Abstract—Logarithmic Sobolev inequalities are a fundamental class of inequalities that play an important role in information theory. They play a key role in establishing concentration inequalities and in obtaining quantitative estimates on the convergence to equilibrium of Markov processes. More recently, deep links have been established between logarithmic Sobolev inequalities and strong data processing inequalities. In this paper we study logarithmic Sobolev inequalities from a computational point of view. We describe a hierarchy of semidefinite programming relaxations which give certified lower bounds on the logarithmic Sobolev constant of a finite Markov operator, and we prove that the optimal values of these semidefinite programs converge to the logarithmic Sobolev constant. Numerical experiments show that these relaxations are often very close to the true constant even for low levels of the hierarchy. Finally, we exploit our relaxation to obtain a sum-of-squares proof that the logarithmic Sobolev constant is equal of-squares proof that the logarithmic Sobolev constant is equal.

Understanding the mixing time of this chain, i.e., how fast \( \mu_t \) converges to \( \pi \), is a subject that has attracted a significant amount of attention in the areas of probability theory, information theory, and dynamical systems. The \( L^2 \) mixing time can be defined as

\[
\tau_2(\epsilon) = \inf \{ t > 0 \mid \sup_{\mu_0} \| (d\mu_t/d\pi) - 1 \|_{2,\pi} \leq \epsilon \}
\]

where the supremum is taken over all initial probability distributions \( \mu_0 \) on \( \mathcal{X} \), and where \( \| x \|_{2,\pi} := \sqrt{\mathbb{E}_\pi [x^2]} \) for \( x \in \mathbb{R}^\mathcal{X} \). By \( d\mu_t/d\pi \) we simply mean the density of \( \mu_t \) with respect to \( \pi \). The total variation mixing time \( \tau_{TV}(\epsilon) \), which measures convergence in the distance \( \| \mu - \pi \|_{TV} := \frac{1}{2} \sum_{x \in \mathcal{X}} |\mu_x - \pi_x| \), is also of great practical interest.

The Logarithmic Sobolev Constant: One way to estimate the mixing time \( \tau_2 \) is via logarithmic Sobolev inequalities [1], which we recall now. For \( x \in \mathbb{R}^\mathcal{X} \), \( x \neq 0 \), we define

\[
\text{Ent}_\pi [x] = \mathbb{E}_\pi [x \log x] - \mathbb{E}_\pi [x] \log \mathbb{E}_\pi [x] = \sum_{i \in \mathcal{X}} \pi_i x_i \log \left( \frac{x_i}{\mathbb{E}_\pi [x]} \right),
\]

where \( \mathbb{E}_\pi [x] = \sum_{i \in \mathcal{X}} \pi_i x_i \). This quantity measures discrepancy between \( \pi \) and the probability distribution \( \sigma \) whose density with respect to \( \pi \) is \( x \). In fact, when \( \mathbb{E}_\pi [x] = 1 \), \( \text{Ent}_\pi [x] = D(\sigma \| \pi) \) is the relative entropy between \( \pi \) and \( \sigma \) whose density with respect to \( \pi \) is \( x \). For \( x, y \in \mathbb{R}^\mathcal{X} \), we define the Dirichlet form corresponding to \( K \)

\[
\mathcal{E}_K(x, y) = -\langle x, L y \rangle_\pi = \sum_{i,j} \pi_{i,j} (L_i y)_j,
\]

in terms of the inner product \( \langle x, y \rangle_\pi = \mathbb{E}_\pi [xy] = \sum_{i \in \mathcal{X}} \pi_i x_i y_i \). When it is clear which kernel \( K \) is in consideration we may simply write \( \mathcal{E} \). The Dirichlet form is bilinear, and the quadratic form it induces,

\[
\mathcal{E}(x, x) = -\langle x, L x \rangle_\pi = \frac{1}{2} \sum_{i,j} \pi_{i,j} (x_i - x_j)^2
\]

is positive semidefinite. Since \( K \) is irreducible, \( \mathcal{E}(x, x) \) vanishes if and only if \( x \) is constant. A logarithmic Sobolev inequality has the form

\[
\mathcal{E}(x, x) \geq \alpha \text{Ent}_\pi [x^2] \quad \forall x \in \mathbb{R}^\mathcal{X} ;
\]

the logarithmic Sobolev constant of the semigroup of the Markov kernel \( K \) is the largest constant \( \alpha \) satisfying (1.2). The
$L^2$ mixing time of the chain can then be bounded in terms of $\alpha$ as follows [1, Theorem 3.7]

$$\tau_2\left(\frac{1}{e}\right) \leq \frac{1}{\alpha} \left(1 + \frac{1}{2} \log \log (\pi_*^{-1})\right) \tag{I.3}$$

where $\pi_* = \min_{x \in \mathcal{X}} \pi_x$. Note that (I.3) also implies a bound in total variation distance

$$\tau_{TV}\left(\frac{1}{2e}\right) \leq \frac{1}{\alpha} \left(1 + \frac{1}{2} \log \log (\pi_*^{-1})\right)$$

since $\|\mu - \pi\|_{TV} \leq \frac{1}{2}\|d\mu/d\pi - 1\|_{2, \pi}$ by the Cauchy-Schwartz inequality.

**Poincaré Inequality:** A similar, but older, tool than the logarithmic Sobolev inequality is the *Poincaré inequality*

$$\mathcal{E}(x, x) \geq \lambda \cdot \text{Var}_\pi[x] \quad \forall x \in \mathbb{R}^\mathcal{X} \tag{I.4}$$

where $\text{Var}_\pi[x] := \mathbb{E}_\pi[x^2] - \mathbb{E}_\pi[x]^2$. The *Poincaré constant* of the Markov kernel $K$ is the largest constant $\lambda$ satisfying (I.4). A classical result is that $\lambda \geq 2\alpha$, see e.g., [1, Lemma 3.1]. The Poincaré constant is sometimes known as the *spectral gap* and, if the semigroup is reversible, it is equal to the second smallest eigenvalue of $-L$ (note that the smallest eigenvalue of $-L$ is 0). Using the Poincaré constant, one can obtain the following bound on the mixing time [2, Corollary 1.6]

$$\tau_2(1/e) \leq \frac{1}{\alpha} \left(1 + \frac{1}{2} \log (\pi_*^{-1})\right). \tag{I.5}$$

The main advantage of the bound (I.3) compared to (I.5) is the dependence on $\pi_*$ which is exponentially smaller in (I.3) vs. (I.5). So if $\frac{\alpha}{\lambda} \gg \frac{2+\log \log (\pi_*^{-1})}{2+\log (\pi_*^{-1})}$, the bound based on the logarithmic Sobolev constant will be much better than the one based on the Poincaré constant. A typical example is the simple random walk on the hypercube $\mathcal{X} = \{-1, +1\}^n$ where $\lambda = 2/n$ and $\alpha = 1/n$, while $\pi_*^{-1} = 2^n$. So for this example, the $L^2$ mixing time bound based on $\alpha$ is better than the bound based on $\lambda$ by a factor of approximately $n/\log_2 n$.

**Strong Data Processing Inequalities:** Logarithmic Sobolev inequalities arise not only in the study of mixing times of Markov chains, but are also connected to the study of strong data processing inequalities for memoryless channels. It can be shown (see [3], [4]) that a strong data processing inequality is implied by a logarithmic Sobolev inequality on a certain Markov kernel derived from the channel. In Section IV-B we give more details on this connection, and sketch how our methods can be modified to prove strong data processing inequalities directly.

### A. Contributions

In this paper we study the problem of *computing* the logarithmic Sobolev constant $\alpha$ of a given Markov kernel $K$. Whereas computing the Poincaré constant of $K$ reduces to an eigenvalue problem which can be solved using standard algorithms, computing the constant $\alpha$ seems much harder. One can read in Saloff-Coste’s lecture notes on finite Markov chains [5, Section 2.2.2]: “The natural question now is: can one compute or estimate the constant $\alpha$? Unfortunately, the present answer is that it seems to be a very difficult problem to estimate $\alpha$”. Indeed $\alpha$ is defined in terms of a nonconvex optimization problem, and finding the exact constant $\alpha$ for small chains can require very lengthy and technical computations, see e.g., [6].

In this work we propose a new method to compute accurate and certified lower bounds on the logarithmic Sobolev constant of a Markov chain $K$. We make use of the powerful *sum-of-squares* framework for global optimisation. Whereas sum-of-squares techniques are most directly applied to *polynomial* inequalities, we show how this framework can be brought to bear on entropy-based functional inequalities by means of rational Padé approximants to the logarithm. For integers $d \geq 2$, we describe a semidefinite program whose solution $\alpha_d$ is a lower bound on $\alpha$. For fixed $d$ the size of the semidefinite program grows polynomially in $|\mathcal{X}|$. The main technical result of the paper can be summarized in the following:

**Theorem 1:** [see Detailed Version in Theorem 8] Let $\mathcal{E}, \pi$ be the Dirichlet form and stationary distribution of an irreducible Markov chain on $\mathcal{X} = \{1, \ldots, n\}$, with logarithmic Sobolev constant $\alpha$. For each $d \geq 2$, there is a number $\alpha_d \leq \alpha$ such that the following is true:

(i) $\alpha_d$ can be computed with a semidefinite program of size $O(n^{2d})$

(ii) $\alpha_d \uparrow \alpha$ as $d \uparrow \infty$.

### B. Overview of Approach

Our approach relies on semidefinite programming, and uses the sum-of-squares paradigm to prove inequalities. Sum-of-squares programming is a general method to search for proofs of polynomial inequalities [7], [8]. The sum-of-squares method cannot, however, be used directly on (I.2) as the right-hand side involves a nonpolynomial function. To circumvent this, we search for stronger inequalities where the term $\text{Ent}_\pi[x^2]$ is upper bounded by a polynomial function. Some simple upper bounds based on Taylor expansions (finite radius of convergence), these methods are not convergent in general and cannot yield the kind of guarantee of Theorem 1. The main novelty in our work is that we consider instead rational Padé approximants which converge pointwise to $\log$ everywhere on the positive line, with an exponential convergence rate on any compact set. This choice of rational bounds, instead of Taylor expansions, is key to obtain the convergence result of Theorem 1.

In addition to the stated convergence result, numerical experiments (see Section V) show that these relaxations are often very close to the true constant, even for low levels of the hierarchy. In fact, as shown in Section VI for the $n$-cycle, the relaxation can even yield the exact value $\alpha$ already for small values of $d$. If the entries of the kernel $K$ are rational, we further explain how to turn the output of the semidefinite program into a formally verified bound on $\alpha$ with rational arithmetic. An implementation of our method is available in the Julia language, with examples at:

https://github.com/oisinfaust/LogSobolevRelaxations.
Importantly, the approach that we develop here is versatile enough to be applied for other entropic functional inequalities. Most notably, we explain in Section IV how similar techniques can be used to obtain bounds on the modified logarithmic Sobolev constant, and the strong data processing constant of a channel. (The latter application appeared in the conference paper [9].)

C. Related Work

As far as we are aware this is the first work that studies the logarithmic Sobolev constant from a computational point of view. In theoretical computer science, the papers [10] and [11] show that certain hypercontractive inequalities, associated to a particular Markov kernel on the hypercube \(-1, 1\)^n, can be proved using sums of squares. It turns out that hypercontractive inequalities are strongly related to logarithmic Sobolev inequalities. A hypercontractive inequality associated to a Markov semigroup \(P_t\) takes the form

\[
\left\| P_t x \right\|_{q, \pi} \leq \left\| x \right\|_{p, \pi}
\]

where \(q = q(t, p) > p\), and we use the notation \(\left\| x \right\|_{p, \pi} = \left( \sum_{i \in \pi} \pi_i |x_i|^p \right)^{1/p} \). Proving a family of inequalities of the form (1.6) can be done by proving a single logarithmic Sobolev inequality of the form (1.2). In fact, if \(P_t\) is reversible, then (1.6) is equivalent to a logarithmic Sobolev inequality with \(q = 1 + (p-1)e^{4\alpha t}\). This was one of the first motivations for the introduction of logarithmic Sobolev inequalities by Gross in [12]. We refer to [13, 5.2.2] for more details on this connection.

Related to our work also are the papers [14] and [15] which formulate the problem of finding the Markov chain with the largest Poincaré constant as a convex optimization problem. In fact the paper [14] concludes by considering the problem of finding the Markov chain with the largest logarithmic Sobolev constant. However no algorithm is proposed as there was no known way of computing/bounding \(\alpha\). The bound we propose in this method can, in fact, be used for this matter. We comment on this application in the discussion section (Section VII).

