Voter model under stochastic resetting

Pascal Grange

Division of Natural and Applied Sciences and Zu Chongzhi Center for Mathematics and Computational Science, Duke Kunshan University, 8 Duke Avenue, Kunshan 215316, Jiangsu, People’s Republic of China

E-mail: pascal.grange@dukekunshan.edu.cn

Received 7 August 2022; revised 7 November 2023
Accepted for publication 10 November 2023
Published 21 November 2023

Abstract

The voter model is a toy model of consensus formation based on nearest-neighbor interactions. A voter sits at each vertex in a hypercubic lattice (of dimension $d$) and is in one of two possible opinion states. The opinion state of each voter flips randomly, at a rate proportional to the fraction of the nearest neighbors that disagree with the voter. If the voters are initially independent and undecided, the model is known to lead to a consensus if and only if $d \leq 2$. In this paper the model is subjected to stochastic resetting: the voters revert independently to their initial opinion according to a Poisson process of fixed intensity (the resetting rate). This resetting prescription induces kinetic equations for the average opinion state and for the two-point function of the model. For initial conditions consisting of undecided voters except for one decided voter at the origin, the one-point function evolves as the probability of presence of a diffusive random walker on the lattice, whose position is stochastically reset to the origin. The resetting prescription leads to a non-equilibrium steady state. For an initial state consisting of independent undecided voters, the density of domain walls in the steady state is expressed in closed form as a function of the resetting rate. This function is differentiable at zero if and only if $d \geq 5$.

Keywords: voter model, out-of-equilibrium systems, stochastic resetting, interacting particle systems

(Some figures may appear in colour only in the online journal)
1. Introduction

The voter model is a toy model of consensus formation in a population. Each vertex of a hypercubic lattice of dimension \( d \) is occupied by a voter, who is in one of two possible opinion states, say \(-1\) and \(+1\). Each voter flips its opinion state at a rate proportional to the fraction of its neighbors that disagree with it. In particular, if the system reached a configuration in which all voters are in the same opinion state, the evolution stops (consensus is reached). The following two problems are particularly natural to consider:

(I) For an initial condition with undecided voters, except one with a positive opinion, how does the opinion state of the initially decided voter evolve in time? Does it relax to the undecided state? To solve this problem, one needs to calculate the one-point function of the model.

(II) For an initial condition consisting of independent undecided voters, is consensus achieved in the long-time limit? To characterize how far from consensus the system is, one needs to estimate the density of domain walls.

In a given configuration, domain walls are quasi-particles sitting at the bond between adjacent vertices occupied by voters in opposite opinion states. For example, in a configuration with independent undecided voters, the opinion states of the voters are independent centered Bernoulli random variables, and the density of domain walls is \( 1/2 \) because two voters at adjacent vertices are in opposite opinion states with probability \( 1/2 \). On the other hand, the density
of domain walls is zero if there is consensus. It is enough to calculate the two-point function of the model to obtain the density of domain walls.

Quite remarkably, Problems (I) and (II) can be solved exactly \([1, 2]\) in any dimension \(d\) (for reviews, see \([3, 4]\) and chapter 8 of \([5]\)). The asymptotic behavior of the one-point function implies that the opinion state of a single decided voter at the origin relaxes to the undecided opinion state: the average opinion state at the origin goes to zero (as \(d/(2\pi t)^{d/2}\) when time \(t\) goes to infinity. For initial conditions corresponding to undecided voters, the system is translationally invariant, so the density of domain walls depends only on time. In \([2]\), the large-time limit of the density was calculated and shown to be zero if and only if \(d \geq 2\) (it decays to zero like the inverse square-root of time in dimension one, and like an inverse logarithm in dimension 2). In higher dimension the limit is strictly positive and expressed in closed form in terms of integrals involving Bessel functions.

In the voter model, individuals are collaborative: they flip their opinion states based solely on the opinion states of their neighbors. In this paper we modify the model to take into account a certain degree of stubbornness in the individuals. We do so by subjecting the voter model to stochastic resetting: we assume voters keep memory of their initial opinion states and revert to it (independently) at random times. For each voter, these times are generated by a Poisson process of intensity \(r\) (the quantity \(r\) is called the resetting rate). The Poisson processes attached to different voters are independent. This kind of resetting prescription, according to which the degrees of freedom in the model are reset independently to their initial value, has been termed local resetting in interacting particle systems \([6]\). More generally, stochastic resetting has received a considerable amount of attention in out-of-equilibrium physics. Indeed, stochastic resetting holds a system away from its long-time stationary state, leading to a non-equilibrium stationary state. The corresponding stationary distribution was first calculated exactly for a diffusive random walker \([7, 8]\). Subsequent developments yielded exact results on the non-equilibrium stationary state of a variety of single-particle dynamics under resetting, including diffusion in potentials \([9]\), Lévy flights \([10]\) and active particles \([11–13]\) (for a review, see \([14, 15]\)). Developments on systems with interacting degrees of freedom under local resetting include models of binary aggregation \([16]\) and exclusion processes \([6, 17]\). In the other broad class of resetting dynamics (global resetting), a system of interacting particles is reset to its initial configuration at Poisson-distributed time. In \([18]\), the Ising model was subjected to stochastic resetting to a ferromagnetic state, which allowed to study the phases of the model (for a field-theoretic treatment of the Ising magnet reset to a paramagnetic state, see \([19]\)). Global resetting has also been studied for exclusion processes \([20–22]\), as well as predator-prey models \([23–25]\), fluctuating interfaces \([26, 27]\), synchronization \([28]\), reaction-diffusion processes \([29]\), glassy systems \([30]\) and zero-range processes \([31, 32]\). For a review of developments on interacting particle systems under stochastic resetting, see \([33]\) and references therein.

Given the wide spectrum of systems for which non-equilibrium stationary states under stochastic resetting can be worked out exactly, and given the simplicity of the dynamics of the voter model, we expect to be able to work out one- and two-point functions in the voter model under stochastic resetting. Intuitively, stochastic resetting should prevent the opinion state of a single initially decided voter to relax to the general undecided state. Moreover, if there is no consensus in the initial configuration, the dynamics of the voter model under resetting does not stop when consensus is reached. Indeed some of the voters will revert to their initial opinion state and break the consensus. We are therefore led to the following two problems, the analogs of Problems (I) and (II) under stochastic resetting:
For an initial condition with undecided voters, except one with a positive opinion at the site labeled $0$ in the lattice, calculate the average opinion state $S(0, \infty)$ of the initially decided voter in the steady state, as a function of the resetting rate $r$. 

For an initial condition consisting of independent undecided voters, calculate the average density of domain walls $\rho_\infty$ in the steady state.

The paper is organized as follows. In section 2 we define and study the model in dimension one. We define the flipping rate of the voter model under stochastic resetting and work out the evolution equation of the average opinion-state profile $S(x, t) = \langle s(x, t) \rangle$. We solve it in Fourier space. The evolution equation of the one-point function with a single initially decided is found to be closely related to the diffusion equation for a random walker on the lattice, with stochastic resetting to the origin. This yields the solution of Problem (I($r$)) in dimension one. We work out the evolution equation of the two-point function, and notice that the resetting prescription induces terms proportional to the two-time two-point function $\langle s(x, t) s(y, 0) \rangle$. However, at fixed $y$, this function satisfies the same evolution equation as the average opinion state with a single decided voter. The steady-state density of domain walls (the solution of Problem (II($r$)) in dimension one) is obtained in closed form by adapting the techniques of [2]. In section 3 we generalize the results to higher dimension and work out the behavior at low resetting rate of the density of domain walls in the steady state. Some of the mathematical details of the derivations are presented in appendices.

2. The voter model under stochastic resetting in dimension one

2.1. Definitions and notations

The one-dimensional voter is defined as follows. At each vertex of a one-dimensional lattice sits a voter who can be in one of two opinion states, say $+1$ and $-1$. At time $t$, the opinion state of the voter at position $x$ is denoted by $s(x, t)$, so that the configuration of the entire system is given by the collection

$$\{s(x, t), x \in aZ\} \in \{-1, +1\}^a, \quad (1)$$

where $a$ denotes the lattice spacing. The opinion state of each voter flips at a rate proportional to the fraction of nearest neighbors that disagree with it. Let us denote the proportionality constant by $\tau^{-1}$ (the time $\tau$ sets the time scale of the system). If the voter at site $na$ is in a given opinion state, its opinion flips:

(i) at a rate of $\tau^{-1}$ if both of its neighbors are in the opposite opinion state,
(ii) at a rate of $\tau^{-1}/2$ if exactly only one of its neighbors is in the opposite opinion state,
(iii) at zero rate if none of its neighbors are in the opposite opinion state.

All the possible cases are listed in table 1. The values of the flipping rate $W(na, t)$ displayed in the third column of the table are reproduced by the formula

$$W(na, t) := \frac{\tau^{-1}}{2} \left[ 1 - \frac{1}{2} s(na, t) \left( s((n + 1)a, t) + s((n - 1)a, t) \right) \right], \quad (n \in Z). \quad (2)$$
Indeed the expression between square brackets takes the values 2, 1, 0 in the cases (i)–(iii) respectively, because of the constraint \( s(na,t)^2 = 1 \). The voters are cooperative in the sense that they change their opinion state based on the opinion states of their neighbors, without taking into account their own initial opinion state. We will refer to the model with flipping rates \( W \) as the ordinary voter model (solved in [1, 2]).

