AN INTRODUCTION TO BUNDLE GERBES

MICHAEL K MURRAY

ABSTRACT. An introduction to the theory of bundle gerbes and their relationship to Hitchin-Chatterjee gerbes is presented. Topics covered are connective structures, triviality and stable isomorphism as well as examples and applications.

1. Dedication

Over the years I have had many interesting mathematical conversations with Nigel and regularly came away with a solution to a problem or a new idea. While preparing this article I was trying to recall when he first told me about gerbes. I thought for awhile that age was going to get the better of my memory as many conversations seemed to have blurred together. But then I discovered that the annual departmental research reports really do have their uses. In July of 1992 I attended the ‘Symposium on gauge theories and topology’ at Warwick and reported in the 1992 Departmental Research Report that:

I . . . had discussions with Nigel Hitchin about ‘gerbes’. These are a generalisation of line bundles . . .

Further searching of my electronic files revealed an order for Brylinski’s book *Loop spaces, characteristic classes and geometric quantization* on the 29th April 1993. I recall that the book took some months to makes its away across the sea to Australia during which time I pondered the advertising material I had which said that gerbes were fibrations of groupoids. Trying to interpret this lead to a paper on bundle gerbes which I submitted to Nigel in his role as a London Mathematical Society Editor on the 25th July 1994. The Departmental Research Report of the same year reports that:

This year I began some work on a geometric construction called a bundle gerbe. These provide a geometric realisation of the three dimensional cohomology of a manifold.

My sincere thanks to Nigel for introducing me to gerbes and for the many other fascinating insights into mathematics that he has given me over the years.

2. Introduction

The theory of gerbes began with Giraud (1971) and was popularised in the book by Brylinski (1993). A short introduction by Nigel Hitchin (2003) in the ‘What is a . . .?’ series can be found in the Notices of the AMS. Gerbes provide a geometric realisation of the three dimensional cohomology of a manifold in a manner analogous to the way a line bundle is a geometric realisation of two dimensional cohomology.

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Part of the reason for their recent popularity is applications to string theory in particular the notion of the $B$-field. Strings on a manifold are elements in the loop space of the manifold and we would expect their quantization to involve a hermitian line bundle on the loop space arising from a two class on the loop space. That two class can arise as the transgression of some three class on the underlying manifold. Gerbes provide a geometrisation of this process. String theory however is not the only application of gerbes and we refer the interested reader to the related work of Hitchin (1999; 2006) which applies gerbes to generalised geometry and to reviews such as (Carey et al., 2000) and (Mickelsson, 2006) which give applications of gerbes to other problems in quantum field theory.

As with everything else in the theory of gerbes, the relationship of bundle gerbes to gerbes is best understood by comparison with the case of hermitian line bundles or equivalently $U(1)$ (principal) bundles. There are basically three ways of thinking about $U(1)$ bundles over a manifold $M$:

1. A certain kind of locally free sheaf on $M$.
2. A co-cycle $g_{\alpha\beta}: U_{\alpha} \cap U_{\beta} \rightarrow U(1)$ for some open cover $\mathcal{U} = \{U_{\alpha} \mid \alpha \in I\}$ of $M$.
3. A principal $U(1)$ bundle $P \rightarrow M$.

In the case of gerbes over $M$ we can think of these as:

1. A certain kind of sheaf of groupoids on $M$ (Giraud, Brylinski).
2. A co-cycle $g_{\alpha\beta\gamma}: U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \rightarrow U(1)$ for some open cover $\mathcal{U} = \{U_{\alpha} \mid \alpha \in I\}$ of $M$ or alternatively a choice of $U(1)$ bundle $P_{\alpha\beta} \rightarrow U_{\alpha} \cap U_{\beta}$ for each double overlap (Hitchin, Chatterjee).
3. A bundle gerbe (Murray).

Note that we are slightly abusing the definition of gerbe here as what we are considering are gerbes with band the sheaf of smooth functions from $M$ into $U(1)$. There are more general kinds of gerbes on $M$ just as there are more general kinds of sheaves on $M$ beyond those arising as the sheave of sections of a hermitian line bundle.

Recall some of the basic facts about $U(1)$ bundles on a manifold $M$.

1. If $P \rightarrow M$ is a $U(1)$ bundle there is a dual bundle $P^* \rightarrow M$ and if $Q \rightarrow M$ is another $U(1)$ bundle there is a product $P \otimes Q \rightarrow M$.
2. If $f: N \rightarrow M$ is a smooth map there is a pullback bundle $f^*(P) \rightarrow N$ and this behaves well with respect to dual and product. That is $f^*(P^*)$ and $(f^*(P))^*$ are isomorphic as also are $f^*(P \otimes Q)$ and $f^*(P) \otimes f^*(Q)$.
3. Associated to a $U(1)$ bundle $P \rightarrow M$ is a characteristic class, $c(P) \in H^2(M, \mathbb{Z})$, which is natural with respect to pullback, that is $f^*(c(P)) = c(f^*(P))$ and additive with respect to product and dual, that is $c(P \otimes Q) = c(P) + c(Q)$ and $c(P^*) = -c(P)$.
4. $P \rightarrow M$ is called trivial if it is isomorphic to $M \times U(1)$ or equivalently admits a global section. $P$ is trivial if and only if $c(P) = 0$.
5. There is a notion of a connection on $P \rightarrow M$. Associated to a connection $A$ on $P$ is a closed two-form $F_A$ called the curvature of $A$ with the property that $F_A/2\pi i$ is a de Rham representative for the image of $c(P)$ in real cohomology.
6. If $\gamma: S^1 \rightarrow M$ is a loop in $M$ and $P \rightarrow M$ a $U(1)$ bundle with connection $A$ then parallel transport around $\gamma$ defines the holonomy, $\text{hol}(A, \gamma)$ of $A$. 

around $\gamma$ which is an element of $U(1)$. If $\gamma$ is the boundary of a disk $D \subset M$ then we have

$$\text{hol}(A, \partial D) = \exp \left( \int_D F_A \right).$$

A gerbe is an attempt to generalise all the above facts about $U(1)$ bundles to some new kind of mathematical object in such a way that the characteristic class is in three dimension cohomology. Obviously for consistency other dimensions then have to change. In particular the curvature should be a three-form and holonomy should be over two dimensional submanifolds. It turns out to be useful to consider the general case of any dimension of cohomology which we call a $p$-gerbe. For historical reasons a $p$-gerbe has a characteristic class in $H^{p+2}(M, \mathbb{Z})$ so the interesting values of $p$ are $-2, -1, 0, 1, \ldots$ with $U(1)$ bundles corresponding to $p = 0$.

A $p$-gerbe then is some mathematical object which represents $p + 2$ dimensional cohomology. To make completely precise what representing $p + 2$ dimensional cohomology means would take us too far afield from the present topic but we give a sketch here to motivate the behaviour we are looking for in $p$-gerbes. To this end we will assume our $p$-gerbes $P$ live in some category $\mathcal{G}$ and there is a (forgetful) functor $\Pi: \mathcal{G} \to \text{Man}$ the category of manifolds. The functor $\Pi$ and the category $\mathcal{G}$ have to satisfy:

1. If $P$ is a $p$-gerbe there is a dual $p$-gerbe $P^*$ and if $Q$ is another $p$-gerbe there is a product $p$-gerbe $P \otimes Q$. In other words $\mathcal{G}$ is monoidal and has a dual operation.
2. If $f: N \to M$ is a smooth map and $\Pi(P) = M$ there is a pullback $p$-gerbe $f^*(P)$ and a morphism $\hat{f}: f^*(P) \to P$ such that $\Pi(f^*(P)) = N$ and $\pi(\hat{f}) = f$. Pullback should behave well with respect to dual and product. That is $f^*(P^*)$ and $(f^*(P))^*$ should be isomorphic as also should be $f^*(P \otimes Q)$ and $f^*(P) \otimes f^*(Q)$.
3. Associated to a $p$-gerbe $P$ is a characteristic class $c(P) \in H^{p+2}(\Pi(P), \mathbb{Z})$, which is natural with respect to pullback, that is $f^*(c(P)) = c(f^*(P))$ and additive with respect to product and dual, that is $c(P \otimes Q) = c(P) + c(Q)$ and $c(P^*) = -c(P)$.
4. As well as the notion of $P$ and $Q$ being isomorphic there is a possibly weaker notion of equivalence where $P$ and $Q$ are equivalent if and only if $c(P) = c(Q)$. We say $P$ is trivial if $c(P) = 0$.
5. There is a notion of a connective structure $A$ on $P$. Associated to a connective structure $A$ on $P$ is a closed $(p + 2)$-form $\omega$ on $\Pi(P)$ called the $(p + 2)$-curvature of $A$ with the property that $\omega/2\pi i$ is a de Rham representative for the image of $c(P)$ in real cohomology.
6. If $X \subset \Pi(P)$ is an oriented $p+1$ dimensional submanifold of $\Pi(P)$ we should be able to define the holonomy of the connective structure $\text{hol}(A, X) \in U(1)$ over $X$. Moreover if $Y \subset \Pi(M)$ is an oriented $p+2$-dimensional submanifold with boundary then we want to have that

$$\text{hol}(A, \partial Y) = \exp \left( \int_Y F_\omega \right).$$

Clearly by construction the category of $U(1)$ bundles, with the forgetful functor which assigns to a $U(1)$ bundle its base manifold, is an example of a $0$-gerbe. Before we consider other examples we need some facts about bundles with structure group

