Quantum Fields in Schwarzschild-de Sitter Space

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Abstract

In the No-Boundary Universe a primordial black hole is created from a constrained gravitational instanton. The black hole created is immersed in the de Sitter background with a positive gravitational constant. The constrained instanton is characterized not only by the external parameters, the mass parameter, charge and angular momentum, but also by one more internal parameter, the identification period in the imaginary time coordinate. Although the period has no effect on the black hole background, its inverse is the temperature of the no-boundary state of the perturbation modes perceived by an observer. By using the Bogoliubov transformation, we show that the perturbation modes of both scalar and spinor fields are in thermal equilibrium with the black hole background at the arbitrary temperature. However, for the two extreme cases, the de Sitter and the Nariai models, the no-boundary state remains pure.

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I. Introduction

The First Cause Problem has been dispelled by the Hawking theory of quantum cosmology. The no-boundary proposal, in principle, has the power to predict the evolution of the universe and its matter content [1]. However, due to the technical difficulty, the calculation has only been carried out for some special models. At this moment it is believed that the Hawking massive scalar model is the most realistic one [2]. In this model, the evolution of the early stage in real time can be approximated by a de Sitter spacetime with an effective cosmological constant $\Lambda$. A de Sitter universe can be thought of as originating through a quantum transition from a Euclidean $S^4$ space. Since the $S^4$ space is the most symmetric manifold, it is conjectured that the de Sitter expansion is the most probable evolution in the Planckian era of the universe.

On the other hand, a black hole is the simplest, i.e. the most beautiful object except for the universe itself in science. The discovery of Hawking radiation is the most dramatic event in gravitational physics for several decades [3]. Unfortunately, the time scale of evaporation for a macroscopic black hole is much longer than the age of the universe. The only hope to observe the radiation is from those microscopic black holes formed in the very early universe. These are so-called primordial black holes. A gravitational collapse of a matter fluctuation can lead to the black hole formation. However, in this paper we shall only consider the most interesting and dramatic way, that is, the black hole is quantum mechanically created at the same moment of the birth of the universe. If the evolution of the very early universe is dominated by the presence of the effective cosmological constant, then the primordial black hole should be immersed in the background of the de Sitter spacetime. Its metric can be described by a member of the Kerr-Newman-de Sitter family [4].

In quantum cosmology it was thought that, at the WKB level, the Lorentzian evolution could be obtained through an analytic continuation from a instanton at its equator $\Sigma$. This is true only for some special cases, like the de Sitter model and the Nariai model which represents a pair of black holes. Here, the black hole mass parameter takes two discrete values 0 and $m_c = \Lambda^{-1/2}/3$, and the charge $Q$ and angular momentum $ma$ are vanishing [5]. The quantum creations of these two models have maximum and minimum probabilities, respectively, in comparison with black hole creation with different parameters $m$, $Q$ and $ma$ and the same cosmological constant. In general, if a wave function represents an ensemble of evolutions with continuous parameters, like the case of a
primordial black hole, then it is unlikely that all these trajectories of the ensemble can be obtained through an analytic continuation from an ensemble of instantons. Unless the action is a constant function of the parameters, then the action cannot be stationary with respect to them in the range of the parameters. On the other hand, the condition for the existence of an instanton is that its action must be stationary.

It was realized recently that a Lorentzian evolution does not need to originate from a regular instanton. Instead, a generalized version of an instanton can provide a seed for the evolution. When one calculates the creation probability through a path integral, the sum is over all no-boundary compact 4-metrics with the given $\Sigma$ as its sector of the quantum transition. The dominating contribution is due to the manifold with a stationary action. Consequently, the manifold should satisfy the Einstein and other field equations everywhere with the possible exception of the equator $\Sigma$. This is derived from the variational calculation with the constraints imposed by the 3-metric and matter fields on $\Sigma$. The stationary action solution is called a constrained gravitational instanton [6].

This is exactly what happens in the case of primordial black hole creation. Here, the topology of the space is $S^2 \times S^1$, instead of $S^3$ in the de Sitter model. The constrained instanton is obtained through a periodic identification of the imaginary time in the Euclidean solution. No period can make the whole manifold regular except for the de Sitter model, the Nariai model or some special models. The identification will lead to at least one conical singularity at the black hole and cosmological horizons. It turns out that although the Euclidean action is not stationary with respect to the three parameters $m$, $Q$ and $ma$ which characterize the 3-metric and matter fields on $\Sigma$, it is independent of the period. Thus the action is stationary under the constraint imposed by the configuration of the wave function at the quantum transition surface. Therefore, we have found the constrained gravitational instanton [6].

The equator passes through the two horizons of the instanton. The Lorentzian evolution emanates from the equator regardless of the identification period. Furthermore, since at the $WKB$ level, the probability is the exponential of the negative of the instanton action, and we know the action is independent of the period, it follows that there is no effect of the period on the background of spacetime. This is in sharp contrast with the case of a black hole immersed in an asymptotically flat space. The instanton action of a asymptotically flat black hole does depend on the period. The
probability of our black hole creation is the exponential of a quarter of the sum of the black hole and cosmological horizons, or the exponential of the total entropy of the universe. The probability is a decreasing function of the magnitudes of these parameters. Therefore, our result supports the claim that the de Sitter evolution is most probable [6].

On the other hand, if we study the quantum state of perturbation modes in the black hole background, then the no-boundary proposal implies that it should be in the minimum excitation state. However, an observer can only measure the configuration within his black hole and cosmological horizons. One has to take the trace of the wave function over the unobserved configuration of the perturbation mode. The quantum state observed is described by a density matrix. This matrix represents a thermal equilibrium state with a temperature represented by the inverse of the identification period in imaginary time. The arbitrariness of the period means that the fields can be in thermal equilibrium with the black hole at any temperature.

It is interesting to note that in the de Sitter model, the black hole horizon disappears, while in the Nariai model, the two horizons degenerate. For both cases, nothing prevents an observer from measuring the quantum state at the whole equator. Therefore the associated temperature is zero. The arbitrariness of the identification period has effect neither on the background spacetime nor on the perturbation modes.

At the WKB level, the Wheeler-DeWitt equation in quantum cosmology can be decomposed into two parts. One is for the background, the other is for the perturbation modes, if we ignore the back reaction of the perturbations to the background. Then a physical time coordinate naturally emerges from the wave packet for the background. The second part of the Wheeler-DeWitt equation takes the form of the Schroedinger equation in the spacetime background. One shall work in the framework of quantum fields in curved spacetime supplemented by the no-boundary condition.

We shall consider a massive scalar field and a massive spin-1/2 field in the Schwarzschild-de Sitter background. By using the Bogoliubov transformation, we show that to an observer whose observation is always restricted by the two horizons, the no-boundary quantum state of bosonic or fermionic fields presents a thermal equilibrium state with temperature being the inverse of the imaginary time period. Therefore, in the Schwarzschild-de Sitter spacetime, the quantum fields can coexist with the black hole at an equilibrium state with an arbitrary temperature. In the two
extreme cases, the de Sitter model and the Nariai model, since the topologies of the equators are different from the nonextreme case, nothing prevents the observer from measuring the fields on the whole equators and the quantum states remain pure.

