A class of Calogero type reductions of free motion on a simple Lie group

L. FEHÉR\textsuperscript{a} and B.G. PUSZTAI\textsuperscript{b}

\textsuperscript{a}Department of Theoretical Physics, MTA KFKI RMKI
1525 Budapest 114, P.O.B. 49, Hungary, and
Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
e-mail: lfeher@rmki.kfki.hu

\textsuperscript{b}Centre de recherches mathématiques, Université de Montréal
C.P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7, and
Department of Mathematics and Statistics, Concordia University
1455 de Maisonneuve Blvd. West, Montréal, Québec, Canada H3G 1M8
e-mail: pusztai@CRM.UMontreal.CA

Abstract

The reductions of the free geodesic motion on a non-compact simple Lie group $G$ based on the $G_+ \times G_+$ symmetry given by left- and right-multiplications for a maximal compact subgroup $G_+ \subset G$ are investigated. At generic values of the momentum map this leads to (new) spin Calogero type models. At some special values the ‘spin’ degrees of freedom are absent and we obtain the standard $BC_n$ Sutherland model with three independent coupling constants from $SU(n+1,n)$ and from $SU(n,n)$. This generalization of the Olshanetsky-Perelomov derivation of the $BC_n$ model with two independent coupling constants from the geodesics on $G/G_+$ with $G = SU(n+1,n)$ relies on fixing the right-handed momentum to a non-zero character of $G_+$. The reductions considered permit further generalizations and work at the quantized level, too, for non-compact as well as for compact $G$.

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1 Introduction

The ‘Calogero type’ integrable models of interacting particles on the line are interesting on account of their physical applications and relationships to important fields of mathematics. Generalizations of the original model [1] can be associated with root systems in correspondence with various admissible interaction potentials and possible couplings to internal ‘spin’ degrees of freedom and to external fields. The richness of these models is demonstrated by the growing number of reviews devoted to them [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. One of the basic models of the family is the hyperbolic $BC_n$ Sutherland model defined classically by the Hamiltonian

$$
H_{BC_n} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \sum_{1 \leq j < k \leq n} \left( \frac{g^2}{\sinh^2(q_j - q_k)} + \frac{g_1^2}{\sinh^2(q_j + q_k)} \right) + \sum_{k=1}^{n} \left( \frac{g_1^2}{\sinh^2(q_k)} + \frac{g_2^2}{\sinh^2(2q_k)} \right)
$$  (1.1)

with arbitrary coupling constants $g, g_1, g_2$. Olshanetsky and Perelomov [14, 15, 2] showed that this model can be viewed as a ‘projection’ of the geodesic system on the symmetric space $SU(n+1, n)/(S(U(n+1) \times U(n))$ if the coupling constants obey the quadratic relation $g_1^2 - 2g^2 + \sqrt{2}gg_2 = 0$. For arbitrary coupling constants, classical and quantum solvability of the model was established by means of different, rather algebraic, methods [16, 17, 18, 19].

Since Hamiltonian reduction is a very effective and general approach to integrable systems, it would be interesting to lift the above quadratic relation of Olshanetsky and Perelomov sticking to this method. Motivated partly by this problem, recently we undertook a systematic study of reductions of the free geodesic motion on Riemannian symmetric spaces, which led to new spin Calogero models as well as to an understanding of the geometric origin of the quadratic relation [20]. Here, we extend this work by going one stage up and explore the reductions of the geodesic system defined on the isometry group of the symmetric space. We shall demonstrate that the classical $BC_n$ model (1.1) with three independent coupling constants can be obtained by Hamiltonian reduction in this extended framework.

The geodesic system on a symmetric space, realized as a coset space $G/G_+$, is a reduction of the geodesic system on the isometry group $G$, belonging to the zero value of the momentum map for the action of the little group $G_+$ on $T^*G$ generated by right-multiplications. This system then can be reduced to spin Calogero models using the residual symmetry generated by the left-multiplications associated with $G_+$. It is clear that more general reduced systems result if one fixes the right-handed momentum to some non-zero value. First, we shall describe the most general reductions of $T^*G$ that rely on the action of $G_+ \times G_+$ through left- and right multiplications. In fact, one obtains (new) spin Calogero type models in general, with the spin degrees of freedom restricted to a trivial one-point space in certain very special cases. Second, we observe that if the space of spin degrees of freedom is trivial for the zero value of the ‘right-handed’ momentum map, then this feature can be ensured also for any non-zero character (one-point coadjoint orbit) of $G_+$. Taking advantage of this observation, we can derive the $BC_n$ model with three independent coupling constants from the geodesic motion on $SU(m,n)$ both for $m = n$ and for $m = n + 1$. The model with two independent coupling constants is obtained from $SU(m,n)$ for any $m \geq (n + 2)$.

The main results of this letter are the characterization of the reductions of the geodesic system on a real simple Lie group $G$ under the $G_+ \times G_+$ symmetry presented in Section 2, where $G$ is non-compact and $G_+$ is a maximal compact subgroup, and the derivation of the model
Our derivation of the classical $BC_n$ model should be compared with the work of Oblomkov [21] treating the quantum mechanical trigonometric $BC_n$ model, in effect, by quantum Hamiltonian reduction. See also Section 4 for further discussion.

2 From free motion to spin Calogero type models

Next we briefly recall some group theoretic background material and introduce our notations, then describe the Hamiltonian reductions of the free particle on a Lie group to spin Calogero type models in a convenient framework. The relevant Lie theoretic results are treated in detail in [22, 23], and we refer to [4, 24] for reviews of symplectic geometry and Hamiltonian reduction.

