CONSTRUCTING ABELIAN EXTENSIONS WITH PRESCRIBED NORMS

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Abstract. Given a number field $K$, a finite abelian group $G$ and finitely many elements $\alpha_1, \ldots, \alpha_t \in K$, we construct abelian extensions $L/K$ with Galois group $G$ that realise all of the elements $\alpha_1, \ldots, \alpha_t$ as norms of elements in $L$. In particular, this shows existence of such extensions for any given parameters.

Our approach relies on class field theory and a recent formulation of Tate’s characterisation of the Hasse norm principle, a local-global principle for norms. The constructions are sufficiently explicit to be implemented on a computer, and we illustrate them with concrete examples.

Contents

1. Introduction 0
2. Constructive proof of Theorem 1 2
3. Computations with characteristic morphisms 10
4. Illustrations 12
References 18

1. Introduction

Attached to each extension $L/K$ of number fields comes the field theoretic norm map $N_{L/K} : L^\times \to K^\times$. Given an extension $L/K$, a classical problem is to study which elements of $K$ are in the image of this norm map. We consider the inverse problem: given elements $\alpha_1, \ldots, \alpha_t \in K^\times$, we construct an extension $L/K$ that realises all of them as norms.

Of course the trivial extension does the trick, so we prescribe moreover a given degree and even a given Galois group $G$. It is easy to prove the existence of a degree-$n$-extension $L/K$ whose normal closure has full Galois group $S_n$, such that a given $\alpha \in K^\times$ is a norm: one may adjoin to $K$ a root of a polynomial $X^n + a_{n-1}X^{n-1} + \cdots + a_1X + (-1)^n\alpha$, with coefficients $a_1, \ldots, a_{n-1}$ sufficiently generic in $K$ so that Hilbert’s irreducibility theorem applies.

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On the opposite end, one can look at Abelian extensions. In this situation, D. Loughran, R. Newton and the first-named author have recently proved the following result.

**Theorem 1.** [5, Theorem 1.1] Let $K$ be a number field, $G$ a finite abelian group and $\alpha_1, \ldots, \alpha_t \in K^\times$. Then there is a normal extension $L/K$ with $\text{Gal}(L/K) \cong G$ and $\{\alpha_1, \ldots, \alpha_t\} \subset N_{L/K}(L^\times)$.

In [5], this follows from an analytic result regarding the density of such extensions. The appendix to [5] by Y. Harpaz and O. Wittenberg provides in addition an algebro-geometric proof of Theorem 1, using descent, a version of the fibration method developed in their work [6], and a version of Hilbert’s irreducibility theorem.

Both of these proofs are not constructive, and the main achievement of our work is the direct construction of an extension $L/K$ that satisfies the conclusion of Theorem 1. Our construction is based on explicit class field theory, the availability of places of $K$ that satisfy certain Chebotarev-type conditions, and a version of Tate’s criterion for the Hasse norm principle (see Theorem 2 in §2.3), which was formulated in [4, 5] and is of central importance for the quantitative arguments in these papers. Some of our constructions are reminiscent of arguments appearing in [8].

A stronger version of Theorem 1 is proved in [5, Corollary 4.11]. There, one can even specify finitely many places at which the extension $L/K$ is required to have a prescribed admissible local structure. Our constructions are sufficiently general to permit the same restrictions, thus also recovering [5, Corollary 4.11]. See §2.4 for precise statements.

Our constructions are explicit enough to be implemented on a computer. In §3, we provide some remarks on how this can be done, though one should note that efficiency is not a central focus of this work. Nevertheless, we illustrate our constructions in §4 with concrete examples obtained by hand and aided by computer.

**Outline of the construction.** By class field theory, we can specify the extension $L/K$ through a continuous epimorphism $\rho$ from the idèle class group $I_K/K^\times$ of $K$ to $G$. We construct the epimorphism $\rho$ in such a way that $\alpha_1, \ldots, \alpha_t$ are local norms at all places, and moreover $L/K$ satisfies the Hasse norm principle, a local global principle for norms reviewed in §2.3. We rely on a criterion for the Hasse norm principle to hold, in terms of the decomposition groups; see Theorem 2. This is especially amenable to class field theory and allows us to reformulate the desired conditions, precisely (3)–(5), in terms of conditions on the local components $\rho_v$ of the epimorphism $\rho$, namely (6)–(9).

The epimorphism $\rho$ will be induced by a simpler object, which we call a characteristic morphism, introduced in §2.5. The former is defined on classes of idèles, and the latter on actual integral $S$-idèles, for a conveniently chosen set of places $S$. Lemmas 5 and 6 explain explicitly how the characteristic morphism inducing $\rho$ determines the corresponding decomposition subgroups $G_v = \rho_v(K_v^\times) \leq G$ of the extension $L/K$.

We actually study in §2.6 a very particular class of characteristic morphisms, of the form $\rho^T_v$ of (13), which are are explicitly determined by the data of a finite set $T$ of places $v_i$ of $K$. To guarantee that $\alpha_1, \ldots, \alpha_t$ are local norms at all places, we give elementary sufficient conditions (11), to be checked only at these places $v_i$. This amounts
In order to also satisfy the Hasse norm principle, we use the mentioned criterion, Theorem 2, which requires us to produce sufficiently many large decomposition groups, and a “spanning type” property of these. At this point it will help to work with auxiliary characteristic morphisms $\rho'_S$ into an auxiliary group $G' \simeq (\Lambda^2 G)^2$, related to $G$ by an explicit map $\Psi : G' \to G$ given in (16). In §2.7 we devise a concrete condition (15) on this auxiliary characteristic morphism $\rho'_S$ to ensure the criterion of Theorem 2 for the characteristic morphism $\Psi \circ \rho'_S$. It involves places $v_i$ and auxiliary places $w_i$. The relevant decomposition groups are those at the places $v_i$, and the auxiliary place $w_i$ is chosen to ensure that the decomposition group $G'_{v_i} \leq G'$ will be large enough. The condition (15) has then to do with relative position of these decomposition groups $G'_{v_i}$, guaranteeing the “spanning type” property for $\Psi \circ \rho'_S$.

We finally explain in §2.9 how one can find a set of places satisfying (15) through an incremental construction. Here, the auxiliary places $w_i$ have to satisfy Chebotarev-type conditions of a form studied earlier in §2.8. If $G$ can be generated by $k'$ elements, then we can stop as soon as we have produced $k = \binom{k'}{2}$ places $v_i$, and as many of the corresponding $w_i$.

Folding up, the set $T = \{v_1; w_1; \ldots; v_k; w_k\}$ determines an auxiliary characteristic morphism (13) into $G'$ which satisfies condition (15), see Theorem 11. Composing with $\Psi$ yields a characteristic morphism into $G$ and an induced epimorphism $\rho : I_K/K^\times \to G$, which corresponds to an extension $L/K$. By Theorem 8 this $\rho$ will satisfy (6)–(9), and hence we are done.

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2. Constructive proof of Theorem 1

2.1. Notation. We fix the number field $K$, the elements $\alpha_1, \ldots, \alpha_t \in K^\times$ and the finite abelian group $G$ henceforth. We write $\Omega_K$ for the set of all places of $K$. By $S$, we will always denote finite subset of $\Omega_K$ that contains all archimedean places. For a non-archimedean place $v$, we write $K_v$ for the completion of $K$ at $v$, $O_v$ for the subring of $v$-adic integers, and $O_v^\times$ for the multiplicative group of $v$-adic units. The ring of $S$-integers of $K$ is defined as $O_S = K \cap \bigcap_{v \notin S} O_v$, and its unit group, the group of $S$-units, is $O_S^\times = K \cap \bigcap_{v \notin S} O_v^\times$. Here and in similar situations, the index $v \notin S$ is understood to run over $\Omega_K \setminus S$. The symbol
will also be used for various objects defining the place \( v \), including the corresponding prime ideals in \( \mathcal{O}_S \) and \( \mathcal{O}_v \), and the exponential valuation at \( v \). We write, for example \( \alpha \mod v \) for the image of \( \alpha \in \mathcal{O}_v \) in the residue field \( F_v \) at \( v \). For any field \( F \) and \( e \in \mathbb{N} \), we will write \( F^{\times e} = (F^\times)^e \) for the group of non-zero \( e \)-th powers in \( F \).

