Integral equations for heat kernel in compound media

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Abstract

By making use of the potentials of the heat conduction equation the integral equations are derived which determine the heat kernel for the Laplace operator $-a^2\Delta$ in the case of compound media. In each of the media the parameter $a^2$ acquires a certain constant value. At the interface of the media the conditions are imposed which demand the continuity of the ‘temperature’ and the ‘heat flows’. The integration in the equations is spread out only over the interface of the media. As a result the dimension of the initial problem is reduced by 1. The perturbation series for the integral equations derived are nothing else as the multiple scattering expansions for the relevant heat kernels. Thus a rigorous derivation of these expansions is given. In the one dimensional case the integral equations at hand are solved explicitly (Abel equations) and the exact expressions for the regarding heat kernels are obtained for diverse matching conditions. Derivation of the asymptotic expansion of the integrated heat kernel for a compound media is considered by making use of the perturbation series for the integral equations obtained. The method proposed is also applicable to the configurations when the same medium is divided, by a smooth compact surface, into internal and external regions, or when only the region inside (or outside) this surface is considered with appropriate boundary conditions.

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I. INTRODUCTION

The heat kernel technique\textsuperscript{1,2,3,4,5} is widely used for constructing the quantum field theory in gravitational background and with allowance for nontrivial boundary conditions. Of a particular interest is the asymptotic expansion of the heat kernel in terms of evolution parameter for its small values. The coefficients of this expansion pertain to divergences and anomalies in the relevant quantum field theory models. Proceeding from this one can develop the renormalization procedure needed.

For well posed spectral problems the heat kernel coefficients are expressed, in a polynomial way, through the local geometric characteristics of the manifold $D$ and its boundary $S$. Not only the contributions of $D$ and $S$ are independent but the contributions of individual regions of $D$ and $S$ are also additive.

The spectral problem is well posed for the goal of constructing the heat kernel if the second order elliptic differential operator in question is close to the Laplace operator defined on a smooth manifold with a smooth boundary, if any.

There are no universal methods for constructing the heat kernel and its asymptotic expansion. The development of different approaches to this problem is the subject of many works (see, for example, the reviews\textsuperscript{1,2,3,4,5} and references therein).

The initial definition of the heat kernel is the Green function of the heat-conduction equation with an elliptic operator under study. In many physical problems it is worth going from the differential equation, defining the solution to be found or the relevant Green’s function, to the equivalent integral equation. In the dynamical evolution problems the integral equations manifestly show the reason-consequence relations governing the physical process under study. Reducing the problem to the integral equation, as a rule, allows one to develop the method of successive approximations (perturbation theory).

Transforming the initial differential equation into the integral form is in the general case a nontrivial problem. A special attention should be paid here to incorporating the boundary conditions into the integral equation. When constructing the integral equations governing the Green’s function of the heat equation we shall use the surface potentials for this equation. The volume potential and the potentials of single and double layers naturally arise in the theory of the Laplace equation. In this case they are referred to as the Newtonian (or electrostatic) potentials. This idea proved to be fruitful also in studies of the Helmholtz
equation describing, for example, steady harmonic oscillations, the wave equation, and the heat-conduction equation. The potentials are particular solutions of these homogeneous equations, and they are constructed in a universal way in terms of the fundamental (or elementary) solution of the initial equation. The potential technique has turned out to be effective both for consideration of the general properties of the equations under study (for example, the prove of the solution uniqueness) and for deriving particular solutions with given properties and for obtaining the Green’s functions.

This paper seeks to demonstrate the efficiency of using the heat potentials when constructing the integral equations for the heat kernel (Green functions) at first in the case of manifolds with boundary. It is worth noting that integration in these equations is spread out over the boundary only. As a result the dimensionality of the initial problem is reduced by co-dimension of the boundary $S$. Further this approach is extended to the compound media, where the principal part of the differential operator has a discontinuity at the interface between different media. A typical example here is the electrodynamics of continuous media. The velocity of light has, in the general case, a jump discontinuity on the border between two media with different characteristics (for example, on the border between dielectric and vacuum). In the both media the Maxwell equations are well defined, and at the interface the matching conditions (or boundary conditions) should be satisfied. The concrete form of these conditions is determined by the physical content of the problem in question. In the same way the conduction of heat in compound media is treated. As far as we are aware, the heat kernel for compound media is not investigated yet.

The layout of the paper is as follows. In Sec. II the essentials of the potential theory are recalled first for the Laplace equation (Newtonian potentials) and then for the heat-conduction equation (heat potentials). By making use of the heat potentials the integral equation for the Green function (heat kernel) is derived for a compact region of Euclidean space bounded by a smooth surface. The perturbative series for this equation is developed which is nothing else as the multiple scattering expansion for the heat kernel. Thus a rigorous derivation of this expansion is presented. The convenience to use here the Laplace transform is shown. In Sec. III the integral equations are derived that determine the heat kernel for compound media. The efficiency of the approach proposed is demonstrated by deriving the $t$-small asymptotics of the first three terms of the perturbation series for the heat kernel in the case of compound media (Section IV). In Sec. V the heat kernel on an infinite line is
constructed in an exact form for diverse matching conditions. In the Conclusion (Section VI) the obtained results are briefly summarized and the possibility of extending the approach proposed is discussed. In Appendix A the general conditions at the interface are found which result in self-adjoint boundary value problem for the Laplace operator considered in compound media.

II. HEAT POTENTIALS

In order to recall the basic facts from the potential theory, we first address the Laplace equation

$$\Delta u = 0 \quad (2.1)$$

considered in the $d$-dimensional Euclidean space $\mathbb{R}_d$, which is divided by a smooth closed surface $S$ into a compact internal domain $D_+$ and external one $D_-$. On the surface $S$ the relevant boundary conditions should be imposed, that depend on the physical content of the problem in question. For example, it may be a mathematical formulation of the electrostatic problem. The surface $S$ is supposed to posses the properties of smoothness needed. In the potential theory this implies that $S$ is the Lyapunov type surface. The points of Euclidean space $\mathbb{R}_d$ are denoted by $x, y, z, \ldots$, and $r_{xy}$ is the Euclidean distance between $x$ and $y$. At any point $x$ on $S$ there exists a unit normal $n_x$ or $n(x)$. For definiteness we chose the inward directed normal. It is possible because we are dealing with closed surfaces $S$.

For a linear homogeneous differential equation the fundamental (or elementary) solution is defined, which is the Green function of this equation in an unbounded space. By definition, the fundamental solution obeys the initial equation with a $\delta$-like source on its right-hand side. In the case of the Laplace equation in $\mathbb{R}_d$ the fundamental solution is

$$E_d(x; x') = -\frac{\Gamma(d/2)}{(2d - 4) \pi^{d/2}} \left(\frac{r_{xx'}}{d + 2}\right)^{d+2}, \quad d \geq 3, \quad E_3(x; x') = -\frac{1}{4\pi r_{xx'}} \quad (2.2)$$

with the properties

$$\Delta_x E_d(x; x') = \Delta_{x'} E_d(x; x') = \delta^{(d)}(x, x'), \quad E_d(x; x') = E_d(x'; x). \quad (2.3)$$

The potentials for the Laplace equation are constructed by making use of the fundamental solution, namely, the volume potential

$$U(x) = -\frac{1}{4\pi} \int_D \frac{w(y)}{r_{xy}} \, dy, \quad (2.4)$$
the single-layer potential

\[ V(x) = -\frac{1}{4\pi} \int_S \frac{\nu(y)}{r_{xy}} dS_y, \]  

(2.5)

and the potential of a double layer

\[ W(x) = -\frac{1}{4\pi} \int_S \mu(y) \frac{\partial}{\partial n_y} \frac{1}{r_{xy}} dS_y. \]  

(2.6)

These formulas are written for \( d = 3 \), and the following notations are used: \( dy \) and \( dS_y \) are, respectively, the elements of the volume and of the surface at the point \( y \), \( w(y) \), \( \mu(y) \), and \( \nu(y) \) are the densities of these potentials. It is convenient to consider the densities \( w(y) \), \( \mu(y) \), and \( \nu(y) \) to be continuous functions.

