Family Symmetry, Fermion Mass Matrices and Cosmic Texture

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Abstract
The observed replication of fermions in three families is undoubtedly a reflection of a deeper symmetry underlying the standard model. In this paper we investigate one very elementary possibility, that physics above the grand unification scale is described by the symmetry group $G \times SU(3)_{\text{fam}}$ with $G$ a gauged grand unified group, and $SU(3)_{\text{fam}}$ a global family symmetry. The breaking of this symmetry at the GUT scale produces global texture, providing a mechanism for structure formation in the universe, and sets strong constraints on the low energy fermion mass matrix. With the addition of a 45 Higgs and certain assumptions about the relative strength of Higgs couplings, the simplest $SU(5)$ theory yields an eight parameter form for the fermion mass matrices, which we show is consistent with the thirteen observable masses and mixing angles. We discuss the natural suppression of flavour changing neutral currents (FCNCs) and emphasise the rich low energy Higgs sector. With minor assumptions, the theory unifies consistently with the recent LEP data. We consider the extension to $G = SO(10)$, where some simplification and further predictiveness emerges. Finally we discuss the dynamics of $SU(3)_{\text{fam}}$ texture and show that the known constraints on unification ‘predict’ a GUT symmetry breaking scale of the order required for cosmic structure formation, and by the recent detection of cosmic microwave anisotropy by COBE.
1. Introduction

The idea that the inhomogeneities required for structure formation in the early universe were generated when a symmetry of nature was broken is one of the most attractive in cosmology. The predictions of such theories depend on the pattern of broken symmetry. One generic possibility involves an unstable defect known as global texture [1], formed whenever a continuous nonabelian global symmetry is completely broken, so that the vacuum manifold has a non-trivial third homotopy group $\Pi_3(M_0)$. The breaking scale, the single tunable parameter in the simplest texture model, is required to be of order $10^{16}$ GeV in order to fit the requirements of structure formation, and to fit the level of microwave background anisotropy recently detected by COBE [2], [3], [4]. This coincidence with the GUT scale is striking. This paper is devoted to an investigation of some of the phenomenological implications of imposing a global family (or “horizontal”) symmetry on a typical grand unified theory of particle interactions. In particular we shall show that these theories can fit the observed low energy phenomenology and do indeed naturally produce the symmetry breaking scale required for cosmic structure formation and to fit the recent COBE results.

The effects of global family symmetries have been discussed at considerable length in the literature [5]. The Goldstone bosons produced when a global symmetry is broken (known as “familons” in the case of a family symmetry) have couplings to the fermions which are inversely proportional to the breaking scale $\phi_o$. The tightest constraints on this scale for a family symmetry come from the FCNCs mediated by the familons ($\phi_0 > 10^{10}$GeV). The constraints from the long range forces mediated by such massless particles
are in fact very weak ($\phi_0 > 10^2$ GeV) because the couplings to the fermions are derivative in the familon fields.

Consider the Yukawa couplings in minimal $SU(5)$

$$Y_{ij}\psi_i^{\alpha T}C\chi_{j\alpha\beta}H_\beta^* + Y'_{ij}\epsilon^{\alpha\beta\gamma\delta\epsilon}\chi^T_{i\alpha\beta}C\chi_{j\gamma\delta}H_\epsilon + h.c.$$ (1)

where $\psi$ and $\chi$ are the fermion 5 and 10, $H$ is the Higgs 5, $i, j$ are family labels and $C$ is the charge conjugation matrix. With $<H_\alpha> = v_5^{\delta_5}$ the first term produces the mass matrices for the $(d, s, b)$ quarks and the charged leptons $(e, \mu, \tau)$ and the second the $(u, c, t)$ mass matrix. If we now postulate a family symmetry, e.g. with the quarks in 3s of $SU(3)$, the nine Yukawa couplings in each term are reduced to a single one. The question of mass relations and mixing angles is therefore a question about what vevs the Higgs potential can produce. In this paper we shall investigate this in some detail.

Such a potential is a function of many variables constrained by the symmetries to just a few terms. We expect this to lead to constraints on the form of the minimum of the potential. Michel’s conjecture [3] states that a potential involving a single irreducible representation of a group has a minimum which leaves unbroken a maximal subgroup. A well known example is the breaking of $SU(5)$ using the hermitian traceless 24. The minimum must be at one of four definite values (up to $SU(5)$ rotations). We are only free to adjust the overall magnitude which is a function of the parameters in the potential. For several representations however there is no general result: one must examine the particular case to see what form the vev takes.

Below we will consider models in which we use a set of GUT scale Higgs fields to break $SU(5) \times G_{fam}$ down to $SU(3)_C \times SU(2)_W \times U(1)_Y$. These heavy Higgs fields couple
to the electroweak Higgs fields which give mass to the fermions. We thus consider an effective low energy Higgs potential which we get by substitution of the GUT scale vevs in the total potential. This effective low energy potential is not completely general because of the constraint of renormalisability imposed on the underlying theory. As a simple illustration consider a theory with two complex fields $\phi$ and $\eta$ with the same U(1) charge. A renormalisable potential contains no terms in $\phi^3$: thus if $\eta$ acquires a vev, and breaks the symmetry, the tree level effective potential for $\phi$ is not completely general. Quantum corrections do generate the ‘missing’ terms, but they are always proportional to powers of the non-trivial couplings between the electroweak and GUT scale fields. This follows to all orders in perturbation theory because in the absence of these couplings the theory is invariant under (global) transformations of the $\phi$ and $\eta$ fields separately. It is technically ‘natural’ to fine tune the $\phi$-$\eta$ couplings to be small.

This means that the quantum corrections do not spoil the hierarchy imposed at tree level. The usual hierarchy problem of GUTs is of course still present, and shall not be addressed in this paper. This problem arises because representations must be split between the two mass scales (e.g. the Higgs 5 must have an electroweak doublet component and a superheavy proton decay mediating color triplet component). This is achieved by fine tuning linear combinations of cross couplings which are not individually fine-tuned. This splitting of the gauge representations can be achieved however with couplings which leave the larger family symmetry on the two mass sectors intact. Provided we not wish to split the family replicas of given gauge components we do not escalate the hierarchy problem of the minimal GUT by adding a family symmetry as we have described. We shall show that
with the tree level potential alone, an acceptable form for the fermion mass matrices is obtainable in the context of a simple perturbative scheme. It would certainly be interesting to extend this work to supersymmetric theories where there is some hope of explaining the electroweak -GUT heirarchy.

One way of phrasing the question we address is this: Does a family symmetry broken at GUT scale leave any mark on the low energy physics? The answer is twofold. The low energy physics reflects the family symmetry in that while the effective Higgs potential is not invariant under family symmetry it is not completely general with respect to the family indices. The low energy fermion masses and mixing angles may or may not reflect the family symmetry. If the Higgs potential is such that it can produce any mass matrices by suitable choice of couplings then we cannot exclude the possibility of such a symmetry existing from low energy phenomenology alone. If on the other hand it produces mass matrices which are predictive and fit the data we may indeed be able to see a ‘shadow’ of this GUT scale symmetry without knowing more of the Higgs sector. With the increasingly accurate measurement of the KM matrix, it seems an opportune time to make a detailed investigation of these issues.

The outline of the paper is as follows. In Section 2 we argue that the simplest choice for a family symmetry is $SU(3)$, and further that this must be a global rather than gauged symmetry for quantum consistency. In Section 3 we perform a detailed minimisation of the potential for the $SU(5) \times SU(3)_{\text{fam}}$ symmetry with a simple choice of Higgs representations and show how mass matrices of the sort required by phenomenology are generated in such a potential. While this is hardly a stunning success, it indicates that the idea of a family
symmetry is viable, and gives us some idea as to how the imposition of further symmetries might make the theory predictive. In Section 4 we discuss flavour changing neutral currents (FCNC). In Section 5 we discuss the running of the tree level mass matrices, and in Section 6 a simple extension of the $SU(5)$ theory to include a $45$, which is needed to obtain the correct fermion masses and makes the theory consistent with the unification of couplings. It also cures the FCNC problem of the minimal theory. In section 7 we fit a particular ansatz to the data to demonstrate the detailed viability of this extended model. In Section 8 we discuss embedding the theory in $SO(10)$, and point out how the observed mass matrices might naturally emerge in the resultant theory. Finally, in Section 9 we discuss the cosmological implications of $SU(3)_{fam}$ texture, and relate the GUT scale to the level of cosmological fluctuations recently detected by COBE. In Section 10 we point out directions for future work.