D. Organization

The rest of this paper is organized as follows. In Section II we review some preliminary results concerning semidefinite programming and sums of squares. In Section III we present our approach to bounding the logarithmic Sobolev constant, and we prove the convergence of the hierarchy of semidefinite programs (Theorem 1). We also discuss some practical aspects related to these bounds, and show how the solution of the semidefinite programs can be turned into formal certificates using exact arithmetic. In Section IV we explain how the techniques used for the log-Sobolev constant can be applied to other entropic functional inequalities, namely the modified logarithmic Sobolev constant, and the strong data processing constant. Finally, in Sections V and VI we illustrate our method on various examples of Markov chains; in particular we use our method to obtain the exact log-Sobolev constant of the simple random walk on the n-cycle for \(n \in \{5, 7, 9, \ldots, 21\}\).

II. Background

Given a variable \(t\), we let \(\mathbb{R}[t] \) (resp. \(\mathbb{R}[t, d]\)) be the vector space of univariate real polynomials (resp. of degree at most \(d\)). Given variables \(x_1, x_2, \ldots, x_n\), \(\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]\) denotes the space of polynomials with real coefficients in \(x = (x_1, x_2, \ldots, x_n)\), and \(\mathbb{R}[x]_d\) is the (finite-dimensional) subspace of \(\mathbb{R}[x]\) containing only polynomials of degree at most \(d\).

A. Sums of Squares

A polynomial \(p \in \mathbb{R}[x]\) is a sum of squares if we can find polynomials \(p_1, \ldots, p_m\) such that \(p = \sum_i p_i^2\). The set of polynomials that can be written as a sum of squares is a convex cone inside \(\mathbb{R}[x]\) which we will denote by \(\Sigma[x]\). Given a subspace \(V \subseteq \mathbb{R}[x]\), we define the cone of polynomials that can be written as a sum of squares of polynomials from \(V\) as:

\[
\Sigma(V) = \left\{ \sum_i q_i^2 : q_i \in V \right\} = \text{cone}(\{q^2 : q \in V\}).
\]

Of particular importance is the case \(V = \mathbb{R}[x]_d\) which corresponds to polynomials that can be written as a sum of squares of polynomials of degree at most \(d\). We reserve the notation \(\Sigma[x]_{2d} := \Sigma(\mathbb{R}[x]_d)\) for this set, which has the property that \(\Sigma[x]_{2d} = \Sigma[x] \cap \mathbb{R}[x]_{2d}\).

A fundamental fact about \(\Sigma(V)\), is that one can decide membership in it using semidefinite programming [7], [16], [17]. This is because a polynomial \(f \in \mathbb{R}[x]\) belongs to \(\Sigma(V)\) if and only if there exists a positive semidefinite Gram matrix \(Q\) satisfying \(f = b(x)^\top Q b(x)\) where \(b(x) = (b_1(x), \ldots, b_{\dim V}(x))\) is a basis of \(V\). Indeed, a positive semidefinite \(Q \succeq 0\) can be equivalently written as \(Q = \sum_i q_i q_i^\top\), and thus

\[
f(x) = b(x)^\top Q b(x), \quad Q \succeq 0 \iff f(x) = \sum_i (q_i^\top b_i(x))^2.
\]

is a sum of squares of polynomials in \(V\).

B. Semidefinite Programming

Recall that a semidefinite program (SDP) is an optimization problem of the form

\[
\min_{X \in \mathbb{S}^n} \text{trace}(CX) \quad \text{s.t.} \quad A(X) = b, X \succeq 0 \tag{II.3}
\]

where \(\mathbb{S}^n\) is the space of \(n \times n\) real symmetric matrices, \(X \succeq 0\) means that \(X\) is positive semidefinite; and \(C \in \mathbb{S}^n, A : \mathbb{S}^n \to \mathbb{R}^m, b \in \mathbb{R}^m\) are given. The set of positive semidefinite matrices is denoted \(\mathbb{S}^n_+\). Observe that, by (II.2), \(\Sigma(V)\) can be described as the feasible set of a semidefinite program

\[
\Sigma(V) = \left\{ f \in \mathbb{R}[x] : \exists Q \succeq 0, \sum_{i,j=1}^{\dim V} Q_{ij} b_i(x) b_j(x) = f(x) \right\}.
\]

(Note that \(\sum_{i,j=1}^{\dim V} Q_{ij} b_i(x) b_j(x) = f(x)\) forms a set of linear equality constraints on \(Q\).)
Whilst it is generally NP-hard to decide whether a polynomial takes only nonnegative values on $\mathbb{R}^n$ [18], there are efficient algorithms for solving SDPs to any desired precision using floating point arithmetic. Therefore, for many computational tasks involving constraints specifying that certain polynomials be nonnegative, we can construct relaxations based on the more tractable condition that these polynomials are sums of squares.

C. Constrained Polynomial Optimization

In many situations, we are interested in certifying nonnegativity of a polynomial $f$ on a particular subset $S \subset \mathbb{R}^n$. If $S$ is described by a finite set of polynomial equations and inequalities

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_j(x) \geq 0, \quad h_1(x) = 0, \ldots, h_k(x) = 0\}$$

then an obvious sufficient condition for $f(x) \in \mathbb{R}[x]$ to be nonnegative on $S$ is that

$$f(x) = \sigma_0(x) + \sigma_1(x)g_1(x) + \cdots + \sigma_j(x)g_j(x) + \phi_1(x)h_1(x) + \cdots + \phi_K(x)h_K(x)$$

where $\sigma_0, \ldots, \sigma_j$ are sums of squares, and $\phi_1, \ldots, \phi_K$ are arbitrary polynomials. If we restrict the degrees of $(\sigma_j)$ and $(\phi_k)$ by $\deg(\sigma_0) \leq 2d$, $\deg(\sigma_j) \leq 2d$ and $\deg(\phi_k) \leq 2d$, then the condition (II.5) can be expressed as a semidefinite feasibility problem having one semidefinite constraint of size $\dim \mathbb{R}[x]_{d} \times \dim \mathbb{R}[x]_{d}$ and $J$ semidefinite constraints of size $\dim \mathbb{R}[x]_{d-\frac{1}{2}\deg g_j} \times \dim \mathbb{R}[x]_{d-\frac{1}{2}\deg g_j}$ each [8].

D. Univariate Polynomials

For univariate polynomials, global nonnegativity is equivalent to being a sum of squares. See for example [19, Theorem 2.3]. We record the following result which will be useful later:

Theorem 2 (E.g. Theorem 2.4 in [19]): Let $p \in \mathbb{R}[t]$ have degree $k$, and suppose $p(t) \geq 0$ for all $t \in [-1, 1]$.

- If $k$ is odd, then $p \in (1-t)\Sigma[t]_{k-1} + (1+t)\Sigma[t]_{k-1}$.
- If $k$ is even, then $p \in \Sigma[t]_{k} + (1-t^2)\Sigma[t]_{k-2}$.

III. OUR APPROACH

Consider a Markov chain on $\mathcal{X} = \{1, \ldots, n\}$ with irreducible transition matrix $K$, (unique) stationary distribution $\pi$, and Dirichlet form $\mathcal{E}(x, \cdot)$. We denote $\pi_{i} = \min\{\pi_{i} : i \in \mathcal{X}\}$. Since the kernel $K$ is irreducible and $\mathcal{X}$ is finite, $\pi_{i} > 0$.

We are interested in a systematic way of determining numbers $\gamma \in \mathbb{R}$ such that

$$\gamma \mathcal{E}(x, x) \geq \text{Ent}_\pi(x^2) := \sum_{i \in \mathcal{X}} \pi_i x_i^2 \log \left( \frac{x_i^2}{\|x\|_\pi^2} \right)$$

holds for every $x \in \mathbb{R}^n$. Then $\gamma^{-1}$ will be a lower bound on $\alpha$, the logarithmic Sobolev constant of the chain. Since (III.1) is homogeneous in $x$, we need only to prove it for $x$ satisfying $\|x\|_\pi^2 = 1$. We may also assume that the components of $x$ are nonnegative, since $\mathcal{E}(\|x\|, \|x\|) \leq \mathcal{E}(x, x)$.

This simplifies the right hand side to $\sum_{i \in \mathcal{X}} 2\pi_i x_i^2 \log x_i$. Let us write $S^\pi$ for the intersection of the hyperellipsoid $\|x\|_\pi^2 = 1$ with the nonnegative orthant

$$S^\pi = \{x \in \mathbb{R}_+^n : \|x\|_\pi^2 = 1\}.$$

Now the inequalities we are interested in have the form

$$\gamma \mathcal{E}(x, x) \geq \sum_{i \in \mathcal{X}} 2\pi_i x_i^2 \log x_i$$

whenever $x \in S^\pi$. The right hand side of (III.2) is not a polynomial in $x$. In order to make use of the sum-of-squares machinery described in Section II, we will replace the right hand side of (III.2) by an upper bound which is a polynomial. Specifically, if $P(x) \in \mathbb{R}[x]$ is a polynomial satisfying $P(x) \geq \text{Ent}_\pi(x^2)$ for all $x \in S^\pi$, then the polynomial inequality

$$\gamma \mathcal{E}(x, x) \geq P(x) \quad \forall x \in S^\pi$$

implies (III.2). Fixing $d \geq \frac{1}{2} \deg P$, we can then find valid values of $\gamma$ by solving the following sum-of-squares program:

$$\min_{\gamma \in \mathbb{R}} \gamma \text{ s.t. } \gamma \mathcal{E}(x, x) - P(x) \in \text{SOS}_d(S^\pi)$$

where $\text{SOS}_d(S^\pi)$ is a degree-$d$ sum-of-squares relaxation of the set of nonnegative polynomials on $S^\pi$ following (II.5), namely:

$$\text{SOS}_d(S^\pi) := \left\{ \sigma_0(x) + \sum_{i=1}^n x_i \sigma_i(x) + \phi(x) (\|x\|_\pi^2 - 1) \right\}.$$  (III.5)

Two questions should be addressed:

- How to choose the polynomial $P$ which approximates $\text{Ent}_\pi(x^2)$ on $S^\pi$?
- Having chosen a polynomial $P$ such that $\text{Ent}_\pi(x^2) \leq P(x) \leq (1+\epsilon) \text{Ent}_\pi(x^2)$ for all $x \in S^\pi$, can we guarantee that the sum-of-squares relaxation (III.4) will be $\epsilon$-close to the log-Sobolev constant $\alpha$, for large enough $d$?

We address these two questions in the following two subsections, starting by the latter.

A. The Sum-of-Squares Relaxation (III.4)

Assume we have a polynomial $P(x)$ which is an $\epsilon$-approximation to the entropy function $\text{Ent}_\pi(x^2)$ on $S^\pi$. In this subsection we show that the sum-of-squares program (III.4) will give, for large enough $d$, an $\epsilon$-approximation of the logarithmic Sobolev constant. This is the object of the next theorem.

Theorem 3: Let $\mathcal{E}, \pi$ be the Dirichlet form and stationary distribution of an irreducible Markov chain on $\mathcal{X} = \{1, \ldots, n\}$.
with log-Sobolev constant $\alpha$. Let $\epsilon \in (0, 1)$, and suppose that $P(x) \in \mathbb{R}[x]$ is a polynomial such that, for all $x \in S^\pi$, we have

$$
\text{Ent}_\pi[x^2] \leq P(x) \leq (1 + \epsilon) \text{Ent}_\pi[x^2].
$$

(III.6)

Let $\gamma_4^\dagger$ be the optimal value of (III.4), and call $\alpha_4^\dagger = 1/\gamma_4^\dagger$. Then $\alpha_4^\dagger \leq \alpha$. Furthermore, there exists $d_0 \in \mathbb{N}$ such that for every $d \geq d_0$, $\alpha_4^\dagger \geq (1 - \epsilon)\alpha$.

It is a foundational result in sum-of-squares programming that any polynomial which is \textit{strictly positive} on a compact basic semialgebraic set (i.e., a compact set of the form (II.4)) has a sum-of-squares representation on that set, provided an additional so-called Archimedean condition is satisfied. This is known as Putinar’s Positivstellensatz [20]. We cannot directly use this result however, as the assumption (III.6) on $P$ implies that $P(\mathbb{1}) = 0$ and so $\gamma_4^\dagger (1, \mathbb{1}) - P(\mathbb{1}) = 0$.

Instead we will use a result from a more recent line of work [21], [22], [23], which guarantees the existence of a sum-of-squares representation provided the second-order sufficient conditions of optimality are satisfied at the vanishing points of the polynomial. The most convenient result for our purposes is [23, Theorem 1.1] which we cite here in the particular case of $S^\pi$ for convenience.

