Quark-Monopole Potentials from Supersymmetric $SL(3, R)$ Deformed IIB Supergravity

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Abstract

We recompute the quark-monopole potential from supersymmetric $SL(3, R)$ deformation of IIB supergravity background dual to deformed Coulomb branch flow of the $\mathcal{N} = 4$ super Yang-Mills theory. The marginal deformations strengthen the Coulombic attraction between quarks and monopoles.
1 Introduction

The $SL(3, R)$ symmetry of type IIB supergravity on a two-torus has been applied to finding new gravity solutions which correspond to marginal deformation of $\mathcal{N} = 4$ super Yang-Mills theory [1]. This method can be used to any solution that has an isometry group which contains $U(1) \times U(1)$. If there exists an extra $U(1)_R$-symmetry in addition to this symmetry, then the deformed solution preserves $\mathcal{N} = 1$ supersymmetry.

In particular, the gravity description of deformed Coulomb branch renormalization group flow of $\mathcal{N} = 4$ super Yang-Mills theory with $SO(2)^3$ global symmetry has been discussed in [2]. The UV limit of the dual gauge theory is the Leigh-Strassler deformation [3] of $\mathcal{N} = 4$ super Yang-Mills theory. This global symmetry corresponds to $U(1) \times U(1)$ global symmetry generated by two angles of two torus and $U(1)_R$ symmetry generated by remaining angle of internal space.

In [4], the Coulombic potential between quark and anti-quark has been reconsidered for supersymmetric $SL(3, R)$ deformed type IIB theory dual to the deformed Coulomb branch flow of $\mathcal{N} = 4$ super Yang-Mills theory with $SO(4) \times SO(2)$ global symmetry. Two of three parameters determining the shape of D3-branes become identical to each other for purely radial string configuration leading to symmetry enhancement $SO(2)^3 \to SO(4) \times SO(2)$. One of the main results is that for certain part of the moduli space, the $\sigma$ deformations induce a transition from Coulombic attraction between quarks and anti-quarks to linear confinement. For the undeformed solution where $\sigma = 0$, there was no confining behavior but as the distance becomes larger, this transition to a regime of linear confinement occurs where the scale of confinement increases with $\sigma$.

Now it is natural to compute the potential between two magnetic monopoles in terms of D-string worldsheet and the result for conformal theory will be the same as the one in [5, 6] but with $g_{YM} \to 4\pi/g_{YM}$ or $g \to 1/g$. One can also compute the interaction between a quark and a magnetic monopole. In this case, the fundamental string ending on a quark will attach to the D-string ending on a magnetic monopole and they will connect to form a $(1, 1)$ string which will go into the horizon [7]. In particular, when the string coupling $g$ is small, the D-string is very rigid and the fundamental string will end on D-string perpendicularly. Then the solution for the fundamental string will be half of the solution for two fundamental strings and leads to $1/4$ in the potential [4] in $AdS_5 \times S^5$ background.

In this paper, we extend the result of [4] to the case of a massive quark and monopole by replacing an anti-quark with a monopole or generalize the result of [7] corresponding to $AdS_5 \times S^5$ background of type IIB theory to the case of computation in the supersymmetric
SL(3, R) deformed type IIB theory by considering more general background. We will see the marginal deformations of \( \mathcal{N} = 4 \) super Yang-Mills theory strengthen the Coulombic attraction between quarks and monopoles. On the gauge theory side, similar discussion can be found in [8].

2 Quark-monopole potentials revisited

We study the effect of \( \sigma \) deformation on the quark and monopole potential for (non)conformal field theories. The quark and monopole potential has been studied for \( \mathcal{N} = 4 \) super Yang-Mills theory in the large \( N \) limit [7] of \( \text{AdS}_5 \times S^5 \) type IIB background. Now we extend this background to supersymmetric \( \text{SL}(3, R) \) deformed type IIB theory dual to the deformed Coulomb branch flow of \( \mathcal{N} = 4 \) super Yang-Mills theory. The UV limit of undeformed (\( \sigma = 0 \)) theory asymptotes to \( \text{AdS}_5 \times S^5 \) where the conformal symmetry is regained.

