Metric decomposability theorems on sets of integers

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Abstract
A set \( \mathcal{A} \subseteq \mathbb{N} \) is called additively decomposable (resp., asymptotically additively decomposable) if there exist sets \( \mathcal{B}, \mathcal{C} \subseteq \mathbb{N} \) of cardinality at least two each such that \( \mathcal{A} = \mathcal{B} + \mathcal{C} \) (resp., \( \mathcal{A} \Delta (\mathcal{B} + \mathcal{C}) \) is finite). If none of these properties hold, the set \( \mathcal{A} \) is called totally primitive. We define \( \mathbb{Z} \)-decomposability analogously with subsets \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) of \( \mathbb{Z} \). Wirsing showed that almost all subsets of \( \mathbb{N} \) are totally primitive. In this paper, in the spirit of Wirsing, we study decomposability from a probabilistic viewpoint. First, we show that almost all symmetric subsets of \( \mathbb{Z} \) are \( \mathbb{Z} \)-decomposable. Then we show that almost all small perturbations of the set of primes yield a totally primitive set. Further, this last result still holds when the set of primes is replaced by the set of sums of two squares, which is by definition decomposable.

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1 | STATEMENT OF THE RESULTS

This article is concerned with sum sets in the integers. Given two subsets \( \mathcal{A}, \mathcal{B} \) of \( \mathbb{Z} \), their sum set \( \mathcal{A} + \mathcal{B} \) is the set \( \{a + b : (a, b) \in \mathcal{A} \times \mathcal{B}\} \). We denote \( 2\mathcal{A} = \mathcal{A} + \mathcal{A} \). Further, we denote by \( \mathcal{A} \sim \mathcal{B} \) the property that \( \mathcal{A} \Delta \mathcal{B} \) is finite. A set \( \mathcal{C} \subseteq \mathbb{N} \) is called additively decomposable (resp., asymptotically additively decomposable) if there exist sets \( \mathcal{A}, \mathcal{B} \subseteq \mathbb{N} \) of cardinality at least two each such that (resp., \( \mathcal{C} \sim \mathcal{A} + \mathcal{B} \)). A set that is not asymptotically additively decomposable is called totally
primitive. Similarly, a set $C \subset \mathbb{Z}$ is called \textit{additively $\mathbb{Z}$-decomposable} (resp \textit{asymptotically additively $\mathbb{Z}$-decomposable}) if there exist sets $A, B \subset \mathbb{Z}$ of cardinality at least two each such that $C = A + B$ (resp., $C \sim A + B$).

An old conjecture of Ostmann [11, p. 13] asserts that the set $P$ of primes is totally primitive. In spite of serious efforts by numerous authors and notable advances (see [2, 3] and the very recent [5] on related problems), the problem remains unsolved. The philosophy supporting this idea is that additive decomposability is a very rare property, so most sets occurring in number theory that are not specifically defined to be a sum set\footnote{However, examples such as the set of sums of two squares, which can be defined through a multiplicative property ($n$ is a sum of two squares if and only if $v_p(n)$ is even for every prime $p$ congruent to 3 modulo 4) or [2, Example 2.2] show that one must be careful with this philosophy.} should not have it. A theorem of Wirsing [12] actually asserts that almost all sets are totally primitive, where “almost all” refers to the construction of a random subset of $\mathbb{N}$ by selecting each integer into the set with probability $1/2$ independently of each other.

On the other hand, Ruzsa recently showed [10] in the Number Theory Web Seminar that the widely believed Hardy–Littlewood prime tuples conjecture implies that the signed set of primes $P \cup (-P)$ is asymptotically additively $\mathbb{Z}$-decomposable, that is, there exist sets $A, B \subset \mathbb{Z}$ such that $P \cup (-P) \sim A - B$. Our first result shows that this property is actually typical. To state it, let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of independent identically distributed Bernoulli variables satisfying $\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = 0) = 1/2$ for each $n \in \mathbb{N}$.

\textbf{Theorem 1.} \textit{Almost all symmetric subsets of $\mathbb{Z}$ are additively $\mathbb{Z}$-decomposable. More precisely, let $D = \{n \in \mathbb{Z} : \xi_{|n| = 1}\}$. Then $\mathbb{P}(D \text{ is } \mathbb{Z}\text{-decomposable}) = 1$.}

This theorem will follow from a finite tuples property that we will prove. Regarding subsets of $\mathbb{N}$, the same finite tuple property yields the following.

