \lambda^{-\frac{4}{3p} + \varepsilon} \|f\|_{L_p(\mathbb{R}^3)}$$

(2.28)

holds for any $\varepsilon > 0$ whenever $\gamma \in \mathcal{C}^D(\varepsilon_0)$ and $\hat{f}$ is supported in $\mathbb{A}_\lambda$.

The critical case $p = 6$ can be included by interpolation with a trivial estimate.
Compared with [23, Theorem 1.4], the range of $p$ is extended to $p \geq 6$ by the aforementioned decoupling inequality [6]. The estimate (2.28) with additional factor $\delta^{-C}$ can be shown by scaling and its uniformity over $\gamma \in C^D(\varepsilon_\circ)$ is also obvious.

Estimates for $A^\gamma[\psi_J](1 - P_J)$ and $A^\gamma[\psi_J]P_e$

The condition (2.26) is satisfied on the support of $\psi_J(s)(1 - \tilde{\chi}_{R_J}(\xi))$. Thus, using Corollary 2.16, we can get a favorable estimate for $A^\gamma[\psi_J](1 - P_J)$. We also obtain a similar estimate for $A^\gamma[\psi_J]P_e$ (see Proposition 2.20 below).

Proposition 2.18 Let $[\omega]_3 \leq 1$, and $J \in \tilde{J}_\circ(\delta)$. If $2 \leq p \leq 6$ and $\varepsilon_\circ > 0$ is small enough,

$$
\|A^\gamma[\psi_J](1 - P_J)f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C_\epsilon \delta^{-C} \lambda^{\frac{1}{2}} \lambda^{\frac{1}{p} - \frac{1}{2}} \|f\|_{L^p(\mathbb{R}^3)} \quad (2.29)
$$

holds for any $\epsilon > 0$ whenever $\text{supp}\, \hat{f} \subset A_\lambda$, $\gamma \in C^D(\varepsilon_\circ)$, and $\psi_J \in \mathcal{N}^D(J)$.

We prove Proposition 2.18 using the next lemma. To this end, we need only to have the inequality (2.30) below for $1 \leq p \leq \infty$. However, for later use (see Sect. 3.4) we prove it for $0 < p \leq \infty$.

Lemma 2.19 Let $0 < p \leq \infty$, $0 < \alpha \leq 4$ and $\omega \in \Omega^\alpha$. Suppose that $F \in L^p(\mathbb{R}^4)$ and $\hat{F}$ is supported on $B^4(0, \lambda)$. Then we have

$$
\|F\|_{L^p(\mathbb{R}^4, \omega)} \leq C[\omega]_{\alpha} \lambda^{\frac{1}{p} - \frac{4 - \alpha}{p}} \|F\|_{L^p(\mathbb{R}^4)}. \quad (2.30)
$$

Proof Since $w \, dx\, dt \ll dx\, dt$ for $\omega \in \Omega^\alpha$, it follows that $\|F\|_{L^\infty(\mathbb{R}^4, \omega)} \leq \|F\|_{L^\infty(\mathbb{R}^4)}$. When $1 \leq p < \infty$, (2.30) is a simple consequence of Hölder’s inequality. Indeed, let us take $\varphi \in \mathcal{S}(\mathbb{R}^4)$ such that $\widehat{\varphi} = 1$ on $B^4(0, 1)$ and $\widehat{\varphi}$ is supported on $B^4(0, 2)$. Then $F = F * \varphi_\lambda$ since $\hat{F}$ is supported on $B^4(0, \lambda)$. Thus, by Hölder’s inequality we have $|F|^p \leq C |F|^p * |\varphi|_\lambda$. So,

$$
\|F\|_{L^p(\mathbb{R}^4, \omega)}^p \leq C \int |F(x)|^p |\varphi|_\lambda * \omega(x) \, dx \leq C \|F\|_{L^p(\mathbb{R}^4)}^p \|\varphi|_\lambda * \omega\|_{\infty}.
$$

This gives (2.30) since $|||\varphi|_\lambda * \omega||_{\infty} \leq C \lambda^{4 - \alpha}[\omega]_{\alpha}$.

When $p \in (0, 1)$, we claim that

$$
|F|^p \leq C |F|^p * |\varphi|_\lambda^p, \quad (2.31)
$$

where we denote $|\varphi|_\lambda^p = (|\varphi|^p)_\lambda$. Once we have (2.31), the proof of (2.30) is straightforward. By (2.31) we have $\|F\|_{L^p(\mathbb{R}^4, \omega)}^p \leq C \|F\|_{L^p(\mathbb{R}^4)}^p \|\varphi|_\lambda^p * \omega\|_{\infty}$. Since $|||\varphi|_\lambda^p * \omega||_{\infty} \leq C \lambda^{4 - \alpha}[\omega]_{\alpha}$, we obtain (2.30).
We now show (2.31). By scaling we may assume \( \lambda = 1 \), otherwise one may replace \( F \) with \( F(\cdot/\lambda) \). To show (2.31) when \( \lambda = 1 \), we first notice that

\[
|F| * |\varphi|(x) = \int |F(y)\varphi(x-y)|dy \leq \|F\varphi(x-\cdot)\|_{L^\infty}^{1-p} |F|^p * |\varphi|^p(x).
\]

Since the Fourier transform of \( F\varphi(x-\cdot) \) is supported in \( \mathbb{R}^d(0,5) \), \( F\varphi(x-\cdot) = (F\varphi(x-\cdot))^*5^4\varphi(5\cdot) \) and thus \( \|F\varphi(x-\cdot)\|_{L^\infty} \leq C\|F\varphi(x-\cdot)\|_1 = C|F|^p * |\varphi|(x) \). Combining this and the inequality above gives \((|F|^p * |\varphi|)^p \leq C|F|^p * |\varphi|^p \). Since \( |F| \leq |F| * |\varphi| \), we get (2.31) with \( \lambda = 1 \).

\( \square \)

**Proof of Proposition 2.18** We set

\[
a(s, t, \xi) = \tilde{\chi}(t)\psi_J(s)(1 - \tilde{\chi}_{R_J}(\xi))\beta(\lambda^{-1}|\xi|),
\]

so that \( A'[a]f = \tilde{\chi}(t)A'[\psi_J](1 - P_J)f \). We claim that (2.26) holds on the support of \( a \) with \( K = C_1 \delta^{-2} > 0 \), \( C_1 = C_1(c_\delta) \).

To see this, it suffices to consider the case \( \xi \in A^{*}_\lambda \) because of (2.20). We first note that \( \xi \in R_J(2^5) \) if \( \sigma(\xi) \in [c_J - |J|, c_J + |J|] \) and \( |R_J(\xi)| \leq 2^3 c_\delta^2 \delta^2 \).

Indeed, since \( |\sigma(\xi) - c_J| \leq 2c_\delta \delta \), we have \( |\gamma'(c_J) \cdot \xi| \leq 2^5 c_\delta^2 \delta^2 \lambda \) by (2.21) and \( |\gamma''(c_J) \cdot \xi| \leq 2^3 c_\delta \delta^2 \) by (2.22). So, it follows \( \xi \in R_J(2^5) \) since \( \xi \in A^{*}_\lambda \). Thus, if \( \xi \in \text{supp}(1 - \tilde{\chi}_{R_J})\beta(\lambda^{-1}|\cdot|) \cap A^{*}_\lambda \), we have \( \sigma(\xi) \notin [c_J - |J|, c_J + |J|] \) or \( |R_J(\xi)| \geq 2^3 c_\delta^2 \). In the first case, by (2.22) we see \( |\gamma''(s) \cdot \xi| \geq 2^2 c_\delta \delta^2 \) for all \( s \in \text{supp} \psi_J \). So, we may assume \( |\xi| \leq 3c_\delta \delta \) and \( |R_J(\xi)| \geq 2^3 c_\delta^2 \).

Then we get \( |\gamma'(s) \cdot \xi| \geq 2c_\delta \delta^2 \). Using (2.21). This shows the claim.

Since (2.26) holds on the support of \( a \), by Corollary 2.16 we have the estimate

\[
\|\tilde{\chi}(t)A'[\psi_J](1 - P_J)f\|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq C_\delta \delta^{-C} \lambda^{-\frac{1}{2} - \frac{1}{2p} + \epsilon} \|f\|_{L^p(\mathbb{R}^3)} \tag{2.32}
\]

for \( 2 \leq p \leq 6 \). We use the estimate to obtain the weighted estimate (2.29). Since the argument is similar to that in the proof of Lemma 2.4, we shall be brief.

As before, let us define an operator \( \tilde{A}_J \) by

\[
\mathcal{F}(\tilde{A}_J h)(\xi, \tau) = \beta_0((\lambda r_0)^{-1} \tau) \beta(\lambda^{-1}|\xi|) \mathcal{F}(\tilde{\chi}(t)A'[\psi_J]h)(\xi, \tau),
\]

where \( r_0 = 1 + 4 \max\{|\gamma(s)| : s \in \text{supp} \psi_J\} \). Then we have \( |(\tilde{\chi}(t)A'[\psi_J] - \tilde{A}_J)h| \leq C\tilde{E}^N_t * |h| \) for any \( N \) if we use Lemma 2.6. Putting together this (e.g., (2.7)), \( [\omega]_3 \leq 1 \) and \( \|(1 - P_J)f\|_p \leq C \|f\|_p \), we see that

\[
\|\tilde{\chi}(t)A'[\psi_J](1 - P_J)f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}, \omega)} \leq \|\tilde{A}_J(1 - P_J)f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}, \omega)} + C\lambda^{-N} \|f\|_p.
\]
The Fourier transform of $\tilde{A}_f(1 - P_J)f$ is supported in $\mathbb{R}^4(0, 2r_0\lambda)$. By Lemma 2.19 we have $\|\tilde{A}_f(1 - P_J)f\|_{L^p(\mathbb{R}^3 \times \mathbb{R}, \omega)} \leq C\lambda^{1/p}\|\tilde{A}_f(1 - P_J)f\|_{L^p(\mathbb{R}^3 \times \mathbb{R})}$. Disregarding the minor contribution of $(\tilde{\chi}(t)A^\gamma[\psi_J] - \tilde{A}_f)(1 - P_J)f$, we only need to consider $\tilde{\chi}(t)A^\gamma[\psi_J](1 - P_J)f$ in $L^p(\mathbb{R}^3 \times \mathbb{R})$. Therefore we obtain the estimate (2.29) by (2.32).

**Proposition 2.20** Under the same assumption as in Proposition 2.18, if $2 \leq p \leq 6$ and $\varepsilon_0 > 0$ is small enough, for any $\varepsilon > 0$,

$$
\|A^\gamma[\psi_J]P_e f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C\varepsilon^{-C\lambda^{1/p}}\|f\|_{L^p(\mathbb{R}^3)}
$$

holds whenever supp $\hat{f} \subset \mathbb{A}_\lambda$, $\gamma \in C^D(\varepsilon_0)$ and $\psi_J \in \mathbb{H}^D(J)$.

**Proof** We set $\pi_1^J(\xi) = \tilde{\chi}_{A_\lambda^*}(\xi)(1 - \beta^0(\delta^{-100}|R_\gamma(\xi)|))$ and $\pi_2^J(\xi) = \beta(\lambda^{-1}|\xi|) - \tilde{\chi}_{A_\lambda^*}(\xi)$, so that $\pi_1 = \pi_1^1 + \pi_2^1$. Then we break $\tilde{\chi}(t)A^\gamma[\psi_J]P_e f = A^\gamma[a_1^J]f + A^\gamma[a_2^J]f$, where

$$
a_j^J(s, t, \xi) = \tilde{\chi}(t)\psi_J(s)\pi_j(\xi), \quad j = 1, 2.
$$

We first consider $A^\gamma[a_1^J]f$. After decomposing $\psi_J$ into the bump functions $\psi_\ell$ supported in finitely overlapping intervals $J_\ell$ such that $\Delta^{100} \leq |J_\ell| \leq 2\Delta^{100}$, $\psi_J = \sum_\ell \psi_\ell$, and $|\psi_\ell^{(k)}| \leq C\varepsilon\delta^{-100}$, we set $a_1^J(s, t, \xi) = \tilde{\chi}(t)\psi_\ell(s)\pi_1^\ell(\xi)$. By (2.22) $|\gamma''(s) \cdot \xi| \geq 2^{-3}\lambda\delta^{100}$ for $s \in \text{supp} \psi_\ell$ if $\sigma(\xi) \notin [c_1 - |J_\ell|, c_1 + |J_\ell|]$. Otherwise, from (2.21) we have $|\gamma'(s) \cdot \xi| \geq 2^2\delta^{100}\lambda$ for $s \in \text{supp} \psi_\ell$ since $|R_\gamma(\xi)| \geq \delta^{100}$ on $\text{supp} \pi_1$. Therefore (2.26) holds with $K = C\delta^{-100}$ for $(s, t, \xi) \in \text{supp}a_1^\ell$. Applying Corollary 2.16 with $a = a_1^\ell$ gives

$$
\|A^\gamma[a_1^J]f\|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq C\varepsilon^{-C\lambda^{-\frac{1}{2}}\frac{1}{2p^+}}\|f\|_{L^p(\mathbb{R}^3)}.
$$

Arguing similarly as in the proof of 2.18, we get the weighted estimate $\|A^\gamma[a_1^J]f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C\varepsilon^{-C\lambda^{-\frac{1}{2}}\frac{1}{2p^+}}\|f\|_{L^p(\mathbb{R}^3)}$. Summation over $\ell$ thus gives the desired estimate since there are at most $C\delta^{-100}$ many $\ell$.

