Linear Regge Trajectories from Worldsheet Lattice Parton Field Theory

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Abstract

We show that unlike conventional field theory, the particle field theory of the string’s constituents produces in the ladder approximation linear Regge trajectories, in accord with its string theory dual. In this theory propagators are Gaussian and this feature facilitates the perturbative evaluation of scattering amplitudes. We develop general techniques for studying their general asymptotic form. We consider radiative corrections to the ladder Regge trajectory and discover that linearity is lost; however, this may be due to certain approximations we have made.

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1 INTRODUCTION

In nonrelativistic quantum mechanics the angular momenta and energies of bound states are related through the language of Regge trajectories [1]. The partial wave Schrödinger wave function (or the scattering amplitude) can be continued to complex values of the angular momentum, and for reasonable potentials has a large domain of meromorphy with poles located at $J = \alpha(E)$. For values of the energy for which the trajectory function $\alpha(E)$ passes through nonnegative (half)integers, the value of $J$ corresponds to the angular momentum of a bound state and $E$ is its energy. In general, for increasing $E$, pole trajectories rise to the right, reach some maximum, and then return to negative values – a situation typical of nonconfining potentials with a finite number of bound states.

In relativistic quantum field theory the scattering amplitude $F(s, t)$ may also exhibit Regge behavior; with suitable analyticity assumptions the partial wave amplitudes can be continued to complex values, with pole trajectories $J = \alpha(s)$ passing through the location of various bound states and resonances. Furthermore, the Regge poles also control the high-energy behavior of the scattering amplitudes in the cross channel, with $F(s, t) \sim \beta(s) t^{\alpha(s)}$ as $t \to \infty$ and $s < 0$. Here the Bethe-Salpeter equation [2] replaces the Schrödinger equation as the basic investigative tool, although its use is relatively limited; the properties of its solution can only be obtained in certain approximations, such as the ladder approximation or perturbative Feynman diagram analysis.

On the experimental side, the data confirms the existence of families of particles lying on rising trajectories $J = \alpha(s)$ that are linear, a fundamental feature, of course, of the Veneziano model and its stringy cousins. However, as in potential scattering, in the various approximations of conventional field theory the trajectories rise for a while and then fall back towards negative values of $J$ for increasing energy. Thus, only a few bound states are produced, as characteristic of a Higgs phase; instead, linearity and an infinite number of bound states are expected to arise as a consequence of confinement, perhaps due to some infrared catastrophe. However, such a catastrophe is absent in the usual calculations, which are always made for massive or off-shell states precisely in order to avoid infrared divergences.

One approach to nonperturbative string theory is quantization on a suitable random lattice representing the worldsheet [3]. The lattice is described by vertices $x_i$ that can be identified with those in a Feynman diagram of an underlying D-dimensional field theory of “partons”. The two theories are “dual” to each other; what is perturbative in one is nonperturbative in the other. In particular, since in the string language the partons are confined, this provides us with a nonperturbative approach to confinement in the corresponding field theory.

Calculations in the lattice theory have been limited mostly to the two-dimensional gravity aspects of string theory; only the dynamics of the worldsheet metric has been studied, and relatively little attention has been devoted to the dynamics of the
corresponding partons. In this paper we consider instead the Feynman diagrams of the particle field theory underlying the bosonic string. The mechanism of “confinement” in this theory differs from that in ordinary field theories because the propagators are Gaussian [4] (but see [5] for a proposal for a string based on ordinary propagators). Both the lack of power-law behavior at large transverse momenta [6] and the absence of poles in the “plasma” phase [7] can be understood as unwelcome symptoms of this feature. Nevertheless, a better understanding of the bound-state mechanism in this model might be helpful in understanding confinement in quantum chromodynamics, or explaining how the graviton can arise as a state in a theory whose only fundamental fields are scalars.

For the bosonic string one starts with the usual functional integral

\[ A = \int Dg DX e^{-S} \] (1)

of the worldsheet continuum action

\[ S = \int \frac{d^2 \sigma}{2\pi} \sqrt{-h} \left[ \frac{1}{2\alpha'} h^{\alpha\beta} (\partial_\alpha X) \cdot (\partial_\beta X) + \mu + \kappa R \right] \] (2)

where the second term is the Liouville (cosmological) term of subcritical string theory, and the last term is the Euler number, identifying \( e^{-\kappa} \) as the string coupling constant.

The worldsheet lattice action is then

\[ S' = \frac{1}{2\alpha'} \sum_{\langle ij \rangle} (x_i - x_j)^2 + \mu \sum_i 1 + \kappa (Euler) \] (3)

where “\( \langle ij \rangle \)” are the links (edges) of the lattice, “\( i \)” are the vertices, and “\( Euler \)” means the Euler number as defined in terms of the numbers of vertices, edges, and faces. This action is integrated as

\[ A = \sum \int \prod dx e^{-S'} = \sum e^{-\mu} \sum_i 1 \int dx \prod_{ij} e^{-\frac{1}{2\alpha'} (x_i - x_j)^2} \] (4)

where the sum over Feynman diagrams replaces functional integration over the worldsheet metric, and the integration is over positions of vertices (except for “external” vertices which are kept fixed; alternatively, the usual external line factors \( e^{ikx} \) can be introduced). Planarity of the lattice worldsheet is enforced by associating the Feynman diagrams with a scalar field that is also an \( N \times N \) matrix, and using the \( 1/N \) expansion. This implies the classical identification

\[ e^{\kappa} = N \] (5)

(which can be modified by worldsheet quantum effects). The action for this scalar field is

\[ \hat{S} = \int d^D x \tr \left[ \frac{1}{2} \phi e^{-\alpha'/2} \phi + G^{a-2} \phi^a \right] \] (6)

where the kinetic operator has been chosen to agree with the propagator \( e^{xp(-x^2/2\alpha')} \) in the amplitude \( A \). The interaction \( \phi^n \) has been chosen arbitrarily; restrictions may
follow from consistency of the worldsheet continuum limit. Its coefficient is again identified by comparison with the amplitude:

\[- NG^2 \sim e^{-\mu}\] (7)

where the proportionality constant depends on normalization of the measure. (This relation can also be modified by quantum effects.) Thus the “perturbative” region of string theory, \(\mu \approx 0\), corresponds to the nonperturbative region of the parton theory, \(-NG^2 \sim 1\), while conversely the perturbative region of the parton theory, \(NG^2 \approx 0\), corresponds to the nonperturbative region of the string theory, \(\mu \approx \infty\). (There is a second string-nonperturbative region, \(\mu \approx -\infty\), corresponding to \(-NG^2 \approx \infty\), which in QCD would be identified with perturbation for the dual “monopole” theory.)

In this work we consider 4-point functions in the parton theory with cubic interaction \(\phi^3\). If we choose “color” singlets in each of the two pairs of external partons, this corresponds to a string propagator; equivalently, we are just analyzing a scattering amplitude to find the Regge trajectory for a two-parton bound state. Another possibility is to introduce for the external states fundamental-representation “quarks” in addition to our adjoint “gluons”, and thus describe open as well as closed strings. In the worldsheet continuum limit, taking the ends of the external propagators on each pair to coincide corresponds to insertion of a ground-state vertex \(exp(ik \cdot X)\). (In the early work on Regge theory the interpretation was a bit different in hadronic physics, where both the scattered particles and their bound states were hadrons, according to the principle of “nuclear democracy”. Here we go back to the original application, where now the constituents are analogous to quarks and gluons, and only the bound states are hadrons.)

We will show that ladder graphs are responsible for a Regge trajectory \(\alpha(s)\) that to lowest order in the coupling is linear, as expected from the string theory associated with the worldsheet continuum limit. However, including some radiative corrections seems to spoil the linearity. This may be due to some approximations we had to make; or it may be that this feature of string theory is not recovered until one includes more corrections or takes the continuum limit.

2 REGGE THEORY

In this section, for the purpose of comparing and contrasting our calculations with those done in the early days of Regge theory, we summarize some of the procedures used, mostly in ordinary \(\phi^3\) theory, to obtain some information about pole location and properties. A good review is contained in ref. [8] and references therein.

Following the original work of Regge, and suggestions that Regge poles might be relevant for the analysis of high-energy scattering, various results were obtained on the basis of analyticity assumptions. The main one was the definition of a suitable continuation to complex angular momentum (the Froissart-Gribov continuation) of
an amplitude which satisfies fixed-energy dispersion relations [9]. It was also shown [10] that a branch cut and/or essential singularity must be present (at \( \ell = -1 \) for the scattering of spinless particles), and that at threshold, \( s = 4m^2 \), an infinite accumulation of poles occurs at \( \ell = -1/2 \). Later on more branch cuts were found from studying particular classes of Feynman diagrams [11].

