Categorical Büchi and Parity Conditions via Alternating Fixed Points of Functors

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Abstract. Categorical studies of recursive data structures and their associated reasoning principles have mostly focused on two extremes: initial algebras and induction, and final coalgebras and coinduction. In this paper we study their in-betweens. We formalize notions of alternating fixed points of functors using constructions that are similar to that of free monads. We find their use in categorical modeling of accepting run trees under the Büchi and parity acceptance condition. This modeling abstracts away from states of an automaton; it can thus be thought of as the “behaviors” of systems with the Büchi or parity conditions, in a way that follows the tradition of coalgebraic modeling of system behaviors.

1 Introduction

Büchi Automata The Büchi condition is a common acceptance condition for automata for infinite words. Let $x_i \in X$ be a state of an automaton $A$ and $a_i \in A$ be a character, for each $i \in \omega$. An infinite run $x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots$ satisfies the Büchi condition if $x_i$ is an accepting state (usually denoted by $\circ$) for infinitely many $i$. An example of a Büchi automaton is shown on the right. The word $(ba)^\omega$ is accepted, while $ba^\omega$ is not. A function that assigns each $x \in X$ the set of accepted words from $x$ is called the trace semantics of the Büchi automaton.

Categorical Modeling The main goal of this paper is to give a categorical characterization of such runs under the Büchi condition. This is in the line of the established field of categorical studies of finite and infinite datatypes: it is well-known that finite trees form an initial algebra, and infinite trees form a final coalgebra; and finite/infinite words constitute a special case. These categorical characterizations offer powerful reasoning principles of (co)induction for both definition and proof. While the principles are categorically simple ones corresponding to universality of initial/final objects, they have proved powerful and useful in many different branches of computer science, such as functional programming and process theory. See the diagram on the right above illustrating coinduction: given a functor $F$, its final coalgebra $\zeta: Z \xrightarrow{\sim} FZ$ has a unique homomorphism to it from an arbitrary $F$-coalgebra $d: Y \rightarrow FY$. In many examples, a final coalgebra is described as a set of “infinite $F$-trees.”
Extension of such (co)algebraic characterizations of data structures to the Büchi condition is not straightforward, however. A major reason is the non-local character of the Büchi condition: its satisfaction cannot be reduced to a local, one-step property of the run. For example, one possible attempt of capturing the Büchi condition is as a suitable subobject of the set $\text{Run}(X) = (A \times X)^{\omega}$ of all runs (including nonaccepting ones). The latter set admits clean categorical characterization as a final coalgebra $\text{Run}(X) \xrightarrow{\sim} F(\text{Run}(X))$ for the functor $F = (A \times X) \times \_$. Specifying its subset according to the Büchi condition seems hard if we insist on the coalgebraic language which is centered around the local notion of transition represented by a coalgebra structure morphism $c : X \rightarrow FX$.

There have been some research efforts in this direction, namely the categorical characterization of the Büchi condition. In [6] the authors insisted on finality and characterize languages of Muller automata (a generalization of Büchi automata) by a final coalgebra in $\text{Sets}$. Their characterization however relies on the lasso characterization of the Büchi condition that works only in the setting of finite state spaces. In [22] we presented an alternative characterization that covers infinite state spaces and automata with probabilistic branching. The key idea was the departure from coinduction, that is, reasoning that relies on the universal property of greatest fixed points. Note that a final coalgebra $\zeta : Z \xrightarrow{\sim} FZ$ is a “categorical greatest fixed point” for a functor $F$.

Our framework in [22] was built on top of the so-called Kleisli approach to trace semantics of coalgebras [17,13,11,12]. There a system is a coalgebra in a Kleisli category $K_{\ell}(T)$, where $T$ represents the kind of branching the system exhibits (nondeterminism, probability, etc.). A crucial fact in this approach is that homsets of the category $K_{\ell}(T)$ come with a natural order structure. Specifically, in [22], we characterized trace semantics under the Büchi condition as in the diagrams (1) below, where i) $X_1$ (resp. $X_2$) is the set of nonaccepting (resp. accepting) states of the Büchi automaton (i.e. $X = X_1 + X_2$), and ii) the two diagrams form a hierarchical equation system (HES), that is roughly a planar representation of nested and alternating fixed points. In the HES, we first calculate the least fixed point for the left diagram, and then calculate the greatest fixed point for the right diagram with $u_1$ replaced by the obtained least fixed point. Note that the order of calculating fixed points matters.

\[ \begin{array}{ccc}
F X & \xrightarrow{\mu} & F Z \\
X_1 & \xrightarrow{u_1} & Z \\
\end{array} \quad \begin{array}{ccc}
F X & \xrightarrow{\nu} & F Z \\
X_2 & \xrightarrow{u_2} & Z \\
\end{array} \]

Contributions: Decorated Trace Semantics by Categorical Datatypes
In this paper we introduce an alternative categorical characterization to the one in [22] for the Büchi conditions, where we do not need alternating fixed points in homsets. This is made possible by suitably refining the value domain, from a

\[ \begin{array}{ccc}
F X & \xrightarrow{\mu} & F Z \\
X_1 & \xrightarrow{u_1} & Z \\
\end{array} \quad \begin{array}{ccc}
F X & \xrightarrow{\nu} & F Z \\
X_2 & \xrightarrow{u_2} & Z \\
\end{array} \]

We write $f : X \rightarrow Y$ for a Kleisli arrow $f \in K_{\ell}(T)(X, Y)$ and $F : K_{\ell}(T) \rightarrow K_{\ell}(T)$ for a lifting of the functor $F$ over $K_{\ell}(T)$, for distinction.
final coalgebra to a novel categorical datatypes $F^{+}0$ and $F^{+}(F^{+}0)$ that have the Büchi condition built in them. Diagrammatically the characterization looks as in (2) below. Note that we ask for the greatest fixed point in both squares.

\[
\begin{array}{c}
F \rightarrow F(F^{+}0 + F^{+}0) \\
\downarrow\quad \downarrow \\
\mathcal{F}(v_1 + v_2) \\
F X \quad F X \\
\mathcal{F}(v_1 + v_2) \\
\downarrow \quad \downarrow \\
\mathcal{F}(v_1 + v_2) \\
\end{array}
\]

The functors $F^{+}$ and $F^{+}0$ used in the datatypes are obtained by applying two operations $(\_)^+$ and $(\_)^0$ to a functor $F$. For an endofunctor $G$ on a category $\mathbb{C}$ with enough initial algebras, $G^{+}$ is given by the carrier object of a (choice of) an initial $G(\_ + X)$-algebra for each $X \in \mathbb{C}$. The universality of initial algebras allows one to define $G^{+}f : G^{+}X \rightarrow G^{+}Y$ for each $f : X \rightarrow Y$ and extend $G^{+}$ to a functor $G^{+} : \mathbb{C} \rightarrow \mathbb{C}$. This definition is much similar to that of a free monad $G^{\star}$, where $G^{\star}X$ is the carrier object of an initial $G(\_ + X)$-algebra for $X \in \mathbb{C}$. The operation $(\_)^{0}$ is defined similarly: for $G : \mathbb{C} \rightarrow \mathbb{C}$ and $X \in \mathbb{C}$, $G^{0}X$ is given by the carrier object of a final $G(\_ + X)$-coalgebra. This construction resembles to that of free completely iterative algebras [15].

The constructions of $F^{+}(F^{+}0)$ and $F^{+}0$ has a clear intuitive meaning. For the specific example of a labeled nondeterministic Büchi automata, $T = \mathcal{P}$, $F = A \times (\_)$, $F^{+}(F^{+}0) \cong F^{+}0 \cong (A^{+})^{0}$. Hence an element in $F^{+}(F^{+}0)$ or $F^{+}0$ is identified with an infinite sequence of finite words. We understand it as an infinite word “decorated” with information about how accepting states are visited, by considering that an accepting state is visited at each splitting between finite words. For example, we regard $(a_{0}a_{1})(a_{2}a_{3}a_{4})(a_{5}a_{6})(a_{7})\ldots \in (A^{+})^{0} \cong F^{+}0$ as an infinite word decorated as follows.

\[
\begin{array}{c}
\circ a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 \ldots \\
\end{array}
\]

An element in $F^{+}(F^{+}0)$ is similarly understood, except that the initial state is regarded as a nonaccepting state. We note that by its definition, the resulting “decorated” word always satisfies the Büchi condition.

Thus the arrows $v_1 : X_1 \rightarrow F^{+}(F^{+}0)$ and $v_2 : X_2 \rightarrow F^{+}0$ in (2) are regarded as a kind of trace semantics that assigns each state $x \in X$ the set of infinite words accepted from $x$ “decorated” with information about the corresponding accepting run. Hence we shall call $v_1$ and $v_2$ a decorated trace semantics for the coalgebra $c$. The generality of the category theory allows us to define decorated trace semantics for systems with other transition or branching types, e.g. Büchi tree automata or probabilistic Büchi automata.

In this paper, we also show the relationship between decorated trace semantics and (ordinary) trace semantics for Büchi automata. For the concrete case of Büchi automata sketched above, there exists a canonical function $(A^{+})^{0} \rightarrow A^{0}$ that flattens a sequence and hence removes the “decorations”. It is easy to see that if we thus remove decorations of a decorated trace semantics then we obtain an ordinary trace semantics. We shall prove its categorical counterpart.
In fact, the framework in [22] also covered the parity condition, which generalizes the Büchi condition. A parity automaton is equipped with a function \( \Omega : X \to [1, 2n] \) that assigns a natural number called a priority to each state \( x \in X \). Our new framework developed in the current paper also covers parity automata. In order to obtain the value domain for parity automata, we repeatedly apply \( (\_)^+ \) and \( (\_)^\oplus \) to \( F \) like \( F^+ \oplus \cdots \oplus 0 \).

Compared to the existing characterization shown in [1], one of the characteristics of our new characterization as shown in (2) is that information about accepting states is more explicitly captured in decorated trace semantics, as in (3). This characteristic would be useful in categorically characterizing notions about Büchi or parity automata. For example, we could use it for categorically characterizing (bi)simulation notions for Büchi automata, e.g. delayed simulation [9], a simulation notion which is known to be appropriate for state space reduction.

To summarize, our contributions in this paper are as follows:

- We introduce a new categorical data type \( F^+\oplus 0 \), an alternating fixed point of a functor, for characterizing the Büchi acceptance condition.
- Using the data type, we introduce a categorical decorated trace semantics, simply as a greatest fixed point.
- We show the categorical relationship with ordinary trace semantics in [22].
- We instantiate the framework to several types of concrete systems.
- We extend the framework to the parity condition (in the appendix).

**Related Work** As we have mentioned, a categorical characterization of Büchi and parity conditions is also found in [6], but adaptation to infinite-state or probabilistic systems seems to be difficult in their framework. There also exist notions which are fairly captured by their characterization but seem difficult to capture in the frameworks in [22] and this paper, such as bisimilarity.

The notion of alternating fixed point of functors is also used in [10,2]. In [10] the authors characterize the set of continuous functions from \( A^\omega \) to \( B^\omega \) as an alternating fixed point \( \nu X. \mu Y. (B \times X) + Y^A \) of a functor. Although the data type and the one used in the current paper are different and incomparable, the intuition behind them is very similar, because the former comes with a Büchi-like flavor: if \( f(a_0a_1 \ldots) = b_0b_1 \ldots \) then each \( b_i \) should be determined by a finite prefix of \( a_0a_1 \ldots \), and therefore \( f \) is regarded as an infinite sequence of such assignments. In [2, §7] a sufficient condition for the existence of such an alternating fixed point is discussed.

**Organization** §2 gives preliminaries. In §3 we introduce a categorical data type for decorated trace semantics as an alternating fixed point of functors. In §4 we define a categorical decorated trace semantics, and show a relationship with ordinary categorical trace semantics in [22]. In §5 we apply the framework to nondeterministic Büchi tree automata. In §6 we briefly discuss systems with other branching types. In §7 we conclude and give future work.
All the discussions in this paper also apply to the parity condition. However, for the sake of simplicity and limited space, we mainly focus on the Büchi condition throughout the paper, and defer discussions about the parity condition to the appendix. We omit a proof if an analogous statement is proved for the parity condition in the appendix. Some other proofs and discussions are also deferred to the appendix.

2 Preliminaries

2.1 Notations

For \( m, n \in \mathbb{N} \), \([m, n]\) denotes the set \( \{ i \in \mathbb{N} \mid m \leq i \leq n \} \). We write \( \pi_i : \prod_j X_j \to X_i \) and \( \kappa_i : X_i \to \prod_j X_j \) for the canonical projection and injection respectively. For a set \( A \), \( A^* \) (resp. \( A^\omega \)) denotes the set of finite (resp. infinite) sequences over \( A \), \( A^\infty \) denotes \( A^* \cup A^\omega \), and \( A^+ \) denotes \( A^* \setminus \{ \varnothing \} \). We write \( \varnothing \) for the empty sequence. For a monotone function \( f : (X, \sqsubset) \to (X, \sqsubset) \), \( \mu f \) (resp. \( \nu f \)) denotes its least (resp. greatest) fixed point (if it exists). We write \( \text{Sets} \) for the category of sets and functions, and \( \text{Meas} \) for the category of measurable sets and measurable functions. For \( f : X \to Y \) and \( A \subseteq Y \), \( f^{-1}(A) \) denotes \( \{ x \in X \mid f(x) \in A \} \).

2.2 Fixed Point and Hierarchical Equation System

In this section we review the notion of hierarchical equation system (HES) \([7,3]\). It is a kind of a representation of an alternating fixed point.

Definition 2.1 (HES) A hierarchical equation system (HES for short) is a system of equations of the following form.

\[
E = \begin{cases} 
  u_1 =_{\eta_1} f_1(u_1, \ldots, u_m) & \in (L_1, \sqsubseteq_1) \\
  u_2 =_{\eta_2} f_2(u_1, \ldots, u_m) & \in (L_2, \sqsubseteq_2) \\
  \vdots \\
  u_m =_{\eta_m} f_m(u_1, \ldots, u_m) & \in (L_m, \sqsubseteq_m) 
\end{cases}
\]

Here for each \( i \in [1, m], (L_i, \leq_i) \) is a complete lattice, \( u_i \) is a variable that ranges over \( L_i \), \( \eta_i \in \{ \mu, \nu \} \) and \( f_i : L_1 \times \cdots \times L_m \to L_i \) is a monotone function.

Definition 2.2 (solution) Let \( E \) be an HES as in Def. 2.1. For each \( i \in [1, m] \) and \( j \in [1, i] \) we inductively define \( f_i^j : L_i \times \cdots \times L_m \to L_i \) and \( l_i^{(j)} : L_{i+1} \times \cdots \times L_m \to L_i \) as follows (no need to distinguish the base case from the step case):

- \( f_i^j(u_i, \ldots, u_m) := f_i(l_{i+1}^{(j-1)}(u_i, \ldots, u_m), \ldots, l_{i-1}^{(j-1)}(u_i, \ldots, u_m), u_i, \ldots, u_m) \); and
- \( l_i^{(j)}(u_{i+1}, \ldots, u_m) := \eta f_i^j(\varnothing, u_{i+1}, \ldots, u_m) \) where \( \eta = \mu \) if \( i \) is odd and \( \eta = \nu \) if \( i \) is even. For \( j < i \), \( l_i^{(j)}(u_{i+1}, \ldots, u_m) := l_{i-1}^{(j-1)}(l_i^{(j-1)}(u_{i+1}, \ldots, u_m), u_{i+1}, \ldots, u_m) \).

If such a least or greatest fixed point does not exist, then it is undefined.

We call \((l_i^{(1)}, \ldots, l_i^{(i)})\) the \( i\)-th intermediate solution. The solution of the HES \( E \) is a family \((u_1^{sol}, \ldots, u_m^{sol}) \in L_1 \times \cdots \times L_m \) defined by \( u_i^{sol} := l_i^{(m)}(\ast) \) for each \( i \).
2.3 Categorical Finite and Infinitary Trace Semantics

We review \[\text{[17][14][13][20]}\] and see how finite and infinitary traces of transition systems are characterized categorically. We assume that the readers are familiar with basic theories of categories and coalgebras. See e.g. \[\text{[5][11]}\] for details.

We model a system as a \((T,F)\)-system, a coalgebra \(c : X \to TFX\) where \(T\) is a monad representing the branching type and \(F\) is an endofunctor representing the transition type of the system. Here are some examples of \(T\) and \(F\):

**Definition 2.3 (\(P, D, L\) and \(\mathcal{G}\))** The powerset monad is a monad \(P = (P, \eta^P, \mu^P)\) on \(\text{Sets}\) where \(PX := \{A \subseteq X\}, Pf(A) := \{f(x) | x \in A\}, \eta^P_X(x) := \{x\}\) and \(\mu^P_X(f) := \bigcup_{A \in f^P} A\). The subdistribution monad is a monad \(D = (D, \eta^D, \mu^D)\) on \(\text{Sets}\) where \(DX := \{\delta : X \to [0,1] | \{x | \delta(x) > 0\}\text{ is countable, and } \sum x \delta(x) \leq 1\}, Df(\delta)(y) := \sum_{x \in f^{-1}(y)} \delta(x), \eta^D_X(x)(x') = 1\text{ if } x = x'\text{ and } 0\text{ otherwise, and } \mu^D_X(\Phi)(x) := \sum_{a \in DX} \Phi(\delta) \cdot \delta(x)\). The lift monad is a monad \(L = (L, \eta^L, \mu^L)\) on \(\text{Sets}\) where \(LX := \{\bot\} + X, Lf(a) = f(a)\text{ if } a \in X\text{ and } +\text{ if } a = \bot, \eta^L_X(x) := x\) and \(\mu^L_X(a) := a\text{ if } a \in X\text{ and } +\text{ if } a = \bot\). The sub-Giry monad is a monad \(\mathcal{G} = (\mathcal{G}, \eta^G, \mu^G)\) on \(\text{Meas}\) where \(\mathcal{G}(X, \mathcal{F}_X)\) is carried by the set of probability measures over \((X, \mathcal{F}_X), \mathcal{G}f(\varphi)(A) := \varphi(f^{-1}(A)), \eta^G_X(x)(A) = 1\text{ if } x \in A\text{ and } 0\text{ otherwise, and } \mu^G_X(\mathcal{E})(A) := \int_{\mathcal{E} \in \mathcal{F}_X} \delta(\mathcal{E})d\mathcal{E}\).

**Definition 2.4 (polynomial functors)** A polynomial functor \(F\) on \(\text{Sets}\) is defined by the following BNF notation: \(F := \text{id} | A \times F | \coprod_{i \in I} F\) where \(A \in \text{Sets}\) and \(I\) is countable. A (standard Borel) polynomial functor \(\hat{F}\) on \(\text{Meas}\) is defined by the following BNF notation: \(\hat{F} := \text{id} | A \times \hat{F} | \coprod_{i \in I} \hat{F}\) where \(A \in \text{Meas}\), \(I\) is countable, and the \(\sigma\)-algebras over products and coproducts are given in the standard manner (see e.g. \[\text{[20]}\text{ Def. 2.2}\]).

A carrier of an initial \(F\)-algebra models a domain of finite traces \[\text{[12]}\] while that of a final \(F\)-coalgebra models a domain of infinitary traces \[\text{[14]}\]. For example, as we have seen in \[\text{[14]}\] for \(F = \{\sqrt{.}\} + A \times (\bot)\) on \(\text{Sets}\), the carrier set of the final \(F\)-coalgebra is \(A^\omega\) while that of the initial \(F\)-algebra is \(A^*\). The situation is similar for a polynomial functor \(F = \{\sqrt{.}\}, \mathcal{P}\{\sqrt{.}\} + (A, \mathcal{PA}) \times (\bot)\) on \(\text{Meas}\). The carrier of an initial algebra is \((A^*, \mathcal{PA}^*)\), and that of a final coalgebra is \((A^\omega, \mathcal{FA}_\omega)\) where \(\mathcal{FA}_\omega\) is the standard \(\sigma\)-algebra generated by the cylinder set.

In general, for a certain class of functors, an initial algebra and a final coalgebra are obtained by the following well-known construction.

**Theorem 2.5 (\[\text{[11]}\]** 1. Let \((A, (\pi_i : F^i0 \to A)_{i \in \omega})\) be a colimit of an \(\omega\)-chain
\[0 \xrightarrow{1} F0 \xrightarrow{F1} F20 \xrightarrow{F21} \ldots\] If \(F\) preserves the colimit, then the unique mediating arrow \(\nu : FA \to A\) from the colimit \((FA, (F\pi_i : F^i+10 \to FA)_{i \in \omega})\) to a cocone \((A, (\pi'_i : F^i0 \to A)_{i \in \omega})\) where \(\pi'_i = \pi_{i+1}\) is an initial \(F\)-algebra.

2. Let \((Z, (\pi_i : A \to F^i1)_{i \in \omega})\) be a limit of an \(\omega\)-chain
\[1 \xleftarrow{1} F1 \xleftarrow{F11} F21 \xleftarrow{F211} \ldots\] If \(F\) preserves the limit, then the unique mediating arrow \(\zeta : Z \to FZ\) from a cone \((Z, (\pi'_i : A \to F^i1)_{i \in \omega})\) where \(\pi'_i = \pi_{i+1}\) to the limit \((FZ, (F\pi_i : FZ \to F^i+11)_{i \in \omega})\) is a final \(F\)-coalgebra. ☐
We next quickly review notions about the Kleisli category $\mathcal{K}(T)$. 

**Definition 2.6** ($\mathcal{K}(T)$, $J$, $U$ and $\mathcal{F}$) Let $T = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. The *Kleisli category* $\mathcal{K}(T)$ is given by $|\mathcal{K}(T)| = |\mathcal{C}|$ and $\mathcal{K}(T)(X,Y) = \mathcal{C}(TX, TY)$ for $X, Y \in |\mathcal{K}(T)|$. An arrow $f \in \mathcal{K}(T)(X,Y)$ is called a *Kleisli arrow*, and we write $f : X \rightarrow Y$ for distinction. Composition of arrows $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is defined by $\mu_Z \circ Tg \circ f$, and denoted by $g \circ f$ for distinction. The *lifting functor* $J : \mathcal{C} \rightarrow \mathcal{K}(T)$ is defined by: $JX := X$ and $J(f) := \eta_Y \circ f$ for $f : X \rightarrow Y$. The *forgetful functor* $U : \mathcal{K}(T) \rightarrow \mathcal{C}$ is defined by: $UX := TX$ and $U(g) := \mu_Y \circ Tg$ for $g : X \rightarrow Y$. A functor $\mathcal{F} : \mathcal{K}(T) \rightarrow \mathcal{K}(T)$ is called a *lifting* of $F : \mathcal{C} \rightarrow \mathcal{C}$ if $\mathcal{F}J = JF$. 