Let $G$ be a non-compact real simple Lie group with finite centre and $\mathfrak{g}$ its Lie algebra. Up to conjugation there exists a unique Cartan involution $\Theta$ of $G$, for which the associated automorphism $\theta$ of $G$ induces the decomposition

$$G = G_+ + G_-,$$

where the restriction of the Killing form $\langle , \rangle$ of $G$ is negative (resp. positive) definite on $G_+$ (resp. on $G_-$. The fixed point set of $\Theta$ is a maximal compact subgroup $G_+ \subset G$ with Lie algebra $\mathfrak{g}_+$. The elements of $\mathfrak{g}_-$ are diagonalizable, with real eigenvalues, in the adjoint representation of $G$ and it is useful to fix a maximal Abelian subspace $A \subset \mathfrak{g}_-$. The choice of $A$ leads to the refined decomposition

$$G_- = A + A^\perp, \quad G_+ = \mathcal{M} + \mathcal{M}^\perp,$$

with

$$\mathcal{M} := \{ X \in \mathfrak{g}_+ | [X,Y] = 0 \ \forall Y \in A \}$$

(2.3)

and the complementary spaces $A^\perp, \mathcal{M}^\perp$ defined with the aid of $\langle , \rangle$. We may write any $X \in \mathfrak{g}$ as $X = X_- + X_+ = X_A + X_{A^\perp} + X_M + X_{M^\perp}$ according to (2.1) and (2.2). We also need the group corresponding to $\mathcal{M}$, the centralizer of $A$ in $G_+$,

$$M := \{ m \in G_+ | m Y m^{-1} = Y \ \forall Y \in A \}.$$  

(2.4)

We remind in passing that the Weyl group of the Riemannian symmetric space $G/G_+$ is $W := \hat{M}/M$, where $\hat{M}$ is the normalizer of $A$ in $G_+$.

Let us call an element of $A$ regular if its kernel in the adjoint representation of $G$ is $A + \mathcal{M}$. The set of regular elements, denoted as $\hat{A} \subset A$, is the union of its connected components and we choose an open Weyl chamber $\hat{A} \subset \hat{A}$ to be such a connected component. The regular elements of $G$ can be characterized by admitting a decomposition of the form

$$g = g_+ e^q h_+ \quad q \in \hat{A}, \quad g_+, h_+ \in G_+,$$

(2.5)

and this decomposition is unique up to replacing $(g_+, h_+)$ by $(g_+ m, m^{-1} h_+)$ for any $m \in M$.

Denote by $\hat{G} \subset G$ the open dense submanifold formed by the regular elements. From now on identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the Killing form $\langle , \rangle$. Then, using the trivialization defined by right-translations on $G$, consider the cotangent bundle $T^*\hat{G}$,

$$P := T^*\hat{G} \cong \hat{G} \times \mathfrak{g} = \{ (g, J) | g \in \hat{G}, \ J \in \mathfrak{g} \},$$

(2.6)

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equipped with the symplectic form $\Omega$ and the Hamiltonian $\mathcal{H}$ of the geodesic system,

$$\Omega = d(J^l, (dg)^{-1}), \quad \mathcal{H}(g, J^l) = \frac{1}{2}(J^l, J^l). \quad (2.7)$$

$J^l$ generates the left-translations and $J^r = -g^{-1}J^l g$ generates the right-translations on $T^*G$. We shall perform Hamiltonian symmetry reduction relying on the subgroup $G_+ \times G_+$ of $G \times G$.

To study the reductions of the geodesic system, it is convenient to first extend it as follows. Take two arbitrary coadjoint orbits of $G_+$, say $(\mathcal{O}^l, \omega^l)$ and $(\mathcal{O}^r, \omega^r)$. The orbits are realized as submanifolds of $\mathcal{G}_+ \simeq \mathcal{G}_+^*$ and $\omega^{l,r}$ denote their own symplectic forms. The extended system is $(P^{\text{ext}}, \mathcal{H}^{\text{ext}}, \Omega^{\text{ext}})$:

$$P^{\text{ext}} := P \times \mathcal{O}^l \times \mathcal{O}^r = \{(g, J^l, \xi^l, \xi^r) \mid g \in \mathcal{G}, J^l \in \mathcal{G}, \xi^l \in \mathcal{O}^l, \xi^r \in \mathcal{O}^r\}, \quad (2.8)$$

$$\Omega^{\text{ext}} := \Omega + \omega^l + \omega^r, \quad \mathcal{H}^{\text{ext}}(g, J^l, \xi^l, \xi^r) := \mathcal{H}(g, J^l). \quad (2.9)$$

Using the Poisson bracket associated with $\Omega^{\text{ext}}$, the corresponding equation of motion reads

$$\dot{g} = \{g, \mathcal{H}^{\text{ext}}\} = J^l g, \quad \dot{J}^l = \{J^l, \mathcal{H}^{\text{ext}}\} = 0, \quad \dot{\xi}^\lambda = \{\xi^\lambda, H^{\text{ext}}\} = 0 \quad \text{for} \quad \lambda = l, r. \quad (2.10)$$

The solution with initial value $(g(0), J^l, \xi^l, \xi^r)$ yields the geodesic $g(t) = e^{tJ^l}g(0)$.