In relativistic field theory the Bethe-Salpeter equation may be used to study the issue of Regge poles. Lee and Sawyer [12] considered its continuation to complex angular momentum, and in ladder approximation (essentially equivalent to the summation of the diagrams of Fig. 1 in the next section) established the existence of Regge poles corresponding to bound states. An equivalent, somewhat simpler, procedure consists in examining directly the high-energy behavior of scattering amplitudes computed in perturbation theory by summing suitable sets of Feynman diagrams [13]. Since we shall use somewhat similar procedures in our present work, we give some details.

One considers an appropriate set of Feynman diagrams (e.g., ladders) for a two-particle scattering amplitude \( A(s, t) \):

\[
A(s, t) = \int d^4k_i \prod_a \frac{1}{p_a^2 + m^2} \sim \int d^4k_i \int_0^\infty \prod_a d\beta_a e^{-\beta_a(p_a^2 + m^2)/2}
\]

where in our conventions the Mandelstam variables are

\[
s = -(q_1 + q_2)^2 = -(q_3 + q_4)^2, \quad t = -(q_1 - q_3)^2; \quad \eta = (-+++)
\]

Here \( k_i \) are independent loop momenta, and we have introduced Schwinger parameters to exponentiate the propagators. We note that at this point the only difference between an ordinary field theory and our theory is the integration over the parameters. In our case they are fixed at \( \beta_a = \alpha' \).

The Gaussian loop momentum integrations can be carried out, and one is led to an expression of the form

\[
A(s, t) \sim \int_0^\infty \prod_a d\beta_a \frac{N(\beta)}{|C(\beta)|^2} e^{-g(\beta)t - d(s, \beta)}
\]

The large \( t \) behavior of the amplitude (which naively vanishes when \( t \to \infty \)) is dominated by the neighborhood of points in \( \beta \)-space where \( g(\beta) = 0 \). (These points are related to the Landau singularities of the graphs; the Coleman-Norton interpretation is that they correspond to classical configurations of point particles, where the Schwinger parameters are their proper times.) Clearly, when the coefficient of \( t \) vanishes one has set to zero parameters which shortcircuit the diagram (eliminating its \( t \) dependence) by removing some of the propagators. These can be determined by starting, for example, at the incoming end of the diagram, and tracing a minimal path (or paths) to the outgoing end, crossing propagators to be removed [14]. (Compare this with the method we shall use for similar purposes in Section 4.) One evaluates the high-energy behavior simply by setting to zero those \( \beta \)'s everywhere except in
$g(\beta)$ and then carrying out the integration [13]\footnote{Frequently in the literature the momentum integrals were performed with Feynman parameters, which correspond to a uniform scaling of all Schwinger parameters. However, evaluating high-energy behavior requires independent scalings of subsets of the Schwinger parameters. In these papers, the authors therefore “unscaled” the Feynman parameters to re-introduce the Schwinger parameters, then performed the required scalings.}. Examples and other possibilities are given in [8], where diagrams which lead to Regge cuts are also presented.

For the ladder graphs one obtains $g(\beta) = 0$ by setting to zero the parameters which eliminate the rungs. After setting them to zero everywhere except in $g(\beta)$ the integrations can be carried out easily and one obtains for the asymptotic behavior of the ladder with $n$ rungs an expression of the form

$$F_n(s, t) \sim g^2 \frac{1}{t} [g^2 K(s) \ln t]^{n-1}$$

where $K(s)$ is just a self-energy diagram evaluated in two dimensions. (The power $K^{n-1}$ comes from the fact that after shortcircuiting the rungs one is left with a product of bubbles.) Finally, the sum of ladder diagrams gives an asymptotic behavior

$$\sum F_n(s, t) = g^2 t^{\alpha(s)}, \quad \alpha(s) = -1 + g^2 K(s)$$

For later comparison we observe that the logarithmic behavior of the individual contributions and the eventual Regge behavior come from the integration over Schwinger parameters.

The calculation above gives a Regge trajectory correct to order $g^2$. Higher order corrections come from considering generalized ladders, which include, in addition to the single particle exchanges, more complicated “blobs” inserted between the simple rungs. The asymptotic behavior is still obtained by shortcircuiting just the rungs (see ref. [8], p. 147). Yet another possibility comes from replacing completely the simple rungs by “H”-insertions as in Fig. 3(d); two Regge poles are then generated [13].

Although the procedure we have outlined above gives the high $t$ behavior of each graph directly from the Feynman amplitude, a more useful and often more powerful method involves the use of the Mellin transform [15]

$$\tilde{F}(z) \equiv \int_{0}^{\infty} d(-t) F(-t)(-t)^{z-1}$$

(We use $-t$ as a variable because the Mandelstam variables as usually defined become negative when continued to Euclidean space, where expressions are most convergent.) We shall discuss this in more detail in the next section, but we mention here its main advantage: In principle it allows the calculation of all the terms in the asymptotic behavior. As we shall see, it allows easy determination of the asymptotic behavior of our ladder amplitudes, a task which would be somewhat difficult otherwise.

Besides the Bethe-Salpeter approach or the investigation of individual Feynman diagrams, a third method for obtaining some information about Regge trajectories...
was provided by the Reggeization program [16]. One considered the scattering of some elementary particles with spin (this is necessary for the program to work) and asked whether among the Regge pole bound states one discovered, the original particles could be found. (This is connected with the idea of “bootstrapping” or “nuclear democracy” mentioned in the Introduction: There is no distinction between the elementary particles and their bound states.) In this program one begins with partial wave helicity amplitudes computed from tree graphs, and builds up ladders by using unitarity. For small values of $J$, unphysical helicity amplitudes (where the helicity exceeds the total angular momentum) have fixed poles, but through unitarity these fixed poles turn into moving Regge poles, and the hope was that the original particles lie on the corresponding trajectories. (Although the use of unitarity is not completely equivalent to the summation of ladder graphs, calculations seem to indicate that the difference is irrelevant.)

The Reggeization program was successful by showing that in the case of QED a Regge pole trajectory in photon-electron scattering does in fact pass, with the correct quantum numbers and mass, through the position corresponding to the electron. It failed to show that the photon Reggeizes, and also failed in some other cases. However, success was achieved with the advent of renormalizable Yang-Mills theory, when it was shown that the vectors, scalars and fermions Reggeize [17]. Furthermore, although these theories are not renormalizable, it was found that in quantum gravity and supergravity certain necessary conditions for Reggeization of gravitons and gravitini hold [18].

3 LADDER GRAPHS

3.1 Amplitude from Recursion Relations

We consider an amplitude for two incoming particles with momenta $q_1$ and $q_2$ and two outgoing particles with momenta $q_3$ and $q_4$; the particles are off shell. We evaluate the amplitude by solving the Bethe-Salpeter equation in the ladder approximation through the iteration procedure depicted in Fig. 1.

![Fig. 1. Summing ladder diagrams](image)

We denote the $n$-loop diagram by $A_n$. The propagators are Gaussian, $\exp\left[-\frac{1}{2}(k_a - k_b)^2\right]$ and $\exp\left[-\frac{1}{2}(k_a - q_i)^2\right]$ (we have set $\alpha' = 1$). Upon integration over the $k_i$ one produces, aside from some numerical factors, exponentials involving squares and
scalar products of the external momenta. Therefore the amplitude will have the form

\[ A_n = C_n \lambda^{2(n+1)} \exp\left[ -\frac{1}{2} (Q_n q_I^2 + Q'_n q_F^2 - S_n s - T_n t) \right] \equiv C_n \lambda^{2(n+1)} \exp\left[ -\frac{1}{2} E_n \right] \]  

(13)

where

\[ q_I^2 = q_1^2 + q_2^2, \quad q_F^2 = q_3^2 + q_4^2, \quad q^2 = \sum_{i=1}^{4} q_i^2 = q_I^2 + q_F^2 \]  

(14)

and \( \lambda \) is the coupling constant appropriately normalized for the measure \( \int \frac{d^D x}{(2\pi)^{D/2}} \) or \( \int d^D p/(2\pi)^{D/2} \). Therefore one can obtain a recursion relation for the different coefficients by performing the \( n + 1 \)st loop integral. We substitute \( A_n \) into the Bethe-Salpeter iteration described by Fig. 1 and obtain

\[ Q_{n+1} = \frac{Q_n + 1}{3 + 2Q_n + T_n} \]  

(15)

\[ T_{n+1} = \frac{T_n}{3 + 2Q_n + T_n} \]  

(16)

\[ Q'_{n+1} = Q'_n + \frac{T_n (Q_n + 1)}{3 + 2Q_n + T_n} \]  

(17)

\[ S_{n+1} = S_n + \frac{(Q_n + 1)^2}{3 + 2Q_n + T_n} \]  

(18)

\[ C_{n+1} = \frac{C_n}{(3 + 2Q_n + T_n)^{D/2}} \]  

(19)

For the ladder diagrams one actually has, by obvious symmetry, \( Q'_n = Q_n \) but for a more general situation with a slightly different input for the right-hand end of the diagram, they could be different\(^5\). For the time being we have also left the dimension \( D \) arbitrary.