$(1)$ bundles to some category $\mathcal{G}$ and there is a (forgetful) functor $\Pi: \mathcal{G} \to \text{Man}$ the category of manifolds. The functor $\Pi$ and the category $\mathcal{G}$ have to satisfy:

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an abelian Lie group $H$. If $P \to M$ is an $H$ bundle on $M$ then by choosing local
sections of $P$ for an open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ we can construct transition
functions $h_{\alpha \beta} : U_\alpha \cap U_\beta \to H$ and in the usual way this defines a class $c(P) \in \to H^1(M,H)$ where here we abuse notation and write $H$ for what is really the sheaf of
smooth functions with values in $H$. It is a standard fact that isomorphism classes of
$H$ bundles are in bijective correspondence with $H^1(M,H)$ in this manner. If $P \to M$ is an $H$ bundle we can define its dual as follows. Let $P^*$ be isomorphic
to $P$ as a manifold with projection to $M$ and for convenience let $p^* \in P^*$ denote
$p \in P$ thought of as an element of $P^*$. Then define a new $H$ action on $P^*$ by
$p^* h = (ph^{-1})^*$. It is obvious that if $h_{\alpha \beta}$ are transition functions for $P$ then $h_{\alpha \beta}^* = h_{\alpha \beta}^\dagger$ are transition functions for $P^*$. In particular we have that $c(P^*) = -c(P)$ if we
write the group structure on $H^1(M,H)$ additively. If $Q$ is another $H$ bundle we can form the fibre product $P \times_M Q$ and let $H$ act on it by $(p,q)h = (ph, qh^{-1})$. Denote the orbit of $(p,q)$ under this action by $[p,q]$ and define an $H$ action by
$[p,q]h = [ph,qh] = [ph,q]$. The resulting $H$ bundle is denoted by $P \otimes Q \to M$. If $h_{\alpha \beta}$ are transition functions for $P$ and $k_{\alpha \beta}$ are transition functions for $Q$ then $h_{\alpha \beta}k_{\alpha \beta}$ are transition functions for $P \otimes Q$ and thus $c(P \otimes Q) = c(P) + c(Q)$. Notice that these constructions will not generally work for non-abelian groups because in
such a case the action of $H$ on $P^*$ is not a right action and the action on $P \otimes Q$ is
not even well-defined.

Example 2.1. The simplest example is that of functions $f : M \to \mathbb{Z}$ where we define
the functor $\Pi$ by $\Pi(f) = M$. The degree of $f$ is the class induced in $H^0(M,\mathbb{Z})$
so functions from $M$ to $\mathbb{Z}$ are $-2$ gerbes over $M$. Product and dual are pointwise
addition and negation. There is no sensible notion of connective structure.

Example 2.2. Consider next principal $\mathbb{Z}$ bundles $P \to M$. Clearly we want the
functor $\Pi$ to be $\Pi(P) = M$ and pull-backs are well known to exist. As $\mathbb{Z}$ is abelian
the constructions above apply and there are duals and products. The isomorphism
class of a bundle is determined by a class in $H^1(M,\mathbb{Z})$ so $\mathbb{Z}$ bundles are $p = -1$
gerbes. A $\mathbb{Z}$ bundle is trivial as a $-1$ gerbe if and only if it is trivial as a $\mathbb{Z}$ bundle.

It is not immediately obvious what a connective structure on a $\mathbb{Z}$ bundle is but it
turns out that the correct notion is that of a $\mathbb{Z}$ equivariant map $\phi : P \to i\mathbb{R}$ where
the action of $n \in \mathbb{Z}$ on $i\mathbb{R}$ is addition of $2\pi in$ so that $\phi(n) = \phi(p) + 2\pi in$. The map
$\phi$ then descends to a map $\phi : M \to S^1$ and the class of the bundle is the degree
of this map. The pull-back of the standard one-form on $\mathbb{R}$, that is $d\phi$ is a one-form
on $P$ which descends to a one-form $\phi^{-1}d\phi$ on $M$. The de Rham class $(\phi^{-1}d\phi)/2\pi i$
is the image of the class of the bundle in real cohomology.

We expect holonomy to be over a $-1 + 1 = 0$ dimensional submanifold. If $m_1, \ldots, m_r$
is a collection of points in $M$ with each $m_i$ oriented by some $\epsilon_i \in \{\pm 1\}$
let us denote by $\sum \epsilon_im_i$ their union as an oriented zero dimensional submanifold
of $M$. Then we define

$$\text{hol} \left( \phi, \sum \epsilon_im_i \right) = \prod_{i=1}^r \phi(m_i)^{\epsilon_i}.$$ 

In the case of an oriented one-dimensional submanifold $X \subset M$ with ends $-X_0$ and
$+X_1$ the fundamental theorem of calculus tells us that

$$\text{hol} \left( \phi, X_1 - X_0 \right) = \exp \left( \int_X d\phi \right)$$
Notice that if we express a \( \mathbb{Z} \) bundle locally in terms of transitions functions these are maps of the form \( f_{\alpha \beta} : U_\alpha \cap U_\beta \to \mathbb{Z} \). That is, over each double overlap we have a \(-2\) gerbe.

**Example 2.3.** It is clear from the above example that maps \( \phi : M \to U(1) \) are also \(-1\) gerbes with a connective structure. The dual and product are just pointwise inverse and pointwise product. The class is the degree and the connective structure is included automatically as part of \( \phi \).

We can also forget that there is a natural connective structure and just regard maps \( \phi : M \to U(1) \) as \(-1\) gerbes. In that case the natural notion of isomorphism between two maps \( \phi, \chi : M \to U(1) \) would be equality. However two such maps have the same degree if and only if they are homotopic. So the notion of equivalence of maps \( \phi : M \to U(1) \), thought of as \(-1\) gerbes (without connective structure) should be homotopy and is different to the notion of isomorphism.

**Example 2.4.** As we have remarked \( U(1) \) bundles are, of course, \( p = 0 \) gerbes. Notice that locally a \( U(1) \) bundle is given by transition functions \( g_{\alpha \beta} : U_\alpha \cap U_\beta \to U(1) \), that is on each double overlap we have a \(-1\) gerbe (with connective structure).

We will see below that this pattern of a \( p \) gerbe being defined as a \( p - 1 \) gerbe on double overlaps of some open cover is exploited by Hitchin and Chatterjee to give a definition of a 1 gerbe. But first we need some additional background material.

### 3. Background

We will be interested in surjective submersions \( \pi : Y \to M \) which we regard as generalizations of open covers. In particular if \( U = \{ U_\alpha \mid \alpha \in I \} \) is an open cover we have the disjoint union

\[
Y_U = \{(x, \alpha) \mid x \in U_\alpha\} \subset M \times I
\]

with projection map \( \pi(x, \alpha) = x \). The surjective morphism \( \pi : Y_U \to M \) is called the **nerve** of the open cover \( U \).

A morphism of surjective submersions \( \pi : Y \to M \) and \( p : X \to M \) is a map \( \rho : Y \to X \) covering the identity, that is \( p \circ \rho = \pi \). Any surjective submersion \( \pi : Y \to M \) admits local sections so there is an open cover \( U \) of \( M \) and local sections \( s_\alpha : U_\alpha \to Y \) of \( \pi \). These local sections define a morphism \( s : Y_U \to Y \) by \( s(x, \alpha) = s_\alpha(x) \). Indeed any morphism \( Y_U \to Y \) will be of this form. If \( V = \{ V_\alpha \mid \alpha \in J \} \) is a refinement of \( U \), that is there is a map \( \rho : J \to I \) such that for every \( \alpha \in J \) we have \( V_\alpha \subset U_\rho(\alpha) \), we have morphism of surjective submersions \( Y_V \to Y_U \) defined by \( (\alpha, x) \mapsto (\rho(\alpha), x) \).