The estimate $\|A^\gamma[a_2^J]f\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C\varepsilon^{-C\lambda^{-\frac{1}{2}}\frac{1}{2p^+}}\|f\|_{L^p(\mathbb{R}^3)}$ can be obtained likewise but more straightforwardly since $|\gamma'(s) \cdot \xi| + |\gamma''(s) \cdot \xi| \geq c_0|\xi|$ on $\text{supp} a_2^J$.

## 3 Multilinear estimates

The main object of this section is to prove the following weighted multilinear (quadrilinear) estimate for $A^\gamma[\psi_J]P_f f$. Throughout this section we assume $\gamma \in C^D(\varepsilon_0)$ with an $\varepsilon_0$ small enough.

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Proposition 3.1 Let $J_k \in J_\circ(\delta)$, $1 \leq k \leq 4$, and $|\omega|_3 \leq 1$. Suppose that $\hat{f}_1, \ldots, \hat{f}_4$ are supported in $A_\lambda$ and $\text{dist}(J_\ell, J_k) \geq \delta$, $\ell \neq k$. If $14/5 < p \leq 6$, there are constants $\varepsilon_p > 0$, $D$, and $C_\delta > 0$ such that

$$\left\| \prod_{k=1}^{4} A^\gamma[\psi_{J_k}](P_nP_{J_k}f_k) \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \leq C_\delta \lambda^{-\varepsilon_p} \prod_{k=1}^{4} \| f_k \|_{L^p(\mathbb{R}^3)}$$

(3.1)

holds whenever $\gamma \in C^D(\varepsilon_\circ)$ and $\psi_{J_k} \in \mathfrak{M}^D(J_k)$, $1 \leq k \leq 4$.

3.1 Expansions of the multiplier

In order to prove Proposition 3.1 we first try to express $A^\gamma[\psi_{J_k}](P_n)$ as a sum of adjoint restriction operators. To do so, we expand the Fourier multiplier of the operator $A^\gamma[\psi_{J_k}](P_n)$ into a series of suitable form. We handle separately $A^\gamma[\psi_{J_k}](P_n)$ (Lemma 3.2) and $A^\gamma[\psi_{J_k}](P_0^j)$, $j = 1, 0$ (Lemma 3.4). The estimates in Lemma 3.2 and 3.4 are somewhat rough, but they are good enough for our purpose. So we do not attempt to make them as efficient as possible.

Multiplier of $A^\gamma[\psi_J]P_n$

Let $J \in J_\circ(\delta)$. For $\psi_J \in \mathfrak{M}^D(J)$ we set

$$m_J(t, \xi) = (2\pi)^{-3} \int e^{-it\gamma(s) \cdot \xi} \psi_J(s) \, ds.$$ 

The worst decay in $\xi$ of the multiplier $m_J(t, \cdot)$ is related to the behavior of $\gamma(s) \cdot \xi$ near $s = \sigma(\xi)$. We define

$$\Phi^c(\xi) = \gamma(\sigma(\xi)) \cdot \xi, \quad \xi \in A^*_\lambda,$$

and an adjoint restriction operator $T^c_\lambda$ by setting

$$T^c_\lambda g(x, t) = \int_{C^c_\lambda(\delta)} e^{i(x \cdot \xi - t\Phi^c(\xi))} g(\xi) \, d\xi,$$

where $C^c_\lambda(\delta) = \{ \xi \in A^*_\lambda : |R\gamma(\xi)| \leq 2\delta^{100} \}$. We note that $\text{supp} \pi_c \subset C^c_\lambda(\delta)$.

Lemma 3.2 Let $0 < \nu \ll 1$ and $J \in J_\circ(\delta)$. Suppose $\gamma \in C^D(\varepsilon_\circ)$, $\psi_J \in \mathfrak{M}^D(J)$, and $\hat{f}$ is supported on $A_\lambda$. Then we have

$$A^\gamma[\psi_J]P_c f = \sum_{\ell \in \mathbb{Z}} \int e^{ite\ell} T^c_\lambda(c_\ell \pi_c \hat{f}) + \mathcal{E}_c f, \quad t \in I,$$

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and the following hold with $C$, $C_N$, and $C_\delta$ independent of $\gamma$ and $\psi_J$:

$$|c_\ell(\xi)| \leq C_N \lambda^{v - \frac{1}{2}} (1 + \lambda^{-3v} |\ell|)^{-N} \quad (3.2)$$

for any $N$ and

$$\|\mathcal{E}c_{\ell}f\|_{L^q(\mathbb{R}^3 \times I)} \leq C_\delta \lambda^{C - \frac{3}{2}vD} \|f\|_p, \quad 1 \leq p \leq q \leq \infty. \quad (3.3)$$

Summation over $\ell$ results from the Fourier series expansion in $t$ of an amplitude function which appears after factoring out $e^{-it\Phi^c(\xi)}$. It simplifies the amplitude function depending both on $\xi$ and $t$ which causes considerable loss in its bound when we attempt to directly apply the multilinear restriction estimate (for example, see [4, Theorem 6.2]).

For the proof of Lemma 3.2 and 3.4 below, we write $m_J(t, \xi)$ in a different form. Changing of variables $s \rightarrow s + \sigma(\xi)$, we have

$$m_J(t, \xi) = (2\pi)^{-3} e^{-it\Phi^c(\xi)} \int e^{-it\phi(s, \xi)} \psi_J(s + \sigma(\xi)) ds, \quad (3.4)$$

where

$$\phi(s, \xi) := \gamma(s + \sigma(\xi)) \cdot \xi - \gamma'(\sigma(\xi)) \cdot \xi.$$ 

We here note that $J \subset (1 + 2c_o)J_o$ and $|\gamma(\xi)| \leq 5c_o$ for $\xi \in \Lambda^*_\lambda$ by Lemma 2.12. Thus $\phi \in C^{D-2}([-1/2, 1/2] \times \Lambda^*_\lambda)$ and supp$\psi_J(\cdot + \sigma(\xi)) \subset 2^3 J_o$. Since $\gamma \in C^D(\varepsilon_o)$ and $\gamma''(\sigma(\xi)) \cdot \xi = 0$, by Taylor expansion of $\phi(\cdot, \xi)$ around $s = 0$ it follows that

$$\phi(s, \xi) = \Lambda_\gamma(\xi)(-R_\gamma(\xi)s + \frac{1}{6}s^3 + \Theta(s, \xi)), \quad (3.5)$$

$$|\partial^k_\xi \Theta(s, \xi)| \leq C_k \varepsilon_o |s|^{\max(4-k, 0)}, \quad 0 \leq k \leq D. \quad (3.6)$$

In what follows we occasionally resort to (3.5) and (3.6) to exploit the properties of the phase function $\phi(\cdot, \xi)$.

**Proof of Lemma 3.2** We need to consider $m_J(t, \xi)$ while $\xi \in \text{supp}\pi_c$. We break

$$\psi_J(s + \sigma(\xi)) = a_m(s, \xi) + a_e(s, \xi),$$

where $a_m(s, \xi) = \psi_J(s + \sigma(\xi))\beta_0(2^{-4\lambda/3 - v}s)$. Then we put

$$\mathcal{I}_\theta(t, \xi) = (2\pi)^{-3} \int e^{-it\phi(s, \xi)} a_\theta(s, \xi) ds, \quad \theta \in \{m, e\}.$$
By (3.4) it follows that

\[ m_J(t, \xi) = e^{-ir\Phi^e(\xi)}(I_m(t, \xi) + I_e(t, \xi)). \]

The major term is \( I_m \) while \( I_e \) decays fast as \( \lambda \to \infty \). Let \( \chi_o \in C_0^\infty([0, 2\pi]) \) such that \( \chi_o = 1 \) on the interval \([2^{-1}, 2^2]\). Expanding \( \chi_o I_m(\cdot, \xi) \) into Fourier series over the interval \([0, 2\pi]\) we have

\[ \chi_o(t)I_m(t, \xi) = \sum_{\ell \in \mathbb{Z}} c_\ell(\xi)e^{it\lambda \ell}. \]

Note that \( F(\chi_o I_m(\cdot, \xi))(\ell) = (2\pi)^{-3} \int \hat{\chi}_o(\ell + \phi(s, \xi))a_m(s, \xi)ds \). Since \(|\phi(s, \xi)| \leq C\lambda^{-3v} \) on \( \text{supp} a_m(\cdot, \xi) \) by (3.5), we have \(|F(\chi_o I_m(\cdot, \xi))(\ell)| \leq C\lambda^{-\frac{1}{2}}|\ell|^{-N} \) for any \( N \) if \(|\ell| \geq C_1\lambda^{-3v} \) for a large \( C_1 \). Thus we get (3.2) for any \( N \). We also note that \(|\partial^\alpha \phi| \leq C \) and \(|\partial^\alpha a_m| \leq C_\delta \) because \(|\partial^\alpha \sigma| \leq C\lambda^{-|\alpha|} \) on \( A_\delta^\varepsilon \) for \(|\alpha| \leq D - 2 \) (see Lemma 2.12). By the same argument we obtain, for any \( N > 0 \),

\[ |\partial^\alpha c_\ell(\xi)| \leq C_\lambda^{-\frac{1}{2}}(1 + \lambda^{-3v}|\ell|)^{-N}. \tag{3.7} \]

We now put

\[ E_\varepsilon g(x, t) = (2\pi)^3 \sum_{|\ell| > \lambda^{10v}} e^{it\ell} F^{-1}_\lambda(c_\ell e^{-ir\Phi^e}\pi e \hat{g}) + (2\pi)^3 F^{-1}_\lambda(I_e(t, \cdot)e^{-ir\Phi^e}\pi e \hat{g}). \]

We shall show (3.3) to complete the proof. The terms \( F^{-1}_\lambda(c_\ell e^{-ir\Phi^e}\pi e \hat{g}) \) in the sum can be handled easily. Combining the estimate (3.7) and \(|\partial^\alpha e^{-ir\Phi^e}| \leq C \) for \(|\alpha| \leq 4 \), we see that \( F^{-1}_\lambda(c_\ell e^{-ir\Phi^e}\pi e \hat{g}) = K_\ell \ast \mathcal{P} \varepsilon g \) and \(|K_\ell| \leq C_\delta\lambda^{-C}(1 + \lambda^{-3v}|\ell|)^{-N}(1 + |x|)^{-4} \). Thus, the convolution inequality gives

\[ \|F^{-1}_\lambda(c_\ell e^{-ir\Phi^e}\pi e \hat{g})\|_{L^q(\mathbb{R}^3 \times I)} \leq C_\delta\lambda^{-C}(1 + \lambda^{-3v}|\ell|)^{-N}\|\mathcal{P} \varepsilon g\|_p \]

for \( 1 \leq p \leq q \leq \infty \). Taking a large \( N \geq D \) and using (2.24), we obtain

\[ \sum_{|\ell| > \lambda^{10v}} \|F^{-1}_\lambda(c_\ell e^{-ir\Phi^e}\pi e \hat{g})\|_{L^q(\mathbb{R}^3 \times I)} \leq C_\delta\lambda^{C-2vD}\|g\|_p. \]