All three potentials are solutions of the Laplace equation (2.1), namely, the volume potential (2.4) is harmonic outside \( D \), \( V(x) \) and \( W(x) \) are harmonic outside \( S \). The single-layer potential \( V(x) \) is continuous everywhere in \( \mathbb{R}^d \), specifically, on passing through \( S \). The potential of a double layer \( W(x) \) has a discontinuity on \( S \), namely:

\[ W_i(x) = W(x) - \frac{1}{2} \mu(x), \]

\[ W_e(x) = W(x) + \frac{1}{2} \mu(x), \quad x \in S. \]  

(2.7)

Here \( W(x) \) is the value of the integral (2.6), when the point \( x \) belongs to \( S \) (\( W(x) \) is a continuous function for \( x \) varying along \( S \)), \( W_i(x) \) is the value of the double-layer potential (2.6), when the point \( x \) tends to \( S \) from \( D_+ \), and \( W_e(x) \) is the same when the point \( x \) approaches at \( S \) from \( D_- \).

In what follows we shall frequently use the derivative along the normal to the surface \( S \) at the point \( y \), which belongs to \( S \). This derivative acts on the function the argument of which is the distance \( r_{xy} \) between the points \( x \) and \( y \), the point \( x \) being not obliged to lay on \( S \). Simple calculation gives

\[ \frac{\partial}{\partial n_y} r_{xy} = \cos \varphi, \]  

(2.8)

where \( \varphi \) is the angle between the vector \( r_{xy} \), which starts at \( x \) and ends at \( y \), and the normal \( n_y \). In the same way we have

\[ \frac{\partial}{\partial n_y} f(r_{xy}) = f'(r_{xy}) \cos \varphi, \quad \frac{\partial}{\partial n_y} \left( \frac{1}{r_{xy}} \right) = -\frac{\cos \varphi}{r_{xy}^2}. \]  

(2.9)

Equations (2.8) and (2.9) are obviously valid for both inward and outward directed normals.
The application of the potentials (2.4) – (2.7) for transforming the boundary problems for the Laplace equation (2.1) to the integral equations can be found in many textbooks on mathematical physics.

Let us proceed to consideration of the heat-conduction equation

$$\frac{\partial u}{\partial t} - a^2 \Delta u = 0. \quad (2.10)$$

The fundamental (or elementary) solution to this equation in $\mathbb{R}_d$ is

$$E_d(x,t;x',t') = \theta(t-t') K_0(x,t;x',t'), \quad (2.11)$$

where

$$K_0(x,t;x',t') = \frac{1}{(2a\sqrt{\pi(t-t')})^d} \exp \left[ -\frac{x'^2}{4a^2(t-t')} \right]. \quad (2.12)$$

The function $K_0(x,t;x',t')$ (the propagator or the heat kernel) obeys the homogeneous heat equations

$$\left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) K_0(x,t;x',t') = 0, \quad (2.13)$$

and inhomogeneous initial condition

$$K_0(x,t;x',t') \to \delta^{(d)}(x-x'), \quad \text{when} \quad t \to t'. \quad (2.14)$$

This condition enables one to construct the solution of the Cauchy problem for the nonhomogeneous heat equation

$$\left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) u(x,t) = f(x,t) \quad (2.15)$$

considered in an unbounded space

$$u(x,t) = \int dx' K_0(x,t;x',t') u_0(x') + \int_{t'}^t d\theta \int dx' K_0(x,t;x',\theta) f(x',\theta), \quad (2.16)$$

where $u_0(x) = u(x,t=t')$. In the classical mathematical physics, the representation (2.16) of the solution to the heat conduction equation (2.15) is known as the Poisson formula. The last term in (2.16) can be considered as an analog of Eq. (2.4) defining the volume potential for the Laplace equation.
In a complete agreement with the definitions (2.5) and (2.6) the heat surface potentials are introduced, namely, the simple-layer potential

\[ V(x, t) = a^2 \int_0^t d\theta \int_S dS_y K_0(x, t; y, \theta) \nu(y, \theta) \]  

(2.17)

and the potential of a double layer

\[ W(x, t) = a^2 \int_0^t d\theta \int_S dS_y \frac{\partial K_0}{\partial n_y}(x, t; y, \theta) \mu(y, \theta) \]

\[ = - \int_0^t \frac{d\theta}{2(t - \theta)} \int_S dS_y r_{xy} \cos \varphi K_0(x, t; y, \theta) \mu(y, \theta), \]  

(2.18)

where the functions \( \nu(y, \theta) \) and \( \mu(y, \theta) \) are the surface densities of these potentials. By the construction the heat potentials \( V(x, t) \) and \( W(x, t) \) vanish at \( t = 0 \).

For bounded density \( \nu(x, t) \) the heat potential of a single layer \( V(x, t) \) is continuous everywhere in \( \mathbb{R}^d \), also on passing through the surface \( S \), and satisfies the homogeneous heat-conduction equation (2.10) outside \( S \), i.e., it is parabolic outside \( S \). The normal derivatives of \( V(x, t) \) have jump discontinuities on \( S \). For continuous in \( S \) density \( \nu(x, t) \) these discontinuities are given by

\[
\begin{align*}
\left( \frac{\partial V(x, t)}{\partial n_x} \right)_i &= \frac{\partial V(x, t)}{\partial n_x} - \frac{1}{2} \nu(x, t), \\
\left( \frac{\partial V(x, t)}{\partial n_x} \right)_e &= \frac{\partial V(x, t)}{\partial n_x} + \frac{1}{2} \nu(x, t), \quad x \in S.
\end{align*}
\]  

(2.19)

For bounded density \( \mu(x, t) \) the heat potential of a double layer (2.18) is continuous everywhere outside of \( S \) (in \( \mathbb{R}^d \setminus S \)) and in \( S \). Outside \( S \) the potential \( W(x, t) \) is parabolic. On passing through \( S \) it has discontinuities. For continuous in \( S \) density \( \mu(x, t) \) these discontinuities are given by \( (d = 3) \)

\[
\begin{align*}
W_i(x, t) &= W(x, t) + \frac{1}{2} \mu(x, t), \\
W_e(x, t) &= W(x, t) - \frac{1}{2} \mu(x, t), \quad x \in S.
\end{align*}
\]  

(2.20)

The normal derivatives of the double-layer potential are continuous on passing through \( S \).

The employment of the Newtonian and heat potentials for transforming the boundary-value problems for Laplace and heat equations into the integral ones is based on the discontinuity properties on the boundary of the double layer potential and the normal derivatives of the single layer potential. Let us consider a simple example, namely, construction of the
solution to the Dirichlet problem for the heat equation in a compact domain $D$ bounded by a smooth surface $S$:

$$
\left( \frac{\partial}{\partial t} - a^2 \Delta \right) u(x,t) = 0, \quad x \in D \quad t > 0,
$$

$$
u(x,0) = 0, \quad x \in D,
$$

$$
u(x,t) = \psi(x,t), \quad x \in S, \quad t > 0,
$$

where the function $\psi(x,t)$ specifies the temperature on the boundary $S$ at different time instants $t$. We shall look for the solution $u(x,t)$ in terms of the heat potential of a double layer (2.18)

$$
u(x,t) = W(x,t).
$$

With account of Eq. (2.20) we have on the boundary $S$

$$
\frac{1}{2} \mu(x,t) = -a^2 \int_{t}^{t'} dt' \int_{S} dS_{x'} \frac{\partial K_0}{\partial n_{x'}}(x,t;x',t') \mu(x',t') + \psi(x,t), \quad x, x' \in S.
$$

Thus, the problem under consideration is reduced to the solution of the linear integral equation of the second kind. With respect to the variable $t$ these equations are of the Volterra type and with respect to the spatial variables $x$ and $x'$ they are of the Fredholm type, the variables $x$ and $x'$ ranging on the boundary $S$.

