Five Definitions of Critical Point at Infinity

Alan H. Durfee

February 11, 2022

Abstract

This survey paper discusses five equivalent ways of defining a “critical point at infinity” for a complex polynomial of two variables.

1 Introduction

A proper smooth map without critical points from one manifold to another is a locally trivial fibration by a well-known theorem of Ehresmann. On the other hand, a nonproper map without critical points may not be a fibration. This phenomenon occurs for complex polynomials. A simple example is provided by \( f : \mathbb{C}^2 \to \mathbb{C} \) defined by the polynomial \( f(x, y) = y(xy - 1) \). This map has no critical points, but the fiber over the origin is different from the other fibers. (In fact, the fiber over the origin is two rational curves, one punctured at two points and the other at one point, whereas the general fiber is a cubic curve, punctured at two points.) One would like to identify these “critical values” where the topology changes and their corresponding “critical points at infinity”. We first review the history of this subject.

Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a complex polynomial. There is a finite set \( \Sigma \in \mathbb{C} \) such that

\[
\mathbb{C}^n - f^{-1}(\Sigma) \to \mathbb{C} - \Sigma
\]

is a fibration. This is a form of Sard’s theorem for polynomials; the set \( \Sigma \) is finite because it is algebraic. For a proof, see [Bro83, Proposition 1] (based on work of Verdier), [Pha83, Appendix A1], [HL84, Theorem 1] or [Ha89]. We let

\[
\Sigma = \Sigma_{fin} \cup \Sigma_{\infty}
\]
where $\Sigma_{fin}$ is the set of critical values coming from critical points in $\mathbb{C}^n$, and $\Sigma_{\infty}$ is the set of critical values "coming from infinity". Of course these two sets may have nonempty intersection.

Broughton in [Bro83, Bro88] calls the polynomial $f$ tame if there is a $\delta > 0$ such that the set \{ $x : |\text{grad} f(x)| \leq \delta$ \} is compact. He proved that if $f$ is tame, then $\Sigma_{\infty}$ is empty.

Thus if the gradient of a polynomial goes to zero along some path going to infinity, then something bad may happen. Topics surrounding the gradient of the polynomial are treated in Section 4 of this paper. The speed at which the gradient of $f$ goes to zero is measured by the Lojasiewicz number at infinity; see [Ha90, Ha91a, Ha94, CNH94, CNH95].

There followed many efforts in the case $n = 2$ to identify the set $\Sigma_{\infty}$ more precisely. Suzuki [Suz74, Corollary 1] provides an estimate on the number of points in $\Sigma$. In [HL84] it is shown that $c \in \Sigma$ if and only if $\chi(f^{-1}(c)) \neq \chi(f^{-1}(t))$, where $f^{-1}(t)$ is a generic fiber of $f$ and $\chi$ denotes Euler characteristic. Further work on identifying $\Sigma_{\infty}$ can be found in [HN89, Ha89, NZ90, NZ92, LO93].

The homology and homotopy of the fibers of the polynomial $f$ were also computed, leading to various numerical invariants which will be discussed in the Section 2 of this paper. Suzuki [Suz74, Proposition 2] shows that

$$\text{rank } H_1(f^{-1}(t)) = \mu + \lambda$$

where $f^{-1}(t)$ is a generic fiber, $\mu$ is the sum of the Milnor numbers at the critical points of $f$ in $\mathbb{C}^2$, and $\lambda$ is the sum of all the "jumps" in the Milnor numbers at infinity. (In the terminology of Section 2, $\lambda = \sum \nu_{p,c}$, where the sum is over $c \in \mathbb{C}$ and $p \in \mathbb{L}_\infty$, and $\nu_{p,c}$ is the jump in the Milnor number at the point $p \in \mathbb{L}_\infty$ and value $c \in \mathbb{C}$.) More on the topology of the fiber can be found in [BCND96].

The polynomial $f$ extends to a function on projective space $\mathbb{P}^2$ which is well defined except at a finite number of points. The points of indeterminacy can be easily resolved, and the structure of the resolution contains information about these points [LW95, LW96]. These topics are discussed in Section 3.

Other topics investigated (but not discussed in this paper) include Newton diagrams [Bro88, Proposition 3.4], [NZ90, CN96], knots [Neu89, Ha91b], and the Jacobian conjecture [LW95].
Papers in higher dimensions (that is, \( n > 2 \)) include \[ \text{Par95, ST95, Tib96} \]. Broughton \[ \text{Bro88, Proposition 3.2} \] shows that the tame polynomials form a dense constructible set in the set of polynomials of a given degree; Cassou-Nogues \[ \text{CN96, Example V} \] gives an example to show that this set is not open in dimension \( n = 3 \).

Although the scene in higher dimensions is not yet settled, the situation in dimension two is now clear. The purpose of this paper is to collect together five definitions of “critical point at infinity” in this low-dimensional case and prove that they are equivalent. These definitions have appeared in the literature in some form or other, usually in a global affine context; the purpose of this paper is to give these definitions and prove their equivalence in a purely local setting near a point on the line at infinity. Many examples are also given. It should be noted that this material can be tricky, despite its apparent simplicity, and one should take care to make precise statements and proofs as well as to check examples.

If \( f(x, y) \) is a polynomial, \( p \in \mathbb{A}_\infty^2 \) is a point through which the level curves of \( f \) pass, and \( c \in \mathbb{C} \), we say that the pair \( (p, c) \) is a regular point at infinity for \( f(x, y) \) if it satisfies any one of the following equivalent conditions. (Otherwise it is a critical point at infinity.)

