The solution to the problem of time in quantum gravity also solves the time of arrival problem in quantum mechanics

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Abstract

We introduced with coauthors some years ago a solution to the problem of time in quantum gravity which consists in formulating the quantum theory in terms of real clocks. It combines Page and Wootters’ relational proposal with Rovelli’s evolving constants of the motion. Time is associated with an operator and not a classical parameter. We show here that this construction provides a natural solution to the time of arrival problem in quantum mechanics and leads to a well defined time-energy uncertainty relation for the clocks.

1. Introduction

In ordinary quantum mechanics there is no operator associated with time, which is a classical parameter. This has led to the ‘time of arrival’ problem: one can make probabilistic predictions about where a particle will be detected at a certain time, but not about when a particle will be detected at a certain position. This problem has been addressed through various approaches in the past and there does not appear to be a consensus answer for it. See [1] for reviews of previous work. We argue that a previous proposal to address the problem of time in constrained systems like quantum gravity naturally leads to a solution of the time of arrival problem. It also leads to a well defined time-energy uncertainty relations for the clocks. Our proposal has elements in common with that of Maccone and Sacha [1] but they do not consider evolving constants of the motion as their focus is not on totally constrained systems like general relativity. In particular, their analysis does not include the computation of the time of arrival measured by real clocks that evolve with a Hamiltonian that is bounded-below.

We introduced with coauthors a few years ago [2, 3], a solution to the problem of time in generally covariant systems like quantum general relativity. In such systems the Hamiltonian is a linear combination of the constraints, and therefore all Dirac observables are constants of the motion, so time evolution has to be introduced in a different way that in traditional classical mechanics. The solution is based in considering the conditional probabilities that Page and Wootters have proposed for solving the problem, but between parameterized Dirac observables (evolving constants of the motion). This provides observable quantities that evolve in time and are well defined in the physical space of solutions of the constraints. It eliminates the objections that Kuchař [5] had made to the Page Wootters construction and leads to correct propagators in model systems. Physical time arises from observing a (quantum) clock. Our proposal has also been shown to be compatible with the approach in terms of positive operator-valued measures of Höhn et al [6]. The idea is to pick a parameterized Dirac observable one wishes to study $O(t)$ and pick another one that will play the role of a clock $T(t)$. In these expressions $t$ is the parameter the Dirac observables depend on. Parameterized Dirac observables were introduced by [7] for the description of the evolution. Generically they are Dirac observables that depend on one or several parameters such that when certain variables in the kinematical space of the constrained system take the value of these parameters the observable reproduce the value of the other variables in the kinematical space. In field theories like general relativity the parameters may be functions [8]. One then computes the conditional probabilities of the observable taking a value...
within an interval of width \(2\Delta O\) around a value \(O_0\) at a time within an interval of width \(2\Delta T\) around a value \(T_0\),

\[
P(O \in [O_0 \pm \Delta O] | T \in [T_0 \pm \Delta T]) = \lim_{T \to \infty} \frac{\int_{T_0}^{T_0 + \Delta T} dt \ \Tr \left( P_{O_0} (t) P_{T_0} (t) \rho P_{T_0} (t) \right)}{\int_{T_0}^{T_0 + \Delta T} dt \ \Tr (P_{T_0} (t) \rho)}.
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In this expression \(\rho\) is the density matrix of the system, \(P_{O_0}\) is the projector on the eigenspace associated with the eigenvalue \(O_0\) of \(O\) and similarly for \(P_{T_0}\). The classical parameter \(t\) is integrated over because it is not an observable quantity, its determination would require the observation of a non observable kinematical variable, and we do not need knowledge of it to compute the probability. Working with evolving constants of the motion is equivalent to working in a Heisenberg like representation where the kinematical variable, and we do not need knowledge of it to compute the probability. Working with non-observable quantities, its determination would require the observation of a non-observable variable.

