CAUCHY-RIEMANN EQUATIONS FOR CAYLEY NUMBERS’ FUNCTIONS

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Abstract

Since the discovery of octonions in 1843 by John T. Graves [1] we seem to be still lacking a satisfactory if any theory of octavevalued functions - satisfactory according to standard requirements or expectation from the side of a theory like a one might look for. Here is a proposal coming back to my twentieth century presentation of a perhaps nonstandard idea hoping to be coping with nonassociativity by an invention.

Key Words: Cauchy-Riemann Equations, alternative algebras

AMS Classification Numbers: 11S80,11R52,17D05

By 1828 George Green (born on 14 July 1793) had written his first and most important paper entitled ”An essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism”.

In this essay, which runs to nearly 80 pages, George Green had formulated what today is called - Greens’ Theorem. This theorem is used to derive immediately the Cauchy Theorem

\[ \oint_{\Gamma} f(z) \, dz = 0, \]

as it leads to Cauchy-Riemann equations involved there.

So we now may celebrate the 180-th anniversary of this great achievement.

In this note - (presented at ”The Polish-Mexican Seminar on Generalized Cauchy -Riemann Structures and Surface Properties of Crystals” - Kazimierz Dolny; August 98 ) - - one proposes the extension of the analyticity notion so that it includes also octonions and in general all composition algebras [2].

We also indicate the possibility of introducing the notion of analyticity for other algebras (suggested by A. Z. Jadczyk in private communication).
The major aim of the note is to formulate the analyticity notion for an octonion algebra in a manner which would enable one to reestablish the main theorems already known for quaternions and for Clifford algebra valued regular functions.

I. A composition algebra $A$ is not necessarily associative, however, if it is, then by means of the Cayley-Dickson procedure on can construct a new composition algebra $(A, \alpha)$ which is a direct sum of vector spaces $A \otimes A$ with the usual addition and multiplication by a real number while the internal product is defined by

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 + \alpha y_2 y_1, x_1 y_2 + y_1 x_2), \quad \alpha \neq 0, a \in \mathbb{R}$$

with the standard notation for conjugate elements in $A$. Conjugation in $(A, a)$ is defined by

$$\overline{(x, y)} = (\overline{x}, -\overline{y}).$$

In the following we restrict our discussion of analyticity concept to the more familiar case of ordinary composition algebras, i.e. complex numbers, quaternions and octonions which we shall call briefly just composition algebras, this being justified by the fact that most of our consideration are valid for the general case of any composition algebra. Let us start with a unified formulation of the algebras of complex numbers $\mathbb{C}$, quaternions $\mathbb{Q}$ and octonions $\Theta$.

From now on Greek indices $\mu, \nu, y, \sigma, \ldots$ will run from 0 to 1 for $\mathbb{C}$, from 0 to 3 for $\mathbb{Q}$ and from 0 to 7 for $\Theta$, while Latin indices $i, j, k, \ldots$ shall take correspondingly values 1, or 2, 3, or 4, 5, 6, 7 (summation convention is used). The algebras $\mathbb{C}$, $\mathbb{Q}$, $\Theta$ can be defined via

$$e_\mu e_\nu = c^\sigma_{\mu\nu} e_\sigma$$

where

$$c^\sigma_{ij} = -\delta_{ij} \delta^\sigma_0 + \epsilon_{ij} \delta^\sigma_k, \quad c^\sigma_{0\mu} = c^\sigma_{\mu0} = \delta^\sigma_\mu$$

For $i = j = k = 1$ we have $\mathbb{C}$, for $i, j, k = 1, 2, 3$ we get $\mathbb{Q}$ and for $i, j, k = 1, \ldots, 7$ we shall obtain the algebra of octonions (see Fig.1) if one defines for which triples $(i, j, k)$ $\epsilon_{ijk} = +1$. In the case of octonions we must add to (2) the following conditions

$$\epsilon_{123} = \epsilon_{145} = \epsilon_{176} = \epsilon_{246} = \epsilon_{347} = \epsilon_{536} = \epsilon_{725} = +1.$$ (3)

