It is shown that a zero curvature representation for $D$-dimensional $p$-brane equations of motion originates naturally in the geometric (Lund–Regge–Omnes) approach.

To study the possibility to use this zero curvature representation for investigation of nonlinear equations of $p$-branes, the simplest case of $D$-dimensional string ($p = 1$) is considered.

The connection is found between the $SO(1,1)$ gauge (world-sheet Lorentz) invariance of the string theory with a nontrivial dependence on a spectral parameter of the Lax matrices associated with the nonlinear equations describing the embedding of a string world sheet into flat $D$-dimensional space–time. Namely, the spectral parameter can be identified with a parameter of constant $SO(1,1)$ gauge transformations, after the deformation of the Lax matrices has been performed.

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**Introduction**

In the recent paper by J. Hoppe [1] a zero curvature representation was constructed for the nonlinear equations of motion of bosonic \((D - 2)\)-branes moving in \(D\)-dimensional space-time. Earlier a Lax pair representation for a particular case of \(D = 4\) membrane was built in Ref. [2]. This zero curvature representation and the associated linear system (without spectral parameter) were used in Ref. [1] for constructing the nonlocal conserved charges.

To emphasize the importance of this result, let us note, that the finding of a zero curvature representation for a self-dual Yang-Mills equation inspires the discovery of the correspondence of self-dual gauge fields with complex holomorphic vector bundles [3] and, hence, was among the achievements which had led to the significant progress in instanton and monopole physics as well as to the development of related fields in mathematics (see, for example, [4] and refs therein). The problem of constructing a zero curvature representation for a given system of (exactly solvable) nonlinear equations was considered previously from the mathematical point of view (see, for example [5]).

In order to write the \((D - 2)\)-brane nonlinear equations in the form of zero curvature representation, the gauge was fixed with respect to world surface diffeomorphism as well as with respect to Lorentz transformations and the pure spatial form of the extrinsic geometry formalism was developed in [1]. However, as it was mentioned already in [1], the lack of manifest Lorentz invariance prohibits the use of the zero curvature representation in full measure.

In the present paper it is shown that the relativistic invariant counterpart of the Hoppe’s construction [1] appears in a natural way in the geometric approach [6]–[9] for bosonic \(p\)-branes. This zero curvature representation

\[ d\Omega^{ab} - \Omega^c_\;a \Omega^{cb} = 0, \]

\[ a, b, c = 0, 1, \ldots, (D - 1) \]

is formulated in terms of \(so(1, D - 1)\) valued 1-forms (connections [2])

\[ \Omega^{ab} = -\Omega^{ba} = \begin{pmatrix} \Omega^{aj} \\ -\Omega^{bi} \end{pmatrix}, \]

\[ a, b, c = 0, 1, \ldots, p \]

\[ i, j = 1, \ldots, (D - p - 1) \]

expressed through an intrinsic world volume vielbein matrix \(e^a_m (e^a = d\xi^m e^a_m, \; m = 0, 1, \ldots, p)\), symmetric and traceless second fundamental form matrix \((K^i)^{ba} = (K^i)^{ba} = (K^i)^{ab} \; (K^i)^{ba} \eta_{ba} = 0)\) and world volume gauge fields \(\Omega^i_m\)

\[ \Omega^{bc} = e^a (e^m_m \partial^b_m e^c_n) - e^m \partial^c_m e^b_n + e^m \partial^c_m e^b_n + e^c_m \partial^b_m e^a_n \]

\[ \Omega^{ai} = e^i_{\beta} \Omega^\beta \]

\[ \Omega^{ij} = d\xi^m \Omega^i_m \]

---

1. \(p\) is the number of space-like dimensions of the world volume of the extended object \((p = 1 \text{ for string, } p = 2 \text{ for membrane etc.})\)
2. Here and below \(d\) is the right external differential and the product of forms is supposed to be external one, i.e. \(\Omega, \Omega_q = (-1)^q \Omega_q \Omega_r, \; d(\Omega, \Omega_q) = \Omega, d\Omega_q + (-1)^q d\Omega \Omega_q \) for a product of any \(r\)– and \(q\)– forms.
It has manifest global $SO(1, D-1)$ (Lorentz) and local (gauge) $SO(1, p) \otimes SO(D-p-1)$ symmetries and it is suitable for a description of any bosonic $p$–brane moving in space time of any dimension $D > p$.

It can be shown that the zero curvature representation (1) – (5) is equivalent to the set of the Peterson– Codazzi, Gauss and Ricci equations of the classical theory of the surfaces [10, 7]. In this sense, it is not new.

Our proposition is to use the manifest $SO(1, p) \otimes SO(D-p-1)$ gauge symmetry of the considered zero– curvature representation (1) – (5) for investigation of the nonlinear equations of motion for $p$–branes with $p \geq 2$.