To solve the recursion relations we need initial conditions which can be obtained from the tree graph:

\[ Q_0 = Q'_0 = S_0 = 0 \]
\[ T_0 = C_0 = 1 \]  

(20)

from which it immediately follows that

\[ C_n = T_n^{D/2} \]  

(21)

The linear combination

\[ I_n \equiv 3 + 2Q_n + T_n \]  

(22)

satisfies a simple recursion relation.

\[ I_{n+1} = 4 - \frac{1}{I_n} \]  

(23)

For large \( n \) the difference between \( Q_n \) and \( Q'_n \) vanishes anyway.
with the initial condition $I_0 = 4$. It can be solved to give

$$I_n = I_+ \frac{\tilde{I} x^{2(n+1)} - 1}{\tilde{I} x^{2n} - 1} \quad (24)$$

where

$$I_{\pm} = 2 \pm \sqrt{3}$$

are the fixed points of the recursion relation (23), $\tilde{I}$ is a constant, and

$$x \equiv I_- = \frac{1}{I_+} = 2 - \sqrt{3} \simeq .27 \quad (25)$$

The recursion relation for $T_n$ can now be solved as the numerators and the denominators cancel:

$$T_n = \frac{1}{(I_+)^n (\tilde{I} x^{2n} - 1)} \quad (26)$$

From the initial conditions we have

$$\tilde{I} = x^2 \text{ and } \tilde{T} = x^2 - 1 \quad (27)$$

and the solutions can be rewritten as

$$I_n = I_+ \left( \frac{1 - x^{2(n+2)}}{1 - x^{2(n+1)}} \right) \quad (28)$$

and

$$T_n = \frac{x^n (1 - x^2)}{(1 - x^{2(n+1)})} \quad (29)$$

From these $Q_n$ is determined as well.

Next, we look at the combination

$$P_n = 4Q_n + 4S_n \quad (30)$$

The recursion relation for $P_n$ is simply

$$P_{n+1} = P_n + 2 \quad (31)$$

once we use the recursion relation for $Q_{n+1}'$ and impose $Q_{n+1}' = Q_{n+1}$. Then

$$P_n = 2n \quad (32)$$

Knowing $P_n$ and $Q_n$ we determine

$$S_n = \frac{1}{2} \left( n + 3 + T_n - I_n \right) \quad (33)$$

which simplifies to

$$S_n = \frac{1}{2} \left[ n + 1 - \left( \frac{I_+ - I_-}{2} \right) \left( \frac{1 - x^{n+1}}{1 + x^{n+1}} \right) \right] \quad (34)$$
Finally, it is easy to obtain

\[ C_n = \left(\frac{x^n(1-x^2)}{1-x^{2(n+1)}}\right)^{\frac{D}{2}} \]  

(35)

where we have used the initial condition

\[ C_0 = 1 \]  

(36)

To summarize we have

\[ A_n = C_n \lambda^{2(n+1)} e^{-\frac{1}{2}E_n} \]  

(37)

with

\[ E_n = \frac{1}{2}q^2n - S_n(s + q^2) - T_n t \]
\[ S_n = \frac{1}{2} \left[ n + 1 - \sqrt{3} \left( \frac{1-x^{n+1}}{1+x^{n+1}} \right) \right] \]
\[ T_n = \frac{x^n(1-x^2)}{(1-x^{2(n+1)})} \]

\[ C_n = T_n^{D/2} \]
\[ x = 2 - \sqrt{3} \]  

(38)

### 3.2 Regge Behavior

For large, negative \( t \) (\( t \) is negative by convention for real Euclidean momenta) the individual amplitudes \( A_n \) vanish. However, the asymptotics is controlled by their large \( n \) behavior. Since \( x < 1 \) we have

\[ I_n \to \frac{1}{x} \]
\[ T_n \to (1-x^2)x^n \]
\[ S_n \to \frac{1}{2}[n + 1 - \sqrt{3}] \]
\[ Q_n \to \frac{\sqrt{3} - 1}{2} \]
\[ C_n \to [(1-x^2)x^n]^{D/2} \]  

(39)

Therefore

\[ A_n \simeq \lambda^{2(n+1)} x^{\frac{nD}{2}} exp \left[ -\frac{1}{2} \left( \frac{1}{2}(\sqrt{3} - 1)q^2 - x^n t - \frac{1}{2}(n + 1 - \sqrt{3})s \right) \right] \]  

(40)
It is of the general form (with positive constants $T$, $\tau$, and $\hat{C}(q^2, s)$, given by the above limits)

$$\hat{A}_n = \hat{C}\lambda^{2(n+1)}x^{\frac{\alpha}{2}}e^{\frac{1}{2} (Tx^n t + \tau ns)}$$ \hspace{1cm} (41)

The total amplitude has the same asymptotic behavior as the series

$$\hat{A} = \sum_1^\infty \hat{A}_n = \hat{C} \sum_1^\infty \lambda^{2(n+1)}x^{\frac{\alpha}{2}}e^{\frac{1}{2} (Tx^n t + \tau ns)}$$ \hspace{1cm} (42)

To sum the series and exhibit the asymptotic behavior we take, term by term, the Mellin transform

$$\tilde{A}(z) \equiv \int_0^\infty d(-t) \hat{A}(-t)(-t)^{-1}$$ \hspace{1cm} (43)

with the inverse transform given by

$$\hat{A}(-t) \equiv \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} dz \tilde{A}(z)(-t)^{-z}$$ \hspace{1cm} (44)

We have

$$\tilde{A}_n = \int_0^\infty d(-t) (-t)^{z-1} \hat{C}\lambda^{2(n+1)}x^{\frac{\alpha}{2}}e^{\frac{1}{2} (Tx^n t + \tau ns)} = \hat{C}\lambda^{2(n+1)}x^{\frac{\alpha}{2}}e^{\frac{1}{2} \tau ns} \Gamma(z) \left(\frac{1}{2T x^n}\right)^z$$ \hspace{1cm} (45)

Summing the series we obtain

$$\tilde{A} = \sum_1^\infty \tilde{A}_n = \sum_1^\infty \hat{C}\Gamma(z) \left(\frac{2x}{T}\right)^z \frac{\lambda^{2(n+1)}x^{\frac{\alpha}{2}}e^{\frac{1}{2} \tau ns}}{x^{nz}}$$

$$= \hat{C}\prime(s)\Gamma(z) \left(\frac{2x}{T}\right)^z \frac{1}{x^z - \lambda^2 x^{\frac{\alpha}{2}}e^{\frac{1}{2} \tau s}}$$ \hspace{1cm} (46)

The asymptotic behavior of the amplitude is determined by poles $z_0$ in its Mellin transform [15]:

$$A(-t) \xrightarrow{t \rightarrow -\infty} (-t)^{-z_0(s)} \equiv (-t)^{\alpha(s)}$$ \hspace{1cm} (47)

where $\alpha(s)$ is the Regge trajectory.

The Mellin transform has a real pole at

$$z_0 = \frac{1}{2\ln x} \tau s + \frac{1}{2} D + \frac{\ln \lambda}{\ln x}$$

Consequently, we obtain a Regge trajectory

$$\alpha(s) = \left(\frac{\tau}{-2\ln x}\right)s + \left(-\frac{1}{2} D + \frac{\ln \lambda}{\ln x}\right)$$ \hspace{1cm} (48)
(For the ladder $\tau = \frac{1}{2}$.) Since $x < 1$, $\ln x$ is negative. Therefore the trajectory is linear, with positive slope.

The real pole in the Mellin transform gave us the asymptotic behavior, but there are also complex poles located parallel to the imaginary axis at

$$z_n = z_0 + \frac{2\pi in}{\ln x} \equiv Z_R + inZ_I$$

(49)

We show in the appendix that they do not affect the Regge trajectory we have found.

