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Relative Fuzzy Set

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Abstract
We propose a fuzzy set of a set characterized by a growth membership function. The elements of the proposed fuzzy set are capable of growing in membership grade over any chosen variable/parameter of the measure chains. It enriches and generalizes the fuzzy set concepts in literature.

Keywords: Fuzzy set, Relative fuzzy set, Time scales, Growth membership function.

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1 Introduction

L. A. Zadeh [25] pioneered research on objects of a set with grades of membership when he published fuzzy sets in 1965. Research efforts and applications of the fuzzy set concept have progressed rapidly ever since. This is because it is an efficient and effective tool in describing and modeling real life scenarios and complex systems.

Zadeh’s paper was followed by several extensions, characterizations, and generalizations of fuzzy sets: Interval-valued fuzzy sets, Intuitionistic fuzzy sets, Type-2-fuzzy sets, Hesitant fuzzy sets, Nonstationary fuzzy sets, Time-dependent fuzzy sets, Soft sets, Fuzzy soft sets e.t.c. See [1, 5, 10, 15, 17, 19, 22] for some definitions and characterizations of some fuzzy related concepts. Molodtsov [17] in 1999 defined soft set as a generalization of fuzzy set by defining on a set of parameters a subset-valued map and Maji et al. [16] introduced fuzzy soft set as a fuzzy set-valued soft set map which describes a multi-criteria soft set as each fuzzy set associated with an element of the parameter defines a criteria. See [17, 6, 7, 13, 20] for some contributions to soft set theory.

Operations, logic, calculus e.t.c. of fuzzy sets, its extensions and generalizations are being developed and their applications are progressing rapidly. But the membership function value of each element is a fixed value in the interval [0, 1] (e.g the membership value of each fuzzy element is a fixed value of [0, 1], the membership value of each intuitionistic fuzzy element is a fixed two-value of [0, 1], the membership value of each interval-valued fuzzy element is a fixed interval of [0, 1], the membership value of each hesitant fuzzy element is a fixed subset of [0, 1]). The snag of these concepts is their inadequacy to represent the theory of dynamic membership grades which characterize many real word scenarios.

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B. P. Lientz [14] defined time-dependent fuzzy sets as fuzzy sets characterized by membership functions of time-dependent sets. It defines fuzzy set of a dynamic set with respect to time such that at any time \( t_i, i = 1, 2, \ldots \), we have fuzzy sets \( A_i = \{(x(t_i), \mu_{A_i}(x(t_i)) : x(t_i) \in X(t_i)\} \) corresponding to states \( X(t_i) \subset X \) for each \( i \). It is a family of fuzzy sets of subsets of \( X \) such that each subset of \( X \) is determined by a state (time) on \( X \). The membership value of element of \( X \) does not change by virtue of state change but by change in membership function. Thus the membership function expression at state \( i \) is not the same as the membership function expression at state \( j \) where \( i \neq j \). Also J. M. Garibaldi et al. [8] defined Non-stationary fuzzy sets as fuzzy sets having membership grades as the time-specific variation of the fuzzy membership values. Thus for each \( t \in T \) there is a membership function defined on the set. Also the elements of a fuzzy soft set is dynamic, the membership values changes as a result of change in membership function (i.e it is a multi-criteria fuzzy set). It reduces to a fuzzy set if it is a single criteria function (i.e characterized by a single membership function for all elements of the parameter set).

Thus fuzzy set and its generalizations so far do not sufficiently describe a holistic representation of imprecision because the objects of a set can grow in membership levels/grades with respect to any variable. Our interest is the consideration of the possibility of a dynamic (continuous and discrete) membership function of the elements of a set with respect to any variable of the measure chains. Obviously, it generalizes the concepts of fuzzy set, time-dependent fuzzy sets, non-stationary fuzzy sets and can be extended to other fuzzy related concepts in literature. Unlike fuzzy soft set, it describes/represents a criteria.

Thus, we introduce a measure chain-dependent dynamic mathematical tool for dealing with uncertainties, imprecision and vagueness. Unlike the time-dependent fuzzy set, a membership function is enough to define the dynamics of the fuzzy set. Our proposed fuzzy set is a single criteria (having one membership function) fuzzy set whose element poses dynamic membership grades/values. We provide examples to show that it is unique in its representation and that it differs or generalizes the fuzzy set and existing fuzzy set generalizations in literature.

Stefan Hilger [11] in his PhD thesis, introduced measure chains with the aim of stating a system that handles both continuous and discrete processes in one. His work has advanced the research in dynamical system and is an efficient tool in modeling hybrid processes. Difference and differential theories on \( Z \) and \( R \) are being merged by defining on a time scales (a measure chain with the relation \( \leq \) or a closed subset of \( R \) ) the time scale differential operator. See [2, 3, 4, 18] for some recent development of measure chains theory so far.

In this study the fuzzy growth values are dependent on a set of variables of the time scale. This is to ensure that both continuous and discrete growths are handled simultaneously.

The paper is structured as follows: Section 2 contains some existing definitions of fuzzy sets and measure chain concepts in literature as well as the motivation for the work. Section 3 contains the introduction of Relative fuzzy set, characterizations and examples while some applications are presented in Section 4.

2 Preliminaries

In this section, we review some existing definitions of the concept of fuzzy sets and measure chains in literature. We refer the reader to [4, 11, 16, 15, 19, 25] for more details.

2.1 Definitions:

Let \( X \) be the universal set. Then

(i) [25] a fuzzy set is a non-empty set \( A \) in \( X \) characterized by a membership function \( \mu_A : X \to [0, 1] \)

(ii) [1] an intuitionistic fuzzy set is a set \( A \) in \( X \) characterized by two membership functions \( \mu_A : X \to [0, 1] \) and \( g_A : X \to [0, 1] \) such that \( 0 \leq \mu_A(x) + g_A(x) \leq 1 \) for all \( x \in X \)

(iii) [22] a hesitant fuzzy set \( A \) in \( X \) is characterized by a membership function \( h_A : X \to 2^{[0,1]} \), where \( 2^{[0,1]} \) is the family of all subsets of \([0,1] \)
A measure chain is a non-empty set of intervals of parameters that is conditionally complete, i.e., for any non-empty bounded subset of the set of parameters, there exists a least upper bound and a greatest lower bound.

### Measure Chains

A measure chain $(M, \preceq, \mu)$ is any non-empty set $M$ equipped with a relation $\preceq$ which is reflexive, transitive, antisymmetric, and total such that:

**MC1.** The relation $\preceq$ is reflexive, transitive, antisymmetric, and total.

**MC2.** The chain $(M, \preceq)$ is conditionally complete, i.e., every non-empty bounded subset has a least upper bound and a greatest lower bound.

**MC3.** The mapping $\mu : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ has the following properties for all $r, s, t \in \mathbb{M}$:

1. **(i):** $\mu(r, s) + \mu(s, t) = \mu(r, t)$
2. **(ii):** If $r \succeq s$, then $\mu(r, s) > 0$
3. **(iii):** $\mu$ is continuous with respect to the product order topology (the order topology is generated by the open intervals of $\mathbb{M}$).

In the context of a fuzzy set $A$ in $X$, characterized by the membership function $\mu_A : X \to [0, 1]$, the set $A(t)$ is a fuzzy set at time $t$ associated with each element of $T \times X$ for a specific variation of $\mu_A(x)$ where $A$ is a fuzzy set in $X$ with membership function $\mu_A : X \to [0, 1]$. For a time-dependent fuzzy set $A(t)$, it is a set characterized by the membership function $\mu_{A(t)} : X(t) \to [0, 1]$ where $X(t)$ is a set of $X$ at time $t$ and $A(t)$ is in $X(t)$.
2.2.1 Remarks:
2.2.1a: Time scales \( \mathbb{T} \) are specific forms of measure chains. It is defined as a nonempty closed subset of \( \mathbb{R} \) ordered by the relation "\( \leq \)". Examples of time scales are \( \mathbb{R}, \mathbb{Z}, h\mathbb{Z}, \{0,1\}, [0,1] \cup [2,3], [0,1] \cup \mathbb{N} \), the Cantor set, e.t.c.

2.2.1b: By MC1, we can define
(i): an order topology
(ii): the forward and backward jump operators i.e the maps \( \sigma, \rho : \mathbb{T} \rightarrow \mathbb{T} \) such that
\[
\sigma(t) = \inf \{ s \in \mathbb{T} : t \leq s \}
\]
and
\[
\rho(t) = \sup \{ s \in \mathbb{T} : s \leq t \}
\]
respectively. This equips measure chains with the property of connectedness.

2.2.1c: Property MC2 ensures the transfer of important features of the real line to other time scales.

2.2.1d: Property MC3 ensures the measure of distances between elements of the measure chains.

2.2.2 Definition
Let \( (\mathbb{T}, \leq, \mu) \) be a time scale and \( \sigma, \rho : \mathbb{T} \rightarrow \mathbb{T} \) be the forward and backward jump operators respectively. Then

2.2.2a: a nonmaximal (nonminimal) element \( t \in \mathbb{T} \) is said to be right (left)-scattered if \( \sigma(t) > t, \rho(t) < t \)

2.2.2b: a nonmaximal (nonminimal) element \( t \in \mathbb{T} \) is said to be right (left)-dense if \( \sigma(t) = t, \rho(t) = t \)

2.2.2c: \( t \in \mathbb{T} \) is said to be isolated (dense) if it is left-scattered and right-scattered (left-dense and right-dense).

