Smoothed particle magnetohydrodynamics with a Riemann solver and the method of characteristics

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ABSTRACT

In this paper, we develop a new method for magnetohydrodynamics (MHD) using smoothed particle hydrodynamics (SPH). To describe MHD shocks accurately, the Godunov method is applied to SPH instead of artificial dissipation terms. In the interaction between particles, we solve a non-linear Riemann problem with magnetic pressure for compressive waves and apply the method of characteristics for Alfvén waves. An extensive series of MHD test calculations is performed. In all test calculations, we compare the results of our SPH code with those of a finite-volume method with an approximate Riemann solver, and confirm excellent agreement.

Key words: magnetic fields – MHD – methods: numerical.

1 INTRODUCTION

It is well known that magnetic fields play an important role in various astrophysical phenomena, such as the formation of stars and planets and high-energy astrophysics. Gas dynamics with a magnetic field is well described by magnetohydrodynamics (MHD). The MHD equations are too complicated to solve analytically, except for special cases, therefore to understand physical phenomena involving magnetic fields numerical simulations are indispensable and powerful tools. A numerical technique to solve MHD equations has been developed successfully in the form of the finite-volume method by many authors.

Smoothed particle hydrodynamics (SPH) is a fully Lagrangian particle method (Lucy 1977; Gingold & Monaghan 1977). This Lagrangian nature has major advantages in problems that have a large dynamic range in spatial scale, such as the formation of large-scale structure, galaxies, stars and planets. Several authors have tried to apply the SPH method to MHD problems. In this paper, we call SPH for MHD ‘smoothed particle magnetohydrodynamics’ (SPMHD). Price & Monaghan (2004a,b) have developed a one-dimensional SPMHD scheme. To describe shock waves, they used artificial dissipation terms proposed by Monaghan (1997) based on an analogy with Riemann solutions of compressible gas dynamics. As the signal velocity they adopted the speed of the fast wave, which is analogous to the sound wave in hydrodynamics (HD). Their SPMHD has been shown to give good results for a wide range of standard one-dimensional problems used in recent finite-volume MHD schemes. Price & Monaghan (2005, hereafter PM05) have developed a multi-dimensional SPMHD scheme based on the above one-dimensional one. Alternatively, Børve, Omang & Trulsen (2001) and Børve, Omang & Trulsen (2006) have implemented an SPMHD scheme using a regularization of the underlying particle distribution and artificial viscosity. Recently, Dolag & Stasyszyn (2009) have implemented MHD in the cosmological SPH code GADGET (Springel, Yoshida & White 2001; Springel 2005), mainly based on PM05. Broad discussions of SPH and SPMHD are found in reviews by Monaghan (1992), Springel (2010a) and Price (2010).

The SPMHD in PM05 can capture fast shocks accurately. However, since they use the fast wave speed as the signal velocity in the artificial viscosity and resistivity terms, Alfvén waves become dissipative. In the finite-volume method, it is well known that Alfvén waves cannot be described accurately if the characteristics of Alfvén waves are not taken into account in the calculation of the numerical flux (e.g. Stone & Norman 1992). This is also the case in SPMHD. In this paper, we apply the Godunov method to SPMHD. The Godunov method was originally developed within the finite-volume method (Godunov 1959; van Leer 1979). Unlike artificial viscosity, the Godunov method can, in principle, take into account the minimum and sufficient amount of dissipation without any free parameters. In HD, the application of the Godunov method to SPH has been carried out by Inutsuka (2002, hereafter I02). In MHD, we can also consider the general Riemann problem (RP) with arbitrary directions of the velocity and magnetic field on both sides. However, it is computationally expensive and complex to solve the RP because the MHD equations have seven characteristics in addition to non-hyperbolicity. Recently, Gaburov & Nitadori (2011) have incorporated the approximate Riemann solver HLLC into a variant particle method (Toro & Spruce 1994; Li 2005). Besides the SPH method, Pakmor, Bauer & Springel (2011) implemented MHD with the approximate Riemann solver HLLD (Miyoshi & Kusano 2005) in...
unstructured, moving-mesh code AREPO code (Springel 2010b). In this paper, we use a simplified approach proposed for the finite-volume method by Sano, Inutsuka & Miyama (1999). In this method, compressible and incompressible parts of the MHD equations are completely divided. The former part is calculated by a non-linear Riemann solver with magnetic pressure, and the latter is calculated by the method of characteristics (MOC) proposed by Stone & Norman (1992).

The paper is organized as follows. In Section 2, we derive the SPMHD equations from the basic equations of MHD. In Section 3, we describe the implementation of the derived SPMHD equations. Various test calculations are demonstrated in Section 4. The paper is summarized in Section 5.

2 SPMHD EQUATIONS

The basic equations of MHD can be written as

\[
\frac{d}{dt} \begin{pmatrix} 1/\rho \\ v^\mu \\ E \end{pmatrix} = \frac{1}{\rho} \nabla \begin{pmatrix} v^\nu \\ T^{\mu\nu} v^\mu \end{pmatrix},
\]

(1)

\[
\frac{d}{dt} \left( \frac{B^\mu}{\rho} \right) = \frac{B^\nu}{\rho} \nabla v^\mu,
\]

(2)

where \( T^{\mu\nu} \) is the stress tensor,

\[
T^{\mu\nu} = - \left( P + \frac{B^2}{2} \right) \delta^{\mu\nu} + B^\mu B^\nu,
\]

(3)

the specific total energy is given by

\[
E = \frac{1}{2} v^2 + e + \frac{B^2}{2\rho},
\]

(4)

e = P[(\gamma - 1)\rho] is the specific internal energy, \( d/dt = \partial/\partial t + v \cdot \nabla \) is the Lagrangian time derivative and we have chosen units so that factor of \( \mu_0 \) does not appear in the equations, where \( \mu_0 \) is the magnetic permeability.

In the SPH method, the density field is expressed as

\[
\rho(x) = \sum_j m_j W(x - x_j, h(x)),
\]

(5)

where the subscripts denote particle labels, \( m_j \) is the mass of the \( j \)th particle, \( W(x, h) \) is a kernel function and \( h \) is the smoothing length that is assumed to depend on \( x \).

There are many choices of kernel functions. In the Godunov SPH (GSPH) schemes, I02 adopted the following Gaussian kernel:

\[
W(x, h) = \left( \frac{1}{\sqrt{2\pi}h} \right)^d e^{-(x/h)^2},
\]

(6)

Figure 1. Results of convergence test for fast (circles), Alfvén (triangles) and slow waves (boxes). The abscissa indicates the average smoothing length, which represents the resolution. The ordinate indicates the norm of the error vector \( \epsilon \). The upper and lower solid lines represent the lines \( \propto h^{-2} \) and \( \propto h^{-1} \), respectively.
where \( d \) is the number of dimension. This is because the Gaussian kernel makes the formulation of the GSPH simpler than other kernel functions. In addition, the results with a Gaussian kernel seem to be better than those with the cubic spline kernel in test calculations shown in Section 4. In this paper, we follow the choice made in I02.

