Strongly Local Hypergraph Diffusions for Clustering and Semi-supervised Learning

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ABSTRACT

Hypergraph-based machine learning methods are now widely recognized as important for modeling and using higher-order and multiway relationships between data objects. Local hypergraph clustering and semi-supervised learning specifically involve finding a well-connected set of nodes near a given set of labeled vertices. Although many methods for local clustering exist for graphs, there are relatively few for localized clustering in hypergraphs. Moreover, those that exist often lack flexibility to model a general class of hypergraph cut functions or cannot scale to large problems. To tackle these issues, this paper proposes a new diffusion-based hypergraph clustering algorithm that solves a quadratic hypergraph cut based objective akin to a hypergraph analog of Andersen-Chung-Lang personalized PageRank clustering for graphs. We prove that, for graphs with fixed maximum hyperedge size, this method is strongly local, meaning that its runtime only depends on the size of the output instead of the size of the hypergraph and is highly scalable. Moreover, our method enables us to compute with a wide variety of cardinality-based hypergraph cut functions. We also prove that the clusters found by solving the new objective function satisfy a Cheeger-like quality guarantee. We demonstrate that on large real-world hypergraphs our new method finds better clusters and runs much faster than existing approaches. Specifically, it runs in few seconds for hypergraphs with a few million hyperedges compared with minutes for flow-based technique. We furthermore show that our framework is general enough that can also be used to solve other $p$-norm based cut objectives on hypergraphs. Our code is available github.com/MengLiuPurdue/LHQC.

KEYWORDS

hypergraph, local clustering, community detection, PageRank

1 INTRODUCTION

Two common scenarios in graph-based data analysis are: (i) What are the clusters, groups, modules, or communities in a graph? and (ii) Given some limited label information on the nodes of the graph, what can be inferred about missing labels? These statements correspond to the clustering and semi-supervised learning problems respectively, and while there exists a strong state of the art in algorithms for these problems on graphs [3, 13, 19, 29, 34, 35, 37, 41], research on these problems is currently highly active for hypergraphs [8, 17, 27, 33, 38, 40] building on new types of results [15, 26, 32] compared to prior approaches [1, 20, 42]. The lack of flexible, diverse, and scalable hypergraph algorithms for these problems limits the opportunities to investigate rich structure in data. For example, clusters can be relevant treatment groups for statistical testing on networks [10] or identify common structure across many types of sparse networks [24]. Likewise, semi-supervised learning helps to characterize subtle structure in the emissions spectra of galaxies in astronomy data through characterizations in terms of biased eigenvectors [23]. The current set of hypergraph algorithms is insufficient for such advanced scenarios.

Hypergraphs, indeed, enable a flexible and rich data model that has the potential to capture subtle insights that are difficult or impossible to find with traditional graph-based analysis [5, 27, 33, 38, 40]. But, hypergraph generalizations of graph-based algorithms often struggle with scalability and interpretation [1, 15] with ongoing questions of whether particular models capture the higher-order information in hypergraphs. Regarding scalability, an important special case for that is a strongly local algorithm. Strongly local algorithms are those whose runtime depends on the size of the output rather than the size of the graph. This was only recently addressed for various hypergraph clustering and semi-supervised learning frameworks [17, 33]. This property enables fast (seconds to minutes) evaluation even for massive graphs with hundreds of millions of nodes and edges [3] (compared with hours). For graphs, perhaps the best known strongly local algorithm is the Andersen-Chung-Lang (henceforth, ACL) approximation for personalized PageRank [3] with applications to local community detection and semi-supervised learning [41]. The specific problem we address is a mincut-inspired hypergraph generalization of personalized PageRank along with a strongly local algorithm to rapidly approximate solutions. Our formulation differs in a number of important ways from existing Laplacian [42] and quadratic function-based hypergraph PageRank generalizations [25, 27, 31].

Although our localized hypergraph PageRank is reasonably simple to state formally (§3.2), there are a variety of subtle aspects to both the problem statement and the algorithmic solution. First, we wish to have a formulation that enables the richness of possible hypergraph cut functions. These hypergraph cut functions are an essential component to rich hypergraph models because they determine when a group of nodes ought to belong to the same cluster or obtain a potential new label for semi-supervised learning. Existing techniques based on star expansions (akin to treating the hypergraph as a bipartite graph) or a clique expansion (creating a weighted graph by adding edges from a clique to the graph for each hyperedge) only model a limited set of cut functions [2, 32].
More general techniques based on Lovász extensions [25, 27, 40] pose substantial computational difficulties. Second, we need a problem framework that gives sparse solutions such that they can be computed in a strongly local fashion and then we need an algorithm that is actually able to compute these—the mere existence of solutions is insufficient for deploying these ideas in practice as we wish to do. Finally, we need an understanding of the relationship between the results of this algorithm and various graph quantities, such as minimal conductance sets as in the original ACL method.

To address these challenges, we extend and employ a number of recently proposed hypergraph frameworks. First, we show a new result on a class of hypergraph to graph transformations [32]. These transformations employ carefully constructed directed graph gadgets, along with a set of auxiliary nodes, to encode the properties of a general class of cardinality based hypergraph cut functions. Our simple new result highlights how these transformations not only preserve cut values, but preserve the hypergraph conductance values as well (§3.1). Then we localize the computation in the reduced graph using a general strategy to build strongly local computations. This involves a particular modification often called a "localized cut" graph or hypergraph [4, 11, 28, 33]. We then use a squared 2-norm (i.e. a quadratic function) instead of a 1-norm that arises in the mincut-graph to produce the hypergraph analogue to strongly local personalized PageRank. Put another way, applying all of these steps on a graph (instead of a hypergraph) is equivalent to a characterization of personalized PageRank vector [12].

Once we have the framework in place (§3.1,§3.2), we are able to show that an adaptation of the push method for personalized PageRank (§3.3) will compute an approximate solution in time that depends only on the localization parameters and is independent of the size of a hypergraph with fixed maximum hyperedge size (Theorem 3.5). Consequently, the algorithms are strongly local.

The final algorithm we produce is extremely efficient. It is a small factor (2-5x) slower than running the ACL algorithm for graphs on the star expansion of the hypergraph. It is also a small factor (2-5x) faster than running an optimized implementation of the ACL algorithm on the clique expansion of the hypergraph (ACL-Star, ACL-Clique, respectively), these algorithms are closely related to ideas proposed in [1, 42] (2), HyperLocal, a recent maximum flow-based hypergraph clustering algorithm [33], (3) quadratic hypergraph PageRank [25, 31] (which is also closely related to [15]), and (4) our Local Hypergraph-PageRank (LHPR). These are all strongly local except for (3), which we include because our algorithm LHPR is essentially the strongly local analogue of (3).

The results of the experiments are show in Figure 1. These show that the flow-based HyperLocal method has a difficult time finding the entire cluster. Flow-based methods are known to have trouble expanding small seed sets [11, 28, 34] and this experiments shows that same behavior. Our strongly local hypergraph PageRank (LHPR) slightly improves on the performance of a quadratic hypergraph PageRank (QHPR) that is not strongly local. In particular, our algorithm has 10k non-zero entries (of 64k) in its solution.

This experiment shows the opportunities with our approach for large hypergraphs. We are able to model a flexible family of hypergraph cut functions beyond those that use clique and star expansions and we equal or outperform all the other methods. (We also tried another recently proposed method [17], although this was unable to finish in a reasonable time (e.g. hours) on this hypergraph as it was designed for hypergraphs with small hyperedge size.)

2 NOTATION AND PRELIMINARIES

Let $G = (V,E,\omega)$ be a directed graph with $|V| = n$ and $|E| = m$. For simplicity, we require weights $\omega_{ij} \geq 1$ for each directed edge $(i,j) \in E$. We interpret an undirected graph as having two directed edges $(i,j)$ and $(j,i)$. For simplicity, we assume the vertices are labeled with indices $1$ to $n$, so that we may use these labels to index matrices and vectors. For instance, we define $d$ as the length-$n$ out-degree vector where its $i$th component $d_i = \sum_{j \in V} \omega_{ij}$. The incidence matrix $B \in \{0, -1, 1\}^m \times n$ measures the difference of adjacent nodes. The $k$th row of $B$ corresponds to an edge, say $(i,j)$, and has exactly two nonzero values, 1 for the node $i$ and -1 for the node $j$. (Recall that we have directed edges, so the origin of the edge is always 1 and the destination is always $-1$.)

Let $H = (V,E)$ be a hypergraph where each hyperedge $e \in E$ is a subset of $V$. Let $\xi = \max_{e \in E} |e|$ be the maximum hyperedge size. With each hyperedge, we associate a splitting function $f_e$ that we use to assess an appropriate penalty for splitting the hyperedge among two labels or splitting the hyperedge between two clusters. Formally, let $S$ be a cluster and let $A = e \cap S$ be the hyperedge’s nodes inside $S$, then $f_e(A)$ penalizes splitting $e$. A common choice in early hypergraph literature was the all-or-nothing split, which assigns a fixed value if a hyperedge is split or zero if all nodes in
Given a set of seeds, or what we commonly think of as a reference set of vertices in the graph. We explain our method: localized hypergraph quadratic diffusions (LHQD) or also localized hypergraph PageRank (LHPR) through two transformations before we formally state the problem and algorithm. We adopted this strategy so that the final proposal is well justified because some of the transformations require additional context to appreciate. Computing the final sweepcut is straightforward for hypergraph conductance, and so we don’t focus on that step.

### 3.1 Hypergraph-to-graph reductions

Minimizing conductance is NP-hard even in the case of simple graphs, though numerous techniques have been designed to approximate the objective in theory and practice [3, 4, 9]. A common strategy for searching for low-conductance sets in hypergraphs is to first reduce a hypergraph to a graph, and then apply existing graph-based techniques. This sounds “hacky” or least “ad-hoc” but this idea is both principled and rigorous. The most common approach is to apply a clique expansion [1, 5, 26, 42, 45], which explicitly models splitting functions of the form \( f_e(A) \propto |A| |e| \). For instance Benson et al. [5] showed that clique expansion can be used to convert a 3-uniform hypergraph into a graph that preserves the all-or-nothing conductance values. For larger hyperedge sizes, all-or-nothing conductance is preserved to within a distortion factor depending on the size of the hyperedge. Later, Li et al. [26] were the first to introduce more generalized notions of hyperedge splitting functions, focusing specifically on submodular functions.

