Disjointed sum of products by a novel technique of orthogonalizing ORing

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Abstract: This work presents a novel combining method called 'orthogonalizing ORing' which enables the building of the union of two conjunctions whereby the result consists of disjointed conjunctions. The advantage of this novel technique is that the results are already presented in an orthogonal form which has a significant advantage for further calculations as the Boolean Differential Calculus. By orthogonalizing ORing two calculation steps - building the disjunction and the subsequent orthogonalization of two conjunctions - are performed in one step. Postulates, axioms and rules for this linking technique are also defined which have to be considered getting correct results. Additionally, a novel equation, based on orthogonalizing ORing, is set up for orthogonalization of every Boolean function of disjunctive form. Thus, disjointed Sum of Products can be easily calculated in a mathematical way by this equation.

Keywords: Disjoint Sum of Products, Orthogonalization, Disjunctive Form, Conjunction, K-map

MSC: 03B99, 03G05, 03G25, 94C10

1 Introduction

A Boolean function or a switching function, respectively, is defined as a mapping \( f(\vec{x}) : \mathcal{B}^n \to \mathcal{B} \) with \( \mathcal{B} = \{0, 1\} \). It can be expressed by using Boolean variables \( x_i = \{x_1, x_2, \ldots, x_n\} \) \[1, 3, 10, 14, 20, 21\]. As shown in Table 1, Boolean variables which are either direct \( x_n \) or negated \( \bar{x}_n \) are connected by operations like conjunction (\( \land \)), disjunction (\( \lor \)), antivalence (\( \oplus \)) and equivalence (\( \circ \)).

Table 1. Boolean operations of two variables

| \( x_i \) | \( x_j \) | \( \bar{x}_i \) | \( \bar{x}_j \) | \( x_i \land x_j \) | \( x_i \lor x_j \) | \( x_i \oplus x_j \) | \( x_i \circ x_j \) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |

There are four standard forms \( f^S(\vec{x}) \) of switching function \[1, 6, 19\] which are either connected by conjunctions \( c_i(\vec{x}) = \bigwedge_{j=1}^n x_j = x_1 \land \cdots \land x_{n-1} \land x_n \) or by disjunctions \( d_i(\vec{x}) = \bigvee_{j=1}^n x_j = x_1 \lor \cdots \lor x_{n-1} \lor x_n \) \[3\]. Conjunctions are connected by disjunctions in the disjunctive form \( DF \) (1) or by antivalence-operations in the antivalence form \( AF \) (3) and disjunctions are connected by conjunctions in the conjunctive form \( CF \) (2) or by equivalence-
operations in the equivalence form \( EF (4) \). The connection of canonical conjunctions or disjunctions is named as normal form: disjunctive normal form \( DNF \), antivalence normal form \( ANF \), conjunctive normal form \( CNF \) and equivalence normal form \( ENF \). Therefore, the orthogonal form of a DF can also be named as disjointed Sum of Products (dSOP) which is the set of products terms (conjunctions) whose disjunction equals \( f(x) \) in non-orthogonal form, such that no two product terms cover the same 1. Consequently, dSOP consist of non intersecting cubes. Deriving an orthogonal form of DF is a classical problem of Boolean theory. In this work, this problem is supported by a contribution of a novel solution.

**Definition 1.1** (Standard Forms with \( m \in \mathbb{N} \setminus \{ 1 \} \)).

1. Disjunctive Form: 
   \[
   f_{DF}(x) = \bigvee_{i=1}^{m} c_i(x).
   \] (1)

2. Conjunctive Form: 
   \[
   f_{CF}(x) = \bigwedge_{i=1}^{m} d_i(x).
   \] (2)

3. Antivalence Form: 
   \[
   f_{AF}(x) = \bigoplus_{i=1}^{m} c_i(x).
   \] (3)

4. Equivalence Form: 
   \[
   f_{EF}(x) = \bigodot_{i=1}^{m} d_i(x).
   \] (4)

2 Characteristic of orthogonality

The characteristic of orthogonality is a special attribute of a switching function \([1, 5, 6, 15–18]\). The orthogonal form of a switching function is characterized by conjunctions or disjunctions which are disjointed to one another in pairs. This means, that for each pair of conjunctions, one of them contains a direct Boolean variable \( (x_i) \) and the other contains the negation \( (\bar{x}_i) \) of the same Boolean variable. Consequently, the intersection of each pair of these conjunctions \( c_{ij}(x) \) results in 0, as shown in Eq. (5). In contrast, the disjunction of each pair of orthogonal disjunctions \( d_{ij}(x) \) results in 1, as shown in Eq. (6).

