A variation of Euler’s approach to values of the Riemann zeta function

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Dedicated to Leonhard Euler on his 296th birthday

1. Introduction

Sometime around 1740, Euler [3, 4, 5] took a decisive step to unraveling the functional
equation of the Riemann zeta function when he discovered a marvelous method of calculating
the values of the (absolutely!) divergent series

\[ 1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}, \]
\[ 1 + 2 + 3 + 4 + 5 + \cdots = -\frac{1}{12}, \]
\[ 1 + 4 + 9 + 16 + 25 + \cdots = 0, \]
\[ 1 + 8 + 27 + 64 + 125 + \cdots = \frac{1}{120}, \text{ etc.} \]

In modern terms, these are the values at non-positive integer arguments of the Riemann zeta
function \( \zeta(s) \), which is defined by the series, absolutely convergent in \( \text{Re}(s) > 1 \):

\[ \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots. \]

Naturally, having no notion of the analytic continuation at that time, to say nothing of
functions of complex variable, Euler had to find a way of giving a meaning to those values
of divergent series. What he actually did proceeds as follows. First, he directs his attention
to “less divergent” alternating series

\[ 1 - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \cdots \]

since its convergent counterpart

\[ 1 - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \frac{1}{5^m} - \frac{1}{6^m} + \frac{1}{7^m} - \frac{1}{8^m} + \cdots \]

does indeed have faster convergence and is linked with the original series by the simple
relation

\[ (1) \quad \tilde{\zeta}(s) = (1 - 2^{1-s}) \zeta(s). \]

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where we have put
\[ \tilde{\zeta}(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \cdots. \]
Then, he observes that the value \( \odot \) is obtained as a “limit” of the power series
\[ (2) \quad 1^m - 2^m x + 3^m x^2 - 4^m x^3 + 5^m x^4 - 6^m x^5 + \text{etc.} \]
as \( x \to 1 \), since, although the series itself converges only for \( |x| < 1 \), it has an expression as a rational function (analytic continuation, as we now put it), finite at \( x = 1 \), which is obtained by a successive application of multiplication by \( x \) and differentiation (or equivalently, applying the Euler operator \( x \cdot d/dx \) successively after once multiplied by \( x \)) to the geometric series expansion
\[ (3) \quad \frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots, \quad (|x| < 1). \]
For instance, if we substitute \( x = 1 \) in (3), we find formally
\[ \frac{1}{2} = 1 - 1 + 1 - 1 - \cdots = \tilde{\zeta}(0) \]
and hence, in view of (1), we have \( \zeta(0) = -\frac{1}{2} \). A few more examples are
\[
\begin{align*}
\frac{1}{(1 + x)^2} &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots, \\
\frac{1 - x}{(1 + x)^3} &= 1 - 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 - \cdots, \\
\frac{1 - 4x + x^2}{(1 + x)^4} &= 1 - 2^3x + 3^3x^2 - 4^3x^3 + 5^3x^4 - \cdots, \text{etc.}
\end{align*}
\]
which give us
\[ \tilde{\zeta}(-1) = \frac{1}{4}, \quad \tilde{\zeta}(-2) = 0, \quad \tilde{\zeta}(-3) = -\frac{1}{8}, \cdots \]
and in turn
\[ \zeta(-1) = -\frac{1}{12}, \quad \zeta(-2) = 0, \quad \zeta(-3) = \frac{1}{120}, \cdots. \]
For all that splendid idea however, this method unfortunately provides no rigorous way to establish the values of \( \zeta(-m) \) as values of the analytically continued function \( \zeta(s) \) at \( s = -m \). (This may only be an afterthought and merely shows the degree to which Euler was ahead of his time and how much our modern point of view owes to him.)

In the present article, aiming to evaluate the value \( \zeta(-m) \) in as elementary, and yet rigorous, way as possible as the value of analytically continued function \( \zeta(s) \), we introduce and investigate a new \( q \)-analogue of the Riemann zeta function. As becomes clear in the course of our study, this function serves very well for the purpose not only of computing \( \zeta(-m) \) but also of giving a nice \( q \)-analogue of \( \zeta(s) \) valid for all \( s \in \mathbb{C} \).
To be more specific, as an alternative for a series like \((2)\), we put \(x = q^t\) (note \(x \cdot d/dx\) is essentially \(d/dt\)) and, instead of repeating differentiation (this inevitably restricts us to looking only at the integer arguments), we replace \(n^n\) by the \(q\)-integer \([n]_q := (1 - q^n)/(1 - q)\) raised by the power \(-s\) (recall that Euler is the grand “Master of \(q^\)!”); namely, we consider the series

\[
f_q(s, t) := \sum_{n=1}^{\infty} \frac{q^{nt}}{[n]_q^n} = \frac{q^t}{[1]_q^s} + \frac{q^{2t}}{[2]_q^s} + \frac{q^{3t}}{[3]_q^s} + \frac{q^{4t}}{[4]_q^s} + \cdots.
\]