The rules:

$$e_1 e_3 = e_2, e_2 e_6 = e_4, e_4 e_5 = e_1, e_3 e_6 = e_5, e_1 e_7 = e_6, e_2 e_7 = e_5, e_4 e_7 = e_3.$$
Let now $A$ be any of the algebras $\mathbb{C}$, $\mathbb{Q}$ or $\Theta$, then $x \in A$ can be represented as

$$x = x^\mu e_\mu, \quad x_\mu \in \mathbb{R},$$

where

$$x^\mu = \frac{1}{2} (x e_\mu, + e_\mu x) \tag{4}$$

and

$$e_0 = \overline{e}_0, \quad e_i = -\overline{e}_i$$

The trace, a linear mapping of $A$ into $\mathbb{R}$, is defined then by

$$\text{Tr} \ e_i = 0, \quad \text{Tr} \ e_0 = N, \tag{5}$$

where $N = \dim A$. Using this trace mapping one may introduce a scalar product in $A$

$$x, y \in A, \quad \langle x | y \rangle = \frac{1}{N} \text{Tr} (\overline{x} y), \tag{6}$$

which has the property

$$\langle x y | x y \rangle = \langle x | x \rangle \langle y | y \rangle \tag{7}$$

Using the definition $[2]$ of the $A$ algebra structure constants one may derive the following properties of $c^\sigma_{\mu \nu}$:

$$c^0_{\mu \nu} = g_{\rho \nu} \quad \text{where} \quad g_{\rho \nu} = \begin{cases} \delta_{0 \nu}, \quad \rho = 0 \\ -\delta_{i \nu}, \quad \rho = i \end{cases} \tag{8}$$
\[ c_{\sigma 0}^\rho = N, \quad c_{\sigma k}^\rho = 0, \quad (9) \]

\[ c_{ij}^k = \epsilon_{ijk}, \quad (10) \]

\[ c_\sigma^{\mu \nu} g_{\rho \sigma} = c_\sigma^{\rho \mu} g_{\sigma \nu}, \quad (11) \]

or

\[ e_\mu e_\nu = 2 \delta_{\mu \nu} e_0. \quad (12) \]

With the help of (8-11) one can prove an important lemma. For that to do let us introduce a differential linear operator of the form

\[ L = L^\sigma e_\sigma \]

acting on \( A \) and a mapping

\[ U : A \rightarrow A \]

for which \( U^\mu(x) (U(x)) = U^\mu(x)e_\mu \) are real differentiable functions [2].

**Lemma 1**

\[ \text{Tr} \{ L(U_q) \} = \text{Tr} \{ (LU)_q \}, \quad \forall q \in A. \quad (13) \]

Another useful lemma can be established using the relations (8-10) [2].

**Lemma 2**

\[ c_\sigma^{\nu} \partial^\sigma \equiv 2 \delta_0^\nu \partial_\sigma - c_\sigma^{\nu \mu} \partial_\sigma, \quad (14) \]

where \( \partial^\sigma \equiv \partial/\partial x_\sigma \).

**II.** In this section we construct a matrix representation of \( A = C, Q, \Theta \) with usual addition and multiplication of matrices as operations in \( A \). The matrices will have operator entries as one of specifications of \( A \) is nonassociative. In the associative cases the operator entries simply become matrices. It is well known that \( C \) can be isomorphically represented by a set of matrices of the form

\[ C \in z, \quad z = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \quad x, y \in \mathbb{R} \quad (15) \]
The conjugation $\sigma : \mathbb{C} \to \mathbb{C}$ can also be represented via matrix multiplication in the following way:

$$\sigma(z) = jzj, \quad \text{where} \quad j^2 = id, \quad jz = -zj$$

(16)

due to commutativity of $\mathbb{C}$. The $j$ matrix is of the form

$$j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

(17)

