COHN PATH ALGEBRAS OF HIGHER-RANK GRAPHS

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Abstract. In this article, we introduce Cohn path algebras of higher-rank graphs. We prove that for a higher-rank graph $\Lambda$, there exists a higher-rank graph $T\Lambda$ such that the Cohn path algebra of $\Lambda$ is isomorphic to the Kumjian-Pask algebra of $T\Lambda$. We then use this isomorphism and properties of Kumjian-Pask algebras to study Cohn path algebras. This includes proving a uniqueness theorem for Cohn path algebras.

1. Introduction

Leavitt path algebras were introduced and studied in [3] and [9] as a generalisation of the class of algebras studied by Leavitt in [25]. Leavitt path algebras are also the natural algebraic analogues of graph $C^*$-algebras studied in [32]; a number of interesting results have been proven by translating between graph $C^*$-algebras and the ring theoretic Leavitt path algebras.

Cohn path algebras were introduced in [6, 7] and generalise the algebras $U_{1,n}$ studied by Cohn in [16]. The idea is to build an algebra out of the path space of a graph; addition and scalar multiplication are defined formally and multiplication of two paths is only nonzero when one can concatenate the paths. Cohn path algebras can also be obtained from Leavitt path algebras by omitting one of the Cuntz-Krieger relations. Hence every Leavitt path algebra can be viewed as a quotient of a Cohn path algebra. On the other hand, Abrams, Ara and Siles Molina show in [2] that for every graph $E$, there exists a graph $TE$ (denoted $E(X)$ in [2]) such that the Cohn path algebra of $E$ is isomorphic to the Leavitt path algebra of $TE$.

Although it has received less attention, the Cohn path algebra of a directed graph is the algebraic analogue of the $C^*$-algebraic Toeplitz algebra of $E$ as defined in [19]. In the $C^*$-algebraic setting, Muhly and Tomforde in [27] and also Sims in [38] each show that for a graph $E$, the Toeplitz algebra of $E$ is isomorphic to the graph $C^*$-algebra of $TE$.

Now we move into the setting of higher-rank graph algebras: In [21], Kumjian and Pask introduced a combinatorial model, called a higher-rank graph, in order to capture the essential features of the $C^*$-algebras studied by Robertson and Steger in [35]. A higher-rank graph, also called a $k$-graph, is a generalisation of the path category of a directed graph where the length of a ‘path’ $\lambda$ in a $k$-graph is an element of $\mathbb{N}^k$. Kumjian and Pask studied the $C^*$-algebras associated to row-finite higher-rank graphs with no sources. Raeburn, Sims and Yeend generalised Kumjian and Pask’s construction by describing the class of $C^*$-algebras associated to more general higher-rank graphs in [30, 31].

1991 Mathematics Subject Classification. 16S99 (Primary); 16S10 (Secondary).

Key words and phrases. Cohn path algebra, Kumjian-Pask algebra, finitely aligned $k$-graph, Steinberg algebra.

This research was done as part of the second author’s PhD thesis at the University of Otago under the supervision of the first author and Iain Raeburn. Thank you to Iain for his guidance. This research was also supported by Marsden grant 15-UOO-071 from the Royal Society of New Zealand.
A few years ago, a higher-rank analogue of Leavitt path algebras, called Kumjian-Pask algebras, was introduced in [10]. The class of Kumjian-Pask algebras includes the class of Leavitt path algebras. In [10] the authors limit their focus to row-finite higher-rank graphs with no sources. Following the generalisation pattern of higher-rank graph $C^*$-algebras, Kumjian-Pask algebras associated to more general higher-rank graphs are described in [14] and [15].

$C^*$-algebraic Toeplitz algebras of higher-rank graphs were introduced in [29]. Thus it seems natural to ask whether there is also an algebraic analogue of these $C^*$-algebras. In this paper, we introduce Cohn path algebras of higher-rank graphs. Our motivation comes from a desire to one day establish an algebraic version of ‘KMS states’ for higher-rank graph algebras (see [21, 22, 23]).

Our strategy is to follow the analysis of [28]. In that paper, Pangalela shows that for every row-finite higher-rank graph $\Lambda$, there exists a higher-rank graph $TA$ such that the Toeplitz algebra of $\Lambda$ is isomorphic to the $C^*$-algebra of $TA$. Although we will start with a row-finite $k$-graph $\Lambda$ with no sources, the $k$-graph $TA$ always has sources and is not ‘locally convex’ so we will need to use the Kumjian-Pask algebra construction given in [15].

Let $\Lambda$ be a row-finite higher-rank graph with no sources and $R$ be a commutative ring with 1. After providing some preliminaries, in Section 3, we define a Cohn $\Lambda$-family (3.1) and show there exists a universal Cohn path algebra $C_R(\Lambda)$ (Proposition 3.5).

In Section 4, we recall the Kumjian-Pask algebras of [15] and the higher-rank graph $TA$ of [28]. We also study properties of the Kumjian-Pask algebra of $TA$ (Proposition 4.1) and show that the Cohn path algebra of $\Lambda$ is isomorphic to the Kumjian-Pask algebra of $TA$ (Theorem 4.13). This isomorphism is an algebraic version of [28, Theorem 4.1]. We then show that every Cohn path algebra is $\mathbb{Z}^k$-graded (see [15, Theorem 3.6]). At the end of the section, we use the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras in [15, Theorem 8.1] to prove the uniqueness theorem for Cohn path algebras (Theorem 4.1).

Our uniqueness theorem for Cohn path algebras is notable for two reasons. The first is that it gives a uniqueness theorem for Cohn path algebras associated to directed graphs. Although in [2], Abrams, Ara and Siles Molina prove that every Cohn path algebra is a Leavitt path algebra and state the Cuntz-Krieger uniqueness theorem for Leavitt path algebras, they do not explicitly investigate uniqueness theorems for Cohn path algebras.

Secondly, we can view the uniqueness theorem for Cohn path algebras as an algebraic analogue of the uniqueness theorem for Toeplitz algebras given in [29]; the proof of our algebraic uniqueness theorem is considerably shorter than the one Toeplitz algebras in [29]. By translating our proof into the $C^*$-algebra setting, we provide an alternative proof of the uniqueness theorem for Toeplitz algebras (see [28, Remark 4.3]).

Finally, we discuss examples and applications in Section 5. First we explicitly demonstrate the relationship between Cohn path algebras and Toeplitz algebras (Proposition 5.2). We also show that our Cohn algebras can be realised as Steinberg algebras (Proposition 5.8).

2. Preliminaries

Let $k$ be a positive integer. We regard $\mathbb{N}^k$ as an additive semigroup with identity 0. For $n \in \mathbb{N}^k$, we write $n = (n_1, \ldots, n_k)$. Meanwhile, for $m, n \in \mathbb{N}^k$, we write $m \leq n$ to...
denote \( m_i \leq n_i \) for \( 1 \leq i \leq k \), and we use expression \( m \lor n \) for their coordinate-wise maximum and \( m \land n \) for their coordinate-wise minimum. We also write \( e_i \) for the usual basis elements in \( \mathbb{N}^k \).

2.1. Higher-rank graphs. A higher-rank graph or \( k \)-graph \( \Lambda = (\Lambda^0, \Lambda, r, s) \) is a countable small category \( \Lambda \) with a functor \( d \) from \( \Lambda \) to \( \mathbb{N}^k \), called the degree map, which satisfies the factorisation property: for every \( \lambda \in \Lambda \) and \( m, n \in \mathbb{N}^k \) with \( d(\lambda) = m + n \), there exist unique elements \( \mu, \nu \in \Lambda \) such that \( \lambda = \mu \nu \) and \( d(\mu) = m \), \( d(\nu) = n \). We then write \( \lambda(m, m+n) \) for their coordinate-wise maximum and \( \lambda(m, m+n) \) for their coordinate-wise minimum. We also write \( e_i \) for the usual basis elements in \( \mathbb{N}^k \).

For \( k = 1 \), we use notation \( E = (E^0, E^*, r, s) \) to denote a 1-graph. In this case, \( E^* \) contains all paths in \( E \) with degree at least 1. We also write \( E^1 \) for the set of all paths with degree 1. Since we view \( E \) as a category, we use different convention from that of Leavitt path algebra and Cohn path algebra literature where people write \( \lambda \mu \) to denote the composition of paths \( \lambda \) and \( \mu \) with \( s(\mu) = r(\lambda) \).

One way to visualise \( k \)-graphs is to use coloured directed graphs, as described in [20]. Suppose \( \Lambda \) is a \( k \)-graph. The coloured graph associated to \( \Lambda \) is a directed graph whose edges are colour coded: Choose \( k \)-different colours \( c_1, \ldots, c_k \). The vertices in the coloured graph are the same as the vertices of \( \Lambda \). Each path \( \lambda \) in \( \Lambda \) with degree \( e_i \) corresponds to an edge of colour \( c_i \) between \( s(\lambda) \) and \( r(\lambda) \). We call this coloured graph the skeleton of \( \Lambda \).

Example 2.1 ([30, Example 2.2.(ii)]). Let \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{\infty\}^k \). We define
\[
\Omega_{k,n} := \left\{ (p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq n \right\}.
\]
This is a category with objects \( \{ p \in \mathbb{N}^k : p \leq n \} \), range map \( r(p, q) = p \), source map \( s(p, q) = q \), and degree map \( d(p, q) = q - p \). Then \( \Omega_{k,n} \) is a \( k \)-graph. The skeleton of \( \Omega_{2,(1,2)} \) is

\[
(1,0) \quad (1,1) \quad (1,2) \\
(0,0) \quad (0,1) \quad (0,2)
\]
where solid edges have degree \( (1,0) \) and dashed edges have degree \( (0,1) \).

We write
\[
W_\Lambda := \bigcup_{n \in (\mathbb{N} \cup \{\infty\})^k} \{ x : \Omega_{k,n} \to \Lambda : x \text{ is a degree preserving functor} \}.
\]
Suppose \( x \in W_\Lambda \). For \( n \in \mathbb{N}^k \) and \( n \leq d(x) \), the path \( \sigma^n x \) is defined by \( \sigma^n x(0, m) = x(n, n+m) \) for all \( m \leq d(x) - n \).

For \( n \in \mathbb{N}^k \), we define
\[
\Lambda^n := \{ \lambda \in \Lambda : d(\lambda) = n \}
\]
and call the elements $\lambda$ of $\Lambda^n$ paths of degree $n$. In particular, we regard elements of $\Lambda^0$ as vertices. We use term edge to denote a path $e \in \Lambda^e$ where $1 \leq i \leq k$, and write

$$\Lambda^1 := \bigcup_{1 \leq i \leq k} \Lambda^e$$

for the set of all edges.

For $v \in \Lambda^0$, $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we define

$$vE := \{\mu \in E : r(\mu) = v\} \quad \text{and} \quad \lambda E := \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}.$$

We say that $\Lambda$ is row-finite if for every $v \in \Lambda^0$, the set $v\Lambda^e$ is finite for $1 \leq i \leq k$. Finally, we say $v \in \Lambda^0$ is a source if there exists $m \in \mathbb{N}^k$ such that $v\Lambda^m = \emptyset$.

**Example 2.2.** Consider the 2-graph $\Lambda_1$ which has skeleton

![Diagram](image)

where the solid edge has degree $(1, 0)$ and the dashed edge has degree $(0, 1)$. It is clear that $w$ is a source since there is no paths going in the vertex. Hence, $\Lambda_1$ is row-finite with sources.

**Example 2.3.** Let $\Lambda_2$ be the 2-graph with skeleton

![Diagram](image)

where $ef = fe$, the solid edge has degree $(1, 0)$ and the dashed edge has degree $(0, 1)$. Since $v\Lambda^m \neq \emptyset$ for all $m \in \mathbb{N}^k$, then $\Lambda_2$ is row-finite with no sources.

For $\lambda, \mu \in \Lambda$, we define

$$\text{MCE} (\lambda, \mu) := \{\tau \in \Lambda : d(\tau) = d(\lambda) \lor d(\mu), \tau(0, d(\lambda)) = \lambda, \tau(0, d(\mu)) = \mu\}$$

and

$$\Lambda^{\text{min}} (\lambda, \mu) := \{(\lambda', \mu') \in \Lambda \times \Lambda : \lambda\lambda' = \mu\mu' \in \text{MCE} (\lambda, \mu)\}.$$

Meanwhile, for $E \subseteq \Lambda$ and $\lambda \in \Lambda$, we write

$$\text{Ext} (\lambda; E) := \bigcup_{\mu \in E} \{\rho : (\rho, \tau) \in \Lambda^{\text{min}} (\lambda, \mu)\}.$$

A set $E \subseteq v\Lambda$ is exhaustive if for all $\lambda \in v\Lambda$, there exists $\mu \in E$ such that $\Lambda^{\text{min}} (\lambda, \mu) \neq \emptyset$.

