Infinitesimal Darboux transformations of the spectral curves of tori in the four-space

P.G. Grinevich † I.A. Taimanov ‡

1 Introduction

The present paper resumes the study of relations between the geometric properties of surfaces in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \) and the spectral properties of the corresponding Dirac operators started in [16]. In the paper we study the behavior of the spectral curve of a torus in \( \mathbb{R}^4 \) under conformal transformations of \( \bar{\mathbb{R}}^4 \) and, in particular, prove that

- the conformal transformations of \( \bar{\mathbb{R}}^4 \) which map a torus \( T \subset \mathbb{R}^4 \) into a compact torus preserve all Floquet multipliers of the corresponding Dirac operator

(see Theorem 1 and Corollary 1).

This generalizes the analogous result for tori in \( \mathbb{R}^3 \) which was proved by M.U. Schmidt and the first author (P.G.G.) [8] and confirmed the conjecture of the second author (I.A.T.) on the conformal invariance of the spectral curves of tori in \( \mathbb{R}^3 \). Therewith by the spectral curve it was understood the analytic set \( \mathcal{M}(\Gamma) \) in \( \mathbb{C}^2 \) formed by the Floquet multipliers on the zero energy level.

There is a more careful and precise definition of the spectral curve compatible with the finite gap integration theory. It reads that this curve \( \Gamma \) gives a one-to-one parameterization of a basis for Floquet–Bloch functions of the corresponding Dirac operator and the multiplier set \( \mathcal{M}(\Gamma) \) is the image of the mapping \( \mathcal{M} \) which corresponds to each point the multipliers of the corresponding Floquet function (see [13]). Therewith we show that

- the spectral curve of a Dirac operator corresponding to a torus \( T \subset \mathbb{R}^4 \) is not always preserved by conformal transformations of \( \mathbb{R}^4 \). However

---

*The work was supported by RFBR (grants no. 05-01-01032a (P.G.G.) and 06-01-0094a (I.A.T.)), by the program “Fundamental problems of nonlinear dynamics” of the Presidium of RAS, and by the program “Leading scientific schools” (grant NS-4182.2006.1). The second author was also supported by the complex integration project 2.15 of SB RAS.

†Landau Institute of Theoretical Physics, Kosygin street 2, 117940 Moscow, Russia; e-mail: pgg@landau.ac.ru.

‡Institute of Mathematics, 630090 Novosibirsk, Russia; e-mail: taimanov@math.nsc.ru
possible deformations consist in gluing together or ungluing points with some fixed multipliers \((\kappa_1, \kappa_2)\) (see Remark 1).

This almost isospectrality effect was not noticed before and we think that it is interesting by itself. It does not relate to the dimension of the ambient space and holds also for tori in the three-space. In fact the nontrivial actions of conformal transformations of \(\mathbb{R}^3\) or \(\mathbb{R}^4\) on the potential of the Dirac operator corresponding to an immersed torus are described by nonlinear systems of the Melnikov type. Thus

- the Melnikov type deformations may be only almost isospectral and we show that in some important geometrical examples (related to the Clifford tori) after reparameterization of the temporary variable such flows reduce to integrable systems on whiskered tori (see §5.1).

To the spectral curve of the Dirac operator with given local parameters near “infinities” there corresponds an infinite family of nonlocal conservation laws which for the case of the one-dimensional potential reduce to the Kruskal–Miura integral. Since their values in fact depend on \(\mathcal{M}(\Gamma)\) these integrals are also preserved by the infinitesimal conformal deformations.

### 2 The Weierstrass representation and the spectral curve

#### 2.1 The Weierstrass representation

The Weierstrass representation of surfaces in \(\mathbb{R}^3\) and \(\mathbb{R}^4\) there corresponds to a surface the Dirac operator with potentials

\[
\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}
\]

(1)

where \(U = U(z, \bar{z})\) and \(z\) is a conformal parameter on surface. In the sequel we shall use the following agreement: we write \(f(z)\) instead of \(f(z, \bar{z})\) and the notation \(f(z)\) does not imply that \(f\) is holomorphic unless the opposite is not stated explicitly.

In fact, \(U\) is defined as a section of some bundle over the surface. In the paper we consider the case of tori. By the uniformization theorem, every torus is conformally equivalent to a flat torus \(\mathbb{R}^2/\Lambda\) where \(\Lambda\) is the period lattice.

We have

- for tori in \(\mathbb{R}^3\) the potential \(U\) is double-periodic: \(U(z + \gamma) = U(z)\) for all \(\gamma \in \Lambda\), and real-valued;
• for tori in $\mathbb{R}^4$ the potential $U$ is double-periodic however is defined up to transformations

$$U \rightarrow U e^{a + \bar{b}z} - a - b$$

where $a, b \in \mathbb{C}$ and $\text{Im}\gamma \in \pi\mathbb{Z}$ for all $\gamma \in \Lambda$.

A surface in $\mathbb{R}^4$ is given by the integral formulas

$$x^k = x^k_0 + \int (x^k_0 dz + \bar{x}^k_0 d\bar{z}), \quad k = 1, 2, 3, 4,$$

with

$$x^1_z = \frac{i}{2}(\bar{\varphi}_2 \psi_2 - \varphi_1 \psi_1), \quad x^2_z = \frac{1}{2}(\bar{\varphi}_2 \psi_2 + \varphi_1 \psi_1),$$

$$x^3_z = \frac{1}{2}(\bar{\varphi}_2 \psi_1 + \varphi_1 \psi_2), \quad x^4_z = \frac{i}{2}(\bar{\varphi}_2 \psi_1 - \varphi_1 \psi_2)$$

where $x^1, \ldots, x^4$ are the Euclidean coordinates in $\mathbb{R}^4$ (see [10, 15]) and $\psi$ and $\varphi$ meet the Dirac equations

$$D\psi = \begin{pmatrix} U & \partial \\ -\bar{\partial} & \bar{U} \end{pmatrix} \psi = 0, \quad D^\vee \varphi = \begin{pmatrix} \bar{U} & \partial \\ -\bar{\partial} & U \end{pmatrix} \varphi = 0$$

(we note that the operators $D$ and $D^\vee$ are Hermitian conjugate).

The case of surfaces in $\mathbb{R}^3$ is obtained as the reduction: $U = V = \bar{U}$, $D = \bar{D}^\vee$, and for $\psi = \varphi$ we obtain a surface lying in the linear subspace $x^4 = 0$.

Although for surfaces in $\mathbb{R}^3$ the derivation of the Weierstrass representation for surfaces in $\mathbb{R}^3$ is straightforward and the vector function $\psi = \varphi$ is easily defined from the geometrical data [16] for surfaces in $\mathbb{R}^4$ the situation is different. This procedure is much more delicate and the functions $\psi$, $\varphi$, and $U$ are defined from nonlinear equations which have to be solved globally on the whole surface [17].

