GENERALIZED LUCAS NUMBERS AND RELATIONS WITH GENERALIZED FIBONACCI NUMBERS

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Abstract. In this paper, we present a new generalization of the Lucas numbers by matrix representation using Generalized Lucas Polynomials. We give some properties of this new generalization and some relations between the generalized order-$k$ Lucas numbers and generalized order-$k$ Fibonacci numbers. In addition, we obtain Binet formula and combinatorial representation for generalized order-$k$ Lucas numbers by using properties of generalized Fibonacci numbers.

1. Introduction

There are various types of generalization of Fibonacci and Lucas numbers. For example Er [1] defined the generalized order-$k$ Fibonacci numbers (GO$k$F), Kılıç [6] defined the generalized order-$k$ Pell numbers (GO$k$P) and Taşçi [3] defined the generalized order-$k$ Lucas numbers (GO$k$L). MacHenry [7] defined Generalized Fibonacci and Lucas Polynomials and MacHenry [8] defined matrices $A_{\infty}^{(k)}$ and $D_{\infty}^{(k)}$ depending on these polynomials. $A_{\infty}^{(k)}$ is reduced to GO$k$F when $t_1 = 1$ and $A_{\infty}^{(k)}$ is reduced to GO$k$P when $t_1 = 2$ and $t_i = 1$ (for $2 \leq i \leq k$). This analogy shows the importance of the matrix $A_{\infty}^{(k)}$ and Generalized Fibonacci and Lucas polynomials in generalizations. However, Lucas generalization of Taşçi [3] is not compatible with the matrix $A_{\infty}^{(k)}$ and Generalization Fibonacci and Lucas polynomials, we studied on generalized order-$k$ Lucas numbers $l_{k,n}$ (GO$k$L) and $k$ sequences of the generalized order-$k$ Lucas numbers $l_{i,k,n}$ (kSO$k$L) with the help of Lucas Polynomials $G_{k,n}$ and the matrix $D_{\infty}^{(k)}$. In this paper, after presenting a matrix representation of $l_{i,k,n}$, we derived a relations between generalized order-$k$ Fibonacci numbers (GO$k$F) and GO$k$L, as well as relation between kSO$k$L and $k$ sequences of the generalized order-$k$ Fibonacci numbers $f_{i,k,n}$ (kSO$k$F). Since many properties of Fibonacci numbers and it’s generalizations are known, these relations are very important. Using these relations, properties of Lucas numbers and properties of it’s generalizations can be obtained. In addition to obtaining these relations, we give a generalized Binet formula and combinatorial representation for kSO$k$L with the help of properties of generalized Fibonacci numbers.

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1.1. Fibonacci and Lucas Numbers and Properties of Fibonacci Generalization.

The well-known Fibonacci sequence \( \{f_n\} \) is defined recursively by the equation,
\[
f_n = f_{n-1} + f_{n-2}, \quad \text{for } n \geq 3
\]
where \( f_1 = 1 \), \( f_2 = 1 \) and Lucas sequence \( \{l_n\} \) is defined recursively by the equation,
\[
l_n = l_{n-1} + l_{n-2}, \quad \text{for } n \geq 2
\]
where \( l_0 = 2 \), \( l_1 = 1 \).

Miles [10] defined generalized order-\( k \) Fibonacci numbers (GO\( k \)F) as,
\[
f_{k,n} = \sum_{j=1}^{k} f_{k,n-j}
\]
for \( n > k \geq 2 \), with boundary conditions: \( f_{k,1} = f_{k,2} = f_{k,3} = \cdots = f_{k,k-2} = 0 \), \( f_{k,k-1} = f_{k,k} = 1 \).

Er [1] defined \( k \)SO\( k \)F as; for \( n > 0 \), \( 1 \leq j \leq k \)
\[
f_{i,k,n} = \sum_{j=1}^{k} c_j f_{i,n-j}
\]
with boundary conditions for \( 1 - k \leq n \leq 0 \),
\[
f_{i,k,n} = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,}
\end{cases}
\]
where \( c_j \) (\( 1 \leq j \leq k \)) are constant coefficients, \( f_{i,n} \) is the \( n \)-th term of \( i \)-th sequence of order \( k \) generalization. \( k \)-th column of this generalization involves the Miles generalization for \( i = k \), i.e. \( f_{k,n} = f_{k,k+n-2} \).

Er [1] showed
\[
F_{n+1} = AF_n
\]
where
\[
A = \begin{bmatrix}
c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]
is \( k \times k \) companion matrix and
\[
F_n^\sim = \begin{bmatrix}
f_{k,n}^1 & f_{k,n}^2 & \cdots & f_{k,n}^k \\
f_{k,n-1}^1 & f_{k,n-1}^2 & \cdots & f_{k,n-1}^k \\
\vdots & \vdots & \ddots & \vdots \\
f_{k,n-k+1}^1 & f_{k,n-k+1}^2 & \cdots & f_{k,n-k+1}^k
\end{bmatrix}
\]
is \( k \times k \) matrix.