The first problem we would like to investigate is whether it is possible to introduce a nontrivial dependence on a spectral parameter [11, 12] into the connections (2) - (5) using the manifest $SO(1, p) \otimes SO(D-p-1)$ gauge invariance of the zero curvature representation (1)– (5) for $p$ –brane equations.

It is reasonable to begin the investigation of the above problem with studying the simplest case of string ($p = 1$) in diverse dimensions $D \geq 3$. This is the main subject of the present paper.

The equations of motion for string theory become linear if a suitable gauge is fixed. However, the embedding of string world– sheet is described by a system of nonlinear equations [6, 7, 8] in the frame of geometric approach. For $D = 3$ string this is the exactly solvable Liouville equation [13, 14].

The components $\Omega_{ab}^m = (\Omega_{ab}^\tau, \Omega_{ab}^\sigma) \equiv (1/2(\Omega_{ab}^{++} + \Omega_{ab}^{--}), 1/2(\Omega_{ab}^{++} - \Omega_{ab}^{--}))$ of the connection 1– form (3) $\Omega^a_b = d\xi^m \Omega^m_{ab} + d\sigma \Omega^a_{\sigma b} = d\xi^{(\pm)} \Omega^a_{(\pm) b}$ are the counterparts of $U$ and $V$ matrices [11, 12] or, equivalently, of $L$ and $A$ operators (see [13, 14] and Refs. therein), which are known for all exactly solvable equations. They can be used to define the associated linear system

$$\hat{D} \Psi^a \equiv d \Psi^a - \Psi^b \Omega^a_b = 0,$$

(6)

However, for all known exactly solvable systems $L$ and $A$ operators do depend on a spectral parameter in a nontrivial way. This dependence is necessary to use the full power of the Inverse Scattering Method (see [11, 13, 12] and Refs. therein) and it can be used to obtain infinite number of conserved currents [11, 1].

The considered zero curvature representation (1) – (5) involves the connections (2)– (5) being independent on spectral parameter. However, it has the manifest $SO(1, p) \otimes SO(D-p-1)$ gauge invariance instead ($SO(1, 1) \otimes SO(D-2)$ for the string case $p = 1$).

Here we will show that the nontrivial dependence of $U = \Omega_\sigma$ and $V = \Omega_\tau$ matrices on the spectral parameter $\alpha$ can be derived by the use of the $SO(1, 1)$ gauge symmetry of the zero curvature representation.

For the case $D = 3$ our prescription reproduces the standard associated linear system with spectral parameter for nonlinear Liouville equation [14] as well as one derived from the continuous limit of the massless limit of the lattice version of the Sine– Gordon model in the recent paper [12].

\textsuperscript{3}The standard way to deal with the geometric approach equations consists in attempts to reduce the number of functions involved as well as the number of equations (5). Thus, in Ref. (3) the number of gauges were fixed and the geometric approach equations for $D$– dimensional bosonic string were reduced to a system of nonlinear equations for $(D - 2)$ physical (transverse) degrees of freedom.

Let us note here that geometric approach had been studied before for string theory mostly.
The prescription makes possible get the similar associated linear systems for nonlinear equations describing the embedding of the string world sheet into space–time of high dimensions $D > 4$, as well as for $n = (1,0)$ and $n = (1,1)$ supersymmetric Liouville systems [15] describing $N = 1$ and $N = 2$ superstrings in $D = 3$.

1 Maurer–Cartan equations as a zero curvature representation for $p$-brane equations of motion

In the standard $p$–brane formulation [16] the (minimal) embedding of the world volume into a flat $D$– dimensional target space is described by the coordinate functions

$$X^\underline{m} = X^\underline{m}(\xi^m), \quad \underline{m} = 0, 1, \ldots, (D - 1), \quad m = 0, 1, \ldots p,$$

satisfying the equation

$$\partial_m(\sqrt{|g|}g^{mn}\partial_nX^\underline{m}) = 0, \quad g_{mn} = \partial_mX^\underline{m}\partial_nX^\underline{m}g^\underline{m}\underline{n},$$

which becomes nonlinear for $p > 1$. Here $g_{mn}$ is the induced world volume metric and $\eta_{mn}$ is the flat target space–time metric $\eta_{mn} = \text{diag}(+, -, \ldots, -)$.

In the geometric approach [6, 7, 8] $p$–brane theory is described by the world–volume vielbein form $e^a = d\xi^m e^a_m$ and the $so(1, D - 1)$ valued Cartan 1–form (2)

$$\Omega^a = -\Omega^b = \left(\begin{array}{cc} \Omega^a_{\phantom{a}b} & \Omega^a_{\phantom{a}i} \\ -\Omega^b_{\phantom{b}i} & \Omega^b_{\phantom{b}i} \end{array}\right)$$

which satisfy the Maurer–Cartan equation (10)

$$d\Omega^a - \Omega^a_{\phantom{a}b}\Omega^b = 0.$$
The left hand parts of Eqs. (14), (16) are differential 2 forms, while Eq. (15) is written in terms of components $\Omega^{ai}_m$ of the form $\Omega^{ai} = d\xi^m \Omega^{ai}_m$ and inverse vielbein matrix $e^a_m$.

