LONG TIME DECAY OF LERAY SOLUTION OF 3D-NSE WITH DAMPING

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ABSTRACT. In [10], the authors show that the Cauchy problem of the Navier-Stokes equations with damping $\alpha|u|^{\beta-1}u (\alpha > 0, \beta \geq 1)$ has global weak solutions in $L^2(\mathbb{R}^3)$. In this paper, we prove the uniqueness, the continuity in $L^2$ for $\beta > 3$, also the large time decay is proved for $\beta \geq \frac{10}{3}$. Fourier analysis and standard techniques are used.

1. Introduction

In this paper we study the global existence of weak solution to the modified incompressible Navier-Stokes equations in $\mathbb{R}^3$

$$(NSD) \begin{align*}
\partial_t u - \nu \Delta u + u.\nabla u + \alpha|u|^{\beta-1}u &= -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) &= u^0(x) \quad \text{in } \mathbb{R}^3, \\
\alpha &> 0, \beta > 1
\end{align*}$$

where $u = u(t, x) = (u_1, u_2, u_3)$, $p = p(t, x)$ denote respectively the unknown velocity and the pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $\nu$ is the viscosity of fluid and $u^0 = (u^0_1(x), u^0_2(x), u^0_3(x))$ the initial given velocity. The damping is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [7, 8, 14, 15] and references therein). The fact that $\text{div } u = 0$, allows to write the term $(u.\nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$ in the following form $\text{div } (u \otimes u) := (\text{div } (u_1 u), \text{div } (u_2 u), \text{div } (u_3 u))$. If the initial velocity $u^0$ is quite regular, the divergence free condition determines the pressure $p$.

In order to simplify the calculations and the proofs of our results, we consider the viscosity unitary (i.e. $\nu = 1$).

The global existence of weak solution of initial value problem of the classical incompressible Navier-Stokes were proved by Leray and Hopf (see [13]-[16]) long before. The uniqueness remains an open problem for the dimensions $d \geq 3$.

The polynomial damping $\alpha|u|^{\beta-1}u$ is studied in [10] by Cai and Jiu, where they proved the global existence of weak solution in

$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)).$

The purpose of this paper is to study the uniqueness, continuity and large time decay of the global solution of the incompressible Navier-Stokes equations with

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damping (NSD). We recall that in [10] we employ the Galerkin approximations to construct the global solution of (NSD) with \( \beta \geq 1 \). But, in our case we use Friedrich method to prove the continuity and the uniqueness of such solution for \( \beta > 3 \). The study of large time decay is studied for \( \beta \geq \frac{10}{3} \). Precisely, our main result is the following:

**Theorem 1.1.**

Let \( \beta > 3 \) and \( u^0 \in L^2(\mathbb{R}^3) \) be a divergence free vector fields, then there is a unique global solution of (NSD): \( u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^+, H^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)) \).

Moreover, for all \( t \geq 0 \)

\[
\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(s)\|^2_{L^2} ds + 2\alpha \int_0^t \|u(s)\|_{L^{\beta+1}}^{\beta+1} ds \leq \|u^0\|^2_{L^2}.
\]

Moreover, if \( \beta \geq \frac{10}{3} \) we have

\[
\limsup_{t \to \infty} \|u(t)\|_{L^2} = 0.
\]

**Remark 1.2.**

In theorem [10], the inequality (1.2) is proved in [10]. The new parts of this theorem is the uniqueness, the continuity of the global solution in \( L^2(\mathbb{R}^3) \) and the asymptotic result (1.2).

2. Notations and Preliminary Results

For a function \( f: \mathbb{R}^3 \to \mathbb{R} \) and \( R > 0 \), the Friedrich operator \( J_R \) is defined by:

\[
J_R f = F^{-1}(\chi_{B_R} f),
\]

where \( B_R \), the ball of center 0 and radius \( R \). If \( L^2(\mathbb{R}^3) \) is the space of divergence-free vector fields in \( L^2(\mathbb{R}^3) \), the operator \( \mathbb{P}: (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3 \) is defined by:

\[
\mathbb{F}(\mathbb{P} f) = \hat{f}(\xi) - (\hat{f}(\xi), \frac{\xi}{|\xi|}) \frac{\xi}{|\xi|} = M(\xi) \hat{f}(\xi).
\]

where \( M(\xi) \) is the matrix \((\delta_{k,l} - \frac{\xi_k \xi_l}{|\xi|^2})_{1 \leq k,l \leq 3} \).

