A short note on the extended Bruinier-Kohnen conjecture

Mohammed Amin Amri ∗ and M’hammed Ziane †

October 17, 2017

Abstract

Let $f$ be a cuspform of integral half-weight $k + 1/2$, whose Fourier coefficients $a(n)$ not necessarily real. We verify partially an extension of a conjecture of Bruinier and Kohnen on the equi-distribution of the signs of $a(n)$ (when are real), conjectured by the first author in [1] for the sequence $\{a(tp^{2\nu})\}_{p,\text{prime}}$, where $\nu$ an odd positive integer and $t$ a square-free integer.

1 Introduction and statement of result

Throughout the paper we denote by $\mathbb{P}$ the set of all prime numbers. Let $S \subset \mathbb{P}$, we denote by $\delta(S)$ the natural density of $S$. We shall denote by $\pi(x)$ the prime-counting function. Let $k, N$ be natural numbers assume that $N$ be an odd and square-free integer, fix a Dirichlet character $\chi$ modulo $4N$. We shall denote by $S_{k+1/2}(4N, \chi)$ the space of cusp forms of weight $k + 1/2$ for the congruence subgroup $\Gamma_0(4N)$ with character $\chi$. When $k = 1$, we shall work only with the orthogonal complement with respect to the Petersson scalar product of the subspace of $S_{3/2}(4N, \chi)$ generated by the unary theta functions. Let $f \in S_{k+1/2}(4N, \chi)$, be a cusp form which lies in the Kohnen’s plus space, and let the Fourier expansion of $f$ at $\infty$ be

$$f(z) = \sum_{n \geq 1} a(n)q^n \quad \text{with} \quad q := e^{2\pi i z}.$$ 

The question about sign changes of $a(n)$ (when are real) was first studied by Bruinier and Kohnen in [4]. After that, there has been more extensive study (cf [2, 1]) which leads the first author [1], to conjectured that

$$\lim_{x \to \infty} \frac{\# \{ n \leq x : \Re \{ a(n)e^{-i\phi} \} \geq 0 \}}{\# \{ n \leq x : \Re \{ a(n)e^{-i\phi} \} \neq 0 \}} = \frac{1}{2},$$

for each $\phi$ blowing to $[0, \pi)$, which can be seen as an extension of an equi-distribution conjecture of Bruinier and Kohnen [1, 8] on the signs of $a(n)$ (when are real). In this note we address the above conjecture, indeed we verify it for the sequence $\{a(tp^{2\nu})\}_{p \in \mathbb{P}}$, we first prove that is oscillatory in the sense of [10, 6] and that no matter how we slice the plane with a straight line going through the origin the proportion of prime from $\{tp^{2\nu}\}_{p,\text{prime}}$ on which the $a(tp^{2\nu})$ are on the same half plane equal to the half of the proportion of prime from $\{tp^{2\nu}\}_{p,\text{prime}}$ on which the $a(tp^{2\nu})$ are not on the line. More precisely we shall prove the following.

∗ACSA Laboratory, Department of Mathematics, Faculty of Sciences, Mohammed First University, Oujda, Morocco amri.amine.mohammed@gmail.com

†ACSA Laboratory, Department of Mathematics, Faculty of Sciences, Mohammed First University, Oujda, Morocco ziane12001@yahoo.fr
Theorem. Let \( f \in S_{k+1/2}(4N, \chi) \) be a cuspidal Hecke eigenform with Dirichlet character \( \chi \) (mod \( 4N \)), squarefree level, and let the Fourier expansion of \( f \) at \( \infty \) be

\[
f(z) = \sum_{n \geq 1} a(n) q^n.
\]

Let \( \nu \) be a positive odd integer, and \( t \) be a square free integer such that \( a(t) \neq 0 \). Let \( \phi \in [0, \pi) \), define the set of primes.

\[
P_{>0}(\phi, \nu) := \{ p \in \mathbb{P} : \Re (a(tp^{2\nu})e^{-i\phi}) > 0 \},
\]

and similarly \( P_{<0}(\phi, \nu), P_\neq 0(\phi, \nu) \). Then the sequence \( \{a(tp^{2\nu})\}_{p, \text{prime}} \) is oscillatory, moreover the sets \( P_{<0}(\phi, \nu) \) and \( P_{>0}(\phi, \nu) \) have natural density equal to the half of the natural density of \( P_\neq 0(\phi, \nu) \).

