BIHARMONIC PROPERLY IMMERSED SUBMANIFOLDS
IN THE EUCLIDEAN SPACES

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Abstract. We consider a complete biharmonic immersed submanifold $M$ in an Euclidean space $E^N$. Assume that the immersion is proper, that is, the preimage of every compact set in $E^N$ is also compact in $M$. Then, we prove that $M$ is minimal. It is considered as an affirmative answer to the global version of Chen’s conjecture for biharmonic submanifolds.

1. Introduction and Main Result

Let $M$ be an $n$-dimensional connected immersed submanifold in the Euclidean $N$-space $E^N$ $(n < N)$ and $x$ its position vector field. Then, it is well known that

\[(1) \quad \Delta x = nH,\]

where $\Delta$ and $H$ denote respectively the (non-positive) Laplace operator and the mean curvature vector field of $M$. The above equation shows particularly that $M$ is minimal, that is, $H = 0$ if and only if the isometric immersion $x : (M, g) \to E^N$ is a harmonic map. Here, $g$ denotes the induced Riemannian metric on $M$ from $x$. $M$ is said to be biharmonic if $H$ satisfies the following:

\[(2) \quad \Delta H = \frac{1}{n} \Delta^2 x = 0.\]

It is obvious that every minimal submanifold is biharmonic. We also note that $M$ is biharmonic if and only if $x$ is a biharmonic map.

For biharmonic submanifolds, there is an interesting problem, namely, Chen’s Conjecture (cf. [1]):

Conjecture 1. Any biharmonic submanifold $M$ in $E^N$ is minimal.

There are many affirmative partial answers to Conjecture 1 (cf. [1] [2] [3] [5] [6] [7]). In particular, there are some complete affirmative answers if $M$ is one of the following: (a) a curve [5], (b) a surface in $E^3$ [1], (c) a hypersurface in $E^4$ [6] [7].

On the other hand, since there is no assumption of completeness for submanifolds in Conjecture 1, in a sense it is a problem in local differential geometry. In this article, we reformulate Conjecture 1 into a problem in global differential geometry as the following (cf. [8] [9]):

Conjecture 2. Any complete biharmonic immersed submanifold in $E^N$ is minimal.
An immersed submanifold $M$ in $\mathbb{E}^N$ is said to be properly immersed if the immersion $M \to \mathbb{E}^N$ is a proper map. Here, we remark that the properness of the immersion implies the completeness of $(M, g)$. Our main result is the following, which gives an affirmative partial answer to Conjecture 2:

**Theorem 1.1.** Any biharmonic properly immersed submanifold $M$ in $\mathbb{E}^N$ is minimal.

For proving Theorem 1.1 the basic tool is the generalized maximum principle technique developed in Cheng-Yau [4] as follows:

Let $(M, g)$ be a complete manifold whose Ricci curvature $\text{Ric}_g$ is bounded from below. Let $u$ be a smooth nonnegative function on $M$. Assume that there exists a positive constant $k > 0$ such that

$$\Delta u \geq ku^2 \text{ on } M.$$  

Then, $u = 0$ on $M$.

The outline of proof of the generalized maximum principle is the following. For a fixed point $x_0 \in M$ and each large positive constant $\rho > 0$, consider the following smooth function

$$f(x) := (\rho^2 - r(x)^2)^2 u(x) \quad \text{for } x \in \overline{B_\rho(x_0)},$$

where $r(x) := \text{dist}_g(x, x_0)$ and $B_\rho(x_0) := \{ x \in M \mid r(x) \leq \rho \}$ denote respectively the distance from $x_0$ and the closed geodesic ball of radius $\rho$ centered at $x_0$. Then, the inequality (3) implies that

$$f(p) \leq c\rho^3 \text{ at a maximum point } p \in B_\rho(x_0) := \{ x \in M \mid r(x) < \rho \},$$

and hence

$$u(x) \leq \frac{c\rho^3}{(\rho^2 - r(x)^2)^2} \quad \text{for } x \in B_\rho(x_0).$$

Letting $\rho \nearrow \infty$ in the above inequality, we then get that $u = 0$ on $M$. Here, $c > 0$ is a positive constant depending only on $k, \dim M$ and the constant $\kappa \geq 0$ satisfying $\text{Ric}_g \geq -\kappa$ on $M$. The assumption of Ricci curvature bound from below is necessary for the estimate of $(\Delta r)(p)$ from above (see [10] for details).

