HYPERBOLIC BALANCE LAWS WITH RELAXATION

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For Peter Lax, with admiration and affection,
on the occasion of his 90th birthday

Abstract. This expository paper surveys the progress in a research program aiming at establishing the existence and long time behavior of \(BV\) solutions to the Cauchy problem for hyperbolic systems of balance laws modeling relaxation phenomena.

1. Introduction. A system of conservation laws in one space dimension has the form
\[
\partial_t U(x,t) + \partial_x F(U(x,t)) = 0,
\]
where \(U\) is the state vector and \(F\) is the flux. The system is strictly hyperbolic when the eigenvalues of the Jacobian matrix of the flux are real and distinct.

Peter Lax has made major contributions in that area of partial differential equations. In fact, it is fair to state that his seminal paper [15], which coined the term “hyperbolic system of conservation laws” set the directions for the active development of this field over the past fifty years.

The root of the difficulties encountered in the analysis of hyperbolic conservation laws with nonlinear flux lies in their distinctive feature of wave breaking: Even when the initial values
\[
U(x,0) = U_0(x), \quad -\infty < x < \infty,
\]
of the state vector are smooth, solutions to the Cauchy problem (1.1), (1.2) develop spontaneously singularities that propagate on as shock waves. The state of the art is that when the total variation of \(U_0\) is sufficiently small there exists a unique admissible weak solution of class \(BV\) on the upper half-plane \((-\infty, \infty) \times [0, \infty)\), which may be constructed by any one of the following three methods: (a) the random choice algorithm of Glimm [14], improved by several authors, most notably Liu [16]; (b) front tracking, developed by the Italian school headed by Bressan [4]; and (c) the vanishing viscosity approach of Bianchini and Bressan [2]. The key estimate is
\[
TVU(\cdot, t) \leq cTVU_0(\cdot), \quad 0 \leq t < \infty.
\]

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The restriction that $TVU_0$ must be small is essential, as there are cases of systems in which specially constructed initial data with large variation generate solutions with variation that explodes in finite time. There are indications that this pathology may be present even in the classical Euler equations of gas dynamics.

Turning from systems (1.1) of conservation laws to systems

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + P(U(x,t)) = 0$$

of balance laws, with source $P$, it is a routine matter to construct local solutions to the Cauchy problem by employing any one of the aforementioned three methods, in conjunction with an operator splitting scheme. However, it is not generally possible to extend local to global solutions, because in the presence of the source the estimate (1.3) must be revised into

$$TVU(\cdot, t) \leq c e^{\mu t} TVU_0(\cdot),$$

where $\mu$ is typically positive. Consequently, as time increases, $TVU(\cdot, t)$ may eventually exceed the range of applicability of the method.

The present paper will survey a research program by the author with objective to establish the existence and long time behavior of $BV$ solutions to the Cauchy problem for hyperbolic systems of balance laws (1.4) modeling relaxation phenomena. In such systems, the effect of the source is dissipative and this raises the expectation for the existence of solutions in the large.

In a parallel direction, there is voluminous literature on the existence of classical solutions in the large to the Cauchy problem for systems in the above class, under initial data that are smooth and small. For the state of the art and references to earlier seminal work, the reader may consult [3]. The theories of classical and $BV$ weak solutions may share goals, but technically are quite distinct, as the former relies on Sobolev space estimates, while the latter hinges on $L^1$ and $BV$ type bounds.

The challenge to the analysis stems from the fact that in the systems of interest the source incurs only partial damping. This can be seen by noting that systems (1.4) modeling relaxation phenomena typically result from the coupling of conservation laws with balance laws and thereby assume the form

$$\begin{cases}
\partial_t V + \partial_x G(V,W) = 0 \\
\partial_t W + \partial_x H(V,W) + C(V,W)W = 0,
\end{cases}$$

It is clear that the source term cannot affect directly the damping of the $V$ component of the state vector. This difficulty may be overcome by exploiting the synergy between source and flux encoded in the so-called Kawashima condition.

The existence and long time behavior of $BV$ solutions to the Cauchy problems for systems (1.4) in the form (1.6) is established in Sections 2,3 and 4, under the assumption that the $V$ component carries zero “mass”. The complications arising when the $V$ component is allowed to carry nonzero “mass” are discussed in Section 5. Section 6 is a survey of similar results for certain systems arising in Physics, modeling continuous media with memory, in which the flux depends not only on the present value but also on the past history of the state vector.

For systems with parametrized source,

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + \mu P(U(x,t)) = 0,$$
a central problem that is still open, at least in the BV setting, is the behavior of solutions as the relaxation parameter $\mu$ tends to zero. An answer, in the context of a special system and very special initial data, is found in [18].

2. The Cauchy problem. We consider systems of strictly hyperbolic balance laws (1.4) under the following assumptions, typical for systems modeling relaxation phenomena:

The state vector $U$ takes values in $\mathbb{R}^n$.

The flux $F$ is a given smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any $U \in \mathbb{R}^n$, the Jacobian matrix $DF(U)$ possesses real eigenvalues $\lambda_1(U) < \cdots < \lambda_n(U)$ and thereby linearly independent sets of left (row) eigenvectors $L_1(U), \ldots, L_n(U)$ and right (column) eigenvectors $R_1(U), \ldots, R_n(U)$, normalized by

$$L_i(U)R_j(U) = \delta_{ij}, \quad i, j = 1, \ldots, n. \quad (2.1)$$

The source $P$, also a given smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$, vanishes at the origin, $P(0) = 0$, so that $U \equiv 0$ is an equilibrium solution of (1.4). Typically, the null space of $DP(0)$ has dimension $k, 1 \leq k < n$, in which case one may assume, without loss of generality, that (1.4) is of the form

$$\begin{cases}
\partial_t V + \partial_x G(V,W) + X(V,W) = 0 \\
\partial_t W + \partial_x H(V,W) + CW + Y(V,W) = 0,
\end{cases} \quad (2.2)$$

with $V$ taking values in $\mathbb{R}^k$, $W$ taking values in $\mathbb{R}^\ell, \ell = n - k$, and $X,Y$ of quadratic order at the origin. In fact, as noted in the Introduction, these systems often appear in the form (1.6).