In an analogous way the integral equations for the Green’s function of the heat equation can be deduced. Let us consider this technique in the case of the first boundary-value problem (the Dirichlet problem) for this equation. The Green function $K(x,t;x',t')$ is specified by the following conditions: it should satisfy the homogeneous heat equation with respect to the first pair of its arguments

$$
\left( \frac{\partial}{\partial t} - a^2 \Delta_x \right) K(x,t;x',t') = 0,
$$

it should obey the inhomogeneous initial condition

$$
K(x,t;x',t) = \delta(x,x'), \quad t \geq 0,
$$

and the homogeneous boundary condition

$$
K(x,t;x',t') = 0, \quad x \in S.
$$

We represent the Green function $K(x,t;x',t')$ as the sum of a free propagator and the heat potential of a double layer

$$
K(x,t;x',t') = K_0(x,t;x',t') + a^2 \int_{t'}^{t} d\theta \int_{S} dS_{y} \frac{\partial K_0}{\partial n_{y}}(x,t;y,\theta) \mu(y,\theta;x',t').
$$
The right-hand side of this equation obviously satisfies Eq. (2.24). When \( t = t' \) the double layer potential in (2.27) (the second term) vanishes. The free propagator in this formula \( K_0 \) enables one to obey the initial condition (2.25). The density of the double layer potential \( \mu(y, \theta; x', t') \) is determined from the boundary condition (2.26). On substituting Eq. (2.27) into (2.26) the following integral equation is obtained for the potential density \( \mu \)

\[
\frac{1}{2} \mu(x, t; x', t') = -K_0(x, t; x', t') - a^2 \int_{\nu} d\theta \int_S dS_y \frac{\partial K_0}{\partial n_y}(x, t; y, \theta) \mu(y, \theta; x', t'), \quad x \in S.
\] (2.28)

For Eq. (2.28) and consequently for Eq. (2.27) the method of successive approximations can be developed, as Eq. (2.28) is an integral equation of the second kind. It has been proved, that the series arising here is uniformly convergent. The first terms of this series for Eq. (2.27) are

\[
K(x, t; x', t') = K_0(x, t; x', t') + (-2a^2)^1 \int_{\nu} d\theta \int_S dS_y \frac{\partial K_0}{\partial n_y}(x, t; y, \theta) K_0(y, \theta; x', t')
\]

\[
+ (-2a^2)^2 \int_{\nu} d\theta \int_S dS_y \frac{\partial K_0}{\partial n_y}(x, t; y, \theta) \int_{\nu} d\theta_1 \int_S dS_{y_1} \frac{\partial K_0}{\partial n_{y_1}}(y, \theta; y_1, \theta_1) K_0(y_1, \theta_1; x', t') + \ldots . \quad (2.29)
\]

Obviously, this series is a result of successive approximations applied to the integral equation for the complete propagator

\[
K(x, t; x', t') = K_0(x, t; x', t') - 2a^2 \int_{\nu} d\theta \int_S dS_y \frac{\partial K_0}{\partial n_y}(x, t; y, \theta) K(y, \theta; x', t'). \quad (2.30)
\]

The series (2.29) is nothing else as the multiple scattering expansion for the heat kernel in the problems under consideration. Thus we have derived this expansion in a rigorous way.

By making use of the Laplace transform in Eq. (2.29) one can remove the integrations over the intermediate time variables \( \theta \)'s

\[
\bar{K}(x, x'; p) = K_0(r_{xx'}; p) + (-2a^2)^1 \int_S dS_y K_1(r_{xy}; p) K_0(r_{yx'}; p)
\]

\[
+ (-2a^2)^2 \int_S dS_y \bar{K}_1(r_{xy}; p) \int_S dS_{y_1} \bar{K}_1(r_{y_1 y}; p) K_0(r_{y_1 x'}; p) + \ldots , \quad (2.31)
\]

where

\[
\bar{K}(x, x'; p) = \int_0^\infty e^{-pt} K(x, t; x', 0) \, dt, \quad (2.32)
\]
\[
\tilde{K}_0(x,y;p) = \int_0^\infty e^{-pt} K_0(x,t;0) \, dt = \frac{1}{2\pi a^2} K_0 \left( \frac{r_{xy}}{a} \sqrt{p} \right), \tag{2.33}
\]
\[
\tilde{K}_1(x,y;p) = \int_0^\infty e^{-pt} \frac{\partial K_0}{\partial n_y}(x,t;0) \, dt = -\frac{\cos \varphi}{2\pi a^2 \sqrt{p}} K_1 \left( \frac{r_{xy}}{a} \sqrt{p} \right), \quad y \in S. \tag{2.34}
\]

In Eq. (2.34) \( \varphi \) is the angle between the vectors \( r_{xy} \) and \( n_y \). The Laplace transforms \( \tilde{K}_0 \) and \( \tilde{K}_1 \) are calculated for \( d = 2 \). They are expressed in terms of the modified Bessel functions \( K_0(z) \) and \( K_1(z) \). We hope that our notations will not lead to confusion because the free propagator \( K_0(x,t;x',t') \) and the Bessel function \( K_0(z) \) have different number of the arguments. The series (2.31) is the perturbative solution to the following integral equation

\[
\tilde{K}(x,x';p) = \tilde{K}_0(r_{xx};p) - 2a^2 \int_S dS_y \tilde{K}_1(r_{xy};p) \tilde{K}(r_{yx};p). \tag{2.35}
\]

The series (2.29) or (2.31) contains complete information about the Green function (heat kernel) in the problem at hand. However, extracting it from here is not a simple task.

The second term at the right hand side of Eq. (2.31) is responsible for one ‘reflection’ from the boundary (the Born approximation). Its contribution into the heat kernel can be expressed in terms of the confluent hypergeometric function \( W_{\alpha/3} \). By making use of the convolution theorem for the Laplace transform \( W_{\alpha/3} \), we obtain

\[
\tilde{K}^{(1)}(x,x';p) = \frac{2a^2 \sqrt{p}}{(2\pi)^2 a^5} \int_S dS_y \cos \varphi K_0 \left( \frac{r_{xy}}{a} \sqrt{p} \right) K_1 \left( \frac{r_{yx'}}{a} \sqrt{p} \right). \tag{2.36}
\]

The inverse Laplace transform gives

\[
K^{(1)}(x,x;t) = \frac{1}{2^{5/2} \pi^{3/2} a^2 t} \int_S dS_y \cos \varphi \frac{1}{r_{xy}} \exp \left( -\frac{r_{xy}^2}{2 a^2 t} \right) W_{\frac{3}{2}} \left( \frac{r_{xy}^2}{a^2 t} \right), \tag{2.37}
\]

where \( \varphi \) is the angle between the vector \( r_{xy} \) and the inward directed normal to the boundary \( S \) at the point \( y \).

The perturbation series (2.31) can be employed, for example, to find the asymptotic expansion of the heat kernel trace.

### III. COMPOUND MEDIA

An important advantage of the heat potential technique for constructing the integral equations is the possibility of applying it to compound media. We show this by considering first the solution of the heat-conduction equation instead of the relevant Green’s function.
Thus, in both the regions $D_+$ and $D_-$ the heat equations are defined

\[
\left( \frac{\partial}{\partial t} - a_+^2 \Delta \right) u_+(x,t) \equiv \hat{T}_{tx}(a_+) u_+(x,t) = 0, \quad x \in D_+, \quad (3.1)
\]

\[
\left( \frac{\partial}{\partial t} - a_-^2 \Delta \right) u_-(x,t) \equiv \hat{T}_{tx}(a_-) u_-(x,t) = 0, \quad x \in D_-
\]

(3.2)

with the matching conditions at the interface $S$, namely, when crossing $S$ the following quantities should be continuous: temperature

\[
u_+(x,t) = u_-(x,t), \quad x \in S
\]

(3.3)

and heat current

\[
\lambda_+ \frac{\partial u_+(x,t)}{\partial n_+(x)} + \lambda_- \frac{\partial u_-(x,t)}{\partial n_-(x)} = 0, \quad x \in S,
\]

(3.4)

where $n_+(x)$ and $n_-(x)$ are inward normals to the surface $S$ at the point $x$ for the regions $D_+$ and $D_-$, respectively. These matching conditions imply, in particular, that there are no heat sources on $S$. The parameters $a_+, a_-, \lambda_+$, and $\lambda_-$ specify the material characteristics of the media.