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We can also consider parameterized Dirac observables (evolving constants of the motion),
\begin{align}
\hat{x}_1 (t) & = \hat{x}_1^{\text{Dirac}} + ct, \\
\hat{x}_2 (t) & = \exp \left( \imath \hat{H}_2 t \right) \hat{x}_2^{\text{Dirac}} \exp \left( -\imath \hat{H}_2 t \right),
\end{align}
where \( t \) is a classical parameter, whose classical counterparts \( x_1 (t) \) and \( x_2 (t) \) satisfy that \( x_1 (t = x_0 / c) = x_1 \) and \( x_2 (t = x_0 / c) = x_2 \) and we choose \( \hbar = 1 \). These operators describe the position of particles 1 and 2.

Notice that \( \hat{x}_1 \) and \( \hat{x}_2 \) are not Dirac observables, but \( \hat{x}_1 (t) \) and \( \hat{x}_2 (t) \) are, they have vanishing commutators with the constraint. They are what Rovelli calls evolving constants of the motion. We will use the position of particle 1 as a clock.

The physical space of states for this system can be constructed straightforwardly. If \( \hat{H}_2 \psi_E (p_2) = E \psi_E (p_2) \) are the eigenfunctions of \( \hat{H}_2 \) with eigenvalue \( E \), we have that a physical state for the complete system (a state that is annihilated by the constraint acting as an operator) is given by,
\begin{equation}
\Psi_{\text{ph}} (p_1, p_2, p_0) = \int dE \delta (p_0 + p_1 + E / c) f (p_1) \psi_E (p_2) C_E,
\end{equation}
where \( C_E \) are the complex coefficients of the expansion of the state in the energy basis, where we have assumed that \( \hat{H}_2 \) has a continuous spectrum and 1 and 2 are independent. A similar construction can be made if the spectrum is discrete, the formulas vary slightly.

The inner product in the physical Hilbert space \( \mathcal{H} \) can be constructed using well known techniques for constrained systems [9, 10],
\begin{equation}
\langle \Psi | \Psi' \rangle_{\text{ph}} = \int dp_1 \, dp_2 \, dE \, df^* \psi_E^* (p_1) \psi_E (p_2) C_E C_E^* f^* (p_1) \psi_E (p_2),
\end{equation}
where we have taken into account that the eigenfunctions of \( \hat{H}_2 \) are orthogonal for different eigenvalues \( E \).

Using the expression for the conditional probabilities (1) and working with \( \hat{x}_1 (t) \) and \( \hat{x}_2 (t) \) we have,
\begin{equation}
P (x_2 \in [y \pm \Delta y / 2] | x_1 = x) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau}^{\tau} dt \ Tr (P (y) \rho_2 (t)) / \Tr (P (x) \rho_1 (t)),
\end{equation}
where we have assumed that \( \Tr (\rho_1) = \Tr (\rho_2) = 1 \) and taken into account that \( P_c (t) = U (t) P_c (0) U (t) \) and that \( \rho_2 (t) = U (t) \rho_2 (0) U^\dagger (t) \). Here \( \rho_1 \) and \( \rho_2 \) are the density matrices of particles 1 and 2 respectively. In this example \( \rho_1 \) would play the role of \( \rho_{\text{sys}} \) and as we discussed in the introduction they are assumed not to interact with each other. As before \( P (y) \) is the projector on the eigenspace associated with the eigenvalues in the interval around \( y \), i.e. \( P (y) = \int_{y-\Delta y / 2}^{y+\Delta y / 2} dE | \langle x | \rangle | \langle x | \rangle \), and \( P (x) = | \langle x | \rangle | \langle x | \rangle \). We do this because normally the description of probabilities in quantum mechanics is for a given time and not for a time interval. Notice that here \( t \) is the parameter previously introduced, that as we have discussed is not observable. All the previous relations were derived without any additional assumption about the nature of \( t \). We shall call \( t \) the ideal time, the explicit connection with the usual definition of time can be obtained by imposing a gauge fixing \( x_0 = ct \). If we did that we would notice that the operators \( \hat{x}_1 (t) \) and \( \hat{x}_2 (t) \) are the usual ones for those systems in the Heisenberg representation. That allows to consider the parameter \( t \) as the usual time in quantum mechanics, that we consider not accessible since we measure times with physical clocks\(^3\). We omit the hats on the projectors and evolution operators to keep in line with notation in our previous papers. We see that in the expression the ideal time \( t \) is integrated over and therefore we do not need to know its value in computing the probability, as we mentioned before.

