2-LOCAL TRIPLE DERIVATIONS ON VON NEUMANN ALGEBRAS

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Abstract. We prove that every (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra $M$ is a triple derivation, equivalently, the set $\text{Der}_t(M)$, of all triple derivations on $M$, is algebraically 2-reflexive in the set $\mathcal{M}(M) = M^M$ of all mappings from $M$ into $M$.

1. Introduction

Let $X$ and $Y$ be Banach spaces. According to the terminology employed in the literature (see, for example, [4]), a subset $\mathcal{D}$ of the Banach space $B(X,Y)$, of all bounded linear operators from $X$ into $Y$, is called algebraically reflexive in $B(X,Y)$ when it satisfies the property:

\begin{equation}
T \in B(X,Y) \text{ with } T(x) \in \mathcal{D}(x), \forall x \in X \Rightarrow T \in \mathcal{D}.
\end{equation}

Algebraic reflexivity of $\mathcal{D}$ in the space $L(X,Y)$, of all linear mappings from $X$ into $Y$, a stronger version of the above property not requiring continuity of $T$, is defined by:

\begin{equation}
T \in L(X,Y) \text{ with } T(x) \in \mathcal{D}(x), \forall x \in X \Rightarrow T \in \mathcal{D}.
\end{equation}

In 1990, Kadison proved that (1.1) holds if $\mathcal{D}$ is the set $\text{Der}(M,X)$ of all (associative) derivations on a von Neumann algebra $M$ into a dual $M$-bimodule $X$ [18]. Johnson extended Kadison’s result by establishing that the set $\mathcal{D} = \text{Der}(A,X)$, of all (associative) derivations from a C*-algebra $A$ into a Banach $A$-bimodule $X$ satisfies (1.2) [17].

Algebraic reflexivity of the set of local triple derivations on a C*-algebra and on a JB*-triple have been studied in [24, 9, 12] and [14]. More precisely, Mackey proves in [24] that the set $\mathcal{D} = \text{Der}_t(M)$, of all triple derivations on a JBW*-triple $M$ satisfies (1.1). The result has been supplemented in [12], where Burgos, Fernández-Polo and the third author of this note prove that

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for each JB*-triple E, the set $\mathcal{D} = \text{Der}_t(E)$ of all triple derivations on $E$ satisfies (1.2).

Hereafter, \textit{algebraic reflexivity} will refer to the stronger version (1.2) which does not assume the continuity of $T$.

In [6], Bre\v{s}ar and \v{S}emrl proved that the set of all (algebra) automorphisms of $B(H)$ is algebraically reflexive whenever $H$ is a separable, infinite-dimensional Hilbert space. Given a Banach space $X$. A linear mapping $T : X \to X$ satisfying the hypothesis at (1.2) for $\mathcal{D} = \text{Aut}(X)$, the set of automorphisms on $X$, is called a local automorphism. Larson and Sourour showed in [22] that for every infinite dimensional Banach space $X$, every surjective local automorphism $T$ on the Banach algebra $B(X)$, of all bounded linear operators on $X$, is an automorphism.

Motivated by the results of \v{S}emrl in [31], references witness a growing interest in a subtle version of algebraic reflexivity called \textit{algebraic 2-reflexivity} (cf. [1, 2, 10, 11, 21, 23, 25, 26] and [29]). A subset $\mathcal{D}$ of the set $\mathcal{M}(X,Y) = Y^X$, of all mappings from $X$ into $Y$, is called algebraically 2-reflexive when the following property holds: for each mapping $T$ in $\mathcal{M}(X,Y)$ such that for each $a, b \in X$, there exists $S = S_{a,b} \in \mathcal{D}$ (depending on $a$ and $b$), with $T(a) = S_{a,b}(a)$ and $T(b) = S_{a,b}(b)$, then $T$ lies in $\mathcal{D}$. A mapping $T : X \to Y$ satisfying that for each $a, b \in X$, there exists $S = S_{a,b} \in \mathcal{D}$ (depending on $a$ and $b$), with $T(a) = S_{a,b}(a)$ and $T(b) = S_{a,b}(b)$ will be called a 2-local $\mathcal{D}$-mapping. If we assume that every mapping $s \in \mathcal{D}$ is $r$-homogeneous (that is, $S(ta) = t^r S(a)$ for every $t \in \mathbb{R}$ or $\mathbb{C}$) with $0 < r$, then every 2-local $\mathcal{D}$-mapping $T : X \to Y$ is $r$-homogeneous. Indeed, for each $a \in X$, $t \in \mathbb{C}$ take $S_{a,ta} \in \mathcal{D}$ satisfying $T(ta) = S_{a,ta}(ta) = t^r S_{a,ta}(a) = t^r T(a)$.

\v{S}emrl establishes in [31] that for every infinite-dimensional separable Hilbert space $H$, the sets $\text{Aut}(B(H))$ and $\text{Der}(B(H))$, of all (algebra) automorphisms and associative derivations on $B(H)$, respectively, are algebraically 2-reflexive in $\mathcal{M}(B(H)) = \mathcal{M}(B(H), B(H))$. Ayupov and the first author of this note proved in [1] that the same statement remains true for general Hilbert spaces (see [20] for the finite dimensional case). Actually, the set $\text{Hom}(A)$, of all homomorphisms on a general C*-algebra $A$, is algebraically 2-reflexive in the Banach algebra $B(A)$, of all bounded linear operators on $A$, and the set $\text{*-Hom}(A)$, of all *-homomorphisms on $A$, is algebraically 2-reflexive in the space $L(A)$, of all linear operators on $A$ (cf. [27]).

