Universal potential estimates for $1 < p \leq 2 - \frac{1}{n}$

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Dedicated to Giuseppe Mingione on the occasion of his 50th birthday

Abstract
We extend the so-called universal potential estimates of Kuusi-Mingione type (J. Funct. Anal. 262: 4205–4269, 2012) to the singular case $1 < p \leq 2 - 1/n$ for the quasilinear equation with measure data

$$\text{div}(A(x, \nabla u)) = \mu$$

in a bounded open subset $\Omega$ of $\mathbb{R}^n$, $n \geq 2$, with a finite signed measure $\mu$ in $\Omega$. The operator $\text{div}(A(x, \nabla u))$ is modeled after the $p$-Laplacian $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$, where the nonlinearity $A(x, \xi)$ ($x, \xi \in \mathbb{R}^n$) is assumed to satisfy natural growth and monotonicity conditions of order $p$, as well as certain additional regularity conditions in the $x$-variable.

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1 Introduction and main results

We are concerned here with the quasilinear elliptic equation with measure data

$$-\text{div}(A(x,\nabla u)) = \mu,$$

in a bounded open subset \(\Omega\) of \(\mathbb{R}^n, n \geq 2\). Here \(\mu\) is a finite signed measure in \(\Omega\) and the nonlinearity \(A = (A_1, \ldots, A_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is vector valued function. Throughout the paper, we assume that there exist \(\Lambda \geq 1\) and \(p > 1\) such that

$$|A(x,\xi)| \leq \Lambda|\xi|^{p-1}, \quad |D_\xi A(x,\xi)| \leq \Lambda|\xi|^{p-2},$$

$$\langle D_\xi A(x,\xi)\eta, \eta \rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2$$

for every \(x \in \mathbb{R}^n\) and every \((\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}\). More regularity assumptions on function \(x \mapsto A(x,\xi)\) will be needed later.

A typical example of (1.1) is the \(p\)-Laplace equation with measure data

$$-\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u) = \mu \quad \text{in} \ \Omega.$$  

Since the seminal work of Kilpeläinen and Mäkly [7] (see also [16] for a different approach), the study of pointwise behaviors of solutions to quasilinear equations with measure data (1.1) has undergone substantial progress. In particular, the series of works [4, 5, 9] (see also [12]) provide interesting pointwise bounds for gradients of solutions to the seemingly unwieldy equation (1.1), at least for \(p > 2 - \frac{1}{n}\). These pointwise gradient bounds have been extended recently in [14, 3, 15] for the more singular case \(1 < p \leq 2 - \frac{1}{n}\).

On the other hand, a more unified approach to pointwise bounds for solutions and their gradients was presented in [8]. The results of [8] give pointwise bounds not only for the size but also for the oscillation of solutions and their derivatives expressed in terms of bounds by linear or nonlinear potentials in certain Calderón spaces. These cover different kinds of pointwise fractional derivative estimates as well as estimates for (sharp) fractional maximal functions of the solutions and their gradients.

However, the treatment of [8] is still confined to the range \(p > 2 - \frac{1}{n}\), and the purpose of this note is to extend it to the singular case \(1 < p \leq 2 - \frac{1}{n}\). Note that, for \(1 < p \leq 2 - \frac{1}{n}\), by looking at the fundamental solution we see that in general distributional solutions of (1.4) may not even belong to \(W^{1,1}_{\text{loc}}(\Omega)\).

Thus in this paper we shall restrict ourselves only to the case

$$1 < p \leq 2 - \frac{1}{n},$$

and note that the main results obtained here also hold in the case \(2 - \frac{1}{n} < p < 2\) thanks to [8]. Moreover, except for the comparison estimates obtained earlier in [13, 15], the methods used in this paper are very much guided by those of [8]. We would also like to point out that there are analogous results in the case \(p \geq 2\) that we refer to [8] for the precise statements.

In some sense our pointwise regularity for the non-homogeneous equation (1.1) is
obtained from perturbation/interpolation arguments involving the associated homogeneous equations. Thus information on the regularity of associated homogeneous equations will play an important role. In this direction, we first recall a quantitative version of the well-known De Giorgi’s result that established \( C^{\alpha_0}, \alpha_0 \in (0, 1) \), regularity for solutions of \( \text{div} (A(x, \nabla w)) = 0 \). Henceforth, by \( Q_r(x_0) \) we mean the open cube \( Q_r(x_0) := x_0 + (-r, r)^n \) with center \( x_0 \in \mathbb{R}^n \) and side-length \( 2r \). In other words,

\[
Q_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0|_\infty := \max_{1 \leq i \leq n} |x_i - x_{0i}| < r \}.
\]

**Lemma 1.1** Under (1.2)-(1.3), let \( w \in W^{1,p}(\Omega) \), \( p > 1 \), be a solution of the equation \( \text{div} (A(x, \nabla w)) = 0 \) in \( \Omega \). Then there exists \( \alpha_0 \in (0, 1) \), depending only on \( n, p \) and \( \Lambda \), such that for any cubes \( Q_\rho(x_0) \subset Q_R(x_0) \subset \Omega \), and \( \epsilon \in (0, 1] \), we have

\[
\int_{Q_\rho(x_0)} |w - (w)_{Q_\rho(x_0)}|^p \, dx \lesssim \left( \frac{\rho}{R} \right)^{\alpha_0 p} \int_{Q_R(x_0)} |w - (w)_{Q_R(x_0)}|^p \, dx, \tag{1.5}
\]

and

\[
\inf_{q \in \mathbb{R}} \int_{Q_\rho(x_0)} |w - q'|^q \, dx \lesssim \left( \frac{\rho}{R} \right)^{\alpha_0 \epsilon} \inf_{q \in \mathbb{R}} \int_{Q_R(x_0)} |w - q'|^q \, dx. \tag{1.6}
\]

We point out that the proof of (1.5) follows from [6, Chapter 7], whereas the proof of (1.3) follows from (1.5) and the reverse Hölder property of \( w \).

In the case the nonlinearity \( A(x, \xi) \) is independent of \( x \), we actually have \( C^{1,\beta_0}, \beta_0 \in (0, 1) \), regularity the the homogeneous equation (see, e.g., [2, 10, 11]). For our purposes, we shall use the following quantitative version of this regularity result (see [5, 9]).

**Lemma 1.2** Let \( v \in W^{1,p}(\Omega) \), \( p > 1 \), be a solution of \( \text{div} (A_0(\nabla v)) = 0 \) in \( \Omega \), where \( A_0(\xi) \) satisfies (1.2)-(1.4) and is independent of \( x \). Then there exists \( \beta_0 \in (0, 1) \), depending only on \( n, p \) and \( \Lambda \), such that for any cubes \( Q_\rho(x_0) \subset Q_R(x_0) \subset \Omega \) and \( \epsilon \in (0, 1] \), we have

\[
\int_{Q_\rho(x_0)} |\nabla w - (\nabla w)_{Q_\rho(x_0)}| \, dx \lesssim \left( \frac{\rho}{R} \right)^{\beta_0} \int_{Q_R(x_0)} |\nabla w - (\nabla w)_{Q_R(x_0)}| \, dx,
\]

and

\[
\inf_{q \in \mathbb{R}^n} \int_{Q_\rho(x_0)} |\nabla v - q'|^q \, dx \lesssim \left( \frac{\rho}{R} \right)^{\beta_0 \epsilon} \inf_{q \in \mathbb{R}^n} \int_{Q_R(x_0)} |\nabla v - q'|^q \, dx. \tag{1.7}
\]

In what follows, we shall use the (maximal) constants \( \alpha_0 \) in Lemma 1.1 and \( \beta_0 \) in Lemma 1.2 as certain thresholds in our regularity theory. Also, henceforth, we reserve the letter \( \kappa \) for the following constant

\[
\kappa := (p - 1)^2 / 2. \tag{1.8}
\]

Our first result provides a De Giorgi’s theory for non-homogeneous equations with measure data, which also includes [15, Theorem 1.4] as an end-point case. For the case \( p > 2 - 1/n \), see [8, Theorem 1.1].
In general, given a nonnegative measure $Q$ and for each cube $\Omega$, defined by

$$Q(\Omega) = \int_{\Omega} \nu \, dx,$$

Note that $W$ uniformity in $\alpha$

(1.2), with $1 < p \leq 2 - \frac{1}{\gamma}$, let $\kappa$ be as in (1.8), and suppose that $u \in C^0(\Omega) \cap W^{1,\gamma}_k(\Omega)$ is a solution of (1.1). Let $Q_R(x_0) \subset \Omega$ and $\bar{\alpha} \in (0, \alpha_0)$, where $\alpha_0$ is as in Lemma 1.1. Then for any $x, y \in Q_R/\bar{\alpha}$, we have

$$|u(x) - u(y)| \lesssim \left[ W_{1-\alpha(p-1)/p,p}(\nu)(x) + W_{1-\alpha(p-1)/p,p}(\nu)(y) \right] |x - y|^\alpha$$

+ \left( \int_{Q_R(x_0)} |u|^\alpha \, dx \right)^{\frac{1}{\alpha}} \left( \frac{|x - y|}{\bar{\alpha}} \right)^{\alpha}

(1.9)

uniformly in $\alpha \in [0, \bar{\alpha}]$. Here the implicit constant depends only on $n, p, \Lambda$, and $\bar{\alpha}$.

In (1.3), the function $W_{1-\alpha(p-1)/p,p}(\nu)(\cdot)$ is a truncated Wolff’s potential of $|\mu|$. In general, given a nonnegative measure $\nu$ and $\rho > 0$, the Wolff’s potential $W_{\alpha,s}^\rho(\nu)$, $\alpha > 0, s > 1$, is defined by

$$W_{\alpha,s}^\rho(\nu)(x) := \int_0^\rho \nu(Q_t(x)) \frac{1}{t^{n-\alpha s}} \, dt, \quad x \in \mathbb{R}^n.$$  

Note that $W_{\alpha,2}^\rho(\nu) = I_{\alpha}^\rho(\nu)$, where $I_{\gamma}^\rho(\nu)$, $\gamma > 0$, is a truncated Riesz’s potential defined by

$$I_{\gamma}^\rho(\nu)(x) := \int_0^\rho \nu(Q_t(x)) \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$  

We remark that, except for (1.2)–(1.3), no further regularity assumption is needed in Theorem 1.3. However, this will force the constant $\bar{\alpha}$ to be small in general.

On the other hand, it is possible to allow $\bar{\alpha}$ to be arbitrarily close to 1 as long as we further impose a ‘small BMO’ condition on the map $x \mapsto A(x, \xi)$. This condition entails the smallness of the limit $\limsup_{\rho \to 0} \omega(\rho)$, where

$$\omega(\rho) := \sup_{y \in \mathbb{R}^n} \left[ \int_{Q_r(y)} \Upsilon(A, Q_r(y))(x)^2 \, dx \right]^{\frac{1}{2}}, \quad \rho > 0,$$

(1.10)

and for each cube $Q_r(y)$ we set

$$\Upsilon(A, Q_r(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|A(x, \xi) - A_{Q_r(y)}(\xi)|}{|\xi|^{p-1}},$$

with $A_{Q_r(y)}(\xi) = \int_{Q_r(y)} A(x, \xi) \, dx$. The precise statement is as follows.