### 2.1 Equation of motion

We define the equation for the evolution of the particle positions as

\[
\ddot{x}_i = \int \frac{1}{\rho} (\nabla^\mu T^\nu) W_i(x)
\]

where \( W_i(x) \equiv W[x - x_i, h(x)] \). Substituting equation (1) into equation (7), one obtains

\[
\ddot{x}_i = \int \frac{1}{\rho} \left( \nabla^\nu \frac{W_i(x)}{\rho} \right) - \int \frac{1}{\rho} \left( \frac{W_i(x)}{\rho} \right),
\]

where we integrate by parts. The first term on the right-hand side in equation (8) becomes the surface integral by the Gauss theorem, and vanishes when \( |x| \to \infty \). With equation (5), the last factor of equation (8) becomes

\[
\nabla^\nu \left( \frac{W_i(x)}{\rho} \right) = \frac{1}{\rho^2} \sum_j m_j \left[ \nabla^\nu W_i(x) W_j(x) - W_i(x) \nabla^\nu W_j(x) \right].
\]

Therefore, we can obtain the equation for the evolution of the particle positions,

\[
\ddot{x}_i = - \sum_j m_j \int \frac{1}{\rho^2} \left\{ \nabla^\nu W_i(x) W_j(x) - W_i(x) \nabla^\nu W_j(x) \right\}.
\]

### 2.2 Total energy equation

We define the equation for the evolution of the particle energy as

\[
\dot{E}_i = \int \frac{1}{\rho} \left( \nabla^\mu v^\nu \right) W_i(x).
\]

Substituting equation (1) into equation (11), one obtains

\[
\dot{E}_i = \int \frac{1}{\rho} \left( \nabla^\mu v^\nu \right) W_i(x) = - \int \frac{1}{\rho} \left( \frac{W_i(x)}{\rho} \right).
\]

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**Figure 2.** Results of the circularly polarized Alfvén wave test after 5 periods. The ordinates indicate the magnetic field perpendicular to the wavenumber, \( k \). The abscissas indicate the projected coordinate in the direction of \( k \). The analytic solution is shown by the solid line in each panel. All particles are plotted by circles. The results are shown at three different resolutions \( N_{tot} = 16 \times 32, 32 \times 64 \) and \( 64 \times 128 \) from bottom to top. The left and right panels correspond to hexagonal lattice and random distributions as the initial condition, respectively.
Using equation (9), we can obtain the energy equation of SPH particles,
\[
\dot{E}_i = - \sum_j m_j \int d^3x \left\{ \frac{T^\mu\nu v^\mu}{\rho^2} \{ \nabla^\nu W_i(x)W_j(x) - W_i(x)\nabla^\nu W_j(x) \} \right\}.
\] (13)

2.3 Induction equation

The \(i\)th SPH particle is assigned its own magnetic field, \(B_i\). The equation for the evolution of the magnetic field is defined as
\[
\frac{d}{dt} \left( \frac{B_i}{\rho} \right) = \int d^3x \frac{d}{dt} \left( \frac{B_i}{\rho} \right) W_i(x).
\] (14)

Substituting equation (2) into equation (14), one obtains
\[
\frac{d}{dt} \left( \frac{B^\mu}{\rho} \right) = \int d^3x \frac{B^\nu}{\rho} \nabla^\nu v^\mu W_i(x).
\] (15)

The right-hand side of equation (15) can be transformed into the following expression:
\[
\int d^3x \frac{B^\nu}{\rho} \nabla^\nu v^\mu W_i(x) = \int d^3x \frac{\nabla^\nu (B^\nu v^\mu)}{\rho} W_i(x) - \int d^3x \frac{v^\mu}{\rho} \nabla^\nu B^\nu W_i(x).
\] (16)

Figure 3. Results of the circularly polarized Alfvén wave test with (a) a cubic spline kernel and (b) a Gaussian kernel after 1 period. The total particle number is 64 \times 128. Half of the calculation region (0 \leq x < 1) is plotted. The solid line indicates the analytic solution in each panel. In each panel, the results are shown with different smoothing length \(C_h = 1.0\) (circles), 1.2 (triangles) and 1.5 (boxes) (see equation 29).

Figure 4. Results of the circularly polarized Alfvén wave test without a monotonicity constraint for \(N_{\text{tot}} = 64 \times 128\) (circles) after 5 periods. The solid line indicates the analytic solution.
We approximate the last term on the right-hand side of equation (15) as follows:
\[ \int v^\mu \frac{\nabla B^\mu}{\rho} W_i(x) d^3x = \dot{x}_i^{\mu} \int \frac{\nabla B^\mu}{\rho} W_i(x) d^3x + \mathcal{O}(h^2). \] (17)

Using equations (15) and (17) and integrating by parts, one obtains the following equation for the evolution of the magnetic field:
\[ \frac{d}{dt} \left( \frac{B^\mu}{\rho} \right)_i = -\sum_j m_j \int d^3x \frac{B^\nu}{\rho^2} \left( v^\mu - \dot{x}_j^{\mu} \right) \left\{ \nabla^\nu W_i(x) W_j(x) - W_i(x) \nabla^\nu W_j(x) \right\}. \] (18)

### 3 IMPLEMENTATION

#### 3.1 Convolution

We define the $s$-axis as being along the vector $x_i - x_j$. The unit vector in the $s$-direction is $n = (x_i - x_j)/|x_i - x_j|$. The distance between the $i$th and $j$th particles is $\Delta s_{ij} = s_i - s_j$, where $s_i$ and $s_j$ are the coordinates of the $i$th and $j$th particles on the $s$-axis, respectively. We need to perform the volume integral in equations (10), (13) and (18). If the smoothing length is spatially constant, the volume integral can

**Figure 5.** Results of the shock-tube problem at $t = 0.2$. The initial condition is $(\rho, P, v_x, v_y, v_z, B_y, B_z) = (1.08, 0.95, 1.2, 0.01, 0.5, 2/\sqrt{4\pi}, 2/\sqrt{4\pi})$ for $x < 0$ and $(1, 1, 0, 0, 0.4/\sqrt{4\pi}, 2/\sqrt{4\pi})$ for $x > 0$ with $B_z = 2/\sqrt{4\pi}$. The circles indicate the results of GSPMHD. The solid grey lines indicate the exact solution.
be performed analytically by interpolating physical variables (Inutsuka 2002). In a similar way, one can derive the GSPMHD equations as follows:

\[
\frac{d}{dt} \left( \frac{\mathbf{E}_i}{\rho_i} \right) = \sum_j m_j F_{ij} \left( \left( T^{\mu\nu} \right)^* n^\nu \left( n^\mu (T^{\mu\nu} v^\nu)^* \right) + \left( B^{\mu\nu} \right)^* n^\nu \left( \left( v^\mu \right)^* - \dot{x}^\mu_i \right) \right),
\]

where

\[
F_{ij} = 2V_{ij}^2(h) \frac{\partial W(\Delta s_{ij}, h)}{\partial s_i},
\]

\((T^{\mu\nu})^*, v^\nu\) and \(B^{\mu\nu}\) are the values at \(s = s^i_j\) (see Inutsuka 2002) and \(\dot{x}^i_j = v_i + \ddot{x}_i \Delta t/2\) is the time-centred velocity of the \(i\)th particle. The quantity \(V_{ij}^2(h)\) is obtained by the following integration:

\[
\int \rho^{-2} W_i(x) W_j(x) d^3 x = V_{ij}^2(h) W(x_i - x_j, \sqrt{2h}),
\]

where I02 used linear and cubic interpolations of \(\rho^{-1}(x)\). The detailed expression for \(V_{ij}^2(h)\) is found in I02. However, for the case with variable smoothing length, the volume integral cannot be easily performed analytically. I02 proposed a simple approximation in which he used \(h_i\) for the half of the integration space that includes \(x_i\) and \(h_j\) for the other half. In this case, \(F_{ij}\) becomes

\[
F_{ij} = V_{ij}^2(h_i) \frac{\partial W(\Delta s_{ij}, \sqrt{2h_i})}{\partial s_i} + V_{ij}^2(h_j) \frac{\partial W(\Delta s_{ij}, \sqrt{2h_j})}{\partial s_i}.
\]

This formulation can capture contact discontinuities accurately (Cha, Inutsuka & Nayakshin 2010; Murante et al. 2011). In this paper, just for simplicity, we use the following crude approximation in the volume integral of equations (10), (13) and (18):

\[
\nabla W_i(x) W_j(x) - W_i \nabla W_j(x) \simeq \nabla W_i(x) \delta(x - x_j) - \delta(x - x_i) \nabla W_j(x).
\]

Using this, one can obtain

\[
F_{ij} = \left( \frac{1}{\rho^*_i} + \frac{1}{\rho^*_j} \right) \frac{\partial W(\Delta s_{ij}, \bar{h}_{ij})}{\partial s_i},
\]

where \(\bar{h}_{ij}\) is an average of \(h_i\) and \(h_j\). This paper adopts \(\bar{h}_{ij} = (h_i + h_j)/2\).

