THE VECTOR FIELD PROBLEM FOR HOMOGENEOUS SPACES

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Dedicated to Professor Peter Zvengrowski with admiration and respect.

Abstract. Let $M$ be a smooth connected manifold of dimension $n \geq 1$. A vector field on $M$ is an association $p \mapsto v(p)$ of a tangent vector $v(p) \in T_pM$ for each $p \in M$ which varies continuously with $p$. In more technical language it is a (continuous) cross-section of the tangent bundle $\tau(M)$. The vector field problem asks: Given $M$, what is the largest possible number $r$ such that there exist vector fields $v_1, \ldots, v_r$ which are everywhere linearly independent, that is, $v_1(x), \ldots, v_r(x) \in T_xM$ are linearly independent for every $x \in M$. The number $r$ is called the span of $M$, written $\text{span}(M)$. It is clear that $0 \leq \text{span}(M) \leq \dim(M)$. The vector field problem is an important and classical problem in differential topology. In this survey we shall consider the vector field problem focussing mainly on the class of compact homogeneous spaces.

1. Introduction

Let $M$ be a smooth connected manifold of dimension $n \geq 1$. All manifolds we consider will be assumed to be paracompact and Hausdorff. If $p \in M$, the tangent space to $M$ at $p$ will be denoted $T_pM$. We denote the tangent bundle of $M$ by $\tau M$ and its total space $\bigcup_{p \in M} T_pM$ by $TM$. The projection of the bundle is denoted $\pi : TM \to M$; thus $\pi$ maps $T_pM$ to $p$.

A vector field $v$ on $M$ is an assignment $p \mapsto v(p) \in T_pM$ of a tangent vector at $p$ for each $p \in M$ which varies continuously with $p$; thus $v : M \to TM$ is continuous and $\pi \circ v = id_M$. In other words, $v$ is a continuous cross-section of the tangent bundle.

We are concerned with the following problem:

The vector field problem: Let $M$ be a smooth manifold. Determine the maximum number $r$ of everywhere linearly independent vector fields on $M$. Thus $r$ is the largest non-negative integer—denoted $\text{span}(M)$—such that there exist (continuous) vector fields $v_1, \ldots, v_r$ on $M$ such that $v_1(p), \ldots, v_r(p) \in T_pM$ are linearly independent for every $p \in M$.

It turns out that, in the vector field problem, if we require the vector fields to be smooth, then the resulting number $r$ is unaltered. This is a consequence of the basic fact that the space of all smooth functions on a (smooth) manifold is dense in the space of all

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continuous functions. So, we may work with smooth vector fields throughout. Observe that $0 \leq \text{span}(M) \leq \dim M$.

In this largely expository article, we address the above problem for an important class of manifolds, namely, homogeneous spaces. After discussing some basic examples, we consider the problem for the spheres $S^{n-1}$ whose solution at various stages brought along with it many new ideas and developments in algebraic topology. Next we survey general results which are applicable to any compact connected smooth manifolds starting with Hopf’s theorem, criterion for existence of a 2-field, the result of Bredon and Kosiński and of Thomas on the possible span of a stably parallelizable manifold, Koschorke’s results on when span and stable span are equal, etc. In §2, we consider the vector field problem for homogeneous spaces for a compact connected Lie group. After elucidating the general results, mainly due to Singhof and Wemmer, for simply connected compact homogeneous spaces, we consider certain special classes of homogeneous spaces (which are not necessarily simply connected) including projective Stiefel manifolds, Grassmann manifolds, flag manifolds, etc. The only new result in this section is Theorem 2.7, due to Sankaran. In §3, we consider homogeneous spaces for non-compact Lie groups. More precisely, we consider the class of solvmanifolds and compact locally symmetric spaces $\Gamma \backslash G/K$ where $G$ is a real semisimple linear Lie group without compact factors, $K$ a maximal compact subgroup of $G$ and $\Gamma$ a uniform lattice in $G$. Theorems 3.3, 3.4 and 3.6 are due to Sankaran (unpublished).

There are already at least two survey articles on the vector field problem. The paper by E. Thomas [88], published in 1968, gives lower bounds for span in a general setup, whereas the main focus of the paper by J. Korbaš and P. Zvengrowski [47], published in 1994, was mostly on flag manifolds and projective Stiefel manifolds. See also [48], [44, S4]. While certain amount of overlap with these papers is unavoidable, the present survey emphasises the vector field problem for homogeneous spaces.

It seems that, in spite of much activity in this area, the determination of span of many families of homogeneous spaces (such as real Grassmann manifolds) remains a wide open problem. I hope it would be useful to young researchers and new entrants to the field.

1.1. First examples. We begin by giving some basic examples of vector fields on manifolds.

If $M$ is an open subspace of $\mathbb{R}^n$ then $\text{span}(M) = \dim(M) = n$. To see this, let $x_1, \ldots, x_n : M \to \mathbb{R}$ be the usual coordinate functions on $\mathbb{R}^n$ restricted to $M$. Then $v_j(p) := \frac{\partial}{\partial x_j}|_p$, $1 \leq j \leq n$ are linearly independent tangent vectors to $M$ at $p \in M$.

Example 1.1. (i) Let $M = S^1$. Then $S^1 \ni (x, y) \mapsto -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \in T_{(x,y)}S^1$ is a (smooth) tangent vector field on $S^1$. So $\text{span}(S^1) = 1 = \dim S^1$.

(ii) Consider the $n$-dimensional sphere $S^n$ consisting of unit vectors in the Euclidean space $\mathbb{R}^{n+1}$. We regard the tangent space to $S^n$ at any point $x = (x_0, \ldots, x_n)$ as the vector subspace $\{x\}^\perp \subset \mathbb{R}^{n+1}$. When $n = 3$, we may regard $\mathbb{R}^4$ as the division algebra
of quaternions over $\mathbb{R}$, generated by $i, j$ where $i^2 = -1 = j^2, k := ij = -ji$. The sphere $S^3$ is the space of unit quaternions. Multiplication (on the left) by the quaternion units $i, j, k$ yields vector fields $v_1, v_2, v_3$ on $S^3$:

$$v_1(q) = (-q_1, q_0, -q_3, q_2) = iq, \quad v_2(q) = (-q_2, q_3, q_0, -q_1) = jq, \quad v_3(q) = (-q_3, -q_2, q_1, q_0) = kq.$$  

for $q = q_0 + q_1i + q_2j + q_3k = (q_0, q_1, q_2, q_3) \in S^3$. Then it is readily checked that $v_r(q) \perp q$, $r = 1, 2, 3$, so that $v_j$ are indeed vector fields on $S^3$. Moreover, $v_r(q) \perp v_s(q)$, $r \neq s$, and $||v_r(q)|| = 1$ for all $q \in S^3$. Thus $v_1, v_2, v_3$ are everywhere linearly independent vector fields on $S^3$ and we conclude that $\text{span}(S^3) = 3 = \dim S^3$.

Using the multiplication in the octonions, one can write down explicitly seven everywhere linearly independent vector fields on $S^7$, as we shall now explain. The algebra of octonions, denoted $\mathbb{O} \cong \mathbb{R}^8$, was first discovered by Graves and shortly thereafter independently by Cayley and is also known as the Cayley algebra. The algebra $\mathbb{O} \cong \mathbb{R}^8$ is a non-commutative, non-associative division algebra generated over $\mathbb{R}$ by $e_i, 1 \leq i \leq 7$, with multiplication defined by $e_ie_{i+1} = e_{i+3}, e_{i+1}e_{i+3} = e_i, e_{i+3}e_i = e_{i+1}, e_i^2 = -1, e_ie_j = -e_je_i$ for $1 \leq i \neq j \leq 7$ where the indices are read mod 7. Denote by $e_0$ the multiplicative identity $1 \in \mathbb{R} \subset \mathbb{O}$. Multiplication by $e_j$ preserves the Euclidean norm on $\mathbb{O}$ where the standard inner product is understood to be with respect to the basis $e_j, 0 \leq j \leq 7$. (To see this, we need only observe that left multiplication by $e_j$ permutes the basis elements up to a sign $\pm 1$.)

Now define $v_j : S^7 \to S^7$ by $v_j(x) = e_j \cdot x, 0 \leq i \leq 7$. Then $v_0(x) = x, ||v_j(x)|| = ||x|| = 1$ and by straightforward verification $v_i(x) \perp v_j(x), 0 \leq i < j \leq 7$, for all $x \in \mathbb{O}$. Thus $v_j, 1 \leq j \leq 7$, are vector fields on $S^7$ which are everywhere linearly independent. Thus $\text{span}(S^7) = 7$.

(iii) Suppose that $G$ is a Lie group and let $v \in T_eG$, where $e$ denotes the identity element. Then we obtain a vector field, again denoted $v$ on $G$ by setting $v(g) := T\lambda_g(v) \in T_gG$ where $\lambda_g : G \to G$ is the left multiplication by $g$, sending $x$ to $gx$. Note that $T\lambda_h(v(g)) = T\lambda_h \circ T\lambda_g(v) = T(\lambda_h \circ \lambda_g)(v) = T\lambda_{hg}(v) = v(hg)$. Thus $v$ is a left-invariant vector field on $G$. Conversely, every left-invariant vector field on $G$ is determined by its value at the identity. Thus $T_eG$ is identified with the vector space of all left vector fields on $G$. If $v_1, \ldots, v_n$ form a basis for $T_eG$, then the left invariant vector fields $v_1, \ldots, v_n$ are everywhere linearly independent. In particular $\text{span}(G) = \dim G$.

The Lie bracket of two left invariant vector fields is again left invariant, making $T_eG$ a Lie algebra; it is the Lie algebra of $G$ and is denoted $\mathfrak{g}$.

(iv) The above example can be generalized to principal $G$-bundles as we shall now explain. Let $\pi : P \to M$ be the projection of a smooth principal bundle over a smooth manifold $M$ with fibre and structure group a Lie group $G$. Let $v \in \mathfrak{g}$ and let $p \in P$. Identifying $G$ with the orbit $Gp \subset P$ through $p$, we obtain a tangent vector $\tilde{v}_q \in T_qP$ that corresponds to $v_q$ where $q.p = q \in Gp \subset P$. Since $v$ is a left invariant vector field, and since the $G$ action on $P$ corresponds to left multiplication in the Lie group $G$, $\tilde{v}_q$ does not depend on the choice of $p$ and so yields a vector field $\tilde{v}$ on $P$. A choice of a basis
v_1, \ldots, v_n$ for $g$ yields everywhere linearly independent vector fields \( \tilde{v}_1, \ldots, \tilde{v}_n \). So we see that $\text{span}(P) \geq \dim(G)$.

(v) Suppose that $p : M \to N$ is a covering projection where $M, N$ are smooth manifolds and $p$ is smooth. Let $\Gamma$ be the deck transformation group (which acts on the left of $M$). Then action of $\Gamma$ on $p$ and $\tilde{p}$ is smooth. Let $\Gamma$ be the deck transformation group (which acts on the left of $M$). Then action of $\Gamma$ on $p$ and $\tilde{p}$ is smooth.

We have seen already that any Lie group is parallelizable as also the spheres $S^1, S^3, S^7$. Bott and Milnor [17] and independently Kervaire [39] showed that these are the only parallelizable spheres (besides $S^0$).

1.2. Span of spheres and projective spaces. Radon [62] and Hurwitz [36] independently obtained the following algebraic result which yields a lower bound for the span of spheres.

A bilinear map $\mu : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is called an orthogonal multiplication if $||\mu(u, v)|| = ||u|| \cdot ||v||$ for all $u \in \mathbb{R}^k, v \in \mathbb{R}^n$. Given an orthogonal multiplication $\mu$ and an orthogonal transformation $\phi$ of $\mathbb{R}^n$ we see that the bilinear map $\phi \circ \mu$ is again an orthogonal multiplication. Also if $u \in \mathbb{R}^k$ is a unit vector, then $\mu_u : \mathbb{R}^n \to \mathbb{R}^n$ defined as $v \mapsto \mu(u, v)$ is an orthogonal transformation.

Using these observations, one may normalize $\mu$ so that $\mu(e_1, y) = y, \forall y \in \mathbb{R}^n$. The proposition below relates the existence of an orthogonal multiplication to the span of $S^{n-1}$.

**Proposition 1.3.** If there exists an orthogonal multiplication $\mu : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$, then $\text{span}(S^{n-1}) \geq k - 1$.

**Proof.** Without loss of generality we assume that $\mu(e_1, y) = y, \forall y \in \mathbb{R}^n$. Let $x \in \mathbb{R}^k$ be a unit vector and let $\mu_x(y) = \mu(x, y)$. As observed already, $\mu_x$ is an orthogonal operator. Wring $\mu_x$ for $\mu_{e_i}, 1 \leq i \leq k$, we claim that if $y \neq 0$, then $\mu_i(y), 1 \leq i \leq k$, are linearly independent. Let, if possible, $\sum a_i \mu_i(y) = 0$ with some $a_i \neq 0$. Multiplying by a scalar if necessary, we assume without loss of generality that $\sum_{1 \leq i \leq k} a_i^2 = 1$ so that $a := \sum_{1 \leq i \leq k} a_i e_i$ is a unit vector. Hence $0 = \sum a_i \mu_i(y) = \mu_a(y)$ implies that $y = 0$ as $\mu_a$ is an orthogonal transformation. This establishes our claim.
Next we claim that for any non-zero \( v \in \mathbb{R}^n \), \( \mu_i(v) \perp \mu_j(v) \) whenever \( i \neq j \). Set \( a := e_i + e_j \in \mathbb{R}^k \). Then, using bilinearity, for \( 2||v||^2 = ||\mu(a, v)||^2 = ||\mu_i(v) + \mu_j(v)||^2 = ||\mu_i(v)||^2 + ||\mu_j(v)||^2 + 2\langle \mu_i(v), \mu_j(v) \rangle \). Since \( ||\mu_i(v)|| = ||\mu_j(v)|| = ||v|| \), we see that \( \mu_i(v) \perp \mu_j(v) \).

Since \( \mu_1 = id \), we have shown that the \( v \mapsto \mu_j(v) \) are vector fields on \( S^{n-1} \) which are everywhere linearly independent. Hence \( \text{span}(S^{n-1}) \geq k - 1 \). \( \square \)

When \( \mu_1 = id \), it is not difficult to show that the \( \mu_i = \mu_{e_i}, 2 \leq i \leq k \), are skew symmetric orthogonal transformations of \( \mathbb{R}^n \) so that \( \mu_i^2 = -id \), and moreover they satisfy the relations \( \mu_i \mu_j = -\mu_j \mu_i, i \neq j, 2 \leq i, j \leq k \). Conversely, if there exist skew symmetric orthogonal transformations \( \mu_i, 2 \leq i \leq k \), satisfying the above relations, then there exists an orthogonal multiplication \( \mu : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \mu_i = \mu_{e_i}, i \geq 2 \), with \( \mu_{e_1} = id \). The transformations \( \mu_2, \ldots, \mu_k \) are known as the Radon-Hurwitz transformations.

A well-known and classical theorem of Hurwitz and Radon gives the maximum value of \( k \) as in the above proposition for any given \( n \). Write \( n = 2^{a+b+c} \times (2c + 1) \) where \( 0 \leq b \leq 3, a \geq 0, c \geq 0 \) are integers. Then the maximum value of \( k \) as in the above proposition is \( k = \rho(n) \) where \( \rho(n) = 8a + 2^b \), the Radon-Hurwitz number of \( n \). See also Eckmann \([23]\).

**Theorem 1.4.** (Radon \([62]\), Hurwitz \([36]\)) Let \( n \geq 2 \) and let \( \rho(n) \) denote the Radon-Hurwitz number defined above. Then \( \text{span}(S^{n-1}) \geq \rho(n) - 1 \).

We now state the celebrated theorem of Adams who showed that the Radon-Hurwitz lower bound is also the upper bound, thereby determining the span of the spheres.

**Theorem 1.5.** (Adams \([1]\)) Let \( n \geq 2 \). Then \( \text{span}(S^{n-1}) = \rho(n) - 1 \). \( \square \)

The proof of this theorem uses \( K \)-theory and Adams operations and is beyond the scope of these notes. The reader may refer to Husemoller’s book \([35]\) for a complete proof.

Note that with notations as in the proof of Proposition 1.3, the vector fields \( \mu_j \) on the sphere \( S^{n-1} \) are odd, that is, \( \mu_j(-v) = -\mu_j(v) \), \( \forall v \in S^{n-1} \). Since \( S^{n-1} \rightarrow \mathbb{R}P^{n-1} \) is a covering projection with deck transformation group \( \mathbb{Z}_2 \) generated by the antipodal map, we see that the \( \mu_j \) define vector fields \( \bar{\mu}_j \) on the quotient space \( S^{n-1}/\mathbb{Z}_2 = \mathbb{R}P^{n-1} \) the \( (n-1) \)-dimensional real projective space. (See Example 1.1 (v).) Thus we have the lower bound \( \text{span}(\mathbb{R}P^{n-1}) \geq \rho(n) - 1 \). On the other hand, \( \text{span}(S^{n-1}) \geq \text{span}(\mathbb{R}P^{n-1}) \) again by the same Example. Hence Adams’ theorem yields the following.

**Corollary 1.6.** \( \text{span}(\mathbb{R}P^{n-1}) = \rho(n) - 1 \). \( \square \)

We will see that the Radon-Hurwitz number arises as the lower bound for span of certain other homogeneous spaces as well.

1.3. Span and characteristic classes. The determination of the span of a manifold is in general a difficult problem. However, techniques and tools of algebraic topology
have been successfully applied to obtain invariants (or obstructions) whose vanishing (or non-vanishing) would lead to lower (or upper) bounds for the span. It is generally the case that obtaining lower bound for span is much harder than finding invariants whose non-vanishing leads to upper bounds. The following result which gives a necessary and sufficient condition for span to be at least one is due to Hopf.

**Theorem 1.7.** (H. Hopf [34]) Let $M$ be a compact connected smooth manifold. Then $\text{span}(M) \geq 1$ if and only if the Euler-Poincaré characteristic $\chi(M)$ of $M$ is zero. \[ \square \]

We merely give an outline of the proof.

First one shows that $M$ admits a smooth vector field $v$ which has only finitely many singularities—points where $v$ vanishes. In fact, put a Riemannian metric on $M$. Then $\text{grad}(f)$, the gradient vector field associated to a Morse function $f : M \to \mathbb{R}$, has only finitely many singularities. To each singular point $p \in M$ one associates an integer called the index of $v$ at $p$ and denoted $\text{index}_p(v)$ obtained as follows. Choose a coordinate chart $(U, \phi)$ around $p$ such that $v_x \neq 0 \ \forall x \in U \setminus \{p\}$. Take a small sphere $S \cong S^{d-1}$ contained in $U$ centred at $p$ where $d = \dim M$. Then $\phi$ induces an orientation on $U$ and hence on $S$ from the standard orientation on $\phi(U) \subset \mathbb{R}^d$. The degree of the map $S \to S^{d-1}$ defined as $x \mapsto v_x/||v_x||$ is defined to be the index of $v$ at $p$. Set $\text{index}(v) := \sum_p \text{index}_p(v)$ (where the sum is over the (finite) set of all singular points of $v$); it is understood that if $v$ has no singularities, the index(v) is zero. It turns out that index(v) is independent of the choice of the vector field $v$.

When $f$ is a Morse function on $M$, the singularities of $\text{grad}(f)$ are precisely the critical points of $f$ and, moreover, the index of $\text{grad}(f)$ at a critical point $p$ is either $+1$ or $-1$ depending on the parity of the index of the function $f$ at $p$. (See [57].) Therefore we see that $\text{index}(\text{grad}(f)) = \sum_{0 \leq q \leq d} (-1)^q c_q$ where $c_q$ is the number of $q$-dimensional cells in the CW structure on $M$ obtained from the Morse function $f$. As is well-known $\sum (-1)^q c_q = \chi(M)$.

