A strategy for simultaneous measurement of the CKM angle $\gamma$ in multiple $B$ meson decays

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Abstract

Several methods exist to measure $CP$ violation observables related to the CKM Unitarity Triangle angle $\gamma$ using $B$ meson decays. These observables are different for every $B$ meson decay considered, although the information they contain on $\gamma$ is encoded in a similar way for all of them. This paper describes a strategy for a simultaneous measurement of $\gamma$ using several $B$ meson decays that takes into account possible correlations between them. Sensitivity studies demonstrate that the simultaneous analysis of several $B$ meson decay modes results in smaller uncertainties and improvement compared to a combination of standalone measurements.
1 Introduction

The angle $\gamma \equiv \arg \left( -\frac{V_{ud} V_{ub}^{*}}{V_{cd} V_{cb}^{*}} \right)$ of the Cabibbo-Kobayashi-Maskawa (CKM) Unitarity Triangle can be measured using tree-level $B$ meson decays that involve interference between $b \to u$ and $b \to c$ quark transitions. Time-integrated measurements can be made to measure $\gamma$ using decays of the type $B \to DX$, where $D$ represents an admixture of the flavour eigenstates $D^0$ and $\bar{D}^0$ and $X$ a final state containing one or more kaons or pions; examples include $B^{\pm} \to D^{(*)} K^{(*)} \pm$, $B^0 \to D K^{*0}$ and $B^{\pm} \to D^{(*)} \pi^{\pm}$ decays. Alongside this, a time-dependent approach can be employed to measure $\gamma$ from decays such as $B_s \to D_s K$.

Several methods can be utilised to measure different CP violation observables in these decays [1–5]; the measurements are then typically used to place tree-level constraints on $\gamma$ without the need for any theoretical input. The current world average value of $\gamma = (71.1^{+4.6}_{-5.3})^\circ$ [6] is dominated by measurements from the LHCb experiment [7].

This paper presents an approach to simultaneously constraining $\gamma$ from multiple $B$ meson decays, allowing for the treatment of candidate decays experimentally reconstructed under different decay hypotheses and the straightforward determination of correlations between systematic uncertainties. The technique employs a reduced set of CP violation parameters and is applicable to all possible measurement approaches.

2 Simultaneous approach for time-integrated measurements with $B \to DX$ decays

2.1 Admixture coefficients $z^m_{\pm}$

As described above, several $B$ meson decays produce admixtures of neutral $D$ mesons that involve $\gamma$. In this paper, $|D^0\rangle$ and $|\bar{D}^0\rangle$ are used to denote the flavour eigenstates of neutral $D$ mesons; $|D_+\rangle$ represents the $D$ meson produced in $B^+$ or $B^0$ meson decays and $|D_-\rangle$ the $D$ meson produced in $B^-$ or $\bar{B}^0$ decays.

In general, one can write

$$
\begin{align*}
|D^m_-\rangle &\sim |D^0\rangle + z^-_m |\bar{D}^0\rangle \\
|D^m_+\rangle &\sim |\bar{D}^0\rangle + z^+_m |D^0\rangle
\end{align*}
\Rightarrow \left\{ \begin{array}{l}
A^m_- \sim A_D + z^-_m A_{\bar{D}} \\
A^m_+ \sim A_{\bar{D}} + z^+_m A_D,
\end{array} \right.
$$

(1)

where $m$ denotes the $B$ decay mode under consideration. The amplitudes $A_D = \langle f |\mathcal{H}|D^0\rangle$, $A_{\bar{D}} = \langle f |\mathcal{H}|\bar{D}^0\rangle$ and $A^m_\pm = \langle f |\mathcal{H}|D^m_\pm\rangle$ define the $D$ meson decay to a final state $f$. The complex coefficients $z^m_{\pm}$ are specific to each $B$ decay, and are typically expressed in either Cartesian $(x^m_{\pm}, y^m_{\pm})$ or polar $(r_m, \delta_m, \gamma)$ coordinates as

$$
z^m_{\pm} = x^m_{\pm} + i y^m_{\pm} = r_m e^{i\delta_m} e^{\pm i\gamma},
$$

(2)

where all parameters with subscript or superscript $m$ are specific to a particular $B$ meson decay. It is apparent that $r_m$ and $\delta_m$ represent the ratio of amplitude magnitudes and their strong phase difference. Using the definition

$$
z_m = r_m e^{i\delta_m},
$$

(3)

leads to

$$
z^m_{\pm} = z_m e^{\pm i\gamma},
$$

(4)
and reveals an invariant for each $B$ meson decay,

$$\frac{z_+^m}{z_-^m} = e^{2i\gamma} \Rightarrow \gamma = \frac{1}{2} \arg \left( \frac{z_+^m}{z_-^m} \right).$$  \hspace{1cm} (5)$$