Now we consider the reduction of the above system based on the symmetry group $G_+ \times G_+$. Any $(g^l_+, g^+_r) \in G_+ \times G_+$ operates by the transformation $T(g^l_+, g^+_r) \in \text{Diff}(P^{\text{ext}})$ defined by

$$T(g^l_+, g^+_r) : (g, J^l, \xi^l, \xi^r) \mapsto (g^l_+ g(g^+_r)^{-1}, g^l_+ J^l (g^+_r)^{-1}, g^l_+ \xi^l (g^+_r)^{-1}, g^l_+ \xi^r (g^+_r)^{-1}). \quad (2.11)$$

The equivariant momentum map, $\Psi = (\Psi^l, \Psi^r) : P^{\text{ext}} \to \mathcal{G}_+^* \oplus \mathcal{G}_+^*$, for this Hamiltonian action is furnished by

$$\Psi(g, J^l, \xi^l, \xi^r) = (J^l_+ + \xi^l, -(g^{-1}J^l g)_+ + \xi^r), \quad (2.12)$$

where the factors $\mathcal{G}_+^*$ are identified with $\mathcal{G}_+$ using the scalar product, the elements of $\mathcal{G}_+ \oplus \mathcal{G}_+$ are denoted as ordered pairs, and $J^l = J^l_+ + J^l_-$ according to (2.1). We are interested in the reduced Hamiltonian system $(P_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}})$ obtained from $(P^{\text{ext}}, \Omega^{\text{ext}}, \mathcal{H}^{\text{ext}})$ at the zero value of the momentum map $\Psi$, i.e.,

$$P_{\text{red}} := P^{\text{ext}}_{\Psi = 0}/(G_+ \times G_+). \quad (2.13)$$

It is easy to see that this is equivalent to the (singular) Marsden-Weinstein reduction [24] of the original system $(P, \Omega, \mathcal{H})$ at an arbitrary value $(-\mu^l, -\mu^r)$ of the corresponding momentum maps, $(J^l_-, J^l_+)$, with $\mu \in \mathcal{O}^l, \mu^r \in \mathcal{O}^r$. We assume in what follows that $P^{\text{ext}}_{\Psi = 0}$ is non-empty, which is a condition on the orbits $\mathcal{O}^l, \mathcal{O}^r$. In fact, the condition that $\Psi(g, J^l, \xi^l, \xi^r) = 0$ admits a solution on $P^{\text{ext}}$ is equivalent to the consistency of (2.13) below for some $\xi^l \in \mathcal{O}^l$ and $\xi^r \in \mathcal{O}^r$.

Now we are ready to characterize the reduced Hamiltonian system defined above. The key step is to utilize that all $G_+ \times G_+$ orbits in the constrained manifold $P^{\text{ext}}_{\Psi = 0}$ intersect the following 'gauge slice':

$$S := \{(e^q, J^l, \xi^l, \xi^r) \in P^{\text{ext}}_{\Psi = 0} \mid q \in \mathcal{A}\}, \quad (2.14)$$

since every regular element of $G$ can be transformed into $\exp(\mathcal{A})$ by means of the action (2.11). The gauge slice $S$ represents only a partial gauge fixing of the gauge transformations defined
by the $G_+ \times G_+$ action \textbf{(2.11)}. The residual gauge transformations (the maps that transform an arbitrarily chosen point of $S$ into $S$) are generated by the subgroup

$$M_{\text{diag}} := \{(m,m) \in G_+ \times G_+ \mid m \in M\}. \tag{2.15}$$

$M_{\text{diag}}$ is naturally isomorphic to, and is below often identified with, $M$. At this point we arrived at the model

$$P_{\text{red}} = P_{\Psi=0}/(G_+ \times G_+) = S/M_{\text{diag}}. \tag{2.16}$$

To describe $P_{\text{red}}$ more explicitly, we use the orthogonal complement of the Lie algebra $M_{\text{diag}} \subset G_+ \oplus G_+$ of $M_{\text{diag}},$

$$M_{\text{diag}}^+ = \{(X_1, X_2) \in G_+ \oplus G_+ \mid \langle X_1 + X_2, V \rangle = 0 \ \forall V \in \mathcal{M}\}, \tag{2.17}$$

with respect to the scalar product $\langle (X_1, X_2), (Y_1, Y_2) \rangle_+ = \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle$ on $G_+ \oplus G_+$. By decomposing $J^l \in \mathcal{G}$ and $\xi^\lambda \in O^\lambda \subset G_+(\lambda = l, r)$ according to \textbf{(2.2)},

$$J^l = J^l_A + J^l_{A^\perp} + J^l_M + J^l_{M^\perp}, \quad \xi^\lambda = \xi^\lambda_M + \xi^\lambda_{M^\perp}, \tag{2.18}$$

and using that $\text{ad}_q (\forall q \in A)$ yields a linear bijection between $M^\perp$ and $A^\perp$, the constraint $\Psi = 0$ on $S$ can be solved as follows. In fact, the condition $\Psi = 0$ on $S$ is equivalent to the equations

$$\xi^\lambda_M + \xi^r_M = 0, \tag{2.19}$$

$$J^l = J^l_A - F(\text{ad}_q)\xi^l_{M^\perp} - w(\text{ad}_q)\xi^l_{M^\perp} - \xi^l, \tag{2.20}$$

where $J^l_A \in A$ is arbitrary and $F, w$ are the analytic functions

$$F(z) = \coth z, \quad w(z) = \frac{1}{\sinh z}. \tag{2.21}$$

Equation \textbf{(2.19)} ensures that $(\xi^l, \xi^r) \in M_{\text{diag}}^+$. Motivated by the parametrization \textbf{(2.20)}, let us introduce the smooth one-to-one map $I : (\hat{A} \times A) \times (O^l \oplus O^r) \cap M_{\text{diag}} \rightarrow S$ by

$$I(q, p, \xi^l, \xi^r) := (e^q, \mathcal{L}(q, p, \xi^l, \xi^r), \xi^l, \xi^r), \quad \mathcal{L}(q, p, \xi^l, \xi^r) := p - F(\text{ad}_q)\xi^l_{M^\perp} - w(\text{ad}_q)\xi^l_{M^\perp} - \xi^l. \tag{2.22}$$