**Theorem 1.4** Under (1.2)–(1.3), with $1 < p \leq 2 - \frac{1}{\gamma}$, let $\kappa$ be as in (1.8), and suppose that $u \in C^0(\Omega)$ is a solution to (1.1). Let $Q_R(x_0) \subset \Omega$. Then for any positive $\bar{\alpha} < 1$ there exists a small $\delta = \delta(n, p, \Lambda, \bar{\alpha}) > 0$ such that

$$\limsup_{\rho \to 0} \omega(\rho) \leq \delta,$$

(1.11)

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then for any \(x, y \in Q_{R/8}(x_0) \subset \Omega\), we have

\[
|u(x) - u(y)| \lesssim \left[ W_{1,\alpha(p-1)/p,p}^1(|\mu|)(x) + W_{1,\alpha(p-1)/p,p}^1(|\mu|)(y) \right] |x - y|^{\alpha} \\
+ \left( \int_{Q_{R}(x_0)} |u|^{1/p} dx \right)^{\frac{1}{p}} \left( \frac{|x - y|}{R} \right)^{\alpha}
\]  

(1.12)

uniformly in \(\alpha \in [0, \bar{\alpha}]\). Here the implicit constant depends on \(n, p, \Lambda, \bar{\alpha}, \omega(\cdot), \) and \(\text{diam}(\Omega)\).

Under a certain Dini-VMO condition, we could also allow \(\bar{\alpha} = 1\) in the above theorem. However, in this case the Wolff’s potential is replaced with a Riesz’s potential raised to the power of \(\frac{1}{p-1}\).

**Theorem 1.5** Under (1.2)–(1.3), with \(1 < p \leq 2 - \frac{1}{n}\), let \(\kappa\) be as in (1.8), and suppose that \(u \in C^1(\Omega)\) is a solution to (1.1). Let \(Q_R(x_0) \subset \Omega\). If for some \(\sigma_1 \in (0, 1)\) such that \(\omega(\cdot)^{\sigma_1}\) is Dini-VMO, i.e.,

\[
\int_0^1 \omega(\rho)^{\sigma_1} \frac{d\rho}{\rho} < +\infty
\]  

(1.13)

then for any \(x, y \in Q_{R/8}(x_0) \subset \Omega\), we have

\[
|u(x) - u(y)| \lesssim \left[ (1^{R}_{p-\alpha(p-1)}(|\mu|)(x))^{\frac{1}{p-1}} + (1^{R}_{p-\alpha(p-1)}(|\mu|)(y))^{\frac{1}{p-1}} \right] |x - y|^{\alpha} \\
+ \left( \int_{Q_{R}(x_0)} |u|^{1/p} dx \right)^{\frac{1}{p}} \left( \frac{|x - y|}{R} \right)^{\alpha}
\]

uniformly in \(\alpha \in [0, 1]\). Here the implicit constant depends on \(n, p, \Lambda, \bar{\alpha}, \sigma_1, \omega(\cdot), \) and \(\text{diam}(\Omega)\).

We remark that, when \(\alpha = 1\), Theorem 1.5 recovers the pointwise gradient estimates of [3] and [13] that were obtained under a slightly different Dini condition.

Finally, under a stronger Dini-Hölder condition we can also bound solution gradients in appropriate Calderón spaces.

**Theorem 1.6** Under (1.2)–(1.3), with \(1 < p \leq 2 - \frac{1}{n}\), let \(\kappa\) be as in (1.8), and suppose that \(u \in C^1(\Omega)\) is a solution to (1.1). Let \(Q_R(x_0) \subset \Omega\). If for some \(\sigma_1 \in (0, 1)\) such that \(\omega(\cdot)^{\sigma_1}\) is Dini-Hölder of order \(\bar{\alpha}\), i.e.,

\[
\int_0^1 \omega(\rho)^{\sigma_1} \frac{d\rho}{\rho^{\alpha}} < +\infty
\]  

(1.14)

for some \(\bar{\alpha} \in [0, \beta_0]\), then for any \(x, y \in Q_{R/4}(x_0) \subset \Omega\), we have

\[
|\nabla u(x) - \nabla u(y)| \lesssim \left[ (1^{R}_{1-\alpha}(|\mu|)(x))^{\frac{1}{1-\alpha}} + (1^{R}_{1-\alpha}(|\mu|)(y))^{\frac{1}{1-\alpha}} \right] |x - y|^{\alpha} \\
+ \left( \int_{Q_{R}(x_0)} |\nabla u|^{1/p} dx \right)^{\frac{1}{p}} \left( \frac{|x - y|}{R} \right)^{\alpha}
\]
uniformly in $\alpha \in [0, \bar{\alpha}]$. Here $\beta_0$ is as in Lemma 1.2, and the implicit constant depends on $n, p, \Lambda, \bar{\alpha}, \sigma_1, \omega(\cdot)$, and $\text{diam}(\Omega)$.

2 Comparison and Poincaré type inequalities

The study of regularity problems for equation (1.1) is based on the following comparison estimate that connects the solution of measure datum problem to a solution of a homogeneous problem.

To describe it, we let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a solution of (1.1). Then for a cube $Q_{2R} = Q_{2R}(x_0) \Subset \Omega$, we consider the unique solution $w \in W^{1,p}_{\text{loc}}(Q_{2R}(x_0)) + u$ to the local interior problem

$$
\begin{cases}
- \text{div}(A(x, \nabla w)) = 0 & \text{in } Q_{2R}(x_0), \\
w = u & \text{on } \partial Q_{2R}(x_0).
\end{cases}
$$

(2.1)

Lemma 2.1 Suppose that $Q_{3R}(x_0) \subset \Omega$ for some $R > 0$. Let $u$ and $w$ be as in (2.1) and let $\kappa$ be as in (1.8), where $1 < p \leq 2 - \frac{4}{\beta_0}$. Then it holds that

$$
\left( \int_{Q_{2R}(x_0)} |\nabla (u - w)|^p \, dx \right)^{\frac{1}{p}} \lesssim \left( \frac{\mu(Q_{3R}(x_0))}{R^{n-1}} \right)^{\frac{1}{p-1}} \left( \int_{Q_{3R}(x_0)} |\nabla u|^\kappa \, dx \right)^{\frac{2-p}{p}}.
$$

(2.2)

Proof. For $1 < p \leq \frac{3n-2}{n-1}$, inequality (2.2) was obtained in [15, Theorem 1.2]. For $\frac{3n-2}{2n-1} < p \leq 2 - \frac{4}{\beta_0}$, by [13, Lemma 2.2], we have

$$
\left( \int_{Q_{2R}(x_0)} |\nabla (u - w)|^{\gamma_0} \, dx \right)^{\frac{1}{\gamma_0}} \lesssim \left( \frac{\mu(Q_{2R}(x_0))}{R^{n-1}} \right)^{\frac{1}{p-1}} \left( \int_{Q_{2R}(x_0)} |\nabla u|^{\kappa} \, dx \right)^{\frac{2-p}{\gamma_0}}
$$

for some $\gamma_0 \in \left[ \frac{2-p}{2-p+\frac{n(p-1)}{n-1}} \right]$. In fact, an inspection of the proof of [13, Lemma 2.2] reveals that we can take any $\gamma_0 \in \left( \frac{n}{2n-1}, \frac{n(p-1)}{n-1} \right)$. Thus we may assume that $\kappa = (p-1)^{2/p} < \gamma_0$. To conclude the proof, it is therefore enough to show that

$$
\left( \int_{Q_{2R}(x_0)} |\nabla u|^{\gamma_0} \, dx \right)^{\frac{1}{\gamma_0}} \lesssim \left( \frac{\mu(Q_{3R}(x_0))}{R^{n-1}} \right)^{\frac{1}{p-1}} + \left( \int_{Q_{3R}(x_0)} |\nabla u|^\kappa \, dx \right)^{\frac{1}{\kappa}}.
$$

(2.3)
To this end, let $\gamma_1 \in (\gamma_0, \frac{n(p-1)}{n-1})$. By [15, Corollary 2.4], we have

$$\left( \int_{Q_\rho(x)} |\nabla u|^{\gamma_1} \, dy \right)^{\frac{1}{\gamma_1}} \lesssim \left( \frac{|\mu|(Q_{9\rho/8}(x))}{\rho^{n-1}} \right)^{\frac{1}{p-1}} + \frac{1}{\rho} \left( \int_{Q_{9\rho/8}(x)} |u - \lambda|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}}$$

(2.4)

for any $\lambda \in \mathbb{R}$ and any cube $Q_\rho(x)$ such that $Q_{9\rho/8}(x) \subset \Omega$.

Now, with $Q_{8\rho/7}(x) \subset \Omega$, let $w_1$ be the unique solution $w_1 \in W_0^{1,p}(Q_{8\rho/7}(x)) + u$ to the problem

$$\begin{cases}
- \text{div} \left( A(x, \nabla w_1) \right) = 0 \quad \text{in} \quad Q_{8\rho/7}(x), \\
w_1 = u \quad \text{on} \quad \partial Q_{8\rho/7}(x).
\end{cases}$$

Then from the proof of [13, Lemma 2.2] (using (2.8) and (2.18) in [13]), we can deduce that

$$\frac{1}{\rho} \left( \int_{Q_{8\rho/7}(x)} |u - w_1|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}} \lesssim \left( \frac{|\mu|(Q_{8\rho/7}(x))}{\rho^{n-1}} \right)^{\frac{1}{p-1}}$$

$$+ \frac{|\mu|(Q_{8\rho/7}(x))}{\rho^{n-1}} \left( \int_{Q_{8\rho/7}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{2-p}{\gamma_0}}.$$  

(2.5)

By Young’s inequality, this yields

$$\frac{1}{\rho} \left( \int_{Q_{8\rho/7}(x)} |u - w_1|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}} \lesssim \left( \frac{|\mu|(Q_{8\rho/7}(x))}{\rho^{n-1}} \right)^{\frac{1}{p-1}} + \left( \int_{Q_{8\rho/7}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}}.$$  

(2.6)

Thus by quasi-triangle and Hölder’s inequalities we get

$$\frac{1}{\rho} \left( \int_{Q_{9\rho/8}(x)} |u - \lambda|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}} \lesssim \left( \frac{|\mu|(Q_{9\rho/8}(x))}{\rho^{n-1}} \right)^{\frac{1}{p-1}} + \left( \int_{Q_{9\rho/8}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}}$$

$$+ \frac{1}{\rho} \left( \int_{Q_{9\rho/8}(x)} |w_1 - \lambda|^{\frac{n}{p-1}} \, dy \right)^{\frac{p-1}{n}},$$

(2.7)
where we choose $\lambda = \int_{Q_{3R}(x_0)} w_1 \, dz$.

