Addendum to “Commensurations and Subgroups of Finite Index of Thompson’s Group $F$”

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We show that the abstract commensurator of $F$ is composed of four building blocks: two isomorphism types of simple groups, the multiplicative group of the positive rationals and a cyclic group of order two. The main result establishes the simplicity of a certain group of piecewise linear homeomorphisms of the real line.

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The purpose of this note is to extend earlier work [2], where we described the commensurator group of Thompson’s group $F$. We prove that an interesting subgroup of $\text{Com}(F)$ is simple and describe the algebraic structure of $\text{Com}(F)$ in terms of short exact sequence of simple groups and the multiplicative group of the positive rationals. For all the details and notation, see the paper [2].

1 The group of eventually periodic maps

Previously [2] we described the commensurator group of $F$ as the group of the eventually integrally periodically affine maps in $P$, which is defined in Section 1 of [2]. These elements may preserve or reverse the orientation of the real line. We also showed that the index-two subgroup $\text{Com}^+(F)$ of orientation-preserving maps fits into the short exact sequence

$$1 \longrightarrow K \longrightarrow \text{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1$$

whose kernel $K$ is exactly those elements $f$ of $P_+$ for which there exists $M > 0$, and two positive integers $p, p'$ such that

$$f(t + p) = f(t) + p \text{ for } t \geq M \text{ and }$$

$$f(t + p') = f(t) + p' \text{ for } t \leq -M.$$ 

Now we can associate to each element $f \in K$ two integrally periodically affine maps $f_+$ and $f_-$, which coincide with $f$ near $\infty$ and $-\infty$, respectively. This property leads to the following definitions.
For \( p \in \mathbb{N} \), we denote by \( H_p \) the subgroup of \( P_+ \) of \( p \)-periodically affine maps, that is

\[
H_p = \{ f \in P_+ \mid f(t + p) = f(t) + p \text{ for all } t \in \mathbb{R} \}.
\]

Clearly, if \( p | q \), then \( H_p \subset H_q \), whence we define the subgroup \( H \) as a direct limit under inclusion by

\[
H = \bigcup_{p=1}^{\infty} H_p.
\]

The maps \( f_+ \) and \( f_- \) now give rise to a homomorphism

\[
\rho: K \longrightarrow H \times H,
\]

given by \( \rho(f) = (f_-, f_+) \). The kernel consists of the eventually trivial elements, and therefore equals \( F' \), the commutator subgroup of \( F \) (see [1] or [2]). In other words, we get the short exact sequence

\[
1 \longrightarrow F' \longrightarrow K \longrightarrow H \times H \longrightarrow 1.
\]

Brin [1] showed that \( \text{Aut}^+(F) = \rho^{-1}(H_1 \times H_1) \) and established the short exact sequence

\[
1 \longrightarrow F \longrightarrow \text{Aut}^+(F) \longrightarrow T \times T \longrightarrow 1,
\]

where \( T \) is Thompson’s group \( T \) (see [4]). Since we clearly have a map \( H_1 \to T \), due to the fact that a map which is \( 1 \)-periodically affine can be viewed as a map on the circle \( S^1 \) given by \( \mathbb{R}/\mathbb{Z} \), an alternative version of this sequence is

\[
1 \longrightarrow F' \longrightarrow \text{Aut}^+(F) \longrightarrow H_1 \times H_1 \longrightarrow 1.
\]

These two sequences are related by the short exact sequence

\[
1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1,
\]

whose kernel \( A_1 \) is the maps \( t \mapsto t + k \) for integers \( k \). Clearly \( A_1 \) is isomorphic to \( \mathbb{Z} \).

It is straightforward to verify that any element \( \alpha \) of \( \text{Com}^+(F) \) which satisfies \( \alpha(t+1) = \alpha(t) + p \) for all \( t \in \mathbb{R} \) conjugates \( H_1 \) to \( H_p \) and \( A_1 \) to \( A_p \), the group of maps of the form \( t \mapsto t + kp \) with \( k \in \mathbb{Z} \). So we clearly have a short exact sequence

\[
1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1.
\]

We note that this extension is, in fact, central, and that one may view this copy of \( T \) as acting on the circle of length \( p \) given by \( \mathbb{R}/p\mathbb{Z} \). We summarise this discussion as follows.
Theorem 1  The structure of the group $\text{Com}(F)$ and its index-two subgroup $\text{Com}^+(F)$ is given by the following short exact sequences and equalities.

$$1 \longrightarrow \text{Com}^+(F) \longrightarrow \text{Com}(F) \longrightarrow C_2 \longrightarrow 1$$

$$1 \longrightarrow K \longrightarrow \text{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1$$

$$1 \longrightarrow F' \longrightarrow K \longrightarrow H \times H \longrightarrow 1, \quad H = \bigcup_{p=1}^{\infty} H_p$$

$$1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1$$

$A_p \cong \mathbb{Z}$ is central in $H_p$

2  Simplicity of the group $H$

Here we exploit the well-known fact that $T$ is simple (eg. [4]) to prove our main result.

