APPROXIMATELY $\pi$ PROOFS THAT THE STOCK MARKET CAN APPROXIMATE $\pi$

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Abstract. We give three derivations of Pólya’s approximation for the expected range of a simple random walk in one dimension. This result allows for an estimation of the volatility of a financial instrument from the difference between the high and low prices, or, equivalently, for an estimation of $\pi$ from the ratio of the volatility to the difference between high and low prices.

1. Computing volatility

The volatility of a financial instrument, for example the S&P 500 stock index or one of its constituent stocks, is a measure of the degree by which the price of the instrument fluctuates over a given period of time. Mathematically, volatility is the variance of the price regarded as a random walk.

Volatility is the key component in options pricing, but it is also vital for determining the underlying risk of a position and for determining optimal asset allocation for a portfolio. Therefore having accurate volatility measurements and forecasts is crucial for the financial sector.

In practice, direct application of the mathematical definition of variance to compute historical volatility is complicated by the sheer volume of trades and inaccurate or missing data. This leads practitioners to compute historical volatility based on a small quantization parameter. For example, one can consider the price every one second and compute variance based on that. Of course, there is an obvious trade off, where smaller parameters offer more accurate estimates but require more intensive computations.

A relatively simple, yet surprisingly effective, method for forecasting future volatility is to take a moving average of historical volatility. For example, a simple moving average of historical volatility is given by

\begin{equation}
\text{SMA}_n(V_t) = \frac{1}{n} \sum_{t=1}^{n} V_t,
\end{equation}

where $V_t$ denotes historical volatility and $n$ is the length of the window.

Now instead consider the range series given by the difference between the daily high and low prices for an instrument. This data is freely available...
at the close of each trading day. Then one can compute a simple moving average for the daily range, denoted \( R_t \), by

\[
(1.2) \quad \text{SMA}_n(R_t) = \frac{1}{n} \sum_{t=1}^{n} R_t.
\]

On any reasonably liquid instrument, that is, something with a high volume of daily trades, one notices that the ratio of these predictors over a trading month (on average 21 days) is constant. The precise approximation is

\[
(1.3) \quad \frac{\text{SMA}_{21}(V_t)}{\text{SMA}_{21}(R_t)^2} \approx \frac{\pi}{8}.
\]

Therefore, if an irrational trader were so inclined, he could use the volatility and the high and low prices of an instrument to estimate \( \pi \). In practice, however, a rational trader is likely more interested in efficiently and accurately estimating future volatility.

Since calculating historical daily volatility is computationally intensive and is dependent upon a timely and accurate (and expensive) data feed, independent traders without access to such a feed, or who are sensitive to the cost of such a feed, as well as traders at larger firms looking to increase efficiency without losing accuracy, can make use of (1.3), or, rather, the following mathematical explanation of it.

**Theorem 1.** Letting \( \sigma^2 \) denote the variance of a random walk and \( \Delta \) denote the range of values, we have

\[
(1.4) \quad \mathbb{E}(\Delta) \sim \sigma \sqrt{\frac{8}{\pi}}.
\]

The prudent trader can use the range series \( R_t \), computed using only 2 data points, as a surrogate for the volatility series \( V_t \), computed using at least 21600 data points (for a 6 hour trading day with 1 second quantization).

2. **Approximating Volatility**

We turn our focus now to determining the accuracy of estimating the variance of a random walk using the range of values attained. To begin, we consider a simple, symmetric, one-dimensional random walk on the integers.