**Example 2.7** Let $T = \mathcal{P}$ and $F = \sum_{n=0}^{\infty} \Sigma_n \times (\_)^n : \text{Sets} \rightarrow \text{Sets}$. A lifting $\mathcal{F}$ over $\mathcal{K}(T)$ is given by $\mathcal{F}X = FX$ for $X \in \text{Sets}$ and $\mathcal{F}f(\sigma, x_0, \ldots, x_{n-1}) = \{(\sigma, y_0, \ldots, y_{n-1}) \mid \forall i, y_i \in f(x_i)\}$ for $f : X \rightarrow Y$, $\sigma \in \Sigma_n$ and $x_0, \ldots, x_{n-1} \in X$. (see e.g. [12]). 

It is well-known that there is a bijective correspondence between a lifting $\mathcal{F}$ and a *distributive law*, a natural transformation $\lambda : FT \Rightarrow TF$ satisfying some axioms [16]. See [13] for the details.

In the rest of this section, let $F$ be an endofunctor and $T$ be a monad on a category $\mathcal{C}$, and assume that a lifting $\mathcal{F} : \mathcal{K}(T) \rightarrow \mathcal{K}(T)$ is given.

In [12], a finite trace semantics of a transition system was characterized as the unique homomorphism to the final $\mathcal{F}$-coalgebra in $\mathcal{K}(T)$, which is obtained by reversing and lifting the initial $F$-algebra in $\mathcal{C}$.

**Definition 2.8** ($\text{tr}(c)$) We say $F$ and $T$ constitute a *finite trace situation* wrt. $\mathcal{F}$ if the following conditions are satisfied:

- An initial $F$-algebra $t^F : FA \rightarrow A$ exists.
- $J(t^F)^{-1} : A \rightarrow \mathcal{F}A$ is a final $\mathcal{F}$-coalgebra.

For $c : X \rightarrow FA$, the unique homomorphism from $c$ to $J(t^F)^{-1}$ is called the *(coalgebraic) finite trace semantics* of $c$ and denoted by $\text{tr}(c) : X \rightarrow A$.

In [12], a sufficient condition for constituting a finite trace situation is given.

**Theorem 2.9** ([12]) Assume each homset of $\mathcal{K}(T)$ carries a partial order $\sqsubseteq$. If the following conditions are satisfied, $F$ and $T$ constitute a finite trace situation.

- The functor $F$ preserves $\omega$-colimits in $\mathcal{C}$.
- Each homset of $\mathcal{K}(T)$ constitutes an $\omega$-cpo with a bottom element $\bot$.
- Kleisli composition $\odot$ is monotone, and the lifting $\mathcal{F}$ is locally monotone, i.e. $f \sqsubseteq g$ implies $\mathcal{F}f \sqsubseteq \mathcal{F}g$.
- Kleisli composition $\odot$ preserves $\omega$-suprema and the bottom element $\bot$. 


Here by Thm. 2.9, the first condition above implies existence of an initial algebra.

In [12] it was shown that \( T \in \{ \mathcal{P}, \mathcal{D}, \mathcal{L} \} \) and a polynomial functor \( F \) satisfy the conditions in Thm. 2.9 wrt. some appropriate orderings and liftings, and hence constitute finite trace situations. We can see the result for \( T = \mathcal{D} \) implies \( T = \mathcal{G} \) and a standard Borel polynomial functor \( F \) also satisfy the conditions.

An infinitary trace semantics was characterized in [13] as the greatest homomorphism to a weakly final coalgebra obtained by lifting a final coalgebra.

**Definition 2.10 (infinitary trace situation)** We further assume that each homset of \( \mathcal{K}(T) \) carries a partial order \( \sqsubseteq \). We say that \( F \) and \( T \) constitute an infinitary trace situation wrt. \( \sqsubseteq \) if the following conditions are satisfied:

- A final \( F \)-coalgebra \( \zeta^F : Z \to FZ \) exists.
- \( J\zeta^F : Z \to ^TFZ \) is a weakly final \( ^TF \)-coalgebra that admits the greatest homomorphism, i.e. for an \( ^TF \)-coalgebra \( c : X \to ^{\mathcal{F}}X \), there exists the greatest homomorphism from \( c \) to \( J\zeta^F \) wrt. \( \sqsubseteq \).

The greatest homomorphism from \( c \) to \( J\zeta^F \) is called the (coalgebraic) infinitary trace semantics of \( c \) and denoted by \( \text{tr}^{\infty}(c) : X \to Z \).

It is known that \( T \in \{ \mathcal{P}, \mathcal{D}, \mathcal{L}, \mathcal{G} \} \) and a polynomial functor \( F \) constitute infinitary trace situations wrt. some orderings and liftings [20]. Differently from finite trace situation, sufficient conditions for infinitary trace situation are not unified. In [20], two families of sufficient conditions are given. One is applicable for \( T = \mathcal{P} \), and the other is for \( T \in \{ \mathcal{L}, \mathcal{G} \} \). No condition is known for \( T = \mathcal{D} \).

**Example 2.11** Let \( T = \mathcal{P} \) and \( F = \{ \checkmark \} + \mathcal{A} \times (\_\_) \). Then a \( TF \)-coalgebra \( c : X \to \mathcal{P}(\{ \checkmark \} + \mathcal{A} \times X) \) is identified with an \( \mathcal{A} \)-labeled nondeterministic automaton whose accepting states are given by \( \{ x \mid \checkmark \in c(x) \} \). The arrow \( \text{tr}(c) \) has a type \( X \to \mathcal{A}^\omega \) and assigns the set of accepted finite words to each state [12]:

\[
\text{tr}(c)(x) = \left\{ a_1a_2\ldots a_n \in \mathcal{A}^\omega \mid \begin{array}{l}
\exists x_0, \ldots, x_n \in X. \forall i \in [1, n-1]. \\
(a_{i+1}, x_{i+1}) \in c(x_i) \text{ and } \checkmark \in c(x_n) 
\end{array} \right\}.
\]

In contrast, \( \text{tr}^{\infty}(c) : X \to \mathcal{A}^\omega \) is given as follows [13]:

\[
\text{tr}^{\infty}(c)(x) = \text{tr}(c)(x) \\
\cup \left\{ a_1a_2\ldots \in \mathcal{A}^\omega \mid \exists x_0, x_1, \ldots \in X. \forall i \in \omega. (a_{i+1}, x_{i+1}) \in c(x_i) \right\}.
\]

**2.4 Büchi \((T,F)\)-systems and its Coalgebraic Trace Semantics**

The results in [2,3] was extended for systems with the parity acceptance condition in [22]. We hereby review the results for the Büchi acceptance condition.

**Definition 2.12 (Büchi \((T,F)\)-system)** Let \( n \in \mathbb{N} \). A Büchi \((T,F)\)-system is a pair \((c, (X_1, X_2))\) of a \( \mathcal{F} \)-coalgebra \( c : X \to ^{\mathcal{F}}X \) in \( \mathcal{K}(T) \) and a partition \((X_1, X_2)\) of \( X \) (i.e. \( X \cong X_1 + X_2 \)). For \( i \in \{1, 2\} \), we write \( c_i \) for \( c\circ\kappa_i : X_i \to ^{\mathcal{F}}X \).
Then for each $i$, we first introduce the categorical datatypes

### 3.1 Categorical Datatypes for Büchi Systems

Their coalgebraic trace semantics is given by a solution of an HES.

**Definition 2.13** ($\text{tr}_i^B(c)$) Assume that each homset of $\mathcal{K}(T)$ carries a partial order $\sqsubseteq$. We say that $F$ and $T$ constitute a Büchi trace situation wrt. $\mathcal{T}$ and $\sqsubseteq$ if they satisfy the following conditions:

- A final $F$-coalgebra $\zeta : Z \to FZ$ exists.
- For an arbitrary Büchi $(T, F)$-system $\mathcal{X} = (c, (X_1, X_2))$, the following HES has a solution.

$$E_c = \begin{cases} u_1 = \mu \zeta^{-1} \circ F[u_1, u_2] \circ c_1 \in (\mathcal{K}(T)(X_1, Z), \sqsubseteq_{X_1, Z}) \\ u_2 = \nu \zeta^{-1} \circ F[u_1, u_2] \circ c_2 \in (\mathcal{K}(T)(X_2, Z), \sqsubseteq_{X_2, Z}) \end{cases}$$

The solution $(u_1^{\text{sol}} : X_1 \to Z, u_2^{\text{sol}} : X_2 \to Z)$ of $E_c$ is called the (coalgebraic) Büchi trace semantics of $\mathcal{X}$. We write $\text{tr}_i^B(c)$ for $u_i^{\text{sol}}$ for each $i$ (see also Eq. (1)).

**Example 2.14** Let $T = \mathcal{P}$ and $F = A \times (\underline{~})$. Then a Büchi $(T, F)$-system $(c : X \to FX, (X_1, X_2))$ is identified with an $A$-labeled Büchi automaton. Following Def. 2.2, we shall sketch how the solution of the HES $E_c$ in Def. 2.13 is calculated. Note that $Z \cong A^\omega$.

- We first calculate an intermediate solution $l_1^{(1)}(u_2) : X_1 \to A^\omega$ as the least fixed point of $u_1 \mapsto \zeta^{-1} \circ F[l_1^{(1)}(u_2)] \circ c_1$.
- We next define $f_2^1 : \mathcal{K}(T)(X_2, Z) \to \mathcal{K}(T)(X_2, Z)$ by $f_2^1(u_2) := \zeta^{-1} \circ F[l_1^{(1)}(u_2)] \circ c_2$.
- We calculate $l_2^{(2)}(*) : X_2 \to A^\omega$ as the least fixed point of $f_2^1$.
- We let $l_1^{(2)}(*) := l_1^{(1)}(l_2^{(2)}) : X_1 \to A^\omega$.

Then for each $i$, the solution $\text{tr}_i^B(c) = l_1^{(2)}(*)$ is given as follows [22]:

$$\text{tr}_i^B(c)(x) := \begin{cases} a_1a_2 \ldots \in A^\omega & \exists x_0, x_1, \ldots \in X, \forall i \in \omega, (a_{i+1}, x_{i+1}) \in c(x_i) \text{ and } x_i \in X_2 \text{ for infinitely many } i \end{cases}.$$

### 3 Alternating Fixed Points of Functors

#### 3.1 Categorical Datatypes for Büchi Systems

We first introduce the categorical datatypes $F^+X$ and $F^\oplus X$, which are understood as least and greatest fixed points of a functor $F$.

**Definition 3.1** ($F^+, F^\oplus$) For $F : \mathcal{C} \to \mathcal{C}$, we define functors $F^+, F^\oplus : \mathcal{C} \to \mathcal{C}$ as follows. Given $X \in \mathcal{C}$, the object $F^+X$ is the carrier of (a choice of) an initial algebra $\iota_X^F : F(F^+X + X) \rightarrow F^+X$ for the functor $F(\underline{~} + X)$. Similarly, the object $F^\oplus X$ is the carrier of a final coalgebra $\zeta_X^F : F^\oplus X \rightarrow F(F^\oplus X + X)$. For
f : X → Y, F+ f : F+ X → F+ Y is given as the unique homomorphism from tF+X to tF+Y o F(IdF+X + f). We define F+ ⊕ : F+ ⊕ X → F+ ⊕ Y similarly.

\[
\begin{align*}
F(F^+ X + X) & \rightarrow F(F^+ Y + X) & F(F^\oplus X + Y) & \rightarrow F(F^\oplus Y + Y) \\
F(F^+ f + f) & \rightarrow F(F^\oplus f + f) \\
\epsilon_F & \rightarrow F(F^+ Y + Y) & \epsilon_F & \rightarrow F(F^\oplus Y + Y) \\
F^+ X & \rightarrow F^+ Y & F^{\oplus} X & \rightarrow F^{\oplus} Y
\end{align*}
\]

**Remark 3.2** The construction F+ resembles the free monad F+ over F. The latter is defined as follows: given X ∈ C, the object F+X is the carrier of an initial algebra F(F+X) + X ≅ F+X for the functor F(−)+ X. The notations generalize the usual distinction between + and *. Indeed, for C = Sets and F = Σ0 ×_ (where Σ0 is an alphabet), we have F+1 = Σ0 (the set of finite words of length ≥ 1) and F+1 = Σ0 (the set of all finite words). Similarly, F+ resembles the free completely iterative monad [15].

**Example 3.3** For F = A × (−), by the construction in Thm 2.3, F+X ≅ A+_X, F^⊕_X ≅ A+_X + A^− and F^⊕+_X ≅ (A^+)+_X + (A^−)^ω. Especially, if we let X = 0 then F^⊕+0 ≅ (A^+)ω. We identify (a_00a_01...a_{0n_0})(a_{10}a_{11}...a_{1n_1})... ∈ F^⊕+0 ≅ (A^+)ω with the following “decorated” sequence:

(a_00, O)(a_01, O)...(a_{0n_0}, O)(a_{10}, O)(a_{11}, O)...(a_{1n_1}, O) ∈ (A × (O, O))ω.

The second component of each element (i.e. decoration) represents a break of a word: it is 2 if it’s the beginning of a word in A+ and 0 if not. It is remarkable that in the sequence above, O always appears infinitely many times. Hence w ∈ (A^+)^ω is understood as an infinite word decorated so that the Büchi condition is satisfied.

We next define Kleisli arrows β1_X and β2_X that are used to define decorated trace semantics (see the diagrams in [2]).

**Definition 3.4** We define natural transformations β1 : F+((F^⊕+ + id) ⇒ F(F+ F^+ ⊕ + F+ ⊕ + id) and β2 : F^+ ⊕ ⇒ F(F+ F^+ ⊕ + F^⊕+ + id) as follows.

\[
\begin{align*}
β_{1X} & := \left( F^+(F^⊕+ X + X) \xrightarrow{(t_{F^⊕+ X}^{-1})} F(F^⊕+ X + F^⊕+ X + X) \right) \\
β_{2X} & := \left( F^⊕+ X \xrightarrow{t_{F^⊕+ X}^{-1}} F(F^⊕+ X + X) \xrightarrow{(t_{F^⊕+ X}^{-1})} F(F^⊕+ X + F^⊕+ X + X) \right)
\end{align*}
\]

**Remark 3.5** As a final coalgebra cF+X is an isomorphism, we can see from Def. 3.3 that F+((F^⊕+ X + X) ≅ F^⊕+ X). For F = A × (−), if we regard F^⊕+ X as (A^+)ω as in Ex. 3.3 F+((F^⊕+ X + X) would be understood as F+((A^+)ω, which is indeed isomorphic to (A^+)ω. However, in this paper, mainly for the sake of simplicity of notations, we explicitly distinguish them and later write types of a decorated trace semantics of a Büchi (T, F)-system as dtr1(c) : X1 → F+(F^⊕+0) and dtr2(c) : X2 → F^⊕+0. Because of this choice, while an element in F^⊕+0 ≅ (A^+)ω...
Example 3.7 Let $\oplus F$.

Example 3.9 $\beta p$

Definition 3.6 (multiplication of those free monads. $F^+$)

We introduce two natural transformations for later use. As mentioned in Rem. 3.2, $3.2$ Natural Transformations Regarding to $F$

Similarly, we define natural transformations $p$ and $p$ respectively, and they are given by the flattening functions. See also Prop. 5.10.

$\frac{\frac{\frac{F(F^\oplus X + F^\oplus X + X) \beta p(u_X \circ \kappa_X)}{\beta \omega X \beta X \omega X}}{\beta \omega X \beta X \omega X}}{\beta \omega X \beta X \omega X}$

is regarded as a decorated word whose first letter is decorated with $\circ$ (Ex. 3.3), an element $a_0 \ldots a_n ((a_0a_1 \ldots a_{n-1})a_0a_1 \ldots a_n)$ is understood as the following decorated sequence:

$\langle a_0, \circ \rangle \ldots (a_n, \circ) \langle a_0, \circ \rangle (a_0, \circ) \ldots (a_{n-1}, \circ) (a_{n-1}, \circ) \ldots (a_1, \circ) \ldots (a_1, \circ)$

3.2 Natural Transformations Regarding to $F^+$ and $F^\oplus$

We introduce two natural transformations for later use. As mentioned in Rem. 3.2, $F^+$ resembles the free monad $F^*$ while $F^\oplus$ is similar to the free completely iterative monad. The first natural transformation we introduce is analogous to the multiplication of those free monads.

Definition 3.6 ($\mu^{F^\oplus}$) We define a natural transformation $\mu^{F^\oplus} : F^\oplus F^\oplus \Rightarrow F^\oplus$ by $\mu^{F^\oplus} := (u_X \circ \kappa_X) \in \mathbb{C}$, where $u_X$ is the unique homomorphism from $[F[\kappa_X, \kappa_X] \circ \xi_F, F[\kappa_X, \kappa_X] \circ \xi_F]$ to $\zeta_X$ (see Fig. 1).

Example 3.7 Let $F = \mathbb{A} \times \bot$. According to the characterizations in Ex. 3.3 and Rem. 3.5, $p^{(1)}_1$ has a type $(A^+) + (A^+) + (A^+) \Rightarrow (A^+) + (A^+) + (A^+)$, and is given by the concatenating function that preserves each finite word.

The second natural transformation is for “removing” decorations.

Definition 3.8 ($p^{(1)}_2$) We define a natural transformation $p^{(1)}_1 : F^+ \Rightarrow F^\oplus$ so that $p^{(1)}_1 : F^+ X \Rightarrow F^\oplus X$ is the unique homomorphism from $J(l_X^F)^{-1}$ to $J\zeta_X^F$.

Similarly, we define natural transformations $p^{(2)}_1 : F^+(F^+ \oplus + \text{id}) \Rightarrow F^\oplus$ and $p^{(2)}_2 : F^+(F^+ \oplus \Rightarrow F^\oplus$ so that $[p^{(1)}_1 X, p^{(2)}_2 X] : F^+(F^+ \oplus X + X) \Rightarrow F^\oplus X$ is the unique homomorphism from $[\beta_1 X, \beta_2 X]$ to $\zeta_X^F$ (see Fig. 2).

Example 3.9 Let $F = \mathbb{A} \times \bot$. According to the characterizations in Ex. 3.3 and Rem. 3.5, $p^{(1)}_1$ has a type $A^+ X \Rightarrow A^+ X + A^\omega$ and is given by the natural inclusion. In contrast, $p^{(2)}_1$ and $p^{(2)}_2$ have types $A^+(A^+)^\omega \Rightarrow A^\omega$ and $(A^+)^\omega \Rightarrow A^\omega$ respectively, and they are given by the flattening functions. See also Prop. 5.10
3.3 Liftings $\overline{F}$ and $\overline{F}^\oplus$ over $\mathcal{K}(T)$

Let $\overline{F}: \mathcal{K}(T) \to \mathcal{K}(T)$ be a lifting of of a functor $F$. We show that under certain conditions, it induces liftings $\overline{F}^{\oplus}: \mathcal{K}(T) \to \mathcal{K}(T)$ of $F^{\oplus}$ and $\overline{F}^\oplus: \mathcal{K}(T) \to \mathcal{K}(T)$ of $F^\oplus$. Note that a lifting $\overline{F}$ induces a lifting $\overline{F}(\_ + \_): \mathcal{K}(T) \to \mathcal{K}(T)$ of $F(\_ + \_)$ which is defined by $\overline{F}(\_ + \_)(f) = \overline{F}(f + \eta_A) = \overline{F}(\eta_{A_1}, \eta_{A_2}) \circ (f + \eta_A)$ using the coproduct in $\mathcal{K}(T)$.

**Definition 3.10** 1. Assume $T$ and $F$ constitute a finite trace situation. For $X \in \mathcal{C}$, we let $\overline{F}X := F^+X$. For $f: X \to Y$, we define $\overline{F}f : \overline{F}X \to \overline{F}Y$ as the unique homomorphism from $\overline{F}(\text{id}_X + f) \circ \overline{F}(\text{id}_Y)$ to $\overline{F}(\text{id}_Y)$. Hence we need an extra assumption to make $\overline{F}$ a functor. We hereby assume a stronger condition than is needed for the sake of discussions in (3).

2. Assume $T$ and $F$ constitute an infinitary trace situation. For $X \in \mathcal{C}$, we let $\overline{F}^\oplus X := F^\oplus X$. For $f: X \to Y$, we define $\overline{F}^\oplus f : \overline{F}^\oplus X \to \overline{F}^\oplus Y$ as the greatest homomorphism from $\overline{F}(\text{id}_X + f) \circ \overline{F}(\text{id}_Y)$ to $\overline{F}(\text{id}_Y)$.

In the rest of this section, we check under which conditions $\overline{F}$ and $\overline{F}^\oplus$ are functors and form liftings of $F^+$ and $F^\oplus$. Funcritality of $\overline{F}$ holds iff for each $f : X \to Y$ and $g : Y \to W$, $\overline{F}g \circ \overline{F}f$ is the unique homomorphism from $\overline{F}(\text{id}_X + g) \circ \overline{F}(\text{id}_Y)$ to $\overline{F}(\text{id}_Y)$. Similarly, functoriality of $\overline{F}^\oplus$ holds iff $\overline{F}^\oplus g \circ \overline{F}^\oplus f$ is the greatest homomorphism from $\overline{F}(\text{id}_X + g) \circ \overline{F}(\text{id}_Y)$ to $\overline{F}(\text{id}_Y)$.

The former always holds by the finality. In contrast, the latter doesn’t necessarily hold: a counterexample is $T = D$ and $F = \{0\} \times (\_)^2$ (see Ex. C.42 for details). Hence we need an extra assumption to make $\overline{F}^\oplus$ a functor. We hereby assume a stronger condition than is needed for the sake of discussions in (3).