### 3.3 Relativistic Harmonic Oscillator

In this subsection we present an alternative, operatorial, derivation of the results obtained above. We show that due to the exponential nature of the one-particle propagators, determining the two-particle propagator can be reduced, after a separation of variables, to solving a harmonic oscillator problem.

We consider, still in ladder approximation, the two-particle propagator $\Delta$ (including the $(2\pi)^D \delta(q_1 - q_3)\delta(q_2 - q_4)$ term), satisfying the Bethe-Salpeter equation

$$\Delta = 1 + e^{-H} \Delta$$

(50)

where $e^{-H}$ sticks an extra rung on the sum of ladders (as in Fig. 1). Explicitly, we can write

$$e^{-H} = (\text{rung propagator}) \times (\text{two “side” propagators})$$

(51)

with integration over either loop momentum (in momentum space) or positions of vertices (in coordinate space). The propagator is given by

$$\Delta = \frac{1}{1 - e^{-H}} = \sum (e^{-H})^n$$

(52)

The Bethe-Salpeter equation corresponds to perturbatively solving a Schrödinger equation with “free” Hamiltonian $1$ and potential $-e^{-H}$ and vanishing total energy. Thus, the Schrödinger equation (on the wave function) is

$$1 - e^{-H} = 0 \quad \rightarrow \quad H = 0$$

(53)

Since $H$ is essentially the sum of $p^2$’s or $x^2$’s for the different propagators, it is separable (unlike in usual field theory) into “center-of-mass” and relative pieces. The trivial center-of-mass term corresponds to the simple $-ns/2$ dependence in $E_n$.

Explicitly, we want to replace integrals with operator expressions. In coordinate space, adding a rung is simply multiplication by the propagator, while adding the two propagators on the side of the ladder involves integration. In momentum space, the reverse is true (because duality between vertices and loops corresponds to Fourier transformation): Adding the two side propagators is just multiplication, while adding
the rung involves integration. So, in operator language adding the rung is simple in terms of the position operators, while adding the two sides is simple in terms of the momentum operators. Thus, adding the two sides followed by adding the rung is performed by the operator

\[
e^{-H} = e^{-(x_1-x_2)^2/2}e^{-(p_1^2+p_2^2)/2}
\]  

(54)

where the \(p\)'s and \(x\)'s are now the operators for the two particles. Separating into average and relative coordinates,

\[
p_{1,2} = \frac{1}{2}P \pm p, \quad x_{1,2} = X \pm \frac{1}{2}x
\]

(55)

this becomes

\[
e^{-H} = e^{-x^2/2}e^{-p^2/4+p^2}
\]

(56)

We can now separate out the \(P^2 = -s\) from the relative parts:

\[
e^{-H} = e^{-x^2/2}e^{-p^2}e^{s/4}
\]

(57)

It is convenient to use a Hermitian expression by a similarity transformation that puts half of one exponential on each side, as

\[
e^{-H} \rightarrow e^{s/4}e^{-x^2/4}e^{-p^2}e^{-x^2/4} \quad \text{or} \quad e^{s/4}e^{-p^2/2}e^{-x^2/2}e^{-p^2/2}
\]

(58)

Since we work in momentum space, we will use the latter choice, whose similarity transformation involves only momentum operators. To determine \(H\) we need to combine the exponentials into a single one. Since the Baker-Campbell-Haussdorf theorem requires using only commutators, it is useful to note that the exponents satisfy the commutation relations of raising and lowering operators, and we can use the representation

\[
\frac{1}{2}x^2 \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{1}{2}p^2 \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad i\frac{1}{2}\{x,p\} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(59)

So we need in general to evaluate expressions of the form

\[
e^{-\alpha p^2/2}e^{-\beta x^2/2}e^{-\alpha p^2/2} \quad \rightarrow \quad e^{-\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}}e^{-\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}}e^{-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}
\]

(60)

\[
= \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 + \alpha \beta & -\beta \\ -\alpha(2 + \alpha \beta) & 1 + \alpha \beta \end{pmatrix} = e^{-\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}}
\]

(61)

We determine \(a, b\) using

\[
\begin{pmatrix} C & -A \\ -B & C \end{pmatrix} = e^{-\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}} = \cosh(\sqrt{ab}) - \frac{\sinh(\sqrt{ab})}{\sqrt{ab}} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}
\]

(62)

\[
\Rightarrow \quad \sqrt{ab} = \ln(C + \sqrt{AB}), \quad \sqrt{\frac{a}{b}} = \sqrt{\frac{A}{B}}
\]

(63)
We find from (58), in harmonic oscillator notation,

\[ H = -\frac{1}{4}s - \ln(\lambda^2) + \omega \left( m\omega \frac{1}{2}x^2 + \frac{1}{m\omega} \frac{1}{2}p^2 \right) \]  

(64)

\[ \omega = \sqrt{ab} = \ln \left( 1 + \alpha\beta + \sqrt{(1 + \alpha\beta)^2 - 1} \right), \quad m\omega = \frac{\beta}{\sqrt{(1 + \alpha\beta)^2 - 1}} \]  

(65)

where we have restored the coupling dependence. In the present case,

\[ \alpha = \beta = 1 \quad \Rightarrow \quad \omega = \ln(2 + \sqrt{3}), \quad m\omega = \frac{1}{\sqrt{3}} \]  

(66)

By similar manipulations, the first choice of \( H \) above (similarity transformation using \( x \)'s instead of \( p \)'s) gives the same result, but with

\[ m\omega = \frac{\sqrt{3}}{2} \]  

(67)

First we look at just the spectrum, and note that \( e^{-H} - 1 = 0 \) is the same as \( H = 2\pi in \). (But the propagators for the two are different, from inverting these operators.) We recognize the harmonic oscillator as a \( D \)-vector, exactly like the oscillators in the usual string Hamiltonian (but now we have only one such vector).\(^6\) We can thus identify the “energy” of the harmonic oscillator Hamiltonian \( m\omega \frac{1}{2}x^2 + \frac{1}{2}p^2 \) as \( (J + D/2)\omega \), where \( (D/2)\omega \) is the ground-state energy of the \( D \) oscillators, and we identify the integer excitation \( J \) with the (maximum) spin for that energy (from acting with \( J \) vector oscillators on the vacuum). The result is then

\[ 2\pi in = -\frac{1}{4}s - \ln(\lambda^2) + \ln(2 + \sqrt{3})(J + \frac{1}{2}D) \]  

(68)

so for the trajectory \( J = \alpha(s) \) we have

\[ \alpha(s) = -\frac{1}{2}D + \frac{1}{\ln(2 + \sqrt{3})} \left( \frac{1}{4}s + \ln(\lambda^2) + 2\pi in \right) \]  

(69)

in agreement with the result of the previous subsection.

We can determine now the two-particle propagator as \( \langle q_3, q_4 | \Delta | q_1, q_2 \rangle \) i.e.

\[ \langle q_3, q_4 | [1 - e^{-H}]^{-1} | q_1, q_2 \rangle = \sum_n \langle q_3, q_4 | e^{-nH} | q_1, q_2 \rangle \]  

(70)

We use the explicit \( H \)

\[ H = -\frac{1}{4}s - \ln(\lambda^2) + \ln(2 + \sqrt{3}) \left( \frac{1}{\sqrt{3}} \frac{1}{2}x^2 + \frac{1}{\sqrt{3}} \frac{1}{2}p^2 \right) \]  

(71)

\(^6\) The interpretation of the single \( D \)-vector harmonic oscillator is as follows: The positions of the two particles in the Bethe-Salpeter equation are two adjacent points on the random lattice, and the relative coordinate represents the first derivative of \( x(\sigma) \), which corresponds to the first oscillator in the expansion of \( x(\sigma) \). (A similar model was considered in ref. [19].)
Because of the similarity transformation, this corresponds to half (really the square root) of the side-of-the-ladder propagators on either side of $e^{-H}$. (If we had used the $x$ transformation, we would instead have half of the rung on either side, which is harder to fix in momentum space.) Since we ultimately want ladders with amputated external propagators, we amputate the other half of the initial and final propagators on our expression to get for each term in the sum

$$A_{n-1} = \lambda^{2n} e^{(ns+q^2)/4} \langle q_3, q_4 | e^{-nH_0} | q_1, q_2 \rangle$$  \hspace{1cm} (72)

where $A_{n-1}$ is the amplitude for the graph with $n$ rungs ($n-1$ loops) and $H_0$ consists of just the harmonic oscillator terms ($x^2$ and $p^2$). (The no-rung graph is a $\delta$-function, with amputation factors for non-existent propagators.) The propagator for a harmonic oscillator in $D$ dimensions (just the product of $D$ one-dimensional ones), with Wick-rotated time, in momentum space, is

$$\langle q_3, q_4 | e^{-nH_0} | q_1, q_2 \rangle = \left[ m\omega \sinh(n\omega) \right]^{-D/2} e^{-[p_I^2+p_F^2]\cosh(n\omega)-2p_Ip_F]/2m\omega \sinh(n\omega)}$$ \hspace{1cm} (73)

where the “time” is now the integer $n$. (This differs from the coordinate space one by $x \rightarrow p$ and $m\omega \rightarrow 1/m\omega$. There is also the usual momentum conservation $\delta$ function $(2\pi)^D/\delta^D(q_1 + q_2 - q_3 - q_4)$. ) We now need to use

$$p_I = \frac{1}{2}(q_1 - q_2), \hspace{1cm} p_F = \frac{1}{2}(q_3 - q_4)$$

$$\Rightarrow \hspace{1cm} p_I^2 + p_F^2 = \frac{1}{2}(s + q^2), \hspace{1cm} 2p_I \cdot p_F = \frac{1}{2}(s + 2t + q^2)$$ \hspace{1cm} (75)