We now use Poincaré and the reverse Hölder’s inequalities for $\nabla w_1$ to obtain that

$$
\frac{1}{\rho} \left( \int_{Q_{3\rho/8}(x)} |w_1 - \lambda|^\frac{\gamma_0}{n-\gamma_1} \, dy \right)^{\frac{n-\gamma_1}{n}} \lesssim \int_{Q_{3\rho/8}(x)} |\nabla w_1| \, dy \lesssim \left( \int_{Q_{3\rho/7}(x)} |\nabla w_1|^\gamma_0 \, dy \right)^{\frac{1}{\gamma_0}}
$$

$$
\lesssim \left( \int_{Q_{3\rho/7}(x)} |\nabla u - \nabla w_1|^\gamma_0 \, dy \right)^{\frac{1}{\gamma_0}} + \left( \int_{Q_{3\rho/7}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}},
$$

where we used [13, Lemma 2.2] and Young’s inequality in the last bound.

Thus combining this result with (2.7) we find

$$
\frac{1}{\rho} \left( \int_{Q_{3\rho/8}(x)} |u - \lambda|^\gamma_0 \, dy \right)^{\frac{1}{\gamma_0}} \lesssim \left( \frac{|\mu|(Q_{3\rho/7}(x))}{\rho^{\gamma_1-1}} \right)^{\frac{1}{\gamma_1}} + \left( \int_{Q_{3\rho/7}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}}.
$$

At this point, plugging this into (2.4) we arrive at

$$
\left( \int_{Q_\rho(x)} |\nabla u|^{\gamma_1} \, dy \right)^{\frac{1}{\gamma_1}} \lesssim \left( \frac{|\mu|(Q_{3\rho/7}(x))}{\rho^{\gamma_1-1}} \right)^{\frac{1}{\gamma_1}} + \left( \int_{Q_{3\rho/7}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}},
$$

which holds for any cube $Q_\rho(x)$ such that $Q_{3\rho/7}(x) \subset \Omega$. Recall that $\gamma_1 > \gamma_0$, and thus by a covering/iteration argument as in [6, Remark 6.12], we have

$$
\left( \int_{Q_\rho(x)} |\nabla u|^{\gamma_1} \, dy \right)^{\frac{1}{\gamma_1}} \lesssim \left( \frac{|\mu|(Q_{3\rho/7}(x))}{\rho^{\gamma_1-1}} \right)^{\frac{1}{\gamma_1}} + \left( \int_{Q_{3\rho/7}(x)} |\nabla u|^{\gamma_0} \, dy \right)^{\frac{1}{\gamma_0}}
$$

(2.8)

for any $\epsilon > 0$. This obviously yields (2.3) as desired and the proof is complete.

**Remark 2.2** Using the above argument, in particular (2.5), we can also show the following comparison estimate for the functions $u$ and $w$: for any $\frac{2n-2}{2n-1} < p \leq 2 - \frac{1}{n}$,

$$
\left( \int_{Q_{2R}(x_0)} |u - w|^p \, dx \right)^{\frac{1}{p}} \lesssim \left( \frac{|\mu|(Q_{3R}(x_0))}{R^{n-p}} \right)^{\frac{1}{p-1}} + \frac{|\mu|(Q_{3R}(x_0))}{R^{n-2}} \left( \int_{Q_{3R}(x_0)} |\nabla u|^\epsilon \, dx \right)^{\frac{2-\epsilon}{n}},
$$

for any $\epsilon > 0$. This obviously yields (2.3) as desired and the proof is complete.
The following Poincaré type inequality was obtained in the case \([15, \text{Corollary 1.3}]\). A similar proof using Lemma 2.1 and inequalities of the form (2.6) and (2.8) also yields the result in the case \(\frac{3n-2}{2n-1} < p \leq \frac{1}{n}\).

**Corollary 2.3** Suppose that \(Q_{3r/2}(x_0) \subset \Omega\) for some \(r > 0\). Let \(u \in W^{1,p}_{\text{loc}}(\Omega),\) \(1 < p \leq 2 - \frac{1}{n}\) be a solution of (1.1). Then for any \(\epsilon > 0\) we have

\[
\inf_{q \in \mathbb{R}} \left( \frac{\int_{Q_r(x_0)} |u - q|^p}{r^n} \right)^{\frac{1}{p}} \lesssim \left( \frac{|\mu|(Q_{3r/2}(x_0))}{r^{n-p}} \right)^{\frac{1}{p}} + r \left( \frac{\int_{Q_{3r/2}(x_0)} |\nabla u|^p}{r^n} \right)^{\frac{1}{p}}.
\]

With \(u\) and \(w\) as in (2.1), we now consider another auxiliary function \(v\) such that \(v \in W^{1,p}_0(Q_R(x_0)) + w\) is the unique solution to the equation

\[
\left\{ \begin{array}{l}
- \text{div} (\overline{A}_{Q_R(x_0)}(\nabla v)) = 0 \quad \text{in} \quad Q_R(x_0), \\
v = w \quad \text{on} \quad \partial Q_R(x_0),
\end{array} \right. \tag{2.9}
\]

where \(\overline{A}_{Q_R(x_0)}(\xi) = \int_{Q_R(x_0)} A(x, \xi) \, dx\).

The following result can be deduced from [8, Lemma 2.3] and an appropriate reverse Hölder’s inequality.

**Lemma 2.4** Let \(p > 1, 0 < \epsilon \leq p,\) and \(u, w,\) and \(v\) be as in (2.1) and (2.9), where \(Q_{2R}(x_0) \subset \Omega\). Then there exists a small positive constant \(\sigma_0 > 0\) such that

\[
\left( \frac{\int_{Q_R(x_0)} |\nabla v - \nabla w|^\epsilon}{r^n} \right)^{\frac{1}{\epsilon}} \lesssim \omega(R)^{\sigma_0} \left( \frac{\int_{Q_{2R}(x_0)} |\nabla w|^\epsilon}{r^n} \right)^{\frac{1}{\epsilon}},
\]

where \(\omega(\cdot)\) is as defined in (1.10).

Likewise, following lemma follows from [8, Lemma 2.5].

**Lemma 2.5** Let \(1 < p < 2, 0 < \epsilon \leq p,\) and \(u, w,\) and \(v\) be as in (2.1) and (2.9), where \(Q_{2R}(x_0) \subset \Omega\). Then for any \(\sigma_1 \in (0,1)\) such that \(\omega(\cdot)^{\sigma_1}\) is Dini-VMO, i.e., (1.13) holds, it follows that

\[
\left( \frac{\int_{Q_R(x_0)} |\nabla v - \nabla w|^\epsilon}{r^n} \right)^{\frac{1}{\epsilon}} \lesssim \omega(R)^{\sigma_1} \left( \frac{\int_{Q_{2R}(x_0)} |\nabla w|^\epsilon}{r^n} \right)^{\frac{1}{\epsilon}}.
\]
3 Pointwise fractional maximal function bounds

As in [3], our proofs of Theorems 1.3–1.6 are based on the corresponding pointwise estimates for the associate fractional and sharp fractional maximal functions, which are interesting in their own right. This section is devoted to such pointwise fractional maximal function bounds.

Given $R > 0$ and $q > 0$, following [1], we define the following truncated sharp fractional maximal function of a function $f \in L^q_{\text{loc}}(\mathbb{R}^n)$:

$$M_{\alpha,q}^\#(f)(x) := \sup_{0 < \rho \leq R} \inf_{m \in \mathbb{R}} \rho^{-\alpha} \left( \frac{1}{Q_{\rho}(x)} \int_{Q_{\rho}(x)} |f - m|^q \, dx \right)^{\frac{1}{q}}, \quad \alpha \geq 0.$$  

Also, we define a truncated fractional maximal function by

$$M_{\beta,q}^R(f)(x) := \sup_{0 < \rho \leq R} \rho^\beta \left( \frac{1}{Q_{\rho}(x)} \int_{Q_{\rho}(x)} |f|^q \, dx \right)^{\frac{1}{q}}, \quad \beta \in [0, n/q].$$  

In the case $q = 1$, we usually drop the index $q$ in the above notation, i.e., we set $M_{\alpha,1}^\#(f) = M_{\alpha}^\#(f)$ and $M_{\beta,1}^R(f) = M_{\beta}^R(f)$. Moreover, the definition of $M_{\beta}^R(f)$ can also be naturally extended to the case where $f = \mu$ is a locally finite signed measure in $\mathbb{R}^n$:

$$M_{\beta}^R(\mu)(x) := \sup_{0 < \rho \leq R} \rho^\beta \left( \frac{1}{Q_{\rho}(x)} \mu(Q_{\rho}(x)) \right), \quad \beta \in [0, n/q].$$  

Note that by Poincaré inequality we have

$$M_{\beta}^R(f)(x) \lesssim M_{1-\beta}(\nabla f)(x), \quad \beta \in [0, 1],$$  

for any $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$.

On the other hand, if $u \in W^{1,p}_{\text{loc}}(\Omega)$, $1 < p \leq 2 - \frac{1}{n}$, then it follows from Corollary 1.3 that

$$M_{\beta}^{R/2}(u)(x) \lesssim \left[ M_{p-\beta(p-1)}^{R/2}(\mu)(x) \right]^{\frac{1}{p-1}} + M_{1-\beta}^{R/2}(\nabla u)(x), \beta \in [0, 1], \quad (3.1)$$  

for any $\epsilon \in (0, 1)$ and any cube $Q_{3R/2}(x) \subset \Omega$.

The following fractional maximal function bound will be needed in the proof of Theorem 1.3:

**Theorem 3.1** Under (1.2)–(1.3), let $1 < p \leq 2 - \frac{1}{n}$, and suppose that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a solution of (1.1). Let $Q_{3R}(x) \subset \Omega$ and $\bar{\alpha} \in (0, \alpha_0)$, where $\alpha_0 \in (0, 1)$ is as in Lemma 1.3. Then we have

$$M_{\alpha,\bar{\alpha}}^{2R}(u)(x) + M_{1-\alpha,\bar{\alpha}}^{3R}(\nabla u)(x)$$

$$\lesssim \left[ M_{p-\alpha(p-1)}^{3R}(\mu)(x) \right]^{\frac{1}{p-1}} + R^{1-\alpha} \left( \int_{Q_{3R}(x)} |\nabla u|^p \, dy \right)^{\frac{1}{p}} \quad (3.2)$$  

uniformly in $\alpha \in [0, \bar{\alpha}]$. Here the implicit constant depends on $n, p, \Lambda$, and $\bar{\alpha}$.  

Proof. The main idea of the proof of (3.2) lies the proof of [8, Proposition 3.1] that treated the case $p > \frac{2}{n}$. Note that by (3.1) it is enough to show

$$M_{1-\alpha,\kappa}^R(\nabla u)(x) \lesssim [M_{p-\alpha(p-1)}^R(\mu)(x)]^{\frac{1}{p-\alpha}} + R^{1-\alpha} \left( \int_{Q_{3R}(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}},$$ (3.3)

for some $\epsilon = \epsilon_1(n, p, \Lambda, \bar{\alpha}) \in (0, 1)$.