The pull-back of $\Omega^{\text{ext}}|_S$ by $I$, where $\Omega^{\text{ext}}|_S$ is the pull-back of $\Omega^{\text{ext}}$ to the submanifold $S \subset P^{\text{ext}}$, turns out to be

$$I^*(\Omega^{\text{ext}}|_S) = d\langle p, dq \rangle + (\omega^l + \omega^r) \mid_{(O^l \oplus O^r) \cap M_{\text{diag}}^+}. \tag{2.23}$$

The first term is the canonical symplectic structure of $T^*\hat{A} \simeq \hat{A} \times A = \{(q, p)\}$. The second term in \textbf{(2.23)} is the restriction of $\omega^l + \omega^r$ to the zero level set of the momentum map for the action of the group $M \simeq M_{\text{diag}}$ on $O^l \oplus O^r$, provided by $(\xi^l, \xi^r) \mapsto (\xi^l_M + \xi^r_M) \in \mathcal{M} \simeq \mathcal{M}^*$. Notice that $I$ is an $M$ equivariant map, where $M$ acts trivially on $T^*\hat{A}$. On account of its equivariance, the map $I$ gives rise to the identification $S/M_{\text{diag}} = T^*\hat{A} \times O_{\text{red}}$ with

$$O_{\text{red}} := (O^l \oplus O^r) \cap M_{\text{diag}}^+/M_{\text{diag}}. \tag{2.24}$$
In terms of the model of $S$ provided by the map $I$ (2.22), the Hamiltonian of the geodesic motion takes the form

$$\{(\mathcal{H}^\text{ext} \circ I)(q,p,\xi^l,\xi^r) = \frac{1}{2}\langle \mathcal{L}(q,p,\xi^l,\xi^r), \mathcal{L}(q,p,\xi^l,\xi^r) \rangle. \quad (2.25)$$

By collecting the above formulae and spelling out the Hamiltonian with the aid of the identity $F(z)w(z) = \frac{1}{2}w^2(\hat{z}) - w^2(z)$, we obtain our

**Main result:** The reduced geodesic system $(P_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}})$ defined above can be identified as

$$P_{\text{red}} = T^*\tilde{A} \times \mathcal{O}_{\text{red}}, \quad \Omega_{\text{red}} = d[p,dq] + \omega_{\text{red}}, \quad (2.26)$$

where $q,p$ are the natural variables on $T^*\tilde{A}$ and $(\mathcal{O}_{\text{red}}, \omega_{\text{red}})$ (2.24) is the symplectic reduction of $\mathcal{O}^l \oplus \mathcal{O}^r$ by the subgroup $M_{\text{diag}} \subset G_+ \times G_+$ at the zero value of its momentum map. The reduced Hamiltonian yields a hyperbolic spin Calogero type model in general, since as an $M$ invariant function on $T^*\tilde{A} \times (\mathcal{O}^l \oplus \mathcal{O}^r) \cap M_{\text{diag}}$ it has the form

$$\mathcal{H}_{\text{red}}(q,p,\xi^l,\xi^r) = \frac{1}{2}\langle p, p \rangle - \frac{1}{2}\langle \xi^l_{\mathcal{M}^+}, w^2(\text{ad}_q)\xi^l_{\mathcal{M}^+} \rangle - \frac{1}{2}\langle \xi^r_{\mathcal{M}^+}, w^2(\text{ad}_q)\xi^r_{\mathcal{M}^+} \rangle + \frac{1}{2}\langle \xi^l_{\mathcal{M}^+}, \xi^l_{\mathcal{M}^+} \rangle + \langle \xi^r_{\mathcal{M}^+}, w^2(\text{ad}_q)\xi^r_{\mathcal{M}^+} \rangle - \frac{1}{2}\langle \xi^r_{\mathcal{M}^+}, w^2(\text{ad}_q)\xi^l_{\mathcal{M}^+} \rangle, \quad (2.27)$$

where $w(z) = \frac{1}{\sinh z}$ and $\xi^l_{\mathcal{M}^+} + \xi^r_{\mathcal{M}^+} = 0$.

Now some remarks are in order. First, note that our spin Calogero models enjoy Weyl group symmetry similarly to the standard Calogero type models. This symmetry is not explicit in the above since the Weyl chambers are permuted by the Weyl group $M$ and we have gauge fixed the coordinate variable $q$ to a single chamber $\tilde{A}$. However, we could have used in our derivation the larger gauge slice, $\tilde{S}$, which differs from $S$ (2.14) only in that $q$ runs over the full set of regular elements $\tilde{A} \subset A$. The corresponding residual gauge transformations belong to the normalizer $\tilde{M}$, and it is easily seen that $P_{\text{red}} = S/M = \tilde{S}/\tilde{M} = \tilde{P}_{\text{red}}/W$ with $\tilde{P}_{\text{red}} := \tilde{S}/M = T^*\tilde{A} \times \mathcal{O}_{\text{red}}$. The point is that the spin Calogero model defined on $\tilde{P}_{\text{red}}$ is invariant with respect to the natural action of $W = \tilde{M}/M$ induced by the action of $\tilde{M} \simeq M_{\text{diag}} \subset G_+ \times G_+$ on $\tilde{S}$.

The structure of the reduced phase space described above is consistent with general results on reduced cotangent bundles derived in [25] under the assumption that only one isotropy type appears for the action of the symmetry group on the configuration space. Indeed, the isotropy group of any element (2.14) of $\tilde{G}$ is conjugate to $M_{\text{diag}}$ for the action of $G_+ \times G_+$. Note that $\mathcal{O}_{\text{red}}$ (2.24) is not a smooth manifold in general. This does not cause any difficulty, since one can define the smooth functions on $P_{\text{red}}$ (2.26) to be the smooth, gauge invariant functions on $P_{\text{red}}$. For a review of singular symplectic reduction, see [24].