Particularly, if \( u \in \mathcal{S}(\mathbb{R}^3)^3 \), we obtain

\[
\mathbb{P}(u) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \delta_{k,j} - \frac{\xi_k \xi_j}{|\xi|^2} \right) \hat{u}(\xi) e^{i \xi \cdot x} d\xi,
\]

where \( \mathcal{S}(\mathbb{R}^n) \) is the Schwartz space. Define also the operator \( A_R(D) \) by:

\[
A_R(D) u = \mathbb{P} J_R(D) u = F^{-1}(M(\xi) \chi_{B_R}(\xi) \hat{u}).
\]

In which follows, we recall some preliminary results:

**Proposition 2.1.** ([9])

Let \( H \) be a Hilbert space.

1. The unit ball is weakly compact, that is: if \( (x_n) \) is a bounded sequence in \( H \), then there is a subsequence \( (x_{n(k)}) \) such that

\[
(x_{n(k)}) |y| \to (x |y|), \quad \forall y \in H.
\]

2. If \( x \in H \) and \( (x_n) \) a bounded sequence in \( H \) such that \( \lim_{n \to +\infty} (x_n |y|) = (x |y|), \)

for all \( y \in H \), then \( \|x\| = \liminf_{n \to \infty} ||x_n|| \).
(3) If \(x \in H\) and \((x_n)\) is a bounded sequence in \(H\) such that
\[
\lim_{n \to +\infty} (x_n, y) = (x, y), \quad \text{for all } y \in H \text{ and } \limsup_{n \to \infty} \|x_n\| \leq \|x\|, \text{ then}
\]
\[
\lim_{n \to \infty} \|x_n - x\| = 0.
\]

We recall the following product law in the homogeneous Sobolev spaces:

**Lemma 2.2.** ([11])

Let \(s_1, s_2\) be two real numbers and \(d \in \mathbb{N}\).

1. If \(s_1 < \frac{d}{2}\) and \(s_1 + s_2 > 0\), there exists a constant \(C_1 = C_1(d, s_1, s_2)\), such that: if \(f, g \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)\), then \(f, g \in \dot{H}^{s_1 + s_2 - \frac{d}{2}}(\mathbb{R}^d)\) and
\[
\|fg\|_{\dot{H}^{s_1 + s_2 - \frac{d}{2}}} \leq C_1(\|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}} \|g\|_{\dot{H}^{s_1}}).
\]

2. If \(s_1, s_2 < \frac{d}{2}\) and \(s_1 + s_2 > 0\) there exists a constant \(C_2 = C_2(d, s_1, s_2)\) such that: if \(f \in \dot{H}^{s_1}(\mathbb{R}^d)\) and \(g \in \dot{H}^{s_2}(\mathbb{R}^d)\), then \(f, g \in \dot{H}^{s_1 + s_2 - \frac{d}{2}}(\mathbb{R}^d)\) and
\[
\|fg\|_{\dot{H}^{s_1 + s_2 - \frac{d}{2}}} \leq C_2(\|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}}).
\]

**Lemma 2.3.**

Let \(\beta > 0\) and \(d \in \mathbb{N}\). Then, for all \(x, y \in \mathbb{R}^d\), we have
\[
\langle |x|^{\beta}x - |y|^{\beta}y, x - y \rangle \geq \frac{1}{2} (|x|^{\beta} + |y|^{\beta})|x - y|^2.
\]

**Proof.**

Suppose that \(|x| > |y| > 0\). For \(u > v > 0\), we have
\[
2(u x - vy, x - y) - (u + v)|x - y|^2 = (u - v)(|x|^2 - |y|^2) \geq 0.
\]

It suffices to take \(u = |x|^{\beta}\) and \(v = |y|^{\beta}\), we get the inequality (2.1).

The following result is a generalization of Proposition 3.1 in [12].

**Proposition 2.4.**

Let \(\nu_1, \nu_2, \nu_3 \in (0, \infty), r_1, r_2, r_3 \in (0, \infty)\) and \(f^0 \in L^2_\sigma(\mathbb{R}^3)\).

