Cubic-quartic optical solitons in Bragg gratings fibers for NLSE having parabolic non-local law nonlinearity using two integration schemes

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Abstract
The optical solitons in Bragg gratings fibers are studied for NLSE having cubic-quartic dispersive reflectivity with parabolic non-local combo law of refractive index. The extended auxiliary equation method and the addendum to Kudryashov’s method are introduced. The existence criteria for such solitons are indicated.

Keywords J acobi elliptic function solutions · The extended auxiliary equation method · The addendum to Kudryashov’s · Fiber Bragg gratings · Cubic-quartic · Parabolic law · Nonlocal law

1 Introduction
In the past few decades, the nonlinear optics is one of the most effective fields of research which study the dynamics of solitons in fiber gratings. Solitons in fiber Bragg gratings (FBG) is one of the many subjects that is studied in nonlinear optics. Recently, many papers have been studied Bragg gratings models by using a lot of numerical methods (Kudryashov 2012, 2020a, b, 2021a, b, c; Biswas et al. 2019a, b, c; Sonmezoglu et al. 2016; Xu 2014; Zayed et al. 2020a, b, c; Bansal et al. 2018; Biswas et al. 2017a, b; Yıldırım et al. 2020a, b; Zayed et al. 2021a, b; Yıldırım et al. 2021a, b, c, d; Zayed et al. 2021c, 2020d; Yıldırım et al. 2021e; Darwish et al. 2020; Xu et al. 2020; Chen et al. 2020; Xia et al. 2020; Chen et al. 2021a, b; Lü and Chen 2021). There are several approaches to handle this project. A great amount of results have been introduced in these papers. In such a situation, Bragg gratings artificially introduces induced dispersion that restores this balance for sustainability of soliton transmission along intercontinental distances. Newly, in the field of nonlinear fiber optics, the concept of cubic–quartic (CQ) optical solitons has been introduced in many papers (Bansal et al. 2018; Biswas et al. 2017a, b; Yıldırım et al. 2020a, b; Zayed et al. 2021a, b; Yıldırım et al. 2021a, b, c, d; Zayed et al. 2021c). When the chromatic dispersion (CD) runs low enough to be
discarded, it is third-order dispersion (3OD) and fourth-order dispersion (4OD) effects that collectivity compensate for this crisis situations. This enable the delicate balance between the effects of dispersion and self-phase modulation, to be restarted. So that the transmission of stable solutions across inter-continental distances is rendered possible. FBG are considered excellent sensor elements, suitable for measuring various engineering parameters such as temperature, strain, pressure, tilt, displacement, acceleration, load as well as the presence of various industrial, biomedical and chemical substances in both static and dynamic modes of operation (Zayed et al. 2021c, 2020d; Yıldırım et al. 2021e; Darwish et al. 2020). It is worth mentioning here that there are also other studies on an interesting kind of exact solutions, such as, lump solution (Lü and Ma 2016), Bäcklund transformation, Pfaffian, Wronskian and Grammian solutions (He et al. 2021), localized characteristics of lump and interaction solutions (Yin et al. 2020), Painleve analysis, soliton solutions, Bäcklund transformation, Laxpair and infinitely many conservation laws (Lü et al. 2021).

In this paper, the coupled system of cubic-quartic nonlinear Schrödinger equation (CQ-NLSE) in Bragg gratings fibers with parabolic non-local combo law nonlinearity will be discussed for the first time by using two different methods mentioned in the abstract. Today’s work will address the same problem with two fresh approaches. These are the extended auxiliary equation method and the addendum to Kudryashov’s method. These methods will lead to the emergence of Jacobi elliptic function solutions, bright, dark and singular solitons from the governing model. The essential details are jotted with the existence criteria for such solitons, after a quick into to the model.

