COMBINATORIAL ASPECTS OF THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRA OF $\mathfrak{sl}_{n+1}$

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ABSTRACT. Quasi-triangular Hopf algebras were introduced by Drinfel’d in his construction of solutions to the Yang–Baxter Equation. This algebra is built upon $\mathcal{H}_h(\mathfrak{sl}_2)$, the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ ($h$ is an indeterminate: in the context of mathematical physics it is Planck’s Constant). In this paper, combinatorial structure in $\mathcal{H}_h(\mathfrak{sl}_2)$ is elicited, and used to assist in highly intricate calculations in this algebra. To this end, a combinatorial methodology is formulated for straightening algebraic expressions to a canonical form in the case $n = 1$. We apply this formalism to the quasi-triangular Hopf algebras and obtain a constructive account not only for the derivation of the Drinfel’d’s $R$-matrix, but also for the arguably mysterious ribbon elements (conventionally denoted by $u$ and $v$) of $\mathcal{H}_h(\mathfrak{sl}_2)$. Finally, we extend these techniques to the higher dimensional algebras $\mathcal{H}_h(\mathfrak{sl}_{n+1})$. While these explicit algebraic results are well-known, our contribution is in our formalism and perspective: our emphasis is on the combinatorial structure of these algebras and how that structure may guide algebraic constructions.

1. INTRODUCTION

1.1. Motivation: Knot Theory. A rich setting in which quasi-triangular Hopf algebras appear is knot theory, so we shall begin by explaining very briefly and informally some of the background to this. A knot is an embedding of the unit circle into $\mathbb{R}^3$ and two knots are equivalent if one may be transformed into the other smoothly: that is, without cutting and re-attaching the ends. An essential question in knot theory is how to construct a map $\theta : \mathcal{K} \rightarrow \mathcal{S}$, from the set $\mathcal{K}$ of all knots to a set $\mathcal{S}$ such that if $a$ and $b$ are knots, then $\theta(a) \neq \theta(b)$ implies that $a$ and $b$ are inequivalent knots. The map $\theta$ is called a knot invariant.

The discovery in the 1990’s that the Yang–Baxter Equation, which appeared in mathematical physics, also arose in knot theory prompted a remarkable resurgence of activity in knot theory and, obiter dictu, marked the beginning of what is now commonly termed Modern Knot Theory. The appearance of the Yang–Baxter Equation may be seen as follows. An oriented knot may be represented in the plane by its regular projection as a four regular graph, together with marks attached to each vertex to indicate whether a crossing is positive or negative. Such an object is called a knot diagram.

To this end, a combinatorial methodology is formulated for straightening algebraic expressions to a canonical form in the case $n = 1$. We apply this formalism to the quasi-triangular Hopf algebras and obtain a constructive account not only for the derivation of the Drinfel’d’s $R$-matrix, but also for the arguably mysterious ribbon elements (conventionally denoted by $u$ and $v$) of $\mathcal{H}_h(\mathfrak{sl}_2)$. Finally, we extend these techniques to the higher dimensional algebras $\mathcal{H}_h(\mathfrak{sl}_{n+1})$. While these explicit algebraic results are well-known, our contribution is in our formalism and perspective: our emphasis is on the combinatorial structure of these algebras and how that structure may guide algebraic constructions.

\begin{equation}
(M \otimes I)(I \otimes M)(M \otimes I) = (I \otimes M)(M \otimes I)(I \otimes M)
\end{equation}

is the image of this relation in the matrix representation. Such a matrix $M$ is called an $R$-matrix. Solutions of the Yang–Baxter equation may be obtained through Ribbon Hopf Algebras. In general, such

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algebras are difficult to construct. The remarkable work of Drinfel’d and Jimbo in the late 1980’s showed that every semisimple Lie algebra over $\mathbb{C}$ gives rise to such an algebra, the starting point of which is the quantized universal enveloping algebra of a semisimple Lie algebra. Consequently, many new knot invariants, generally contained in the class of quantum invariants, were discovered. Readers interested in reading further about the connexions with knot theory are referred to [Oht02].

The three algebras which will be encountered are:

(i) the quantised universal enveloping algebra $U_h(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$; this is a Hopf algebra;
(ii) a quasi-triangular Hopf algebra; this is a Hopf algebra with an invertible element $R$ called a universal $R$-matrix;
(iii) a ribbon Hopf algebra; this is a quasi-triangular Hopf algebra with a particular element $v$ called a ribbon element (determined from $R$).

1.2. Purpose. We include a self-contained introduction to quantized universal enveloping algebras of semisimple Lie groups from a particular perspective: namely, that they contain a rich combinatorial structure which may be used as a guide to highly intricate algebraic calculations within these algebras. We demonstrate the efficacy of this approach by deriving several fundamental results that may be found in [CP94; Kas95; Oht02] and original sources such as [Dri87; RT91; RT90; Bur90; KR90; LS91]. These results include:

- straightening in $U_h(\mathfrak{sl}_2)$, but from a constructive approach; (§3.2)
- a direct derivation of an $R$-matrix for $U_h(\mathfrak{sl}_2)$ without recourse to Drinfel’d’s quantum double or to the quantum Weyl group; (§4.2)
- the same for $U_h(\mathfrak{sl}_{n+1})$, $n \geq 2$; (§§5.4-6.5)
- a direct, essentially combinatorial, construction of the ribbon Hopf structure on $U_h(\mathfrak{sl}_2)$; (§4.3).

After completing this investigation, it came to our attention that the article [KT91] studied universal enveloping algebras and a universal $R$-matrix for quantized super algebras. They did so through the combinatorics of root systems. While there are some similarities, our approach is through the combinatorics of straightening and the combinatorics of $q$-series.

1.3. Organization.

§2 contains general comments on the quantized universal enveloping algebra $U_h(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$. We discuss a Poincaré-Birkhoff-Witt basis, straightening in $U_h(\mathfrak{sl}_2)$ and establish the technical lemmas which are crucial to all that follows. These are applied to straighten the monomial $x^a y^b$ in $U_h(\mathfrak{sl}_2)$ so that it is a sum of monomials of the form $y^c x^d$.

In §3 we discuss some $q$-identities in the combinatorial context of inversions in bimodal permutations, and then prove an extension of a classic identity of Cauchy that is crucial to our approach to the construction of the ribbon Hopf structure on $U_h(\mathfrak{sl}_2)$.

In §4 we return to the structure of $U_h(\mathfrak{sl}_2)$. We discuss the notion of a quasi-triangular Hopf algebra, constructively derive an $R$-matrix for this algebra, and then give an explicit construction for the associated ribbon Hopf structure on $U_h(\mathfrak{sl}_2)$.

In §5 the approach is extended to $U_h(\mathfrak{sl}_{n+1})$ for $n \geq 2$. These higher dimensional studies further clarify, and amplify, the essential features of our technique.

In §6 we derive a $R$-matrix for $U_h(\mathfrak{sl}_{n+1})$.

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2. Quantized Universal Enveloping Algebra of $\mathfrak{sl}_2$

This section gives a short introduction to the Drinfel’d-Jimbo quantized universal enveloping algebra $U_h(\mathfrak{sl}_2)$ for the Lie algebra $\mathfrak{sl}_2$. Although this object is well-known, our aim is twofold: first, to highlight the rich combinatorial structure contained within this object and, second, to begin systematizing the
study of this structure through straightening. For completeness, we begin by recalling some standard definitions, which may also be found in books such as [Hum72, Ser01, Fi91, Kas95, CP94].

### 2.1. Background.

#### 2.1.1. Lie Algebra. A Lie algebra over the complex numbers \( \mathbb{C} \) is a vector space \( g \) over \( \mathbb{C} \) equipped with a Lie bracket, i.e. a bilinear map \([-,-]\) : \( g \times g \rightarrow g \) satisfying antisymmetry \([x,y]=-[y,x]\) and the Jacobi identity \([x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0\).

For an \( n \)-dimensional Lie algebra \((g,[-,-])\), the data of the Lie bracket \([-,-]\) can be described concretely via its structure constants. Namely, let \( x_1, \ldots, x_n \) be a basis for \( g \). Then for each pair \( 1 \leq i < j \leq n \), there are unique \( c_{ij}^k \in \mathbb{C} \), \( k=1,\ldots,n \), such that

\[
[x_i,x_j] = \sum_{k=1}^{n} c_{ij}^k x_k.
\]

Structure constants give a concrete construction of an associative algebra that best approximates the Lie algebra \( g \). This associative algebra is called the universal enveloping algebra \( U(g) \) of the Lie algebra \( g \) and can be constructed as the free associative \( \mathbb{C} \)-algebra with generators \( x_1, \ldots, x_n \), subject to the relations

\[
(2.1) \quad x_i x_j - x_j x_i = \sum_{k=1}^{n} c_{ij}^k x_k, \quad \text{for each } i, j = 1, \ldots, n.
\]

#### 2.1.2. A Poincaré–Birkhoff–Witt basis. An ordered basis \( x_1 < \cdots < x_n \) for \( g \) induces a distinguished linear basis for the algebra \( \mathcal{U}(g) \): the set

\[
\mathcal{B} := \{ x_1^{e_1} \cdots x_n^{e_n} : e_1,\ldots,e_n \in \mathbb{Z}_{\geq 0} \}
\]

is a basis of \( \mathcal{U}(g) \). This is the Poincaré–Birkhoff–Witt Theorem, (see [Hum72, §17.3]), and the basis \( \mathcal{B} \) is called a Poincaré–Birkhoff–Witt (PBW) basis. Linear independence of \( \mathcal{B} \) comes from a careful degree argument. That \( \mathcal{B} \) spans \( U(g) \), and that any such monomial can be straightened so as to be expressed as a linear combination of elements of \( \mathcal{B} \) by using the relations (2.1).

#### 2.1.3. The Lie algebra \( \mathfrak{sl}_2 \). For the early sections of this article, we shall be concerned with the Lie algebra \( \mathfrak{sl}_2 \). It consists of a three-dimensional vector space with basis \( x, y \) and \( h \), together with Lie bracket

\[
[h,x] = 2x, \quad [h,y] = -2y, \quad [x,y] = h.
\]

The universal enveloping algebra \( \mathcal{U}(\mathfrak{sl}_2) \) of \( \mathfrak{sl}_2 \) is the associative algebra with generators \( x, y \) and \( h \), subject to the relations

\[
hx - xh = 2x, \quad hy - yh = -2y, \quad xy -yx = h.
\]

With the ordering \( h < y < x \) of generators, the PBW basis of \( \mathcal{U}(\mathfrak{sl}_2) \) is the set \( \{ h^a y^b x^c : a, b, c \in \mathbb{Z}_{\geq 0} \} \).

#### 2.1.4. Quantized Universal Enveloping Algebras. A quantized universal enveloping algebra \( \mathcal{U}_h(g) \) for a Lie algebra \( g \) is an associative \( \mathbb{C}[\![h]\!] \)-algebra which is a deformation of \( \mathcal{U}(g) \) in that specializing the indeterminate \( h \) to 0 in \( \mathcal{U}_h(g) \) recovers \( \mathcal{U}(g) \). In the case of \( \mathfrak{sl}_2 \), we have the following, first due to [KR81].

**Definition 2.1.1.** The quantized universal enveloping algebra \( \mathcal{U}_h(\mathfrak{sl}_2) \) is \( \mathbb{C}[\![h]\!] \)-algebra with underlying \( \mathbb{C}[\![h]\!] \)-module \( \mathcal{U}_h(\mathfrak{sl}_2)[\![h]\!] \) of formal power series in \( h \) with coefficients in \( \mathcal{U}_h(\mathfrak{sl}_2) \), and such that \( x, y, h \in \mathcal{U}_h(\mathfrak{sl}_2) \) satisfy the relations

\[
(2.2) \quad hx - xh = 2x, \quad hy - yh = -2y, \quad xy -yx = \frac{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}.
\]

The analogue of the PBW basis in \( \mathcal{U}_h(\mathfrak{sl}_2) \) for the ordered basis \( h < y < x \) of \( \mathfrak{sl}_2 \) is the following.

**Theorem 2.1.2.** [CP94, p.199] The set \( \mathcal{B}_h := \{ h^a y^b x^c : a, b, c \in \mathbb{Z}_{\geq 0} \} \) is a \( \mathbb{C}[\![h]\!] \)-basis for \( \mathcal{U}_h(\mathfrak{sl}_2) \).
We shall also write
\[ q := e^{\frac{h}{2}}, \quad k := e^{\frac{h}{4}}, \quad \overline{q} := q^{-1}, \quad \overline{k} := k^{-1}, \quad [h + n] := \frac{q^n k^2 - \overline{q}^n \overline{k}^2}{q - \overline{q}}. \]
With this notation, the commutation relation for \( x \) and \( y \) from (2.2) is simply expressed as
\[ [x, y] := xy - yx = \frac{k^2 - \overline{k}^2}{q - \overline{q}} = [h]. \]
We shall also write
\[ [n]_q := \frac{q^n - \overline{q}^n}{q - \overline{q}}, \quad [n]!_q := \prod_{i=1}^{n} [i]!_q, \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]!_q [n-k]!_q} \]
for a quantum integer, the quantum factorial function and the quantum binomial function, respectively, where \( k \) is a non-negative integer. Related to the quantum factorial function is the quantum lower factorial
\[ [n]_{q, i} := \frac{[n]_q!}{[i]!_q}. \]
Note that subscript \( q \) in each notation is to be thought of as the argument in its definition. That is, for any function \( f(q) \) of \( q \), we write
\[ [n]_{f(q)} := \frac{f(q)^n - f(\overline{q})^n}{f(q) - f(\overline{q})}, \]
and similarly with the other definitions. When an explicit subscript is omitted, \( f(q) = q \) by convention.