Let \( X \) be a discrete, symmetric, one-dimensional random variable. For example, let \( X \) take values \( \{\pm 1\} \) each with probability 1/2, i.e.

\[
(2.1) \quad \text{Prob}\{X = +1\} = \frac{1}{2} \quad \text{Prob}\{X = -1\} = \frac{1}{2}.
\]

The expectation of \( X \) is \( \mathbb{E}(X) = \sum_x x \text{Prob}\{X = x\} \). For the example,

\[
(2.2) \quad \mathbb{E}(X) = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0.
\]
In general, the expectation of a symmetric random walk is always 0 since
\[ E(X) = \sum_{x<0} x\text{Prob}\{X = x\} + \sum_{x>0} x\text{Prob}\{X = x\} = \sum_{x>0} (x-x)\text{Prob}\{X = x\} = 0. \]

The variance of \( X \) is \( \text{Var}(X) = E\left((X - E(X))^2\right) = E(X^2) - E(X)^2 \). Again, for the example we have
\[(2.3) \quad \text{Var}(X) = 1 - 0 = 1. \]

Define a new random variable \( S \) by summing successive independent trials of \( X \), i.e.
\[ S_k = X_1 + X_2 + \cdots + X_k \]
with the convention \( S_0 = 0 \). Since the trials are independent, the expectation of \( S_k \) is
\[(2.4) \quad E(S_k) = E\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} E(X_i) = 0. \]

and the variance of \( S_k \) is
\[(2.5) \quad \text{Var}(S_k) = \text{Var}\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} \text{Var}(X_i). \]

For the example, since \( \text{Var}(X_i) = 1 \), we have \( \text{Var}(S_k) = k \).

We say that the \( n \)th trial \( X_n \) occurs at epoch \( n \). We call the successive partial sums \( S_k \) the positions of a particle performing a random walk and mark these values on the vertical axis. A particular point on the vertical axis will be referred to as a site. For example, Figure 1 depicts a 40 step random walk for \( S_k \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{random_walk.png}
\caption{A geometric depiction of the random walk \( S_k \).}
\end{figure}

Define a new random variable \( \Delta_k \) to be the number of distinct sites visited during the random walk up to epoch \( k \). That is,
\[(2.6) \quad \Delta_k = \#\{S_0, \ldots, S_k\} = \max\{S_j\}_{j=0}^{k} - \min\{S_j\}_{j=0}^{k} + 1. \]

For the example, Theorem 1 becomes the following, first (foot)noted by Pólya \cite{polya}.

\footnote{Following Feller\cite{feller} who follows Riordan, the word epoch is used to denote points on the time axis because some contexts use the alternative terms (such as moment, time, point) in different meanings.}
**Theorem 2.** For $n$ large, the expectation of $\Delta_n$ is approximately given by

$$
E(\Delta_n) \sim \sqrt{\frac{8n}{\pi}} \approx 1.5958\sqrt{n}.
$$

Therefore an accurate estimate for the expectation of the range of the walk (the daily high minus low of an instrument) yields an estimate for the variance (the volatility of the instrument).

The relevance of Theorem 2 to the stock market hinges upon one’s belief that the market behaves like a symmetric random walk. An alternative but equally reasonable assumption is that the market behaves like a persistent random walk. In this case, there is a single parameter $\alpha$ taken between 0 and 1, and the probabilities for the random variable $X$ become

$$
\text{Prob}\{X_i = X_{i-1}\} = \alpha \quad \text{Prob}\{X_i = -X_{i-1}\} = 1 - \alpha,
$$

with the usual convention that $X_1$ satisfies (2.1). For this case, Theorem 1 becomes the following.

**Theorem 3.** For $n$ large, the expectation of $\Delta_n$ for a persistent random walk with parameter $\alpha$ is approximately given by

$$
E(\Delta_n) \sim \sqrt{\frac{8n\alpha}{\pi(1 - \alpha)}}.
$$

Since the variance of a persistent walk is proportional to the variance of a symmetric walk with constant of proportionality $\alpha/(1 - \alpha)$, Theorem 3 follows as a corollary to Theorem 2.

2.1. **Expected errors.** It is important to note that (2.7) holds only for the expectation of $\Delta$ and not for any particular instance of $\Delta$.