The flux and the source are coupled by the Kawashima condition

$$DP(0)R_i(0) \neq 0, \quad i = 1, \ldots, n, \quad (2.3)$$

which guarantees that the linearized system

$$\partial_t U + DF(0)\partial_x U + DP(0)U = 0 \quad (2.4)$$

does not admit solutions in the form $u(x - \lambda_i(0)t)R_i(0)$, manifesting undamped propagating fronts.

The system (1.4) is endowed with an entropy-entropy flux pair $(\eta, q)$,

$$D^2\eta(U)DF(U) = DF(U)^\top D^2\eta(U), \quad (2.5)$$

with $\eta$ normalized by $\eta(0) = 0, D\eta(0) = 0$. Furthermore, the Hessian $D^2\eta(0)$ is positive definite, so $\eta$ is convex on some neighborhood of the origin. Consequently, admissible solutions of (1.4) with values in a small neighborhood of the origin must satisfy the entropy inequality

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + D\eta(U(x,t))P(U(x,t)) \leq 0. \quad (2.6)$$

The source is dissipative in that the entropy production $D\eta P$ satisfies

$$D\eta(U)P(U) \geq a|P(U)|^2, \quad (2.7)$$

with $a > 0$, for $U$ in some neighborhood of the origin. Since the null space of $DP(0)$ is nontrivial, the entropy production is merely positive semidefinite and the damping exerted unilaterally by the source is only partial. To achieve full damping will require the synergy between source and flux expressed by the Kawashima condition (2.3).
When the system (1.4) is in the form (1.6), the assumption (2.7) reduces to the statement that the \( k \times \ell \) matrix \( \eta_{WW}(0,0) \) vanishes while the \( \ell \times \ell \) matrix \( \eta_{WW}(0,0)C(0,0) \) is positive definite.

We assign initial data \( U_0 \) with bounded variation,

\[
T V U_0(\cdot) = \delta,
\]

that decay as \( |x| \to \infty \) sufficiently fast to render the integral

\[
\int_{-\infty}^{\infty} (1 + x^2) |U_0(x)|^2 \, dx = \sigma^2
\]

finite.

The analysis is simplified substantially when the system (1.4) is in the form (1.6) and the \( V_0 \) component of \( U_0 \) carries zero “mass”:

\[
\int_{-\infty}^{\infty} V_0(x) \, dx = 0.
\]

**Theorem 2.1.** Assume that the system (1.4) satisfies the conditions stated above and is in the form (1.6). Then there are positive constants \( \delta_0, \sigma_0, c_0, c_1, \nu \) and \( b \) with the following properties. For any initial data \( U_0 = (V_0, W_0) \) that satisfy (2.8), with \( \delta < \delta_0, (2.9) \), with \( \sigma < \sigma_0 \), and (2.10), the Cauchy problem (1.4), (1.2) possesses an admissible \( BV \) solution \( U \) on \((-\infty, \infty) \times [0, \infty)\) and

\[
\int_{-\infty}^{\infty} |U(x,t)| \, dx \leq b \sigma, \quad 0 \leq t < \infty,
\]

\[
T V U(\cdot, t) \leq c_0 \sigma + c_1 \delta e^{-\nu t}, \quad 0 \leq t < \infty,
\]

\[
\int_{-\infty}^{\infty} |U(x,t)| \, dx \to 0, \quad \text{as } t \to \infty,
\]

\[
T V U(\cdot, t) \to 0, \quad \text{as } t \to \infty.
\]

As already noted in the Introduction, the existence of a local solution to the Cauchy problem (1.4), (1.2) follows from a tedious, but fairly routine, application of the standard algorithms for treating hyperbolic systems of conservation laws, in conjunction with operator splitting, so as to account for the effects of the source. This analysis is found, for instance, in [13]. Thus the challenge for proving Theorem 2.1 lies in establishing the estimate (2.12), which will allow for extending the local solution into a global one. The derivation of this estimate will be the task of the following two sections, 3 and 4. In Section 5 we shall consider the complications arising when the system (1.4) is in its most general form (2.2) and/or the restriction (2.10) on the initial data is removed.

3. **\( L^1 \) estimates.** Under the conditions of Theorem 2.1, assuming that \( U = (V, W) \) is a solution to the Cauchy problem for (1.6), with initial values \( U_0 = (V_0, W_0) \), we establish here the assertions (2.11) and (2.13).

We will employ the “potential” function

\[
\Psi(x, t) = \int_{-\infty}^{x} V(y, t) \, dy.
\]

Clearly, \( \Psi \) is Lipschitz, with derivatives

\[
\partial_x \Psi = V, \quad \partial_t \Psi = -G(V, W).
\]
The role of the assumption (2.10) is to secure that the initial value \( \Psi_0(x) \) of \( \Psi \) satisfies \( \Psi_0(\pm \infty) = 0 \). Then, integrating by parts,

\[
\int_{-\infty}^{\infty} |\Psi_0(x)|^2 \, dx = \int_{-\infty}^{0} \left[ \int_{-\infty}^{x} \Psi_0(y) \, dy \right]^2 \, dx + \int_{0}^{\infty} \left[ \int_{-\infty}^{x} \Psi_0(y) \, dy \right]^2 \, dx
\]

\[
= -2 \int_{-\infty}^{0} x \Psi_0^\top(x) \Psi_0(x) \, dx - 2 \int_{0}^{\infty} \Psi_0^\top(x) \Psi_0(x) \, dx
\]

\[
\leq 2 \int_{-\infty}^{\infty} x^2 |\Psi_0(y)|^2 \, dy + \frac{1}{2} \int_{-\infty}^{\infty} |\Psi_0(x)|^2 \, dx,
\]

so that, by virtue of (2.9),

\[
\int_{-\infty}^{\infty} |\Psi_0(x)|^2 \, dx \leq 4\sigma^2.
\]  

(3.4)

We will be operating under the ansatz that, for some \( \omega > 0 \),

\[ |\Psi(x, t)| < \omega, \quad -\infty < x < \infty, \quad 0 \leq t < \infty, \]

which will be verified later.