We will always assume that \( S \) is large enough, in particular we will require that

\[
S \text{ contains all archimedean places,}
\]

\[
\mathcal{O}_S \text{ is a principal ideal domain, and}
\]

\[
\{\alpha_1, \ldots, \alpha_t\} \subset \mathcal{O}_S^\times.
\]

2.2. Class field theory. By global class field theory, extensions \( L/K \) with an isomorphism \( \text{Gal}(L/K) \to G \) are parameterised by epimorphisms \( \rho : I_K/K^\times \to G \). Here, \( I_K \) is the idèle group of \( K \), in which \( K^\times \) is embedded diagonally, and a morphism between topological groups is a continuous group homomorphism. We endow the finite group \( G \) with the discrete topology. The extension given by such an epimorphism \( \rho \) will be denoted by \( L_{\rho}/K \). By the interplay between local and global class field theory, the local behaviour of the extension \( L_\rho \) corresponding to \( \rho \) at \( v \) is described by the restriction \( \rho_v \) of \( \rho \) to the subgroup \( K_v^\times \subset I_K/K^\times \), embedded by sending \( \alpha \in K_v^\times \) to the class of the idèle \( (1, \ldots, 1, \alpha, 1, \ldots) \) that has component \( \alpha \) at place \( v \) and 1 at every other place.

2.3. Hasse norm principle. Our goal is to construct an epimorphism \( \rho \) whose corresponding extension \( L = L_\rho \) satisfies \( \{\alpha_1, \ldots, \alpha_t\} \subset N_{L/K}(L^\times) \). Since it is hard to detect global norms directly, we rely on local norms and a suitable local-global principle. Recall that the norm \( N_{L/K} : L^\times \to K^\times \) extends to a homomorphism \( N_{L/K} : I_L \to I_K \). By definition, the Hasse norm principle holds for the extension \( L/K \), if

\[
N_{L/K}(L^\times) = K^\times \cap N_{L/K}(I_L).
\]

We call the elements of the right-hand set local norms at all places of \( K \), so the Hasse norm principle, if valid for \( L/K \), allows us to detect global norms as local norms at all places. The Hasse norm theorem [7] asserts the validity of the Hasse norm principle when \( \text{Gal}(L/K) \) is cyclic. However, it may fail already in the case of biquadratic extensions, with the first counter-example of \( \mathbb{Q}(\sqrt{-3}, \sqrt{13})/\mathbb{Q} \) also due to Hasse.

We rely on the following criterion for the Hasse norm principle, which is based Tate’s cohomological description of the knot group \((K^\times \cap N_{L/K}(I_L))/N_{L/K}(L^\times)\) (e.g. [10, §11.4]).

**Theorem 2** ([5, Lemma 4.2], [4, §6]). Let \( G \) be a finite abelian group and \( \rho : I_K/K^\times \to G \) a surjective homomorphism. Then the Hasse norm principle holds for the extension \( L_\rho/K \) if and only if the natural map

\[
\bigoplus_{v \in \Omega_K} \bigwedge^2 \rho_v(K_v^\times) \to \bigwedge^2 G
\]

is surjective.
2.4. **Local conditions.** For additional flexibility, we allow ourselves to impose arbitrary local conditions at the finite set of places $S$, as long as these conditions are realised by some sub-$G$-extension that admits $\alpha_1,\ldots,\alpha_t$ as local norms at all places. More precisely, assume we are given a morphism $\psi : I_K/K^\times \to G$, not necessarily surjective, such that 
\[ \{\alpha_1,\ldots,\alpha_t\} \subset N_{L_v/K}(I_{L_v}). \]
We will construct an epimorphism $\varphi : I_K/K^\times \to G$ with the following properties:

1. \[ \{\alpha_1,\ldots,\alpha_t\} \subset N_{L_v/K}(I_{L_v}), \]
2. the extension $L_v/K$ satisfies the Hasse norm principle, and
3. the local restriction $\varphi_v$ agrees with $\psi_v$ on $K_v^\times$ for all $v \in S$.

In particular, by (3) and (4) the $\alpha_i$ are global norms from $L_v$, and thus we obtain a constructive proof of Theorem 1 and $[5, \text{Corollary 4.11}]$.

Possibly enlarging $S$, we will assume that it contains all places at which $\psi$ is ramified. Let $\text{Spl}_S(\psi)$ be the set of all places $v \notin S$ that are split completely in $L_\psi$, i.e. $\psi_v(K_v^\times) = \{e_G\}$.

In the rest of this section, we will explain how to construct an epimorphism $\rho : I_K/K^\times \to G$ that satisfies the following:

1. \[ \text{for all } v \in \Omega_K, \{\alpha_1,\ldots,\alpha_t\} \subset \ker(\rho_v : K_v^\times \to I_K/K^\times \to G), \]
2. \[ \text{the natural map } \bigoplus_{v \in \text{Spl}_S(\psi)} \bigwedge^2 \rho_v(K_v^\times) \to \bigwedge^2 G \text{ is surjective,} \]
3. \[ G = \sum_{v \in \text{Spl}_S(\psi)} \rho_v(K_v^\times), \]
4. \[ \text{for all } v \in S, \text{ the local morphism } \rho_v : K_v^\times \to G \text{ is trivial.} \]

Once we have constructed such an epimorphism $\rho$, we can take $\varphi : I_K/K^\times \to G$ to be the product of $\psi$ and $\rho$. The following lemma justifies this.

**Lemma 3.** If $\rho : I_K/K^\times \to G$ is an epimorphism with the properties (6)–(9), then the product $\varphi = \psi \rho : I_K/K^\times \to G$ is an epimorphism with the properties (3)–(5).

**Proof.** For the local morphisms at all places $v$, we have $\varphi_v = \psi_v \rho_v : K_v^\times \to G$. For $v \in \text{Spl}_S(\psi)$, the morphism $\psi_v$ is trivial, and hence $\varphi_v = \rho_v$. This shows that 
\[ G = \sum_{v \in \text{Spl}_S(\psi)} \varphi_v(K_v^\times) \]
and that 
\[ \bigoplus_{v \in \text{Spl}_S(\psi)} \bigwedge^2 \varphi_v(K_v^\times) \to \bigwedge^2 G \text{ is surjective.} \]

In particular, $\varphi$ is surjective and (4) holds by Theorem 2. For $v \in S$, the local morphism $\rho_v$ is trivial and thus $\varphi_v = \psi_v$, as desired in (5). For each $\alpha_i$ and $v \in S$, we see by local class field theory that $\alpha_i \in \ker(\psi_v) = \ker(\varphi_v)$. Let $v \notin S$, then $\psi$ is unramified at $v$ and thus $\psi_v$ vanishes on $\mathcal{O}_v^\times$, which contains all of the $\alpha_i$ by (1). Hence, $\varphi_v(\alpha_i) = \rho_v(\alpha_i) = e_G$. We have shown that all the local morphisms $\varphi_v$ vanish on all $\alpha_i$, hence (3) holds by local class field theory. \qed
2.5. **Characteristic morphisms.** Here we describe a hands-on way of constructing an epimorphism $\rho : I_K/K^\times \to G$ through the specification of a simpler object which we call a characteristic morphism.

Consider the $S$-idèles $I_S = K_S^\times \times \hat{O}_S^\times \subset I_K$, where

$$K_S^\times = \prod_{v \in S} K_v^\times \text{ and } \hat{O}_S^\times = \prod_{v \notin S} O_v^\times.$$

Since the $S$-class group Pic($O_S$) is trivial by (1), we have $I_SK^\times = I_K$. Inclusion of the open subgroup $I_S$ into $I_K$ thus induces an open epimorphism $I_S \to I_K/K^\times$ with kernel $I_S \cap K^\times = \hat{O}_S^\times$, and hence an isomorphism

$$I_S/\hat{O}_S^\times \cong I_K/K^\times.$$

With $p : I_S \to \hat{O}_S^\times$ the natural projection, we define the following.

**Definition 4.** A characteristic morphism for $G$ is a continuous epimorphism $\rho_S : \hat{O}_S^\times \to G$ that vanishes on $p(\hat{O}_S^\times)$.

Since $\rho_S$ is by definition continuous, it vanishes on the closure of $p(\hat{O}_S^\times)$. Note that, by the Dirichlet $S$-unit theorem (e.g. [9, Chapter VI, Prop. 1.1]), $\hat{O}_S^\times$ is finitely generated of $\mathbb{Q}$-rank $r = |S| - 1$, so it has generators $\gamma_0, \ldots, \gamma_r$, with $\gamma_0$ a root of unity. For $\rho_S$ to vanish on $p(\hat{O}_S^\times)$, it is thus enough that it vanishes on the images of $\gamma_0, \ldots, \gamma_r$ under the diagonal embedding $\hat{O}_S^\times \to \hat{O}_S^\times$.

A characteristic morphism $\rho_S : \hat{O}_S^\times \to G$ induces an epimorphism $\rho : I_K/K^\times \to G$ as follows: as the epimorphism $\rho_S \circ p : I_S \to G$ vanishes on $\hat{O}_S^\times$, it induces an epimorphism $I_S/\hat{O}_S^\times \to G$, which by (10) gives the desired $\rho$.

We can also describe the restriction $\rho_v$ of $\rho$ to $K_v^\times \subset I_K/K^\times$ in terms of the characteristic morphism $\rho_S$. To do so, we need the following simple lemma.