Let us subject the voter model to stochastic resetting as follows. We assume each voter reverts to its initial opinion state at Poisson-distributed times. A Poisson process of intensity denoted by \( r \) (the resetting rate) is attached to the site labeled \( na \). It generates resetting times. At each resetting time, the variable \( s(na,t) \) is reset to its initial value \( s(na,0) \), if it differs from this initial value (otherwise it does not change). The Poisson processes attached to different sites are independent. Hence, between times \( t \) and \( t + dt \), the opinion state \( s(na,t) \) reverts to \( s(na,0) \) with probability \( r dt \) if \( s(na,t) \neq s(na,0) \). This resetting prescription induces an additional flipping rate at site \( na \) and time \( t \), denoted by \( R(na,t) \). The possible values of \( R(na,t) \) are listed in the fourth column of Table 1. They are reproduced by the formula

\[
R(na,t) = \frac{r}{2} \left[ 1 - s(na,0)s(na,t) \right], \quad (n \in \mathbb{Z}),
\]

because the term \( 1 - s(na,0)s(na,t) \) takes the value 2 if the opinion state \( s(na,t) \) differs from its initial value (and the value zero otherwise). With these notations, the total flipping rate \( w(na,t) \) of the binary variable \( s(na,t) \) in the voter model under resetting is expressed as

| \([s((n−1)a,t),s(na,t),s((n+1)a,t)]\) | \(s(na,0)\) | \(W(na,t)\) | \(R(na,t)\) | \(w(na,t)\) |
|-----------------|--------|--------|--------|--------|
| \([+1,+1,+1]\)  | +1     | 0      | 0      | 0      |
| \([+1,+1,-1]\)  | +1     | \(\tau^{-1}\) | 0      | \(\tau^{-1}\) |
| \([-1,+1,+1]\)  | +1     | \(\tau^{-1}\) | 0      | \(\tau^{-1}\) |
| \([-1,+1,-1]\)  | +1     | \(\tau^{-1}\) | 0      | \(\tau^{-1}\) |
| \([+1,-1,+1]\)  | −1     | \(\tau^{-1}\) | 0      | \(\tau^{-1}\) |
| \([-1,-1,+1]\)  | −1     | \(\tau^{-1}\) | 0      | \(\tau^{-1}\) |
| \([-1,-1,-1]\)  | −1     | 0      | 0      | 0      |

Table 1. The values of the flipping rate at site \( na \) and time \( t \) in each of the possible configurations of the four opinion states \( s((n−1)a,t),s(na,t),s((n+1)a,t) \) and \( s(na,0) \). The third column contains the flipping rate of the ordinary voter model, which depends only on the opinion states at time \( t \) listed in the first column. The fourth column contains the additional flipping rate induced by resetting, which depends only on \( s(na,t) \) and \( s(na,0) \). The flipping rate in the voter model under stochastic resetting is \( w(na,t) \), defined in equation (4).
Figure 1. A flipping process at site $na$ and time $t$. The colored arrows symbolize the current opinion states at sites $(n-1)a, na, (n+1)a$. The vertical black arrows symbolize the opinion state at site $na$ at time 0. The tips of the black arrows are not shown at sites $(n-1)a, (n+1)a$, as the values of $s((n-1)a, 0)$ and $s((n+1)a, 0)$ are irrelevant to the flipping rate of $s(na, t)$. The opinion state at site $na$ is different from its initial value, and from the opinion state of one of the neighbors. The rate of the process is therefore $r + \tau^{-1}/2$.

$$w(na, t) := W(na, t) + R(na, t)$$
$$= \frac{1}{2} \left[ \tau^{-1} \left( 1 - \frac{1}{2} \sum_{\epsilon = \pm 1} s((n + \epsilon)a, t) \right) + r \left( 1 - s(na, 0) s(na, t) \right) \right]. \quad (4)$$

The possible values of $w(na, t)$ depend on the opinion states at time $t$ at sites $(n-1)a, na, (n+1)a$, and on the initial opinion state at site $na$. The possible values are listed and mapped to the associated configurations in table 1. To represent the configuration of the system graphically, we can symbolize opinion states by arrows (upward-pointing if the opinion state is positive and downward-pointing if the opinion state is negative). To calculate the flipping rates, we need to show at every vertex a colored arrow symbolizing the current opinion state (say a red upward-pointing arrow if the opinion state is positive, and a blue downward-pointing arrow if the opinion state is negative), and a black arrow symbolizing the initial opinion state. The rates of the flipping processes of the opinion state at site $na$ listed in table 1 are represented diagrammatically in figure 1.

The average opinion at time $t$ of the voter at site $x$ is the ensemble average of $s(x, t)$. Let us denote it by

$$S(x, t) := \langle s(x, t) \rangle. \quad (5)$$

This average opinion state is also called average magnetization (if the binary variables are interpreted as spins, the dynamics of the ordinary voter model is the zero-temperature Glauber dynamics). Another quantity of interest is the spatial two-point function of the opinion state:

$$G(x, y, t) := \langle s(x, t) s(y, t) \rangle, \quad (6)$$

which describes the extent to which two distant voters agree at time $t$.

The configuration of the system at a given time can be described by a sequence of opinions as in equation (1), or equivalently by the list of pairs of nearest neighbors that have opposite opinions (together with the opinion of one voter, say at the origin). These pairs are called the domain walls of the configuration. They separate domains on the lattice where the voters have a uniform opinion. Consider the average density of domain walls at position $x$ at time $t$, which we denote by $\rho(x, t)$. It is the probability that the spin variables at site $x$ and $x+1$ have opposite values

$$\rho(x, t) = \text{Prob}(\{s(x, t) = -1, s(x + a, t) = +1\}) + \text{Prob}(\{s(x, t) = +1, s(x + a, t) = -1\}). \quad (7)$$
In the particular case where the sites $x$ and $y$ are nearest neighbors, the two-point function $G(x, y, t)$ allows to express the density of domain walls. Indeed,

\[
G(x, y, t) = \langle s(x, t) s(x + a, t) \rangle
\]

\[
= \text{Prob}(s(x, t) = +1, s(x + a, t) = +1) + \text{Prob}(s(x, t) = -1, s(x + a, t) = -1)
\]

\[
- \text{Prob}(s(x, t) = -1, s(x + a, t) = +1) - \text{Prob}(s(x, t) = -1, s(x + a, t) = +1)
\]

\[
= 1 - 2 \rho(x, t), \quad x \in a\mathbb{Z}.
\]

We combined equation (7) and normalization of probability in the last step.

We will use the following notations for the average opinion state and two-point function in the initial configuration:

\[
S_0(x) := \langle s(x, 0) \rangle, \quad G_0(x, y) := \langle s(x, 0) s(y, 0) \rangle, \quad x, y \in a\mathbb{Z}.
\]

The functions $S_0$ and $G_0$ are considered data of the problem: the initial configuration $\{s(x, 0), x \in a\mathbb{Z}\}$ is random but drawn from a distribution that yields a fixed average opinion state and a fixed two-point function.

2.2. Average opinion state

2.2.1. Evolution equation. When a binary opinion state $s(na, t)$ in $\{ -1, +1 \}$ changes, its value is shifted by $-2s(na, t)$. The evolution equation of the average opinion state follows as

\[
\frac{\partial S}{\partial t}(na, t) = -2(s(na, t) w(na, t)).
\]

Substituting the flipping rate defined in equation (2) and using the constraint $s(na, t)^2 = 1$ yields

\[
\frac{\partial S}{\partial t}(na, t) = -\left( \tau^{-1} \left( s(na, t) - \frac{1}{2} s(na, t)^2 \sum_{j=\pm 1} s(na + ja, t) \right) \right.
\]

\[
+ r \left( s(na, t) - s(na, 0) s(na, t)^2 \right)
\]

\[
= \tau^{-1} \left[ -S(na, t) + \frac{1}{2} (S(na - a, t) + S(na + a, t)) \right]
\]

\[
+ r |S_0(na) - S(na, t)|, \quad n \in \mathbb{Z}.
\]

The terms proportional to $\tau^{-1}$ on the r.h.s. correspond to the ordinary voter model, whose average opinion state satisfies a diffusion equation on the lattice. The $r$-dependent terms are analogous to a heat transfer according to the Newton law of cooling: the initial opinion-state profile $S_0$ is analogous to the temperature of a thermostat, and the resetting rate is analogous to the rate of thermal transfer between the system and the thermostat.

Given a function $f$ of the discrete space coordinate (and possibly other variables), let us denote by $\hat{f}$ its Fourier transform:

\[
\hat{f}(k) := \sum_{n \in \mathbb{Z}} f(na) e^{ika}, \quad (k \in \mathbb{R}).
\]
The Fourier transform is inverted by integrating over the first Brillouin zone \([-\pi/a, \pi/a]\):

\[
f(na) = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} \hat{f}(k) e^{-i kna} dk, \quad (n \in \mathbb{Z}).
\]  

(13)

With these notations, the Fourier transform of equation (11) reads

\[
\frac{\partial \hat{S}(k,t)}{\partial t} = \left[\tau^{-1} \cos (ka) - (r + \tau^{-1})\right] \hat{S}(k,t) + r \hat{S}_0(k).
\]  

(14)

For a fixed momentum \(k\), this ordinary differential equation is readily solved with the initial condition \(\hat{S}(k,0) = \hat{S}_0(k)\):

\[
\hat{S}(k,t) = \exp \left(\left[\tau^{-1} \cos (ka) - (r + \tau^{-1})\right] t - ikna\right) \hat{S}_0(k)
+ r \int_0^t du \hat{S}_0(k) \exp \left(\left[\tau^{-1} \cos (ka) - (r + \tau^{-1})\right] u - ikna\right), \quad n \in \mathbb{Z}.
\]  

(15)

Fourier inversion yields the average opinion state at position \(na\) and time \(t\) as

\[
S(na,t) = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \exp \left(\left[\tau^{-1} \cos (ka) - (r + \tau^{-1})\right] t - ikna\right) \hat{S}_0(k)
+ r \int_0^t du \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \hat{S}_0(k) \exp \left(\left[\tau^{-1} \cos (ka) - (r + \tau^{-1})\right] u - ikna\right), \quad n \in \mathbb{Z}.
\]  

(16)

Let us inject the generating function of the modified Bessel functions of the first kind:

\[
e^{-\cos(ka)} = \sum_{m \in \mathbb{Z}} I_m(ka) e^{ima}.
\]  

(17)

Permuting the summations and rearranging yields

\[
S(na,t) = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk e^{-\left(r+\tau^{-1}\right) t - i k n a} \sum_{m \in \mathbb{Z}} I_m(\tau^{-1} t) S_0(k) e^{i m a}
+ r \int_0^t du \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \hat{S}_0(k) e^{-\left(r+\tau^{-1}\right) u - i k n a} \sum_{m \in \mathbb{Z}} I_m(\tau^{-1} u) e^{i m a}
= e^{-\left(r+\tau^{-1}\right) t} \sum_{m \in \mathbb{Z}} I_m(\tau^{-1} t) \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \hat{S}_0(k) e^{i (m-a) ka}
+ r \int_0^t du \sum_{m \in \mathbb{Z}} I_m(\tau^{-1} u) \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \hat{S}_0(k) e^{i (m-a) ka}
= e^{-\left(r+\tau^{-1}\right) t} \sum_{m \in \mathbb{Z}} I_m(\tau^{-1} t) S_0(na - ma)
+ r \int_0^t du \sum_{m \in \mathbb{Z}} I_m(\tau^{-1} u) S_0(na - ma).
\]  

(18)
In the last step we used the identity \( S_0(Na) = \frac{a}{2\pi} \int_{-\pi}^{\pi} dk \tilde{S}_0(k) e^{-iNka} \), which is the Fourier representation of the initial average opinion-state profile. The average opinion-state profile at time \( t \) is therefore the discrete convolution of \( S_0 \) and the time-dependent kernel \( K_t \) defined as follows:

\[
S(na, t) = \sum_{m \in \mathbb{Z}} K_t(ma) S_0(na - ma),
\]

with \( K_t(na) := e^{-(r+\tau^{-1})t} I_n(\tau^{-1}t) + r \int_0^t du e^{-(r+\tau^{-1})u} I_n(\tau^{-1}u), \quad (n \in \mathbb{Z}). \tag{19} \]

Let us denote with a tilde the Laplace transform of any quantity depending on time (and possibly other variables):

\[
\tilde{f}(s) := \int_0^\infty \, du e^{-st} f(t).
\tag{20}
\]

