Thermodynamic Analysis of a Self-Gravitating Gas in Astrophysical Contexts

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The thermodynamics of a self-gravitating gas cloud of particles interacting only via their gravitational potential is an interesting problem with peculiarities arising due to the long-ranged nature of the gravitational interaction. Based on our recent work on the properties of such a configuration, we extend the system to contain a central gravitational field in which the particles are moving, to mimic the potential of a central compact object exerting an external force on the gas cloud. After an introduction to the general problem, including the aforementioned peculiarities and possible solutions, we will discuss the particular properties of the self-gravitating gas in a central field and its thermodynamic analysis.

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1. Introduction

The topic of thermodynamic systems in the presence of gravity has been discussed in many occasions and forms\cite{1, 2, 3}, and has raised many questions on how to deal with the long-range effects of gravity in the thermodynamic analysis of systems, where concepts such as the isolation of a system in one or the other regard are important. Non-stationary equilibrium situations, negative heat capacities or simply divergences in the thermodynamic limit have been plaguing the analyses, and the conventional, very successful thermodynamic framework of Boltzmann-Gibbs statistics had to be adapted and modified in order to account for the peculiarities of the thermodynamics of a gravitational system.

Based on a Boltzmann-Gibbs analysis of the self-gravitating gas\cite{4}, the statistical analysis and subsequent calculation of thermodynamic properties have been carried out\cite{4} assuming a generalized framework intended to describe a system with non-extensive properties, due to the presence of long-range forces such as gravity. The adopted generalized framework, i.e., Tsallis generalized $q$-statistics, has been developed in order to consider non-extensive effects, entailing an additional parameter $q$ in the statistical analysis.

This work is an extension of these previous investigations which generalizes the self-gravitating gas to a more realistic system featuring a centrally placed compact object, like e.g. a black hole, around which the gas is extending. Due to some peculiarities and open questions, we will not continue using the non-extensive $q$-statistics, but rather return to the conventional Boltzmann-Gibbs statistics, in order to get a first impression of the results. Other generalizations can be thought of, which will be commented on in the last section.
2. Statistical mechanics and thermodynamics of a self-gravitating gas

I will briefly review the most important steps in the analysis of a self-gravitating gas\textsuperscript{4,5}, from the system’s properties to the peculiarities of the thermodynamic analysis and some of its outcomes.

The governing force of the self-gravitating gas is the gravitational attraction between its $N$ identical constituent particles which are otherwise moving freely, and thus the Hamiltonian of the system is

$$H = T + U = \sum_{i=1}^{N} \frac{p_{i}^{2}}{2m} - Gm^{2} \sum_{1 \leq i < j \leq N} \frac{1}{|\mathbf{q}_{i} - \mathbf{q}_{j}|_{A}},$$

where $G$ is the gravitational constant, $m$ the mass of an individual particle, and $A$ represents a short-range cutoff imposed in order to avoid the unphysical collapse of the system to a point. This Hamiltonian is the basis for a thermodynamic analysis which can be done in principle in different ensembles, like the microcanonical one, where the energy of the system is kept constant, or the canonical one, where instead the temperature is fixed, and energy can be exchanged with a reservoir. In the microcanonical ensemble, the most important thermodynamic quantity from which everything is derived is the entropy, i.e., the logarithm of this sum over microstates $\Omega(E, V, N)$,

$$S = k_{B} \ln \Omega(E, V, N),$$

where

$$\Omega = \frac{(2\pi m)^{3N/2}}{N! \hbar^{3N} \Gamma \left( \frac{3N}{2} + 1 \right)} \int d^{3N} \mathbf{q} \int d^{3N} \mathbf{p} \exp \left( -\beta H(\mathbf{p}, \mathbf{q}) \right).$$

From the entropy, you can obtain important thermodynamic quantities such as the temperature of the gas, or the equation of state, i.e., the relation between pressure, temperature and volume.

In the case of the canonical ensemble, the starting point is the partition function, defined as

$$Z = \frac{1}{N! \hbar^{3N}} \int d^{3N} \mathbf{q} d^{3N} \mathbf{p} \exp \left( -\beta H(\mathbf{p}, \mathbf{q}) \right),$$

and everything else is derived from that quantity, like the equation of state. The temperature in this ensemble is fixed, so it cannot be calculated.

Following the definition of these basic thermodynamic functions, calculations can be simplified by the assumption of a weak gravitational interaction, i.e., the gravitational contribution can be treated as a small correction to the ideal gas, and results can be obtained analytically in this case.

