Correlation Functions of the Energy Momentum Tensor 
on Spaces of Constant Curvature

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An analysis of one and two point functions of the energy momentum tensor on homogeneous spaces of constant curvature is undertaken. The possibility of proving a c-theorem in this framework is discussed, in particular in relation to the coefficients $c, a$, which appear in the energy momentum tensor trace on general curved backgrounds in four dimensions. Ward identities relating the correlation functions are derived and explicit expressions are obtained for free scalar, spinor field theories in general dimensions and also free vector fields in dimension four. A natural geometric formalism which is independent of any choice of coordinates is used and the role of conformal symmetries on such constant curvature spaces is analysed. The results are shown to be constrained by the operator product expansion. For negative curvature the spectral representation, involving unitary positive energy representations of $O(d - 1, 2)$, for two point functions of vector currents is derived in detail and extended to the energy momentum tensor by analogy. It is demonstrated that, at non coincident points, the two point functions are not related to $a$ in any direct fashion and there is no straightforward demonstration obtainable in this framework of irreversibility under renormalisation group flow of any function of the couplings for four dimensional field theories which reduces to $a$ at fixed points.

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1. Introduction

The energy momentum tensor is a universal probe in any relativistic quantum field theory. Parameters which are defined through correlation functions of the energy momentum tensor, or its expectation value when the space on which the theory is defined has non trivial topology or non zero curvature, may serve to specify the theory independent of any particular formulation in terms of elementary fields. Such parameters should be well defined at any renormalisation group fixed point where the theory becomes conformal, which should also include the case of free theories. The cardinal example of course is the Virasoro central charge \( c \) for two dimensional conformal field theories which may be defined through the trace of the energy momentum tensor on a curved background, the coefficient of the energy momentum tensor two point function on flat space \( \mathbb{R}^2 \) or a universal term in the dependence of the Casimir energy on the circumference when the underlying space is compactified on a cylinder \( S^1 \times \mathbb{R} \). Furthermore away from critical points \( c \) may be generalised to a function of the couplings \( g^i \), \( \mathcal{C}(g) \), which monotonically decreases under RG flow as the basic scale of the theory is evolved to large distances and the couplings are attracted to any potential infra red fixed point. This is the content of the celebrated Zamolodchikov \( c \)-theorem \([2]\), where \( \mathcal{C}(g) \) was constructed so as to satisfy

\[
\beta^i(g) \frac{\partial}{\partial g^i} \mathcal{C}(g) = \mathcal{G}_{ij}(g) \beta^j(g) \beta^i(g),
\]

with \( \mathcal{G}_{ij}(g) \), for unitary theories, positive definite. Since, in a two dimensional conformal field theory \( c \) may be interpreted as a measure of the degrees of freedom, the \( c \)-theorem incorporates the physical intuition that RG flow is irreversible as a consequence of loss of information concerning details at short distances in any infra red limit. \[2\]

Many efforts have been made to generalise such ideas beyond two dimensions to realistic four dimensional field theories \([4,5,6,7,8,9,10,11,12]\), even making connections to modern ideas of holography \([13,14]\). In particular Cardy \([4]\) discussed a possible generalisation for \( c \) to four dimensions in terms of the energy momentum tensor trace on a curved background. At a conformal fixed point there are two parameters associated with two independent scalars formed from the curvature which may apear in the energy momentum tensor trace, and are denoted by \( c, a \). Cardy’s conjecture for a four dimensional generalisation of \( c \) involved \( a \), the coefficient of the four dimensional Euler density. An analysis of this proposal for general four dimensional renormalisable quantum field theories demonstrated \([4]\) irreversibility of the RG flow of a quantity \( \tilde{a} \), which is equal to \( a \) for vanishing \( \beta \)-functions, in some neighbourhood of weak coupling. Recently non perturbative formulae

\[1\] A selection of papers discussing the \( c \)-theorem using statistical mechanical methods is given in \([3]\).
for the flow of both $c$ and $a$ between UV asymptotically free fixed points and non trivial IR fixed points in $\mathcal{N} = 1$ supersymmetric gauge theories were proposed, on the basis of anomaly calculations, which demonstrated that the RG flow of $a$ is monotonic, so long as the anomalous dimensions of the basic chiral fields are not too large [15,16]. The supersymmetric results were subsequently shown to be in accord with the previous perturbative discussions for general theories in [17].

It therefore appears natural to try to analyse further how far two dimensional results for flat space may be generalised to curved backgrounds. In this investigation we restrict attention to homogeneous spaces of constant curvature, for theories continued to a space with positive definite metric, either the $d$-dimensional sphere $S^d$ or the negative curvature hyperboloid $H^d$. For such spaces

$$R_{\mu\nu\sigma\rho} = \pm \rho^2 (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \quad \Rightarrow \quad R = \pm d(d - 1)\rho^2,$$

(1.2)

with $1/\rho$ a length scale. The dependence on the metric is then reduced to just the single parameter $\rho$ and we may write

$$\rho \frac{d}{d\rho} g_{\mu\nu} = -2g_{\mu\nu} + \mathcal{L}_u g_{\mu\nu}, \quad \mathcal{L}_u g_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu,$$

(1.3)

where $u^\mu$ is a vector field which depends on the particular choice of coordinates. The simplicity of considering homogeneous spaces of constant curvature is that the isometry group is as large as on flat space and therefore a group theoretic analysis is possible which, once the appropriate basis functions are introduced, is not essentially more complicated than for flat space [18,19]. The virtues of considering field theories on negative curvature spaces were advocated in [20].

For the case of negative curvature an analysis of two point correlation functions, where the positivity properties arising from unitarity are most evident, for the energy momentum tensor was first given in [6]. More recently Forte and Latorre [11] have endeavoured to use these results to prove a four dimensional version of the $c$-theorem by considering as a candidate for $C(g)$, which naturally interpolates between $a$ at fixed points, a function $C(g)$ determined by the one point function of the energy momentum tensor on a space of constant negative curvature when $\langle T_{\mu\nu} \rangle$ is proportional to $g_{\mu\nu}$. It is important to recognise that although we may have $\Delta a < 0$ under RG flow between fixed points it is in general necessary to add extra terms of $O(\beta)$ to any interpolating function $C(g)$ to obtain a $C(g)$ satisfying (1.1). We show later that there is a freedom of definition for $C(g)$ of $O(\beta^2)$, and correspondingly for $G_{ij}(g)$, which preserves (1.1) and which may be necessary to ensure that $G_{ij}(g)$ is positive.

The work in [11] formed part of the stimulus for this investigation although we attempt to provide a complete set of results for the energy momentum tensor two point function on constant curvature spaces. The salient results obtained here are,
• An analysis of the Ward identities following from the conservation equation for the energy momentum tensor and the presence of anomalous terms in its trace, together with consistency conditions following from renormalisation group equations, applied to the associated one and two point functions on spaces of constant curvature. These relations are further restricted for the particular cases of two and four dimensions.

• The derivation of the conditions flowing from the conservation equations on a general expression of the two point function together with the calculation of its explicit form for massless scalar, spinor and vector theories.

• A derivation in detail of the unitary positive energy representations of $O(d - 1, 2)$ for a spin one lowest energy state. These are applied to construct the spectral representation for a vector two point function on a negative curvature space. This is extended to the spin two case appropriate to the energy momentum tensor two point function.

• An analysis of the implications of these results for the derivation of a $c$-theorem along similar lines to that in [11] (and also the extension of the Zamolodchikov proof to curved space). The justification of positivity conditions through the spectral representation for the two point function is carefully considered. We are not able to obtain an equation of the form (1.1) for $C(g)$ although a related equation of the form $\beta^i \partial_i C = G - dC$ with $G > 0$ at least in two dimensions is found.

• A discussion of the possible conditions which would imply irreversibility of RG flow and the constraints that a general $c$-function should satisfy.

• An application to free massive scalar fields when all contributions to the various identities may be explicitly calculated.

An analysis of consistency conditions related to the energy momentum tensor trace anomaly was also previously undertaken in [21] for general curved space backgrounds. The consistency conditions obtained subsequently are a subset of those in [21] but are here directly related to physical correlation functions.

In more detail the outline of this paper is as follows. In the next section a general framework for Ward identities, together with RG equations, for the energy momentum tensor two point function is described. This is then specialised to two and four dimensions in sections 3 and 4. In section 5 we discuss a geometric approach appropriate for describing two point functions of tensor fields on spaces of constant curvature. In section 6 the conformal Killing equation is solved, independently of any choice of coordinates in $d$-dimensions, and the corresponding conformal group $O(d + 1, 1)$ identified as well as the appropriate isometry groups. A scalar function $s$ of two points $x, y$, which transforms homogeneously under conformal transformations, is constructed. An associated bi-vector,
related to inversions, which also transforms simply under conformal transformations is found. These results allow the construction of conformally covariant two point functions for any tensor fields. The geometric formalism is extended to spinor fields in section 7 and the corresponding inversion matrix as well as explicit forms for Killing spinors are obtained. The formalism of section 5 is applied in section 8 to determine a general expression for the two point function of the energy momentum tensor \( \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle \) and the necessary conditions required to satisfy the conservation equation for the energy momentum tensor are obtained. On \( S^2 \) or \( H^2 \) for a traceless energy momentum tensor these have simple solutions with only an undetermined overall scale for \( S^2 \). Expressions which satisfy the conservation equations automatically are found in terms of two independent scalar functions which can be interpreted as corresponding to spin-0 and spin-2 contributions. The spin-2 function gives a form for the two point function appropriate to a traceless energy momentum tensor. It is shown how to determine generally each of the scalar functions for any expression for the two point function obeying the conservation equations. In section 9 the arbitrariness in this decomposition, arising when the the spin-0 scalar function is a Green function for \(-\nabla^2 - \frac{1}{d-1} R\) and the resulting expression for the energy momentum tensor two point function is traceless, is discussed. In two dimensions, when the spin-2 function is absent, \( \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle \) is determined uniquely for conformal theories with the overall scale set by the Virasoro central charge \( c \). In section 10 we calculate the form of the two point function in the conformal limit for free scalar and spinor fields in general dimensions and also for free vectors in four dimensions. For \( S^d \) the results are proportional to the unique conformally covariant form with the same overall coefficient \( C_T \) as on flat space. On \( H^d \) the results are not unique but there is a simple expression for the leading singular term with the same coefficient \( C_T \). Some aspects of these results are understood in section 11 using the operator product expansion although its form on spaces of non zero curvature is not yet fully clear. In section 12 we discuss the positive energy unitary representations of \( O(d - 1, 2) \) and their significance for unitary quantum field theories on \( H^d \). We obtain in detail the representation for a spin 1 lowest weight state. The technicalities of this section are then used in section 13 to obtain the spectral representation for the two point function of a vector field when the intermediate states are decomposed into representations of \( O(d - 1, 2) \). This work motivates a natural extension giving the spectral representation of the energy momentum tensor two point function. Finally in section 14 general aspects of the \( c \)-theorem are discussed and difficulties in deriving it for a field theory defined on a space of constant curvature are described. In appendix A the crucial results obtained in section 8 for positive curvature are listed in the negative curvature case while the calculation of Green functions in terms of hypergeometric functions is described in appendix B. Some detailed results for the spin one representation of \( O(d - 1, 2) \), and the calculation of the norms of the basis states, are deferred to appendix C. This also
contains a summary of some properties of the arbitrary dimension spherical harmonics used in section 12. In appendix D we discuss the example of a free massive scalar field $\phi$, following [10], where the mass is the sole coupling. This involves the explicit construction of the spectral representation for $\phi^2$.

### 2. Ward Identities and Consistency Conditions

Our subsequent discussion depends crucially on the Ward and trace identities satisfied by correlation functions involving the energy momentum tensor. These are not necessarily unique since there may be ambiguous local contact term, involving $\delta$-functions, in any two or higher point function. In consequence it is useful to first give a precise derivation of the identities in a consistent framework and later take account of the potential freedom of contact terms. To this end we consider the vacuum functional $W$ for a quantum field theory defined on an arbitrarily curved space with metric $g_{\mu\nu}(x)$ and also local sources $g^i(x)$ coupled to a set of scalar fields $O_i(x)$. The expectation values of the quantum operator fields in the background for an arbitrary metric but with $g^i(x) = g^i$ the physical coupling constants (which are taken to be dimensionless by introducing an appropriate power of a scale $\rho$ on which the metric depends) are given by

$$\sqrt{g(x)}\langle T_{\mu\nu}(x) \rangle = -2 \frac{\delta}{\delta g^{\mu\nu}(x)} W\bigg|_{g^i(x) = g^i}, \quad \sqrt{g(x)}\langle O_i(x) \rangle = -\frac{\delta}{\delta g^i(x)} W\bigg|_{g^i(x) = g^i}, \quad (2.1)$$

where the functional derivatives, for $d$-dimensions, are defined by

$$\frac{\delta}{\delta g^{\mu\nu}(x)} g^{\alpha\beta}(y) = \delta^{(\mu}_\alpha \delta^{\nu)}_{\beta} \delta^d(x - y), \quad \frac{\delta}{\delta g^i(x)} g^j(y) = \delta^i_j \delta^d(x - y). \quad (2.2)$$

With this prescription the associated two point functions are given by

$$\sqrt{g(x)}\sqrt{g(y)}\langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle = 4 \frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\alpha\beta}(y)} W\bigg|_{g^i(x) = g^i}, \quad (2.3)$$

$$\sqrt{g(x)}\sqrt{g(y)}\langle O_i(x)O_j(y) \rangle = \frac{\delta^2}{\delta g^i(x)\delta g^j(y)} W\bigg|_{g^i(x) = g^i},$$

which are manifestly symmetric. We may also similarly define $\langle T_{\mu\nu}(x)O_j(y) \rangle$. In general the correlation functions $\langle T_{\mu\nu}(x)\ldots O_j(y) \ldots \rangle$ formed from $T_{\mu\nu}$ and the scalar fields $O_j$ form a basic set related by Ward identities and obeying RG equations which are the subject of discussion here.

The Ward identities may be derived from the condition that $W$ is a scalar functional, corresponding to the requirement that any regularisation preserves invariance under diffeomorphisms, which implies

$$\int d^d x \left( - (\nabla^\mu \phi^\nu + \nabla^\nu \phi^\mu) \frac{\delta}{\delta g^{\mu\nu}} + \nu^\mu \partial_\mu g^i \frac{\delta}{\delta g^i} \right) W = 0. \quad (2.4)$$
It is easy to see that this gives
\[ \nabla^\mu \langle T_{\mu\nu} \rangle = 0, \quad (2.5) \]
and also
\[ \nabla^\mu \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle = \nabla_\nu \left( \delta_\alpha^\sigma \delta_\beta^\rho \delta^d(x, y) \right) \langle T_{\sigma\rho}(x) \rangle + 2 \nabla_\sigma \left( \delta_\alpha^\rho \delta_\beta^\mu \delta^d(x, y) \langle T_{\rho\mu}(x) \rangle \right), \quad (2.6a) \]
\[ \nabla^\mu \langle T_{\mu\nu}(x)O_i(y) \rangle = \partial_\nu \delta^d(x, y) \langle O_i(x) \rangle. \quad (2.6b) \]

In these equations, which contain \( \delta^d(x, y) = \delta^d(x - y) / \sqrt{g(x)} \), the appropriate connections to appear in covariant derivatives need to be clearly specified. Any ambiguities are here resolved by using the convention that the indices \( \nu, \sigma, \rho \) are regarded as ‘at \( x \)’ while \( \alpha, \beta \) are ‘at \( y \)’, covariant derivatives such as \( \nabla_\nu \) involve differentiation with respect to \( x \) and have the required connection necessary according to the tensorial structure at \( x \).

We also consider associated trace identities which take the form
\[ g^{\mu\nu} \langle T_{\mu\nu} \rangle - \langle \Theta \rangle = A, \quad \Theta = \beta^i O_i, \quad (2.7) \]
where \( A \) is the anomalous contribution to the trace present in field theories on a curved background. \( A \) is a local scalar formed from the the Riemann curvature and its derivatives and obeys consistency conditions which correspond to integrability conditions for \( W \). In \( (2.7) \) we have assumed that the operators \( O_i \) form a basis for the trace of the energy momentum tensor with coefficients the \( \beta \)-functions corresponding to the couplings \( g^i \). Derivatives of lower dimensional operators are thus neglected in \( (2.7) \) but if present they do not change the essential results of the analysis, as discussed in \([21]\). As a consequence of \( (2.7) \) with the definitions \( (2.3) \) we have
\[ g^{\mu\nu}(x) \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle - \langle \Theta(x)T_{\alpha\beta}(y) \rangle - 2 \langle T_{\alpha\beta}(y) \rangle \delta^d(x, y) = A_{\alpha\beta}(x, y), \quad (2.8) \]
where \( A_{\alpha\beta}(x, y) \) is formed from \( \delta^d(x, y) \) and derivatives and is explicitly given by
\[ A_{\alpha\beta}(x, y) = -\frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g_{\alpha\beta}(y)} \left( \sqrt{g} A(x) \right). \quad (2.9) \]

We further have
\[ g^{\mu\nu}(x) \langle T_{\mu\nu}(x)O_i(y) \rangle - \langle \Theta(x)O_i(y) \rangle + \partial_\nu \beta^j \langle O_j(y) \rangle \delta^d(x, y) = B_i(x, y). \quad (2.10) \]
where \( B_i(x, y) \) also has support only for \( x = y \). The appearance of \( \partial_\nu \beta^j \) reflects the fact that this is the anomalous dimension matrix for the operators \( O_j \). It is easy to see that we must have for consistency
\[ \nabla^\alpha A_{\alpha\beta}(x, y) = \partial_\beta \delta^d(x, y) A(y), \quad \int d^d y \sqrt{g} B_i(x, y) = -\partial_i A(x), \quad (2.11) \]
and also
\[ \mathcal{H}(x, y) = g^{\alpha \beta}(y) A_{\alpha \beta}(x, y) + B_i(y, x) \beta^i = \mathcal{H}(y, x). \]  
(2.12)
which is an integrability condition. The necessity of (2.12) may be seen by combining (2.8) and (2.10) to give
\[ g^{\mu \nu}(x) g^{\alpha \beta}(y) \langle T_{\mu \nu}(x) T_{\alpha \beta}(y) \rangle - \langle \Theta(x) \Theta(y) \rangle = 2 g^{\alpha \beta}(y) \delta^d(x, y) - \beta^i \partial_i \beta^j \langle O_j(y) \rangle \delta^d(x, y) + \mathcal{H}(x, y). \]  
(2.13)

To obtain further consistency conditions it is necessary also to consider the linear RG equations for the one and two point functions. First we define the derivative operator which generates constant rescalings of the metric and the associated flow of the couplings by
\[ \mathcal{D} = -2 \int d^d x g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} + \beta^i \partial_i. \]  
(2.14)
For a constant curvature metric depending on a single parameter \( \rho \) as in (1.2) we have, up to the effects of the reparameterisation generated by \( u \) as in (1.3),
\[ 2 \int d^d x g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} \sim \rho \frac{d}{d \rho}. \]  
(2.15)
With the definition (2.14) the RG equations for \( \langle T_{\mu \nu} \rangle \) and \( \langle O_i \rangle \) are then
\[ (\mathcal{D} + d - 2) \langle T_{\mu \nu} \rangle = C_{\mu \nu}, \quad C_{\mu \nu}(x) = \int d^d y \sqrt{g} A_{\mu \nu}(y, x), \]  
(2.16a)
\[ (\mathcal{D} + d) \langle O_i \rangle + \partial_i \beta^j \langle O_j \rangle = C_i, \quad C_i(x) = \int d^d y \sqrt{g} B_i(y, x). \]  
(2.16b)
For the two point functions we have
\[ (\mathcal{D} + 2d - 4) \langle T_{\mu \nu}(x) T_{\alpha \beta}(y) \rangle = \mathcal{E}_{\mu \nu, \alpha \beta}(x, y), \]  
(2.17)
where
\[ \mathcal{E}_{\mu \nu, \alpha \beta}(x, y) = \frac{4}{\sqrt{g(x)} \sqrt{g(y)}} \frac{\delta^2}{\delta g^{\mu \nu}(x) \delta g^{\alpha \beta}(y)} \int d^d x \sqrt{g} A, \]  
(2.18)
and
\[ (\mathcal{D} + 2d - 2) \langle T_{\mu \nu}(x) O_i(y) \rangle + \partial_i \beta^j \langle T_{\mu \nu}(x) O_j(y) \rangle = \mathcal{F}_{\mu \nu, i}(x, y), \]  
(2.19)
for
\[ \mathcal{F}_{\mu \nu, i}(x, y) = - \frac{2}{\sqrt{g(x)} \sqrt{g(y)}} \frac{\delta}{\delta g^{\mu \nu}(x)} \left( \sqrt{g} C_i(y) \right), \]  
(2.20)
together with

\[
(D + 2d)\langle O_i(x)O_j(y) \rangle + \partial_i \beta^k \langle O_k(x)O_j(y) \rangle \\
+ \partial_j \beta^k \langle O_i(x)O_k(y) \rangle - \partial_i \partial_j \beta^k \langle O_k(x) \rangle \delta^d(x, y) = G_{ij}(x, y) .
\]

Given an expression for the trace anomaly \( A \) on a general curved space \( C_{\mu\nu} \) and \( E_{\mu\nu,\alpha\beta} \), as well as \( A_{\alpha\beta} \), may be directly calculated. For \( d = 2 \) and \( 4 \) general forms for \( F_{\mu\nu,i}(x, y) \) and \( G_{ij}(x, y) = G_{ji}(y, x) \) may be constructed as a sum of terms involving the curvature and derivatives of \( \delta^d(x, y) \) with the appropriate dimension. From (2.21) we must have

\[
\nabla^\mu F_{\mu\nu,i}(x, y) = \partial_\nu \delta^d(x, y) C_i(x) ,
\]

and (2.21) gives a relation between \( E_{\mu\nu,\alpha\beta} \) and \( C_{\mu\nu} \).

Requiring consistency of (2.16a,b) and (2.17), (2.19), (2.21) with (2.8), (2.10) leads to

\[
(D + 2d - 2) A_{\alpha\beta}(x, y) = g^{\mu\nu}(x) E_{\mu\nu,\alpha\beta}(x, y) - F_{\alpha\beta,i}(y, x) \beta^i - 2C_{\alpha\beta}(y) \delta^d(x, y) ,
\]

and

\[
(D + 2d) B_i(x, y) + \partial_i \beta^j (B_j(x, y) - C_j(y) \delta^d(x, y)) = g^{\mu\nu} F_{\mu\nu,i}(x, y) - G_{ij}(x, y) \beta^j .
\]

The identities obtained above are explored in the following sections in the particular cases of two and four dimensions after restricting to spaces of constant curvature, as given in (1.2). On such homogeneous spaces \( \langle O_i \rangle \) and also the curvature trace anomaly \( A \) in (2.7) are just constants and also

\[
\langle T_{\mu\nu}(x) \rangle = -\frac{1}{d} C \rho^d g_{\mu\nu}(x) ,
\]

with \( C(g) \) dimensionless. The result (2.25) of course trivially satisfies (2.5). Furthermore (2.6a) simplifies in this case to

\[
\nabla^\mu \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle = -\frac{1}{d} C \rho^d \left( \partial_\nu \delta^d(x, y) g_{\alpha\beta}(y) + 2\nabla_\sigma \delta^d(\delta_{(\alpha}^\sigma \delta_{\beta)}^\rho g_{\mu\nu}(x) \delta^d(x, y)) \right) .
\]

If we define

\[
\langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle_{\text{con}} = \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle + \frac{1}{d} C \rho^d (g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} + g_{\mu\nu} g_{\alpha\beta}) \delta^d(x, y) ,
\]

then

\[
\nabla^\mu \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle_{\text{con}} = 0 .
\]
Using the definition (2.3) and also (2.13) we find in general
\[
\frac{\rho}{\rho} \frac{d}{d\rho} C = \frac{1}{\rho^d} \int d^d y \sqrt{g} g^{\mu\nu}(x) g^{\alpha\beta}(y) \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_{\text{con}} - dC \\
= \frac{1}{\rho^d} \int d^d y \sqrt{g} \left( \langle \Theta(x) \Theta(y) \rangle + \mathcal{H}(x, y) \right) - \frac{1}{\rho^d} \beta^i \partial_i \langle \mathcal{O}_j \rangle. 
\]

Applying the basic definitions in (2.1) with (2.2), for the theory defined on a homogeneous space of constant curvature, we have the consistency conditions
\[
\frac{\partial}{\partial g^{i}} C = -\rho \frac{d}{d\rho} \left( \frac{1}{\rho^d} \langle \mathcal{O}_i \rangle \right). 
\]

Since from (2.7) \( C = - (\beta^i \langle \mathcal{O}_i \rangle + \mathcal{A}) \rho^{-d} \), (2.30) implies the RG equation (2.16a) which now takes the form
\[
\left( -\rho \frac{d}{d\rho} + \beta^i \frac{\partial}{\partial g^{i}} \right) C = \rho \frac{d}{d\rho} \left( \frac{1}{\rho^d} \mathcal{A} \right) = - \frac{1}{\rho^d} g^{\mu\nu} C_{\mu\nu}. 
\]

3. Two Dimensions

In two dimensions the curved space trace anomaly for an arbitrary metric is just proportional to the scalar curvature so that in (2.7) we may write
\[
2\pi \mathcal{A} = \frac{1}{12} c R. 
\]

In this case in (2.8) and (2.10) we now take
\[
2\pi \mathcal{A}_{\alpha\beta}(x, y) = \frac{1}{6} c \left( \nabla_{\alpha} \nabla_{\beta} - g_{\alpha\beta} \nabla^2 \right) \delta^2(x, y), \\
2\pi \mathcal{B}_{i}(x, y) = - \frac{1}{12} \partial_{i} c R \delta^2(x, y) - \frac{1}{6} w_{i} \nabla^2 \delta^2(x, y), 
\]
which are in accord with (2.11) and where \( w_{i}(g) \) is a vector function of the couplings. Furthermore in two dimensions \( C_{\mu\nu} \) and \( E_{\mu\nu,\alpha\beta} \) are both zero so that (2.16a) and (2.17) now become
\[
\mathcal{D} \langle T_{\mu\nu} \rangle = 0, \quad \mathcal{D} \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle = 0, 
\]
while in (2.16b) and (2.19) we now assume
\[
2\pi \mathcal{C}_{i} = - \frac{1}{12} \partial_{i} c R, \quad 2\pi \mathcal{F}_{\mu\nu,i}(x, y) = - \frac{1}{6} \partial_{i} c \left( \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \nabla^2 \right) \delta^2(x, y), 
\]
so that (2.16b) now reads
\[
2\pi \left( \mathcal{D} + 2 \right) \langle \mathcal{O}_{i} \rangle + \partial_{i} \beta^{j} \langle \mathcal{O}_{j} \rangle = - \frac{1}{12} \partial_{i} c R. 
\]
Finally in (2.21) we take, for \( G_{ij}(g) \) a symmetric tensor,

\[
2\pi G_{ij}(x, y) = \frac{1}{6} G_{ij} \nabla^2 \delta^2(x, y) + \frac{1}{12} \partial_i \partial_j c R \delta^2(x, y).
\] (3.6)

With the expressions in (3.2) and (3.4) the consistency condition (2.23) is identically satisfied. Inserting the appropriate forms into (2.24) leads to the single relation

\[
\partial_i c - G_{ij} \beta^j = -\mathcal{L}_\beta w_i = -\beta^j \partial_j w_i - \partial_i \beta^j w_j.
\] (3.7)

If we define

\[
\tilde{c} = c + w_i \beta^i,
\] (3.8)

then (3.7) may be rewritten as

\[
\partial_i \tilde{c} = (G_{ij} + \partial_i w_j - \partial_j w_i) \beta^j \quad \Rightarrow \quad \beta^i \partial_i \tilde{c} = G_{ij} \beta^i \beta^j.
\] (3.9)

We now show how the Ward and trace identities may be solved, after restricting to a homogeneous space of constant curvature, to give explicit forms for the two point functions with relations between them. Assuming (2.25) we may take

\[
\frac{(2\pi)^2}{2} \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_{\text{con}} = \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{2} g_{\mu\nu} R \right) F(x, y) \left( \nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right),
\] (3.10)

which satisfies (2.26) identically. In a similar fashion

\[
\frac{(2\pi)^2}{2} \left( \langle T_{\mu\nu}(x) \mathcal{O}_i(y) \rangle - g_{\mu\nu} \langle \mathcal{O}_i \rangle \delta^2(x, y) \right) = \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{2} g_{\mu\nu} R \right) G_i(x, y) \left( -\nabla^2 - R \right),
\] (3.11)

automatically satisfies (2.6b). Applying (2.8) with (3.2) gives the relations

\[
F(x, y) = G_i(x, y) \beta^i + \frac{1}{3} c K_0(x, y),
\] (3.12)

assuming \( K_0 \) is a solution of

\[
(-\nabla^2 - R) K_0(x, y) = 2\pi \delta^2(x, y).
\] (3.13)

Explicit solutions of this equation are discussed later. For the case of the positive curvature sphere (3.13) has to be modified although this does not change (3.12) or (3.10). The RG equations for the two point functions in (3.10) and (3.11) may now be reexpressed in terms of \( F \) and \( G_i \),

\[
\mathcal{D} F(x, y) = 0, \quad \mathcal{D} G_i(x, y) + \partial_i \beta^j G_j(x, y) = -\frac{1}{6} \partial_i c K_0(x, y).
\] (3.14)
If we write
\[ (2\pi)^2 \langle O_i(x)O_j(y) \rangle = \left( -\nabla^2 - R \right) H_{ij}(x,y) \left( -\tilde{\nabla}^2 - R \right) , \] (3.15)
then we have from (2.10)
\[ G_i(x,y) = H_{ij}(x,y)\beta^j + \frac{1}{6} w_i K_0(x,y) + V_i K_1(x,y) , \] (3.16)
\[ V_i = \frac{1}{6} (w_i - \frac{1}{2} \partial_i c) R - 2\pi \left( 2\langle O_i \rangle + \partial_i \beta^j \langle O_j \rangle \right) , \]
if
\[ (-\nabla^2 - R) K_1(x,y) = K_0(x,y) . \] (3.17)
Furthermore (2.21) gives
\[ \mathcal{D} H_{ij}(x,y) + \partial_i \beta^k H_{kj}(x,y) + \partial_j \beta^k H_{ik}(x,y) = -\frac{1}{6} G_{ij} K_0(x,y) - S_{ij} K_1(x,y) . \] (3.18)
for
\[ S_{ij} = \frac{1}{6} (G_{ij} - \frac{1}{2} \partial_i \partial_j c) R - 2\pi \partial_i \partial_j \beta^k \langle O_k \rangle . \] (3.19)
Consistency of (3.18) with (3.16) and (3.14) depends on (3.7) and also
\[ (\mathcal{D} + 2) V_i + \partial_i \beta^j V_j = S_{ij} \beta^j , \] (3.20)
which follows from (3.7) and (3.5).