To work out a similar construction for quaternions it is sufficient to notice the essence of the above representation which was the Cayley-Dickson procedure applied to $\mathbb{R}$. $\mathbb{Q}$ can be then isomorphically represented as a set of matrices of the form

$$q \in \mathbb{Q}, \quad q = \begin{bmatrix} z_1 & -z_2j \\ z_2j & z_1 \end{bmatrix}, \quad z_1, z_2 \in \mathbb{C}$$

(18)

This form again is a manifestation of the Cayley-Dickson procedure. The conjugation $\sigma(q) = \overline{q}$ also can be realized by

$$\sigma(q) = \epsilon \circ q \circ \epsilon, \quad \epsilon^2 = id, \quad \epsilon \cdot q = \overline{q} \cdot \epsilon$$

but this time, because of noncommutativity of $\mathbb{Q}$, $\epsilon \circ q \circ \epsilon$ does not denote simply matrix multiplication as we must have

$$\epsilon \circ (q_1 \circ q_2) \circ \epsilon = (\epsilon \circ q_2 \circ \epsilon)(\epsilon \circ q_1 \circ \epsilon).$$

However, it is enough to say that the $\epsilon$ operator acts by matrix multiplication under the condition that this multiplication reverses the order of the product of $q$-matrices whenever they are sandwiched between two $\epsilon$ operators (matrices). With this in mind

$$\epsilon = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}.$$  \hspace{1cm} (19)

Let us identify

$$z_1 \in \mathbb{Q}, \quad z_1 = \begin{bmatrix} z_1 & 0 \\ 0 & z_1 \end{bmatrix}, \quad z_1 \in \mathbb{C}.$$  

Then

$$q \in \mathbb{Q}, \quad q = z_1 + z_2j,$$

(20)

where
Now the action of $\epsilon$ on $Q$ can be defined as follows:

$$\epsilon \circ q \circ \epsilon = \epsilon z_1 \epsilon + \epsilon j z_2 \epsilon \equiv \overline{q} \tag{21}$$

and as

$$\epsilon j = -je, \quad zj = j\overline{z}$$

one has

$$\epsilon \circ q = \overline{q} \circ \epsilon.$$

The natural representation of imaginary units $e_1, e_2, e_3 \in Q$ is then given by

$$e_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -j \\ -j & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -ij \\ ij & 0 \end{bmatrix} \tag{22}$$

Note that the product of $q_1, q_2 \in Q$ is realized by the usual matrix multiplication ($q_1 \circ q_2 = q_1 q_2$) while $\epsilon \notin Q$.

$\epsilon$ is a specific operator acting on $Q$, $\epsilon$ is in a sense a "square root" of the conjugation operator $s$ and can be thought of as the matrix (19) but then one must remember that though it acts by matrix multiplication ..., it reverses the order of $q$-matrices - if sandwiched between two $\epsilon$ matrices. Similarly to previous cases, octonions can be represented by

$$\theta \in \Theta, \quad \theta = \begin{bmatrix} q_1 & -q_2 \epsilon \\ q_2 \epsilon & q_1 \end{bmatrix} \equiv q_1 + q_1 E, \quad \tag{23}$$

where

$$E = \begin{bmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{bmatrix}, \quad q_1, q_2, E \in \Theta.$$

The representation is given once the multiplication law in $\Theta$ is defined. It is given by

$$(a) \quad (q_1 \epsilon) \circ (q_2 \epsilon) = \epsilon \circ (\overline{q}_1 q_2) \circ \epsilon = \overline{q}_2 q_1,$$

$$(b) \quad (q_1 \epsilon) \circ q_2 = (q_1 \overline{q}_2) \epsilon,$$

$$(c) \quad q_1 \circ (q_2 \epsilon) = (q_2 q_1) \epsilon$$

$$(d) \quad \epsilon \cdot \epsilon = \epsilon \epsilon = 1. \tag{24}$$
The rules (24) can be derived (for that form of matrix representation (23)) from the Moufang identities [3]. For example (24a) can be derived from

\[ x(yz)x = (xy)(zx) \quad \forall x, y, z \in A; \quad \text{where } A \text{— alternative algebra.} \]

