On the Four-Dimensional Effective Action of Strongly Coupled
Heterotic String Theory

André Lukas\textsuperscript{1\#}, Burt A. Ovrut\textsuperscript{1,3\§} and Daniel Waldram\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of Pennsylvania
Philadelphia, PA 19104–6396, USA

\textsuperscript{2}Department of Physics
Joseph Henry Laboratories, Princeton University
Princeton, NJ 08544, USA

\textsuperscript{3}Institut für Physik, Humboldt Universität
Invalidenstraße 110, 10115 Berlin, Germany

Abstract

The low-energy $D=4$, $N=1$ effective action of the strongly coupled heterotic string is explicitly computed by compactifying Hořava-Witten theory on the deformed Calabi-Yau threefold solution due to Witten. It is shown that, to order $\kappa^{2/3}$, the Kähler potential is identical to that of the weakly coupled theory. Furthermore, the gauge kinetic functions are directly computed to order $\kappa^{4/3}$ and shown to receive a non-vanishing correction. Also, we compute gauge matter terms in the Kähler potential to the order $\kappa^{4/3}$ and find a nontrivial correction to the dilaton term. Part of those corrections arise from background fields that depend on the orbifold coordinate and are excited by four-dimensional gauge field source terms.

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1 Introduction

The simplest evidence that models of particle physics might best be described by the strong-coupling limit of heterotic string theory comes from the string tree-level predictions of the four-dimensional Planck scale and the grand unification scale and coupling. Generically, it is difficult to match the experimentally inferred values without a string coupling many orders of magnitude larger than unity. Witten [1] has investigated this possibility in the case of the $E_8 \times E_8$ heterotic string, using the identification of the strongly coupled limit with M theory compactified on an $S^1/Z_2$ orbifold [2, 3].

The effective action of M theory can be formulated as a momentum expansion in terms of powers of the eleven-dimensional gravitational coupling $\kappa^2$. Remarkably, Witten finds a model where the four-dimensional Planck scale and the inferred grand unification scale and coupling can just be matched, without corrections in the expansion becoming too large [1, 4]. In addition, Hořava has shown that in this model supersymmetry can be broken via an interesting topological version of gluino condensation [5].

The purpose of the present paper is to derive, directly from M theory, the $N = 1$ four-dimensional effective action which arises from Witten’s strongly coupled model. As a preparation, we prove several useful properties of Witten’s solution. In particular, we introduce a simple reparameterization of this solution and find its explicit form in terms of a harmonic expansion. Armed with these results, we calculate the corrections due to terms of higher order in $\kappa^2$, which are the analog of loop-corrections in the weakly coupled string. We present the Kähler potential to order $\kappa^{2/3}$ for the moduli fields and to order $\kappa^{4/3}$ in the gauge matter fields. For the gauge fields we calculate the gauge kinetic functions to order $\kappa^{4/3}$. In general, we take care to identify contributions from background fields excited by four-dimensional gauge and gauge matter field sources. We find these background fields contribute to various corrections at order $\kappa^{4/3}$. In particular, they generate the imaginary part of the threshold, with the real part arising from the metric distortion due to the internal gauge fields. For the Kähler potential, we show that it is possible to choose a parameterization of the universal moduli, such that their Kähler potential receives no corrections to the order $\kappa^{2/3}$. To the order $\kappa^{4/3}$, we find a gauge matter field term correcting the dilaton part of the Kähler potential, which is the direct analog of the order $\kappa^{4/3}$ threshold corrections to the gauge kinetic functions.

Some of these results have been anticipated in the literature. The Kähler potential, to zeroth order in $\kappa$, was computed in ref. [7, 8, 9, 10] but these authors did not discuss possible order $\kappa^{2/3}$ modifications. Such modifications, and their absence to this order, were argued for indirectly in ref. [4]. The modification of the gauge coupling was first discussed by Witten in his presentation of the strongly coupled model. Arguments for the form of the gauge kinetic functions, based on results in the weakly coupled limit, symmetry and holomorphicity, were presented in ref. [11, 13]. It appears that a related calculation to the one presented here, for the real part of the gauge
threshold correction, was carried out in ref. [6], although details and the exact coefficient were not presented. Phenomenological aspects of the zeroth order effective action of the strongly coupled heterotic string have been discussed in ref. [12, 13, 14]. Gaugino condensation in this model has been considered in ref. [5, 6, 9].

It is useful to understand the scales in Witten’s model and also the exact nature of the expansion in $\kappa^2$. To do so, let us be a little more explicit about the Hořava-Witten description of the strongly coupled heterotic string, and also about Witten’s background solution. To zeroth order in a momentum expansion in $\kappa^2$, M theory is simply eleven-dimensional supergravity [15]. When compactified on $S^1/Z_2$, the first correction appears at order $\kappa^{2/3}$. One finds that a set of $E_8$ gauge fields must be included on each of the two fixed hyperplanes of the orbifold and the Bianchi identity for the supergravity four-form field $G$ gets a correction. These terms are required if the theory is to be free of anomalies.

To make contact with low-energy physics, Witten considered a compactification which, to zeroth order in $\kappa^2$, was simply the $S^1/Z_2$ orbifold times a Calabi-Yau three-fold. Two universal moduli appear, which are the analogs of the $S$ and $T$ moduli of the weakly coupled limit, namely, the volume of the Calabi-Yau three-fold $V$ and the radius of the orbifold $\rho$. One finds that the four-dimensional Newton constant and the grand-unified (GUT) coupling constant are given by

\[
G_N = \frac{\kappa^2}{16\pi^2 V \rho}, \quad \alpha_{GUT} = \frac{(4\pi\kappa^2)^{2/3}}{2V},
\]

while the GUT scale (about $10^{16}$ Gev) is set by the size of the Calabi-Yau manifold $V^{1/6}$. To match the known values of $G_N$ and $\alpha_{GUT}$, Banks and Dine [4] find that $V^{1/6}/\kappa^{2/9} \sim 2$ while $\pi \rho/\kappa^{2/9} \sim 8$. That is to say, the Calabi-Yau radius is about twice as large as the eleven-dimensional Planck scale $\kappa^{2/9}$, while the orbifold radius is about an order of magnitude larger. (These numbers can vary somewhat depending on how exactly the scales are defined.) Thus in such a scenario, with increasing energy, the universe appears first five- and then eleven-dimensional, and the four-dimensional Planck scale is relegated to a low-energy parameter, with no direct relevance to the scale of quantum gravitational effects.

To this order, other than the difference in scales, the low-energy theory is indistinguishable from that for the tree-level weakly coupled string. However, Witten calculates the modification in the M theory background due to the next order corrections in $\kappa^2$, which appear at $\kappa^{2/3}$, and which still preserves $N = 1$ supersymmetry in four dimensions. In analogy to the weakly coupled case, the spin connection is embedded in one of the $E_8$ groups. However, the supergravity four-form can no longer be taken to be zero because of the correction to its Bianchi identity. This in turn means that the internal space becomes deformed and is no longer a simple product of the $S^1/Z_2$ orbifold with a Calabi-Yau three-fold. It is the modification to the low-energy action due to this deformation in which we are interested.
To understand the form of potential corrections to the low-energy action, one needs to identify what exactly is the expansion parameter in Witten’s solution. Since there are two dimensionless scales, related to the size of the Calabi-Yau space and the orbifold in eleven-dimensional Planck units, we do not \textit{a priori} know if the expansion is in $\kappa^{2/3}/V^{1/2}$ or $\kappa^{2/3}/\rho^3$ or some combination thereof. In fact, one finds that the metric deformation in Witten’s solution is of order $\epsilon = \kappa^{2/3}/V^{2/3}$. Using the Banks and Dine values for $V$ and $\rho$ given above, one finds $\epsilon \sim \frac{1}{2}$, and we are right at the limit of the validity of the expansion. However, partly since $\epsilon$ depends somewhat on exactly how the scales are defined, we shall assume in this paper that the expansion is reasonable. From the form of $\epsilon$, we find that the first-order corrections to the low-energy action should depend on the universal moduli in the combination $\rho/V^{2/3}$.

Let us conclude this section by summarizing our conventions. We denote the eleven-dimensional coordinates by $x^0, \ldots, x^9, x^{11}$ and the corresponding indices by $I, J, K, \ldots = 0, \ldots, 9, 11$. The orbifold $S^1/Z_2$ is chosen in the $x^{11}$-direction, so we assume that $x^{11} \in [-\pi\rho, \pi\rho]$ with the endpoints identified as $x^{11} \sim x^{11} + 2\pi\rho$. The $Z_2$ symmetry acts as $x^{11} \rightarrow -x^{11}$. Then there exist two ten-dimensional hyperplanes, $M_{i}^{10}$ with $i = 1, 2$, locally specified by the conditions $x^{11} = 0$ and $x^{11} = \pi\rho$, which are fixed under the action of the $Z_2$ symmetry. We will sometimes use the “downstairs” picture where the orbifold is considered as an interval $x^{11} \in [0, \pi\rho]$ with the fixed hyperplanes forming boundaries to the eleven-dimensional space. In the “upstairs” picture the eleventh coordinate is considered as the full circle with singular points at the fixed hyperplanes. We will use barred indices $\bar{I}, \bar{J}, \bar{K}, \ldots = 0, \ldots, 9$ to label the ten-dimensional coordinates. When we later further compactify the theory on a Calabi-Yau three-fold, we will use indices $A, B, C, \ldots = 4, \ldots, 9$ for the Calabi-Yau coordinates, and indices $\mu, \nu, \ldots = 0, \ldots, 3$ for the coordinates of the remaining, uncompactified, four-dimensional space. Holomorphic and antiholomorphic coordinates on the Calabi-Yau space will be labeled by $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$.

2 The strongly coupled heterotic string and compactification on Calabi–Yau spaces

In the first part of this section, we review the formulation of the low-energy limit of strongly coupled heterotic string theory as the effective action for M theory compactified on the orbifold $S^1/Z_2$. The effective theory is given as a momentum expansion in the eleven-dimensional gravitational coupling $\kappa^{2/3}$. As an example of techniques we will use in deriving effective actions in four-dimensions, we then consider the limit where the fields are assumed to vary slowly over the scale of the orbifold interval. The resulting ten-dimensional theory, though calculated in the strongly coupled limit, is identical to the low-energy effective action for the weakly coupled heterotic string. In particular, we demonstrate how the ten-dimensional Chern-Simons and Green-Schwarz terms appear through a somewhat novel mechanism, arising from components of the supergravity four-form which depend
non-trivially on the orbifold dimension. This mechanism will reappear when we come to derive the effective action in four dimensions. Finally, we review the form of a solution of the eleven-dimensional theory, due to Witten, which has a spacetime geometry \( M^{11} = S^1/Z_2 \times X \times M^4 \), where \( X \) is a deformed Calabi–Yau three-fold and \( M^4 \) is the Minkowski four-space. This background is the starting point for the construction of a \( D = 4 \) effective action with \( N = 1 \) supersymmetry.

2.1 M theory on \( S^1/Z_2 \)

To zeroth order in \( \kappa^2 \), the effective action for M theory on \( S^1/Z_2 \) is simply that of eleven-dimensional supergravity. In the upstairs picture, the bosonic parts are given by

\[
S_{SG} = \frac{1}{2\kappa^2} \int_{M^{11}} \sqrt{-g} \left[ -R - \frac{1}{24} G_{IJKL} G^{IJKL} - \frac{\sqrt{2}}{1728} \epsilon^{I_1 \ldots I_{11}} C_{I_1I_2I_3} G_{I_4 \ldots I_7} G_{I_8 \ldots I_{11}} \right].
\]  

(2)

The field \( C_{IJK} \) is the three-form potential for the field strength \( G_{IJKL} = 24 \partial I [C_{JKL}] \). In addition, the fields are restricted to respect the \( Z_2 \) orbifold symmetry. For the bosonic fields, \( g_{I\bar{J}}, g_{11I} \) and \( C_{11\bar{I}J} \) must be even under \( Z_2 \), while \( g_{11\bar{I}} \) and \( C_{I\bar{J}\bar{K}} \) must be odd. For the eleven dimensional gravitino the condition is

\[
\Psi_I(x^{11}) = \Gamma_{11} \Psi_I(-x^{11}) \quad \Psi_{11}(x^{11}) = -\Gamma_{11} \Psi_{11}(-x^{11}).
\]  

(3)

This constraint means that the gravitino is chiral from a ten-dimensional perspective and so, unlike the usual case in eleven-dimensions, the theory has a gravitational anomaly, localized on the fixed hyperplanes.

The way to cancel this anomaly is to introduce two ten-dimensional \( E_8 \) gauge supermultiplets, one on each of the two fixed hyperplanes of the orbifold. The leading correction to the supergravity action enters at order \( \kappa^{2/3} \). If we restrict ourselves for the moment to terms at most quadratic in derivatives, one finds the action is given by

\[
S = S_{SG} + S_{YM} = S_{SG} - \frac{1}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^{10}} \sqrt{-g} \ tr(F^{(1)})^2 - \frac{1}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^{10}} \sqrt{-g} \ tr(F^{(2)})^2
\]  

(4)

where \( F_{i\bar{j}}^{(1,2)} \) are the field strengths of the two \( E_8 \) gauge fields \( A^{(1,2)}_i \). Hidden in the above expression is a further order \( \kappa^{2/3} \) correction. Namely, to preserve supersymmetry, it is necessary to correct the Bianchi identity for \( G_{IJKL} \), introducing source terms localized on the hyperplanes so that

\[
(dG)_{1\bar{I}J\bar{K}L} = -\frac{1}{2\sqrt{2}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left( J^{(1)}_{IJKL} \delta(x^{11}) + J^{(2)}_{IJKL} \delta(x^{11} - \pi \rho) \right).
\]  

(5)

where the sources are given by

\[
J^{(i)}_{IJKL} = \left( \text{tr} F^{(i)} \wedge F^{(i)} - \frac{1}{2} \text{tr} R \wedge R \right)_{IJKL} = 6 \left( \text{tr} F^{(i)}_{[IJKL]} - \frac{1}{2} \text{tr} R_{[IJKL]} \right).
\]  

(6)
Note, that the gravitational contribution to this Bianchi identity is distributed equally between the two hyperplanes, giving rise to the factor 1/2 in front of $\text{tr} R^2$. As Hořava and Witten point out [3], if $G$ is to be free from delta functions, this Bianchi identity is equivalent to a boundary condition on the value of $G$ at the two fixed hyperplanes. One finds

\[ G_{IJKL}\big|_{x^{11}=0} = \left( \frac{1}{4\sqrt{2\pi}} \right)^{2/3} \frac{(\kappa/4\pi)^{2/3}}{2} J^{(1)}_{IJKL} \]

\[ G_{IJKL}\big|_{x^{11}=\pi\rho} = \left( \frac{1}{4\sqrt{2\pi}} \right)^{2/3} \frac{(\kappa/4\pi)^{2/3}}{2} J^{(2)}_{IJKL} \]

at the two boundaries.

In what follows, we will see that the action must also have additional terms at order $\kappa^{2/3}$ which are higher-order in derivatives. In particular, we will find evidence for terms quadratic in the curvature localized on the two hyperplanes. These enter the action as the Gauss-Bonnet combination,

\[ S_{R^2} = \frac{1}{16\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M_1^{10}, M_2^{10}} \sqrt{-g} \left( R_{IJKL} R^{IJKL} - 4 R_{Ij} R^{IJ} + R^2 \right), \]

where the integral is over both hyperplanes. (In fact, we will not be able to fix the coefficients of the Ricci tensor and scalar terms in (8). This is typically also the case in the calculating perturbative string effective actions. As is often done, we will assume it is the Gauss-Bonnet combination which appears since this is free of ghosts.)

