AN INTEGRAL REPRESENTATION AND PROPERTIES OF
BERNOULLI NUMBERS OF THE SECOND KIND

FENG QI

Abstract. In the paper, the author establishes an integral representation
and properties of Bernoulli numbers of the second kind and reveals that the
generating function of Bernoulli numbers of the second kind is a Bernstein
function on $(0, \infty)$.

1. Introduction

The Bernoulli numbers of the second kind $b_0, b_1, b_2, \ldots, b_n, \ldots$ are defined by
\[ \frac{x}{\ln(1 + x)} = \sum_{n=0}^{\infty} b_n x^n. \] (1.1)
They are also known as Cauchy numbers, Gregary coefficients, or logarithmic num-
bers. The first few Bernoulli numbers are
\[ b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24}, \quad b_4 = -\frac{19}{720}, \quad b_5 = \frac{3}{160}. \] (1.2)

Can one establish an explicit formula for computing $b_n$ for $n \in \mathbb{N}$?

In [14], by establishing an explicit formula for the $n$-th derivative of $\frac{1}{\ln x}$, an
explicit formula for calculating $b_n$ was obtained as follows.

Theorem 1.1 ([14]). For $n \geq 2$, Bernoulli numbers of the second kind $b_n$ can be
computed by
\[ b_n = (-1)^n \frac{1}{n!} \left( \frac{1}{n+1} + \sum_{k=2}^{n} \frac{a_{n,k} - na_{n-1,k}}{k!} \right), \] (1.3)
where $a_{n,k}$ are defined by
\[ a_{n,2} = (n - 1)! \] (1.4)
and
\[ a_{n,i} = (i - 1)!(n - 1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}} \] (1.5)
for $n + 1 \geq i \geq 3$.

In this paper, we will establish an integral representation and properties for
Bernoulli numbers of the second kind $b_n$ and show that the generating function
$\frac{x}{\ln(1 + x)}$ in the left-hand side of (1.1) is a Bernstein function on $(0, \infty)$.

2010 Mathematics Subject Classification. Primary 11B68; Secondary 11R33, 11S23, 26A48,
30E20, 33B99.

Key words and phrases. integral representation; property; bernoulli numbers of the second kind;
completely monotonic sequence; minimal; generating function; Bernstein function.

This paper was typeset using \texttt{AMS-LaTEX}.
2. SOME DEFINITIONS, NOTIONS, AND PROPERTIES

We first collect some necessary definitions and notations.

**Definition 2.1** ([13, 28]). A function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and

$$(−1)^{k−1}f^{(k)}(t) ≥ 0$$

for $x ∈ I$ and $k ∈ \mathbb{N}$.

**Definition 2.2** ([16, 19]). A function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

$$(−1)^{k}[\ln f(t)]^{(k)} ≥ 0$$

for $k ∈ \mathbb{N}$ on $I$.

**Definition 2.3** ([26, 28]). A function $f : I ⊆ (−∞, ∞) → [0, ∞)$ is called a Bernstein function on $I$ if $f(t)$ has derivatives of all orders and $f'(t)$ is completely monotonic on $I$.

**Definition 2.4** ([26, p. 19, Definition 2.1]). A Stieltjes function is a function $f : (0, ∞) → [0, ∞)$ which can be written in the form

$$f(x) = \frac{a}{x} + bx + \int_0^∞ \frac{1}{s+x}dμ(s),$$

where $a, b ≥ 0$ are nonnegative constants and $μ$ is a nonnegative measure on $(0, ∞)$ such that

$$\int_0^∞ \frac{1}{1+s}dμ(s) < ∞.$$

**Definition 2.5** ([5, Definition 1]). Let $f(t)$ be a function defined on $(0, ∞)$ and have derivatives of all orders. A number $r ∈ \mathbb{R} ∪ \{±∞\}$ is said to be the completely monotonic degree of $f(t)$ with respect to $t ∈ (0, ∞)$ if $t^rf(t)$ is a completely monotonic function on $(0, ∞)$ but $t^{r+ε}f(t)$ is not for any positive number $ε > 0$.