The above theorem improve the result of the first author in [1, Theorem 4.1]. The proof which will be given in the next section rely on Sato-Tate conjecture [3, Theorem B].

2 Proof of the result

We start by recalling some basic properties of Shimura lift and introduce some notation. Let

\[
f(z) = \sum_{n \geq 1} a(n) q^n,
\]

be a non-zero cuspidal Hecke eigenform, and let \( t \) be a squarefree integer such that \( a(t) \neq 0 \). The Shimura correspondence [11, 12] lifts \( f \) to a Hecke eigenform \( f_t \) of weight \( 2k \) for the group \( \Gamma_0(2N) \) with character \( \chi^2 \). Let

\[
f_t(z) = \sum_{n \geq 1} A_t(n) q^n,
\]

be its expansion at \( \infty \). According to [11], the \( n \)-th Fourier coefficient of \( f_t \) is given by

\[
A_t(n) = \sum_{d|n} \chi_{t,N}(d) d^{k-1} a \left( \frac{n^2 t}{d^2} \right),
\]

where \( \chi_{t,N} \) denotes the character \( \chi_{t,N}(d) := \chi(d) \left( \frac{(-1)^k N^2 t}{d} \right) \), we let \( \chi_0(d) := \left( \frac{(-1)^k N^2 t}{d} \right) \). Since \( f \) is a Hecke eigenform, then so is the Shimura lift. Indeed we have \( f_t = a(t)f \) where \( f \) is a normalized Hecke eigenform, write

\[
f(z) = \sum_{n \geq 1} \lambda(n) n^{k-1/2} q^n,
\]

for its Fourier expansion at \( \infty \). We shall assume that \( a(t) = 1 \), in the general case we may apply the proof to \( \frac{f}{a(t)} \). Since \( 2N \) is square-free, it follows that \( f \) is a Hecke eigenform without complex multiplication.

Let \( \zeta \) be a root of unity blowing to \( \Im(\chi) \), and let \( p \) be a prime number such that \( \chi(p) = \zeta \), it is clear that \( \frac{\lambda(p)}{\zeta} \) is real. By Deligne’s bound we have \( |\frac{\lambda(p)}{\zeta}| \leq 2 \), thus we may write

\[
\lambda(p) = 2\zeta \cos(\theta_p),
\]

for a uniquely defined angel \( \theta_p \in [0, \pi] \). At this point we state the following theorem which will be crucial for our purpose.
Theorem 1. (Barnet-Lamb, Geraghty, Harris, Taylor) With the above setup. The sequence \( \{\theta_p\}_p \) is equidistributed in \([0, \pi]\) as \( p \) varies over primes satisfying \( \chi(p) = \zeta \), with respect to the Sato-Tate measure \( \mu_{ST} := \frac{2}{\pi} \sin^2 \theta d\theta \). In particular, for any sub-interval \( I \subset [0, \pi] \) we have

\[
\lim_{x \to \infty} \frac{\# \{ p \leq x : \chi(p) = \zeta, \theta_p \in I \}}{\frac{2}{\pi} \int_I \sin^2 \theta d\theta} = \frac{1}{r_{\chi}}.
\]

We shall need the following technical lemmas.

Lemma 2.1. Let \( \zeta \) a root of unity blowing to \( \text{Im}(\chi) \), then the set

\[
\{ p \in \mathbb{P} : \chi(p) = \zeta \},
\]

have natural density equal to \( \frac{1}{r_{\chi}} \), where \( r_{\chi} \) denotes the order of the character \( \chi \).