1.1 Governing model

The coupled CQ-NLSE in Bragg gratings fibers with parabolic non-local law nonlinearity is written as:

\[iu_t + i\alpha_1 v_{xxx} + b_1 v_{xxxx} + (c_1 |u|^2 + d_1 |v|^2)u + (\xi_1 |u|^4 + \eta_1 |u|^2|v|^2 + \zeta_1 |v|^4)u\]
\[+ \left[ f_1(|u|^2)_{xx} + g_1(|v|^2)_{xx} \right]u + i\alpha_1 u_x + \beta_1 v + \sigma_1 u^*v^2 = 0, \tag{1}\]

and

\[iv_t + i\alpha_2 u_{xxx} + b_2 u_{xxxx} + (c_2 |v|^2 + d_2 |u|^2)\nu + (\xi_2 |v|^4 + \eta_2 |v|^2|u|^2 + \zeta_2 |u|^4)\nu\]
\[+ \left[ f_2(|v|^2)_{xx} + g_2(|u|^2)_{xx} \right]\nu + i\alpha_2 v_x + \beta_2 u + \sigma_2 v^*u^2 = 0, \tag{2}\]

where $\alpha_i, b_i, c_i, d_i, \xi_i, \eta_i, \zeta_i, f_i, g_i, \alpha_l, \beta_l$ and $\sigma_l, (l = 1, 2)$ are real parameters such that $i = \sqrt{-1}$. Here, $u(x, t), v(x, t)$ are the complex wave profiles. The coefficients of $\alpha_i, b_i$ are 3OD and 4OD, respectively. The parameters $c_i, \xi_i, f_i$ represent the self-phase modulation (SPM) coefficients, while the cross-phase modulation (XPM) effect comes from the coefficients $d_i, \eta_i, \zeta_i$ and $g_i$. The parameters $\alpha_l, \beta_l$ and $\sigma_l$ are the coefficients of inter–modal dispersion (IMD), detuning parameter and four-wave mixing effect (4WM) for Kerr part of the nonlinearity, respectively. The system (1) and (2) is a manifested version of the standard model. That is the well-known NLSE model in FBG with CD, that is structured as Zayed et al. (2020d):
Cubic-quartic optical solitons in Bragg gratings fibers for…

\[iu_x + E_1 v_{xx} + (c_1 |u|^2 + d_1 |v|^2) u + (\xi_1 |u|^4 + \eta_1 |u|^2 |v|^2 + \zeta_1 |v|^4) u + \left[ f_1 (|u|^2)_{xx} + g_1 (|v|^2)_{xx} \right] u + i \alpha_1 u_x + \beta_1 v + \sigma_1 u^* v^2 = 0,\]

(3)

and

\[iv_x + E_2 u_{xx} + (c_2 |v|^2 + d_2 |u|^2) v + (\xi_2 |v|^4 + \eta_2 |u|^2 |v|^2 + \zeta_2 |u|^4) v + \left[ f_2 (|v|^2)_{xx} + g_2 (|u|^2)_{xx} \right] v + i \alpha_2 v_x + \beta_2 u + \sigma_2 v^* u^2 = 0,\]

(4)

where \(E_1\) and \(E_2\) are the coefficients of CD. In the system (1) and (2), it is this CD that is replaced by 3OD and 4OD, which formulate the dispersion effects.

The objective of this paper is to apply the extended auxiliary equation method and the addendum to Kudryashov’s method to find the bright, dark and singular solitons solutions as well as the Jacobi elliptic function solutions of the coupled system (1) and (2).

The organization of this article can be written as: the mathematical preliminaries are discussed in Sect. 2. The extended auxiliary equation method is applied to the coupled system (1) and (2) in Sect. 3. The addendum to Kudryashov’s method is applied to the same coupled system in Sect. 4. Lastly, conclusions are given in Sect. 5.

## 2 Mathematical preliminaries

In order to recover solitons of the CQ-NLSE in fiber Bragg gratings with parabolic non-local law nonlinearity, we set

\[u(x, t) = P_1(\xi) \exp \{i \phi(x, t)\},\]

\[v(x, t) = P_2(\xi) \exp \{i \phi(x, t)\},\]

(5)

and

\[\xi = x - Ct, \phi(x, t) = -\kappa x + \omega t + \theta_0,\]