- **Condition M** (2.15): There is no jump in the Milnor number (2.1):
  \[ \nu_{p,c} = 0. \]
- **Condition E** (2.16): The family of germs \( f(x, y) = tz^d \) at \( p \) is equisingular at \( t = c \).
- **Condition F** (2.17): The map \( f \) is a smooth fiber bundle near \( p \) and the value \( c \).
- **Condition R** (3.1): There is a resolution \( \tilde{f} : M \to \mathbb{P}^1 \) with \( \pi : M \to \mathbb{P}^2 \) and a neighborhood \( U \) of \( p \in \mathbb{P}^2 \) such that \( \{ \tilde{f} = c \} \cap \pi^{-1}(U) \) is smooth and intersects the exceptional set \( \pi^{-1}(p) \) transversally.
- **Condition G** (4.1): There does not exist a sequence of points \( \{p_k\} \in \mathbb{C}^2 \) with \( p_k \to p \), \( \text{grad } f(p_k) \to 0 \) and \( f(p_k) \to c \) as \( k \to \infty \).

Most of these equivalences are well known; we give either proofs or references for proofs in the pages that follow.
There are several new results in this paper. First, we define an invariant \( \nu_{p,\infty} \) which measures the number of vanishing cycles at a point \( p \) on the line at infinity for the critical value infinity, and show that this invariant has many of the same properties that \( \nu_{p,c} \) does for \( c \in \mathbb{C} \). Secondly, we define \( g_{p,c} \) to be the number of isotopy classes of paths \( \alpha : \mathbb{R} \to \mathbb{C}^2 \) such that \( \alpha(t) \to p \), \( \text{grad} \, f(\alpha(t)) \to 0 \) and \( f(\alpha(t)) \to c \) as \( t \to +\infty \). We use this to give a new proof that Condition M implies Condition G. In fact, we will show (Proposition 4.12) that \( \nu_{p,c} \geq g_{p,c} \).

The work described in this paper started in 1989 when the author supervised a group of undergraduates in the Mount Holyoke Summer Research Institute in Mathematics who were working on corresponding problems for real polynomials. These results are described in [DKM’93], with further results in [Dur]. The work for this paper was carried out at the Tata Institute, Bombay; Martin-Luther University, Halle (with support from IREX, the International Research and Exchanges Board), the University of Nijmegen, Warwick University, the Massachusetts Institute of Technology and the University of Bordeaux. The author would like to thank all of them for their hospitality.

Earlier versions of this paper included results on deformations of critical points at infinity; these will appear elsewhere.

2 Numerical Invariants

We will use coordinates \((x, y)\) for the complex plane \( \mathbb{C}^2 \), and coordinates \([x, y, z]\) for the projective plane \( \mathbb{P}^2 \). We let

\[ \mathbb{L}_\infty = \{[x, y, z] \in \mathbb{P}^2 : z = 0\} \]

be the line at infinity.

We let \( d \) be the degree of the polynomial \( f(x, y) \). We let \( f_d \) denote the homogeneous term of degree \( d \) in \( f \). If \( p = [a, b, 0] \in \mathbb{L}_\infty \), we let \( d_p \) be the multiplicity of the factor \((bx - ay)\) in \( f_d \).

Suppose that the level sets of \( f \) intersect \( \mathbb{L}_\infty \) at \( p \). Let

\[ F_t(x, y, z) = z^d f(x/z, y/z) - tz^d \]

be the homogenization of the polynomial \( f(x, y) - t \), where \( t \in \mathbb{C} \), and let \( g_{p,t} \) be the local equation of \( F_t \) at \( p \). If \( p = [1, 0, 0] \), then \( g_{p,t} \) is given in local
coordinates
\[(u, v) = (y/x, 1/x)\]
by
\[g_{p,t}(u, v) = F_t(1, u, v) = v^d f(1/v, u/v) - tv^d\]

Note that the multiplicity of \(g_{p,t}\) at \((0, 0)\) is at most \(d_p\).

**Definition 2.1.** The Milnor number \(\mu_{p,t}\) of \(f(x, y)\) at \((p, t)\) \(\in \mathbb{L}_\infty \times \mathbb{C}\) is the Milnor number of the germ \(g_{p,t}\) at \((0, 0)\) in the usual sense. The generic Milnor number \(\mu_{p,\text{gen}}\) is the Milnor number \(\mu_{p,t}\) for generic \(t\). The number of vanishing cycles at \((p, t)\) is
\[\nu_{p,t} = \mu_{p,t} - \mu_{p,\text{gen}}\]

**Example 2.2.** Let \(f(x, y) = y(xy - 1)\) and \(p = [1, 0, 0]\). Then \(g_{p,t}(u, v) = u^2 - uv^2 - tv^3\). We have \(\mu_{p,\text{gen}} = 2\), \(\nu_{p,0} = 1\), and all other \(\nu_{p,t} = 0\). In fact, for \(t \neq 0\), the singularity is of type \(A_2\), and for \(t = 0\), the singularity is of type \(A_3\). This well-known example is the simplest “critical point at infinity”. More generally, if \(f(x, y) = y(x^a y - 1)\) then for \(t \neq 0\), \(\mu_{p,t} = a + 1\) and there is a singularity of type \(A_{a+1}\). For \(t = 0\), \(\mu_{p,0} = 2a + 1\) and there is a singularity of type \(A_{2a+1}\).

**Example 2.3.** Let \(f(x, y) = x(y^2 - 1)\) and \(p = [1, 0, 0]\). Then \(g_{p,t}(u, v) = u^2 - v^2 - tv^3\). For all \(t\), \(\mu_{p,t} = 1\); the family is equisingular, and there is no “critical point at infinity”. This is another basic example.