**Definition 3.1.** A splitting function \( f_e \) is **submodular** if

\[
f_e(A) + f_e(B) \geq f_e(A \cup B) + f_e(A \cap B) \quad \forall A, B \subseteq e. \tag{2}
\]

These authors showed that for this submodular case, clique expansion could be used to define a graph preserving conductance to within a factor \( O(\zeta) \) (\( \zeta \) is the largest hyperedge size).

More recently, Veldt et al. [32] introduced graph reduction techniques that exactly preserve submodular hypergraph cut functions which are cardinality-based.

**Definition 3.2.** A splitting function \( f_e \) is **cardinality-based** if

\[
f_e(A) = f_e(B) \quad \text{whenever } |A| = |B|. \tag{3}
\]
Cardinality-based splitting functions are a natural choice for many applications, since node identification is typically irrelevant in practice, and the cardinality-based model produces a cut function that is invariant to node permutation. Furthermore, most previous research on applying generalized hypergraph cut penalties implicitly focused on cut functions that are naturally cardinality-based [5, 15, 20, 25, 27, 45]. Because of their ubiquity and flexibility, in this work we also focus on hypergraph cut functions that are submodular and cardinality-based. We briefly review the associated graph transformation and then build on previous work by showing that these hypergraph reductions can be used to preserve the hypergraph conductance objective, and not just hypergraph cuts.

**Reduction for Cardinality-Based Cuts**

Veldt et al. [32] showed that the cut properties of a submodular, cardinality-based hypergraph could be preserved by replacing each hyperedge with a set of directed graph gadgets. Each gadget for a hyperedge $e$ is constructed by introducing a pair of auxiliary nodes $a$ and $b$, along with a directed edge $(a, b)$ with weight $\delta_e > 0$. For each $e \in E$, two unit-weight directed edges are introduced: $(a, a)$ and $(b, b)$. The entire gadget is then scaled by a weight $\epsilon_e \geq 0$. The resulting gadget represents a simplified splitting function of the following form:

$$f_e(A) = \epsilon_e \cdot \min(|A|, |e \setminus A|, \delta_e).$$  

(4)

Figure 2(b) illustrates the process of replacing a hyperedge with a gadget. The cut properties of any submodular cardinality-based splitting function can be exactly modeled by introducing a set of $O(|e|)$ or fewer such splitting functions [32]. If an approximation suffices, only $O(\log |e|)$ gadgets are required [6].

An important consequence of these reduction results is that in order to develop reduction techniques for any submodular cardinality-based splitting functions, it suffices to consider hyperedges with splitting functions of the simplified form given in (4). In the remainder of the text, we focus on splitting functions of this form, with the understanding that all other cardinality-based submodular splitting functions can be modeled by introducing multiple hyperedges on the same set of nodes with different edge weights.

In Figure 2, we illustrate the procedure of reducing a small hypergraph to a directed graph, where we introduce a single gadget per hyperedge. Formally, for a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, this procedure produces a directed graph $G = (\bar{V}, \bar{E})$, with directed edge set $\bar{E}$, and node set $\bar{V} = \mathcal{V} \cup \mathcal{V}_a \cup \mathcal{V}_b$, where $\mathcal{V}$ is the set of original hypergraph nodes. Sets $\mathcal{V}_a, \mathcal{V}_b$ store auxiliary nodes, in such a way that for each pair of auxiliary nodes $a, b$ where $(a, b)$ is a directed edge, we have $a \in \mathcal{V}_a$ and $b \in \mathcal{V}_b$. This reduction technique was previously developed as a way of preserving minimum cuts and minimum $s$-$t$ cuts for the original hypergraph. Here, we extend this result to show that for a certain choice for node degree, this reduction also preserves hypergraph conductance.

**Theorem 3.3.** Define a degree vector $d$ for the reduced graph $G = (\bar{V}, \bar{E})$ such that $d(v) = d_0$ is the out-degree for each node $v \in \mathcal{V}$, and $d(u) = d_0 = 0$ for every auxiliary node $u \in \mathcal{V}_a \cup \mathcal{V}_b$. If $T^*$ is the minimum conductance set in $G$ for this degree vector, then $S^* = T^* \cap \mathcal{V}$ is the minimum hypergraph conductance set in $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

**Proof.** From previous work on these reduction techniques [6, 32], we know that the cut penalty for a set $S \subseteq \mathcal{V}$ in $\mathcal{H}$ equals the cut penalty in the directed graph, as long as auxiliary nodes are arranged in a way that produces the smallest cut penalty subject to the choice of node set $S \subseteq \mathcal{V}$. Formally, for $S \subseteq \mathcal{V}$,

$$\text{cut}_H(S) = \min_{T \subseteq \mathcal{V}} \text{cut}_G(T),$$  

(5)

where $\text{cut}_G$ denotes the weight of directed out-edges originating inside $S$ that are cut in $G$. By our choice of degree vector, the volume of nodes in $G$ equals the volume of the non-auxiliary nodes in $\mathcal{H}$. That is, for all $T \subseteq \mathcal{V}$, $\text{vol}_G(T) = \sum_{v \in V} d_0 + \sum_{u \in \mathcal{V}_a \cup \mathcal{V}_b} d_0 = \text{vol}_G(T \cap \mathcal{V}) = \text{vol}_H(T \cap \mathcal{V})$. Let $T^* \subseteq \mathcal{V}$ be the minimum conductance set in $G$, and $S^* = T^* \cap \mathcal{V}$. Without loss of generality we can assume that $\text{vol}_G(T^*) \leq \text{vol}_G(T^*)$. Since $T^*$ minimizes conductance, and auxiliary nodes have no effect on the volume of this set, $\text{cut}_G(T^*) = \min_{T \subseteq \mathcal{V}: T \cap \mathcal{V} \neq \emptyset} \text{cut}_G(T) = \text{cut}_H(S^*)$, and so $\text{cut}_G(T^*) / \text{vol}_G(T^*) = \text{cut}_H(S^*) / \text{vol}_H(S^*)$. Thus, minimizing conductance in $G$ minimizes conductance in $\mathcal{H}$. $hd$

### 3.2 Localized Quadratic Hypergraph Diffusions

Having established a conductance-preserving reduction from a hypergraph to a directed graph, we now present a framework for detecting localized clusters in the reduced graph $G$. To accomplish this, we first defined a localized directed cut graph, involving a source and sink nodes and new weighted edges. This approach is closely related to previously defined localized cut graphs for local graph clustering and semi-supervised learning [4, 7, 12, 28, 34, 43], and a similar localized cut hypergraph used for flow-based hypergraph clustering [33]. The key conceptual difference is that we apply this construction directly to the reduced graph $G$, which by Theorem 3.3 preserves conductance of the original hypergraph $\mathcal{H}$. Formally, we assume we are given a set of nodes $R \subseteq \mathcal{V}$ around which we wish to find low-conductance clusters, and a parameter $\gamma > 0$. The localized directed cut graph is defined by applying the following steps to $G$:

- Introduce a source node $s$, and for each $r \in R$ define a directed edge $(s, r)$ of weight $\gamma d_o$.
- Introduce a sink node $t$, and for each $v \in \bar{V}$ define a directed edge $(v, t)$ with weight $\gamma d_o$.

We do not connect auxiliary nodes to the source or sink, which is consistent with the fact that their degree is defined to be zero in order for Theorem 3.3 to hold. We illustrate the construction of the localized directed cut graph in Figure 2(d). It is important to note that in practice we do not in fact form this graph and store it in memory. Rather, this provides a conceptual framework for finding localized low-conductance sets in $G$, which in turn correspond to good clusters in $\mathcal{H}$.

**Definition: Local hypergraph quadratic diffusions.** Let $B$ and $w$ be the incidence matrix and edge weight vector of the localized directed cut graph with $\gamma$. The objective function for our hypergraph clustering diffusion, which we call local hypergraph quadratic diffusion or simply local hypergraph PageRank, is

$$\min_x \frac{1}{2} w^T (Bx)^2 + \gamma \sum_{i \in \mathcal{V}} x_i d_i$$  

subject to $x_0 = 1, x_t = 0, x \geq 0$.

(6)

We use the function $(x)_+ = \max(x, 0)$, applied element-wise to $Bx$, to indicate we only keep the positive elements of this product. This is analogous to the fact that we only view a directed edge as being cut if it crosses from the source to the sink side; this is similar to previous directed cut minorants on graphs and hypergraphs [39].
The first term in the objective corresponds to a 2-norm minorant of the minimum s-t cut objective on the localized directed cut graph. (In an undirected regular graph, the term \(w^T(Bx)\) turns into an expression with the Laplacian, which can in turn be formally related to PageRank [12]). If instead, we replace exponent 2 with a 1 and ignore the second term, this would amount to finding a minimum s-t cut (which can be solved via a maximum flow). The second term in the objective is included to encourage sparsity in the solution, where \(\kappa \geq 0\) controls the desired level of sparsity. With \(\kappa > 0\) we are able to show in the next section that we can compute solutions in time that depends only on \(\kappa, \gamma,\) and \(\text{vol}(R)\), which allows us to evaluate solutions to (6) in a strongly local fashion.

### 3.3 A strongly local solver for LHQD (6)

In this section, we will provide a strongly local algorithm to approximately satisfy the optimality conditions of (6). We first state the optimality conditions in Theorem 3.4, and then present the algorithm to solve them. The simplest way to understand this algorithm is as a generalization of the Andersen-Chung-Lang push procedure for PageRank [3], which we will call ACL as well as the more recent nonlinear push procedure [28]. Two new challenges about this new algorithm are: (1) the new algorithm operates on a directed graph, which means unlike ACL there is no single closed form update at each iteration (2) there is no sparsity regularization for auxiliary nodes, which will break the strongly local guarantees for existing analyses of the push procedure.

We begin with the optimality conditions for (6).