To end up: multiplication in \( \Theta \) is just matrix multiplication where the rules of dealing with expressions involving \( \epsilon \) symbols are given by (24). These rules apply to any specification of \( A \). The “square root” \( \epsilon \) of the conjugation operator \( \sigma \) (with respect to (24) multiplication) is given by the matrix

\[
\epsilon = \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix}
\]

and again \( \epsilon \notin \Theta \); it is an operator acting on \( \Theta \) similarly as \( E \) does on \( Q \). In the natural representation, generators of \( \Theta \) have the form

\[
e_1 = \begin{bmatrix} e_1 & 0 \\ 0 & e_1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} e_2 & 0 \\ 0 & e_2 \end{bmatrix}, \quad e_3 = \begin{bmatrix} e_3 & 0 \\ 0 & e_3 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{bmatrix},
\]

\[
e_5 = \begin{bmatrix} 0 & -e_1 \epsilon \\ e_1 \epsilon & 0 \end{bmatrix}, \quad e_6 = \begin{bmatrix} 0 & -e_2 \epsilon \\ e_2 \epsilon & 0 \end{bmatrix}, \quad e_7 = \begin{bmatrix} 0 & -e_3 \epsilon \\ e_3 \epsilon & 0 \end{bmatrix}.
\]

Using this representation one proves the following [2]

**Lemma 3**

\( \forall \Theta, u \in A \quad \Theta(\overline{\Theta}u) = (\Theta\overline{\Theta})u. \)

Introducing then the operators defined on functions on \( Q \)

\[
\partial_{\overline{\mu}} \equiv e^\mu \partial_\mu
\]

where \( Q \ni q = x^\mu e_\mu \) and \( \partial_q = \overline{e}^\mu \partial_\mu \) we have for octonions

\[
\partial_\theta = \begin{bmatrix} \partial_{\overline{q}_1} & \partial_{\overline{q}_2} \epsilon \\ -\partial_{\overline{q}_2} \epsilon & \partial_{\overline{q}_1} \end{bmatrix}, \quad \partial_{\overline{\mu}} = \begin{bmatrix} \partial_{\overline{\gamma}_1} & \partial_{\overline{\gamma}_2} \epsilon \\ -\partial_{\overline{\gamma}_2} \epsilon & \partial_{\overline{\gamma}_1} \end{bmatrix},
\]

\[
\partial_{\overline{\mu}} \circ \partial_\theta = \partial_\theta \circ \partial_{\overline{\mu}} = \begin{bmatrix} \diamondsuit & 0 \\ 0 & \diamondsuit \end{bmatrix}.
\]

where \( \square_8 \equiv \diamondsuit_\infty = e^\mu \partial_\mu \) and the \( \infty \)-sign stands for horizontal 8, because of my editorial limitations.
III. The Cauchy-Riemann (C-R) equations for $C$ can be written in the form
\[
\begin{bmatrix}
\partial_0 - \partial_1 \\
\partial_1 - \partial_0
\end{bmatrix}
\begin{bmatrix}
u_0 & -u_1 \\
u_1 & u_0
\end{bmatrix} = 0
\]
or $\partial_\tau U = 0$, where $\partial_\mu = \partial/\partial x^\mu$, $U = U^\mu e_\mu$; $\mu = 0, 1$.

This definition of analytic function $U$ can be extended to any algebra $A = C, Q, \Theta$. Let $U$ be an $A$-valued function on $A$ with $U^\mu(x_0, x_1, ..., x_{N-1})$ functions differentiable with respect to $x_\nu$. The $U$ can be represented as
\[
U = \begin{bmatrix} a & -b\alpha \\
b\alpha & a \end{bmatrix}
\]
with $\alpha = 1, j, \epsilon$ correspondingly to the chosen case; $A = C, Q, \Theta$.