If both $z_\pm^m$ coefficients are multiplied by any complex coefficient $\xi$, the result will contain exactly the same information on $\gamma$. In particular, it is always possible to relate the $z_\pm^m$ coefficients for channel $m$ to those for $B^\pm \to DK^\pm$ decays, denoted $z_\pm$:

$$z_\pm^m = \xi_m z_\pm, \hspace{1cm} (6)$$

where

$$\xi_m = \frac{z_-^m}{z_-^{DK}}. \hspace{1cm} (7)$$

By definition, from Equations (5) and (7), the $\xi_m$ coefficients do not depend on $\gamma$; they can therefore be considered as nuisance parameters.

Using the $\xi_m$ coefficients to perform a simultaneous fit for the Cartesian parameters in $N$ distinct $B$ meson decay modes, reduces the number of independent parameters in the fit from $4N$ to $2(N + 1)$: 4 parameters for $B^\pm \to DK^\pm$, and 2 for each of the other decays. Although this is only one more parameter than a simultaneous fit for the polar coordinates, it has the advantage that the real and imaginary components of $z_\pm$ and $\xi_m$ are expected to exhibit Gaussian behaviour.

### 2.2 The $\eta$ function

In order to simplify later notation, it is useful to define the $\eta$ function as

$$\eta(a, b, \kappa) = |a|^2 + |b|^2 + 2\kappa \text{Re}(a^*b),$$  \hspace{1cm} (8)

where $a, b \in \mathbb{C}$ and $\kappa \in \mathbb{R}$. This function is symmetric ($\eta(a, b, \kappa) = \eta(b, a, \kappa)$) and scales as $\eta(a, b, \kappa) = |a|^2 \eta(1, \frac{b}{a}, \kappa)$. The $\kappa$ coefficient, commonly known as the coherence factor, indicates the fraction of coherent sum that contributes to $\eta$,

$$\eta(a, b, \kappa) = \kappa |a + b|^2 + (1 - \kappa) (|a|^2 + |b|^2). \hspace{1cm} (9)$$

In this paper, when the coherence factor argument is omitted, it should be assumed that it is implicit; if one of the complex arguments is omitted, it should be assumed that it is 1 (for example, $\eta(a) \equiv \eta(a, 1, \kappa)$).

### 2.3 Signal amplitude

The probability distribution function for a given $B$ meson decay, $p^m_\pm$, is proportional to the squared amplitude $|A^m_\pm|^2$ integrated over the $B$ decay phase space,

$$p^m_\pm \sim \int dP_B |A^m_\pm|^2.$$ \hspace{1cm} (10)

For a specific $B$ decay mode, defining $A_c$ as the decay amplitude corresponding to a $b \to c$ transition and $A_u e^{\pm i\gamma}$ as the decay amplitude corresponding to a $b \to u$ transition leads to

$$A_- \sim A_c A_D + A_u e^{-i\gamma} A_{\bar{D}}, \hspace{1cm} (11)$$

$$A_+ \sim A_c A_D + A_u e^{+i\gamma} A_{\bar{D}}. \hspace{1cm} (12)$$
For a 2-body $B$ meson decay, such as $B^\pm \rightarrow DK^\pm$, the amplitudes $A_c$ and $A_u$ are constants, and one can write
\begin{align}
A_- &\sim A_D + z_- A_D, \\
A_+ &\sim A_{\bar{D}} + z_+ A_D,
\end{align}
where
\begin{equation}
z_\pm = \frac{A_u}{A_c} e^{\pm i\gamma}.
\end{equation}
This implies that, for 2-body decays,
\begin{align}
p_- &\sim \eta(A_D, z_- A_D), \\
p_+ &\sim \eta(A_{\bar{D}}, z_+ A_D).
\end{align}

In the case of a multi-body $B$ meson decay with 3 or more particles in the final state, such as $B^0 \rightarrow DK\pi$, the amplitude $|A_m^m|^2$ may be integrated over a reduced part of the $B$ decay phase space, for example around the $K^{*0}(892)$ resonance for $B^0 \rightarrow DK\pi$.