The supergravity dual of the deformed Coulomb branch flow in the string frame is [2] [4]

\[
ds^2 = \alpha' \sqrt{Hf} R^2 \left( \frac{r^2}{fR^4} dx^2 \mu + \frac{dr^2}{fr^2 L_1 L_2 L_3} + R^2 ds_5^2 \right) \tag{2.1}
\]

where the length scale is \( R^4 = g_Y^2 M N \), the metric of deformed five-sphere \( \tilde{S}^5 \) depends on the internal coordinates \( \alpha, \theta \) and the various functions are given by

\[
L_i = 1 + \frac{\ell_i^2}{r^2}, \quad H = 1 + \hat{\sigma}^2 \frac{f}{gh} s_\alpha^2, \quad \hat{\sigma} \equiv \sigma R^2 / 2.
\]

The three parameters \( \ell_i \) specify the ellipsoidal shape of D3-brane distribution [9]. We assume that the classical supergravity description is valid with the same spirit of [4]. The explicit form for functions \( f, g \) and \( h \) is given in [2] [3] but we do not need them in this paper. For the purely radially oriented string configuration, they have simple expression. Moreover, \( H \) and \( f \) can be written in terms of \( L_i \)'s. See the footnote 1. The metric of deformed five-sphere depends on the modulus of \( \beta = \gamma - \tau_s \sigma \) where \( \gamma \) and \( \sigma \) are real deformation parameters and \( \tau_s \) is a complex structure.

Now a probe D3-brane has been taken at large distance (\( r = \infty \)) and let us consider the behavior of string configurations ending on this brane. The \( N \) D3-branes are located at \( r = 0 \). To find a static configuration we set \( \tau = T \) and \( \sigma = x \) where \( x \) is a direction along D3-branes. Then string action can be simplified to [4]

\[
S = \frac{T}{2\pi} \int dx \sqrt{H} \sqrt{\frac{r^4}{fR^4} + \frac{r^2}{fL_1 L_2 L_3}} \tag{2.2}
\]
where $t$ denotes a derivative with respect to $x$ and $T$ denotes the time interval. We put the
the solution to the equations of motion : $\alpha = \pi/2, \theta = \pi/4$ and $\ell_2 = \ell_3$. When there are
no D3-brane distributions ($f = 1 = L_i$) and there are no $\sigma$ deformations ($H = 1$), then the
above geometry is exactly the same as the conformal case of $\mathcal{N} = 4$ super Yang-Mills theory
[6, 5]. For the deformed solution with no D3-brane distributions, as in the case of quark-
anti quark potential, the conformal factor $H$ is a constant ($H = 1 + \hat{\sigma}^2$). Then the Wilson
loop computation implies that the marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theory
enhance the Colombic attraction.

There exist a heavy quark at $x = 0$ and a heavy monopole at $x = L \ [7]$. These transform
under the fundamental representation of $SU(N)$. In the $(x, r)$-plane, a fundamental string(or
$(1, 0)$ string) is attached to the D3-brane at $(0, \infty)$ and a D-string(or $(0, 1)$ string) is attached
to D3-brane at $(L, \infty)$. Moreover, there should be another $(1, 1)$ string attached to the other
strings at $(\Delta L, r_0)$ and the other end of this $(1, 1)$ string is attached to one of the D3-branes
at $(\Delta L, 0)$. The detailed configuration was given in [7] or similar configuration where there
exists a shift by $\Delta L$ in the $x$-axis is given in Figure 1.

As for the $(1, 0)$ and $(0, 1)$ strings, the minimization of the action for each string $^1$ satisfies

$$\sqrt{\frac{H}{f}} \frac{r^4}{\sqrt{\frac{r^4}{R^4} + \frac{r^2}{L_1 L_2}}} = \sqrt{\frac{H_i}{f_i} r_i^2 R^2}, \quad i = 1, 2 \quad (2.3)$$

where $i = 1$ is for the fundamental string $(1, 0)$ while $i = 2$ is for the D-string $(0, 1)$ and
$H_i \equiv H(r_i)$ and $f_i \equiv f(r_i)$. The $r_i$’s are determined later. Using these equations (2.3), one
can write down $x$ in terms of $r$. The solutions for the lengths of two strings are

$$\Delta L = R^2 \int_{r_0}^{r_1} \frac{dr}{r^2 \sqrt{L_1 L_2} \sqrt{\frac{f_i H r^4}{f_i H r_i^4} - 1}},$$

$$L - \Delta L = R^2 \int_{r_0}^{r_2} \frac{dr}{r^2 \sqrt{L_1 L_2} \sqrt{\frac{f_2 H r^4}{f_2 H r_2^4} - 1}}. \quad (2.4)$$

By adding these, one gets the length $L$ which is a function of $r_0, r_1$ and $r_2$ as well as $\ell_i$ and $\hat{\sigma}$.