Let us analyze the behavior of the clock when it is perfectly synchronized with the ideal time \( t \). That is, when the system 1 behaves like an ideal clock. In that case we would have that \( \Psi (x_1, t) = \Psi (x_1 - ct) \) since,
\begin{align}
\hat{x}_1 (t) | x, t \rangle & = x | x, t \rangle = (\hat{x}_1^{\text{Dirac}} + ct) | x, t \rangle, \\
\hat{x}_1^{\text{Dirac}} | x, t \rangle & = (x - ct) | x, t \rangle.
\end{align}

Since \( p_1 \) is the Hamiltonian of the first particle we have that \( \langle x, t | \psi_\text{ph} \rangle = \langle x | \exp(-\imath p_1 t) \rangle \). Therefore \( \psi (x, t) = \langle x, t | \psi_\text{ph} \rangle = \langle x - ct | \psi_\text{ph} \rangle \).

\(^3\) We live in a quantum mechanical generally covariant Universe where all observable quantities are constants of the motion represented by Dirac observables.
In order to have a reasonably behaved clock we need a localized wavefunction, for instance a Gaussian, that advances as \( t \) grows,
\[
\Psi_1 (x, t) = \left( \frac{1}{2\pi} \right)^{1/4} \exp \left( \frac{-|x-x_0|^2}{4\sigma_x^2} \right),
\]
and
\[
\text{Tr} \left( P(x) \rho_1 (t) \right) = \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-|x-x_0|^2}{2\sigma_x^2} \right).
\]
The position of system 1 behaves like an ideal clock when \( \sigma_x \to 0 \), where one has that,
\[
\text{Tr} \left( P(x) \rho_1^{\text{ideal}} (t) \right) = \delta (x - ct).
\]

As a consequence, the denominator of equation (12) becomes \( 1/(2\tau) \) for all \( x \in [-\tau, \tau] \) and therefore,
\[
P \left( x_2 \in [y \pm \Delta y/2] | x_1 = x \right) = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} dt \text{Tr} \left( P(y) \rho_2 (t) \right) \text{Tr} \left( P(x) \rho_1 (t) \right)
\]
and the conditional probability coincides (up to a factor) with the simultaneous probability of measuring \( x_1 \) and \( x_2 \) at certain instant. It should be noted that the conditional probability (18), for \( \Delta y \) sufficiently small and \( \rho_2(t) = |\psi_2(t)\rangle \langle \psi_2(t)| \) is nothing else but \( |\psi_2(y,t)|^2 \Delta y \), that is, the Born rule applied at \( t = x/c \). We have therefore shown how the covariant description in terms of evolving Dirac observables allows to recover, when the clock is perfectly correlated with \( t \) the usual Born rule.

2. Time of arrival in terms of conditional probabilities

The use of real clocks allows to assign observables to the time variable. In the previous example \( \tilde{x}_1(t) \), the evolving constant of the motion representing the position of particle 1, plays the role of time. In this section we will define the probabilistic distribution of the time of arrival of a particle at a given point. Different requirements to assign probabilities to the time of arrival have been developed through the years. All of them have questionable aspects and lead to paradoxes. We will see that the solution here proposed not only has a very simple and clear origin but it also solves the paradoxes. We will concentrate in the comparison with the ‘standard approach to time of arrival’ advocated by Egusquiza et al [11] and discussed by several of the authors in [1], among them the derivations of Allcock, of Kijowski, of Grot, Rovelli and Tate, and of Delgado and Muga are for the special case of free evolution. They [11] titled their distillation of the various approaches leading to the two equations that appear bellow ‘standard’ quantum mechanical approach to times of arrival, emphasizing their claim that (2) can be derived ‘without in any way distorting the standard framework of quantum mechanics’. However, Leavens [12] observed that even for the case of free evolution, they need to associate the direction of arrival with sign of \( k \), which is an unjustified assumption that is not a part of conventional quantum mechanics. As we will discuss in detail in section 3, several paradoxical behaviors emerge.