Proof. To establish the above lemma, it suffices to see

\[
\{ p \in \mathbb{P} : \chi(p) = \zeta \} = \bigcup_{\nu \in (\mathbb{Z}/4N) \ast} \{ p \in \mathbb{P} : p \equiv a \pmod{4N} \}.
\]

Note that the number of sets in the union is \( \# \ker(\chi) = \frac{\phi(4N)}{r_{\chi}} \), hence by Dirichlet’s theorem on arithmetic progressions we have

\[
\lim_{x \to \infty} \frac{\# \{ p \leq x : \chi(p) = \zeta \}}{\pi(x)} = \frac{\# \ker(\chi)}{\phi(4N)} = \frac{1}{r_{\chi}}.
\]

\( \square \)

Lemma 2.2. Assume the set-up above. Let \( \nu \) an odd positive integer, and \( \zeta \) be a root of unity such that \( \zeta \in \text{Im}(\chi) \). Define the set of primes

\[
\mathbb{P}_{>0}(\zeta, \nu) := \left\{ p \in \mathbb{P} : \chi(p) = \zeta, \frac{a(tp^{2\nu})}{\zeta^\nu} > 0 \right\},
\]

and similarly \( \mathbb{P}_{\geq 0}(\zeta, \nu) \), \( \mathbb{P}_{< 0}(\zeta, \nu) \) and \( \mathbb{P}_{\leq 0}(\zeta, \nu) \). Then the sets

\[
\mathbb{P}_{>0}(\zeta, \nu), \mathbb{P}_{\geq 0}(\zeta, \nu), \mathbb{P}_{< 0}(\zeta, \nu), \mathbb{P}_{\leq 0}(\zeta, \nu)
\]

have a natural density equal to \( \frac{1}{2r_{\chi}} \), where \( r_{\chi} \) denotes the order of the character \( \chi \).

Proof. Fix \( \phi \in [0, \pi) \). Denote by \( \pi_{>0}(x, \zeta) := \# \{ p \leq x : p \in \mathbb{P}_{>0}(\zeta, \nu) \} \) and similarly \( \pi_{<0}(x, \zeta), \pi_{\leq 0}(x, \zeta), \pi_{\geq 0}(x, \zeta) \). Applying the Möbius inversion formula to (2.1), we derive that

\[
a(tn^2) = \sum_{d|n} \mu(d)\chi_{t,N}(d)d^{k-1}A_t\left(\frac{n}{d}\right).
\]

The above equality specialises to

\[
\frac{a(tp^{2\nu})}{\zeta^\nu} = \frac{\lambda(p^\nu)}{\zeta^\nu} - \frac{\chi_0(p)}{\sqrt{p}} \frac{\lambda(p^{\nu-1})}{\zeta^{\nu-1}},
\]

(2.3)
by taking \( n = p^\nu \) and normalizing by \( p^{\nu(k-1/2)} \zeta^\nu \). An elementary calculation yields to the following trigonometric identity

\[
\lambda(p^\nu) = \frac{\sin((\nu+1)\theta_p)}{\sin \theta_p} \zeta^\nu \quad \text{with} \quad \theta_p \in (0, \pi), \tag{2.4}
\]

and in the limiting cases when \( \theta_p = 0 \) and \( \theta_p = \pi \) respectively we have \( \lambda(p^\nu) = (\nu+1)\zeta^\nu \) and \( \lambda(p^\nu) = (-1)^\nu(\nu+1)\zeta^\nu \), which can happen for at most finitely many primes \( p \) only. Thus we may and do assume that \( \theta_p \in (0, \pi) \).