We shall look for the solution to this problem in terms of the heat potentials of single layer and double layer. Here the following feature proves to be important. If the solution $u_+(x,t)$ in the internal region $D_+$ is represented as the heat potential of a single layer

\[
u_+(x,t) = a_+^2 \int_0^t dt' \int_S dS_{x'} K_0^{(+)}(x,t; x', t') \nu(x', t'),
\]

(3.5)

then the solution $u_-(x,t)$ in the external region $D_-$ should be looked for in terms of the heat potential of a double layer

\[
u_-(x,t) = a_-^2 \int_0^t dt' \int_S dS_{x'} \frac{\partial K_0^{(-)}}{\partial n_-(x')} (x,t; x', t') \mu(x', t'),
\]

(3.6)

where $K_0^{(+)}$ and $K_0^{(-)}$ are the fundamental solutions of the heat equations (3.1) and (3.2), which are defined by the formula (2.12) with $a = a_+$ and $a = a_-$, respectively.

Substituting Eqs. (3.5) and (3.6) into the first matching condition (3.3) we obtain

\[
a_+^2 \int_0^t dt' \int_S dS_{x'} K_0^{(+)}(x,t; x', t') \nu(x', t')
\]

\[
= \frac{1}{2} \mu(x,t) + a_-^2 \int_0^t dt' \int_S dS_{x'} \frac{\partial K_0^{(-)}}{\partial n_-(x')} (x,t; x', t') \mu(x', t'), \quad x, x' \in S.
\]

(3.7)
The second matching condition (3.4) results in another integral equation

\[ \lambda_+ a_+^2 \int_0^t dt' \int_S dS x' \frac{\partial K_0^{(+)}(x, t; x', t')}{\partial n_+(x)} \nu(x', t') - \frac{1}{2} \lambda_+ \nu(x, t) \]

\[ + \lambda_- a_-^2 \int_0^t dt' \int_S dS x' \frac{\partial^2 K_0^{(+)}(x, t; x', t')}{\partial n_-(x) \partial n_-(x')} \mu(x', t') = 0, \quad x, x' \in S. \quad (3.8) \]

Thus the problem under consideration is reduced to the solution of the system of two linear integral equations of the second kind (3.7) and (3.8). It is worth noting that we have obtained homogeneous equations, because there are no any heat sources in the problem under study. Hence, we are dealing here with the eigenfunctions only.

Let us proceed to the Green’s function \( K(x, t; x', t') \) in this problem. In what follows, it is convenient to represent this function in terms of the following four components depending on the range of the arguments:

\[ K(x, t; x', t') = \begin{cases} 
K_{++}(x, t; x', t'), & x, x' \in D_+, \\
K_{+-}(x, t; x', t'), & x \in D_+, x' \in D_-, \\
K_{-+}(x, t; x', t'), & x \in D_-, x' \in D_+, \\
K_{--}(x, t; x', t'), & x, x' \in D_. 
\end{cases} \quad (3.9) \]

The conditions, which specify the Green function \( K(x, t; x', t') \), can be found in the following way. This function should provide the solution \( \bar{u}(x, t) \) of the inhomogeneous boundary-value problem (3.1), (3.2), (3.3), and (3.4) with the heat source \( f(x, t) \) in the form

\[ \bar{u}(x, t) = -\theta(t - t') \int_{t'}^t d\tau \int_{\mathbb{R}_d} K(x, t; \xi, \tau) f(\xi, \tau) d\xi, \quad (3.10) \]

where \( \theta(t - t') \) is the step function. The function \( \bar{u}(x, t) \) will satisfy the inhomogeneous heat-conduction equation

\[ f(x, t) = \begin{cases} 
\hat{T}_{tx}(a_+) \bar{u}(x, t), & x \in D_+, \\
\hat{T}_{tx}(a_-) \bar{u}(x, t), & x \in D_- 
\end{cases} \quad (3.11) \]

if the Green function \( K(x, t; x', t') \) obeys the corresponding homogeneous heat equations with respect to the first pair of its arguments

\[ \hat{T}_{tx}(a_+) K(x, t; x', t') = 0, \quad x \in D_+, \]

\[ \hat{T}_{tx}(a_-) K(x, t; x', t') = 0, \quad x \in D_- \quad (3.12) \]
and the inhomogeneous initial condition (3.25) with respect to the both pairs of its arguments. In terms of the components (3.9) the initial condition (2.25) acquires the form

\[ K_{++}(x, t; x', t) = \delta(x, x'), \quad x, x' \in D_+, \]  
\[ K_{--}(x, t; x', t) = \delta(x, x'), \quad x, x' \in D_-, \]  
\[ K_{+-}(x, t; x', t) = K_{-+}(x, t; x', t) = 0. \]  

(3.13)  
(3.14)  
(3.15)

On the right-hand side of the initial conditions (3.15) the delta function \( \delta(x, x') \) with \( x, x' \in S \) is absent. Thus in our consideration we eliminate the treatment of heat sources at the interface. The point is such sources alter the matching conditions instead of the initial conditions (3.4). In the next section the solution to the heat conduction equation, defined on a line, will be constructed for such a configuration by making use of the heat potential technique.

The matching conditions (3.3) and (3.4) are directly transformed into the conditions for the Green function \( K(x, t; x', t') \) with respect to the first pair of its arguments

\[ K_{++}(x, t; x', t') = K_{--}(x, t; x', t') \]  
\[ \lambda_+ \frac{\partial K_{++}}{\partial n_+}(x, t; x', t') + \lambda_- \frac{\partial K_{--}}{\partial n_-}(x, t; x', t') = 0, \]  
\[ K_{+-}(x, t; x', t') = K_{-+}(x, t; x', t') \]  
\[ \lambda_+ \frac{\partial K_{+-}}{\partial n_+}(x, t; x', t') + \lambda_- \frac{\partial K_{-+}}{\partial n_-}(x, t; x', t') = 0, \]  
\[ x \in S. \]  

(3.16)  
(3.17)  
(3.18)  
(3.19)

It turns out that the heat equations (3.12), the initial conditions (3.13) – (3.15), and the matching conditions (3.16) – (3.19) are enough for construction of the Green function in the case of compound media in a unique way.

We shall look for the components of the Green function (3.9) in terms of the heat potentials of a single and double layers with respect to the first pair of their arguments, the components \( K_{++} \) and \( K_{+-} \) being expressed through the heat potentials of single layers and the components \( K_{--} \) and \( K_{-+} \) through the heat potentials of double layers. In order to take into account the inhomogeneous initial conditions (3.13) and (3.14) for the components \( K_{++} \) and \( K_{--} \) we add to the chosen heat potentials (nonsingular part of the Green function) the free propagator \( K_0^{(+)} \) or \( K_0^{(-)} \) (singular part of this function)

\[ K_{++}(x, t; x', t') = K_0^{(+)}(x, t; x', t') + a_+^2 \int_{t'}^t d\theta \int_S dS_y K_0^{(+)}(x, t; y, \theta) \nu_{++}(y, \theta; x', t'). \]  

(3.20)
\[ K_{+-}(x, t; x', t') = a_+^2 \int_{t'}^t d\theta \int_S dS_y K_0^{(+)}(x, t; y, \theta) \nu_{+-}(y, \theta; x', t'), \quad (3.21) \]

\[ K_{--}(x, t; x', t') = K_0^{(-)}(x, t; x', t') + a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial K_0^{(-)}}{\partial n_-(y)}(x, t; y, \theta) \nu_{--}(y, \theta; x', t'), \quad (3.22) \]

\[ K_{-+}(x, t; x', t') = a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial K_0^{(-)}}{\partial n_-(y)}(x, t; y, \theta) \nu_{-+}(y, \theta; x', t'). \quad (3.23) \]