In this article, we focus on row-finite $k$-graphs. For further discussion about row-finite $k$-graphs and their generalisations, see [24, 30, 31, 32, 40].
2.2. Graded rings. Let $G$ be an additive abelian group. If $A$ is a ring, we say that $A$ is $G$-graded if there are additive subgroups $\{A_g : g \in G\}$ satisfying:

$$A = \bigoplus_{g \in G} A_g$$

and for $g, h \in G$, $A_gA_h \subseteq A_{g+h}$.

For $g \in G$, the subgroup $A_g$ is called the homogeneous component of $A$ of degree $g$.

3. Cohn $\Lambda$-families

Throughout this section, suppose that $\Lambda$ is a row-finite $k$-graph with no sources and $R$ is a commutative ring with $1$. For each $\lambda \in \Lambda$, we introduce a formal symbol $\lambda^*$ called a ghost path; if $v \in \Lambda^0$, we identify $v^* := v$. We then write $G(\Lambda)$ the set of ghost paths and define $r$ and $s$ on $G(\Lambda)$ by

$$r(\lambda^*) := s(\lambda) \quad \text{and} \quad s(\lambda^*) := r(\lambda).$$

We also define composition in $G(\Lambda)$: set $\lambda^* \mu^* = (\mu \lambda)^*$ for $\lambda, \mu \in \Lambda$ with $s(\mu) = r(\lambda)$; and finally, we write $G(\Lambda^{\neq 0}) := \{\lambda^* : \lambda \in \Lambda \setminus \Lambda^0\}$.

**Definition 3.1.** A Cohn $\Lambda$-family $\{T_\lambda, T_\mu^* : \lambda, u \in \Lambda\}$ in an $R$-algebra $A$ consists of a map $T : \Lambda \cup G(\Lambda^{\neq 0}) \to A$ such that:

(CP1) $\{T_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal idempotents;

(CP2) for $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, we have $T_\lambda T_\mu = T_{\lambda \mu}$ and $T_\mu^* T_\lambda^* = T_{(\lambda \mu)^*}$;

(CP3) $T_\lambda T_\mu = \sum_{(\nu, \gamma) \in \Lambda_{\text{min}}(\lambda, \mu)} T_\nu T_\gamma^*$ for all $\lambda, \mu \in \Lambda$.

**Remark 3.2.**

(i) For 1-graph $E$, people usually write $\{v, e, e^* : v \in E^0, e \in E^1\}$ instead of $\{T_\lambda, T_\mu^* : \lambda, u \in E^*\}$ (see [11, 13, 15, 16, 18]). We do not use this notation because we want to distinguish the paths in $E$ and the elements of the algebra $A$.

(ii) Since $\Lambda$ is row-finite, $|\Lambda_{\text{min}}(\lambda, \mu)|$ is finite and the sum in (CP3) makes sense. We also interpret the empty sum as 0, so $\Lambda_{\text{min}}(\lambda, \mu) = \emptyset$ implies $T_\lambda T_\mu = 0$.

Since (CP1-3) are the same as (KP1-3) of [15, Definition 3.1], Proposition 3.3 of [15] also applies to Cohn $\Lambda$-families as stated in the following proposition.

**Proposition 3.3.** Suppose that $\Lambda$ is a row-finite $k$-graph with no sources, $R$ is a commutative ring with 1, and $\{T_\lambda, T_\mu^* : \lambda, u \in \Lambda\}$ is a Cohn $\Lambda$-family in an $R$-algebra $A$. Then

(a) $T_\lambda T_\mu T_\mu^* = \sum_{\lambda \nu \in \text{MCE}(\lambda, \mu)} T_{\lambda \nu} T_{(\lambda \nu)^*}$ for $\lambda, \mu \in \Lambda$; and $\{T_\lambda T_\mu^* : \lambda \in \Lambda\}$ is a commuting family.

(b) The subalgebra generated by $\{T_\lambda, T_\mu^* : \lambda, u \in \Lambda\}$ is

$$\text{span}_R \{T_\lambda T_\mu^* : \lambda, u \in \Lambda, s(\lambda) = s(\mu)\}.$$

Now we give an example of a Cohn $\Lambda$-family. We use this example later to study properties of “the universal Cohn $\Lambda$-family” (Theorem 3.5).

**Proposition 3.4.** Suppose that $\Lambda$ is a row-finite $k$-graph with no sources and $R$ is a commutative ring with 1. Suppose that $F_R(W_\Lambda)$ is the free $R$-module with basis $W_\Lambda$. Then there exists a Cohn $\Lambda$-family $\{T_\lambda, T_\mu^* : \lambda, u \in \Lambda\}$ in the $R$-algebra $\text{End}(F_R(W_\Lambda))$ such that for $v \in \Lambda^0$, $\lambda, u \in \Lambda$ and $x \in W_\Lambda$, we have

$$T_v(x) = \begin{cases} x & \text{if } r(x) = v; \\ 0 & \text{otherwise,} \end{cases}$$
Proof. We modify the construction of the infinite-path representation of \([10]\). Take \(v \in \Lambda^0\) and \(\lambda, \mu \in \Lambda \setminus \Lambda^0\). Define functions \(f_v, f_\lambda,\) and \(f_\mu^* : W_\Lambda \to \mathbb{F}_R(W_\Lambda)\) by

\[
\begin{align*}
T_\lambda(x) &= \begin{cases} 
\lambda x & \text{if } s(\lambda) = r(x); \\
0 & \text{otherwise},
\end{cases} \\
T_\mu^*(x) &= \begin{cases} 
\sigma^{d(\mu)}x & \text{if } x(0, d(\mu)) = \mu; \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

Furthermore, \(rT_v \neq 0\) and \(r \prod_{e \in v\Lambda^1} (T_v - T_e^*T_e) \neq 0\) for all \(r \in R \setminus \{0\}\) and \(v \in \Lambda^0\).

By the universal property of free modules, there exist nonzero endomorphisms \(S_v, S_\lambda, S_\mu^* : \mathbb{F}_R(W_\Lambda) \to \mathbb{F}_R(W_\Lambda)\) extending \(f_v, f_\lambda,\) and \(f_\mu^*\).

We claim that \(\{T_\lambda, T_\mu^* : \lambda, u \in \Lambda\}\) is a Cohn \(\Lambda\)-family. First we show (CP1). Take \(v \in \Lambda^0\). Then \(T_v^2(x) = x = T_v(x)\) if \(r(x) = v\), and \(T_v^2(x) = 0 = T_v(x)\) otherwise. Hence \(T_v^2 = T_v\). Now take \(v, w \in \Lambda^0\) with \(v \neq w\). Then \(x \in wW_\Lambda\) implies \(x \notin vW_\Lambda\). Thus \(T_vT_w(x) = 0\) for every \(x \in W_\Lambda\), and then \(T_vT_w = 0\).

Next we show (CP2). Take \(\lambda, \mu \in \Lambda\) with \(s(\lambda) = r(\mu)\). Then \(T_\lambda T_\mu(x) = \lambda \mu x = T_{\lambda \mu}(x)\) if \(x \in s(\mu)W_\Lambda\), and \(T_\lambda T_\mu(x) = 0 = T_{\lambda \mu}(x)\) otherwise. Hence \(T_\lambda T_\mu = T_{\lambda \mu}\). On the other hand, we have

\[
T_\mu^*T_\lambda^*(x) = T_\mu^*\sigma^{d(\lambda)}x = \sigma^{d(\lambda)+d(\mu)}x = \sigma^{d(\lambda \mu)}x = T_{(\lambda \mu)^*}(x)
\]

if \(x(0, d(\lambda \mu)) = \lambda \mu\), and \(T_\mu^*T_\lambda^*(x) = 0 = T_{(\lambda \mu)^*}(x)\) otherwise. Therefore, \(T_\mu^*T_\lambda^* = T_{(\lambda \mu)^*}\).

Next we show (CP3). Take \(\lambda, \mu \in \Lambda\). If \(r(\lambda) \neq r(\mu)\), then \(T_\lambda T_\mu = 0 = \Lambda^{\min}(\lambda, \mu)\). Suppose \(r(\lambda) = r(\mu)\). We have

\[
T_\lambda T_\mu(x) = \begin{cases} 
(\mu x)(d(\lambda), d(\mu x)) & \text{if } x \in s(\mu)W_\Lambda \text{ and } (\mu x)(0, d(\lambda)) = \lambda; \\
0 & \text{otherwise}.
\end{cases}
\]

Take \(x \in s(\mu)W_\Lambda\). Note that \(s(\mu) = r(\gamma)\) for \((\nu, \gamma) \in \Lambda^{\min}(\lambda, \mu)\). First suppose \((\mu x)(0, d(\lambda)) \neq \lambda\). Then for \((\nu, \gamma) \in \Lambda^{\min}(\lambda, \mu)\),

\[
(\mu x)(0, d(\lambda \nu)) \neq \lambda \nu \text{ and } (\mu x)(0, d(\mu \gamma)) \neq \mu \gamma.
\]

Hence \((\mu x)(0, d(\gamma)) \neq \gamma\) and \(T_\nu T_{\gamma^*}(x) = T_\nu(0) = 0\). Therefore

\[
\sum_{(\nu, \gamma) \in \Lambda^{\min}(\lambda, \mu)} T_\nu T_{\gamma^*}(x) = 0.
\]

Next suppose \((\mu x)(0, d(\lambda)) = \lambda\). Since \((\mu x)(0, d(\lambda)) = \lambda\) and \(\mu x)(0, d(\mu)) = \mu\), then there is \(\gamma \in s(\mu)\Lambda\) such that \((\nu, \gamma) \in \Lambda^{\min}(\lambda, \mu)\) and \((\mu x)(0, d(\mu \gamma)) = \mu \gamma\). Therefore
\(x(0, d(\gamma)) = \gamma\). Note that this \(\gamma\) is unique by the factorisation property. Hence for \((\nu', \gamma') \in \Lambda^{\min}(\lambda, \mu)\) such that \((\nu', \gamma') \neq (\nu, \gamma)\), we have \(T_{\nu'} T_{\gamma'}(x) = 0\). Since \(x(0, d(\gamma)) = \gamma\), then

\[
T_{\nu'} T_{\gamma'}(x) = T_{\nu'}\left(x(\omega(\gamma), d(x))\right) = \nu\left[x(\omega(\gamma), \nu(x))\right]
\]

\[
= \nu\left[(\mu x) (\omega(\gamma), d(x))\right]
\]

\[
= \nu\left[(\mu x) (\omega(\gamma), d(x))\right] \text{ (since } \mu \gamma = \lambda \nu)
\]

\[
= (\mu x) (\omega(\gamma), d(x))
\]

and

\[
\sum_{(\nu', \gamma') \in \Lambda^{\min}(\lambda, \mu)} T_{\nu'} T_{\gamma'}(x) = T_{\nu} T_{\gamma'}(x) = (\mu x) (\omega(\lambda), d(x)) = T_{\lambda} T_{\mu}(x),
\]

as required. Thus \(\{T_{\lambda}, T_{\mu^*} : \lambda, \mu \in \Lambda\}\) is a Cohn \(\Lambda\)-family, as claimed.

Finally we show that \(rT_v \neq 0\) and \(\prod_{e \in v\Lambda^1} (T_v - T_e) \neq 0\) for all \(r \in R \setminus \{0\}\) and \(v \in \Lambda^0\). Take \(r \in R \setminus \{0\}\) and \(v \in \Lambda^0\). Then \(v \in W_\Lambda\). Hence \(rT_v(v) = rv\) and \(rT_v \neq 0\).

On the other hand, for \(e \in v\Lambda^1\), we have \(T_e(v) = 0\) and then

\[
r \prod_{e \in v\Lambda^1} (T_v - T_e) (v) = rT_v(v) = rv.
\]

Hence, \(r \prod_{e \in v\Lambda^1} (T_v - T_e) \neq 0\), as required. \(\square\)

Next we show that there is an \(R\)-algebra which is universal for Cohn \(\Lambda\)-families.

**Theorem 3.5.** Suppose that \(\Lambda\) is a row-finite \(k\)-graph with no sources and \(R\) is a commutative ring with 1.

(a) There is a universal \(R\)-algebra \(C_R(\Lambda)\) generated by a Cohn \(\Lambda\)-family \(\{t_\lambda, t_{\mu^*} : \lambda, \mu \in \Lambda\}\) such that if \(\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}\) is a Cohn \(\Lambda\)-family in an \(R\)-algebra \(A\), then there exists an unique \(R\)-algebra homomorphism \(\phi_T : C_R(\Lambda) \to A\) such that \(\phi_T(t_\lambda) = T_\lambda\) and \(\phi_T(t_{\mu^*}) = T_{\mu^*}\) for \(\lambda, \mu \in \Lambda\).

(b) We have \(rt_v \neq 0\) and \(r \prod_{e \in v\Lambda^1} (T_v - T_e) \neq 0\) for all \(r \in R \setminus \{0\}\) and \(v \in \Lambda^0\).