At a fixed point, when \( \beta^i = 0 \), then
\[ 2\pi \langle T_{\mu\nu} \rangle |_{\beta^i=0} = \frac{1}{2\pi} c R g_{\mu\nu} , \] (3.21)
so that in (2.25) \( 2\pi C |_{\beta^i=0} = \mp \frac{1}{6} c \). From (3.12) the two point function in (3.10) is also determined to be,
\[ \left( 2\pi \right)^2 \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle \big|_{\beta^i=0} \bigg|_{\text{con}} = \frac{1}{6} c \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{2} g_{\mu\nu} R \right) K_0(x,y) \left( \tilde{\nabla}_\alpha \tilde{\nabla}_\beta - \tilde{\nabla}^2 g_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) , \] (3.22)
neglecting contact terms. The overall coefficient is just \( c \) and of course the result is non local, not just a potentially ambiguous contact term. Explicit expressions are obtained later in section 8. In the same way from (3.16) \( G_i = \frac{1}{3} w_i G_0 + V_i G_1 \). Substituting in (3.11), disregarding contact terms and using (3.7) in the expression for \( V_i \) gives
\[ \left( 2\pi \right)^2 \langle T_{\mu\nu}(x)O_i(y) \rangle |_{\beta^i=0} = \left( \nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \nabla^2 \right) K_0(x,y) \Gamma^j_i \left( \frac{1}{12} w_j R - 2\pi \langle O_j \rangle \right) , \] (3.23)
\[ \Gamma^j_i = 2\delta^j_i + \partial_i \beta^j . \]
\( \Gamma^j_i \) is the matrix defining the dimensions of the fields \( O_i \). The result (3.23) demonstrates that \( w_i \) is well defined at a critical point although it should be recognised that \( \langle O_i \rangle \) is arbitrary up to terms \( \propto \partial_i f R \) for any function \( f(g) \) of the couplings which leads to a corresponding freedom \( w_i \sim w_i + \partial_i f \).
4. Four Dimensions

As is well known in any even dimension beyond two dimensions there are several possible curvature dependent terms which may contribute to the energy momentum tensor trace. For four dimensions we take
\[ 16\pi^2 A = cF - aG - bR^2, \] (4.1)
neglecting a term \( \propto \nabla^2 R \) which may be cancelled by a local redefinition of \( W \), and where
\[ G = \frac{1}{4} \epsilon^{\mu\nu\sigma\rho} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\sigma\rho} = 6 R_{\alpha\beta}^{\gamma\delta} R^{\gamma\delta}_{\alpha\beta}, \]
\[ F = C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}, \] (4.2)
with \( C \) the Weyl tensor. \( G \) is a topological density which is reflected by its variation being a total divergence,
\[ \delta(\sqrt{g}G) = \sqrt{g} \nabla \gamma V^\gamma, \quad V^\gamma = 24 R^{[\alpha\beta}_{\gamma\delta} \nabla^\delta g^{\epsilon\delta} g_{\epsilon\delta}. \] (4.3)
For general metrics then (2.13) gives
\[ 16\pi^2 g^{\alpha\beta}(y) A_{\alpha\beta}(x, y) = - 8a \nabla_\alpha \nabla_\beta (G^{\alpha\beta}_{\gamma\delta}(x, y)) + 12b R(x) \nabla^2 \delta^4(x, y), \]
\[ G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R, \] (4.4)
so that, for \( R \) not constant, the \( b \) term is not symmetric and hence the condition (2.12) requires \( b = 0 \) if \( \beta^i = 0 \). On a homogeneous space with (1.2) then \( C_{\alpha\beta\gamma\delta} = 0 \) and
\[ G = \frac{1}{6} R^2, \quad V^\gamma = \frac{1}{3} R \left( - \nabla_\beta \delta^\gamma g_{\beta\gamma} + \nabla^\gamma (g_{\alpha\beta} \delta^\alpha g_{\beta\gamma}) \right), \] (4.5)
so that in this case the anomaly reduces to just
\[ 2\pi^2 A = - \frac{1}{12} \hat{a} R^2, \quad \hat{a} = a + 6b, \] (4.6)
and using (4.5)
\[ 2\pi^2 A_{\alpha\beta}(x, y) = - \frac{1}{12} \hat{a} R \left( \nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2 \right) \delta^4(x, y). \] (4.7)
Similarly we require in this case
\[ 2\pi^2 B_i(x, y) = \frac{1}{48} \partial_i \hat{a} R^2 \delta^4(x, y) - Y_i R \nabla^2 \delta^4(x, y) - U_i \nabla^2 \nabla^2 \delta^4(x, y). \] (4.8)
Since now \( C_{\mu\nu} = 0 \) and
\[ 2\pi^2 C_i = \frac{1}{48} \partial_i \hat{a} R^2, \] (4.9)
\[ \text{\footnote{2 For free fields } c = \frac{1}{120}(12n_V + 6n_F + n_S), a = \frac{1}{360}(62n_V + 11n_F + n_S).} \]
the RG eqs. (2.16a, b) become

\[(\mathcal{D} + 2)\langle T_{\mu\nu} \rangle = 0, \quad (4.10a)\]
\[2\pi^2((\mathcal{D} + 4)\langle O_i \rangle + \partial_i \beta^j \langle O_j \rangle) = \frac{1}{4\pi^2} \partial_i \hat{a} R^2. \quad (4.10b)\]

With the result (4.1) the definition (2.18) may be expressed as

\[2\pi^2 \mathcal{E}_{\mu\nu,\alpha\beta}(x,y) = 4c \nabla^\sigma \nabla^\rho (\mathcal{E}^C_{\mu\rho\nu,\alpha\gamma\delta\beta} \delta^4(x,y)) \nabla^\gamma \nabla^\delta \]
\[b (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4} g_{\mu\nu} R) \delta^4(x,y) \left(\nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R\right) \]
\[+ \frac{1}{2} b R H_{\mu\nu,\alpha\beta}(x,y), \quad (4.11)\]

where \(\mathcal{E}^C_{\mu\rho\nu,\alpha\gamma\delta\beta}(x)\) is the projector at \(x\) onto tensors with the symmetry and traceless properties of the Weyl tensor \(C_{\mu\rho\nu}\),

\[C_{\mu\rho\nu} = C_{[\mu\rho]}[\nu], \quad C_{\mu[\sigma\rho\nu]} = 0, \quad g^{\sigma\rho} C_{\mu\sigma\rho\nu} = 0. \quad (4.12)\]

An explicit form for \(\mathcal{E}^C\) is given later by (8.21) for \(y = x\). The term involving \(\mathcal{E}^C\) in (4.11) is automatically conserved and traceless since, on spaces of constant curvature when (1.2) holds, for any \(C_{\mu\rho\nu}\) satisfying (4.12)

\[T_{\mu\nu} = \nabla^\sigma \nabla^\rho C_{\mu\sigma\rho\nu} \Rightarrow T_{\mu\nu} = T_{\nu\mu}, \quad g^{\mu\nu} T_{\mu\nu} = 0, \quad \nabla^\mu T_{\mu\nu} = 0. \quad (4.13)\]

The remaining term \(H_{\mu\nu,\alpha\beta}(x,y)\) in (4.11) is then defined by

\[H_{\mu\nu,\alpha\beta}(x,y) = \nabla_\mu \nabla_\sigma \left(\delta^\sigma_{(\alpha} \delta^\rho_{\beta)} g_{\rho\nu} \delta^4(x,y)\right) + \nabla_\nu \nabla_\sigma \left(\delta^\sigma_{(\alpha} \delta^\rho_{\beta)} g_{\rho\mu} \delta^4(x,y)\right) \]
\[- \nabla^2 \left(\delta^\sigma_{(\alpha} \delta^\rho_{\beta)} g_{\sigma\rho\nu} \delta^4(x,y)\right) \]
\[- g_{\alpha\beta} \nabla_\mu \nabla_\nu \delta^4(x,y) - g_{\mu\nu} \nabla_\alpha \nabla_\beta \delta^4(x,y) + g_{\mu\nu} g_{\alpha\beta} \nabla^2 \delta^4(x,y) \]
\[+ \frac{1}{12} R \left(g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} + g_{\mu\nu} g_{\alpha\beta}\right) \delta^4(x,y), \quad (4.14)\]

and, with the conventions on covariant derivatives described above, this is symmetric

\[H_{\mu\nu,\alpha\beta}(x,y) = H_{\alpha\beta,\mu\nu}(y,x), \quad (4.15)\]

and, for constant curvature, satisfies

\[\nabla^\mu H_{\mu\nu,\alpha\beta}(x,y) = 0, \quad H_{\mu\nu,\alpha\beta}(x,y) g^{\alpha\beta} = -2(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4} g_{\mu\nu} R) \delta^4(x,y). \quad (4.16)\]

As a consequence of (4.13) and (4.16) the form (4.11) satisfies

\[\nabla^\mu \mathcal{E}_{\mu\nu,\alpha\beta}(x,y) = 0, \quad 2\pi^2 g^{\mu\nu}(x) \mathcal{E}_{\mu\nu,\alpha\beta}(x,y) = 3b \nabla^2 \delta^4(x,y) \left(\nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R\right). \quad (4.17)\]
In addition the corresponding quantities entering into (2.19) and (2.21) may now be
assumed to be given by (for a general metric then $2\pi^2C_i$ may contain a term $\frac{1}{2}D_i\nabla^2 R$, its
variation according to (2.20) gives rise to the corresponding term below)

$$2\pi^2 F_{\mu\nu, i}(x, y) = \frac{1}{12} \partial_i \hat{a} R (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) \delta^4(x, y)$$

$$+ D_i (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4} g_{\mu\nu} R) \nabla^2 \delta^4(x, y),$$

and

$$2\pi^2 G_{ij}(x, y) = G_{ij} \nabla^2 \delta^4(x, y) + L_{ij} R \nabla^2 \delta^4(x, y) - \frac{1}{48} \partial_i \partial_j \hat{a} R^2 \delta^4(x, y).$$

Inserting the results (4.7), (4.17) and (4.18) into (2.23) then gives

$$3b = D_i \beta^i.$$  (4.20)

This provides an alternative demonstration of the vanishing of the coefficient of the $R^2$
trace anomaly at a fixed point. Imposing now (2.24) leads to two relations,

$$L_\beta U_i = G_{ij} \beta^j + 3D_i,$$  \hspace{1cm}  $$L_\beta Y_i = L_{ij} \beta^j + D_i + \frac{1}{4} \partial_i \hat{a},$$

and eliminating $D_i$ then gives an analogous formula to (3.7),

$$\frac{3}{4} \partial_i \hat{a} = \hat{G}_{ij} \beta^j - L_\beta \hat{U}_i,$$  \hspace{1cm}  $$\hat{G}_{ij} = G_{ij} - 3L_{ij},$$  \hspace{1cm}  $$\hat{U}_i = U_i - 3Y_i.$$  (4.22)

In order to apply these results we consider explicit forms for the two point functions
for a homogeneous space of constant curvature given by

$$(2\pi^2)^2 \langle O_i(x)O_j(y) \rangle = \left( - \nabla^2 - \frac{1}{3} R \right) H_{ij}(x, y) \left( - \nabla^2 - \frac{1}{3} R \right),$$

and

$$(2\pi^2)^2 \langle T_{\mu\nu}(x)O_i(y) - g_{\mu\nu} \langle O_i \rangle \delta^4(x, y) \rangle = \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4} g_{\mu\nu} R \right) G_{ij}(x, y) \left( - \nabla^2 - \frac{1}{3} R \right),$$

which satisfies (2.64). With the result (4.3) the relation (2.10) leads to

$$3G_j(x, y) = \beta^i H_{ij}(x, y) - U_j 2\pi^2 \delta^4(x, y) + (Y_j - \frac{2}{3} U_j) R K_0(x, y) + V_j K_1(x, y),$$

$$V_j = \left( \frac{1}{48} \partial_j \hat{a} - \frac{1}{9} U_j \right) - 2\pi^2 \left( 4 \langle O_j \rangle + \partial_j \beta^k \langle O_k \rangle \right),$$

where

$$(- \nabla^2 - \frac{1}{3} R) K_0(x, y) = 2\pi^2 \delta^4(x, y),$$  \hspace{1cm}  $$(- \nabla^2 - \frac{1}{3} R) K_1(x, y) = K_0(x, y).$$  (4.26)
As in two dimensions the equation for $K_0$ must be modified in the positive curvature case as demonstrated for its explicit solution found later. The RG equations (2.21) and (2.13) now become

$$(\mathcal{D} + 4)H_{ij}(x, y) + \partial_i \beta^k H_{kj}(x, y) + \partial_j \beta^k H_{jk}(x, y) = G_{ij} 2\pi^2 \delta^4(x, y) - (L_{ij} - \frac{2}{3} G_{ij}) RK_0(x, y) - S_{ij} K_1(x, y),$$

and

$$(\mathcal{D} + 4) G_i(x, y) + \partial_i \beta^j G_j(x, y) = -D_i 2\pi^2 \delta^4(x, y) + \frac{1}{3} (\frac{1}{3} \partial_i \hat{a} - D_i) RK_0(x, y).$$

Compatibility of (4.28) and (4.27) with (4.25), which requires

$$(\mathcal{D} + 4) V_j + \partial_j \beta^k V_k = S_{jk} \beta^k,$$

is guaranteed as a consequence of (4.21).

It remains to find a general form for the energy momentum tensor two point function. For homogeneous spaces in general dimensions other than two this involves two independent tensor structures which represent spin 2 and spin 0. For the conserved two point function defined by (2.28)

$$(2\pi^2)^2 \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_{\text{con}} = \nabla^\sigma \nabla^\rho (\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x, y) F_2(x, y)) \nabla^\gamma \nabla^\delta$$

$$+ (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4} g_{\mu\nu} R) F_0(x, y) \left( \nabla_{\alpha} \nabla_{\beta} - \nabla^2 g_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R \right),$$

where $\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x, y)$ is a bi-tensor, constructed explicitly later, which satisfies the symmetry and traceless conditions of the Weyl tensor separately at $x$ and $y$ and which for $x = y$ reduces to the projector $\mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x)$ introduced above. Both expressions on the right hand side of (4.30) are automatically conserved, the first term, involving $F_2$, using (1.13) is in addition traceless. It is important to recognize that the decomposition in (4.30) is not unique. If $F_0 \rightarrow K_0$, satisfying (1.26), then this term is also both conserved and traceless for non coincident $x, y$ and, as shown in section 8, any conserved traceless two point function can always be written in terms of an appropriate $F_2$ for $x \neq y$. This therefore leads to the relation

$$2\pi^2 H_{\mu\nu,\alpha\beta}(x, y) = 8 \nabla^\sigma \nabla^\rho \left( \mathcal{E}^C_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x, y) K_2(x, y) \right) \nabla^\gamma \nabla^\delta$$

$$- \frac{2}{3} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4} g_{\mu\nu} R) K_0(x, y) \left( \nabla_{\alpha} \nabla_{\beta} - \nabla^2 g_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R \right),$$

for a suitable $K_2(x, y)$ where $H_{\mu\nu,\alpha\beta}(x, y)$ is purely a local contact term. Due to the derivatives the right hand side of (4.31) is well defined as a distribution. By considering
the divergence and trace of both sides of this relation $H_{\mu\nu,\alpha\beta}(x,y)$ may be identified with the previous definition in (1.14) since it is the unique form satisfying (4.13) and (1.16). On flat space $\mathcal{E}^C$ is a constant tensor and $K_0$ and $K_2$ are identical.

Applying the decomposition in (4.30) we may first note, by using the general relation (2.8) with (4.24) and (4.7), that

$$3F_0(x,y) = G_i(x,y)\beta^i - \frac{1}{12}\delta R K_0(x,y).$$

(4.32)

In order to implement the RG equation (2.17) it is necessary to rewrite the result for $\mathcal{E}_{\mu\nu,\alpha\beta}(x,y)$, which is given by (4.11), by using (1.34) in the more convenient form,

$$2\pi^2\mathcal{E}_{\mu\nu,\alpha\beta}(x,y) = 4 \nabla^\sigma \nabla^\rho \left( \mathcal{E}_{\mu\nu,\alpha\gamma\delta\beta}^C(x,y) (c 2\pi^2 \delta^4(x,y) + b RK_2(x,y)) \right) \nabla^\gamma \nabla^\delta - b \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - \frac{1}{4}g_{\mu\nu}R \right) (\delta^4(x,y) + \frac{1}{3}RK_0(x,y)) \left( \nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}R \right).$$

(4.33)

With this result the RG equations reduce to

$$(D + 4)F_2(x,y) = 4 \left( c 2\pi^2 \delta^4(x,y) + b RK_2(x,y) \right),$$

(4.34a)

$$(D + 4)F_0(x,y) = - b \left( 2\pi^2 \delta^4(x,y) + \frac{1}{3}RK_0(x,y) \right).$$

(4.34b)

It is important to note that (4.34b) follows directly from the expression (4.32) and (4.28) so long as (4.20) holds.

The general results simplify if we restrict to a fixed point where $\beta$-functions vanish. The expectation value of a single energy momentum tensor becomes

$$16\pi^2 \langle T_{\mu\nu} \rangle|_{\beta^i = 0} = - \frac{1}{24} aR^2 g_{\mu\nu},$$

(4.35)

or in (2.25) $2\pi^2 C|_{\beta^i = 0} = 3a$. From (4.25) we must also have

$$(2\pi^2)^2 \langle T_{\mu\nu}(x) O_i(y) \rangle|_{\beta^i = 0} = - \left( \nabla_\mu \nabla_\nu - \frac{1}{4}g_{\mu\nu} \nabla^2 \right) K_0(x,y) \frac{1}{3} \Gamma^i_j \left( \frac{1}{36} \dot{U}_j R^2 + 2\pi^2 \langle O_j \rangle \right),$$

$$\Gamma^i_j = 4\delta^i_j + \partial_i \beta^j,$$

(4.36)

which is the extension of (3.23) to four dimensions. This provides a definition of $\dot{U}_i$ at a fixed point. In the energy momentum tensor two point function $F_0 = - \frac{3}{2} a R K_0$, from (4.32), and using (4.31) we may write

$$(2\pi^2)^2 \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle|_{\beta^i = 0} = \nabla^\sigma \nabla^\rho \left( \mathcal{E}_{\mu\nu,\alpha\gamma\delta\beta}^C(x,y) \hat{F}_2(x,y) \right) \nabla^\gamma \nabla^\delta + \frac{1}{24} aR 2\pi^2 H_{\mu\nu,\alpha\beta}(x,y),$$

$$\hat{F}_2(x,y) = F_2(x,y) - \frac{1}{3} a RK_2(x,y).$$

(4.37)

Hence in this case, apart from a contact term, there is only the manifestly conserved and traceless spin 2 contribution to this two point function.
5. Geometrical Results for Spaces of Constant Curvature

In order to find more explicit expressions for the two point functions considered above we discuss here some geometrical results which allow natural expressions for the two point functions on homogeneous spaces of constant curvature to be found. A related discussion was given by Allen and Jacobsen [22] but there are differences in derivation and also application.

Our starting point is the geodetic interval \( \sigma(x,y) \) which is a bi-scalar defined for any curved manifold which is unique for \( x \) in the neighbourhood of \( y \) and which satisfies \[ g^{\mu \nu} \partial_\mu \sigma \partial_\nu \sigma = 2\sigma. \] (5.1)

On flat space \( \sigma(x,y) = \frac{1}{2}(x - y)^2 \). For a homogeneous space of constant curvature then any \( T_{\mu \nu \ldots } (x, y) \), which transforms covariantly under all isometries as a tensor at \( x \) and a scalar at \( y \), may be expanded in a basis formed by \( \partial_\mu \sigma(x, y) \) and \( g_{\mu \nu}(x) \) with coefficients functions of \( \sigma(x, y) \) [22]. Thus we may write

\[
\nabla_\mu \nabla_\nu \sigma = g_{\mu \nu} f(\sigma) + \partial_\mu \sigma \partial_\nu \sigma g(\sigma). \tag{5.2}
\]

Since, from (5.1), \( \nabla_\mu \nabla_\nu \sigma = \partial_\mu \sigma, \) we have

\[
f(\sigma) + 2\sigma g(\sigma) = 1. \tag{5.3}
\]

Imposing \( [\nabla_\sigma, \nabla_\mu] \nabla_\nu \sigma = -R^{\rho \nu \sigma \mu} \partial_\rho \sigma \) on (5.2) gives using (1.2)

\[
f'(\sigma) - f(\sigma) g(\sigma) = \mp \rho^2, \tag{5.4}
\]

or defining \( \sigma = \frac{1}{2} \theta^2 / \rho^2 \), so that \( \theta / \rho \) is the geodesic distance from \( x \) to \( y \),

\[
\theta \frac{d}{d\theta} f - f(1 - f) = \mp \theta^2. \tag{5.5}
\]

With the boundary condition \( f(0) = 1 \) the solutions are

\[
f = \frac{\theta}{\tan \theta}, \quad \frac{\theta}{\tanh \theta}, \quad g = \frac{\rho^2}{\theta^2} (1 - f). \tag{5.6}
\]

For the sphere \( \theta \) is the angular separation of \( x \) and \( y \) and we may restrict \( 0 \leq \theta \leq \pi, \theta = \pi \) corresponds to \( x \) and \( y \) being antipodal points.\(^3\)

\(^3\) If we define the bi-scalar \( \Delta(x, y) = \det(-\partial_\mu \sigma(x, y) \tilde{\partial}_\alpha)/\sqrt{g(x) \sqrt{g(y)}} \) then solving the equations in [23] gives \( \Delta = (\theta / \tan \theta)^{d-1}, (\theta / \tanh \theta)^{d-1}. \) For the spherical case the divergence at \( \theta = \pi \) is a reflection of this being a caustic.
For any bi-tensor, \( T_{\mu \nu \ldots \alpha \beta \ldots} (x, y) \), then the basis for expansion may be extended to include \( \sigma(x, y) \partial_\alpha \), which transforms as a covariant vector at \( y \), and also the bi-vector \( g_{\mu \alpha} (x, y) = -\partial_\mu \sigma(x, y) \partial_\alpha \) as well as \( g_{\alpha \beta} (y) \). In practice it is more convenient to introduce \( I^\mu_\alpha (x, y) \) which gives parallel transport of vectors along the geodesic from \( y \) to \( x \). This is defined by

\[
\sigma^\rho \nabla_\rho I^\mu_\alpha = 0, \quad I^\mu_\alpha (x, x) = \delta^\mu_\alpha . \tag{5.7}
\]

For homogeneous spaces as considered here we may write the general form

\[
I_{\mu \alpha} = g_{\mu \alpha} a(\sigma) - \partial_\mu \sigma \partial_\alpha b(\sigma) . \tag{5.8}
\]

Inserting this into (5.7) and using (5.1),(5.2),(5.3) gives

\[
a'(\sigma) + g(\sigma)a(\sigma) = 0, \quad \frac{d}{d\sigma} (a(\sigma) + 2\sigma b(\sigma)) = 0 . \tag{5.9}
\]

Solving these with the boundary condition \( a(0) = 1 \) gives

\[
a = \frac{\sin \theta}{\theta} , \quad \frac{\sinh \theta}{\theta} , \quad b = \frac{\rho^2}{\theta^2} (1 - a) . \tag{5.10}
\]

An important consistency check, which follows directly from (5.7), is that

\[
I^\mu_\alpha I^\mu_\beta = g_{\alpha \beta} , \quad I_{\mu \alpha} I^\nu_\alpha = g_{\mu \nu} , \tag{5.11}
\]

and further we have

\[
\partial_\mu \sigma(x, y) I^\mu_\alpha (x, y) = -\sigma(x, y) \partial_\alpha . \tag{5.12}
\]

In the following sections these results are used to obtain a natural form for the tensorial expansions of two point functions whose coefficients are functions of the biscalar \( \theta(x, y) \). It is convenient to use, as well as the parallel transport bi-vector \( I_{\mu \alpha} \), a basis formed by \( \hat{x}_\mu \) and \( \hat{y}_\alpha \), which are unit vectors at \( x, y \), given by

\[
\partial_\mu \theta = \rho \hat{x}_\mu , \quad \theta \partial_\alpha = \rho \hat{y}_\alpha , \quad \hat{x}_\mu I^\mu_\alpha = -\hat{y}_\alpha , \quad I^\mu_\alpha \hat{y}_\alpha = -\hat{x}_\mu , \tag{5.13}
\]

using (5.11) and (5.12). With these definitions (5.2) and (5.8) then become

\[
\nabla_\mu \hat{x}_\nu = \rho \cot(\theta(g_{\mu \nu} - \hat{x}_\mu \hat{x}_\nu)) , \quad \rho \coth(\theta(g_{\mu \nu} - \hat{x}_\mu \hat{x}_\nu)) , \tag{5.14}
\]

and also from (5.8) and (5.10) we have,

\[
\partial_\mu \hat{y}_\alpha = -\rho \cosec(\theta(I_{\mu \alpha} + \hat{x}_\mu \hat{y}_\alpha)) , \quad -\rho \cosech(\theta(I_{\mu \alpha} + \hat{x}_\mu \hat{y}_\alpha)) , \quad \nabla_\mu I^\nu_\alpha = \rho \tan \frac{1}{2} \theta(g_{\mu \nu} \hat{y}_\alpha + \hat{x}_\nu I^\nu_\alpha) , \quad -\rho \tanh \frac{1}{2} \theta(g_{\mu \nu} \hat{y}_\alpha + \hat{x}_\nu I^\nu_\alpha) . \tag{5.15}
\]
For the spherical case $\hat{x}_\mu, \hat{y}_\alpha$ and $I_{\mu\alpha}$ are singular when $\theta = \pi$ and $x, y$ are antipodal points. Using (5.14) it is easy to verify

$$\nabla^2 F(\theta) = \rho^2 \left( \frac{d^2}{d\theta^2} + \frac{d-1}{\tan \theta} \frac{d}{d\theta} \right) F(\theta), \quad \rho^2 \left( \frac{d^2}{d\theta^2} + \frac{d-1}{\tanh \theta} \frac{d}{d\theta} \right) F(\theta).$$

(5.16)

6. Conformal Symmetries

At a renormalisation group fixed point quantum field theories are additionally constrained by conformal invariance. Conformal symmetry may be extended to spaces of non zero curvature by seeking conformal Killing vectors which satisfy

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = 2w g_{\mu\nu}, \quad w = \frac{1}{d} \nabla \cdot v.$$  

(6.1)

On flat space the solutions are well known. We show here how it is similarly possible to solve (6.1) on homogeneous spaces of constant curvature without any restriction to particular choices of coordinates. From (6.1) and (1.2) we first derive an expression for two covariant derivatives of $v$

$$\nabla_\sigma \nabla_\mu v_\nu = \mp \rho^2 (g_{\sigma\mu} v_\nu - g_{\sigma\nu} v_\mu) - g_{\sigma\mu} \partial_\nu w + g_{\sigma\nu} \partial_\mu w + g_{\mu\nu} \partial_\sigma w.$$  

(6.2)

Also by contracting with $\nabla_\mu \nabla_\nu$ we may obtain \[4\]

$$(d-1)\nabla^2 w + Rw = 0,$$  

(6.3)

and by using this with (6.1) together with $(d-2)\nabla_\nu w = -(\nabla^2 + \frac{1}{d} R)v_\nu,$ which also follows from (6.1), we find for $d \neq 2$

$$\nabla_\mu \nabla_\nu w = \mp \rho^2 g_{\mu\nu} w.$$  

(6.4)

This is easily extended to arbitrarily many derivatives

$$\nabla_{(\mu_1} \ldots \nabla_{\mu_{2n})} w = (\mp \rho^2)^n g_{(\mu_1\mu_2} \ldots g_{\mu_{2n-1}\mu_{2n})} w,$$

$$\nabla_{(\mu_1} \ldots \nabla_{\mu_{2n+1})} w = (\mp \rho^2)^n g_{(\mu_1\mu_2} \ldots g_{\mu_{2n-1}\mu_{2n}} \partial_{\mu_{2n+1})} w,$$

and then applying the covariant Taylor expansion, which for any scalar $F$ takes the form [24]

$$F(x) = \sum_{n=0} \frac{1}{n!} \sigma^{\alpha_1} \ldots \sigma^{\alpha_n} (-1)^n \nabla_{(\alpha_1} \ldots \nabla_{\alpha_n)} F(y), \quad \sigma^\alpha(x, y) = g^{\alpha\beta}(y) \sigma(x, y) \delta^\beta,$$

(6.6)

---

4 However writing the metric in a conformally flat form $g_{\mu\nu} = \Omega^2 \delta_{\mu\nu}$ then if $v^\mu$ is a solution of the flat space conformal Killing equation it remains a solution for the metric $g_{\mu\nu}$.