The solutions of the reduced system $(P_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}})$ can be obtained algebraically, by projecting the obvious solution curves of $(P_{\text{ext}}^{{\Psi}^\text{ext}}, \Omega_{\text{ext}}^{{\Psi}^\text{ext}}, \mathcal{H}_{\text{ext}}^{{\Psi}^\text{ext}})$ (2.10) that satisfy the constraint $\Psi = 0$. All spin Calogero models that arise by reduction are integrable in this direct sense. These models naturally possess many constants of motion, too. Indeed, $J^\lambda$ and $\xi^\lambda$ ($\lambda = l,r$) are conserved quantities for the dynamics (2.10) on $P_{\text{ext}}$, and any combination of them that is invariant with respect to the $G_+ \times G_+$ symmetry transformations (2.11) induces a constant of motion for the reduced system. For example, consider the function on $P_{\text{red}}$ induced by

$$h(K^\lambda(v)) \quad \text{with} \quad K^\lambda(v) := J^\lambda - v\xi^\lambda, \quad (2.28)$$
where $v$ is any real parameter, $\lambda \in \{l, r\}$, and $h$ is a $G$ invariant real function on $G$. A straightforward calculation, similar to Section 4 in [20], shows that all constants of motion of the form (2.28) are in involution on $P_{\text{red}}$. The Liouville integrability of the reduced systems could be shown starting from these remarks.

If desired, one may also construct Lax pairs as follows. Let $\sigma \subseteq S$ (2.14) denote a gauge slice (of a partial or complete gauge fixing) and for any $v \in \mathbb{R}$ define $L^\lambda(v) : \sigma \to G$ by

$$L^\lambda(v) := K^\lambda|_\sigma.$$  \hspace{1cm} (2.29)

With respect to the projection of the Hamiltonian vector field (2.10) to $\sigma$, $L^\lambda(v)$ is found to satisfy a Lax equation

$$\dot{L}^\lambda(v) = [Y^\lambda, L^\lambda(v)], \quad \lambda = l, r.$$  \hspace{1cm} (2.30)

In fact, proceeding like in [20] we find that

$$Y^l = Y_M + \frac{1}{2} \xi^l M - w^2 (\text{ad}_q)\xi^l M - (wF)(\text{ad}_q)\xi^r M,$$

$$Y^r = Y_M + \frac{1}{2} \xi^r M - w^2 (\text{ad}_q)\xi^r M - (wF)(\text{ad}_q)\xi^l M,$$  \hspace{1cm} (2.31)

where $w, F$ appear in (2.21) and $Y_M \in \mathcal{M}$ can be determined by the consistency of the gauge fixing conditions imposed on $\sigma$. Equation (2.10) also implies that $\dot{q} = p$ and by using this one can verify that the two Lax equations in (2.30) are actually equivalent to each other.

3 Spinless $BC_n$ Sutherland models from $SU(m, n)$

Let us begin by noting that the symmetry reductions based on $G \times G$ can be implemented also as a two step process, say imposing first the momentum map constraint on $J^l$. If one chooses $\mathcal{O} = \{0\}$ in this first step, then one obtains the geodesic system on the symmetric space $G/G^+$, which is subsequently reduced in the second step imposing the constraint on $J^l$. The $\mathcal{O} = \{0\}$ special case of the result given by (2.26), (2.27) reproduces a result in [20], where we studied the reductions of the geodesic system on $G/G^+$ taking an arbitrary orbit for $\mathcal{O}$. In this reference we also examined the cases for which the reduced phase space is isomorphic to $T^*\tilde{A}$, which means that the reduced system gives a spinless Calogero model. Next we outline a mechanism whereby the models obtained in [20] can be deformed whenever $G^+$ admits a one-point coadjoint orbit consisting of a non-zero (infinitesimal) character.

Let $C \in G^+_\perp \simeq G_+$ be a non-zero character, i.e., an element invariant under conjugation by $G_+$. Starting from $\mathcal{O}_{\text{red}}$ (2.24), we can define a shifted space of spin degrees of freedom by

$$\mathcal{O}^y_{\text{red}} := \left((\mathcal{O}^l - yC) \oplus (\mathcal{O}^r + yC)\right) \cap \mathcal{M}_{\text{diag}}^+ / \mathcal{M}_{\text{diag}}, \quad \forall y \in \mathbb{R},$$  \hspace{1cm} (3.1)

where $(\mathcal{O}^l - yC)$ and $(\mathcal{O}^l + yC)$ are one parameter families of coadjoint orbits of $G_+$. This is possible since the constraint (2.19) is invariant under replacing $(\xi^l, \xi^r)$ by $(\xi^l - yC, \xi^r + yC)$. A crucial point to notice is that if $\mathcal{O}_{\text{red}} = \mathcal{O}^y_{\text{red}}$ is a one-point space, then this feature holds for any $y \in \mathbb{R}$ with the reduced Hamiltonian $H_{\text{red}}$ (2.27) acquiring a dependence on the ‘deformation parameter’ $y$. It is well-known [22, 23] that non-trivial characters exist if and only if $G/G_+$ is a
Hermitian symmetric space, which holds for example if $G = SU(m, n)$, and in these cases the space of characters is one-dimensional.