**Proof.** Let \(X := \Lambda \cup G(\Lambda^0)\) and \(F_R(w(X))\) be the free algebra on the set \(w(X)\) of words on \(X\). Let \(I\) be the ideal of \(F_R(w(X))\) generated by elements of the following sets:

(i) \(\{vw - \delta_{v,w}v : v, w \in \Lambda^0\}\),

(ii) \(\{\lambda - \mu \nu, \lambda^* - \nu^* \mu^* : \lambda, \mu, \nu \in \Lambda\}\) and \(\lambda = \mu \nu\) and

(iii) \(\{\lambda^* \mu - \sum_{(\nu, \gamma) \in \Lambda^{\min}(\lambda, \mu)} \nu^* \gamma^* : \lambda, \mu \in \Lambda\}\).

Set \(C_R(\Lambda) := F_R(w(X))/I\) and write \(q : F_R(w(X)) \to F_R(w(X))/I\) for the quotient map. Define \(t_\lambda := q(\lambda)\) for \(\lambda \in A\), and \(t_{\mu^*} := q(\mu^*)\) for \(\mu^* \in G(\Lambda^0)\). Then \(\{t_\lambda, t_{\mu^*} : \lambda, \mu \in \Lambda\}\) is a Cohn \(\Lambda\)-family in \(C_R(\Lambda)\).

Now suppose that \(\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}\) is a Cohn \(\Lambda\)-family in an \(R\)-algebra \(A\). Define \(f : X \to A\) by \(f(\lambda) := T_\lambda\) for \(\lambda \in \Lambda\), and \(f(\mu^*) := T_{\mu^*}\) for \(\mu^* \in G(\Lambda^0)\). By the universal property of \(F_R(w(X))\), there exists an unique \(R\)-algebra homomorphism \(\pi : F_R(w(X)) \to A\) such that \(\pi|_X = f\). Since \(\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}\) is a Cohn \(\Lambda\)-family, then \(I \subseteq \ker(\pi)\). Thus there exists an \(R\)-algebra homomorphism \(\phi_T : C_R(\Lambda) \to A\) such that \(\phi_T \circ q = \pi\). The homomorphism \(\phi_T\) is unique since the element in \(X\) generate \(F_R(w(X))\) as an algebra. Furthermore, we have \(\phi_T(t_\lambda) = T_\lambda\) and \(\phi_T(t_{\mu^*}) = T_{\mu^*}\) for \(\lambda, \mu \in \Lambda\), as required.
For (b), suppose that \( \{ T_\lambda, T_\mu^* : \lambda, u \in \Lambda \} \) is the Cohn \( \Lambda \)-family as in Proposition 3.4. Then \( rT_v \neq 0 \) and \( r \prod_{e \in v \Lambda^1} (T_v - T_e T_e^*) \neq 0 \) for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \). Since \( \phi_T (rt_v) = rT_v \neq 0 \) and \( \phi \left( r \prod_{e \in v \Lambda^1} (t_v - t_e t_e^*) \right) \neq 0 \) for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \).

4. The uniqueness theorem for Cohn path algebras

In this section, we establish a uniqueness theorem for Cohn path algebras (Theorem 4.1). This uniqueness theorem can be viewed as an algebraic analogue of the uniqueness theorem for Toeplitz algebras [29, Theorem 8.1]. Note that in [29], Raeburn and Sims state the uniqueness theorem for Toeplitz algebras in terms of the more general product systems of graphs over \( \mathbb{N}^k \); thus their result includes Toeplitz algebras associated to \( k \)-graphs. In [28, Theorem 2.2], Pangalela states the uniqueness theorem in the \( k \)-graph setting explicitly. For further discussion, see Remark 2.3 and Remark 2.4 of [28]. This uniqueness theorem also does not require any hypothesis on the \( k \)-graph and thus applies generally.

**Theorem 4.1** (The uniqueness theorem for Cohn path algebras). Let \( \Lambda \) be a row-finite \( k \)-graph with no sources and \( R \) be a commutative ring with 1. Suppose that \( \phi : C_R(\Lambda) \to A \) is a ring homomorphism such that

\[
\phi (rt_v) \neq 0 \quad \text{and} \quad \phi \left( r \prod_{e \in v \Lambda^1} (t_v - t_e t_e^*) \right) \neq 0
\]

for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \). Then \( \phi \) is injective.

The rest of this section is devoted to proving Theorem 4.1. To help readers follow our proofs, we divide the arguments into three subsections. In Subsection 4.1, we recall the Kumjian-Pask \( \Lambda \)-families of [15] and study some of their properties. In Subsection 4.2, we recall the \( k \)-graph \( TA \) of [28] and investigate the Kumjian-Pask algebra of \( TA \). Finally, in Subsection 4.3, we show that every Cohn \( \Lambda \)-family is isomorphic to the Kumjian-Pask \( TA \)-family (Theorem 4.13). Once we have this isomorphism, we show that Theorem 4.1 is a consequence of the Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras [15, Theorem 8.1].

4.1. Kumjian-Pask algebras. Suppose that \( \Lambda \) is a row-finite \( k \)-graph. Recall from [15, Definition 3.1] that a Kumjian-Pask \( \Lambda \)-family \( \{ S_\lambda, S_\mu^* : \lambda, u \in \Lambda \} \) in an \( R \)-algebra \( A \) is a family which satisfies (CP1-3) and

\[
(KP) \prod_{v \in E} (S_v - S_\lambda S_\lambda^*) = 0 \quad \text{for all} \quad v \in \Lambda^0 \quad \text{and} \quad \text{finite exhaustive} \quad E \subseteq v \Lambda.
\]

**Remark 4.2.** We are careful to not say that a Kumjian-Pask \( \Lambda \)-family is a Cohn \( \Lambda \)-family which satisfies (KP). This is because in Definition 3.1, we define Cohn \( \Lambda \)-family of row-finite \( k \)-graphs with no sources; however, the above definition of Kumjian-Pask \( \Lambda \)-family allows for more general row-finite \( k \)-graphs (in particular, to \( k \)-graphs with sources). We will need this level of generality later on.
For a row-finite $k$-graph $\Lambda$, there exists an $R$-algebra $\text{KP}_R(\Lambda)$ generated by the universal Kumjian-Pask $\Lambda$-family $\{s_\lambda, s_\mu^* : \lambda, u \in \Lambda\}$.

**Remark 4.3.** For a row-finite $k$-graph $\Lambda$ with no sources, the set $v\Lambda^i$ is exhaustive for all $v \in \Lambda^0$ and $1 \leq i \leq k$ (see [31, Proof of Lemma B.2]). Hence, $\text{KP}_R(\Lambda)$ is a nontrivial quotient of $C_R(\Lambda)$ and then $C_R(\Lambda)$ is not simple.

There is a powerful uniqueness theorem for Kumjian-Pask algebras, called the Cuntz-Krieger uniqueness theorem which we prove Theorem 8.1 of [15]. Since we shall use this theorem in the proof of Theorem 4.1 (see Subsection 4.3), we state it explicitly:

**Theorem 4.4.** Let $\Lambda$ be a row-finite $k$-graph which satisfies the aperiodicity condition:

for every pair of distinct paths $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$,

there exists $\eta \in s(\lambda)\Lambda$ such that $\text{MCE}(\lambda\eta, \mu\eta) = \emptyset$.

Let $R$ be a commutative ring with $1$. Suppose that $\pi : \text{KP}_R(\Lambda) \rightarrow A$ is a ring homomorphism such that $\pi(rs_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$. Then $\pi$ is injective.

**Remark 4.5.** Our aperiodicity condition is from [26, Definition 3.1] and applies to very general (not necessarily row-finite) $k$-graphs. In the setting of row-finite higher-rank graphs, our definition is equivalent to various aperiodicity definitions, including Condition (B') of [17, Remark 7.3] and [36, Definition 2.1.(ii)] (see [26, 33, 34, 36]).

**Remark 4.6.** The aperiodicity condition is a higher-rank analogue of Condition (L) for $1$-graphs. Using our path convention, Condition (L) says that every cycle has an entry (see [1, 2, 32]).

The following proposition will be useful to simplify calculations in Kumjian-Pask algebras. In essence gives an alternate formulation of (KP).

**Proposition 4.7.** Let $\Lambda$ be a row-finite $k$-graph and $R$ be a commutative ring with $1$. Suppose that $\{S_\lambda, S_\mu^* : \lambda, u \in \Lambda\}$ is a Cohn $\Lambda$-family in an $R$-algebra $A$. Then

$$\{S_\lambda, S_\mu^* : \lambda, u \in \Lambda\}$$

is a Kumjian-Pask $\Lambda$-family if and only if

$$\prod_{v \in E} (S_v - S_eS_e^*) = 0$$

for all $v \in \Lambda^0$ and exhaustive $E \subseteq v\Lambda^1$.

Before proving Proposition 4.7, we establish the following helper lemma.

**Lemma 4.8.** Let $\Lambda$ be a row-finite $k$-graph and $R$ be a commutative ring with $1$. Suppose that $\{S_\lambda, S_\mu^* : \lambda, u \in \Lambda\}$ is a Cohn $\Lambda$-family in an $R$-algebra $A$. Suppose $v \in \Lambda^0$, $\lambda \in v\Lambda$ and $E \subseteq s(\lambda)\Lambda$ is finite and satisfies $\prod_{v \in E} (S_{s(\lambda)} - S_vS_v^*) = 0$. Then

$$S_v - S_\lambda S_\lambda^* = \prod_{v \in E} (S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}) .$$

**Proof.** We follow the $C^*$-algebraic argument of [31, Lemma C.7]. For $v \in s(\lambda)\Lambda$, we have

$$(S_v - S_\lambda S_\lambda^*) (S_v - S_{\lambda\nu}S_{(\lambda\nu)^*}) = S_v - S_\lambda S_\lambda^*. $$
so
\[
(S_v - S_{\lambda} S_{\lambda^*}) \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}) = S_v - S_{\lambda} S_{\lambda^*}.
\]

On the other hand,
\[
\begin{align*}
(S_v - S_{\lambda} S_{\lambda^*}) \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}) &= S_v \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}) - S_{\lambda} S_{\lambda^*} \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}) \\
&= \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}) - \prod_{\nu \in E} (S_{\lambda} S_{\lambda^*} - S_{\lambda\nu} S_{(\lambda\nu)^*}) \\
&= \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*}) - S_{\lambda} \left( \prod_{\nu \in E} (S_{s(\lambda)} - S_{\nu} S_{\nu^*}) \right) S_{\lambda^*} \\
&= \prod_{\nu \in E} (S_v - S_{\lambda\nu} S_{(\lambda\nu)^*})
\]

since \( \prod_{\nu \in E} (S_{s(\lambda)} - S_{\nu} S_{\nu^*}) = 0 \) by the hypothesis. The conclusion follows. \( \Box \)

Proof of Proposition \([\text{4.7}] \). We use a similar argument to the \( C^* \)-algebraic version in \([\text{31}] \) Proposition C.3. If \( \{S_{\lambda}, S_{\mu^*} : \lambda, u \in \Lambda \} \) is a Kunimasa-Pask \( \Lambda \)-family, then it satisfies \( \prod_{e \in E} (S_v - S_{e} S_{e^*}) = 0 \) for all \( v \in \Lambda^0 \) and exhaustive set \( E \subseteq v \Lambda^1 \). Now we show the reverse implication. First for \( E \subseteq \Lambda \), we write
\[
I (E) := \bigcup_{i=1}^{k} \{ \lambda (0, e_i) : \lambda \in E, d (\lambda)_i > 0 \} \text{ and } L (E) := \sum_{i=1}^{k} \max d (\lambda)_i.
\]
We have to show \( \prod_{\lambda \in E} (S_v - S_{\lambda} S_{\lambda^*}) = 0 \) for all \( v \in \Lambda^0 \) and exhaustive set \( E \subseteq v \Lambda \). We show this by induction on \( L (E) \). If \( L (E) = 1 \), then \( E \subseteq v \Lambda^1 \) for some \( v \in \Lambda^0 \) and \( \prod_{e \in E} (S_v - S_{e} S_{e^*}) = 0 \) by assumption.