**Definition 3.11 ($\Phi_{c, \sigma}$)** Let $c: X \to \overline{F}X$ and $\sigma: \overline{F}Y \to Y$. We define a function $\Phi_{c, \sigma}: \mathcal{K}(T)(X, Y) \to \mathcal{K}(T)(X, Y)$ by $\Phi_{c, \sigma}(f) := \sigma \circ \overline{F}f \circ c$.

**Definition 3.12** Assume that $T$ and $F$ constitute an infinitary trace situation. Let $\overline{F}^\oplus X = \overline{F}X$ and $\overline{F}^\oplus$ be a final $F$-coalgebra. We say that $T$ and $F$ satisfy the gfp-preserving condition wrt. an $F$-algebra $\sigma: MY \to Y$ if for each $X \in \mathcal{C}$ and $c: X \to FX$, if $l : X \to Z$, then $m \circ l : X \to Y$ is the greatest fixed point of $\Phi_{c, \sigma}$. We next check if $\overline{F}$ and $\overline{F}^\oplus$ are liftings of $F^+$ and $F^\oplus$. It is immediate by definition that $\overline{F}^\oplus X = \overline{F}X$ and $\overline{F}^\oplus JX = \overline{F}^\oplus X$ for each $X \in \mathcal{C}$. Let $f : X \to Y$. By definition, $\overline{F}f : \overline{F}X \to \overline{F}Y$ holds iff $\overline{F}f$ is a unique homomorphism from $\overline{F}(\text{id}_X + Jf) \circ J(\text{id}_Y)$ to $\overline{F}(\text{id}_Y)$.
The former is easily proved by the finality of $J(\xi_F)^{-1}$, while the latter requires an assumption again.

**Definition 3.13** Assume $T$ and $F$ constitute an infinitary trace situation. Let $\xi^F : Z \to FZ$ be a final $F$-coalgebra. We say that $T$ and $F$ satisfy the deterministic-greatest condition if for $c : X \to FX$ in $\mathbb{C}$, if $u : X \to Z$ is the unique homomorphism from $c$ to $\xi^F$ then $Ju$ is the greatest homomorphism from $Jc$ to $J\xi^F$.

Concluding the discussions so far, we obtain the following proposition.

**Proposition 3.14** 1. If $T$ and $F(\_+\_A)$ constitute a finite trace situation for each $A \in \mathbb{C}$, the operation $F^+$ is a functor and is a lifting of $F^+$.

2. If $T$ and $F(\_+\_A)$ constitute an infinitary trace situation and satisfy the gfp-preserving condition wrt. an arbitrary algebra and the deterministic-greatest condition for each $A \in \mathbb{C}$, then $F^{\oplus}$ is a functor and is a lifting of $F^{\oplus}$. □

Hence under appropriate conditions, a lifting $\overline{F} : \mathcal{K}(T) \to \mathcal{K}(T)$ of $F$ gives rise to liftings of $F^+$ and $F^{\oplus}$. By repeating this, we can define $F^{(i)}_{\_j}$ for each $i$ and $j$.

See [4] for the distributive laws corresponding to the liftings defined above.

**Example 3.15** Let $F = A \times (\_)$ and $T = P$. As we have seen in Ex. 3.6, $F^+X \cong (A^+)^+X + (A^+)\omega$. Let $\overline{F}$ be a lifting that is given as in Ex. 2.7. We can construct a lifting $F^{\oplus}$ using Prop. 3.14 and for $f : X \to Y$ in $\mathcal{K}(P)$, $F^{\oplus}f : (A^+)^+X + (A^+)\omega \to (A^+)^+Y + (A^+)\omega$ is given by $F^{\oplus}f(w) = \{w'y \mid y \in f(x)\}$ if $w = w'x$ where $w' \in (A^+)^+$ and $x \in X$, and $\{w\}$ if $w \in (A^+)\omega$.

## 4 Decorated Trace Semantics of Büchi $(T, F)$-systems

### 4.1 Definition

**Assumption 4.1** Throughout this section, let $T$ be a monad and $F$ be an endofunctor on $\mathbb{C}$, and assume that each homset of $\mathcal{K}(T)$ carries a partial order $\sqsubseteq$. We further assume the following conditions for each $A \in \mathbb{C}$.

1. $F^+ : \mathcal{C} \to \mathcal{C}$ are well-defined and liftings $\overline{F}, F^+, F^{\oplus} : \mathcal{K}(T) \to \mathcal{K}(T)$ are given.

2. $T$ and $F(\_+\_A)$ satisfy the conditions in Thm. 2.9 wrt. $F^+$ and hence constitute a finite trace situation.

3. $T$ and $F^+\_A$ constitute an infinitary trace situation wrt. $F^+(\_+\_A)$ and $\sqsubseteq$.

4. $T$ and $F^+(\_+\_A)$ satisfy the gfp-preserving condition wrt. an arbitrary $\sigma$.

5. $T$ and $F^+(\_+\_A)$ satisfy the deterministic-greatest condition.

6. The liftings $\overline{F}^+$ and $F^{\oplus}$ are obtained from $\overline{F}$ and $\overline{F}^+$ using the procedure in Def. 3.10 respectively.

7. $F^+(\_+\_A)$ and $F^{\oplus}(\_+\_A)$ are locally monotone.

8. $T$ and $F$ constitute a Büchi trace situation wrt. the same $\sqsubseteq$ and $\overline{F}$.
Using the categorical data type defined in [3] we now introduce a decorated Büchi trace semantics $dtr_1(c) : X_1 \rightarrow F^+(F^{+\oplus}0)$ and $dtr_2(c) : X_2 \rightarrow F^{+\oplus}0$.

**Definition 4.2 (dtr$_i$(c))** For a Büchi $(T, F)$-system $(c, (X_1, X_2))$, the decorated Büchi trace semantics is a solution $(dtr_1(c) : X_1 \rightarrow F^+(F^{+\oplus}0), dtr_2(c) : X_2 \rightarrow F^{+\oplus}0)$ of the following HES (see also Eq. (2)).

\[
\begin{align*}
v_1 &= \nu J(\beta_1^0)^{-1} \odot \overline{F}(v_1 + v_2) \circ c_1 \in (\mathcal{K}(T)(X_1, F^+(F^{+\oplus}0)), \sqsubseteq) \\
v_2 &= \nu J(\beta_2^0)^{-1} \odot \overline{F}(v_1 + v_2) \circ c_2 \in (\mathcal{K}(T)(X_2, F^{+\oplus}0)), \sqsubseteq)
\end{align*}
\]

Existence of a solution will be proved in the next section.

### 4.2 Trace Semantics vs. Decorated Trace Semantics

This section is devoted to sketching the proof of the following theorem, which relates decorated trace semantics $dtr_i(c)$ and Büchi trace semantics $tr^B_i(c)$ in [22] via the natural transformation in Def. 3.8.

**Theorem 4.3** For each $i \in \{1, 2\}$, $tr^B_i(c) = p_{i,0}^2 \circ dtr_i(c)$.

To prove this, we introduce Kleisli arrows $c_2^\dagger, \tilde{\ell}_1^{(1)}, \tilde{\ell}_1^{(2)}, \tilde{\ell}_2^{(2)}$. They are categorical counterparts to $\overline{F}_2, l_1^{(1)}, l_1^{(2)}$ and $l_2^{(2)}$ (see Def. 2.2) for the HES defining $tr^B_i(c)$ (see Def. 2.13), and bridge the gap between $dtr_i(c)$ and $tr^B_i(c)$.

**Definition 4.4** ($c_2^{\dagger}, \tilde{\ell}_1^{(1)}, \tilde{\ell}_1^{(2)}, \tilde{\ell}_2^{(2)}$) We define Kleisli arrows $\tilde{\ell}_1^{(1)} : X_1 \rightarrow F^+X_2$, $c_2^{\dagger} : X_2 \rightarrow F^+X_2$, $\tilde{\ell}_1^{(2)} : X_2 \rightarrow F^{+\oplus}0$ and $\tilde{\ell}_2^{(2)} : X_1 \rightarrow F^{+\oplus}0$ as follows:

- We define $\tilde{\ell}_1^{(1)} : X_1 \rightarrow F^+X_2$ as the unique homomorphism from an $F(\_ + X_2)$-coalgebra $c_1$ to $J(JF_{X_2})^{-1}$ (see the left diagram in Eq. (4) below).
- We define $c_2^{\dagger} : X_2 \rightarrow F^+X_2$ by:

\[
c_2^{\dagger} := \left(X_2 \xrightarrow{c_2} F(X_1 + X_2) \xrightarrow{F(\odot)} F(F^+X_2 + X_2) \xrightarrow{\overline{F}(\odot)} F^+X_2 \xrightarrow{\nu J_{\ell_2}} F^+(F^{+\oplus}0) \right).
\]

- We define $\tilde{\ell}_1^{(2)} : X_2 \rightarrow F^{+\oplus}0$ as the greatest homomorphism from $c_2^{\dagger}$ to $J_{\ell_0}^{F^+}$ (see the right diagram below).

- We define $\tilde{\ell}_1^{(2)} : X_1 \rightarrow F^+(F^{+\oplus}0)$ as follows:

\[
\tilde{\ell}_1^{(2)} := \left(X_1 \xrightarrow{\tilde{\ell}_1^{(2)}} F^+(F^{+\oplus}0) \xrightarrow{F(\odot)} F^+(F^{+\oplus}0) \right).
\]
We explain an intuition why Kleisli arrows defined above bridge the gap between $\text{tr}_i^B(c)$ and $\text{dtr}_i(c)$. One of the main differences between them is that $\text{tr}_1^B(c)$ is calculated from $\iota_1^{(1)}(u_2)$ which is the least fixed point of a certain function, while $\text{dtr}_1(c)$ is defined as the greatest fixed point. The arrow $\tilde{\ell}_1^{(1)}$ fills the gap because it is defined as the unique fixed point, which is obviously both the least and the greatest fixed point.

We shall prove Thm. 4.3 following the intuition above. The lemma below, which is easily proved by the finality of $a$, shows that not only $\tilde{\ell}_1^{(1)}$ but also $\tilde{\ell}_2^{(2)}$ is characterized as the unique homomorphism.

**Lemma 4.5** The Kleisli arrow $\tilde{\ell}_1^{(2)} : X_1 \to F^+(F^+ \circ_0 f)$ is the unique homomorphism from $\text{Tr}((\text{id} + \tilde{\ell}_2^{(2)}) \circ c_1)$ to $J(\ell_{F^++0})^{-1}$. □

Together with the definition of $\tilde{\ell}_2^{(2)}$, we have the following proposition.

**Proposition 4.6** For each $i \in \{1, 2\}$, $\tilde{\ell}_i^{(2)} = \text{dtr}_i(c)$. □

This proposition implies the existence of a solution of the HES in Def. 1.2.

It remains to show the existence of the $\ell_j^{(i)}$ and $\text{tr}_i^B(c)$. By using that $\tilde{\ell}_1^{(1)}$ is the unique fixed point (and hence the least fixed point), we can prove the following equality for an arbitrary $u_2 : X_2 \to F^{\oplus 0}$.

$$\iota_1^{(1)}(u_2) = \left( X_1 \xrightarrow{\ell_1^{(1)}} F^+ X_2 \xrightarrow{F^+ u_2} F^+ F^{\oplus 0} \xrightarrow{J(\ell_{F^++0})^{-1} \circ F^+ F^{\oplus 0}} F^{\oplus 0} \right)$$

The following equalities are similarly proved using the equality above.

$$\iota_1^{(2)}(\ast) = \left( X_1 \xrightarrow{\iota_1^{(2)}} F^+ F^{\oplus 0} \xrightarrow{F^+ F^{\oplus 0}} F^+ F^{\oplus 0} \xrightarrow{J(\iota_{F^++0}^{(2)})^{-1} \circ F^+ F^{\oplus 0}} F^{\oplus 0} \right)$$

$$\iota_2^{(2)}(\ast) = \left( X_2 \xrightarrow{\iota_2^{(2)}} F^{\oplus 0} \xrightarrow{F^{\oplus 0} F^{\oplus 0}} F^{\oplus 0} \xrightarrow{J(\iota_{F^++0}^{(2)})^{-1} \circ F^{\oplus 0} F^{\oplus 0}} F^{\oplus 0} \right)$$

By the definition of $\text{tr}_i^B(c)$, these equalities imply the following proposition.

**Proposition 4.7** For each $i \in \{1, 2\}$, $\text{tr}_i^B(c) = \iota_i^{(2)} \circ \tilde{\ell}_i^{(2)}$. □

Prop. 4.6 and Prop. 4.7 immediately imply Thm. 4.3.

## 5 Decorated Trace Semantics for Nondeterministic Büchi Tree Automata

We apply the framework developed in [34] to **nondeterministic Büchi tree automata** (NBTA), systems that nondeterministically accept trees wrt. the Büchi condition (see e.g. [19]). We show what datatypes $F^+(F^+ \circ_0 F^+ \circ_0)$ and $\text{dtr}_i(c)$ characterize for an NBTA. We first review some basic notions.
5.1 Preliminaries on Büchi Tree Automaton

**Definition 5.1 (ranked alphabet)** A ranked alphabet is a set \( \Sigma \) equipped with an arity function \(|-| : \Sigma \to \mathbb{N} \). We write \( \Sigma_n \) for \( \{ a \in \Sigma \mid |a| = n \} \).

For a set \( X \), we regard \( \Sigma + X \) as a ranked alphabet by letting \(|x| = 0\). We also regard \( \Sigma \times X \) as a ranked alphabet by letting \(|(a,x)| = |a|\).

**Definition 5.2 (\( \Sigma \)-labeled tree, [8])** A tree domain is a set \( D \subseteq \mathbb{N}^* \) s.t.: i) \( \langle \rangle \in D \), ii) for \( w, w' \in \mathbb{N}^* \), \( ww' \in D \) implies \( w \in D \) (i.e. it is prefix-closed), and iii) for \( w \in D \) and \( i, j \in \mathbb{N} \), \( wi \in D \) and \( j \leq i \) imply \( wj \in D \) (i.e. it is downward-closed). A \( \Sigma \)-labeled (infinitary) tree is a pair \( t = (D, l) \) of a tree domain \( D \) and a labeling function \( l : D \to \bigcup_{n \in \mathbb{N}} \Sigma_n \) s.t. \( w \in D \), \(|l(w)| = n \) implies \( \{ i \in \mathbb{N} \mid wi \in D \} = [0, n - 1] \). A \( \Sigma \)-labeled tree \( t = (D, l) \) is finite if \( D \) is a finite set. We write \( \text{Tree}_\infty(\Sigma) \) (resp. \( \text{Tree}_{\text{fin}}(\Sigma) \)) for the set of \( \Sigma \)-labeled infinitary (resp. finite) trees. For \( w \in D \), the \( w \)-th subtree \( t_w \) of \( t \) is defined by \( t_w = (D_w, l_w) \) where \( D_w := \{ w' \in \mathbb{N}^* \mid ww' \in D \} \) and \( l_w(w') := l(ww') \). A branch of \( t \) is a possibly infinite sequence \( i_1i_2\ldots \in \mathbb{N}^\infty \) s.t. \( i_1i_2\ldots i_k \in D \) for each \( k \in \mathbb{N} \), and if it is a finite sequence \( i_1i_2\ldots i_k \) then \(|l(i_0i_1\ldots i_k)| = 0\). We sometimes identify a branch \( i_0i_1\ldots \in \mathbb{N}^\infty \) with a sequence \( l(\langle \rangle)i_1l(i_1i_2)\ldots \in \Sigma^\infty \).

**Remark 5.3** For the sake of notational simplicity, we identify a \( \Sigma \)-labeled tree with a \( \Sigma \)-term in a natural manner. For example, a \( \{a, b, c\}\)-term \( \{a, b, b\} \) denotes an \( \{a, b\}\)-labeled finite tree \( t = \langle \langle \rangle, 0, 1 \rangle, \langle \rangle \mapsto a, 0 \mapsto b, 1 \mapsto b \rangle \). Moreover, for \( \{a, b, c\}\)-labeled trees \( t_0 = (D_0, l_0) \) and \( t_1 = (D_1, l_1) \), we write \( (c, t_0, t_1) \) for a tree \( t = \langle \langle \rangle \cup \{ 0w \mid w \in D_0 \} \cup \{ 1w \mid w \in D_1 \} \rangle, \langle \rangle \mapsto c, 0w \mapsto l_0(w), 1w \mapsto l_1(w) \rangle \).

**Definition 5.4 (NBTA)** A nondeterministic Büchi tree automaton (NBTA) is a tuple \( \mathcal{A} = (X, \Sigma, \delta, \text{Acc}) \) of a state space \( X \), a ranked alphabet \( \Sigma \), a transition function \( \delta : X \to \mathcal{P}(\bigcup_{n \in \mathbb{N}} \Sigma_n \times X^n) \) and a set \( \text{Acc} \subseteq X \) of accepting states.

**Definition 5.5 (\( L^\Sigma_B \))** Let \( \mathcal{A} = (X, \Sigma, \delta, \text{Acc}) \) be an NBTA. A run tree over \( \mathcal{A} \) is a \( (\Sigma \times X) \)-labeled tree \( \rho \) such that for each subtree \((\langle a, x \rangle, (a_0, x_0), t_{00}, \ldots, t_{0n_0}), \ldots, (\langle a_n, x_n \rangle, t_{n0}, \ldots, t_{nn_0})\), \( (a, x_0, \ldots, x_n) \in \delta(x) \) holds. A run tree is accepting if for each branch \((a_0, x_0)(a_1, x_1)\ldots \in (\Sigma \times X)^\infty, x_i \in \text{Acc} \) for infinitely many \( i \). We write Run(\( \mathcal{A} \))(\( x \)) (resp. \( \text{AccRun}(\mathcal{A})(\mathcal{A}) \)) for the set of run trees (resp. accepting run trees) whose root node is labeled by \( x \in X \). For \( A \subseteq X \), Run\( \mathcal{A}(A) \) denotes \( \cup_{x \in A} \text{Run}(x) \). We define \( \text{AccRun}(\mathcal{A})(A) \) similarly. If no confusion is likely, we omit the subscript \( \mathcal{A} \). We define DelSt : Run(\( X \)) \to Tree_\infty(\( \Sigma \)) by DelSt\( (D, l) := (D, l') \) where \( l'(w) := \pi_1(l(w)) \). The language \( L^\Sigma_B : X \to \mathcal{P} \text{Tree}_\infty(\Sigma) \) of \( \mathcal{A} \) is defined by \( L^\Sigma_B(\mathcal{A})(x) = \text{DelSt}(\text{AccRun}(\mathcal{A})(x)) \).

5.2 Decorated Trace Semantics of NPTA

A ranked alphabet \( \Sigma \) induces a functor \( F_\Sigma = \prod_{n \in \mathbb{N}} \Sigma_n \times (\underline{\_})^n : \text{Sets} \to \text{Sets} \). In [22], an NBTA \( \mathcal{A} \) was modeled as a Büchi \( \langle \mathcal{P}, F_\Sigma \rangle \)-system, and it was shown that \( L^\Sigma_B \) is characterized by a coalgebraic Büchi trace semantics \( \text{tr}_B(c) \).
Proposition 5.6 ([22]) For $X, Y \in \text{Sets}$, we define an order $\sqsubseteq$ on $\mathcal{K}\ell(P)(X, Y)$ by $f \sqsubseteq g \overset{\text{def}}{\iff} \forall x \in X. f(x) \sqsubseteq g(x)$. We define $\overline{T}_\Sigma : \mathcal{K}\ell(P) \to \mathcal{K}\ell(P)$ by $\overline{T}_\Sigma X := X$ for $X \in \mathcal{K}\ell(P)$ and $\overline{T}_\Sigma f(a, x_1, \ldots, x_n) := \{ (a, y_1, \ldots, y_n) \mid y_i \in f(x_i) \}$ for $f : X \to Y$. It is easy to see that $\overline{T}_\Sigma$ is a lifting of $T_\Sigma$. Then we have:

1. $\mathcal{P}$ and $F_\Sigma$ constitute a Büchi situation (Def. 2.13) wrt. $\sqsubseteq$ and $\overline{T}_\Sigma$.
2. The carrier set of the final $F_\Sigma$-coalgebra is isomorphic to $\text{Tree}_\infty(\Sigma)$.
3. For an NBTA $A = (X, \Sigma, \delta, \text{Acc})$, we define a Büchi $(\mathcal{P}, F_\Sigma)$-system $(c : X \to F_\Sigma X, (X_1, X_2))$ by $c := \delta$, $X_1 := X \setminus \text{Acc}$ and $X_2 := \text{Acc}$. Then we have: $[\text{tr}^A(c), \text{tr}^A_\infty(c)] = L^A : X \to \mathcal{P}\text{Tree}_\infty(\Sigma)$.

In the rest of this section, for an NBTA $A = (X, \Sigma, \delta, \text{Acc})$ modeled as a $(\mathcal{P}, F_\Sigma)$-system ($c : X \to \mathcal{P}F_\Sigma X, (X_1, X_2)$), we describe $\text{dtr}_i(c)$ and show the relationship with $\text{tr}^A_i(c)$ using Thm. 4.3.

We first describe datatypes $F_\Sigma^+ (F_\Sigma^+ \oplus 0)$ and $F_\Sigma^+ \oplus 0$ referring to the construction of a final coalgebra in Thm. 2.5. We can easily see that $F_\Sigma^+ A \cong \text{Tree}_\infty(\Sigma + A) := \text{Tree}_\infty(\Sigma + A) \setminus \{ x \mid x \in A \}$. Hence for each $i \in \omega$, by a similar characterization to Ex. 3.3 we have:

$$(F_\Sigma^+ (- + 0))^i \cong \text{Tree}_\infty(\Sigma, \text{Tree}_\infty(\Sigma, \ldots, \text{Tree}_\infty(\Sigma, \{\{\} \ldots) \} \cong \left\{ \xi \in \text{Tree}_\infty(\Sigma \times \{\{\}, \{\} \} \mid \text{the root node is labeled by } \{\}, \text{ and for each branch } \{ + \} \} \right\}.$$

Therefore $F_\Sigma^+ \oplus 0$, a limit of the above sequence by Thm. 2.5 and $F_\Sigma^+ (F_\Sigma^+ \oplus 0)$ are characterized as follows:

Proposition 5.7 We define $\text{AccTree}_i (\Sigma) \subseteq \text{Tree}_\infty(\Sigma \times \{\{\}, \{\} \})$ by:

$$\text{AccTree}_i (\Sigma) := \left\{ \xi \in \text{Tree}_\infty(\Sigma \times \{\{\}, \{\}) \mid \text{the root node is labeled by } \bullet, \text{ and for each infinite branch } \{\} \text{ appears infinitely often} \right\}.$$

where $i \in \{1, 2\}$ and $\bullet$ is $\{\}$ if $i = 1$ and $\{\}$ if $i = 2$. Then $\text{AccTree}_1 (\Sigma) \cong F_\Sigma^+ (F_\Sigma^+ \oplus 0)$ and $\text{AccTree}_2 (\Sigma, A) \cong F_\Sigma^+ \oplus 0$.

