Vertical stress under vertical pressure by extended Mindlin's equation

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ABSTRACT

Analytical elasticity solutions provide an efficient means of performing a first approximate analysis in foundation engineering. One of the well-known basic solutions is Mindlin’s solution to the stress and displacement induced by a point load at an embedment depth in a half-space. This solution is more superior but less widely used than Boussinesq’s solution for a point load at only the boundary of half-space. To promote the applications of Mindlin’s solution, its vertical stress equation is integrated to obtain an explicit formula for calculating vertical stresses at any arbitrary point. The stresses are induced by uniformly and triangularly distributed vertical pressure, which is exerted over a rectangular area in the interior of a homogeneous, isotropic, elastic half-space. It shows that the vertical stress decreases as the embedment depth of loaded area increases.

Keywords: stress, elasticity, footings/foundations, embedment depth

1 INTRODUCTION

Analytical elasticity solutions play an important role in preliminary predictions and in calibrating any sophisticated numerical schemes that are ultimately used for solving more complex practical problems in geotechnical engineering (Geddes 1966; Ng and Lei 2003; Selvadurai 2007; Lei et al. 2014). Many solutions have been proposed for stress in the fields of soil and rock mechanics (Poulos and Davis 1974). Among them, the solutions for loading over a rectangular area in a homogeneous, isotropic, elastic half-space are the most frequently applied ones in foundation engineering analysis (Skopek 1961; Giroud 1970; Griffiths 1984; Sharma and Venkatappa Rao 1984; Vitone and Valsangkar 1986; Li 1991; Rossa and Auvinet 1992; De Jaeger 1994; Align 2000, 2001; Sun et al. 2013). These solutions are commonly expressed in the form of graphs and tables. They are, however, derived only for the computing points at or below the corners of the loaded rectangle. The method of superposition has to be used to calculate other points.

This paper focuses on the derivation of vertical stress solution for linearly distributed vertical pressure over a rectangular area embedded in the interior of the half-space. A solution for calculating the stresses at any arbitrary point is obtained by double integration of Mindlin (1936)’s vertical stress equation. The solution is expressed in an explicit form for practical engineering applications, e.g. calculations of vertical stress under eccentrically loaded buried rectangular footings.

2 COORDINATE SYSTEM AND NOTATIONS

The rectangular coordinate system for a half-space is defined in Fig. 1. The origin of the coordinate system is located on the boundary surface of the half-space. A vertical point load V is applied at an arbitrary point M(u, v, w) away from the origin. The stresses at any point N(x, y, z) due to the point load can be obtained by using Mindlin (1936)’s stress equations. Nevertheless, it should be noted that the coordinate system adopted here is slightly different from that of Mindlin (1936), where the point load V is applied at a point (u = 0, v = 0, w = c) under the origin.

For the purpose of presenting the derived stress solutions in a compact form, the following notations are assumed the same as those used for deriving the displacements under linearly distributed loading over a rectangular area by Sun et al. (2013).

\[ X = x - u \]  
\[ Y = y - v \]
vertical stress solution as follows:

\[
\sigma_z = \frac{\lambda_1 \left( (1 - 2(1 - \nu) Y (T_{z1} + T_{z2}) + (1 - 2\nu) Z_l \ln \frac{Y + R_l}{Y + R_i} \right) - \frac{YZ_l (Z_l^2 + Y_l^2) + xZ_l (R_l^2 + Z_l^2)}{R_l (X_l^2 + Z_l^2)(Y_l^2 + Z_l^2)} - \frac{(3 - 4\nu) Z_l - w(5z - w)}{Z_l} \times \frac{Y (Z_l^2 + Y_l^2) + xZ_l (R_l^2 + Z_l^2)}{R_l (X_l^2 + Z_l^2)(Y_l^2 + Z_l^2)} - 2wz \left[ \frac{Z_l^3}{X_l^3} \right] \left[ \frac{2x^2}{X_l^3} \left( \frac{3X^2 - Z_l^2}{Z_l^2 (Y_l^2 + Z_l^2)} \right)^3 + \frac{x (3X^2 - Z_l^2)}{Z_l^2 (Y_l^2 + Z_l^2)} \right] - \frac{uX_l^2}{(X_l^2 + Z_l^2)^2 R_l^2} - \frac{2X_l^3}{X_l^2 (X_l^2 + Z_l^2)^2} \right] \]^{u_{\text{min}}}_{u_{\text{max}}} \right]^{v_{\text{max}}}_{v_{\text{min}}} \]}

where \( \lambda_1 \) is Poisson’s ratio.

\[
\lambda_1 = \frac{-p_2 - p_1}{\eta (u_2 - u_1)} \quad \lambda_2 = \frac{-p_2 - p_1}{\eta (u_2 - u_1)}
\]

where \( \eta \) is the shear modulus.

