Schur-Weyl Categories and non-quasiclassical Weyl type Formula

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Abstract

To a vector space $V$ equipped with a non-quasiclassical involutary solution of the quantum Yang-Baxter equation and a partition $\lambda$, we associate a vector space $V_\lambda$ and compute its dimension. The functor $V \mapsto V_\lambda$ is an analogue of the well-known Schur functor. The category generated by the objects $V_\lambda$ is called the Schur-Weyl category. We suggest a way to construct some related twisted varieties looking like orbits of semisimple elements in $sl(n)^*$. We consider in detail a particular case of such “twisted orbits”, namely the twisted non-quasiclassical hyperboloid and we define the twisted Casimir operator on it. In this case, we obtain a formula looking like the Weyl formula, and describing the asymptotic behavior of the function $N(\lambda) = \{ \sum \lambda_i \leq \lambda \}$, where $\lambda_i$ are the eigenvalues of this operator.

0 Introduction

It is well-known that the motivation to introduce and develop the theory of quantum groups arose from the theory of integrable system. Quantum groups (QG) and their dual objects (“quantum cogroups”) supplied us an adequate language to describe symmetries of some integrable models. However, these objects have found another important application. It turned out
that they provided us with a natural way to enlarge the framework of classical geometry. Namely, it was recognized that the usual flip $\sigma$ occurring in numerous constructions of commutative and non-commutative\footnote{Note that the term “non-commutative geometry” is now used abusively in different senses. We prefer to reserve it for non-commutative geometry in the sense of A.Connes. This type geometry deals with non-commutative algebras looking like operator algebras in non-twisted categories. These algebras are usually equipped with a commutative trace and an involution satisfying the classical property $(A B)^* = B^* A^*$ and their derivations are usually defined by means of the usual Leibniz rule. Cyclic (co)homology is also defined by means of the classical flip $\sigma$. In contrast, “twisted” geometry deals with algebras equipped with a twist different from the usual flip. Such algebras can be “twisted commutative” or “twisted non-commutative”. In the latter case they are realized as operator algebras in twisted categories in the spirit of \cite{8}.} geometry can be replaced by some other twists, in particular, those arising from the well known QG $U_q(\mathfrak{g})$.

By a twist we mean a solution of the so-called quantum Yang-Baxter equation (QYBE)
\begin{equation}
S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}
\end{equation}
where
\begin{equation}
S : V^{\otimes 2} \to V^{\otimes 2}
\end{equation}
is a linear operator, $V$ is a vector space over a basic field $k$ and as usual
\begin{equation}
S^{12} = S \otimes \text{Id} \quad \text{and} \quad S^{23} = \text{Id} \otimes S.
\end{equation}
(In a more general context, $S^{12}$ and $S^{23}$ are treated as operators acting on the tensor product of three vector spaces $U \otimes V \otimes W$.)

Thus, the twists arising from the QG $U_q(\mathfrak{g})$ are defined by
\begin{equation}
S = \sigma(\rho_U \otimes \rho_U) R,
\end{equation}
where $R$ is the quantum universal R-matrix corresponding to the QG in question and $\rho_U$ is a representation of $U_q(\mathfrak{g})$ into a given $U_q(\mathfrak{g})$-module $U$.

Let us mention two properties of this twist. In the first place, it is a deformation of the flip $\sigma$. The category whose twists possess this property as well as their objects will be called \textit{quasiclassical}. Secondly, it is not involutory ($S^2 \neq \text{Id}$) and it cannot be made involutory by a rescaling $S \to aS$, $a \in k$.

This second property gives rise to some difficulties, that do not occur in the case of an involutory twist (in the sequel called a \textit{symmetry}).
following question appears from the very beginning: which algebras can be considered as twisted analogues of commutative algebras and, in particular, which system of equations compatible with the action of the QG $U_q(\mathfrak{g})$ gives rise to a “twisted variety”? (We refer the reader to the paper [8] for a detailed discussion of this problem.)

Nevertheless, the fact that the twist (0.1) is quasiclassical gives us a criterion of a “raison d’être” for $U_q(\mathfrak{g})$-covariant algebras: such an algebra is of interest if it is a flat deformation of its classical counterpart\(^2\).

However, for non-quasiclassical twists this criterion is no longer valid. Thus, it is not so evident what algebra arising from such a twist can be considered as a “twisted variety”. In the present paper we suggest a way to construct some twisted non-quasiclassical varieties looking like the orbits of semisimple elements in $sl(n)^\dagger$. (By abusing the language we use the term “variety” for the corresponding function algebra.) Nevertheless, we restrict ourselves to algebras connected with symmetries (a way to generalize our scheme to some non-quasiclassical twists of Hecke type is discussed in the last section).

To construct such a twisted variety we need first a tensor category possessing a “sufficiently large” supply of objects. We construct such categories looking like that of $SL(n)$-modules and we call them Schur-Weyl (SW) categories. Hopefully, such a category can also be treated as the one consisting of $sl(V_S)$-modules, where $sl(V_S)$ is a twisted non-quasiclassical analogue of the Lie algebra $sl(n)$. However, these categories can be introduced directly without any (usual or twisted) Hopf structure or twisted Lie algebra. The objects of such a category are labeled by Young diagrams (up to some identification) in a way similar to the classical one but their dimensions are different from the classical ones.

This implies some drastic modifications in the well-known asymptotic Weyl formula. In its classical form it says that the function
\[
N(\lambda) = \{ \# \lambda_i \leq \lambda \}
\]  

where $\lambda_n$ are eigenvalues of the Laplace-Beltrami operator on a smooth com-

\(^2\)Let us recall that a $k[[[\hbar]]]$-module $A_\hbar$ is called a flat deformation of a $k$-module $A$ if $A_\hbar$ and $A[[[\hbar]]]$ are isomorphic as $k[[[\hbar]]]$-modules and $A_\hbar/\hbar A_\hbar = A$. Here $\hbar$ is a formal parameter, $k$ is the basic field and $A[[[\hbar]]]$ stands for the completion of $A \otimes k[[[\hbar]]]$ in the $\hbar$-adic topology.
pact (pseudo)Riemannian variety $M$ has the following asymptotic behavior

$$N(\lambda) \sim c\lambda^{n/2}, \quad n = \dim M$$

(0.4)

with some constant $c$ depending on the volume of $M$.

However, this formula is no longer valid in a non-quasiclassical case. In the present paper we show that on a twisted non-quasiclassical hyperboloid (which is the simplest example of a “twisted non-quasiclassical orbit”) the function (0.3) of the twisted Casimir operator has an exponential growth w.r.t. $\sqrt{\lambda}$. (Let us observe that on symmetric orbits in $\mathfrak{g}^*$ the Casimir operator is equal up to a factor to the Laplace-Beltrami one if the latter is $\mathfrak{g}$-invariant.)

To explain the reason of this phenomenon let us recall first some aspects of “twisted linear algebra” developed essentially in [12] and [16]. Let us fix a symmetry (0.1) and associate to it symmetric and skew-symmetric algebras in a natural way

$$\text{Sym}(V) = \wedge_+(V) = T(V)/\{\text{Im} (\text{Id} - S)\}, \quad \wedge_-(V) = T(V)/\{\text{Im} (\text{Id} + S)\}$$

(0.5)

where $T(V)$ is the free tensor algebra of the space $V$ and $\{I\}$ stands for the ideal generated by a subset $I \subset T(V)$. (These algebras are also well defined for the so-called Hecke symmetries, i.e. twists satisfying the relation

$$(q \text{Id} - S)(\text{Id} + S) = 0, \quad q \in k$$

if $q$ is generic.)

Let us remark that the algebra $\wedge_+(V)$ is $S$-commutative (or simply commutative) in the following sense. We say that an algebra $\mathcal{A}$ equipped with a symmetry $S : \mathcal{A} \otimes^2 \rightarrow \mathcal{A} \otimes^2$ is commutative if

$$\mu = \mu S \quad \text{and} \quad S\mu^{12} = \mu^{23} S^{12} S^{23}$$

(0.6)

where $\mu : \mathcal{A} \otimes^2 \rightarrow \mathcal{A}$ is the product in $\mathcal{A}$ (the second relation implies a similar one with interchanged couples of indexes 12 and 23).

So, identifying as usual

$$\text{Fun}(V^*) \approx \wedge_+(V)$$

(0.7)

we can treat $V^*$ as an example of a twisted variety which is not, however interesting from the geometrical viewpoint. Nevertheless, we want to point
out the main peculiarity of this “variety”: the supply of elements of the algebra (1.7) differs drastically from that in the classical case. The very useful tool allowing us to measure this supply is the so-called Poincaré (or Poincaré-Hilbert) series. These series are defined for symmetric \( \wedge_+(V) \) and skew-symmetric \( \wedge_-(V) \) algebras by

\[
P_{\pm}(t) = \sum \dim \wedge^l_\pm(V) t^l
\]

where \( \wedge^l_\pm(V) \) is the degree \( l \) homogeneous component of the algebra in question. However, the classical relation

\[
P_+(t)P_-(\frac{1}{t}) = \text{Id}
\]

is valid for any symmetry (or a Hecke symmetry for a generic \( q \), cf. [12]).

As shown in [12], there exist a lot of symmetries (1.1) with \( \dim V = n \geq 3 \) such that the corresponding Poincaré series \( P_-(t) \) is a monic polynomial of degree \( p < n \). The degree \( p \) is called rank of the space \( V_S \) and denoted \( \text{rank} V = \text{rank} V_S \). (We use the notation \( V_S \) for a vector space \( V \) equipped with a twist (0.1).)