Figure 1 shows string configurations for $\ell_1 = 0, \ell_2 = \ell_3 = 10$ for a uniform distribution
of D3-branes on a three-dimensional spherical shell. For a given a distance $L$ between quark
and monopole, the $\sigma$ deformations increase the energy scale probed by a string. The behavior
of $x = x(r)$ for the positive $x$ characterized by $(1, 0)$ string can be obtained from the first
integral (2.4) with the upper limit replaced by $r$. On the other hand, the curve $x = x(r)$ with

\[\text{In the below, we use the reduced relations } H = 1 + \hat{\sigma}^2 \frac{L_4}{L_1} \text{ and } f = \frac{1}{L_1 L_2} \text{ for purely radial string configuration all the time.}\]
Figure 1: Various string configurations for $\ell_1 = 0$, $\ell_2 = \ell_3 = 10$, $t = 0.5$ and $\hat{\sigma} = 1$ (black), $2$ (red) and $3$ (blue) using (2.4). For a given $L$, the $\sigma$ deformations increase the energy scale probed by a string. At $x = 0$, $r = r_0$, $(1,1)$ string coming out of this junction is attached to one of $N$ D3-branes at the origin $x = 0$, $r = 0$ and $(1,0)$ and $(0,1)$ strings located at positive $x$ and negative $x$ respectively are attached to the probe D3-brane at infinity ($r = \infty$). Note that the $x = 0$ of this figure is different from the one given in [7].

The total regularized energy combined by the three kinds of string configurations is given by

$$E = \frac{1}{2\pi} \left[ \int_{r_0}^{\infty} dr \left( \sqrt{\frac{f_1}{H_1}} \frac{H r^2}{f r^2 \sqrt{L_1 L_2} \sqrt{\frac{f_1 H r^4}{f H_1 \gamma^4} - 1}} - \sqrt{1 + \hat{\sigma}^2} \right) - \int_0^{r_0} dr \sqrt{1 + \hat{\sigma}^2} \right]$$

$$+ \frac{t}{2\pi} (r_1 \to r_2) + \frac{1}{2\pi} \sqrt{1 + t^2} \int_0^{r_0} dr \sqrt{\frac{H}{L_2}}$$

(2.5)
string coupling $g$

$$t \equiv 1/g.$$ 

The last term of (2.5) comes from the contribution of $(1, 1)$ string where the main contribution arises from $r'$ term inside the square root of the action (2.2).

The length $L$ by adding the contributions (2.4) and the energy $E$ from (2.5) can be written in terms of elliptic integrals for four separate cases. Note that the integrands of (2.4) are the same as the case of quark-anti quark potential [4] except that $r_0$ is replaced by $r_i$'s. So one can perform them without any difficulty. It turns out that elliptic integrals have more general arguments due to the fact that lower limit $r_0$ is different from $r_i$. For the energy $E$, the integrands from the contributions $(1, 0)$ string and $(0, 1)$ string look similar to the ones in [4] and can be computed straightforwardly. The expressions are more complicated due to the presence of $r_i$ as well as $r_0$. The contribution from $(1, 1)$ string can be written as an elliptic integral also.

The first case $^2$ is given by

$$L = 2R^2(1 + \dot{\sigma}^2)\beta_1 \sqrt{(r_1^2 + \ell_2^2)\alpha_1 [\Pi(\alpha_0, \alpha_1, \sqrt{\alpha_2}) - F(\alpha_0, \sqrt{\alpha_2})]} + (r_1 \rightarrow r_2),$$

$$E = E_{1,0} + E_{0,1} + E_{1,1}$$

(2.6)

where the contributions to the energies from $(1, 0), (0, 1)$ and $(1, 1)$ strings are given by

$$E_{1,0} = \frac{\sqrt{\beta_2}}{2\pi} \left[ (1 + \dot{\sigma}^2)(r_1^2 + \ell_2^2) \left( K(\sqrt{\alpha_2}) - \frac{1}{1 - \alpha_2} E(\sqrt{\alpha_2}) - \Pi(\beta_0, 1, \sqrt{\alpha_2}) \right) + (\ell_1^2 + \ell_2^2\dot{\sigma}^2) F(\alpha_0, \sqrt{\alpha_2}) - (1 + \dot{\sigma}^2)\ell_1^2 (K(\sqrt{\alpha_2}) - F(\beta_0, \sqrt{\alpha_2})) \right],$$

$$E_{0,1} = tE_{1,0}(r_1 \rightarrow r_2),$$

$$E^+_{1,1} = \frac{1}{2\pi} \sqrt{1 + \dot{\sigma}^2} \sqrt{1 + t^2} \left[ \frac{\sqrt{\ell_2^2 - \ell_1^2}}{1 + \dot{\sigma}^2} F(\mu, \frac{1}{\sqrt{1 + \dot{\sigma}^2}}) \right.