We remark that Definition 2.5 slightly but essentially modifies [11, Definition 1.5].

These classes of functions have the following properties and relations.

**Proposition 2.1** ([28, p. 161, Theorem 12b]). A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < ∞$ is that

$$f(x) = \int_0^∞ e^{-xt}dα(t),$$

where $α(t)$ is non-decreasing and the integral converges for $0 < x < ∞$.

**Proposition 2.2** ([26, p. 15, Theorem 3.2]). A function $f : (0, ∞) → \mathbb{R}$ is a Bernstein function if and only if it admits the representation

$$f(x) = a + bx + \int_0^∞ (1 − e^{-xt})dμ(t),$$

where $a, b ≥ 0$ and $μ$ is a measure on $(0, ∞)$ satisfying

$$\int_0^∞ \min\{1, t\}dμ(t) < ∞.$$

**Proposition 2.3** ([1, 6, 16, 19]). Any logarithmically completely monotonic function must be completely monotonic.
Proposition 2.4 ([1]). The set of all Stieltjes functions is a subset of logarithmically completely monotonic functions on $(0, \infty)$.

Proposition 2.5 ([3, pp. 161–162, Theorem 3] and [26, p. 45, Proposition 5.17]). The reciprocal of any Bernstein function is logarithmically completely monotonic.

For the history and survey of the notion “logarithmically completely monotonic function”, please refer to [21, p. 2154, Remark 8], [22, Introduction], [23, Remark 4.8], and a lot of closely related references therein.

For convenience, the notation $\deg^t_{cm}[f(t)]$ was designed in [5] to stand for the completely monotonic degree $r$ of $f(t)$ with respect to $t \in (0, \infty)$.

It is obvious that the completely monotonic degree of any non-trivial Bernstein function on $(0, \infty)$ is greater than 0.

Bernstein functions have the following properties.

Theorem 2.1. Let $f(x)$ be a Bernstein function on $(0, \infty)$. Then

$$\deg^x_{cm}[f(x)] \geq -1. \quad (2.6)$$

In other words, the completely monotonic degree of any Bernstein function on $(0, \infty)$ is not less than $-1$.

Proof. Differentiating on both sides of the formula (2.5) gives

$$f'(x) = b + \int_0^\infty e^{-xt} t \, d\mu(t).$$

By Definition 2.3, the derivative $f'(x)$ is a completely monotonic function on $(0, \infty)$. In virtue of Proposition 2.1, it is derived that the measure $\mu(t)$ in (2.5) is non-decreasing on $(0, \infty)$. Dividing by $x$ on both sides of the formula (2.5) leads to

$$\frac{f(x)}{x} = \frac{a}{x} + b + \int_0^\infty q(xt) t \, d\mu(t), \quad (2.7)$$

where

$$q(u) = \frac{1 - e^{-u}}{u}$$

for $u \in (0, \infty)$. By some closely related knowledge in the papers [7, 8, 9, 10, 15, 17, 18, 20, 24, 25, 29], we find that

$$q(u) = \int_{1/e}^1 s^{u-1} \, ds$$

and

$$q^{(i)}(u) = \int_{1/e}^1 (\ln s)^i s^{u-1} \, ds,$$

which implies that the function $q(u)$ is completely monotonic on $(0, \infty)$. Consequently, we have

$$\frac{d^i}{dx^i} q(xt) = t^i q^{(i)}(xt),$$

which means that the function $q(xt)$ is completely monotonic with respect to $x \in (0, \infty)$. Accordingly, the very right term in (2.7) is a completely monotonic function of $x$. As a result, the function $x^{-1}f(x)$ is completely monotonic on $(0, \infty)$, that is,

$$\deg^x_{cm}[f(x)] \geq -1.$$ 

The proof of Theorem 2.1 is complete. \hfill $\square$
Theorem 2.2. The completely monotonic degree of the reciprocal of a Bernstein function on \((0, \infty)\) is non-negative and less than 1.