**Definition 2.1** (Orthogonality of Conjunctions or Disjunctions). Two conjunctions \( c_i(x) \) and \( c_j(x) \) are orthogonal to each other if the following applies:

\[
    c_i(x) \land c_j(x) = 0 \quad i \neq j.
\] (5)

Two disjunctions \( d_i(x) \) and \( d_j(x) \) are orthogonal to each other if the following applies:

\[
    d_i(x) \lor d_j(x) = 1 \quad i \neq j.
\] (6)

The orthogonal form of a \( f^S(x) \) enables its transformation in another form, which will have equivalent function values. This means, that the native form and the transformed form have the same function values if the same input values are used in each case. Therefore, orthogonalization simplifies the handling for further calculation steps, especially in the application of electrical engineering, e.g. for further calculation step as the Boolean Differential Calculus (BDC) \([2, 3]\) by which all possible test patterns for a combinational circuit can be determined. Test patterns are used to detect feasible faults in combinational circuits. Additionally, it facilitates the calculation of BDC particularly in Ternary-Vector-List (TVL) arithmetic \([11, 12, 16–18]\). TVL is a kind of matrix which simplifies the computational representation of Boolean functions and their computational handling of tasks in a facilitated way.

**Theorem 2.2** (Orthogonality of DF and AF). If the intersection of every two conjunctions \( c_{ij}(x) \) of a given DF is 0 then \( DF^{orth} = AF^{orth} \) applies. That means, an orthogonal disjunctive form \( DF^{orth} \) is equivalent to the orthogonal
antivalence form $A_{F^{orth}}$ including the same conjunctions $[1, 3, 10, 14, 15, 21]$:

$$
\bigvee_{k=1}^{m} c_k = \bigoplus_{k=1}^{m} c_k.
$$

(7)

**Proof of Theorem 2.2.** By using (5) for orthogonal conjunctions $c_{i,j}(x)$ the relation in (7) applies. The respective proof is brought by the following relation. The disjunction of two conjunctions $c_{i,j}(x)$ on the right side is equivalent to the antivalence operation of the same two conjunctions on the right side. This is the procedure of reshaping of a disjunctive form in the antivalence form. For the case that both conjunctions are orthogonal the last term on the right side results in 0. An antivalence-operation with 0 is to be neglected, as $x_i \oplus 0 = x_i$ applies. Consequently, this leads to the relation in (7).

$$
c_i(x) \lor c_j(x) = c_i(x) \oplus c_j(x) \oplus (c_i(x) \land c_j(x)).
$$

(8)

**Theorem 2.3** (Orthogonality of CF and EF). *If the union of every two disjunctions $d_{i,j}(x)$ of a given CF is 1 then $C_{F^{orth}} = E_{F^{orth}}$ applies. Thus, an orthogonal conjunctive form $C_{F^{orth}}$ is equivalent to the orthogonal equivalence form $E_{F^{orth}}$ including the same conjunctions $[1, 3, 10, 14, 15, 21]$:

$$
\bigwedge_{k=1}^{m} d_k = \bigodot_{k=1}^{m} d_k.
$$

(9)

**Proof of Theorem 2.3.** A CF can be represented as an EF by using the following condition in (10). By using (6) for orthogonal disjunctions $d_{i,j}(x)$ in this case the relation in (9) applies. The conjunction of two disjunctions $d_{i,j}(x)$, that means a CF, on the left side is equivalent to the right side which illustrates the equivalence operation of the same two disjunctions. If these both disjunctions $d_{i,j}(x)$ are orthogonal then the union of them results in 1, as shown by the last term on the right side. An equivalence-operation with 1 is to be neglected, as $x_i \odot 1 = x_i$ applies. Consequently, the relation in (9) applies.

$$
d_{i}(x) \land d_{j}(x) = d_{i}(x) \odot d_{j}(x) \odot (d_{i}(x) \lor d_{j}(x)).
$$

(10)

3 Elementary operations of two conjunctions

In this section the elementary operations (intersection, union, difference-building) of conjunctions, which are deduced out of the set theory due to the isomorphism, are defined for the switching algebra. These formulas for different operations of conjunctions specify the order in which the variables of the given conjunction have to be linked. That means, if a variable is displayed negated, the corresponding literal of the given conjunction must be negated at this point. The number of variables in their respective conjunction is defined by $n$ or $n'$.

**Theorem 3.1** (Intersection of two conjunctions). *The intersection of any two conjunctions $c_{i,j}(x)$ with $n, n' \in N$ is calculated by:

$$
c_{i}(x) \land c_{j}(x) = \bigwedge_{i=1}^{n} x_i \land \bigwedge_{j=1}^{n'} x_j.
$$

(11)

**Theorem 3.2** (Union of two conjunctions). *The union of any two conjunctions $c_{i,j}(x)$ with $n, n' \in N$ is given by:

$$
c_{i}(x) \lor c_{j}(x) = \bigwedge_{i=1}^{n} x_i \lor \bigwedge_{j=1}^{n'} x_j.
$$