In order to show the estimate for \( F^{-1}_\lambda(I_e(t, \cdot)e^{-ir\Phi^e}\pi e \hat{g}) \) we claim

\[ |\partial^\alpha I_e(t, \xi)| \leq C_\delta\lambda^{-\frac{3v}{2}D}|\alpha|, \quad |\alpha| \leq 4 \tag{3.8} \]

for \( \xi \in \text{supp} \pi_e \). Using (3.8), similarly as before, we see \( F^{-1}_\lambda(I_e(t, \cdot)e^{-ir\Phi^e}\pi e \hat{g}) = K_t \ast \mathcal{P} \varepsilon g \) with \(|K_t| \leq C_\delta\lambda^{C-\frac{3v}{2}D}(1 + |x|)^{-4} \). Therefore, the convolution inequality and (2.24) give
\[ \left\| \mathcal{F}_x^{-1}(\mathcal{I}_e(t, \cdot)e^{-it\Phi}(\pi \xi)\widehat{g}) \right\|_{L^p(\mathbb{R}^3 \times I)} \leq C_5 \lambda^{C_3^{-\frac{3}{2}}vD} \|g\|_p, \quad 1 \leq p \leq q \leq \infty. \]

Now it remains to show (3.8). We recall \( a_e(s, \xi) = \psi_j(s + \sigma(\xi))(1 - \beta_0(2^{-4}4^{\frac{1}{3}} - v)s)) \). Since \(|s| \geq 2^4 \lambda^{v - \frac{1}{3}}\) on \( \text{supp} a_e(\cdot, \xi) \) and \(|R_\gamma(\xi)| \leq 2^4 \lambda^{-\frac{3}{2}v - \frac{3}{2}}\) for \( \xi \in \text{supp} \pi_c \), by (3.5) and (3.6) it follows that \( C_1 \lambda|s|^2 \leq |\partial_\xi \phi(s, \xi)| \leq C_2 \lambda|s|^2 \) and

\[ C_3 \lambda|s|^{3-k} \leq |\partial^k_\xi \phi(s, \xi)| \leq C_4 \lambda|s|^{3-k}, \quad k = 2, 3, \]
\[ |\partial^k_\xi \phi(s, \xi)| \leq C_5 \varepsilon_0 \lambda, \quad 4 \leq k \leq D \]

for some positive constants \( C_1, \ldots, C_5 \). Thus, noting \(|\partial^k_\xi a_e(s, \xi)| \leq C_6 \lambda^{(\frac{1}{3} - v)k}\) for \( 0 \leq k \leq D \), we have

\[ b_{\ell+1} := \frac{|\partial^{\ell+1}_\xi \phi(s, \xi)|}{|\partial^\ell_\xi \phi(s, \xi)|^{\ell+1}} \leq C_6 \lambda^{-\frac{3}{2}v(\ell+1)}, \]
\[ b'_\ell := \frac{|\partial^\ell_\xi a_e(s, \xi)|}{|\partial^\ell_\xi \phi(s, \xi)|^{\ell}} \leq C_6 \lambda^{-3v\ell} \]

for \( \ell \geq 1 \) if \( \xi \in \text{supp} \pi_c \) and \(|s| \geq 2^4 \lambda^{v - \frac{1}{3}}\). After integration by parts \( D - 1 \) times we see that \(|\mathcal{I}_e(t, \xi)|\) is bounded by a finite sum of the integrals

\[ C \int \prod_{j=1}^m M_{\ell_j} ds \text{ where } M_{\ell_j} \in \{b_{\ell_j+1}, b'_j\}, \sum_{j=1}^m \ell_j = D - 1, \text{ and } \ell_j \geq 1. \]

Using (3.10) we get \(|\mathcal{I}_e(t, \xi)| \leq C_6 \lambda^{3/2vD}\) for \( \xi \in \text{supp} \pi_c \). Furthermore, since \( \partial_\xi^k \partial_\xi^\alpha \phi, \alpha \neq 0 \) are bounded, the same argument shows (3.8). \( \square \)

**Multipliers of** \( A^Y[\psi_j]P^1_0 \) **and** \( A^Y[\psi_j]P^0_0 \)

We obtain similar expansions for \( m_j \pi^j_0 \), \( j = 0, 1 \). As we shall see, \( m_j \pi^0_0 \) is decaying rapidly as \( \lambda \to \infty \) (see (3.21) below). We concentrate on the case \( \xi \in \text{supp} \pi^0_0 \) for the moment.

Let \( \rho_1 \in C^\infty([-4^2, 4^2]), \rho_0 \in C^\infty_c([0, 2^{-4}]), \) and \( \rho_2 \in C^\infty_c((2^4, \infty)) \) such that \( \rho_1 = 1 \) on \([-4^2, 2^4]\) and \( \rho_0 + \rho_1 + \rho_2 = 1 \) on \([0, \infty)\). For \( j = 0, 1, 2, \) we set

\[ a_j(s, \xi) = \psi_j(s + \sigma(\xi))\rho_j \left(R^{-1/2}_\gamma(\xi)|s|\right), \]
\[ \mathcal{I}_j(t, \xi) = (2\pi)^{-3} \int e^{-it\Phi(s, \xi)} a_j(s, \xi) ds, \]

and then we have

\[ m_j(t, \xi) = e^{-i t \Phi_\xi(\xi)} \left( \mathcal{I}_0(t, \xi) + \mathcal{I}_1(t, \xi) + \mathcal{I}_2(t, \xi) \right). \]
The main term is $I_1$ while $I_0$ and $I_2$ are rapidly decaying as $\lambda \to \infty$ (see (3.22) below). The second derivative of the phase function does not vanish on $\text{supp}a_1(\cdot, \xi)$, so we may apply the method of stationary phase for $I_1(t, \xi)$. For the purpose we set

$$\tilde{\phi}(s, \xi) = L^{-1}(\xi)\phi(R_1^{1/2}(\xi)s, \xi)$$

(3.12)

where $L(\xi) = \Lambda_\gamma(\xi)R_1(\xi)^{3/2}$, and set

$$a^\pm(s, \xi) = \psi J(R_1/2)(s + \sigma(\xi))\rho_1(\pm s),$$

$$\mathcal{I}_1^\pm(t, \xi) = (2\pi)^{-3}R_1^{1/2}(\xi)\int e^{-itL(\xi)\tilde{\phi}(s, \xi)}a^\pm(s, \xi)ds.$$

By scaling $s \to R_1/2$ we have

$$I_1(t, \xi) = I_1^+(t, \xi) + I_1^-(t, \xi).$$

(3.13)

We try to find the stationary points of the function $\tilde{\phi}(\cdot, \xi)$, which give rise to two different phase functions $\Phi^\pm$ (see (3.15) below). As we shall see later, it is important for application of the multilinear restriction estimate how smooth these phase functions are. So, we treat the matter carefully.

**Lemma 3.3** There are $\tau_+, \tau_- \in C^{D-4}(A_\lambda^* \times [-\delta^{10}, \delta^{10}])$, homogeneous of degree zero, such that $\pm \tau_\pm(\xi, \theta) \in [2^{-1}, 2]$ and, if $R_\gamma(\xi) \geq 0$,

$$\partial_s\tilde{\phi}(\tau_\pm(\xi, R_\gamma^{1/2}(\xi)), \xi) = 0.$$

(3.14)

**Proof** Recalling (3.5), we set

$$\Theta_0(s, \xi) = s^{-3}\Theta(s, \xi),$$

which is homogeneous of degree zero in $\xi$. One can see $\Theta_0 \in C^{D-3}([-1/2, 1/2] \times A_\lambda^*)$ without difficulty because $\Theta_0(s, \xi) = (s/3!) \int_0^1 (1 - t)^3\gamma^{(4)}(st + \sigma(\xi)) \cdot \xi \Lambda_\gamma^{-1}(\xi)dt$ by Taylor’s theorem with integral remainder. Then we consider the function

$$\tilde{\phi}_0(s, \xi, \theta) = -s + \frac{s^3}{3!} + s^3\Theta_0(\theta s, \xi)$$

with $(s, \xi, \theta) \in \Omega^\pm := (\pm[2^{-5}, 2^5]) \times A_\lambda^* \times [-\delta^{10}, \delta^{10}]$. It is clear that $\tilde{\phi}_0 \in C^{D-3}(\Omega^\pm)$.

Since $\Theta_0, \partial_s\Theta_0$ and $\partial_s^2\Theta_0$ are $O(\varepsilon_0)$ as can be seen using (3.5) and (3.6), we have $\partial_s\tilde{\phi}_0(s, \xi, \theta) = -1 + s^2/2 + O(\varepsilon_0)$ and $\partial_s^2\tilde{\phi}_0(s, \xi, \theta) = s + O(\varepsilon_0)$.
We now note that \( \partial_s \tilde{\phi}_0(\cdot, \xi, \theta) \) has two distinct zeros which are respectively close to \( \sqrt{2} \) and \( -\sqrt{2} \), thus by the implicit function theorem there are \( \tau_+(\xi, \theta) \) and \( \tau_-(\xi, \theta) \) such that \( \partial_s \tilde{\phi}_0(\tau_\pm(\xi, \theta), \xi, \theta) = 0 \) and \( \pm \tau_\pm(\xi, \theta) \in [2^{-1}, 2] \) if \( \varepsilon_0 \) is small enough. Additionally, \( \tau_+ \) and \( \tau_- \) are \( D - 4 \) times continuously differentiable since so is \( \partial_s \tilde{\phi}_0 \). By (3.5) and (3.12) we have \( \tilde{\phi}_0(s, \xi, R_\gamma^{1/2}(\xi)) = \tilde{\phi}(s, \xi) \), thus it follows that \( \partial_s \tilde{\phi}_0(s, \xi, R_\gamma^{1/2}(\xi)) = \partial_s \tilde{\phi}(s, \xi) \) when \( R_\gamma(\xi) \geq 0 \). Therefore we obtain (3.14).

We set
\[
s_\pm(\xi) = R_\gamma^{1/2}(\xi) \tau_\pm(\xi, R_\gamma^{1/2}(\xi)).
\]
Then from (3.12) it follows that \( \gamma'(s_\pm(\xi) + \sigma(\xi)) \cdot \xi = 0 \). We define
\[
\Phi^\pm(\xi) = \gamma(s_\pm(\xi) + \sigma(\xi)) \cdot \xi
\] (3.15)
for \( \xi \in \mathbb{A}_0^* \cap \{ \xi : R_\gamma(\xi) \geq 0 \} \). If \( R_\gamma(\xi) = 0 \) for some \( \xi \), \( \nabla \Phi^\pm(\xi) \) may not exist because \( R_\gamma^{1/2} \) is not differentiable at \( \xi \). However, \( \nabla \Phi^\pm \) can be defined to be a continuous function on \( \mathbb{A}_0^* \cap \{ \xi : R_\gamma(\xi) \geq 0 \} \). Indeed, differentiating (3.15) gives
\[
\nabla \Phi^\pm(\xi) = \gamma'(s_\pm(\xi) + \sigma(\xi))
\] (3.16)
if \( R_\gamma(\xi) > 0 \). Thus \( \nabla \Phi^\pm \) becomes continuous on \( \mathbb{A}_0^* \cap \{ \xi : R_\gamma(\xi) \geq 0 \} \) if we set \( \nabla \Phi^\pm(\xi) = \gamma'(\sigma(\xi)) \) when \( R_\gamma(\xi) = 0 \) since \( \gamma, \sigma \) are continuous.

We define the adjoint restriction operators \( T^\pm_\lambda \) by
\[
T^\pm_\lambda g(x, t) = \int_{C^\theta_\lambda(\delta)} e^{i(x \cdot \xi - t \Phi^\pm(\xi))} g(\xi) \, d\xi,
\]
where \( C^\theta_\lambda(\delta) := \{ \xi \in \mathbb{A}_0^* : 0 \leq R_\gamma(\xi) \leq 2\delta^{100} \} \). Putting together the discussion so far with the method of stationary phase we can obtain

**Lemma 3.4** Let \( 0 < \nu \ll 1 \), \( M = [\frac{D-1}{3}] \), and \( J \in \mathfrak{J}_0(\delta) \). Suppose \( \gamma \in \mathcal{C}^D(\varepsilon_\circ) \), \( \psi_J \in \mathcal{M}^D(J) \), and \( \hat{f} \) is supported on \( \mathbb{A}_\varepsilon^\theta \). Then, we have
\[
A^\gamma[\psi_J](P^1_o + P^0_o) f = \sum_{\pm} \sum_{\ell=0}^{M-1} t^{-\frac{2\ell+1}{2}} T^\pm_\lambda(\gamma^\pm_\ell \pi_o^1 \hat{f}) + \mathcal{E}_o f, \quad t \in I,
\] (3.17)
and the following hold with \( C \) and \( C_\delta \) independent of \( \gamma \) and \( \psi_f \):

\[
|y_\pm^\ell(\xi)| \leq C_\delta \lambda^{-\frac{1}{2} - \frac{\nu}{2}} \lambda^{-3\ell v}
\]

for \( 0 \leq \ell \leq M - 1 \) and

\[
\|E_0 f\|_{L^q(\mathbb{R}^3 \times I)} \leq C_\delta \lambda^{C - 3vM} \|f\|_p, \quad 1 \leq p \leq q \leq \infty.
\]

It should be noted that the expansion in (3.17) is obtained only on the support of \( \pi^1_o \) but not on the larger set \( C_\lambda^0(\delta) \).