The first step is to show

\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} |U(x, t)|^2 \, dx \, dt \leq c\sigma^2.
\]  

(3.6)

By virtue of (2.7), integration of the inequality (2.6) over \((-\infty, \infty) \times [0, \infty)\) readily yields that \( \int \int |W|^2 \, dx \, dt \) is bounded by \( c\sigma^2 \). However, showing that this will also be the case for the complementary component \( V \) will require considerable effort, as it rests on the synergy between the partially dissipative source and the flux, encoded in the Kawashima condition. The following argument has been adapted from [17].

We introduce the notations

\[
B = G_V(0, 0), \quad J = G_W(0, 0), \quad E = H_V(0, 0), \quad D = H_W(0, 0),
\]

\[
K = \eta_V V(0, 0), \quad M = \eta_W W(0, 0),
\]

\[
Q = [\eta_W W(0, 0) C(0, 0)]^{-1}.
\]  

(3.7)

(3.8)

(3.9)

The \( k \times k \) matrix \( K \) and the \( \ell \times \ell \) matrix \( M \) are symmetric and positive definite. Furthermore, since \( D^2 \eta(U) D F(U) \) is symmetric and \( \eta_V V(0, 0) = 0 \),

\[
(K B) \top = K B, \quad (M D) \top = M D, \quad (M E) \top = K J.
\]

(3.10)

Finally, the \( \ell \times \ell \) matrix \( Q \) is positive definite.

If \( N \) is any eigenvector of the matrix \( B \), then \( EN \neq 0 \), since \( EN = 0 \) would imply that \( R = \begin{pmatrix} N \\ 0 \end{pmatrix} \) is an eigenvector of \( D F(0) \) with \( D P(0) R = 0 \), in contradiction to the Kawashima condition. It may then be shown (see [17]) that there exists a \( k \times k \) matrix \( \Omega \) such that \( \Omega K \) is skew-symmetric and \( \Omega K B \) is positive on the kernel of \( M E \).

We now define the following functions:

\[
\Theta(V,W,\Psi) = \Psi^\top K \Psi - 2\Psi^\top K J Q M W - \kappa \Psi^\top \Omega K V + \gamma \eta(V,W),
\]

(3.11)

\[
\Xi(V,W,\Psi) = \Psi^\top K B \Psi - 2\Psi^\top K J Q M H(V, W) - \kappa \Psi^\top \Omega K G(V, W) + \gamma q(V, W),
\]

(3.12)

\[
\Pi(V,W,\Psi) = 2\Psi^\top K G(V, W) - 2\Psi^\top K B V - 2G^\top(V, W) K J Q M W + 2V^\top K J Q M H(V, W) - 2\Psi^\top K J Q M C(V, W) W - \kappa G^\top(V, W) \Omega K V
\]

(3.13)
\[ +\kappa V^\top \Omega KG(V,W) + \gamma \eta_W(V,W)C(V,W)W, \]

where \( \kappa \) and \( \gamma \) are positive constants to be fixed below.

A lengthy but straightforward calculation, using (1.6), (3.2) and (2.6), yields

\[ \partial_t \Theta(V,W,\Psi) + \partial_x \Xi(V,W,\Psi) + \Pi(V,W,\Psi) \leq 0. \]  

(3.14)

We perform a (finite) Taylor expansion of \( \Pi(V,W,\Psi) \) about the origin. Using the symmetry relations (3.10) and recalling that \( |U| < \rho \) and \( |\Psi| < \omega \), we obtain

\[ \Pi = V^\top \Lambda V + V^\top \Gamma W + W^\top \Delta W + O(\rho + \omega)(|V|^2 + |W|^2), \]  

(3.15)

where

\[ \Lambda = 2(ME)^\top QME + 2\kappa \Omega KB, \]  

(3.16)

\[ \Delta = -2J^\top KJQM + \gamma Q^{-1}, \]  

(3.17)

\[ \Gamma = -2KBJQM + 2KJQMD + 2\kappa \Omega KJ. \]  

(3.18)

The crucial observation is that the second term on the right-hand side of (3.16) is positive on the kernel of \( ME \) and the first term is positive on the complementary space, whence, for \( \kappa \) sufficiently small, \( \Lambda \) is positive definite. We thus fix \( \rho \) and \( \omega \) sufficiently small and \( \gamma \) sufficiently large so that both \( \Theta \) and \( \Pi \) become positive definite at the origin. Then, integrating (3.14) over \(( -\infty, \infty) \times [0, \infty) \) and using (2.9) and (3.4), we arrive at (3.6).

The next step is to show that, always under the ansatz (3.5), for \( \omega \) as fixed above,

\[ t \int_{-\infty}^{\infty} |U(x,t)|^2 dx \leq c\sigma^2, \quad 0 \leq t < \infty, \]  

(3.19)

\[ t \int_{-\infty}^{\infty} |U(x,t)|^2 dx \to 0, \quad \text{as } t \to \infty. \]  

(3.20)

On account of (2.6) and (2.7),

\[ \partial_t [t\eta(U)] + \partial_x [tq(U)] \leq \eta(U). \]  

(3.21)

Integrating the above inequality over \(( -\infty, \infty) \times [0,t] \) and using (3.6), we arrive at (3.19).