For the example in Figure 1, $\Delta_{40} = 7$ whereas the right hand side of (2.7) is 10.09. The actual value of $E(\Delta_{40})$ is approximately 10.16. Therefore error from using $\Delta$ in place of $E(\Delta)$ is due to the variance of $\Delta$.

**Theorem 4.** For $n$ large, the variance of $\Delta_n$ is approximately given by

$$
\text{Var}(\Delta_n) \sim 4n \left( \ln 2 - \frac{2}{\pi} \right) \approx 0.2181n.
$$

For $n = 40$, the right hand side of (2.10) is 8.724, so this example is more illustrative than exceptional.

When using the range as a surrogate for volatility in a simple moving average, having a small window can lead to large errors due to the variance of the range. For instance, if a trader chooses to use a moving average over a trading month, then he can expect little error from the variance of the range. However, if he instead chooses to use only the prior day’s historical volatility, then using the range instead will likely lead to significant errors in the forecast.
2.2. **Approximate errors.** For the sake of concreteness, the following table gives the estimates and errors for $n \leq 7$.

| $n$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\text{E}(\Delta_n)$ | 1.0000 | 2.0000 | 2.5000 | 3.0000 | 3.3750 | 3.7500 | 4.0625 | 4.3750 |
| $\sqrt{\frac{8}{n}} \pi$ | 0.0000 | 1.5958 | 2.2568 | 2.7640 | 3.1915 | 3.5682 | 3.9088 | 4.2220 |
| Error | 1.0000 | 0.4040 | 0.2432 | 0.2360 | 0.1835 | 0.1818 | 0.1537 | 0.1530 |
| % Error | 100.00 | 20.21 | 9.73 | 7.87 | 5.44 | 4.85 | 3.78 | 3.50 |

In order to understand the rate of convergence more precisely, we present three proofs: using Stirling’s formula, using the Tauberian Theorem, and using properties of the $\Gamma$-function.

### 3. An elementary approach

In order to prove Theorem 2, we refine $\Delta_k$ with a new random variable $\delta_k$ defined by

$$
\delta_k = \begin{cases} 
1 & \text{a new site is visited at epoch } k, \\
0 & \text{otherwise}.
\end{cases}
$$

Then clearly we have

$$
\Delta_k = \delta_0 + \delta_1 + \cdots + \delta_k.
$$

If we can derive a formula for $\text{E}(\delta_k)$, then (3.2) will allow us transform it into a formula for $\text{E}(\Delta_k)$.

To begin, note that since $\delta_k \in \{0, 1\}$, we have

$$
\text{E}(\delta_k) = \text{Prob}\{\delta_k = 1\}.
$$

Dvoretzky and Erdős [1] gave the following alternative interpretation for $\text{E}(\delta_k)$.

**Lemma 5.** The expectation of $\delta_k$ is given by

$$
\text{E}(\delta_k) = \text{Prob}\{\text{the origin has not been revisited by epoch } n\}.
$$

**Proof.** The event $\delta_k = 1$ occurs if and only if there is no loop beginning at epoch $i$ and returning at epoch $k$ for any $i$. Reversing time, this is equivalent to stating that the particle does not return to the origin at epoch $k-i$ for any $i$. The result now follows from (3.3). \(\square\)

To make use of this result, we now derive a formula for the probability that the particle is at the origin at epoch $n$. At this point we restrict our attention to the running example in order to make the problem more
concrete. As often is the case, it is simpler to derive a more general formula. For an epoch and a site, define \( p_{n,r} \) by

\[
p_{n,r} = \text{Prob}\{\text{at epoch } n, \text{ the particle is at site } r\}.
\]

Since a particle can only return to the origin after an even number of steps, we shall always take \( n \) even when \( r = 0 \).

**Proposition 6.** For an epoch and a site, we have

\[
p_{n,r} = \frac{1}{2^n} \binom{n}{(n+r)/2},
\]

where the binomial coefficient is 0 if the lower term is not an integer.