Away from the hyperplanes, the theory at order $\kappa^{2/3}$ is still simple eleven-dimensional supergravity. Including the contributions from the hyperplanes, the Einstein equation gets additional source terms localized on the boundaries. The corresponding equations of motion are then

\[ D_I G^{IJKL} = \frac{\sqrt{2}}{1152} e^{IJKL\ldots I_8} G_{I_1\ldots I_4} G_{I_5\ldots I_8} \]

\[ R_{IJ} - \frac{1}{2} g_{IJ} R = \frac{1}{24} \left( 4 G_{IJKLM} G_{J}^{JKLM} - \frac{1}{2} g_{IJ} \frac{1}{2} g^{IJKLMN} G_{KLNM} \right) - \frac{1}{2\pi} (\kappa/4\pi)^{2/3} \left( \delta(x^{11}) T_{I,J}^{(1)} + \delta(x^{11} - \pi\rho) T_{I,J}^{(2)} \right), \]

where $T_{I,J}^{(i)}$ is the energy momentum tensor of the gauge field $A^{(i)}$ plus the contribution from the $R^2$ terms [3]. Concentrating on the gauge fields, its only nonvanishing components are given by

\[ T^{(i)}_{I,J} = (g_{11,11})^{-1/2} \left( \text{tr} F_{I,K}^{(i)} F_{J}^{(i)\bar{K}} - \frac{1}{4} g_{IJ} \text{tr} (F^{(i)})^2 \right). \]

The supersymmetry transformation of the gravitino $\Psi_I$ to this order in $\kappa$ is unchanged from the supergravity expression,

\[ \delta \Psi_I = D_I \eta + \frac{\sqrt{2}}{288} (\Gamma_{IJKLM} - 8 g_{IJ} \Gamma_{KLM}) G^{JKLM} \eta + \cdots, \]
where the dots indicate the omitted fermionic terms and $\eta$ is an eleven-dimensional Majorana spinor. This spinor should be restricted by the condition

$$\eta(x^{11}) = \Gamma_{11}\eta(-x^{11})$$

(12)

for the supersymmetry variation to be compatible with the $Z_2$ symmetry. This constraint means that the theory has the usual 32 supersymmetries in the bulk but only 16 (chiral) supersymmetries on the 10–dimensional orbifold hyperplanes.

While some terms were calculated by Hořava and Witten \[3\], the full M theory action to the next order in expansion, namely $\kappa^{4/3}$, is not known. In general, this will mean that we will not trust background solutions or correction terms in a four-dimensional effective action beyond $\kappa^{2/3}$. However, we will show that certain terms of order $\kappa^{4/3}$, namely those involving gauge or gauge matter fields, can be reliably computed.

### 2.2 The ten-dimensional limit

One can ask what form the low-energy effective action of the previous section takes when reduced to an effective ten-dimensional theory. That is, when one considers fields which vary slowly on the scale of the size of the orbifold interval. This is a different limit from the one we will consider in deriving the effective four-dimensional action. As we discussed in the introduction, in the latter case the theory is further compactified on a six-dimensional Calabi-Yau manifold, where the size of the Calabi-Yau space is smaller than the size of the orbifold. Thus there is no scale at which the universe appears ten-dimensional. However, going to the ten-dimensional limit does highlight many of the particular techniques we will use in deriving an effective action in four-dimensions. A more detailed discussion of this procedure will appear in \[16\].

It is also interesting to compare the ten-dimensional limit to the weakly coupled theory. The ten-dimensional theory derived in this subsection describes the effective M-theory action for fields with momenta below the scale set by the orbifold interval. The interval is still assumed to be large compared with the eleven-dimensional Planck length and so the theory still describes the strongly coupled string. Nonetheless, we will find that the low-energy theory exactly reproduces the one-loop effective action of the weakly coupled heterotic string. This is, in fact, not surprising since this form is completely fixed by anomaly heterotic string. It is essentially the statement that the form of the lowest-dimension terms of the low-energy weakly coupled action cannot get corrections. This serves as simple evidence that the M-theory action describes the low-energy strongly coupled limit of the heterotic string. A related calculation was performed by Dudas and Mourad \[19\]. However, these authors made a projection onto one hyperplane of the orbifold rather than a Kulaza-Klein reduction, where one averages over the orbifold interval. As such, they were unable to exactly identify the coefficients of the Chern-Simons and Green-Schwarz terms. Here we find that the exact ten-dimensional coefficients appear.
The ten-dimensional theory arises as the familiar Kaluza-Klein truncation of the eleven-dimensional effective action. We have already noted that only the $g_{\bar{I}\bar{J}}$ and $g_{1111}$ components of the metric, together with the $C_{\bar{I}\bar{J}11}$ components of the four-form field, are even under the orbifold $Z_2$ symmetry. Since the massless Kaluza-Klein modes are independent of $x^{11}$, only these components can survive in ten dimensions. The bosonic fields are thus given by

$$ds^2 = e^{-c/4}g_{\bar{I}\bar{J}}dx^{\bar{I}}dx^{\bar{J}} + e^{2c}(dx^{11})^2 \quad C_{\bar{I}\bar{J}11} = \frac{1}{6}B_{\bar{I}\bar{J}},$$

where the conformal factor in the ten-dimensional metric is chosen to put the truncated action in canonical form. Reducing the zeroth-order action (2), we find the bosonic part of the ten-dimensional theory is given by

$$S_{10} = \frac{\pi \rho}{\kappa^2} \int_{M^{10}} \sqrt{-g} \left\{ -R - \frac{9}{8} (\partial c)^2 - \frac{1}{6} e^{-3c/2} H^2 \right\},$$

where $H_{\bar{I}\bar{J}\bar{K}} = 3\partial_{[\bar{I}}B_{\bar{J}\bar{K}]}$. We immediately recognize equation (14) as the action for ten-dimensional $N = 1$ supergravity.

Turning to the order $\kappa^{2/3}$ corrections one must now consider gauge fields. Furthermore the source terms in the Bianchi identity (5) no longer vanish or, equivalently, $G_{\bar{I}\bar{J}\bar{K}\bar{L}}$ is, in general, non-zero at the boundary. This leads to an interesting effect. To zeroth order, since the field $G_{\bar{I}\bar{J}\bar{K}\bar{L}}$ is not even under the $Z_2$ symmetry, we expect it to be set to zero and play no role in the reduced ten-dimensional theory. Now, however, in order to match the two boundary conditions, $G_{\bar{I}\bar{J}\bar{K}\bar{L}}$ is not zero. In fact, it must vary across the orbifold interval since, in general, the sources are not equal. Thus, we find that, although we try to consider only fields which depend on ten dimensions, exciting gauge fields or curvature necessary leads to exciting a component of $G$ which depends on the internal $x^{11}$ coordinate.

We can try and calculate the effect of these sources by finding an approximate solution for $G$. Consider the theory in the downstairs picture. We can always write $G$ in terms of a potential $C$. However, $G$ must also satisfy the boundary conditions (5), together with the equation of motion (6). Since the boundary conditions imply $G$ is proportional to $\kappa^{2/3}$, working to first order in $\kappa^2$, the equation of motion reduces to

$$D^IG_{\bar{I}\bar{J}\bar{K}\bar{L}} = 0$$

In the limit we are considering, the orbifold interval is much smaller than the scale on which the ten-dimensional fields vary. Thus we can look for a solution as a momentum expansion in derivatives of the boundary sources $J^{(i)}$. To do this, we first note that the sources can be integrated as

$$J^{(i)} = \text{tr}F^{(i)} \wedge F^{(i)} - \frac{1}{2}\text{tr}R \wedge R = d\omega^{(i)}_3,$$

where we define

$$\omega^{(i)}_3 = \omega^{YM,(i)}_3 - \frac{1}{2}\omega^{(i)}_3,$$
as the sum of the Yang-Mills and Lorentz Chern-Simons forms $\omega_3^{YM(i)}$ and $\omega_3^L$. The first-order solution is then given by
\[
C_{IJK} = - \frac{1}{24\sqrt{2\pi}} (\kappa/4\pi)^{2/3} \left\{ \omega_3^{(1)} - (x^{11}/\pi\rho)\omega_3^{(2)} \right\}_{IJK},
\]
where we are dropping correction terms of the form $\rho \partial J^{(i)}$ and higher as an expansion in derivatives. The corresponding field strengths $G = 6dC$ are then
\[
G_{IJKL} = - \frac{1}{4\sqrt{2\pi}} (\kappa/4\pi)^{2/3} \left\{ J^{(1)} - (x^{11}/\pi\rho)(J^{(1)} + J^{(2)}) \right\}_{IJKL}
\]
\[
G_{IJK11} = - \frac{1}{4\sqrt{2\pi}^2\rho} (\kappa/4\pi)^{2/3} \left\{ \omega_3^{(1)} + \omega_3^{(2)} \right\}_{IJK}.
\]
We see that the solution is a simple linear extrapolation between the two values of $G_{IJKL}$ at the two boundaries. However, the Bianchi identity links $G_{IJKL}$ to other components of $G$ and, in particular, we find $G_{IJK11}$ is non-zero. A similar phenomenon has been discussed in the context of soliton solutions in M-theory on $S^1/Z_2$ [21]. There, the full momentum expansion including all higher derivative terms has been worked out for the gauge five-brane solution.

In summary, at this order in $\kappa^2$, the boundary conditions on $G$ have forced us to include a non-zero $C_{IJK}$, depending linearly on $x^{11}$, which would have been set to zero in a typical dimensional reduction. Similar $x^{11}$-dependent background fields arise for the metric due to the gauge field stress energy on the boundary. For a complete dimensional reduction, these metric backgrounds have to be taken into account as well, and we will do so in the reduction to four dimensions later on. In this subsection, however, we will concentrate on the effect of the three-form background, which already has a number of consequences. A complete reduction to ten dimensions will be presented elsewhere [10].

As before, the ten-dimensional fields are given by equation (13) together with the $E_8 \times E_8$ gauge fields. However, the field strength $G_{IJK11}$ now has two contributions
\[
G_{IJK11} = H_{IJK} = 3\partial_{[J} B_{K]} - \frac{1}{4\sqrt{2\pi}^2\rho} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ \omega_3^{(1)} + \omega_3^{(2)} \right\}_{IJK}
\]
\[
= 3\partial_{[J} B_{K]} - \frac{1}{4\sqrt{2\pi}^2\rho} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ \omega_3^{YM(1)} + \omega_3^{YM(2)} - \omega_3^L \right\}_{IJK},
\]
where the new second term comes from the non-zero $C_{IJK}$. We see that the effect of this term is to generate exactly the same gauge and Lorentz Chern-Simons terms that appear in the effective action for the weakly coupled heterotic string.

If we continue the dimensional reduction, including the gauge and $R^2$ boundary terms, we find the ten-dimensional action to order $\kappa^{2/3}$
\[
S_{10} = \frac{\pi \rho}{\kappa^2} \int_{M^{10}} \sqrt{-g} \left\{ -R - \frac{9}{8} (\partial c)^2 - \frac{1}{6} e^{-3c/2} H^2 - \frac{1}{8\pi^2\rho} \left( \frac{\kappa}{4\pi} \right)^{2/3} e^{-3c/4} \left( \text{tr}(F^{(1)})^2 + \text{tr}(F^{(1)})^2 \right)
\]
\[
+ \frac{1}{8\pi^2\rho} \left( \frac{\kappa}{4\pi} \right)^{2/3} e^{-3c/4} \left( R_{IJKL} R^{IJKL} - 4 R_{IJ} R^{IJ} + R^2 \right) \right\},
\]
\[
(21)
\]
where $H$ is the field strength with Chern-Simons terms as given in equation (20), and we have dropped higher-order terms with involving the scalar $c$. We see that we have exactly reproduced the terms of the one-loop effective action for the weakly coupled heterotic string. We note that the Gauss-Bonnet $R^2$ terms are required by supersymmetry [17], pairing with the Lorentz Chern-Simons terms in $H$. Unless such terms are included explicitly as boundary terms in the eleven-dimensional theory, they will not appear upon dimensional reduction. (No bulk $R^2$ terms are allowed in eleven-dimensional supergravity.) This provides our first evidence that such terms are necessary. We shall see more evidence in the following section. We expect that the presence of Gauss-Bonnet boundary terms can also be demonstrated explicitly in eleven-dimensions by the requirement of supersymmetry together with the boundary condition on $G$, in a calculation similar to the calculation demonstrating their presence in ten-dimensions [17].

Finally, we come to the question of the Green-Schwarz terms. We know that the ten-dimensional theory, as written, is anomalous without Green-Schwarz terms. Hořava and Witten have argued how the anomaly is cancelled in eleven-dimensions, but how do the required terms appear here? Concentrating on the gauge terms, as expected from the eleven-dimensional argument, we find that they appear from the reduction of the bulk $C \wedge G \wedge G$ in eleven-dimensional supergravity, entering at order $\kappa^{4/3}$. Using the form of $C_{IJK}$ given in (18) and the ten-dimensional fields (13), after integrating over $x^{11}$, these terms contribute

$$S_{GS} = \frac{1}{12\sqrt{2\pi^2}} \pi \rho \left( \frac{\kappa}{4\pi} \right)^{4/3} \int_{M^{10}} B \wedge X_8^{YM}$$

(22)

to the effective action, where

$$X_8^{YM} = -\frac{1}{4} \left\{ \left( \text{tr} F^{(1)} \right)^2 - \text{tr} F^{(1)} \text{tr} F^{(2)} \right\}$$

(23)

and wedge products are understood in this expression. We see that $X_8$ is precisely the Yang-Mills contribution to the Green-Schwarz term. This calculation can be extended to include the gravitational contributions. These come partly from the dimensional reduction of the $C \wedge G \wedge G$ term and partly from an explicit bulk M theory term which couples $C$ to four curvature tensors. (Such a term has been discussed in various places [18] and is necessary for anomaly cancellation in the Hořava-Witten formulation of the strongly coupled heterotic string.) While we do not give the details of either of the calculation here, it is possible to show that through this mechanism, one reproduces exactly the full Green-Schwarz term as appears in the effective action of the weakly coupled heterotic string [17].

We have seen that reducing the strongly coupled theory to ten dimensions gives the same effective action as the weakly coupled string to one-loop, including Chern-Simons and Green-Schwarz terms. The appearance of these latter terms depended on the boundary conditions on $G$ which led to a non-zero $C_{IJK}$ depending on $x^{11}$. In general, boundary sources for the Einstein equation means that the metric also has components which depend non-trivially on $x^{11}$, although we have
not discussed them in detail here. In deriving an effective action in four-dimensions, we will find an identical mechanism leading to gauge field couplings at order $\kappa^{4/3}$.

2.3 Review of Witten’s solution

In order to make contact with low-energy physics, one would like to consider compactifications of the strong-coupled theory which have $N = 1$ supersymmetry in four dimensions. Such backgrounds have been constructed by Witten. To zeroth order the compactification is simple. The $G$ equation of motion and zeroth-order Bianchi identity can be satisfied by setting $G = 0$. Supersymmetry is then preserved by compactifying on a Calabi-Yau three-fold. The background metric then has the form

$$ds^2 = g^{(0)}_{IJ} dx^I dx^J = \eta_{\mu\nu} dx^\mu dx^\nu + \Omega_{AB} dx^A dx^B + (dx^{11})^2 .$$  

(24)

$\Omega_{AB}$ is the metric of the Calabi-Yau space, which, in holomorphic coordinates, is related to the Kähler form $\omega_{AB}$ by $\omega_{a\bar{b}} = -i \Omega_{a\bar{b}}$. As we have already mentioned, to match to physical values of the grand-unified and gravitational couplings and grand-unified scale, it is necessary to take the size of the Calabi-Yau space smaller than the size of the orbifold interval. Thus we are in a very different limit from the previous subsection, where the orbifold was taken smaller than the scale of the remaining ten dimensions.