Finally, for \( i \in \mathbb{Z}_{\geq 0} \),
\[ [h + n]_{(i)} := \prod_{r=0}^{i-1} [h + n - r], \quad \text{and} \quad \left[ \begin{array}{c} h + n \\ i \end{array} \right] := \frac{1}{[i]!} [h + n]_{(i)}. \]

3. Straightening in \( \mathcal{U}_h(\mathfrak{sl}_2) \)

3.1. Basic Straightening Rules. The proofs of the following are straightforward and are largely omitted.

**Lemma 3.1.1** (Separation Lemma). Let \( x, y, a \) be indeterminates. Then
\[
\begin{aligned}
(i) & \quad [-x] = [-x], \\
(ii) & \quad [x] \cdot [y] - [x - a] [y + a] = [a] [y - x + a], \\
(iii) & \quad [x] [y] + [a] [x + y + a] = [x + a] [y + a].
\end{aligned}
\]

**Proof.** Identity (iii) comes by noting \( (q - \overline{q})^2 [x] [y] = (q^{x+y} + \overline{q}^{x+y}) - (q^{y-x} + \overline{q}^{y-x}) \). Applying this twice,
\[ (q - \overline{q})^2 [x] [y] = q^{y-x} (q^{2a} - 1) + \overline{q}^{y-x} (\overline{q}^{2a} - 1). \]
Finally, (iii) follows from (ii) upon replacing \( x \) by \( -x \). \( \square \)

**Remark 3.1.2.** These identities are valid when either:
\( (a) \) both \( x \) and \( y \) are integers; or
\( (b) \) one of \( x \) or \( y \) is an integer and the other is an expression involving \( h \).
Lemma 3.1.3(ii) may be viewed as a device for separating the $x$ and $y$ in $[y - x - a]$.

Lemma 3.1.3 (Straightening). Let $a \in \mathbb{Z}$, $b, c \in \mathbb{Z}_{\geq 0}$ and $f(x)$ be a formal power series in $x$. Then the following hold in $\mathcal{U}_b(\mathcal{S}_k)$.

(i) $[a + 1][h + a] = [h] + [a][h + a + 1]$

(ii) $f(h)x^b = x^bf(h + 2b), f(h)y^b = y^bf(h - 2b),$

(iii) $k_ay^b = q_ay^bk_ay, k_ay^b = \overline{q}_ay^b\overline{y}^bk_ay,$

(iv) $(kx)^b = \frac{q}{q_2}k^b(n-1)k^b, (ky)^b = q^{\frac{1}{b}(n-1)}k^by^b,$

(v) $xy^b = y^bx + [b][h + b - 1]y^{-1}, x^by = y^bx + [b][h - b + 1]x^{-1},$

(vi) $(kk)^bf(h) = f(h - 2b)(kx)^b, (k\overline{y})f(h) = f(h + 2b)(k\overline{y})^b,$

(vii) $y^b(kx)^c = \frac{q}{q_2}y^b\overline{c}+2bc-c)k^c, y^b(k\overline{y})^c = q^{\frac{1}{b}(c+2bc-c)}k^c.$

Proof. We only prove (v). Let $A_b := xy^b$. Then, from (2.2) and part (ii),

(3.1) $A_b = (xy)^y^b = yA_{b-1} + [h]y^b$.

Iterating this gives

(3.2) $A_b = y^b\sum_{i=0}^{b-1} + f_0(h)y^b$ where $f_0(h) = 0$

which, when substituted into (3.1) and part (ii) is applied, gives $f_b(h) = f_{b-1}(h + 2) + [h]$. Thus

$f_b(h) = \sum_{i=0}^{b-1} h + 2i + \sum_{i=0}^{b-1} h + 2b - 2i$. 

Lemma 3.1.4 with $a = 1, x = b = 2$, and $y = h + b - 2$ gives $[h + 2b - 2] = [b][h + b - 1] - [b - 1][h + b - 2]$, so

$f_b(h) = [h + b - 1] = f_{b-1}(h) - [b - 1][h + b - 2] = c$

where $c$ is therefore independent of $b$. Setting $b = 0$ in the left hand side gives $c = f_0(h) = 0$, so $f_b(h) = [b][h + b - 1]$ and the result follows from (3.2).

When constructing an R-matrix in §4.2, a term $e^{\frac{h}{b}x^y}$ will appear. The following result will be useful for commuting terms past this exponential.

Lemma 3.1.4. Let $f(x)$ be a formal power series in $x$. Then, for any integer $m$,

(i) $f(1 \otimes k^m)x)e^{\frac{h}{b}x^y} = e^{\frac{h}{b}x^y}f(k^2 \otimes k^m)x$;

(ii) $f(1 \otimes k^m)y)e^{\frac{h}{b}x^y} = e^{\frac{h}{b}x^y}f(k^2 \otimes k^m)y$.

3.2. Straightening of $x^ay^b$. We now straighten $x^ay^b$, $a, b \in \mathbb{Z}_{\geq 0}$, with respect to the ordering $h < y < x$ by a constructive method. From Lemma 3.1.3(v), the straightening of the premultiplication of $y^b$ by $x$ is

$x^ay^b = y^bx + [b][h + b - 1]y^{-1}.$

Iterating this $a$ times, and noting that $x$ may be moved through quantum brackets containing only $h$ by means of Lemma 3.1.3(ii),

(3.3) $x^ay^b = \sum_{0 \leq i : j \leq \min(a, b)} F_{a,b,i,j}(h)x^iy^j$.

But $e^{h}x^ay^b = x^ay^be^{h-2a+2b}$ by Lemma 3.1.3(ii) and so, applying this to (3.3), gives

$x^ay^be^{h-2a+2b} = \sum_{0 \leq i : j \leq \min(a, b)} F_{a,b,i,j}(h)x^iy^je^{h-2j+2i},$

and so

$x^ay^b = \sum_{0 \leq i : j \leq \min(a, b)} F_{a,b,i,j}(h)x^iy^je^{2(a-j)^2 - 2(b-1)}.$
Equating coefficients of $y^i x^j$ on the right hand side of this and (3.3), we have $i = b - k$ and $j = a - k$ for some non-negative integer $k$, whence we conclude that

$$x^a y^b = \sum_{0 \leq k \leq \min(a,b)} G_{a,b,k}(h) y^{b-k} x^{a-k}$$

where $G_{a,b,k}(h) := F_{a,b,a-k,b-k}(h)$. Since commuting $x$ from the left of $y^b$ yields a single term of top degree with coefficient 1, the boundary condition is

$$G_{a,b,0}(h) = 1.$$  

A recursion for $G_{a,b,k}$ is obtained from the identity $x^a y^b = x^{a-1} (xy)^b$. First, from Lemma 3.1.3(ii) and (v),

$$x^a y^b = (x^{a-1} y^{b-1}) (yx) + [b] [h + b - 2a + 1] (x^{a-1} y^{b-1}).$$

Then, substituting (3.4) into this,

$$\sum_{k \geq 0} G_{a,b,k} y^{b-k} x^{a-k} = \sum_{k \geq 0} G_{a-1,b-1,k} \left( y^{b-k-1} x^{a-k-1} (xy) + [b] [h + b - 2a + 1] y^{b-k-1} x^{a-k-1} \right).$$

From Lemma 3.1.3(v), $x^{a-k} y = x^{a-k-1} + [a-k] [h-a+k+1] x^{a-k-2}$, Substituting this into the above, and then equating the coefficients of $y^{b-k} x^{a-k}$ gives the recurrence equation

$$G_{a,b,k} = G_{a-1,b-1,k} + [a-k] [h+b-a-k+1] + [b] [h+b-2a+1] G_{a-1,b-1,k}.$$ 

Then from Lemma 3.1.3(ii) with $x \mapsto b$, $y \mapsto h + b - 2a + 1$ and $a \mapsto a-k$,

$$G_{a,b,k} = G_{a-1,b-1,k} + [a+b-k] [h+b-a-k+1] G_{a-1,b-1,k-1}.$$ 

Each instance of $G_{i,j}$ in this recurrence equation satisfies $i - j = a-b$. The only term that does not contain $a-b$ is $[a+b-k]$. This suggests using $[k] [a+b-k] = [a] [b] - [a-k] [b-k]$ from the Separation Lemma 3.1.1 to separate $a$ and $b$ in this quantum bracket and then transforming $G_{a,b,k}$ to form a new recurrence equation in which $a-b$ is an invariant. Let

$$G_{a,b,k} = \frac{[a] [b]_k}{[k]!} B_{a,b,k}.$$ 

Then, substituting (3.7) into (3.6) gives

$$[a] [b] B_{a,b,k} - [a-k] [b-k] B_{a-1,b-1,k} = \left( [a] [b] - [a-k] [b-k] \right) [h+b-a-k+1] B_{a-1,b-1,k-1}.$$ 

Suppose that $B_{a,b,k}$ depends only on the difference $b-a$. Then $B_{a,b,k} = B_{a-1,b-1,k}$ and (3.6) becomes

$$B_{a,b,k} = [h+b-a-k+1] B_{a,b,k-1}$$ 

for $k \geq 1$ and $B_{a,b,0} = 1$ from (3.5). This suggests the solution $B_{a,b,k} = [h+b-a]_k$. Indeed, it is readily checked that this does indeed satisfy (3.8), from which we have

$$G_{a,b,k} = \frac{[a] [b]_k}{[k]!} \begin{pmatrix} h + b - a \\ k \end{pmatrix}.$$ 

So, from (3.4), we have therefore (both derived and) proved the following lemma.

**Lemma 3.2.1.** Let $a$ and $b$ be non-negative integers. Then

$$\frac{x^a y^b}{[a]! [b]!} = \sum_{i \geq 0} \begin{pmatrix} h + b - a \\ i \end{pmatrix} \frac{y^{b-i}}{[b-i]!} \frac{x^{a-i}}{[a-i]!}.$$ 

An important special case of Lemma 3.2.1 is where the powers of $x$ and $y$ coincide.

**Lemma 3.2.2.** Let $n$ be a non-negative integer. Then

$$x^n y^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^{n-i}.$$
3.3. Straightening q-commuting variables. Indeterminates $a$ and $b$ are said to $q$-commute if they satisfy

$$ab = qba.$$ 

Certain combinations of elements in $\mathcal{Y}_q(\mathfrak{S}_n)$ q-commute and so we will find use for straightening rules involving series in q-commuting variables. In this section, we collect straightening rules involving abstract q-commuting variables. We view these identities as arising from combinatorial properties of inversions in permutations.

3.3.1. Basic q-series. For $n, k \in \mathbb{Z}$, define the q-integer $n$, the q-factorial of $n$ and the q-binomial coefficient of $n$ and $k$ as

$$(n)_q := \frac{1 - q^n}{1 - q}, \quad (n)_q! := (1)_q \cdot (2)_q \cdots (n)_q, \quad \binom{n}{k}_q := \frac{(n)_q!}{(k)_q! (n-k)_q!},$$

respectively. The q-exponential series is

$$\exp_q(x) := \sum_{n \geq 0} \frac{x^n}{(n)_q!} \in \mathbb{Q}(q)[[x]]$$

as a formal power series in $x$ with coefficients that are rational functions of $q$. This series has a multiplicative q-property for q-commuting indeterminates.

The proof uses the observation that $q$ is associated with a combinatorial property of sets, as follows. An ordered bipartition $(1, \ldots, n)$ of type $(r, n-r)$ is $(\alpha, \beta)$, where $\alpha$ and $\beta$ are disjoint subsets of $\{1, \ldots, n\}$ of size $r$ and $n-r$, respectively. A between-set inversion of $(\alpha, \beta)$ is a pair $(i, j) \in \alpha \times \beta$ such that $i > j$. An inversion in a permutation $\pi \in \mathfrak{S}_n$ is a pair $(i, j)$ with $1 \leq i < j \leq n$ such that $\pi(i) > j$.

**Lemma 3.3.1.** Let $a, b$ be such that $ab = qba$. Then $\exp_q(a + b) = \exp_q(a) \exp_q(b)$.

**Proof.** There is clearly an expression for $(a + b)^n$ of the form

$$(a + b)^n = \sum_{r=0}^{n} f_{r, n-r}(q) a^r b^{n-r}$$

where $f_{r, n-r}(q)$ is a polynomial in $q$. Then $|q^k| f_{r, n-r}(q)$ is immediately identified as the number of ordered bi-partitions of $\{1, \ldots, n\}$ of type $(r, n-r)$ with precisely $k$ between-set inversions.