5 This demonstrates that there is no solution with $w = 1$ if $R \neq 0$ so that there is no dilation current $j^\mu = T^{\mu\nu} v_\nu$ satisfying $\nabla_\mu j^\mu = g_{\mu\nu} T^{\mu\nu}$ except in flat space.
leads to
\[ w(x) = \cos \theta w(y) - \frac{1}{\rho} \sin \theta \hat{y} \cdot \partial w(y), \quad \cosh \theta w(y) - \frac{1}{\rho} \sinh \theta \hat{y} \cdot \partial w(y). \tag{6.7} \]

This result demonstrates that \( w(x) \) is determined completely by the values of \( w \) and \( \partial_\alpha w \) at any arbitrary \( y \) so that there are \( d + 1 \) independent forms for \( w \) (for \( R > 0 \) from (6.3) these correspond to the \( d + 1 \) normalisable eigenvectors of \( -\nabla^2 \) with eigenvalue \( d\rho^2 \)). If we define
\[ \nabla_{[\mu} v_{\nu]} = -\omega_{\mu\nu}, \tag{6.8} \]
then from (6.2) and (6.4) we get
\[ \nabla_{(\mu_1} \cdots \nabla_{\mu_{2n})} v_{\nu)} = (\mp \rho^2)^n \left\{ (g_{(\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n})} v_{\nu)} - v_{(\mu_1} g_{\mu_2\mu_3} \cdots g_{\mu_{2n})\nu)} \right. \]
\[ \pm \frac{1}{\rho^2} (g_{(\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n})} \partial_{\nu} w - 2\partial_{(\mu_1} w g_{\mu_2\mu_3} \cdots g_{\mu_{2n})\nu)}) \right\}, \tag{6.9} \]
\[ \nabla_{(\mu_1} \cdots \nabla_{\mu_{2n+1})} v_{\nu)} = - (\mp \rho^2)^n (g_{(\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}} \omega_{\mu_{2n+1})\nu} - g_{(\mu_2\mu_3} \cdots g_{\mu_{2n+1})\nu}) w), \tag{6.10} \]

Applying the Taylor expansion (6.6) to \( I_{\mu}^\alpha(x, y) v_\alpha(y) \) leads to
\[ v_\mu(x) = I_{\mu}^\alpha \left\{ \frac{\cos \theta}{\cosh \theta} v_\alpha(y) + \left( \frac{1 - \cos \theta}{\cosh \theta - 1} \right) \left( \hat{y} \cdot v(y) \hat{y}_\alpha + \frac{1}{\rho^2} \partial_\alpha w(y) \pm \frac{2}{\rho^2} \hat{y} \cdot \partial w(y) \hat{y}_\alpha \right) \right. \]
\[- \frac{1}{\rho} \sin \theta \left( \omega_{\alpha\beta}(y) \hat{y}^\beta + w(y) \hat{y}_\alpha \right) \right\}, \tag{6.11} \]

It is straightforward to verify, using (5.14) and (5.15), that (6.10) and (6.7) satisfy (6.1). Besides \( w \) and \( \partial_\alpha w \) the general expression for \( v_\mu(x) \) is determined by \( v_\alpha \) and \( \omega_{\alpha\beta} \) at some arbitrary \( y \), giving \( \frac{1}{2}(d + 1)(d + 2) \) linearly independent vectors. Taking \( y = 0 \) it is easy to see that (6.10) and (6.7) reduce to the standard results for flat space with \( v_\mu(x) \) quadratic and \( w(x) \) linear in \( x \). From the definition (5.8) we may also derive
\[ \omega_{\mu\nu}(x) = I_{\mu}^\alpha I_{\nu}^\beta \left\{ \omega_{\alpha\beta}(y) \mp \sin \theta \sinh \theta 2\hat{y}_{[\alpha} \left( \rho v_{\beta]}(y) \pm \frac{1}{\rho} \partial_{\beta]} w(y) \pm \frac{1 - \cos \theta}{\cosh \theta - 1} \right) 2\hat{y}_{[\alpha} \omega_{\beta]} \hat{y}^\gamma \right\}. \tag{6.11} \]

Using the above solutions for conformal Killing vectors, which are specified by \( v^\alpha, \omega^\alpha_{\beta}, w, \partial^\alpha w \), it is straightforward to calculate the Lie algebra of the associated vector fields,
\[ [v_1, v_2]^\mu = -v_3^\mu \quad \Rightarrow \quad v_3^\mu = \omega^\mu_{1\nu} v_2^\nu - v_1^\mu w_2 - (1 \leftrightarrow 2). \tag{6.12} \]

Using (6.10), (6.7) and (6.11) this is identical with the algebra of matrices \( W \),
\[ [W_1, W_2] = W_3, \tag{6.13} \]
where
\[
W^A_B = \begin{pmatrix}
\omega^\alpha{}_{\beta} & \rho v^\alpha \pm \frac{1}{\rho} \partial^\alpha w & \pm \frac{1}{\rho} \partial^\alpha w \\
\mp \rho v_{\beta} - \frac{1}{\rho} \partial_{\beta} w & 0 & w \\
\frac{1}{\rho} \partial_{\beta} w & w & 0
\end{pmatrix}.
\] (6.14)

Since, for
\[
G_{AB} = \begin{pmatrix}
\delta_{\alpha\beta} & 0 & 0 \\
0 & \pm 1 & 0 \\
0 & 0 & \mp 1
\end{pmatrix},
\]
we have
\[
W_{AB} = G_{AC} W^C_B = -W_{BA},
\] (6.15)

it is clear that (6.13) corresponds in both cases to the Lie algebra of the \(d\)-dimensional conformal group \(O(d + 1, 1)\). If we restrict to Killing vectors for which \(w = 0\) it is also evident that the algebra reduces to that for the isometry groups \(O(d + 1)\) or \(O(d, 1)\) for \(S^d\) or \(H^d\) respectively as expected.

For construction of conformally covariant expressions for correlation functions it is necessary to construct bi-scalar functions of \(x, y\) which transform homogeneously under such conformal transformations. In consequence we consider the generalisation \(s\) of the flat space \((x - y)^2\) which is required to satisfy
\[
\left( v^\mu(x) \partial_\mu + v^\alpha(y) \partial_\alpha \right) s = \left( w(x) + w(y) \right)s.
\] (6.17)

Writing \(s(\theta)\) the left hand side of (6.17) involves
\[
\rho \left( v^\mu(x) \hat{x}_\mu + v^\alpha(y) \hat{y}_\alpha \right) = \tan \frac{1}{2} \theta \left( w(x) + w(y) \right), \quad \tanh \frac{1}{2} \theta \left( w(x) + w(y) \right),
\] (6.18)

from (6.10) and (6.7). Hence (6.17) becomes
\[
\frac{d}{d\theta} s = \cot \frac{1}{2} \theta s, \quad \coth \frac{1}{2} \theta s.
\] (6.19)

Imposing the boundary condition that \(s \sim \theta^2/\rho^2\) as \(x \to y\) gives the solution
\[
\rho^2 s = 2(1 - \cos \theta), \quad 2(\cosh \theta - 1),
\] (6.20)

so that \(\sqrt{s}\) may be interpreted as the chordal distance between \(x\) and \(y\). For the hyperbolic case it is useful also to define
\[
\rho^2 \tilde{s} = \rho^2 s + 4 = 2(\cosh \theta + 1).
\] (6.21)

Any power \(s^{-\lambda}\) also transforms homogeneously under conformal transformations, as in (6.17), and (5.16) gives
\[
(\nabla^2 \pm \rho^2 \lambda(\lambda - d + 1)) s^{-\lambda} = \lambda(2\lambda + 2 - d)s^{-\lambda - 1}.
\] (6.22)
Using the bi-scalar $s$ we may further define a bi-vector by

$$-\partial_\mu \ln s(x, y) \tilde{\partial}_\alpha = \frac{2}{s(x, y)} I_{\mu\alpha}(x, y), \quad (6.23)$$

which gives

$$I_{\mu\alpha} = I_{\mu\alpha} + 2\hat{x}_\mu \hat{y}_\alpha. \quad (6.24)$$

$I_{\mu\alpha}$ generalises the inversion tensor to spaces of constant curvature and, from (5.11) and (5.12), we have $I_{\mu\alpha} I_{\mu\beta} = g_{\alpha\beta}$. From its definition (6.23) and (6.17), with (6.8), we have

$$(v^\mu(x) \partial_\mu + v^\alpha(y) \partial_\alpha) I_{\mu\alpha}(x, y) = \omega^\nu_\mu(x) I_{\nu\alpha}(x, y) + \omega^\beta_\alpha(y) I_{\mu\beta}(x, y). \quad (6.25)$$

For the positive curvature case since $s(x, y)$ is single valued for arbitrary $x, y$ (6.23) also ensures that $I_{\mu\alpha}$ is well defined at $\theta = \pi$.

For later reference it is useful also to define related bi-vector

$$\hat{I}_{\mu\alpha} = -\frac{1}{2} \partial_\mu s \tilde{\partial}_\alpha = I_{\mu\alpha} + (1 - \cos \theta) \hat{x}_\mu \hat{y}_\alpha, \quad I_{\mu\alpha} = (\cosh \theta - 1) \hat{x}_\mu \hat{y}_\alpha, \quad (6.26)$$

which satisfies

$$\nabla_\mu \hat{I}_{\nu\alpha} = g_{\mu\nu} \sin \theta \hat{y}_\alpha, \quad -g_{\mu\nu} \sinh \theta \hat{y}_\alpha. \quad (6.27)$$

7. Spinors

For spinor fields we may define, using vielbeins as usual, Dirac matrices $\gamma_{\mu}(x)$ satisfying $\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}$. The essential geometrical object for our purposes is the bispinor $I(x, y)$ which acting on spinor at $y$ parallel transports it along the geodesic from $y$ to $x$. This satisfies

$$\hat{x}_\mu \nabla_\mu I = 0, \quad I(x, x) = 1, \quad (7.1)$$

where $\nabla_\mu = \partial_\mu + \omega_\mu$ is the spinor covariant derivative. For a homogeneous space of constant curvature we follow a similar approach to that of Allen and Lütken in four dimensions [25] and express the covariant derivative in a form compatible with (7.1),

$$\nabla_\mu I(x, y) = -\frac{1}{2} \rho \alpha(\theta)(\gamma_{\mu} \gamma \cdot \hat{x} - \hat{x}_\mu) I(x, y). \quad (7.2)$$

Using (5.14) we may then find

$$\nabla_\mu \nabla_\nu I(x, y) = \frac{1}{2} \alpha(\alpha + \cot \theta) \gamma_{[\mu} \gamma_{\nu]} I(x, y) - \frac{1}{2} (\alpha' - \alpha^2 - 2 \alpha \cot \theta) \hat{x}_{[\mu} \gamma \cdot \hat{x} I(x, y),$$

$$= \frac{1}{2} \alpha(\alpha + \coth \theta) \gamma_{[\mu} \gamma_{\nu]} I(x, y) - \frac{1}{2} (\alpha' - \alpha^2 - 2 \alpha \coth \theta) \hat{x}_{[\mu} \gamma \cdot \hat{x} I(x, y). \quad (7.3)$$
For a spinor $\psi$ the commutator of covariant derivatives is given by

$$\left[\nabla_\mu, \nabla_\nu\right] \psi = \frac{1}{2} R_{\rho\mu\nu} s^\rho \psi = \pm \rho^2 s_{\mu\nu} \psi, \quad s_{\mu\nu} = \frac{1}{2} \gamma_{[\mu} \gamma_{\nu]}, \quad (7.4)$$

using (1.2). Applying this integrability condition to (7.3) leads to equations for $\alpha$ which are readily solved giving

$$\alpha = \tan \frac{1}{2} \theta, \quad \tanh \frac{1}{2} \theta, \quad (7.5)$$

and hence (7.2) becomes

$$\nabla_\mu I(x, y) = -\rho \cot \frac{1}{2} \theta \frac{1}{4} \gamma_{[\mu} \gamma_{\cdot \hat{x}] I(x, y)}, \quad -\rho \coth \frac{1}{2} \theta \frac{1}{4} \gamma_{[\mu} \gamma_{\cdot \hat{y}] I(x, y)},$$

$$I(x, y) \nabla_\alpha = \rho \frac{1}{2} \theta I(x, y) \frac{1}{4} \gamma_{[\alpha} \gamma_{\cdot \hat{y]}}, \quad \rho \frac{1}{2} \theta I(x, y) \frac{1}{4} \gamma_{[\alpha} \gamma_{\cdot \hat{y}]}.$$

(7.6)

The spinor parallel transport $I(x, y)$ is easily seen to satisfy

$$I(x, y) I(y, x) = 1, \quad (7.7)$$

and also

$$I(x, y) \gamma_\alpha I(y, x) = \gamma_\nu I^\nu_\alpha (x, y),$$

(7.8)

which may be verified by applying a covariant derivative to both sides, using (7.6) with (7.5) as well as (5.15).

It is also useful to define

$$I(x, y) = \gamma_{\cdot \hat{x}} I(x, y) = -I(x, y) \gamma_{\cdot \hat{y}}, \quad (7.9)$$

which plays the role of the inversion matrix on spinors analogous to the inversion tensor (6.24). It is easy to see, using (7.8), that

$$\text{tr}(\gamma_\mu I(x, y) \gamma_\alpha I(y, x)) = 2^d \delta I_{\mu\alpha}(x, y), \quad \text{tr}(\gamma_\mu I(x, y) \gamma_\alpha I(y, x)) = 2^d I_{\mu\alpha}(x, y). \quad (7.10)$$

and from (7.9)

$$I(x, y) I(y, x) = -1. \quad (7.11)$$

From (7.4) and (7.5) we may also easily obtain

$$\nabla_\mu I(x, y) = \rho \cot \frac{1}{2} \theta \frac{1}{4} \gamma_{[\mu} \gamma_{\cdot \hat{x}] I(x, y)}, \quad \rho \coth \frac{1}{2} \theta \frac{1}{4} \gamma_{[\mu} \gamma_{\cdot \hat{y}] I(x, y)},$$

$$I(x, y) \nabla_\alpha = -\rho \cot \frac{1}{2} \theta I(x, y) \frac{1}{4} \gamma_{[\alpha} \gamma_{\cdot \hat{y]}}, \quad -\rho \coth \frac{1}{2} \theta I(x, y) \frac{1}{4} \gamma_{[\alpha} \gamma_{\cdot \hat{y}]}.$$

(7.12)

The importance of $I$ is that it transforms homogeneously under conformal transformations similarly to (6.25) since from (7.12), using (3.11) and (3.11) and the spin matrices defined in (7.4),

$$v^\mu(x) \nabla_\mu I(x, y) + I(x, y) \nabla_\alpha v^\alpha(y) = \frac{1}{2} \omega_{\mu\nu}(x) s^{\mu\nu} I(x, y) - I(x, y) \frac{1}{2} \omega_{\alpha\beta}(y) s^{\alpha\beta}. \quad (7.13)$$
Using the above results it is easy to construct a spinor Green function satisfying
\[ \gamma^\mu \nabla_\mu S(x, y) = \delta^d(x, y), \quad (7.14) \]
which are then given by
\[
S(x, y)_+ = \frac{1}{S_d} \frac{1}{s^\frac{1}{2}(d-1)} I(x, y), \\
S(x, y)_- = \frac{1}{S_d} \left( \frac{1}{s^\frac{1}{2}(d-1)} I(x, y) \pm \frac{1}{s^\frac{1}{2}(d-1)} I(x, y) \right),
\]
where
\[ S_d = \frac{2\pi^{\frac{1}{2}d}}{\Gamma(\frac{1}{2}d)}. \quad (7.16) \]
In the negative curvature case there are two inequivalent Green functions which correspond to two alternative boundary conditions and are appropriate for different spinor representations [25]. The boundary conditions in either case violate chiral symmetry.

Following [26] and [25] we may also construct a Killing bispinor which satisfies the Killing spinor equation at \( x \) and at \( y \). Consistent with (7.4) a Killing spinor \( \epsilon \) satisfies
\[ \nabla_\mu \epsilon^\pm = \pm \frac{1}{2} i \gamma^\mu \epsilon^\pm, \pm \frac{1}{2} \gamma^\mu \epsilon^\pm. \quad (7.17) \]
Such spinors allow the construction of solutions of the Dirac equation in terms of those for a scalar field since if
\[ \psi = \left( (-\gamma \cdot \partial + \lambda^\pm \rho) \phi \right) \epsilon^\pm, \quad (7.18) \]
then
\[ \left( -\nabla^2 + \frac{d-2}{4(d-1)} R \right) \phi = 0 \Rightarrow \gamma \cdot \nabla \psi = 0 \text{ if } \lambda^\pm = \mp \frac{1}{2} i (d-2), \mp \frac{1}{2} (d-2). \quad (7.19) \]
Writing the Killing bispinor in the general form \( S(x, y) = p(\theta) I(x, y) + q(\theta) I(x, y) \) and using (7.6), (7.5) and (7.12) then leads to differential equations for \( p, q \) which are easily solved,
\[ S^\pm = \cos \frac{1}{2} \theta I \pm i \sin \frac{1}{2} \theta I, \cosh \frac{1}{2} \theta I \pm \sinh \frac{1}{2} \theta I. \quad (7.20) \]
These satisfy
\[ \nabla_\mu S^\pm = \pm \rho \frac{1}{2} i \gamma^\mu S^\pm, \pm \rho \frac{1}{2} \gamma^\mu S^\pm, \quad S^\pm \nabla_\alpha = \mp \rho \frac{1}{2} i S^\pm \gamma_\alpha, \mp \rho \frac{1}{2} S^\pm \gamma_\alpha, \quad (7.21) \]
and hence for any solution of (7.17) we may write
\[ \epsilon^\pm(x) = S^\pm(x, y) \epsilon^\pm(y), \quad (7.22) \]

\[ ^6 \text{In [20] the } \pm \text{ is replaced in four dimensions by an arbitrary phase } e^{i \xi \gamma_5}. \]
so that the general solution is determined by a fixed spinor at any arbitrary point \( y \). This gives solutions coinciding with those found with particular coordinate choices \( [27] \). It is of course possible to form Killing vectors from Killing spinors, this is exemplified here by

\[
\nabla_\mu S^-(y,x)\gamma_\mu S^+(x,y) = g_{\mu\nu} \rho S^-(y,x)S^+(x,y), \\
\nabla_{(\mu}S^{+(x,y)\gamma_{\nu})S^+(x,y) = 0.
\]

Furthermore from (7.20)

\[
S^-(y,x)\gamma_\mu S^+(x,y) = I_\mu^\alpha \left\{ \gamma_\alpha \mp \frac{(1 - \cos \theta)}{\cosh \theta - 1} \gamma^\cdot \hat{y} \gamma_\alpha - \frac{i \sin \theta}{\sinh \theta} \hat{y} \gamma_\alpha \right\}, \\
S^+(y,x)\gamma_\mu S^+(x,y) = I_\mu^\alpha \left\{ \cos \theta \gamma_\alpha \pm \frac{(1 - \cos \theta)}{\cosh \theta - 1} \gamma^\cdot \hat{y} \gamma_\alpha - \frac{i \sin \theta}{\sinh \theta} \left[ \frac{1}{2} \gamma_\alpha, \gamma_\beta \right] \hat{y} \gamma_\beta \right\},
\]

which are in accord with the general form exhibited in (6.10).

8. Two Point Functions on Spaces of Constant Curvature

In this section we obtain a general decomposition for the two point function of the energy momentum tensor for arbitrary dimension \( d \) using the geometrical results of the previous section. For simplicity we confine our attention here to the positive curvature sphere \( S^d \) although they are easily extended to the negative curvature case using the usual correspondence between trigonometric and hyperbolic functions. The critical formulae are displayed in appendix A.

It is convenient first to consider \( \Gamma_{\mu\nu}(x,y) \) which is a symmetric tensor at \( x \) and has the form

\[
\Gamma_{\mu\nu}(x,y) = \hat{x}_\mu \hat{x}_\nu A(\theta) + g_{\mu\nu} B(\theta).
\]

Imposing the conservation equation \( \nabla^\mu \Gamma_{\mu\nu}(x,y) = 0 \), using (5.14) easily gives

\[
A' + B' + (d - 1) \cot \theta A = 0.
\]

Imposing also the traceless condition \( g^{\mu\nu}(x)\Gamma_{\mu\nu}(x,y) = 0 \), or \( A + dB = 0 \), leads to a solution

\[
\Gamma_{\mu\nu}(x,y) = K(\sin \theta)^{-d}(d\hat{x}_\mu \hat{x}_\nu - g_{\mu\nu}).
\]

However such a solution is unacceptable due to the singularity at \( \theta = \pi \), although there is no difficulty with the corresponding solution in the hyperbolic case when \( \sin \theta \to \sinh \theta \).
The corresponding results for the two point function of the energy momentum tensor are less trivial. The general form for a bi-tensor $\Gamma_{\mu\nu,\alpha\beta}(x, y)$, symmetric in $\mu\nu$ and $\alpha\beta$, may be reduced to six independent functions of $\theta$

$$
\Gamma_{\mu\nu,\alpha\beta} = \hat{x}_\mu \hat{x}_\nu \hat{y}_\alpha \hat{y}_\beta R + (I_{\mu\alpha} \hat{x}_\nu \hat{y}_\beta + \mu \leftrightarrow \nu, \alpha \leftrightarrow \beta) S + (I_{\mu\alpha} I_{\nu\beta} + I_{\mu\beta} I_{\nu\alpha}) T + (\hat{x}_\mu \hat{x}_\nu g_{\alpha\beta} U_1 + \hat{y}_\alpha \hat{y}_\beta g_{\mu\nu} U_2) + g_{\mu\nu} g_{\alpha\beta} V .
$$

(8.4)

For symmetry, $\Gamma_{\mu\nu,\alpha\beta}(x, y) = \Gamma_{\alpha\beta,\mu\nu}(y, x)$, it is clearly necessary that

$$U_1 = U_2 = U ,
$$

(8.5)

and imposing tracelessness, $g^{\mu\nu} \Gamma_{\mu\nu,\alpha\beta} = \Gamma_{\mu\nu,\alpha\beta} g^{\alpha\beta} = 0$, further requires

$$P_1 = R - 4S + dU = 0 , \quad P_2 = 2T + U + dV = 0 .
$$

(8.6)

In two dimensions the basis used in (8.4) is overcomplete as a consequence of the identity

$$I_{\mu\alpha} \hat{x}_\nu \hat{y}_\beta + \mu \leftrightarrow \nu, \alpha \leftrightarrow \beta + I_{\mu\alpha} I_{\nu\beta} + I_{\mu\beta} I_{\nu\alpha} + 2 (\hat{x}_\mu \hat{x}_\nu g_{\alpha\beta} + \hat{y}_\alpha \hat{y}_\beta g_{\mu\nu} - g_{\mu\nu} g_{\alpha\beta}) = 0 .
$$

(8.7)

In order to impose the conservation equation $\nabla^\mu \Gamma_{\mu\nu,\alpha\beta}(x, y) = 0$ we make use of (5.14) as well as (5.15) giving in terms of the expansion (8.4),

$$R' - 2S' + U'_2 + (d - 1) (\cot \theta R + 2 \tan \frac{1}{2} \theta S) + 2 \cot \frac{1}{2} \theta S - 2 \cosec \theta U_2 = 0 ,
$$

$$S' - T' + d \cot \theta S + d \tan \frac{1}{2} \theta T - \cosec \theta U_2 = 0 ,
$$

$$U'_1 + V' + (d - 1) \cot \theta U_1 - 2 \cosec \theta S + 2 \tan \frac{1}{2} \theta T = 0 .
$$

(8.8)

Requiring the symmetry condition (8.5) of course guarantees that $\Gamma_{\mu\nu,\alpha\beta} \hat{\nabla}^\alpha = 0$ as well. For subsequent use it is convenient to rewrite (8.8). With the definitions (8.6), and assuming (8.3), we have

$$P'_1 + P'_2 = -(d - 1) \cot \theta P_1 .
$$

(8.9)

In addition defining

$$Q = 2T + \frac{d - 1}{d} (R - 4S) ,
$$

(8.10)

then gives (equivalent equations for $d = 4$ were found in [28]),

$$Q' + d \cot \theta Q + \frac{1}{d} P'_1 = -2d \cosec \theta (S - T) ,
$$

(8.11a)

$$(S - T)' + d \cot \theta (S - T) = - \cosec \theta (Q + (d - 2)(d + 1)T + \cosec \theta \frac{1}{d} P_1 .
$$

(8.11b)
Hence, if $P_1 = P_2 = 0$ there are just two independent equations and in this case knowing $Q$ in general determines $R, S, T$.

In two dimensions these relations provide stronger constraints since on the right hand side of (8.11) $4T + R - 4S = 2Q$ (it should be noted that $R$ and $T - S$ are independent of ambiguities that arise from (8.7)). Taking the sum and difference of (8.11a,b) then gives

$$R' + 2 \cot \frac{1}{2} \theta R + P'_1 = 2 \csc \theta P_1, \quad (8.12a)$$
$$R' - 2 \tan \frac{1}{2} \theta (R - 8S + 8T) + P'_1 = -2 \csc \theta P_1. \quad (8.12b)$$

If $P_1 = 0$ there is a straightforward solution

$$R = C \frac{\rho^4}{\sin^4 \frac{1}{2} \theta}, \quad R - 8S + 8T = 0, \quad (8.13)$$

where we have imposed the condition that there are no singularities at $\theta = \pi$.

Explicit forms satisfying (8.8) may be found in terms of the two independent terms displayed for $d = 4$ in (4.30). Initially we consider the contribution corresponding to spin zero intermediate states

$$\Gamma_{\mu, \nu, \alpha, \beta}(x, y) = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 - (d - 1) \rho^2 g_{\mu\nu}) F_0(\theta) (\nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} - (d - 1) \rho^2 g_{\alpha\beta}). \quad (8.14)$$

To evaluate this we first write

$$F_0(\theta) (\nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} - (d - 1) \rho^2 g_{\alpha\beta}) = \hat{y}_\alpha \hat{y}_\beta A + g_{\alpha\beta} B, \quad (8.15)$$

where $A(\theta), B(\theta)$ are given by

$$A = \rho^2 (F_0'' - \cot \theta F_0'), \quad B = -\rho^2 (F_0'' + (d - 2) \cot \theta F_0' + (d - 1) F_0). \quad (8.16)$$

These expressions automatically satisfy (8.2). Inserting (8.15) into (8.14), for general $A, B$, further gives

$$R_0 = \rho^2 (A'' - (\cot \theta + 4 \csc \theta) A' + 4 \csc \theta \cot \frac{1}{2} \theta A),$$
$$S_0 = \rho^2 (- \csc \theta A' + \csc \theta \cot \frac{1}{2} \theta A), \quad T_0 = \rho^2 \csc 2 \theta A,$$
$$U_{0,1} = \rho^2 (B'' - \cot \theta B'), \quad U_{0,2} = -\rho^2 (A'' + (d - 2) \cot \theta A' - (d - 1)(2 \csc^2 \theta - 1) A),$$
$$V_0 = -\rho^2 (B'' + (d - 2) \cot \theta B' + (d - 1) B + 2 \csc^2 \theta A). \quad (8.17)$$

It is easy to verify that these satisfy (8.8) and, given the results for $A, B$ in (8.16), that the symmetry condition (8.5) also holds. With the definitions in (8.6) and also (8.3) we may note that

$$P_1 + P_2 = -\rho^2 (d - 1)(\cot \theta (A' + dB') + A + dB), \quad (8.18)$$
which may easily solved for $A + dB$ up to $A + dB \sim A + dB + k \cos \theta$, reflecting the freedom of adding a solution of the homogeneous equation. Consistency of (8.18) with (8.17) and (8.3) requires (8.9). Furthermore we may obtain from (8.16) $x$.

For (8.14) requires (8.9). Furthermore we may obtain from (8.16) of adding a solution of the homogeneous equation. Consistency of (8.18) with (8.17) and traceless conditions of the Weyl tensor for general $d$.

Besides (8.14) we may use (4.13) to obtain a second solution of the conservation equations by taking, as in (4.30), $F$ where the bi-tensor $\theta$ chosen to make this orthogonal to $\cos \theta$ as this is a zero mode for $\nabla^2 + d \rho^2$. Hence from (8.16) we may find $A, B$ separately and then from (8.17) obtain $R_0, S_0, T_0, U_0, V_0$ satisfying the conservation equations. Since $P_1$ is determined by (8.3) these functions reproduce the same $P_1, P_2$ in (8.1) as obtained from the original $R, S, T, U, V$.