Given a surjective morphism \( \pi : Y \to M \) we can form the \( p \)-fold fibre product

\[
Y^{[p]} = \{(y_1, \ldots, y_p) \mid \pi(y_1) = \cdots = \pi(y_p)\} \subset Y^p.
\]

The submersion property of \( \pi \) implies that \( Y^{[p]} \) is a submanifold of \( Y^p \). There are smooth maps \( \pi_i : Y^{[p]} \to Y^{[p-1]} \), for \( i = 1, \ldots, p \), defined by omitting the \( i \)th element. We will be interested in two particular examples.

**Example 3.1.** If \( U \) is an open cover of \( M \) then the \( p \)th fibre product \( Y_{U}^{[p]} \) is the disjoint union of all the ordered \( p \)-fold intersections. For example if \( U = \{ U_1, U_2 \} \) is an open cover of \( M \) then \( Y_{U}^{[2]} \) is the disjoint union of \( U_1 \cap U_2, U_2 \cap U_1, U_1 \cap U_1 \) and \( U_2 \cap U_2 \).
Example 3.2. If $P \to M$ is a principal $G$ bundle then $P \to M$ is a surjective submersion. It is easy to show that $P^{[p]} = P \times G^{p-1}$. In particular $P^{[2]} = P \times G$ and we shall need later the related fact that there is a map $g : P^{[2]} \to G$ defined by $p_1g(p_1, p_2) = p_2$.

Let $\Omega^q(Y^{[p]})$ be the space of differential $q$-forms on $Y^{[p]}$. Define

$$\delta : \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]}) \quad \text{by} \quad \delta = \sum_{i=1}^{p}(-1)^{p-1}\pi_i^*.$$ 

These maps form the fundamental complex

$$0 \to \Omega^q(M) \xrightarrow{\pi_1^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \ldots$$

and from (Murray, 1996) we have:

**Proposition 3.1.** The fundamental complex is exact for all $q \geq 0$.

Note that if $Y = Y_{\delta}$ then this Proposition is a well-known result about the Čech de Rham double complex. See, for example Bott and Tu’s book (1982).

Finally we need some notation. Let $H$ be an abelian group. If $g : Y^{[p-1]} \to H$ we define $\delta(g) : Y^{[p]} \to H$ by

$$\delta(g) = (g \circ \pi_1) - (g \circ \pi_2) + (g \circ \pi_3) \cdots .$$

If $P \to Y^{[p-1]}$ is an $H$ bundle we define an $H$ bundle $\delta(P) \to Y^{[p]}$ by

$$\delta(P) = \pi_1^*(P) \otimes (\pi_2^*(P))^* \otimes \pi_3^*(P) \otimes \cdots .$$

It is easy to check that $\delta(\delta(g)) = 1$ and that $\delta(\delta(P))$ is canonically trivial.

4. **Bundle gerbes**

**Definition 4.1.** A bundle gerbe (Murray, 1996) over $M$ is a pair $(P, Y)$ where $Y \to M$ is a surjective submersion and $P \to Y^{[2]}$ is a $U(1)$ bundle satisfying:

1. There is a bundle gerbe multiplication which is a smooth isomorphism

$$m : \pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$$

of $U(1)$ bundles over $Y^{[3]}$.

2. This multiplication is associative, that is, if we let $P_{(y_1, y_2)}$ denote the fibre of $P$ over $(y_1, y_2)$ then the following diagram commutes for all $(y_1, y_2, y_3, y_4) \in Y^{[4]}$:

$$\begin{array}{ccc}
P_{(y_1, y_2)} \otimes P_{(y_2, y_3)} \otimes P_{(y_3, y_4)} & \to & P_{(y_1, y_5)} \otimes P_{(y_3, y_4)} \\
\downarrow & & \downarrow \\
P_{(y_1, y_2)} \otimes P_{(y_2, y_4)} & \to & P_{(y_1, y_4)}
\end{array}$$

We remark that for any $(y_1, y_2, y_3) \in Y^{[3]}$ the bundle gerbe multiplication defines an isomorphism:

$$m : P_{(y_1, y_2)} \otimes P_{(y_2, y_3)} \to P_{(y_1, y_3)}$$

of $U(1)$ spaces.

We can show using the bundle gerbe multiplication that there are natural isomorphisms $P_{(y, y_2)} \cong P_{(y_2, y_1)}^*$ and $P_{(y, y_2)} \cong Y^{[2]} \times U(1)$.

We can rephrase the existence and associativity of the bundle gerbe multiplication to an equivalent pair of conditions in the following way. The bundle gerbe
multiplication gives rise to a section $s$ of $\delta(P) \to Y^{[3]}$. Moreover $\delta(s)$ is a section of $\delta(\delta(P)) \to Y^{[4]}$. But $\delta(\delta(P))$ is canonically trivial so it makes sense to ask that $\delta(s) = 1$. This is the condition of associativity. The family of spaces $\{Y^{[p]} \mid p = 1, 2, \ldots\}$ is an example of a simplicial space (Dupont, 1978) and by comparing to (Brylinski and McLaughlin, 1994) we see that a bundle gerbe is the same thing as a simplicial line bundle over this particular simplicial space.

Example 4.1. If we replace $Y$ in the definition by $Y_U$ for some open cover $U$ of $M$ we obtain the definition of gerbe given by Hitchin (1999) and by his student Chatterjee (1998). This consists of choosing an open cover $U$ of $M$ and a family of $U(1)$ bundles $P: U_\alpha \cap U_\beta$ such that over triple overlaps we have sections

$$s_{\alpha\beta\gamma} \in \Gamma(U_\alpha \cap U_\beta \cap U_\gamma \mid P_{\beta\gamma} \otimes P^*_{\alpha\gamma} \otimes P_{\alpha\beta})$$

and we require that $\delta(s) = 1$ in the appropriate way.

Example 4.2. The simplest example of a line bundle is given by the clutching construction on the two sphere $S^2$. If $U_0$ and $U_1$ are the open neighbourhoods of the north and south hemispheres we take the transition function $g: U_0 \cap U_1 \to U(1)$ to have winding number one. As there are only two open sets there is no condition on triple overlaps and we obtain the $U(1)$ bundle over $S^2$ of Chern class one. In a similar fashion we can consider $U_0$ and $U_1$ to be open neighbourhoods of the north and south hemispheres of the three-sphere $S^3$. Their intersection is retractable to the two-sphere so we can choose over this the $U(1)$ bundle $P$ of Chern class one. Again there are no additional conditions and we obtain the gerbe of degree one over $S^3$.

Example 4.3. Hitchin and Chatterjee also consider a gerbe as in Example 4.1 but with the added requirement that each $P_{\alpha\beta}$ is trivial in the form $P_{\alpha\beta} = U_\alpha \cap U_\beta \times U(1)$. Writing elements of the disjoint union $Y_U^{[2]}$ as $(\alpha, \beta, x)$ where $x \in U_\alpha \cap U_\beta$ we see that the bundle gerbe multiplication must take the form

$$((\alpha, \beta, x), z) \otimes ((\beta, \gamma, x), w) \mapsto ((\alpha, \gamma, x), zw \gamma_{\alpha\beta\gamma}(x))$$

for some $\gamma_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to U(1)$ and will be associative precisely when $\gamma_{\alpha\beta\gamma}$ is a cocycle.

We will refer to gerbes of the forms in Examples 4.1 or 4.3 as Hitchin-Chatterjee gerbes. The connection with bundle gerbes is simple. For clarity we define:

**Definition 4.2.** A bundle gerbe $(P,Y)$ over $M$ is called local if $Y = Y_U$ for some open cover $U$ of $M$.

We then obviously have:

**Proposition 4.3.** A Hitchin-Chatterjee gerbe is the same thing as a local bundle gerbe.

If $(P,Y)$ is a bundle gerbe over $M$ then associated to every point $m$ of $M$ we have a groupoid constructed as follows. The objects are the elements of the fibre $Y_m$ and the morphisms between $y_1$ and $y_2$ in $Y_m$ are $P(y_1, y_2)$. Composition comes from the bundle gerbe multiplication. If we call a groupoid a $U(1)$ groupoid if it is transitive and the group of morphisms of a point is isomorphic to $U(1)$, then the algebraic conditions on the bundle gerbe (that is the multiplication and its associativity) are
captured precisely by saying that a bundle gerbe is a bundle of $U(1)$ groupoids over $M$.