Altogether from (2.3) and (2.4) we have

\[
a(tp^\nu) = \frac{\sin((\nu+1)\theta_p)}{\zeta^\nu} > 0 \iff \sin((\nu+1)\theta_p) > \frac{\chi_0(p)}{\sqrt{p}} \sin(\nu \theta_p). \tag{2.5}
\]

Let \( \epsilon > 0 \) (small enough). Since for all \( p > \frac{1}{\epsilon^2} \) we have \( \left| \frac{\chi_0(p)}{\sqrt{p}} \sin(\nu \theta_p) \right| < \epsilon \), we infer

\[
\{ p \leq x : \chi(p) = \zeta, \sin((\nu+1)\theta_p) > \epsilon \} \subset \left\{ p \leq \frac{1}{\epsilon^2} : \chi(p) = \zeta \right\} \cup \{ p \leq x : p \in \mathbb{P}_{>0}(\zeta, \nu) \}. \tag{2.6}
\]

On the other hand

\[
\sin((\nu+1)\theta_p) > \epsilon \iff \theta_p \in I_\epsilon := \bigcup_{j=1}^{\mu+1} \frac{(2j-2)\pi + \arcsin(\epsilon)}{\nu+1}, \quad \frac{(2j-1)\pi - \arcsin(\epsilon)}{\nu+1},
\]

Thus (2.6) is equivalent to

\[
\{ p \leq x : \chi(p) = \zeta, \theta_p \in I_\epsilon \} \subset \left\{ p \leq \frac{1}{\epsilon^2} : \chi(p) = \zeta \right\} \cup \{ p \leq x : p \in \mathbb{P}_{>0}(\zeta, \nu) \}.
\]

Consequently we have

\[
\pi_{>0}(x, \zeta) + \pi_\zeta \left( \frac{1}{\epsilon^2} \right) \geq \# \{ p \leq x : \chi(p) = \zeta, \theta_p \in I_\epsilon \},
\]

where, \( \pi_\zeta(x) := \# \{ p \leq x : \chi(p) = \zeta \} \). Now divide the above inequality by \( \pi_\zeta(x) \), we obtain

\[
\frac{\pi_{>0}(x, \zeta)}{\pi_\zeta(x)} + \frac{\pi_\zeta \left( \frac{1}{\epsilon^2} \right)}{\pi_\zeta(x)} \geq \# \{ p \leq x : \chi(p) = \zeta, \theta_p \in I_\epsilon \}. \tag{2.7}
\]

Since \( \pi_\zeta \left( \frac{1}{\epsilon^2} \right) \) is finite the term \( \frac{\pi_\zeta \left( \frac{1}{\epsilon^2} \right)}{\pi_\zeta(x)} \) tends to zero as \( x \to \infty \). By Theorem \( \mathbb{I} \) we have

\[
\lim_{x \to \infty} \frac{\# \{ p \leq x : \chi(p) = \zeta, \theta_p \in I_\epsilon \}}{\pi_\zeta(x)} = \mu_{ST}(I_\epsilon).
\]

Hence a passage to the limit in (2.7) yields

\[
\liminf_{x \to \infty} \frac{\pi_{>0}(\zeta, x)}{\pi_\zeta(x)} \geq \mu_{ST}(I_\epsilon).
\]

Letting \( \epsilon \) to zero in the above inequality we find

\[
\liminf_{x \to \infty} \frac{\pi_{>0}(\zeta, x)}{\pi_\zeta(x)} \geq \mu_{ST}(I),
\]

4
where

\[ I := \bigcup_{j=1}^{\nu+1} \left( \frac{(2j-2)\pi}{\nu+1}, \frac{(2j-1)\pi}{\nu+1} \right). \]

Now from [9] we know that \( \mu_{\text{ST}}(I) = \frac{1}{2} \), therefore

\[ \liminf_{x \to \infty} \frac{\pi_{>0}(\zeta, x)}{\pi_\zeta(x)} \geq \frac{1}{2}. \]