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**Figure 6.** Results of the shock-tube test with initial condition \((\rho, P, v_x, v_y, v_z, B_x, B_y) = (1, 20, 10, 0, 0, 5/\sqrt{4\pi}, 0)\) for \(x < 0\) and \((1, -10, 0, 0, 5/\sqrt{4\pi}, 0)\) for \(x > 0\) with \(B_z = 5/\sqrt{4\pi}\) at \(t = 0.06\). The circles indicate the results of GSPMHD. The solid grey lines indicate the exact solution.
3.2 The usage of the Riemann solver

The equation (19) do not include a dissipative process, which is required to describe shock waves. The Godunov method uses the exact Riemann solver to include the minimum and sufficient amount of dissipation into the scheme. In the finite-volume method, the result of the Riemann problem at cell interfaces is used in the calculation of numerical flux. In the GSPH in I02, the values \( P^* \) and \( (Pv)^* \) in equations (58) and (59) in his paper are replaced by the results of the RP between the \( i \)th and \( j \)th particles. In the same way, \( (T^\mu{}^\nu)^* \) and \( (T^\mu{}^\nu{}^v)^* \) are replaced by the results of the RP between the \( i \)th and \( j \)th particles. In equation (19), the projection of \( T^\mu{}^\nu \) on the \( s \)-axis is found as follows:

\[
T^\mu{}^\nu n^\nu = - \left( P_1 - \frac{B_1^2}{2} \right) n^\mu + B_1 B_1^\mu,
\]

where \( P_1 = P + B_1^2 / 2 \) and the component parallel (perpendicular) to \( n \) is represented by using the subscript \( \parallel \) (\( \perp \)).

The first term on the right-hand side of equation (25) represents the compressive term working along the \( s \)-axis. In contrast, the second term represents the incompressible term working in the perpendicular direction. In the compressible part, we use the result of the non-linear RP without \( B_1 \), which contains fast shocks, fast rarefaction waves and one contact discontinuity. From the RP, one can obtain \( P_1^* \) and \( v_1^* \). The detailed description is shown in Appendix A. In the incompressible term, the MOC is used (Stone & Norman 1992). From the MOC, one can obtain \( B_1^* \) and \( v_1^* \). The detailed description is shown in Appendix B.

Figure 7. Contour maps of the Orszag–Tang vortex test at \( t = 0.5 \) for (a) density, (b) pressure, (c) magnetic energy \( B^2 / 2 \) and (d) specific kinetic energy \( v^2 / 2 \).
In the calculation of the RP and the MOC, the initial values on each side at \( s_{ij}^* \) are required. In this paper, we adopt \( s_{ij}^* = 0 \) for simplicity. It is confirmed that the value of \( s_{ij}^* \) does not affect the results. To make a spatially second-order method, we consider the piecewise linear distribution of the physical variables. Using the gradients, the initial values on each side of a one-dimensional RP are the average values of each domain of dependence:

\[
U_R = U_i - \frac{1}{2} \left( \frac{\partial U}{\partial s} \right)_i \left[ \Delta s_{ij} - C_i \Delta t \right],
\]

\[
U_L = U_j + \frac{1}{2} \left( \frac{\partial U}{\partial s} \right)_j \left[ \Delta s_{ij} - C_j \Delta t \right],
\]

where \( U = (\rho, P, v, B) \) and \( C \) is a characteristic speed. In the compressible RP, the speed of the fast wave \( C = \sqrt{(\gamma P + B^2)/\rho} \) is adopted. In the incompressible RP, the speed of the Alfvén wave \( C = |B|/\sqrt{\rho} \) is adopted.

In finite-volume methods with higher spatial accuracy, we need to impose a monotonicity constraint on the gradients of the physical variable to obtain a stable description of discontinuities. This is the case in the GSPMHD scheme. The detailed description of the monotonicity constraint adopted in this paper is shown in Appendix C.

In actual calculations, we solve the following equation:

\[
\frac{d}{dt} \begin{pmatrix} \dot{x}_i \\ E_i \\ (B/\rho)_i \end{pmatrix} = \sum_j m_j F_{ij} \begin{pmatrix} - \left( [P]_{RP} - P_{B.,i} \right) + B^*_{\perp MOC} \cdot (v_{\perp MOC} - \dot{x}_i) \\ - \left( [P]_{RP} - P_{B.,i} \right) (v_{\parallel i} + B^*_{\perp MOC} \cdot (v_{\perp MOC} - \dot{x}_i) \right) \\ B^*_{\parallel i} \left( (v_{\parallel i})_{RP} n + (v_{\perp MOC})_{MOC} - \dot{x}_i \right) \end{pmatrix},
\]

where subscripts ‘RP’ and ‘MOC’ indicate values evaluated using the RP and MOC, respectively, \( P_{B.} = (B_{\parallel,i}^2 + B_{\perp,i}^2)/4 \) and \( B^*_{\parallel} = (B_{\parallel,i} + B_{\perp,i})/2 \).

Figure 8. (a) Horizontal slices of the temperature in Orszag–Tang vortex at \( t = 0.5 \) taken at (a) \( y = 0.5 \), (b) \( y = 0.427 \) and (c) \( y = 0.3125 \). The black and grey lines in each panel denote results with GSPMHD and the finite-volume method, respectively.

Figure 9. Time evolution of the error of (a) total energy \( |E_{tot}/E_{tot}(t = 0) - 1| \) and (b) divergence error \( \delta_B \) for the Orszag–Tang Vortex test. (c) The spatial distribution of SPH particles for \( h_i |V \cdot B|/|B_i| > 0.05 \).
3.3 Variable smoothing length

In this paper, the variable smoothing length is used to obtain large dynamic ranges. The smoothing length of the \(i\)th particle is determined iteratively by

\[
h_i = C_h \left( \frac{m_i}{\sum_j m_j W(x_i - x_j, h_i)} \right)^{1/d},
\]

where \(C_h\) is a parameter. The density of the \(i\)th particle is evaluated by

\[
\rho_i = \sum_j m_j W(x_i - x_j, h_i).
\]

The Gaussian kernel is not truncated at a finite radius but has infinite range. In practical calculations, we ignore the contribution from the \(j\)th to \(i\)th particles if \(|x_i - x_j| > 3.1h\), because \(\exp(-3.1^2) \approx 6.7 \times 10^{-5}\) is sufficiently small. The number of neighbours becomes \(\sim 6C_h\) in one-dimensional (1D), \(\sim 30C_h^2\) in two-dimensional (2D) and \(\sim 124C_h^3\) in three-dimensional (3D) schemes. In this paper, we present the results of 2D test calculations and adopt \(C_h = 1.2\), indicating that the average neighbour number is \(\sim 43\).

3.4 Corrections for avoiding tensile instability due to magnetic force

From equation (25), one can see that the stress tensor can be negative when the plasma beta, \(\beta \equiv 2P/B^2\), is low. This causes unphysical clumping of SPH particles (Monaghan 1992). This numerical instability is called ‘tensile instability’ (Swegle, Hicks & Attaway 1995). The tensile instability arises from the fact that the SPH expression \(\nabla \cdot B\) is not completely zero. In finite-volume methods, it is well known that

![Figure 10. Contour maps of rotor problem at \(t = 0.15\) for (a) density, (b) gas pressure, (c) Mach number, \(|v|/\sqrt{\gamma p}r\) and (d) magnetic energy \(B^2/2\). The contour lines are the same as those in Tóth (2000).](https://academic.oup.com/mnras/article-abstract/418/3/1668/1061511)
non-zero $\nabla \cdot B$ produces an unphysical force along the magnetic field and causes large errors in the simulations when using the conservative form (Brackbill & Barnes 1980). Many authors have proposed methods for vanishing $\nabla \cdot B$, such as the projection method (Brackbill & Barnes 1980), the constrained transport (Evans & Hawley 1988), and so on.

