Analysis of Dynamic Axial-Symmetric Shells

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The aim of this work is to analyze the dynamical behavior of relativistic infinite axial-symmetric shells with flat interior and a radiation filled curved exterior spacetimes. It will be proven, by the use of conservation equations of Israel, that the given configuration does not let expansion or collapse of the shell which was proposed before, but rather the shell stays at constant radius. The case of null-collapse will also be considered in this work and it will be shown that the shell collapses to zero radius, and moreover, if cylindrical flatness is imposed a boundary layer is obtained still contrary to previous works.

I. INTRODUCTION

Dynamic relativistic shells were studied by many authors for many different applications. Israel, who first formulated the tools for analyzing shell models [1] also worked on the collapse of spherically symmetric shells [2] that separate an interior Minkowski spacetime with an exterior Schwarzschild spacetime and the possible violations of causality in such systems. Later Barrabés and Israel [3]-[4] developed the null shell formalism and considered collapse of spherically symmetric shells along with Poisson [5]. Later, Olea and Crisóstomo treated the dynamics of shell models by the use of the Hamiltonian formalism and considered collapse of spherically symmetric shells [6]. Recently, Jezierski [7] studied the geometry of crossing null shells which is another important application. Similar dynamic shell models prove to be very important in wormhole constructions and such models are extensively studied by Visser and he collected and summarized these in his book [8]. These wormhole models were constructed by the so called cut and paste technique where two copies of the same spacetime are identified on a hypersurface which is the boundary of two regions cut from the spacetimes. This junction hypersurface in the spherically symmetric case that forms the throat of the wormhole was given a radial degree of freedom in order to minimize the use of stress-energy which violates the weak energy condition. Recently, Eiroa and Simeone analyzed the case of cylindrical cut and paste wormholes [9] which is the direct generalization of Visser’s approach to axial-symmetric spacetimes. Axial-symmetric dynamic shells were also studied by Geršl, Klepáč and Horský who considered a charged cylindrical shell that separates an interior Bonnor-Melvin Universe and an exterior Datta-Raychaudhuri spacetime [10]. Other examples of dynamic axial-symmetric shells were studied by Pereira and Wang [11] and then later by Seriu [12], who considered an axial-symmetric dynamic shell which separates an interior flat spacetime and an exterior radiation-filled curved spacetime.

Our consideration is based on the same configuration analyzed by Pereira, Wang [11] and Seriu [12] before. Since the most recent one is the work by Seriu, let us recall this work [12]. First he considered a general axial-symmetric spacetime and calculated the elements of the second fundamental form and the surface stress-energy tensor from the Israel equations. Then he considered the more special case analyzed before by Pereira and Wang and obtained dynamical equations concerning the motion of the shell from the components of the surface stress-energy tensor. By analyzing these dynamical equations he concluded with the result that the configuration at hand does not let the collapse of the shell even in the limits where the radial velocity of the shell approaches to the speed of light. (i.e. the null limit) Finally he compared his work with the previous one by Pereira and Wang.

In our work we also use the same configuration used previously by Pereira-Wang [11] and Seriu [12], but the method we use to obtain the dynamical equations concerning the motion of the shell is totally different. After we calculate the elements of the second fundamental form for both the interior and exterior spacetimes, we directly use the conservation equations of Israel and we prove that in the given configuration the shell stays at constant radius and does not move at all. We also compare our results with those obtained by Seriu in two ways. We first use the null-shell formalism [3]-[4]-[5] directly and show that the results obtained are different from those obtained by him. Secondly, we showed that his results are inconsistent with the conservation equations.

Our work possesses the use of conservation equations for determining the dynamical behavior of thin shells which has a non-vacuum exterior spacetime for the first time. Previously, only the vacuum conservation equations were used by Israel [1]-[2] for considering dynamic spherical shells, so this work is an example where the full conservation equations are used.

In section II, we start directly by considering the null shell limit in the configuration given by Seriu, Pereira and Wang [12]-[11]. We first calculate the elements of the transverse curvature in both spacetimes and then apply the Barrabés-Israel null shell formalism and obtain the surface stress-energy tensor. We show that in this case the shell collapses to zero radius, which is contrary to the recent work by Seriu [12]. Moreover, when cylindrical flatness is imposed, the shell turns into a boundary layer which is also contrary to the above mentioned work.

In section III, we analyze the timelike case by using the Israel formalism directly. We first calculate the elements of
the second fundamental form in both spacetimes and then use them in the conservation equations which proves that the shell must stay stationary at some constant radius. In addition, we compare our results with the previous ones and show that the previous ones do not reach the conclusions they made, when conservation equations are used.

In section IV, we give a conclusion of our results.

Throughout this work, we use the sign conventions of MTW \[12\] except the sign of the second fundamental form and transverse curvature is Israel’s \[1\], \[2\]. We use $\Sigma$ to denote timelike hypersurfaces and $\Xi$ to denote null hypersurfaces. $k$ denotes the normal vector to $\Xi$, $N$ denotes the transverse vector, $\sigma_{AB}$ denotes the intrinsic metric on $\Xi$, $y^a = (\lambda, \theta^C)$ denotes the intrinsic coordinates on $\Xi$, $C_{ab}$ denotes the transverse curvature and finally upper case Latin indices runs from 2 to 3.