Theorem 2  The group $H = \{f \in P_+ \mid f(t+p) = f(t) + p \text{ for some } p \in \mathbb{N}\}$ is simple.

Note that for $p, q \in \mathbb{N}$ with $p|q$, we have $H_p \subset H_q$ and $A_p \supset A_q$. So the theorem says that in the union $H$ the groups $A_p$ cease to be normal. This is due to the following.

Lemma 3  A normal subgroup of $H_p$ is either $H_p$ or it is contained in $A_p$.

Proof  In the light of the isomorphism between $H_p$ and $H_1$ which carries $A_p$ to $A_1$, it suffices to consider the case $p = 1$. Let $N$ be a normal subgroup of $H_1$ and consider its image in $T$. Since $T$ is simple, the image of $N$ is either $\{1\}$ or the whole $T$. If the image is $\{1\}$, then $N \subset A_1$. So we assume that the image is $T$, which yields the exact sequence

$$1 \longrightarrow B \longrightarrow N \longrightarrow T \longrightarrow 1$$

with kernel $B = N \cap A_1 \subset A_1$. It follows that $B = A_r$ for some $r$, and we find that

$$H_1/N \cong A_1/A_r \cong \mathbb{Z}/r\mathbb{Z}.$$ 

In particular $H_1/N$ is abelian. The proof will be complete once we show that $H_1$ is equal to its commutator subgroup, because then $N = H_1$. In order to establish this, we recall from [4] that $T$ is generated by three elements $x_0, x_1$ and $c$ subject to the relators
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\[ [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], \quad [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2], \quad x_1 x_0^{-1} c x_1 c^{-1}, \]
\[ (x_0^{-1} c x_1)^2 x_0^{-1} c^{-1}, \quad x_1 x_0^{-2} c x_1 x_0^{-1} x_1 x_0^{-1} c x_1^{-1} x_0 \quad \text{and} \quad c^3. \]

This easily gives rise to a finite presentation for \( H_1 \) with three generators \( x_0, x_1 \) and \( c \) subject to the same relators, except for \( c^3 \) which has to be replaced by the two relators \([c^3, x_0]\) and \([c^3, x_1]\). Here \( x_0, x_1 \) and \( c \) are the preimages of the corresponding generators for \( T \), as defined in [4], with \( x_0(0) = x_1(0) = 0 \) and \( c(0) = -1/4 \); composition is then to be read from right to left, as in [4]. In this case \( c^3 \) is the map \( t \mapsto t - 1 \) which generates \( A_1 \). Modulo the commutator subgroup of \( H_1 \), the third, fourth and fifth relators yield the relators \( x_0^{-1} x_1^2, x_0^{-3} x_1^2 c \) and \( x_0^{-1} x_1 \), respectively, which in turn imply \( x_0 = x_1 = c = 1 \). This proves that \([H_1, H_1] = H_1\).

**Proof of Theorem 2.** Let \( N \) be a non-trivial normal subgroup of \( H \). According to Lemma 3, for each \( p \), we have that \( N \cap H_p \) is either \( H_p \) or it is contained in \( A_p \).

We claim that if \( N \cap H_p = H_p \) for some \( p \), then this happens for all \( p \in \mathbb{N} \). We take \( q \in \mathbb{N} \). Then \( N \cap H_{pq} \) is a normal subgroup of \( H_{pq} \), and

\[ N \cap H_{pq} \supset N \cap H_p = H_p \supset A_p \supset A_{pq}, \]

which shows that \( N \cap H_{pq} = H_{pq} \), by the lemma. Thus, in this case \( N \) contains all \( H_q \), and hence \( N = H \).

The only case left now is that \( N \cap H_p \subset A_p \) for all \( p \). Since \( N \) is non-trivial and the \( A_p \) are infinite cyclic, there exists a \( p \) with \( N \cap H_p = A_{rp} \) for some \( r \geq 1 \). But then

\[ A_{2pr} \supset N \cap H_{2pr} \supset N \cap H_p = A_{pr} \supset A_{2pr}, \]

which is a contradiction. Thus the only normal subgroups of \( H \) are \( H \) and the identity as claimed.

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