We now show what $\text{dtr}_i(c)$ characterizes for an NBTA wrt. the characterization in Prop. 5.7. Firstly, the assumptions in the previous section are satisfied.

Proposition 5.8 Asm. 4.7 is satisfied by $(T, F) = (\mathcal{P}, F_\Sigma)$.

By Prop. 5.7, for $i \in \{1, 2\}$, $\beta_{i0}$ (see Def. 3.4) has a type

$$\beta_{i0} : \text{AccTree}_i (\Sigma) \to \prod_{n \in \omega} \Sigma_n \times (\text{AccTree}_1 (\Sigma) + \text{AccTree}_2 (\Sigma)),$$

and is given by $\beta_{iA}(\xi) = (a, \xi_0, \ldots, \xi_{n-1})$ if the root of $\xi$ is labeled by $(a, \bullet) \in \Sigma_n \times \{\{\}, \{\} \}$. Using this, we can show the following characterization of $\text{dtr}_i(c)$.
Proposition 5.9 Let \( A = (X, \Sigma, \delta, \text{Acc}) \) be an NBTA. We define \( \Omega : \text{Run}(X) \to \text{Tree}_\infty(\Sigma \times \{\top, \bot\}) \) by \( \Omega(D, l) := (D, l') \) where for \( w \in D \) s.t. \( l(w) = (a, x) \), \( l'(w) := (a, \top) \) if \( x \notin \text{Acc} \) and \( (a, \bot) \) if \( x \in \text{Acc} \). We define a Büchi \((\mathcal{P}, \mathcal{F}_X)\)-system \((c : X \to \mathcal{F}_X \cdot X, (X_1, X_2))\) as in Prop. 5.8. Then for \( i \in \{1, 2\} \) and \( x \in X_i \),

\[
dtr_i(c)(x) = \{ \Omega(\rho) \in \text{AccTree}_i(\Sigma) \mid \rho \in \text{AccRun}_i(\mathcal{A}) \}.
\]

We conclude this section by instantiating \( p_{i,A}^{(2)} \) (Def. 3.8) for NBTA.

Proposition 5.10 We overload \( \text{DelSt} \) and define \( \text{DelSt} : \text{AccTree}_1(\Sigma) + \text{AccTree}_2(\Sigma) \to \text{Tree}_\infty(\Sigma) \) by \( \text{DelSt}(D, l) := (D, l') \) where \( l'(w) := \pi_1(l(w)) \). Then with respect to the isomorphism in Prop. 2.7 \( \text{DelSt}(\xi) = p_{i,A}^{(2)}(\xi) \) for each \( i \in \{1, 2\} \) and \( \xi \in \text{AccTree}_i(\Sigma) \).

Hence Thm. 4.3 results in the following (obvious) equation for NBTA:

\[
\{ \text{DelSt}(\Omega(\rho)) \mid \rho \in \text{AccRun}_i(\mathcal{A}) \} = L_{i,A}^\infty(x).
\]

6 Systems with Other Branching Types

In this section we briefly discuss other monads than \( T = \mathcal{P} \). As we have discussed in [3.3] the framework does not apply to \( T = \mathcal{D} \).

Let \( T = \mathcal{L} \) and \( F = \mathcal{F}_X \). A Büchi \((\mathcal{L}, \mathcal{F}_X)\)-system \((c : X \to \mathcal{F}_X \cdot X, (X_1, \ldots, X_{2n}))\) is understood as a \( \Sigma \)-labeled deterministic Büchi tree automaton with an exception. In a similar manner to \( T = \mathcal{P} \) we can prove that they satisfy Asm. 1.1. The resulting decorated trace semantics has a type \( \text{dtr}_i(c) : X_i \to \{\bot\} + \text{AccTree}_i(\Sigma) \).

Note that once \( x \in X \) is fixed, either of the following occurs: a decorated tree is determined according to \( c \) or \( \bot \) is reached at some point. The function \( \text{dtr}_i(c) \) assigns \( \bot \) to \( x \in X_i \) if \( \bot \) is encountered from \( x \) or the resulting decorated tree does not satisfy the Büchi condition: otherwise, the generated tree is assigned to \( x \). See [4.1] for detailed discussions, which includes the case of parity automata.

We next let \( T = \mathcal{G} \). A Büchi \((\mathcal{G}, \mathcal{F}_X)\)-system is understood as a probabilistic Büchi tree automaton. In fact, it is open if \( T = \mathcal{G} \) and \( F = \mathcal{F}_X \) satisfy Asm. 1.1. The challenging part is the gfp-preserving condition (Asm. 4.11). However, by carefully checking the proofs of the lemmas and the propositions where the gfp-preserving condition is used (i.e. Prop. 3.14, Lem. 4.5, and Prop. 4.7), we can show that Asm. 4.11 can be relaxed to the following weaker but more complicated conditions:

\textbf{H1-1.} \( T \) and \( F^+(\bot + A) \) satisfy the gfp-preserving condition wrt. an algebra \( \mathcal{F}^+(F^+ \oplus B + A) \) \(
\xymatrix{F^+(\bot + f \cdot i) \ar[r]^j & \mathcal{F}^+(F^+ \oplus B + B) \ar[r]^-{(\xi_{B,F^+})^{-1}} & F^+ \oplus B}
\) for each \( f : A \to B \);

\textbf{H1-2.} \( T \) and \( F^+(\bot + A) \) satisfy the gfp-preserving condition wrt. an algebra \( F^+(F^+ \oplus A + A) \) \(
\xymatrix{F^+(F^+ \oplus A + A) \ar[r]^j & F^+ \oplus F^+ A \ar[r]^-{(\xi_{A,F^+})^{-1}} & F^+ \oplus A}
\) where \( \tau \) is the unique homomorphism from \((\nu_{F^+ \oplus A + A})^{-1}\) to \( \xi_{F^+ \oplus A + A} \); and
4.3. $T$ and $F(\_+A)$ satisfy the gfp-preserving condition wrt. an algebra $F(F^\oplus A + F^\oplus A + A) \xrightarrow{JF([id,id]+id)} F(F^\oplus A + A) \xrightarrow{J(\zeta)^{-1}} F^\oplus A$.

In fact, only the first condition is sufficient to prove Prop. 4.4 and Lem. 4.5.

We can show that $T = \mathcal{G}$ and $F = F^\Sigma$ on Meas satisfy the above weakened gfp-preserving condition, and hence we can consider a decorated trace semantics $dT(c)$ for a Büchi $(\mathcal{G}, F^\Sigma)$-system $(c : X \rightarrow T^\Sigma X, (X_1, X_2))$ and use Thm. 4.3.

Assume $X$ is a countable set equipped with a discrete $\sigma$-algebra for simplicity. Then the resulting decorated trace semantics $dT(c)$ has a type $X_i \rightarrow \mathcal{G}(\text{AccTree}_i(\Sigma), \mathfrak{F}_{\text{AccTree}_i}(\Sigma))$ where $\mathfrak{F}_{\text{AccTree}_i}(\Sigma)$ is the standard $\sigma$-algebra generated by cylinders. The probability measure assigned to $x \in X_i$ by $dT(c)$ is defined in a similar manner to the probability measure over the set of run trees generated by a probabilistic Büchi tree automaton (see e.g., [20]).

The situation is similar for parity $(\mathcal{G}, F^\Sigma)$-systems. See §4.2 for the details.

7 Conclusions and Future Work

We have introduced a categorical data type for capturing behavior of systems with Büchi acceptance conditions. The data type was defined as an alternating fixed point of a functor, which is understood as the set of traces decorated with priorities. We then defined a notion of coalgebraic decorated trace semantics, and compared it with the coalgebraic trace semantics in [22]. We have applied our framework for nondeterministic Büchi tree automata, and showed that decorated trace semantics is concretized to a function that assigns a set of trees decorated with priorities so that the Büchi condition is satisfied in every branch. We have focused on the Büchi acceptance condition for simplicity, but all the results can be extended to the parity acceptance condition (see §A).

Future Work There are some directions for future work. In this paper we focused on systems with a simple branching type like nondeterministic or probabilistic. Extending this so that we can deal with systems with more complicated branching type like two-player games (systems with two kinds of nondeterministic branching) or Markov decision processes (systems with both nondeterministic and probabilistic branching) is a possible direction of future work.

Another direction would be to use the framework developed here to categorically generalize a verification method. For example, using the framework of coalgebraic trace semantics in [22], a simulation notion for Büchi automata is generalized in [21]. Searching for an existing verification method that we can successfully generalize in our framework would be interesting.

Finally, it was left open in [10] if Asm. 4.1.4 is satisfied by $T = \mathcal{G}$ and $F = F^\Sigma$. Investigating this is clearly a future work.

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Appendix

A Categorical Parity Conditions via Alternating Fixed Points of Functors

The parity condition is a generalization of the Büchi condition. A parity automaton is equipped with a priority function $\Omega : X \rightarrow [1, 2n]$ that assigns a natural number called a priority to each state $x \in X$. An infinite run $x_0 \trans a x_1 \trans a \cdots$ satisfies the parity condition if $\limsup_{i \rightarrow \infty} \Omega(x_i)$ is even. For example, the parity automaton on the right above accepts an infinite word iff it contains infinitely many $b$ but only finitely many $c$.

A.1 Parity $(T, F)$-systems and its Coalgebraic Trace Semantics

Analogous results to those in [22] hold for the parity acceptance condition [22]. In [22] a priority function of a parity automaton was captured by a partition $X = X_1 + \cdots + X_2n$ of a state space $X$ so that $X_i$ collects states with priority $i$, and trace semantics was modeled by a solution of a hierarchical equation system that is similar to [1] but consists of $2n$ diagrams.

Definition A.1 (parity $(T, F)$-system) Let $n \in \mathbb{N}$. A parity $(T, F)$-system is a pair $(c, (X_1, \ldots, X_{2n}))$ of a $\overline{F}$-coalgebra $c : X \rightarrow \overline{F}X$ in $\mathcal{K}(T)$ and a partition $(X_1, \ldots, X_{2n})$ of $X$ (i.e. $X \cong X_1 + \cdots + X_{2n}$). For $i \in [1, 2n]$, we write $c_i$ for $c \circ \kappa_i : X_i \rightarrow \overline{F}X$.

Their coalgebraic trace semantics is given by a solution of an HES.

Definition A.2 (tr$^p_i(c)$) Assume that each homset of $\mathcal{K}(T)$ carries a partial order $\subseteq$. We say that $F$ and $T$ constitute a parity trace situation wrt. $\overline{F}$ and $\subseteq$ if they satisfy the following conditions:

- A final $F$-coalgebra $\zeta : Z \rightarrow FZ$ exists.
- For an arbitrary parity $(T, F)$-system $X = (c, (X_1, \ldots, X_{2n}))$, the following HES has a solution:

$$E_c = \begin{cases} u_1 =_{\mu} J\zeta^{-1} \circ \overline{F}[u_1, \ldots, u_{2n}] \circ c_1 \in (\mathcal{K}(T)(X_1, Z), \subseteq X_1, Z) \\ u_2 =_{\nu} J\zeta^{-1} \circ \overline{F}[u_1, \ldots, u_{2n}] \circ c_2 \in (\mathcal{K}(T)(X_2, Z), \subseteq X_2, Z) \\ \vdots \\ u_{2n} =_{\nu} J\zeta^{-1} \circ \overline{F}[u_1, \ldots, u_{2n}] \circ c_{2n} \in (\mathcal{K}(T)(X_{2n}, Z), \subseteq X_{2n}, Z) \end{cases}$$

The solution $(u^\text{sol}_i : X_i \rightarrow Z)_{1 \leq i \leq 2n}$ of $E_c$ is called the (coalgebraic) parity trace semantics of $X$. We write $\text{tr}^p_i(c)$ for $u^\text{sol}_i$ for each $i \in [1, 2n]$ (see also Eq. (1)).

Note that the notions of Büchi $(T, F)$-system and Büchi trace semantics are special cases of those of parity $(T, F)$-system and parity trace semantics respectively.
Similarly, for parity \((T, F)\)-system (\(c : X \rightarrow FX, (X_1, \ldots, X_{2n})\)) is identified with an \(A\)-labeled parity automaton. Each \(\text{tr}^p_i(c)\) has a type \(X_i \rightarrow A^\omega\), and it is given as follows:

\[
\text{tr}^p_i(c)(x) := \left\{ a_1a_2\ldots \in A^\omega \mid \exists x_0, x_1, \ldots \in X, \forall i \in \omega. x_i \in X_{p_i}, (a_{i+1}, x_{i+1}) \in c(x_i) \text{ and } \lim sup_{i \rightarrow \infty} p_i \text{ is even} \right\}.
\]

### A.2 Categorical Datatypes for parity Systems

By repeatedly applying the operations \((\_)^+\) and \((\_)^\oplus\) to \(F\), we can obtain functors \(F^{\oplus i}\) and \(F^{(+\oplus)^i}\) where \((+\oplus)^i\) denotes \(i\)-repetition of \(\oplus\). We introduce notations for them for simplicity.

**Definition A.4** \((F_i^\dagger)\) For \(i \in \mathbb{N}\), we define \(F_i^\dagger : C \rightarrow C\) by \(F_i^\dagger := F^{(+\oplus)^i}\) if \(i = 2l\) and \(F_i^\dagger := F^{(+\oplus)^i}\) if \(i = 2l + 1\).

**Example A.5** We continue Ex. 3.3 In general, for \(i > 0\),

\[
F_i^\dagger X \cong \{ (a_0, p_0) \ldots (a_k, p_k) x \in (A \times [1, i]^+)X \mid k \in \mathbb{N}, p_0 = i \}
\]

\[
\cup \{ (a_0, p_0)(a_1, p_1) \ldots \in (A \times [1, i]^\omega) \mid p_0 = i, \lim sup_{i \rightarrow \infty} p_i \text{ is even} \}. \quad (5)
\]

For Büchi \((T, F)\)-systems, we have distinguished the following datatypes that are isomorphic to each other, and we wrote types of decorated trace semantics as \(dtr_2(c) : X_2 \rightarrow F^{+\oplus 0}\) and \(dtr_1(c) : X_1 \rightarrow F^+F^{+\oplus 0}\) for the sake of simplicity (Rem. 3.3):

\[
F^{+\oplus} X \cong F^+(F^{+\oplus} X + X) \cong F(F^+(F^{+\oplus} X + X) + F^{+\oplus} X + X).
\]

Similarly, for parity \((T, F)\)-systems, we distinguish the following datatypes.

\[
F^{(+\oplus)^i} X \cong F^{(+\oplus)^n-i+(F^{(+\oplus)^n} X + X)} \cong F^{(+\oplus)^{n-i}(F^{(+\oplus)^{2n-i}}(F^{(+\oplus)^n} X + X) + F^{(+\oplus)^n} X + X)} \cong \cdots.
\]

We hereby introduce a short notation for each of the above.

**Definition A.6** \((F_j^{(i)}, \alpha_j^{(i)}, \beta_j^{(i)})\) For \(i \in \mathbb{N}\) and \(j \in [0, i]\), we inductively define a functor \(F_j^{(i)} : C \rightarrow C\) as follows: i) \(F_0^{(i)} := F^\dagger\); and ii) \(F_j^{(i)} := F_1^{(i)}(\coprod_{k=j+1}^{j} F_k^{(i)}(\_))\) for \(j < i\). Moreover, for \(i \in \mathbb{N}\) and \(j \in [1, i]\), we define a natural transformation \(\alpha_j^{(i)} : F_j^{(i)} \Rightarrow F_{j-1}^{(i)}\) by \(\alpha_j^{(i)} := (\coprod_{k=j+1}^{j-1} F_k^{(i)}X + X)^{-1}\) if \(j\) is odd and \(\coprod_{k=j+1}^{j-1} F_k^{(i)}X + X\) if \(j\) is even. By its definition, each \(\alpha_j^{(i)}\) is an isomorphism. Furthermore, we define a natural transformation \(\beta_j^{(i)} : F_j^{(i)} \Rightarrow F_0^{(i)}\) by:

\[
\beta_j^{(i)} := (F_j^{(i)} \xrightarrow{\alpha_j^{(i)}} F_{j-1}^{(i)} \xrightarrow{\alpha_{j-1}^{(i)}} \cdots \xrightarrow{\alpha_1^{(i)}} F_1^{(i)} \xrightarrow{\alpha_0^{(i)}} F_0^{(i)}).
\]
A.3 Natural Transformation \( p_j^{(i)} \)

We introduce a natural transformation \( p_j^{(i)} \), which is a generalization of the transformations introduced in Def. 3.8 and removes decorations.

**Definition A.7** \((p_j^{(i)})\) For \( i \in \mathbb{N} \) and \( j \in [1, i] \), we define a natural transformation \( p_j^{(i)} : F_j^{(i)} \Rightarrow F^{\oplus} \) so that \([p_l^{(i)}X, \ldots, p_{k}^{(i)}X] : \prod_{j=1}^{i} F_j^{(i)}(X) \rightarrow F^{\oplus}X\) is the unique homomorphism from \([\beta_1^{(i)}X, \ldots, \beta_i^{(i)}X]\) to \( \xi_X^{\oplus} \).

\[
F(\prod_{j=1}^{i} F_j^{(i)}X + X) = - - \rightarrow F(F^{\oplus}X + X)
\]

\[
\prod_{j=1}^{i} F_j^{(i)}X = - - \rightarrow F^{\oplus}X
\]

A.4 Decorated Trace Semantics for Parity \((T, F)\)-systems

In order to deal with parity \((T, F)\)-systems, we modify Asm. 4.1 as follows.

**Assumption A.8** Throughout this section, let \( T \) be a monad and \( F \) be an endofunctor on \( C \), and assume that each homset of \( \mathcal{K}(T) \) carries a partial order \( \sqsubseteq \). We further assume the following conditions for each \( n \in \mathbb{N} \) and \( A \in C \).

1. \( F_n^l : C \rightarrow C \) is well-defined and a lifting \( \overline{F_n^l} : \mathcal{K}(T) \rightarrow \mathcal{K}(T) \) of \( F_n^l \) is given.
2. If \( n \) is even, \( T \) and \( F_n^l(\_ + A) \) satisfy the conditions in Thm. 2.9.
3. If \( n \) is odd, \( T \) and \( F_n^l(\_ + A) \) constitute an infinitary trace situation wrt. \( F_n^l \) and \( \sqsubseteq \).
4. If \( n \) is odd, \( T \) and \( F_n^l(\_ + A) \) satisfy the gfp-preserving condition wrt. an arbitrary \( \sigma \).
5. If \( n \) is odd, \( T \) and \( F_n^l(\_ + A) \) satisfy the deterministic-greatest condition.
6. The lifting \( F_{n+1}^l \) is obtained from \( F_n^l \) using the procedure in Def. 3.10.
7. For \( n \in \mathbb{N} \) and \( A \in C \), \( F_n^l(\_ + A) \) is locally monotone.
8. \( T \) and \( F \) constitute a parity trace situation wrt. the same \( \sqsubseteq \) and \( \overline{F} \).

**Definition A.9** \((dtr_i(c))\) For a parity \((T, F)\)-system \((c, (X_1, \ldots, X_{2n}))\), the **decorated parity trace semantics** is a solution \( (dtr_i(c) : X_i \rightarrow F_i^{(2n)} 0)_{1 \leq i \leq 2n} \) of the following HES, all of whose equal symbols are subscripted by \( \nu \).

\[
\begin{align*}
v_1 &= \nu \quad J(\beta_1^{(2n)})^{-1} \circ \overline{F}(v_1 + \cdots + v_{2n}) \circ c_1 \quad \in (\mathcal{K}(T)(X_1, F_1^{(2n)} 0), \sqsubseteq) \\
v_2 &= \nu \quad J(\beta_2^{(2n)})^{-1} \circ \overline{F}(v_1 + \cdots + v_{2n}) \circ c_2 \quad \in (\mathcal{K}(T)(X_2, F_2^{(2n)} 0), \sqsubseteq) \\
&\vdots \\
v_{2n} &= \nu \quad J(\beta_{2n}^{(2n)})^{-1} \circ \overline{F}(v_1 + \cdots + v_{2n}) \circ c_{2n} \quad \in (\mathcal{K}(T)(X_{2n}, F_{2n}^{(2n)} 0), \sqsubseteq)
\end{align*}
\]
A.5 Trace Semantics vs. Decorated Trace Semantics: Parity Case

We shall prove the following theorem, which generalizes Thm. 4.3

**Theorem A.10** For each $i \in [1, 2n]$, $\text{tr}_i^p(c) = P_{i0}^{(2n)} \circ \text{dtr}_i(c)$.

We prove the above theorem in a way that was sketched in 4.2 for Büchi ($T, F$)-systems. Def. A.11, Lem. A.13, Prop. A.14 and Prop. A.16 in the below generalize Def. 4.3, Lem. 4.5, Prop. 4.8 and Prop. 4.7 respectively.