What if we try to find a solution to the next order in $\kappa^2$? We immediately see that the presence of $\text{tr} R \wedge R$ terms in the Bianchi identity, or equivalently in the boundary conditions, which are non-zero for a general Calabi-Yau space, mean that we can no longer set $G = 0$. We can, however, take a lead from the weakly coupled limit and try embedding the spin connection in the gauge group. In the weakly coupled case, the orbifold dimension is small and the low-energy theory is effectively ten-dimensional. There is an analogous Bianchi identity for the antisymmetric tensor, $dH \sim \text{tr} F^{(1)} \wedge F^{(1)} + \text{tr} F^{(2)} \wedge F^{(2)} - \text{tr} R \wedge R$. If we embed the spin connection in one of the $E_8$ groups, say $F^{(1)}$, we can set $\text{tr} F^{(1)} \wedge F^{(1)} = \text{tr} R \wedge R$ and so set $H$ to zero. In the strongly coupled limit, however, the same trick does not quite work. Because the gravitational contribution to the Bianchi identity is distributed between the two hyperplanes, we do not find $dG = 0$. However, suppose we do embed the spin-connection in, say, the $F^{(1)}$ group, and choose the internal background

$$\text{tr} F^{(1)} \wedge F^{(1)} = \text{tr} R \wedge R \quad F_{AB}^{(2)} = 0 ,$$  

(25)

breaking the symmetry of one $E_8$ to $E_6$, while leaving the other $E_8$ unbroken. We find there is then an equal and opposite contribution to the Bianchi identity from the two hyperplanes, $\frac{1}{2} \text{tr} R \wedge R$ from $M_{10}^1$ and $-\frac{1}{2} \text{tr} R \wedge R$ from $M_{10}^2$. Witten’s procedure is to solve the Bianchi identity to this order, together with the equations of motion, with the additional constraint that the solution continues to preserve $N = 1$ supersymmetry in four dimensions. To do this requires distorting the six-dimensional manifold so that it is no longer a Calabi-Yau three-fold.
The distortion turns out to be of the form
\[ ds^2 = g_{IJ}^{(0)} dx^I dx^J + k_{IJ} dx^I dx^J \]
\[ = g_{IJ}^{(0)} dx^I dx^J + \left( b \eta_{\mu \nu} dx^\mu dx^\nu + h_{AB} dx^A dx^B + \gamma (dx^{11})^2 \right), \] (26)
where the corrections \( b, h_{AB} \) and \( \gamma \) depend on the Calabi-Yau coordinates and \( x^{11} \), and only the \( h_{ab} \) terms are non-zero.

The result of solving for vanishing supersymmetry variation of the gravitino to this order can be summarized as follows. Define the quantities
\[ \beta_A = \omega^{BC} G_{ABC11} \]
\[ \theta_{AB} = w^{CD} G_{ABCD} \]
\[ \alpha = \omega^{AB} \omega^{CD} G_{ABCD}, \] (27)
where we raise and lower all indices with the metric \( \Omega_{AB} \). The equation of motion and the Bianchi identity lead to the following set of relations for the above quantities, as given by Witten [1],
\[ D_{\bar{a}} \beta_{\bar{b}} - D_{\bar{b}} \beta_{\bar{a}} = 0 \]
\[ -\frac{i}{2} D^A \theta_{\bar{A}b} + \frac{1}{4} D_b \alpha - \frac{1}{2} D_{11} \beta_b = 0 \]
\[ D_{11} \beta_{\bar{b}} - D^b \theta_{\bar{b}b} = 0 \]
\[ D_{11} \beta_{\bar{a}} + \frac{i}{4} D_{\bar{a}} \alpha = 0 \]
\[ D^A \beta_A = 0 \] (28)
together with two more which come from contractions of the \((dG)_{ABCD11}\) identity with \( \omega^{AB} \omega^{AB} \) and \( \omega^{CD} \) respectively,
\[ D_{11} \alpha + 4 i \left( D^\bar{a} \beta_{\bar{a}} - D^a \beta_a \right) = -\frac{1}{2 \sqrt{2 \pi}} \left( \frac{\kappa}{4 \pi} \right)^{2/3} J \left\{ \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right\} \] (29)
\[ D_{11} \theta_{a\bar{a}} + (D_{\bar{a}} \beta_{\bar{a}} - D_{\bar{a}} \beta_{\bar{a}}) + i \left( D^\bar{b} G_{\bar{b}a11} - D^b G_{ba11} \right) \]
\[ = -\frac{1}{2 \sqrt{2 \pi}} \left( \frac{\kappa}{4 \pi} \right)^{2/3} J_{a\bar{a}} \left\{ \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right\} . \] (30)
The sources \( J_{AB} \) and \( J = \omega^{AB} J_{AB} \) are given in terms of \( J^{(i)} \) defined in equation (2) as follows. With the spin connection embedded in the gauge connection we have
\[ J_{ABCD}^{(1)} = -J_{ABCD}^{(2)} \equiv J_{ABCD} = \frac{1}{2} \left( \text{tr} R^{(\Omega)} \wedge R^{(\Omega)} \right)_{ABCD}, \] (31)
where \( R^{(\Omega)}_{AB} \) is the curvature of the zeroth-order metric on the Calabi-Yau space \( \Omega_{AB} \). We then have
\[ J_{AB} = J_{ABCD} \omega^{CD} = 3 \text{tr} R^{(\Omega)}_{AB} R^{(\Omega)}_{CD} \omega^{CD} . \] (32)
Note that it is only these last two equations for $G$ (29) and (30) which receive contributions from the hyperplane terms in the Bianchi identity (5).

Introducing a corrected spinor $\tilde{\eta} = e^{-\psi} \eta$ and substituting into the supersymmetry variation (11), to order $\kappa^{2/3}$ one finds that the variation vanishes if the following conditions on the metric distortion are satisfied,

$$\sqrt{2} i \beta_a = 6 \partial_a b = -24 \partial_a \psi = -3 \partial_a \gamma$$

$$\frac{1}{2 \sqrt{2}} \alpha = 6 \partial_{11} b = -24 \partial_{11} \psi$$

$$\partial_a h_{b\tilde{a}} - \partial_{b\tilde{a}} h_a = -\sqrt{2} \left( G_{b\tilde{a}11} + \frac{i}{6} (\Omega_{b\tilde{a}} \beta - \Omega_{b\tilde{a}} \beta) \right)$$

$$\Omega^{b\tilde{a}} D_{b\tilde{a}} h = -\frac{1}{3} \sqrt{2} \beta$$

$$\partial_{11} h_{ab} = -\frac{1}{\sqrt{2}} \left( i \theta_{ab} - \frac{1}{12} \alpha \Omega_{ab} \right).$$

The full set of equations is compatible and thus a solution exists which preserves the $N = 1$ supersymmetry of the background.

3 Properties of Witten’s solution

3.1 Explicit form of the solution

It is possible to summarize Witten’s solution in a more compact and explicit form. To do this, we first dualize the four-form field strength $G$ to a seven-form field $H$. This is possible because, for the configurations we are considering, the $G \wedge G$ terms do not contribute to the equation of motion for $G$. We define

$$H_{I_1...I_7} = *G_{I_1...I_7} = \frac{1}{4!} \epsilon_{I_1...I_8 I_{10} I_{11}} G^{I_8 I_{10} I_{11}}.$$ (38)

If we include the sources which live at the boundaries of the orbifold, the Bianchi identity and equation of motion for $H$ read

$$dH = 0 \quad \text{and} \quad D_{I_1} H_{I_2...I_6} = -\frac{1}{2 \sqrt{2} \pi} \left( \frac{\kappa}{4 \pi} \right)^{2/3} \left\{ *J^{(1)}_{I_1...I_6} \delta(x^{11}) + *J^{(2)}_{I_1...I_6} \delta(x^{11} - \pi \rho) \right\},$$

where $*J^{(i)}$ is the dual of $J^{(i)}$ defined in equation (38), so that

$$*J^{(i)}_{I_1...I_6} = \frac{1}{4!} \epsilon_{I_1...I_8 I_{10} I_{11}} \left\{ \text{tr} F^{(i)}_{I_1 I_8} F^{(i)}_{I_9 I_{10}} - \frac{1}{2} \text{tr} R_{I_7 I_8} R_{I_9 I_{10}} \right\}.$$ (40)

The Bianchi identity is solved by writing $H_{I_1...I_7} = 7 \partial_{[I_1} B_{I_2...I_7]}$. We are then also, as usual, free to choose a harmonic gauge where $D_{I_1} B_{I_2...I_6} = 0$. The equation of motion then reduces to

$$\Delta_{I_1} B_{I_1...I_6} = -\frac{1}{2 \sqrt{2} \pi} \left( \frac{\kappa}{4 \pi} \right)^{2/3} \left\{ *J^{(1)}_{I_1...I_6} \delta(x^{11}) + *J^{(2)}_{I_1...I_6} \delta(x^{11} - \pi \rho) \right\},$$

(41)
where $\Delta_{11}$ is the eleven-dimensional Laplacian.

Now let us specialize to the particular solution considered by Witten. As we have already discussed, the zeroth-order solution is simply the product of a Calabi-Yau three-fold and a $S^1/Z_2$ orbifold. As before, embedding the spin connection in the gauge connection for $F^{(1)}_{IJ}$, we find that the non-zero components of the sources \((40)\) are

$$
* J_{\mu
u\rho\sigma AB}^{(1)} = -* J_{\mu
u\rho\sigma AB}^{(2)} = \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} \epsilon_{ABCDEF} \text{tr} R^{CD} R^{EF}.
$$

Thus we find that the only component of $B$ which is excited is a $(1,1)$-form on the Calabi-Yau space, of the form

$$
B_{\mu\nu\rho\sigma a\bar{b}} = \epsilon_{\mu
u\rho\sigma} B_{a\bar{b}}.
$$

We can effectively ignore the external four-space and simply consider a two-form potential in the internal seven-space. The corresponding three-form field strength $\mathcal{H}$ has the nonvanishing components $\mathcal{H}_{11AB} = \partial_{11} B_{AB}$ and $\mathcal{H}_{ABC} = 3 \partial_{[A} B_{BC]}$. Then, the relation \((38)\) between the four-form $G$ and $\mathcal{H}$ turns into

$$
\begin{align*}
\mathcal{H}_{11AB} & = \frac{1}{24} \epsilon_{ABCDEF} G^{CDEF} \\
\mathcal{H}_{ABC} & = \frac{1}{6} \epsilon_{ABCDEF} G^{DEF11} \, .
\end{align*}
$$

We can then express the fields $G_{a\bar{b}11}$, $\theta_{a\bar{b}}$, $\beta_a$ and $\alpha$ defined in the previous section in terms of $\mathcal{H}$ as

$$
egin{align*}
G_{a\bar{b}11} & = i \mathcal{H}_{a\bar{b}} + \frac{1}{2} (\Omega_{ab} \mathcal{H}_{c\bar{b}} - \Omega_{cb} \mathcal{H}_{a\bar{b}}) \\
\theta_{a\bar{b}} & = -2 \mathcal{H}_{11ab} + \Omega_{ab} \mathcal{H}_{11} \\
\beta_a & = i \mathcal{H}_a \\
\alpha & = 4 \mathcal{H}_{11} \, .
\end{align*}
$$

where $\mathcal{H}_A = \omega^{BC} \mathcal{H}_{ABC}$ and $\mathcal{H}_{11} = \omega^{BC} \mathcal{H}_{11BC}$.

Since the sources only depend on the internal coordinates, the equation of motion for the $(1,1)$-form $B$ reduces, at order $\kappa^{2/3}$, to

$$
\left( \Delta_X + D_1^2 \right) B_{AB} = -\frac{1}{16 \sqrt{2 \pi} \left( \frac{\kappa}{4 \pi} \right)^{2/3}} \epsilon_{ABCDEF} \text{tr} R^{(O)}_{CD} R^{(O)}_{EF} \left\{ \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right\} \, .
$$

where $\Delta_X$ is the Laplacian on the Calabi-Yau space, while the gauge condition on $B$ gives

$$
D^A B_{AB} = 0 \, .
$$

These equations are completely equivalent to the set of equations of motion and Bianchi identities given for $G$ in equations \((28)\) above.
As we have reviewed in the previous section, the correction to the metric is fixed by requiring $N = 1$ supersymmetry in four dimensions, which leads to the relations (33)–(37). In terms of the two-form $B_{ab}$, we find that these are equivalent to the simple relations

$$
\begin{align*}
    h_{ab} &= \sqrt{2} i \left( B_{ab} - \frac{1}{3} \omega_{ab} B \right) + h'_{ab}, \\
    b &= \frac{\sqrt{2}}{6} B + b', \\
    \gamma &= -\frac{\sqrt{2}}{3} B + \gamma',
\end{align*}
$$

and

$$
\psi = -\frac{\sqrt{2}}{24} B + \psi',
$$

where $B = \omega^{AB} B_{AB}$. Here $h'_{ab}$ is a zero mode of the Laplacian on the Calabi-Yau space, $b'$ and $\psi'$ are constants and $\gamma'$ is an arbitrary function of $x^{11}$. Taking a trace, one finds the useful relation

$$
h = -\sqrt{2} B + h',
$$

for $h = \Omega^{AB} h_{AB}$, where $h' = \Omega^{AB} h'_{AB}$. To see the structure of this solution, we can solve the equation for $B_{AB}$ explicitly in terms of eigenmodes of the Laplacian on the Calabi-Yau space. We define

$$
\Delta_X \pi^i_{ab} = -\lambda_i^2 \pi^i_{ab}.
$$

The eigenvalues are zero or negative since the Calabi-Yau three–fold is compact. We will usually write the subset of eigenmodes with zero eigenvalue as $\omega^i_{ab}$. The modes form an orthonormal set normalized by

$$
\int_X \pi^{iAB} \pi^{jAB} = 6V \delta^{ij},
$$

where $V$ is the volume of the Calabi-Yau space. We can then decompose the sources into eigenmodes, such that

$$
-\frac{3}{2\sqrt{2}\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \epsilon_{ABCDE} \text{tr} R^{(\Omega)CD} R^{(\Omega)EF} = \sum_i \alpha_i \pi^i_{AB}
$$

where the coefficients $\alpha_i$ are given by

$$
\alpha_i = -\frac{1}{\sqrt{2}\pi V} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_X \pi^i \wedge \text{tr} R^{(\Omega)} \wedge R^{(\Omega)}.
$$

Matching the sources at $x^{11} = 0$ and $x^{11} = \pi \rho$, and recalling that $B_{ab}$ must be an even function under $Z_2$, we find an explicit solution of the form

$$
B_{ab} = \sum_{\text{massive}} \frac{\alpha_i \sinh \lambda_i (\pi^{11} - \pi \rho/2)}{24 \lambda_i \sinh (\pi \rho \lambda_i/2)} \pi^i_{ab} + \sum_{\text{massless}} \frac{1}{24} \alpha_i \left( \pi^{11} - \pi \rho/2 \right) \omega^i_{ab} + B'_{ab}.
$$
Here $\mathcal{B}'_{ab}$ is a general zero mode of the $\Delta_X + D^2_{11}$ operator. As such, it is pure gauge and will be set to zero from here on.