In recent contributions, Burgos, Fernández-Polo and the third author of this note prove that the set $\text{Hom}(M)$ (respectively, $\text{Hom}_t(M)$), of all homomorphisms (respectively, triple homomorphisms) on a von Neumann algebra (respectively, on a JBW*-triple) $M$, is an algebraically 2-reflexive subset of $\mathcal{M}(M)$ (cf. [10], [11], respectively), while Ayupov and the first author of this note establish that set $\text{Der}(M)$ of all derivations on $M$ is algebraically 2-reflexive in $\mathcal{M}(M)$ (see [2]).
In this paper, we consider the set $\text{Der}_t(A)$ of all triple derivations on a $C^*$-algebra $A$. We recall that every $C^*$-algebra $A$ can be equipped with a ternary product of the form
\[ \{a, b, c\} = \frac{1}{2}(ab^* c + cb^* a). \]
When $A$ is equipped with this product it becomes a JB$^*$-triple in the sense of [19]. A linear mapping $\delta : A \to A$ is said to be a triple derivation when it satisfies the (triple) Leibnitz rule:
\[ \delta\{a, b, c\} = \delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}. \]
It is known that every triple derivation is automatically continuous (cf. [3]). We refer to [3, 15] and [28] for the basic references on triple derivations. According to the standard notation, 2-local $\text{Der}_t(A)$-mappings from $A$ into $A$ are called 2-local triple derivations.

The goal of this note is to explore the algebraic 2-reflexivity of $\text{Der}_t(A)$ in $\mathcal{M}(A)$. Our main result proves that every (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra $M$ is a triple derivation (hence linear and continuous) (see Theorem 2.14), equivalently, $\text{Der}_t(M)$ is algebraically 2-reflexive in $\mathcal{M}(M)$.

2. 2-LocAl triple Derivations on von Neumann Algebras

We start by recalling some generalities on triple derivations. Let $A$ be a $C^*$-algebra. For each $b \in A$, we shall denote by $M_b$ the Jordan multiplication mapping by the element $b$, that is $M_b(x) = b \cdot x = \frac{1}{2}(bx + xb)$. Following standard notation, given elements $a, b$ in $A$, we denote by $L(a, b)$ the operator on $A$ defined by $L(a, b)(x) = \{a, b, x\} = \frac{1}{2}(ab^* x + xb^* a)$. It is known that the mapping $\delta(a, b) : A \to A$, given by
\[ \delta(a, b)(x) = L(a, b)(x) - L(b, a)(x), \]
is a triple derivation on $A$ (cf. [3, 15]), called an inner triple derivation.

Let $\delta : A \to A$ be a triple derivation on a unital $C^*$-algebra. By [15, Lemmas 1 and 2], $\delta(1)^* = -\delta(1)$, and $M_{\delta(1)} = \delta$$\left(\frac{1}{2}\delta(1), 1\right)$ is an inner triple derivation on $A$ and the difference $D = \delta - \delta(\frac{1}{2}\delta(1), 1)$ is a Jordan $*$-derivation on $A$, more concretely,
\[ D(x \circ y) = D(x) \circ y + x \circ D(y), \text{ and } D(x^*) = D(x)^*, \]
for every $x, y \in A$. By [3, Corollary 2.2], $\delta$ (and hence $D$) is a continuous operator. A widely known result, due to B.E. Johnson, states that every bounded Jordan derivation from a $C^*$-algebra $A$ to a Banach $A$-bimodule is an associative derivation (cf. [16]). Therefore, $D$ is an associative $*$-derivation in the usual sense. When $A = M$ is a von Neumann algebra, we can guarantee that $D$ is an inner derivation, that is there exists $a \in A$ satisfying $D(x) = [a, x] = ax - xa$, for every $x \in A$ (cf. [30, Theorem 4.1.6]). Further, from the condition $D(x^*) = D(x)^*$, for every $x \in A$, we
deduce that \((\bar{a}^* + \bar{a})x = x(\bar{a}^* + \bar{a})\). Thus, taking \(a = \frac{1}{2}(\bar{a} - \bar{a}^*)\), it follows that \([a, x] = [\bar{a}, x]\), for every \(x \in M\). We have therefore shown that for every triple derivation \(\delta\) on a von Neumann algebra \(M\), there exist skew-hermitian elements \(a, b \in M\) satisfying
\[
\delta(x) = [a, x] + b \circ x,
\]
for every \(x \in M\).

Our first lemma is a direct consequence of the above arguments (see [15, Lemmas 1 and 2]).

**Lemma 2.1.** Let \(T : A \to A\) be a (not necessarily linear nor continuous) 2-local triple derivation on a unital \(C^*\)-algebra. Then

(a) \(T(1)^* = -T(1)\);

(b) \(M_{T(1)} = \delta \left( \frac{1}{2}T(1), 1 \right)\) is an inner triple derivation on \(A\);

(c) \(\hat{T} = T - \delta \left( \frac{1}{2}T(1), 1 \right)\) is a 2-local triple derivation on \(A\) with \(\hat{T}(1) = 0\).

\(\square\)

In what follows, we denote by \(A_{sa}\) the hermitian elements of the \(C^*\)-algebra \(A\).

**Lemma 2.2.** Let \(T : A \to A\) be a (not necessarily linear nor continuous) 2-local triple derivation on a unital \(C^*\)-algebra satisfying \(T(1) = 0\). Then \(T(x) = T(x)^*\) for all \(x \in A_{sa}\).