**Theorem 3.4.** Fix a seed set \(R, \gamma > 0, \kappa > 0\), define a residual function \(r(x) = -\frac{1}{y}B^T \text{diag}((Bx)_+)w\). A necessary and sufficient condition to satisfy the KKT conditions of (6) is to find \(x^*\) where \(x^* \geq 0, r(x^*) = [r_s, g^T, r_t]^T\) with \(g_i \leq \kappa d_i\) (where \(d\) reflects the graph before adding s and t but does include the 0 degree nodes), \((kd_i - g_i)^T x_i^* = 0\) for \(i \in V\) and \(g_t = 0\) for all auxiliary nodes added.

The proof of this can be found in the appendix. We further note that solutions \(x^*\) are unique because the problem is strongly convex due to the quadratic.

In §3.1, we have shown that the reduction technique of any cardinality submodular-based splitting function suffices to introduce multiple directed graph gadgets with different \(\delta_e\) ad \(c_e\). In order to simplify our exposition, we assume that each hyperedge has a \(\delta\)-linear threshold splitting function [33] \(f_e = \min\{|A|, |e|, |A|, \delta\}\) with \(\delta \geq 1\) to be a tunable parameter. This splitting function can be exactly modeled by replacing each hyperedge with one directed graph gadget with \(c_e = 1\) and \(\delta_e = \delta\). (This is what is illustrated in Figure 2.) Also when \(\delta = 1\), it models the standard unweighted all-or-nothing cut [14, 18, 22] and when \(\delta\) goes to infinity, it models star expansion [45]. Thus this splitting function can interpolate these two common cut objectives on hypergraphs by varying \(\delta\).

By assuming that we have a \(\delta\)-linear threshold splitting function, this means we can associate exactly two auxiliary nodes with each hyperedge. We call these a and b for simplicity. We also let \(V_a\) be the set of all a auxiliary nodes and \(V_b\) be the set of all b nodes.

At a high level, the algorithm to solve this proceeds as follows: whenever there exists an graph node \(i \in V\) that violates optimality, i.e. \(r_i > \kappa d_i\), we first perform a hyperpush at \(i\) to increase \(x_i\) so that the optimality condition is approximately satisfied, i.e., \(r_i = \rho d_i\), where \(0 < \rho < 1\) is a given parameter that influences the approximation. This changes the solution \(x\) only at the current node \(i\) and residuals at adjacency auxiliary nodes. Then for adjacent auxiliary we immediately push on them, which means to increase their value so that the residuals remain zero. After pushing each pair \((a, b)\) of associated auxiliary nodes, we then update residuals for adjacent nodes in \(V\). Then we search for another optimality violation. (See Algorithm 1 for a formalization of this strategy.) When \(\rho < 1\), we only approximately satisfy the optimality conditions; and this approximation strategy has been repeatedly and successfully used in existing local graph clustering algorithms [3, 12, 28].

**Notes on optimizing the procedure.** Algorithm 1 for a formalization of this strategy. Describes a general strategy to approximately solve these diffusions. We now note a number of optimizations that we have found to greatly accelerate this strategy. First, note that \(x\) and \(r\) can be kept as sparse vectors with only a small set of entries stored. Second, note that we can maintain a list of optimality violations because each update to \(x\) only causes \(r\) to increase, so we can simply check if each coordinate increase creates a new violation and add it to a queue. Third, to find the value that needs to be “pushed” to each node, a general strategy is to use a binary
Algorithm 1 LHQD($\gamma, \kappa, \rho$) for set $R$ and hypergraph $H$ with $\delta$-linear penalty where $0 < \rho < 1$ determines accuracy

1: Let $x = 0$ except for $x_a = 1$ and set $r = -y^T B_x \text{diag}(B(x)) w(\delta, \gamma)$.
2: While there is any vertex $i \in V$ where $r_i > \kappa d_i$, or stop if none exists (find an optimality violation)
3: Perform LHQD-hyperpush at vertex $i$ so that $r'_i = \rho x d_i$, updating $x$ and $r$. (satisfy optimality at $i$)
4: For each pair of adjacent auxiliary nodes $a, b$ where $a \in V_a$, $b \in V_b$ and $a \rightarrow b$, perform LHQD-auxpush at $a$ and $b$ so that $r'_a = r'_b = 0$, then update $x$ and $r$ each auxpush.
5: Return $x$.

The key to understand the strong locality of the algorithm is that after each LHQD-hyperpush, the decrease of $\Delta x_i$ and $\Delta x_b$ is $\Delta x_i = (r_i - \rho x d_i) / ((s_a^{(i)} + s_b^{(i)})/y + d_i)$. (9)

Then we need to check whether the assumption holds by checking $x_i + \Delta x_i \leq \min \{ x^{(i)}_a, x^{(i)}_b \}$ (10)

Now assume the ordering is the same, $r'_i$ can be written as $r'_i = r_i + o_i - \frac{1}{y} (s_a^{(i)} + s_b^{(i)}) \Delta x_i - d_i \Delta x_i = \rho x d_i$, so

$$\Delta x_i = (r_i - \rho x d_i) / ((s_a^{(i)} + s_b^{(i)})/y + d_i).$$

If not, we need to use binary search to find the new location of $x_i + \Delta x_i$ (Note $\Delta x_i$ here is the true value that is still unknown), update $s_a^{(i)}, s_b^{(i)}, o^{(i)}_a, o^{(i)}_b$ and $x^{(i)}_min$ and recompute $\Delta x_i$. This process is summarized in LHQD-hyperpush.

Similarly, when the pushed nodes $a \in V_a, b \in V_b$ where $a \rightarrow b$, are a pair of auxiliary nodes, for its adjacent nodes $i \in V$, we record:

- $z_a$: the sum of edge weights $w_{ia}$ where $x_a < x_i$
- $z_b$: the sum of edge weights $w_{ib}$ where $x_b > x_i$
- $x^{(a)}_min$: the minimum $x_i$ where $x_a < x_i$
- $x^{(b)}_min$: the minimum $x_i$ where $x_b < x_i$

Then we solve $\Delta x_a, \Delta x_b$ by solving the following linear system (here we assume $x_b \geq x_i$, otherwise $\Delta x_a = \Delta x_b = 0$)

$$w_{ab}(\Delta x_b - \Delta x_a) + \frac{w_{ia}}{y}((x_i - x_a)_+ - (x_i - x_a)_+) - z_a \Delta x_a = 0$$
$$w_{ab}(\Delta x_a - \Delta x_b) - \frac{w_{ib}}{y}((x_b - x'_b)_+ - (x_b - x_i)_+) + z_b \Delta x_b = 0$$

(11)

where $x'_j$ refers to the updated $x_i$ after applying LHQD-hyperpush at node $i$. And the assumption will hold if and only if the following inequalities are all satisfied:

$$x'_i \leq x_b, \quad x_a + \Delta x_a \leq x^{(a)}_min, \quad x_b + \Delta x_b \leq x^{(b)}_min$$

(12)

If not, we also need to use binary search to update the locations of $x_a + \Delta x_a$ and $x_b + \Delta x_b$, update $z_a, z_b, x^{(a)}_min, x^{(b)}_min$ and recompute $\Delta x_a$ and $\Delta x_b$.

Establishing a runtime bound. The key to understand the strong locality of the algorithm is that after each LHQD-hyperpush, the decrease of $||g||_1$ can be lower bounded by some number that is independent of the total size of the hypergraph, while LHQD-auxpush doesn’t change $||g||_1$. Formally, we have the following theorem:

**Theorem 3.5.** Given $g > 0, x > 0, \delta > 0$ and $0 < \rho < 1$. Suppose the splitting function $f_\rho$ is submodular, cardinality-based and satisfies $1 \leq f_\rho(i) \leq \delta$ for any $i \in e_i$. Then calling LHQD-auxpush doesn’t
Algorithm 3 LQHD-auxpush(\(i, a, b, y, x, r, \Delta x\))

1. Solve \(\Delta x_a, \Delta x_b\) with \(z_a, z_b, x_{\min}^{(a)}, x_{\min}^{(b)}\) using (11).
2. if (12) doesn’t hold. (\(\Delta x_a, \Delta x_b\) altered the order) then
3. Binary search on \(\Delta x_a\) until we find the smallest interval among all adjacent original nodes of \(a\) that will include \(x_a + \Delta x_a, x_b + \Delta x_b\), similarly for \(z_a, z_b, x_{\min}\) using (11).
4. Use (8) and the found intervals to solve \(\Delta x_a, \Delta x_b\) by setting \(r_a = r_b = 0\).
5. end if
6. Change the following entries in \(x\) and \(r\) to update the solution and the residual
7. (a) \(x_a \leftarrow x_a + \Delta x_a\) and \(x_b \leftarrow x_b + \Delta x_b\)
8. (b) For each neighboring node \(i \rightarrow a\) where \(i \in V, r_i \leftarrow r_i + \frac{1}{\Gamma} w_{ia}(x_i - x_a) + \frac{1}{\Gamma} w_{ia}(x_i - x_a - \Delta x_a) + \frac{1}{\Gamma} w_{ia}(x_b - x_i) + \frac{1}{\Gamma} w_{ia}(x_b + \Delta x_b - x_i)\)
9. end for

change \(||g||_1\) while calling LQHD-hyperpush on node \(i \in V\) will decrease \(||g||_1\) by at least \(\gamma k(1 - p)\) \(d_i / (\gamma k + \delta)\).

Suppose LQHD stops after \(T\) iterations and \(d_i\) is the degree of the original node updated at the \(i\)-th iteration, then \(T\) must satisfy:

\[\sum_{i=1}^{T} d_i \leq (\gamma k + \delta) \text{Vol}(R) / (\gamma k(1 - p)) = O(\text{Vol}(R)).\]

The proof is in the appendix. Note that this theorem only upper bounds the number of iterations Algorithm 1 requires. Each iteration of Algorithm 1 will also take \(O(\sum_{e \in E, i \in e} |e|)\) amount of work. This ignores the binary search, which only scales it by \(\log(\max\{d_i, \max_{e \in E, i \in e} |e|\})\) factor in the worst case. Putting these pieces together shows that if we have a hypergraph with independently bounded maximum hyperedge size, then we can treat this additional work as a constant. Consequently, our solver is strongly local for graphs with bounded maximum hyperedge size; this matches the interpretation in [33].