For \(n \in \mathbb{N}\), let \(F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3\) be a measurable function in \(C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))\) such that
\[
A_n(D)F_n = F_n, \quad F_n(0, x) = A_n(D)f^0(x)
\]
and
\[
(E1) \quad \partial_t F_n + \sum_{k=1}^{3} \nu_k |D_k|^{2r_k} F_n + A_n(D)\text{div}(F_n \otimes F_n) + A_n(D)h(|F_n|)F_n = 0.
\]
\[
(E2)
\]
\[
\|F_n(t)\|_{L^2}^2 + 2 \sum_{k=1}^{3} \nu_k \int_0^t \|D_k|^{r_k} F_n(s)\|_{L^2}^2 ds
\]
\[
+ 2a \int_0^t \|h(|F_n(s)|)\|F_n(s)^2\|_{L^1} ds \leq \|f^0\|_{L^2}^2.
\]

where \(h(z) = \alpha z^{\beta - 1}\), with \(\alpha > 0\) and \(\beta > 3\). Then: for every \(\varepsilon > 0\) there is \(\delta = \delta(\varepsilon, \alpha, \beta, \nu_1, \nu_2, \nu_3, r_1, r_2, r_3, \|f^0\|_{L^2}) > 0\) such that: for all \(t_1, t_2 \in \mathbb{R}^+\), we have
\[
|t_2 - t_1| < \delta \implies \|F_n(t_2) - F_n(t_1)\|_{H^{-\gamma_0}} < \varepsilon, \quad \forall n \in \mathbb{N},
\]
with \(s_0 \geq \max(3, 2r_1, 2r_2, 2r_3)\).
Proof. The proof is similar to that of Proposition 2.4 in [6].

3. Proof of Theorem 1.1

3.1. Existence of Solution.
Consider the approximate system:

\[
\begin{aligned}
\partial_t u - \Delta J_n u + J_n (J_n u \nabla J_n u) + \alpha J_n [||J_n u||^{\beta-1} J_n u] &= -\nabla p_n \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
(p_n) &= (-\Delta)^{-1} \left( \text{div} J_n (J_n u \nabla J_n u) + \alpha \text{div} J_n [||J_n u||^{\beta-1} J_n u] \right) \\
\text{div} u &= 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) &= J_n u^0(x) \text{ in } \mathbb{R}^3,
\end{aligned}
\]

where \(J_n\) is the Friedrichs operator defined by \(J_n(D) f = \mathcal{F}^{-1}(\chi_{B_n} \hat{f})\) and \(B_n\) the ball of center 0 and radius \(n \in \mathbb{N}\).

- By Cauchy-Lipschitz theorem, there exists a unique solution \(u_n \in C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))\) of the system \((NSD_n)\) such that \(J_n u_n = u_n\) and

\[
\|u_n(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u_n(s)\|_{L^2}^2 ds + 2\alpha \int_0^t ||u_n(s)||_{L^{\beta+1}}^{\beta+1} ds \leq ||u^0||_{L^2}^2.
\]

- The sequence \((u_n)\) is bounded in \(L^2(\mathbb{R}^3)\) and on \(H^1(\mathbb{R}^3)\).

Using Proposition 2.4 and the interpolation method, we deduce that the sequence \((u_n)\) is equicontinuous on \(H^{-1}(\mathbb{R}^3)\).

- For \((T_q)\) a strictly increasing sequence such that \(\lim_{q \to +\infty} T_q = \infty\), consider a sequence of functions \((\theta_q)\) in \(C_0^\infty(\mathbb{R}^3)\) such that

\[
\begin{cases}
\theta_q(x) = 1, & \text{for } |x| \leq q + \frac{5}{4} \\
\theta_q(x) = 0, & \text{for } |x| \geq q + 2 \\
0 \leq \theta_q \leq 1.
\end{cases}
\]

Using the energy estimate (3.1), the equicontinuity of the sequence \((u_n)\) on \(H^{-1}(\mathbb{R}^3)\) and classical argument by combining Ascoli’s theorem and the Cantor diagonal process, there exists a subsequence \((u_{\varphi(n)})\) and \(u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))\) such that: for all \(q \in \mathbb{N}\),

\[
\lim_{n \to \infty} ||\theta_q(u_{\varphi(n)}(t)) - u(t)||_{L^\infty([0, T_q], H^{-1})} = 0.
\]

In particular, the sequence \((u_{\varphi(n)}(t))\) converges weakly in \(L^2(\mathbb{R}^3)\) to \(u(t)\) for all \(t \geq 0\).

- Using the same method in [2], we obtain:

\[
\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2\alpha \int_0^t ||u(s)||_{L^{\beta+1}}^{\beta+1} ds \leq ||u^0||_{L^2}^2.
\]

for all \(t \geq 0\), and \(u\) is a solution of the system \((NSD)\).
3.2. Continuity of the solution in $L^2$.