\textbf{Theorem 4} ([23, Theorem 1.1]): Assume $f(x) \in \mathbb{R}[x]$ is a polynomial such that $f(x) \geq 0$ for all $x \in S^\pi$. Let $h(x) = \|x\|_\pi^2 - 1$, and assume that any zero $x^*$ of $f$ in $S^\pi$ satisfies the following:

- $x_i^* > 0$ for each $i \in \{1, \ldots, n\}$
- $\nabla f - \kappa h(x^*) = 0$ for some $\kappa \in \mathbb{R}$ (which can depend on $x^*$)
- $\nabla^2 f(x^*)$ is positive definite on the subspace $\{v \in \mathbb{R}^n : \sum_{i=1}^n \pi_i x_i^* v_i = 0\}$ (the tangent space to $S^\pi$ at $x^*$).

Then there exists large enough $d$ such that $f \in \text{SOS}_d(S^\pi)$.

Equipped with this theorem, we are ready to prove Theorem 3.

\textbf{Proof of Theorem 3}: The fact that $\alpha_4^\dagger \leq \alpha$ is obvious by construction. We therefore focus on the second claim of the theorem.

Let $f_1(x) := \frac{1}{1 - \epsilon^2} \epsilon E(x, x) - P(x)$. Our goal is to show that for large enough $d$, $f_1 \in \text{SOS}_d(S^\pi)$. By definition of the log-Sobolev constant and (III.6), the polynomial $f_0(x) := (1 - \epsilon^2) \epsilon E(x, x) - P(x)$ is nonnegative on $S^\pi$. Note that $1 + \epsilon < (1 - \epsilon)^{-1}$ and, by irreducibility, $E(x, x)$ is strictly positive on $S^\pi \setminus \{\mathbb{1}\}$ (this follows from Perron-Frobenius). Since

$$
f_1(x) = f_0(x) + \frac{2\epsilon}{(1 - \epsilon^2)\alpha} E(x, x),
$$

the polynomial $f_1$ is positive on $S^\pi \setminus \{\mathbb{1}\}$, and zero at $x = \mathbb{1}$.

It remains to check that the conditions of Theorem 4 hold for $f_1$ at $x^* = \mathbb{1}$. Certainly $\mathbb{1} > 0$, and since $x^* = \mathbb{1}$ is a minimum of $f_1$ on $S^\pi$, the first-order optimality condition also holds for some $\kappa \in \mathbb{R}$.

Let $A_0(x) = f_0(x) - \kappa(\|x\|_\pi^2 - 1)$, and $A_1(x) = f_1(x) - \kappa(\|x\|_\pi^2 - 1)$. The second-order sufficiency condition for $f_1$, which it remains to verify, is that for any $v \neq 0$ s.t. $\langle v, \mathbb{1} \rangle_\pi = 0$, $v^T \nabla^2 A_0(\mathbb{1}) v > 0$. By the second-order \textit{necessary} condition for $f_0$ to be nonnegative, $v^T \nabla^2 A_0(\mathbb{1}) v \geq 0$ for any such $v$, so it suffices to check that $v^T \nabla^2 E(\mathbb{1}, \mathbb{1}) v > 0$. Since $A_1(x) = A_0(x) + \frac{2\epsilon}{(1 - \epsilon^2)\alpha} E(x, x)$, we have $v^T \nabla^2 E(\mathbb{1}, \mathbb{1}) v = 2E(v, v)$, and by (I.1), $2E(v, v) \leq 0 \iff \sum_{i,j} \pi_i K_{i,j} (v_i - v_j)^2 \leq 0$. Since $K$ is irreducible, the latter inequality implies $v_i = v_j$ for each pair $i, j$, which is impossible since $v \neq 0$ and $\langle v, \mathbb{1} \rangle_\pi = 0$.

\textbf{Remark 1 (Rate of Convergence):} It would be satisfying to prove an upper bound on $d_0$ in terms of $n$, or to characterize the rate of convergence of the solutions $\gamma$ to $\sup_{x \in S^\pi} P(x)/E(x, x)$, as was done e.g., in [24], [25], and [26]. Unfortunately, as far as we can anticipate, results in Section V suggest that finite convergence often occurs already at $d = 3$, at least for small chains.

Whilst Theorem 3 asserts that for large enough $d$, $\alpha_4^\dagger \geq (1 - \epsilon)\alpha$, it is not a priori obvious that for small values of $d$, the value of the relaxation $\alpha_4^\dagger$ is even strictly positive, i.e., that (III.4) is feasible. We show below, in Proposition 5, that under some mild conditions on $P$ the program (III.4) is indeed feasible for any $d$, and hence can already yield nontrivial bounds on $\alpha$ for small values of $d$.

\textbf{Proposition 5}: Let $E, \pi$ be the Dirichlet form and stationary distribution of an irreducible Markov chain on $\mathcal{X} = \{1, \ldots, n\}$ with log-Sobolev constant $\alpha$. Suppose that $P(x) \in \mathbb{R}[x]$ is a polynomial such that, for all $x \in S^\pi$, we have

$$
\text{Ent}_\pi[x^2] \leq P(x) \leq (1 + \epsilon) \text{Ent}_\pi[x^2]
$$

(III.7)

for some $\epsilon > 0$. Then for any $d \geq (\deg P)/2$, $\alpha^\dagger \geq 0$, i.e., (III.4) is feasible.

\textbf{Proof}: We only include a brief sketch, and defer the technical details to the appendix. First we note that, from (III.7), $P(x) \geq 0$ for $x \in S^\pi$ and furthermore $P(\mathbb{1}) = 0$. Thus, by the first-order conditions of optimality we must have $\nabla P(\mathbb{1}) = 2\kappa \pi$ for some $\kappa \in \mathbb{R}$, or equivalently $\nabla \Lambda(\mathbb{1}) = 0$ where

$$
\Lambda(x) = P(x) - \kappa(\|x\|_\pi^2 - 1).
$$

Let $b_1(x), \ldots, b_N(x)$ be a basis for the subspace of polynomials $\{p \in \mathbb{R}[x] : p(\mathbb{1}) = 0\}$. By Lemma 12 in the Appendix we know that

$$
\{b_i b_j : 1 \leq i, j \leq N\} = \{p \in \mathbb{R}[x] \text{ s.t. } p(\mathbb{1}) = 0\}.
$$

This means that we can write

$$
\Lambda(x) = P(x) - \kappa(\|x\|_\pi^2 - 1) = b(x)^T B b(x)
$$

(III.8)

for some $N \times N$ symmetric matrix $B$, where $b(x) = [b_i(x)]_{1 \leq i, j \leq N}$.

To complete the proof, we will show there exists a positive \textit{definite} matrix $A > 0$ such that

$$
E(x, x) = b(x)^T A b(x) \text{ mod } (\|x\|_\pi^2 - 1).
$$

(III.9)

This will conclude the proof, because then by choosing $\gamma$ large enough such that $\gamma A - B \succeq 0$, we get

$$
\gamma E(x, x) - P(x) = b(x)^T (\gamma A - B) b(x) \text{ mod } (\|x\|_\pi^2 - 1).
$$

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Since $\gamma A - B \succeq 0$, the term $b(x)^T (\gamma A - B) b(x)$ is a sum-of-squares of polynomials, and so this establishes that $\gamma \mathcal{E}(x, x) - P(x) \in \text{SOS}_d(S^n)$ which is what we wanted.

We focus on proving (III.9). From the Poincaré inequality, we know that $\mathcal{E}(x, x) - \lambda \text{Var}_x [x] \succeq 0$ with $\lambda > 0$, for all $x \in S^n$. Since $\mathcal{E}(x, x) - \lambda \text{Var}_x [x]$ is a quadratic form, this means that it is necessarily a sum-of-squares, i.e., $\mathcal{E}(x, x) - \lambda \text{Var}_x [x] \in \text{SOS}_1(S^n)$. By writing $\mathcal{E}(x, x) = (\mathcal{E}(x, x) - \lambda \text{Var}_x [x]) + \lambda \text{Var}_x [x]$ it suffices to prove that $\text{Var}_x [x] = \|x - 1\|^2_\pi$ has the required decomposition (III.9). This is proved in Lemma 13 in the Appendix. \(\square\)

Remark 2: In the proof of Proposition 5 the only property we actually use about $P$ is that $P(1) = 0$ and that $\nabla P(1) \propto \pi$, which is a consequence of (III.6).

None of our results so far give us any indication of how to choose $P(x)$, nor indeed whether it is possible for a polynomial to approximate $\text{Ent}_+ [x^2]$ in the manner that Theorem 3 demands. This will be the topic of our next section.

B. Choice of Polynomial Upper Bound

The focus of this section is on obtaining polynomial upper bounds for the relative entropy $\text{Ent}_+ [x^2] = \sum_i \pi_i x_i^2 \log(x_i^2)$ which are valid on $S^n$. In this work we will only consider separable bounds of the form $P(x) = \sum_i \pi_i p_i(x_i)$, where we will require $p_i(t) \geq t^2 \log(t^2)$ for all $t \in [0, 1/\sqrt{\pi_1}]$. We describe two ways of obtaining polynomial upper bounds on $q(t) = t^2 \log(t^2)$ for $t > 0$. The first is elementary and consists simply of a truncated Taylor approximation. Even though this method yields reasonably good bounds on small chains, it cannot yield a convergent sequence of lower bounds to $\alpha$. The second is based on rational approximations of the logarithm. This more sophisticated method will allow us to obtain a hierarchy of sum-of-squares programs which provably converge to the logarithmic Sobolev constant of a given Markov process (Theorem 8).

1) Taylor Series: For $k \geq 2$, the Taylor expansion to odd order $2k - 1$ of $q(t) := t^2 \log(t^2)$ about $t = 1$ gives an upper bound on $q(t)$ valid for every $t > 0$. To see this, write $p_{T/2}^{2k-1}(t)$ for the Taylor expansion of $q(t)$ to order $2k - 1$ about $t = 1$, and write $q^{(3)}(t)$ for the $j$th derivative of $q$ at $t$. By Taylor’s theorem the remainder term (in Lagrange form) is

$$q(t) - p_{T/2}^{2k-1}(t) = \frac{q^{(2k+2)}(\zeta_t)}{(2k+2)!}(t - 1)^{2k+2}$$

for some $\zeta_t$ between 0 and $t$. We have $q^{(3)}(t) = 4/t$, hence

$$q^{(2k+2)}(t) = \frac{4(2k)!}{t^2k}$$

for $k \geq 2$. It follows that

$$q(t) - p_{T/2}^{2k-1}(t) = 4(-1)^{2k-1}\frac{(t - 1)^{2k+2}}{(2k + 1)(2k+2)!},$$

which is nonpositive.

Using $P(x) = \sum_i \pi_i p_{T/2}^i(x_i)$ in the SoS program (III.4) will allow us to obtain the exact logarithm Sobolev constant for the simple random walk on the $n$-cycle for small $n$ (see Section VI). However, in view of the divergence of the truncated Taylor series for $|t - 1| > 1$, the resulting sequence of sum-of-squares relaxations do not in general converge to the logarithmic Sobolev constant $\alpha$.

2) Adaptive Bounds via Padé Approximants: The $(m, k)$ Padé approximant of a smooth function $g(t)$ around $t = 1$ is the rational function

$$r_{m,k}(t) = \frac{R(t)}{S(t)}$$

where $R$ and $S$ are polynomials of degree $m$ and $k$ respectively, such that $S(1) = 1$ and such that derivatives of $r_{m,k}(t)$ at $t = 1$ agree with those of $g$ to as high an order as possible. For a generic smooth $g$, this means that derivatives of order $(0, 1, \ldots, m + k)$ agree, i.e. $g(t) - r_{m,k}(t) = O((t - 1)^{m+k+1})$. Despite being defined by the same category of data as the truncated Taylor series’ (namely the first few derivatives of $g$ at $t = 1$), in many situations Padé approximants can be considered “better” approximations of $g$, in the sense that convergence can occur outside of the radius of convergence of the Taylor series, and moreover this convergence may be faster. As the following result shows, this is indeed the case for the diagonal Padé approximants of the logarithm function.

Theorem 6: For each $m \geq 0$ let $\overline{r}_{m}(t)$ be the $(m + 1, m)$ Padé approximant to $\log(t)$ at $t = 1$. (i)

1) The denominator $S_m(t)$ of the rational function $\overline{r}_{m}(t)$ has no roots in $(0, \infty)$, hence $S_m$ is positive on this interval.