Eq. (16) means that the world volume vielbein is covariantly constant with respect to the parallel transport defined by the connection $\Omega^{ab}$.

Eq. (15) is the only equation of the set (10) – (16) which has dynamical content. From the standpoint of surface theory, it defines the embedding as being minimal one. On the other hand it can be reduced to Eq. (8) if eliminating the auxiliary variables.

In order to show this one needs, first of all, to solve the Maurer–Cartan equations (10) (= (11) - (13)) expressing so(1, D−1) valued connection $\Omega^{a b}$ in terms of SO(1, D−1) valued matrix $\lfloor u^a_m, u^i_m \rceil \in SO(1, D−1)$ (17)

$$
\iff
\lfloor u^a_m, u^b_m \rceil = \eta^{ab} = \text{diag}(+, -, ..., -) \iff \begin{cases} 
  u^a_m u^b_m = \eta^{ab}, \\
  u^a_m u^m_j = 0, \\
  u^i_m u^m_j = -\delta^{ij} 
\end{cases}
$$

The solution has the form

$$
\Omega^{a b} = u^a_m du^b_m \iff \begin{cases} 
  \Omega^{ai} = u^a_m d\xi^m_i, \\
  \Omega^{ab} = u^a_m d\xi^m_b, \\
  \Omega^{ij} = u^i_m d\xi^m_j
\end{cases}
$$

(19)

The vectors $u^a_m$ and $u^i_m$ can be considered as $(p + 1) \times D$ and $(D − p − 1) \times D$ rectangular blocks of the Lorentz group valued matrix (17). They can be identified with Lorentz harmonics or, equivalently, with multidimensional generalization of Newman–Penrose dyades (see [17, 8] and refs. therein).

Using the unity matrix decomposition in terms of moving frame vectors $\delta^{a m}_n = u^a_m u^n_m = u^a_m r^a_n - u^i_m u^i_n$ (20)

and Eqs. (19), Eqs. (16) and (14) may be combined to form one equation

$$
d(e^a u^m_m) = 0
$$

(21)

The closed forms $e^a u^m_m$ (21) can be represented locally as external differentials of some scalar (with respect to world volume) functions

$$
e^a u^m_m = dX^m
$$

(22)

In such a way the coordinate functions (7) appear in geometric approach [3, 7].

Taking into account Eq. (19) and the relation

$$u^m_a = e^a_n \partial_n X^m,$$

which is equivalent to (22), Eq. (13) acquires the form of Eq.(8) with $g^{mn} = e^a_n \eta^{ab} e^b_m$.

---

4 The detailed consideration of relation between the standard p-brane description (8) and that of the geometric approach is given in Ref. [9], where the supersymmetric generalization of the geometric approach is performed.
To get a zero curvature representation for arbitrary bosonic p–brane on the base of the Maurer–Cartan equation (14), we shall only prove the following statement: All the equations of the geometric approach besides those included in the Maurer–Cartan equation (10) can be solved with respect to the Cartan forms $\Omega^{ab}$ and $\Omega^{ai}$.

The proof of this statement is very simple: Eq. (16) can be solved with respect to $\Omega^{ab} = d\xi^m \Omega^{ab}_m$. Such solution is known from General Relativity and has the form (3).

General solution of Eqs. (14), (15) has the form (4)

$$\Omega^{ai} = e^b_{\space k} (K^i)^{ba}_{\space k} (23)$$

where symmetric and traceless matrices $(K^i)^{ba} = (K^i)^{ab}$, $(K^i)^{ba} \eta_{ba} = 0$) are related to the second fundamental form (see, for example, [9]).

Thus, the pull–backs of the Cartan forms $\Omega^{ai}$ and $\Omega^{ab}$ are expressed in terms of second fundamental form matrix $K^{ab}$ and intrinsic world volume vielbein matrix $e^a_{\space m}$ by Eqs. (3), (23). The pull–back of $SO(D−p−1)$ connection form $\Omega^{ij} = d\xi^m \Omega^{ij}_m$ remains independent and gives rise to the world volume gauge fields $\Omega^{ij}_m = \Omega^{ij}_m(\xi^n) (24)$.

The very significant fact is that all the equations for these variables are included into the set of Maurer–Cartan equations (14) (or (11) – (13)) having the form of zero curvature condition.