We determine the generating series $g_n(q)$, where $|q^n| g_n(q)$ is the number of inversions in $\pi \in \mathfrak{S}_n$, in two different ways. First, by considering the contribution to inversions by the symbol $n$ in $\pi$, we have the recursion

$$g_n(q) = g_{n-1}(q) (1 + q + \cdots + q^{n-1})$$

for $n \geq 1$ with $g_0(q) = 1$, so $g_n(q) = n!q$. On the other hand, by considering a fixed bi-partition $(\alpha, \beta)$ of type $(r, n-r)$, we have $(n!)q = g_r(q) g_{n-r}(q) f_{r, n-r}(q)$ since each inversion of $\pi$ occurs within $\alpha$, or within $\beta$, or between $\alpha$ and $\beta$. Then $f_{r, n-r}(q) = n!q / (r!q \cdot (n-r)_q)$ and the result then follows immediately from [3.9].

3.3.2. An extension of a finite product identity of Cauchy.

**Theorem 3.3.2 (Bimodal Permutation [GJ83].)** Let $q, w, x, y, z$ be indeterminates, let $n$ be a non-negative integer and let $Q_n(x, y) := \prod_{i=0}^{n-1} (y + xq^i)$. Then

$$\sum_{k=0}^{n} \binom{n}{k}_q Q_k(x, y) Q_{n-k}(w, z) = \sum_{k=0}^{n} \binom{n}{k}_q Q_k(w, y) Q_{n-k}(x, z).$$

The q-analogue of the Binomial Theorem along with several classical q-identities are now easily obtained. For more, see, for example, [GJ04].
Corollary 3.3.3 \textit{(}q\text{-analogue of the Binomial Theorem).} Let $q, x, y, z$ be indeterminates and let $n$ be a non-negative integer. Then

$$Q_n(-x, z) = \sum_{k=0}^{n} \binom{n}{k}_q Q_k(-x, y) Q_{n-k}(-y, z).$$

An immediate consequence of this is a finite product identity due to Cauchy \textit{[Cau09]}, and its inverse.

Lemma 3.3.4. Let $z$ and $q$ be indeterminates. Then

\begin{enumerate}[(i)]
  \item $z^n = \sum_{k=0}^{n} \binom{n}{k}_q \prod_{i=0}^{k-1} (z-q^i)$,
  \item $\prod_{i=0}^{n-1} (z-q^i) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\frac{1}{2}k(k-1)} z^{n-k}$.
\end{enumerate}

Proof. Part (i) follows from Corollary 3.3.3 by setting $y = 1$ and $x = 0$; for part (ii) set $x = 1$ and $y = 0$. \hfill $\Box$

Taking $z$ to be an exponential related to $q$, Cauchy’s finite product identity can be extended to an identity of certain formal power series.

Lemma 3.3.5. Let $h, t$ and $x$ be indeterminates such that $q = e^{\frac{h}{t}}$. Then, in $\mathbb{Q}[x, t][[h]]$,

\begin{equation}
  e^{xht} = \sum_{k=0}^{\infty} \binom{x}{k} \prod_{i=0}^{k-1} (e^{ht} - q^{2i}).
\end{equation}

Proof. The coefficient of $h^m$ on the left is a polynomial in $t$ and $x$. On the right hand side, note that

$$\text{val}_h \left( \prod_{i=0}^{k-1} (e^{ht} - q^{2i}) \right) = \text{val}_h \left( \prod_{i=0}^{k-1} (t - i)h \right) = k,$$

where $\text{val}_h$ extracts the exponent of the smallest power of $h$ with with nonzero coefficient. So only the finitely many indices $0 \leq k \leq m$ contribute to the coefficient of $h^m$ and each index contributes a binomial coefficient $\binom{k}{i} q^{2i}$. But, as a power series in $h$, $\left( \frac{h}{t} \right) q^{2i}$ has coefficients which are polynomial in $x$. Therefore the coefficient of $h^m$ on the right hand side of (3.10) is polynomial in $t$ and $x$.

In particular, the coefficient of $h^m t^n$ is a polynomial in $x$ on both sides. By Lemma 3.3.4(b) these polynomials in $x$ agree for each positive integer and thus they must be equal as polynomials. \hfill $\square$

3.3.3. \textit{The quantum exponential function.} The functions defined in (2.5) and §3.3.1 are related through

\begin{equation}
  [n]_q = q^{-\frac{n}{2}} (n)_q^2, \quad [n]!_q = q^{-\frac{1}{2} n(n-1)} (n!)_q^2, \quad \binom{n}{k}_q = q^{k(n-k)} \binom{n}{k}. 
\end{equation}

The quantum exponential function defined by

\begin{equation}
  \text{Exp}_q(x) := \sum_{n \geq 0} \frac{q^{\frac{n}{2} n(n-1)}}{[n]!_q} x^n
\end{equation}

enjoys an analogous multiplicative property as the $q$-exponential series under quantum commutation.

Lemma 3.3.6. Let $a$ and $b$ be such that $ab = q^2 ba$. Then $\text{Exp}_q(a + b) = \text{Exp}_q(a) \text{Exp}_q(b)$.

Proof. From (3.11), $[n]!_q = q^{\frac{1}{2} n(n-1)} (n)!_q^2$. Then $\text{Exp}_q(x) = \text{Exp}_q^2(x)$. Thus, from Lemma 3.3.1 $\text{Exp}_q(a + b) = \text{Exp}_q(a) \text{Exp}_q(b)$, and the result follows. \hfill $\square$

4. \textit{An R-matrix for $\mathcal{U}_h(\mathfrak{sl}_2)$}

In this section, we use the straightening framework to give a direct and constructive approach to the construction of the $R$-matrix for $\mathcal{U}_h(\mathfrak{sl}_2)$. We begin with a few standard definitions. Further details can be found in \textit{[Kas95, CP94, Oht02]}. 
4.1. The algebras.

4.1.1. Hopf Algebras. Recall that a Hopf algebra over, say, $\mathbb{C}$ is an associative $\mathbb{C}$-algebra $\mathcal{A}$, with product $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and unit $\eta: \mathbb{C} \to \mathcal{A}$, equipped with algebra homomorphisms $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, a co-product, and $\varepsilon: \mathcal{A} \to \mathbb{C}$, a co-unit, and an algebra anti-homomorphism $S: \mathcal{A} \to \mathcal{A}$, an antipode, satisfying

\[(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta, \quad (\varepsilon \otimes \text{id}) \circ \Delta(a) = 1_k \otimes a, \quad (\text{id} \otimes \varepsilon) \circ \Delta(a) = a \otimes 1_k,\]

for all $a \in \mathcal{A}$.

4.1.2. Quasi-Triangular Hopf Algebras. A quasi-triangular Hopf algebra is a Hopf algebra $\mathcal{A}$ equipped with an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$, called a universal $R$-matrix, satisfying

- (i) $(\tau \circ \Delta)(a) = R \cdot \Delta(a) \cdot R^{-1}$ for every $a \in \mathcal{A}$,
- (ii) $(\Delta \otimes \text{id})(R) = R_{13} \cdot R_{23}$,
- (iii) $(\text{id} \otimes \Delta)(R) = R_{13} \cdot R_{12},$

where $\tau: a \otimes b \mapsto b \otimes a$ is the twist map, and

\[R_{12} := \sum_i a_i \otimes \beta_i \otimes 1, \quad R_{13} := \sum_i \alpha_i \otimes 1 \otimes \beta_i, \quad R_{23} := \sum_i 1 \otimes \alpha_i \otimes \beta_i\]

where $\alpha_i, \beta_i \in \mathcal{A}$ and are defined through writing $R$ as $R := \sum_i a_i \otimes \beta_i$.

4.1.3. Hopf Structure on $\mathcal{U}_h(sl_2)$. Sklyanin [Skl85] showed how $\mathcal{U}_h(sl_2)$ is a Hopf algebra.

**Theorem 4.1.1.** The quantized universal enveloping algebra $\mathcal{U}_h(sl_2)$ is a Hopf algebra with structure maps defined on algebra generators by

\[
\Delta: \mathcal{U}_h(sl_2) \to \mathcal{U}_h(sl_2) \otimes \mathcal{U}_h(sl_2), \quad S: \mathcal{U}_h(sl_2) \to \mathcal{U}_h(sl_2), \quad \varepsilon: \mathcal{U}_h(sl_2) \to \mathbb{C},
\]

- $x \mapsto x \otimes k + k \otimes x$,
- $y \mapsto y \otimes k + k \otimes y$,
- $h \mapsto h \otimes 1 + 1 \otimes h$,
- $x \mapsto -qx$,
- $y \mapsto -qy$,
- $h \mapsto -h$,

for $k \geq 0$. Since $\Delta$ is an algebra map

\[\Delta(k) = e^{\frac{b}{2} \Delta(h)} = e^{\frac{b}{2} \Delta(\hbar + \ell + \hbar)} = \left(e^{\frac{b}{2} \hbar} \otimes 1 - 1 \otimes e^{\frac{b}{2} \hbar}\right) = e^{\frac{b}{2} \hbar} \otimes e^{\frac{b}{2} \hbar} = k \otimes k\]

since $h \otimes 1$ and $1 \otimes h$ commute. Similarly,

\[S(k) = S\left(e^{\frac{b}{4} \hbar}\right) = e^{\frac{b}{4} \hbar} S(h) = e^{\frac{b}{4} \hbar} = k, \quad \text{and} \quad \varepsilon(k) = \varepsilon\left(e^{\frac{b}{4} \hbar}\right) = e^{\frac{b}{4} \varepsilon(h)} = e^0 = 1.\]

Collecting these evaluations:

\[\Delta(k) = k \otimes k, \quad S(k) = k, \quad \varepsilon(k) = 1.\]

4.2. Constructing an $R$-matrix. We construct a universal $R$-matrix for $\mathcal{U}_h(sl_2)$ in two steps. First, by considering the dependency on $x$, $y$, and $h$ in an $R$-matrix, we propose an ansatz. Second, coefficients and parameters in the ansatz are determined through the requirements on $R$.

4.2.1. An ansatz for $R$. In general, $R$ is a sum of elements of $\mathcal{U}_h(sl_2) \otimes \mathcal{U}_h(sl_2)$, so after straightening each tensor component, $R$ can be expressed as a sum of terms of the form $x^m y^n \otimes x^t y^u$, $m, t, u, n \in \mathbb{Z}_{\geq 0}$. Factors of $h$ appear through powers of $k$ and $\overline{k}$ when straightening, say in applying Lemma [3.2.1]. So a general term in $R$ might look like

\[k^r x^m y^n \otimes k^t x^u y^v = e^{\frac{b}{2} (r(\hbar + 1) + s(\ell + 1)} x^m y^n \otimes x^u y^v \text{ for } r, s \in \mathbb{Z} \text{ and } m, t, u, n \in \mathbb{Z}_{\geq 0}.\]

Now consider condition (i) of Definition 4.1.2 for $a = x$:

\[(k \otimes x + x \otimes k) \cdot R = R \cdot (x \otimes k + k \otimes x). \]
This suggests that an asymmetric change in powers of \( k \) and \( \tau \) in the general term of \( R \) needs to be related to one another by straightening. Ultimately, we require a device which can introduce additional powers of \( k \otimes 1 \) and \( 1 \otimes k \) via straightening. A solution for this is to include powers of \( e^{\frac{1}{2}(h \otimes h + h \otimes 1 + 1 \otimes h)} \), as straightening with expressions involving \( x \) and \( y \) creates terms of the form \( e^{\frac{1}{2}(h \otimes h + h \otimes 1 + 1 \otimes h)} \) through Lemma 3.1.3(ii).

This argument leads to

\[
R = \sum_{k,m,n,r,s,t,u} a_{k,r,s,m,n,t,u}(q) e^{\frac{1}{2}(k \otimes h + 1 \otimes h + 1 \otimes h)} x^m y^t \otimes u^y_n.
\]

as a conjectural form for \( R \), where \( a_{k,r,s,m,n,t,u}(q) \) is a function of \( q \).

To simplify further, begin by considering the simpler form in which \( x \) and \( y \) occur exclusively in the first and second tensor components, respectively. Namely, set \( t = 0 \) and \( u = 0 \) in (4.2) and write \( a_{k,r,s,m,n}(q) := a_{k,r,s,m,n,0,0}(q) \) to obtain the ansatz

\[
R = \sum_{k,m,n,r,s} a_{k,r,s,m,n}(q) e^{\frac{1}{2}(h \otimes h)} x^m \otimes k^r y^n.
\]

For brevity, explicit mention of the dependence of \( a_{k,r,s,m,n}(q) \) on \( q \) will henceforth be suppressed.

### 4.2.2. Condition (i) of Definition 4.1.1

Since \( \Delta \) is an algebra morphism it is sufficient to show that this condition holds for the generators \( h, x \) and \( y \).

