A Note on the Filtered Decomposition Theorem

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Abstract
We generalize the logarithmic decomposition theorem of Deligne–Illusie to a filtered version. There are two applications. The easier one provides a mod \(p\) proof for a vanishing theorem in characteristic zero. The deeper one gives rise to a positive characteristic analogue of a theorem of Deligne on the mixed Hodge structure attached to complex algebraic varieties.

Keywords
Decomposition theorem · Mixed Fontaine–Laffaille complex · Spectral sequence · Vanishing theorem · Weight filtration

Mathematics Subject Classification
14F40 · 14G17 · 18G40

1 Introduction

Let \(k\) be a perfect field of characteristic \(p > 0\), and let \(X\) be a smooth \(k\)-scheme equipped with a normal crossing divisor (NCD) \(D \subset X\). Assume that \((X, D)/k\) is \(W_2(k)\)-liftable. By \(\tau_{< p}, F\), we mean the canonical truncation of complexes at degrees \(< p\) and the relative Frobenius of \(X\) over \(k\), respectively. Let \((X', D')/k\) be the base change of \((X, D)/k\) via the Frobenius automorphism \(F_k\) of \(\text{Spec}(k)\). Denote by \(D^+(X')\) the bounded below derived category of \(\mathcal{O}_{X'}\)-modules. Deligne–Illusie [3, Theorem 2.1] (see also [9, Theorem 5.1], [4, Theorem 10.16]) showed the following cohomological Hodge decomposition in positive characteristic.

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Theorem 1.1 (Deligne–Illusie) \( \tau_\tau p F_\star \Omega^*_{X/k}(\log D) \) is decomposable in \( D^+(X') \), i.e., we have

\[
\tau_\tau p F_\star \Omega^*_{X/k}(\log D) \cong \bigoplus_{i=0}^{p-1} \Omega^i_{X'/k}(\log D')[-i] \quad \text{in} \quad D^+(X').
\]  

(1.1)

There are various extensions of Theorem 1.1, such as Ogus-Vologodsky [14, Corollary 2.27], Schepler [15, Theorem 5.7]. Recently, Illusie kindly informed us that he has generalized Theorem 1.1 to a lci scheme over \( k \) by the derived de Rham complex. In this paper, we extend Theorem 1.1 to a filtered version, following the spirit of Deligne–Illusie [3]. As shown in classical Hodge theory, one can define a weight filtration \( W \) on the logarithmic de Rham complex \( \Omega^*_{X/k}(\log D) \) in positive characteristic. On the other hand, one can use the similar way to construct a weight filtration on \( \bigoplus_i \Omega^i_{X'/k}(\log D')[-i] \), again denoted by \( W \). Note that both sides of (1.1) inherit a filtration, by abuse of notation, we still write them by the same letter \( W \). Denote by \( D^+ F(X') \) the bounded below derived category of filtered \( \mathcal{O}_{X'} \)-modules. We prove a filtered version of Theorem 1.1, which can be regarded as a cohomological mixed Hodge decomposition in positive characteristic.

Theorem 1.2 \((\tau_\tau p F/\star \Omega^*_{X/k}(\log D), W) \) is decomposable in \( D^+ F(X') \). More precisely, for any \( W_2(k) \)-lifting \((\tilde{X}, \tilde{D})/W_2(k) \) of \((X, D)/k \), we have an isomorphism in \( D^+ F(X') \):

\[
\Psi_{(\tilde{X}, \tilde{D})} : (\bigoplus_{i=0}^{p-1} \Omega^i_{X'/k}(\log D')[-i], W) \rightarrow (\tau_\tau p F/\star \Omega^*_{X/k}(\log D), W).
\]

There is an easy application of Theorem 1.2, which gives rise to a mod \( p \) proof of the following theorem.

Theorem 1.3 Let \( X \) be a smooth projective variety over a field \( K \) of characteristic 0, \( D \subset X \) a NCD, \( L \) an ample line bundle on \( X \). Then, for any \( l \), we have

\[
\mathbb{H}^j(X, W_l \Omega^l_{X/k}(\log D) \otimes L) = 0 \quad \text{for} \quad i + j > \text{dim}(X).
\]

Note that if \( D \) is a simple normal crossing divisor (SNCD), then this theorem is an easy consequence of Kodaira–Akizuki–Nakano vanishing theorem (see [9, Theorem 6.10]).

Another deeper application of Theorem 1.2 concerns a de Rham analogue of the following theorem in positive characteristic.

Theorem 1.4 (Deligne [2, Theorem 8.2.2]) The cohomology of smooth complex algebraic varieties is endowed with a mixed Hodge structure (MHS) functorial with respect to algebraic maps.

We briefly review Deligne’s proof. Let \( X \) be temporarily a smooth complex algebraic variety. By Nagata’s compactification theorem and Hironaka’s desingularization
Theorem, there exists a smooth proper complex algebraic variety $\tilde{X}$ endowed with a simple normal crossing divisor $D$ such that $X = \tilde{X} - D$. It can be shown that the cohomology of $X$ is computed by the hypercohomology of the logarithmic de Rham complex $\Omega^*_X / \mathbb{C}$ (log $D$). This complex carries two filtrations, i.e., the weight filtration $W$ and the Hodge filtration Fil. A key result says that the bifiltered complex $(\Omega^*_X / \mathbb{C}(\log D), W, \text{Fil})$ is a cohomological mixed Hodge complex. It follows that

$$(K^*, W, \text{Fil}) := R\Gamma(X, (\Omega^*_X / \mathbb{C}(\log D), W, \text{Fil}))$$

is a Hodge complex. Denote by $\{E^{i,j}_r, d^{i,j}_r\}_{r \geq 0}$ the spectral sequence of $(K^*, W)$. Then,

- The filtration Fil on $K^*$ induces a recursive filtration $\text{Fil}_{rec}$ on $E^{i,j}_r$ such that for any $r \geq 1$, $(E^{i,j}_r, \text{Fil}_{rec})$ is a pure Hodge structure and $d^{i,j}_r$ is a morphism of Hodge structures;
- This spectral sequence degenerates at $E_2$;
- $\{E^{i,j}_r\}$ gives rise to the mixed Hodge structure of the cohomology of $X$.

The discussion above is functorial with respect to algebraic maps.

In positive characteristic $p$, we try to follow the line of Deligne’s proof. Let $\text{Sch}^\text{sp}_{W_2(k), \text{log}}$ be the category of smooth proper $W_2(k)$-schemes of dimension less than $p$ endowed with a normal crossing divisor. Pick $(\tilde{X}, \tilde{D}) \in \text{Sch}^\text{sp}_{W_2(k), \text{log}}$ with closed fiber $(X, D)/k$. We introduce the notion of cohomological mixed Fontaine–Laffaille complexes over $X$, which can be regarded as a positive characteristic analogue of the notion of cohomological mixed Hodge complexes. In particular, we show that $(\Omega^*_X / k(\log D), W, \text{Fil}, \Psi(\tilde{X}, \tilde{D}))$ is such a bifiltered complex. Set

$$(K^*_{\text{dR}}, W, \text{Fil}, \psi) := R\Gamma(X, (\Omega^*_X / k(\log D), W, \text{Fil}, \Psi(\tilde{X}, \tilde{D}))),$$

then we obtain a mixed Fontaine–Laffaille complex over $k$. Denote by

$$\{E^{-i,j}_{r, \text{dR}}, d^{-i,j}_{r, \text{dR}}\}_{r \geq 1}, \{E^{-i,j}_{r, \text{Hig}}, d^{-i,j}_{r, \text{Hig}}\}_{r \geq 1} \quad (1.2)$$

the spectral sequence of $(K^*_{\text{dR}}, W), (K^*_{\text{Hig}}, W) := F^*_k \text{GrFil}(K_{\text{dR}}, W)$, respectively. Then:

- The filtration Fil on $K^*_{\text{dR}}$ induces a recursive filtration $\text{Fil}_{rec}$ on $E^{i,j}_{r, \text{dR}}$ such that for any $r \geq 1$, $F^*_k \text{GrFil}_{rec}$ sends $d^{i,j}_{r, \text{dR}}$ to $d^{i,j}_{r, \text{Hig}}$;
- $\psi$ induces an isomorphism of spectral sequences

$$\psi : \{E^{-i,j}_{r, \text{Hig}}, d^{-i,j}_{r, \text{Hig}}\}_{r \geq 1} \rightarrow \{E^{-i,j}_{r, \text{dR}}, d^{-i,j}_{r, \text{dR}}\}_{r \geq 1}.$$

Unfortunately, we cannot prove the $E_2$-degeneration of $\{E^{-i,j}_{r, \text{dR}}, d^{-i,j}_{r, \text{dR}}\}_{r \geq 1}$ as in classical Hodge theory, since there has no appropriate positive characteristic analogue of
the action of complex conjugation on Hodge structures. Using Lemma 3.14, we show that the discussion above is functorial with respect to morphisms in $\text{Sch}_{W_2(k), \log}$.