(12)
Theorem 3.3 (Difference-building of two conjunctions). The difference-building of a conjunction \( c_m(x) \) as minuend and another conjunction \( c_s(x) \) as subtrahend with \( n, n' \in \mathbb{N} \) is calculated by following equation, which is deduced out of the set theory \([4, 13]\). That means, for the difference of two sets \( M - S \) it applies \( M \cap \bar{S} \) which is transferred to the switching algebra due to the isomorphism. Consequently, the difference-building of two conjunctions is the intersection of the minuend and the complement of the subtrahend. By building the difference several conjunctions arise which are not disjointed (orthogonal) to each other.

\[
\begin{align*}
c_m(x) - c_s(x) &= \bigwedge_{m=1}^{n} x_m - \bigwedge_{s=1}^{n'} x_s = \bigwedge_{m=1}^{n} x_m \land \bigwedge_{s=1}^{n'} x_s = \bigwedge_{m=1}^{n} x_m \lor \bigvee_{s=1}^{n'} \bar{x}_s. 
\end{align*}
\]  

(13)

4 Orthogonalizing difference-building

The technique of orthogonalizing difference-building \( \oplus \) is used to calculate the orthogonal difference of two conjunctions \( (c_m(x)) \) whereby the result is orthogonal. This method is generally valid and equivalent to the usual method of difference-building \([6–9]\). Orthogonalizing difference-building \( \oplus \) is the composition of two calculation steps - the difference-building and the subsequent orthogonalization.

Definition 4.1 (Orthogonalizing difference-building). Orthogonalizing difference-building \( \oplus \) corresponds to the removal of the intersection which is formed between the minuend \( c_m(x) \) and the subtrahend \( c_s(x) \) from the minuend \( c_m(x) \), which means \( c_m(x) - (c_m(x) \land c_s(x)) \); the result is orthogonal. Orthogonalizing difference-building of two conjunctions with \( n, n' \in \mathbb{N} \) is defined as:

\[
\begin{align*}
c_m(x) \oplus c_s(x) &= \left( \bigwedge_{m=1}^{n} x_m \right) \oplus \left( \bigwedge_{s=1}^{n'} x_s \right) := \bigwedge_{m=1}^{n} x_m \land \bigvee_{s=1}^{n'} \bar{x}_s = \left( x_1 \cdot \ldots \cdot x_{n-1} \cdot x_n \right)_m \land \left( \bar{x}_{\tilde{1}} \lor x_1 \lor \bar{x}_2 \lor \ldots \lor x_1 \cdot x_2 \cdot \ldots \cdot \bar{x}_n \right)_{n'}.
\end{align*}
\]  

(14)

In this case, the formula \( \left( \bigvee_{j=1}^{n'} \bar{x}_{\tilde{j}} = \bar{x}_1 \lor x_1 \lor \bar{x}_2 \lor \ldots \lor x_1 \cdot x_2 \cdot \ldots \cdot \bar{x}_n \right) \) from \([10]\) is used to describe the orthogonalizing difference-building in a mathematically easier way, where \( x_1, x_2, \ldots, x_n \) are literals of the subtrahend conjunction \( c_s(x) \).

This method is explained by the following example and description. Additionally, this example is also illustrated in a K-map (Figure 1).

Example 4.2. A subtrahend \( (c_s(x) = x_1 x_2 x_3) \) is subtracted from a minuend \( (c_m(x) = x_3) \). It is a result of several conjunctions \( (1^{\text{st}} \text{ cube}, 2^{\text{nd}} \text{ cube}, 3^{\text{rd}} \text{ cube}) \) which are orthogonal to each other and cover all of the remaining 1s.

\[
\begin{array}{ccc}
X_3 & \oplus & X_1 X_2 X_3 \\
\text{Minuend} & \text{Subtrahend} & \text{1st cube} \lor \text{2nd cube} \lor \text{3rd cube}
\end{array}
\]

- The first literal of the subtrahend, here \( x_1 \), is complemented and builds the intersection with the minuend, here \( x_3 \). Consequently, the first conjunction of the difference is \( \bar{x}_1 x_3 \).
- Then the second literal, here \( x_2 \), is complemented and forms the intersection with the minuend, and the first literal \( x_1 \) of the subtrahend is built. Therefore, the second conjunction is \( x_1 \bar{x}_2 x_3 \).
- The next literal, here \( x_3 \), is complemented and forms the intersection with the minuend, and the first literal \( x_1 \) and second literal \( x_2 \) of the subtrahend is built. Thus, the third term of the difference is \( x_1 x_2 x_3 \bar{x}_3 \).
- This process is continued until all literals of the subtrahend are singly complemented and linked by building the intersection with the minuend in a separate conjunction.
5 Orthogonalizing ORing

By a further novel technique ‘orthogonalizing ORing’, which is based on the orthogonalizing difference-building ⊙, the union of two conjunctions \( (c_{\text{1st}}(x)) \) can be calculated of two conjunctions \( (c_{\text{2nd}}(x)) \) whereby the result is orthogonal.