2.2.2d: the graininess function \( \mu^* : \mathbb{T} \rightarrow [0,\infty) \) is defined by
\[
\mu^*(t) = \sigma(t) - t
\]

2.3 Motivation
In the metamorphosis process of an insect, we approximately have five stages: egg, lava, pupa, caterpillar and adult. These stages are time dependent (all other factors constant). So, by the definition of fuzzy set, we can describe a typical insect population as thus:

If we let the insect population be represented by the universal set \( U \) and \( i = 1,2,3,4,5 \) corresponds to the stages: egg, lava, pupa, caterpillar and adult stages respectively. Suppose the membership values of these are assigned the values \( 0,2,0,4,0,6,0,8,1 \) respectively, then the fuzzy set of a sub-population \( P \) of \( U \) with the criteria insect membership in a population be defined by the function \( \mu_P : U \rightarrow [0,1] \) such that \( \mu_P(x) \) is the membership value of \( x \) in \( P \) of \( U \). That is, the function value reveals the membership value of an insect \( x \) at a particular stage \( i \). Thus if an insect \( x \) is in the lava stage then \( \mu_P(x) = 0.4 \).

Now suppose an offspring insect \( x \in P \subset U \) grows over time from say lava to pupa, then its membership grade changes. Thus an insect \( x \) at lava stage with membership value \( \mu_P(x) = 0.4 \) grows into pupa stage with membership value \( \mu_P(x) = 0.6 \). That is, the same insect progressing in membership grades. This is remarkable and cannot be ignored in the study of such processes. However, the fuzzy set and extensions of it in literature do not give representation to the population subset above characterized by a membership function whose values for an insect in the population changes across the stages of development.

Also, in Example 2.1a of [26], a realtor wants to classify the houses he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in it. Let \( X = \{1,2,3,4,\ldots,10\} \) be the set of available types of houses described by \( x \) number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a four-person family" may be described as
\[
A = \{(1,0.2),(2,0.5),(3,0.8),(4,1),(5,0.7),(6,0.3)\}
Suppose the criteria is "a comfortable and affordable house for an average income earning four-person family". If the prices are graded "$c = \text{cheap}, m = \text{moderate} \text{ and } e = \text{expensive}". Then the membership function of a house with $x$ bedroom(s) at a price $c$ differs from the membership function of a house with the same number of bedrooms at price $m$. The fuzzy set concept in literature is inadequate to describe the cost-dependent comfortable house. Our proposed fuzzy concept adequately handles this case sufficiently.

Therefore in order to give sufficient account of the subset of a universal set characterized by membership grade growth dynamics of the element(s) membership value(s) with respect to an element of a measure chain (not restricted to time).

### 3 Main Results

We begin by defining a Relative fuzzy set as a fuzzy set whose membership function is defined on the subset of the Cartesian products of the universal set of discourse and a time scale such that for any point of the set, we have changing membership values as the points of the time scale changes. The time scale represents any variable.

#### 3.1 Relative Fuzzy Set

Let $X$ be the universal set of discourse, $T$ any time scale, $R$ a subset of $X$ and $\sigma : T \to T$ a forward difference operator. Then a Relative fuzzy set $R$ on $X \times T$ is a set equipped with the membership growth function $\mu_R : X \times T \to [0, 1]$ such that

$$
\mu_R(x, \sigma(t)) = \begin{cases} 
0 & \text{if } x \notin R \\
\in (0, 1] & \text{if } x \in R \text{ and } t \in T 
\end{cases}
$$

for all $x \in X$ and all $t \in T$

Thus $R$ is completely determined by the set of tuples

$$R = \{(x, t), \mu_R(x, \sigma(t)) : x \in X \text{ and } t \in T\}$$

Similarly, we can define a Relative fuzzy set for a backward difference operator.

#### 3.1.1 Remarks

(i) If $t$ is nonmaximal and dense in $T$ then $\mu_R(x, \sigma(t)) = \mu_R(x, t)$. Similarly, if $t$ is nonminimal and dense in $T$ then $\mu_R(x, \rho(t)) = \mu_R(x, t)$.

(ii) If $t$ is nonmaximal and isolated in $T$ then $\mu_R(x, \sigma(t)) \neq \mu_R(x, t)$ and if $t$ is nonminimal and isolated in $T$ then $\mu_R(x, \rho(t)) \neq \mu_R(x, t)$.

(iii) If $T = \mathbb{R}$ then $\mu_R(x, \sigma(t)) = \mu_R(x, t) = \mu_R(x, \rho(t))$ for all $t \in T$.

(iv) If $T = \mathbb{Z}$ then $\mu_R(x, \sigma(t)) = \mu_R(x, t + 1)$ and $\mu_R(x, \rho(t)) = \mu_R(x, t - 1)$ for all $t \in T$.

(v) If $T = h\mathbb{Z}$ then $\mu_R(x, \sigma(t)) = \mu_R(x, t + h)$ and $\mu_R(x, \rho(t)) = \mu_R(x, t - h)$ for all $t \in T$.

(vi) Unlike the time-dependent fuzzy set, the Relative fuzzy subset of a set is not determined by time but it a fuzzy subset of the product of the set $X$ and $T$. Also while the time-dependent fuzzy set defines a fuzzy set for each subset of $X$ determined by time, a Relative fuzzy is characterized by a single membership function defining the dynamics of the element(s) membership value(s) with respect to an element of a measure chain.

(vii) Unlike the non-stationary fuzzy set, the Relative fuzzy set is not a time-varying fuzzy sets but evolutionary with respect to an element of a measure chain.
3.1.2 Examples

(i) Let $\mathbb{R}$ be the universal set of discourse, $X \subset \mathbb{R}$ with $X = \{x : -10 \leq x \leq 10\}$ and $\sigma$ a forward difference operator on $\mathbb{T}$ onto $\mathbb{T}$. Then a set $X$ characterized by a Relative fuzzy membership function

$$\mu_X : \mathbb{R} \times \mathbb{T} \rightarrow [0, 1]$$

such that

$$\mu_X(x, t) = \frac{1}{|\sigma(t) + x|}$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{T}$ is a relative fuzzy set.

(ii) Let $\mathbb{R}$ be the universal set of discourse, $X \subset \mathbb{R}$ with $X = \{x : 10 \leq x \leq 20\}$ and $\mathbb{T} = \{0, 1\} \cup [3, 7]$ with $\sigma$ a forward difference operator on $\mathbb{T}$. Define a Relative fuzzy set $R$ by a Relative fuzzy function

$$\mu_X : \mathbb{R} \times \mathbb{T} \rightarrow [0, 1]$$

such that

$$\mu_X(x, t) = e^{-\frac{1}{|\sigma(t) + x|}}$$

for all $x \in X$.

| $X$ | $\mu_X(x, t_0)$ | $\mu_X(x, t_1)$ | $\mu_X(x, t_2)$ | $\mu_X(x, t_3)$ | $\mu_X(x, t_4)$ | $\mu_X(x, t_5)$ | $\mu_X(x, t_6)$ | $\mu_X(x, t_7)$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 10  | 0.905          | 0.913          | 0.926          | 0.931          | 0.936          | 0.939          | 0.943          |               |
| 11  | 0.913          | 0.926          | 0.931          | 0.936          | 0.939          | 0.943          | 0.946          |               |
| 12  | 0.926          | 0.931          | 0.936          | 0.939          | 0.943          | 0.946          | 0.949          |               |
| 13  | 0.931          | 0.900          | 0.999          | 1              | 1              | 1              | 1              |               |
| 14  | 0.900          | 0.941          | 0.999          | 1              | 1              | 1              | 1              |               |
| 15  | 0.941          | 0.961          | 0.999          | 1              | 1              | 1              | 1              |               |
| 16  | 0.961          | 0.972          | 0.999          | 1              | 1              | 1              | 1              |               |
| 17  | 0.972          | 0.980          | 1.000          | 1              | 1              | 1              | 1              |               |
| 18  | 0.980          | 0.984          | 1.000          | 1              | 1              | 1              | 1              |               |
| 19  | 0.984          | 0.987          | 1.000          | 1              | 1              | 1              | 1              |               |
| 20  | 0.987          | 0.000          | 0.999          | 1              | 1              | 1              | 1              |               |