Karaduman [5] showed \( F_i^\sim = A \) and \( F_n^\sim = A^n \) for \( c_j = 1 \), \( 1 \leq j \leq k \).

Kalman [2] derived the Binet formula by using Vandermonde matrix, for \( \lambda_i \) \( (1 \leq i \leq k) \) are roots of the polynomial
\[
P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k
\]
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\[(t_1, \ldots, t_k \text{ are constants})\]

\[
f_{k,n}^k = \sum_{i=1}^k \frac{(\lambda_i)^n}{P' (\lambda_i)}
\]

where \(f_{k,n}^k\) is (for \(c_j = 1, 1 \leq j \leq k\) and \(i = k\)) \(k\)-th sequences of kSOkF and \(P(x)\) is derivative of the polynomial (1.4).

Kılıç [5] studied \(F_n^\sim\) and \(f_{k,n}^k\) and gave some formulas and properties concerning kSOkF. One of these is Binet formula for kSOkF. For roots of (1.4) named as \(\lambda_i\) \((1 \leq i \leq k)\),

\[
V = \begin{bmatrix}
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \\
\lambda_1^{k-2} & \lambda_2^{k-2} & \cdots & \lambda_k^{k-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}
\quad \text{and} \quad
d_k' = \begin{bmatrix}
\lambda_1^{k-i+n} \\
\lambda_2^{k-i+n} \\
\vdots \\
\lambda_k^{k-i+n}
\end{bmatrix}
\]

where \(V\) is a \(k \times k\) Vandermonde matrix and \(V^{(i)}_j\) is a \(k \times k\) matrix obtained from \(V\) by replacing \(j\)-th column of \(V\) by \(d_k'\), Binet formula of \(f_{k,n}^k\) is;

\[
f_{k,n}^k = t_{1k} = \frac{\det(V^{(1)}_k)}{\det(V)}.
\]

1.2. Generalized Fibonacci and Lucas Polynomials. MacHenry [7] defined generalized Fibonacci polynomials \((F_{k,n}(t))\), Lucas polynomials \((G_{k,n}(t))\) and obtained important relations between generalized Fibonacci and Lucas polynomials, where \(t_i\) \((1 \leq i \leq k)\) are constant coefficients of the core polynomial (1.4). \(F_{k,n}(t)\) defined inductively by

\[
\begin{align*}
F_{k,n}(t) &= 0, \quad n < 0 \\
F_{k,0}(t) &= 1 \\
F_{k,1}(t) &= t_1 \\
F_{k,n+1}(t) &= t_1F_{k,n}(t) + \cdots + t_kF_{k,n-k+1}(t)
\end{align*}
\]

where \(t = (t_1, t_2, \ldots, t_k)\), \(k \in \mathbb{N}\), \(n\) is an integer and \(G_{k,n}(t_1, t_2, \ldots, t_k)\) defined by

\[
\begin{align*}
G_{k,n}(t) &= 0, \quad n < 0 \\
G_{k,0}(t) &= k \\
G_{k,1}(t) &= t_1 \\
G_{k,n+1}(t) &= t_1G_{k,n}(t) + \cdots + t_kG_{k,n-k+1}(t).
\end{align*}
\]

In addition, in [9] authors obtained \(F_{k,n}(t)\) and \(G_{k,n}(t)\) \((n, k \in \mathbb{N}, n \geq 1)\) as

\[
F_{k,n}(t) = \sum_{a=n} \binom{|a|}{a_1, \ldots, a_k} t_1^{a_1} \cdots t_k^{a_k}
\]

and
where \(a_i\) are nonnegative integers for all \(1 \leq i \leq k\), with initial conditions given by

\[
F_{k,0}(t) = 1, \quad F_{k,-1}(t) = 0, \quad \ldots, \quad F_{k,-k+1}(t) = 0.
\]

and

\[
G_{k,0}(t) = k, \quad G_{k,-1}(t) = 0, \quad \ldots, \quad G_{k,-k+1}(t) = 0.
\]

In this paper, the notations \(a \vdash n\) and \(|a|\) are used instead of \(\sum_{a \vdash n} a_{a_1,\ldots,a_k} t_1^{a_1} \ldots t_k^{a_k}\), respectively. A combinatorial representation for Fibonacci polynomials is given in [9] as

\[
(1.11) \quad F_{2,n}(t) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n-j}{j} F_{n-2j}(-t_2)^j
\]

for \(n \in \mathbb{Z}\), where \(\left\lfloor \frac{n}{2} \right\rfloor = k\), either \(n = 2k\) or \(n = 2k - 1\).