The matching conditions (3.16) and (3.17) give

\[ K_0^{(+)}(x, t; x', t') + a_+^2 \int_{t'}^t d\theta \int_S dS_y K_0^{(+)}(x, t; y, \theta) \nu_{++}(y, \theta; x', t') \]

\[ = \frac{1}{2} \nu_{--}(x, t; x', t') + a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial K_0^{(-)}}{\partial n_-(y)}(x, t; y, \theta) \nu_{+-}(y, \theta; x', t'), \quad (3.24) \]

\[ \lambda_+ \frac{\partial K_0^{(+)}}{\partial n_+(x)}(x, t; x', t') + \lambda_+ a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial K_0^{(+)}}{\partial n_+(y)}(x, t; y, \theta) \nu_{++}(y, \theta; x', t') \]

\[ -\frac{1}{2} \lambda_+ \nu_{++}(y, \theta; x', t') + \lambda_- a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial^2 K_0^{(-)}}{\partial n_-(x) \partial n_-(y)}(x, t; y, \theta) \nu_{--}(y, \theta; x', t') = 0, \quad (3.25) \]

In the same way we deduce from (3.18) and (3.19)

\[ a_+^2 \int_{t'}^t d\theta \int_S dS_y K_0^{(+)}(x, t; y, \theta) \nu_{+-}(y, \theta; x', t') \]

\[ = K_0^{(-)}(x, t; x', t') + \frac{1}{2} \nu_{--}(x, t; x', t') + a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial K_0^{(-)}}{\partial n_-(y)}(x, t; y, \theta) \nu_{--}(y, \theta; x', t'), \quad (3.26) \]

\[ -\frac{1}{2} \lambda_+ \nu_{+-}(x, t; x', t') + \lambda_+ a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial K_0^{(+)}}{\partial n_+(x)}(x, t; x', t') \nu_{++}(y, \theta; x', t') \]

\[ + \lambda_- a_+^2 \int_{t'}^t d\theta \int_S dS_y \frac{\partial^2 K_0^{(-)}}{\partial n_-(x) \partial n_-(y)}(x, t; y, \theta) \nu_{--}(y, \theta; x', t') + \lambda_- \frac{\partial K_0^{(-)}}{\partial n_-(x)}(x, t; x', t') = 0, \quad (3.27) \]

The sets of integral equations of the second kind (3.24), (3.25) and (3.26), (3.27) define the heat kernel for compound media in full. With respect to spatial variables these equations are of Fredholm type while regarding time variable they are of Volterra type. It is essential that the integration over the spatial variables is restricted by the interface \( S \) only. Hence the dimension of the initial problem is reduced by 1. By making use of the Laplace transform one can remove the integration over the time variables in Eqs. (3.24) – (3.27) as it has been done in Sec. II.
Obviously the integral equations for the heat kernel derived here can be also applied when the surface $S$ divides the same medium into the regions $D_+$ and $D_+$, i.e. when the constants $a_+^2$ and $a_-^2$ equal.

For constructing the solutions to the integral equations (3.24) – (3.27) the perturbation theory can be employed (see Sec. II). The expansion parameters in this case prove to be the constants $a_+^2$ and $\lambda_\pm$. The perturbation series generated here are nothing else as the multiple scattering expansion for the heat kernel. Thus we have proposed a rigorous derivation of these expansions both for homogeneous media and compact regions and for compound media.

IV. ASYMPTOTIC EXPANSION OF HEAT KERNEL FROM PERTURBATION SERIES

In practical applications, especially in QFT, the asymptotic expansion of the integrated heat kernel when $t \to +0$ proves to be important. It has the form

$$K(t) \equiv \int dxK(x, t; x, 0) = (4\pi t)^{-d/2} \sum_{n=0,1,2,...} t^{n/2} B_{n/2} + ES. \quad (4.1)$$

In this expansion $d$ is the dimension of the configuration space and ES stands for the exponentially small corrections as $t \to +0$. We show how to derive this expansion proceeding from the perturbation series for the integral equations (3.24) – (3.27). The functions $K_{-+}$ and $K_{+-}$ do not contribute to integrated heat kernel (see subsection V C), thus we have to consider only $K_{++}$ and $K_{--}$.

For our purposes it is convenient to use such coordinates that in the vicinity of the surface $S$ the metric is $g_{ij} dx^i dx^j = (dx^3)^2 + g_{ab} dx^a dx^b$ where $x^3$ is a coordinate on the normal to $S$, $x^3=0$ on $S$. In view of the exact form of the free propagator (2.12), one can infer that in each term of the perturbation series for integral equations (3.24) – (3.27) the power in $t$ contributions are given only at the following conditions: when evaluating the heat kernel trace, the integration over $dx$ should be spread over the region immediately adjacent to the boundary $S$ and in the course of the multiple integration over the boundary $S$ the respective distances $r_{yy'}$ should be also small. Therefore in the vicinity of $S$ we may replace the squared distance $(x - z)^2$ by several terms of its expansion in powers of the corresponding geodesic
distance $\sigma$ on the surface $S$

$$(x-z)^2 = (x_3-z_3)^2 + \sigma^2 \{1 - (x_3 + y_3)k_1 + x_3 z_3(k_1^2 + k_2^2)\}$$

$$+ \sigma^2 \left\{-\frac{1}{3}(2z_3 + x_3)k_1' + x_3 z_3(k_1'k_1 + k_2'k_1)\right\} + \ldots \quad (4.2)$$

$$k_1 = L_{ab}\xi^a\xi^b, \quad k_2 = \frac{1}{2}(\varepsilon_{ac}L^c_b + \varepsilon_{bc}L^c_a)\xi^a\xi^b, \quad \varepsilon_{ac} = -\varepsilon_{ac}, \quad k_1' \equiv \frac{dk_1}{d\sigma}, \quad k_2' \equiv \frac{dk_2}{d\sigma}.$$  

The surface area element is $dS = (1 - \frac{1}{12}R_{ab}c^a\xi^b\sigma^2 + \ldots)\sigma d\sigma d\Omega, \Omega$ parameterizes a unit sphere, $L_{ab}$ is the second fundamental form on $S$, $R_{ab}$ is intrinsic Ricci curvature, $\xi$ is a unit tangent vector at $x$ to the geodesics with the length $\sigma$ joining $z$ to $x$ on $S$ (see, for example, Ref. 23).

Here we present the first three terms of the perturbation series under consideration when $t \to +0 \ (d = 3)$

$$K^{(0)}(t) = K^{(0)}_{++}(t) + K^{(0)}_{--}(t) = \frac{t^{-3/2}}{(4\pi a_+^2)^{3/2}}D_+ + \frac{t^{-3/2}}{(4\pi a_-^2)^{3/2}}D_-,$$

$$K^{(1)}_{++}(t) = \frac{t^{-1}S}{8\pi a_+^2} + \frac{t^{-1/2}}{8\pi^{3/2}a_+} \int_S dS L_+^a + \frac{t^0}{2^8\pi} \int_S dS \left[5(L_+^a)^2 + L_+^a L_+^b - \frac{2}{3}R_+^a\right] + \ldots,$$

$$K^{(2)}_{++}(t) = -\frac{t^{-1}}{8\pi} \frac{S}{\lambda_+ a_+^2} - \frac{1}{8\pi^{3/2}} \lambda_+ t^{-1/2} \int_S dS L_+^a$$

$$+ \frac{t^0}{32\pi} \left\{a_+ \frac{\lambda_-}{\lambda_+} (a_+ + a_-) \frac{a_-}{a_+ + a_-} \int S dS \left[-(L_+^a)^2 + 4L_+^b L_+^b - \frac{1}{3}R_+^a\right] + \frac{1}{8} - \frac{1}{\lambda_+ (a_+ + a_-)^2} \frac{35}{12}a_-^3 + \frac{11}{4}a_-^2 a_+ + 2a_+^2 a_-$$

$$+ \frac{2}{3}a_+^3 + \frac{9}{4}a_+^4 + \frac{3}{4}a_+^5\right\} \int S dS \left[(L_+^a)^2 + 2L_+^b L_+^b\right]\right\} + \ldots \quad (4.3)$$

To obtain $K^{(1)}_{--}(t)$ and $K^{(2)}_{--}(t)$ one should replace $a_+ \leftrightarrow a_-, \lambda_+ \leftrightarrow \lambda_-, L_+^b \to -L_+^b$. The asymptotics of the subsequent terms of perturbation series may be found in a similar way. After that all factors appearing with the same powers of $t$ are added up to give the heat kernel coefficients. The latter are expressed through the integrals of the surface geometric invariants. The asymptotics (4.3) were presented in Ref. 24 without considering the derivation of the relevant integral equations.