**Proof.** Let \(x \in A_{sa}\). By assumptions,
\[
T(x)^* = \{1, T(x), 1\} = \{1, \delta_{x,1}(x), 1\} = \delta_{x,1}(1, x, 1) - 2\{\delta_{x,1}(1), x, 1\}
\]
\[
= \delta_{x,1}(x^*) - 2\{T(1), x, 1\} = \delta_{x,1}(x) = T(x).
\]
The proof is complete. \(\square\)

**Lemma 2.3.** Let \(T : M \to M\) be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra satisfying \(T(1) = 0\). Then for every \(x, y \in M_{sa}\) there exists a skew-hermitian element \(a_{x,y} \in M\) such that
\[
T(x) = [a_{x,y}, x], \quad \text{and,} \quad T(y) = [a_{x,y}, y].
\]

**Proof.** For every \(x, y \in M_{sa}\) we can find skew-hermitian elements \(a_{x,y}, b_{x,y} \in M\) such that
\[
T(x) = [a_{x,y}, x] + b_{x,y} \circ x, \quad \text{and,} \quad T(y) = [a_{x,y}, y] + b_{x,y} \circ y.
\]
Taking into account that \(T(x) = T(x)^*\) (see Lemma 2.2) we obtain
\[
[a_{x,y}, x] + b_{x,y} \circ x = T(x) = T(x)^* = [a_{x,y}, x]^* + (b_{x,y} \circ x)^*
\]
\[
= [x, a_{x,y}^*] + x \circ b_{x,y}^* = [x, -a_{x,y}] - x \circ b_{x,y} = [a_{x,y}, x] - b_{x,y} \circ x,
\]
i.e. \(b_{x,y} \circ x = 0\), and similarly \(b_{x,y} \circ y = 0\). Therefore \(T(x) = [a_{x,y}, x]\), \(T(y) = [a_{x,y}, y]\), and the proof is complete. \(\square\)

We state now an observation, which plays an useful role in our study.
Lemma 2.4. Let \( a \) and \( b \) be skew-hermitian elements in a \( C^* \)-algebra \( A \). Suppose \( x \in A \) is self-adjoint with \( ax - xa + bx + xb = 0 \). Then \( ax = xa \), and \( bx = -xb \).

Proof. Since \( 0 = ax - xa + bx + xb \). Passing to the adjoint, we obtain \( ax - xa - (bx + xb) = 0 \). Conclude the proof by adding and subtracting these two equalities. The proof is complete. \( \square \)

Let \( M \) be a von Neumann algebra. If \( x \in M_{sa} \), we denote by \( s(x) \) the support projection of \( x \) — that is, the projection onto \( (\ker(x))^\perp = \text{ran}(x) \). We say that \( x \) has full support if \( s(x) = 1 \) (equivalently, \( \ker(x) = \{0\} \)).

Lemma 2.5. Let \( M \) be a von Neumann algebra. Suppose \( u \in M_+ \) has full support, \( c \in M \) is self-adjoint, and \( \sigma(c^2u) \cap (0, \infty) = \emptyset \). Then \( c = 0 \).

Consequently, if \( u \) and \( c \) are as above, and \( uc + cu = 0 \) (or \( c^2u = -cuc \leq 0 \)), then \( c = 0 \).

Proof. For the fist statement of the lemma, suppose \( \sigma(c^2u) \cap (0, \infty) = \emptyset \). Note that

\[
(-\infty, 0] \supseteq \sigma(c^2u) \cup \{0\} = \sigma(c \cdot cu) \supseteq \sigma(cuc).
\]

However, \( cuc \) is positive, hence \( \sigma(cuc) \subset [0, \|cuc\|] \), with \( \max_{\lambda \in \sigma(cuc)} = \|cuc\| \). Thus, \( cu^{1/2}u^{1/2}c = cuc = 0 \), which means that \( cu^{1/2} = u^{1/2}c = 0 \) and hence \( s(c) \subset 1 - (u^{1/2}) = 1 - s(u) = 0 \), which leads to \( c = 0 \).

To prove the second part, we have \( c^2u = -cuc \leq 0 \), hence in particular, \( \sigma(c^2u) \subset (-\infty, 0] \). The proof is complete. \( \square \)

In [2, Lemma 2.2], Ayupov and the first author of this note prove that for every (not necessarily linear nor continuous) 2-local derivation on a von Neumann algebra \( \Delta : M \to M \), and every self-adjoint element \( z \in M \), there exists \( a \in M \) satisfying

\[
\Delta(x) = [a, x],
\]
for every \( x \in \mathcal{W}^*(z) \), where \( \mathcal{W}^*(z) = \{z\}'' \) denotes the abelian von Neumann subalgebra of \( M \) generated by the element \( z \), and the unit element and \( \{z\}'' \) denotes the bicommutant of the set \( \{z\} \). We prove next a ternary version of this result.

**Lemma 2.6.** Let \( T : M \to M \) be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. Let \( z \in M \) be a self-adjoint element and let \( \mathcal{W}^*(z) = \{z\}'' \) be the abelian von Neumann subalgebra of \( M \) generated by the element \( z \) and the unit element. Then there exist skew-hermitian elements \( a_z, b_z \in M \), depending on \( z \), such that

\[
T(x) = [a_z, x] + b_z \circ x = a_z x - xa_z + \frac{1}{2} (b_z x + xb_z)
\]

for all \( x \in \mathcal{W}^*(z) \). In particular, \( T \) is linear on \( \mathcal{W}^*(z) \).
Proof. We can assume that \( z \neq 0 \). Note that the abelian von Neumann subalgebras generated by \( 1 \) and \( z \) and by \( 1 \) and \( 1 + \frac{z}{2\|z\|} \) coincide. So, replacing \( z \) with \( 1 + \frac{z}{2\|z\|} \) we can assume that \( z \) is an invertible positive element.