We immediately notice some important properties of the solution. First, the contribution from the massive modes decays as one moves a distance of order $V^{1/6}$ away from the orbifold planes. This is to be expected, since far from the planes the field does not see the details of how the sources vary over the Calabi-Yau space, but only their average properties characterized by their zero-mode decomposition. Further, with these particular sources, which are equal and opposite on the two planes, there is a gauge where the correction $\mathcal{B}'_{ab}$ is exactly zero at the middle of the orbifold interval $x^{11} = \pi \rho / 2$. If one would not use the standard embedding as a starting point for the construction of the solution, the sources would not be equal and opposite. Then, the $\alpha_i$ generally would not be the same for the two sources and there would be no point in the orbifold interval where the contributions of the massive modes to $\mathcal{B}'_{ab}$ could all be zero. Finally, one notes that in order to match the massless modes to the sources, the values of the massless $\alpha_i$–coefficients must be the same for each source. This is simply a realization of the condition, noted by Hořava and Witten, that the sources must be cohomologically zero.

Let us now turn to discussing the structure of the metric correction which, as we have seen, is essentially completely specified in terms of $\mathcal{B}'_{ab}$. The only remaining freedom are the functions $h'$, $b'$ and $\gamma'$. However, these simply reflect an ambiguity in how one defines the zeroth-order solution. There is always a freedom in the first-order solution corresponding to zero-mode deformations of the original space, such as rescaling the volume of the Calabi-Yau or the length of the orbifold interval. However, these corrections are most naturally included in the definition of the zeroth-order solution, and so set to zero. To see this explicitly, consider defining a new zeroth-order metric by

$$ds^2 = (1 + b') \eta_{\mu\nu} dx^\mu dx^\nu + (\Omega_{AB} + h'_{AB}) dx^A dx^B + (1 + \gamma') (dx^{11})^2.$$  \hspace{1cm} (56)

It is easy to see that this has exactly the same form as the original zeroth-order metric (24). First we note that, since $b'$ is a constant, it can be absorbed by rescaling the $x^\mu$ coordinates. Since we are free to refine the $x^{11}$ coordinate by $x^{11} \to f(x^{11})$, we can similarly remove the arbitrary function $\gamma'(x^{11})$. Finally, since $h'_{AB}$ is a zero mode of the Calabi-Yau Laplacian, the new metric $\Omega' = \Omega + h'$ still describes a Calabi-Yau manifold. We might as well redefine the zeroth-order metric by equation (55) and then set $b' = \gamma' = h'_{AB} = 0$ in the first-order solution.

Of course, other definitions are possible. One could, for instance, keep part of $h'_{AB}$ in the zeroth-order metric and part in the first-order correction. However, there is one sense in which the above prescription is natural. We have noted that in the gauge $\mathcal{B}'_{AB} = 0$, the two-form $\mathcal{B}_{AB}$ vanishes at the midpoint of the orbifold interval. Further, with the prescription $b' = \gamma' = h'_{AB} = 0$, we find that the metric correction also vanishes at this point. That is to say, at $x^{11} = \pi \rho / 2$ the full first-order metric is simply that of a Calabi-Yau cross an orbifold interval. Our prescription
corresponds to setting the zeroth-order solution equal to the metric at this midpoint of the orbifold. We note that this property is special to Witten’s particular solution. Generically, when the sources on the two hyperplanes are not equal and opposite, then there is no point in the orbifold interval where the first-order metric reduces to that of simply a Calabi-Yau space cross an orbifold interval.

In summary, we can choose a gauge for $B_{AB}$ and a prescription for the splitting of the metric into zeroth- and first-order pieces such that Witten’s solution is completely specified by the solution for $B_{AB}$ given in equations (54) and (55). The metric distortion is related to $B_{AB}$ by the expressions (48) and the various components of $G_{IJKL}$ can be obtained from the eqs. (44) and (45). These corrections all vanish at the midpoint of the orbifold interval. Furthermore, we see from the form of the solution for $B_{AB}$ (55), that the average of the corrections over the orbifold interval all vanish. That is

$$\langle B_{AB} \rangle_{11} = \langle k_{IJ} \rangle_{11} = 0$$  \hspace{1cm} (57)

where we define the orbifold average by

$$\langle F \rangle_{11} = \frac{1}{\pi \rho} \int_0^{\pi \rho} dx^{11} F.$$  \hspace{1cm} (58)

This property is peculiar to Witten’s solution. More general sources lead to first-order corrections which do not average to zero.

3.2 The metric as a solution of Einstein’s equations

Let us now prove that Witten’s background solves the equations of motion to linear order. In doing so, we will find the solution matches to a set of source terms, localized on the orbifold hyperplanes, which exactly match the sources which would arise from the ten-dimensional gauge fields and putative $R^2$ terms. This gives further evidence for Gauss-Bonnet curvature terms in the eleven-dimensional action, localized on the orbifold hyperplanes.

We recall that the relations (48) were derived by requiring the solution had $N = 1$ supersymmetry in four dimensions working to order $\kappa^2/3$. Let us, for the moment, work away from the orbifold hyperplanes. For consistency, the metric should satisfy the Einstein equations (9) to the same order in $\kappa^2$. Since $G$ is zero to first order, the first non-zero contribution to the stress-energy, which is quadratic in $G$, is at second-order, proportional to $\kappa^4/3$. Thus we see that, to first order in $\kappa^2/3$, the metric must satisfy the free Einstein equations, namely

$$R_{IJ} = 0.$$  \hspace{1cm} (59)

The zeroth-order metric $g_{IJ}^{(0)}$ is Ricci flat. Thus we are left with the familiar condition that the first-order perturbation $k_{IJ}$ satisfies the linearized Einstein equations [21]

$$D^2 k_{IJ} + D_I D_J k - D^K D_I k_{JK} - D^K D_J k_{IK} = 0$$  \hspace{1cm} (60)
where $D_I$ is the covariant derivative with respect to the zeroth-order metric and all index contractions are with $g^{(0)}_{IJ}$, so that, for instance, $k = g^{(0)IJ}k_{IJ}$. Thus, for consistency of Witten's solution, the relations (18) on $b$, $\gamma$ and $h_{AB}$ derived from the supersymmetry condition, together with the equation of motion (16) and gauge condition (17) for $B_{AB}$, must imply that the linearized Einstein equations (60) are satisfied.

To see this, we first note that the gauge condition for $B_{AB}$ (17) implies a gauge condition for $k_{IJ}$

$$D^I \left( k_{IJ} - \frac{1}{2} g^{(0)}_{IJ} k \right) = 0. \quad (61)$$

We can use this to simplify the linearized Einstein equation (60). Together with the fact that the zeroth-order metric is Ricci-flat, it means that the equation can be rewritten as

$$D^2 k_{IJ} + 2 R^{(0)}_{IJKL} k^{KL} = 0. \quad (62)$$

Substituting from the expressions (18) for $k_{IJ}$ in terms of $B_{\bar{a}b}$, we find that the $b$ and $\gamma$ equations reduce to

$$\left( \Delta_X + D^2_{11} \right) B = 0, \quad (63)$$

while the $h_{a\bar{b}}$ equation implies that

$$\left( \Delta_X + D^2_{11} \right) B_{a\bar{b}} = 0. \quad (64)$$

But from the equation of motion for $B_{\bar{a}b}$, we know that both of these conditions are satisfied, at least away from the fixed orbifold planes. Thus, the first-order solution for $k_{IJ}$, satisfying the supersymmetry conditions (18), is also the first-order solution to the eleven-dimensional equations of motion.

What if, however, we include the source terms which appear in the equation of motion for $B_{\bar{a}b}$? Calculating the linearized components of the Einstein tensor $G_{IJ} = R_{IJ} - \frac{1}{2} g_{IJ} R$, one finds the only non-zero components are

$$G_{AB} = -\frac{1}{8\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ 2 \text{tr} R^{(1)}_{AC} R^{(1)}_{CD} \frac{C}{B} - \frac{1}{2} g_{AB} \text{tr} R^{(0)}_{CD} R^{(0)CD} \right\} \left( \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right)$$

$$G_{\mu\nu} = -\frac{1}{8\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ -\frac{1}{2} \eta_{\mu\nu} \text{tr} R^{(0)}_{CD} R^{(0)CD} \right\} \left( \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right). \quad (65)$$

We recall that embedding the gauge connection in the spin connection implies that $\text{tr} R^{(0)}_{AC} R^{(0)CD} = \text{tr} F^{(1)}_{AC} F^{(1)CD}$. Furthermore, the zeroth order curvature $R^{(0)}_{AB}$ is Ricci flat. One then sees that these sources match exactly to the sources that would arise from boundary gauge and curvature Gauss-Bonnet terms of the form given in equations (4) and (8) above. (Strictly, we only constrain the Riemann curvature terms in the Gauss-Bonnet combination, since the zeroth-order Ricci curvature vanishes.) This is the second piece of evidence for $R^2$ terms at order $\kappa^{2/3}$ in the effective action of M theory on an $S^1/Z_2$ orbifold.
3.3 Setup for the dimensional reduction

In this final subsection, we will prepare all the relevant information which we are going to need for the computation of the effective four-dimensional action. We will perform this computation up to linear terms in the distortion of the background; that is, generally, up to terms of order $\kappa^{2/3}$ from the bulk and of order $\kappa^{4/3}$ from the boundary. Given that the eleven-dimensional theory is defined up to the order $\kappa^{2/3}$ only, it is difficult to reliably determine terms quadratic in the distortion and we will not attempt to do so in this paper.

One set of fields which necessarily survive as massless fields in a four-dimensional action are the moduli of the background solution. To calculate the kinetic terms for these moduli, as well as their interactions with other four-dimensional fields, we need to understand how they enter the background solution. To zeroth-order, the moduli simply correspond to the moduli of the Calabi-Yau manifold together with a modulus describing a rescaling of the size of the orbifold. We will generally concentrate in what follows on the generic four-dimensional fields, which are independent of the particular form of the Calabi-Yau manifold. For the zeroth-order metric, there are only two such moduli controlling the overall size of the Calabi-Yau space and the length of the orbifold interval. We write

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2a} \Omega_{AB} dx^A dx^B + e^{2c} (dx^{11})^2,$$

so that the Calabi-Yau volume is now $e^{6a} V$ and the length of the orbifold interval is $e^c \pi \rho$. Corresponding moduli are present in the first-order background, but they enter the metric in a much more complicated way, since the correction does not depend on the form of the zeroth-order solution in a simple way. Nonetheless, we can derive the dependence on $a$ and $c$ using the explicit solution for $B_{ab}$ \[^{[48]}\]. Since we take $h' = b' = \gamma' = 0$, the metric correction can be computed directly from $B_{ab}$ using the relations \[^{[48]}\]. We find the explicit expression

$$B_{ab} = e^{c-4a} \sum_{\text{massive}} \frac{\alpha_i \sinh e^{c-a} \lambda_i (|x^{11}| - \pi \rho/2)}{24 e^{c-a} \lambda_i \sinh (\pi \rho e^{c-a} \lambda_i / 2)} \omega_{ab}^i + e^{c-4a} \sum_{\text{massless}} \frac{1}{24} \alpha_i \left(|x^{11}| - \pi \rho/2\right) \omega_{ab}^i. \quad (67)$$

As we have already noted, we would like to calculate the effective action to linear order in the background distortion. Further, reducing to four dimensions means integrating over the Calabi-Yau space and, for terms coming from the bulk, also integrating over the orbifold interval. Thus we are going to need averages of the correction over the Calabi-Yau space. Let us define such an average by

$$\langle F \rangle_{\text{CY}} = \frac{1}{V} \int d^d x \sqrt{\Omega} F. \quad (68)$$

Then, recalling that $\omega_{AB}$ is a zero mode of the Calabi-Yau Laplacian, from the orthogonality of the eigenmodes we find that

$$\langle B \rangle_{\text{CY}} = \langle \omega^{AB} B_{AB} \rangle_{\text{CY}} = \frac{1}{4} e^{c-4a} \alpha_0 \left(|x^{11}| - \pi \rho/2\right) \quad (69)$$

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where $\alpha_0$ is the coefficient of the $\omega_{AB}$ zero mode in the expansion of the source. That is
\[
\alpha_0 = -\frac{1}{\sqrt{2\pi V}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_X \omega \wedge \text{tr} R^{(\Omega)} \wedge R^{(\Omega)}.
\] (70)

Note that $\alpha_0$ also equals the Calabi–Yau average of the quantity $\alpha$ defined in eq. (27); that is,
\[
\alpha_0 = \langle \alpha \rangle_{\text{CY}}.
\] (71)

We see that the moduli enter in exactly the combination of scales depending on the ratio of the length of the orbifold interval to the volume of the Calabi-Yau to the power 2/3, which appeared in the expansion parameter $\epsilon$ defined in the introduction. This is confirming that Witten’s solution is an expansion in $\epsilon$. Other relevant Calabi–Yau averages are those for $h$, $b$ and $\gamma$, which can be directly obtained from eq. (69) and the relations (48), (50). One finds that
\[
\langle b \rangle_{\text{CY}} = \frac{\sqrt{2}}{24} e^{-4a} \alpha_0(|x^{11}| - \pi \rho/2),
\]
\[
\langle \gamma \rangle_{\text{CY}} = -\frac{\sqrt{2}}{12} e^{-4a} \alpha_0(|x^{11}| - \pi \rho/2),
\]
\[
\langle h \rangle_{\text{CY}} = -\frac{1}{2\sqrt{2}} e^{-4a} \alpha_0(|x^{11}| - \pi \rho/2).
\] (72)

The reason for the simple moduli dependence of the above averages is that, by orthogonality, we have projected out the Kähler form term in the zero mode part of the solution (55). The fact that the heavy modes drop out is actually a general property of the effective action up to linear terms in the distortion. This can be seen as follows. Consider inserting the distorted background, which, since we have set $h' = b' = \gamma' = 0$ in eq. (48), can be completely expressed in terms of $B_{AB}$, into the action and expand up to linear terms in $B_{AB}$. Then the action will consist of zeroth order, $B_{AB}$–independent pieces and terms of the form $L_{AB} B_{AB}$ where $L_{AB}$ is some expression which depends on the zeroth order background only. Consequently, $L_{AB}$ can be expanded in terms of the zero modes $\omega^i_{AB}$ on the Calabi–Yau space as $L_{AB} = \sum_i L_i \omega^i_{AB}$, where the $L_i$ do not depend on Calabi–Yau coordinates. After integrating over the Calabi–Yau space we have, therefore, $\langle L_{AB} B_{AB} \rangle_{\text{CY}} = \frac{\sqrt{2}}{3} e^{-4a} \sum_i L_i \alpha_i (|x^{11}| - \pi \rho/2)$; that is, we project onto the zero mode part of $B_{AB}$ in eq. (57). In addition, if the term under consideration results from the bulk action, we have to perform another average over the orbifold direction so that $\langle L_{AB} B_{AB} \rangle_{\text{CY},11} = 0$, as long as the $L_i$ are $x^{11}$ independent. Though the latter condition will be ordinarily fulfilled for a reduction to four dimensions (since all the fields just depend on $x^\mu$), we will find that the Bianchi identity forces us to introduce $x^{11}$–dependent background fields. In those cases, a nonzero result (from background zero mode pieces) remains after performing the orbifold average. As we have seen previously, the vanishing of the orbifold average is related to the specific choice of the solution we have made by setting $h' = b' = \gamma' = 0$ in the relations (48). For this choice, the orbifold average of the metric correction $k_{IJ}$ vanishes; that is, $\langle k_{IJ} \rangle_{11} = 0$. Another way of characterizing
this choice is therefore the following. The moduli $a$ and $c$ are defined in such a way that the volume of the original Calabi-Yau space $e^{aV}$ and the length of the original orbifold interval $e^{c\pi\rho}$ equal the corresponding quantities for the distorted background averaged over the orbifold and the Calabi-Yau space respectively.