Now fix \( l \geq 1 \) and suppose that \( \prod_{\lambda \in F} (S_v - S_{\lambda} S_{\lambda^*}) = 0 \) for all \( v \in \Lambda^0 \) and exhaustive set \( F \subseteq v \Lambda \) with \( L (F) \leq l \). Take \( v \in \Lambda^0 \) and exhaustive set \( E \subseteq v \Lambda \) with \( L (E) = l + 1 \). If \( v \in E \), then \( \prod_{\lambda \in E} (S_v - S_{\lambda} S_{\lambda^*}) = 0 \). So suppose \( v \notin E \). Note that \( I (E) \subseteq v \Lambda^1 \). Since \( E \) is exhaustive, then by \([31] \) Lemma C.6], \( I (E) \) is also exhaustive. So
\[
(4.1) \quad I (E) \subseteq v \Lambda^1 \text{ is exhaustive.}
\]
Take \( e \in I (E) \) and by \([31] \) Lemma C.5], \( \text{Ext} (e; E) \) is exhaustive. By \([31] \) Lemma C.8], \( L (\text{Ext} (e; E)) < L (E) = l + 1 \) and then \( L (\text{Ext} (e; E)) \leq l \). So by the inductive hypothesis,
\[
\prod_{e \in \text{Ext}(e;E)} (S_v - S_{e} S_{e^*}) = 0.
\]
and then by Lemma \([\text{4.8}] \) we get
\[
(4.2) \quad S_v - S_{e} S_{e^*} = \prod_{e \in \text{Ext}(e;E)} \left( S_v - S_{ev} S_{(ev)^*} \right).
\]
Now note that for \( \nu \in \text{Ext} (e; E) \), there exists \( \lambda \in E \) with \( e\nu = \lambda\nu' \), and then
\[
(S_v - S_{\lambda} S_{\lambda^*}) \left( S_v - S_{e\nu} S_{(e\nu)^*} \right) = S_v - S_{\lambda} S_{\lambda^*}.
\]
Hence
\[
\prod_{\lambda \in E} (S_v - S_{\lambda} S_{\lambda^*}) = \left( \prod_{\lambda \in E} (S_v - S_{\lambda} S_{\lambda^*}) \right) \left( \prod_{e \in I (E) \nu \in \text{Ext}(e;E)} \left( S_v - S_{e\nu} S_{(e\nu)^*} \right) \right)
= \left( \prod_{\lambda \in E} (S_v - S_{\lambda} S_{\lambda^*}) \right) \left( \prod_{e \in I (E)} (S_v - S_{e} S_{e^*}) \right) \text{ (by (4.2))}
\]
\[(\prod_{\lambda \in E} (S_\lambda - S_\Lambda \lambda^*)) (0)\) (by Proposition 4.9 and the inductive hypothesis)

= 0,

as required. \qed

4.2. The $k$-graph $T\Lambda$ and Kumjian-Pask $T\Lambda$-families. As in [28 Proposition 3.1], for a row-finite $k$-graph with no sources $\Lambda$, $T\Lambda$ is the $k$-graph where

\[T\Lambda^0 := \{\alpha(v), \beta(v) : v \in \Lambda^0\}, \quad T\Lambda := \{\alpha(\lambda), \beta(\lambda) : \lambda \in \Lambda\},\]

and for $\alpha(\lambda), \beta(\lambda) \in T\Lambda \setminus T\Lambda^0$,

\[r(\alpha(\lambda)) := r(\alpha(\lambda)), \quad s(\alpha(\lambda)) := s(\alpha(\lambda)), \quad r(\beta(\lambda)) := r(\beta(\lambda)), \quad s(\beta(\lambda)) := s(\beta(\lambda)).\]

Note that every vertex $\beta(v)$ satisfies $\beta(v) T\Lambda = \{\beta(v)\}$ and by [28 Proposition 3.4], $T\Lambda$ is row-finite and aperiodic. The next proposition characterises exhaustive sets of $T\Lambda$.

**Proposition 4.9.** Suppose that $\Lambda$ is a row-finite $k$-graph with no sources. Then for every $\alpha(v) \in T\Lambda^0$, the only exhaustive set contained in $\alpha(v) T\Lambda^1$ is $\alpha(v) T\Lambda^1$ itself.

**Proof.** Fix an exhaustive set $E \subseteq \alpha(v) T\Lambda^1$. We have to show $E = \alpha(v) T\Lambda^1$. Since $E$ is exhaustive, for $\beta(e) \in \alpha(v) T\Lambda^1$, there exists an edge $\tau_e \in E$ such that $T\Lambda^{\text{min}}(\beta(e), \tau_e) \neq \emptyset$. Since $s(\beta(e)) T\Lambda = \{s(\beta(e))\}$, then MCE$(\beta(e), \tau_e) = \{\beta(e)\}$. Hence, $\tau_e = \beta(e)$ because both $\tau_e$ and $\beta(e)$ are edges. Thus $\beta(e) \in E$ and $E$ contains $\beta(v) T\Lambda^1$.

Now we claim $\alpha(v) T\Lambda^1 \subseteq E$. Suppose for a contradiction that there exist $1 \leq i \leq k$ and $e \in v\Lambda^e$ such that $\alpha(e) \notin E$. Since $\Lambda$ has no sources, there exists an edge $f \in s(\alpha) \Lambda^e$. Now consider the path $\tau = \alpha(e) \beta(f)$. This is a path with degree $2e_i$ whose range at $\alpha(v)$ and $s(\tau) T\Lambda = \{\beta(s(f))\}$. Since $E$ is exhaustive, there exists $\omega \in E$ such that $T\Lambda^{\text{min}}(\tau, \omega) \neq \emptyset$. Since $\tau$ is a path with length $2e_i$ and $s(\tau) T\Lambda = \{\beta(s(f))\}$, then $\omega$ is either equal to $\tau$ or $\alpha(e)$. Since $\alpha(e) \notin E$, then $\tau = \omega \in E$, which contradicts that $E$ only contains edges. The conclusion follows. \qed

A consequence of Proposition 4.9 is the following:

**Lemma 4.10.** Let $\Lambda$ be a row-finite $k$-graph with no sources and $R$ be a commutative ring with 1. Suppose that $\{S_{\tau}, S_{\omega} : \tau, \omega \in T\Lambda\}$ is a Cohn $T\Lambda$-family in an $R$-algebra $A$. Then the collection is a Kumjian-Pask $T\Lambda$-family if and only if for every $\alpha(v) \in T\Lambda^0$,

\[\prod_{g \in \alpha(v) T\Lambda^1} (S_{\alpha(v)} - S_g S_{g^*}) = 0.\]

**Proof.** If $x = \beta(v)$, then $\beta(v) T\Lambda = \{\beta(v)\}$ and there is no exhaustive set contained in $x T\Lambda^1$. On the other hand, if $x = \alpha(v)$, by Proposition 4.9, the only exhaustive set contained in $\alpha(v) T\Lambda^1$ is $\alpha(v) T\Lambda^1$. Therefore, by Proposition 4.7, $\{S_{\tau}, S_{\omega} : \tau, \omega \in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family if and only if $\prod_{g \in \alpha(v) T\Lambda^1} (S_{\alpha(v)} - S_g S_{g^*}) = 0$ for all $\alpha(v) \in T\Lambda^0$, as required. \qed
4.3. Relationship between Cohn $\Lambda$-families and Kumjian-Pask $\mathcal{T}\Lambda$-families. In this section, we start out by investigating the relationship between Cohn $\Lambda$-families and Kumjian-Pask $\mathcal{T}\Lambda$-families (Theorem 4.13). Once we have this, we are then ready to prove Theorem 4.11.

First we establish some stepping stone results (Lemma 4.11 and Lemma 4.12).

**Lemma 4.11.** Suppose that $\{\lambda, \mu, \nu, \lambda, u \in \Lambda\}$ is a Cohn $\Lambda$-family in an $R$-algebra $A$. For $v \in \Lambda^0$, define

$$F_{T,v} := T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) .$$

Then

(a) For $v \in \Lambda^0$, we have

$$F_{T,v} = F_{T,v}^2 \text{ and } T_v - F_{T,v} = (T_v - F_{T,v})^2 .$$

(b) For every $v, w \in \Lambda^0$ with $v \neq w$, we have

$$F_{T,v} F_{T,w} = 0 = F_{T,v} F_{T,w} \text{ and } T_w F_{T,v} = 0 = F_{T,v} T_w .$$

(c) For $v \in \Lambda^0$ and $\lambda \in v\Lambda \setminus \{v\}$, we have

$$T_v F_{T,v} = F_{T,v} T_v ,$$

$$F_{T,v} T_{\lambda} = T_{\lambda} \text{ and } T_{\lambda} F_{T,v} = T_{\lambda^*} .$$

(d) Furthermore, $F_{T,v} \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$ if and only if $T_v \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$.

**Proof.** First we show (a). Take $v \in \Lambda^0$. Note that $(T_v - T_e T_e^*)^2 = (T_v - T_e T_e^*)$ for $e \in v\Lambda^1$. Hence

$$(T_v - F_{T,v})^2 = \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*)^2 = \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) = T_v - F_{T,v}$$

and

$$F_{T,v}^2 = (T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*))^2 = T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) = F_{T,v} .$$

To show (b), we take $v, w \in \Lambda^0$ with $v \neq w$. Then $T_w T_v = 0$ and $T_w T_e = 0$ for all $e \in v\Lambda^1$. Hence, $T_w F_{T,v} = T_w \left( T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \right) = 0$ and by using a similar argument, we also get $F_{T,v} F_{T,w} = 0$, as required. On the other hand, we also have

$$F_{T,w} F_{T,v} = \left( T_w - \prod_{f \in w\Lambda^1} (T_w - T_f T_f^*) \right) \left( T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \right) = 0$$

and a similar argument also applies to get $F_{T,v} F_{T,w} = 0$.

Next we show (c). We take $v \in \Lambda^0$. Then

$$T_v F_{T,v} = T_v \left( T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \right) = T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) = F_{T,v}$$

and since $T_v = T_v^*$, then by using a similar argument, we also get $F_{T,v} T_v = F_{T,v}$.
Hence, $(\prod_{e \in v\Lambda^1} (T_v - T_eT_{e^*}))T_\lambda = 0$

and

$F_{T,v}T_\lambda = \left( T_v - \prod_{e \in v\Lambda^1} (T_v - T_eT_{e^*}) \right)T_\lambda = T_\lambda$.

By using a similar argument, we also get $T_\lambda \cdot F_{T,v} = T_\lambda \cdot$.

Finally, we show (d). First suppose that there exists $r \in \mathbb{R} \setminus \{0\}$ and $v \in \Lambda^0$ with $rT_v = 0$. Then $rT_v = rT_eT_e = 0$ for all $e \in v\Lambda^1$ and then $rF_{T,v} = rT_v - r\prod_{e \in v\Lambda^1} (T_v - T_eT_{e^*}) = 0$.

To show the reverse implication, suppose that there exists $r \in \mathbb{R} \setminus \{0\}$ and $v \in \Lambda^0$ with $rF_{T,v} = 0$. Take $f \in v\Lambda^1$, then

$$T_f T_{f^*} (T_v - T_f T_{f^*}) = T_f (T_{f^*} T_f) T_{f^*} - T_f T_{f^*} = 0.$$  \hfill (4.3)

Hence,

$$rT_f T_{f^*} = rT_f (T_v - T_{f^*}) T_f = rT_f T_{f^*} \left( F_{T,v} + \prod_{e \in v\Lambda^1} (T_v - T_eT_{e^*}) \right) = rT_f T_{f^*} \prod_{e \in v\Lambda^1} (T_v - T_eT_{e^*}) = 0$$

since $f \in v\Lambda^1$ and (4.3). Therefore,

$$rT_f = rT_f T_{s(f)} = rT_f (T_{f^*} T_f) = (rT_f T_{f^*}) T_f = (0) T_f = 0$$

and then

$$rT_{s(f)} = rT_{f^*} T_f = T_{f^*} (rT_f) = T_{f^*} (0) = 0,$$

as required. \hfill \Box

**Lemma 4.12.** Let $\Lambda$ be a row-finite $k$-graph with no sources and $R$ be a commutative ring with 1. Suppose that $\{T_\lambda, T_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn $\Lambda$-family in an $R$-algebra $A$. For $\tau, \omega \in T\Lambda$, define

$$S_{\tau} := \begin{cases} 
T_\lambda F_{T,s(\lambda)} & \text{if } \tau = \alpha(\lambda); \\
T_\lambda (T_{s(\lambda)} - F_{T,s(\lambda)}) & \text{if } \tau = \beta(\lambda), 
\end{cases}$$

and

$$S_{\omega^*} := \begin{cases} 
F_{T,s(\mu)} T_{\mu^*} & \text{if } \omega = \alpha(\mu); \\
(T_{s(\mu)} - F_{T,s(\mu)}) T_{\mu^*} & \text{if } \omega = \beta(\mu). 
\end{cases}$$

Then

(a) $\{S_{\tau}, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ is a Kumjian-Pask $T\Lambda$-family.

(b) Suppose that $rT_v \neq 0$ and $r\prod_{e \in v\Lambda^1} (T_v - T_eT_{e^*}) \neq 0$ for all $r \in \mathbb{R} \setminus \{0\}$ and $v \in \Lambda^0$. Suppose that $\pi_S : KP_R(T\Lambda) \to A$ is the $R$-algebra homomorphism such that $\pi_S (s_\tau) = S_{\tau}$ and $\pi_S (s_{\omega^*}) = S_{\omega^*}$ for $\tau, \omega \in T\Lambda$. Then $\pi_S$ is injective.