We shall review the approach of constrained gravitational instanton in Sec. II. This method will be used in Sec. III to study the quantum state of the Kerr-Newman-de Sitter metric family, which represents a single primordial black hole immersed in de Sitter spacetime background. Sec. IV investigates the no-boundary quantum state of the scalar field in the background. A similar calculation is carried out for the spinor field in Sec. V. Sec. VI is a discussion of this.

II. The constrained gravitational instanton

In the No-Boundary Universe, the wave function of the universe is given by the Hartle-Hawking ground state [1]

\[
\Psi(h_{ij}, \phi) = \int_C d[g_{\mu\nu}] d[\phi] \exp(-\bar{I}(g_{\mu\nu}, \phi)),
\]

where the path integral is over the class \(C\) of all compact Euclidean 4-metrics and matter field configurations which agree with the given 3-metrics \(h_{ij}\) of the only boundary and matter configuration \(\phi\) on it. Here \(\bar{I}\) means the Euclidean action.

The Euclidean action for the gravitational part of a smooth spacetime manifold \(M\) with boundary \(\partial M\) is

\[
\bar{I} = -\frac{1}{16\pi} \int_M (R - 2\Lambda) - \frac{1}{8\pi} \int_{\partial M} K,
\]

where \(\Lambda\) is the cosmological constant, \(R\) is the scalar curvature and \(K\) is the trace of the second fundamental form of the boundary.

The dominant contribution to the path integral comes from some stationary action manifolds with matter fields on them, which are the saddle points for the path integral. Then the wave function takes a superposition form of wave packets

\[
\Psi \approx C \exp(-S/h),
\]

where we have written \(h\) explicitly; \(C\) is a slowly varying prefactor; and \(S \equiv S_+ + iS_\times\) is a complex phase. \(S\) is identified with the action \(\bar{I}\) here.
Since the wave packets of form (3) are not independent in the decomposition of the wave function, one more restriction should be imposed. That is, the wave packets themselves should obey the Wheeler-DeWitt equation. Classically, this means that the evolutions represented by the wave packet should satisfy the Einstein equation with some quantum corrections, as will be shown below.

From the Hartle-Hawking proposal one can derive the probability of the 3-surface $\Sigma$ with the matter field $\psi$ on it

$$P = \Psi^* \Psi = \int_C d[g_{\mu\nu}] d[\psi] \exp(-\bar{I}([g_{\mu\nu}, \psi])), \quad (4)$$

where the class $C$ is composed of all no-boundary compact Euclidean 4-metrics and matter field configurations which agree with the given 3-metric $h_{ij}$ and the matter field $\psi$ on $\Sigma$. We shall concentrate on the 3-geometries at which a quantum transition from a so-called Euclidean sector to a Lorentzian sector occurs at the WKB level. These two sectors are represented by the wave packets with purely real and imaginary phases. These kinds of transitions are quite special and are called real tunneling.

Here, we do not restrict the class $C$ to regular metrics only, since the derivation from Eq. (1) to Eq. (4) has already led to some jump discontinuities in the extrinsic curvature at $\Sigma$.

The main contribution to the path integral in Eq. (4) comes from the stationary action 4-metric, which meets all requirements on the 3-surface $\Sigma$. At the WKB level, the exponential of the negative of the stationary action is the probability of the corresponding Lorentzian trajectory.

From the above viewpoint, extending the class $C$ to include metrics with some mild singularities is essential. Indeed, it is recognized that, in some sense, the set of all regular metrics is not complete. For many cases, under the usual regularity conditions and the requirements at the equator $\Sigma$, there may not exist any stationary action metric, i.e. gravitational instanton. Therefore, it seems reasonable to include metrics with jump discontinuities of extrinsic curvature and their degenerate cases, that is, the conical or pancake singularities [7].

If we lift the requirement on the 3-metric of the equator, then the stationary action solution becomes the regular gravitational instanton, as it satisfies the Einstein equation everywhere. Then the following Gibbons-Hartle condition should hold at the equator [8]

$$K_{ij} = 0, \quad (5)$$
where $K_{ij}$ is the extrinsic curvature of the equator. The probability of the corresponding trajectory takes stationary value; it may be maximum, minimum or neither [9].

If the regular gravitational instanton has minimum action, then the Lorentzian evolution emanating from it is singled out as the most probable trajectory. Therefore, quantum cosmology fully realizes its prediction power: there is no degree of freedom left [9]. Without using the instanton theory, the degree of freedom is only reduced to half by the ground state proposal. Roughly speaking, this is due to the regularity condition at the south pole of the Euclidean manifold in the path integral.

If in addition, one needs to calculate the probability of a trajectory emanating from a given 3-metric, then the extra requirement on $\Sigma$ is so strong that no gravitational instanton exists, unless the 3-metric coincides with a sector of a regular gravitational instanton. However, one can still find a stationary action trajectory from variational calculus. From the variational principle, the trajectory obeys the Einstein equation and other field equations on the 4-manifold with some exceptions on $\Sigma$, and the Gibbons-Hartle condition is no longer valid there.

In this way one can find a irregular gravitational instanton with some mild singularities within the class $C$. It is not a gravitational instanton in the conventional sense, and is called a constrained gravitational instanton. Even if the action is stationary under the constraint of $\Sigma$, the action or the probability of the real tunneling associated with it will not be stationary when lifting the 3-metric requirements. This is in comparison within the set of all classical trajectories.

In summary, the action of a constrained instanton is not necessarily stationary with respect to the so-called external parameters which characterize the 3-metric of quantum transition and matter field on it unless one seeks the most probable Lorentzian trajectory. However, the action should be stationary with respect to the so-called internal parameters which are the rest of the continuous parameters. In fact, if there exist some internal parameters, the action should be independent of them. The reason is as follows: if the action depends on some of the internal parameters, then the instantons can be singled out from them by using the stationary action condition, and these parameters are not qualified and should be excluded from the very beginning.

The occurrence of singularities here can be considered as a purely quantum effect at the $WKB$ level. One should welcome the situation, if he admits that Nature is quantum. Classically, one can
say that the solution obeys the generalized Einstein equation in the whole manifold.

III. The quantum state of the spacetime background

It is believed that the universe at the Euclidean and inflation stage can be approximated by an $S^4$ space and a de Sitter space with an effective cosmological constant $\Lambda$. In the Hawking massive scalar model [2], one can set $\Lambda = 3m_0^2\phi_0^2$, where $\phi_0$ is the initial value of the scalar field. A primordial black hole is sitting in the background of the de Sitter spacetime. A chargeless and non-rotating black hole with the de Sitter background can be described by the Schwarzschild-de Sitter spacetime. It is the unique spherically symmetric vacuum solution to the Einstein equation with a cosmological constant $\Lambda$. The $S^2 \times S^2$ Nariai spacetime is its degenerate case, and it represents a pair of black holes.