4 LOCAL Conductance APPROXIMATION

We give a local conductance guarantee that results from solving (6). Because of space, we focus on the case \(k = 0\). We prove that a sweepcut on the solution \(x\) of (6) leads to a Cheeger-type guarantee for conductance of the hypergraph \(H\) even when the seed-set size \(|R|\) is 1. It is extremely difficult to guarantee a good approximation property with an arbitrary seed node, and so we first introduce a seed sampling strategy \(\mathcal{P}\) with respect to a set \(S^*\) that we wish to find. Informally, the seed selection strategy says that the expected solution mass outside \(S^*\) is not too large, and more specifically, not too much larger than if you had seeded on the entire target set \(S^*\).

Definition 4.1. Denote \(x(y, R)\) as the solution to (6) with \(k = 0\). A good sampling strategy \(\mathcal{P}\) for a target set \(S^*\) is

\[
\mathbb{E}_{\mathcal{P}} \left[ \frac{1}{\delta V} \sum_{u \in \mathcal{S}, y, \{v\}} d_{\mathcal{P}} x_u(y, \{v\}) \right] \leq \frac{c}{\text{Vol}(S^*)} \sum_{u \in \mathcal{V}, y, \{v\}} d_{\mathcal{P}} x_u(y, S^*)
\]

for some positive constant \(c\).

Note that \(\text{Vol}(S^*)\) is just to normalize the effect of using different numbers of seeds. For an arbitrary \(S^*\), a good sampling strategy \(\mathcal{P}\) for the standard graph case with \(c = 1\) is to sample nodes from \(S^*\) proportional to their degree. Now, we provide our main theorem and show its proof in Appendix D.

Theorem 4.2. Given a set \(S^*\) of vertices s.t. \(\text{Vol}(S^*) \leq \frac{\text{Vol}(H)}{2}\) and \(\varphi_H(S^*) \leq \frac{\gamma}{100}\) for some positive constant \(\gamma\). If we have a seed sampling strategy \(\mathcal{P}\) that satisfies Def. 4.1, then with probability at least \(\frac{1}{2}\), sweepcut on (6) with find \(S^*_x\) with

\[
\varphi(S^*_x) \leq \delta \sqrt{32\ln(100\text{Vol}(S^*) / d_0)},
\]

where \(\delta = \max_{e \in S^*_x} \min|e| / 2\) where \(S^*_x = \{e \in \mathcal{E} | e \cap S^*_x \neq \emptyset, e \cap S^*_x \neq \emptyset\}\) and \(v\) is the seeded node.

The proof is in the appendix. This implies that for any set \(S^*\), if we have a sampling strategy that matches \(S^*\) and tune \(\gamma\), our method can find a node set with conductance \(O(\delta \sqrt{\varphi_H(S^*)} \log(\text{Vol}(S^*))\).

The term \(\delta\) is the additional cost that we pay for performing graph reduction. The dependence on \(\delta\) essentially generalizes the previous works that analyzed the conductance with only all-or-nothing penalty [25, 31], as our result matches these when \(\delta = 1\). But our method gives the flexibility to choose other values \(\delta\) and while \(\delta\) in the worst case could be as large as \(|e|/2\), in practice, \(\delta\) can be chosen much smaller (See §7). Also, although we reduce \(\mathcal{H}\) into a directed graph \(G\), the previous conductance analysis for directed graphs [25, 39] is not applicable as we have degree zero nodes in \(G\). Those degree zero nodes introduce challenges.

5 DIRECTLY RELATED WORK

We have discussed most related work in-situ throughout the paper. Here, we address a few related hypergraph PageRank vectors directly. First, Li et al. [25] defined a quadratic hypergraph PageRank by directly using Lovász extension of the splitting function \(f_e\) to control the diffusion instead of a reduction. Both Li et al. [25] and Takai et al. [31] simultaneously proved that this PageRank can be used to partition hypergraphs with an all-or-nothing penalty and a Cheeger-type guarantee. Neither approach gives a strongly local algorithm and they have complexity \(O(|\mathcal{E}| |\mathcal{V}| \text{min}(|\mathcal{E}|, |\mathcal{V}|) \text{poly}(\delta))\) or terms of Euler integration or subgradient descent.

6 GENERALIZATION TO P-NORMS

In the context of the local graph clustering, the quadratic cut objective can sometimes “over-expand” or “bleed out” over natural boundaries in the data. (This is the opposite problem to the maxflow-based clustering.) To solve this issue, [28] proposed a more general p-norm based cut objective, where \(1 < p \leq 2\). The corresponding p-norm diffusion algorithm can not only grow from small seed set, but also capture the boundary better than 2-norm cut objective. Moreover, [37] proposed a related p-norm flow objective that shares similar characteristics. Our hypergraph diffusion framework easily adapts to such a generalization.

Definition: p-norm local hypergraph diffusions. Given a hypergraph \(\mathcal{H} = (\mathcal{V}, \mathcal{E})\), seeds \(R\), and values \(y, x\). Let \(B, w\) again be the incidence matrix and weight vector of the localized reduced directed cut graph. A p-norm local hypergraph diffusion is:

\[
\text{minimize} \quad w^T f((Bx)_i + y) \sum_{i \in \mathcal{V}} x_i d_i
\]

subject to \(x_i = 1, x_i = 0, x \geq 0\).
Here \( \ell(x) = \frac{1}{2}x^p \), \( 1 < p \leq 2 \). And the corresponding residual function is \( r(x) = \frac{1}{2}B^2 \text{diag}((\ell'(Bx)))w \).

The idea of solving (13) is similar to the quadratic case, where the goal is to iteratively push values to \( x_i \) as long as node \( i \) violates the optimality condition, i.e. \( r_i > \varepsilon d_i \). The challenge of solving a more general p-norm cut objective is that we no longer have a closed form solution even if the ordering of adjacent nodes is known. Thus, we need to use binary search to find \( \Delta x_i \), \( \Delta x_a \) and \( \Delta x_b \) up to \( \epsilon \) accuracy at every iteration. This means that in the worst case, the general push process can be slower than 2-norm based push process by a factor of \( O(\log(1/\epsilon)) \).

**Algorithm 4**

1. Use binary search to find \( \Delta x_i \) such that the 4th coordinate of residual after adding \( \Delta x_i \) to \( x_i \), \( r_i' = p xdi \), the binary search stops when the range of \( \Delta x_i \) is smaller than \( \epsilon \) (satisfy KKT at \( i \)).

2. Update \( x \) and \( r \), \( x_i \leftarrow x_i + \Delta x_i \), \( r_i \leftarrow r_i + \rho \Delta x_i \).

3. For each pair of adjacent auxiliary nodes \( a \in V_a \), \( b \in V_b \) where \( a \rightarrow b \), update the corresponding residuals

   \[
   \begin{align*}
   r_a & \leftarrow \frac{a}{b} w_{ad} \ell'((x_i-x_a)\omega) + \frac{a}{b} w_{bd} \ell'((x_i+\Delta x_i-x_a)\omega) \quad r_b \leftarrow r_b + \frac{a}{b} w_{bd} \ell'((x_b-x_i)\omega) + \frac{a}{b} w_{bd} \ell'((x_b-x_i-\Delta x_i)\omega)
   \end{align*}
   \]

In the appendix, we will show that the generalized push algorithm for p-norm based cut objective is also strongly local with the number of push iterations also upper bounded by \( O(\text{vol}(R)) \).

7 EXPERIMENTS

In the experiments, we will investigate both the LHQD (2-norm) and 1.4-norm cut objectives with the \( \delta \)-linear threshold as the splitting function (more details about this function in §3.3). Our focus in this experimental investigation is on the use of the methods for semi-supervised learning. Consequently, we consider how well the algorithms identify “ground truth” clusters that represent various known labels in the datasets when given a small set of seeds.

In the plots and tables, we use LH-2.0 to represent our LHQD or LHPR method and LH-1.4 to represent the 1.4 norm version from §6. The other methods we are comparing against include: (i) ACL [3], which is initially designed to compute approximated PageRank on graphs. Here we transform each hypergraph to a graph using three different techniques, which are star expansion (star+ACL), unweighted clique expansion (UCE+ACL) and weighted clique expansion (WCE+ACL) where a hyperedge \( e \) is replaced by a clique where each edge has weight \( 1/|e| \) [42]. We chose to run ACL on the expanded graphs because ACL is known as one of the fastest and most successful local graph clustering algorithm in several benchmarks [28, 34] and has a similar quadratic guarantee on local graph clustering [3, 44], (ii) flow [33], which is the maxflow-mincut based local method designed for hypergraphs. Since the flow method has difficulty growing from small seed set as illustrated in the yelp experiment in §1, we will first use the one hop neighborhood to grow the seed set. (OneHop+flow) To limit the number of neighbors included, we will order the neighbors using the BestNeighbors as introduced in [33] and only keep at most 1000 neighbors. (Given a seedset R, BestNeighbors orders nodes based on the fraction of hyperedges incident to \( \sigma \) that are also incident to at least one node from \( R \)). (iii) LH-2.0-flow, this is a simple combination of LH-2.0 and flow where we use the output of LH-2.0 as the input of the flow method to refine it. (iv) HGCARD, using this is a hypergraph generalization of CRD [36], which is a hybrid diffusion and flow.1

In order to select an appropriate \( \delta \) for different datasets, Veldt et al. found that the optimal \( \delta \) is usually consistent among different clusters in the same dataset [33]. Thus, the optimal \( \delta \) can be visually approximated by varying \( \delta \) for a handful of clusters if one has access to a subset of ground truth clusters in a hypergraph. We adapt the same procedure in our experiments and report the results in App. E.

Other parameters are in the reproduction details footnote.2

7.1 Detecting Amazon Product Categories

In this experiment, we use different methods to detect Amazon product categories [30]. The hypergraph is constructed from Amazon product review data where each node represents a product and each hyperedge is set of products reviewed by the same person. It has 2,268,264 nodes and 4,285,363 hyperedges. The average size of hyperedges is around 17. We select 6 different categories with size between 100 and 10000 as ground truth clusters used in [33]. We set \( \delta = 1 \) for this dataset (more details about this choice in §E). We select 1% nodes as seed set for each cluster and report median F1 scores and median runtime over 30 trials in Table 1 and 2. Overall, LH-1.4 has the best F1 scores and LH-2.0 has the second best F1 scores. The two fastest methods are LH-2.0 and star+ACL. While achieving better F1 scores, LH-2.0 is 20x faster than HyperLocal (flow) and 2-5x faster than clique expansion based methods.