\textbf{Theorem 2.} \textit{Almost all subsets of $\mathbb{N}$ contain a sumset $A + B$ where both summands are infinite. More precisely, let $C = \{n \in \mathbb{N} : \xi_n = 1\}$. Then with probability 1, $C$ contains a sumset $A + B$ where both summands are infinite.}

A theorem of Granville [4] shows that the Hardy–Littlewood conjecture implies that set of primes contains a sumset $A + B$ where both summands are infinite. Thus, what is known of the primes, resp., the signed primes, under the Hardy–Littlewood conjecture, is true of almost every subset of $\mathbb{N}$, resp., almost every symmetric subset of $\mathbb{Z}$. Further, let us observe that a much stronger statement than Theorem 2 holds, namely that every dense subset of $\mathbb{N}$ contains a sumset $A + B$ where both summands are infinite; this was conjectured by Erdős and proven by Moreira, Richter, and Robertson [8] (see also [7] for a simpler proof). Almost every subset being dense, this statement implies Theorem 2, which may be thought of as a “cheap” version of the theorem of Moreira, Richter, and Robertson.

Regarding decomposability of subsets of $\mathbb{N}$, we consider new probability distributions whose mass is concentrated “near” a fixed set of interest such as the set of the primes. First we introduce a standard notational convention: a set (or equivalently an increasing sequence) of integers is denoted by a calligraphic letter (e.g., $\mathcal{A}$), its elements are denoted by the corresponding lower case letter (e.g., $a_n$), its counting function by the corresponding upper case letter (e.g., $A(x) = |\mathcal{A} \cap [1,x]|$).
Let $S$ be an infinite set and define a function $f = f_S$ by $f(x) = x/S(x)$. We make the following two hypotheses.

**S1** $f(x)$ tends to infinity as $x$ does.

**S2** The number of $s_k \leq x$ satisfying $s_{k+2} - s_k \leq h$ is $o_h(S(x)/\log f(x))$.

Then we fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ that has the following properties.

**D1** $\delta_n \geq 2$ for all but $o(x)$ integers $n \leq x$.

**D2** There exists an integer $\ell$ such that $\sum_{s_k \leq x} \prod_{i=0}^{\ell-1} \delta_{k+i}^{-1} = o(S(x)/\log f(x))$.

**D3** $1 \leq \delta_n \leq (s_{n+1} - s_n - 1)/2$ for all $n \geq 2$.

An obvious consequence of **D2** is that $\limsup_n \delta_n = \infty$. Also we may assume that $\ell$ is even. We will often denote $\delta_n^{-1}$ by $\eta_n$. Moreover, we consider a sequence $\varepsilon_n$ of independent random integers such that

**E1** $-(s_n - s_{n-1})/2 < \varepsilon_n \leq (s_{n+1} - s_n)/2$ for all $n \in \mathbb{N}$;

**E2** For every $k \in \mathbb{Z}$, we have $\mathbb{P}(\varepsilon_n = k) \leq \delta_n^{-1}$.

Thanks to **D3**, such a sequence exists: we may take $\varepsilon_n$ to be uniformly distributed on the interval of integers $(-(s_n - s_{n-1})/2, (s_{n+1} - s_n)/2]$ for all $n \in \mathbb{N}$, for instance. Note that **E2** implies that $\Var(\varepsilon_n) \gg \delta_n^2$, so we require a certain amount of dispersion; otherwise the problem is too close to the deterministic question of whether $S$ itself is asymptotically additively decomposable, which is completely different.

Finally we consider the random sequence $c_n = \varepsilon_n + s_n$ and the random set $C_n = \{c_n : n \in \mathbb{N}\} \subset \mathbb{N}$. Observe that the definition of $\varepsilon_n$ ensures that $c_n < c_{n+1}$ for all $n$, so that the sequence $\varepsilon_n$ uniquely determines $C$.

**Theorem 3.** The set $C$ is almost surely totally primitive.

The following theorem shows that we can take $S$ to be the set of the primes or the set of the sums of two squares. Basically, it suffices to disturb the $n$th prime or sum of two squares by a random integer of standard deviation a small power of $\log \log n$.