2. Choice of Group and Fermion Representations

We choose to take $G_{fam}$ as a simple Lie group. The breaking of a discrete family symmetry group would produce the familiar cosmological domain wall problem, unless the symmetry were anomalous $[7]$, in which case one might question the logic of imposing a symmetry at the classical level when it is broken anyway by quantum effects. We choose further to constrain the possible $SU(5)$ fermion representations by insisting that we can embed the model in $SO(10)$ with its attractive feature that all the fermions fit in a single $16$. Since the $SU(5) \times G_{fam}$ decomposition is

$$(16, 3) = (1, 3) + (10, 3) + (\bar{5}, 3)$$

the representations of the fermions are then specified.
The only simple groups with three dimensional representations are $SU(2)$ and $SU(3)$. $SU(3)$ has several features which make us favour it:

(i) Consider again the Yukawa couplings in (1). Taking the assignments of (2) for the 10 and $\overline{5}$ we see that if the Higgs 5 is to give the fermions a mass, it must be in the following representations

$$SU(2): 1, 3, 5 \quad \text{(first term)}$$

$$1, 5 \quad \text{(second term)}$$

$$SU(3): \overline{3}, 6 \quad \text{(first term)}$$

$$\overline{6} \quad \text{(second term)}$$

Note that the $(5, 1)$ and $(5, 5)$ under $SU(5) \times SU(2)$ couple to both terms whereas this does not happen for the $SU(5) \times SU(3)$ case, because the relevant representations of $SU(3)$ are complex. This has the beneficial effect of suppressing FCNCs which we will argue happens quite naturally in the simplest $SU(3)$ model partly because of this feature.

(ii) We will see below that the zero-order form of the $SU(3)$ 6 and $\overline{6}$ vevs (by “zero-order” we mean their values when different representations are not coupled) reproduces that of all three fermion mass matrices

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & M
\end{pmatrix}
$$

i.e. there is one fermion in each triplet much heavier than the others. To produce this approximate form in $SU(2)$ requires a fine tuning between the singlet and the 5 as the latter is traceless [8]. It is not difficult to see that this also means that the minimal number of Higgs we need in $SU(2)$ is three (1, 3, 5) to have any chance of fitting the mass matrices.
Even then we have 5 independent Yukawa couplings as opposed to only 2 in the minimal $SU(3)$ model which we will consider.

(iii) The $SU(3)$ imposed on these GUTs is anomalous because of its simple chirality. If all fifteen left handed fermions are placed in 3's of $SU(3)$, it is easy to check that $\text{Tr}(T^a\{T^b, T^c\})$ is nonzero so the $SU(3)$ symmetry, if gauged, would be anomalous. In contrast, the 3 of $SU(2)$ is a real representation so the generators are antisymmetric, and $\text{Tr}(T^a\{T^b, T^c\})$ is zero. Thus $SU(3)$ in contrast to $SU(2)$ cannot be gauged. This might traditionally be regarded as a flaw, but with the production of cosmic texture in mind, we regard it as a virtue - the $SU(3)$ family symmetry must be global. Note also that there is no anomalous coupling between the Goldstone boson of $SU(3)$ and the QCD gauge fields, because the generators of $SU(3)$ are traceless. So the $SU(3)$ Goldstone bosons remain exactly massless.

So consider $SU(5) \times SU(3)_{fam}$. We take these representations:

$$\Phi_1^{\alpha ab} \ (5, 6) \quad \Phi_2^{\alpha \beta ab} \ (5, \bar{6}) \quad \Sigma^{1\beta \alpha}_a \ (24, 3) \quad \Sigma^{2\beta \alpha}_a \ (24, 3)$$  \hspace{1cm} (5)

$\Phi_1$ and $\Phi_2$ are the electroweak Higgs fields which produce the mass matrices. We take two $(24,3)$s as we must completely break $SU(3)_{fam}$ at the GUT scale (it is easy to see that a single $(24,3)$ cannot achieve this). We will make the two line up to be perpendicular in family space.

Our mass matrices will be symmetric. This in itself is not a restriction on the resultant fermion mass phenomenology as any mass matrices related by the transformation

$$M_d \rightarrow TM_d S_1^\dagger \quad M_u \rightarrow TM_u S_2^\dagger$$  \hspace{1cm} (6)
where $T, S_1$ and $S_2$ are unitary matrices produce the same masses and KM parameters. In particular the matrices can always be made symmetric by the transformation

$$M_d \rightarrow T M_d S_1^\dagger = M_d^{diag} \quad M_u \rightarrow T T' M_u S_2^\dagger T'^* T^T = V_{KM}^\dagger M_u^{diag} V_{KM}^*$$

where $(T, S_1)$ and $(T', S_2)$ diagonalise $M_d$ and $M_u$ respectively and the KM matrix $V_{KM} = T T'^\dagger$.

3. Perturbative Minimisation of Higgs Potential

We consider the most general renormalizable potential of the representations (5):

$$\lambda_1^1 (tr(\Phi_{1\alpha} \Phi_1^\alpha) - C_1^2)^2 + \lambda_1^2 ((tr\Phi_{1\alpha} \Phi_1^\alpha)^2 - tr(\Phi_{1\alpha} \Phi_1^\alpha)^2)$$

$$+ \lambda_2^1 ((tr\Phi_{1\alpha} \Phi_1^\alpha)^2 - tr(\Phi_{1\alpha} \Phi_1^\beta) tr(\Phi_{1\beta} \Phi_1^\alpha)) + (1 \rightarrow 2)$$

$$+ \lambda_1^2 tr(\Phi_{1\alpha} \Phi_{2\beta} tr(\Phi_1^\alpha \Phi_2^\beta) - tr(\Phi_{1\alpha} \Phi_1^\beta) tr(\Phi_{2\beta} \Phi_2^\alpha))$$

$$+ \lambda_1^2 tr(\Phi_{1\alpha} \Phi_{2\beta} tr(\Phi_1^\alpha \Phi_2^\beta) + \lambda_1^2 tr(\Phi_{1\alpha} \Phi_{2\beta} tr(\Phi_1^\beta \Phi_2^\alpha))$$

$$+ \lambda_1^2 tr(\Phi_{1\alpha} \Phi_{2\beta} tr(\Phi_1^\alpha \Phi_2^\beta) + \lambda_1^2 tr(\Phi_{1\alpha} \Phi_{2\beta} tr(\Phi_1^\beta \Phi_2^\alpha))$$

$$+ \mu_1^{11} \Phi_{1ab} \Sigma^{1b} \Sigma_c^1 \Phi_{1}^{ac} + \mu_1^{22} \Phi_{1ab} \Sigma^{2b} \Sigma_c^2 \Phi_{1}^{ac} + \mu_1^{11} \Phi_{2ab} \Sigma^{1b} \Sigma_c^1 \Phi_{2}^{ac} + \mu_2^{22} \Phi_{2ab} \Sigma^{2b} \Sigma_c^2 \Phi_{2}^{ac}$$

$$+ \mu_1^{12} \Phi_{1ab} \Sigma^{1b} \Sigma_c^2 \Phi_{1}^{ac} + \mu_2^{12} \Phi_{2ab} \Sigma^{1b} \Sigma_c^2 \Phi_{2}^{ac} + h.c.$$ (12)

$$+ \lambda_1^{ij} \Phi_{1ab} \Sigma_i^j \Phi_{2ef} \delta^{ame} \epsilon^{bnf} + h.c.$$ (13)

The traces are over $SU(3)$ indices and the indices raised relative to (5) indicate complex conjugation. We have ignored terms like $\Phi_{1ab} \Sigma^{1c} \Sigma_c^1 \Phi_{1}^{ab}$ which contribute to the overall magnitude of the vevs but do not affect their relative orientation. We have also not explicitly written the different $SU(5)$ contractions in the $\Sigma - \Phi$ terms. To simplify slightly
we have relegated the analysis of the \((24,3)\)'s potential to an appendix. We show there that the vevs can give the desired breaking pattern

\[
\Sigma^1 = (v_1 \text{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2}), 0, 0) \quad \Sigma^2 = (0, v_2 \text{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2}), 0)
\]

When we substitute these vevs in the potential above the different \(SU(5)\) contractions give the same term. In the same way as in minimal \(SU(5)\) a fine tuning between these two contractions is used to make the colour triplets heavy.