**Definition 5.1** (Orthogonalizing ORing). The intersection of a conjunction \( c_{\text{1st}}(x) \), called as the first summand, and a second conjunction \( c_{\text{2nd}}(x) \), called as the second summand, is removed by the method \( \bigodot \) from the first summand \( c_{\text{1st}}(x) \), and the second summand \( c_{\text{2nd}}(x) \) is linked to that subtraction by a disjunction; the result is orthogonal. This procedure is labeled as orthogonalizing ORing and is defined with \( n, n' \in \mathbb{N} \) as:

\[
c_{\text{1st}}(x) \bigodot c_{\text{2nd}}(x) = \bigwedge_{s_1=1}^{n} x_{s_1} \bigodot \bigvee_{s_2=1}^{n'} x_{s_2} := \bigwedge_{s_1=1}^{n} x_{s_1} \bigvee \bigwedge_{s_2=1}^{n'} x_{s_2} = \\
= \bigwedge_{s_1=1}^{n} x_{s_1} \bigvee \bigwedge_{s_2=1}^{n'} \overline{x_{s_2}} = \bigwedge_{s_1=1}^{n} x_{s_1} \bigvee \bigwedge_{s_2=1}^{n'} x_{s_2} = \\
= \bigwedge_{s_1=1}^{n} \left( x_{1} \ldots \cdot x_{n_{s_1}} \right)_{s_1} \bigwedge \left( \overline{x_{1}} \cup \ldots \cup x_{1} \cdot \ldots \cdot \cdot \overline{x_{n_{s_2}}} \right)_{s_2} \bigvee \left( x_{1} \ldots \cdot x_{n} \right)_{s_2}.
\]

This method of orthogonalizing ORing is explained by the Example 5.2 which is also illustrated in a K-map (Fig. 2).

**Example 5.2.** The orthogonalizing ORing is built of two conjunctions \( c_{\text{1st}}(x) = \overline{x}_{2} \) and \( c_{\text{2nd}}(x) = x_{1}x_{3} \) by using Eq. (15).

\[
\overline{x}_{2} \bigodot x_{1}x_{3} = \overline{x}_{1}\overline{x}_{3} \bigvee \overline{x}_{1}\overline{x}_{2}\overline{x}_{3} \bigvee \overline{x}_{1}x_{3} \cdot \\
\text{1st sum.} \quad \text{2nd sum.} \quad \text{1st cube} \quad \text{2nd cube} \quad \text{disjointed conjunctions}
\]

It is a result of several conjunctions or cubes which are disjointed to each other in pair (Fig. 2). The following points explain this unprecedented technique:

- The first literal of the second summand \( c_{\text{2nd}}(x) \), here \( x_1 \), is complemented and ANDed to the first summand \( c_{\text{1st}}(x) \). Consequently, the first conjunction of the orthogonal result arises, \( \overline{x}_1x_2 \).
Then the second literal of the second summand \( c_2(x) \) is complemented, here \( x_1 \), and ANDed with the first literal of the second summand \( c_2(x) \) to the first summand \( c_1(x) \). Therefore, the second conjunction of the orthogonal result is developed, \( x_1\bar{x}_2\bar{x}_3 \).

This process is continued until all literals of the second summand \( c_2(x) \) are singly complemented and linked by ANDing to the first summand \( c_1(x) \) in a separate term.

At last the second summand \( c_2(x) \) is added to the heretofore calculated conjunctions.

By swapping the position of the summands, the result changes as shown in the following:

\[
x_1x_2\otimes\bar{x}_2 = x_1x_2x_3 \lor x_1\bar{x}_2.
\]

However, both solutions are equivalent because the same 1s are covered. They only differ in the form of their coverage. But in order to represent an orthogonal result with a fewer number of conjunctions, the conjunction with more literals has to be accepted as the first summand. That is possible because orthogonalizing ORing has the property of commutativity. By the mathematical induction the general validity of Eq. (15) is given in Proof 1.

The number of conjunctions in the result, called \( n_x \), corresponds to the number of literals presented in the second summand \( c_2(x) \) and not presented in the first summand \( c_1(x) \) at the same time; in addition, a 1 is added to \( n_x \), which stands for the second summand \( c_2(x) \) as the last linked term. It applies:

\[
n_x + 1 \quad \text{with} \quad n_x \in \mathbb{N}.
\]

Furthermore, the number of possible results, which primarily depends on \( n_x \), can be charged by:

\[
n_x! \quad \text{for} \quad n_x > 0.
\]
Proof. 1

General validity of Eq. (15):
\( \forall (n, n') \in \mathbb{N} \setminus \{1\}, \ (n, n') \geq n_0 \) applies \( A(n, n') \):

if \( A(n_0) \land (\forall (n, n') \in \mathbb{N}, \ (n, n') \geq n_0 : A(n, n') \rightarrow A((n, n') + 1)) \) \( \Rightarrow (\forall (n, n') \in \mathbb{N}, \ (n, n') \geq n_0 : A(n, n') \)