7.2 Detecting Stack Overflow Question Topics

In the Stack Overflow dataset, we have a hypergraph with each node representing a question on “stackoverflow.com” and each hyperedge representing questions answered by the same user [33]. For HGCRD, we set \( \rho \), \( \gamma \), and \( \kappa \) once the paper is non-anonymized. We fix the LH locality parameter \( \kappa \) for Stack Overflow based on cluster size. For ACL, we use \( \rho = 0.1 \). We select 1% nodes as seed set for each cluster and report median F1 scores and median runtime over 30 trials in Table 1 and 2. Overall, LH-1.4 has the best F1 scores and LH-2.0 has the second best F1 scores. The two fastest methods are LH-2.0 and star+ACL. While achieving better F1 scores, LH-2.0 is 20x faster than HyperLocal (flow) and 2-5x faster than clique expansion based methods.

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1 Another highly active topic for clustering and semi-supervised learning involves graph neural networks (GNN). Prior comparisons between GNNs and diffusions shows mixed results in the small seed set regime we consider [16, 28] and complicates doing a fair comparison. As such, we focus on comparing with the most directly related work.

2 Reproduction details. We will release the full algorithm and evaluation codes on github once the paper is non-anonymized. We fix the LH locality parameter \( \gamma \) to be 0.1, approximation parameter \( \rho \) to be 0.5 in all experiments. We set \( \kappa = 0.00025 \) for Amazon and \( \kappa = 0.00025 \) for Stack Overflow based on cluster size. For ACL, we use the same set of parameters as LH. For LH-2.0-flow, we set the flow method’s locality parameter to be 0.1. For OneHop+flow, we set the locality parameter to be 0.05, 0.005 on Amazon and Stack Overflow accordingly. For HGCARD, we set \( \Upsilon = 0 \) (maximum flow that can be send out of a node), \( \beta = 3 \) (maximum flow that an edge can handle), \( \omega = 2 \) (multiplicative factor for increasing the capacity of the nodes at each iteration), \( \alpha = 1 \) (controls the eligibility of hyperedge), \( \tau = 0.5 \) and 6 maximum iterations.

---

8
We select 40 clusters with 2,000 to 10,000 nodes and a conductance
we set

Table 3: This table summarizes the median of median runtime in seconds for the Stack Overflow experiments as well as median Precision, Recall, F1 and all-or-nothing conductance over the 40 clusters.

| Alg    | LH2 | LH1.4 | LH2 | ACL | ACL | Flow | HG-1Hop CRD |
|--------|-----|-------|-----|-----|-----|------|-------------|
| Time   | 3.69| 39.89 | 43.84| 1.54| 15.25| 13.71| 48.28       |
| Pr     | 0.65| 0.66  | 0.74 | 0.66| 0.65 | 0.66 | 0.83        |
| Re     | 0.67| 0.67  | 0.59 | 0.6  | 0.66 | 0.65 | 0.11        |
| F1     | 0.66| 0.66  | 0.66 | 0.63| 0.65 | 0.65 | 0.19        |
| Cond   | 0.12| 0.11  | 0.09 | 0.16| 0.13 | 0.12 | 0.07        |

given target tag. This hypergraph is much larger with 15,211,989
nodes and 1,103,243 edges. The average hyperedge size is around 24.
We select 40 clusters with 2,000 to 10,000 nodes and a conductance score below 0.2 using the all-or-nothing penalty. (There are 45 clusters satisfying these conditions, 5 of them are used to select δ in §E.) In this dataset, a large threshold can give better results. For diffusion based method, we set the δ−linear threshold to be 0.1% to 10%. At each seed ratio, denoted as r, we set r = 0.025r.

For the flow-based method, we also consider removing the OneHop
Alg 12 18 17 25 15 24

F1 & Med. F1 & Med. F1 & Med. F1 & Med. F1 & Med. F1 & Med.

LH-2.0 0.77 0.65 0.25 0.19 0.22 0.22
LH-1.4 0.9 0.87 0.32 0.22 0.27 0.77
LH-2.0+flow 0.95 0.82 0.15 0.16 0.46 0.87
star+ACL 0.64 0.51 0.19 0.15 0.2 0.49
WCE+ACL 0.64 0.51 0.2 0.14 0.21 0.51
UCE+ACL 0.27 0.09 0.06 0.05 0.11 0.14
OneHop+flow 0.52 0.6 0.16 0.12 0.09 0.22
HGCRD 0.11 0.16 0.05 0.06 0.07 0.17

Table 2: Median runtime in seconds on detecting Amazon product categories

Table 3: This table summarizes the median of median run-
times in seconds for the Stack Overflow experiments as well as median Precision, Recall, F1 and all-or-nothing conductance over the 40 clusters.

7.3 Varying Number of Seeds

Comparing to the flow-based method, an advantage of our diffusion method is that it expands from a small seed set into a good enough cluster that detects many labels correctly; LH-1.4 can sometimes be slower than the flow method when the target cluster contains many large hyperedges.

8 DISCUSSION

This paper studies the opportunities for strongly local quadratic and \( p \)-norm diffusions in the context of local clustering and semi-
supervised learning.
In closing, flow-based algorithms have often been explained or used as refinement operations to the clusters produced by spectral methods [21] as in LH-2.0+flow. As a final demonstration of this usage on the Yelp experiment from §1, we have precision, recall, and F1 result of 0.87, 0.998, 0.93, respectively, which demonstrates how these techniques may be easily combined to even more accurately find target clusters.

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Since $\overline{T}(Bx - u) = 0$, we also have that for each edge $i \to j$, 
\[
    f_{ij} \cdot ((x_j - x_i) - (x_i - x_j)_+) = 0
\]
\[
\Longleftrightarrow (-z_{ij} + w_{ij} \cdot f((x_i - x_j)_+)) \cdot ((x_i - x_j) - (x_i - x_j)_+) = 0.
\]
(18)

Similarly, $z^T u = 0$ means that for edge $i \to j$, 
\[
    z_{ij} \cdot (x_j - x_i)_+ = 0.
\]
(19)

If $x_j \geq x_i$, then (18) holds automatically, but we need $z_{ij} = 0$ for (19) to hold. If $x_i < x_j$, then (19) holds automatically, but we need $z_{ij} = 0$ in order for (18) to hold. Either way, at optimality we simply have $z = 0$. So line 2 of KKT condition can be simplified into $f = \text{diag}(f'(u)) w$, which implies $f \geq 0$. Then line 4 is automatically satisfied by combining $f = \text{diag}(f'(u)) w$ and $u = (Bx)_+$. Therefore, the simplified KKT condition can be written as
\[
    \kappa y + B^T f((Bx)_+) w + \lambda x_+ + \lambda \psi^T - k = 0
\]
\[
k^T x = 0
\]
\[
k \geq 0, x_+ = 1, x_0 = 0
\]

Since the residual vector $r$ is defined as $-\frac{1}{2} B^T \text{diag}(f'(u)(Bx)_+) w$, from $k \geq 0$ and $d_i = 0$ for $i \in V_0 \cup V_p$, $d_i = d_i$ for $i \in V$, we have $g_{rV} \leq x_0$ and $g_{V_0V_p} = 0$. Also for $i \in V_0 \cup V_p$, from $g_{a} = d_{a} = 0$, we have $k_{i} = 0$. Thus $k^T x = 0$ can be reduced to $(x_0 - g_{rV})^T x_V = 0$. Now by replacing $f(x)$ with $x^2/2$, we can get theorem 3.4.

\section*{B PROOF OF THEOREM 3.5}
First we have the following observations on $r$ and $x$.

\textbf{LEMMA B.1.} At any iteration of Algorithm 1, for each pair of auxiliary nodes, $a \in V_a$, $b \in V_b$ and $a \to b$, $x_a \geq x_b$.

\textbf{PROOF.} If $x_a < x_b$, from $g_a = w_{ab}(x_a - x_b)_+ + \sum_{i \in V} w_{bi}(x_b - x_i)_+ = 0$, we have for any $i \in V$ where $b \to i$, $x_i \leq x_i$. This means $g_a = -w_{ab}(x_a - x_b)_+ + \sum_{i \in V} w_{ai}(x_i - x_a)_+ > 0$, which is a contradiction because $g_a$ is zero for all iterations.

\textbf{LEMMA B.2.} At any iteration of Algorithm 1, for any $i \in V \cup V_0 \cup V_b$, $g_i$ will stay nonnegative and $0 \leq x_i \leq 1$.

\textbf{PROOF.} The nonnegativity of $g_i$ is obvious. At any iteration of Algorithm 1, we either change $g_i$ to $p x_i$ or add some nonnegative values to the residual of adjacent nodes or keep $g_i$ as zero. For each original node $i \in V$, to prove $0 \leq x_i \leq 1$, recall $x_i = g_i$ expands to $g_i = \frac{1}{y} \sum_{b \in V_b} w_{bi}(x_b - x_i)_+ - \frac{1}{y} \sum_{a \in V_a} w_{ai}(x_i - x_a)_+ + d_i [\text{Ind}(i \in R) - x_i]$.

Suppose $x_i$ is the largest element and $x_i > 1$, then $\text{Ind}(i \in R) - x_i < 0$. Since $g_i \geq 0$, this means there exists $b \in V_b$ such that $x_b > x_i > 1$. Assume $a \in V_a$ is adjacent to $b$, then
\[
g_a = -w_{ab}(x_a - x_b)_+ + \sum_{i \in V} w_{bi}(x_i - x_a)_+ = 0
\]
\[
g_b = w_{ab}(x_a - x_b)_+ + \sum_{i \in V} w_{bi}(x_i - x_b)_+ = 0
\]
Since $x_i$ is the largest among all nodes $i \in V$, the only way to satisfy both equations is $x_a = x_b = x_i$. Thus, we have $x_i = x_b > x_i$, which is a contradiction. To prove $x_i \leq 1$ for $i \in V_0 \cup V_b$, from Lemma B.1, we only need to show $x_i \leq 1$ for $i \in V_a$. If there exists $a \in V_a$ such
that $x_a > 1$, since $x_i \leq 1$ for any $i \in V$, then from $g_a = 0$, we have $x_a = x_b > 1$ where $a \rightarrow b$. This means $g_f < 0$, which is a contradiction.