**V. HEAT KERNEL ON A LINE**

In this section we demonstrate the efficiency of our approach based on integral equations for constructing the heat kernel on a line. In this case the interface between the media
reduces to a point. As a result we are dealing with the Volterra integral equations in respect of one (time) variable. These equations are of a special type (Abel equations), and their solutions can be found in an exact form.

A. Homogeneous media with gluing conditions

By making use of the heat potential technique we construct here, in an exact form, the heat kernel $K(x, y; t)$ for the Laplace operator on an infinite line for homogeneous medium with a nonstandard gluing conditions at the origin (these conditions will be specified below). From the physical point of view $K(x, y; t)$ is the temperature at the point $x$ which is generated by a unit instantaneous heat source placed at the point $y$ at the moment $t = 0$.

As in previous sections we first formulate the conditions that define the heat kernel in the problem under consideration. With respect to the first argument $K(x, y; t)$ should satisfy the one-dimensional heat-conduction equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) K(x, y; t) = 0, \quad t > 0, \quad -\infty < x < \infty$$

and special conditions at the interface $x = 0$

$$l K(-0, y; t) = l^{-1} K(+0, y; t),$$

$$l^{-1} \frac{\partial}{\partial x} K(x, y; t) \bigg|_{x=-0} = l \frac{\partial}{\partial x} K(x, y; t) \bigg|_{x=+0},$$

where $l$ is a dimensionless parameter. We shall refer to these conditions as to gluing ones. In the Appendix A it is shown that such conditions lead to a selfadjoint spectral problem for the Laplace operator in any dimension. The initial condition for $K(x, y; t)$ involves its both space arguments

$$K(x, y; 0) = \delta(x, y).$$

We shall seek for $K(x, y; t)$ in terms of free heat kernel and single layer heat potentials. The solution is decomposed in four components related to different positions of the heat source and the observer

$$K_{-+}(x, y; t) = \int_0^t d\tau K_0(x, 0; t - \tau) \alpha_1(\tau, y), \quad x < 0, \quad y > 0,$$

$$K_{++}(x, y; t) = K_0(x, y; t) + \int_0^t d\tau K_0(x, 0; t - \tau) \alpha_2(\tau, y), \quad x, y > 0,$$
$$K_+(x, y; t) = \int_0^t d\tau K_0(x, 0; t - \tau) \alpha_3(\tau, y), \quad x > 0, \; y < 0,$$

$$K_-(x, y; t) = K_0(x, y; t) + \int_0^t d\tau K_0(x, 0; t - \tau) \alpha_4(\tau, y), \quad x, \; y < 0,$$

where \(K_0(x, y; t)\) is the free heat kernel (propagator) on an infinite line

$$K_0(x, y; t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}}.$$  \(5.9\)

First we substitute Eqs. (5.5) and (5.6) into the gluing conditions (5.2) and (5.3). Then we take into account that the single layer potential changes smoothly across the boundary, while the normal derivative of this potential undergoes a jump

$$K_+(0, y; t) = \frac{1}{2\sqrt{\pi}} \int_0^t d\tau \frac{\alpha_1(y, \tau)}{\sqrt{t - \tau}},$$

$$K_-(0, y; t) = K_0(0, y; t) + \frac{1}{2\sqrt{\pi}} \int_0^t d\tau \frac{\alpha_2(y, \tau)}{\sqrt{t - \tau}},$$

$$\frac{\partial}{\partial x} K_+(x, y; t) \bigg|_{x=0} = -\frac{1}{2} \alpha_1(y, t),$$

$$\frac{\partial}{\partial x} K_-(x, y; t) \bigg|_{x=0} = -\frac{1}{2} \alpha_2(y, t) + \frac{\partial}{\partial x} K_0(x, y; t) \bigg|_{x=0}.$$  \(5.13\)

Inserting Eqs. (5.10) and (5.11) into Eq. (5.2) one obtains the Abel integral equation

$$\frac{1}{2\sqrt{\pi}} \int_0^t d\tau \frac{1}{\sqrt{t - \tau}} \left[l^2 \alpha_1(y, \tau) - \alpha_2(y, \tau)\right] = K_0(0, y; t)$$

with the solution

$$\alpha_1(y, t) - l^2 \alpha_2(y, t) = \frac{2}{l^2 \sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{K_0(0, y; \tau)}{\sqrt{t - \tau}} d\tau = \frac{1}{2\sqrt{\pi l^2}} \frac{y}{t^{3/2}} e^{-\frac{y^2}{4t}}.$$  \(5.15\)

The substitution of Eqs. (5.12) and (5.13) into Eq. (5.3) gives

$$\alpha_1(y, t) = -l^2 \alpha_2(y, t) + \frac{l^2}{2\sqrt{\pi}} \frac{y}{t^{3/2}} e^{-\frac{y^2}{4t}}.$$  \(5.16\)

From (5.15) and (5.16) it follows that

$$\alpha_1(y, t) = \frac{l^2}{l^4 + 1} \frac{y}{\sqrt{\pi t^{3/2}}} e^{-\frac{y^2}{4t}}, \quad \alpha_2(y, t) = \frac{l^4}{l^4 + 1} \frac{y}{2\sqrt{\pi t^{3/2}}} e^{-\frac{y^2}{4t}}.$$  \(5.17\)

And finally

$$K_+(x, y; t) = \frac{l^2}{l^4 + 1} \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}},$$

$$K_-(x, y; t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} + \frac{l^4 - 1}{l^4 + 1} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x+y)^2}{4t}}.$$  \(5.19\)
In a similar way one gets

\[
K_{+-}(x, y; t) = \frac{l^2}{l^4 + 1} \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}},
\]

\[
K_{-+}(x, y; t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} - \frac{l^4 - 1}{l^4 + 1} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x+y)^2}{4t}},
\]

(5.20) (5.21)

These formulas are in complete agreement with the result of a combined employment of the Lemma 5.2 argued in Ref. 25 and Lemma 4.1 from Ref. 26.

\[
K_{++}(x, y; t) = \cos^2 \theta K_N(x, y; t) + \sin^2 \theta K_D(x, y; t),
\]

\[
K_{--}(x, y; t) = \sin^2 \theta K_N(x, y; t) + \cos^2 \theta K_D(x, y; t),
\]

\[
K_{+-}(x, y; t) = \sin \theta \cos \theta \left[ K_N(x, y; t) - K_D(x, y; t) \right],
\]

(5.22) (5.23) (5.24)

where

\[
\cos^2 \theta = \frac{l^4}{l^4 + 1}, \quad \sin^2 \theta = \frac{1}{l^4 + 1},
\]

and \(K_D(x, y; t)\) and \(K_N(x, y; t)\) are the heat kernels for Dirichlet and Neumann boundary conditions, respectively.

**B. Dielectric-like conditions on a line**

We construct here the heat kernel for the Laplace operator defined on an infinite line with dielectric-like matching conditions at the origin \(x = 0\). The heat kernel is defined by the heat conduction equation

\[
P \frac{\partial}{\partial t} - a^2(x) \frac{\partial^2}{\partial x^2} K(x, y; t) = 0, \quad t > 0, \quad -\infty < x < +\infty, \quad x \neq 0,
\]

(5.25)

where

\[
a^2(x) = \begin{cases} \ a_+^2, & x < 0, \\ \ a_-^2, & x > 0, \end{cases}
\]

\(a_+^2\) and \(a_-^2\) being positive constants. At the interface of dielectric media the matching conditions

\[
K(-0, y; t) = K(+0, y; t),
\]

\[
\lambda_- \frac{\partial}{\partial x} K(x, y; t) \bigg|_{x=-0} = \lambda_+ \frac{\partial}{\partial x} K(x, y; t) \bigg|_{x=+0}
\]

(5.26) (5.27)

should be met. As usual, the initial condition for \(K(x, y; t)\) is given by [54].
We call the boundary conditions (5.26) and (5.27) the dielectric-like conditions. The use of this term requires some explanations. When two dielectric media $D_+$ and $D_-$ possessing different characteristics are separated by the interface $S$ of an arbitrary form then in Maxwell theory we have on the surface $S$ the set of coupled boundary conditions involving all the components of the electromagnetic potential $A_\mu(t, x), \, \mu = 0, \ldots, d$. If one disregards the vector character of the electromagnetic field and confine oneself to oscillations described by a sole scalar potential (for example, sound waves which are described by a scalar velocity potential) then at the interface between different media the conditions (5.26) and (5.27) should be satisfied. In other words, these boundary conditions hold in the theory of scalar ‘photons’ in compound media.