(6)

where \(C, \kappa, \omega \) and \(\theta_0\) are all non zero parameters. Here, \(C\) is the velocity of soliton, \(\kappa\) is the frequency of soliton, \(\omega\) is the wave number of the soliton and finally, \(\theta_0\) is the phase parameter, while \(P_1(\xi), P_2(\xi)\) and \(\phi(x, t)\) are real functions representing the amplitude portion of the soliton and the phase component of the soliton, respectively. If we substitute (5) and (6) into Eqs. (1) and (2) and separate the real and imaginary parts, we deduce that the real parts are

\[b_1 P_1^{(+)} + 3 \kappa (a_1 - 2 b_2 \kappa) P_1^{(+)} + (a_1 \kappa - \omega) P_1 + \left[ \kappa^3 (b_1 \kappa - a_1) + \beta_1 \right] P_2 + c_1 P_1^2 + 2 \left[ f_1 (P_1 P_2^2 + P_1^2 P_2^0) + g_1 (P_1 P_2^2 + P_1^2 P_2^0) \right] + (d_1 + \sigma_1) P_1 P_2^2 + \xi_1 P_1^3 + \eta_1 P_1^2 P_2^0 + \zeta_1 P_1 P_2^4 + 0 = 0,\]

(7)

\[b_2 P_1^{(+)} + 3 \kappa (a_2 - 2 b_2 \kappa) P_1^{(+)} + (a_2 \kappa - \omega) P_2 + \left[ \kappa^3 (b_2 \kappa - a_2) + \beta_2 \right] P_1 + c_2 P_2^3 + 2 \left[ f_2 (P_2 P_2^2 + P_2^2 P_2^0) + g_2 (P_2 P_2^2 + P_2^2 P_2^0) \right] + (d_2 + \sigma_2) P_2 P_1^2 + \xi_2 P_2^3 + \eta_2 P_2^2 P_1^0 + \zeta_2 P_2 P_1^4 = 0,\]

(8)

and the imaginary parts are...
\[(a_1 - 4b_1 \kappa)P''_2 - (3a_1 - 4b_1 \kappa)\kappa^2P'_2 + (a_1 - C)P'_1 = 0, \quad (9)\]

\[(a_2 - 4b_2 \kappa)P''_1 - (3a_2 - 4b_2 \kappa)\kappa^2P'_1 + (a_2 - C)P'_2 = 0. \quad (10)\]

Setting

\[P_2(\xi) = \chi P_1(\xi), \quad (11)\]

where \(\chi\) is a constant such that \(\chi \neq 0, 1\). Now, Eqs. (7)–(10) become

\[b_1\chi P_1^{(4)} + 3\kappa \chi (a_1 - 2b_1 \kappa)P''_1 + 2\left(f_1 + \chi^2 g_1\right)\left(P_1 P''_1 + P''_1 P''_1\right) + \left[a_1 \kappa - \omega + \chi \kappa^2 (b_1 \kappa - a_1) + \chi \beta_1\right]P_1 + \left[c_1 + \chi^2 (d_1 + \sigma_1)\right]P_1^3 + \left(\xi_1 + \eta_1 \kappa^2 + \chi^4 \xi_1\right)P_1^5 = 0, \quad (12)\]

\[b_2 P_1^{(4)} + 3\kappa (a_2 - 2b_2 \kappa)P''_1 + 2\chi \left(\chi^2 f_2 + g_2\right)\left(P_1 P''_1 + P''_1 P''_1\right) + \left[\chi (a_2 \kappa - \omega) + \kappa^3 (b_2 \kappa - a_2) + \beta_2\right]P_1 + \chi \left[\chi^2 c_2 + d_2 + \sigma_2\right]P_1^3 + \chi \left(\chi^4 \xi_2 + \chi^2 \eta_2 + \xi_2\right)P_1^5 = 0, \quad (13)\]

and

\[\chi (a_1 - 4b_1 \kappa)P''_1 + \left[a_1 - C - \chi (3a_1 - 4b_1 \kappa)\kappa^2\right]P'_1 = 0, \quad (14)\]

\[(a_2 - 4b_2 \kappa)P''_1 + \left[\chi (a_2 - C) - (3a_2 - 4b_2 \kappa)\kappa^2\right]P'_1 = 0. \quad (15)\]

Integrating Eqs. (14) and (15) with zero-integration constants, we have

\[\chi (a_1 - 4b_1 \kappa)P'_1 + \left[a_1 - C - \chi (3a_1 - 4b_1 \kappa)\kappa^2\right]P_1 = 0, \quad (16)\]

\[(a_2 - 4b_2 \kappa)P'_1 + \left[\chi (a_2 - C) - (3a_2 - 4b_2 \kappa)\kappa^2\right]P_1 = 0. \quad (17)\]