Its Euclidean metric can be written as

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)dr^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2d\Omega_2^2. \quad (\tau = it)$$  \hspace{1cm} (6)

The black hole and cosmological horizons $r_2$ and $r_3$ are determined by the factorization of the potential

$$\Delta = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} = -\frac{\Lambda}{3r}(r - r_0)(r - r_2)(r - r_3),$$  \hspace{1cm} (7)

where $r_0$ is the horizon for the negative $r$. We are interested in the Euclidean sector $r_2 \leq r \leq r_3$ for $0 \leq m \leq m_c = \Lambda^{-1/2}/3$. For the extreme case $m = m_c$ the sector degenerates into the $S^2 \times S^2$ space, or the Euclidean version of the Nariai metric [5].

The surface gravities $\kappa_2$ and $\kappa_3$ on the two horizons are [4]

$$\kappa_2 = \frac{\Lambda}{6r_2}(r_3 - r_2)(r_2 - r_0),$$  \hspace{1cm} (8)

$$\kappa_3 = \frac{\Lambda}{6r_3}(r_3 - r_2)(r_3 - r_0).$$  \hspace{1cm} (9)

A constrained gravitational instanton can be constructed as follows [6]. In the $(\tau - r)$ plane $r = r_2$ is an axis of symmetry, the imaginary time coordinate $\tau$ is identified with period $\beta_2 = 2\pi \kappa_2^{-1}$, and $\beta_2^{-1}$ is the Hawking temperature. This makes the Euclidean manifold regular at the black hole horizon. One can also apply this procedure to the cosmological horizon with period $\beta_3 = 2\pi \kappa_3^{-1}$, and
$\beta_3^{-1}$ is the Gibbons-Hawking temperature [4]. For the $S^2 \times S^2$ case, these two horizons are identical, and one obtains a regular instanton. Except for the $S^2 \times S^2$ spacetime, one cannot simultaneously regularize the whole manifold at both horizons due to the inequality of the two temperatures.

One can make two cuts along $\tau = const.$ between $r = r_2$ and $r = r_3$ and then identify them. Then a $f_2$-fold cover turns the $(\tau - r)$ plane into a cone with a deficit angle of $2\pi(1 - f_2)$ at the black hole horizon. In a similar way, one can have an $f_3$-fold cover at the cosmological horizon. To construct a symmetric manifold, $f_2$ and $f_3$ can be any pair of real numbers satisfying the relation

$$f_2 \beta_2 = f_3 \beta_3.$$  \hspace{1cm} (10)

Consequently, the parameter $f_2$ or $f_3$ is the only degree of freedom left. One only needs to see whether the above action is stationary with respect to this parameter in order to make sure the manifold obtained is truly a constrained gravitational instanton. We set $\Delta \tau = |f_2 \beta_2|$ below. If $f_2$ or $f_3$ is different from 1, then the cone at the black hole or cosmological horizon will have an extra contribution to the action. The integral of $K$ with respect to the 3-area in the boundary term of the action (2) is the area increase rate along its normal. Thus, the extra contribution due to the conical singularities can be considered as the degenerate form

$$\bar{I}_2,\text{deficit} = -\frac{1}{8\pi} \cdot 4\pi r_2^2 \cdot 2\pi (1 - f_2),$$  \hspace{1cm} (11)

$$\bar{I}_3,\text{deficit} = -\frac{1}{8\pi} \cdot 4\pi r_3^2 \cdot 2\pi (1 - f_3).$$  \hspace{1cm} (12)

The addition of the boundary term to the volume action is equivalent to the change from the extrinsic curvature representation to the metric representation. For our case, at the classical level, the momentum at the equator vanishes except for the two horizons. At the horizons, the pair of canonical conjugate variables can be described by the horizon area and the deficit angle. Taking into account the fact that the extrinsic curvature vanishes elsewhere, the terms $\bar{I}_{i,\text{deficit}}$ are the only nonvanishing Legendre terms.

The volume term of the action for the manifold can be calculated as

$$\bar{I}_{vol} = -\frac{A}{6} (r_3^3 - r_2^3) f_2 \beta_2.$$  \hspace{1cm} (13)

Using Eqs. (10) - (13), one can get the total action

$$\bar{I}_{\text{total}} = -\pi (r_2^2 + r_3^2).$$  \hspace{1cm} (14)
This is one quarter of the negative of the sum of the two horizon areas.

One readily notices that the action is independent of the choice of \(f_2\) or \(f_3\). Our result (14) shows that the constructed manifold is indeed a constrained gravitational instanton, and \(f_2\) or \(f_3\) is identified as the internal parameter. We can set \(\tau = \pm \Delta \tau / 4\) for the equator of the quantum transition. Our calculation shows that no matter which flat fragment of the constrained gravitational instanton is chosen, the same black hole should be created with the same probability. Of course, the most dramatic case is that of no volume, i.e. \(f_2 = f_3 = 0\).

The probability of the black hole creation is

\[
P_m \approx \exp(\pi(r_2^2 + r_3^2)).
\]

This result interposes two special cases [5]. The first is the de Sitter model with \(m = 0\),

\[
P_0 \approx \exp\left(\frac{3\pi}{\Lambda}\right)
\]

and the second is the Nariai model, or pair black hole creation, with \(m = m_c\),

\[
P_c \approx \exp\left(\frac{2\pi}{\Lambda}\right).
\]

The probability is an exponentially decreasing function in terms of the mass parameter. The de Sitter case has the maximum probability and the Nariai case has the minimum probability.

If one includes an electromagnetic field into the model, one would be able to carry out a similar calculation [10]. One just simply replaces the potential by

\[
\Delta = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} = -\frac{\Lambda}{3r^2}(r - r_0)(r - r_1)(r - r_2)(r - r_3),
\]

where \(Q\) is the charge parameter of the black hole.

For the magnetically charged black hole case, the configuration of the wave function is the 3-metric and magnetic charge. However, the configuration for the wave function of an electrically charged black hole is not well defined [5][6][11][12], if one naively uses the folding and gluing techniques described above. For the electric case, the configuration of the wave function is the 3-metric and the canonical momentum conjugate to the charge. In order to get the wave function for the charge, one has to appeal to a Fourier transformation, by which the duality between electric and magnetic black holes is recovered.
One can also investigate the problem of creation of a rotating black hole, that is, of a Kerr-Newman-de Sitter black hole. The configuration of the wave function from the naive cutting, folding and gluing is the 3-metric, matter field and the differentiation rotation of the two horizons. One has to use another Fourier transformation to obtain the wave function for the angular momentum. More crucially, only under this consideration, is the Euclidean action stationary, and our construction becomes meaningful.

The probability of black hole creation is the exponential of a quarter of both black hole and cosmological horizon areas, or the exponential of the total entropy. It is an exponentially decreasing function with respect to the mass, charge amplitude and angular momentum. The de Sitter spacetime without black hole is the most probable evolution at the Planckian era. Due to the No-Hair theorem, the problem of a single black hole creation in quantum cosmology has been completely resolved [6].

