The Moyal-Weyl quantization procedure is embedded into the twist formalism of vector fields on phase space. Double application of twists provide most general deformations of Minkowskian Heisenberg-algebras and corresponding quantizations of the Lorentz-algebra. Such deformations deliver high-energy extensions of standard relativistic quantum mechanics. These are required to obtain minimal uncertainty properties for high-energy spacetime measurements that standard quantum mechanics lacks. The procedure of double twist application is outlined. We give an instructive and genuine example.
1 Introduction

The scheme of canonical quantization, presented in textbooks of quantum mechanics, is the most simple quantization one might perform. Noncommutative geometry is considered as some enhancement of this scheme. There are two basic ideas of how noncommutative geometry can be interpreted in physics. From the side of effective theories, we hope for some alternatives to standard perturbative treatment of field theories and their renormalization. Such alternatives would be required by quantum chromodynamics and gravity such as \[6, 5\] already suggests. On the other hand one might stick to a more fundamental point of view. Noncommutative geometry is then regarded as a gravity effect itself. Such approaches can be found in gravity motivated canonical noncommutative geometry \[10, 9\], but also within discussions of minimal uncertainty theories such as in \[18, 17, 16, 15\]. Moreover there are close relations of noncommutative geometry as well as of doubly special relativity to loop quantum gravity \[2, 1, 3, 4\]. Within such a fundamental approach, noncommutative geometry should not be expected as a static noncommutative background for field theories anymore. Instead, noncommutative geometry itself should become subject to gravity by making it dependent on energy and momentum. After all we expect, Planck scale effects at high energy-momentum densities and thus a grainy structure of spacetime, obtained from noncommutative geometry, can only be mediated by operators of energy and momentum. This is nothing else than a more general deformation of phase space than obtained by canonical quantization. Moreover in such an approach, noncommutative geometry should become localized to those space volumes, where densities of energy and momentum enter the actual high energy regime. Standard problems such as IR-UV-Mixing effects should thus not occur in such a setup. A first and actually most prominent example of such a general quantization is the well known Quantum-Spacetime of Snyder \[12, 13, 11, 25, 24, 28\]. Canonical quantization can be understood as a deformation-quantization of the phase space towards the Heisenberg-algebra. Weyl and Moyal \[22, 27\] performed this deformation by means of starproducts. In this paper we formalise this setup by introducing a Hopf-algebra of vector fields on phase space. We use these vector fields to twist the phase space to the standard Heisenberg-algebra. We further apply twists to deform the Heisenberg-algebra itself. These two twists can be merged to a single one. The paper is organized as follows. In the first section we introduce the 2n-dimensional Heisenberg-algebra \(\mathfrak{h}_{2n}\) and its universal enveloping algebra \(U(\mathfrak{h}_{2n})\). We then recall how this algebra is obtained by deformation-quantization of a commutative phase space algebra. This is due to Weyl and Moyal. We formalise and introduce a Hopf-algebra of vector fields on the phase space. In a second step we further apply twists to deform the Heisenberg-algebra itself. These two twists can be merged to a single one. The paper is organized as follows. In the first section we introduce the 2n-dimensional Heisenberg-algebra \(\mathfrak{h}_{2n}\) and its universal enveloping algebra \(U(\mathfrak{h}_{2n})\). We then recall how this algebra is obtained by deformation-quantization of a commutative phase space algebra. This is due to Weyl and Moyal. We formalise and introduce a Hopf-algebra of vector fields on the phase space. In a second step we further apply twists to deform the Heisenberg-algebra itself. These two twists can be merged to a single one. The paper is organized as follows. In the first section we introduce the 2n-dimensional Heisenberg-algebra \(\mathfrak{h}_{2n}\) and its universal enveloping algebra \(U(\mathfrak{h}_{2n})\). 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conclusion.
Before we actually come to general matters, we first have to do some remarks that clarify and motivate the directions pursued in the following constructions and that indeed go hand in hand with the formalism chosen by Weyl and Moyal.

In textbooks on field theory, we often find the representation of the Lorentz-algebra in terms of generators of $U(\mathfrak{h}_{2n})$. In particular the generators $m^{\mu\nu}$ of the Lorentz-algebra are represented in $U(\mathfrak{h}_{2n})$ by

$$m^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}.$$  

Using the commutation relation

$$[p^\mu, x^\rho] = -i\eta^{\mu\rho},$$  \hspace{1cm} (1.1)

the action of $m^{\mu\nu}$ on basis elements $x^\rho$ and $p^\sigma$ of $U(\mathfrak{h}_{2n})$ is then evaluated by commutators

$$[m^{\mu\nu}, x^\rho] = [x^{\mu}p^{\nu} - x^{\nu}p^{\mu}, x^\rho] = x^{\mu} [p^\nu, x^\rho] - x^{\nu} [p^\mu, x^\rho]$$
$$= -i\eta^{\nu\rho} x^{\mu} + i\eta^{\mu\rho} x^{\nu},$$  \hspace{1cm} (1.2)

$$[m^{\mu\nu}, p^\sigma] = [x^{\mu}p^{\nu} - x^{\nu}p^{\mu}, p^\sigma] = [x^{\mu}, p^\sigma] p^\nu - [x^{\nu}, p^\sigma] p^\mu$$
$$= i\eta^{\mu\sigma} p^{\nu} - i\eta^{\nu\sigma} p^{\mu}. $$  \hspace{1cm} (1.3)

There are several pictures how this setup can be interpreted in physics. At first we can stick to the Poincaré-algebra, generated by $m^{\mu\nu}$ and $p^\rho$, that is represented on Minkowski-space. In this scheme we do not consider the Lorentz-algebra to be represented in terms of generators of $U(\mathfrak{h}_{2n})$, as we did above - but nevertheless consider the "representation" of the Lorentz-algebra in terms of commutators $[m^{\mu\nu}, x^\rho]$ or $[p^\nu, x^\rho]$ although this already incorporates a multiplicative structure between the symmetry algebra and its representation space. For the commutative case this is alright - but deformations to noncommutative geometry modify the commutation relations in such a way that they do not close on the representation space anymore. There is actually a mixing of the symmetry algebra and the representation space. This phenomenon is also described in [29]. To fix this problem we might thus argue that we have actually to stay within the Heisenberg-algebra $U(\mathfrak{h}_{2n})$. Then, with $m^{\mu\nu}$ represented in $U(\mathfrak{h}_{2n})$ as performed above, we do not care anymore if a mixing occurs. In this case the commutator $[p^\nu, x^\rho]$ manages everything that is represented on Minkowski-space. At first this argumentation makes perfect sense and in the case of deformations of Minkowskian $U(\mathfrak{h}_{2n})$ it has been reasoned a long such a way [19, 18, 17, 16]. Algebraically the subalgebra of momenta in $U(\mathfrak{h}_{2n})$ does not differ from that of coordinates and thus if the commutator $[p^\nu, x^\rho]$ is considered to represent the subalgebra of momenta on the coordinates, we might as well argue that in turn $[x^\mu, p^\sigma]$ is some sort of representation of coordinates on the momenta.
as also performed in our computation in [1.3] from above. But this as well rises the question how a coordinate would possibly act on products of generators of momenta. Or in other words, what is the coproduct of a coordinate? This argumentation is of course too naive and these issues actually do not become a question for the commutative case - but if we are to consider deformations, we have to know about such coproducts, at least in principle. We have to have a neat bialgebra or Hopf-algebra as a framework to consider any deformation. In fact it is not possible to endow the coordinates with the same primitive type of coproduct as we use it for the momenta. Such an introduction of a coproduct contradicts the property of the coproduct to be an algebra-homomorphism. Nevertheless there are examples that neatly and quite elegantly endow a phase space with proper coproducts on momenta and coordinates [21]. However these also incorporate some specific structure that already accommodates some physics. The solution to this dilemma can be found in the introduction of vector fields on the entire phase space that we are presenting here. This had been performed first by Moyal and Weyl in [22, 27]. We thus first concentrate on their work in a Minkowskian setting and formalize this to our requirements. In particular we lift these vector fields to a Hopf-algebra as presented in [19]. We are then able to fit in the Lorentz-symmetry and consider further deformations.