We will follow the latter in order to derive the expression of the distribution of times of arrival for an ensemble of quantum particles whose initial state is \( \psi \) \( (x, t = 0) \).

The standard approach starts from the probability distribution of observing the particle arrival at \( X \) at time \( T \),
\[
\Pi(T, X) = \Pi_+ (T, X) + \Pi_- (T, X),
\]
with,
\[
\Pi_\pm (T, X) = \frac{1}{2\pi m} \int_{-\infty}^{\infty} dk |\Theta(\pm k)| r_{\phi(k)}^{1/2} \exp (ikX) \phi(k, T) |^2,
\]
where \( \Pi_+ \) and \( \Pi_- \) represent the contributions to \( \Pi(T, X) \) from particles that come from the left and right respectively, \( \phi(k, T) \) is the momentum space representation of the quantum state (Leavens [12] calls it \( \phi \), we call it \( \psi \) in the rest of this paper). The expression was initially derived for free particles and was recently extended [13] for particles in an arbitrary potential \( V(X, T) \). Leavens [12] has analyzed in several concrete examples the counter-intuitive and paradoxical aspects of the standard approach. We will revisit them here using the technique of conditional probabilities and we will see that the issues are resolved. We will leave for a future publication the comparison of these results with those of Bohmian mechanics that also appear to solve some of the observed problems.
we are working in units such that $\hbar = 1$ and distances, velocities and times are in compatible units, for instance cm, cm s\(^{-1}\), s.

The probability density corresponding to the state as a function of time is, 

\[
\rho(x) = \frac{(2\pi\sigma_x^2)^{-1/4}}{\sqrt{1+i\alpha t}} \exp \left[ -\frac{(y - y_0)^2}{2\sigma_x^2} - \frac{\alpha y}{1 + i\alpha t} \right] \Theta(t),
\]

with $\alpha = 1/(2m\sigma_x^2)$, $\hbar = 1$ and $\Theta(t)$ a Heaviside function equal to 0 for $t < 0$ and 1 otherwise. The initial expectation value of the momentum of the particle is $p_0$. We have chosen zero as the instant of the preparation of the state. The probability density corresponding to the state as a function of time is,

\[
P(y,t) = \frac{(2\pi\sigma_y^2)^{-1/2}}{\sqrt{1+\alpha^2 t^2}} \exp \left[ -\frac{2(y - y_0 t)^2}{2\sigma_y^2(1 + \alpha^2 t^2)} \right] \Theta(t),
\]

with $y_0 = p_0/m$ and $\text{Tr}(P_y \rho_2(t)) = P(y,t)\Delta y$.

In the limiting case in which particle 1 behaves like an ideal clock, one has that $\text{Tr}(P_T \rho_1(t)) = \delta(T - t)\Delta T$ and the probability density that the time of arrival to $y$ be $T$ is,

\[
P \left( x_2 \in [cT \pm \varepsilon \Delta T] \mid x_2 \in [y \pm \Delta y] \right) = \frac{P(y,T)}{\int_0^\infty dt \ P(y,t)},
\]

with $P(y, T = t)$ given by (23). The expression obtained differs from that of the standard approach [12], which is given by (20) with,

\[
\phi(k,T) = \left( \frac{2\pi\sigma_y^2}{\pi} \right)^{1/4} \exp \left( -\frac{(k - p_0)^2}{2m}\Delta T \right).
\]
Figure 2. Time of arrival at $y = 1$ for a packet prepared at $y = 3$ with velocity $v = -4$ in the case of an infinite potential barrier at $y = 0$. The two peaks corresponding to the first passage of the particle at time $x = 1/2$ and the passage of its reflection at time $x = 1$.