In this paper we study quantum fields in the Schwarzschild-de Sitter black hole background only.

The whole scenario of the Schwarzschild-de Sitter black hole creation is shown in Fig. 1. The $S^2$ space ($\theta - \phi$) is represented by a $S^1$ space around the vertical axis. The radius of $S^2$ is $r$. The bottom part is the instanton, the upper part shows the black hole created. It shows the collapsing of the internal edge of the doughnut from the black hole horizon into the singularity $r = 0$ and the expansion of the external edge from the cosmological horizon to $r = \infty$. The $S^1$ equator in space ($\tau - r$) and the conical singularities are not explicitly shown here. The observer can only measure the fields at one half of the equator between the two horizons.

The global aspect of the scenario is depicted by the Penrose-Carter diagram in Fig. 2. The $S^2$ space ($\theta - \phi$) is depressed. There is an infinite sequence of diamond shape regions, singularities $r = 0$ and spacelike infinities $r = \infty$. The scenario of the black hole creation can be obtained by an identification, for instance, with lines $ABC$ and $DEF$ [13]. The equator in the instanton is identified as the closed line $BE$ here. We shall set $t = 0$ and $\tau = \Delta \tau/4$ for line $\Sigma_1$ or $OE$ and $t = 0$ and $\tau = -\Delta \tau/4$ for the line $\Sigma_2$ or $BO$.

IV. The quantum state of the bosonic field

Now we are going to discuss the quantum state of perturbation modes in the Schwarzschild-de

11
Sitter spacetime background. The quantum fields in the Schwarzschild spacetime background has been studied [14][15][16][17][18][19].

We shall show that the quantum state of the matter fields in the same black hole background is characterized by the parameter of the identification period in the imaginary time.

If we neglect the back reaction of the perturbations to the background, then the wave function of the universe for a fixed $\Delta \tau$, at the WKB level, can be written as a product [20]

$$\Psi(h_{ij}, \varphi, \psi) = \Psi(h_{ij})\Psi(h_{ij}, \varphi)\Psi(h_{ij}, \psi),$$

(19)

where $\Psi(h_{ij})$ is the wave function for the spacetime background, $\Psi(h_{ij}, \varphi)$ is the wave function for the scalar perturbation modes and $\Psi(h_{ij}, \psi)$ is the wave function for the spinor perturbation modes. The contributions to the wave functions come from the classical trajectories which satisfy the no-boundary condition. The internal time coordinate emerges naturally from the wave packet $\Psi(h_{ij})$ for the background. The Wheeler-DeWitt equation can be decomposed into several parts. The parts for the perturbation modes take the form of the Schroedinger equation in the spacetime background. It means, at the level of our approximation, the study of perturbation modes in quantum cosmology can be reduced to that of quantum fields in curved spacetime supplemented by the no-boundary condition. This is exactly what we are doing now. We shall defer the spinor case to the next section.

The action of the scalar field $\varphi$ is

$$\bar{I}_s = \frac{1}{2} \int_M \left( g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m_0^2 \varphi^2 \right).$$

(20)

From the action one can derive the Euclidean equation of motion

$$[\Box - m_0^2] \varphi = 0.$$  

(21)

Its basis of solutions are

$$\omega_{\omega l m A} = \exp(\pm \omega \tau) R_{\omega l m A}(r, \theta, \phi) \quad (\omega > 0).$$

(22)

The eigenfunctions for the scalar field takes the form

$$R_{\omega l m A}(r, \theta, \phi) = R_{\omega l A}(r) Y_{lm}(\theta, \phi),$$

(23)
where \( Y_{l\bar{m}} \) are the real spherical harmonics, \( l, \bar{m} \) denote the angular momentum magnitude and its component along a given direction, and \( A \) is the label of the two independent radial functions. The radial function satisfies the equation

\[
\frac{d}{dr} \left( r^2 - 2mr + \frac{\Lambda r^4}{3} \right) \frac{d}{dr} - l(l + 1) + \omega^2 \left( \frac{r^2}{1 - \frac{2m}{r} + \frac{\Lambda}{r^4}} - m_0^2 \right) R_{\omega l A}(r) = 0. \tag{24}
\]

The eigenfunctions satisfy the orthonormality and completeness conditions in the space \((r, \theta, \phi)\) with the weight factor \( g^{r\tau} g^{1/2} \).

If one rescales the radial function by \( \bar{R}_{\omega l A}(r) = \sqrt{\frac{2\pi}{r}} R_{\omega l A}(r) \), then the equation takes the form

\[
\frac{d^2 \bar{R}_{\omega l A}}{dr^2} + (\omega^2 - V_l) \bar{R}_{\omega l A} = 0, \tag{25}
\]

where

\[
r^* = \frac{3}{\Lambda} \left( \frac{r_0 \ln|r - r_0|}{(r_0 - r_2)(r_0 - r_3)} + \frac{r_2 \ln|r - r_2|}{(r_2 - r_0)(r_2 - r_3)} + \frac{r_3 \ln|r - r_3|}{(r_3 - r_2)(r_3 - r_0)} \right) \tag{26}
\]

and

\[
V_l = \left( 1 - \frac{2m}{r} + \frac{\Lambda r^2}{3} \right) \left( \frac{l(l + 1)}{r^2} + \frac{2m}{r^3} + \frac{2\Lambda}{3} + m_0^2 \right). \tag{27}
\]

For the given scalar field \( \varphi_{\pm}(r, \theta, \phi) \) at the boundary \( \Sigma \), the scalar solution can be written as a linear combination

\[
\varphi(\tau, r, \theta, \phi) = \sum_{\lambda} \left[ \varphi_{\lambda, +} \varphi_{\lambda, -}(\tau, r_+, \theta, \phi) + \varphi_{\lambda, -} \varphi_{\lambda, +}(\tau, r_-, \theta, \phi) \right], \tag{28}
\]

where \( r_+ \) and \( r_- \) are to distinguish the \( r \) coordinate at \( \Sigma_1 \) and \( \Sigma_2 \), and the basis functions are

\[
u_{\lambda, \pm}(\tau, r, \theta, \phi) = \frac{\sinh[(\Delta \tau/4 \mp \tau)\omega]}{\sinh(\Delta \tau \omega/2)} R_{\omega l m A}(r, \theta, \phi), \tag{29}
\]

\[
\lambda = (\omega, l, \bar{m}, A), \quad \sum_{\lambda} = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \sum_{l, \bar{m}, A}. \tag{30}
\]

and the coefficients \( \varphi_{\lambda, \pm} \) are determined by the decomposition at the boundary \( \Sigma \)

\[
\varphi_{\pm}(r, \theta, \phi) = \sum_{\lambda} \varphi_{\lambda, \pm} R_{\lambda}(r, \theta, \phi). \tag{31}
\]