In [8], matrices \(A_{(k)}^\infty\) and \(D_{(k)}^\infty\) are defined by using the following matrix,

\[
A_{(k)} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
t_k & t_{k-1} & t_{k-2} & \ldots & t_1 
\end{bmatrix}
\]

They also record the orbit of the \(k\)-th row vector of \(A_{(k)}\) under the action of \(A_{(k)}\), below \(A_{(k)}\), and the orbit of the first row of \(A_{(k)}\) under the action of \(A_{(k)}^{-1}\) on the first row of \(A_{(k)}\) is recorded above \(A_{(k)}\), and consider the \(\infty \times k\) matrix whose row vectors are the elements of the doubly infinite orbit of \(A_{(k)}\) acting on any one of them. For \(k = 3\), \(A_{(3)}^\infty\) looks like this

\[
A_{(3)}^\infty = \begin{bmatrix}
\cdots & \cdots & \cdots \\
S_{(-n,1,2)} & -S_{(-n,1)} & S_{(-n)} \\
\cdots & \cdots & \cdots \\
1 & S_{(-3,1,2)} & -S_{(-3,1)} & S_{(-3)} \\
0 & 1 & 0 \\
0 & 0 & 1 \\
t_3 & t_2 & t_1 \\
\cdots & \cdots & \cdots \\
S_{(n-1,1,2)} & -S_{(n-1,1)} & S_{(n-1)} \\
S_{(n,1,2)} & -S_{(n,1)} & S_{(n)} \\
\cdots & \cdots & \cdots 
\end{bmatrix}
\]

and
\[ A_{(k)}^n = \begin{bmatrix}
(-1)^{k-1}S_{(n-k+1,1^{k-1})} & \cdots & (-1)^{k-j}S_{(n-k+1,1^{k-j})} & \cdots & S_{(n-k+1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
(-1)^{k-1}S_{(n,1^{k-1})} & \cdots & (-1)^{k-j}S_{(n,1^{k-j})} & \cdots & S_{(n)}
\end{bmatrix} \]

where

\[ S_{(n-r,1^r)} = (-1)^r \sum_{j=r+1}^{n} t_j S_{(n-j)}, \quad 0 \leq r \leq n. \]

Derivative of the core polynomial \( P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k \) is \( P(x) = k x^{k-1} - t_1 (k-1) x^{k-2} - \cdots - t_{k-1} \), which is represented by the vector \((-t_{k-1}, \ldots, -t_1 (k-1), k)\) and the orbit of this vector under the action of \( A_{(k)} \) gives the standard matrix representation \( D_{(k)}^{\infty} \).

Right hand column of \( A_{(k)}^{\infty} \) contains sequence of the generalized Fibonacci polynomials \( F_{k,n}(t) \) and \( tr(A_{(k)}^{n}) = G_{k,n}(t) \) for \( n \in \mathbb{Z} \), where \( G_{k,n}(t) \) is the sequence of the generalized Lucas polynomials, which is also a \( t \)-linear recursion. In addition, the right hand column of \( D_{(k)}^{\infty} \) contains sequence of the generalized Lucas polynomials \( G_{k,n}(t) \).

It is clear that, for \( t_i = 1 \) and \( c_i = 1 \) (\( 1 \leq i \leq k \)) \( S_{(n)} = f_{k,n}^1 \) where \( f_{k,n}^1 \) is the \( n \)-th term of the first sequence of \( k \) SOkF. Moreover, the matrix \( A_{(k)}^{\infty} \) involves the generalization (1.2).

**Example 1.1.** We give matrix \( A_{(3)}^{\infty} \) for \( k=3 \) and the matrix \( D_{(4)}^{\infty} \) for \( k=4 \), while \( t_1 = t_2 = \cdots = t_k = 1 \)

\[
A_{(3)}^{\infty} = \begin{bmatrix}
\cdots & \cdots & \cdots \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\cdots & \cdots & \cdots
\end{bmatrix} \quad \text{and} \quad D_{(4)}^{\infty} = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
7 & 1 & 0 & -1 \\
-1 & 6 & 0 & -1 \\
-1 & -2 & 5 & -1 \\
-1 & -2 & -3 & 4 \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

2. **Generalizations of Lucas Numbers**

For \( t_s = 1, 1 \leq s \leq k \), the Lucas polynomials \( G_{k,n}(t) \) and \( D_{(k)}^{\infty} \) together are reduced to

\[ l_{k,n} = \sum_{j=1}^{k} l_{k,n-j} \]

with boundary conditions

\[ l_{k,1-k} = l_{k,2-k} = \ldots = l_{k,-1} = -1 \quad \text{and} \quad l_{k,0} = k, \]

which is called generalized order-\( k \) Lucas numbers (GO\( k \)L). When \( k=2 \), it is reduced to ordinary Lucas numbers.