**Lemma 5.** Every place $v \in \Omega_K \setminus S$ admits a uniformiser $\pi_v \in O^\times_{S \cup \{v\}}$, and $\pi_v$ is unique up to multiplication by $S$-units.

*Proof.* The admissible choices for $\pi_v$ are the generators of the prime ideal $(O_v \setminus O_v^\times) \cap O_S$ of $O_S$, which is principal due to (1). □

For $v \notin S$, we embed $K_v^\times$ into $\hat{O}_S^\times$ as follows: we identify $O_v^\times$ with the $v$-component of $\hat{O}_S^\times$. In addition, we fix a uniformiser $\pi_v \in O^\times_{S \cup \{v\}}$ at $v$ as in Lemma 5, which we map to the element of $\hat{O}_S^\times$ that is 1 at its $v$-component and $\pi_v^{-1} \in O^\times_{S \cup \{v\}} \subset O_w^\times$ at the component indexed by $w \notin S \cup \{v\}$.

**Lemma 6.** Let $\rho_S : \hat{O}_S^\times \to G$ be a characteristic morphism and $\rho$ the induced epimorphism $I_K/K^\times \to G$. Then $\rho$ vanishes identically on $K_v^\times$ for $v \in S$. For $v \notin S$, let the embedding $K_v^\times \to \hat{O}_S^\times$ be as specified above. Then the following diagram commutes:
Proof. The key case is the image of $\pi_v$ for $v \not\in S$. In this case,

$$(1, \ldots, 1, \pi_v, 1, \ldots)K^\times = (\pi_v^{-1}, \ldots, \pi_v^{-1}, 1, \pi_v^{-1}, \ldots)K^\times,$$

and the latter idèle is in $I_S$ by our choice of $\pi_v$. Hence,

$$\rho(\pi_v) = (\rho_S \circ p)((\pi_v^{-1}, \ldots, \pi_v^{-1}, 1, \pi_v^{-1}, \ldots)) = \rho_S(\pi_v),$$

by definition of the embedding of $K_v^\times$ in $\hat{O}_S^\times$.

\[\square\]

2.6. Local norms. Here we construct characteristic morphisms whose induced epimorphisms $I_K/K^\times \to G$ have the properties (6), (8) and (9). In particular, $\alpha_1, \ldots, \alpha_t$ are local norms from the corresponding extensions at all places. When $G$ is cyclic, then $\bigwedge^2 G = 0$, so (7) holds automatically and we are done. We will return to (7) for non-cyclic $G$ in §2.7.

For a non-archimedean place $v \in \Omega_K$, we write $F_v$ for the residue field at $v$ and $q_v = |F_v|$. Given $e \in \mathbb{N}$ and a finite set $\{x_0, \ldots, x_n\} \subseteq K^\times$, we denote by $T(S; e; \psi; \{x_0, \ldots, x_n\})$ the set of places $v$ of $K$ that satisfy the following:

$$v \in \text{Spl}_S(\psi),$$

$$q_v \equiv 1 \mod e,$$

$$\{x_0, \ldots, x_n\} \subset O_v^\times,$$

$$x_0, \ldots, x_n \text{ are } e\text{-th powers in the residue field } F_v^\times.$$

Note that $T(S; e; \psi; \{x_0, \ldots, x_n\})$ contains precisely those places in $\Omega_K \setminus S$ that split completely in the normal extension $L_\psi(\zeta_e, \sqrt{x_0}, \ldots, \sqrt{x_n})/K$, where $\zeta_e$ is a primitive $e$-th root of unity in $\bar{K}$. Hence, by the Chebotarev density theorem, $T(S; e; \psi; \{x_0, \ldots, x_n\})$ is infinite and of density $1/[L_\psi(\zeta_e, \sqrt{x_0}, \ldots, \sqrt{x_n}) : K]$ in $\Omega_K$. Every $v \in T(S; e; \psi; \{x_0, \ldots, x_n\})$ satisfies in particular that

$$(12) \quad F_v^\times \otimes \mathbb{Z}/e\mathbb{Z} \simeq \mathbb{Z}/e\mathbb{Z},$$

as $e | |F_v^\times|$. Recall that $\{\gamma_0, \ldots, \gamma_r\}$ is a set of generators of $O_S^\times$. For any subset $T = \{v_1, \ldots, v_k\} \subset T(S; e; \psi; \{\gamma_0, \ldots, \gamma_r\})$ of $k$ places, and any epimorphism $\Phi : (\mathbb{Z}/e\mathbb{Z})^k \to G$, we define the epimorphism

$$(13) \quad \rho_S^T : \hat{O}_S^\times \to F_{v_1}^\times \times \cdots \times F_{v_k}^\times \to (F_{v_1}^\times \times \cdots \times F_{v_k}^\times) \otimes \mathbb{Z}/e\mathbb{Z} \xrightarrow{\Phi} (\mathbb{Z}/e\mathbb{Z})^k \xrightarrow{\Phi} G,$$

where the first two maps are the natural epimorphisms, and the third one comes from a choice of the isomorphisms (12).

Lemma 7. The morphism $\rho_S^T$ defined in (13) is a characteristic morphism, and the induced epimorphism $\rho_T : I_K/K^\times \to G$ satisfies (6), (8) and (9).
Proof. For \( \gamma \in K \) and \( v \notin S \), write \( \gamma_v \) for the local embedding \( \gamma \in K_v^\times \subset O_v^\times \). We start by showing that \( \gamma_v \in \ker \rho_S^T \) for all \( v \notin S \) and \( \gamma \in \{ \gamma_0, \ldots, \gamma_r \} \). If \( v \notin T \), this is obviously true by the construction of \( \rho_S^T \). For \( v \in T \), our choice of \( T \) ensures that \( \gamma \) is an \( e \)-th power in \( F_v^\times \), and hence vanishes in \( F_v^\times \otimes \mathbb{Z}/e\mathbb{Z} \).

Since \( \gamma_0, \ldots, \gamma_r \) generate \( O_S^\times \), we have shown that \( \rho_S^T \) vanishes at all local components \( \gamma_v \), for \( v \notin S \) and \( \gamma \in O_S^\times \). This implies in particular that \( p(O_S^\times) \subset \ker \rho_S^T \), so \( \rho_S^T \) is a characteristic morphism. Moreover, as in (1) and since the induced epimorphism \( \rho^T \) vanishes on all of \( K_v^\times \) for \( v \in S \), we immediately get (6) and (9). By the construction of \( \rho_S^T \), it is also obvious that the condition (8) is satisfied. \( \square \)

2.7. Forcing validity of the Hasse norm principle. Here we discuss how to choose the morphism \( \rho_S^T \) from the previous section in order to guarantee that the induced epimorphism \( \rho^T \) satisfies (7). We will assume that \( G \) is not cyclic, as otherwise there is nothing to do.

Fix an epimorphism \( \Phi': (\mathbb{Z}/e\mathbb{Z})^{k'} \rightarrow G \), with some \( k' \geq 2 \). Write \( k = \binom{k'}{2} \) and take a set of \( 2k \) distinct places \( T = \{ v_1, w_1, v_2, w_2, \ldots, v_k, w_k \} \) such that

\[
\{v_1, \ldots, v_k\} \subset T(S; e; \psi; \{\gamma_0, \ldots, \gamma_r\}) \text{ and } \{w_1, \ldots, w_k\} \subset T(S; e; 1; \{\gamma_0, \ldots, \gamma_r\}).
\]

We write

\[
G' = (F_{v_1}^\times \times F_{w_1}^\times \times \cdots \times F_{v_k}^\times \times F_{w_k}^\times) \otimes \mathbb{Z}/e\mathbb{Z} \simeq ((\mathbb{Z}/e\mathbb{Z})^2)^k
\]

and consider the auxiliary morphism

\[
(14) \quad \rho_S': \hat{O}_S^\times \rightarrow F_{v_1}^\times \times F_{w_1}^\times \times \cdots \times F_{v_k}^\times \times F_{w_k}^\times \rightarrow G'
\]
as in (13). (Formally, to fit the definition of \( \rho_S^T \) in (13), take \( \Phi : (\mathbb{Z}/e\mathbb{Z})^{2k} \rightarrow G' \) to be the inverse of the isomorphism coming from (12).) By Lemma 7 (with \( \psi = 1 \)), this is a characteristic morphism for \( G' \), so it induces an epimorphism \( \rho : I_K/K^\times \rightarrow G' \). We will show later in \( \S 2.9 \) that the places in \( T \) can be chosen in such a way that the following holds:

\[
(15) \quad \text{There is a } \mathbb{Z}/e\mathbb{Z}-\text{basis } e_1, e'_1, \ldots, e_k, e'_k \text{ of } G', \text{ such that } e_i, e'_i \text{ is a basis of the decomposition group } G'_{v_i} = \rho'(K_{v_i}^\times) \text{ for all } 1 \leq i \leq k.
\]

Let \( e_1, e'_1, \ldots, e_k, e'_k \) be a basis of \( G' \) as in (15), and choose any basis \( f_1, \ldots, f_{k'} \) of \( (\mathbb{Z}/e\mathbb{Z})^{k'} \), and any enumeration \((f_{m_1}, f_{n_1})_{1 \leq i \leq k}\) of the \( k = \binom{k'}{2} \) pairs \((f_m, f_n)\) with \( 1 \leq m < n \leq k' \).