Now fix any \( \epsilon > 0 \). By virtue of (3.6), there exists \( \tau > 0 \) such that

\[ \tau \int_{-\infty}^{\infty} \eta(U(x,\tau))dx < \frac{\epsilon}{4}, \]  

(3.22)

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(U(x,t))dxdt < \frac{\epsilon}{2}. \]  

(3.23)

For any \( t \in (\tau, \infty) \), integrating (3.21) over \(( -\infty, \infty) \times [\tau,t] \) yields

\[ t \int_{-\infty}^{\infty} \eta(U(x,t))dx < \epsilon, \]  

(3.24)

whence (3.20) follows.

We have now laid the preparation for establishing (2.11) and (2.13), while also verifying, in the process, the ansatz (3.5).

We identify \( \lambda > 0 \) such that

\[ |q(U)| \leq \lambda \eta(U) \]  

(3.25)

holds for all \( U \) in a neighborhood of the origin.
On account of (3.19) and Schwarz’s inequality,
\[
\int_{-2\lambda t}^{2\lambda t} |U(x,t)| dx \leq \left\{ 4\lambda t \int_{-2\lambda t}^{2\lambda t} |U(x,t)|^2 dx \right\}^{1/2} \leq c\sigma.
\] (3.26)
Furthermore, (3.26) and (3.20) imply
\[
\int_{-2\lambda t}^{2\lambda t} |U(x,t)| dx \rightarrow 0, \quad \text{as } t \rightarrow \infty.
\] (3.27)

We now proceed to estimate the integral of \(|U(x,t)|\) over the interval \((2\lambda t, \infty)\). For \(k = 1, 2, \ldots\), we integrate the entropy inequality (2.6) over the trapezoid with vertices \((2^k \lambda t, t), (2^{k+1} \lambda t, t), ((2^k - 1) \lambda t, 0)\) and \(((2^{k+1} + 1) \lambda t, 0)\). On account of (3.25), this yields
\[
\int_{2^k \lambda t}^{2^{k+1} \lambda t} |U(x,t)|^2 dx \leq \beta \int_{(2^k - 1) \lambda t}^{(2^{k+1} + 1) \lambda t} |U_0(x)|^2 dx.
\] (3.28)
Therefore,
\[
\int_{2^k \lambda t}^{2^{k+1} \lambda t} \left( x^2 + 1 \right) |U(x,t)|^2 dx \leq \beta (4^{k+1} \lambda t^2 + 1) \int_{(2^k - 1) \lambda t}^{(2^{k+1} + 1) \lambda t} |U_0(x)|^2 dx \leq 16 \beta \int_{(2^k - 1) \lambda t}^{(2^{k+1} + 1) \lambda t} (x^2 + 1) |U_0(x)|^2 dx.
\] (3.29)

Summing the above inequalities over \(k = 1, 2, \ldots\), we deduce
\[
\int_{2\lambda t}^{\infty} (x^2 + 1) |U(x,t)|^2 dx \leq 32 \beta \int_{\lambda t}^{\infty} (x^2 + 1) |U_0(x)|^2 dx \leq c\sigma^2.
\] (3.30)
Finally, by Schwarz’s inequality,
\[
\int_{2\lambda t}^{\infty} |U(x,t)| dx \leq \left[ \int_{2\lambda t}^{\infty} (x^2 + 1)^{-1} dx \right]^{1/2} \left[ \int_{2\lambda t}^{\infty} (x^2 + 1) |U_0(x)|^2 dx \right]^{1/2} \leq c\sigma(\lambda t + 1)^{-\frac{1}{2}}.
\] (3.31)

A similar bound is obtained for the integral of \(|U(x,t)|\) over \((-\infty, -2\lambda t)\). These bounds together with (3.26) and (3.27) establish (2.11), for some \(b\), and (2.13), albeit subject to (3.5). In order to show that this restriction is redundant, we first note that, upon increasing, if necessary, the size of \(b\), (2.11) holds at \(t = 0\), regardless of (3.5). We thus set \(\sigma_0 = \omega/b\) and fix \(\sigma < \sigma_0\). Since \(|\Psi(x,t)| \leq \|V(\cdot, t)\|_{L^1}\), we deduce that when (2.11) is satisfied for some \(t\), (3.5) must hold on \((-\infty, \infty) \times [t + \epsilon]\), by virtue of (3.2). A simple continuation argument then establishes that, for any \(\sigma < \sigma_0\), (3.5) holds on \((-\infty, \infty) \times [0, \infty)\). This completes the proof of (2.11) and (2.13).

4. Redistribution of damping and BV bounds. To complete the proof of Theorem 2.1, we need to verify (2.12) and (2.14). For guidance, we turn to the linearized system (2.4). We assemble the left eigenvectors \(L_1(0), \ldots, L_n(0)\) and the right eigenvectors \(R_1(0), \ldots, R_n(0)\) as rows and columns of \(n \times n\) matrices \(L(0)\) and \(R(0)\). In terms of the new state vector \(V = L(0)R\), (2.4) becomes
\[
\frac{\partial}{\partial t} V_i(x,t) + \lambda_t(0) \partial_x V_i(x,t) + \sum_{j=1}^n A_{ij} V_j(x,t) = 0, \quad i = 1, \ldots, n,
\] (4.1)
with
\[ A = L(0)DP(0)R(0). \]  \hspace{1cm} (4.2)

It is easily seen that if \( A \) is column-diagonally dominant, namely
\[ A_{ii} - \sum_{j \neq i} |A_{ji}| > 0, \quad i = 1, \ldots, n, \]  \hspace{1cm} (4.3)
then the \( L^1 \) norm, and thereby also the total variation, of the solution \( V \) to the Cauchy problem for (4.1) are time-decreasing. In fact, it is known \([1,5,13]\) that when (4.3) holds, the Cauchy problem for the nonlinear system (1.4), with initial data of sufficiently small total variation, possesses a unique admissible \( BV \) solution \( U \) on \((-\infty, \infty) \times [0, \infty)\) and
\[ TVU(\cdot, t) \leq c e^{-\nu t} TVU_0(\cdot), \quad 0 \leq t < \infty, \]  \hspace{1cm} (4.4)
for some \( \nu > 0 \).