In this paper, we study on positive direction of \( D_{(k)}^{\infty} \) for \( t_s = 1, 1 \leq s \leq k \), which can be written explicitly as
where

\[ I \]

for \( i \leq k \), with boundary conditions

\[ L^i_{k,n} = \begin{cases} 
-i & \text{if } i - n < k, \\
-2n + i & \text{if } i - n = k, \\
k - i - 1 & \text{if } i - n > k 
\end{cases} \]

for \( 1 - k \leq n \leq 0 \), where \( L^i_{k,n} \) is the \( n \)-th term of \( i \)-th sequence. This generalization is called \( k \) sequences of the generalized order- \( k \) Lucas numbers (kSOkL).

Although names are the same, the initial conditions of this generalization are different from the generalizations in [3]. These initial conditions arise from Lucas Polynomials and \( D_{(k)}^{\infty} \).

When \( i = k = 2 \), we obtain ordinary Lucas numbers and \( L^2_{k,n} = l_{k,n} \).

**Example 2.1.** Substituting \( k = 3 \) and \( i = 2 \) we obtain the generalized order-3 Lucas sequence as;

\[ l_{3,-2}^2 = 0, \ l_{3,-1}^2 = 4, \ l_{3,0}^2 = -2, \ l_{3,1}^2 = 2, \ l_{3,2}^2 = 4, \ l_{3,3}^2 = 4, \ldots \]

**Lemma 2.2.** Matrix multiplication and (2.2) can be used to obtain

\[ L_{n+1}^\sim = A_1 L_n^\sim \]

where

\[ A_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{k \times k} = \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ & \cdots & 1 \end{bmatrix}_{k \times k} \]

where \( I \) is \((k - 1) \times (k - 1)\) identity matrix and we define a \( k \times k \) matrix \( L_n^\sim \) as;

\[ L_n^\sim = \begin{bmatrix} l_{k,n}^1 & l_{k,n}^2 & \cdots & l_{k,n}^k \\ l_{k,n-1}^1 & l_{k,n-1}^2 & \cdots & l_{k,n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ l_{k,n-k+1}^1 & l_{k,n-k+1}^2 & \cdots & l_{k,n-k+1}^k \end{bmatrix}_{k \times k} \]

which is contained by \( k \times k \) block of \( D_{(k)}^{\infty} \) for \( t_i = 1, 1 \leq i \leq k \).

**Lemma 2.3.** Let \( A_1 \) and \( L_n^\sim \) be as in (2.3) and (2.4), respectively. Then

\[ L_{n+1}^\sim = A_1^{n+1} L_0^\sim, \]

where

\[ L_0^\sim = \begin{bmatrix} -1 & -2 & -3 & \cdots & -(k - 2) & -(k - 1) & k \\ -1 & -2 & -3 & \cdots & -(k - 2) & k + 1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -2 & 2k - 3 & \cdots & 1 & 0 & -1 \\ -1 & 2k - 2 & k - 4 & \cdots & \vdots & \vdots & \vdots \\ 2k - 1 & k - 3 & k - 4 & \cdots & 1 & 0 & -1 \end{bmatrix}_{k \times k} \]
Lemma 2.4. Let $F_n^\sim$ and $L_n^\sim$ be as in (1.3) and (2.4), respectively. Then
\[ L_n^\sim = F_n^\sim L_0^n. \]

Proof. Proof is trivial from $F_n^\sim = A^n I$ (see [4]) and Lemma 2.3. \qed

Example 2.5. From Lemma 2.4 for $k = 2$, we have
\[ \begin{pmatrix} \ell_{2,n}^1 & \ell_{2,n}^2 \\ \ell_{2,n-1}^1 & \ell_{2,n-1}^2 \end{pmatrix} = \begin{pmatrix} f_{2,n}^1 & f_{2,n}^2 \\ f_{2,n-1}^1 & f_{2,n-1}^2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}. \]
Therefore, $\ell_{2,n}^2 = 2f_{2,n}^2 - f_{2,n}^1$. Since $f_{2,n}^1 = f_{2,n+1}^2$ for all $n \in \mathbb{Z}$, then we have
\[ \ell_{2,n}^2 = 2f_{2,n+1}^2 - f_{2,n}^2, \]
where $\ell_{2,n}^2$ and $f_{2,n}^2$ are ordinary Lucas and Fibonacci numbers, respectively.