**Example 2.4.** Here is a more complicated example (see [DKM+93, Dur]): Let \(f(x, y) = (xy^2 - y - 1)^2 + (y^2 - 1)^2\). At \(p = [1, 0, 0]\) we have \(\mu_{\text{gen}} = 15\), \(\nu_{p,1} = 2\), \(\nu_{p,2} = 1\) and \(\nu_{p,c} = 0\) for all other \(c\).

Next we relate \(\nu_{p,t}\) to homological vanishing cycles. Fix \(p \in \mathbb{L}_\infty\) and \(c \in \mathbb{C} \cup \{\infty\}\). Let \(U \subset \mathbb{C}^2\) be an open set such that the closure in projective space of the set
\[\{(x, y) \in \mathbb{C}^2 : (x, y) \in U \text{ and } f(x, y) = t\}\]
is \(p\) for \(t\) near \(c\). Choose \(C > 0\) large. We define the Milnor fiber of \(f\) at \((p, c)\) to be
\[\tilde{F}_{p,c} = \{(x, y) \in \mathbb{C}^2 : (x, y) \in U \text{ and } |(x, y)| \geq C \text{ and } f(x, y) = t\}\]
where the overbar indicates closure in projective space, and, if \( c \in \mathbb{C} \), then \( t \) is near, but not equal to, \( c \), and if \( c = \infty \), then \( t \) is large.

**Proposition 2.5.** For \( p \in L_\infty \) and \( c \in \mathbb{C} \),

\[
\nu_{p,c} = \text{rank } H_1(\tilde{F}_{p,c})
\]

**Proof.** Without loss of generality, we may assume that \( p = [1, 0, 0] \). The number \( \nu_{p,c} \) is the difference of the Milnor number \( \mu_{p,c} \) and the generic Milnor number \( \mu_{p,\text{gen}} \). The number \( \mu_{p,\text{gen}} \) is the Milnor number of \( g_{p,t} \) for \( t \) near, but not equal to, \( c \). By the usual argument, this difference is \( \text{rank } H_1(\{g_{p,c} = 0\} \cap B_0) \) where \( B_0 \) is the small ball for the Milnor number of \( g_{p,t} \). We may replace \( \{g_{p,c} = 0\} \cap B_0 \) by

\[
F'_{p,c} = \{(u, v) \in \mathbb{C}^2 : |v| \leq \epsilon' \text{ and } g_{p,t}(u, v) = 0\}
\]

We may replace \( \tilde{F}_{p,c} \) by

\[
\tilde{F}'_{p,c} = \{(x, y) \in \mathbb{C}^2 : (x, y) \in \mathcal{U} \text{ and } |x| \geq C \text{ and } f(x, y) = t\}
\]

The change of coordinates \( x = 1/v \) and \( y = u/v \) takes \( \tilde{F}_{p,c} \) to \( F'_{p,c} \).

To define “vanishing cycles” for the critical value \( c = \infty \), we take the above proposition to be a definition:

**Definition 2.6.** For \( p \in L_\infty \) we let

\[
\nu_{p,\infty} = \text{rank } H_1(\tilde{F}_{p,\infty})
\]

**Remark 2.7.** Here is a topological interpretation of the number of vanishing cycles at infinity: Suppose the level curves of the polynomial \( f \) of degree \( d \) intersect \( L_\infty \) at \( k \) points (counted without multiplicities). Then \( \nu_{p,\infty} = 0 \) for all \( p \in L_\infty \) if and only if \( \{f(x, y) = t\} \) for \( t \) large is homeomorphic to a \( d \)-fold cover of \( L_\infty \) branched at \( k \) points. For example, \( y(xy - 1) = t \) (where \( \nu_{p,\infty} = 0 \) for all \( p \)) is a three-fold cover of \( \mathbb{P}^1 \) branched at two points, but \( y^2 - x = t \) (where \( \nu_{[1,0,0],\infty} = 1 \)) is not a two-fold cover of \( \mathbb{P}^1 \) branched at one point.
Next we describe three ways of computing the number of vanishing cycles \( \nu_{p,c} \). The first is to compute (perhaps with a computer algebra program) \( \mu_{p,c} \) and \( \mu_{p,\text{gen}} \) and subtract. The second is by counting nondegenerate critical points, as described in the proposition below. This is similar to computing the usual Milnor number by counting the number of nondegenerate critical points in a Morsification (see [AGZV85, vol II, p. 31]), and the proof is similar.

**Proposition 2.8.** Let \( p \in L_\infty \) and \( c \in \mathbb{C} \cup \{\infty\} \). The number \( \nu_{p,c} \) is equal to the number of critical points (assumed nondegenerate) \( q \neq (0,0) \) of the function \( g_{p,t} \) such that \( q \to (0,0) \) as \( t \to c \).

**Example 2.9.** Let \( f(x, y) = y(xy - 1) \) and \( p = [1, 0, 0] \). Then \( g_{p,t}(u, v) = u^2 - uv^2 - tv^3 \). For \( t \neq 0 \) the function \( g_{p,t} \) has a (degenerate) critical point at \((0,0)\) with critical value 0, and a nondegenerate critical point at \(((9/2)t^2, -3t)\) with critical value \((27/4)t^4\). As \( t \to 0 \) the second critical point approaches \((0,0)\). Thus \( \nu_{p,0} = 1 \).