**Proof.** Now we show (a). First we show that $\{S_{\tau}, S_{\omega^*} : \tau, \omega \in T\Lambda\}$ satisfies (CP1). Take $x \in T\Lambda^0$. We have to show $S_x = S_{x^*} = S_x^2$. Note that $S_x = F_{T,v}$ if $x = \alpha(v)$; and $S_x = T_v - F_{T,v}$, otherwise. In both cases, by Lemma 4.11(a), we have $S_x = S_{x^*} = S_x^2$, as required.
Now take \( x, y \in T \Lambda^0 \) with \( x \neq y \). We have to show \( S_x S_y = 0 \). Since \( S_x \) is either \( F_{T,v} \) or \( T_v - F_{T,v} \), and \( S_y \) is also either \( F_{T,w} \) or \( T_w - F_{T,w} \), then Lemma 4.11(b) tells that \( x \neq y \) implies \( S_x S_y = 0 \). Therefore, \( \{ S_x, S_{\omega^*} : \tau, \omega \in T \Lambda \} \) satisfies (CP1).

Next we show that \( \{ S_{\tau}, S_{\omega^*} : \tau, \omega \in T \Lambda \} \) satisfies (CP2). Take \( \tau, \omega \in T \Lambda \) where \( s(\tau) = r(\omega) \). We have to show \( S_{\tau} S_{\omega} = S_{\tau \omega} \) and \( S_{\tau \omega} S_{\tau^2} = S_{(\tau \omega)^*} \). Note that each \( \tau \) and \( \omega \) is either in the form \( \alpha(\lambda) \) or \( \beta(\mu) \). So we give a separate argument for each case.

First suppose \( \tau = \beta(\lambda) \). Since \( s(\tau) T \Lambda = \beta(s(\lambda)) \) and \( s(\tau) = r(\omega) \), then \( \omega = s(\beta(\lambda)) \).

Hence, (4.4)

\[
S_{\beta(\lambda)} S_{\beta(s(\lambda))} = (T_{\lambda} (T_{s(\lambda)} - F_{T,s(\lambda)}) (T_{s(\lambda)} - F_{T,s(\lambda)}) = T_{\lambda} (T_{s(\lambda)} - F_{T,s(\lambda)})^2 = T_{\lambda} (T_{s(\lambda)} - F_{T,s(\lambda)}) = S_{\beta(\lambda)}.
\]

Next suppose \( \tau = \alpha(\lambda) \) and \( \omega = \beta(\mu) \). Then \( s(\tau) = r(\omega) \) implies \( \mu \in s(\lambda) \Lambda \setminus \{ s(\lambda) \} \) and by Lemma 4.11(c), \( F_{T,s(\lambda)} T_{\mu} = T_{\mu} \). Hence, (4.5)

\[
S_{\alpha(\lambda)} S_{\beta(\mu)} = (T_{\lambda} F_{T,s(\lambda)}) (T_{\mu} (T_{s(\mu)} - F_{T,s(\mu)})) = T_{\lambda} T_{\mu} (T_{s(\lambda)} - F_{T,s(\lambda)}) = T_{\lambda} - T_{\mu} = S_{\beta(\mu)}.
\]

Finally suppose \( \tau = \alpha(\lambda) \) and \( \omega = \alpha(\mu) \). Then \( S_{\beta(\lambda)} S_{\alpha(\mu)} = (S_{\beta(\lambda)} S_{\beta(s(\lambda))}) (S_{\alpha(s(\lambda))} S_{\alpha(\mu)}) = 0 \) (since \( S_{\beta(s(\lambda))} S_{\alpha(s(\lambda))} = 0 \)) and (4.6)

\[
S_{\alpha(\lambda)} S_{\alpha(\mu)} = (S_{\alpha(\lambda)} + S_{\beta(\lambda)}) (S_{\alpha(\mu)} + S_{\beta(\mu)}) = S_{\alpha(\lambda)} S_{\beta(\mu)} - S_{\beta(\lambda)} S_{\alpha(\mu)} S_{\beta(\mu)} - S_{\beta(\lambda)} S_{\beta(\mu)} = T_{\lambda} T_{\mu} - S_{\alpha(\lambda)} S_{\alpha(\mu)} - S_{\beta(\lambda)} S_{\beta(\mu)}.
\]

If \( \mu = s(\lambda) \), then \( S_{\alpha(\lambda)} S_{\beta(\mu)} = (S_{\alpha(\lambda)} S_{\alpha(s(\lambda))}) S_{\beta(s(\lambda))} = 0 \) (since \( S_{\alpha(s(\lambda))} S_{\beta(s(\lambda))} = 0 \)) and by (4.4), (4.6) becomes

\[
S_{\alpha(\lambda)} S_{\alpha(s(\lambda))} = T_{\lambda} - T_{\mu} = S_{\alpha(\lambda)}.
\]

On the other hand, if \( \mu \neq s(\lambda) \), then \( S_{\beta(\lambda)} S_{\beta(\mu)} = (S_{\beta(\lambda)} S_{\beta(s(\lambda))}) (S_{\beta(s(\lambda))} S_{\beta(\mu)}) = 0 \) (since \( \beta(s(\lambda)) = r(\beta(\mu)) \) and \( S_{\beta(s(\lambda))} S_{\beta(\mu)} = 0 \)) and by (4.5), (4.6) becomes

\[
S_{\alpha(\lambda)} S_{\alpha(\mu)} = T_{\lambda} - T_{\mu} = S_{\alpha(\lambda)}.
\]

Therefore, \( S_x S_\omega = S_{\tau \omega} \) and by using a similar argument, we get \( S_\omega S_{\tau^2} = S_{(\tau \omega)^*} \). Thus \( \{ S_{\tau}, S_{\omega^*} : \tau, \omega \in T \Lambda \} \) satisfies (CP2).

Now we show that \( \{ S_{\tau}, S_{\omega^*} : \tau, \omega \in T \Lambda \} \) satisfies (CP3). Take \( \tau, \omega \in T \Lambda \). We have to show \( S_{\tau \omega} = \sum_{(\rho, \zeta) \in T \Lambda^{\min}(\tau, \omega)} S_{\rho} S_{\zeta} \). Note that each \( \tau \) and \( \omega \) is either in the form \( \alpha(\lambda) \) or \( \beta(\mu) \). So we give a separate argument for each case.

First suppose \( \tau = \beta(\lambda) \). Since \( s(\tau) T \Lambda = \beta(s(\lambda)) \), then \( T \Lambda^{\min}(\tau, \omega) \neq \emptyset \) implies \( \text{MCE}(\tau, \omega) = \{ \tau \} \). Hence, if \( T \Lambda^{\min}(\tau, \omega) \neq \emptyset \), then we have \( \tau = \omega \beta(\nu) \) for some \( \nu \in \Lambda \) and

\[
S_{\tau \omega} = S_{\beta(\nu)^*} S_{\omega} = S_{\beta(\nu)^*} S_{\beta(\nu)} = S_{\beta(\nu)^*} = \sum_{(\rho, \zeta) \in T \Lambda^{\min}(\tau, \omega)} S_{\rho} S_{\zeta}.
\]

So suppose \( T \Lambda^{\min}(\tau, \omega) = \emptyset \) and we have to show \( S_{\tau \omega} = 0 \). First note that regardless of whether \( \omega \) is equal to \( \alpha(\mu) \) or \( \beta(\mu) \), \( S_{\tau \omega} \) has the form \( (S_{\beta(\lambda)^*} T_{\mu}) b \). So it suffices to show \( S_{\beta(\lambda)^*} T_{\mu} = 0 \). We have

(4.7)

\[
S_{\beta(\lambda)^*} T_{\mu} = (T_{s(\lambda)} - F_{T,s(\lambda)}) T_{\lambda} T_{\mu} = (T_{s(\lambda)} - F_{T,s(\lambda)}) \sum_{(\nu, \gamma) \in T \Lambda^{\min}(\lambda, \mu)} T_{\nu} T_{\gamma^*}.
\]
If \( \Lambda_{\min}(\lambda, \mu) = \emptyset \), then \( S_{\beta(\lambda)} T_\mu = 0 \), as required. So suppose \( \Lambda_{\min}(\lambda, \mu) \neq \emptyset \). Since \( T \Lambda_{\min}(\tau, \omega) = \emptyset \) and \( \Lambda_{\min}(\lambda, \mu) \neq \emptyset \), then \( \lambda \notin \text{MCE}(\lambda, \mu) \). Hence, for every \((\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)\), we have \( \nu \in s(\lambda) \Lambda \setminus \{s(\lambda)\} \) and by Lemma 4.11(c), \( F_{T,s(\lambda)} T_\nu = T_\nu \). Hence, we can rewrite (4.7) as

\[
S_{\beta(\lambda)} T_\mu = \sum_{(\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)} (T_\nu - T_\gamma) T_{\tau}, \quad = 0,
\]

as required.

Next suppose \( \tau = \alpha(\lambda) \) and \( \omega = \beta(\mu) \). By using a similar argument as in the case \( \tau = \beta(\lambda) \), we get \( S_{\alpha(\lambda)} S_{\beta(\mu)} = \sum_{(\rho, \zeta) \in T \Lambda_{\min}(\alpha(\lambda), \beta(\mu))} S_{\rho} S_{\zeta}^* \).

Finally suppose \( \tau = \alpha(\lambda) \) and \( \omega = \alpha(\mu) \). We give a separate argument for whether \( \alpha(\lambda) \) or \( \alpha(\mu) \) belongs to \( \text{MCE}(\alpha(\lambda), \alpha(\mu)) \). First suppose that at least one of \( \alpha(\lambda) \) and \( \alpha(\mu) \) belongs to \( \text{MCE}(\alpha(\lambda), \alpha(\mu)) \). Without loss of generality, we suppose \( \alpha(\lambda) \in \text{MCE}(\alpha(\lambda), \alpha(\mu)) \). [A similar argument also applies when \( \alpha(\mu) \in \text{MCE}(\alpha(\lambda), \alpha(\mu)) \).] Then \( \alpha(\lambda) = \alpha(\mu) \) for some \( \nu \in \Lambda \) and

\[
S_{\alpha(\lambda)} S_{\alpha(\mu)} = S_{\alpha(\nu)} S_{\alpha(\mu)}^* S_{\alpha(\mu)} = S_{\alpha(\nu)}^* = \sum_{(\rho, \zeta) \in T \Lambda_{\min}(\alpha(\lambda), \alpha(\mu))} S_{\rho} S_{\zeta}^*,
\]

as required.

So suppose \( \alpha(\lambda), \alpha(\mu) \notin \text{MCE}(\alpha(\lambda), \alpha(\mu)) \). Hence \( \lambda, \mu \notin \text{MCE}(\lambda, \mu) \). Then for every \((\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)\), we have \( \nu \in s(\lambda) \Lambda \setminus \{s(\lambda)\} \) and \( \gamma \in s(\mu) \Lambda \setminus \{s(\mu)\} \), and by Lemma 4.11(c), \( F_{T,s(\lambda)} T_\nu = T_\nu \) and \( T_{\tau} F_{T,s(\mu)} = T_{\tau} \). Therefore,

\[
(4.8) \quad S_{\alpha(\lambda)} S_{\alpha(\mu)} = (F_{T,s(\lambda)} T_\nu) (T_\mu F_{T,s(\mu)}) = F_{T,s(\lambda)} \left( \sum_{(\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)} T_\nu T_{\tau} \right) F_{T,s(\mu)} = \sum_{(\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)} F_{T,s(\lambda)} T_\nu F_{T,s(\mu)} = \sum_{(\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)} T_\nu T_{\tau}.
\]

Since \( s(\nu) = s(\gamma) \) for every \((\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)\), then by Lemma 4.11(a), (4.8) becomes

\[
S_{\alpha(\lambda)} S_{\alpha(\mu)} = \sum_{(\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)} \left( (T_\nu F_{T,s(\nu)}) \left( F_{T,s(\gamma)} T_{\tau} \right) + T_\nu \left( T_{s(\nu)} - F_{T,s(\nu)} \right) \left( T_{s(\gamma)} - F_{T,s(\gamma)} \right) T_{\tau} \right)
\]

\[
= \sum_{(\nu, \gamma) \in \Lambda_{\min}(\lambda, \mu)} (S_{\alpha(\nu)} S_{\alpha(\gamma)} + S_{\beta(\nu)} S_{\beta(\gamma)}) = \sum_{(\rho, \zeta) \in T \Lambda_{\min}(\alpha(\lambda), \alpha(\mu))} S_{\rho} S_{\zeta}^*,
\]

as required. Therefore, \( S_{\tau} S_{\nu} = \sum_{(\rho, \zeta) \in T \Lambda_{\min}(\tau, \nu)} S_{\rho} S_{\zeta}^* \) for all \( \tau, \nu \in TA \). Thus the collection \( \{S_{\tau}, S_{\omega} : \tau, \omega \in TA\} \) satisfies (CP3).