(iii) Considering the insect population example above. In a particular generation (egg $\rightarrow$ lava $\rightarrow$ pupa $\rightarrow$ caterpillar $\rightarrow$ adult), the growth values of insects $x, y, z \in P \subset U$ are functions of their weight and temperature at egg stage. The following Relative fuzzy functions

$$\mu_R(x, t) = w_x \sigma(t)$$

and

$$\mu_K(x, t) = \left| \frac{\sigma(t)}{c_x} + 0.8 \right|$$

for $t \in \{0, 1, 2, 3, 4, 5\}$ with $w_x$ and $c_x$, the weight and temperature of an insect $x$ at egg stage respectively define the growth behaviors of the insects over the time interval $\{0, 1, 2, 3, 4, 5\}$. 
Insect \( t \) \( \sigma(t) \) Weight\( (w) \) \( \mu_R \) Temperature\( (c) \) \( \mu_R' \)
\begin{array}{|c|c|c|c|c|}
\hline
\text{x} & 0 & 1 & 0.2 & 0.2 & 30 & 0.833 \\
& 1 & 2 & \ 0.4 & \ & \ & 0.867 \\
& 2 & 3 & \ 0.6 & \ & \ & 0.9 \\
& 3 & 4 & \ 0.8 & \ & \ & 0.933 \\
& 4 & 5 & \ 1 & \ & \ & 1 \\
\hline
\text{y} & 0 & 1 & 0.025 & 0.025 & -7.4 & 0.665 \\
& 1 & 2 & \ 0.05 & \ & \ & 0.53 \\
& 2 & 3 & \ 0.075 & \ & \ & 0.4 \\
& 3 & 4 & \ 0.1 & \ & \ & 0.26 \\
& 4 & 5 & \ 0.13 & \ & \ & 0.13 \\
\hline
\text{z} & 0 & 1 & 0.22 & 0.22 & 25 & 0.84 \\
& 1 & 2 & \ 0.44 & \ & \ & 0.88 \\
& 2 & 3 & \ 0.66 & \ & \ & 0.92 \\
& 3 & 4 & \ 0.88 & \ & \ & 0.96 \\
& 4 & 5 & \ 1 & \ & \ & 1 \\
\hline
\end{array}

(iv) The periodical Cicada [12, 24] is an insect with an amazing life cycle (egg \( \rightarrow \) nymph \( \rightarrow \) adult). Literature has
that an insect Cicada spends approximately 3 weeks at egg stage before hatching, 882 weeks at nymph stage, 1 week as adult before mating and 1 (male) and 2 (female) weeks as adult before dying out.
Using the concept of a Relative fuzzy set, we can describe the life cycle of a periodical Cicada insect as thus:
Let \( X \) be the set of periodical Cicada population, \( T = [0, 889] \) and the membership values at egg, nymph, mating adult and dying adult stages as 0.2, 0.4, 0.8, 1 respectively. So a Cicada insect membership growth values over time (life cycle time) is given as

\[
\mu_X(x, t) = \begin{cases} 
0.2 & \text{egg} \\
0.4 & \text{nymph} \\
0.8 & \text{mating adult} \\
1 & \text{dying adult}
\end{cases} \quad \begin{align*}
0 \leq t & \leq 3 \quad 3 \text{weeks} \\
4 < t & \leq 886 \quad 882 \text{weeks} \\
886 < t & \leq 887 \quad 1 \text{week} \\
887 < t & \leq 889 \quad 2 \text{weeks}
\end{align*}
\]

for the female insect Cicada

\[
\mu_X(x, t) = \begin{cases} 
0.2 & \text{egg} \\
0.4 & \text{nymph} \\
0.8 & \text{mating adult} \\
1 & \text{dying adult}
\end{cases} \quad \begin{align*}
0 \leq t & \leq 3 \quad 3 \text{weeks} \\
4 < t & \leq 886 \quad 882 \text{weeks} \\
886 < t & \leq 887 \quad 1 \text{week} \\
887 < t & \leq 888 \quad 1 \text{week}
\end{align*}
\]

(v) Let \( X = \{w_x : x \in X\} \) represent any set of roads of equal distances represented by their widths linking two destinations and \( t \) be any hour from 6:00 to 18:00. We define a Relative fuzzy function on \( X \times [6, 18] \) such that

\[
\mu_X(x, t) = 1 - \frac{w_x}{n_x(t)}
\]

where \( n_x \in \mathbb{R}, n_x \neq 0, n_x(t) \) is the average number of vehicles per kilometer on road \( x \) at time \( t \) and \( w_x \) is the width of road \( x \) in meters. Thus, \( \mu_X(x, t) \) represent how free is the traffic flow on road \( x \) at an hour \( t \) where \( \mu_X(x, t) = 1 \) indicates absolutely free traffic flow and \( \mu_X(x, t) = 0 \) represent complete hold up situation. With this representation, a motorist can determine which road to use at a particular hour between 6 and 18 hours in other to reach the destination timely.

Suppose \( x, y, z \) are any three such roads with road width 50, 100, 150 respectively linking destinations \( A \) and \( B \) with the road record given in the table below:
Then we can use the idea of the Relative fuzzy set to represent the traffic situation given the traffic records from 6:00 to 18:00 daily:

| $T$ | $n_x$ | $n_y$ | $n_z$ |
|-----|-------|-------|-------|
| 6   | 50    | 30    | 50    |
| 7   | 55    | 25    | 70    |
| 8   | 60    | 20    | 90    |
| 9   | 70    | 30    | 100   |
| 10  | 70    | 30    | 120   |
| 11  | 100   | 25    | 150   |
| 12  | 145   | 35    | 150   |
| 13  | 180   | 55    | 170   |
| 14  | 190   | 70    | 130   |
| 15  | 180   | 80    | 80    |
| 16  | 200   | 90    | 40    |
| 17  | 180   | 90    | 20    |
| 18  | 190   | 90    | 10    |

Then we can use the idea of the Relative fuzzy set to represent the traffic situation given the traffic records from 6:00 to 18:00 daily:

| $T$ | $\mu_X(x,t)$ | $\mu_X(y,t)$ | $\mu_X(z,t)$ |
|-----|---------------|---------------|---------------|
| 6   | 1             | 1             | 0.8           |
| 7   | 0.9           | 0.5           | 0.8           |
| 8   | 0.6           | 0.2           | 0.6           |
| 9   | 0.2           | 0.2           | 0.6           |
| 10  | 0.1           | 0.3           | 0.5           |
| 11  | 0.2           | 0.6           | 0.4           |
| 12  | 0.3           | 0.7           | 0.4           |
| 13  | 0.5           | 0.7           | 0.4           |
| 14  | 0.4           | 0.4           | 0.4           |
| 15  | 0.6           | 0.5           | 0.4           |
| 16  | 0.7           | 0.4           | 0.6           |
| 17  | 0.8           | 0.3           | 0.8           |
| 18  | 0.8           | 0.3           | 0.9           |

(vi) A typical periodical Cicada predator bird has about 160 weeks life cycle [12] (egg $\rightarrow$ chick $\rightarrow$ adult). Using the concept of a Relative fuzzy set, we can describe the life cycle as thus:

Let $Y$ be the set of periodical Cicada predator bird population, $T = [0, 160]$ and the membership values at egg, chick, and adult stages as 0.2, 0.6, 1 respectively. So a periodical Cicada predator bird membership growth values over time (life cycle time) is given as

$$
\mu_Y(y,t) = \begin{cases} 
0.2 & \text{egg } 0 \leq t \leq 4 & 4\text{weeks} \\
0.6 & \text{mating adult } 4 < t \leq 10 & 6\text{week} \\
1 & \text{dying adult } 10 < t \leq 160 & 150\text{weeks} 
\end{cases}
$$

(vii) A realtor wants to classify the houses he offers to his clients. Suppose the criteria is a comfortable and affordable house for an average income earning four-person family. Let $X = \{1, 2, 3, 4, \ldots, 10\}$ be the set of available types of houses described by $x$ number of bedrooms in a house and the prices $P = \{p_1, p_2, p_3\}$ are graded as "cheap ($p_1 = 100$ dollars), moderate ($p_2 = 500$ dollars) and expensive ($p_3 = 750$ dollars)". Then the Relative fuzzy function $\mu_X$ with the criteria "comfortable and affordable type of house for an average income earning four-
person family” can be defined as:

\[
\mu_X(x, p_i) = \begin{cases} 
\frac{10x}{p_i} & x \leq 4 \\
\frac{1}{25} \frac{p_i}{150} & x > 4
\end{cases} \quad i = 1
\]

\[
\begin{align*}
1 - \frac{125x}{p_i} & \quad 4 \leq x < 8 \\
2 - \frac{15x}{p_i} & \quad x \geq 8
\end{align*} \quad i = 2
\]

\[
\begin{align*}
1 - \frac{250x}{p_i} & \quad x \leq 3 \\
2 - \frac{250x}{p_i} & \quad 3 < x < 6
\end{align*} \quad i = 3
\]

\[
\begin{align*}
1 - \frac{350x}{p_i} & \quad 6 \leq x < 9 \\
2 - \frac{350x}{p_i} & \quad 9 \leq x < 12
\end{align*} \quad i = 4
\]

(viii) Let \( X = \{15, 20, 25, 30, 35, 40, 45\} \) be a set of room temperatures in \( ^0C \) in an experiment and room sizes in the interval \( S = [5, 70] \) in \( m^2 \). Suppose the criteria is “room temperature \( x \) is considerable higher than 25\(^0C \) with room size \( s \) approximately 26\( m^2 \)”. Then the fuzzy set

\[ X = \{(x, s), \mu_{X \times S}(x, s)) : (x, s) \in X \times S\} \]

characterized by

\[
\mu_X(x, s) = \begin{cases} 
0 & x \leq 25 \text{ and } |s - 26| \geq 15 \\
1 & x > 25 \text{ and } |s - 26| < 15
\end{cases}
\]

is a Relative fuzzy set.