Unfortunately, the condition (4.3) is far too restrictive, as it presumes that the damping is widely distributed among the equations of the system, and is thus incompatible with systems in the form (1.6). The plan here is to devise a transformation of the state vector that redistributes the damping more equitably among the equations of the system.

The first step is to show that (2.7) together with (2.3) imply
\[ A_{ii} > 0, \quad i = 1, \ldots, n. \]  \hspace{1cm} (4.5)
Indeed, by (2.7), the function
\[ \theta(U) = D\eta(U)P(U) - a|P(U)|^2 \]  \hspace{1cm} (4.6)
is minimized at \( U = 0 \), and hence the Hessian matrix
\[ D^2\theta(0) = D^2\eta(0)DP(0) + DP(0)^TD^2\eta(0) - 2aDP(0)^TDP(0) \]  \hspace{1cm} (4.7)
is positive semidefinite. Multiplying, from the left, (2.5) by \( R_i^\top \) yields
\[ R_i^\top D^2\eta DF = \lambda_i R_i^\top D^2\eta, \]  \hspace{1cm} (4.8)
which shows that \( R_i^\top D^2\eta \) is collinear to \( L_i \), and in particular
\[ R_i^\top D^2\eta = (R_i^\top D^2\eta R_i) L_i. \]  \hspace{1cm} (4.9)

We now multiply (4.7), from the left by \( R_i^\top \) and from the right by \( R_i \). Using that \( D^2\theta(0) \) is positive semidefinite, together with (4.8) at \( U = 0 \), we conclude
\[ [R_i^\top (0)D^2\eta(0)R_i(0)]A_{ii} \geq a|DP(0)R_i(0)|^2. \]  \hspace{1cm} (4.10)

Let us now assume that \( U \) is an admissible \( BV \) solution of the Cauchy problem (1.4), (1.2), on the upper half-plane, taking values in a ball of small radius \( \rho \), centered at the origin. We introduce the functions
\[ \Phi(x,t) = \int_{-\infty}^x NU(y,t)dy, \quad Z(x,t) = \int_{-\infty}^x NP(U(y,t))dy, \]  \hspace{1cm} (4.11, 4.12)
where \( N \) is a \( n \times n \) matrix to be specified below. We note that \( \Phi \) is Lipschitz with
\[ \partial_x \Phi(x,t) = NU(x,t), \quad \partial_t \Phi(x,t) = -NF(U(x,t)) - Z(x,t). \]  \hspace{1cm} (4.13)
We replace \( U \) by the new state vector
\[ \dot{U} = U - \Phi \]  \hspace{1cm} (4.14)
and rewrite (1.4) as a system for \( \hat{U} \), namely
\[
\partial_t \hat{U}(x,t) + \partial_x \hat{F}(\hat{U}(x,t),\Phi(x,t)) + \hat{P}(\hat{U}(x,t),\Phi(x,t),Z(x,t)) = 0,
\]
where
\[
\hat{F}(\hat{U},\Phi) = F(\hat{U} + \Phi) - F(\Phi),
\]
\[
\hat{P}(\hat{U},\Phi,Z) = P(\hat{U} + \Phi) - NF(\hat{U} + \Phi) + DF(\Phi)N[\hat{U} + \Phi] - Z.
\]

The motivation for switching from the relatively simple (1.4) to the cumbersome (4.15), which is not even a closed system, is that if one presumes that \( \Phi \) and \( Z \) are somehow known, and regards (4.15) as an inhomogeneous system of balance laws for \( \hat{U} \), then, in the place of (4.2), one gets the matrix \( \hat{A} \) with entries
\[
\hat{A}_{ij} = L_j(0)D\hat{P}(0,0,0)R_j(0) = A_{ij} + [\lambda_i(0) - \lambda_j(0)]\Delta_{ij},
\]
(4.18)
where \( \Delta = L(0)NR(0) \). This presents the opportunity of making \( \hat{A} \) diagonally dominant by properly selecting the matrix \( N \). In particular, the choice \( N = R(0)\Delta L(0) \), with \( \Delta_{ii} = 0 \), for \( i = 1,\ldots,n \) and
\[
\Delta_{ij} = \frac{A_{ij}}{\lambda_i(0) - \lambda_j(0)}, \quad \text{for } i \neq j,
\]
(4.19)
renders a diagonal \( \hat{A} \), with \( \hat{A}_{ii} = A_{ii} \), for \( i = 1,\ldots,n \), and \( \hat{A}_{ij} = 0 \), for \( i \neq j \). In that case, when (4.5) holds, \( \hat{A} \) is diagonally dominant.