It is remarkable that the zero curvature representation under consideration can be written in spinor notations

$$d\Omega^{\beta}_{\xi^\beta} - \Omega^{\beta}_{\xi^\alpha} \Omega^{\alpha}_{\xi^\beta} = 0, \quad (24)$$

where $\Omega^{\beta}_{\xi^\beta}$ is spin$(1, D−1)$ valued Cartan form. It is related to $so(1, D−1)$ valued 1–form $\Omega^{ba}$ by $D$– dimensional $\Gamma$– matrices

$$\Omega^{\beta}_{\xi^\beta} \propto \Omega^{ba}(\Gamma^{ba}_{\xi^\beta}) \propto \Omega^{ba}(\Gamma^b_{\xi^\alpha}) \Omega^{\alpha}_{\xi^\beta} \propto (25)$$

The coefficients in Eq. (25) are dependent on the number of target space– time dimensions $D$.

The associated linear system, which reproduce the considered zero curvature representation (14), (9), (3), (23) as integrability conditions, has the evident form (4)

$$\hat{D}\Psi^a \equiv d\Psi^a - \Psi^b \Omega^{ab}_b = 0, \quad (26)$$

with $\Omega^{ab}_b = \eta_{bc} \Omega^{ca}$ determined by (3), (3), (23). Otherwise, it can be identified with equivalent form $du^{am} = u^{am} \Omega^{am}_b$ of Eq. (19) or with spinor counterparts of the above equations.

In conclusion, the relativistic invariant zero curvature representation for $D$-dimensional p–brane equations of motion is given by the Maurer–Cartan equation (14) for $so(1, D−1)$ valued Cartan 1– forms (4) with $\Omega^{ai} = d\xi^m \Omega^{ai}_m(\xi)$ and $\Omega^{ab} = d\xi^m \Omega^{ab}_m(\xi)$ specified by Eqs. (23) and (3).

Of course, this zero curvature representation is equivalent to the set of Peterson–Codazzi, Gauss and Ricci equations and, in this sense, is not new. Our proposition is to

\[\text{see [8] for reformulation of string theory in terms of two dimensional gauge fields.}\]
use its manifest global \( SO(1, D - 1) \) and local \( SO(1, p) \otimes SO(D - p - 1) \) symmetries for investigation of the \( p \)-brane equations of motion.

One of the intriguing possibilities is to use these manifest symmetries to include a spectral parameter into the associated linear system (6).

If this is impossible for a general case, then the problem can be reformulated as one of classification of all the particular cases of \( p \)-brane motions in \( D \)-dimensional space time, where the introduction of spectral parameter is possible.

Below we will consider the zero curvature representation (10) for nonlinear equations describing bosonic string theory in geometric approach. We will present an explicit prescription for inclusion of spectral parameter into the associated linear system (26) starting from \( SO(1, 1) \) gauge symmetry of (10). For the simplest case of string in \( D = 3 \) dimensional space– time, which is described by nonlinear Liouville equation, we reproduce in such a way the standard associated linear system with spectral parameter [14] as well as one derived from the lattice limit in Ref. [12].

2 Geometric approach for bosonic string.

\( SO(1, 1) \) gauge invariance and spectral parameter

For \( p = 1 \) case the world sheet vielbein can be splitted into \( SO(1, 1) \) covariant light–like components \( e^a = (e^+, e^-) \).

The set of \( SO(1, D - 1) \) Cartan forms \( \Omega^{ab} \equiv -\Omega^{ba} \)

\[
\Omega^{ab} = \eta_{ab} \Omega^{ab} \equiv \begin{pmatrix}
\Omega^{(0)} & 0 & 2^{-1/2} \Omega^{--i} \\
0 & -\Omega^{(0)} & 2^{-1/2} \Omega^{++i} \\
2^{-1/2} \Omega^{++j} & 2^{-1/2} \Omega^{--j} & \Omega^{ij}
\end{pmatrix}
\tag{27}
\]

splits naturally into the \( SO(1, 1) \) connection \( \Omega^{ab} \propto \epsilon^{ab} \Omega^{(0)} \), \( SO(D - 2) \) connection \( \Omega^{ij} \) and two \( \frac{SO(1, D - 1)}{SO(1, 1) \times SO(D - 2)} \) vielbein forms \( \Omega^{ai} = (\Omega^{++i}, \Omega^{--i}) \).