By definition, there exist skew-hermitian elements \( a_z, b_z \in M \) (depending on \( z \)) such that
\[
T(z) = [a_z, z] + b_z \circ z.
\]
Define a mapping \( T_0 : M \to M \) given by
\[
T_0(x) = T(x) - ([a_z, z] + b_z \circ z),
\]
for every \( x \in M \). Clearly, \( T_0 \) is a 2-local triple derivation on \( M \). We shall show that \( T_0 \equiv 0 \) on \( W^*(z) \). Let \( x \in W^*(z) \) be an arbitrary element. By assumptions, there exist skew-hermitian elements \( c_{z,x}, d_{z,x} \in M \) such that
\[
T_0(z) = [c_{z,x}, z] + d_{z,x} \circ z, \quad \text{and,} \quad T_0(x) = [c_{z,x}, x] + d_{z,x} \circ x.
\]
Since
\[
0 = T_0(z) = [c_{z,x}, z] + d_{z,x} \circ z,
\]
we get
\[
[c_{z,x}, z] + d_{z,x} \circ z = 0.
\]
Taking into account that \( z \) is a hermitian element and Lemma 2.4 we get
\[
c_{z,x}z = zc_{z,x} \quad \text{and} \quad d_{z,x}z = -zd_{z,x}.
\]
Since \( z \) has a full support, and \( d_{z,x}^2z = -zd_{z,x} \), Lemma 2.5 implies that \( d_{z,x} = 0 \). Further
\[
c_{z,x} \in \{z\}' = \{z\}'' = W^*(z)',
\]
i.e. \( c_{z,x} \) commutes with any element in \( W^*(z) \). Therefore
\[
T_0(x) = [c_{z,x}, x] + d_{z,x} \circ x = 0
\]
for all \( x \in W^*(z) \). The proof is complete. \( \square \)

2.1. Complete additivity of 2-local derivations and 2-local triple derivations on von Neumann algebras.

Let \( \mathcal{P}(M) \) denote the lattice of projections in a von Neumann algebra \( M \). Let \( X \) be a Banach space. A mapping \( \mu : \mathcal{P}(M) \to X \) is said to be finitely additive when
\[
\mu \left( \sum_{i=1}^{n} p_i \right) = \sum_{i=1}^{n} \mu(p_i),
\]
for every family \( p_1, \ldots, p_n \) of mutually orthogonal projections in \( M \). A mapping \( \mu : \mathcal{P}(M) \to X \) is said to be bounded when the set
\[
\{ \|\mu(p)\| : p \in \mathcal{P}(M) \}
\]
is bounded.

The celebrated Bunce-Wright-Mackey-Gleason theorem ([7, 8]) states that if \( M \) has no summand of type \( I_2 \), then every bounded finitely additive mapping \( \mu : \mathcal{P}(M) \to X \) extends to a bounded linear operator from \( M \) to \( X \).
According to the terminology employed in [32] and [13], a completely additive mapping \( \mu : \mathcal{P}(M) \to \mathbb{C} \) is called a charge. The Dorofeev–Sherstnev theorem ([32, Theorem 29.5] or [13, Theorem 2]) states that any charge on a von Neumann algebra with no summands of type \( I_n \) is bounded.

We shall use the Dorofeev-Shertsnev theorem in Corollary 2.8 in order to be able to apply the Bunce-Wright-Mackey-Gleason theorem in Proposition 2.9. To this end, we need Proposition 2.7, which is implicitly applied in [2, proof of Lemma 2.3] for 2-local associative derivations. A proof is included here for completeness reasons.

First, we recall some facts about the strong*-topology. For each normal positive functional \( \varphi \) in the predual of a von Neumann algebra \( M \), the mapping \( x \mapsto \|x\|_\varphi = \left( \varphi \left( \frac{xx^* + x^*x}{2} \right) \right)^{\frac{1}{2}} \) defines a prehilbertian seminorm on \( M \). The strong* topology of \( M \) is the locally convex topology on \( M \) defined by all the seminorms \( \| - \|_\varphi \), where \( \varphi \) runs in the set of all positive functionals in \( M_* \) (cf. [30, Definition 1.8.7]). It is known that the strong* topology of \( M \) is compatible with the duality \( (M, M_*) \), that is a functional \( \psi : M \to \mathbb{C} \) is strong* continuous if and only if it is weak* continuous (see [30, Corollary 1.8.10]). We also recall that the product of every von Neumann algebra is jointly strong* continuous on bounded sets (see [30, Proposition 1.8.12]).

Suppose \( X = W \) is another von Neumann algebra, and let \( \tau \) denote the norm-, the weak*- or the strong*-topology of \( W \). The mapping \( \mu \) is said to be \( \tau \)-completely additive (respectively, countably or sequentially \( \tau \)-additive) when

\[
(2.1) \quad \mu \left( \sum_{i \in I} e_i \right) = \tau- \sum_{i \in I} \mu(e_i)
\]

for every family (respectively, sequence) \( \{e_i\}_{i \in I} \) of mutually orthogonal projections in \( M \).