**II. CYLINDRICAL SHELLS: THE LIGHTLIKE LIMIT**

We will work on the two spacetimes $\mathcal{M}^−$ and $\mathcal{M}^+$ given in \[12\] separated by the hypersurface $\Sigma$ in the lightlike limit. Let us give the metrics corresponding to $\mathcal{M}^−$ and $\mathcal{M}^+$:

\[
ds_+^2 = e^{2\gamma}(t_+ - r)(-dt_+^2 + dr^2) + dz^2 + r^2d\phi^2
\]

\[
ds_-^2 = -dt_-^2 + dr^2 + dz^2 + r^2d\phi^2
\]

In our case, the shell moves only radially so with respect to $\mathcal{M}^\pm$, the null shell $\Xi$ is defined by the equation $r = \rho(t_\pm)$.

Since the shell moves at the speed of light, the radial component of 4-velocity satisfies,

\[
\frac{d\rho}{d\eta} = -1
\]

where $\eta$ is an affine parameter for the null geodesic of our interest. The minus sign reflects the fact that the shell is collapsing. Therefore, in $\mathcal{M}^−$, the collapsing shell satisfies the equation,

\[
t_+ - \rho = \rho_+ = \text{constant}
\]

On the other hand, in $\mathcal{M}^+$ the equation for the null geodesic that describes the collapsing shell $\Xi$ can be calculated by solving the Euler-Lagrange Equations. Defining,

\[
2K \equiv e^{2\gamma}t_+^{\rho_+^2} - e^{2\gamma}\rho^2 = 0
\]

where $q^o$ denotes $\frac{dq}{d\eta}$. The Euler-Lagrange equations are given by,

\[
\frac{\partial K}{\partial q} - \frac{d}{d\eta} \frac{\partial K}{\partial q^o} = 0
\]

where $q$ represents the coordinates $t_+, r, z, \phi$. Solving this equation for $q = t_+$,

\[
\dot{\gamma}(t_+^{\rho_+^2} - \rho^{\rho_+^2}) - \frac{d}{d\eta}(e^{2\gamma}t_+^{\rho_+^2}) = 0
\]

where $\dot{\gamma}$ represents $\frac{d\gamma}{dt_+}$. Then we are left only with the term $e^{2\gamma}t_+^{\rho_+^2}$ which is equal to a constant $C$ because its derivative with respect to $\eta$ is zero. Then we have,

\[
t_+^{\rho_+^2} = Ce^{-2\gamma}
\]

Using the above equality in \[3\] we get,

\[
\rho^o = \pm Ce^{-2\gamma} \rightarrow \rho^o = -Ce^{-2\gamma} \quad \text{(since we are considering collapse)}
\]

Then noting the following relation,

\[
\rho_+^{\rho_+^2} = \frac{d\rho}{dt_+} = -1
\]

where in the last step we used \[3\] and \[4\]. Solving this trivial equation gives the equation of $\Xi$ in $\mathcal{M}^+$ as,

\[
t_+ + \rho = \rho_+ = \text{constant}
\]

similar to its counterpart in $\mathcal{M}^−$. If we assume that as $r \to \infty$, $\mathcal{M}^+$ approaches to the flat Minkowski metric (i.e as $r \to \infty$, $\gamma \to 0$ so $e^{2\gamma} \to 1$), the constant $C$ should be chosen as 1. This case we refer to as the cylindrically flat exterior spacetime.

Now let us use the Null-Shell Formalism for the current case:
A. Transverse Curvature in $\mathcal{M}^-$

Since $\frac{d\rho}{d\eta} = -1$ let us choose the parameter $\lambda$ that describes the shell’s null character as $\lambda = \eta$, then intrinsic metric on the hypersurface $\Xi$ is given by,

$$ds^2_\Xi = dz^2 + \lambda^2 d\phi^2 \tag{8}$$

so $\Xi$ is described by the equation,

$$t_- = \lambda + \vartheta_-$$
$$r = -\lambda$$
$$z = z$$
$$\phi = \phi$$

where in the above equations, right hand sides correspond to intrinsic coordinates on $\Xi$, while the left hand sides show their relation to coordinates in $\mathcal{M}^-$, so they together define $\Xi$. Therefore, the tangent-normal vector of $\Xi$ is given by,

$$k^\alpha = \frac{\partial x^\alpha}{\partial \lambda} = (1, -1, 0, 0) \tag{9}$$

by using the equations we derived above that describes $\Xi$. Now following [5], we have to find the transverse vector $\mathbf{N}$ that points out of $\Xi$ and different from $\mathbf{k}$ which must satisfy the following equations,

1) $N_\alpha k^\alpha = -1$
2) $N_\alpha N^\alpha = 0$
3) $N_\alpha e^\alpha_A = 0$

Then the 3rd condition indicates a form $N_\alpha = (x, y, 0, 0)$ where $x$ and $y$ are to be found by solving conditions 1 and 2 by using (9), which gives,

$$N_\alpha = (-\frac{1}{2}, 1, \frac{1}{2}, 0, 0) \tag{10}$$

Then using the definition of Transverse Curvature,

$$C_{ab} = -\mathbf{N} \cdot (\nabla_a e_b) = -N_\alpha e^\beta_A \nabla_\beta e^\alpha_B = -N_\alpha \left( \frac{\partial^2 x^\alpha}{\partial y^\beta \partial y^\rho} + \Gamma^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial y^\gamma} \frac{\partial x^\gamma}{\partial y^\rho} \right) \tag{11}$$

Therefore, the only non-vanishing component the transverse curvature becomes,

$$C^-_{\phi\phi} = -N_\gamma \Gamma^\gamma_{\phi\phi} = \frac{r}{2} = \frac{r^2}{2r} = \frac{\sigma_{\phi\phi}}{2r}$$