In SPMHD, $\nabla \cdot B$ inevitably has some amount of numerical noise that causes tensile instability to occur even if divergence-cleaning methods are used in the conservation form (Price & Monaghan 2005). Therefore, several methods have been proposed to suppress tensile instability. Phillips & Monaghan (1985) proposed that the stress tensor be made positive by subtracting a constant value from the stress tensor (also see Price & Monaghan 2005). As another approach, Børve et al. (2001) suggested that the monopole source terms should be explicitly subtracted from the equation of motion and the energy equation, as follows:

$$\frac{dv^\mu}{dt} = \frac{1}{\rho} \nabla^\nu T^\mu_{\nu} - \frac{1}{\rho} B^\mu \nabla \cdot B$$

and

$$\frac{dE}{dt} = \frac{1}{\rho} \nabla^\nu (T^\mu_{\nu} v^\mu) - \frac{1}{\rho} (B \cdot v) \nabla \cdot B.$$  

The left-hand side of equation (31) corresponds to the Lorentz force $[(\nabla \times B) \times B]/\rho$. This source term significantly stabilizes the tensile instability. This formulation is the same as the so-called ‘8-wave’ formulations proposed by Powell et al. (1999) in the finite-volume method. This formulation is numerically stable, and the value of $\nabla \cdot B$ remains zero within the truncation error without any cleaning methods for simple test problems. However, in realistic 3D problems satisfaction of the divergence constraint is not guaranteed. Thus, we need to monitor

Figure 11. Horizontal slices of the rotor problem at (a) $y = 0$ (first and second columns) and (b) $x = 0$ (third and fourth columns). The density, pressure, velocities and magnetic fields are plotted. In each panel, the black and grey lines indicate results with GSPMHD and the finite-volume method, respectively.
the value of $\nabla \cdot \mathbf{B}$ in actual calculations. To derive the GSPMHD equations, we take the convolution of the source term in equation (31),

$$\int \frac{1}{\rho} \mathbf{B} \cdot \nabla \cdot \mathbf{B} W_i(x) d^3x \simeq -\mathbf{B}_i \int \mathbf{B} \cdot \nabla \left( \frac{W_i(x)}{\rho} \right) d^3x. \quad (33)$$

Using equation (9), one obtains

$$\ddot{x}^\mu_i = \sum_j m_j \left\{ (T^{\mu\nu})^n - B^\mu_i B^\nu_i \right\} F_{ij}. \quad (34)$$

In a similar way, the energy equation may also be obtained as

$$\dot{E}^\mu_i = \sum_j m_j \left\{ (T^{\mu\nu})^n - \mathbf{B}_i \cdot \dot{x}_i^\nu B^\nu_i \right\} F_{ij}. \quad (35)$$

The major disadvantage of this correction is violation of momentum and total energy conservation. Actually, in all SPMHD schemes without tensile instability conservation of momentum and/or total energy is sacrificed.

### 3.5 Divergence error estimate

In SPMHD, there are many choices of $\nabla \cdot \mathbf{B}$ estimate. In this paper, we use the following expression:

$$\left( \frac{1}{\rho} \nabla \cdot \mathbf{B} \right)_i = \int \frac{1}{\rho} \mathbf{B} \cdot \nabla W(x - x_i) d^3x = \sum_j m_j B^\mu_j F_{ij}. \quad (36)$$

In order to estimate errors in the $\nabla \cdot \mathbf{B} = 0$ constraint, we monitor $\delta_B$, defined as

$$\delta_B = \frac{1}{N_{tot}} \sum_i h_i | \rho_i (\nabla \cdot \mathbf{B} / \rho_i) |, \quad (37)$$

where $N_{tot}$ is the total number of SPH particles. This estimate is the same as that in the correction terms in Section 3.4. Note that PM05 and Børve et al. (2006) adopted a different choice,

$$(\nabla \cdot \mathbf{B})_i = \frac{1}{\rho_i} \sum_j m_j (B_i - B_j) \cdot \nabla_i W(x_i - x_j, h_i). \quad (38)$$

The divergence error estimated in equation (38) tends to be smaller than that in equation (36).

### 4 NUMERICAL TESTS

In this section, we show the results of test calculations in the two-dimensional GSPMHD method, starting with the convergence test.

#### 4.1 Convergence test

In this section we check the accuracy of the GSPMHD. A good problem for the convergence test is the propagation test of linear MHD waves in a uniform medium. First, we define the unperturbed state. A uniform and at rest gas is considered ($\rho = 1$, $P = 1$, $v = 0$). The magnetic field is parallel to the $x$-direction and its magnitude is $1/\sqrt{2}$. Simulations are performed in the square domain, $x, y \in [0, \sqrt{2}]$. The particles are uniformly spaced on a cubic lattice with sides parallel to the $x$- and $y$-axes. A periodic boundary condition is imposed.

We consider linear MHD waves having a wavenumber of $k = 2\pi(1/\sqrt{2}, 1/\sqrt{2}, 0)$, indicating that the simulation domain contains two wavelengths. It is well known that MHD linear waves consist of fast, Alfvén and slow waves with phase velocities given by 1.4, 0.5 and 0.46, respectively in this configuration. We add the initial perturbation based on the eigenmode. In the fast and slow modes, all fluctuations lie in...
the \((x, y)\) plane. The amplitude of the density fluctuation is set to \(10^{-3}\). In the Alfvén wave, only \(v_z\) and \(B_z\) fluctuate. The amplitude of the fluctuation of \(B_z\) is set to \(10^{-3}\). The vector component parallel (perpendicular) to the wavenumber is expressed in terms of the subscripts \(\xi\) (\(\psi\)) in the \((x, y)\) plane.

As a measure of the error we introduce the error vector, defined as

\[
epsilon = \frac{1}{N_{\text{tot}}} \sum_{i=1}^{N_{\text{tot}}} |U_{\text{ref}}(x_i) - U_i(x_i)|,
\]

where \(U = (\rho, v_{\xi}, v_{\psi}, B_{\psi}, E)\) for the fast and slow waves and \(U = (v_z, B_z)\) for the Alfvén wave. As a reference solution, \(U_{\text{ref}}\), we adopt the results with \(N_{\text{tot}} = 512 \times 512\). To eliminate the error coming from \(\Delta t\), it is set to the small value of \(3 \times 10^{-4}\) for all resolutions. The error vector is evaluated after 100 time-steps at various resolutions. In Fig. 1, the norm of the error vector is plotted as a function of the average smoothing length, which represents the resolution for the fast (circles), Alfvén (triangles) and slow waves (boxes). In the scheme having second-order spatial accuracy, \(|\epsilon|\) is expected to scale as \(h^2\). Fig. 1 shows that the error is proportional to \(h^2\) for all wave modes. Therefore, it is confirmed that our GSPMHD is spatially a second-order scheme.

4.2 Non-linear circularly polarized Alfvén wave

Tóth (2000) investigated a non-linear circularly polarized Alfvén wave that is one of the exact solutions of non-linear MHD. Following Tóth (2000), we set the following initial condition. The Alfvén wave propagates toward an angle \(\alpha = \pi/6\) with respect to the \(x\)-axis. The initial condition is \(\rho = 1\), \(P = 0.1\), \(B_{\xi} = 1\), \(v_{\phi} = B_{\phi} = 0.1 \sin(2\pi x_{\xi})\) and \(v_z = B_z = 0.1 \cos(2\pi x_{\xi})\), where \(x_{\xi} = x \cos \alpha + y \sin \alpha\). In order

![Figure 13. Contour maps of the moderate-\(\beta\) case at \(t = 0.15\) for (a) density, (b) gas pressure, (c) specific kinetic energy \((v^2/2)\) and (d) magnetic energy \((B^2/2)\). The 30 contour lines are shown for ranges \(0.14 < \rho < 2.78\), \(0 < P < 0.95\), \(0 < v^2/2 < 0.37\) and \(0.105 < B^2/2 < 1.4\).](#)
to investigate the effect of the particle distribution, two kinds of initial particle distributions are considered. One is an ordered distribution (a hexagonal packed lattice) and the other is a random distribution that is relaxed until the density dispersion is sufficiently small. In each particle distribution, we calculate this test at three resolutions, $16 \times 32, 32 \times 64$ and $64 \times 128$ particles.