Throughout the paper, we always assume \(0 < q < 1\), so the series \((4)\) converges absolutely for any \(s \in \mathbb{C}\) and \(\text{Re}(t) > 0\). If \(\text{Re}(s) > 1\) (and \(\text{Re}(t) > 0\)), the series obviously converges to \(\zeta(s)\) when \(q \uparrow 1\). This suggests that we should regard the function \(f_q(s, t)\) as a \(q\)-analogue of the Riemann zeta function \(\zeta(s)\), but we reserve this until we make the specialization \(t = s - 1\) which turns out to be utterly crucial. Before going into the specialization, we establish below the meromorphic continuation of \(f_q(s, t)\) as a function of two variables \(s\) and \(t\), which is carried out quite easily by using the binomial theorem.

In the next section, we specialize \(t = s - 1\) and establish a formula for \(s = -m \in \mathbb{Z}_{\leq 0}\) (Proposition 2) as well as its limit when \(q \uparrow 1\) (Theorem 1). Then we give the result concerning the limit as \(q \uparrow 1\) for any \(s\) (Theorem 2).

**Proposition 1.** Let \(0 < q < 1\). As a function of \((s, t) \in \mathbb{C}^2\), \(f_q(s, t)\) is continued meromorphically via the series expansion

\[
f_q(s, t) = (1 - q)^s \sum_{r=0}^{\infty} \binom{s + r - 1}{r} \frac{q^{t+r}}{1 - q^{t+r}}
\]

having poles of order 1 at \(t \in \mathbb{Z}_{\leq 0} + 2\pi i \mathbb{Z}/ \log q := \{a + 2\pi ib/ \log q | a, b \in \mathbb{Z}, a \leq 0\}\).

**Proof.** We just apply the binomial expansion \((1 - q^n)^{-s} = \sum_{r=0}^{\infty} (s r - 1) q^{nr} \) and change the order of summations to get

\[
f_q(s, t) = (1 - q)^s \sum_{n=1}^{\infty} \frac{q^{nt}}{(1 - q^n)^s} = (1 - q)^s \sum_{n=1}^{\infty} q^{nt} \sum_{r=0}^{\infty} \binom{s + r - 1}{r} q^{nr}
\]

\[
= (1 - q)^s \sum_{r=0}^{\infty} \binom{s + r - 1}{r} \sum_{n=1}^{\infty} q^{nt(r+1)} = (1 - q)^s \sum_{r=0}^{\infty} \binom{s + r - 1}{r} \frac{q^{t+r}}{1 - q^{t+r}}.
\]

The other assertions follows readily from this. \(\blacksquare\)

**Remark.** It is worth noting that the function \(f_q(s, t)\) can be expressed as the (beta-like) Jackson integral. In fact, we have

\[
q^{-t}(1 - q)^{1-s} f_q(s, t) = (1 - q) \sum_{j=0}^{\infty} \frac{q^{jt}}{(1 - q^{j+1})^s} = \int_0^1 x^{t-1}(1 - qx)^{-s} dq x.
\]
2. Main results

Now we put \( t = s - 1 \). When \( s = -m \in \mathbb{Z}_{\leq 0} \), the point \((s, t) = (-m, -m - 1)\) lies on the pole divisor \( t = -m - 1 \) of \( f_q(s, t) \). Nevertheless, a sort of “miracle” happens that the point turns out to be what is called “the point of indeterminacy”, the function \( f_q(s, s - 1) \) having a finite limit as \( s \to -m \) and moreover the limit approaches to the “correct” value \( \zeta(-m) \) as \( q \uparrow 1 \). What is more, the function \( f_q(s, s - 1) \) converges as \( q \uparrow 1 \) to \( \zeta(s) \) for any \( s \)!

These results, to be proved in quite elementary ways (certainly with only devices of which Euler could avail himself), well justify the function \( f_q(s, s - 1) \) being referred to as the “true” \( q \)-analogue of the Riemann zeta function, and we label it hereafter as

\[
\zeta_q(s) := f_q(s, s - 1) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{[n]_q^s} = \frac{q^{s-1}}{1} + \frac{q^{2(s-1)}}{[2]_q^s} + \frac{q^{3(s-1)}}{[3]_q^s} + \frac{q^{4(s-1)}}{[4]_q^s} + \cdots.
\]

**Remark.** 1) Proper choice of \( t \) seems to be essential. For example, the choice \( t = s \) adopted in \([3]\) needed an extra term to adjust the convergence when \( q \uparrow 1 \) and gave no nice values at negative integers. The choices \( t = s - 2, s - 3, s - 4, \ldots \) seem as good as the value \( \zeta(-m) \) is concerned, but extra poles at \( s = 2, 3, 4, \ldots \) emerge. However, these poles disappear at the limit \( q \uparrow 1 \). For example, with \( t = s - 2 \) the residue at the simple pole \( s = 2 \) is \(-1/(q^2 - 1)\) which goes to 0 as \( q \uparrow 1 \). How things become different depending on the choice of \( t \) still seems to be mysterious.