We now proceed to apply to \( \mathcal{I}^\pm_1 \) the method of stationary phase. We first note that \( \text{supp} a^\pm(\cdot, \xi) \subset \pm [2^{-5}, 2^5] \) and, as seen in the above, the phase \( \tilde{\phi}(\cdot, \xi) \) has the stationary points \( \tau_\pm(\xi, R_\gamma^{1/2}(\xi)) \) while \( \partial_s^2 \tilde{\phi}(\cdot, \xi) = s + O_3(\varepsilon_3) \) for \( \xi \in \Lambda_\gamma^\pm \cap \{ \xi : 0 \leq R_\gamma(\xi) \leq 2\delta^{100} \} \). We also note that \( |L(\xi)| \geq 2^{-1} \lambda^{3v} \) for \( \xi \in \text{supp} \pi^1_o \) and that \( L(\xi) \tilde{\phi}(\tau_\pm(\xi, R_\gamma^{1/2}(\xi)), \xi) = \gamma(s_\pm(\xi) + \sigma(\xi)) \cdot \xi - \Phi^\varepsilon(\xi) \).

Bring all these observations together, we now apply [12, Theorem 7.7.5] (also see [12, Theorem 7.7.6]) and obtain

\[
\mathcal{I}^\pm_1(t, \xi) = e^{it(\Phi^\varepsilon(\xi) - \Phi^\pm(\xi))} R_\gamma^{1/2}(\xi) \sum_{\ell=0}^{M-1} d^\pm_\ell(\xi)(tL(\xi))^{-\frac{1}{2} - \ell} + e^\pm_M(t, \xi)
\]

for \( \xi \in \text{supp} \pi^1_o \) where \( M = \lceil \frac{D-1}{3} \rceil \) and \( e^\pm_M(t, \xi) = O(|tL(\xi)|^{-M}) \). The functions \( d^\pm_\ell \) are bounded on the support of \( \pi^1_o \) since so are \( \partial_{\xi}^k \tilde{\phi} \) and \( \partial_{\xi}^k a^\pm \).

**Proof of Lemma 3.4** Recalling (3.11) and (3.13), we write

\[
m_J(\pi^1_o + \pi^0_o) = e^{-it\Phi^\varepsilon}(\mathcal{I}^+_1 + \mathcal{I}^-_1)\pi^1_o + e^{-it\Phi^\varepsilon}(\mathcal{I}_0 + \mathcal{I}_2)\pi^1_o + m_J\pi^0_o.
\]

Using (3.20), we now put

\[
E(t, \cdot) = e^{-it\Phi^\varepsilon}(e^+_M(t, \cdot) + e^-_M(t, \cdot))\pi^1_o + e^{-it\Phi^\varepsilon}(\mathcal{I}_0(t, \cdot) + \mathcal{I}_2(t, \cdot))\pi^1_o + m_J(t, \cdot)\pi^0_o,
\]

and then we set \( E_0 f = (2\pi)^3 F_{\xi}^{-1}(E(t, \cdot) \hat{f}) \) and \( \gamma_\ell^\pm(\xi) = R_\gamma^{1/2}(\xi) d_\ell^\pm(\xi) (L(\xi))^{-\frac{1}{2} - \ell} \). Thus we have (3.17). Recalling \( L(\xi) = \Lambda_\gamma(\xi) R_\gamma(\xi)^{\frac{3}{2}} \), we see (3.18) holds since \( C\lambda^{2v - 2/3} \leq |R_\gamma(\xi)| \) and \( d_\ell^\pm \) are bounded on \( \text{supp} \pi^1_o \).

To show (3.19) we use the following:

\[
|\partial_{\xi}^\alpha m_J(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2}v(D-|\alpha|)}, \quad \xi \in \text{supp} \pi^0_o,
\]
and
\[
|\tilde{\partial}_\xi^\alpha \mathcal{I}_0(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2} \nu(D - |\alpha|)}, \quad \xi \in \text{supp} \pi_0^1.
\]
\[
|\tilde{\partial}_\xi^\alpha \mathcal{I}_2(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2} \nu(D - |\alpha|)},
\]
Assuming this for the moment we obtain (3.19). Note that \( |\tilde{\partial}_\xi^\alpha \Phi^c| \leq C \lambda^{1 - |\alpha|} \) and \( |\tilde{\partial}_\xi^\alpha e_M^\pm| \leq C_1 \lambda^{C - 3 \nu} M \) for \( |\alpha| \leq 4 \). Combining this, (3.21) and (3.22) for \( |\alpha| \leq 4 \) and using the estimate (2.24), we get (3.19) in the same manner as before.

To complete the proof, we are left to prove (3.21) and (3.22). Let us first consider (3.21), which is easier. Since \( R_\gamma(\xi) \leq -\lambda^{2v - \frac{2}{3}} \) for \( \xi \in \text{supp} \pi_0^1 \), by (3.5) we see that \( |\tilde{\partial}_\xi \phi| \geq C_1 \lambda \left( \gamma(R_\gamma(\xi) + s^2(1/2 - e_o|s|)) \right) \geq C_2 \lambda \max(s^2, \lambda^{2v - \frac{2}{3}}) \) for some \( C_1, C_2 > 0 \). Combining this with (3.9), we have (3.10) for \( \ell \geq 1 \) with \( \gamma(s, \xi) \) replaced by \( \gamma(s + \sigma(\xi)) \). Thus integration by parts gives \( |m_J(t, \xi)| \leq C_\delta \lambda^{-\frac{3}{2} \nu D} \) since \( R_\gamma(\xi) \leq -\lambda^{2v - \frac{2}{3}} \). The same argument also works for \( |\tilde{\partial}_\xi^\alpha m_J(t, \xi)| \), so we obtain (3.21).

We now show (3.22) only with \( \alpha = 0 \), and the derivatives \( \tilde{\partial}_\xi^\alpha \mathcal{I}_0 \) and \( \tilde{\partial}_\xi^\alpha \mathcal{I}_2 \) can be handled likewise. We consider \( \mathcal{I}_0 \) first. By (3.5) we have \( |\tilde{\partial}_\xi \phi| \geq C_\lambda R_\gamma(\xi) \) for \( |s| \leq 2^{-4} R_\gamma^{-1/2}(\xi) \). Combining this with (3.9), we get the first estimate in (3.10) for \( \ell \geq 1 \) when \( |s| \leq 2^{-4} R_\gamma^{-1/2}(\xi) \) because \( \lambda^{2v - \frac{2}{3}} \leq R_\gamma(\xi) \) for \( \xi \in \text{supp} \pi_0^1 \). Note that \( |\tilde{\partial}_\xi^\alpha a_0(s, \xi)| \leq C_\delta R_\gamma^{-\ell/2}(\xi) \), hence for \( \ell \geq 1 \) we have the second estimate in (3.10) with \( a_e \) replaced by \( a_0 \). Therefore repeated integration by parts gives the estimate for \( \mathcal{I}_0 \). We can handle \( \mathcal{I}_2 \) in the same manner. Since \( |s| \geq 2^{-4} R_\gamma^{-1/2}(\xi) \), by (3.5) we have \( C_1 \lambda |s|^2 \leq |\tilde{\partial}_\xi \phi(s, \xi)| \leq C_2 \lambda |s|^2 \) and obviously \( |\tilde{\partial}_\xi^\alpha a_2(s, \xi)| \leq C_\delta R_\gamma^{-\ell/2}(\xi) \). So, we get the estimate (3.10) for \( |s| \geq 2^{-4} R_\gamma^{-1/2}(\xi) \) and \( \ell \geq 1 \) while \( a_e \) is replaced by \( a_2 \). Thus integration by parts gives the estimate for \( \mathcal{I}_2 \).

In contrast to \( \Phi^c \) the 2nd derivatives of \( \Phi^\pm \) are no longer bounded. However, a computation with \( \gamma = \gamma_o \) leads us to expect that \( \Phi^\pm \in C^{1,1/2} \). What follows shows this holds true for \( \gamma \in \mathcal{C}^D(\epsilon_o) \).

**Lemma 3.5** For \( \xi_1, \xi_2 \in C^0(\delta) \), there is a constant \( C \) independent of \( \gamma \) such that
\[
|\nabla \Phi^\pm(\xi_1) - \nabla \Phi^\pm(\xi_2)| \leq C|\xi_1 - \xi_2|^{\frac{1}{2}}.
\]

\[\text{If } \gamma = \gamma_o, \Phi^c(\xi) = -\xi_1 \xi_2 / \xi_3 + \xi_3^3 / (3 \xi_3^2) \quad \text{and} \quad \Phi^\pm(\xi) = \Phi^c(\xi) + 3^{-1} \xi_3(\xi_2^2 / \xi_3 - 2 \xi_1 / \xi_3)^{3/2}.\]
Proof Let us set \( \tau_{0}^{\pm}(\xi) = \tau_{\pm}(\xi, R_{1}^{1/2}(\xi)) \), so \( s_{\pm}(\xi) = R_{1}^{1/2}(\xi) \tau_{0}^{\pm}(\xi) \). Using (3.16) and applying the mean value inequality, one can easily see

\[
|\nabla \Phi^\pm(\xi_1) - \nabla \Phi^\pm(\xi_2)| \leq C |s_{\pm}(\xi_1) - s_{\pm}(\xi_2)| + C |\sigma(\xi_1) - \sigma(\xi_2)|.
\]

Since \( \sigma \in C^{D-2}(\mathbb{A}_1^*) \) from Lemma 2.12, we only have to consider the first one on the right hand side, which is in turn bounded by

\[
|R_{1}^{1/2}(\xi_1) - R_{1}^{1/2}(\xi_2)||\tau_{0}^{\pm}(\xi_1)| + R_{1}^{1/2}(\xi_2)|\tau_{0}^{\pm}(\xi_1) - \tau_{0}^{\pm}(\xi_2)|.
\]

It is easy to see \( |R_{1}^{1/2}(\xi_1) - R_{1}^{1/2}(\xi_2)| \leq C |\xi_1 - \xi_2|^{1/2} \). Since \( \tau_{\pm} \) is \( D - 4 \) times continuously differentiable in a region containing \( C_{1}(\delta) \) (Lemma 3.3) and \( \tau_{0}^{\pm}(\xi) = \tau_{\pm}(\xi, R_{1}^{1/2}(\xi)) \), by the mean value inequality it follows that \( |\tau_{0}^{\pm}(\xi_1) - \tau_{0}^{\pm}(\xi_2)| \leq C |R_{1}^{1/2}(\xi_1) - R_{1}^{1/2}(\xi_2)| + C |\xi_1 - \xi_2| \). Consequently, we get the inequality (3.23).

\[\square\]

3.2 Multilinear restriction estimate

In this section, we obtain a form of multilinear restriction estimate, which we need to prove (3.1). The surfaces associated with \( \Phi^c \) and \( \Phi^\pm \) have some curvature property, so it is possible to get an \( L^2-L^q \) smoothing estimate using the typical \( TT^* \) argument. However, the consequent estimate is not so strong enough to be useful for controlling the maximal operator. Instead, we utilize 4-linear estimates which we deduce from the multilinear restriction estimate under transversality assumption ([4]).

Multilinear restriction estimate for \( C^{1,\alpha} \) hypersurfaces

For the adjoint restriction estimate, the surfaces are typically assumed to be compact and twice continuously differentiable. The same assumption was also made for the multilinear restriction estimate in [4, Theorem 1.16], but the phase functions \( \Phi^\pm \) no longer have bounded second derivatives. Nevertheless, it is not difficult to see that the argument in [4] continues to work with \( C^{1,\alpha} \) surface, \( \alpha > 0 \). It seems to the authors that there is no proper reference concerning this matter, so we provide a brief discussion on the multilinear restriction estimate for the less regular \( C^{1,\alpha} \) surfaces.