When $(M, g)$ is a Riemannian immersed submanifold in $\mathbb{E}^N$, it is impossible to get such Ricci curvature bound from below without an assumption of boundedness for the second fundamental form $h$ of $M$. However, for Conjecture 2, any assumption for $h$ is artificial in some sense. To overcome this difficulty, we consider the function

$$F(x) := (\rho^2 - |x(x)|^2)^2 u(x) \quad \text{for } x \in M \cap x^{-1}(\overline{B_\rho})$$

instead of $f(x)$, where $|x(x)|^2 := \langle x(x), x(x) \rangle$ denotes the square-norm of the position vector $x(x)$ of $x \in M$ in $\mathbb{E}^N$ and $\overline{B_\rho} := \{ x \in \mathbb{E}^N \mid |x(x)| \leq \rho \}$. From the formula (1), we then get

$$|\Delta x(x)| \leq n|H(x)|.$$

Moreover if $M$ is biharmonic, by the harmonicity [2] combined with the above estimate, one can obtain a similar estimate to (1) for $u(x) := |H(x)|^2$ especially (see Section 3 for details).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. Section 3 is devoted to the proof of Theorem 1.1.
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2. Preliminaries

Let $M$ be an $n$-dimensional immersed submanifold in $\mathbb{E}^N$, $\mathbf{x} : M \to \mathbb{E}^N$ its immersion and $g$ its induced Riemannian metric. For simplicity, we often identify $M$ with its immersed image $\mathbf{x}(M)$ in every local arguments. Let $\nabla$ and $D$ denote respectively the Levi-Civita connections of $(M, g)$ and $\mathbb{E}^N = (\mathbb{R}^N, \langle\ ,\ \rangle)$. For any vector fields $X, Y \in \mathfrak{X}(M)$, the Gauss formula is given by

$$D_X Y = \nabla_X Y + h(X, Y),$$

where $h$ stands for the second fundamental form of $M$ in $\mathbb{E}^N$. For any normal vector field $\xi$, the Weingarten map $A_\xi$ with respect to $\xi$ is given by

$$D_X \xi = -A_\xi X + \nabla_{\nabla^\perp X} \xi,$$

where $\nabla^\perp$ stands for the normal connection of the normal bundle of $M$ in $\mathbb{E}^N$. It is well known that $h$ and $A$ are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

For any $x \in M$, let $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_N\}$ be an orthonormal basis of $\mathbb{E}^N$ at $x$ such that $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $T_x M$. Then, $h$ is decomposed as at $x$

$$h(X, Y) = \sum_{\alpha = n+1}^N h_{\alpha}(X, Y)e_{\alpha}.$$

The mean curvature vector $\mathbf{H}$ of $M$ at $x$ is also given by

$$\mathbf{H}(x) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i) = \sum_{\alpha = n+1}^N H_{\alpha}(x)e_{\alpha}, \quad H_{\alpha}(x) := \frac{1}{n} \sum_{i=1}^n h_{\alpha}(e_i, e_i).$$

It is well known that the necessary and sufficient conditions for $M$ in $\mathbb{E}^N$ to be biharmonic, namely $\Delta \mathbf{H} = 0$, are the following (cf. [1, 2, 3]):

$$\begin{cases} 
\Delta \mathbf{H} - \sum_{i=1}^n h(A_\mathbf{H}e_i, e_i) = 0, \\
n \nabla|\mathbf{H}|^2 + 4 \text{ trace } A_{\nabla^\perp \mathbf{H}} = 0,
\end{cases}$$

(5)

where $\Delta \mathbf{H}$ is the (non-positive) Laplace operator associated with the normal connection $\nabla^\perp$.

From the first equation of (5), we have the following.

Lemma 2.1. Let $M = (M, g)$ be a biharmonic immersed submanifold in $\mathbb{E}^N$. Then, the following inequality for $|\mathbf{H}|^2$ holds

$$\Delta |\mathbf{H}|^2 \geq \frac{2}{n} |\mathbf{H}|^4.$$

Proof. Under the above notations, the first equation of (5) implies that, at each $x \in M$,