**Theorem 4.** The set $P$ of the primes and the set $2Q$ of the sums of two squares fulfill the hypotheses S1,S2. When $S$ is either of these two sets and the sequence $\delta$ satisfies $\delta_n \leq (s_{n+1} - s_n - 1)/2$ for all $n \in \mathbb{N}$ and $\delta_n \gg \min((s_{n+1} - s_n - 1)/2, (\log \log n)^i)$ for $n \geq 2$ and some $i > 0$ arbitrarily small, the properties **D1-3** hold.

In this sense, almost every small perturbation of either of these two sets is totally primitive. Regarding sums of two squares, this statement is a kind of inverse Atkin theorem: Atkin [1] proved that for almost every small perturbation $Q'$ of the set $Q$ of the squares, the sumset $2Q'$ is dense, in sharp contrast with $2Q$. Here we perturb the sumset instead of perturbing the summands, and lose almost surely the decomposability property.

### PROOF OF THEOREMS 1 AND 2

Both results rely on the finite tuples property, which is the property of a set $C \subset \mathbb{Z}$ such that for any finite $H \subset \mathbb{Z}$, there exist infinitely many $n \in \mathbb{N}$ such that $n + H \subset C$. 
Here let \((\xi_n)_{n \in \mathbb{N}}\) be a sequence of independent identically distributed Bernoulli variables satisfying \(P(\xi_n = 1) = P(\xi_n = 0) = 1/2\) for each \(n \in \mathbb{N}\) and let \(C = \{n \in \mathbb{N} : \xi_n = 1\}\).

**Lemma 5.** Almost all subsets of \(\mathbb{N}\), and therefore almost all symmetric subsets of \(\mathbb{Z}\), have the finite tuples property. That is, with probability 1, \(C\) and therefore \(C \cup -C\) have the finite tuples property.

**Proof.** Fix a particular finite non-empty set \(H \subset \mathbb{Z}\) of cardinality \(k\). Then for any given \(n \in \mathbb{N}\) large enough, the probability that a random subset \(C \subset \mathbb{N}\) is such that \(n + H \subset C\) is \(2^{-k}\). One can extract an infinite sequence \(n_j\) of integers such that the events \(n_j + H \subset C\) are pairwise independent, for instance \(n_j = j(diam H + 1)\) where \(diam\) denotes the diameter (difference between maximum and minimum). Therefore, by the Borel–Cantelli lemma, with probability 1, there exist infinitely many \(n\) such that \(n + H \subset C\). Now this is true for any particular \(H\), but as there are only countably many finite tuples \(H \subset \mathbb{N}\), we can take the intersection and conclude.

**Lemma 6.** Any symmetric subset of \(\mathbb{Z}\) satisfying the finite tuples property is additively \(\mathbb{Z}\)-decomposable; it is in fact the difference set of two infinite subsets of \(\mathbb{N}\).

Lemmas 5 and 6 immediately imply Theorem 1.

**Proof.** Indeed, let \(D\) be a symmetric subset of \(\mathbb{Z}\) satisfying the finite tuple property (and therefore infinite), and let \(D = \{d_1, d_2, \ldots\}\) be an ordering of \(D\). We will construct iteratively increasing sequences \(a_k, b_k\) of positive integers such that \(D = \{a_k\} - \{b_k\}\) and \(d_k = a_k - b_k\). To achieve this, start with any pair \((a_1, b_1)\) such that \(d_1 = a_1 - b_1\). Assuming finite increasing sequences of positive integers \(a_1, \ldots, a_k\) (forming a set \(A_k\)) and \(b_1, \ldots, b_k\) (forming a set \(B_k\)) have already been constructed and satisfy \(a_i - b_i = d_i\) and \(A_k - B_k \subset D\), let us construct \(a_{k+1} \notin A_k\) and \(b_{k+1} \notin B_k\) such that \(A_k \cup \{a_{k+1}\} - B_k \cup \{b_{k+1}\} \subset D\). Let us look for a positive integer \(x\) such that \(x - B_k \subset D\) and \(x - d_{k+1} - A_k \subset D\) (by symmetry equivalently \(-x + d_{k+1} + A_k \subset D\)). There exist infinitely many such \(x\), due to the finite tuples property applied to the tuple \(-B_k \cup (d_{k+1} + A_k)\). So, there exists such an \(x\) outside of the finite set \(A_k \cup (d_{k+1} + B_k)\), and we pick \(a_{k+1} = x\) and \(b_{k+1} = x - d_{k+1}\) and we are done.