Proof. Let \(f(x)\) be a Bernstein function on \((0, \infty)\). Then, by Theorem 2.1, the function \(x^{-1}f(x)\) is completely monotonic, so its reciprocal \(\frac{1}{x f(x)}\) is non-decreasing, that is, the reciprocal \(\frac{1}{f(x)}\) is not completely monotonic. Thus, the degree of \(\frac{1}{f(x)}\) is less than 1. However, by Propositions 2.5 and 2.3, the reciprocal \(\frac{1}{f(x)}\) is completely monotonic on \((0, \infty)\), so its completely monotonic degree is non-negative.

3. A Lemma

For establishing integral representations for Bernoulli numbers of the second kind \(b_n\), we need the following lemma.

Lemma 3.1 ([2, p. 2130]). The function \(\frac{1}{\ln(1+z)}\) is a Stieltjes function and has the integral representation

\[
\frac{1}{\ln(1+z)} = \frac{1}{z} + \int_{1}^{\infty} \frac{1}{\frac{1}{\ln(t-1)^2 + \pi^2} z + t} \, dt
\]

(3.1)

for \(z \in \mathbb{C} \setminus (-\infty, 0]\).

Remark 3.1. We note that the integral representation (3.1) is a correction of the equation (34) on page 2130 in [2].

4. Integral representations for Bernoulli numbers

We are now in a position to establish integral representations of Bernoulli numbers of the second kind \(b_n\).

Theorem 4.1. The Bernoulli numbers of the second kind \(b_n\) for \(n \in \mathbb{N}\) may be calculated by

\[
b_n = (-1)^{n+1} \int_{1}^{\infty} \frac{1}{\frac{1}{\ln(t-1)^2 + \pi^2} t^n} \, dt.
\]

(4.1)

Proof. By (3.1), we have

\[
\frac{x}{\ln(1+x)} = 1 + \int_{1}^{\infty} \frac{1}{\frac{1}{\ln(t-1)^2 + \pi^2} x + t} \, dt
\]

(4.2)

and

\[
\left[ \frac{x}{\ln(1+x)} \right]^{(k)} = \int_{1}^{\infty} \frac{1}{\frac{1}{\ln(t-1)^2 + \pi^2} \left( \frac{x}{x+t} \right)^{(k)}} \, dt
\]

(4.3)

for \(k \in \mathbb{N}\). On the other hand, by (1.1), we also have

\[
\left[ \frac{x}{\ln(1+x)} \right]^{(k)} = \sum_{n=k}^{\infty} b_n \frac{n!}{(n-k)!} x^{n-k}.
\]

(4.4)
Combining (4.3) with (4.4) leads to
\[
\sum_{n=k}^{\infty} b_n \frac{n!}{(n-k)!} x^{n-k} = (-1)^{k+1} k! \int_{1}^{\infty} \frac{t}{[\ln(t-1)]^2 + \pi^2} \frac{1}{(x+t)^{k+1}} \, dt.
\] (4.5)

Letting \( x \to 0^+ \) on both sides of the above equation produces
\[
k! b_k = (-1)^{k+1} k! \int_{1}^{\infty} \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{1}{t^k} \, dt.
\]

Thus, the formula (4.1) is proved. \( \square \)

5. Properties of Bernoulli numbers of the second kind

Basing on the integral representation (4.1) in Theorem 4.1 for Bernoulli numbers of the second kind \( b_n \), we now turn our attention to investigate the complete monotonicity and other properties of Bernoulli numbers \( b_n \) for \( n \in \mathbb{N} \).

We recall from [28, p. 108, Definition 4] that a sequence \( \{\mu_n\}_{n=0}^{\infty} \) is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, that is
\[
(-1)^k \Delta^k \mu_n \geq 0 \quad (5.1)
\]
for \( n, k \geq 0 \), where
\[
\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}. \quad (5.2)
\]

Theorem 4a in [28, p. 108] reads that a necessary and sufficient condition that the sequence \( \{\mu_n\}_{n=0}^{\infty} \) should have the expression
\[
\mu_n = \int_{0}^{1} t^n \, d\alpha(t) \quad (5.3)
\]
for \( n \geq 0 \), where \( \alpha(t) \) is non-decreasing and bounded for \( 0 \leq t \leq 1 \), is that it should be completely monotonic.