We now consider the properties given in Section 2 which we would like a 1-gerbe to satisfy and show how they are satisfied by bundle gerbes.

4.1. **Pullback.** If $f: N \to M$ then we can pullback $Y \to M$ to $f^*(Y) \to N$ with a map $f: f^*(Y) \to Y$ covering $f$. There is an induced map $f^*[2]: f^*(Y)^[2] \to Y^[2]$. Let

$$f^*(P, Y) = (f^*[2], f^*(Y)).$$

To see this is a bundle gerbe notice that all this is doing is pulling back the $U(1)$ groupoid at $f(n) \in M$ and placing it at $n \in N$ so we have a bundle of $U(1)$ groupoids over $N$ and thus a bundle gerbe.

4.2. **Dual and product.** If $(P, Y)$ is a bundle gerbe then $(P, Y)^* = (P^*, Y)$ is also a bundle gerbe called the dual of $(P, Y)$.

If $(P, Y)$ and $(Q, X)$ are bundle gerbes we can form the fibre product $Y \times_M X \to M$, a new surjective submersion and then define a $U(1)$ bundle

$$P \otimes Q \to (Y \times_M X)^[2] = Y^[2] \times_M X^[2]$$

by

$$(P \otimes Q)_{((y_1, x_1), (y_2, x_2))} = P_{y_1, y_2} \otimes Q_{x_1, x_2}.$$  

We define $(P, Y) \otimes (Q, X) = (P \otimes Q, Y \times_M X)$.

4.3. **Characteristic class.** The characteristic class of a bundle gerbe is called the Dixmier-Douady class. We construct it as follows. Choose a good cover $\mathcal{U}$ of $M$ (Bott and Tu, 1982) with sections $s_\alpha: U_\alpha \to Y$. Then

$$(s_\alpha, s_\beta): U_\alpha \cap U_\beta \to Y^2$$

is a section. Choose a section $\sigma_{\alpha\beta}$ of $P_{\alpha\beta} = (s_\alpha, s_\beta)^*(P)$. That is some

$$\sigma_{\alpha\beta}: U_\alpha \cap U_\beta \to P$$

such that $\sigma_{\alpha\beta}(x) \in P_{(s_\alpha(x), s_\beta(x))}$. Over triple overlaps we have

$$m(\sigma_{\alpha\beta}(x), \sigma_{\beta\gamma}(x)) = g_{\alpha\beta\gamma}(x) \sigma_{\alpha\gamma}(x) \in P_{(s_\alpha(x), s_\gamma(x))}$$

for $g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \to U(1)$. This defines a co-cycle which is the Dixmier-Douady class

$$DD((P, Y)) = [g_{\alpha\beta\gamma}] \in H^2(M, U(1)) = H^3(M, \mathbb{Z}).$$

**Example 4.3.** If $\mathcal{U}$ is an open cover and $g_{\alpha\beta\gamma}$ a $U(1)$ co-cycle then we can build a Hitchin-Chatterjee gerbe or local bundle gerbe of the type considered in Example 4.3. It is easy to see that this has Dixmier-Douady class given by the Čech class $[g_{\alpha\beta\gamma}]$.

Notice that this example shows that every class in $H^3(M, \mathbb{Z})$ arises as the Dixmier-Douady class of some Hitchin-Chatterjee gerbe or of some (local) bundle gerbe.

It is straightforward to check that if $f: N \to M$ and $(P, Y)$ is a bundle gerbe over $M$ then $f^*(DD(P, Y)) = DD(f^*(P, Y))$. Moreover we have

1. $DD((P, Y)^*) = -DD((P, Y))$, and
2. $DD((P, Y) \otimes (Q, X)) = DD((P, Y)) + DD((Q, X))$. 

We will defer the question of triviality of a bundle gerbe until the next section and consider next the notion of a connective structure on a bundle gerbe.

4.4. Connective structure. As \( P \to Y^{[2]} \) is a \( U(1) \) bundle we can pick a connection \( A \). Call it a bundle gerbe connection if it respects the bundle gerbe multiplication. That is if the section \( s \) of \( \delta(P) \to Y^{[3]} \) satisfies \( s^*(\delta(A)) = 0 \), i.e.

is flat for \( \delta(A) \). We would like bundle gerbe connections to exist. This is a straightforward consequence of the fact that the fundamental complex is exact. Indeed if \( A \) is any connection consider \( s^*(\delta(A)) \); we have \( \delta(s^*(\delta(A))) = \delta(s)^*(\delta\delta(A)) = 0 \) because \( \delta\delta(A) \) is the flat connection on the canonically trivial bundle \( \delta\delta(P) \). Hence there is a one-form \( a \) on \( Y^{[2]} \) such that \( \delta(a) = s^*(\delta(A)) \) and thus \( A - a \) is a bundle gerbe connection.

If \( A \) is a bundle gerbe connection then the curvature \( F_A \in \Omega^2(Y^{[2]}) \) satisfies \( \delta(F_A) = 0 \). From the exactness of the fundamental complex there must be an \( f \in \Omega^1(Y) \) such that \( F_A = \delta(f) \). As \( \delta \) commutes with \( d \) we have \( \delta(df) = d\delta(f) = df_A = 0 \). Hence \( df = \pi^*(\omega) \) for some \( \omega \in \Omega^3(M) \). So \( \pi^*(d\omega) = d\pi^*(\omega) = df = 0 \) and \( \omega \) is closed. In fact it is a consequence of standard Čech de Rham theory that:

\[
\frac{1}{2\pi i} \omega = r(DD((P,Y))) \in H^3(M,\mathbb{R}).
\]

We call \( f \) a curving for \( A \), the pair \( (A,f) \) a connective structure for \( (P,Y) \) and \( \omega \) is called the three-curvature of the connective structure \( (A,f) \). In string theory the two-form \( f \) is called the B-field.

We can give a local description of the connective structure as follows. Assume we have an open cover \( U \) of \( M \) with local sections \( s_\alpha : U_\alpha \to Y \) and sections over double overlaps \( \sigma_{\alpha\beta} \) of \( (s_\alpha, s_\beta)^*(P) \to U_\alpha \cap U_\beta \). We define

\[
A_{\alpha\beta} = (s_\alpha, s_\beta)^*(A) \in \Omega^1(U_\alpha \cap U_\beta)
\]

and

\[
f_\alpha = s_\alpha^*(f) \in \Omega^2(U_\alpha).
\]

These satisfy

\[
A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} = g_{\alpha\beta\gamma} df_{\alpha\beta}\]

and the three-curvature \( \omega \) restricted to \( U_\alpha \) is \( df_\alpha \).

Example 4.5. We can use this local description of the connective structure to calculate the Dixmier-Douady class of the Hitchin-Chatterjee gerbe on the three sphere defined in Example 12. Stereographic projection from either pole identifies \( S^3 \setminus \{(1,0,0)\} \) and \( S^3 \setminus \{(-1,0,0)\} \) with \( \mathbb{R}^3 \) and maps the equator to the unit sphere \( S^2 \subset \mathbb{R}^3 \). Let \( U_0 \) and \( U_1 \) be the pre-images of the interior of a ball of radius two in \( \mathbb{R}^3 \) under both stereographic projections. We can identify \( U_0 \cap U_1 \) with \( S^2 \times (-1,1) \). Pull back the line bundle of chern class \( k \) on \( S^2 \), with connection \( A \) and curvature \( F \), to \( U_0 \cap U_1 \). Because there are no triple overlaps this is a bundle gerbe connection. If we choose a partition of unity \( \psi_0 \) and \( \psi_1 \) for \( U_0 \) and \( U_1 \) then \( f_0 = -\psi_1 F \) and \( f_1 = \psi_0 F \) define two-forms on \( U_0 \) and \( U_1 \) respectively satisfying \( F = f_1 - f_0 \) on \( U_0 \cap U_1 \). These two forms define a curving for the bundle gerbe connection. The curvature is the globally defined three-form \( \omega \) whose restriction to \( U_0 \) and \( U_1 \) is \( -d\psi_1 \wedge F \) and \( d\psi_0 \wedge F \) respectively. The integral of \( \omega \) over the three
sphere reduces, by Stokes theorem, to the integral of $F$ over the two-sphere. Hence this bundle gerbe has Dixmier-Douady class $k \in H^3(M, \mathbb{Z}) = \mathbb{Z}$.

Holonomy will need to wait until we have considered the notion of triviality which we turn to now.