**Lemma 7.** Any subset \(D\) of \(\mathbb{N}\) satisfying the finite tuples property contains a sumset \(A + B\) where both summands are infinite.

Lemmas 5 and 7 imply Theorem 2.

**Proof.** We will construct iteratively increasing sequences \(a_k, b_k\) of positive integers such that \(\{a_k : k \in \mathbb{N}\} + \{b_k : k \in \mathbb{N}\} \subset D\). We start with any pair \((a_1, b_1)\) such that \(a_1 + b_1 \in D\). Assuming pairwise distinct \(a_1, \ldots, a_k\) (forming a set \(A_k\)) and \(b_1, \ldots, b_k\) (forming a set \(B_k\)) have already been constructed and satisfy \(A_k + B_k \subset D\), we first select \(a_{k+1}\) to be an integer outside \(A_k\) satisfying \(a_{k+1} + B_k \subset D\), which exists by the finite tuple property. We then set \(A_{k+1} = \{a_{k+1}\} \cup A_k\) and take \(b_{k+1} \in \mathbb{N}\) outside \(B_k\) such that \(b_{k+1} + A_{k+1} \subset D\). □

### 3 PROOF OF THEOREMS 3 AND 4

**Proof of Theorem 3.** Let Dec be the set of additively decomposable subsets of \(\mathbb{N}\).
As an asymptotically additively decomposable set differs from an element of $\mathcal{A}$ by finitely many editions, of which there are countably many, it suffices to show that $P(C \in \text{Dec}) = 0$. Let $\text{Dec}_1$ be the set of sets $C$ of the form $C = \mathcal{A} + \mathcal{B}$ for some $\mathcal{A}, \mathcal{B} \subset \mathbb{N}$ satisfying $\text{min}(|\mathcal{A}|, |\mathcal{B}|) \geq 2$ and $A(x) + B(x) < \frac{\log x}{2f(x)\log f(x)}$ for infinitely many integers $x$. Let $\text{Dec}_2$ be the set of sets $C$ of the form $C = \mathcal{A} + \mathcal{B}$ for some $\mathcal{A}, \mathcal{B} \subset \mathbb{N}$ satisfying $2 \leq \text{min}(|\mathcal{A}|, |\mathcal{B}|) < \infty$. Let $\text{Dec}_3 = \text{Dec} \setminus (\text{Dec}_1 \cup \text{Dec}_2)$ so that $\text{Dec} = \text{Dec}_1 \cup \text{Dec}_2 \cup \text{Dec}_3$.

First we show that $P(C \in \text{Dec}_1) = 0$. We note that $\text{Dec}_1 = \bigcap_{x_0 \geq 1} \bigcup_{x \geq x_0} \text{Dec}_1(x)$ where $\text{Dec}_1(x)$ is the set of sets $C$ of the form $C = \mathcal{A} + \mathcal{B}$ for some $\mathcal{A}, \mathcal{B} \subset \mathbb{N}$ satisfying $|\mathcal{A}|, |\mathcal{B}| \geq 2$ and $A(x) + B(x) < \frac{\log x}{2f(x)\log f(x)}$. Let $x$ be a large integer and $r \leq x/2$. We note that if $\mathcal{A} + \mathcal{B} \subset [0, x]$ then $\mathcal{A} + \mathcal{B}$ is one of at most $r(2 \cdot x/r) = r^2$ sets. But for any set $\mathcal{A} \subset [0, x]$, the probability that $\mathcal{A} \subset [0, x]$ is at most $\prod_{n : s_n + 1 < x} \frac{\alpha}{\beta - 1} n \leq 2^{-\Omega(x/\log x)}$ by definition of $\mathcal{A}$ and property $D_1$. Taking $r = \lfloor \frac{\log x}{2f(x)\log f(x)} \rfloor$, we infer that $P(C \in \text{Dec}_1(x)) = \exp(-\Omega(x/\log x)) = o(1)$. This implies that $P(C \in \text{Dec}_1) = 0$.