A similar reasoning yields \( \liminf_{x \to \infty} \frac{\pi_{<0}(\zeta, x)}{\pi_\zeta(x)} \geq \frac{1}{2} \), in combination with \( \pi_{\leq 0}(x, \zeta) = \pi_\zeta(x) - \pi_{>0}(x, \zeta) \), one sees that \( \lim_{x \to \infty} \frac{\pi_{=0}(\zeta, x)}{\pi_\zeta(x)} \) exist and equal to \( \frac{1}{2} \). Now from Lemma 2.1 we conclude

\[ \lim_{x \to \infty} \frac{\pi_{>0}(\zeta, x)}{\pi_\zeta(x)} = \frac{1}{2r_\chi}. \]

Similar arguments apply to the other cases. \( \square \)

Now we proceed to prove our main theorem. Let \( \zeta \) be a root of unity blowing to \( \text{Im}(\chi) \). If \( p \) a prime number satisfying \( \chi(p) = \zeta \), from Lemma 2.2 we know that \( \frac{a(tp^{2\nu})}{\zeta} \) is real, thus we may write

\[ \text{Re} \left( a(tp^{2\nu})e^{-i\phi} \right) = \frac{a(tp^{2\nu})}{\zeta} \text{Re} \left( \zeta' e^{-i\phi} \right) \quad \text{with} \quad \chi(p) = \zeta. \quad (2.8) \]

If \( \arg(\zeta') = \phi + \frac{\pi}{2} \mod 2\pi \), the sequence \( \{ \text{Re} \left( a(tp^{2\nu})e^{-i\phi} \right) \}_{p, \chi(p) = \zeta} \) is trivial. Assume \( \arg(\zeta') \neq \phi + \frac{\pi}{2} \mod 2\pi \), in this case the sequence \( \{ \text{Re} \left( a(tp^{2\nu})e^{-i\phi} \right) \}_{p, \chi(p) = \zeta} \) is not trivial. Without restriction of generality we can assume \( \text{Re} \left( \zeta' e^{-i\phi} \right) > 0 \). Thus for any fixed \( \epsilon > 0 \) (but small) from (2.3) we have

\[ \left\{ p \in \mathbb{P} : p > \frac{1}{\epsilon^2}, \chi(p) = \zeta, \theta_p \in I_\epsilon \right\} \subset \left\{ p \in \mathbb{P} : \chi(p) = \zeta, \text{Re} \left( a(tp^{2\nu})e^{-i\phi} \right) > 0 \right\}, \quad (2.9) \]

and

\[ \left\{ p \in \mathbb{P} : p > \frac{1}{\epsilon^2}, \chi(p) = \zeta, \theta_p \in I'_\epsilon \right\} \subset \left\{ p \in \mathbb{P} : \chi(p) = \zeta, \text{Re} \left( a(tp^{2\nu})e^{-i\phi} \right) < 0 \right\}, \quad (2.10) \]

where

\[ I_\epsilon := \bigcup_{j=1}^{\nu+1} \left( \frac{(2j-2)\pi + \arcsin(\epsilon)}{\nu+1}, \frac{(2j-1)\pi - \arcsin(\epsilon)}{\nu+1} \right) \subset [0, \pi], \]

and

\[ I'_\epsilon := \bigcup_{j=1}^{\nu+1} \left( \frac{(2j-1)\pi + \arcsin(\epsilon)}{\nu+1}, \frac{2j\pi - \arcsin(\epsilon)}{\nu+1} \right) \subset [0, \pi]. \]