2 Quantum Mechanics according to Weyl and Moyal

This section is intended as a basic review and outline that constitutes the actual input and fundaments of our constructions. The section is divided in two parts. In the first subsection we introduce $n$-dimensional Minkowski-space and the corresponding representation of the Poincaré-algebra. This is the only input we require for all of our considerations in this work. Based on this we build the $2n$-dimensional Minkowskian phase space and the Heisenberg Lie-algebra by taking direct sums of copies of Minkowski-space. These three vector spaces are further more enhanced to algebras of universal enveloping algebra type. The second subsection then reviews the deformation-quantization of Minkowskian phase space towards the Heisenberg-algebra according to Weyl and Moyal using the starproduct. In mathematical terms this is a deformation-quantization of a Poisson-Manifold. For completeness we shortly review this latter notion. We thereby obtain the required setup for further deformations with the double application of twists that is discussed in the next sections. As a textbook we recommend [8] as reference for this section.
2.1 The Minkowskian Heisenberg-Algebra

The $n$-dimensional Minkowski-space $\mathbb{R}^{(1,n-1)}$ is a vector space with scalar product

$$<x, y> = \eta_{\mu\nu} x^\mu y^\nu, \quad x, y \in \mathbb{R}^{(1,n-1)},$$

(2.1)

that is left invariant under the action of the Lorentz-group $\text{SO}(1, n-1)$. Within a specific coordinate system, the invariance of (2.1) under matrix representations of transformations $\Lambda \in \text{SO}(1, n-1)$ is given by

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma, \quad \mu, \nu, \rho, \sigma \in 0, \ldots, (n-1).$$

The signature of the metric tensor $\eta_{\mu\nu}$ has not to be specified within our consideration. We consider Minkowski space to be generated by a basis $(x^\mu)_{\mu \in 0,1,\ldots,n-1}$. Apart from isotropy of spacetime, homogeneity of $\mathbb{R}^{(1,n-1)}$ is generated by the action of the $n$-dimensional translational group $T_n$. The Poincaré group $\mathcal{P}$ is the semi-direct product $\text{SO}(1, n-1) \rtimes T_n$. The Lie groups $\text{SO}(1, n-1)$ and $T_n$ are generated by Lie-algebras $\mathfrak{so}_{1,n-1}$ and $\mathfrak{t}_n$ respectively to constitute the Poincaré-algebra $\mathfrak{p}$. In particular for representations we actually consider the universal enveloping algebras $U(p)$, $U(\mathfrak{so}_{1,n-1})$ and $U(\mathfrak{t}_n)$. In order to endow Minkowski-space with a commutative algebraic structure, we enhance it to a Lie-algebra by the introduction of a trivial bracket

$$[\ , \ ] : \mathbb{R}^{(1,n-1)} \times \mathbb{R}^{(1,n-1)} \rightarrow \mathbb{R}^{(1,n-1)},$$

that for $x^\rho, x^\sigma \in \mathbb{R}^{(1,n-1)}$ is given by

$$[x^\rho, x^\sigma] = 0.$$

(2.2)

On this basis we consider the universal enveloping algebra $U(\mathbb{R}^{(1,n-1)})$. The generators $m^{\mu\nu} \in U(\mathfrak{so}_{1,n-1})$ and $\pi^\rho \in U(\mathfrak{t}_n)$ of the Poincaré-algebra $U(p)$ are subject to commutation relations

$$[m^{\mu\nu}, m^{\rho\sigma}] = i\eta^{\mu\rho} m^{\nu\sigma} - i\eta^{\nu\rho} m^{\mu\sigma} + i\eta^{\nu\sigma} m^{\mu\rho} - i\eta^{\mu\sigma} m^{\nu\rho},$$

$$[m^{\mu\nu}, \pi^\rho] = i\eta^{\mu\rho} \pi^\nu - i\eta^{\nu\rho} \pi^\mu,$$

$$[\pi^\rho, \pi^\sigma] = 0.$$

(2.3)

that generate its two-sided ideal. The Poincaré-algebra $U(p)$ becomes a Hopf-algebra with the following coproduct, counit and antipode:

$$\Delta(m^{\mu\nu}) = m^{\mu\nu} \otimes 1 + 1 \otimes m^{\mu\nu}, \quad \epsilon(m^{\mu\nu}) = 0, \quad S(m^{\mu\nu}) = -m^{\mu\nu},$$

$$\Delta(\pi^\rho) = \pi^\rho \otimes 1 + 1 \otimes \pi^\rho, \quad \epsilon(\pi^\rho) = 0, \quad S(\pi^\rho) = -\pi^\rho.$$

(2.4)
The Hopf-algebra $U(p)$ is represented on $U(\mathbb{R}^{(1,n-1)})$ as a left action by
\[ m^{\mu\nu} \triangleright x^\rho = -i\eta^{\nu\rho} x^\mu + i\eta^{\mu\rho} x^\nu, \]
\[ \pi^\mu \triangleright x^\rho = -i\eta^{\mu\rho}, \]
\[ 1_p \triangleright x^\rho = x^\rho, \] (2.5)
such that relations (2.3) are realized on the vector space $\mathbb{R}^{(1,n-1)}$, i.e.
\[ (m^{\mu\nu} m^{\rho\sigma} - m^{\rho\sigma} m^{\mu\nu} + i\eta^{\nu\rho} m^{\mu\sigma} - i\eta^{\mu\sigma} m^{\nu\rho} + i\eta^{\mu\sigma} m^{\nu\rho}) \triangleright x^\lambda = 0, \]
\[ (m^{\mu\nu} \pi^\rho - \pi^\rho m^{\mu\nu} - i\eta^{\mu\rho} \pi^\nu + i\eta^{\nu\rho} \pi^\mu) \triangleright x^\lambda = 0, \]
\[ (\pi^\rho \pi^\sigma - \pi^\sigma \pi^\rho) \triangleright x^\lambda = 0. \] (2.6)
The action of generators $m^{\mu\nu}, \pi^\mu \in U(p)$ on products of coordinates in $U(\mathbb{R}^{(1,n-1)})$ is given by
\[ m^{\mu\nu} \triangleright (x^\rho x^\sigma) = \Delta(m^{\mu\nu}) \triangleright (x^\rho x^\sigma) = (m^{\mu\nu} \triangleright x^\rho) x^\sigma + x^\rho (m^{\mu\nu} \triangleright x^\sigma), \]
\[ \pi^\mu \triangleright (x^\rho x^\sigma) = \Delta(\pi^\mu) \triangleright (x^\rho x^\sigma) = (\pi^\mu \triangleright x^\rho) x^\sigma + x^\rho (\pi^\mu \triangleright x^\sigma), \]
\[ m^{\mu\nu} \triangleright 1 = \epsilon(m^{\mu\nu}), \quad \pi^\mu \triangleright 1 = \epsilon(p^\mu), \] (2.7)
such that the generating relations (2.2) of $U(\mathbb{R}^{(1,n-1)})$ are respected by their action according to
\[ m^{\mu\nu} \triangleright (x^\rho x^\sigma - x^\sigma x^\rho - [x^\rho, x^\sigma]) = 0, \]
\[ \pi^\mu \triangleright (x^\rho x^\sigma - x^\sigma x^\rho - [x^\rho, x^\sigma]) = 0. \] (2.8)
As a next step we introduce Minkowskian phase space $\Gamma$ as the direct sum of two copies of Minkowski-space $\mathbb{R}^{(1,n-1)}$, i.e. we obtain
\[ \Gamma = \mathbb{R}^{(1,n-1)} \oplus \mathbb{R}^{(1,n-1)}. \] (2.9)
As for Minkowski-space, we enhance $\Gamma$ with a commutative Lie-algebraic structure. Within a specific coordinate system we thus take $(x^\mu, p^\nu)_{\mu,\nu \in 0,1,\ldots,n-1}$ as a basis and introduce the brackets
\[ [x^\mu, x^\nu] = 0, \]
\[ [x^\mu, p^\nu] = 0, \]
\[ [p^\mu, p^\nu] = 0, \] (2.10)
We then obtain the universal enveloping algebra $U(\Gamma)$ by once more taking these brackets as the generating relations for the corresponding two-sided ideal of $U(\Gamma)$. Concerning covariance under the action of $U(p)$, we can replace coordinates $x$ by
2.2 Phase Space Quantization with Starproducts

In the last subsection we considered the phase space algebra as the universal enveloping algebra $U(\Gamma)$. Dually we have the algebra of complex-valued functions $\mathcal{F}(\Gamma)$ on $\Gamma$. Defining the Poisson-bracket on functions $\mathcal{F}(\Gamma)$, we turn $\Gamma$ into a Poisson-manifold.

momenta $p$ in conditions (2.5) and (2.6), i.e. on the vector space $\Gamma = \mathbb{R}^{(1,n-1)} \oplus \mathbb{R}^{(1,n-1)}$ the Lorentz group $SO(1,n-1)$ is represented by block-diagonal matrices

$$\Lambda p = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}.$$  

(2.11)

With respect to the covariance of the algebraic structure of $U(\Gamma)$ we can replace products of coordinates $x^\rho x^\sigma$ in (2.7) and (2.7) by products of coordinates and momenta $x^\rho p^\sigma$ and products of momenta $p^\rho p^\sigma$. We thereby obtained a left action of $U(p)$ on $U(\Gamma)$.