Putting this together (with $\omega = \ln(2 + \sqrt{3})$ and $m\omega = 1/\sqrt{3}$) gives the result of subsection 2.1 (where $x = e^{-\omega}$).

As another interesting example, consider a “cylindrical” ladder: again two long lines for the sides of the ladder, but with circular rungs, equivalent to double rungs, from a $g\phi^4$ coupling instead of $g\phi^3$. The only difference in the above calculation is the replacement of the $e^{-x^2}$ factor with $e^{-2x^2}$. The only difference in the result is

$$\omega = \ln(3 + 2\sqrt{2}), \hspace{1cm} m\omega = \frac{1}{\sqrt{2}}$$ \hspace{1cm} (76)

In particular, this gives a different Regge slope.

### 4 GENERAL GRAPHS

We have shown by exact calculations that ladder graphs corresponding to a Gaussian field theory indeed produce a linear Regge trajectory, but what can be said about more general graphs? Do they give rise to radiative corrections to the Regge trajectory we found for the simple ladders and do they give rise to additional Regge poles? Is it possible to see that ladder graphs give the leading asymptotic behavior for large $t$? To try to answer any of these questions we have to be able to compute the
asymptotic behavior of an amplitude coming from an arbitrary graph. As compared to old calculations on Regge behavior we are helped by two features: a) The diagrams in our theory are all planar and b) the dependence on $s$ and $t$ is always exponential for any graph. We show in this section how this dependence can be determined in principle in terms of the adjacency matrix of the dual graph. The more difficult task, how to determine the asymptotic behavior of sums of general graphs, is beyond the scope of this paper.

4.1 Adjacency Matrix, Edge and Path Weights

We start by looking at the dual graph of a given momentum space graph where each simplex is replaced by a point corresponding to a loop momentum, $k_a$. In the case of the ladder this is shown in Fig. 2.

![Fig. 2. Momentum labeling and dual diagram](image)

The external four points correspond to the external loop momenta, which we label as $p_i$, $i = 1, ..., 4$. They are related to the external particle momenta in the following way

$$
q_1 = p_2 - p_1 \\
q_2 = p_1 - p_4 \\
q_3 = p_2 - p_3 \\
q_4 = p_3 - p_4
$$

(77)

In the literature one often chooses to start with triangulation of the worldsheet, then dualizes to $\phi^3$ Feynman diagrams. We choose to start instead with the Feynman diagrams because it relates directly to our calculation, has the more physical interpretation, and the unitary choice of integration measure is obvious from the usual Feynman rules.
The Mandelstam variables are

\[ s \equiv -(q_1 + q_2)^2 = -(p_2 - p_4)^2 \]
\[ t \equiv -(q_3 - q_1)^2 = -(p_1 - p_3)^2 \]

(78)

We note the “gauge” invariance \( p_i \rightarrow p_i + r \).

Now, if in the original graph two simplices (loops) are adjacent to each other then in the dual graph they are connected by an edge. Thus the dual graph is bounded by four edges connecting the \( p_i \) along with internal momentum points \( k_a \) variously connected to each other. Let \( A^{ij} = (0, 1, 2, ...) \) denote the adjacency matrix in the dual graph, where \( i = i, a \). Our first objective is to compute the exponent \( E \) corresponding to a particular graph. (For the time being we will not use the Einstein summation convention unless stated explicitly.) We have

\[ E = \frac{1}{2} \sum_{ij} A^{ij} (p_i - p_j)^2 = \frac{1}{2} \sum_{ij} A^{ij} (p_i - p_j)^2 + \frac{1}{2} \sum_{ab} A^{ab} (k_a - k_b)^2 + \sum_{ib} A^{ib} (p_i - k_b)^2 \]  

(79)

The first term is a constant, \( E_1 \), the same for all graphs. Expanding the second and third terms we find

\[ E = E_1 + \sum_{ab} A_{ab} k_a \cdot k_b + \sum_{ib} A^{ib} p_i^2 + \sum_{ib} A^{ib} k_b^2 - 2 \sum_{ib} A^{ib} p_i \cdot k_b \]
\[ = E_1 + \sum_i p_i^2 \left( \sum_b A_{ib} \right) + \sum_b k_b^2 \left( \sum_i A^{ib} \right) - \left[ \sum_{ab} A^{ab} k_a \cdot k_b - 2 \sum_{ib} A^{ib} p_i \cdot k_b \right] \]

(80)

The second term in the last equation contains only squares of the external loop momenta and we will soon argue that for large \( t \) behavior it is unimportant. For the third term we note that the sum within parentheses is the “degree” \( d_b \) of the internal momentum (the number of lines meeting at the point \( k_b \) or, in the original diagram, the number of propagator lines bounding the corresponding loop; \( d = 4 \) for the ladder diagram). We have

\[ E = E_1 + E_2 (p_i^2) + \sum_b k_b^2 d_b - \left[ \sum_{ab} A^{ab} k_a \cdot k_b - 2 \sum_{ib} A^{ib} p_i \cdot k_b \right] \]

(81)

Let us rescale the internal momenta and the adjacency matrix as

\[ k_a \rightarrow \sqrt{d_a} k_a \]
\[ A^{ab} \rightarrow \frac{A^{ab}}{\sqrt{d_a d_b}} , \quad A^{ib} \rightarrow \frac{A^{ib}}{\sqrt{d_b}} \]

(82)

(83)

The exponent becomes

\[ E = E_1 + E_2 (p_i^2) + \sum_b (\delta^{ab} - A^{ab}) k_a \cdot k_b - 2 \sum_b k_b \cdot \left( \sum_i A^{ib} p_i \right) \]

(84)

\[ ^8 \text{Since we are only interested in the dependence of the external momenta we do not keep track of the Jacobian.} \]
This procedure essentially attaches a “weight” to each edge. For the case of the ladder diagram the rescaled adjacency matrix has the following appearance

\[
A = \begin{pmatrix}
0 & a & 0 & 0 \\
0 & 0 & a & 0 \\
0 & a & 0 & a \\
0 & 0 & a & 0 \\
\vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

(85)

and \(a = 1/4\) is the edge weight.

Our next task is to perform the (Gaussian) loop integrals. Using (we reintroduce the summation convention)

\[
\int [dx] e^{-\frac{1}{2}(A^{ij}x^ix^j + B_ix^i)} \sim e^{-\frac{1}{2}\tilde{E}}, \quad \tilde{E} = -\frac{1}{4}B_lA^{lk}B_k
\]

we obtain, after some simplifications,

\[
\tilde{E} = E_1 + E_2(p_1^2) - p_iA^{ic}(I - A)^{-1}A^{dj}p_j \equiv E_1 + E_2(p_1^2) + E_3(p_i)
\]

(87)

We can now expand the third term in a power series of the rescaled adjacency matrix

\[
E_3 = -p_iA^{ic}(\delta^{cd} + A^{cd} + A^{ce}A^{ed} + \ldots)A^{dj} \cdot p_j
\]

(88)

and its interpretation is clear. It picks up a contribution only when there exists a path between the external points \(p_i\) and \(p_j\), with the contribution becoming smaller and smaller with each additional internal point introduced in the path since the edge weights are less than 1. Henceforth we refer to the product of all the edge weights along a path as the “path weight”. The coefficient of \(p_ip_j\) \((i \neq j)\) in the exponent \(E_3\) is then given by the sum of all path weights connecting \(p_i\) and \(p_j\) (including smaller and smaller contributions from paths that involve back and forth retracing through the vertices).