Denote by $\pi_{d-1}S^{d-1} = \mathbb{Z}$ the local coefficient system associated to the unit tangent bundle $S(\tau M) \to M$. If $M$ is orientable it is the constant coefficient system $\mathbb{Z}$; otherwise it is given by the homomorphism $\pi_1(M) \to \text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ with kernel the index 2 subgroup corresponding to the orientation double cover of $M$. In any case one has the Poincaré duality isomorphism $H^d(M; \mathbb{Z}) \cong H_0(M; \mathbb{Z}) = \mathbb{Z}$. The obstruction to the existence of a cross-section of $S(\tau M) \to M$ is the Euler class $\xi(M) \in H^d(M; \mathbb{Z})$. The class $\xi(M)$ corresponds, under Poincaré duality, to the index of a vector field $v$ on $M$ with isolated singularities. Since $\text{index}(v) = \chi(M)$, vanishing of $\chi(M)$ implies the existence of a nowhere vanishing vector field. We refer the reader to [82], [59], [58] for further details.

When $\dim M = d$ is odd, the Euler-Poincaré characteristic of $M$ vanishes (by Poincaré duality) and so we have

**Corollary 1.8.** Suppose that $\dim(M)$ is odd. Then $\text{span}(M) \geq 1$. \[ \square \]
The notion of span extends in a natural way to any vector bundle over an arbitrary topological space. The span of a real vector bundle \( \xi \) over \( X \) with projection \( p : E(\xi) \to X \) is the maximum number \( r \), denoted \( \text{span}(\xi) \), such that there exist everywhere linearly independent cross sections \( s_1, \ldots, s_r : X \to E(\xi) \). The number \( \text{rank}(\xi) - \text{span}(\xi) \) is called the geometric dimension of \( \xi \). Note that \( 0 \leq \text{span}(\xi) \leq d \) where \( d \) denotes the rank of \( \xi \).

If \( X \) is a \( d \)-dimensional CW-complex (where \( d \) is finite) or has the homotopy type of such a space, and if \( \xi \) is a real vector bundle over \( X \) such that \( \text{rank}(\xi) \geq d \), then \( \text{span}(\xi) \geq \text{rank}(\xi) - d \); also the geometric dimension of \( \xi \oplus k\epsilon \) is independent of the choice of \( k \geq 1 \). See [35]. Here, and in what follows, \( \epsilon \) denotes a trivial line bundle and \( k\eta \) denotes the \( k \)-fold Whitney sum \( \eta \oplus \cdots \oplus \eta \) of \( \eta \) with itself. The notion of stable span of a manifold is defined as follows:

**Definition 1.9.** Let \( M \) be a connected smooth manifold. The stable span of \( M \), denoted \( \text{span}^0(M) \), is defined to be the largest integer \( r \) such that \( \tau M \oplus k\epsilon = \eta \oplus (k + r)\epsilon \) where \( k \geq 1 \). We say that \( M \) is stably trivial if \( \tau M \oplus k\epsilon \) is trivial for some \( k \geq 1 \).

In view of the observation preceding the definition, we may take always \( k = 1 \) to obtain \( \text{span}^0(M) \). In particular \( M \) is stably trivial if and only if \( \tau M \oplus \epsilon \) is trivial. Stably parallelizable manifolds are also known as \( \pi \)-manifolds.

We shall assume familiarity with the definitions and properties of characteristic classes associated to vector bundles such as Stiefel-Whitney classes, Pontrjagin classes, etc. The standard reference for these is the book by Milnor and Stasheff [59].

The Stiefel-Whitney classes of a smooth manifold \( M \), denoted \( w_j(M) \in H^j(M; \mathbb{Z}_2) \), are by definition, the Stiefel-Whitney classes \( w_j(\tau M) \) of the tangent bundle of \( M \). Similar convention holds for Pontrjagin classes. It turns out that the Stiefel-Whitney classes of a compact connected smooth manifold are independent of the smoothness structure and depend only on the underlying topological manifold. This is because the total Stiefel-Whitney class \( w(M) = \sum w_j(M) \) of \( M \) can be described purely in terms of the cohomology algebra \( H^*(M; \mathbb{Z}_2) \) and the action of the Steenrod algebra \( \mathcal{A}_2 \) on it. So \( w_j(M) \) are even homotopy invariants; see [59, Ch. 11]. In contrast, it is known that the Pontrjagin classes are not homotopy invariants. (Note the Stiefel-Whitney classes are not homotopy invariants when the manifold is not compact. For example, \( w_1(M) \neq 0 \) when \( M \) is the Möbius strip as it is not orientable whereas the cylinder \( S^1 \times \mathbb{R} \) is parallelizable and so \( w_1(S^1 \times \mathbb{R}) = 0 \).)

Recall that Stiefel-Whitney classes are ‘stable’ classes: \( w_j(\xi \oplus r\epsilon) = w_j(\xi) \) for all \( j \geq 0 \), \( \forall r \geq 1 \), and that \( w_k(\xi) = 0 \) if \( k > \text{rank}(\xi) \) for any vector bundle \( \xi \). It follows that, \( \text{span}^0(M) \leq r \) if \( w_{d-r}(M) \neq 0 \) for some \( r \geq 0 \). Likewise, the Pontrjagin classes \( p_j(\xi) \) are also stable classes, and, \( p_j(\xi) = 0 \) if \( j > \text{rank}(\xi) \). So the non-vanishing of \( p_j(M) \) implies that \( \text{span}^0(M) \leq \dim M - j \).

All spheres are stably parallelizable and so \( \text{span}^0(S^n) = \text{span}(S^n) \) if and only if \( n = 1, 3, 7 \). On the other hand we have the following result, which is a special case of a more general result due to James and Thomas [37].
Theorem 1.10. (Cf. [37, Corollary 1.10]) For any $n \geq 1$, \( \text{span}^0(\mathbb{R}P^n) = \text{span}(\mathbb{R}P^n) \).

Proof. If $n$ is even, the Stiefel-Whitney class $w_n(\mathbb{R}P^n)$ is not equal to $0$ which shows that the stable span of $\mathbb{R}P^n$ vanishes. Also, trivially, the statement is valid if $n = 1, 3, 7$.

So assume that $n$ is odd and that $n \neq 1, 3, 7$. James and Thomas [37] have shown that, for such an $n$, if $\eta$ is an $n$-plane bundle such that $\eta \oplus \varepsilon \cong (n+1)\xi \cong \tau \mathbb{R}P^n \oplus \varepsilon$, then $\eta$ is isomorphic to $\tau \mathbb{R}P^n$. This readily implies that the geometric dimension of $\tau \mathbb{R}P^n$ and of $\tau \mathbb{R}P^n \oplus \varepsilon$ are equal—equivalently span$^0(\mathbb{R}P^n) = \text{span}(\mathbb{R}P^n)$. \( \square \)

Definition 1.11. Let $M$ be a closed connected orientable manifold of dimension $n$ where $n = 2m + 1$ is odd. The Kervaire mod 2 semi-characteristic of $M$ is defined as $\hat{\chi}_2(M) := \sum_{0 \leq j \leq m} \dim \mathbb{Z}_2 H^{2j}(M; \mathbb{Z}_2)$ mod 2. Likewise the Kervaire real semi-characteristic of $M$ is defined as $\kappa(M) =: \sum_{0 \leq j \leq m} b_{2j}(M)$ mod 2 where $b_k(M)$ denotes the $k$-th Betti number of $M$.

Suppose that $M$ is not orientable but satisfies the weaker condition that $w_1(M)^2 = 0$. Assume that $n \equiv 1 \pmod{4}$. Atiyah and Dupont [5] defined the twisted Kervaire semi-characteristic, denoted $R_L(M)$, using cohomology with coefficient in a local system $L$ of the field of complex numbers. One may view $L$ as the complex line bundle associated to a covering projection $\tilde{M} \to M$ with deck transformation group $\mathbb{Z}_4$. (Thus the total space of $L$ is $\tilde{M} \times_{\mathbb{Z}_4} \mathbb{C}$ where the action of $\mathbb{Z}_4$ on $\mathbb{C}$ is generated by multiplication by $i \in \mathbb{S}^1$.) Such a cover corresponds to a homomorphism $\pi_1(M) \to \mathbb{Z}_4$ or equivalently an element $u \in H^1(M; \mathbb{Z}_4)$. The element $u$ is chosen so that $w_1(M) = u \mod 2$. Such an element exists since $w_1(M)^2 = 0$. The cohomology $H^*(M; L)$, which is the same as the de Rham cohomology with coefficients in $L$, admits a non-degenerate Poincaré pairing $H^{n-p}(M; L) \times H^p(M; L) \to H^n(M; \Omega^n \otimes \mathbb{C}) \cong \mathbb{C}$ in view of the isomorphism $L \otimes L \cong \Omega^n \otimes \mathbb{C}$. (Here $\Omega^n$ is the determinant of the cotangent bundle of $M$.) The twisted semi-characteristic is defined as $R_L(M) = (1/2)\sum_{0 \leq k \leq n} \dim \mathbb{C}(H^k(M; L))) \mod 2$. When $w_1(M) = 0$, that is, when $M$ is orientable, then $L$ and $\Omega^n$ are trivial and we have $R_L = \kappa(M)$.

We now state a result which gives necessary and sufficient conditions for the span to be at least 2 (under mild restrictions on the manifold), similar in spirit to Hopf’s theorem 1.7. We refer the reader to [47] and [88, §2] for a detailed discussion and relevant references.

Recall that the signature $\sigma(M)$ of a compact connected oriented manifold of dimension $4m$ is the signature of the symmetric bilinear pairing $H^{2m}(M; \mathbb{R}) \times H^{2m}(M; \mathbb{R}) \to \mathbb{R}$ given by $(\alpha, \beta) \mapsto \langle \alpha \cup \beta, \mu_M \rangle$ where $\mu_M \in H_{4m}(M; \mathbb{Z}) \hookrightarrow H_{4m}(M; \mathbb{R}) \cong \mathbb{R}$ denotes the fundamental class of $M$.

Theorem 1.12. (See [88, §2]) Suppose that $M$ is a compact connected oriented smooth manifold of dimension $d \geq 5$. Then $\text{span}(M) \geq 2$ if and only if one of the following holds (depending on the value of $d$ mod 4):
(a) $d \equiv 1 \mod 4$ and $w_{d-1}(M) = 0, \kappa(M) = 0$;
(b) $d \equiv 2 \mod 4$ and $\chi(M) = 0$;
(c) \( d \equiv 3 \mod 4 \);  (d) \( d \equiv 0 \mod 4 \), and \( \chi(M) = 0, \sigma(M) \equiv 0 \mod 4 \).

We shall now explain the approach of Koschorke \([50]\) who regarded a sequence of \( r \) vector fields on \( M \) as a vector bundle homomorphism \( r \epsilon \rightarrow \tau M \) and constructed obstruction classes \( \omega_r \) which live in the normal bordism group \( \Omega_r^{-1}(\mathbb{R}P^{r-1} \times M; \phi_M) \) for a suitable virtual vector bundle \( \phi_M = \phi^+ - \phi^- \) over \( \mathbb{R}P^{r-1} \times M \).

Let \( r < n/2 \) and let \( X = X_1, \ldots, X_r \) be a sequence of vector fields on a smooth manifold \( M \).  \(^1\)  A point \( p \in M \) is a singularity of \( X \) if \( X_{1,p}, \ldots, X_{r,p} \in T_p M \) is linearly dependent. Denote by \( S = S(X) \) the singularity set, that is, \( S := \{ p \in M \mid X_{1,p}, \ldots, X_{r,p} \text{ is linearly dependent} \} \). We say that \( X \) is non-degenerate if the following conditions hold: (a) \( \forall p \in S \), the vectors \( X_{1,p}, \ldots, X_{r,p} \) span a subspace of \( T_p M \) of dimension \( r - 1 \), (b) \( S \) is a compact smooth submanifold of dimension \( (r - 1) \), (c) the map \( M \ni p \mapsto (X_{1,p}, \ldots, X_{r,p}) \in E(r \tau M) \) is transverse to the (closed) subspace \( D_{r-1} := \cup_{p \in M} D_{r-1}^p \) where \( D_{r-1}^p := \{ (u_1, \ldots, u_r) \mid u_j \in T_p M, 1 \leq j \leq r, \text{ span a linear space of dimension } \leq r - 1 \} \). It turns out that when \( 2r < n \), there always exists a non-degenerate sequence \( X \).

Note that \( S \) meets \( D_{r-1} \) along the submanifold \( A_{r-1} = D_{r-1} \setminus D_{r-2} \) of \( M \). Non-degeneracy guarantees a well-defined embedding \( g : S \rightarrow \mathbb{R}P^{r-1} \times M \) obtained as \( p \mapsto ([a_1, \ldots, a_r], p) \) where \( \sum a_j X_{j,p} = 0 \) where not all \( a_j \) are zero. Consider the virtual bundle \( \phi_M := \phi^+ - \phi^- \) over \( \mathbb{R}P^{r-1} \times M \) where \( \phi^+ = \xi \otimes \tau M, \phi^- = r \xi \oplus \tau M \). (Here \( \xi \) denotes the Hopf bundle over the projective space \( \mathbb{R}P^{r-1} \).) Then \( g^* (\phi_M) \) is a stable normal bundle over \( S \); more precisely, there is a vector bundle isomorphism \( \tilde{g} : \epsilon \oplus \tau S \oplus g^* (\phi^+) \cong \epsilon \oplus g^* (\phi^-) \) that covers \( g \) where \( \phi_M = \phi^+ - \phi^- \). This leads to a well-defined obstruction class \( \omega_r(M) := [S, g, \tilde{g}] \in \Omega_r^{-1}(\mathbb{R}P^{r-1} \times M, \phi_M) \) in the normal bordism ring \( \Omega_r(\mathbb{R}P^{r-1} \times M, \phi_M) \). (If \( S \) is empty, it is understood that \( [S, g, \tilde{g}] = 0 \).) The element \( \omega_r(M) \) is independent of the choice of \( X \).

Koschorke \([50, \text{ Theorem 13.3}]\) showed that \( \text{span}(M) \geq r \) if and only if \( \omega_r(M) = 0 \). We point out some important applications to span and stable span.

The theorem below gives criterion for span to be at least 3. Koschorke considers all values of \( \dim M \geq 7 \), but we confine ourselves to the case when \( \dim M \equiv 2 \mod 4 \).

**Theorem 1.13.** (U. Koschorke \([50, \text{§14}]\)) Let \( M \) be a \( d \)-dimensional manifold where \( d \geq 10 \). Suppose that \( \chi(M) = 0, w_{d-2}(M) = 0 \) and that \( d \equiv 2 \mod 4 \). Then \( \text{span}(M) \geq 3 \).

**Theorem 1.14.** (U. Koschorke \([50, \text{§20}]\), V. Eagle \([22]\).) Let \( M \) be a smooth compact connected manifold of dimension \( d \).

(a) If \( d \equiv 0 \mod 2 \), and \( \chi(M) = 0 \), then \( \text{span}^0(M) = \text{span}(M) \).

(b) If \( d \equiv 1 \mod 4 \) and if \( w_1(M)^2 = 0 \), then \( \text{span}^0(M) = \text{span}(M) \) if the twisted Kervaire semi-characteristic \( R_L(M) \) vanishes; if \( R_L \neq 0 \), then \( \text{span}(M) = 1 \).

(c) If \( d \equiv 3 \mod 8 \) and \( w_1(M) = w_2(M) = 0 \), then \( \text{span}^0(M) = \text{span}(M) \) if \( \hat{\chi}_2 = 0 \); if \( \hat{\chi}_2(M) \neq 0 \), then \( \text{span}(M) = 3 \).

\(^1\) Such a sequence is referred to as an \( r \)-field in \([50]\), but in the literature it is also often used to mean one which is everywhere linearly independent. So we avoid this terminology altogether.
Koschorke had noted that the above results were obtained by Eagle in his PhD thesis using entirely different methods.

We now state without proof the following theorem which determines the span of a stably parallelizable but non-parallelizable manifold.

**Theorem 1.15.** (G. Bredon and A. Kosiński [19], E. Thomas [86]) Let $M$ be a compact connected manifold of dimension $d$. Suppose that $M$ is stably parallelizable. Then either $M$ is parallelizable or \( \text{span}(M) = \text{span}(S^d) = \rho(d + 1) - 1 \). If $d$ is odd and $d \notin \{1, 3, 7\}$, then $M$ is parallelizable if and only if the Kervaire semi-characteristic $\hat{\chi}_2(M) = 0$. If $d$ is even, $M$ is parallelizable if and only if $\chi(M) = 0$. \( \square \)

Note that $S^d$ is parallelizable when $d = 1, 3, 7$ although $\hat{\chi}_2(S^d) \neq 0$.

**Remark 1.16.** In view of Theorems 1.15 and 1.14, it is important to have criteria for the vanishing of the Kervaire semicharacteristics $\hat{\chi}_2(M)$ and $\kappa(M)$ of a compact connected orientable smooth manifold $M$ of dimension $d = 2m + 1$. Note that the orientability assumption implies that the twisted semicharacteristic $R_L(M)$ equals $\kappa(M)$. Lusztig, Milnor and Peterson [55] showed that $\hat{\chi}_2(M) - \kappa(M)$ equals the Stiefel-Whitney number $w_2w_{d-2}[M] \in \mathbb{Z}_2$. In particular $\kappa(M) = \hat{\chi}_2(M)$ if $M$ is a spin manifold or if $M$ is nullcobordant. Stong [84] proved that if $M$ admits a free smooth $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $M$, then $\hat{\chi}_2(M) = 0$.

We shall now give several examples, starting from elementary ones.

**Example 1.17.** (i) Let $S$ be a compact orientable connected surface of genus $g$. Its Euler-Poincaré characteristic is $\chi(S) = 2 - 2g$. Thus when $g \neq 1$, span of $S$ equals zero. When $g = 1$, $S$ equals the torus $S^1 \times S^1$ which is parallelizable. When $g = 0$, the surface $S = S^2$. Fix an imbedding $j : S \rightarrow \mathbb{R}^3$ and denote by $\nu$ the normal bundle (over $S$) with respect to $j$. Then we obtain that $3\epsilon = j^*(\tau\mathbb{R}^3) = \tau S \oplus \nu$. Since $S$ is orientable, the normal bundle $\nu$ is trivial and we conclude that $S$ is stably parallelizable.

(ii) Suppose that $M$ is a non-orientable surface. Then it has an orientable double covering $p : S \rightarrow M$. One has $\chi(M) = (1/2)\chi(S)$. It follows that $\text{span}(M) = 0$ except when $S$ is a torus $S^1 \times S^1$. When $S$ is a torus, $M$ is the Klein bottle and we have $\chi(M) = 0$. By Hopf’s theorem 1.7, we have $\text{span}(M) \geq 1$. Since $M$ is not orientable, $\text{span}(M) < \text{dim}(M) = 2$ and hence $\text{span}(M) = 1$. Since $M$ is non-orientable, it is not stably parallelizable.

(iii) Any orientable compact connected manifold $M$ of dimension 3 is parallelizable. This was first observed by Stiefel. The proof involves obstruction theory and uses the fact that $\pi_2(SO(3)) = \pi_2(\mathbb{R}P^3) = 0$. See [59, Problem 12-B].

(iv) Let $M = S \times S^1$, where $S$ is a non-orientable surface. Then $M$ is non-orientable and hence not parallelizable. So $1 \leq \text{span}(M) \leq 2$. If $S = K$, the Klein bottle, we have $\text{span}(M) = 2$. This follows from the isomorphism of vector bundles $\tau(M_1 \times M_2) \cong \tau M_1 \times \tau M_2 \cong pr_1^*(\tau M_1) \oplus pr_2^*(\tau M_2)$, where $pr_j : M \rightarrow M_j$ is the $j$-th projection. If
$S = \mathbb{R}P^2$, or more generally if $S$ has odd Euler-Poincaré characteristic, then the Stiefel-Whitney class $w_2(S) \in H^2(S; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is non-zero. This implies that $w_2(M) \neq 0$. It follows that $\text{span}(M) = 1$.

(v) Becker [14] has determined the span of quotient spaces $M := \Sigma^{n-1}/G$ when $n \neq 8, 16$, where $G$ is a finite group that acts freely and smoothly on a homotopy sphere $\Sigma^{n-1}$. In the special case when $\Sigma^{n-1}$ is the standard sphere and $G$ is a group of odd order that acts orthogonally, it was shown by Yoshida [91] that $\text{span}(M) = \text{span}(\Sigma^{n-1}) = \rho(n) - 1$ for $n \neq 8$, settling a conjecture of Sjerve [78]. The following general result which is of independent interest and also due to Becker, is a crucial step in the determination of span of $M$.