Set $E_{r \geq 1}/\text{Vec}_k$ to be the additive category of spectral sequences starting from $E_1$-page over the category $\text{Vec}_k$ of $k$-vector spaces. Note that the Frobenius automorphism $F^*_k$ of $k$ induces an automorphism of $E_{r \geq 1}/\text{Vec}_k$, again denoted by $F^*_k$. The following theorem is a positive characteristic analogue of Theorem 1.4.

**Theorem 1.5** (1.2) defines functors

$$\rho_{\text{dR}}, \rho_{\text{Hig}} : (\text{Sch}_{W_2(k), \log})^{\text{op}} \rightarrow E_{r \geq 1}/\text{Vec}_k.$$  

There have a natural Hodge filtration $\text{Fil}$ on $\rho_{\text{dR}}$ such that

$$\rho_{\text{Hig}} = F^*_k \circ \text{GrFil} \rho_{\text{dR}}$$

and an isomorphism of functors

$$\psi : \rho_{\text{Hig}} \rightarrow \rho_{\text{dR}}.$$  

We call $(\rho_{\text{dR}}, \rho_{\text{Hig}}, \text{Fil}, \psi)$ a Fontaine–Laffaille quadruple over the category $\text{Sch}_{W_2(k)\log}$.

**2 Filtered Decomposition Theorem**

**2.1 The Decomposition Theorem of Deligne–Illusie**

In this subsection, we briefly recall the decomposition theorem of Deligne–Illusie. For more details, we refer the reader to [4, Sect. 8–10], [9, Sect. 5]. Let $k$ be a perfect field of positive characteristic $p$, $X$ a smooth variety over $k$, $D \subset X$ a normal crossing divisor. Assume that there exists a $W_2(k)$-lifting $(\tilde{X}, \tilde{D})$ of $(X, D)$. Consider the following famous diagram:

![Diagram](image)

where the right square is cartesian, $F_k$ and $F_X$ the absolute Frobenius of $\text{Spec}(k)$ and $X$, respectively. Set $F := F_{X/k}$. Note that we may take $X' := X$ (i.e., $\pi_{X/k} = \text{id}_X$) as abstract scheme $^1$, hence $(\tilde{X}, \tilde{D})$ induces a $W_2(k)$-lifting $(\tilde{X}', \tilde{D}')$ of $(X', D' := D)$. Choose an open affine covering $\mathcal{U} = \{U_{a}\}$ of $X$, and let $\mathcal{U}' = \{U'_{a}\}, \tilde{\mathcal{U}} = \{\tilde{U}_{a}\}, \tilde{\mathcal{U}}' = \{\tilde{U}'_{a}\}$.

---

$^1$ The structure morphism $X' \rightarrow \text{Spec}(k)$ is the composite of $X \xrightarrow{F_k^{-1}} \text{Spec}(k) \xrightarrow{\pi_{X/k}} \text{Spec}(k)$.
\{\tilde{U}_\alpha'\} be the corresponding coverings of \(X', \tilde{X}, \tilde{X}'\), respectively. By smoothness and affineness, there exists a family of log liftings \(\{\tilde{F}_{U_\alpha/k} \to \tilde{U}_\alpha'\}\), i.e., \(F_{U_\alpha/k} = \tilde{F}_{U_\alpha/k} \times \mathbb{W}_2(k)\) and \(\tilde{F}_{U_\alpha/k} \tilde{D}' = p\tilde{D}\) for any \(\alpha\).

**Definition 2.1** The family \(\{U_\alpha, \tilde{F}_{U_\alpha/k}\}\) constructed above is said to be a D-I atlas of \((\tilde{X}, \tilde{D})/\mathbb{W}_2(k)\).

**Lemma 2.2** ([3, Sect. 2]) Using a D-I atlas \(\{U_\alpha, \tilde{F}_{U_\alpha}\}\) of \((\tilde{X}, \tilde{D})/\mathbb{W}_2(k)\), one can construct two families

\[
\{\zeta_\alpha : \Omega_{X'/k}(\log D')|_{U'_\alpha} \to F_*\Omega_{X/k}(\log D)|_{U'_\alpha}\},
\]

\[
\{h_{\alpha\beta} : \Omega_{X'/k}(\log D)|_{U'_\alpha \cap U'_\beta} \to F_*\Omega_{X}|_{U'_\alpha \cap U'_\beta}\}
\]

such that:

(i) \(\zeta_\beta - \zeta_\alpha = dh_{\alpha\beta}\);

(ii) (cocycle condition) \(h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}\).

**Proof** We briefly recall the constructions of \(\zeta_\alpha\) and \(h_{\alpha\beta}\). Let \(x \in \mathcal{O}_{U'_\alpha}\) with a lifting \(\tilde{x} \in \mathcal{O}_{\tilde{U}_\alpha'}\). It is easy to see that \(\tilde{F}_{U_\alpha}(\tilde{x}) = \tilde{x}p + p\tilde{\lambda}\), where \(\tilde{\lambda} \in \mathcal{O}_{\tilde{U}_\alpha'}\) is a lifting of some \(\lambda \in \mathcal{O}_{U'_\alpha}\). We set

\[\zeta_\alpha(dx) := x^{p-1}dx + d\lambda.\]

Furthermore, we assume that \(x \in \mathcal{O}_{U'_\alpha \cap U'_\beta}\). Write \(\tilde{F}_{U_\beta}(\tilde{x}) = \tilde{x}p + p\tilde{\mu}\), where \(\tilde{\mu} \in \mathcal{O}_{\tilde{U}_\alpha' \cap \tilde{U}_\beta'}\) is a lifting of some \(\mu \in \mathcal{O}_{U'_\alpha \cap U'_\beta}\). We set

\[h_{\alpha\beta}(dx) := \mu - \lambda.\]

One can check that \(\zeta_\alpha, h_{\alpha\beta}\) are well-defined and they satisfy the conditions (i), (ii).

The trick for showing Theorem 1.1 can be summarized by the following diagram of quasi-isomorphisms:

\[
\tau_{<p} F_*\Omega_{X/k}(\log D) \xrightarrow{i} \tilde{\mathfrak{C}}(U', \tau_{<p} F_*\Omega_{X/k}(\log D)) \xrightarrow{\varphi_x} \bigoplus_{i=0}^{p-1} \Omega_{X'/k}(\log D')[-i].
\]

This diagram involves many notations; we quickly review them.

- For a complex \(K^*\) over some abelian category, we set \(\tau_{<p} K^*\) to be the subcomplex of \(K^*\) with components \(K^i\) for \(i \leq p - 2\), \(\ker(d : K^{p-1} \to K^p)\) for \(i = p - 1\) and 0 for \(i \geq 0\).
\[ \Omega_{X/k}^*(\log D) \text{ is the logarithmic de Rham complex of } X \text{ over } k \text{ with pole along } D \) (see, for instance, [9, Sect. 7.1]).

- Let \( \tilde{\mathcal{C}}(\mathcal{U}', \tau_{<p} F_* \Omega^*_{X/k}(\log D)) \) be the Čech resolution of \( \tau_{<p} F_* \Omega^*_{X/k}(\log D) \) by \( \mathcal{U}' \), and let \( \tilde{\mathcal{C}}(\mathcal{U}', \tau_{<p} F_* \Omega^*_{X/k}(\log D)) \) be the associated single complex [9, Pages 121-122].

- For any \( i, \Omega^i_{\mathcal{X}'/k}(\log D')[-i] \) is a complex with components \( \Omega^i_{\mathcal{X}'/k}(\log D') \) for degree \( i \) and 0 for other degrees.

- Let \( f : K^*_1 \to K^*_2 \) be a morphism between complexes over some abelian category. Then, we say \( f \) is a quasi-isomorphism if \( H^i(f) : H^i(K^*_1) \to H^i(K^*_2) \) is an isomorphism for any \( i \).

- \( \iota \) is the natural augmentation which is a quasi-isomorphism (see [8, Lemma 4.2, Chapter III]).

- \( \varphi_X \) is an \( \mathcal{O}_X \)-linear morphism of complexes constructed below, which is a quasi-isomorphism (see [3, Theorem 2.1]).