(ix) The membership function of the relative fuzzy set of real numbers close to 1 relative to the interval \([1, 4]\), can be defined as \( \mu_R(x, t) = \exp(-\beta(x-1)^2) \) where \( \beta \) is a positive real number.

The Figures below are a graphical representation of the membership function of copies of \( \mu_R \) of \( x \) in the interval \([-1, 3]\) relative to \( t \in [1, 4] \) or a chosen \( \beta \)

![Figure 1: Copies of the membership function for \( x \in [-1, 3] \) w.r.t. \( t \) with \( \beta = 1.2 \).](image-url)
3.1.3 Remarks on Relative Fuzzy Set

(i) A Relative fuzzy set $R$ of $X$ characterized by $\mu : X \times T \to [0, 1]$ is a fuzzy set of $X$ if for each $x \in X$, $\mu(x, t)$ is constant for all $t \in T$ i.e membership level/value of an element is the same over the chosen time scale values.

(ii) While the time dependent fuzzy set defines fuzzy sets of a dynamic set (w.r.t. time), the Relative fuzzy set defines a dynamic fuzzy set (w.r.t. points of a time-scales interval) on any set.

(iii) The examples above are different from a fuzzy soft set as the Relative fuzzy set criteria is defined by one membership function.

3.2 Characterization of Relative Fuzzy Set

3.2.1 Ordering of Relative Fuzzy Sets

Let $T$ be a time scale and $R$ be a set of Relative fuzzy sets of sets of $X$. Let $R, R' \in R$ with $\mu_R : X \times T \to [0, 1]$ and $\mu_{R'} : X \times T \to [0, 1]$ respectively be Relative fuzzy sets. Then

(i) $R \leq_T R'$ iff $\mu_R(x, t) \leq \mu_{R'}(x, t)$ for all $x \in X$ and each $t \in T$.

(ii) $R \leq_t R'$ iff $\mu_R(x, t) \leq \mu_{R'}(x, t)$ for all $x \in X$ and a fixed $t \in T$.

3.2.2 Equality of Relative Fuzzy Sets

Let $X$ be a universal set and $R$ and $R'$ Relative fuzzy sets of $X$ with Relative fuzzy functions $\mu_R, \mu_{R'}$ respectively. Then $R$ and $R'$

(i) $R =_T R'$ iff $\mu_R(x, t) = \mu_{R'}(x, t)$ for all $x \in X$ and each $t \in T$.

(ii) $R =_t R'$ iff $\mu_R(x, t) \leq \mu_{R'}(x, t)$ for all $x \in X$ and a fixed $t \in T$.

From the Example 3.1.2(iii),

$R =_5 R'$

when $t = 5$ as

$\mu_R(x, 5) = \mu_{R'}(x, 5) = 1$

$\mu_R(y, 5) = \mu_{R'}(y, 5) = 0.13$

and

$\mu_R(z, 5) = \mu_{R'}(z, 5) = 1$

But $R \neq R'$ for all $t \in \{1, 2, 3, 4, 5\}$. 

Figure 2: The relative membership function for $x$ w.r.t. $t$ with $\beta = 1.2$. 
3.2.3 Subset

Let $X$ be a universal set and $R$ and $R'$ Relative fuzzy sets of $X$ with Relative fuzzy functions $\mu_R, \mu_{R'}$ respectively. Then

(i) $R \subseteq T \ R'$ iff $\mu_R(x, t) \leq \mu_{R'}(x, t)$ for all $x \in X$ and each $t \in T$.

(ii) $R \subseteq T \ R'$ iff $\mu_R(x, t)^F \leq \mu_{R'}(x, t)$ for all $x \in X$ and a fixed $t \in T$.

From Example 3.1.2(iii), $R \subseteq T \ R'$ at a membership growth rate interval $[1, 2, 3, 4, 5]$ as

$$\mu_R(x, t) \leq \mu_{R'}(x, t)$$

for all $x \in U$ and $t \in [1, 2, 3, 4, 5]$.

3.2.4 Self-subset of a Relative Fuzzy Set

Let $R$ be a Relative fuzzy set w.r.t $T = T$ with Relative fuzzy function

$$\mu_R : X \times T \rightarrow [0, 1].$$

Then for any $p \in T$, the Relative fuzzy set $R^p$ with the Relative fuzzy function

$$\mu_{R^p} : X \times [t_1, p] \rightarrow [0, 1]$$

is an upper self-subset of $R$ with respect to $p$ (denoted $R^p \subset R$).

Similarly, we define a lower self-subset $R_p$ of $R$ (denoted $R_p \subset R$) with respect to $p$ as a Relative fuzzy set with the Relative fuzzy function

$$\mu_{R_p} : X \times [p, t_2] \rightarrow [0, 1]$$

3.2.5 Remarks

(i) $R^p \cup R_p = R$ for any $p \in T$

(ii) $R^p \cap R_p = x$ such that $\mu_R(x, p) = \mu_{R^p}(x, p) = \mu_{R_p}(x, p)$ for any $p \in T$

(iii) $R^p \cap R^p \subset R \cap R'$

(iv) $R_p \cap R_p \subset R \cap R'$

(v) $(R^p \cap R_p) \cup R_p \cap R' = R \cap R'$

(vi) $R_p \cup R^p \subset R \cup R'$

(vii) $R_p \cup R' \subset R \cup R'$

3.2.6 The Lattice Structure for Relative Fuzzy Sets

Since

$$\mu_R(x, t)_{T} \leq \mu_R'(x, t)_{T}$$

if

$$\mu_R(x, t) \leq \mu_R'(x, t)$$

for all $x \in R$, each $t \in T$. Then

(i) $\mu_R(x, t)_{T} \lor \mu_R'(x, t)_{T}$

if

$$\mu_R(x, t) \lor \mu_R'(x, t)$$

for all $x \in R$, each $t \in T$ and
(ii) 
\[ \mu_R(x,t) \land \mu_R'(x,t) \]
if
\[ \mu_R(x,t) \land \mu_R'(x,t) \]
for all \( x \in R \) and \( t \in T \)

Thus the De Morgan algebra for the set of all Relative fuzzy subsets of \( S \) is given as
\[
\begin{align*}
(R \land R')_{(x,t)} &= R_{(x,t)} \land R'_{(x,t)} \\
(R \lor R')_{(x,t)} &= R_{(x,t)} \lor R'_{(x,t)} \\
R^c_{(x,t)} &= (R_{(x,t)})^c
\end{align*}
\]

3.2.7 Fuzzy Support of Relative Fuzzy Sets

Let \( R \) be a Relative fuzzy subset of \( X \). Then the support of \( R \) at

(i) \( t \) denoted \( \text{supp}(R_t) \) is the crisp subset of \( X \) whose elements have nonzero membership grades in \( R \) i.e
\[
\text{supp}(R_t) = \{ x \in X : \mu_R(x,t) > 0, \text{ for any } t \}
\]

(ii) \( t \) for all \( t \leq t_a \in T \) denoted \( \text{supp}(R\leq t_a) \) is the crisp subset of \( X \) whose elements have nonzero membership grades in \( R \) for all \( t \leq t_a \) i.e
\[
\text{supp}(R\leq t_a) = \{ x \in X : \mu_R(x,t) > 0, \text{ for all } t \leq t_a \}
\]

(iii) \( T \) denoted \( \text{supp}(R_T) \) is the crisp subset of \( X \) whose elements have nonzero membership grades in \( R \) for all \( t \in T \) i.e
\[
\text{supp}(R_T) = \{ x \in X : \mu_R(x,t) > 0, \text{ for all } t \in T \}
\]

3.2.8 Remark

For any Relative set \( R \) of \( X \), \( \alpha, \beta \in [0,1] \) and any \( t, t_a \in T \), then we have that

(i) \( \text{supp}(R_T) \subseteq \text{supp}(R_t) \)

(ii) \( \text{supp}(R_T) \subseteq \text{supp}(R\leq t_a) \)

(iii) \( \text{supp}(R\leq t_a) \subseteq \text{supp}(R_t) \) if \( t_a \geq t \)

3.2.9 \( \alpha \)-level sets of Relative Fuzzy Sets

Let \( R \) be a Relative fuzzy subset of \( X \). Then

(i) the \( \alpha \)-level sets for the Relative fuzzy function \( \mu_R \) at a membership growth rate \( t \) is defined as
\[
R(\alpha)_t = \{ x \in X : \mu_R(x,t) \geq \alpha, \text{ for any } t \}
\]
where \( \alpha \in (0,1] \)

(ii) the \( \alpha\leq t_a \)-level sets for the Relative fuzzy function \( \mu_R \) at a membership growth over \( t \leq t_a \) is defined as
\[
R(\alpha)\leq t_a = \{ x \in X : \mu_R(x,t) \geq \alpha, \text{ for all } t \leq t_a \}
\]
where \( \alpha \in (0,1] \) and \( t_a \in T \)
(iii) the $\alpha_T$-level sets for the Relative fuzzy function $\mu_R$ at a membership growth over $\mathbb{T}$ is defined as

$$R(\alpha)_T = \{x \in X : \mu_R(x,t) \geq \alpha, \text{ for all } t \in \mathbb{T}\}$$

where $\alpha \in (0,1]$

From Example 3.1.2(iii),

$$R(0.75)_3 = \{\}$$

and

$$R'(0.84)_1 = \{z\}$$

$R(0.025)_{[1,2,3,4,5]} = \{x,y,z\}$

and

$$R(0.2)_{[1,2,3,4,5]} = \{x,z\}$$

3.2.10 Remark

For any Relative set $R$ of $X$, $\alpha, \beta \in [0,1]$ and any $t \in \mathbb{T}$, then we have that