The large-time limit \( \tilde{K}_\infty \) of the kernel \( K_t \) is expressed in terms of the Laplace transforms of modified Bessel functions of the first kind. Using tabulated formulas listed in appendix A (equation (131)) yields:

\[
\tilde{K}_\infty(na) = r \tau \int_0^\infty \, dv e^{-(r+\tau^{-1})v} I_n(v) = r \tau I_n(\tau + 1) = \frac{r \tau}{\sqrt{(r \tau + 1)^2 - 1}} \left[ (\tau + 1) + \sqrt{(r \tau + 1)^2 - 1} \right]^{-1}. \tag{21} \]

The approach to the steady state of the kernel is described by the large-time behavior of the following quantity:

\[
K_t(na) - \tilde{K}_\infty(na) = e^{-(r+\tau^{-1})t} I_n(\tau^{-1}t) - r \int_0^\infty \, du e^{-(r+\tau^{-1})u} I_n(\tau^{-1}u). \tag{22} \]

Let us take the Fourier representation:

\[
K_t(na) - \tilde{K}_\infty(na) = \frac{a}{2\pi} \int_{-\pi}^{\pi} \, dk \left[ \tilde{K}_t(k) - \tilde{K}_\infty(k) \right] e^{-iNka}. \tag{23} \]

Using equation (17) we can calculate \( \tilde{K}_t(k) \) using the generating function of the modified Bessel functions:

\[
\tilde{K}_t(k) = e^{-(r+\tau^{-1})t} \sum_{n \in \mathbb{Z}} I_n(\tau^{-1}t) e^{inka} + r \int_0^t \, du e^{-(r+\tau^{-1})u} \sum_{n \in \mathbb{Z}} I_n(\tau^{-1}u) e^{inka}
\]

\[
= e^{-[\tau + \tau^{-1}(1 - \cos(ka))]t} + r \int_0^t \, du e^{-[\tau + \tau^{-1}(1 - \cos(ka))]u}
\]

\[
= \frac{r + \tau^{-1}(1 - \cos(ka)) [e^{-\tau^{-1}(1 - \cos(ka))]t + \tau^{-1}(1 - \cos(ka))]. \tag{24} \]

The large-time limit of the Fourier transform of the kernel follows as

\[
\tilde{K}_\infty(k) = \frac{r}{r + \tau^{-1}(1 - \cos(ka))}. \tag{25} \]
Using the equivalent form at time average opinion state:

\[
\frac{\tau^{-1} (1 - \cos (ka)) e^{-r + \tau^{-1} (1 - \cos (ka)) t}}{r + \tau^{-1} (1 - \cos (ka))} \]

yields

\[
\frac{\tau^{-1} (1 - \cos (ka)) e^{-r + \tau^{-1} (1 - \cos (ka)) t}}{r + \tau^{-1} (1 - \cos (ka))} = \frac{e^{-r + \tau^{-1} (1 - \cos (ka)) t}}{r + \tau^{-1} (1 - \cos (ka))}.
\]

Using the equivalent

\[
I_n(z) \sim \frac{e^z}{\sqrt{2\pi} z}, \quad (n \in \mathbb{Z}),
\]

we notice that the upper bound is equivalent to \((2\pi \tau^{-1} t)^{-\frac{1}{2}} e^{-nt}\) when time \(t\) is large compared to the time scale \(\tau\).

Moreover, we can obtain an upper bound on \(|S(na, t) - S(na, \infty)|\) by using the Fourier transform of equation (19). As the average opinion state profile is a convolution, its Fourier transform at time \(t\) is given by the product of the Fourier transforms of the kernel \(K_t\) and the initial average opinion state:

\[
\hat{S}(k, t) = \hat{S}_0(k) \hat{K}_t(k), \quad (t \geq 0).
\]

Combining equations (24), (25) and (28) and using the Fourier representation of the kernel as in equation (26) yields

\[
|S(na, t) - S(na, \infty)| = \left| \frac{a}{2\pi} \int_{-\pi}^{\pi} dk \left[ \hat{S}_0(k) \left( \hat{K}_t(k) - \hat{K}_\infty(k) \right) \right] e^{-inka} \right|.
\]

Combining equations (24), (25) and (28) and using the Fourier representation of the kernel as in equation (26) yields

\[
S(na, t) - S(na, \infty) = \frac{a}{2\pi} \int_{-\pi}^{\pi} dk \left[ \hat{S}_0(k) \left( \hat{K}_t(k) - \hat{K}_\infty(k) \right) \right] e^{-inka}.
\]

Combining equations (24), (25) and (28) and using the Fourier representation of the kernel as in equation (26) yields

\[
\hat{S}(k, t) = \hat{S}_0(k) \hat{K}_t(k), \quad (t \geq 0).
\]
Moreover, the upper bound is uniform in the integer $n$ labeling the site. It goes to zero if $\hat{S}_0$ is integrable on $[-\pi/a, \pi/a]$.

2.2.2. Example: a single decided voter at the origin. Consider an initial condition with independent undecided voters, except for a single decided voter with positive opinion state at the origin. If $n \neq 0$, the opinion state $s(na, 0)$ is a centered Bernoulli random variable. On the other hand $s(0, 0) = 1$. Hence

$$S_0(ma) := \delta_{m,0}, \quad (m \in \mathbb{Z}).$$

(30)

In this case the average opinion state at site labeled $n$ coincides with the quantity $K_c(na)$ defined in equation (19):

$$S(na, t) = e^{-(r+\tau^{-1})t}I_n(\tau^{-1}t) + r \int_0^t \text{d}u e^{-(r+\tau^{-1})u}I_n(\tau^{-1}u), \quad (n \in \mathbb{Z}).$$

(31)

The r.h.s. does not depend on the lattice spacing $a$, which is expected because the parameter $a$ does not appear in the definition of the flipping rates. The steady state of the average opinion state follows from equation (21) as an exponentially-decaying function of the distance to the origin (plotted in figure 2):

$$S(na, \infty) = K_c(na) = r\tau I_n(r\tau + 1) = \frac{r\tau}{\sqrt{(r\tau + 1)^2 - 1}} \lambda(r, \tau)^{-|n|}$$

with $\lambda(r, \tau) := r\tau + 1 + \sqrt{(r\tau + 1)^2 - 1}$.

(32)
In dimension one, the above result solves Problem (I(r)) defined in the introduction. Indeed, the average opinion state of the initially decided voter does not relax to zero in the steady state as it does in the ordinary model. In the limit of low resetting rate, it is proportional to the square root of the resetting rate:

\[ S(0, \infty) \sim r^{-1/2} \sqrt{\frac{r^*}{2}}. \]  

In the initial state, the average opinion-state profile is nonnegative and normalized, indeed

\[ \sum_{n \in \mathbb{Z}} S(na, 0) = 1. \]  

Moreover, equation (11) becomes in this case

\[ \frac{\partial S}{\partial t}(na, t) = \tau^{-1} \left[ -S(na, t) + \frac{1}{2} (S(na - a, t) + S(na + a, t)) \right] - rS(na, t) + r0, \quad (n \in \mathbb{Z}). \]  

The terms carrying a factor of \( \tau^{-1} \) describe the motion of a diffusive random walker on the lattice \( a \mathbb{Z} \). The \( r \)-dependent terms \( r(\delta_{n,0} - S(na, t)) \) in equation (11) correspond to resetting the position of the random walker to the origin between times \( t \) and \( t + dt \) with probability \( r dt \).

The quantity \( S(na, t) = K_t(na) \) is the probability of presence at site \( na \) and time \( t \) of a diffusive random walker on the lattice, starting at the origin at time 0 and stochastically reset to its initial position at rate \( r \). The average opinion-state profile is therefore normalized at all times:

\[ \sum_{n \in \mathbb{Z}} S(na, t) = \sum_{n \in \mathbb{Z}} K_t(na) = 1. \quad (t \geq 0). \]  

Continuum limit. Let us take the continuum limit as follows: the lattice spacing \( a \) and the characteristic time \( \tau \) both go to zero, while the resetting rate \( r \) and the diffusion constant

\[ D := \frac{a^2}{2\tau} \]  

are kept fixed.

Let us consider a large position \( x \) (on the scale of the lattice spacing \( a \)):

\[ x := na, \quad |n| \gg 1. \]  

Expanding the expression of the quantity \( \lambda(r, \tau) \) given in equation (32) in powers of \( r \tau \) yields

\[ \lambda(r, \tau) = 1 + \sqrt{2r\tau} + o(\sqrt{2r\tau}). \]  

In the large-distance regime, the spatial decay of the average opinion state is obtained as

\[ \lambda(r, \tau)^{-|n|} = \exp (-|n| \log \lambda(r, \tau)) \sim \exp \left( -\frac{|n|}{a} \sqrt{2r\tau} \right) = e^{-|n|\sqrt{D}}. \]  

which is the simplest exponential decay that can be expressed using the position, resetting rate and diffusion coefficient, for dimensional reasons. Introducing a factor of \( a^{-1} \), we obtain the diffusive limit of the average opinion state per unit of length as

\[ a^{-1} S(na, \infty) = \frac{r^*}{\sqrt{a^2 [(r^* + 1)^2 - 1]}} \lambda(r, \tau)^{-|n|} \mid_{a, \tau \to 0, x = na, D = \frac{a^2}{2\tau}} \sim \frac{1}{2} \sqrt{\frac{r}{D}} e^{-\sqrt{\frac{D}{2}}} \sqrt{e^{-\sqrt{\frac{D}{2}}}}. \]  

Continuum limit. Let us take the continuum limit as follows: the lattice spacing \( a \) and the characteristic time \( \tau \) both go to zero, while the resetting rate \( r \) and the diffusion constant

\[ D := \frac{a^2}{2\tau} \]  

are kept fixed.

Let us consider a large position \( x \) (on the scale of the lattice spacing \( a \)):

\[ x := na, \quad |n| \gg 1. \]  

Expanding the expression of the quantity \( \lambda(r, \tau) \) given in equation (32) in powers of \( r \tau \) yields

\[ \lambda(r, \tau) = 1 + \sqrt{2r\tau} + o(\sqrt{2r\tau}). \]  

In the large-distance regime, the spatial decay of the average opinion state is obtained as

\[ \lambda(r, \tau)^{-|n|} = \exp (-|n| \log \lambda(r, \tau)) \sim \exp \left( -\frac{|n|}{a} \sqrt{2r\tau} \right) = e^{-|n|\sqrt{D}}. \]  

which is the simplest exponential decay that can be expressed using the position, resetting rate and diffusion coefficient, for dimensional reasons. Introducing a factor of \( a^{-1} \), we obtain the diffusive limit of the average opinion state per unit of length as

\[ a^{-1} S(na, \infty) = \frac{r^*}{\sqrt{a^2 [(r^* + 1)^2 - 1]}} \lambda(r, \tau)^{-|n|} \mid_{a, \tau \to 0, x = na, D = \frac{a^2}{2\tau}} \sim \frac{1}{2} \sqrt{\frac{r}{D}} e^{-\sqrt{\frac{D}{2}}} \sqrt{e^{-\sqrt{\frac{D}{2}}}}. \]
As expected, the diffusive limit of the random walk on the lattice with resetting reproduces the steady-state probability density of a diffusive random random walker on a line under stochastic resetting to the origin ([7, 8], for a review see section 2.3 of [14]).