**Proof.** The number of lattice paths with \( p \) northeast steps (\( \nearrow \)) and \( q \) southeast steps (\( \searrow \)) is given by \( \binom{p+q}{p} \). For such a path to go from \((0,0)\) to \((n,r)\), we must have \( n = p + q \) and \( r = p - q \). Therefore the number of lattice paths from \((0,0)\) to \((n,r)\) is \( \binom{n}{(n+r)/2} \). Dividing by \( 2^n \), the total number of lattice paths with \( n \) steps, yields the result. \( \square \)

In order to apply this result to Lemma 5, we must either take the sum of probabilities \( p_{k,0} \) for \( 1 \leq k \leq n \) or use the following result.

**Lemma 7.** The probability that the origin has not been revisited by epoch \( 2k \) is equal to the probability that a return to the origin occurs at epoch \( 2k \).

**Proof.** We will construct a bijection between paths from \((0,0)\) to \((2k,0)\) and paths with \( 2k \) steps that remain weakly above the horizontal axis. Given a path from \((0,0)\) to \((2k,0)\), let \((j,m)\) be the leftmost occurrence of the minimum site visited. That is, \( m \leq S_i \) for all \( i \) and if equality holds then \( j \leq i \). Reflect the portion of the path from \((0,0)\) to \((j,m)\) across the vertical line \( t = j \), and slide the right endpoint of the reflected segment to \((2k,0)\). Consider \((j,m)\) to be the resulting path so that it now ends at \((2k,2|m|)\). This new path clearly remains weakly above the horizontal axis, and the process is easily reversible.

![Figure 2. A bijection between paths ending at the origin and paths staying weakly above the origin.](image)

We now claim that the number of paths of length \( 2k \) that lie strictly above the horizontal axis except for the origin is equal to one half the number of paths of length \( 2k \) that lie weakly above the horizontal axis. The former paths must all pass through the point \((1,1)\) after which they never fall below
the horizontal line $S = 1$. Thus resetting the origin to $(1, 1)$ yields a path of length $2k - 1$ that lies weakly above the horizontal axis. Since $2k - 1$ is odd, the final point is at least 1, and so adding another step, either northeast ($\nearrow$) or southeast ($\searrow$), the resulting path is of length $2k$ and still remains weakly above the horizontal axis. Since the last step has probability $1/2$ of either option, the claim is proved.

The lemma now follows from the observation that the number of paths of length $2k$ that never return to the origin is twice the number of paths of length $2k$ that lie strictly above the horizontal axis except for the origin. □

The final ingredient is an approximation for central binomial coefficients, which we can derive easily from Stirling’s formula [4].

**Theorem 8** (Stirling’s formula). We have

\[
(3.7) \quad n! = \sqrt{2\pi n^{n+1/2}}e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O(n^{-3})\right) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.
\]

**Theorem 9.** For $n$ large, the expectation of $\delta_n$ is approximately given by

\[
(3.8) \quad E(\delta_n) \sim \sqrt{\frac{2}{\pi n}}.
\]

**Proof.** By Lemma 5 the expectation of $\delta_n$ is the probability of no return to the origin by epoch $n$. By Lemma 7 for $n$ even this is equal to the probability that the particle is at the origin at epoch $n$. Therefore, by Proposition 6 we have

\[
(3.9) \quad E(\delta_{2k}) = \frac{1}{2^k} \binom{2k}{k}.
\]

Letting $n = 2k$, Stirling’s formula yields the result. □

Finally, to derive Theorem 2 from Theorem 9 we use (3.2):

\[
(3.10) \quad E(\Delta_n) = \sum_{j=0}^{n} E(\delta_j) \sim \sqrt{\frac{2}{\pi}} \sum_{j=0}^{n} j^{-1/2} = \sqrt{\frac{2}{\pi}} \left(2\sqrt{n}\right) = \sqrt{\frac{8n}{\pi}}.
\]