**Example 2.10.** Let \( f(x, y) = x - y^2 \) and \( p = [1, 0, 0] \). Then \( g_{p,t}(u, v) = v - u^2 - tu^2 \). The function \( g_{p,t} \) has a single nondegenerate critical point at \((0, 1/(2t))\) with critical value \(1/(4t)\). As \( t \to \infty \) this critical point approaches \((0,0)\), so \( \nu_{p,\infty} = 1 \).

The next proposition describes the result of computing \( \nu_{p,\infty} \) by similar methods.

**Proposition 2.11.** For \( p \in L_\infty \),

\[
\nu_{p,\infty} = (d_p - 1)(d - 1) - \mu_{p,\text{gen}}
\]

**Proof.** Without loss of generality \( p = [1, 0, 0] \). The intersection multiplicity of the curves \((g_{p,t})_u\) and \((g_{p,t})_v\) at \((0,0)\) for \( t = \infty \), where \((u, v)\) are local coordinates at \((0,0)\), can be computed using the algorithm in [Ful69], and is found to be \((d_p - 1)(d - 1)\). (To compute the intersection multiplicity at \( t = \infty \), we let \( s = 1/t \) and compute it at \( s = 0 \).) For large \( t \neq \infty \), the intersections split into those at \((0,0)\), the number of which is \( \mu_{p,\text{gen}} \), and those not at \((0,0)\), the number of which is \( \nu_{p,\infty} \). \( \square \)
Example 2.12. The polynomial \( f(x, y) = y^a + x^{a-2}y + x \) has \( \nu_{p, \infty} = a^2 - 2a \) at the point \( p = [1, 0, 0] \), and all other \( \nu_{p, c} = 0 \).

Finally, \( \nu_{p, t} \) can computed a third way by using polar curves, as described below. (See [HN89, 1.6, 1.8].) This method also shows that some vanishing cycles are easy to “see” from a contour plot, since they are where the level curves of the polynomial have a vertical tangent.

Proposition 2.13. Suppose \( p = [1, 0, 0], c \in \mathbb{C} \cup \{\infty\} \) and the level sets of \( f \) pass through \( p \). Then \( \nu_{p, c} \) is the number of points of intersection \( q \in \mathbb{C}^2 \) (assumed transverse) of the curves \( f = t \) and \( f_y = 0 \) in \( \mathbb{C}^2 \) such that \( q \to p \) as \( t \to c \).

Proof. The set \( F'_{p, c} \) from the proof of Proposition 2.5 is a connected branched cover of the disk \( |v| \leq \epsilon' \) in the \( uv \)-plane. Two sheets come together at each branch point, and all the sheets come together over \( p \). The result follows from Hurwitz’s formula.

Example 2.14. If \( f(x, y) = x(y^2 - 1) \), the curves \( f = t \) and \( f_y = 0 \) intersect at \( (-t, 0) \). As \( t \to \infty \), the intersection point \( (-t, 0) \to [1, 0, 0] \) and \( f(t, 0) \to \infty \). Thus \( \nu_{[1,0,0],\infty} = 1 \). All other \( \nu_{[1,0,0],c} = 0 \).

Next we give three definitions of “critical point at infinity”.

Definition 2.15. The polynomial \( f(x, y) \) satisfies Condition M at the point \( p \in L_\infty \) and \( c \in \mathbb{C} \cup \{\infty\} \) if \( \nu_{p, c} = 0 \).

Definition 2.16. The polynomial \( f(x, y) \) satisfies Condition E at the point \( (p, c) \in L_\infty \times \mathbb{C} \) if the family of germs \( g_{p, t} \) at \( (0,0) \) is equisingular at \( t = c \).

A proof that Condition M for \( c \in \mathbb{C} \) is equivalent to Condition E may be found at the end of [LR76].

There are various equivalent ways of specifying equisingularity; see for instance the papers by Zariski in volume IV of [Zar79]. One that will be useful for us is the following: The family of germs \( g_{p, t} \) is equisingular if the germs \( g_{p, t} = 0 \) at \( (0,0) \) form a fiber bundle near \( t = c \).

Definition 2.17. The polynomial \( f(x, y) \) satisfies Condition F at a point \( (p, c) \in L_\infty \times \mathbb{C} \) if the map \( f \) is a smooth fiber bundle near \( p \) and the value \( c \).

(More precisely, a polynomial satisfies Condition F if there is a \( U \subset \mathbb{C}^2 \) with
Let \( p \) in the closure of \( U \) in projective space and \( C > 0 \) and \( \beta > 0 \) such that, letting
\[
B = \{ t \in \mathbb{C} : |t - c| \leq \beta \}
\]
and
\[
N = \{ (x, y) \in \mathbb{C}^2 : (x, y) \in U \text{ and } |(x, y)| \geq C \text{ and } f(x, y) \in B \}
\]
then
\[
f : N \rightarrow B
\]
is a smooth fiber bundle.}

**Proposition 2.18.** A polynomial \( f(x, y) \) satisfies Condition E at a point \((p, c) \in L_\infty \times \mathbb{C}\) if and only if it satisfies Condition F at that point.

**Proof.** The proof is straightforward, and just involves replacing the “spherical” Milnor fiber by one in a “box”: Without loss of generality, we may assume that \( p = [1, 0, 0] \). We may replace Condition E by the following: There is an \( \epsilon' > 0 \) and a \( \delta' > 0 \) such that, letting
\[
D' = \{ t \in \mathbb{C} : |t - c| < \delta' \}
\]
and
\[
M' = \{ (u, v, t) \in \mathbb{C}^2 \times \mathbb{C} : |v| \leq \epsilon' \text{ and } t \in D' \text{ and } g_{p, t}(u, v) = 0 \}
\]
then the restriction of the projection to the third coordinate
\[
\pi : M' \rightarrow D'
\]
is a fiber bundle. We may do this since the germs \( g_{p, t}(u, v) = 0 \) never have \( v = 0 \) as a component.