We therefore see that the approach we advocate reproduces the traditional results for the case without a potential. That case does not involve paradoxes and contradictions, so both approaches can be considered satisfactory. To illustrate the advantages of our approach, we will discuss other situations in the next section.

3. Freedom from usual paradoxes

When the standard approach is applied to an infinite potential barrier, it leads to paradoxical predictions. A good discussion of the problem can be found in Leavens [12]. He shows that, applied in that case, the standard approach leads to a non-vanishing distribution of arrival times in the forbidden regions where potentials are infinite. The infinite barrier is a particular case of an infinite well when its width tends to infinity. Leavens’ analysis also leads to non-trivial arrival times in regions where the probability of finding the particle is strictly zero in the case of the well. As a consequence, one can easily see that the same will happen to the infinite barrier. Figure 1 in that paper show the arrival times in the forbidden region for arbitrary values of the well’s width $a$.

Let us start from equation (24) and consider the case of a potential barrier at $y = 0$. The particle is in the region $y > 0$. Let us study a wavepacket initially centered in $p_0, y_0$, confined to move in the region $y > 0$,

$$\psi_2(y, t) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} \sin(py) \Theta(y) \exp\left(-\left(p - p_0\right)^2 \sigma^2\right) \exp\left(-\frac{p^2 t}{2m}\right) dp \Theta(t),$$

where we have chosen the origin of the $t$ parameter at the instant of preparation of the state. The conditional probability of observing the clock at $x_1$ in $T$ when it goes through $x_2 = y$ is given by,

$$P(x_1 = T | x_2 = y) = \frac{P(y, t = T)}{P(y, t)dt},$$

with,

$$P(y, t) = \frac{1}{2\sqrt{\alpha^2 + 1}} \left[ \sqrt{\frac{\pi m}{\alpha}} \Theta(y) \left[ \exp\left(\frac{\beta_+}{\rho}\right) + \exp\left(-\frac{\beta_+}{\rho}\right) \right] - \exp\left(\frac{\gamma + \delta}{\rho}\right) - \exp\left(\frac{\gamma - \delta}{\rho}\right) \right] \Theta(t)$$

(28)
with $\Theta(y)$ and $\Theta(t)$ Heaviside functions and,

$$\alpha = \frac{1}{2m\sigma^2},$$  \hfill (29)

$$\beta_\pm = \alpha (m(y \pm y_0) \pm tp_0)^2,$$  \hfill (30)

$$\rho = m (\alpha^2 t^2 + 1),$$

$$\gamma = \left( (-y^2 - y_0^2) m^2 - 2mp_0ty_0 - p_0^2 t^2 \right) \alpha,$$  \hfill (32)

$$\delta = 2im_p0y - 2im^2t^2y_0.$$  \hfill (33)

One can immediately check that for $y < 0$ the expression yields $0/0$ and is ill defined. As a consequence, there is no prediction of a time of arrival in the forbidden region.

Let us analyze in some detail two cases: (1) an initially well localized packet at $y = 3$ that propagates towards the barrier with $p/m = v = -4$. The time of arrival is shown in figure 2.

The probability distribution of measuring $t$ has two peaks, one at $T = 0.5$ that represents the direct propagation from $y = 3$ to $y = 1$ and another at $T = 1$ which represents the arrival after the reflection of the packet on the barrier. The packet widens with time so the latter arrival time will have more dispersion in $T = 1$. Figure 3 shows the arrival time for a particle that starts at $y = 3$ to $y = 1$ that moves with speed $-0.1$. Due to lower speed, the widening of the packet generates interference between the wave that travels towards the barrier and the reflected one. A classical particle would have arrival times $T = 20$ and $T = 40$. The dispersion allows to find the particle in $y = 1$ for large times. The dispersion stemming from the standard approach not only yields arrival times in the forbidden region but also differs significantly from the probability distribution computed with our approach. For the same values of the parameters that describe an incoming particle with $v = -0.1$, the resulting distribution is shown in figure 4. Notice that the standard approach only provides non vanishing probabilities for values of $T > 0$. The paradox arises because for negative $y$, in the forbidden region, it provides a non trivial distribution for $T$.