The problem of finding the minimum of this potential is analytically intractable for general values of the couplings. What we do now is show that we can produce approximate minima of the form required to fit the phenomenology in some chosen region of coupling parameter space.

Consider first the self couplings of the \(\Phi\) fields \((8)\). For positive couplings we have a global minimum at \(\Phi_{\alpha a b} = C \delta^5_\alpha \delta^3_a \delta^3_b\). The only degeneracy on the minimum is given by the action of \(SU(5) \times SU(3)\). This can be seen by expanding to quadratic order about the minimum and checking that the only massless degrees of freedom are the Goldstone modes generated by the action of the group. We can see qualitatively that the third term pushes the six \(SU(5)\) vectors parallel. If we then rotate them into one direction (the "5" direction, say) the second term is of the form \((trA)^2 - trA^2\) where \(A = \Phi_5 \Phi^5\) is an hermitian matrix with positive eigenvalues. To make this zero we must have

\[
A = v^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \Phi_5 = C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

We can next use the positive definite term \((9)\) to make the two representations parallel
in $SU(5)$ and, rotating them into the 5 direction, we have vevs

$$
\Phi_1 = C_1 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Phi_2 = C_2 U \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} U^T 
$$

(17)

where $C_1, C_2 \in IR$ and $U \in SU(3)$. If we now couple to the $\Sigma$ vevs in (15) through the first three terms in (12) we are left with only an $SU(2)$ degeneracy on the minimum and the vevs can be written

$$
\Phi_1 = C_1 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Phi_2 = C_2 e^{2i\phi} \begin{pmatrix}
0 & 0 & 0 \\
0 & \beta^2 e^{2i\psi} & 0 \\
0 & \alpha e^{i\psi} & \alpha^2
\end{pmatrix} 
$$

(18)

where $\alpha^2 + \beta^2 = 1$ where $\alpha, \beta \in IR$. The terms (10) and (11) each reduce to just one when we fix the $SU(5)$ direction and they are

$$
tr(\Phi_1 \Phi_2)tr(\Phi_1^* \Phi_2^*) \quad tr(\Phi_1 \Phi_2 \Phi_2^* \Phi_1^*) 
$$

(19)

These together with the last term in (12) produce a quadratic in $\beta^2$

$$
(\mu_2^2 C_2^2 v_2^2 - \lambda_4^{12} C_1^2 C_2^2 - 2\lambda_2^{12} C_1^2 C_2^2) \beta^2 + \lambda_2^{12} C_1^2 C_2^2 \beta^4 
$$

(20)

which we can use to fix $\beta$. ($\lambda_2^{12}$ and $\lambda_4^{12}$ are here the sums of the appropriate couplings).

All the other parameters we shall need to produce acceptable mass matrices may be obtained by considering fluctuations about the above form, and balancing linear corrections coming from terms in the potential not so far considered against quadratic mass terms.

We consider all perturbations to the leading order vevs together and write them in the form

$$
\Phi_1 \rightarrow C_1 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} + C_1 \begin{pmatrix}
\alpha_1 & \rho_1 & \rho_2 \\
\rho_1 & \alpha_2 & \rho_3 \\
\rho_2 & \rho_3 & \alpha_3
\end{pmatrix} 
$$

(21)
\[ \Phi_2 \to C_2 e^{2i\phi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta^2 e^{2i\psi} & \alpha \beta e^{i\psi} \\ 0 & \alpha \beta e^{i\psi} & \alpha^2 \end{pmatrix} + C_2 e^{2i\phi} \begin{pmatrix} \beta_1 & \delta_1 & \delta_2 \\ \delta_1 & \beta_2 & 0 \\ \delta_2 & 0 & \beta_3 \end{pmatrix} \] (22)

where \( \beta_3 \in \mathbb{IR} \).

We now calculate the linear and quadratic corrections to the potential order by order in the small parameter \( \epsilon \sim \sqrt{\beta} \). We take the terms (19) to be of order \( \epsilon \) smaller than the leading terms which gave us (18) and the remaining terms (13) and (14) to be of order \( \epsilon^2 \) i.e.

\[ \lambda_1^1 C_1^4 \sim \lambda_1^2 C_2^4 \sim \lambda_1^2 C_2^2 C_1^2 \sim \mu_1^{11} C_1^2 v_1^2 \sim \mu_1^{12} C_2^2 v_2^2 \sim \mu_2^{11} C_2^2 v_1^2 \]

\[ \lambda_2^{12} C_1^2 C_2^2 \sim \lambda_1^{12} C_1^2 C_2^2 \sim \mu_2^{22} C_1^2 v_2^2 \sim \epsilon \lambda_1^{11} C_1^4 \] (23)

\[ \mu_1^{12} C_1^2 v_1 v_2 \sim \mu_1^{12} C_2^2 v_1 v_2 \sim \lambda^{ij} v_i v_j C_1 C_2 \sim \epsilon^2 \lambda_1^{11} C_1^4 \]

The leading linear corrections are

\[ (\lambda_2^{12} + \lambda_4^{12}) C_1^2 C_2^2 (\alpha_3 + \beta_3 + h.c.) \] (24)

which are just corrections to the entries which we are already free to fix at leading order.

To order \( \epsilon^2 \) the linear corrections are

\[ \lambda_1^{11} C_1 C_2 e^{-2i\phi} v_1^2 (\alpha_2 + \beta_2^*) + h.c. \]

\[ -\lambda_1^{12} C_1 C_2 e^{-2i\phi} v_1 v_2 (\rho_1 + \delta_1^*) + h.c. \] (25)

\[ \lambda_2^{22} C_1 C_2 e^{-2i\phi} v_2^2 (\alpha_1 + \beta_1^*) + h.c. \]

The quadratic corrections to the leading terms are

\[ \lambda_1^1 C_1^4 (\alpha_3^* + \alpha_3)^2 + \lambda_1^2 C_2^4 (\beta_3^* + \beta_3)^2 \]

\[ + 2\lambda_2^1 C_1^4 (|\alpha_1|^2 + |\alpha_2|^2 + 2|\rho_1|^2) + 2\lambda_2^2 C_2^4 (|\beta_1|^2 + |\beta_2|^2 + 2|\delta_1|^2) \]

\[ + \mu_1^{11} v_1^2 C_1^2 (|\alpha_1|^2 + |\rho_1|^2 + |\rho_2|^2) + \mu_1^{22} v_2^2 C_1^2 (|\rho_1|^2 + |\alpha_2|^2 + |\rho_3|^2) \]

\[ + \mu_2^{11} v_1^2 C_2^2 (|\beta_1|^2 + |\delta_1|^2 + |\delta_2|^2) \] (26)
Fixing the perturbations in (23) with (26) we get to order $\epsilon^2$

$$
< \Phi_1 > = C_1 \begin{pmatrix}
\alpha_1 e^{i(2\phi - \theta_{22})} & \rho_1 e^{i(2\phi - \theta_{12})} & 0 \\
\rho_1 e^{i(2\phi - \theta_{12})} & \alpha_2 e^{i(2\phi - \theta_{11})} & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
< \Phi_2 > = C_2 e^{2i\phi} \begin{pmatrix}
\beta_1 e^{-i(2\phi - \theta_{22})} & \delta_1 e^{-i(2\phi - \theta_{12})} & 0 \\
\delta_1 e^{-i(2\phi - \theta_{12})} & \beta_2 e^{-i(2\phi - \theta_{11})} & \beta e^{i\psi} \\
0 & 0 & 1
\end{pmatrix}
$$

(27)

where we now use $\alpha_1, \alpha_2$ etc. to denote the magnitudes of the perturbations and $\theta_{ij}$ are the phases in the $\lambda_{ij}$ of (14).