We define the epimorphism of \( \mathbb{Z}/e\mathbb{Z} \)-modules \( \Psi : G' \rightarrow (\mathbb{Z}/e\mathbb{Z})^{k'} \) by

\[
(16) \quad \Psi(e_i) = f_{m_i} \text{ and } \Psi(e'_i) = f_{n_i} \text{ for } 1 \leq i \leq k,
\]
and correspondingly the epimorphism \( \Phi = \Phi' \circ \Psi : G' \rightarrow G \). With this \( \Phi \) and the set \( T \) from above, we define the morphism

\[
\rho_S^T = \Phi \circ \rho_S' : \hat{O}_S^\times \xrightarrow{\rho_S'} G' \xrightarrow{\Phi} G.
\]

Theorem 8. Assume that (15) holds. Then the morphism \( \rho_S^T : \hat{O}_S^\times \rightarrow G \) defined above is a characteristic morphism. The induced epimorphism \( \rho : I_K/K^\times \rightarrow G \) satisfies (6)–(9). In particular, the extension \( L_\rho/K \) has Galois group \( G \) and \( \{\alpha_1, \ldots, \alpha_t\} \subset N_{L_\rho/K}(L_\rho^\times) \).
Proof. Since $\rho^T_S$ is of the form (13), Lemma 7 (with $\psi = 1$) shows that $\rho^T_S$ is a characteristic morphism and that $\rho$ satisfies (6) and (9). For the induced morphisms, note that $\rho = \Phi \circ \rho'$. Since $v_1, \ldots, v_k \in \text{Spl}_S(\psi)$, condition (8) follows immediately from our assumption (15) and the fact that $\Phi$ is surjective.

It remains to verify (7). Since $f_1, \ldots, f_k$ generate $(\mathbb{Z}/e\mathbb{Z})^k$ and $\Phi' : (\mathbb{Z}/e\mathbb{Z})^k \rightarrow G$ is an epimorphism, the group $\bigwedge^2 G$ is generated by the elements $\Phi'(f_{m_1}) \wedge \Phi'(f_{m_i})$ for $1 \leq i \leq k$. By our hypothesis (15) and the definition of $\Psi$ in (16), we see that $\{\Phi'(f_{m_1}), \Phi'(f_{m_i})\} \subset \rho_v(K_v^\times)$. Hence, $\Phi'(f_{m_1}) \wedge \Phi'(f_{m_i})$ is in the image of the natural map $\bigwedge^2 \rho_v(K_v^\times) \rightarrow \bigwedge^2 G$ for all $1 \leq i \leq k$. This is enough to show that

$$\bigoplus_{1 \leq i \leq k} \bigwedge^2 \rho_v(K_v^\times) \rightarrow \bigwedge^2 G$$

is surjective. As $v_1, \ldots, v_k \in \text{Spl}_S(\psi)$, we have proved (7).

\[ \square \]

2.8. Supply of places. We still need to specify how to choose the set $\{v_1, w_1, \ldots, v_k, w_k\}$ of places from §2.7 in order to guarantee that $\rho'$ satisfies (15).

For $e \in \mathbb{N}$, a finite set $\{x_0, \ldots, x_n\} \subset K^\times$ and $y \in K^\times$, we define $T(S; e; \{x_0, \ldots, x_n\}; y)$ as the set of all places $w \in T(S; e; 1; \{x_0, \ldots, x_n\})$ such that $y \in \mathcal{O}_w^\times$ and the polynomial $X^e - y$ modulo $w$ is irreducible over the residue field $F_w$.

Let us show that these sets provide an ample supply of places in those cases that will be of interest to us.

Lemma 9. Suppose there is a place $v \in T(S; e; 1; \{x_0, \ldots, x_n\})$ such that $y$ is a uniformiser at $v$. Then the set $T(S; e; \{x_0, \ldots, x_n\}; y)$ is infinite and of natural density

$$\phi(e) = \frac{e[K(\zeta_v, \sqrt[1]{x_1}, \ldots, \sqrt[1]{x_n}) : K]}{e[K(\zeta_v, \sqrt[1]{x_1}, \ldots, \sqrt[1]{x_n}) : K]} > 0$$

in the set of all places $v$ of $K$ when ordered by $q_v$.

Proof. Write $F = K(\sqrt[1]{x_1}, \ldots, \sqrt[1]{x_n})$. Since $v$ splits completely in $F$, the element $y$ is a uniformiser for ever place $\tilde{v}$ of $F$ above $v$. Therefore, $f = X^e - y$ is irreducible over $F$ by Eisenstein’s criterion in $F_{\tilde{v}}$, and hence the extension $F(\sqrt[1]{y})/F$ is cyclic of degree $e$.

The set $T(S; e; \{x_0, \ldots, x_n\}; y)$ is precisely the set of places $w \in \text{Spl}_S(F)$, such that all places $\tilde{w}$ of $F$ above $w$ are inert in $F(\sqrt[1]{y})$.

These are precisely the places $w \notin S$ that do not divide $y$ and whose Frobenius class in $\text{Gal}(E(\sqrt[1]{y})/K)$ consists of generators of the normal subgroup $\text{Gal}(F(\sqrt[1]{y})/F) \cong \mathbb{Z}/e\mathbb{Z}$. The desired result follows from the Chebotarev density theorem. \[ \square \]

Lemma 10. If $w \in T(S; e; \{x_0, \ldots, x_n\}; y)$, then $(y \bmod w) \otimes 1$ generates $F_w^\times \otimes \mathbb{Z}/e\mathbb{Z}$.

Proof. Let $a \in F_w^\times$. Then, $a \otimes 1$ generates $F_w^\times \otimes \mathbb{Z}/e\mathbb{Z}$ if and only if none of the elements $a, a^2, \ldots, a^{e-1}$ are in $F_w^\times$. But $a^j \in F_w^\times$ if and only if $a \in F_w^\times/e\mathbb{Z}$, and thus $a \otimes 1$ generates $F_w^\times$ if and only if $a \notin F_w^\times$ for all primes $p \mid e$. The latter condition is clearly satisfied if $X^e - a$ is irreducible over $F_w$. \[ \square \]
2.9. **Choosing the right places.** With our supply of places guaranteed, we choose the places \( v_1, w_1, \ldots, v_k, w_k \) as follows. Take \( v_1, \ldots, v_k \) to be any \( k \) distinct elements of \( T(S; e; \psi; \{ \gamma_0, \ldots, \gamma_r \}) \). We fix a uniformiser \( \pi_i \in \mathcal{O}_{S \cup \{v_i\}}^x \) for each \( v_i \), which exists by Lemma 5.

Moreover, we choose each place \( w_i \in T(S; e; \{ \gamma_0, \ldots, \gamma_r \}; \{ u_i \pi_i \}) \setminus \{ v_1, \ldots, v_k \} \), with certain local units \( u_i \in K \cap \mathcal{O}_{v_i}^x \), to be determined inductively. As \( u_i \pi_i \) is a uniformiser at \( v_i \), the hypothesis of Lemma 9 is satisfied.

Recall from the discussion leading up to Lemma 6 that each \( K_{v_i}^x \) is embedded into \( \hat{\mathcal{O}}_S^x \) by sending \( \mathcal{O}_{v_i}^x \) to its factor in \( \hat{\mathcal{O}}_S^x \), and \( \pi_i \) to \( (\pi_i^{-1}, \ldots, \pi_i^{-1}, 1, \pi_i^{-1}, \ldots) \). We will now determine the \( u_i \) and \( w_i \) inductively, such that the following property holds for all \( i \):

\[
\text{(17)} \quad \text{The images of } \pi_1, \ldots, \pi_i \in \hat{\mathcal{O}}_S^x \text{ in } (F_{w_1}^x \times \cdots \times F_{w_i}^x) \otimes \mathbb{Z}/e\mathbb{Z}
\]

generate \( (F_{w_1}^x \times \cdots \times F_{w_i}^x) \otimes \mathbb{Z}/e\mathbb{Z} \).