We shall assume that the observer is at the cut \( \Sigma_1 \).
The Euclidean action can be evaluated by substituting (29) into (20), yielding
\[ \bar{I}_s = \frac{1}{2} \sum_{\lambda} \left[ \omega_\lambda \coth(\Delta r \omega_\lambda/2)(\varphi^2_{\lambda,+} + \varphi^2_{\lambda,-}) - 2\omega_\lambda \operatorname{cosech}(\Delta r \omega_\lambda/2)\varphi_{\lambda,+}\varphi_{\lambda,-} \right]. \] (32)

The action can be rewritten into the form
\[ \bar{I}_s = \frac{1}{2} \sum_{\lambda} \omega_\lambda \left[ \eta^2_{\lambda,+} + \eta^2_{\lambda,-} \right], \] (33)
where the new variables \( \eta_{\lambda,\pm} \) are defined as follows
\[ \varphi_{\lambda,\pm} = \left(2 \sinh \frac{\Delta r \omega_\lambda}{2}\right)^{-1/2} \left[e^{\Delta r \omega_\lambda/4} \eta_{\lambda,\pm} + e^{-\Delta r \omega_\lambda/4} \eta_{\lambda,\mp}\right]. \] (34)

The no-boundary wave function at the WKB level is due to the regular solution with the boundary condition (31), it can be evaluated using the action (33)
\[ \Psi_{NB}(\varphi_+,\varphi_-) \approx \exp -\bar{I}_s = \exp \left(-\frac{1}{2} \sum_{\lambda} \omega_\lambda (\eta^2_{\lambda,+} + \eta^2_{\lambda,-})\right), \] (35)
the wave function takes the Gaussian minimum excitation form. Thus, if one can simultaneously measure the quantum state within the whole transition sector, he will find that the scalar field is in the ground state of these modes, in the sense of usual quantum mechanics. However, the observation is restricted by the two horizons and the quantum state will appear to be mixed due to the information loss as we shall show later.

The operator version of the scalar field \( \varphi \) can be expanded in the Lorentzian spacetime as
\[ \varphi(t, r, \theta, \phi) = \sum_{\lambda} \frac{1}{\sqrt{2\omega_\lambda}} \left[a_{\lambda,+}u_{\lambda,-}(t, r, \theta, \phi) + a_{\lambda,+}^\dagger u_{\lambda,-}(t, r, \theta, \phi) \right. \]
\[ \left. + a_{\lambda,-}u_{\lambda,+}(t, r, \theta, \phi) + a_{\lambda,-}^\dagger u_{\lambda,+}^\ast(t, r, \theta, \phi) \right]. \] (36)

where \( a_{\lambda,\pm}, a_{\lambda,\pm}^\dagger \) are the annihilation and creation operators associated with the modes \( \varphi_{\lambda,\pm} \) at \( \Sigma_1 \) and \( \Sigma_2 \). Each mode can be considered as a linear oscillator. Here the modes \( u_{\lambda,\pm} \) are not of purely positive or negative frequency due to the acausal propagation of the particles in the spacetime with nontrivial topology.

In the \( \varphi_{\lambda,\pm} \) representation, one has [21]
\[ a_{\lambda,\pm} + a_{\lambda,\pm}^\dagger = (2\omega_{\lambda,\pm})^{1/2} \varphi_{\lambda,\pm}, \] (37)

14
\[ a_{\lambda,\pm} - a_{\lambda,\pm}^\dagger = \left( \frac{2}{\omega_{\lambda,\pm}} \right)^{1/2} \frac{1}{\partial \varphi_{\lambda,\pm}}. \]  \hspace{1cm} (38)

If one expands the scalar field in terms of the modes \( \eta_{\lambda,\pm} \), then the associated annihilation and creation operators \( c_{\lambda,\pm}, c_{\lambda,\pm}^\dagger \) in a similar way satisfy
\[
c_{\lambda,\pm} + c_{\lambda,\pm}^\dagger = (2 \omega_{\lambda,\pm})^{1/2} \eta_{\lambda,\pm}, \hspace{1cm} (39)
\]
\[
c_{\lambda,\pm} - c_{\lambda,\pm}^\dagger = \left( \frac{2}{\omega_{\lambda,\pm}} \right)^{1/2} \frac{1}{\partial \eta_{\lambda,\pm}}. \hspace{1cm} (40)
\]

From (34) one can derive
\[
\frac{1}{\partial \varphi_{\lambda,\pm}} = \left( 2 \sinh \frac{\Delta \tau \omega_{\lambda}}{2} \right)^{-1/2} \left[ e^{\Delta \tau \omega_{\lambda}/4} \frac{1}{\partial \eta_{\lambda,\pm}} - e^{-\Delta \tau \omega_{\lambda}/4} \frac{1}{\partial \eta_{\lambda,\mp}} \right]. \hspace{1cm} (41)
\]

From Eqs. (34), (37)-(41) it follows that
\[
a_{\lambda,\pm} = \left( 2 \sinh \frac{\Delta \tau \omega_{\lambda}}{2} \right)^{-1/2} \left[ e^{\Delta \tau \omega_{\lambda}/4} c_{\lambda,\pm} + e^{-\Delta \tau \omega_{\lambda}/4} c_{\lambda,\mp}^\dagger \right]. \hspace{1cm} (42)
\]

This is the familiar Bogoliubov transformation for bosonic field [22]. Therefore for the observer, who can only observe the modes \( \varphi_{\lambda,\pm} \) within the two horizons, the no-boundary state or vacuum state \( \left| 0_{\text{NB}} \right> \) in (35) will present the Planck spectrum for radiation at temperature \( \Delta \tau^{-1} \). Indeed the observer will detect
\[
\langle 0_{NB} | a_{\lambda,\mp}^\dagger a_{\lambda,\mp} | 0_{NB} \rangle = \frac{1}{e^{\Delta \tau \omega_{\lambda}/2} - 1} \hspace{1cm} (43)
\]
particles in the mode \( \varphi_{\lambda,\mp} \). To the observer the quantum state should be described by a density matrix, which represents a thermal equilibrium state at temperature \( \Delta \tau^{-1} \), as in the Schwarzschild case [17]. If the observer is in \( \Sigma_2 \), then the situation will be the same.

For comparison, we show the so-called Boulware vacuum state observed at the quantum transition surface of the spacetime background [19]
\[
\Psi_{B,\pm} \approx \exp \left[ -\frac{1}{2} \sum_{\lambda} \omega_{\lambda} \varphi_{\lambda,\pm}^2 \right], \hspace{1cm} (44)
\]
which are separately defined on \( \Sigma_1 \) and \( \Sigma_2 \). The Boulware modes are eigenfunctions of the Killing vector \( \partial / \partial t \).

Our result is well expected, since the constrained gravitational instanton with parameter \( f_2 \) is the seed of the Schwarzschild-de Sitter spacetime background, the period of the imaginary time is
\( \Delta \tau \). It is noticed from Eq. (15) that the probability of the temperature in the perturbation modes is constant.