The phases $\psi$ and $\phi$ are still undetermined as the dependence on them vanishes when we substitute back the perturbations in (27) into (23) and (26). To fix them we must go to higher order in our perturbative minimization. At next order we get the additional linear correction

$$(2\lambda_2^{12} + \lambda_4^{12})C_1^2 C_2^2 \beta e^{i\psi} \rho_3 + h.c.$$

(28)

and quadratic corrections

$$
\lambda_2^{12} C_1^2 C_2^2 ((\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 + \alpha_3 \beta_3^* + 2\rho_1 \delta_1 + 2\rho_2 \delta_2 + h.c.) + |\alpha_3|^2 + |\beta_3|^2)
$$

$$
+ \lambda_4^{12} C_1^2 C_2^2 (|\rho_2|^2 + |\delta_2|^2 + |\rho_3|^2 + |\alpha_3|^2 + |\beta_3|^2 + (\rho_2 \delta_2 + \alpha_3 \beta_3 + \alpha_3 \beta_3^* + h.c.))
$$

$$
+ \mu_2^{22} C_2^2 \nu_2^2 (|\delta_1|^2 + |\beta_2|^2)
$$

(29)

If we consider all terms to this order we have for each pair of corrections a quadratic potential of the form

$$
\lambda \delta + \lambda' \rho^* + h.c. + \lambda_1 |\delta|^2 + \lambda_2 |\rho|^2 + \lambda_3 (\delta \rho + h.c.)
$$

(30)

which has minima at

$$
\rho = -\frac{1}{\lambda_0} (\lambda_1 \lambda' - \lambda_3 \lambda) \quad \delta = -\frac{1}{\lambda_0} (\lambda_2 \lambda^* - \lambda_3 \lambda^*)
$$

(31)
and terms which depend on the phases of the couplings which can be written

\[ \frac{\lambda_3}{\lambda_0} (\lambda \lambda^* + h.c.) \]  

(32)

where \( \lambda_0 = \lambda_1 \lambda_2 - \lambda_3^2 \). In (25) and (28) \( \lambda \lambda^* \) is real and we again fail to find contributions to the potential to fix the phases.

At order \( \epsilon^4 \) we get the linear terms

\[ -2 \lambda^{11} C_1 C_2 e^{-2i\phi} v_1^2 \beta e^{-i\psi} \rho_3 + h.c. \]

\[ \lambda^{12} C_1 C_2 e^{-2i\phi} v_1 v_2 \beta e^{-i\psi} \rho_2 + h.c. \]  

(33)

\[ \mu_2^{12} C_2^2 v_1 v_2 \beta e^{-i\psi} \delta_2 + h.c. \]

and quadratic terms

\[ -2 \lambda_2^2 C_2^4 \beta e^{i\psi} \delta_2 \delta_1^* + h.c. \]

\[ + \mu_1^{12} v_1 v_2 C_1^2 (\alpha_1 \rho_1^* + \rho_1 \alpha_2^* + \rho_2 \rho_3^*) + \mu_2^{12} v_1 v_2 C_2^2 (\beta_1 \delta_1^* + \delta_1 \beta_2^*) + h.c. \]

\[ \lambda^{11} C_1 C_2 e^{-2i\phi} v_1^2 (+\alpha_3 \beta_2^* + \alpha_2 \beta_3^*) + h.c. \]  

(34)

\[ + \lambda^{12} C_1 C_2 e^{-2i\phi} v_1 v_2 (+\rho_3 \delta_2^* - \alpha_3 \delta_1^* - \rho_1 \beta_3^*) + h.c. \]

\[ \lambda^{22} C_1 C_2 e^{-2i\phi} v_1^2 (-2 \rho_2 \delta_2^* + \alpha_1 \beta_3^* + \alpha_3 \beta_1^*) + h.c. \]

When we substitute the corrections fixed at higher orders into these terms the dominant contribution which depends on the appropriate phases is

\[ 4 \beta^2 C_1 C_2^3 \frac{(v_1/v_2)^2 \lambda^{11} (2 \lambda_2^{12} + \lambda_4^{12})}{\mu_1^{22}} \cos(2\psi + 2\phi - \theta_{11}) \]  

(35)

The other terms which depend on these phases are smaller by \( \epsilon \) or more. Thus to this order the argument of the cosine is fixed to be 0 or \( \pi \) at the minimum depending on the sign of the coefficient.
We find however that we only get terms which fix $\psi + \phi$. The reason for this is accidental $U(1)$ symmetries in the low energy tree level Lagrangian. Without the terms (14) these are

\[
\Phi_1 \to e^{i\alpha} U_\theta \Phi_1 U_\theta^T \quad \Phi_2 \to e^{i\beta} e^{-2i\theta} U_\theta^* \Phi_1 U_\theta^T \quad \Psi \to e^{i(\alpha + \beta)} e^{-i\theta} U_\theta \Psi \quad \chi \to e^{-i\frac{\beta}{2}} e^{i\theta} U_\theta \chi
\]

(36)

where $\Psi$ and $\chi$ are the fermion 3s and $U_\theta$ is the diagonal family transformation

\[
U_\theta = \begin{pmatrix}
e^{i\theta} & 0 & 0 \\
0 & e^{i\theta} & 0 \\
0 & 0 & e^{-2i\theta}
\end{pmatrix}
\]

(37)

The terms (14) break (36) to a remaining two $U(1)$s by forcing $\alpha = \beta$. The first $U(1)$ which is an accidental symmetry of the whole theory is then just the usual one which becomes B-L when combined with the appropriate $SU(5)$ generator. The second “$\theta$” symmetry will presumably be broken by higher order quantum corrections.

We do not need to consider these contributions however as the action on the fermions in (36) is just a quark phase redefinition and can be used to transform the mass matrices into the form

\[
M_d = m_b \begin{pmatrix}
\alpha_1 e^{i(2\phi + 2\psi - \theta_{22})} & \rho_1 e^{i(2\phi + 2\psi - \theta_{12})} & 0 \\
\rho_1 e^{i(2\phi + 2\psi - \theta_{12})} & \alpha_2 e^{i(2\phi + 2\psi - \theta_{11})} & \rho_3 \\
0 & \rho_3 & 1
\end{pmatrix}
\]

\[
M_u = m_t \begin{pmatrix}
\beta_1 e^{i(2\phi + 2\psi - \theta_{22})} & \delta_1 e^{i(2\phi + 2\psi - \theta_{12})} & 0 \\
\delta_1 e^{i(2\phi + 2\psi - \theta_{12})} & \beta_2 e^{i(2\phi + 2\psi - \theta_{11})} & \beta \\
0 & \beta & 1
\end{pmatrix}
\]

(38)

(to order $\epsilon^3$) where $\rho_3$ is now also real. Note that $M_u \propto \Phi_2^*$. A further quark phase redefinition yields

\[
M_d = m_b \begin{pmatrix}
\alpha_1 e^{i\theta'} & \rho_1 & 0 \\
\rho_1 & \alpha_2 e^{i\phi'} & \rho_3 \\
0 & \rho_3 & 1
\end{pmatrix} \quad M_u = m_t \begin{pmatrix}
\beta_1 e^{i\theta'} & 0 & \delta_1 \\
0 & \beta_2 e^{i\phi'} & \beta \\
\delta_1 & \beta & 1
\end{pmatrix}
\]

(39)

where $\theta' = -(2\phi + 2\psi - 2\theta_{12} + \theta_{22})$ and $\phi' = 2\phi + 2\psi - \theta_{11}$.
The leading terms which fix \( \beta^2 \) are just those in (20) and we thus have

\[
\beta^2 = \frac{(2\lambda_2^{12}C_1^2C_2^2 - \mu_2^{22}C_2^2v_2^2 + \lambda_1^{12}C_1^2C_2^2)}{2\lambda_2^{12}C_1^2C_2^2}
\]

and we require \( 0 < \mu_2^{22}C_2^2v_2^2 - \lambda_1^{12}C_1^2C_2^2 < 2\lambda_2^{12}C_1^2C_2^2 \) to make this a minimum. Using (35) we can now see that the 22 corrections (which are proportional to \( \lambda^{11} \)) in both mass matrices in (39) are determined to be real quantities which are positive or negative depending on whether the sign of \( 2\lambda_2^{12} + \lambda_1^{12} \) is positive or negative. The sign of \( \rho_3 \) is then determined by (28) to be opposite to that of \( \alpha_2 \).

Choosing the negative sign for \( \alpha_2 \) we arrive at the form

\[
M_d = m_b \begin{pmatrix}
\alpha_1e^{i\theta^\prime} & \rho_1 & 0 \\
\rho_1 & -\alpha_2 & \rho_3 \\
0 & \rho_3 & 1
\end{pmatrix} \quad M_u = m_t \begin{pmatrix}
\beta_1e^{i\theta^\prime} & \delta_1 & 0 \\
\delta_1 & -\beta_2 & \beta \\
0 & \beta & 1
\end{pmatrix}
\]

(41)

Although there are ten real parameters and one phase this is a constrained form of the mass matrices. We will outline the procedure below which can be used to diagonalize these matrices and extract the phenomenological predictions. We will not consider this particular form in detail however as we need to add further Higgs fields to this model to make it realistic. We will only note that a full calculation shows the CP violation in these mass matrices to be very small as the only complex elements are forced very small by the hierarchy in the quark masses.