For $k = 3$, we have
\[ \begin{pmatrix} \ell_{3,n}^1 & \ell_{3,n}^2 & \ell_{3,n}^3 \\ \ell_{3,n-1}^1 & \ell_{3,n-1}^2 & \ell_{3,n-1}^3 \\ \ell_{3,n-2}^1 & \ell_{3,n-2}^2 & \ell_{3,n-2}^3 \end{pmatrix} = \begin{pmatrix} f_{3,n}^1 & f_{3,n}^2 & f_{3,n}^3 \\ f_{3,n-1}^1 & f_{3,n-1}^2 & f_{3,n-1}^3 \\ f_{3,n-2}^1 & f_{3,n-2}^2 & f_{3,n-2}^3 \end{pmatrix} \begin{pmatrix} -1 & -2 & 3 \\ -1 & 4 & -1 \\ 5 & 0 & -1 \end{pmatrix}. \]
Therefore, $\ell_{3,n}^3 = 3f_{3,n}^3 - f_{3,n}^2 - f_{3,n}^1$. Since for $k=3$, $f_{3,n}^1 = f_{3,n+1}^3$ and $f_{3,n}^2 = f_{3,n-1}^3 + f_{3,n-2}^3$, then for all $n \in \mathbb{Z}$, we have
\[ \ell_{3,n}^3 = 3f_{3,n+1}^3 - 2f_{3,n}^2 - f_{3,n}^1. \]

Theorem 2.6. For $i = k$, $n \geq 0$ and $c_1 = \cdots = c_k = 1$,
\[ (2.5) \quad t_{k,n}^i = kf_{k,n+1}^i - (k-1)f_{k,n}^i - \cdots - f_{k,n-k+2}^i = kf_{k,n+1}^i - \sum_{j=2}^{k} (k-j+1)f_{k,n+2-j} \]
where $t_{k,n}^i$ and $f_{k,n}^i$ are $k$-SOKL and $k$-SOKF, respectively.

Proof. We use mathematical induction to prove the following equality
\[ t_{k,n}^i = kf_{k,n+1}^i - \sum_{j=2}^{k} (k-j+1)f_{k,n+2-j}. \]
It is easy to obtain $t_{k,0}^i = k$, $f_{k,0}^i = 0$ and $f_{k,1}^i = 1$ for all $k \in \mathbb{Z}^+$ with $k \geq 2$, from the definition of $k$-SOKL and $k$-SOKF. So, the equation (2.5) is true for $n = 0$, i.e.,
\[ t_{k,0}^i = kf_{k,1}^i - (k-1)f_{k,0}^i - \cdots - f_{k,-k+2}^i = k.1 + 0 = k. \]
Suppose that the equation holds for all positive integers less than or equal to $n$ i.e., for integer $n$,
\[ t_{k,n}^i = kf_{k,n+1}^i - \sum_{j=2}^{k} (k-j+1)f_{k,n+2-j} \]
then from (1.2) and (2.2), for $c_1 = \cdots = c_k = 1$, we get;
\[
\begin{align*}
\ell_{k,n+1}^k &= \ell_{k,n}^k + \ell_{k,n+1}^k + \ell_{k,n+2}^k + \cdots + \ell_{k,n-k+1}^k \\
&= (kf_{k,n+1}^k - (k-1)f_{k,n}^k - \cdots - f_{k,n-k+2}^k) + \\
&\quad (kf_{k,n}^k - (k-1)f_{k,n-1}^k - \cdots - f_{k,n-k+1}^k) + \\
&\quad \cdots + (kf_{k,n-k+2}^k - (k-1)f_{k,n-k+1}^k - \cdots - f_{k,n-k+3}^k) \\
&= kf_{k,n+2}^k - (k-1)f_{k,n+1}^k - \cdots - f_{k,n-k+3}^k \\
&= kf_{k,n+2}^k - \sum_{j=2}^{k} (k-j+1)f_{k,n+3-j}^k.
\end{align*}
\]

So, the equation holds for \((n+1)\) and proof is complete. \qed

Since \(f_{k,n}^k = f_{k,n+k-2}\) and \(\ell_{k,n}^k = \ell_{k,n}\) the following relation is obvious

\[
\ell_{k,n} = kf_{k,n+k-1} - \sum_{j=2}^{k} (k-j+1)f_{k,n+k-j}
\]

where \(f_{k,n}\) is the \(n\)-th GO\(k\)F as in (1.1), \(\ell_{k,n}\) is GO\(k\)L as in (2.1) and \(\ell_{k,n}^k\) is the \(n\)-th term of \(k\)-th sequences of the GO\(k\)L as in (2.2).

The following theorem shows that equation (2.5) is valid for Generalized Fibonacci and Lucas Polynomials as well.

**Theorem 2.7.** For \(k \geq 2\) and \(n \geq 0\),

\[
G_{k,n}(t) = kF_{k,n}(t) - \sum_{j=2}^{k} (k-j+1)t_{j-1}F_{k,n+1-j}(t)
\]

where \(F_{k,n}(t)\) and \(G_{k,n}(t)\) are the Generalized Fibonacci and Lucas Polynomials, respectively.