We also recall from [28, p. 163, Definition 14a] that a completely monotonic sequence \( \{a_n\}_{n=0}^{\infty} \) is minimal if it ceases to be completely monotonic when \( a_0 \) is decreased. Theorem 4a in [28, p. 164] states that a completely monotonic sequence \( \{\mu_n\}_{n=0}^{\infty} \) is minimal if and only if the equality (5.3) is valid for \( n \geq 0 \) and \( \alpha(t) \) is a non-decreasing bounded function continuous at \( t = 0 \).

**Theorem 5.1.** The sequence \( \{(-1)^n b_{n+1}\}_{n=0}^{\infty} \) of Bernoulli numbers of the second kind is completely monotonic and minimal.

*Proof.* This follows from setting in the equality (5.3)
\[
\alpha(t) = \int_{0}^{t} \frac{1}{s [\ln(1/s-1)]^2 + \pi^2} \, ds \quad (5.4)
\]
for \( t \in [0,1] \) and \( \alpha(1) = b_1 = \frac{1}{2} \). The proof of Theorem 5.1 is complete. \( \square \)

**Theorem 5.2.** Let \( m \in \mathbb{N} \) and \( a_i \) for \( 1 \leq i \leq m \) be nonnegative integers. Then
\[
\left| (a_i + a_j)! b_{a_i + a_j + 1} \right|_m \geq 0 \quad (5.5)
\]
and
\[
\left| (-1)^{a_i + a_j} (a_i + a_j)! b_{a_i + a_j + 1} \right|_m \geq 0, \quad (5.6)
\]
where \( |a_{ij}|_m \) denotes a determinant of order \( m \) with elements \( a_{ij} \).
Proof. From the proofs of Theorem 4.1, we observe that
\[ b_n = (-1)^{n+1} \lim_{x \to 0^+} h_n(x), \] (5.7)
where the function
\[ h_n(x) = \int_1^\infty \frac{1}{(\ln(t-1)^2 + x^2)(t+x)^n} \, dt \] (5.8)
is completely monotonic on \([0, \infty)\).

In [12], or see [13, p. 367], it was obtained that if \( f \) is a completely monotonic function, then
\[ |f^{(a_i+a_j)}(x)|_m \geq 0 \] (5.9)
and
\[ |(-1)^{a_i+a_j} f^{(a_i+a_j)}(x)|_m \geq 0, \] (5.10)
where \(|a_{ij}|_m\) denotes a determinant of order \( m \) with elements \( a_{ij} \) and \( a_i \) for \( 1 \leq i \leq m \) are nonnegative integers. Applying \( f \) in (5.9) and (5.10) to the function \( h_n(x) \) yields
\[ |h_n^{(a_i+a_j)}(x)|_m \geq 0 \] (5.11)
and
\[ |(-1)^{a_i+a_j} h_n^{(a_i+a_j)}(x)|_m \geq 0, \] (5.12)
that is,
\[ |(-1)^{a_i+a_j} \frac{(n+a_i+a_j-1)!}{(n-1)!} h_{n+a_i+a_j}(x)|_m \geq 0 \] (5.13)
and
\[ \frac{(n+a_i+a_j-1)!}{(n-1)!} h_{n+a_i+a_j}(x)|_m \geq 0. \] (5.14)
Letting \( x \to 0^+ \) in (5.13) and (5.14) and making use of (5.7) produce
\[ |(-1)^{a_i+a_j} \frac{(n+a_i+a_j-1)!}{(n-1)!} (-1)^{n+a_i+a_j+1} b_{n+a_i+a_j}|_m \geq 0 \] (5.15)
and
\[ |(n+a_i+a_j-1)! (-1)^{n+a_i+a_j+1} b_{n+a_i+a_j}|_m \geq 0. \] (5.16)
Further simplifying (5.15) and (5.16) leads to
\[ |(-1)^{n+1} (n+a_i+a_j-1)! b_{n+a_i+a_j}|_m \geq 0 \]
and
\[ |(-1)^{n+a_i+a_j+1} (n+a_i+a_j-1)! b_{n+a_i+a_j}|_m \geq 0, \]
which are equivalent to (5.5) and (5.6). Theorem 5.2 is thus proved. \( \square \)