The parts (14), (12) and (13) of the Maurer–Cartan equation (10) have the form

\[
D\Omega^{\pm \pm i} = d\Omega^{\pm \pm i} \pm \Omega^{(0)} \Omega^{\pm \pm i} + \Omega^{\pm \pm i} \Omega^{ij} = 0
\tag{28}
\]

\[
\mathcal{F} = d\Omega^{(0)} = \frac{1}{2} \Omega^{-i} \Omega^{++i},
\tag{29}
\]

\[
R^{ij}(d, d) = d\Omega^{ij} + \Omega^{ij} \Omega^{ij} = -\Omega^{--[i} \Omega^{++j]},
\tag{30}
\]

and Eqs. (16), (14) and (15) take the forms

\[
T^{\pm \pm} \equiv De^{\pm \pm} = de^{\pm \pm} \pm \Omega^{(0)} e^{\pm \pm} = 0,
\tag{31}
\]

\[
e^{+i} \Omega^{-i} + e^{-i} \Omega^{++i} = 0,
\tag{32}
\]

\[
e_{+}^{+} \Omega^{++i} + e_{-}^{-} \Omega^{-i} = 0
\tag{33}
\]
2.1 String in $D = 3$ and Liouville equation.

For the simplest case of $D = 3$ bosonic string internal index $i$ has only one value $i = \perp$ and internal connection $\Omega^{ij}$ is absent.

Eqs. (32) and (33) have simple solution
\[
\Omega^{++} = e^{-\Omega^{+-}}, \quad \Omega^{-+} = e^{++}\Omega_{++}^{-}
\] (34)

$\Omega^{++}$ and $\Omega^{-+}$ can be regarded as the nonvanishing components of symmetric and traceless second fundamental form matrix $K^{ab}$ (23).

Then we can solve Eq. (28) with respect to (induced) spin connection $\Omega^{(0)}$
\[
\Omega^{(0)} \equiv e^{\pm\Omega^{(0)}_{\pm\pm}} = \frac{1}{2} e^{++}(\Omega^{++}_{\pm\pm})^{-1} e_{++}^{m} \partial_{m} \Omega^{++}_{\pm\pm} = \frac{1}{2} e^{--}(\Omega^{--}_{++})^{-1} e_{--}^{m} \partial_{m} \Omega^{--}_{++}
\] (35)

Using (35), Eq. (31) can be written in the form
\[
d((\Omega^{++}_{--})^{1/2} e^{--}) = 0, \quad d((\Omega^{--}_{++})^{1/2} e^{++}) = 0
\]
or
\[
e^{--} = (\Omega^{++}_{--})^{-1/2} d\xi^{(-)}, \quad e^{++} = (\Omega^{--}_{++})^{-1/2} d\xi^{(+)}
\]

This means the conformal flatness of any 2–dimensional geometry. Further we will consider $\xi^{++}$, $\xi^{--}$ as functions of world sheet coordinates $\xi^{\mu} = (\tau, \sigma)$. By this we exclude from the consideration the possible contributions from a nontrivial world sheet topology. The influence of such inputs is an interesting problem for further investigations.

It is convenient to use $\xi^{(\pm\pm)}$ as the local world sheet coordinates (and thus to fix the gauge with respect to reparametrization symmetry) and rewrite the expressions for the Cartan form pull backs in term of the holonomic basis $d\xi^{(\pm\pm)}$ of the cotangent space $\left( \partial_{(\pm\pm)} = \frac{\partial}{\partial \xi^{(\pm\pm)}} \right)$
\[
\Omega^{++} = \exp(W - L) d\xi^{(-)}
\] (36)
\[
\Omega^{-+} = \exp(W + L) d\xi^{(++)}
\] (37)
\[
\Omega^{(0)} = (d\xi^{(++)} \partial^{(++)} - d\xi^{(-)} \partial^{(-)}) W - dL
\] (38)

Here the gauge invariant degree of freedom of the fields $\Omega^{++}$ and $\Omega^{--}$ is denoted by $W$
\[
\exp(4W) \equiv \Omega^{++}_{--},
\]
and another (pure gauge) degree of freedom is denoted by $L$
\[
\exp(4L) \equiv \Omega^{--}_{++} / \Omega^{++}_{--}
\]

The Gauss equation (29) for the forms (33) – (38) gives rise to the Liouville equation
\[
\partial_{(++)} \partial^{(-)} W = \frac{1}{4} \exp(2W)
\] (39)

while the Peterson–Codazzi equations (28) are satisfied identically.

The field $L$ remains unrestricted. This reflects the local $SO(1, 1)$ (world sheet Lorentz) invariance of the string theory.
Thus we have reproduced the known result [3, 4]: in the framework of geometric approach \( D = 3 \) string is described by Liouville equation (39).

However, in the proposed approach Liouville equation appears in the form of manifestly \( SO(1,1) \) gauge invariant zero curvature representation (10) (or (24)) for the \( so(1,2) \) valued Cartan forms (1) (or (24)), (33) – (38).

Below we will find an explicit relation between the spectral parameter of the linear system associated with the Liouville equation [14] and \( SO(1,1) \) gauge invariance of the zero curvature representation (i.e. world sheet Lorentz invariance of string theory). Namely, we will show that the spectral parameter can be identified with a parameter of constant \( SO(1,1) \) transformations after a deformation of the Lax matrices (i.e. \( SO(1,2) \) connections) has been performed.