5. Triviality

Recall that a $U(1)$ bundle $P \to M$ is trivial if it is isomorphic to the bundle $M \times U(1)$ or, equivalently has a global section. This occurs if and only if $P \to M$ has zero Chern class. If $s_\alpha : U_\alpha \to P$ are local sections then $P$ is determined by a transition function $g : U_\alpha \cap U_\beta \to U(1)$ given by $s_\alpha = s_\beta g_{\alpha \beta}$ and $P \to M$ is trivial if and only if there exist $h_\alpha : U_\alpha \to U(1)$ such that

$$g_{\alpha \beta} = h_\beta h_\alpha^{-1}.$$  

In an analogous way Hitchin and Chatterjee (1998) define a gerbe $P_{\alpha \beta} \to U_\alpha \cap U_\beta$ to be trivial if there are $U(1)$ bundles $R_\alpha \to U_\alpha$ and isomorphisms $\phi_{\alpha \beta} : R_\alpha \otimes R_\beta^* \to P_{\alpha \beta}$ on double-overlaps in such a way that the multiplication becomes the obvious contraction

$$R_\alpha \otimes R_\beta^* \otimes R_\beta \otimes R_\alpha^* \to R_\alpha \otimes R_\alpha^*.$$

In the bundle gerbe formalism this idea takes the following form (Murray and Stevenson, 2000). Let $R \to Y$ be a $U(1)$ bundle and let $\delta(R) \to Y^{[2]}$ be defined as above. Note that $\delta(R)$ has a natural associative bundle gerbe multiplication given by

$$\delta(R)_{(y_1,y_2)} \otimes \delta(R)_{(y_2,y_3)} = R_{y_1} \otimes R_{y_2}^* \otimes R_{y_2} \otimes R_{y_3}^* \simeq R_{y_1} \otimes R_{y_3}^* = \delta(R)_{(y_1,y_3)}.$$  

**Definition 5.1.** A bundle gerbe $(P,Y)$ over $M$ is called trivial if there is a $U(1)$ bundle $R \to Y$ such that $(P,Y)$ is isomorphic to $(\delta(R),Y)$. In such a case we call a choice of $R$ and the isomorphism $\delta(R) \simeq P$ a trivialisation of $(P,Y)$.

**Example 5.1.** Let $(P,Y)$ be a bundle gerbe and assume that $Y \to M$ admits a global section $s : M \to Y$. Define $R \to Y$ by $R_y = P_{s(\pi(y),y)}$. Then we have an isomorphism

$$\delta(R)_{(y_1,y_2)} = P_{s(\pi(y_2),y_2)} \otimes P_{s(\pi(y_1),y_1)}^*$$  

$$= P_{s(\pi(y_2),y_2)} \otimes P_{(y_1,s(\pi(y_1))}^*$$  

$$\simeq P_{(y_1,y_2)}$$

using the bundle gerbe multiplication and the fact that $s(\pi(y_1)) = s(\pi(y_2))$. It is easy to check that this isomorphism preserves the respective bundle gerbe multiplications and we have shown that if $Y$ admits a global section then any bundle gerbe $(P,Y)$ is trivial. Notice that the converse is not true. Just take an open cover with more than one element and $g_{\alpha \beta \gamma} = 1$ to obtain a Hitchin-Chatterjee gerbe which has zero Dixmier-Douady class but for which $Y_U \to M$ has no global section.

Consider the Dixmier-Douady class of $\delta(R)$. If $s_\alpha : U_\alpha \to Y$ are local sections for a good cover choose local sections $\eta_\alpha$ of $s_\alpha^* (R)$. Then we can take as local sections of $(s_\alpha, s_\beta)^* (\delta(R))$ the sections $\sigma_{\alpha \beta} = \eta_\alpha \otimes \eta_\beta^*$ and it follows that the corresponding $g_{\alpha \beta \gamma} = 1$ and $\delta(R)$ has Dixmier-Douady class equal to zero. The converse is also true. Consider a bundle gerbe with Dixmier-Douady class zero. So we have an open cover and $\sigma_{\alpha \beta}$ such that

$$g_{\alpha \beta \gamma} = h_\beta h_\gamma h_\alpha^{-1}.$$
By replacing $\sigma_{\alpha\beta}$ by $\sigma_{\alpha\beta}/h_{\alpha\beta}$ we can assume that $g_{\alpha\beta\gamma} = 1$. Let $Y_\alpha = \pi^{-1}(U_\alpha)$ and define $R_\alpha \to Y_\alpha$ by letting the fibre of $R_\alpha$ over $y \in Y_\alpha$ be $P_{(y,s_\alpha(\pi(y)))}$. Construct an isomorphism $\chi_{\alpha\beta}(y)$ from the fibre of $R_\alpha$ over $y$ to the fibre of $R_\beta$ over $y$ by noting that

$$P_{(y,s_\alpha(\pi(y)))} = P_{(y,s_\beta(\pi(y)))} \otimes P_{(s_\alpha(\pi(y))s_\beta(\pi(y)))}$$

and using $\sigma_{\alpha\beta}(\pi(y)) \in P_{(s_\alpha(\pi(y))s_\beta(\pi(y)))}$. Because $\sigma_{\beta\gamma}\sigma_{\alpha\beta}\sigma_{\alpha\beta} = 1$ we can show that $\chi_{\alpha\beta}(y) \circ \chi_{\beta\gamma}(y) = \chi_{\alpha\gamma}(y)$ and hence the $R_\alpha$ clutch together to form a global $U(1)$ bundle $R \to Y$. It is straightforward to check that $\delta(R) = P$.

Consider now a $U(1)$ bundle $R \to Y$ and assume that $\delta(R) \to Y$ has a section $s$ with $\delta(s) = 1$ with respect to the canonical trivialisation of $\delta(\delta(R)) \to Y$. The section $s$ is called descent data for $R$ and is equivalent to $R$ being the pull-back of a $U(1)$ bundle on $M$. Indeed $s$ constitutes a family of isomorphisms

$$s(y_1, y_2) : R_{y_1} \to R_{y_2}$$

and $\delta(s) = 1$ is equivalent to $s(y_2, y_3) \circ s(y_1, y_2) = s(y_1, y_3)$ from which it is easy to define a bundle on $M$ whose pull-back is $R$.

Assume that a bundle gerbe $(P, Y)$ over $M$ is trivial and that $R_1$ and $R_2$ are two trivialisations. Then we have $\delta(R_1) \simeq P \simeq \delta(R_2)$ and hence a section of $\delta(R_1 \otimes R_2)$ which is descent data for $R_1 \otimes R_2$. Thus $R_1 = R_2 \otimes \pi^*(Q)$ for some $U(1)$ bundle $Q \to M$. It is easy to show the converse that if $R$ is a trivialisation and $Q \to M$ a $U(1)$ bundle then $R \otimes \pi^*(Q)$ is another trivialisation. We have now proved

**Proposition 5.2.** Let $(P, Y)$ be a bundle gerbe over $M$. Then:

1. $(P, Y)$ is trivial if and only if $DD(P, Y) = 0$, and
2. if $DD(P, Y) = 0$ then any two trivialisations of $(P, Y)$ differ by a $U(1)$ bundle on $M$.

This should be compared to the case of $U(1)$ bundles (0-gerbes) where two trivialisations or sections of the bundle differ by a map into $U(1)$ which is a $-1$-gerbe. The general pattern is that we expect two trivialisations of a $p$-gerbe to differ by a $p - 1$ gerbe. Notice also that whereas any two trivial $U(1)$ bundles are isomorphic there are many trivial bundle gerbes which are not isomorphic. This leads us to the notion of stable isomorphism.

**Definition 5.3.** If $(P, Y)$ and $(Q, X)$ are bundle gerbes over $M$ we say they are stably isomorphic (Murray and Stevenson, 2000) if $(P, Y)^* \otimes (Q, X)$ is trivial. A choice of a trivialisation is called a stable isomorphism from $(P, Y)$ to $(Q, X)$.

We have:

**Proposition 5.4.** Bundle gerbes $(P, Y)$ and $(Q, X)$ over $M$ are stably isomorphic if and only if $DD(P, Y) = DD(Q, X)$. The Dixmier-Douady class defines a bijection between stable isomorphism classes of bundle gerbes on $M$ and $H^3(M, \mathbb{Z})$.

**Proof.** Bundle gerbes $(P, Y)$ and $(Q, X)$ over $M$ are stably isomorphic if and only if $(P, Y)^* \otimes (Q, X)$ is trivial which occurs if and only if $-DD(P, Y) + D(Q, X) = 0$. We have already seen that every three class arises as the Dixmier-Douady class of some bundle gerbe on $M$. 