Remark 5.1. Taking \( m = 3 \) and \( a_i = i \) for \( i = 0, 1, 2 \) in Theorem 5.2 and using values of \( b_i \) for \( 1 \leq i \leq 5 \) in (1.2) show us that
\[
\begin{vmatrix}
0b_1 & 1b_2 & 2b_3 \\
1b_2 & 2b_3 & 3b_4 \\
2b_3 & 3b_4 & 4b_5 \\
\end{vmatrix}
= \begin{vmatrix}
\frac{1}{2} & \frac{1}{12} & \frac{1}{12} \\
-\frac{1}{12} & \frac{17}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{3}{5} \\
\end{vmatrix}
= \frac{857}{86400}
\]
and
\[
\begin{vmatrix}
0b_1 & -1b_2 & 2b_3 \\
-1b_2 & 2b_3 & -3b_4 \\
2b_3 & -3b_4 & 4b_5 \\
\end{vmatrix}
= \begin{vmatrix}
\frac{1}{2} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{3}{5} \\
\end{vmatrix}
= \frac{857}{86400}.
\]
Further taking $x$ which can be simplified as $\mu$ This implies the required logarithmic convexity.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n$. A sequence $\lambda$ is said to be majorized by $\mu$ (in symbols $\lambda \preccurlyeq \mu$) if

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$$

for $k = 1, 2, \ldots, n - 1$ and

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i,$$

where $\lambda[1] \geq \lambda[2] \geq \cdots \geq \lambda[n]$ and $\mu[1] \geq \mu[2] \geq \cdots \geq \mu[n]$ are rearrangements of $\lambda$ and $\mu$ in a descending order.

A sequence $\lambda$ is said to strictly majorized by $\mu$ (in symbols $\lambda \prec \mu$) if $\lambda$ is not a permutation of $\mu$.

In [27, p. 106, Theorem A], a correction of [4, Theorem 1] which was collected in [13, p. 367, Theorem 2], it was obtained that if $f$ is a completely monotonic function on $(0, \infty)$ and $\lambda \preceq \mu$, then

$$\prod_{i=1}^{n} f^{(\lambda_i)}(x) \leq \prod_{i=1}^{n} f^{(\mu_i)}(x). \quad (5.17)$$

**Theorem 5.3.** Let $m \in \mathbb{N}$ and let $\lambda$ and $\mu$ be two $m$-tuples of nonnegative numbers such that $\lambda \preceq \mu$. Then

$$\prod_{i=1}^{m} \lambda_i! b_{\lambda_i+1} \leq \prod_{i=1}^{m} \mu_i! b_{\mu_i+1}. \quad (5.18)$$

**Proof.** Employing the inequality (5.17) applied to $h_n(x)$ defined by (5.8) creates

$$\prod_{i=1}^{m} (-1)^{\lambda_i} \frac{(n + \lambda_i - 1)!}{(n - 1)!} h_{n+\lambda_i}(x) \leq \prod_{i=1}^{m} (-1)^{\mu_i} \frac{(n + \mu_i - 1)!}{(n - 1)!} h_{n+\mu_i}(x),$$

which can be simplified as

$$\prod_{i=1}^{m} (n + \lambda_i - 1)! b_{n+\lambda_i}(x) \leq \prod_{i=1}^{m} (n + \mu_i - 1)! b_{n+\mu_i}(x).$$

Further taking $x \to 0^+$ and utilizing (5.7) turn out

$$\prod_{i=1}^{m} (n + \lambda_i - 1)!(-1)^{n+\lambda_i+1} b_{n+\lambda_i} \leq \prod_{i=1}^{m} (n + \mu_i - 1)!(-1)^{n+\mu_i+1} b_{n+\mu_i},$$

which is equivalent to (5.18). The proof of Theorem 5.3 is complete. \hfill \Box

**Theorem 5.4.** The sequence $\{i!b_{i+1}\}_{0}^{\infty}$ is logarithmically convex.

**Proof.** It is clear that $(i, i+2) \succ (i+1, i+1)$ for $i \geq 0$. Therefore, by virtue of (5.18), we have

$$(i!b_{i+1})[(i+2)!b_{i+3}] \geq [(i+1)!b_{i+2}]^2. \quad (5.19)$$

This implies the required logarithmic convexity.