Now we seek to show that $P(C \in \text{Dec}_2) = 0$. Note that every set $\mathcal{A} \subset [0, x]$ satisfies $\limsup \text{min}(d_{k+1} - d_k, d_k - d_{k-1}) < \infty$. Indeed, if $\mathcal{A} = \mathcal{A} + \mathcal{B}$ where $\mathcal{B}$ is finite, let $H$ be the largest gap between two consecutive elements of $\mathcal{B}$; then $\text{min}(d_{k+1} - d_k, d_k - d_{k-1}) = H$ for every $k$. Further we claim that for any $e$ and $k$ integers,

$$P(c_k + h = c_{k+1}) \leq h^2 \eta_k \eta_{k+1}. \quad (1)$$

To see this, note that $c_k + h = c_{k+1}$ implies that $e_k \in [(s_{k+1} - s_k)/2 - h, (s_{k+1} - s_k)/2)$ and similarly $e_{k+1} \in [-(s_{k+1} - s_k)/2, -(s_{k+1} - s_k)/2 + h]$. We infer that $P(c_{k+1} - c_k \leq H) \leq H^3 \eta_k \eta_{k+1}$. Therefore,

$$P(\text{min}(c_{k+1} - c_k, c_k - c_{k-1} \leq H) \leq H^3 \eta_k \eta_{k+1} \eta_{k+1}).$$

As $\inf \eta_k = 0$, it follows that

$$P(\forall k \text{ min}(c_{k+1} - c_k, c_k - c_{k-1} \leq H) \leq H \leq \inf_k P(\text{min}(c_{k+1} - c_k, c_k - c_{k-1} \leq H) = 0),$$

hence $P(C \in \text{Dec}_2) = 0$.

We turn to $\text{Dec}_3$. We note that $\text{Dec}_3 = \bigcup_{x_0 \geq 1} \bigcup_{x \geq x_0} \text{Dec}_3(x)$ where $\text{Dec}_3(x)$ is the set of sets $C$ of the form $C = \mathcal{A} + \mathcal{B}$ for some infinite sets $\mathcal{A}, \mathcal{B} \subset \mathbb{N}$ satisfying $A(x) + B(x) \geq \frac{\log x}{2f(x)\log f(x)}$. Suppose $C = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are infinite and $A(x) + B(x) \gg x/(f(x)\log f(x))$, in particular $\text{max}(A(x), B(x)) \gg xx/(f(x)\log f(x))$ for some constant $c > 0$ and every $x$ large enough. Let $x$ be a large integer and assume without loss of generality that $A(x) \geq xx/(f(x)\log f(x))$. Let $b_1 < b_2 < \ldots < b_\epsilon$ be in $B$. Then we know that for any $n \in \mathcal{A}$ the set $\{n + b_i : i \in [\epsilon]\}$ lies in $C$. So, the number of $n \in [0, x]$ such that $\{n + b_i : i \in [\epsilon]\} \subset C$ is at least $A(x)$. Therefore,

$$P(C \in \text{Dec}_3(x)) \leq P([0, x] \cap \bigcap_{i \in [\epsilon]} (C - b_i) \geq xx/(f(x)\log f(x))). \quad (2)$$

Now suppose $\{n + b_i : i \in [\epsilon]\} \subset C$. Then $n + b_i \in C$ means $n + b_i = c_{j_i} = s_{j_i} + \epsilon_{j_i}$ for some $j_1 < j_2 < \ldots < j_\epsilon$. However, the first property of $\epsilon_n$ implies easily that either $j_i = j_1 + i - 1$ for
every \( i \in [\ell] \), or \( s_{j_i+2} - s_{j_i} < 2(b_{i+1} - b_{i}) \) for some \( i \in [\ell] \). Recall that, by hypothesis, the number of \( k \) such that \( s_{k+1} \leq x \) and \( s_{k+2} - s_k < 2h \) is \( o_h(x/(f(x) \log f(x)) \).