In a similar manner, as for $\Gamma$, we obtain the Minkowskian Heisenberg-algebra $h_{2n}$ by taking the direct sum of two copies of $\mathbb{R}^{(1,n-1)}$ and the real numbers, i.e.

$$h_{2n} = \mathbb{R}^{(1,n-1)} \oplus \mathbb{R}^{(1,n-1)} \oplus i\mathbb{R}.$$  

(2.12)

This vector space becomes a Lie-algebra by introducing a bracket

$$[,] : h_{2n} \times h_{2n} \rightarrow h_{2n}$$

that for $X_1, Y_1, X_2, Y_2 \in \mathbb{R}^{(1,n-1)}$ and $c_1, c_2 \in \mathbb{R}$ is defined by

$$[(X_1, Y_1, c_1), (X_2, Y_2, c_2)] = (0, 0, i \cdot (\langle X_1, Y_2 \rangle - \langle Y_1, X_2 \rangle)).$$  

(2.13)

Through the scalar product (2.1) used in this definition we obtain $h_{2n}$ to be covariant under the action of $U(p)$. Besides this, Lorentz-covariance is equally intruduced as for the phase space $\Gamma$. By the identification

$$X^\mu \equiv (e^\mu, 0, 0) \in h_{2n}, \quad P_\nu \equiv (0, e_\nu, 0) \in h_{2n},$$

we obtain the bracket-relations between coordinates $X^\mu$ and momenta $P^\nu$

$$[X^\mu, X^\nu] = 0, \quad [X^\mu, P^\nu] = i\hbar \eta^{\mu\nu}, \quad [P^\mu, P^\nu] = 0.$$  

(2.14)

These relations once more generate the two-sided ideal that is required to formulate the universal enveloping algebra $U(h_{2n})$ of the Heisenberg-algebra $h_{2n}$. We are now prepared to consider deformation-quantization of $U(\Gamma)$ towards $U(h_{2n})$ as it had been introduced by Moyal.
As such we deform it to $U(\mathfrak{h}_{2n})$ according to the quantization procedure applied by Moyal [22]. This is more generally known as a deformation of Poisson manifolds. We recall these notions here. In order to perform this quantization we switch between the dual pictures of $U(\Gamma)$ and $\mathcal{F}(\Gamma)$. We begin by introducing the algebra of functions $\mathcal{F}(\Gamma)$ on $\Gamma$.

On the vector space $\Gamma = \mathbb{R}^{(1,n-1)} \oplus \mathbb{R}^{(1,n-1)}$ we consider the subset $\mathcal{F}(\Gamma) \subset C^\infty(\Gamma, \mathbb{C})$ of smooth complex-valued functions, that we endow with a Poisson-bracket

$$\{ \omega, \varphi \} : \mathcal{F}(\Gamma) \times \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\Gamma),$$

that in particular is defined for $\omega, \varphi \in \mathcal{F}(\Gamma)$ by

$$\{ \omega, \varphi \} := \frac{\partial \omega}{\partial p_\mu} \cdot \frac{\partial \varphi}{\partial x^\mu} - \frac{\partial \omega}{\partial x^\mu} \cdot \frac{\partial \varphi}{\partial p_\mu}. \quad (2.15)$$

In addition to this bracket, the vector space of functions $\mathcal{F}(\Gamma)$ is endowed with point-wise multiplication that is induced from the product within the complex numbers, i.e. for $\omega, \varphi \in \mathcal{F}(\Gamma)$ we have

$$(\omega \cdot \varphi)(x^\mu, p_\nu) = \omega(x^\mu, p_\nu) \cdot \varphi(x^\mu, p_\nu)$$

By the introduction of the Poisson-bracket (2.15), we turn the vector space $\Gamma$ into what is called a Poisson manifold that is more generally defined as follows.

2.1 Definition Let $\mathcal{M}$ be a $d$-dimensional manifold and $C^\infty(\mathcal{M}, \mathbb{C})$ be the set of complex-valued smooth functions on $\mathcal{M}$. Then $\mathcal{M}$ is called a Poisson Manifold, if there exists a bracket $\{\cdot, \cdot\}$

$$\{\cdot, \cdot\} : C^\infty(\mathcal{M}, \mathbb{C}) \times C^\infty(\mathcal{M}, \mathbb{C}) \rightarrow C^\infty(\mathcal{M}, \mathbb{C}),$$

such that the following properties hold:

$$\forall \omega, \varphi, \psi \in C^\infty(\mathcal{M}, \mathbb{C}) : \{ \varphi, \omega \} = -\{ \omega, \varphi \}$$

$$\{ \varphi \cdot \omega, \psi \} = \varphi \cdot \{ \omega, \psi \} + \{ \varphi, \psi \} \cdot \omega$$

$$\{ \{ \varphi, \omega \}, \psi \} + \{ \{ \omega, \psi \}, \varphi \} + \{ \{ \psi, \varphi \}, \omega \} = 0$$

We thus have two distinct algebraic structures on $\Gamma$, i.e. on $\mathcal{F}(\Gamma)$. The original problem considered by Weyl and Moyal in [22, 27] had been to grasp the procedure of quantization as mathematical term. The procedure of quantization in particular sends the Poisson-bracket of $\mathcal{F}(\Gamma)$ to the commutator of $U(\mathfrak{h}_{2n})$ according to

$$\{\cdot, \cdot\} \rightarrow \frac{i}{\hbar} [\cdot, \cdot].$$
This procedure agitates the former algebraic structures of $\Gamma$. It "maps" the commutative algebra of functions $\mathcal{F}(\Gamma)$ to the noncommutative $U(\mathfrak{g}_{2n})$. The solution is to consider quantization to be the deformation of the product of the algebra of functions $\mathcal{F}(\Gamma)$ performed in such a way that the commutator of the deformed algebra of functions corresponds to the structure implied by the Poisson-bracket. More generally this is known to be a quantization of a Poisson-manifold that more precisely is defined as follows.

2.2 Definition Let a Poisson manifold $(\mathcal{M}, \{\cdot, \cdot\}, \mathbf{K})$ over the field $\mathbf{K}$ be given. A quantization of $\mathcal{M}$ with deformation parameter $h \in \mathbf{K}$ is a manifold $\mathcal{M}_h = (\mathcal{M}, [\cdot *^h \cdot], \mathbf{K})$, such that to first order in the deformation parameter $h$ the commutator $[f_1 *^h f_2]_h = \{f_1, f_2\} \mod (h)$ satisfies the following property:

$$\forall f_1, f_2 \in \mathcal{F}(\mathcal{M}) : \frac{[f_1 *^h f_2]}{h} = \frac{f_1 *^h f_2 - f_2 *^h f_1}{h} = \{f_1, f_2\} \mod (h)$$

The quantization of the algebra of functions is typically performed in terms of starproducts. To this purpose it is convenient to consider $U(\mathfrak{g}_{2n})$ instead of $\mathcal{F}(\Gamma)$. Since $\mathcal{F}(\Gamma) \subset C^\infty(\Gamma, \mathbb{C})$ and $\mathcal{F}(\Gamma)$ is commutative, this duality merely means that functions $\varphi \in \mathcal{F}(\Gamma)$ can be represented in terms of formal power series in $U(\mathfrak{g}_{2n})$ that moreover can be regarded as power series of a real parameters and thus can converge locally. We thus express functions $\varphi \in \mathcal{F}(\Gamma)$ as power series

$$\varphi(x^\mu, p_\nu) = \sum_{r,s} C_{r,s} \cdot (x^0)^{r_0} \cdot \ldots \cdot (x^{(n-1)})^{r_{(n-1)}} \cdot (p_0)^{s_0} \cdot \ldots \cdot (p^{(n-1)})^{s_{(n-1)}}$$

$C_{r,s} \in \mathbb{C}; \quad r, s \in \mathbb{N}_0^n$.