**Theorem 1.18.** (Becker) Suppose that $N$ is a compact connected orientable smooth $d$-dimensional manifold and $\tilde{N} \to N$ is covering projection of odd degree. Let $k \leq (d - 1)/2$ is a positive integer. Then $\text{span}(N) \geq k$ if and only if $\text{span}(\tilde{N}) \geq k$.

### 1.4. Span of products of manifolds.

If $M, N$ are compact connected smooth manifolds, then it is clear that $\text{span}(M) + \text{span}(N) \leq \text{span}(M \times N)$ and that equality holds when $\text{span}(M) = 0 = \text{span}(N)$ by Hopf’s Theorem 1.7. If $M, N$ are stably parallelizable, then so is $M \times N$. The converse is also valid; this is because the normal bundle to the inclusion of each factor into the product $M \times N$ is trivial. On the other hand, if $M \times N$ is parallelizable, we cannot conclude that $M$ and $N$ are parallelizable. See Theorem 1.19 below. There is no general ‘formula’ that expresses $\text{span}(M \times N)$ in terms of the span of $M, N$. In this section we shall obtain some bounds for the span of $M \times N$, when one of the factors is stably parallelizable.

We begin with the following result whose proof, due to E. B. Staples [81], is surprisingly simple, considering that the solution to the vector field problem for spheres is highly non-trivial.

**Theorem 1.19.** (Staples [81].) The manifold $S^m \times S^n$ is parallelizable if at least one of the numbers $m, n \geq 1$ is odd. If both $m, n$ are even, then $\text{span}(S^m \times S^n) = 0$.

**Proof.** Assume that $m$ is odd. Then $\tau(S^m) = \epsilon \oplus \eta$ for some subbundle $\eta \subset \tau(S^m)$. Let $p_i$ denote the projection to the $i$-th factor of $S^m \times S^n$. Using $\epsilon \oplus \tau(S^n) = (n + 1)\epsilon$, we have $\tau(S^m \times S^n) = p_1^*(\tau(S^m)) \oplus p_2^*(\tau(S^n)) = p_1^*(\eta) \oplus \epsilon \oplus \tau(S^n) = p_1^*(\eta) \oplus (n + 1)\epsilon = p_1^*(\eta \oplus 2\epsilon) \oplus (n - 1)\epsilon = (m + 1)\epsilon \oplus (n - 1)\epsilon = (m + n)\epsilon$.

If both $m, n$ are even, then $\chi(S^m \times S^n) = \chi(S^m) \times \chi(S^n) = 4$ and so by Hopf’s Theorem 1.7 $\text{span}(S^m \times S^n) = 0$. \qed

Note that in the above proof we exchanged, repeatedly, $\tau S^n \oplus \epsilon$ for $(n + 1)\epsilon$. This is often referred to as *boot-strapping*. We will have several occasions in the sequel to use it. The next theorem, essentially due to Staples, is a generalization of the above.

**Theorem 1.20.** (Staples [81].) Suppose that $M, N$ are smooth compact connected positive dimensional manifolds. Assume that $\chi(N) = 0$ and that $\text{span}^0(M) \geq 1$. Then $\text{span}^0(M) +
span^0(N) \leq \text{span}(M \times N) \leq \text{span}^0(M) + \text{dim} N, \text{dim} M + \text{span}^0(N)\}.

Moreover, if \(M, N\) are stably parallelizable, then \(M \times N\) is parallelizable.

**Proof.** By Hopf’s theorem 1.7, \(r := \text{span}(N) \geq 1\). Write \(\tau N = \theta \oplus r\epsilon\) and \(\tau(N) \oplus \epsilon = \eta \oplus (s + 1)\epsilon\) where \(s := \text{span}^0(N)\). Similarly write \(\tau M \oplus \epsilon = \xi \oplus (p + 1)\epsilon\), where \(p = \text{span}^0(M) \geq 1\). To simplify notations, we will denote the pull-back bundle \(pr_1^* (\tau M)\) also by the same symbol \(\tau M\) where \(pr_1 : M \times N \to M\) is the first projection. Similar notational conventions will be followed for \(pr_2^*(\theta), pr_2^*(\tau N)\), etc.

We have the chain of bundle isomorphisms by boot-strapping:

\[
\tau (M \times N) = \tau M \oplus \tau N
\]
\[
= \tau M \oplus \theta \oplus r\epsilon
\]
\[
= (\tau M \oplus \epsilon) \oplus \theta \oplus (r - 1)\epsilon
\]
\[
= \xi \oplus (p + 1)\epsilon \oplus \theta \oplus (r - 1)\epsilon
\]
\[
= \xi \oplus (p - 1)\epsilon \oplus \theta \oplus (r + 1)\epsilon
\]
\[
= \xi \oplus (p - 1)\epsilon \oplus \tau N \oplus \epsilon
\]
\[
= \xi \oplus (p - 1)\epsilon \oplus \eta \oplus (s + 1)\epsilon
\]
\[
= \xi \oplus \eta \oplus (p + s)\epsilon.
\]

Hence \(\text{span}(M \times N) \geq p + s = \text{span}^0(M) + \text{span}^0(N)\).

Fix a point \(q \in N\). The natural inclusion \(M \subset M \times N\) pulls back the tangent bundle of \(M \times N\) to \(\tau M \oplus (\text{dim} N)\epsilon\). This implies that \(\text{span}^0(M \times N)\) cannot exceed \(\text{span}^0(M) + \text{dim} N\). Similarly \(\text{span}^0(M \times N) \leq \text{span}^0(N) + \text{dim} M\). \(\square\)

**Example 1.21.** (i) Let \(M = \mathbb{R}P^m \times S^n\). If \(m\) is odd, \(\text{span}(M) = \text{span}^0(\mathbb{R}P^m) + n = \rho(m + 1) + n - 1\). For the last equality, see Theorem 1.10. If \(m, n\) are both even, \(\text{span}(M) = 0\), \(\text{span}^0(M) = n\), since \(w_m(M) \neq 0\). If \(m\) is even and \(n\) odd, then \(\text{span}(S^n) \leq \text{span}(M) \leq \text{span}^0(M) = n\). But the exact value of \(\text{span}\) seems to be unknown in general. When \(m = 2\) and \(n \equiv 1 \mod 8, n \geq 9\), it turns out that \(\text{span}(M) = 3\) whereas \(\text{span}^0(M) = n\); see [50, Exercise 20.18].

(ii) Suppose that \(M\) is the boundary of a parallelizable manifold-with-boundary \(W\). Then \(M\) is stably parallelizable. This is because \(W\) is necessarily orientable and the normal bundle \(\nu\) to the inclusion \(M \hookrightarrow W\) is a trivial line bundle. (One may take ‘outward pointing’ unit normal at each point of \(M\) with respect to a Riemannian metric on \(W\).) Hence \(\tau W|_M \cong \tau M \oplus \epsilon\). Since \(\tau W\) is trivial, so is \(\tau M \oplus \epsilon\), that is, \(M\) is stably parallelizable.

(iii) A well-known result of Kervaire and Milnor [40, Theorem 3.1] says that any smooth homotopy sphere is stably parallelizable. An immediate corollary is that a product of two or more smooth homotopy spheres is parallelizable if and only if at least one of them is odd-dimensional. If all the homotopy spheres are of even dimension, then their span is zero (in view of Hopf’s theorem 1.7).

(iv) J. Roitberg [65] has constructed smooth \((4k - 2)\)-connected manifolds \(M_1, M_2\) of dimension \(d = 8k + 1\) for each \(k \geq 2\), having the following properties: (a) \(M_1\) and \(M_2\) are homeomorphic (in fact \(M_1, M_2\) admit PL-structures and are PL-homeomorphic).
with \( \text{span}(M_1) = 1 = \text{span}(M_2) \), (b) \( M_2 \) is stably parallelizable, but \( M_1 \) is not, (c) \( \text{span}(M_2 \times N) > \text{span}(M_1 \times N) \) for any stably parallelizable manifold \( N \) of dimension \( n \geq 1 \). The construction of the manifolds \( M_1, M_2 \) involves deep machinery which goes far beyond the scope of these notes. We shall be content with some remarks. It turns out that \( M_2 \) has the same \( \mathbb{Z}_2 \)-homology groups as the sphere \( S^n \). It follows that the Betti numbers \( b_j(M_2) \) of \( M_2 \) vanish for \( 1 \leq j \leq d - 1 = 8k \). Hence the real Kervaire semi-characteristic \( \kappa(M_2) = \sum_{0 \leq j \leq 4k} b_j(M_2) = 1 \). (See Remark 1.16.) By Theorem 1.12(a), \( \text{span}(M_2) \leq 1 \). Since \( M_2 \) is odd-dimensional, equality must hold. Since \( M_1 \) is homeomorphic to \( M_2 \), the same argument applies to \( M_1 \) as well and so \( \text{span}(M_1) = 1 \). For the assertion (c), note that \( M_2 \times N \) is parallelizable (by Theorem 1.20) but \( M_1 \times N \) is not stably parallelizable since \( M_1 \) is not.

(v) Crowley and Zvengrowski [20] have extended the results of Roitberg to dimensions \( \geq 9 \). More precisely, for each \( d \geq 9 \), they have shown the existence of manifolds \( M_1, M_2 \) which are PL-homeomorphic but \( \text{span}(M_1) \neq \text{span}(M_2) \). They also showed that there can be no such examples in dimensions up to 8.

In contrast to the case of spheres, the span of the product \( M = \mathbb{R}P^{m-1} \times \mathbb{R}P^{n-1} \) of real projective spaces for general \( m, n \) is unknown. Of course, the span is zero when both \( m, n \) are odd since in that case \( \chi(M) = 1 \). Using the formula for Stiefel-Whitney classes of projective spaces, one obtains that \( w_{m+n-k-l}(M) = w_{m-k}(\mathbb{R}P^{m-1}) \times w_{n-l}(\mathbb{R}P^{n-1}) \neq 0 \) where \( k = 2^r, l = 2^s \) are highest powers of 2 which divide \( m, n \) respectively. It follows that \( \text{span}(M) \leq k + l - 2 \). This is uninteresting when \( m, n \) are both powers of 2 and is strong when \( k, l \leq 2 \). On the other hand one has the lower bound \( \text{span}(M) \geq \text{span}^0(\mathbb{R}P^{m-1}) + \text{span}^0(\mathbb{R}P^{n-1}) = \rho(m) + \rho(n) - 2 \). (See Theorem 1.10.)

The following result is due to Davis.

**Theorem 1.22.** (Davis [21]) Suppose that 16 \( \not| m \) and 16 \( \not| m \), or, \( m = 2, 4, 8 \). Then \( \text{span}(\mathbb{R}P^{m-1} \times \mathbb{R}P^{n-1}) = \rho(m) + \rho(n) - 2 \).

*Proof.* Let \( m = 2, 4, \) or \( 8 \). Then \( \rho(m) = m \) and for any manifold \( N \) we have \( \text{span}(\mathbb{R}P^{m-1} \times N) = m - 1 + \text{span}^0(N) \). Taking \( N = \mathbb{R}P^{n-1} \), we get \( \text{span}^0(N) = \text{span}(N) = \rho(n) - 1 \) which proves the assertion in this case.

Suppose that 16 divides neither \( m \) nor \( n \). In this case, \( \rho(m) = 2^r, \rho(n) = 2^s \) where \( m = 2^r \cdot m', n = 2^s \cdot n' \) with \( m', n' \) being odd. Using the formula \( w_j(\mathbb{R}P^{m-1}) = \binom{m}{j} a^j \in H^j(\mathbb{R}P^{m-1}; \mathbb{Z}_2) = \mathbb{Z}_2 a^j \), we obtain that \( w_{m-2^r}(M) \neq 0 \), and \( w_j(M) = 0 \) for \( j > m - 2^r \). Similar statement holds for \( N \) and so we obtain that \( w_{m+n-2^r-2^s}(M \times N) = w_{m-2^r}(M) \times w_{n-2^s}(N) \neq 0 \). Hence \( \text{span}(M \times N) \leq 2^r + 2^s - 2 \). Since \( \text{span}(M \times N) \geq \text{span}(M) + \text{span}(N) = \rho(m) - 1 + \rho(n) - 1 = 2^r + 2^s - 2 \), the assertion follows. \( \square \)

When 16 \( | m \), the Stiefel-Whitney upper bound is rather weak. Using BP-cohomology, Davis [21] has obtained an upper bound for the span of \( \mathbb{R}P^{m-1} \times \mathbb{R}P^{n-1} \) which is sharper than the previously known ones. No example of a pair of numbers \( (m, n) \) seems to be
known where the span of $\mathbb{R}P^{m-1} \times \mathbb{R}P^{n-1}$ is strictly bigger than the Radon-Hurwitz lower bound $\rho(m) + \rho(n) - 2$.

2. Vector fields on homogeneous spaces

In this section we shall consider the vector field problem for homogeneous spaces, mostly focusing on Stiefel manifolds, Grassmann manifolds, and related spaces. We will assume familiarity with Lie groups and representation theory of compact Lie groups. As we proceed further, acquaintance with (topological) K-theory will also be assumed.

Let $G$ be any Lie group and let $H$ be a closed subgroup. We consider the natural differentiable structure on the homogeneous space $M = G/H$. Thus the quotient map $G \to G/H$ is smooth.

We begin with the following well-known result. (Compare Example 1.1(v).)

**Theorem 2.1.** (Borel-Hirzebruch [16].) Let $\Gamma \subset G$ be a discrete subgroup of a Lie group $G$. Then $G/\Gamma$ is parallelizable.

*Proof.* We shall work with the space of right cosets $\Gamma \backslash G$ instead of $G/\Gamma$. The tangent bundle $\tau(\Gamma \backslash G)$ has the following description: $T(\Gamma \backslash G) = G \times g/ \sim$ where $(x, v) \sim (hx, d\lambda_h(v))$, $x \in G, v \in g, h \in \Gamma$. Let $v_1, \ldots, v_n$ be everywhere linearly independent $G$-invariant vector fields on $G$ where $n = \dim G$. Since $d\lambda_h(v_j(x)) = v_j(hx)$ for all $h \in H$, we see that $\bar{v}_j(\Gamma x) = [x, v_j(x)] \in T_{Hx}(\Gamma \backslash G)$ is a well-defined (smooth) vector field on $\Gamma \backslash G$ for $1 \leq j \leq n$. See Example 1.1(v). \qed

2.1. Homogeneous spaces of compact Lie group. In this section we consider homogeneous spaces $G/H$ where $G$ is a compact connected Lie group and $H$ a closed subgroup.

First suppose that $T$ is a maximal torus of a compact connected Lie group $G$. That is, $T \subset G$ is isomorphic to $(S^1)^r$ with $r$ largest. The number $r$ is called the rank of $G$. It is well-known that $G$ is a union of its maximal tori and that any two maximal tori in $G$ are conjugates in $G$. Let $N_G(T)$ denote the normalizer of $H$ in $G$. Then $W(G, T) := N_G(T)/T$ is a finite group known as the Weyl group of $G$ with respect to $T$. It is known that the Euler-Poincaré characteristic of $G/T$ equals $|W(G, T)|$, the cardinality of the Weyl group. To see this, first note that an element $gT$ is a $T$-fixed point for the action of $T$ on $G/T$ if and only if $g \in N(T)$. Since $g_0^{-1}g_1 T = g_1 T$ if and only if $g_0^{-1}g_1 \in T$, we have a bijection between $T$-fixed points of $G/T$ and $W(G, T)$. Applying [18, Theorem 10.9] we see that $\chi(G/T) = |W(G, T)|$. If $H$ is a closed connected subgroup of $G$ such that $T \subset H \subset G$, then $W(H, T)$ is a subgroup of $W(G, T)$ and the coset space $W(G, T)/W(H, T)$ will be denoted $W(G, H)$.

Let $H \subset G$ be any connected subgroup having the same rank as $G$. If $T \subset H$ is a maximal torus of $H$, then

$$\chi(G/H) = |W(G, H)|. \quad (2)$$
To see this, observe that one has a fibre bundle with fibre space $H/T$ and projection $G/T \to G/H$. The required result then follows from the multiplicative property of the Euler-Poincaré characteristic and the formula for $\chi(G/T)$.

Suppose that $S \cong (S^1)^s$ is a toral subgroup of $G$ where $s < r = \text{rank}(G)$. Then $S$ is properly contained in a maximal torus $T$ of $G$. Considering the fibre bundle with fibre $T/S \cong (S^1)^{r-s}$ and projection $G/S \to G/T$, we see that $\chi(G/S) = \chi(G/T) \cdot \chi(T/S) = 0$ since $\chi(T/S) = (\chi(S^1)^{r-s}) = 0$. It follows that if the rank of $H$ is less than the rank of $G$, then, taking $S$ to be a maximal torus of $H$ and using the $H/S$-bundle with projection $G/S \to G/H$, we have $\chi(G/H) = \chi(G/S)/\chi(H/S) = 0$. If $H \subset G$ is not connected, denoting the identity component $H_0$, the natural map $G/H_0 \to G/H$ is a covering projection of degree $|H/H_0|$ and so $\chi(G/H) = \chi(G/H_0)/|H/H_0|$. The following result is an immediate consequence of Hopf’s Theorem 1.7.

**Theorem 2.2.** Let $G$ be a compact connected Lie group and let $H$ be a closed subgroup of $G$. Let $H_0$ denote the identity component of $H$. If $\text{rank}(H_0) = \text{rank}(G)$, then $\text{span}(G/H) = 0$. If $\text{rank}(H_0) < \text{rank}(G)$, then $\text{span}(G/H) > 0$. □

Next we describe the tangent bundle of $G/H$ in terms of the adjoint representation. We do not assume that $G$ is compact. Also $H$ is not assumed to be connected.

The conjugation $g \mapsto t_g$ defined as $t_g(x) = gxg^{-1}, x \in G$, defines an action of $G$ on itself. Clearly the identity element $e \in G$ is fixed under this action. Hence, we obtain a representation $G \to GL(g)$ defined as $g \mapsto dt_g|_e$. This is referred to as the adjoint representation, denoted $Ad_G$. By restricting the action to the subgroup $H \subset G$ we obtain a representation of $Ad_G|_H$. Note that since $H \subset G$, the adjoint representation of $H$ on $T_eH = h$ is a subrepresentation of $Ad_G|_H$ and moreover we obtain a representation of $H$ on $g/h$. Further, the tangent space to $G/H$ at the identity coset $H$ may be identified with $g/h$, as can be seen by considering the differential $d\pi|_e : g = T_eG \to T_{\pi(e)}(G/H)$ of the projection of the $H$-bundle $\pi : G \to G/H$, whose kernel is $h = T_eH$. It turns out that the tangent bundle $\tau(G/H)$ has the following description:

$$T(G/H) = G \times_H g/h$$

(3)

where the right hand side denotes the quotient of $G \times g/h$ by the relation $(g, v + h) \sim (gh^{-1}, Ad(h)(v) + h)$. The projection $G \times_H g/h \to G/H$ defined as $[g, v + h] \mapsto gH \in G/H$ is the projection of a vector bundle with fibre $g/h$ which is isomorphic to the tangent bundle of $G/H$.

The exact sequence of $H$-representations

$$0 \to h \to g \to g/h \to 0$$

induces an exact sequence of vector bundles over $G/H$:

$$0 \to \nu \to E \to \tau_{G/H} \to 0$$

(4)

where $E = G \times_H g$. The bundle $E$ is isomorphic to the trivial bundle $d\epsilon$ of rank $d := \text{dim} G$ since the action of $H$ on $g$ extends to an action of $G$ on $g$ (namely the adjoint action).
**Proposition 2.3.** Let $H \subset G$ be a toral subgroup of a compact connected Lie group $G$. Then $G/H$ is stably parallelizable; it is parallelizable if and only if $\text{rank}(G) > \dim H$.