Let us remark that in the classical case (\( S = \sigma \)) we have \( P_-(t) = (1 + t)^n \) and consequently, \( \text{rank} V_S = \dim V_S \). This is also valid for any symmetry (or a Hecke symmetry) being a deformation of the classical flip. A (Hecke) symmetry and corresponding vector space \( V_S \) whose Poincaré series \( P_-(t) \) is a monic polynomial will be called even.

Thus, even symmetries whose Poincaré series \( P_-(t) \) are different from \( (1 + t)^n \), \( n = \dim V \) cannot be obtained by a deformation of the flip \( \sigma \). We call them non-quasiclassical. Other varieties, different from \( V^* \), and more interesting from a geometric point, are discussed in this paper. They are also non-quasiclassical, i.e., they are not a deformation of a classical variety, and the “supply of elements” in the corresponding algebras differs drastically from the classical one. As a measure of this supply we consider the function (1.3) corresponding to the “twisted Casimir operator”. (Let us emphasize that this operator arises from a Casimir element, which is not analogue of the Casimir element from \( U_q(g) \), see Section 4).

This paper is organized as follows. In Section 1, we recall some facts from [12] about a possible form of an even symmetry (1.1). The aim is to show that the family of non-quasiclassical symmetries whose so-called determinant is central is big enough. In Section 2, we introduce the Schur-Weyl
category generated by a vector space $V_S$ equipped with such a symmetry. This category is formed by objects $V_{\lambda}$ arising from twisted analogues of the Schur functor and their direct sums. The main result of this Section is the computation of $\dim V_{\lambda}$. In Section 3 we introduce a twisted Lie algebra of $sl(n)$ type which plays the role of symmetries of the SW category and define a twisted analogue of the Casimir element. In section 4 we consider a particular example of a twisted non-quasiclassical variety, namely the twisted non-quasiclassical hyperboloid, and give an estimation of the function $N(\lambda)$ corresponding to the Casimir operator on it. We end the paper with a discussion of those aspects of our approach that can be generalized to Hecke symmetries, and we suggest a possible way to obtain other “twisted non-quasiclassical orbits”.

Throughout the whole of the paper the basic field $k$ is $\mathbb{C}$ or $\mathbb{R}$.

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1 Even symmetries

We begin with the following observation. Usually, we assume that a tensor category (or a quasitensor category, in the terminology of \[\]) is given, for example the category of $U_q(g)$-modules, and we study the properties of the objects of such a category. Our approach will be completely opposite: We begin with a basic object $V_S$ and generate some tensor category from it.

Moreover, this category can be constructed without any Hopf or twisted Hopf algebra. We use a twisted Lie algebra (its enveloping algebra can be equipped with a twisted Hopf algebra structure) only to describe twisted orbits and define twisted Casimir operator. From this viewpoint, Hopf or twisted Hopf algebras are derived objects themselves and they can be found from the reconstruction theorems (cf. \[8\]), although their explicit description is not always easy (cf. \[] where an attempt is made to describe an analogue of the QG $U_q(g)$ for some non-quasiclassical Hecke symmetries).

Let us pass now to describing a possible form of an even symmetry. Let us fix a space $V = V_S$. Let $T^m(V) = V^\otimes m$ be the $m$-th tensor power of the space $V$ (with $T^0(V) = k$) and $T(V) = \bigoplus_{m=0}^{\infty} T^m(V)$ its free tensor algebra. The symmetry $S$ can be naturally extended to the tensor algebra: we have

$$S : T^m(V) \otimes T^n(V) \to T^n(V) \otimes T^m(V)$$
and therefore $S : T(V)^{\otimes 2} \to T(V)^{\otimes 2}$ (we keep the notation $S$ for the extended symmetry). Moreover, we assume that

$$S(a \otimes x) = x \otimes a \quad \forall x \in T^m(V), \quad \forall a \in k.$$  

In fact, a symmetry allows us to equip the space $T^m(V)$ with a representation of the symmetric group $S(m)$ in the natural way. We associate to an elementary transposition $s_{i\,i+1} \in S(m)$ the operator

$$S_{i\,i+1} = \text{Id}_{i-1} \otimes S \otimes \text{Id}_{m-i-2}$$  

where $\text{Id}_i$ is the identity operator on $T^i(V)$. Any element of the symmetric group can be expressed as a monomial of the elementary transpositions. By substituting in this monomial the operators $S_{i\,i+1}$ we get a representation of the symmetric group $S(m)$ into the space $T^m(V)$. Consequently, we have a representation of the group algebra $k[S(m)]$. It will be denoted $\rho_S$. (Note that we treat the space $T^m(V)$ as a left $k[S(m)]$-module, i.e., $(AB)x = A(Bx)$ for any $x \in T^m(V)$ and $A, B \in k[S(m)]$.)

**Remark 1.1** This representation of the symmetric group in tensor powers of a linear space has a very particular property: the operators $S_{i\,i+1}$ and $S_{i+1\,i+2}$ are related by the formula

$$S_{i+1\,i+2} = \sigma_{i+1} \sigma_{i+1\,i+2} S_{i\,i+1} \sigma_{i+1\,i+2} \sigma_{i\,i+1} \quad 1 \leq i \leq m - 2.$$  

More precisely, the space $T^m(V)$ is equipped with an action of $S(m) \times S(m)$, one copy of $S(m)$ being represented by $\rho_S$ and the other one by $\rho_{\sigma}$ and they are related by the above formula.

Let us consider the corresponding symmetric $\wedge_+(V)$ and skew-symmetric $\wedge_-(V)$ algebras defined in the Introduction, and the corresponding Poincaré series $P_{\pm}(t)$.

**Definition 1.2** We say that a symmetry (0.1) or the corresponding space $V = V_S$ is even (resp., odd) if the Poincaré series $P_-(t)$ (resp., $P_+(t)$) is a monic polynomial (i.e. a polynomial with leading coefficient 1). For an even symmetry $S$, we call the degree of the polynomial $P_-(t)$ the rank of $V$, and we denote this by $\text{rank} V$. 

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Remark 1.3 As shown in [19], the Poincaré series of a Hecke symmetry is a rational function (a proof in the case of symmetries also appeared in [6]). Let us assume that $P_-(t)$ is a rational function with monic numerator and denominator, and no common factors in numerator and denominator. We introduce the bi-rank $(p, q)$ as the ordered pair consisting of the degrees of the numerator and denominator of $P_-(t)$. Thus, for any even (resp., odd) symmetry, we have bi-rank $V_S = (p, 0)$ (resp., bi-rank $V_S = (0, q)$). Let us remark that the notion of bi-rank is a generalization of super-dimension (see also Remark 1.5).

In the sequel, we deal with even symmetries. Our next aim is to introduce the dual space $V^*$. A space $V^*$ is called right dual if there exists an extension of $S$ to

$$(V \oplus V^*)^{\otimes 2} \rightarrow (V \oplus V^*)^{\otimes 2}$$

and an invariant pairing

$$V \otimes V^* \rightarrow k.$$  

"Invariant" means that this pairing commutes with $S$ in the following sense

$$< , >^{12} S^{23} S^{12} = S < , >^{23}$$

where

$$< , >^{12} = < , > \otimes \text{Id}, \quad < , >^{23} = \text{Id} \otimes < , >.$$ 

Both sides of this formula are treated as operators acting on $V \otimes V \otimes V^*$. Hereafter we index the operators in question from left to right (for example, in the above formula, the operator $S^{12}$ acts on $V^{\otimes 2}$ and $S^{23}$ acts on $V \otimes V^*$).

In a similar sense we will speak about invariance of other linear maps $V^{\otimes i} \rightarrow V^{\otimes j}$.

Let us show that for any even (Hecke) symmetry the right dual space exists.

To an even symmetry $S$, we associate the projector

$$P^p_\alpha : T^p(V) \rightarrow \Lambda^p_\alpha(V).$$

In view of Definition 1.2, we have $\dim \text{Im } P^p_\alpha = 1$.

Fix a base

$$\{x_i\}, \quad 1 \leq i \leq n = \dim V, \quad x_i \in V.$$
Then the projector $P_p$ can be described as follows

$$P_p : x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_p} \rightarrow u_{i_1i_2...i_p}v \quad (1.10)$$

where

$$v = v^{j_1j_2...j_p} x_{j_1} \otimes x_{j_2} \otimes ... \otimes x_{j_p} \in T^p(V) \quad \text{and} \quad u_{i_1i_2...i_p} v^{i_1i_2...i_p} = 1. \quad (1.11)$$

Any index that appears as an upper and a lower index is assumed to be a summation index; we omit the summation symbol $\Sigma$.

**Definition 1.4** *The element $v$ (resp., $u = u_{i_1i_2...i_p} x^{i_1} \otimes x^{i_2} \otimes ... \otimes x^{i_p}$) is called determinant (resp., codeterminant) and denoted $\det$ (resp., $\text{codet}$).*

Let us remark that the couple $(\det, \text{codet})$ is defined up to a change

$$\det \rightarrow a \det, \quad \text{codet} \rightarrow a^{-1} \text{codet}, \quad a \neq 0.$$

In the sequel, we assume that such a couple is fixed.

We say that the space $W \subset T^m(V)$ is invariant if

$$S(W \otimes V) \subset V \otimes W.$$

It is not difficult to see that the subspace $\text{Im} P_p \subset T^p(V)$ is invariant. This follows from the fact that the projector $P_p$ can be expressed as a polynomial in $S^{12}, S^{23}, ..., S^{p-1p}$. However, the determinant itself is in general not central. Remark that the determinant and codeterminant become simultaneously central.