We may also replace Condition F by the following: There is a \( U' \subset \mathbb{C}^2 \) with \( p \) in the closure of \( U' \) in projective space and \( C' > 0 \) and \( \beta' > 0 \) such that, letting
\[
B' = \{ t \in \mathbb{C} : |t - c| \leq \beta' \}
\]
and
\[
N' = \{ (x, y) \in \mathbb{C}^2 : (x, y) \in U' \text{ and } |x| \geq C' \text{ and } f(x, y) \in B' \}
\]
then
\[ f : N' \to B' \]
is a smooth fiber bundle.

The change of coordinates \( x = 1/v \) and \( y = u/v \) takes \( f(x, y) = t \) to \( g_{p,t}(u, v) = 0 \) and \( N' \) to \( M' \). □

### 3 Resolutions

The polynomial
\[ f : \mathbb{C}^2 \to \mathbb{C} \]
extends to a map
\[ \hat{f} : \mathbb{P}^2 \to \mathbb{P} \]
which is undefined at a finite number of points on the line at infinity \( L_\infty \). By blowing up these points one gets a manifold \( M \) and a map
\[ \pi : M \to \mathbb{P}^2 \]
such that the map
\[ \tilde{f} : M \to \mathbb{P} \]
which is the lift of \( \hat{f} \) is everywhere defined. We call the map \( \tilde{f} \) a resolution of \( f \). Some interesting results on the structure of resolutions are announced in [LW95, Theorems 2, 3, 4].

For example, a resolution (the minimal resolution) of \( y(xy - 1) \) is given in Figure 1. The symbol \( c^m \) next to a divisor means that at each smooth point of the divisor there are local coordinates \((z, w)\) in a neighborhood of the point such that the divisor is \( z = 0 \) and \( \tilde{f}(z, w) = (z - c)^m \). The proper transform of level curves of \( f \) have arrowheads on them; the exceptional sets do not.

Resolution are easy compute. For example, starting with \( f(x, y) = y(xy - 1) \) which we wish to resolve near \([1, 0, 0]\), the function in local coordinates at \([1, 0, 0]\) is \( u(u - v^2)/v^3 \), and we blow up in the standard fashion until it is everywhere defined. More examples are shown in Figures 2 and 3.

Next we give a condition for “regular point at infinity” in terms of a resolution. (See also [LW95, Theorem 5].)
Figure 1: Resolution of $y(xy - 1)$

Figure 2: Resolution of $x(y^2 - 1)$ at $[1, 0, 0]$
Figure 3: Resolution of $(xy^2 - y - 1)^2 + (y^2 - 1)^2$ at $[1,0,0]$
Figure 4: Resolution of $y - (xy - 1)^2$ at $[1, 0, 0]$

**Definition 3.1.** The polynomial $f(x, y)$ satisfies Condition R at a point $(p, c) \in L_\infty \times \mathbb{C}$ if there is a resolution $\tilde{f} : M \to \mathbb{P}^1$ with $\pi : M \to \mathbb{P}^2$ and a neighborhood $U$ of $p \in \mathbb{P}^2$ such that $\{\tilde{f} = c\} \cap \pi^{-1}(U)$ is smooth and intersects the exceptional set $\pi^{-1}(p)$ transversally.

**Example 3.2.** Let $f(x, y) = y - (xy - 1)^2$ near $[1, 0, 0]$ (See [Kra91]). In this example the level curve of the function $\tilde{f} = 0$ is smooth, but it does not intersect the exceptional divisor transversally; see Figure 4. Hence $(p, c) = ([1, 0, 0], 0)$ does not satisfy Condition R. (Here $\nu_{[1,0,0],0} = 1$.)

**Proposition 3.3.** A point $p \in L_\infty$ and a value $c \in \mathbb{C}$ for a polynomial $f(x, y)$ satisfies Condition E if and only if it satisfies Condition R.

**Proof.** Suppose $(p, c)$ satisfies Condition E. Let $U$ be a neighborhood of $p$ in $\mathbb{P}^2$ containing no critical points of $f$ in $\mathbb{C}^2$ or points on $L_\infty$ though which
the level curves of \( f \) pass. Find a resolution \( \tilde{f} \) of \( f \). By further blowing up (if necessary), we may assume that \( \tilde{f}^{-1}(c) \) is a divisor with normal crossings transversally intersecting the exceptional set where \( \tilde{f} \) is not constant. Equisingularity in the form of Zariski's (b)-equivalence [Zar65, p. 513] implies that the functions \( g_{p,t} \) for \( t \) near \( c \) have the same resolution as the function \( g_{p,c} \). This can only happen if \( \{ \tilde{f}^{-1}(t) \cap \pi^{-1}(U) \} \) is smooth and transversally intersects the exceptional set \( \pi^{-1}(p) \). Thus \( p \) and \( c \) satisfy Condition R.

Conversely, if \( p \) and \( c \) satisfy Condition R, then the resolutions of \( \{ g_{p,t} = 0 \} \) for \( t \) near \( c \) are (b)-equivalent and hence equisingular. □

4 The Gradient

If \( f \) is a complex polynomial, we define \( \text{grad } f \) as in [Mil68] to be the complex conjugate of the vector of partial derivatives.