If \( m = 0 \), our model is reduced to the de Sitter case, the black hole horizon disappears and the topology of the 3-geometry becomes \( S^3 \). The observer can measure the quantum field in the whole equator \( \Sigma \). If one analytically continues the basis functions (29) into real time at the quantum transition, then one obtains

\[
u_{\lambda, \pm}(t, r, \theta, \phi) = (\cos \omega t \mp i \coth(\Delta \tau \omega/2) \sin \omega t) R_{\omega l m A}(r, \theta, \phi), (45)\]

since the scalar field operator is Hermitian, the appearance of the imaginary part of (45) is spurious. One can use the prescription for the analytic continuation from the imaginary time to the real time at \( \Sigma_1 \), that is, to set \( \tau = \Delta \tau/4 \pm it \) and then take the average. Then the imaginary part in (45) is cancelled, the perturbation modes behave as standing waves, and the dependence of the later development on the parameter \( \Delta \tau \) is therefore completely erased.

So the choice of the period in imaginary time used in constructing the instanton has effect neither on the background, nor on the perturbation modes. At the initial moment the quantum state of the perturbation modes is strictly at the minimum excitation state allowed by the Heisenberg Uncertainty Principle, or the temperature is zero then [20]. If we keep using the Killing time, then the situation will remain the same forever.

However, it is more appropriate to use the so-called synchronous coordinates in cosmology [10], at least for the background. In this coordinate, it would be convenient to decompose the perturbation in terms of the comoving modes. Then, the state will remain in the ground state with a time-dependent frequency which varies inversely with respect to the scale of the universe, as long as the wave length of the mode is smaller than the horizon. If the mode goes out the horizon, then its wave function will freeze, it becomes a superposition of a number of excited states due to the background evolution. This phenomenon leads to the formation of structures in the universe [20]. The calculation can be carried out more quantitatively by using the technique of the so-called squeezed vacuum state developed in quantum optics [23]. The quantum state observed is described by a density matrix obtained by taking trace of the wave function over the field configuration beyond the cosmological horizon.

One can straightforward apply the above argument to the Nariai case. Since the two horizons
degenerate, no trace has to be taken if one uses the coordinate with the Killing time. Again, if one uses the synchronous coordinates, the quantum state will behave in a similar way as that in the de Sitter model. In summary, the constrained instanton approach keeps the de Sitter and Nariai models intact.

In principle, one can also analyze the quantum state in the Schwarzschild-de Sitter background using the synchronous coordinates in a similar way.

V. The quantum state of the fermionic field

Now we are going to analyze the spin-1/2 fermionic perturbation modes. The theory of a quantized spin-1/2 field in the Schwarzschild space \cite{24} will be generalized to the case in the Schwarzschild-de Sitter space.

The Dirac spinor $\psi$ can be globally defined in the flat Minkowski space. However, in a curved space, one has to locally introduce a set of orthonormal basis vectors represented by the tetrad $e^a_\mu$ with respect to which the spinors $\psi$ are defined. The action of the spinor field is

$$I_f = \int_M \left( \frac{i}{2} \tilde{\psi} \gamma^a \nabla_a \psi - m_0 \tilde{\psi} \psi \right) - \frac{i}{2} \int_{\partial M} \tilde{\psi} \gamma \psi$$

where $m_0$ is the mass of the spinor. In the usual form, one has $\tilde{\psi} = \psi^\dagger \gamma^0$. Here instead, we shall free the daggered variables from being the Hermitian conjugates of undaggered variables by twofold reason. First, it is due to the variational calculation technique. Second, it will leave room for the calculation below in Euclidean approach. The boundary term depends on the variational condition posed. Like in the purely gravitational case, we specify the 3-geometry on the boundary for action (2), here the boundary term is for the condition that the fermionic variables $\tilde{\psi}$ and $\psi$ are specified on the final surface $S_f$ and initial surface $S_i$ if the two surfaces are properly defined.

The Dirac equation can be derived from the action

$$(-i\gamma^a \nabla_a + m_0) \psi = 0,$$

$$\tilde{\psi} \left( i\gamma^a \nabla_a + m_0 \right) = 0.$$  

For the Schwarzschild-de Sitter background, one can choose the tetrad along the coordinate basis \cite{24} and a proper spinor representation. Then the Euclidean Dirac equation for $\psi$ in the sector
$r_2 < r < r_3$ can be written

$$\left( -\frac{\gamma^0 \partial}{\Delta \partial r} + \frac{\gamma^1 \Delta^{1/2} \partial \Delta^{1/2} r}{ir \partial r} + \frac{\gamma^2 \partial \sin^{1/2} \theta}{ir \sin^{1/2} \theta \partial \theta} + \frac{\gamma^3 \partial}{ir \sin \theta \partial \phi} + m_0 \right) \psi = 0. \quad (49)$$

The Dirac equation for $\tilde{\psi}$ can be treated in a similar way.

One can choose the representation of the Dirac matrices to be a direct product $\vec{\rho} \otimes \vec{\sigma}$ of two-dimensional Pauli matrices [24]

$$\gamma^0 = \rho_2, \quad \gamma^1 = i \rho_1, \quad \gamma^2 = -i \rho_3 \sigma_3, \quad \gamma^3 = -i \rho_3 \sigma_1, \quad (50)$$
then the basis of solutions of Eq. (49) are

$$\eta^\pm_\lambda = \exp(\mp i \omega \tau) \psi^\pm_\lambda (r, \theta), \quad \lambda = (\omega, k', m'), \quad (51)$$
where the upper index $\pm$ means the positive (negative) frequency part. Then the eigenfunctions for the spinor field can be written as

$$\psi^\pm_\lambda (r, \theta, \phi) = r^{-1} \Delta^{-1/2} Y_{k'm'}(\theta, \phi) \psi^\pm_\lambda (r), \quad (52)$$
where the radial functions $\psi^\pm_\lambda (r)$ live in the $\rho$ subspace, the spinor harmonics $Y_{k'm'}$ live in the $\sigma$ subspace, and $k'$ and $m'$ denote the eigenvalue of the following operators [21][24]

$$k = \frac{\sigma^3 \partial \sin^{1/2} \theta}{i \sin^{1/2} \theta \partial \theta} + \frac{\sigma^1 \partial}{i \sin \theta \partial \phi} \quad (53)$$
and

$$J_3 = \frac{\partial}{\partial \phi}. \quad (54)$$

For the harmonics with a given total angular momentum $j(j + 1)$ the degeneracy is $2(2j + 1)$.