2) For each $m \in \mathbb{N}$ there is a constant $B_m$ such that

$$0 \leq \overline{r}_{m}(t) - \log(t) \leq B_m(t - 1)^2/t \quad \forall t > 0,$$

and such that $B_m \downarrow 0$ as $m \to \infty$. In fact, for $m \geq 2$ we can take $B_m = \frac{4}{(m+1)^2\pi^2}$.

3) For $t > 0$ and $m \geq 0$,

$$\overline{r}_{m+1}(t) \leq \overline{r}_{m}(t).$$

Proof: See [27, Prop. 2.2]. We remark that the cited proposition shows that $\overline{r}_{m}(t) - \log(t) \leq B_m (t - 1)^2(t - 1)^{2m}$ so that convergence is exponentially fast on any compact interval of $(0, \infty)$. \(\square\)

It remains to specify how to use the Padé approximants $\overline{r}_{m}(t)$ in the sum-of-squares relaxation (III.4). The obvious difficulty is that $\overline{r}_{m}(t)$ is a rational function, and not a polynomial. To overcome this difficulty, we let the polynomial $P$ (i.e., its coefficients) be itself a decision variable in the optimization problem, subject to the constraint that $P(x)$ upper bounds $\text{Ent}_+ [x^2]$ on $S^n$. If we consider separable polynomials $P(x) = \sum_i \pi_i p_i(x_i)$, then this gives us the following optimization problem:

$$\min_{\gamma \in \mathbb{R}, \quad p_i \in \mathbb{R}[t]_{2d}} \gamma \quad \text{subject to}$$

$$\\begin{cases}
\gamma \mathcal{E}(x, x) - \sum_i \pi_i p_i(x_i) \in \text{SOS}_d(S^n), \\
p_i(t) \geq t^2 \log(t^2) \quad \forall t \in [0, 1/\sqrt{\pi_1}], \forall i \in \mathcal{X}.
\end{cases}$$

(III.10)

The coefficients of $p_i$ are now variables of the optimization problems, which are constrained so that the inequality $p_i(t) \geq t^2 \log(t^2)$ holds on the appropriate interval. Because we do not
have a method to enforce this constraint directly, we replace 
log(t^2) by its rational Padé approximant. The key observation is 
that given a rational function \( R(t)/S(t) \) which is an upper 
bound for \( t^2 \log(t^2) \), we can enforce the constraint \( p_i(t) \geq \)
\( R(t)/S(t) \) on the interval \([0,1/\sqrt{\pi}]\) by demanding a sum-of-
quares certificate for the univariate polynomial \( S(t)p_i(t) - R(t) \). 
Such a certificate will naturally imply the stronger condition 
\( p_i(t) \geq t^2 \log(t^2) \). If we let
\[
\ell_m(t) = \frac{(t-1)R_m(t)}{S_m(t)}
\]
be the \((m+1,m)\) Padé approximant of \( \log \) at \( t = 1 \) (where \( \deg R_n = \deg S_m = m \)), then this yields the following optimization problem:
\[
\alpha_{d,m} := \min_{\gamma \in \mathbb{R}, p_i \in \mathbb{R}[t]_{2d}} \gamma \text{ subject to }
\left\{ \begin{align*}
\gamma \mathcal{E}(x,x) &- \sum_{i=1}^{n} \pi_i p_i(x_i) \in \text{SOS}_d(S^\pi), \\
S_m(t)p_i(t) - 2\ell(t-1)R_n(t) &\in \text{SOS}_{d+[m/2]}([0,\pi_i^{-1/2}]), \ i \in \mathcal{X} \end{align*} \right\} (H_{d,m})
\]
where \( \text{SOS}_k([a,b]) \) is the set of polynomials of degree at 
most \( 2k \) that are nonnegative on the interval \([a,b] \). Since any 
such polynomial admits a sum-of-squares representation, this 
set can be described using semidefinite programming (see 
Section II-D).

We note that the semidefinite program \((H_{d,m})\) is indexed 
by two parameters: \( d \), which is the degree of the sum-of-
quares relaxation, and \( m \) which is the degree of the rational 
approximation to the log function.

The next theorem shows that with large enough \( m \) and \( d \), 
one can find a polynomial \( p \) of degree \( 2d \) such that \( p_1 = \cdots = 
p_n = p \) is feasible for \((H_{d,m})\) and such that 
\( \sum \pi_i p(x_i) \leq (1 + \epsilon) \text{Ent}_\pi[x^2] \) on \( S^\pi \).

**Theorem 7:** Let \( \pi_1, \ldots, \pi_n > 0 \) be such that \( \sum_{i=1}^{n} \pi_i = 1 \), 
and write \( \pi^* = \min \pi_i \). For any \( \epsilon \in (0,1) \), there exists \( m \in \mathbb{N} \) 
and a univariate polynomial \( p \in \mathbb{R}[t] \) such that the following 
is true:

- \( p(t) \geq 2\ell(t) \) whenever \( t \in [0,1/\sqrt{\pi}] \), and for all 
  \( x \in S^\pi \); and
- \( \sum_{i=1}^{n} \pi_i p(x_i) \leq (1 + \epsilon) \text{Ent}_\pi[x^2] \) for all \( x \in S^\pi \).

**Proof:** Write \( N_\epsilon = \pi^*^{-1/2} \), and let \( \epsilon > 0 \) be a constant 
depending on \( \epsilon \) and \( N_\epsilon \) whose value we will determine later. 
Choose \( m \in \mathbb{N} \) large enough that \( B_m < \epsilon_1 \) (see Theorem 6). 
We obtain that for every \( t \geq 0 \)
\[
2\ell(t) - t^2 \log(t^2) \leq 2\epsilon_1(t-1)^2. 
\]
(III.11)

Applying the Weierstrass Approximation Theorem to the continuous function \( t \mapsto \frac{t \log(t) - (t-1)^2}{2} \), we deduce the existence of a polynomial \( q \) satisfying
\[
0 \leq q(t) - \frac{t \ell(t) - (t-1)^2}{2} \leq \epsilon_1.
\]

Note that \( \ell(t) = (t-1) + O(t^{-1}) \) as \( t \to 1 \) since \( \ell(t) - \log(t) = 
O(t^{-1/2}) \).

**Remark 3:** [Size of the Semidefinite Program \((H_{d,m})\)] The 
semidefinite program corresponding to \( \alpha_{d,m} \), is over the 
product of positive semidefinite cones
\[
\Pi \text{SOS}_{d+[m/2]}([0,\pi_i^{-1/2}])
\]
which has dimension \( O(n^d + m^2 n) \) for fixed \( d \).
Remark 4 (Monotonicity in $d$ and $m$): For fixed $d$, $\alpha_{d,m}$ is nondecreasing in $m$ (because the $(m + 1, m)$ Padé approximants are nonincreasing in $m$, see Theorem 6). Similarly for fixed $m$, $\alpha_{d,m}$ is nondecreasing in $d$, because any feasible $(\gamma, (p_i))$ for $(H_{d,m})$ is also valid for $(H_{d',m})$ with $d' > d$. In particular, this means we can (in principle) consider only the diagonal steps of the hierarchy to get a hierarchy $(\alpha_{d,x})_d$ (indexed by a single parameter $d$) which converges monotonically to $\alpha$. In practice, since the size of the underlying semidefinite program for $H_{d,m}$ scales exponentially in $d$, but only polynomially in $m$ (see Remark 3 above), it is desirable to keep the freedom to increase $m$ and $d$ independently.

Remark 5: [Strict Feasibility of $(H_{d,m})$] Even though we show in Theorem 8 that $(H_{d,m})$ is feasible as soon as $d \geq 2$ and $m \geq 0$, the semidefinite program is not, as written, strictly feasible. Strict feasibility of a semidefinite program is a very useful property to have both in theory (e.g., for strong duality), and in practice, to avoid numerical issues. Thankfully, one can easily modify the semidefinite program $(H_{d,m})$ to make it strictly feasible. There are two reasons why $(H_{d,m})$ is not strictly feasible. The first reason is that the polynomial $\gamma E(x, x) - \sum_i \pi_i p_i(x_i)$ vanishes at $x = 1$ for any feasible $(\gamma, (p_i))$. To make the first constraint of the semidefinite program strictly feasible, it suffices to slightly change the definition of $\text{SOS}_d(S^\pi)$ in (III.5) and require that $\sigma_0 \in \Sigma(V_d)$ and $\sigma_1, \ldots, \sigma_n \in \Sigma(V_{d-1})$ where $V_k = \{ p \in \mathbb{R}[x] : p(1) = 0 \}$, and $\Sigma(V_k) = \text{cone} \{ q^2 : q \in V_k \}$ (see (II.1)). The second reason that $(H_{d,m})$ is not strictly feasible, is that the univariate polynomials $S_m(t)p_1(t) - 2t^2(t-1)R_m(t)$ all vanish at $t = 1$, so they cannot be in the interior of $\text{SOS}_{d+[m/2]}(0, \pi_i^{-1/2})$. This can be easily fixed by factoring out a term $(t - 1)$ in the definition of $\text{SOS}_{d+[m/2]}(0, \pi_i^{-1/2})$. We omit the details here for brevity.

Remark 6 (A Possible Alternative Approach): Instead of searching over polynomials $p_i$ which are upper bounds for $2t^2R_m(t)$, one could consider simply substituting $P(x) = 2 \sum \pi_i x_i^2R_m(x_i)$ in (III.4) and searching for a sum-of-squares nonnegativity certificate for $\gamma E(x, x) - \sum_i \pi_i x_i^2(1 - R_m(x_i) \prod_{j \neq i} S_m(x_j)$ on the hyperellipsoid $S^\pi$. The main drawback of this approach is that the above polynomial has degree $nm + 3$, so we would need to search for certificates in $\text{SOS}_{n^m+3}(S^\pi)$. This means the size of the resulting semidefinite program grows like $\Omega(n^{m+3})$. Even for very small values of $m, n$, this is impossible to implement in practice. In contrast, in our proposed method the semidefinite program has size $O(n^{d+2}m^2n)$, where $d$ is fixed in advance independent of $n$ and $m$. As we will see in section V, this approach works well in practice.

D. Exact Rational Lower Bounds
Computing the lower bounds $\alpha_{d,m}$ requires solving semidefinite programs numerically and at present, all efficient algorithms for solving semidefinite programs use floating point arithmetic. Hence the solutions they return are infeasible by a small margin. In this section we briefly describe how these numerical solutions can be turned into formal rational lower bounds on the logarithmic Sobolev constant. The techniques we use are similar to the rational rounding approach of [28].

The sum-of-squares program $(H_{d,m})$ can be put in standard semidefinite programming form as follows:

$$\begin{align*}
\text{minimize} & \langle c, y \rangle \text{ s.t. } A \begin{pmatrix} y \\ z \end{pmatrix} = b, \\
& \text{VecToSymMat}(z) \succeq 0, \\
& y \in \mathbb{R}^{N_1}, z \in \mathbb{R}^{N_2(N_2+1)/2}
\end{align*}$$

where $A, b, c$ encodes the problem data, namely the transition kernel $K$ and the invariant distribution $\pi$. Here $\text{VecToSymMat}$ denotes the vectorization of the upper triangular part of a symmetric matrix. In most chains of interest the transition probabilities (and hence also the stationary probabilities) are rational numbers, so the corresponding data in $A, b, c$ are rational too.

When a strictly feasible, bounded below SDP in the form (III.14) is provided as input to an interior-point solver, we can typically expect to obtain a floating point output $(\hat{y}, \hat{z})$ satisfying

$$|| A \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} - b ||_\infty \leq (1 + ||b||_\infty) \varepsilon_{\text{aff}}$$

for some constant $\varepsilon_{\text{aff}}$ (typically $\approx 10^{-8}$). Because the linear equations are only satisfied approximately, we cannot expect that $\langle c, \hat{y} \rangle$ is a true upper bound on $\alpha(K)^{-1}$.

In this case, following [28] we seek to round $(\hat{y}, \hat{z})$ to rational numbers satisfying the affine constraints exactly. This can be done in two steps:

1. First, we convert the floating-point solution $(\hat{y}, \hat{z})$ to rational numbers. Every floating-point number is already rational, with denominator some power of 2, so this can be done exactly by simply changing the internal representation our computer uses to represent these numbers.