2) If we introduce the \( q \)-analogue \( \tilde{\zeta}_q(s) \) of the alternating \( \tilde{\zeta}(s) \) in the introduction by

\[
\tilde{\zeta}_q(s) = \sum_{n=1}^{\infty} (-1)^n \frac{q^{n(s-1)}}{[n]_q^s},
\]

the identity corresponding to (1) takes the form

\[
\tilde{\zeta}_q(s) = \zeta_q(s) - 2(1 + q)^{-s}\zeta^2_q(s).
\]

In contrast to the situation of Euler, this does not help much and indeed even makes things worse because of the occurrence of another base \( q^2 \). It may be said that once \( q \) is introduced, the acceleration of convergence is fully achieved and nothing more is needed.

The formula in Proposition 1 when specialized to \( t = s - 1 \) becomes

\[
(5) \quad \zeta_q(s) = (1 - q)^s \sum_{r=0}^{\infty} \frac{s + r - 1}{r} \frac{q^{s+r-1}}{1 - q^{s+r-1}} = (1 - q)^s \left( \frac{q^{s-1}}{1 - q^{s-1}} + \frac{s}{2} \frac{q^s}{1 - q^s} + \frac{s(s+1)}{2} \frac{q^{s+1}}{1 - q^{s+1}} + \cdots \right).
\]

**Proposition 2.** 1) The function \( \zeta_q(s) \) has a simple pole at points in \( 1 + 2\pi i \mathbb{Z}/\log q \) and in the set \( \{a + 2\pi ib/\log q \mid a, b \in \mathbb{Z}, a \leq 0, b \neq 0\} \). In particular, \( s = 1 \) is a simple pole of \( \zeta_q(s) \) with residue \( (q - 1)/\log q \).
2) For \( m \in \mathbb{Z}, \ m \geq 0 \), the limiting value \( \lim_{s \to -m} \zeta_q(s) \) exists (which we write \( \zeta_q(-m) \)) and is given explicitly by

\[
\zeta_q(-m) = (1 - q)^{-m} \left\{ \sum_{r=0}^{m} (-1)^r \left( \begin{array}{c} m \\ r \end{array} \right) \frac{1}{q^{m+1-r} - 1} + \frac{(-1)^{m+1}}{(m+1) \log q} \right\}.
\]

Proof. Assertion 1) is straightforward from (5), the formula \( \lim_{y \to 0} y/(1 - q^y) = -1/\log q \) being used for the residue at \( s = 1 \). For 2), note the terms with \( r \geq m + 2 \) in the sum vanishes when \( s \to -m \) since \( (-m+r-1) = 0 \) and \( 1 - q^{-m+r-1} \neq 0 \). On the other hand, for \( r = m + 1 \) we have \( \lim_{s \to -m} (s+m)/(1 - q^{s+m}) = -1/\log q \) and hence

\[
\lim_{s \to -m} \left( \frac{s+m}{m+1} \right) \frac{q^{s+m}}{1-q^{s+m}} = \frac{(-1)^m m!}{(m+1)!} \left( -\frac{1}{\log q} \right) = \frac{(-1)^{m+1}}{(m+1) \log q}.
\]

The rest of the computation is clear.

Before giving our general formula for \( \lim_{q \uparrow 1} \zeta_q(-m) \) (with expected value), let us look at the first few examples.

Example 1. As stated in Proposition 2, \( \zeta_q(s) \) has a simple pole at \( s = 1 \) with residue \( (q-1)/\log q \), which converges to 1 as \( q \to 1 \). This agrees with the well-known fact (reviewed later) that \( \zeta(s) \) has a simple pole at \( s = 1 \) with residue 1.

Example 2. By (6) we have \( \zeta_q(0) = \frac{1}{1 - q} - \frac{1}{\log q} \).

Since

\[
\frac{1}{\log q} = \frac{1}{\log(1 + (q-1))} = \frac{1}{(q-1) - (q-1)^2/2 + \cdots} = \frac{1}{q-1} + \frac{1}{2} + O(q-1),
\]

we find

\[
\lim_{q \to 1} \zeta_q(0) = -\frac{1}{2}.
\]

This agrees with Euler’s computation \( \zeta(0) = -1/2 \).