**Construction 2.3** Let \( \{\xi_{\alpha_0}\}, \{h_{\alpha_0\alpha_1}\} \) be as in Lemma 2.2. The morphism \( \varphi_X \) in (2.1) is constructed as follows:

- \( \varphi^0_X \) is clear;

- \( \varphi^i_X = (\varphi^1_X)^\cup_i \delta_i \) for \( 0 < i < p \), where \( \delta_i \) is the standard section of the natural projection \( \Omega^\oplus_{\mathcal{X}'/k}(\log D') \to \Omega^i_{\mathcal{X}'/k}(\log D') \) and

\[
(\varphi^1_X)^\cup_i : \Omega^\oplus_{\mathcal{X}'/k}(\log D') \to \tilde{\mathcal{C}}(\mathcal{U}', \tau_{<p} F_{\mathcal{X}'/k*} \Omega^*_{\mathcal{X}/k}(\log D))
\]

is given by

\[
(\varphi^1_X)^\cup_i (\omega_1 \otimes \cdots \otimes \omega_i) = \varphi^1_X(\omega_1) \cup \cdots \cup \varphi^1_X(\omega_i), \quad \omega_1, \cdots, \omega_i \in \Omega_{\mathcal{X}/k}(\log D).
\]

- \( \varphi^1_X = \varphi_X(0, 1) \oplus \varphi_X(1, 0) \), where

\[
\varphi_X(0, 1) : \Omega_{\mathcal{X}'/k}(\log D') \to \tilde{\mathcal{C}}^0(\mathcal{U}', F_{\mathcal{X}'/k*} \Omega_{\mathcal{X}/k}(\log D))
\]

is given by \( \varphi_X(0, 1)(\omega) = [\xi_\omega(\omega)] \), and

\[
\varphi_X(1, 0) : \Omega_{\mathcal{X}'/k}(\log D') \to \tilde{\mathcal{C}}^1(\mathcal{U}', F_{\mathcal{X}'/k*} \mathcal{O}_X)
\]

is given by \( \varphi_X(1, 0)(\omega) = [h_{\alpha_0\alpha_1}(\omega)] \).

To understand Theorem 1.1, it remains to recall \( D(X') \).

**Definition 2.4** ([9, Sect. 4]) Let \( \mathbb{A} \) be an abelian category. We define \( C^+(\mathbb{A}) \) to be the category of bounded below complexes over \( \mathbb{A} \). Set \( K^+(\mathbb{A}) \) to be the homotopy category obtained by \( C^+(\mathbb{A}) \) modulo the homotopy equivalence. Let \( D^+(\mathbb{A}) \) be the derived category obtained by inverting quasi-isomorphisms in \( K^+(\mathbb{A}) \). Set \( D^+(X') := D^+(\text{Mod}_{\mathcal{O}_{\mathcal{X}'}}) \), where \( \text{Mod}_{\mathcal{O}_{\mathcal{X}'}} \) is the abelian category of \( \mathcal{O}_{\mathcal{X}'} \)-modules on \( X' \).
2.2 Filtered Derived Category

For a thorough treatment of this subject, we refer the reader to [1, Sect. 3.3.1] or [2, Sect. 7]. Let \( \mathbb{A} \) be an abelian category. A filtered object of \( \mathbb{A} \) is an object of \( \mathbb{A} \) together with a finite increasing filtration. A filtered morphism in \( \mathbb{A} \) is a morphism \( f : (A_1, W^1) \to (A_2, W^2) \) between two filtered objects of \( \mathbb{A} \), i.e., \( f : A_1 \to A_2 \) is a morphism of \( \mathbb{A} \) subjects to \( f(W_i^1 A_1) \subset W_i^2 A_2 \) for any \( i \). Denote by \( F\mathbb{A} \) the category of filtered objects of \( \mathbb{A} \) and filtered morphisms in \( \mathbb{A} \). It is an additive category.

Let \( C^+(\mathbb{A}) \) be the category of bounded below complexes over \( F\mathbb{A} \), and let \( K^+(\mathbb{A}) \) be the homotopy category obtained by \( C^+(\mathbb{A}) \) modulo the homotopy equivalence. A morphism \( f : (A^1_i, W^1) \to (A^2_i, W^2) \) in \( C^+(\mathbb{A}) \) is said to be a quasi-isomorphism if \( f : A^1_i \to A^2_i \) is a filtered quasi-isomorphism with respect to the filtrations \( W^1, W^2 \), i.e., the induced morphism \( f : W^1_i A^1_i \to W^2_i A^2_i \) is a quasi-isomorphism in \( C^+(\mathbb{A}) \) for any \( i \). Let \( D^+(\mathbb{A}) \) be the derived category obtained by inverting quasi-isomorphisms in \( K^+(\mathbb{A}) \).

2.3 The Weight Filtration

Let \( S \) be a scheme, \( X \) a smooth \( S \)-scheme, \( D \) a normal crossing divisor on \( X \) relative to \( S \). Recall that besides the decreasing Hodge filtration \( \text{Fil} \) on the logarithmic de Rham complex \( \Omega^*_X/(\log D) \), it carries another increasing weight filtration \( W \).

**Definition 2.5** ([1, Sect. 3.4.1.2]) For any \( i \geq 0 \), there is an increasing weight filtration \( W \) on \( \Omega^i_{X/S}(\log D) \) defined as follows:

\[
W_i \Omega^i_{X/S}(\log D) := \begin{cases} 
\Omega^i_{X/S}(\log D) \wedge \Omega^{i-l}_{X/S}, & l \leq i; \\
\Omega^i_{X/S}(\log D), & l > i.
\end{cases}
\]

The weight filtration \( W \) on each term \( \Omega^i_{X/S}(\log D) \) induces a weight filtration on \( \Omega^*_{X/S}(\log D) \) or \( \bigoplus_i \Omega^i_{X/S}(\log D)[-i] \), again denoted by \( W \).

2.4 \( \varphi_X \) is an Isomorphism in \( D^+F(X') \)

Let \( (X, D)/k, (\tilde{X}, \tilde{D})/W_2(k), \varphi_X \) be as in Sect. 2.1.

Set

\[
C^+(X') := C^+(F\text{Mod}_{O_{X'}}), \quad K^+(X') := K^+(F\text{Mod}_{O_{X'}})
\]

and \( D^+(X') := D^+(F\text{Mod}_{O_{X'}}) \). Note that the weight filtration \( W \) on \( \Omega^*_{X/k}(\log D) \) induces a filtration on the complex \( \tau_{<p} F_* \Omega^*_{X/k}(\log D) \) or \( \tilde{C}(U', \tau_{<p} F_* \Omega^*_{X/k}(\log D)) \), again denoted by \( W \). By definition, the following

\[
(\tau_{<p} F_* \Omega^*_{X/k}(\log D), W), \quad (\tilde{C}(U', \tau_{<p} F_* \Omega^*_{X/k}(\log D)), W),
\]
Lemma 2.7 \( \phi \) are objects of \( C^+ F(X') \). It is obvious that \( \iota \) is a quasi-isomorphism in \( C^+ F(X') \) and \( \varphi_X \) is a morphism in \( C^+ F(X') \). Moreover, we have

**Proposition 2.6** \( \varphi_X \) is an isomorphism in \( D^+ F(X') \).

We prove this proposition by four steps. Since the problem is étale local, we may assume that \( D \) is a simple normal crossing divisor on \( X \) and \( F \) admits a global log lifting \( \tilde{F} \). Write \( D = \sum_{i=1}^{m} D_i \), where each \( D_i \) is a smooth component of \( D \). For any \( \emptyset \neq I \subset \{1, \ldots, m\} \), set \( D_I := \cap_{i \in I} D_i \). Set \( D_{\emptyset} = X \).

**Step 1.** Recall that \( \varphi_X \) is constructed by a D-I atlas \( \{U_\alpha, \tilde{F}_{U_\alpha}\} \) of \( (\tilde{X}, \tilde{D})/W_2(k) \). Using the global log lifting \( \tilde{F} \), there is another D-I atlas \( \{U_\alpha, \tilde{F}|_{U_\alpha}\} \) of \( (\tilde{X}, \tilde{D})/W_2(k) \). Running Construction 2.3 for this new D-I atlas, we obtain a new morphism which is denoted by \( \varphi_{\tilde{F}} \).

**Lemma 2.7** \( \varphi_X = \varphi_{\tilde{F}} \) in \( K^+ F(X') \).

**Proof** This lemma is a special case of Lemma 3.14: we take \( (\tilde{Y}, \tilde{E}) = (\tilde{X}, \tilde{D}), \tilde{f} = id_{(\tilde{X}, \tilde{D})}, \) D-I atlases \( \{U_\alpha, \tilde{F}_{U_\alpha}\}, \{U_\alpha, \tilde{F}|_{U_\alpha}\} \) of \( (\tilde{X}, \tilde{D})/W_2(k), (\tilde{Y}, \tilde{E})/W_2(k) \), respectively, and \( \chi = id \).