Therefore,

$$C^-_{\phi\phi} |_{\Xi} = \frac{\sigma_{\phi\phi}}{2\rho} \tag{12}$$

B. Transverse Curvature in $\mathcal{M}^+$

The treatment in $\mathcal{M}^+$ is totally similar. We again set $\lambda = \eta$. The intrinsic metric on $\Xi$ is given in $\mathcal{M}^+$ as,

$$ds^2_\Xi = dz^2 + \lambda^2 d\phi^2$$
since $e^{2\gamma}(-dt^2 + dr^2) = 0$, same with $[5]$ as it should be since the continuity of the metric must be satisfied. Therefore $\Xi$ is described by the set of equations,

\[
\begin{align*}
  t_+ &= -\rho(\lambda) + \vartheta_+ \\
  r &= \rho(\lambda) \\
  z &= z \\
  \phi &= \phi
\end{align*}
\]

Then the tangent-normal vector $k^\alpha$ is given as,

\[
k^\alpha = \frac{\partial x^\alpha}{\partial \lambda} = (-\frac{d\rho}{d\lambda}, \frac{d\rho}{d\lambda}, 0, 0)
\]

Since we calculated $\frac{d\rho}{d\lambda}$ in $[6]$ as $\rho^o = -Ce^{-2\lambda}$, $k^\alpha$ becomes,

\[
k^\alpha = Ce^{-2\gamma}(1, -1, 0, 0)
\]  

(13)

Again we have to find the transverse vector $N$ by the before given properties,

\[
N_\alpha^+ N_\alpha^o = 0 \quad N_\alpha^+ k_\lambda^o = -1 \quad N_\alpha^+ e_\lambda^o = 0
\]

which can also in view of 3rd equation can be found as putting $N_\alpha^o = (x, y, 0, 0)$ into the first two equations which yields,

\[
N_\alpha^+ = \frac{e^{2\gamma}}{2C}(-1, 1, 0, 0)
\]  

(14)

Then the non-vanishing components of the transverse curvature are found also by $[10]$. 

\[
C_{\lambda\lambda}^+ = -N_t^+ \left( \frac{\partial^2 t^+}{\partial \lambda^2} + \Gamma^t_{tt}(\frac{\partial t^+}{\partial \lambda})^2 + \Gamma^t_{rr}(\frac{\partial r^+}{\partial \lambda})^2 + 2\Gamma^t_{rt}(\frac{\partial r^+}{\partial \lambda})(\frac{\partial t^+}{\partial \lambda}) \right)
\]

\[
+ -N_t^+ \left( \frac{\partial^2 r^+}{\partial \lambda^2} + \Gamma^r_{tt}(\frac{\partial t^+}{\partial \lambda})^2 + \Gamma^r_{rr}(\frac{\partial r^+}{\partial \lambda})^2 + 2\Gamma^r_{rt}(\frac{\partial r^+}{\partial \lambda})(\frac{\partial t^+}{\partial \lambda}) \right)
\]

(15)

Now using the fact that $\gamma \equiv \gamma(t_+ - r)$, we have $\dot{\gamma} = -\gamma'$ where prime denotes differentiation with respect to $r$. So the Christoffel symbols become for $[2]$, 

\[
\Gamma^r_{rr} = -\Gamma^t_{tt} = \Gamma^t_{tr} = -\Gamma^r_{tt} = \Gamma^r_{tr} = \gamma'
\]  

(16)

Now since we have $\frac{\partial t^+}{\partial \lambda} = -\frac{\partial r}{\partial \lambda} = Ce^{-2\lambda}$, we get,

\[
\frac{\partial^2 t^+}{\partial \lambda^2} = -2\frac{d\gamma}{d\lambda}Ce^{-2\gamma} = -2C(\frac{d\gamma}{d\lambda} + \frac{dr}{dl}(\frac{dt^+}{d\lambda}))e^{-2\gamma}
\]

\[
= 4C^2\gamma e^{-4\gamma} = -\frac{\partial^2 r}{\partial \lambda^2}
\]  

(17)

Thus putting $[17]$ and $[16]$ into $[15]$ one gets simply,

\[
C_{\lambda\lambda}^+ = 0
\]  

(18)

The only non-vanishing component of $C^+_{ab}$ then becomes,

\[
C^+_{\phi\phi} = -N_r^+ \Gamma^r_{\phi\phi} = \frac{r^2}{2rC}
\]

Thus,

\[
C^+_{\phi\phi} |_{z} = \frac{\sigma_{\phi\phi}}{2\rho C}
\]  

(19)
Then we have,

\[ \mu = -\frac{1}{8\pi} \sigma^{AB}[C_{AB}] \]
\[ j^A = -\frac{1}{8\pi} \sigma^{AB}[C_{\lambda B}] \]
\[ p = -\frac{1}{8\pi}[C_{\lambda\lambda}] \]

(20)

with the surface stress-energy tensor given as,

\[ S^{\alpha\beta} = \mu k^{\alpha}k^{\beta} + j^A (k^{\alpha}e^{\beta}_A + k^{\beta}e^{\alpha}_A) + p\sigma^{AB}e^{\alpha}_Ae^{\beta}_B \]

(21)

Since

\[ [C_{\lambda\lambda}] = 0, \quad [C_{\lambda B}] = 0, \quad [C_{\phi\phi}] = \frac{\sigma_{\phi\phi}}{2}(1 - \frac{C}{C}) \]