Setting the coefficients of the linearly independent functions of Eqs. (16) and (17) to zero, yields

\[\kappa = \frac{a_1}{4b_1}, \quad (18)\]

or

\[\kappa = \frac{a_2}{4b_2}, \quad (19)\]

and

\[C = a_1 - \chi (3a_1 - 4b_1 \kappa)\kappa^2, \quad (20)\]

or
Cubic-quartic optical solitons in Bragg gratings fibers for…

\[ C = \frac{\chi \alpha_2 - (3a_2 - 4b_2 \kappa) \kappa^2}{\chi}. \]  

(21)

From (18) and (19), one gets

\[ a_1b_2 = a_2b_1. \]  

(22)

From (20) and (21), one gets the following constraint condition:

\[ \chi (\alpha_1 - \alpha_2) - 3\kappa^2 (a_1 \alpha^2 - a_2) + 4\kappa^3 (b_1 \chi^2 - b_2) = 0. \]  

(23)

Eqs. (12) and (13) have the same form under the constraint conditions:

\[ b_1 \chi = b_2, \]  

(24)

\[ \chi (a_1 - 2b_1 \kappa) = a_2 - 2b_2 \kappa, \]  

(25)

\[ f_1 + \chi^2 g_1 = \chi (\kappa f_2 + g_2), \]  

(26)

\[ \alpha_1 \kappa - \omega + \chi \kappa^3 (b_1 \kappa - a_1) + \chi \beta_1 = \chi (\alpha_2 \kappa - \omega) + \kappa^3 (b_2 \kappa - a_2) + \beta_2, \]  

(27)

\[ c_1 + \chi^2 (d_1 + \sigma_1) = \chi [\kappa c_2 + d_2 + \sigma_2], \]  

(28)

\[ \xi_1 + \eta_1 \chi^2 + \chi^4 \xi_1 = \chi (\kappa^4 \xi_2 + \kappa^2 \eta_2 + \xi_2). \]  

(29)

Consequently, we deduce that

\[ \omega = \frac{(a_1 - \chi a_2) \kappa + \chi \beta_1 - \beta_2}{1 - \chi}, \]  

(30)

\[ \chi = \frac{b_2}{b_1} = \frac{a_2 - 2b_2 \kappa}{a_1 - 2b_1 \kappa}, \]  

provided \( a_1 \neq a_2 \) and \( b_1 \neq b_2 \). Eq. (12) can be rewritten in the form:

\[ P_1^{(4)} + L_1 P_1'' + L_2 (P_1 P_2' + P_2' P_1') + L_3 P_1 + L_4 P_1^3 + L_5 P_1^5 = 0, \]  

(31)

where

\[ L_1 = 6 \kappa^2, L_2 = \frac{2(c_1 + \chi^2 \xi_2)}{b_1 \chi}, L_3 = \frac{a_1 \kappa - \omega + \chi \kappa^3 (b_1 \kappa - a_1) + \chi \beta_1}{b_1 \chi}, \]  

(32)

\[ L_4 = \frac{c_1 + \chi^2 (d_1 + \sigma_1)}{b_1 \chi}, L_5 = \frac{\xi_1 + \eta_1 \chi^2 + \chi^4 \xi_1}{b_1 \chi}, \]  

provided \( b_1 \chi \neq 0 \) and \( L_1 > 0 \). Balancing \( P_1^{(4)} \) with \( P_1^5 \) in (31), yields the balance number \( N = 1 \). The problem now is to solve Eq. (31) using the following two schemes.
3 The extended auxiliary equation method

According to this method (Xu 2014; Zayed et al. 2020a, b, c), we assume that Eq. (31) has the formal solution

\[ P_1(\xi) = A_0 + A_1 F(\xi) + A_2 F^2(\xi), \] (33)

where \( A_0, A_1 \) and \( A_2 \) are constants to be determined, such that \( A_2 \neq 0 \), while the function \( F(\xi) \) satisfies the following first order equation:

\[ F^2(\xi) = C_0 + C_2 F^2(\xi) + C_4 F^4(\xi) + C_6 F^6(\xi), \] (34)

where \( C_j (j = 0, 2, 4, 6) \) are constants to be determined. It is well known that Eq. (34) has the following solution:

\[ F(\xi) = \frac{1}{2} \left[ -\frac{C_4}{C_6} (1 \pm f(\xi)) \right]^{\frac{1}{3}}, \] (35)