Besides (8.14) we may use (4.13) to obtain a second solution of the conservation equations by taking, as in (4.30),

$$\Gamma_{2,\mu\nu,\alpha\beta}(x, y) = \nabla^\sigma \nabla^\rho (\mathcal{E}^{C}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x, y) F_2(\theta)) \nabla^\gamma \nabla^\delta, \quad (8.20)$$

where for general $d$ the bi-tensor $\mathcal{E}^{C}$ may now be expressed explicitly in terms of the parallel transport matrix $I$ defined in (5.7) by (if $d = 3$ then $\mathcal{E}^{C} = 0$),

$$\mathcal{E}^{C}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} = \frac{1}{12} (I_{\mu\alpha} I_{\nu\beta} I_{\sigma\gamma} I_{\rho\delta} + I_{\mu\delta} I_{\sigma\beta} I_{\rho\alpha} I_{\nu\gamma} - \mu \leftrightarrow \sigma, \nu \leftrightarrow \rho)$$

$$+ \frac{1}{27} \left( I_{\mu\alpha} I_{\nu\beta} I_{\sigma\gamma} I_{\rho\delta} - \mu \leftrightarrow \sigma, \nu \leftrightarrow \rho, \alpha \leftrightarrow \gamma, \delta \leftrightarrow \beta \right)$$

$$- \frac{1}{d - 2} \frac{1}{8} \left( g_{\mu\rho} g_{\alpha\delta} I_{\sigma\gamma} I_{\nu\beta} + g_{\mu\rho} g_{\alpha\delta} I_{\sigma\beta} I_{\nu\gamma} - \mu \leftrightarrow \sigma, \nu \leftrightarrow \rho, \alpha \leftrightarrow \gamma, \delta \leftrightarrow \beta \right)$$

$$+ \frac{1}{(d - 2)(d - 1)} \frac{1}{2} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma} \right) \left( g_{\alpha\delta} g_{\beta\gamma} - g_{\alpha\beta} g_{\gamma\delta} \right). \quad (8.21)$$

For $x = y$, when $I_{\mu\alpha} = g_{\mu\alpha}$, this reduces to the projector onto tensors with the symmetries and traceless conditions of the Weyl tensor for general $d$ given (4.12). To evaluate (8.20) we may first obtain

$$\mathcal{F}_{\mu\sigma\rho\nu,\alpha\beta}(x, y) G(\theta) = \left( \mathcal{E}^{C}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x, y) F_2(\theta) \right) \nabla^\gamma \nabla^\delta, \quad (8.22)$$

where the bi-tensor $\mathcal{F}_{\mu\sigma\rho\nu,\alpha\beta}$, which is a symmetric traceless tensor at $y$, is given by

$$\mathcal{F}_{\mu\sigma\rho\nu,\alpha\beta} = \hat{x}_\mu \hat{x}_\nu \left( I_{\sigma\alpha} I_{\rho\beta} + I_{\sigma\beta} I_{\rho\alpha} - \frac{2}{d} g_{\sigma\rho} g_{\alpha\beta} \right) - \mu \leftrightarrow \sigma, \nu \leftrightarrow \rho$$

$$- \frac{1}{d - 2} \left( g_{\mu\nu} X_{\sigma\rho,\alpha\beta} - \mu \leftrightarrow \sigma, \nu \leftrightarrow \rho \right)$$

$$+ \frac{4}{(d - 2)(d - 1)d} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma} \right) \left( d \hat{y}_\alpha \hat{y}_\beta - g_{\alpha\beta} \right), \quad (8.23)$$

$$X_{\sigma\rho,\alpha\beta} = I_{\sigma\alpha} I_{\rho\beta} + I_{\sigma\beta} I_{\rho\alpha} + \left( I_{\alpha\beta} \hat{x}_\rho \hat{y}_\gamma + \sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta \right) + \frac{2}{d} \left( 2 \hat{x}_\sigma \hat{x}_\rho - g_{\sigma\rho} \right) g_{\alpha\beta},$$

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and
\[ 8G = \rho^2 (F_2'' - \cot \theta F_2' - 2(d-1) \tan \frac{1}{2} \theta F_2 + (d-1)(d-2) \tan^2 \frac{1}{2} \theta F_2) . \] (8.24)

Using (8.22) in (8.20) gives an expression which automatically satisfies the symmetry and traceless conditions (8.5) and (8.6) for any \( G \) and corresponding to the general form (8.4) we have solution of (8.8) given in terms of

\[ R_2 = \rho^2 \frac{d^2 - 3}{d^2 - 1} \left\{ G'' + (2d - 3) \cot \theta G' - 2(d+1) \cosec \theta G' \right. \\
+ \left. \left( (d-1)(d-2) \cot^2 \theta - 2(d+1)(d-2) \cot \theta \cosec \theta + (d^2 + d + 2) \cosec^2 \theta \right) G \right\} , \]

\[ S_2 = \rho^2 \frac{d^2 - 3}{d^2 - 1} \left\{ G'' + (2d - 3) \cot \theta G' - \frac{(d+1)(d-2)}{d-1} \cosec \theta G' \right. \\
+ \left. \left( (d-1)(d-2) \cot^2 \theta - (d-2) \frac{d+1}{d-1} \cot \theta \cosec \theta - \frac{d^2 - d + 2}{d-1} \cosec^2 \theta \right) G \right\} , \]

\[ T_2 = \rho^2 \frac{d^2 - 3}{d^2 - 1} \left\{ G'' + (2d - 3) \cot \theta G' \right. \\
+ \left. \left( (d-1)(d-2) \cot^2 \theta - \frac{d^2 - d + 2}{d-1} \cosec^2 \theta \right) G \right\} . \] (8.25)

If the traceless conditions (8.6) are satisfied then the expressions given by (8.25) are generally valid. This is easily seen since, with the definition (8.10) in terms of \( R_2, S_2, T_2, \) we have

\[ Q = \rho^2 2(d-3) d \frac{d+1}{d-1} \cosec^2 \theta G , \] (8.26)

and (8.11a, b), with \( P_1 = 0, \) are equivalent to (8.25). Conversely for arbitrary \( R, S, \ldots \) satisfying (8.8) we may first solve (8.18) and (8.19) to obtain \( R_0, S_0, \ldots \) and then define \( R_2 = R - R_0, S_2 = S - S_0, \ldots \) which must necessarily satisfy (8.8) as well as \( P_1 = P_2 = 0. \) Hence defining \( Q \) in (8.10) in terms of these \( R_2, S_2, \ldots \) ensures through (8.26) that they may all be expressed as in (8.25). Furthermore \( F_2 \) may be found by solving (8.24) in conjunction with (8.26) which may alternatively be expressed as

\[ \rho^4 \frac{d^2}{dw^2} (w^{d-1} F_2) = \frac{16(d-1)}{(d-3)(d+1)} w^{d-1} Q , \quad w = \frac{1}{2}(1 + \cos \theta) . \] (8.27)

Assuming \( Q \) and \( F_2 \) are non singular at \( w = 0 \) ensures a unique solution. In consequence the general bi-tensor \( \Gamma_{\mu\nu,\alpha\beta} \) as given by (8.4) subject to the conservation equation \( \nabla^\mu \Gamma_{\mu\nu,\alpha\beta} = 0 \) may be decomposed into spin 0 and spin 2 pieces,

\[ \Gamma_{\mu\nu,\alpha\beta}(x, y) = \Gamma_{0,\mu\nu,\alpha\beta}(x, y) + \Gamma_{2,\mu\nu,\alpha\beta}(x, y) , \] (8.28)
where the two independent expressions are given by \((8.14)\) and \((8.20)\).

As an illustration which is relevant for the case of conformally invariant theories, for which \(P_1 = P_2 = 0\), we may consider

\[
\frac{1}{4} R = S = 2T = Q = C \frac{1}{s^d} = C \left(\frac{1}{4} \rho^2\right)^d (1 - w)^{-d},
\]  

(8.29)

when \((8.27)\) may be integrated to give

\[
F_2 = C \left(\frac{1}{4} \rho^2\right)^{d-2} \frac{d - 1}{(d - 3)d^2(d + 1)^2} w^2 F(d, d; d + 2; w).
\]  

(8.30)

9. Ambiguities in Spin 0 - Spin 2 Decomposition

In the previous section we showed how the energy momentum tensor can be decomposed into a spin 2 traceless part and a spin 0 contribution determined by the trace, as in \((8.28)\) with \((8.20)\) and \((8.4)\). However the spin 0 part may also result in a traceless expression for \(\langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle\) if \(F_0\) is proportional to the Green function \(K_0\) for \(-\nabla^2 - \frac{1}{d-1} R\),

\[
F_0(\theta) = C_0 \rho^{d-2} K_0(\theta),
\]  

(9.1)

This Green function is constructed in appendix B. In the positive curvature case, since \(d\rho^2\) is an eigenvalue of \(-\nabla^2\), this satisfies

\[
(-\nabla^2 - d\rho^2)K_0(\theta)_+ = S_d \delta^d(x, y) - k_d \rho^d \cos \theta.
\]  

(9.2)

Determining \(A, B\) from \((8.15)\) and \((8.16)\) ensures they satisfy \((8.2)\) and from \((9.2)\)

\[
A + dB = -(d - 1)k_d \cos \theta.
\]  

(9.3)

Although the traceless condition \(A + dB = 0\) is not satisfied the corresponding expression for \(\Gamma_{0,\mu;\alpha;\beta}\) is since \((\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu}(\nabla^2 + \rho^2))\cos \theta = 0\). For general \(d\) \(A, B\) are not expressible in terms of elementary functions but for \(d = 2, 4\), either by solving \((9.3)\) with \(k_2 = 3/2, k_4 = 15/4\) or by using the explicit form for \(K_0\) given in appendix B, we have

\[
A = C_0 \left(\frac{2}{s} + \frac{1}{2} \rho^2 \cos \theta\right), \quad d = 2,
\]

\[
A = C_0 \left(\frac{2}{s^2} (2 + \rho^2 s) + \frac{3}{4} \rho^4 \cos \theta\right), \quad d = 4.
\]  

(9.4)

Applying \((8.17)\) for \(d = 2\) then gives, if we make use of \((8.7)\) to set \(U_0 = 0\),

\[
\frac{1}{8} R_0 = \frac{1}{2} S_0 = T_0 = -V_0 = \frac{6C_0}{s^2}.
\]  

(9.5)
Assuming (3.22) requires taking $C_0 = \frac{1}{6}c$ and, using (5.24), gives the following simple form

$$4\pi^2\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle|_{\beta^i=0} = \frac{c}{s^2} \left(I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta}\right). \quad (9.6)$$

In two dimensions this the unique expression, as expected from the solution obtained in (8.13), since the contribution corresponding to $\Gamma_{2,\mu\nu,\alpha\beta}$ is absent. When $d = 4$ we may also obtain

$$T_0 = C_0 \frac{\rho^2}{s^3} \left(4 + 3\rho^2 s + \frac{3}{2}(\rho^2 s)^2\right), \quad S_0 = C_0 \frac{\rho^2}{s^3} \left(24 + 8\rho^2 s + \frac{3}{2}(\rho^2 s)^2\right), \quad R_0 = C_0 \frac{2\rho^2}{s^3} \left(96 + 12\rho^2 s + (\rho^2 s)^2\right).$$

(9.7)

In this case if we use (8.10) we find

$$Q_0 = 80C_0 \frac{\rho^2}{s^3},$$

(9.8)

and inserting in (8.27) gives

$$\rho^2 \frac{d^2}{dw^2} \left(w^3F_2\right) = 3C_0 \frac{w^3}{(1-w)^3}. \quad (9.9)$$

In consequence the two point function may be equivalently be expressed in terms of $F_2$, i.e. for $F_2$ determined by (9.9) inserted in (8.24) and hence (8.27) we have $R_2 = R_0$ etc. This corresponds to the existence of $K_2$ in (1.31) and by solving (9.9) we may find

$$K_2 = \frac{\rho^2}{8w^3} \left(\frac{w}{1-w} + 11w - w^2 + 6(2-w)\ln(1-w)\right). \quad (9.10)$$

For the negative curvature case there are no complications due to zero modes as reflected in (9.2). In this case $A, B$ satisfy the traceless condition $A + dB = 0$ and the dependence on $\theta$ is simple for any $d$,

$$A = C_0 \rho^{2d-2} \frac{d}{(\sinh \theta)^d}. \quad (9.11)$$

For $d = 2$ the results, setting $U_0 = 0$ as before, involve essentially two independent terms

$$\frac{1}{4}R_0 = S_0 = 12C_0 \frac{1}{s^2}, \quad T_0 = -V_0 = 6C_0 \left(\frac{1}{s^2} + \frac{1}{s^2}\right). \quad (9.12)$$

Inserting these expressions into (3.22) with $C_0$ determined in terms of $c$ as above now gives

$$4\pi^2\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle - |_{\beta^i=0} = \frac{c}{s^2} \left(I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta}\right) + \frac{c}{s^2} \left(I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta}\right). \quad (9.13)$$
From (9.11) for any \( d \) we may find using (8.10)

\[
Q_0 = C_0 \rho^{2d} d(d + 1) \frac{d + (d - 1) \sinh^2 \theta}{(\sinh \theta)^{d+2}},
\]

(9.14)

and for \( d = 4 \) the complete expression for \( \Gamma_{2,\mu\nu,\alpha\beta} \) is given by

\[
T_0 = 4C_0 \rho^8 \frac{1}{\sinh^6 \theta}, \quad S_0 = 4C_0 \rho^8 \frac{5 \cosh \theta + 1}{\sinh^6 \theta}, \quad R_0 = 16C_0 \rho^8 \frac{5 \cosh^2 \theta + 5 \cosh \theta + 2}{\sinh^6 \theta}.
\]

(9.15)

As in the positive curvature case we may solve the equivalent equation to (8.27), given in (A.10), to determine a corresponding \( F_2 \) which gives identical forms for \( R, S, T \) with \( U, V \) determined by (8.6). In our later discussion of the spectral representation we show that the resulting \( F_2 \) cannot be accommodated by imposing the unitarity bound on possible spin 2 intermediate states.

10. Free Fields

It is important in order to understand what results may be found from an analysis of the two point function on spaces of constant curvature to calculate the possible forms at renormalisation group fixed points. To this end we first consider the results for conformally invariant free field theories, free scalar and fermion theories in general dimension \( d \) and abelian gauge fields in dimension four. In each case we initially consider the theory on a sphere, with \( R > 0 \), avoiding boundary conditions which are necessary for the hyperbolic case when \( R < 0 \).

The free conformally coupled scalar field \( \phi \) satisfies, on a space of constant curvature where (1.2) holds, \( \Delta \phi \equiv (-\nabla^2 \pm \frac{1}{4} d(d - 2) \rho^2) \phi = 0 \). The associated energy momentum tensor may be written in various equivalent forms but here we take

\[
T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} \frac{1}{d - 1} \left( (d - 2) \nabla_\mu \nabla_\nu + g_{\mu\nu} \nabla^2 \mp (d - 1)(d - 2) \rho^2 g_{\mu\nu} \right) \phi^2,
\]

(10.1)

which is conserved and traceless on the equations of motion. Correlation functions for the energy momentum tensor and other composite fields are determined in terms of the basic scalar field two point function which for the sphere may be taken, with \( s \) the chordal distance defined in (5.20) and \( S_d \) given in (7.16), as

\[
\langle \phi(x)\phi(y) \rangle_+ = \frac{1}{(d - 2)S_d} \frac{1}{s^{\frac{d}{2} - 1}}.
\]

(10.2)

\[ g^{\mu\nu}T_{\mu\nu} = \phi \Delta \phi, \quad \nabla^\mu T_{\mu\nu} = \frac{1}{2} \partial_\nu \Delta \phi - \frac{1}{2} \Delta \phi \partial_\nu \phi. \]
It is first useful to verify
\[ \langle T_{\mu\nu}(x)\phi^2(y) \rangle_+ = 0, \quad (10.3) \]
and then we may obtain for the two point function of the energy momentum tensor \( S^2_d \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle_+ \) the form given in (8.4) with
\[ 2T_+ = S_+ = \frac{1}{d} R_+ = -\frac{1}{d} V_- = \frac{d}{d-1} \frac{1}{s^d}, \quad U_+ = 0. \quad (10.4) \]
The actual result may be written simply, using (6.24), in the form
\[ S^2_d \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle_{+,\text{conformal}} = C_T \frac{1}{s^d} \left( \frac{1}{s^{d-1}} - \frac{1}{s^{\frac{d}{2}+1}} \right), \quad (10.5) \]
where
\[ C_{T,\phi} = \frac{d}{d-1}. \quad (10.6) \]
Comparing with (9.4) for \( d = 2 \) gives \( c = 1 \) as expected. the expression (10.5) is just what would be expected by conformal transformation from flat space.

When the scalar fields are considered on the negative curvature hyperboloid \( H^d \) boundary conditions are necessary. For Dirichlet boundary conditions the basic two point function of the conformally coupled scalar field becomes
\[ \langle \phi(x)\phi(y) \rangle_- = \frac{1}{(d-2)S_d} \left( \frac{1}{s^{d-1}} - \frac{1}{s^{\frac{d}{2}+1}} \right). \quad (10.7) \]
Unlike (10.2) this is well defined as \( d \to 2 \). In this case, in contrast to (10.3), we have
\[ S^2_d \langle T_{\mu\nu}(x)\phi^2(y) \rangle_- = \frac{4}{d-1} \frac{1}{\rho^2(s^d)} (d\hat{x}_\mu\hat{x}_\nu - g_{\mu\nu}). \quad (10.8) \]
Using this result then we may find, after some calculation, the corresponding expressions to (10.4) for \( \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle_- \) in this case which are given by
\[ (d-1)T_- = \frac{1}{2} d \left( \frac{1}{s^d} + \frac{1}{s^d} - \frac{2}{(s^d)^{\frac{d}{2}}} \right) - d \frac{d-2}{d-1} \frac{4}{\rho^4(s^d)^{\frac{d}{2}+1}}, \]
\[ (d-1)S_- = d \left( \frac{1}{s^d} - \frac{1}{(s^d)^{\frac{d}{2}}} \right) - d \frac{d-2}{d-1} ((d+1)\cosh \theta + 1) \frac{4}{\rho^4(s^d)^{\frac{d}{2}+1}}, \]
\[ (d-1)R_- = d \left( \frac{4}{s^d} + (d^2 - 4) \frac{1}{(s^d)^{\frac{d}{2}}} \right) - d \frac{d-2}{d-1} (d(d+1)\cosh^2 \theta + 4(d+1)\cosh \theta + d + 4) \frac{4}{\rho^4(s^d)^{\frac{d}{2}+1}}, \]
\[ (d-1)U_- = -d^2 \frac{1}{(s^d)^{\frac{d}{2}}} + d \frac{d-2}{d-1} ((d+1)\cosh^2 \theta + 1) \frac{4}{\rho^4(s^d)^{\frac{d}{2}+1}}, \]
\[ (d-1)V_- = -\left( \frac{1}{s^d} + \frac{1}{s^d} - (d+2) \frac{1}{(s^d)^{\frac{d}{2}}} \right) - d \frac{d-2}{d-1} ((d+1)\cosh^2 \theta - 1) \frac{4}{\rho^4(s^d)^{\frac{d}{2}+1}}. \quad (10.9) \]
These results satisfy the traceless conditions (8.6), as expected, and further for \( d = 2 \) the terms involving \( s \bar{s} \) disappear as a consequence of (8.7) and the result is compatible with (9.13) for \( c = 1 \). The expression written in (10.9) is such that the different terms separately obey the conservation equations. The resulting expression is no longer in the simple form given by (10.5) in the positive curvature case but if the terms involving \( s \bar{s} \), which vanish if \( d = 2 \), are dropped we have

\[
S_d^2 \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_{\text{conformal}} = C_T \left\{ \frac{1}{4} d g_{\mu\nu} g_{\alpha\beta} \right\},
\]

(10.10)

with the same result for \( C_T \) as in (10.6). The remaining terms present in (10.9) can be understood in terms of the operator product expansion but it is clear that the form given in (10.10) cannot be the unique expression for conformal field theories.

For massless vector fields, satisfying the free equations \( \nabla^{\mu} F_{\mu\nu} + \partial_{\nu} \nabla^{\mu} A_\mu / \xi = 0 \) with \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) and \( \xi \) a gauge fixing parameter, we restrict to \( d = 4 \) when the theory is conformally invariant on flat space. Neglecting terms which are irrelevant in gauge invariant correlation functions the energy momentum tensor is

\[
T_{\mu\nu} = F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}.
\]

(10.11)

From the two point functions of the gauge field \( A_\mu \), which have been obtained in [22] and more recently in [29] and are discussed in appendix B, the two point correlation functions of the field strength \( F_{\mu\nu} \) are simple

\[
\langle F_{\mu\nu}(x) F_{\alpha\beta}(y) \rangle_+ = \frac{1}{4 \pi^2 s^2} (\mathcal{I}_{\mu\alpha} \mathcal{I}_{\nu\beta} - \mathcal{I}_{\mu\beta} \mathcal{I}_{\nu\alpha}) ,
\]

\[
\langle F_{\mu\nu}(x) F_{\alpha\beta}(y) \rangle_- = \frac{1}{4 \pi^2 s^2} (\mathcal{I}_{\mu\alpha} \mathcal{I}_{\nu\beta} - \mathcal{I}_{\mu\beta} \mathcal{I}_{\nu\alpha}) + \frac{1}{4 \pi^2 s^2} (I_{\mu\alpha} I_{\nu\beta} - I_{\mu\beta} I_{\nu\alpha}) .
\]

(10.12)

With the explicit form (10.11) it is straightforward to calculate in this case that the energy momentum tensor two point function is just as in (10.5) or (10.10) for \( d = 4 \) with

\[
C_{T,V} = 8 .
\]

(10.13)

We may also easily show that

\[
\langle T_{\mu\nu}(x) F_{\alpha\beta}^{\alpha\beta}(y) \rangle_+ = 0, \quad (2\pi^2)^2 \langle T_{\mu\nu}(x) F_{\alpha\beta}^{\alpha\beta}(y) \rangle_- = \frac{32}{s^2} (4 \hat{x}_\mu \hat{x}_\nu - g_{\mu\nu}) .
\]

(10.14)

For massless spinor fields, satisfying \( \gamma^{\mu} \nabla_\mu \psi = 0 \), the energy momentum tensor may be taken as

\[
T_{\mu\nu} = \bar{\psi} \gamma(\mu) \nabla_\nu \psi ,
\]

(10.15)
which is conserved and traceless for any \( d \). The formalism for spinors on constant curvature spaces was described in section 7 and the basic two point functions were obtained in (7.15) in terms of the inversion bispinor \( I(x, y) \) and parallel transport bispinor \( I(x, y) \) giving

\[
\langle \psi(x) \bar{\psi}(y) \rangle_+ = \frac{1}{S_d} \frac{1}{s^\frac{d}{2}(d-1)} I, \\
\langle \psi(x) \bar{\psi}(y) \rangle_- = \frac{1}{S_d} \left( \frac{1}{s^\frac{d}{2}(d-1)} I \pm \frac{1}{s^\frac{d}{2}(d-1)} I \right).
\]

From this we may obtain

\[
\langle \nabla_\mu \psi(x) \bar{\psi}(y) \nabla_\alpha \rangle_+ = \frac{1}{S_d} \frac{1}{s^\frac{d}{2}(d+1)} \left\{ \gamma_\mu I \gamma_\alpha + dI_\mu \alpha I + \cos \frac{2}{1} \theta (\gamma_\mu \gamma \cdot \hat{x} - d \hat{x}_\mu) I (\gamma \cdot \hat{y} \gamma_\alpha - d \hat{y}_\alpha) \right\}, \\
\langle \nabla_\mu \psi(x) \bar{\psi}(y) \nabla_\alpha \rangle_- = \frac{1}{S_d} \frac{1}{s^\frac{d}{2}(d+1)} \left\{ \gamma_\mu I \gamma_\alpha + dI_\mu \alpha I + \cosh \frac{2}{1} \theta (\gamma_\mu \gamma \cdot \hat{x} - d \hat{x}_\mu) I (\gamma \cdot \hat{y} \gamma_\alpha - d \hat{y}_\alpha) \right\},
\]

and then straightforwardly calculate expressions for \( \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_\pm \) which are just as in (10.5) or (10.10) with

\[
C_{T,\psi} = 2^\frac{d}{2} d - 1.
\]

It is also useful to note that

\[
\langle T_{\mu\nu}(x) \bar{\psi}(y) \rangle_+ = 0, \quad S_d^2 \langle T_{\mu\nu}(x) \psi(y) \rangle_- = \pm 2^\frac{d}{2} + 1 \frac{1}{\rho(s^s)^{\frac{d}{2}}} (d \hat{x}_\mu \hat{x}_\nu - g_{\mu\nu}).
\]

11. Operator Product Expansion

The singular behaviour of the above results for two point functions for free field theories may be understood in terms of the operator product expansion. This may also be applied for non trivial interacting theories to determine the various possible contributions to two, and higher, point functions. We restrict here to the situation of theories at a fixed point, so that the flat space theory is conformally invariant, and further consider just the operator product expansions involving the energy momentum tensor and a scalar field \( \mathcal{O} \) with an arbitrary dimension \( \eta \). On curved space the leading term in any operator product expansion coefficient is determined by the corresponding results on flat space [30,31]. Using
the notation of section 5 we consider first the operator product expansion of $T_{\mu\nu}$ and $O$ which may be written as

$$T_{\mu\nu}(x)O(y) \sim -\frac{\eta}{d-1} \frac{1}{S_d} \frac{1}{s^{\frac{d}{2}}} (d\hat{x}_\mu \hat{x}_\nu - g_{\mu\nu})O(y). \quad (11.1)$$

The overall coefficient is determined through Ward identities [30,31]. Secondly the contribution of the scalar $O$ to the operator product expansion of two energy momentum tensors has the form

$$T_{\mu\nu}(x)T_{\alpha\beta}(y) \sim \frac{1}{s^{d-\frac{d}{2}}} \left( a \hat{x}_\mu \hat{y}_\alpha \hat{y}_\beta + b(\hat{x}_\mu \hat{y}_\alpha I_{\nu\beta} + \mu \leftrightarrow \nu, \alpha \leftrightarrow \beta) \right. \\
\left. + c(I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha}) - \text{traces}(\mu\nu, \alpha\beta) \right)O(y), \quad (11.2)$$

where $a, b, c$ satisfy the relations, derived essentially by imposing the conservation equations on the expansion coefficient,

$$(a - 4b)(1 - \frac{1}{d}(d - \eta)(d - 1)) + d\eta b = 0, \quad a - 4b - d(d - \eta)b + d(2d - \eta)c = 0. \quad (11.3)$$

In this case there remains a single undetermined scale reflecting the arbitrariness in the normalisation of $O$.

The operator product (11.1) is in accord with the the leading singular behaviour as $s \to 0$ of the expressions (10.8), (10.14) and (10.19) in the hyperbolic case taking then

$$\langle \phi^2 \rangle_- = -\frac{1}{(d-2)S_d} (\frac{1}{2\rho})^{d-2}, \quad \langle F_{\alpha\beta}F^{\alpha\beta} \rangle_- = -\frac{3}{4\pi^2} \rho^4, \quad \langle \bar{\psi}\psi \rangle_- = \mp \frac{2^d}{S_d} (\frac{1}{4\rho})^{d-1}. \quad (11.4)$$

These results for the one-point functions follow directly from (10.7), (10.12) and (10.16) when the coincident limit is regularised by dropping the singular contributions in $s$ (which is consistent with the vanishing of these one-point functions in the spherical case). As usual with dimensional regularisation this prescription is essentially unambiguous for $d \neq \text{integer}$, but for the fermion case the result for $\langle \bar{\psi}\psi \rangle$ is unique and well defined even at $d = 4$ since it is non zero only if chirality is violated so that the first term in (10.16) cannot contribute. The form of the two point function for the energy momentum tensor in the case of scalar fields given by (10.9) may also be understood by using the operator product expansion in (11.2). For a free scalar theory taking $O \to \phi^2$, which has dimension $d - 2$, we have

$$c = \frac{d(d - 2)^2}{4(d - 1)^2} \frac{1}{S_d}, \quad b = (d + 2)c, \quad a = (d + 2)(d + 4)c, \quad (11.5)$$

which satisfy (11.3). With the result (11.4) for $\langle \phi^2 \rangle$ (11.2) and (11.5) generate the terms in (10.9) $\propto s^{-\frac{d}{2}}$ as $s \to 0$. There are no corresponding contributions in the case of free
vectors or fermions from $\langle F_{\alpha\beta}F^{\alpha\beta} \rangle$ or $\langle \bar{\psi}\psi \rangle$ since these operators do not appear in the operator product expansion of two energy momentum tensors, for $\bar{\psi}\psi$ by chirality and for $F_{\alpha\beta}F^{\alpha\beta}$ for free vector theories in $d = 4$ by direct calculation \[31\]. Although the overall normalisation of $a, b, c$ in (11.3) is arbitrary such freedom is absent from the associated contributions to $\langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle$ since it is cancelled by a corresponding factor in $\langle \mathcal{O} \rangle$.

Although the above results demonstrate the relevance of the operator product expansion to understanding the form of the two point function on spaces of constant curvature there are also areas where its role is less transparent. It was our initial hope, based on the fact that the operator product expansion of two energy momentum tensors always contains the energy momentum tensor itself and that its one point function, which has the form shown in (2.25), is in general non zero, that expression for the two point function for the energy momentum tensor on such spaces would involve its coefficient $C$. In particular in four dimensions at a fixed point, as a consequence of (4.35), this would require a dependence on the parameter $a$, which is defined initially through the energy momentum tensor trace on general curved backgrounds, in some evident fashion. Such a result would be significant since $a$ is the favourite candidate for a four dimensional generalisation of the $c$-theorem \[4\] (there is now strong theoretical support for this conjecture in supersymmetric theories \[15\]) and it could be hoped that a proof of the $c$-theorem in four dimensions might be found in terms of the two point functions of the energy momentum tensor on curved space following a similar approach to the original discussion of Zamolodchikov \[2\]. Nevertheless, despite the arguments based on the operator product expansion for the appearance of the parameter $a$ in the two point function at non coincident points, this expectation is not supported by the explicit free field expressions obtained in section 10. This is most apparent from the results for fermions and vectors which are have the identical form (10.10) when $d = 4$ with the coefficient $C_T$ determined by $c$ in the trace anomaly (4.1). On the other hand $a$, which is also initially defined in terms of (11.1), has no direct relation to $c$ as demonstrated by results for free fields.

It appears therefore that there may be some unresolved questions when applying the operator product expansion on spaces with non zero curvature. For a conformal field theory on flat space all one point functions vanish (except trivially for the identity operator). For quasi-primary operators, which transform homogeneously under conformal transformations, two point functions, which are non zero only if both operators have the same spin and scale dimension, are unique up to an overall constant and three point functions are also given by a finite number of linearly independent forms. Together these determine completely the operator product expansions for any pair of quasi-primary operators in terms of all quasi-primary operators and their descendents or derivatives for which the associated three point functions are non zero and also the identity operator if their two
point function is non zero. If the operator product expansion is extended to a curved space background the leading singular contributions to the coefficient functions in the expansion are of course determined by those for flat space, but there may now be less singular contributions involving the curvature which can naturally appear associated with derivative terms. The contribution of the identity operator to the product of two operators may also be expected in general now to be a function of the separation rather than just a power determined by the scale dimension of the operators, since a non zero curvature introduces a scale. Moreover such a function cannot just be identified with the associated two point function if other operators present in the expansion have a non zero expectation value on curved space. In the particular case of the contribution of the energy momentum tensor to the operator product expansion of two energy momentum tensors the coefficient functions found from flat space calculations correspond to purely traceless energy momentum tensors \[31,32\] so that the straightforward generalisation to curved space does not generate terms associated with its expected trace anomaly. The necessity of a non zero trace for \(\langle T_{\mu\nu} \rangle\) on a general curved space for a flat space conformal theory arises in \(d = 2, 4, \ldots\) dimensions as a consequence of the requirement that the regularisation procedure necessary to determine \(\langle T_{\mu\nu} \rangle\) should maintain the conservation equation (2.5). [33]. The trace is then unambiguous, independent of boundary conditions and non zero. However, on a homogeneous space of constant curvature, the conservation equation is trivially satisfied as a consequence of (2.25). Hence, as remarked in [26], there is no calculation uniquely determining the anomaly, or the coefficient \(C\) in (2.25), which is entirely intrinsic to the theory on the constant curvature space (although calculations adapted to such spaces are described in [34,26,35]).

When \(d = 4\) the contribution of the energy momentum tensor, of dimension 4, to the operator product expansion of two energy momentum tensors will in general mix with terms quadratic in the Riemann tensor and such terms will contribute to the two point function. This may be illustrated by the associated example of the two point function of the energy momentum tensor and the scalar field \(\mathcal{O}\). (4.36) for the hyperbolic case gives

\[
(2\pi^2)^2 \langle T_{\mu\nu}(x)\mathcal{O}_i(y) \rangle \bigg|_{\beta^i = 0} = -\frac{1}{3(\frac{1}{4} \rho^2 s s')^2} (4\hat{x}_\mu \hat{x}_\nu - g_{\mu\nu}) \Gamma^j_i \left( \frac{1}{36} \hat{U}_j R^2 + 2\pi^2 \langle \mathcal{O}_j \rangle \right). \tag{11.6}
\]

Since \(\Gamma^j_i\) is the dimension matrix for the fields \(\mathcal{O}_j\) this is exactly in accord with (11.1) but the presence of a non zero \(\hat{U}_j\) in general demonstrates such a mixing with curvature terms.