\textbf{Proof of Theorem 3.5.} By using Lemma B.2, $\|g\|$ becomes

$$\|g\|_1 = \sum_{i \in V} g_i = \sum_{i \in R} d_i(1 - x_i) - \sum_{i \in R} d_i x_i$$

This implies that any change to the auxiliary nodes will not affect $\|g\|_1$. Thus calling LHQ-approx does not change $\|g\|_1$. When there exists $i \in V$ such that $g_i > k d_i$, then hyper-push will find $\Delta x_i$ such that $g'_i = \rho k d_i$. Then the new $g'_i$ can be written as

$$g'_i = \frac{1}{Y} \sum_{b \in V_b} w_b(x_b - x_i - \Delta x_i)_+ - \frac{1}{Y} \sum_{a \in V_a} w_{ia}(x_i + \Delta x_i - x_a)_+ + k d_i (\text{Ind}(i \in R) - x_i - \Delta x_i) \quad (20)$$

Note $g'_i$ is a decreasing function of $\Delta x_i$ and $g'_i > 0$ when $\Delta x_i = 0$, $g'_i < 0$ when $\Delta x_i = 1$ by using Lemma B.2. This suggests that there exists a unique $\Delta x_i$ that satisfies $g'_i = \rho k d_i$. Moreover, $(x_b - x_i - \Delta x_i)_+ \geq (x_b - x_i - \Delta x_i) \geq (x_i + \Delta x_i - x_a)_+ \leq (x_i - x_a + \Delta x_i)$, thus we have

$$\rho k d_i = g'_i \geq g_i - \frac{1}{Y} \sum_{b \in V_b} w_b + \sum_{a \in V_a} w_{ia} |\Delta x_i| - k d_i |\Delta x_i|$$

From equation (4.9) of [32],

$$\sum_{b \in V_b} w_b = \sum_{a \in V_a} w_{ia} = \sum_{e \in E, i \in e} f_e(\{i\}) \leq \delta d_i$$

Thus, we have $d_i |\Delta x_i| \geq \frac{\rho k d_i}{\max(\rho, \delta)}$. So the decrease of $\|g\|_1$ will be at least $\gamma (1 - \rho) d_i / (\gamma + \delta)$. Since $\|g\|_1 = \|\text{vol}(R)\|$ initially, we have $\sum_{i=1}^T d_i \leq (\gamma + \delta) \|\text{vol}(R)\| / (1 - \rho) = O(\|\text{vol}(R)\|)$.\qed

\section{Runtime Guarantee in Section 6}

Proof. The procedure of bounding the number of times to call (4) is similar. Whenever there exists $g'_i > k d_i$, we need to find $\Delta x_i$ such that $g'_i = \rho k d_i$. And $g'_i$ can be rewritten as:

$$g'_i = \frac{1}{Y} \sum_{b \in V_b} w_b (x_b - x_i - \Delta x_i)^{p-1} - \frac{1}{Y} \sum_{a \in V_a} w_{ia} (x_i + \Delta x_i - x_a)^{p-1} + k d_i (1_{1_R} - 1_{1_R} - x_i - \Delta x_i)^{p-1}$$

Here $1_{1_R}$ is a binary vector where $(1_{1_R}) = 1$ if $R \in R$ and $(1_{1_R}) = 0$ otherwise. For $1 < p < 2$, from Lemma 4 of [28], we know $(x_b - x_i - \Delta x_i)^{p-1} \geq (x_b - x_i)^{p-1} - 2^{2-p}(\Delta x_i)^{p-1}$, $(x_i + \Delta x_i - x_a)^{p-1} \geq (x_i - x_a)^{p-1} + 2^{2-p}(\Delta x_i)^{p-1}$ and $(1_{1_R} - x_i - \Delta x_i)^{p-1} \geq (1_{1_R} - x_i)^{p-1} + 2^{2-p}(\Delta x_i)^{p-1}$. Thus, we have the following inequality:

$$g'_i \geq g_i - \frac{2^{2-p}}{Y} \sum_{b \in V_b} w_b + \sum_{a \in V_a} w_{ia} (\Delta x_i)^{p-1} - 2^{2-p} k d_i (\Delta x_i)^{p-1}$$

which gives $\Delta x_i \geq (\gamma (1 - \rho) / (\gamma + \delta))^{1/(p-1)}$. On the other hand, $\|g\|_1$ can now be written as:

$$\|g\|_1 = \sum_{i \in R} d_i (1 - x_i)^{p-1} - \sum_{i \in R} d_i x_i^{p-1}$$

Also from Lemma 4 of [28], $(1 - x_i - \Delta x_i)^{p-1} \leq (1 - x_i)^{p-1} - (p - 1) \Delta x_i$ and $(x_i + \Delta x_i)^{p-1} \geq x_i^{p-1} + (p - 1) \Delta x_i$ so the decrease of $\|g\|_1$ will be at least $(p - 1) d_i ((\gamma (1 - \rho) / (\gamma + \delta))^{1/(p-1)}$. If we need to call (4) $T$ times, then

$$\sum_{i=1}^T d_i \leq \frac{(\gamma (1 - \rho) / (\gamma + \delta))^{1/(p-1)} \text{vol}(R)}{(p - 1) ((\gamma (1 - \rho) / (\gamma + \delta))^{1/(p-1)} \text{vol}(R))} = O(\|\text{vol}(R)\|)$$\qed

\section{Proof of Theorem 4.2}

Similar to the last section, we use $1_v$ to denote the canonical basis with $v$-th component as 1 and others as 0 and $1_S = \sum_{e \in E} 1_e$. Without loss of generality, we assume each gadget reduced from a hyperedge has weight $\epsilon_e = 1$. Otherwise one can simply add $\epsilon_e$ as the coefficients and nothing needs to change. We omit the degree regularization term in (6). We also remove the constraints $x_v = 1$ and $x_0 = 0$ by directly pushing them into the objective. Now the objective in (6) becomes isotropic with respect to $x$. Therefore, we are able to normalize $x$ to guarantee $\sum_{e \in E} d_e x_e = 0$. Note that this step can significantly simplify the notations in our following analysis. We can remove such normalization by multiplying $x$ with $\|\text{vol}(S)\|$, which we will do in the following step. Then, the objective in (6)

$$\min_{x} Q(x) \leq \gamma \sum_{e \in E} d_e (x_v - \frac{1}{\|\text{vol}(S)\|} (1_v))^2 + \sum_{e \in E} Q_e(x) \quad (22)$$

where

$$\max_{\epsilon} Q_e(x) \leq \min_{\epsilon \in [\epsilon_e, \epsilon_e]} \sum_{e \in E} (x_v - x_v^{(e)})^2 + (x_v^{(e)} - x_v)^2 + \epsilon(x_v^{(e)} - x_v^{(e)})^2. \quad (23)$$

We denote $M = \|\text{vol}(H)\| = \sum_{e \in E} d_e$. We denote the solution $x$ with the parameter $y$ and the seed set $S$ of the optimization (22) as $x(y, S)$ and its component for node $u$ as $x_u(y, S)$. We also define another degree-weighted vector $p = D x$, where $D$ is the diagonal degree matrix. For a vector $p$ and a node set $S$, we use $p(S')$ to denote $p(S') = \sum_{e \in E} p_e$. It is easy to check that $p(y, S) = 1$ for any $S$. For a node set $S$, define $S_e = \{ e \in E | e \cap S \neq \emptyset, e \cap V \neq \emptyset \}$. We now define our main tool: the Lovász-Simonovits Curve.

\textbf{Definition 6.1 (Lovász-Simonovits Curve (LSC)).} Given an $x$, we order its components from large to small by breaking equality arbitrarily, say $x_1, x_2, \ldots, x_n$ w.l.o.g. and define $S_1 = \{ x_1, \ldots, x_j \}$. LSC defines a corresponding piece-wise linear function $I_x : \{0, 1\} \rightarrow R$ s.t. $I_x(0) = 0$, $I_x(\text{vol}(G)) = 1$ and $I(\text{vol}(S^{(j)})) = p(S^{(j)})$. And for any $k \in [\text{vol}(S^{(j)}), \text{vol}(S^{(j+1)})]$:

$$I_x(k) = I_x(\text{vol}(S^{(j)})) + (k - \text{vol}(S^{(j)})) x_j.$$ 

LSC can tell us some meaningful information about conductance. Note that LSC is completely determined by the values at breakpoints. Intuitively, when there exists a big disparity between slopes at breakpoints, the distribution is very different from the stationary distribution $\pi : V \rightarrow \mathbb{R}_{\geq 0}$ where $\pi_v = d_v / M$. This in turn indicates the existence of a small cut, which inhibits the flow of probability mass.

Some properties of LSC will be used throughout the proof. First, LSC is piece-wise linear with decreasing slope and therefore concave. Second, for any set $S \subseteq V$, we have $p(S) \leq I_x(\text{vol}(S))$. Therefore, in many cases we can first obtain results about $I_x(\text{vol}(S^{(j)}))$ i.e.
breakpoints and then extend them through concavity to arbitrary \( I_x(m) \) where \( m \in [0, M] \), and at last to arbitrary \( S \subset V \).

Our proof depends on the following two Lemma D.2 and D.3 that will characterize the upper and lower bound of \( I_x \), which finally leads to the main theorem.

**Lemma D.2.** For a set \( S \), given an \( x = x(y, S) \), let \( \phi_x \) and \( S_x \) be the minimal conductance and the node set obtained through a sweep-cut over \( x \). For any integer \( t > 0 \) and \( k \in [0, M] \), the following bound holds

\[
I_x(k) \leq \frac{k}{M} + \frac{yt}{2+y} + \sqrt{\frac{\min\{k, M-k\}}{\min e \in S} \left(1 - \frac{\sigma_x^2 \phi_x^2}{8}\right)} t := f(t)(k)
\]

where \( \sigma_x = (2 \max_{e \in S_x} \min\{\delta_e, |e|/2\})^{-1} \).