**Definition A.11** $(c_i^+, \bar{c}_i^{(i)})$ For $i \in [1, 2n]$ and $j \in [1, i]$, we inductively define Kleisli arrows $c_i^+ : X_i \to F_{i-1}^+(X_1 + \cdots + X_{2n})$ and $\bar{c}_i^{(i)} : X_j \to F_j(X_{i+1} + \cdots + X_{2n})$ as follows (no need to distinguish the base case from the step case):

- $c_i^+ : X_i \to F_{i-1}^+(X_1 + \cdots + X_{2n})$ is defined by:

$$c_i^+ := \begin{cases} X_i \xrightarrow{e_i} F(X_1 + \cdots + X_{2n}) \xrightarrow{P_{i-1}^{(i-1)} \bar{c}_i^{(i-1)} + id_{X_1+\cdots+X_{2n}}} & 
F(\bigcup_{j=i+1}^{2n} F^{(i-1)}(X_1 + \cdots + X_{2n} + X_i + \cdots + X_{2n}) \\
= F_0^{(i-1)}(X_i + \cdots + X_{2n}) \xrightarrow{\beta_j^{(i-1)} X_i + \cdots + X_{2n}^{-1}} & 
F_{i-1}^+(X_1 + \cdots + X_{2n}) \end{cases}.$$ 

- $\bar{c}_i^{(i)} : X_i \to F_i(X_{i+1} + \cdots + X_{2n})$ is defined as follows:

- If $i = 2k + 1$, we define $\bar{c}_i^{(i)} : X_i \to F_i^{(i)}(X_{i+1} + \cdots + X_{2n})$ as the unique homomorphism from $c_i^+$ to $J(F_i^{(i)}(X_{i+1} + \cdots + X_{2n})^{(i-1)}$.

$$F^{(i)}(X_{i+1} + \cdots + X_{2n}) \xrightarrow{\bar{c}_i^{(i)}(F_i^{(i)}(X_{i+1} + \cdots + X_{2n}))} \cong J(F_i^{(i)}(X_{i+1} + \cdots + X_{2n})^{(i-1)}.$$ 

- If $i = 2k$, we define $\bar{c}_i^{(i)} : X_i \to F_i^{(i)}(X_{i+1} + \cdots + X_{2n})$ as the greatest homomorphism from $c_i^+$ to $J(F_i^{(i)}(X_{i+1} + \cdots + X_{2n})^{(i-1)}$.

$$F_i^{(i)}(X_{i+1} + \cdots + X_{2n}) \xrightarrow{\bar{c}_i^{(i)}(F_i^{(i)}(X_{i+1} + \cdots + X_{2n}))} \cong J(F_i^{(i)}(X_{i+1} + \cdots + X_{2n})^{(i-1)}.$$ 

- For $j < i$, $\bar{c}_j^{(i)} : X_j \to F_j(X_{i+1} + \cdots + X_{2n})$ is defined by:

$$\bar{c}_j^{(i)} := \begin{cases} X_j \xrightarrow{\bar{c}_j^{(i-1)} F_j^{(i-1)}(X_i + X_{i+1} + \cdots + X_{2n})} & 
F^{(i-1)}(X_i + X_{i+1} + \cdots + X_{2n} + X_i + \cdots + X_{2n}) \\
= F_j^{(i)}(X_{i+1} + \cdots + X_{2n}) \end{cases}.$$ 

The result above gives $\text{tr}_i^p(c) = P_{i0}^{(2n)} \circ \text{dtr}_i(c)$.
Here the last equality is by the following lemma.

**Lemma A.12** For \( i \in \mathbb{N} \) and \( j \in [0, i] \), \( F_j^{(i)}(F_{i+1}^{(i+1)}(\_)+\_)+\_ = F_j^{(i+1)} \).

**Proof.** We prove the statement by the induction on \( j \).

If \( j = i \) then the statement is immediate by definition.

If \( j = 2l - 1 < i \), we have:

\[
F_j^{(i)}(F_{i+1}^{(i+1)}(\_)+\_) = F^{(i)}\left(\prod_{k=2l}^i F_k^{(i)}(F_{i+1}^{(i+1)}(\_)+\_)+F_{i+1}^{(i+1)}(\_)+\_\right) \quad \text{(by definition)}
\]

\[
= F^{(i)}\left(\prod_{k=2l}^{i+1} F_k^{(i+1)}(\_)+F_{i+1}^{(i+1)}(\_)+\_\right) \quad \text{(by IH)}
\]

\[
= F^{(i+1)}(\_)+\_ \quad \text{(by definition)}.
\]

We can similarly prove the statement when \( j = 2l - 2 < i \).

The lemma below shows that if \( j \) is odd (resp. even), not only \( \bar{\ell}_j^{(i)} \) but also \( \bar{\ell}_j^{(i)} \) with \( i > j \) is characterized as the least (resp. greatest) homomorphism.

**Lemma A.13** Let \( i \in [1, 2n] \) and \( j \in [1, i] \). For simplicity, we write \( X_j^{(i)} \) for \( \bigl(\prod_{k=j+1}^i X_{k+1} + \cdots + X_{2n}\bigr) + X_{i+1} + \cdots + X_{2n} \).

1. If \( j \) is odd, \( \bar{\ell}_j^{(i)} : X_j \to F_j^{(i)}(X_{i+1} + \cdots + X_{2n}) \) is the unique homomorphism from \( F_{j-1}^{(i)}(\id_{X_j} + \prod_{k=j+1}^i \bar{\ell}_k^{(i)} + \id_{X_{i+1} + \cdots + X_{2n}}) \odot c_j^i \) to \( J\left(F_{j}^{(i)}\right)^{-1} \).

2. If \( j \) is even, \( \bar{\ell}_j^{(i)} : X_j \to F_j^{(i)}(X_{i+1} + \cdots + X_{2n}) \) is the greatest homomorphism from \( F_{j-1}^{(i)}(\id_{X_j} + \prod_{k=j+1}^i \bar{\ell}_k^{(i)} + \id_{X_{i+1} + \cdots + X_{2n}}) \odot c_j^i \) to \( J\left(F_{j}^{(i)}\right)^{-1} \).

**Proof.** Item 1 is easily proved by the finality of \( J(F_{j+1}^{(i)}\cap X_{n+1} + \cdots + X_{2n})^{-1} \). We prove Item 2 by the induction on \( i \).

If \( i = j \), then the statement is immediate by the definition of \( \bar{\ell}_j^{(i)} : X_j \to F_j^{(i)}(X_{i+1} + \cdots + X_{2n}) \) (Def. A.11).

Let \( i > j \) and assume that \( \bar{\ell}_j^{(i-1)} : X_j \to F_j^{(i-1)}(X_i + \cdots + X_{2n}) \) is the greatest homomorphism from \( F_{j-1}^{(i)}(\id_{X_j} + \prod_{k=j+1}^{i-1} \bar{\ell}_k^{(i-1)} + \id_{X_{i+1} + \cdots + X_n}) \odot c_j^i \) to \( J\left(F_{j}^{(i-1)}\right)^{-1} \).

By the definition of \( F_{j}^{(i-1)} \), we have the following equation.

\[
F_{j}^{(i-1)}(\bar{\ell}_i^{(i)} + \id) = F_{j}^{(i)}(\prod_{k=j+1}^{i-1} F_{k}^{(i-1)}(\bar{\ell}_k^{(i)} + \id) + \bar{\ell}_i^{(i)} + \id)
\]
By the definition of a lifting $F^i_j$, this means that $F^i_j(\tilde{\ell}^{(i)}_i + \text{id})$ is the greatest homomorphism from $F^i_{j-1}(\text{id} + \bigsqcup_{k=j+1}^{i-1} F^{(i)}_k(\tilde{\ell}^{(i)}_i + \text{id}) + \tilde{\ell}^{(i)}_i)$ to $J_{\zeta^i_{X^i_j}}$. Hence by the gfp-preserving condition, $\tilde{\ell}^{(i)}_i = F^i_j(\tilde{\ell}^{(i-1)}_i + \text{id}) \circ \tilde{\ell}^{(i-1)}_i$ is the greatest homomorphism from $F^i_{j-1}(\text{id} + \bigsqcup_{k=j+1}^{i-1} F^{(i)}_k(\tilde{\ell}^{(i)}_i + \text{id}) + \tilde{\ell}^{(i)}_i)$ to $J_{\zeta^i_{X^i_j}}$. As we have

\[
F_{j-1}^i(\text{id} + \bigsqcup_{k=j+1}^{i-1} F^{(i)}_k(\tilde{\ell}^{(i)}_i + \text{id}) + \tilde{\ell}^{(i)}_i) \circ F_{j-1}^i(\text{id} + \bigsqcup_{k=j+1}^{i-1} \tilde{\ell}^{(i-1)}_k + \text{id}) \circ c^+_j
\]

\[
= F_{j-1}^i(\text{id} + \bigsqcup_{k=j+1}^{i} F^{(i)}_k(\tilde{\ell}^{(i)}_i + \text{id}) \circ \tilde{\ell}^{(i-1)}_k) + \tilde{\ell}^{(i)}_i + \text{id}) \circ c^+_j
\]

by the definition of $\tilde{\ell}^{(i)}_i$, the statement is proved. See also Fig. 3

\[\square\]

**Proposition A.14** For each $i \in [1, 2n]$, $\tilde{\ell}^{(2n)}_{i} = \text{dtr}_i(c)$.

**Proof.** Assume that $j$ is odd. By Lem. A.13, $\tilde{\ell}^{(2n)}_{i}$ is the unique homomorphism from $F^i_{j-1}(\tilde{\ell}^{(i)}_i + \text{id}) \circ c^+_j$ to $J((F^i_{j-1} \bigsqcup_{k=j+1}^{2n} \tilde{\ell}_k^{(2n)})^{-1}$. This means that it is the greatest homomorphism.

\[
F^i_{j-1}(X_j + \bigsqcup_{k=j+1}^{2n} F^{(i)}_k(0) + \bigsqcup_{k=j+1}^{2n} \tilde{\ell}_k^{(2n)}) \xrightarrow{\text{id} + \bigsqcup_{k=j+1}^{2n} \tilde{\ell}_k^{(2n)}} F^i_{j-1}(F^{(2n)}_0 + \bigsqcup_{k=j+1}^{2n} \tilde{\ell}_k^{(2n)}) \xrightarrow{\text{v}} F^i_{j-1}(X_j + \cdots + X_{2n}) = u \xrightarrow{J(\text{id} + \bigsqcup_{k=j+1}^{2n} \tilde{\ell}_k^{(2n)})^{-1}} F^{(2n)}_0
\]

By the definition of $c^+_j$ (Def. A.11), this means that $\tilde{\ell}^{(2n)}_{i}$ is the greatest fixed point of the following function.

\[
f \mapsto J((F^i_{j-1} \bigsqcup_{k=j+1}^{2n} \tilde{\ell}_k^{(2n)})^{-1} \circ (T(\tilde{\ell}^{(i-1)}_i + \cdots + \tilde{\ell}^{(i-1)}_i + \text{id}) \circ c_j)
\]
The text is not readable due to the distortion. It appears to be a mathematical or scientific diagram, possibly related to algebra or another branch of mathematics. Without clearer visibility, the specific content or details cannot be accurately transcribed.
Note here that the right hand side can be transformed as follows:

\[
J^{F_j}_{j+1} \circ F_{i+1}^{(2n)}_0 \circ F^{(j-1)}_{j-1} (f + \bar{\ell}_{j+1}^{(2n)} + \cdots + \bar{\ell}_{2n}^{(2n)}) \circ J^{(j-1)}_{j-1} \circ (f + \bar{\ell}_{j+1}^{(2n)} + \cdots + \bar{\ell}_{2n}^{(2n)}) \circ c_j
\]

\[
= J^{F_j}_{j+1} \circ F_{i+1}^{(2n)}_0 \circ J^{(j-1)}_{j-1} \circ (f + \bar{\ell}_{j+1}^{(2n)} + \cdots + \bar{\ell}_{2n}^{(2n)}) \circ c_j
\]

(by naturality of \( \beta^{(j-1)}_{j-1} \))

\[
= J^{F_j}_{j+1} \circ F_{i+1}^{(2n)}_0 \circ J^{(j-1)}_{j-1} \circ (f + \bar{\ell}_{j+1}^{(2n)} + \cdots + \bar{\ell}_{2n}^{(2n)}) \circ c_j
\]

(by Def. \( A.6 \))

\[
= J^{(j-1)}_{j-1} \circ F_{i+1}^{(2n)}_0 \circ (f + \bar{\ell}_{j+1}^{(2n)} + \cdots + \bar{\ell}_{2n}^{(2n)}) \circ c_j
\]

(by Def. \( A.6 \)).

Hence \( \bar{\ell}_j^{(2n)} \) is the greatest fixed point of the following function:

\[
f \mapsto J^{(2n)}_0 \circ \bar{F}_1^{(2n)} + \cdots + \bar{F}_{j-1}^{(2n)} + f + \bar{\ell}_{j+1}^{(2n)} + \cdots + \bar{\ell}_{2n}^{(2n)} \circ c_j.
\]

We can similarly prove the same statement when \( j \) is even. Hence \((\bar{\ell}_1^{(2n)}, \ldots, \bar{\ell}_{2n}^{(2n)})\) is the solution of the HES in Def. \( A.9 \), and this concludes the proof. 

This proposition implies the existence of a solution of the HES in Def. \( A.9 \).

**Lemma A.15** For the HES in Def. \( A.2 \) we define the intermediate solution \( i_j^{(i)} : \mathcal{K}(T)(X_{i+1}, F^0) \times \cdots \times \mathcal{K}(T)(X_{2n}, F^0) \rightarrow \mathcal{K}(T)(X_i, F^0) \) as in Def. \( A.2 \) (note that \( Z \) in Def. \( A.2 \) is \( F^0 \)). Then for \( (u_k : X_k \rightarrow F^0)_{k \in [i+1, 2n]} \), we have:

\[
i_j^{(i)}(u_{i+1}, \ldots, u_{2n}) = J_0^{F^0} \circ J_{i}^{F^0} \circ \bar{F}^{(i)}_{j+1} \circ \bar{F}^{(i)}_{j+1} | u_{i+1}, \ldots, u_{2n} | \circ \bar{\ell}_j^{(i)}.
\]

The proof of the above lemma is very long, so we defer the proof to \( A.2 \).

**Proposition A.16** For each \( i \in [1, 2n] \), \( \text{tr}^{i}_c = \mu_0^{(2n)} \circ \bar{\ell}_i^{(2n)} \).

**Lemma A.17** For each \( j \in [1, i] \), \( \mu_j^{F^0} \circ \bar{F}_j^{(i)} | F^0 \circ \bar{F}_j^{(i)} | F^0 = \mu_j^{(i)} \).

**Proof.** By definition, it suffices to prove that the following arrow is a homomorphism from \([\beta_1^{(i)}, \ldots, \beta_i^{(i)}]_0 \) to \( \epsilon_F^0 \).

\[
[\mu_0^{F^0} \circ P_1^{(i)} | F^0, \ldots, \mu_0^{F^0} \circ P_1^{(i)} | F^0, \ldots, \mu_0^{F^0} \circ P_1^{(i)} | F^0, \ldots, \mu_0^{F^0} \circ P_1^{(i)} | F^0] : \prod_{j=1}^{i} F_j^{(i)} | F^0 \rightarrow F^0
\]
We have:

\[ C_F \circ [\mu_0 F^\oplus \circ p_1^{(i)} F^\oplus \circ F_1^{(i)} \circ i_{F^\oplus 0} \circ \ldots \circ \mu_0 F^\oplus \circ p_1^{(i)} F^\oplus \circ F_i^{(i)} \circ i_{F^\oplus 0}] = F[\mu_0 F^\oplus , \text{id}_{F^\oplus 0}] \circ C_F \circ [p_1^{(i)} F^\oplus \circ F_1^{(i)} \circ i_{F^\oplus 0} \circ \ldots \circ p_i^{(i)} F^\oplus \circ F_i^{(i)} \circ i_{F^\oplus 0}] \] (by Def. 3.30)

\[ = F[\mu_0 F^\oplus , \text{id}_{F^\oplus 0}] \circ F([p_1^{(i)} F^\oplus \circ i_{F^\oplus 0} \circ \ldots \circ p_i^{(i)} F^\oplus \circ i_{F^\oplus 0}]) + \text{id}_{F^\oplus 0} \] (by Def. 3.17)

\[ = F[\mu_0 F^\oplus , \text{id}_{F^\oplus 0}] \circ F([p_1^{(i)} F^\oplus \circ i_{F^\oplus 0} \circ \ldots \circ p_i^{(i)} F^\oplus \circ i_{F^\oplus 0}] \circ [\beta_1^{(i)} F^\oplus \circ \ldots \circ \beta_i^{(i)} F^\oplus \circ i_{F^\oplus 0}] \] (by naturality)

\[ = F([\mu_0 F^\oplus \circ p_1^{(i)} F^\oplus \circ F_1^{(i)} \circ i_{F^\oplus 0} \circ \ldots \circ \mu_0 F^\oplus \circ p_i^{(i)} F^\oplus \circ F_i^{(i)} \circ i_{F^\oplus 0}] \circ [\beta_1^{(i)} F^\oplus \circ \ldots \circ \beta_i^{(i)} F^\oplus \circ i_{F^\oplus 0}]. \]

This concludes the proof.

Proof (Prop. A.16). Immediate by Lem. A.15 and Lem. A.17

Proof (Thm. A.10). Immediate from Prop. A.14 and Prop. A.16

A.6 Decorated Trace Semantics for Nondeterministic Parity Tree Automata

We extend the discussions in §29 for nondeterministic parity tree automata (NPTA).

Definition A.18 (NPTA, see e.g. [19]) A nondeterministic parity tree automaton (NPTA for short) is a quadruple \( \mathcal{A} = (X, \Sigma, \delta, \Omega) \) of a set \( X \) of states, a ranked alphabet \( \Sigma \), a transition function \( \delta : X \rightarrow \mathcal{P}(\prod_{n \in \mathbb{N}} \Sigma_n \times X^n) \) and a priority function \( \Omega : X \rightarrow [1, 2n] \). For \( i \in [1, 2n] \), we write \( X_i \) for \( \{ x \in X \mid \Omega(x) = i \} \).

Definition A.19 (L_\Sigma^\mathcal{P}) Let \( \mathcal{A} = (X, \Sigma, \delta, \Omega) \) be an NPTA where \( \Omega : X \rightarrow [1, 2n] \). A run tree over \( \mathcal{A} \) is defined in a similar manner to that over an NBTA (Def. 5.3). A run tree is accepting if for each branch \((a_0, x_0)(a_1, x_1) \ldots \in (\Sigma \times X)^\omega \), \( \lim \sup_{k \rightarrow \infty} \Omega(x_k) \) is even. We write Run_\mathcal{A}(x) (resp. AccRun_\mathcal{A}(x)) for the set of run trees (resp. accepting run trees) whose root node is labeled by \( x \in X \). For \( A \subseteq X \), \( \text{Run}_\mathcal{A}(A) \) denotes \( \cup_{x \in A} \text{Run}_\mathcal{A}(x) \). We define \( \text{AccRun}_\mathcal{A}(A) \) similarly. If no confusion is likely, we omit the subscript \( \mathcal{A} \). We define DelSt : Run_\mathcal{A}(X) \rightarrow \text{Tree}_\infty(\Sigma) in a similar manner to that for NBTA. The language \( L_\mathcal{A}^\mathcal{P} : X \rightarrow \mathcal{P}\text{Tree}_\infty(\Sigma) \) of \( \mathcal{A} \) is defined by \( L_\mathcal{A}(x) = \text{DelSt}(\text{AccRun}_\mathcal{A}(x)). \)

Proposition A.20 (22) We assume the situation in Prop. 5.6.

1) \( \mathcal{P} \) and \( F_\Sigma \) constitute a parity trace situation (Def. A.2) wrt. \( \subseteq \) and \( F_\Sigma \).

2) For an NPTA \( \mathcal{A} = (X, \Sigma, \delta, \Omega) \) where \( \Omega : X \rightarrow [1, 2n] \), we define a parity \( (\mathcal{P}, F_\Sigma) \)-system \( (c : X \rightarrow F_\Sigma X, (X_1, \ldots, X_{2n})) \) by \( c := \delta \) and \( X_i := \{ x \in X \mid \Omega(x) = i \} \). Then we have: \( [\text{tr}_1^\mathcal{P}(c), \ldots, \text{tr}_{2n}^\mathcal{P}(c)] = L_\mathcal{A}^\mathcal{P} : X \rightarrow \mathcal{P}\text{Tree}_\infty(\Sigma) \). □
In the rest of this section, as in [6.2], we describe $F_j^{(i)} 0$ and $dtr_i(c)$ for $F = F_\Sigma$, and show the relationship with $tr_p(c)$ in accordance with Thm. A.11 for an NPTA $A = (X, \Sigma, \delta, \Omega)$ modeled as a $(P, F_\Sigma)$-system \( (c : X \rightarrow \mathcal{P} F_\Sigma X, (X_1, \ldots, X_{2n})) \).

The following proposition generalizes Prop. 5.7. It is proved in a similar manner to Ex. A.5.

**Proposition A.21** We define a set $\text{AccTree}^{(i)}_j(\Sigma, A) \subseteq \text{Tree}_\infty(\Sigma \times [1, i] + A)$ by:

$$\text{AccTree}^{(i)}_j(\Sigma, A) := \left\{ \xi \in \text{Tree}_\infty(\Sigma \times [1, i] + A) \middle| \begin{array}{l}
\text{the root node is labeled by } j, \text{ and for each infinite branch, the maximum priority appearing infinitely is even} \\
\text{or}
\text{the priority appearing infinitely is even}
\end{array} \right\}.$$  

Moreover, we define a function $\text{decomp}^{(i)}_j : \text{AccTree}^{(i)}_j(\Sigma, A) \rightarrow \text{AccTree}^{(i)}_{j-1}(\Sigma, A)$ by $\text{decomp}^{(i)}_j(D, l) := (D, l')$ where

$$l'(w) := \begin{cases} 
  (a, j - 1) & (w = \langle \rangle) \text{ and } l(\langle \rangle) = (a, j) \\
  l(w) & \text{(otherwise)}.
\end{cases}$$

Then $\text{AccTree}^{(i)}_j(\Sigma, A) \cong (F_\Sigma)^{(i)}_j A$, and

$$\left(\text{decomp}^{(i)}_j : \text{AccTree}^{(i)}_j(\Sigma, A) \rightarrow \text{AccTree}^{(i)}_{j-1}(\Sigma, A)\right) \cong \left(\alpha^{(i)}_j : (F_\Sigma)^{(i)}_j A \rightarrow (F_\Sigma)^{(i)}_{j-1} A\right)$$

where $\alpha^{(i)}_j$ is defined as in Def. A.6 and $\text{AccTree}^{(i)}_0(\Sigma, A)$ is defined as follows:

$$\text{AccTree}^{(i)}_0(\Sigma, A) := \left\{ \xi \in \text{Tree}_\infty(\Sigma \times [0, i] + A) \middle| \begin{array}{l}
\text{only the root node is labeled by } 0, \text{ and for each infinite branch, the maximum priority appearing infinitely is even} \\
\text{or}
\text{the priority appearing infinitely is even}
\end{array} \right\}.$$  

By using the characterizations in the proposition above, we can concretely prove that the assumptions required in the previous sections are satisfied by $\mathcal{P}$ and $F_\Sigma$.