By squaring the modulus of expressions (13) and (14) and defining
\begin{align}
N_{\alpha\beta} &= \int dP_B A^*_\alpha A_\beta, \\
X_{\alpha\beta} &= \frac{N_{\alpha\beta}}{\sqrt{N_{\alpha\alpha} N_{\beta\beta}}},
\end{align}
one can write
\begin{align}
p_- &\sim |A_D|^2 + \frac{N_{uu}}{N_{cc}} |A_D|^2 + 2 |X_{cu}| \text{ Re} \left( \sqrt{\frac{N_{uu}}{N_{cc}}} \frac{X_{cu}}{|X_{cu}|} e^{-i\gamma} A^*_D A_D \right), \\
p_+ &\sim |A_{\bar{D}}|^2 + \frac{N_{uu}}{N_{cc}} |A_D|^2 + 2 |X_{cu}| \text{ Re} \left( \sqrt{\frac{N_{uu}}{N_{cc}}} \frac{X_{cu}}{|X_{cu}|} e^{+i\gamma} A^*_D A_D \right).
\end{align}

It should be noted that $|X_{\alpha\beta}| \leq 1$, because of the Cauchy-Schwarz inequality. Defining
\begin{align}
\kappa &= |X_{cu}|, \\
r &= \sqrt{\frac{N_{uu}}{N_{cc}}}, \\
e^{i\delta} &= \frac{X_{cu}}{|X_{cu}|}, \\
z &= r e^{i\delta}, \\
z_\pm &= z e^{\pm i\gamma},
\end{align}
the signal amplitude probability distribution for $B$ meson decay mode $m$ can then be expressed as
\begin{align}
p^m_- &\sim \eta(A_D, \xi_m z_- A_{\bar{D}}, \kappa_m) = |A_D|^2 + |\xi_m z_-|^2 |A_D|^2 + 2 \kappa_m \text{ Re} (\xi_m z_- A^*_D A_{\bar{D}}), \\
p^m_+ &\sim \eta(A_D, \xi_m z_+ A_{\bar{D}}, \kappa_m) = |A_D|^2 + |\xi_m z_+|^2 |A_D|^2 + 2 \kappa_m \text{ Re} (\xi_m z_+ A^*_D A_{\bar{D}}).
\end{align}

These expressions describe the physics of the neutral $D$ meson admixture that leads to the different $CP$ observables used to measure $\gamma$, but they are not specific to any measurement method.
3 Specific formalism for established methodologies

3.1 The GLW approach

For a specific $B \to DX$ meson decay mode, the GLW method \cite{1,2} uses two sets of final states from the $D$ meson decay. The first are those states that are accessible from only one of the $D$ meson flavour eigenstates, either $|D^0\rangle$ or $|\bar{D}^0\rangle$, such that $A^D_D = \langle f_D | H | D^0 \rangle$, $A^D_{\bar{D}} = \langle f_{\bar{D}} | H | \bar{D}^0 \rangle$, and $\langle f_D | H | D^0 \rangle = \langle f_{\bar{D}} | H | \bar{D}^0 \rangle = 0$. The second are those states that are accessible from one of the $CP$ eigenstates $|D^{\pm}_{CP}\rangle$, such that $A^D_{\pm CP} = \langle f_{\pm} | H | D^{\pm}_{CP} \rangle$ and $\langle f_{\pm} | H | D^{\pm}_{CP} \rangle = 0$.

The observables of interest for a given $B$ decay mode $m$ are ratios and asymmetries that can be used to constrain $z_\pm$ and $\xi_m$. For example, for $B^\pm \to D\pi^\pm$ decays,

$$R^{D\pi}_{CP} = \frac{\Gamma (B^\pm \to D^{\pm}_{CP} \pi^-) + \Gamma (B^\pm \to D^{\pm}_{CP} \pi^+)}{\Gamma (B^\pm \to D^0 \pi^-) + \Gamma (B^\pm \to D^0 \pi^+)} = \frac{1}{2} \left| \frac{A^{D_{CP}}}{A^D} \right|^2 \frac{\eta (\pm \xi_{D\pi} z_-) + \eta (\pm \xi_{D\pi} z_+)}{2}, \quad (29)$$

$$A^{D\pi}_{CP} = \frac{\Gamma (B^\pm \to D^{\pm}_{CP} \pi^-) - \Gamma (B^\pm \to D^{\pm}_{CP} \pi^+)}{\Gamma (B^\pm \to D^0 \pi^-) + \Gamma (B^\pm \to D^0 \pi^+)} = \frac{\eta (\pm \xi_{D\pi} z_-) - \eta (\pm \xi_{D\pi} z_+)}{\eta (\pm \xi_{D\pi} z_-) + \eta (\pm \xi_{D\pi} z_+)}. \quad (30)$$