With exponential functions

$$e^{i(\eta_\mu x^\mu + \xi_\nu p_\nu)}, \quad \eta_\mu, \xi_\nu \in \mathbb{R}^{(1,n-1)}$$

as a basis for $\mathcal{F}(\Gamma)$ we can also consider $\varphi \in \mathcal{F}(\Gamma)$ as a linear combination in terms of its Fourier-transformation

$$\hat{\varphi}(\eta_\mu, \xi_\nu) = \int d^n x \ d^n_p \ \hat{\varphi}(\eta_\mu, \xi_\nu) e^{-i(\eta_\mu x^\mu + \xi_\nu p_\nu)}$$

$$\hat{\varphi}(\eta_\mu, \xi_\nu) = \frac{1}{(2\pi)^{2n}} \int d^n x \ d^n p \ \varphi(x^\mu, p_\nu) e^{i(\eta_\mu x^\mu + \xi_\nu p_\nu)}.$$
algebra of functions over the Heisenberg Lie-group. The Poincaré-Birkhoff-Witt theorem enables us to map $U(\mathfrak{h}_{2n})$ to $\mathcal{F}(\Gamma)$ by an isomorphism $W$ of vector spaces. In particular this statement reads as follows.

2.3 Theorem Let $\mathfrak{g}$ be an $n$-dimensional Lie-algebra with basis $(g_i)_{i \in \{1...n\}}$ over the field $\mathbf{K}$. Furthermore let

$$
\pi : \{1...n\} \subset \mathbb{N} \rightarrow \{1...n\}
$$

$$
k \mapsto i_k
$$

be any permutation, then the ordered monomials

$$(g_{i_1})^{m_{i_1}} \cdots (g_{i_k})^{m_{i_k}} \cdots (g_{i_n})^{m_{i_n}} \in U(\mathfrak{g}), \ m_{i_k} \in \mathbb{N}$$

constitute a basis of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and there exists an isomorphism $W$ of vector spaces

$$
W : U(\mathfrak{g}) \rightarrow U(\mathbb{R}^n)
$$

$$(g_{i_1})^{m_{i_1}} \cdots (g_{i_k})^{m_{i_k}} \cdots (g_{i_n})^{m_{i_n}} \mapsto (x_{i_1})^{m_{i_1}} \cdots (x_{i_k})^{m_{i_k}} \cdots (x_{i_n})^{m_{i_n}}.
$$

Introducing a star product on $\mathcal{F}(\Gamma)$, i.e. performing the quantization of the Poisson-manifold as described, actually enhances the isomorphism $W$ of vector spaces to an isomorphism of corresponding algebras. In particular we therefore consider how basis elements are mapped, i.e. we obtain

$$
W : U(\mathfrak{h}_{2n}) \rightarrow \mathcal{F}(\Gamma)
$$

$$
e^{i(\eta_\mu X^\mu + \xi^\nu P_\nu)} \mapsto e^{i(\eta_\mu x^\mu + \xi^\nu p_\nu)}.
$$

By application of the inverse map $W^{-1}$ we receive for two functions $\varphi, \omega \in \mathcal{F}(\Gamma)$ the corresponding objects within $U(\mathfrak{h}_{2n})$. In particular we obtain

$$
W^{-1}(\varphi)(X^\mu, P_\nu) = \int d^m \eta \ d^n \xi \ \hat{\varphi}(\eta_\mu, \xi^\nu) \ e^{-i(\eta_\mu X^\mu + \xi^\nu P_\nu)}
$$

$$
W^{-1}(\omega)(X^\mu, P_\nu) = \int d^m \eta \ d^n \xi \ \hat{\omega}(\eta_\mu, \xi^\nu) \ e^{-i(\eta_\mu X^\mu + \xi^\nu P_\nu)}.
$$

In order to endow the vector space $\Gamma$ with a deformed multiplication map $*_{\hbar}$ we require that

$$
W^{-1}(\varphi *_{\hbar} \omega)(X^\mu, P_\nu) := W^{-1}(\varphi)(X^\mu, P_\nu) \cdot W^{-1}(\omega)(X^\mu, P_\nu)
$$

$$
= \int d^m \eta \ d^n \xi \ d^n \kappa \ d^n \lambda \ \hat{\varphi}(\eta_\mu, \xi^\nu) \hat{\omega}(\kappa_\mu, \lambda^\nu) \times e^{-i(\eta_\mu X^\mu + \xi^\nu P_\nu)} \ e^{-i(\kappa_\mu X^\mu + \lambda^\nu P_\nu)}
$$

$$
= \int d^m \eta \ d^n \xi \ d^n \kappa \ d^n \lambda \ \hat{\varphi}(\eta_\mu, \xi^\nu) \hat{\omega}(\kappa_\mu, \lambda^\nu)
$$

$$
\times e^{-i((\eta_\mu + \kappa_\mu) X^\mu + (\xi^\nu + \lambda^\nu) P_\nu - i\frac{\hbar}{2} \eta_\mu \lambda_\nu - \xi_\nu \kappa_\mu)} 1.
$$
The final step we performed by the use of the Baker-Campbell-Hausdorff formula
\[ e^A e^B = e^{A+B + \frac{1}{2} [A,B] + \frac{1}{12}([A,[A,B]]-[B,[A,B]]) + \frac{1}{12}([A,B,[A,B]]-[B,[A,[A,B]]]) + \ldots} \]

We transform back by the use of the isomorphism \( W \) and thus obtain
\[
(\varphi \ast h \omega)(x^\mu, p_\nu) = \int d^n \eta \ d^n \xi \ d^n \kappa \ d^n \lambda \ \hat{\varphi}(\eta_\mu, \xi_\nu) \hat{\omega}(\kappa_\mu, \lambda_\nu) \\
\times e^{-i((\eta_\mu + \kappa_\mu) X^\mu + (\xi_\nu + \lambda_\nu) P_\nu) - i \frac{\hbar}{2} \eta^\mu \eta_\nu (\eta_\mu \lambda_\nu - \xi_\mu \kappa_\nu)} \\
= \int d^n \eta \ d^n \xi \ d^n \kappa \ d^n \lambda \ \varphi(\eta_\mu, \xi_\nu) \ e^{-i(\eta_\mu x^\mu + \xi_\nu p_\nu)} \\
\times \hat{\omega}(\kappa_\mu, \lambda_\nu) \ e^{-i(\kappa_\mu x^\mu + \lambda_\nu p_\nu)} e^{-i \frac{\hbar}{2} \eta^\mu \eta_\nu (\eta_\mu \lambda_\nu - \xi_\mu \kappa_\nu)}
\]

Replacing \( \eta_\mu \rightarrow i \frac{\partial}{\partial x^\mu}, \xi_\nu \rightarrow i \frac{\partial}{\partial p^\nu} \) and \( \kappa_\mu \rightarrow i \frac{\partial}{\partial \mu}, \lambda_\nu \rightarrow i \frac{\partial}{\partial p^\nu} \), we finally received the starproduct
\[
(\varphi \ast h \omega)(x^\mu, p_\nu) = e^{i \frac{h}{2} \eta^\mu \eta_\nu (\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial p^\nu} - \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial x^\nu})} \varphi(x^\mu, p_\nu) \ \omega(\hat{x}^\mu, \hat{p}_\nu) \big|_{(x^\mu, \omega \rightarrow (x^\mu, x^\mu, \omega))}.
\]

(2.16)

In particular for \( \varphi(x^\rho, p_\sigma) = x^\rho \) and \( \omega(x^\rho, p_\sigma) = p_\sigma \) we recover the second relation of (2.16), distinguishing the generating relations of \( U(\mathfrak{h}_{2n}) \) from those of \( U(\Gamma) \).

\[
[x^\rho \ast h p_\sigma] = x^\rho \ast h p_\sigma - p_\sigma \ast h x^\rho \\
= e^{i \frac{\hbar}{2} \eta^\mu \eta_\nu (\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial p^\nu} - \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial x^\nu})} x^\rho \cdot \hat{p}_\sigma |_{p^\sigma \rightarrow p^\sigma} - e^{i \frac{\hbar}{2} \eta^\mu \eta_\nu (\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial p^\nu} - \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial x^\nu})} p_\sigma \cdot \hat{x}^\rho |_{x^\rho \rightarrow x^\rho} \\
= x^\rho \cdot p_\sigma + i \frac{\hbar}{2} \eta^\rho_\sigma - p_\sigma \cdot x^\rho + i \frac{\hbar}{2} \eta^\rho_\sigma \\
= i \eta^\rho_\sigma.
\]

This final computation closes our short review of Weyl-Moyal deformation-quantization. We are now prepared to formalise this procedure.