Again we seek for the solution in terms of a relevant free propagator and single layer heat potentials:

$$K_{-+}(x, y; t) = a_-^2 \int_0^t d\tau K_0(x, 0; a_-^2(t - \tau)) \beta_1(\tau, y), \, x < 0, y > 0, \quad (5.28)$$

$$K_{++}(x, y; t) = K_0(x, 0; a_+^2t) + a_+^2 \int_0^t d\tau K_0(x, 0; a_+^2(t - \tau)) \beta_2(\tau, y), \, x > 0, \quad (5.29)$$

$$K_{+-}(x, y; t) = a_+^2 \int_0^t d\tau K_0(x, 0; a_+^2(t - \tau)) \beta_3(\tau, y), \, x > 0, y < 0, \quad (5.30)$$

$$K_{--}(x, y; t) = K_0(x, 0; a_-^2t) + a_-^2 \int_0^t d\tau K_0(x, 0; a_-^2(t - \tau)) \beta_4(\tau, y), \, x < 0. \quad (5.31)$$

First we insert Eqs. (5.28) and (5.29) into the matching conditions (5.20) and (5.21). The single layer potential changes smoothly across the boundary, while its normal derivative undergoes a jump

$$K_{-+}(-0, y; t) = \frac{a_-}{2\sqrt{\pi}} \int_0^t d\tau \frac{\beta_1(y, \tau)}{\sqrt{t - \tau}}, \quad (5.32)$$

$$K_{++}(+0, y; t) = K_0(0, y; a_+^2t) + \frac{a_+}{2\sqrt{\pi}} \int_0^t d\tau \frac{\beta_2(y, \tau)}{\sqrt{t - \tau}}, \quad (5.33)$$

$$\left. \frac{\partial}{\partial x} K_{-+}(x, y; t) \right|_{x=-0} = -\frac{1}{2} \beta_1(y, t), \quad (5.34)$$

$$\left. \frac{\partial}{\partial x} K_{++}(x, y; t) \right|_{x=+0} = -\frac{1}{2} \beta_2(y, t) + \left. \frac{\partial}{\partial x} K_0(x, y; a_+^2t) \right|_{x=0}. \quad (5.35)$$

Substituting Eqs. (5.32) and (5.33) into Eq. (5.26) one obtains the Abel integral equation

$$\frac{1}{2\sqrt{\pi}} \int_0^t d\tau \frac{\beta_1(y, \tau)}{\sqrt{t - \tau}} = K_0(0, y; a_+^2t) \quad (5.36)$$
with the solution
\[ a_+ \beta_1(y, t) - a_+ \beta_2(y, t) = \frac{2}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} K_0(0, y; a_+^2 \tau) \]
\[ = \frac{1}{2\sqrt{\pi}} \frac{y}{a_+^2 t^{3/2}} \exp \left( -\frac{y^2}{4a_+^2 t} \right). \]  
(5.37)

The substitution of Eqs. (5.34) and (5.35) into Eq. (5.27) gives
\[ \lambda_+ \beta_1(y, t) + \lambda_- \beta_2(y, t) + \lambda_+ \frac{1}{2\sqrt{\pi}} \frac{y}{a_+ t^{3/2}} e^{-\frac{y^2}{4a_+^2 t}}. \]  
(5.38)

From Eqs. (5.37) and (5.38) it follows that
\[ \beta_1(y, t) = \frac{\lambda_+}{\lambda_+ + \lambda_-} \frac{y}{a_+^2 t^{3/2}} \exp \left( -\frac{y^2}{4a_+^2 t} \right), \quad \beta_2(y, t) = \frac{\lambda_-}{\lambda_+ + \lambda_-} \frac{y}{2\sqrt{\pi} a_+ t^{3/2}} e^{-\frac{y^2}{4a_+^2 t}}. \]  
(5.39)

And finally
\[ K_{-+}(x, y; t) = \frac{\lambda_+}{\lambda_+ + \lambda_-} \frac{1}{\sqrt{\pi} a_+^2 t} e^{-\frac{(x-y a_+/a_-)^2}{4a_+^2 t}}, \]  
(5.40)
\[ K_{++}(x, y; t) = \frac{1}{2\sqrt{\pi} a_+^2 t} e^{-\frac{(x-y)^2}{4a_+^2 t}} + \frac{\lambda_+}{\lambda_+ + \lambda_-} \frac{\lambda_-}{\lambda_+ + \lambda_-} \frac{1}{2\sqrt{\pi} a_-^2 t} e^{-\frac{(x+y)^2}{4a_-^2 t}}. \]  
(5.41)

In a similar way one gets
\[ K_{+-}(x, y; t) = \frac{\lambda_-}{\lambda_+ + \lambda_-} \frac{1}{\sqrt{\pi} a_-^2 t} e^{-\frac{(x-y a_+/a_-)^2}{4a_-^2 t}}, \]  
(5.42)
\[ K_{--}(x, y; t) = \frac{1}{2\sqrt{\pi} a_-^2 t} e^{-\frac{(x-y)^2}{4a_-^2 t}} + \frac{\lambda_+}{\lambda_+ + \lambda_-} \frac{\lambda_-}{\lambda_+ + \lambda_-} \frac{1}{2\sqrt{\pi} a_+^2 t} e^{-\frac{(x+y)^2}{4a_+^2 t}}. \]  
(5.43)

The solution obtained here exactly reproduces the results obtained in this problem by other methods.  

C. δ-Like heat source at the interface

In preceding considerations we excluded the configuration when the δ-like heat source is placed at the interface of the media \( y = 0 \). For completeness we have to check whether this configuration contributes to the trace \( \int_{-\infty}^{\infty} K(x, x; t) dx \). To this end we use the approach of Ref. 28. The idea is to modify the boundary conditions (5.26) and (5.27) so that they allow for the heat source placed at the interface
\[ K_-(0; t) = K_+(0; t), \]  
(5.44)
\[
\lambda_\frac{\partial}{\partial x} K_-(x; t) \bigg|_{x=-0} - \lambda_+ \frac{\partial}{\partial x} K_+(x; t) \bigg|_{x=+0} = Q \delta(t) , \tag{5.45}
\]

The condition (5.45) means that the heat \( Q \) instantly generated by the source is divided into two flows which are proportional to \( \lambda_- \) and \( \lambda_+ \). We represent the solution in the form

\[
K_-(x; t) = a_-^2 \int_0^t d\tau K_0(x, 0; a_-^2(t - \tau)) \beta_1(\tau), \quad x < 0 , \tag{5.46}
\]

\[
K_+(x; t) = a_+^2 \int_0^t d\tau K_0(x, 0; a_+^2(t - \tau)) \beta_2(\tau), \quad x > 0 . \tag{5.47}
\]

The functions \( \beta_1(\tau) \) and \( \beta_2(\tau) \) can be determined by making use of the Laplace transform. We denote by \( \bar{K}(x; s) \) the transform of \( K(x; t) \), i.e.

\[
\bar{K}(x; p) = \int_0^\infty dt e^{-pt} K(x; t).
\]

Transforming (5.46) and (5.47) we obtain

\[
\bar{K}_-(x; p) = a_- \bar{\beta}_-(p) \sqrt{\pi/p} \exp(-x\sqrt{p}/a_-), \tag{5.48}
\]

\[
\bar{K}_+(x; p) = a_+ \bar{\beta}_+(p) \sqrt{\pi/p} \exp(-x\sqrt{p}/a_+). \tag{5.49}
\]

Then Eq. (5.44) leads to the relation between the Laplace transforms \( \bar{\beta}_- \) and \( \bar{\beta}_+ \)

\[
a_- \bar{\beta}_-(p) - a_+ \bar{\beta}_+(p) = 0 . \tag{5.50}
\]