Basis \( A(n_0) \): \( n_0, n_0' = 2 \)

\[
\left[ \bigwedge_{s_1=1}^{2} x_{s_1} \land \bigvee_{s_2=1}^{2} x_{s_2} \right] \lor \left[ \bigvee_{s_2=1}^{2} x_{s_2} = \left[ \left( x_1 x_2 \right)_{s_1} \lor \left( \tilde{x}_1 \lor x_1 \tilde{x}_2 \right)_{s_2} \right] \lor \left( x_1 x_2 \right)_{s_2} \right]
\]

\[
\left[ \left( x_1 x_2 \right)_{s_1} \land \left( \tilde{x}_1 \lor x_1 \tilde{x}_2 \right)_{s_2} \right] \lor \left( x_1 x_2 \right)_{s_2} = \left[ \left( x_1 x_2 \right)_{s_1} \lor \left( \tilde{x}_1 \lor x_1 \tilde{x}_2 \right)_{s_2} \right] \lor \left( x_1 x_2 \right)_{s_2}
\]

Inductive step \( A(n, n') \rightarrow A(n+1, n'+1) \): \( n \rightarrow n+1 \) and \( n' \rightarrow n' + 1 \)

\[
\left[ \bigwedge_{s_1=1}^{n+1} x_{s_1} \land \bigvee_{s_2=1}^{n+1} x_{s_2} \right] \lor \left[ \bigwedge_{s_2=1}^{n+1} x_{s_2} = \left[ \left( x_1 \cdots x_{n+1} \right)_{s_1} \land \left( \tilde{x}_1 \lor \cdots \lor x_1 \tilde{x}_n \lor \cdots \lor x_n \tilde{x}_{n+1} \right)_{s_2} \right] \lor \right.
\]

\[
\left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \right]
\]

\[
\left[ \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_1} \land \left( \tilde{x}_1 \lor x_1 \tilde{x}_1 \cdots \lor x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \right.
\]

\[
\left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \right]
\]

\[
\left[ \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_1} \land \left( \tilde{x}_1 \lor x_1 \tilde{x}_1 \cdots \lor x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \right.
\]

\[
\left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \right]
\]

\[
\left[ \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_1} \land \left( \tilde{x}_1 \lor x_1 \tilde{x}_1 \cdots \lor x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \right.
\]

\[
\left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left. \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \lor \left( x_1 \cdots x_n \tilde{x}_{n+1} \right)_{s_2} \right]
\]
5.1 Orthogonalizing ORing between a DF and a conjunction

The orthogonalizing ORing of an orthogonal DF $f(x)^{orth}$ and a conjunction $c_{s_i}(x)$ can be reached by Eq. (19).

With $l_{s_i} \in \mathbb{N}^{*}$ as the number of conjunctions in the given function the following applies:

\[
f_{s_i}(x)^{orth} \otimes c_{s_j}(x) = \bigvee_{i=1}^{l_{s_i}} \left[ \bigwedge_{s_{i1}=1}^{n} x_{i,s_{i1}} \bigwedge_{s_{i2}=1}^{n'} x_{s_{i2}} :=\right.
\]

\[
= \bigvee_{i=1}^{l_{s_i}} \left[ \bigwedge_{s_{i1}=1}^{n} x_{i,s_{i1}} \bigvee_{s_{i2}=1}^{n'} x_{s_{i2}} \right] \bigvee_{s_{i1}=1}^{n} x_{s_{i1}} =
\]

\[
= \bigvee_{i=1}^{l_{s_i}} \left[ \bigwedge_{s_{i1}=1}^{n} x_{i,s_{i1}} \bigwedge_{s_{i2}=1}^{n'} x_{s_{i2}} \right] \bigvee_{s_{i2}=1}^{n'} x_{s_{i2}}.
\]

Since the general validity for orthogonalizing ORing is given, there is no need to prove the general validity for Eq. (19) in this case. As shown in Example 5.3 the use of (19) is illustrated.

**Example 5.3.** Let DF $f_1(x) = x_1 \vee x_2 x_3$ and a conjunction $c_1(x) = \bar{x}_2 \bar{x}_3$. The orthogonalizing ORing between both has to be calculated.

In this case the orthogonal form of $f_1(x)$ has to be calculated by Eq. (15) otherwise by Eq. (36):

\[
x_{1} \otimes x_2 x_3 = x_1 \bar{x}_2 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3.
\]

The orthogonal form is generated:

\[
f_1(x)^{orth} = x_1 \bar{x}_2 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3.
\]

Next step is the application of Eq. (19). Each conjunction of $f_1(x)^{orth}$ is combined by orthogonalizing ORing with $c_1(x)$. However, the adding of the second summand at last is fulfilled after each combining step.

\[
f_1(x)^{orth} \otimes c_1(x) = (x_1 \bar{x}_2 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3) \otimes \bar{x}_2 \bar{x}_3
\]

\[
= x_1 \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee x_2 x_3 \vee \bar{x}_2 \bar{x}_3.
\]

In the K-map on the left side the cubes of $f_1(x)$ and $c_1(x)$ are represented, and on the K-map on the right side the result after the procedure of orthogonalizing ORing is illustrated (Fig. 3).