### 2.2.3. Example with infinite total opinion state: decided voters on a half line.

Not all the possible initial configurations of the voter model satisfy \( \sum_{n \in \mathbb{Z}} |S_0(na)| < \infty \). As an example, consider an initial condition with independent undecided voters at sites with negative indices, and decided voters with positive opinion state occupying at sites with nonnegative indices:

\[
S_0(m) := 1 \quad (m \geq 0), \quad S(m) := 1 \quad (m \in \mathbb{Z}).
\]

This configuration has infinite total opinion state, but it is still amenable to explicit calculation. From the discrete convolution in equation (18) we obtain

\[
S(na, t) = e^{-(r+\tau^{-1})t} \sum_{m=-\infty}^{a} I_m(\tau^{-1}t) + r \int_{0}^{t} du e^{-(r+\tau^{-1})u} \sum_{m=-\infty}^{a} I_m(\tau^{-1}u).
\]

Using the identities

\[
\sum_{m=1}^{\infty} I_m(z) = \frac{1}{2}(e^z - I_0(z)), \quad I_n = I_{-n}, \quad (n \in \mathbb{Z}),
\]

we obtain the expression of the average opinion state at site \( na \) as a sum of a finite number of terms:

\[
S(0, t) = \frac{1}{2} + \frac{1}{2} e^{-(r+\tau^{-1})t} I_0(\tau^{-1}t) + \frac{r}{2} \int_{0}^{t} du e^{-(r+\tau^{-1})u} I_0(\tau^{-1}u),
\]

\[
S(na, t) = S(0, t) + e^{-(r+\tau^{-1})t} \sum_{m=1}^{[n]} I_m(\tau^{-1}t) + r \int_{0}^{t} du e^{-(r+\tau^{-1})u} \sum_{m=1}^{[n]} I_m(\tau^{-1}u), \quad (n > 0),
\]

\[
S(na, t) = S(0, t) - e^{-(r+\tau^{-1})t} \sum_{m=0}^{[n]} I_m(\tau^{-1}t) - r \int_{0}^{t} du e^{-(r+\tau^{-1})u} \sum_{m=0}^{[n]} I_m(\tau^{-1}u), \quad (n < 0).
\]

The average opinion state is plotted as a function of \( n \) in figure 3, together with the result of direct numerical simulations performed for a system with a finite number of sites\(^1\).

\(^1\) The algorithm is as follows. The system consists of \( 2K + 1 \) voters. The configuration is initialized: \( s(na, 0) = 1 \) for \( 0 \leq n \leq K \), and the value of \( s(na, 0) \) is drawn from a centered Bernoulli distribution for \( -K \leq n \leq -1 \). Time is initialized as \( t = 0 \). The flipping rates and their sum \( \Phi := \sum_{n=-K}^{K} w(na, t) \) are computed. The time \( t_1 \) of the first flipping event is drawn from an exponential distribution with parameter \( \Phi \). The voter whose opinion state flips at \( t_1 \) is then decided from a discrete distribution (the voter at position \( n \) changes its opinion state with probability \( w(na, t)/\Phi \)). The opinion state of this voter is flipped, which yields the configuration of the system at time \( t_1 \). Time is changed to \( t = t_1 \), the new flipping rates are calculated, and the process is iterated. This algorithm is run for \( N \) independent samples of the system and the estimated average opinion state \( S(na, t) \) is the average of \( s(na, t) \) over the samples.
Figure 3. Average opinion state for initial conditions with undecided voters in the negative half line and decided voters in the positive half line. The parameters are the resetting rate $r = 10^{-3}$ and the time $\tau = 1$. Dots correspond to a direct simulation of the model (with $-200 \leq n \leq 200$), averaged over $N = 100000$ samples of the initial configuration with independent undecided voters in the negative half-line. The predicted value $S(na, t)$ is based on equation (44).

Let us take the large-time limit. The equivalents of the Bessel functions equation (27) imply that the terms in equation (44) that are not in integral form go to zero. The steady state is again expressed in terms of Laplace transform of Bessel functions:

$$S(0, \infty) = \frac{1}{2} + \frac{r\tau}{2} \int_0^\infty dv e^{-\frac{r\tau}{2}v} I_0(v) = \frac{1}{2} + \frac{r\tau}{2\sqrt{(r\tau+1)^2-1}},$$

$$S(na, \infty) = S(0, \infty) + r\tau \sum_{n=1}^\infty I_n\left(\frac{r\tau}{\sqrt{(r\tau+1)^2-1}}\right), \quad (n > 0),$$

$$S(na, \infty) = S(0, \infty) - r\tau \sum_{n=0}^{-1} I_n\left(\frac{r\tau}{\sqrt{(r\tau+1)^2-1}}\right), \quad (n < 0).$$

This steady state is shown as a green dashed line on figure 3.
2.3. Two-point function and density of domain walls

2.3.1. Evolution equation of the two-point function. Consider the two-point function \( G(x, y, t) = \langle s(x, t) s(y, t) \rangle \). If \( x \) and \( y \) are equal, \( G \) is constant because of the constraint \( s(x, t)^2 = 1 \). On the other hand, if \( x = na \) and \( y = ma \) are two distinct sites on the lattice, the value of \( G(x, y, t) \) changes when any of the voters at position \( x \) or \( y \) changes its opinion state:

\[
\frac{\partial G}{\partial t} (na, ma, t) = -2\langle s(na, t) s(ma, t) [w(na, t) + w(ma, t)] \rangle, \quad (n, m \in \mathbb{Z}, n \neq m). \tag{46}
\]

Substituting the expression of the flipping rate given in equation (4) and using the constraint \( s(na, t)^2 = 1 \) we obtain

\[
2s(na, t) s(ma, t) w(na, t) = \tau^{-1} s(na, t) s(ma, t) - \frac{\tau^{-1}}{2} s(ma, t) \sum_{z=na\pm a} s(z, t)
\]

\[
+ rs(na, t) s(ma, t) - rs(ma, t) s(na, 0). \tag{47}
\]

Taking the ensemble averages of both sides, we notice that one term proportional to the resetting rate involves the following two-time two-point function:

\[
H(na, ma, t) := \langle s(na, t) s(ma, 0) \rangle. \tag{48}
\]

We will use this notation for any integer values of \( m \) and \( n \), but the terms that appear in the evolution equation of the two-point function \( G \) correspond to \( n \neq m \).

With this notation, equation (47) implies

\[
2\langle s(na, t) s(ma, t) w(na, t) \rangle = \tau^{-1} G(na, ma, t) - \frac{\tau^{-1}}{2} \sum_{z=na\pm a} G(z, ma, t)
\]

\[
+ rG(na, ma, t) - rH(ma, na, t), \quad (n, m \in \mathbb{Z}, n \neq m). \tag{49}
\]

The quantity \( 2\langle s(x, t) s(y, t) w(y, t) \rangle \) is obtained by permuting \( n \) and \( m \) in the above expression, which yields the following evolution equation:

\[
\frac{\partial G}{\partial t} (na, ma, t) = -2(r + \tau^{-1}) G(na, ma, t)
\]

\[
+ \frac{\tau^{-1}}{2} [G((n-1)a, ma, t) + G((n+1)a, ma, t) + G(na, (m-1)a, t)
\]

\[
+ G(a, (m+1)a, t)] + r[H(na, ma, t) + H(ma, na, t)], \quad (m, n \in \mathbb{Z}, n \neq m). \tag{50}
\]

On the other hand, the two-point function \( G \) at coincident points is constant:

\[
G(na, na, t) = \langle s(na, t)^2 \rangle = 1, \quad (n \in \mathbb{Z}). \tag{51}
\]

To solve the evolution equation of \( G \), we will need to work out the evolution equation of the two-time two-point function \( H \) defined in equation (48). The evolution is induced by the flipping rate of the opinion state at site \( na \):

\[
\frac{\partial H}{\partial t} (na, ma, t) = -2\langle w(na, t) s(na, t) s(ma, 0) \rangle, \quad (m, n \in \mathbb{Z}). \tag{52}
\]
This equation holds even for \( n = m \), because the quantity \( s(y, 0) = S_0(y) \) does not evolve in time. Substituting the expression of the flipping rate given in equation (2) and using again the constraint \( s(na, t)^2 = 1 \) yields

\[
\frac{\partial H}{\partial t}(na, ma, t) = - \left( s(na, t) \left[ \tau^{-1} \left( 1 - \frac{1}{2}s(na, t) \sum_{z:|z|\geq na} s(z, t) \right) + r \left( 1 - s(na, 0)s(na, t) \right) \right] s(na, 0) \right)
\]

\[
= - \tau^{-1} (s(na, t)s(ma, 0)) + \tau^{-1} \sum_{z:|z|\geq na} (s(z, t)s(ma, 0))
\]

\[
- r(s(na, t)s(ma, 0)) + r(s(na, 0)s(ma, 0))
\]

\[
= - \left( r + \tau^{-1} \right) H(na, ma, t) + \tau^{-1} \sum_{z:|z|\geq na} H(na, ma, t) + r \delta_{nm},
\]

with \( H(na, ma, 0) = \langle s(na, 0)s(ma, 0) \rangle = \delta_{nm}, \ (n, m \in \mathbb{Z}). \)

(53)

For any fixed \( m \), we notice that \( H(na, ma, t) \) satisfies the evolution equation of a diffusive random walker on the lattice, stochastically reset to its initial position \( ma \). At this point, let us specialize to the case of initially-undecided voters.