Example 3. Again by (6) we have

\[
\zeta_q(-1) = (1 - q)^{-1} \left( \frac{1}{q^2 - 1} - \frac{1}{q-1} + \frac{1}{2 \log q} \right)
\]

\[
= \frac{1}{1-q} \left( \frac{1}{q-1} \cdot \frac{1}{2 + q-1} - \frac{1}{q-1} + \frac{1}{2 \log q} \right)
\]

\[
= \frac{1}{1-q} \left( \frac{1}{2(q-1)} - \frac{1}{4} + \frac{q-1}{8} + \cdots - \frac{1}{q-1} + \frac{1}{2(q-1)} + \frac{1}{4} - \frac{q-1}{24} + \cdots \right)
\]

\[
\to -\frac{1}{12} \quad \text{as} \quad q \to 1,
\]

in accordance with \( \zeta(-1) = -1/12 \).
Let the Bernoulli numbers $B_k$ be defined by the generating series
\[
\frac{te^t}{e^t - 1} = \frac{t}{1 - e^{-t}} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.
\]

First several values are
\[
B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \ldots
\]

Now the general formula is the following

**Theorem 1.** For each non-negative integer $m$, we have
\[
\lim_{q \to 1} \zeta_q(-m) = -\frac{B_{m+1}}{m+1}.
\]

**Proof.** On account of formula (6), we have to show
\[
\lim_{q \to 1} (1 - q)^{-m} \left\{ \sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{1}{q^{m+1-r} - 1} + \frac{(-1)^{m+1}}{(m+1) \log q} \right\} = -\frac{B_{m+1}}{m+1}.
\]
(Note here that since the sum on the left is finite so we may replace the limit $q \uparrow 1$ by $q \to 1$.) Multiplying both sides by $(-1)^{m+1}(m+1)$ and changing $r \to m+1-r$, we see this is equivalent to
\[
\lim_{q \to 1} (1 - q)^{-m} \left\{ (m+1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r-1} \frac{1}{q^r - 1} + \frac{1}{\log q} \right\} = (-1)^m B_{m+1}.
\]
Writing
\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} (-1)^k B_k \frac{t^k}{k!},
\]
and using
\[
\frac{1}{q^r - 1} = \frac{1}{r} \cdot \frac{r \log q}{e^{r \log q} - 1} \cdot \frac{1}{\log q}
\]
we have
\[
(m + 1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r-1} \frac{1}{q^r - 1}
\]
\[
= (m + 1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r-1} \frac{1}{r} \sum_{k=0}^{\infty} (-1)^k B_k \frac{(r \log q)^k}{k!} \frac{1}{\log q}
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{r=1}^{m+1} (-1)^r \binom{m+1}{r} r^k \right) (-1)^k B_k \frac{(\log q)^{k-1}}{k!}.
\]
Since the inner sum on the right can be calculated as
\[
\sum_{r=1}^{m+1} (-1)^r \binom{m+1}{r} r^k = \left( \left. \left( x \frac{d}{dx} \right)^k ((1-x)^{m+1} - 1) \right|_{x=1} \right.
\]
\[
= \begin{cases} 
-1 & \text{if } k = 0, \\
0 & \text{if } 0 < k < m+1, \\
(-1)^{m+1} (m+1)! & \text{if } k = m+1,
\end{cases}
\]

Thus
\[
\sum_{r=1}^{m+1} (-1)^r \binom{m+1}{r} r^k = (-1)^{m+1} (m+1)!.
\]
we find
\[(m + 1) \sum_{r=1}^{m+1} (-1)^r \binom{m}{r} \frac{1}{q^r - 1} = -\frac{1}{\log q} + B_{m+1}(\log q)^m + O((\log q)^{m+1}) \quad \text{(as } q \to 1)\]  

From this and the expansion \(\log q = 1 + O((q - 1)^2) \quad (q \to 1)\), we obtain the desired result.

Remark. In view of Theorem 1, we may define the \(q\)-Bernoulli number \(B_m(q)\) by
\[B_m(q) := -m\zeta_q(1 - m) \quad (m \geq 1)\]

By (8) (letting \(m \to m - 1\) and \(r \to m - r\)) we have the closed formula
\[B_m(q) = (q - 1)^{-m+1} \left( \sum_{r=1}^{m} (-1)^r \binom{m}{r} \frac{r}{q^r - 1} + \frac{1}{\log q} \right)
= (q - 1)^{-m+1} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \frac{r}{q^r - 1} \quad (m \geq 1).\]

Here, the term with \(r = 0\) in the last sum should be read as \(1/\log q\) (the limiting value when \(r \to 0\)). This suggests to put
\[B_0(q) = \frac{q - 1}{\log q}\]

With this, the \(q\)-Bernoulli numbers \(\{B_m(q)\}_{m \geq 0}\) satisfy the recursion
\[\sum_{m=0}^{n} (-1)^m \binom{n}{m} q^m B_m(q) = (-1)^n B_n(q) + \delta_{1n} \quad (n \geq 0),\]

where \(\delta_{1n} = 1\) if \(n = 1\) and 0 otherwise, and the generating function
\[F_q(t) := \sum_{m=0}^{\infty} B_m(q) \frac{t^m}{m!}\]

satisfies the relation
\[F_q(qt) = e^t F_q(t) - te^t.\]

This \(q\)-Bernoulli number is essentially (i.e., up to the sign \((-1)^m\)) the same as the one introduced in Tsumura [7].