One comment on the perturbative method used to generate these mass matrices is required. We have shown that we can generate from the potential under the given assumptions mass matrices with entries at certain orders in the expansion parameter \( \epsilon \)

\[
M_d \sim \begin{pmatrix}
e^2 & e^2 & e^4 \\
e^2 & e^2 & e^3 \\
e^4 & e^3 & 1
\end{pmatrix} \quad M_u \sim \begin{pmatrix}
e^2 & e^2 & e^4 \\
e^2 & e^2 & e^2 \\
e^4 & e^2 & 1
\end{pmatrix}
\]

(42)
This perturbation expansion in $\epsilon$ can however be altered easily to give the corrections at the orders needed to fit phenomenological mass matrices. For example if we assume

$$\frac{(v_2^2)}{v_1^2} \sim \epsilon \quad \frac{C_1}{C_2} \sim \epsilon \quad \lambda_1^2 \sim \lambda_2^2 \sim \lambda_3^2 \sim \lambda_4^{12} \sim \lambda_5^{12}$$

$$\mu_1^{11} C_1^2 v_1^2 \sim \lambda_2^2 C_2 \quad \lambda_1^{11} \sim \lambda_1^{12} \sim \lambda_2^{12} \sim \lambda_3^{12} \sim \lambda_4^{22} \sim \lambda_5^{22}$$

$$\mu_2^{22} \sim \epsilon^2 \mu_1^{11} \mu_1^{12} \sim \mu_2^{12} \sim \epsilon^6 \mu_1^{11} \lambda_1^{11} \sim \epsilon^4 \mu_1^{11}$$

we get

$$M_d \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^5 \\ \epsilon^3 & \epsilon^2 & \epsilon^3 \\ \epsilon^5 & \epsilon^3 & 1 \end{pmatrix} \quad M_u \sim \begin{pmatrix} \epsilon^6 & \epsilon^5 & \epsilon^7 \\ \epsilon^5 & \epsilon^3 & \epsilon^2 \\ \epsilon^7 & \epsilon^2 & 1 \end{pmatrix}$$

(44)

It is not difficult to see that the phase analysis is also unchanged with these assumptions.

4. FCNCs and Higgs Phenomenology

Tree level flavour changing neutral currents (FCNCs) arise generically in models of this type. From the Yukawa couplings e.g.

$$Y_{\psi_\alpha}^{aT} C \chi_{ba,\beta} \Phi_{1}^{\beta ab}$$

(45)

we see that they are mediated by the off diagonal elements of the $6$ and $\bar{6}$. When we include the effects of quark mixing there will also be an induced mixing in these currents.

The obvious way of suppressing such processes below their phenomenological limits is by appropriately raising the masses of the offending particles. In a single Higgs doublet model the neutral Higgs after SSB has mass $m$ with $m^2 \approx \lambda v^2$ where $v$ is the vev, fixed by the gauge boson mass. To make $m$ large we must move into the strong coupling region which is unattractive and may violate perturbative unitarity bounds. However we see that in (26) there are many other terms which contribute to the Higgs masses. In particular the $\mu$ terms provide mass contributions which are not related to a 4 point low energy coupling.
In fact precisely the sort of choices we made in (43) to produce the elements in the mass matrices at the appropriate order are those needed to suppress the FCNCs. For example \( \mu_{11} C_1^2 v_1^2 \sim \lambda_2^2 C_2^4 \) implies \( \mu_{11} v_1^2 \sim \epsilon^{-2} \lambda_2^2 C_2^2 \). \( \mu_{11} v_1^2 \) is precisely the mass squared picked up through this term in the potential by the perturbations mediating the \( d \to q \) currents and the term on the right side of the relation is the mass squared associated with the four point coupling \( \lambda_2^2 \).

The only components which do not pick up a mass in this way are the Higgs predominantly mediating the flavour neutral processes in the \( b \) and \( t \) quarks. Even with mixing they only mediate the \( u \to t \) and \( c \to t \). Thus the suppression of FCNCs and the existence of one heavy quark of each charge are tied together. The same terms in the potential which make the FCNC mediating Higgs heavy single out the direction of the heavy family and suppress the masses of the other quarks. The heaviest Higgs particles mediate processes between the lightest quarks.

In the particular model we discussed above this feature is frustrated by the requirements of generating \( \beta \). We chose to put this parameter where we did because the Higgs particle which gets mass from \( \mu_{22}^2 \) is the \( c \to t \) mediating one. We required \( \mu_{22}^2 < \lambda_4^{12} + 2 \lambda_2^{12} \) (dropping vevs). This had the effect of making the dominant contribution to the mass of the \( \beta_2 \) fluctuation come from the \( \lambda_2^2 \) term. Then the requirement that \( \alpha_2 \) and \( \beta_2 \) be generated at \( \epsilon^2 \) and \( \epsilon^3 \) respectively forced us to choose \( \mu_{22}^2 C_1^2 v_2^2 \sim \epsilon \lambda_2^2 C_2^4 \) which leads to \( \mu_{11}^2 v_2^2 \sim \epsilon^{-1} \lambda_2^2 C_2^4 \). The \( \alpha_2 \) mediates significant \( d \to s \) when we take into account the quark mixing in the mass eigenstate basis and we will again be forced to making the four point couplings large to suppress them to the desired level. In the extensions of this model which
we discuss below this is not the case.

One caveat is required here. As noted in the introduction these terms which may give extra mass to the Higgs particles are also crucial in the generation of quantum corrections. It is not difficult to see that the radiative corrections become important if $\mu v^2 \sim \lambda C^2$. If one allows these cross terms to become large we would have to address the question in detail of the corrections to this tree level analysis and what sort of additional fine-tuning might be required. We shall not do this here - we merely note that there are terms present in the Lagrangian which may be used to give masses to the FCNC Higgs perhaps an order of magnitude larger than the electroweak scale. A more detailed analysis would be required to determine to what extent fine tuning is necessary to preserve this mild hierarchy of mass scales.

It is worth emphasising that the low energy Higgs sector of this theory would have a rich and distinctive phenomenology. Up to mixing each of the neutral components of the twelve Higgs doublets mediates a different neutral current. The gauge boson couplings are also interesting - at leading order only the doublets mediating $tt$ and $bb$ processes have linear couplings to the gauge bosons.

5. Radiative Corrections and Running

The mass matrices we extract from the potential are valid at the GUT scale. We implicitly assumed a simple scaling of these matrices to 1GeV. This is in fact exact. To see this consider the one loop 1PI corrections to the Yukawa couplings, masses and mixing angles involving one internal Higgs line. The Yukawa couplings of the neutral and charged
Higgs to the quarks are

\[ Yq'_{aR}q'_{bL} \Phi^5_{1ab} + Y^*q_{aL}q_{bR} \Phi^1_{15ab} + Y'q_{aR}q_{bL} \Phi^{25}_{ab} + Y'^*q_{aL}q_{bR} \Phi^5_{a b} \]

\[ + Yq'_{aR}q_{bL} \Phi^4_{1ab} + Y^*q_{aL}q_{bR} \Phi^{14ab} - Y'q_{aR}q_{bL} \Phi^{24}_{ab} - Y'^*q_{aL}q_{bR} \Phi^{4}_{a b} \]  

(46)

where \( q \) and \( q' \) are the u and d type quarks respectively. It is easy to see from these couplings that in the unbroken phase these diagrams are not allowed because they do not conserve global charge. If they are non zero in the broken phase they must derive from a term in the unbroken effective action with more \( \Phi \) or \( \Sigma \) external legs. Any such term is finite however just because of renormalizability. (The mass self energy with one extra leg is just the first term). Therefore the mass matrices do not run at all due to Higgs couplings. All the running comes from the gauge fields which distinguish only between the three mass matrices but not their elements.

Running and finite radiative corrections are particularly relevant if one extracts a predictive ansatz by imposing symmetries on the Higgs potential. Since the running just causes a simple scaling in these models we must turn to finite corrections to generate anything we do not have at tree level. We have considered such corrections and found them to be far too small to serve this purpose. To generate a correction \( \approx \frac{1}{48\pi^2} \approx 10^{-3} \) we find that we need precisely the term in the potential which would generate the parameter at tree level. Any other potentially interesting corrections are generated only at \( \approx \frac{1}{48\pi^2}\beta \).

One noteworthy feature which could perhaps be exploited usefully in a different model is that the corrections contribute differently to the \( M_d \) and \( M_e \) matrices (e.g. if the tree level vev giving the radiative correction did not couple to the fermions because of some symmetry).