We introduce two operators $M = (M^j_i)$ and $N = (N^j_i)$ acting on $V$, defined by their matrices with respect to the basis $\{x_i\}$:

$$M^j_i = u_{i_1i_2...i_{p-1}} v^{j_1j_2...j_{p-1}} \quad \text{and} \quad N^j_i = u_{i_1i_2...i_{p-1}} v^{i_1i_2...i_{p-1}j}$$

(we use the notation of [12]). In [12] it is shown that

$$M^j_i N^k_j = p^{-2} \delta^k_i \quad (1.12)$$

($\delta^k_i$ is the Kronecker symbol), and

$$S(v \otimes x_i) = (-1)^{p-1} p M^j_i (x_j \otimes v), \quad S(x_i \otimes v) = (-1)^{p-1} p N^j_i (v \otimes x_j). \quad (1.13)$$
Let $\det^{-1}$ be a new formal generator such that

$$\det^{-1} \det = \det \det^{-1} = 1.$$  

This implies that the commutation rule of $\det^{-1}$ with the elements of the space $V$ is the inverse of (1.13), namely

$$S(\det^{-1} \otimes x_i) = (-1)^{p-1} pN_i^j (x_j \otimes \det^{-1})$$

$$S(x_i \otimes \det^{-1}) = (-1)^{p-1} pM_i^j (\det^{-1} \otimes x_j).$$

Let us define the dual space $V^*$ by fixing a base $\{x^j\}$, $1 \leq j \leq n$ such that $x^j$ is identified with

$$\det^{-1} \otimes (v^{j_1j_2...j_{p-1}j} x_{j_1} \otimes x_{j_2} \otimes ... \otimes x_{j_{p-1}}).$$

(1.14)

We leave it to the reader to verify that the pairing

$$< , >: V \otimes V^* \to k, \quad < , > x_i \otimes x^j \mapsto \delta_i^j$$

is invariant; it suffices to verify that the element $x^i \otimes x_i$ is invariant, i.e.

$$S(y \otimes (x^i \otimes x_i)) = (x^i \otimes x_i) \otimes y \quad \forall y \in V.$$  

This completes construction of the right dual space.

**Remark 1.5** There exists another base $\{y^j\}$ of the space $V^*$, such that the pairing

$$< , >: V^* \otimes V \to k, \quad < , > y^j \otimes x_i \mapsto \delta_i^j$$

is invariant. Thus, the space $V^*$ can also be treated as the left dual space of $V$, if we equip it with an appropriate pairing.

Such a base can be introduced as follows: let $y^j = C^j_i x^i$, where $C^j_i = T^{ijk}_{ik}$ and the operator $T = (T^{ij}_{ik})$ is defined by

$$S_{ij}^{kl} T_{km}^{in} = \delta_i^l \delta_j^m.$$  

We say that a twist $S$ is “invertible by column” if such an operator $T$ exists (it is easy to see that this is independent of the choice of basis). For any twist invertible by column the right and left dual spaces can be introduced
and they can be identified. In particular, this means that the extension (1.9) exists.

In the sequel, we also need the operator defined by the matrix $B^j_i = T_{kj}^i$, which is the inverse of the one defined by $C^j_i$, cf. [12].

Moreover, for any Hecke symmetry ($q$ is assumed to be generic), there exists a complex consisting of the terms $V^i \otimes (V^*)^j$ and a differential arising from the operator which is inverse to the pairing between $V$ and $V^*$. Such a complex was called in [12] Koszul complex of the second kind.

If the cohomology of this complex is one-dimensional, then its generator is called determinant. The above determinant is a particular case of the latter one. However, up to now it is not clear whether any (Hecke) symmetry invertible by column has a determinant. Hopefully, the Poincaré series of a Hecke symmetry invertible by column is a rational function with monic numerator and denominator (see Remark 1.3).

In [12], this complex was used to show that, for any even Hecke symmetry with a generic $q$, the polynomial $P^-(t)$ is reciprocal. The case of symmetries was considered previously in [10].

**Definition 1.6** We say that the determinant $\det$ is central if

\[
(-1)^{p-1}pM = \text{Id} \quad \text{and} \quad (-1)^{p-1}pN = \text{Id}.
\]

In fact, in view of (1.12), the first relation implies the second one and vice versa.

If $\det$ is central, then the dual space $V^*$ can be identified with $\wedge^{p-1}(V)$ since in the formula (1.14) we can omit the factor $\det^{-1}$. In other words, if $\det$ is central, then the map

\[
\wedge^p (V) \to k, \quad \det \mapsto 1
\]

is invariant.

Now we want to describe a family of even symmetries with non-quasiclassical Poincaré polynomial $P^-(t)$ and central determinant.

Let us begin with the simplest case $p = \text{rank} V = 2$. In this case the polynomial $P^-(t)$ is equal to $1 + nt + t^2$ where $n = \dim V \geq 2$ (the case $n = 2$ corresponds to the quasiclassical case). Then $S$ can be represented as

\[
S_{ij}^{kl} = \delta_{i}^{k} \delta_{j}^{l} - 2u_{ij} v^{kl}
\]
with

\[ u_{ij} v^{ij} = 1. \]  

(1.17)

Thus, the determinant in this case is \( v = v^{kl} x_k x_l \). If no confusion is possible, we omit the sign \( \otimes \). It is not difficult to see that if \( S \) is of the form (1.16), then the QYBE for it is equivalent to the relation

\[ u v u^t v^t = \frac{1}{4} \text{Id} \]

or, in a more detailed form,

\[ u_{ij} v^{jk} u_{lk} v^{ml} = \frac{1}{4} \delta^m_i \]

(1.18)

where \( u^t \) is the transposed of \( u \).

In [12], a classification of all solutions \((u, v)\) of the equations (1.17)-(1.18) is given, including a more general case of Hecke symmetries\(^3\). However, without any classification, it is easy to see that the system (1.18)-(1.17) possesses a large set of skew-diagonal solutions. These are solutions \((u, v)\) for which the only non-trivial elements appear on the skew-diagonals of \( u \) and \( v \), or

\[ u_{ij} = 0 = v^{ij} \quad \text{if} \quad i + j \neq n + 1. \]

Some of them satisfy a complementary condition of centrality of the determinant. In the case \( p = 2 \), this condition takes the following form

\[ 2u_{ij} v^{jk} = -\delta^k_i \quad \text{in particular,} \quad 2u_{i \, n+1-i} v^{n+1-i,i} = -1, \quad i = 1, \ldots, n \]

(1.19)

if \( u \) and \( v \) are skew-diagonal (in the latter formula no summation is assumed).

We leave it to the reader to describe the family of solutions of the system (1.17)-(1.19). We restrict attention to the case \( n = 3 \). In this case the family of the couples \((u, v)\) satisfying the system (1.17)-(1.19) is parameterized by two indeterminates: if we choose \( u_{13} = a \) and \( u_{22} = b \), then we have

\[ u_{31} = -a/x, \quad v^{13} = x/(2a), \quad v^{22} = -1/2b, \quad v^{31} = -1/2a \]

where \( x \) is a solution of the equation \( x + x^{-1} = 3 \).

\(^3\)Some symmetries of this type were discovered independently in [10].
Our next aim is to describe a way to construct even symmetries of rank greater than 2 with central determinant.

First, assume that two symmetries

\[ S_1 : V_1^\otimes 2 \to V_1^\otimes 2 \quad \text{and} \quad S_2 : V_2^\otimes 2 \to V_2^\otimes 2 \]

are given. We present a procedure that allows us to construct a symmetry \( S \) acting on the space \((V_1 \oplus V_2)^\otimes 2\). In [MM], such a procedure is called gluing; for Hecke symmetries it was suggested earlier in [12]. Let us assume that \( S \) transposes \( V_1 \) and \( V_2 \) by means of the usual flip \( \sigma \). Then \( S \) is a symmetry and we have

\[ P_{\pm}(t, V) = P_{\pm}(t, V_1) P_{\pm}(t, V_2). \]

Therefore, if \( V_1 \) and \( V_2 \) are even, then \( V \) is also even, and \( \text{rank} \, V = \text{rank} \, V_1 + \text{rank} \, V_2 \). Moreover, the matrices \( p(-1)^p M \) and \( p(-1)^p N \) related to the symmetry \( S \) and measuring non-centrality of the determinant connected to \( S \) are equal to the tensor product of the corresponding matrices related to \( S_1 \) and \( S_2 \). The first statement is shown in [12, Proposition 4.4] and the second one is obvious.

This procedure enables us to construct a big family of even non-quasiclassical symmetries of higher rank and with central determinant by starting with symmetries of rank 2 possessing this property. For example, if we take two symmetries

\[ S_1 : V_1^\otimes 2 \to V_1^\otimes 2 \quad \text{and} \quad S_2 : V_2^\otimes 2 \to V_2^\otimes 2. \]

with Poincaré polynomials \( P_\pm(t, V_i) = 1 + n_i t + t^2 \), where \( n_i = \dim V_i \geq 2, \ i = 1, 2, \) then

\[ P_\pm(t, V) = 1 + nt + (n_1n_2 + 2)t^2 + nt^3 + t^4, \quad n = n_1 + n_2. \]

Thus, for a fixed \( n = \dim V \geq 4 \) we can construct symmetries \([1.1]\) such that \( \text{rank} \, V_S = 4 \) and the middle coefficient of \( P_\pm(t) \) is equal to \( a(n - a) + 2, \ a = 2, 3, ..., n - 2 \) (for other coefficients of \( P_\pm(t) \) there is no choice).

These examples show that the Poincaré polynomial \( P_\pm(t) \) of an even symmetry is not determined by the couple \((\dim V, \text{rank} V)\).