We have,

\[ \mu = \frac{C - 1}{16\pi C\rho} \]

(22)

Therefore,

\[ S^{\alpha\beta} = \frac{C - 1}{16\pi C\rho} k^{\alpha}k^{\beta} \]

From the above equations, we see that \( C^{\pm}_{\lambda\lambda} = 0 \) which proves that the parameter chosen is affine both in \( \mathcal{M}^+ \) and in \( \mathcal{M}^- \) by \( \mathbb{F} \). So we get a null shell with nonzero energy density, which collapses with the speed of light to zero radius. On the other hand, if cylindrical flatness (i.e. as \( r \to \infty, \gamma \to 0 \)) is imposed as a special case (\( C = 1 \)), then we get \( \mu = j^A = p = 0 \) so the hypersurface \( \Xi \) becomes a boundary layer rather than a thin shell.

### III. COLLAPSE OF CYLINDRICAL SHELL IN TIMELIKE CASE

In this section, we will work again on the spacetime described by the metrics (1) and (2) but this time the hypersurface connecting \( \mathcal{M}^- \) and \( \mathcal{M}^+ \) will be timelike (i.e its normal vector \( n \) is spacelike), so will be denoted by \( \Sigma \). Again \( \Sigma \) is defined by the equation \( \Sigma : r = \rho_{\pm}(t_{pm}) \), which gives the induced metric on \( \Sigma \) as,

\[ dS^2 = -d\tau^2 + dz^2 + r^2d\phi^2 \]

(23)

where

\[ d\tau^2 = e^{2\gamma}(1 - \dot{\rho}_+^2)dt_+^2 = (1 - \dot{\rho}_-^2)dt_-^2 \]

again dot represents derivative with respect to \( t_+ \) or \( t_- \). Defining the quantity \( \Delta^{-1} = \frac{dt_+}{dt_-} \) and noting that \( \dot{\rho}_- = \Delta^{-1}\dot{\rho}_+ \), we find from the above equation,

\[ e^{2\gamma}(1 - \dot{\rho}_+^2)\Delta^{-2} = 1 - \Delta^{-2}\dot{\rho}_+^2 \]

\[ \Delta^{-1} = [(1 - e^{2\gamma})\dot{\rho}_+^2 + e^{2\gamma}]^{-1/2} \]

(24)

by imposing the continuity of the metric at \( \Sigma \). In the future, we will need to write \( \dot{\rho}_- \) in terms of the quantities of \( \mathcal{M}^+ \). For this purpose, we know derive some useful equations as follows (remember that \( \gamma' = -\dot{\gamma} \)):

From equation (24),

\[ \frac{d}{dt_+}\Delta = \frac{1}{2}\Delta^{-1}\{2(1 - e^{2\gamma})\dot{\rho}_+\ddot{\rho}_+ + 2\gamma' e^{2\gamma}\dot{\rho}_+^2 - 2\gamma'e^{2\gamma}\} \]

\[ = \frac{1}{\Delta}\dot{\rho}_+\ddot{\rho}_+ - \frac{e^{2\gamma}}{\Delta}[\dot{\rho}_+^2 - \gamma'(1 - \dot{\rho}_+^2)] \]

(25)
Therefore,
\[ \ddot{\rho}_- = \frac{d}{dt_-}(\rho_-) \]

Using (25) above we get
\[ \frac{d}{dt_-}(\rho_-) = \Delta^{-1} \frac{d}{dt_+}(\Delta^{-1} \rho_+) \]
\[ = \frac{\dot{\rho}_+}{\Delta^2} - \frac{\dot{\rho}_+^2}{\Delta^4} \left( \rho_+^2 (1 - e^{2\gamma}) + e^{2\gamma} \right) - \frac{\dot{\rho}_+}{\Delta^2} \left\{ -e^{2\gamma} + \gamma'(1 - \rho_+^2)\dot{\rho}_+ e^{2\gamma} \right\} \]

Thus,
\[ \ddot{\rho}_- = \frac{e^{2\gamma}}{\Delta^4} [\dot{\rho}_+ + \gamma'(1 - \rho_+^2)\dot{\rho}_+] \] (26)

Now let us look at the behavior of Σ by first calculating its 4-velocity \( u^\alpha \) and then the normal vectors \( n_\pm \). Clearly in \( M^+ \) \( u^\alpha = (X_+, \dot{\rho}^+ , 0, 0) \) where \( X_+ = \frac{dX_+}{dt_+} \). Then from \( u^\alpha u_\alpha = -1 \) we get,
\[ e^{2\gamma} X^2 (1 - \rho_+^2) = 1 \rightarrow X = \frac{e^{-\gamma}}{\sqrt{1 - \rho_+^2}} \] (27)

where we chose the plus sign for \( X \). Then the normal vector \( n_+ \) pointing from \( M^- \) to \( M^+ \) can be found by letting \( n_+^\alpha = (k, l, 0, 0) \) and inserting this to the equations which \( n_+ \) must satisfy which are,
\[ n_+^\alpha n_+^\alpha = 1, \quad n_+^\alpha u_\alpha = 0 \]
which then gives,
\[ n_+^\alpha = e^{2\gamma} X_+ (-\dot{\rho}^+, 1, 0, 0) \] (28)