![Fig. 3. Before and after the process of orthogonalizing ORing](image)

5.2 Axioms and Postulates

5.2.1 Postulates

The following postulates are necessary for getting correct results after each operation of orthogonalizing ORing.
– If two conjunctions are already orthogonal to each other \( (c_i(x) \perp c_j(x)) \) the result corresponds to the disjunction of both conjunctions:

\[
c_i(x) \lor c_j(x) = c_i(x) \lor c_j(x).
\]  
(20)

– If the first conjunction is the subset of the second conjunction \( (c_i(x) \subseteq c_j(x)) \) it follows:

\[
c_i(x) \lor c_j(x) = c_j(x).
\]  
(21)

– and in the reverse case \( (c_j(x) \subseteq c_i(x)) \) it follows:

\[
c_i(x) \lor c_j(x) = c_i(x).
\]  
(22)

\subsection*{5.2.2 Axioms for variables}

The following rules apply for the linking of variables and constants.

– The orthogonalizing ORing of a variable and constant 0 results in the variable itself (23) and the orthogonalizing ORing of a variable and constant 1 results in 1 (24).

\[
x_i \lor 0 = x_i, \quad (23)
\]

\[
x_i \lor 1 = 1. \quad (24)
\]

– Furthermore, the orthogonalizing ORing of a variable with the same variable leads to the variable itself (25) and the orthogonalizing ORing of a variable with its negated form leads to the union of both (26). Consequently, this results in 1 at last

\[
x_i \lor x_i = x_i, \quad (25)
\]

\[
x_i \lor \overline{x}_i = x_i \lor \overline{x}_i \quad (= 1). \quad (26)
\]

\subsection*{5.2.3 Axioms for conjunctions}

Following axioms for conjunctions are deduced out of the axioms for variables.

– The neutral element of orthogonalizing ORing is 0:

\[
c_i(x) \lor 0 = c_i(x). \quad (27)
\]

– The result of orthogonalizing ORing between 0 and a conjunction \( c_i(x) \) is this conjunction \( c_i(x) \) itself:

\[
0 \lor c_i(x) = c_i(x). \quad (28)
\]

– The orthogonalizing ORing of a conjunction \( c_i(x) \) with the unit-term 1 leads to 1:

\[
c_i(x) \lor 1 = 1. \quad (29)
\]

– The result of orthogonalizing ORing between 1 and a conjunction \( c_i(x) \) is 1:

\[
1 \lor c_i(x) = 1. \quad (30)
\]

– The result of the orthogonalizing ORing between two the same conjunctions \( c_i(x) \) results to this conjunction itself:

\[
c_i(x) \lor c_i(x) = c_i(x). \quad (31)
\]

– The orthogonalizing ORing of a conjunction \( c_i(x) \) with its complement \( \overline{c_i(x)} \) results in an unit-term 1:

\[
c_i(x) \lor \overline{c_i(x)} = 1. \quad (32)
\]
5.2.4 Commutativity

Commutativity is the property of operation which allows the changing of the terms in their position in such that the value of the expression will not change. As the novel method of orthogonalizing ORing is commutative, the sequence of its execution can be changed.

\[ c_1(x) \lor c_3(x) = c_2(x) \lor c_1(x). \]  (33)

The value of both sides are equivalent and orthogonal. They can only differ in the form of coverage. The following Example 5.4 gives an overview about the commutative property.

Example 5.4.

\[ x_1 \lor x_2 \bar{x}_3 = x_2 \bar{x}_3 \lor x_1 \]
\[ x_3 \bar{x}_2 \lor x_1 x_2 x_3 \lor x_2 \bar{x}_3 = \bar{x}_1 x_2 \bar{x}_3 \lor x_1. \]

The left side differs only in the form of coverage in contrast to the right side, which is shown in the corresponding K-maps (Fig. 4). Both sides consist of disjointed cubes.

5.2.5 Associativity

Associativity is the property of an operation which allows the rearranging of the parentheses in such that the value of the expression will not change. As orthogonalizing ORing is associative, the position of the parentheses can be changed.

\[ (c_1(x) \lor c_2(x)) \lor c_3(x) = c_1(x) \lor (c_2(x) \lor c_3(x)). \]  (34)

The value of both sides are equivalent and orthogonal. Only the form of their coverage can be different. Following Example 5.5 illustrates this characteristic of associativity.