**For the generator \( h \):** The condition asserts that \((\tau \circ \Delta)(h) \cdot R = R \cdot \Delta(h)\), so

\[
(\tau \circ \Delta)(h) \cdot R - R \cdot \Delta(h) = (h \otimes 1 + 1 \otimes h) \cdot R - R \cdot (h \otimes 1 + 1 \otimes h) = 0.
\]

Since \( e^{\frac{1}{2}(h \otimes h + h \otimes 1 + 1 \otimes h)} \) commutes with \( h \otimes 1 + 1 \otimes h \), the condition is equivalent to \((h \otimes 1 + 1 \otimes h)(k^r x^m \otimes k^r y^n) = (k^r x^m \otimes k^r y^n)(h \otimes 1 + 1 \otimes h)\). By Lemma 3.1.3(ii), the left hand side is

\[
(k^r x^m \otimes k^r y^n)(h \otimes 1 + 1 \otimes h) + 2(m - n)(k^r x^m \otimes k^r y^n)
\]

so the condition implies that \( m = n \). Let \( a_{k,r,s,n} \) denote \( a_{k,r,s,m,n,n} \). Then

\[
R = \sum_{k,n,r,s} a_{k,r,s,n} e^{\frac{1}{2}(h \otimes h)} (k^r x^n \otimes k^r y^n).
\]

**For the generator \( x \):** By Lemma 3.1.3(ii),

\[
(\tau \circ \Delta)(x) \cdot R = \sum_{k,n,r,s} a_{k,r,s,n} e^{\frac{1}{2}(h \otimes h)} \left( k^{k-1-r} x^n \otimes x k^r y^n + x k^r x^n \otimes k^{k+1-s} y^n \right),
\]

\[
R \cdot \Delta(x) = \sum_{k,n,r,s} a_{k,r,s,n} e^{\frac{1}{2}(h \otimes h)} \left( k^r x^{n+1} \otimes k^r y^n k + k^r x^n k \otimes k^r y^n x \right).
\]

The equation \((\tau \circ \Delta)(x) \cdot R = R \cdot \Delta(x)\) may be rearranged as

\[
A := \sum_{k,n,r,s} a_{k,r,s,n} e^{\frac{1}{2}(h \otimes h)} \left( k^{k-1-r} x^n \otimes x k^r y^n + x k^r x^n \otimes k^{k+1-s} y^n \right) = B.
\]

To simplify \( A \), note that \( x k^r = (k+1) x \) and \( x k^r \) appears from the second tensor factors of this expression. Doing so and writing \( e_{r,n} \) for \( a_{1,r,-n,n} \), Lemma 3.1.3(iv) allows us to compute the commutator to yield

\[
A = e^{\frac{1}{2}(h \otimes h)} \sum_{n,r} e_{r,n+1} (1 + q) q^{n+1} k^r x^{n+1} \otimes [h + n] k^{r+1} y^n.
\]
To simplify $B$, set $s = -m$ and $k = 1$ as above and observe, from Lemma 3.1.4, that $y^n k = q^n k^r y^n$ and $xk^r = q^r k^r x$. This gives

$$B = e^{\frac{h}{\hbar} s} \sum_{n,r} a_{r,n}(q) k^r x^{n+1} \otimes \left( q^n k^{r-1} y^n - q^n k^{r+3} y^n \right).$$

Set $r = n$ and denote $a_{n,n}$ by $b_n$ to get

$$B = e^{\frac{h}{\hbar} s} \sum_{n \geq 0} a_n (q - \theta) k^n x^{n+1} \otimes | [h+n] k^{n+1} y^n. $$

Setting $r = n$ for $A$ in (4.6), we have

$$A = e^{\frac{h}{\hbar} s} \sum_{n \geq 0} a_{n+1}[n+1] q^{n+1} k^n x^{n+1} \otimes | [h+n] k^{n+1} y^n. $$

Comparing $A$ and $B$ shows that the $a_n = a_n(q)$ must satisfy the two-term recurrence equation

$$a_{n+1}(q) \cdot [n+1] q^{n+1} = a_{n}(q) \cdot (q - \theta)$$

with initial condition $a_0(q) = 1,$

whence $a_n = \frac{(q - \theta)\theta^{n-1}}{\theta^n} q^{n(n+1)}$. Substituting these settings into $(4.4)$ gives

$$R = e^{\frac{h}{\hbar} s} \sum_{n \geq 0} (q - \theta)\theta^n [n!] (k^n x^n \otimes k^n y^n).$$

Now $k^n x^n = q^{\frac{1}{2}n(n-1)}(kx)^n$ and $\overline{k} y^n = q^{\frac{1}{2}n(n-3)}(ky)^n$ from Lemma 3.1.3, so

$$R = e^{\frac{h}{\hbar} s} \sum_{n \geq 0} (q - \theta)\theta^n [n!] q^{\frac{1}{2}n(n-3)} (kx^n \otimes (ky)^n).$$

For the generator: It is readily shown that this condition is satisfied by the expression for $R$ given in (4.7).

4.2.3. Condition [iii] of Definition 4.1.2 It is immediate from the definition of the quantum exponential function in (3.12) and the expression for $R$ given in (4.7) that

$$R = e^{\frac{h}{\hbar} s} \text{Exp}_q \left( \lambda_q (kx \otimes \overline{k}y) \right) \text{ where } \lambda_q := \theta(q - \theta).$$

Condition [iii] of Definition 4.1.2 requires that $(\Delta \otimes \text{id})(R) = R_{13} \cdot R_{23}$. It is readily seen that

$$(\Delta \otimes \text{id})(R) = e^{\frac{h}{\hbar} s} (\text{Exp}_q \left( \lambda_q (kx \otimes k^2 + 1 \otimes kx) \otimes \overline{k} \right))$$

since $\Delta$ is an algebra morphism, and $\Delta(k) = k \otimes k$ from (4.1).

On the other hand, setting $h_1 := e^{\frac{h}{\hbar} s} \overline{k} \otimes k$ and $h_2 := e^{\frac{h}{\hbar} s} k \otimes k$ in Definition 4.1.2 gives

$$R_{13} = h_2 \text{Exp}_q \left( \lambda_q (kx \otimes 1 \otimes \overline{k}y) \right) \text{ and } R_{23} = h_1 \text{Exp}_q \left( \lambda_q (1 \otimes kx \otimes \overline{k}y) \right)$$

so

$$R_{13} \cdot R_{23} = h_2 \text{Exp}_q \left( \lambda_q (kx \otimes 1 \otimes \overline{k}y) \right) h_1 \text{Exp}_q \left( \lambda_q (1 \otimes kx \otimes \overline{k}y) \right).$$

Now, by Lemma 3.1.4, $\text{Exp}_q \left( \lambda_q (kx \otimes 1 \otimes \overline{k}y) \right) h_1 = h_1 \text{Exp}_q \left( \lambda_q (kx \otimes k^2 \otimes \overline{k}y) \right)$ and since $h_1$ and $h_2$ commute,

$$R_{13} \cdot R_{23} = e^{\frac{h}{\hbar} s} \text{Exp}_q \left( \lambda_q (kx \otimes k^2 \otimes \overline{k}y) \right) \cdot \text{Exp}_q \left( \lambda_q (1 \otimes kx \otimes \overline{k}y) \right).$$

Let $C := kx \otimes k^2 \otimes \overline{k}$ and $D := 1 \otimes kx \otimes \overline{k}$. Then, from Lemma 3.1.3, we have $CD = \left( kx \otimes k^3 \otimes (\overline{k})^2 \right)$ and $DC = \overline{k} \left( kx \otimes k^3 \otimes (\overline{k})^2 \right)$ whence $CD = q^2 DC$. Then, from Lemma 3.3.6

$$R_{13} \cdot R_{23} = e^{\frac{h}{\hbar} s} \text{Exp}_q \left( \lambda_q (kx \otimes k^2 \otimes \overline{k}y + 1 \otimes kx \otimes \overline{k}y) \right).$$

It follows from (4.9) that this is equal to $(\Delta \otimes \text{id})(R)$, so the Condition is satisfied.
4.2.4. Condition [iii] of Definition 4.1.2. It may be shown similarly that this condition is also satisfied. We have therefore proved the following:

**Theorem 4.2.1** (An R-matrix).

\[
R = e^{\frac{h}{2}}\sum_{n=0}^{\infty} \frac{(q-\bar{q})^n}{n!} q^{\frac{1}{2}n(n-3)} (k\chi)^n \otimes (\bar{k}\chi)^n.
\]

is a universal R-matrix for \(\mathcal{U}_h(sl_2)\).

**Remark 4.2.2.** Three comments on the derivation of R:

(A) Condition [iii] of Definition 4.1.2 was required only for the generators \(h\) and \(x\) to fully define the unknown series \(q_{k,r,s,m,n}(q)\), in Ansatz (4.3) and thence \(R\). It was then confirmed that this Condition also held for the other generator \(\gamma\), and that Conditions [ii] and [iii] also held. This suggests that Ansatz (4.3) is a particularly restrictive one, in that Conditions [ii] and [iii] were forced.

(B) The salient aspects of the derivation of \(R\) are (i) the appearance of the commutator \([x,y]^n\) in (4.9), and (ii) the fact that \(R\) may be expressed succinctly in terms of the quantum exponential function in (4.8). The factorization property given in Lemma 3.3.6 for the quantum exponential is crucial in showing that Conditions [ii] and [iii] of Definition 4.1.2 hold.

(C) The above argument may be used to show that if the form for \(R\) proposed in (4.2) is an R-matrix then it is the one given in Theorem 4.2.1.

4.3. **Ribbon Hopf algebra structure on \(\mathcal{U}_h(sl_2)\).** In this section, we compute the ribbon Hopf algebra structure on \(\mathcal{U}_h(sl_2)\) equipped with the R-matrix computed in Theorem 4.2.1. These calculations are similar to those in [Oht02, Appendix A], but are simpler and more direct. For instance, our calculation of \(S(u)\), based on the combinatorial identity in Lemma 3.3.5, is significantly shorter than the corresponding calculation in [Oht02].

4.3.1. **Ribbon Hopf Algebras.** Let \(\mathcal{A}\) be a quasi-triangular Hopf algebra with R-matrix \(R = \sum \alpha_i \otimes \beta_i\). Consider the element

\[
u := \sum \alpha_i \otimes \beta_i \in \mathcal{A}.
\]

Drinfel’d observed in [Dri89] that this element satisfies the following properties:

(i) For all \(x \in \mathcal{A}\), the square of the antipode \(S^2 : \mathcal{A} \rightarrow \mathcal{A}\) acts by conjugation \(S^2(x) = u x u^{-1}\);

(ii) \(\Delta(u) = (\tau(R) \cdot R)^{-1}(u \otimes u) = (u \otimes u)(\tau(R) \cdot R)^{-1}\).

(iii) \(u\) is invertible, with

\[
u^{-1} = \sum \alpha_i S^{-1}(\beta_i) \otimes \beta_i, \quad \text{where } R^{-1} = \sum \alpha_i \otimes \beta_i.
\]

See [Kas95, pp.180–184] for details.

Ribbon Hopf algebras are quasi-triangular Hopf algebras which admit a sort of square root to the element \(u\). They were introduced by Reshetikhin and Turaev [RT90] in order to construct a polynomial invariant for framed links.

**Definition 4.3.1.** A ribbon Hopf Algebra is quasi-triangular Hopf algebra \((\mathcal{A}, R)\) equipped with an element \(v \in \mathcal{A}\), called a ribbon element, satisfying:

(i) \(v\) is central;

(ii) \(v^2 = S(u) \cdot u\);

(iii) \(S(v) = v\);

(iv) \(\varepsilon(v) = 1\);

(v) \(\Delta(v) = (\tau(R) \cdot R)^{-1}(v \otimes v)\).

We now explicitly compute a ribbon element for \(\mathcal{U}_h(sl_2)\). First we compute the element \(u\) for the R-matrix in Theorem 4.2.1. Condition [iii] of Definition 4.3.1 then allows us to compute \(v\). These are referred to as the ribbon elements.
4.3.2. The element $u$. We compute $u$ from its definition given in (4.10).

Lemma 4.3.2. $u = e^{-\frac{h}{4}y^2} \sum_{n \geq 0} (-1)^n \frac{(q - \overline{q})^n}{[n]!} \overline{q}^{n(n+3)} k^n y^n x^n$.

Proof. From Theorem 4.1.1 and Lemma 3.1.3(iv),

$$R = \sum_{m,n \geq 0} \frac{h^n}{m^n m!} c_n \overline{q}^{n(n-1)} (h^m k^n x^n) \otimes (h^m k^n x^n)$$

where

$$(4.12) c_n := \frac{(q - \overline{q})^n}{[n]!} \overline{q}^{n(n-3)}$$

and so, from (4.10),

$$u = \sum_{m,n \geq 0} \frac{h^n}{m^n m!} c_n \overline{q}^{n(n-1)} \left( \left( h^m k^n x^n \right) \otimes \left( h^m k^n x^n \right) \right).$$

To straighten the summand, we note that $S(k) = k$ from (4.1), and that $h$ and $k$ commute. Using the action of $S$ defined in Theorem 4.1.1, we have

$$\left( S(h^m k^n x^n) \right) \otimes \left( h^m k^n x^n \right) = (-1)^m n_q h^m k^n x^n = (-1)^{m+n} q^m h^m k^n x^n$$

from Lemma 3.1.3(vi) and (vi). So

$$u = \sum_{n \geq 0} (-1)^n c_n q^m e^{-\frac{h}{4}(h+2n)^2} k^{2n} y^n x^n.$$ 

The result then follows by observing that $e^{-\frac{h}{4}(h+2n)^2} = \overline{q}^{2n} e^{-\frac{h}{4} h^2} \cdot 1^n$. \hfill $\square$

Using Lemma 4.3.2 we compute the action of the antipode on $u$.