This lemma shows that if \( \varphi_{\tilde{F}} \) is a quasi-isomorphism in \( C^+ F(X') \), then Proposition 2.6 follows. Let

\[
\zeta_{\tilde{F}} : \Omega_{X'/k}(\log D') \to F_* \Omega_{X/k}(\log D)
\]

be the morphism induced by \( \tilde{F} \). Then, we can construct a morphism

\[
\tilde{\varphi}_{\tilde{F}} : \left( \bigoplus_{i=0}^{p-1} \Omega_{X'/k}^i(\log D')[-i], W \right) \to (F_* \Omega_{X/k}^*(\log D), W)
\]

in \( C^+ F(X') \) as follows: for \( \omega_1, \ldots, \omega_s \in \Omega_{X'/k}(\log D') \), we set

\[
\tilde{\varphi}_{\tilde{F}}(\omega_1 \wedge \cdots \wedge \omega_s) := \zeta_{\tilde{F}}(\omega_1) \wedge \cdots \wedge \zeta_{\tilde{F}}(\omega_s);
\]

for \( x' \in \mathcal{O}_{X'} \), we set \( \tilde{\varphi}_{\tilde{F}}(x') := F^* x' \). Let

\[
\tau_{<p} \tilde{\varphi}_{\tilde{F}} : \left( \bigoplus_{i=0}^{p-1} \Omega_{X'/k}^i(\log D')[-i], W \right) \to (\tau_{<p} F_* \Omega_{X/k}^*(\log D), W)
\]

be the morphism obtained by truncating \( \tilde{\varphi}_{\tilde{F}} \) at degrees \( < p \), then one can check that \( \varphi_{\tilde{F}} = \iota \circ \tau_{<p} \tilde{\varphi}_{\tilde{F}} \) holds in \( C^+ F(X') \). Since \( \iota \) is a quasi-isomorphism in \( C^+ F(X') \), it is enough to show that

\[ \Box \] Springer
Lemma 2.8  \( \tau_{< p} \tilde{\phi}_F \) is an isomorphism in \( D^+ F(X') \).

**Step 2.** When take grading \( Gr^W_l \) on \( \tau_{< p} F_* \Omega^*_{X/k} (\log D) \), the truncation \( \tau_{< p} \) may make trouble. the following lemma dispel this worry.

**Lemma 2.9** The natural morphism

\[
\mu : Gr^W_l \tau_{< p} F_* \Omega^*_{X/k} (\log D) \rightarrow \tau_{< p} Gr^W_l F_* \Omega^*_{X/k} (\log D)
\]  

is a quasi-isomorphism in \( C^+ (\text{Mod}_X) \).

**Proof** It is enough to check that \( \mathcal{H}^{p-1} (\mu) \) is an isomorphism. Since the problem is Zariski local, we may assume that \( X \) admits a global coordinate system \( t_1, \ldots, t_n \) such that \( D = (t_1 \cdots t_m = 0) \). A direct computation shows that

\[
\mathcal{H}^{p-1} \left( \frac{W_l \tau_{< p} F_* \Omega^*_{X/k} (\log D)}{W_{l-1} \tau_{< p} F_* \Omega^*_{X/k} (\log D)} \right) = \frac{\left\{ x \in F_* W_l \Omega^*_{X/k} (\log D) : dx = 0 \right\}}{\left\{ x \in F_* W_{l-1} \Omega^*_{X/k} (\log D) : dx = 0 \right\} + dW_l \Omega^*_{X/k} (\log D)}
\]

and

\[
\mathcal{H}^{p-1} (Gr^W_l F_* \Omega^*_{X/k} (\log D)) = \frac{\left\{ x \in F_* W_l \Omega^*_{X/k} (\log D) : d x \in F_* W_{l-1} \Omega^*_{X/k} (\log D) \right\}}{F_* W_{l-1} \Omega^*_{X/k} (\log D) + d F_* W_l \Omega^*_{X/k} (\log D)}.
\]

The injectivity of \( \mathcal{H}^{p-1} (\mu) \) is clear. Let us show the surjectivity of \( \mathcal{H}^{p-1} (\mu) \). For any \( \emptyset \neq I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, m\} \) with \( i_1 < \cdots < i_s \), write

\[
t_I = \prod_{i \in I} t_j, \quad d_I \log t = d \log t_1 \wedge \cdots \wedge d \log t_i.
\]

Let \( x \in F_* W_l \Omega^*_{X/k} (\log D) \), then it can be locally expressed as the following form:

\[
x = \sum_{|I| = l} d_I \log t \wedge x_I + x', \quad x_I \in \Omega^*_{X/k} (\log D), \quad x' \in W_{l-1} \Omega^*_{X/k} (\log D).
\]

Assume that \( dx \in W_{l-1} \Omega^*_{X/k} (\log D) \). By Lemma 2.12 (i), we have \( d_{D_I/k}(x|_{D_I}) = 0 \) for any \( |I| = l \). Using the Cartier isomorphism [4, Theorem 9.14], we locally have

\[
x_I|_{D_I} = \sum_{K \cap I = \emptyset, |K| = p-1-l} (\lambda_K^p t_K^p d_K \log t)|_{D_I} + d_{D_I/k}(x'_I|_{D_I}),
\]

\[
\lambda_K \in \mathcal{O}_X, \quad x'_I \in \Omega^*_{X/k} (\log D).
\]
Consequently, we locally get
\[ x = \sum \lambda_K p_K dK \log t + dx'' + x''', \]
where \( x'' \in W_l \Omega^{p-2}_X/k (\log D), x''' \in W_{l-1} \Omega^{p-1}_X/k (\log D) \). From which the surjectivity of \( \mathcal{H}^{p-1}(\mu) \) follows, this completes the proof. □

Combining this lemma with 5-lemma, it is enough to show that

**Lemma 2.10** \( \text{Gr}^W \hat{\phi}_F \) is an isomorphism in \( D^+(X') \) for any \( l \), or equivalently, \( \hat{\phi}_F \) is an isomorphism in \( D^+ F(X') \).

**Step 3.** We introduce the Poincaré residue map.

**Definition 2.11** (i) Let \( I \subseteq \{1, \ldots, m\} \), and \( i_{D_I} : D_I \hookrightarrow X \) be the natural closed embedding. For any \( 0 \leq l \leq q \) with \( |I| = l \), we define a Poincaré residue map ([1, Sect. 3.4.1.3])

\[ \text{Res}_{D_I} : W_l \Omega^q_{X/k}(\log D) \to i_{D_I*} \Omega^{q-l}_{D_I/k} \]

as follows. Let \( I = \{i_1, \ldots, i_l\} \) with \( i_1 < \cdots < i_l \), and let \( \omega \in W_l \Omega^q_{X/k}(\log D) \). Locally, we can write

\[ \omega = d \log t_1 \wedge \cdots \wedge d \log t_l \wedge \omega' + \eta, \]

where \( t_1, \ldots, t_l \) are local defining functions of \( D_{i_1}, \ldots, D_{i_l} \) and

\[ \omega' \in \Omega^{q-l}_{X/k}, \eta \in \sum_{i \in I} \Omega^q_{X/k}(\log D - D_i). \]

Then, we locally define

\[ \text{Res}_{D_I}(\omega) = \omega'|_{D_I} \in i_{D_I*} \Omega^{q-l}_{D_I/k}. \]

(ii) Let \( (K^*, d) \) be a complex over abelian category \( A \), and let \( l \) be an integer. We define \( (K^*, d)[l] \) as follows:

\[ (K^*, d)[l] := (K^*[l], d[l]), \quad K^*[l] := K^{i+l}, \quad d[l] := (-1)^l d^{i+l}. \]

**Lemma 2.12** Let \( i_{D'_I} : D'_I \hookrightarrow X' \) be the base change of \( i_{D_I} : D_I \hookrightarrow X \) via \( F_k \). Then, we have the following decompositions:

(i) \( \text{Gr}^W_{l*} \Omega^q_{X/k}(\log D) \cong \bigoplus_{|I| = l} i_{D_I*} \Omega^q_{D_I/k}[-l]; \)
(ii) \( \text{Gr}^W_{l*} \bigoplus_q \Omega^q_{X'/k}(\log D')[q] \cong \bigoplus_{|I| = l} i_{D_I*} \bigoplus_q \Omega^q_{D_I/k}[-q][l]. \)

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\textbf{Proof} It is easy to see that \( \text{Res}_{D_I} \) induces a morphism
\[
Gr^W_I \Omega^q_{X/k}(\log D) \to \Omega^q_{D_I}^{q-|I|},
\]
again denoted by \( \text{Res}_{D_I} \). Moreover, one can check that
\[
\bigoplus_{|I|=l} \text{Res}_{D_I} : Gr^W_I \Omega^q_{X/k}(\log D) \to \bigoplus_{|I|=l} i_{D_I*}\Omega^q_{D_I/k}^{q-|I|}
\]
is an isomorphism, from which (i) follows. (ii) is similar to (i).