The results are shown in Fig. 2 after five periods. The abscissa and the ordinate denote the projected coordinate $x_c$ and the perpendicular magnetic field $B_x$ in the $(x, y)$ plane, respectively. The exact solution is shown by the solid line in each panel. All particles are plotted by circles. The results are shown at three different resolutions $N_{\text{tot}} = 16 \times 32, 32 \times 64$ and $64 \times 128$ from bottom to top. The left and right panels correspond to the hexagonal lattice and random distributions as the initial condition, respectively. Fig. 2 also shows that the results with different initial particle configurations agree extremely well in each resolution. Price & Monaghan (2005) performed the same test. In their fig. 6, a phase error is found even at the highest resolution. In all panels of Fig. 2, the phase error is not seen in our GSPMHD. The phase error may come from the fact that PM05 used the cubic spline kernel. The results with cubic spline and Gaussian kernels after one period are shown in Figs 3(a) and (b) for $N_{\text{tot}} = 64 \times 128$, respectively. To see the phase error clearly, Fig. 3 is plotted for $0 \leq n \leq 1$. In each panel, we show the results with different smoothing length $C_h = 1.0$ (circles), 1.2 (triangles) and 1.5 (boxes) (see equation 29). The solid line indicates the analytic solution in each panel. From Fig. 3(a), even after one period one can see that the cubic spline kernel gives relatively large phase errors, the values of which depend on the smoothing length, $C_h$. On the other hand, the Gaussian kernel shows sufficiently small phase errors compared with the cubic spline kernel and the phase errors are nearly independent of $C_h$. Therefore, the Gaussian kernel is superior to the cubic spline kernel in the propagation of Alfvén waves.

Fig. 2 shows some errors around the extremal points of $B_x$ at $x_c = 0.25, 0.75, 1.25$ and 1.75. This error comes from the strange ‘clipped’ shape of the wave compared with the exact solution in Fig. 2, while it was not found in PM05. This is caused by the monotonicity constraint on the gradients of the physical variables, required for a stable description of discontinuities (see Section 3.2). Because the monotonicity constraint makes the scheme’s spatial accuracy first-order around extremal points, the profile around extremal points dissipates preferentially and is flattened as seen in Fig. 2. Fig. 4 shows the results without the monotonicity constraint for $N_{\text{tot}} = 64 \times 128$ (circles) after five periods. One can see that no deformation arises. Deformation of the wave shape is also found in finite-volume methods using the Monotone upstream-centered scheme for conservative laws (MUSCL) method (van Leer 1979). Apart from the wave shape, the waves in GSPMHD are less dissipated than those in PM05, who adopted an artificial resistivity.

### 4.3 Shock-tube problems

MHD shock-tube problems are widely used to test numerical codes. In this section, we calculate two shock-tube tests.

First, we define a shock tube where initial states are given by $(\rho, P, v_x, v_y, B_x, B_y) = (1.08, 0.95, 1.2, 0.01, 0.5, 3.6/\sqrt{4\pi}, 2/\sqrt{4\pi})$ for $x < 0$ and $(1, 1, 0, 0, 0, 4/\sqrt{4\pi}, 2/\sqrt{4\pi})$ for $x > 0$ with $B_z = 2/\sqrt{4\pi}$. Initially, SPH particles are distributed in a hexagonal lattice with a particle separation of $4 \times 10^{-3}$. The particle separation in the $x$-direction for $x > 0$ widens slightly to obtain the initial density discontinuity. The rectangular domain is $[-0.74, 0.5] \times [-3.2 \times 10^{-2}, 3.2 \times 10^{-2}]$. The same shock-tube problem was presented by Dai & Woodward (1994) and Ryu & Jones (1995) in finite-volume methods and PM05 in SPMHD. The exact solution consists of two fast shocks, two rotational discontinuities, two slow shocks and one contact discontinuity. Fig. 5 shows the results of GSPMHD at $t = 0.2$. The solid grey lines indicate the exact solution. One can see that GSPMHD describes all discontinuities very well. Our GSPMHD can resolve rotational discontinuities and slow shocks, although they are smeared out in fig. 9 of PM05. This may illustrate the contrast that our GSPMHD takes into account the characteristics of Alfvén waves and PM05 use an artificial resistivity in the induction equation.

![Figure 14. Horizontal slices of the blast wave in the relatively low-$\beta$ case at (a) $\alpha = \pi/4$ (first column) and (b) $\alpha = -\pi/4$ (second column). The density, pressure, velocities and magnetic fields are plotted. In each panel, the black and grey lines indicate results with GSPMHD and the finite-volume method, respectively.](https://academic.oup.com/mnras/article-abstract/418/3/1668/1061511)
Next, we perform a shock-tube test containing stronger shocks than the previous one. The initial states are given by \( (\rho, P, v_x, v_y, B_x, B_y) = (1, 20, 10, 0, 0, 5/\sqrt{4\pi}, 0) \) for \( x < 0 \) and \( (1, -10, 0, 0, 5/\sqrt{4\pi}, 0) \) for \( x > 0 \) with \( B_z = 5/\sqrt{4\pi} \). The same shock-tube problem was presented by Dai & Woodward (1994), Ryu & Jones (1995) and Tóth (2000) in finite-volume methods. The exact solution consists of two fast shocks, a left-propagating slow rarefaction wave, a right-propagating slow shock and one contact discontinuity. The initial particle distribution is a hexagonal lattice with average particle separation \( 5.4 \times 10^{-2} \) in \([-1, 1] \times [-1, 1] \). Fig. 6 shows the results of GSPMHD at \( t = 0.06 \), with solid grey lines indicating the exact solution. This figure shows that GSPMHD can reproduce the exact solution and describes all discontinuities better than PM05 and Børve et al. (2006). Fast shocks can be resolved by a small number of particles. In the finite-volume method, Tóth (2000) reported a relatively large error of \( B_z \) in his fig. 13, where the non-conservative method (Powell et al. 1999) is used. However, even in this kind of shock-tube problem with strong shocks our scheme shows the error in \( B_z \) to be less than 1 percent, except for the vicinity of the discontinuities.

### 4.4 Orszag–Tang vortex

The next test is the Orszag–Tang vortex problem, which was originally investigated by Orszag & Tang (1979) in incompressible MHD flows. This problem is a standard two-dimensional test for compressible MHD schemes (Tóth 2000). This calculation is performed in a \([0, 1] \times [0, 1] \) domain. At all boundaries, periodic boundary conditions are imposed. The initial conditions are given by \( \rho = 25/(36\pi), \quad P = 5/(12\pi), \quad \v(\mathbf{r}) = (\sin(2\pi y), \cos(2\pi x), 0) \) and \( \mathbf{B}(\mathbf{r}) = \frac{1}{\sqrt{4\pi}} (-\sin(2\pi y), \sin(4\pi x), 0) \).

Although the initial velocity and magnetic field are not random, the system moves into turbulence through non-linear interaction of MHD waves. Fig. 7 shows contour maps of the test at \( t = 0.5 \) for (a) density, (b) pressure, (c) magnetic energy \( \mathbf{B}^2/2 \) and (d) specific kinetic energy \( v^2/2 \). Fig. 7 can be directly compared with fig. 22 of Stone et al. (2008) and one can see that the agreement is excellent. To view the results quantitatively, we compare the horizontal cuts of temperature for GSPMHD and a finite-volume method with the approximate Riemann solver HLLD and the constrained transport (provided by Dr T. Matsumoto) in Fig. 8 at (a) \( y = 0.5 \), (b) \( y = 0.427 \) and (c) \( y = 0.3125 \). The black and grey solid lines denote the results with GSPMHD and the finite-volume method with a resolution of \( 256 \times 256 \). One can see that the profiles between the two methods show good agreement except for the peak at \( x = 0 \) in the \( y = 0.5 \) slice. Our scheme does not strictly conserve the total energy, \( E_\text{tot} = \sum m_i \v_i \). The time evolution of the relative error of \( E_\text{tot} \) is presented in Fig. 9(a). One can see that the error in \( E_\text{tot} \) is sufficiently small. Fig. 9(b) shows the time evolution of the divergence error, which is maintained at an acceptable level \( \sim 1 \) per cent. The distribution of the divergence error is localized at shock fronts. The spatial distribution of SPH particles for \( h_i | \nabla \cdot \mathbf{B} | / |\mathbf{B}| > 0.05 \) is shown in Fig. 9(c). The divergence error is highly localized at discontinuities. This error comes from the irregular particle distribution.