Since we are looking for surface stress-energy on Σ, we now will calculate the Second Fundamental Forms in \( M^- \) and \( M^+ \). Recall that,
\[ K_{ab} = -n_\alpha \left( \frac{\partial^2 x^\alpha}{\partial \xi^a \partial \xi^b} + \Gamma^\beta_\alpha_\gamma \frac{\partial x^\beta}{\partial \xi^a} \frac{\partial x^\gamma}{\partial \xi^b} \right) \]

where \( \xi^a = \{ \tau, z, \phi \} \) are the intrinsic coordinates on Σ. The non-vanishing components of the Christoffel symbols in \( M^- \) is given by,
\[ \Gamma^\tau_\phi_\phi = -r, \quad \Gamma^\phi_\tau_\tau = \frac{1}{r} \] (29)

and the non-vanishing components of the Christoffel symbols in \( M^+ \) is given by,
\[ \Gamma^r_\tau_\tau = -\Gamma^t_\tau_\tau = \Gamma^t_\tau_\tau = -\Gamma^r_\tau_\tau = \Gamma^t_\tau_\tau = \gamma', \quad \Gamma^r_\phi_\phi = -\frac{r}{e^{2\gamma}}, \quad \Gamma^\phi_\phi_\tau = \frac{1}{r} \] (30)

Thus, the non-vanishing components of \( K_{ab}^+ \) is given by,
\[ K_{\phi\phi}^+ = X_+ r, \rightarrow K_{\phi\phi}^+ \mid _\Sigma = \frac{X_+}{\rho_+}, \quad K_{zz}^+ = 0 \] (31)

Note that we convert our tensorial quantities to their counterparts in local Minkowski frame by \( K_{ab} = K_{ab} e^a_a e^b_b \). Now
calculation of $K_\tau^+$ is a little tedious, but straightforward. Using the definition,

$$K_\tau^+ = -n_\tau^+ \left\{ \frac{\partial^2 t_+}{\partial \tau^2} + \Gamma^t_{tt}(\frac{\partial t_+}{\partial \tau})^2 + \Gamma^t_t(\frac{\partial \tau}{\partial \tau})^2 + 2\Gamma^t_r(\frac{\partial \tau}{\partial \tau})(\frac{\partial t_+}{\partial \tau}) \right\}$$

$$- n_\tau^+ \left\{ \frac{\partial^2 r_+}{\partial \tau^2} + \Gamma^r_t(\frac{\partial t_+}{\partial \tau})^2 + \Gamma^r_r(\frac{\partial \tau}{\partial \tau})^2 + 2\Gamma^r_r(\frac{\partial \tau}{\partial \tau})(\frac{\partial r_+}{\partial \tau}) \right\}$$

$$= e^{2\gamma} X_+ \left\{ \frac{dX_+}{d\tau} - \gamma' X_+^2 + 2\gamma' X_+^2 \rho_+^3 - \gamma X_+^2 \rho_+^2 \right\}$$

$$- e^{2\gamma} X_+ \left\{ \frac{d(\rho_+ X_+)}{d\tau} + \gamma' X_+^2 - 2\gamma' X_+^2 \rho_+^3 + \gamma X_+^2 \rho_+^2 \right\}$$

$$= e^{2\gamma} X_+ \left\{ \rho_+ \frac{dX_+}{d\tau} - \frac{d(\rho_+ X_+)}{d\tau} + \gamma' X_+^2 \rho_+^2 (1 - \rho_+^2) \right\}$$

where in the last line we used $u_\alpha u^\alpha = e^{2\gamma} X_+^2 (\rho_+^2 - 1) = -1$. Thus finally,

$$K_\tau^+ = -e^{2\gamma} X_+^3 \rho_+ - \gamma' X_+ (1 - \rho_+^2)$$

(32)

We have calculated the components of the second fundamental form in $\mathcal{M}^+$, let us now calculate its components in $\mathcal{M}^-$. To do this we have to first find $n_-$. Clearly the 4-velocity of the shell $u^\alpha$ is the same in $\mathcal{M}^+$ and $\mathcal{M}^-$, but the normal vectors clearly change sign, since second fundamental form is a measure of how the normal vector pointing towards the desired spacetime changes on the hypersurface. Therefore with the same considerations that lead us to $\rho_\phi^\tau$ with the sign reversed, we get,

$$n^\alpha_\sim = X_- (\rho_-^3, -1, 0, 0)$$

(33)

(Note that $X_-$ has the same sign with $X_+$ since $X_- = \Delta X_+$ and $\Delta > 0$) Thus the from the Christoffel symbols we calculated for $\mathcal{M}^-$ above, the non-vanishing components of the second fundamental form becomes,

$$K^\sim_\phi^\phi = -X_- r \rightarrow K^\sim_\phi^\phi|_\Sigma = \frac{\Delta X_+}{\rho_+}$$

$$K^\sim_\tau^\tau = 0$$

(34)

The important term $K^\sim_\tau^\tau$ becomes,

$$K^\sim_\tau^\tau = -n_\tau^\sim \frac{\partial^2 \tau_\sim}{\partial \tau^2} - n_\tau^\sim \frac{\partial^2 r_\sim}{\partial \tau^2}$$

$$= -X_- (\rho_- \frac{dX_-}{d\tau} - \frac{d(\rho_- - X_-)}{d\tau})$$

$$- X_- \rho_-$$

Therefore,

$$K^\sim_\tau^\tau = X_- \rho_-$$

Now using $X_- = \Delta X_+$ and (26), the above equation becomes,

$$K^\sim_\tau^\tau = \Delta^3 X_+ ^3 \frac{e^{2\gamma}}{\Delta} (\rho_+ + \gamma' \rho_+ (1 - \rho_+^2))$$

$$= \Delta^3 X_+ ^3 \frac{e^{2\gamma}}{\Delta} X_+ \rho_+ + \frac{\gamma'}{\Delta} X_+ \rho_+ e^{2\gamma} X_+^2 (1 - \rho_+^2)$$

$$= \frac{X_+^3 }{\Delta} e^{2\gamma} \rho_+ + \frac{\gamma'}{\Delta} X_+ \rho_+$$