3 Vector Fields \( \mathfrak{W}(\Pi, \Gamma) \) on Minkowskian Phase Space

Beginning with this section we formalise the presented constructions of Weyl and Moyal. In particular we intend to absorb the starproduct (2.16) into the modern setup of twists of vector fields, as presented in [19]. We thus make a step beyond mere quantizations of Poisson manifolds because the twist formalism also enables us to further deform the Heisenberg-algebra \( U(\mathfrak{h}_{2n}) \) itself. Furthermore the twist
formalism also provides us with the opportunity to make required deformations of the Poincaré-algebra such that we can preserve spacetime covariance under deformations. In this section we therefore introduce the required Hopf-algebra of vector fields \( \mathcal{W}(\Pi, \Gamma) \) on \( U(\Gamma) \) that provides us with the necessary tools to express the starproduct (2.16) as a twist within \( \mathcal{W}(\Pi, \Gamma) \otimes \mathcal{W}(\Pi, \Gamma) \). In the next section we accommodate \( U(p) \) within \( \mathcal{W}(\Pi, \Gamma) \) as a subalgebra. In this way, the starproduct turned into a twist thus also manages the deformation of \( U(p) \). In the mean time twists in \( \mathcal{W}(\Pi, \Gamma) \otimes \mathcal{W}(\Pi, \Gamma) \) enable us, as already mentioned, to go beyond the quantization of Poisson manifolds. As already announced, double application of such twists then provides us with desired deformations of the Heisenberg-algebra \( U(\hbar) \), covariant under corresponding deformations of \( U(p) \). In order to undertake this step of formalisation, we first consolidate our formulation of \( U(\Gamma) \) by setting

\[
\xi^R = \begin{cases} 
  x^\rho & : \rho = R \land R = 0, \ldots, (n-1) \\
  p^\mu & : \mu = R - n \land R = n, \ldots, (2n-1) 
\end{cases} 
R \in 0, \ldots, (n-1), n, \ldots, (2n-1). 
\] (3.1)

The generating relations (2.10) of \( U(\Gamma) \) are then reduced to the single equation

\[
\xi^R \xi^S - \xi^S \xi^R = 0. 
\] (3.2)

As a first step towards the Hopf-algebra of vector fields \( \mathcal{W}(\Pi, \Gamma) \), we introduce an algebra of momenta \( U(\Pi) \) in the following subsection.

### 3.1 The Algebra of Momenta \( U(\Pi) \) represented on \( U(\Gamma) \)

In order to obtain a \( 2n \)-dimensional Hopf-algebra of momenta \( U(\Pi) \), we take a copy of \( U(\Gamma) \) and enhance it to a Hopf-algebra. In particular we consider \( (\pi_N)_{N \in 0, \ldots, 2n-1} \) as a basis for \( U(\Pi) \). The generating relations, analogous to (2.10), are then given by

\[
\pi_M \cdot \pi_N - \pi_N \cdot \pi_M = 0, \quad M, N \in 0, \ldots, (n-1), n, \ldots, (2n-1). 
\]

The Hopf-structure on \( U(\Pi) \) is given by the following coproduct, counit and antipode

\[
\Delta(\pi_M) = \pi_M \otimes 1_\pi + 1_\pi \otimes \pi_M, \quad \epsilon(\pi_M) = 0, \quad S(\pi_M) = -\pi_M. 
\]

The Hopf-algebra axioms are easily verified. The Hopf-algebra of momenta \( U(\Pi) \) is represented by a left action on \( U(\Gamma) \), as follows.

\[
\pi_M \triangleright \xi^R = -iE^{MR}, \\
\pi_M \triangleright 1 = \epsilon(\pi_M), \\
1 \triangleright \xi^R = \xi^R. 
\] (3.3)
To this purpose we introduce the $2n$-dimensional tensor

$$E^{MR} = \begin{cases} 
\eta^{MR} & : M = 0, \ldots, (n-1) \land R = 0, \ldots, (n-1) \\
0 & : M = 0, \ldots, (n-1) \land R = n, \ldots, (2n-1) \\
0 & : M = n, \ldots, (2n-1) \land R = 0, \ldots, (n-1) \\
\eta^{(M-n)(R-n)} & : M = n, \ldots, (2n-1) \land R = n, \ldots, (2n-1)
\end{cases}$$

Alternatively we can also formulate (3.3) in the form

$$\pi_M \triangleright \xi^R = -i\Delta^R_M,$$

with

$$\Delta_M^R = \begin{cases} 
\delta_M^R & : M = 0, \ldots, (n-1) \land R = 0, \ldots, (n-1) \\
0 & : M = 0, \ldots, (n-1) \land R = n, \ldots, (2n-1) \\
0 & : M = d, \ldots, (2n-1) \land R = 0, \ldots, (n-1) \\
\delta_{M-n}^R & : M = d, \ldots, (2n-1) \land R = d, \ldots, (2n-1)
\end{cases}$$

We further verify that $U(\Pi)$ is realized on the vector space $\Gamma$ by

$$(\pi_M \cdot \pi_N - \pi_N \cdot \pi_M) \triangleright \xi^R = \pi_M \triangleright (\pi_N \triangleright \xi^R) - \pi_N \triangleright (\pi_M \triangleright \xi^R) = -i\Delta^R_N \epsilon(\pi_M) + i\Delta^R_M \epsilon(\pi_N) = 0. \quad (3.4)$$

Moreover the action of $U(\Pi)$ respects the algebraic structure (3.2) of $U(\Gamma)$, i.e. we have

$$\pi_M \triangleright (\xi^R \cdot \xi^S - \xi^S \cdot \xi^R) = \Delta(\pi_M) \triangleright (\xi^R \cdot \xi^S) - \Delta(\pi_M) \triangleright (\xi^S \cdot \xi^R) = (\pi_M \triangleright \xi^R)\xi^S + \xi^R(\pi_M \triangleright \xi^S)$$

$$- (\pi_M \triangleright \xi^S)\xi^R - \xi^S(\pi_M \triangleright \xi^R) = -i\Delta^R_M \xi^S - i\Delta^S_M \xi^R + i\Delta^R_M \xi^R + i\Delta^S_M \xi^R = 0. \quad (3.5)$$

We thus obtained a valid representation of $U(\Pi)$ on $U(\Gamma)$ and can join them now to a single cross-product algebra.