Another way to construct even higher rank symmetries arises from Schur functors described in the next Section; hopefully, the determinant corresponding to the space \( V_\lambda \) is central as well.
Remark 1.7 If $S$ is an even symmetry, then $-S$ is an odd symmetry, i.e., the series $P_+(t)$ is a monic polynomial. Using the above gluing procedure we can produce mixed symmetries, whose Poincaré series $P_-(t)$ are rational functions with monic numerators and denominators. However, there exist symmetries possessing such a type Poincaré series $P_-(t)$ and such that the corresponding space $V$ cannot be split into a direct sum of an even subspace and an odd one. The simplest example of such a symmetry is the following one ([16]):

\[ V = \text{span}(x, y), \]

with

\[ S(x \otimes x) = x \otimes x + by \otimes y, \quad S(x \otimes y) = y \otimes x, \quad S(y \otimes y) = -y \otimes y, \quad b \in k \]

This space can be split into an even and an odd subspaces iff $b = 0$, and in this case it becomes a super-symmetry.

2 Schur-Weyl categories

Let

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k, \quad |\lambda| = \lambda_1 + \ldots + \lambda_k = m \]

be a partition of an integer $m$. The corresponding Young diagram will be also denoted by $\lambda$.

We consider the right regular representation of $k[S(m)]$, i.e. equip the algebra $k[S(m)]$ with the natural structure of a right $k[S(m)]$-module. Then this algebra can be presented as

\[ k[S(m)] = \bigoplus [M_{\lambda}]^{\otimes n_{\lambda}}, \quad n_{\lambda} = \dim [M_{\lambda}] \]

where $[M_{\lambda}]$ is the class of the pairwise isomorphic irreducible $k[S(m)]$-module corresponding to the partition $\lambda$ and $\lambda$ runs over all partitions of the integer $m$. We keep the notation $M_{\lambda}$ for a representative of this class. Thus, $M_{\lambda}$ is understood to be a $k[S(m)]$-module equipped with an embedding $M_{\lambda} \hookrightarrow k[S(m)]$.

In particular, such an embedding arises from the following procedure. Let us convert the diagram $\lambda$ into a tableau by arranging the integers 1, ..., $m$ by columns. This means that we put in the first column the numbers 1, 2, ..., $\lambda'_1$, in the second one $\lambda'_1 + 1, \lambda'_1 + 2, ..., \lambda'_2$ and so on, where

\[ \lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{k'}) \quad |\lambda'| = \lambda'_1 + \ldots + \lambda'_{k'} = m \]
is the partition dual to that \( \lambda \). We assign to this tableau the Young symmetrizor

\[ p_\lambda = c_\lambda r_\lambda \]

where \( r_\lambda \) (resp., \( c_\lambda \)) is the symmetrizor by lines (resp., the skew-symmetrizor by columns). Let us generate by the elements \( p_\lambda \in k[S(m)] \) the right \( k[S(m)] \)-module, i.e., consider the set

\[ p_\lambda q, \quad \forall q \in k[S(m)]. \tag{2.22} \]

This \( k[S(m)] \)-module is just a representative of the family \([M_\lambda]\). This module and all related objects will be called canonical (we will explain this choice of “canonical” embedding \( M_\lambda \hookrightarrow k[S(m)] \) later).

To a \( k[S(m)] \)-module \( M_\lambda \hookrightarrow k[S(m)] \), we associate the space

\[ V(M_\lambda) = V_\lambda = \text{Im} \rho_S(M_\lambda) \]

with \( \rho_S : k[S(m)] \to \text{End} (T^m(V)) \) as above. Thus, the space \( V_\lambda \) is equipped with an embedding \( V_\lambda \hookrightarrow T^m(V) \) depending on the embedding \( M_\lambda \hookrightarrow k[S(m)] \). Let \([V_\lambda]\) be the class of all such spaces \( V_\lambda \) embedded in \( T^m(V) \) in one or another way.

The image of the canonical tableau is just the set

\[ \text{Im} \rho_S(p_\lambda q) \quad \forall q \in k[S(m)] \]

Let \( \overline{M}_\lambda \) denote the two-sided module in \( k[S(m)] \) generated by all \( M_\lambda \) (in other words, \( \overline{M}_\lambda = [M_\lambda]^{\oplus n_\lambda} \)). Its image in \( T^m(V) \) will be denoted \( \overline{V}_\lambda \). Thus, we have

\[ \overline{V}_\lambda = \text{Im} \rho_S(q_1 p_\lambda q_2), \quad \forall q_1, q_2 \in k[S(m)] \]

In contrast to the space \( V_\lambda \), which depends on the chosen embedding \( M_\lambda \hookrightarrow k[S(m)] \), \( \overline{V}_\lambda \) depends only on \( \lambda \).

**Proposition 2.1** We have

\[ \dim \overline{V}_\lambda = \dim V_\lambda \dim M_\lambda. \]

**Proof** The statement follows immediately from [11, Lemma 6.22].

Let us consider the classical case \((S = \sigma)\) in more detail.
Let \( k = l(\lambda) \) be the length of the partition \( \lambda \), i.e., number of lines in the corresponding diagram. It is obvious that the space \( V_\lambda \) is trivial if \( l(\lambda) > n = \dim V \). It is well known that if we equip the initial space \( V \) with an action of the groups \( GL(n) \) the spaces \( V_\lambda \) become irreducible \( GL(n) \)-modules as well as their products with 

\[
(det g)^p, \ p \in \mathbb{Z}, \ g \in GL(n).
\]

The family of irreducible \( SL(n) \)-modules (considered up to isomorphisms) coincides with \( \{[V_\lambda], \ l(\lambda) \leq n\} \), up to the following identification. If two partitions differ by a shift, i.e., \( \mu = \lambda + a \) (this means that \( \mu_i = \lambda_i + a, \ 1 \leq i \leq n \)) then the corresponding irreducible \( SL(n) \)-modules are identified. This is motivated by the fact that a column consisting of \( n \) entries (\( n \)-column for short) corresponds to the representation defined by the determinant and is trivial for the group \( SL(n) \).

So, in the category of \( SL(n) \)-modules we can always reduce \( \lambda \) with \( \lambda_n \neq 0 \) to \( \mu \) with \( \mu_n = 0 \) by means of this identification. In what follows this operation will be called the reduction procedure.

Dimensions of the spaces \( V_\lambda \) can be found from the well known formula

\[
\dim V_\lambda = \frac{V_\lambda(z_1, z_2, ..., z_n)}{V(z_1, z_2, ..., z_n)}. \quad (2.23)
\]

Here \( V(z_1, z_2, ..., z_n) \) is the ordinary Vandermonde determinant in \( n \) indeterminates and \( V_\lambda(z_1, z_2, ..., z_n) \) is the generalized Vandermonde determinant corresponding to the partition \( \lambda \), and defined as follows

\[
\begin{vmatrix}
  z_1^{\lambda_1+n-1} & z_2^{\lambda_1+n-1} & \cdots & z_n^{\lambda_1+n-1} \\
  z_1^{\lambda_2+n-2} & z_2^{\lambda_2+n-2} & \cdots & z_n^{\lambda_2+n-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  z_1^{\lambda_n} & z_2^{\lambda_n} & \cdots & z_n^{\lambda_n}
\end{vmatrix}
\]

Observe that

\[
V(z_1, z_2, ..., z_n) = V_\lambda(z_1, z_2, ..., z_n) \quad \text{if} \quad \lambda = (0, 0, ..., 0).
\]

The quotient \( V_\lambda(z_1, z_2, ..., z_n)/V(z_1, z_2, ..., z_n) \) is called the Schur function (polynomial) in \( n \) indeterminates corresponding to the partition \( \lambda \) and is usually denoted \( s_\lambda = s_\lambda(z_1, z_2, ..., z_n) \). We do not consider Schur functions in
an infinite number of indeterminates, which can be defined as limits of the above ones for \( n \to \infty \). Thus, in virtue of (2.23) we have

\[
\dim V_\lambda = s_\lambda(1,1,...,1)
\] (2.24)

Note that if \( l(\lambda) = l < n \) we put in the above formula \( \lambda_{l+1} = ... = \lambda_n = 0 \).

Let us consider now the category \( SL(n)\text{-Mod} \) of all finite dimensional \( SL(n)\)-modules. The classes \([V_\lambda]\) with \( l(\lambda) < n \) form a base of this category, i.e. any object of \( SL(n)\text{-Mod} \) is isomorphic to a direct sum of irreducible \( SL(n)\)-modules \([V_\lambda]\). Thus, the tensor product of any two objects of this category is determined by that of two basic objects. The latter product is given by the formula

\[
[V_\lambda] \cdot [V_\mu] = c^{\nu}_{\lambda,\mu}[V_\nu].
\] (2.25)

The coefficients \( c^{\nu}_{\lambda,\mu} \) occurring in this decomposition can be found by means of the Littewood-Richardson rule. However, it is necessary to keep in the mind that if for some \( \nu \) entering this sum we have \( l(\nu) > n \) the component \([V_\nu]\) disappears and if \( l(\nu) = n \) we reduce \( \nu \) as above.

The algebra consisting of finite sums with integer coefficients of formal objects \([V_\lambda]\) equipped with the product (2.25) is called the fusion ring of the group \( G = SL(n) \) (or \( SU(n) \)).

Now we pass to the general case \( S \neq \sigma \). Although we do not have any object of Hopf algebra type (it will be introduced in the next Section), we can introduce a category looking like that of \( SL(n)\)-modules directly. Its objects are finite sums of the spaces \( V_\lambda \). In order to treat this category as a twisted tensor category, we have to explain how to decompose the tensor product

\[
V_\lambda \otimes V_\mu \hookrightarrow T^m(V) \otimes T^n(V) = T^{m+n}(V) \quad |\lambda| = m, \ |\mu| = n
\]

into a direct sum of \( V_\nu \). This can be done as follows.