**Proof.** The bundle $\nu$ with total space $G \times_H \mathfrak{h}$ is trivial since the adjoint representation of $H$ is trivial. (This is because the group $H \cong (S^1)^s$ is abelian.) The above sequence of vector bundles splits (after choosing a Euclidean metric on $\mathcal{E}$), and so we have $d\nu \cong \tau(G/H) \oplus \nu = \tau(G/H) \oplus se$ where $s = \dim H$. This proves our first assertion. To prove the last assertion, note that if $\dim H = \text{rank}(G)$, then $\chi(G/H) = |W(G, H)| \neq 0$ and so $\text{span}(G/H) = 0$. If $s = \dim H < \text{rank}(G) =: r$, choose a maximal torus $T \supset H$. Consider the $T/H$-bundle with projection $\pi : G/H \to G/T$. This is a principal bundle with fibre and structure group $T/H$. Hence the vertical bundle is trivial (see Example 1.1(iv)) and we have $\pi^*(\tau(G/T)) \oplus (r - s)e \cong \tau(G/H)$. Since $\tau(G/T)$ is stably trivial and $r > s$, it follows that $\tau G/H$ is trivial, i.e., $G/H$ is parallelizable. \qed

Let $\psi : H \to GL(V)$ be a representation of a Lie group $H$ on a real vector space $V$. Suppose that $H$ is a closed subgroup of a Lie group $G$. Denote by $\alpha(\psi)$ the associated vector bundle $G \times_H V \to G/H$. (Here $G \times_H V = G \times V/\sim$ where $(gh, v) = (g, \psi(h)(v)), \forall (g, v) \in G \times V, h \in H$.) Bundles over $G/H$ associated to representations of $H$ are referred to as homogeneous vector bundles. For example, as we have seen already, the tangent bundle $\tau(G/H)$ is a homogeneous vector bundle associated to the representation on $\mathfrak{g}/\mathfrak{h}$ induced by $Ad_G|_H$, the adjoint representation of $G$ (on $\mathfrak{g}$) restricted to $H$ and the adjoint representation $Ad_H$ (on $\mathfrak{h} \subset \mathfrak{g}$). This so-called $\alpha$-construction defines a ring homomorphism $\alpha : RO(H) \to KO(G/H)$ from the real representation ring of $G$ to the $KO$-theory of $G/H$. Analogously, one has the $\alpha$-construction on complex representations leading to $\alpha_c : R(H) \to K(G/H)$. The kernel of $\alpha$ (resp. $\alpha_c$) contains the ideal of $RO(H)$ (resp. $R(H)$) generated by the elements of the form $[E] - \dim[E]$ where $E$ is the restriction to $H$ of a real (resp. complex) representation of $G$.

Denoting the complexification homomorphisms $RO(H) \to R(H)$ and $KO(G/H) \to K(G/H)$ by the same symbol $c$, one has $c \circ \alpha = \alpha_c \circ c$. Similarly we have the ‘realification’ homomorphisms $r : R(H) \to RO(H)$ and $r : K(G/H) \to KO(G/H)$ which forgets the complex structure. Note that $c$ is a ring homomorphism whereas $r$ is only a homomorphism of abelian groups. One has $r \circ c = 2$ and $c \circ r = 1 +^\mathbb{C}$, where the notation $^\mathbb{C}$ stands for the complex conjugation. These relations hold on the real and complex representation rings and also on the real and complex $K$-theoretic rings. We refer the reader to [6] for detailed discussion and further results on the relation between representation rings $G, H$ and the $K$-theory $G/H$.

Singhof and Wemmer [76] established Theorem 2.4 given below. The sufficiency part is immediate from the exact sequence (4) of vector bundles and has been noted earlier (see [75, p. 103]). The proof of the necessity part involves verification using the classification of compact simple Lie groups. Recall that a connected Lie group $G$ is said to be simple if $G$ is not abelian and has no proper connected normal subgroups. For example, $SU(n)$ is simple, although its centre is a cyclic group of order $n$. One says that $G$ is semisimple if
its universal cover is a product of simple Lie groups. A compact connected Lie group is semisimple if and only if its centre is finite.

One has also the Grothendieck group $RSp(G)$ of (virtual) $G$-representations of left $\mathbb{H}$-vector spaces. The restriction homomorphisms $RO(G) \to RO(H), R(G) \to R(H), RSp(G) \to RSp(H)$ will all be denoted by the same symbol $\rho$. Note that $\rho$ is a ring homomorphism in the case of real and complex representation rings. Although $RSp(G)$ is only an abelian group, one can form the tensor product of a right and a left $\mathbb{H}$-representation to obtain a real representation. If $W$ (resp. $U$) is a left (resp. right) $\mathbb{H}$-vector space, then $U \otimes_{\mathbb{H}} W$ has only the structure of a real vector space of dimension $4 \dim_{\mathbb{H}} U \dim_{\mathbb{H}} W$. If $H$ acts on $U, W$ $\mathbb{H}$-linearly, then $U \otimes_{\mathbb{H}} W$ is naturally a real representation of $H$. Its isomorphism class determines an element, denoted $[U \otimes_{\mathbb{H}} W]$, in $RO(H)$. If $V$ is a left $\mathbb{H}$-vector space, denote by $V^*$ the right $\mathbb{H}$-vector space where $v \cdot q = \bar{q}v, v \in V, q \in \mathbb{H}$. We have a $\mathbb{Z}$-bilinear map $\beta : RSp(H) \times RSp(H) \to RO(H)$ defined as $([V], [W]) \mapsto [V^* \otimes_{\mathbb{H}} W]$. If $V = \mathbb{H}$ is a trivial $\mathbb{H}$-representation, then $\beta([V], [W]) = [W_{\mathbb{R}}]$ where $W_{\mathbb{R}}$ stands for the same $H$-representation $W$ with scalar multiplication restricted to $\mathbb{R} \subset \mathbb{H}$.

We denote by $J = J(G, H)$ the ideal of $RO(H)$ generated by elements of the form
\begin{enumerate}[(i)]    \item $\rho(x) - \dim x, x \in RO(G)$,
    \item $\beta(\rho(x - \dim_{\mathbb{H}}(x)[\mathbb{H}]), y), x \in RSp(G), y \in RSp(H)$.
\end{enumerate}
It is easy to see that, if $x \in RO(G)$, then $\rho(x) - \dim x$ is contained in the kernel of $\alpha : RO(H) \to KO(G/H)$. In fact we have $J(G, H) \subset \ker(\alpha)$.

**Theorem 2.4.** (See [76], [77]) (i) Let $G$ be a simply connected compact connected Lie group and $H$ a closed connected subgroup. Then $G/H$ is stably parallelizable if $[Ad_H]$ is in the image of the restriction homomorphism $\rho : RO(G) \to RO(H)$.

(ii) Conversely, suppose that $G/H$ is stably parallelizable and that $G$ is simple. (a) If $G \neq Sp(n)$, then $[Ad_H]$ is in the image of $\rho$. (b) If $G = Sp(n)$, then $[Ad_H] - \dim H$ is in the ideal $J(Sp(n), H)$ of $RO(H)$.

The first part of the above theorem holds for any connected Lie group. If $G$ acts linearly on a real vector space $W$, then the associated vector bundle $\alpha(W)$ on $G/H$ with projection $G \times_H W \to G/H$ is trivial, without any condition on $G$. If the $H$ action on $\mathfrak{h} \oplus \mathbb{R}^k = W$ (where the adjoint action on the first summand and the trivial action on $\mathbb{R}^k$ is understood) extends to a linear action of $G$, then $\nu \oplus k \epsilon \cong \alpha(W)$ is a trivial vector bundle on $G/H$ where $\nu$ is as in the exact sequence (4). It follows that $\tau G/H$ is trivial.

We state, without proofs, the following results of Singhof [75].

**Theorem 2.5.** (Singhof [75].) Let $G$ be a connected compact simple Lie group and let $H$ be a closed connected subgroup of $G$ such that $H$ is neither a torus nor semisimple. Then the first Pontrjagin class $p_1(G/H)$ is non-zero. In particular, $G/H$ is not stably parallelizable.

**Theorem 2.6.** (Singhof [75]) Let $H \cong SU(k_1) \times \cdots \times SU(k_r)$ be a closed subgroup of $G = SU(n)$. Then the following are equivalent:

(i) $G/H$ is stably parallelizable.
(ii) \( H \) equals one of the following subgroups: (a) \( k_j = 2,1 \leq j \leq r \leq n/2 \) and \( H \) is embedded block diagonally, (b) \( n = 4, H = SU(2) \) is the diagonal copy of \( SU(2) \times SU(2) \subset SU(4) \), (c) \( H = SU(k) \), with standard embedding. Moreover, if \( SU(n)/H \) is stably parallelizable and is not a sphere, then it is parallelizable. \( \square \)

Singhof and Wemmer [76] completely determined all pairs \((G,H)\) where \( H \) is a closed connected subgroup of a compact simply connected simple Lie group \( G \) such that \( G/H \) is (stably) parallelizable. Let \( H = SU(k_1) \times \cdots \times SU(k_r) \subset SU(n), r \geq 2 \), with \( k := \sum k_j < n \). Set \( M := SU(n)/H, N := SU(k)/H \) and \( B := SU(n)/SU(k) \). One has a fibre bundle with fibre space \( N := SU(n)/H \) and projection \( p : M \to B \) since \( H \subset SU(k) \). Since \( k < n \), the base space \( B \) is the complex Stiefel manifold if \( k \leq n - 2 \) and is the sphere \( S^{2n-1} \) if \( k = n - 1 \). In any case \( B \) is stably parallelizable. (The parallelizability results for Stiefel manifolds will be discussed in detail in \( \S 2.2 \).) Denote by \( F \) the complex flag manifold \( SU(k)/K \) where \( K := S(U(k_1) \times \cdots \times U(k_r)) \). We have the following result due to Sankaran (unpublished).

**Theorem 2.7.** With notations as above, let \( r \geq 2 \) and let \( k = \sum_{1 \leq j \leq r} k_j \). Then:

(i) \( \text{span}(N) \geq r - 1 \).

(ii) If \( k < n \), then \( r - 1 + n^2 - k^2 \leq \text{span}(M) \leq r - 1 + n^2 - k^2 + \text{span}^0(F) \); in particular, if \( \chi(F) = k!/(k_1! \cdots k_r!) \) is odd, then \( \text{span}(M) = r - 1 + n^2 - k^2 \).

*Proof.* We shall only obtain the bounds for \( \text{span}(M) = SU(n)/H \). Let \( V := SU(n)/K \) where \( K = S(U(k_1) \times \cdots \times U(k_r)) \). One has a principal fibre bundle \( \pi : M \to V \) with fibre and structure group the torus \( K/H \cong (S^1)^{r-1} \). Hence we see that \( \tau M = (r - 1)\epsilon + \pi^*(\tau SU(n)/K) \). Again \( q : V \to B \) is a fibre bundle projection with fibre \( F := SU(k)/K \) and so, by Example 1.1(iv) we obtain a splitting \( \tau SU(n)/K \cong q^*(\tau B) \oplus \eta \) where \( \eta \) restricts to the tangent bundle of \( F \) along any fibre of \( q \). Hence

\[
\tau(M) = (r - 1)\epsilon + \pi^*(q^*(\tau B)) \oplus \pi^*(\eta)
\]

\[
\cong (r - 1 + \text{dim } B)\epsilon + \pi^*(\eta)
\]

since \( B \) is stably parallelizable and \( r \geq 2 \). Therefore \( r - 1 + n^2 - k^2 \leq \text{span}(M) \leq r - 1 + n^2 - k^2 + \text{span}^0(F) \) as \( \text{dim } B = n^2 - k^2 \). Finally, if \( \chi(F) \) is odd, then \( \text{span}(F) = \text{span}^0(F) = 0 \) as the top Stiefel-Whitney class of \( F \) is non-zero and so the last assertion follows. Note that the equality \( \chi(F) = k!/(k_1! \cdots k_r!) \) follows from (2) and the fact that the Weyl group of \( SU(k) \) is the permutation group \( S_k \). \( \square \)

Next we shall discuss some important special cases of compact homogeneous spaces.

### 2.2. Stiefel manifolds

Let \( 1 \leq k < n \). Recall that the Stiefel manifold \( V_{n,k} \) is the space of all ordered \( k \)-tuples \( (v_1, \ldots, v_k) \) of unit vectors in \( \mathbb{R}^n \) which are pairwise orthogonal (with respect to the standard inner product). When \( k = 1 \), \( V_{n,1} \) is the sphere \( S^{n-1} \). The group \( SO(n) \) acts transitively on \( V_{n,k} \) with isotropy at \( (e_1, \ldots, e_k) \) being \( I_k \times SO(n-k) = SO(n-k) \). Hence \( V_{n,k} \cong SO(n)/SO(n-k) \). The complex and quaternionic Stiefel manifolds are defined analogously using the standard Hermitian product on \( \mathbb{C}^n \) and the standard
‘quaternionic’ product \( \mathbb{H}^n \) defined as \( q \cdot q' = \sum_{1 \leq r \leq n} \bar{q}_r q'_r, q, q' \in \mathbb{H}^n \). We have the following description of \( W_{n,k}, Z_{n,k} \) as coset spaces: \( W_{n,k} \cong U(n)/U(n-k) = SU(n)/SU(n-k) \) and \( Z_{n,k} \cong Sp(n)/Sp(n-k) \). Note that \( V_{n,1} = S^{n-1}, W_{n,1} = S^{2n-1}, Z_{n,1} = S^{4n-1} \). We call an element of \( V_{n,k}, W_{n,k}, Z_{n,k} \) an orthonormal, hermitian, quaternionic \( k \)-frame (or more briefly a \( k \)-frame) respectively.

Let \( \beta_{n,k} \) (or more briefly \( \beta \)) denote the real vector bundle over \( V_{n,k} \) whose fibre over any \( k \)-frame \( v = (v_1, \ldots, v_k) \in V_{n,k} \) is the real vector space \( \{v_1, \ldots, v_k\}^\perp \subset \mathbb{R}^n \). The complex vector bundle of rank \( n-k \) over \( W_{n,k} \) and the quaternionic (left) vector bundle of rank \( n-k \) over \( Z_{n,k} \) are defined similarly. One has the \( \mathbb{F} \)-vector bundle isomorphism

\[
 k \epsilon_{\mathbb{F}} \oplus \beta_{n,k} \cong n \epsilon_{\mathbb{F}} \tag{5}
\]

where \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) according as the base space is \( V_{n,k}, W_{n,k}, Z_{n,k} \); here \( \epsilon_{\mathbb{F}} \) denotes the trivial \( \mathbb{F} \)-vector bundle. As always, \( \epsilon \) would denote the trivial real line bundle.

**Theorem 2.8.** (W. Sutherland [83], K. Y. Lam [53], D. Handel [28].) The real, complex, and quaternionic Stiefel manifolds \( V_{n,k}, W_{n,k}, Z_{n,k} \) are parallelizable when \( k \geq 2 \).

**Proof.** We shall only consider the case of the real Stiefel manifolds; The following description of the tangent bundle is due to Lam:

\[
 \tau V_{n,k} \cong k \beta \oplus \binom{k}{2} \epsilon, \quad \tau W_{n,k} \cong k^2 \epsilon \oplus 2k \beta, \quad \tau Z_{n,k} \cong (2k^2 + k) \epsilon \oplus 4k \beta, \tag{6}
\]

where the isomorphisms are, of course, of real vector bundles; by abuse of notation, \( \beta \) stands for the underlying real vector bundle (in the complex and quaternionic cases). If \( k \geq 3 \), then \( \binom{k}{2} \geq k \). Using the isomorphism \( \beta \oplus k \epsilon \cong n \epsilon \) on \( V_{n,k} \), we obtain

\[
 \tau V_{n,k} \cong k \beta \oplus \binom{k}{2} \epsilon \\
 \cong \beta \oplus k \epsilon \oplus (k-1) \beta \oplus \left( \binom{k-1}{2} - 1 \right) \epsilon \\
 \cong (k-1) \beta \oplus \left( \binom{k-1}{2} + n - 1 \right) \epsilon.
\]

A boot-strapping argument leads to the triviality of \( \tau V_{n,k} \). The case of the complex and quaternionic Stiefel manifolds can be handled in an analogous manner. In fact, in the case of \( W_{n,2} \) and \( Z_{n,2} \), boot-strapping is still possible. Since \( 4\epsilon \oplus \beta \cong 2n \epsilon \) we have

\[
 \tau W_{n,2} = 4\epsilon \oplus 2\beta \\
 = 2n \epsilon \oplus \beta \\
 = (2n-4)\epsilon \oplus 2n \epsilon \\
 = (4n-4)\epsilon.
\]

The proof in the case of \( Z_{n,2} \) is similar and hence omitted.

When \( k = 2 \), boot-strapping fails for \( V_{n,2} \). However, it allows us to show that \( \tau V_{n,k} \oplus \epsilon \) is trivial. Thus \( V_{n,2} \) is stably parallelizable.

There does not seem to be any easy argument to show the parallelizability of \( V_{n,2} \) although boot-strap proof is still possible when \( n \) is even using the isomorphism \( \tau S^{n-1} \cong \xi \oplus \epsilon \). The general case requires obstruction theory. We refer the reader to [83] for
details, where the more general case of the total space of a sphere bundle over sphere is considered.

The stable parallelizability of the Stiefel manifolds also follows from the sufficiency part of Theorem 2.4 as noted by Singhof [75, p.103].

2.3. The projective Stiefel manifolds. We begin by recalling the definition of projective Stiefel manifolds. Although one has the notion of quaternionic projective Stiefel manifolds, not much is known about their span. (See [53].) For this reason we shall be content with defining them, but discuss the vector field problem only for real and complex projective Stiefel manifolds.

The real projective Stiefel manifold $PV_{n,k}$ is defined as the quotient of $V_{n,k}$ under the antipodal identification: $v \sim -v$. Note that $PV_{n,1}$ is the real projective space $\mathbb{RP}^{n-1}$. The manifold $PV_{n,k}$ is the homogenous space $O(n)/(\mathbb{Z}_2 \times O(n-k))$ where the factor $\mathbb{Z}_2$ is the subgroup $\{I_k, -I_k\} \subset O(k) \subset O(n)$. Evidently, the quotient map $V_{n,k} \to PV_{n,k}$ is the double covering map which is universal except when $k = n-1$ as $V_{n,n-1} \cong SO(n)$.

The complex projective Stiefel manifolds are defined similarly as $PW_{n,k} := U(n)/(S^1 \times U(n-k))$ where the factor $S^1 \subset U(n)$ is the subgroup $\{zI_k \mid |z| = 1\} \subset U(k)$. Evidently $PW_{n,k}$ is the quotient of $W_{n,k}$ by the action of $S^1$ where $z \cdot (w_1, \ldots, w_k) = (zw_1, \ldots, zw_k)$ and in fact the quotient map $W_{n,k} \to PW_{n,k}$ is the projection of a principal $S^1$-bundle.

Analogously, the quaternionic projective Stiefel manifold $PZ_{n,k}$ is the homogeneous space $Sp(n)/Sp(1) \times Sp(n-k)$ where the factor $Sp(1)$ is subgroup $\{qI_k \mid q \in \mathbb{H}, \|q\| = 1\} \subset Sp(k)$. It is the quotient of $Z_{n,k}$ under the action of $Sp(1)$ where $q \cdot (v_1, \ldots, v_k) = (v_1\bar{q}, \ldots, v_k\bar{q})$, $(v_1, \ldots, v_k) \in Z_{n,k}, q \in Sp(1)$. The quotient map $Z_{n,k} \to PZ_{n,k}$ is evidently the projection of a principal $Sp(1)$-bundle.

We denote by $\zeta_{n,k}$, or more briefly $\zeta$, the real (resp. complex) line bundle over $PV_{n,k}$ (resp. $PW_{n,k}$) associated to the double cover $V_{n,k} \to PV_{n,k}$ (resp. the principal $U(1)$-bundle $W_{n,k} \to PW_{n,k}$). We shall denote by $\beta_{n,k}$ (more briefly $\beta$) the bundle over $PV_{n,k}$ whose fibre over a point $[v_1, \ldots, v_k] \in PV_{n,k}$ is the orthogonal complement of $\mathbb{R}v_1 + \cdots + \mathbb{R}v_k$ in $\mathbb{R}^n$. The similarly defined complex vector bundle of rank $n-k$ over $PW_{n,k}$ will also be denoted by the same symbol $\beta_{n,k}$ (or $\beta$).