In [20] we explained that one-point reduced orbits (2.24) with $O^r = \{0\}$ result if one takes $G = SU(m, n)$ and chooses $O^l$ in a very special manner utilizing minimal coadjoint orbits of an $SU(k)$ factor of $G_+$. This is the essential point behind the derivation of the $BC_n$ Sutherland model from the geodesic system of the symmetric space of $SU(n + 1, n)$ due to Olshanetsky and Perelomov [14, 15, 2]. However, the three coupling constants of the model resulting from their procedure are necessarily subject to a quadratic relation. Here, we utilize the one parameter family of characters of $G_+$ to increase the number of independent coupling constants in the reduced Hamiltonian by one. In fact, we show below that in this way the classical $BC_n$ Sutherland model with three independent coupling constants can be obtained as a reduction of the geodesics on $SU(n, n)$ and on $SU(n + 1, n)$.

We need some further notations. Consider the joint eigensubspaces of the elements of $\mathcal{A}$,

$$\mathcal{G}_\alpha := \{ X \in \mathcal{G} \mid [Y, X] = \alpha(Y)X \ \forall Y \in \mathcal{A} \}. \quad (3.2)$$

The linear functions $\alpha \in A^* \setminus \{0\}$ with $\dim(\mathcal{G}_\alpha) \neq 0$ are called restricted roots. They form a crystallographic root system, denoted by $R$. The subspaces in (2.2) satisfy $\mathcal{M}^\perp + \mathcal{A}^\perp = \bigoplus_{\alpha \in R} \mathcal{G}_\alpha$. We fix a polarization $R = R_+ \cup R_-$ and choose $E^a_\alpha \in \mathcal{G}_\alpha \ (a = 1, \ldots, \nu_a := \dim(\mathcal{G}_\alpha))$ so that

$$\theta(E^a_\alpha) = -E_{-a}^a, \quad \langle E^a_\alpha, E^b_\beta \rangle = \delta_a, -\beta \delta_{ab}. \quad (3.3)$$

Then $\mathcal{M}^\perp$ and $\mathcal{A}^\perp$ are spanned by

$$E^+, a = \frac{1}{\sqrt{2}}(E^a_\alpha + \theta(E^a_\alpha)) \in \mathcal{M}^\perp, \quad E^- a = \frac{1}{\sqrt{2}}(E^a_\alpha - \theta(E^a_\alpha)) \in \mathcal{A}^\perp \quad \forall \alpha \in R_+. \quad (3.4)$$

Let us now focus on $SU(m, n)$ and its Lie algebra $su(m, n)$, given by

$$SU(m, n) = \{ g \in SL(m + n, \mathbb{C}) \mid g^\dagger I_{m,n} g = I_{m,n} \}, \quad (3.5)$$

$$su(m, n) = \{ X \in sl(m + n, \mathbb{C}) \mid X^\dagger I_{m,n} + I_{m,n} X = 0 \}, \quad (3.6)$$

where $I_{m,n} := \text{diag}(1_m, -1_n) \ (m \geq n)$ and $1_k \ (k = m, n)$ is the $k \times k$ identity matrix. A block matrix $X \in \mathcal{G} = su(m, n)$ reads

$$X = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}, \quad (3.7)$$

where $B \in \mathbb{C}^{m \times n}, \ A \in u(m), \ D \in u(n)$ and $\text{tr} A + \text{tr} D = 0$. The Cartan involution of $G = SU(m, n)$ is $\Theta : g \mapsto (g^\dagger)^{-1}$. Thus

$$G_+ = S(U(m) \times U(n)), \quad (3.8)$$

$$\mathcal{G}_+ = su(m) \oplus su(n) \oplus \mathbb{R} C_{m,n} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + x C_{m,n} \ \mid \ A \in su(m), \ D \in su(n), \ x \in \mathbb{R} \right\} \quad (3.9)$$

with the central element

$$C_{m,n} := \text{diag}(in1_m, -im1_n), \quad (3.10)$$
which spans the space of characters. A maximal Abelian subspace of \( \mathcal{G}_- \) is furnished by

\[
\mathcal{A} := \left\{ q := \begin{pmatrix} 0_n & 0 & Q \\ 0 & 0 & 0 \\ Q & 0 & 0_n \end{pmatrix} \left| Q = \text{diag}(q^1, \ldots, q^n), \; q^j \in \mathbb{R} \right. \right\}.
\]

(3.11)

Taking \( \chi := \text{diag}(\chi_1, \ldots, \chi_n) \) with any \( \chi_j \in \mathbb{R} \), the centralizer of \( \mathcal{A} \) in \( \mathcal{G}_+ \) is

\[
\mathcal{M} = \{ \text{diag}(i\chi, \gamma, i\chi) \mid \gamma \in u(m - n), \; \text{tr } \gamma + 2i\text{tr } \chi = 0 \},
\]

(3.12)

and the subgroup \( M = G_+ \) is

\[
M = \{ \text{diag}(e^{i\chi}, \Gamma, e^{i\chi}) \mid \Gamma \in U(m - n), \; (\det \Gamma)(\det e^{2i\chi}) = 1 \}.
\]

(3.13)

One may define the functionals \( e_k \in \mathcal{A}^* \) \((k = 1, \ldots, n)\) by \( e_k(q) := q^k \). The system of restricted roots is of \( BC_n \) type if \( m > n \) and of \( C_n \) type if \( m = n \). Indeed, we have

\[
\mathcal{R}_+ := \{ e_j \pm e_k \mid 1 \leq j < k \leq n \}, \quad 2e_k, e_k \mid 1 \leq k \leq n \} \quad \text{if} \; m > n,
\]

(3.14)

\[
\mathcal{R}_+ := \{ e_j \pm e_k \mid 1 \leq j < k \leq n \}, \quad 2e_k, e_k \mid 1 \leq k \leq n \} \quad \text{if} \; m = n,
\]

(3.15)

with the multiplicities

\[
\nu_{e_j \pm e_k} = 2 \quad (1 \leq j < k \leq n), \quad \nu_{2e_k} = 1 \quad \text{and} \; \nu_{e_k} = 2(m - n) \quad (1 \leq k \leq n).
\]

(3.16)

We adopt the convention described explicitly in [20], where the basis vectors of \( \mathcal{M}^\perp \) (3.4) are denoted as

\[
E_{e_j \pm e_k}^+, \quad E_{e_j \pm e_k}^{+, i}, \quad E_{2e_k}^{+, i}, \quad E_{e_k}^{+, i, d}, \quad E_{e_k}^{+, i, d} \quad \text{for} \; 1 \leq d \leq (m - n).
\]

(3.17)

The superscripts \( r \) or \( i \) refer to purely real or imaginary matrices.