The following fundamental relation, apart from its own importance, guarantees that our computation at negative integers above does give us the correct values which we intended to obtain on a rigorous basis.

Theorem 2. For any \(s \in \mathbb{C}, s \neq 1\), we have
\[\lim_{q \to 1} \zeta_q(s) = \zeta(s).\]
What we understand as the right-hand side for arbitrary \( s \) is the value of the function analytically continued to the whole \( s \)-plane. (We give the analytic continuation by using the Euler-Maclaurin summation formula, see the proof below.) On the left-hand side, \( q \) should avoid the values with which \( \zeta_q(s) \) has a pole at \( s \), but this is achieved once \( q \) gets close enough to 1.

**Example.** We give some numerical examples. Take \( s = 1/2 \) and \( q = 0.999 \) in (5). Sum of the first \( 10^5 \) terms gives us the value \(-1.46014527395 \cdots\). Replacing \( q \) by \( q = 0.99999 \) and taking the first \( 10^7 \) terms we get the value \(-1.460352417 \cdots\), which agrees with the actual value \( \zeta(1/2) = -1.4603545088 \cdots \) up to 5 decimal points. Take the point \( s = 1/2 + 14.1347i \) near the first non-trivial zero (= 1/2 + 14.134725141734693790457251983562 \cdots i) of \( \zeta(s) \). For \( q = 0.9999 \), the first \( 10^5 \) terms gives the absurdly large \( 10^{835}552 \cdots + 10^{270}.785 \cdots i \), while \( 10^6 \) terms gives \(-0.000306477 \cdots + 0.0000794677 \cdots i\) (the actual value is \( \zeta(1/2 + 14.134725i) = 0.000000017674 \cdots - 0.00000011102 \cdots i \)).

Combining Theorem 1 and Theorem 2, we readily obtain

**Corollary.** For each non-negative integer \( m \), we have

\[
"1^m + 2^m + 3^m + 4^m + 5^m + \cdots" = \zeta(-m) = -\frac{B_{m+1}}{m+1}.
\]

**Remarks.** 1) We can also define a \( q \)-analogue of the Hurwitz zeta function \( \zeta(s;a) = \sum_{n=0}^{\infty} 1/(n+a)^s \) by

\[
\zeta_q(s;a) = \sum_{n=0}^{\infty} \frac{q^{(n+a)(s-1)}}{[n+a]^s_q}
\]

and prove the identity

\[
\lim_{q \uparrow 1} \zeta_q(s;a) = \zeta(s;a)
\]

for any \( s \neq 1 \), as well as the formula

\[
\lim_{q \uparrow 1} \zeta_q(-m;a) = -\frac{B_{m+1}(a)}{m+1}
\]

for integers \( m \leq 0 \). Here, the Bernoulli polynomial \( B_k(x) \) is defined by the generating series

\[
\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.
\]

As in the remark after Theorem 1, we can also define the \( q \)-Bernoulli polynomial similarly and derive elementary formulas. But to make our presentation as concise as possible, we restrict ourselves to the case of the Riemann zeta function.
2) It would be amusing to note that the limit
\[
\lim_{q \uparrow 1} (1 - q)^k \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n} = (k - 1)! \zeta(k) \quad (\forall k \geq 2, \ k \in \mathbb{Z})
\]
is derived easily from
\[
\lim_{q \uparrow 1} \zeta_q(k) = \zeta(k).
\]
(The latter directly follows from the definition without appealing to Theorem 2 because we are in the region of absolute convergence.) In fact, if we put \(s = 2\) in (8) and make \(r+1 \to n\), we have
\[
\zeta_q(2) = (1 - q)^2 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n},
\]
which gives the desired limit for \(k = 2\). For general \(k\), we similarly put \(s = k\) in (8) and make \(k + r - 1 \to n\) to find
\[
\zeta_q(k) = (1 - q)^k \sum_{n=1}^{\infty} \binom{n}{k-1} \frac{q^n}{1 - q^n}.
\]
(Observe \({n \choose k-1} = 0\) for \(n = 1, 2, \ldots, k - 2\).) We note that
\[
\binom{n}{k-1} = \frac{n^{k-1}}{(k-1)!} + \text{lower degree terms}
\]
and, on taking the limit \(q \uparrow 1\), sums coming from lower terms vanish inductively, hence we obtain the conclusion.

