The maximum number of perfect matchings in graphs
with a given degree sequence

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17 March, 2008

Abstract

We show that the number of perfect matching in a simple graph $G$ with an even number of vertices and degree sequence $d_1, d_2, \ldots, d_n$ is at most $\prod_{i=1}^{n} (d_i)!^{\frac{1}{2d_i}}$. This bound is sharp if and only if $G$ is a union of complete balanced bipartite graphs.

2000 Mathematics Subject Classification: 05A15, 05C70.
Keywords and phrases: Perfect matchings, permanents.

1 Introduction

Let $G = (V, E)$ be an undirected simple graph. For a vertex $v \in V$, let $\deg v$ denote its degree. Assume that $|V|$ is even, and let perfmat $G$ denote the number of perfect matchings in $G$. The main result of this short note is:

Theorem 1.1

$\text{perfm} G \leq \prod_{v \in V} (\frac{\deg v)!}{\deg v},$ (1.1)

where $0^0 = 0$. If $G$ has no isolated vertices then equality holds if and only if $G$ is a disjoint union of complete balanced bipartite graphs.

For bipartite graphs the above inequality follows from the Bregman-Minc Inequality for permanents of $(0, 1)$ matrices, mentioned below.

The inequality (1.1) was known to Kahn and Lovász, c.f. [2, (7)], but their proof was never published, and it was recently stated and proved independently by the second author in [3]. Here we show that it is a simple consequence of the Bregman-Minc Inequality.

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2 The proof

Let $A$ be an $n \times n$ $(0,1)$ matrix, i.e. $A = [a_{ij}]_{i,j=1}^n \in \{0,1\}^{n \times n}$. Denote $r_i = \sum_{j=1}^n a_{ij}, i = 1,\ldots,n$. The celebrated Bregman-Minc inequality, conjectured by Minc [4] and proved by Bregman [1], states

$$\text{perm} A \leq \prod_{i=1}^n (r_i!)^{\frac{1}{r_i}}, \quad (2.1)$$

where equality holds (if no $r_i$ is zero) iff up to permutation of rows and columns $A$ is a block diagonal matrix in which each block is a square all-1 matrix.

Proof of Theorem 1.1: The square of the number of perfect matchings of $G$ counts ordered pairs of such matchings. We claim that this is the number of spanning 2-regular subgraphs $H$ of $G$ consisting of even cycles (including cycles of length 2 which are the same edge taken twice), where each such $H$ is counted $2^s$ times, with $s$ being the number of components (that is, cycles) of $H$ with more than 2 vertices. Indeed, every union of a pair of perfect matchings $M_1, M_2$ is a 2-regular spanning subgraph $H$ as above, and for every cycle of length exceeding 2 in $H$ there are two ways to decide which edges came from $M_1$ and which from $M_2$.

The permanent of the adjacency matrix $A$ of $G$ also counts the number of spanning 2-regular subgraphs $H'$ of $G$, but now we allow odd cycles as well. Here, too, each such $H'$ is counted $2^s$ times, where $s$ is the number of cycles of $H'$ with more than 2 vertices (as there are 2 ways to orient each such cycle as a directed cycle and get a contribution to the permanent.) Thus the square of the number of perfect matchings is at most the permanent of the adjacency matrix, and the desired inequality follows from Bregman-Minc by taking the square root of (2.1), where the numbers $r_i$ are the degrees of the vertices of $G$.

It is clear that if $G$ is a vertex-disjoint union of balanced complete bipartite graphs then equality holds in (1.1). Conversely, if $G$ has no isolated vertices and equality holds, then equality holds in (2.1), and no $r_i$ is zero. Therefore, after permuting the rows and columns of the adjacency matrix of $G$ it is a block diagonal matrix in which every block is an all-1 square matrix, and as our graph $G$ has no loops, this means that it is a union of complete balanced bipartite graphs, completing the proof. □

References

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