(i) $R(\alpha)_T \subseteq R(\alpha)_t$

(ii) $R(\alpha)_T \subseteq R(\alpha)_{\leq t_a}$

(iii) $R(\alpha)_{\leq t_a} \subseteq R(\alpha)_t$ if $t_a \geq t$

(iv) $R(\alpha)_T \subseteq supp(R_T)$

(v) $R(\alpha)_T \subseteq supp(R_{t_a})$

(vi) $R(\beta)_T \subseteq R(\alpha)_T$ if $\alpha \leq \beta$

(vii) $R(\beta)_{\leq t} \subseteq R(\alpha)_{\leq t_a}$ if $\alpha \leq \beta$

3.2.11 $\alpha_T$ function-level sets of a Relative Fuzzy Set

Let $R$ be a Relative fuzzy subset of $X$ over $\mathbb{T}$ and $\alpha : \mathbb{T} \to (0,1]$ such that for any $t \in \mathbb{T}$, $\alpha(t) \leq 1$. Then the $\alpha_T$ function-level sets for the Relative fuzzy function $\mu_R$ at a membership growth rate $t$ is defined as

$$R(\alpha(t))_t = \{x \in X : \mu_R(x,t) \geq \alpha(t), \text{ for any } t\}$$

Suppose in Example 3.1.2(viii), above we define a function $\alpha : S \to [0,1]$ such that $\alpha(x) = \frac{1}{x}$. Then we have that $27 \in R(\alpha(30))_{30}$ as $\mu_{\chi_S}(27,30) > \alpha(30)$

3.2.12 $\alpha_T$ function-level sets of a Relative Fuzzy Set

Let $R$ be a Relative fuzzy subset of $X$ over $\mathbb{T}$ and $\alpha : \mathbb{T} \to (0,1]$ such that for any $t \in \mathbb{T}, \alpha(t) \leq 1$. Then the $\alpha_T$ function-level sets for the Relative fuzzy function $\mu_R$ at a membership growth interval rate $T$ is defined as

$$R(\alpha(t))_T = \{x \in X : \mu_R(x,t) \geq \alpha(t), \text{ for all } t \in \mathbb{T}\}$$

From Example 3.1.2(iii), let $\alpha : [1,2,3,4,5] \to (0,1]$ such that for any $t \in [1,2,3,4,5], \alpha(t) = |0.9 - \frac{1}{t}|$,

$$R(\alpha(t))_{[1,2,3,4,5]} = R(0.1,0.4,0.66,0.65,0.7)_{[1,2,3,4,5]} = \{x,z\}$$
3.2.13 Remark
(i) If \( \alpha(t) = \alpha \in (0, 1] \) for all \( t \in T \) where \( \alpha : T \to (0, 1] \), and \( R \) be a Relative fuzzy subset of \( X \) over \( T \), then
\[
R(\alpha(t))_T = R(\alpha)_T
\]
(ii) If \( \alpha(t) \leq \alpha \in (0, 1] \) for all \( t \in T \) where \( \alpha : T \to (0, 1] \), and \( R \) be a Relative fuzzy subset of \( X \) over \( T \), then
\[
R(\alpha)_T \subset R(\alpha(t))_T
\]
(iii) \( R(\alpha)_t = R(\alpha(t))_t \) if \( \alpha(t) = \alpha \) for any \( t \)

3.2.14 \( \alpha \)-level point of a Relative Fuzzy Set
Let \( R \) be a Relative fuzzy subset of \( X \).
(i) Let \( \mu_R \) the Relative membership function such that \( \mu_R(x, t) \neq \mu_R(y, t) \) for any \( t \) with \( x \neq y \) for all \( x, y \in X \). Then the \( \alpha_i \)-level point for the Relative fuzzy function \( \mu_R \) at a membership growth rate \( t \) is the element \( x \in X \) such that \( \mu_R(x, t) = \alpha \), for any \( t \) where \( \alpha \in (0, 1] \).
(ii) Let \( \mu_R \) the Relative membership function such that \( \mu_R(x, t) \neq \mu_R(y, t) \) for all \( t \in T \) with \( x \neq y \) for all \( x, y \in X \). Then the \( \alpha_i \)-level point for the Relative fuzzy function \( \mu_R \) at a membership growth over \( T \) is the element \( x \in X \) such that \( \mu_R(x, t) = \alpha \) for any \( t \in T \) where \( \alpha \in (0, 1] \).

From Example 3.1.2(iii), \( R(0.13)_4 = y, R'(0.84)_0 = z, R(0.025)_{1,2,3,4,5} = \emptyset \) and \( R(0.2)_{1,2,3,4,5} = \emptyset \).

3.2.15 \( \alpha \)-level function-level point of a Relative Fuzzy Set
Let \( R \) be a Relative fuzzy subset of \( X \) over \( T \) and \( \rho : T \to (0, 1] \) such that for any \( t \in T, \alpha(t) \leq 1 \). Then the \( \alpha_i \)-level point for the Relative fuzzy function \( \mu_R \) at a membership growth rate \( t \) such that \( \mu_R(x, t) \neq \mu_R(y, t) \) for any \( t \in T \) with \( x \neq y \) for all \( x, y \in X \), is the element \( x \in X \) such that \( \mu_R(x, t) = \alpha(t) \) for any \( t \in T \).

From Example 3.1.2(iii), let \( \alpha : [1, 2, 3, 4, 5] \to (0, 1] \) such that for any \( t = 4 \), \( \alpha(t) = |0.9 - \frac{1}{t}| \). Then \( R(\alpha(2))_2 = y \)

3.2.16 \( \alpha \)-level function-level point of a Relative Fuzzy Set
Let \( R \) be a Relative fuzzy subset of \( X \) over \( T \) and \( \alpha : T \to (0, 1] \) such that for any \( t \in T, \alpha(t) \leq 1 \). Then the \( \alpha \)-level function-level point for the Relative fuzzy function \( \mu_R \) at a membership growth over \( T \) such that \( \mu_R(x, t) \neq \mu_R(y, t) \) for all \( t \in T \) with \( x \neq y \) for all \( x, y \in X \), is the element \( x \in X \) such that \( \mu_R(x, t) = \alpha(t) \), for all \( t \in T \).

From Example 3.1.2(iii), let \( \alpha : [1, 2, 3, 4, 5] \to (0, 1] \) such that for any \( t \in [1, 2, 3, 4, 5], \alpha(t) = |0.9 - \frac{1}{t}|, R(\alpha(t))_{1,2,3,4,5} = \emptyset \)

3.2.17 Smallest \( \alpha \)-level Set
The smallest \( \alpha \)-level set for the Relative fuzzy function at a membership growth over \( T \) is defined as
\[
SR(\alpha)_T = \bigcap_{T=t}^{T} \{ x \in X : \mu_R(x, t) \geq \alpha \text{ for any } t \}
\]
where \( \alpha \in (0, 1] \)

3.2.18 Largest \( \alpha \)-level Set
The largest \( \alpha \)-level set for the Relative fuzzy function at a membership growth over \( T \) is defined as
\[
LR(\alpha)_T = \bigcup_{T=t}^{T} \{ x \in X : \mu_R(x, t) \geq \alpha \text{ for any } t \}
\]
where \( \alpha \in (0, 1] \)

3.2.19 Remark
(i) \( R(\alpha)_T \subset SR(\alpha)_T \subset LR(\alpha)_T \)
3.2.20 Example

Let \( X = \{ x : -1 \leq x \leq 10, x \in \mathbb{Z}\} \) and \( T = \{ 3, 4, 5 \} \) and \( \sigma : T \to T \) a forward difference operator. Define a Relative fuzzy set \( R \) with a Relative fuzzy function 

\[
\mu_X : \mathbb{Z} \times T \to [0, 1]
\]

such that

\[
\mu_R(x, t) = \frac{1}{|\sigma(t) + x|}
\]

for all \( x \in X, \mu_X(x, t) \in [0, 1] \). The table below shows the Relative fuzzy set

| \( X \) | \( T \) | \( \mu_R \) | \( X \) | \( T \) | \( \mu_R \) |
|---|---|---|---|---|---|
| -1 | 3 | 0.5 | 5 | 3 | 0.125 |
| 4 | 0.33 | 4 | 4 | 0.11 | 5 | 0.25 |
| 5 | 0.25 | 5 | 0.1 | |
| 0 | 3 | 0.33 | 5 | 3 | 0.11 |
| 4 | 0.25 | 4 | 0.1 | 5 | 0.09 |
| 5 | 0.2 | 5 | 0.09 |
| 1 | 3 | 0.25 | 7 | 3 | 0.1 |
| 4 | 0.2 | 4 | 0.09 | 5 | 0.08 |
| 5 | 0.17 | 5 | 0.08 |
| 2 | 3 | 0.2 | 8 | 3 | 0.09 |
| 4 | 0.17 | 4 | 0.08 | 5 | 0.07 |
| 5 | 0.14 | 5 | 0.07 |
| 3 | 3 | 0.17 | 9 | 3 | 0.08 |
| 4 | 0.14 | 4 | 0.07 | 5 | 0.07 |
| 5 | 0.126 | 5 | 0.07 |
| 4 | 3 | 0.14 | 10 | 3 | 0.07 |
| 4 | 0.12 | 4 | 0.07 | 5 | 0.06 |
| 5 | 0.11 | 5 | 0.06 |

i.e

(i) fuzzy set \( R \) of \( X \) for \( t = 3 \), is

\[
R_3 = \{ (-1, 0.5), (0, 0.33), (1, 0.25), (2, 0.2), \\
(3, 0.17), (4, 0.14), (5, 0.125), (6, 0.11), (7, 0.1), \\
(8, 0.09), (9, 0.08), (10, 0.07) \}
\]