By the inequality (3.3), we have $\limsup_{t \to 0} \|u(t)\|_{L^2} \leq \|u^0\|_{L^2}$ and using proposition 2.1 (3), we get $\limsup_{t \to 0} \|u(t) - u(t)\|_{L^2} = 0$. This ensures the continuity of the solution $u$ at $0$. To prove the continuity on $\mathbb{R}$, consider the functions $v_{n,\varepsilon}(t) = u_{\varphi}(n)(t + \varepsilon)$, $p_{n,\varepsilon}(t) = p_{\varphi}(n)(t + \varepsilon)$, for $n \in \mathbb{N}$ and $\varepsilon > 0$. We have:

$$\partial_t u_{\varphi}(n) - \Delta u_{\varphi}(n) + J_{\varphi}(n)\langle u_{\varphi}(n) \rangle + \alpha J_{\varphi}(n)(|u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n)) = -\nabla p_{\varphi}(n)$$

$$\partial_t v_{n,\varepsilon} - \Delta v_{n,\varepsilon} + J_{\varphi}(n)(v_{n,\varepsilon} \nabla v_{n,\varepsilon}) + \alpha J_{\varphi}(n)(|v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}) = -\nabla p_{n,\varepsilon}.$$  

The function $w_{n,\varepsilon} = u_{\varphi}(n) - v_{n,\varepsilon}$ fulfills the following:

$$\partial_t w_{n,\varepsilon} - \Delta w_{n,\varepsilon} + \alpha J_{\varphi}(n)\left(|u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n) - |v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}\right) = -\nabla (p_{\varphi}(n) - p_{n,\varepsilon}) + J_{\varphi}(n)(w_{n,\varepsilon} \nabla w_{n,\varepsilon}) - J_{\varphi}(n)(\langle u_{\varphi}(n) \rangle - \langle u_{\varphi}(n) \rangle \nabla w_{n,\varepsilon}).$$

Taking the scalar product with $w_{n,\varepsilon}$ in $L^2(\mathbb{R}^3)$ and using the fact that $\langle w_{n,\varepsilon} \nabla w_{n,\varepsilon}, w_{n,\varepsilon}\rangle = 0$ and $\text{div } w_{n,\varepsilon} = 0$, we get

$$\frac{1}{2} \frac{d}{dt} \|w_{n,\varepsilon}(t)\|^2 + \|\nabla w_{n,\varepsilon}(t)\|^2_{L^2} + \alpha \langle J_{\varphi}(n)(|u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n) - |v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}); w_{n,\varepsilon}\rangle_{L^2}$$

$$= -\langle J_{\varphi}(n)(\langle u_{\varphi}(n) \rangle - \langle u_{\varphi}(n) \rangle \nabla w_{n,\varepsilon}); w_{n,\varepsilon}\rangle_{L^2}. \quad (3.4)$$

By inequality 2.1, we have

$$\langle J_{\varphi}(n)(|u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n) - |v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}); w_{n,\varepsilon}\rangle_{L^2}$$

$$= \langle \langle |u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n) - |v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}; J_{\varphi}(n) w_{n,\varepsilon}\rangle_{L^2} \rangle_{L^2}$$

$$= \langle \langle |u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n) - |v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}; w_{n,\varepsilon}\rangle_{L^2} \rangle_{L^2}$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^3} \left(|u_{\varphi}(n)|^{\beta-1} + |v_{n,\varepsilon}|^{\beta-1}\right)|w_{n,\varepsilon}|^2$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^3} |u_{\varphi}(n)|^{\beta-1}|w_{n,\varepsilon}|^2,$$

which implies

$$\alpha \langle J_{\varphi}(n)(|u_{\varphi}(n)|^{\beta-1} u_{\varphi}(n) - |v_{n,\varepsilon}|^{\beta-1} v_{n,\varepsilon}); w_{n,\varepsilon}\rangle_{L^2} \geq \frac{\alpha}{2} \int_{\mathbb{R}^3} |u_{\varphi}(n)|^{\beta-1}|w_{n,\varepsilon}|^2. \quad (3.5)$$

Also, we have

$$\langle J_{\varphi}(n)(w_{n,\varepsilon} \nabla u_{\varphi}(n)); w_{n,\varepsilon}\rangle_{L^2} \leq \int_{\mathbb{R}^3} |w_{n,\varepsilon}| \cdot |u_{\varphi}(n)| \cdot |\nabla w_{n,\varepsilon}|$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^3} |w_{n,\varepsilon}|^2 |u_{\varphi}(n)|^2 + \frac{1}{2} \|\nabla w_{n,\varepsilon}\|^2_{L^2}.$$