To summarize, we have seen that, to linear order in the distortion, the heavy modes in the expansion for $B_{AB}$, eq. (67) do not contribute to the effective action. In addition, for the part of the effective action arising from the bulk, the zero mode pieces also vanish for the definition of moduli we have adopted. We expect, however, corrections from the boundary actions, since no orbifold average has to be performed in this case. The second source of corrections are the $x^{11}$–dependent background fields mentioned above. The fact that both types of corrections depend on the massless piece of the solution (67) only, simplifies some of the calculations for the effective action considerably. Therefore, it is useful to analyze which of the quantities describing the full background really depend on the massless part of $B_{AB}$. Since the metric corrections $k_{IJ}$ are given directly in terms of $B_{AB}$, they depend on massless as well as massive modes. For the form field we note that, while $\mathcal{H}_{11AB}$ contains all modes, $\mathcal{H}_{ABC}$ depends on heavy modes only. Therefore, $G_{ABCD}$, $\theta_{AB}$, $\alpha$ consist of the full spectrum whereas $G_{ABC11}$, $\beta_A$ contain massive Calabi–Yau modes only. This observation allows us to neglect $G_{ABC11}$ and $\beta_A$ wherever necessary.

4 The $D = 4$ effective action

We now turn to deriving the four-dimensional effective action by reducing the general eleven-dimensional action (4) on Witten’s background solution. By construction, the low-energy massless excitations fall into four-dimensional $\mathcal{N} = 1$ chiral or vector supermultiplets. The low-energy theory is then characterized by specifying three types of functions. The Kähler potential $K$ describes the structure of the kinetic energy terms for the chiral fields while the holomorphic superpotential $W$ encodes the potential for the chiral fields. The holomorphic gauge kinetic functions $f$ represent the coupling of the chiral multiplets to the $E_6$ and $E_8$ gauge fields. For a set of chiral fields $Y^i$, the relevant terms in the low-energy action are

$$S = -\frac{1}{16\pi G_N} \int_{M^4} \sqrt{-g} \left[ R + 2K_{ij}\partial_i Y^i \partial^j Y^j + 2e^K \left( K_{ij} D_i W D_j \bar{W} - 3|W|^2 \right) + \text{D-terms} \right]$$

$$-\frac{1}{16\pi G_{\text{GUT}}} \int_{M^4} \sqrt{-g} \left[ \text{Re} f(Y^i) \text{tr} F^2 + \text{Im} f(Y^i) \text{tr} \tilde{F} \tilde{F} \right],$$

(73)

Here $K_{ij} = \frac{\partial^2 K}{\partial Y^i \partial Y^j}$ is the Kähler metric and $D_i W = \partial_i W + \frac{\partial K}{\partial Y^i} W$ is the Kähler covariant derivative acting on the superpotential. The dual field strength $\tilde{F}_{\mu\nu}$ is defined as $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. Note that in the above expression all fields $Y^i$ are chosen to be dimensionless.

In this section, we will derive these functions, first to zeroth-order and then include the $\kappa^{2/3}$ corrections. Finally, we will deduce some of the terms arising from gauge fields at order $\kappa^{4/3}$. In
each case, one must first identify the zero mode excitations for the given background and then calculate the effective action. We will concentrate on the generic modes, which are independent of the particular form of the Calabi-Yau.

### 4.1 Zeroth order action

To zeroth-order the background solution completely factorizes into a Calabi-Yau space times an orbifold. The relative size of the two spaces does not enter the solution. For this reason, the effective action is identical to the weakly coupled limit, which corresponds to taking the orbifold size to be small. The derivation is consequently well known. We will, nevertheless, repeat the essential steps for completeness and to set the notation.

We start by giving the zeroth-order background and identifying the corresponding zero modes. The metric has the form

\[ ds^2 = \bar{g}_{\mu \nu} dx^\mu dx^\nu + e^{2a} \Omega_{AB} dx^A dx^B + e^{2c} (dx^{11})^2. \] (74)

Here \( \Omega_{AB} \) is the metric of the background Calabi-Yau space, while \( \bar{g}_{\mu \nu} \) is the metric in the external four-dimensional space. The universal moduli are \( a \), measuring the volume of the Calabi-Yau, and \( c \), measuring the size of the orbifold interval.

The components \( G_{ABCD}, G_{ABC11} \) of Witten’s background can be neglected to this order, while the gauge fields do not enter until we go to order \( \kappa^2/3 \). The remaining low-energy bosonic fields come from components of \( G \) which survive the \( Z_2 \) orbifold projection. As there are no harmonic one-forms on a generic Calabi-Yau, the only relevant components are \( G_{\mu AB11} \) and \( G_{\mu \nu \rho 11} \). Since the sources vanish to zeroth order, the Bianchi identity (5) implies that

\[ \partial_\mu [G_{\nu}]_{AB11} = \partial_\mu [G_{\nu \rho}]_{11} = 0, \]

so that we can write the field strengths in terms of the three-form potential as

\[ G_{\mu AB11} = 6 \partial_\mu C_{AB11}^{(0)}, \quad G_{\mu \nu \rho 11} = 18 \partial_\mu C_{\nu \rho 11}^{(0)} . \] (75)

The equations of motion then imply that the zero modes for \( C_{AB11}^{(0)} \) correspond to harmonic two-forms in the Calabi-Yau. Since the Kähler form is harmonic, we always have the modes

\[ C_{AB11}^{(0)} = \frac{1}{6} \chi \omega_{AB}, \quad C_{\mu \nu 11}^{(0)} = \frac{1}{6} B_{\mu \nu} . \] (76)

Thus the universal low-energy moduli are the scalars \( a, c, \chi \) and the two–form \( B_{\mu \nu} \).

With the given expression for the metric and the components of \( G_{IJKL} \), the eleven-dimensional supergravity action (2) truncates to

\[ S^{(0)} = \frac{\pi \rho V}{\kappa^2} \int_M \sqrt{-g} \left[ -R - 18 \partial_\mu a \partial^\mu a - \frac{3}{2} \partial_\mu \tilde{c} \partial^\mu \tilde{c} - 3 e^{-2c} \partial_\mu \chi \partial^\mu \chi - \frac{1}{6} e^{12a} H_{\mu \nu \rho} H^{\mu \nu \rho} \right], \] (77)

where we have already performed the Weyl rotation

\[ \bar{g}_{\mu \nu} = e^{-6a - c} g_{\mu \nu} . \] (78)
to the Einstein frame metric $g_{\mu\nu}$. The modulus $\hat{c}$ is defined by

$$\hat{c} = c + 2a,$$

while $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$. By adding the term $(\pi\rho V/6\kappa^2) \int d^4x \sqrt{-g} \sigma \epsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma}$ to the action, we can dualize the two-form $B_{\mu\nu}$ to a scalar field $\sigma$. From the equation of motion for $H_{\mu\nu\rho}$, one then finds

$$H_{\mu\nu\rho} = e^{-12a} \epsilon_{\mu\nu\rho} \sigma \partial_\sigma \sigma,$$

which leads to

$$S^{(0)} = \frac{\pi\rho V}{\kappa^2} \int_{M^4} \sqrt{-g} \left[ -R - 18 \partial_\mu a \partial^\mu a - 3e^{-2\hat{c}} \partial_\mu \chi \partial^\mu \chi - e^{-12a} \partial_\mu \sigma \partial^\mu \sigma \right].$$

We note the appearance of the expression for the four-dimensional gravitational coupling first given in equation (1) above. By comparison with eq. (73), it is easy to see that this action can be derived as the bosonic part of an $N = 1$ supergravity action with the familiar Kähler potential

$$K = -\ln(S + \bar{S}) - 3\ln(T + \bar{T})$$

if the field identification

$$S = e^{6a} + i\sqrt{2}\sigma, \quad T = e^{\hat{c}} + i\sqrt{2}\chi$$

is made.

### 4.2 Distorted background and zero modes

In this section, we will derive the complete solution which we are going to use for the calculation of the order $\kappa^{2/3}$ and $\kappa^{4/3}$ corrections. Several new features with respect to the zeroth-order calculation of the previous subsection have to be considered. Clearly, the corrections to the Calabi–Yau background metric have to be taken into account. Similar corrections can be expected for the zero modes coming from the three–form. At the order $\kappa^{2/3}$, we also have to consider the gauge fields which give rise to four–dimensional gauge fields and gauge matter. As we will see, these additional nonzero components of the gauge fields complicate the picture even further. Via the nontrivial Bianchi identity and the source terms in the Einstein equation, these gauge fields switch on a new background configuration of both the four-form field strength and the metric, which will play a crucial rôle in deriving the effective action.

Let us start by considering the zero modes from the gauge fields; that is, the four–dimensional gauge fields $A_\mu^{(1)}$, $A_\mu^{(2)}$ with field strengths $F_{\mu\nu}^{(1)}$, $F_{\mu\nu}^{(2)}$ and the gauge–matter fields. For the latter, we will concentrate on the generic zero mode, so that our Ansatz for the internal part of the gauge field in the zeroth-order background Calabi-Yau is given by

$$A_\mu^{(1)} = \bar{A}_b + w_b c T_{cp} C^p.$$
Here $\bar{A}_b$ is the background field resulting from the standard embedding (it equals the spin connection), $T_{cp}$ are the broken $E_8$ generators transforming in the $(3,27)$ representation of $SU(3) \times E_6$ and $C^p$ is the generic matter field transforming in the $27$ representation of $E_6$. The generators are normalized so that $\mathrm{tr}(T_{cp} T_{dq}) = \delta_c^d \delta_p^q$. Another useful relation is $\mathrm{tr}(T_{ap} T_{bq} T_{cr}) = \epsilon_{abc} d_{pqr}$, where $d_{pqr}$ is the tensor which projects out the singlet in $27^3$. Since the gauge fields appear at order $\kappa^{2/3}$, one does not need to worry about corrections to eq. (84), due to the distortion of the background metric, for a calculation of the effective action to this order. We will argue below, however, that for terms involving gauge fields we can reliably calculate the effective action to the order $\kappa^{4/3}$. Clearly, for those terms, order $\kappa^{2/3}$ corrections to eq. (84) are relevant and, in general, such corrections can be expected since the $\kappa^{2/3}$ metric distortion modifies the equation of motion for $A_b^{(1)}$. We can account for this modification by replacing the Kähler form $\omega_{ab}$ in eq. (84) with a corrected $(1,1)$ form $\tilde{\omega}_{ab} + \omega_{ab}$, which is harmonic with respect to the distorted metric. Note that, since we are dealing with the gauge field at the boundary $x^{11} = 0$, this distorted metric corresponds to the deformed Calabi–Yau space at $x^{11} = 0$. Therefore, if we expand the correction $\tilde{\omega}_{ab}$ as $\tilde{\omega}_{ab} = \sum_{\text{massive}} \lambda_i \pi_i^{ab} + \sum_{\text{massless}} \lambda_i \omega_i^{ab}$ into harmonics on the Calabi–Yau space, the expansion coefficients $\lambda_i$ are $x^{11}$–independent. Consequently, the massless terms can be absorbed by a redefinition of the moduli. As a result, the Ansatz (84) only includes corrections corresponding to heavy Calabi–Yau modes. The massive coefficients $\lambda_i$ in this expansion, though not needed in the following, can be computed explicitly by using the solution (57) for Witten’s background. These, however, can be neglected since they are orthogonal to the zeroth-order zero-mode given in (84) and so, after integrating over the Calabi-Yau, their linear-order contribution vanishes. In the following, we can, therefore, simply work with eq. (84). The nonvanishing components of the field strength are then given by

\begin{equation}
\begin{align*}
F^{(1)}_{ab} &= \omega_a c_{b} d_{[T_{cp}, T_{dq}]} C^p C^q \\
F^{(1)}_{(a} &= \tilde{F}_{ab} + \omega_a c_{b} d_{[T_{cp}, T_{dq}]} C^p C_q \\
F^{(1)}_{(a} &= \omega_b c_{T_{cp}} (D_{(a} C)^p .
\end{align*}
\end{equation}

Another useful relation for the background $F_{AB}$ is

\begin{equation}
\langle \mathrm{tr} F_{AB} F^{AB} \rangle_{\text{CY}} = \langle \mathrm{tr} R_{AB}^{(\Omega)} R^{(\Omega)AB} \rangle_{\text{CY}} = 2\sqrt{2}\pi (4\pi / \kappa)^{2/3} \alpha_0 .
\end{equation}

We now turn to a discussion of the corrections to the metric. Generally, we split the metric into three pieces

\begin{equation}
g_{IJ} = g_{IJ}^{(0)} + g_{IJ}^{(1)} + g_{IJ}^{(B)}
\end{equation}

Here, $g_{IJ}^{(0)}$ and $g_{IJ}^{(1)}$ are the familiar zeroth and first order pieces which we have discussed previously. Introducing the generic moduli $a$ and $c$ in eq. (28), they read

\begin{equation}
\begin{align*}
g_{\mu\nu}^{(0)} &= \bar{g}_{\mu\nu} , & g_{AB}^{(0)} &= e^{2a} \Omega_{AB} , & g_{11,11}^{(0)} &= e^{2c} \\
g_{\mu\nu}^{(1)} &= b(a,c) \bar{g}_{\mu\nu} , & g_{AB}^{(1)} &= e^{2a} h_{AB}(a,c) , & g_{11,11}^{(1)} &= e^{2c} \gamma(a,c) .
\end{align*}
\end{equation}
As we have explained in section 3, the distortion $b$, $h_{AB}$ and $\gamma$ have an implicit dependence on $a$, $c$ which we have indicated in eq. (88). Its explicit form can be obtained from the previous results by expressing the distortion in terms of the two–form $B_{AB}$ via eq. (48), and using the solution (67) for $B_{AB}$. This solution contains the full dependence on the generic moduli $a$ and $c$. It is with these replacements understood, that eq. (88) should be used for the calculation of the effective action. For the effective action, we will consider corrections linear in the background distortion. In such a situation, we have shown that the heavy Calabi–Yau modes in the solution for $B_{AB}$ do not contribute to the effective action, which simplifies the problem considerably. Correspondingly, from eq. (67), the moduli dependence of the massless part of $b$, $h_{AB}$ and $\gamma$ is simply a scaling with $e^{c-4a}$.