2. Next, we project $(\hat{y}, \hat{z})$ onto the affine subspace defined by the constraints of (III.14):

$$A = \left\{ (y, z) \in \mathbb{Q}^{N_1} \times \mathbb{Q}^{N_2(N_2+1)/2} \mid A \begin{pmatrix} y \\ z \end{pmatrix} = b \right\}.$$ 

The orthogonal projection map onto this space is given by

$$\Pi_A \left( \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left( \begin{pmatrix} y \\ z \end{pmatrix} - A^\top (AA^\top)^{-1} A \begin{pmatrix} y \\ z \end{pmatrix} - b \right).$$

Note that this projection operator uses only rational arithmetic, and can be implemented exactly in a computer algebra system. Therefore when applied to $(\hat{y}, \hat{z})$, this projection produces a rational point

$$\begin{pmatrix} \hat{y}^* \\ \hat{z}^* \end{pmatrix} = \Pi_A \left( \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} \right)$$

satisfying the affine constraints exactly.
However the new point \((y^*, z^*)\) may not be feasible for the semidefinite program (III.14), in particular \(\text{VecToSymMat}(z^*)\) may not be positive semidefinite.

To remedy this situation, consider the slightly stronger semidefinite program:

\[
\min \{c, y\} \text{ s.t. } A \begin{pmatrix} y \\ z \end{pmatrix} = b,
\]

\[
\text{VecToSymMat}(z) \succeq \varepsilon I,
\]

\[
y \in \mathbb{R}^{N_1}, \quad z \in \mathbb{R}^{N_2(N_2+1)/2},
\]

(III.17)

where \(\varepsilon > 0\) is a parameter. The next proposition shows that by rounding a floating-point solution of (III.17) with \(\varepsilon > \sqrt{\lambda_{\min}(AA^*)}\) and \(\|b\|_\infty \varepsilon_{\text{aff}}\), one can obtain a rational feasible point to (III.14) which satisfies the constraints exactly.

**Proposition 9:** Consider the semidefinite program (III.17) where \(A \in \mathbb{R}^{r \times (N_1+N_2(N_2+1)/2)}\) is assumed to have full row rank \(r\). Assume \((\tilde{y}, \tilde{z})\) is an approximate rational feasible point of (III.17) with

\[
\varepsilon = \sqrt{\frac{2r}{\lambda_{\min}(AA^*)}} (1 + \|b\|_\infty \varepsilon_{\text{aff}})
\]

i.e., \(\text{VecToSymMat}(\tilde{z}) \succeq \varepsilon I\) and \(\|A (\begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix}) - b\|_\infty \leq (1 + \|b\|_\infty \varepsilon_{\text{aff}})\). Then \((y^*, z^*)\) defined by (III.16) is a rational feasible point to (III.14).

**Proof:** See Appendix B.

**IV. Beyond Logarithmic Sobolev Inequalities**

In this section we show how the techniques developed in the previous section can be applied for other entropic functional inequalities. We show how to use sum-of-squares relaxations combined with Padé rational approximants to obtain bounds on modified logarithmic Sobolev constants (also known as the entropy constant), and strong data processing inequalities.

**A. Algorithmic Bounds on the Entropy Constant**

Given an irreducible Markov kernel \(K\) with stationary distribution \(\pi\), as an alternative to (I.2), one can try to prove modified logarithmic Sobolev inequalities (MLSI) [29] of the form

\[
\mathcal{E}(x, \log x) \geq \rho \text{Ent}_\pi[x] \quad \forall x \in \mathbb{R}^+_\pi.
\]

(IV.1)

The largest constant \(\rho\) for which (IV.1) holds is called the entropy constant of the Markov semigroup \(P_t\) associated to \(K\) via \(P_t = e^{t(K - I)}\). Modified log-Sobolev inequalities are important because it can be shown [30, Theorem 2.4] that the entropy constant has an equivalent characterization as the best constant \(\rho\) such that

\[
D(\mu || \pi) \leq e^{-\rho t} D(\mu_0 || \pi)
\]

(IV.2)

holds for all initial distributions \(\mu_0\) on \(\mathcal{X}\), where \(\mu_t := \mu_0 P_t\). Therefore, the entropy constant of a Markov semigroup bounds the relative entropy mixing time of the corresponding Markov process.

For every irreducible kernel \(K\), one has \(2\alpha(K) \leq \rho(K) \leq 2\lambda(K)\), and if \(K\) is reversible, the first inequality can be strengthened to \(4\alpha(K) \leq \rho\) [2, Proposition 1.10 & Remark 1.11]. Thus the log-Sobolev constant always provides a lower bound on the entropy constant. However, one can have \(\alpha < \rho\) for example in the case of the random walk on the group of permutations \(S_n\) of \(\{1, \ldots, n\}\) which at each jump moves by a uniformly sampled transposition, see [30, Example 3.12].

Note that for any \(x \in \mathbb{R}^+_\pi\) and \(\rho > 0\), we have

\[
\mathcal{E}(\gamma x, \log \gamma x) = \gamma \mathcal{E}(x, \log x + \log(\gamma) I) = \gamma \mathcal{E}(x, \log x),
\]

so by homogeneity we may restrict the variable \(x\) appearing in (IV.1) to the simplex

\[
\Delta \pi := \{x \in \mathbb{R}^+_\pi : \text{Ent}_\pi[x] = 1\}.
\]

We can now rearrange (IV.1), using the definitions of \(\mathcal{E}\) and \(\text{Ent}_\pi\), to obtain the equivalent inequality

\[
\sum_{i,j \in \mathcal{X}} \pi_i K_{ij} x_i (\bar{\rho} \log x_i - \log x_j) \geq 0 \quad \forall x \in \Delta \pi,
\]

(IV.3)

where \(\bar{\rho} := 1 - \rho\).

Suppose that for each pair \((i, j) \in \mathcal{X} \times \mathcal{X}\), \(p_{ij}(t, s)\) is a bivariate polynomial satisfying

\[
p_{ij}(t, s) \leq \bar{\rho} t \log t - t \log s \quad \forall (t, s) \in [0, \pi_i^{-1}] \times [0, \pi_j^{-1}],
\]

(IV.4)

for some fixed \(\bar{\rho} > 0\). Then the polynomial inequality

\[
\sum_{i,j \in \mathcal{X}} \pi_i K_{ij} p_{ij}(x_i, x_j) \geq 0 \quad \forall x \in \Delta \pi
\]

(IV.5)

implies (IV.3). In order to construct sum-of-squares relaxations of this type of inequality, we introduce the set

\[
\text{SOS}_d(\Delta \pi) := \left\{ \sigma_0(x) + \sum_{i=1}^n x_i \sigma_i(x) + \phi(x) (\text{Ent}_\pi[x] - 1) \right\}
\]

s.t. \(\sigma_0 \in \Sigma[x]_{2d}, \sigma_i \in \Sigma[x]_{2d-2} i \in \mathcal{X}\), \(\phi \in \mathbb{R}[x]_{2d-1}\).

(IV.6)

\(\text{SOS}_d(\Delta \pi)\) is the degree-\(d\) sum-of-squares relaxation of the set of polynomials which are nonnegative on \(\Delta \pi\). The following sum-of-squares programs, indexed by \(d \geq \frac{1}{2} \max_{i,j} \{\deg p_{ij}\}\), are a sequence of increasingly general sufficient conditions for (IV.5):

\[
\sum_{i,j \in \mathcal{X}} \pi_i K_{ij} p_{ij}(x_i, x_j) \in \text{SOS}_d(\Delta \pi).\]

(IV.7)

Now we consider how to obtain sum-of-squares certificates of the bivariate inequality (IV.4). In Section III-B2, we saw that the rational functions \(f_m(s) = \frac{(s-1)B_m(s)}{S_{m+1}(s)}\) the \((m+1, m)\) Padé approximants to \(\log(s)\) around \(s = 1\) are a convergent sequence of upper bounds for \(\log(s)\). In order to prove inequalities of the form (IV.4), we will also need rational
lower bounds on \( \log(t) \). Observe that \( \log(t) = -\log(1/t) \geq -r_m(1/t) \). Therefore
\[
t \log(t) \geq -t r_m(1/t) = \frac{(t-1)r_m(t)}{S_m(t)},
\]
where \( S_m(t) := t^m S_m(t) \) and \( R_m(t) := t^m R_m(1/t) \) are degree-\( m \) polynomials.

A sufficient condition for (IV.4) is therefore that
\[
p_{ij}(t, s) \leq -\hat{\rho} r_m(1/t) - t r_m(s)
\]
\[
= \frac{\hat{\rho}(t-1)R_m(t)}{S_m(t)} - \frac{(t-1)R_m(s)}{S_m(s)}
\]
for all \((t, s) \in [0, \pi^{-1}_t] \times [0, \pi^{-1}_s]\). Clearing denominators (recall that the polynomial \( S_m(t) \) is positive for \( t \geq 0 \), this is equivalent to the polynomial inequality
\[
\hat{\rho}(t-1)R_m(t)S_m(s) - t(s-1)R_m(s)S_m(t)
\]
\[
- p_{ij}(t, s)S_m(t)S_m(s) \geq 0
\]
for all \((t, s) \in [0, \pi^{-1}_t] \times [0, \pi^{-1}_s]\). This polynomial inequality can be enforced as a sum-of-squares constraint using the set
\[
\text{SOS}_k \left( [0, \pi^{-1}_t] \times [0, \pi^{-1}_s] \right) := \left\{ \sigma_0 + (1 - \pi_1 t)\sigma_1 + s(1 - \pi_2 s)\sigma_2 \right. \\
\text{s.t. } \sigma_0, \sigma_1, \sigma_2 \in \Sigma[t, s]_{2k-2} \}
\]
of degree-\( k \) sum-of-squares relaxations of the set of polynomials which are nonnegative on \([0, \pi^{-1}_t] \times [0, \pi^{-1}_s]\).

We arrive at the following sum-of-squares optimization problem, parametrized by positive integers \( d, m \):
\[
\rho_{d, m} := 1 - \min_{\hat{\rho} \in \mathbb{R}, \ p_{ij} \in \left[ [0, \pi^{-1}_t] \right]_{2d}} \hat{\rho} \text{ subject to }
\]
\[
\sum_{i, j \in \mathcal{X}} \pi_i K_{ij} p_{ij}(x, x_{-}) \in \text{SOS}_{d}(\Delta^\pi),
\]
\[
\hat{\rho}(t-1)R_m(t)S_m(s) - t(s-1)R_m(s)S_m(t)
\]
\[
- p_{ij}(t, s)S_m(t)S_m(s) \in \text{SOS}_{d+m} \left( [0, \pi^{-1}_t] \times [0, \pi^{-1}_s] \right) \quad i, j \in \mathcal{X}.
\]
For every \( m, d \), the numbers \( \rho_{d, m} \) are upper bounds for the true entropy constant \( \rho(K) \), and \( \rho_{d, m} \) can be computed as the solution to a semidefinite program of size \( O(n^{2d} + m^d n^2) \) for fixed \( d \).

B. Strong Data Processing Inequalities

A fundamental inequality in information theory is the data processing inequality, which states that if \( W : \mathcal{X} \to \mathcal{Y} \) is a channel, then for any two input distributions \( \mu, \pi \) on \( \mathcal{X} \) we have
\[
D(\mu W || \pi W) \leq D(\mu || \pi).
\]
For a specific channel \( W \) and reference input distribution \( \pi \), such an inequality can usually be strengthened. We say that the pair \((\pi, W)\) satisfies a strong data processing inequality (SDPI) if there is a constant \( \delta < 1 \) such that
\[
D(\mu W || \pi W) \leq \delta D(\mu || \pi) \tag{IV.8}
\]
for all probability distributions \( \mu \) on \( \mathcal{X} \) [31, 32]. The smallest such \( \delta = \delta^*(\pi, W) \) is the SDPI constant of \((\pi, W)\). For example, the binary symmetric channel with noise \( \epsilon \) and a uniform source is known to have SDPI constant \( \delta^*(\text{Bern}(\epsilon/2), \text{BSC}_\epsilon) = (1 - 2\epsilon)^2 \) [31]. Strong data processing inequalities, and more generally the contraction properties of discrete channels, have received a lot of attention recently in the information theory community [3], [32], [33], [34], [35], [36], [37], and have been applied e.g., to obtain various converse results.

The similarity of (IV.8) and (IV.2) suggests a connection between LSI/MLSI and SDPI. Indeed, such a connection was made precise by Raginsky in [3]. First, one defines the cascade channel \( W^2 : \mathcal{X} \to \mathcal{Y} \) by \( W^2 j[i] = \sum_{k \in \mathcal{Y}} W(k[j]) W(k[i]) \pi_k \) for \( i, j \in \mathcal{X} \). \( W^2 \) satisfies the detailed balance equations with respect to \( \pi \), so it is a \( \pi \)-reversible Markov kernel on \( \mathcal{X} \). Then one has \( \alpha(W^2 \pi) \leq 1 - \delta^*(\pi, W) \leq \rho(W^2 \pi) \) [3, Theorems 3 \\& 4].