This conclusion can also be deduced from Theorem 5.2. The proof of Theorem 5.4 is thus complete. \hfill \Box
Remark 5.2. Letting $i = 2$ in (5.19) and using three corresponding values of $b_i$ for $3 \leq i \leq 5$ in (1.2) give

$$(2!b_3)(4!b_5) = \frac{3}{80} = 0.0375 \geq (3!b_4)^2 = \frac{361}{14400} = 0.0250 \ldots$$

6. A Bernstein function

Finally, we prove that the generating function $\frac{x^r}{\ln(1+x)}$ is a Bernstein function.

**Theorem 6.1.** The generating function $\frac{x^r}{\ln(1+x)}$ of Bernoulli numbers of the second kind $b_k$ is a Bernstein function on $(0, \infty)$.

**First proof.** The integral representation (4.2) shows us that the function $x \ln(1+x)$ is positive and increasing on $(0, \infty)$. The integral representation (4.3) reveals that the first derivative of $x \ln(1+x)$ is completely monotonic on $(0, \infty)$. So, by Definition 2.3, the function $x \ln(1+x)$ is a Bernstein function on $(0, \infty)$. The proof of Theorem 6.1 is complete.

**Second proof.** It is not difficult to see that

$$\frac{x}{\ln(1+x)} = \int_0^1 (1+x)^t \, dt$$

(6.1)

and the function $(1+x)^t$ for $t \in (0, 1)$ is a Bernstein function. Theorem 6.1 is thus proved.

**REFERENCES**

[1] C. Berg, Integral representation of some functions related to the gamma function, Mediterr. J. Math. 1 (2004), no. 4, 433–439; Available online at http://dx.doi.org/10.1007/s00009-004-0022-6.

[2] C. Berg and H. L. Pedersen, A one-parameter family of Pick functions defined by the Gamma function and related to the volume of the unit ball in $n$-space, Proc. Amer. Math. Soc. 139 (2011), no. 6, 2121–2132; Available online at http://dx.doi.org/10.1090/S0002-9939-2010-10636-6.

[3] C.-P. Chen, F. Qi, and H. M. Srivastava, Some properties of functions related to the gamma and psi functions, Integral Transforms Spec. Funct. 21 (2010), no. 2, 153–164; Available online at http://dx.doi.org/10.1080/10652460903064216.

[4] A. M. Fink, Kolmogorov-Landau inequalities for monotone functions, J. Math. Anal. Appl. 90 (1982), 251–258; Available online at http://dx.doi.org/10.1016/0022-247X(82)90057-9.

[5] B.-N. Guo and F. Qi, A completely monotonic function involving the tri-gamma function and with degree one, Appl. Math. Comput. 218 (2012), no. 19, 9890–9897; Available online at http://dx.doi.org/10.1016/j.amc.2012.03.075.

[6] B.-N. Guo and F. Qi, A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 72 (2010), no. 2, 21–30.

[7] B.-N. Guo and F. Qi, A simple proof of logarithmic convexity of extended mean values, Numer. Algorithms 52 (2009), no. 1, 89–92; Available online at http://dx.doi.org/10.1007/s11075-008-9259-7.

[8] B.-N. Guo and F. Qi, Properties and applications of a function involving exponential functions, Commun. Pure Appl. Anal. 8 (2009), no. 4, 1231–1249; Available online at http://dx.doi.org/10.3934/cpaa.2009.8.1231.

[9] B.-N. Guo and F. Qi, The function $(b^x - a^x)/x$: Logarithmic convexity and applications to extended mean values, Filomat 25 (2011), no. 4, 63–73; Available online at http://dx.doi.org/10.2298/FIL1004063G.