The projection onto the $j$th coordinate $p_j : PV_{n,k} \to PV_{n,1} = \mathbb{RP}^{n-1}$ is covered by a bundle map of $\zeta$ on $PV_{n,k}$ and the Hopf line bundle $\xi$ on $\mathbb{RP}^{n-1}$. Hence $p_j^*(\zeta) \cong \zeta$ for $1 \leq j \leq k$. Using this one obtains the following isomorphism of real (resp. complex) vector bundles over $PV_{n,k}$ (resp. $PW_{n,k}$):

$$k\zeta_{n,k} \oplus \beta_{n,k} \cong n\varepsilon_\mathbb{F},$$

(7)

where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ as appropriate. Equivalently, upon tensoring with $\bar{\zeta}$ and using the isomorphism $\zeta \otimes_\mathbb{F} \bar{\zeta} \cong \varepsilon_\mathbb{F}$ we obtain

$$k\varepsilon_\mathbb{F} \oplus \beta_{n,k} \otimes_\mathbb{F} \bar{\zeta}_{n,k} \cong n\bar{\zeta}_{n,k}.$$

(8)

When $\mathbb{F} = \mathbb{R}$, we have $\bar{\zeta} \cong \zeta$. 

\end{document}
The Hopf line bundles over the real and complex projective Stiefel manifolds have the following universal property. This has been observed by S. Gitler and D. Handel [25, p.40] and also by L. Smith [79] for $PV_{n,k}$ where the universal property is established for real line bundles over finite complexes. The paper [11] removed the restriction on the base space. We note that the formulation and proof also works for complex line bundles. We merely state the result and omit its proof.

**Theorem 2.9.** Let $\xi$ be any real (resp. complex) line bundle over a topological space $X$. Then there exist a positive integer $n$ and a real (resp. complex) vector bundle $\eta$ such that $n\xi \sim \eta \oplus k\epsilon$ as real (resp. complex) vector bundles if and only if there exists a continuous map $f : X \to PV_{n,k}$ (resp. $X \to PW_{n,k}$) such that $f^*(\zeta_{n,k}) \cong \xi$.  

A description of the mod 2 cohomology algebra was obtained by Gitler and Handel [25] which we shall now recall. Let $N := \min_{1 \leq j \leq k} \{n - k + j \mid \binom{n}{n-k+j} \equiv 1 \mod 2\}$. Denote by $V = V(x_1, \ldots, x_m)$ a $\mathbb{Z}_2$-algebra generated by homogeneous elements $x_j$, $1 \leq j \leq m$, such that $\{x_1^{e_1} \cdots x_m^{e_m} \mid e_j \in \{0,1\}\}$ form a basis for the $\mathbb{Z}_2$ vector space $V(x_1, \ldots, x_m)$.

**Theorem 2.10.** (Gitler and Handel [25]) With notations as above, the mod 2-cohomology algebra of $PV_{n,k}$ is isomorphic to $\mathbb{Z}_2[y]/\langle y^N \rangle \otimes V(y_{n-k}, \ldots, y_{n-2}, y_{n-1})$, where $\deg(y) = 1, \deg(y_j) = j, n - k \leq j \leq n - 1, (j \neq N - 1)$ for a suitable algebra $V$. Furthermore, $w_1(\zeta) = y$.

Gitler and Handel also determined, almost completely, the action of the Steenrod algebra on $H^*(PV_{n,k}; \mathbb{Z}_2)$. See also [13] and [2].

The following descriptions of the tangent bundle of real and complex projective Stiefel manifolds was obtained by Lam [53].

$$
\tau PV_{n,k} \cong \begin{pmatrix} k \\ 2 \end{pmatrix} \epsilon \oplus k\zeta \otimes \beta,
$$

$$
\tau PW_{n,k} \cong (k^2 - 1)\epsilon_\mathbb{R} \oplus k\bar{\zeta} \otimes \mathbb{C} \beta,
$$

where we have denoted by the same symbol $\bar{\zeta} \otimes \mathbb{C} \beta$ to denote its underlying real vector bundle.

Using the isomorphism (8) one obtains the following description for the stable tangent bundle:

$$
\tau PV_{n,k} \oplus \begin{pmatrix} k + 1 \\ 2 \end{pmatrix} \epsilon \cong nk\zeta,
$$

$$
\tau PW_{n,k} \oplus (k^2 + 1)\epsilon_\mathbb{R} \cong kn\bar{\zeta} \cong nk\zeta,
$$

where, again in (12), we have used $\zeta$ also to denote its underlying real vector bundle; note that $\bar{\zeta} \cong \zeta$ as real vector bundles.

**Theorem 2.11.** (i. (Zvengrowski [94], Antoniano, Gitler, Ucci, Zvengrowski [3].)

(a) $PV_{n,k}$ is parallelizable in the following cases: $n = 2, 4, 8; k = n - 1; k = 2m - 2, n = 2m; (n, k) = (16, 8)$.
(b) $PV_{n,k}$ is not stably parallelizable in all the other cases, except possibly when $(n, k) =$...
Proof. (i) (a) The parallelizability of $PV_{n,n-1} = V_{n,n-1}/\{\pm I\} = SO(n)/\{\pm I\}$ is implied by the Borel-Hirzebruch Theorem 2.1.

We shall now show the parallelizability of $PV_{n,n-2}$ where $n = 2m$. Let $d = \dim PV_{n,n-2} = \binom{n}{2} - 1$. It is easy to see that $\binom{n-2}{2} > \rho(d)$ if $n \geq 4$. In view of the bundle isomorphism (9) and Bredon-Kosiński’s theorem, we see that it suffices to show that $PV_{n,n-2}$ is stably parallelizable. Note that $PV_{n,n-2} = SO(n)/Z \cdot SO(2)$, where $Z = \{I_n, -I_n\} \subset SO(n)$ is the centre of $SO(n)$ since $n = 2m$ is even. Let $H = Z \cdot SO(2) \cong Z \times SO(2)$. Then the adjoint representation of $H$ is trivial since $H$ is abelian. It follows that, in the exact sequence (4), the bundle $\nu$ is trivial. So $PV_{n,n-2}$ is stably parallelizable, as was to be shown.

In the remaining cases, consider the projection $q : PV_{n,k} \to \mathbb{R}P^n$ which pulls back the Hopf bundle $\xi$ over $\mathbb{R}P^n$ to $\xi$. By a well-known result of Adams [1], the order of $\xi = \zeta_{n,1}$ is known: $2^{\varphi(n-1)}\xi = 2^{\varphi(n-1)}\epsilon$ where the function $\varphi$ is defined as $\varphi(n)$ is the number of numbers $r$ such that $1 \leq r \leq n$ such that $r \equiv 0, 1, 2, 4 \mod 8$. We have $2^{\varphi(n-1)} = n$ if and only if $n = 2, 4, 8$. Since $q^*(\zeta_{n,1}) = \zeta_{n,k}$, we see that $2^{\varphi(n-1)}\zeta_{n,k} \cong 2^{\varphi(n-1)}\epsilon$ for all $k < n$. Since $\varphi(15) = 7$ we have $2^7\zeta_{16,8} = 2^7\epsilon$. Therefore $\tau V_{16,8} \oplus \binom{9}{2}\epsilon \cong 16 \cdot 8\zeta \cong 2^7\epsilon$ using the isomorphism (11). Also $\dim V_{16,8} = 120 - 28 = 92$, $\rho(93) = 1$ whereas span of $PV_{16,8}$ is at least $\binom{8}{2} = 28$. So by the Bredon-Kosiński theorem, $PV_{16,8}$ is parallelizable. The case when $n = 4, 8$ are similarly handled and in fact easier.

(i)(b). In several cases, fairly elementary arguments can be used to decide whether $PV_{n,k}$ is stably parallelizable or not. For example, if both $n, k$ are odd, then $nk\zeta_{n,k}$ is not orientable. So, by (11), we conclude that $PV_{n,k}$ is also not orientable and hence not stably parallelizable. However, such simple arguments leave infinitely many cases unsettled. In [3], the authors compute the complex K-theory of $PV_{4q,k}$ which leads to the determination of the (additive) order of $[\zeta_{4q,k} \otimes \mathbb{C}] - [\epsilon_C] \in K(PV_{4q,k})$. This readily leads to the determination of the order of $\zeta_{4q,k}$ up to a factor of 2. (By the order of a real line bundle $\xi$ we mean the smallest positive integer $m$ (if it exists) such that $m\xi$ is trivial; if the base space is a finite CW complex, it is always finite and is a power of 2.) Moreover, using the inclusion $PV_{4q,k} \to PV_{4q+t,k}$, $1 \leq t \leq 3$, leads to estimation of the order of $\zeta_{n,k}$ for any $n$. This is then used to show that $nk\zeta_{n,k}$ is not trivial for almost all the manifolds not covered in (i), still leaving out $PV_{n,k}$ where $(n, k) = (10, 4), (12, 8)$ and a few others (when $5 \leq n \leq 7$). When $m$ is odd, $m\xi$ is non-orientable so we may assume that $nk$ is even. Thus only the cases $(7, 4), (7, 2), (6, 3), (6, 2), (5, 2)$ remain, leaving out the case $(12, 8)$ which remains at this time unresolved. Of these, only the cases $(n, k) = (7, 4), (6, 3), (6, 2)$ are ‘critical’ and were proven to be non-stably parallelizable by a computation of the order of $\zeta_{n,k}$, using the Atiyah-Hirzebruch spectral sequence for $KO$-theory.

(12, 8): $PV_{12,8}$ is parallelizable if it is stably parallelizable.
In the case of $PV_{12,8}$ it was shown that $32\zeta_{12,8} \otimes C = 32\epsilon_C$ which implies that $64\zeta_{12,8} \cong 64\epsilon_{\mathbb{R}}$ but it is unknown whether $32\zeta_{12,8} \cong 32\epsilon_C$. Since $\tau PV_{12,8} \oplus 36\epsilon \cong 96\epsilon$, it remains unknown whether it is parallelizable or not.

Since (9) implies that $\text{span} PV_{12,8} \geq 28$, and since $\dim PV_{12,8} = 60$, $\rho(61) = 0$, by the Bredon-Kosiński Theorem again we see that $PV_{12,8}$ is parallelizable if it is stably parallelizable.

(ii) Suppose that $k < n - 1$. By Singhof’s theorem (Theorem 2.5), we know that $PW_{n,k}$ is not stably parallelizable. Since we did not give proof that theorem, we now proceed to give a proof of it in the special case of complex Stiefel manifolds. Using (12) we compute the Pontrjagin class $p_1(PW_{n,k})$. Since $\tau PW_{n,k} \otimes \mathbb{C}$ is stably equivalent to $nk\zeta_{n,k} \otimes \mathbb{R} \cong nk(\zeta_{n,k} \oplus \overline{\zeta}_{n,k})$, a straightforward computation yields $p_1(PW_{n,k}) = nkc_1(\zeta_{n,k})^2$. Using the Gysin sequence of the principal $S^1$-bundle $W_{n,k} \to PW_{n,k}$ and the fact that $W_{n,k} = SU(n)/SU(n-k)$ is 4-connected when $1 \leq k \leq n - 2$, it is easily seen that $c_1(\zeta_{n,k})^2$ generates $H^4(PW_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$. Hence $p_1(PW_{n,k}) \neq 0$ and so $PW_{n,k}$ is not stably parallelizable.

Note that $PW_{n,n-1} = U(n)/(Z.U(1))$ is the quotient of a compact connected Lie group modulo $S = Z.U(1) \cong S^1 \times S^1$, which is a torus of rank 2. By Lemma 2.3 $PW_{n,n-1}$ is parallelizable if $n > 2$. The remaining part of (ii) is clearly valid.

Determination of the span of a real projective manifold $PV_{n,k}$, for general values of $n, k$, is largely an open problem. For certain infinite set of values of $(n, k)$ the span has been determined. When $k$ is in the so-called upper range (roughly $k > n/2$) very good estimates for the span of $PV_{n,k}$ have been obtained by Korbaš and Zvengrowski. (See [46], [48], [49].) It turns out that the estimates are sharp whenever span and stable span are known to be equal. (See Theorem 1.14.) Usually it is easier to obtain bounds for stable span since it is possible to approach this using the tools of homotopy theory and K-theory.

A major source of estimates for the lower bound for stable span of $span^0(PV_{n,k})$ is the known estimate for the solution to the generalized vector field problem. The generalized vector field problem asks: What is the largest value $r$ so that $m\zeta_{n,1}$ is isomorphic as a vector bundle to $r \epsilon \oplus \eta$? That is, it asks for the determination of span($m\zeta_{n,1}$). It appears that the best known estimate for the solution to this problem general values of $m, n$ is due to Lam [52]. Note that if $m < n$, then $w_m(\zeta_{n,1}) \neq 0$ and so $r = 0$. When $m \geq n$, we have $r \geq m - n + 1$ since any vector bundle $\xi$ of rank $m$ over any CW complex of dimension $d$ is isomorphic to $(m - d)\epsilon \oplus \eta$ for a suitable vector bundle $\eta$. Since $\zeta_{n,k} \cong q^*(\zeta_{n,1})$ where $q$ is the projection $PV_{n,k} \to PV_{n,1} = \mathbb{R}P^{n-1}$, we see that span($m\zeta_{n,k}$) $\geq$ span($m\zeta_{n,1}$). This gives us, using (11), lower bounds for the stable span of $PV_{n,k}$. Combining with Theorem 1.14 which provides sufficient conditions for span to equal span results in the following.

**Theorem 2.12.** (Korbaš, Sankaran, Zvengrowski [46], Korbaš, Zvengrowski [48].) One has span($PV_{n,k}$) = span$^0(PV_{n,k})$ in the following cases:

(a) $n \equiv 0 \mod 2$ and $k \equiv 0, 2, 3, 4, 7 \mod 8$, 
(b) $n \equiv 1 \mod 2$ and $k \equiv 0, 1, 4, 5, 6 \mod 8$.
(c) $(n, k) = (4m, 8l + 5), (4m + 2, 8l + 1), (4m, 16l + 6), (8m, 16l + 9), (8m - 1, 16l + 7)$. \[ \square \]

For example, using (9) we saw in the course of the proof of Theorem 2.11 that $\text{span}(PV_{12,8}) \geq 28$. From Koschorke’s Theorem 1.14, one knows that whenever $\chi(M) = 0$ and $\dim(M) \equiv 0 \mod 2$, the span of $M$ equals the stable span of $M$. Hence we may use (11) to obtain $\text{span}(PV_{12,8}) \geq \text{span}(96\zeta_{12,8}) - 36 \geq 85 - 36 = 49$. Here we used the estimate $\text{span}(96\zeta_{12,8}) \geq \text{span}(96\zeta_{12,1}) \geq 96 - \dim \mathbb{R}P^{11} = 85$. However, one can improve this lower bound using the work of Lam [52] which implies that $\text{span}(96\zeta_{12,1}) = 91$ to obtain $\text{span}(PV_{12,8}) \geq 55$. See [46] for similar estimates for the span of $PV_{n,k}$ for $n \leq 16$.

As for upper bounds for the stable span of $PV_{n,k}$, an obvious tool is the Stiefel-Whitney classes. They generally work only for small values of $k$ and $n$ not a power of 2. In any case, the bounds so obtained are generally weak. Another source of upper bounds uses the structure of $K$-ring of $PV_{n,k}$ using Theorem 2.9. Suppose that $\text{span}^{0}(\tau PV_{n,k}) \geq r$. Then we see that $\text{span}(nk\zeta_{n,k}) \geq \binom{k}{2} + r$. This implies, by Theorem 2.9 the existence of a continuous map $f : PV_{n,k} \to PV_{nk,\binom{k}{2}+r}$ such that $f^{*}(\zeta_{nk,\binom{k}{2}+r}) \cong \zeta_{n,k}$. By considering the map induced by $f$ between the $K$-rings of the spaces the following result was obtained.

The structure of the ring $K(PV_{n,k})$ had been determined for $n \equiv 0 \mod 4$ in [3] and for all values of $n \mod 4$ in [10].

**Theorem 2.13.** (Sankaran and Zvengrowski [73,]). Let $2 < k < \lfloor (n - 1)/2 \rfloor$. Write $m : = \lfloor n/2 \rfloor$, $s : = \lfloor k/2 \rfloor$, $d : = \text{dim} PV_{n,k} = nk - \binom{k+1}{2}$.

(i) Suppose that $n \equiv 0 \mod 2$. Then $\text{span}^{0}(PV_{n,k}) \leq d - 2q - 2$ if $(-1)^q \binom{nk-1}{q}$ is not a quadratic residue modulo $2^{m-2q}$.

(ii) Suppose that $n \equiv 1 \mod 2$, $k = 2s$. Then $\text{span}^{0}(PV_{n,k}) \leq d - 2q - 2$ if $(-1)^s \binom{ns-1}{q}$ is not a quadratic residue modulo $2^{m-2q}$.

(iii) Suppose that $n \equiv 1 \mod 2$, $k = 2s + 1$ and $1 \leq q < s - 1, m \geq 3q$. Then $\text{span}^{0}(PV_{n,k}) \leq d - 2q$ if $(-1)^{r-q} \binom{r}{q}$ is not a quadratic residue modulo $2^{m-3q}$ where $r = (nk - 1)/2$.

Part (i) of the above result was stated without proof in [46].

We point out here two conjectures of Korbaš and Zvengrowski [48, p. 100]:

**Conjecture A:** $\text{span}^{0}(PV_{n,k}) = \text{span}(PV_{n,k})$ for all $n, k$.

**Conjecture B:** $\text{span}(PV_{n,k}) \geq \kappa_{n,k}$ where $\kappa_{n,k} : = \text{span}(nk\zeta_{n,1} - \binom{k+1}{2})$.

Note that Conjecture A is stronger than Conjecture B. Indeed if conjecture A holds, then $\text{span} PV_{n,k} = \text{span}^{0}(PV_{n,k}) = \text{span}(nk\zeta_{n,k}) - \binom{k+1}{2}$. Since $\zeta_{n,k} = p^{*}(\zeta_{n,1})$ where $p : PV_{n,k} \to PV_{n,1} = \mathbb{R}P^{n-1}$ pulls back $\zeta_{n,1}$ to $\zeta_{n,k}$, we see that $\text{span}(nk\zeta_{n,k}) \geq \text{span}(nk\zeta_{n,1})$ and so we conclude that $\text{span}(PV_{n,k}) \geq \kappa_{n,k}$. Conjecture A has been verified in many cases by Korbaš and Zvengrowski [48] using the work of Koschorke [50] (Theorem 1.14 above). They also verified Conjecture B in all cases except when $n$ is odd and $k = 2$ by using a boot-strapping argument.
2.4. Quotients of $W_{n,k}$ by cyclic groups. The complex Stiefel manifold $W_{n,k} = U(n)/U(n-k)$ is acted on by the circle group $S^1 = Z(U(n))$. Therefore for any $m \geq 2$, one has a natural action of the cyclic group $\mathbb{Z}_m \subset S^1$ which is free. The quotient space is denoted $W_{n,k;m}$ and is called the $m$-projective (complex) Stiefel manifold. It is clear that one has a principal $S^1$-bundle with projection $W_{n,k;m} \to PW_{n,k}$ and a covering projection $W_{n,k} \to W_{n,k;m}$ with deck transformation group $\mathbb{Z}_m$. Let $\xi_{n,k;m}$ (or more briefly $\xi$) denote the complex line bundle which is the pull-back of the bundle $\xi_{n,k}$ over $PW_{n,k}$. The smooth manifolds $W_{n,k;m}$ were studied in [26]. We merely state here without proof the results obtained therein concerning the span and parallelizability of $W_{n,k;m}$. We leave out the case $k = 1$ which is the standard lens space $L_m$ of dimension $2n - 1$.

The tangent bundle of $W_{n,k;m}$ satisfies the following isomorphism of (real) vector bundles as can be seen using (12):

$$\tau W_{n,k;m} \oplus k^2 \epsilon = nk\xi_{n,k;m} \quad (13).$$

We state without proof the following result due to Gondhali and Sankaran [26].

**Theorem 2.14.** Let $2 \leq k < n$ and $m \geq 2$. Then

(i) $\text{span}(W_{n,k;m}) > \text{span}^0(PW_{n,k}) \geq \dim(W_{n,k;m}) - 2n + 1$; moreover, when $n$ is even $\text{span}(W_{n,k;m}) > \dim(W_{n,k;m}) - 2n + 3$.

(ii) $\text{span}(W_{n,k;m}) > \text{span}^0(W_{n,k-1;m})$.

(iii) $W_{n,n-1;m}$ is parallelizable.

Using Koschorke’s Theorem 1.14 the following result was obtained in [26].