For \( i = 1 \), we take \( u_1 = 1 \) and choose \( w_1 \in T(S; e; \{ \gamma_0, \ldots, \gamma_r \}; \pi_1) \setminus \{ v_1, \ldots, v_k \} \). Then the image of \( \pi_1 \) under

\[
\mathcal{O}_{S \cup \{v_1\}}^x \to \mathcal{O}_{w_1}^x \to F_{w_1}^x \to F_{w_1}^x \otimes \mathbb{Z}/e\mathbb{Z}
\]

generates \( F_{w_1}^x \otimes \mathbb{Z}/e\mathbb{Z} \), as \( X^e - \pi_1 \) is irreducible over \( F_{w_1} \). Therefore, we see that \( F_{w_1}^x \otimes \mathbb{Z}/e\mathbb{Z} \) is also spanned by the image \((\pi_1^{-1} \text{ mod } w_1) \otimes 1 \) of \( \pi_1 \in K_{v_1} \subset \hat{\mathcal{O}}_S^x \) under \( \hat{\mathcal{O}}_S^x \to F_{w_1}^x \otimes \mathbb{Z}/e\mathbb{Z} \).

For \( j > 1 \), assume that we have determined \( u_1, \ldots, u_{j-1} \) and chosen \( w_1, \ldots, w_{j-1} \) such that (17) holds for \( i = j - 1 \). Then there are \( c_1, \ldots, c_{j-1} \in \{ 0, \ldots, e - 1 \} \) such that the elements \( \pi_j \) and \( \pi_1^{c_1} \cdots \pi_{j-1}^{-c_{j-1}} \in \hat{\mathcal{O}}_S^x \) have the same image in \( (F_{w_1}^x \times \cdots \times F_{w_{j-1}}^x) \otimes \mathbb{Z}/e\mathbb{Z} \).

We set \( u_j = \pi_1^{-c_1} \cdots \pi_{j-1}^{-c_{j-1}} \in K \cap \mathcal{O}_{v_j}^x \) and choose \( w_j \in T(S; e; \{ \gamma_0, \ldots, \gamma_r \}; u_j \pi_j) \setminus \{ v_1, \ldots, v_k, w_1, \ldots, w_{j-1} \} \). Consider the projection

\[
P : (F_{w_1}^x \times \cdots \times F_{w_j}^x) \otimes \mathbb{Z}/e\mathbb{Z} \to (F_{w_1}^x \times \cdots \times F_{w_{j-1}}^x) \otimes \mathbb{Z}/e\mathbb{Z}.
\]

Then the images of \( \pi_1, \ldots, \pi_{j-1} \in \hat{\mathcal{O}}_S^x \) generate the range of \( P \) by our inductive hypothesis. By construction, the image of \( \pi_1^{-c_1} \cdots \pi_{j-1}^{-c_{j-1}} \pi_j \) in \( \hat{\mathcal{O}}_S^x \) is in ker \( P \). It even generates ker \( P \), since \( X^e - u_j \pi_j \) is irreducible over \( F_{w_j} \), so \( ((u_j \pi_j)^{-1} \text{ mod } w_j) \otimes 1 \) generates \( F_{w_j} \otimes \mathbb{Z}/e\mathbb{Z} \). This shows that (17) holds for \( i = j \), as desired.

Finally, let us show that the places chosen above achieve what we want to satisfy the hypothesis of Theorem 8.

**Theorem 11.** Let the set \( T = \{ v_1, w_1, \ldots, v_k, w_k \} \subset T(S; e; 1; \{ \gamma_0, \ldots, \gamma_r \}) \) be chosen as described above. Then the epimorphism \( \rho' : I_K/K^x \to G' \), induced by the auxiliary morphism \( \rho'_S \) defined in (14), satisfies (15).

**Proof.** For each \( 1 \leq i \leq k \), we have \( K_{v_i}^x = \mathcal{O}_{v_i}^x \oplus (\pi_i) \). Using Lemma 6, we see that \( \rho'(\mathcal{O}_{v_i}^x) \) is the factor \( F_{v_i}^x \otimes \mathbb{Z}/e\mathbb{Z} \simeq \mathbb{Z}/e\mathbb{Z} \) of \( G' \). Let \( e_i \) be a generator of this factor.

Moreover, take \( e'_i \) to be the image of \( \pi_i \) in \( G' \). Then \( e_i, e'_i \) is a basis of \( \rho'(K_{v_i}^x) \). Consider the projection \( P : G' \to (F_{v_1}^x \times \cdots \times F_{v_k}^x) \otimes \mathbb{Z}/e\mathbb{Z} \). Then clearly \( e_1, \ldots, e_k \) is a basis of ker \( P \simeq (F_{v_1}^x \times \cdots \times F_{v_k}^x) \otimes \mathbb{Z}/e\mathbb{Z} \). Moreover, due to (17), the images \( P(e'_1), \ldots, P(e'_k) \) form a basis of the image of \( P \). \( \square \)
3. Computations with characteristic morphisms

Let us discuss how to compute a characteristic morphism of the form $\rho^T_S$ as in (14) and the map $\Phi : G' \to G$ in practice, and how to infer from this data some information about the $G$-extension defined by it. We will focus on the case where $\psi = 1$, in order to avoid having to specify how $\psi$ is represented. We assume to be given the set $S$ of places satisfying (1), the epimorphism $\Phi' : (\mathbb{Z}/e\mathbb{Z})^{k'} \to G$, and $k = \left(\frac{q}{2}\right)$.

We need to find places $v_1, w_1, \ldots, v_k, w_k$ such that (17) is satisfied for all $1 \leq i \leq k$.

3.1. Finding the places $v_i$. First, we need a set of generators $\{\gamma_0, \ldots, \gamma_r\}$ for the $S$-units $\mathcal{O}_S^\times$, which can be computed by well-known algorithms implemented in many computer algebra systems.

As $\psi = 1$, the places $v_1, \ldots, v_k$ have to be members of the set $T(S; e; 1; \{\gamma_0, \ldots, \gamma_r\})$, so they need to be in $\Omega_K \setminus S$ and such that

$$q_v \equiv 1 \mod e, \text{ and the residues of } \gamma_0, \ldots, \gamma_r \text{ are } e\text{-th powers in } \mathbb{F}_v^\times.\tag{18}$$

These places have positive density among all places of $K$ by the Chebotarev density theorem, so we can produce enough of them by an exhaustive search. Effective lower bound versions of Chebotarev’s theorem (e.g. [11, (3.2)]) guarantee that we will not be unlucky for too long. Note that the condition of $\gamma$ being an $e$-th power in $\mathbb{F}_v^\times$, with $q_v \equiv 1 \mod e$, can be easily checked using Euler’s criterion: it holds if and only if $\gamma^{(q_v - 1)/e} \equiv 1 \mod v$.

Once $v_i$ has been specified, we fix a residue class $b_i \in \mathbb{F}_{v_i}^\times$ whose order is a multiple of $e$, for example a primitive root in $\mathbb{F}_{v_i}^\times$.

We also need a uniformiser $\pi_i \in \mathcal{O}_{S_{\cup\{v_i\}}}^\times$ for each of the places $v_i$, whose existence is guaranteed by Lemma 5. It can be found, for example, by computing generators $\{g_0, \ldots, g_{r+1}\}$ of $\mathcal{O}_{S_{\cup\{v_i\}}}^\times$ and solving the linear Diophantine equation $c_0 v_i(g_0) + \cdots + c_{r+1} v_i(g_{r+1}) = 1$ for the exponents in $\pi_i = g_0^c \cdots g_{r+1}^c$, with $v_i(\cdot)$ the exponential valuation at $v_i$.

3.2. Finding the places $w_i$. The place $w_i$ has to be in $T(S; e; \{\gamma_0, \ldots, \gamma_r\}, u\pi_i)$. The element $u_i \in \mathcal{O}_{S_{\cup\{v_i\}}}^\times$ is specified by $u_1 = 1$ and inductively for $i > 1$, as explained in §2.9.