It follows that the correct notion of equivalence for bundle gerbes is stable isomorphism. It is actually possible to compose stable isomorphisms and the details
are given in work of Stevenson (2000) where the structure of the two category of all bundles gerbes on \( M \) is discussed. See also (Waldorf, 2007).

Note that we also have:

**Proposition 5.5.** Every bundle gerbe is stably isomorphic to a Hitchin-Chatterjee gerbe.

### 5.1. Holonomy.

Consider now a bundle gerbe \((P,Y)\) with connective structure \((A,f)\) over a surface \(\Sigma\). Because \(H^3(\Sigma, \mathbb{Z}) = 0\) we know that \((P,Y)\) is trivial. So there is a \(U(1)\) bundle \(R \to Y\) with \(\delta(R) = P\). Choose a connection \(\alpha\) for \(R\) and note that \(\delta(\alpha)\) is a connection for \(P\) using the isomorphism \(\delta(R) = P\). But \(\delta(\delta(\alpha))\) is flat so \(\delta(\alpha)\) is a bundle gerbe connection. Hence \(A = \delta(\alpha) + \alpha\) for a one-form \(\alpha\) on \(Y\). Using the exactness of the fundamental complex we can solve \(\delta(\alpha) = \delta(\alpha')\) and hence show that \(\delta(\alpha + \alpha') = A\). So without loss of generality we can choose a connection \(\alpha\) on \(R\) with \(\delta(\alpha) = A\). Consider the two-form \(f - F_A\). This satisfies \(\delta(f - F_A) = F_A - F_{\delta(\alpha)} = 0\) so \(f - F_A = \pi^*(\mu_\alpha)\) for some two form \(\mu_\alpha\) on \(\Sigma\). Define the holonomy of \((A,f)\) over \(\Sigma\) by

\[
\text{hol}((A,f),\Sigma) = \exp\left(\int_\Sigma \mu_\alpha\right)
\]

and note that this is independent of the choice of trivialisation \(R\) and connection \(\alpha\). Indeed any two trivialisations with connection will differ by a \(U(1)\) bundle on \(M\) with connection and the corresponding \(\mu_\alpha\) will differ by the curvature of the connection on that \(U(1)\) bundle. But the integral of the curvature of a \(U(1)\) bundle over a closed surface is in \(2\pi i \mathbb{Z}\) so the two definitions of holonomy agree.

If \((P,Y)\) is a bundle gerbe with connective structure \((A,f)\) on a general manifold \(M\) and \(\Sigma \subset M\) is a submanifold we can pull \((P,Y)\) and \((A,f)\) back to \(\Sigma\) and define hol((\(A,f),\Sigma\)) as above. In this more general setting if \(X \subset M\) is a three dimensional submanifold with boundary \(\partial X\) also a submanifold of \(M\) we can trivialise \((P,Y)\) over all of \(X\) and repeat the construction above. We then have \(d\mu_\alpha = \omega\), the three-curvature of \((A,f)\), and thus

\[
\text{hol}((A,f),\partial X) = \exp\left(\int_X \omega\right)
\]

the final property in Section 2 which we wanted a 1-gerbe to satisfy.

Assume that we have a local description for \((P,Y)\) and \((A,f)\) as in Section 4.4 in terms of \(g_{\alpha\beta\gamma}\), \(A_{\alpha\beta}\) and \(f_\alpha\). Then there is a remarkable formula (Gawędzki and Reis, 2002) for the holonomy, first proposed in (Alvarez, 1985) and also (Gawędzki, 1988) and subsequently derived by a number of authors, which can be described as follows. Choose a triangulation \(\Delta\) of \(\Sigma\) and a map \(\chi: \Delta \to I\) such that for any simplex \(\sigma \in \Delta\) we have \(\sigma \subset U_{\chi(\sigma)}\). Write \(\sigma^2\), \(\sigma^1\) and \(\sigma^0\) for two, one and zero dimensional simplices, that is, faces, edges and vertices. Then

\[
\text{hol}((A,f),\Sigma) = 
\exp\left(\sum_{\sigma^2} \int_{\sigma^2} f_{\chi(\sigma^2)}\right) 
\exp\left(\sum_{\sigma^1 \subset \sigma^2} \int_{\sigma^1} A_{\chi(\sigma^2)\chi(\sigma^1)}\right) 
\prod_{\sigma^0 \subset \sigma^1 \subset \sigma^2} g_{\chi(\sigma^2)\chi(\sigma^1)\chi(\sigma^0)}(\sigma^0).
\]
5.2. **Obstructions to certain kinds of** \(Y \to M\). When we consider the examples in the next section it will become apparent that they tend to cluster into two kinds: either \(Y \to M\) has infinite dimensional fibres or \(Y \to M\) has discrete fibres as in the case of Hitchin-Chatterjee gerbes. There is a reason for this which is the following result.

**Proposition 5.6.** Let \((P, Y)\) be a bundle gerbe over \(M\) with \(Y \to M\) a finite dimensional fibration with both \(M\) and the fibres of \(Y \to M\) one connected. Then the Dixmier-Douady class of \((P, Y)\) is torsion.

**Proof.** The proof uses the result from (Gotay et al., 1983) which shows that if \(Y \to M\) satisfies the hypothesis we have stated in the Proposition and moreover there is a smooth two-form \(\mu\) defined on the vertical tangent bundle of \(Y \to M\) which is closed on each fibre then we can extend \(\mu\) to a global, closed two form on \(Y \to M\). To apply this choose a connection and curving \(f\) for \((P, Y)\). If we restrict \(f\) to any fibre of \(Y \to M\) it agrees with \(F\) and hence is closed in the fibre directions. So there exists a global two form \(\mu\) on \(Y \to M\) which agrees with \(f\) in the vertical directions. Consider \(\rho = f - \mu\). Both \(\rho\) and \(d\rho\) are zero in the vertical directions so that \(\rho = \pi^*(\chi)\) for some two-form \(\chi\) on \(M\). But then \(\pi^*(d\rho) = d(f - \mu) = \pi^*(\omega)\) where \(\omega\) is the three curvature. Hence the image of the Dixmier-Douady class in real cohomology is zero so the Dixmier-Douady class is torsion. \(\Box\)

**6. Examples of bundle gerbes**

We expect to find bundle gerbes on manifolds \(M\) which have three dimensional cohomology. There are two, related, cases we will discuss here. The first comes from lifting problems for principal bundles, the so-called lifting bundle gerbe and the second is to take \(M\) a simple, compact Lie group.

### 6.1. Lifting bundle gerbe

Consider a central extension of Lie groups

\[0 \to U(1) \to \hat{G} \xrightarrow{\rho} G \to 0.\]

so that \(U(1)\) is the kernel of \(p\) and is in the center of \(\hat{G}\). If \(Y \to M\) is a \(G\) principal bundle then we can ask if it lifts to a \(\hat{G}\) bundle \(\hat{Y} \to M\). That is can we find a \(\hat{G}\) bundle \(\hat{Y} \to M\) and a bundle morphism \(f: \hat{Y} \to Y\) such that \(f(pg) = f(p)\rho(g)\) for all \(p \in \hat{Y}\) and \(g \in \hat{G}\). There is a well known topological obstruction to the existence of \(\hat{Y}\) which we can calculate as follows. Choose a good open cover \(U\) of \(M\) and local sections of \(Y\) that give rise to a transition function

\[g_{\alpha \beta}: U_{\alpha} \cap U_{\beta} \to G\]

in the usual way. Because these double overlaps are all contractible we can choose lifts of each \(g_{\alpha \beta}\) to \(\hat{g}_{\alpha \beta}: U_{\alpha} \cap U_{\beta} \to \hat{G}\). Note that

\[\epsilon_{\alpha \beta \gamma} = g_{\beta \gamma} g_{\alpha \gamma}^{-1} g_{\alpha \beta}\]

takes values in \(U(1)\). It is not difficult to show that \(\epsilon_{\alpha \beta \gamma}\) is a \(U(1)\) valued co-cycle and that the class

\[\epsilon_{\alpha \beta \gamma} \in H^2(M, U(1) \simeq H^3(M, \mathbb{Z})\]

vanishes if and only if \(P\) lifts to a \(\hat{G}\) bundle.
Given a $\mathcal{G}$ bundle $Y \to M$ there is a map $g: Y^{[2]} \to \mathcal{G}$ defined by $y_1 g(y_1, y_2) = y_2$. Notice that $\hat{\mathcal{G}} \to \mathcal{G}$ is a $U(1)$ bundle so we can pull it back to define a $U(1)$ bundle $Q \to Y^{[2]}$ whose fibres are cosets of $U(1)$ in $\hat{\mathcal{G}}$ defined by