6. Alterations to the minimal \( SU(5) \) model
The model we discussed above has several shortcomings:

(i) We ignored the lepton mass matrices. In $SU(5)$ the Yukawa couplings produce the mass relation $M_d = M_e$ at the GUT scale. When renormalized to the electroweak scale this gives the successful relation $m_b \approx 3m_{\tau}$ [10]. However the other relations are not right. Georgi and Jarlskog [11] proposed alternative mass relations between the charge $-\frac{1}{3}$ quarks and the leptons which they showed could be produced with the addition of a Higgs $45$ of $SU(5)$. We thus add a $(45,6)$ which couples to the first term in (1) only.

(ii) The model does not unify as it is at low energy just the standard model with extra doublets [12]. We could consider many types of additions of heavier particles to remedy this as in [13]. However the addition of the $(45,6)$ prompted by (i) is already adequate for this purpose.

In [12] it was shown that the addition of a single $45$ to the one Higgs doublet minimal $SU(5)$ model is sufficient to bring about unification if some of the components of this $45$ are put at intermediate mass scales e.g. if the $(3,3)$ ($SU(3)_c \times SU(2)_w$) component is at $10^8 - 10^9$ GeV and the $(8,2)$ component at some scale below this. One must calculate beta functions with contributions from an “effective number” $n_{eff}$ of copies of each multiplet, given by

$$n_{eff} = \Sigma_i n_i \log \frac{M_G}{M_i} \log \frac{M_G}{M_Z}$$  \hspace{1cm} (47)$$

where $n_i$ are the number of the appropriate multiplet at mass $M_i$, $M_G$ is the unification scale (the X boson mass) and $M_Z$ is the Z mass. With the eighteen electroweak doublets and six $45$s in our model we unify with components at intermediate mass scales closer to the GUT scale. We have found, for example, that with the $(\overline{6},1)$ component at $10^{10}$GeV
and the \((8, 2)\) component at \(10^{13}\)GeV (for all six copies) we have \(M_G\) at \(10^{15}\)GeV. For a higher unification scale of \(4 \times 10^{16}\) we put both components at \(10^{12}\)GeV \([14]\). We could also investigate the possibility of using some of the components of the \((24, 3)\) to alter the running. The main point here is that because there are many copies of the \(SU(5)\) representations significant contributions to the beta functions can come from fields not far below the GUT scale.

(iii) In the analysis of the simplest model above we noted that we must make Higgs self-couplings large to suppress FCNCs. With an extra \(45\) however we expect the \(\alpha_2\) correction to come predominantly from the \(45\), to generate the G-J mass relations. Then there is no problem if we make \(\mu_1^{22}\) larger. In general the smallness of a correction to the leading order form is associated with a large Higgs mass for the corresponding degree of freedom.

We will only briefly discuss the effects of adding a \((45, 6)\) to the Higgs potential as the analysis is just as in the case we have looked at in detail.

A \(45\) of \(SU(5)\) can be represented as a three index tensor \(H_{\alpha\beta\gamma}\) with the properties

\[ H_{\alpha\beta\gamma} = -H_{\beta\alpha\gamma} \quad H_{\alpha\beta\alpha} = 0 \]

It breaks \(SU(5) \downarrow SU(3) \times SU(2) \times U(1)\) with the vev

\[ <H_{\alpha\beta\gamma}> \propto \delta_\beta^\gamma (\delta_\alpha^\gamma - 4\delta_1^\gamma \delta_4^\alpha) - (\beta \leftrightarrow \alpha) \]

The \((45, 6)\) couples to the first term in \([1]\) only and for a contribution \(m\) to an element of the \(M_d\) mass matrix gives \(-3m\) for the corresponding element in the lepton mass matrix.

We can construct a Higgs potential for the \((45, 6)\) alone which has zero vev. Coupling to the \((5, 6)\) and \((5, \bar{6})\) we find that the \((45, 6)\) is generated in the appropriate \(SU(5)\) direc-
tions in (49). Then considering the $SU(3)$ perturbations we find that the only corrections generated are
\[ \hat{\lambda}^{11} C_1 C_2 e^{-2i\phi} v_2^2 \hat{\alpha}_2 + h.c. \]
\[ -\hat{\lambda}^{12} C_1 C_2 e^{-2i\phi} v_1 v_2 \hat{\rho}_1 + h.c. \]
\[ \hat{\lambda}^{22} C_1 C_2 e^{-2i\phi} v_2^2 \hat{\alpha}_1 + h.c. \]
and at $\epsilon^2$ smaller
\[ -2\hat{\lambda}^{11} C_1 C_2 e^{-2i\phi} v_1^2 \beta e^{-i\psi} \hat{\rho}_3 + h.c. \]
\[ \hat{\lambda}^{12} C_1 C_2 e^{-2i\phi} v_1 v_2 \beta e^{-i\psi} \hat{\rho}_2 + h.c. \]
where the hats are used everywhere to denote the replacement of the $(5, 6)$ by the $(45, 6)$ (which has the same $SU(5)$ quintality and couples in the same ways to the other fields). The $(45, 6)$ is taken to be zero to zeroth order in $\epsilon$ - the perturbations are written with the same overall scale $C_1$ to be simply compared with the $(5, 6)$ perturbations.

The only quadratic corrections involving cross couplings between the electroweak Higgs fields are
\[ \hat{\mu}_1^{11} v_1^2 C_1^2 (\alpha_1 \hat{\alpha}_1^* + \rho_1 \hat{\rho}_1^* + \rho_2 \hat{\rho}_2^*) + h.c. \]
\[ \hat{\mu}_1^{22} v_2^2 C_1^2 (\rho_1 \hat{\rho}_1^* + \alpha_2 \hat{\alpha}_2^* + \rho_3 \hat{\rho}_3^*) + h.c. \]
\[ \hat{\mu}_1^{12} v_1 v_2 C_1^2 (\alpha_1 \hat{\rho}_1^* + \rho_1 \hat{\alpha}_2^* + \rho_2 \hat{\rho}_3^*) \]
The other terms which we might expect to see by analogy with the previous analysis turn out not to arise because of the $SU(5)$ contractions. The analysis of the phases still holds as given because these new terms do not give new relevant contributions.

Thus we see that we can easily produce a G-J type ansatz
\[
M_u = m_t \begin{pmatrix} 0 & \delta_1 & 0 \\ \delta_1 & -\beta_2 & \beta \\ 0 & \beta & 1 \end{pmatrix} 
M_d = m_b \begin{pmatrix} 0 & \rho_1 & 0 \\ \rho_1 & \hat{\alpha}_2 e^{i\theta} & \rho_3 \\ 0 & \rho_3 & 1 \end{pmatrix} 
M_l = m_b \begin{pmatrix} 0 & \rho_1 & 0 \\ \rho_1 & -3\hat{\alpha}_2 & \rho_3 \\ 0 & \rho_3 & 1 \end{pmatrix}
\]
simply by making $\lambda^{22}$ appropriately small and the masses of the $\hat{\alpha}_1$ and $\hat{\rho}_1$ perturbations very large to suppress these corrections to the mass matrix.

7. Analysis of Masses and Mixing Angles

Symmetric mass matrices may be diagonalized with a single unitary matrix

$$M_d^{\text{diag}} = V_L M_d V_L^T \quad M_u^{\text{diag}} = U_L M_u U_L^T \quad V_{KM} = U_L V_L^†$$

(54)

To diagonalize (53) and fit it to the phenomenology we follow the procedure used in [15] and [16]. Assuming that $\rho_3 \sim \rho_1 \sim \epsilon^3$, $\alpha_2 \sim \epsilon^2$, $\beta_2 \sim \epsilon^3$, $\delta_1 \sim \epsilon^5$ we can diagonalize each mass matrix approximately with a product of two $SU(2)$ matrices

$$V_L = \begin{pmatrix} c_1 & s_1 e^{i\xi_1} & 0 \\ -s_1 e^{-i\xi_1} & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_4 & s_4 e^{i\xi_4} \\ 0 & -s_4 e^{-i\xi_4} & c_4 \end{pmatrix}$$

$$U_L = \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix}$$

(55)

where the $c_1, s_1$ etc. are sines and cosines which will be given below. Using these we get the KM matrix

$$V_{KM} = \begin{pmatrix} c_1 c_2 - s_1 s_2 f e^{-i\xi_1} & s_1 c_2 + c_1 s_2 f e^{-i\xi_1} & s_2 g e^{-i\xi_1} \\ s_1 g & c_1 c_2 f - s_1 s_2 e^{-i\xi_1} & 0 \\ s_1 g & -c_1 g^* & f \end{pmatrix}$$

(56)

where $f = c_3 c_4 + s_3 s_4 e^{i\xi_4}$, $g = s_3 c_4 - c_3 s_4 e^{i\xi_4}$. We have multiplied the first column by $e^{i\xi_1}$ and the first row by $e^{-i\xi_1}$ (quark phase redefinitions) to get to this form.