By using the convex inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq a^p + b^q$$

with $p = \frac{\beta - 1}{2}$, $q = \frac{\beta - 1}{\beta - 3}$, $a = |w_{n,\varepsilon}|^2 \left(\frac{\alpha}{2}\right)^{\frac{2}{\beta - 3}}$, $b = \left(\frac{2}{\alpha}\right)^{\frac{2}{\beta - 3}}$, 
we get
\[ |(J_{\varphi(n)}(w_{n, \varepsilon}, \nabla u_{\varphi(n)}); w_{n, \varepsilon})| \leq \frac{\alpha}{4} \int_{\mathbb{R}^n} |w_{n, \varepsilon}|^{\beta - 1} |u_{\varphi(n)}|^2 + C_{\alpha, \beta} \|w_{n, \varepsilon}\|_{L^2}^2 + \frac{1}{2} \|\nabla w_{n, \varepsilon}\|_{L^2}^2, \]
with \( C_{\alpha, \beta} = \frac{1}{4} \alpha \beta^{\frac{1}{\beta - 1}}. \) Combining this inequality and inequalities (3.4), (3.5) and (3.6), we get
\[ \frac{d}{dt}\|w_{n, \varepsilon}\|_{L^2}^2 + \|\nabla w_{n, \varepsilon}\|_{L^2}^2 \leq 2C_{\alpha, \beta}\|w_{n, \varepsilon}\|_{L^2}^2. \]
By Gronwall Lemma, we deduce the following:
\[ \|w_{n, \varepsilon}(t)\|_{L^2} \leq \|w_{n, \varepsilon}(0)\|_{L^2} e^{C_{\alpha, \beta} t}, \]
and
\[ \|u_{\varphi(n)}(t + \varepsilon) - u_{\varphi(n)}(t)\|_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2} e^{C_{\alpha, \beta} t}. \]
For \( t_0 > 0 \) and \( \varepsilon \in (0, t_0) \), we have
\[ \|u_{\varphi(n)}(t_0 + \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2} e^{C_{\alpha, \beta} t_0}. \]
\[ \|u_{\varphi(n)}(t_0 - \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2} \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2} e^{C_{\alpha, \beta} t_0}. \]
So
\[ \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 = \|J_{\varphi(n)}u_{\varphi(n)}(\varepsilon) - J_{\varphi(n)}u_{\varphi(n)}(0)\|_{L^2}^2 \]
\[ = \|J_{\varphi(n)}(u_{\varphi(n)}(\varepsilon) - u^0)\|_{L^2}^2 \]
\[ \leq \|u_{\varphi(n)}(\varepsilon) - u^0\|_{L^2}^2 \]
\[ \leq \|u_{\varphi(n)}(\varepsilon)\|_{L^2}^2 + \|u^0\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u^0 \rangle \]
\[ \leq 2\|u^0\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u^0 \rangle. \]
But \( \lim_{n \to +\infty} \langle u_{\varphi(n)}(\varepsilon), u^0 \rangle = \langle u(\varepsilon), u^0 \rangle \), hence
\[ \liminf_{n \to +\infty} \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \leq 2\|u^0\|_{L^2}^2 - 2Re\langle u(\varepsilon), u^0 \rangle. \]
Moreover, for all \( q, N \in \mathbb{N} \)
\[ \|J_N \left( \theta_q(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)) \right) \|_{L^2}^2 \leq \|\theta_q(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2 \]
\[ \leq \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2. \]
Using (3.2) we get, for \( q \) big enough,
\[ \|J_N \left( \theta_q(u(t_0 \pm \varepsilon) - u(t_0)) \right) \|_{L^2} \leq \liminf_{n \to +\infty} \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}. \]
Then
\[ \|J_N \left( \theta_q(u(t_0 \pm \varepsilon) - u(t_0)) \right) \|_{L^2}^2 \leq 2\left( \|u^0\|_{L^2}^2 - Re\langle u(\varepsilon), u^0 \rangle \right) e^{2C_{\alpha, \beta} t_0}. \]
By applying the Monotone Convergence Theorem in the order \( N \) and \( q \), we get
\[ \|u(t_0 \pm \varepsilon) - u(t_0)\|_{L^2}^2 \leq 2\left( \|u^0\|_{L^2}^2 - Re\langle u(\varepsilon), u^0 \rangle \right) e^{2C_{\alpha, \beta} t_0}. \]
Using the continuity at \( 0 \) and make \( \varepsilon \to 0 \), we get the continuity at \( t_0 \).
3.3. Uniqueness.
Let \( u, v \) be two solutions of \((NSD)\) in the space
\[
C_b(\mathbb{R}^+,L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+,L^{\beta+1}(\mathbb{R}^3)).
\]
The function \( w = u - v \) satisfies the following:
\[
\partial_t w - \Delta w + \alpha \big( |u|^{\beta-1} u - |v|^{\beta-1} v \big) = -\nabla (p - \bar{p}) + w \nabla w - w \nabla u - u \nabla w.
\]
Taking the scalar product in \( L^2 \) with \( w \), we get
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2_{L^2} + \| \nabla w \|^2_{L^2} + \alpha \left( \| |u|^{\beta-1} u - |v|^{\beta-1} v \|_{L^2} \right) \langle w, w \rangle_{L^2} = -\langle w, \nabla u; w \rangle_{L^2}.
\]
By adapting the same method for the proof of the continuity of such solution in \( L^2(\mathbb{R}^3) \), with \( u, v, w \) instead of \( u_{\varepsilon(n)}, v_{\varepsilon}, w_{n,\varepsilon} \) in order, we find
\[
\alpha \left( \| |u|^{\beta-1} u - |v|^{\beta-1} v \|_{L^2} \right) \langle w, w \rangle_{L^2} \geq \frac{\alpha}{2} \int_{\mathbb{R}^3} |u|^{\beta-1} |w|^2.
\]
and
\[
\| \langle w, \nabla u; w \rangle_{L^2} \| \leq \frac{\alpha}{4} \int_{\mathbb{R}^3} |w|^2 |u|^2 + C_{\alpha,\beta} \| w \|_{L^2}^2 + \frac{1}{2} \| \nabla w \|^2_{L^2}.
\]
Combining the above inequalities, we find the following energy estimate:
\[
\frac{d}{dt} \| w(t) \|^2_{L^2} + \| \nabla w(t) \|^2_{L^2} \leq 2C_{\alpha,\beta} \| w(t) \|^2_{L^2}.
\]
By Gronwall Lemma, we obtain
\[
\| w(t) \|^2_{L^2} + \int_0^t \| \nabla w \|^2_{L^2} \leq \| w^0 \|^2_{L^2} e^{2C_{\alpha,\beta} t}.
\]
As \( w^0 = 0 \), then \( w = 0 \) and \( u = v \), which implies the uniqueness.