2.2 The spectral curve

In the framework of differential geometry of surfaces the spectral curves appear as the spectral curves of integrable surfaces (constant mean curvature tori in $\mathbb{R}^3$ [3] and harmonic tori in $S^3$ [9]) and were used for constructing the explicit formulas for such tori in terms of theta functions corresponding to these spectral curves which appear to be of finite genus. Recently they were used for obtaining the lower estimates for the areas of minimal tori in $S^3$ [6].

For general tori in $\mathbb{R}^3$ the spectral curve was defined via the Weierstrass representation by the second author as the spectral curve of the Dirac operator associated, i.e. with the potential $U = e^{a}H/2$ where $e^{2a}dzd\bar{z}$ is the induced metric and $H$ is the mean curvature.

First the spectral curve on the zero energy level was introduced by Dubrovin, Krichever, and Novikov for the two-dimensional Schrödinger operator [4].

The general definition of the Floquet–Bloch spectrum of a multi-periodic operator was introduced in [13]. For the two-dimensional case it is as follows.
Given a double-periodic operator $L$, its Floquet (–Bloch) function $\psi$ is defined as the formal solution to the equation

$$L\psi = E\psi$$

meeting the periodicity conditions

$$\psi(z + \gamma_i) = \kappa_i \psi(z), \quad i = 1, 2,$$

where $\gamma_1, \gamma_2$ are the generators of the period lattice. It is said that $E$ is the energy level and $\kappa_1, \kappa_2 \in \mathbb{C} \setminus \{0\}$ are the (Floquet) multipliers of $\psi$. Let us put $E = 0$, i.e., let us consider the zero energy level, and assume that the possible values of the multipliers meet some analytical dispersion relation

$$F(\kappa_1, \kappa_2) = 0$$

which defines a one-dimensional complex manifold $\mathcal{M}(\Gamma)$ (complex curve).

The dispersion relations do exist not for all operators. However this picture is true for elliptic operators and some other operators closed to them. In the middle of 1980s the problem of rigorous confirmation of this picture, in particular, for the two-dimensional Schrödinger operator and the heat operator was addressed by two different ways:

1) by perturbation methods Krichever did construct the spectral curve and the Floquet–Bloch eigenfunction as perturbations of their counterparts for the operator with the zero potential \[11\]. Therewith for the case of the Schrödinger operator we have two infinite ends at which the eigenfunction asymptotically behaves as a holomorphic function at one end and as a antiholomorphic function at another end and the perturbation of the spectral curve consists in opening resonant pairs into handles outside some compact part at which the perturbation may result in more complicated topological surgery. This geometrical picture rising to the spectral theory initiated the development of the analytical theory of such Riemann surfaces (non only hyperelliptic) of infinite genus \[5\].

2) the second author (I.A.T.) demonstrated how to obtain the analytical dispersion relation for hyperelliptic periodic operators by using the Fredholm alternative for analytical pencils of operators (the Keldysh theorem, for the Dirac operator such a proof is exposed in \[15\]). Therewith $F(\kappa_1, \kappa_2)$ is a (regularized) determinant of the operator $L(\kappa_1, \kappa_2)$ which is the operator $L$ defined on the space of functions meeting the boundary conditions

$$\psi(z + \gamma_1) = \kappa_1 \psi(z), \quad \psi(z + \gamma_2) = \kappa_2 \psi(z), \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$  

(3)

In general to several points of this manifold there correspond not a one-dimensional family of Floquet eigenfunctions with these multipliers. Therefore to obtain the spectral curve $\Gamma$ we have to consider a partial normalization of $\mathcal{M}(\Gamma)$. If it is of finite genus it admits a compactification by finitely many points (the “infinities”) to an algebraic curve.

Therefore, we have to make difference between
• the spectral curve $\Gamma$ such that there is a rank one bundle, over $\Gamma$, which is formed by the corresponding Floquet functions such that any Floquet function is a linear composition of the generators of fibres $^1$.

• the multiplier set $\mathcal{M} = \mathcal{M}(\Gamma)$ which parameterizes the pairs of Floquet multipliers $(\kappa_1, \kappa_2)$. This set is the image of the multiplier mapping

$$\mathcal{M} : \Gamma \rightarrow \mathbb{C}^2, \quad \mathcal{M}(\psi) = (\kappa_1, \kappa_2);$$

• the algebraic spectral curve which is obtained by a compactification by finitely many points from the curve $\Gamma$ of finite genus.

The difference between these three spectral curves is demonstrated in §5.2 for the one-dimensional Schrödinger operator with a complex-valued potential.

We refer for the detailed explanation of this picture for the Dirac operator to [18]. This survey also explains the approach which was proposed by the second author (I.A.T.) to proving the Willmore conjecture by using the spectral curves. In part, this approach comes from the observation that the Willmore functional

$$W(M) = \int_M H^2 d\mu$$

for surfaces $M$ in $\mathbb{R}^3$ is related to their Weierstrass representation and, in particular, to the spectral curve (here $H$ is the mean curvature and $d\mu$ is the induced measure on the surface).

For closed surfaces the Willmore functional is invariant under conformal transformations of $\overline{\mathbb{R}^3}$ which preserve the compactness of the surface. This led to another conjecture that the spectral curve for tori is invariant with respect to these transformations and this conjecture was confirmed in [8]. On the modern language we may say that in [8] it was proved that the multiplier set $\mathcal{M}$ is invariant.

Hence, given the potential $U$ for a torus in $\mathbb{R}^4$, we have a pair of spectral curves $\Gamma$ and $\Gamma^\vee$, i.e., the spectral curves of the operators $\mathcal{D}$ and $\mathcal{D}^\vee$. These spectral curves are closely related and, in particular, we have the evident

**Lemma 1** The operators $\mathcal{D}(\kappa_1, \kappa_2)$ and $\mathcal{D}^\vee(\bar{\kappa}_1, \bar{\kappa}_2)$ are Hermitian conjugate for all $\kappa_1, \kappa_2$ and their indices are equal to 0.

Therefore

1) $\dim \ker \mathcal{D}(\kappa_1, \kappa_2) = \dim \ker \mathcal{D}^\vee(\bar{\kappa}_1, \bar{\kappa}_2)$ for all $\kappa_1, \kappa_2$;

2) The multiplier sets for the operators $\mathcal{D}$ and $\mathcal{D}^\vee$ are complex conjugate.