**Lemma D.3.** For a set \( S \), if a node \( v \in S \) is sampled according to a distribution \( P \) s.t.

\[
\mathbb{E}_{v \sim s}[p(y, \{v\})(S)] \leq c p(y, S)(S),
\]

where \( c \) is a constant, then with probability at least \( \frac{1}{2} \), one has

\[
p(y, \{v\})(S) \leq 1 - 2c\phi(S)/y.
\]

Lemma D.3 gives the lower bound \( L_x(y(x), (v)) \) \( \text{vol}(S) \) as this value is no less than \( p(y, \{v\})(S) \). Note that the sampling assumption of Lemma D.3 is natural in the standard graph case, when \( P \) samples each node proportionally to its degree, we have an equality with \( c = 1 \) in (24). Combining Lemma D.2 and D.3, we arrive at

**Theorem D.4.** Given a set \( S^* \) of vertices s.t. \( \text{vol}(S^*) \leq \frac{M^2}{2} \) and \( \phi(S^*) \leq \frac{\phi_{\text{avg}}}{\phi(S)} \) for some positive constants \( \gamma, \phi(S) \). If there exists a distribution \( P \) s.t. \( \mathbb{E}_{v \sim s}[p(y, \{v\})(S)] \leq c p(y, S)(S) \), then with probability at least \( \frac{1}{2} \), the obtained conductance satisfies

\[
\phi_x \leq \left( \frac{\min\{k, M-k\}}{\min e \in S} \right)^{1/2} \sqrt{32y \ln \left( \frac{\text{vol}(S^*)}{d_x} \right)}.
\]

where \( x = x(y, \{v\}) \) and \( v \) is sampled from \( P \). \( S_x \) is the node set obtained via the sweep-cut over \( x \).

**Proof.** The proof is in the same spirit as Thm. 17 in [25] (Sec. 7.7.3). Since \( \phi(S^*) \leq \frac{\phi_{\text{avg}}}{\phi(S)} \), then we have \( \mathbb{E}[p(y, \{v\})(S)] \geq \frac{1}{2} \). Now combining the upper bound of \( I_x \) in Lemma D.2 and its lower bound in Lemma D.3 yields

\[
\frac{3}{4} \leq \frac{1}{2} + \frac{yt}{2+y} + \sqrt{\frac{\text{vol}(S^*)}{d_x} \left(1 - \frac{\sigma_x^2 \phi_x^2}{8}\right)} t
\]

for any \( t > 0 \).

Specifically, we know \( 1 - \frac{\sigma_x^2 \phi_x^2}{8} \leq e^{-\frac{\sigma_x^2 \phi_x^2}{8}} \). Then by setting

\[
t = \frac{8}{\sigma_x^2 \phi_x^2} \ln \left( \frac{\text{vol}(S^*)}{d_x} \right),
\]

we transform the inequality to

\[
\frac{3}{4} \leq \frac{5}{8} + \frac{yt}{2+y} \sqrt{\frac{\text{vol}(S^*)}{d_x} \left(1 - \frac{\sigma_x^2 \phi_x^2}{8}\right)}
\]

Therefore,

\[
\phi_x \leq \frac{1}{\sigma_x} \sqrt{32y \ln \left( \frac{\text{vol}(S^*)}{d_x} \right)}
\]

and this completes the proof. \( \Box \)

By removing the normalizing on the number of seeded nodes, Thm. D.4 becomes Thm. 4.2.

**D.1 Proof of Lemma D.2**

Define \( L_x(x) = \nabla_x \frac{1}{2} Q_x(x) \) (23) and with some algebra, we have

\[
L_x(x) = \sum_{e \in E} \left( x_a(e) - x_a^*(e) \right)_+ + \left( x_b(e) - x_b^*(e) \right)_+ \text{vol}(S)
\]

where \( x_a^*(e) \) and \( x_b^*(e) \) are the optimal values in (23). In the following, we will first prove Lemma D.5 and further use it to prove Lemma D.6. The same proof in Thm. 16 in [25] (Sec. 7.7.2) can be used to leverage Lemma D.6 to prove Lemma D.2.

**Lemma D.5.** Given an \( x \), we order its components from large to small, say \( x_1, x_2, ..., x_n \) w.l.o.g., and define \( S_j = \{x_1, ..., x_j\} \). Define \( \sigma_j^2 = (1 + 2 \max_{e \in S_j} \min\{\delta_e, |e|/2\})^{-1} \), and we have

\[
2I_x(\text{vol}(S_j)) - \sum_{e \in E} L_x((x), 1_S^e) \leq I_x(\text{vol}(S_j^*)) - \sigma_j^2 \text{cut}(S_j^*) + I_x(\text{vol}(S_j^*) + \sigma_j^2 \text{cut}(S_j^*)).
\]

**Lemma D.6.** Suppose \( x = x(y, S) \), \( x_0 = 1_S/y \) and \( \sigma_j^2 = (1 + 2 \max_{e \in S_j} \min\{\delta_e, |e|/2\})^{-1} \). We have

\[
I_x(\text{vol}(S_j^*)) \leq \frac{Y}{2+y} I_{x_0}(\text{vol}(S_j^*)) + \frac{1}{2+y} \left( I_x(\text{vol}(S_j^*)) - \sigma_j^2 \text{cut}(S_j^*) + I_x(\text{vol}(S_j^*) + \sigma_j^2 \text{cut}(S_j^*) \right).
\]

Furthermore, for \( k \in [0, M] \), \( I_x(k) \leq I_{x_0}(k) \).

**Proof of Lemma D.5.** Given a hyper-edge \( e \), we order \( \{x_a \setminus e \} \) from large to small by breaking equality arbitrarily and obtain \( x_a^*(e) \), \( x_a^{(e)} \), ..., \( x_a^{(e)} \). Suppose \( x_a^{(e)} \) \( \notin S_j \) and \( x_a^{(e)} \notin S_j \). Then,

\[
\langle L_x(x), 1_S^e \rangle \leq \frac{1}{k} \sum_{i=1}^k \left( x_a^{(e)} - x_a^{(e)} \right)_+ + (x_b^{(e)} - x_b^{(e)})_+
\]

Next, we will bound (26) by analyzing three cases. We only focus on \( x_a^{(e)} \) and \( x_a^{(e)} \) and otherwise \( x_a^{(e)} \) and \( x_a^{(e)} \) are constant for all \( v \in e \). Also denote \( k_n = \max\{1 \setminus |x_a^{(e)} - x_a^{(e)}|\}, k_n = \min\{|e| - i \setminus |x_a^{(e)} - x_a^{(e)}|\}, k_n = |e| - 1 - k \). The optimality of \( x_a^{(e)} \), \( x_a^{(e)} \), we have

\[
x_a^{(e)} = \left( k_n + \delta_e \right)x_a^{(e)} + \delta_e x_a^{(e)} - k_n \delta_e x_a^{(e)}
\]

where \( x_a^{(e)} = (k_n + \delta_e)x_a^{(e)} + \delta_e x_a^{(e)} - k_n \delta_e x_a^{(e)} \). Then,

\[
\textrm{Thus, we have}
\]

Through some algebra we can see that the sum of positive coefficients in each case of (27) will correspond to \( -\sigma_j^2 \text{cut}(S_j^*) \) in (25), whereas the sum of negative coefficients will correspond to \( +\sigma_j^2 \text{cut}(S_j^*) \). Due to the concavity of \( I_x \), when trying to establish upper bound as in (25) we only need to consider the smaller one of the two sums in each case, which we analyze below.
By using the definition of \( X_{t_i}^{(e)} \), noticing that all coefficients on \( X_{t_i}^{(e)} \leq 1 \) in the left hand side of (27), and a good deal of algebra (Sec. D.3), we can further show in each case the smaller sum satisfies

\[
\begin{align*}
&\frac{k(k_0-k)(k_0-k_0)+k_0-k_0}{k_0k_0-k_0k_0} \geq \frac{1}{4\sigma^2} \min\{k,|e|-k,\delta_e\} \\
&\text{if } k \leq k_0, \\
&\frac{k(k_0-k)(k_0-k_0)+k_0-k_0}{k_0k_0-k_0k_0} \geq \frac{1}{2\sigma^2(1+1)} \min\{k,|e|-k,\delta_e\}, \\
&\text{if } k_0 \leq k \leq k_0^*, \\
&\frac{k(k_0-k)(k_0-k_0)+k_0-k_0}{k_0k_0-k_0k_0} \geq \frac{1}{2\sigma^2} \min\{k,|e|-k,\delta_e\}, \\
&\text{if } k \geq k_0^*. \\
\end{align*}
\]

where \( \delta_e' = \min\{\delta_e,|e|/2\} \).

Notice that \( \min\{k,|e|-k,\delta_e\} \) is exactly the splitting cost of \( S_f^y \) in \( e \). Then by linearity we can establish

\[
2L_k(\mathbb{vol}(S_f^y)) - \sum_{e \in E} (L_e(x), 1_S_f^y) \leq \left( I_k(\mathbb{vol}(S_f^y) - \sigma^2\mathbb{cut}(S_f^y)) + I_k(\mathbb{vol}(S_f^y) + \sigma^2\mathbb{cut}(S_f^y)) \right)
\]

where \( \sigma^2 = 1/(2 \max_{e \in \mathbb{A}} \delta_e' + 1) \), which concludes the proof.

**Proof of Lemma D.6.** Compute the derivative of \( Q(x) \) (22) w.r.t. \( x \) and use the optimality of \( x = x(y,S) \) to get \( 0 = yD(x-x_0) + L_e(x) \). Therefore, taking the inner product with \( 1_S_f^y \) yields

\[
0 = y(I_k(\mathbb{vol}(S_f^y)) - I_{\mathbb{vol}}(\mathbb{vol}(S_f^y))) + \langle L_e(x), 1_S_f^y \rangle.
\]

Plug in Lemma D.5 and we achieve (25). By using the concavity of \( I_k \), we obtain \( I_k(\mathbb{vol}(S_f^y)) \leq I_{\mathbb{vol}}(\mathbb{vol}(S_f^y)) \) and therefore \( I_k(k) \leq I_{\mathbb{vol}}(k) \) for any \( k \in [0, M] \).

**Proof of Lemma D.2.** We prove the lemma using the same technique as in[25] (Sec. 7.7.2). The only difference is that we need to replace \( \delta_e \) with \( \sigma_e \). We include the proof here for completeness.

Notice that since \( f(t)(k) \), we only need to prove the claim for the breakpoints of LSC. Fix some \( k = \mathbb{vol}(S_f^y) \). Now we will induct on \( t \).