2.3.2. Solution for initially undecided voters. Let us assume that the voters are independent and undecided in the initial state: the initial opinion states of the voters are independent centered Bernoulli random variables. Hence

\[
S_0(na) = 0,
\]

\[
G_0(na, ma) = \langle s(na, 0)s(ma, 0) \rangle = \delta_{nm} + (1 - \delta_{nm}) S_0(na) S_0(ma) = \delta_{nm}, \ (m, n \in \mathbb{Z}). \)

(54)

This choice of initial conditions implies that the system is translationally invariant. The value of the two-point function depends only on the separation of the voters (and on time). Hence there exist functions \( \mathcal{H} \) and \( \mathcal{G} \) of two variables such that:

\[
\mathcal{H}(na, ma, t) = \mathcal{H}((m-n)a, t),
\]

\[
\mathcal{G}(na, ma, t) = \mathcal{G}((m-n)a, t), \quad (m, n \in \mathbb{Z}, t \geq 0).
\]

(55)

The initial condition \( G_0(na, ma) = \delta_{nm} \) can be rewritten as

\[
\mathcal{G}(na, 0) = \delta_{n,0}, \quad (n \in \mathbb{Z}).
\]

(56)

The evolution equation (50) derived above holds for two distinct points, \( n \neq m \), hence

\[
\frac{\partial \mathcal{G}}{\partial t}(na, t) = -2 \left( r + \tau^{-1} \right) \mathcal{G}(na, t) + \tau^{-1} \left[ \mathcal{G}((n-1)a, t) + \mathcal{G}((n+1)a, t) \right]
\]

\[
+ r \mathcal{H}(na, t) + r \mathcal{H}(-na, t), \quad (n \in \mathbb{Z}, n \neq 0).
\]

(57)

The constraint expressed in equation (51) reads

\[
\mathcal{G}(0, t) = 1, \quad (t \geq 0).
\]

(58)
To ensure this condition holds throughout the evolution of the system, let us add an (unknown) source term $J(t)$ at the origin:

$$
\frac{\partial G}{\partial t}(na,t) = -2 \left( r + \tau^{-1} \right) G(na,t) + \tau^{-1} \left[ G((n-1)a,t) + G((n+1)a,t) \right] + \tau H(na,t) + rH(-na,t) + J(t) \delta_n, \quad (n \in \mathbb{Z}).
$$

(59)

This technique was introduced in [2] to solve the voter model without resetting (see chapter 8.2 in [5] for a review), which is recovered by setting the resetting rate $r$ to zero in the above equation.

As equation (59) holds for all values of the discrete space coordinate, we can take its Fourier transform. As we have noticed from equation (53), the quantity $H(na,ma,t)$ is the probability of presence at $na$ of a random walker stochastically reset to its initial position $ma$. It satisfies the parity property $H((m-n)a,ma,t) = H((m+n)a,ma,t)$ for every integer $n$, which implies $H(na,t) = H(-na,t)$. The Fourier transform of equation (59) therefore reads

$$
\frac{\partial \hat{G}}{\partial t}(k,t) = -2 \left( r + \tau^{-1} \right) \hat{G}(k,t) + 2\tau^{-1} \cos(ka) \hat{G}(k,t) + 2r \hat{H}(k,t) + J(t),
$$

(60)

with $\hat{G}(k,0) = 1$, $k \in \left[ -\frac{\pi}{a}, \frac{\pi}{a} \right]$. Solving this ordinary differential equation yields the expression of the two-point function in Fourier space at time $t$ in terms of the unknown current $J$ (and the unknown function $H$):

$$
\hat{G}(k,t) = e^{2\alpha(k)t} + 2r \int_0^t du e^{2\alpha(k)(t-u)} \hat{H}(k,u) + \int_0^t du e^{2\alpha(k)(t-u)} J(u),
$$

(61)

where $\alpha(k) = \tau^{-1} \cos(ka) - (r + \tau^{-1})$.

At this point we need the Fourier transform of the function $H$. The evolution equation of the two-time two-point function $H$ (equation (53)) is readily rewritten in terms of the function $H$ as

$$
\frac{\partial H}{\partial t}(na,t) = - \left( r + \tau^{-1} \right) H(na,t) + \tau^{-1} \left[ H((n+1)a,t) + H((n-1)a,t) \right] + r \delta_{n0}.
$$

(62)

with $H(na,0) = \delta_{n0}$, $(n \in \mathbb{Z})$. This equation is identical to the one satisfied by the average opinion state (equation (11)) with a single decided voter at the origin in the initial state. The Fourier transform with the initial condition $\hat{H}(k,0) = 1$ is therefore obtained directly from equation (15) as

$$
\hat{H}(k,t) = \exp \left( \left[ \tau^{-1} \cos(ka) - (r + \tau^{-1}) \right] t \right) + r \int_0^t du \exp \left( \left[ \tau^{-1} \cos(ka) - (r + \tau^{-1}) \right] u \right).
$$

(63)

Coming back to equation (61) and inverting the Fourier transform yields

$$
G(na,t) = e^{-2(r+\tau^{-1})t} I_n(2\tau^{-1}t) + 2r \frac{a}{2\pi} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} dk \int_0^t du e^{2\alpha(k)(t-u) - ika} \hat{H}(k,u)
$$

$$
+ \int_0^t du e^{-2(r+\tau^{-1})(t-u)} I_n(2\tau^{-1}(t-u)) J(u),
$$

(64)
where in the first and last term we have used the identity \( e^{2\pi^{-1}(t-u)} \cos(ka) = \sum_{n \in \mathbb{Z}} I_n(2\pi^{-1}(t-u)) e^{i n a} \), which is the generating function of the modified Bessel functions of the first kind defined in equation (17). We have to adjust the current \( J \) so that the condition \( \mathcal{G}(0,t)=1 \) is satisfied at all times:

\[
1 = e^{-2(\alpha+\pi^{-1})/I_0(2\pi^{-1}t)} + 2r^{\alpha} \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} dk \int_0^t du e^{2\alpha(k)(t-u)} \tilde{H}(k,u)
+ \int_0^t du e^{-2(\alpha+\pi^{-1})(t-u)} I_0(2\pi^{-1}(t-u)) J(u) \tag{65}
\]

Let us take the Laplace transform, which maps the convolution product to an ordinary product, leading to

\[
\frac{1}{s} = \tilde{\varphi}_0(s) + 2r \Psi_0(s) + \tilde{\varphi}_0(s) \tilde{J}(s),
\tag{66}
\]

with the notations

\[
\varphi_n(t) := e^{-2(\alpha+\pi^{-1})} I_n(2\pi^{-1}t),
\quad \text{and } \Psi_n(s) := \frac{a}{2\pi} \int_{-\pi/2}^{\pi/2} dk e^{-i k a} \frac{1}{s-2\alpha(k)} \tilde{H}(k,s), \quad (n \in \mathbb{Z}).
\tag{67}
\]

Moreover, the expression of the two-point function in equation (64) can be used to relate the Laplace transform of the current \( J \) to the one of the density of domain walls.

2.3.3. The steady-state density of domain walls. In a translationally-invariant system, the density of domain walls is a function of time only, call it \( \varrho \):

\[
\rho(na,t) = \varrho(t), \quad (n \in \mathbb{Z}).
\tag{68}
\]

According to equation (8) the density of domain walls is related to the two-point function, which yields an expression of the density of domain walls \( \varrho \) in terms of the value \( \mathcal{G}(a,t) \):

\[
\varrho(t) = \frac{1}{2} \left( 1 - G(na,(n+1)a,t) \right) = \frac{1}{2} \left( 1 - \mathcal{G}(a,t) \right).
\tag{69}
\]

From the expression of the two-point function in equation (64) in the particular case \( n = 1 \) we obtain

\[
1 - 2\varrho(t) = e^{-2(\alpha+\pi^{-1})} I_1(2\pi^{-1}t) + 2r^{\alpha} \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} dk \int_0^t du e^{2\alpha(k)(t-u)} \tilde{H}(k,u)
+ \int_0^t du e^{-2(\alpha+\pi^{-1})(t-u)} I_1(2\pi^{-1}(t-u)) J(u).
\tag{70}
\]

The Laplace transform \( \tilde{\varrho} \) of the density of domain walls is therefore related to the Laplace transform of the current \( J \) by

\[
\frac{1}{s} - 2\tilde{\varrho}(s) = \tilde{\varphi}_1(s) + 2r \tilde{\Psi}_1(s) + \tilde{\varphi}_1(s) \tilde{J}(s).
\tag{71}
\]
Combining with the value of $\tilde{J}(s)$ obtained in equation (66) yields

$$\tilde{\varrho}(s) = \frac{1}{2} \left( \frac{1}{s} - 2r \Psi_1(s) + 2r \frac{\tilde{\varphi}_1(s)}{\tilde{\varphi}_0(s)} \Psi_0(s) - \frac{1}{s} \frac{\tilde{\varphi}_1(s)}{\tilde{\varphi}_0(s)} \right). \quad (72)$$

Applying the final-value theorem, we obtain the steady-state value $\varrho_\infty$ of the density of domain walls for an initial state with undecided voters:

$$\varrho_\infty = \lim_{s \to 0} (s \tilde{\varrho}(s)) = \frac{1}{2} \lim_{s \to 0} \left( 1 - 2r \Psi_1(s) + 2r \frac{\tilde{\varphi}_1(s)}{\tilde{\varphi}_0(s)} \Psi_0(s) - \frac{\tilde{\varphi}_1(s)}{\tilde{\varphi}_0(s)} \right), \quad (73)$$

where we used again the notations introduced in equation (67). The quantities $\Psi_0(s)$ and $\Psi_1(s)$ and the needed Laplace transforms are worked out in appendix A.1 (equations (133) and (136)). Substitution yields:

$$\varrho_\infty = \frac{1}{2} \left[ 1 - \frac{(rt)^2}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(x)}{(-\cos(x) + rt + 1)^2} \right. \right.$$

$$- \left. \frac{1}{rt + 1 + \sqrt{(rt + 1)^2 - 1}} \left( 1 - \frac{(rt)^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{(-\cos(x) + rt + 1)^2} \right) \right]. \quad (74)$$

In dimension one, the above result is the solution of Problem (II_c(1)) defined in the introduction. The density $\varrho_\infty$ is plotted as a function of $rt$ on figure 4. Consensus is not achieved in the voter model under resetting in one dimension. However, the steady-state density goes to
Figure 5. Results of numerical simulation of the average density of domain walls. The initial condition consists of independent undecided voters. The parameter $\tau$ was set to 1.

zero when the resetting rate goes to zero. This limit corresponds to the ordinary voter model, which is known to lead to a consensus in dimension one \cite{2}. The vertical tangent at the origin corresponds to the equivalent

$$\varrho_\infty \sim \frac{\sqrt{\tau r}}{2},$$

which is worked out in appendix C.5. On the other hand, in the limit of large resetting rate, the density $\varrho_\infty$ goes to $1/2$, which is the average density of domain walls in a system of undecided voters. Direct numerical simulations of the evolution of the density of domain walls are shown together with the predicted steady-state value on figure 5.

3. Generalization to higher dimension

Consider the voter model on a hypercubic $d$-dimensional lattice, with lattice spacing $a$. Let us pick an orthonormal basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$. There is a voter at every vertex in the lattice, carrying a binary opinion. At time $t$, the voter at the site $x = a n = \sum_i a_n e_i$ (where $n$ is an element of $\mathbb{Z}^d$) is in the opinion state $s(x, t)$ in $\{-1, +1\}$.