We can determine $u_i$ by linear algebra with the images of $\pi_1, \ldots, \pi_i \in \hat{\mathcal{O}}_S^\times$ in

$$(\mathbb{F}_{w_1}^\times \times \cdots \times \mathbb{F}_{w_{i-1}}^\times) \otimes \mathbb{Z}/e\mathbb{Z} \cong (\mathbb{Z}/e\mathbb{Z})^{i - 1}.$$

To specify the above isomorphism, fix elements $b'_j \in \mathbb{F}_{w_j}^\times$ of order divisible by $e$, and identify

$$(b'^{a_1}_{1}, \ldots, b'^{a_{i-1}}_{i-1}) \otimes 1 \quad \text{with} \quad (a_1, \ldots, a_{i-1}) \in (\mathbb{Z}/e\mathbb{Z})^{i - 1}.$$

Let $l'_{m,j} \in \mathbb{Z}/e\mathbb{Z}$ be such that $\pi_j^{-1} \mod w_m$ and $(b'_m)^{l'_{m,j}}$ differ only by an $e$-th power in $\mathbb{F}_{w_m}^\times$. For example, if $b'_m$ is a primitive root of $\mathbb{F}_{w_m}^\times$, then $l'_{m,j}$ can be taken as the discrete logarithm of $\pi_j^{-1}$ with respect to the generator $b'_m$ of $\mathbb{F}_{w_m}^\times$, modulo $e$. Recalling how the uniformiser $\pi_j \in K_S^\times$ is embedded into $\hat{\mathcal{O}}_S^\times$, we see that the image of $\pi_j$ in $(\mathbb{Z}/e\mathbb{Z})^{i - 1}$ is then $(l'_{1,j}, \ldots, l'_{i-1,j})$. Hence, the exponents in $u_i = \pi_1^{-c_1} \cdots \pi_{i-1}^{-c_{i-1}}$ can be determined by
Let $g \in \mathbb{R}$ and the matrix characteristic morphism $\Phi : G \to T(S; e; \{\gamma_0, \ldots, \gamma_r\}, u_i \pi_i)$ by an exhaustive search. In addition to (18), it has to satisfy that the reduction of $X^e - u_i \pi_i$ is irreducible over $\mathbb{F}_{w_i}$. Actually, we only need the potentially weaker (if $4 \mid e$) condition that $(u_i \pi_i \mod w_i) \otimes 1 \text{ generates } \mathbb{F}_{w_i}^\times \otimes \mathbb{Z}/e\mathbb{Z}$. As explained in the proof of Lemma 10, it is enough to verify that

$$u_i \pi_i \mod w_i \notin \mathbb{F}_{w_i}^{\times p} \text{ for all primes } p \mid e,$$

which can again be checked by Euler’s criterion.

As the conditions for the $w_i$ are more restrictive than for the $v_i$, we can potentially get smaller places by searching for $w_i$ instead of computing first all the $v_i$. Computing $\pi_i$ generates $\hat{G}$, and similarly for the $b_i$. We can then use the resulting basis $b_1, b'_1, \ldots, b_k, b'_k$ the standard basis of $G'$. Let us compare this basis to the basis $e_1, e'_1, \ldots, e_k, e'_k$ of $G'$, satisfying (15), that is constructed in the proof of Theorem 11. The matrix describing the change of basis from the standard basis to the $e_i, e'_i$ is

$$A = \begin{pmatrix}
1 & l_{1,1} & 0 & l_{1,2} & \cdots & 0 & l_{1,k} \\
0 & l'_{1,1} & 0 & l'_{1,2} & \cdots & 0 & l'_{1,k} \\
0 & l_{2,1} & 1 & l_{2,2} & \cdots & 0 & l_{2,k} \\
0 & l'_{2,1} & 0 & l'_{2,2} & \cdots & 0 & l'_{2,k} \\
: & : & : & : & : & \cdots & : \\
0 & l_{k,1} & 0 & l_{k,2} & \cdots & 1 & l_{k,k} \\
0 & l'_{k,1} & 0 & l'_{k,2} & \cdots & 0 & l'_{k,k}
\end{pmatrix}$$

Let $B$ be the matrix representing the epimorphism $\Psi : G' \to (\mathbb{Z}/e\mathbb{Z})^{k'}$ from (16) with respect to the bases $e_i, e'_i$ and $f_1, \ldots, f_{k'}$, and $C$ a matrix representing $\Phi' : (\mathbb{Z}/e\mathbb{Z})^{k'} \to G$ with respect to $f_1, \ldots, f_{k'}$ and any system $g_1, \ldots, g_{k'}$ of generators for $G$. Then the epimorphism $\Phi : G' \to G$ is represented with respect to the standard basis of $G'$ and the generators $g_1, \ldots, g_{k'}$ of $G$ by the matrix $R = CBA^{-1}$.

With the places $v_1, w_1, \ldots, v_k, w_k$ defining $G'$, the standard basis $b_1, b'_1, \ldots, b_k, b'_k$ of $G$ and the matrix $R$ describing $\Phi$, we have now computed all the data required to describe the characteristic morphism $\rho^S_{\mathbb{K}} : \hat{G}_{\mathbb{S}} \to G$ from Theorem 8, and thus the induced epimorphism $\rho : I_{\mathbb{K}}/K^\times \to G$ in a convenient way.
3.4. Conductor, Artin map and equations. The epimorphism \( \rho : I_K/K^\times \to G \) induced by \( \rho_S^T \) defines an extension \( L_\rho/K \) together with an isomorphism \( \text{Gal}(L_\rho/K) \to G \). Let us see how to easily infer some information about \( L_\rho \) from our description of \( \rho_S^T \).

First of all, the places \( v \) of \( K \) that ramify in \( L_\rho \) are those, for which the inertia group \( \rho(\mathcal{O}_v^\times) \) does not vanish in \( G \). By our construction of \( \rho_S^T \) in (14), only places in \( \{v_1, w_1, \ldots, v_k, w_k\} \) can ramify. More precisely, the ramified places are all those \( v_i, w_i \) for which the corresponding standard basis element \( b_i \) or \( b_i' \) is not in the kernel of \( \Phi \). If we write \( (r_1, \ldots, r_k)^T \) for the corresponding column of \( R \) and write \( G \) additively, this means that \( r_1 g_1 + \cdots + r_k g_k \neq 0 \) in \( G \). All these places are tamely ramified, so the conductor ideal of \( L_\rho \) is just the product of the prime ideals corresponding to these places.

Every place \( v \in S \) is split completely, as then \( \rho(K^\times_v) \) vanishes in \( G \). For places \( v \notin S \cup \{v_1, w_1, \ldots, v_k, w_k\} \), we can compute the Artin symbol \( (L_\rho/K, v) \in G \) as follows: compute a uniformiser \( \pi_v \in \mathcal{O}_{S,v}^\times \) at \( v \) and \( l_{v,i}, l'_{v,i} \in \mathbb{Z}/e\mathbb{Z} \) such that \( \pi_v^{-1} = b_i^{l_{v,i}} \) in \( F_{v_i} \otimes \mathbb{Z}/e\mathbb{Z} \) and \( \pi_v^{-1} = (b_i')^{l'_{v,i}} \) in \( F_{w_i} \otimes \mathbb{Z}/e\mathbb{Z} \). Then the coordinates of \( \rho_S^T(\pi_v) \in G' \) with respect to the standard basis are given by \( l_v = (l_{v,1}, l'_{v,1}, \ldots, l_{v,k}, l'_{v,k})^T \). Hence, the Artin symbol is given by \( (L_\rho/K,v) = \rho(\pi_v) = (g_1, \ldots, g_{w}) \cdot R \cdot l_v \). If \( v \in \{v_1, w_1, \ldots, v_k, w_k\} \) is unramified, we can compute \( (L_\rho/K, v) \) in the same way as above, except that we take \( l_{v,i} = 0 \) if \( v = v_i \) and \( l'_{v,i} = 0 \) if \( v = w_i \).

Given the conductor \( \mathfrak{f} \) and a method to evaluate the Artin symbol \( (L_\rho/K, v) \), we have enough information to specify the congruence subgroup of the ray class group modulo \( \mathfrak{f} \) corresponding to \( L_\rho \) through an exhaustive search for generating splitting primes. From this data, one can compute polynomials defining \( L_\rho \) using algorithms of computational class field theory, see, e.g. [1–3]. Such algorithms are implemented, for example, in the computer algebra system Magma.

3.5. Subextensions. If \( \Pi : G \to H \) is an epimorphism, then \( \Pi \circ \rho \) describes a subextension \( L_\Pi \) of \( L_\rho \) together with an isomorphism \( \text{Gal}(L_\Pi/K) \to H \). The epimorphism \( \Pi \circ \rho \) is induced by the characteristic morphism \( \Pi \circ \rho_S^T : \hat{\mathcal{O}}_S^\times \to H \), and, in a similar way as described in \( \S 3.4 \), we can compute the ramified primes, splitting primes and equations for \( H \).

Given a presentation \( G \simeq \mathbb{Z}/n_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/n_r \mathbb{Z} \), we can apply this in particular to the projections \( \Pi_i : G \to H_i = \mathbb{Z}/n_i \mathbb{Z} \). Then \( L_\rho \) is the compositum of the fields \( L_{\Pi_i} \), which we can exploit, for example, to get equations for \( L_\rho \) with lower computational effort.

4. Illustrations

4.1. A biquadratic extension of \( \mathbb{Q} \) with norm 37/16. In this first example, we take \( G = (\mathbb{Z}/2\mathbb{Z})^2 \) and find an epimorphism \( \rho : I_{\mathbb{Q}}/\mathbb{Q}^\times \to G \) with \( \alpha = 37/16 \in \mathbb{N}_{L_\rho/\mathbb{Q}}(L_\rho^\times) \).