\( \Box \)

\textbf{Step 4.} We study the mixed structure of \( \tilde{\phi}_F \). Set \( \tilde{F}_I \) to be the restriction of \( \tilde{F} \) on \( \tilde{D}_I \), then it induces a morphism
\[
\zeta_{\tilde{F}_I} : \Omega^i_{D_I^0/k} \to F_*\Omega^i_{D_I/k}.
\]
We similarly define \( \tilde{\varphi}_{\tilde{F}_I} \) as \( \varphi_{\tilde{F}} \). For any \( |I| = l \) and \( i \geq l \), the following diagram can be easily checked:
\[
\begin{array}{ccc}
W_l \Omega^i_{X'/k}(\log D) & \xrightarrow{\tilde{\phi}_F} & W_l F_*\Omega^i_{X/k}(\log D) \\
\downarrow \text{Res}_{D'_I} & & \downarrow \text{Res}_{D_I} \\
i_{D'_I*}\Omega^{i-|I|}_{D'_I/k} & \xrightarrow{\tilde{\varphi}_{\tilde{F}_I}} & i_{D_I*} F_*\Omega^i_{D_I/k}.
\end{array}
\]
Combining Lemma 2.12 with this diagram, we get
\[
\text{Gr}^W_I \tilde{\phi}_F = \bigoplus_{|I|=l} i_{D'_I*}\tilde{\varphi}_{\tilde{F}_I} [-l].
\]
By the Cartier isomorphism loc.cit., we obtain that \( \tilde{\phi}_F \) is a quasi-isomorphism in \( C^+ F(D'_I) \) for any \( I \). This implies that \( \text{Gr}^W_I \tilde{\varphi}_F \) is an isomorphism in \( D^+(X') \), from which Lemma 2.10 follows. The proof of Proposition 2.6 is completed.

As an easy consequence of Proposition 2.6, Theorem 1.2 follows.

Using Proposition 2.6 and an argument which is similar to the proof of Raynaud’s vanishing theorem [9, Theorem 5.8], we have the following.

\textbf{Theorem 2.13} \textit{Let} \( X \) \textit{be a smooth projective variety over} \( k \), \textit{and let} \( D \) \textit{be a normal crossing divisor on} \( X \). \textit{Assume that} \( \dim(X) < p \). \textit{Then, for any ample line bundle} \( L \) \textit{on} \( X \), \textit{we have}
\[
\mathbb{H}^{i+j}(X, W_l \Omega^i_{X/k}(\log D) \otimes L) = 0 \text{ for } i + j > \dim(X).
\]

Note that when \( D \) is a simple normal crossing divisor on \( X \), it is an easy consequence of [9, Theorem 5.8].
Combining Theorem 2.13 with spreading-out technical (see [9, Sect. 6]), Theorem 1.3 follows.

3 Proof of Theorem 1.5

This section aims to the proof of Theorem 1.5.

3.1 Mixed Fontaine–Laffaille Complexes

In this subsection, we establish a positive characteristic analogue of the notion of mixed Hodge complexes (see [1, Definition 3.3.17], [2, (8.1.5)]).

Definition 3.1 We call a quadruple \((K^*_d, W, \text{Fil}, \psi)\) a mixed Fontaine–Laffaille complex over \(k\) if:
- \(K^*_d\) is a bounded below complex of \(k\)-vector spaces;
- \(W\) is a finite increasing weight filtration on \(K^*_d\);
- \(\text{Fil}\) is a finite decreasing Hodge filtration on \(K^*_d\);
- \(H^j(\text{Gr}_i^W K^*_d) = 0\) for \(i + j \gg 0\) and it is a finite-dimensional \(k\)-vector space for any \(i, j\);
- Let \((K^*_\text{Hig}, W) := F^*_k \text{GrFil}(K^*_d, W)\), then

\[
\psi : (K^*_\text{Hig}, W) \to (K^*_d, W)
\] (3.1)

is an isomorphism in \(D^+ F(\text{Vec}_k)\).

A morphism between two mixed Fontaine–Laffaille complexes

\[
f : (K^*_d, W^1, W^2, \text{Fil}_1, \psi_1) \to (K^*_d, W^2, \text{Fil}_2, \psi_2)
\]
is a morphism of complexes of \(k\)-vector spaces

\[
f : K^*_d, W^1 \to K^*_d, W^2
\]
satisfying the following conditions:
- \(f\) preserves the filtrations \(W^1, W^2\) and \(\text{Fil}_1, \text{Fil}_2\);
- The following diagram is commutative in \(D^+ F(\text{Vec}_k)\):

\[
\begin{array}{ccc}
(K^*_\text{Hig}, W^1) & \xrightarrow{F^*_k \text{Gr}(f)} & (K^*_\text{Hig}, W^2) \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
(K^*_d, W^1) & \xrightarrow{f} & (K^*_d, W^2).
\end{array}
\] (3.2)

Denote by MFLC\((k)\) the category of mixed Fontaine–Laffaille complexes over \(k\).
Remark 3.2 When $K^0_{dR} = V$, $K^i_{dR} = 0$ for $i \neq 0$ and $W_1 K^*_dR = 0$, $W_0 K^*_dR = K^*_dR$, the mixed Fontaine–Laffaille complex $(K^*_{dR}, W, \Fil, \psi)$ becomes a Fontaine–Laffaille module $(V, \Fil, \psi)$ over $k$. In this case, we notice that $\psi : F^k_k \GrFil V \to V$ and (3.2) live in $\Vec_k$. Denote by $\MF(k)$ the category of Fontaine–Laffaille modules over $k$.

It is well known that the category of Hodge structures is an abelian category and the morphisms are strict, i.e., if $f : (H_1, \Fil_1) \to (H_2, \Fil_2)$ is a morphism of Hodge structures, then

$$f(\Fil^i H_1) = f(H_1) \cap \Fil^i H_2.$$ 

For the category $\MF(k)$, we have similar properties.

Theorem 3.3 ([5, Theorem 2.1]) $\MF(k)$ is an abelian category and morphisms in $\MF(k)$ are strict.

In what follows, let us fix a mixed Fontaine–Laffaille complex $(K^*_{dR}, W, \Fil, \psi)$ and study its spectral sequences. We briefly recall the relevant notions.

Definition 3.4 Let $\mathbb{A}$ be an abelian category. For any $r_0 \geq 0$, a spectral sequence starting from $E_{r_0}$-page over $\mathbb{A}$ is a family $\{E^i,j_r, d^i,j_r\}_{r \geq r_0}$ subjects to the following conditions:

- For any $i, j$ and $r \geq r_0$, $E^i,j_r$ is an object of $\mathbb{A}$ and $d^i,j_r : E^i,j_r \to E^{i+r,j-r+1}_r$ is a morphism in $\mathbb{A}$;
- $d^i,j_r \circ d^i,r_{r,j+r-1} = 0$ and $E^{i,j}_{r+1} \cong \ker(d^i,j_r) / \im(d^i,r_{r,j+r-1})$.

A morphism between spectral sequences starting from $E_{r_0}$-page over $\mathbb{A}$

$$f : \{E^i,j_r, d^i,j_r\}_{r \geq r_0} \to \{E'^i,j_r, d'^i,j_r\}_{r \geq r_0}$$

consisting of the following data:

- For any $i, j$ and $r \geq r_0$, there is a morphism $f : E^i,j_r \to E'^i,j_r$ in $\mathbb{A}$ which fits into a commutative diagram

$$
\begin{array}{ccc}
E^i,j_r & \xrightarrow{d^i,j_r} & E^{i+r,j-r+1}_r \\
\downarrow{f} & & \downarrow{f} \\
E'^i,j_r & \xrightarrow{d'^i,j_r} & E'^{i+r,j-r+1}_r.
\end{array}
$$

- $f : E^{i,j}_{r+1} \to E'^{i,j}_{r+1}$ is induced from the $E_r$-page.

Denote by $E_{r \geq r_0}/\mathbb{A}$ the category of spectral sequences starting from $E_{r_0}$-page over $\mathbb{A}$, which is an additive category.