Since \( G_+ \) (3.8) contains factors of \( SU(k) \) type, we can use the minimal coadjoint orbits of \( SU(k) \) in our reduction procedure, which underlie also the derivation [26] of the \( k \)-particle Sutherland model from the geodesic motion on \( SU(k) \). For any \( u \in \mathbb{C}^k \), viewed as a column vector, we define

\[
\eta_{\pm}(u) := \pm i \left( uu^\dagger - \frac{u^\dagger u}{k} \mathbf{1}_k \right) \in \mathfrak{su}(k).
\]

(3.18)

The minimal coadjoint orbits of \( SU(k) \) are provided by

\[
\mathcal{O}_{k, \kappa, \pm} := \{ \xi \in \mathfrak{su}(k) \mid \exists u \in \mathbb{C}^k, \; u^\dagger u = k \kappa, \; \xi = \eta_{\pm}(u) \},
\]

(3.19)

where \( \kappa > 0 \) is a constant. For definiteness, we below take the plus sign.

For \( G = SU(n, n) \), we now consider the following coadjoint orbits of \( G_+ \):

\[
\mathcal{O}^i := \mathcal{O}_{n, \kappa, +} + \{ xC_{n, n} \}, \quad \mathcal{O}^r := \{ yC_{n, n} \},
\]

(3.20)

where \( x \) and \( y \) are real constants and \( \mathcal{O}_{n, \kappa, +} \) is embedded say in the upper \( su(n) \) block of \( \mathcal{G}_+ \). Since \( C_{n, n} \in \mathcal{M}^\perp \), no restriction on \( x, y, \kappa \) arises from the constraint (2.19). One may confirm
in the standard manner [25, 20] that the reduced orbit $O_{\text{red}}$ (2.24) consists of a single point, and as a representative one can take

$$
\xi^i := \kappa \sum_{1 \leq j < k \leq n} \left( E_{e_j+e_k}^{+,i} + E_{e_j-e_k}^{+,i} \right) + \sqrt{2} \kappa \epsilon n \sum_{k=1}^n E_{2e_k}^{+,i}, \quad \xi^r := yC_{n,n} = \sqrt{2} \kappa \epsilon n \sum_{k=1}^n E_{2e_k}^{+,i}. \quad (3.21)
$$

Upon substitution into (2.27) using the normalization (3.3), $\langle X, Y \rangle := \text{tr}(XY)$, the reduced Hamiltonian (2.27) now gives

$$
\frac{1}{2} H_{\text{red}}^{SU(n,n)}(q,p,\xi^i,\xi^r) = H_{BC_n}(q,p) \quad \text{with} \quad g_1^2 = \frac{\kappa^2}{4}, \quad g_1^2 = \frac{xy\epsilon n^2}{2}, \quad g_2^2 = \frac{(x-y)^2\epsilon n^2}{2}, \quad (3.22)
$$

where we use the notation (1.1). The coupling constants $g_1^2, g_1^2, g_2^2$ can take arbitrary positive values, and we may even change the sign of $g_1^2$ by changing the sign of $xy$. This association of the classical $BC_n$ Sutherland model with $SU(n,n)$ appears to be a new result. By setting $y = 0$ we reproduce the $C_n$ type Hamiltonian previously known to arise from $SU(n,n)$ [21, 27] and $x = y \neq 0$ (resp. $x = y = 0$) yields the $B_n$ (resp. $D_n$) type Sutherland Hamiltonian.

For $G = SU(n+1,n)$, we take $O^l$ and $O^r$ to be

$$
O^l := O_{n+1,\kappa,+} + \{xC_{n+1,n}\}, \quad O^r := \{yC_{n+1,n}\}, \quad (3.23)
$$

where $O_{n+1,\kappa,+}$ is embedded into the $su(n+1)$ factor of $G_+$. An analysis similar to [20] shows that the consistency of the constraint (2.19) requires

$$
\kappa + x + y \geq 0 \quad \text{and} \quad \kappa - n(x+y) \geq 0. \quad (3.24)
$$

The reduced orbit (2.24) again consists of a single point, and for a representative one can use

$$
\xi^l := -\xi^r_M + 2g \sum_{1 \leq j < k \leq n} \left( E_{e_j+e_k}^{+,i} + E_{e_j-e_k}^{+,i} \right) \quad \text{with} \quad g_1^2 = \frac{\kappa + x + y}{2}, \quad g_2^2 = \frac{2(n+1)x+y}{\sqrt{2}}, \quad (3.25)
$$

$$
g = \frac{\kappa + x + y}{2}, \quad h_1 = \frac{\kappa - n(x+y)}{\sqrt{2}}, \quad h_2 = \frac{2(n+1)x+y}{\sqrt{8}}, \quad (3.26)
$$

$$
\xi^r_M = -\frac{iy}{2} \text{diag}(1_n, -2n, 1_n), \quad \xi^r_M = 2\tilde{h}_2 \sum_{k=1}^n E_{2e_k}^{+,i}, \quad \tilde{h}_2 = \frac{y(2n+1)}{\sqrt{8}}. \quad (3.27)
$$

Referring to $H_{BC_n}$ in (1.1), in the present case we find

$$
\frac{1}{2} H_{\text{red}}^{SU(n+1,n)}(q,p,\xi^l,\xi^r) = H_{BC_n}(q,p) - \frac{y^2(2n^2 + n)}{8} \quad \text{with} \quad g_1^2 = h_1^2 + h_2\tilde{h}_2, \quad g_2^2 = (h_2 - \tilde{h}_2)^2. \quad (3.28)
$$

The coupling constants $g_1^2, g_1^2, g_2^2$ of $H_{BC_n}$ depend on the three parameters $x, y, \kappa$ subject to (3.24), and one recovers the result of [14, 20] upon setting $y = 0$.