Other paradoxical behaviors of the standard approach are eliminated by our proposal. Leavens [12] observed that a free particle in an antisymmetric superposition around $y = 0$ has several anomalous behaviors. The wavefunction he obtains is,

$$\psi^f(y, 0) = N \left[ \exp \left( -\frac{(y - y_0)^2}{4(\Delta y)^2} + ik_0y \right) - \exp \left( -\frac{(y + y_0)^2}{4(\Delta y)^2} - ik_0y \right) \right],$$  \hfill (34)
Figure 4. Time of arrival at $y = 1$ for a packet prepared at $y = 3$ in the case of an infinite potential barrier at $y = 0$ computed in the standard approach. We only plot a limited number of points, as they are computationally costly. As can be seen, the predictions differ significantly from the approach presented in this paper.

with,

$$
N = \left\{ \frac{2^{3/2}}{\pi^{1/2}} \Delta y \left[ 1 - \exp \left( -2(\Delta y)^2 k_0^2 - \frac{y_0^2}{2(\Delta y)^2} \right) \right] \right\}^{-1/2}.
$$

That is, a coherent linear superposition of Gaussians with centroids at $\pm y_0$, mean wavenumbers $\pm k_0$ and equal spatial widths $\Delta y$. Although the probability density $|\psi(0)(y, t)|^2$ vanishes at $y = 0$, the standard approach assigns finite probabilities to the times of arrival at $y = 0$. Moreover, in spite that the evolution of the wavepacket $|\psi(0)(y, t)|^2$ for $y > 0$ is identical to the case of the infinite barrier, the distribution for the time of arrival is different in both cases. This is due to the Fourier transform of the wavefunctions for the infinite barrier $\psi(k, T)$ and for the superposition $\psi(0)(k, T)$ are different and therefore $\Pi_{\pm}$ will be too. When one applies the approach we present in this paper both paradoxes disappear. The time of arrival to $x = 0$ is not defined (it is $0/0$) and the probability distribution for arrival at the infinite barrier and the one resulting from the superpositions given in (34) are identical. The result cannot be surprising since our distribution is computed from the probability density stemming from the wavefunction $\psi(0)(y, t) = \psi_2(y, t)$ (with $\psi_2$ given by (26)) for $y > 0$ without ever requiring a Fourier transform.

4. Time of arrival with real clocks

We have discussed how to compute the time of arrival using conditional probabilities in the limit of an ideal clock. Physical systems evolve with Hamiltonians that are bounded-below. In particular, that would apply to the Hamiltonian of a real physical clock. The previous sections’ analyses can be applied to real clocks. For instance, one can study a free particle using another free particle as a physical clock. Starting from equation (12) with $\rho_1$ and $\rho_2$ states of the free particle for particles 1 (clock) and 2 respectively we have that the denominator of equation (12) is given by,

$$
\int_{-\infty}^{\infty} dt \, \text{Tr} \left( \rho(y) \rho(t) \right) = \int_{-\infty}^{\infty} dt \sqrt{\frac{2}{\pi}} \sigma_y m_2^{1/2} \chi^2 \exp \left( -\frac{2\sigma_y^2 (y - y_0)^2 m_2 - t p_2^{(0)} - \Gamma)^2}{\chi^2} \right) \Theta(t) \tag{36}
$$

with $y_0$ the initial position and $y$ the arrival point and $\chi^2 = 4\sigma_y^4 m_2^2 + t^2$. We have assumed that the state was prepared at $t_0$ with a packet width $\sigma_y$ and $p_2^{(0)}$ is the initial momentum and $m_2$ the mass.
Figure 5. Probability of times of arrival with a real clock for the case of a particle at $y_0 = 3, v_1 = 1, m_2 = 200, p_2^{(0)} = -20, \sigma_y = 0.2, v_2 = -0.1$ and for the clock at $x_0 = 0, m_1 = 5000, p_1^{(0)} = 5000, \sigma_x = 0.02$.