The substitution of Eqs. (5.46) and (5.47) into Eq. (5.45) gives

\[
\lambda_- \beta_-(t) + \lambda_+ \beta_+(t) = 2 Q \delta(t) . \tag{5.51}
\]

After the Laplace transform one arrives at the second relation between \( \bar{\beta}_1 \) and \( \bar{\beta}_2 \)

\[
\lambda_1 \bar{\beta}_-(p) + \lambda_2 \bar{\beta}_+(p) = 2 Q \bar{\delta}(p) . \tag{5.52}
\]

How to apply the Laplace transform to the singular \( \delta \) function can be found in appropriate handbooks.\(^{20,21}\) The essence of the matter comes to defining the integration rule

\[
\int_0^\infty f(t) \delta(t) dt = f(0) ,
\]

whence it follows in particular

\[
\bar{\delta}(p) = \int_0^\infty e^{-pt} \delta(t) dt = 1 .
\]
In the problems treated by the integral Laplace transform the semiaxis \( t > 0 \) (or \( t > t_0 \)) is usually considered. Therefore the \( \delta \)-function should be defined here in a nonsymmetric way, for example, as the limit when \( \varepsilon \to +0 \) of the function

\[
\delta_\varepsilon(t) = \begin{cases} 
0, & t < 0, \ t > \varepsilon, \\
1/\varepsilon, & 0 < t < \varepsilon.
\end{cases}
\]

The solution of the system (5.50), (5.52) is

\[
\bar{\beta}_-(p) = \frac{2a_+ Q \delta(p)}{\lambda_+ a_- + \lambda_- a_+}, \quad \bar{\beta}_+(p) = \frac{2a_- Q \tilde{\delta}(p)}{\lambda_+ a_- + \lambda_- a_+}.
\] (5.53)

By the inverse Laplace transform we find from (5.53)

\[
\beta_-(t) = \frac{2a_+ Q \delta(t)}{\lambda_+ a_- + \lambda_- a_+}, \quad \beta_+(t) = \frac{2a_- Q \delta(t)}{\lambda_+ a_- + \lambda_- a_+}.
\] (5.54)

Having inserted (5.54) into (5.46) and (5.47) we derive

\[
K_-(x; t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4a_+^2 t}} \frac{Q}{\lambda_+ a_- + \lambda_- a_+}, \quad x < 0,
\] (5.55)

\[
K_+(x; t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4a_-^2 t}} \frac{Q}{\lambda_+ a_- + \lambda_- a_+}, \quad x > 0.
\] (5.56)

These formulae show that for \( t > 0 \) the heat kernel is finite notwithstanding the \( \delta \)-like heat source situated at the interface between two media. Therefore the neighborhood of the point \( x = 0 \) gives no contribution to the heat kernel trace \( \int_{-\infty}^{\infty} K(x, x; t) \, dx \).

**VI. CONCLUSION**

For a broad set of boundary conditions the finding of the heat kernel is reduced to the solution of integral equations defined on the boundary (or at the interface) of the manifolds. As a result the dimension of the initial problem is brought down by 1. Remarkably this technique is applicable to compound media where the standard methods for the investigation of heat kernel do not work because in this case the principal part of the elliptic operator in question is not smooth.

The perturbation series for the integral equations derived are nothing else as the multiple scattering expansions for the relevant heat kernels. Thus a rigorous derivation of these expansions both for homogeneous media and compact regions and for compound media has been done.
The efficiency of this approach is convincingly demonstrated by constructing, in an exact form, the heat kernel on an infinite line with diverse matching conditions and by deriving the first terms of the asymptotic expansion for integrated heat kernel in the case of three dimensional compound media.

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APPENDIX A: SELF-ADJOINTNESS OF BOUNDARY VALUE PROBLEMS FOR COMPOUND MEDIA

Often it is helpful to use the spectral representation for the Green’s function of the heat conduction equation (2.10)

\[ K(x, t; x', 0) = \sum_k e^{-\omega_k t} f_k(x) f_k^*(x'), \]  

where \( f_k(x) \) are the eigenfunctions of the spectral problem at hand

\[ -a^2 \Delta f_k(x) = \omega_k f_k(x), \quad x \in D \]  

obeying the relevant boundary conditions on \( S \). Apparently, the representation (A1) is well defined if the spectral problem (A2) is hermitian and positive definite. In this connection it is worth elucidating the boundary conditions for compound media that lead to the self-adjoint spectral problem (see Eqs. (3.1) – (3.4)). In this case we have instead of (A2)

\[ -a^2(x) \Delta f_k(x) = \omega_k f_k(x), \]  

where

\[ a^2(x) = \begin{cases} a_+^2, & x \in D_+, \\ a_-^2, & x \in D_-, \end{cases} \]  

\( a_+^2 \) and \( a_-^2 \) being constants. At the interface \( S \) the natural modes \( f_k(x) \) obey the dielectric like conditions

\[ f_{k+}(x) = f_{k-}(x), \]
Here we are using the same notations as in Eqs. (3.1) – (3.4).

Since the outer region $D_-$ is not bounded the spectrum $\omega_k$ is continuous. For example, we can assume that at the spatial infinity $|x| \to \infty$ the natural modes $f_k(x)$ satisfy the scattering problem conditions and decrease sufficiently fast. The explicit form of these conditions will not be needed below.

The differential operator in Eq. (A3) apparently coincides with its adjoint

\[
[-a^2(x) \Delta]^\dagger = \begin{cases} 
-a^2_+ \Delta, & x \in D_+, \\
-a^2_- \Delta, & x \in D_-.
\end{cases}
\]

The boundary value problem with the operator $-a^2(x) \Delta$ will be self-adjoint when the integral for two sufficiently smooth functions $u(x)$ and $v(x)$

\[
I = \int_{D_+ \cup D_-} dx a^2(x) (v \Delta u - u \Delta v)
\]

vanishes. Applying the Green integral formula for the domains $D_+$ and $D_-$ separately we obtain

\[
I = \int_S dS \left[ a^2_+ \left( u_+ \frac{\partial v_+}{\partial n_+} - v_+ \frac{\partial u_+}{\partial n_+} \right) + a^2_- \left( u_- \frac{\partial v_-}{\partial n_-} - v_- \frac{\partial u_-}{\partial n_-} \right) \right].
\]

The functions $u(x)$ and $v(x)$ are assumed to diminish at the infinity in such a way that the region $|x| \to \infty$ does not contribute to the integral (A8). The integral $I$ is equal to zero, for example, for the following conditions at the interface $S$

\[
u_+(x) = u_-(x),
\]

\[a^2_+ \frac{\partial u_+}{\partial n_+} + a^2_- \frac{\partial u_-}{\partial n_-} = 0, \quad x \in S.
\]

The function $v(x)$ should satisfy the same boundary conditions on $S$. Thus the dielectric-like conditions (A5) lead to self-adjoint boundary value problem if

\[
\frac{\lambda_+}{a^2_+} = \frac{\lambda_-}{a^2_-}.
\]

Of course, conditions (A9) do not exhaust all the cases when the boundary value problem under consideration is self-adjoint. Let the surface $S$ divides the same medium into the
domains $D_+$ and $D_-$, i.e. $a_+^2 = a_-^2$. We get self-adjoint spectral problem if impose at the interface $S$ the following gluing conditions

$$l u_+(x) = l^{-1} u_-(x),$$

$$l^{-1} \frac{\partial u_+(x)}{\partial n_+(x)} + l \frac{\partial u_-(x)}{\partial n_-(x)} = 0, \quad x \in S,$$  \hfill (A10)

where $l$ is a dimensionless constant. The one-dimensional version of this problem has been considered in Sec. V.

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29. Strictly speaking, the function $K(x, t; x', t')$ is the propagator of the heat equation. It should be multiplied by the step function $\theta(t - t')$ in order to get the Green function. This point should be kept in mind when dealing with the parabolic equations.

30. In multi-dimensional problems we may also choose single layer potential for internal region and double layer potential for external region. However when $d = 1$ such a choice may lead to divergent integrals.

31. When considering the gluing conditions the use of the single layer potentials leads to the exactly
solvable (no iterations needed!) Abel equation only in 1-dimensional case.