In the paper [9], the authors present a hierarchy of semidefinite programming relaxations which give certified upper bounds on the strong data processing (SDPI) constant of a discrete channel. Moreover it is shown that the hierarchy converges to the true SDPI constant. The relaxations developed in [9] should be seen as the SDPI analogues of the relaxations developed in the present work for log-Sobolev and modified log-Sobolev inequalities.

V. Numerical Results

In this section we numerically illustrate the ideas developed in the previous sections. In cases where the exact log-Sobolev constant \( \alpha \) is not known (which are all cases in this section apart from subsection V-B) we use a method developed in section V-A to find an upper bound \( \overline{\pi} \) for \( \alpha \) in order to quantify approximation accuracy.

In each experiment (sections V-B through V-E) we set \( 2d = 6 \) (the degree of the polynomials \( p_i \) bounding the logarithmic term) and use the Padé relaxation approach of section III-B2 with \( m = 5 \) (the degree of the Padé upper bound).

A. Obtaining Upper Bounds for Comparison

The exact log-Sobolev constant is unknown for most finite state Markov chains. In order to judge the accuracy of our sum-of-squares method, we will need a way of obtaining good upper bounds on the log-Sobolev constant of a given Markov chain. One way to do this is to search for \( x \in \mathbb{R}^n \) for which
\[
\Psi(x) := \frac{\mathcal{E}(x, x)}{\text{Ent}_\pi(x^2)} = \frac{\sum_{i, j \in \mathcal{X}} \pi_i K_{ij}(x_i - x_j)^2}{\sum_{i \in \mathcal{X}} \pi_i x_i^2 \log(x_i^2/\|x^2\|)}
\]
is small. Indeed the log-Sobolev constant of the chain under consideration is exactly \( \alpha = \inf_{x \in \mathbb{R}^n} \Psi(x) \).

This minimization problem is nonconvex and difficult in general. We use Newton’s method to perform a local optimization, initialized at a randomly chosen initial point \( x^0 \), to obtain a local minimizer \( x^\star \). This is repeated \( R \) times, with different initializations, and the algorithm returns the best
relaxation with $d$, the relative error of this approximation, listed, along with the relative error of this approximation, is usually sufficient for random Markov chains on $n = 10$ vertices (see the numerical results in section V-D).

B. The Complete Graph

The log-Sobolev constant for the simple random walk on the complete graph with $n$ vertices is known to be $\alpha = \frac{1}{(n-1)\log(n-1)}$ (see [1, Corollary A.4]). In Table I lower bounds obtained from a Padé relaxation with $d = 3$ and $m = 5$ are listed, along with the relative error of this approximation, defined in this case as $\epsilon_{\text{rel}} = \alpha / \alpha_{3.5} - 1$. For comparison, we have also tabulated the relative errors of a truncated Taylor series based approximation, as described in section III-B1.

While performing the computations, no attempt was made to take advantage of the large amount of symmetry in the structure of these Markov chains. Doing this would have sped up computations significantly, but requires some effort to implement. In section VI, we will see an example where symmetry is exploited in order to compute the log-Sobolev constant for chains on larger state spaces than we consider here.

C. The Petersen Graph

The Petersen graph is a famous graph with 10 vertices and 15 edges. It exhibits a lot of symmetry – its automorphism group is isomorphic to $S_5$. We consider the simple random walk on this graph. Its spectral gap is $\lambda = 2/3$. Using a Padé relaxation with $d = 3$ and $m = 5$, we obtain

$$\alpha_{3.5} = 0.306638 \ldots \approx 0.459957 \cdot \lambda.$$
in which case it is equal to half the spectral gap. Its value was not yet known for odd $n$ larger than 5. We will show how the technique developed in Section III of this paper for obtaining lower bounds on logarithmic Sobolev constants can use to prove the following:

**Theorem 10:** The logarithmic Sobolev constant for the simple random walk on the $n$-cycle is equal to $\frac{1}{2}(1 - \cos \frac{2\pi}{n})$ for $n \in \{5, 7, 9, \ldots, 21\}$.

The proofs are too long to be checked manually, but are made available online, along with the small amount of code required to carry out a verification with the open-source computer algebra system SageMath [39].

Showing that the log-Sobolev constant of the $n$-cycle is equal to $\frac{\lambda}{2}$ amounts to proving the inequality

$$\frac{2}{\lambda} \mathcal{E}(x, x) - \frac{2}{n} \sum_{i=1}^{n} x_i^2 \log(x_i) \geq 0$$

for all $x \in \mathbb{R}^n$ satisfying $\|x\|_\infty = \frac{1}{n} \|x\|^2 = 1$. Following the approach described in Section III, we will prove instead a stronger inequality where the logarithmic term $2x_i^2 \log(x_i)$ is upper bounded by its Taylor expansion to order 5, namely

$$p_{5\text{tay}}(t) = 2(t - 1) + 3(t - 1)^2 + \frac{2}{3}(t - 1)^3 - \frac{1}{6}(t - 1)^4 + \frac{1}{15}(t - 1)^5.$$

If we define the polynomial

$$F(x) = \frac{1}{1 - \cos(2\pi/n)} \sum_{i \in \mathbb{Z}_n} (x_i - x_{i+1})^2 - \sum_{i=1}^{n} p_{5\text{tay}}(x_i),$$

our goal is to show that $F(x)$ is a sum-of-squares on the sphere $\{x \in \mathbb{R}^n : \|x\|^2 = n\}$. In other words, we want to solve the sum-of-squares feasibility problem:

$$\exists h \in \mathbb{R}[x]_{n,4} \text{ s.t. } F(x) + (\|x\|^2 - n) h(x) \in \Sigma(W)$$

(VI.1)

where $W$ is a subspace of $\mathbb{R}[x]_3$. Recall that $\Sigma(W)$ is the set of polynomials which are sums of squares of polynomials in $W$. This relaxation is most powerful when $W$ is the full space $\mathbb{R}[x]_3$, but in practice this leads to extremely large SDPs as $n$ grows. Another reason why we may want to have $W \subsetneq \mathbb{R}[x]_3$ is that when $W = \mathbb{R}[x]_3$, (VI.1) has no strictly feasible solutions (more details in subsection VI-A). The feasibility problem (VI.1) can be phrased as a semidefinite feasibility problem and solved numerically much like the examples in Section V.

### A. Facial Reduction

There are two barriers to extracting a proof from a floating-point SDP solution using the rational rounding method described in Subsection III-D.

Firstly, the data defining the problem are not in general rational, since they include the number $\cos(2\pi/n)$. However this can be easily remedied by working in the number field $\mathbb{Q}[\cos(2\pi/n)]$ instead. Secondly, in order to produce
formal certificates, we want the sum-of-squares feasibility problem (VI.1) to be strictly feasible (see Section III-D). Note that Remark 5 about strict feasibility does not apply here because the variable $\gamma$ from (H$_{d,m}$) is now fixed to be $2/\lambda$. The next theorem gives necessary conditions on the subspace $W$ so that the feasibility problem (VI.1) is strictly feasible.

**Theorem 11:** If (VI.1) is strictly feasible, then necessarily

$$W \subseteq W_{\text{max}} := \{ q \in \mathbb{R}[x]_3 \mid q(1) = 0, \phi^T \nabla q(1) = 0, \psi^T \nabla q(1) = 0 \}$$

where $\phi, \psi \in \mathbb{R}^n$ are the vectors $\phi_i = \cos(2\pi i/n)$, $\psi_i = \sin(2\pi i/n)$.

**Proof:** For $h \in \mathbb{R}[x]_{n,4}$ let

$$F_h(x) = F(x) + \left( \|x\|^2 - n \right) h(x).$$

The theorem follows from the fact that for any $h$ feasible for (VI.1), $F_h(1) = 0$ and $\phi^T \nabla^2 F_h(1) \phi = 0$ and $\psi^T \nabla^2 F_h(1) \psi = 0$. The details are deferred to Appendix C.

In practice, we used the following choice of subspace $W$, which is a strict subspace of $W_{\text{max}}$:

$$W = \left\{ q \in \mathbb{R}[x]_3 \mid \begin{align*} q(1) &= 0, \\ \phi^T \nabla q(1) &= 0, \\ \psi^T \nabla q(1) &= 0, \\ \frac{d^3 q}{dx_i dx_j dx_k} (1) &= 0 \end{align*} \right\}.$$  

The last condition is introduced in order to reduce the size of the semidefinite program. It demands that when expressed in terms of the variables $\tilde{x}_i = x_i - 1$, every term from a polynomial in $W$ involves at most two different variables. For $n \geq 3$ we have $\dim W = \binom{n+3}{3} - \binom{3}{3} - 3 = 3n(n+1)/2 - 2$, while $\dim W_{\text{max}} = \binom{n+5}{3} - 3$. When $n = 17$ these dimensions are 457 and 1137 respectively.

Given this choice of $W$, we formulate (VI.1) as a semidefinite feasibility problem and we use an approach very similar to that given in Section III-D to obtain an exact feasible point. The only difference is that the computations are done in the number field $\mathbb{Q}[\cos(2\pi/n)]$ instead of $\mathbb{Q}$. If the obtained Gram matrix is strictly feasible, an $LDL^T$ factorization can be performed to obtain a sum-of-squares certificate for the nonnegativity of $F(x)$, proving that $\alpha = \lambda/2$ for the $n$-cycle. Such a proof can be readily verified in a computer algebra system, by first squaring and adding together the polynomials in the certificate, and then checking that the result is equal to $F(x)$.

**B. Additional Practical Considerations**

In this subsection, we will describe some further details of the method used to find exact proofs for the $n$-cycle. Most importantly, we show how to use representation theory-based symmetry reduction techniques, described in [40], to determine a polynomial basis for $W$ with respect to which the Gram matrix in a sum-of-squares representation of $F_h(x)$ can be chosen to be block diagonal. By using such a basis, we are able to find more compact certificates than we would otherwise have done. The amount of computation required is also reduced, allowing us to search for sum-of-squares proofs for larger $n$ than would otherwise be possible.

1) **Symmetry Reduction:** The automorphism group of the $n$-cycle is $D_{2n}$, the dihedral group of order $2n$ generated by the cyclic shift $i \in \mathbb{Z}_n \mapsto i + 1$ and the reflection $i \in \mathbb{Z}_n \mapsto -i$. It is easy to see that the function $F(x)$ is invariant under the action of permuting the indices of the $x_i$ according to elements of $D_{2n}$. In other words, $F(x) = F(x_\sigma)$ for each $\sigma \in D_{2n}$, where we use the notation $(x_\sigma)_i := x_{\sigma(i)}$. Furthermore, if (VI.1) is feasible for some $h \in \mathbb{R}[x]_4$, then $h$ can always be chosen to be invariant under this action as well. This can be seen by considering

$$\frac{1}{D_{2n}} \sum_{\sigma \in D_{2n}} h(x_\sigma).$$

The group $D_{2n}$ naturally acts on $\mathbb{R}[x]$ by permuting the variables. The subspace $W$ defined in Equation (VI.4) happens to be an invariant subspace for this action. By Maschke’s theorem, $W$ can be written as a direct sum of irreducible
The irreducible representations of $D_{2n}$ are well known: for $n$ odd (which is the case of interest to us), $D_{2n}$ has two 1-dimensional irreducible representations and $\binom{n+1}{2}$ 2-dimensional irreducible representations (so $\frac{n+3}{2}$ in total). A standard analysis allows us to get the multiplicities $m_i$ with which each irreducible representation appears in $W$, and the explicit subspaces $W_{ij}$ in (VI.5), see Appendix C-B.

In particular each 2-dimensional representation of $D_{2n}$ appears with multiplicity $\frac{3n+1}{2}$ or $\frac{3n+3}{2}$, the trivial 1-dimensional representation appears with multiplicity $n+2$, and the sign 1-dimensional representation appears with multiplicity $\frac{n-1}{2}$.