(ii) \( \alpha \)-level sets for the Relative fuzzy function \( \mu_R \) at a membership growth rate 3 and \( \alpha = 0.15 \) is

\[
R(0.15)_3 = \{ -1, 0, 1, 2, 3 \}
\]

(iii) fuzzy set \( R \) of \( X \) for \( t = 4 \), is

\[
R_4 = \{ (-1, 0.33), (0, 0.25), (1, 0.2), (2, 0.17), \\
(3, 0.14), (4, 0.12), (5, 0.11), (6, 0.1), (7, 0.09), \\
(8, 0.08), (9, 0.07), (10, 0.07) \}
\]

(iv) \( \alpha \)-level sets for the Relative fuzzy function \( \mu_R \) at a membership growth rate 4 and \( \alpha = 0.15 \) is

\[
R(0.15)_4 = \{ -1, 0, 1, 2 \}
\]
(v) fuzzy set \( R \) of \( X \) for \( t = 5 \), is
\[
R_5 = \{(-1,0.25), (0,0.2)(1,0.17), (2,0.14),
(3,0.12), (4,0.11), (5,0.09), (7,0.08),
(8,0.07), (9,0.07), (10,0.06)\}
\]

(vi) \( \alpha \)-level sets for the Relative fuzzy function \( \mu_R \) at a membership growth rate 5 and \( \alpha = 0.15 \) is
\[
R(0.15)_5 = \{-1,0,1\}
\]

(vii) Relative fuzzy set is
\[
R_{[3,5]} = \{(-1,0.5), (-1,0.33), (-1,0.25), (0,0.33), (0,0.25),
(0,0.2), (1,0.25), (1,0.2), (1,0.17), (2,0.2), (2,0.17), (2,0.14), (3,0.17),
(3,0.14), (3,0.12), (4,0.14), (4,0.12), (4,0.11), (5,0.125), (5,0.11), (5,0.1),
(6,0.11), (6,0.1), (6,0.09), (7,0.1), (7,0.09), (7,0.08), (8,0.09), (8,0.08),
(8,0.07), (9,0.08), (9,0.07), (10,0.07), (10,0.07), (10,0.06)\}
\]

(viii)
\[
R(0.15)_{[3,5]} = \{-1,0,1\}
\]

(ix) the smallest and largest \( \alpha \)-level sets are
\[
SR(0.15)_{[3,5]} = \{-1,0,1\}
\]
and
\[
LR(0.15)_{[3,5]} = \{-1,0,1,2,3\}
\]
respectively.

(x) If \( \rho : [3,4,5] \to (0,1] \) such that \( \rho(t) = 0.4 - \frac{1}{t} \). Then
\[
R(\rho) = \{-1,0\}
\]

3.2.21 Proposition
Let \( U \) be a universal set, \( R \) Relative fuzzy sets of \( U \) and \( \alpha \in (0,1] \). If
(i) \( \mu_R \) is decreasing then
\[
R(\alpha)_{[5]} = SR(\alpha)_{T}
\]
and
\[
R(\alpha)_{[1]} = LR(\alpha)_{T}
\]
(ii) \( \mu_R \) is increasing then
\[
R(\alpha)_{[1]} = SR(\alpha)_{T}
\]
and
\[
R(\alpha)_{[5]} = LR(\alpha)_{T}
\]

Proof. Let \( x \in R(\alpha)_{[5]} \) then \( \mu_R(x,t_2) \geq \alpha \). If \( \mu_R \) is decreasing then \( \mu_R(x,t_1) \geq \mu_R(x,t) \geq \mu_R(x,t_2) \). This implies that \( x \in \bigcap_1^5 R(\alpha)_T \). So
\[
R(\alpha)_{[5]} \subseteq SR(\alpha)_{T}
\]
Let \( x \in SR(\alpha)_T \) then \( \mu_R(x, t) \geq \alpha \) for all \( t \in T \). Since \( \mu_R \) is decreasing then \( \mu_R(x, t_2) \geq \alpha \). So

\[
SR(\alpha)_T \subset R(\alpha)_{t_2}
\]

Thus 1a holds. Similarly (1b) and (2) hold.

### 3.3 Operations of Relative Fuzzy Set

#### 3.3.1 Complement

The complement of a Relative fuzzy set \( R \) of \( U \) at a membership growth rate \( t \) is the Relative fuzzy set \( R' \) with Relative fuzzy membership function \( \mu_{R'} \) defined as

\[
\mu_{R'}(x, t) = 1 - \mu_R(x, t)
\]

for all \( x \in U \) and a fixed \( t \).

The complement of a Relative fuzzy set \( R \) of \( U \) at a membership growth over \( T \) is the Relative fuzzy set \( R' \) with Relative fuzzy membership function \( \mu_{R'} \) defined as

\[
\mu_{R'}(x, t) = 1 - \mu_R(x, t)
\]

for all \( x \in U \) and \( t \in T \).

In Example 3.1.2(iv), above \( \mu_{X'}(x, 6) = 1 - \mu_X(x, 6) = 1 - 0.4 = 0.6 \) and

\[
\mu_{R'}(x, t) = \begin{cases} 
0.8 & \text{egg} \quad 0 < t \leq 3 \\
0.6 & \text{nymph} \quad 4 < t \leq 887 \\
0.2 & \text{mating adult} \quad 887 < t \leq 889 \\
0 & \text{dying adult} \quad 888 < t \leq 889
\end{cases}
\]

#### 3.3.2 Union

Let \( U \) be a universal set and \( R \) and \( R' \) Relative fuzzy sets of \( U \) with Relative fuzzy functions \( \mu_R, \mu_{R'} \) respectively. Then the union of \( R \) and \( R' \) at a membership growth rate \( t \) is the Relative fuzzy set with Relative membership function defined by

\[
\mu_{(R \cup R')_T}(x, t) = \max\{\mu_R(x, t), \mu_{R'}(x, t)\}
\]

for all \( x \in U \).

The union of \( R \) and \( R' \) at a membership growth over \( T \) is the Relative fuzzy set with Relative fuzzy function defined as

\[
\mu_{(R \cup R')_T}(x, t) = \max\{\mu_R(x, t), \mu_{R'}(x, t)\} : t \in T \subset T
\]

for all \( x \in U \).

From Example 3.1.2(iii), \( R \cup R' = R' \) at a membership growth rate interval \([1, 2, 3, 4, 5]\) as

\[
\mu_{(R \cup R')_{[1,2,3,4,5]}}(x, t) = \begin{cases} 
\max\{\mu_R(x, t), \mu_{R'}(x, t)\} & t \in [1, 2, 3, 4, 5] \\
\mu_{R'}(x, t) & t \in [1,2,3,4,5]
\end{cases}
\]

(3.1)

(3.2)

for all \( x \in U \).

#### 3.3.3 Maximum

Let \( U \) be a universal set and \( R \) Relative fuzzy sets of \( U \) with Relative fuzzy functions \( \mu_R \) respectively. Then the maximum of \( R \) at a membership growth over \( T \) is the Relative fuzzy set with Relative membership function defined by

\[
\max\{\mu_R\} = \max\{\mu_R(x, t), \mu_R(y, t)\}
\]

for all \( x, y \in U \) and \( t \in T \).

From Example 3.1.2(v), \( \max\{\mu_X\} \) represent the best option(s) of road to use at a particular time. It is given as
Then the sum of $R$ is defined as

$$\max \{ \mu_R \} = \min \{ \mu_R(x, t), \mu_R'(y, t) \}$$

for all $x, y \in U$ and $t \in \mathbb{T}$.

3.3.4 Intersection

Let $U$ be a universal set, $R$ and $R'$ Relative fuzzy sets of $U$ with Relative fuzzy functions $\mu_R$ and $\mu_R'$, respectively. Then the intersection of $R$ and $R'$ at a membership growth rate $t$ is the Relative fuzzy set with fuzzy function defined by

$$\mu_{(R \cap R')_t}(x, t) = \min \{ \mu_R(x, t), \mu_R'(x, t) \}$$

for all $x \in U$.