3.2 The ADS approach

The ADS method \cite{3} uses final states that are accessible from both neutral $D$ meson flavour eigenstates, enhancing the possible $CP$ asymmetry by considering the interference between a favoured $B$ meson decay followed by a doubly CKM-suppressed $D$ decay, and a suppressed $B$ meson decay followed by a CKM-favoured $D$ decay. For the example of $B^\pm \to D\pi^\pm$ decays, with the convention $CP | D^0 \rangle = | \bar{D}^0 \rangle$, and assuming no direct $CP$ violation in the $D$ decay,

$$\Gamma^{\pm}_{fav} = \Gamma (B^\pm \to D^{\pm}_{fav} \pi^\pm) = \left| A^D_{D^{\pm}_{CP}} \right|^2 \eta (1, \rho \xi_{D\pi} z_\pm), \quad (31)$$

$$\Gamma^{\pm}_{sup} = \Gamma (B^\pm \to D^{\pm}_{sup} \pi^\pm) = \left| A^D_{\bar{D}^{\pm}_{CP}} \right|^2 \eta (1, \rho \xi_{D\pi} z_\pm), \quad (32)$$

where $\rho = \frac{A^D_{\bar{D}}}{A^D_D}$, and the subscripts “sup” and “fav” refer to the suppressed and favoured decay modes of the produced $D$ meson, respectively.

The $CP$ observables of interest are

$$R^{D\pi}_{ADS} = \frac{\Gamma^{\pm}_{sup}}{\Gamma^{\pm}_{fav}} = \frac{\eta (\rho, \xi_{D\pi} z_\pm)}{\eta (1, \rho \xi_{D\pi} z_\pm)}. \quad (33)$$

Other ratios and asymmetries are also commonly used,

$$R^{D\pi}_{ADS} = \frac{\Gamma^{\pm}_{sup} + \Gamma^{\pm}_{fav}}{\Gamma^{\pm}_{sup} - \Gamma^{\pm}_{fav}} = \frac{\eta (\rho, \xi_{D\pi} z_-) + \eta (\rho, \xi_{D\pi} z_+)}{\eta (1, \rho \xi_{D\pi} z_-) + \eta (1, \rho \xi_{D\pi} z_+)}, \quad (34)$$

$$A^{D\pi}_{ADS} = \frac{\Gamma^{\pm}_{sup} - \Gamma^{\pm}_{fav}}{\Gamma^{\pm}_{sup} + \Gamma^{\pm}_{fav}} = \frac{\eta (\rho, \xi_{D\pi} z_-) - \eta (\rho, \xi_{D\pi} z_+)}{\eta (\rho, \xi_{D\pi} z_-) + \eta (\rho, \xi_{D\pi} z_+)}, \quad (35)$$

$$A^{D\pi}_{ADS} = \frac{\Gamma^{\pm}_{sup} - \Gamma^{\pm}_{fav}}{\Gamma^{\pm}_{sup} + \Gamma^{\pm}_{fav}} = \frac{\eta (1, \rho \xi_{D\pi} z_-) - \eta (1, \rho \xi_{D\pi} z_+)}{\eta (1, \rho \xi_{D\pi} z_-) + \eta (1, \rho \xi_{D\pi} z_+)}, \quad (36)$$

again showing the relationship between the observables and $\xi_{D\pi}$ and $z_\pm$. 

6
3.3 The GGSZ approach

The GGSZ approach [4] uses $D$ meson decays to three or more final state particles that can be accessed from both $|D^0\rangle$ or $|\bar{D}^0\rangle$. In contrast to the GLW or ADS approaches, this method does not involve intermediate observables, and the goal is to fit for $\xi_m$ and $z_\pm$ directly, using Equations (27-28).