- \ell_2^2 - \ell_1^2 E(\mu, \frac{1}{\sqrt{1 + \dot{\sigma}^2}}) + \sqrt{\frac{\alpha_0}{\beta_2} (1 + \dot{\sigma}^2)} - (r_0 \rightarrow 0) \right],$$

$$E^-_{1,1} = \frac{1}{2\pi} \sqrt{1 + \dot{\sigma}^2} \sqrt{1 + t^2} \times \left[ -\frac{\sqrt{\ell_2^2 - \ell_1^2}}{\sqrt{1 + \dot{\sigma}^2}} E(\nu, \frac{\sqrt{1 + \dot{\sigma}^2}}{\sigma}) + \sqrt{\frac{\alpha_0}{\beta_2} (1 + \dot{\sigma}^2)} (r_0 \rightarrow 0) \right].$$

(2.7)

$^2$This case applies when either $r_i^2 \geq \ell_1^2 - 2\ell_2^2$ for any $\dot{\sigma}$, or else $r_i^2 < \ell_1^2 - 2\ell_2^2$ and $\dot{\sigma}^2 < \frac{r_i^2}{r_1^2 + \ell_2^2 - r_i^2}$. The symbols $F, E, \Pi$ denote the elliptic integrals of the first, second and third kind, respectively and $K$ denotes the complete elliptic integral of first kind.
D3-branes on a two-dimensional disk (ℓ it has the effect of rescaling ˆσ is shown for the case for ℓ quark and monopole. At asymptotically large distance, this force vanishes. In Figure 2, this perpendicular to a uniform distribution of D3-branes on a particular five-dimensional ellipsoid (all [4].

the quark-monopole potential $E$ is enhanced by $\sqrt{H}$ where $H = 1 + 2$. The energy is $E = c/\sqrt{H}/L$. The strength parameter $c$ is invariant under the S-duality transformation $g \rightarrow 1/g$. Since $L$ is also invariant under this transformation, $E$ is invariant under the $g \rightarrow 1/g$ [7].

Figure 2 is a parametric plot of the distance between the quark and monopole $L$ versus the quark-monopole potential $E$ by using [240] and (2.7). For trajectories which are perpendicular to a uniform distribution of D3-branes on a particular five-dimensional ellipsoid (all $\ell_i$ nonvanishing and $\ell_2 = \ell_3$), the $\sigma$ deformations enhance the Coulombic force between the quark and monopole. At asymptotically large distance, this force vanishes. In Figure 2, this is shown for the case for $\ell_1 = 1$, $\ell_2 = \ell_3 = 0.1$. We set $R$ to unity for convenience because it has the effect of rescaling $\sigma$ and $L$. One can see similar behavior for the distribution of D3-branes on a two-dimensional disk($\ell_1 \neq 0, \ell_2 = \ell_3 = 0$), as in quark-anti quark potential [4].

Assuming that the dominant contributions arise from the regions near $r = r_1$ for $\Delta L$ and near $r = r_2$ for $L - \Delta L$ in (2.4), we find that

$$L \approx \frac{R^2}{r_1^2 \sqrt{L_1(r_1)} L_2(r_1)} I_1 + \frac{R^2}{r_2^2 \sqrt{L_1(r_2)} L_2(r_2)} I_2,$$

$$E \approx \frac{1}{2\pi} \sqrt{\frac{H_1}{L_2(r_1)}} I_1 + \frac{t}{2\pi} \sqrt{\frac{H_2}{L_2(r_2)}} I_2, \quad I_i \equiv \int_{r_0}^{\infty} \frac{dr}{\sqrt{\frac{r}{H_i(r)}} - 1}. \quad (2.8)$$