**Proposition A.22** Asm. A.8 is satisfied by $T, F = (\mathcal{P}, F_\Sigma)$.

**Proof.** Cond. 2 is proved in a similar manner to [12]. Cond. 3 is proved in a similar manner to [20] using Prop. A.21.

We prove that Cond. 3 is satisfied. Let $c : X \rightarrow (F_\Sigma)^{(i)}_j(X + A)$ and $\sigma : (F_\Sigma)^{(i)}_j(Y + A) \rightarrow Y$. Let $l : X \rightarrow ((F_\Sigma)^{(i)}_j)\oplus A$ be the greatest homomorphism from $c$ to $J_{c_A}((F_\Sigma)^{(i)}_j)$, and $m : ((F_\Sigma)^{(i)}_j)\oplus A \rightarrow Y$ be the greatest fixed point of $\Phi_{J_{c_A}((F_\Sigma)^{(i)}_j)\oplus \sigma}$. 


It is easy to see that \( m \circ l \) is a fixed point of \( \Phi_{\omega, \sigma} \). We show that it is the greatest fixed point. Let \( t : X \to ((F_{\Sigma})^i)^\oplus A \) be a fixed point of \( \Phi_{\omega, \sigma} \).

For each \( k \in \omega \), we inductively define \( \pi_k : ((F_{\Sigma})^i)^\oplus A \to ((F_{\Sigma})^i(\_ + A))^k \) as follows: \( \pi_0 := !((F_{\Sigma})^i)^\oplus A \) and \( \pi_{k+1} := ((F_{\Sigma})^i(\_ + \text{id}_A))^k \circ \zeta_{\sigma_k}((F_{\Sigma})^i)^1 \).

By Thm. \( \text{A.23} \) \(((F_{\Sigma})^i)^\oplus A, (\pi_k)_{k \in \omega} \) is a limit over a final sequence \( 1 \leftarrow (F_{\Sigma})^i(1 + A) \leftarrow (F_{\Sigma})^i((F_{\Sigma})^i(1 + A) + A) \leftarrow (F_{\Sigma})^i(((F_{\Sigma})^i(1 + \text{id}_A) + \text{id}_A) + A) \ldots \). Hence we can identify \(((F_{\Sigma})^i)^\oplus A\) with the following set:

\[
\left\{ (z_k \in (F_{\Sigma})^i(\_ + A)^k)_{k \in \omega} \mid \forall k \in \omega. F_{\Sigma}^i(\_ + \text{id}_A)^k(z_{k+1}) = z_k \right\}.
\]

For each \( k \in \omega \), we inductively define \( t_k : X \to (F_{\Sigma})^i(\_ + A)^k \) as follows: i) \( t_0 := J ! \circ t \) and ii) \( t_{k+1} := (F_{\Sigma})^i(t_k + A) \odot \omega. \) Moreover we define a function \( l' : X \to ((F_{\Sigma})^i)^\oplus A \) as follows:

\[
l'(x) := \{(z_k \in (F_{\Sigma})^i(\_ + A)^k)_{k \in \omega} \in ((F_{\Sigma})^i)^\oplus A \mid \forall k \in \omega. z_k \in t_k(x) \}.
\]

Then \( l' \) is a homomorphism from \( c \) to \( J_{\zeta_{\sigma_1}}((F_{\Sigma})^i)^1 \), and moreover \( m \circ l' = t \). The former implies \( l' \subseteq l \). Hence we have \( t = m \circ l' \subseteq m \circ l \). Therefore \( m \circ l \) is the greatest fixed point of \( \Phi_{\omega, \sigma} \). Hence Cond. \( \text{A.3} \) is satisfied.

We prove that Cond. \( \text{A.5} \) is satisfied. Let \( c : X \to (F_{\Sigma})^i_n(X + A) = (F_{\Sigma})^i_n(\_ + A) \)-coalgebra and \( u : X \to ((F_{\Sigma})^i)^\oplus A \) be the unique homomorphism from \( c \) to \( \zeta_{\sigma_1}((F_{\Sigma})^i)^1 \). Let \( f : X \to ((F_{\Sigma})^i)^\oplus A \) be a homomorphism from \( Jc \) to \( J_{\zeta_{\sigma_1}}((F_{\Sigma})^i)^1 \) and \( x \in X \), and assume that \( t \in f(x) \). Then as \( f \) and \( Ju \) are homomorphism from \( Jc \) to \( J_{\zeta_{\sigma_1}}((F_{\Sigma})^i)^1 \), we can prove \( t = u(x) \) by the induction on the structure of \( t \). Hence \( Ju \) is the greatest homomorphism.

By Cond. \( \text{A.4} \) we can inductively define a lifting \( (F_{\Sigma})^i_n : \mathcal{KL}(T) \to \mathcal{KL}(T) \) for each \( n \in \mathbb{N} \). Then Cond. \( \text{A.4} \) and Cond. \( \text{A.6} \) are satisfied.

It is proved in a similar manner to \( \text{A.21} \) that Cond. \( \text{A.7} \) is satisfied using Prop. \( \text{A.21} \).

Using Prop. \( \text{A.21} \) Cond. \( \text{A.8} \) is proved in a similar manner to \( \text{A.22} \). \( \square \)

We now show what \( \text{dtr}((c) \) characterizes for an NPTA wrt. the characterization in Prop. \( \text{A.21} \) By Prop. \( \text{A.21} \) \( \beta_j^i \) is isomorphic to the following type

\[
\beta_j^i : \text{AccTree}_j^i(\Sigma, A) \to \prod_{n \in \omega} \Sigma_n \times (\prod_{k=1}^i \text{AccTree}_j^i_k(\Sigma, A) + A),
\]

and is given by \( \beta_j^i(\xi) = (a, \xi_0, \ldots, \xi_{n-1}) \) if the root node of \( \xi \) is labeled by \( (a, j) \in \Sigma_n \times [1, i] \). We write \( \text{AccTree}_j^i(\Sigma) \) for \( \text{AccTree}_j^i(\Sigma, \emptyset) \).

**Proposition A.23** Let \( A = (X, \Sigma, \delta, \Omega) \) be an NPTA where \( \Omega : X \to [1, 2n] \). We overload \( \Omega \) and define \( \Omega : \text{Run}(X) \to \text{Tree}_{\infty}(\Sigma \times [1, 2n]) \) by \( \Omega(D, l) := (D, l') \) where \( l'(w) := (a, \Omega(x)) \) if \( l(w) = (a, x) \). We define a parity \((P, F_{\Sigma})\)-system
\( c : X \to \overrightarrow{\Sigma} X, (X_1, \ldots, X_{2n}) \) as in Prop. \textbf{A.20}.3. Then for \( i \in [1, 2n] \) and \( x \in X_i \),

\[
\text{dtr}_i(c)(x) = \{ \Omega(\rho) \in \text{AccTree}_{i}(2n) \mid \rho \in \text{AccRun}_A(x) \}.
\]

**Proof.** By Prop. \textbf{A.21} and the definition of \( \text{AccRun}_A(x) \), it is easy to see that \( \{ \Omega(\rho) \mid \rho \in \text{AccRun}_A(x) \} \subseteq F_{i}(2n)0 \). For each \( i \in [1, 2n] \), we define \( f_i : X_i \to \text{PAccTree}_{i}(2n) \) by \( f_i(x) := \{ \Omega(\rho) \mid \rho \in \text{AccRun}_A(x) \} \). We show that a family \( (f_i)_{i \in [1, 2n]} \) is the solution of the HES as in Def. \textbf{A.9}.

We first prove \( f_i = J(\beta(2n)_{i})^{-1} \circ \overrightarrow{\Sigma}(f_1 + \cdots + f_{2n}) \circ c_i \) for each \( i \). For each \( x \in X_i \), we have:

\[
\begin{align*}
J(\beta(2n)_{i})^{-1} \circ \overrightarrow{\Sigma}(f_1 + \cdots + f_{2n}) \circ c_i(x) \\
= J(\beta(2n)_{i})^{-1} \circ \overrightarrow{\Sigma}(f_1 + \cdots + f_{2n})(\{(a, x_0, \ldots, x_{m-1}) \in \tau(x)\}) \\
= J(\beta(2n)_{i})^{-1}(\{(a, \Omega(\rho_0), \ldots, \Omega(\rho_{m-1})) \in \Sigma_m \times \prod_{k=1}^{2n} \text{AccTree}_{k}(2n) \mid (a, x_0, \ldots, x_{m-1}) \in \tau(x), \rho_t \in \text{AccRun}_A(x_t) \text{ for each } t \in [0, m-1]\}) \\
= \{ (a, \Omega(\rho_0), \ldots, \Omega(\rho_{m-1})) \in \text{AccTree}_{i}(2n) \mid (a, x_0, \ldots, x_{m-1}) \in \tau(x), \rho_t \in \text{AccRun}_A(x_t) \text{ for each } t \in [0, m-1]\} \\
= \{ \Omega(\rho) \in \text{AccTree}_{i}(2n) \mid \rho \in \text{AccRun}_A(x) \} \\
= f_i(x).
\end{align*}
\]

We next show that it is the greatest fixed point. Let \( (g_i : X_i \to \text{PAccTree}_{i}(2n))_{1 \leq i \leq 2n} \) be a family of functions such that \( g_i = J(\beta(2n)_{i})^{-1} \circ \overrightarrow{\Sigma}(g_1 + \cdots + g_{2n}) \circ c_i \) for each \( i \). It suffices to show that \( g_i(x) \subseteq f_i(x) \) for each \( i \in [1, 2n] \) and \( x \in X_i \).

Let \( i \in [1, 2n] \) and \( x \in X_i \), and assume that \( \xi = (D, l) \in g_i(x) \). We write \( l_1(w) \) and \( l_2(w) \) for \( \pi_1(l(w)) \) and \( \pi_2(l(w)) \) respectively. We hereby define a function \( l' : D \to \Sigma \times X \) by \( l'(w) := (l_1(w), l'_2(w)) \), where \( l'_2(w) \in X \) is inductively defined as follows so that the following condition is satisfied: for each \( w \in D, \xi_w \in [g_1, \ldots, g_{2n}](l'_2(w)) \) (recall that \( \xi_w \) denotes the \( w \)-th subtree of \( \xi \)).

- For \( w = \emptyset \), we let \( l'_2(\emptyset) = x \).
- Let \( w \in D, \pi_1(l(w)) = m \). Assume that we have fixed \( l'_2(w) \) so that the condition above is satisfied. Assume \( l'_2(w) \in X_i \). Assume that the root node of the \( w \)-th subtree \( \xi_w \) of \( \xi \) is labeled by \( a \in \Sigma_m \). Then \( \xi_w \) has a shape \( ((a, i), \xi_{w0}, \ldots, \xi_{w(m-1)}) \). Here by the assumption on \( g_i \), we have:

\[
\xi_w = ((a, i), \xi_{w0}, \ldots, \xi_{w(m-1)}) \\
\in g_i(l'_2(w)) \\
= J(\beta(2n)_{i})^{-1} \circ \overrightarrow{\Sigma}(g_1 + \cdots + g_{2n}) \circ c_i(x') \\
= J(\beta(2n)_{i})^{-1} \circ \overrightarrow{\Sigma}(g_1 + \cdots + g_{2n})(\{(a', x_0, \ldots, x_{m'-1}) \in \tau(l'_2(w))\})
\]
\( \Omega \in \xi \) above means that Lemma B.1

B.1 Proof of Prop. 3.14

By the characterization of \( F \)

\( \) satisfies the gfp-preserving condition (Def. 3.12). For each \( X, A, B \)

Proof (Prop. A.24). It is easy to see that \( \Omega (\rho) = \xi \).

In contrast, by the construction, \( \rho \) is a run tree over \( A \), and moreover, as \( \xi \in \text{AccTree}_{j}^{(2n)} (\Sigma) \), \( \rho \) is accepting. Therefore by the definition of \( f_{i} \), we have \( \Omega (\rho) \in f_{i} (x) \). This concludes the proof. \( \square \)

The following proposition generalizes Prop. 5.10

**Proposition A.24** We define \( \text{DelSt}_{j}^{(i)} : \text{AccTree}_{j}^{(i)} (\Sigma, A) \to \text{Tree}_{\infty} (\Sigma + A) \)

by \( \text{DelSt}_{j}^{(i)} (D, l) := (D, l') \) where \( l'(w) := \pi_{1} (l (w)) \). Then with respect to the isomorphism in Prop. A.22 \( \text{DelSt}_{j}^{(i)} (\xi) = p_{j}^{(i)} A (\xi) \).

**Proof** (Prop. A.24). It is easy to see that \( \text{DelSt}_{j}^{(i)} : \text{AccTree}_{j}^{(i)} (\Sigma, A) \to \text{Tree}_{\infty} (\Sigma + A) \) satisfies the following equality for each \( \xi = ((a, i), (\xi_{0}, \ldots, \xi_{m-1})) \in \text{AccTree}_{j}^{(i)} (\Sigma, A) \).

\[
\text{DelSt}_{j}^{(i)} (\xi) = \left( a, ([\text{DelSt}_{1}^{(i)} , \ldots, \text{DelSt}_{1}^{(i)}] (\xi_{0}), \ldots, [\text{DelSt}_{1}^{(i)} , \ldots, \text{DelSt}_{1}^{(i)}] (\xi_{m-1})) \right) .
\]

By the characterization of \( F_{A}^{\infty} \) and \( \zeta_{F}^{A} \) in [5.2] and by Prop. A.21 the equation above means that \( [\text{DelSt}_{1}^{(i)} , \ldots, \text{DelSt}_{2n}^{(i)}] \) is a homomorphism from \( [\beta_{1}^{A}, \ldots, \beta_{1}^{A}] \) to \( \zeta_{F}^{A} \). Therefore immediate by the definition of \( p_{j}^{(i)} . \)

Hence Thm. A.10 results in the following equation, which is again obvious.

\[
\{ [\text{DelSt}_{1}^{(2n)}, \ldots, \text{DelSt}_{2n}^{(2n)}] (\Omega (\rho)) \mid \rho \in \text{AccRun}_{A} (x) \} = L^{p} (x) .
\]

**B Omitted Proofs**

**B.1 Proof of Prop. 5.14**

**Lemma B.1** Assume that \( T \) and \( F \) constitute an infinitary trace situation and satisfy the gfp-preserving condition (Def. 5.12). For each \( X, A, B \in \mathbb{C} \), \( c : X \to F (X + A) \) and \( f : A \to B \), if \( l : X \to F^{\otimes} A \) is the greatest homomorphism
From $c$ to $J_{A}^{F}$, then $\overline{F} \circ f \circ 1 : X \rightarrow F \circ \overline{B}$ is the greatest homomorphism from $F(id_X + f) \circ c$ to $J_{B}^{F}$.

\[
\begin{align*}
F(X + B) & \xrightarrow{\overline{F}(id + f)} F(F \oplus A + B) \xrightarrow{\overline{F}(\nu + f)} F(F \oplus B + B) \\
F(X + A) & \xrightarrow{\overline{F}(id + f)} F(F \oplus A + A) \xrightarrow{\nu} J_{B}^{F} \\
X & \xrightarrow{\lambda} F \oplus A \xrightarrow{\overline{F} f} F \oplus B
\end{align*}
\]

Proof. It is easy to see that $\overline{F} \circ f$ is the greatest fixed point of $\Phi_{J_{B}^{F} \circ \overline{F}(id + f)}$, and the greatest homomorphism from $F(id_X + f) \circ c$ to $J_{B}^{F}$ is the greatest fixed point of $\Phi_{c, J_{B}^{F} \circ \overline{F}(id + f)}$.

\[
\begin{align*}
F(X + A) & \xrightarrow{\overline{F}(id + f)} F(F \oplus A + A) \xrightarrow{\overline{F}(\mu + id)} F(F \oplus B + A) \\
F \oplus A & \xrightarrow{\nu} J_{A}^{F} \xrightarrow{\nu} F \oplus B \xrightarrow{\overline{F} f} F \oplus B
\end{align*}
\]

Hence immediate by the gfp-preserving condition.

Proof (Prop.[X.14]). Item. [1] is immediate by the finality. Item. [2] is easily proved by the gfp-preserving condition, the deterministic-greatest condition and Lem. [B.1]

\[\square\]

B.2 Proof of Lem. [A.15]

Sublemma B.2 For $A \in C$, the unique homomorphism from $[\overline{F}[\kappa_1, \kappa_2] \circ \zeta_{A}^{F}, F[\kappa_2, \kappa_3] \circ \zeta_{A}^{F}]$ to $\zeta_{A}^{F}$ is given by $[\mu_{A}^{\oplus_1}, id_A]$.

Proof. Let $u : F \oplus F \oplus A + F \oplus A \rightarrow F \oplus A$ be the unique homomorphism from $[\overline{F}[\kappa_1, \kappa_2] \circ \zeta_{A}^{F}, F[\kappa_2, \kappa_3] \circ \zeta_{A}^{F}]$ to $\zeta_{A}^{F}$.

Note that $u = [u \circ \kappa_1, u \circ \kappa_2]$. By Def. [A.8] $\mu_{A}^{\oplus_1}$ we shall show that $u \circ \kappa_2 = id_A$.

It is easy to see that $\kappa_2 : F \oplus A \rightarrow F \oplus F \oplus A + F \oplus A$ is a homomorphism from $\zeta_{A}^{F}$ to $[\overline{F}[\kappa_1, \kappa_2] \circ \zeta_{A}^{F}, F[\kappa_2, \kappa_3] \circ \zeta_{A}^{F}]$. Therefore $u \circ \kappa_2$ is a homomorphism from $\zeta_{A}^{F}$ to itself, on the one hand. On the other hand, $id_A$ is also a homomorphism from $\zeta_{A}^{F}$. Hence by the finality of $\zeta_{A}^{F}$, we have $u \circ \kappa_2 = id_{F \oplus A}$.

\[
\begin{align*}
F(\overline{F} \oplus A + A) & \xrightarrow{\nu} F(\overline{F} \oplus F \oplus A + F \oplus A + A) \xrightarrow{\nu} F(\overline{F} \oplus A + A) \\
F \oplus A & \xrightarrow{\kappa_2} F \oplus \overline{F} \oplus A + F \oplus A \xrightarrow{\nu} F \oplus A
\end{align*}
\]
Hence we have $u = [u \circ \kappa_1, u \circ \kappa_2] = [\mu_A^{\otimes}, \id_A]$.

\[\text{Sublemma B.3} \quad \text{We define an } F\text{-coalgebra } \gamma_i : \coprod_{j=1}^{i-1} F^{(i)} \otimes 0 + F \otimes 0 \to F(\coprod_{j=1}^{i-1} F^{(i)} \otimes 0 + F \otimes 0) \text{ as follows:}
\]

\[
\gamma_i \triangleq \coprod_{j=1}^{i-1} F^{(i)} \otimes 0 + F \otimes 0 \xrightarrow{[\beta^{(i)}_1 \otimes 0, \ldots, \beta^{(i)}_{i-1} \otimes 0, \kappa_{i+1} \circ \zeta_0]} F(\coprod_{j=1}^{i} F^{(i)} \otimes 0 + F \otimes 0)
\]

Then the unique homomorphism from $\gamma_i$ to $\zeta_0^F : F \otimes 0 \to FF \otimes 0$ is given by the following arrow:

\[
[\mu_0^{\otimes} \circ p^{(i)}_1 F \otimes 0, \ldots, \mu_0^{\otimes} \circ p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0}] : \coprod_{j=1}^{i-1} F^{(i)} \otimes 0 + F \otimes 0 \to F \otimes 0.
\]

\[\text{Proof.} \quad \text{It suffices to show that it is a homomorphism. We have:}
\]

\[
\zeta_0^F \circ [\mu_0^{\otimes} \circ p^{(i)}_1 F \otimes 0, \ldots, \mu_0^{\otimes} \circ p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0}]

\]

\[
= \zeta_0^F \circ [\mu_0^{\otimes} \circ p^{(i)}_1 F \otimes 0, \ldots, \mu_0^{\otimes} \circ p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0}] + \id_{F \otimes 0}
\]

\[
= F[\mu_0^{\otimes}, \id_{F \otimes 0}] \circ F([p^{(i)}_1 F \otimes 0, \ldots, p^{(i)}_{i-1} F \otimes 0] + \id_{F \otimes 0})
\]

\[
\text{(by Sublem. B.3)}
\]

\[
= F[p^{(i)}_1 F \otimes 0, \ldots, p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0}] \circ [\beta^{(i)}_1 F \otimes 0, \ldots, \beta^{(i)}_{i-1} F \otimes 0, F \kappa_2 \circ \zeta_0^F]
\]

\[
\text{(by Def. A.7)}
\]

\[
= F[p^{(i)}_1 F \otimes 0, \ldots, p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0}] \circ [\beta^{(i)}_1 F \otimes 0, \ldots, \beta^{(i)}_{i-1} F \otimes 0, F \kappa_1 \circ \zeta_0^F]
\]

\[
= F[p^{(i)}_1 F \otimes 0, \ldots, p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0}] \circ (F[p^{(i)}_1 F \otimes 0, \ldots, p^{(i)}_{i-1} F \otimes 0, F \kappa_1 \circ \zeta_0^F])
\]

This concludes the proof.