**Proof.** Proof is by induction as Theorem 2.6. \qed

**Theorem 2.8.** For \(i = k\) and \(n \geq 0\),

\[
\ell_{k,n}^k = \sum_{j=1}^{k} jf_{k,n+1-j}^k
\]

where \(\ell_{k,n}^k\) and \(f_{k,n}^i\) are the GO\(k\)L and GO\(k\)F respectively.

**Proof.** Proof is by induction as Theorem 2.6. \qed

**Lemma 2.9.** For \(k \geq 2\), \(i\)-th sequences of GO\(k\)L in terms of \(k\)-th sequences of GO\(k\)L is

\[
l_{k,n}^i = \begin{cases} 
\ell_{k,n-1}^i & \text{if } i = 1 \\
\sum_{m=1}^{i} \ell_{k,n-m}^i & \text{if } 1 < i < k \\
\ell_{k,n}^k & \text{if } i = k
\end{cases}
\]

**Theorem 2.10.** \(i\)-th sequences of GO\(k\)L can be written in terms of \(k\)-th sequences of GO\(k\)F (which is GO\(k\)F with index iteration) in different ways;
Example 2.11. Let us obtain \( l^i_{k,n} \) for \( k = 4, n = 4 \) and \( i = 3 \) by using Theorem (2.10 (iii)).

\[
l^3_{4,4} = 3 \sum_{m=1}^{4} j \cdot f^4_{4,4-m-j+1} = 3 \sum_{m=1}^{4} (f^4_{4,4-m} + 2f^4_{4,3-m} + 3f^4_{4,2-m} + 4f^4_{4,1-m})
= f^4_{4,3} + 2f^4_{4,2} + 3f^4_{4,1} + 4f^4_{4,0} + f^4_{4,2} + 2f^4_{4,1} + f^4_{4,1} = 11
\]

since \( f^4_{4,0} = 0, f^4_{4,1} = f^4_{4,2} = 1 \) and \( f^4_{4,3} = 2 \).

Theorem 2.12. Let \( l^i_{k,n} \) and \( f^i_{k,n} \) be the kSOkL and kSOkF, respectively. Then, for \( m, n \in \mathbb{Z} \) and \( 1 \leq i \leq k-1 \),

\[
l^i_{n+m} = \sum_{j=1}^{i} (l^k_{m-j} \sum_{s=1}^{j} f^s_{n}) + \sum_{j=1}^{k} (l^k_{m-j} \sum_{s=j-i+1}^{j} f^s_{n}) + \sum_{j=k+1}^{k+i-1} (l^k_{m-j} \sum_{s=j-i+1}^{k} f^s_{n})
\]

where we assume that, the sum is equal to zero, if the subscript is greater than the superscript in the sum.
Proof. We know that $L_{n}^{\sim} = F_{n}^{\sim} L_{0}^{\sim}$ (Lemma 2.4), so we can write that

$$L_{n+m}^{\sim} = F_{n+m}^{\sim} L_{0}^{\sim} = A_{1}^{n+m} L_{0}^{\sim} = A_{1}^{n} L_{m}^{\sim} = F_{n} L_{m}^{\sim}.$$  

From this matrix product and Lemma 2.9 we obtain

$$l_{k,n+m}^{i} = f_{k,n}^{1} l_{k,m}^{1} + \cdots + f_{k,n}^{k} l_{k,m-k+1}^{i} = f_{k,n}^{1} (l_{k,m-1}^{i} + \cdots + l_{k,m-i}^{i}) + \cdots + f_{k,n}^{k} (l_{k,m-k}^{i} + \cdots + l_{k,m-k+i}^{i})$$

$$= l_{k,m-1}^{i} f_{k,n}^{1} + l_{k,m-2}^{i} f_{k,n}^{2} + \cdots + l_{k,m-i}^{i} f_{k,n}^{i} + l_{k,m-i}^{i} (f_{k,n}^{i+1} + \cdots + f_{k,n}^{k}, k) + l_{k,m-k}^{i} (f_{k,n}^{k} + \cdots + f_{k,n}^{k}) + l_{k,m-k+i}^{i} f_{k,n}^{k}$$

$$= \sum_{j=1}^{i} \left( l_{k,m-j}^{i} \sum_{i=1}^{j} f_{k,n}^{i} \right) + \sum_{j=i+1}^{k} \left( l_{k,m-j}^{i} \sum_{j=i+1}^{k} f_{k,n}^{i} \right) + \sum_{j=k+1}^{k+i-1} \left( l_{k,m-j}^{i} \sum_{i=1}^{k} f_{k,n}^{i} \right).$$