### 2.2 Spectral parameter and gauge invariance.

Below it is convenient to use a spinor representation (24) of the Maurer – Cartan equation (14). For \( D = 3 \), case \( so(1,D-1) = so(1,2) = sl(2,R) \) and \( sl(2,R) \) valued connection \( \Omega^\beta_\alpha \) has the form

\[
\Omega^\beta_\alpha = \frac{1}{2} \begin{pmatrix} \Omega^{(0)} & \Omega^{--} \\ \Omega^{++} & -\Omega^{(0)} \end{pmatrix}
\]

where \( \Omega^{(0)} \), \( \Omega^{\pm\pm} \) are still defined by eqs. (38), (39), (37).

The Liouville equation (33) appears as a diagonal element of the matrix equation (24). In order to reproduce the associated linear system (i.e. \( U \) and \( V \) operators [11, 12]) being dependent nontrivially on a spectral parameter [14, 12] the following trick shall be done.

Let us deform \( SL(2,R) (= SO(1,2)) \) connection (40) replacing the coset form \( \Omega^{++} \) by its sum with the other coset form \( \Omega^{--} \) multiplied by a function \( f^{++--}(\xi) \): \( \Omega^{++} \rightarrow (\Omega^{++} + f^{++--}\Omega^{--}) \)

\[
\Omega^\beta_\alpha \rightarrow \Omega^\beta_\alpha(f) \equiv \frac{1}{2} \begin{pmatrix} \Omega^{(0)} & \Omega^{--} \\ \Omega^{++} + f^{++--}\Omega^{--} & -\Omega^{(0)} \end{pmatrix}
\]

In general case the curvature of the deformed connections (41) does not vanish

\[
d\Omega^\beta_\alpha(f) - \Omega^\beta_\alpha(f)\Omega^\beta_\alpha(f) = \frac{1}{2}\Omega^{--}\mathcal{D}f^{++}(0 0)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

In Eq. (42) the Liouville equation (33) as well as the explicit form of \( \Omega^{\pm\pm}, \Omega^{(0)} \) (36) – (38) are taken into account.

The class of the deformed connections with zero curvature is defined by the functions \( f^{++--}(\xi) \) which satisfy

\[
\Omega^{--}\mathcal{D}f^{++} \equiv \Omega^{--}(df^{++} - 2f^{++\Omega^{(0)}}) = 0 \quad \Rightarrow \quad \partial^{(++)}(e^{2(W+L)}f^{++}) = 0
\]

The general solution of Eq. (43) has the form

\[
f^{++}(\xi^m) = h(\xi^{(++)})e^{-2(W+L)}
\]

\(^6\) Eq. (43) means that the deformed connection \( \Omega(f) \) is related with \( \Omega = \Omega(0) \) by a local \( SL(2,R) \) transformation.
where $h$ is a function of $\xi^{(++)}$ only ($\partial_{(--)} h = 0$). The later can be gauged to a constant using conformal symmetry of the considered system. Thus we could consider (44) with a constant $h$ as a general solution of Eq. (43).

The simplest kind of deformations is characterized by a constant function $f^{-+}$ ($df^{-+} = 0$). In this case Eq. (43) acquire the form

$$f^{-+} \Omega^{(0)} \Omega^{--} = f^{-+} d\Omega^{--} = 0$$

(45)

If one fixes the gauge with respect to $SO(1, 1)$ symmetry

$$L(\xi) = -W(\xi) + \ln \alpha,$$

(46)

where $\alpha$ is a constant, then the form $\Omega^{--}$ becomes closed

$$\Omega^{--} = \alpha d\xi^{(++)} \Rightarrow d\Omega^{--} = 0$$

Hence, the right hand side of eq. (42) vanishes for constant $f$.

In this gauge the flat $SO(1, 2)$ connection (40) acquires the form

$$\Omega^\beta_{\alpha}(\alpha, f) = \left( \begin{array}{cc} d\xi^{(++)} \partial^{(++)} W & \frac{a}{2} d\xi^{(++)} \\
\frac{1}{2a} d\xi^{(--)} \exp\{2W\} + f^{--} \frac{a}{2} d\xi^{(++)} & -d\xi^{(++)} \partial^{(++)} W \end{array} \right)$$