We have not yet discussed the last piece, $g^{(B)}_{IJ}$, in the metric (87). It arises through a mechanism analogous to that which leads to the Chern-Simons and Green-Schwarz terms in the reduction from eleven to ten dimensions discussed in section 2.2. As explained there, this piece is explicitly $x^{11}$–dependent and originates from the gauge field source terms in the Einstein equation (9). The physical picture is that small fluctuation of the observable gauge and gauge matter fields cause boundary source terms which force the metric (and the four–form as we will see below) to interpolate between those sources. This can be viewed as a kind of back–reaction, where every fluctuation of a low energy gauge field causes a small distortion of the background on which the reduction is carried out. Clearly, to arrive at a sensible purely four–dimensional effective action, this back–reaction has to be taken into account. To determine $g^{(B)}_{IJ}$, we write the Einstein equation (9) to the order $\kappa^{2/3}$ in the form

$$R_{IJ} + \frac{1}{6} \left( G_{IKLM} G^{KLMI} - \frac{1}{12} g_{IJ} G^2 \right) = -\frac{1}{2\pi} (\kappa/4\pi)^{2/3} \left( \delta(x^{11}) S_{IJ}^{(1)} + \delta(x^{11} - \pi \rho) S_{IJ}^{(2)} \right)$$  \hspace{1cm} (89)$$

where the sources $S_{IJ}^{(i)}$ are given by

$$S_{IJ}^{(i)} = (g_{11,11})^{-1/2} \left( \text{tr} F^{(i)}_I F^{(i)*}_K - \frac{1}{12} g_{IJ} \text{tr}(F^{(i)})^2 \right)$$  \hspace{1cm} (90)$$

$$S_{11,11}^{(i)} = \frac{1}{6} \sqrt{g_{11,11}} \text{tr}(F^{(i)})^2.$$  \hspace{1cm} (91)$$

We are now going to solve this equation in a linearized form for the correction $g^{(B)}_{IJ}$ to the metric, writing $g_{IJ} = g^{(0)}_{IJ} + g^{(B)}_{IJ}$. In order for the zero mode fields in $g^{(B)}_{IJ}$ to be well defined, we require in addition that $\langle g^{(B)}_{IJ} \rangle_{11} = 0$. To zeroth-order we neglect the source terms and the zeroth-order metric satisfies the left-hand-side of (89). Expanding to first order, we then get an equation for the contribution $g^{(B)}_{IJ}$ which now contains the extra source term. Furthermore, since the source terms arise from slowly varying low energy fields, we can neglect derivatives $D_\mu$ in this equation. Calabi–Yau derivatives, on the other hand, cannot be neglected, since the radius of the Calabi–Yau
space is smaller than the orbifold length. We will, however, see later on that the Calabi–Yau part of the source terms and, consequently, \(g_{AB}^{(B)}\) is proportional to \(\Omega_{AB}\). Therefore, the Calabi–Yau part of the linearized Einstein equation vanishes for this solution. Having said all this, the boundary value problem we have to solve can be formulated in the “downstairs” picture as

\[
D_{11}^2 g_{IJ}^{(B)} = \frac{1}{2\pi^2 \rho} (\kappa/4\pi)^{2/3} \left( S_{IJ}^{(1)} + S_{IJ}^{(2)} \right)
\]

\[
D_{11} g_{IJ}^{(B)} \big|_{x^{11}=0} = -\frac{1}{2\pi} (\kappa/4\pi)^{2/3} S_{IJ}^{(1)}.
\]

\[
D_{11} g_{IJ}^{(B)} \big|_{x^{11}=\pi \rho} = \frac{1}{2\pi} (\kappa/4\pi)^{2/3} S_{IJ}^{(2)}.
\]

The solution is given by

\[
g_{IJ}^{(B)} = \frac{1}{2\pi} (\kappa/4\pi)^{2/3} e^{2c} \left[ \left( \frac{1}{2\pi \rho} (x^{11})^2 - x^{11} + \frac{\pi \rho}{3} \right) S_{IJ}^{(1)} + \left( \frac{1}{2\pi \rho} (x^{11})^2 - \frac{\pi \rho}{6} \right) S_{IJ}^{(2)} \right].
\]

where we have set \(\langle g_{IJ}^{(B)} \rangle_{11} = 0\), as required. For the calculation of the effective action we are going to need explicit expressions for the sources \(S^{(i)}\). Inserting the field strengths (85) into the definition (83) we find

\[
S_{\mu\nu}^{(1)} = e^{-c} \left[ 3e^{-2a}(D_\mu C D_\nu \tilde{C} + D_\mu \tilde{C} D_\nu C) + \text{tr} F_{\mu \rho}^{(1)} F_{\nu \rho}^{(1)} \right.
\]

\[
- \frac{1}{12} g_{\mu \nu} \left( 6e^{-4a}(8 |d_{r p q} C^p C^q|^2 + (\tilde{C} T^q C)^2) + 12e^{-2a} |D_\mu C |^2 + \text{tr} F_{\rho \sigma}^{(1)} F^{(1) \rho \sigma} \right)
\]

\[
S_{AB}^{(1)} = e^{-c} \left[ \frac{1}{2} e^{-2a} (8 |d_{r p q} C^p C^q|^2 + (\tilde{C} T^q C)^2) - \frac{1}{12} e^{2a} \text{tr} F_{\rho \sigma}^{(1)} F^{(1) \rho \sigma} + \frac{1}{12} \text{tr} F_{\rho \sigma}^{(2)} F^{(2) \rho \sigma} \right] \Omega_{AB}
\]

\[
S_{\mu\nu}^{(2)} = e^{-c} \left[ \text{tr} F_{\mu \rho}^{(2)} F_{\nu \rho}^{(2)} - \frac{1}{12} g_{\mu \nu} \text{tr} F_{\rho \sigma}^{(2)} F^{(2) \rho \sigma} \right]
\]

\[
S_{AB}^{(2)} = e^{-c} \left[ - \frac{1}{12} e^{2a} \text{tr} F_{\mu \rho}^{(2)} F_{\mu \rho}^{(2)} \right] \Omega_{AB}.
\]

Here \(T^i, i = 1, \ldots, 78\) are the \(E_6\) generators in the fundamental representation 27. One notes that, as claimed above, both \(S_{AB}^{(1)}\) and \(S_{AB}^{(2)}\) are proportional to \(\Omega_{AB}\). These results, together with eq. (83), determine the background part \(g_{IJ}^{(B)}\) of the metric completely.

Finally, we should discuss the structure of the three–form field \(C_{IJK}\). To do this as systematically as possible, we split \(C_{IJK}\) into three pieces as

\[
C_{IJK} = C_{IJK}^{(0+1)} + C_{IJK}^{(B)} + \tilde{C}_{IJK}
\]

and correspondingly

\[
G_{IJKL} = G_{IJKL}^{(0+1)} + G_{IJKL}^{(B)} + \tilde{G}_{IJKL}
\]

which we discuss separately. Each of these pieces will satisfy the equation of motion and the Bianchi identity separately. The first piece, \(G_{IJKL}^{(0+1)}\), is the part which contains the actual zero mode fields.
of the low energy theory, and it includes potential corrections to those zero modes of the order $\kappa^{2/3}$. The second piece, $G_{IJKL}^{(B)}$, corresponds to the components of the form which are switched on by the gauge and gauge matter fields via the source terms in the Bianchi identity. The last piece, $\tilde{G}_{IJKL}$, is simply the form–field part of Witten’s background which we have discussed at length in section 2 and 3. Its nonvanishing components are $\tilde{G}_{ABCD}$ and $\tilde{G}_{ABC11}$. Note that, while the zero mode piece $C_{IJK}^{(0+1)}$ contains zeroth order and order $\kappa^{2/3}$ contributions, the background pieces $C_{IJK}^{(B)}$ and $\tilde{C}_{IJK}$ are both of order $\kappa^{2/3}$.

Let us start to compute the zero mode piece $C_{IJK}^{(0+1)}$. Its two nonvanishing components can be written as

$$C_{\mu\nu11}^{(0+1)} = \frac{1}{6} B_{\mu\nu}, \quad C_{AB11}^{(0+1)} = \frac{1}{6} \chi (\omega_{AB} + \omega'_{AB}) .$$

(97)

For the field strength we get

$$G_{\mu\nu\rho\sigma}^{(0+1)} = 3 \partial_{[\mu} B_{\nu\rho\sigma]}, \quad G_{\mu AB11}^{(0+1)} = \partial_\mu \chi (\omega_{AB} + \omega'_{AB}) .$$

(98)

The only modification, as compared to the zeroth order expressions (94), is the correction $\omega'_{AB}$ which has to be added in order to make $C_{AB11}^{(0+1)}$ a zero mode of the distorted background metric. This is in analogy to what we have discussed for the gauge matter zero mode. A crucial difference is, however, that we now have to deal with the full 11–dimensional metric since $C_{AB11}$ is a bulk field. Consequently, an expansion of $\omega'_{AB}$ in terms of harmonics on the Calabi–Yau space will lead to $x^{11}$–dependent expansion coefficients. This means that the massless part of this expansion cannot be absorbed by a redefinition of moduli fields (which are $x^{11}$–independent), unlike in the case of gauge matter. From the equation of motion for $C$, the metric (88), the relations (33)–(37) and the equations (48) we find

$$\omega'_{AB} = \frac{\sqrt{2}}{3} \left( B_{AB} - \frac{1}{4} \omega_{AB} B \right) + \text{massive terms} .$$

(99)

The key in arriving at this result is the observation that the quantity $\beta_A$ defined in eq. (27) has no contributions from massless Calabi–Yau modes. The massive terms in eq. (99) can be computed as well, but, as argued before, will not be needed in the following — they are orthogonal to the zeroth-order expression for $C_{AB11}$ and so give a vanishing contribution after integration over the Calabi-Yau space.

We now turn to the background $C_{IJK}^{(B)}$, which is in analogy to the metric background $g_{IJ}^{(B)}$. It originates from the source terms in the Bianchi identity, as has been already explained in section 2.2 in a somewhat different context. For convenience, let us repeat some of the essential steps here. The basic problem is to solve the equation of motion $D_I G^{(B)IJKL} = 0$ and the Bianchi identity $dG^{(B)} = 0$ subject to the boundary conditions

$$G_{IJKL}^{(B)} \bigg|_{x^{11}=0} = -\frac{1}{4\sqrt{2\pi}} (\kappa/4\pi)^{2/3} J^{(1)}_{IJKL} ,$$

$$G_{IJKL}^{(B)} \bigg|_{x^{11}=-\pi R} = \frac{1}{4\sqrt{2\pi}} (\kappa/4\pi)^{2/3} J^{(2)}_{IJKL} .$$

(100)
where the sources $J^{(i)}$ are defined as

$$
J^{(i)}_{IJKL} = 6 \left( \text{tr} F^{(i)}_{IJ} R^{(i)}_{KL} - \frac{1}{2} \text{tr} R_{IJ} R_{KL} \right). \tag{101}
$$

A solution to this boundary value problem, for source terms $J^{(i)}$ varying slowly over scales comparable to the separation $\rho$ of the hyperplanes, is given by

$$
C^{(B)}_{IJK} = - \frac{1}{24\sqrt{2}\pi} (\kappa/4\pi)^{2/3} \left\{ \omega^{(1)}_3 - (x^{11}/\pi\rho)(\omega^{(2)}_3 + \omega^{(1)}_3) \right\}_{IJK}, \tag{102}
$$

where the Chern–Simons three–forms $\omega^{(i)}_3$ are defined by $J^{(i)} = d\omega^{(i)}_3$ and can be expressed in terms of the Yang–Mills and Lorentz Chern–Simons forms as in eq. (17). Note that this approximation is well justified in the case under consideration, since the sources are generated by low energy fields.

The field strengths $G^{(B)} = 6dC^{(B)}$ are then given by

$$
G^{(B)}_{IJKL} = - \frac{1}{4\sqrt{2}\pi} (\kappa/4\pi)^{2/3} \left\{ J^{(1)} - (x^{11}/\pi\rho)(J^{(2)} + J^{(1)}) \right\}_{IJKL}
$$

$$
G^{(B)}_{IJK11} = - \frac{1}{4\sqrt{2}\pi^2\rho} (\kappa/4\pi)^{2/3} \left( \omega^{(1)}_3 + \omega^{(2)}_3 \right)_{IJK}. \tag{103}
$$

The solution is a simple linear interpolation between the given values at the two boundaries. We see that the $x^{11}$–independent component $G^{(B)}_{IJK11}$ plays a rôle similar to the weakly coupled Chern–Simons form and will, therefore, give rise to terms in the effective action which are familiar from the weakly coupled case. The $x^{11}$–dependent component $G^{(B)}_{IJKL}$ clearly has no direct analog in the weakly coupled theory. We have seen in section 2.2 that it is needed to properly reproduce certain weakly coupled Green–Schwarz terms in the 10–dimensional limit. Correspondingly, in the computation of the effective action it will lead to terms which, in the weakly coupled theory, arise from those Green–Schwarz terms, as we will see explicitly later on.

Let us now be more specific about the nonvanishing components of $G^{(B)}_{IJKL}$. The components $F^{(1)}_{\mu
u}$ and $F^{(1)}_{BC}$ of the gauge field strength lead to a nonvanishing source $J^{(1)}_{\muABC}$, which affects the components $G^{(B)}_{\muABC}$ and $G^{(B)}_{ABC11}$. Since we do not consider hidden sector matter, the source term at $x^{11} = \pi\rho$ vanishes and we have

$$
G^{(B)}_{\muABC} = - \frac{1}{4\sqrt{2}\pi} (\kappa/4\pi)^{2/3} \left( 1 - \frac{x^{11}}{\pi\rho} \right) J^{(1)}_{\muABC}, \tag{104}
$$

$$
G^{(B)}_{ABC11} = - \frac{1}{4\sqrt{2}\pi^2\rho} (\kappa/4\pi)^{2/3} \omega^{(1)}_{3ABC}. \tag{105}
$$

Explicitly, we find for the sources

$$
J^{(1)}_{\mu abc} = 12i\epsilon_{abc} d_{ppp} (D_{\mu} C)^{p} C^{q} C^{r}, \quad \omega^{(1)}_{3abc} = 4i\epsilon_{abc} d_{ppp} C^{p} C^{q} C^{r}. \tag{106}
$$

Furthermore, the square of $F^{(1)}_{\mu\nu}$ leads to sources $J^{(1)}_{\mu\nu AB}$ at $x^{11} = 0$. As before, the source at $x^{11} = \pi\rho$ vanishes so that

$$
G^{(B)}_{\mu\nu AB} = - \frac{1}{4\sqrt{2}\pi} (\kappa/4\pi)^{2/3} \left( 1 - \frac{x^{11}}{\pi\rho} \right) J^{(1)}_{\mu\nu AB}. \tag{107}
$$

27
with

\[ J^{(1)}_{\mu\nu AB} = 2i\omega_{AB} (D_\mu CD_\nu C - D_\mu \bar{C}D_\nu C) , \quad \omega^{(1)}_{3\mu AB} = i\omega_{AB} (CD_\mu \bar{C} - \bar{C}D_\mu C) . \]

Finally, the squares of the external gauge field strengths \( F^{(i)}_{\mu\nu} \) give rise to sources \( J^{(i)}_{\mu\nu\rho\sigma} \) on both hyperplanes. The solution then reads

\[ G^{(B)}_{\mu\nu\rho\sigma} = -\frac{1}{4\sqrt{2\pi^2} \rho} (\kappa/4\pi)^{2/3} \omega^{(1)}_{3\mu\nu} \]

\[ G^{(B)}_{\mu\nu,11} = -\frac{1}{4\sqrt{2\pi^2} \rho} (\kappa/4\pi)^{2/3} \omega^{(1)}_{3\mu\nu} \]

with

\[ J^{(i)}_{\mu\nu\rho\sigma} = 6 \left( \text{tr} F^{(i)}_{[\mu\nu} F^{(i)}_{\rho\sigma]} - \frac{1}{2} \text{tr} R_{[\mu\nu} R_{\rho\sigma]} \right) , \]

and \( J^{(i)}_{\mu\nu\rho\sigma} = (d\omega^{(i)}_{3})_{\mu\nu\rho\sigma} \). This completes our survey of zero modes and background fields, and we are now ready to discuss the effective action including correction terms.