### 4.5 Rotor

The MHD rotor problem was introduced by Balsara & Spicer (1999) to test propagation of strong torsional Alfvén waves. The computation domain is a square unit \([-0.5, 0.5] \times [-0.5, 0.5] \). This problem consists of a dense and rapidly rotating cylinder (rotor) embedded in a rarefied uniform medium. The initial conditions are given by

\[
\begin{align*}
\rho &= 10, \quad \mathbf{v} = (-2y/r_0, 2x/r_0, 0) \quad \text{for} \quad r < \sqrt{x^2 + y^2} < r_0, \\
\rho &= 1 + 9f(r), \quad \mathbf{v} = f(r)(-2y/r, 2x/r, 0) \quad \text{for} \quad r_0 < r < r_1, \\
\rho &= 1, \quad \mathbf{v} = 0
\end{align*}
\]

![Figure 15](https://www.nature.com/mnras/article/418/3/1668/1061511)

Figure 15. The same as Fig. 9 but for the blast-wave test.
for $r > r_1$, where $f(r) = (r_1 - r)/(r_1 - r_0)$, $r_0 = 0.1$ and $r_1 = 0.115$. The pressure and the magnetic field $P = 1$, $B = (5/\sqrt{4\pi}, 0, 0)$ are uniform and the adiabatic index is $\gamma = 1.4$. All SPH particles are assumed to have the same mass. Therefore, the number density of SPH particles in the rotor is larger than that in the ambient gas. The SPH particle mass is determined so that the resolution in the ambient gas is as large as $256 \times 256$, leading to a total particle number of 86,968. The initial particle distribution is constructed using a relaxation method presented in Whitworth et al. (1995).

Fig. 10 shows contour maps of (a) density, (b) gas pressure, (c) Mach number $|v|/\sqrt{\gamma P/\rho}$ and (d) magnetic energy $B^2/2$ at $t = 0.15$. The contour levels are the same as those in fig. 18 of Tóth (2000). From Fig. 10, the results agree with Tóth (2000) quite well. Compared with other SPMHD schemes, such as that of Price & Monaghan (2005); Børve et al. (2006), the contours appear to be smoother.

For comparison with the finite-volume method in more detail, we plot horizontal slices of the rotor problem at $y = 0.5$ (first row) and at $x = 0.5$ (second row) in Fig. 11. In each panel, the black and grey lines indicate results with GSPMHD and the finite-volume method, respectively. One can see that the agreement between the two methods is quantitatively excellent in all variables. Since GSPMHD is a Lagrangian method, the resolution is better in the dense ring, while GSPMHD gives more diffusive results in the narrow region with low density ahead of the dense ring. Fig. 12 is the same as Fig. 9 but for the rotor test. In this test, the energy and the divergence error are maintained at a sufficiently low level.

![Figure 16](https://academic.oup.com/mnras/article-abstract/418/3/1668/1061511)

**Figure 16.** The same as Fig. 13 but for the low-$\beta$ case at $t = 0.02$. The 30 contour lines are shown for ranges $0.233 < \rho < 3.31$, $32.1 < P < 1.1$, $0 < v^2/2 < 13$ and $24.5 < B^2/2 < 76$. 

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4.6 Blast wave in a strongly magnetized gas

The next test is blast waves that propagate into a strongly magnetized gas (Balsara & Spicer 1999; Londrillo & Del Zanna 2000). The calculation region is a square domain of \([-0.5, 0.5] \times [-0.5, 0.5]\). In this problem, an overpressured hot region with a pressure of \(P_{\text{hot}}\) is set within \(r < r_0\), where \(r = \sqrt{x^2 + y^2}\). Around the hot central region there is a rarefied ambient gas with a pressure of \(P_{\text{amb}}\). The density \(\rho = 1\) is spatially uniform. The initial uniform magnetic field \(B = B_0(1/\sqrt{3}, 1/\sqrt{3}, 0)\) makes the angle \(\pi/4\) with the x-axis. Here, \(r_0, P_{\text{hot}}, P_{\text{amb}}\) and \(B_0\) are parameters. We consider two cases: a moderate-\(\beta\) case and a low-\(\beta\) case.

4.6.1 Moderate-\(\beta\) case

We adopt \(r_0 = 0.1, P_{\text{hot}} = 10, P_{\text{amb}} = 0.1\) and \(B_0 = 1\). Therefore, the plasma \(\beta\) of the ambient gas is as low as 0.2. These parameters were adopted in Gardiner & Stone (2005). Fig. 13 shows the contour maps of the blast wave at \(t = 0.15\) for (a) \(\rho\), (b) \(P\), (c) \(v^2/2\) and (d) \(B^2/2\). The total particle number is as large as 256 × 256. One can see the shock structures around the elongated hot bubble along \(B_0\). The fast (slow) shock propagates toward the direction parallel (perpendicular) to \(B_0\). This figure can be directly compared with the bottom column of Fig. 28 in Stone et al. (2008). One can see that the contour map is quite similar to those in Stone et al. (2008) except for the central rarefied hot bubble, where GSPMHD is more diffusive owing to its Lagrangian nature.

For comparison with the finite-volume method in detail, we consider slices of physical variables passing through the centre (0, 0). To characterize the direction of the slice, we introduce an angle \(\alpha\) that is the angle between the slice and the x-axis. Fig. 14 shows the results for the cases with \(\alpha = \pi/4\) and \(\alpha = -\pi/4\), which correspond to the directions parallel and perpendicular to the initial magnetic field, respectively. The subscripts || and \(\perp\) represent the components parallel and perpendicular to the direction of the slice. The black and grey lines indicate results with GSPMHD and the finite-volume method, respectively. One can see that the results with GSPMHD agree with those of the finite-volume method very well except in the central region, as mentioned above. In the profile of the parallel magnetic field \(B_\parallel\) for \(\alpha = \pi/4\), there are some wiggles near the contact discontinuity. This comes from the pressure jump in the initial condition and is not serious, because \(B_\parallel\) agrees with the finite-volume method in the other places. If the pressure jump is smoother, the wiggle becomes small. Fig. 15 is the same as Fig. 9 but for the blast-wave test. In this test, the energy and the divergence error are maintained at a sufficiently low level.

4.6.2 Low-\(\beta\) case

We adopt \(r_0 = 0.125, P_{\text{hot}} = 100, P_{\text{amb}} = 1\) and \(B_0 = 10\). Therefore, the plasma \(\beta\) of the ambient gas is as low as 0.02. These parameters were adopted in Londrillo & Del Zanna (2000) and Gardiner & Stone (2005). Fig. 16 is the same as Fig. 13 but for the low-\(\beta\) case. One can see that there is a stronger slow shock along the initial magnetic field than in the previous case. Fig. 17 shows the slices of the physical variables along \(\alpha = \pm\pi/4\) with respect to the x-axis, and is the same as Fig. 14 but for the low-\(\beta\) case. One can see that the results of GSPMHD coincide very well with those of the finite-volume method in the low-\(\beta\) case also.

5 SUMMARY

In this paper, we developed a new SPMHD scheme with the Godunov method. To take physical dissipation into account, we consider a non-linear RP with magnetic pressure and the MOC in the interaction between SPH particles, instead of the artificial dissipation used...
in previous works. Using the MUSCL method, spatially our scheme attains second-order accuracy $O(h^2)$, which is confirmed in the convergence test of linear MHD waves (see Section 4). From several test calculations in Section 4, it is confirmed that our method can capture all MHD discontinuities more accurately than previous proposed methods. GSPMHD can provide results comparable to finite-volume methods with approximate Riemann solvers. We will apply GSPMHD to astrophysical problems for which the Lagrangian description has advantages.