Lemma 4.3.3. $S(u) = k^2 u$.

Proof. Let $d_n := (-1)^n \frac{(q - \overline{q})^n}{[n]!} \overline{q}^{n(n+3)}$. Then

$$k^2 u = e^{-\frac{h}{4}h^2} \sum_{n \geq 0} d_n k^{2(n+2)} y^n x^n.$$ 

From the definition of $S$ in Theorem 4.1.1 and from its action on $k$ given in (4.1),

$$S(u) = \sum_{n \geq 0} d_n x^n y^n k^{2n} e^{-\frac{h}{4}h^2}.$$ 

Now $k^{2n}$ and $e^{-\frac{h}{4}h^2}$ commute and, by Lemma 5.1.3(ii), both of these commute with $x^n y^n$, so

$$S(u) = e^{-\frac{h}{4}h^2} \sum_{n \geq 0} d_n k^{2n} x^n y^n.$$ 

Using Theorem 2.1.2 we can establish the equality of $k^2 u$ and $S(u)$ by comparing coefficients. First we compare the coefficients of $y^n x^n$ with the aid of Lemma 3.2.2. Doing so, the assertion of the Lemma is therefore equivalent to the identity

$$d_n k^{2(n+2)} = \sum_{m \geq n} d_m \frac{[m]^2}{[n]^2} k^{2m} \left[ \frac{h}{m-n} \right]$$

and so, by changing the index of summation to $s := m - n$, to showing that

$$k^{4(n+1)} = A_n \quad \text{where} \quad A_n := \sum_{s \geq 0} d_{n+s} \frac{[n+s]^2}{[s]^2} k^{2s} \left[ \frac{h}{s} \right].$$

It is in this form that we shall prove the lemma. From (2.3) and (2.6),

$$\left[ \frac{h}{s} \right] = 1 \sum_{r=0}^{s-1} \frac{q^r}{[s]!} \frac{q^{s-r}}{[r]!} = 1 \sum_{r=0}^{s-1} \frac{q^r}{[s]!} \frac{q^{s-r}}{[r]!} \prod_{r=0}^{s-1} (k^4 - q^{2r})$$
Let

\[
A_n = \sum_{s \geq 0} (-1)^s q^{-s(n+s+1)} \binom{n+s+1}{s} \prod_{r=0}^{s-1} (k^4 - q^{2r}).
\]

We shall now transform the quantum binomial coefficient into a \(q\)-binomial coefficient through \(3.11\) and then to a negative \(q\)-binomial coefficient to obtain

\[
A_n = \sum_{s \geq 0} (-1)^s q^{s(2n+s+1)} \binom{n+s}{s} \prod_{r=0}^{s-1} (k^4 - q^{2r}) = \sum_{s \geq 0} \binom{-n-1}{s} \prod_{r=0}^{s-1} (k^4 - q^{2r}).
\]

Using the expressions \(q^2 = e^h\) and \(k^4 = e^{hh}\) for these elements in \(\mathcal{U}_h(\mathfrak{sl}_2)\), we have \(A_n = e^{-(n+1)hh}\).

4.3.3. Construction of \(v\). We begin by constructing a putative expression for \(v\) from Definition \(4.3.1\). Lemma \(4.3.3\) says \(v^2 = k^4 u^2\). But property \(\mathbf{[ii]}\) of \(u\) says \(S^2(k)u = u k\). Using \(4.1\), \(u k = k u\), so

\[(i) \quad u \text{ and } k \text{ commute}; \quad (ii) \quad v = k^2 u.\]

Therefore, from Lemma \(4.3.2\) we have the putative expression

\[
e^{-\frac{h}{2}} \cdot \sum_{n \geq 0} \frac{(q - q)^n}{n!} q^{-n(n+3)} k^{2(n+1)} y^n x^n
\]

for \(v\). We show next that this expression satisfies the appropriate conditions.

**Theorem 4.3.4.** \((\mathcal{U}_h(\mathfrak{sl}_2), R, v)\) is a Ribbon Hopf algebra, where

\[
v := e^{-\frac{h}{2}} \cdot \sum_{n \geq 0} \frac{(q - q)^n}{n!} q^{-n(n+3)} k^{2(n+1)} y^n x^n
\]

and

\[
R := e^{\frac{h}{2} hh} \cdot \sum_{n \geq 0} \frac{(q - q)^n}{n!} q^{-n(n-3)} (kx)^n \otimes (ky)^n.
\]

**Proof.** We have already shown that \((\mathcal{U}_h(\mathfrak{sl}_2), R)\) is a quasi-triangular Hopf algebra. It remains to check that the conditions \((i)-(v)\) of Definition \(4.3.1\) are satisfied.

Condition \(\mathbf{[i]}\) asks for \(v\) to be central. It is enough to show \(v\) commutes with the generators \(h, x\) and \(y\).

The generator \(h\): Lemma \(3.1.3\) \(\mathbf{[ii]}\) implies \(h\) and \(y^x x^y\) commute, so \(h\) commutes with \(v\). The generator \(x\): Let \(T_n := e^{-\frac{h}{2}} q^\frac{n^2}{2} k^{n(n+1)} y^n x^n\) be a general term of \(v\). From Lemma \(3.1.3\) \(\mathbf{[ii]}\),

\[
x e^{-\frac{h}{2}} q^\frac{n^2}{2} k^{n(n+1)} (y^n x^n + [n] [h + n - 1] y^{n-1} x^n)
\]

and so, using the notation from \(4.12\),

\[
x v = e^{-\frac{h}{2}} \sum_{n \geq 0} c_n q^\frac{n^2}{2} k^{n(n+1)} (y^n x^n + [n] [h + n - 1] y^{n-1} x^n)
\]

by shifting the summation index for the term containing \(y^n x^n\) by one. The bracketed term is equal to

\[
\frac{(q-q)^{n-1}}{[n-1]!} \cdot \frac{q^{-n-1} q^2}{k^{n^2+3n-4}}.
\]

Therefore

\[
x v = e^{-\frac{h}{2}} q^\frac{(n^2+n-1)}{2} \frac{(q-q)^{n-1}}{[n-1]!} \cdot \frac{q^n}{k^n} y^{n-1} x^n.
\]

The right hand side is readily seen to be equal to \(v x\) by shifting the summation index for the expression for \(v\) to start at 0.
The generator $y$: This is proved similarly. Condition (iii) holds by construction of $v$. Conditions (iii) and (iv) are immediate by definition of the maps $S$ and $e$ from Theorem 4.1.1. To check condition (v), substitute $v$ with $\kappa^u$ and rearrange. Now we show that $(\tau(R) \cdot R) \cdot (\kappa^u \otimes \kappa^u) \cdot \Delta(u) = (\kappa^u \otimes \kappa^u) \cdot (u \otimes u)$. Since each term of $R$ (and thus $\tau(R)$) commutes with $\kappa^u \otimes \kappa^u$, it is sufficient to show $(\tau(R) \cdot R) \cdot \Delta(u) = u \otimes u$. But this is property (iii) of $u$.

5. Extension to $\mathcal{O}_h(\mathfrak{sl}_{n+1})$

In this section, we generalize the discussion in the earlier part of this article and consider straightening in $\mathcal{O}_h(\mathfrak{sl}_{n+1})$ for $n \geq 2$. We then apply our formalism to construct $R$-matrix for $\mathcal{O}_h(\mathfrak{sl}_{n+1})$, following the ideas of §4. For other calculations of $R$-matrix, see, for example, [CP94; Ram98].

The quantized universal enveloping algebra $\mathcal{O}_h(\mathfrak{sl}_{n+1})$, $n \geq 1$, of the Lie algebra $\mathfrak{sl}_{n+1}$ is defined as follows. For a general treatment of quantized universal algebras, see, for example, [CP94; Ram98].

**Definition 5.0.1.** The quantized universal enveloping algebra $\mathcal{O}_h(\mathfrak{sl}_{n+1})$ is the $\mathbb{C}[\hbar]$-algebra generated by $n \mathfrak{sl}_2$ triples $(x_i, h_i, y_i)$, $i = 1, \ldots, n$, subject to the relations $[h_i, h_j] = 0$, $[x_i, y_j] = \delta_{ij} \frac{k_i^2 - k_i^2}{q - \bar{q}}$, where $k_i := e^{\hbar h_i}$, and for each $i, j$ with $|i - j| = 1$, $q$-Serre relations

$$
(5.1) \quad [h_i, x_j] = \begin{cases} 2x_j & j = i, \\ -x_j & j = i + 1, \\ 0 & \text{otherwise,} \end{cases} \quad [h_i, y_j] = \begin{cases} -2y_j & j = i, \\ y_j & j = i + 1, \\ 0 & \text{otherwise,} \end{cases} \quad [x_i, y_j] = \delta_{ij} \frac{k_i^2 - k_i^2}{q - \bar{q}},
$$

where $k_i := e^{\hbar h_i}$, and for each $i, j$ with $|i - j| = 1$, $q$-Serre relations

$$
x_i^2 x_j - (q + \bar{q}) x_i x_j x_i + x_j x_i^2 = 0, \quad y_j^2 y_i - (q + \bar{q}) y_j y_i y_j + y_i y_j^2 = 0,
$$

A Hopf algebra structure on $\mathcal{O}_h(\mathfrak{sl}_{n+1})$ may be defined on the generators $(x_i, h_i, y_i)$ by taking the maps from Theorem 4.1.1 on each $\mathfrak{sl}_2$-triple.

5.1. **Simplifying the Serre relations.**

5.1.1. **Cubic $q$-Serre relations.** Straightening methods may be applied also to quadratic relations, viewed as commutation relations. As such, the cubic $q$-Serre relations must be simplified to quadratic relations. This suggests we perform the following manipulation on the Serre relation for $i$ and $i + 1$:

$$
0 = x_i^2 x_{i+1} - (q + \bar{q}) x_i x_{i+1} x_i + x_{i+1} x_i^2 = x_i(x_i x_{i+1} - \bar{q} x_{i+1} x_i) - (q x_i x_{i+1} - x_{i+1} x_i x_i) = \bar{q} x_i(q x_i x_{i+1} - \bar{q} x_{i+1} x_i) - q x_i(q x_i x_{i+1} - \bar{q} x_{i+1} x_i x_i).
$$

Similarly, we may manipulate the Serre relation for $i + 1$ and $i$ to get:

$$
q x_{i+1}^2 x_i - q x_{i+1} x_i x_{i+1} = -q x_{i+1}(q x_i x_{i+1} - \bar{q} x_{i+1} x_i) x_i.
$$

Thus, defining $x_{i,i+1}$ by

$$
x_{i,i+1} := q \frac{q}{2} x_{i} x_{i+1} - \frac{q}{2} x_{i+1} x_{i},
$$

the $q$-Serre relations involving $i$ and $i + 1$ may be rewritten as

$$
\bar{q} x_{i,i+1} x_i - q x_{i,i+1} x_{i+1} = 0, \quad \text{and} \quad q x_{i,i+1} x_{i+1} - q x_{i,i+1} x_{i+1} = 0.
$$

**Example 5.1.1.** When $n = 2$, the ordered set of elements

$$
\{h_1, h_2, y_1, y_2, x_1, x_2\}
$$

of $\mathcal{O}_h(\mathfrak{sl}_3)$ form a set of algebra generators in which all relations are quadratic relations amongst pairs of generators. Moreover, the set of generators gives rise to a PBW basis

$$
\mathcal{B} := \{(h_1, h_2, y_1, y_2, x_1, x_2) : r_i, s_i, t_i \in \mathbb{Z}_{\geq 0}\}
$$

and straightening can be used effectively to perform computations.
5.1.2. Higher Degree Generators in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. For $n \geq 3$, the elements $x_{i,j+1}$ are not enough to transform all algebra relations of $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ to commutation relations. Consider, for example, the relations between $x_{i,j+1}$ and $x_{i+2}$. Writing out the relevant Serre relations shows that they are related by a cubic relation as well. Thus we must introduce elements $x_{i,j}$ for each pair of indices $1 \leq i < j \leq n$. So, for each $i = 1, \ldots, n$, set $x_{i,j} := x_i$ and for each pair of indices $1 \leq i < j \leq n$, inductively define
\begin{equation}
 x_{i,j} := q^\frac{1}{2} x_{i} x_{i+1,j} - q^{-\frac{1}{2}} x_{i+1,j} x_{i}.
\end{equation}

A short induction argument shows that if we take
\begin{equation}
 h_1 < \cdots h_2 < \cdots < h_n < y_1 < y_2 < \cdots < y_{1,n} < y_2 < \cdots < y_{n-1,n} < y_n < x_1 < x_{12} < \cdots < x_{n-1,n} < x_n
\end{equation}
as an ordered set of algebra generators for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$, then all the algebra relations of $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ are generated by quadratic relations amongst these generators. Moreover, this set of generators gives rise to a PBW basis as in Example [5.1.1]. This final statement can be established directly as in [Ros89] or via the general theory of quantum groups, as developed in [Lus93, Chapter 40].