The principal task is to solve the Cauchy problem for the inhomogeneous system (4.15) of balance laws, estimating, in the process, the variation of the solution \( \hat{U} \). If \( \hat{F} \) and \( \hat{P} \) did not depend on \( \Phi \) and \( Z \), the diagonal dominance of \( \hat{A} \) would induce exponential decay in the variation of \( \hat{U} \), as in (4.4). The difficulty stems from the fact that \( \Phi \) and \( Z \) are not known in advance, as they depend on the solution. However, the saving grace is that the variation of these terms is already under control. Indeed, on account of (4.11), (4.12) and (2.11), we have
\[
TV \Phi(\cdot,t) \leq c\sigma, \quad TV Z(\cdot,t) \leq c\sigma, \quad 0 \leq t < \infty.
\]
(4.20)
In particular, (4.14) and (4.20) allow us to relate the variations of \( U \) and \( \hat{U} \):
\[
TVU(\cdot,t) \leq TV\hat{U}(\cdot,t) + c\sigma.
\]

The construction of \( \hat{U} \) is attained by combining the random choice method of Glimm with an operator splitting algorithm, and it is lengthy and technical. It can be found in [6,7,12]. The conclusion is summed up in the following estimate:
\[
TV U(\cdot,t) \leq c\delta e^{-2\nu t} + c\sigma + c\rho \int_0^t e^{-2\nu(t-\xi)} TV U(\cdot,\xi)d\xi.
\]
(4.21)
When \( \rho \) is sufficiently small, (4.21) implies the desired estimate (2.12). The assertion (2.14) follows easily from (2.12) and (2.13).

5. **Nonzero mass.** The proof of Theorem 2.1, and in particular the analysis in Section 3, rely heavily on the assumption that the system is in the form (1.6) and the mass carried by the \( V \) component is zero. The last condition is induced by (2.10). The decay (2.13) of the \( L^1 \) norm is a direct consequence of the above. The situation is substantially more complicated for systems (1.4) in the more general form (2.2), or even for systems in the form (1.6) when the assumption (2.10) is relaxed. We shall illustrate this here in the context of an example.
We consider the Cauchy problem for the simple system
\[
\begin{cases}
\partial_t u - \partial_x v + \alpha v^2 = 0 \\
\partial_t v + \partial_x p(u) + v = 0,
\end{cases}
\tag{5.1}
\]
where \(p'(u) < 0\), and \(\alpha\) is a constant, positive, negative or zero. In the place of Theorem 2.1, we here have

**Theorem 5.1.** Assume \((u_0, v_0)\) are given functions on \((-\infty, \infty)\) satisfying
\[
\int_{-\infty}^{\infty} (1 + x^2)[u_0^2(x) + v_0^2(x)]dx = \sigma^2, \tag{5.2}
\]
\[
TV u_0(\cdot) + TV v_0(\cdot) = \delta, \tag{5.3}
\]
with \(\sigma\) and \(\delta\) sufficiently small. Then there exists an admissible \(BV\) solution \((u, v)\) of the system \((5.1)\) on \((-\infty, \infty) \times [0, \infty)\), with initial data \((u_0, v_0)\). Furthermore,
\[
\int_{-\infty}^{\infty} ||u(x,t) - \theta(x,t)|| + |v(x,t)||dx \leq b\sigma(t+1)^{-\frac{1}{4}}, \quad 0 \leq t < \infty, \tag{5.4}
\]
\[
TV u(\cdot, t) + TV v(\cdot, t) \leq c_0\sigma(t+1)^{-\frac{1}{4}} + c_1\delta e^{-\nu t}, \quad 0 \leq t < \infty, \tag{5.5}
\]
where \(c_0, c_1, \nu\) and \(b\) are positive constants, independent of the initial data, and \(\theta\) is the solution
\[
\theta(x, t) = M(4\pi t)^{-\frac{1}{2}} \exp \left( -\frac{x^2}{4t} \right), \tag{5.6}
\]
to the heat equation, with \(M\) some constant depending on \((u_0, v_0)\).

Sketching the proof of the above theorem, we discuss, for simplicity, only the special case \(\alpha = 0\), so that \((5.1)\) is still in the form \((1.6)\). However, we no longer impose \((2.10)\), assuming instead
\[
\int_{-\infty}^{\infty} u_0(x)dx = M. \tag{5.7}
\]
The details of the proof, together with the treatment of the general case \(\alpha \neq 0\), are found in [11].

The objective is to demonstrate that, for any \(0 < t < \infty\),
\[
\int_{-\infty}^{\infty} ||u(x,t) - \hat{u}(x,t)|| + |v(x,t) - \hat{v}(x,t)||dx \leq c\sigma(t+1)^{-\frac{1}{4}}, \tag{5.8}
\]
where \((\hat{u}, \hat{v})\) is the solution to the system
\[
\begin{cases}
\partial_t \hat{u} - \partial_x \hat{v} = 0 \\
\hat{v} = -\partial_x p(\hat{u}),
\end{cases}
\tag{5.9}
\]
with the same initial values \((u_0, v_0)\) as \((u, v)\). Thus \(\hat{u}\) satisfies the porous media equation
\[
\partial_t \hat{u} + \partial_x^2 p(\hat{u}) = 0. \tag{5.10}
\]
By means of lengthy but straightforward analysis, involving elementary “energy” estimates, it is possible to establish bounds for the solution \(\hat{u}\) of \((5.10)\) and thereby for \(\hat{v}\), including the following:
\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} \hat{v}^2(x,t)dxdt \leq c\sigma^2, \tag{5.11}
\]
where θ is defined by (5.6). Thus proving (5.8) will establish the assertion (5.4). The asserted decay (5.5) in the variation will then easily follow.