A typical example of spectral sequences is the following.
**Example 3.5** ([7, page 440]) Let $K^*$ be a complex over an abelian category $A$ with differential $d$ and $F$ a decreasing filtration on $K^*$. We construct the spectral sequence \( \{E_r^{i,j}(K^*, W), d_r^{i,j}\}_{r \geq 0} \) of \((K^*, F)\) starting from $E_0$-page as follows:

\[
E_r^{i,j}(K^*, W) = Z_r^{i,j}/B_r^{i,j}, \quad Z_r^{i,j} = \{x \in F^i K^{i+j}: dx \in F^{i+r} K^{i+j+1}\}, \\
B_r^{i,j} = d F^{i-r+1} K^{i+j-1} \cap F^i K^{i+j} + \{x \in F^{i+1} K^{i+j}: dx \in F^{i+r} K^{i+j+1}\}
\]

and $d_r^{i,j}: E_r^{i,j} \to E_r^{i+r-j+r+1}$ is induced by $d$. Note that we can take $r = \infty$. We say this spectral sequence degenerates at $E_r$ if $d_r^{i,j} = 0$ for any $i, j$ and $s \geq r$.

Let $W$ be another finite increasing filtration on $K^*$. Set $\hat{W}^i := W_{-i}$. Then, we take the spectral sequence of $(K^*, W)$ to be the spectral sequence of $(K^*, \hat{W})$.

**Lemma 3.6** The spectral sequences of $(K_{dR}^*, \text{Fil}), \text{Gr}_{i}^W (K_{dR}^*, \text{Fil})$ degenerate at $E_1$.

**Proof** It is obvious that

\[
\dim_k H^m(K_{dR}^*) = \dim_k H^m(F^*_k \text{Fil} K_{dR}^*) = \dim_k H^m(\text{Fil} K_{dR}^*),
\]

from which the $E_1$-degeneration of $(K_{dR}^*, \text{Fil})$ follows. Note that

\[
F^*_k \text{Fil} \text{Gr}_{i}^W K_{dR}^* = \text{Gr}_{i}^W F^*_k \text{Fil} K_{dR}^*,
\]

then the $E_1$-degeneration of $\text{Gr}_{i}^W (K_{dR}^*, \text{Fil})$ follows similarly. \( \square \)

We proceed to study the spectral sequence of $(K_{dR}^*, W)$. Before doing this, it is necessary to review Deligne’s technical result.

**Definition 3.7** ([1, Definition 3.2.26], [2, (7.2.4)]) Let $K^*$ be a bounded below complex over an abelian category $A$ with differential $d$, $W$ a finite increasing filtration and $F$ a finite decreasing filtration on $K^*$. Assume that $H^j(\text{Gr}_{i}^W K^*) = 0$ and $H^j(\text{Gr}_F^j K^*) = 0$ for $i + j \gg 0$. Then, there are three decreasing filtrations $F_d, F_{d^*}, F_{\text{rec}}$ on $E_r^{i,j}(K^*, W)$ which are defined as follows:

- $F_d^{i,j} E_r^{i,j}(K^*, W) := \text{im}(E_r^{i,j}(F^i K^*, W) \to E_r^{i,j}(K^*, W));$
- Dually, we define $F_{d^*}^{i,j} E_r^{i,j}(K^*, W) := \text{ker}(E_r^{i,j}(K^*, W) \to E_r^{i,j}(K^*/F^i K^*, W));$
- It is easy to check that $F_d = F_{d^*}$ on $E_0^{i,j}(K^*, W)$, and that $F_{\text{rec}}$ is defined on $E_r^{i,j}(K^*, W)$, and that $F_{\text{rec}}$ is defined on $E_r^{i,j}(K^*, W)$. Since $E_{r+1}^{i,j}(K^*, W)$ is a subquotient of $E_r^{i,j}(K^*, W)$, then it induces a filtration $F_{\text{rec}}$ on $E_r^{i,j}(K^*, W)$.

Now we can state Deligne’s technical lemma.

**Lemma 3.8** (Deligne [1, Theorem 3.2.30], [2, Proposition 7.2.5]) Use the notation as in the definition above.

(i) The differential $d$ is strictly compatible with $F$ if and only if the spectral sequence of $(K^*, F)$ degenerates at $E_1$.\( \square \)
(ii) Let \( r_0 \geq 0 \). Assume that \( d^{i,j}_r \) is strictly compatible with \( F_{rec} \) for any \( i, j \) and \( r < r_0 \). Then, we have the following exact sequence:

\[
0 \to E^{i,j}_{r_0} (F^l K^*, W) \to E^{i,j}_{r_0} (K^*, W) \to E^{i,j}_{r_0} (K^*/F^l K^*, W) \to 0.
\]

Denote by

\[
\begin{align*}
E^{i,j}_{r,dR} &= \frac{Z^{i,j}_{r,dR}}{B^{i,j}_{r,dR}}, \quad d^{i,j}_{r,dR} \\
E^{i,j}_{r,Hig} &= \frac{Z^{i,j}_{r,Hig}}{B^{i,j}_{r,Hig}}, \quad d^{i,j}_{r,Hig}
\end{align*}
\]

the spectral sequences of \( (K^*_{dR}, W), (K^*_{Hig}, W) := F^*_k \text{GrFil}(K^*_{dR}, W) \), respectively. They are related by the following theorem.

**Theorem 3.9** Let \( (K^*_{dR}, W, \psi) \) be as above.

(i) \( \psi \) induces an isomorphism of spectral sequences

\[
\psi : \left\{ E^{i,j}_{r,Hig}, d^{i,j}_{r,Hig} \right\}_{r \geq 1} \to \left\{ E^{i,j}_{r,dR}, d^{i,j}_{r,dR} \right\}_{r \geq 1}.
\]

(ii) \( \text{Fil}_d = \text{Fil}_{rec} \) on \( E^{i,j}_{r,dR} \) and \( F^*_k \text{GrFil}_{rec} \) sends \( d^{i,j}_{r,dR} \) to \( d^{i,j}_{r,Hig} \) for \( r \geq 0 \).

**Proof** (i) is obvious. For (ii), we restate it in a more subtle form, i.e., for any \( i, j \) and \( r \geq 0 \), we have:

- \( \text{Fil}_d = \text{Fil}_{rec} = \text{Fil}_{d^*} \) on \( E^{i,j}_{r,dR} \) and \( d^{i,j}_{r,dR} \) is strict with respect to \( \text{Fil}_{rec} \);
- There is an isomorphism

\[
\mu : F^*_k \text{GrFil}_{rec} E^{i,j}_{r,dR} \to E^{i,j}_{r,Hig}
\]

in \( \text{Vec}_k \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\bigoplus_i F^*_k \text{Fil}^l Z^{i,j}_{r,dR} & \longrightarrow & Z^{i,j}_{r,Hig} \\
\downarrow & & \downarrow \\
F^*_k \text{GrFil}_{rec} E^{i,j}_{r,dR} & \longrightarrow & E^{i,j}_{r,Hig}
\end{array}
\] (3.3)

where the upper and two vertical morphisms are natural morphisms. Moreover, the two vertical maps are surjective, from which \( \mu \) is uniquely determined.

- The following diagram is commutative:

\[
\begin{array}{ccc}
F^*_k \text{GrFil}_{rec} E^{i,j}_{r,dR} & \longrightarrow & E^{i,j}_{r,Hig} \\
\downarrow & & \downarrow d^{i,j}_{r,Hig} \\
F^*_k \text{GrFil}_{rec} [E^{i,j}_{r,dR}]_{r+1} & \longrightarrow & E^{i,j}_{r,Hig}_{r+1}
\end{array}
\] (3.4)
We prove this subtle form by induction on $r$. The case $r = 0$. According to Lemma 3.6 and Lemma 3.8 (i), $d_{i,j,0}^{i,j}$ is strict with respect to $\Fil_{\rec}$ for any $i, j$. We set

$$
\mu : F_{k}^{*} \Gr_{\Fil_{\rec}} W_{r}^{i+j} \to \Gr_{r_{r_{D}}}^{W} F^*_{k} \Gr_{\Fil_{\rec}} K_{r_{r_{D}}}^{i+j} = \Gr_{r_{r_{D}}}^{W} K_{Hig}^{i+j}
$$

to be the natural isomorphism. The other statements are obvious; hence, this case follows.