To show that \( \{S_{\tau}, S_{\omega} : \tau, \omega \in TA\} \) is a Kumjian-Pask \( TA \)-family, by Lemma 4.10 it suffices to show \( \prod_{g \in \alpha(v) TA} (S_{\alpha(v)} - S_{g} S_{g}^*) = 0 \) for \( \alpha(v) \in TA \). Take \( \alpha(v) \in TA \) and we have

\[
(4.9) \quad \prod_{g \in \alpha(e) TA} (S_{\alpha(v)} - S_{g} S_{g}^*)
\]

\[
= \prod_{e \in \alpha(\nu) \Lambda} (S_{\alpha(v)} - S_{e} S_{\alpha(e)}^*) (S_{\alpha(v)} - S_{\beta(e)} S_{\beta(e)}^*)
\]
Then \( \{ \pi \text{ and } 1 \} \) is an \( \ast \)-family. Since \( F_{T,s}(e) = 0 \) (by Lemma 4.11(c)), then (4.9) becomes

\[
\prod_{g \in \alpha(v)T_{\Lambda}} (S_{\alpha(v)} - S_{g}S_{g^*}) = \prod_{e \in e\Lambda^1} (F_{T,v} - T_{e}T_{e^*})
\]

(by Lemma 4.11(a,c)). Since \( F_{T,s}(e) = 0 \) (by Lemma 4.11(c)), then (4.9) becomes

\[
\prod_{g \in \alpha(v)T_{\Lambda}} (S_{\alpha(v)} - S_{g}S_{g^*}) = \prod_{e \in e\Lambda^1} (F_{T,v} - T_{e}T_{e^*})
\]

(by Lemma 4.11(a,c))

\[
= F_{T,v} \prod_{e \in e\Lambda^1} (T_{v} - T_{e}T_{e^*}) \text{ (by Lemma 4.11(a,c))}
\]

\[
= F_{T,v}(T_{v} - F_{T,v}) = F_{T,v}T_{v} - F_{T,v}^2
\]

\[
= F_{T,v} - F_{T,v} \text{ (by Lemma 4.11(a,c))}
\]

\[
= 0.
\]

Then \( \{ S_{\tau}, S_{\omega^*} : \tau, \omega \in T_{\Lambda} \} \) is a Kumjian-Pask \( T_{\Lambda} \)-family, as required.

Next we show (b). Suppose that \( r T_{v} \neq 0 \) and \( r \prod_{e \in e\Lambda^1} (T_{v} - T_{e}T_{e^*}) \neq 0 \) for all \( v \in \Lambda^0 \), and \( \pi_{S} : KP_{R}(T_{\Lambda}) \rightarrow A \) is the \( R \)-algebra homomorphism such that \( \pi_{S}(s_{\tau}) = S_{\tau} \) and \( \pi_{S}(s_{\omega^*}) = S_{\omega^*} \) for \( \tau, \omega \in T_{\Lambda} \). We have to show \( \pi_{S} \) is injective. Since \( r T_{v} \neq 0 \) for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \), then by Lemma 4.11(d), \( r F_{T,v} \neq 0 \) for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \). Therefore, for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \),

\[
r S_{\alpha(v)} = rT_{v}F_{T,v} = rF_{T,v} \neq 0
\]

and

\[
r S_{\beta(v)} = rT_{v}(T_{v} - F_{T,v}) = r(T_{v} - F_{T,v}) = r \prod_{e \in e\Lambda^1} (T_{v} - T_{e}T_{e^*}) \neq 0.
\]

Hence, \( r S_{x} \neq 0 \) for all \( r \in R \setminus \{0\} \) and \( x \in T_{\Lambda} \). Since \( T_{\Lambda} \) is aperiodic, then by Theorem 4.3, \( \pi_{S} \) is injective.

One immediate application of Lemma 4.12 is:

**Theorem 4.13.** Let \( \Lambda \) be a row-finite \( k \)-graph with no sources and \( R \) be a commutative ring with 1. Suppose that \( \{ t_{\lambda}, t_{\mu^*} : \lambda, \mu \in \Lambda \} \) is the universal Cohn \( \Lambda \)-family and \( \{ s_{\tau}, s_{\omega^*} : \tau, \omega \in T_{\Lambda} \} \) is the universal Kumjian-Pask \( T_{\Lambda} \)-family. For \( \tau, \omega \in T_{\Lambda} \), define

\[
S_{\tau} := \begin{cases} 
 t_{\lambda} F_{t_{\lambda},(\lambda)} & \text{if } \tau = \alpha(\lambda); \\
 t_{\lambda} (t_{\lambda} - F_{t_{\lambda},(\lambda)}) & \text{if } \tau = \beta(\lambda),
\end{cases}
\]

\[
S_{\omega^*} := \begin{cases} 
 F_{t_{\mu},(\mu)} t_{\mu^*} & \text{if } \omega = \alpha(\mu); \\
 (t_{\mu} - F_{t_{\mu},(\mu)}) t_{\mu^*} & \text{if } \omega = \beta(\mu).
\end{cases}
\]

Then

(a) There exists an \( R \)-algebra homomorphism \( \pi : KP_{R}(T_{\Lambda}) \rightarrow C_{R}(\Lambda) \) such that \( \pi(s_{\tau}) = S_{\tau} \) and \( \pi(s_{\omega^*}) = S_{\omega^*} \) for \( \tau, \omega \in T_{\Lambda} \). Furthermore, \( \pi \) is an isomorphism.

(b) The subsets

\[
C_{R}(\Lambda)_{n} := \text{span}_{R} \{ t_{\lambda} t_{\mu^*} : \lambda, \mu \in \Lambda, d(\lambda) - d(\mu) = n \}
\]

form a \( \mathbb{Z}^{k} \)-grading of \( C_{R}(\Lambda) \).
Proof. First we show part (a). By Lemma 4.12(a), \{S_T, S_{s\tau} : \tau, \omega \in TA\} is a Kumjian-Pask TA-family and by the universal property of Kumjian-Pask TA-family [13, Theorem 3.7(a)], there exists an \(R\)-algebra homomorphism \(\pi : KP_R(TA) \to CR(\Lambda)\) such that \(\pi (s_\tau) = S_T\) and \(\pi (s_{s\tau}) = S_{s\tau}\) for \(\tau, \omega \in TA\). On the other hand, Theorem 4.12(b) tells that \(rt_v \neq 0\) and \(r \prod_{e \in \Lambda} (t_v - t_e) \neq 0\) for all \(r \in R\setminus \{0\}\) and \(v \in \Lambda^0\). Hence, by Lemma 4.12(b), \(\pi\) is injective.

Now we show the surjectivity of \(\pi\). Since

\[
C_R(\Lambda) = \text{span}_R \{t_\lambda t_{s\tau} : \lambda, u \in \Lambda, s (\lambda) = s (\mu)\}
\]

(Proposition 4.3(b)), it suffices to show that for \(\lambda, \mu \in \Lambda\), both \(t_\lambda\) and \(t_{s\mu}\) belong to the image of \(\pi\). Take \(\lambda, \mu \in \Lambda\), then we have

\[
t_\lambda = t_\lambda t_{s\lambda} = t_\lambda F_{t,s}(\lambda) + t_\lambda (t_{s\lambda} - F_{t,s}(\lambda)) = \pi (s_\alpha(\lambda)) + \pi (s_\beta(\lambda))
\]

and

\[
t_{s\mu} = t_{s(\mu)} t_{s\mu} = F_{t,s}(\mu) t_{s\mu} + t_{s(\mu)} t_{s\mu} - F_{t,s}(\mu) t_{s\mu} = \pi (s_{\alpha(\mu)}^\ast) + \pi (s_{\beta(\mu)}^\ast),
\]

as required. Therefore, \(\pi\) is an isomorphism.

Next we show part (b). Recall from [15, Theorem 3.6(c)] that the subsets

\[
KP_R(TA)_n := \text{span}_R \{s_{s\tau} : \tau, \omega \in TA, d(\tau) - d(\omega) = n\}
\]

forms a \(Z^k\)-grading of \(KP_R(\Lambda)\). Note that for every \(v \in \Lambda^0\), \(d (t_{s\lambda} - F_{t,s}(\lambda)) = 0 = d (F_{t,s}(\lambda))\). Hence regardless of whether \(\tau\) and \(\omega\) are in the form \(\alpha (\lambda)\) or \(\beta (\mu)\), we have \(d (\tau) - d (\omega) = d (\lambda) - d (\mu)\) and \(s_{s\tau} \in C_R (\Lambda)_n\) which implies \(\pi (s_{s\tau}) \in KP_R (TA)_n\).

Since \(\pi\) is an isomorphism, then \(C_R(\Lambda)_n\) forms a grading for \(C_R(\Lambda)\), as required.

Remark 4.14. Our Theorem 4.13 generalises results about Cohn path algebras associated to 1-graphs. In particular, Theorem 4.13(a) generalises [3, Theorem 5] (which is also stated in [2, Theorem 1.5.18]); Theorem 4.13(b) generalises [2, Corollary 2.1.5.(ii)].

Proof of Theorem 4.17. Since \(\{t_\lambda, t_{s\mu} : \lambda, u \in \Lambda\}\) is a Cohn \(A\)-family, then so is \(\{\phi (t_\lambda), \phi (t_{s\mu}) : \lambda, u \in \Lambda\}\). For \(\tau, \omega \in TA\), define

\[
S_\tau := \begin{cases} 
\phi (t_\lambda) F_{t,s}(\lambda) & \text{if } \tau = \alpha (\lambda) \\
\phi (t_\lambda) (\phi (t_{s\lambda}) - F_{t,s}(\lambda)) & \text{if } \tau = \beta (\lambda) 
\end{cases}
\]

\[
S_{s\tau} := \begin{cases} 
F_{t,s}(\mu) \phi (t_{s\mu}) & \text{if } \omega = \alpha (\mu) \\
(\phi (t_{s\mu}) - F_{t,s}(\mu)) \phi (t_{s\mu}) & \text{if } \omega = \beta (\mu) 
\end{cases}
\]

Lemma 4.12(a) tells that \(\{S_\tau, S_{s\tau} : \tau, \omega \in TA\}\) is a Kumjian-Pask TA-family and by the universal property of Kumjian-Pask TA-family, there exists an \(R\)-algebra homomorphism \(\pi_S : KP_R(\Lambda) \to A\) such that \(\pi_S (s_\tau) = S_\tau\) and \(\pi_S (s_{s\tau}) = S_{s\tau}\) for \(\tau, \omega \in TA\). On the other hand, by the hypothesis, \(\phi (rt_v) \neq 0\) and \(\phi (r \prod_{e \in \Lambda} (t_v - t_e)) \neq 0\) for all \(r \in R \setminus \{0\}\) and \(v \in \Lambda^0\). Hence, by Lemma 4.12(b), \(\pi_S\) is injective.

Now recall from Theorem 4.13(a) that \(\pi : KP_R(TA) \to CR(\Lambda)\) is an isomorphism with

\[
\pi (s_\tau) = \begin{cases} 
t_\lambda F_{t,s}(\lambda) & \text{if } \tau = \alpha (\lambda) \\
t_\lambda (t_{s\lambda} - F_{t,s}(\lambda)) & \text{if } \tau = \beta (\lambda) 
\end{cases}
\]
Note that for \( \lambda, \mu \in \Lambda \), we have \( t_\lambda = \pi (s_{\alpha(\lambda)}) + \pi (s_{\beta(\lambda)}) \) and \( t_{\mu^*} = \pi (s_{\alpha(\mu^*)}) + \pi (s_{\beta(\mu)^*}) \) (see (4.10) and (4.11)). Hence,

\[
(\pi_S \circ \pi^{-1})(t_\lambda) = (\pi_S \circ \pi^{-1})\left(\pi (s_{\alpha(\lambda)}) + \pi (s_{\beta(\lambda)})\right) = \pi_S (s_{\alpha(\lambda)}) + \pi_S (s_{\beta(\lambda)}) = S_{\alpha(\lambda)} + S_{\alpha(\lambda)} = \phi(t_\lambda) F_{\phi(t),s(\lambda)} + \phi(t_\lambda) \left(\phi(t_{s(\lambda)}) - F_{\phi(t),s(\lambda)}\right)
\]

and

\[
(\pi_S \circ \pi^{-1})(t_{\mu^*}) = (\pi_S \circ \pi^{-1})\left(\pi (s_{\alpha(\mu^*)}) + \pi (s_{\beta(\mu)^*})\right) = \pi_S (s_{\alpha(\mu^*)}) + \pi_S (s_{\beta(\mu)^*}) = S_{\alpha(\mu^*)} + S_{\alpha(\mu^*)} = F_{\phi(t),s(\mu)} \phi(t_{\mu^*}) + \phi(t_{s(\mu)}) - F_{\phi(t),s(\mu)} \phi(t_{\mu^*}) = \phi(t_{\mu^*})
\]

These imply \( \phi = \pi_S \circ \pi^{-1} \) since \( C_R(\Lambda) = \text{span}\{t_v t_{\mu^*} : \lambda, u \in \Lambda, s(\lambda) = s(\mu)\} \) (Proposition 3.3(b)). Furthermore, the injectivity of both \( \pi^{-1} \) and \( \pi_S \) imply that \( \phi \) is also injective.