For \( k = 1, \ldots, d \), let \( U_k \) be a compact subset of an open set \( U'_k \subset \mathbb{B}^{d-1}(0, 2^2) \) and \( \Phi_k \) be a real valued function on \( U'_k \) which satisfies \( \|\Phi_k\|_{C^{1,\alpha}(U'_k)} \leq B \) for some \( 0 < \alpha \leq 1 \). Let us set

\[
T_k g_k(x, t) = \int_{U_k} e^{i(x \cdot \xi - t\Phi_k(\xi))} g_k(\xi) \, d\xi.
\]
Theorem 3.6 Let $d \geq 2$, $\theta \in (0, 1]$, and let $N_k(\xi) = \frac{(\nabla \Phi_k(\xi), 1)}{|(\nabla \Phi_k(\xi), 1)|}$. Suppose $|\det(N_1(\xi_1), \ldots, N_d(\xi_d))| \geq \theta$ for $\xi_k \in U_k$, $k = 1, \ldots, d$. Then, for any $\varepsilon > 0$,
\[
\| \prod_{k=1}^{d} T_k g_k \|_{L^{2-\varepsilon}(B^d(0, R))} \leq C_\varepsilon(\theta) R \prod_{k=1}^{d} \| g_k \|_{L^2(U_k)}
\]
holds whenever $R \geq 1$. The constant $C_\varepsilon(\theta)$ takes the form $C \theta^{-C_\varepsilon}$ for some constants $C, C_\varepsilon > 0$.

When $d = 2$, the theorem holds true with a $C^1$ curve even with a Lipschitz curve, but it is unknown whether the same continues to be true in higher dimensions. Once one makes a couple of crucial observations concerning $C^{1, \alpha}$ surfaces, it is not difficult to prove Theorem 3.6 through routine adaptation of the arguments in [4, Proposition 2.1]. Instead of reproducing them in detail, we provide a sketch of the proof. We refer the reader to [3, 4] for the details.

For the proof of Theorem 3.6, first of all, we observe that
\[
|\Phi_k(\xi + h) - \Phi_k(\xi) - \nabla \Phi_k(\xi) \cdot h| \leq CB|h|^{\alpha + 1}
\]
for $\xi + h, \xi \in U_k$. If $\Phi_k$ is assumed to be in $C^{1, \alpha}(U_k)$ instead of $C^{1, \alpha}(U_k')$, this can not be completely clear. In such a case we need to impose an additional condition such that $U_k$ has a $C^{1, \alpha}$ boundary (e.g. see [8, pp. 136–137]). On the other hand, if $U_k$ is convex, (3.25) is a simple consequence of the mean value theorem. Since $U_k$ is compact, there is a positive number $\rho_k$ such that $x, y$ are contained in a ball which is a subset of $U_k'$ whenever $x, y \in U_k$ and $|x - y| \leq \rho_k$. Therefore, we get (3.25) for $|h| \leq \rho_k$ and this is enough to show (3.25) for any $\xi, \xi + h \in U_k$ because $U_k$ is compact and $\nabla \Phi_k$ is continuous.

Sketch of proof

Let us denote $\Sigma_k = \{ (\xi, -\Phi_k(\xi)) : \xi \in U_k \}$. We consider the estimate
\[
\| \prod_{k=1}^{d} \widehat{G}_k \|_{L^{2-\varepsilon}(B^d(0, R))} \leq C_0 R^{-\frac{d}{2}} \prod_{k=1}^{d} \| G_k \|_{L^2(B^d)}
\]
for $R \geq 1$ when $G_k$ is supported in $\Sigma_k(1/R) := \{ (\xi, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R} : \text{dist}((\xi, \tau), \Sigma_k) < 1/R \}$. The estimate (3.24) is equivalent to (3.26) with $C_0 = CR^\varepsilon$ (see [4]). Let $C(R)$ be the infimum of $C_0$ with which (3.26) holds. The key part of the proof is to establish the implication
\[
C(R) \leq R^b \implies C(R) \leq C(\theta, \varepsilon) R^{\frac{b}{1+\alpha} + \varepsilon}
\]
for any $\varepsilon > 0$ where $b$ is a positive constant. Via iteration, the exponent of $R$ can be suppressed to be arbitrarily small and hence we get the estimate (3.24).

Using (3.25) we see that the set $\Sigma_k(1/R) \cap \mathbb{B}^d(\xi, R^{-1/(1+\alpha)})$, $\xi \in \Sigma_k$ is contained in a $C/R$ neighborhood of the tangent plane to $\Sigma_k$ at $\xi$. Thus $\Sigma_k(1/R)$ can be covered with a collection $\{\mathcal{H}_j\}$ of finitely overlapping rectangles of dimensions about $R^{-1} \times R^{-1/(1+\alpha)} \times \ldots \times R^{-1/(1+\alpha)}$ which are essentially tangential to $\Sigma_k(1/R)$. These rectangles provide a decomposition of $G_k = \sum_j G^k_j$ while $\text{supp} G^k_j \subset \mathcal{H}_j$. Thus, after applying the assumption $C(R) \leq R^b$ to the integrals over the balls of radius $R^{1/(1+\alpha)}$ which finitely overlap and cover $\mathbb{B}^d(0, R)$, one can get the implication (3.27) using the multilinear Kakeya estimate [4,9] for the transversal collection $\mathcal{I}_k$ of the tubes of width $R^{1/(1+\alpha)}$ and length $R$ which have their axes parallel to the normal vector of the surface $\Sigma_k$.

Making use of Theorem 3.6 we obtain the following.

**Proposition 3.7** Let $\theta_1, \ldots, \theta_4 \in \{c, +, -, \}$ and let $J_k \in \mathfrak{J}_o(\delta)$, $1 \leq k \leq 4$. Suppose that $\gamma \in \mathcal{C}^D(\varepsilon_o)$ and $\text{dist}(J_{\ell}, J_k) \geq \delta$, $\ell \neq k$. Then, for $\varepsilon > 0$ and $R \geq 1$, there is a constant $C_\varepsilon$ such that

$$\left\| \prod_{k=1}^{4} |T^\theta_k(\tilde{\gamma} R_{J_k}(\lambda \cdot) g_k)|^{1/4} \right\|_{L^2(\mathbb{B}^4(0, R))} \leq C_\delta^{-C_\varepsilon} R^{\varepsilon} \prod_{k=1}^{4} \|g_k\|_{L^2}^{1/2}. \tag{3.28}$$

**Proof** We begin with recalling that $\tilde{\gamma} R_{J_k}(\lambda \cdot)$ is supported in $\lambda^{-1} R_{J_k}(2^6)$ and that $|R_\gamma(\xi)| \leq 2^10^0$ if $\xi \in \mathcal{C}_1^1(\delta)$ or $\mathcal{C}_0^0(\delta)$. Since $\nabla_\xi \Phi^\varepsilon(\xi) = \gamma(\sigma(\xi)) + \gamma'(\sigma(\xi)) \cdot \xi \nabla(\xi)$, we have $\nabla_\xi \Phi^\varepsilon(\xi) = \gamma(\sigma(\xi)) + O(10^0)$ for $\xi \in \mathcal{C}_1^1(\delta)$. If $\xi \in \mathcal{C}_1^0(\delta)$, by (3.16) we have $\nabla_\xi \Phi^\pm(\xi) = \gamma(\sigma(\xi)) + O(2^3 5^{10})$ because $|R_\gamma(\xi)| \leq 2^3 10^0$. Thus

$$N_k(\xi) := |(\nabla \Phi^\theta_k(\xi), 1)|^{-1}(\nabla \Phi^\theta_k(\xi), 1)$$

which is normal to the surface $(\xi, -\Phi^\theta_k(\xi))$ satisfies

$$N_k(\xi) = \frac{(\gamma(\sigma(\xi)), 1)}{\sqrt{|\gamma(\sigma(\xi))|^2 + 1}} + O(2^3 5^{10}), \quad \xi \in \mathcal{C}_1^0(\delta), \quad k = 1, \ldots, 4,$$

where we denote $\mathcal{C}_{1+}^\pm(\delta) = \mathcal{C}_1^0(\delta)$.

Let $\xi_k \in \lambda^{-1} R_{J_k}(2^6) \cap \mathcal{C}_1^{\pm}(\delta)$, $k = 1, \ldots, 4$. Then we have $\sigma(\xi_k) \in [-3c_0, 3c_0]$ since $J_k \subset (1+2c_0) J_o$. Let $\Gamma$ denote the matrix whose $k$-th column is the vector $(\gamma(\sigma(\xi_k)), 1)$, $k = 1, \ldots, 4$. By the generalized mean value theorem (see for example [22, Part V, Ch.1, 95]) there exists $u_k \in [-3c_0, 3c_0]$
such that
\[
\det \Gamma = \det \begin{pmatrix}
\gamma'(u_1) & \gamma'(u_2) & \gamma''(u_3) & \gamma''(u_4) \\
1 & 0 & 0 & 0
\end{pmatrix} \prod_{1 \leq \ell < k \leq 4} |\sigma(\xi_\ell) - \sigma(\xi_k)|.
\]

Since \( \gamma \in \mathcal{C}^D(\epsilon_0) \) and \( u_1, \ldots, u_4 \in [-3c_0, 3c_0] \), the determinant on the right hand side has its absolute value \( 1 + \mathcal{O}_s(\epsilon_0) \) regardless of \( \gamma \) (for example see (2.20)). On the other hand, using (2.22) with \( \mathcal{X}_{\ell} = c_{J_\ell} \), for \( \xi_k = \lambda^{-1} R_{J_\ell}(2^6) \cap C_1(\delta) \) we have \( |c_{J_\ell} - \sigma(\xi_k)| \leq 2^{-2}\delta \) with a small enough \( \epsilon_0 \), and we also have \( |c_{J_\ell} - c_{J_k}| \geq (1 + 2c_0)\delta \), \( \ell \neq k \) because \( \text{dist}(J_\ell, J_k) \geq \delta \). So, \( |\sigma(\xi_\ell) - \sigma(\xi_k)| > 2^{-1}\delta \) if \( \ell \neq k \), and we thus have \( \prod_{1 \leq \ell < k \leq 4} |\sigma(\xi_\ell) - \sigma(\xi_k)| > 2^{-6}\delta^6 \).

Consequently, we obtain
\[
|\det(N_1(\xi_1), \ldots, N_4(\xi_4))| > 2^{-7}\delta^6
\]
provided that \( \xi_k \in \lambda^{-1} R_{J_\ell}(2^6) \cap C_1(\delta) \) for \( k = 1, \ldots, 4 \). That is to say, the transversality condition holds uniformly regardless of the choice of \( \theta_1, \ldots, \theta_4 \in \{\epsilon, +, -\} \).

We now note that \( \Phi^\epsilon \) is continuously differentiable at least twice in a region containing \( C_1(\delta) \) and that \( \|\Phi^\pm\|_{C^{1.1/2}(C_1(\delta))} \leq C \) by Lemma 3.5. To apply Theorem 3.6 we need only to make it sure that \( \Phi^\pm \) extends as a \( C^{1.1/2} \) function to an open set containing \( C_1(\delta) \). The only part of the boundary which can be problematic is \( S := \{\xi : R_{\gamma}(\xi) = 0\} \cap C_1(\delta) \) since \( \Phi^\pm \) is homogenous and \( D - 4 \) times continuously differentiable on \( \{\xi : R_{\gamma}(\xi) = 2^6 100\} \cap C_1(\delta) \) (see Lemma 2.12 and 3.3). We note that \( R_{\gamma}(\xi) = 0 \) if and only if \( g(\xi) := \gamma'(\sigma(\xi)) \cdot \xi = 0 \). Since \( \nabla g(\xi) = \gamma'(\sigma(\xi)) = e_1 + \mathcal{O}_s(6c_0) \) for \( \xi \in \Lambda_1^* \) by Lemma 2.12 and since \( g \in \mathcal{C}^{D-2}(\Lambda_1^*) \), by the implicit function theorem it follows that \( S \) is a part of a \( \mathcal{C}^{D-2} \) boundary. Thus we can extend \( \Phi^\pm \) to be a \( C^{1.1/2} \) function across \( S \) (e.g., [8, pp. 136–137]). Therefore we may apply Theorem 3.6 and get the estimate (3.28). \( \square \)

As \( \Phi^\epsilon, \Phi^\pm \) are homogeneous of degree 1, the following is an immediate consequence of Proposition 3.7 by means of scaling and Plancherel’s theorem.

**Corollary 3.8** Under the same assumption as in Proposition 3.7, for \( \epsilon > 0 \), there is a \( C_\epsilon = C_\epsilon(\delta) > 0 \) such that
\[
\left\| \prod_{k=1}^4 |\mathcal{X}_{\ell}^\theta_k(\widetilde{\gamma} R_{J_\ell} \widetilde{f}_k)|^{1/2} \right\|_{L^2(\mathbb{R}^3(0,2^6))} \leq C_\epsilon \lambda^{\epsilon} \prod_{k=1}^4 \|f_k\|_{L^2}^{1/2}.
\]
3.3 Multilinear estimate for $A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k}$

We are ready to prove Proposition 3.1. We first show quadrilinear estimates without weight, from which we deduce the weighted estimates.