We also have

**Lemma 2** The spectral curves for operators $\mathcal{D}$ and $\mathcal{D}^\vee$ are both real.

$^1$This is exactly the complex curve which is coming into the finite gap integration scheme.
Proof. Let 
\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]
be a zero Floquet eigenfunction of \( D \) with multipliers \( \kappa_1, \kappa_2 \). Then the function 
\[ \psi^* = \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix} \]
is also a zero Floquet eigenfunction of \( D \) with multipliers \( \bar{\kappa}_1, \bar{\kappa}_2 \). Therefore \( \Gamma \) admits the antiholomorphic involution \( \psi \to \psi^*, (\kappa_1, \kappa_2) \to (\bar{\kappa}_1, \bar{\kappa}_2) \).

We remark that the potential \( U \) of a torus in \( \mathbb{R}^4 \) is defined up to transformations (2) which imply the following simple transformations of the Floquet–Bloch eigenfunctions \( \psi \) of \( D \) and their multipliers:
\[ \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \to \left( \begin{array}{c} e^{a+ib\xi} \psi_1 \\ e^{\bar{a}+\bar{b}\xi} \psi_2 \end{array} \right), \quad \kappa_i \to e^{b\gamma_i} \kappa_i, \quad i = 1, 2 \]
(see [17]).

Therefore the Dirac operator \( D \) corresponding to a conformal immersion of a torus \( \mathbb{R}^2/\Lambda \) into \( \mathbb{R}^4 \) is not unique: we may put for it \( D \) or \( D^\lor \) or any of their transformations of the form (2). If the torus lies in \( \mathbb{R}^3 \subset \mathbb{R}^4 \) there is a normalization condition \( U = \bar{U} \) which fixes the operator uniquely. However transformations of the form (2) as well as the changes of the base for the period lattice \( \Lambda \) does not change the spectral curve and only transforms the multiplier mapping \( M \).

3 The infinitesimal Darboux transformation

Let
\[ L = \begin{pmatrix} \partial & -p \\ -q & \partial \end{pmatrix}, \quad L^\lor = \begin{pmatrix} -\partial & -q \\ -p & \partial \end{pmatrix}, \quad p = p(z), \quad q = q(z), \quad (4) \]
be a pair of linear operators with periodic coefficients:
\[ p(z + \gamma_1) = p(z + \gamma_2) = p(z), \quad q(z + \gamma_1) = q(z + \gamma_2) = q(z), \]
i.e. the functions \( p, q \) are defined on a torus \( T \) with the periods \( \gamma_1 \) and \( \gamma_2 \).

Let us define the infinitesimal Darboux transformation.

Let us consider
• \( \Psi^D \) and \( \Phi^D \) be a pair of zero Floquet eigenfunctions of \( L \) and \( L^\lor \) respectively such that
\[ L \Psi^D = 0, \quad L^\lor \Phi^D = 0, \]
\[ \Psi^D(z + \gamma_i) = \bar{\kappa}_i \Psi^D(z), \quad \Phi^D(z + \gamma_i) = 1/\bar{\kappa}_i \Phi^D(z), \quad i = 1, 2, \]
\[ \Psi^D = \begin{pmatrix} \Psi^D_1(z) \\ \Psi^D_2(z) \end{pmatrix}, \quad \Phi^D = \begin{pmatrix} \Phi^D_1(z) \\ \Phi^D_2(z) \end{pmatrix}. \]
• a family of zero Floquet–Bloch eigenfunctions \(\psi(\lambda, z)\) of \(L\) with multipliers \(\kappa_1(\lambda), \kappa_2(\lambda)\) respectively, \(\lambda \in \Gamma\);

• a family of zero Floquet–Bloch eigenfunctions of \(L'\): \(\phi(\mu, z)\) with multipliers \(\kappa_1'(\mu), \kappa_2'(\mu)\) respectively, \(\mu \in \Gamma'\).

Let us define the following pair of forms:

\[
d\omega(\lambda, z) = \Phi_1^D(z)\psi_1(\lambda, z)dz - \Phi_2^D(z)\psi_2(\lambda, z)d\bar{z} \quad (5)
\]

\[
d\omega'(\mu, z) = \phi_1(\mu, z)\Psi_1^D(z)d\bar{z} - \phi_2(\mu, z)\Psi_2^D(z)dz \quad (6)
\]

**Lemma 3** The both forms \(d\omega(\lambda, z)\) and \(d\omega'(\mu, z)\) are closed. Therefore formulas (5) and (6) defines the functions \(\omega(\lambda, z)\) and \(\omega'(\mu, z)\) up to integration constants \(c(\lambda)\) and \(c'(\mu)\).

**Proof** is straightforward:

\[
d(d\omega) = -(\partial_\bar{z}[\Phi_1^D \psi_1(\lambda)] + \partial_z[\Phi_2^D \psi_2(\lambda)])dz \wedge d\bar{z} =
-(p\Phi_1^D \psi_1(\lambda) - q\Phi_2^D \psi_2(\lambda) + q\Phi_1^D \psi_2(\lambda) - p\Phi_2^D \psi_1(\lambda))dz \wedge d\bar{z} = 0.
\]

**Lemma 4** Let either \(\kappa_1(\lambda)/\hat{\kappa}_1 \neq 1\) or \(\kappa_2(\lambda)/\hat{\kappa}_2 \neq 1\). Then the function \(\omega(\lambda, z)\) is defined uniquely by the Floquet–Bloch condition:

\[
\omega(\lambda, z + \gamma_i) = \frac{\kappa_i(\lambda)}{\hat{\kappa}_i}\omega(\lambda, z).
\]

If either \(\kappa_1'(\mu)/\hat{\kappa}_1 \neq 1\) or \(\kappa_2'(\mu)/\hat{\kappa}_2 \neq 1\), the function \(\omega'(\mu, z)\) is defined uniquely by the Floquet–Bloch condition:

\[
\omega'(\mu, z + \gamma_i) = \frac{\kappa_i'(\mu)}{\hat{\kappa}_i}\omega'(\mu, z).
\]

**Proof**. Let us consider the basic parallelogram \(0, \gamma_1, \gamma_1 + \gamma_2, \gamma_2\). Let us denote

\[
I_1(\lambda) = \int_0^{\gamma_1} d\omega(\lambda), \quad I_2(\lambda) = \int_0^{\gamma_2} d\omega(\lambda).
\]

We have

\[
\int_{\gamma_2}^{\gamma_1 + \gamma_2} d\omega(\lambda) = I_1(\lambda) \cdot \kappa_2(\lambda)/\hat{\kappa}_2, \quad \int_{\gamma_1}^{\gamma_1 + \gamma_2} d\omega(\lambda) = I_2(\lambda) \cdot \kappa_1(\lambda)/\hat{\kappa}_1.
\]