4 Sensitivity studies

In the first study, toy Monte Carlo simulation is used to evaluate the impact of applying a simultaneous measurement technique on the sensitivity to $z_\pm$ with the GGSZ approach, where a model description of the $D$ meson decay amplitude over its phase space is implemented [8]. One thousand pseudo-experiments are generated, with each pseudo-experiment including four $B$ meson decay modes $B^\pm \to DK^\pm$, $B^0 \to DK^{*0}$, $B^\pm \to D^{*0}K^\pm$ and $B^\pm \to D\pi^\pm$, and two $D$ meson final states of $K_s\pi^+\pi^-$ and $K_sK^+K^-$, using world average branching fractions for the various decay modes [9]. The size of each pseudo-experiment is equivalent to 3 fb$^{-1}$ of LHCb data. Where possible, world average values [6, 10] are also used for the CP violation parameters, $r_m$ and $\delta_m$, in each $B$ meson decay mode, with $\gamma = 70^\circ$. Since current experimental constraints on the admixture coefficients for $B^\pm \to D\pi^\pm$ decays are rather loose [11], a value of $\xi_D = 0.08 + 0.06i$ is assumed, as $CP$-violating effects are expected to be one order of magnitude smaller than those in $B^\pm \to DK^\pm$ decays.

A standalone fit to the pseudo-data with $B^\pm \to DK^\pm$ decays only is used as a reference, and several simultaneous fits are performed, progressively adding other $B$ meson decay modes to the set of decays considered. In each case, $z_\pm = x_\pm + iy_\pm$ and the appropriate $\xi_m$ parameters are determined. The obtained statistical uncertainties and biases on these parameters are summarised in Table 1.

With the pseudo-experiment size considered in this study, the inclusion of $B^\pm \to D\pi^\pm$ decays does not improve the sensitivity to $\gamma$. However, the addition of each other decay mode contributes significantly to a smaller uncertainty on $z_\pm$. All biases are at least one order of magnitude smaller than the statistical uncertainty for the relevant parameter.

A second study is performed to investigate the difference in statistical sensitivity to the key parameter of interest, the CKM angle $\gamma$, when using the simultaneous approach compared to that obtained from the conventional method of combining standalone measurements from different decay modes [11]. In the first study, the value and uncertainty on $\gamma$ is determined for each pseudo-experiment using the central values and uncertainties (as well as their correlations) on $z_\pm$, $\xi_{DK^\pm}$, $\xi_{DK^{*0}}$ and $\xi_{D^{*0}K^\pm}$ (in other words using the configuration labelled “all” in Table 1). In this second study, the value and uncertainty on $\gamma$ is determined using the same pseudo-experiments that have been fitted independently for $z_\pm^m$ in each separate decay mode; $B^\pm \to DK^\pm$, $B^\pm \to D\pi^\pm$, $B^0 \to DK^{*0}$ and $B^\pm \to D^{*0}K^\pm$. The distribution of the determined values and uncertainties on CKM angle $\gamma$ are shown in Fig. 1, showing that there is minimal bias on CKM angle $\gamma$ when using either fit strategy. In addition, the simultaneous approach, for the particular values of $r_m$, $\delta_m$ and $\gamma$ assumed, demonstrates the potential for considerable statistical improvement compared to the combination of standalone measurements. It should be noted that the statistical precision obtained in any future analyses employing this technique will depend on the exact results obtained. Nevertheless, the techniques outlined here provide a rigorous and straight-forward treatment of the uncertainties, in particular allowing for simple inclusion of correlations between the various decay modes used in the combination.
Table 1: Statistical uncertainties (upper part) and biases (lower part) of all the parameters involved in each fit.

| Decay modes | $\sigma_x$ | $\sigma_y$ | $\sigma_{x+}$ | $\sigma_{x-}$ | $\sigma_{R\theta(D_{x+})}$ | $\sigma_{R\theta(D_{x-})}$ | $\sigma_{I\theta(D_{x+})}$ | $\sigma_{I\theta(D_{x-})}$ | $\sigma_{R\theta(D_{x^0})}$ | $\sigma_{R\theta(D_{x^0})}$ |
|-------------|-----------|-----------|-------------|-------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $D^+_K$     | 0.0172    | 0.0186    | 0.0210      | 0.0240      | —             | —             | —             | —             | —              | —              |
| $D^+_K$, $D^+_K$ | 0.0175 | 0.0185 | 0.0212 | 0.0245 | 0.0673 | 0.0794 | — | — | — | — |
| $D^+_K$, $D^+_K$ | 0.0149 | 0.0169 | 0.0211 | 0.0236 | — | — | 2.11 | 2.26 | — | — |
| $D^+_K$, $D^+_K$ | 0.0151 | 0.0170 | 0.0211 | 0.0235 | 0.0648 | 0.0750 | 2.09 | 2.26 | — | — |
| $D^+_K$, $D^+_K$ | 0.0146 | 0.0169 | 0.0211 | 0.0239 | — | — | — | 0.442 | 0.426 |
| all         | 0.0132 | 0.0159 | 0.0208 | 0.0233 | 0.0654 | 0.0739 | 2.02 | 2.28 | 0.429 | 0.418 |