The flipping rule is adapted from the one-dimensional case (equation (2)) as follows. The transition rate at site $x$ consists of two contributions. The first one is the flipping rate $W$ of the ordinary voter model in dimension $d$, which is the fraction of disagreeing neighbors, with a factor of $\tau^{-1}$:
\[ W(x, t) := \frac{\tau - 1}{2} \left[ 1 - \frac{1}{2d} s(x, t) \sum_{i=1}^{d} \left[ s(x + e_i, t) + s(x - e_i, t) \right] \right] , \quad \left( x \in a\mathbb{Z}^d \right). \] (76)

The only difference with equation (2), apart from the vector-valued spatial variables, is the factor of \( d^{-1} \) in front of the sum over nearest neighbors, which ensures that the maximum flipping rate is \( \tau^{-1} \) (achieved in configurations in which all of the \( 2d \) neighbors of the site \( x \) are occupied by voters with opinion state \( -s(x, t) \)). This can immediately be checked by using the constraint \( s(x, t)^2 = 1 \). The second contribution corresponds to resetting. It is directly obtained from equation (3) by substituting vector-valued space variables to the positions of the sites:

\[ R(x, t) := \frac{r}{2} \left( 1 - s(x, 0) s(x, t) \right) , \quad \left( x \in a\mathbb{Z}^d \right). \] (77)

The resetting times are Poisson-distributed with intensity \( r \), at each resetting time the opinion state of the voter at site \( x \) reverts to its initial value \( s(x, 0) \) if \( s(x, t) \neq s(x, 0) \).

The transition rate \( w \) of the voter model under stochastic resetting in dimension \( d \) is therefore expressed as

\[ w(x, t) := W(x, t) + R(x, t) \]
\[ = \frac{1}{2} \left( \tau^{-1} - \frac{1}{2d} s(x, t) \sum_{i=1}^{d} s(x + e_i, t) + s(x - e_i, t) + r (1 - s(x, 0) s(x, t)) \right). \] (78)

To study the evolution of the one- and two-point functions of the voter model in dimension \( d \), let us adapt the reasoning of the previous section.

3.1. Average opinion state

The average opinion state at site \( x \) and time \( t \) is again denoted by \( S \), and defined as the ensemble average

\[ S(x, t) := \langle s(x, t) \rangle. \] (79)

The opinion state at site \( x \) changes by \( -2s(x, t) \) when the voter at site \( x \) flips its opinion state. This induces the following evolution equation for the average opinion state (generalizing equation (11)):

\[ \frac{\partial S(x, t)}{\partial t} = -2(s(x, t) w(x, t)) \]
\[ = \tau^{-1} \left( -S(x, t) + \frac{1}{2d} \sum_{i=1}^{d} [S(x - a e_i, t) + S(x + a e_i, t)] \right) + r (S(x, 0) - S(x, t)) , \quad \left( x \in a\mathbb{Z}^d \right). \] (80)

We are instructed to solve this equation with an initial condition consisting of a given function \( S_0 \) defined on the hypercubic lattice:

\[ S(x, 0) = S_0(x) , \quad \left( x \in a\mathbb{Z}^d \right). \] (81)
In dimension $d$ the Fourier transform of a function $f$ of the position in the hypercubic lattice (and possibly other variables) is defined as

$$\hat{f}(\mathbf{k}) := \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n} a) e^{i\mathbf{k} \cdot \mathbf{n} a}, \quad \mathbf{k} \in \mathbb{R}^d.$$  \hfill (82)

The Fourier transform is inverted by integrating over the first Brillouin zone $[-\pi/a, \pi/a]^d$:

$$f(\mathbf{n} a) = \left( \prod_{i=1}^d \int_{-\pi}^{\pi} \frac{a}{2\pi} d\mathbf{k}_i \right) \hat{f}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{n} a}, \quad (\mathbf{n} \in \mathbb{Z}^d).$$  \hfill (83)

The expressions given in equations (12) and (13) are recovered if $d = 1$. The Fourier transform of equations (80) and (81) reads

$$\frac{\partial \hat{S}(\mathbf{k}, t)}{\partial t} = \left[ - (r + \tau^{-1}) + \frac{1}{\tau d} \sum_{i=1}^d \cos(k_i a) \right] \hat{S}(\mathbf{k}, t) + r \hat{S}_0(\mathbf{k}),$$

$$\hat{S}(\mathbf{k}, 0) = \hat{S}_0(\mathbf{k}).$$  \hfill (84)

This initial-value problem is readily solved for any vector $\mathbf{k}$ in $\mathbb{R}^d$:

$$\hat{S}(\mathbf{k}, t) = \exp \left( \left[ - (r + \tau^{-1}) + \frac{1}{\tau d} \sum_{i=1}^d \cos(k_i a) \right] t \right) \hat{S}_0(\mathbf{k}) + r \int_0^t du \hat{S}_0(\mathbf{k}) \exp \left( \left[ - (r + \tau^{-1}) + \frac{1}{\tau d} \sum_{i=1}^d \cos(k_i a) \right] u \right).$$  \hfill (85)

Fourier inversion yields the opinion state profile at position $\mathbf{n} a$ and time $t$ as

$$S(\mathbf{n} a, t) = \left( \prod_{i=1}^d \int_{-\pi}^{\pi} \frac{a}{2\pi} d\mathbf{k}_i \right) \exp \left( \left[ - (r + \tau^{-1}) + \frac{1}{\tau d} \sum_{j=1}^d \cos(k_j a) \right] t - i\mathbf{k} \cdot \mathbf{n} a \right) \hat{S}_0(\mathbf{k})$$

$$+ r \int_0^t du \left( \prod_{j=1}^d \int_{-\pi}^{\pi} \frac{a}{2\pi} d\mathbf{k}_j \right) \hat{S}_0(\mathbf{k}) \exp \left( \left[ - (r + \tau^{-1}) + \frac{1}{\tau d} \sum_{j=1}^d \cos(k_j a) \right] u - i\mathbf{k} \cdot \mathbf{n} a \right).$$  \hfill (86)

Let us use the identity $e^{i(\tau d)^{-1} \cos(k_j a)} = \sum_{m \in \mathbb{Z}} I_m((\tau d)^{-1} t)e^{im\theta}$ every $j$ in $[1..d]$ (which is the generating function of the modified Bessel functions of the first kind, given in equation (17)). Taking the same steps as in equation (18), we obtain...
\[ S(\mathbf{n}a, t) = \exp(- (r + \tau^{-1}) t) \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{a}{2\pi} dk_j \right) \tilde{S}_0(\mathbf{k}) \prod_{l=1}^{d} \exp \left( \frac{t}{\tau d} \cos(k_l a) - iak_l n_l \right) \]

\[ + r \int_{0}^{t} du \exp(- (r + \tau^{-1}) u) \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{a}{2\pi} dk_j \right) \tilde{S}_0(\mathbf{k}) \prod_{l=1}^{d} \exp \left( \frac{u}{\tau d} \cos(k_l a) - iak_l n_l \right) \]

\[ = \exp(- (r + \tau^{-1}) t) \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{a}{2\pi} dk_j \right) \tilde{S}_0(\mathbf{k}) \prod_{l=1}^{d} \sum_{m_l \in \mathbb{Z}} \lambda_{m_l}((\tau d)^{-1} t) e^{im_l u} e^{-iak_l n_l} \]

\[ + r \int_{0}^{t} du \exp(- (r + \tau^{-1}) u) \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{a}{2\pi} dk_j \right) \tilde{S}_0(\mathbf{k}) \prod_{l=1}^{d} \sum_{m_l \in \mathbb{Z}} \lambda_{m_l}((\tau d)^{-1} u) e^{im_l u} e^{-iak_l n_l} \]

\[ = \exp(- (r + \tau^{-1}) t) \sum_{m \in \mathbb{Z}^d} \lambda_m((\tau d)^{-1} t) S_0(\mathbf{n}a - \mathbf{m}a), \quad (\mathbf{n} \in \mathbb{Z}^d), \] (87)

where the multi-index modified Bessel function \( I_m \) is defined as:

\[ I_m(t) = \prod_{j=1}^{d} I_{m_j}(t), \quad (\mathbf{m} \in \mathbb{Z}^d). \] (88)

The average opinion state at time \( t \) is therefore the convolution product of the initial average opinion state and a time-dependent kernel denoted again by \( \mathcal{K}_t \) (which generalizes equation (19) to dimension \( d \)):

\[ S(\mathbf{n}a, t) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \mathcal{K}_t(\mathbf{m}a) S_0(\mathbf{n}a - \mathbf{m}a), \]

with \( \mathcal{K}_t(\mathbf{m}a) := \exp(- (r + \tau^{-1}) t) I_m((\tau d)^{-1} t) \) (89)

\[ + r \int_{0}^{t} du \exp(- (r + \tau^{-1}) u) I_m((\tau d)^{-1} u), \quad (\mathbf{m} \in \mathbb{Z}^d). \]

Taking the Fourier transform of the discrete convolution product yields the product

\[ \hat{S}(\mathbf{k}, t) = \hat{\mathcal{K}}_t(\mathbf{k}) \hat{S}_0(\mathbf{k}). \] (90)

Moreover, we can read off the Fourier transforms \( \hat{S}(\mathbf{k}, t) \) from the integrand on the r.h.s. of equation (86). We deduce the Fourier transform of the kernel and its large-time limit:
\[ \mathcal{K}_r(\mathbf{k}) = \exp \left( \left[ - (r + \tau^{-1}) + \frac{1}{\tau^d} \sum_{j=1}^d \cos(k_j) \right] t \right) \]

\[ + r \int_0^t \exp \left( \left[ - (r + \tau^{-1}) + \frac{1}{\tau^d} \sum_{j=1}^d \cos(k_j) \right] u \right) du \]

\[ = \frac{r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right) e^{r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right)t}}{r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right) \mathcal{K}_\infty(\mathbf{k})} \]

\[ \mathcal{K}_\infty(\mathbf{k}) = \frac{r}{r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right)} \quad (\mathbf{k} \in \mathbb{R}^d). \]

We have thus generalized equations (24) and (25). Let us derive an upper bound on \( |S(na, t) - S(na, \infty)| \) by generalizing equation (29):

\[ |S(na, t) - S(na, \infty)| \]

\[ = \left| \left( \prod_{j=1}^d \frac{\pi}{2} a dk_j \right) \left[ \hat{S}_0(\mathbf{k}) \left( \hat{K}_r(\mathbf{k}) - \hat{K}_\infty(\mathbf{k}) \right) \right] e^{-i \mathbf{k} \cdot \mathbf{n}} \right| \]

\[ = e^{-\tau t} \left| \left( \prod_{j=1}^d \frac{\pi}{2} a dk_j \right) \hat{S}_0(\mathbf{k}) \left[ \frac{\tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right) e^{-r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right)t}}{r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right) \mathcal{K}_\infty(\mathbf{k})} \right] e^{-i \mathbf{k} \cdot \mathbf{n}} \right| \]

\[ \leq e^{-\tau t} \left| \left( \prod_{j=1}^d \frac{\pi}{2} a dk_j \right) \hat{S}_0(\mathbf{k}) e^{-r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right)t} \right| \left| \frac{\tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right)}{r + \tau^{-1} \left( 1 - d^{-1} \sum_{j=1}^d \cos(k_j) \right) \mathcal{K}_\infty(\mathbf{k})} \right| \]

\[ \leq e^{-\tau t} \left| \left( \prod_{j=1}^d \frac{\pi}{2} a dk_j \right) \hat{S}_0(\mathbf{k}) \right| e^{-r + (1 - d^{-1} \sum_{j=1}^d \cos(k_j))t}, \quad (\mathbf{n} \in \mathbb{Z}^d). \]

Again the upper bound on the r.h.s. is independent of the position \( na \), and goes to zero if \( \hat{S}_0 \) is integrable on the first Brillouin zone.