(47)

and the associated linear system with the spectral parameter $\alpha$

$$d\Psi^\alpha - \Omega^\beta_{\alpha}(\alpha, f) d\Psi^\beta = 0,$$

(48)

coinsides with one presented in [14] for the choice of the constant $f^{-+} = -1$. The constants $f$ and $h$ (44) are related by

$$f^{-+} = h \alpha^2.$$ 

If one comes back to the general solution (44) of Eq. (43) and fixes the gauge

$$L(\xi) = \ln \alpha = \text{const}$$

(49)

with respect to $SO(1, 1)$ symmetry, then the flat $SL(2, R)$ connections (41) takes the form

$$\Omega^\beta_{\alpha}(\alpha, h) = d\xi^{(++)} \Omega^{(++)}_{(++)} \frac{\alpha}{2a} (\alpha, h)$$

$$\Omega^{(+-)}(\alpha, h) = \frac{1}{2} \left( \begin{array}{cc} \partial^{(++)} W & \alpha e^W \\
\frac{1}{a} e^{-W} \partial^{(++)} W & 0 \end{array} \right), \quad \Omega^{(--)}(\alpha, h) = \frac{1}{2} \left( \begin{array}{cc} -\partial^{(--)} W & 0 \\
\frac{1}{a} e^{-W} \partial^{(--)} W & \partial^{(++)} W \end{array} \right)$$

(50)

For the choice $h = -\alpha^2$ ($f^{-+} = -e^{-2W}$), the matrices $U = 2i(\Omega^{(--)} - \Omega^{(++)})$, $V = -2i(\Omega^{(--)} + \Omega^{(++)})$ coincide with the Lax pair for Liouville equation obtained in Ref. [12].

The nontrivial dependence of these $U$ and $V$ matrices on the spectral parameter $\alpha$ is

7Note, that all elements of the matrix (40) have well defined properties with respect to $SO(1, 1)$ gauge transformations, which suggested to be conserved after the deformation (41).

8 To get the associated linear system in the form presented in Ref. [14] the following redefinition of the variables and the spectral parameter shall be done: $\xi^{(--)} = \tau$, $\xi^{(++)} = 4\sigma$, $W(\xi) = \phi(\tau, \sigma)$, $-2^{-1/2} \alpha = \lambda$.

9 The relation between the variables and the spectral parameter of the present work with ones from Ref. [12] are $\xi^{\pm} = -4(\sigma \pm \tau)$, $W(\xi) = -i \Phi(\sigma, \tau)$, $\alpha = -ie^{-\lambda}$.
were used in [12] to apply the Quantum Inverse Scattering Method [18] for studying the quantum Liouville system.

For both cases the parameter $\alpha$ of the global $SO(1,1)$ transformations, which appears as the reminder of the gauge $SO(1,1)$ symmetry of the Gauss and Peterson–Codazzi equations ([12], [14]), becomes the spectral parameter of the associated linear system (48).

Thus there is the close relation of the spectral parameter with the gauge symmetries of the zero curvature representations.

However, we should stress, that this relation should not be taken literally. So, if one puts $f = h = 0$ and identifies the spectral parameter $\alpha$ with constant $SO(1,1)$ gauge transformation parameter, then Eq. (50) reproduces the $U$ and $V$ operators being dependent on the spectral parameter $\alpha$ in a trivial way [19, 12].

So, the possibility of constructing the deformations similar to (41) without breaking the Maurer–Cartan equation (24) is the crucial factor for a nontrivial introduction of a spectral parameter into the associated linear system.

In the same manner the spectral parameter can be introduced into the linear system associated with nonlinear system of equations describing minimal embedding of string world sheet into the flat space time of any dimension $D > 3$, which we will describe shortly in the next section.

2.3 Bosonic string in diverse dimensions and the sets of nonlinear equations

As in the simplest $D = 3$ case, all equations for the bosonic string in $D \geq 4$ can be represented as the Maurer–Cartan equation (11) for 1–form (27) with

\[
\Omega^{++i} = d\xi^{(-)} \exp(W + L) G^{ij}_L M^j
\]

(51)

\[
\Omega^{--i} = d\xi^{(++)} \exp(W - L) G^{ij}_R N^j
\]

(52)

\[
\Omega^{(0)} = (d\xi^{(++)} \partial_{(++)} - d\xi^{(--)} \partial_{(--)})(W - dL)
\]

(53)

\[
\Omega^{ij} = d\xi^{(--)} \partial_{(--)} G^{ik}_R G^{jk}_R + d\xi^{(++)} \partial_{(++)} G^{ik}_L G^{jk}_L
\]

(54)

Here $M$ and $N$ are $SO(D - 2)$ vector fields of opposite chirality

\[
\partial_{++} M^i = 0 \quad \partial_{--} N^i = 0,
\]

(55)

whose norms can be fixed to be unity

\[
M^i M^i = 1, \quad N^i N^i = 1
\]

(56)

using the conformal symmetry of the system under consideration. $G_R$, $G_L$ are orthogonal ($SO(D - 2)$ valued) matrix fields

\[
G^{ik}_R G^{jk}_R \equiv (G_R G_R^T)^{ij} = \delta^{ij} \quad G^{ik}_L G^{jk}_L \equiv (G_L G_L^T)^{ij} = \delta^{ij}
\]