The intersection of $R$ and $R'$ at a membership growth over $\mathbb{T}$ is the Relative fuzzy set with a Relative fuzzy function defined as

$$\mu_{(R \cap R')_\mathbb{T}}(x, t) = \min \{ \mu_R(x, t), \mu_R'(x, t) \} : t \in \mathbb{T} \subset \mathbb{T}$$

for all $x \in U$.

From Example 3.1.2(iii), $R \cap R' = R$ at a membership growth rate interval $[1, 2, 3, 4, 5]$ as

$$\mu_{(R \cap R')_{[1,2,3,4,5]}}(x, t) = \min \{ \mu_R(x, t), \mu_R'(x, t) \} : t \in [1, 2, 3, 4, 5]$$

(3.3)

$$\mu_{R(x, t)_{[1,2,3,4,5]}} = \mu_R(x, t)$$

(3.4)

for all $x \in U$.

3.3.5 Sum

Let $X$ be a universal set and $R$ and $R'$ Relative fuzzy sets of $X$ with Relative fuzzy functions $\mu_R, \mu_R'$ respectively. Then the sum of $R$ and $R'$ at a membership growth rate $t$ is the Relative fuzzy set with a fuzzy function defined by

$$\mu_{(R + R')_t}(x, t) = \mu_R(x, t) + \mu_R'(x, t)$$

for any $x \in X$.

The sum of $R$ and $R'$ at a membership growth over $\mathbb{T}$ is the Relative fuzzy set with a Relative fuzzy function defined as

$$\mu_{(R + R')_\mathbb{T}}(x, t) = \max \{ \mu_R(x, t) + \mu_R'(x, t) : t \in \mathbb{T} \subset \mathbb{T} \}$$

for any $x \in U$.

Note that sum is defined when $\mu_R(x, t) + \mu_R'(x, t) \leq 1$ for any $x \in X$ and $t \in \mathbb{T}$.
From Example 3.1.2(iii),
\[ \mu_{(R+R')}^1(y, 2) = \mu_R(y, 2) + \mu_{R'}(y, 2) = 0.475 \]
but
\[ \mu_{(R+R')}^2(z, 2) = \mu_R(z, 2) + \mu_{R'}(z, 2) = 1.58 > 1 \]
for \( y \in X \).
Thus \( \mu_{(R+R')}^T(y, t) \) is not defined.

3.3.6 Absolute Difference

Let \( X \) be a universal set and \( R \) and \( R' \) Relative fuzzy sets of \( U \) with Relative fuzzy functions \( \mu_R, \mu_{R'} \) respectively. Then the absolute difference of \( R \) and \( R' \) at a membership growth rate \( t \) is the Relative fuzzy set with Relative fuzzy function defined by
\[ \mu_{(R-R')}^T(x, t) = |\mu_R(x, t) - \mu_{R'}(x, t)| \]
for any \( x \in X \).
The absolute difference of \( R \) and \( R' \) at a membership growth over \( T \) is defined as
\[ \mu_{(R-R')}^T(y, t) = \min\{ |\mu_R(y, t) - \mu_{R'}(y, t)| : t \in T \subset T \} \]
for any \( x \in X \). From Example 3.1.2(iii),
\[ \mu_{(R-R')}^0(y, 0) = |\mu_R(y, 0) - \mu_{R'}(y, 0)| = 0.640 \]
\[ \mu_{(R-R')}^1(y, 1) = |\mu_R(y, 1) - \mu_{R'}(y, 1)| = 0.48 \]
\[ \mu_{(R-R')}^2(y, 2) = |\mu_R(y, 2) - \mu_{R'}(y, 2)| = 0.325 \]
\[ \mu_{(R-R')}^3(y, 3) = |\mu_R(y, 3) - \mu_{R'}(y, 3)| = 0.16 \]
\[ \mu_{(R-R')}^4(y, 4) = |\mu_R(y, 4) - \mu_{R'}(y, 4)| = 0 \]
Thus \( \mu_{(R-R')}^T(y, t) = 0 \)

3.3.7 Multiple

Let \( U \) be a universal set and \( R \) and \( R' \) Relative fuzzy sets of \( U \) with Relative fuzzy functions \( \mu_R, \mu_{R'} \) respectively. Then the Relative fuzzy set \( R \) is said to a multiple of \( R' \) at membership growth rate \( t \) if for any \( x, y \in U \) we have that
\[ \mu_R(x, t) = \mu_{R'}(y, m) \]
where \( m \) is a multiple of \( t \).
Let \( m = at \), then we refer to \( a \) as the generation of equivalence.

Let \( U \) be a universal set and \( R \) and \( R' \) Relative fuzzy sets of \( U \) with Relative fuzzy functions \( \mu_R, \mu_{R'} \) respectively. Then the Relative fuzzy set \( R \) is said to a multiple of \( R' \) at membership growth over \( T \) if for any \( x, y \in U \) we have that
\[ \mu_R(x, t) = \mu_{R'}(y, m) \]
for all \( t \in T \) where \( m \) is a multiple of \( t \).
Let \( m = at \), for all \( t \in T \), then we refer to \( a \) as the generation of equivalence.

3.3.8 Example

In Examples 3.1.2(iv), and 3.1.2(vi), above, we have that the Relative fuzzy set \( Y \) is multiple of \( X \) at membership growth rate \( t = 156 \) as
\[ g(y, 156) = f(x, 780) \]
Now \( 780 = 5 \times (156) \). Thus, 5 is a generation of equivalence. So if an insect cicada and cicada’s predator bird are at egg state at a particular time then the insect cicada \( x \) will be available as prey for the fifth generation of the cicada’s predator bird \( y \).

3.3.9 Remark
(i) The Relative fuzzy functions are equal at the growth rate \( t \) iff \( a = 1 \)
(ii) The Relative fuzzy functions are equal at the growth rate interval \( T \) iff \( a = 1 \) for all \( t \in T \).

3.4 Limit Relative Fuzzy Membership Value

We can associate with each element of a Relative fuzzy set a limiting membership value if it exists. We call this the limit Relative fuzzy membership value for any element of the fuzzy set. This is essential as such value as a limit can approximately represent the growth membership values. This is actually more realistic instead of choosing a value in the set (membership values set) arbitrarily. So in the foregoing, we define a limiting membership value for the set of membership values in a set. Furthermore, we define \( \alpha \)-level set of the Relative fuzzy set \( R \) of \( X \).

3.4.1 Definition
Let \( R \) be a Relative fuzzy set of \( X \) over \( T \), with \( t_1, t_2, \ldots \), a sequence in \( T \). Then the limit Relative fuzzy value of an element \( x \in X \) is the value \( l_x \in [0, 1] \) such that for any \( k \in \mathbb{N} \), \( \mu_R(x, t_k) \to l_x \) as \( |t_i - t_k| \to 0 \) for any \( i > k, i \in \mathbb{N} \). We denote it as
\[
\lim_{|t_i - t_k| \to 0} \mu_R(x, t_k) = l_x
\]

3.4.2 Definition
Let \( R \) be a Relative fuzzy set of \( X \) over \( T \), and \( l_{x_1}, l_{x_2}, \ldots, l_{x_n} \), a sequence of limit Relative fuzzy value of the elements \( x_1, x_2, \ldots, x_n \in X \) for all \( x \in X \). We define \( \alpha \)-level set for \( R \) as
\[
R(\alpha)_{l_k} = \{ x \in X : l_x \geq \alpha, \alpha \in (0, 1] \}
\]

3.4.3 Example
Let \( R \subset \mathbb{R}^+ \), and \( T \subset \mathbb{R}, 2 \leq t \leq 10, 1 \leq x \leq 10 \). Define a Relative fuzzy set \( R \) by a Relative fuzzy function
\[
\mu_R : \mathbb{R}^+ \times T \to [0, 1]
\]

such that
\[
\mu_R(x, t) = \frac{1}{\sigma(t)x^2} + 0.4
\]

for all \( x \in R, \mu_R(x, t) \in (0, 1] \).
Then \( l_1 = 0.5 \) and \( l_{10} = 0.4 \).
Thus \( R(0.45) = [1, \frac{3}{2}] \)

If
\[
\mu_R(x, t) = \frac{1}{\sigma(t)x} + 0.4
\]

Then \( l_1 = 0.5 \) and \( l_{10} = 0.4 \).
But \( R(0.45) = [1, 2] \)
4 Applications

4.1 Application of Relative Fuzzy Set for Maintaining Room Temperature

An expert wants to design an equipment (with heater and cooler components) that ensures the room temperature specified by the user is maintained irrespective of external/internal temperature altering agent(s). A simple Relative fuzzy function that depends on the values of the room temperature and the user specified room temperature is useful in this type of equipment design. The equipment function (output) depends on the Relative membership function of the variables, where the Relative membership value 1 indicate maximum output and 0 indicate minimum or no output(i.e the equipment sleeps). This mechanism re-detects room temperature periodically and repeats the process. Equipment over-use, energy waste and user discomfort amongst others are avoided.