Assume this subtle form holds for any $r \leq r_{0}$. Observing that the isomorphism $\psi : (K_{Hig}^{r}, W) \to (K_{D}^{r}, W)$ induces an isomorphism $E_{r_{,Hig}}^{i,j} \to E_{r_{,D}}^{i,j}$ for any $i, j$ and $r \geq 1$, again denoted by $\psi$. By Theorem 3.3, we know that

$$
d_{r_{,D}}^{i,j} : (E_{r_{,D}}^{i,j}, \Fil_{\rec}, \psi) \to (E_{r_{,D}}^{i+r,j-r+1}, \Fil_{\rec}, \psi)
$$

(3.5)
is a morphism in $MF(k)$ for any $i, j$ and $1 \leq r \leq r_{0}$. It follows that

$$
F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{+1,D}}^{i,j} \cong H(F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{,D}}^{i-r,j-r+1} \to F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{,D}}^{i,j} \to F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{,D}}^{i+r,j-r+1})
$$

holds for any $i, j$ and $1 \leq r \leq r_{0}$. For any $i, j$, let

$$
\mu : F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{0+1,D}}^{i,j} \to E_{r_{0+1,D}}^{i,j}
$$

(3.6)
be the cohomology of

$$
\begin{array}{cccc}
F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{,D}}^{i-r,j+r-1} & \to & F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{,D}}^{i,j} & \to & F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{,D}}^{i+r,j-r+1} \\
\mu & & \mu & & \mu \\
E_{r_{,Hig}}^{i-r,j+r-1} & \to & E_{r_{,Hig}}^{i,j} & \to & E_{r_{,Hig}}^{i+r,j-r+1}.
\end{array}
$$

One can check that (3.6) satisfies (3.3) for $r = r_{0} + 1$. By assumption, $d_{r_{,D}}^{i,j}$ is strict for any $i, j$ and $r \leq r_{0}$. Using Lemma 3.8 (ii), it implies that $\Fil_{d} = \Fil_{\rec}$ on $E_{r_{0+1,D}}^{i,j}$ for any $i, j$, from which the surjectivity of the vertical morphisms in (3.3) for $r = r_{0} + 1$ follows. Combining the following easily checked commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{i} F_{k}^{*} \Fil_{d} Z_{r_{0+1,D}}^{i,j} & \to & Z_{r_{0+1,Hig}}^{i,j} \\
\downarrow & & \downarrow \\
\bigoplus_{i} F_{k}^{*} \Fil_{d} Z_{r_{0+1,D}}^{i+r,j-r+1} & \to & Z_{r_{0+1,Hig}}^{i+r,j-r+1}
\end{array}
$$

with the surjectivity of $\mu : \bigoplus_{i} F_{k}^{*} \Fil_{d} Z_{r_{0+1,D}}^{i,j} \to F_{k}^{*} \Gr_{\Fil_{\rec}} E_{r_{0+1,D}}^{i,j}$, the commutativity of (3.4) for $r = r_{0} + 1$ follows. It remains to show the strictness of $d_{r_{0+1,D}}^{i,j}$. 

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with respect to \( Fil_{rec} \) for any \( i, j \). This is easy, because the commutativity of (3.4) for \( r = r_0 + 1 \) says that (3.5) is a morphism in \( MF(k) \) for \( r = r_0 + 1 \). The proof is completed.

Note that \( F^*_k \) induces an automorphism of \( E_{r \geq 1}/Vec_k \). As an application of the theorem above, one has

**Lemma-Definition 3.10** There are functors

\[
\rho_{dR}^{MFLC}, \quad \rho_{Hig}^{MFLC} : MFLC(k) \to E_{r \geq 1}/Vec_k
\]

with

\[
\rho_{dR}^{MFLC}(K^*_dR, W, Fil, \psi) := \{E_{r, dR}^{i,j}(K^*_dR, W), d_{r, dR}^{i,j}\}_{r \geq 1},
\]

\[
\rho_{Hig}^{MFLC}(K^*_dR, W, Fil, \psi) := \{E_{r, Hig}^{i,j}(K^*_Hig, W), d_{r, Hig}^{i,j}\}_{r \geq 1}.
\]

The recursive filtration \( Fil_{rec} \) on \( E_{r, dR}^{i,j}(K^*_dR, W) \) induces a Hodge filtration \( Fil \) on \( \rho_{dR}^{MFLC} \) with \( \rho_{Hig}^{MFLC} = F^*_k \circ GrFil\rho_{dR}^{MFLC} \). Let \( \psi : \rho_{Hig}^{MFLC} \to \rho_{dR}^{MFLC} \) be the isomorphism of functors induced by (3.1). We call \( (\rho_{dR}^{MFLC}, \rho_{Hig}^{MFLC}, Fil, \psi) \) a Fontaine–Laffaille quadruple over \( MFLC(k) \).

**Proof** Using Theorem 3.9, we know that \( d_{r, dR}^{i,j} \) is strict with respect to \( Fil_{rec} \) for any \( i, j \) and \( r \geq 0 \). It follows that

\[
\{Fil_{rec}E_{r, dR}^{i,j}, d_{r, dR}^{i,j}\}_{r \geq 1}, \{GrFil_{rec}E_{r, dR}^{i,j}, d_{r, dR}^{i,j}\}_{r \geq 1} \in E_{r \geq 1}/Vec_k, \forall l.
\]

In other words, the Hodge filtration \( Fil \) on \( \rho_{dR}^{MFLC} \) is well defined. By Theorem 3.9 (ii), \( \rho_{Hig}^{MFLC} = F^*_k \circ GrFil\rho_{dR}^{MFLC} \) follows. The other statements are obvious.

**3.2 Cohomological Mixed Fontaine–Laffaille Complexes**

We introduce a positive characteristic analogue of the notion of cohomological mixed Hodge complexes (see [1, Definition 3.3.18], [2, (8.1.6)]).

**Definition 3.11** Let \( X \) be a scheme over \( k \). Denote by \( Mod_{k_X} \) (resp. \( Mod_{k_X'} \)) the abelian category of sheaves of \( k \)-vector spaces on \( X \) (resp. \( X' \)). We call a quadruple \( (K^*_dR, W, Fil, \Psi) \) a cohomological mixed Fontaine–Laffaille complex over \( X \) if:

- \( K^*_dR \in C^+(Mod_{k_X}) \);
- \( W \) is a finite increasing weight filtration on \( K^*_dR \);
- \( Fil \) is a finite decreasing Hodge filtration on \( K^*_dR \);
- \( R^j\Gamma(X, Gr_i^W K^*_dR) = 0 \) for \( i + j \gg 0 \) and it has finite dimension over \( k \) for any \( i, j \);
- Set \( (K^*_Hig, W) := \pi^*_X/k GrFil(K^*_dR, W) \in C^+(Mod_{k_X'}) \), then
  \[
  \Psi : (K^*_Hig, W) \to F_*(K^*_dR, W)
  \]
Lemma 3.12 Let \((\mathcal{K}^\ast_{dR}, W, \text{Fil, } \psi)\) be a cohomological mixed Fontaine–Laffaille complex over \(X\). Then, \((R\Gamma(X, (\mathcal{K}^\ast_{dR}, W, \text{Fil})), R\Gamma(X', \Psi))\) is a mixed Fontaine–Laffaille complex over \(k\).

Proof We abbreviate \(R\Gamma(X, \text{Fil})\) as Fil. For the proof of this lemma, it is enough to notice that

\[
F^*_k \text{GrFil} \Gamma(X, (\mathcal{K}^\ast_{dR}, W)) = R\Gamma(X', \pi^\ast_X \text{GrFil}(\mathcal{K}^\ast_{dR}, W)) = R\Gamma(X', (\mathcal{K}^\ast_{\text{Hig}}, W))
\]

and \(R\Gamma(X, (\mathcal{K}^\ast_{dR}, W)) = R\Gamma(X', F_*(\mathcal{K}^\ast_{dR}, W)).\)

Taking account Theorem 1.2, we immediately obtain the following.

Theorem 3.13 Notation as in Theorem 1.2 and additionally assume that \(X\) is smooth proper over \(k\) with dimension less than \(p\). Then, \((\Omega^\ast_{X/k}(\log D), W, \text{Fil, } \Psi(\tilde{X}, \tilde{D}))\) is a cohomological mixed Fontaine–Laffaille complex over \(X\).

3.3 The Functoriality of \(\Psi\)

Let \(f : (\tilde{X}, \tilde{D}) \rightarrow (\tilde{Y}, \tilde{E})\) be a morphism in the category \(\text{Sch}_{sp, \log}^{\text{sp}}\), and let \(f : (X, D) \rightarrow (Y, E)\) be its closed fiber in \(\text{Sch}_{k, \log}^s\). Choosing D-I atlases \(\{U_\alpha, \tilde{U}_\alpha/k\}_{\alpha \in A}, \{V_\beta, \tilde{V}_\beta/k\}_{\beta \in B}\) of \((\tilde{X}, \tilde{D})/W_2(k), (\tilde{Y}, \tilde{E})/W_2(k)\), respectively, and set \(\mathcal{U} := \{U_\alpha\}_{\alpha \in A}, \mathcal{V} := \{V_\beta\}_{\beta \in B}\). By suitably choose \(A, B\), we may assume that there exists a map \(\chi : A \rightarrow B\) such that \(f(U_\alpha) \subset V_{\chi(\alpha)}\) for any \(\alpha \in A\).