### 4.3 Order \( \kappa^{2/3} \)

Let us turn to calculating the effective action at order \( \kappa^{2/3} \). As we have seen, background fields \( G \) and gauge fields are excited at this order and the background metric gets a correction. Furthermore we can now consider zero modes of the gauge fields.

By inserting the field configuration discussed in the previous subsection into the action \( \mathcal{I} \), we find the contributions

\[ S^{(1)} = 2\pi \rho V \int_{M^4} \langle k_{IJ} \rangle_{CY,11} \frac{\delta S_{SG}}{\delta g_{IJ}} \mid_{\delta g_{IJ}} = \frac{\pi \rho V}{\kappa^2} \int_{M^4} \sqrt{-g} (\omega^{AB}_{J}\omega'_{AB})_{CY,11} e^{-2\hat{c}} \partial_{\mu} \chi \partial^{\mu} \chi \]

and

\[ S^{(1)'} = \frac{\pi \rho V}{\kappa^2} \int_{M^4} \sqrt{-g} \left[ -3e^{-\hat{c}} D_\mu C D^{\mu} \bar{C} - \frac{3i}{\sqrt{2}} e^{-2\hat{c}} (\bar{C}D_\mu C - D_\mu \bar{C}) \partial^{\mu} \chi \\
- \frac{3k^2}{4} e^{-2\hat{c} - 6a} |d_{pq} C^p C^q|^2 - \frac{3k^2}{32} e^{-2\hat{c} - 6a} (\bar{C}T^i C)^2 \right] \]

\[ -\frac{V}{8\pi \kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int d^4 x e^{6a} \sqrt{-g} \left[ \text{tr}(F^{(1)}_{\mu\nu})^2 + \text{tr}(F^{(2)}_{\mu\nu})^2 \right] . \]

(114)

to the four–dimensional effective action. One notes the appearance of the GUT coupling constant first given in equation \( \mathcal{I} \) above. Here the constant \( k \) is given by \( k = 4\sqrt{2} \rho \pi (4\pi/\kappa)^{1/3} \) and \( T^i, \ i = 1, ..., 78 \) are the \( E_6 \) generators in the fundamental representation \( 27 \). In order to normalize the kinetic term for the gauge matter field \( C \), we have applied the redefinition

\[ C^p \rightarrow \pi \sqrt{2} \rho \left( \frac{4\pi}{\kappa} \right)^{1/3} C^p . \]

(115)
The $R^2$ terms in the 10–dimensional boundary theories contribute higher derivative terms at the order $\kappa^{2/3}$, which we have omitted in the above expressions. An explanation about the origin and the meaning of the various terms is in order. The first part, $S^{(1)}$, of the corrections results from the change in the bulk supergravity action induced by the metric distortion in eq. (88) and the distortion of the components $G_{\mu AB}^{11}$ in eq. (98). The expression $\delta S_{SG}/\delta g_{IJ}|$ denotes the metric variation of the supergravity action with the zeroth order fields inserted. It is multiplied by $\langle k_{IJ} \rangle_{11}$ which, as shown in eq. (57), vanishes. The second term in eq. (113) originates from the part $G_{\mu AB}^{11}G_{\mu AB}^{11}$ of the four–form kinetic term with the zero mode (98) inserted. However, from the eqs. (99) and (57), we see, as before, that the orbifold average of the correction $\omega'_{AB}$ is zero and, therefore, this term vanishes too. Consequently, $S^{(1)} = 0$ and we conclude that there are no order $\kappa^{2/3}$ bulk corrections to the effective action induced by the metric distortion or the distortion of the form field moduli.

As we have shown previously, this fact is directly related to the dropping out of massive Calabi–Yau modes for terms linear in the distortion, and to our specific definition of the moduli fields.

The second piece, $S^{(1)'}$, of the correction contains the terms involving gauge fields. All terms in $S^{(1)'}$ are standard ones and are familiar from the weakly coupled case. The kinetic term for $C$ originates from the Yang–Mills term $tr F^{(1)2}$ on the boundary at $x^{11} = 0$. The second term in eq. (114) comes from a mixing between the zero mode $G_{\mu AB}^{(0+1)}$, eq. (98), and the background $G_{\mu AB}^{(B)}$, eq. (108). The second and the third line of eq. (114) represent the scalar field potential and the Yang–Mills action for the four–dimensional gauge fields, respectively.

The full bosonic effective action to the order $\kappa^{2/3}$ is given by the sum of eq. (114) and the zeroth order action (77). We have to be careful, however, about the definition of the three-form field strength $H$ in eq. (77). Since we have not included the background field $G_{\mu AB}^{(B)}$ in $S^{(1)'}$, we should identify $H_{\mu\nu\rho}$ with the full component $G_{\mu\nu\rho}^{11}$ of the four–form; that is, we should define

$$H_{\mu\nu\rho} = G_{\mu\nu\rho}^{(B)} + G_{\mu\nu\rho}^{(0+1)}.$$  

The additional background $G_{\mu\nu\rho}^{(B)}$ in this definition then leads, via eq. (111), to the familiar non-trivial Bianchi identity

$$4\partial_{[\mu}H_{\nu\rho]} = \frac{3}{2\sqrt{2}\pi^2 \rho} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left[ tr F^{(1)}_{[\mu} F^{(1)}_{\rho]} + tr F^{(2)}_{[\mu} F^{(2)}_{\rho]} - tr R_{[\mu} R_{\rho]} \right]$$

for $H$. If we dualize $H$ to a scalar field $\sigma$, we pick up additional order $\kappa^{2/3}$ terms due to the source terms in this Bianchi identity. More explicitly, we add the term $\left( \pi\rho V/6\kappa^2 \right) \int d^4 x \sqrt{-g} \epsilon^{\mu\nu\rho\sigma} (\partial_{[\mu}H_{\nu\rho]} - \tilde{J}_{[\mu\nu\rho\sigma]})$, where $4\tilde{J}_{[\mu\nu\rho\sigma]}$ represents the terms on the right-hand side of equation (117) to the action (77). Integrating out $H_{\mu\nu\rho}$ leads to a new term in the low-energy action, namely

$$S^{(1)''} = \frac{V}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^4} \sqrt{-g} \left( \sqrt{2}\sigma \right) \left[ tr F^{(1)} \tilde{F}^{(1)} + tr F^{(2)} \tilde{F}^{(2)} - tr R \tilde{R} \right],$$

29
where the dual field strengths are defined as usual by \( \tilde{F}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma} \) and similarly for \( \tilde{R}_{\mu \nu} \). As we will see in a moment, the gauge-field terms are precisely those required to pair with the \exp(6a) terms in eq. (114) to give a chiral field in the holomorphic gauge kinetic functions. Similarly, the \( \text{tr} R \tilde{R} \) term will be paired by supersymmetry with the \( \text{tr} R^2 \) resulting from the boundary Gauss–Bonnet terms. Both terms will be consistently neglected since they are higher derivative.

To summarize, the effective action to order \( \kappa^2/3 \), after dualizing the three–form field \( H \), is given by

\[
S = S^{(0)} + S^{(1)'} + S^{(1)''}
\]

with the three parts specified in eqs. (81), (114) and (118). Comparison with the supergravity action (73) shows that this implies the standard expressions

\[
K = -\ln(S + \bar{S}) - 3 \ln(T + \bar{T} - |C|^2) \\
W = k d_{pqpr} C^p C^q C^r \\
f^{(1)} = f^{(2)} = S
\]

for the Kähler potential, the superpotential and the gauge kinetic functions if we identify the fields as

\[
S = e^{6a} + i\sqrt{2}\sigma, \quad T = e^{\hat{c}} + i\sqrt{2}\chi + \frac{1}{2} |C|^2.
\]

Why, after all, did none of these functions receive modifications from the distortion of the background as compared to the weakly coupled case? We have argued before that all massive Calabi–Yau modes drop out in linear order in the distortion, and contributions from massless modes vanish for our definition of the moduli after performing the orbifold average. The only two potential sources of corrections, therefore, arise from massless Calabi–Yau modes on the boundary (since no orbifold average is performed) and from the \( x^{11} \)–dependent background fields \( g^{(B)}_{IJ} \) and \( G^{(B)}_{IJKL} \) inserted in the bulk action. The first source of corrections cannot be seen at the order \( \kappa^2/3 \), since the boundary theories are already suppressed by this amount. Those correction will, however, become relevant at the order \( \kappa^4/3 \). As for the background field \( G^{(B)} \), we would need a cross term between \( G^{(B)}_{IJKL} \) and the moduli fields in the form field kinetic term. The moduli fields, however, are entirely contained in \( G_{IJK11} \) and can, therefore, only have cross terms with the \( x^{11} \)–independent part \( G^{(B)}_{IJK11} \) of the background. As argued before, this part of the background is the analog of the weakly coupled Chern–Simons term and, consequently, does not lead to any “unconventional” terms in the effective action. At the order \( \kappa^4/3 \), we will find that the background \( G^{(B)}_{IJKL} \) comes into play. Finally, the background \( g^{(B)}_{IJ} \) for the metric appears linearly at the order \( \kappa^2/3 \) and, therefore, vanishes after averaging over the orbifold.

### 4.4 Some order \( \kappa^4/3 \) terms

We would now like to discuss some terms of order \( \kappa^4/3 \). Clearly, we cannot compute the full effective action to that order since the original eleven-dimensional theory is generally constructed up to terms of the order \( \kappa^3/3 \) only. Additional terms in the eleven-dimensional theory, like, for example, \( R^4 \)
terms \(\kappa^{4/3}\), can be expected to appear at the order \(\kappa^{2/3}\). Correspondingly, the supersymmetric background is known to order \(\kappa^{2/3}\) only. Consequently, for any term of order \(\kappa^{4/3}\) in the effective action which we want to reliably compute, we should be able to control the effect of possible \(\kappa^{4/3}\) terms in the original action or of order \(\kappa^{4/3}\) distortions of the background. Since the boundary Yang–Mills theories are already suppressed by \(\kappa^{2/3}\), this can be achieved for terms in the low energy action involving gauge fields or gauge matter fields, but it seems very hard for terms which contain bulk fields only. In this section, we will, therefore, compute the order \(\kappa^{4/3}\) terms involving gauge or gauge matter fields only.

Let us discuss the question of unknown contributions to those terms in more detail. Clearly, unknown order \(\kappa^{4/3}\) corrections to the original theory involving gauge fields can occur on the boundary only. Many of those putative terms, such as, for example, \(F^4\), are, on dimensional grounds, suppressed by powers in \(\kappa\) larger than \(4/3\). All other terms, such as, for example, \(G^2F^2\), lead to “unconventional” powers in \(\kappa\), which are not integer powers of \(\kappa^{2/3}\), and we will, therefore, assume that they do not occur in the eleven-dimensional theory. For the same reason these terms are probably forbidden by supersymmetry. (Even if they did appear they would generate terms in the low energy effective action with the same unconventional power in \(\kappa\) which, though lower in order than the \(\kappa^{4/3}\) terms, would consequently not interfere with those we calculate.) Next, we should think about unknown order \(\kappa^{4/3}\) corrections to the background. Clearly, for the boundary theories those terms are irrelevant. For the bulk theory, on the other hand, they have to involve gauge fields in order to contribute to gauge field terms in the low energy action. We have seen that such corrections involving gauge fields already occur at order \(\kappa^{2/3}\); namely, the background fields \(G^{(B)}\) and \(g^{(B)}\) generated by the boundary sources. Clearly, those backgrounds will have gauge field dependent order \(\kappa^{4/3}\) contributions as well which we have not determined. They enter the low energy effective action linearly at order \(\kappa^{4/3}\). From our previous experience, the metric pieces vanish after taking the orbifold average. This is not quite true for the form background \(G^{(B)}\). It has however, to be paired in the action with the form field zero modes to contribute at order \(\kappa^{4/3}\). This is possible only for the components \(G^{(B)}_{\mu\rho 11}\) and \(G^{(B)}_{\mu AB 11}\) which can contract with the zero mode fields (98) in the \(G^2\) term. In conclusion, we have identified the two \(\kappa^{4/3}\) components \(G^{(B2)}_{\mu\rho 11}\) and \(G^{(B2)}_{\mu AB 11}\) of the background as the only unknown sources of \(\kappa^{4/3}\) terms involving gauge fields. Since they have to contract with the zero mode form field, their contributions to the low energy action will always be proportional to the imaginary parts \(\rho, \chi\) of the moduli \(S, T\). Other terms, which do not contain \(\rho, \chi\) cannot be affected. We will find that our results are consistent with the contributions from those backgrounds vanishing. This is also supported by studying the analogous problem in the reduction to 10 dimensions [16]. For practical purposes, we impose two further constraints on the types of terms in the low energy action we will be calculating. These are that we will not consider higher derivative terms and terms of mass dimension larger than six (where we count \(C, a, c\) as dimension one).
With the above remarks in mind, we use the field configuration of section 4.2 to find the following order $\kappa^{4/3}$ terms with gauge or gauge matter fields

$$S^{(2)} = \frac{\pi \rho V}{\kappa^2} \int_{M^4} \sqrt{-g} \left[ \frac{3}{8} e^{-2\xi D_\mu C D^\mu C + \bar{C}^2 D_\mu C D^\mu C - 2|C|^2 D_\mu C D^\mu C} \right]$$

As before the constant $k$ is given by $k = 4\sqrt{2}\pi(4\pi/\kappa)^{1/3}$ and the gauge matter field $C$ has been redefined according to eq. (112). In addition, we have introduced $\xi = \sqrt{2}\pi \rho/16$. In eq. (121), we have two types of terms; namely, “conventional” ones and those proportional to the Calabi–Yau distortion $\alpha_0$ defined in eq. (70). The conventional terms arise from certain components in $G^{(B)}_{1J11} G^{(B)}_{1J11}$ and, therefore, appear in the same way as the corresponding terms in the weakly coupled case. More specifically, the terms in the first line arise from $G^{(B)}_{\mu AB1} G^{(B)}_{\mu AB1}$ and serve to complete the $|C|^2$ piece in the $-\ln(T + \bar{T} - |C|^2)$ part of the Kähler potential. The $|C|^3$ term in the third line arises from $G^{(B)}_{ABC11} G^{(B)}_{ABC11}$ and accounts for the $|W|^2$ part of the scalar potential in eq. (73).

As for the terms proportional to the Calabi–Yau distortion $\alpha_0$, let us start to explain terms involving the four–dimensional gauge fields. The terms in the second to last line of eq. (121) represent a threshold correction to the gauge coupling proportional to $\pm \xi \alpha_0 \Re(T)$. There are two distinct sources which potentially contribute to this threshold. The first source is just the distortion $g^{(1)}$ of the background metric, corresponding to Witten’s original calculation [1]. Its contribution has been computed using eqs. (24). The reason why this contribution does not vanish in the same way the bulk correction terms did is, simply, the aforementioned fact that we do not average the metric distortion over the orbifold, but rather use the boundary values of this distortion only. Note also that the Calabi–Yau averages (72) that determined the magnitude of this contribution to the threshold, have been determined precisely for our definition of the moduli; that is, the definition which leads to the standard Kähler potential. This fact is essential for a reliable calculation of the threshold. If we had not determined the Kähler potential to order $\kappa^{2/3}$, the moduli $S$, $T$ would be unnormalized to that order, allowing for arbitrary $\kappa^{2/3}$ field redefinitions. This would then lead to an ambiguity in the threshold. In addition, there is a second source for the threshold; namely, the background metric $g^{(B)}$, eq. (73), which, from eq. (14), contains the gauge field kinetic terms. Inserted into the boundary action with the explicit $F^2$ and $R^2$ terms in this action replaced by
their internal value using eq. \((86)\) they lead to a second contribution to the threshold which has exactly the same size as the first one. There is, however, a third contribution which arises from the expansion of the bulk curvature term up to second order in the metric distortion with one distortion being replaced by \(g^{(1)}\) (which leads to \(\alpha_0\)) and the other one being replaced by \(g^{(B)}\) (which leads to \(F^2\)). It turns out that this contribution exactly cancels the second one. Finally, therefore, the threshold is entirely due to the deformation \(g^{(1)}\) of the metric which arises from the internal gauge fields and coincides with the threshold that results from the pure background calculation done in ref. \([1]\).