Of course \( p \in \mathbb{C}^2 \) is a regular point for a function \( f \) with regular value \( c \in \mathbb{C} \) if \( f(p) = c \) and \( \text{grad } f(p) \neq 0 \). An equivalent definition would be to say that there is no sequence of points \( \{ p_k \} \) with \( p_k \to p \), \( \text{grad } f(p_k) \to 0 \) and \( f(p_k) \to c \) as \( k \to \infty \). We can now imitate this definition for \( p \in \mathbb{L}_\infty \) as follows:

**Definition 4.1.** The polynomial \( f(x, y) \) satisfies Condition G at a point \( p \in \mathbb{L}_\infty \) and \( c \in \mathbb{C} \cup \{ \infty \} \) if there does not exist a sequence of points \( \{ p_k \} \in \mathbb{C}^2 \) with \( p_k \to p \), \( \text{grad } f(p_k) \to 0 \) and \( f(p_k) \to c \) as \( k \to \infty \).

If \( (p, c) \) does not satisfy Condition G, then a version of Milnor's curve selection lemma (see for instance [Ha91a, Lemma 3.1] or [NZ92, Lemma 2]) implies that the sequence of points can be replaced by a curve:

**Lemma 4.2.** If \( (p, c) \) does not satisfy Condition G, then there is a smooth real algebraic curve \( \alpha : \mathbb{R}^+ \to \mathbb{C}^2 \) such that \( \alpha(t) \to p \), \( \text{grad } f(\alpha(t)) \to 0 \) and \( f(\alpha(t)) \to c \) as \( t \to +\infty \).

By “real algebraic curve” we mean that the image of \( \alpha \) in \( \mathbb{C}^2 \) is contained in an irreducible component of the zero locus of a real polynomial.

**Example 4.3.** Let \( f(x, y) = y(xy - 1) \). Let \( \alpha(t) = (t, 1/(2t)) \). As \( t \to +\infty \), \( \alpha(t) \to [1, 0, 0] \), the gradient of \( f \) goes to 0 and the value of the function approaches 0.
Example 4.4. (c.f. Example 2.4.) Let \( f(x, y) = (xy^2 - y - 1)^2 + (y^2 - 1)^2 \).
Let \( \alpha(t) = (t + t^2, \pm 1/t) \). As \( t \to +\infty \), \( \alpha(t) \to [1, 0, 0] \), the gradient of \( f \) goes to 0 and the function approaches the value 1.

If \( \beta(t) = (t/2, 1/t) \), then \( \beta(t) \to [1, 0, 0] \), the gradient of \( f \) goes to 0 and the function approaches the value 2. (These paths were found by Ian Robertson in the Mount Holyoke REU program in the summer of 1992.)

Example 4.5. If \( f(x, y) = x^2y + xy^2 + x^5y^3 + x^3y^5 \) along the curve \( y^2x^3 = -1/3 \), then \( \nabla f(q) \to 0 \) and \( f(q) \to \infty \). Here \( v_{p,\infty} = 1 \). This polynomial is “quasi-tame” but not “tame” [NZ92]. It would be interesting to find more examples like this.

The following proposition is well-known. It was first proved in the global case by Broughton [Bro88]; see also [NZ90], proof of Theorem 1, and [ST95], proof of Proposition 5.5. The idea of the proof is to use integral curves of the vector field \( \nabla f/|\nabla f|^2 \) to identify the fibers.

Proposition 4.6. If a polynomial \( f(x, y) \) satisfies Condition G at \((p, c) \in L_\infty \times \mathbb{C}\), then it satisfies Condition F at this point.

Next we will show that Condition M implies Condition G; this has been shown in [Ha90, Ha91a, ST95, Par95]. Here we will prove a stronger result by different methods.

Definition 4.7. For \( p \in L_\infty \) and \( c \in \mathbb{C} \cup \{\infty\} \), let \( g_{p,c} \) be the number of isotopy classes of smooth real algebraic curves \( \alpha : \mathbb{R} \to \mathbb{C}^2 \) such that \( \alpha(t) \to p \), \( \nabla f(\alpha(t)) \to 0 \) and \( f(\alpha(t)) \to c \) as \( t \to +\infty \).

Example 4.8. If \( f(x, y) = y^5 + x^2y^3 - y \) and \( p = [1, 0, 0] \), then \( v_{p,0} = 2 \). There are two isotopy classes of curves approaching \( p \) along which \( \nabla f \) goes to zero, namely the ones containing the two branches of the curve \( f_y = 0 \) at \( p \). Hence \( g_{p,0} = 2 \). (This example is from [DKM+93].)

Clearly \((p, c) \in L_\infty \times (\mathbb{C} \cup \{\infty\})\) satisfies Condition G if and only if \( g_{p,c} = 0 \).
Now let \( \pi : M \to \mathbb{P}^2 \) be a resolution of \( f, f_x, \) and \( f_y \) (so that \( \tilde{f}, \tilde{(f_x)} \) and \( \tilde{(f_y)} \) are defined on \( M \)), and let \( G_{p,c} = \{q \in M : \pi(q) = p, \tilde{f}(q) = c, \tilde{(f_x)}(q) = 0 \text{ and } \tilde{(f_y)}(q) = 0\} \)
Definition 4.9. For \( p \in \mathbb{L}_\infty \) and \( c \in \mathbb{C} \cup \{\infty\} \), let \( \tilde{g}_{p,c} \) be the number of connected components of \( G_{p,c} \).

The number \( \tilde{g}_{p,c} \) is independent of the resolution by the usual argument.