where \( f(\xi) \) could be expressed through the Jacobi elliptic functions \( sn(\xi, m) \), \( cn(\xi, m) \), \( dn(\xi, m) \) and so on. Here \( 0 < m < 1 \) is the modulus of the Jacobi elliptic functions. Substituting (33) along with (34) into Eq.(31), collecting the coefficients of each power \( F^l(\xi) (F'(\xi))^l \), \( (l = 0, 1, 2, \ldots, 10, \; j = 0, 1) \) and setting these coefficients to zero, we have a set of algebraic equations which can be solved using the Maple to obtain the following results:

\[ L_2 = -\frac{5\sqrt{6L_5}}{3}, \quad L_3 = -A_0^2 L_5 + \frac{1}{3} A_0 (L_1 A_0 + 4 A_2 C_0) \sqrt{6L_5}, \quad L_4 = -\frac{L_1 \sqrt{6L_5}}{3}, \quad C_0 = C_0, \]

\[ C_2 = \frac{A_0^2 \sqrt{6L_5}}{4}, \quad C_4 = \frac{A_0 A_2 \sqrt{6L_5}}{6}, \quad C_6 = \frac{A_2^2 \sqrt{6L_5}}{24}, \quad A_0 = A_0, A_1 = 0, A_2 = A_2, \] (36)

provided \( L_5 > 0 \). From (33), (35) and (36), then we have the solutions:

\[ P_1(\xi) = \mp A_0 f(\xi). \] (37)

We have the following families of Jacobi elliptic functions solutions of Eqs. (1) and (2):

**Family-1.** If \( C_0 = \frac{C_4^3(m^2 - 1)}{32C_6 m^2}, \quad C_2 = \frac{C_4^2(5m^2 - 1)}{16C_6 m^2}, \quad C_6 > 0, \) then

\[ u(x, t) = \left( \frac{6m^4 L_i^2}{L_5 (m^2 + 1)^2} \right)^{\frac{1}{4}} \text{sn} \left( \frac{\sqrt{L_1}}{m^2 + 1} \xi, m \right) e^{(-k x + \omega t + \theta_i)}, \] (38)

and

\[ v(x, t) = \chi \left( \frac{6m^4 L_i^2}{L_5 (m^2 + 1)^2} \right)^{\frac{1}{4}} \text{sn} \left( \frac{\sqrt{L_1}}{m^2 + 1} \xi, m \right) e^{(-k x + \omega t + \theta_i)}, \] (39)

or

\[ \text{sn} \left( \frac{\sqrt{L_1}}{m^2 + 1} \xi, m \right). \] (40)
and

$$v(x, t) = \chi \left( \frac{6L_i^2}{L_5} \right)^{\frac{1}{4}} \text{ns} \left( \sqrt{\frac{L_1}{m^2 + 1}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}.$$  \hfill (41)

In particular, if $m \to 1^-$, then we have the dark soliton solutions:

$$u(x, t) = \left( \frac{3L_i^2}{2L_5} \right)^{\frac{1}{4}} \text{tanh} \left( \sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \hfill (42)$$

and

$$v(x, t) = \chi \left( \frac{3L_i^2}{2L_5} \right)^{\frac{1}{4}} \text{tanh} \left( \sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \hfill (43)$$

as well as the singular soliton solution:

$$u(x, t) = \left( \frac{3L_i^2}{2L_5} \right)^{\frac{1}{4}} \text{coth} \left( \sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \hfill (44)$$

and

$$v(x, t) = \chi \left( \frac{3L_i^2}{2L_5} \right)^{\frac{1}{4}} \text{coth} \left( \sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \hfill (45)$$

while if $m \to 0^+$, then we have the periodic solutions:

$$u(x, t) = \left( \frac{6L_i^2}{L_5} \right)^{\frac{1}{4}} \text{csc} \left( \sqrt{\frac{L_1}{L_5}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \hfill (46)$$

and

$$v(x, t) = \chi \left( \frac{6L_i^2}{L_5} \right)^{\frac{1}{4}} \text{csc} \left( \sqrt{\frac{L_1}{L_5}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \hfill (47)$$

provided $L_5 > 0$ (Fig. 1).