When \(k\) is even and \(k \geq 4\), the series \(\sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n} = \sum_{n=1}^{\infty} \left( \sum_{d \mid n} d^{k-1} \right) q^n\) constitutes the Fourier series of the Eisenstein series \(G_k(\tau)\) of weight \(k\) on the modular group, with constant term \(-B_k/2k = (1 - k)/2\). Here \(\tau\) is a variable in the upper-half plane and is linked with \(q\) by \(q = e^{2\pi i \tau}\). The modularity amounts to the transformation formula \(G_k(-1/\tau) = \tau^k G_k(\tau)\), which can be derived from, as Hecke [6] showed, the functional equation of the corresponding Dirichlet series \(\varphi(s) := \zeta(s) \zeta(s+1-k)\):
\[
(2\pi)^{-s} \Gamma(s) \varphi(s) = (-1)^{k/2} (2\pi)^{s-k} \Gamma(k-s) \varphi(k-s).
\]
(When \(k\) is odd, the functional equation of \(\varphi(s)\) fails to take this form and so the series \(\sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}\) cannot be a Fourier series of a modular form.) Hecke also showed that the residue of \(\varphi(s)\) at the simple pole \(s = k\) is equal to \((2\pi i)^k c_0/(k-1)!\) where \(c_0\) is the constant term of the corresponding modular form. In our case, the residue \(\zeta(k)\) and thus the constant term of \(G_k(\tau)\) is \((k-1)! \zeta(k)/(2\pi i)^k = -B_k/2k\), as expected. As an alternative way, we may use (8) to determine the constant term as follows: Put \(\tau = it\) with \(t > 0\). Then \(e^{2\pi i (-1/it)} \to 0\) as \(t \to 0\) and so
\[
c_0 = \lim_{t \to 0} G_k(-\frac{1}{it}) = \lim_{t \to 0} (it)^k G_k(it) = \lim_{q \uparrow 1} \zeta_q(k) = \frac{1}{(2\pi i)^k (k-1)!} \zeta(k).
\]
Proof of Theorem 2. Recall the celebrated summation formula of Euler [1, 2] (and Maclaurin, cf. [3, §7.21], obtained simply by repeating integration by parts): For a \(C^\infty\)-function \(f(x)\) on \([1, \infty)\) and arbitrary integers \(M \geq 0\), \(N \geq 1\), we have

\[
\sum_{n=1}^{N} f(n) = \int_{1}^{N} f(x) \, dx + \frac{1}{2} (f(1) + f(N)) + \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!} (f^{(k)}(N) - f^{(k)}(1))
\]

\[
- \frac{(-1)^{M+1}}{(M+1)!} \int_{1}^{N} \tilde{B}_{M+1}(x) f^{(M+1)}(x) \, dx,
\]

where \(\tilde{B}_{M+1}(x)\) is the “periodic Bernoulli polynomial” defined by

\[\tilde{B}_k(x) = B_k(x - [x]) \quad ([x] \text{ is the largest integer not exceeds } x).\]

Recall the Bernoulli polynomial \(B_k(x)\) is defined by the generating series (7):

\[B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x, \ldots\]

As is well-known, by taking \(f(x) = x^{-s}\) and letting \(N \to \infty\), we obtain the analytic continuation of \(\zeta(s)\) to the region \(\text{Re}(s) > -M:\)

\[
\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!} (s)_k = \frac{(s)_{M+1}}{(M+1)!} \int_{1}^{\infty} \tilde{B}_{M+1}(x)x^{-s-M-1} \, dx,
\]

where \((s)_k := s(s+1) \cdots (s+k-1)\). Since we may choose \(M\) arbitrary large, this gives the analytic continuation of \(\zeta(s)\) to the whole \(s\)-plane, revealing the (unique) simple pole at \(s = 1\) with residue 1.

Now we take \(f(x) = q^{x(s-1)}/(1 - q^x)^s\) and \(M = 1\) in (3). Assuming \(\text{Re}(s) > 1\) and noting

\[
f'(x) = \log q \cdot q^{x(s-1)} \frac{s-1 + q^x}{(1 - q^x)^{s+1}},
\]

\[
f''(x) = (\log q)^2 q^{x(s-1)} \frac{s(s+1) - 3s(1 - q^x) + (1 - q^x)^2}{(1 - q^x)^{s+2}},
\]

and in general \(f^{(k)}(x) = (\log q)^k q^{x(s-1)}(1 - q^x)^{-s-k} \times \text{a polynomial in } s \text{ and } q^x\), we see that we can take the limit \(N \to \infty\) and obtain

\[
\sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{(1 - q^n)^s} = \int_{1}^{\infty} \frac{q^{x(s-1)}}{(1 - q^x)^s} \, dx + \frac{1}{2} \cdot \frac{q^{s-1}}{(1 - q)^s} - \frac{1}{12} (\log q) q^{s-1} \frac{s-1 + q}{(1 - q)^{s+1}}
\]

\[
- \frac{(\log q)^2}{2} \int_{1}^{\infty} \tilde{B}_2(x) q^{x(s-1)} \frac{s(s+1) - 3s(1 - q^x) + (1 - q^x)^2}{(1 - q^x)^{s+2}} \, dx
\]