Example 5.5.

\[ (x_1 \lor x_2 \bar{x}_3) \lor \bar{x}_1 x_2 = x_1 \lor (x_2 \bar{x}_3 \lor \bar{x}_1 x_2) \]
\[ (x_1 \bar{x}_2 \lor x_1 x_2 x_3 \lor x_2 \bar{x}_3) \lor \bar{x}_1 x_2 = x_1 \lor (x_1 x_2 \bar{x}_3 \lor \bar{x}_1 x_2) \]
\[ x_3 \bar{x}_2 \lor x_1 x_2 x_3 \lor x_1 x_2 \bar{x}_3 \lor \bar{x}_1 x_2 = x_1 \lor x_1 \lor \bar{x}_1 x_2 \]
\[ x_1 \bar{x}_2 \lor x_1 x_2 x_3 \lor x_1 x_2 \bar{x}_3 \lor \bar{x}_1 x_2 = x_1 \lor \bar{x}_1 x_2. \]

Both sides are homogeneous and orthogonal as shown in the K-maps in Figure 5. They only differ in their form of coverage.
The general validity of (36) is proven by the following mathematical induction:

**Example 5.6.**

The validity of the distributive property is given by the following proof:

\[
c_1(x) \cdot (c_2(x) \cap c_3(x)) = (c_1(x) \cdot c_2(x)) \cap (c_1(x) \cdot c_3(x)).
\]  

(35)

The distributive property of an operation allows the exclusion of the same term. That means, that a term can be factored out. Hereby, the orthogonality of both sides has to be insisted. In this case, the distributive law for ANDing out applies for left and right side.

\[
c_1(x) \cdot (c_2(x) \cap c_3(x)) = (c_1(x) \cdot c_2(x)) \cap (c_1(x) \cdot c_3(x)).
\]

Both sides are equivalent and orthogonal. They can only differ in the form of their coverage. This characteristic of distributivity is demonstrated by the following Example 5.6 whereby both sides result to the same term.

**Example 5.6.**

\[
x_1 \cdot (x_2 \cdot x_3 \cap x_1 \cdot x_2) = (x_1 \cdot x_2 \cdot x_3) \cap (x_1 \cdot x_1 \cdot x_2)
\]

\[
x_1 \cdot (x_1 \cdot x_2 \cdot x_3 \cap x_1 \cdot x_2) = x_1 \cdot x_2 \cdot x_3 \cap x_1 \cdot x_2
\]

\[
x_1 \cdot x_2 = x_1 \cdot x_2.
\]

**6 Disjointed sum of products**

Based on this technique of orthogonalizing ORing a novel Equation (36) is formed which enables the orthogonalization of every disjunctive form \(DF\). That means, this formula enables the transformation of a SOP in a homogeneous dSOP. With \(m \in \mathbb{N}\) as the number of conjunctions \(c_i(x)\) that are included in the given \(DF\), which has to be orthogonalized, it follows that:

\[
f_{DF}(x)^{orth} = \bigvee_{i=1}^{m} c_i(x) = c_1(x) \cap c_2(x) \cap \ldots \cap c_{m-1}(x) \cap c_m(x).
\]  

(36)

The explanation for Equation (36) is provided as follows:

- Orthogonalizing ORing is realized between the first and the second conjunction, \(c_1(x) \cap c_2(x)\), by Eq. (15). After that, orthogonalizing ORing is calculated between the result of \((c_1(x) \cap c_2(x))\) and the third conjunction, \((c_1(x) \cap c_2(x)) \cap c_3(x)\), by Eq. (19).
- This procedure is continued until the last conjunction \(c_m(x)\).

The general validity of (36) is proven by the following mathematical induction:
Proof. 2

- \( \forall (m) \in \mathbb{N}, \ (m) \geq m_0 \) applies \( A(m) \):

\[
\bigvee_{i=1}^{m} c_i(x) = c_1(x) \uplus c_2(x) \uplus \ldots \uplus c_m(x).
\]

- if \( A(m_0) \land (\forall (m) \in \mathbb{N}, \ m \geq m_0 : A(m) \to A(m + 1)) \Rightarrow (\forall (m) \in \mathbb{N}, \ m \geq m_0 : A(m)) \)

- Basis \( A(m_0) \): \( m_0 = 2 \)

\[
\bigvee_{i=1}^{2} c_i(x) = c_1(x) \uplus c_2(x)
\]

\[
c_1(x) \uplus c_2(x) = c_1(x) \uplus c_2(x).
\]

- Inductive step \( A(m) \to A(m + 1) : m \to m + 1 \)

\[
\bigvee_{i=1}^{m+1} c_i(x) = c_1(x) \uplus c_2(x) \uplus \ldots \uplus c_m(x) \uplus c_{m+1}(x)
\]

\[
\bigvee_{i=1}^{m} c_i(x) \uplus c_{m+1}(x) = c_1(x) \uplus c_2(x) \uplus \ldots \uplus c_m(x) \uplus c_{m+1}(x).
\]

The orthogonal result may differ depending on the order of the conjunctions because of the property of commutativity. Thus, the conjunctions can be changed if necessary. However, all solutions are equivalent and orthogonal. In Example 6.1 it is given an overview of the use of Eq. (36)

**Example 6.1.** Function \( f_2(x) = x_3 \lor x_1 \lor x_2 \bar{x}_3 \) has to be orthogonalized by Eq. (36).

\( f_2(x) \text{orth} = x_3 \uplus x_1 \uplus x_2 \bar{x}_3 \) is the orthogonal form of function \( f_2(x) \), illustrated in the K-maps (Fig. 6).