$$\Delta |\mathbf{H}|^2 = 2 \sum_{i=1}^n \langle \nabla^\perp_{e_i} \mathbf{H}, \nabla^\perp_{e_i} \mathbf{H} \rangle + 2 \langle \Delta \mathbf{H}, \mathbf{H} \rangle \geq 2 \sum_{i=1}^n \langle h(A_\mathbf{H}e_i, e_i), \mathbf{H} \rangle = 2 \sum_{i=1}^n \langle A_\mathbf{H}e_i, A_\mathbf{H}e_i \rangle.$$
When \( \mathbf{H}(x) \neq 0 \), set \( e_N := \frac{\mathbf{H}(x)}{|\mathbf{H}(x)|} \). Then, \( \mathbf{H}(x) = H_N(x)e_N \) and \( |\mathbf{H}(x)|^2 = H_N(x)^2 \). From (9), we have at \( x \)
\[
\Delta|\mathbf{H}|^2 \geq 2 H_N^2 \sum_{i=1}^n (A_{\sigma e_i} e_i, A_{\sigma e_i} e_i) \\
= 2 |\mathbf{H}|^2 |h_N|^2 \\
\geq \frac{2}{n} |\mathbf{H}|^2 H_N^2 \\
= \frac{2}{n} |\mathbf{H}|^4.
\]
Even when \( \mathbf{H}(x) = 0 \), the above inequality (9) still holds at \( x \). This completes the proof.

3. Proof of Main Theorem

Proof of Theorem 1.1 If \( M \) is compact, applying the standard maximum principle to the elliptic inequality (8), we have that \( \mathbf{H} = 0 \) on \( M \). Therefore, we may assume that \( M \) is noncompact. Suppose that \( \mathbf{H}(x_0) \neq 0 \) at some point \( x_0 \in M \). Then, we will lead a contradiction.

Set
\[
u(x) := |\mathbf{H}(x)|^2 \quad \text{for } x \in M.
\]
For each \( \rho > 0 \), consider the function
\[
F(x) = F_\rho(x) := (\rho^2 - |x(x)|^2)u(x) \quad \text{for } x \in M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho).
\]
Then, there exists \( \rho_0 > 0 \) such that \( x_0 \in \mathbf{x}^{-1}(\mathbf{B}_{\rho_0}) \). For each \( \rho \geq \rho_0 \), \( F = F_\rho \) is a nonnegative function which is not identically zero on \( M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho) \). Take any \( \rho \geq \rho_0 \), and fix it. Since \( M \) is properly immersed in \( \mathbb{E}^N \), \( M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho) \) is compact.

By this fact combined with \( F = 0 \) on \( M \cap \mathbf{x}^{-1}(\partial \mathbf{B}_\rho) \), there exists a maximum point \( p \in M \cap \mathbf{x}^{-1}(\mathbf{B}_\rho) \) of \( F = F_\rho \) such that \( F(p) > 0 \). We have \( \nabla F = 0 \) at \( p \), and hence
\[
\nabla u = \frac{2 \nabla |x(x)|^2}{\rho^2 - |x(x)|^2} \quad \text{at } p.
\]
We also have that \( \Delta F \leq 0 \) at \( p \). Combining this with (8), we obtain
\[
\Delta u \leq \frac{6 |\nabla |x(x)|^2|^2}{\rho^2 - |x(x)|^2} + \frac{2 \Delta |x(x)|^2}{\rho^2 - |x(x)|^2} \quad \text{at } p.
\]
From (2), we note
\[
\Delta |x(x)|^2 = 2 \sum_{i=1}^n \nabla e_i x(x)|^2 \quad \text{and } (\Delta x(x), x(x)) \leq 2n + 2n |\mathbf{H}||x(x)|,
\]
\[
|\nabla |x(x)|^2|^2 \geq 4n |x(x)|^2.
\]

It then follows from (6), (9) and (10) that
\[
u(p) \leq \frac{12n^2 |x(p)|^2}{(\rho^2 - |x(p)|^2)^2} + \frac{2n^2(1 + \sqrt{u(p)}|x(p)|)}{\rho^2 - |x(p)|^2},
\]
and hence
\[
F(p) \leq 12n^2|x(p)|^2 + 2n^2(\rho^2 - |x(p)|^2) + 2n^2\sqrt{F(p)}|x(p)|.
\]
Therefore, there exists a positive constant \( c(n) > 0 \) depending only on \( n \) such that
\[
F(p) \leq c(n)p^2.
\]
Since $F(p)$ is the maximum of $F = F_\rho$, we have
\[ F(x) \leq F(p) \leq c(n)\rho^2 \quad \text{for} \quad x \in M \cap x^{-1}(B_\rho), \]
and hence
\[ |H(x)|^2 = u(x) \leq \frac{c(n)\rho^2}{(\rho^2 - |x(x)|^2)^2} \quad \text{for} \quad x \in M \cap x^{-1}(B_\rho) \quad \text{and} \quad \rho \geq \rho_0. \]
Letting $\rho \to \infty$ in (11) for $x = x_0$, we have that
\[ |H(x_0)|^2 = 0. \]
This contradicts our assumption that $H(x_0) \neq 0$. Therefore, $M$ is minimal. \hfill \Box

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