**Family-2.** If $C_0 = \frac{C_4^3}{32C_6^2(1 - m^2)}$, $C_2 = \frac{C_4^2(4m^2 - 5)}{16C_6(m^2 - 1)}$, $C_6 > 0$, then
and

\[ \frac{0}{\frac{1}{2} - \frac{2}{m^2}} < 1 \] and \( L_5 > 0 \). In particular, if \( m \to 0^+ \), in (48) and (49), then we have the periodic solutions:

\[ u(x, t) = \left( \frac{6L_1^2(m^2 - 1)^2}{L_5(2m^2 - 1)^2} \right)^{\frac{1}{4}} \text{nc} \left( \sqrt{\frac{L_1}{1 - 2m^2}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{50} \]

and

\[ v(x, t) = \chi \left( \frac{6L_2^2(m^2 - 1)^2}{L_5(2m^2 - 1)^2} \right)^{\frac{1}{4}} \text{ds} \left( \sqrt{\frac{L_1}{1 - 2m^2}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{51} \]

where \( 0 < (1 - 2m^2) < 1 \) and \( L_5 > 0 \). In particular, if \( m \to 0^+ \), in (48) and (49), then we have the periodic solutions:

\[ u(x, t) = \left( \frac{6L_1^2}{L_5} \right)^{\frac{1}{4}} \sec \left( \sqrt{\frac{L_1}{2m^2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \tag{52} \]

and
while if $m \to 0^+$, in (50) and (51), then we have the same periodic solutions (46) and (47).

4 The addendum to Kudryashov’s method

According to this method (Kudryashov 2020a; Zayed et al. 2021a, b, c), we balance $P_1^{(4)}$ with $P_1^5$ in Eq. (31), we get

$$M + 4p = 5M \implies M = p.$$  (54)

Now, the following cases can be considered.  

**Case-1.** Choose $p = 1$, then $M = 1$. Now, we have the formal solution

$$P_1(\xi) = B_0 + B_1 R(\xi),$$  (55)

where $B_0$ and $B_1$ are parameters, provided $B_1 \neq 0$. Here $R(\xi)$ satisfies the differential equation:

$$R''(\xi) = R^2(\xi) \left[1 - \lambda R^2(\xi)\right] \ln^2 K, \quad 0 < K \neq 1,$$  (56)

where $\lambda$ is a constant. Substituting (55), (56) in (31) and setting all the coefficients of $[R(\xi)]^u_1 [R'(\xi)]^u_2, (u_1 = 0, \ldots, 5, u_2 = 0, 1)$ to zero, we have the results

$$B_0 = 0, B_1 = \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}},$$  (57)

and

$$L_3 = -(L_1 + \ln^2 K) \ln^2 K, L_5 = \frac{3(L_1 L_2 + 2L_2 \ln^2 K - 4L_4)(L_4 + 2L_2 \ln^2 K)}{2(L_1 + 10 \ln^2 K)^2}.$$  (58)

provided $\lambda(L_4 + 2L_2 \ln^2 K) > 0$. Substituting (57) along with the well-known solution of Eq. (56) obtained in Kudryashov (2020a), Zayed et al. (2021a), Zayed et al. (2021b), Yıldırım et al. (2021a) in (55), we have the solutions:

$$u(x, t) = \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left(\frac{4S}{4S^2 K^\xi + \lambda K^{-\xi}}\right) e^{(-\kappa x + \omega t + \theta_0)},$$  (59)

and

$$v(x, t) = \lambda \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left(\frac{4S}{4S^2 K^\xi + \lambda K^{-\xi}}\right) e^{(-\kappa x + \omega t + \theta_0)}.$$  (60)
The solutions (59), (60) can be rewritten in the form:

\[
u(x, t) = \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left[ \frac{4S}{(4S^2 + \lambda \cosh (\xi \ln K) + (4S^2 - \lambda) \sinh (\xi \ln K)} e^{(-\kappa x + \omega t + \theta_0)} \right]
\]

and

\[
v(x, t) = \chi \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left[ \frac{4S}{(4S^2 + \lambda \cosh (\xi \ln K) + (4S^2 - \lambda) \sinh (\xi \ln K)} e^{(-\kappa x + \omega t + \theta_0)} \right]
\]

which represent the combo bright-singular soliton solution. In particular, when \( \lambda = 4S^2 \), we have the bright soliton solution:

\[
u(x, t) = \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \text{sech} (\xi \ln K) e^{(-\kappa x + \omega t + \theta_0)},
\]