12. Unitary Representations of \(O(d - 1, 2)\) for Spin One

In a quantum field theory with a unique vacuum the correlation functions may be expressed in terms of the vacuum state expectation values of products of local field op-
For unitarity the operators are assumed to act on a Hilbert space with positive norm. A further essential requirement is that there is a hermitian Hamiltonian operator $\hat{H}$, which annihilates the vacuum but otherwise with positive spectrum, generating translations in a coordinate $\tau$. This ensures that the theory may be analytically continued, by letting $\tau \to i\tau$, from an underlying $d$-dimensional with positive metric to define a unitary quantum field theory on the associated space with a metric with signature $(-,+,...)$. We here consider field theories on the homogeneous constant curvature spaces $S^d$ and $H^d$. In the positive curvature case the analytic continuation of $S^d$, when $\tau$ is periodic, leads to de Sitter space $dS^d$ and the correlation functions of the resulting quantum field theory are interpreted as corresponding to a thermal bath of non zero temperature \[33\], and so are not given by vacuum matrix elements of field operators.\[34\] In consequence we restrict our attention here to the negative curvature case when the homogeneous space $H^d$ is continued to $AdS^d$. Hence we consider states which form unitary representations of the isometry group $O(d-1,2)$ where the vacuum $|0\rangle$ is a unique state forming a trivial singlet representation. The general unitary positive energy representations are constructed from lowest weight states which form a basis for a representation space for $O(d-1)$. The representations are used subsequently to obtain spectral representations for the two point correlation functions. For scalar fields the unitary representations are formed from a spin 0 or singlet lowest weight state and the spectral representations are well known \[19,6\]. However the extension to fields with spin involves further complications so in this section, and appendix C, we construct in detail the representation for a spin one lowest weight state.

In order to identify convenient coordinates, with a privileged choice for $\tau$, for $H^d$ we consider first its embedding in $\mathbb{R}^{d+1}$ as the hypersurface given by the constraint 
$$\rho^2 g_{ab} \eta^a \eta^b = -1$$
where with $\eta^a = (\eta^0, \eta^1, \eta^d)$ we have $g_{00} = -g_{dd} = 1, g_{ij} = \delta_{ij}.$

Global coordinates $x = (\tau, r, \xi_i)$ for $H^d$ are then given by
$$\rho \eta^0 = \sinh \tau \sec r, \rho \eta^i = \tan r \xi_i, \rho \eta^d = -\cosh \tau \sec r,$$
where $\xi_i \xi_i = 1, \xi_i \in S^{d-2}$ and $\tau, r$ are in the ranges $-\infty < \tau < \infty, 0 \leq r \leq \frac{\pi}{2}$. On continuing $\tau \to i\tau$ these coordinates then cover the simply connected covering space for $AdS_d$. For two points, represented by $\eta^a, \eta'^a$, 
$$\cosh \theta = -\rho^2 g_{ab} \eta^a \eta'^b = \cosh(\tau - \tau') \sec r \sec r' - \tan r \tan r' \xi \cdot \xi'.$$
The associated metric $ds^2 = g_{ab} d\eta^a d\eta^b$ becomes

$$\rho^2 ds^2 = \sec^2 r (d\tau^2 + dr^2 + \sin^2 r d\eta^2_{S^{d-2}}).$$

The isometry group $O(d,1)$ generators $L_{ab} = \eta_a \partial_b - \eta_b \partial_a$ obey the algebra

$$[L_{ab}, L_{cd}] = g_{bc} L_{ad} - g_{ac} L_{bd} - g_{bd} L_{ac} + g_{ad} L_{bc}.$$  \tag{12.2}

Writing

$$H = L_{0d}, \quad L_{\pm i} = L_{0i} \mp L_{di},$$ \tag{12.3}

\[8\] For a relevant discussion see \[36\].
the algebra (12.2) decomposes as

\[
[H, L_{\pm i}] = \pm L_{\pm i}, \quad [L_{-i}, L_{+j}] = 2\delta_{ij}H - 2L_{ij}, \quad [L_{+i}, L_{+j}] = [L_{-i}, L_{-j}] = 0,
\]

\[
[L_{ij}, H] = 0, \quad [L_{ij}, L_{\pm k}] = \delta_{jk}L_{\pm i} - \delta_{ik}L_{\pm j},
\]

with \(L_{ij}\) the generators of \(O(d-1)\) obeying an algebra of the same form as (12.2) with \(g_{ab} \to \delta_{ij}\). With these coordinates we have

\[
H = -\frac{\partial}{\partial \tau}, \quad L_{\pm i} = -e^{\mp r}(\sin r \frac{\partial}{\partial \tau} \pm \cos r \frac{\partial}{\partial r}) \pm \text{cosec} r D_i, \quad L_{ij} = \xi_i D_j - \xi_j D_i,
\]

where \(D_i\) is the tangential derivative

\[
[D_i, D_j] = L_{ij}, \quad \xi_i D_i = 0, \quad D_i \xi_j = \delta_{ij} - \xi_i \xi_j,
\]

so that we may write \(D_j = \xi_i L_{ij}\) and also \([L_{ij}, D_k] = \delta_{jk}D_i - \delta_{ik}D_j\).

In a quantum field theory which is unitary on continuation to \(AdS_d\) the isometry group is represented by operator generators \(\hat{L}_{ab}\) obeying (12.2) which on decomposing as in (12.3) obey the hermeticity properties

\[
\hat{H}^\dagger = \hat{H}, \quad \hat{L}_{-i}^\dagger = \hat{L}_{+i}, \quad \hat{L}_{ij}^\dagger = -\hat{L}_{ij}.
\]

These conditions correspond to restricting to unitary representations of the algebra of \(O(d-1,2)\). The vacuum state \(|0\rangle\) of course satisfies \(\hat{L}_{ab}|0\rangle = 0\), forming the trivial singlet representation. The quadratic Casimir has the form

\[
\hat{C}_2 = -\frac{1}{2} \hat{L}_{ab} \hat{L}_{ab} = -\frac{1}{2} \hat{L}_{ij} \hat{L}_{ij} - \hat{L}_{\pm i} \hat{L}_{\mp i} + \hat{H}^2 \mp (d-1)\hat{H},
\]

while

\[
-\frac{1}{2} L_{ab} L_{ab} = \frac{1}{\rho^2} \nabla^2_H \nabla^2_S + \cos^2 r \left( \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial r^2} \right) + (d-2) \cot r \frac{\partial}{\partial r} + \cot^2 r \nabla^2_{S_{d-2}},
\]

\[
\nabla^2_{S_{d-2}} = D_i D_i = \frac{1}{2} L_{ij} L_{ij}.
\]

For a scalar field \(\phi(x)\) the action of the generators is simply

\[
[\hat{L}_{ab}, \phi] = -L_{ab} \phi.
\]

To determine the extension to vector fields we consider the transformation of a vector field \(\mathcal{A}_a(\eta)\) on \(\mathbb{R}^{d+1}\) for which the corresponding action of the generators is \([\hat{L}_{ab}, \mathcal{A}_c] = -L_{ab} \mathcal{A}_c - g_{ac} \mathcal{A}_b + g_{bc} \mathcal{A}_a\). This may be reduced to a \(d\)-component field on the embedded
The representation space is then spanned by linear combinations of states of the form $[\hat{H},A_{\pm}] = -HA_{\pm} \pm A_{\pm}$, $[\hat{H},A_i] = -HA_i$, $[\hat{L}_{ij},A_{\pm}] = -L_{ij}A_{\pm}$, $[\hat{L}_{ij},A_k] = -L_{ij}A_k - \delta_{ik}A_j + \delta_{jk}A_i$, (12.11)

and

$[\hat{L}_{\pm i},A_{\pm}] = -L_{\pm i}A_{\pm}$,

$[\hat{L}_{\pm i},A_j] = -L_{\pm i}A_j \pm e^{\mp\tau} \cosec r \xi_j A_i + (\delta_{ij} - \xi_i \xi_j)(A_{\pm} - \frac{1}{2} \cosec^2 r (A_{\pm} - e^{\mp 2\tau} A_{\mp}))$,

$[\hat{L}_{\pm i},A_{\mp}] = -L_{\mp i}A_{\pm} - 2A_i + \xi_i \cosec r (e^\tau A_+ - e^{-\tau} A_-)$, (12.12)

For a scalar field $\phi(\eta)$ there is a corresponding vector $A_{\alpha} = \eta^b L_{ab} \phi$ satisfying $\eta^a A_a = 0$. In the same fashion as previously when $A_a(\eta) \rightarrow A_a(x)$ on $H^d$ we may therefore define the vector field $\nabla_a \phi(x) = e_a^{\mu}(x) \partial_\mu \phi(x)$ where explicitly in terms of the coordinates $x = (\tau, r, \xi_i)$

$$\nabla_- \phi = e^{\mp\tau} \left( \cos r \partial_\tau \phi \mp \sin r \partial_r \phi \right), \quad \nabla_+ \phi = \cot r D_i \phi.$$ (12.13)

Similarly for a vector $A_a$ there is an associated scalar $-\eta^a L_{ab} A^b$ which may be used to defined the divergence $\nabla \cdot A$ and which is then given by,

$$\nabla \cdot A = \frac{1}{2} \cos r \left( e^\tau \partial_\tau A_+ + e^{-\tau} \partial_\tau A_- \right)
- \frac{1}{2} \cot r \left( \cos r \partial_r + (d-3) \cosec r \left( e^\tau A_+ - e^{-\tau} A_- \right) + \cot r D_i A_i. \right.$$ (12.14)

For scalar fields the appropriate representation is defined in terms of a spin-0 lowest weight state $|\lambda\rangle$, $\langle \lambda | \lambda \rangle = 1$, satisfying

$$\hat{H} |\lambda\rangle = \lambda |\lambda\rangle, \quad \hat{L}_{-i} |\lambda\rangle = \hat{L}_{ij} |\lambda\rangle = 0.$$ (12.15)

The representation space is then spanned by linear combinations of states of the form $\prod_i [\hat{L}_{+i}]^{n_i} |\lambda\rangle$. These may be decomposed into representations of $O(d-2)$ by using symmetric traceless rank $\ell$ tensors $C_{i_1 \ldots i_\ell}$, $\ell = 1, 2, \ldots$, satisfying

$$C_{i_1 \ldots i_\ell} = C_{(i_1 \ldots i_\ell)} \quad \text{and} \quad C_{i_1 i_2 \ldots i_\ell-2} = 0,$$ (12.16)

to define, taking for $\ell = 0, C = 1$,

$$|n, \ell, C\rangle = \hat{K}_+^n \hat{L}_{+i_1} \ldots \hat{L}_{+i_\ell} |\lambda\rangle C_{i_1 \ldots i_\ell}, \quad \hat{K}_+ = \hat{L}_{+i} \hat{L}_{+i}, \quad n, \ell = 0, 1, 2, \ldots.$$ (12.17)
It is easy to see that \( \hat{H}|n \ell, C\rangle = (\lambda + 2n + \ell)|n \ell, C\rangle \) and for all states the Casimir operator, given by (12.3), takes the value \( \lambda(\lambda - d + 1) \) while for each \( \ell \) they transform irreducibly under \( O(d - 2) \). These states satisfy the orthogonality condition

\[
\langle n' \ell', C'|n \ell, C\rangle = \delta_{n' n} \delta_{\ell' \ell} C' \cdot C N_{n \ell}, \quad C' \cdot C = C'_{i_1 \ldots i_\ell} C_{i_1 \ldots i_\ell},
\]

where, as shown in appendix C, the norms are

\[
N_{n \ell} = 2^{4n + \ell} n! \ell!(\lambda)_{n+\ell}(\mu + \ell)_n(\lambda + 1 - \mu)_n, \quad \mu = \frac{1}{2}(d - 1), \quad (\lambda)_{n} = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}. \tag{12.19}
\]

A unitary representation therefore requires as usual \( \lambda > 0, \geq \frac{1}{2}(d - 3) \) or \( \lambda = 0 \) when there is just the singlet vacuum state \(|0\rangle\). An complete orthogonal basis of states, \( \{|n \ell, I\rangle\} \), of the form given by (12.17) may be obtained by introducing for each \( \ell \) a basis of symmetric traceless tensors \( C^l_{i_1 \ldots i_\ell} \), satisfying (12.16), such that

\[
C'^l \cdot C^l = \delta_{l'I}, \quad \sum_I C^l_{j_1 \ldots j_I} C^l_{i_1 \ldots i_\ell} = \mathcal{P}^{(\ell)}_{j_1 \ldots j_I, i_1 \ldots i_\ell}, \tag{12.20}
\]

where \( \mathcal{P}^{(\ell)} \) is here the projector onto symmetric traceless tensors of rank \( \ell \).

The action of \( \hat{L}_{+i} \) on the basis states (12.17) takes \( (n \ell) \to (n+1 \ell-1), (n \ell+1) \). It is convenient to define the \( \ell \pm 1 \) rank symmetric traceless tensors

\[
C^+_{i_1 \ldots i_{\ell+1}} = \delta_{i_1} C_{i_2 \ldots i_{\ell+1}} - \frac{\ell}{d + 2\ell - 3} \delta_{i_1 i_2} C_{i_3 \ldots i_{\ell+1}} i, \quad \ell = 0, 1, \ldots, \tag{12.21}
\]

\[
C^-_{i_1 \ldots i_{\ell-1}} = C_{i_1 \ldots i_{\ell-1}}, \quad \ell = 1, 2, \ldots,
\]

so that

\[
\hat{L}_{+i}|n \ell, C\rangle = |n \ell+1, C^+_{i_1}\rangle + \frac{\ell}{d + 2\ell - 3} |n+1 \ell-1, C^-_{i}\rangle. \tag{12.22}
\]

To obtain the spectral representation of the two point function of the scalar field \( \phi \) we need to determine the matrix elements of \( \phi \) between \(|0\rangle\) and arbitrary states in the representation. For the lowest weight state

\[
\langle 0|\phi(x)|\lambda\rangle = e^{-\lambda x} f(r), \tag{12.23}
\]

and \( \hat{L}_{-i}\lambda\rangle = 0 \) leads to, from (12.5), to \( \cos rf'(r) + \lambda \sin rf(r) = 0 \) so that

\[
f(r) = N(\cos r)^\lambda. \tag{12.24}
\]

A general matrix element has the form

\[
\langle 0|\phi(x)|n \ell, C\rangle = e^{-\lambda n x} f_{\ell n}(r) Y^n_C(\xi), \quad \lambda n \ell = \lambda + 2n + \ell. \tag{12.25}
\]
where we define appropriate spherical harmonics (for further properties see appendix C, a useful summary of spherical harmonics in arbitrary dimensions is given in \[37\]) by

\[
Y^C_\ell (\xi) = C_{i_1 \ldots i_\ell} \xi_{i_1} \ldots \xi_{i_\ell} .
\] (12.26)

From (12.8), (12.9) and (12.10) with \(\hat{L}_{ab}|0\rangle = 0\) it is easy to see that the matrix elements (12.25) satisfy \(\nabla_d^2 (e^{-\lambda \xi^2} f_{n\ell}(r) Y^C_\ell (\xi)) = r^2 \lambda (\lambda - d + 1) e^{-\lambda \xi^2} f_{n\ell}(r) Y^C_\ell (\xi)\) which may be reduced to a simple equation for \(f_{n\ell}\). In order to find \(f_{n\ell}\) with the overall scale determined we use instead (12.10) for \(\hat{L}_{++}\), with the expression (12.5) for \(L_{++}\), and from (12.21),

\[
\xi_i Y^C_\ell (\xi) = Y^{C^+}_{\ell+1} (\xi) + \frac{\ell}{d+2\ell-3} Y^{C^-}_{\ell-1} (\xi),
\]

\[
D_i Y^C_\ell (\xi) = \ell Y^{C^-}_{\ell-1} (\xi) - \ell \xi_i Y^C_\ell (\xi) = -\ell Y^{C^+}_{\ell+1} (\xi) + \frac{\ell}{d+2\ell-3} Y^{C^-}_{\ell-1} (\xi),
\] (12.27)

to obtain

\[
-\lambda_{n\ell} \sin rf_{n\ell}(r) + \cos rf_{n\ell}(r) - \ell \cosec rf_{n\ell}(r) = -f_{n\ell+1}(r), \quad \ell = 0, 1, \ldots ,
\]

\[
-\lambda_{n\ell} \sin rf_{n\ell}(r) + \cos rf_{n\ell}(r) + (d + \ell - 3) \cosec rf_{n\ell}(r) = -f_{n+1\ell-1}(r), \quad \ell = 1, 2, \ldots ,
\] (12.28)

relating \(f_{n\ell}\) for differing \(n, \ell\). The solutions are well known, involving Jacobi polynomials \(P_{n}^{(\alpha, \beta)}\) which satisfy the crucial, for our purposes, recurrence relations,

\[
\cos r \frac{d}{dr} P_{n}^{(\alpha, \beta)}(\cos 2r) - 2(n + \alpha + \beta + 1) \sin r P_{n}^{(\alpha, \beta)}(\cos 2r)
\]

\[
= -2(n + \alpha + \beta + 1) \sin r P_{n}^{(\alpha + 1, \beta)}(\cos 2r)
\]

\[
= -2\alpha \cosec r P_{n}^{(\alpha, \beta)}(\cos 2r) + 2(n + 1) \cosec r P_{n+1}^{(\alpha - 1, \beta)}(\cos 2r).
\] (12.29)

Since (12.24) with \(P_{0}^{(\alpha, \beta)}(x) = 1\) gives \(f_{00}(r) = N(\cos r)^\lambda\) we may then find from (12.28)

\[
f_{n\ell}(r) = N (-1)^n 2^n + \ell n! (\lambda)_{n+\ell} (\cos r)^\lambda (\sin r)^\ell P_{n}^{(\ell+\mu-1, \lambda-\mu)}(\cos 2r).
\] (12.30)

For subsequent application it is sufficient to consider only \(r = 0\) when, using \(P_{n}^{(\alpha, \beta)}(1) = (\alpha + 1)_{n}/n!\), the results reduce to just

\[
f_{n\ell}(0) = N (-1)^n 2^n (\lambda)_{n} (\mu)_{n} \delta_{\ell 0}.
\] (12.31)

The above results are designed to set the scene for the more involved analysis when spin is involved. For spin one we consider a lowest weight state \(|\lambda, i\rangle, \langle \lambda, i| \lambda, j\rangle = \delta_{ij}\), with (12.13) replaced by,

\[
\hat{H}|\lambda, i\rangle = \lambda|\lambda, i\rangle, \quad \hat{L}_{-i}|\lambda, i\rangle = 0, \quad \hat{L}_{ij}|\lambda, k\rangle = \delta_{jk}|\lambda, i\rangle - \delta_{ik}|\lambda, j\rangle.
\] (12.32)

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In order to build a basis of states analogous to (12.17) it is necessary besides (12.16) to introduce tensors $\hat{C}_{ji_1...i_\ell}$, $\ell = 1, 2, \ldots$ belonging to mixed symmetry, described by $(\ell, 1, 0 \ldots)$ Young tableaux, representations which satisfy the properties

$$\hat{C}_{ji_1...i_\ell} = \hat{C}_{j(i_1...i_\ell)}, \quad \hat{C}_{(ji_1...i_\ell)} = 0, \quad \hat{C}_{jji_1...i_{\ell-1}} = \hat{C}_{jii_1...i_{\ell-2}} = 0.$$  \hspace{1cm} (12.33)

Corresponding to (12.17) we now define the states,

$$|n\ell+,C\rangle = \hat{K}_{\lambda,\mu}^n \hat{L}_{i_1} \ldots \hat{L}_{i_{\ell-1}} |\lambda, k\rangle \mathcal{C}_{ki_1...i_{\ell-1}}, \quad n = 0, 1, \ldots, \ell = 1, 2, \ldots,$$

$$|n\ell-,C\rangle = \hat{K}_{\lambda,\mu}^n \hat{L}_{i_1} \ldots \hat{L}_{i_{\ell-1}} |\lambda, k\rangle \mathcal{C}_{i1...i_{\ell-1}}, \quad n = 0, 1, \ldots, \ell = 1, 2, \ldots,$$

$$|n\ell,\hat{C}\rangle = \hat{K}_{\lambda,\mu}^n \hat{L}_{i_1} \ldots \hat{L}_{i_{\ell-1}} |\lambda, k\rangle \hat{\mathcal{C}}_{i1...i_{\ell}} , \quad n = 0, 1, \ldots, \ell = 1, 2, \ldots,$$  \hspace{1cm} (12.34)

with the associated eigenvalues of $\hat{H}$ taking the values,

$$\lambda_{n\ell\mp} = \lambda + 2n + \ell \mp 1, \quad \lambda_{n\ell} = \lambda + 2n + \ell.$$  \hspace{1cm} (12.35)

For $\ell = 1, 2, \ldots$ the scalar products of the states $|n\ell+,C\rangle$ are described by a $2 \times 2$ matrix,

$$\begin{pmatrix} \langle n' + 1\ell' -, C'|n+1\ell -, C\rangle & \langle n' + 1\ell' -, C'|n + 1\ell +, C\rangle \\ \langle n'\ell' +, C'|n+1\ell -, C\rangle & \langle n'\ell' +, C'|n + 1\ell +, C\rangle \end{pmatrix} = \delta_{n'n'} \delta_{\ell'\ell} C' \cdot C \mathcal{N}_{n\ell},$$  \hspace{1cm} (12.36)

while also

$$\langle n'\ell', \hat{C}|n\ell, \hat{C}\rangle = \delta_{n'n'} \delta_{\ell'\ell} \hat{C}\cdot \hat{C} \mathcal{N}_{n\ell}, \quad \hat{C}' \cdot \hat{C} = \hat{C}_{ji_1...i_\ell} \hat{C}_{j(i_1...i_\ell)} ,$$

$$\langle n' 0 +, 1|n 0 +, 1\rangle = \delta_{n'n'} \mathcal{N}_{n+}.$$  \hspace{1cm} (12.37)

From appendix C

$$\mathcal{N}_{n\ell} = 2^{4n+\ell}n!(\lambda + 1)n_{\ell-1}(\mu + \ell)n(\lambda + 1 - \mu)n(\lambda - 1),$$

$$\mathcal{N}_{n+} = 2^{4n+2}n!(\lambda + 1)n_{\mu+1}(\lambda + 1 - \mu)n(\lambda - d + 2),$$  \hspace{1cm} (12.38)

so that unitarity requires $\lambda \geq 1, d - 2$. The expressions for the elements of $\mathcal{N}_{n\ell}$ are more involved and so are deferred to appendix C.

In a similar fashion to (12.21) we define

$$\hat{C}^+_{i,j\ell_1...i_{\ell+1}} = \delta_{i(i_1} \hat{C}_{j|i_2...i_{\ell+1})} - \frac{1}{d + 2\ell - 3} \left( \delta_{(i_1i_2} \hat{C}_{j|i_3...i_{\ell+1})i} - \delta_{(i_1i_2} \hat{C}_{j|i_3...i_{\ell+1})ji} \right),$$

$$\hat{C}^-_{i,j\ell_1...i_{\ell-1}} = \hat{C}_{j(i_1...i_{\ell-1})i} - \hat{C}_{(i_1...i_{\ell-1})ji}, \quad \hat{C}_{i,1...i_{\ell}} = \hat{C}_{i1...i_{\ell}},$$  \hspace{1cm} (12.39)
where \( \hat{C}_{i,j_1...j_\ell} \) are mixed symmetry tensors satisfying (12.33) while \( \hat{C}_{i_1...i_\ell} \) is a symmetric tensor obeying (12.16). Furthermore from \( C_{i_1...i_\ell} \) we may also define a mixed symmetry tensor satisfying (12.33) by

\[
C_{i,j_1...j_\ell} = \frac{1}{\ell + 1} \left( \delta_{i(i_1} C_{j|i_2...i_\ell)} - \delta_{ij} C_{i_1...i_\ell} - \frac{\ell - 1}{d + \ell - 4} \left( \delta_{i_1,i_2} C_{j|i_3...i_\ell} - \delta_{j(i_1} C_{i_2...i_\ell)} \right) \right).
\]

(12.40)

With the states defined by (12.34) we then have

\[
\hat{L}_{+i}|n \ell +, C\rangle = |n \ell +1 +, C^+_i\rangle + \frac{\ell}{d + 2\ell - 3} |n\ell +1 - , C^-_i\rangle,
\]

\[
\hat{L}_{+i}|n \ell -, C\rangle = |n \ell +1 - , C^+_i\rangle + |n \ell , C_i\rangle + \frac{(\ell - 1)(d + \ell - 3)}{(d + 2\ell - 3)(d + \ell - 4)} |n\ell -1 + , C^-_i\rangle + \frac{d - 3}{(d + 2\ell - 3)(d + \ell - 4)} |n\ell -1 - , C^-_i\rangle,
\]

\[
\hat{L}_{+i}|n \ell , \hat{C}\rangle = |n \ell +1 , \hat{C}^+_i\rangle + \frac{\ell - 1}{d + 2\ell - 3} |n\ell -1 - , \hat{C}^-_i\rangle + \frac{1}{d + \ell - 3} \left( |n \ell + , \hat{C}_i\rangle - |n\ell -1 - , \hat{C}_i\rangle \right).
\]

(12.41)

Following a similar route to the case of scalar fields we consider now the matrix elements of a vector current \( A_a(x) \) between the singlet \( |0\rangle \) and the states of this spin one representation.\(^9\) For the lowest weight state we take, consistent with \( \xi_j A_j = 0 \),

\[
\langle 0|A_-(x)|\lambda, k\rangle = 0, \quad \langle 0|A_+(x)|\lambda, k\rangle = \xi_k e^{-(\lambda + 1)\tau} g(r),
\]

\[
\langle 0|A_j(x)|\lambda, k\rangle = (\delta_{jk} - \xi_j \xi_k) e^{-\lambda \tau} f(r).
\]

(12.42)

Imposing the equations following from \( \hat{L}_{-i}|\lambda, k\rangle = 0 \), which also requires the vanishing of the \( A_+ \) matrix element, and using (12.5) we get

\[
\lambda \sin rf + \cos rf' = 0, \quad f + \frac{1}{2} \cosec rg = 0, \quad (\lambda + 1)g + \cos rg' - \cosec rg = 0,
\]

(12.43)

for which the solution is

\[
f(r) = N(\cos r)^\lambda, \quad g(r) = -2N(\cos r)^\lambda \sin r.
\]

(12.44)

It is easy to verify from (12.14) that this solution implies\(^10\)

\[
\langle 0|\nabla \cdot A|\lambda, k\rangle = 0.
\]

(12.45)

\(^9\) In four dimensions the spin one representation was discussed by Fronsdal [38] who obtained equivalent results for \( \langle 0|A_a|\lambda, k\rangle \) to those obtained below.

\(^{10}\) Alternatively for a spin 0 lowest weight state we find \( \langle 0|A_-(x)|\lambda\rangle = N e^{-(\lambda - 1)\tau}(\cos r)^{\lambda - 1}, \langle 0|A_j(x)|\lambda\rangle = 0, \langle 0|A_+(x)|\lambda\rangle = N e^{-(\lambda + 1)\tau}(\cos r)^{\lambda - 1} \cos 2\tau \) but in this case \( \langle 0|A_a(x)|\lambda\rangle = -\nabla_a N e^{-\tau}(\cos r)^\lambda /\lambda \). It is useful to also note that \( \langle 0|\nabla \cdot A(x)|\lambda\rangle = -(\lambda - d + 1)N e^{-\tau}(\cos r)^\lambda \) so the current is conserved if \( \lambda = d - 1 \).
For other states the matrix elements may be determined in a similar fashion to previously.