Let $N = \dim W = 3n(n+1)/2 - 2$ and fix a basis $b(x) = (b_1(x), \ldots, b_N(x))$ of $W$ adapted to the decomposition (VI.5), and such that the action of $D_{2n}$ on $W$ is orthogonal. By this we mean that for any $\sigma \in D_{2n}$, we can write $b(x_\sigma) = P_\sigma b(x)$ for some orthogonal matrix $P_\sigma$. Our semidefinite feasibility problem (VI.1) can be written as

$$\text{find } h \in \mathbb{R}[x]^\text{inv}_N \text{ and } Q \in S^N_+ \text{ s.t. } F_h(x) = b(x)^T Q b(x)$$

(VI.6)

where $\mathbb{R}[x]^\text{inv}$ is the ring of polynomials invariant under the action of $D_{2n}$, and $F_h$ is the polynomial defined in (VI.3). Since $F_h$ is invariant under the action of $D_{2n}$, an averaging argument tells us that we can choose $Q$ to satisfy $P_\sigma^T Q P_\sigma = Q$ for all $\sigma \in D_{2n}$. A standard application of Schur’s lemma tells us that such $Q$ must be block diagonal, with one block for each isotypic component. With a little care, it can be actually ensured that each of these isotypic blocks of $Q$ is itself block diagonal, with a number of blocks equal to the dimension of the corresponding irreducible representation. This analysis allows us to reduce the single semidefinite constraint of size $N \times N$ in (VI.6), to smaller block constraints of size $m_i \times m_i$.

C. Outcome

For the cases $n \in \{5, 7, 9, \ldots, 21\}$, we were able to find exact sum-of-squares certificates of (VI.1). Together with the discussion at the beginning of Section III, these certificates prove that the log-Sobolev constant of the $n$-cycle is half of its spectral gap for odd $n > 3$ up to 21. The decompositions are made available online, along with the Julia code used to generate them and the few lines of SageMath code required to verify them. The exact computer algebra necessary to convert a numerical SDP solution into a sum-of-squares certificate of (VI.1) beyond the case $n = 21$ is beyond the capabilities of the computer used to perform the computations for this paper; however, experiments carried out entirely in floating point arithmetic suggest that (VI.1) continues to hold for larger $n$ (at least up to $n = 35$).

VII. DISCUSSION

In this paper we have presented a computational method, based on semidefinite programming, to estimate the logarithmic Sobolev constant of a Markov kernel on a finite state space. Our method provably converges to the true value of the logarithmic Sobolev constant as the level of the hierarchy $d \to \infty$. Moreover, we have shown how to extract rigorous certificates of the validity of these bounds. The accuracy of the method was investigated on several example chains. Finally, we have used this method to derive the exact logarithmic Sobolev constant for some Markov chains for which it was not previously known (to the best of our knowledge). We hope this can be used by others as a tool to guide future exploration of logarithmic Sobolev inequalities on discrete spaces.

A. Implementation

The code necessary to reproduce the results in this paper is made available online. It is written in the Julia programming language [41] and makes use of JuMP [42], a popular mathematical optimization package. Nemo, a computer algebra package for Julia, is used for exact computations in rational numbers and in cyclotomic fields (necessary for the results in Section VI).

B. Further Work

There are many possible directions for future work.

• Unless the Markov kernel under consideration has a large symmetry group, the method we have proposed is generally capable of yielding accurate results on a typical personal computer only for Markov chains whose state space has size $n \lesssim 13$. It would of course be of interest to develop a method which remains performant for chains supported on much larger state spaces, or to improve the method presented in this paper in this direction.

• The works [14] and [15] proposed a semidefinite programming-based algorithm for finding the fastest-mixing Markov process on a graph by maximizing the spectral gap, and compared the resulting optimal chains with popular heuristics such as the maximum-degree method and the Metropolis-Hastings algorithm. The problem of searching for a fastest-mixing process in terms of the logarithmic Sobolev constant was actually suggested in the former paper as a future line of research. The approach described in the present paper not only allows us to lower bound the logarithmic Sobolev constant of a particular chain, but it can also be used to search for fast-mixing Markov processes in the sense of maximizing the logarithmic Sobolev constant. Indeed, given a graph with vertices $\mathcal{X}$ and edge set $E$, and a target distribution $\pi$, we can consider a modified version of the sum-of-squares programs considered in this paper:

$$\min_{\gamma} \text{ s.t. } \frac{1}{2} \sum_{i,j \in \mathcal{X}} \pi_i K_{ij}(x_i - x_j)^2 - \sum_{i \in \mathcal{X}} \pi_i p_i(x_i) \in \text{SOS}_d(S^\pi),$$

2https://github.com/oisinfaust/LogSobolevRelaxations/releases/download/v0.1.0/proofs.zip

3https://github.com/oisinfaust/LogSobolevRelaxations
\[
\overline{K} = \gamma \mathbb{1},
\]
\[
\pi K = \gamma \pi,
\]
\[
\overline{K} \geq 0,
\]
\[
\overline{K}_{ij} = 0 \forall (i, j) \notin E.
\]

This is a sum-of-squares relaxation of the problem of maximizing the log-Sobolev constant \( \gamma^{-1} \) over all Markov kernels \( \gamma^{-1} \overline{K} \) supported on \( E \) and having stationary distribution \( \pi \). In the future, it would be interesting to compare the properties of the chains derived in [14] and [15] with those obtained from the optimization problem above.

**APPENDIX A**

**PROOFS OF LEMMATA FOR PROPOSITION 5**

**Lemma 12:** Let \( a \in \mathbb{R}^n \), and let \( b_1(x), \ldots, b_N(x) \) be a basis for the subspace \( \{ p \in \mathbb{R}[x]_{2d} : p(a) = 0 \} \). Then

\[
\text{span} \{ b_i(x)b_j(x) : 1 \leq i, j \leq N \} = \{ p \in \mathbb{R}[x]_{2d} : p(a) = 0 \text{ and } \nabla p(a) = 0 \}. \tag{A.1}
\]

**Proof:** It is immediate to verify that the inclusion \( \subseteq \) holds in (A.1). To prove the reverse inclusion, we assume, without loss of generality that \( a = 0 \). In this case, the right-hand side corresponds to degree \( 2d \) polynomials with no constant or linear monomials. Thus it suffices to show that any monomial \( x^\gamma \) where \( 2 \leq |\gamma| \leq 2d \) is in \( \text{span} \{ b_i(x)b_j(x) : 1 \leq i, j \leq N \} \). We can write \( x^\gamma = x^\alpha x^\beta \) where \( 1 \leq |\alpha|, |\beta| \leq d \), and so, by assumption \( x^\alpha = \sum \lambda_i b_i \) and \( x^\beta = \sum \mu_j b_j \) so that \( x^\gamma = \sum \lambda_i \mu_j b_i b_j \) is a linear combination of the \( b_i b_j \).

**Lemma 13:** Let \( b_1(x), \ldots, b_N(x) \) be a basis for the subspace of polynomials

\[
V_d = \{ p \in \mathbb{R}[x]_d : p(\mathbb{1}) = 0 \}. \tag{A.2}
\]

Let \( \pi_1, \ldots, \pi_n > 0 \) such that \( \sum_{i=1}^n \pi_i = 1 \). Then there exists a positive definite matrix \( A \) of size \( N \times N \) such that

\[
\sum_{d'=0}^{d-1} \| x \|_\pi^{2d'} \| x - 1 \|_\pi^2 = \| b_i(x) \|_\pi^2 A b_i(x)). \tag{A.3}
\]

**Proof:** The existence of a positive semidefinite matrix \( A \) that satisfies (A.3) is straightforward because the left-hand side is a sum-of-squares.

The main point of the lemma is that one can choose \( A \) to be positive definite. Note that it suffices to prove the lemma for one choice of spanning set of (A.2), say \( \sigma_1, \ldots, \sigma_M \) with \( M \geq N \). This is because if \( b_1, \ldots, b_N \) is a basis of (A.2) then we can write \( \sigma_i(x) = R[b_i(x)] \) where \( R \) is a \( M \times N \) matrix with rank(\( R \)) = \( N \), and so \( \sigma_i(x)^T A \sigma_i(x) = \overline{b_i(x)^T R^T A R b_i(x)] \) where \( R^T A R > 0 \).

For each \( d' > 0 \) define the collection of polynomials indexed by \( \alpha \in [n]^{d'} \),

\[
\sigma^d_\alpha(x) = (\pi_{\alpha_1} \cdots \pi_{\alpha_{d'}}) \frac{1}{2}(x_{\alpha_1} - 1)x_{\alpha_2} \cdots x_{\alpha_{d'}}.
\]

Clearly, we have

\[
\| x - 1 \|_\pi^2 \| x \|_\pi^{2d'} = \sum_{\alpha \in [n]^{d'}} \sigma^d_\alpha(x)^2.
\]

To prove the lemma, we need to show that the collection of \( \{ \sigma^d_\alpha \}_{d'=1}^d \) spans the subspace \( V_d = \{ p \in \mathbb{R}[x]_{d} : p(\mathbb{1}) = 0 \}. \) We prove this by induction. The base case \( d = 1 \) is certainly true, since \( \{ x_1 - 1, \ldots, x_n - 1 \} \) is a basis for \( V_1 \). Let \( p \in V_d \) for \( d > 1 \). We can always write \( p(x) = x_1 p_1(x) + \ldots + x_n p_n(x) + p_0 \), where \( p_0 \) is a constant and \( p_i \in \mathbb{R}[x]_{d-1} \) for \( i \in [n] \). Then rewrite \( p \) as

\[
p(x) = \sum_{i=1}^n (x_i - 1)p_i(x) + \sum_{i=0}^n p_i(x). \tag{A.4}
\]

Note that \( r(\mathbb{1}) = p(\mathbb{1}) = 0 \), so we have \( r \in \mathbb{V}_{d-1} \). Using the induction hypothesis, \( r \in \mathbb{R}^{d-1} \text{span}\{\sigma^d_\alpha\} \). On the other hand, \( q \) is a sum of terms of the form \( (x_i - 1)x^\beta \) where \( x^\beta \) is a monomial of degree at most \( d-1 \), i.e. \( q \in \mathbb{R}^{d-1} \text{span}\{\sigma^d_\alpha\} \). Therefore, \( p \in \mathbb{R}^{d-1} \text{span}\{\sigma^d_\alpha\} \).

**APPENDIX B**

**PROOF OF PROPOSITION 9 ON Rounding**

**Proof of Proposition 9:** Let \( \hat{Z}, Z^* \) be the symmetric matrices corresponding to \( \hat{z} \), \( z^* \). By assumption we know that \( \lambda_{\text{min}}(Z) \geq \varepsilon \). We need to prove that \( \lambda_{\text{min}}(Z^*) \geq \varepsilon \). To do so it suffices to show that \( \| \hat{Z} - Z^* \|_\sigma \leq \varepsilon \), where \( \| \cdot \|_\sigma \) denotes the spectral norm. We have the following sequence of inequalities, where \( \| \cdot \|_F \) is the Frobenius norm:

\[
\| \hat{Z} - Z^* \|_\sigma \leq \| \hat{Z} - Z^* \|_F \\
\leq \sqrt{2} \| \begin{pmatrix} \hat{y} - y^* \\ \hat{z} - z^* \end{pmatrix} \|_2 \\
= \sqrt{2} \| \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} \|_2 - \| \Pi_A \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} \|_2 \\
= \sqrt{2} \| A^T (A A^T)^{-1} \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} - b \|_2 \\
\leq \sqrt{2} \| A^T (A A^T)^{-1} \|_\sigma \| A \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} - b \|_2 \\
\leq \frac{2r}{\lambda_{\text{min}}(A A^T)} (1 + \| b \|_\infty) \varepsilon \text{aff} = \varepsilon.
\]

The last inequality follows by observing that

\[
\| A^T (A A^T)^{-1} \|_\sigma = \sqrt{\lambda_{\text{max}}((A A^T)^{-1} A^T (A A^T)^{-1})} \\
= \lambda_{\text{max}}((A A^T)^{-1}) \\
= 1/\lambda_{\text{min}}(A A^T),
\]

and that

\[
\| A \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} - b \|_2 \leq \sqrt{r} \| A \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix} - b \|_\infty \leq \sqrt{r} (1 + \| b \|_\infty) \varepsilon \text{aff}.
\]

This completes the proof. \( \square \)
APPENDIX C
THE CYCLE GRAPH

A. Proof of Theorem 11

If (VI1) is strictly feasible then there is \( h \in \mathbb{R}[x]_4 \) and a basis \( q_1, \ldots, q_N \) of \( W \) such that

\[
F_h(x) := F(x) + \left( \| x \|^2 - n \right) h(x) = \sum_{j=1}^N q_j(x)^2. \tag{C.1}
\]

(Indeed, one can check that the interior of \( \Sigma(W) \) is the precisely the set of polynomials that can be written as \( \sum_i q_i^2 \) where \( q_i \) forms a basis of \( W \).) We recall for convenience

\[
F(x) = \frac{2}{\lambda} \sum_{i \in \Sigma_n} (x_i - x_{i+1})^2 - \sum_{i=1}^n p(x_i),
\]

where \( \lambda = (1 - \cos((2\pi/n))/2 \) and \( p(t) = 2(t - 1) + 3(t - 1)^2 + 2(t - 1)^3 - \frac{1}{3!}(t - 1)^4 + \frac{1}{4!}(t - 1)^5 \) is the Taylor expansion of \( 2t^2 \log(t) \) to order 5.