**Remark 4.15.** The Cohn \( \Lambda \)-family \( \{T_\lambda, T_{\mu^*} : \lambda, u \in \Lambda\} \) as constructed in Proposition 8.4 satisfies \( r T_v \neq 0 \) and \( r \prod_{v \in \rho \lambda^1} (T_v - T_v T_v^*) \neq 0 \) for all \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \). Hence Theorem 4.1 tells that the \( R \)-algebra homomorphism \( \phi_T : C_R(\Lambda) \to \text{End}(\mathcal{F}_R(W_\Lambda)) \) such that \( \phi_T(t_\lambda) = T_\lambda \) and \( \phi_T(t_{\mu^*}) = T_{\mu^*} \) for \( \lambda, \mu \in \Lambda \), is injective.

**Remark 4.16.** Note that when \( \Lambda \) is a 1-graph, that is, when \( k = 1 \), then \( \Lambda \) is the path category of a directed graph \( E \). One consequence of Theorem 3.5 and Theorem 4.4 is that the universal Cohn algebra \( C_R(\Lambda) \) that we have constructed is isomorphic to the Cohn path algebra associated to \( E \) as defined in [2] Definition 1.5.1. Since [2] Definition 1.5.1 only considers the situation where \( R \) is a field, our construction gives a generalisation of the Cohn path algebra to the setting where \( R \) is an arbitrary commutative ring with 1.

## 5. Examples and Applications

### 5.1. Higher-rank graph Toeplitz algebras

As mentioned in the introduction of Section 4, the uniqueness theorem for Cohn path algebras (Theorem 4.1) is an analogue of the uniqueness theorem for Toeplitz algebras [29] Theorem 8.1. We show that if \( \Lambda \) is a row-finite \( k \)-graph with no sources, then its Cohn path algebra over the complex numbers is dense in the Toeplitz algebra associated to \( \Lambda \) (Proposition 5.2). First we give some preliminaries on Toeplitz-Cuntz-Krieger \( \Lambda \)-families and Toeplitz algebras.

Suppose that \( \Lambda \) is a row-finite \( k \)-graph with no sources. A **Toeplitz-Cuntz-Krieger \( \Lambda \)-family** is a collection \( \{Q_\lambda : \lambda \in \Lambda\} \) of partial isometries in a \( C^* \)-algebra \( B \) satisfying:

- **(TCK1)** \( \{Q_v : v \in \Lambda^0\} \) is a collection of mutually orthogonal projections;
- **(TCK2)** \( Q_\lambda Q_\mu = Q_\lambda \) whenever \( s(\lambda) = r(\mu) \); and
- **(TCK3)** \( Q_\lambda Q_\mu = \sum_{(\nu,\gamma) \in \Lambda^{\min}(\lambda,\mu)} Q_\nu Q_\gamma^* \) for all \( \lambda, \mu \in \Lambda \).

**Remark 5.1.** In [29] Lemma 9.2, a Toeplitz-Cuntz-Krieger \( \Lambda \)-family is defined to be a collection of partial isometries \( \{Q_\lambda : \lambda \in \Lambda\} \) satisfying (TCK1-3) and an additional condition: for all \( m \in \mathbb{N}^k \setminus \{0\} \), \( r \in \Lambda^0 \) and every set \( E \subseteq r \Lambda^m \), \( Q_v \geq \sum_{\lambda \in E} Q_\lambda Q_\lambda^* \). However, by [31] Lemma 2.7 (iii), this additional condition is a direct consequence of (TCK1-3) and hence our definition is equivalent to that of [29].
For a row-finite $k$-graph $\Lambda$, there exists a $C^*$-algebra $TC^* (\Lambda)$, called the Toeplitz algebra of $\Lambda$, generated by the universal Toeplitz-Cuntz-Krieger $\Lambda$-family $\{q_\lambda : \lambda \in \Lambda\}$. Furthermore, for $v \in \Lambda^0$, we have $q_v \neq 0$ and $\prod_{e \in \mathcal{E}(\Lambda)} (q_v - q_v e^*) \neq 0$ \cite[Corollary 3.7.7]{37].

**Proposition 5.2.** Let $\Lambda$ be a row-finite $k$-graph with no sources. Suppose that $\{q_\lambda : \lambda \in \Lambda\}$ is the universal Toeplitz-Cuntz-Krieger $\Lambda$-family and $\{t_\lambda, t_{\mu^*} : \lambda, \mu \in \Lambda\}$ is the universal (complex) Cohn $\Lambda$-family. Then there is an isomorphism

$$\phi_q : C_\mathbb{C} (\Lambda) \to \text{span}\{q_\lambda q_{\mu^*} : \lambda, \mu \in \Lambda\}$$

such that $\phi_q (t_\lambda) = q_\lambda$ and $\phi_q (t_{\mu^*}) = q_{\mu^*}$ for $\lambda, \mu \in \Lambda$. In particular, $C_\mathbb{C} (\Lambda)$ is isomorphic to a dense subalgebra of $TC^* (\Lambda)$.

**Proof.** Since $\{q_\lambda : \lambda \in \Lambda\}$ satisfies (TCK1-3), then by putting $q_\lambda := q_\lambda$ and $q_{\mu^*} := q_{\mu^*}$, the collection $\{q_\lambda, q_{\mu^*} : \lambda, \mu \in \Lambda\}$ is a Cohn $\Lambda$-family in $TC^* (\Lambda)$. Thus the universal property of $C_\mathbb{C} (\Lambda)$ gives a homomorphism $\phi_q$ from $C_\mathbb{C} (\Lambda)$ onto the dense subalgebra

$$A := \text{span}_\mathbb{C}\{q_\lambda q_{\mu^*} : \lambda, \mu \in \Lambda\}$$

of $TC^* (\Lambda)$.

On the other hand, for all $r \in \mathbb{C} \setminus \{0\}$ and $v \in \Lambda^0$, we have $\frac{1}{r} \phi_q (rt_v) = q_v \neq 0$ and

$$\frac{1}{r} \phi_q \left( r \prod_{e \in \mathcal{E}(\Lambda)} (t_v - t_e e^*) \right) = \prod_{e \in \mathcal{E}(\Lambda)} (q_v - q_v e^*) \neq 0.$$

So $\phi_q (rt_v) \neq 0$ and $\phi_q \left( r \prod_{e \in \mathcal{E}(\Lambda)} (t_v - t_e e^*) \right) \neq 0$ for all $r \in \mathbb{C} \setminus \{0\}$ and $v \in \Lambda^0$. Then by Theorem 5.1, $\phi_q$ is injective. \hfill \Box

**Remark 5.3.** For $k = 1$, Proposition 5.2 tells that the Cohn path algebra of 1-graph $E$ is isomorphic to a dense subalgebra of $TC^* (E)$.

### 5.2. Groupoids and Steinberg algebras.

In \cite[Proposition 5.4]{15}, the authors show that each Kumjian-Pask algebra is isomorphic to a Steinberg algebra. Thus Theorem 4.13 implies that the Cohn path algebra of $\Lambda$ is also isomorphic to a Steinberg algebra associated to $T\Lambda$. However, this is somewhat obscure because one has to go through $T\Lambda$. We improve this result by showing that there exists a groupoid associated to $\Lambda$ such that its Steinberg algebra is isomorphic to the Cohn path algebra of $\Lambda$ (Proposition 5.3). We start out with an introduction to groupoids and Steinberg algebras in general.

A groupoid $\mathcal{G}$ is a small category with inverses. We write $\mathcal{G}^{(0)}$ for the set $\{aa^{-1} : a \in \mathcal{G}\}$, and we denote $r$ and $s$ the range and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$. We also write $\mathcal{G}^{(2)}$ for the set of pairs $(a, b) \in \mathcal{G} \times \mathcal{G}$ with $s (a) = r (b)$, and for $A, B \subseteq \mathcal{G}$,

$$AB := \{ab : a \in A, b \in B, (a, b) \in \mathcal{G}^{(2)}\}.$$

We say $\mathcal{G}$ is topological if $\mathcal{G}$ is endowed with a topology such that the range, source, and composition maps are continuous. We call an open set $U \subseteq \mathcal{G}$ open bisection if $s|_U$ and $r|_U$ are homeomorphisms. Finally, a groupoid $\mathcal{G}$ is ample if it has a basis of compact open bisections.

Suppose that $\mathcal{G}$ is Hausdorff ample groupoid and $R$ is a commutative ring with 1. The Steinberg algebra of $\mathcal{G}$, denoted $A_R (\mathcal{G})$, is the set of all functions from $\mathcal{G}$ to $R$ that
are locally constant and have compact support (see [12, 13, 39]). Addition and scalar multiplication of $A_R(G)$ are defined pointwise, and convolution is given by
\[
(f \ast g)(a) := \sum_{r(a) = r(b)} f(b)g(b^{-1}a).
\]
Furthermore, for compact open bisections $U$ and $V$, we have
\[1_U \ast 1_V = 1_UV.\]

**Example 5.4.** Suppose that $\Lambda$ is a row-finite $k$-graph with no sources. Following [41, Definition 3.4], we construct an ample groupoid as follows. Write $\Lambda^\ast s \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = s(\mu)\}$.

For $(\lambda, \mu) \in \Lambda^\ast s \Lambda$ and finite subset $G \subseteq s(\lambda) \Lambda$, we write
\[
TZ_\Lambda(\lambda) := \lambda W_\Lambda,
\]
\[
TZ_\Lambda(\lambda \setminus G) := TZ_\Lambda(\lambda) \setminus \left( \bigcup_{\nu \in G} TZ_\Lambda(\lambda \nu) \right),
\]
\[
TZ_\Lambda(\lambda \ast_s \mu \setminus G) := TZ_\Lambda(\lambda \ast_s \mu) \setminus \left( \bigcup_{\nu \in G} TZ_\Lambda(\lambda \nu \ast_s \mu \nu) \right).
\]
Then $T G_\Lambda$ is a groupoid, called the path groupoid, with object set $\text{Obj}(T G_\Lambda) := W_\Lambda$,
morphism set
\[
\text{Mor}(T G_\Lambda) := \{(\lambda z, d(\lambda) - d(\mu), \mu z) \in W_\Lambda \times Z^k \times W_\Lambda : (\lambda, \mu) \in \Lambda^\ast s \Lambda, z \in s(\lambda) W_\Lambda\}
\]
\[
= \{(x, m, y) \in W_\Lambda \times Z^k \times W_\Lambda : \text{there exists } p, q \in \mathbb{N}^k \text{ such that } p \leq d(x), q \leq d(y), p - q = m \text{ and } \sigma^p x = \sigma^q y\},
\]
range and source maps $r(x, m, y) := x$ and $s(x, m, y) := y$, composition
\[
((x_1, m_1, y_1), (y_1, m_2, y_2)) \mapsto (x_1, m_1 + m_2, y_2),
\]
and inversion $(x, m, y) \mapsto (y, -m, x)$. Furthermore, the sets $TZ_\Lambda(\lambda \setminus G)$ form a basis of compact open sets for $T G_\Lambda^0$, and the sets $TZ_\Lambda(\lambda \ast_s \mu \setminus G)$ form a basis of compact open sets for locally-compact, second-countable, Hausdorff topology on $T G_\Lambda$ under which it is an étale topological groupoid. Since for $(\lambda, \mu) \in \Lambda^\ast s \Lambda$ and finite subset $G \subseteq s(\lambda) \Lambda$, $TZ_\Lambda(\lambda \ast_s \mu \setminus G)$ is a bijection, then $T G_\Lambda$ is also ample.

**Remark 5.5.** We think of $T G_\Lambda^0 = W_\Lambda$ as a subset of $T G_\Lambda$ under the correspondence $x \mapsto (x, 0, x)$.  

Proposition 5.6. Let $\Lambda$ be a row-finite $k$-graph with no sources. Then the path groupoid $\mathcal{T}\mathcal{G}_\Lambda$ is effective, in the sense that the interior of
\[ \text{Iso} (\mathcal{T}\mathcal{G}_\Lambda) := \{ a \in \mathcal{T}\mathcal{G}_\Lambda : s (a) = r (a) \} \]
is $\mathcal{T}\mathcal{G}_\Lambda^{(0)}$.