From Theorem 1 we know that the sequence \( \{ \theta_p \}_{p, \chi(p) = \zeta} \) is equidistributed in \( [0, \pi] \) with respect to the Sato-Tate measure \( \mu_{\text{ST}} \), it follows that there are infinitely many primes \( p \) satisfying \( \chi(p) = \zeta \) such that \( \theta_p \in I'_\epsilon \) and infinitely many primes \( p \) satisfying \( \chi(p) = \zeta \) such that \( \theta_p \in I_\epsilon \), thus the sets in (2.9) and (2.10) are infinite. Hence for all root of unity \( \zeta \) blowing to \( \text{Im}(\chi) \) the sequence \( \{ a(tp^{2\nu}) \}_{p, \chi(p) = \zeta} \) is oscillatory, then the sequence \( \{ a(tp^{2\nu}) \}_{p \in \mathbb{P}} \) so is.
Now we proceed to calculate the natural density of the sets $P_{>0}(\phi, \nu)$ and $P_{<0}(\phi, \nu)$. From (2.8) one observes that

$$P_{>0}(\phi, \nu) = \prod_{\zeta, \text{root of unity}} P_{>0}(\zeta, \nu) \bigcup \prod_{\zeta, \text{root of unity}} P_{<0}(\zeta, \nu),$$

and

$$P_{<0}(\phi, \nu) = \prod_{\zeta, \text{root of unity}} P_{>0}(\zeta, \nu) \bigcup \prod_{\zeta, \text{root of unity}} P_{<0}(\zeta, \nu),$$

up to finitely many primes. Then the above combined with Lemma 2.2 gives

$$\delta(P_{>0}(\phi, \nu)) = \lim_{x \to \infty} \sum_{\zeta, \text{root of unity}} \frac{\pi_{>0}(x, \zeta)}{\pi(x)} + \lim_{x \to \infty} \sum_{\zeta, \text{root of unity}} \frac{\pi_{<0}(x, \zeta)}{\pi(x)}$$

$$= \frac{1}{2} \sum_{\zeta, \text{root of unity}} \frac{1}{r_{\chi}},$$

$$= \frac{\delta(P_{\neq 0}(\phi, \nu))}{2}.$$ 

Likewise we get $\delta(P_{<0}(\phi, \nu)) = \frac{\delta(P_{\neq 0}(\phi, \nu))}{2}$, which concludes the proof.

References

[1] Amri M. A.: Oscillatory behavior and equidistribution of signs of Fourier coefficients of cusp forms. arXiv:1710.06211 (2017).

[2] Amri, M. A., Ziane, M.: Angular changes of complex Fourier coefficients of cusp forms. arXiv preprint arXiv:1704.00982v4 (2017).

[3] Barnet-Lamb, Geraghty, D., Harris, M., Taylor, R.: A Family of Calabi-Yau Varities and Potential Automorphy II, Pub. Res. Inst. Math. Sci., 47, (2011), 29-98.

[4] Bruinier, J.-H., Kohnen, W.: Sign changes of coefficients of half integral weight modular forms. In: Edixhoven, B. (ed.) Modular forms on Schiermonnikoong, (2008), pp. 57–66. Cambridge University Press, Cambridge

[5] Inam, I., Wiese, G.: Equidistribution of Signs for Modular Eigenforms of Half Integral Weight. Archiv der Mathematik, 101, (2013), 331-339.

[6] Knopp, M., Kohnen, W., Pribitkin, W.: On the signs of Fourier coefficients of cusp forms, The Ramanujan Journal, 7(1), (2003),269-277.

[7] Kohnen, W.: Newforms of half-integral weight. Journal für die reine und angewandte Mathematik. 333, (1982), 32-72.

[8] Kohnen, W., Lau, YK. and Wu, J.: Fourier coefficients of cusp forms of half-integral weight. J. Math. Z. (2013), 273, 29-41.
[9] Meher, J., Shankhadhar, K. D., Viswanadham, G. K.: On the coefficients of symmetric power $L$-functions. to appear in International Journal of number theory.

[10] Pribitkin, W.: On the oscillatory behavior of certain arithmetic functions associated with automorphic forms. Journal of Number Theory, 131(11), (2011), 2047-2060.

[11] Shimura, G.: On Modular Forms of Half-Integral Weight, Annals of Math., 97, (1973), 440-481.

[12] Niwa, S.: Modular forms of half integral weight and the integral of certain theta-functions. Nagoya Mathematical Journal, 56, (1975), 147-161.