Let $0 < \rho < r \leq R$, and choose $w$ as in (2.1) with $Q_{2r}(x)$ in place of $Q_{2R}(x_0)$. We have

$$\int_{Q_{\rho}(x)} |\nabla u|^\kappa dy \lesssim \int_{Q_{\rho}(x)} |\nabla w|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy$$

$$\lesssim \left( \frac{\rho}{r} \right)^{(\alpha_0-1)\kappa} \int_{Q_{2r}(x)} |\nabla w|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy$$

$$\lesssim \left( \frac{\rho}{r} \right)^{(\alpha_0-1)\kappa} \int_{Q_{2r}(x)} |\nabla w|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy,$$

where we used the inequality

$$\int_{Q_{\rho}(x)} |\nabla w|^\kappa dy \lesssim \left( \frac{\rho}{r} \right)^{(\alpha_0-1)\kappa} \int_{Q_{2r}(x)} |\nabla w|^\kappa dy,$$

which is a modified version of [8, Theorem 2.2], in the second inequality. Thus by Lemma 2.1 we get

$$\left( \int_{Q_{\rho}(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}} \lesssim \left( \frac{\rho}{r} \right)^{(\alpha_0-1)\kappa} \left( \int_{Q_{2r}(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}}$$

$$+ \left( \frac{r}{\rho} \right)^n \left[ \frac{\mu((Q_{3r}(x)) \right]^{\frac{1}{p-1}}$$

$$+ \left( \frac{r}{\rho} \right)^n \left( \frac{\mu((Q_{3r}(x)) \right] \left( \int_{Q_{3r}(x)} |\nabla u|^\kappa dy \right)^{(2-p)/\kappa}}.$$

Let $\epsilon \in (0, 1)$, and choose $\rho = \epsilon r$. Then by Young’s inequality we have

$$\left( \int_{Q_{r\epsilon}(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}} \leq C(\epsilon) \left[ \frac{\mu((Q_{3r}(x)) \right]^{\frac{1}{p-1}}$$

$$+ [C \epsilon^{\alpha_0-1} + 1] \left( \int_{Q_{3r}(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}}.$$
Multiplying both sides by \((\epsilon r)^{1-\alpha}\), \(0 < \alpha \leq \hat{\alpha} < \alpha_0\), and taking the supremum with respect to \(r \in (0, R]\), we find

\[
\sup_{0 < r \leq \epsilon R} r^{1-\alpha} \left( \int_{Q_r(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa} \leq C(\epsilon) \sup_{0 < r \leq R} \left[ \frac{|\mu|(Q_{3r}(x))}{r^{-n-p+\alpha(p-1)}} \right]^{\frac{1}{p-1}} + |C\epsilon^{\alpha_0-1} + 1|(\epsilon/3)^{1-\alpha} \sup_{0 < r \leq R} (3r)^{1-\alpha} \left( \int_{Q_{3r}(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa}.
\]

We now choose \(\epsilon \in (0, 1)\) such that

\[
|C\epsilon^{\alpha_0-1} + 1|(\epsilon/3)^{1-\hat{\alpha}} \leq 1/2,
\]

to deduce that

\[
\sup_{0 < r \leq \epsilon R} r^{1-\alpha} \left( \int_{Q_r(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa} \leq C(\epsilon) \sup_{0 < r \leq R} \left[ \frac{|\mu|(Q_{3r}(x))}{r^{-n-p+\alpha(p-1)}} \right]^{\frac{1}{p-1}} + \sup_{\epsilon R < r \leq 3R} r^{1-\alpha} \left( \int_{Q_{3r}(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa} \lesssim \left[ M^{3R}_{\alpha, \kappa}(\mu)(x) \right]^{\frac{1}{p-1}} + R^{1-\alpha} \left( \int_{Q_{3R}(x)} |\nabla u|^\kappa \, dy \right)^{\frac{1}{\kappa}}.
\]

This is (3.3) and the proof is complete.

The following result will be needed for the proof of Theorem 1.4.

**Theorem 3.2** Let \(1 < p \leq 2 - \frac{1}{n}\) and \(u \in C^0(\Omega)\) be a solution to (1.1). Suppose that \(Q_{3R}(x) \subset \Omega\). Then for any positive \(\hat{\alpha} < 1\) there exists a small \(\delta = \delta(n, p, \Lambda, \hat{\alpha}) > 0\) such that if (1.1) holds, then the estimate

\[
M^{3R}_{\alpha, \kappa}(u)(x) + M^{3R}_{1-\alpha, \kappa}(\nabla u)(x)
\]

\[
\lesssim \left[ M^{3R}_{p-\alpha(p-1)}(\mu)(x) \right]^{\frac{1}{p-1}} + R^{1-\alpha} \left( \int_{Q_{3R}(x)} |\nabla u|^\kappa \, dy \right)^{\frac{1}{\kappa}}
\]

holds uniformly in \(\alpha \in [0, \hat{\alpha}]\). Here the implicit constant depends on \(n, p, \Lambda, \hat{\alpha}, \omega(\cdot), \) and \(\text{diam}(\Omega)\).

**Proof.** The proof is similar to that of Theorem 3.1 but this time we need to use Lemma 2.4. As above, by (5.11) it is enough to show (4.3) for some \(\epsilon = \epsilon_1(n, p, \Lambda, \hat{\alpha}) \in (0, 1)\). Let \(0 < \rho \leq r \leq R\), and choose \(w\) as in (2.1) with \(Q_{2r}(x)\) in place of \(Q_{2R}(x_0)\).
Then choose \( v \) as in (2.9) with \( Q_r(x) \) in place of \( Q_R(x_0) \). This time we have

\[
\int_{Q_r(x)} |\nabla u|^\kappa dy \\
\lesssim \int_{Q_r(x)} |\nabla v|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_r(x)} |\nabla v - \nabla w|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy
\]

\[
\lesssim \int_{Q_r(x)} |\nabla v|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_r(x)} |\nabla v - \nabla w|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy
\]

\[
\lesssim \int_{Q_r(x)} |\nabla v|^\kappa dy + \left( 1 + \left( \frac{r}{\rho} \right)^n \right) \left( \int_{Q_r(x)} |\nabla v - \nabla w|^\kappa dy + \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy \right)
\]

\[
\lesssim \int_{Q_r(x)} |\nabla v|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_r(x)} |\nabla v - \nabla w|^\kappa dy + \left( \frac{r}{\rho} \right)^n \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy.
\]

Here we used

\[
\int_{Q_r(x)} |\nabla v|^\kappa dy \lesssim \int_{Q_r(x)} |\nabla v|^\kappa dy,
\]

which is a modified version of (2.6) in [8, Theorem 2.1] in the second inequality. Then by Lemma 2.4 we get

\[
\left( \int_{Q_r(x)} |\nabla u|^\kappa dy \right)^{1/\kappa} \lesssim \left( \int_{Q_r(x)} |\nabla u|^\kappa dy \right)^{1/\kappa} + \left( \frac{r}{\rho} \right)^{n/\kappa} \omega(r)^{\sigma_0} \left( \int_{Q_{2r}(x)} |\nabla w|^\kappa dy \right)^{1/\kappa}
\]

\[
+ \left( \frac{r}{\rho} \right)^{n/\kappa} \left( \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy \right)^{1/\kappa}
\]

\[
\lesssim \left\{ 1 + \left( \frac{r}{\rho} \right)^{n/\kappa} \omega(r)^{\sigma_0} \right\} \left( \int_{Q_r(x)} |\nabla u|^\kappa dy \right)^{1/\kappa}
\]

\[
+ \left\{ \left( \frac{r}{\rho} \right)^{n/\kappa} \omega(r)^{\sigma_0} + \left( \frac{r}{\rho} \right)^{n/\kappa} \right\} \left( \int_{Q_{2r}(x)} |\nabla u - \nabla w|^\kappa dy \right)^{1/\kappa},
\]

for a small constant \( \sigma_0 > 0 \). Thus using Lemma 2.1 and the fact that \( \omega(r) \leq 2\Lambda \), we find

\[
\left( \int_{Q_r(x)} |\nabla u|^\kappa dy \right)^{1/\kappa} \lesssim \left\{ 1 + \left( \frac{r}{\rho} \right)^{n/\kappa} \omega(r)^{\sigma_0} \right\} \left( \int_{Q_r(x)} |\nabla u|^\kappa dy \right)^{1/\kappa}
\]

\[
+ \left( \frac{r}{\rho} \right)^{n/\kappa} \left[ \frac{|\mu|(Q_{3r}(x))}{r^{n-1}} \right]^{1/\kappa} + \left( \frac{r}{\rho} \right)^{n/\kappa} \left( \frac{|\mu|(Q_{3r}(x))}{r^{n-1}} \right)^{(2-p)/\kappa}.
\]
Let \( \epsilon \in (0, 1) \), and choose \( \rho = \epsilon r \). Then by Young’s inequality we have
\[
\left( \int_{Q_{r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa} \leq C_{\epsilon} \left[ \frac{|\mu(Q_{3r}(x))|}{\rho^{n-1}} \right]^{1/\kappa}
\]
\[+ \left[ c_{1} \epsilon^{-n/\kappa} \omega(r)^{\alpha_{0}} + c_{2} \right] \left( \int_{Q_{3r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa}.
\]
Multiplying both sides by \((\epsilon r)^{1-\alpha}\), \(0 < \alpha \leq \bar{\alpha} < 1\), and taking the supremum with respect to \( r \in (0, R] \), we find
\[
\sup_{0 < r \leq \epsilon R} r^{1-\alpha} \left( \int_{Q_{r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa} \leq C_{\epsilon} \sup_{0 < r \leq R} \left[ \frac{|\mu(Q_{3r}(x))|}{\rho^{n-\alpha(p+1)}} \right]^{1/\kappa}
\]
\[+ \left[ c_{1} \epsilon^{-n/\kappa} \sup_{0 < r \leq R} \omega(r) + c_{2} \right] (\epsilon/3)^{1-\alpha} \sup_{0 < r \leq R} (3r)^{1-\alpha} \left( \int_{Q_{3r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa}.
\]
We now choose \( \epsilon \in (0, 1) \) such that
\[c_{2}(\epsilon/3)^{1-\alpha} \leq 1/4,
\]
and then choose \( \bar{R} = \bar{R}(n, p, \Lambda, \omega(\cdot)) > 0 \) and a small \( \delta = \delta(n, p, \Lambda, \bar{\alpha}) > 0 \) in (1.1) such that
\[c_{1} \epsilon^{-n/\kappa} \sup_{0 < r \leq R} \omega(r)(\epsilon/3)^{1-\alpha} \leq c_{1} \epsilon^{-n/\kappa}(2\delta)(\epsilon/3)^{1-\alpha} \leq 1/4.
\]
Then it follows that
\[\left[ c_{1} \epsilon^{-n/\kappa} \sup_{0 < r \leq R} \omega(r) + c_{2} \right] (\epsilon/3)^{1-\alpha} \leq 1/2,
\]
provided \( R \leq \bar{R} \). Hence, for \( R \leq \bar{R} \), we deduce that
\[
\sup_{0 < r \leq R} r^{1-\alpha} \left( \int_{Q_{r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa} \leq C(\epsilon) \sup_{0 < r \leq 3R} \left[ \frac{|\mu(Q_{3r}(x))|}{\rho^{n-\alpha(p+1)}} \right]^{1/\kappa}
\]
\[+ \sup_{\epsilon R < r \leq 3R} r^{1-\alpha} \left( \int_{Q_{r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa}
\]
\[\leq \left[ M_{p-\alpha(p-1)}^{3R} (\mu)(x) \right]^{1/\kappa} + R^{1-\alpha} \left( \int_{Q_{3r}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa}.
\]
This proves (3.3) in the case \( R \leq \bar{R} \). For \( R > \bar{R} \), we observe that
\[
M^{\alpha}(\nabla u)(x) \leq M^{\alpha}(\nabla u)(x) + \left( \frac{R}{R} \right)^{n/\kappa} (\epsilon R)^{1-\alpha} \left( \int_{Q_{\epsilon R}(x)} |\nabla u|^{\kappa} dy \right)^{1/\kappa}.
\]
Thus we also obtain \( 3.3 \) in the case \( R > \tilde{R} \) as long as we allow the implicit constant to depend on \( \text{diam}(\Omega) \), and \( n, p, \Lambda, \alpha, \omega(\cdot) \).