$$Q(y_1, y_2) = U(1) g(y_1, y_2)$$

Because $g(y_1, y_2) g(y_2, y_3) = g(y_1, y_3)$ the product of an element in the coset containing $g(y_1, y_2)$ with an element in the coset containing $g(y_2, y_3)$ will be an element in the coset containing $g(y_1, y_3)$ which defines a bundle gerbe multiplication

$$Q(y_1, y_2) \otimes Q(y_2, y_3) \to Q(y_1, y_3).$$

The bundle gerbe $(Q, Y)$ is called the lifting bundle gerbe of $Y$ (Murray, 1996). It is easy to check that the lifting bundle gerbe has Dixmier-Douady class precisely the obstruction to lifting the bundle $Y \to M$. Indeed if we follow through the construction in Section 4.3 we find that $\hat{\sigma}_{\alpha\beta} = \hat{g}_{\alpha\beta}$. It follows from the discussion above that the lifting bundle gerbe is trivial if and only if the bundle $Y \to M$ lifts to $\hat{\mathcal{G}}$. In fact this follows directly because a lift $\hat{Y} \to Y$ will be a $U(1)$ bundle over $Y$ and actually a trivialization of the lifting bundle gerbe defined above.

### 6.2. Projective bundles.

Let $H$ be a Hilbert space, possibly finite dimensional. A Hilbert bundle with fibre $H$ can be regarded locally as a collection of transition functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to U(H)$$

satisfying the co-cycle condition

$$g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1.$$

In some situations we have slightly less than this, namely a collection of transition functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to U(H)$$

satisfying

$$g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} 1_{U(H)}.$$

If we denote by $\rho: U(H) \to PU(H)$ the projection onto the projective unitary group we see that have

$$\rho(g_{\beta\gamma}) \rho(g_{\alpha\gamma}^{-1}) \rho(g_{\alpha\beta}) = \rho(\epsilon_{\alpha\beta\gamma} 1_{U(H)})$$

so there is a well-defined bundle of projective spaces or a principal $PU(H)$ bundle. Given a projective bundle a natural question is to ask when it is the projectivisation of a global Hilbert bundle. This is equivalent to lifting the $PU(H)$ bundle to $U(H)$. The obstruction to this lifting is the class is

$$[\epsilon_{\alpha\beta\gamma}] \in H^2(M, U(1))$$

arising from the sequence

$$0 \to U(1) \to U(H) \to PU(H) \to 0.$$

In the finite dimensional class we can also consider the obstruction to lifting from $PU(n)$ to $SU(n)$ via the exact sequence

$$0 \to \mathbb{Z}_n \to SU(n) \to PU(n) \to 0$$

In this case the lifting bundle gerbe is really a $\mathbb{Z}_n \subset U(1)$ bundle gerbe and the Dixmier-Douady class is a torsion class in the image of the Bockstein map

$$H^2(M, \mathbb{Z}_n) \to H^3(M, \mathbb{Z}).$$
For further details on \(\mathbb{Z}_n\) bundle gerbes see (Carey et al., 2000). Note finally that if \(Y \to M\) is a \(PU(n)\) bundle then the lifting bundle gerbe has finite dimensional fibres so the fact that its Dixmier-Douady class is torsion is implied by Proposition 5.6.

6.3. Bundle gerbes on Lie groups. If \(G\) is a compact, simple, Lie group then \(H^3(G, \mathbb{Z}) \cong \mathbb{Z}\) so we expect to find bundle gerbes on \(G\). There are, in fact, a number of constructions of bundle gerbes on \(G\) and we have seen one already for \(SU(2) \cong S^3\) in Section 4.2. For convenience let us fix an orientation of \(G\) and call the bundle gerbe over \(G\) of Dixmier-Douady class one the basic bundle gerbe on \(G\).

**Example 6.1.** Let \(PG\) be the based path space of all smooth maps \(g\) of the interval \([0,1]\) into \(G\) with \(g(0) = 1\). Let \(ev: PG \to G\) be the evaluation map \(ev(g) = g(1)\). The kernel of \(ev\) is the space of based loops \(\Omega(G)\), that is all smooth maps \(g: [0,1] \to G\) for which \(g(0) = 1 = g(1)\). In the smooth Fréchet topology \(PG \to G\) is a principal \(\Omega(G)\) bundle. Moreover there is a well-known central extension (Pressley and Segal, 1986)

\[
0 \to U(1) \to \widehat{\Omega}(G) \to \Omega(G) \to 0
\]

where \(\Omega(G)\) is the Kac-Moody group. Hence there is a corresponding lifting bundle gerbe over \(G\) which is the basic bundle gerbe. It is possible to give an explicit construction of this gerbe over \(G\), see (Murray, 1996) and also (Stevenson, 2000).

If we wish to avoid the infinite dimensional spaces there is a construction of the basic bundle gerbe over \(G\) in the simply connected case due to Meinrenken (2003) and see also (Gawedzki and Reis, 2004) for the non-simply connected case. This construction has disconnected fibres for \(Y \to G\) and uses the standard structure theory of compact, simple, simply-connected Lie groups.

**Example 6.2.** For \(G = SU(n)\) there is a simple construction of the basic bundle gerbe due to Meinrenken (2003) (see also (Mickelsson, 2003)) using an open cover of \(G\) which can be presented without the cover as follows. Let

\[
Y = \{(g, z) \mid \det(g - z1) \neq 0\} \subset G \times U(1).
\]

For convenience let us write an element \((g, z), (g, w)\) \(\in Y^2\) as \((g, w, z)\). If \(u \in U(1)\) and \(u \neq w\) and \(u \neq z\) let us say that \(u\) is between \(w\) and \(z\) if an anti-clockwise rotation of \(z\) into \(w\) passes through \(u\). Then let \(W_{(g,w,z)}\) be the sum of all the eigenspaces of \(g\) for eigenvalues between \(z\) and \(w\) and define \(P_{(g,w,z)}\) to be the \(U(1)\) frame bundle of \(\det W_{(g,w,z)}\). To define the bundle gerbe product notice that if \(u\) is between \(w\) and \(z\) then

\[
W_{(g,w,u)} \oplus W_{(g,u,z)} \oplus W_{(g,z,w)} = \mathbb{C}^n
\]

so that

\[
\det W_{(g,w,u)} \otimes \det W_{(g,u,z)} \otimes \det W_{(g,z,w)} = \mathbb{C}
\]

and thus

\[
\det W_{(g,w,u)} \otimes \det W_{(g,u,z)} = \det W_{(g,z,w)}^*
\]

Similarly

\[
W_{(g,w,z)} \oplus W_{(g,z,w)} = \mathbb{C}^n
\]

so that

\[
\det W_{(g,w,z)} \otimes \det W_{(g,z,w)} = \mathbb{C}
\]
and
\[ \det W_{(g,w,z)} = \det W_{(g,z,w)}. \]
There are a number of other cases that can be dealt with in a similar fashion and putting all these facts together gives a bundle gerbe multiplication on \( P \rightarrow Y[2] \). A construction of the curving on \((P,Y)\) will appear in (Murray and Stevenson, 2007).

7. Applications of bundle gerbes

7.1. The Wess-Zumino-Witten term. The Wess-Zumino-Witten term associates to a smooth map \( g \) of a surface \( \Sigma \) into a compact, simple, Lie group \( G \) an invariant \( \Gamma(g) \in U(1) \). As noted by a number of authors (Carey et al., 2000; Hitchin, 2002; Gawędzki and Reis, 2002) this can be understood as the holonomy of a connection and curving on the basic gerbe on the group. The original definition of Witten (1984) is that we choose a three-manifold \( X \) with \( \partial X = \Sigma \) and extend \( g \) to \( \hat{g}: X \rightarrow G \). We then consider
\[ \int_X \hat{g}^*(\omega) \]
where \( \omega \) is a three form on \( G \) representing a generator of \( H^3(G, 2\pi i \mathbb{Z}) \). If we choose a different extension \( \tilde{g}: X \rightarrow G \) then the pair can be combined to define a map from the manifold \( X \cup_{\Sigma} X \), formed by joining two copies of \( X \) (with opposite orientations) along \( \Sigma \), into \( G \). Call this map \( \hat{g} \cup \tilde{g} \). Then
\[ \int_X \hat{g}^*(\omega) - \int_X \tilde{g}^*(\omega) = \int_{X \cup_{\Sigma} X} (\hat{g} \cup \tilde{g})^*(\omega) \in 2\pi i \mathbb{Z}. \]
It follows that
\[ \Gamma(g) = \exp \left( \int_X \hat{g}^*(\omega) \right) \in U(1) \]
depends only on \( g \).