As an example, let us work out the solution for an initial condition corresponding to undecided voters, except for a single decided voter at the origin with positive opinion. At every site in the lattice it coincides with the time-dependent kernel \( \mathcal{K}_r \). Indeed, working out the discrete convolution in equation (89) yields

\[ S_0(na) = \delta_{n, 0}, \]

\[ S(na, t) = \mathcal{K}_r(na) \]

\[ = \exp \left( -(r + \tau^{-1}) t \right) I_n \left( (\tau d)^{-1} t \right) \]

\[ + r \int_0^t \exp \left( -(r + \tau^{-1}) u \right) I_n \left( (\tau d)^{-1} u \right), \quad n \in \mathbb{Z}^d, \]
which generalizes equation (31). The average opinion state is equal to the probability of presence of a diffusive random walker on a $d$-dimensional hypercubic lattice starting from the origin, and reset to its initial position at times generated by a Poisson process of intensity $r$.

The average opinion state in the steady state of the system is therefore (up to a factor or $rT$), the Laplace transform at $rT + 1$ of the probability of presence of a random diffusive walker starting from the origin on a hypercubic lattice.

Consider the average opinion state at the origin in the steady state. Let us can rewrite its expression obtained in equation (94) to obtain the solution of Problem (I(0)) defined in the Introduction:

$$S(0, \infty) = 2rT \mathcal{I}_d(2rr),$$

where in the third step we inserted the generating function of modified Bessel functions defined in equation (17). Moreover, we used parity to replace $e^{in\theta}$ with $\cos(nq_j)$ in the integrand. We may recover this expression by using the Fourier representation of $K_r(na)$ and using equation (91). Indeed, in this step we inserted the generating function of modified Bessel functions defined in equation (17). Moreover, we used parity to replace $e^{in\theta}$ with $\cos(nq_j)$ in the integrand. We may recover this expression by using the Fourier representation of $K_r(na)$.

$$S(na, \infty) = K_{\infty}(na) = \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \right) K_{\infty}(na) e^{-i\sum_{j=1}^{d} k_j n_j}$$

$$= \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \right) K_{\infty}(na) e^{-i\sum_{j=1}^{d} k_j n_j}$$

$$= \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \right) \frac{r}{r + r^{-1} \left( 1 - d^{-1} \sum_{j=1}^{d} \cos(k_j a) \right)} e^{-i\sum_{j=1}^{d} k_j n_j}$$

$$= \left( \prod_{j=1}^{d} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} \right) \frac{\cos(nq_j)}{r + 1 - \frac{1}{d} \sum_{j=1}^{d} \cos(q_j) }.$$

The stationary state of the average opinion state is therefore (up to a factor or $rT$), the Laplace transform at $rT + 1$ of the probability of presence of a random diffusive walker starting from the origin on a hypercubic lattice.

The Laplace transform of functions of time and other variables has been defined in equation (20).
with the notation (borrowed from chapter 8 of [5]):

\[
I_d(\epsilon) := \left( \prod_{i=1}^{d} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi_i \right) \frac{1}{\epsilon + \frac{1}{2} \sum_{i=1}^{d} (1 - \cos(\phi_i))}.
\]  

Let us check consistency with the results reported in section 2. In dimension one, equation (32) yield

\[
S(0, \infty) = r\tau I_0(r\tau + 1).
\]

Using the Fourier representation of the Bessel function \(I_0\), we calculate:

\[
2\pi \left( \int_{-\pi}^{\pi} d\phi \right) \frac{1}{r\tau + 1 - \cos(q)} = 2\pi \int_{-\pi}^{\pi} d\phi \frac{1}{2\pi r\tau + 2(1 - \cos(q))}.
\]

The general result obtained in equation (94) is therefore consistent with the solution of the model in dimension one. In the limit of zero resetting rate, the voter model under stochastic resetting reduces to the ordinary voter model and the opinion of the initially-decided voter should relax to the undecided state of the rest of the population. On the other hand, the behavior of the average opinion state in the steady state at low resetting rate depends on the behavior of \(I_d(2r\tau)\) when \(r \ll \tau^{-1}\). This behavior depends on the dimension and is worked out in the appendices: \(I_d(0)\) is finite if \(d \geq 3\). In particular, the quantity \(I_3(0)\) is known exactly, as one of the Watson integrals (see equation (158)). We obtain the equivalent \(I_2(2\tau\tau) \sim -2(2\pi)^{-1} \ln(r\tau)\) (see equation (162)). Moreover, the equivalent \(I_1(2\tau\tau) \sim (2\sqrt{2}r\tau)^{-1}\) (see equation (169)) allows to recover the result reported in equation (33). Substituting into equation (94) yields the equivalents

\[
S(0, \infty) \sim \begin{cases} 
2\pi \tau I_0(0), & (d \geq 3), \\
-\pi \ln(r\tau), & (d = 2), \\
\sqrt{\frac{\pi}{2}}, & (d = 1).
\end{cases}
\]  

In particular, the average opinion state at the origin has a finite susceptibility \(\frac{\partial S(0, \infty)}{\partial r}\) if and only if \(d \geq 3\).

**Continuum limit.** Let us take again the continuum limit by sending the lattice spacing \(a\) and the time \(\tau\) to zero, with a fixed diffusion coefficient \(D = a^2/(2\tau)\). Consider a position \(x\) in the lattice, with a large norm on the scale of the lattice spacing:

\[
x = na, \quad \text{with} \quad ||n|| \gg 1.
\]
On the other hand, the identity shifted by $K$ the identity in $\mathbb{R}^d$ probability density where in the last step we have worked out a Gaussian integral. We recognize the steady-state dimension $d$. To calculate density of domain walls, let us consider the two-point function of the model in $3.2$. Two-point function and density of domain walls

Changing integration variables to $t := \tau v$ and $k_j := a_j^{-1} q_j$, ($1 \leq j \leq d$) in the third row of equation (94), we obtain the density of opinion state in the diffusive limit as

$$a^{-d} S(\mathbf{a}, \infty) = r \int_0^\infty dw \left( \prod_{j=1}^d \int_{-\infty}^\infty \frac{dk_j}{2\pi} \right) \exp \left( -rw + \frac{w}{\tau d} \sum_{j=1}^d (1 + \cos(ak_j)) - i|k| \cdot \mathbf{x} \right)$$

$$a \to 0, \ |\mathbf{a}| \to \infty, \ x = \mathbf{n}, \ D = \frac{a^2}{t}$$

$$= r \int_0^\infty dwe^{-rw} \frac{1}{(4\pi Dw)^{\frac{d}{2}}} \exp \left( -\frac{||\mathbf{x}||^2}{4Dw} \right),$$

(101)

where in the last step we have worked out a Gaussian integral. We recognize the steady-state probability density $p^*(\mathbf{x})$ of a diffusive random walker under stochastic resetting to the origin in $\mathbb{R}^d$ [34]. The relevant Laplace transform can be expressed in terms of the modified Bessel function of the second kind $K_{1/2}$ (which appears because of the identity $\int_0^\infty dt e^{-\beta t} e^{-\gamma t} = 2(\gamma^{-1})^{\frac{1}{2}} K_{\frac{1}{2}}(2\sqrt{\beta\gamma})$). We just quote the result from [34]:

$$p^*(\mathbf{x}) = \left( \frac{r}{2\pi D} \right) \left( \frac{r}{D} ||\mathbf{x}|| \right)^{1-\frac{d}{2}} K_{1-\frac{d}{2}} \left( \sqrt{D^{-1}} ||\mathbf{x}|| \right).$$

(102)

The identity $K_{1/2}(y) = \sqrt{(2y)^{-1}} e^{-y}$ allows to recover equation (40) in dimension $d = 1$. The density $p^*(\mathbf{x})$ diverges at the origin in dimension $d \geq 2$, even though it integrates to 1 over $\mathbb{R}^d$ (see section 2.6 and figure 3 in [14] for a review and a plot).

3.2. Two-point function and density of domain walls

To calculate density of domain walls, let us consider the two-point function of the model in dimension $d$:

$$G(\mathbf{m}, \mathbf{n}, t) := \langle s(\mathbf{n}, t) s(\mathbf{m}, t) \rangle, \quad (\mathbf{m}, \mathbf{n}) \in \mathbb{Z}^d. \quad (103)$$

Let us derive its evolution equation. When the opinion state $s(\mathbf{x}, t)$ is flipped, its value is shifted by $-2s(\mathbf{x}, t)$. If $\mathbf{n} \neq \mathbf{m}$, the transition rates at sites $\mathbf{n}\mathbf{a}$ and $\mathbf{m}\mathbf{a}$ induce

$$\frac{\partial}{\partial t} G(\mathbf{m}, \mathbf{n}, t) = -2\langle s(\mathbf{n}, t) s(\mathbf{m}, t) [w(\mathbf{n}, t) + w(\mathbf{m}, t)] \rangle, \quad (\mathbf{m}, \mathbf{n}) \in \mathbb{Z}^d, \mathbf{m} \neq \mathbf{n}. \quad (104)$$

On the other hand, the identity $s(\mathbf{x}, t)^2 = 1$ implies that the two-point function at coinciding points is constant:

$$G(\mathbf{n}, \mathbf{n}, t) = 1, \quad (\mathbf{n} \in \mathbb{Z}, t \geq 0). \quad (105)$$

27
Substituting into equation (104) the transition rates defined in equation (78) and using the constraint $s(na)^2 = 1$ yields

$$
\frac{\partial}{\partial t} G(na, ma, t) = -2\left( r + \tau^{-1} \right) G(na, ma, t) \\
+ \frac{\tau^{-1}}{d} \sum_{i=1}^{d} \left[ G(na - ae_i, ma, t) + G(na + ae_i, ma, t) \\
+ G(na - ae_i, t) + G(na - ae_i, t) \right] \\
+ \frac{r}{2d} [H(na, ma, t) + H(ma, na, t)], \quad (m, n \in \mathbb{Z}^d, m \neq n).
$$

(106)

This equation generalizes equation (50). We used again the symbol $H$ to denote the two-time two-point function

$$
H(na, ma, t) := \langle s(na, t)s(ma, 0) \rangle, \quad (m, n \in \mathbb{Z}^d).
$$

(107)

The evolution equation of $H$ follows from the flipping rule as

$$
\frac{\partial H}{\partial t} (na, ma, t) = -2\langle w(na, t)s(na, t)s(ma, 0) \rangle \\
= - \left( r + \tau^{-1} \right) H(na, ma, t) + \frac{\tau^{-1}}{2d} \sum_{i=1}^{d} [H(na + e_i, ma, t) + H(na - e_i, ma, t)] \\
+ rG(na, ma, 0), \quad (m, n \in \mathbb{Z}^d),
$$

(108)

which generalizes equation (53). As in dimension one, we notice that the evolution equation of the two-time two-point function (for any fixed value of the second argument), satisfies the same evolution equation as the average opinion state (which in dimension $d$ is equation (80)).