5.2. Combinatorial description of generators. There is a more symmetric description of $x_{i,j}$.

**Lemma 5.2.1** (Splitting). Let $1 \leq i < j \leq n$ and let $s \in \{i, \ldots, j-1\}$. Then
\begin{equation}
 x_{i,j} = q^\frac{1}{2} x_{i} x_{i+s+1,j} - q^{-\frac{1}{2}} x_{i+s+1,j} x_{i}.
\end{equation}

To establish this Lemma, we give a combinatorial description of $x_{i,j}$ in terms of monomials in the $x_{i}, \ldots, x_{j}$ indexed by orientations on a path of length $j - i$. Specifically, let $P_n$ denote the Dynkin diagram for $\mathfrak{sl}_{n+1}$, i.e. a path consisting of $n$ vertices labelled from left to right by the integers 1 to $n$. For $1 \leq i < j \leq n$, let $P_{i,j}$ denote the induced subgraph of $P$ obtained by taking the vertices labelled $i, i+1, \ldots, j$. Let $\mathcal{D}_{i,j}$ denote the set of orientations on $P_{i,j}$. For each orientation $D \in \mathcal{D}_{i,j}$, let
\begin{align*}
 D_+ & := \#(\ell \rightarrow \ell + 1 \in D), \\
 D_- & := \#(\ell \leftarrow \ell + 1 \in D)
\end{align*}
be the number of right- and left-pointing arrows in the orientation $D$.

For each $D \in \mathcal{D}_{i,j}$, set
\begin{equation}
 q^{D} := (-1)^{D_-} q^{\frac{1}{2}(D_+ - D_-)}.
\end{equation}

For $D \in \mathcal{D}_{i,j}$, let $x_{D}$ be the monomial $x_{i}, \ldots, x_{j}$ constructed as follows. Begin by writing $x_{j}$. Next, if $j - 1 \rightarrow j \in D$, then place $x_{j-1}$ to the left of $x_{j}$; otherwise, $j - 1 \leftarrow j \in D$ and so place $x_{j-1}$ on the right of $x_{j}$. Next, if $j - 2 \rightarrow j - 1 \in D$, then place $x_{j-2}$ at the leftmost end; otherwise, place it on the rightmost. At each step, regard a right-pointing arrow $\ell - 1 \rightarrow \ell$ as indicating that $x_{\ell-1}$ and $x_{\ell}$ are to be positioned “in order”, so that $x_{\ell-1}$ appears before $x_{\ell}$; similarly, a left-pointing arrow $\ell - 1 \leftarrow \ell$ indicates that $x_{\ell-1}$ and $x_{\ell}$ appear “out of order”. Continue this process until all of $x_{i}$ to $x_{j}$ have been placed and call the result $x_{D}$.

**Example 5.2.2.** Suppose $n = 8$, $i = 2$ and $j = 6$. Then
\begin{align*}
 P_{7} & = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7, \\
 P_{2,6} & = 2 \quad 3 \quad 4 \quad 5 \quad 6.
\end{align*}
The set $\mathcal{D}_{2,6}$ has $2^4 = 16$ elements, one being $D = 2 \quad 3 \quad 4 \quad 5 \quad 6$. Then $D_+ = 3, D_- = 1, q^{D} = -q$. The construction of $x_{D}$ proceeds through the following steps: $x_{6}, x_{5}x_{6}, x_{4}x_{5}x_{6}, x_{4}x_{5}x_{6}x_{3}$, and finally $x_{D} = x_{2}x_{4}x_{3}x_{5}x_{6}$. We now have the following combinatorial formula for $x_{i,j}$.

**Lemma 5.2.3.** Let $1 \leq i < j \leq n$. Then $x_{i,j} = \sum_{D \in \mathcal{D}_{i,j}} q^{D} x_{D}$.

**Proof.** We use induction on the difference $j - i$. When $j = i + 1$, this formula reduces to the definition of $x_{i,i+1}$ in [5.2]. In general,
\begin{equation}
 x_{i,j} = q^\frac{1}{2} x_{i} x_{i+1,j} - q^{-\frac{1}{2}} x_{i+1,j} x_{i} = \sum_{D \in \mathcal{D}_{i+1,j}} (-1)^{D_-} q^{\frac{1}{2}(D_+ + 1 - D_-)} x_{i} x_{D} + (-1)^{D_- + 1} q^{\frac{1}{2}(D_+ - (D_- + 1))} x_{D} x_{i}.
\end{equation}
Since \( i \) is the smallest index and by the construction of \( x_{Dv} \) with \( D' \in \mathcal{D}_{i,j} \), the first terms in the sum correspond to orientations \( D' \) with right-pointing arrow \( i \to i+1 \). Similarly the second terms correspond to orientations with left-pointing \( i \to i+1 \).

Note that the \( x_D \) are simply convenient representatives of commutation equivalence class of monomials in \( x_i \) — recall that \( x_i x_j = x_j x_i \) whenever \( |i-j| > 1 \). The important data encoded in the orientation \( D \in \mathcal{D}_{i,j} \) is the relative position of adjacent generators. It is clear that any product of \( x_i, \ldots, x_j \) satisfying \( x_{i+1} \) is right of \( x_i \) if and only if \( i \to i+1 \) in \( D \) is equivalent to \( x_D \).

**Proof of Lemma 5.2.1.** This follows immediately from Lemma 5.2.3 and the discussion above after observing that an orientation on \( P_{i,j} \) consists of orientations on \( P_{i+1} \) and \( P_{i,j-1} \) and an orientation on the edge \((s, s+1)\). The term \( q^2 x_{si+1} x_{si+1} \) accounts for all orientations with \( s \to s+1 \) where as \(-q^{-2} x_{si+1} x_{si+1} \) accounts for orientations with \( s \to s+1 \).

### 5.3. Straightening in \( \mathcal{U}_h(\mathfrak{s}(n+1)) \)

When straightening generators with the same subscript, all the straightening laws developed in §3.1 apply. For straightening terms with different subscripts, we have the following set of “mixed” straightening laws. The proofs of these are straightforward and are omitted.

**Lemma 5.3.1.** Let \( a \in \mathbb{Z}_{>0}, b \in \mathbb{Z}, i, j \in \{1, \ldots, n\}, l \in \{1, \ldots, n\}, \) and \( f(x) \) a formal power series in \( x \). Then the following identities hold in \( \mathcal{U}_h(\mathfrak{s}(n+1)) \):

1. \( x_i f(h_i) = \begin{cases} f(h_i+1)x_i, & \text{if } i \neq j, \\ f(h_i)x_i, & \text{if } i = j, \end{cases} \)
   \[ y_i f(h_i) = \begin{cases} f(h_i+1)y_i, & \text{if } i \neq j, \\ f(h_i)y_i, & \text{if } i = j, \end{cases} \]

2. \( \begin{aligned} x_i^a y_i^b = & q^a b_{i+1} x_i^a y_i^b, \\ y_i^a x_i^b = & q^{-a} b_{i+1} y_i^a x_i^b, \end{aligned} \)

3. \( \begin{aligned} x_i x_j = & q^{\delta_{i,j}} x_i x_j, \\ y_i y_j = & q^{-\delta_{i,j}} y_i y_j, \end{aligned} \)

4. \( \begin{aligned} x_i^a y_i^b = & q^{a b_{i+1}} x_i^a y_i^b, \\ y_i^a x_i^b = & q^{-a b_{i+1}} y_i^a x_i^b, \end{aligned} \)

Interesting combinatorial structure appears when commuting \( x_i \) through terms like \( x_{i+1,j} \) or \( y_{i,j} \). In doing so, the relations of \( \mathcal{U}_h(\mathfrak{s}(n+1)) \) either lengthen or shorten the interval indexing \( x_{i+1,j} \) or \( y_{i,j} \). More precisely, the following pair of lemmas hold.

**Lemma 5.3.2 (Lengthening).** Let \( 1 \leq i < j \leq n \). Then the following hold in \( \mathcal{U}_h(\mathfrak{s}(n+1)) \).

1. \( x_i^a x_{i+1,j} = q^{a b_{i+1}} x_i^a x_{i+1,j} - q^{a b_{i+1}} x_i x_{i+1,j} \)
2. \( x_j x_{i+1,j} = q^{a b_{i+1}} x_j x_{i+1,j} - q^{a b_{i+1}} x_j x_{i+1,j} \)

**Lemma 5.3.3 (Shortening).** Let \( 1 \leq i < j \leq n \). Then the following hold in \( \mathcal{U}_h(\mathfrak{s}(n+1)) \).

1. \( y_i^a y_{i,j} = q^{a b_{i+1}} y_i^a y_{i,j} - q^{a b_{i+1}} y_i y_{i,j} \)
2. \( x_j y_{i,j} = q^{a b_{i+1}} x_j y_{i,j} - q^{a b_{i+1}} x_j y_{i,j} \)
If there is no ambiguity, we may suppress the argument $K$ for some matrix of coefficients $U$ for ordered products of the generators in $U$.

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Lemma 5.3.4 (Passing). Let $1 \leq i < j \leq n$. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$.

(i) \[ x_i^{m_i} x_j^{m_j} = x_j^{m_j} x_i^{m_i}, \]

Moreover, for any $i < s < j$, \[ x_s^{m_s} x_i^{m_i} x_j^{m_j} = x_i^{m_i} x_s^{m_s} x_j^{m_j}, \]

Finally, we have the following analogue of Lemma 3.1.4 for moving generators past the exponential factor in the $R$-matrix.

Lemma 5.3.5. Let $f(x)$ be a formal power series in $x$, $a \in \mathbb{Z}_{>0}$, $\kappa \in \mathbb{Z}$ any integer and $i, j, l \in \{1, \ldots, n\}$. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$:

\[
\begin{align*}
  f(1 \otimes x_i^a) e^{\frac{\kappa}{2} h_i \otimes h_j} &= \begin{cases} 
    e^{\frac{\kappa}{2} h_i \otimes h_j} f(k_i^2 \otimes x_i^a), & \text{if } j = i, \\
    e^{\frac{\kappa}{2} h_i \otimes h_j} f(k_i^2 \otimes x_i^a), & \text{if } |j - i| = 1, \\
    e^{\frac{\kappa}{2} h_i \otimes h_j} f(1 \otimes x_i^a), & \text{otherwise},
  \end{cases} \\
  f(1 \otimes y_i^a) e^{\frac{\kappa}{2} h_i \otimes h_j} &= \begin{cases} 
    e^{\frac{\kappa}{2} h_i \otimes h_j} f(k_i^2 \otimes y_i^a), & \text{if } j = i, \\
    e^{\frac{\kappa}{2} h_i \otimes h_j} f(k_i^2 \otimes y_i^a), & \text{if } |j - i| = 1, \\
    e^{\frac{\kappa}{2} h_i \otimes h_j} f(1 \otimes y_i^a), & \text{otherwise},
  \end{cases}
\end{align*}
\]

and similarly for $f(x_i^a \otimes 1)$ and $f(y_i^a \otimes 1)$.

5.4. Constructing an R-matrix for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. We carry out the program of §4 to construct an $R$-matrix for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. An ansatz for $R$ can be constructed from first principles through general reasoning about straightening relations, as was done for $\mathcal{U}_h(\mathfrak{sl}_2)$. Alternatively, and more efficiently, we use the general principle that objects in semisimple Lie algebras can be built by appropriately combining ingredients from constituent $\mathfrak{sl}_2$ to obtain an ansatz for $R$ here. This principle, together with the form of $R$ given in Theorem 4.2.1 leads us to propose the following Ansatz for the form of $R$ in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$:

\begin{equation}
R := e^K \sum_m \alpha(m) X(m) \otimes \overline{K}(m) Y(m)
\end{equation}

where $m := (m_1, m_2, m_{12}, \ldots, m_{n-1,n}, m_n)$ is a vector of integers ordered as generators are ordered, the coefficient $\alpha(m)$ is a rational function in $q$, and the $K, \overline{K}, X$ and $Y$ are defined by

\[
X(m) := x_1^{m_{12}} \cdots x_{n-1,n}^{m_{n-1,n}} x_n^{m_n}, \\
K(m) := k_1^{m_1} k_{12}^{m_{12}} \cdots k_n^{m_n}, \\
Y(m) := y_1^{m_{12}} \cdots y_{n-1,n}^{m_{n-1,n}} y_n^{m_n}, \\
\overline{K}(m) := \overline{k}_1^{m_1} \overline{k}_{12}^{m_{12}} \cdots \overline{k}_n^{m_n},
\]

for ordered products of the generators in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$, $k_{ij} := k_i k_{i+1} \cdots k_j$, and

\[
e^K := \exp \left( \sum_{i,j=1}^n x_{i,j} h_i \otimes h_j \right)
\]

for some matrix of coefficients $K := (k_{ij})$. In using this notation, use the abbreviated form $X(m_{ij} - 1; m) := X(m_1, m_{12}, \ldots, m_{ij-1}, \ldots, m_n)$, for example, to indicate a change in exponent of some monomial. If there is no ambiguity, we may suppress the argument $m$ and simply write $X(m_{ij} - 1)$. Now the coefficients and the exponents in (5.4) are determined by imposing Conditions (i), (ii) and (iii) of Definition 4.1.2. As before, it Condition (ii) completely specifies the free parameters in (5.4) and the
remaining conditions need to be verified with the resulting expression. Here, we only perform the first step and leave the latter two conditions to the interested reader.