3.4. Asymptotic Study of the Global Solution.
To prove the asymptotic behavior \( (1.2) \), we need some preliminaries lemmas:

**Lemma 3.1.**
If \( u \) is a global solution of \((NSD)\) with \( \beta \geq \frac{10}{3} \), then \( u \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \).

**Proof.**
Let \( E_1 = \{(t,x) : |u(t,x)| \leq 1\} \) and \( E_2 = \{(t,x) : |u(t,x)| > 1\} \), \( L_1 = \int_{E_1} |u(s,x)|^\beta dxds \) and \( L_2 = \int_{E_2} |u(s,x)|^\beta dxds \). We have
\[
L_1 = \int_{E_1} |u(s,x)|^\beta dxds = \int_{E_1} |u(s,x)|^\beta - \frac{10}{3} |u(s,x)|^\frac{10}{3} dxds \leq \int_0^\infty \| u(s) \|_\frac{10}{3} dxds.
\]
By using the Sobolev injection \( \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^{\frac{10}{3}}(\mathbb{R}^3) \), we get
\[
(3.6) \quad L_1 \leq C \int_0^\infty \| u(s) \|_\frac{10}{3} dxds.
\]
By interpolation inequality \( \|u(s)\|_{H^\frac{3}{4}} \leq \|u(s)\|^2_{H^{\frac{3}{2}}} \), we obtain
\[
L_1 \leq C \int_0^\infty \|u(s)\|^4_{L^2} \|\nabla u(s)\|^2_{L^2} \leq C \|u^0\|^4_{L^2} \int_0^\infty \|\nabla u(s)\|^2_{L^2}.
\]

For the term \( L_2 \), we have
\[
L_2 = \int_{X_2} |u(s, x)|^\beta \, dx \leq \int_0^\infty \int_{\mathbb{R}^3} |u(s, x)|^\beta + 1 \, dx \, ds.
\]
Hence
\[
\|u\|_{L^\beta((\mathbb{R}^+ \times \mathbb{R}^3)} \leq C \|u^0\|^\frac{4}{3} \int_0^\infty \|\nabla u(s)\|^2_{L^2} \, ds + \int_0^\infty \int_{\mathbb{R}^3} |u(s, x)|^\beta + 1 \, dx \, ds.
\]
Therefore \( u \in L^\beta(\mathbb{R}^+ \times \mathbb{R}^3) \).