In order to prove Theorem \( 1.5 \) we need the following pointwise fractional maximal function bound.

**Theorem 3.3** Let \( 1 < p \leq 2 - \frac{1}{n} \) and \( u \in C^1(\Omega) \) be a solution to \( 1.1 \). Suppose that \( Q_{3\tilde{R}}(x) \subset \Omega \). If for some \( \sigma_1 \in (0,1) \) such that \( \omega(\cdot)^{\sigma_1} \) is Dini-VMO, i.e., \( 1.13 \) holds, then the estimate

\[
M_{\alpha, \kappa}^R(u)(x) + M_{1-\alpha, \kappa}^R(\nabla u)(x) \\
\lesssim \left[ \int_{p-\alpha(p-1)}^R (|\mu|(x)) \right]^{\frac{1}{p-1}} + R^{1-\alpha} \left( \int_{Q_{3\tilde{R}}(x)} |\nabla u|^{\kappa} \, dy \right)^{\frac{1}{\kappa}}
\]

holds uniformly in \( \alpha \in [0,1] \). Here the implicit constant depends on \( n, p, \Lambda, \alpha, \omega(\cdot), \sigma_1, \) and \( \text{diam}(\Omega) \).

**Proof.** As in the proof of Theorem \( 3.2 \) it is enough to show

\[
M_{1-\alpha, \kappa}^R(\nabla u)(x) \lesssim \left[ \int_{p-\alpha(p-1)}^R (|\mu|(x)) \right]^{\frac{1}{p-1}} + R^{1-\alpha} \left( \int_{Q_{\tilde{R}}(x)} |\nabla u|^{\kappa} \, dy \right)^{\frac{1}{\kappa}}.
\]

Moreover, we may assume that \( R \leq \tilde{R} \), where \( \tilde{R} = \tilde{R}(n, p, \Lambda, \sigma_1, \omega(\cdot)) > 0 \) is to be determined.

Arguing as in the proof of \( 3.3 \), but this time using \( 1.7 \) (in Lemma \( 1.2 \)) instead of \( 3.4 \) and Lemma \( 2.5 \) instead of Lemma \( 2.4 \) we have for \( Q_{\rho}(x) \subset Q_{r}(x) \subset Q_{3\tilde{R}}(x) \subset \Omega \),

\[
\left( \int_{Q_{\rho}(x)} |\nabla u - q_{Q_{\rho}(x)}|^{\kappa} \, dy \right)^{1/\kappa} \lesssim \left( \frac{\rho}{r} \right)^{\beta_0} \left( \int_{Q_{3\tilde{R}}(x)} |\nabla u - q_{Q_{3\tilde{R}}(x)}|^{\kappa} \, dy \right)^{1/\kappa} \\
+ \left( \frac{r}{\rho} \right)^{n/\kappa} \omega(r)^{\sigma_1} \left( \int_{Q_{3\tilde{R}}(x)} |\nabla u|^{\kappa} \, dy \right)^{1/\kappa} \\
+ \left( \frac{r}{\rho} \right)^{n/\kappa} \left( \frac{|\mu|(Q_{3\tilde{R}}(x))}{r^{n-1}} \right)^{(2-p)/\kappa} \left( \int_{Q_{3\tilde{R}}(x)} |\nabla u|^{\kappa} \, dy \right)^{(2-p)/\kappa}.
\]

Here \( q_{Q_{\rho}(x)} \in \mathbb{R}^n \) is defined by

\[
q_{Q_{\rho}(x)} := \arg \min_{q \in \mathbb{R}^n} \left( \int_{Q_{\rho}(x)} |\nabla u - q|^{\kappa} \, dy \right)^{1/\kappa}, \quad Q_{\rho}(x) \subset \Omega.
\]

That is, \( q_{Q_{\rho}(x)} \) is a vector such that

\[
\inf_{q \in \mathbb{R}^n} \left( \int_{Q_{\rho}(x)} |\nabla u - q|^{\kappa} \, dy \right)^{1/\kappa} = \left( \int_{Q_{\rho}(x)} |\nabla u - q_{Q_{\rho}(x)}|^{\kappa} \, dy \right)^{1/\kappa}.
\]
Note that for $Q_\rho(x) \subset Q_s(x) \in \Omega$, one has

$$|q_{Q_s(x)}| = \left( \int_{Q_s(x)} |q_{Q_s(x)}|^\kappa dy \right)^{1/\kappa}$$

$$\lesssim \left( \int_{Q_s(x)} |\nabla u - q_{Q_s(x)}|^\kappa dy \right)^{1/\kappa} + \left( \int_{Q_s(x)} |\nabla u|^\kappa dy \right)^{1/\kappa},$$

and also

$$|q_{Q_\rho(x)} - q_{Q_s(x)}| = \left( \int_{Q_\rho(x)} |q_{Q_\rho(x)} - q_{Q_s(x)}|^\kappa dy \right)^{1/\kappa}$$

$$\lesssim \left( \int_{Q_\rho(x)} |\nabla u - q_{Q_\rho(x)}|^\kappa dy \right)^{1/\kappa} + \left( \int_{Q_\rho(x)} |\nabla u - q_{Q_s(x)}|^\kappa dy \right)^{1/\kappa}$$

$$\lesssim \left( \frac{s}{\rho} \right)^{\n/\kappa} \left( \int_{Q_s(x)} |\nabla u - q_{Q_s(x)}|^\kappa dy \right)^{1/\kappa}.$$  (3.7)

For brevity, for any $j = 0, 1, 2, \ldots$, and $Q_{3R}(x) \subset \Omega$, we now define

$$Q_j = Q_{R_j}(x), \quad R_j = \epsilon^j R,$$

where $\epsilon \in (0, 1/3)$ is to be determined, and

$$A_j = \left( \int_{Q_j} |\nabla u - q_j|^\kappa dy \right)^{1/\kappa}, \quad q_j = q_{Q_j}.$$

Then applying (3.6) with $\rho = \epsilon R_j < r = R_j/3$ we have

$$A_{j+1} \leq c_1 \epsilon^\beta_0 A_j + c_2 \epsilon^{-n/\kappa} \omega(R_j/3)^{\sigma_1} \left( \int_{Q_j} |\nabla u|^\kappa dy \right)^{1/\kappa} + C_\epsilon \left[ \frac{|\mu|(Q_j)}{R_j^{n-1}} \right]^{1/\kappa}$$

$$+ C_\epsilon \left( \frac{|\mu|(Q_j)}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa dy \right)^{(2-p)/\kappa}.  \quad (3.9)$$

By quasi-triangle inequality, this yields

$$A_{j+1} \leq c_1 \epsilon^\beta_0 A_j + c_2 \epsilon^{-n/\kappa} \omega(R_j/3)^{\sigma_1} A_j$$

$$+ c_2 \epsilon^{-n/\kappa} \omega(R_j/3)^{\sigma_1} |q_j| + C_\epsilon \left[ \frac{|\mu|(Q_j)}{R_j^{n-1}} \right]^{1/\kappa}$$

$$+ C_\epsilon \left( \frac{|\mu|(Q_j)}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa dy \right)^{(2-p)/\kappa}.$$
We now choose $\epsilon$ sufficiently small so that $c_1 \epsilon^{3\alpha} \leq 1/4$ and then restrict $R \leq \bar{R}$, where $\bar{R} = \bar{R}(n, p, \Lambda, \sigma_1, \omega(\cdot)) > 0$ is such that
\[
 c_2 \epsilon^{-n/\kappa} \sup_{0 < \rho \leq \bar{R}} \omega(\rho)^{\sigma_1} \leq 1/4.
\]

Then we have
\[
 A_{j+1} \leq \frac{1}{2} A_j + C \omega(R_j/3)^{\sigma_1} |q_j| + C \left[ \frac{|\mu(Q_j)|}{R_j^{n-1}} \right]^{\frac{1}{p-1}}
+ C \left( \frac{|\mu(Q_j)|}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{\frac{2-p}{p}}.
\] (3.10)

Summing this up over $j \in \{0, 1, \ldots, m - 1\}$, $m \in \mathbb{N}$, we get
\[
 \sum_{j=1}^m A_j \leq \frac{1}{2} \sum_{j=0}^{m-1} A_j + C \sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} |q_j| + C \sum_{j=0}^{m-1} \left[ \frac{|\mu(Q_j)|}{R_j^{n-1}} \right]^{\frac{1}{p-1}}
\]
\[
+ C \sum_{j=0}^{m-1} \left( \frac{|\mu(Q_j)|}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{\frac{2-p}{p}}.
\]

Hence,
\[
 \sum_{j=1}^m A_j \leq A_0 + C \sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} |q_j| + C \sum_{j=0}^{m-1} \left[ \frac{|\mu(Q_j)|}{R_j^{n-1}} \right]^{\frac{1}{p-1}}
\]
\[
+ C \sum_{j=0}^{m-1} \left( \frac{|\mu(Q_j)|}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{\frac{2-p}{p}}.
\]

On the other hand, for any $m \in \mathbb{N}$, by (3.10) we can write
\[
 |q_{m+1}| = \sum_{j=0}^m (|q_{j+1}| - |q_j|) + |q_0| \leq C \sum_{j=0}^m A_j + |q_0|,
\]
and therefore in view of (3.7),
\[
 |q_{m+1}| \leq c A_0 + |q_0| + C \sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} |q_j| + C \sum_{j=0}^{m-1} \left[ \frac{|\mu(Q_j)|}{R_j^{n-1}} \right]^{\frac{1}{p-1}}
\]
\[
+ C \sum_{j=0}^{m-1} \left( \frac{|\mu(Q_j)|}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{\frac{2-p}{p}}
\]
\[
\leq C \left( \int_{Q_{m+1}} |\nabla u|^\kappa \, dy \right)^{1/\kappa} + C \sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} |q_j| + C \sum_{j=0}^{m-1} \left[ \frac{|\mu(Q_j)|}{R_j^{n-1}} \right]^{\frac{1}{p-1}}
\]
\[
+ C \sum_{j=0}^{m-1} \left( \frac{|\mu(Q_j)|}{R_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{\frac{2-p}{p}}.
\]
At this point, multiplying both sides of the above inequality by $R_{m+1}^{1-\alpha}$, $m \in \mathbb{N}$, we deduce that

$$R_{m+1}^{1-\alpha}|q_{m+1}| \leq R_{m+1}^{1-\alpha} \left( \int_{Q_R(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa} + C \sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} R_{m+1}^{1-\alpha}|q_j|$$

$$+ \sum_{j=0}^{m-1} \left[ \frac{|\mu|(Q_j)}{R_j^{n-p+\alpha(p-1)}} \right]^{1/(p-1)} R_{m+1}^{1-\alpha} \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{(2-p)/\kappa}.$$  