\[\square\]

\[\text{Sublemma B.4} \quad \text{For each } j \in [1, i - 1], \text{ we have the following equality.}
\]

\[
\mu_0^{\otimes} \circ p^{(i)}_j F \otimes 0 = \mu_0^{\otimes} \circ p^{(i-1)}_j F \otimes 0 \circ F_j^{-1} \circ [\mu_0^{\otimes} \circ p^{(i)}_j F \otimes 0, \id_{F \otimes 0}].
\]

\[\text{Proof.} \quad \text{By Sublem. B.3, it suffices to show that the following arrow is a homomorphism from } \gamma_i \text{ to } \zeta_0^F.
\]

\[
[\mu_0^{\otimes} \circ p^{(i-1)}_1 F \otimes 0, \ldots, \mu_0^{\otimes} \circ p^{(i-1)}_{i-1} F \otimes 0, \id_{F \otimes 0}] \circ [\beta^{(i)}_1 F \otimes 0, \ldots, \beta^{(i)}_{i-1} F \otimes 0, F \kappa_1 \circ \zeta_0^F]
\]

\[
= \mu_0^{\otimes} \circ p^{(i)}_1 F \otimes 0, \ldots, \mu_0^{\otimes} \circ p^{(i)}_{i-1} F \otimes 0, \id_{F \otimes 0]}
\]

\[
\mu_0^{\otimes} \circ p^{(i-1)}_j F \otimes 0 \circ F_j^{-1} \circ [\mu_0^{\otimes} \circ p^{(i)}_j F \otimes 0, \id_{F \otimes 0}].
\]
We have:

\[
\begin{align*}
\zeta^F \circ [\mu^F_0 \circ p_{(i-1)}^F \circ F_{(i-1)}^F \circ [\mu^F_0 \circ p_{(i)}^F \circ \text{id}_{F_{(0)}}]], \ldots, \\
\mu^F_0 \circ p^F_{(i)} \circ F^F_{(i-1)} \circ \text{id}_{F_{(0)}}], \ldots, \\
= \zeta^F \circ [\mu^F_0, \text{id}_{F_{(0)}}] \circ \left( [p_{(i-1)}^F \circ F_{(i-1)}^F \circ \text{id}_{F_{(0)}}]], \ldots, \\
\right)
\end{align*}
\]

\[
\begin{align*}
= F[\mu^F_0, \text{id}_{F_{(0)}}] \circ \left[ \zeta^F \circ F_{(i)} \circ \text{id}_{F_{(0)}}] \circ p^F_{(i)} \circ F^F_{(i-1)} \circ \text{id}_{F_{(0)}}]], \ldots, \\
\right)
\end{align*}
\]

(by Sublem. B.2)

\[
\begin{align*}
= F[\mu^F_0, \text{id}_{F_{(0)}}] \circ \left[ \zeta^F \circ F_{(i)} \circ \text{id}_{F_{(0)}}] \circ p^F_{(i)} \circ F^F_{(i-1)} \circ \text{id}_{F_{(0)}}]], \ldots, \\
\right)
\end{align*}
\]

(by Def. A.7)

\[
\begin{align*}
= F[\mu^F_0, \text{id}_{F_{(0)}}] \circ \left[ \zeta^F \circ F_{(i)} \circ \text{id}_{F_{(0)}}] \circ p^F_{(i)} \circ F^F_{(i-1)} \circ \text{id}_{F_{(0)}}]], \ldots, \\
\right)
\end{align*}
\]

(by naturality)

\[
\begin{align*}
= F[\mu^F_0, \text{id}_{F_{(0)}}] \circ \left[ \zeta^F \circ F_{(i)} \circ \text{id}_{F_{(0)}}] \circ p^F_{(i)} \circ F^F_{(i-1)} \circ \text{id}_{F_{(0)}}]], \ldots, \\
\right)
\end{align*}
\]

(by Def. A.6)
It suffices to that the arrow is a homomorphism. We have:

$$F[\mu_{F\otimes 0} \circ p_1^{(i)} \circ p_{i+1}^{(i)}, \mu_{F\otimes 0} \circ p_{i+1}^{(i)}, \text{id}_{F\otimes 0}] + [\mu_{F\otimes 0} \circ p_{i+1}^{(i)}, \text{id}_{F\otimes 0}]$$

This concludes the proof. □

**Sublemma B.5** The unique homomorphism from an $F$-coalgebra

$$[\beta_1^{(i)}_{F\otimes 0}, \ldots, \beta_{r-1}^{(i)}_{F\otimes 0}, F\kappa_{i+1} \circ \zeta_0^F] : \prod_{j=1}^{i} F_j^{(i)} \to F^{(i)} \otimes 0 + F^{(i)} \otimes 0$$

to $\zeta_0^F : F^{(i)} \otimes 0 \to F F^{(i)} \otimes 0$ is given by the following arrow:

$$F[\mu_0^{F\otimes 0} \circ p_1^{(i)} \circ \ldots \circ p_{i+1}^{(i)}, \mu_0^{F\otimes 0} \circ p_{i+1}^{(i)}, \text{id}_{F\otimes 0}] : \prod_{j=1}^{i} F_j^{(i)} \otimes 0 + F^{(i)} \otimes 0 \to F^{(i)} \otimes 0.$$  

**Proof.** It suffices to that the arrow is a homomorphism. We have:

$$\zeta_0^F \circ [\mu_0^{F\otimes 0} \circ p_1^{(i)} \circ \ldots \circ p_{i+1}^{(i)}, \mu_0^{F\otimes 0} \circ p_{i+1}^{(i)}, \text{id}_{F\otimes 0}]$$

$$= \zeta_0^F \circ [\mu_0^{F\otimes 0} \circ p_1^{(i)} \circ \ldots \circ p_{i+1}^{(i)}, \text{id}_{F\otimes 0}] \circ (p_1^{(i)}, \ldots, p_{i+1}^{(i)}) + \text{id}_{F\otimes 0})$$

$$= F[\mu_0^{F\otimes 0}, \text{id}_{F\otimes 0}] \circ [\zeta_0^F, F\kappa_2 \circ \zeta_0^F] \circ (p_1^{(i)}, \ldots, p_{i+1}^{(i)}) [\text{id}_{F\otimes 0}]$$

(by Sublem. B.4)

$$= F[\mu_0^{F\otimes 0}, \text{id}_{F\otimes 0}] \circ [F(\mu_0^{F\otimes 0}) \circ (p_1^{(i)}, \ldots, p_{i+1}^{(i)}) + \text{id}_{F\otimes 0}] \circ [\beta_1^{(i)}_{F\otimes 0}, \ldots, \beta_{r-1}^{(i)}_{F\otimes 0}, F\kappa_{i+1} \circ \zeta_0^F].$$

This concludes the proof. □

**Sublemma B.6** Let $i > 0$ and $A \in \mathbb{C}$. We define an $F_{i-1}^{(i)} + F^{(i)}$-algebra $\sigma_i : F_{i-1}^{(i)}(F^{(i)} + F^{(i)}) \to F^{(i)}$ as follows:

$$\sigma_i := F_{i-1}^{(i)}(F^{(i)} + F^{(i)}) \to F_{i-1}^{(i)}(\text{id}_{F^{(i)}} + \text{id}_{F^{(i)}}) \to F_{i-1}^{(i)}(F^{(i)} + F^{(i)}) \to F_{i-1}^{(i)}(\text{id}_{F^{(i)}} + \text{id}_{F^{(i)}}) \to F^{(i)}.$$
Then if \( i \) is even (resp. odd), \( J_{\mu_0}^{\oplus} \odot J_{P_{\oplus}^{i}}^{(i)} : F_{\oplus}^{i} F_{\oplus}^{i} 0 \rightarrow F_{\oplus}^{i} 0 \) is the greatest fixed point of \( \Phi_{J_{\sigma_i}^{i-1}, J_{\sigma_i}}^{J_{\sigma_i}^{i-1}, J_{\sigma_i}} \).

Proof. Assume that \( i \) is even. We write \( \Phi \) for \( \Phi_{J_{\sigma_i}^{i-1}, J_{\sigma_i}}^{J_{\sigma_i}^{i-1}, J_{\sigma_i}} \), for simplicity. For \( f : F_{\oplus}^{i} F_{\oplus}^{i} 0 \rightarrow F_{\oplus}^{i} 0 \), we have:

\[
\Phi(f) = J_{\sigma_i} \circ F_{i-1}^{-1}(f + \text{id}_{F_{\oplus}^{i}}) \circ J_{\sigma_i} F_{i-1}^{i-1}
\]

(by Def. \ref{def:J})(\ref{def:J})

\[
= J(\zeta_0^{F_{\oplus}})^{-1} \odot JF[\mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \text{id}_{F_{\oplus}^{0}}] \odot J\beta_{i-1}^{(i-1)}
\]

(by definition)

\[
= J(\zeta_0^{F_{\oplus}})^{-1} \odot JF[\mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \text{id}_{F_{\oplus}^{0}}] \odot F_{i-1}^{i-1}[f, \text{id}_{F_{\oplus}^{0}}]
\]

(by the naturality of \( \beta_{i-1}^{(i-1)} \))

\[
= J(\zeta_0^{F_{\oplus}})^{-1} \odot JF[\mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \text{id}_{F_{\oplus}^{0}}] \odot F_{i-1}^{i-1}[f, \text{id}_{F_{\oplus}^{0}}]
\]

(by Def. \ref{def:J})(\ref{def:J})

\[
= J_{\sigma_i} \circ F_{i-1}^{-1}(f + \text{id}_{F_{\oplus}^{i}}) \circ J_{\sigma_i} F_{i-1}^{i-1}
\]

(7)

We now show that \( J_{\mu_0}^{\oplus} \odot P_{i,F_{\oplus}^{0}}^{(i)} \) is a fixed point of \( \Phi \). By Eq. (\ref{df:J}) above, we have:

\[
\Phi(J_{\mu_0}^{\oplus} \odot P_{i,F_{\oplus}^{0}}^{(i)})
\]

\[
= J(\zeta_0^{F_{\oplus}})^{-1} \odot JF[\mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \mu_0^{\oplus} \odot P_{1,F_{\oplus}^{0}}^{(i-1)}, \text{id}_{F_{\oplus}^{0}}], \ldots
\]

\[
J_{\mu_0}^{\oplus} \odot J_{P_{i,F_{\oplus}^{0}}^{(i)}}^{(i-1)} \odot F_{i-1}^{i-1} [f, \text{id}_{F_{\oplus}^{0}}], f, \text{id}_{F_{\oplus}^{0}}] \odot J_{\beta_{i,F_{\oplus}^{0}}}^{(i)}
\]

(by Sublem. \ref{sublem:J})(\ref{sublem:J})

\[
= J_{\mu_0}^{\oplus} \odot J_{P_{i,F_{\oplus}^{0}}^{(i)}}^{(i-1)} \odot F_{i-1}^{i-1} [f, \text{id}_{F_{\oplus}^{0}}]
\]

(7)

Hence \( J_{\mu_0}^{\oplus} \odot P_{i,F_{\oplus}^{0}}^{(i)} \) is a fixed point of \( \Phi \).

It remains to show that it is the greatest fixed point. Let \( f \) be a fixed point of \( \Phi \). For each \( j \in [1, i - 1] \), we have:

\[
J_{\zeta_0^{F_{\oplus}}} \circ (J_{\mu_0}^{\oplus} \odot J_{P_{j,F_{\oplus}^{0}}^{(i)}}^{(i-1)} \odot F_{j}^{(i-1)} [f, \text{id}_{F_{\oplus}^{0}}])
\]
\[ = JF[\mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \ldots, \mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \text{id}_{F \circ 0}] \circ J\beta^{(i-1)}_j F \circ 0 \circ \overline{F^{(i-1)}_j}[f, \text{id}_{F \circ 0}] \]

(by Sublem. B.5)

\[ = JF[\mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \ldots, \mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \text{id}_{F \circ 0}] \circ J\beta^{(i-1)}_j F \circ 0 \circ \overline{F^{(i-1)}_j}[f, \text{id}_{F \circ 0}] \]

\[ = JF[\mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \ldots, \mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \text{id}_{F \circ 0}] \circ J\beta^{(i)}_j \]

(by Def. A.6)

\[ = J[\mu^\oplus_0 \circ Jp^{(i-1)}_1 F \circ 0 \circ \overline{F^{(i-1)}_j}[f, \text{id}_{F \circ 0}], \ldots, \mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \text{id}_{F \circ 0}] \circ J\beta^{(i)}_j \]

Therefore, together with Eq. (7), we can see that the following arrow is a homomorphism from \( [\beta^{(i)}_1 F \circ 0, \ldots, \beta^{(i)}_i F \circ 0, F \circ \sigma] \) to \( \zeta^F_0 \).

\[ [J\mu^\oplus_0 \circ Jp^{(i-1)}_1 F \circ 0 \circ \overline{F^{(i-1)}_j}[f, \text{id}_{F \circ 0}], \ldots, \mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \text{id}_{F \circ 0}] \]

Hence by Sublem. B.5 and the deterministic-greatest condition (Asm. A.8), we have:

\[ [J\mu^\oplus_0 \circ Jp^{(i-1)}_1 F \circ 0 \circ \overline{F^{(i-1)}_j}[f, \text{id}_{F \circ 0}], \ldots, \mu^\oplus_0 \circ p^{(i-1)}_1 F \circ 0, \text{id}_{F \circ 0}] \]

This immediately implies \( f \subseteq J\mu^\oplus_0 \circ Jp^{(i)}_1 F \circ 0 \). Hence \( J\mu^\oplus_0 \circ Jp^{(i)}_1 F \circ 0 \) is the greatest fixed point of \( \Phi \).

The proof when \( i \) is odd is similar. \( \square \)

**Sublemma B.7** Let \( T \) be a monad and \( F \) be an endofunctor on \( C \). Assume that a lifting \( \overline{F} : \mathcal{K}(T) \to \mathcal{K}(T) \) of \( F \) is given and each homset of \( \mathcal{K}(T) \) carries a partial order \( \subseteq \). Assume further that they satisfy the conditions in Thm. 2.3. Let \( \nu : F \circ A \to A \) be an initial \( F \)-algebra. For each \( X, Y \in \mathcal{C}, c : X \to FX \) and \( \sigma : FY \to Y \), if \( u : X \to A \) is the unique homomorphism from \( c \) to \( J(\nu)^{-1} \) and a function \( \Phi_{J(\nu), \sigma} \) (see Def. 3.12) has a fixed point \( m : A \to Y \), then \( m \circ u : X \to Y \) is the least fixed point of \( \Phi_{c, \sigma} \).

**Proof.** It is easy to see that \( m \circ u \) is a fixed point of \( \Phi_{c, \sigma} \). We shall show that it is the least fixed point. Let \( f : X \to Y \) be a fixed point of \( \Phi_{c, \sigma} \).
By the conditions in Thm. 2.9, a homset $\mathcal{K}(T)(X, A)$ is $\omega$-complete and has the least element $\bot$, and a function $\Phi_{c, J_i} : \mathcal{K}(T)(X, A) \to \mathcal{K}(T)(X, A)$ is monotone and $\omega$-continuous. Hence by the Kleene fixed point theorem, the unique fixed point $u : X \to A$ of $\Phi_{c, J_i}$ is given by $\bigsqcup_{i \in \omega} \Phi_{c, J_i}(\bot)$.

We now prove $m \odot \Phi_{c, J_i}(\bot) \sqsubseteq f$ by the induction on $i$.

- If $i = 0$, by the conditions in Thm. 2.9 we have:

$$m \odot \Phi_{c, J_i}(\bot) = m \odot \bot = \bot \sqsubseteq f.$$ 

- Assume that $m \odot \Phi_{c, J_i}(\bot) \sqsubseteq f$. Then we have:

$$m \odot \Phi_{c, J_i+1}(\bot) = m \odot \Phi_{c, J_i}(\bot) \odot c \quad \text{(by definition)}$$

$$= \sigma \odot Fm \odot J(t^F)^{-1} \odot J(t^F) \odot \Phi_{c, J_i}(\bot) \odot c \quad \text{(m is a fixed point)}$$

$$= \sigma \odot F(m \odot \Phi_{c, J_i}(\bot)) \odot c$$

$$\sqsubseteq \sigma \odot Ff \odot c \quad \text{(by IH)}$$

$$= f \quad \text{(f is a fixed point)}.$$ 

Hence we have $m \odot \Phi_{c, J_i}(\bot) \sqsubseteq f$ for each $i \in \omega$. Therefore we have:

$$m \odot u = m \odot \bigsqcup_{i \in \omega} \Phi_{c, J_i}(\bot) = \bigsqcup_{i \in \omega} (m \odot \Phi_{c, J_i}(\bot)) \sqsubseteq f.$$ 

Hence $m \odot u$ is the least fixed point. 

**Proof (Lem. A.15).** We prove Eq. (6) by the induction on $i$. We don’t have to distinguish the base case from the step case.

We first prove Eq. (6) for $j = i$. Assume that $i$ is even. By the definition of intermediate solutions (Def. 2.2), it suffices to show that $J\mu^{F \oplus}_{\circ} \circ Jp^{(i-1)}_{J_i} \circ F^{(i)}_{\circ}[u_{i+1}, \ldots, u_{2n}] \odot \tilde{t}^{(i)}_{i-1}$ is the greatest fixed point of the following function:

$$f \mapsto J(\zeta_{0}^{F})^{-1} \odot F \left[ t_{1}^{(i-1)}(f, u_{i+1}, \ldots, u_{2n}), \ldots, t_{i-1}^{(i-1)}(f, u_{i+1}, \ldots, u_{2n}), f, u_{i+1}, \ldots, u_{2n} \right] \odot c_{i}. $$

Here the right hand side can be deformed as follows:

$$J(\zeta_{0}^{F})^{-1} \odot F \left[ t_{1}^{(i-1)}(f, u_{i+1}, \ldots, u_{2n}), \ldots, t_{i-1}^{(i-1)}(f, u_{i+1}, \ldots, u_{2n}), f, u_{i+1}, \ldots, u_{2n} \right] \odot c_{i}$$

$$= J(\zeta_{0}^{F})^{-1} \odot F \left[ Jp_{0}^{F \oplus} \circ J_{p_{1}}^{(i-1)} \circ F_{1}^{(i-1)}[f, u_{i+1}, \ldots, u_{2n}] \odot \tilde{t}_{i-1}^{(i-1)}, f, u_{i+1}, \ldots, u_{2n} \right] \odot c_{i}$$

(by IH)

$$= J(\zeta_{0}^{F})^{-1} \odot F \left[ Jp_{0}^{F \oplus} \circ J_{p_{1}}^{(i-1)} \circ F_{1}^{(i-1)}[f, u_{i+1}, \ldots, u_{2n}], \ldots, f, u_{i+1}, \ldots, u_{2n} \right] \odot c_{i},$$
Therefore by the definition of $c_i^+$ we have:

$$J \mu_0^{i_{\oplus}} \circ J p_{i-1}^{(i-1)} \circ F_{i-1}^{(i-1)}[f, u_{i+1}, \ldots, u_{2n}],$$

$$f, u_{i+1}, \ldots, u_{2n}] \circ J \beta_{i-1}^{(i-1)} \circ c_i^+ \quad \text{(by the definition of $c_i^+$)}$$

$$= J(F^{-1})_{i-1} \circ F [J p_{i-1}^{(i-1)}, \ldots, J p_{i-1}^{(i-1)}, \text{id}_{F \oplus 0}]$$

$$\circ F([\Pi_{j=1}^{i-1} F_{j}^{(i-1)}][f, u_{i+1}, \ldots, u_{2n}] + [f, u_{i+1}, \ldots, u_{2n}])$$

$$= J(F^{-1})_{i-1} \circ F [J p_{i-1}^{(i-1)}, \ldots, J p_{i-1}^{(i-1)}, \text{id}_{F \oplus 0}] \circ F_{i-1}^{(i-1)}[f, u_{i+1}, \ldots, u_{2n}]$$

$$\circ J \beta_{i-1}^{(i-1)} \circ J \xi_{i-1}^{(i-1)} \circ c_i^+ \quad \text{(by the definition of $c_i^+$)}$$

$$= J(F^{-1})_{i-1} \circ J F [\mu_0^{i_{\oplus}} \circ p_{i-1}^{(i-1)}, \ldots, \mu_0^{i_{\oplus}} \circ p_{i-1}^{(i-1)}, \text{id}_{F \oplus 0}] \circ J \beta_{i-1}^{(i-1)} \circ J \xi_{i-1}^{(i-1)}$$

$$\circ F_{i-1}^{(i-1)}[f + \text{id}_{F \oplus 0}] \circ F_{i-1}^{(i-1)}(\text{id}_{X_i} + [u_{i+1}, \ldots, u_{2n}]) \circ c_i^+$$

$$= J \sigma_i \circ F_{i-1}^{(i-1)}(f + \text{id}_{F \oplus 0}) \circ (F_{i-1}^{(i-1)}(\text{id}_{X_i} + [u_{i+1}, \ldots, u_{2n}]) \circ c_i^+)$$

(by the definition of $\sigma_i$).  \hspace{2em} (8)

Therefore we have to show that $J \mu_0^{i_{\oplus}} \circ J p_i^{(i)} \circ F_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{\xi}_i^{(i)}$ is the greatest fixed point of $F_i^{(i)}(\text{id}_{X_i} + [u_{i+1}, \ldots, u_{2n}]) \circ c_i^+ \cdot J \sigma_i$ (see also Fig. 4). We have:

$$\tilde{\xi}_i^{(i)}$$

is the greatest homomorphism from $c_i^+$ to $J \xi_{i-1}^{(i-1)} + \ldots + X_{2n}$.

By definition, $\tilde{\xi}_i^{(i)}$ is the greatest homomorphism from $c_i^+$ to $J \xi_{i-1}^{(i-1)} + \ldots + X_{2n}$.

Therefore by Lem. [B.3] and Sublem. [B.6] instead of the gfp-preserving condition respectively.

It remains to prove Eq. [6] for $j < i$. We have:

$$l_j^{(i)}(u_{i+1}, \ldots, u_{2n})$$

$$= l_j^{(i-1)}(l_i^{(i)}(u_{i+1}, \ldots, u_{2n}), u_{i+1}, \ldots, u_{2n}) \quad \text{(by definition)}$$
\[
= l_j^{(i-1)}(\mathcal{J}_\eta^{F_0} \circ P_j^{(i-1)} \circ F_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{g}_j^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_j^{(i-1)})
\]
(by the discussion above)

\[
= \mu_0^{F_0} \circ P_j^{(i-1)}
\]
\[
\circ F_j^{(i-1)}[\mu_0^{F_0} \circ P_j^{(i-1)} \circ F_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{g}_j^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_j^{(i-1)}]
\]
(by IH)

\[
= \mu_0^{F_0} \circ P_j^{(i-1)}
\]
\[
\circ F_j^{(i-1)}[\mu_0^{F_0} \circ P_j^{(i-1)} \circ F_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_i^{(i)}]
\]
(by Def. A.11)

\[
= \mu_0^{F_0} \circ P_j^{(i-1)}
\]
\[
\circ F_j^{(i-1)}[\mu_0^{F_0} \circ P_j^{(i-1)} \circ F_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_i^{(i)}]
\]
(by Lem. A.12)

\[
= \mu_0^{F_0} \circ P_j^{(i-1)}
\]
\[
\circ F_j^{(i-1)}[\mu_0^{F_0} \circ P_j^{(i-1)} \circ F_i^{(i)}[u_{i+1}, \ldots, u_{2n}] \circ \tilde{f}_i^{(i)}]
\]
(by Sublem. [B.3].)