Example 2.13. Let us obtain $l_{k,n+m}^{i}$ for $k = 5$, $i = 3$, $n = 3$ and $m = 4$ by using Theorem 2.12;

$$l_{5,3+4}^{3} = \sum_{j=1}^{3} \left( l_{5,4-j}^{5} \sum_{i=1}^{j} f_{5,3}^{i} \right) + \sum_{j=4}^{5} \left( l_{5,4-j}^{5} \sum_{i=1}^{j} f_{5,3}^{i} \right) + \sum_{j=6}^{7} \left( l_{5,4-j}^{5} \sum_{i=1}^{j} f_{5,3}^{i} \right)$$

$$= l_{5,3}^{3} f_{5,3}^{1} + l_{5,2}^{3} f_{5,3}^{2} f_{5,3}^{2} + l_{5,1}^{3} f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3} + f_{5,3}^{3}$$

$$= 28 + 24 + 12 + 55 - 9 - 5 - 2 = 103.$$

\[ \square \]

2.0.1. Binet Formula. We have the following corollary by (1.5) and (Theorem 2.10 (iii)).

Corollary 2.14. For $1 \leq i \leq k$ and $m, n \in \mathbb{Z}^{+},$

$$l_{k,n}^{i} = \begin{cases} \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{(\lambda_{j})^{n-j}}{P(\lambda_{j})} & \text{for } i = 1 \\ \sum_{m=1}^{i} \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{(\lambda_{j})^{n-m-j+1}}{P(\lambda_{j})} & \text{for } 1 < i < k \\ \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{(\lambda_{j})^{n-i+1}}{P(\lambda_{j})} & \text{for } i = k \end{cases}.$$  

where $l_{k,n}^{i}$ is the $kSO_{k}L.$

We have the following corollary by (1.7) and (Theorem 2.10 (iii)).

Corollary 2.15. Let $l_{k,n}^{i}$ be the $kSO_{k}L.$ Then, for $1 \leq i \leq k$ and $m, n \in \mathbb{Z}^{+},$

$$l_{k,n}^{i} = \begin{cases} \sum_{j=1}^{k} \frac{\det(V_{k,n-j}^{(i)})}{\det(V)} & \text{for } i = 1 \\ \sum_{m=1}^{i} \sum_{j=1}^{k} \frac{\det(V_{k,n-m-j+1}^{(i)})}{\det(V)} & \text{for } 1 < i < k \\ \sum_{j=1}^{k} \frac{\det(V_{k,n-j+1}^{(i)})}{\det(V)} & \text{for } i = k \end{cases}.$$
respectively. Then, for all \( m, n \) \( (2.8) \)

\[
\begin{pmatrix}
\lambda_1^{k-1+n-s} \\
\lambda_2^{k-1+n-s} \\
\vdots \\
\lambda_k^{k-1+n-s}
\end{pmatrix}
\]

2.1. Combinatorial Representation of the Generalized Order-\( k \) Fibonacci and Lucas Numbers. In this subsection, we obtain some combinatorial representations of \( i \)-th sequences of kSO\( k \)F and kSO\( k \)L with the help of combinatorial representations of Generalized Fibonacci and Lucas Polynomials. \( i \)-th sequences of kSO\( k \)F can be stated in terms of \( k \)-th sequences of kSO\( k \)F as follows. For \( c_i = 1 \) \( (1 < i < k) \),

\[
f_{k,n}^i = \sum_{m=1}^{k-i+1} f_{k,n-m+1}^k.
\]

For \( t_i = 1 \) \( (1 < i < k) \), \( F_{k,n-1}(t) \) is reduced to sequence \( f_{k,n}^1 \). So for \( t_i = 1 \) \( (1 < i < k) \), \( f_{k,n}^i = \sum_{m=1}^{k-i+1} F_{k,n-m}(t) \) and using (1.9) we have

\[
f_{k,n}^i = \sum_{m=1}^{k-i+1} \sum_{a^{(n-m)}} \binom{|a|}{a_1, \ldots, a_k}.
\]

It is obvious that, for \( t_i = 1 \) \( (1 < i < k) \), \( F_{k,n}(t) = F_{k,n}^1 \) and \( F_{k,n}(t) = F_{k,n}^{k+1} \), respectively. Then, for all \( m, n \in \mathbb{Z}^+ \),

\[
(2.8) \quad f_{k,n}^i = \begin{cases} 
\sum_{a^{(n-1)}} \binom{|a|}{a_1, \ldots, a_k} & \text{if } i = 1 \\
\sum_{m=1}^{k-i+1} \sum_{a^{(n-m)}} \binom{|a|}{a_1, \ldots, a_k} & \text{if } 1 < i < k \\
\sum_{a^{(n)}} \binom{|a|}{a_1, \ldots, a_k} & \text{if } i = k
\end{cases}
\]