(57)

The field $W$ as well as the matrix valued field $G^{ij} \equiv G^{ki}_L G^{kj}_R$ are invariant under the natural $SO(1,1) \otimes SO(D - 2)$ gauge symmetry of the considered system.
The system of nonlinear equations for the above fields, which follows from Eq. (10), is rather complicated. It involves, besides the chirality conditions for $M$ and $N$ fields, the Liouville-like equation

$$\partial_{(++)}\partial_{(−−)}W = \frac{1}{4}e^{2W}N^iG^{ij}M^j$$

and the $\sigma$-model-like equation for the orthogonal matrix fields

$$\partial_{(−−)}(\partial_{+++}G_L \ G_L^T)^{ij} - \partial_{++}(\partial_{−−}G_R \ G_R^T)^{ij} + [\partial_{+++}G_L G_L^T, \partial_{−−}G_R G_R^T]^{ij} = -e^{2W}(G_R N)^{[i}(G_L M)^{j]}$$

The field $L$ is not restricted by any equation and transforms as a compensator under the world-sheet Lorentz transformations.

As for the simplest $D = 3$ case, the system of nonlinear equations is presented in the form of zero curvature condition and the spectral parameter can be introduced into the associated linear system by the trick described for $D = 3$ case. \[10\]

3 Conclusion and discussion

We have demonstrated that the zero curvature representation for the nonlinear equations of motion of bosonic $p$-brane moving in $D$ dimensional space–time originates in the geometric approach \[4, 1, 8, 9\]. It is given by Maurer–Cartan equation (1) (or (24)) involving (2) ((25)) independent forms $\Omega^{ij}$ and the forms $\Omega^{ab}$, $\Omega^{ai}$ expressed in terms of the world-sheet vielbein and the second fundamental form matrix by the solution (3) of Eq.(16) and by Eq.(4) respectively. This representation is manifestly Lorentz invariant. It is vailed for any $p$–brane in $D$–dimensional space–time.

The considered zero curvature representation is based on the map of the $p$–brane world volume into the coset $SO(1,D − 1)/(SO(1,p) \otimes SO(D − p − 1))$ in distinction with Hoppe’s construction \[1\] based on the map of $(D − 2)$–brane world volume into the Euclidean $(D − 1)$–dimensional hyperplane of the $D$– dimensional flat target space.

So, the $D$– dimensional bosonic $p$–branes shall supplement the list of dynamical systems, whose equations of motion can be naturally presented in the form of zero curvature conditions. Let us remind, that such list includes, in addition to exactly solvable two-dimensional models, the self–dual Yang–Mills system \[3\] as well as the full $N = 3$ \[20\] and $N = 2$ \[21\] super–Yang–Mills theories. (See also Refs. \[23, 24\].)

It is important to investigate the possibility to introduce a spectral parameter into a linear system (4) associated with $p$–brane equations of motion. As it was noted in \[1\], this should result, in particular, in the possibility to obtain infinitely many conserved currents.

If this is impossible for general case, the problem can be reformulated as one of classification of all particular cases of $p$–brane motions in $D$–dimensional space time, where an infinite number of conserved currents exists.

As a first step we have investigated here the relation between the spectral parameter and the natural $SO(1,1)$ gauge invariance of the zero curvature representation for the

\[10\] So, in the gauge $G_R = I$, $L = W + \text{const}$, the form $\Omega^{−1}$ becomes closed (see Eq.\(\[5\]\))
case of string theory in $D \geq 3$. We have presented the prescription of the deformation of the $SO(1, D-1)$ connections, which do not break the zero curvature condition, and have demonstrated that then the spectral parameter appears in the associated linear system as a parameter of constant $SO(1,1)$ gauge transformations.

For $D = 3$ case such associated linear system (18), (14), (13), (36), (37) reproduces the Lax matrices (17) for nonlinear Liouville equations obtained in Ref. [14] as well as one (51) obtained in Ref. [12] for an appropriate choice of the $SO(1,1)$ gauge and the values of the constants involved.

The form of the geometric approach used here is most suitable for (doubly) supersymmetric generalization, which was performed in [1] (see also [22]). It seems to be of interest to construct a zero curvature representation and associated linear system for equations of motion of supersymmetric extended object. Now such supersymmetric generalization has been built for the simplest cases of $D = 3$, $N = 1$ and 2 superstrings. This leads to a new version of $n = (1,0)$ and $n = (1,1)$ supersymmetric Liouville equations [15]. It can be shown that the nontrivial dependence on the spectral parameters can be introduced into the linear systems associated with these supersymmetric Liouville models by the above prescription.

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