For a holomorphic gauge kinetic function, the gauge coupling corrections have to be paired with terms proportional to \(\text{Im}(T)\text{tr}F\tilde{F}\), which are just the terms in the last line of eq. \((121)\). Their origin is very different from the one of the gauge coupling corrections. They result from a component of the \(CGG\) term in the 11–dimensional action, namely from \(\epsilon^{ABCDEF\mu\nu\rho\sigma}C_{11AB}^{(0+1)}G^{(B)}_{CDEF}G_{\mu\nu\rho\sigma}^{(B)}\).

Recall that \(G_{CDEF}\) is just a component of Witten’s background which, upon contraction with the \(\epsilon\) tensor, gives rise to the quantity \(\alpha_0\). The zero mode piece \(C_{11AB}^{(0+1)}\) given in eq. \((97)\), contains the field \(\chi\), whereas the background field \(G_{\mu\nu\rho\sigma}^{(B)}\) from eq. \((101)\) leads to \(\text{tr}F\tilde{F}\). The size of this imaginary part is exactly what is needed for a holomorphic gauge kinetic function. This indicates that the order \(\kappa^{4/3}\) background field \(G_{\mu\nu\rho\sigma}^{(B\mu\nu)}\) discussed in the beginning of this subsection which could potentially modify the imaginary part of the threshold does, in fact, not contribute.

In analogy to the real and imaginary part of the correction to the gauge kinetic function, we have two contributions to the kinetic terms of the gauge matter field \(C\) in the second line of eq. \((121)\). The first of those terms (the analog of the real part of the gauge kinetic function) again has two potential sources. The first source is the metric distortion \(g^{(1)}\) at \(x^{11} = 0\) inserted in \(\sqrt{-g}\text{tr}F^{(1)2}\). The second source is the metric distortion \(g^{(B)}\) inserted into the boundary action and into the second order expansion of the bulk curvature term. In complete analogy with the threshold calculation the two latter contributions cancel against each other and we remain with the first one generated by \(g^{(1)}\). The second term in the second line of eq. \((121)\), which is the analog of the \(\text{Im}(T)\text{tr}F\tilde{F}\) piece, comes from the \(CGG\) component \(\epsilon^{\mu\nu\rho\sigma ABCDEF}C_{\mu\nu\rho\sigma}^{(0+1)}G^{(B)}_{CDEF}G_{\mu\nu\rho\sigma}^{(B)}\). Again, \(G_{CDEF}\) contracts to \(\alpha_0\). The zero mode \(C_{\mu\nu}^{(0+1)}\) in eq. \((97)\) contains the two–form \(B_{\mu\nu}\), which dualizes to \(\sigma\), and \(G_{\mu\nu\rho\sigma}^{(B)}\) is the background given in eq. \((107)\), which leads to the \(C\partial C\) terms. Comparison with eq. \((114)\) (first line) shows that these terms are similar in structure to those that gave rise to the \(|C|^2\) piece in the \(-\ln(T + \bar{T} - |C|^2)\) part of the Kähler potential, but with \(T\) replaced by \(S\). Consequently, they modify the \(S\) part of the Kähler potential to \(-\ln(S + \bar{S} - \xi\alpha_0)|C|^2\). The potential terms proportional to \(\alpha_0\) in the third and fourth line of eq. \((121)\), are precisely those necessary to account for this change of the Kähler potential such that the superpotential remains unmodified.

To summarize, the bosonic action to order \(\kappa^{2/3}\) in all generic fields, and to the order \(\kappa^{4/3}\) in gauge fields and generic gauge matter fields, excluding higher derivative terms and terms with mass dimension larger than six is given by \(S = S^{(0)} + S^{(1)'} + S^{(1)''} + S^{(2)}\). The various parts of this action
are defined in eqs. (81), (114), (118) and (121). With the field identifications

\[ S = e^{6a} + i\sqrt{2\sigma} + \frac{1}{2} \xi \alpha_0 |C|^2, \quad T = e^{\hat{c}} + i\sqrt{2\chi} + \frac{1}{2} |C|^2, \quad (122) \]

we find by comparison with the supergravity action (73)

\[
K = - \ln(S + \bar{S} - \xi \alpha_0 |C|^2) - \ln(T + \bar{T} - |C|^2)
\]

\[
W = k d_{pqrs} C^p C^q C^r
\]

\[
f^{(1)} = S + \xi \alpha_0 T
\]

\[
f^{(2)} = S - \xi \alpha_0 T
\]

with \( k = 4\sqrt{2}\pi(4\pi/\kappa)^{1/3} \) and \( \xi = \sqrt{2}\pi \rho/16 \). The Calabi-Yau distortion \( \alpha_0 \) is a constant, that can be computed from eq. (70) for a given Calabi-Yau space. Clearly, at the order we are working we cannot really decide whether \( |C|^2 \) in the \( S \) part of the Kähler potential should be inside the logarithm, but we write it in the above form for convenience.

As compared to the \( \kappa^{2/3} \) result (119), we have found two new terms, the threshold correction \( \pm \xi \alpha_0 T \) to the gauge kinetic functions and the \( |C|^2 \) in the \( S \) part of the Kähler potential. Both corrections are proportional to the deformation of the background measured by \( \alpha_0 \). We have seen that the origin of the \( |C|^2 \) term and the threshold is very similar, so that the former can be viewed as a “gauge matter field threshold”. Furthermore, the real and imaginary parts arise in a very different way for both terms. The real part is obtained directly from the distortion of the background metric. The imaginary part results from certain components of the eleven-dimensional “Chern–Simons” term \( C G G \) once the \( x^{11} \)-dependent background field \( G^{(B)} \) is taken into account.

### 5 Conclusion

In this paper, we have systematically derived the four-dimensional effective action of strongly coupled heterotic string theory starting from the eleven-dimensional effective theory as constructed by Hořava and Witten. In this derivation, we have taken Witten’s background solution for \( N = 1 \) supersymmetry in four dimensions, good to order \( \kappa^{2/3} \). The solution has a distorted Calabi-Yau space as well as a non-zero four-form. We have proven several useful properties of this solution which were needed in the derivation of the effective action. Specifically, we have shown that Witten’s solution can be completely expressed, in a simple way, in terms of a harmonic two–form \( \mathcal{B}_{AB} \) on the internal seven–dimensional space. An explicit solution for this two–form has been given in terms of an expansion in harmonics of the Calabi–Yau space. Furthermore, we have demonstrated that Witten’s solution, which has been originally derived by requiring \( N = 1 \) supersymmetry, is indeed a solution of the equations of motion; that is, it satisfies the linearized Einstein equation in the bulk. A calculation of the source terms, which are needed to support the solution on the boundaries,
provided evidence for Gauss–Bonnet $R^2$–terms in the ten-dimensional boundary actions at order $\kappa^{2/3}$. Further evidence was provided by the reduction of the theory to ten dimensions. Such terms were necessary to ensure supersymmetry of the ten-dimensional theory.

A feature of the theory, which we have emphasized in this paper, is that the source terms in the Bianchi identity and the Einstein equation provided by low energy gauge or gauge matter fields switch on $x^{11}$–dependent components of the metric and the four-form. This is a significant deviation from the conventional dimensional reduction where one would take only fields independent of $x^{11}$. Further, some of the new components would, in an ordinary dimensional reduction, be projected out by $Z_2$–invariance. These backgrounds have been approximately determined using a momentum expansion scheme, valid if the source terms fluctuate on scales much larger than the radius of the orbifold. Clearly, these backgrounds are important for any reduction of the theory to lower dimensions, and for the derivation of the four–dimensional effective action in particular. As an application, we have explained how the eleven-dimensional theory can be properly reduced to its ten-dimensional limit. The $x^{11}$–dependent backgrounds played an important rôle in reproducing the Green–Schwarz terms of the ten-dimensional theory.

The central issue of this paper was the derivation of the four–dimensional $N = 1$–supersymmetric effective action. This has been done to order $\kappa^{2/3}$ in all generic fields, and to order $\kappa^{4/3}$ in the generic matter fields. We have excluded higher-derivative terms and, at order $\kappa^{4/3}$, terms of mass dimension larger than six. One main result is that we find the Kähler potential does not receive any corrections of order $\kappa^{2/3}$, provided we make an appropriate definition of the moduli fields to this order. At order $\kappa^{4/3}$, an unconventional matter field correction to the dilaton part of the Kähler potential appears. Altogether we find

$$ K = -\ln \left( S + \bar{S} - \frac{\sqrt{2}\pi\rho}{16}\alpha_0|C|^2 \right) - 3\ln \left( T + \bar{T} - |C|^2 \right). \tag{124} $$

Crucial in the derivation of this result was the definition of the moduli $S$ and $T$ in terms of the underlying eleven-dimensional geometry. The real part of $S$, for example, should be chosen as the average Calabi-Yau volume in order to arrive at the above Kähler potential. It is precisely this definition of the moduli which we used in the further computations. We stress that, for a meaningful computation of the threshold, it is necessary to compute the Kähler potential to order $\kappa^{2/3}$, since, otherwise, the field $S$, $T$ would be ambiguous to that order.

With the definition of moduli fields as explained above, we have derived the superpotential and the gauge kinetic functions. The superpotential remains unchanged, and is given by

$$ W = k d_{pqrs} C^p C^q C^r. \tag{125} $$

For the gauge kinetic functions we find

$$ f^{(1)} = S + \frac{\sqrt{2}\pi\rho}{16}\alpha_0 T \quad f^{(2)} = S - \frac{\sqrt{2}\pi\rho}{16}\alpha_0 T, \tag{126} $$
where the indices (1), (2) refer to the gauge groups $E_6$ and $E_8$. The threshold part of this expression, as well as the $|C|^2$ correction to the Kähler potential above, are proportional to the constant $\rho a_0$. This is of order $\epsilon = \kappa^{2/3} \rho / V^{2/3}$ ($V$ is the average volume of the Calabi-Yau space, while $\rho$ is the orbifold radius) and, as the dimensionless expansion parameter of Witten’s solution, measures the distortion of the Calabi-Yau space. The precise value of $a_0$ for a given Calabi-Yau manifold can be explicitly computed from eq. (70). It is interesting to see how the real and imaginary parts of the threshold arise in our M–theory calculation. The real part is directly related to the linear increase of the Calabi-Yau volume along the orbifold direction due to the internal gauge background field, in the sense first explained in ref. [1]. We have seen that the background metric induced by the external gauge fields can also potentially contribute to the threshold. There are, however, two terms arising from this background, one from the boundary and the other one from the bulk curvature term, which exactly cancel each other. The low energy threshold is, therefore, entirely determined by the Calabi–Yau deformation due to internal gauge fields and corresponds to the result of ref. [1].

The imaginary part of the threshold arises from the eleven-dimensional $C \wedge G \wedge G$ “Chern–Simons”-term, but with a nontrivial background $G_{\mu
u\rho\sigma}$ inserted. This background is required by the five-dimensional Bianchi identity with the four-dimensional gauge fields and the four-dimensional metric as the source terms, in much the same way that the internal Bianchi identity leads to a distortion of the Calabi-Yau space. The origin of the two terms in the action which account for the $|C|^2$ correction in the Kähler potential is in complete analogy with this. While the “real” part comes from the distortion of the metric, the “imaginary” part results from the $C \wedge G \wedge G$ term.

How are we to interpret these results as compared to the effective low energy theory of the weakly coupled heterotic string? To zeroth order in the expansion in $\kappa^2$, we certainly expect the strongly and weakly coupled limits to give the same effective action in four-dimensions, since nothing in the solution to this order is sensitive to the relative sizes of the Calabi-Yau space and the orbifold. The fact that the form of gauge kinetic functions also agrees up to order $\kappa^{4/3}$ with the weakly coupled result [23, 24, 25], can be interpreted as an example of the power of holomorphy, as has been argued by Banks and Dine [4]. Such an argument, however, does not apply to the Kähler potential. We have traced the vanishing of order $\kappa^{2/3}$ corrections to the Kähler potential to the fact that those correction would arise from terms linear in the background distortion. Therefore, the part of the distortion corresponding to massive Calabi–Yau modes drops out because of orthogonality whereas the massless part can always be absorbed into a redefinition of the moduli. From the generality of this argument, it is clear that the non–correction statement for the Kähler potential to that order extends to non–generic moduli as well. It is also clear that bulk corrections quadratic in the distortion, which are of order $\kappa^{4/3}$, cannot vanish in general. Therefore, at this order, one can expect the Kähler potential for $S$ and $T$ to receive correction terms which depend heavily on the distortion of the background. Those terms could potentially distinguish the effective theory of the strongly coupled heterotic string from its weakly coupled counterpart. Unfortunately, at the
present stage, these terms are not accessible to computation since the eleven-dimensional theory is generally constructed up to the order $\kappa^{2/3}$ only.

What is the meaning of the threshold correction and the $|C|^2$ Kähler correction in comparison to the weakly coupled case? First, we should point out that the $|C|^2$ piece is compatible with the weakly coupled scaling symmetries [26, 27, 28], and can arise as a one–loop term in the weakly coupled effective theory. There, the imaginary part of both terms can be found by dimensional reduction of the ten-dimensional Green–Schwarz terms. This is well known for the imaginary part of the threshold [23, 24, 25] and, though less well known, it is true for the imaginary part of the $|C|^2$ term as well. Given that, in our context, these imaginary parts arise from the $CGG$ term which, in turn, gives rise to some of the weakly coupled Green–Schwarz terms upon reduction to 10 dimensions, this is not surprising. The real part of the threshold can be found from $R^4$ and $F^4$ terms in the 10–dimensional weakly coupled effective theory or, more directly, from a certain large radius limit of a string one–loop calculation. This has been explicitly demonstrated in ref. [11]. In the strongly coupled case, it is obtained from the distortion of the background which, therefore, encodes some of the string one–loop information. Similarly, the real part of the $|C|^2$ Kähler correction arises from $R^4$ and $F^4$ terms in the weakly coupled theory and represents a one–loop correction. It is important to note that while the same threshold terms are present in the weakly-coupled theory, the size of the corrections are quite different. In the strongly coupled case, the correction is proportional to $\epsilon$ and so can be appreciable, while the analogous terms in the weakly coupled limit are generally rather small.

In summary, we have seen that, to the order in $\kappa$ which we can address at present, the form of the four–dimensional effective theories for the weakly and strongly coupled heterotic string cannot be distinguished from each other. We believe that this can be systematically understood by a reduction of the eleven–dimensional theory to ten dimensions with all backgrounds taken into account [16]. This presumably reproduces much, or all, of the one–loop structure of the weakly coupled heterotic string. Partial evidence for this has been given by showing that the ten–dimensional Green–Schwarz term can be properly reproduced from eleven dimensions.

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