(35)

where in the 2nd line we used again $u_\alpha u^\alpha = e^{2\gamma} X_+^2 (\rho_+^2 - 1) = -1$. Thus in summary,

1) $K_\tau^+ = -e^{2\gamma} X_+^3 \rho_+ - \gamma' X_+ (1 - \rho_+^2), \quad K^\sim_\tau^\tau = \frac{X_+^3 }{\Delta} e^{2\gamma} \rho_+ + \frac{\gamma'}{\Delta} X_+ \rho_+$

2) $K^\sim_\phi^\phi|_\Sigma = \frac{X_+}{\rho_+}, \quad K^\sim_\phi^\phi|_\Sigma = -\frac{\Delta X_+}{\rho_+}$
3) \( K^\pm_{\pm\pm} = 0, \quad K^\mp_{\pm\mp} = 0 \)

since we have now finished the calculation of second fundamental forms, now we can use the famous formula connecting \([K_{ab}]\) to the surface stress-energy \(S_{ab}\) where \([\ ]\) represents for an quantity \(\Psi\), we have \([\Psi] = \Psi^+ - \Psi^-\).

\[
S_{ab} = -\frac{1}{8\pi}\{[K_{ab}] - g_{ab}[K]\} \tag{36}
\]

Therefore, from the second fundamental forms we obtained above, we see that the surface stress-energy tensor is diagonal with its elements \(S_{\phi\phi} = diag[S_{\tau\tau}, S_{\phi\phi}, S_{\bar{z}\bar{z}}]\). Calculation of these elements are trivial. The traces of the second fundamental forms and the discontinuity is given by,

\[
K^+ = -K^+_{\tau\tau} + K^+_{\phi\phi}
\]

\[
K^- = -K^-_{\tau\tau} + K^-_{\phi\phi}
\]

\[
[K] = -[K_{\tau\tau}] + [K_{\phi\phi}] \tag{37}
\]

Then using (37) in (36), we have the following equations,

\[
S_{\tau\tau} = -\frac{[K_{\phi\phi}]}{8\pi} \tag{38}
\]

\[
S_{\phi\phi} = -\frac{[K_{\tau\tau}]}{8\pi} \tag{39}
\]

\[
S_{\bar{z}\bar{z}} = -\frac{1}{8\pi}\{[K_{\tau\tau}] - [K_{\phi\phi}]\} \tag{40}
\]

Thus in full form by letting \(S_{\tau\tau} = \epsilon, \quad S_{\phi\phi} = p_{\phi}, \quad S_{\bar{z}\bar{z}} = p_{\bar{z}}\), we have,

1) \(\epsilon = -\frac{1}{8\pi}\frac{X_{\tau}}{\rho_+}(\Delta + 1)\)

2) \(p_{\phi} = \frac{1}{8\pi}\{\epsilon^2\gamma X^3_{\tau}\rho_+(1 + \frac{1}{\Delta}) + \gamma^\prime X_{\tau}\}\)

3) \(p_{\bar{z}} = \frac{1}{8\pi}\{\epsilon^2\gamma X^3_{\tau}\rho_+(1 + \frac{1}{\Delta}) + \gamma^\prime X_{\tau} + \frac{\Delta}{\rho_+}(\Delta + 1)\}\)

Instead of following the route of [12] and [11], in which the dynamical behavior of the above equations are investigated to see how \(\rho_+\) changes to comment on whether the shell collapses or not, we will consider conservation equations of Israel [1] which were derived for vacuum and in [14] for arbitrary spacetimes. The conservation equations are,

1) \(\vec{K}^b_{ab} - \vec{K}_{\alpha\alpha} = 8\pi(T_{\alpha\beta}\xi^\alpha n^\beta)\)

2) \(S^b_{ab} = -[T_{\alpha\beta}\xi^\alpha n^\beta]\)

3) \(S^{ab}\vec{K}_{ab} = [T_{\alpha\beta}n^\alpha n^\beta]\)

4) \((3)\vec{R} + \vec{K}_{ab}\vec{K}^{ab} - \vec{K}^2 = -\frac{1}{4}(8\pi)^2(S_{ab}S^{ab} - \frac{S^2}{2}) - 8\pi(T_{\alpha\beta}n^\alpha n^\beta)\)

Let us consider the 3rd conservation equation which will prove an incredible result. Since the right hand side of this equation contains the discontinuity of the stress-energy tensor on \(\mathcal{M}^+\) and \(\mathcal{M}^-\) we have to calculate it from the Einstein Field Equations \(G^\pm_{\alpha\beta} = 8\pi T^\pm_{\alpha\beta}\). The interior spacetime \(\mathcal{M}^-\) is flat therefore making \(T^-_{\alpha\beta} = 0\). On the other hand we have,

\[
G^+_{\alpha\beta} = \frac{\gamma^\prime}{r}\zeta_\alpha \zeta_\beta \tag{41}
\]

where \(\zeta_\alpha = (1, -1, 0, 0)\) is a null vector. This clearly represents a spacetime with radiation of null-particles. Therefore, the stress-energy tensors become,

\[
T^+_{\alpha\beta} = \frac{\gamma^\prime}{8\pi r}\zeta_\alpha \zeta_\beta \tag{42}
\]

\[
T^-_{\alpha\beta} = 0 \tag{43}
\]
Thus, given (28) and (12) we get,

\[
[T_{\alpha\beta} n^\alpha n^\beta] = \frac{\gamma'}{8\pi \rho_+} X_+^2 (\rho_+ - 1)^2
\]