To write expressions for matrix elements we need besides (12.26) vector spherical harmonics which may be defined in terms of tensors satisfying (12.33),

\[ Y_{\ell,j}^c (\xi) = \hat{C}_{ij_1 \ldots i_{\ell}} \ldots \xi_{i_{\ell}} \]  

(12.46)

Like \( D_j Y_{\ell}^C (\xi) \) this satisfies \( n_j Y_{\ell,j}^C (\xi) = 0 \) and also we have \( D_j Y_{\ell,j}^C (\xi) = 0 \). Using the above spherical harmonics the matrix elements may then be written in general as

\[
\langle 0| A_j(x)| n \ell \pm, C \rangle = e^{-\lambda_{n\ell\pm} \tau} f_{n\ell\pm} (r) D_j Y_{\ell}^C (\xi), \quad \langle 0| A_j(x)| n \ell, \hat{C} \rangle = e^{-\lambda_{n\ell} \tau} \hat{f}_{n\ell}(r) Y_{\ell,j}^C (\xi),
\]

\[
\langle 0| A_+ (x)| n \ell \pm, C \rangle = e^{-(\lambda_{n\ell\pm}+1) \tau} g^+_{n\ell\pm} (r) Y_{\ell}^C (\xi), \quad \langle 0| A_+ (x)| n \ell, \hat{C} \rangle = 0,
\]

\[
\langle 0| A_- (x)| n \ell \pm, C \rangle = e^{-(\lambda_{n\ell\pm}-1) \tau} g^-_{n\ell\pm} (r) Y_{\ell}^C (\xi), \quad \langle 0| A_- (x)| n \ell, \hat{C} \rangle = 0. \tag{12.47}
\]

In order to use (12.41) we need besides (12.27)

\[
\xi_i D_j Y_{\ell}^C (\xi) = D_j (\xi_i Y_{\ell}^C (\xi)) - (\delta_{ij} - \xi_i \xi_j) Y_{\ell}^C (\xi) = \ell Y_{\ell,j}^C (\xi) + D_j \left( \frac{\ell}{\ell + 1} Y_{\ell+1,j}^C (\xi) + \frac{\ell (d + \ell - 3)}{(d + 2\ell - 3)(d + \ell - 4)} Y_{\ell-1,j}^C (\xi) \right),
\]

\[
D_i D_j Y_{\ell}^C (\xi) + \xi_j D_i Y_{\ell}^C (\xi) = D_j D_i Y_{\ell}^C (\xi) + \xi_i D_j Y_{\ell}^C (\xi) = \ell Y_{\ell,j}^C (\xi) - D_j \left( \frac{\ell^2}{\ell + 1} Y_{\ell+1,j}^C (\xi) - \frac{\ell (d + \ell - 3)^2}{(d + 2\ell - 3)(d + \ell - 4)} Y_{\ell-1,j}^C (\xi) \right), \tag{12.48}
\]

using

\[
(\delta_{ij} - \xi_i \xi_j) Y_{\ell}^C (\xi) = D_j \left( \frac{1}{\ell + 1} Y_{\ell+1,j}^C (\xi) - \frac{\ell}{(d + 2\ell - 3)(d + \ell - 4)} Y_{\ell-1,j}^C (\xi) \right) - \ell Y_{\ell,j}^C (\xi), \tag{12.49}
\]

and furthermore

\[
\xi_i Y_{\ell,j}^C (\xi) = Y_{\ell+1,j}^C (\xi) + \frac{\ell - 1}{d + 2\ell - 3} Y_{\ell-1,j}^C (\xi) - \frac{1}{d + \ell - 3} \frac{1}{\ell} D_j Y_{\ell,j}^C (\xi),
\]

\[
D_i Y_{\ell,j}^C (\xi) + \xi_j Y_{\ell,j}^C (\xi) = -\ell Y_{\ell+1,j}^C (\xi) + (\ell - 1) \frac{d + \ell - 3}{d + 2\ell - 3} Y_{\ell-1,j}^C (\xi) - \frac{\ell}{d + \ell - 3} \frac{1}{\ell - 3} D_j Y_{\ell,j}^C (\xi). \tag{12.50}
\]

With these results we may obtain equations determining \( f_{n\ell \pm} (r) \), \( \hat{f}_{n\ell}(r) \), \( g^+_{n\ell \pm} (r) \), \( g^-_{n\ell \pm} (r) \) by using the results for the commutators of \( \hat{L}_{++} \) in (12.12) with, from (12.42) and (12.44),

\[
f_{01-} (r) = N (\cos r)^\lambda, \quad g^+_{01-} (r) = -2N (\cos r)^\lambda \sin r, \quad g^-_{01-} (r) = 0. \tag{12.51}
\]
The details are quite lengthy so we restrict ourselves here to the main results. The essential equations are similar in general form to (12.28) along with various algebraic relations necessary for consistency. From \( (0|A_{\pm}(x)|n,\ell,C) = 0 \) we may obtain, using \( Y^\xi_{\ell,i}(\xi) = Y_{\ell,i}^C(\xi) \), for \( \ell = 1, 2, \ldots, \)

\[
g^+_{n+1,\ell-}(r) - g^+_{n\ell+}(r) = 0, \quad g^-_{n+1,\ell-}(r) - g^-_{n\ell+}(r) = -2(d + \ell - 3)f_{n\ell}(r), \quad (12.52)
\]

and from the analysis of \([\hat{L}_{++}, A_j]\) we may obtain

\[
\hat{f}_{n+1}(r) + \hat{f}_{n+1,\ell-1}(r) = \ell(f_{n+1,\ell-}(r) - f_{n\ell+}(r)), \quad \ell = 2, 3, \ldots. \quad (12.53)
\]

The results, apart from \( g^+_{n\ell+}(r) \) which is easily obtained from (12.52), are then

\[
g^+_{n\ell-}(r) = -N(-1)^n2^{2n+\ell}n!(\lambda + 1)_{n+\ell-1}(\cos r)^\lambda (\sin r)^\ell P_n^{(\ell+\mu-1,\lambda-\mu)}(\cos 2r),
\]

\[
\hat{f}_{n\ell}(r) = N(\lambda - 1)(-1)^n2^{2n+\ell}n!(\lambda + 1)_{n+\ell-1}(\cos r)^\lambda (\sin r)^\ell P_n^{(\ell+\mu-1,\lambda-\mu)}(\cos 2r),
\]

\[
f^-_{n\ell-}(r) = N(-1)^n2^{2n+\ell-1}n!(\lambda)_{n+\ell-1}(\cos r)^\lambda (\sin r)^{\ell-1} \times \frac{1}{\lambda\ell} \left( \ell P_n^{(\ell+\mu-1,\lambda-\mu-1)}(\cos 2r) + (\lambda - 1)P_n^{(\ell+\mu-2,\lambda-\mu)}(\cos 2r) \right),
\]

\[
f^-_{n\ell+}(r) = -N(-1)^n2^{2n+\ell}n!(\lambda + 1)_{n+\ell-1}(\cos r)^\lambda (\sin r)^\ell \times \left( 2(n + 1)P_n^{(\ell+\mu-1,\lambda-\mu-1)}(\cos 2r) + (\lambda - 1)P_n^{(\ell+\mu-1,\lambda-\mu)}(\cos 2r) \right),
\]

\[
g^-_{n\ell-}(r) = -N(-1)^n2^{2n+\ell}n!(\lambda + 1)_{n+\ell-2}(\cos r)^\lambda (\sin r)^\ell \times (\lambda + \mu + n + \ell - 2)P_n^{(\ell+\mu-1,\lambda-\mu)}(\cos 2r),
\]

\[
g^-_{n\ell+}(r) = N(-1)^n2^{2n+\ell+1}n!(\lambda + 1)_{n+\ell-1}(\cos r)^\lambda (\sin r)^\ell \times (2(\lambda + n)(\mu + n + \ell) - (\lambda - 1)\ell)P_n^{(\ell+\mu-1,\lambda-\mu)}(\cos 2r), \quad (12.54)
\]

where also \( g^-_{0\ell-}(r) = 0 \). Otherwise, except for \( f^-_{n\ell+}(r) \) which is defined only for \( \ell \geq 1 \), the formulae given by (12.54) are valid for all \( n, \ell \) required by the definition of the basis in (12.34). As a consequence of (12.45) the solutions for all matrix elements given by (12.47) and (12.54) are consistent with \( \nabla \cdot A = 0 \).

For \( r = 0 \), in a similar fashion to (12.31), the non zero results reduce to just

\[
g^+_{n0+}(0) = g^-_{n0+}(0) = N(-1)^n2^{2n+2}(\lambda + 1)_{n}(\mu)_{n+1},
\]

\[
f^-_{n-1}(0) = N(-1)^n2^{2n}(\lambda + 1)_{n-1}(\mu + 1)_{n-1}(\mu \lambda + n),
\]

\[
f^-_{n1+}(0) = -N(-1)^n2^{2n+1}(\lambda + 1)_{n}(\mu + 1)_{n}(2n + \lambda + d). \quad (12.55)
\]

13. Spectral Representations

The spectral representation for two point functions encodes completely the analyticity requirements following from locality for any quantum field theory together with the
conditions of unitarity which reduce to positivity of the spectral weight function. Such representations may also be found for spaces of constant curvature [19] although they are significantly less straightforward to determine in specific cases. For two dimensional field theories on flat space the proof of the c-theorem may be recast with added insight in terms of the spectral representation for the energy momentum tensor two point function [3]. It is therefore natural to also consider the spectral representation for the energy momentum tensor two point function on spaces of constant curvature.

For the scalar field $\phi$ we first compute the two point function corresponding to summing over the intermediate states $|n \ell, I\rangle$ labelled by $\lambda$ and for which the Casimir operator has the value $\lambda(\lambda-d+1)$. It is simplest to set $r=0$ and then only $\ell=0$ states contribute, in a similar fashion to [3], and using (12.31) we have

$$
\langle 0|\phi(\tau, 0, \xi)\phi(0, 0, \xi)|0\rangle = \sum_{n=0}^{\infty} \frac{f_{n0}(0)^2}{N_{n0}} e^{-\lambda n_0^2 \tau} = N^2 e^{-\lambda \tau} F(\lambda, \mu; \lambda + 1 - \mu; e^{-2\tau}) .
$$

(13.1)

For $x = (\tau, 0, \xi)$, $y = (0, 0, \xi)$ we have $\theta = \tau$. With the appropriate choice for $N^2$ this is the standard Green function for $-\nabla^2 + \lambda(\lambda - d + 1)\rho^2$, as verified in appendix B,

$$
G_\lambda(\theta) = \frac{\rho^{d-2}}{2\pi \mu} \frac{\Gamma(\lambda)}{\Gamma(\lambda + 1 - \mu)} e^{-\lambda \rho^2} F(\lambda, \mu; \lambda + 1 - \mu; e^{-2\theta}) .
$$

(13.2)

For the conformally coupled case the two choices of $\lambda$ are $\frac{1}{2}d$ for Dirichlet boundary conditions and $\frac{1}{2}(d-2)$ for the Neumann case. The spectral representation for the scalar two point function can then be written as

$$
\langle \phi(x)\phi(y) \rangle = \int_{\frac{1}{2}(d-3)}^{\infty} d\lambda \rho_\phi(\lambda) G_\lambda(\theta) .
$$

(13.3)

In a similar fashion to the discussion of the scalar two point function in [13.1] we may now determine the contribution of a single spin 1 irreducible representation to the vector two point function. Choosing the same configuration as previously the non zero components are

$$
\langle 0|A_\pm(\tau, 0, \xi)A_\pm(0, 0, \xi)|0\rangle = \langle 0|A_\pm(\tau, 0, \xi)A_\mp(0, 0, \xi)|0\rangle = e^{+\tau} D_\lambda(\tau) ,
$$

$$
\langle 0|A_j(\tau, 0, \xi)A_k(0, 0, \xi)|0\rangle = (\delta_{jk} - \xi_j \xi_k) E_\lambda(\tau) .
$$

(13.4)

Using the results (12.35) with (12.38) then (13.1) adapted to this case gives

$$
D_\lambda(\tau) = \sum_{n=0}^{\infty} \frac{f_{n0}(0)^2}{N_{n0}^2} e^{-(\lambda + 2n + 1)\tau}
$$

$$
= \frac{4N^2}{\lambda - d + 2} e^{-(\lambda + 1)\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(\lambda + 1, \mu)_{n+1}}{\Gamma(\lambda + 1 - \mu)} e^{-2n\tau}
$$

$$
= \frac{4N^2 \mu}{\lambda - d + 2} e^{-(\lambda + 1)\tau} F(\lambda + 1, \mu + 1; \lambda + 1 - \mu; e^{-2\tau}) ,
$$

(13.5)

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and also, by summing over intermediate states \(|n \, 1 \pm, C\rangle = |n \, 1 \pm, i\rangle C_i\),

\[
E_\lambda(\tau) = e^{-\lambda \tau} \sum_{n=0} A_n e^{-2n\tau},
\]

where

\[
A_0 = N^2, \quad A_{n+1} = (f_{n+1} - (0) \quad f_{n+1} + (0)) \sum_{n=0} \frac{N_{n+1}^{-1}}{n!} \left( f_{n+1} - (0) \over f_{n+1} + (0) \right).
\]

Using (12.35) for \(f_{n1\pm}(0)\) and the results of appendix C we have

\[
A_n = N^2 \frac{1}{n!} \left( \frac{\lambda n \mu n}{\lambda + 1 - \mu} \right) \left( 1 + \frac{2n(\lambda + n)}{\lambda \mu (\lambda - d + 2)} \right),
\]

so that

\[
E_\lambda(\tau) = \left( 1 + \frac{1}{2\lambda \mu (\lambda - d + 2)} \frac{d^2}{d\tau^2} - \lambda^2 \right) e^{-\lambda \tau} F(\lambda, \mu; \lambda + 1 - \mu; e^{-2\tau}).
\]

The functions \(D_\lambda\) and \(E_\lambda\) are constrained by conservation equations. To obtain these and to show the relation to the general formalism used in section 8 we may write in general for the correlation function of two vector currents

\[
\langle A_\mu(x) A_\alpha(y) \rangle = \hat{x}_\mu \hat{y}_\alpha D(\theta) + (I_{\mu\alpha} + \hat{x}_\mu \hat{y}_\alpha) E(\theta).
\]

Choosing, as in (13.4), \(x = (\tau, 0, \xi), y = (0, 0, \xi)\), when \(\theta = \tau\), then the non zero components of \(\hat{x}, \hat{y}\) are just \(\hat{x}_\tau = 1, \hat{y}_\tau = -1\) and also we have \(I_{\tau\tau} = 1\). Using (12.13) for these \(x,y\) to obtain the equivalent results in the basis used in (13.4) given by \(A_a = e_a^\mu A_\mu\) gives then

\[
\hat{x}_\pm = e^{\mp\tau}, \quad \hat{y}_\pm = 1, \quad I_{\pm\pm} = I_{\mp\mp} = e^{\mp\tau}, \quad I_{ij} = \delta_{ij} - \xi_i \xi_j.
\]

Applying (13.11) in (13.10) then leads to identical expressions for each components as in (13.4) where \(D, E\) are given by just the single spin one irreducible representation specified by \(\lambda\). The conservation equation \(\nabla^\mu \langle A_\mu(x) A_\alpha(y) \rangle = 0\) using the results in section 5 is easily seen to give

\[
\sinh \theta D'(\theta) + (d - 1) \cosh \theta D(\theta) = -(d - 1) E(\theta).
\]

It is straightforward to check that this is satisfied by (13.5) and (13.9). By extension of (13.2) we define

\[
G_{\lambda,n}(\theta) = \frac{\rho^{d-2+2n}}{2\pi \mu} \frac{\Gamma(\lambda + n)}{\Gamma(\lambda + 1 - \mu)} e^{-(\lambda + n)\theta} F(\lambda + n, \mu + n; \lambda + 1 - \mu; e^{-2\theta}),
\]

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and then the representation for a general vector two point function may then be obtained by expressing $D$ as, if $d > 3$,

$$D(\theta) = \int_{d-2}^{\infty} d\lambda \rho_V(\lambda) G_{\lambda,1}(\theta),$$

where $\rho_V(\lambda)$ is a positive weight function, with $E$ determined by (13.12).

It is of interest to consider separately the contribution from $\lambda = d - 2$ when we have

$$D_0(\theta) = C G_{d-2,1}(\theta) = C \frac{\rho^d}{2S_d} \frac{\cosh \theta}{(\sinh \theta)^d},$$
$$E_0(\theta) = \frac{C}{d-1} \frac{\rho^d}{2S_d} \frac{1}{(\sinh \theta)^d}.$$

With these expressions the vector two point function (13.10) may be written just in terms of the scalar $G_S$

$$\langle A_{\mu}(x)A_{\alpha}(y) \rangle_S = \partial_{\mu} G_S(\theta) \overline{\partial_{\alpha}},$$

where

$$G_S(\theta) = C \frac{d-2}{(d-1)^2} G_{d-1}(\theta).$$

Thus $G_S$ is proportional to the scalar Green function, as given by (13.2), for $\lambda = d - 1$ which since then $\nabla^2 G_S(\theta) = 0$ for $x \neq y$ ensures that (13.16) satisfies the conservation equation. In this special case the contribution of the spin one representation intermediate states is therefore identical with those of spin zero. This is in accord with the representation theory since, if $\lambda = d - 2$ as may be seen in (C.7), the states $\{ |n 0 +, C\}$ span an invariant subspace with $|0 0 +, 1\rangle = \hat{L}_{+k}|\lambda, k\rangle$, annihilated by $\hat{L}_{-i}$, the lowest weight state. This representation formed by this subspace is identical with a spin zero representation corresponding to $\lambda = d - 1$. Summing over intermediate states in this space gives just the result (13.16) with (13.17).

Our primary interest here is of course the energy momentum tensor two point function. The unitary representation of the isometry group $O(d - 1, 2)$ for a spin two lowest weight state, $|\lambda, ij\rangle = |\lambda, ji\rangle, |\lambda, ii\rangle = 0$ which satisfies analogous conditions to (12.32), may in principle be constructed along similar lines to the spin one case considered above and in appendix C, but this would be tedious to carry out. The norm of the state $\hat{L}_{+j}|\lambda, ij\rangle$ is proportional to $\lambda - d + 1$ so for unitarity it is necessary that $\lambda \geq d - 1$ in this case. Although we do not here undertake the detailed relevant calculations the results for spin one nevertheless suggest a natural conjecture for the contribution of the representation built on the lowest weight state $|\lambda, ij\rangle$ to the energy momentum tensor two point function.
To describe this we first decompose the bi-tensor $\Gamma_{\mu\nu,\alpha\beta}(x,y)$ given by (8.4), with (8.3), for $x = (\tau, 0, \xi)$, $y = (0, 0, \xi)$ in the basis given by (13.11) when we obtain

$$
\Gamma_{\pm\pm,\pm\pm} = e^{\mp 2\tau}(R - 4S + 2T + 2U + V),
\Gamma_{j,\pm\pm} = e^{\mp \tau} \delta_{j l} (S - T),
\Gamma_{ij,\pm\pm} = \hat{\delta}_{ij} (U + V),
\Gamma_{ijkl} = (\hat{\delta}_{ik} \hat{\delta}_{jl} + \hat{\delta}_{il} \hat{\delta}_{jk}) T + \delta_{ij} \delta_{kl} V,
\hat{\delta}_{ij} = \delta_{ij} - \xi_i \xi_j,
$$

and other components are trivially related to those in (13.18). With the definitions (8.10) and (8.6) we have

$$
R - 4S + 2T + 2U + V = \frac{d - 1}{d} Q + \frac{2d - 1}{d} P_1 + \frac{1}{d} P_2.
$$

The conservation equations (A.2) demonstrate that $\Gamma_{\pm\pm,\pm\pm}$ determines the other components, assuming the traceless conditions (8.6), just like $D$ determines $E$ in (13.12) for the vector current two point function.

Based on analogy with (13.14) we conjecture the spectral representation for spin 2 intermediate states can therefore be written for $Q$ in the form, with the definition (13.13),

$$
Q(\theta) = S_d \int_{d-1}^{\infty} d\lambda \rho_2(\lambda) G_{\lambda,2}(\theta).
$$

Of course, by virtue of the conservation equations, this determines the spectral representation for the whole energy momentum tensor two point function since for intermediate spin 2 states it is automatically traceless. The general conserved energy momentum tensor two point function, as given by (2.27), can then be expressed as in (8.28),

$$
S_d^2 \langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_{\text{con}} = \Gamma_{0,\mu\nu,\alpha\beta}(x,y) + \Gamma_{2,\mu\nu,\alpha\beta}(x,y),
$$

where the spin 0 piece is determined by $F_0(\theta)$,

$$
\Gamma_{0,\mu\nu,\alpha\beta}(x,y) = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 + (d - 1) \rho^2 g_{\mu\nu}) F_0(\theta) (\nabla_\alpha \nabla_\beta - \nabla^2 g_{\alpha\beta} + (d - 1) \rho^2 g_{\alpha\beta}),
$$

and the spin 2 piece is traceless and may be written as in (8.20) in terms of $F_2(\theta)$. Since $Q$ determines $\Gamma_{2,\mu\nu,\alpha\beta}$ (13.20) implies therefore a spectral representation for the spin 2 part of $\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle$ determined by the positive weight function $\rho_2(\lambda)$. Equivalently by integrating (A.10) with $Q$ given by (13.20) a spectral representation for $F_2$ may be found. The spectral representation for the spin 0 part $\Gamma_{0,\mu\nu,\alpha\beta}$ is also obtained directly by writing a representation analogous to (13.3) for $F_0$,

$$
F_0(\theta) = S_d \int_{\lambda_0}^{\infty} d\lambda \rho_0(\lambda) G_{\lambda}(\theta),
$$

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where \( \lambda_0 \) is in general restricted just by the unitarity bound \( \lambda_0 > \frac{1}{2}(d-3) \) if \( d > 3 \). In [3] it is suggested that \( \lambda_0 = d \) in order to ensure that \( \int d^d x \sqrt{g} g^{\mu \nu} T_{\mu \nu} \) is well defined, in the coordinates corresponding to the metric in \([12,1] \), when \( \sqrt{g_{H^2}} = (\cos r)^{-d}(\sin r)^{d-2} \sqrt{g_{S^{d-2}}} \), as \( x \) approaches the boundary \( r = \frac{\pi}{2} \), \( e^\theta \sim 2 \cosh \tau / \cos r \). Directly from \([13,22] \) we have

\[
S_d^2 g^{\mu \nu}(x) g^{\alpha \beta}(y) \langle T_{\mu \nu}(x) T_{\alpha \beta}(y) \rangle_{\text{con}} = (d - 1)^2( - \nabla^2 + d \rho^2)^2 F_0(\theta),
\]

and since for \( x \neq y \) this is identical with \( \langle \Theta(x) \Theta(y) \rangle \) we must have, neglecting possible subtractions,

\[
S_d \langle \Theta(x) \Theta(y) \rangle = \int_{\lambda_0}^{\infty} d\lambda \rho_\Theta(\lambda) G_\lambda(\theta), \quad \rho^A(d - 1)^2(\lambda + 1)^2(\lambda - d)^2 \rho_0(\lambda) = \rho_\Theta(\lambda).
\]

so that this determines \( \rho_0(\lambda) \) in terms of \( \rho_\Theta(\lambda) \) except for \( \lambda = d \). As an illustrative example we consider free massive scalars in appendix D. However, although \([13,22] \) is valid in general for \( \theta > 0 \), it may be extended to give a well defined distribution, for test functions non zero at \( x = y \), only if, for large \( \lambda \), \( \rho_0(\lambda) = O(\lambda^\alpha) \) with \( \alpha < 1 \). Assuming \([13,23] \) and \([13,24] \) we have, if \( \lambda_0 \geq d \),

\[
G_0 = S_d \frac{1}{\rho^d} \int d^d y \sqrt{g} g^{\mu \nu}(x) g^{\alpha \beta}(y) \langle T_{\mu \nu}(x) T_{\alpha \beta}(y) \rangle_{\text{con}}
= (d - 1)^2 \frac{1}{\rho^{d-2}} \int_{\lambda_0}^{\infty} d\lambda \rho_0(\lambda) \frac{1}{\lambda(\lambda - d + 1)},
\]

which ensures that \( G_0 > 0 \). The spectral representation may be modified to allow for a large \( \lambda \) behaviour with \( \alpha < 3 \) by introducing a subtraction,

\[
F_0(x, y) = G_0 \frac{\rho^{d-4}}{(d - 1)^2 d^2} S_d \delta^d(x, y) + S_d \frac{1}{\rho^2} \nabla^2 \int_{\lambda_0}^{\infty} d\lambda \frac{\rho_0(\lambda)}{\lambda(\lambda - d + 1)} G_\lambda(\theta),
\]

if \( \lambda_0 > d - 1 \). Clearly this depends on the subtraction constant \( G_0 \), for which there is then no necessary positivity constraint, as well as \( \rho_0(\lambda) \). In \([13,25] \) it is necessary to require \( \alpha < -1 \) is subtractions are to be avoided. In four dimensions the expected behaviour in renormalisable field theories away from fixed points is \( \alpha = 1 \), apart from powers of \( \ln \lambda \), so that it is then necessary to use \([13,27] \). Of course unless there is a well defined representation for \( \langle \Theta(x) \Theta(y) \rangle \) such as given by \([13,24] \) and \([13,27] \) relations involving local contact terms like \([2.13] \) are without significance.

An important role is played by the contribution to \( F_0(\theta) \) for \( \lambda = d \) for which \( \Gamma_{0,\mu \nu,\alpha \beta}(x, y) \) satisfies the traceless conditions for \( x \neq y \). In this case, following \([9.1] \) since \( K_0 = S_d G_d \),

\[
\rho_0(\lambda) = C_0 \rho^{d-2} \delta(\lambda - d) \quad \Rightarrow \quad F_0(\theta) = C_0 \rho^{d-2} S_d G_d(\theta),
\]

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leads to a form for $\Gamma_{\alpha\mu,\alpha\beta}$ which may be expressed in terms of the result calculated in (9.14) for $Q_0$. From the definition of $G_{\lambda,2}$ in (13.13) we have

$$Q_0(\theta) = 4d(d+1)C_0\rho^{d-2} S_dG_{d-2,2}(\theta).$$

This result shows exact agreement with the contribution expected from a spin 2 representation, as in (13.20), for $\lambda = d - 2$. Although this is outside the unitarity bound it corresponds to the invariant subspace present for this value of $\lambda$ and which may be constructed from the spin zero lowest weight state $\hat{L}_{+k}\hat{L}_{+l}|\lambda, kl\rangle$, which is then annihilated by $\hat{L}_{-i}$.

If we assume the minimal form given by (10.10) for conformally invariant theories we find

$$Q(\theta)_{\text{conformal}} = C_T \left( \frac{1}{s^d} + \frac{1}{\bar{s}^d} \right) = C_T\rho^{2d} e^{-d\theta} \left( (1 - e^{-\theta})^{-2d} + (1 + e^{-\theta})^{-2d} \right).$$

(13.30)

The spectral representation (13.20) can be written in this case in the form

$$Q(\theta)_{\text{conformal}} = 2C_T\rho^{2d} \sum_{r=0} A_r e^{-(d+2r)\theta} F(d + 2r, \mu + 2; \mu + 2r; e^{-2\theta}).$$

(13.31)

By expanding both (13.30) and (13.31) in powers of $e^{-2\theta}$ we find, by matching the first ten terms,

$$A_0 = 1, \quad A_r = 2 \frac{(d+2r)(d-2)2r-1}{(2r)! (\mu)_{2r-1}}, \quad r = 1, 2, \ldots$$

(13.32)

For $d = 3$, when $A_r = 2(r + 1)(2r + 1)$ if $r > 0$, the hypergeometric functions may be reduced to elementary functions and the summation carried out explicitly. As required $A_r \geq 0$ for $d > 2$ and $A_r = 0$ if $r > 0$ and $d = 2$. For $r > 0$ the coefficients $A_r$ determine the appropriate $\rho_2(\lambda)$ in (13.20) while from (13.29) we must have\footnote{The coefficient $C_0$ was calculated directly in \cite{[6]} for free scalar fields with exact agreement, with due regard for conventions, with this result.}

$$2^{d-1}d(d+1)(d-1)C_0 = C_T.$$

(13.33)

The remaining contribution is then given by taking

$$S_d\rho_2(\lambda)_{\text{conformal}} = 4\pi^\mu C_T\rho^{d-2} \sum_{r=1} \frac{\Gamma(\mu + 2r)}{\Gamma(d + 2r)} A_r \delta(\lambda - d + 2 - 2r).$$

(13.34)

Asymptotically $\rho_2(\lambda)_{\text{conformal}} \sim (d - 2)\rho C_T(\frac{1}{2} \rho \lambda)^{d-3}/\Gamma(d)$. 

For scalar fields the simple form given by (10.10), and hence (13.30), is not the whole story in the conformal limit. The extra terms present in (10.9) inserted into (8.10) give, writing

\[ Q(\theta) = Q_{\text{conformal}} + Q_1(\theta) \]

with \(C_T\) given by (10.6),

\[ Q_1(\theta) = -4C_{T,\phi} \frac{d(d-2)(d+1)}{d-1} \frac{1}{\rho^4(s\bar{s})^{\frac{1}{2}d+1}} \]

This corresponds to the contribution with only \(\lambda = d\) in the spectral representation. Added to (13.31), and using the result (13.32), leads just to the cancellation of the term involving \(A_1 = 2d(d+1)(d-2)/(d-1)\). In particular the coefficient \(A_0\) is unaffected.

In general \(C_0\) determines the leading large distance behaviour of the energy momentum tensor two point function while \(C_T\) is related to its singular form at short distances. It is unclear whether the relation (13.33) survives in interacting conformal field theories.

### 14. Implications for a Possible \(c\)-Theorem in Four Dimensions

The initial stimulus for this paper was to investigate the possibility of deriving a \(c\)-theorem by considering one and two point functions of the energy momentum tensor on spaces of constant curvature. Within the framework described here this does not seem to be feasible. A possible \(C\)-function need not necessarily reproduce all the attractive features uncovered by Zamolodchikov in two dimensions but may satisfy some or all the following properties:

1) \(C(\mu \ell; g)\) should be a physically measurable positive function of the couplings \(g^i\) and some length scale \(\ell\) such that at a fixed point, \(\beta^i(g_*^i) = 0\), \(C(\mu \ell; g_*) = C_*\) is independent of \(\ell\) and unambiguously and universally defined in terms of the properties of the conformal field theory which is obtained at the fixed point. The minimal condition of irreversibility of RG flow is that for a unitary quantum field theory in which there are both UV and IR fixed points we require

\[ C_*(UV) - C_*(IR) > 0. \tag{14.1} \]

Ideally \(C(\mu \ell; g)\) should be a function of the couplings for all relevant and irrelevant operators in the space of cut-off quantum field theories since its essential definition is independent of perturbation theory, \(C = 0\) should correspond to a totally trivial theory with no finite energy degrees of freedom. It may, although this does not seem essential, be independent of any strictly marginal couplings whose \(\beta\)-functions vanish.\(^{12}\) The \(C\) function should be

\(^{12}\) In this case \(C\) is just a constant equal to its free field value in \(N = 4\) SYM.
extensive so that if \( \{g\} \) can be separated into two distinct sets \( \{g_1\}, \{g_2\} \) corresponding to two decoupled theories then
\[
\mathcal{C}(\mu \ell; g) = \mathcal{C}_1(\mu \ell; g_1) + \mathcal{C}_2(\mu \ell; g_2). \tag{14.2}
\]

2) The \( \mathcal{C} \)-function should obey the usual RG equation expressing its independence of the arbitrary RG scale \( \mu \)
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta^i \frac{\partial}{\partial g^i} \right) \mathcal{C}(\mu \ell; g) = 0. \tag{14.3}
\]
Irreversibility of RG flow, at least in some finite domain of couplings \( \{g\} \) near a fixed point, is then entailed by the requirement
\[
\ell \frac{\partial}{\partial \ell} \mathcal{C}(\mu \ell; g) \begin{cases} < 0 & \text{on } \{g\} \setminus g_* , \\ = 0 & \text{if } g = g_* . \end{cases} \tag{14.4}
\]
With these conditions \( \mathcal{C} \) defines a Liapunov function in the region \( \{g\} \) for the RG flow
\[
\dot{\mathcal{C}}(\mu \ell; g_t) < 0 \text{ if } g_t \neq g_* , \quad \dot{g}_t^i = -\beta^i(g_t). \tag{14.5}
\]

3) A natural condition which clearly entails (14.4) is to require
\[
\ell \frac{\partial}{\partial \ell} \mathcal{C}(\mu \ell; g) = -\mathcal{G}_{ij}(\mu \ell; g)\beta^i(g)\beta^j(g), \tag{14.6}
\]
where \( \mathcal{G}_{ij}(\mu \ell; g) \) is independently defined as a positive symmetric tensor on the space of couplings and can be regarded as playing the role of a metric. \( \mathcal{G}_{ij}(\mu \ell; g) \) should satisfy a homogeneous RG equation,
\[
\mu \frac{\partial}{\partial \mu} \mathcal{G}_{ij}(\mu \ell; g) + (\mathcal{L}_\beta \mathcal{G})_{ij}(\mu \ell; g) = 0 , \quad (\mathcal{L}_\beta \mathcal{G})_{ij} = \beta^i \partial_j \mathcal{G}_{ij} + \partial_i \beta^k \mathcal{G}_{kj} + \partial_j \beta^k \mathcal{G}_{ik} , \tag{14.7}
\]
where \( \partial_i \beta^j \) is the anomalous dimension matrix for the operators \( \mathcal{O}_j \). Trivially from (14.3) and (14.6) we may obtain the essential Zamolodchikov equation (1.1),
\[
\beta^i(g) \frac{\partial}{\partial g^i} \mathcal{C}(\mu \ell; g) = \mathcal{G}_{ij}(\mu \ell; g)\beta^i(g)\beta^j(g). \tag{14.8}
\]

4) Assuming (1.1) the definition of \( \mathcal{C} \) is still ambiguous to the extent that
\[
\mathcal{C}(\mu \ell; g) \to \tilde{\mathcal{C}}(\mu \ell; g) = \mathcal{C}(\mu \ell; g) + D_{ij}(\mu \ell; g)\beta^i(g)\beta^j(g) , \quad \mathcal{G}_{ij}(\mu \ell; g) \to \tilde{\mathcal{G}}_{ij}(\mu \ell; g) = \mathcal{G}_{ij}(\mu \ell; g) + (\mathcal{L}_\beta D)_{ij}(\mu \ell; g) , \tag{14.9}
\]
\[
\]
leaves (1.1) invariant. This defines an equivalence amongst C-functions and associated metrics which ensures that the precise value of \( \ell \) chosen in (1.1) is irrelevant but of course all such functions give the same value \( C_* \) at a fixed point. To ensure the desired properties of irreversible RG flow it is only necessary that there is a \( D_{ij} \) such that \( \tilde{G}_{ij} \) is positive.\(^{13}\)

Introducing the correct terms linear in \( \beta^i \), as in (3.8), are in general necessary to find a \( C \) satisfying (1.1) although they also do not change \( C_* \).