The diagonalization of the $2 \times 2$ matrices yields

$$s_4 \sim -\rho_3 \quad \xi_4 \sim \hat{\alpha}_2 \sin \theta \sim 0 \quad s_1 \sim \sqrt{\frac{m_d}{m_s}} \quad \xi_1 \sim -\theta \quad s_3 \sim \beta \quad s_2 \sim \sqrt{\frac{m_u}{m_c}}$$

$$m_s \sim \hat{\alpha}_2 \quad m_d \sim \frac{\rho_1^2}{\hat{\alpha}_2} \quad m_c \sim \beta^2 + \beta_2 \quad m_u \sim \frac{\delta_1^2}{m_c}$$

$$m_\tau = m_b \quad m_\mu \sim 3\hat{\alpha}_2 \sim 3m_s \quad m_\tau \sim \frac{\rho_1^2}{3\hat{\alpha}_2} \sim \frac{1}{3} m_d$$

(57)
where the last three are the GUT scale G-J mass relations which give the correct lepton-quark mass relations when renormalized to the electroweak scale (multiplication of the quark masses by a factor of approximately 3).

The KM matrix in the Wolfenstein parametrization is

\[
V_{KM} = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\
\lambda & 1 - \frac{1}{2} \lambda^2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & A\lambda^2 & 1
\end{pmatrix}
\] (58)

and we make the identifications as in [16]

\[
\lambda = (s_1^2 + s_2^2 + 2s_1s_2\cos\theta)^{\frac{1}{2}}
\]

\[
\lambda^2A = s_3 - s_4 \quad \lambda\sqrt{\rho^2 + \eta^2} = s_2 \quad \eta = \frac{s_2\sin\theta}{\lambda}
\] (59)

(In [16] the elements \(\rho_3\) and \(\beta_2\) are generated as radiative corrections to a GUT scale ansatz which has them set to zero). These relations are consistent with the present phenomenology of quark and lepton masses and the KM matrix [17].

Our derived form (53) has seven real parameters and one phase (\(\rho_3\) is small and makes little contribution to the observable parameters) and fits the nine masses, three mixing angles and one phase of the observed fermion mass matrices. There are the usual three G-J predictions for the lepton masses and two extra ones arising from the relations which follow from (57) and (59) between the quark masses and KM parameters which fix \(\rho\) and \(\eta\) once the other parameters are specified.

8. Embedding in SO(10)

The simplest possible embedding of the model we have been considering is

\[
(16, 3) = (1, 3) + (10, 3) + (\bar{5}, 3)
\]

\[
(10, 6) = (5, 6) + (\bar{5}, 6)
\]

\[
(126, 6) = (1, 6) + (\bar{5}, 6) + (45, 6) + ...
\]

\[
(45, 3) = (1, 3) + (24, 3) + ...
\]
This is also a minimal choice. We must take the $126$ to accommodate the right handed neutrinos and it must be in a $6$ of $SU(3)$ to couple to the fermions. We cannot however take the $5$s in the $10$ to be the only ones to acquire electroweak vevs as they are then forced to have the same vevs or one to be zero by the self potential - this would not be consistent with our perturbative scheme. The only other components available which can acquire vevs are precisely the ones we need - the $(\bar{5}, 6)$ to make $m_\ell$ large and the $(45, 6)$ to give the G-J relations.

The restrictions on our $SU(5)$ analysis which result are surprisingly weak. This is simply because we have more electroweak fields so that any restriction arising from the larger symmetry group is more than compensated for. Again we can fit any mass matrices at tree level. The parameter $\beta$ could be generated between the $(\bar{5}, 6)$ and GUT scale $(1, 6)$ so circumventing any constraints on the low energy four point couplings.

It is interesting however to speculate in this model about one striking feature of mass matrix phenomenology - the smallness of the $\alpha_1, \beta_1$ components. We showed above that it was possible to fit the low energy phenomenology with them set to zero. In this $SO(10)$ model the two $(24, 3)$s are replaced by a $(1, 3)$ and a $(24, 3)$. One of the three couplings of the type $(\mathbb{13})$ between the $(45, 6)$ and $(\bar{5}, 6)$ is now not allowed. If the $(1, 3)$ lies in the $2$ direction these terms will generate contributions to the $12$ and $22$ components in the mass matrices but none to the $11$ component. If we suppose that the $(45, 6)$ and the $(5, 6)$ give the dominant contributions to the corrections through these terms we get mass matrices
of the form
\[
M_u = m_t \begin{pmatrix} 0 & \delta_1 & 0 \\ \delta_1 & \beta_2 & \delta_3 \\ 0 & \delta_3 & 1 \end{pmatrix} \quad M_d = m_b \begin{pmatrix} 0 & \rho_1 & 0 \\ \rho_1 & \alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_e = m_\tau \begin{pmatrix} 0 & -3\rho_1 & 0 \\ -3\rho_1 & -3\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
which is successful in all but the lightest lepton-quark mass relation.

9. Implications for Cosmic Texture

The investigations of cosmological structure formation and microwave anisotropies produced by global symmetry breaking have so far been performed only for the case of an \(O(N)\) symmetry broken by a scalar \(N\) to \(O(N - 1)\). The case of \(N = 4\) is easily seen to be equivalent to the complete breaking of an \(SU(2)\) global symmetry by a scalar \(2^{3}\). However, as we have argued here, as far as family symmetry goes a global \(SU(3)\) symmetry appears a better bet. In this section we shall give the evolution equations for \(SU(3)\) cosmic texture and discuss how the GUT scale calculated above is directly related to the parameter governing the amplitude of cosmological perturbations.

The nonlinear sigma model governing the dynamics of the 8 \(SU(3)\) Goldstone bosons is easily described in terms of the two complex triplets \(\Sigma^1\) and \(\Sigma^2\) discussed above. For the appropriate range of parameters in the Higgs potential, the minimum of the potential is given by
\[
\Sigma^1 = v_1(\psi_1 + i\psi_2, \psi_3 + i\psi_4, \psi_5 + i\psi_6)
\]
\[
\Sigma^2 = v_2(\chi_1 + i\chi_2, \chi_3 + i\chi_4, \chi_5 + i\chi_6)
\]
\[
\bar{\chi}^2 = \bar{\psi}^2 = 1 \quad \bar{\psi}^T \bar{\chi} = \bar{\psi}^T \mathbf{M} \bar{\chi} = 0
\]
where we suppress the matrix \(\text{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2})\) in \(SU(5)\) space, and the \(6 \times 6\) matrix
\[
\mathbf{M} = \text{diag}(i\sigma_2, i\sigma_2, i\sigma_2) \quad \text{with} \quad \sigma_2 \quad \text{the usual Pauli matrix.}
\]

Following the usual procedure of imposing the constraints with Lagrange multipliers,
one finds the following equations of motion for the unit vectors \( \chi \) and \( \psi \):

\[
\nabla_\mu \partial_\mu \vec{\psi} + (\partial \vec{\psi})^2 \vec{\psi} + (\partial \vec{\psi} \partial \vec{\chi}) \vec{\chi} + (\partial \vec{\psi} \partial M \partial \vec{\chi}) M \vec{\chi} = 0
\]

\[
\nabla_\mu \partial_\mu \vec{\chi} + (\partial \vec{\chi})^2 \vec{\chi} + (\partial \vec{\chi} \partial \vec{\psi}) \vec{\psi} + (\partial \vec{\chi} \partial M \partial \vec{\psi}) M \vec{\psi} = 0
\]

(63)

It is easy to see that the simplest \( SU(2) \) scaling solution also solves the \( SU(3) \) equations: if one of \( \vec{\psi} \) or \( \vec{\chi} \) is constant it may be rotated into \((0, 0, 0, 0, 1)\). The other can then only have the first four components nonzero, and the equations (63) reduce to the \( SU(2) \) sigma model equation with known scaling solution [4]. It would be interesting to examine more general scaling solutions corresponding to other embeddings of \( SU(2) \) in \( SU(3) \). It will also be interesting to perform full three dimensional simulations of the ordering dynamics of (63).