Thus,

$$R_{m+1}^{1-\alpha}|q_{m+1}| \leq c_3 R_{m+1}^{1-\alpha} \left( \int_{Q_R(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa}$$

$$+ c_3 \sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} R_{m+1}^{1-\alpha}|q_j| + c_3 \left[ \int_{R_j/3}^R (|\mu|)(x) \right]^{1/(p-1)}$$

$$+ c_3 \left[ \int_{R_j/3}^R (|\mu|)(x) \frac{M_{R_{m+1}}^{\alpha,\kappa}(\nabla u)(x)}{R_{m+1}^{1-\alpha}} \right]^{2-p}.$$  

(3.11)

We next further restrict $\bar{R}$ so that for any $R \leq \bar{R}$,

$$\sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} \leq \frac{1}{2c_3}.$$  

This is possible because we have

$$\sum_{j=0}^{m-1} \omega(R_j/3)^{\sigma_1} = \omega(R/3) + \sum_{j=1}^{m-1} \omega(R_j/3)^{\sigma_1}$$

$$\leq c \int_{R/3}^R \omega(\rho)^{\sigma_1} \frac{d\rho}{\rho} + c \sum_{j=1}^{m-1} \int_{R_j/3}^{R_j-1/3} \omega(\rho)^{\sigma_1} \frac{d\rho}{\rho}$$

$$\leq c \int_0^R \omega(\rho)^{\sigma_1} \frac{d\rho}{\rho}.$$  

(3.12)

where we used the fact that $\omega(\rho_1) \leq c \omega(\rho_2)$ provided $\rho_1 \leq \rho_2 \leq C\rho_1$, $C > 1$. Then by an induction argument we deduce from (3.11) that

$$R_{m+1}^{1-\alpha}|q_{m+1}| \leq c_3 R_{m+1}^{1-\alpha} \left( \int_{Q_R(x)} |\nabla u|^\kappa \, dy \right)^{1/\kappa}$$

$$+ \left[ \int_{R_j/3}^R (|\mu|)(x) \right]^{1/(p-1)} + \left[ \int_{R_j/3}^R (|\mu|)(x) \frac{M_{R_{m+1}}^{\alpha,\kappa}(\nabla u)(x)}{R_{m+1}^{1-\alpha}} \right]^{2-p}.$$  

(3.13)
for every integer $m \geq 0$.

Let us call the right-hand side of (3.13) by $Q$. Then from (3.10) and simple manipulations we obtain

$$A_{m+1} \leq \frac{1}{2} A_m + c|q_m| + cR_m^{-1}Q,$$

which by (3.13) yields

$$R_m^{1-\alpha}A_{m+1} \leq \frac{1}{2} R_m^{1-\alpha}A_m + cR_m^{1-\alpha}q_m \leq \frac{1}{2} R_m^{1-\alpha}A_m + cQ.$$

As $R_0^{1-\alpha}A_0 \leq cQ$, by iteration we get

$$R_m^{1-\alpha}A_m \leq CQ,$$

for every integer $m \geq 0$.

To conclude the proof, we observe that

$$M^{\frac{1}{p}(\nabla u)}(x) \leq C \sup_{m \geq 0} R_m^{1-\alpha} \left( \int_{Q_m} |\nabla u|^\kappa dy \right)^{1/\kappa}$$

$$\leq C[R_m^{1-\alpha}A_m + R_m^{1-\alpha}q_m] \leq CQ,$$

where we used (3.13) and (3.14) in the last inequality. Then recalling the definition of $Q$ and using Young’s inequality we obtain

$$M^{\frac{1}{p}(\nabla u)}(x) \leq CR^{1-\alpha} \left( \int_{Q_0} |\nabla u|^\kappa dy \right)^{1/\kappa} + C \left[ I_{2R}^{1-\alpha}(\mu)(x) \right]^{1/p-1} + \frac{1}{2} M^{\frac{1}{p}(\nabla u)}(x).$$

This completes the proof of the theorem.

The following pointwise sharp fractional maximal function bound will be used in the proof of Theorem 1.6.

**Theorem 3.4** Let $1 < p \leq 2 - \frac{1}{n}$ and $u \in C^1(\Omega)$ be a solution to (1.1). Suppose that $Q_{3R}(x) \subset \Omega$. If for some $\sigma_1 \in (0,1)$ such that

$$\sup_{0 < \rho \leq 1} \frac{\omega(\rho)^{\sigma_1}}{\rho^\alpha} \leq K,$$

for some $\bar{\alpha} \in [0, \beta_0)$, then the estimate

$$M^{\frac{1}{p}(\nabla u)}(x) \lesssim \left[ M^{\frac{1}{p}(\nabla u)}(\mu)(x) \right]^{1/p-1}$$

$$+ \left[ I^{1-\alpha}(\mu)(x) \right]^{1/p-1} + R^{-\alpha} \left( \int_{Q_{3R}(x)} |\nabla u|^\kappa dy \right)^{1/\kappa}$$

holds uniformly in $\alpha \in [0, \bar{\alpha}]$. Here $\beta_0$ is as in Lemma 1.2, and the implicit constant depends on $n, p, \Lambda, \bar{\alpha}, \omega(\cdot), \sigma_1, K$, and $\text{diam}(\Omega)$.
Remark 3.5 Condition (3.15) implies the Dini-VMO condition (1.13). In turns, (1.13) implies (1.11), whereas (3.15) is implied by the Dini-Hölder condition (1.14).

Proof. It suffices to show
\[
M_{\alpha, \kappa}^R(\nabla u)(x) \lesssim [M_{1-a, \kappa}^R(\mu)(x)]^{\frac{1}{p-1}} + [I_1^R(\{\mu\})(x)]^{\frac{1}{p-1}} + R^{-\alpha} \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + R^{-\alpha} \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C \left[ \frac{\mu((Q_j))}{R_{n(x)}} \right]^{\frac{1}{p-1}} + C [I_{R_{n(x)}}^R(\{\mu\})(x)]^{\frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}}.
\]
for \( R \leq 1 \), where the implicit constant depends on \( n, p, \Lambda, \bar{\alpha}, \omega(\cdot), \sigma_1, K, \) and \( \text{diam}(\Omega) \).

With the notation used in proof of Theorem 3.3, multiplying both sides of (3.9) by \( R^{-\alpha} j + 1 \), \( j \geq 0 \), we have
\[
R_{j+1}^{-\alpha} A_{j+1} \leq c_1 e^{\beta_0 - \alpha} R_j^{-\alpha} A_j + C e R_j^{-\alpha} \left( \frac{1}{R_j^{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C \left( \frac{\mu((Q_j))}{R_{n(x)}} \right) \left( \frac{1}{R_j^{n(x)}} \right)^{\frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C [I_{R_{n(x)}}^R(\{\mu\})(x)]^{\frac{1}{p-1}} + C [I_{R_{n(x)}}^R(\{\mu\})(x)]^{\frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}}.
\]
This time we choose \( \epsilon \in (0, 1/3) \) such that
\[
c_1 e^{\beta_0 - \alpha} \leq c_1 e^{\beta_0 - \bar{\alpha}} \leq \frac{1}{2},
\]
and employ (3.15) together with the restriction \( R_j \leq 1 \), to deduce
\[
R_{j+1}^{-\alpha} A_{j+1} \leq \frac{1}{2} R_j^{-\alpha} A_j + CK \left( \frac{1}{R_j^{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C \left( \frac{\mu((Q_j))}{R_{n(x)}} \right) \left( \frac{1}{R_j^{n(x)}} \right)^{\frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C \left[ \frac{\mu((Q_j))}{R_{n(x)}} \right]^{\frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}}.
\]
On the other hand, applying Theorem 3.3 in the case \( \alpha = 1 \), we can bound
\[
\left( \frac{1}{R_j^{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}}
\]
for every integer \( j \geq 0 \). Thus, using (3.15) and Young’s inequality we get
\[
R_{j+1}^{-\alpha} A_{j+1} \leq \frac{1}{2} R_j^{-\alpha} A_j + C \left[ M_{1-a, \kappa}^R(\mu)(x) \right]^{\frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}} + C \left( \frac{1}{R_{n(x)}} \right)^{\frac{\kappa}{\kappa} + \frac{1}{p-1}}.
\]
Iterating this inequality, we find for any $m \in \mathbb{N}$,
\[
R_m^{-\alpha} A_m \leq 2^{-m} R_0^{-\alpha} A_0 + C \left[ M_{1-\alpha,\kappa}^R(\mu(x)) \right]^{\frac{1}{p-1}} \\
+ C \left[ I_1^R(|\mu|)(x) \right]^{\frac{1}{p-1}} + C \left( \int_{Q_R(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}} \\
\leq C \left[ M_{1-\alpha,\kappa}^R(\mu(x)) \right]^{\frac{1}{p-1}} \\
+ C \left[ I_1^R(|\mu|)(x) \right]^{\frac{1}{p-1}} + CR^{-\alpha} \left( \int_{Q_R(x)} |\nabla u|^\kappa dy \right)^{\frac{1}{\kappa}}.
\]

In view of the fact that
\[
M_{\alpha,\kappa}^R(\nabla u)(x) \lesssim \sup_{m \geq 0} R_m^{-\alpha} A_m,
\]
this completes the proof of the theorem.