4. **TWO GENERATING FUNCTION APPROACHES**

Another approach to Theorem 2 is to derive a closed form for the generating function of $E(\Delta_n)$. Define the generating functions of $\delta_n$ and $\Delta_n$ by

\[
(4.1) \quad \delta(z) = \sum_{n\geq 0} E(\delta_n)z^n \quad \text{and} \quad \Delta(z) = \sum_{n\geq 0} E(\Delta_n)z^n.
\]

From (3.2), we have

\[
(4.2) \quad \Delta(z) = \sum_{n\geq 0} z^n \sum_{j=0}^{n} E(\delta_j) = \sum_{j=0}^{\infty} E(\delta_j) \sum_{n\geq j} z^n = \frac{\delta(z)}{1 - z}.
\]
Therefore it suffices to find a closed formula for $\delta(z)$.

Define $f_{n,r}$ to be the probability that the particle has first reached site $r$ at epoch $n$. Then we have

$$(4.3) \quad E(\delta_k) = \text{Prob}\{\delta_k = 1\} = \sum_r f_{k,r}. $$

Letting $F_r(z) = \sum_{n \geq 0} f_{n,r}z^n$ be the generating function for $f_{n,r}$, we have

$$(4.4) \quad \delta(z) = \sum_r F_r(z). $$

Recall $p_{n,r}$ denotes that probability that the particle is at site $r$ at epoch $n$. This probability may be decomposed into the first arrival at site $r$, say at epoch $j$, followed by a loop back $n-j$ steps later. That is,

$$(4.5) \quad p_{n,r} = \sum_{j=0}^n f_{j,r}p_{n-j,0}. $$

Letting $P_r(z) = \sum_{n \geq 0} p_{n,r}z^n$ be the generating functions for $p_{n,r}$, this becomes

$$(4.6) \quad P_r(z) = F_r(z)P_0(z). $$

Returning now to $\delta(z)$, solving (4.6) for $F_r(z)$ and combining the result with (4.4) yields

$$\delta(z) = \sum_r \frac{P_r(z)}{P_0(z)} = \frac{1}{P_0(z)} \sum_r \sum_{n \geq 0} p_{n,r}z^n = \frac{1}{P_0(z)} \sum_{n \geq 0} z^n \left( \sum_r p_{n,r} \right). $$

The inner sum on the right hand side is the probability that some site is visited at epoch $n$, which is a certainty. Therefore

$$(4.7) \quad \delta(z) = \frac{1}{P_0(z)} \sum_{n \geq 0} z^n = \frac{1}{P_0(z)} \frac{1}{1 - z}. $$

Substituting back into (4.2), we have proved the following.

**Theorem 10.** The generating function for $E(\Delta_n)$ is given by

$$(4.8) \quad \Delta(z) = \sum_{n \geq 0} E(\Delta_n)z^n = \frac{1}{P_0(z)} \frac{1}{(1 - z)^2}. $$

By Proposition 6, we deduce a closed form for $P_0(z)$ for the simple walk,

$$P_0(z) = \sum_{n \geq 0} p_{2n,0}z^{2n} = \sum_{n \geq 0} \frac{1}{2^{2n}} \binom{2n}{n} z^{2n} = \sum_{n \geq 0} \frac{1/2}{n} (z^2)^n = \frac{1}{\sqrt{1 - z^2}}. $$

Here we have used the basic identity $(1 - \zeta)^m = \sum_{k \geq 0} \binom{-m}{k} \zeta^k$. Substituting this into (4.2) gives

$$(4.9) \quad \Delta(z) = \frac{\sqrt{1 - z^2}}{(1 - z)^2}. $$
With a closed form for the generating function, we can get an estimate on the coefficients. To do this, it is helpful to rewrite (4.8) as

\[
\Delta(z) = \frac{\sqrt{2}}{(1-z)^{3/2}} \left( 1 + \frac{1}{2}(1-z) \right)^{1/2}.
\]

4.1. **Using the Tauberian Theorem.** Our first estimate uses the powerful Tauberian Theorem [5].