Therefore

\[
I_1(\lambda) + I_2(\lambda) \cdot \kappa_1(\lambda)/\hat{\kappa}_1 = I_2(\lambda) + I_1(\lambda) \cdot \kappa_2(\lambda)/\hat{\kappa}_2,
\]

and

\[
I_1(\lambda)[\kappa_2(\lambda)/\hat{\kappa}_2 - 1] = I_2(\lambda)[\kappa_1(\lambda)/\hat{\kappa}_1 - 1]. \quad (7)
\]
The periodicity condition for $\omega(\lambda)$ implies

$$
\omega(\lambda, \gamma_1) = \omega(\lambda, 0) \kappa_1(\lambda)/\kappa_1 = \omega(\lambda, 0) + I_1(\lambda),
$$
$$
\omega(\lambda, \gamma_2) = \omega(\lambda, 0) \kappa_2(\lambda)/\kappa_2 = \omega(\lambda, 0) + I_2(\lambda),
$$
$$
\omega(\lambda, 0) \cdot [\kappa_1(\lambda)/\kappa_1 - 1] = I_1(\lambda),
$$
$$
\omega(\lambda, 0) \cdot [\kappa_2(\lambda)/\kappa_2 - 1] = I_2(\lambda).
$$

From (7) it follows, that under the assumptions of Lemma the system (8) is compatible and has an unique solution.

**Theorem 1** Consider the following variation of the space of Floquet–Bloch functions:

$$
\begin{align*}
\delta \psi_1(\lambda, z) &= \omega(\lambda, z) \Psi_D^1(z) \\
\delta \psi_2(\lambda, z) &= \omega(\lambda, z) \Psi_D^2(z) \\
\delta \Phi_D^1(\mu, z) &= \omega(\mu, z) \Phi_D^1(z) \\
\delta \Phi_D^2(\mu, z) &= \omega(\mu, z) \Phi_D^2(z)
\end{align*}
$$

(9)

1. This deformation corresponds to the following variation of the operators $L$, $L'$:

$$
\begin{align*}
\delta L &= \begin{pmatrix} 0 & \Psi_D^1(z) \Phi_D^2(z) \\ -\Psi_D^2(z) \Phi_D^1(z) & 0 \end{pmatrix}, \\
\delta L' &= \begin{pmatrix} 0 & -\Psi_D^2(z) \Phi_D^1(z) \\ \Psi_D^1(z) \Phi_D^2(z) & 0 \end{pmatrix}
\end{align*}
$$

(10)

Therefore it is self-consistent (variations of different wave functions result in the same variation of potentials), and respects the symmetry between $L$, $L'$. In terms of $p$, $q$ we get Melnikov-type variations of potentials (11):

$$
\delta p(z) = -\Psi_D^1(z) \Phi_D^2(z), \quad \delta q(z) = \Psi_D^2(z) \Phi_D^1(z).
$$

(11)

2. For all $\lambda$, $\mu$ such, that the conditions of Lemma are fulfilled it is natural to normalize the kernels $\omega(\lambda)$, $\omega(\mu)$ by the Floquet–Bloch conditions. Then deformation (9) respects the Floquet–Bloch properties of the functions $\psi(\lambda)$, $\phi(\mu)$, and does not change the multipliers.

**Proof.** From (4) it follows:

$$
\begin{align*}
p &= \partial_z \psi_1(\lambda)/\psi_2(\lambda), \quad q = \partial_z \psi_2(\lambda)/\psi_2(\lambda), \\
\delta p &= \frac{\partial_z [\delta \psi_1(\lambda)]}{\psi_2(\lambda)} - \frac{[\partial_z \psi_1(\lambda)]\delta \psi_2(\lambda)}{\psi_2(\lambda)} = \\
&= \frac{\partial_z [\omega(\lambda) \Psi_D^1]}{\psi_2(\lambda)} - \frac{\omega(\lambda) \Psi_D^1}{\psi_2(\lambda)} = \frac{[\partial_z \omega(\lambda)] \Psi_D^1}{\psi_2(\lambda)} + \frac{\omega(\lambda) \partial_z \Psi_D^1}{\psi_2(\lambda)} - \frac{\omega(\lambda) \Psi_D^1}{\psi_2(\lambda)} = \\
&= -\frac{\psi_2(\lambda) \Phi_D^2 \Phi_D^1}{\psi_2(\lambda)} + \frac{\omega(\lambda) p \Phi_D^2}{\psi_2(\lambda)} - \frac{\omega(\lambda) \Phi_D^2}{\psi_2(\lambda)} = -\Phi_D^1 \Phi_D^2
\end{align*}
$$
\[
\delta \eta = \frac{\partial_z [\delta \psi_2(\mu) \Psi^D_2]}{\psi_1(\mu)} - \frac{[\partial_z, \psi_2(\mu)] \delta \psi_1(\mu)}{\psi_1^2(\mu)} = \frac{\psi_1(\mu) \Psi^D_1 \psi_1(\mu)}{\omega^\vee(\mu) \Psi^D_1} - q \frac{\omega^\vee(\mu) \Psi^D_1}{\psi_1(\mu)} = \psi_1(\mu) \Phi^D_1.
\]

Theorem 1 and, in particular, part 2 shows that the deformation respects the Floquet–Bloch properties of the function \(\psi(\lambda, z)\) if the condition of Lemma 4 is fulfilled which, in particular, means that \(\kappa_1(\lambda)/\tilde{\kappa}_1 \neq 1\) or \(\kappa_2(\lambda)/\tilde{\kappa}_2 \neq 1\). These inequalities are fulfilled at a generic point however there is a discrete set of points of the spectral curve at which we have

\[
\kappa_1(\lambda)/\tilde{\kappa}_1 = \kappa_2(\lambda)/\tilde{\kappa}_2 = 1.
\]

Hence this deformation preserves the multiplier set of \(L\) outside of a discrete set of points satisfying (12). Since the multiplier set is analytic, it is preserved. Of course, the same becomes valid for \(L^\vee\) and its Floquet–Bloch functions \(\phi(\mu, z)\) after replacing (12) by the condition

\[
\kappa_1^\vee(\mu)\tilde{\kappa}_1 = \kappa_2^\vee(\mu)\tilde{\kappa}_2 = 1.
\]

Corollary 1 The infinitesimal deformation (10) preserves the multiplier sets \(\mathcal{M}(\Gamma)\) (on the zero energy level) for the operators \(L\) and \(L^\vee\).

Remark 1. Theorem 1 does not imply that the spectral curve is preserved. Indeed, the forms \(\omega\) and \(\omega^\vee\) are not defined at points meeting the condition (12). Given a multiple point meeting this condition the analytic continuation of \(\omega\) and \(\omega^\vee\) a priori gives its own limit at each branch. Therewith the deformation is correctly defined on the normalization of \(\Gamma\) and the corresponding Floquet function evolves differently and this leads to decreasing the multiplicity of a singular point on the spectral curve. Of course, the converse is also possible. The examples from \(\S\) shows that that may take place and how that happens.