(44)

The left hand side of the 3\textsuperscript{rd} conservation equations can be calculated by just inserting the equations (32)-(35) just derived above yielding,

\[
\tilde{K}_{ab} S^{\alpha\beta} = \tilde{K}_{\tau\tau} S^{\tau\tau} + \tilde{K}_{\phi\phi} S^{\phi\phi}
\]

\[
= \frac{1}{16\pi}(K^+_{\tau\tau} + K^-_{\tau\tau})(K^+_{\phi\phi} - K^-_{\phi\phi}) + \frac{1}{16\pi}(K^+_{\phi\phi} + K^-_{\phi\phi})(K^+_{\tau\tau} - K^-_{\tau\tau})
\]

Thus we have,

\[
-K^+_{\tau\tau} K^+_{\phi\phi} + K^-_{\tau\tau} K^-_{\phi\phi} = \frac{\gamma'X_+^2 (\rho_+ - 1)^2}{\rho_+}
\]

(45)

Thus, inserting (32)-(35) in the above equation we get,

\[
\{ e^{2\gamma} X_+^3 \dot{\rho}_+ + \gamma' X_+(1 - \rho_+) \}(X_+ / \rho_+) - \frac{\Delta X_+}{\Delta} \left\{ e^{2\gamma} X_+^3 \dot{\rho}_+ + \gamma' X_+ \dot{\rho}_+ \right\} = \frac{\gamma'X_+^2 (\rho_+ - 1)^2}{\rho_+}
\]

Then we get the trivial solution for \( \dot{\rho}_+ \) (note that \( \gamma' \neq 0 \) is assumed, otherwise both metrics would be flat),

\[
\dot{\rho}_+ = 0
\]

(46)

Thus the shell never collapses and just stays at constant radius. This means that there can not be a shell motion, which is contrary to the results of [10]-[11].

The results we obtained above can also be obtained by the use of Hamiltonian formalism as done by Olea and Crisóstomo [6] for the spherically symmetric case. The variation of the gravitational action yields the Hamiltonian and the momentum constraints which can be calculated for both \( \mathcal{M}^+ \) and \( \mathcal{M}^- \). Then by integrating these constraints across \( r = \rho_+ \) one obtains the same equations that have been obtained above by the conservation equations so both approaches yield the same results. (Actually the conservation equations of Israel are obtained by adding and subtracting the Hamiltonian and momentum constraints corresponding to \( \mathcal{M}^+ \) and \( \mathcal{M}^- \). The distinction is that in Israel formalism, these constraints are obtained by the Gauss-Codazzi Equations rather than the variational principle).

Now let us look at the 2\textsuperscript{nd} conservation equation. First of all, the right hand side is given by using \( u^\alpha = (X_+, \rho_+, 0, 0) \) and equation (28)

\[
T_{\alpha\beta} e^\alpha e^\beta = \frac{\gamma'}{8\pi \rho_+} (\zeta^\alpha \zeta^\beta) (\zeta^e u^\alpha) = -\frac{\gamma'X_+^2}{8\pi \rho_+} \text{ for } a = \tau
\]

\[
= 0 \quad \text{for } a = \phi, z
\]

since \( e^\alpha \zeta_\alpha = e_\phi \zeta_\phi = 0 \) and \( \dot{\rho}_+ = 0 \). Then the nontrivial part of 2\textsuperscript{nd} conservation equation becomes,

\[
S^b_{\tau \mid b} = S^\tau_{\tau} = \frac{\gamma'X_+^2}{8\pi \rho_+}
\]

Using (38) we have

\[
S^\tau_{\tau} = \frac{[K_{\phi\phi}]}{8\pi} = \frac{X_+ (1 + \Delta)}{8\pi \rho_+}
\]

Note that from (24) when we have for \( \rho_+ = 0 \) we get \( \Delta = e^\gamma \) and from (27), we have \( X_+ = e^{-\gamma} \). Therefore, noting that the intrinsic covariant derivative becomes just the ordinary derivative for \( \tau \)-component from (28), we get,

\[
S^\tau_{\tau} = \left( \frac{e^{-\gamma} + 1}{8\pi \rho_+} \right)_{\tau} = \frac{\gamma' e^{-2\gamma}}{8\pi \rho_+}
\]

Where we used the fact that \( \frac{dX_+}{d\tau} = X_+ \dot{\gamma} = -e^{-\gamma} \gamma' \) since we had \( \gamma' = -\dot{\gamma} \). Thus finally we get

\[
S^\tau_{\tau} = \frac{\gamma' e^{-2\gamma}}{8\pi \rho_+} = -[T_{\alpha\beta} e^\alpha n^\beta]
\]

(47)
by which we get trivially $1 = 1$ which justifies our calculations.

Now we have not used the $1^{st}$ and $4^{th}$ conservation equations, since the $1^{st}$ one when applied results in equation 47 and the $4^{th}$ one results in equation 46.