for \(\text{Re}(s) > 1\). The first integral on the right is computed as

\[
\int_{1}^{\infty} \frac{q^{x(s-1)}}{(1 - q^x)^s} \, dx = \int_{1}^{\infty} \frac{q^{-x}}{(q^{-x} - 1)^s} \, dx = \left[ \frac{(q^{-x} - 1)^{1-s}}{(s-1) \log q} \right]_{1}^{\infty} = -\frac{q^{s-1}(1 - q)^{1-s}}{(s-1) \log q}.
\]
We therefore obtain

\[\zeta_q(s) = (1 - q)^s \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{(1 - q^n)^s}\]

\[= \frac{q^{s-1}}{s-1} \cdot \frac{q - 1}{\log q} + \frac{q^{s-1}}{2} \cdot \frac{q - 1}{\log q} (s - 1 + q)\]

\[- (1 - q)^s (\log q)^2 \int_1^{\infty} \frac{\bar{B}_2(x)q^{x(s-1)}}{(1 - q^{x})^{s+2}} dx.\]

Unlike the classical case (10), just to let \(M\) larger does not make the convergence of the integral better, since the factor \(q^{x(s-1)}\) in \(f^{(M+1)}(x)\) always forces \(\text{Re}(s) > 1\). Instead, we use in (11) the Fourier expansion of the periodic Bernoulli polynomials\(^\dagger\) (cf. [8, Ch.IX, Misc. Ex. 12])

\[\bar{B}_k(x) = -k! \sum_{n \in \mathbb{Z}, n \neq 0} \frac{e^{2\pi inx}}{(2\pi in)^k}.\]

The equality is valid for all real numbers \(x\) when \(k \geq 2\), the sum being absolutely and uniformly convergent. Putting this (for \(k = 2\)) into (11) and interchanging the summation and the integration, we find

\[\zeta_q(s) = \frac{q^{s-1}}{s-1} \cdot \frac{q - 1}{\log q} + \frac{q^{s-1}}{2} \cdot \frac{q - 1}{\log q} (s - 1 + q)\]

\[+ (1 - q)^s (\log q)^2 \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2\pi in)^2} \int_1^{\infty} e^{2\pi inx} q^{x(s-1)} \frac{s(s+1)-3s(1-q^x)+(1-q^x)^2}{(1-q^x)^{s+2}} dx.\]

Further we make a change of variable \(q^x = u\) to obtain

\[\zeta_q(s) = \frac{q^{s-1}}{s-1} \cdot \frac{q - 1}{\log q} + \frac{q^{s-1}}{2} \cdot \frac{q - 1}{\log q} (s - 1 + q)\]

\[-(1 - q)^s \log q \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2\pi in)^2} \left\{ s(s+1)b_q(s - 1 + \delta n, -s - 1)\right\} + 3sb_q(s - 1 + \delta n, -s) + b_q(s - 1 + \delta n, -s + 1)\},\]

where we have put \(\delta = 2\pi i / \log q\) and

\[b_t(\alpha, \beta) = \int_0^t u^{\alpha-1}(1 - u)^{\beta-1} du\]

(referred to as the incomplete beta function\(^\dagger\)). Note that each of the incomplete beta integrals in (13) converges absolutely for \(\text{Re}(s) > 1\) and uniformly bounded with respect to \(n\);

\[|b_q(s - 1 + \delta n, -s + \nu)| \leq \int_0^q u^{\sigma - 2}(1 - u)^{-\sigma + \nu - 1} du \quad (\forall n, \sigma = \text{Re}(s), \nu = -1, 0, 1),\]

\(^\ast\text{We owe Ueno-Nishizawa \[8\] the idea of replacing }\bar{B}_2(x)\text{ in the integral by its Fourier expansion. However, our argument that follows, which uses only integration by parts and no confluent hypergeometric functions or the like, seems considerably different from the one in \[8\].}\)

\(^\dagger\text{We are tempted to remind the reader that the beta integral is often called the Euler integral.}\)
hence the sum converges absolutely.

Now, repeated use of integration by parts provides us the formula

\[ b_t(\alpha, \beta) = \int_0^t \left( \frac{u^\alpha}{\alpha} \right)' (1 - u)^{\beta - 1} du = \frac{1}{\alpha} t^\alpha (1 - t)^{\beta - 1} - \frac{1 - \beta}{\alpha} \int_0^t \left( \frac{u^{\alpha+1}}{\alpha + 1} \right)' (1 - u)^{\beta - 2} du \]

\[ = \frac{1}{\alpha} t^\alpha (1 - t)^{\beta - 1} - \frac{1 - \beta}{\alpha} \int_0^t \left( \frac{u^{\alpha+1}}{\alpha + 1} \right)' (1 - u)^{\beta - 2} du \]

\[ = \ldots \]

\[ = \sum_{k=1}^{M-1} (-1)^{k-1} \frac{(1 - \beta)k-1}{(\alpha)_k} t^{\alpha+k-1} (1 - t)^{\beta - k} \]