We apply the set of operators \( \rho_S(M_\nu) \) to the product \( V_\lambda \otimes V_\mu \). This defines a projection of this product onto the component \( V_\nu \).

Observe that the twist

\[
S : V_\lambda \otimes V_\mu \to V_\mu \otimes V_\lambda
\] (2.26)

is well defined. By definition, it is the restriction of the twist \( S : T^l(V) \otimes T^m(V) \to T^m(V) \otimes T^l(V) \), where \( V_\lambda \) (resp. \( V_\mu \)) is embedded into \( T^l(V) \)
(resp., $T^m(V)$). It is left as an exercise to the reader to show that the image of $V_\lambda \otimes V_\mu$ belongs to $V_\mu \otimes V_\lambda$.

It is easy to see that the formula (2.25) is valid with the same coefficients as in the classical case, up to some modifications. The role of $n = \dim V$ is played by $p = \text{rank } V$. More precisely, a component $[V_\nu]$ occurring in the formula (2.25) is replaced by 0 if $l(\nu) > p = \text{rank } V$ and it is reduced as above if $l(\nu) = p$, assuming that the determinant defined in the previous section is central. Finally, we recover just the same fusion ring as in the classical case but with $n$ replaced by $p$. This fact has been already mentioned in the mathematical literature (cf. [3]), even in more general situation related to Hecke symmetries.

However, dimensions of the spaces $V_\lambda$ and the corresponding Clebsch-Gordan coefficients (which are defined if we fix some bases in the components $V_\lambda$) are drastically different from the classical ones. We will now calculate these dimensions.

Let $\beta_1, \beta_2, ..., \beta_p$ be the roots of the Poincaré polynomial $P_-(t)$ corresponding to an even space $V_S$ such that $\text{rank } V_S = p$ and let $\alpha_i = -\beta_i$. Then

$$P_-(t) = \prod (t + \alpha_i) = \prod (\alpha_i t + 1)$$

(in the latter equality we use the fact that this polynomial is reciprocal).

The following proposition is a generalization of the formula (2.24).

**Proposition 2.2** Assuming that $l(\lambda) \leq p$ we have

$$\dim V_\lambda = s_\lambda(\alpha_1, ..., \alpha_p).$$

This results immediately from Proposition 2.1 and the following.

**Proposition 2.3** ([16]) The multiplicity of the irreducible $k[S(m)]$-module $[M_\lambda]$ related to the partition $\lambda$ in the $k[S(m)]$-module $T^m(V)$ is equal to $s_\lambda(\alpha_1, ..., \alpha_p)$.

**Proof** Let $\chi_\lambda$ be the character of the $S(m)$-module $T^m(V)$ (as above, the algebra $k[S(m)]$ is represented by $\rho_S$), $\chi^\lambda$ the character of $[M_\lambda]$ and $\eta_m$ the character of the trivial representation of the group $S(m)$. Then the multiplicity of the irreducible $k[S(m)]$-module $[M_\lambda]$ is measured by the following
quantity

\[ \langle \chi^\lambda, \chi_m \rangle = \det(\langle \eta_{\lambda_1-i+j}, \chi_m \rangle S(m)) \]

\[ = \sum \text{sgn} \pi \]

\[ \langle \eta_{\lambda_1-1+\pi(1)} \cdots \eta_{\lambda_n-n+\pi(n)}, \chi_m \rangle S(m) \]

\[ = \sum \text{sgn} \pi \]

\[ \langle \text{ind}_{S(m)} \eta_{\lambda_1-1+\pi(1)} \times \cdots \times S(\lambda_m-1+\pi(m)) \eta_{\lambda_1-1+\pi(1)} \cdots \eta_{\lambda_n-n+\pi(n)}, \chi_m \rangle S(m) \]

\[ \text{res}_{S(m)} S(\lambda_1-1+\pi(1)) \times \cdots \times S(\lambda_m-1+\pi(m)) \]

\[ \sum \text{sgn} \pi \langle \eta_{\lambda_1-1+\pi(1)}, \chi_{\lambda_1-1+\pi(1)} S(\lambda_1-1+\pi(1)) \cdots \rangle \]

\[ \langle \eta_{\lambda_m-1+\pi(m)}, \chi_{\lambda_m-1+\pi(m)} S(\lambda_m-1+\pi(m)) \rangle \]

\[ = \det(\langle \eta_{\lambda_1-i+j}, \chi_{\lambda_1-i+j} S(\lambda_1-i+j) \rangle). \]

Here the pairing in question is \( S(m) \)-invariant and the Frobenius reciprocity is used.

Since \( \langle \eta_k, \chi_k \rangle_{S(k)} \) is the multiplicity of the trivial module in \( T^k(V) \), it is equal to \( \dim \wedge^k_+ (V) \). Thus, we have

\[ \langle \chi, \chi_m \rangle = \det(\langle h_{\lambda_1-i+j}(\alpha_1, \ldots, \alpha_p) \rangle) = s_\lambda(\alpha_1, \ldots, \alpha_p) \]

where \( h_k(x_1, \ldots, x_p) \) are complete symmetric polynomials. Here we use the relations

\[ s_\lambda(x_1, \ldots, x_p) = \det(\langle h_{\lambda_1-i+j}(x_1, \ldots, x_p) \rangle) \]

and

\[ \langle \eta_{\lambda_1-i+j}, \chi_{\lambda_1-i+j} S(\lambda_1-i+j) \rangle = \dim \wedge^k_+ (V) = h_k(\alpha_1, \ldots, \alpha_p). \]

This completes the proof.

Let us observe that if two partitions \( \lambda \) and \( \mu \) such that \( l(\lambda) \leq p, \ l(\mu) \leq p \) differ by a shift we have \( \dim V_{\lambda} = \dim V_{\mu} \).

The previous result makes very plausible the following.

**Conjecture 2.4** Let root \( P_-(t) \) denote the set of the roots of the polynomial \( P_-(t) \) and \( -\text{root } P_-(t) \) be the set of the opposite numbers. Then

\[ -\text{root } P_-(t, V_\lambda) = W_\lambda(-\text{root } P_-(t, V)) \]
where
\[ W_\lambda(z), \ z = \{z_1, z_2, \ldots, z_p\}, \ z_i \in \mathbb{C}, \ z_1 \cdots z_p = 1 \]
is defined in the following way. To \( z \) we associate the diagonal matrix
\[
\text{diag} (z_1, z_2, \ldots, z_p).
\]
Then the matrix corresponding to \( z \) in the \( SL(p) \)-module \( V_\lambda \) is also diagonal. The set of its diagonal elements is denoted \( W_\lambda(z) \).

**Remark 2.5** Besides dimensions of the objects in any twisted rigid category (i.e. a category closed with respect to the functor \( V \mapsto V^* \)), there are also the so-called inner (or quantum) dimensions, given by
\[
\dim V = \text{tr} \text{Id}_V
\]
where \( \text{tr} : \text{End}(V, V) \to k \) is the trace which is well defined in any rigid twisted tensor category (see \[5\] and the Section 4) and \( \text{Id}_V : V \to V \) is the identity operator considered as an element of \( \text{End}(V, V) \). From results of \[12\], it follows that
\[
\dim V = \text{rank} V
\]
for any even symmetry. Indeed, in view of \[12\] Proposition 2.12 we have
\[
\text{tr} \text{Id} = T_{ij}^{in} = p
\]
where \( T_{km}^{in} \) is the operator mentioned in Remark \[12\].

If Conjecture 2.4 holds, then this implies that for any even symmetry
\[
\dim V_\lambda = s_\lambda(1, 1, \ldots, 1)
\]
where the unity is taken \( p \) times. If \( p = n \), then this is just the classical formula. Thus, the inner dimension of an even object depends only on the rank, i.e., on the degree of the polynomial \( P_\lambda(t) \), while the ordinary dimension depends on the roots of this polynomial, i.e., on the whole polynomial.

---

4 The space \( \text{End}(U, V) \) is identified with \( V \otimes U^* \) (here \( U^* \) is the left dual of \( U \)) and is called the space of (left) inner morphisms from \( U \) to \( V \).
Definition 2.6 Let $V_S$ be a vector space equipped with a symmetry $S$ such that \( \text{rank } V = p \) and the determinant is central. We call Schur-Weyl (SW) category and denote $SW(V)$ the twisted symmetric category whose objects are the spaces $V_\lambda$, $l(\lambda) \leq p$ and their direct sums and whose morphisms of the objects $V_\lambda$ are of two types.

The first type morphism is by definition a linear map of the form

\[ \rho_S(p) : T^m(V) \to T^m(V), \quad p \in k[S(m)]. \]

Such morphisms give rise to a change of embedding of a given object $V_\lambda \hookrightarrow T^m(V)$. The second type morphism arises from the reduction procedure as follows.

Let $M_\lambda$ be the right $k[S(m)]$-module canonically embedded into $k[S(m)]$ and $V_\lambda$ be the corresponding subspace of $T^m(V)$. If the diagram $\lambda$ contains a $p$-column we apply the map (1.15) and kill it. (This is just the motivation of the above “canonical” arrangement.) The inverse linear map which is well defined is also a morphism by definition. A morphism of two direct sums of objects $V_\lambda$ is by definition a map being a morphism on each component. As usual we say that a morphism is an isomorphism if its inverse exists and is a morphism as well.

Remark that the map $V \mapsto V_\lambda$ is a twisted analogue of the well-known Schur functor defined in the case $S = \sigma$ (cf. [11]).

Let us also observe that for two different embeddings $V_\lambda \hookrightarrow T^m(V)$, there exists a morphism sending one of them to the other one.