**Theorem 2.15.** Let $2 \leq k < n$ and $m \geq 2$. Then $\text{span}(W_{n,k}) = \text{span}^0(W_{n,k})$ in each of the following cases: (i) $k$ is even, (ii) $n$ is odd, (iii) $n \equiv 2 \mod 4.$

Let $2 \leq k < n$ and $m \geq 2, 1 \leq r < n$. Define positive integers $m_r$ as follows: $m_r := m$ if $r < n-k$; $m_r := \gcd\{m, \binom{n}{j} \mid n-k < j \leq r\}$.

It is easily seen that $H^2(W_{n,k;m};\mathbb{Z}) \cong \mathbb{Z}_m$ generated by the first Chern class of the complex line bundle associated to the $S^1$-extension of the universal covering projection $W_{n,k} \to W_{n,k;m}$. Denoting this generator by $y_2$, it turns out that the order of $y_2^r \in H^{2r}(W_{n,k;m};\mathbb{Z})$ is $m_r$. In particular, the height of $y_2$ is the largest $r \leq n$ such that $m_r > 1$. By computing the Pontrjagin class of $W_{n,k;m}$ one obtains the following result.

**Theorem 2.16.** Let $2 \leq k < n$ and let $m \geq 2$. With notation as above, if there exists an $r \geq 1$ such that $m_{2r}$ does not divide $\binom{n}{r}$, then $W_{n,k;m}$ is not stably parallelizable. In particular, if $W_{n,k;m}, k < n-1$, is stably parallelizable, then $m$ divides $nk$. The manifold $W_{n,n-1;m}$ is parallelizable for all $m$.

**Remark 2.17.** Gondhali and Subhash [27] introduced a generalization of complex projective Stiefel manifolds, which depend on a $k$-tuple $l := (l_1, \ldots, l_k)$ of positive integers with $\gcd\{l_1, \ldots, l_k\} = 1$. These are homogeneous spaces $P_l W_{n,k} = U(n)/S^1 \times U(n)$ where the group $S^1 \subset U(k)$ consists of diagonal matrices $\text{diag}(z^{l_1}, \ldots, z^{l_k}), |z| = 1$. They obtained results on the (stable) parallelizability of these homogeneous spaces. Basu and Subhash [12] obtained, among other things, upper bounds for the span of $P_l W_{n,k}$.
2.5. Grassmann manifolds, flag manifolds and related spaces. Let $\mathbb{F}G_{n,k}$ denote the space of $k$-dimensional $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}, \mathbb{C},$ or $\mathbb{H}$. One has the following description of $\mathbb{F}G_{n,k}$ as a homogeneous space: $\mathbb{R}G_{n,k} = O(n)/(O(k) \times O(n-k)) = SO(n)/S(O(k) \times O(n-k)), \mathbb{C}G_{n,k} = U(n)/(U(k) \times U(n-k))$, and $\mathbb{H}G_{n,k} = Sp(n)/(Sp(k) \times Sp(n-k))$. It is clear that $\mathbb{F}G_{n,1}$ is the $\mathbb{F}$-projective space $\mathbb{P}^{n-1}$.

More generally, one has the $\mathbb{F}$-flag manifold defined as follows: Suppose that $\mu := (n_1, \ldots, n_r)$ is a sequence of positive numbers with sum $|\mu| := \sum_{1 \leq j \leq r} n_j =: n$. Then the real flag manifold of type $\mu$ is the coset space $O(n)/(O(n_1) \times \cdots \times O(n_r)) := \mathbb{R}G(\mu)$. The complex (resp. quaternionic) flag manifold are defined as $\mathbb{C}G(\mu) = U(n)/(U(n_1) \times \cdots \times U(n_r))$ (resp. $\mathbb{H}G(\mu) := Sp(n)/(Sp(n_1) \times \cdots \times Sp(n_r))$) respectively. Clearly $\mathbb{F}G(n_1, n_2)$ is just the Grassmann manifold $\mathbb{F}G(n_1, n_2)$. One may identify $\mathbb{F}G(\mu)$ with the space of flags $V := (V_1, \ldots, V_r)$ where $V_j \subset \mathbb{F}^n$ is a (left) $\mathbb{F}$-vector space of dimension $n_j$ and $V_i \perp V_j$ if $1 \leq i < j \leq r$. (Note that $V_r$ is determined by the rest of the $V_j$.) It is clear that $\mathbb{F}G(\mu) \cong \mathbb{F}G(\lambda)$ if $\lambda$ is a permutation of $\mu$. For this reason, one may assume that $n_1, \ldots, n_r$ is an increasing (or a decreasing) sequence. It is readily verified that $\dim_\mathbb{R} \mathbb{F}G(\mu) = (\dim_\mathbb{F})\sum_{1 \leq i < j \leq r} n_i n_j$.

It turns out that any complex flag manifold has the structure of a complex projective variety. When $r > 2$, one has an obvious fibre bundle projection $p_j : \mathbb{F}G(\mu) \to \mathbb{F}G(\mu,j), 1 \leq j < r$, where $\mu(j)$ is the sequence obtained from $\mu$ by replacing $n_j, n_{j+1}$ by $n_j + n_{j+1}$. The fibre of this bundle is readily seen to be the Grassmann manifold $\mathbb{F}G(n_j, n_{j+1})$.

The complex and quaternionic flag manifolds are simply connected. However, this is not true of the real flag manifolds. Indeed, $\pi_1(\mathbb{R}G(\mu)) \cong (\mathbb{Z}_2)^{r-1}$ (where $r$ is the length of $\mu$), except when $n = 2, \mu = (1,1)$ which corresponds to the case of the circle, $\mathbb{R}P^1$. The oriented flag manifold of type $\mu$, denoted $\tilde{G}(\mu)$ is defined as the coset space $SO(n)/(SO(n_1) \times \cdots \times SO(n_r))$. It may be identified with the space of all oriented flags $(V; \sigma), V \in \mathbb{F}G(\mu)$ and $\sigma = (\sigma_1, \ldots, \sigma_r)$ where $\sigma_j$ is an orientation on $V_j, 1 \leq j \leq r$ with the restriction that these orientations induce the standard orientation on $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_r$. (Thus $\sigma_r$ is determined by $\sigma_j, 1 \leq j < r$.) The natural projection $q : \tilde{G}(\mu) \to \mathbb{R}G(\mu), (V, \sigma) \mapsto V$ which forgets the orientations on the flags is a covering map. It is universal except when $\mu = (1, 1)$, in which case it is the double covering of the circle. The deck transformation group is generated by the elements $t_j : \tilde{G}(\mu) \to \tilde{G}(\mu), 1 \leq j < r$, which reverses the orientation on $V_j$ and on $V_r$.

Let $\gamma_j(\mu)$ (more briefly $\gamma_j$) denote the canonical $n_j$-plane bundle over $\mathbb{F}G(\mu)$ whose fibre over a flag $V = (V_1, \ldots, V_r) \in \mathbb{F}G(\mu)$ is the vector space $V_j$. Evidently we have the $\mathbb{F}$-bundle isomorphism

$$\gamma_1(\mu) \oplus \cdots \oplus \gamma_r(\mu) \cong n \mathbb{F}. \quad (14)$$

The tangent bundle of $\mathbb{F}G(\mu)$ has the following description: Recall that $\Hom_\mathbb{F}(\xi, \eta) \cong \xi \otimes \eta$ as $Z(\mathbb{F})$-vector bundles. Here $\xi$ denotes the same underlying real vector bundle but with conjugate $\mathbb{F}$-structure; when $\mathbb{F} = \mathbb{R}$, $\xi = \xi$.

$$\tau \mathbb{F}G(\mu) \cong \bigoplus_{1 \leq i < j \leq r} \Hom_\mathbb{F}(\gamma_i, \gamma_j) \cong \bigoplus_{1 \leq i < j \leq r} \gamma_i \otimes_\mathbb{F} \gamma_j \quad (15)$$
as \(Z(\mathbb{F})\)-vector bundles where \(Z(\mathbb{F})\) denotes the centre of the division ring \(\mathbb{F}\). We refer the reader to [53] for a proof.

We shall denote the bundle \(q^*(\gamma_j)\) by \(\tilde{\gamma}_j\). Then, from (15), we see that tangent bundle \(\tau\tilde{G}(\mu)\) is isomorphic to \(\oplus_{1 \leq i < j < r} \tilde{\gamma}_i \otimes \tilde{\gamma}_j\). The bundle \(\tilde{\gamma}_j\) is canonically oriented: the orientation on the fibre of \(\tilde{\gamma}_j\) over \((V, \sigma) \in G(\mu)\) is the oriented vector space \((V, \sigma_j)\); it follows that the tangent bundle of \(\tilde{G}(\mu)\) is also canonically oriented. The deck transformation \(t_j\) induces a bundle isomorphism \(Tt_j : \tau\tilde{G}(\mu) \rightarrow \tau\tilde{G}(\mu)\) which preserves the summands \(\tilde{\gamma}_k \otimes \tilde{\gamma}_l\). It preserves the orientation on \(\tilde{\gamma}_k \otimes \tilde{\gamma}_l, k < l\), if and only if one of the following holds: (a) \((k, l) = (j, r), n_j \equiv n_r\) mod 2, (b) \(k = j, l < r, n_i\) is even, (c) \(k \neq j, l = r, n_k\) is even, or (d) \(\{k, l\} \cap \{j, r\} = \emptyset\). As \(t_j\) preserves the orientation on \(\tau\tilde{G}(\mu)\) if and only if it reverses the orientation on an even number of summands \(\tilde{\gamma}_k \otimes \tilde{\gamma}_l, 1 \leq k < l \leq r\), it follows that \(t_j\) is orientation preserving on \(\tilde{G}(\mu)\) if and only if \(n_j \equiv n_r\) mod 2. Hence it follows that \(\mathbb{R}G(\mu)\) is orientable if and only if \(n_j \equiv n_r\) for every \(j, 1 \leq j < r\). This fact may also be verified by computing the first Stiefel-Whitney class. As remarked already, the complex and quaternionic flag manifolds are simply connected. It follows that they are orientable.

We have the following theorem concerning the (stable) parallelizability of \(\mathbb{F}\)-flag manifolds. The case of Grassmann manifolds \(\mathbb{F}G(n_1, n_2)\) was settled in full generality by Trew and Zvengrowski [89]. See also [31], [92], [9]. The (stable) parallelizability of \(\mathbb{F}\)-flag manifolds was settled by Sankaran and Zvengrowski [74]. It turns out that the proof in most of the cases \(r \geq 3\) follows easily from the results on Grassmann manifolds. The result for the class of complex flag manifolds is a special case of a more general result, namely Theorem 2.6, due to Singhof [75]. See also [76], [77] where the result for quaternionic flag manifolds was obtained. Korbaˇs [42] obtained the results for real flag manifolds using Stiefel-Whitney classes.

**Theorem 2.18.** Let \(\mu = (n_1, \ldots, n_r)\) where \(n_1 \geq \ldots \geq n_r \geq 1, r \geq 2\), and let \(n := \sum_{1 \leq j \leq r} n_j\). Let \(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}\). Then \(\mathbb{F}G(\mu)\) is stably parallelizable in the following cases:

(i) \(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}\) and \(n_1 = 1\) for all \(j\). Moreover, when \(n_1 = 1\), \(\mathbb{F}G(\mu)\) is parallelizable only when \(\mathbb{F} = \mathbb{R}\). When \(\mathbb{F} = \mathbb{C}, \mathbb{H}\), the remaining \(\mathbb{F}\)-flag manifolds are not stably parallelizable.

(ii) Let \(\mathbb{F} = \mathbb{R}\) and \(n_1 > 1, r \geq 2\). Other than the projective spaces \(\mathbb{R}P^{n_1} = \mathbb{R}G(n_1, 1)\) when \(n_1 = 3, 7\), none of the flag manifolds \(\mathbb{R}G(\mu)\) are stably parallelizable.

**Proof.** We shall first consider the case \(r = 2\) namely that of the Grassmann manifold. Since \(\mathbb{F}G_{n,k} \cong \mathbb{F}G_{n,n-k}\), we may assume that \(1 \leq k \leq n/2, n \geq 4\). Since the case \(k = 1\) corresponds to the \(n - 1\)-dimensional projective space which is well-known and classical, we shall assume that \(k \geq 2\). As is customary, we shall denote the canonical bundles \(\gamma_1, \gamma_2 = \gamma_1^\perp\) over \(\mathbb{R}G_{n,k}\) by \(\gamma_{n,k}, \beta_{n,k}\) respectively.

One has an inclusion \(h_j : \mathbb{F}G_{n-j,k-j} \subset \mathbb{F}G_{n,k}, 1 < j < k\), induced by the inclusion of \(\mathbb{F}^{n-j} \subset \mathbb{F}^n\). Explicitly, \(h_j(V) = V + \mathbb{F}e_{n-j+1} + \cdots + \mathbb{F}e_n \in \mathbb{F}G_{n,k}, \forall V \in \mathbb{F}G_{n-j,k-j}\). (Here \(e_i\) denotes the standard basis element.) Now \(h_j^*(\gamma_{n,k}) = \gamma_{n-j,k-j} \oplus j \epsilon_{\mathbb{F}}, h_j^*(\beta_{n,k}) = \beta_{n-j,k-j}\).
Therefore, we have the following $Z(F)$-bundle isomorphisms:

$$h^*_j(\tau F G_{n,k}) \cong h^*_j(\gamma_{n,k}) \otimes_F h^*_j(\beta_{n,k}) = \left(\gamma_{n-j,k-j} \oplus j \varepsilon\right) \otimes \beta_{n-j,k-j} = \tau F G_{n-j,k-j} \oplus j \beta_{n-j,k-j}.$$ 

Put $j = k - 1$ and use the isomorphism $\tau F G_{n,1} \oplus \varepsilon_F \cong n\gamma_{n,1}$ to obtain $h^*_{k-1}(\tau F G_{n,k}) \oplus \varepsilon_F = (n - k + 1)\gamma_{n-k+1,1} \oplus (k - 1)\beta_{n-k+1,1}$. Now use the isomorphism $\gamma_{n-k+1,1} \oplus \beta_{n-k+1,1} \cong (n - k + 1)\varepsilon_F$ and the fact that, for any $F$-vector bundle $\xi$, we have the isomorphism $\xi \cong \xi$ of real vector bundles, we obtain, in $KO(F G_{n-k+1,1})$ the following:

$$h^*(\tau F G_{n,k}) = (n - k + 1)[\gamma_{n-k+1,1}] + (k - 1)[\beta_{n-k+1,1}] = (n - k + 1)[\gamma_{n-k+1,1} + (k - 1)\varepsilon_F] - (k - 1)[\gamma_{n-k+1}] = (n - 2k + 2)[\gamma_{n-k+1,1}] + d(n - k + 1)(k - 1)\varepsilon_F$$

where $d = \dim_F F$.

When $F = \mathbb{R}$, it follows from the known order of the Hopf bundle $\gamma_{n-k+1,1}$, that $(n - 2k + 2)[\gamma_{n-k+1,1}] \neq 0$ in $KO(\mathbb{R} P^{n-k})$ since $(n - k) \geq k \geq 2$. When $F = \mathbb{C}, \mathbb{H}$, an easy computation of the first Pontrjagin class (of the underlying real vector bundle) shows that $(n - 2k + 2)\gamma_{n-k+1,1}$ is not stably trivial as a real vector bundle. (Trew and Zvengrowski [89] altogether avoided computation of Pontrjagin classes, but used information about $KO$-theory in the case of complex and quaternionic projective spaces.) Thus in all cases, $F G_{n,k}$ is not stably parallelizable.

It remains to consider the case $r \geq 3$. In this case, we have a fibre inclusion $F G(n_1, n_2) \hookrightarrow F G(\mu)$ of the fibre bundle projection $F G(\mu) \to F G(n_1 + n_2, \ldots, n_r)$. Since the normal bundle to fibre inclusion is trivial, we see that the tangent bundle of $F G(\mu)$ restricts to the stable tangent bundle of $F G(n_1, n_2) = F G_{n_1+n_2,n_2}$. Therefore $F G(\mu)$ is not stably parallelizable except, possibly, when $F G(n_1, n_2)$ is stably parallelizable. Thus $F G(\mu)$ is not stably parallelizable, except possibly in the following cases: (i) $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $n_1 = 1$; (b) $F = \mathbb{R}$ and $n_2 = 1, n_1 = 3, 7$.

Case (i): The parallelizability of $RG(1, \ldots, 1)$ follows from Thermen 2.1. Note that $CG(1, \ldots, 1) = U(n)/(U(1) \times \cdots \times U(1))$ is stably parallelizable by Theorem 2.3. The stable parallelizability of $\mathbb{H}G(1, \ldots, 1)$ was first proved by Lam [53], making essential use of the functor $\mu^2$, which is an analogue of the second exterior power in the real and complex case.

Case (ii): We need only show that $RG(3, 1, 1)$ and $RG(7, 1, 1)$ are not stably parallelizable. Note that one has a double covering projection $PV_{n,2} \to RG(n - 2, 1, 1)$ defined as $[v_1, v_2] \mapsto (\{v_1, v_2\}, [\nu v_1, \nu v_2])$. Since $PV_{n,2}$ and $PV_{q,2}$ are not stably parallelizable by Theorem 2.11, it follows that $RG(3, 1, 1), RG(7, 1, 1)$ are also not stably parallelizable. □

We shall write $G(\mu)$ to denote the real flag manifold of type $\mu$. We assume that $n_1 \geq \cdots \geq n_r$. We observed already that the map $q : \tilde{G}(\mu) \to G(\mu)$ that forgets the orientations on the flags is a covering projection with deck transformation group $(\mathbb{Z}_2)^{r-1}$ generated by $t_j, 1 \leq j < r$. Recall that $\tilde{\gamma}_i(\mu)$ or more briefly $\tilde{\gamma}_i$ is the pull-back bundle $q^*(\gamma_i(\mu))$. In
the case of the oriented Grassmann manifolds, we have \( r = 2 \) and we write \( \tilde{\gamma}_{n,k}, \tilde{\beta}_{n,k} \) to denote \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) respectively. From (14) we obtain that \( \tilde{\tau}(\mu) := \tau(\tilde{G}(\mu)) = \bigoplus_{1 \leq i < j \leq r} \tilde{\gamma}_i \otimes \tilde{\gamma}_j \).

Next we have the following result concerning the oriented flag manifolds. As usual, \( \mu = (n_1, \ldots, n_r) \), \( n = \sum_{1 \leq j \leq r} n_j \).

**Theorem 2.19.** ([56], [70], [72]).

(i) Let \( 2 \leq k \leq n/2 \). The oriented Grassmann manifold \( \tilde{G}_{n,k} \) is stably parallelizable if and only if \( (n, k) = (4, 2), (6, 3) \). The manifold \( \tilde{G}_{6,3} \) is parallelizable but \( \tilde{G}_{4,2} \cong S^2 \times S^2 \) is not.

(ii) Let \( r \geq 3 \). Then \( \tilde{G}(\mu) \) is stably parallelizable if and only if any one of the following holds: \( \{n_1, \ldots, n_r\} \) is contained in \( \{1, 2\} \) or in \( \{1, 3\} \).

**Proof.** Consider the inclusion \( \tilde{G}_{n-2,k-1} \overset{i}{\hookrightarrow} \tilde{G}_{n-1,k-1} \overset{j}{\hookrightarrow} \tilde{G}_{n,k} \) induced by the inclusion of \( \mathbb{R}^{n-2} \hookrightarrow \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n \). Explicitly \( i(V) = V \) and \( j(U) = U + \mathbb{R}e_n \) with the direct sum orientation, got by adjoining \( e_n \) to an oriented basis of \( U \). Denote by \( q \) the composition \( j \circ i \). It is easily verified that the normal bundle to the embeddings \( i, j, q \) are respectively \( \tilde{\beta}_{n-2,k-1}, \tilde{\gamma}_{n-1,k-1}, (n-2k-2) \epsilon \) respectively. Therefore the composition of the embeddings of type \( q \) leads to embeddings \( f : \tilde{G}_{n-2k+4,2} \overset{q}{\to} \tilde{G}_{n,k} \) when \( n > 2k \), and \( g : \tilde{G}_{8,4} \to \tilde{G}_{n,k} \) when \( n = 2k \geq 8 \), with trivial normal bundle in each case.