### 3.2 The Hopf-Algebra $\mathcal{W}(\Pi, \Gamma)$ of Vector Fields

In order to obtain the Hopf-algebra of vector fields $\mathcal{W}(\Pi, \Gamma)$ on $U(\Gamma)$, we have to consider the associative left cross-product algebra $U(\Gamma) \triangleright < U(\Pi)$ that is build on the tensor product $U(\Gamma) \otimes U(\Pi)$. Additional division of this cross-product enables
us to lift $\mathcal{W}(\Pi, \Gamma)$ itself to a Hopf-algebra that is once more represented on $U(\Gamma)$. The left cross-product in $U(\Gamma) \otimes U(\Pi)$ is given by
\[
(\xi^R \otimes \pi^M) \circ (\xi^S \otimes \pi^N) = \sum \xi^R(\pi^{M(1)} \triangleright \xi^S) \otimes \pi^{M(2)} \pi^N
= \xi^R(\pi^M \triangleright \xi^S) \otimes \pi^N + \xi^R(1 \triangleright \xi^S) \otimes \pi^M \pi^N
= -iE^MS(\xi^R \otimes \pi^N) + \xi^R \xi^S \otimes \pi^M \pi^N
\]
\[
\Delta(\pi^M) = \sum \pi^{M(1)} \otimes \pi^{M(2)}. \quad (3.6)
\]
Within $U(\Gamma) \triangleright\triangleright U(\Pi)$ the former subalgebras $U(\Gamma)$ and $U(\Pi)$ are also accomodated. They are identified by elements $\xi^R \equiv \xi^R \otimes 1$ and $\pi^M \equiv 1 \otimes \pi^M$ respectively. We introduce the following elements
\[
w^0_0 := \xi^R \otimes \pi^M, \quad w^+_M := 1 \otimes \pi^M,
\]
\[
w^-_R := \xi^R \otimes 1, \quad 1 = 1 \otimes 1, \quad (3.7)
\]
that generate $U(\Gamma) \triangleright\triangleright U(\Pi)$, i.e. we obtain
\[
U(\Gamma) \triangleright\triangleright U(\Pi) = \frac{T(U(\Gamma) \otimes U(\Pi))}{\mathcal{I}_{\Gamma, \Pi}},
\]
where $T(U(\Gamma) \otimes U(\Pi))$ is the tensor algebra of $U(\Gamma) \otimes U(\Pi)$ and $\mathcal{I}_{\Gamma, \Pi}$ is the two-sided ideal generated by relations
\[
[w^0_0^{RM}, w^{SN}_0]_\circ = -iE^MS w^0_0^{RN} + iE^NR w^0_0^{SM}, \quad [w^+_M, w^-_R]_\circ = -iE^{RM} 1
\]
\[
[w^0_0^{RM}, w^-_R]_\circ = -iE^SM w^-_R, \quad [w^0_0^{RM}, w^+_M]_\circ = iE^{RN} w^+_M,
\]
\[
[w^+_M, w^+_N]_\circ = 0, \quad [w^-_R, w^-_S]_\circ = 0. \quad (3.8)
\]
These are induced by (3.6) and (3.7). We further enhance the ideal $\mathcal{I}_{\Gamma, \Pi}$ by setting $w^-_R = 0$ such that we receive a new two-sided ideal $\mathcal{I}_{2\Pi}$ that is generated by relations
\[
[w^0_0^{RM}, w^0_0^{SN}]_\circ = -iE^MS w^0_0^{RN} + iE^NR w^0_0^{SM}, \quad [w^0_0^{RM}, w^+_N]_\circ = iE^{RN} w^+_N,
\]
\[
[w^+_M, w^+_N]_\circ = 0, \quad (3.9)
\]
such that we finally obtain the algebra of vector fields $\mathcal{W}(\Pi, \Gamma)$ by
\[
\mathcal{W}(\Pi, \Gamma) = \frac{T(U(\Gamma) \otimes U(\Pi))}{\mathcal{I}_{2\Pi}}.
\]
The algebra $\mathcal{W}(\Pi, \Gamma)$ is lifted to a Hopf-algebra by introducing coproducts, counits and antipodes on its generators $w^0_0^{RM}$ and $w^+_M$ according to
\[
\Delta(w^0_0^{RM}) = w^0_0^{RM} \otimes 1 + 1 \otimes w^0_0^{RM}, \quad \epsilon(w^0_0^{RM}) = 0, \quad S(w^0_0^{RM}) = -w^0_0^{RM},
\]
\[
\Delta(w^+_M) = w^+_M \otimes 1 + 1 \otimes w^+_M, \quad \epsilon(w^+_M) = 0, \quad S(w^+_M) = -w^+_M.
\]
It is easy to verify the axioms of Hopf-algebras and homomorphism. A detailed proof can be found in [19]. The Hopf-algebra of vector fields \( \mathfrak{W}(\Pi, \Gamma) \) is represented by a left action on \( U(\Gamma) \) according to

\[
\begin{align*}
\mathfrak{w}_0^{RM} \triangleright \xi^S &= -iE^{SM} \xi^R \\
\mathfrak{w}_+^{M} \triangleright \xi^R &= -iE^{RM} 1.
\end{align*}
\]

We verify that the generating relations of \( \mathfrak{W}(\Pi, \Gamma) \) are realized on \( \Gamma \), i.e. for the first relation in \( \text{[3.9]} \) we obtain that

\[
\begin{align*}
(w_0^{RM} \cdot w_0^{SN} - w_0^{SN} \cdot w_0^{RM} + iE^{MS}w_0^{RN} - iE^{NR}w_0^{SM}) \triangleright \xi^V \\
&= w_0^{RM} \triangleright (w_0^{SN} \triangleright \xi^V) - w_0^{SN} \triangleright (w_0^{RM} \triangleright \xi^V) \\
&\quad + iE^{MS}(w_0^{RN} \triangleright \xi^V) - iE^{NR}(w_0^{SM} \triangleright \xi^V) \\
&= -iE^{VN}(w_0^{RM} \triangleright \xi^S) + iE^{VM}(w_0^{SN} \triangleright \xi^R) \\
&\quad + E^{MS}E^{VN}\xi^R - E^{NR}E^{VM}\xi^S \\
&= -E^{VN}E^{SM}\xi^R + E^{VM}E^{RN}\xi^S + E^{MS}E^{VN}\xi^R - E^{NR}E^{VM}\xi^S = 0.
\end{align*}
\]

For the second relation we further compute that

\[
\begin{align*}
(w_0^{RM} \cdot w_+^{N} - w_+^{N} \cdot w_0^{RM} - iE^{RN}w_+^{M}) \triangleright \xi^V \\
&= w_0^{RM} \triangleright (w_+^{N} \triangleright \xi^V) - w_+^{N} \triangleright (w_0^{RM} \triangleright \xi^V) - iE^{RN}(w_+^{M} \triangleright \xi^V) \\
&= -iE^{VN}(w_0^{RM} \triangleright 1) + iE^{VM}(w_+^{N} \triangleright \xi^R) - E^{RN}E^{VM} \\
&= E^{VM}E^{RN} - E^{RN}E^{VM} = 0.
\end{align*}
\]

The third relation is already satisfied by \( \text{[3.4]} \). We further check that \( \mathfrak{W}(\Pi, \Gamma) \) respects the generating relations of \( U(\Gamma) \). For \( w_{+}^{M} \) this is already verified by \( \text{[3.5]} \). We thus consider

\[
\begin{align*}
w_0^{RM} \triangleright (\xi^V \xi^W - \xi^W \xi^V) \\
&= \Delta(w_0^{RM} \triangleright (\xi^V \xi^W) - \Delta(w_0^{RM} \triangleright (\xi^W \xi^V) \\
&= (w_0^{RM} \triangleright \xi^V) \xi^W + \xi^V(w_0^{RM} \triangleright \xi^W) \\
&\quad - (w_0^{RM} \triangleright \xi^W) \xi^V - \xi^W(w_0^{RM} \triangleright \xi^V) \\
&= -iE^{VM}\xi^R\xi^W - iE^{WM}\xi^V\xi^R + iE^{WM}\xi^R\xi^V + iE^{VM}\xi^W\xi^R = 0.
\end{align*}
\]

We are now prepared to take the next step that embeds the Poincaré-algebra \( U(\mathfrak{p}) \) within \( \mathfrak{W}(\Pi, \Gamma) \).

4 The Vector Field Representation of the Lorentz-Algebra

The previous preparations of the last sections enable us to represent the Poincaré-algebra \( U(\mathfrak{p}) \) within \( \mathfrak{W}(\Pi, \Gamma) \). As a corresponding representation of the Lorentz-
generators $M^{LN} \in U(so_{1,n-1})$ we introduce

$$M^{LN} = \begin{cases}
\bar{w}_0^{LN} - w_0^{NL} : & L = 0, \ldots, (n-1) \quad \land \quad N = 0, \ldots, (n-1) \\
0 : & L = 0, \ldots, (n-1) \quad \land \quad N = n, \ldots, (2n-1) \\
0 : & L = n, \ldots, (2n-1) \quad \land \quad N = 0, \ldots, (n-1) \\
\bar{w}_0^{LN} - w_0^{NL} : & l = n, \ldots, (2n-1) \quad \land \quad N = n, \ldots, (2n-1)
\end{cases}$$

(4.1)

Translational operators are already given by the algebra of momenta $U(\Pi)$, i.e. we have

$$P^N := w^N_1.$$

(4.2)

With relations (3.9) we compute the generating relations (2.3) of $U(p)$ in their block-diagonal form (2.11) to be

$$[M^{LN}, M^{IP}] = -iE^{NI}M^{LP} + iE^{PL}M^{IN} + iE^{NP}M^{IL} - iE^{IL}M^{PN},$$

$$[M^{LN}, P^M] = iE^{LM}P^N - iE^{NM}P^L.$$

Due to the linear combination of the Lorentz generators (4.1) in terms of generators of $W(\Pi, \Gamma)$, the Hopf-structure of the vector fields is carried over to the expected Hopf-structure in this representation of $U(p)$, i.e. we have

$$\Delta(M^{LN}) = M^{LN} \otimes 1 + 1 \otimes M^{LN}, \quad \epsilon(M^{LN}) = 0, \quad S(M^{LN}) = -M^{LN}.$$

The representation of $W(\Pi, \Gamma)$ on $U(\Gamma)$ determines that of $U(p)$, i.e.

$$M^{LN} \triangleright \xi^R = iE^{NR}\xi^L - iE^{LR}\xi^N.$$

According to (3.1), we receive the corresponding $n + n$-decomposition being

$$m^{\mu\nu} \triangleright x^\rho = -i\eta^{\rho\sigma}x^\mu + i\eta^{\rho\sigma}x^\nu,$$

$$m^{\mu\nu} \triangleright p^\sigma = -i\eta^{\rho\sigma}p^\mu + i\eta^{\rho\sigma}p^\nu,$$

that is in accordance with (2.5). Since $U(p)$ is a sub-Hopf-algebra of $W(\Pi, \Gamma)$, we do not require to further verify properties of the representation of $U(p)$ on $U(\Gamma)$. Before we turn to actual twist-deformations of $U(\Gamma)$ and $U(\eta_{2n})$, we have to consider the twist formalism as such. In particular we have to discuss now double application of twists.