Remark 2.7 It is worth saying that the definitions of SW category and Schur functor can be naturally generalized to Hecke symmetries (cf. [19]). However, the corresponding twists (2.26) in a particular case $\lambda = \mu$ are not Hecke symmetries anymore.

Remark 2.8 Note that, in the classical case (when the above category is just that of $SL(n)$-modules) there exists another way to introduce a decomposition of the product $V_\lambda \otimes V_\mu$ into a direct sum of irreducible modules, using the notion of a highest weight element with respect to a triangular decomposition of the algebra $sl(n)$. So, we can study the above decomposition without any embedding irreducible $SL(n)$-modules into tensor powers of the
basic space $V$. Unfortunately, in the general case such an approach is not yet elaborated, it is not even clear what a triangular decomposition of the corresponding twisted algebra (considered in the next Section) should be. So, the only way to introduce a category looking like that of $SL(n)$-modules is the Weyl type scheme developed above.

3 The twisted Lie algebra $sl(V_S)$ and the twisted Casimir operator

Consider a vector space $V = V_S$ equipped with a symmetry $S$ that is invertible by column, and that has thereafter a well-defined (say left) dual space $V^*$. Identifying $\text{End} V$ with $V \otimes V^*$, we can extend the symmetry $S$ to (see Remark 1.3)

$$S = S_{\text{End} V} : (\text{End} V)^{\otimes 2} \to (\text{End} V)^{\otimes 2}.$$  

Here we treat the elements of $\text{End} (V)$ as left (inner) morphisms, and in this setting the space $V$ becomes a left $\text{End} (V)$-module.

Moreover, $\text{End} (V)$ can be equipped in a natural way by a twisted (generalized or S-) Lie bracket as follows

$$[\ , \ ] = \circ (\text{Id} - S), \ S = S_{\text{End} V}$$

where $\circ$ is the operator product in $\text{End} (V)$. The space $V$ equipped with such a bracket will be denoted by $gl(V_S)$.

Now consider the S-trace (or simply the trace), defined on the algebra $\text{End} (V)$ by

$$\text{tr} : \text{End} V \to k, \ \text{tr} = <\ , \ > S, \ \text{End} V = V \otimes V^*.$$  

Remark that the trace is invariant and symmetric. Thus, we have

$$\text{tr} [\ , \ ] = 0. \quad (3.28)$$

Let $e^i_j$ be the element of $\text{End} V$ for which

$$e^i_j(x_k) = \delta^i_k x_j,$$

i.e., we identify $e^i_j$ with $x_i \otimes x^j$. Then $\text{tr} (e^i_j) = C^i_j$, where the operator $C$ is defined as in Remark 1.3. It follows from (3.28) that the traceless elements of
End \((V)\) form a subalgebra with respect to the twisted Lie bracket mentioned above. This subalgebra will be denoted by \(sl(V_S)\).

It is worthwhile to mention the bracket \([\ , \ ]\) can be expressed in terms of \(C^i_j\) in the case where \(\text{rank}(V) = 2\) (cf. [12]).

The algebras \(gl(V_S)\) and \(sl(V_S)\) are particular cases of twisted (generalized or S-) Lie algebras defined as follows.

**Definition 3.1** ([12]) \(g = (V_S, [\ , \ ] : V_S^\otimes 2 \rightarrow V_S)\) is called a twisted (generalized or S-)Lie algebra if the bracket \([\ , \ ]\) is invariant and skew-symmetric and if the following twisted analogue of the Jacobi relation holds:

\[
[\ , ][\ , ]^{12}(\text{Id} + S^{12}S^{23} + S^{23}S^{12}) = 0.
\]

Assume that \(V_S\) is equipped with an invariant and symmetric (resp., skew-symmetric) pairing. Then we can introduce twisted Lie algebras of \(so\) (resp., \(sp\)) type as the subalgebra of \(sl(V_S)\) consisting of elements preserving this pairing.

The enveloping algebra of a twisted Lie algebra \(g\) is defined in the following natural way

\[
U(g) = T(g)/\{x_i \otimes x_j - S(x_i \otimes x_j) - [x_i, x_j]\}.
\]

This enveloping algebra can be made into a cocommutative Hopf algebra (see [12]). The comultiplication is given by the formula

\[
\Delta x_i = x_i \otimes 1 + 1 \otimes x_i.
\]

There is a version of the PBW Theorem for the enveloping algebra \(U(g)\).

**Proposition 3.2** There exists a natural isomorphism

\[
\Lambda_+(g) \cong \text{Gr} U(g)
\]

where \(\text{Gr} U(g)\) is the graded quadratic algebra associated to the filtered algebra \(U(g)\).

**Proof** The algebra \(\Lambda_+(g)\) is Kozsul (cf. [4] for the definition). It follows from the exactness of the Koszul complex of the first kind from [12]. Then by [4] we have the result.
We say that a linear space $W$ is a $g$-module if there exists a twist $S : W^\otimes 2 \to W^\otimes 2$ which can be extended to

$$S = S_{\text{End}} : (\text{End } W)^\otimes 2 \to (\text{End } W)^\otimes 2$$

and an invariant linear map $\rho : g \to \text{End } (W)$ such that the operators associated to the elements $x_i$ via $\rho$ satisfy the same relations as the elements $x_i$ themselves in the enveloping algebra. The map $\rho$ is called a representation of the algebra $g$. (It defines a representation of the algebra $U(g)$ as well.)

Our next aim is to realize the category $\mathcal{SW}(V)$ as that of $g$-modules (with $g = \text{sl}(V_S)$). It is obvious that there is only one way to do this. We have already defined the action of $g$ on the base space $V = V_S$. We can extend this action to any tensor power of $V$ by means of the above comultiplication. Observe that this action commutes with the symmetry, because the comultiplication is cocommutative. Therefore this action commutes with any morphism

$$\rho_S(p), \forall p \in k[S(m)] \ \forall m.$$  

This implies that all elements of $g$ map any space $V_\lambda \hookrightarrow T^m(V)$ into itself. Passing from one embedding to another one corresponds to passing from one representation of $g$ to an isomorphic representation. Otherwise stated, we can say that the the first type morphisms commute with the action of $g$.

A similar statement holds for the morphisms of the second type. This ensues from the following result.

**Proposition 3.3** The twisted Lie algebra $g = \text{sl}(V_S)$ maps the determinant $v$ into 0.

**Proof** The statement results from the following Proposition.

**Proposition 3.4** The following formula holds

$$X(v) = p \left( \text{tr } X \right) v, \quad X \in g = \text{sl}(V_S).$$

**Proof** The map

$$X \otimes x \to X(x), \quad X \in g, \quad x \in V_S$$

is invariant and $v \in \wedge^p$, so we have

$$X(v) = X(v^{i_1i_2...i_p}x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_p}) = Q v^{i_1i_2...i_p}(X(x_{i_1})) \otimes x_{i_2} \otimes \cdots \otimes x_{i_p}$$

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\[ Q = \text{Id} - S^{12} + S^{23}S^{12} + \cdots + (-1)^{p-1}S^{p-1}p \cdots S^{23}S^{12}. \]

It is easy to see that \( X(v) \in \wedge^p \). Indeed,
\[ v^{i_1i_2\cdots i_p}(X(x_{i_1})) \otimes x_{i_2} \otimes \cdots \otimes x_{i_p} \in V \otimes \wedge^{p-1}(V) \]
and the operator \( Q \) maps the space \( V \otimes \wedge^{p-1}(V) \) into \( \wedge^p(V) \).

Thus the element \( X(v) \) does not change if we apply the projection \( P^p \) to it. Setting \( X = a^i_i e_j \), we have
\[ P^p(X(v)) = p v^{i_1i_2\cdots i_p}a^j_{i_1}x_j \otimes x_{i_2} \otimes \cdots \otimes x_{i_p} = p C^{i_j}_{a_{i_1}^j} u^{i_1i_2\cdots i_p}v = p C^{i_j}_{a_{i_1}^j} v \]
since \( C^{i_j}_{a_{i_1}^j} = v^{i_2\cdots i_p}u^{i_1i_2\cdots i_p} \) (cf. [12]). The proof is complete after we observe that
\[ \text{tr} (X) = C^{i_j}_{a_{i_1}^j} \]
because of the equality \( \text{tr} e_i^j = C^{i_j}_{i_1} \).

**Conjecture 3.5** All the \( g \)-modules \( V_\lambda \) are irreducible and any irreducible finite dimensional \( g \)-module is isomorphic to one of them. Moreover, a linear map between two objects of the category \( SW(V) \) is a morphism in the above sense if and only if it is a \( sl(V_S) \)-morphism, that is, it commutes with the action of \( sl(V_S) \). Thus, hopefully, the category \( SW(V) \) can be treated as the category of \( sl(V_S) \)-(or \( U(sl(V_S)) \)-) modules.

**Remark 3.6** Beside the above twisted Hopf algebra \( U(g) \), the category in question can be treated as the category of modules over a usual Hopf algebra \( H \). Its dual Hopf algebra (quantum cogroup) \( H^* \) has been constructed in [14]. An explicit description of the algebra \( H \) (which is also well defined for Hecke symmetries) is not so easy (cf. [1]). Let us mention also the papers [4] and [19] where the algebra \( H \) is considered.