5) A stronger condition, analogous to (3.9), which implies (1.1) is

\[
\partial_i C(\mu \ell; g) = T_{ij}(\mu \ell; g) \beta^j(g), \quad T_{ij} = \mathcal{G}_{ij} + \partial_i \mathcal{W}_j - \partial_j \mathcal{W}_i \quad (14.10)
\]

If \( T_{[ij]} = 0 \) this ensures that the RG flow is a gradient flow. In general (14.10) shows that \( C \) is independent of marginal coupling if \( T_{ij} \) has no off diagonal pieces. Under variations as in (14.9) then (14.10) still holds if at the same time

\[
\tilde{\mathcal{W}}_j(\mu \ell; g) = \mathcal{W}_j(\mu \ell; g) + D_{jk}(\mu \ell; g) \beta^k(g), \quad (14.11)
\]

which demonstrates that \( \mathcal{W}_j \) cannot be zero in general and that the assumed form for \( T_{[ij]} \) in (14.10) is consistent.

To illustrate some aspects of the above we first attempt to rederive the Zamolodchikov c-theorem \([2]\) (for a recapitulation see \([39]\)). The basic inequality is similarly obtained using previous results applied to the two point function of the energy momentum tensor in two dimensions on \( S^2 \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle = \Gamma_{\mu\nu,\alpha\beta}(x,y) \) as in (8.4). The positivity condition is provided by \( g^{\mu\nu} g^{\alpha\beta} \langle T_{\mu\nu}(x)T_{\alpha\beta}(y) \rangle = P_1 + 2P_2 \geq 0 \). Using (8.12a) and (8.9) for \( d = 2 \) we require

\[
\rho^4 C(\theta) = 2 \sin^4 \frac{1}{2} \theta (R + P_1) + f(\theta)(P_1 + P_2), \quad (14.12)
\]

to satisfy

\[
\rho^4 C'(\theta) = \frac{1}{2} f'(\theta)(P_1 + 2P_2). \quad (14.13)
\]

This provides a differential equation for \( f \) which is readily solved,

\[
f(\theta) = -4 \sin^4 \frac{1}{2} \theta + 4 \sin^2 \theta \ln \cos \frac{1}{2} \theta. \quad (14.14)
\]

As \( \theta \to 0 \) \( f(\theta) \sim -\frac{3}{4} \theta^4 \) so that \( f'(\theta) < 0 \) for some finite region near \( \theta = 0 \) (\( f'(\theta) < 0 \) for \( \theta \lesssim 2.5 \)) in which then \( C'(\theta) < 0 \). Nevertheless \( C \) is in general a function of the dimensionless \( \mu/\rho \) as well as \( \theta \) and the couplings \( g^i \) and the \( \theta \) derivative is not linked

\(^{13}\) In a perturbative context \([21]\) we may obtain \( C(\mu \ell; g) = C(g) + \Omega_{ij}(\mu \ell; g) \beta^i(g) \beta^j(g) \)

\[
\mathcal{G}_{ij}(\mu \ell; g) = G_{ij}(g) + (\mathcal{L}_\beta \Omega)_{ij}(\mu \ell; g) \quad \text{where } \beta^i \partial_i C = G_{ij} \beta^j \quad \text{and } \mu \frac{\partial}{\partial \mu} \Omega_{ij}(\mu \ell; g) + (\mathcal{L}_\beta \Omega)_{ij} = -G_{ij}. \]
to the dependence on the RG scale \( \mu \). Hence there appears to be no demonstration of irreversibility of RG flow for general \( \rho \) from such an inequality. Only with the assumption of a sensible flat space limit which requires \( C(\theta) \sim C(\theta \mu / \rho) \) as \( \rho \to 0 \) may such a result be obtained. In this case the resulting \( C \)-function, which satisfies (14.6) as well as (14.3) with \( \ell = \sqrt{s} = \theta / \rho \), is equivalent to that found by Zamolodchikov.\(^{14}\)

More recently \(^{11}\) an alternative derivation of a \( c \)-theorem for general \( d \) in terms of quantum field theories on spaces of constant negative curvature, where the \( c \)-function \( C \) is defined by \( g^{\mu \nu} \langle T_{\mu \nu} \rangle = -C \rho^d \), has been suggested which is based on equations akin to (2.29a,b). Defining

\[
\dot{C} = S_d C ,
\]

(2.29a) may be written, using (13.26), as

\[
\frac{d}{d \rho} \dot{C} = G_0 - d \dot{C} ,
\]

(14.16)

where, if the unsubtracted representation (13.23) is valid, (13.26) implies that \( G_0 > 0 \). If we assume \( A \propto \rho^d \), as in sections three and four for both two and four dimensions, then from (2.31) we have

\[
\beta^i \frac{\partial}{\partial g^i} \dot{C} = \rho \frac{d}{d \rho} \dot{C} .
\]

(14.17)

In this case the right hand side of (14.16) must vanish for \( \beta^i = 0 \). This becomes more apparent by using (2.29a) to now write

\[
\beta^i \partial_i \dot{C} = G_\Theta - \frac{1}{\rho^d} S_d \beta^i (\partial_i A + \partial_i \beta^j \langle O_j \rangle) , \quad G_\Theta = \frac{1}{\rho^d} S_d \int d^d y \sqrt{g} \langle \Theta(x) \Theta(y) \rangle . \quad (14.18)
\]

In \(^{11}\) the additional terms beyond \( G_\Theta \), which is \( O(\beta^2) \), are missing (there is no discernible redefinition of \( \dot{C} \) which achieves this). As it stands, given the definition of \( \dot{C} \), (14.18) is just an identity. If these additional terms (14.18) are disregarded then taking \( \dot{C}(\mu / \rho, g) \), which satisfies (14.3) with \( \ell = 1 / \rho \), as a possible \( C \)-function hinges on supposing \( G_\Theta > 0 \) in order to obtain the essential inequality (14.6). For \( x \neq y \) in unitary theories \( \langle \Theta(x) \Theta(y) \rangle > 0 \). However the regularisation of potential singularities in the integration defining \( G_\Theta \) at \( x = y \) is much less clear.\(^{15}\)

It is important to recognise that \( G_\Theta \) may be defined as a finite quantity

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\(^{14}\) To make the comparison clear, using a similar notation to that in \(^{39}\), we may define \( F = \frac{1}{16} s^2 R \), \( G = \frac{1}{4} s^2 P_1 \), \( H = s^2 (P_1 + 2 P_2) \) and then \( C = 2F - G - \frac{1}{4} s H + g(G + \frac{1}{4} s H) \), from (14.12), for \( g(s) = 1 + (1 - \frac{1}{4} \rho^2 s)/\frac{1}{4} \rho^2 s \ln(1 - \frac{1}{4} \rho^2 s) \) satisfies \( s C'(s) = -\frac{1}{4} (3 - h) H \) where \( h(s) = 1 + (1 - \frac{1}{4} \rho^2 s)/\frac{1}{4} \rho^2 s \ln(1 - \frac{1}{4} \rho^2 s) \). Neglecting \( g, h \), which is justified as \( s \to 0 \), this is identical with the standard flat space result.

\(^{15}\) The prescription in \(^{11}\) of subtracting \( \delta \)-function contact terms is not unambiguous and does not provide a well defined regularisation in general. Introducing additional couplings into the quantum field theory involving curvature terms does not change the essential argument.
through (14.18) but the regularisation of the integral need not preserve any positivity conditions. These also do not apply to contact terms that may be present in \( \langle \Theta(x)\Theta(y) \rangle \) (and which may be arbitrary although any such ambiguity cancels on both sides of (14.18)).

In general \( G_\Theta \) depends on possible subtraction constants in the spectral representation of the two point function whose positivity need not be entailed by that of the spectral weight function.

A perhaps convincing argument as to the essential difficulties of such an approach, independent of intricacies of the precise definition of \( G_\Theta \), is that in four dimensions (4.33) shows that we expect \( \hat{C} > 0 \) for either positive or negative curvature, and at a fixed point it is equal to \( a \) up to a factor, and then \( G_\Theta > 0 \) would be sufficient to prove the desired \( c \)-theorem for \( \hat{C} \) implying (14.1), so long as the other terms on the r.h.s of (14.18) are neglected. However in two dimensions from (3.21) \( \hat{C} \) has no definite sign although it is proportional to the Virasoro central charge \( c \) at a fixed point. In particular for the positive curvature case \( \hat{C} < 0 \) and then \( G_\Theta > 0 \) would be the wrong sign inequality to generate the required irreversible RG flow. It is nevertheless difficult to see why any putative derivation along these lines should not apply for both positive and negative curvature.

It is worth emphasising that considering a field theory on a space of constant non-zero curvature of course introduces an extra scale, here denoted by \( \rho \). This complicates the discussion of physical consequences from the RG equations except in some flat space limit as became apparent in the attempt to generalise the Zamolodchikov derivation of the \( c \)-theorem. Another way of appreciating the difference is that on flat space the RG equation may be regarded as implementing broken scale invariance identities. On flat space the full conformal group \( O(d+1,1) \) is reduced, except at fixed points, to \( O(d) \times T_d \). The broken generators decompose under \( O(d) \) into singlet generating scale transformations and a \( d \)-dimensional vector corresponding to special conformal transformations. The scale invariance Ward identities may still be implemented in a quantum field theory away from fixed points if they are associated with a flow in the space of couplings generated by the usual \( \beta \)-functions. Although such linear equations for the correlation functions do not involve any arbitrary scale \( \mu \) they are equivalent to the standard RG equations. There are corresponding identities relevant for special conformal transformations (40) but these involve the insertion of operators which are not generated by derivatives with respect to the couplings. In the absence of closed equations such identities are not then of much practical significance. If a quantum field theory is defined on \( S^d \) or \( H^d \), as considered in this paper, then the manifest symmetry group is \( O(d+1) \) or \( O(d,1) \). In either case the remaining generators of \( O(d+1,1) \) transform as a \( d + 1 \)-dimensional vector. The broken conformal identities are then similar to those for special conformal transformations on flat space and there are no associated linear equations relating directly the \( x \)-dependence and
the dependence on the couplings through the $\beta$-functions.

Although, as illustrated in (7.15), (10.2), (10.12), and (10.5), with (10.6), (10.13), (10.18), the results for free field theories on $S^d$ are in accord with expectation from conformal invariance, using (6.17), (6.25) and (7.13), this does not apply to the corresponding results on $H^d$. This arises since in order to derive the identities expressing conformal symmetry it is necessary to integrate by parts and surface terms on the boundary cannot be dropped due to lack of sufficient fall off of the Killing vector fields as the boundary of $H^d$ is approached. In consequence the results are less constrained in this case.

In the end the analysis of the energy momentum tensor on spaces of constant curvature has not apparently led to new insight concerning a proof of a possible $c$-theorem away from two dimensions. Nevertheless we have rederived (4.22) which is the four dimensional analogue of (3.7) in two dimensions which then gave (3.9), a perturbative analogue of the $c$-theorem. This result can be shown to be directly connected with the full $C$-function which entails irreversible RG flow [21]. The result (4.22) may be expected to be related to similar equations involving three point functions at non coincident points. In such cases positivity is no longer manifest but might be linked to positivity conditions on the energy momentum tensor [41]. It is interesting to note that a recent demonstration of irreversible RG flow in an ADS/CFT context depended on a positive energy condition [14], albeit for classical supergravity.

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Appendix A. Results for Negative Curvature

For the negative curvature case the conservation equations in (8.8) become

\[ R' - 2S' + U_2' + (d - 1)(\coth \theta R - 2 \tanh \frac{1}{2} \theta S) + 2 \coth \frac{1}{2} \theta S - 2 \cosech \theta U_2 = 0, \]
\[ S' - T' + d \coth \theta S - d \tanh \frac{1}{2} \theta T - \cosech \theta U_2 = 0, \]
\[ U_1' + V' + (d - 1) \coth \theta U_1 - 2 \cosech \theta S - 2 \tanh \frac{1}{2} \theta T = 0. \]  

(A.1)

With the definition (8.10), (8.9) and (8.11a, b) become

\[ P_1' + P_2' = -(d - 1) \coth \theta P_1, \]
\[ Q' + d \coth \theta Q + \frac{1}{d} P_1' = -2d \cosech \theta (S - T), \]
\[ (S - T)' + d \coth \theta (S - T) = - \cosech \theta \frac{1}{d - 1}((Q + (d - 2)(d + 1)T) \]
\[ + \cosech \theta \frac{1}{d} P_1. \]  

(A.2)

For \( d = 2 \) the equations become

\[ R' + 2 \coth \frac{1}{2} \theta R + P_1' = 2 \cosech \theta P_1, \]
\[ (R - 8S + 8T)' + 2 \tanh \frac{1}{2} \theta (R - 8S + 8T) + P_1' = -2 \cosech \theta P_1, \]  

(A.3)

and if \( P_1 = 0 \) instead of the solution in (8.13) we have

\[ R = C \frac{\rho^4}{\sinh^4 \frac{1}{2} \theta}, \quad R - 8S + 8T = C' \frac{\rho^4}{\cosh^4 \frac{1}{2} \theta}. \]  

(A.4)

The expression obtained in (9.13) corresponds to \( C = C' \).

For the spin zero contribution given by an analogous formula to (8.14) we have replacing (8.16) and (8.17)

\[ A = \rho^2 (F_0'' - \coth \theta F_0'), \quad B = -\rho^2 (F_0'' + (d - 2) \coth \theta F_0' - (d - 1)F_0), \]  

(A.5)

and

\[ R_0 = \rho^2 (A'' - (\coth \theta + 4 \cosech \theta)A' + 4 \cosech \theta \coth \frac{1}{2} \theta A), \]
\[ S_0 = \rho^2 (- \cosech \theta A' + \cosech \theta \coth \frac{1}{2} \theta A), \quad T_0 = \rho^2 \cosech^2 \theta A, \]
\[ U_{0,1} = \rho^2 (B'' - \coth \theta B'), \quad U_{0,2} = -\rho^2 (A'' + (d - 2) \coth \theta A' - (d - 1)(2 \cosech^2 \theta + 1)A), \]
\[ V_0 = -\rho^2 (B'' + (d - 2) \coth \theta B' - (d - 1)B + 2 \cosech^2 \theta A). \]  

(A.6)
Instead of (8.18)

\[ P_1 + P_2 = \rho^2 (d - 1) \left( - \coth \theta (A' + dB') + A + dB \right), \quad (A.7) \]

and from

\[ A + dB = -(d - 1) \left( \nabla^2 F_0 - d \rho^2 F_0 \right). \quad (A.8) \]

Solutions of the homogeneous equation (A.7), \( A + dB \propto \cosh \theta \), may now be discarded by requiring appropriate boundary conditions and (A.8) may be solved to determine \( F_0 \).

For the spin two contribution, which satisfies the traceless conditions (8.6), we have instead of (8.24), (8.26) in this case

\[ 8 \rho^2 G = \rho^4 (F_2'' - \coth \theta F_2' + 2(\frac{1}{2}) \theta F_2' + (d - 1)(d - 2) \tanh^{\frac{1}{2}} \theta F_2) \]

\[ = \frac{4(d - 1)}{(d - 3)d(d + 1)} \sinh^2 \theta Q. \quad (A.9) \]

As in (8.27) this may be simplified to the form

\[ \rho^4 \frac{d^2}{du^2} (u^{d-1} F_2) = \frac{16(d - 1)}{(d - 3)d(d + 1)} u^{d-1} Q, \quad u = \cosh^{\frac{1}{2}} \theta. \quad (A.10) \]

For \( P_1 = P_2 = 0 \) (A.2) demonstrates how \( R, S, T \) can then be determined from \( Q \).

In two dimensions if we modify the discussion in (14.12), (14.13) and (14.14) to the negative curvature case instead of (14.12) we may define

\[ \rho^4 C(\theta) = 2 \sinh^4 \frac{1}{2} \theta (R + P_1) + f(\theta)(P_1 + P_2), \quad (A.11) \]

and then (14.13) holds if

\[ f(\theta) = -4 \sinh^4 \frac{1}{2} \theta - 4 \sinh^2 \theta \ln \cosh \frac{1}{2} \theta. \quad (A.12) \]

In this case \( f'(\theta) < 0 \) for all \( \theta \) and \( C(\theta) \) is monotonically decreasing.

Appendix B. Construction of Green Functions

We here discuss the various Green functions for particular differential operators which were used in the text. For homogeneous spaces of constant curvature these all depend just on the single variable \( \theta(x, y) \) and the differential equations for \( S^d \) or \( H^d \) are similar although the regularity or boundary conditions in each case lead to different solutions.
First we determine the Green function \( G_\lambda(\theta) \) for the operator \(-\nabla^2 \mp \lambda(\lambda - d + 1)\rho^2\) for \( R \geq 0 \). For the negative curvature case, writing \( G_\lambda(\theta)_- = e^{-\lambda\theta}H(e^{-2\theta}) \), then 
\((-\nabla^2 + \lambda(\lambda - d + 1)\rho^2)G_0(\theta)_- = 0 \) gives using (5.16),
\[
z(1 - z)H''(z) + ((\lambda + 1)(1 - z) - \mu(1 - z))H'(z) - \lambda \mu H(z) = 0, \quad \mu = \frac{1}{2}(d - 1), \quad (B.1)
\]
which is of standard hypergeometric form. Imposing boundedness as \( \theta \to \infty \) requires the solution \( \propto F(\lambda, \mu; \lambda + 1 - \mu; z) \). For \( G_\lambda(\theta)_\pm \) to be a Green function it must have the singular behaviour as as \( \theta \to 0 \), \( G_\lambda(\theta)_\mp \sim (\theta/\rho)^{-d+2}/S_d(d-2) \), so as to generate \( \delta^d(x,y) \) under the action of \(-\nabla^2\). It is easy to see that this gives the result (13.2). For \( \lambda = d \) we have for the Green function \( K_0 \) defined by (3.13) and (4.26) when \( d = 2 \) and \( d = 4 \),
\[
K_0(\theta)_- = S_dG_d(\theta)_- = \rho^{d-2} \frac{2d}{d+1} z^{\frac{d}{2}}F(d, \mu; \mu + 2; z). \quad (B.2)
\]
For \( d = 2, 4 \) this can be reduced to elementary functions giving respectively
\[
K_0(\theta)_- = -\frac{1}{2} \left( (1 + \frac{1}{2} \rho^2 s) \ln \frac{s}{\bar{s}} + 2 \right), \quad (B.3)
\]
For \( R > 0 \) we write \( G_\lambda(\theta)_+ = F(w) \) for \( w = \frac{1}{2}(1 + \cos \theta) \) and then the homogeneous equation becomes
\[
w(1 - w)F''(w) + d(\frac{1}{2} - w)F'(w) + \lambda(\lambda + d + 1)F(w) = 0. \quad (B.4)
\]
By requiring a solution which is regular at \( w = 0 \) and imposing the required behaviour as \( \theta \to 0 \) the Green function becomes
\[
G_\lambda(\theta)_+ = \frac{\rho^{d-2}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \frac{\Gamma(\frac{d}{2} - d)}{\Gamma(\frac{d}{2})} \frac{\Gamma(\lambda - d)}{\Gamma(\frac{d}{2} - d)} (1 + \frac{1}{2}w) \quad (B.5)
\]
In the text \( K_0 \) is defined as the Green function for \(-\nabla^2 - d\rho^2\), corresponding to taking \( \lambda = d \). However
\[
G_{d+\epsilon}(\theta)_+ \sim \frac{1}{\epsilon} \frac{\rho^{d-2}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} (1 + \frac{1}{2}w) \quad \text{as} \quad \epsilon \to 0, \quad (B.6)
\]
which is a reflection of the existence of normalisable eigenvectors of \(-\nabla^2\) with eigenvalue \( d \). By subtracting this singular piece and then taking the limit \( \epsilon \to 0 \) we may define
\[
K_0(\theta)_+ = S_d \frac{\rho^{d-2}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \left( \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \frac{(d)_n}{(\frac{d}{2})_n} w^n - 2(1 + \frac{1}{d})w \right). \quad (B.7)
\]
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This then satisfies
\begin{equation}
(-\nabla^2 - d\rho^2)K_0(\theta) = S_d \delta^d(x,y) - k_d \rho^d \cos \theta, \quad k_d = \frac{d + 1}{(4\pi)^{d/2}} \frac{\Gamma(d)}{\Gamma(1/2)^d} S_d.
\end{equation}

Again $K_0(\theta)_+$ may be found explicitly for $d = 2, 4$. Dropping some terms proportional to $\cos \theta$, which satisfy the homogeneous equation, we have in each case,
\begin{align}
K_0(\theta)_+ &= -\frac{1}{2} \left( (1 - \frac{1}{2}\rho^2 s) \ln \frac{1}{4} \rho^2 s + 1 \right), \\
K_0(\theta)_+ &= \frac{1}{2s} - \frac{3}{4} \rho^2 \left( (1 - \frac{1}{2}\rho^2 s) \ln \frac{1}{4} \rho^2 s + 1 \right),
\end{align}

For the vector field the basic equation is
\begin{equation}
-\nabla^\mu \langle F_{\mu\nu}(x)A_\alpha(y) \rangle - \frac{1}{\xi} \partial_\nu \nabla^\mu \langle A_\mu(x)A_\alpha(y) \rangle = g_{\mu\alpha} \delta^d(x,y).
\end{equation}

To solve this we adapt the methods of ref. [29] to our notation. Using the definition (6.26) we may write
\begin{equation}
\langle A_\mu(x)A_\alpha(y) \rangle = F(\theta) \hat{I}_{\mu\alpha} + \partial_\mu H(\theta) \hat{\partial}_\alpha,
\end{equation}

and, since as a consequence of (6.27) $\partial_{[\hat{I}_\nu]}\alpha = 0$, we then have
\begin{equation}
\rho \langle F_{\mu\nu}(x)A_\alpha(y) \rangle = F'(\theta)(\hat{x}_\mu I_{\nu\alpha} - \hat{x}_\nu I_{\mu\alpha}).
\end{equation}

In order to solve (B.10) we first require
\begin{equation}
-\nabla^\mu \langle F_{\mu\nu}(x)A_\alpha(y) \rangle = g_{\nu\alpha} \delta^d(x,y) + \partial_\nu (S(\theta)\hat{g}_\alpha).
\end{equation}

This decomposes into two equations in either case
\begin{align}
S_+ &= \sin \theta \nabla^2 F_+ - \rho^2 \cos \theta F'_+, \quad S'_+ = \rho^2 (d - 1) \cosec \theta F'_+, \\
S_- &= \sinh \theta \nabla^2 F_- - \rho^2 \cosh \theta F'_-, \quad S'_- = \rho^2 (d - 1) \coth \theta F'_-,
\end{align}

which may be satisfied, along with (B.13), by imposing
\begin{equation}
-\nabla^2 F(\theta)_\pm \pm (d - 2) \rho^2 F(\theta)_\pm = \delta^d(x,y).
\end{equation}

This is identical to the equation for the scalar Green function for $\lambda = d - 2$ so that from the above solutions we have (for $R > 0$ the result is essentially given in [12])
\begin{equation}
F(\theta)_+ = \rho^{d-2} \frac{\Gamma(d-2)}{(4\pi)^{d/2} \Gamma(1/2)^d} F(d-2, 1; 1/2; w), \quad F(\theta)_- = \frac{\rho^{d-2}}{(d-2) \Gamma(1/2)^d} \frac{1}{(\sin \theta)^{d-2}}.
\end{equation}

The gauge dependent part in (B.11) can be found by solving, in the positive curvature case,
\begin{equation}
\frac{1}{\rho} \nabla^2 H_+ \hat{\partial}_\alpha = (1 - \xi) S_+ \hat{g}_\alpha + 2 \rho^2 \sin \theta F_+ \hat{g}_\alpha,
\end{equation}

with a similar equation if $R < 0$, although knowing $H$ is unnecessary to obtain (10.12) when in addition we use $F(2, 1; 2; w) = (1 - w)^{-1}$. 63
Appendix C. Calculation of Norms and Spherical Harmonics

The calculation of the norms of the basis states defined in (12.17) or (12.34) for the scalar or vector representations follows from computing the action of \( \hat{L}_{-i} \) on these states. In the standard fashion we use the basic commutator \([\hat{L}_{-i}, \hat{K}_+] = 2\delta_{ij}\hat{H} - 2\hat{L}_{ij}\), as well as those involving \( \hat{H}, \hat{L}_{ij} \) with \( \hat{L}_{+j} \), until they act on the lowest weight state and we may use (12.15) or (12.32). For the scalar case, with the aid of,

\[
[\hat{L}_{-i}, \hat{K}_+] = 4\hat{L}_{+i}(\hat{H} - \mu + 1) - 2\hat{L}_{+j}\hat{L}_{ij}, \quad \mu = \frac{1}{2}(d - 1),
\]
we find

\[
\hat{L}_{-i}|n\ell, \mathcal{C}\rangle = 4n(\lambda - \mu + n)\hat{L}_{+i}|n-1\ell, \mathcal{C}\rangle + 2\ell(\lambda + 2n + \ell - 1)|n\ell-1, \mathcal{C}^-\rangle
\]

\[
= 4n(\lambda - \mu + n)|n-1\ell+1, \mathcal{C}^+\rangle
\]

\[
+ \frac{2\ell}{\mu + \ell - 1}(\lambda + n + \ell - 1)(\mu + n + \ell - 1)|n\ell-1, \mathcal{C}^-\rangle,
\]

after using (12.22). With the scalar products in (12.18) and using the hermeticity conditions (12.7) with (C.2) and (12.22) again we have

\[
\langle n-1\ell+1, \mathcal{C}'|\hat{L}_{-i}|n\ell, \mathcal{C}\rangle = 4n(\lambda - \mu + n)\mathcal{N}_{n-1\ell+1}\mathcal{C}'\cdot \mathcal{C}^+_i = \frac{\ell + 1}{2(\mu + \ell)} \mathcal{N}_{n\ell}\mathcal{C}''_i\cdot \mathcal{C}, \quad (C.3)
\]

where

\[
\mathcal{C}'\cdot \mathcal{C}^+_i = \mathcal{C}_{i_1\ldots i}\cdot \mathcal{C}_{i_1\ldots i}. \quad (C.4)
\]

Hence (C.3) gives relation between \( \mathcal{N}_{n-1\ell+1} \) and \( \mathcal{N}_{n\ell} \). Similarly

\[
\langle n\ell-1, \mathcal{C}'|\hat{L}_{-i}|n\ell, \mathcal{C}\rangle = \frac{2\ell}{\mu + \ell - 1}(\lambda + n + \ell - 1)(\mu + n + \ell - 1)\mathcal{N}_{n\ell-1}\mathcal{C}'\cdot \mathcal{C}''_i = \mathcal{N}_{n\ell}\mathcal{C}''_i\cdot \mathcal{C}, \quad (C.5)
\]

with \( \mathcal{C}'\cdot \mathcal{C}_i = \mathcal{C}''_i\cdot \mathcal{C} \) in this case, relates \( \mathcal{N}_{n\ell-1} \) and \( \mathcal{N}_{n\ell} \). Solving the recurrence relations given by (C.3) and (C.5), with \( \mathcal{N}_{00} = 1 \), then gives (12.19).