For the sake of convenience, let us define \( \bar{k} = \min\{k, M - k\} \) and \( d_{\min} = \min_{e \in \mathbb{S}^e} d_t \). Then we write

\[
f(t)(k) = \frac{k}{M} + \frac{y}{2 + y} t + \sqrt{\frac{k}{d_{\min}}} \left( 1 - \frac{\sigma^2\Phi^2}{8} \right)^{t/2}.\]

We start from the base case of \( t = 1 \). From Lemma D.6 we know that \( I_{\mathbb{vol}}(k) \leq I_{\mathbb{vol}}(k) \). When \( k = 0 \), we have \( I_{\mathbb{vol}}(0) = 0 = I^{(0)}(0) \). For \( k \in [d_{\min}, M] \), we have

\[
I_{\mathbb{vol}}(k) \leq 1 \leq I^{(0)}(k).
\]

Thus the base case is verified.

Assume the claim holds for \( t \). Now we need to show that the claim also holds for \( t + 1 \). From Lemma D.6, we have

\[
I_{\mathbb{vol}}(k) \leq \frac{y}{2 + y} I_{\mathbb{vol}}(k) + \frac{1}{2 + y} \left[ I_1(k - \bar{k}\sigma, \Phi(S_f^y)) + I_1(k + \bar{k}\sigma, \Phi(S_f^y)) \right].
\]

By using the definition of \( X_{t_i}^{(e)} \), noticing that all coefficients on \( X_{t_i}^{(e)} \leq 1 \) in the left hand side of (27), and a good deal of algebra (Sec. D.3), we can further show in each case the smaller sum satisfies

\[
\begin{align*}
&\frac{k}{k_0k_0-k_0k_0} \geq \frac{1}{4\sigma^2} \min\{k,|e|-k,\delta_e\} \\
&\text{if } k \leq k_0, \\
&\frac{k}{k_0k_0-k_0k_0} \geq \frac{1}{2\sigma^2(1+1)} \min\{k,|e|-k,\delta_e\}, \\
&\text{if } k_0 \leq k \leq k_0^*, \\
&\frac{k}{k_0k_0-k_0k_0} \geq \frac{1}{2\sigma^2} \min\{k,|e|-k,\delta_e\}, \\
&\text{if } k \geq k_0^*. \\
\end{align*}
\]

where \( \delta_e' = \min\{\delta_e,|e|/2\} \).

Notice that \( \min\{k,|e|-k,\delta_e\} \) is exactly the splitting cost of \( S_f^y \) in \( e \). Then by linearity we can establish

\[
2L_k(\mathbb{vol}(S_f^y)) - \sum_{e \in E} (L_e(x), 1_S_f^y) \leq \left( I_k(\mathbb{vol}(S_f^y) - \sigma^2\mathbb{cut}(S_f^y)) + I_k(\mathbb{vol}(S_f^y) + \sigma^2\mathbb{cut}(S_f^y)) \right)
\]

where \( \sigma^2 = 1/(2 \max_{e \in \mathbb{A}} \delta_e' + 1) \), which concludes the proof.

**Proof of Lemma D.6.** Compute the derivative of \( Q(x) \) (22) w.r.t. \( x \) and use the optimality of \( x = x(y,S) \) to get \( 0 = yD(x-x_0) + L_e(x) \). Therefore, taking the inner product with \( 1_S_f^y \) yields

\[
0 = y(I_k(\mathbb{vol}(S_f^y)) - I_{\mathbb{vol}}(\mathbb{vol}(S_f^y))) + \langle L_e(x), 1_S_f^y \rangle.
\]

Plug in Lemma D.5 and we achieve (25). By using the concavity of \( I_k \), we obtain \( I_k(\mathbb{vol}(S_f^y)) \leq I_{\mathbb{vol}}(\mathbb{vol}(S_f^y)) \) and therefore \( I_k(k) \leq I_{\mathbb{vol}}(k) \) for any \( k \in [0, M] \).

**Proof of Lemma D.2.** We prove the lemma using the same technique as in[25] (Sec. 7.7.2). The only difference is that we need to replace \( \delta_e \) with \( \sigma_e \). We include the proof here for completeness.

Notice that since \( f(t)(k) \), we only need to prove the claim for the breakpoints of LSC. Fix some \( k = \mathbb{vol}(S_f^y) \). Now we will induct on \( t \).

For the sake of convenience, let us define \( \bar{k} = \min\{k, M - k\} \) and \( d_{\min} = \min_{e \in \mathbb{S}^e} d_t \). Then we write

\[
f(t)(k) = \frac{k}{M} + \frac{y}{2 + y} t + \sqrt{\frac{k}{d_{\min}}} \left( 1 - \frac{\sigma^2\Phi^2}{8} \right)^{t/2}.\]

We start from the base case of \( t = 1 \). From Lemma D.6 we know that \( I_{\mathbb{vol}}(k) \leq I_{\mathbb{vol}}(k) \). When \( k = 0 \), we have \( I_{\mathbb{vol}}(0) = 0 = I^{(0)}(0) \). For \( k \in [d_{\min}, M] \), we have

\[
I_{\mathbb{vol}}(k) \leq 1 \leq I^{(0)}(k).
\]

Thus the base case is verified.

Assume the claim holds for \( t \). Now we need to show that the claim also holds for \( t + 1 \). From Lemma D.6, we have

\[
I_{\mathbb{vol}}(k) \leq \frac{y}{2 + y} I_{\mathbb{vol}}(k) + \frac{1}{2 + y} \left[ I_1(k - \bar{k}\sigma, \Phi(S_f^y)) + I_1(k + \bar{k}\sigma, \Phi(S_f^y)) \right].
\]
Here we use that $\sum_{e \in S} \sum_{e \in \mathcal{E}} [(\hat{x}_a - \hat{x}_a(\epsilon)) + (\hat{x}_b(\epsilon) - \hat{x}_a)] = 0$. Plug (29) into (30) and we have

$$0 \leq \frac{1}{\phi(S)} \left( \sum_{e \in \mathcal{E}} d_e \hat{e} - 1 \right) + \sum_{e \in \mathcal{E}} \min\{ |e \cap S|, |\epsilon(S)| + |e_\epsilon| - 1 \} \frac{\text{vol}(S)}{S},$$

which concludes the proof.

### D.3 Algebraic details of Lemma D.3

This section contains the good deal of algebra for (28). We also want to give the readers intuition about when the equities are achieved.

**Case 1:** $k \leq k^*$

For convenience, let us write

$$C_1 = \frac{(k_+ - k)(k_- + \delta_k) + k_- \delta_k}{k_+ k_- + \delta_k k_+ + \delta_k k_-}.$$ 

Now let us lower bound $t = \frac{C_1}{\min(k, |e| - k, \delta_k)}$. Notice that

$$C_1 = 1 - \frac{kk_- + k\delta_k}{k_k_+ + \delta_k k_+ + \delta_k k_-} \geq 1 - \frac{\delta_k}{k_+ + \delta_k k_- + \delta_k k_+} \geq \frac{\delta_k}{\delta_k k + \delta_k k_- + \delta_k k_+}$$

where (1) comes from $k \leq k^*$ and (2) comes from $k \geq \delta_k$. Therefore, we have

$$t = \frac{\min(k, |e| - k, \delta_k)}{k_+ k_- + \delta_k k_+ + \delta_k k_-}.$$

If $k, \delta_k \geq |e|/2$, we can calculate

$$t \geq \frac{1}{|e|/2} = \frac{1}{\delta_k + 2}.$$

If $k < |e|/2$ or $\delta_k < |e|/2$, we can calculate

$$t \geq \frac{\delta_k k + \delta_k k_- + \delta_k k_+}{\min(k, \delta_k) + 2} \geq \frac{\min(|e|/2, \delta_k) + 2}{\delta_k + 2}.$$

**Case 2:** $k^* < k \leq k^-$

In this case, we can calculate

$$\frac{k_+ k_- + \delta_k k_+ + \delta_k k_-}{k_+ k_- + \delta_k k_+ + \delta_k k_-} \geq \frac{1}{1 + 2\delta_k},$$

where the lower bound is achieved when $k = \delta_k = |e|/2, k_- = k_+ = 1$.

**Case 3:** $k \geq k^-$

This case follows a symmetric analysis as Case 1.

For convenience, let us write

$$C_2 = \frac{k_+ \delta_k + (k - k^*)(k_+ + \delta_k)}{k_+ k_- + \delta_k k_+ + \delta_k k_-}.$$ 

Now let us lower bound $u = \frac{|e|/2}{\min(k, |e| - k, \delta_k)}$. Notice that

$$C_2 \geq 1 - \frac{|e|/2}{\min(k, |e| - k, \delta_k)} C_2 = 1 - \frac{(k_+ + \delta_k)(|e| - k)}{k_+ k_- + \delta_k k_+ + \delta_k k_-} \geq \frac{(k_+ + \delta_k)(|e| - k)}{k_+ (|e| - k) + \delta_k (|e| - k) + \delta_k k_+} \geq \frac{\delta_k}{\delta_k (|e| - k) + \delta_k (|e| - k) + \delta_k k_+}.$$ 

where (1) comes from the definition $k_- = |e| - k^-$, (2) comes from $k \geq k^-$ and (3) comes from $k_+ \geq 1$.

Therefore, we have

$$u = \frac{\delta_k (|e| - k)}{\delta_k (|e| - k) + (|e| - k) + \delta_k min(k, |e| - k, \delta_k)}.$$ 

If $|e| - k, \delta_k \geq |e|/2$, we can calculate

$$u \geq \frac{1}{|e|/2} = \frac{1}{\delta_k + 2} \geq \frac{1}{\delta_k + 2}.$$ 

If $|e| - k < |e|/2$ or $\delta_k < |e|/2$, we can calculate

$$u \geq \frac{\min(|e| - k, \delta_k) + \delta_k}{\min(|e|/2, \delta_k) + 2} \geq \frac{1}{\delta_k + 2}.$$ 

Here, we conclude the entire proof.

### E SELECTING $\delta$.

To select $\delta$ for each dataset, we run LH-2.0 on a handful of alternative clusters as we vary $\delta$. Below, we show F1 scores on those clusters and pick $\delta = 1$ for Amazon and $\delta = 1000$ for Stack Overflow.