**Proposition 3.9** Let $J_k \in \mathcal{J}_\sigma(\delta)$, $1 \leq k \leq 4$. Suppose that $\text{dist}(J_\ell, J_k) \geq \delta$, $\ell \neq k$. If $1/q = 5/(8p) + 1/16$ and $2 \leq p \leq 6$, then for $\varepsilon > 0$, there are constants $C_\varepsilon = C_\varepsilon(\delta)$ and $D = D(\varepsilon)$ such that

$$
\left\| \prod_{k=1}^{4} |A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k} f_k|^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^3 \times I)} \leq C_\varepsilon \lambda^{-1} \frac{1}{\delta^6} + \varepsilon \prod_{k=1}^{4} \|f_k\|_p^{\frac{1}{2}} (3.30)
$$

holds whenever $\gamma \in \mathcal{C}^D(\varepsilon_0)$, $\psi_{J_k} \in \mathcal{M}^D(J_k)$, and $\hat{f}_k$ is supported on $\mathbb{A}_\lambda$.

By the localization argument it is sufficient for the estimate (3.30) to show its local counterpart. In fact, we have

**Lemma 3.10** Let $1 \leq p \leq q \leq \infty$ and $b \in \mathbb{R}$, and let $I' \subset I$ be an interval. Let $\gamma \in \mathcal{C}^D(\varepsilon_0)$, $\omega \in \Omega^\alpha$, $0 < \alpha \leq 4$, and $\psi_{J_k} \in \mathcal{M}^D(J_k)$, $J_k \in \mathcal{J}_\sigma(\delta)$, $1 \leq k \leq 4$. If

$$
\left\| \prod_{k=1}^{4} |A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k} f_k|^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^3 \times I') \times \omega} \leq B \lambda^{\frac{1}{4}} [\omega]^{\frac{1}{q}} \prod_{k=1}^{4} \|f_k\|_p^{\frac{1}{4}} (3.31)
$$

holds for a large enough $D = D(b)$, then we have

$$
\left\| \prod_{k=1}^{4} |A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k} f_k|^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^3 \times I', \omega)} \leq C_\delta B \lambda^{\frac{1}{4}} [\omega]^{\frac{1}{q}} \prod_{k=1}^{4} \|f_k\|_p^{\frac{1}{4}}. (3.32)
$$

**Proof** Let $K_k(\cdot, t)$ denote the kernel of the operator $A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k}$. We note that the multiplier of $\mathcal{P}_n P_{J_k}$ is given by $m(\hat{\xi}) = \hat{\chi}_{\mathcal{K}}^\gamma (\hat{\xi}) \beta_0(\delta^{-100} |R_\gamma (\hat{\xi})|) \hat{\chi}_{\mathcal{K}_\delta}(\hat{\xi})$ and $\|m(\cdot)\|_{C_M} \leq C_\delta^{-CM}$ for $M \leq D - 2$. Since $|\gamma(s)| \leq 2(c_\sigma + \varepsilon_0)$ for $s \in J_k$, by Lemma 2.3 we have $|K_k(x, t)| \leq C_\delta |E_M(x)|$ for $M \leq (D - 5)/2$ if $|x| \geq 2$ and $t \in I$. For $k \in \mathbb{Z}^3$ set $B_k = \mathbb{B}^3(k, 1)$ and $B'_k = \mathbb{B}^3(k, 3)$. Then we have

$$
|A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k} f| \leq \sum_{k \in \mathbb{Z}^3} \chi_{B_k} |A^\gamma [\psi_{J_k}] \mathcal{P}_n P_{J_k} (\chi_{B'_k} f)| + C_\delta |E_M * |f| |.$$

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Taking \( M = 4N + 9 \) above, we combine the inequality with the trivial estimate

\[
|A^\gamma[J_k]P_nP_kg| \leq C_\delta \lambda^3(1 + | \cdot |)^{-N} |g|.
\]

Then we see that \( \prod_{k=1}^4 |A^\gamma[J_k]P_nP_kf_k| \) is bounded by

\[
\sum_{k \in \mathbb{Z}^3} \chi_{B_k} \prod_{k=1}^4 |A^\gamma[J_k]P_nP_k(\chi_{B'_k}f_k)| + C_\delta \prod_{k=1}^4 (E_N * |f_k|).
\]

Since \( \|E_N * |f|\|_{L^q(\mathbb{R}^3 \times I', \omega)} \leq C[\omega]^{1/q} \lambda^{-N}\|f\|_p \) for \( 1 \leq p \leq q \), taking a large \( N \geq -b \), we may disregard the second term. We now use (3.31) to get

\[
\left\| \prod_{k=1}^4 |A^\gamma[J_k]P_nP_k(\chi_{B'_k}f_k)|^{\frac{1}{2}} \right\|_{L^q(B_k \times I', \omega)} \leq B \lambda^b [\omega]^{\frac{1}{q}} \prod_{k=1}^4 \|\chi_{B'_k}f_k\|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}}.
\]

Thus the desired estimate (3.32) follows by summation over \( k \) and Hölder’s inequality since \( B'_k \) overlap each other at most \( 6^2 \) times.

Thanks to Lemma 3.10, the proof of Proposition 3.9 is reduced to showing

\[
\left\| \prod_{k=1}^4 |A^\gamma[J_k]P_nP_kf_k|^{\frac{1}{2}} \right\|_{L^\infty(B_k \times I', \omega)} \leq C_\epsilon \lambda^{-\frac{1}{2p} - \frac{1}{6} + \epsilon} \prod_{k=1}^4 \|f_k\|_p^{\frac{1}{2}} \quad (3.33)
\]

for \( p, q \) satisfying \( 1/q = 5/(8p) + 1/16 \) and \( 2 \leq p \leq 6 \). Since \( \|P_nP_kg\|_p \leq C_\delta \|g\|_p \) by (2.25), using the estimate (2.28) with \( p = 6 \) after Hölder’s inequality, we get the estimate (3.33) with \( p = 6 \). Thus in view of interpolation we only have to obtain

\[
\left\| \prod_{k=1}^4 |A^\gamma[J_k]P_nP_kf_k|^{\frac{1}{2}} \right\|_{L^8(\mathbb{R}^3 \times (0,1) \times I)} \leq C_\epsilon \lambda^{-\frac{1}{2} + \epsilon} \prod_{k=1}^4 \|f_k\|_p^{\frac{1}{2}} \quad (3.34)
\]

Proof of (3.34) For a given \( \epsilon > 0 \) we fix \( v \) such that \( 10v = 2^{-1} \epsilon \) and then take an integer \( D \) such that \( D \geq C_1/v \) with a large constant \( C_1 \). For simplicity let us set

\[
F_k = A^\gamma[J_k]P_nP_kf_k, \quad k = 1, \ldots, 4.
\]

By Lemma 3.2 and 3.4, we have

\[
F_k = F_k^c + F_k^+ + F_k^- + E f_k, \quad k = 1, \ldots, 4,
\]

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where \( E \) satisfies \( \| E f_k \|_q \leq C \delta \lambda^{C-vD} \| f_k \|_p \) for \( 1 \leq p \leq q \leq \infty \), and

\[
F^c_k = \sum_{|\ell| \leq \lambda^{10v}} e^{it\ell} T^c_\lambda (c_\ell \pi_c \tilde{\chi}_{R_k} f_k),
\]
\[
F^\pm_k = \sum_{0 \leq m \leq M-1} t^{-\frac{2m+1}{2}} T^\pm_\lambda (Y_m^{\pm} \pi_o \tilde{\chi}_{R_k} f_k).
\]

We thus need to handle the product terms \( \prod_{k=1}^4 h_k \) where \( h_k \in \{ F^c_k, F^\pm_k, E f_k \} \), \( 1 \leq k \leq 4 \). Any product which has \( E f_k \) as one of its factors is easily handled by taking \( C_1 \) large enough if one uses Hölder’s inequality and the trivial estimates \( \| T^c_\lambda (\pi_c g) \|_q \leq C \delta \lambda \| g \|_p \) and \( \| T^\pm_\lambda (\pi_o \tilde{\chi} R_k f_k) \|_q \leq C \delta \lambda \| g \|_p \), which hold for \( 1 \leq p \leq q \leq \infty \). So, it suffices to obtain the estimates for the products which consist only of the terms \( F^c_k, F^\pm_k \).

By (3.2) and (3.18) we have
\[
\sum_{|\ell| \leq \lambda^{10v}} \| c_\ell \|_\infty \leq C \lambda^{3\nu} \quad \text{and} \quad \sum_{k=0}^{M-1} \| \gamma_\ell^{\pm} \|_\infty \leq C \delta \lambda^{-\frac{1}{3}-\frac{\nu}{2}}.
\]
Thus, using the estimate (3.29) and Plancherel’s theorem, we obtain
\[
\left\| \prod_{k=1}^4 |F^c_k|^{\frac{1}{4}} \right\|_{L^8(\mathbb{R}^4(0,2^3))} \leq C \varepsilon \lambda^{-\frac{1}{2}+10v+\frac{\nu}{2}} \prod_{k=1}^4 \| f_k \|_{L^2}(\mathbb{R}^3)\]
for \( 4/5 < p \leq 6 \). Here we keep using the simpler notation (3.35).

### 3.4 Proof of Proposition 3.1

We are in a position to prove Proposition 3.1. By Lemma 3.10, it suffices to show that
\[
\left\| \prod_{k=1}^4 |\tilde{\chi} F_k|^{\frac{1}{4}} \right\|_{L^p(\mathbb{R}^3(0,1) \times I,\omega)} \leq C \delta \lambda^{-\frac{\nu}{p}} \prod_{k=1}^4 \| f_k \|_{L^p(\mathbb{R}^3)}
\]
for \( 14/5 < p \leq 6 \). Here we keep using the simpler notation (3.35).

We deduce the weighted estimate from Proposition 3.9 in the same way as in the proof of Proposition 2.18. The difference is that we are dealing with a multilinear estimate and the exponent \( p/4 \) can be less than 1. Nonetheless, Lemma 2.19 works as before. To apply Lemma 2.19, we break \( \tilde{\chi} F_k = \tilde{\chi} f_k + \tilde{\chi} E_k f_k \) where
\[
\mathcal{F}(\tilde{\chi} f_k)(\xi, \tau) = \beta_0((\lambda r_0)^{-1} \tau) \mathcal{F}(\tilde{\chi} F_k)(\xi, \tau)
\]
and \( r_0 = 1 + 4 \max \{ |\gamma(s)| : s \in \text{supp} \psi_{J_k}, k = 1, \ldots, 4 \} \). Since \([\omega]_3 \leq 1\) and \( \| P_n P_k f \|_p \leq C_\delta \| f \|_p \) and since \( |E_k f_k(x, t)| \leq C \tilde{E}_t^M \| P_n P_k f_k(x) \|_p \)
Maximal estimates for averages over space curves

by Lemma 2.6, we see \( \| \mathcal{E}_k f_k \|_{L^q(\mathbb{R}^3 \times \mathbb{R}, \omega)} \leq C_D \lambda^{-M} \| f_k \|_p \) for any \( M > 0 \). Using the trivial estimate \( |\tilde{\mathcal{X}} F_k| \leq C_D \lambda^3 (1 + | \cdot |)^{-M} \| f_k \| \), we also have \( \| \tilde{\mathcal{X}} F_k \|_{L^q(\mathbb{R}^3 \times \mathbb{R}, \omega)} \leq C_D \lambda^3 \| f_k \|_p \). Making use of those estimates and taking a large \( M \), one can easily see

\[
\left\| \prod_{k=1}^{4} |\tilde{\mathcal{X}} F_k|^{\frac{1}{4}} \right\|_{L^q(\mathbb{B}^3(0,1) \times I)} \leq C \left\| \prod_{k=1}^{4} |\tilde{\mathcal{X}} F_k|^{\frac{1}{4}} \right\|_{L^q(\mathbb{R}^4, \omega)} + C_D \lambda^{-N} \prod_{k=1}^{4} \| f_k \|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}}
\]

for a large \( N \) and \( q \geq p \).