Remark 2. The idea of using kernels analogous to (5) and (6) for calculating the Floquet–Bloch functions deformations for all values of spectral parameter was first suggested in [7] by A.Yu. Orlov and the first author (P.G.G.).

4 Proof of the conformal invariance

Let us apply Theorem 4 to conformal transformations of tori in \(\mathbb{R}^4\) induced by conformal transformations of the ambient space \(\mathbb{R}^4\).

We assume we have a conformal immersion of torus \(\mathbb{R}^2/\Lambda\) into \(\mathbb{R}^4\) defined by:

\[
\begin{align*}
\partial_z x^1 &= \frac{i}{2} (\Phi_2 \Psi_2 + \Phi_1 \Psi_1), \\
\partial_z x^2 &= \frac{1}{2} (\Phi_2 \Psi_2 - \Phi_1 \Psi_1), \\
\partial_z x^3 &= \frac{1}{2} (\Phi_2 \Psi_1 + \Phi_1 \Psi_2), \\
\partial_z x^4 &= \frac{i}{2} (\Phi_2 \Psi_1 - \Phi_1 \Psi_2).
\end{align*}
\]
The periodicity of the quantities $x^k, k = 1, 2, 3, 4$, with respect to $\Lambda$ only implies that

$$
\Psi_1(z + \gamma_i) = \kappa_i \Psi_1(z), \quad \Psi_2(z + \gamma_i) = \bar{\kappa}_i \Psi_2(z), \\
\Phi_1(z + \gamma_i) = \frac{1}{\kappa_i} \Phi_1(z), \quad \Phi_2(z + \gamma_i) = \frac{1}{\bar{\kappa}_i} \Phi_2(z), \quad i = 1, 2.
$$

However these vector functions satisfy differential equations with periodic coefficients:

$$
D \Psi = 0, \quad D^\vee \Phi = 0
$$

which implies that $\kappa_i = \bar{\kappa}_i, i = 1, 2$, i.e. the multipliers are real-valued. Therefore we have

$$
\Psi(z + \gamma_i) = \kappa_i \Psi(z), \quad \Phi(z + \gamma_i) = \frac{1}{\kappa_i} \Phi(z), \quad i = 1, 2,
$$

and we may apply the results from the previous section.

Let a deformation of $L, L^\vee$ be a sum of the following infinitesimal Darboux transformations:

$$
\partial_\tau = \partial_{\tau_1} + \partial_{\tau_2},
$$

where $\partial_{\tau_1}$ is generated by the following pair of solutions:

$$
\Psi^D = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Phi^D = \begin{pmatrix} \Phi_2 \\ -\Phi_1 \end{pmatrix}
$$

and $\partial_{\tau_2}$ is generated by:

$$
\Psi^D = \begin{pmatrix} \bar{\Psi}_2 \\ -\bar{\Psi}_1 \end{pmatrix}, \quad \Phi^D = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}.
$$

**Theorem 2 1.** The flow $\partial_\tau$ is isospectral and respects the reality conditions. Moreover

$$
\partial_\tau U = \Phi_1 \Psi_1 - \Phi_2 \Psi_2, \quad \partial_\tau \bar{U} = \bar{\Phi}_1 \Psi_1 - \bar{\Phi}_2 \Psi_2,
$$

2. Let us assume, that the kernels $\omega$ and $\omega^\vee$ for the functions $\Psi$ and $\Phi$, are normalized by: $\omega(0) = 0, \omega^\vee(0) = 0$. Then

$$
\partial_\tau \Psi_1 = (x^3 - ix^4) \Psi_1 - i(x^1 - ix^2) \bar{\Psi}_2 \\
\partial_\tau \Psi_2 = (x^3 - ix^4) \Psi_2 + i(x^1 - ix^2) \bar{\Psi}_1 \\
\partial_\tau \Phi_1 = (x^3 + ix^4) \Phi_1 - i(x^1 - ix^2) \bar{\Phi}_2 \\
\partial_\tau \Phi_2 = (x^3 + ix^4) \Phi_2 + i(x^1 - ix^2) \bar{\Phi}_1.
$$

(15)

**Proof.** The first part follows directly from Theorem 1. For the derivatives of the coordinate functions we have

$$
-i(x^1 - ix^2)_z = \Phi_1 \Psi_1 \\
-i(x^1 + ix^2)_z = \Phi_2 \Psi_2 \\
(x^3 + ix^4)_z = \Phi_1 \bar{\Psi}_2 \\
(x^3 - ix^4)_z = \bar{\Phi}_2 \Psi_1,
$$

(16)
and for the flow $\partial_{\tau_1}$

$$\partial_2 \omega = \Phi_2 \Psi_1, \quad \partial_2 \omega^\vee = \Phi_1 \Psi_1.$$ 

Therefore

$$\omega = (x^3 - ix^4), \quad \omega^\vee = -i(x^1 - ix^2),$$

$$\partial_{\tau_1} \Psi_1 = (x^3 - ix^4) \Psi_1^D = (x^3 - ix^4) \Psi_1$$

$$\partial_{\tau_1} \Psi_2 = (x^3 - ix^4) \Psi_2^D = (x^3 - ix^4) \Psi_2$$

$$\partial_{\tau_1} \Phi_1 = -i(x^1 - ix^2) \Phi_1^D = -i(x^1 - ix^2) \Phi_1$$

$$\partial_{\tau_1} \Phi_2 = -i(x^1 - ix^2) \Phi_2^D = i(x^1 - ix^2) \Phi_2.$$ 

For the flow $\partial_{\tau_2}$

$$\partial_2 \omega = \Phi_1 \Psi_1, \quad \partial_2 \omega^\vee = \Phi_1 \Psi_2.$$ 

Therefore

$$\omega = -i(x^1 - ix^2), \quad \omega^\vee = (x^3 + ix^4),$$

$$\partial_{\tau_2} \Psi_1 = -i(x^1 - ix^2) \Psi_1^D = -i(x^1 - ix^2) \Psi_2$$

$$\partial_{\tau_2} \Psi_2 = -i(x^1 - ix^2) \Psi_2^D = i(x^1 - ix^2) \Phi_1$$

$$\partial_{\tau_2} \Phi_1 = (x^3 + ix^4) \Phi_1^D = (x^3 + ix^4) \Phi_1$$

$$\partial_{\tau_2} \Phi_2 = (x^3 + ix^4) \Phi_2^D = (x^3 + ix^4) \Phi_2.$$ 

**Theorem 3** This action generate the following infinitesimal conformal transformations of the immersed surface:

$$\partial_1 x^1 = 2x^1 x^3$$

$$\partial_1 x^2 = 2x^2 x^3$$

$$\partial_1 x^3 = (x^3)^2 - (x^1)^2 - (x^2)^2 - (x^4)^2$$

$$\partial_1 x^4 = 2x^2 x^3.$$ 

**Proof.** It is sufficient to check that

$$\partial_1 \partial_2 x^1 = 2[\partial_2 x^1] x^3 + 2x^1[\partial_2 x^3]$$

$$\partial_1 \partial_2 x^2 = 2[\partial_2 x^2] x^3 + 2x^2[\partial_2 x^3]$$

$$\partial_1 \partial_2 x^3 = 2x^3[\partial_2 x^3] - x^1[\partial_2 x^1] - x^2[\partial_2 x^2] - x^4[\partial_2 x^4]$$