Clearly, we can not go to the null limit as was done in 12 since we have found that the shell is stationary. Therefore, we can state that the null case must be directly considered by the null-shell formalism separately as we did, rather than looking at the null limit of the timelike case.

Here we would like to remark that we are at a variance of the results in 12-11. For example in 12, it was found that both in the timelike case and in the lightlike limit, the shell first approaches to a minimum radius and then expands infinitely. However, we found out that in the timelike case the shell is stationary and in the lightlike limit we get a collapsing shell. In addition, if we impose cylindrical flatness, for the null case we get a boundary layer so the shell is lost.

The difference of our work and previous ones is due to the fact that they did not considered the conservation equations at all and some difference in the calculated values of the second fundamental forms. We can compare our results with the results of 12 now since it is the most recent work. The extrinsic curvatures for the spacetimes 11 and 2 joined at $\Sigma$ was found as,

$$K_{\tau\tau}^+ = -X_+^3 e^{2\gamma} \dot{\rho}_+ + \gamma' X_+^3 \rho_+^2 (1 - \dot{\rho}_+)$$

$$K_{\tau\tau}^- = -X_-^3 \rho_- = -\frac{X_-^3}{\Delta} e^{2\gamma} \dot{\rho}_+ + \frac{\gamma'}{\Delta} X_+^3 e^{2\gamma} (1 - \rho_+^2) \dot{\rho}_+$$

$$K_{\tau\tau}^\pm = 0$$

$$K_{\phi\phi}^+ = \frac{X_+}{\rho_+}$$

$$K_{\phi\phi}^- = \frac{\Delta X_+}{\rho_+}$$

Using the fact that $X_+^2 e^{2\gamma} (1 - \dot{\rho}_+^2) = 1$, equation 49 reduces to $K_{\tau\tau}^- = -\frac{X_-^3}{\Delta} e^{2\gamma} \dot{\rho}_+ - \frac{\gamma'}{\Delta} X_+ \rho_+$. First of all, one can see the most apparent difference with our results with Seriu’s 12 is that the sign of the normal vector $n \perp \Sigma$ is taken to be (+) in both $M^+$ and in $M^-$ (note that $X_+, X_- > 0$) which the correct selections should be as ours (i.e negative sign in $M^-$ and positive sign in $M^+$). Secondly, the equation for $K_{\tau\tau}^+$ is different as can be compared with our previous calculations. Let us show that these results lead to a contradiction in the conservation equations. Putting these values in 49 we get,

$$[e^{2\gamma} X_+ \dot{\rho}_+ - \gamma' X_+ e^{2\gamma} \rho_+^2 (1 - \dot{\rho}_+)](\frac{X_+}{\rho_+}) = \frac{\Delta X_+}{\rho_+} [e^{2\gamma} X_+ \dot{\rho}_+ + \gamma' X_+ \rho_+] = \frac{\gamma' X_+^2 (\rho_+^2 - 1)^2}{\rho_+}$$

which reduces to

$$-(X_+^2 e^{2\gamma} \dot{\rho}_+ (1 - \rho_+^2) + \rho_+) = (\dot{\rho}_+ - 1)^2 \rho_+$$

(52)

Now using $X_+^2 e^{2\gamma} (1 - \dot{\rho}_+^2) = 1$ once more in the above equation we get,

$$\frac{\dot{\rho}_+^2}{1 + \rho_+} + \dot{\rho}_+ = -\frac{(\dot{\rho}_+ - 1)^2}{\rho_+}$$

(53)

In principle, the above expression can be rewritten by solving the cubic equation for $\dot{\rho}_+$, then this equation can be integrated. Doing so will prove to be very difficult, since the integral to be evaluated is very complicated. Rather than doing this, we can analyze the equation 53 without solving it. Clearly, we must have $|\rho_+| < 1$ since the shell is assumed to be timelike. For the expansion of the shell, $0 < \dot{\rho}_+ < 1$, there is no solution, since the right hand side is always negative, but the left hand side is always positive which is contradictory. For the collapsing shell $-1 < \dot{\rho}_+ < 0$, the contradiction can only be removed if $-1 < \dot{\rho}_+ < -0.5$ is imposed. However, this result is a contradiction since the only possible solution Seriu 12 obtains is the expanding shell, where $\dot{\rho}_+ > 0$.

IV. CONCLUSION

In this work we considered an axial-symmetric hypersurface separating an interior flat and an exterior curved radiation filled spacetimes both in the null and the timelike cases. We found that in the null case the shell collapses
to zero radius at the speed of light but when cylindrical flatness is imposed the shell is lost (i.e. surface stress-energy becomes identically zero). and a boundary layer is obtained. For the timelike case, we used the full conservation equations to determine the dynamical behavior of a thin shell for the first time, and showed that the axial-symmetric shell in consideration stays stationary at some constant radius. The Hamiltonian formalism presented in [6] for spherically symmetric shells, also gives the same results when used for the configuration we analyze.

With these results, we are at a variance with the predictions of [11] and [12] since we considered the conservation equations to determine the dynamical behavior rather than the equations on stress-energy. Seriu [12] found that in the timelike case and its null limit, the shell first contracts to a minimum nonzero radius and then expands infinitely. Pereira and Wang [11] also obtained similar, but they also found collapsing shell configurations in some special cases. Both of these results are different from ours for the reasons we discussed above. However, one should consider the conservation equations when analyzing the dynamical behavior of thin shells since they are powerful constraints on the equations that are obtained from the surface stress-energy.

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