3 DERIVATION PROCEDURES

In keeping with the conventional sign convention used in soil mechanics, compressive stresses are taken as positive. Following the adopted sign convention, the coordinate system in Fig. 1 and the assumed notations, Mindlin (1936)’s original equation for obtaining the vertical stress at any point \( N(x, y, z) \) can be readily rewritten as follows by a shift of origin and sign.

\[
\sigma_z = \frac{-V}{\eta} \left[ \frac{(1 - 2\nu) Z_l}{R_l^2} + \frac{(1 - 2\nu) Z_l}{R_i^2} - \frac{3Z_l^3}{R_l^2} \right]
\]

The vertical stress due to vertical pressure over a rectangle \((u = u_1 \text{ to } u_2, v = v_1 \text{ to } v_2, w = \text{constant}; \text{Fig. 2})\) can be derived by double integration of Eq. (9) with respect to \(u\) and \(v\). In doing so, the point load on a rectangular element at the point \((u, v, w)\) can be readily derived as follows:

\[
V = \frac{1}{(u_2 - u_1)} \left[ (u_2 p_1 - u_1 p_2) + (p_2 - p_1) u \right] \text{d}u \text{d}v
\]

where \( p_1 \) and \( p_2 \) are the pressures at the points \((u_1, v = v_1 \text{ to } v_2, w)\) and \((u_2, v = v_1 \text{ to } v_2, w)\), respectively. Substituting this point load into Eq. (9) and taking double integration with respect to \(v\) and \(w\) gives the double-precision arithmetic is used to calculate the stresses. It should be noted that for computing stresses at the boundary including at corners of the rectangular area, the proposed stress solution is singular. This is
because some denominator terms in Eq. (11) are singular when the values of $X$, $Y$, $Z_1$ and $R_1$ defined in Eqs. (1)-(3) and (5) become zero.

4 VERIFICATION

Derivation of Eq. (11) is complicated. Therefore, the accuracy of this equation should be verified by the available stress solutions for some special loading cases. For comparison purpose, Poisson’s ratio $\nu$ is assumed to be 0.5.

Vertical stresses at points below the corner of a surface rectangular area subjected to uniform vertical pressure have been given by Giroud (1970). Figure 3 shows the results calculated by Giroud (1970) and by this study using Eq. (11) and applying $p_1 = p_2 = p$ and $w = 0$ m. It can be seen that good agreement is obtained.

Vertical stresses at points below the corner of a surface rectangular area due to trapezoidal vertical pressure has been given by Algin (2000, 2001). The calculated results from this study (by applying $w = 0$ m) are compared with the results from Algin (2000), as shown in Fig. 5. Again, it can be seen that they are in good agreement.

Vertical stresses at points below the corner of a surface rectangular area subjected to trapezoidal vertical pressure, parametric study is conducted using different embedded depths $w = 0, 5, 10$ m and different pressure ratios of $pz/p_1 = 1, 5, 10$. The length and width of the loaded area are assumed to be $l = b = 10$ m. Poisson’s ratio $\nu$ is assumed to be 0.5.

Figure 6(a) shows the distributions of normalized vertical stresses (by $p_1$) along depth ($z = w$) below the corners at which the minimum pressure $p_1$ is applied. It can be seen that under uniform pressure ($p_2/p_1 = 1$), the maximum vertical stress appears at the embedment depth $z = w$. However, under non-uniform pressure, the maximum vertical stress appears at a certain depth below the embedment depth. This is believed to be caused by the stress dispersion effect due to nearby pressures higher than $p_1$. The maximum vertical stress increases as the pressure ratio increases. This implies that the higher the pressure ratio, the more significant the stress dispersion effect is, as is expected. It can also be observed that the maximum vertical stress decreases as the embedment depth increases. This is believed to be caused by the boundary effect due to the distance of the loaded area away from the free surface boundary of half-space. The applied load is partially shared by the stresses in the medium included between the embedment depth and the free surface. The deeper the embedment depth, the higher the load sharing of that medium is.

Figure 6(b) shows the distributions of normalized
vertical stresses (by $p_2$) along depth ($z-w$) below the corners at which the maximum pressure $p_2$ is applied. It can be seen that the maximum vertical stresses for different pressure ratios appear at the embedment depth. By comparing Fig. 6(b) with Fig. 6(a), it can be found that the influence of pressure ratio on the vertical stress under the maximum pressure is less significant than that under the minimum pressure. However, the maximum vertical stress decreases as the embedment depth increases, like the case shown in Fig. 6(a). This is also caused by the boundary effect as explained above.

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