In the above we have seen how the spinless $BC_n$ Sutherland model with three arbitrary coupling constants arises from $SU(n,n)$ and from $SU(n+1,n)$. What happens if $m \geq (n+2)$? Briefly, in these cases we can obtain a one-point reduced orbit $O_{\text{red}}$ (2.24) if

$$
O^l = O_{n,\kappa,+} + \{xC_{m,n}\}, \quad O^r = \{yC_{m,n}\}, \quad x = -y. \quad (3.29)
$$
The orbit $\mathcal{O}_{n,\kappa,+}$ is embedded in the $su(n)$ factor of $\mathcal{G}_+$. The condition $(x + y) = 0$ is now enforced by the constraint (2.19). This leads again to the $BC_n$ model, but with only two independent coupling parameters. Concretely, we find that
\[
\frac{1}{2} \mathcal{H}_{\text{red}}^{SU(m,n)} = \mathcal{H}_{BC_n} - \frac{y^2(m^2 - n^2)n}{8} \quad \text{with} \quad g^2 = \frac{\kappa^2}{4}, \quad g_1^2 = -\frac{g_2^2}{4} = -\frac{y^2(m + n)^2}{8}. \quad (3.30)
\]
In the $y = 0$ case $[20]$ the model $\mathcal{H}_{\text{red}}^{SU(m,n)}$ becomes of type $D_n$. Finally, we note that the choice (3.29) is available for $m = n + 1$ as well.

One can spell out the Lax matrices (2.29) for all the above cases and can also determine the explicit form of $\mathcal{Y}_M$ in (2.31). The Lax pairs derived in this way appear to be closely related to the Lax pairs of the $BC_n$ model (1.1) obtained in [10] by a different method.

4 Discussion

The main results of this letter are the general description of the reduced geodesic system $(\mathcal{P}_{\text{red}}, \mathcal{G}_{\text{red}}, \mathcal{H}_{\text{red}})$ presented in Section 2 and the realization that this contains the spinless $BC_n$ Sutherland models (1.1) with three independent coupling constants as explained in Section 3. The results can be extended to compact Lie groups straightforwardly, in correspondence with the trigonometric version of the hyperbolic (spin) Calogero models encoded by (2.27).

It could be interesting to investigate generalizations based on replacing the group $G_+ \times G_+$ by suitable groups $G'_+ \times G''_+$, where the factors are fixed by two commuting involutions of $G$. One can proceed as in Section 2 whenever a ‘good decomposition’ analogous to (2.5) is available.

The models (2.27) can be quantized by quantum Hamiltonian reduction as follows. One starts by replacing the coadjoint orbits $\mathcal{O}^\lambda$ in (2.8) by irreducible unitary representations $\rho^\lambda$ of $G_+$ on vector spaces $V^\lambda$ for $\lambda = l, r$ and considers also the associated representation $\rho$ of $G_+ \times G_+$ on $V = V_l \otimes V_r$. The quantum analogue of $P_{\text{ext}}$ (2.8) is the Hilbert space of $V$ valued square integrable wave functions on $\mathcal{G}$ and quantum Hamiltonian reduction amounts to allowing only those wave functions $\psi$ that are equivariant in the sense that $\psi(g_1^l g_1^r) = \rho(g_1^l, g_1^r) \psi(g)$ holds. Because of the equivariance property, these functions are determined by their restrictions to the domain $\exp(\mathbb{A})$ and the restricted wave functions take their values in the subspace $V^M$ of $V$ spanned by the vectors invariant under the subgroup $M_{\text{diag}}$ of $G_+ \times G_+$. The allowed representations must therefore satisfy $\dim(V^M) > 0$. Spinless Calogero type models arise at the quantum mechanical level if $\dim(V^M) = 1$. The reduced Hilbert space naturally comes equipped with a commuting family of self-adjoint operators induced by the centre of the universal enveloping algebra of $\mathcal{G}$. This perspective on quantum Calogero type models originates from [28], where the trivial representation was taken for the $\rho^\lambda$ above. Many interesting results obtained in this framework can be found in [3, 5, 11, 12, 29] and references therein.

We plan to elaborate the consequences of the quantum Hamiltonian reduction in a future publication, where we shall also deal with the relationship between the reduction procedure proposed in Section 3 and the interpretation of $BC_n$ type Jacobi polynomials as generalized spherical functions on $GL(m+n, \mathbb{C})/(GL(m, \mathbb{C}) \times GL(n, \mathbb{C}))$ put forward by Oblomkov [21]. It is well-known (see e.g. [5]) that these polynomials give the eigenstates of the $BC_n$ type trigonometric Sutherland model. However, the natural compact analogue of Oblomkov’s construction,
obtained by substituting $SU(m+n)$ for $GL(m+n, \mathbb{C})$, does not seem to coincide with the quantized version of our classical Hamiltonian reduction, except in the $m=n$ case. For $m \neq n$, his construction and ours may produce the eigenstates of the $BC_n$ Hamiltonian for different discrete sets of the coupling constants, but it requires further work to clarify this issue.

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