**Definition 1** $U$ is called left $A$-analytic iff
\[
\partial_\tau U = 0 \quad \text{or} \quad \begin{bmatrix}
\partial_{\tau_1} - \partial_{\tau_2} \epsilon \\
-\partial_{\tau_2} \epsilon - \partial_{\tau_1}
\end{bmatrix}
\begin{bmatrix} a & -b\alpha \\
b\alpha & a \end{bmatrix} = 0 \quad (29)
\]

**Definition 2** $U$ is called left $A$-antianalytic iff
\[
\partial_\gamma U = 0 \quad \text{or} \quad \begin{bmatrix}
\partial_{\gamma_1} - \partial_{\gamma_2} \epsilon \\
-\partial_{\gamma_2} \epsilon - \partial_{\gamma_1}
\end{bmatrix}
\begin{bmatrix} a & -b\alpha \\
b\alpha & a \end{bmatrix} = 0 \quad (30)
\]

As a conclusion from Section II we get [2]

**Lemma 4** An $A$-analytic or $A$-antianalytic function is a harmonic function, i.e.
\[
\diamondsuit_N U = 0.
\]

There exists a lot of $A$-analytic functions. The infinite number of examples is given by simple combinations of $B$-analytic and $B$-antianalytic functions where $A = (B, -1)$ (see Section I). The Cayley-Dickson procedure inherent in this definition allows us to relate octonion analyticity to quaternion or via quaternion to complex, "usual" analyticity. Let us consider in more detail octonion-antianalyticity as an example. The octonion function can be written in three equivalent forms

\[
U = a + bE, \quad a, b, E \in \Theta,
\]
\[
U = A_1 + B_1j + (A_2 + B_2j)E, \quad A_1, A_2, B_1, B_2 \in C,
\]
\[
U = U^\mu e_\mu, \quad U^0, U^1, ..., U^\mu \in R.
\]
Introducing the notation

\[
\partial_b = \begin{bmatrix}
\frac{\partial y_1}{\partial y_1} & \frac{\partial y_2}{\partial y_1} \\
-\frac{\partial y_2}{\partial y_2} & \frac{\partial y_1}{\partial y_2}
\end{bmatrix}, \quad \partial_{q_1} = \begin{bmatrix}
\frac{\partial y_1}{\partial y_1} & \frac{\partial y_2}{\partial y_1} \\
-\frac{\partial y_2}{\partial y_2} & \frac{\partial y_1}{\partial y_2}
\end{bmatrix}, \quad \partial_{\tau_2} = \begin{bmatrix}
\frac{\partial y_2}{\partial y_2} & \frac{\partial y_1}{\partial y_2} \\
-\frac{\partial y_1}{\partial y_1} & \frac{\partial y_2}{\partial y_1}
\end{bmatrix}, \quad \partial_{\tau_1} = \begin{bmatrix}
\frac{\partial y_1}{\partial y_1} & \frac{\partial y_2}{\partial y_1} \\
-\frac{\partial y_2}{\partial y_2} & \frac{\partial y_1}{\partial y_2}
\end{bmatrix}
\]

we can write octonionic antianalyticity C-R equations in three equivalent forms:

**Quaternionic form:**
\[
\partial_{q_1}a + b\partial_{\tau_2} = 0 \\
-\partial_{\tau_2} + b\partial_{q_1} = 0
\]
where \(\partial^j\) means action to the left.

**Complex form:**
\[
\partial_{q_1}A_1 + \partial_{q_2}B + \partial_{\tau_2} = 0 \\
\partial_{q_1}B_1 = 0 \\
\partial_{q_2}A_1 = 0 \\
\partial_{q_1}B_2 = 0 \\
\partial_{q_1}B_2 = 0
\]