The radial function satisfies the equation

$$\left( \mp \frac{\rho^2 \omega}{\Delta} + \frac{\rho^1 \partial}{\partial \rho^*} - \frac{i \rho^3 k'}{r} + m_0 \right) \psi^\pm_\lambda (r) = 0. \quad (55)$$

Let us study the spinor field at the surface of the quantum transition or $\Sigma$. One can decompose the spinor field as follows

$$\psi(t, r, \theta, \phi) = \sum_\lambda [m_{\lambda^+} \eta^+_\lambda (it, r, \theta, \phi) + \bar{n}_{\lambda^+} \eta^-_\lambda (it, r, \theta, \phi)$$

18
+ m_{\lambda,-}\eta_\lambda^\dagger (it,r_-,\theta,\phi) + \bar{n}_{\lambda,-}\eta_\lambda^- (it,r_-,\theta,\phi), \quad (56)

\psi^\dagger (t,r,\theta,\phi) = \sum_{\lambda} [\bar{m}_{\lambda,+}\eta_\lambda^\dagger (it,r_+,\theta,\phi) + n_{\lambda,+}\bar{\eta}_\lambda^- (it,r_+,\theta,\phi) + \bar{m}_{\lambda,-}\eta_\lambda^\dagger (it,r_-,\theta,\phi) + n_{\lambda,-}\bar{\eta}_\lambda^- (it,r_-,\theta,\phi)], \quad (57)

where

\sum_{\lambda} = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \sum_{k',m'} \quad (58)

and the field \( \psi, \psi^\dagger \) and the coefficients \( m_{\pm}, n_{\pm}, \bar{m}_{\pm}, \bar{n}_{\pm} \) are taken to be odd elements of a Grassmann algebra. It is noted that in the expansions (56) (57) we use the Killing time coordinate \( t \). We implicitly assume that the Lorentzian evolution is along the \( t \) increasing (decreasing) direction at \( \Sigma_1 \) (\( \Sigma_2 \)).

There is interference neither between different modes nor between those at \( \Sigma_1 \) and \( \Sigma_2 \). Thus it is enough to study individual modes separately. The Lagrangian for one mode is

\[ L_\lambda = \bar{m}_\lambda (i\frac{\partial}{\partial t} - \omega_\lambda) m_\lambda + \bar{n}_\lambda (i\frac{\partial}{\partial t} + \omega_\lambda) n_\lambda \quad (59) \]

and the Hamiltonian is

\[ H_\lambda = \omega_\lambda (\bar{m}_\lambda m_\lambda - \bar{n}_\lambda n_\lambda). \quad (60) \]

The quantum versions of \( m_\lambda, n_\lambda, \bar{m}_\lambda, \bar{n}_\lambda \) obey the anti-commutation relations

\[ \{ m_\lambda, \bar{m}_\lambda \} = 1, \quad \{ n_\lambda, \bar{n}_\lambda \} = 1. \quad (61) \]

In the \( m_\lambda, n_\lambda \) representations, one has

\[ \bar{m}_\lambda = \frac{\partial}{\partial m_\lambda}, \quad \bar{n}_\lambda = \frac{\partial}{\partial n_\lambda}. \quad (62) \]

The Hamiltonian operator can be written as

\[ H_\lambda = \omega_\lambda \left( -m_\lambda \frac{\partial}{\partial m_\lambda} + n_\lambda \frac{\partial}{\partial n_\lambda} \right). \quad (63) \]

Within the framework one can define the creation and annihilation operators for the spinor field at \( \Sigma_1 \) and \( \Sigma_2 \)

\[ a_{\lambda,\pm} = \frac{\partial}{\partial m_{\lambda,\pm}}, \quad a_{\lambda,\pm}^\dagger = m_{\lambda,\pm}. \quad (64) \]
\[ b_{\lambda, \pm} = n_{\lambda, \pm}, \quad b_{\lambda, \pm}^\dagger = \frac{\partial}{\partial n_{\lambda, \pm}}. \] \tag{65} 

Then the vacuum state of the spinor field with respect to these operators, or the wave function of the minimum excitation state for the field defined only at \( \Sigma_1(\Sigma_2) \), is

\[ \Psi_{B, \pm}(h_{ij}, \psi) = n_{\lambda, \pm}. \] \tag{66} 

Now we are going to evaluate the no-boundary wave function for the fermionic perturbation modes. At the WKB level, the wave function is the exponential of the negative of the Euclidean action for the classical solution. The action (46) is suitable for the solution with the boundary condition that the fermionic variables \( \tilde{\psi} \) and \( \psi \) are specified at the final and initial surfaces, or \( \Sigma_1 \) and \( \Sigma_2 \) respectively.

The dominating contribution to the no-boundary wave function is due to the Euclidean solution satisfying the above boundary condition. Since \( \Sigma \) is also the surface of quantum transition, at which or the moment \( t = 0 \) the Lorentzian evolution will emanate, the spinor solution in the Lorentzian regime can be obtained through an analytic continuation from that in the Euclidean regime and it can be written as a linear combination

\[ \psi(t, r, \theta, \phi) = \sum_{\lambda} [m_{\lambda, -} \eta_+^\lambda (it, r_+, \theta, \phi) \exp(-\Delta \tau \omega_\lambda/2) + n_{\lambda, -} \eta_-^\lambda (it, r_-, \theta, \phi) \exp(\Delta \tau \omega_\lambda/2)] 
+ m_{\lambda, +} \eta_+^\lambda (-it, r_-, \theta, \phi) + n_{\lambda, +} \eta_-^\lambda (-it, r_+, \theta, \phi)], \] \tag{67} 

\[ \tilde{\psi}(t, r, \theta, \phi) = \sum_{\lambda} [m_{\lambda, +} \bar{\eta}_+^\lambda (it, r_+, \theta, \phi) + n_{\lambda, +} \bar{\eta}_-^\lambda (it, r_+, \theta, \phi) 
+ m_{\lambda, -} \bar{\eta}_+^\lambda (-it, r_-, \theta, \phi) \exp(-\Delta \tau \omega_\lambda/2) + n_{\lambda, -} \bar{\eta}_-^\lambda (-it, r_-, \theta, \phi) \exp(\Delta \tau \omega_\lambda/2)]. \] \tag{68} 

To comply with the boundary term of the action (46), the coefficients \( m_{\lambda, +}, n_{\lambda, +} (m_{\lambda, -}, n_{\lambda, -}) \) are determined by the decomposition of the field at the final and initial surfaces, or \( \Sigma_1 \) and \( \Sigma_2 \) respectively.

The Euclidean action can be evaluated by substituting (67) and (68) into (46). For the classical solution, only the boundary term of the action survives,

\[ I_f = \sum_{\lambda} [m_{\lambda, +} m_{\lambda, -} \exp(-\Delta \tau \omega_\lambda/2) + n_{\lambda, +} n_{\lambda, -} \exp(\Delta \tau \omega_\lambda/2)]. \] \tag{69}
The no-boundary wave function for the fermionic perturbations can be written as a product of that for each mode. Therefore, the no-boundary wave function of mode $\lambda$ is

$$\Psi_{NB}(h_{ij}, m_{\lambda,+}, m_{\lambda,-}, n_{\lambda,+}, n_{\lambda,-}) \approx \exp^{-\bar{I}_{f,\lambda}}$$

$$= 1 - m_{\lambda,+} m_{\lambda,-} \exp(-\Delta \omega_{\lambda}/2) - n_{\lambda,+} n_{\lambda,-} \exp(\Delta \tau_{\lambda}/2) + m_{\lambda,+} m_{\lambda,-} n_{\lambda,+} n_{\lambda,-}. \quad (70)$$