**Remark 5.4.1.** Verification of Conditions (11) and (11) can be done in a manner similar to the $\Psi_{lh}(sl_2)$ case. That being said, the $\Psi_{lh}(sl_{n+1})$ case is much more complicated due to non-commuting $q$-exponentials in (6.2). Ultimately, this problem is solved by studying in detail how $q$-exponentials commute in the presence of Lengthening and Shortening. This is done, for example, in [KT91] through their approach.

6. DERIVING THE TERMS IN THE ANSATZ (5.4)

We determine the terms in the Ansatz (5.4) for the R-matrix. These are the $K$ terms, the $A$ terms, the $B$ terms, the $a$ term and the exponential prefactor. These are determined in separate subsections clarifying the separate course of the construction.

6.1. Deriving the Coefficients. It suffices to impose Condition (11) for $x_i$, $h_i$ and $y_i$, $i = 1, \ldots, n$. In fact, since all relations involving the $y_i$ are mirror to those involving the $x_i$, the calculations required for the $y_i$ are completely analogous to those of the $x_i$. So we only need to perform computations for the $h_i$ and $x_i$.

6.1.1. For the generators $h_i$. Since the $h_i$ commute with $k_1, \ldots, k_n$, we need only consider the $x_j$ and $y_j$ components of each general term in (5.4). Write a general term as $x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} \otimes y_1^{t_1} y_2^{t_2} \cdots y_n^{t_n}$. Using Lemma (5.3.1) and (3), Condition (1) for the $h_i$ are equivalent to the system of equations

\[
\begin{align*}
2(s_1 - t_1) + \sum_{j=2}^{n} (s_j - t_j) - \sum_{j=2}^{n} (s_j - t_j) &= 0, \\
- \sum_{i=1}^{n-2} (s_{i,n-1} - t_{i,n-1}) + \sum_{i=1}^{n-1} (s_{i,n} - t_{i,n}) + 2(s_n - t_n) &= 0.
\end{align*}
\]

These equations are satisfied when $s_{ij} = t_{ij} = m_{ij}$ as in (5.4).

6.1.2. For the generators $x_i$. Recall that $\Delta(x_i) = x_i \otimes k_i + k_i \otimes x_i$. Thus

\[
(1) \quad (k_i \otimes x_i + x_i \otimes k_i) \cdot R = R \cdot (x_i \otimes k_i + k_i \otimes x_i).
\]

Following Remark 4.2.3, decompose the general terms of (6.1) as follows. For $i = 1, \ldots, n$ and for each $m = (m_1, m_2, \ldots, m_n)$, set

\[
\begin{align*}
A_i^+(m) := (k_i \otimes x_i) \cdot e^K K(m) X(m) \otimes \overline{K}(m) Y(m), \\
B_i^-(m) := (x_i \otimes k_i) \cdot e^K K(m) X(m) \otimes \overline{K}(m) Y(m), \\
A_i^-(m) := e^K K(m) X(m) \otimes \overline{K}(m) Y(m) (k_i \otimes x_i), \\
B_i^+(m) := e^K K(m) X(m) \otimes \overline{K}(m) Y(m) (x_i \otimes k_i),
\end{align*}
\]

so (6.1) is the assertion

\[
\sum_m A_i^+(m) + B_i^-(m) = \sum_m A_i^-(m) + B_i^+(m).
\]

As in (4.5), we rearrange the sums so that equation (6.1) is equivalent to

\[
A_i := \sum_m A_i^+(m) - A_i^-(m) = \sum_m B_i^+(m) - B_i^-(m) =: B_i.
\]

We wish to simplify the series on both sides. In the $\Psi_{lh}(sl_2)$ case, the essential simplification to the $A$ side came about in identifying a commutator $[x_i, y_i^{m_1}]$ in (4.6). Analogously, if we straighten and apply the Shortening Lemma (5.3.3) to the terms in $A_i^+(m) - A_i^-(m)$, we find a single term with $\cdots \cdot x_i y_i^{m_1} \cdots$ in $A_i^+$ and a single term with $\cdots y_i^{m_1} x_i \cdots$ in $A_i^-$ which should be combined to form a commutator $[x_i, y_i^{m_1}]$. 
6.1.3. _Exponential Prefactor_. As a first step to combining \( A_i^+(m) \) with \( A_i^-(m) \), we should have that the exponential factor \( e^K \) of both terms coincide after straightening. From (6.2), this means that we should have
\[
(k_i \otimes x_i) e^K = e^K (k_i \otimes x_i).
\]
Using Lemma 5.3.1(ii) and Lemma 5.3.5, we obtain the following system of \( n \) linear equations:
\[
\kappa_{i, i-1} - 2\kappa_{i, j} + \kappa_{i, i+1} = -2\delta_{ij}, \quad i = 1, \ldots, n,
\]
where \( \kappa_{i, i-1} := 0 \) and \( \kappa_{i, n+1} := 0 \). In matrix form,
\[
KC_n := K \cdot 
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & -1 & 2
\end{pmatrix} = 2I_n
\]
where \( I_n \) is the \( n \times n \) identity matrix. It is readily seen that \( C_n \) is invertible, so \( K = 2C_n^{-1} \). The inverse can be explicitly calculated, yielding
\[
(6.3)
\kappa_{ij} = \begin{cases} 
\frac{2}{n+1} i(n-j+1) & \text{if } j \leq i, \\
\frac{2}{n+1} j(n-i+1) & \text{if } i \leq j.
\end{cases}
\]

For what follows, we may pull the exponential factor out of \( A_i \) and \( B_i \), at which point it remains to straighten the terms of the form \((k_i \otimes x_i)K(m)X(m) \otimes \bar{K}(m)Y(m)\), in order to deduce a recurrence relation on the coefficients \( \alpha(m) \). Because of Shortening and Lengthening phenomena, each summand of \( A_i \) and \( B_i \) will itself be a sum of terms indexed by certain segments of \([1, \ldots, n]\). To be clear, _straightening_ the term \( x_iY \) means that we need to move \( x_i \) through the \( Y \) until it is immediately left of \( y_i \); similarly, _straightening_ \( Yx_i \) means that \( x_i \) needs to be moved until it is immediately right of \( y_i \).

6.2. _The \( A_i^- \) terms._

6.2.1. _Terms of \( A_i^- = (k_i \otimes x_i) \mathbf{R} \)._ When straightening \( A_i^+ \), \( x_i \) needs to be moved past the \( \mathbf{R} \) in front of the \( Y \). From Lemma 5.3.1(ii), commuting \( x_i \) past \( \bar{k}_{a,b} \) has a contribution whenever the segment \([a, a+1, \ldots, b]\) contains either one or two of \([i-1, i, i+1]\). Precisely, the coefficient contribution when \( x_i \) is moved past a \( \bar{k}_{a,b} \) indexed by a segment
- \([a, \ldots, i-1]\) for \( 1 \leq a < i \) is \( q^{\frac{1}{2}m_{a,i-1}} \);
- \([a, \ldots, i]\) for \( 1 \leq a < i \) is \( q^{\frac{1}{2}m_{a,i}} \);
- \([i]\) is \( q^{m_i} \);
- \([i, \ldots, b]\) for \( i < b \leq n \) is \( q^{\frac{1}{2}m_{i,b}} \); and
- \([i+1, \ldots, b]\) for \( i < b \leq n \) is \( q^{\frac{1}{2}m_{i+1,b}} \).

This leads to a combined coefficient contribution of
\[
(6.4)
\left(\prod_{a=1}^{i-1} q^{\frac{1}{2}m_{a,i-1}} - q^{\frac{1}{2}m_{a,i-1}} \right) q^{m_i} \left(\prod_{b=i+1}^{n} q^{\frac{1}{2}m_{i,b}} - q^{\frac{1}{2}m_{i,b}} \right)
\]
to the terms of \( A_i^- \).

Since \( x_i \) is being straightened starting from the left of \( A_i^+ \), shortening occurs for each interval \([s, \ldots, i]\), where \( s = 1, \ldots, i \); denote by \( A_i^- \) the term where \( y_{s,i}^{m_{s,i}} \) is shortened by \( x_i \). For \( s < i \), the Shortening Lemma 5.3.3(ii) shows that the \( Y \) term in \( A_i^- \) has the form
\[
\left( \cdots y_{s,i-1}^{m_{s,i-1}} y_{s,i}^{m_{s,i}} \cdots \right) \rightarrow \left( \cdots y_{s,i-1}^{m_{s,i-1}} - q^{\frac{1}{2}m_{s,i-1}} [m_{s,i}] K_i y_{s,i-1} y_{s,i}^{m_{s,i-1}} \right) \cdots
\]

The \( k_i^2 \) must now be moved left through the \( Y \). Powers of \( q \) arising here are indexed by segments.
For $6.3.1$, Terms of $B_i^\pm$: Setting $\alpha$ by \ref{5.3.4}, $\beta$ is simply equal to (6.5).

Moreover, the exponents of the generators in $A_i^+(m)$ change as

\[
\text{K}(m)x(m) \otimes Y(m) \to \text{X}(m) \otimes \mathcal{K}(m)Y(m_i+1, m_{i+1}; 1; m).
\]

Finally, the term $A_i^+$ arises from commuting $x_j$ through the $Y$ until it is to the left of $y_j$, i.e. this comes from repeatedly taking the first term in Lemma \ref{5.3.3}. The coefficient of $A_i^+$ does not acquire any additional factors and thus is simply equal to (6.4).

6.2.2. Terms of $A_i^-$: The overall coefficient in $A_i^-$ comes from straightening a $\mathcal{K}_i$ through $X$ from the right and is again (6.4). When moving the $x_j$ through the $Y$ from the right, Shortening occurs for segments $\{j, \ldots, t\}, i \leq t \leq n$; denote these terms by $A_i^-$.

When $i < t$, the Shortening Lemma shows that the $Y$ part of $A_i^-$ is now

\[
\left(\cdots y_{i,t}^{-1} y_{i,t+1} \cdots \right) \to \left(\cdots -q^{-1} y_{i,t}^{-1} y_{i,t+1} \cdots \right).
\]

This $k_i^2$ needs to be moved through the $Y$ to the left and the $y_{i,t+1}$ needs to be straightened to the right. From the Passing Lemma \ref{5.3.4} we find the coefficient of $A_i^-(m), i < t, n$, to be

\[
-\alpha(m)[m_{i,t}]q^{3n_i-2} \prod_{a=1}^{l-1} q^{\frac{1}{\beta} m_{a,i-1}} \prod_{b=1}^{l-1} q^{\frac{1}{\beta} m_{b,i+1} + \frac{1}{\beta} m_{i+1,b}} \prod_{b=t}^{n} q^{\frac{1}{\beta} m_{b,i+1} + \frac{1}{\beta} m_{i+1,b}}
\]

and that the exponents in $A_i^-(m)$ change as

\[
\text{K}(m)x(m) \otimes \mathcal{K}(m)Y(m) \to \text{K}(m)x(m) \otimes \mathcal{K}(m)Y(m_{i+1}, m_{i+1}+1; m).
\]

As before, when $t = i$, no additional powers of $q$ are accrued while commuting $x_i$ through the $Y$ until it is right of $y_i$. Thus $A_i^-$ has coefficient (6.4).