$$\partial_1 \partial_2 x^4 = 2[\partial_2 x^4] x^3 + 2x^4[\partial_2 x^3].$$ 

We obtain by straightforward computations that

$$\partial_1 \partial_2 x^1 = \frac{i}{2} ([\partial_1 \Phi_2] \Psi_2 + \Phi_2 [\partial_2 \Psi_2] + [\partial_1 \Phi_1] \Psi_1 + \Phi_1 [\partial_2 \Psi_1]) =$$

$$= \frac{i}{2} ([(x^3 - ix^4) \Phi_2 - i(x^1 + ix^2) \Phi_1] \Psi_2 + \Phi_2 [(x^3 + ix^4) \Psi_2 - i(x^1 + ix^2) \Psi_1] +$$

$$+ [(x^3 + ix^4) \Phi_1 - i(x^1 - ix^2) \Phi_2] \Psi_1 + \Phi_1 [(x^3 - ix^4) \Psi_1 - i(x^1 - ix^2) \Psi_2]) =$$
Conformal transformations of $\bar{G}$ action of $\text{T}_4$ particular, any continuous family $f$ factors for a given torus $T$ operator. Into a compact surface preserves the multiplier set of the corresponding Dirac operator.

Theorem 4 Any conformal transformation of $\bar{G}$ preserves the multiplier set of the corresponding Dirac operator.

It is known that the group of conformal transformations of $\mathbb{R}^4$ is generated by rotations, dilations, translations and inversions. The spectral curve is evidently preserved by rotations, translations and dilations. Any inversion is conjugate by these transformations to the inversion with the generator of the form $(19)$. It is clear that the set of conformal transformations which map a given torus into a compact surface is path-connected. Therefore Theorems 2 and 3 imply

Theorem 4 Any conformal transformation of $\mathbb{R}^4$ which maps a torus $T \subset \mathbb{R}^4$ into a compact surface preserves the multiplier set of the corresponding Dirac operator.

As we mentioned in §22 there is some freedom in choosing the Dirac operators for a given torus $T \subset \mathbb{R}^4$. The possible potentials form a discrete set $P_T$. Conformal transformations of $\mathbb{R}^4$ which map the torus $T$ into a compact surface form a path-connected component $G_T$ of the conformal group $SO(5,1)$ and the action of $G_T$ on $P_T$ is defined. Therewith Theorem 4 reads that this action and, in particular, any continuous family $f_t \subset G_T$ preserves the multiplier sets.
5 Examples

5.1 Spectral curves of the Clifford tori

Following [18] we present a pair of tori in $\mathbb{R}^4$ which are related by a conformal transformation, have the same multiplier set however their spectral curves are different.

In fact these are the Clifford tori in $S^3$ and $\mathbb{R}^3$ which are related by a stereographical projection. the famous Willmore conjecture reads that for tori in $\mathbb{R}^3$ the Willmore functional attains its minimum on the Clifford torus and its images under conformal transformations. This conjecture is generalized for tori in $\mathbb{R}^4$ and reads actually the same: the Willmore functional

$$W(M) = \int_M |H|^2 d\mu$$

attains its minimum on the Clifford torus in $S^3$ and its conformal images (here $H$ is the mean curvature vector).

A) The Clifford torus in the unit three-sphere $S^3 \subset \mathbb{R}^4$ is defined by the equations

$$(x^1)^2 + (x^2)^2 = \frac{1}{2}, \quad (x^3)^2 + (x^4)^2 = \frac{1}{2}$$

where the sphere is defined by the equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$ 

Its spectral curve is $\Gamma$ is the sphere $\mathbb{C}P^1 = \mathbb{C}$ with two marked points $\lambda = 0, \infty$ added at the compactification.

The potential of the Dirac operator $D$ equals

$$U = \frac{1 + i}{4}.$$ 

The zero Floquet–Bloch functions are parameterized by the points of $\Gamma' = \Gamma \setminus \{\lambda = 0, \infty\}$ and are glued into the Baker–Akhiezer function $\psi$ which is is meromorphic on $\Gamma'$ and has at the marked points ("infinities") the following asymptotics:

$$\psi \approx \left( \begin{array}{c} e^{\lambda z} \\ 0 \end{array} \right) \text{ as } \lambda \to \infty; \quad \psi \approx \left( \begin{array}{c} 0 \\ e^{-i\lambda^2 \bar{z}} \end{array} \right) \text{ as } \lambda \to 0$$

where $u = \frac{1+i}{4}$.

The Clifford torus is defined via the Weierstrass representation by the functions

$$\psi_1 = \psi_2 = \frac{1}{\sqrt{2}} \exp \left( -\frac{i(x+y)}{2} \right), \quad \varphi_1 = -\varphi_2 = -\frac{1}{2\sqrt{2}} \exp \left( \frac{i(y-x)}{2} \right).$$
Although the Dirac operator for a torus in $\mathbb{R}^4$ is defined up to transformations and replacing $D$ by $D^\vee$ it is easy to check that for all such choices the spectral curve of the Clifford torus stays smooth and coincide with $\mathbb{C}P^1$.

B) The Clifford torus in $\mathbb{R}^3$ is the image of the Clifford torus in $S^3 \subset \mathbb{R}^4$ under a stereographic projection

$$(x^1, x^2, x^3) \rightarrow \left( \frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4} \right), \quad \sum_k (x^k)^2 = 1,$$

which is extended to a conformal transformation of $\bar{\mathbb{R}}^4$.