So, in total

\[
\mathbb{E}[\{|n \leq x : \{n + b_i : i \in [\ell] \} \subset C\}|] = \sum_{n \leq x} \mathbb{P}(\{n + b_i : i \in [\ell] \} \subset C)
\]

\[
\leq \sum_{k : s_{k+1} \leq x+b_\ell} 1_{s_k+2 - s_k < 2h}
\]

\[
+ \sum_{k : s_{k+1} \leq x+b_1} \mathbb{P}(\forall i \in [\ell] : c_k + b_i - b_1 = c_{k+i-1}).
\]

Recall that \( \mathbb{P}(c_k + h = c_{k+1}) \leq h^2 \eta_k \eta_{k+1} \) by Equation (1). Assume \( \ell = 2\ell' \). By independence,

\[
\mathbb{P}(\forall i \in [\ell] : c_k + b_i - b_1 = c_{k+i-1}) \leq \prod_{i=0}^{\ell'-1} \eta_{k+i}.
\]

As \( \sum_{s_k \leq x} \prod_{i=0}^{\ell'-1} \eta_{k+i} = o(x/(f(x) \log f(x))) \), we obtain that

\[
\mathbb{E}[\{|n \leq x : \{n + b_i : i \in [\ell] \} \subset C\}|] = o(x/(f(x) \log f(x))).
\]

By Markov’s inequality, we find that

\[
\mathbb{P}(\{|n \leq x : \{n + b_i : i \in [\ell] \} \subset C\}| \geq \kappa x/(f(x) \log f(x))) = o(1).
\]

In view of Equation (2), infer that \( \mathbb{P}(C \in \text{Dec}_3(x)) = o(1) \) whence \( \mathbb{P}(C \in \text{Dec}_3) = 0 \).

Proof of Theorem 4. By the prime number theorem, when \( S \) is the set \( P \) of the primes, we have \( f_P(x) \sim \log x \) that proves S1. Further, let \( \pi_m(x) \) be the number of primes \( p \leq x \) such that \( p + m \in P \). Then by Selberg’s sieve \( \pi_m(x) \ll \frac{x}{\log^2 x} \prod_{p|m} (1 + 1/p) \), where the implied constant is absolute; see, for instance, [6]. Then the number of \( p_k \leq x \) such that \( p_{k+1} \leq p_k + M \) is at most

\[
\sum_{m \leq M} \pi_m(x) \ll \frac{x}{\log^2 x} \sum_{m \leq M} \prod_{p|m} (1 + 1/p) = \frac{x}{\log^2 x} \sum_{d \leq M} \frac{\mu(d)^2}{d} \ll M \frac{x}{\log^2 x}. \tag{3}
\]

In particular, the number of primes \( p_k \leq x \) such that \( p_{k+2} - p_k \leq h \) is \( O_h(x/(f(x) \log f(x))) \) (in fact it is \( O_h(x/(f(x)^3)) \)). Thus, S2 holds.

Let the sequence \( \delta \) satisfy \( \delta_n \leq (s_{n+1} - s_{n-1})/2 \) for all \( n \in \mathbb{N} \) and \( \delta_n \gg \min((s_{n+1} - s_{n-1})/2, (\log \log n)^i) \) for \( n \geq 2 \) and some \( i > 0 \). Let us check that this sequence satisfies the properties required above. D1,3 are obvious. Applying (3) with \( M = (\log \log x)^i \), we see that \( \delta_n \gg (\log \log n)^i \) for all but \( o(x/(f(x) \log f(x))) \) primes \( p_n \leq x \). As a result, D2 is satisfied for any \( \ell > i^{-1} \).

Similarly, for the set \( S = 2Q \) of sums of two squares, we have \( f_S(x) \sim \sqrt{\log x} \) by a classical result of Landau, which proves S1. Further, let \( \vartheta_m(x) \) be the number of sums of two squares \( s \leq x \)
such that $s + m \in S$. Then again $\vartheta_m(x) \ll \frac{x}{\log x} \prod_{p|m, p \equiv 3 \mod 4}(1 + 1/p)$, where the implied constant is absolute; this may be achieved via Selberg’s sieve, see [9]. Arguing like in Equation (3), we find that the number of $s_k \leq x$ such that $s_{k+1} \leq s_k + M$ is at most $M \frac{x}{\log x}$. In particular, the number of sums of two squares $s_k \leq x$ such that $s_{k+2} - p_k \leq h$ is $O_h(x/f(x)^2) = o_h(x/(f(x)\log f(x)))$ (in fact here too, it is $O_h(x/(f(x)^3))$). Thus $S2$ holds.

Simultaneously, this proves $D1$. $D3$ holds by definition and $D2$ may be proven along the same lines as above. 

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