**Real form:**
\[
\begin{bmatrix}
\partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 & \partial_7 \\
-\partial_1 & \partial_0 & \partial_3 & -\partial_2 & \partial_5 & -\partial_4 & -\partial_7 & \partial_6 \\
-\partial_2 & -\partial_3 & \partial_0 & \partial_1 & \partial_6 & \partial_7 & -\partial_4 & -\partial_5 \\
-\partial_3 & \partial_2 & -\partial_1 & \partial_0 & \partial_7 & -\partial_6 & \partial_5 & -\partial_4 \\
-\partial_4 & -\partial_5 & -\partial_6 & -\partial_7 & \partial_0 & \partial_1 & \partial_2 & \partial_3 \\
-\partial_5 & \partial_4 & -\partial_7 & \partial_0 & -\partial_1 & \partial_0 & -\partial_3 & \partial_2 \\
-\partial_6 & \partial_7 & -\partial_5 & -\partial_2 & -\partial_3 & \partial_0 & -\partial_1 & \partial_0 \\
-\partial_7 & -\partial_6 & \partial_5 & \partial_4 & -\partial_3 & -\partial_2 & \partial_1 & \partial_0
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6 \\
U_7
\end{bmatrix} = \bar{O}
\]

Left octonion-analyticity C-R eqs. are obtained by replacing \(a\) by \(-a\); in (31). Correspondingly, complex and real forms of quaternion-analyticity conditions in the analogous notation \((u = u^\mu e_\mu)\)

\[
U = A + Bj, \quad y = x_0 + x_1i, \quad q = y + zj \quad z = x_2 + x_3i,
\]
are given by:

\[
\partial_{q_2}A - \partial_{\tau_2}B = 0, \quad \partial_{q_1}B - \partial_{\tau_1} = 0, \quad \text{(complex form)}
\]
\[
\begin{pmatrix}
\partial_0 & -\partial_1 & -\partial_2 & -\partial_3 \\
\partial_1 & \partial_0 & -\partial_3 & \partial_2 \\
\partial_2 & \partial_3 & \partial_0 & -\partial_1 \\
\partial_3 & \partial_2 & \partial_1 & \partial_0 \\
\end{pmatrix}
\begin{pmatrix}
U_0 \\
U_1 \\
U_2 \\
U_3 \\
\end{pmatrix} = 0 \quad \text{(real form).} \quad \text{(35)}
\]

The above formulation of analyticity coincides for \( A = \mathbb{C} \) with Cauchy-Riemann and for \( A = \mathbb{Q} \) with Fueter’s analyticity.

To see the latter we shall write C-R equations for quaternions in another form.

Let us introduce the notation

\[ \vec{U} = (U_1, U_2, U_3), \quad \vec{V} = (\partial_1, \partial_2, \partial_3) \]

for quaternions and

\[ \vec{U} = (U_1, U_2, ..., U_7), \quad \vec{V} = (\partial_1, \partial_2, ..., \partial_7) \]

for octonions. Then (35) can be written in the form

\[ \partial_0 U_0 = \vec{V} \cdot \vec{U}, \quad \partial_0 \vec{U} = -\vec{V} U_0 - \vec{V} \times \vec{U}, \quad \text{(36)} \]

while (33) is equivalent to

\[ \partial_0 U_0 = \vec{V} \cdot \vec{U}, \quad \partial \vec{U} = -\vec{V} U_0 - \vec{V} \otimes \vec{U}, \quad \text{(37)} \]

where the "octonionic vector product \( \otimes \)" is defined by

\[ (\vec{V} \otimes \vec{U})_j = \sum_{i,k} \epsilon_{jki} \partial_k U_i \quad \text{(38)} \]

with \( \epsilon_{jki} \) satisfying (3).

One component of the \( \otimes \)-vector product is an algebraic sum of six terms because the \( (k,i) \) pair index takes six values for an index \( j \) being fixed.

A more straightforward real form of C-R equations for \( \mathbb{C}, \mathbb{Q} \) or \( \Theta \) is the equation

\[ C^\rho_{\nu \sigma} \partial^\nu U^\sigma = 0 \quad \text{(39)} \]

equivalent to (29). This however does not exhibit the structure originating from the Cayley-Dickson procedure. In the representation of \( e_\mu \), we have given before, \( \partial_{\mu} = e^\mu \partial_\mu \) acting as a linear operator on \( A \), can be represented by a matrix in a \( \{e_\mu\} \) basis. In view of Lemma 2 of Section I the matrix elements of this operator are given by the expression