Due to the explicit nature of biquadratic fields, arguments simpler than ours would suffice here. A construction yielding distinct primes \( p, q \) such that a given rational number \( c \) is the norm of an element of \( \mathbb{Q}(\sqrt{p}, \sqrt{q}) \) was communicated to D. Loughran and the first-named author by J.-L. Colliot-Thélène, in response to a question raised at the workshop “Rational Points 2017” in Schloss Schney (Germany). Similarly to our approach here, the primes \( p, q \)
are chosen in such a way as to guarantee that the extension $\mathbb{Q}(\sqrt{p}, \sqrt{q})/\mathbb{Q}$ satisfies the Hasse norm principle and $c$ is a local norm at every place.

Since this example should serve as a first illustration of our approach, we will nevertheless use our setup relying on characteristic morphisms. The set $S = \{\infty, 2, 37\} \subset \Omega_{\mathbb{Q}}$ satisfies (1), and $O_S^\times = \mathbb{Z}[1/2, 1/37]^\times = (-1, 2, 37)$. We can take $e = k' = 2$ and $\Phi' : G \to G$ the identity. With $k = (\frac{2}{3}) = 1$, we are aiming at a set $T$ consisting of a pair $v, w$ of places. The places in $T(S; e; 1; \{-1, 2, 37\})$ are those primes $p$ that satisfy

$$\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{37}{p}\right) = 1,$$

with $(\cdot)$ the Legendre symbol. We take $v = 41$, the smallest such prime. In addition to the above, the place $w$ then has to satisfy

$$\left(\frac{41}{p}\right) = -1,$$

and the smallest choice is $w = 137$. Hence, we have found our auxiliary morphism

$$\rho'_S : \mathbb{Z}_S^\times \to (\mathbb{F}_{41}^\times \times \mathbb{F}_{137}^\times) \otimes \mathbb{Z}/2\mathbb{Z} = G' \simeq (\mathbb{Z}/2\mathbb{Z})^2 = G.$$

By our construction, we get $\rho'_S(\mathbb{Q}_{41}^\times) = G'$, so we can take $\rho_S = \Phi \circ \rho'_S$ for, e.g., the isomorphism $\Phi : G' \to G$, that identifies $\mathbb{F}_{41}^\times \otimes \mathbb{Z}/2\mathbb{Z}$ with the first $\mathbb{Z}/2\mathbb{Z}$-factor of $G$ and $\mathbb{F}_{137}^\times \otimes \mathbb{Z}/2\mathbb{Z}$ with the second. We have thus achieved that $\bigwedge^2 \rho(\mathbb{Q}_{41}^\times) = \bigwedge^2 G$, which confirms the validity of the Hasse norm principle for $\rho$. All elements of $\mathbb{Z}[1/2, 1/37]^\times$, including in particular our $\alpha = 37/16$, are local norms at all places, and therefore in $N_{L_{\rho}/\mathbb{Q}}(L_{\rho}^\times)$.

Which biquadratic extension does $\rho$ describe? Via the projections $\Pi_1, \Pi_2 : G \to \mathbb{Z}/2\mathbb{Z}$ to the two $\mathbb{Z}/2\mathbb{Z}$-factors of $G$, we can describe $L_{\rho}$ as the compositum of two quadratic subfields with conductors 41 and 137. Hence, $\rho$ defines the number field $L_{\rho} = \mathbb{Q}(\sqrt{41}, \sqrt{137})$.

We know now that $37/16$ is a norm from this field. In other words, using the norm form coming from the basis $1, \sqrt{41}, \sqrt{137}, \sqrt{5617}$ of $L_{\rho}$, the equation

$$x_1^4 - 82x_1^2x_2^2 + 1681x_2^4 - 274x_2^2x_3^2 - 11234x_2^2x_3^2 + 18769x_3^4 + 44936x_1x_2x_3x_4$$

$$-11234x_2^2x_4^2 - 460594x_2^2x_4^2 - 1539058x_2^2x_4^2 + 31550689x_4^4 = 37/16$$

has rational solutions $(x_1, x_2, x_3, x_4) \in \mathbb{Q}^4$.

4.2. A $\mathbb{Z}/6\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^3$-extension of $\mathbb{Q}(\sqrt{-47})$ from which $2 + 3\sqrt{-47}$ is a norm.

Here we consider an example that is sufficiently generic to demonstrate all the features of our construction. Our computations are assisted by the computer algebra system Sage$^1$. Towards the end, we will compute concrete polynomials giving our extension $L$. For this, we use the Magma Calculator$^2$. None of the computations took longer than a few seconds.

Take the number field $K = \mathbb{Q}(\sqrt{-47})$ with class number 5. We construct an extension $L/K$ with Galois group $G = \mathbb{Z}/6\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^3$, such that $2 + 3\sqrt{-47} \in N_{L/K}(L^\times)$.

1. https://www.sagemath.org/
2. http://magma.maths.usyd.edu.au/calc/
The principal ideal \((2+3\sqrt{-47})\) factors as the product of two prime ideals of \(\mathcal{O}_K\), namely
\[
(2+3\sqrt{-47}) = \left(7, \frac{3+\sqrt{-47}}{2}\right) \left(61, \frac{21+\sqrt{-47}}{2}\right).
\]

We take \(S\) to consist of the archimedean place of \(K\) and these two prime ideals. Then \(\mathcal{O}_S\) has class number one, so (1) is satisfied. A quick computation reveals the following set of generators \(\{\gamma_0, \gamma_1, \gamma_2\}\) for the \(S\)-units \(\mathcal{O}^\times_S\): \(\gamma_0 = -1, \gamma_1 = 2+3\sqrt{-47}\) and \(\gamma_2 = 128+3\sqrt{-47}\).

Take \(\Phi' : (\mathbb{Z}/6\mathbb{Z})^4 \to G\) to be the epimorphism mapping each factor \(\mathbb{Z}/6\mathbb{Z}\) to the corresponding factor \(\mathbb{Z}/6\mathbb{Z}\) or \(\mathbb{Z}/3\mathbb{Z}\) of \(G\) in the obvious way. With \(k' = 4\), we are looking for \(k = \left(\frac{2}{3}\right) = 6\) pairs of places \(v_1, w_1, \ldots, v_6, w_6\) of \(K\) to define our auxiliary morphism \(\rho'_S\) from (14).

Our places need to be chosen from \(T(S; 6; 1; \{\gamma_0, \gamma_1, \gamma_2\})\), so they must be in \(\Omega_K \setminus S\) and satisfy the conditions that
\[
-1, 2+3\sqrt{-47} \text{ and } 128+2\sqrt{-47} \text{ are 6-th power residues in } \mathbb{F}_p^\times.
\]

Here is a list of the (prime ideals corresponding to) the first few places satisfying these conditions, in order of the rational primes lying below:

\[
\begin{align*}
(97, (27 + \sqrt{-47})/2), & \quad (569), \quad (809), \quad (1033), \quad (1381, (1445 + \sqrt{-47})/2), \\
(1913), & \quad (2281, (619 + \sqrt{-47})/2), \quad (2377, (3677 + \sqrt{-47})/2), \\
(2887), & \quad (4621), \quad (4789, (2537 + \sqrt{-47})/2), \quad (5227), \quad (6101).
\end{align*}
\]
Hence, we take $v_1$ as the first of these places, choose $b_1$ to be a primitive root in $\mathbf{F}_{v_1}^\times$ and compute a uniformiser $\pi_1 \in \mathcal{O}_{S}^{\times}$ at $v_1$ as in §3.1. The precise values are recorded in Table 1.

As $u_1 = 1$, the place $w_1$ has to satisfy in addition to (21) the condition that $X^6 - \pi_1$ is irreducible over $\mathbf{F}_{w_1}^\times$, so $\pi_1$ may not be a quadratic or cubic residue in $\mathbf{F}_{w_1}^\times$. We choose $w_1$ as the first place satisfying these conditions and $b_1'$ as a primitive root for $\mathbf{F}_{w_1}^\times$. See Table 2 for the values.

Next, we pick $v_2$ as the next available place from our list and choose a corresponding primitive root $b_2$ and uniformiser $\pi_2 \in \mathcal{O}_{S}^{\times}$ at $v_2$. As $v_2$ is generated by an inert rational prime, the choice of uniformiser is simple.

Before choosing $w_2$, we need to find $u_2$. To this end, we compute the discrete logarithms $l_{1,1}' = 5$ and $l_{1,2}' = 0$ (modulo 6) of $\pi_1^{-1}$ and $\pi_2^{-1}$ in $\mathbf{F}_{w_1}^\times$, with respect to the chosen primitive root $b_1'$. Hence, the images of $\pi_1$ and $\pi_2$ in $\mathbf{F}_{w_1}^\times \otimes \mathbb{Z}/6\mathbb{Z}$ have coordinates 5 and 0 with respect to the basis $b_1' \otimes 1$, and thus $\pi_2$ has the same image as $\pi_1^0$. We take $u_2 = (\pi_1^0)^{-1} = 1$.