The GSPMHD described in this paper loses its strict conservation property with respect to both momentum and energy. This is to avoid tensile instability in strong magnetic fields. Although conservation errors are sufficiently small in the test calculations (see Section 4), the non-conservative formulation can be problematic in long-term calculations. Thus, further investigations are needed to improve the conservation property, together with achieving better performance in low-$\beta$ plasma cases.

In GSPMHD, the Gaussian kernel is used. Other kernel functions (e.g. the cubic spline kernel) can be applied easily in our GSPMHD if we use equation (24) or some other symmetrization of the kernel function. However, in SPMHD the choice of kernel functions may be important, in contrast to the HD case. In Section 4.2, it is shown that the cubic spline kernel brings relatively large phase errors into the propagation of Alfvén waves. In shock-tube tests, we confirm that results with the cubic spline kernel are worse than with the Gaussian kernel. To obtain reasonable results with the cubic spline kernel, one needs large neighbours $h_i > 1.5(m_i/\rho_i)^{1/2}$ in the two-dimensional code, as suggested in PM05. Thus we recommend the Gaussian kernel in GSPMHD.

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APPENDIX A: RIEMANN SOLVER WITH MAGNETIC PRESSURE

In this Appendix we present the non-linear Riemann solver. Fig. A1 shows the schematic picture of the non-linear RP. Initially, we consider two uniform states $U_L$, $U_R$ that are separated by a discontinuity at $m = 0$, where $U = (\rho, P, \mathbf{v}, B)\perp$ and $m \equiv \int_0^m \rho \, ds$ is the mass coordinate. The magnetic field in the $s$-direction, $B_s$, is assumed to be zero. The RP depends only on $B_\perp$. Since $\partial \mathbf{v}_\perp / \partial t = 0$ for $B_\parallel = 0$, $\mathbf{v}_\perp$ is constant spatially and temporally at each side even if $\mathbf{v}_\perp$ has a discontinuity at $m = 0$. Therefore, $\mathbf{v}_\perp$ does not affect the RP, suggesting that we can set $\mathbf{v}_\perp = 0$ without loss of generality.

In this RP, fast shock (FS) or fast rarefaction (FR) waves propagate outward as shown in Fig. A1. This configuration is the same as the RP used in the Godunov method of HD (Godunov 1959). The RP is separated into left and right intermediate states, $U_L^*$ and $U_R^*$, by the contact discontinuity (CD). At the CD, since the total pressure and velocity are continuous, one can obtain the following relations:

$$P_{t}^* \equiv P_{tL}^* + \frac{(B_{sL}^*)^2}{2} = P_{tR}^* + \frac{(B_{sR}^*)^2}{2},$$

(A1)

$$v_{\parallel}^* = v_{\parallel L}^* = v_{\parallel R}^*.$$  

(A2)

In order to take into account the FR exactly, we need numerical integration, which is computationally expensive. Therefore, we treat the FR as a ‘rarefaction shock’. This means that the shock-jump condition is used in the case of $P_{t}^* < P_{tL}$ or $P_{tR}$. This treatment is reasonably accurate because the tangential lines of the Hugoniot curve and the adiabatic curve coincide at any point in the $(\rho, P_t)$ plane.

From equation (1), the relations between the jump of total pressure and velocity across the right-facing fast shock and the left-facing fast shock are given by

$$P_{t}^* - P_{tR} = M_R (v_{\parallel}^* - v_{\parallel R}) \quad \text{and} \quad P_{t}^* - P_{tL} = -M_L (v_{\parallel}^* - v_{\parallel L})$$

(A3)

respectively, where $M_R$ and $M_L$ are the Lagrangian speeds of the right-facing and left-facing shocks, respectively. From equation (A3), the total pressure and velocity in the intermediate state are given by

$$P_{t}^* = \frac{1}{1/M_R + 1/M_L} \left[ \frac{P_{tR}}{M_R} + \frac{P_{tL}}{M_L} - (v_{\parallel R} - v_{\parallel L}) \right],$$

(A4)

$$v_{\parallel}^* = \frac{1}{M_R + M_L} \left[ M_R v_{\parallel R} + M_L v_{\parallel L} - (P_{tR} - P_{tL}) \right].$$

(A5)

The Lagrangian shock speeds $M_L, M_R$ are derived from the jump conditions across the shock:

$$[\rho \mathbf{v}_\parallel] = 0, \quad \left[ \frac{\rho v_{\parallel}^2 + P + \frac{B_\perp^2}{2}}{2} \right] = 0,$$

(A6)

$$[v_{\parallel} B_\perp] = 0, \quad \left[ \frac{\rho v_{\parallel}^2 + \gamma P}{\gamma - 1} \right] v_{\parallel} + v_{\parallel} B_\perp = 0.$$  

(A7)

Figure A1. Schematic picture of the Riemann problem. ‘CD’, ‘FS’ and ‘FR’ denote the contact discontinuity, fast shock and fast rarefaction wave.
Using equations (A6) and (A7), one can obtain
\[ M_c^2 = \frac{\rho_c}{4} \left[ (\gamma - 3) P_a + (\gamma + 3) P^* = (\gamma - 2) B_{\perp a}^2 + \sqrt{D_s} \right], \]  
(A8)
\[ D_s = \left\{ (\gamma + 1) P_a + (\gamma - 1) P^* \right\}^2 - (\gamma - 2) B_{\perp a}^2 \left\{ 2 (\gamma - 3) P_a + 2(\gamma + 3) P^* = (\gamma - 2) B_{\perp a}^2 \right\}, \]  
(A9)
where \( a \) is one of L and R. From equation (A8), since \( M_L \) and \( M_R \) depend on \( P^* \), equation (A4) is non-linear with respect to \( P^* \). Therefore, we solve equation (A4) iteratively by the following procedure. First, the total pressure \( P^{\text{eq}}(1) = (P_R + P_L)/2 \) is inserted into the right-hand side of equation (A4). Then we can obtain \( P^{\text{eq}}(2) \), which is also inserted into equation (A4) to get \( P^{\text{eq}}(3) \). The iteration is continued until the desired accuracy is reached. Finally, the velocity \( v^* \) is obtained from equation (A5).

**APPENDIX B: METHOD OF CHARACTERISTICS**

In this Appendix, the MOC is briefly reviewed. We consider the propagation of an Alfvén wave along the \( s \)-axis. For simplicity, we consider the case with \( B_I > 0 \). A more general expression including the case with \( B_I < 0 \) is presented later. The MHD equations for a one-dimensional (\( s \)-direction) incompressible fluid are given by
\[ \frac{d\mathbf{v}_\perp}{dt} = \frac{B_i}{\rho} \frac{\partial \mathbf{B}_\perp}{\partial s} \quad \text{and} \quad \frac{dB_i}{dt} = B_i \frac{\partial v_\perp}{\partial s}. \]  
(B1)
Equations (B1) can be written as
\[ \left( \frac{d\mathbf{v}_\perp}{dt} - \frac{1}{\sqrt{\rho}} \frac{dB_i}{dt} \right) + \frac{\mathbf{B}_\parallel}{\sqrt{\rho}} \left( \frac{\partial \mathbf{v}_\perp}{\partial s} - \frac{1}{\sqrt{\rho}} \frac{\partial B_i}{\partial s} \right) = 0 \]  
(B2)
and
\[ \left( \frac{d\mathbf{v}_\perp}{dt} + \frac{1}{\sqrt{\rho}} \frac{dB_i}{dt} \right) - \frac{\mathbf{B}_\parallel}{\sqrt{\rho}} \left( \frac{\partial \mathbf{v}_\perp}{\partial s} + \frac{1}{\sqrt{\rho}} \frac{\partial B_i}{\partial s} \right) = 0. \]  
(B3)
From equations (B2) and (B3), one can see that
\[ d\mathbf{J}_+ = d\mathbf{v}_\perp - \frac{dB_i}{\sqrt{\rho}} \quad \text{and} \quad d\mathbf{J}_- = d\mathbf{v}_\perp + \frac{dB_i}{\sqrt{\rho}} \]  
(B4)
are constant on trajectories with \( ds/dt = B_I/\sqrt{\rho} \) and \( ds/dt = -B_I/\sqrt{\rho} \), respectively. Fig. B1 shows the schematic picture. The partially updated values \( v^*_I \) and \( B^*_I \) at \( t + \Delta t/2 \) can be obtained by extrapolating back in time along \( C_+ \) and \( C_- \) to the present time-step, where all variables are known. The positions \( s_+ \) and \( s_- \) are the footpoints where \( C_+ \) and \( C_- \) intersect at \( s_0 \) on \( t + \Delta t/2 \), respectively (see Fig. B1). Using the Riemann invariant \( d\mathbf{J}_\pm \), the characteristic equations along \( C_+ \) and \( C_- \) are given by
\[ v^*_I - v^*_I - \frac{B^*_I - B^*_I}{\sqrt{\rho^*}} = 0 \quad \text{and} \quad v^*_I - v^*_I + \frac{B^*_I - B^*_I}{\sqrt{\rho^*}} = 0, \]  
(B5)
respectively.