On the other hand, the numerator of (12) is given by,

$$
\int_{-\infty}^{\infty} \frac{dt}{\pi} \chi_2^{-1/2} \sigma_m^2 \chi_1^{-1/2} \sigma_y m_2 \\
\times \exp \left( -2\sigma_y^2 \left( \frac{(y - y_0) m_2 - t p_2^{(0)}}{\chi_2} \right)^2 \right) \exp \left( -2\sigma_x^2 \left( \frac{(x - x_0) m_1 - t p_1^{(0)}}{\chi_1} \right)^2 \right) \Theta(t)
$$

with $\chi_1 = 4\sigma_x^4 m_1^2 + t^2$ and $p_1^{(0)}$ the initial momentum of particle 1 and $m_1$ its mass.

Figure 5 shows the distribution of probabilities of arrival times. With the parameters chosen for the figure, the probability distribution of arrival times is not symmetric anymore, as is was in figure 1. Due to the dispersion of the physical clock it decreases more slowly than it grows as is shown in the figure. This is due to the widening of the clock’s wavepacket as it propagates to larger values of $t$.

5. Time-energy uncertainty relation

It is straightforward to show, within this framework, that time-energy uncertainty relations for the clock hold at the instant at which an event occurs, like the arrival of a particle to a given position. This can be done with the same techniques as in ordinary quantum mechanics one does for conjugate observables like $\hat{x}$ and $\hat{p}$. It is also possible to obtain analogous relations for arbitrary observables $\hat{A}$ and $\hat{B}$ of the clock such that $[\hat{A}, \hat{B}] = i\hat{C}$.

Taking into account that any Dirac observable, with either discrete or continuous spectrum, can be written as,

$$
\hat{A} = \sum_i a_i |a_i\rangle \langle a_i| \quad \text{or} \quad \int da |a\rangle \langle a|,
$$

equation (21) allows to compute the expectation value of $\hat{A}$ (an observable associated with a clock) when $\hat{y}$ is observed as,

$$
\langle \hat{A} \rangle_y = \lim_{\tau \to \infty} \frac{\int_{-\tau}^{\tau} dt \langle \hat{A}_t \rangle_2 \text{ Tr} \left( P_{y}^2(t) \rho_2 \right)}{\int_{-\tau}^{\tau} dt \text{ Tr} \left( P_{y}^2(t) \rho_2 \right)}.
$$

From this definition it is straightforward to derive uncertainty relations for the clock when an event, like the arrival of a particle to a point $y$, occurs. This is done using standard techniques (see for instance [14].
page 286) and to show that if

$$\Delta A^2_y \equiv \langle (\hat{A} - \langle \hat{A} \rangle_y)^2 \rangle_y, \quad (40)$$

$$\Delta B^2_y \equiv \langle (\hat{B} - \langle \hat{B} \rangle_y)^2 \rangle_y, \quad (41)$$

$$[\hat{A}, \hat{B}] = i\hat{C}, \quad (42)$$

then,

$$\Delta A_y \Delta B_y \geq \frac{1}{2} \langle \hat{C} \rangle_y. \quad (43)$$

For the ideal clock with $\hat{H} = \hat{p} c$ and therefore $[T, H] = i$ we have that $\Delta T \Delta E \geq 1/2$ and the usual expression is recovered. For real clocks the analysis is more delicate, and will be analyzed in detail in a separate paper.

6. Conclusions

We have applied the relational definition of time proposed as a solution to the problem of time in quantum gravity to the problem of time of arrival in quantum mechanics. We show that it yields satisfactory results, avoiding all of the problems and paradoxes that previous approaches had encountered. Since each event gets assigned a projector the present solution of the time of arrival problem allows to assign a time to the occurrence of any event in a system. It has the advantage of being the only approach applicable in generally covariant systems and of treating the clock as any physical system.

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Data availability statement

No new data were created or analyzed in this study.

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