To interpret (70), it is convenient to introduce the operators

$$c_{\lambda,\pm} = \left(2 \cosh \frac{\Delta \tau_{\omega_{\lambda}}}{2}\right)^{-1/2} \left[e^{\frac{\Delta \tau_{\omega_{\lambda}}}{4} \frac{\partial}{\partial m_{\lambda,\pm}}} \pm e^{-\frac{\Delta \tau_{\omega_{\lambda}}}{4} m_{\lambda,\mp}}\right], \quad (71)$$

$$d_{\lambda,\pm} = \left(2 \cosh \frac{\Delta \tau_{\omega_{\lambda}}}{2}\right)^{-1/2} \left[e^{\frac{\Delta \tau_{\omega_{\lambda}}}{4} n_{\lambda,\mp}} \pm e^{-\frac{\Delta \tau_{\omega_{\lambda}}}{4} \frac{\partial}{\partial n_{\lambda,\pm}}}\right], \quad (72)$$

and their adjoint

$$c_{\lambda,\pm}^\dagger = \left(2 \cosh \frac{\Delta \tau_{\omega_{\lambda}}}{2}\right)^{-1/2} \left[e^{\frac{\Delta \tau_{\omega_{\lambda}}}{4} m_{\lambda,\mp}} \pm e^{-\frac{\Delta \tau_{\omega_{\lambda}}}{4} \frac{\partial}{\partial m_{\lambda,\pm}}}\right], \quad (73)$$

$$d_{\lambda,\pm}^\dagger = \left(2 \cosh \frac{\Delta \tau_{\omega_{\lambda}}}{2}\right)^{-1/2} \left[e^{\frac{\Delta \tau_{\omega_{\lambda}}}{4} n_{\lambda,\pm}} \pm e^{-\frac{\Delta \tau_{\omega_{\lambda}}}{4} \frac{\partial}{\partial n_{\lambda,\mp}}}\right]. \quad (74)$$

These operators are the creation and annihilation operators which satisfy the corresponding anti-commutative relations. It is noted that if one sets $\Delta \tau = \infty$, then Eqs. (71) to (74) will be reduced to Eqs. (64)(65). One can verify that the no-boundary wave function (70) is the ground state with respect to these operators. That is, one has

$$c_{\lambda,\pm} \Psi_{NB}(h_{ij}, \psi) = 0 \quad (75)$$

and

$$d_{\lambda,\pm} \Psi_{NB}(h_{ij}, \psi) = 0. \quad (76)$$

Therefore, if there exists an omniscient being who can perceive the quantum state on the whole surface of quantum transition, he would find that the spinor modes are in a pure state, and even more than this, that is the state is in vacuum. However, the reality is that one can only observe the quantum state within his black hole and cosmological horizons, say at $\Sigma_1$. Since the information beyond these horizons is lost, he would find that the quantum state is described by a density matrix. In fact, one can even expect that it is in a thermal equilibrium state, as we shall show below.
From Eqs. (64)-(65)(71)-(74) one can derive the following relations:

\[
a_{\lambda,\pm} = \left(2 \cosh \frac{\Delta \tau \omega_{\lambda}}{2}\right)^{-1/2} \left[e^{\Delta \tau \omega_{\lambda}/4} c_{\lambda,\pm} \mp e^{-\Delta \tau \omega_{\lambda}/4} c_{\lambda,\mp} \right]
\]  

(77)

and

\[
b_{\lambda,\pm} = \left(2 \cosh \frac{\Delta \tau \omega_{\lambda}}{2}\right)^{-1/2} \left[e^{\Delta \tau \omega_{\lambda}/4} d_{\lambda,\pm} \pm e^{-\Delta \tau \omega_{\lambda}/4} d_{\lambda,\mp} \right].
\]  

(78)

These are the Bogoliubov transformations for fermionic fields. The sign differences are due to the fact that \(m\) (\(n\)) modes are associated with the positive (negative) frequency part of the solution. Since the direction of Killing time vector flips at surfaces \(\Sigma_1\) and \(\Sigma_2\), there exists a duality of creation and annihilation processes between surfaces \(\Sigma_1\) and \(\Sigma_2\). Therefore, the vacuum state observed by the omniscient being appears as a thermal equilibrium state for the ordinary observer like us who is restricted by the two horizons. The associated temperature is the inverse of \(\Delta \tau\) as shown by (77)-(78). Indeed the observer will detect

\[
\langle 0_{NB} \mid a_{\lambda,\pm}^\dagger a_{\lambda,\pm} \mid 0_{NB} \rangle = \frac{1}{e^{\Delta \tau \omega_{\lambda} / 2} + 1}
\]  

(79)

particles in \(m_{\lambda,\pm}\) mode and

\[
\langle 0_{NB} \mid b_{\lambda,\pm}^\dagger b_{\lambda,\pm} \mid 0_{NB} \rangle = \frac{1}{e^{\Delta \tau \omega_{\lambda} / 2} + 1}
\]  

(80)

particles in \(n_{\lambda,\pm}\) mode. If the observer is in \(\Sigma_2\), then the situation will be the same.

The alternative way to show this thermal property of the state observed at \(\Sigma_1\) (\(\Sigma_2\)) is to reduce the wave function into a density matrix by taking the trace over the unobserved configuration at \(\Sigma_2\) (\(\Sigma_1\)) [17].

The argument about the two special cases, the de Sitter and the Nariai models in the last section remains intact for the spinor field. If one sets \(m = 0\), then the model is reduced to the case of the spinor field in the de Sitter background [25].

VI. Discussion

The possibility of quantum creation of a single black hole in the de Sitter background has been argued for a long period of time. The reason is that there does not exist any regular gravitational instanton which is the seed of the created hole. However, in quantum theory, one is mainly concerned
about the stationary action solution. If one can find a stationary action solution, then the wave
function can be approximated by a classical trajectory. The stationary action solutions are not
always the solutions to the field equation. The set of all solutions to the classical field equations is
only a subset of the class of all stationary action solutions. In this sense, we introduce the concept
of constrained instanton. In quantum cosmology we are particularly interested in the constrained
instantons for which the 3-geometry of the quantum transition surface is given.

We find that there exists a free parameter in the constrained gravitational instanton for the
black hole creation in the de Sitter background. This is the identification period of imaginary time
in constructing the instanton. It is naturally expected that the inverse of this parameter should
be the temperature of quantum fields in the spacetime background. We studied both bosonic and
fermionic perturbation modes and confirmed that this was indeed the case.

It was thought that in the Schwarzschild-de Sitter space, the quantum state of the perturbation
could never be in thermal equilibrium, since the temperatures associated with the two horizons are
different. It turns out that Nature is more generous than we expected. The quantum fields can
coexist with the black hole spacetime in an equilibrium state at an arbitrary temperature.

When we discussed the quantum state of the perturbation modes, we treated spacetime as a
classical background. We ignored both the interaction between the gravitational and the matter
field modes and the back reaction of the perturbation modes to the background. We assumed that
their effects were negligible.

Quantum fields in curved spacetime is an incomplete and temporary theory. Practically, it can
only be useful for some very symmetric backgrounds. Conceptually, one can foresee a very dramatic
revolution in this direction in the not so distant future.

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