\[ + (-1)^{M-1} \frac{(1 - \beta)M-1}{(\alpha)_M-1} \beta_t(\alpha + M - 1, \beta - M + 1) \]

for any \( M \geq 2 \). Applying this to \( b_q(s - 1 + \delta n, -s - 1) \), we have (note \( q^{\delta n} = 1 \))

\[ b_q(s - 1 + \delta n, -s - 1) = \sum_{k=1}^{M-1} (-1)^{k-1} \frac{(s + 2)k-1}{(s - 1 + \delta n)_k} q^{s+k-2}(1 - q)^{-s-1-k} \]

\[ + (-1)^{M-1} \frac{(s + 2)M-1}{(s - 1 + \delta n)_M-1} b_q(s - 2 + M + \delta n, -s - M). \]

This accomplishes the analytic continuation of \( b_q(s - 1 + \delta n, -s - 1) \) as a function of \( s \) into the region \( \text{Re}(s) > 2 - M \). From this we have

\[ \sum_{n \in \mathbb{Z}} \frac{s(s + 1)}{(2\pi in)^2} b_q(s - 1 + \delta n, -s - 1) \]

\[ = \sum_{k=1}^{M-1} (-1)^{k-1} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2\pi in)^2} \frac{(s)_{k+1}}{(s - 1 + \delta n)_k} q^{s+k-2}(1 - q)^{-s-1-k} \]

\[ + (-1)^{M-1} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2\pi in)^2} \frac{(s)_{M+1}}{(s - 1 + \delta n)_{M-1}} \int_0^q u^{s-3+M+\delta n}(1 - u)^{-s-1-M-1} du \]

\[ = \sum_{k=1}^{M-1} (-1)^{k-1} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2\pi in)^2} \frac{(s)_{k+1}}{(s - 1 + \delta n)_k} q^{s+k-2}(1 - q)^{-s-1-k} \]

\[ + (-1)^{M-1} \log q \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(2\pi in)^2} \frac{(s)_{M+1}}{(s - 1 + \delta n)_{M-1}} \int_1^\infty e^{2\pi in x} q^{x(s-2+M)}(1 - q^x)^{-s-1-M-1} dx. \]

Using

\[ \lim_{q \to 1} \frac{\log q}{1 - q} = -1, \quad \lim_{q \to 1} (1 - q)^k(s - 1 + \delta n)_k = (-2\pi i)^k, \quad \lim_{q \to 1} \frac{1 - q^x}{1 - q} = x, \]

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we obtain, for $\text{Re}(s) > 2 - M$,

$$\lim_{q \uparrow 1} (1 - q)^{s} \log q \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{s(s + 1)}{(2\pi in)^2} b_q(s - 1 + \delta n, -s - 1)$$

$$= \sum_{k=1}^{M-1} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(s)_{k+1}}{(2\pi in)^{k+2}} - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(s)_{M+1}}{(2\pi in)^{M+1}} \int_{1}^{\infty} e^{2\pi inx} x^{-s-M-1} \, dx$$

$$= -\sum_{k=1}^{M-1} \frac{B_{k+2}}{(k+2)!} (s)_{k+1} + \frac{(s)_{M+1}}{(M+1)!} \int_{1}^{\infty} \tilde{B}_{M+1}(x) x^{-s-M-1} \, dx.$$

In the last equality, we have used (12) and its specialization ($x = 1$)

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(2\pi in)^k} = -\frac{B_k}{k!}$$

valid for all $k \geq 2$. We do exactly the same for the terms containing $b_q(s - 1 + \delta n, -s)$ and $b_q(s - 1 + \delta n, -s + 1)$. As it turns out however, the contributions from these two vanish when we take $q \uparrow 1$, for the powers of $1 - q$ involved are lower than those from $b_q(s - 1 + \delta n, -s - 1)$.

We therefore obtain, for $\text{Re}(s) > 2 - M$,

$$\lim_{q \uparrow 1} \zeta_q(s) = \frac{1}{s - 1} + \frac{1}{2} + \frac{s}{12} + \sum_{k=2}^{M} \frac{B_{k+1}}{(k+1)!} (s)_{k} - \frac{(s)_{M+1}}{(M+1)!} \int_{1}^{\infty} \tilde{B}_{M+1}(x) x^{-s-M-1} \, dx$$

$$= \frac{1}{s - 1} + \frac{1}{2} + \sum_{k=1}^{M} \frac{B_{k+1}}{(k+1)!} (s)_{k} - \frac{(s)_{M+1}}{(M+1)!} \int_{1}^{\infty} \tilde{B}_{M+1}(x) x^{-s-M-1} \, dx.$$

This coincides with formula (10) for $\zeta(s)$ valid in $\text{Re}(s) > -M$, and thus the theorem is established since the integer $M$ can be arbitrary large.

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