By (2.30) \( \| \prod_{k=1}^{4} |\tilde{\mathcal{X}} f_k|^{\frac{1}{4}} \|_{L^q(\mathbb{R}^4, \omega)} \leq C_D \lambda^{1/q} \| \prod_{k=1}^{4} |\tilde{\mathcal{X}} f_k|^{\frac{1}{4}} \|_{L^q(\mathbb{R}^4)} \) since \( [\omega]_3 \leq 1 \) and the support of \( \mathcal{F}(\prod_{k=1}^{4} \tilde{\mathcal{X}} f_k) \) is contained in a ball of radius \( 2^4 r_0 \lambda \). To estimate \( \| \prod_{k=1}^{4} |\tilde{\mathcal{X}} f_k|^{\frac{1}{4}} \|_{L^q(\mathbb{R}^4)} \), using the estimate \( \| \mathcal{E}_k f_k \|_{L^q(\mathbb{R}^3 \times \mathbb{R}, \omega)} \leq C_D \lambda^{-M} \| f_k \|_p \) again, we may disregard the minor contributions. So, it is sufficient to consider \( \| \prod_{k=1}^{4} |\tilde{\mathcal{X}} F_k|^{\frac{1}{4}} \|_{L^q(\mathbb{R}^4)} \). Since \( \text{supp} \tilde{\mathcal{X}} \subset I \), by the estimate (3.30) we get

\[
\left\| \prod_{k=1}^{4} |\tilde{\mathcal{X}} F_k|^{\frac{1}{4}} \right\|_{L^q(\mathbb{B}^3(0,1) \times I)} \leq C_{\varepsilon} (\delta) \lambda^{\frac{7}{24}(\frac{1}{p} - \frac{5}{16}) + \varepsilon} \prod_{k=1}^{4} \| f_k \|_{L^p(\mathbb{R}^3)}^{\frac{1}{4}}
\]

for \( 1/q = 5/(8p) + 1/16 \) and \( 2 \leq p \leq 6 \). Finally, we obtain (3.36) for \( 14/5 < p \leq 6 \) by Hölder’s inequality since \( q \geq p \) and \( \| \omega \|_{L^1(\mathbb{B}^3(0,1) \times I)} \leq C[\omega]_3 \). \( \square \)

Remark 1 In the above we try to obtain the estimate (3.1) on a range of \( p \) as large as possible by suppressing \( \nu \) arbitrarily small (Proof of (3.34)). This forces us to take a large \( D \geq C_1 \nu \). However, to obtain the maximal estimate it is enough to have the estimate (3.1) on a smaller range \( 3 < p \leq 6 \) instead of \( 14/5 < p \leq 6 \). For \( 3 < p \leq 6 \), we can prove (3.1) with a fixed \( \nu \) and \( D \). For example, optimizing the estimates at various places, we can take \( \nu = 1/397 \) and \( D = 720 \). In other words, Theorem 1.1 holds true for \( \gamma \in C^{720}(\mathbb{J}) \).

4 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. We prove the sufficiency and the necessity parts in separate sections.

4.1 Sufficiency

By the reduction in Sect. 2.4, Lemma 2.4 and Lemma 2.2, it suffices to prove Proposition 2.10, which also proves Theorem 2.1.
Decomposition

We first decompose the averaging operator $A^\gamma[\psi]$ in such a way that we can use the multilinear estimate obtained in Sect. 3. The following Lemma 4.1 is a slight modification of [11, Lemma 2.8]. Let us set

$$\mathcal{J}_4^\delta(\delta) = \{(J_1, \ldots, J_4) : J_1, \ldots, J_4 \in \mathcal{J}_\circ(\delta), \min_{\ell \neq k} \text{dist}(J_\ell, J_k) \geq \delta\}.$$  

Lemma 4.1 Let $\psi \in \mathcal{M}^D(J_\circ)$ and $\gamma \in \mathcal{E}^D(\varepsilon_\circ)$. There is a constant $C = C(D)$ independent of $z = (x, t)$, $\gamma$, and $\delta$ such that

$$|A^\gamma[\psi]f(z)| \leq C \max_{J \in \mathcal{J}_3(\delta)} |A^\gamma[\psi_J]f(z)| + C\delta^{-1} \sum_{(J_1, \ldots, J_4) \in \mathcal{J}_4^\delta(\delta)} \prod_{k=1}^4 |A^\gamma[\psi_{J_k}]f(z)|^{1/4}, \tag{4.1}$$

where $\psi_J \in \mathcal{M}^D(J)$ and $\psi_{J_k} \in \mathcal{M}^D(J_k)$.

Proof Let us recall (2.14). It is clear that there is a constant $C_D > 0$ such that $C_D^{-1}\psi \xi_J \in \mathcal{M}^D(J)$ for $J \in \mathcal{J}_3(\delta)$. Setting $\psi_J = C_D^{-1}\psi \xi_J$ we have

$$A^\gamma[\psi]f(z) = C_D \sum_{J \in \mathcal{J}_3(\delta)} A^\gamma[\psi_J]f(z).$$

Let us set $\mathcal{J}_1 = \mathcal{J}_3(\delta)$. For a fixed $z$, define $J_1^*$ to be an interval in $\mathcal{J}_1$ such that $|A^\gamma[\psi_{J_1^*}]f(z)| = \max_{J \in \mathcal{J}_1} |A^\gamma[\psi_J]f(z)|$. For $k = 2, 3, 4$, we recursively define $\mathcal{J}_k$ and $J_k^* \in \mathcal{J}_k$. Let $\mathcal{J}_k = \{J \in \mathcal{J}_{k-1} : \text{dist}(J, J_{k-1}^*) \geq \delta\}$ and let $J_k^* \in \mathcal{J}_k$ denote an interval such that $|A^\gamma[\psi_{J_k^*}]f(z)| = \max_{J \in \mathcal{J}_k} |A^\gamma[\psi_J]f(z)|$. Thus, if $\text{dist}(J, J_k^*) \geq \delta$ for all $1 \leq k \leq 3$, we have $|A^\gamma[\psi_J]f| \leq |A^\gamma[\psi_{J_k^*}]f|$ for $1 \leq k \leq 4$.

Let us denote $\mathcal{J} = \bigcup_{k=1}^3 \{J \in \mathcal{J}_k(\delta) : \text{dist}(J, J_k^*) < \delta\}$. Splitting the sum into the cases $J \in \mathcal{J}$ and $J \notin \mathcal{J}$, we have

$$C_D^{-1}|A^\gamma[\psi]f(z)| \leq \sum_{J \in \mathcal{J}} |A^\gamma[\psi_J]f(z)| + \sum_{J \notin \mathcal{J}} |A^\gamma[\psi_J]f(z)|.$$ 

The first sum on the right hand side is apparently bounded by a constant times $\max_{J \in \mathcal{J}_3(\delta)} |A^\gamma[\psi_J]f(z)|$ and the second by $C\delta^{-1} \prod_{k=1}^4 |A^\gamma[\psi_{J_k^*}]f(z)|^{1/4}$. This gives (4.1) since $\text{dist}(J_k^*, J_\ell^*) \geq \delta$ if $k \neq \ell$. \hfill \Box

In the next lemma, using $K_\delta(\lambda)$ given in Lemma 2.11, we get a bound on the first one on the right hand side of (4.1).
Lemma 4.2 Let \( 2 < p \leq 6 \), and let \([\omega]_3 \leq 1\) and \( \psi_J \in \mathcal{N}^D(J) \) for each \( J \in \mathcal{J}_3(\delta) \). If \( \delta^3 \lambda \geq 2^2 \) and \( \varepsilon_\circ > 0 \) is sufficiently small, there is an \( \varepsilon_p > 0 \) such that

\[
\left\| \max_{J \in \mathcal{J}_3(\delta)} |A^Y[\psi_J]f| \right\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C \left( \delta^{1-\frac{3}{p}} K_\delta(\lambda) + C \delta \lambda^{-\varepsilon_p} \right) \| f \|_{L^p(\mathbb{R}^3)}
\]

holds whenever \( \gamma \in \mathcal{C}^D(\varepsilon_\circ) \) and \( \hat{f} \) is supported on \( \mathbb{A}_\lambda \).

Proof of Lemma 4.2 By the embedding \( \ell^p \subset \ell^\infty \) and Minkowski’s inequality,

\[
\left\| \max_{J \in \mathcal{J}_3(\delta)} |A^Y[\psi_J]f| \right\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)}^p \leq 2^p (I + II),
\]

where

\[
I = \sum_{J \in \mathcal{J}_3(\delta)} \left\| A^Y[\psi_J]P_J f \right\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)}^p,
\]

\[
II = \sum_{J \in \mathcal{J}_3(\delta)} \left\| A^Y[\psi_J](f - P_J f) \right\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)}^p.
\]

For \( II \) we apply Proposition 2.18. Taking \( \varepsilon_p = \frac{1}{4} \left( \frac{1}{2} - \frac{1}{p} \right) \) and using the estimate (2.29) with \( \varepsilon = \varepsilon_p/2 \), we have \( II \leq C \delta \lambda^{-\varepsilon_p} \| f \|_{L^p(\mathbb{R}^3)} \) since there are at most \( C \delta^{-1} \) many \( J \). To handle \( I \), we invoke Lemma 2.11 and then use Lemma 2.14 to obtain

\[
I \leq C \delta^{p-3} K_\delta(\lambda)^p \sum_{J \in \mathcal{J}_3(\delta)} \| P_J f \|_p^p \leq C \delta^{p-3} K_\delta(\lambda)^p \| f \|_p^p.
\]

Therefore the desired bound follows. \( \square \)

Now we consider the product terms appearing in (4.1).

Lemma 4.3 Let \( \frac{14}{5} < p \leq 6 \), \( [\omega]_3 \leq 1 \), and \((J_1, \ldots, J_4) \in \mathcal{J}_4^4(\delta) \). If \( \delta^3 \lambda \geq 2^2 \) and \( \varepsilon_\circ > 0 \) is small enough, there are positive constants \( \varepsilon_p, c, D \) such that

\[
\left\| \prod_{k=1}^4 |A^Y[\psi_{J_k}]f|^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^3 \times [1, 2], \omega)} \leq C \delta \left( \lambda^{-\varepsilon_p} + \lambda^{-c} K_\delta(\lambda) \right) \| f \|_{L^p(\mathbb{R}^3)} \quad (4.2)
\]

holds whenever \( \gamma \in \mathcal{C}^D(\varepsilon_\circ) \), \( \psi_{J_k} \in \mathcal{N}^D(J_k) \), \( k = 1, \ldots, 4 \), and \( \hat{f} \) is supported in \( \mathbb{A}_\lambda \).
Proof For each $1 \leq k \leq 4$ we split $f = b_k + g_k$, where

$$b_k = \mathcal{P}_n P_{J_k} f, \quad g_k = \mathcal{P}_n (1 - P_{J_k}) f + \mathcal{P}_e f.$$ 

We here use $f = \mathcal{P}_n f + \mathcal{P}_e f$ because $\hat{f}$ is supported on $A_\lambda$. Thus, the left hand side of (4.2) is bounded by a constant times

$$\mathcal{M} = \sum_{h_k \in \{b_k, g_k\}} \left\| \prod_{k=1}^{4} |A^v[\psi_{J_k}] h_k|^{\frac{1}{4}} \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)}.$$

We consider the cases $(h_1, \ldots, h_4) = (b_1, \ldots, b_4)$ and $(h_1, \ldots, h_4) \neq (b_1, \ldots, b_4)$. For the former case we use Proposition 3.1 and the estimate (2.25). Since $14/5 < p \leq 6$, there is an $\varepsilon_p > 0$ such that

$$\left\| \prod_{k=1}^{4} |A^v[\psi_{J_k}] b_k|^{\frac{1}{4}} \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \leq C \delta \lambda^{-\varepsilon_p} \| f \|_{L^p(\mathbb{R}^3)}.$$

For the other case we combine Proposition 2.18, 2.20, and Lemma 2.11. In fact, Proposition 2.18 and 2.20 followed by (2.25) yield

$$\| A^v[\psi_{J_k}] g_k \|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \leq C \delta - C \lambda^{\frac{1}{2}(\frac{1}{p} - \frac{1}{2})+\varepsilon} \| f \|_{L^p(\mathbb{R}^3)}$$

for $2 \leq p \leq 6$. If we consider a particular case $(h_1, \ldots, h_4) = (b_1, b_2, b_3, g_4)$, by Hölder’s inequality and the above estimate we have

$$\left\| \prod_{k=1}^{4} |A^v[\psi_{J_k}] h_k|^{\frac{1}{4}} \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \leq C_\delta \lambda^{-c} \| f \|_{L^p(\mathbb{R}^3)} \prod_{k=1}^{3} \| A^v[\psi_{J_k}] b_k \|_{L^p(\mathbb{R}^3 \times [1,2], \omega)}$$

for a constant $c > 0$ because $p > 14/5$. We apply Lemma 2.11 to handle the last three factors. Since $\| b_k \|_{L^p(\mathbb{R}^3)} \leq C_1 \delta^{-C} \| f \|_{L^p(\mathbb{R}^3)}$ from (2.25), the inequality (2.19) gives

$$\left\| \prod_{k=1}^{4} |A^v[\psi_{J_k}] h_k|^{\frac{1}{4}} \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \leq C_\delta \lambda^{-c} K_\delta(\lambda)^{\frac{3}{2}} \| f \|_{L^p(\mathbb{R}^3)}.$$
We can deal with the remaining products similarly. As a result, we obtain
\[ M \leq C \delta \left( \lambda^{-\varepsilon_p} + \sum_{\ell=1}^{3} \lambda^{- (4-\ell) c} K_\delta (\lambda)^{\ell/4} \right) \| f \|_{L^p(\mathbb{R}^3)} \]
and therefore the bound (4.2) after a simple manipulation since we may assume \( \varepsilon_p \leq c \) taking a smaller \( \varepsilon_p \) if necessary. \( \square \)

We now conclude the proof of (2.18) putting together the previous estimates.