and

\[
v(x, t) = \chi \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left[ \frac{4S}{(4S^2 + \lambda \cosh (\xi \ln K) + (4S^2 - \lambda) \sinh (\xi \ln K)} e^{(-\kappa x + \omega t + \theta_0)} \right]
\]

provided \((L_4 + 2L_2 \ln^2 K) > 0\), while if \( \lambda = -4S^2 \), we have the singular soliton solutions:

\[
u(x, t) = \sqrt{\frac{2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \text{csch} (\xi \ln K) e^{(-\kappa x + \omega t + \theta_0)},
\]

and

\[
v(x, t) = \chi \sqrt{\frac{-2\lambda(L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \text{csch} (\xi \ln K) e^{(-\kappa x + \omega t + \theta_0)},
\]

provided \((L_4 + 2L_2 \ln^2 K) < 0\). The solutions (59)–(66) exist under the conditions (58) (Fig. 2).

Case-2. Choose \( p = 2 \), then \( M = 2 \). Now, we have the formal solution

\[
P_1(\xi) = B_0 + B_1 R(\xi) + B_2 R^2(\xi),
\]

where \( B_0, B_1 \) and \( B_2 \) are parameters, provided \( B_2 \neq 0 \). Here \( R(\xi) \) satisfies the differential equation:

\[
R^\prime(\xi) = R^2(\xi) [1 - \lambda R^4(\xi) \ln^2 K], \quad 0 < K \neq 1,
\]

where \( \lambda \) is a constant. Substituting (67) and (68) into (31) and setting all the coefficients of \([R(\xi)]^{u_1} [R^\prime(\xi)]^{u_2}\), \((u_1 = 0, ..., 10, u_2 = 0, 1)\) to zero, we have the results.
provided $\lambda(L_4 + 8L_2 \ln^2 K) > 0$. Now, we have the results:

$$u(x, t) = \sqrt{\frac{8\lambda(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left( \frac{4S}{4S^2 K \lambda^2 + \lambda - K^{-2}} \right) e^{i(-\kappa x + \omega t + \theta_0)},$$

(71)

and

$$v(x, t) = \sqrt{\frac{8\lambda(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left( \frac{4S}{4S^2 K \lambda^2 + \lambda - K^{-2}} \right) e^{i(-\kappa x + \omega t + \theta_0)}.$$  

(72)

The solutions (71), (72) can be rewritten in the form

$$u(x, t) = \sqrt{\frac{8\lambda(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left[ \frac{4S}{(4S^2 + \lambda) \cosh(2\xi \ln K) + (4S^2 - \lambda) \sinh(2\xi \ln K)} \right] e^{i(-\kappa x + \omega t + \theta_0)},$$

(73)

and
which represent the combo bright-singular soliton solution. In particular, when $\lambda = 4S^2$, we have the bright soliton solution:

$$u(x, t) = \sqrt{\frac{8(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \sech (2\xi \ln K) \, e^{i(-\kappa x + \omega t + \theta_0)},$$

(74)

and

$$v(x, t) = \sqrt{\frac{8(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \sech (2\xi \ln K) \, e^{i(-\kappa x + \omega t + \theta_0)},$$

(75)

provided $(L_4 + 8L_2 \ln^2 K) > 0$, while if $\lambda = -4S^2$, we have the singular soliton solution:

$$u(x, t) = \sqrt{-\frac{8(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \csch (2\xi \ln K) \, e^{i(-\kappa x + \omega t + \theta_0)},$$

(76)

and

$$v(x, t) = \sqrt{-\frac{8(L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \csch (2\xi \ln K) \, e^{i(-\kappa x + \omega t + \theta_0)},$$

(77)

provided $(L_4 + 8L_2 \ln^2 K) < 0$. The solutions (71)–(78) exist under the conditions (70).

Similarly, we can find many other solutions by choosing other values for $p$ and $M$.

## 5 Conclusions

Many exact solutions have been obtained for the coupled system of cubic-quartic nonlinear Schrödinger equation (CQ-NLSE) in fiber Bragg gratings with parabolic non-local combo law of nonlinearity. The two integrated schemes, namely the extended auxiliary equation method and the addendum to Kudryashov’s method are applied. The Jacobi elliptic function solutions, bright, dark and singular solitons from the governing model are obtained. These soliton solutions depend on certain constraint conditions and are given together with their existence criteria. Finally, our solutions have been checked using the Maple by putting them back into the original equations.

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