4 Proof of Theorem 1.3

Proof of Theorem 1.3 For any cube $Q_\rho(x) \subset \Omega$, let $q_{Q_\rho(x)} \in \mathbb{R}$ be defined by
\[
q_{Q_\rho(x)} := \arg \min_{q \in \mathbb{R}} \left( \int_{Q_\rho(x)} |u - q|\kappa dy \right)^{1/\kappa},
\]
i.e., $q_{Q_\rho(x)}$ is a real number such that
\[
\inf_{q \in \mathbb{R}} \left( \int_{Q_\rho(x)} |u - q|\kappa dy \right)^{1/\kappa} = \left( \int_{Q_\rho(x)} |u - q_{Q_\rho(x)}|\kappa dy \right)^{1/\kappa}.
\]
Then using quasi-triangle inequality a few times and Lemma 1.1, we have for $Q_\rho(x) \subset Q_r(x) \subset Q_{3r}(x) \subset \Omega$,
\[
\left( \int_{Q_\rho(x)} |u - q_{Q_\rho(x)}|\kappa dy \right)^{1/\kappa} \lesssim \left( \frac{\rho}{r} \right)^{\alpha_0} \left( \int_{Q_{2r}(x)} |u - q_{Q_{2r}(x)}|\kappa dy \right)^{1/\kappa} \\
+ \left( \frac{\rho}{r} \right)^{-n/\kappa} \left( \int_{Q_{2r}(x_0)} |u - w|\kappa dy \right)^{1/\kappa}.
\]

Here we choose $w$ as in (2.1) with $Q_{2r}(x)$ in place of $Q_{2R}(x_0)$.
We now apply Remark 2.2 to bound the second term on the right-hand side of the
above inequality. This yields that
\[
\left( \int_{Q_{\rho r}(x)} |u - q_{Q_{\rho r}(x)}|^\kappa \, dy \right)^{1/\kappa} \lesssim \left( \frac{\rho^\alpha}{r} \right)^{\alpha_0} \left( \int_{Q_{2r}(x)} |u - q_{Q_{2r}(x)}|^\kappa \, dy \right)^{1/\kappa} + \left( \frac{\rho}{r} \right)^{-n/\kappa} \left( \frac{\mu(Q_{3r}(x))}{r^{n-p}} \right)^{\frac{1}{p-1}} + \left( \frac{\rho}{r} \right)^{-n/\kappa} |\mu| Q_{3r}(x) \left( \int_{Q_{3r}(x)} |u - q_{Q_{3r}(x)}|^\kappa \, dy \right)^{\frac{2-n}{p}}.
\]

Letting \( \rho = \epsilon r \), \( \epsilon \in (0, 1) \), and using Young’s inequality we find
\[
\left( \int_{Q_{\epsilon r r}(x)} |u - q_{Q_{\epsilon r r}(x)}|^\kappa \, dy \right)^{1/\kappa} \lesssim C \epsilon^{\alpha_0} \left( \int_{Q_{3r}(x)} |u - q_{Q_{3r}(x)}|^\kappa \, dy \right)^{1/\kappa} + C \epsilon \left( \frac{\mu(Q_{3r}(x_0))}{r^{n-p}} \right)^{\frac{1}{p-1}}. \tag{4.1}
\]

Next, we choose \( \epsilon \in (0, 1/3) \) small enough so that \( C \epsilon^{\alpha_0} \leq \frac{1}{2} \), where \( C \) is the constant in (4.1). Let \( Q_R(x_0) \subset \Omega \) be as given in the theorem. Then for any cube \( Q_\delta(x) \subset Q_R(x_0) \) we set \( \delta_j = \epsilon^j \delta \), \( Q_j = Q_{\delta_j}(x) \), \( q_j = q_{Q_j} \), \( j \geq 0 \), and define
\[
B_j := \left( \int_{Q_j} |u - q_{Q_j}|^\kappa \, dy \right)^{1/\kappa}.
\]

Applying (4.1) with \( r = \delta_j/3 \) yields
\[
B_{j+1} \leq \frac{1}{2} B_j + C \left( \frac{\mu(Q_j)}{\delta_j^{n-p}} \right)^{\frac{1}{p-1}}.
\]

Summing this up over \( j \in \{1, 3, ..., m - 1\} \), we obtain
\[
\sum_{j=1}^{m} B_j \leq C B_1 + C \sum_{j=1}^{m-1} \left( \frac{\mu(Q_j)}{\delta_j^{n-p}} \right)^{\frac{1}{p-1}}.
\]

As in (3.8), we have
\[
|q_{j+1} - q_j| \leq C B_j
\]

for all integers \( j \geq 1 \), and thus
\[
|q_m| \leq |q_m - q_1| + q_1 \leq q_1 + C \sum_{j=1}^{m-1} B_j
\]
\[
\leq q_1 + C B_1 + C \sum_{j=1}^{m-1} \left( \frac{\mu(Q_j)}{\delta_j^{n-p}} \right)^{\frac{1}{p-1}}
\]
\[
\leq C \left( \int_{Q_1} |u|^\kappa \, dx \right)^{\frac{1}{\kappa}} + C \sum_{j=1}^{m-1} \left( \frac{\mu(Q_j)}{\delta_j^{n-p}} \right)^{\frac{1}{p-1}} \tag{4.2}
\]
holds for every integer \( m \geq 2 \). Here we use the simple fact (see (3.7)) that
\[
B_1 + q_1 \leq C \left( \int_{Q_1} |u|^p \, dx \right)^{\frac{1}{p}}.
\]

Now for \( x, y \in Q_{R/8}(x_0) \) we choose
\[
\delta = \frac{1}{2}|x - y|_{\infty} = \frac{1}{2} \max_{1 \leq i \leq n} |x_i - y_i|.
\]

Note that \( \delta < R/8 \) and \( Q_\delta(y) \subset Q_{3\delta}(x) \subset Q_{R/2}(x_0) \). Then applying (4.2), we have
\[
|q_m| \leq C \left( \int_{Q_\delta(x)} |u|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha \sum_{j=1}^{m-1} \left( |\mu|_{(Q_{\delta_j}(x))} \right)^{\frac{1}{p-1}}.
\]

Sending \( m \to \infty \) and using [1, Lemma 4.1], we get
\[
|u(x)| \leq C \left( \int_{Q_\delta(x)} |u|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha W_{1-\alpha/p,p}^{R} (|\mu|)(x).
\]

Since \( u - m, m \in \mathbb{R} \), is also a solution of (1.1), it follows that
\[
|u(x) - m| \leq C \left( \int_{Q_\delta(x)} |u - m|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha W_{1-\alpha/p,p}^{R} (|\mu|)(x)
\]
\[
\leq C \left( \int_{Q_{3\delta}(x)} |u - m|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha W_{1-\alpha/p,p}^{R} (|\mu|)(x).
\]

Likewise, we have
\[
|u(y) - m| \leq C \left( \int_{Q_\delta(y)} |u - m|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha W_{1-\alpha/p,p}^{R} (|\mu|)(y)
\]
\[
\leq C \left( \int_{Q_{3\delta}(x)} |u - m|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha W_{1-\alpha/p,p}^{R} (|\mu|)(y).
\]

Now choosing \( m = q_{Q_{3\delta}(x)} \) we find
\[
|u(x) - u(y)| \leq C \left( \int_{Q_{3\delta}(x)} |u - q_{Q_{3\delta}(x)}|^p \, dz \right)^{\frac{1}{p}} + C\delta^\alpha \left[ W_{1-\alpha/p,p}^{R} (|\mu|)(x) + W_{1-\alpha/p,p}^{R} (|\mu|)(y) \right].
\]
On the other hand, by Theorem 3.1 and the fact that $3\delta < 3R/8$, we have
\[
\left( \frac{1}{Q_{3\delta}(x)} \right) |u - q_{Q_{3\delta}}(x)|^\alpha d\bar{z} \right)^{\frac{1}{\alpha}} 
\lesssim \delta^\alpha \left[ M^{9R/16}_{p-\alpha(p-1)}(\mu)(x) \right]^{\frac{1}{p-1}} + \left( \frac{\delta}{R} \right)^\alpha \left( \frac{1}{Q_{9R/16}(x)} |\nabla u|^\alpha d\bar{z} \right)^{\frac{1}{\alpha}} 
\lesssim \delta^\alpha W_{1-\alpha(p-1)/p,p}(|\mu|)(x) + \left( \frac{\delta}{R} \right)^\alpha \left( \frac{1}{B(x,\delta)} |u|^\alpha d\bar{z} \right)^{\frac{1}{\alpha}},
\]
where we used a Caccioppoli type inequality of [15, Corollary 2.4] in the last bound. Combining inequalities (4.3) and (4.4), we complete the proof of the theorem. ■

5 Proof of Theorems 1.4 and 1.5

Proof of Theorems 1.4. The main idea of the proof of Theorem 1.4 lies in the proof of [3, Theorem 1.2]. First, in view of Theorem 1.3, it suffices to prove (1.12) uniformly in $\alpha \in [\alpha_0/2, \tilde{\alpha}]$, $\tilde{\alpha} < 1$, for all $x, y \in Q_{R/8}(x_0)$. On the other hand, for a.e. $x, y \in Q_{R/8}(x_0)$ and $f \in L^\infty(Q_R(x_0))$, we have the inequality
\[
|f(x) - f(y)| \leq \left( \frac{c}{\alpha} \right) |x - y|^\alpha \left[ A_{1-\alpha(p-1)}^R(\mu)(x) + A_{1-\alpha(p-1)}^R(\mu)(y) \right]^{\frac{1}{p-1}} 
+ \left( \frac{c}{\alpha} \right) |x - y|^\alpha R^{1-\alpha} \left\{ \left( \frac{1}{Q_{3R/4}(x)} |\nabla u|^\alpha d\bar{z} \right)^{\frac{1}{\alpha}} + \left( \frac{1}{Q_{3R/4}(y)} |\nabla u|^\alpha d\bar{z} \right)^{\frac{1}{\alpha}} \right\}.
\]
Then invoking the Caccioppoli type inequality of [15, Corollary 2.4] we obtain (1.12) uniformly in $\alpha \in [\alpha_0/2, \tilde{\alpha}]$ as desired. ■

Proof of Theorem 1.5. The proof of Theorem 1.5 is similar to that of Theorems 1.4 but this time we use Theorem 3.3 instead of Theorem 3.2. ■

6 Proof of Theorem 1.6

Proof of Theorem 1.6. Let $Q_R(x_0) \subset \Omega$ be as given in the theorem. For any $x, y \in Q_{R/4}(x_0)$, we set $\delta = \frac{1}{2}|x - y|_\infty$. Note that $\delta < R/4$ and $Q_{\delta}(y) \subset Q_{3\delta}(x) \subset Q_R(x_0)$. We shall keep the notation in the proof of Theorem 3.3 except that we replace $R$ with $\delta$ so that $R_j = \delta_j = e^j \delta$, $Q_j = Q_{e^j \delta}(x)$, $q_j = q_{Q_{e^j \delta}(x)}$, and
\[
A_j = \left( \frac{1}{Q_{e^j \delta}(x)} |\nabla u - q_{Q_{e^j \delta}(x)}|^\alpha d\bar{z} \right)^{1/\alpha} = \left( \frac{1}{Q_j} |\nabla u - q_j|^\alpha d\bar{z} \right)^{1/\alpha}.
\]
for all integers \( j \geq 0 \).
Then by choosing \( \epsilon \) in (3.9) such that \( c_1 \epsilon^2 \leq 1/2 \), we have

\[
A_{j+1} \leq \frac{1}{2} A_j + C \omega(\delta_j/3)^{\sigma_1} \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{1/\kappa} + C \epsilon \left[ \frac{\mu(Q_j)}{\delta_j^{n-1}} \right]^{p-1}
\]

\[+ \quad C \epsilon \left( \frac{\mu(Q_j)}{\delta_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{(2-p)/\kappa}.
\]

Summing this up over \( j \in \{0, 1, \ldots, m-1\} \), \( m \in \mathbb{N} \), and then simplifying, we get

\[
\sum_{j=1}^{m} A_j \leq A_0 + C \sum_{j=0}^{m-1} \omega(\delta_j/3)^{\sigma_1} \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{1/\kappa} + C \sum_{j=0}^{m-1} \left[ \frac{\mu(Q_j)}{\delta_j^{n-1}} \right]^{1/\kappa}
\]

\[+ \quad C \sum_{j=0}^{m-1} \left( \frac{\mu(Q_j)}{\delta_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{(2-p)/\kappa}.
\]