This concludes the proof. \(\square\)

C Omitted Example

Example C.1 We define \(F : \text{Sets} \rightarrow \text{Sets}\) by \(F = \{o\} \times (\_\_) \times (\_\_).\) Let \(X = \{x\}\) and \(Y = \{y_1, y_2\},\) and define \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\) in \(\mathcal{K}(\mathcal{D})\) by \(f(x) = [y_1 \rightarrow \frac{1}{2}, y_2 \rightarrow \frac{3}{2}]\) and \(f(y_1) = f(y_2) = [x \rightarrow 1].\) In a similar manner to Ex. 3.3 we can show that \(F^\otimes X\) is identified with the set of infinitary binary trees whose depth is greater than 1, nodes are labeled with \(o\) and leaves are labeled with \(x.\) A set \(F^\otimes Y\) is similarly characterized. Let \(t_X \in F^\otimes X\) be an element identified with a tree \(o(x, o(x, o(x, \ldots))).\) For each \(t_Y \in F^\otimes Y,\)

\[
F^\otimes f(t_X)(t_Y) = J(\zeta^+_{F^\otimes})^{-1} \circ T(F^\otimes f + \text{id}_Y) \circ T(\text{id} + f) \circ \zeta^+_{F^\otimes}(t_X)(t_Y) = \frac{1}{2} F^\otimes f(t_X)(t_Y).
\]

This implies \(F^\otimes f(t_X)(t_Y) = 0,\) and therefore \(F^\otimes g \circ F^\otimes f(t_X)(t_Y) = 0.\) In contrast, \(\text{id}_X : X \rightarrow X\) is a homomorphism from \(F(\text{id} + g) \circ F(\text{id} + f) \circ J\zeta^+_{F^\otimes} = J\zeta^+_{F^\otimes}\) to itself, and \(\text{id}_X(t_X)(t_X) = 1 \neq 0.\) Hence \(F^\otimes (g \circ f)(t_X)(t_X) \geq 1,\) and this means that the operation \(F^\otimes\) does not satisfy the functoriality.

D Distributive Laws from \(T\) to \(F^+\) and \(F^\otimes\)

Definition D.1 A distributive law from \(T\) to \(F\) is a natural transformation \(\lambda : FT \Rightarrow TF\) that makes the following diagrams commute for each \(X.\)

\[
\begin{array}{ccc}
FTX & \xrightarrow{\lambda_X} & TFX \\
\downarrow{F\eta_X} & & \downarrow{\eta_{TX}} \\
FX & \xrightarrow{\eta_{FX}} & TX
\end{array}
\]  \hspace{1cm} \begin{array}{ccc}
FTX & \xrightarrow{\lambda_X} & TFX \\
\uparrow{F\mu_X} & & \uparrow{\mu_{TX}} \\
FT^2X & \xrightarrow{\lambda_{TX}} & TFX
\end{array}
\]
Fig. 4: $J\mu F_i^0 \odot J\rho_{j,F_i^0} \odot F_i^F[u_{i+1},\ldots,u_{2n}] \odot \hat{e}_i^{(i)}$ is the greatest fixed point.
Proposition D.2 Let $\lambda : FT \Rightarrow TF$ be a distributive law from $T$ to $F$. For $A, X \in \mathbb{C}$, we write $\lambda_{A, X}$ for $\lambda_{A, X} \circ F[T_{K_1 \eta A, K_2}] : F(A + TX) \rightarrow TF(A + X)$.

1. Assume $T$ and $F(\_ + A)$ constitute a finite trace situation for each $A \in \mathbb{C}$.

For $X \in \mathbb{C}$ we define $\lambda_{+, X} : F^+TX \rightarrow TF^+X$ as the unique homomorphism from $\lambda_{F^+TX, X} \circ J(i_{F^+X})^{-1}$ to $J(i_{F^+X})^{-1}$. Then $\lambda_+ := (\lambda_{+, X})_{X \in \mathbb{C}}$ is a natural transformation $F^+T \Rightarrow TF^+$, and is a distributive law from $T$ to $F^+$.

2. Assume $T$ and $F(\_ + A)$ constitute an infinitary trace situation and satisfy the gfp-preserving condition and the deterministic-greatest condition for each $A \in \mathbb{C}$. For $X \in \mathbb{C}$, let $\lambda_{\oplus, X} : F^\oplus TX \rightarrow TF^\oplus X$ be the greatest homomorphism from $\lambda_{F^\oplus TX, X} \circ Jc_{F^\oplus X}$ to $Jc_{F^\oplus X}$. Then $\lambda_{\oplus} := (\lambda_{\oplus, X})_{X \in \mathbb{C}}$ is a natural transformation $F^\oplus T \Rightarrow TF^\oplus$, and is a distributive law from $T$ to $F^\oplus$.

\[
\begin{align*}
F(F^+TX + X) & \xrightarrow{\mathcal{F}(\lambda_{+, X} + id)} F(F^+X + X) \\
F(F^\oplus TX + X) & \xrightarrow{\mathcal{F}(\lambda_{\oplus, X} + id)} F(F^\oplus X + X)
\end{align*}
\]

\[
\begin{align*}
\lambda_{F^+TX, X} & \xrightarrow{\mathcal{J}(\iota_{F^+X})^{-1}} JF(F^+TX) = \mathcal{J}c_{F^+X} \\
\lambda_{F^\oplus TX, X} & \xrightarrow{\mathcal{J}(\iota_{F^\oplus X})^{-1}} JF(F^\oplus TX) = \mathcal{J}c_{F^\oplus X}
\end{align*}
\]

E Decorated Trace Semantics for Other Branching Types

E.1 Decorated Trace Semantics for $T = \mathcal{L}$

Definition E.1 A parity tree automaton with an exception is a quadruple $A = (X, \Sigma, \delta, \Omega)$ consisting of a state space $X$, a ranked alphabet $\Sigma$, a transition function $\delta : X \rightarrow \{\bot\} + \prod_{n=0}^{\infty} \Sigma^n \times X^n$ and a priority function $\Omega : X \rightarrow [1, 2n]$.

A run tree over $A$ is a $\mathcal{L}$-labeled tree $\rho$ such that for each subtree $\rho_w = ((a, x), ((a_0, x_0), l_0, \ldots, l_{0n}), \ldots, ((a_n, x_n), t_0, \ldots, t_{nn}))$ of $\rho$, $\delta(x)$ is defined and $(a, x_0, \ldots, x_n) = \delta(x)$ holds.

Lemma E.2 For each $x \in X$, if a run tree of $A$ whose root node is labeled with $x$ exists, then it is unique.

Proposition E.3 For each $X, Y \in \text{Sets}$, we define a partial order $\subseteq$ over $K(\mathcal{L})(X, Y)$ by $f \subseteq g \iff \forall x \in X, g(x) = \bot \Rightarrow f(x) = \bot$. Then the conditions in Asm. [A.8] are satisfied by $(T, F) = (\mathcal{L}, F_\Sigma)$ wrt. $\subseteq$.

Proof. Proved in a similar manner to Prop. [A.22]

Proposition E.4 Let $A = (X, \Sigma, \delta, \Omega)$ be a parity tree automaton with an exception where $\Omega : X \rightarrow [1, 2n]$. We define a parity $(\mathcal{L}, F_\Sigma)$-system $(c : X \rightarrow LF_{\Sigma}X, (X_1, \ldots, X_{2n}))$ by $c := \delta$ and $X_i := \{x \mid \Omega(x) = i\}$ for each $i \in [1, 2n]$. Then for each $i \in [1, 2n]$ and $x \in X_i$, with respect to the isomorphism in Prop. [A.21] we have:

\[
d_{\text{tr}}(c)(x) = \begin{cases} 
\Omega(p) & \text{(an accepting run tree $p$ whose root node is labeled with $x$ exists)} \\
\bot & \text{(otherwise)}
\end{cases}
\]

Proof. Proved in a similar manner to Prop. [A.23]
E.2 Decorated Trace Semantics for $T = G$

For parity $(T,F)$-systems, the weakened conditions in [6] becomes as follows.

Assumption E.5 When $n$ is odd, the following conditions are satisfied.

1. $T$ and $F^i_n (\_ + A)$ satisfy the gfp-preserving condition wrt. an algebra $F^1_n ((F^1)^\oplus B + A)$ for each $f : A \rightarrow B$;

2. $T$ and $F^+(\_ + A)$ satisfy the gfp-preserving condition wrt. an algebra $F^+(F^\oplus A + A)$ where $\tau$ is the unique homomorphism from $\zeta^F_{F^\oplus A + A}$ to $\zeta^F_{F^\oplus A + A}$; and

3. $T$ and $F(\_ + A)$ satisfy the gfp-preserving condition wrt. an algebra $F(F^\oplus A + F^\oplus A + A)$.

By carefully checking the proofs of Prop. 3.14, Lem. A.13 and Lem. A.15 where the gfp-preserving condition is used, we can prove the following proposition.

Proposition E.6 Thm. A.16 still holds if we replace Cond. A of Asm. A.8 with the conditions in Asm. E.7.

Definition E.7 Let $\Sigma$ be a ranked alphabet. A $\Sigma$-labeled partial tree is a finite $\Sigma + \{\ast\}$-labeled tree. Hence the set of $\Sigma$-labeled partial tree is denoted by $\text{Tree}_{\text{fin}} (\Sigma + \{\ast\})$. We say that $t = (D,l) \in \text{Tree}_{\text{fin}} (\Sigma + \{\ast\})$ is a prefix of $t' = (D',l') \in \text{Tree}_{\infty} (\Sigma)$ and write $t \preceq t'$ if $D \subseteq D'$ and $l(w) \neq \ast$ implies $l(w) = l'(w)$.

Definition E.8 For each $t \in \text{Tree}_{\text{fin}} (\Sigma + \{\ast\})$, we define $\text{Cyl}(t) \subseteq \text{Tree}_{\infty} (\Sigma)$ by: $\text{Cyl}(t) := \{ t' : t \in \text{Tree}_{\infty} (\Sigma) | t \preceq t' \}$. We define a $\sigma$-algebra $\mathcal{F}_{\text{Tree}_{\infty} (\Sigma)} \subseteq \mathcal{P}\text{Tree}_{\infty} (\Sigma)$ over $\text{Tree}_{\infty} (\Sigma)$ as the smallest $\sigma$-algebra that contains $\text{Cyl}(t)$ for each $t \in \text{Tree}_{\text{fin}} (\Sigma + \{\ast\})$.

Definition E.9 A probabilistic parity tree automaton (PPTA) is a quadruple $A = (X, \Sigma, \delta, \Omega)$ consisting of a countable state space $X$, a ranked alphabet $\Sigma$, a transition function $\delta : X \rightarrow [0,1]^{\Sigma \times X^n}$ such that $\sum_{x \in X^n} \delta(x)(x) \leq 1$ for each $x \in X$, and a priority function $\Omega : X \rightarrow [1,2n]$. We identify the set $X$ with a measurable space $(X, \mathcal{P}\mathcal{X})$.

We inductively define a function $f : X \times \text{Tree}_{\text{fin}} (\Sigma \times X + \{\ast\}) \rightarrow [0,1]$ as follows:

- A function $f(\_, \text{Tree}_{\infty} (\Sigma \times X)) : X \rightarrow [0,1]$ is defined as the greatest fixed-point of the following function:

$$g \mapsto \int_{(a,x_0,\ldots,x_{m-1}) \in \prod_{m=0}^{m} \Sigma_m \times X^m} \prod_{t=0}^{m-1} g(x_t) \, d\delta(x).$$

Here $\mathcal{F}_{\prod_{m=0}^{m} \Sigma_m \times X^m}$ denotes the $\sigma$-algebra over $\prod_{m=0}^{m} \Sigma_m \times X^m$. 


− If \( t = ((a, x), t_0, \ldots, t_{m-1}) \), then we let \( f(t) := \sum_{(x_0, \ldots, x_{m-1}) \in X} \prod_{k=0}^{m-1} \delta(x)(a, x_0, \ldots, x_{m-1}) \cdot f(x_k, t_k) \).

By the Carathéodory theorem (see e.g. [11]), for each \( x \in X \), there exists a unique probability measure \( \text{Prob}_{\text{Run}_{\mathcal{A}}}(x) \) over \((\text{Tree}_{\infty}(\Sigma \times X), \mathcal{F}_{\text{Tree}_{\infty}}(\Sigma \times X))\) such that \( \text{Prob}_{\text{Run}_{\mathcal{A}}}(x)(\text{Cyl}(t)) = f(x, t) \) for each \( t \in \text{Tree}_{\infty}(\Sigma \times X + \{\star\}) \).

**Proposition E.10** For each \((X, \mathcal{G}_X), (Y, \mathcal{G}_Y) \in \text{Meas}\), we define a partial order \( \subseteq \) over \( \mathcal{K}(\mathcal{G})(\Sigma, \mathcal{G}_X, \Sigma, \mathcal{G}_Y) \) by \( f \subseteq g \iff \forall x \in X, \forall A \in \mathcal{G}_Y, f(x)(A) \leq g(x)(A) \). Then Cond. 7-10 in Assn. 1-3 and Cond. 7-1 in [10] are satisfied by \((T, F) = (\mathcal{G}, F_2)\) wrt. \( \subseteq \).

**Proof.** Cond. 7-2-3-4-7 and 8 are proved in a similar manner to Prop. A.22.

We next prove that Cond. 7-1 is satisfied. Let \( c : X \to F^1_i(X + A) \). Let \( l : X \to (F^1_i)^+ A \) be the greatest homomorphism from \( c \) to \( J\mathcal{G}_{\mathcal{A}}^{F^1_i} \). Let \( m : (F^1_i)^+ A \to (F^1_i)^+ B \) be the greatest arrow such that \( m = (J\mathcal{G}_{\mathcal{B}}^{(F^1_i)^+ A} - \cdots \circ F^1_i(m + \id)) \circ (F^1_i)^+ B \).

For each \( k \in \omega \), we inductively define \( \pi_k : (F^1_i)^+ A \to (F^1_i)^+ (\mathcal{A} + A) \) as follows: \( \pi_0 := !_{(F^1_i)^+ A} \) and \( \pi_{k+1} := (F^1_i)^+ (\mathcal{A} + \id_A) \circ F^1_i(\pi_k + \id_A) \). We define \( \pi_k : (F^1_i)^+ B \to (F^1_i)^+ (\mathcal{B} + B) \).

By Thm. 2.5, \((F^1_i)^+ A, (\pi_k)_{k \in \omega}\) is a limit over a final sequence \( 1^1 \leftarrow F^1_i(1 + A) \leftarrow F^1_i(F^1_i(1 + A) + A) \leftarrow F^1_i(F^1_i(F^1_i(1 + A) + A + A) + A) \leftarrow \cdots \). Similarly, \((F^1_i)^+ B, (\pi_k)_{k \in \omega}\) is a limit over \( 1^1 \leftarrow F^1_i(1 + B) \leftarrow F^1_i(F^1_i(1 + B) + B) \leftarrow F^1_i(F^1_i(F^1_i(1 + B) + B + B) + B) \leftarrow \cdots \).

It is known that \( \mathcal{G} : \text{Meas} \to \text{Meas} \) preserves a limit of an \( \omega^{\text{op}} \)-sequence that consists of standard Borel sets [18]. This means that \(((F^1_i)^+ A, (\pi_k)_{k \in \omega})\) is a limit over a sequence \( 1^1 \leftarrow F^1_i(1 + A) \leftarrow F^1_i(F^1_i(1 + A) + A) \leftarrow F^1_i(F^1_i(F^1_i(1 + A) + A + A) + A) \leftarrow \cdots \).

It is easy to see that it is also a 2-limit, i.e. for two cones \((X, (\gamma_k)_{k \in \omega})\) and \((X, (\gamma_k')_{k \in \omega})\) over \( 1^1 \leftarrow F^1_i(1 + A) \leftarrow F^1_i(F^1_i(1 + A) + A) \leftarrow F^1_i(F^1_i(F^1_i(1 + A) + A + A) + A) \leftarrow \cdots \), if \( \gamma_k^1 \subseteq \gamma_k'^1 \) for each \( k \in \omega \), then we have \( l^1 \subseteq l^2 \) where \( l^1 \) (resp. \( l^2 \)) is a mediating arrow from \((X, (\gamma_k)_{k \in \omega})\) (resp. \((X, (\gamma_k')_{k \in \omega})\)) to \(((F^1_i)^+ A, (\pi_k)_{k \in \omega})\).

Similarly, \(((F^1_i)^+ B, (\pi_k)_{k \in \omega})\) is a 2-limit over \( 1^1 \leftarrow F^1_i(1 + B) \leftarrow F^1_i(F^1_i(1 + B) + B) \leftarrow F^1_i(F^1_i(F^1_i(1 + B) + B + B) + B) \leftarrow \cdots \).

We inductively define a cone \((X, (\gamma_k) : X \to (F^1_i(\mathcal{A} + A)) \leftarrow (F^1_i(1 + A) + A) \leftarrow (F^1_i(F^1_i(1 + A) + A + A) + A) \leftarrow \cdots \) as follows:

− For each \( t \in \omega \), we define an arrow \( f_t : X \to 1 \) as follows: i) \( f_0 := J ! X \) and ii) \( f_{t+1} := J ! \circ f_t \circ (\mathcal{A} + \id_A) \). It is easy to see that \( f_0 \supseteq f_1 \supseteq \ldots \).

We define \( f_\omega : X \to 1 \) by \( f_\omega := \prod_{t \in \omega} f_t \). As the composition \( \circ \) in \( \mathcal{K}(T) \) is \( \omega^{\text{op}} \)-continuous, by the Kleene fixed point theorem, \( f_\omega \) is the greatest fixed point of a function \( f \to J ! \circ (\mathcal{A} + \id_A) \circ c \). We let \( \gamma_0 := f_\omega \).
Proof. Proved in a similar manner to [20, Thm. A.13].

For an arbitrary cone \((X, (\gamma'_k : X \to (F^+_i(\_ + A))^k)_{k \in \omega})\) over \(1 \xleftarrow{l^\_} F^+_i(1 + A) \xrightarrow{F^+_i(l + id_A)} F^+_i(F^+_i(1 + A) + A) \xrightarrow{F^+_i(F^+_i(l + id_A) + id_A)} \ldots\), we have \(\gamma_k' \subseteq \gamma_k\) for each \(k\). Hence the mediating arrow from \((X, (\gamma_k)_{k \in \omega})\) to \(((F^+_i)^{\oplus}A, (J\pi'_k)_{k \in \omega})\) is the greatest homomorphism \(l\).

We inductively define a cone \((X, (\delta_k : X \to (F^+_i(\_ + B))^k)_{k \in \omega})\) over \(1 \xleftarrow{m^\_} F^+_i(1 + B) \xrightarrow{F^+_i(l + id_B)} F^+_i(F^+_i(1 + B) + B) \xrightarrow{F^+_i(F^+_i(l + id_B) + id_B)} \ldots\) in a similar manner. The mediating arrow \(l' : X \to (F^+_i)^{\oplus}B\) from \((X, (\gamma_k)_{k \in \omega})\) to \(((F^+_i)^{\oplus}B, (J\pi'_k)_{k \in \omega})\) is the greatest fixed point of \(\Phi := \coprod_{\xi} f^\_ \circ l^\_\) on the one hand.

On the other hand, we can easily show that \((F^+_i(id + \_))^k f \circ \gamma_k = \delta_k \circ l\). This implies that \(m \circ l\) is a mediating arrow from \((X, (\gamma_k)_{k \in \omega})\) to \(((F^+_i)^{\oplus}B, (J\pi'_k)_{k \in \omega})\).

Hence we have \(l' = m \circ l\).

Cond. 4'-2 and Cond. 4'-3 are similarly proved.

It is easy to see that if \(Jf \subseteq g : X \to Y\), then \(g = Jf\). Hence Cond. \(\Box\) holds. This concludes the proof.

Lemma E.11 For \(i \in \mathbb{N}\) and \(j \in [1, i]\), \((F_\Sigma^\_ j)^{(i)} A \cong (\text{AccTree}^{(i)}_j(\Sigma, A), \mathcal{F}_{\text{AccTree}^{(i)}_j}(\Sigma, A))\)

where \(\mathcal{F}_{\text{AccTree}^{(i)}_j}(\Sigma, A) := \mathcal{F}_{\text{AccTree}^{(i)}_j}(\Sigma, A) \cap \text{AccTree}^{(i)}_j(\Sigma, A)\). Moreover, if \(i\) is odd then the function \(\text{decomp}^{(i)}_j\) in Prop. [A.22] is given by \((i^{(F\_\Sigma^\_ j)^{(i)} A + A})^{-1}\).

If \(i\) is even, then it is given by \(\coprod_{i = j+2}^{(F\_\Sigma^\_ j)^{(i)} A + A}\).

Proposition E.12 Let \(A = (X, \Sigma, \delta, \Omega)\) be a probabilistic parity tree automaton where \(\Omega : X \to [1, 2n]\). We define a parity \((\mathcal{G}, F\_\Sigma)\)-system \((c : X \to LF\_\Sigma X, (X_1, \ldots, X_{2n}))\) by \(X := (X, prob\,X), c(x)(\{x\}) := \delta(x)(x)\) and \(X_i := \{x \mid \Omega(x) = i\}\) for each \(i \in [1, 2n]\). Then for each \(i \in [1, 2n]\), \(x \in X_i\), and \(A \in \mathcal{F}_{\text{AccTree}^{(i)}_j}(\Sigma, A)\), with respect to the isomorphism in Lem. [E.11]

\[dtr_i(c)(x)(A) = \text{ProbRun}_A(x)(\Omega^{-1}(A))\]

Proof. Proved in a similar manner to [20, Thm. A.13]. □