Proof. For $x \in \mathcal{T}\mathcal{G}_\Lambda^{(0)}$, we have $(x, 0, x) \in \text{Iso} (\mathcal{T}\mathcal{G}_\Lambda)$ and then $\mathcal{T}\mathcal{G}_\Lambda^{(0)}$ belongs to the interior of $\text{Iso} (\mathcal{T}\mathcal{G}_\Lambda)$. Now we show the reverse inclusion. Take $a$ an interior point of $\text{Iso} (\mathcal{T}\mathcal{G}_\Lambda)$. Then there exists $T Z_A (\lambda * s, \mu \backslash G)$ such that $T Z_A (\lambda * s, \mu \backslash G) \subseteq \text{Iso} (\mathcal{T}\mathcal{G}_\Lambda)$ and $a \in T Z_A (\lambda * s, \mu \backslash G)$. We have to show $\lambda = \mu$. Since $a \in T Z_A (\lambda * s, \mu \backslash G)$, then $T Z_A (\lambda * s, \mu \backslash G)$ is not empty; so $s (\lambda) \notin G$. Hence, $(\lambda, d (\lambda) - d (\mu), \mu) \in T Z_A (\lambda * s, \mu \backslash G)$ and since $T Z_A (\lambda * s, \mu \backslash G) \subseteq \text{Iso} (\mathcal{T}\mathcal{G}_\Lambda)$, this implies $\lambda = \mu$. Therefore, $\mathcal{T}\mathcal{G}_\Lambda$ is effective.

Remark 5.7. Our definition of effectiveness is from Lemma 3.1 of [11]. That lemma gives several equivalent characterisations of effective.

Proposition 5.8. Let $\Lambda$ be a row-finite $k$-graph with no sources, $\mathcal{T}\mathcal{G}_\Lambda$ be its path groupoid and $R$ be a commutative ring with $1$. Then there is an isomorphism $\phi_Q : C_R (\Lambda) \rightarrow A_R (\mathcal{G}_\Lambda)$ such that $\phi_Q (t_\lambda) = 1_{T Z_A (\lambda * s, (\lambda))}$ and $\phi_Q (t_{\mu, r}) = 1_{T Z_A (s (\mu) * s, \mu)}$ for $\lambda, \mu \in \Lambda$.

Before proving Proposition 5.8 we first note that the argument of [13, Lemma 5.6] also applies to the path groupoid $\mathcal{T}\mathcal{G}_\Lambda$ and we get the following result.

Lemma 5.9. Let $\{ T Z_A (\lambda_i * s, \mu_i \backslash G_i) \}_{i=1}^n$ be a finite collection of compact open bisection sets and
\[ U := \bigcup_{i=1}^n T Z_A (\lambda_i * s, \mu_i \backslash G_i) . \]
Then
\[ 1_U \in \text{span}_R \{ 1_{T Z_A (s (\mu), \Lambda)} : (\lambda, \mu) \in \Lambda * s, \Lambda, G \subseteq s (\lambda) \Lambda \} . \]

Proof of Proposition 5.8. By [17, Theorem 6.9] and [11, Example 7.1], $Q_\lambda := 1_{T Z_A (s (\mu), \Lambda)}$ determines a Toeplitz-Cuntz-Krieger $\Lambda$-family. Now define $Q_{\lambda, \mu} := 1_{T Z_A (s (\mu), \Lambda)}$ and $Q_{\mu, r} := 1_{T Z_A (s (\mu), \Lambda)}$ for $\lambda, \mu \in \Lambda$. Then $\{ Q_{\lambda, \mu} : \lambda, \mu \in \Lambda \}$ is a Cohn $\Lambda$-family in $A (\mathcal{T}\mathcal{G}_\Lambda)$.

Hence there exists a homomorphism $\phi_Q : C_R (\Lambda) \rightarrow A_R (\mathcal{T}\mathcal{G}_\Lambda)$ such that $\phi_Q (t_\lambda) = Q_\lambda$ and $\phi_Q (t_{\mu, r}) = Q_{\mu, r}$ for $\lambda, \mu \in \Lambda$.

Now we show that $\phi_Q$ is injective. Note that for all $r \in R \setminus \{ 0 \}$ and $v \in \Lambda^0$, we have
\[ \phi_Q (r t_v) = r Q_v = r 1_{T Z_A (s (v), \Lambda)} \neq 0 \]
and
\[ \phi_Q \left( r \prod_{e \in v \Lambda^1} (t_v - t_e) \right) = r \prod_{e \in v \Lambda^1} (Q_v - Q_e) = r \prod_{e \in v \Lambda^1} (1_{T Z_A (s (v), \Lambda)} - 1_{T Z_A (s (e), \Lambda)}) \]
\[ = r \prod_{e \in v \Lambda^1} 1_{T Z_A (s (v), \Lambda \setminus \{ e \})} = r 1_{\prod_{e \in v \Lambda^1} T Z_A (s (v), \Lambda \setminus \{ e \})} \neq 0. \]

Hence, by Theorem [11], $\phi_Q$ is injective, as required.

Finally we show the surjectivity of $\phi_Q$. Take $f \in A_R (\mathcal{T}\mathcal{G}_\Lambda)$. By [12, Lemma 2.2], $f$ can be written as $\sum_{U \in F} a_U 1_U$ where $a_U \in R$, each $U$ is in the form $\bigcup_{i=1}^n T Z_A (\lambda_i * s, \mu_i \backslash G_i)$ for
some \( n \in \mathbb{N} \), and \( F \) is finite set of mutually disjoint elements. Hence to show \( f \in \text{im} (\phi_Q) \), it suffices to show

\[
1_U \in \text{im} (\phi_Q)
\]

where \( U := \bigcup_{i=1}^n T Z_\Lambda (\lambda_i \* s \mu_i \setminus G_i) \) for some \( n \in \mathbb{N} \) and collection \( \{ T Z_\Lambda (\lambda_i \* s \mu_i \setminus G_i) \}_{i=1}^n \). By Lemma 5.9, \( 1_U \) can be written as the sum of elements in the form \( 1_{T Z_\Lambda (\lambda \* s \mu \setminus G)} \). On the other hand, by following the argument of [15, Equation 5.5], we have

\[
1_{T Z_\Lambda (\lambda \* s \mu \setminus G)} = Q_\lambda \left( \prod_{\nu \in G} \left( Q_\lambda s(\lambda) - Q_\nu Q_\nu^* \right) \right) Q_\mu
\]

for all \( (\lambda, \mu) \in \Lambda \* s \Lambda \) and finite \( G \subseteq s(\lambda) \Lambda \). Hence, every \( 1_{T Z_\Lambda (\lambda \* s \mu \setminus G)} \) belongs to \( \text{im} (\phi_Q) \) and so does \( 1_U \), as required. Hence \( \phi_Q \) is surjective and then is an isomorphism. \( \square \)

**Remark 5.10.** Proposition 5.5 of [15] shows that the Kumjian-Pask algebra of \( T \Lambda \) is isomorphic to the Steinberg algebra associated to the boundary-path groupoid \( \mathcal{G}_{T \Lambda} \) of [41]. Indeed, we could have shown that the path groupoid \( \mathcal{T \mathcal{G}}_\Lambda \) of Example 5.4 is topologically isomorphic to the boundary-path groupoid \( \mathcal{G}_{T \Lambda} \), and deduced Proposition 5.8. However, the direct argument above takes about the same amount of effort.

**References**

[1] G. Abrams, *Leavitt path algebras: the first decade*, Bull. Math. Sci. 5 (2015), 59–120.

[2] G. Abrams, P. Ara and M. Siles Molina, Leavitt path algebras, A primer and handbook, Springer, to appear. https://www.dropbox.com/s/gqqx735jddrip8f/Abra msAraSiles_BookC1C2.pdf?dl=0

[3] G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra 293 (2005), 319–334.

[4] G. Abrams and G. Aranda Pino, *The Leavitt path algebras of arbitrary graphs*, Houston J. Math. 34 (2008), 423–442.

[5] G. Abrams and M. Kanuni, *Cohn path algebras have Invariant Basis Number*, Commun. Alg. 44 (2016), 371–380.

[6] G. Abrams and Z. Mesyan, *Simple Lie algebras arising from Leavitt path algebras*, J. Pure Applied Algebra 216 (2012), 2302–2313.

[7] P. Ara and K.R. Goodearl, *C*-algebras of separated graphs, J. Funct. Anal. 261 (2011), 2540–2568.

[8] P. Ara and K.R. Goodearl, *Leavitt path algebras of separated graphs*, J. Reine Angew. Math. 669 (2012), 165–224.

[9] P. Ara, M.A. Moreno and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory 10 (2007), 157–178.

[10] G. Aranda Pino, J. Clark, A. an Huef and I. Raeburn, *Kumjian-Pask algebras of higher-rank graphs*, Trans. Amer. Math. Soc. 365 (2013), 3613–3641.

[11] J. Brown, L.O. Clark, C. Farthing and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum 88 (2014), 433–452.

[12] L.O. Clark and C. Edie-Michell, *Uniqueness theorems for Steinberg algebras*, Algebr. Represent. Theory 18 (2015), 907–916.

[13] L.O. Clark, C. Farthing, A. Sims and M. Tomforde, *A groupoid generalisation of Leavitt path algebras*, Semigroup Forum 89 (2014), 501–517.

[14] L.O. Clark, C. Flynn and A. an Huef, *Kumjian-Pask algebras of locally convex higher-rank graphs*, J. Algebra 399 (2014), 445–474.

[15] L.O. Clark and Y.E.P. Pangalela, *Kumjian-Pask algebras of finitely aligned higher-rank graphs*, arXiv:1512.06547 [math.RA], 2015.

[16] P.M. Cohn, *Some remarks on the invariant basis property*, Topology 5 (1966), 215–228.

[17] C. Farthing, P. S. Muhly and T. Yeend, *Higher-rank graph C*-algebras: an inverse semigroup and groupoid approach*, Semigroup Forum 71 (2005), 159–187.

[18] N.J. Fowler, M. Laca and I. Raeburn, *The C*-algebras of infinite graphs*, Proc. Amer. Math. Soc. 128 (2000), 2319–2327.
[19] N.J. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J. 48 (1999), 155–181.

[20] R. Hazlewood, I. Raeburn, A. Sims and S.B.G. Webster, *Remarks on some fundamental results about higher-rank graphs and their C*-algebras*, Proc. Edinb. Math. Soc. 56 (2013), 575–597.

[21] A. an Huef, S. Kang and I. Raeburn, *Spatial realisations of KMS states on the C*-algebras of higher-rank graphs*, J. Math. Anal. Appl. 427 (2015), 977–1003.

[22] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on C*-algebras associated to higher-rank graphs*, J. Funct. Anal. 266 (2014), 265–283.

[23] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C*-algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal. 268 (2015), 1840–1875.

[24] A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. 6 (2000), 1–20.

[25] W.G. Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. 42 (1962), 113–130.

[26] P. Lewin and A. Sims, *Aperiodicity and cofinality for finitely aligned higher-rank graphs*, Math. Proc. Cambridge Philos. Soc. 149 (2010), 333–350.

[27] P. Muhly and M. Tomforde, *Adding tails to C*-correspondences*, Documenta Math. 9 (2004), 79–106.

[28] Y.E.P. Pangalela, *Realising the Toeplitz algebra of a higher-rank graph as a Cuntz-Krieger algebra*, New York J. Math. 22 (2016), 277–291.

[29] I. Raeburn and A. Sims, *Product systems of graphs and the Toeplitz algebras of higher-rank graphs*, J. Operator Theory 53 (2005), 399–429.

[30] I. Raeburn, A. Sims and T. Yeend, *Higher-rank graphs and their C*-algebras*, Proc. Edinb. Math. Soc. 46 (2003), 99–115.

[31] I. Raeburn, A. Sims and T. Yeend, *The C*-algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. 213 (2004), 206–240.

[32] I. Raeburn, *Graph Algebras*, CBMS Regional Conference Series in Math., vol. 103, American Mathematical Society, 2005.

[33] D. Robertson and A. Sims, *Simplicity of C*-algebras associated to higher-rank graphs*, Bull. London Math. Soc. 39 (2007), 337–344.

[34] D. Robertson and A. Sims, *Simplicity of C*-algebras associated to row-finite locally convex higher-rank graphs*, Israel J. Math. 172 (2009), 171–192.

[35] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher-rank Cuntz-Krieger algebras*, J. Reine Angew. Math. 513 (1999), 115–144.

[36] J. Shotwell, *Simplicity of finitely-aligned k-graph C*-algebras*, J. Operator Theory 67 (2012), 335–347.

[37] A. Sims, *C*-algebras associated to higher-rank graphs*, PhD Thesis, Univ. Newcastle, 2003.

[38] A. Sims, *The co-universal C*-algebra of a row-finite graph*, New York J. Math. 16 (2010), 507–524.

[39] B. Steinberg, *A Groupoid approach to discrete inverse semigroup algebras*, Adv. Math. 223 (2010), 689–727.

[40] S. B. G. Webster, *The path space of a higher-rank graph*, Studia Math., 204 (2011), 155–185.

[41] T. Yeend, *Groupoid models for the C*-algebras of topological higher-rank graphs*, J. Operator Theory 51 (2007), 95–120.

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