Comparing these two Hopf algebras (the usual one and twisted one) we want to emphasize that the twisted Hopf algebra \( U(g) \) is more suitable for our aims because namely in terms of this algebra we can describe tangent space of a twisted variety and introduce the twisted Casimir operator playing the role of the Laplace-Beltrami operator in our approach (see Remark [1]).
Now we equip the algebra $gl(V_S)$ with the pairing arising from the trace, namely,
\[
\langle e^i_j, e^m_n \rangle = \delta^i_n \text{tr} e^i_j = \delta^i_j C^i_j.
\]
A direct computation shows that the element $\text{Id} = e^i_i$ is orthogonal to the algebra $sl(V_S)$. Moreover, the operator
\[
\text{Id} \mapsto \left( e^i_i \mapsto f^i_j = e^i_j - p^{-1} C^i_j \text{Id} \right),
\]
p = rank ($V_S$) = tr Id
is a projection onto the algebra $sl(V_S)$. The elements \{f^i_j, 1 \leq i, j \leq n\} generate this algebra but they are not free ($f^i_i = 0$).

Now we define the (quadratic) Casimir element in the algebra $U(gl(V_S))$ (resp., $U(sl(V_S))$) as follows
\[
\mathcal{C} = \mathcal{C}_{gl} = B^i_j e^i_j e^j_i \quad \text{(resp., } \mathcal{C}_{sl} = B^i_j f^i_j f^j_i)\).
\]
The operator $B = (B^i_j)$ is defined in Remark 1.5. These two Casimir elements are related by the formula
\[
\mathcal{C}_{sl} = \mathcal{C} - \text{Id} \otimes \text{Id} / p. \tag{3.29}
\]
It is easy to see that these elements are invariant. Now we will show that their images in $\text{End} (V_\lambda)$ are scalar (this also follows from Conjecture 3.5, but we will not make use of it) and compute the corresponding eigenvalues. The operators arising from the Casimir elements $\mathcal{C}$ and $\mathcal{C}_{sl}$ will be called Casimir operators and they will be denoted by the same letters.

Let us begin with the Casimir operator $\mathcal{C}$. We have
\[
\mathcal{C} \triangleright e_k = B^i_j e^i_j e^j_i \triangleright e_k = B^i_j e^i_k \triangleright e_j = B^i_j e^i_j = p e_k.
\]
The symbol $\triangleright$ stands for the action of the operator in question on an element. Applying this operator to the product $e_k e_l \in V_S \otimes 2$, we obtain
\[
\mathcal{C} \triangleright (e_k e_l) = (\mathcal{C} \triangleright e_k) e_l + e_k (\mathcal{C} \triangleright e_l) + 2 \mathcal{C} \triangleright (e_k e_l)
\]
where $\mathcal{C}$, the so-called split Casimir operator is defined by the formula
\[
\mathcal{C} \triangleright (e_k e_l) = \ev^{12} \ev^{34} S^{23}(B^i_j e^i_n e^j_n e_k e_l).
\]
From here on, $\ev$ is the evaluation operator defined by $\ev(A \otimes x) = Ax$ where $A$ is an operator and $x$ is an element.

Using the properties of the tensor $B^i_j$ (cf. [12, Section 1]), we can easily prove the following result.
Proposition 3.7 We have $\mathcal{C} \triangleright (e_k e_l) = S(e_k e_l)$ and therefore the formula

$$\mathcal{C}|_{V_S^2} = 2p \text{Id} + 2S$$

holds.

This implies the following relation

$$\mathcal{C}|_{V_S^m} = mp \text{Id} + 2 \sum_{1 \leq i < j \leq m} S_{ij}.$$  

Restricting the operator $\mathcal{C}$ to the component $V_\lambda \hookrightarrow T^m(V)$, we find

$$\mathcal{C}|_{V_\lambda} = mp \text{Id} + 2Q_\lambda,$$

where $Q_\lambda = Q|_{M_\lambda}$ and $Q = \sum_{1 \leq i < j \leq m} S_{ij}$.

Observe that the element $Q$ is central in the algebra $k[S(m)]$. So, being applied to $M_\lambda$ (we consider it as an operator $p \to Qp$) it becomes a scalar operator. Thus, we have $Q_\lambda = \gamma_\lambda \text{Id}$. Since all summands of $Q$ are of the same cyclic type $(m-2, 1, 0, \ldots, 0)$ we have

$$\gamma_\lambda = (m^2 - m) \chi_\lambda(C_{(m-2,1,0\ldots,0)}/(2 \dim M_\lambda) \quad (3.31)$$

where $C_{(i_1, i_2, \ldots, i_l)}$ is the conjugacy class with the cyclic type $(i_1 i_2 \cdots i_l)$ and $\chi_\lambda$ is the character of $M_\lambda$.

Now we can conclude the following.

Proposition 3.8 The Casimir element $\mathcal{C}$ being applied to $V_\lambda$ as above becomes a scalar operator and it is given by formula (3.30) with $Q_\lambda = \gamma_\lambda \text{Id}$ and $\gamma_\lambda$ defined by (3.31).

Let us now consider two particular cases of the formula (3.30). If $\lambda = (m, 0, \ldots, 0)$ then

$$\mathcal{C}|_{V_\lambda} = mp \text{Id} + (m^2 - m) \text{Id} = (m^2 + m(p - 1)) \text{Id} \quad (3.32)$$

and if $\lambda = (1, 1, \ldots, 1)$ ($m$ times) then

$$\mathcal{C}|_{V_\lambda} = mp \text{Id} - (m^2 - m) \text{Id}.$$ 

By (3.29) we have

$$\mathcal{C}_{sl}|_{V_\lambda} = \mathcal{C}|_{V_\lambda} - m^2/p \text{Id} = (mp + 2\gamma_\lambda - m^2/p) \text{Id}. \quad (3.33)$$

We can conclude that the eigenvalues of the Casimir operators depend on $\text{rank} V_S$, but not on $\dim V_S$. 

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4 Non-quasiclassical hyperboloid and Weyl type formula

The principle aim of this Section is to find a twisted non-quasiclassical analogue of the asymptotic Weyl formula (0.4). The role of the Laplace-Beltrami operator will be played by the Casimir operator $C_{sl}$. First, we will describe a “twisted non-quasiclassical variety”, namely the “twisted non-quasiclassical hyperboloid”. The drastic difference between the behaviour of the function $N(\lambda)$ in the classical case and the non-quasiclassical will be clear from this example. As usual, we assume that the determinant is central.

First we observe that the space $sl(V_S)$ is itself an object $V_\lambda$ of the category $\mathcal{SW}(V)$ corresponding to the diagram $\lambda = (2, 1^{p-2})$ where $p = \text{rank } V_S$. If $p = \text{rank } V_S = 2$, then the diagram corresponding to $sl(V_S)$ is $\lambda = (2)$. In this Section, we restrict attention to this case.

Let $g = sl(V_S)$, and decompose the space $g \otimes 2$ into a direct sum of objects $V_\lambda$ in the category $\mathcal{SW}(V)$. This sum contains three components $V_\lambda$ with

$$\lambda = (4), \quad \lambda = (3, 1), \quad \lambda = (2, 2).$$

If we carry out the reduction procedure from above, we can reduce the diagrams $\lambda = (3, 1)$ and $\lambda = (2, 2)$ to respectively $\lambda = (2)$ and $\lambda = (0)$. But instead of doing this, we consider the symmetric algebra of $g$ and impose some equations which are compatible with the action of the twisted Lie algebra $sl(V_S)$.

Namely, we consider the following quotient algebra

$$\mathcal{A}_c = T(g)/\{ f_i^j f_k^l - S(f_i^j f_k^l), \quad B_i^j f_k^l f_j^k - c \}, \quad c \in k.$$ 

If $c \neq 0$, then the algebra $\mathcal{A}_c$ is called a (twisted non-quasiclassical) hyperboloid. The algebra $\mathcal{A}_0$ is called a (twisted non-quasiclassical) cone.

It is not difficult to see that the latter equation is compatible with the $sl(V_S)$ action. Thus, this twisted variety is introduced by means of a unique equation in the symmetric algebra $\wedge_+(sl(V_S))$ similar to a classical hyperboloid (or cone).

Moreover, similarly to the classical case, it is possible to show that the algebra $\mathcal{A}_c$ is a direct sum of the components

$$V_{(2m)} \subset sl(V_S)^{\otimes m} \quad m = 0, 1, 2, ...$$
By using the results of the previous Section we are able to estimate the function $N(λ)$ for the Casimir operator $C_{sl}$. But first we want to realize this operator as a second order twisted differential operator. In order to do this we will say some words on twisted differential operators on the hyperboloid in question. Connected to this is the paper [13] where some aspects of differential calculus arising from symmetries are considered.

Recall that a twisted vector field (or $S$-vector field) on a twisted commutative algebra $A$ is an operator $X : A → A$ satisfying the Leibniz rule,

$$X(a ◦ b) = X(a) ◦ b + ◦ ev (X ⊗ Id ) S (a ⊗ b), \quad a, b ∈ A.$$  \hspace{1cm} (4.34)

If $A = ∧_+(V)$, then we can identify the space $\text{Vect}(A)$ of all (left) vector fields with $A ⊗ V^*$; here $V^*$ is the (left) dual space, with action extended to the full algebra $A$ by means of the Leibniz rule (4.34)).

We say that a vector field $X ∈ \text{Vect}(A)$ is a vector field on a factor algebra $A/\{I\}$ if

$$X(a) ∈ \{I\} \quad ∀ a ∈ \{I\}.$$  

For $a ∈ A$, we consider the operator $a ◦ b = ab$. Operators of the form $X + a$, with $X ∈ \text{Vect}(A)$ and $a ∈ A$ are called first order differential operators on $A$. In a similar way, we can define differential operators of order $n$ on $A$ and its quotients.