With an eye to verifying (5.8), we set \( w = u - \hat{u} \) and \( z = v - \hat{v} \), noting that \((w, z)\) satisfies the system

\[
\begin{align*}
&\partial_t w - \partial_x z = 0 \\
&\partial_t z + \partial_x \hat{p}(w, \hat{u}) + z + \partial_t \hat{v} = 0,
\end{align*}
\]

with zero initial data. In (5.16), \( \hat{p} \) stands for the “relative pressure” defined by

\[
\hat{p}(w, \hat{u}) = p(w + \hat{u}) - p(\hat{u}).
\]

The admissibility of solutions to (5.16) is encoded in the “relative entropy” inequality

\[
\partial_t [\hat{\psi}(w, \hat{u}) + \frac{1}{2} \hat{z}^2] + \partial_x [\hat{p}(w, \hat{u}) \hat{z}] + \hat{z}^2 \leq -[\hat{p}(w, \hat{u}) - p'(\hat{u})w] \partial_t \hat{u} - \hat{z} \partial_t \hat{v},
\]

where \( \hat{\psi} \) denotes the “relative internal energy” defined by

\[
\hat{\psi}(w, \hat{u}) = - \int_{\hat{u}}^{w+\hat{u}} p(\xi) d\xi + p(\hat{u}) w.
\]

A priori bounds on \((w, z)\) will be derived by combining (5.18) with the balance law

\[
\partial_t \left[ \frac{1}{2} \Phi^2 + (z + \hat{v}) \Phi \right] + \partial_x [\hat{p}(w, \hat{u}) \Phi] - \hat{p}(w, \hat{u}) w = \hat{z}^2 + \hat{z} \hat{v},
\]

where \( \Phi \) is the “potential function”

\[
\Phi(x, t) = \int_{-\infty}^{x} w(y, t) dy.
\]

Terms with the “good sign” appearing in (5.18) and (5.20) include \( \hat{\psi}(w, \hat{u}), \hat{z}^2, \Phi^2 \) and \(-\hat{p}(w, \hat{u}) w\). It is easy to see that, when \( \sigma \) is sufficiently small, the terms of indefinite sign may be balanced, with the help of (5.11), (5.12) and (5.13), against the terms with the good sign, yielding bounds

\[
\int_{-\infty}^{\infty} [w^2(x, t) + \hat{z}^2(x, t)] dx \leq c\sigma^2,
\]

\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} [w^2(x, t) + \hat{z}^2(x, t)] dx dt \leq c\sigma^2.
\]

To get the next round of estimates, we multiply (5.18), first by \( t \) and then by \( x^2 \), which yields

\[
\partial_t \{ t[\hat{\psi}(w, \hat{u}) + \frac{1}{2} \hat{z}^2] \} + \partial_x \{ t\hat{p}(w, \hat{u}) \hat{z} \} + tz^2 \\
\leq \hat{\psi}(w, \hat{u}) + \frac{1}{2} \hat{z}^2 - t[\hat{p}(w, \hat{u}) - p'(\hat{u})w] \partial_t \hat{u} - tz \partial_t \hat{v},
\]
\[ \partial_t \{ x^2 [\psi(w, \dot{u}) + \frac{1}{2} z^2] \} + \partial_x \{ x^2 \hat{\rho}(w, \dot{u}) z \} + x^2 z^2 \leq 2x\hat{\rho}(w, \dot{u}) z - x^2 [\hat{\rho}(w, \dot{u}) - \rho'(\dot{u}) w] \partial_t \dot{u} - x^2 z \partial_t \dot{v}. \]  

With the help of (5.22) and (5.23), together with (5.11), (5.12) and (5.13), one may balance the terms of indefinite sign in (5.24) and (5.25) against the terms \( t\dot{\psi}, x^2 \dot{\psi}, tz^2 \) and \( x^2 z^2 \), with the good sign, to obtain the estimate
\[ \int_{-\infty}^{\infty} (x^2 + t + 1)[w^2(x,t) + z^2(x,t)]dx \leq \sigma^2. \]  

Finally, we apply Schwarz’s inequality to (5.26) to get
\[ \left[ \int_{-\infty}^{\infty} [w(x,t)]^2 + [z(x,t)]^2 \right]^{1/2} \leq 2 \int_{-\infty}^{\infty} (x^2 + t + 1)^{-1} dx \int_{-\infty}^{\infty} (x^2 + t + 1)[w^2(x,t) + z^2(x,t)]dx \leq \sigma^2(t + 1)^{-\frac{1}{2}}, \]
whence we obtain (5.8).

The fact that at the \( L^1 \) level solutions to systems (5.1) behave asymptotically as heat kernels is not unexpected, as similar behavior of classical solutions is familiar. The counterpart of Theorem 5.1 for general systems (1.4) is still unknown.

6. **Viscous relaxation.** The methodology expounded in the previous sections also applies to systems of conservation laws modeling viscous relaxation of the Boltzmann type, in which the flux depends not only on the present value but also on the past history of the state vector:
\[ \partial_t U(x,t) + \partial_x F(U(x,t)) + \int_{-\infty}^{t} K(t - \tau) \partial_x G(U(x,\tau))d\tau = 0. \]  

We shall illustrate this by means of two examples.

First we consider the system
\[ \begin{cases} 
\partial_t u(x,t) - \partial_x v(x,t) = 0 \\
\partial_t v(x,t) - \partial_x f(u(x,t)) - \int_{0}^{t} a'(t - \tau) \partial_x u(x,\tau)d\tau = 0,
\end{cases} \tag{6.2} \]
which governs longitudinal oscillations of viscoelastic bars and shearing motions of viscoelastic slabs.

To avoid technicalities, we assume that the relaxation kernel \( a \) is in the form
\[ a(t) = \sum_{i=1}^{n} \alpha_i e^{-\lambda_i t}, \tag{6.3} \]
with \( \alpha_i > 0 \) and \( 0 < \lambda_1 < \cdots < \lambda_n \).

The functions \( f(u) \) and \( g(u) = f(u) - a(0)u \), which encode the instantaneous and the relaxed elastic response of the medium must be increasing, \( f'(u) > 0, g'(u) > 0 \). Consequently, the system (6.2) is of “hyperbolic” type.