**Fig. 6.** \( f_2(x), f_2(x) \text{orth} \) and \( f_2(x) \text{sort} \) in K-maps

By rearrangement of the order of the consisting conjunctions of a DF we obtain fewer number of conjunctions in the derived orthogonal form. This procedure of sorting is carried out from large to small. That means, it takes place from conjunctions of higher number of variables to conjunctions of fewer number of variables. The following Example clarifies this advantage of sorting.
Example 6.2. Function \( f_2(x) = x_3 \lor x_1 \lor x_2 \bar{x}_3 \) has to be orthogonalized after resorting.

\[
f_2(x)_{orth} = x_2 \bar{x}_3 \lor x_3 \lor x_1 = (x_2 \bar{x}_3) \lor x_1 = \bar{x}_1 x_2 \bar{x}_3 \lor \bar{x}_1 x_3 \lor x_1.
\]

The orthogonalized form of the sorted DF contains fewer number of conjunctions (see Fig. 6).

Fig. 7. Average number of conjunctions in the orthogonal result

The analysis of a measurement, as shown in Fig. 7, gives an overview of the comparison of the orthogonalization process depending on sorting. The orthogonalization of unsorted DF called as \( ORTH[\lor] \) and the sorted of the same DFs called as \( sortORTH[\lor] \) are compared. Thereby, 50 different functions with five conjunctions were orthogonalized by the novel technique with regards to the tuple length (number of variables) which runs from 1 to 50. Out of these 50 calculations per number of variables the number of conjunctions were determined, from which an average value was calculated for each tuple length. These range of averages were subsequently plotted in the diagram to show the deviation of \( sortORTH[\lor] \) from \( ORTH[\lor] \). This comparison illustrates a reduction of conjunctions in the orthogonal form when the DF was sorted before. This reduction is approximately 19% on average (see Table 2). Additionally, this feature allows the reducing of operation for subsequent calculation steps. Hereby, the relation in (37) is confirmed by the comparison in Fig. 7. The number of conjunctions of an orthogonalized DF \( P_{c\ell}(f(x)_{orth}) \) which was sorted before (\( sortORTH[\lor] \)) is smaller than the number of conjunctions of the orthogonalized DF \( P_{c\ell}(f(x)_{orth}) \) which was not sorted before (\( ORTH[\lor] \)):

\[
P_{c\ell}(f(x)_{orth}) \leq P_{c\ell}(f(x)_{orth}). \tag{37}
\]

7 Conclusion

This work showed a novel technique for building a union of disjointed conjunctions which is called as orthogonalizing ORing. Its results are orthogonal. Orthogonalizing ORing is used to calculate the orthogonal form of building the union of two conjunctions. This linking technique replaces two calculation steps - building a union and the subsequent orthogonalization - by one step. Orthogonalizing ORing is valid in general, which was proven by the mathematical induction, and is also equivalent to the usual method of union \( \lor \). Additionally, postulates related to commutativity, distributivity and associativity and axioms for this method are also defined. Furthermore, every Boolean function of disjunctive form or every Sum of Products, respectively, can easily be orthogonalized mathematically by a novel equation which is based on this linking technique of orthogonalizing ORing. By this orthogonalization a disjointed Sum of Products
can be reached in a simpler way. The general validity was also proven by the mathematical induction. An additional step of sorting before the step of orthogonalization achieves a reduction of approximately 19% of the number of conjunctions in the orthogonal result. This feature was illustrated by a measurement whereby the orthogonalization took place before and after sorting.

Table 2. Average number of conjunctions in the orthogonal result

| $x_n$ | sortORTH[$\vee$] | ORTH[$\vee$] | deviation in % |
|-------|------------------|--------------|----------------|
| 1     | 1                | 1.48         | 48.0           |
| 2     | 1.38             | 1.84         | 33.3           |
| 3     | 1.88             | 2.54         | 35.1           |
| 4     | 2.54             | 3.24         | 27.6           |
| 5     | 2.84             | 4.4          | 54.9           |
| ...   | ...              | ...          | ...            |
| 26    | 29.76            | 35.88        | 20.6           |
| 27    | 30.54            | 36.06        | 18.1           |
| 28    | 33.16            | 38.44        | 15.9           |
| 29    | 31.26            | 34.98        | 11.9           |
| ...   | ...              | ...          | ...            |
| 46    | 63.96            | 66.72        | 4.3            |
| 47    | 61.26            | 66.52        | 8.6            |
| 48    | 72.28            | 76.84        | 6.3            |
| 49    | 69.02            | 74.06        | 7.3            |
| 50    | 69.02            | 77.46        | 12.2           |
|       |                  |              | average deviation in % 19.3 |

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