## 5 Twisting

In this section we first shortly review basic definitions and properties of twists. Our primary aim however is to show that a double application of twists in turn
can be treated as a twist as well. This comes in handy when we first deform the \(2n\)-dimensional commutative phase space algebra \(U(\Gamma)\) to the Heisenberg-algebra \(U(h_{2n})\) and in a second step twist once more in order to obtain some deformation of \(U(h_{2n})\) itself. These two twists of course can be merged to a single expression by the use of the Baker-Campbell-Hausdorff formula. But application of this formula might turn out to be complicated by the computation of higher order terms in the exponent. It might thus be a better choice not to evaluate this product of twists, although the application then becomes a little bulky. Thus, up to the double application of twists, the first subsection of this section is rather a review to keep everything clear. The second subsection further embeds the starproduct of Weyl and Moyal \(^{(2.16)}\) into the vector field formalism.

\section*{5.1 Double Twisting}

As announced, we begin with a little review of the definition of twists and basic properties. We thus define twists for a general Hopf-algebra \(H\) to be given by

\begin{definition}
Let \((H, \mu, \eta, \Delta, \epsilon, S; K)\) be a Hopf-algebra over the field \(K\). Then an invertible object \(F \in H \otimes H\) is called a twist, if the following two conditions hold

\begin{align*}
F_{12} (\Delta \otimes \text{id}) (F) &= F_{23} (\text{id} \otimes \Delta) (F) \\
(\epsilon \otimes \text{id}) (F) &= 1 = (\text{id} \otimes \epsilon) (F).
\end{align*}

For \(F = \sum F^{(1)} \otimes F^{(2)}\) the objects \(F_{12}\) and \(F_{23}\) are defined by

\begin{align*}
F_{12} &= \sum F^{(1)} \otimes F^{(2)} \otimes 1 \\
F_{23} &= \sum 1 \otimes F^{(1)} \otimes F^{(2)}.
\end{align*}

\end{definition}

This definition is the basic ingredient to perform deformations. That these twists in turn provide the desired deformations of Hopf-algebras \(H_F\) is stated within the following proposition.

\begin{proposition}
Let \((H, \mu, \eta, \Delta, \epsilon, S; K)\) be a Hopf-algebra and let furthermore the objects \(\eta, \eta^{-1} \in H\) be given by

\begin{align*}
\eta &= \mu (\text{id} \otimes S) (F) \\
\eta^{-1} &= \mu (S \otimes \text{id}) (F).
\end{align*}

Then \((H, \mu, \eta, \Delta_F, \epsilon, S_F; K)\) with

\begin{align*}
\Delta_F(h) &= F \Delta(h) F^{-1} \\
S_F(h) &= \eta S(h) \eta^{-1}
\end{align*}

is a Hopf-algebra.
\end{proposition}
and h ∈ H is the Hopf-algebra \( H_\mathcal{F} \) that is called the twist of H.

The crucial point we have to emphasize within the next step is that the Hopf-algebra H is not necessarily cocommutative. And in this respect H might already be the outcome of a preceding twist, applied to a Hopf-algebra that actually might have been cocommutative. Let us thus assume that we have a twist \( J \in H \otimes H \) in the tensor product of a Hopf-algebra H. In particular it satisfies conditions (5.1) and (5.2) of above definition, i.e.

\[
J_{12} (\Delta \otimes \text{id}) (J) = J_{23} (\text{id} \otimes \Delta) (J) \quad (5.3)
\]

\[
(\epsilon \otimes \text{id}) (J) = 1 = (\text{id} \otimes \epsilon) (J).
\]

We then receive a Hopf-algebra \( H_J \) according to above proposition. We can now go ahead and twist once more. Thus let \( G \in H \otimes H \) be a twist of \( H_J \), i.e. we have

\[
G_{12} (\Delta_J \otimes \text{id}) (G) = G_{23} (\text{id} \otimes \Delta_J) (G) \quad (5.4)
\]

With \( \Delta_J (h) = J \Delta(h) J^{-1} \) for \( h \in H \) the first of these two relations can be written as

\[
G_{12} J_{12} (\Delta \otimes \text{id}) (G) J_{12}^{-1} = G_{23} J_{23} (\text{id} \otimes \Delta) (G) J_{23}^{-1}.
\]

We thus claim that \( F = G \cdot J \) is a twist of H as well. Relation (5.2) is directly satisfied by the homomorphism property of the counit \( \epsilon \). Relation (5.1) in turn is verified by direct computation. In particular we obtain by the use of (5.4) and (5.3) that

\[
F_{12} (\Delta \otimes \text{id}) (F) = G_{12} \cdot J_{12} (\Delta \otimes \text{id}) (G \cdot J)
\]

\[
= G_{12} \cdot J_{12} (\Delta \otimes \text{id}) (G) (\Delta \otimes \text{id}) (J)
\]

\[
= G_{12} \cdot J_{12} (\Delta \otimes \text{id}) (G) J_{12}^{-1} J_{23} (\text{id} \otimes \Delta) (J)
\]

\[
= G_{23} J_{23} (\text{id} \otimes \Delta) (G) J_{23}^{-1} J_{23} (\text{id} \otimes \Delta) (J)
\]

\[
= G_{23} J_{23} (\text{id} \otimes \Delta) (G) (\text{id} \otimes \Delta) (J)
\]

\[
= F_{23} (\text{id} \otimes \Delta) (F).
\]

We thus collected all required ingredients to proceed to actual deformations of \( U(h_{2n}) \).

5.2 Twists, Starproducts and Vector Fields

The Hopf-algebra of vector fields \( \mathfrak{W}(\Pi, \Gamma) \) enables us to express the starproduct (2.16) as the inverse of a twist \( G \in \mathfrak{W}(\Pi, \Gamma) \otimes \mathfrak{W}(\Pi, \Gamma) \) that in the mean time is capable to deform the Poincaré-algebra \( U(p) \). The twist \( G \) corresponding to starproduct
(2.16) is given by
\[
G = e^{i\frac{\hbar}{2} \Xi_{MN} w^M_N \otimes w^N_M},
\]
where we define the antisymmetric tensor \(\Xi_{MN}\) by
\[
\Xi_{MN} = \begin{cases} 
0 & : M = 0, \ldots, (n-1) \quad \wedge \quad N = 0, \ldots, (n-1) \\
\eta_M, (N-n) & : M = 0, \ldots, (n-1) \quad \wedge \quad N = n, \ldots, (2n-1) \\
-\eta_M, (n-N) & : M = n, \ldots, (2n-1) \quad \wedge \quad N = 0, \ldots, (n-1) \\
0 & : M = n, \ldots, (2n-1) \quad \wedge \quad N = n, \ldots, (2n-1)
\end{cases}
\]