6.3. The $B_i^\pm$ terms.

6.3.1. Terms of $B_i^+$: Setting $\alpha$ through $X$ from the gives a coefficient of (6.4) for $B_i^+$. Terms in $B_i^+$ come from Lengthening of monomials $x_{i+1,t}$, $i \leq t \leq n$; denote the resulting term by $B_i^+$. For $i < t$, Lengthening Lemma \ref{5.3.4} shows $B_i^+$ looks like

\[
\left(\cdots x_{i+1,t}^{-1} x_{i+1,t+1} \cdots \right) \to \left(\cdots x_{i+1,t}^{-1} \left(-q^{3} x_{i+1,t} x_{i+1,t+1} \cdots \right) \right).
\]

Straightening this form means commuting $x_{i,t}$ to the left through some $x_{i+1,t}, t < t'$, then through $x_{i,t'}$, $t < t'$. By Passing Lemma \ref{5.3.4}, there are no additional factors of $q$ when moving past the $x_{i+1,b}$ terms; by Passing Lemma \ref{5.3.4}, a factor of $q^{\frac{1}{\beta} m_{i+1,b}}$ is obtained when moving past $x_{i+1,b}$. The coefficient of $B_i^+$ is thus

\[
-\alpha(m)[m_{i,t}]q^{m_i-1} \prod_{a=1}^{l-1} q^{\frac{1}{\beta} m_{a,i-1}} \prod_{b=1}^{l-1} q^{\frac{1}{\beta} m_{b,i+1} + \frac{1}{\beta} m_{i+1,b}} \prod_{b=t}^{n} q^{\frac{1}{\beta} m_{b,i+1} + \frac{1}{\beta} m_{i+1,b}}
\]

and the exponents in $B_i^+(m)$ change as

\[
\text{K}(m)x(m) \otimes \mathcal{K}(m)Y(m) \to \text{K}(m)x(m_{i,t}+1, m_{i+1,t}-1; m) \otimes \mathcal{K}(m)Y(m).
\]
When \( t = i \), additional contributions of \( q \) come from the first summand in the Lengthening Lemma 5.3.2 and from Passing Lemma 5.3.4(i). Thus the coefficient of \( B_{j,i}^{+}(m) \) is

\[
\alpha(m) q^{m_i} \prod_{a=1}^{l-1} q^2 m_{a,i} - \frac{1}{2} m_{a,i-1} \prod_{b=i+1}^{n} q^2 m_{i,b} - \frac{1}{2} m_{i+1,b}.
\]

### 6.3.2. Terms of \( B_{i,j}^{-} = (x_i \otimes \overline{x}_j) \cdot R \)

Commuting \( x_i \) through \( K \) from the left yields an overall coefficient of the inverse of (6.4) to all terms of \( B_{i,j}^{-} \). When straightening \( x_i \) from the left through \( X \), Lengthening occurs for every segment \( \{s, \ldots, i-1\} \), \( 1 \leq s \leq i-1 \); the resulting term is denoted by \( B_{i,j}^{+} \). Note that this does not include the term \( B_{j,i}^{-} \) arising from moving the new \( x_i \) to the \( x_i \) term in \( X \).

By the Passing Lemma, moving \( x_i \) through the \( X \) to \( x_{s,i-1} \) picks up powers of \( q \) whenever \( x_i \) passes \( x_{a,i-1} \), contributing \( q^{m_{a,i}} \), and \( x_{a,i} \), contributing \( q^{-m_{a,i}} \), respectively. From Lengthening Lemma 5.3.2(ii), the change from Lengthening is

\[
\left( \cdots x_{s,i-1} x_{s,i} \right) \to \left( \cdots (-q^2)^{m_{s,i-1}} x_{s,i} \right) x_{s,i}^{m_{s,i}} \cdots .
\]

No further straightening needs to take place. Thus the coefficient of \( B_{j,i}^{-}(m) \) is

\[
-a_m [m_{s,i-1}] q^{m_{i-1} - \frac{1}{2} m_{i-1} - \frac{1}{2} m_{i-1}} \prod_{b=i+1}^{n} q^2 m_{i,b} - \frac{1}{2} m_{i+1,b}
\]

and the exponents in \( B_{j,i}^{-}(m) \) change as

\[
-k(m) X(m) \otimes \overline{K}(m) Y(m) \to k(m) X(m_{s,i-1} - 1, m_{s,i} + 1; m) \otimes \overline{K}(m) Y(m).
\]

For \( s = i \), considerations similar to those for \( B_{j,i}^{-} \) show that the coefficient of \( B_{i,j}^{+}(m) \) is

\[
\alpha(m) q^{-m_i} \prod_{a=1}^{l-1} q^2 m_{a,i} - \frac{1}{2} m_{a,i-1} \prod_{b=i+1}^{n} q^2 m_{i,b} - \frac{1}{2} m_{i+1,b}.
\]

### 6.4. Combining Diagonal Terms

Finally, we combine the diagonal terms of \( A_i \) and \( B_i \). These simplifications should be compared to those made when computing \( R \) for \( \mathcal{U}_h(\mathfrak{sl}_2) \) in §1. First, the diagonal A terms \( A_{i,i}^{+}(m) \) and \( A_{i,i}^{-}(m) \) differ only in the \( Y \) component as

\[
A_{i,i}^{+} = KX \otimes \overline{K}(\cdots x_{i} y_{i}^{m_{i}} \cdots), \quad A_{i,i}^{-} = KX \otimes \overline{K}(\cdots y_{i}^{-m_{i}} x_{i} \cdots).
\]

These terms may be combined in \( A_i \) to obtain a commutator \([x_i, y_i^{m_{i}}] \), which can be simplified using Lemma 5.1.3(ii); denote the resulting term by \( A_{i,i} \). This term has coefficient

\[
\alpha(m) [m_i] q^{m_{i-1}} \prod_{a=1}^{l-1} q^2 m_{a,i} - \frac{1}{2} m_{a,i-1} \prod_{b=i+1}^{n} q^2 m_{i,b} - \frac{1}{2} m_{i+1,b}
\]

and exponent change of

\[
-k(m) X(m) \otimes \overline{K}(m) Y(m) \to k(m) X(m) \otimes \overline{K}(m) Y(m_{i} - 1; m).
\]

Similarly, \( B_{i,i}^{+}(m) \) and \( B_{i,i}^{-}(m) \) can be combined by factoring out

\[
(q - \overline{q}) \prod_{a=1}^{l-1} q^2 m_{a,i} - \frac{1}{2} m_{a,i-1} \prod_{b=i+1}^{n} q^2 m_{i,b} - \frac{1}{2} m_{i+1,b}
\]

from each of \( B_{i,i}^{+}(m) \). This allows us to combine the generators in \( B_{i,i}^{+}(m) \), with the difference

\[
\frac{1}{q - \overline{q}} \left( \prod_{a=1}^{l-1} q^{m_{a,i} - m_{a,i-1}} \right) q^{-m_{i} k_{i}^{-1}} - \left( \prod_{a=1}^{l-1} q^{m_{a,i} - m_{a,i-1}} \right) q^{-m_{i} k_{i}^{m_{i} + 1}}
\]

in place of the \( k_i \) in \( K \). After factoring out \( k_{i}^{m_{i} + 1} \), this can be simplified to the quantum number

\[
\left[ h_{i} + m_{i} + \sum_{a=1}^{l-1} m_{a,i} - m_{a,i-1} \right].
\]
After straightening, this will contribute the same quantum number as obtained from simplifying the commutator in $A_{i,j}$. Overall, the coefficient of $B_{i,j}$ is

\begin{equation}
\alpha(m)(q - q^{-1}) \prod_{a=1}^{i-1} q^{m_{a,i}} \prod_{b=i+1}^{n} q^{m_{i+1,b}}
\end{equation}

and the change in exponent is due only to the additional $x_i$ term on the right

\begin{equation}
K(m)X(m) \otimes R(m)Y(m) \rightarrow K(m)X(m+1) \otimes R(m)Y(m).
\end{equation}

### 6.4.1. Recurrence for Coefficients

Finally, a recurrence for the $\alpha(m)$ is constructed by comparing like terms in $A_i = B_i$. From (6.14) and (6.16), the terms of $A_{i,j}$ agree in shape with those of $B_{i,j}$ so coefficients can be compared after making the shift $m_i \rightarrow m_i + 1$ in $B_{i,j}$. Equations (6.13) and (6.15) then give a recursion relation for $\alpha(m)$ with respect to the index $m_i$:

\begin{equation}
\alpha(m) = \alpha(m-1; m) \frac{(q - q^{-1})}{m} \prod_{a=1}^{i-1} q^{m_{a,i} - m_{a,i-1}} \prod_{b=i+1}^{n} q^{m_{i+1,b}}.
\end{equation}

From (6.9) and (6.12), the coefficients of $A^{s}_{k,s}$ and $B^{t}_{k,t}$ can be compared upon shifting $m_{k,i-1} \rightarrow m_{k,i-1} + 2$ for $A^{s}_{k,s}$ and $m_{k,i} \rightarrow m_{k,i} - 1$ for $B^{t}_{k,t}$. Equations (6.3) and (6.11) then yield a recursion of the form

\begin{equation}
\alpha(m_{k,i} - 1; m) = -\alpha(m_{k,i} - 1; m) \frac{m_{k,i}}{m_{k,i-1}} q^{m_{k,i} - m_{k,i-1}} \prod_{a=1}^{i-1} q^{m_{a,i} - m_{a,i-1}} \prod_{b=i+1}^{n} q^{m_{i+1,b} - m_{i+1,b}}.
\end{equation}

Comparing (6.8) and (6.10) shows that the coefficients of $A^{s}_{k,s}$ and $B^{t}_{k,t}$ can be compared, after making an appropriate exponent shift, and similar type of recursion relation can be constructed using (6.7) and (6.9). However, the two sets of relations (6.17) and (6.18) are sufficient to solve for $\alpha(m(i))$.

### 6.4.2. Solving Recurrences

The recursion (6.17) can be solved directly. For (6.18), notice that the relation expresses the changes in $\alpha(m)$ in terms of changes with respect to $m_{k,i}$, an exponent indexed by a shorter interval. Thus, by iterating (6.18) relates $\alpha(m_{k,i} - 1)$ with $\alpha(m_{k,i-1} - 1)$, which is related to $\alpha(m_{k,i-2} - 1)$, and continuing on in this fashion, is related to $\alpha(m_{k,i} - 1) = \alpha(m_{k,i} - 1)$. Then (6.17) can be applied to relate $\alpha(m_{k,i} - 1)$ with $\alpha(m)$. This process yields the following recursion:

\begin{equation}
\alpha(m) = (-1)^{i-s} \alpha(m_{k,i} - 1; m) \frac{q - q^{-1}}{m_{k,i}} \prod_{a=1}^{i} q^{m_{a,i}} \prod_{b=i+1}^{n} q^{m_{i+1,b}} \prod_{a=1}^{s} q^{m_{a,i+1}} \prod_{b=s+1}^{n} q^{m_{i+1,b}}.
\end{equation}

Solving the recursions (6.17) and (6.19) separately and putting the results together give

\begin{equation}
\alpha(m) = \prod_{a=1}^{i} q^{m_{a,i}} \prod_{b=i+1}^{n} q^{m_{i+1,b}} \prod_{a=1}^{s} q^{m_{a,i+1}} \prod_{b=s+1}^{n} q^{m_{i+1,b}} \times \text{(cross terms)}
\end{equation}

where the cross terms are indexed by pairs $s < i$ and are of the form

\begin{equation}
\left( \prod_{a=1}^{i} q^{m_{a,i}} \prod_{b=i+1}^{n} q^{m_{i+1,b}} \prod_{a=1}^{s} q^{m_{a,i+1}} \prod_{b=s+1}^{n} q^{m_{i+1,b}} \right).
\end{equation}

These cross terms can be eliminated by rearranging the general term of $R$ so that

\begin{equation}
K(m)X(m) \otimes R(m)Y(m) \rightarrow (k_{1}x_{1})_{m_{1}}(k_{2}x_{12})_{m_{2}} \cdots \otimes (k_{1}y_{1})_{m_{1}}(k_{2}y_{12})_{m_{2}} \cdots
\end{equation}

the monomials so that powers of $k_{a,b}, x_{a,b}$ and $k_{a,b}, y_{a,b}$ occur together. The factors of $q$ arising from this rearrangement cancel the cross terms, leaving only the first products in the expression of $\alpha(m)$ above.
6.5. **An R-matrix for \( \mathcal{W}_h(\mathfrak{sl}_{n+1}) \).** The computations of §6.1 show that

\[
R = \exp \left( \frac{\hbar}{4} \sum_{i,j=1}^{n} k_{ij} h_i \otimes h_j \right) \prod_{1 \leq a, b \leq n} \left( (-1)^{b-a} \frac{q-q^{-1}}{m_{a,b}} \right) q^{\frac{1}{2}m_{a,b}(m_{a,b} - 3)}
\]

\[
\times \left( (k_1 x_1)^{m_1} (k_{12} x_{12})^{m_{12}} \cdots (k_n x_n)^{m_n} \right) \otimes \left( (\overline{k}_1 y_1)^{m_1} (\overline{k}_{12} y_{12})^{m_{12}} \cdots (\overline{k}_n y_n)^{m_n} \right)
\]

where the \( k_{ij} \) are as in (6.3). We recognize each summand as a product of \( q \)-exponential functions, as we did in §4.2.3 so we find

\[
(6.20) \quad R = \exp \left( \frac{\hbar}{4} \sum_{i,j=1}^{n} k_{ij} h_i \otimes h_j \right) \prod_{1 \leq a, b \leq n} \left( (-1)^{b-a} \lambda_q k_{a,b} x_{a,b} \otimes \overline{k}_{a,b} y_{a,b} \right)
\]

for an R-matrix for \( \mathcal{W}_h(\mathfrak{sl}_{n+1}) \). Here, \( \lambda_q = \overline{q}(q-q^{-1}) \) as in (4.8) and the \(<\) on the product signifies that terms in the product are taken with respect to the ordering of indices in (6.3).

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