It is known that every family \( (p_i)_{i \in I} \) of mutually orthogonal projections in a von Neumann algebra \( M \) is summable with respect to the weak*-topology of \( M \) and \( p = \text{weak}^* - \sum_{i \in I} p_i \) is a projection in \( M \) (cf. [30, Definition 1.13.4]).

Further, for each normal positive functional \( \phi \) in \( M_* \) and every finite set \( F \subset I \), we have

\[
\left\| p - \sum_{i \in F} p_i \right\|_\phi^2 = \phi \left( p - \sum_{i \in F} p_i \right),
\]

which implies that the family \( (p_i)_{i \in I} \) is summable with respect to the strong*-topology of \( M \) with the same limit, that is, \( p = \text{strong}^* - \sum_{i \in I} p_i \).
Proposition 2.7. Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. Then the following statements hold:

(a) The restriction $T|_{\mathcal{P}(M)}$ is sequentially strong$^*$-additive, and consequently sequentially weak$^*$-additive;

(b) $T|_{\mathcal{P}(M)}$ is weak$^*$-completely additive, i.e.,

$$
(2.2) \quad T \left( \sum_{i \in I} p_i \right) = \sum_{i \in I} T(p_i)
$$

for every family $(p_i)_{i \in I}$ of mutually orthogonal projections in $M$.

Proof. (a) Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of mutually orthogonal projections in $M$. Let us consider the element $z = \sum_{n \in I} \frac{1}{n} p_n$. By Lemma 2.6 there exist skew-hermitian elements $a_z, b_z \in M$ such that $T(x) = [a_z, x] + b_z \circ x$ for all $x \in W^*(z)$. Since $\sum_{n=1}^{\infty} p_n, p_m \in W^*(z)$, for all $m \in \mathbb{N}$, and the product of $M$ is jointly strong$^*$-continuous, we obtain that

$$
T \left( \sum_{n=1}^{\infty} p_n \right) = \left[ a_z, \sum_{n=1}^{\infty} p_n \right] + b_z \circ \left( \sum_{n=1}^{\infty} p_n \right)
$$

$$
= \sum_{n=1}^{\infty} [a_z, p_n] + \sum_{n=1}^{\infty} b_z \circ p_n = \sum_{n=1}^{\infty} T(p_n),
$$

i.e. $T|_{\mathcal{P}(M)}$ is a countably or sequentially strong$^*$-additive mapping.

(b) Let $\varphi$ be a positive normal functional in $M_*$, and let $\| \|_\varphi$ denote the prehilbertian seminorm given by $\| z \|_\varphi^2 = \frac{1}{2} \varphi(z z^* + z^* z)$ ($z \in M$). Let $\{p_i\}_{i \in I}$ be an arbitrary family of mutually orthogonal projections in $M$. For every $n \in \mathbb{N}$ define

$$
I_n = \{ i \in I : \| T(p_i) \|_\varphi \geq 1/n \}.
$$

We claim, that $I_n$ is a finite set for every natural $n$. Otherwise, passing to a subset if necessary, we can assume that there exists a natural $k$ such that $I_k$ is infinite and countable. In this case the series $\sum_{i \in I_k} T(p_i)$ does not converge with respect to the semi-norm $\| \|_\varphi$. On the other hand, since $I_k$ is a countable set, by (a), we have

$$
T \left( \sum_{i \in I_k} p_i \right) = \text{strong}^* - \sum_{i \in I_k} T(p_i),
$$

which is impossible. This proves the claim.

We have shown that the set

$$
I_0 = \left\{ i \in I : \| T(p_i) \|_\varphi \neq 0 \right\} = \bigcup_{n \in \mathbb{N}} I_n
$$
is a countable set, and \( \|T(p_i)\|_\varphi = 0 \), for every \( i \in I \setminus I_0 \).

Set \( p = \sum_{i \in I \setminus I_0} p_i \in M \). We shall show that \( \varphi(T(p)) = 0 \). Let \( q \) denote the support projection of \( \varphi \) in \( M \). Having in mind that \( \|T(p_i)\|_\varphi^2 = 0 \), for every \( i \in I \setminus I_0 \), we deduce that \( T(p_i) \perp q \) for every \( i \in I \setminus I_0 \).

Replacing \( T \) with \( \hat{T} = T - \delta(\frac{1}{2} T(1), 1) \) we can assume that \( T(1) = 0 \) (cf. Lemma 2.1) and \( T(x) = T(x)^* \), for every \( x \in M_{sa} \) (cf. Lemma 2.2). By Lemma 2.3, for every \( i \in I \setminus I_0 \) there exists a skew-hermitian element \( a_i = a_{p_i} \in M \) such that

\[
T(p) = a_i p - p a_i, \quad \text{and} \quad T(p_i) = a_i p_i - p_i a_i.
\]

Since \( T(p_i) \perp q \) we get \( (a_i p_i - p_i a_i)q = q(a_i p_i - p_i a_i) = 0 \), for all \( i \in I \setminus I_0 \). Thus, since \( p a_i p_i q = p_i a_i q \),

\[
(T(p)p_i)q = (a_i p - p a_i)p_i q = a_i p_i q - p a_i p_i q
\]

and similarly

\[
q(p_i T(p)) = 0,
\]

for every \( i \in I \setminus I_0 \). Consequently,

\[
(2.3) \quad (T(p)p)q = T(p) \left( \sum_{i \in I \setminus I_0} p_i \right) q = 0 = q \left( \sum_{i \in I \setminus I_0} p_i \right) T(p) = q(pT(p)).
\]