For example given possible room temperatures $X$ in the interval $[x_1, x_2]$ and user choice of room temperature $T$ in the interval $T$. We can represent the equipment output with a membership function $m_X: X \times T \rightarrow [0, 1]$ such that

$$\mu_X(x,t) = \frac{|x-t|}{x_2 - x_1}$$

where $\mu_X(x,t) = 0$ indicate minimum output and $\mu_X(x,t) = 1$ indicates maximum output for all $x \in [x_1, x_2]$ and $t \in T$. Obviously, the time-dependent and nonstationary fuzzy sets concept in literature are not suitable for this equipment design.

Suppose the criteria is to maintain user/choice room temperature given the prevailing room temperature. Then the Relative membership function of room temperature $x \in [x_1, x_2]$ with a specified/choice room temperature $t_i \in T$ differs from the Relative membership function of the same room temperature but with a specified/choice room temperature of $t_j \in T$ ($t_i \neq t_j$). We develop an algorithm for the behavior of the equipment using the Relative fuzzy set concept.

4.2 An Algorithm for Room Temperature Maintenance Equipment Using the Concept of Relative Fuzzy Set

(i) Begin

(ii) Input detection period $p$

(iii) Input desired room temperature $t$

(iv) Assign $p$ value to $d$

(v) Detect room temperature $x$

(vi) If $x = t$, sleep or no output

(vii) Elseif $x > t$ compute $\mu_X(x,t)$, Cooler works to the grade $\mu_X(x,t)$

(viii) Elseif $x < t$ compute $\nu_X(x,t)$, Heater works to the grade $\nu_X(x,t)$

(ix) Compute $d = d - 1$

(x) If $d > 0$ goto 5

(xi) Else goto 4

(xii) Stop
4.3 Application of Relative Fuzzy Set in Drug Prescription

In medicine, disease diagnosis and drug prescription usually depend on available symptoms and/or test results. Suppose the effectiveness of certain drugs for treating a particular illness depends on a number of factors like age, weight, sex, blood group, genotype etc., then effective diagnosis and prescription is needed. To aid the medical personnel a Relative fuzzy rule based logic system can be designed. Below is a simple suggested algorithm

(i) Input variables like age, weight, time of the day, genotype etc. of patients.

(ii) Testing: the software compares input variables using an IF-THEN statement using the Relative fuzzy rule based logic

(iii) Result: based on the result of (2) above, the software suggest the appropriate drug to the consultant

For an effective design of the software program/algorithm, an expert/observed knowledge base is required. Also updating the knowledge base is very important.

Example

A survey was carried out that showed the response to two kinds of drugs (D1 and D2) suitable for treating a particular illness (N) in patients with different genotype (AA, AS and SS) and age ranges (0 – 10, 11 – 20, 21 – 30, 31 – 40, 41 – 50 and 51 – 60). The table below gives the data of the average amount in mg of disease parasite remaining in patients blood after treatment with the drugs out of about 0.0078 mg discovered during diagnosis.

**After Treatment with Drug D1**

| G/A | 0 – 10 | 11 – 20 | 21 – 30 | 31 – 40 | 41 – 50 | 51 – 60 |
|-----|--------|---------|---------|---------|---------|---------|
| AA  | 0.0028 | 0.0026  | 0.001   | 0.0017  | 0.0012  | 0.0003  |
| AS  | 0.0015 | 0.0022  | 0.0015  | 0.0013  | 0.0001  | 0       |
| SS  | 0.00012| 0.0004  | 0.0016  | 0.0017  | 0.005   | 0.0002  |

**After Treatment with Drug D2**

| G/A | 0 – 10 | 11 – 20 | 21 – 30 | 31 – 40 | 41 – 50 | 51 – 60 |
|-----|--------|---------|---------|---------|---------|---------|
| AA  | 0.0013 | 0.0025  | 0.0001  | 0.0034  | 0.0002  | 0.0055  |
| AS  | 0.0065 | 0       | 0.0005  | 0.0023  | 0.0001  | 0.0002  |
| SS  | 0.00011| 0.0003  | 0.0011  | 0       | 0.0005  | 0.006   |

We can represent the obtained data using the Relative fuzzy set idea by computing the Relative fuzzy function for each combination (genotype/age) where \( f(g_1, a_1) = 1 \) indicates maximum parasite level in the body while \( f(g_1, a_1) = 0 \) indicates minimum parasite level in the body of patients with genotype \( g_1 \) and age range \( a_1 \).

The following rule based logic system is developed to identify which of the two drugs can be prescribed for a patient if the threshold of the illness is 0.0018 mg

(i) Begin

(ii) \( i = 1 \) to 3

(iii) \( j = 1 \) to 6

(iv) If \( \mu_{D1}(G = g, and A = a_j) \leq \text{threshold} \) then suggest drug D1

(v) \( j = j + 1 \) goto 3

(vi) Else if \( \mu_{D2}(G = g, and A = a_j) \leq \text{threshold} \) then suggest drug D2

(vii) If \( i = i + 1 \leq 3 \) goto 2

(viii) Else stop
Using the Relative fuzzy rule above for the data, we have the prospective prescription of drug D1 and drug D2 as shown in the tables below:

| G/A | 0 – 10 | 11 – 20 | 21 – 30 | 31 – 40 | 41 – 50 | 51 – 60 |
|-----|--------|---------|---------|---------|---------|---------|
| AA  | N      | N       | Y(0.001) | Y(0.0017) | Y(0.0012) | Y(0.0003) |
| AS  | Y(0.0015) | N       | Y(0.0015) | Y(0.0013) | Y(0.0001) | Y(0) |
| SS  | Y(0.00012) | Y(0.0004) | Y(0.0016) | Y(0.0017) | N | Y(0.0002) |

respectively.

The Relative fuzzy rule for the final prescription is

(i) \( i = 1 \) to 3

(ii) \( j = 1 \) to 6

(iii) If \( \mu_{D1}(G = g, and A = a_j) = \mu_{D2}(G = g, and A = a_j) \) prescribe D1 or D2

(iv) Else If \( \min\{\mu_{D1}(G = g, and A = a_j), \mu_{D2}(G = g, and A = a_j)\} = \mu_{D1}(G = g, and A = a_j), \mu_{D2} \) prescribe D1

(v) Else If \( \min\{\mu_{D1}(G = g, and A = a_j), \mu_{D2}(G = g, and A = a_j)\} = \mu_{D2}(G = g, and A = a_j), \mu_{D2} \) prescribe D2

(vi) IF \( j = j + 1 \leq 6 \) goto 2

(vii) Else If \( i = i + 1 \leq 3 \) goto 1

(viii) Else stop

The table below shows the final prescription

| G/A | 0 – 10 | 11 – 20 | 21 – 30 | 31 – 40 | 41 – 50 | 51 – 60 |
|-----|--------|---------|---------|---------|---------|---------|
| AA  | D2     | None    | D2      | D1      | D2      | D1      |
| AS  | D1     | D2      | D2      | D1/D2   | D1      |
| SS  | D2     | D2      | D2      | D2      | D1      |

4.4 Application of Relative Fuzzy Set in Fuzzy Optimization problems

The fuzzy set of the objective function "attractive dividend relative to units of shares" could, for instance, be defined by:

\[
\mu_{o}(x,u) = \begin{cases} 
1 & x \geq 2, u \geq \max\{\mu_{o}(x) < 1\} + 0.1 \\
-\frac{9u+0.9}{2100} (29x^3 - 366x^2 - 877x + 540) & 1 < x < 5, u \in [0, 0.5] \\
0 & x \leq 0.5 
\end{cases}
\]

while the fuzzy set (constraint) "modest dividend" could be represented by

\[
\mu_{c}(x) = \begin{cases} 
1 & x \leq 1 \\
\frac{1}{2100} [-29x^3 - 243x^2 + 16x + 2388] & 1.2 < x < 6 \\
0 & x \geq 2.8 
\end{cases}
\]
The relative fuzzy set "decision" at any \( u \) is then characterized by its membership function

\[
\mu_D(x, u) = \min \{ \mu_o(x, u), \mu_c(x) \}
\]

i.e the fixed fuzzy point (optimal value) w.r.t. \( \alpha \) is \( x_{\alpha} \) such that \( \max \mu_D(x, u) = \alpha \) and the relative fuzzy set "decision" over \( U \) is then characterized by its membership function

\[
\mu_D(x, U) = \min \{ \mu_o(x, u), \mu_c(x) \}
\]

for all \( u \in U \). i.e the fixed fuzzy points (optimal values) w.r.t. \( \alpha \) is the set \( x_{\alpha} \) such that \( D^\alpha_u = \max \mu_D(x, u) = \alpha \), for each \( u \in U \). Therefore the fixed fuzzy point (optimal value) w.r.t. \( \alpha \) for all \( x \) and all \( u \in U \) is \( x_{\alpha}^* \) such that \( D^\alpha_{x^*} \) is the minimum of the \( D^\alpha_u \)'s.

![Figure 3: The relative fuzzy set "decision".](image)

5 Conclusion

In this study, we introduce an evolutionary fuzzy set, the Relative fuzzy set distinct from existing fuzzy set and fuzzy set related concepts in literature. It is a fuzzy set of a set characterized by a growth membership value which depends on the set value and a time scale value. Our time scale value is any variable (scalar quantity e.g age, temperature, time e.t.c. or even linguistic) and not necessarily time. Its properties are defined and discussed. Examples are provided conceptually and practically. Finally, possible ways of applications are presented.

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