Let us choose (cf. eq. (77)) the “gauge” \(p_2 = 0\). In this gauge

\[
\begin{align*}
p_1 \cdot p_3 &= \frac{1}{2}(q_1^2 + q_3^2 + t) \\
p_1 \cdot p_4 &= \frac{1}{2}(q_1^2 - s - q_2^2) \\
p_3 \cdot p_4 &= \frac{1}{2}(q_3^2 - s - q_1^2)
\end{align*}
\]

(89)

We note that, with the possible exception of the tree graphs, \(E_1\) and \(E_2\) do not contain any \(p_i \cdot p_j\) cross term. Therefore, in this gauge the coefficient of \(t\) in the exponent of the amplitude is directly proportional to the coefficient of \(p_1 \cdot p_3\) and we have, showing just these terms,

\[
\tilde{E} = -A^{1c}(\delta^{cd} + A^{cd} + \ldots)A^{d3}p_1 \cdot p_3 + \ldots = -\frac{1}{2}A^{1c}(\delta^{cd} + A^{cd} + A^{cf}A^{fd} + \ldots)A^{d3}t + \ldots
\]

(90)

i.e., the coefficient is \(-\frac{1}{2}\) the total path weight between the points \(p_1\) and \(p_3\). Obviously this is a gauge independent statement. In a similar fashion, picking instead the gauge
\( p_4 = 0 \), one can show that the coefficient of \( s \) is \( -\frac{1}{2} \) the total path weight between the points \( p_2 \) and \( p_4 \).

Thus we have a simple expression for the \( t \) and \( s \) exponents in terms of the rescaled adjacency matrix or edge and path weights, providing us with significant insight into the amplitude contributions coming from arbitrary graphs. For example, using the notion of path weights it is clear why at each loop level (i.e., fixed number of internal points) the ladder graphs have the leading high \( t \) behavior, i.e., smallest \([p_1,p_3]\) path contribution. There is only a single path connecting \( p_1 \) to \( p_3 \) and moreover it contains all the points in the path making the path weight the smallest\(^9\). Thus it may be tempting to conclude that the large \( t \) asymptotic behavior will indeed be dominated by the ladder graph contributions. Unfortunately, while summing an infinite number of graphs the properties of the amplitude may change, potentially invalidating such an argument. Thus we will take up a more modest position and compare, in the next subsection, only a particular class of graphs to the ladder graphs.

### 4.2 Thick Ladder Diagrams

We will look at diagrams that can be obtained from the ladder diagrams by replacing the rungs with more complicated 4-point subdiagrams, such as those depicted in Fig. 3, in other words by making the vertical lines “thicker”.

\[
\begin{align*}
\begin{array}{c}
\text{Fig. 3. “Thick” ladders}
\end{array}
\end{align*}
\]

The method we have used in the previous subsections can be generalized to determine the asymptotic behavior of these diagrams. Because of the Gaussian nature

---

\(^9\)There are some complications coming from subleading paths which allows for back and forth motion. Also, some of the edge-weights in a general graph can be smaller than that of the ladder graphs because some of the points can have \( d > 4 \). However, one can still argue that the net path-weight for the ladder graphs is indeed the smallest.
of propagators the exact expressions for the subdiagrams can be obtained by direct integration; they are again exponentials of form similar to the original one for the simple rungs. They can be inserted as kernels into the Bethe-Salpeter equations and recursion relations similar to the ones we have already considered can be obtained and solved. In particular the large $n$ behavior of these thick ladders is the same as before. Only the values of the constants $\hat{C}, T, \tau$ and $x$ change. Also, the power of $\lambda$ associated with the $n$th diagram increases

$$\lambda^2 \rightarrow \lambda^{2(1+\delta)}$$

where $\delta$ is the number of loops in the subgraph. One obtains again Regge behavior of the form

$$A(-t) \xrightarrow{t \rightarrow \infty} (-t)^{\alpha(s)}$$

with

$$\alpha(s) = \left(\frac{\tau}{-2\ln x}\right) s + \left(-\frac{1}{2}D + 2\left(1+\delta\right)\ln \lambda\right)$$

where the values of $x$ and $\tau$ are given in the table below.

| Case | Type     | $x$  | $\tau$ | slope |
|------|----------|------|--------|-------|
| (a)  | Ladder   | .27  | 1/2    | 0.191 |
| (b)  | Propagator | .38  | 1/2    | 0.250 |
| (c)  | Vertex   | .27  | 5/6    | 0.318 |
| (d)  | H-Ladder | .15  | 9/10   | 0.237 |

We also have given the numerical values of the Regge slopes from each class of ladders. However, on their own they don’t have much significance; one should consider instead diagrams which are combinations with thin and thick rungs, so as to provide radiative corrections to the pure ladder trajectories. (We consider this in the next section.) We have presented them here for comparison with the approximate results we can obtain from the adjacency matrix methods that we consider now.

As a warm-up exercise let us again look at a ladder graph. There is only one path connecting $p_1$ and $p_3$ and it includes all the points in the path. For a ladder with $n$-loops, or $n$ edges with edge-weight $a \equiv a_L = 1/4$ (the loops are bounded by four propagators; equivalently, in the dual diagram, four edges meet at one point) the path-weight $P$ is naively given by

$$P = a_L^n = (.25)^n$$

One immediately notices a discrepancy between (94) and the exact result (25) directly obtained using recursion relations. This is because we have considered only the “shortest-path” contribution ($A_{p_1k_1}A_{k_1k_2}A_{k_2k_3}...A_{k_n p_3}$) without including contributions coming from paths which allow for back and forth motion along the edges. However, these subleading contributions can be computed and only rescale the edge-weight giving the correct value (39)

$$P = x_L^n \simeq (.27)^n$$
We will discuss this rescaling due to back and forth motion in more detail in the next section.

Let us now replace the vertical lines of the ladder graph with something more complicated e.g., incorporating vertex or propagator corrections such as those in Fig. 3. We will refer to the simplices (loops) separated by the thick lines as big simplices or “b-simplices”. For the example with vertex corrections, Fig. 3c, the shortest path approximation gives for \( n \) b-simplices

\[
P = \left( \frac{1}{8} \right)^n = (0.125)^n = \tilde{a}_V^n
\]

where the factor 8 comes from the fact that the b-simplices (loops separated by the vertical rungs) have 8 edges. This expression should be compared with the corresponding expression for ladders (94); the edge-weight \( a_L \) has been replaced by an effective edge-weight \( \tilde{a}_V \) between the b-simplices. (Actually the \( a_V \) obtained from (96) is only a rather crude approximation to the exact effective edge weight \( a_V \) between the b-simplices. One should include contributions from paths that cut through the top and bottom triangles, as well as some back and forth motions to be described later.) We will show in the next section that such a replacement with an exact effective edge-weight works for any general thick ladder graph, even when one accounts for all possible paths, not just the shortest one.

For the propagator corrections in Fig. 3b, we similarly find

\[
\tilde{a}_p = \frac{2}{8} = 0.25
\]  

(97)

(the 2 because there are two shortest paths) and for the H diagram in Fig. 3d

\[
\tilde{a}_H = \frac{2}{6.4} \approx 0.08
\]  

(98)

As we will explain in the next section, the back and forth motion between the b-simplices further rescales the effective edge weights \((a \rightarrow x)\) in a manner similar to the ladders.

5 RADIATIVE CORRECTIONS

It is evident that one consequence of having exponential propagators is that for individual graphs the dependence on \( s \) and \( t \) will always be exponential, \( \sim \exp(St + Tt) \). The coefficients \( S, T \) are computable exactly in principle, and approximately in terms of truncations to short path weights. For example, in the case of the thick ladders in Fig. 3, we could add to the shortest path contribution computed above also contributions from paths which enter the small loops in the thick rungs but do not include paths involving back and forth motions. Finally, we could include back and forth motions, within a thick rung or within different b-simplices. In this section we will
define the exact edge weight $a_E$ of a thick rung and show how to pass from $a_E$ to a corresponding $x_E$ which gives $T = x_E^n$. We will then use the analysis to estimate the radiative corrections to the ladder Regge trajectory from two classes of thick subdiagrams.