| Decay modes | $\delta x$ | $\delta y$ | $\delta x_+$ | $\delta y_+$ | $\delta \text{Re}(\xi_{D_{x+}})$ | $\delta \text{Im}(\xi_{D_{x+}})$ | $\delta \text{Im}(\xi_{D_{x^0}})$ | $\delta \text{Re}(\xi_{D_{x^0}})$ | $\delta \text{Im}(\xi_{D_{x^0}})$ |
|-------------|-----------|-----------|-------------|-------------|----------------|----------------|----------------|----------------|----------------|
| $D^+_K$     | —0.000198 | —0.000209 | —0.000225 | —0.00178 | — | — | — | — | — |
| $D^+_K$, $D^+_K$ | —0.00207 | —0.00077 | —0.00154 | —0.00109 | 0.0133 | 0.00917 | — | — | — |
| $D^+_K$, $D^+_K$ | —0.000184 | —0.00141 | —0.00185 | —0.00117 | — | — | — | 0.149 | 0.411 |
| $D^+_K$, $D^+_K$ | —0.00162 | —0.00059 | —0.00154 | —0.00096 | 0.0137 | 0.00782 | — | 0.150 | 0.448 |
| $D^+_K$, $D^+_K$ | —0.000207 | 0.00018 | —0.00138 | —0.00072 | — | — | — | — | 0.00980 | 0.0206 |
| all         | —0.00165 | 0.00065 | —0.00143 | —0.00058 | 0.0144 | 0.00739 | — | 0.126 | 0.394 | 0.00078 | 0.0091 |

Figure 1: Comparison between the simultaneous approach (blue) and combination of standalone measurements (red) for the central value (left) and uncertainty (right) on the CKM angle $\gamma$ obtained from pseudo-experiments. The solid lines in the left-hand figure show a Gaussian fit to each distribution, with the mean and width of the fitted Gaussian shown in the top left and top right, respectively.
5 Extension to time-dependent measurements

The time evolution of the amplitude of the decay $B_s \to D_s K$ is governed by the equation

$$A_{B_s}^D(t) = A_{B_s}^D [g_+(t) + \lambda_+ g_-(t)],$$

(37)

where $t$ is the $B_s$ meson lifetime and $g_{\pm}(t)$ are functions that describe the mixing of the $B_s$ meson flavour eigenstates. The $\lambda_\pm$ parameters can be expressed as

$$\lambda_\pm = \xi_m z_{\pm} e^{\pm i \phi},$$

(38)

where in this case, $m$ specifically denotes $B_s \to D_s K$.

Existing measurements of $\gamma$ in decays with $B$ meson mixing introduce intermediate observables instead of targeting the $z_{\pm}^m$ parameters themselves. In these cases, the squared amplitude of the time-dependent probability distribution function is expressed as

$$e^{-\Gamma t} \frac{1 + |\lambda_-|^2}{2} \left[ \frac{\cosh(x \Gamma t) + 1 - |\lambda_-|^2 \cos(y \Gamma t)}{1 + |\lambda_-|^2} \cos(x \Gamma t) - \frac{-2 \Re(\lambda_-)}{1 + |\lambda_-|^2} \sinh(x \Gamma t) - \frac{2 \Im(\lambda_-)}{1 + |\lambda_-|^2} \sin(y \Gamma t) \right],$$

(39)

where $x$ and $y$ are parameters describing the $B$ meson mixing and $\Gamma$ is its decay rate; there is a similar expression for the conjugated amplitude. Although this approach [5] is observable based, it is possible to include the $C_f, A_f^{\Delta R}$ and $S_f$ parameters, and equivalent parameters for the conjugate decay, in a simultaneous fit that shares the $z_{\pm}$ parameters with other $B$ decays and only introduces $\xi_m$ as a new coefficient.

6 Conclusions

An approach to simultaneously measuring the $CP$ parameters $z_{\pm}$ sensitive to the CKM angle $\gamma$ in multiple $B$ meson decays has been presented. The formalism reduces the number of free parameters, allows for the consideration of experimentally reconstructed decays that are signal in one case but background in another, and allows for a common treatment of systematic uncertainties. Sensitivity studies show that including additional $B$ meson decay modes contributes significantly to a smaller uncertainty on $z_{\pm}$, and thus has the potential to offer a considerable statistical improvement compared to a combination of standalone measurements. This is likely to be further enhanced when systematic uncertainties and their correlations are also considered, particularly for large future data sets.

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