Example 4.10. In the minimal resolution of \( f(x, y) = y(xy - 1) \) at \( p = [1, 0, 0] \) (Figure 1), the functions \( f_x \) and \( f_y \) are defined. The zero locus of the lift of \( f_x \) contains the exceptional set where the lift of \( f \) is zero, and the zero locus of the lift of \( f_y \) intersects this set transversally. Thus \( G_{p,0} \) consists of a single point, and \( \tilde{g}_{p,0} = 1 \). If \( f(x, y) = y^5 + x^2y^3 - y \) and \( p = [1, 0, 0] \), one finds similarly that \( G_{p,0} \) consists of two points.

The two definitions are equivalent by the following proposition.

Proposition 4.11. For \( p \in \mathbb{L}_\infty \) and \( c \in \mathbb{C} \cup \{\infty\} \), \( g_{p,c} = \tilde{g}_{p,c} \).

Proof. Let \( \pi : M \to \mathbb{P}^2 \) be a resolution of \( f, f_x \) and \( f_y \). We will show that there is a one-one correspondence between isotopy classes of curves satisfying (4.7) and connected components of \( G_{p,c} \). Suppose that \( \alpha : \mathbb{R}^+ \to \mathbb{C}^2 \) is a smooth real algebraic curve satisfying the conditions of (4.7). Since \( \alpha \) is real algebraic, it lifts to a map \( \tilde{\alpha} : \mathbb{R}^+ \cup \{\infty\} \to M \) with \( \tilde{\alpha}(\infty) \in \pi^{-1}(\mathbb{L}_\infty) \). Let \( q = \tilde{\alpha}(\infty) \). Then \( \tilde{f}(q) = c \) and \( (\tilde{f}_x)(q) = 0 \) and \( (\tilde{f}_y)(q) = 0 \). Thus \( q \in G_{p,c} \).

If \( \alpha_0 \) is isotopic to \( \alpha_1 \) through curves \( \alpha_t \) satisfying (4.7), then the curves \( \alpha_t \) lift to \( M \) and are isotopic. In particular, \( \tilde{\alpha}_0(\infty) \) and \( \tilde{\alpha}_1(\infty) \) are in the same connected component of \( G_{p,c} \).

For each \( q \in G_{p,c} \) there is an algebraic curve \( \tilde{\alpha} : \mathbb{R}^+ \cup \{\infty\} \to M \) with \( \tilde{\alpha}(\infty) = q \) and \( \tilde{\alpha}(\mathbb{R}^+) \subset \pi^{-1}(\mathbb{C}^2) \). Let \( \alpha = \pi \circ \tilde{\alpha} : \mathbb{R}^+ \to \mathbb{C}^2 \). Then \( \alpha \) satisfies the conditions of (4.7). If \( q_0, q_1 \in G_{p,c} \), then there are two such curves \( \tilde{\alpha}_0, \tilde{\alpha}_1 \). If \( q_0 \) and \( q_1 \) are in the same connected component of \( G_{p,c} \), then \( \tilde{\alpha}_0 \) is isotopic to \( \tilde{\alpha}_1 \) through a family of such curves \( \tilde{\alpha}_t \). Hence \( \alpha_0 \) is isotopic to \( \alpha_1 \) through curves satisfying (4.7). If \( q_0 \) and \( q_1 \) are in different connected components, then \( \alpha_0 \) is not isotopic to \( \alpha_1 \) through curves satisfying (4.7). Thus \( g_{p,c} = \tilde{g}_{p,c} \).

\[ \square \]

Proposition 4.12. For \( p \in \mathbb{L}_\infty \) and \( c \in \mathbb{C} \cup \{\infty\} \),
\[ \nu_{p,c} \geq g_{p,c} \]
Proof. We will show that $\nu_{p,c} \geq \tilde{g}_{p,c}$ and will use Proposition 2.13 to compute $\nu_{p,c}$. We may assume without loss of generality that $p = [1, 0, 0]$. Pick a connected component $G'$ of $G_{p,c}$. Let $t$ be near $c$. We will show that $f = t$ intersects $f_y = 0$ in $\mathbb{C}^2$ near $G'$.

There is a $q \in G'$ and a component $C$ of $f_y = 0$ in $\mathbb{C}^2$ such that $q$ is in the closure of $C$ in $M$: We have that $\tilde{f}_y = 0$ on $G'$. Blow down $G'$ to a point $q'$, and let $E'$ be the image of $\pi^{-1}(p)$. Then the lift of $f_y$ is not constant on $E'$ near $q'$, so there is a component of $f_y = 0$ passing through $q'$. Lift this component back to $M$.

Next, $f$ is not constant on $C$: If it were, then the gradient vector of $f$ would be horizontal, so $C$ would be of the form $x = \text{const}$, and $p$ would not be in the closure of $C$.

Thus $f = t$ intersects $C$ near $q$ for small $\epsilon \neq 0$, and the intersection points are in $\mathbb{C}^2$. □

Remark 4.13. The inequality of the proposition can be strict, as it is for the polynomial $y(x^2y - 1)$ at $p = [1, 0, 0]$ and $c = 0$, where $\nu_{p,c} = 2$ and $g_{p,c} = 1$.

Corollary 4.14. If $p \in \mathbb{L}_\infty$ and $c \in \mathbb{C} \cup \{\infty\}$ satisfy Condition $M$, then they satisfy Condition $G$.

Remark 4.15. The converse to this corollary is not true for $c = \infty$: For example, the polynomial $x(y^2 - 1)$ has a gradient whose magnitude is bounded below for large $x$ and hence satisfies Condition $G$ at $p = [1, 0, 0]$, yet $\nu_{p,\infty} = 1$.

References

[AGZV85] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. Singularities of Differentiable Maps. Birkhauser, Boston, 1985.