The defining conditions (5.1) and (5.2) for twists are easily checked. It is also easily verified that the generating relations (2.14) of \(U(\mathfrak{h}_{2n})\) are reproduced by the inverse \(G^{-1}\) of (5.5). We can thus use (5.5) in order to deform the algebraic sector of \(U(\Gamma)\) to that of \(U(\mathfrak{h}_{2n})\). We further concentrate on the deformation of coproducts (2.4) in \(U(\mathfrak{p})\) within the representation (4.1). Due to commutativity between \(P^M\) and \(G\) we only expect possible deformations for the coproduct of \(M^{LN}\). With the undeformed coproduct
\[
\Delta(M^{LN}) = \Delta(w^L_0 - w^N_0) = (w^L_0 - w^N_0) \otimes 1 + 1 \otimes (w^L_0 - w^N_0)
\]
and with the help of the formula
\[
e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, [A, \ldots [A, B]]]],
\]
we compute the deformed coproduct to be
\[
\Delta(M^{LN}) = M^{LN} \otimes 1 + 1 \otimes M^{LN} + \frac{\hbar}{2} \Xi_{RS} (E^{RL} w^N_+ - E^{RN} w^L_+) \otimes w^S_+ + \frac{\hbar}{2} \Xi_{RS} w^R_+ \otimes (E^{SL} w^N_+ - E^{SN} w^L_+)
\]
This corresponds to results presented in [7, 20, 23, 26]. However, we should give some comments to this particular deformed coproduct in respect to the discussion of the introduction. Textbooks on field theory of course never mention the existence of a deformed coproduct of the Poincaré-algebra \(U(\mathfrak{p})\) in order to respect the commutation relations of the Heisenberg-algebra \(U(\mathfrak{h}_{2n})\). Within the introduction we argued that \(U(\mathfrak{p})\) could be embedded in \(U(\mathfrak{h}_{2n})\) - without the requirement to explicitly introduce any deformed coproducts. In fact the coproducts of \(U(\mathfrak{p})\) actually are deformed without being manifest. This can be seen as follows. We freely choose the
upper part of the block-diagonal generator $M^{LN}$ and consider its coproduct, i.e.

$$\Delta(M^{\lambda\nu}) = M^{\lambda\nu} \otimes 1 + 1 \otimes M^{\lambda\nu} + \frac{\hbar}{2} \Xi_{RS} (E^{R\lambda} w^{\nu}_+ - E^{R\nu} w^{\lambda}_+) \otimes w^{S}_+$$

$$+ \frac{\hbar}{2} \Xi_{RS} w^R_+ \otimes (E^{S\lambda} w^{\nu}_+ - E^{S\nu} w^{\lambda}_+)$$

$$= M^{\lambda\nu} \otimes 1 + 1 \otimes M^{\lambda\nu} + \frac{\hbar}{2} \eta_{\rho\sigma} (\eta^{\rho\lambda} w^{\nu}_+ - \eta^{\rho\nu} w^{\lambda}_+) \otimes w^{(\sigma+n)}_+$$

$$- \frac{\hbar}{2} \eta_{\rho\sigma} w^{(\sigma+n)}_+ \otimes (\eta^{\rho\lambda} w^{\nu}_+ - \eta^{\rho\nu} w^{\lambda}_+)$$

We see that the coproduct of $M^{\lambda\nu}$ is nearly cocommutative - up to a minus sign in the deformed part. A cocommutative deformation would be trivial and thus we have a true but hidden deformation for the case we embed $U(p)$ in $U(h)$ as we did in the introduction. This particular minus sign distinguishes the naive "action" of the momentum on a coordinate $[p^\mu, x^\rho]$ from the "action" of a coordinate on momentum $[x^\mu, p^\rho]$. This comes into account when we determine the representation of $m^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ on a coordinate or a momentum operator by the commutator $[ , ]$ as in (1.2) and (1.3).

6 An Example for a Twisted Heisenberg-Algebra

In this final section we intend to outline the presented construction for a specific example. In particular we concentrate on how the Heisenberg-algebra $U(h)$ is further deformed by a an additional twist $I$. This corresponds to a second deformation of $U(\Gamma)$. In this section we merely wish to give some guidance to the presented apparatus and thus stick to a very simple but nontrivial example. We leave it to the reader to find more interesting or even more realistic deformations. We sketch an example that corresponds to a twist presented earlier in [19] and [14] and adapt it to our context. This specific twist is given by

$$I = e^{i a w^{(2n-1)(2n-1)}_0 \otimes w^{(n-1)}_+}, \quad a \in \mathbb{R}.$$ 

With the total twist

$$F = G \cdot I = e^{i \frac{\hbar}{2} \Xi_{MN} w^M_+ \otimes w^N_+} \cdot e^{i a w^{(2n-1)(2n-1)}_0 \otimes w^{(n-1)}_+},$$

we obtain the starproduct

$$F^{-1} = I^{-1} \cdot G^{-1} = e^{-i a w^{(2n-1)(2n-1)}_0 \otimes w^{(n-1)}_+} \cdot e^{-i \frac{\hbar}{2} \Xi_{MN} w^M_+ \otimes w^N_+},$$

$20$
that provides us with a deformation of \( U(\mathfrak{h}_{2n}) \). With the starproduct \( \mathcal{I}^{-1} \) only some of the generating relations of \( U(\mathfrak{h}_{2n}) \) actually become deformed. We first generally consider the starproduct of the product of two generators \( \xi^R, \xi^S \in U(\Gamma) \), i.e.

\[
\xi^R \ast \xi^S = \xi^R \xi^S + i a E^{(2n-1)R} E^{(n-1)S} \xi^{(2n-1)} + i \frac{\hbar}{2} R S .
\]

In particular we thus obtain for the choice \( R \to 2n - 1 \) and \( S \to n - 1 \) that

\[
\xi^{(2n-1)} \ast \xi^{(n-1)} = \xi^{(2n-1)} \xi^{(n-1)} + i a \xi^{(2n-1)} - i \frac{\hbar}{2} \eta^{(n-1)(n-1)},
\]

such that within the \( n + n \)-separation we obtain the commutator

\[
[x^{(n-1)} \ast p^{(n-1)}] = i \hbar \eta^{(n-1)(n-1)} - i a p^{(n-1)} .
\]

We thus obtained the expected deformation of the Heisenberg-algebra \( U(\mathfrak{h}_{2n}) \) for one of its most characteristic relations. We further compute an example for a deformation of the coproduct of \( M^{LN} \) such that we obtain manifest covariance with respect to deformed \( U(\mathfrak{p}) \). In particular we choose the coproduct \( \Delta_{\mathcal{F}}(M^{(2n-1)\ast n}) \) and to this purpose we first compute the corresponding twisted coproducts of \( w_0^{(2n-1)n} \) and \( w_0^{n(2n-1)} \), i.e. we have

\[
\Delta_{\mathcal{F}}(w_0^{(2n-1)n}) = G \cdot \mathcal{I} \left( w_0^{(2n-1)n} \otimes 1 + 1 \otimes w_0^{(2n-1)n} \right) \mathcal{I}^{-1} \cdot G^{-1}
\]

and

\[
\Delta_{\mathcal{F}}(w_0^{(2n-1)n}) = G \cdot \mathcal{I} \left( w_0^{(2n-1)n} \otimes 1 + 1 \otimes w_0^{(2n-1)n} \right) \mathcal{I}^{-1} \cdot G^{-1}
\]
We thus obtain that
\[
\Delta_F(M^{(2n-1)n}) = M^{(2n-1)n} \otimes e^{+a \eta^{(n-1)(n-1)} P^{(n-1)}} + 1 \otimes M^{(2n-1)n} \\
+ 2w_0^{(2n-1)} \otimes \sinh(+a \eta^{(n-1)(n-1)} P^{(n-1)}) \\
- \frac{\hbar}{2} \eta^{(n-1)(n-1)} P^{n} \otimes \left(P^{(2n-1)} e^{+a \eta^{(n-1)(n-1)} P^{(n-1)}} - P^{(2n-1)}\right) \\
+ \frac{\hbar}{2} \eta^{(n-1)(n-1)} P^{(2n-1)} \otimes \left(P^{n} - P^{n} e^{-a \eta^{(n-1)(n-1)} P^{(n-1)}} \right)
\]

There are of course several more deformed coproducts for this specific example of deformation. However, since we merely wish to give some idea of how the constructions in this work are applied to specific examples, we close our considerations at this point.

7 Conclusion

Providing the formalism to perform deformations of the Heisenberg-algebra and the corresponding twists of the Poincaré-algebra is certainly only one step of several that have to be mastered in order to obtain some enhanced version of relativistic quantum mechanics. In order to receive useful representations of the deformed Heisenberg-algebra on states of a Hilbert-space, it is for example a crucial point to discuss hermiticity and self-adjointness of the generators in deformed $U(\hbar^{2n})$. It is moreover not yet clear what further implications for the interpretation of quantum mechanics might result from such algebraic mixture of coordinates and momenta. However, quantum mechanics as we apply it in field theories, does not discribe high-energy measurements in such a way as we would expect them from Planck-scale physics. Thus, regarding noncommutative geometry as a high-energy approach, we should als take into account that gravity might not only provide a static form of noncommutativity - but one that is caused by the properties of matter itself that exists within such backgrounds.

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