**Theorem 11** (Tauberian Theorem). Let \( \{q_n\} \) be a monotone, nonnegative sequence with generating function \( Q(z) = \sum_{n \geq 0} q_n z^n \). Then for \( \rho > 0 \) and \( L \) a slowly varying function, we have

\[
Q(z) \sim \frac{1}{(1-z)^\rho} L \left( \frac{1}{1-z} \right) \text{ as } z \to 1^- \quad \text{iff} \quad q_n \sim \frac{1}{\Gamma(\rho)} n^{\rho-1} L(n) \text{ as } n \to \infty
\]

*Here slowly varying means that for all \( \lambda \), \( L(\lambda x)/L(x) \to 1 \) as \( x \to \infty \).*

We may now derive Theorem [2] from (4.10) and the Tauberian Theorem with \( \rho = 3/2 \) and \( L(x) = \sqrt{2} \):

\[
E(\Delta_n) \sim \frac{1}{\Gamma(3/2)} n^{1/2} \sqrt{2} = \sqrt{\frac{8n}{\pi}}.
\]

Unfortunately, with such a general and powerful theorem, there are not good estimates on the error of the approximation.

4.2. **Using the \( \Gamma \)-function.** Since the closed form of \( \Delta(z) \) is fairly simple, we can use properties of the \( \Gamma \) function directly to find the error terms. We begin by expanding (4.10) as follows:

\[
\Delta(z) = \frac{\sqrt{2}}{(1-z)^{3/2}} \left( 1 + \frac{1}{2}(1-z) \right)^{1/2} = \frac{\sqrt{2}}{(1-z)^{3/2}} \sum_{k \geq 0} \binom{-1/2}{k} \left( \frac{1}{2} \right)^{k} (1-z)^k
\]

\[
= \sqrt{2} \sum_{k \geq 0} \binom{-1/2}{k} \left( \frac{1}{2} \right)^{k} (1-z)^k \cdot \sum_{n \geq 0} \binom{3/2-k}{n} z^n.
\]
Isolating the coefficient of $z^n$ and manipulating using the $\Gamma$ function gives
\[
E(\Delta_n) = \sqrt{2} \sum_{k \geq 0} \left( \frac{-1/2}{k} \right) \left( \frac{3/2 - k}{n} \right) \left( \frac{1}{2} \right)^k \\
= \sqrt{2} \sum_{k \geq 0} \left( \frac{-1/2}{k} \right) \left( \frac{1}{2} \right)^k \frac{\Gamma(3/2 - k + n)}{\Gamma(3/2 - k)n!} \\
= \frac{\sqrt{2}}{\Gamma(3/2 + n)} \frac{\Gamma(3/2 + n)n!}{n!} \sum_{k \geq 0} \left( \frac{-1/2}{k} \right) \left( \frac{1}{2} \right)^k \frac{\Gamma(3/2 - k + n)\Gamma(3/2)}{\Gamma(3/2 - k)\Gamma(3/2 + n)}
\]

We can manipulate the summand using the fundamental property of $\Gamma$
\[\Gamma(z + 1) = z\Gamma(z),\]
and we can approximate the outer term using the estimate
\[\frac{\Gamma(n + a)}{n!} = n^{a-1} \left( 1 + \frac{a(a-1)}{2n} + \frac{a(a-1)(a-2)(3a-1)}{24n^2} + O(n^{-3}) \right).\]
Combining all of these gives
\begin{equation}
E(\Delta_n) = \sqrt{\frac{8n}{\pi}} \left( 1 + \frac{1}{4n} - \frac{1}{32n^2} + O(n^{-3}) \right).
\end{equation}

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