[BCND96] E. Artal Bartolo, Pierrette Cassou-Nogues, and Alexandru Dimca. Sur la topologie des polynomes a deux variables complexes. Preprint, University of Bordeaux, 1996.
[Bro83] S. A. Broughton. On the topology of polynomial hypersurfaces. In P. Orlik, editor, *Proceedings of Symposia in Pure Mathematics, Vol. 40*, pages 167–178, Providence RI, 1983. American Mathematical Society.

[Bro88] S. A. Broughton. Milnor numbers and the topology of polynomial hypersurfaces. *Inventiones Math.*, 92:217–241, 1988.

[CN96] Pierrette Cassou-Nogues. Sur la generalisation d’un theoreme de Kouchnirenko. Preprint, University of Bordeaux, 1996.

[CNH94] Pierrette Cassou-Nogues and Huy Vui Ha. Theoreme de Kuiper-Kuo-Bochnak-Lojasiewicz a l’infini. Preprint, Universite Bordeaux, 1994.

[CNH95] Pierrette Cassou-Nogues and Huy Vui Ha. Sur le nombre de Lojasiewicz a l’infiniti d’un polynome. *Annales Polonici Math.*, 62:23–44, 1995.

[DKM+93] A. Durfee, N. Kronenfeld, H. Munson, J. Roy, and I. Westby. Counting critical points of a real polynomial in two variables. *American Math Monthly*, 100:255–271, 1993.

[Dur] Alan H. Durfee. The index of $\operatorname{grad} f(x,y)$. Duke alg-geom preprint 9506002. To appear (in revised form), *Topology*.

[Ful69] William Fulton. *Algebraic Curves*. Benjamin, New York, 1969.

[Ha89] Huy Vui Ha. Sur la fibration globale des polynomes de deux variables complexes. *C. R. Acad. Sci Paris*, 309:231–234, 1989.

[Ha90] Huy Vui Ha. Nombres de Lojasiewicz et singularites a l’infini des polynomes de deux variables complexes. *C. R. Acad. Sci Paris*, 311:429–432, 1990.

[Ha91a] Huy Vui Ha. On the irregular at infinity algebraic plane curves. Preprint, Mathematical Institute, Hanoi, 1991.

[Ha91b] Huy Vui Ha. Sur l’irregularite du diagramme splice pour l’entrelacement a l’infini des courbes planes. *C. R. Acad. Sci Paris*, 313:277–280, 1991.
[Ha94] Huy Vui Ha. A version at infinity of the Kuiper-Kuo theorem. _Acta Math Vietnamica_, 19:3–12, 1994.

[HL84] Huy Vui Ha and Dung Trang Le. Sur la topologie des polynômes complexes. _Acta Math Vietnamica_, 9:21–32, 1984.

[HN89] Huy Vui Ha and Le Anh Nguyen. Le comportement géométrique à l’infini des polynômes de deux variables complexes. _C. R. Acad. Sci Paris_, 309:183–186, 1989.

[Kra91] T. Krasinski. On branches at infinity of a pencil of polynomials in two complex variables. _Ann. Polon. Math._, 55:213–220, 1991.

[LO93] Van Thanh Le and Mutsuo Oka. Estimation of the number of the critical values at infinity of a polynomial function $f : C^2 \to c$. Preprint, Tokyo Institute of Technology, 1993.

[LR76] Dung Trang Le and C. P. Ramanujam. The invariance of Milnor number implies the invariance of the topological type. _Amer. J. Math_, 98:67–78, 1976.

[LW95] Dung Trang Le and Claude Weber. Polynomes a fibres rationnelles et conjecture jacobienne a 2 variables. _C. R. Acad. Sci Paris_, 320:581–584, 1995.

[LW96] Dung Trang Le and Claude Weber. Equisingularité dans les pinceaux de germes de courbes planes et $c^0$-suffisance. Preprint, University of Geneva, 1996.

[Mil68] John Milnor. _Singular points of complex hypersurfaces_. Princeton University Press, Princeton, 1968.

[Neu89] Walter D. Neumann. Complex algebraic plane curves via their links at infinity. _Invent. math._, 98:445–489, 1989.

[NZ90] A. Némethi and A. Zaharia. On the bifurcation set of a polynomial function and Newton boundary. _Publ. Math RIMS Kyoto_, 26:681–689, 1990.

[NZ92] A. Némethi and A. Zaharia. Milnor fibration at infinity. _Indag Math_, 3:323–335, 1992.
Adam Parusinski. A note on singularities at infinity of complex polynomials. Preprint no. 426, University of Nice-Sophia-Antipolis, 1995.

Frederic Pham. Vanishing homologies and the $n$ variable saddle-point method. In P. Orlik, editor, Proceedings of Symposia in Pure Mathematics, Vol. 40, pages 319–333, Providence RI, 1983. American Mathematical Society.

Dirk Siersma and Mihai Tibar. Singularities at infinity and their vanishing cycles. Duke Math. J., 80:771–783, 1995.

Masakazu Suzuki. Proprietes topologiques des polynomes de deux variables complexes, et automorphismes algebriques de l’espace $c^2$. J. Math.Soc. Japan, 26:141–157, 1974.

Mihai Tibar. Topology at infinity of polynomial mappings and Thom regularity condition. Preprint, University of Angers, 1996.

O. Zariski. Studies in equisingularity i. Amer. J. Math., 87:507–536, 1965.

Oscar Zariski. Collected Papers. The MIT Press, Cambridge, 1979.

Department of Mathematics
Mount Holyoke College
South Hadley, MA 01075
email: adurfee@mtholyoke.edu