Note that the evolution equations (63) have the attractive feature that there are no free parameters - the equations are purely geometrical, just as in the simpler \( O(N) \) theories. (It is not hard to see that had we chosen a range of parameters in the potential so that the minimum was at a nonzero value of \( \Sigma_1 \Sigma_2 \) there would be an extra free dimensionless ‘angle’ parameter in the evolution equations.) However the stress energy tensor of these fields, which determines the final density fluctuations, does depend on the value of \( v_1 \) and \( v_2 \) separately.

Finally let us note that the relation between the unification scale \( M_X \) and \( v_1 \) and \( v_2 \) may be used to predict the magnitude of the cosmological density perturbations. From the standard relations (see e.g. [18]) we find \( M_X^2 = \frac{25}{8} g^2 (v_1^2 + v_2^2) \). Substituting the fields (62) into the scalar kinetic terms and performing the trace over the \( SU(5) \) 24 matrices, one finds that the theory describes two real 6-component scalar fields with vacuum field strength \( \sqrt{15} v_1 \) and \( \sqrt{15} v_2 \). If we choose \( v_1 = v_2 \) for example, then the dimensionless
parameter $\epsilon = 8\pi^2 G\phi_0^2$ governing the magnitude of cosmological perturbations is related to the unification scale $M_X$ by

$$\frac{M_X^2}{M_{pl}^2} = \frac{5\alpha_{GUT}}{24\pi} \epsilon^5 \alpha_{GUT}^2 \pi^2 (64)$$

If we set $\epsilon = 10^{-4}$, which is the value of the field strength required for the $SU(2)$ theory to fit the COBE microwave anisotropy [4], then we deduce that $M_X \approx 4 \times 10^{16}$ GeV, consistent with the values obtained by requiring unification and a sufficiently long proton lifetime in Section 6. Conversely, from the requirement that $M_X = 10^{15-16}$ GeV, one sees that $\epsilon$ is constrained to be within an order of magnitude of the value required by COBE.

As far as we are aware, there is no other theoretical framework which comes this close to a prediction of the level of primordial density perturbations based on particle physics considerations alone.

**Conclusions**

In this paper we have examined in some detail the possibility of a fundamental symmetry relating the three families of elementary particles. We would be the first to acknowledge that the scheme we have explored is simplistic, and makes no attempt to explain why such a symmetry should exist. The fact that the symmetry is in the form of a direct product $G_{fam} \times G_{GUT}$ is ugly - the gauge and global symmetries are not unified. We have done nothing to ameliorate the heirarchy problem, and have made no attempt to include supersymmetry or indeed gravity - we might well be criticised for ignoring possible violations of global symmetries by quantum gravity (see e.g. [19]).

Nevertheless we have made a case for a simple family symmetry group - $SU(3)$ - and argued that it has to be a global symmetry. We have shown in detail that a renormalisable
symmetry breaking Higgs potential can produce vevs with sufficient parameters to match the measured fermion masses and quark mixing matrix. We have begun to explore the low energy phenomenology of the theory - the rich low energy Higgs sector - and shown how flavour changing neutral currents might be avoided while remaining in the weak coupling regime. We have pointed out how the simplest $SU(5)$ theory simplifies when it is embedded in $SO(10)$, and shown how unification of the coupling constants is rather economically achieved. Finally we showed how the global symmetry breaking at the GUT scale leads to the production of cosmic texture with the correct symmetry breaking scale to produce structure in the universe. While the ideas in this paper might be criticised as being naive, they cannot be criticised for not being testable!

There are several further developments of this work which we believe could be fruitful.

a) The detailed phenomenology of the eighteen (!) low energy electroweak doublets should be examined. In particular it would be interesting to know in general which of the charged or neutral Higgs bosons would be the easiest to detect. The consequences of $CP$ violation in the Higgs sector should also be investigated - the neutron and electron dipole moments, and the baryon asymmetry produced at the electroweak phase transition. Of course the main problem is that the parameter space is huge, but it would be interesting to know whether there are any generic predictions.

b) Full cosmological simulations of the $SU(3)$ nonlinear sigma model described in Section 8 may be performed to compute structure formation and cosmic microwave anisotropies in the theory.

c) The $SO(10)$ theory should be constructed in detail. Predictions of relations between
neutrino masses and mixing angles may be possible.

d) The theory given here produces stable magnetic monopoles, which, in the absence of inflation, are cosmologically disastrous. Nevertheless it may be possible that for some range of parameters and temperatures the finite temperature potential has a minimum without a $U(1)$ unbroken gauge group factor. As Kibble and Weinberg have recently argued [20], in this case magnetic monopoles would either never be formed at all, or might be connected by strings and disappear by the Langacker-Pi mechanism.

e) One might search for a fundamental origin of the $SU(3)$ symmetry invoked here. The fermion kinetic term has an accidental $SU(3)$ global symmetry, and if one adopted the ‘technicolor’ approach to symmetry breaking i.e. insisted that there should be no fundamental scalar fields, it is conceivable that one might preserve this as an exact symmetry of the theory, spontaneously broken by fermion bilinears as happens in massless $QCD$. The Higgs fields we need can all be obtained as bilinears in fermion fields in the $16$ of $SO(10)$, plus the adjoint representation, which arises most naturally as the extra space components of a gauge field in Kaluza Klein theoires.

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Appendix

Consider the potential of the $(24,3)$s. We want to show that we can get the minimum
in (15). As in our analysis of the potential above we will not attempt a general solution but simply choose certain terms and values of the couplings.

Consider the terms

\[-\frac{\mu^2}{2} tr (\Sigma_a \Sigma_a^\dagger) + \frac{\alpha_1}{4} tr (\Sigma_a \Sigma_a^\dagger \Sigma_b \Sigma_b^\dagger) + \frac{\alpha_2}{4} tr (\Sigma_a \Sigma_a^\dagger) tr (\Sigma_b \Sigma_b^\dagger) + \frac{\alpha'_2}{4} tr (\Sigma_a \Sigma_b) tr (\Sigma_a^\dagger \Sigma_b^\dagger) \]  

(65)

where the traces are now over the SU(5) indices. We expand explicitly in the six hermitian 24s in each complex vector (24,3). For example the first term gives

\[-\frac{\mu^2}{2} \Sigma tr (A_a^2 + B_a^2) \]  

(66)

where we have substituted \( \Sigma_a = A_a + iB_a \) and \( A_a^\dagger = A_a, B_a^\dagger = B_a \). Doing the same for every term we get a sum of terms which consists of two parts:

(i) groups of terms which are the same as the potential of a single real 24 i.e.

\[-\frac{\mu^2}{2} tr A^2 + \frac{\alpha_1}{4} tr A^4 + \frac{\alpha_2}{4} (tr A^2)^2 \]  

(67)

which has a stationary point at \( A = 0 \) and at \( A = diag(1,1,1,-\frac{3}{2},-\frac{3}{2}) \) if

\[ v^2 = \frac{4\mu^2}{7\alpha_1 + 30\alpha_2} \]  

(68)

(ii) remaining terms which are stationary at (15) simply because they involve more than one of the six real matrices in the decomposition, and are at least quadratic in any matrix contributing to a given term.

Thus (65) is stationary at (15) if

\[ v^2 = \frac{4\mu^2}{7\alpha_1 + 30(\alpha_2 + \alpha'_2)} \]  

(69)
When we expand in perturbations about (15) we find that the only massless modes are the Goldstone modes generated by action of the symmetry group. All the other perturbations can be shown to have positive masses provided \(0 < \frac{13}{60} \alpha_1 < |\alpha'_2| < \alpha_2\) (and \(\alpha'_2 < 0\)).

When we now consider this potential for each of the two \((24,3)\)s and couple them together with the positive definite terms

\[
tr(\Sigma^1_a \Sigma^2_a)tr(\Sigma^2_b \Sigma^1_b) \quad (\perp \text{ in } SU(3))
\]

\[
tr(\Sigma^1_a \Sigma^1_a)tr(\Sigma^2_b \Sigma^2_b) - tr(\Sigma^1_a \Sigma^2_b)tr(\Sigma^2_b \Sigma^1_a) \quad (\parallel \text{ in } SU(5))
\]

we get the required minimum. One can check easily that these coupling terms (70) are sufficient to completely break the symmetry as desired.
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