Lemma 2.16. [5] Let \( f_{k,n}^i \) be the \( k \)-th sequences of kSO\( k \)F, then,

\[
f_{k,n}^i = \sum_{m^{(n-1+k)}} \frac{m_k}{m} \times \binom{|m|}{m_1, \ldots, m_k}
\]

where \( m = (m_1, m_2, \ldots, m_k) \) nonnegative integers satisfying \( m_1 + 2m_2 + \ldots + km_k = n - 1 + k \). In addition for \( 0 \leq i \leq n - 1 \)

\[
f_{k,n-i}^i = \sum_{m^{(n-i+k-1)}} \frac{m_k}{m} \times \binom{|m|}{m_1, \ldots, m_k}
\]

where the summation is over nonnegative integers satisfying \( m_1 + 2m_2 + \ldots + km_k = n - 1 - i + k \).

Then we have the following corollary using (Theorem 2.10. \( \text{iii} \)).
Corollary 2.17. Let $l^1_{k,n}$ be the $k$SO$k$L, then, for $m, n \in \mathbb{Z}^+$,

$$l^1_{k,n} = \begin{cases} \sum_{i=1}^{k} j \sum_{m=1}^{n} \frac{m_{jk}}{|m|} \times (m_{j1}, \ldots, m_{jk}) & \text{if } i = 1 \\ \sum_{m=1}^{k} \sum_{j=1}^{n} \frac{t_{jk}}{|m|} \times (t_{m1}, \ldots, t_{mj}) & \text{if } 1 < i < k \\ \sum_{j=1}^{k} \sum_{m=1}^{n} \frac{m_{jk}}{|m|} \times (m_{j1}, \ldots, m_{jk}) & \text{if } i = k \end{cases}$$

where $t = (t_{m1}, \ldots, t_{mj})$ and $m = (m_{j1}, m_{j2}, \ldots, m_{jk})$.

Corollary 2.18. Let $l^1_{k,n}$ be the $k$SO$k$L, then, for all $m, n \in \mathbb{Z}^+$

$$l^1_{k,n} = \begin{cases} \sum_{a^{(n-1)}} i \frac{n-1}{|a|} (|a|_{a_1, \ldots, a_k}) & \text{if } i = 1 \\ \sum_{m=1}^{k} \sum_{a^{(n-m)}} \frac{n-m}{|a|} (|a|_{a_1, \ldots, a_k}) & \text{if } 1 < i < k \\ \sum_{a^{(n)}} \frac{n}{|a|} (|a|_{a_1, \ldots, a_k}) & \text{if } i = k \end{cases}$$

Proof. For $t_i = 1 (1 \leq i \leq k)$, $G_{k,n}$ is reduced to $l^1_{k,n}$. Since $l^k_{k,n} = \sum_{a^{(n)}} \frac{n}{|a|} (|a|_{a_1, \ldots, a_k})$ from (1.10) and by using (2.7) the proof is completed. \qed

Corollary 2.19. Let $l^1_{k,n}$ be the $k$SO$k$L, then, for $1 \leq i \leq k$ and $m, n \in \mathbb{Z}^+$

$$l^1_{k,n} = \begin{cases} \sum_{j=1}^{k} \sum_{a^{(n-1-j)}} j \frac{n-1}{|a|} (|a|_{a_1, \ldots, a_k}) & \text{if } i = 1 \\ \sum_{m=1}^{k} \sum_{j=1}^{n-m} \sum_{a^{(n-m-j)}} j \frac{n-m}{|a|} (|a|_{a_1, \ldots, a_k}) & \text{if } 1 < i < k \\ \sum_{j=1}^{k} \sum_{a^{(n-j)}} j \frac{n}{|a|} (|a|_{a_1, \ldots, a_k}) & \text{if } i = k \end{cases}$$

Proof. Proof is trivial from (1.9), (2.7). \qed

Corollary 2.20. Let $l^2_{2,n}$ be the second sequence of the $2$SO$2$L, then,

$$l^2_{2,n} = \sum_{j=1}^{2} \left\lceil \frac{n-j}{s} \right\rceil \sum_{s=0}^{n-j} \binom{n-j-s}{s}$$

where $\binom{n}{s}$ is combinations $s$ of $n$ objects, such that $\binom{n}{s} = 0$ if $n < s$.

Proof. In (1.11), $F_{2,n}(t) = \sum_{j=0}^{\lceil \frac{n}{2} \rceil} (-1)^j \binom{n-j}{j} F^{n-2j}(t)(-t_2)^j$ and for $t_i = 1$ and $c_i = 1$ ($1 \leq i \leq k$), $F_{2,n-1}(t)$ is reduced to sequence $f^2_{k,n}$. Proof is completed by using $f^2_{k,n}(c_i = 1$ for $1 \leq i \leq k$) and (2.10. iii). \qed

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