5.1 Relating Recursion Relations and Path Weights

Can one explain the transition $a \rightarrow x$ by properly accounting for all the paths for a general diagram? We discuss this here, starting again with the ladder graph with $n$ loops and the matrix

$$1 - A = \begin{pmatrix}
1 & -a & 0 & 0 & \ldots \\
-a & 1 & -a & 0 & \ldots \\
0 & -a & 1 & -a & \ldots \\
0 & 0 & -a & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix} \quad (99)$$

where $A$ is the rescaled adjacency matrix of eq. (85) and $a$ is the edge-weight. For ladder graphs $a = 1/4$ but as we will see, for thick ladders $a$ will be given by the effective edge-weight between two b-simplices.

Let $\Delta_n$ denote the determinant of $I - A$. Then, according to (87) the exact path weight (aside from the “external” factors $A^{1c}, A^{d3}$ in (90) is

$$P = (I - A)_n^{-1} = \frac{a^{n-1}}{\Delta_n} \quad (100)$$

The determinant satisfies a Fibonacci type recursion relation

$$\Delta_n = \Delta_{n-1} - a^2 \Delta_{n-2} \quad (101)$$

The most general solution has the form

$$\Delta_n = c_+ \mu_+^n + c_- \mu_-^n \quad (102)$$

with

$$\mu_\pm = \frac{1 \pm \sqrt{1 - 4a^2}}{2} \quad (103)$$

Initial conditions determine

$$c_\pm = \frac{1}{2} \left(1 \pm \frac{1}{\sqrt{1 - 4a^2}}\right) \quad (104)$$

The roots satisfy

$$\mu_+ \mu_- = a^2 \quad (105)$$
For large $n$ clearly $\mu_+$ dominates $\Delta_n$ (recall also that the large $t$ behavior is dominated by the smallest $P$) so that we have

$$ P \sim \frac{a^n}{\mu_+^n} = \left( \frac{\mu_-}{a} \right)^n $$

(106)

Specifically for ladder graphs we have

$$ \frac{\mu_-}{a} = \frac{1}{4} (2 - \sqrt{3}) = \frac{1}{4} x $$

(107)

and we reproduce our previous result (25). Thus, we have learned how to pass from the shortest path approximation (just crossing edges with edge weight $a_L$ and a total weight $a_L^n$) to the exact value $x_L$ that determines the $t$-coefficient $T = x^n$.

Let us now investigate how one goes from thin to thick lines. Suppose there are $n$ b-simplices. Note that any path from 1 to $n$ can be broken up into steps of one (i.e., a step from one b-simplex to one of its neighbours). For simplicity we focus first on a path which does not contain any back and forth motion between different b-simplices but only, possibly, within the “thick” rungs bordering a given b-simplex. The path weight is then given by

$$ P = \prod_{i=1}^{n-1} P_{i,i+1} $$

(108)

where $P_{i,i+1}$ consists of products of edge-weights in the sub-path from $i$ to $i+1$. These edge weights are formed by starting in the $i$th b-simplex, entering thick rungs either to its left or right, proceeding back and forth anywhere within these regions and eventually emerging in the $i+1$st b-simplex; see Fig. 4 for examples.

Fig. 4. Illustrating various paths: (ab) – shortest path; (cd) – a longer path; (ef) – a path involving back and forth motion; (gj) – a more complicated path; the weight of the subpath (hi) is an element of the effective edge weight of the middle “thick” rung.

Now, consider a very similar path which only differs from the earlier path in how it goes from, say, 1 to 2. The path weight for such a path will be given by

$$ P' = P'_{1,2} \prod_{i=2}^{n-1} P_{i,i+1} $$

(109)
so that the sum of the two paths yields

$$P + P' = (P_{1,2} + P'_{1,2}) \prod_{i=2}^{n-1} P_{i,i+1}$$

It is clear that this process can be continued to include all the subpaths originating in 1 and ending in 2, so that

$$P + P' + \ldots = a_{e12} \prod_{i=2}^{n-1} P_{i,i+1}$$

where $a_{e12}$, the sum of all subpath weights between 1 and 2, is defined to be the effective edge-weight between 1 and 2. It is also clear that this process can be carried out between any two adjacent b-simplices. Thus the sum of all paths with no back and forth motion between the b-simplices is given by

$$\sum P = \prod_{i=1}^{n-1} a_{e_i,i+1}$$

We have defined the effective edge weight $a_{e_i,i+1}$ as the sum of all path-weights originating in $i$ and ending in $i+1$ without encountering any other, i.e., $i-1$ or $i+1$, b-simplex. Essentially in obtaining (112) we have replaced the sum of products with a product of sums. It is evident that one can do this for any arbitrary path, i.e., also for those which contains back and forth motion between the b-simplices themselves. Thus the problem essentially reduces to the original ladder graph problem except that now we have replaced $a_L \rightarrow a_e$.

For the graphs in Fig. 3 all the $a_e$’s are the same and the expressions (96-98) give us their shortest path approximations. It is clear now that once the $a_e$ have been determined (exactly or approximately), going over to the corresponding $x$ proceeds in exactly the same way as for the ladder graphs (103-106); in the adjacency matrix one simply replaces the ladder edge weights by the effective edge weights. In this fashion we obtain the exact (or approximate, if the effective edge weight has been computed approximately) asymptotic behavior of any thick ladder graph.

5.2 Corrected trajectories

In principle, the Bethe-Salpeter equation provides us with a means of obtaining the exact scattering amplitude and therefore the exact form of the Regge trajectories. One only needs to know the kernel function, which consists of the sum of all 2PI diagrams, planar for our theory. The simplest approximation to the kernel is just the exchange diagram that leads to pure ladders, and a better approximation would be obtained by adding to it the simplest radiative corrections corresponding to Figs. 3b,c.
Fig. 5. Thick insertion at i-th position

Let us start with a ladder graph with \( n \) internal loops or equivalently \( n + 1 \) vertical lines\(^{10}\). Now suppose that we “thicken” the \( i \)-th rung as shown in Fig. 5 by replacing it with a complicated subgraph. The ladder edge weight has to be then replaced by the effective edge weight for the \( i \)-th line as discussed above,

\[
a_L \to a_e
\]

so that in the “shortest” path approximation the path weight becomes

\[
P_n = a_L^{n-1} a_e
\]

We expect this to get modified once the back and forth motions are included

\[
P_n \to P_n = x^{n-1} y
\]

where \( x \) and \( y \) are the rescaled edge weights for the ladder and the thick line respectively. (This is not exact; see the discussion below.) In a similar fashion the path weight which determines the coefficient of the Mandelstam variable \( s \) will change. Since in eq. (42) the factor \( n \) in the coefficient counts the number of simple paths from \( p_2 \) to \( p_4 \) we might guess that \( \tau n \to \tau (n - 1) + \sigma \). (Again, we don’t expect this to be the full story.) Thus we may guess that the corresponding amplitude looks like

\[
\sim \lambda^{2(n+\delta)} e^\frac{1}{2} \left\{ x^{n-1} y t + \tau(n-1) + \sigma s \right\}
\]

There are \( n \) places where such a subgraph could be inserted. So the overall contribution to the amplitude is

\[
A_{n,1} \sim n \lambda^{2(n+\delta)} e^\frac{1}{2} \left\{ x^{n-1} y t + \tau n s \right\}
\]

We can extend the argument to \( p \) insertions:

\[
P_{n,p} \simeq x^{n-p} y^p
\]

One can have these \( p \) insertions in \( \binom{n}{p} \) ways so that

\[
A_{n,p} = \binom{n}{p} \lambda^{2(n+p\delta)} e^\frac{1}{2} \left\{ T x^{n-p} y^p t + \tau(n-p) + \sigma \right\}
\]

\(^{10}\)There are \( n - 1 \) internal vertical lines, and the two boundary lines effectively act as an extra edge weight.
\[ A_{n,p} = \lambda^{2n} e^{2\tau s} \binom{n}{p} \lambda^{2p} e^{2T x^n y^p p} \frac{1}{2} e^{\frac{1}{2} \sigma - \tau s} \]

Then taking the Mellin transformation we get

\[ \tilde{A}_{n,p} = \lambda^{2n} e^{2\tau ns} \binom{n}{p} \frac{\Gamma(z)}{\left( \frac{1}{2} T x^n y^p p \right)^z} \lambda^{2p} e^{\frac{1}{2} \sigma - \tau s} \]

(120)

Summing over \( p \) gives

\[ \tilde{A}_{n} = \Gamma(z) \left( \frac{2}{T} \right)^z \frac{\lambda^{2n} e^{2\tau ns}}{x^{nz}} \left[ 1 + \left( \frac{x}{y} \right)^z \lambda^{2\delta} e^{\frac{1}{2} \sigma - \tau s} \right]^n \]

(121)