Therefore,

\[
T(p) = \delta_{p,1}(p) = \delta_{p,1}(p, p, p) = 2\{\delta_{p,1}(p), p, p\} + \{p, \delta_{p,1}(p), p\}
\]

\[
= 2\{T(p), p, p\} + \{p, T(p), p\} = pT(p) + T(p)p + pT(p)^* p
\]

\[
= pT(p) + T(p)p + pT(p)p,
\]

which implies that

\[
\varphi(T(p)) = \varphi(pT(p) + T(p)p + pT(p)p)
\]

\[
= \varphi(qpT(p)q) + \varphi(qT(p)pq) + \varphi(qpT(p)pq) = (by\ (2.3)) = 0.
\]

Finally, by (a) we have

\[
T \left( \sum_{i \in I_0} p_i \right) = \| \cdot \|_\varphi \sum_{i \in I_0} T(p_i).
\]

Two more applications of (a) give:

\[
\varphi \left( T \left( \sum_{i \in I} p_i \right) \right) = \varphi \left( T \left( p + \sum_{i \in I_0} p_i \right) \right) = \varphi \left( T(p) + T \left( \sum_{i \in I_0} p_i \right) \right)
\]
= ϕ(T(p)) + \varphi \left( T \left( \sum_{i \in I_0} p_i \right) \right) = \sum_{i \in I_0} \varphi(T(p_i)).$

By the Cauchy-Schwarz inequality, $0 \leq |\varphi(T(p))|^2 \leq \|T(p_i)\|_{\varphi}^2 = 0$, for every $i \in I \setminus I_0$, and hence $\sum_{i \in I_0} \varphi(T(p_i)) = \sum_{i \in I} \varphi(T(p_i))$. The arbitrariness of $\varphi$ shows that $T\left(\operatorname{weak}^*-\sum_{i \in I} p_i\right) = \operatorname{weak}^*-\sum_{i \in I} T(p_i).$ \hfill \square

Let $\phi$ be a normal functional in the predual of a von Neumann algebra $M$. Our previous Proposition 2.7 assures that for every (not necessarily linear nor continuous) 2-local triple derivation $T : M \to M$ the mapping $\phi \circ T|_{\mathcal{P}(M)} : \mathcal{P}(M) \to \mathbb{C}$ is a completely additive mapping or a charge on $M$. Under the additional hypothesis of $M$ being a continuous von Neumann algebra or, more generally, a von Neumann algebra with no Type I$_n$-factors ($1 < n < \infty$) direct summands (i.e. without direct summand isomorphic to a matrix algebra $M_n(\mathbb{C})$, $1 < n < \infty$), the Dorofeev–Sherstnev theorem ([32, Theorem 29.5] or [13, Theorem 2]) imply that $\phi \circ T|_{\mathcal{P}(M)}$ is a bounded charge, that is, the set $\{|\phi \circ T(p)| : p \in \mathcal{P}(M)\}$ is bounded. The uniform boundedness principle gives:

**Corollary 2.8.** Let $M$ be a von Neumann algebra with no Type I$_n$-factor direct summands ($1 < n < \infty$) and let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then the restriction $T|_{\mathcal{P}(M)}$ is a bounded weak$^*$-completely additive mapping. \square

### 2.2. Additivity of 2-local triple derivations on hermitian parts of von Neumann algebras.

Suppose now that $M$ is a von Neumann algebra with no Type I$_n$-factor direct summands ($1 < n < \infty$), and $T : M \to M$ is a (not necessarily linear nor continuous) 2-local triple derivation. By Corollary 2.8 combined with the Bunce-Wright-Mackey-Gleason theorem [7, 8], there exits a bounded linear operator $G : M \to M$ satisfying that $G(p) = T(p)$, for every projection $p \in M$.

Let $z$ be a self-adjoint element in $M$. By Lemma 2.6, there exist skew-hermitian elements $a_z, b_z \in M$ such that $T(x) = [a_z, x] + b_z \circ x$, for every $x \in \mathcal{W}^*(z)$. Since $G|_{\mathcal{W}^*(z)}, T|_{\mathcal{W}^*(z)} : \mathcal{W}^*(z) \to M$ are bounded linear operators, which coincide on the set of projections of $\mathcal{W}^*(z)$, and every self-adjoint element in $\mathcal{W}^*(z)$ can be approximated in norm by finite linear combinations of mutually orthogonal projections in $\mathcal{W}^*(z)$, it follows that $T(x) = G(x)$ for every $x \in \mathcal{W}^*(z)$, and hence

$T(a) = G(a)$, for every $a \in M_{sa}$,

in particular, $T$ is additive on $M_{sa}$.

The above arguments materialize in the following result.
Proposition 2.9. Let \( T : M \to M \) be a \( ( \text{not necessarily linear nor continuous} \) \) 2-local triple derivation on a von Neumann algebra with no Type \( I_n \)-factor direct summands \( (1 < n < \infty) \). Then the restriction \( T|_{M_{s.a}} \) is additive. \( \square \)

Corollary 2.10. Let \( T : M \to M \) be a \( ( \text{not necessarily linear nor continuous} \) \) 2-local triple derivation on a properly infinite von Neumann algebra. Then the restriction \( T|_{M_{s.a}} \) is additive.

Next we shall show that the conclusion of the above corollary is also true for a finite von Neumann algebra.

First we show that every 2-local triple derivation on a von Neumann algebra “intertwines” central projections.

Lemma 2.11. If \( T \) is a \( ( \text{not necessarily linear nor continuous} \) \) 2-local triple derivation on a von Neumann algebra \( M \), and \( p \) is a central projection in \( M \), then \( T(Mp) \subset Mp \). In particular, \( T(px) = pT(x) \) for every \( x \in M \).