Observe that \( F_h(1) = 0 \), and so this implies that \( q_j(1) = 0 \) for all \( j = 1, \ldots, N \). If we differentiate the identity (C.1) we get \( \nabla F_h(1) = 0 \) i.e., \(-21 + 2h(1)1 = 0 \) which implies \( h(1) = 1 \). Furthermore

\[
\nabla^2 F_h(1) = \frac{4}{\lambda}(I - K) - 6I + 2\ h(1)I
\]

\[
+ \nabla h(1)1^T + \nabla h(1)1^T
\]

where \( K \) is the transition matrix of the simple walk on the \( n \)-cycle. The vectors \( \phi = \{(\cos(2\pi i/n))_{1 \leq i \leq n} \) and \( \psi = (\sin(2\pi i/n))_{1 \leq i \leq n} \) are eigenvectors of \((I - K)\) with eigenvalue \( \lambda \), and are both orthogonal to \( 1 \), so we have that \( \phi^T \nabla^2 F_h(1) \phi = \psi^T \nabla^2 F_h(1) \psi = 0 \). Using Lemma 14 (to follow) this implies that \( \phi^T \nabla q_j(1) = \psi^T \nabla q_j(1) = 0 \).

We have shown that the \( q_j \) must all satisfy the linear relations

\[
q_j(1) = 0, \ \phi^T q_j(1) = \psi^T q_j(1) = 0 \ \forall j = 1, \ldots, N.
\]

Since by assumption the \( q_j \) span the subspace \( W \) we get the desired inclusion (VI2).

It remains to prove the lemma below.

Lemma 14: Let \( F(x) \in \mathbb{R}[x]_{2d} \) with \( F(1) = 0 \) and suppose that \( F \) is a sum of squares of polynomials:

\[
F(x) = \sum_{j=1}^N q_j(x)^2.
\]

Let \( \nabla^2 F(x) \) be the Hessian matrix of second derivatives of \( F \). Then for any \( v \in \ker \nabla^2 F(1), 1 \leq j \leq N \) we have

\[
v^T \nabla q_j(1) = 0.
\]

Proof: For a real variable \( t \) define the function \( f(t) = F(1 + tv) \). One the one hand,

\[
f''(0) = \frac{d}{dt} \left( v^T \nabla F(1 + tv) \right)|_{t=0} = v^T \nabla^2 F(1)v = 0.
\]

On the other,

\[
f''(0) = \sum_{j=1}^N \left( \frac{d^2}{dt^2} q_j(1 + tv) \right)|_{t=0}^2
\]

\[
= 2 \sum_{j=1}^N \left( \frac{d}{dt} q_j(1 + tv) \right)|_{t=0}^2
\]

\[
+ q_j(1) \frac{d^2}{dt^2} q_j(1 + tv)|_{t=0}
\]

\[
= 2 \sum_{j=1}^N \left( \frac{d}{dt} q_j(1 + tv) \right)|_{t=0}^2
\]

\[
+ q_j(1) \frac{d^2}{dt^2} q_j(1 + tv)|_{t=0}
\]

This implies \( v^T \nabla q_j(1) = 0 \) for all \( 1 \leq j \leq N \) as desired. \( \square \)

This completes the proof of Theorem 11.

B. Decomposition of the Subspace \( W \) Into Irreducibles

In order to decompose \( W \) (defined in (VI4)) into irreducible representations, it will be helpful to first consider an enlarged space \( \hat{W} \). This space is defined as \( \hat{W} = \{ q \in \mathbb{R}[x]_3 \mid \frac{dq}{dx_1 dx_2 dx_3} (1) = 0 \ \forall i, j, k \ \text{distinct} \}. \) Note that \( W = \hat{W} \cap W_{max} \). In the rest of this section, we will work in the more convenient variables \( \hat{x}_i := x_i - 1 \). From this point of view, \( \hat{W} \) has a nice interpretation as the space of degree 3 polynomials, no term of which involves 3 different variables.

There is an obvious way to coarsely decompose \( \hat{W} \) into invariant subspaces: (1)

1) \( \text{span}\{1\} \), the space of constant polynomials,

2) \( \text{span}\{\hat{x}_1, \ldots, \hat{x}_n\}, \ \text{span}\{\hat{x}_1^2, \ldots, \hat{x}_n^2\} \) and \( \text{span}\{\hat{x}_1^3, \ldots, \hat{x}_n^3\} \) are invariant subspaces which each induce a copy of the permutation representation. Moreover, for \( \Delta \in \{1, \ldots, \frac{n-1}{2}\} \), the spaces

\[
\text{span}\{\hat{x}_i \hat{x}_j \mid i, j \in \{1, \ldots, n\}, d_n(i, j) = \Delta \}
\]

are also invariant subspaces equivalent to the permutation representation via the isomorphism

\[
\hat{x}_i \hat{x}_j \rightarrow \begin{cases} e_i + \frac{\hat{x}_j}{n} \ (\text{mod } n) & \text{if } \Delta \text{ even} \\ e_i + \frac{\hat{x}_j}{n} - \hat{x}_j \ (\text{mod } n) & \text{if } \Delta \text{ odd} \end{cases}
\]

Here \( d_n(i, j) := \min\{|i - k| \text{ s.t. } k \equiv j \ (\text{mod } n)\} \).

3) For \( \Delta \in \{1, \ldots, \frac{n-1}{2}\} \), the spaces

\[
\text{span}\{\hat{x}_i^2 \hat{x}_j \mid i, j \in \{1, \ldots, n\}, d_n(i, j) = \Delta \}
\]

are invariant subspaces which each induce a copy of the regular representation. Indeed, writing \( D_{2n} = \langle r, s \mid \ r^n = s^2 = 1, srs = r^{-1} \rangle \), the map

\[
\hat{x}_i^2 \hat{x}_j \rightarrow \begin{cases} e_{ri} \ j \equiv i + \Delta \ (\text{mod } n) \\ e_{ri}s \ j \equiv i - \Delta \ (\text{mod } n) \end{cases}
\]

is an isomorphism which respects the action of \( D_{2n} \). In order to find a full decomposition of \( \hat{W} \) into irreducible representations as in (VI5), it remains only to decompose the regular and permutation representations of \( D_{2n} \) into irreducibles. We omit the details (this is standard material in representation theory) except to say that the resulting symmetry-adapted basis \( \hat{b}(\hat{x}) \) for \( \hat{W} \) can be chosen so that each basis element is a polynomial whose coefficients lie in \( \mathbb{Q}[\cos(2\pi/n)] \). It may be necessary to rescale certain basis elements by a constant factor outside of this field, e.g., the coefficients of \( \sum_i \psi_i \hat{x}_i = \sum_i \sin(2\pi i/n) \hat{x}_i \) are not in \( \mathbb{Q}[\cos(2\pi/n)] \), but those of \( \sin(2\pi i/n) \sum_i \psi_i \hat{x}_i \) are.
We now aim to obtain a decomposition of \( W \) into irreducible representations. It is easily verified that 
\[
\text{span}\{\phi_i \tilde{x}_i, \psi_j \tilde{x}_i\}_{i=1}^n
\]
is an invariant subspace of 
\[
\text{span}\{\tilde{x}_1, \ldots, \tilde{x}_n\}.
\]
In fact, the span of these two polynomials is irreducible, and not equivalent to any other subrepresentation of 
\[
\text{span}\{\tilde{x}_1, \ldots, \tilde{x}_n\}.
\]
It follows that both 
\[
\phi_i \tilde{x}_i
\]
and 
\[
\psi_j \tilde{x}_i
\]
are elements of \( \tilde{b}(\tilde{x}) \), the basis of \( W \) obtained from the procedure sketched out above. Therefore a symmetry-adapted basis for \( W = \bigwedge \cap W_{\text{max}} \) can be obtained by dropping from \( \tilde{b}(\tilde{x}) \) the polynomials 
\[
\{\phi_i \tilde{x}_i\}_{i=1}^n \quad \text{and} \quad \{\psi_j \tilde{x}_i\}_{j=1}^n,
\]
as well as the constant polynomial.

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REFERENCES

[1] P. Diaconis and L. Saloff-Coste, “Logarithmic Sobolev inequalities for finite Markov chains,” Ann. Appl. Probab., vol. 6, no. 3, pp. 695–750, Aug. 1996. [Online]. Available: http://www.jstor.org/stable/22452210

[2] R. Montenegro and P. Tetali, “Mathematical aspects of mixing times in Markov chains,” Found. Trends® Theor. Comput. Sci., vol. 1, no. 3, pp. 237–354, 2005, doi: 10.1561/0400000003.

[3] M. Raginsky, “Logarithmic sobolev inequalities and strong data processing theorems for discrete channels,” in Proc. IEEE Int. Symp. Inf. Theory, Jul. 2013, pp. 419–423.

[4] M. Raginsky, “Strong data processing inequalities and \( \Phi \)-Sobolev inequalities for discrete channels,” IEEE Trans. Inf. Theory, vol. 62, no. 6, pp. 3355–3389, Jun. 2016.

[5] L. Saloff-Coste, “Lectures on finite Markov chains,” in Lectures on Probability Theory and Statistics. Berlin, Germany: Springer, 1997, pp. 301–413, doi: 10.1007/BFb0092621.

[6] G.-Y. Chen, W.-W. Liu, and L. Saloff-Coste, “The logarithmic Sobolev constant of some finite Markov chains,” Annales Fac. Sci. Toulouse, Mathématiques, vol. 17, no. 2, pp. 239–290, Dec. 2008.

[7] J. B. Lasserre, “Global optimization with polynomials and the problem of moments,” SIAM J. Optim., vol. 11, no. 3, pp. 796–817, Jan. 2001.

[8] P. A. Parrilo, “Semidefinite programming relaxations for semialgebraic problems,” Math. Program., vol. 96, no. 2, pp. 293–320, May 2003.

[9] O. Faust and H. Fawzi, “Strong data processing inequalities for channels and Bayesian networks,” in Convexity and Concentration, E. Carlen, M. Madiman, and E. M. Werner, Eds. New York, NY, USA: Springer, 2017, pp. 211–249.

[10] T. A. Courtade, “Outer bounds for multiterminal source coding via a strong data processing inequality,” in Proc. IEEE Int. Symp. Inf. Theory, Jul. 2013, pp. 559–563.

[11] Y. Polyanskiy and Y. Wu, “Strong data-processing inequalities for channels and Bayesian networks,” in Convexity and Concentration, E. Carlen, M. Madiman, and E. M. Werner, Eds. New York, NY, USA: Springer, 2017, pp. 211–249.

[12] T. A. Courtade, “Outer bounds for multiterminal source coding via a strong data processing inequality,” in Proc. IEEE Int. Symp. Inf. Theory, Jul. 2013, pp. 559–563.

[13] Y. Polyanskiy and Y. Wu, “Application of the information-percolation method to reconstruction problems on graphs,” Math. Statist. Learn., vol. 2, no. 1, pp. 1–24, Feb. 2020.

[14] O. Ordentlich and Y. Polyanskiy, “Strong data processing constant is achieved by binary inputs,” IEEE Trans. Inf. Theory, vol. 68, no. 3, pp. 1480–1481, Mar. 2022.

[15] G.-Y. Chen and Y.-C. Su, “On the log-Sobolev constant for the simple random walk on the n-cycle: The even cases,” J. Funct. Anal., vol. 202, no. 2, pp. 473–485, Aug. 2003.

[16] The Sage Developers. (2020). SageMath, the Sage Mathematics Software System (Version 9.2). [Online]. Available: https://www.sagemath.org

[17] K. Gatermann and P. A. Parrilo, “Symmetry groups, semidefinite programs, and sums of squares,” J. Pure Appl. Algebra, vol. 192, nos. 1–3, pp. 95–128, Sep. 2004.

[18] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, “Julia: A fresh approach to numerical computing,” SIAM Rev., vol. 59, no. 1, pp. 65–98, Jan. 2017.

[19] I. Dunning, J. Huchette, and M. Lubin, “JuMP: A modeling language for mathematical optimization,” SIAM Rev., vol. 59, no. 2, pp. 295–320, Jan. 2017.

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