For the vector case the results may be obtained in an analogous fashion albeit the expressions are more lengthy. For the states defined in (12.34) we have

\[
\hat{L}_{-i}|n\ell, \hat{\mathcal{C}}\rangle = 4n(\lambda - \mu + n)\hat{L}_{+i}|n-1\ell, \hat{\mathcal{C}}\rangle + 4n|n-1\ell+, \hat{\mathcal{C}}_i\rangle + 2\ell |n\ell+, \hat{\mathcal{C}}_i\rangle
\]

\[
+ 2\ell(\lambda + 2n + \ell - 1)\hat{K}_+^n\hat{L}_{+i_1}\ldots \hat{L}_{+i_{\ell-1}}|\lambda, k\rangle \hat{\mathcal{C}}_{i_1\ldots i_{\ell-1}i}
\]

\[
= 4n(\lambda - \mu + n)|n-1\ell+1, \hat{\mathcal{C}}^+_i\rangle
\]

\[
+ \frac{2(\ell - 1)}{\mu + \ell - 1}(\lambda + n + \ell - 1)(\mu + n + \ell - 1)|n\ell-1, \hat{\mathcal{C}}^-_i\rangle,
\]

\[
+ \frac{2}{d + \ell - 3}\left(2n(\lambda + \mu + n + \ell - 2)|n-1\ell+, \hat{\mathcal{C}}_i\rangle
\]

\[
- (2(\lambda + n - 1)(\mu + n - 1) + \ell(\lambda + 2n - 1))|n\ell-, \hat{\mathcal{C}}_i\rangle\right),
\]

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\[ \hat{L}_{-i}|n \ell+, C \rangle = 4n(\lambda - \mu + n + 1) \hat{L}_{+i}|n-1 \ell+, C \rangle + 2\ell(\lambda + 2n + \ell)|n \ell-1+, C^{-i} \rangle \\
+ 2(\lambda - d + 2) K_+ n \hat{L}_{+i_1} \ldots \hat{L}_{+i} \lambda, i \rangle C_{i_1 \ldots i_\ell} \\
= 4n(\lambda - \mu + n + 1)|n-1 \ell+1+, C^{-i} \rangle \\
+ \frac{2\ell}{\mu + \ell - 1}(\lambda + n + \ell)(\mu + n + \ell - 1)|n \ell-1+, C^{-i} \rangle, \quad (C.7) \\
+ 2(\lambda - d + 2)(|n \ell+1-, C^{+i} \rangle - \ell|n \ell, C \rangle) \\
+ \frac{\ell(\lambda - d + 2)}{(\mu + \ell - 1)(d + \ell - 4)} \left(2(\mu + \ell - 2)|n \ell-1+, C^{-i} \rangle \\
- (\ell - 1)|n+1 \ell-1-, C^{-i} \rangle \right), \]

and also

\[ \hat{L}_{-i}|n \ell-, C \rangle = 4n(\lambda - \mu + n) \hat{L}_{+i}|n-1 \ell-, C \rangle + 2(\ell - 1)(\lambda + 2n + \ell - 1)|n \ell-1-, C^{-i} \rangle \\
- 4n K_+ n^{-1} \hat{L}_{+i_1} \ldots \hat{L}_{+i} \lambda, i \rangle C_{i_1 \ldots i_\ell} + 4n|n-1 \ell-1+, C^{-i} \rangle \\
= 4n(\lambda - \mu + n)|n-1 \ell+1-, C^{+i} \rangle + 4n(\lambda - \mu + n + \ell)|n-1 \ell, C \rangle \\
+ \frac{2(\ell - 1)}{\mu + \ell - 1}(\lambda + n + \ell - 1)(\mu + n + \ell - 1)|n \ell-1-, C^{-i} \rangle, \quad (C.8) \\
+ \frac{2n}{(\mu + \ell - 1)(d + \ell - 4)} \left((\ell - 1)(\lambda - \mu + n + \ell)|n \ell-1-, C^{-i} \rangle \\
+ (d - 3)(\lambda + \mu + n + 2\ell - 3)|n-1 \ell-1+, C^{-i} \rangle \right). \]

With these formulae we may find the norms of the basis vectors defined in (12.36) and (12.37) by following the same procedure as in the spinless case. First for \( \hat{N}_{n \ell} \) we obtain

\[ \langle n-1 \ell+1, \hat{C}'|\hat{L}_{-i}|n \ell, \hat{C} \rangle = 4n(\lambda - \mu + n) \hat{N}_{n-1 \ell+1} \hat{C}' \cdot \hat{C}^{+i} = \frac{\ell}{2(\mu + \ell)} \hat{N}_{n \ell} \hat{C}' \cdot \hat{C}, \quad (C.9) \]

and

\[ \langle n \ell-1, \hat{C}'|\hat{L}_{-i}|n \ell, \hat{C} \rangle = \frac{2(\ell - 1)}{\mu + \ell - 1}(\lambda + n + \ell - 1)(\mu + n + \ell - 1) \hat{N}_{n \ell-1} \hat{C}' \cdot \hat{C}^{-i} = \hat{N}_{n \ell} \hat{C}' \cdot \hat{C}, \quad (C.10) \]

by using (C.6) and (12.41). In (C.9) we have the relation

\[ \hat{C}' \cdot \hat{C}^{+i} = \frac{\ell}{\ell + 1} \hat{C}' \cdot \hat{C} = \hat{C}'_{j i_1 \ldots i_\ell} C_{i_1 \ldots i_\ell}, \quad (C.11) \]

and similarly in (C.10) with \( \ell \rightarrow \ell - 1 \). Thus \( \hat{N}_{n \ell} \) obeys identical recurrence relations to \( \hat{N}_{n \ell} \) and (12.38) is obtained starting from \( \hat{N}_{01} = 2(\lambda - 1) \).
The $2 \times 2$ matrix $\mathbf{N}_{n\ell}$ defined by (12.36) may now be obtained directly with the aid of algebraic relations expressing each element in terms of $\mathbf{N}_{n\ell}$. Using (C.8) and (C.7) we may find, with an obvious notation for the components of $\mathbf{N}_{n\ell}$,

\[
\langle n\ell, \hat{C}|\hat{L}_{-i}|n+1\ell-, \mathcal{C} \rangle = 4(n + 1)(\lambda - \mu + n + \ell + 1)\mathbf{N}_{n\ell} \hat{C} \cdot \mathbf{C}_i
\]

\[
= \frac{1}{d + \ell - 3}(\mathbf{N}_{n\ell}^{++} - \mathbf{N}_{n\ell}^{--}) \hat{C}_i \cdot \mathbf{C}, \tag{C.12}
\]

\[
\langle n\ell, \hat{C}|\hat{L}_{-i}|n\ell+, \mathcal{C} \rangle = -2\ell(\lambda - d + 2)\mathbf{N}_{n\ell} \hat{C} \cdot \mathbf{C}_i
\]

\[
= \frac{1}{d + \ell - 3}(\mathbf{N}_{n\ell}^{++} - \mathbf{N}_{n\ell}^{--}) \hat{C}_i \cdot \mathbf{C},
\]

where

\[
\hat{C}_i \cdot \mathbf{C} = -\ell \hat{C} \cdot \mathbf{C}_i = \hat{C}_{ii1...i} \mathbf{C}_{i1...i,} \tag{C.13}
\]

Furthermore we have using (C.6)

\[
\langle n\ell+, \mathcal{C}|\hat{L}_{-i}|n+1\ell, \hat{C} \rangle = 0
\]

\[
= \frac{2}{d + \ell - 3} \left(2(n + 1)(\lambda + \mu + n + \ell - 1)\mathbf{N}_{n\ell}^{++} - (2(\lambda + n)(\mu + n) + \ell(\lambda + 2n + 1))\mathbf{N}_{n\ell}^{--}\right) \mathbf{C} \cdot \hat{C}_i. \tag{C.14}
\]

Although a further relation may be obtained by considering $\langle n+1\ell-, \mathcal{C}|\hat{L}_{-i}|n+1\ell, \hat{C} \rangle$ (C.12) with (C.13) and (C.14), since $\mathbf{N}_{n\ell}^{--} = \mathbf{N}_{n\ell}^{++}$, are sufficient to determine $\mathbf{N}_{n\ell}$. Writing

\[
\mathbf{N}_{n\ell} = 2^{4n+\ell+1} n!(\ell - 1)!((\lambda + 1)_{n+\ell-1}(\mu + \ell)n(\lambda + 1 - \mu)_n \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{C.15}
\]

we then find

\[
a = 4(n + 1)((\lambda - \mu)(\lambda + \ell - 1)(\mu + \ell - 1) + (n + 1)((\lambda - 1)(\mu - 1) + \ell(\lambda - \mu))),
\]

\[
b = 2(n + 1)\ell(\lambda + \mu + n + \ell - 1)(\lambda - d + 2), \tag{C.16}
\]

\[
c = \ell(2(\lambda + n)(\mu + n) + \ell(\lambda + 2n + 1))(\lambda - d + 2).
\]

The result for $\mathbf{N}_{n+}$ in (12.38) may be obtained by the result for $\mathbf{N}_{n\ell}^{++}$ given by (C.13) and (C.16) by setting $\ell = 0$. Since we require an expression for $\mathbf{N}_{n\ell}^{-1}$ it is useful to note that

\[
\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = 4(n+1)\ell(\lambda-1)(\lambda+n+\ell)(\mu+n+\ell)(\lambda-\mu+n+1)(d+\ell-3)(\lambda-d+2), \tag{C.17}
\]

which for a unitary representation must of course be positive.
We also summarise the essential results for the spherical harmonics defined as in (12.26) by
\[ Y_{l}^{j}(\xi) = C_{l}^{j} \xi_{i_{1}} \cdots \xi_{i_{l}} \]
with \( C_{l}^{j} \) a basis of symmetric traceless tensors of rank \( l \) so that \( \sum_{l} 1 = (d - 2) \ell_{-1} (d - 3 + 2 \ell) / \ell! \). Using
\[ \int_{S_{d-2}} d\xi_{i_{1}} \cdots \xi_{i_{2\ell}} = \frac{(2\ell)!}{2^{2\ell} \ell!(\mu)_{\ell}} S_{d-1} \delta_{(i_{1}i_{2}) \cdots \delta_{(i_{2\ell - 1}i_{2\ell})}} , \] (C.18)
and (12.20) we have
\[ \int_{S_{d-2}} dv Y_{\ell}^{I}(\xi) Y_{\ell}^{J}(\xi) = \frac{\ell!}{2^\ell (\mu)_{\ell}} S_{d-1} \delta_{IJ} . \] (C.19)
Furthermore we have
\[ \sum_{l} Y_{l}^{I}(\xi_{1}) Y_{l}^{J}(\xi_{2}) = \frac{\ell!}{2^\ell (\mu - 1)_{\ell}} C_{l}^{\mu - 1}(\xi_{1}, \xi_{2}) , \quad C_{l}^{\mu - 1}(1) = \frac{(d - 3)\ell}{\ell!} , \] (C.20)
with \( C_{l}^{\mu - 1} \) a Gegenbauer polynomial.

Appendix D. Results for Free Massive Scalar Fields

Forte and Latorre \cite{10} have discussed the case of free massive scalar fields on a space of constant negative curvature. We here re-examine this case, which despite being a free field theory is non trivial, in the light of the main discussion in this paper.

For \( \phi \) a free scalar field satisfying \( (\Delta + m^2)\phi = 0 \) then if
\[ m^2 = (\lambda_{\phi} - \frac{1}{2} d)(\lambda_{\phi} - \frac{1}{2} d + 1) \rho^2 , \] (D.1)
the basic \( \phi \) two point function is \( G_{\lambda_{\phi}}(\theta) \). For such free massive fields the energy momentum tensor may still be taken to be given by (10.1) but this is no longer traceless giving
\[ \Theta = -m^2 \phi^2 . \] (D.2)
In terms of the general formalism set up earlier \( m^2 \) may be regarded as a coupling with the associated operator \( O_{m^2} = \frac{1}{2} \phi^2 \) and \( \beta m^2 = -2m^2 \).

In order to determine the spectral representation for \( \phi^2 \) in this case we make use of
\[ F(\lambda_{1}, \mu; 1 - \mu; e^{-2\theta}) F(\lambda_{2}, \mu; 1 - \mu; e^{-2\theta}) = \sum_{n=0} B_{\lambda_{1}, \lambda_{2}, n} e^{-2n\theta} F(\lambda_{1} + \lambda_{2} + 2n, \mu; \lambda_{1} + \lambda_{2} + 1 - \mu + 2n; e^{-2\theta}) , \] (D.3)
where, in a similar fashion to (13.32),
\[
B_{\lambda_1,\lambda_2,n} = \frac{(\lambda_1)_n(\lambda_2)_n}{(\lambda_1 + 1 - \mu)_n(\lambda_2 + 1 - \mu)_n} \frac{(\mu)_n(\lambda_1 + \lambda_2 + 1 - 2\mu + n)_n}{n!(\lambda_1 + \lambda_2 + n - \mu)_n}. \tag{D.4}
\]

For \( d = 3 \), \( B_{\lambda_1,\lambda_2,n} = 1 \), when (D.3) is easily checked. For the free field \( \phi \) we therefore have
\[
\langle \phi^2(x)\phi^2(y) \rangle = 2G_{\lambda_\phi}(\theta)^2 = \int_2^{\infty} d\lambda \rho_{\phi^2}(\lambda) G_\lambda(\theta), \tag{D.5}
\]
where from (D.3)
\[
\rho_{\phi^2}(\lambda) = \frac{\rho^{d-2}}{\pi^\mu} \frac{\Gamma(\lambda_\phi)^2}{\Gamma(\lambda_\phi + 1 - \mu)^2} \sum_{n=0}^{\infty} \frac{\Gamma(2\lambda_\phi + 2n + 1 - \mu)}{\Gamma(2\lambda_\phi + 2n)} B_{\lambda_\phi,\lambda_\phi,n}\delta(\lambda - 2\lambda_\phi - 2n). \tag{D.6}
\]

Asymptotically \( \rho_{\phi^2}(\lambda) \sim 2(d - 1)\rho(\frac{1}{2}\rho\lambda)^{d-3}/\Gamma(d)S_d \). Using the relation (13.25) we have
\[
\rho_0(\lambda) = \frac{m^4S_d}{\rho^4(d - 1)^2(\lambda + 1)^2(\Lambda - d)^2\rho_{\phi^2}(\lambda)} \quad \overset{m^2 \to 0}{\longrightarrow} \quad \frac{1}{2d-1}(d - 1)^2(d + 1) \delta(\lambda - d), \tag{D.7}
\]
where the limit \( m^2 \to 0 \) may be taken either by \( \lambda_\phi \to \frac{1}{2}d \) or \( \lambda_\phi \to \frac{1}{2}d - 1 \) in (D.1). Comparing with (13.28) the result for \( C_0 \) is in agreement with (13.33) where \( C_\tau \) is given by (10.6).

For general \( d \), \( G_\lambda(\theta) \) may be separated, by using standard hypergeometric identities, into two pieces one of which contains terms of the form \( \theta^{d+2+2n}, n = 0, 1, 2, \ldots \), and the other which is analytic in \( \theta \). Discarding the former we may easily obtain using (10.1)
\[
\langle \phi^2 \rangle = \frac{\rho^{d-2}}{(4\pi)^{d/2}} \Gamma(1 - \frac{1}{2}d) \frac{\Gamma(\lambda_\phi)}{\Gamma(\lambda_\phi + 2 - d)}, \quad \langle T_{\mu\nu} \rangle = -\frac{1}{d}m^2g_{\mu\nu}\langle \phi^2 \rangle. \tag{D.8}
\]
The result for \( \langle T_{\mu\nu} \rangle \) is of course as expected from (D.2). With the expression in (D.8) for \( \langle \phi^2 \rangle \) we have, noting that \( \psi(x) = \Gamma'(x)/\Gamma(x) \),
\[
-2\frac{\partial}{\partial m^2}\langle \phi^2 \rangle = \langle \phi^2 \rangle \frac{\psi(\lambda_\phi) - \psi(\lambda_\phi + 2 - d)}{(\lambda_\phi - \mu)\rho^2} = \int d^dy\sqrt{g}\langle \phi^2(x)\phi^2(y) \rangle = \int_2^{\infty} d\lambda \rho_{\phi^2}(\lambda) \frac{1}{\lambda(\lambda - d + 1)\rho^2}. \tag{D.9}
\]
The representation in terms of \( \rho_{\phi^2} \) may be verified directly for \( d = 3 \) and is convergent for \( d < 4 \) when it demonstrates \( \partial\langle \phi^2 \rangle/\partial m^2 < 0 \). For \( d = 2 \) (D.4) gives
\[
\rho^2 \int d^2y\sqrt{g}\langle \phi^2(x)\phi^2(y) \rangle = \frac{1}{2\pi} \frac{\psi(\lambda_\phi)}{\lambda_\phi - \frac{1}{2}} = \int_2^{\infty} d\lambda \rho_{\phi^2}(\lambda) \frac{1}{\lambda(\lambda - 1)}. \tag{D.10}
\]
As expected the one point functions in (D.8) are divergent when \( d = 2, 4 \). Subtracting the poles at \( d = 2, 4 \) as usual we have

\[
2\pi \langle \phi^2 \rangle_{\text{reg}}|_{d=2} = -\ln \frac{\rho}{\mu} - \psi(\lambda_{\phi}) - r, \\
8\pi^2 \langle \phi^2 \rangle_{\text{reg}}|_{d=4} = m^2 \left( \ln \frac{\rho}{\mu} + \psi(\lambda_{\phi} - 1) + r \right) + \rho^2 s, \tag{D.11}
\]

where \( \mu \) is a regularisation scale and \( r \), which may be absorbed into the definition of \( \mu \), and \( s \) are arbitrary parameters reflecting the precise choice of renormalisation scheme (to obtain the result for \( d = 4 \) it is essential to subtract a pole term \( \propto m^2/\varepsilon \), \( \varepsilon = 4 - d \), where \( m^2 \) is given by (D.4) including \( O(\varepsilon) \) terms). The \( \ln \rho/\mu \) terms reflect the mixing of the operator \( \phi^2 \) with \( 1, m^2 \) for \( d = 2, 4 \) respectively. By using (2.31) we may obtain \( \partial C/\partial m^2 \) in terms of \( \langle \phi^2 \rangle_{\text{reg}} \) and integrating this gives

\[
g^{\mu\nu}\langle T_{\mu\nu} \rangle = -C \rho^d = -m^2 \langle \phi^2 \rangle_{\text{reg}} + A, \\
2\pi A|_{d=2} = -\frac{1}{2} m^2 - \frac{1}{6} c \rho^2, \\
2\pi^2 A|_{d=4} = \frac{1}{2} m^4 - 3a \rho^4. \tag{D.12}
\]

The free parameters \( r, s, \) for \( d = 2, 4 \), correspond to the potential freedom of adding to the action terms of the form \( \int d^2 x \sqrt{g} m^2 \) and \( \int d^4 x \sqrt{g} m^4, \int d^4 x \sqrt{g} m^2 R \) respectively. The undetermined integration constants \( c, a \) in (D.12), which are independent of \( m^2 \), cannot be so modified and are therefore renormalisation scheme independent. The \( \ln \rho/\mu \) terms demonstrate the mixing of \( T_{\mu\nu} \) with \( g_{\mu\nu} m^2, g_{\mu\nu} m^4 \) for \( d = 2, 4 \) (even for free theories and defining the energy momentum tensor through normal ordering for instance there are such terms if the mass used to define normal ordering is varied from the physical \( m \)).

Forte and Latorre effectively choose \( r, s \), as well as \( \mu \), by imposing the natural decoupling condition that \( \langle T_{\mu\nu} \rangle \), and also \( \langle \phi^2 \rangle \), should vanish as \( m^2 \to \infty \).\(^{13}\) This further determines the integration constants \( c, a \). With \( \hat{C} \) defined by (14.13) and using the expansion of \( \psi(x) \) for large \( x \), the decoupling condition leads to

\[
\hat{C}_{FL}|_{d=2} = \frac{m^2}{\rho^2} 2\pi \langle \phi^2 \rangle_{FL} + \frac{1}{6} = \frac{m^2}{\rho^2} (\psi(\lambda_{\phi}) - \ln \frac{m}{\rho} - \frac{1}{6} \frac{\rho^2}{m^2}) \quad \Rightarrow \quad c = 1, \tag{D.13}
\]

for \( d = 2 \) and for \( d = 4 \)

\[
\hat{C}_{FL}|_{d=4} = \frac{m^4}{\rho^4} 2\pi^2 \langle \phi^2 \rangle_{FL} + \frac{1}{120} = \frac{m^4}{4\rho^4} (\psi(\lambda_{\phi} - 1) - \ln \frac{m}{\rho} - \frac{1}{6} \frac{\rho^2}{m^2} + \frac{1}{30} \frac{\rho^4}{m^4}) \quad \Rightarrow \quad a = \frac{1}{300}. \tag{D.14}
\]

\(^{16}\) However for \( d = 3 \) where there are no ambiguities \( 4\pi \langle \phi^2 \rangle = -(m^2 + \frac{1}{4} \rho^2)^{\frac{3}{2}} \) for \( R < 0 \) and \( 4\pi \langle \phi^2 \rangle = -(\frac{1}{4} \rho^2 - m^2)^{\frac{3}{2}} \cot \pi \sqrt{\frac{1}{4} - m^2/\rho^2} \) if \( R > 0 \).
These results for $c, a$ determine $C$ in the conformal limit $m^2 \to 0$ and are just as expected for free scalar theories in 2, 4 dimensions. This prescription also ensures that $A$ is independent of $m^2$, in contrast to (D.12). Both results for $\hat{C}_{FL}$ decrease monotonically to zero as $m^2$ increases from 0 to $\infty$. For $d = 2$ the result for the $m^2$ derivative can be expressed, in agreement with (14.16) and (14.17), as
\[-2m^2 \frac{\partial}{\partial m^2} \hat{C}_{FL} = -2\hat{C}_{FL} + \mathcal{G}_0,\]
where explicitly
\[\mathcal{G}_0 = \frac{m^4}{\rho^6} \frac{\psi'(\lambda_\phi - \frac{3}{2})}{\lambda_\phi} - \frac{m^2}{\rho^2} + \frac{1}{3} = 4 \int_{2\lambda_\phi}^\infty d\rho_0(\lambda) \frac{1}{\lambda(\lambda - 1)}.\]
The $\lambda$ integral, with $\rho_0$ given by (D.7), (D.6) and (D.4) for $d = 2$, may be verified numerically and in special cases analytically. $\mathcal{G}_0$ is equal to $2\pi \rho^{-2} m^4 \int d^2 y \sqrt{g} \langle \phi^2(x) \phi^2(y) \rangle$ with the $O(m^2)$ and $O(1)$ terms at large $m^2$ subtracted. For $d = 4$ the corresponding result is again in accord with the general expression given by (14.16) and (14.17),
\[-2m^2 \frac{\partial}{\partial m^2} \hat{C}_{FL} = -4\hat{C}_{FL} + \mathcal{G}_0,\]
if we now take
\[\mathcal{G}_0 = \frac{1}{4} \left( -\frac{m^6}{\rho^6} \frac{\psi'(\lambda_\phi - \frac{3}{2})}{\lambda_\phi} + \frac{m^4}{\rho^4} - \frac{1}{3} \frac{m^2}{\rho^2} + \frac{2}{15} \right) = 144 \frac{1}{\rho^2} \int_{2\lambda_\phi}^\infty d\rho_0(\lambda) \frac{1}{\lambda(\lambda - 3)} - \frac{1}{15}.\]
The additional constant present in (D.18) in the expression for $\mathcal{G}_0$ beyond that given by (13.26) in terms of the integral over $\rho_0(\lambda)$ is necessary since the right hand sides of (D.13),(D.17) should vanish as $m^2 \to 0$ while the integral in this limit is restricted to $\lambda = d$, according to (D.4), and is determined by $C_0 \propto C_T \propto c$ while in four dimensions $\hat{C}_{FL} \propto a$. From (13.26) and (D.7) we would have $\mathcal{G}_0|_{m^2=0} = d/(2^{d-1}(d+1))$.

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17 For $R > 0$ a similar approach gives for $d = 2$, $4\pi \langle \phi^2 \rangle = \ln m^2/\rho^2 - \psi(\alpha_+) - \psi(\alpha_-)$ and for $d = 4$, $16\pi^2 \langle \phi^2 \rangle = m^2 (-\ln m^2/\rho^2 + \frac{1}{3} \rho^2/m^2 + \psi(\alpha_+) + \psi(\alpha_-))$ where in both cases $\alpha_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - m^2/\rho^2}$. By integrating (2.30) with the boundary condition that $\hat{C}$ vanishes as $m^2 \to \infty$ we get for $d = 2$, $\hat{C} = m^2 2\pi \langle \phi^2 \rangle/\rho^2 - \frac{1}{6}$ and for $d = 4$, $\hat{C} = m^2 2\pi \langle \phi^2 \rangle/\rho^4 + \frac{1}{120}$. The integration constants are just those expected from the trace anomaly. When $d = 2$ $\hat{C}$ decreases monotonically with increasing $m^2$, despite the sign of the anomaly in this case, as a consequence of the result $4\pi \langle \phi^2 \rangle \sim \rho^2/m^2$ as $m^2 \to 0$ with the singularity arising from the existence of constant normalisable zero modes for $-\nabla^2$ on a sphere. When $d = 4$ $\hat{C}$ changes sign at $m^2 \approx 0.2\rho^2$ before tending to zero.
References

[1] H.W.J. Blöte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 273; I. Affleck, Phys. Rev. Lett. 56 (1986) 746.

[2] A.B. Zamolodchikov, JETP Lett. 43 (1986) 730; Sov. J. Nucl. Phys. 46 (1988) 1090.

[3] D. Boyanovsky and R. Holman, Phys. Rev. D40 (1989) 1964, Nucl. Phys. B332 (1990) 641; N.E. Mavromatos, J.L. Miramontes and J.M. Sánchez de Santos, Phys. Rev. D40 (1989) 535; N.E. Mavromatos and J.L. Miramontes, Phys. Lett. B226 (1989) 291; J. Gaite, Phys. Rev. Lett. 81 (1998) 3587, Entropic C theorems in free and interacting two-dimensional field theories, hep-th/9810107.

[4] J.L. Cardy, Phys. Lett. B215 (1988) 749.

[5] H. Osborn, Phys. Lett. B222 (1989) 97; I. Jack and H. Osborn, Nucl. Phys. B343 (1990) 647.

[6] A. Cappelli, D. Friedan and J.I. Latorre, Nucl. Phys. B352 (1991) 616.

[7] A. Cappelli, J.I. Latorre and X. Vilasís-Cardona, Nucl. Phys. B376 (1992) 510.

[8] G.M. Shore, Phys. Lett. B253 (1991) 380; B256 (1991) 407.

[9] A.H. Castro Neto and E. Fradkin, Nucl. Phys. B400 (1993) 525.

[10] S. Forte and J.I. Latorre, A Proof of the Irreversibility of Renormalization Group Flows in Four Dimensions, Nucl. Phys. B535 [FS] (1998) 709, hep-th/9805013.

[11] T. Appelquist, A.G. Cohen and M. Schmaltz, A new constraint on strongly coupled field theories, Phys. Rev. D60 (1999) 045003, hep-th/9901109.

[12] D. Anselmi, Anomalies, Unitarity and Quantum Irreversibility, Ann. Phys. 276 (1999) 361, hep-th/9903059.

[13] E. Álvarez and C. Gómez, Geometric Holography, the Renormalization Group and the c-Theorem, Nucl. Phys. B441 (1999) 441, hep-th/9807226.

[14] D.Z. Freedman, S.S. Gubser, K. Pilch and N. Warner, Renormalization Group Flows from Holography - Supersymmetry and a c-Theorem, hep-th/9904017.

[15] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, Nucl. Phys. B526 (1998) 543, hep-th/9708042.

[16] D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, Phys. Rev. D57 (1998) 7570, hep-th/9711035.
[17] D.Z. Freedman and H. Osborn, *Constructing a c-function for SUSY Gauge Theories*, Phys. Lett. B432 (1998) 353, [hep-th/9804101](https://arxiv.org/abs/hep-th/9804101).

[18] S.L. Adler, Phys. Rev. D6 (1972) 3445; D8 (1973) 2400.

[19] D.W. Düsedau and D.Z. Freedman, Phys. Rev. D33 (1986) 389.

[20] C.G. Callan and F. Wilczek, Nucl. Phys. B340 (1990) 366.

[21] H. Osborn, Nucl. Phys. B363 (1991) 486.

[22] B. Allen and T. Jacobson, Comm. Math. Phys. 103 (1986) 669.

[23] B.S. DeWitt, *Dynamical Theory of Groups and Fields*, (Gordon and Breach, 1965).

[24] S.A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time*, (Cambridge University Press, 1989).

[25] B. Allen and C.A. Lütken, Comm. Math. Phys. 106 (1986) 201.

[26] C.J.C. Burges, D.Z. Freedman, S. Davis and G.W. Gibbons. Ann. Phys. 167 (1986) 285.

[27] H. Lü, C.N. Pope and J. Rahmfeld, *A construction of Killing spinors on S^n*, [hep-th/9805151](https://arxiv.org/abs/hep-th/9805151).

[28] I. Antoniadis and E. Mottola, J. Math. Phys. 32 (1991) 1037.

[29] E. D’Hoker and D.Z. Freedman, *Gauge boson exchange in AdS_d+1*, Nucl. Phys. B544 (1999) 612, [hep-th/9809179](https://arxiv.org/abs/hep-th/9809179);
E. D’Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Graviton and gauge boson propagators in AdS_d+1*, [hep-th/9902042](https://arxiv.org/abs/hep-th/9902042).

[30] J.L. Cardy, Nucl. Phys. B290 (1987) 355.

[31] H. Osborn and A. Petkou, Ann. Phys. 231 (1994) 311, [hep-th/9307010](https://arxiv.org/abs/hep-th/9307010).

[32] J. Erdmenger and H. Osborn, Nucl. Phys. B483 (1997) 431, [hep-th/9605009](https://arxiv.org/abs/hep-th/9605009).

[33] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, (Cambridge University Press, 1982).

[34] I.T. Drummond and G.M. Shore, Phys. Rev. D19 (1979) 1134;
G.M. Shore, Phys. Rev. D21 (1980) 2226.

[35] R. Camporesi and A. Higuchi, Phys. Rev. D45 (1992) 3591.

[36] E. Mottola, Phys. Rev. D33 (1986) 2136.

[37] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, Adv. Theor. Math. Phys. 2 (1998) 697, [hep-th/9806074](https://arxiv.org/abs/hep-th/9806074).
[38] C. Fronsdal, Phys. Rev. D12 (1975) 3819.

[39] J.L. Cardy, Phys. Rev. Lett. 60 (1988) 2709;
   John Cardy, Scaling and Renormalization in Statistical Physics, (Cambridge University Press, 1996).

[40] N.K. Nielsen, Nucl. Phys. B65 (1973) 413; B97 (1975) 527;
   S. Sarker, Nucl. Phys. B83 (1974) 108;
   R.J. Crewther, Phys. Lett. 397 (1997) 137.

[41] J.I. Latorre and H. Osborn, Nucl. Phys. B511 (1998) 737.

[42] I.T. Drummond and G.M. Shore, Ann. Phys. 117 (1979) 89.