\[ (\partial_{\mu})^\mu_{\nu} = C^\mu_{\nu \sigma} \partial_\mu - C^\sigma_{\nu \mu} \partial_\sigma. \]
In this section we introduce the definition of analyticity for any algebra with unit element as was proposed by A. Z. Jadczyk (private communication); then we show that for an ordinary composition algebras it is exactly the same notion as the one we have introduced in previous sections. Let \( A \) be now any algebra with unit element and let \( U \) be a differentiable mapping \( U : A \rightarrow A \). The derivative of \( U \) at \( x \in A \) is then an \( \mathbb{R} \)-linear transformation \( U'_x \) of \( A \) and can be written

\[
U'_x(h) = h^\sigma \partial_\sigma U^\mu e_\mu, \tag{40}
\]

where \( A \ni h = h^\sigma e_\mu, U^\mu e_\mu = U \in A, U^\nu \in \mathbb{R} \) and \( \{e_\nu\} \) is a basis of \( A \).

C-R equations are conditions on such a linear transformation \( U'_x \): conditions related to the algebraic structure of \( A \). The requirement of \( A \)-linearity though seemingly natural is very naive and yields a notion void of content already for quaternions [2].

A. Z. Jadczyk proposed a weakened condition. Let \( f \) be any linear mapping \( f : A \rightarrow A \). A trace of that mapping is then defined by

\[
\text{Tr} f = f(e_\mu)^\mu, \tag{41}
\]

where \( \{e_\mu\} \) is a certain basis of \( A \) and for \( a \in A \), \( a^\mu \) denotes the \( \mu \)-th coordinate of \( a \) in a basis \( \{e_\mu\} \). Then as a generalization of C-R equations he proposes the equality of traces of the following two linear mappings

\[
f_{1,q} : A \rightarrow A, \quad f_{1,q}(a) = U'_x(qa), \quad f_{2,q} : A \rightarrow A, \quad f_{2,q}(a) = U'_x(q)a, \quad q, a \in A. \tag{42}
\]

Generalized C-R eqs. for an algebra \( A \) then have the form

\[
\forall q \in A \quad \{U'_x(qe_\mu)\}^\mu = \{U'_x(q)e_\mu\}^\mu \tag{43}
\]

or equivalently:

\[
\{U'_x(e_\mu e_\nu)\}^\nu = \{U'_x(e_\mu)e_\nu\}^\nu. \tag{44}
\]

One then easily finds the C-R eqs. in terms of structure constants

\[
C^\sigma_{\mu\nu} \partial_\sigma U^\nu = C^\sigma_{\nu\sigma} \partial_\mu U^\nu \tag{45}
\]

In what follows we show that this definition for \( A = \mathbb{C}, \mathbb{Q}, \Theta \) is equivalent to [39].

At first let us notice that the definition [41] of the trace of a linear mapping for composition algebras coincides with that of Section I because of [39]. Secondly, [45] educes for composition algebras to
\[ C^\sigma_{\mu\nu} \partial_\sigma U^\nu = N \partial_\mu U^0 \] (46)

and this definition is equivalent to ours because of Lemma 2 of Section I. The fact that we have a factor \( N \) instead of 2 on the right-hand side of (46) is not important in view of the lemma (due to A. Z. Jadczyk).

**Lemma 5** Let \( F_R \) denote the vector space of functions \( f : A \rightarrow A \) satisfying the \( \kappa - C - R \) eqs.

\[ C^\sigma_{\mu\nu} \partial_\sigma f^\nu = \kappa \partial_\mu f^0, \] (47)

where \( \kappa, \kappa' \in \mathbb{R} \) and \( A \) is a composition algebra. Then for \( \kappa, \kappa' \in \mathbb{R}, \kappa \kappa' - 1 \neq 0 \) there exists an isomorphism \( T_{\kappa \kappa'} : F_\kappa \rightarrow F_{\kappa'} \).

**Proof:** The isomorphism is defined by \( (T_{\kappa \kappa'} f)^i = f^i, (T_{\kappa \kappa'} f)^0 = \mu f^0 \), where \( \mu = (1 - \kappa)/(1 - \kappa \kappa') \). One may check that \( T_{\kappa \kappa'} f \) satisfies (47) with \( \kappa' \) instead of \( \kappa \) (use 9).