If we choose a suitable connection and curving \((A, f)\) on the basic gerbe on \( G \), so that it has curvature \( \Omega \) then we see that
\[ \Gamma(g) = \text{hol}(\Sigma, g^*(A, f)). \]
The gerbe approach to the Wess-Zumino-Witten term has two advantages. Firstly it removes the topological restriction on \( M \) of 2-connectedness necessary in Witten’s definition so that the map \( g \) can be extended to the three-manifold \( X \). Secondly we can use the local formula for the holonomy given in Section 5.1. For details see (Gawędzki and Reis, 2002).

7.2. The Faddeev-Mickeilsson anomaly. We follow (Carey and Murray, 1996) and (Segal, 1985). Let \( X \) be a compact, Riemannian, spin, three-manifold and denote by \( \mathcal{A} \) the space of connections on a complex vector bundle over \( M \) and by \( \mathcal{G} \) the space of gauge transformations. For any \( A \in \mathcal{A} \) the chiral Dirac operator \( D_A \) coupled to \( A \) has discrete spectrum. Let
\[ Y = \{(A, t) \mid t \notin \text{spec}(D_A)\} \]
considered as a submersion over \( \mathcal{A} \). Note that \( \mathcal{G} \) acts on \( Y \) and \( Y/\mathcal{G} \rightarrow \mathcal{A}/\mathcal{G} \) is another submersion. Following Segal (1985) for \((A, s) \in Y\) we can decompose the Hilbert space \( H \) of coupled spinors into a direct sum of eigenspaces of \( D_A \) for eigenvalues greater than \( s \) and a direct sum of eigenspaces of \( D_A \) for eigenvalues
less than \( s \). Denote these by \( H^{-}_{(A,s)} \) and \( H^{+}_{(A,s)} \) respectively. We can then form the Fock space

\[
F_{(A,s)} = \bigwedge H^{+}_{(A,s)} \otimes \bigwedge (H^{-}_{(A,s)})^*
\]

which is a bundle \( F \to Y \). Notice that \( G \) acts on \( F \) and gives rise to a bundle \( F/G 
\to Y/G \).

If we choose another \( t < s \) we have

\[
H = H^{-}_{(A,t)} \oplus V_{(A,t,s)} \oplus H^{+}_{(A,s)}
\]

where \( V_{(A,t,s)} \) is the sum of all the eigenspaces of \( D_A \) for eigenvalues between \( t \) and \( s \). If we use the canonical isomorphism

\[
\bigwedge V_{(A,t,s)}^* \otimes \det V_{(A,t,s)} = \bigwedge V_{(A,t,s)}
\]

then we can show that

\[
F_{(A,s)} = F_{(A,t)} \otimes \det V_{(A,t,s)}.
\]

It follows that the projective spaces of \( F_{(A,s)} \) and \( F_{(A,t)} \) are canonically isomorphic and descend to a projective bundle \( P \to A \) and as \( G \) acts there is also a projective bundle \( P/G \to A/G \). The question of interest (Segal, 1985) is whether there is a Hilbert bundle \( H \to A/G \) whose projectivisation is \( P/G \). Notice that as \( A \) is contractible the answer to the equivalent question on \( A \) is clearly positive.

As noted in Section 6.2 we could give a bundle gerbe interpretation of this question via the lifting bundle gerbe for the central extension

\[
0 \to U(1) \to PU(H) \to U(H) \to 0
\]

for a suitable Hilbert space \( H \). However there is a more direct approach as follows: Let \( P_{(A,s,t)} \) be the unitary frame bundle of \( \det V_{(A,t,s)} \). Notice that if \( r < s < t \) then

\[
V_{(A,r,t)} \oplus V_{(A,t,s)} = V_{(A,r,s)}
\]

so that

\[
\det V_{(A,r,t)} \otimes \det V_{(A,t,s)} = \det V_{(A,r,s)}
\]

gives rise to a bundle gerbe multiplication similar to the \( SU(n) \) case Example 6.2. Again \( G \) acts so this descends to a bundle gerbe on \( A/G \). If this bundle gerbe is trivial then there is a line bundle \( L \to Y \) such that

\[
\det V_{(A,t,s)} = L_{(A,s)} \otimes L^*_{(A,t)}
\]

and moreover as it is the bundle gerbe on \( A/G \) which is trivial these are \( G \) equivariant isomorphisms. It follows that

\[
F_{(A,s)} = F_{(A,t)} \otimes L_{(A,s)} \otimes L^*_{(A,t)}
\]

or

\[
F_{(A,s)} \otimes L^*_{(A,s)} = F_{(A,t)} \otimes L^*_{(A,t)}.
\]

and these are \( G \) equivariant isomorphisms. Hence \( F_{(A,s)} \otimes L^*_{(A,s)} \) descends to a \( G \) equivariant Hilbert bundle on \( A \) whose projectivisation is \( P \). It follows that there is a Hilbert bundle on \( A/G \) whose projectivisation is \( P/G \).
7.3. String structures. In (1987) Killingback introduced the notion of a string structure. Given a $G$ bundle $Q \to X$ he considers the corresponding loop bundle $LQ \to LX$ which is an $LG$ bundle. As noted above we have the Kac-Moody central extension

$$0 \to U(1) \to \hat{L}G \to LG \to 0.$$ 

Killingback says that $Q \to X$ is string if $LQ \to LX$ lifts to $\hat{L}G$, calls a choice of such a lift a string structure and defines the obstruction class to the lift to be the string class. In (Murray and Stevenson, 2003) we consider the more general situation of an $LG$ bundle on a manifold $M$ and the question of whether it lifts to $\hat{L}G$. There is an equivalence between $LG$ bundles on $M$ and $G$ bundles on $S^1 \times M$ which is exploited to define a connection and curving on the lifting bundle gerbe associated to the $LG$ bundle. This enables the derivation of a de Rham representative for the image of the string class in real cohomology. The same approach could be applied to the other level Kac-Moody central extensions.

8. Other matters

In the interests of brevity nothing has been said about the original approach to gerbes as sheaves of groupoids. Details are given in (Brylinski, 1993) and the relationship with bundle gerbes is discussed in (Murray, 1996) and in more detail in (Stevenson, 2000). In this same context I would like to thank Larry Breen for pointing out that in the work of Ulbrich (1990) and (1991) the notion of cocycle bitorsors can be interpreted as a form of a bundle gerbe.

In the definitions and theory above we could replace $U(1)$ by any abelian group $H$ and there would only be obvious minor modifications such as the Dixmier-Douady class being in $H^2(M,H)$. If we want to replace $H$ by a non-abelian group things become more difficult as we mentioned in the introduction because we cannot form the product of two $H$ bundles if $H$ is non-abelian. To get around this difficulty we need to replace principal bundles by principal bibundles which have a left and right group action. The resulting theory becomes more complicated although closer to Giraud’s original aim of understanding non-abelian cohomology. For details see (Aschieri et al., 2005) and (Breen and Messing, 2005).

We have motivated gerbes by the idea of replacing the transition function $g_{\alpha\beta}$ by a $U(1)$ bundle $P_{\alpha\beta}$ on double overlaps. It is natural to consider what happens if we take the next step and replace $P_{\alpha\beta}$ by a bundle gerbe on each double overlap. This gives rise to the notion of bundle two gerbes whose characteristic class is a four class. In particular there is associated to any principal $G$ bundle $P \to M$ a bundle two gerbe whose characteristic class is the pointrjagin class of $P$. For details of the theory see (Stevenson, 2000; Stevenson, 2004) and for an application to Chern-Simons theory see (Carey et al., 2005).

It is clear that we can continue on in this fashion and consider bundle two gerbes on double overlaps and more generally inductively use $p$ gerbes on double overlaps to define $p + 1$ gerbes. However the theory becomes increasingly complex for a reason that we have not paid much attention to in the discussion above. In the case of $U(1)$ bundles the transition function has to satisfy one condition, the cocycle identity. In the case of gerbes the $U(1)$ bundles over double overlaps satisfy two conditions, the existence of a bundle gerbe multiplication and its associativity. In the case of bundle two gerbes there are three conditions and the complexity continues to grow in this fashion.
IntroducTion

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(Michael K. Murray) School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
*E-mail address:* michael.murray@adelaide.edu.au