Its spectral curve is $\Gamma$ is the sphere $\mathbb{C}P^1 = \bar{\mathbb{C}}$ with two marked points $\lambda = 0, \infty$ added at the compactification and two double points obtained by gluing together the points from the following pairs:

$$\left( \frac{1+i}{4}, -\frac{1+i}{4} \right) \quad \text{and} \quad \left( -\frac{1+i}{4}, \frac{1-i}{4} \right).$$

The potential of the Dirac operator $D$ equals

$$U = \frac{\sin y}{2\sqrt{2}(\sin y - \sqrt{2})}$$

The zero Floquet–Bloch functions are parameterized by the points of $\Gamma' = \Gamma \setminus \{ \lambda = 0, \infty \}$ and are glued into the Baker–Akhiezer function which is uniquely defined by the following conditions:

1) it is meromorphic on $\Gamma'$ and has the following asymptotics

$$\psi \approx \begin{pmatrix} e^{\lambda z} \\ 0 \end{pmatrix} \quad \text{as} \quad \lambda \rightarrow \infty; \quad \psi \approx \begin{pmatrix} 0 \\ e^{-i\sqrt{2} z} \end{pmatrix} \quad \text{as} \quad \lambda \rightarrow 0$$

where $u = \frac{1+i}{4}$;

2) it has three poles $\Gamma'$ which are independent on $z$ and have the form

$$p_1 = -1 + i + \frac{\sqrt{2i - 4}}{4\sqrt{2}}, \quad p_2 = -1 + i - \frac{\sqrt{2i - 4}}{4\sqrt{2}}, \quad p_3 = \frac{1}{\sqrt{8}}.$$

The Clifford torus in $\mathbb{R}^3$ is constructed via the Weierstrass representation from the function

$$\psi = \psi \left( z, \bar{z}, \frac{1-i}{4} \right)$$

which has the form

$$\psi_1(z, \bar{z}, \lambda) = e^{\lambda z - \frac{1+i}{\lambda} \bar{z}} \left( q_1 \frac{\lambda}{\lambda - p_1} + q_2 \frac{\lambda}{\lambda - p_2} + (1 - q_1 - q_2) \frac{\lambda}{\lambda - p_3} \right),$$

$$\psi_2(z, \bar{z}, \lambda) = e^{\lambda z - \frac{1+i}{\lambda} \bar{z}} \left( t_1 \frac{p_1}{p_1 - \lambda} + t_2 \frac{p_2}{p_2 - \lambda} + (1 - t_1 - t_2) \frac{p_3}{p_3 - \lambda} \right).$$
where \( u = \frac{1 + i}{4} \). The functions \( q_1, q_2, t_1, t_2 \) depend only on \( y \) and 2\( \pi \)-periodic with respect to \( y \) and found from the following conditions

\[
\psi(z, \bar{z}, \frac{1 + i}{4}) = \psi(z, \bar{z}, -\frac{1 + i}{4}), \quad \psi(z, \bar{z}, -\frac{1 + i}{4}) = \psi(z, \bar{z}, \frac{1 - i}{4}).
\]

We see that the spectral curve of the Clifford torus in \( S^3 \) is smooth and the spectral curve of the Clifford torus in \( \mathbb{R}^3 \) has a pair of double points. Their multiplier sets are the same. Therefore the spectral curves can be deformed by conformal transformations however, by Theorem 1, such a deformation may only consist in gluing together or ungluing multiple points with the multipliers \( \kappa_i/\hat{\kappa}_i = 1 \) for \( \Gamma \) and \( \kappa_i\hat{\kappa}_i = 1 \) for \( \Gamma^\vee \).

In particular, we see that in a completely conformal setting adopted, for instance, in [6] it is impossible to define the spectral curve \( \Gamma \) as we did above and its definition always needs the analytical theory of differential operators.

Let us point out, that in the both cases discussed above the potential does not depend on the variable \( x \): \( U(x, y) = U(y) \). For such potentials the Floquet solution has the following form:

\[
\psi(\lambda, x, y) = \left( \frac{\tilde{\psi}_1(\lambda, y)}{\tilde{\psi}_2(\lambda, y)} \right) e^{kx}, \quad \varphi(\lambda, x, y) = \left( \frac{\tilde{\psi}_2(\sigma\lambda, y)}{\tilde{\psi}_1(\sigma\lambda, y)} \right) e^{-kx},
\]

where \( \tilde{\psi}(\lambda, y) \) denotes the Floquet solution for the 1-dimensional Dirac operator

\[
\tilde{L} \tilde{\psi}(\lambda, y) = -k \tilde{\psi}(\lambda, y), \quad \tilde{L} = \begin{bmatrix} i\partial_y & -2U \\ -2U & -i\partial_y \end{bmatrix}, \quad \tilde{\psi}(\lambda, y) = \begin{pmatrix} \tilde{\psi}_1(\lambda, y) \\ \tilde{\psi}_2(\lambda, y) \end{pmatrix}.
\]

\( \tilde{L} \) is the auxiliary operator for the self-focusing nonlinear Schrödinger equation (NLS). The spectral curve \( \Gamma \) for \( \tilde{L} \) is a two-sheeted covering of the \( k \)-plane. All branching points of \( \Gamma \) lie outside the real line and form complex conjugate pairs. Here \( \sigma \) denotes the transposition of sheets.

The kernels \( \omega(\lambda, z) \), \( \omega^\vee(\mu, z) \) can be easily calculated explicitly. Assuming \( \Psi^D(z) = e^{k_1 x} \tilde{\Psi}^D(y) \), \( \Phi^D(z) = e^{-k_1 x} \tilde{\Phi}^D(y) \), and integrating over \( x \) we obtain

\[
\omega(\lambda, z) = \frac{e^{(k(\lambda)-k_1)x}}{k(\lambda) - k_1} \left( \tilde{\Phi}_1^D(y)\tilde{\psi}_1(\lambda, y) - \tilde{\Phi}_2^D(y)\tilde{\psi}_2(\lambda, y) \right)
\]

\[
\omega^\vee(\mu, z) = \frac{e^{(k_1-k(\mu))x}}{k_1 - k(\mu)} \left( \tilde{\psi}_2(\sigma\mu, y)\tilde{\Psi}_1^D(y) - \tilde{\psi}_1(\sigma\mu, y)\tilde{\Psi}_2^D(y) \right)
\]

The spectral curve of the Clifford torus corresponds to the so-called whiskered torus of the NLS equation [12]. It means, that the Liouville torus is the product \( S^1 \times T \), where \( T \) denotes the one-point compactification of the product \( S^1 \times \mathbb{R} \). The Liouville torus is represented as a compact set in the phase space. The \( y \)-dynamics of the NLS equation and the phase gauging flow \( U \to U e^{i\phi} \) are
orthogonal to the infinite direction, and the time-evolution corresponds to the motion along the infinite cycle. The limiting point corresponds to the Clifford torus in $S^3$. The conformal flow is proportional to the time evolution plus phase gauging, but the coefficient between these flows became infinite near the fixed point, therefore using the conformal transformations one reaches the fixed point at finite time.

The whiskered tori are very important in the theory of the NLS equation, because they are the principal source of instability with respect to small perturbations, and, as a corollary, generate numerical chaos [1, 2, 12].