**Remark.** For \( \kappa = N \) as in (46) one should take \( \mu = (N - 1)/(2N - 1) \) to get \( \kappa' = 2 \) in (47) which then coincides with (39) because of Lemma 2, Section I.

We have already argued in Sec. III that there are many examples of \( A \)-analytic functions, where \( A \) is a composition algebra. However, this set of functions does not include \( U(x) = x^n(n > 1) \) functions (except for \( N = 2 \)) although it does include other \( R \)-homogeneous functions of degree \( n \) and these \( A \)-analytic homogeneous functions play the role similar to \( x^n \) in complex analytic functions theory.

To illustrate the above statement we quote [1] the following

**Lemma 6** Let \( x \in A = C, Q, \Theta \); \( U(x) = x^2 \); then \( U \) satisfies C-R eqs. iff \( N = 2 \).

To end up let us make three remarks.

1. There exists a formulation of C-R eq. based on the analogy with Clifford algebra product. Let \( U \) be a function on \( A = Q, \Theta(U^\mu e_\mu = U) \). We shall call \( U_0 \) the scalar part and \( \vec{U} = (U_1, ..., U_{N-1}) \) the vector part of \( U \) and we shall represent \( U \) by a pair \( U = (U_0, \vec{U}) \) and similarly \( \partial_x = (\partial_0, \vec{\nabla}) \), \( x \in A \).

Then \( \kappa - C - R \) eqs. for \( \kappa = 2 \) can be written in the form

\[ \left( \partial_0, \vec{\nabla} \right) \cdot \left( U_0, \vec{U} \right) = 0, \] (48)

where
\[(a_0, \vec{a}) \circ (b_0, \vec{b}) = (a_0 b_0 - \vec{a} \vec{b}, a_0 \vec{b} + \vec{a} b_0 + \vec{a} \otimes \vec{b});\]

\(\otimes\) denotes the octonionic vector product coinciding with the usual one for the quaternionic subalgebra of \(\Theta\). One easily notices that eqs. (37-38) are just specifications of (48). C-R eq. (46) can also be cast in the form (48) or equivalently with slight modification, namely

\[
\left( \partial_0, \vec{\nabla} \right) \circ \left( (N - 1)U_0, \vec{U} \right) = 0.
\]  

(49)

2. If for a linear transformation \(L\) on \(A\) there exists \(a \in A = C, Q, \Theta\) such that

\[
\forall u \in A \quad L(u) = au
\]

then of course

\[
\forall u, q \quad \text{Tr} \{L(uq)\} = \text{Tr} \{(Lu)q\}
\]

(51)

Whether (51) implies for \(L\) representability in the form (50) or not is an open question at present. A positive answer would provide us with an algebraic interpretation of the \(A\)-analyticity concept as introduced via (43) C-R eqs.

3. The formulation (43) of C-R eqs. (equivalent to ours for composition algebras) is appropriate for extension to any algebra with unit element though the question immediately arises, whether this extension is equivalent in some sense to the straightforward extension of (48).

Finally, let us remark that the Fueter analyticity is a special case of Clifford analyticity. For a Clifford algebra the \(\partial_0\) operator may be regarded as a kind of square root of Laplace operator with respect to Clifford algebra multiplication. In this sense \(\partial_0\) and \(\partial_{\Theta}\) operators are the "square roots" of the "\(\circ\)" product introduced in Section III. This product becomes a Clifford one for \(A = Q\).

Acknowledgements

This ten years later article was born in TeX due to kindness and abilities of my undergraduate Student of computer science Maciej Dziemiańczuk. Thanks and glory Unto Him.

The author thanks Prof. J. Lawrynowicz for invitation to participate in "The Polish-Mexican Seminar on Generalized Cauchy-Riemann Structures and Surface Properties of Crystals" - Kazimierz Dolny; August 98, and for His kind hospitality there as well as for discussions on the subject.
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