Proof. Consider \( x \in Mp \), then \( x = pxp = \{x, p, p\} \). \( T \) coincides with a triple derivation \( \delta_{x,p} \) on the set \( \{x, p\} \), hence \( T(x) = \delta_{x,p}(x) = \delta_{x,p}\{x, p, p\} = \{\delta_{x,p}(x), p, p\} + \{x, \delta_{x,p}(p), p\} + \{x, p, \delta_{x,p}(p)\} \) lies in \( Mp \).

For the final statement, fix \( x \in M \), and consider skew-hermitian elements \( a_{x, xp}, b_{x, xp} \in M \) satisfying

\[
T(x) = \begin{bmatrix} a_{x,xp}, x \end{bmatrix} + b_{x, xp} \circ x, \text{ and } T(xp) = \begin{bmatrix} a_{x,xp}, xp \end{bmatrix} + b_{x, xp} \circ (xp).
\]

The assumption \( p \) being central implies that \( pT(x) = T(px). \) \( \square \)

Proposition 2.12. Let \( T : M \to M \) be a \( ( \text{not necessarily linear nor continuous} \) \) 2-local triple derivation on a finite von Neumann algebra. Then the restriction \( T|_{M_{s.a}} \) is additive.

Proof. Since \( M \) is finite there exists a faithful normal semi-finite trace \( \tau \) on \( M \). We shall consider the following two cases.

Case 1. Suppose \( \tau \) is a finite trace. Replacing \( T \) with \( \tilde{T} = T - \delta\left(\frac{1}{2}T(1), 1\right) \) we can assume that \( T(1) = 0 \) (cf. Lemma 2.1) and \( T(x) = T(x)^* \), for every \( x \in M_{s.a} \) (cf. Lemma 2.2). By Lemma 2.3, for every \( x, y \in M_{s.a} \) there exists a skew-hermitian element \( a_{x,y} \in M \) such that \( T(x) = \begin{bmatrix} a_{x,y}, x \end{bmatrix} \) and \( T(y) = \begin{bmatrix} a_{x,y}, y \end{bmatrix} \). Then

\[
T(x)y + xT(y) = \begin{bmatrix} a_{x,y}, x \end{bmatrix}y + x\begin{bmatrix} a_{x,y}, y \end{bmatrix} = \begin{bmatrix} a_{x,y}, xy \end{bmatrix},
\]

that is,

\[
[a_{x,y}, xy] = T(x)y + xT(y).
\]

Further

\[
0 = \tau(\begin{bmatrix} a_{x,y}, xy \end{bmatrix}) = \tau(T(x)y + xT(y)),
\]
i.e. \( \tau(T(x)y) = -\tau(xT(y)) \), for every \( x, y \in M_{sa} \). For arbitrary \( u, v, w \in M_{sa} \), set \( x = u + v \), and \( y = w \). The above identity implies
\[
\tau(T(u+v)w) = -\tau((u+v)T(w)) = \\
= -\tau(uT(w)) - \tau(vT(w)) = \tau(T(u)w) + \tau(T(v)w) = \tau((T(u) + T(v))w),
\]
and so
\[
\tau((T(u+v) - T(u) - T(v))w) = 0
\]
for all \( u, v, w \in M_{sa} \). Take \( w = T(u + v) - T(u) - T(v) \). Then \( \tau(wu^*) = 0 \).

Since the trace \( \tau \) is faithful it follows that \( uw^* = 0 \), and hence \( w = 0 \). Therefore
\[
T(u + v) = T(u) + T(v).
\]

**Case 2.** As in **Case 1**, we may assume \( T(1) = 0 \). Suppose now that \( \tau \) is a semi-finite trace. Since \( M \) is finite there exists a family of mutually orthogonal central projections \( \{z_i\} \) in \( M \) such that \( z_i \) has finite trace for all \( i \) and \( \bigvee z_i = 1 \) (cf. [30, §2.2 or Corollary 2.4.7]). By Lemma 2.11, for each \( i \), \( T \) maps \( z_iM \) into itself. From Case 1, \( T|_{z_iM} : z_iM \to z_iM \) is additive. Furthermore,
\[
z_iT(x + y) = T|_{z_iM}(z_ix + z_iy) = T|_{z_iM}(z_ix) + T|_{z_iM}(z_iy) = ziT(x) + ziT(y),
\]
for every \( x, y \in M \) and every \( i \). Therefore
\[
T(x + y) = \left( \sum_i z_i \right) T(x + y) = \sum_i z_iT(x + y) = \sum_i (z_iT(x) + z_iT(y))
\]
\[
= \left( \sum_i z_i \right) T(x) + \left( \sum_i z_i \right) T(y) = T(x) + T(y),
\]
for every \( x, y \in M \). The proof is complete. \( \square \)

Let \( T : M \to M \) be a (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra. In this case there exist orthogonal central projections \( z_1, z_2 \in M \) with \( z_1 + z_2 = 1 \) such that:

1. \( z_1M \) is a finite von Neumann algebra;
2. \( z_2M \) is a properly infinite von Neumann algebra,

(cf. [30, §2.2]).

By Lemma 2.11, for each \( k = 1, 2 \), \( z_kT \) maps \( z_kM \) into itself. By Corollary 2.10 and Proposition 2.12 both \( z_1T \) and \( z_2T \) are additive on \( M_{sa} \). So \( T = z_1T + z_2T \) also is additive on \( M_{sa} \).