In order to see this, we need to use some properties between the elliptic integrals [10]. When $\cos \alpha \tan \phi \tan \psi = 1$, then there are two “addition” relations: $F(\phi, \alpha) + F(\psi, \alpha) = K(\alpha)$ and $E(\phi, \alpha) + E(\psi, \alpha) = E(\alpha) + \sin^2 \alpha \sin \phi \sin \psi$. Then our reduced expressions for $L$ and $E$ are exactly same as the ones in [11] where the zero-temperature limit was given in terms of elliptic integrals rather than hypergeometric functions [7].
Figure 2: Quark-monopole potential $E = E(L)$ for $\ell_1 = 1$, $\ell_2 = \ell_3 = 0.1$, $t = 0.5$ and $\hat{\sigma} = 1$ (black), 1.5 (red) and 2 (blue) using a parametric plot for (2.6). In this part of the Coulomb branch, the $\sigma$ deformations simply enhance the Coulombic force. As we increase $t$, the whole curves are shifted to the lower right direction.

When $t = 1$, then $r_1 = r_2$ from (2.10). Since the $L$ and $E$ have common factor $I \equiv I_1 = I_2$, the energy $E$ is proportional to the length $L$, as in quark-anti quark case [4]. In other words, the range of parameters ($\alpha_2 \to 1, r_1, r_2 \to 0$ and $\ell_1 \to 0$) provides linear behavior of quark-monopole. However, this range of parameters implies that it takes an infinite amount of energy to separate quark and anti-quark. Therefore, one cannot have a quark available to find the potential between quark and monopole. When $t$ is not equal to 1 ($r_1 \neq r_2$), it is impossible to see any simple analytic expression between $E$ and $L$ because $r_1$ and $r_2$ are complicated functions of $\ell_1, \ell_2, r_0$ and $\hat{\sigma}$ through (2.10).

Following the procedure by [7], we want to compute $r_i$ from the vanishing of net forces at the junction $r = r_0$ rather than differentiating the energy $E$ with respect to $r_i$. For the $(1,0)$ string and $(0,1)$ string, the derivatives $r'$ at $r = r_0$ can be obtained from (2.3) and the infinitesimal lengths squared along the strings can be read off from the metric (2.1). The $ds^2$ can be written in terms of $dx^2$. Then according to [7] by recognizing that the integrand of an action (2.2) is equal to a tension $T_{p,q}$ multiplied by $ds$, the tensions of the strings at $r = r_0$ are given by

$$T_{1,0} = \frac{1}{2\pi\sqrt{\alpha'R}} \left( \frac{H_0}{f_0} \right)^{1/4} r_0, \quad T_{0,1} = \frac{1}{2\pi\sqrt{\alpha'R}} t \left( \frac{H_0}{f_0} \right)^{1/4} r_0,$$

$$T_{1,1} = \frac{1}{2\pi\sqrt{\alpha'R}} \sqrt{1 + t^2} \left( \frac{H_0}{f_0} \right)^{1/4} r_0.$$

Of course, the conformal limit [7] is recovered since the extra factors $\left( \frac{H_0}{f_0} \right)^{1/4}$ become 1 due
to the fact that $H_0 = f_0 = 1$. From these, the vertical and horizontal components of forces exerted by each of the strings in the $(x,r)$-plane are set to zero [12] [13]:

\[
\sqrt{H_1 f_0} r_1^2 T_{1,0} - \sqrt{H_2 f_0} r_2^2 T_{0,1} = 0,
\]

\[
\sqrt{H_1 f_0} r_1^4 T_{1,0} + \sqrt{H_2 f_0} r_2^4 T_{0,1} - T_{1,1} = 0.
\]  

(2.9)

Although the common extra factors $\left(\frac{H_i}{f_0}\right)^{1/4}$ appearing the tension above do not change the relations between the forces, the slopes of $(1,0)$ string or $(0,1)$ string at $r = r_0$ do depend on $H_i$ and $f_i$ where $i = 0,1,2$. This will lead to more complicated expressions for $r_i$ where $i = 1,2$. By simplifying (2.9), the solution for these are given by

\[
\left(\frac{H_0 r_0^4}{H_1 f_1 r_1^4}\right) = 1 + \frac{1}{t^2}, \quad \left(\frac{H_0 r_0^4}{H_2 r_2^4}\right) = 1 + t^2.
\]  

(2.10)

Therefore, these allow us to write $r_1$ and $r_2$ in terms of $r_0, \ell_i, \hat{\sigma}$ and $t$ by substituting $H_i$ and $f_i$ explicitly. It is easy to see the conformal limit [14] can be seen from the fact that $H_i = 1 = f_i$ where $i = 0,1,2$. By substituting these expressions (2.10) with (2.4) and (2.5), all the previous plots on $x$ versus $r$ and $E$ versus $L$ are drawn for the values of $r_0, \ell_i, \hat{\sigma}$ and $t$.