One can now also sum over \( n \) to get

\[ \tilde{A} = \Gamma(z) \left( \frac{2}{T} \right)^z \left\{ 1 - \lambda^{2\epsilon \tau s} \left[ 1 + \left( \frac{x}{y} \right)^z \lambda^{2\delta} e^{\frac{1}{2} \sigma - \tau s} \right] \right\}^{-1} \]

(122)

The Mellin transform has a pole (or perhaps more) at

\[ 1 = \frac{\lambda^{2\epsilon \tau s}}{x^z} \left[ 1 + \left( \frac{x}{y} \right)^z \lambda^{2\delta} e^{\frac{1}{2} \sigma - \tau s} \right] \]

(123)

or

\[ z \ln x = \ln \lambda^2 + \frac{1}{2} \tau s + \ln \left[ 1 + \left( \frac{x}{y} \right)^z \lambda^{2\delta} e^{\frac{1}{2} \sigma - \tau s} \right] \]

\[ \simeq \ln \lambda^2 + \frac{1}{2} \tau s + \left( \frac{x}{y} \right)^z \lambda^{2\delta} e^{\frac{1}{2} \sigma - \tau s} \]

\[ \simeq \ln \lambda^2 + \frac{1}{2} \tau s + \frac{\lambda^{2\delta}}{y^z} e^{z \ln x + \frac{1}{2} \sigma - \tau s} \]

\[ \simeq \ln \lambda^2 + \frac{1}{2} \tau s + \lambda^{2(1+\delta)} e^{\frac{\sigma}{2} - z \ln y} \]

(124)

where we have expanded the logarithm (for small coupling constant) and iterated using the small coupling solution.

The radiative corrections seem to spoil the linearity of the original Regge trajectory. However, in reaching the result we have made some approximations which are not completely justified. For example, we have assumed that the expression in (118) is valid irrespective of the ordering of thin and thick rungs and this is definitely not the case since the effective edge weight of a thick line depends on the nature of its neighbors.

As a simple exercise we have looked at the exact expression for the path weight in the case of \( n+1 \) ordinary rungs followed by \( m \) thick rungs. The procedure follows that of Subsection 4.1 where now the matrix \( 1 - A \) has entries with \( n \) rows and columns.
containing the simple ladder weight $a$ and $m$ rows and columns where an effective edge weight $d$ for thick ladders appears, and in between an additional, transitional row and column where different weights yet, $b$ and $c$ appear (because at the loop which is bordered by a thin line on one side and a thick line on the other side the effective edge weights are different; also, we have ignored a similar effect where the thick rungs end). The total path weight, $[(1 - A)^{-1}]_{ij}$ between the initial and the final rungs can be calculated exactly and one finds a result

$$P \sim x^a y^m \left[1 - ba \frac{\Delta_{n-1}(a)}{\Delta_n(a)} - cd \frac{\Delta_{m-1}(d)}{\Delta_m(d)}\right]^{-1}$$

(125)

where, as before, $x = a^n/\Delta_n(a)$ and $y = d^m/\Delta_m(d)$ and the $\Delta$’s are the corresponding determinants for the $n \times n$ and $m \times m$ submatrices.

This is the simplest case and it already shows deviations from the approximate expression we have assumed in (118)$^{11}$; if the two kinds of rungs are intermingled the result is expected to be different as well. Furthermore, the expression we assumed for the path weight controlling the coefficient of $s$ is also not accurate. (In the limit of large $n, m$ the expression in brackets in (125) reduces to a constant so that the overall powers of $x, y$ are not affected; however, similar and independent constants from other orderings of thin and thick ladders could change the result of the summations that led to (122).) Thus, it is somewhat difficult without further work to judge the reliability of the trajectory corrections we have found.

6 CONCLUSIONS

In this work we have studied a model field theory defined by the lattice approximation to the relativistic string. Unlike ordinary field theory, one is dealing with exponential propagators and this feature significantly modifies the usual properties of theories with conventional propagators. Ultraviolet divergences are in general absent and the calculation of amplitudes is very much simpler.

We have concentrated on the Regge behavior of scattering amplitudes. By solving the Bethe-Salpeter equation we showed that in ladder approximation the four-particle amplitude exhibits Regge behavior with a trajectory which is a linear function of the energy (see (48)). We also presented an alternative, operatorial method for obtaining this result. We have developed general techniques for obtaining the high energy behavior of any diagram and we have attempted to determine the effect of radiative corrections on the linear trajectory we found for the simple ladder. We found that these corrections spoil the linearity of the trajectory; however we used a rather crude approximation and it is conceivable that more precise calculations would change this conclusion.

$^{11}$Note, for large values of $n, m$ the ratios $\Delta_{n-1}(a)/\Delta_n(a)$ and $\Delta_{m-1}(d)/\Delta_m(d)$ are constants and then indeed the expression reduces to (118), but they depend on $n, m$ when their values are small.
There are other calculations that could be carried out in the field theory described in this paper. The exponential nature of the propagators implies that no UV or IR divergences are encountered and because of the planar nature of the diagrams other problems are avoided. For example, it is not difficult to compute in standard fashion the one-loop effective potential; other features can be investigated as well.

Acknowledgments

The work of T. Biswas and M. Grisaru is supported by NSERC Grant No. 204540. The work of M. Grisaru is also supported by NSF Grant No. PHY-0070475. The work of W. Siegel is supported in part by NSF Grant No PHY-0354776.

7 Appendix: Imaginary Poles

We have pointed out in the main text the existence of complex poles in the Mellin transform, located at $z_n = z_0 + \frac{i 2\pi}{\ln x} = Z_R + i n Z_I$. In the inverse Mellin transform

$$A(-t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} d(-t) (-t)^{-\tilde{z}} \tilde{A}(z)$$

(126)

each of them contributes to the integral when one translates the contour towards the left in order to obtain the asymptotic behavior [8]. We show that these additional pole contributions do not alter the behavior we found earlier.

We begin by computing the residues at these poles,

$$\text{Res}(A_{z=z_n}) = \hat{C}'(s) \Gamma(z_n) \left( \frac{2x}{T} \right)^{z_n} \lim_{z \to z_n} \frac{z - z_n}{x^z - \lambda^2 x^2 e^{2\tau s}}$$

(127)

We evaluate

$$\lim_{z \to z_n} \frac{x^z - \lambda^2 x^2 e^{2\tau s}}{z - z_n} = \lim_{z' \to 0} \frac{x^{z'} x^{z_n} - \lambda^2 x^2 e^{2\tau s}}{x^{z'} - 1} = \lambda^2 x^2 e^{2\tau s} \ln x$$

(128)

and obtain

$$\text{Res}(A_{z=z_n}) = \hat{C}'(s) \Gamma(z_n) \left( \frac{2x}{T} \right)^{z_n} \frac{1}{\lambda^2 x^2 e^{2\tau s} \ln x}$$

(129)

Thus the asymptotic behavior of $A(-t)$ is given by

$$A(-t) \sim \sum_{n=\infty}^{\infty} (-t)^{-Z_R - i n Z_I} \hat{C}'(s) \Gamma(z_n) \left( \frac{2x}{T} \right)^{z_n} \frac{1}{\lambda^2 x^2 e^{2\tau s} \ln x}$$

$$\left[ \hat{C}'(s) \left( \frac{2x}{T} \right)^{Z_R} \frac{1}{\lambda^2 x^2 e^{2\tau s} \ln x} \right] (-t)^{-Z_R} \sum_{n=\infty}^{n=-\infty} (-t)^{-i n Z_I} \left( \frac{2x}{T} \right)^{i n Z_I} \Gamma(Z_R + i n Z_I)$$

(130)
We will show that, except for the $n = 0$ term, the sum is indeed bounded so that the asymptotic behavior is as prescribed by the real pole at $z = z_0$.

We note that

$$\sum_{n=-\infty}^{\infty} (-t)^{-inZ_I} \left( \frac{2x}{T} \right)^{inZ_I} \Gamma(Z_R + inZ_I) < \sum_{n=-\infty}^{\infty} \left| t^{-inZ_I} \left( \frac{2x}{T} \right)^{inZ_I} \Gamma(Z_R + inZ_I) \right|$$

$$= \sum_{n=-\infty}^{\infty} |\Gamma(Z_R + inZ_I)| \quad (131)$$

For large $n$

$$|\Gamma(Z_R + inZ_I)| = \sqrt{2\pi} e^{-\frac{1}{4} |nZ_I|^2} |nZ_I|^{Z_R - \frac{1}{2}} \quad (132)$$

Therefore the bounding series converges and the imaginary poles simply modify the coefficient of the Regge-behaved term $(-t)^\alpha(s)$.

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