**Lemma 3.2.**
If \( u \) is a global solution of \((NSD)\), with \( \beta \geq \frac{10}{3} \), then \( \lim_{t \to \infty} \|u(t)\|_{H^2} = 0 \).

**Proof.**
For \( \varepsilon > 0 \), using the energy inequality (1.1) and Lemma 3.1, there exists \( t_0 \geq 0 \) such that
\[
\|\nabla u\|_{L^2((t_0, \infty) \times \mathbb{R}^3)} < \frac{\varepsilon}{4},
\]
and
\[
\|u\|_{L^\beta((t_0, \infty) \times \mathbb{R}^3)} < \frac{\varepsilon}{4}.
\]

Now, consider the following system
\[
\begin{align*}
\partial_t v - \nu \Delta v + v.\nabla v + \alpha |v|^{\beta-1}v & = -\nabla q & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } v & = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
v(0, x) & = u(t_0, x) & \text{in } \mathbb{R}^3.
\end{align*}
\]
By the existence and uniqueness part, the system \((NSD')\) has a unique global solution \( v \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)) \) such that \( v(t_0) = u(t_0, x) \) and \( q(t) = p(t_0 + t) \). The energy estimate for this system is as follows:
\[
\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla v(s)\|^2_{L^2} \, ds + 2a \int_0^t \|v(s)\|_{L^{\beta+1}} \leq \|u(0)\|^2_{L^2} \leq \|v^0\|^2_{L^2}.
\]
By the Duhamel formula, we obtain
\[
v(t, x) = e^{t\Delta} v^0(x) + f(t, x) + g(t, x),
\]
where
\[
f(t, x) = -\int_0^t e^{(t-s)\Delta} \text{div} (v \otimes v)(s, x) \, ds
\]
and
\[
g(t, x) = -\alpha \int_0^t e^{(t-s)\Delta} \text{div } |v(s, x)|^{\beta-1} v(s, x) \, ds.
\]
By Dominated Convergence Theorem, \( \lim_{t \to \infty} \| e^{t \Delta} v^0 \|_{L^2} = 0 \) and hence \( \lim_{t \to \infty} \| e^{t \Delta} v^0 \|_{H^{-2}} = 0 \).

Moreover,

\[
\| f(t) \|_{H^{-2}}^2 \leq \| f(t) \|_{H^{-2+}}^2 \leq \int_{\mathbb{R}^3} |\xi|^{-1} \left( \int_0^t e^{-s} |\xi|^2 |\mathcal{F} \text{div}(v \otimes v)(s, \xi)| ds \right)^2 d\xi \\
\leq \int_{\mathbb{R}^3} |\xi| \left( \int_0^t e^{-s} |\xi|^2 |\mathcal{F}(v \otimes v)(s, \xi)| ds \right)^2 d\xi.
\]

Since

\[
\left( \int_0^t e^{-s} |\mathcal{F}(v \otimes v)(s, \xi)| ds \right)^2 \leq \int_0^t e^{-2s} |\mathcal{F}(v \otimes v)(s, \xi)|^2 ds \\
\leq |\xi|^{-2} \int_0^t |\mathcal{F}(v \otimes v)(s, \xi)|^2 ds,
\]

then

\[
\| f(t) \|_{H^{-2}}^2 dt \leq \int_{\mathbb{R}^3} |\xi|^{-1} \int_0^t |\mathcal{F}(v \otimes v)(s, \xi)|^2 ds d\xi \\
\leq \int_0^t \left( \int_{\mathbb{R}^3} |\xi|^{-1} |v \otimes v)(s, \xi)|^2 d\xi \right) ds = \int_0^t \| v \otimes v)(s, \xi)|^2_{H^{-2+}} ds.
\]

Using the product law in homogeneous Sobolev spaces, with \( s_1 = 0, s_2 = 1 \), we get

\[
\| f(t) \|_{H^{-2}}^2 dt \leq C \int_0^t \| v(s) \|^2_{L^2} \| \nabla v(s) \|^2_{L^2} ds.
\]