Replacing $V$ by $g$ in the previous example, we obtain a definition of the (left) vector fields on the algebra $∧_+(g)$ and its quotients. A particular case of such a vector field is given by those arising from the (left) adjoint action of the algebra $sl(V_S)$ onto itself

$$X → \text{ad}_X \quad \text{with} \quad \text{ad}_X Y = [X, Y].$$

Thus, the twisted Lie algebra $sl(V_S)$ is represented in the algebra $∧_+(g)$ in two ways. The first one is given via the map introduced in the previous Section and the second one is realized by the adjoint action. It is worth saying that these two actions of the algebra $sl(V_S)$ on $∧_+(g)$ coincide. We do not need this statement in the sequel. Let us only observe that by realizing the elements of $sl(V_S)$ as twisted vector fields we can treat the operator $C_{sl}$ as a second order differential operator on the algebra $A_c$. Moreover, if $c ≠ 0$, then it is the unique second order operator which is $sl(V_S)$-invariant. In the
case \( c = 0 \) there exists another second order invariant operator, namely the square of the invariant vector field defined by \( X(f) = m f \) where \( f \) is a degree \( m \) element of \( \mathcal{A}_0 \).

**Remark 4.1** Let us describe briefly a way to introduce the tangent space on the hyperboloid in question. Denote \( F^j_i \) the vector field corresponding to the element \( f^j_i \in \text{sl}(V_\mathbb{S}) \). It is possible to show that the vector fields \( F^j_i \) generate the space \( \text{Vect}(\mathcal{A}_c) \) as a (left) \( \mathcal{A}_c \)-module if \( c \neq 0 \). Moreover, these vector fields satisfy the relation \( B^j_i f^i_k F^k_j = 0 \). So, it is natural to define the tangent space on the twisted non-quasiclassical hyperboloid as the quotient of the free left \( \mathcal{A}_c \)-module generated by the formal generators \( F^j_i \) such that \( F^i_i = 0 \) over the left submodule generated by the element \( B^j_i f^i_k F^k_j \). Moreover, this module (called tangent) is projective and the corresponding projection is an \( \text{sl}(V_\mathbb{S}) \)-morphism.

Let us emphasize that the operators coming from the Hopf algebra \( H \) mentioned in Remark 4.6 are rather useless for describing the tangent space.

A similar situation takes place for a quantum \( U_q(\text{sl}(2)) \)-covariant hyperboloid. Its tangent space can be described in terms of “braided vectors fields” which are completely different from those coming from \( U_q(\text{sl}(2)) \), in spite of a tradition assigning the meaning of vector fields to the images of the elements \( X, Y, H \in U_q(\text{sl}(2)) \) (see [2]).

Now we assume that \( n = \dim V_\mathbb{S} > 2 \) (the case \( n = 2 \) corresponds to the classical hyperboloid).

**Proposition 4.2** On the hyperboloid in question the eigenvalues \( \lambda_l \) of the Casimir operator \( \mathcal{C}_{st} \) and their multiplicities \( m_l \) are

\[
\lambda_l = (2l)^2/2 + 2l, \quad m_l = (\alpha^2_{l+1} - \alpha^2_1)/(\alpha_2 - \alpha_1), \quad l = 0, 1, 2, ...
\]

where \( \alpha_i, \ i = 1, 2 \) are the roots of the equation \( P_\alpha(-t) = 1 - nt + t^2 = 0 \).

**Proof** The result follows immediately from the formulae (3.33) and (2.27).

Let \( \alpha_2 > \alpha_1 \). Then we have the following.

**Proposition 4.3** The function (0.3) possesses the upper and low limits given by

\[
\lim N(\lambda) = \beta \alpha_2^{\sqrt{2}x}, \quad \lim N(\lambda) = \beta \alpha_2^{\sqrt{2}x},
\]

where \( \beta \) is a positive constant.
Let us emphasize two principle difference between the behavior of the function $N(\lambda)$ in the classical and non-quasiclassical cases. First, in the non-quasiclassical case the function $N(\lambda)$ has an exponential growth with respect to $\sqrt{\lambda}$ and, secondly, it does not have a limit but only an upper and a lower limit.

**Remark 4.4** We are not able to give any estimation for the constant $\beta$ in the spirit of the classical Weyl formula since we do not know any twisted analogue of the notion of volume.

It is interesting to compare this result with the analysis of the spectrum of an “exotic harmonic oscillator” arising from non-quasiclassical symmetries introduced in \[14\].

Let us discuss now the case $p = \text{rank} V_S > 2$. In this case it is not so easy to find a system of equations which would define a twisted non-quasiclassical variety. We restrict attention to the “twisted orbits” looking like the projective space $\mathbb{CP}^n$ embedded as an orbit $O$ in $su(n)^*$ (this means that the decomposition of the corresponding algebra into a sum $\oplus V_\lambda$ looks like that of $\text{Fun} (\mathbb{CP}^n)$). To describe such a twisted orbit we should impose some system of equations on the space $sl(V_S)$. It is not difficult to guess their general form using $sl(V_S)$-covariance of the system but the problem is to find some factors occurring in this system (cf. \[8\] where this problem is discussed w.r.t. to $U_q(sl(n))$-covariant “orbits”). To find such a system we can use a scheme close to that considered in \[8\]. Let us describe its classical version.

Let $g$ be a simple Lie algebra and $h + n_+ + n_-$ its triangular decomposition. Fix an element $\omega \in h^*$ and extend it to $g$ by setting $\omega(n_\pm) = 0$. Let $O_\omega$ be the $G$-orbit of $\omega$, where $G$ acts on $g^*$ by $Ad^*$ action. Consider the algebra $A = \text{Fun} (O_\omega)$ defined as the restriction of the algebra $\Lambda^+ (g)$ to the orbit $O_\omega$. Its quantization can be realized in the following way. We associate to the element $\omega$ some infinite dimensional $g$-module $M_\omega$ called the (generalized) Verma module - its construction is described in \[8\]. Let

$$\rho_\omega : T(g) \to \text{End} (M_\omega)$$

be the corresponding representation of the tensor algebra $T(g)$. Set $\rho_h = h \rho_\omega / h$. Then the algebra

$$A_h = T(g)[h]/\{\text{Ker} \rho_h\} = \text{Im} \rho_h T(g)[h] \subset \text{End} (V_\omega)[[h]]$$

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is a flat deformation of the initial algebra in the sense of footnote 2 (cf. [3] for detail). Let us remark that the quantum object is realized as an operator algebra.

The passage from the algebra \( \mathcal{A}_h \) to \( \mathcal{A} \) is usually called “dequantization”, and can be used to find system of equations describing the orbit in question (compare to [8]).

Unfortunately, for a twisted Lie algebra \( \mathfrak{g} \) corresponding to a non-quasiclassical symmetry, it is not clear what its (generalized) Verma module is. However, we can suggest some discrete analogue of this method dealing with finite dimensional \( \mathfrak{g} \)-modules.

Fix an object \( V_\lambda \) of the Schur-Weyl category (in particular, the generating space \( V = V_S \) itself) and set

\[
V_l = \wedge_+^l (V_\lambda), \ l = 1, 2, ... \quad \text{thus,} \quad V_1 = V_\lambda.
\]

Let 

\[
\rho_l : T(\mathfrak{g}) \to \text{End} (V_l)
\]

be the representation of the twisted Lie algebra \( \mathfrak{g} = sl(V_S) \), which is the extension of the representation \( T(\mathfrak{g}) \to \text{End} (V_S) \). Consider the representation \( l^{-1}\rho_l : T(\mathfrak{g}) \to \text{End} (V_l) \) (the passage from \( \rho_1 \) to \( l^{-1}\rho_l \) is an analogue of the above passage from \( \rho_\omega \) to \( \hbar \rho_\omega/\hbar \)).

Put

\[
I_l = \text{Ker} \ l^{-1}\rho_l T(\mathfrak{g}), \quad \mathcal{A}_l = T(\mathfrak{g})/\{I_l\}.
\]

Hopefully, the algebras \( \mathcal{A}_l \) converge to a commutative algebra \( \mathcal{A} \) which is considered as a “twisted orbit”. The system of equations describing this “orbit” can be found from this limit (cf. [3]).

For the \( \mathbb{C}P^n \)-type twisted orbits mentioned above, the corresponding “function algebra” should be decomposed into a direct sum of components

\[
V_\lambda \quad \text{with} \quad \lambda = (0), (2, 1^{p-2}), (4, 2^{p-2}), (6, 3^{p-2}), ...
\]

By using the results of the previous Section, it is not difficult to obtain an estimation of the function \( N(\lambda) \) for the Casimir operator \( C_{\text{cl}} \) in this case.

Let us indicate now which aspects of the above theory can be generalized to the Hecke symmetries. As we have already said the construction of a category \( \mathcal{SW}(V) \) and that of Schur functor have natural analogues in the case when the symmetry ([1,4]) is of Hecke type since the representation theory of
the Hecke algebra for a generic $q$ looks like that of the symmetric group. If a Hecke symmetry $S_q$ is a deformation of a symmetry $S$, the dimensions of the spaces $V_\lambda$ arising from corresponding Schur functors are stable during the deformation $S \to S_q$.

The problem is to find a reasonable way to define corresponding twisted non-quasiclassical varieties, to introduce their tangent spaces and to define an analogue of the Casimir operator. We hope to treat this problem elsewhere. We refer the reader to the paper [2] where the problem is solved for a quantum hyperboloid related to $U_q(sl(2))$.

We conclude by saying that, in order to construct “reasonable” twisted varieties related to non-involutoriy non-quasiclassical twists, a criterion of flatness of deformation can be very useful. By considering twists $S_q$ which are deformations of a symmetry $S$, we should first define a twisted variety arising from the latter symmetry (as we said in the Introduction, an involutory case is easier to study) and then deform it to a variety related to the twist $S_q$. This scheme looks like the one used in [8] for introducing some $U_q(sl(n))$-covariant algebras, but a commutative algebra replaces as initial point is replaced by an $S$-commutative one with an involutory $S$.

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