5.2 The spectral curve of the one-dimensional Schrödinger operator with a complex potential

We consider the one-dimensional Schrödinger operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + u(x)$$

associated with the KdV equation. We assume that $u(x)$ is a real periodic finite-gap potential: $u(x + T) = u(x)$. Let us consider the $g$-gap potential. The spectral gaps are $(-\infty, E_0), (E_1, E_2), \ldots, (E_{2g-1}, E_{2g})$. The surface $\Gamma$ parameterizing the Floquet–Bloch solutions is the two-sheeted covering of the complex plane ramified at the branch points $\infty, E_k, k = 0, \ldots, 2g$. Compactifying $\Gamma$ by adding an infinite point one obtains a smooth algebraic curve of genus $g$. The Floquet–Bloch function $\psi(x, P), P \in \Gamma$, is defined as a function on the Riemann surface $\Gamma$:

$$w^2 = Q(E) = (E - E_0) \ldots (E - E_{2g}),$$

with the asymptotic

$$\psi(P, x) = e^{i\sqrt{E}x} \left( 1 + O \left( \frac{1}{\sqrt{E}} \right) \right).$$

It is assumed that it is meromorphic for finite $E$ and it is uniquely fixed by its $g$ poles lying at the points $(\gamma_j, \sqrt{Q(\gamma_j)}), j = 1, \ldots, g$.

The operator $\mathcal{L}$ has infinitely many resonant points $\tilde{E}_j \in \mathbb{C}$, namely the energy levels at which the monodromy operator becomes equal to $\pm I$, where $I$ is the unit $2 \times 2$ matrix. These points can be treated as degenerate gaps of zero length. Denote the preimages of $\tilde{E}_j$ in $\Gamma$ by $\tilde{E}_j^\pm$. The pairs $\tilde{E}_j^\pm$ are called resonant pairs.

The immersion of $\Gamma$ into $\mathbb{C}^2$ is defined by $\mathcal{M}(\gamma) = (E, \kappa(\gamma))$, where $\kappa$ is the multiplier with respect to the shift $x \to x + T$. The curve $\mathcal{M}(\Gamma)$ has infinitely many self-intersections: $\mathcal{M}(E_j^+) = \mathcal{M}(E_j^-)$ for all $j$.

The set of all real potentials corresponding to this curve $\mathcal{M}(\Gamma)$ is a real $g$-dimensional torus, and the spectral curve parameterizing the variety of Floquet-Bloch functions coincides with $\Gamma$ for all members of this family. But if we

\footnote{In the terms of [2, 12] this complex curve may be considered as the complex curve of the two-dimensional Schrödinger operator $\partial_y - \partial_x^2 + a(x)$ on the zero energy level.}
consider the complex potentials corresponding to the curve $\mathcal{M}(\Gamma)$, the situation drastically changes. These potentials form an infinite-parametric family, and for generic members of this family the Floquet-Bloch solutions are parameterized by a curve, obtained from $\Gamma$ by gluing together all resonant pairs. From the algebro-geometrical point of view these potentials are infinite-gap. If only $k$ resonant pairs are glued, the corresponding potentials form a family of complex dimension $g+k$, and the arithmetic genus of the parameterizing spectral curve is equal to $g+k$.

At the analytic level the procedure of gluing of a resonant pair means, that an extra pole is added to the divisor, and simultaneously an extra linear relation is imposed on the wave function $\psi(\gamma,x)$: $\psi(E_j^+,x) = \psi(E_j^-,x)$ for all $x$.

The opposite operation of ungluing the pair $(E_j^+,E_j^-)$ corresponds to the following degeneration: one of the divisor points $\gamma_l$ tends to either $E_j^+$ or $E_j^-$. It is easy to check, then the residue at $\gamma_l$ is proportional to $\gamma_l - E_j$, and after taking the limit the pole vanishes, but the values of the wave functions at the points $E_j^+$ and $E_j^-$ become different.

References

[1] Ablowitz, M.J., Herbst, B.: Numerically induced chaos in the nonlinear Schrödinger equation. Phys. Rev. Lett. 62, (1989), 2065–2069

[2] Ablowitz, M. J., Herbst, B. M.: On homoclinic structure and numerically induced chaos for the nonlinear Schrödinger equation. SIAM J. Appl. Math. 50 (1990), 339–351

[3] Bobenko, A.I.: All constant mean curvature tori in $\mathbb{R}^3, S^3, H^3$ in terms of theta functions. Math. Ann. 290 (1991), 209–245.

[4] Dubrovin, B.A., Krichever, I.M., Novikov, S.P.: The Schrödinger equation in a periodic field and Riemann surfaces. Soviet Math. Dokl. 17 (1976), 947–952.

[5] Feldman, J., Knörer, H., Trubowitz, E.: Riemann surfaces of infinite genus. CRM Monograph Series, 20. American Mathematical Society, Providence, RI, 2003.

[6] Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori. Invent. Math. 146 (2001), 505–593.

[7] Grinevich P.G., Orlov A.Yu.: Virasoro action on Riemann surfaces, Grassmanians, $\det \bar{\partial}$ and Segal-Wilson $\tau$-function. In: Problems of modern quantum field theory. Eds. A.A.Belavin, A.U.Klimyk, A.B.Zamolodchikov, Springer-Verlag, 1989, 86-106.
[8] Grinevich, P.G., Schmidt, M.U.: Conformal invariant functionals of immersions of tori into $\mathbb{R}^3$. J. Geom. Phys. 26 (1997), 51–78.

[9] Hitchin, N.J.: Harmonic maps from a 2-torus to the 3-sphere. J. Differential Geom. 31 (1990), 627–710.

[10] Konopelchenko, B.G.: Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy. Annals of Global Anal. and Geom. 16 (2000), 61–74.

[11] Krichever, I.M.: Spectral theory of two-dimensional periodic operators and its applications. Russian Math. Surveys 44:2 (1989), 145–225.

[12] McLaughlin, D. W.: Whiskered tori for NLS equations. Important developments in soliton theory, 537–558, Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993.

[13] Melnikov, V.K.: New method for deriving nonlinear integrable systems. J. Math. Phys. 31:5 (1990), 1106–1113.

[14] Novikov, S.P.: Two-dimensional Schrödinger operators in periodic fields. Journal of Soviet Mathematics 28 (1985), 1–20.

[15] Pedit, F., Pinkall, U.: Quaternionic analysis on Riemann surfaces and differential geometry. Doc. Math., J.DMV Extra Vol. ICM II (1998), 189–200.

[16] Taimanov, I.A.: Modified Novikov–Veselov equation and differential geometry of surfaces. Amer. Math. Soc. Transl., Ser. 2, V. 179, 1997, pp. 133–151.

[17] Taimanov, I.A.: Surfaces in the four-space and the Davey–Stewartson equations. J. Geom. Phys. 56 (2006), 1235–1256.

[18] Taimanov, I.A.: Two-dimensional Dirac operator and surface theory. Russian Math. Surveys 61:1 (2006), 85–164.