Using inequalities (3.8) and (3.9), we get

\[
\| f(t) \|_{H^{-2}}^2 dt \leq C \int_0^t \| u(s) \|^2_{L^2} \| \nabla u(t) \|^2_{L^2} ds \\
\leq C \| u^0 \|^2_{L^2} \int_0^\infty \| \nabla u(t) \|^2_{L^2} ds \\
\leq C \| u^0 \|^2_{L^2} \int_{t_0}^\infty \| \nabla u(s) \|^2_{L^2} ds \\
\leq C \| u^0 \|^2_{L^2} \frac{\varepsilon^2}{9(C \| u^0 \|^2_{L^2} + 1)},
\]

which implies that

\[
\| f(t) \|_{H^{-2}} < \frac{\varepsilon}{3}, \; \forall t \geq 0.
\]

For an estimation of \( \| g(t) \|_{H^{-2}} \) and using

\[
L^1(\mathbb{R}^3) \hookrightarrow H^{-s}(\mathbb{R}^3), \; \forall s > 3/2,
\]
with \( s = 2 \), we get
\[
\|g(t)\|_{H^{-2}}^2 dt \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-2} \left( \int_0^t e^{-(t-s)}|\xi|^2 |F(|v|^{-1}v)(s, \xi)| ds \right)^2 d\xi
\]
\[
\leq C \left( \int_0^t \|(|v|^{-1}v)(s, \cdot)\|_{L^1(\mathbb{R}^3)} ds \right)^2
\]
\[
\leq C \left( \int_0^t \|v(s, \cdot)\|_{L^1(\mathbb{R}^3)} ds \right)^2
\]
\[
\leq C \|v\|_{L^2(\mathbb{R}^3)}^2,
\]
where \( C = \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-2} d\xi \).

Also using inequality \((3.9)\), we get
\[
\|g(t)\|_{H^{-2}}^2 dt \leq C \|u(t_0 + \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2
\]
\[
\leq C \|u\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \leq C \epsilon^2 \frac{2}{3},
\]
which implies that \(\|g(t)\|_{H^{-2}} < \frac{\epsilon}{3}\), \(\forall t \geq 0\).

Combining the above inequalities, we obtain
\[
\lim_{t \to \infty} \|u(t)\|_{H^{-2}} = 0.
\]

**Lemma 3.3.**
If \( u \) is a global solution of \((NSD)\) and \( \beta \geq \frac{10}{3} \), then \( \lim_{t \to \infty} \|u(t)\|_{L^2} = 0. \)

**Proof.**
Let
\[
w_1 = 1_{|D| < 1} u = F^{-1}(1_{|\xi| < 1} \tilde{u}) \quad \text{and} \quad w_2 = 1_{|D| \geq 1} u = F^{-1}(1_{|\xi| \geq 1} \tilde{u}).
\]

Using the second step, we get
\[
\|w_1(t)\|_{L^2} = c_0 \|w_1(t)\|_{H^\alpha} \leq 2c_0 \|w_1(t)\|_{H^{-2}} \leq 2 \|u(t)\|_{H^{-2}},
\]
which implies
\[
\lim_{t \to \infty} \|w_1(t)\|_{L^2} = 0.
\]

For \( \epsilon > 0 \), there is a \( t_1 > 0 \) such that
\[
\|w_1(t)\|_{L^2} < \frac{\epsilon}{2}, \forall t \geq t_1.
\]

We have
\[
\int_{t_1}^\infty \|w_2(t)\|_{L^2}^2 dt \leq \int_{t_1}^\infty \|\nabla w_2(t)\|_{L^2}^2 dt \leq \int_{t_1}^\infty \|\nabla u(t)\|_{L^2}^2 dt < \infty.
\]
Since the map \( t \mapsto \|w_2(t)\|_{L^2} \) is continuous, there exists \( t_2 \geq t_1 \) such that
\[
\|w_2(t_2)\|_{L^2} < \frac{\epsilon}{2}.
\]
Hence
\[
\|u(t_2)\|_{L^2}^2 = \|w_1(t_2)\|_{L^2}^2 + \|w_2(t_2)\|_{L^2}^2 < \frac{\epsilon^2}{2}.
\]
Using the following energy estimate
\[
\|u(t)\|_{L^2}^2 + 2 \int_{t_2}^{t} \|\nabla u(s)\|_{L^2}^2 ds + 2\alpha \int_{t_2}^{t} \|u(s)\|_{L^{k+1}} ds \leq \|u(t_2)\|_{L^2}^2, \quad \forall t \geq t_2,
\]
we get
\[
\|u(t)\|_{L^2} < \varepsilon, \quad \forall t \geq t_2,
\]
and the proof is completed.

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