We have thus proved the following result:

**Proposition 2.13.** Let \( T : M \to M \) be a (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra. Then the restriction \( T|_{M_{sa}} \) is additive. \( \square \)
2. Main result.

We can state now the main result of this paper.

**Theorem 2.14.** Let $M$ be an arbitrary von Neumann algebra and let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then $T$ is a triple derivation (hence linear and continuous). Equivalently, the set $\text{Der}_T(M)$, of all triple derivations on $M$, is algebraically 2-reflexive in the set $M(M) = M^M$ of all mappings from $M$ into $M$.

We need the following two Lemmata.

**Lemma 2.15.** Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra with $T(1) = 0$. Then there exists a skew-hermitian element $a \in M$ such that $T(x) = [a, x]$, for all $x \in M_{sa}$.

**Proof.** Let $x \in M_{sa}$. By Lemma 2.3 there exist a skew-hermitian element $a_{x,x^2} \in M$ such that

$$T(x) = [a_{x,x^2}, x], \quad T(x^2) = [a_{x,x^2}, x^2].$$

Thus

$$T(x^2) = [a_{x,x^2}, x^2] = [a_{x,x^2}, x]x + x[a_{x,x^2}, x] = T(x)x + xT(x),$$

i.e.

$$T(x^2) = T(x)x + xT(x),$$

for every $x \in M_{sa}$.

By Proposition 2.13 and Lemma 2.2, $T|_{M_{sa}} : M_{sa} \to M_{sa}$ is a real linear mapping. Now, we consider the linear extension $\hat{T}$ of $T|_{M_{sa}}$ to $M$ defined by

$$\hat{T}(x_1 + ix_2) = T(x_1) + iT(x_2), \quad x_1, x_2 \in M_{sa}.$$

Taking into account the homogeneity of $T$, Proposition 2.13 and the identity (2.4) we obtain that $\hat{T}$ is a Jordan derivation on $M$. By [5, Theorem 1] any Jordan derivation on a semi-prime algebra is a derivation. Since $M$ is von Neumann algebra, $\hat{T}$ is a derivation on $M$ (see also [33] and [16]). Therefore there exists an element $a \in M$ such that $\hat{T}(x) = [a, x]$ for all $x \in M$. In particular, $T(x) = [a, x]$ for all $x \in M_{sa}$. Since $T(M_{sa}) \subseteq M_{sa}$, we can assume that $a^* = -a$, which completes the proof.

**Lemma 2.16.** Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. If $T|_{M_{sa}} \equiv 0$, then $T \equiv 0$.

**Proof.** Let $x \in M$ be an arbitrary element and let $x = x_1 + ix_2$, where $x_1, x_2 \in M_{sa}$. Since $T$ is homogeneous, if necessary, passing to the element $(1 + \|x_2\|)^{-1}x$, we can suppose that $\|x_2\| < 1$. In this case the element $y =
$1 + x_2$ is positive and invertible. Take skew-hermitian elements $a_{x,y}, b_{x,y} \in M$ such that

\[
T(x) = [a_{x,y}, x] + b_{x,y} \circ x, \\
T(y) = [a_{x,y}, y] + b_{x,y} \circ y.
\]

Since $T(y) = 0$, we get $[a_{x,y}, y] + b_{x,y} \circ y = 0$. By Lemma 2.4 we obtain that $[a_{x,y}, y] = 0$ and $ib_{x,y} \circ y = 0$. Taking into account that $ib_{x,y}$ is hermitian, $y$ is positive and invertible, Lemma 2.5 implies that $b_{x,y} = 0$.

We further note that

\[
0 = [a_{x,y}, y] = [a_{x,y}, 1 + x_2] = [a_{x,y}, x_2],
\]

i.e.

\[
[a_{x,y}, x_2] = 0.
\]

Now,

\[
T(x) = [a_{x,y}, x] + b_{x,y} \circ x = [a_{x,y}, x_1 + ix_2] = [a_{x,y}, x_1],
\]

i.e.

\[
T(x) = [a_{x,y}, x_1].
\]

Therefore,

\[
T(x)^* = [a_{x,y}, x_1]^* = [x_1, a_{x,y}^*] = [x_1, -a_{x,y}] = [a_{x,y}, x_1] = T(x).
\]

So

(2.5) \hspace{1cm} T(x)^* = T(x).

Now replacing $x$ by $ix$ on (2.5) we obtain from the homogeneity of $T$ that

(2.6) \hspace{1cm} T(x)^* = -T(x).

Combining (2.5) and (2.6) we obtain that $T(x) = 0$, which finishes the proof. \hfill \Box

**Proof of Theorem 2.14.** Let us define $\widehat{T} = T - \delta \left( \frac{1}{2} \Theta(1), 1 \right)$. Then $\widehat{T}$ is a 2-local triple derivation on $M$ with $\widehat{T}(1) = 0$ (cf. Lemma 2.1) and $\widehat{T}(x) = \widehat{T}(x)^*$, for every $x \in Msa$ (cf. Lemma 2.2). By Lemma 2.15 there exists an element $a \in M$ such that $\widehat{T}(x) = [a, x]$ for all $x \in Msa$. Consider the 2-local triple derivation $\widehat{T} - [a, \cdot]$. Since $(\widehat{T} - [a, \cdot])|_{Msa} \equiv 0$, Lemma 2.16 implies that $\widehat{T} = [a, \cdot]$, and hence $T = [a, \cdot] + \delta \left( \frac{1}{2} \Theta(1), 1 \right)$, witnessing the desired statement. \hfill \Box

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