The memory has a dissipative effect. In particular, admissible solutions of (6.2) satisfy the entropy inequality
\[ \partial_t \eta(x,t) + \partial_x q(x,t) + r(x,t) \leq 0, \tag{6.4} \]
\[ \eta(x,t) = \frac{1}{2} v^2(x,t) + h(u(x,t)) - \frac{1}{2} \int_{0}^{t} a'(t - \tau)[u(x,t) - u(x,\tau)]^2d\tau, \tag{6.5} \]
\[ \begin{align*}
h(u) &= \int_0^u g(\omega) d\omega, \\
q(x,t) &= -v(x,t) \left[ f(u(x,t)) + \int_0^t a'(t-\tau) u(x,\tau) d\tau \right], \\
r(x,t) &= \frac{1}{2} \int_0^t a''(t-\tau)(u(x,t) - u(x,\tau))^2 d\tau.
\end{align*} \]

Notice that, as in earlier sections, the entropy production is positive semidefinite but not positive definite.

For simplicity, we assign initial conditions
\[ u(x,0) = u_0(x), \quad v(x,0) = 0, \quad -\infty < x < \infty, \]
where \( u_0 \) is a function of bounded variation
\[ TV u_0(\cdot) = \delta, \]
that decays as \( |x| \to \infty \) sufficiently fast to render the integral
\[ \int_{-\infty}^{\infty} (1 + x^2) u_0^2(x) dx = \sigma^2 \]
finite.

**Theorem 6.1.** Under the assumptions on \( f, g \) and \( a \) recorded above, there are positive constants \( \delta_0, \sigma_0, c_1, c_2 \) and \( b \) such that when \( u_0 \) satisfies (6.10) and (6.11) with \( \delta < \delta_0 \) and \( \sigma < \sigma_0 \), then there exists an admissible \( BV \) solution \((u,v)\) to (6.2), (6.9) on \( (-\infty, \infty) \times [0, \infty) \) and
\[ \int_{-\infty}^{\infty} |u(x,t)| + |v(x,t)| dx \leq b \sigma, \quad 0 \leq t < \infty, \]
\[ TV u(\cdot,t) + TV v(\cdot,t) \leq c_1 \sigma + c_2 \delta, \quad 0 \leq t < \infty. \]
Furthermore,
\[ TV u(\cdot,t) + TV v(\cdot,t) \to 0, \quad \text{as } t \to \infty. \]

The proof to the above theorem (under slightly different assumptions) is found in [8,10]. It follows the same pattern as the proof of Theorem 2.1 but requires new estimates.

The next example of damping induced by memory dependence is provided by the equation
\[ \partial_t \theta(x,t) = \int_0^t a(t-\tau) \partial_x f(\partial_x \theta(x,\tau)) d\tau, \]
which governs heat flow in one-dimensional media with fading memory. The function \( f(u) \) is increasing, \( f'(u) > 0 \), and for simplicity, the relaxation kernel is assumed to be in the form (6.3).

We prescribe initial conditions
\[ \theta(x,0) = \theta_0(x), \quad -\infty < x < \infty, \]
where
\[ \int_{-\infty}^{\infty} \theta_0^2(x) dx = \rho^2. \]

Upon setting
\[ u(x,t) = \partial_x \theta(x,t), \quad v(x,t) = \partial_t \theta(x,t), \]

where
\[ h(u) = \int_0^u g(\omega) d\omega, \]

\[ q(x,t) = -v(x,t) \left[ f(u(x,t)) + \int_0^t a'(t-\tau) u(x,\tau) d\tau \right], \]

\[ r(x,t) = \frac{1}{2} \int_0^t a''(t-\tau)(u(x,t) - u(x,\tau))^2 d\tau. \]
(6.15), (6.16) reduce to
\[
\begin{aligned}
\partial_t u(x,t) - \partial_x v(x,t) &= 0 \\
\partial_t v(x,t) - \partial_x f[u(x,t)] - \int_0^t a'(t-\tau)\partial_x f(u(x,\tau))d\tau &= 0
\end{aligned}
\]  
(6.19)

where \( u_0(x) = \theta_0(x) \). Then

**Theorem 6.2.** Under the assumptions on \( f \) and \( a \) listed above there are positive constants \( \rho_0, \delta_0, \sigma_0, c_1, c_2 \) and \( c_3 \) such that when \( \theta_0 \) and \( u_0 \) satisfy (6.16), (6.10) and (6.11), with \( \rho < \rho_0, \delta < \delta_0 \) and \( \sigma < \sigma_0 \), then there exists an admissible BV solution \((u,v)\) to (6.19), (6.20) on \(( -\infty, \infty ) \times [0, \infty ) \) and \( TVu(\cdot, t) + TVv(\cdot, t) \leq c_1 \rho + c_2 \sigma + c_3 \delta, \quad 0 \leq t < \infty \).

(6.21)

The proof of the above theorem (under slightly different hypotheses) is shown in [9]. It should be noted that even though the systems (6.2) and (6.19) look very similar, they require different treatments. In particular, the bounds for solutions to (6.2) are derived with the help of “energy” estimates, and the entropy inequality (6.4) plays a central role. By contrast, (6.19) is treated by writing its second equation in the equivalent form
\[
\begin{aligned}
\partial_t v(x,t) - \partial_x f[u(x,t)] + \partial_t \int_0^t k(t-\tau)v(x,\tau)d\tau &= 0,
\end{aligned}
\]
(6.22)

where \( k \) is the resolvent kernel of \( a' \),
\[
k(t) = \kappa_0 + \sum_{i=1}^{n-1} \kappa_i e^{\mu_i t},
\]
(6.23)

with \( \kappa_i > 0 \) and \( 0 < \lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n \).

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