So we need only show that \( \tau \tilde{G}_{m,2}, m \geq 5 \), and \( \tau \tilde{G}_{8,4} \) are not stably trivial. It turns out \( w_2(\tilde{G}_{5,2}) \neq 0 \).

Let \( m \geq 6 \). Using the embedding \( i \) repeatedly, we obtain an embedding \( \tilde{G}_{6,2} \to \tilde{G}_{m,2} \) we get \( \tau \tilde{G}_{m,2} = \tau \tilde{G}_{6,2} \oplus (m-6)\tilde{\gamma}_{6,2} \). Similarly, under the embedding \( \tilde{G}_{6,2} \hookrightarrow \tilde{G}_{7,3} \hookrightarrow \tilde{G}_{8,4} \), \( \tau \tilde{G}_{8,4} \) restricts to \( \tau \tilde{G}_{6,2} \oplus 2\tilde{\beta}_{6,2} = \tilde{\gamma}_{6,2} \otimes \tilde{\beta}_{6,2} \oplus 2\tilde{\beta}_{6,2} \).

Working in the ring \( KO(\tilde{G}_{6,2}) \) we have \( [\tilde{\beta}_{6,2}] = 6 - [\tilde{\gamma}_{6,2}] \) and we must show that the elements \( x_m := \lbrack \tilde{\gamma}_{6,2} \rbrack (6 - \lbrack \tilde{\gamma}_{6,2} \rbrack) + (m-6) \lbrack \tilde{\gamma}_{6,2} \rbrack - 2(m-2) \) and \( y := \lbrack \tilde{\gamma}_{6,2} \rbrack \cdot (6 - \lbrack \tilde{\gamma}_{6,2} \rbrack) + 2 \cdot (6 - \lbrack \tilde{\gamma}_{6,2} \rbrack) - 16 \) are not zero. Set \( z = \lbrack \tilde{\gamma}_{6,2} \rbrack - 2 \). We have \( x_m := (z+2)(4-z) + (m-6)(z+2) - 2(m-2) = -z^2 + (m-4)z \) and \( y = (z+2)(4-z) + 2(4-z) - 16 = -z^2 \).

Now consider the map \( h : \mathbb{C}P^2 \to \tilde{G}_{6,2} \) obtained by regarding any complex line \( L \subset \mathbb{C}^3 \cong \mathbb{R}^6 \) as a real vector space of dimension 2 with its natural orientation. Then \( h^*(\tilde{\gamma}_{6,2}) = \mathbb{C} \gamma_{3,1} =: \zeta \). The ring homomorphism \( h^* : KO(\tilde{G}_{6,2}) \to KO(\mathbb{C}P^2) \) maps \( z \) to \( [\zeta] - 2 \). Since \( [\zeta] - 2, ([\zeta] - 2)^2 \) generate a free abelian group of rank 2 in \( KO^*(\mathbb{C}P^2) \) by \([24]\), it follows that the same is true of \( z, z^2 \in KO(\tilde{G}_{6,2}) \) and so we conclude that \( x_m \neq 0 \) \( \forall m \geq 6 \), and \( y \neq 0 \).

The stable parallelizability of \( \tilde{G}_{4,2} = SO(4)/SO(2) \times SO(2) \) is immediate from Proposition 2.3. By the same proposition, if \( n_1 \leq 2 \), then \( \tilde{G}(\mu) = SO(n)/S \) where \( S \) is a torus and hence it is stably parallelizable. It is parallelizable precisely if \( S \) is not a maximal torus.

Next suppose that \( n_i = 3, 1 \leq i \leq s \), and \( n_j = 1 \) for \( s < j \leq r \) for some \( s \). For any oriented vector bundle \( \xi \) of rank \( m \), we have \( \Lambda^p(\xi) \cong \Lambda^{m-p}(\xi) \). Hence we have \( \Lambda^2(\tilde{\gamma}_i) \cong \tilde{\gamma}_i, 1 \leq i \leq s \). Also \( \Lambda^2(\tilde{\gamma}_j) = 0 \) for \( j > s \). Applying \( \Lambda^2 \) to both sides of the isomorphism,
isomorphism $\oplus_{1 \leq j \leq r} \tilde{\gamma}_j = n\epsilon$, and using (15) we obtain $\tau \tilde{G}(\mu) = \oplus_{1 \leq i \leq s, j > s} \tilde{\gamma}_i \otimes \tilde{\gamma}_j$, we obtain $\binom{n}{s} \epsilon = \tau \tilde{G}(\mu) \oplus (\oplus_{1 \leq j \leq s} \Lambda^2(\tilde{\gamma}_j)) = \tau \tilde{G}(\mu) \oplus (\oplus_{1 \leq i \leq s} \tilde{\gamma}_i)$. Since $\tilde{\gamma}_j \cong \epsilon$ as $n_j = 1$ for $j > s$, we obtain $\binom{n}{s} + (r - s) \epsilon = \tau \tilde{G}(\mu) \oplus (\oplus_{1 \leq j \leq s} \tilde{\gamma}_j) = \tau \tilde{G}(\mu) \oplus n\epsilon$ and so $\tilde{G}(\mu)$ is stably parallelizable. As for the parallelizability of $\tilde{G}(\mu)$, we apply the Bredon-Kosinski Theorem 1.15. Evidently $\chi(\tilde{G}(\mu)) = 0$, implying the parallelizability when the dimension is even. When the dimension is odd, leaving out the case $\tilde{G}(3, 1) = S^3$ and the 7-dimensional manifold $\tilde{G}(3, 1, 1)$ which are parallelizable, we must show that the mod 2 Kervaire semicharacteristic $\hat{\chi}_2(\tilde{G}(\mu))$ vanishes. This follows from the fact that $\tilde{G}(\mu)$ admits a fixed point free $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (see Remark 1.16).

Finally, suppose that there exists $i, j \leq r$ such that $n_i \geq 3, n_j \geq 2$ but $n_j \neq 3$, then the oriented Grassmann manifold $\tilde{G}(n_i, n_j)$ is not stably parallelizable. Since $\tilde{G}(\mu)$ is fibred by $\tilde{G}(n_i, n_j)$, it follows that $\tilde{G}(\mu)$ is not stably parallelizable. \qed

**Remark 2.20.** One has the universal double cover $\text{Spin}(4) \to \text{SO}(4)$ under which the maximal torus $\text{SO}(2) \times \text{SO}(2)$ lifts to a maximal torus $T$. One has an isomorphism of Lie groups $\text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3)$ under which $T$ corresponds to $\tilde{T} = \text{Spin}(2) \times \text{Spin}(2)$. So

$$
\tilde{G}_{4,2} = \frac{\text{SO}(4)}{(\text{SO}(2) \times \text{SO}(2))} \\
\cong \frac{\text{Spin}(4)}{\tilde{T}} \\
\cong \frac{\text{Spin}(3)/\text{Spin}(2) \times \text{Spin}(3)/\text{Spin}(2)}{\text{Spin}(2)} \\
= S^2 \times S^2.
$$

Note that span of complex and quaternionic Grassmann manifolds are zero since they have non-vanishing Euler-Poincaré characteristic; see (2). We have the following general result concerning the span of real Grassmann manifolds. This is essentially due to Leite and Miatello [54] who considered oriented Grassmann manifolds. The proof given here is due to Zvengrowski (unpublished).

**Theorem 2.21.** When $k$ is even or $n$ is odd, $\text{span}(\mathbb{R}G_{n,k}) = 0$. When $k$ is odd and $n$ even,

$$
\text{span}(\mathbb{R}G_{n,k}) \geq \text{span}(S^{n-1}) = \rho(n) - 1.
$$

**Proof.** The rank of $G := \text{SO}(n)$ equals $\lfloor n/2 \rfloor$. Let $H = S(O(k) \times O(n - k))$. Then $H_0 = \text{SO}(k) \times \text{SO}(n - k)$ has the same rank as $\text{SO}(n)$ if and only if $n$ is odd or $k$ is even. Since $\mathbb{R}G_{n,k} = G/H$, it follows from Theorem 2.2 that $\text{span}(G_{n,k}) > 0$ if and only if $n$ is even and $k$ is odd.

Let $n$ be even and $k$ odd. Let $r = \rho(n)$ and let $\mu_1 = id, \mu_2, \ldots, \mu_r : \mathbb{R}^n \to \mathbb{R}^n$ be the Radon-Hurwitz transformations. Thus $\mu_i \mu_j = -\mu_j \mu_i, \mu_j^2 = -id, 2 \leq i < j \leq r$. (See §1.2.)

Let $v_j(X) \in \text{Hom}(X, X^\perp)$ be the composition $X \hookrightarrow \mathbb{R}^n \xrightarrow{\mu_i} \mathbb{R}^n \xrightarrow{p} X^\perp$, $2 \leq j \leq r$. Here $p$ denotes the orthogonal projection. Then $v_j(X) \in T_X \mathbb{R}G_{n,k}$, $2 \leq j \leq r$, and we obtain smooth vector fields $v_2, \ldots, v_r$ on $\mathbb{R}G_{n,k}$. We claim that these are everywhere linearly independent on $\mathbb{R}G_{n,k}$. To see this, note that if $(a_2, \ldots, a_r) \in S^{r-1}$, then $\mu :=
Remark 2.22. (i) Write \(\sum_{2 \leq j \leq r} a_j \mu_j\) is a skew-symmetric orthogonal transformation of \(\mathbb{R}^n\). Hence it does not have an odd-dimensional invariant subspace in \(\mathbb{R}^n\). It follows that, for any \(X \in \mathbb{R}G_{n,k}\), the composition \(X \hookrightarrow \mathbb{R}^n \overset{\rho}{\to} \mathbb{R}^n \overset{\rho}{\to} X^\perp\) is non-zero. Therefore \(\sum_{2 \leq j \leq r} a_j \nu_j(X) \neq 0\). Thus span(\(\mathbb{R}G_{n,k}\)) \(\geq r - 1 = \rho(n) - 1\). \(\square\)

(ii) It is known that equality holds in (16) in infinitely many cases. We point out a sample of such results obtained in [67]. For example, span(\(\mathbb{R}G_{n,3}\)) = 3 when \(n\) is of the form \(4(2^r + 1)\). This follows by showing that \(w_{d-3}(\mathbb{R}G_{n,k}) \neq 0\) where \(d = k(n - k) = \dim \mathbb{R}G_{n,k}\). Also when \(n \equiv 2 \mod 4, k \equiv 1 \mod 2\), we have \(\dim \mathbb{R}G_{n,k} \equiv 1 \mod 4\). Since \(n\) is even and \(k\) is odd, \(\mathbb{R}G_{n,k}\) is orientable and is an unoriented boundary (see [68]). Using Remark 1.16 it can be seen that the Kervaire semicharacteristic \(\kappa(\mathbb{R}G_{n,k})\) equals 0 or 1 according as \(n/k\) \(\equiv 0\) or \(2 \mod 4\). So it follows from Theorem 1.12 that span(\(\mathbb{R}G_{n,k}\)) = 1 (resp. span(\(\mathbb{R}G_{n,k}\)) \(\geq 2\)) if \(n/k\) \(\equiv 2 \mod 4\) (resp. \((n/k)\) \(\equiv 0 \mod 4\)).

The last remark should be compared with the following result.

Theorem 2.23. (Leite and Miatoello [54] ) Let \(n - k = 2r + 1, k = 2s + 1,\) where \(s \geq 1,\) and \(r \geq 1\) is odd. Suppose that \(r + s\) is even so that \(n \equiv 2 \mod 4\). Then:

\[
\text{span}(\tilde{G}_{n,k}) = \begin{cases} 
1 & \text{if } \left(\frac{r+s}{r}\right) \equiv 1 \mod 2, \\
\geq 2 & \text{if } \left(\frac{r+s}{r}\right) \equiv 0 \mod 2.
\end{cases}
\] \(\square\)

In general, the determination of span of real Grassmann manifolds is a wide open problem. The first ‘non-trivial’ case is that of the Grassmann manifold \(\mathbb{R}G_{6,3}\). In this case the Radon-Hurwitz bound yields span(\(\mathbb{R}G_{6,3}\)) \(\geq 1\). We have the following result:

Theorem 2.24. (Korbaš-Sankaran [45].) span(\(\mathbb{R}G_{6,3}\)) = 7.

Proof. Recall that \(\tau(G_{n,k}) \cong \gamma_{6,3} \otimes \beta_{6,3}\). Since \(\gamma_{6,3} \otimes \beta_{6,3} = 6\xi\), taking the second exterior power on both sides we obtain \(\Lambda^2(\gamma_{6,3} \otimes \beta_{6,3}) = 15\xi\). Expanding the left hand side we obtain \(\Lambda^2(\gamma_{6,3} \otimes \beta_{6,3}) = \Lambda^2(\gamma_{6,3}) \otimes \gamma_{6,3} \otimes \beta_{6,3} \otimes \Lambda^2(\beta_{6,3}) = \gamma_{6,3} \otimes \xi \otimes \tau \mathbb{R}G_{6,3} \otimes \beta_{6,3} \otimes \eta\), where
in the last equality \( \xi := \det(\gamma_{6,3}) \), \( \eta := \det(\beta_{6,3}) \) and we have used the bundle isomorphism \( \Lambda^r(\omega) \cong \Lambda^{m-r}\omega \otimes \det(\omega) \) for any vector bundle \( \omega \) of rank \( m \). Since a real line bundle is determined by its first Stiefel-Whitney class, it is readily seen that \( \xi \cong \eta \) and so we obtain

\[
15\epsilon = \tau \mathbb{R}G_{6,3} \oplus (\gamma_{6,3} \oplus \beta_{6,3}) \otimes \xi = \tau \mathbb{R}G_{6,3} \oplus 6\xi. \tag{17}
\]

It is known that there exists a 3-fold *vector product* \( \mu : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8 \); see [90]. An explicit formula was given by Zvengrowski [93]. The map \( \mu \) has the following properties:

(a) \( \mu \) is multilinear, (b) \( \mu(u, v, w) = 0 \) if \( u, v, w \) are linearly dependent, (c) \( \mu(u, v, w) \in \mathbb{R}^8 \) is a unit vector that depends only on the oriented 3-dimensional vector space spanned by \( u, v, w \) if they are pairwise orthogonal. The map \( V \mapsto \mathbb{R}\mu(u, v, w) \) is a well-defined continuous map \( f : \mathbb{R}G_{8,3} \to \mathbb{R}P^7 \) where \( u, v, w \) is any orthonormal basis \( V \in \mathbb{R}G_{6,3} \). It is not difficult to see that \( f \) induces an isomorphism of fundamental groups. From this it follows easily that \( \det(\gamma_{8,3}) \cong f^*(\gamma_{8,1}) \). Restricting to \( \mathbb{R}G_{6,3} \) and using the fact that \( 8\gamma_{8,1} \cong 8\epsilon \), we obtain that \( 8\xi \cong 8\epsilon \) whence \( 15\epsilon = 7\epsilon \oplus 8\xi \). Therefore, using (17) we conclude that \( \tau \mathbb{R}G_{6,3} \) is stably isomorphic to \( 7\epsilon \oplus 2\xi \). Hence \( \text{span}^0(\mathbb{R}G_{6,3}) \geq 7 \). By a straightforward computation we have \( w_2(\mathbb{R}G_{6,3}) \neq 0 \) and so \( \text{span}^0(\mathbb{R}G_{6,3}) = 7 \).

Since by Remark 2.22(ii), \( R_L(\mathbb{R}G_{6,3}) = \kappa(\mathbb{R}G_{6,3}) = 0 \), by appealing to Theorem 1.14 we conclude that \( \text{span}(\mathbb{R}G_{6,3}) = \text{span}^0(\mathbb{R}G_{6,3}) = 7 \). \( \square \)

For results on the (stable) parallelizability of *partially oriented flag manifolds* the reader is referred to [70], [71]. For the orientability of a generalized real flag manifolds see [61].

3. Homogeneous spaces for non-compact Lie groups

We now turn to the case where \( G \) is a connected non-compact Lie group.

For convenience we will assume that \( G \) is a connected linear Lie group, that is, \( G \) is a closed connected subgroup of \( GL(N, \mathbb{R}) \) for some \( N \). Let \( R = \text{rad}(G) \) be the *radical* of \( G \), i.e., the maximal connected normal solvable subgroup of \( G \). Then \( \bar{G} := G/R \) is semisimple and \( R \) is a semidirect product \( R_u \rtimes T \) where \( R_u \) is the *unipotent radical* of \( G \), namely, the maximal connected normal nilpotent subgroup of \( G \), and \( T \) is an abelian subgroup which is diagonalizable. We will consider two separate cases: (a) \( G = R \) is solvable, and (b) \( G \) is semisimple, i.e., \( \text{rad}(G) \) is trivial. We refer the reader to [64] and to [7] for general facts concerning lattices in Lie groups and the structure of solvmanifolds respectively.

3.1. Solvmanifolds. First suppose that \( G \) is nilpotent and \( M = G/H \) where \( H \) is a closed subgroup. Such a space is called a *nilmanifold*. Then \( M \) is diffeomorphic to a product \( \mathbb{R}^s \times M_0 \) where \( M_0 \) is a *compact* smooth manifold of the form \( N/\Gamma \) where \( N \) is a connected nilpotent Lie group and \( \Gamma \) is a discrete subgroup of \( N \). By Theorem 2.1, \( M_0 \) is parallelizable and so \( M \) itself is parallelizable.

Suppose that \( G \) is a solvable Lie group. In this case a homogeneous space \( M = G/H \) is known as a solvmanifold. For basic facts about the structure of solvmanifolds,
some of which will be recalled below, see [7]. Auslander and Tolimieri showed that $M$ is diffeomorphic to the total space of a vector bundle over a compact solvmanifold, as conjectured by Mostow. Unlike in the case of nilmanifolds, solvmanifolds are not even stably parallelizable in general. For example the Klein bottle and the M"obius band are solvmanifolds. It turns out that any solvmanifold is an Eilenberg-MacLane space $K(\pi, 1)$ and that if it is compact, then its diffeomorphism type is determined by its fundamental group. Thus the span of a solvmanifold is an invariant of its fundamental group. The fundamental group $\Gamma$ of a compact solvmanifold is strongly polycyclic, that is, there is a filtration

$$\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_n > \Gamma_{n+1} = 1$$

(18)

where each $\Gamma_{i+1}$ is normal in $\Gamma_i$ and $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$. Choose an element $\alpha_i \in \Gamma_i$ which maps to the generator of $\Gamma_i/\Gamma_{i+1}$. Then $\Gamma_i \cong \Gamma_{i+1} \rtimes \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\Gamma_{i+1}$ is given by the restriction to $\Gamma_{i+1}$ of the conjugation by $\alpha_i$.

While any finitely generated torsionless nilpotent group is a uniform lattice in a connected nilpotent Lie group, the analogous statement for solvable groups is false in general. For example, the fundamental group $\Gamma = \langle x, y \mid xyx^{-1}y \rangle = \mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle cannot be a lattice in a connected Lie group $G$. (Otherwise the Klein bottle would be parallelizable.) This makes the vector field problem for (compact) solvmanifolds nontrivial and interesting.

The following well-known result due Auslander and Szczarba [8] says that the structure group of the tangent bundle of a $d$-dimensional solvable manifold can be reduced to the diagonal subgroup of the orthogonal group $O(d)$. Thus the manifold is close to being parallelizable.

**Theorem 3.1.** (Auslander and Szczarba [8].) Let $M$ be a compact solvmanifold of dimension $d$. Then there exists line bundles $\xi_1, \ldots, \xi_d$, such that $\tau M \cong \xi_1 \oplus \cdots \oplus \xi_d$. In particular all Pontrjagin classes of $M$ are trivial.

Using the fact that $M$ fibres over a circle, it can be seen easily that one of the line bundles, say $\xi_1$ may be taken to be trivial. As an immediate corollary, one obtains the following

**Theorem 3.2.** Let $M$ be a compact solvmanifold of dimension $d$. Then there exists a smooth covering $\widetilde{M} \to M$ of degree $2^k$, $k < d$, such that $\widetilde{M}$ is parallelizable.

Next we obtain a criterion for the (stable) parallelizability in a special case and a criterion for the orientability in the general case.