From equations (B5), \( B^*_I \) and \( v^*_I \) are given by
\[ B^*_I = \left( \frac{1}{\sqrt{\rho^*}} + \frac{1}{\sqrt{\rho}} \right)^{-1} \left[ \frac{B^+_I}{\sqrt{\rho^*}} + \frac{B^-_I}{\sqrt{\rho}} - v^*_I + v^*_I \right] \]  
(B6)
and
\[ v^*_I = \frac{\sqrt{\rho^*} v^*_I + \sqrt{\rho} v^-_I - B^+_I + B^-_I}{\sqrt{\rho^*} + \sqrt{\rho}}, \]  
(B7)

*Figure B1.* Schematic picture of the method of characteristics for \( B_I > 0 \).
respectively. So far, we consider only the case with $B_0 > 0$. For the case with $B_0 < 0$, the Alfvén wave propagates in the opposite direction of the $s$-axis. Therefore, the positions of $s^+$ and $s^-$ replace each other. The general expressions are given by

$$B^*_\perp = \left( \frac{1}{\sqrt{\rho^+}} + \frac{1}{\sqrt{\rho^-}} \right)^{-1} \left[ \frac{B^+}{\sqrt{\rho^+}} + \frac{B^-}{\sqrt{\rho^-}} + \text{sgn}(B_0) \left( -v^+_{\perp} + v^-_{\perp} \right) \right]$$

and

$$v^*_{\perp} = \frac{\sqrt{\rho^+} v^+_{\perp} + \sqrt{\rho^-} v^-_{\perp} + \text{sgn}(B_0) \left( -B^+_{\perp} + B^-_{\perp} \right)}{\sqrt{\rho^+} + \sqrt{\rho^-}},$$

where $\text{sgn}(B_0)$ is the sign of $B_0$. In actual calculations, for $B_0$ we use the following simple average value: $(B_{0,i} + B_{0,j})/2$.

**APPENDIX C: MONOTONICITY CONSTRAINT**

To solve the RP and the MOC in the interaction between the $i$th and $j$th particles, we need to evaluate physical variables $U_L$ and $U_R$ at $s = (s_i + s_j)/2$, as shown in Section 3.2. In GSPMHD, second-order spatial accuracy can be achieved using a piecewise linear interpolation of the physical variables to determine $U_L$ and $U_R$. Fig. C1 shows the schematic picture of the linear interpolation of a physical variable $Q$ that is one of the components of $U$. The slope of the linear interpolation of $Q$ is simply assigned by the gradient at each particle’s position, which is evaluated as

$$\left( \frac{\partial Q}{\partial s} \right)_i = n \cdot \left( \sum_k m_k \nabla W(x_i - x_k, h_i) \right)$$

for $Q = \rho$, otherwise

$$\left( \frac{\partial Q}{\partial s} \right)_i = n \cdot \left( \sum_k \rho_k \frac{Q_k - Q_i}{\Delta s_{ij}} \nabla W(x_i - x_k, h_i) \right).$$

Using the gradient, the values on the left- and right-hand sides in the RP are given by

$$Q_L = Q_j + \frac{1}{2} \Delta Q_j \quad \text{and} \quad Q_R = Q_i - \frac{1}{2} \Delta Q_i,$$

respectively, where

$$\Delta Q_j = \left( \frac{\partial Q}{\partial s} \right)_i \Delta s_{ij} \quad \text{and} \quad \Delta Q_i \equiv \left( \frac{\partial Q}{\partial s} \right)_j \Delta s_{ij}.$$

If equation (C1) is used directly in the derivation of $Q_L$ and $Q_R$, unphysical numerical oscillations arise (van Leer 1979). In order to obtain a stable second-order scheme, we need to impose monotonicity constraints on $\Delta Q$. In the finite-volume method, van Leer (1979) proposed several monotonicity constraints. We apply one of them to GSPMHD as follows:

$$(\Delta Q'_j)^{\text{mono}} = \begin{cases} \min \{ 2|Q - Q_j|, \, |\Delta Q_j|, \, 2|\Delta Q'_j| \} \text{sgn}(\Delta Q'_j) \quad & \text{if} \quad \text{sgn}(Q - Q_j) = \text{sgn}(\Delta Q_j) = \text{sgn}(\Delta Q'_j), \\ 0 & \text{otherwise}, \end{cases}$$

where $\Delta Q'_j$ satisfies

$$\Delta Q'_j = \frac{(Q_j - Q_{j-1}) + \Delta Q'_j}{2}.$$
Here, we consider the case with $\Delta Q_i > 0$ and $Q_i - Q_j > 0$ as shown in Fig. C1. The first two terms in equation (C5), $\min (2(Q_i - Q_j), \Delta Q_i)$, ensure the condition $Q_R > Q_j$ (see Fig. C1). This is the lower bound of $Q_R$ for $\Delta Q_i > 0$. In the finite-volume method, the upper bound of $Q_R$ is determined by $Q$ for $s > s_i$. On the other hand, in the SPH method we do not know the distribution of $Q$ for $s > s_i$ explicitly at the instant of calculation of the interaction between the $i$th and $j$th particles. However, it can be estimated by the fact that $\frac{\partial Q_i}{\partial s}$ is calculated using all particles for $s < s_i$ and $s > s_i$, suggesting that $\frac{\partial Q_i}{\partial s}$ can be regarded as the average gradient around $x_i$. If the gradient for $s_j < s < s_i$ is simply approximated by $(Q_i - Q_j) / \Delta s_{ij}$, the gradient for $s > s_i$, $\Delta Q / \Delta s_{ij}$, can be guessed by equation (C6). In this paper, we set the upper bound of $Q_R$ ad $\Delta Q_i$ (see equation C5).

In the actual calculation, we take into account the domain of dependence as follows:

$$Q_L = Q_j + \frac{\Delta Q_j}{2} \left( 1 - \frac{C_j \Delta t}{\Delta s_{ij}} \right) \quad \text{and} \quad Q_R = Q_i - \frac{\Delta Q_i}{2} \left( 1 - \frac{C_i \Delta t}{\Delta s_{ij}} \right).$$

(C7)

In the RP, we use the monotonicity constraint with respect to $\rho, \mathbf{v} \cdot \mathbf{n}, P, B^2 \perp$. In the MOC, the monotonicity constraint is used with respect to the Riemann invariants $\Delta J_{\pm,i}$ (see equation B4). Using them, we derive $\Delta B_{L,i}$ and $\Delta v_{L,i}$. Equation (C5) suppresses numerical oscillations reasonably well. However, many improvements could still be made to the monotonicity constraint in the SPH method.

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