Under this additional assumption, further quantities that are of thermodynamic interest can be calculated, like the heat capacity and other response functions of the
system. Both equation of state and heat capacity have been calculated and compared in the framework of Tsallis statistics, and the details can be found there. An important point for further investigations is the question of the statistical framework, which is closely connected to the choice of thermodynamic limit. Tsallis’ non-extensive statistics naturally features a modification of the thermodynamic limit, in which the thermodynamic state variables result in convergent functions. In the case the conventional Boltzmann-Gibbs statistics, another modification of the thermodynamic limit has to be adopted in order to obtain convergent results. In the following, we will employ Boltzmann-Gibbs statistics with the modified thermodynamic limit.

3. Addition of the central gravitational potential

As a modification to the basic setup of a simple self-gravitating gas many complications can be thought of. The simplest gas perhaps is the addition of a central potential, to model the situation of a self-gravitating gas around a black hole. We will start with the assumption of an external gravitational field caused by a mass \( M \) of size \( r_S = 2GM/c^2 \) in the center of the configuration, restricting the movement of the gas between the radius of the innermost stable circular orbit (ISCO) at \( r_{\text{ISCO}} = 3r_S \) and infinity. This will make a difference in the integrals contained in the sum over microstates and the partition function, respectively. Moreover, the central potential will have its influence on every particle in the gas. The generalized Hamiltonian thus reads

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} - Gm^2 \sum_{1\leq i<j\leq N} \frac{1}{|q_i - q_j|_A} - GmM \sum_{1\leq i<j\leq N} \frac{1}{|q_i - r|_A},
\]

with \( r \) denoting the center of mass of the compact object. To simplify calculations, we choose \( r = 0 \).

The computational procedures in order to extract the thermodynamic equation of state is fairly analogous to the case of a simple self-gravitating gas, and differs only in the restriction of the range of integration, due to the fact that we consider a ring-like structure, or even a flat two-dimensional disk shape. This restriction will manifest itself in the definition of the virial coefficients \( b_i \), which will be slightly different.

The interesting question is whether the modification of the system will lead to differences in the thermodynamic limit, i.e., facilitate the calculation of otherwise divergent functions, or modify the qualitative dependence on the number of particles in the thermodynamic limit. Preliminary results indicate that this is not the case, and that modifications are limited to the virial coefficients of the problem.

4. Outlook

We have here discussed the thermodynamic properties of a self-gravitating gas under the influence of a central gravitational field caused by a heavy mass at the
center of the configuration. Basing on the analysis of a self-gravitating gas cloud consisting of ideal particles, an additional term accounting for the central gravitational potential was added to the analysis, and the resulting thermodynamic state variables were calculated. Preliminary results indicate slight modifications of the state variables, depending on the new parameter, the mass of the central object. The goal is to generalize the analysis of a simple self-gravitating gas to eventually be able to make predictions on the thermodynamic behavior of matter around a compact object, i.e., an accretion disk of sorts.

Besides the inclusion of a central compact object, the gas itself can be modified in its properties, e.g., by considering non-ideal interactions between the particles. This could be accounted for in an exact way by adding additional particle-particle interactions to the Hamiltonian, with the corresponding coupling constant, like for example an electromagnetic charge. The different strengths of gravity and any other interactions that may be added have to be weighed against each other, and approximations could be applied. Another possibility would be to include effective potentials which are used in condensed matter systems, Mie-type potential like the Lennard-Jones case or others, in order to describe different variations of the gas. Investigations in this direction would represent the first steps towards the description of non-ideal fluids in gravitational contexts - either gas clouds of interacting particles, or non-ideal fluids constituting accretion disks or clouds around a central compact object.

Further generalizations include rotation of the system, or charge of the central object. Importantly, these results should then be connected to results of other calculations on accretion disks, in particular the accretion of (charged) dust particles in a spherical shell or torus structure.

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References

1. D. Lynden-Bell and R. Wood, *Mon. Not. Roy. Astron. Soc.* **138**, p. 495 (1968).
2. J. Oppenheim, *Phys. Rev. E* **68**, p. 016108 (2003).
3. A. Campa, T. Dauxois and S. Ruffo, *Phys. Rep.* **480**, 57 (2009).
4. H. J. de Vega and N. Sanchez, *Nuclear Physics B* **625**, 409 (2002).
5. L. F. Escamilla-Herrera, C. Gruber, V. Pineda-Reyes and H. Quevedo, *Submitted for publication* (2018).
6. K. Schroven, E. Hackmann and C. Lämmerzahl, *Phys. Rev. D* **96**, p. 063015 (2017).