Note that the pullback

\[
f^* : (\Omega^\ast_{Y'/k}(\log E), W) \rightarrow f_*(\Omega^\ast_{X/k}(\log D), W)
\]

together with \(\chi\) induce a morphism in \(C^+ F(X')\):

\[
f^* : (\tilde{C}(\mathcal{V}', F_* \Omega^\ast_{Y/k}(\log E)), W) \rightarrow f'_*(\tilde{C}(\mathcal{U}', F_* \Omega^\ast_{X/k}(\log D)), W).
\]

The following lemma can be regarded as a sort of functoriality of \(\Psi\).

Lemma 3.14 The following diagram is commutative in \(D^+ F(X')\):

\[
\begin{array}{ccccc}
(\bigoplus_{i} \Omega^\ast_{Y'/k}(\log E')[-i], W) & \xrightarrow{f'^*} & (\bigoplus_{i} \Omega^\ast_{X'/k}(\log D')[-i], W) & \xrightarrow{f^*} & (F_* \Omega^\ast_{Y/k}(\log E), W) \\
\Psi_{(\tilde{X}, \tilde{D})} & & \Psi_{(\tilde{X}, \tilde{D})} & & \Psi_{(\tilde{X}, \tilde{D})} \\
\end{array}
\]

\[
\begin{array}{ccccc}
(\tilde{C}(\mathcal{V}', F_* \Omega^\ast_{Y/k}(\log E)), W) & \xrightarrow{f'^*} & (\tilde{C}(\mathcal{U}', F_* \Omega^\ast_{X/k}(\log D)), W) & \xrightarrow{f^*} & (F_* \Omega^\ast_{Y/k}(\log E), W) \\
\phi_Y & & \phi_X & & \phi_Y \\
\end{array}
\]

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Proof It is enough to show that the upper square is commutative in \( K^+ F(X') \). For this purpose, we need to construct a morphism

\[
\eta_i : (\Omega^{\otimes i}_{Y'/k}(log E'), W) \to f'_*(\tilde{\mathcal{C}}(U', F_X\Omega^*_X/k(log D))^{i-1}, W)
\]  

(3.7)

for any \( 0 \leq i < p \) such that

\[
f^* \varphi_Y^i - \varphi_X^i f'^* = d \eta_i \delta_i.
\]

There we set \( d \) to be the differential of \( \tilde{\mathcal{C}}(U', F_X\Omega^*_X/k(log D)) \)

\[
W_j \Omega^i_{Y'/k}(log E') := \left\{ \begin{array}{ll}
\sum_{\sigma \in S_i} \sigma \cdot \Omega^{\otimes (j)}_{Y'/k}(log E') \otimes \Omega^{\otimes (i-j)}_{Y'/k}, & j \leq i; \\
\Omega^{\otimes i}_{Y'/k}(log E'), & j > i,
\end{array} \right.
\]

\(^2\) and \( \delta_i \) the standard section of the projection \( \Omega^i_{X'/k}(log D') \to \Omega^i_{Y'/k}(log D') \).

We proceed to the construction of \( \eta_i \). The case \( i = 0 \) is obvious with \( \eta_0 = 0 \).

Consider the case \( i = 1 \). Choosing \( \alpha \in \mathcal{A} \) and set \( \beta = \chi(\alpha) \). Note that though the following diagram

\[
\begin{array}{ccc}
\tilde{U}_\alpha & \xrightarrow{\tilde{f}} & \tilde{V}_\beta \\
\downarrow \tilde{f} & & \downarrow \tilde{f}' \\
\tilde{V}_\beta & \xrightarrow{\tilde{F}_V/k} & \tilde{U}_\alpha
\end{array}
\]

is not necessarily commutative at the level of schemes, but it is commutative at the level of topological spaces. Hence we can consider the difference

\[
\tilde{f}' \tilde{F}_V^* - \tilde{f}' \tilde{F}_{U/k}^* \tilde{f}'^* : \mathcal{O}_{\tilde{V}'_\beta} \to p(\tilde{f}' \tilde{F}_{U/k}^* \mathcal{O}_{\tilde{U}_\alpha}).
\]

Similar to the construction of \( h_{\alpha_0 \alpha_1} \) in Lemma 2.2, the difference induces a morphism

\[
\eta_{1\alpha}^\beta := (f' \tilde{F}_V^* - \tilde{F}_U^* \tilde{f}'^*) / p : \Omega_{Y'/k}(log D') \mid_{\tilde{V}'_\beta} \to f'_*(F_{U/k} \mathcal{O}_{\tilde{U}_\alpha}).
\]

We define

\[
\eta_1 : \Omega_{Y'/k}(log E') \to f'_*(\mathcal{O}(U', F_X\mathcal{O}_X), \alpha' \mapsto \{ \eta_{1\alpha}^\beta(\omega') \in \mathcal{O}_{\tilde{U}_\alpha} \}_{\alpha \in \mathcal{A}}.
\]

\(^2\) The permutation group \( S_i \) acts on \( \Omega^{\otimes i}_{Y'/k}(log E') \) as follows: \( \sigma \cdot \omega_1 \otimes \cdots \otimes \omega_i := \text{sgn}(\sigma) \cdot \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)} \) for \( \omega_1, \cdots, \omega_i \in \Omega_{Y'/k}(log E') \).
One can check that $\eta_1$ gives rise to the claimed morphism in (3.7) when $i = 1$. Proceeding to the case $1 < i < p$. Since

$$f^*(\varphi_Y^1)^{\cup i} = (f^*\varphi_Y^1)^{\cup i} = (\varphi_X^1 f^* + d\eta_1)^{\cup i} = (\varphi_X^1 f^* + \sum_{j=0}^{i-1} (-1)^j (\varphi_X^1 f^* + \eta_1 + (f^*\varphi_Y^1)^{\cup (i-j-1)},$$

where we set $(\varphi_X^1 f^* + \eta_1)^{\cup 0} = (f^*\varphi_Y^1)^{\cup 0} := 1$. Hence

$$\eta_i := \sum_{j=0}^{i-1} (-1)^j (\varphi_X^1 f^* + \eta_1 + (f^*\varphi_Y^1)^{\cup (i-j-1)})$$

is a claimed morphism in (3.7). This completes the proof. \qed

This lemma allows us to make the following definition.

**Definition 3.15** There is a derived functor

$$R\Gamma : (\text{Sch}^{sp}_{W_2(k), \log})^{op} \to \text{MFLC}(k)$$

defined as follows:

- Let $(\tilde{X}, \tilde{D}) \in \text{Sch}^{sp}_{W_2(k), \log}$ with closed fiber $(X, D)$, and set

$$R\Gamma(\tilde{X}, \tilde{D}) := (R\Gamma(X, (\Omega^*_X/\kappa)_{\log} D, W, \text{Fil})), R\Gamma(X', \Psi(\tilde{X}, \tilde{D}));$$

- For any morphism $\tilde{f} : (\tilde{X}, \tilde{D}) \to (\tilde{Y}, \tilde{E})$ in $\text{Sch}^{sp}_{W_2(k), \log}$ with closed fiber $f : (X, D) \to (Y, E)$, we set

$$R\Gamma(\tilde{f}) := (f^* : R\Gamma(Y, (\Omega^*_Y/\kappa)_{\log} E)) \to R\Gamma(X, (\Omega^*_X/\kappa)_{\log} D)).$$

The following easily checked lemma finishes the proof of Theorem 1.5.

**Lemma 3.16** The functors $\rho^{\text{MFLC}}_{\text{dR}} \circ R\Gamma, \rho^{\text{MFLC}}_{\text{Hig}} \circ R\Gamma$ coincide with the functors $\rho^{\text{dR}}, \rho^{\text{Hig}}$ in Theorem 1.5. Together with the Hodge filtration $\text{Fil} := \{\text{Fil}^l \rho^{\text{MFLC}}_{\text{dR}} \circ R\Gamma\}_{l}$ on $\rho^{\text{MFLC}}_{\text{dR}} \circ R\Gamma$ and an isomorphism of functors

$$\psi \circ R\Gamma : \rho^{\text{MFLC}}_{\text{Hig}} \circ R\Gamma \to \rho^{\text{MFLC}}_{\text{dR}} \circ R\Gamma,$$

Theorem 1.5 follows.

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