**Proof of (2.18)**

Since \( \| Af \|_{L^\infty(\mathbb{R}^3 \times [1,2], \omega)} \leq C \| f \|_{L^\infty(\mathbb{R}^3)} \), by interpolation it is sufficient to show (2.18) for \( 3 < p < 6 \). Let \( p \in (3,6) \) and take an \( \varepsilon_0 > 0 \) small enough and a large \( D \) such that the estimates in Lemma 4.2 and 4.3 hold whenever \( \gamma \in \mathcal{C}^D(\varepsilon_0) \) and \( \psi J \in \mathcal{N}^D(J), J \in \mathcal{F}_0(\delta) \).

Let \( \gamma \in \mathcal{C}^D(\varepsilon_0), \omega \in \Omega^3 \) with \( [\omega]_3 \leq 1 \) and \( \psi \in \mathcal{N}^D(J_0) \), and let \( f \) be a function such that \( \text{supp} \hat{f} \subset A_\lambda \) and \( \| f \|_p \leq 1 \). By (4.1) and Minkowski’s inequality we see that \( \| A^{\gamma} [\psi] f \|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \) is bounded by
\[
C \left\| \max_{J \in \mathcal{F}_0(\delta)} | A^\gamma [\psi J] f | \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} + C_\delta \sum_{(J_1,\ldots,J_4) \in \mathcal{F}_0(\delta)} \left\| \prod_{k=1}^{4} | A^{\gamma} [\psi J_k] f |^{1/4} \right\|_{L^p(\mathbb{R}^3 \times [1,2], \omega)}.\]

Then, Lemma 4.2 and 4.3 give
\[
\| A^{\gamma} [\psi] f \|_{L^p(\mathbb{R}^3 \times [1,2], \omega)} \leq C \left( \delta^{1-3/p} + \lambda^{-c} \right) K_\delta (\lambda) + C_\delta \lambda^{-\varepsilon_p} \]
if \( 2^2 \delta^{-3} \leq \lambda \). Taking supremum over \( f, \omega, \psi, \gamma \), we obtain
\[
Q(\lambda) \leq C \left( \delta^{1-3/p} + \lambda^{-c} \right) K_\delta (\lambda) + C_\delta \lambda^{-\varepsilon_p} \tag{4.3}\]
for \( 2^2 \delta^{-3} \leq \lambda \). In order to close the induction we need to modify \( Q(\lambda) \) slightly. Fix \( 0 < b \), which is to be chosen later. We define
\[
\overline{Q}_b(\lambda) = \sup_{1 \leq r \leq \lambda} r^b Q(r).\]

We observe \( \lambda^b K_\delta (\lambda) \leq 2^{2b} \delta^{-3b} \delta^{2b} \leq 2^b \delta^{-b} Q(2^b) \), and hence we have \( \lambda^b K_\delta (\lambda) \leq C | \log \delta | \delta^{3b} \overline{Q}_b(2^2 \delta \lambda) \). Multiplying \( \lambda^b \) to both sides of (4.3),
we get
\[
\lambda^b Q(\lambda) \leq C(\delta^{1-\frac{3}{p}} + \lambda^{-c}) |\log \delta| \delta^{-3b} Q_b(2^2 \delta \lambda) + C_\delta \lambda^{b-\varepsilon_p}
\]
for \(2^2 \delta^{-3} \leq \lambda\). We now choose a small \(b\) such that \(1 - \frac{3}{p} - 3b > 0\) and \(b - \varepsilon_p < 0\), then fix a small enough \(\delta > 0\) such that \(C\delta^{1-\frac{3}{p}} |\log \delta| \delta^{-3b} \leq 2^{-2}\) and \(2^2 \delta \leq 1\). Such a choice is clearly possible because \(p > 3\). Let \(\lambda_0\) be a large number such that \(\delta^{1-\frac{3}{p}} \geq \lambda_0^{-c}\) and \(2^2 \delta^{-3} \leq \lambda_0\). Then we have the inequality \(\lambda^b Q(\lambda) \leq 2^{-1} Q_b(\lambda) + C_\delta\) for \(\lambda \geq \lambda_0\) since \(Q_b\) is increasing. This obviously implies
\[
\lambda^b Q(\lambda) \leq 2^{-1} Q_b(r) + C_\delta
\]
for \(\lambda_0 \leq \lambda \leq r\). Note \(Q_b(\lambda_0) \leq \lambda_0^b C_2\) for some constant \(C_2\) (because of the trivial estimate \(Q(\lambda) \leq C\lambda^2\)). Taking supremum over \(\lambda \in [1, r]\) we get \(Q_b(r) \leq 2^{-1} Q_b(r) + \lambda_0^b C_2 + C_\delta\). Therefore we have \(Q_b(r) \leq C_3\) for a constant \(C_3\) and conclude \(Q(\lambda) \leq C_3 \lambda^{-b}\) for \(\lambda \geq 1\).

**Remark 2** Routine adaptation of our argument also proves \(L^p\) improving property of the localized maximal operator \(M f(x) := \sup_{1 \leq t \leq 2} |A f(x, t)|\). In fact, the estimate \(||M f||_{L^q(\mathbb{R}^3)} \leq C ||f||_{L^p(\mathbb{R}^3)}\) holds provided that \((p, 1/q)\) is contained in the interior of the triangle with vertices (0, 0), (1/3, 1/3), and (19/66, 8/33). It is possible to extend the range slightly making use of the estimate (2.28) for \(p > 6\). Furthermore, one can show that \(M\) is bounded from \(L^p\) to \(L^p(d\mu)\) for \(p > 9 - 2\alpha\) when \(\mu\) is an \(\alpha\) dimensional measure and \(3 > \alpha \geq \frac{65 - \sqrt{865}}{12} = 2.9657\ldots\).

### 4.2 Necessity

To prove that \(L^p\) boundedness of \(M\) fails for \(p \leq 3\), it is sufficient to show the next proposition. Our construction below is a modification of Stein’s example in [32].

**Proposition 4.4** Let \(p \leq 3\) and \(\psi \neq 0\) be a nonnegative continuous function supported in \(J\). Suppose \(\gamma : J \rightarrow \mathbb{R}^3\) is a smooth curve satisfying (1.1). Then there is an \(h \in L^p(\mathbb{R}^3)\) such that \(Mh = \infty\) on a nonempty open set.

**Proof** Since \(\psi \geq 0\) and \(\psi \neq 0\), we may assume that \(\psi(s) \geq c\) on an interval \(J \subset \mathbb{R}\) for some \(c > 0\). By (1.1) we may additionally assume that \(|\gamma(s)| \geq c\) on \(J\) taking a subinterval of \(J\) if necessary because the condition (1.1) can not be satisfied if there is no such a subinterval.
Since \( \gamma'(s), \gamma''(s), \gamma'''(s) \) are linearly independent, we can write

\[
\gamma(s) = c_1(s)\gamma'(s) + c_2(s)\gamma''(s) + c_3(s)\gamma'''(s), \quad s \in J
\]

(4.4)

for some smooth functions \( c_1, c_2, \) and \( c_3 \). We claim that there is an \( s_0 \in J \) such that \( c_3(s_0) \neq 0 \). Suppose that there is no such \( s_0 \in J \), that is to say, \( c_3(s) \equiv 0 \) for all \( s \in J \). Differentiating both side of (4.4), we have

\[
(c_1'(s) - 1)\gamma'(s) + [c_1(s) + c_2'(s)]\gamma''(s) + c_2(s)\gamma'''(s) = 0,
\]

which implies \( c_2(s) \equiv 0, c_1(s) + c_2'(s) \equiv 0, \) and \( c_1'(s) \equiv 1 \) for \( s \in J \). This leads to a contradiction and proves the claim. Therefore there are \( s_0 \in J \) and \( \delta > 0 \) such that

\[
|c_3(s)| \geq c, \quad s \in [s_0 - \delta, s_0 + \delta] \subset J
\]

for some \( c > 0 \). We only consider the case \( c_3(s) \geq c \) since the other case can be handled similarly.

For \( x \in \mathbb{R}^3 \) let \( y = (y_1, y_2, y_3) \) denote the coordinate of \( x \) with respect to the basis \( \{\gamma'(s_0), \gamma''(s_0), \gamma'''(s_0)\} \), i.e., \( x = y_1\gamma'(s_0) + y_2\gamma''(s_0) + y_3\gamma'''(s_0) \), and set \( \overline{y} = (y_1, y_2) \). For some \( \varepsilon \in (0, 1/3) \) we take \( g(t) = \chi_{[0, 2^{-1}]}(t)|t|^{-\frac{1}{2}} \log |t|^{-\frac{1}{2}-\varepsilon} \) and then we consider

\[
h(x) = \chi_0(|\overline{y}|)g(y_3),
\]

where \( \chi_0 \in C^\infty_{\text{c}}([-2, 2]) \) is a nonnegative function such that \( \chi_0 = 1 \) on \([-1, 1]\). It is easy to see \( h \in L^p(\mathbb{R}^3) \) for \( p \leq 3 \) because \( g \in L^p(\mathbb{R}) \) for \( p \leq 3 \). Thus we only have to show that \( \sup_{0 < t} Ah = \infty \) on a nonempty open set.

We write \( \gamma(s) = a_1(s)\gamma'(s_0) + a_2(s)\gamma''(s_0) + a_3(s)\gamma'''(s_0) \) and \( \overline{a}(s) = (a_1(s), a_2(s)) \). Since \( c_j(s_0) = a_j(s_0), j = 1, 2, 3, \) by a Taylor expansion we have

\[
\gamma(s) = (c_1(s_0) + (s - s_0))\gamma'(s_0) + (c_2(s_0) + (s - s_0)^2/2!)\gamma''(s_0)
+ (c_3(s_0) + (s - s_0)^3/3!)\gamma'''(s_0) + \mathcal{O}((s - s_0)^4).
\]

So, \( y_3 - ta_3(s) = y_3 - tc_3(s_0) - t(6^{-1}(s - s_0)^3 + \mathcal{O}((s - s_0)^4)) \). For \( y_3 > 0 \) we take \( t = y_3/c_3(s_0) > 0 \). Then it follows that \( C_1y_3|s - s_0|^3 \leq |y_3 - ta_3(s)| \leq C_2y_3|s - s_0|^3 \) for some \( C_1, C_2 > 0 \), so \( |g(y_3 - ta_3(s))| \geq Cy_3^{-\frac{1}{2}}|s - s_0|^{-1} \log(y_3|s - s_0|^3)|^{-\frac{1}{2}-\varepsilon} \) provided that \( |s - s_0| < c' \) for a small \( c' > 0 \) and \( 0 < y_3 \leq 1 \). Thus, by our choice of \( \delta \) and \( s_0 \) we have

\[
Ah(x, \frac{y_3}{c_3(s_0)}) \geq Cy_3^{-\frac{1}{2}}\int_{|s - s_0| \leq \delta'} \chi_0(y, s)|s - s_0|^{-1} |\log(y_3|s - s_0|^3)|^{-\frac{1}{2}-\varepsilon} ds
\]
for $0 < y_3 \leq 1$ where $\delta' = \min(\delta, c')$ and $\tilde{\chi}_0(y, s) = \chi_0(|\vec{y} - \frac{y_3}{c_3(s_o)}\vec{a}(s)|)$. Since $\tilde{\chi}_0(y, s) \geq 1$ if $|y| \leq r_o$ for a small enough $r_o > 0$, we have

$$Ah\left(x, \frac{y_3}{c_3(s_o)}\right) \geq Cy_3^{-\frac{1}{4}} \int_{|s - s_o| \leq \min(\delta', y_3^3/10)} |s - s_o|^{-1} |\log |s - s_o||^{-\frac{1}{3} - \epsilon} ds$$

$$= \infty$$

for $y \in \mathbb{R}^3(0, r_o) \cap \{y : 0 < y_3 < 1\}$ as desired. \hfill \Box

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