On the other hand, by (3.8),

\[
|q_{m+1} - m| = \sum_{j=0}^{m} (|q_{j+1} - m| - |q_j - m|) + |q_0 - m|
\]

\[\leq \sum_{j=0}^{m} (|q_{j+1} - q_j| + |q_0 - m|) \leq C \sum_{j=0}^{m} A_j + |q_0 - m|.
\]

which holds for any \( m \in \mathbb{R}^n \) and integer \( m \geq 0 \).
Hence, it follows that

\[
|q_{m+1} - m| \leq C \sum_{j=0}^{m-1} \omega(\delta_j/3)^{\sigma_1} \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{1/\kappa} + C \sum_{j=0}^{m-1} \left[ \frac{\mu(Q_j)}{\delta_j^{n-1}} \right]^{1/\kappa}
\]

\[+ \quad C \sum_{j=0}^{m-1} \left( \frac{\mu(Q_j)}{\delta_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{(2-p)/\kappa}.
\]

Then using

\[
|q_0 - m| \leq \left( \int_{Q_{\delta(x)}} |\nabla u - m|^\kappa \, dy \right)^{1/\kappa},
\]

which can be proved as in (3.7), we get

\[
|q_{m+1} - m| \leq \sum_{j=0}^{m-1} \omega(\delta_j/3)^{\sigma_1} \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{1/\kappa} + \sum_{j=0}^{m-1} \left[ \frac{\mu(Q_j)}{\delta_j^{n-1}} \right]^{1/\kappa}
\]

\[+ \quad \sum_{j=0}^{m-1} \left( \frac{\mu(Q_j)}{\delta_j^{n-1}} \right) \left( \int_{Q_j} |\nabla u|^\kappa \, dy \right)^{(2-p)/\kappa}
\]

\[+ \quad \left( \int_{Q_{\delta(x)}} |\nabla u - q_{\delta(x)}|^\kappa \, dy \right)^{1/\kappa} + \left( \int_{Q_{\delta(x)}} |\nabla u - m|^\kappa \, dy \right)^{1/\kappa}.
\]
We next set
\[ M(x, r) := \left[ \mathbf{I}^1_1(|\mu|(x)) \right]^{\frac{1}{p-1}} \left( \frac{1}{Q_{j}(x)} \right) |\nabla u|^\kappa dz \right)^{\frac{1}{\kappa}}, \quad r > 0. \]

Then applying Theorem 3.3 with \( \alpha = 1 \) and \( 3R = \delta \), we have
\[ \left( \frac{1}{Q_{j}} |\nabla u|^\kappa \right)^{\frac{1}{\kappa}} \lesssim M(x, \delta), \quad \forall j \geq 0. \]

Plugging this into the last bound for \( |q_{m+1} - m| \) we deduce that
\[ \left| q_{m+1} - m \right| \lesssim \delta^\alpha \sum_{j=0}^{m-1} \delta_j^{-\alpha} \omega(\delta_j/3)^{\alpha_1} M(x, \delta) + \delta^\alpha \sum_{j=0}^{m-1} \frac{|\mu|(Q_j)}{\delta_j^{n-1+\alpha}} \right]^{\frac{1}{p-1}} \frac{\alpha(2-p)}{\delta} \left( \frac{\kappa}{p-1} \right) \]
\[ + \left( \frac{1}{Q_{\delta}(x)} |\nabla u - qQ_{\delta}(x)|^\kappa dy \right)^{\frac{1}{\kappa}} + \left( \frac{1}{Q_{\delta}(x)} |\nabla u - m|^\kappa dy \right)^{\frac{1}{\kappa}}. \]

Also, note that as in (3.12) we have
\[ \sum_{j=0}^{m-1} \delta_j^{-\alpha} \omega(\delta_j/3)^{\alpha_1} \lesssim \sum_{j=0}^{m-1} \delta_j^{-\alpha} \omega(\delta_j/3)^{\alpha_1} \lesssim \int_0^\delta \frac{\omega(\rho)^{\alpha_1}}{\rho^\alpha} d\rho \lesssim \int_0^{R/4} \frac{\omega(\rho)^{\alpha_1}}{\rho^\alpha} d\rho. \]

At this point, using the Dini-Hölder condition (1.14), we obtain, after some simple manipulations,
\[ \left| q_{m+1} - m \right| \lesssim \delta^\alpha M(x, \delta) + \delta^\alpha \left[ \mathbf{I}^R_{1-\alpha}(|\mu|(x)) \right]^{\frac{1}{p-1}} \delta^\alpha \mathbf{I}^R_{1-\alpha}(|\mu|(x)) M(x, \delta)^{2-p} \]
\[ + \left( \frac{1}{Q_{\delta}(x)} |\nabla u - qQ_{\delta}(x)|^\kappa dy \right)^{\frac{1}{\kappa}} + \left( \frac{1}{Q_{\delta}(x)} |\nabla u - m|^\kappa dy \right)^{\frac{1}{\kappa}}. \]

Here we also used that \( \delta < R/4 < \text{diam}(\Omega) \) and the implicit constants are allowed to depend on \( \text{diam}(\Omega) \).

Thus letting \( m \to \infty \) and using Young’s inequality we obtain
\[ |\nabla u(x) - m| \lesssim \delta^\alpha M(x, \delta) + \delta^\alpha \left[ \mathbf{I}^R_{1-\alpha}(|\mu|(x)) \right]^{\frac{1}{p-1}} \]
\[ + \left( \frac{1}{Q_{\delta}(x)} |\nabla u - qQ_{\delta}(x)|^\kappa dz \right)^{\frac{1}{\kappa}} + \left( \frac{1}{Q_{\delta}(x)} |\nabla u - m|^\kappa dz \right)^{\frac{1}{\kappa}}. \]
Likewise, we also have

\[ |\nabla u(y) - m| \lesssim \delta^\alpha M(y, \delta) + \delta^\alpha \left[ I_{1-\alpha}^R (|\mu|)(y) \right]^{\frac{1}{p-1}} \]

\[ + \left( \int_{Q_3(y)} |\nabla u - q_{Q_3}(y)|^\kappa dz \right)^{1/\kappa} + \left( \int_{Q_3(y)} |\nabla u - m|^\kappa dz \right)^{1/\kappa} \]

\[ \lesssim \delta^\alpha M(y, \delta) + \delta^\alpha \left[ I_{1-\alpha}^R (|\mu|)(y) \right]^{\frac{1}{p-1}} \]

\[ + \left( \int_{Q_3(x)} |\nabla u - q_{Q_3}(x)|^\kappa dz \right)^{1/\kappa} + \left( \int_{Q_3(x)} |\nabla u - m|^\kappa dz \right)^{1/\kappa} , \]

where we used that \( Q_3(y) \subset Q_3(x) \).

Combining these two estimates and choosing \( m = q_{Q_3}(x) \), we find

\[ |\nabla u(x) - \nabla u(y)| \lesssim \delta^\alpha \left\{ \left[ I_{1-\alpha}^R (|\mu|)(x) \right]^{\frac{1}{p-1}} + \left[ I_{1-\alpha}^R (|\mu|)(y) \right]^{\frac{1}{p-1}} \right\} \]

\[ + \delta^\alpha [M(x, \delta) + M(y, \delta)] + \left( \int_{Q_3(x)} |\nabla u - q_{Q_3}(x)|^\kappa dz \right)^{1/\kappa} . \] (6.1)

As \( \delta < R/4 \) and \( Q_{R/4}(x) \cup Q_{R/4}(y) \subset Q_R(x_0) \), we can apply Theorem 3.3 with \( \alpha = 1 \) to have the bound

\[ M(x, \delta) + M(y, \delta) \lesssim \left[ I_{1}^{R/4} (|\mu|)(x) \right]^{\frac{1}{p-1}} + \left( \int_{Q_{R/4}(y)} |\nabla u|^\kappa dz \right)^{1/\kappa} \]

\[ + \left[ I_{1}^{R/4} (|\mu|)(y) \right]^{\frac{1}{p-1}} + \left( \int_{Q_{R/4}(y)} |\nabla u|^\kappa dz \right)^{1/\kappa} \]

\[ \lesssim \left[ I_{1-\alpha}^{R/4} (|\mu|)(x) \right]^{\frac{1}{p-1}} + \delta^\alpha \left[ I_{1-\alpha}^{R/4} (|\mu|)(y) \right]^{\frac{1}{p-1}} \]

\[ + R^{-\alpha} \left( \int_{Q_R(x_0)} |\nabla u|^\kappa dz \right)^{1/\kappa} . \] (6.2)

Similarly, we can use Theorem 3.4 to bound the last term on the right-hand of (6.1) as follows:

\[ \left( \int_{Q_{3\alpha}(x)} |\nabla u - q_{Q_3}(x)|^\kappa dz \right)^{1/\kappa} \lesssim \delta^\alpha M_{1-\alpha,\kappa}^{3R/4} (\nabla u)(x) \]

\[ \lesssim \delta^\alpha \left[ M_{1-\alpha,\kappa}^{3R/4} (\mu)(x) \right]^{\frac{1}{p-1}} + \delta^\alpha \left[ I_{1}^{3R/4} (|\mu|)(x) \right]^{\frac{1}{p-1}} \]

\[ + \left( \frac{\delta}{R} \right)^\alpha \left( \int_{Q_{3\alpha}(x)} |\nabla u|^\kappa dz \right)^{\frac{1}{\kappa}} \]

\[ \lesssim \delta^\alpha \left[ I_{1-\alpha}^{R/4} (|\mu|)(x) \right]^{\frac{1}{p-1}} + \left( \frac{\delta}{R} \right)^\alpha \left( \int_{Q_{R}(x_0)} |\nabla u|^\kappa dz \right)^{\frac{1}{\kappa}} . \] (6.3)
We now plug estimates (6.2) and (6.3) into (6.1) to arrive at
\[ |\nabla u(x) - \nabla u(y)| \lesssim \delta^\alpha \left\{ \left[ R^{\alpha \mu}(x) \right]^{\frac{1}{p-1}} + \left[ R^{\alpha \mu}(y) \right]^{\frac{1}{p-1}} \right\} \]
\[ + \left( \frac{\delta}{R} \right)^\alpha \left( \int_{Q_R(x_0)} |\nabla u|^p \, dz \right)^{\frac{1}{p}}. \]

This completes the proof because \( \delta \leq \frac{1}{2} |x - y| \).

\[ \text{Remark 6.1} \quad \text{In Theorems 1.5-1.6 and 3.3-3.4 we may take } \sigma_1 = 1 \text{ in (1.13), (1.14)} \]
\[ \text{and (3.15), provided we replace } \omega \text{ with a non-decreasing function } \tilde{\omega} : [0, 1] \to [0, \infty) \text{ such that} \]
\[ \lim_{\rho \to 0} \tilde{\omega}(\rho) = 0, \quad \text{and} \quad |A(x, \xi) - A(y, \xi)| \leq \tilde{\omega}(|x - y|)|\xi|^{p-1} \]
for all \( x, y, \xi \in \mathbb{R}^n, |x - y| \leq 1 \). The reason is that in this case solutions to (2.1) are locally Lipschitz, and we can also take \( \sigma_1 = 1 \) in Lemma 2.5; see [8, Section 8].

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