Note that there exists a relation $r_1 \leftrightarrow r_2$ under the S-duality transformation $g \leftrightarrow 1/g$.

The second case [4] for which $L$ and $E$ can be written in terms of elliptic integrals is

\[
L = \frac{R^2}{2(\ell_1^2 - \ell_2^2)} \sqrt{\frac{r_1^2 + \ell_2^2}{2\beta_1(1 + \hat{\sigma}^2)(r_1^2 + \ell_1^2)}} \left[\Pi \left(q_0, \frac{\ell_1^2 - \ell_2^2}{r_1^2 + \ell_1^2}, \sqrt{q}\right) - F(q_0, \sqrt{q})\right] + (r_1 \rightarrow r_2),
\]

\[
E = E_{1,0} + E_{0,1} + E_{1,1}
\]  

(2.11)

where the energy from $(1,0)$ string is given by

\[
E_{1,0} = \frac{1}{2\pi \sqrt{(1 + \hat{\sigma}^2)(r_1^2 + \ell_1^2)}} \left[(\ell_1^2 + \ell_2^2\hat{\sigma}^2)F(q_0, \sqrt{q})\right]
\]

\[
+ \frac{1}{\beta_2} (K(\sqrt{q}) - \frac{1}{1-q}E(\sqrt{q}) - \Pi(p_0, 1, \sqrt{q}))
\]

\[
- (1 + \hat{\sigma}^2) \left[\frac{1}{\beta_2(1 + \hat{\sigma}^2)} - r_1^2\right] (K(\sqrt{q}) - F(p_0, \sqrt{q}))\right],
\]  

(2.12)

This case applies when $r_1^2 < \ell_1^2 - 2\ell_2^2$ and $\hat{\sigma}^2 > \frac{\ell_1^2 + \ell_2^2}{r_1^2 - 2\ell_2^2 - r_1^2}$.
the energy from (0, 1) string $E_{0,1}$ is equal to $tE_{1,0}(r_1 \to r_2)$ and finally the energy from (1, 1) string $E_{1,1}$ is the same as previous one (2.7). The parameters in this case are defined by

$$q_0 \equiv \sin^{-1}\sqrt{\frac{r_1^2 + \ell_1^2}{r_0^2 + \ell_1^2}}, \quad p_0 \equiv \sin^{-1}(\cos \alpha_0), \quad q \equiv \frac{\hat{\sigma}^2 \ell_1^2 - (1 + 2\hat{\sigma}^2)\ell_2^2 - (1 + \hat{\sigma}^2)r_2^2}{(1 + \hat{\sigma}^2)(r_1^2 + \ell_1^2)}$$

where $\alpha_0$ was defined previously and $r_1$ and $r_2$ are given by (2.10). In this case, since $\hat{\sigma}$ cannot be zero, there is no undeformed result. One expects that similar behavior for various $\ell_i, \hat{\sigma}$ and $t$ can be obtained but it is not too much interested because all the quark-monopole potentials are described in the nonzero $\hat{\sigma}$ and we cannot see any phase transition between undeformed theory and deformed theory.

There are also two other cases. When we compute the $L$, for example, we can take the first part of $L$ (2.6) as an integral $\Delta L$ and the second part of $L$ (2.11) as an integral $L - \Delta L$. Similarly, we can compute $L$ as a combination of the second part of $L$ in (2.6) for (1, 0) string configuration and first part of $L$ (2.11) for (0, 1) string configuration. Also one can compute the energies $E$ with same regions of parameter space. In each case, there exist some restrictions on the parameters. As in second case above, since there is no undeformed solution due to the nonzero of $\hat{\sigma}$, we do not know much about any big difference between deformed solution and undeformed solution.

In summary, we have studied the gravity dual of deformed Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills theory and applied to Wilson loop calculations by minimizing the action for three kinds of strings on the type IIB supergravity background.

It would be interesting to consider the nonradial string configuration [6, 14] by considering more general solution to equations of motion of the action and to study how these $\sigma$ deformations change the confining properties and the phase structure of finite temperature [11]. For the quark-monopole-dyon system [15] where a dyon is added to quark-monopole, one can apply $\sigma$ deformation to see how it reflects the energy of the system.

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