The condition that $w_2$ needs to satisfy, in addition to (21), is that $X^6 - \pi_2$ is irreducible over $\mathbf{F}_{w_2}^\times$. We take $w_2$ as the first place that satisfies this condition and compute a primitive root $b_2'$ of $\mathbf{F}_{w_2}^\times$.

We choose $v_3$ as the next available place in our list, together with a primitive root $b_3$ and uniformiser $\pi_3$.

For $u_3$, the $\mathbb{Z}/6\mathbb{Z}$-linear system (19) takes the form

$$
\begin{pmatrix}
5 & 0 \\
4 & 5
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
=
\begin{pmatrix}
0 \\
3
\end{pmatrix},
$$

with solution $(c_1, c_2) = (0, 3)$. Hence, we take $u_3 = \pi_2^{-3}$, and choose $w_3$ as the smallest place that satisfies (21) and moreover that $X^6 - u_3 \pi_3$ is irreducible over $\mathbf{F}_{w_3}^\times$.

We continue in the same fashion to compute the remaining places $v_4, w_4, v_5, w_5, v_6, w_6$ as well as the corresponding primitive roots and uniformisers (for the $v_i$). The data is collected in Tables 1 and 2.

This gives us everything that we need to define the auxiliary morphism

$$
\rho_S' : \hat{\mathcal{O}}_S^\times \to G' = (\mathbf{F}_{v_1}^\times \times \mathbf{F}_{w_1}^\times \times \cdots \times \mathbf{F}_{v_6}^\times \times \mathbf{F}_{w_6}^\times) \otimes \mathbb{Z}/6\mathbb{Z}
$$

and the standard basis $b_1 \otimes 1, b_1' \otimes 1, \ldots, b_6 \otimes 1, b_6' \otimes 1$. The change of basis from the standard basis to the $e_i, e'_i$ satisfying (15) is described by the matrix (20), which takes in our case the form
An epimorphism $\Psi : G' \to (\mathbb{Z}/6\mathbb{Z})^4$ as in (16) is given, with respect to the basis $e_i, e'_i$ of $G'$ and the standard basis $f_1, f_2, f_3$ of $(\mathbb{Z}/6\mathbb{Z})^4$ by the matrix

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 1 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 4 & 0 & 5 & 0 & 3 & 0 & 1 & 0 & 4 & 0 \\
0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 4 & 0 & 2 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 0 & 1 \\
0 & 1 & 0 & 4 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 2 & 0 & 2 & 0 & 4 & 0 & 5 & 0 \\
0 & 3 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 4 & 1 \\
0 & 4 & 0 & 1 & 0 & 5 & 0 & 5 & 0 & 0 & 3
\end{pmatrix}.$$

The epimorphism $\Phi' : (\mathbb{Z}/6\mathbb{Z})^4 \to G$ just reduces the last three coordinates modulo 3, so it is represented with respect to the $f_i$ and the standard generators $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ of $G$ by the identity matrix. In total, our map $\Phi = \Phi' \circ \Psi : G' \to G$ is given, with respect to the standard basis of $G'$ and the standard generators of $G$, by the matrix

$$B = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The extension $L_j$ is thus induced by the characteristic morphism $\rho_{S,j} = \Phi_j \circ \rho_S$, where the map $\Phi_j = \Pi_j \circ \Phi$ is represented by the $j$-th row $R_j$ of the matrix $R$. We can read off the places of $K$ ramified in $L_j$ as those $v_i, w_i$ whose entry in the $j$-th row of $R$ is not equal
to 0 (modulo 3 for \( j = 2, 3, 4 \)). Hence, the conductor ideal of \( L_j \) is \( \mathfrak{f}_j \), with
\[
\begin{align*}
\mathfrak{f}_1 &= v_1 w_1 v_2 w_2 v_3 w_3 w_4 w_5 w_6, \\
\mathfrak{f}_2 &= w_1 w_2 w_3 v_4 v_5 w_5 w_6, \\
\mathfrak{f}_3 &= w_1 w_2 w_4 v_5 v_6 w_6, \\
\mathfrak{f}_4 &= w_1 w_2 w_3 w_4 w_5.
\end{align*}
\]
All places in \( S \) split completely in all of the \( L_j \). For a place \( v \notin S \), we compute a uniformiser \( \pi_v \in \mathcal{O}^\times_{S,j(v)} \) as in §3.1 and the vector \( l_v = (l_{1,v}, l'_{1,v}, \ldots, l_{6,v}, l'_{6,v}) \), with \( l_{i,v} \) (or \( l'_{i,v} \)) the discrete logarithm of \( \pi_v^{-1} \) in \( \mathbb{F}_v^\times \) (or \( \mathbb{F}_w^\times \)) with respect to \( b_1 \) (or \( b'_1 \)), taken modulo 6. Then \( l_v \) consists of the coordinates of \( \rho^v_S(\pi_v) \) in the standard basis of \( G' \), so \( \rho^v_S(\pi_v) = 0 \) if and only if \( R_j l_v = 0 \) (modulo 3 for \( j = 2, 3, 4 \)). These \( v \) are exactly the places not in \( S \) that split completely in \( L_j \).

In each of the extensions \( L_j \), after an exhaustive search for places satisfying these conditions, we present the first few places of \( K \) that split completely: for \( L_1 \), these are the prime ideals
\[
\begin{align*}
(7, (3 + \sqrt{-47})/2), (7, (11 + \sqrt{-47})/2), (53, (71 + \sqrt{-47})/2), (59, (37 + \sqrt{-47})/2), \\
(61, (21 + \sqrt{-47})/2), (67), (97, (167 + \sqrt{-47})/2), (103, (57 + \sqrt{-47})/2), \\
(131, (79 + \sqrt{-47})/2), (149, (129 + \sqrt{-47})/2), (157, (253 + \sqrt{-47})/2).
\end{align*}
\]
For \( L_2 \), we get the prime ideals
\[
\begin{align*}
(2, (1 + \sqrt{-47})/2), (5), (7, (3 + \sqrt{-47})/2), (11), (41), (43), \\
(53, (71 + \sqrt{-47})/2), (59, (37 + \sqrt{-47})/2), (61, (21 + \sqrt{-47})/2), \\
(61, (101 + \sqrt{-47})/2), (67), (73), (79, (43 + \sqrt{-47})/2).
\end{align*}
\]
For \( L_3 \), the first few splitting places are
\[
\begin{align*}
(2, (-1 + \sqrt{-47})/2), (2, (1 + \sqrt{-47})/2), (3, (1 + \sqrt{-47})/2), \\
(7, (3 + \sqrt{-47})/2), (17, (19 + \sqrt{-47})/2), (29), (43), (53, (35 + \sqrt{-47})/2), \\
(61, (21 + \sqrt{-47})/2), (71, (109 + \sqrt{-47})/2), (73), (79, (43 + \sqrt{-47})/2).
\end{align*}
\]
Finally, the first few splitting primes for \( L_4 \) are
\[
\begin{align*}
(2, (-1 + \sqrt{-47})/2), (2, (1 + \sqrt{-47})/2), (5), (7, (3 + \sqrt{-47})/2), \\
(17, (19 + \sqrt{-47})/2), (19), (23), (29), (37, (45 + \sqrt{-47})/2).
\end{align*}
\]
In each case, the presented list of splitting primes is barely enough to generate the correct congruence subgroup of the ray class group modulo \( \mathfrak{f}_j \). From this information, Magma obtains the following polynomials defining the \( L_j \):

The field \( L_1 \) is generated by roots of the polynomials
\[
X^2 + 1625554186831677234132\sqrt{-47} + 1811860086730297979035
\]
and
\[ X^3 - 3748167037906625162481707496423982188817923353276910868587X \\
+ 20932667006810986711572641003097069272613523 \\
90891328392614557353058433435300040785407916 \\
- 4892889242489320858461993956906597337627420 \\
962014982904896214316984337417374211250173\sqrt{-47}. \]

The field \( L_2 \) is generated by a root of
\[ X^3 - 51425250005413458725291124378250069242533908387X \\
+ \frac{1}{2}\left(-29846938917972463681140245645712648 \\
301029855802019621285424376904094635 \\
+ 1662861756262325040866728524283288871452 \\
494124982673546015261665625225\sqrt{-47}\right). \]

For \( L_3 \), we get the generating polynomial
\[ X^3 - 913736091749824689643505040786706198418283X \\
- 23307321750072728541912828122253866623543300542376262114304064 \\
+ 22955375656018471299613470304327792587793595929519410170152870\sqrt{-47}, \]

and for \( L_4 \) the polynomial
\[ X^3 - 4720036166349902544210196434745323X \\
- 142867906955019411285258390809644752015601994134080 \\
- 6327448763258896081623032545540797957509248167898\sqrt{-47}. \]

Hence, the compositum \( L \) is generated over \( K \) by roots of all of these polynomials.

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