[10] B.-N. Guo and F. Qi, The function $(b^x - a^x)/x$: Ratio’s properties, Available online at http://arxiv.org/abs/0904.1115.
INTEGRAL REPRESENTATION OF BERNOULLI NUMBERS

[11] S. Koumandos and H. L. Pedersen, Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function, J. Math. Anal. Appl. 355 (2009), no. 1, 33–40; Available online at http://dx.doi.org/10.1016/j.jmaa.2009.01.042.

[12] D. S. Mitrinović and J. E. Pečarić, On two-place completely monotonic functions, Anzeiger Öster. Akad. Wiss. Math.-Naturwiss. Kl. 126 (1989), 85–88.

[13] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.

[14] F. Qi, Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind, available online at http://arxiv.org/abs/1301.6845.

[15] F. Qi, Logarithmic convexity of extended mean values, Proc. Amer. Math. Soc. 130 (2002), no. 6, 1787–1796; Available online at http://dx.doi.org/10.1090/S0002-9939-01-06275-X.

[16] F. Qi and C.-P. Chen, A complete monotonicity property of the gamma function, J. Math. Anal. Appl. 296 (2004), no. 2, 603–607; Available online at http://dx.doi.org/10.1016/j.jmaa.2004.04.026.

[17] F. Qi and J.-X. Cheng, Some new Steffensen pairs, Anal. Math. 29 (2003), no. 3, 219–226; Available online at http://dx.doi.org/10.1023/A:1025467221664.

[18] F. Qi, J.-X. Cheng and G. Wang, New Steffensen pairs, Inequality Theory and Applications, Volume 1, 273–279, Nova Science Publishers, Huntington, NY, 2001.

[19] F. Qi and B.-N. Guo, Complete monotonicities of functions involving the gamma and digamma functions, RGMIA Res. Rep. Coll. 7 (2004), no. 1, Art. 8, 63–72; Available online at http://rgmia.org/v7n1.php.

[20] F. Qi and B.-N. Guo, On Steffensen pairs, J. Math. Anal. Appl. 271 (2002), no. 2, 534–541; Available online at http://dx.doi.org/10.1016/S0022-247X(02)00120-8.

[21] F. Qi, S. Guo, and B.-N. Guo, Complete monotonicity of some functions involving polygamma functions, J. Comput. Appl. Math. 233 (2010), no. 9, 2149–2160; Available online at http://dx.doi.org/10.1016/j.cam.2009.09.044.

[22] F. Qi, Q.-M. Luo, and B.-N. Guo, Complete monotonicity of a function involving the divided difference of digamma functions, Sci. China Math. (2013), in press; Available online at http://dx.doi.org/10.1007/s11425-012-4662-0.

[23] F. Qi, C.-F. Wei, and B.-N. Guo, Complete monotonicity of a function involving the ratio of gamma functions and applications, Banach J. Math. Anal. 6 (2012), no. 1, 35–44.

[24] F. Qi and S.-L. Xu, Refinements and extensions of an inequality, II, J. Math. Anal. Appl. 211 (1997), no. 2, 616–620; Available online at http://dx.doi.org/10.1006/jmaa.1997.5318.

[25] F. Qi and S.-L. Xu, The function \((b^x - a^x)/x\): Inequalities and properties, Proc. Amer. Math. Soc. 126 (1998), no. 11, 3355–3359; Available online at http://dx.doi.org/10.1090/S0002-9939-98-04442-6.

[26] R. L. Schilling, R. Song, and Z. Vondraček, Bernstein Functions, de Gruyter Studies in Mathematics 37, De Gruyter, Berlin, Germany, 2010.

[27] H. van Haeringen, Completely monotonic and related functions, J. Math. Anal. Appl. 204 (1996), no. 2, 389–408; Available online at http://dx.doi.org/10.1006/jmaa.1996.0443.

[28] D. V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1946.

[29] S.-Q. Zhang, B.-N. Guo, and F. Qi, A concise proof for properties of three functions involving the exponential function, Appl. Math. E-Notes 9 (2009), 177–183.

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.wordpress.com