Let $A \in GL(n, \mathbb{Z})$ and let $\Gamma = \Gamma(A) := \mathbb{Z}^n \rtimes \mathbb{Z}$ be the extension of $\mathbb{Z}$ by $\mathbb{Z}^n$ where the $\mathbb{Z}$-action on $\mathbb{Z}^n$ is generated by $A$. The smooth compact solvmanifold $M = M(A)$ with fundamental group $\Gamma$ may be described as follows: Let $\alpha : \mathbb{T} \to \mathbb{T}$ be the diffeomorphism of the $n$-dimensional torus $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ defined by the linear automorphism of $\mathbb{R}^n$, $v \to Av$. The ‘mapping circle’ $M = \mathbb{T}^n \times I/\sim$, where $(a, 0)$ is identified to $(\alpha(a), 1)$, is a smooth manifold. We have $T_{(a,x)}(\mathbb{T} \times I) = T_a\mathbb{T} \times \mathbb{R} \cong \mathbb{R}^n \times \mathbb{R}$ for all $a \in \mathbb{T}, x \in I$. The total
space of the tangent bundle of $M$ has the following description: $TM = T(\mathbb{T} \times I)/\sim$ where $(a, 0; v, s) \in T_{(a, 0)}(\mathbb{T} \times I)$ is identified with $(\alpha(a), 1; Av, s) \in T_{(\alpha(a), 1)}(\mathbb{T} \times I)$ for $v \in T_a \mathbb{T}, s \in T_0 I = T_1 I = \mathbb{R}$. The projection $\mathbb{T} \times I \to I$ induces a fibre bundle projection $\pi : M \to S^1$ with fibre $\mathbb{T}$. Hence we obtain an isomorphism $\tau M \cong \pi^*(\tau S^1) \oplus \eta = \epsilon \oplus \eta$, where $\eta$ is the vertical bundle that restricts to the tangent bundle on the fibres of $\pi$.

**Theorem 3.3.** (Sankaran, unpublished.) Let $A \in \text{GL}(n, \mathbb{Z})$. With the above notation, the manifold $M = M(A)$ is parallelizable if $\det(A) = 1$ and is not orientable—hence not stably parallelizable—if $\det(A) = -1$.

**Proof.** First assume that $\det(A) = 1$. We shall show that $\eta$ is trivial. Let $\sigma : I \to \text{GL}(n, \mathbb{R})$ be a smooth path such that $\sigma(0) = I_n$, the identity matrix, and, $\sigma(1) = A$. We will write $A_t$ to denote $\sigma(t)$. We have $\tau \mathbb{T} = n\epsilon$ with total space $\mathbb{T} \times \mathbb{R}^n$. The standard basis of $\mathbb{R}^n$ yields vector fields $X_1, \ldots, X_n$ on $\mathbb{T} \times I$ defined as follows: $X_j(a, x) = (a, x; Ae_j, 0) \in \mathbb{T} \times I \times \mathbb{R}^n \times \mathbb{R}$. Note that $X_j(\alpha(a), 1) = (\alpha(a), 1; Ae_j, 0) \sim (a, 0; e_j, 0) = X_j(a, 0)$. Hence $X_j$ descends to a well-defined smooth vector field, again denoted $X_j$ on $M$. Since $T\pi(X_j(p)) = 0$ for every $j$, it follows that $X_j, 1 \leq j \leq n$, are cross-sections of $\eta$. Since $A_t \in \text{GL}(n, \mathbb{R})$ for all $t$, it is evident that $X_1, \ldots, X_n$ are everywhere linearly independent. So $\eta$ is trivial, as was to be shown.

Now suppose that $\det(A) = -1$. Fix a point $a_0 \in \mathbb{T}$ and choose a path $\sigma : I \to \mathbb{T}$ from $a_0$ to $\alpha(a_0)$. Let $\theta : S^1 \to M$ be the embedding $\exp(2\pi it) \mapsto [\sigma(t), t] \in M, 0 \leq t \leq 1$. Consider the pull-back line bundle $\xi := \theta^*(\Lambda^n(\eta))$ over $S^1$. We assert that $\xi$ is not orientable. This readily implies that $\eta$ is non-orientable and hence $M$ itself is not orientable. To prove the assertion, we need only observe that the total space of $\xi$ is obtained by identifying in $I \times \mathbb{R}$, the point $(0, t) \in I \times \mathbb{R}$ with $(1, (\det(A))t) = (1, -t)$ for all $t \in \mathbb{R}$. Thus $E(\xi)$ is homeomorphic to the Möbius band and so $\xi$ is non-orientable. $\square$

We remark that when $\det(A) = -1$, the double cover $\widetilde{M}$ of $M = M(A)$ corresponding to the subgroup $\mathbb{Z}^n \times (2\mathbb{Z}) \subset \Gamma$ is just the group $\Gamma(A^2)$ and hence is parallelizable.

One may generalize one part of the above theorem so as to obtain a criterion for the orientability of a solvmanifold.

Let $\Gamma$ be a strongly polycyclic group and write $\Gamma \cong \Gamma_1 \times \mathbb{Z}$ where $\Gamma_1$ is as in (18). We let $A$ be the automorphism of $\Gamma_1$ that defines the action of $\mathbb{Z}$ on $\Gamma_1$. Let $N := M(\Gamma_1)$ be a compact solvmanifold with fundamental group $\Gamma_1$. Then $M = M(\Gamma)$, a solvmanifold with fundamental group $\Gamma$, may be obtained as the mapping circle of a diffeomorphism $\alpha : N \to N$ that induces $A$. Now $M$ fibres over the circle $\pi : M \to S^1$ with fibre $N$ and we have a splitting $\tau M = \tau S^1 \oplus \eta$ where $\eta$ is the vertical bundle. If $N$ is non-orientable, neither is $M$ since the normal bundle to the fibre inclusion $N \hookrightarrow M$ is trivial.

Assume that $N$ is orientable. If $\alpha : N \to N$ is orientable, then $T\alpha : T_a N \to T_{\alpha(a)} N$ is orientation preserving. As before, for any $a \in N$, we have $[a, 0] = [a, 1]$ in $M$ and the tangent space $T_{[a, 0]}M$ is obtained from $T(N \times I) = TN \times I \times \mathbb{R}$ by identifying $(u, 0; t) \in T_{(a, 0)}(N \times I)$ with $(T_a\alpha(u), 1; t) \in T_{(\alpha(a), 1)}(N \times I)$ where $u \in T_a N, t \in \mathbb{R}$. 


Since \( \tau M = \eta \oplus \epsilon \), the total space \( E(\eta) \subset TM \) is the space of all vectors with vanishing last coordinate. Since \( T_a \alpha : T_a N \to T_{\alpha(a)} N \) is orientation preserving, we see that \( \eta \) is orientable. Hence \( M \) is orientable. On the other hand, if \( \alpha \) is orientation reversing, then choosing a path \( \sigma : I \to N \) from a point \( a \) to \( \alpha(a) \) we obtain an imbedding \( \bar{\sigma} : S^1 \to M, (\exp(2\pi it), t) \in [0, 1] \). The bundle \( \Lambda^n(\eta) \) pulls-back via \( \bar{\sigma} \) to a line bundle \( \xi \) which is seen to be non-orientable. It follows that \( \eta \) is non-orientable. Hence \( M \) is non-orientable. Repeated application of this argument yields the following theorem.

**Theorem 3.4.** (Sankaran, unpublished) Let \( \Gamma \) be a strongly polycyclic group. Let \( \Gamma_i, 1 \leq i \leq n + 1 \), be as in (18). Then \( M(\Gamma) \) is orientable if each \( M(\Gamma_i) \) is orientable and the action of \( \Gamma_i/\Gamma_{i+1} \cong \mathbb{Z} \) on \( M(\Gamma_i) \) is orientation preserving for \( 1 \leq i \leq n \).

### 3.2. Homogeneous spaces for non-compact semisimple Lie groups.

Suppose that \( G \) is a connected non-compact semisimple Lie group with finite centre. Let \( K \) be a maximal compact subgroup of \( G \). Then it is a consequence of Cartan (or Iwasawa) decomposition that \( X := G/K \) is diffeomorphic to \( \mathbb{R}^n \) for some \( n \). So the vector field problem for \( G/K \) is uninteresting. Suppose that \( H \) is any connected compact subgroup of \( G \). Then \( H \) is contained in a maximal compact connected subgroup \( K \). One has a smooth fibre bundle \( G/H \to G/K \) with fibre \( K/H \). Since \( G/K \cong \mathbb{R}^n \), the bundle is trivial and we have \( G/H \cong \mathbb{R}^n \times K/H \). Therefore \( \text{span}(G/H) = \text{span}(\mathbb{R}^n \times K/H) = \text{span}^0(K/H) + n = \text{span}^0(G/H) \) since \( n \geq 1 \).

The manifold \( X = G/K \) admits a \( G \)-invariant metric with respect to which it becomes a globally symmetric space. One may express \( X \) as \( \bar{G}/\bar{K} \) where \( \bar{G} = G/Z(G) \) since the centre \( Z(G) \subset K \). Note that \( \bar{G} \) is a linear Lie group (via the adjoint representation). Thus we may assume, to begin with, that \( G \) itself is linear. Also, we will assume that \( G \) has no (nontrivial) compact connected normal subgroup \( N \). (Any compact normal group is contained in \( K \) and so \( X \cong (G/N)/(K/N) \). So there is no loss of generality in such an assumption.)

Let \( \Gamma \) be a uniform lattice in \( G \), that is, \( \Gamma \) is a discrete group subgroup of \( G \) such that \( X_\Gamma := \Gamma \backslash G/K \) is compact. We will assume that \( \Gamma \) is torsionless, that is, no element other than the identity has finite order. Then \( X_\Gamma \) is a smooth manifold and quotient map \( X \to X_\Gamma \) is a covering projection. The space \( X_\Gamma \) is called a locally symmetric space. Note that since \( G \) is connected, given any element \( g \in G \) the left translation by \( g \) on \( X \) is orientation preserving. Hence \( X_\Gamma \) is orientable.

In a more general setting, one allows \( \Gamma \) to be a (torsionless) lattice in the group \( I(X) \) of all isometries of \( X \). In this generality, a locally symmetric space \( X_\Gamma = I(X)/\Gamma \) is not necessarily orientable. The case considered above corresponds to the case where the lattice is contained in the identity component \( G = I_0(G) \) of \( I(X) \).

Before proceeding further, we pause for an example.

Let \( G = SL(2, \mathbb{R}) \), we have \( X = \mathcal{H} \), the upper half space \( \{ z = x + iy \in \mathbb{C} \mid y > 0 \} \) (with the Poincaré metric). If \( \Gamma \) is a uniform torsionless lattice in \( G \), then \( X_\Gamma \) is a compact Riemann surface of genus \( g \geq 2 \). By the uniformization theorem every compact Riemann
surface arises in this manner. Although $SL(2, \mathbb{Z})$ is a lattice in $SL(2, \mathbb{R})$, no finite index subgroup of it is uniform. Explicit construction of a uniform lattice in $SL(2, \mathbb{R})$ requires some preparation and will take us too far afield.

Borel [15] has shown that every (non-compact) semisimple Lie group $G$ admits both uniform and non-uniform lattices. If $G$ is linear, then any lattice in $G$ has a finite index subgroup $\Gamma$ which is torsionless so that $\Gamma \backslash G/K$ is a smooth manifold.

The globally symmetric space $X$ has a compact dual $X_\nu := U/K$ where $U$ is a maximal compact subgroup of the ‘complexification’ of $G$, denoted $G_C$, that contains $K$. The group $G_C$ is characterized by the requirements that its Lie algebra is the complexification $g_C = g \otimes \mathbb{R} \mathbb{C}$ and $G \subset G_C$. Such a group $G_C$ exists in view of our assumption that $G$ is linear. When $G = SL(n, \mathbb{R})$, we take $K = SO(n) \subset G$. Then $G_C = SL(n, \mathbb{C})$ and $U = SU(n)$, the special unitary group. Hence the compact dual of $X$ is $X_\nu = SU(n)/SO(n)$. We shall refer to $X_\nu$ also as the compact dual of a locally symmetric space $X_\Gamma$.

Returning to the general case of a compact locally symmetric space $X_\Gamma$, the well-known Hirzebruch proportionality principle says that the Pontrjagin numbers of $X_\Gamma$ are proportional to the corresponding Pontrjagin numbers of the compact dual $X_\nu$, the proportionality constant being dependent only on $X_\Gamma$. Thus vanishing of the latter implies the vanishing of the former. See [33]. What we need is a stronger version of the converse, namely: the non-vanishing of a characteristic class implies the non-vanishing of the corresponding characteristic class of $X_\Gamma$. This has been established by T. Kobayashi and K. Ono [41] in a more general setting. The following theorem and its proof is essentially due to Lafont and Ray [51], although they stated their result for characteristic numbers. The assertion concerning the Euler characteristic is well-known (cf. [29]).

**Theorem 3.5. ([41], [51, Theorem A].) With the above notation, if $\Gamma \subset G$ is a uniform lattice, then: (i) $\chi(X_\Gamma) = c\chi(X_\nu)$ for some $c \neq 0$.
(ii) If $\text{rank}(K) = \text{rank}(U)$, and if some Pontrjagin class $p_i(X_\nu) \neq 0$, then $p_i(X_\Gamma) \neq 0$.

**Proof.** (i). We may assume that $X_\Gamma$ is even-dimensional. If $\chi(X_\nu) = 0$, then the Euler class $e(X_\nu)$ of $X_\nu$ vanishes. Hence by [41] it follows that $e(X_\Gamma) = 0$ and so $\chi(X_\Gamma) = 0$. On the other hand, suppose that $\chi(X_\nu) \neq 0$. Then $\text{rank}(U) = \text{rank}(K)$. In this case, a result of Okun [63] says that there exist a finite index subgroup $\Lambda \subset \Gamma$ and tangential map $f : X_\Lambda \to X_\nu$ of non-zero degree. That is, $f^*(\tau X_\nu) \cong \tau X_\Lambda$ and $f_* : H_n(X_\nu; \mathbb{R}) \to H_n(X_\Lambda; \mathbb{R})$ is non-zero where $n = \dim X$. By the naturality of the Euler class, $f^*(e(X_\nu)) = e(X_\Lambda)$; see [59]. Now $\chi(X_\Lambda) = \langle e(X_\Gamma), \mu_{X_\Lambda} \rangle = \langle f^*(e(X_\nu)), \mu_{X_\Lambda} \rangle = \langle e(X_\nu), f_*(\mu_{X_\nu}) \rangle = \langle e(X_\nu), \deg(f) \mu_{X_\nu} \rangle = \deg(f) \chi(X_\nu) \neq 0$. This shows that $\chi(X_\Lambda) \neq 0$. Since $\Lambda$ has finite index in $\Gamma$, we have a covering projection $X_\Lambda \to X_\Gamma$ and so $\chi(X_\Lambda) = |\Gamma/\Lambda| \chi(X_\Gamma)$. Therefore $\chi(X_\Gamma) = \deg(f)\chi(X_\nu)/|\Gamma/\Lambda|$. This proves (i).

(ii). Suppose that $p_i(X_\nu) \neq 0$. We proceed as in (i) above and use the same notations. Note that the tangent bundle of $X_\Gamma$ pulls back under the covering projection to that of $X_\Lambda$. By the naturality of Pontrjagin classes, it suffices to show that $p_i(X_\Lambda) \neq 0$. Since $f : X_\Lambda \to X_\nu$ is tangential $f^*(p_i(X_\nu)) = p_i(X_\Lambda)$. Since $\deg(f) \neq 0$, the induced
map in rational cohomology \( f^*: H^*(X_u; \mathbb{Q}) \to H^*(X_\Lambda; \mathbb{Q}) \) is a monomorphism. Hence \( p_1(X_\Lambda) \neq 0 \).

The group \( G_C \) is also the complexification of \( U \). In particular \( U \) is semisimple. The rank of \( U \) is also called the (complex) rank of \( G \). However \( K \subset G \) is not necessarily semisimple. For example, when \( G = SL(2, \mathbb{R}) \), \( K = SO(2) \) is abelian. When the centre of \( K \) is not discrete and \( G \) is simple, the homogeneous space \( X = G/K \) has the structure of a Hermitian symmetric domain. Also the compact dual \( X_u = U/K \) has the structure of a complex projective variety. When \( G \) is semisimple it is an almost direct product \( G = G_1 \cdots G_k \) where each \( G_i \) is a simple normal subgroup of \( G \). By our assumption, none of the \( G_i \) is compact. Any maximal compact subgroup \( K \) is likewise an almost product \( K = K_1 \cdots K_k \) where \( K_i \subset G_i \) is a maximal compact subgroup of \( G_i \). Moreover \( X = G/K \) is diffeomorphic to the Cartesian product \( X_1 \times \cdots \times X_k \) where \( X_i = G_i/K_i \). The \( X_i \) are called the irreducible factors of \( X \). Correspondingly, one has a factorization \( X_u \) of the compact dual into a product \( X_{1,u} \times \cdots \times X_{k,u} \) where \( X_{i,u} = U_i/K_i \), \( U_i \) being the maximal compact subgroup of \( G_{i,C} \) that contains \( K_i \).

As an application of Theorem 2.5 we obtain the following result.

**Theorem 3.6.** (Sankaran, unpublished.) Let \( \Gamma \) be a torsionless uniform lattice in a linear connected semisimple Lie group \( G \). With the above notations, (i) \( \text{span}(X_\Gamma) > 0 \) if and only if \( \text{rank}(G) > \text{rank}(K) \).

(ii) Suppose that an irreducible factor \( X_i \) of \( X \) is a Hermitian symmetric space where \( G_i \) is not locally isomorphic to \( SL(2, \mathbb{R}) \). Then \( X_\Gamma \) is not stably parallelizable.

(iii) Suppose that each simple factor of \( G \) is either a complex Lie group or is locally isomorphic to \( SO_0(1,k) \). Then there exists a finite index subgroup \( \Lambda \subset \Gamma \) such that \( X_\Lambda \) is stably parallelizable. Such an \( X_\Lambda \) is parallelizable if and only if \( \text{rank}(G) > \text{rank}(K) \).

**Proof.** (i) This is a direct consequence of Theorem 3.5(i), since \( \chi(X_\Gamma) = 0 \) if and only if \( \chi(U/K) = 0 \) if and only if \( \text{rank}(G) > \text{rank}(K) \).

(ii). Consider the factor \( X_{i,u} = U_i/K_i \). The assumption that \( G_i \) is not locally isomorphic to \( SL(2, \mathbb{R}) \) implies that \( K_i \) is neither semisimple nor a torus. Since \( U_i \) is simple, by Theorem 2.5, we see that \( p_1(X_{i,u}) \neq 0 \). It follows that \( p_1(X_u) \neq 0 \) since \( X_{i,u} \) is a direct factor of \( X_u \). By Theorem 3.5(ii), it follows that \( p_1(X_\Gamma) \neq 0 \).

(iii). Let \( H \) be a simple factor of \( G \) with \( L \subset H \) being a maximal compact subgroup. Let \( Y = H/L \) and \( Y_u \) its compact dual. When \( H \) is a simple complex Lie group with maximal compact subgroup \( L \), its complexification is the product \( H \times H \) with maximal compact subgroup \( L \times L \). Hence the compact dual of \( Y \) is the homogeneous space \( L \times L/L \) where \( L \) is embedded diagonally. Consequently \( Y_u \) is diffeomorphic to the Lie group \( L \) and hence is parallelizable. When \( H \) is locally isomorphic to \( SO_0(1,k) \), the symmetric space \( Y = H/L \) is the hyperbolic space \( \mathcal{H}^k \) and its compact dual is the sphere \( S^k \). Our hypothesis implies that \( X_u \) is a product of spheres and a Lie group \( M \) (possibly trivial). Thus \( X_u \) is stably parallelizable. By Okun’s theorem [63], there exists a finite index subgroup \( \Lambda \subset \Gamma \) such that there exists a tangential map \( f : X_\Lambda \to X_u \); thus \( f^*(\tau(X_u)) \cong \tau(X_\Lambda) \). It follows
that $X_A$ is stably parallelizable and that it is parallelizable if and only if either one of the spheres is odd dimensional or $M$ is positive dimensional (see Theorems 1.19, 1.20). The last condition is equivalent to $\chi(X_u) = 0$, which is itself equivalent to the requirement that $\text{rank}(G) > \text{rank}(K)$. □

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