Consider an initial condition with independent undecided voters. The system is again translationally invariant: the values of the functions $G$ and $H$ depend only the separation of the voters (and on time). Hence there exist two functions (denoted again by $G$ and $H$) such that

$$
G(na, ma, t) = G([m - n]a, t), \\
H(na, ma, t) = H([m - n]a, t), \quad (m, n \in \mathbb{Z}^d).
$$

(109)

Let us rewrite equations (106) and (108) in terms of the functions $G$ and $H$:

$$
\frac{\partial G}{\partial t} (na, t) = -2\left( r + \tau^{-1} \right) G(na, t) + \frac{1}{d} \sum_{i=1}^{d} [G((n + e_i)a, t) + G((n - e_i)a, t)] + rH(na, t) \\
+ rH(-na, t), \quad (n \in \mathbb{Z}^d, n \neq 0),
$$

(110)

$$
\frac{\partial H}{\partial t} (na, t) = - \left( r + \tau^{-1} \right) H(na, t) + \frac{\tau^{-1}}{2d} \sum_{i=1}^{d} [H((n + e_i)a, t) + H((n - e_i)a, t)] \\
+ rG(na, 0), \quad (n \in \mathbb{Z}^d).
$$

(111)

The above two equations generalize equations (57) and (62) to dimension $d$. The initial condition $H(na, ma, 0) = \langle s(na, 0)s(ma, 0) \rangle = \delta_{m,n}$ induces $H(na, 0) = \delta_{m,0}$. Hence $H(na, t)$ is
the probability of presence at site $na$ and time $t$ of a diffusive random walker on the hyper-
cubic lattice, subjected to stochastic resetting to its initial position at the origin. In particular $H(na,t) = H(-na,t)$ for all $n$ in $\mathbb{Z}^d$, and the last two terms in equation (110) are equal.

Moreover the constraint $s(x,t)^2 = 1$, which holds for every vertex $x$ in the lattice, induces

$$\mathcal{G}(0,t) = 1, \quad (t \geq 0). \tag{112}$$

The initial condition with independent undecided voters reads

$$\mathcal{G}(na,0) = \delta_{n,0}, \quad (n \in \mathbb{Z}^d). \tag{113}$$

Let us add a source term $J$ at the origin (an unknown function of time) so that the two-point function $\mathcal{G}$ satisfies the constraint of equation (112) at all times:

$$\frac{\partial \hat{\mathcal{G}}}{\partial t}(na,t) = -2 \left( r + \tau^{-1} \right) \hat{\mathcal{G}}(na,t) + \frac{1}{\tau^d} \sum_{i=1}^{d} \left[ \hat{G}(\langle n + e_i \rangle a,t) + \hat{G}(\langle n - e_i \rangle a,t) \right]$$

$$+ 2r \hat{H}(na,t) + J(t) \delta_{n,0}, \quad (n \in \mathbb{Z}^d), \tag{114}$$

This generalizes equation (59) to dimension $d$.

Taking the Fourier transform yields an ordinary differential equation

$$\frac{\partial \hat{\mathcal{G}}}{\partial t}(k,t) = -2 \left( r + \tau^{-1} \right) \hat{\mathcal{G}}(k,t) + \frac{2}{\tau^d} \sum_{j=1}^{d} \cos(k_j a) \hat{\mathcal{G}}(k,t) + 2r \hat{H}(k,t) + J(t), \tag{115}$$

together with the initial condition $\hat{\mathcal{G}}(k,0) = 1$. This initial-value problem is readily solved with the as

$$\hat{\mathcal{G}}(k,t) = e^{2\alpha(k)t} + 2r \int_{0}^{t} dt e^{2\alpha(k)(t-u)} \hat{H}(k,u) + \int_{0}^{t} dt e^{2\alpha(k)(t-u)} J(u),$$

with $\alpha(k) := - (r + \tau^{-1}) + (\tau d)^{-1} \sum_{i=1}^{d} \cos(k_i a), \quad (k \in \mathbb{R}^d). \tag{116}$

We need the Fourier transform of the function $H$. It is obtained by taking the Fourier transform of equation (111) and solving with the initial condition $\hat{H}(k,0) = 1$:

$$\hat{H}(k,t) = e^{\alpha(k)t} + r \int_{0}^{t} dt e^{\alpha(k)(t-u)} \hat{H}(k,u) + \int_{0}^{t} dt e^{\alpha(k)(t-u)} J(u), \tag{117}$$

Substituting into equation (116) yields

$$\hat{\mathcal{G}}(k,t) = e^{2\alpha(k)t} + 2r \int_{0}^{t} dt e^{2\alpha(k)(t-u)} \hat{H}(k,u) + \int_{0}^{t} dt e^{2\alpha(k)(t-u)} J(u). \tag{118}$$

Inverting the Fourier transform yields the two-point function at time $t$ for initially undecided voters in dimension $d$ as

$$\mathcal{G}(na,t) = \left( \prod_{i=1}^{d} \int_{-\frac{\pi}{a}}^{0} \frac{d\xi_i}{2\pi} \right) e^{-in\cdot k a} \hat{\mathcal{G}}(k,t). \tag{119}$$
We can express the inverse Fourier transform of $e^{2\alpha(k)t}$ in terms of modified Bessel functions of the first kind, indeed

$$
\left( \prod_{j=1}^{d} \int_{ \mathbb{R} } \frac{ \alpha }{ 2\pi } dk \right) e^{-i\sum_{j=1}^{d} \eta_j k_j + \alpha(k)t} = e^{-2(r+r^{-1})} \left( \prod_{j=1}^{d} \int_{ \mathbb{R} } \frac{ \alpha }{ 2\pi } dk \right) e^{-i\sum_{j=1}^{d} \eta_j k_j} e^{\frac{2\alpha}{t} \sum_{j=1}^{d} \cos(k_j)}
$$

$$
= e^{-2(r+r^{-1})} \left( \prod_{i=1}^{d} \left( \int_{ -\pi }^{ \pi } \frac{ \alpha }{ 2\pi } dk_i e^{-imk_i} \sum_{m \in \mathbb{Z}} I_m \left( 2(\tau d)^{-1} \right) e^{imk_i} \right) \right)
$$

$$
= e^{-2(r+r^{-1})} \sum_{m \in \mathbb{Z}} I_m \left( 2(\tau d)^{-1} \right) \delta_{nm}
$$

$$
= e^{-2(r+r^{-1})} \prod_{i=1}^{d} I_n \left( 2(\tau d)^{-1} \right),
$$

where in the third step we used the generating function of modified Bessel functions defined in equation (17). The two-point function in real space

$$
\mathcal{G}(\mathbf{n}a,t) = e^{-2(r+r^{-1})} \prod_{i=1}^{d} I_{n} \left( 2(\tau d)^{-1} \right) + 2r \left( \prod_{i=1}^{d} \int_{ -\pi }^{ \pi } \frac{ \alpha }{ 2\pi } dk_i \right) \int_{0}^{\tau} du e^{2\alpha(k)(t-u)} I_m \left( 2(\tau d)^{-1} \right) \bar{\mathcal{H}}(k,u)
$$

$$
+ \int_{0}^{\tau} du J(u) e^{-2(r+r^{-1})(t-u)} \prod_{i=1}^{d} I_{n} \left( 2(\tau d)^{-1} \right) (t-u).
$$

We have not yet imposed the constraint $\mathcal{G}(0,t) = 1$, or calculated the source $J$. Let us do so in Laplace space. The Laplace transform of the constraint is $\mathcal{G}(0,s) = s^{-1}$. On the other hand, we can express $\mathcal{G}(0,s)$ by considering equation (121) in the special case $\mathbf{n} = 0$, and taking the Laplace transform. The convolution product is mapped to an ordinary product by the Laplace transform, hence

$$
\frac{1}{s} = \varphi_{0,d}(s) + 2r \Psi_{0,d}(s) + \varphi_{0,d}(s) \mathcal{J}(s),
$$

with the notations

$$
\varphi_{n,d}(t) := e^{-2(r+r^{-1})t} \left[ I_n(2(\tau d)^{-1}t) \right]^{d-1} \left( I_n(2(\tau d)^{-1}t) \right),
$$

$$
\Psi_{n,d}(s) := \left[ \prod_{i=1}^{d} \left( \frac{ \alpha }{ 2\pi } \right) \frac{1}{ s - 2\alpha(k) } \right] e^{-inkd} \left( \frac{1}{ s - 2\alpha(k) } \right), \quad (n \in \mathbb{Z}).
$$

These notations generalize the ones introduced in equation (67) for the one-dimensional model, with $\varphi_{n,d} = \varphi_{n}$ and $\Psi_{n,d} = \Psi_{n}$.

The system is translationally invariant, moreover it is isotropic, hence (by applying the reasoning of equation (8) in dimension $d$) we can express the density of domain walls at time
\( t \), denoted again by \( \varrho(t) \), in terms of the function \( \mathcal{G} \). Indeed, for any vertex \( \mathbf{x} \) in the hypercubic lattice,

\[
1 - 2\varrho(t) = \text{Prob}(s(\mathbf{x}, t) = +1, s(\mathbf{x} + \mathbf{e}_1, t) = +1) + \text{Prob}(s(\mathbf{x}, t) = -1, s(\mathbf{x} + \mathbf{e}_1, t) = -1) \\
- \text{Prob}(s(\mathbf{x}, t) = -1, s(\mathbf{x} + \mathbf{e}_1, t) = +1) - \text{Prob}(s(\mathbf{x}, t) = +1, s(\mathbf{x} + \mathbf{e}_1, t) = -1) \\
= \langle s(\mathbf{x}, t) s(\mathbf{x} + \mathbf{e}_1, t) \rangle \\
= \mathcal{G}(\mathbf{a}\mathbf{e}_1, t).
\]  

(124)

The above equation generalizes equation (69) to dimension \( d \).

Using equation (121) in the special case \( \mathbf{n} = \mathbf{e}_1 \), the Laplace transform of the above equation is obtained as

\[
\frac{1}{s} - 2\check{\varrho}(s) = \mathcal{G}(\mathbf{e}_1, s) = \check{\varphi}_{1,d}(s) + 2r\check{\varphi}_{1,d}(s) + \check{\varphi}_{1,d}(s)\check{J}(s).
\]  

(125)

Combining equations (122) and (125) yields a linear system in \( \check{J}(s) \), \( \check{\varrho}(s) \). The solution reads

\[
\check{J}(s) = \frac{1}{s\check{\varphi}_{0,d}(s)} - \frac{2r}{\check{\varphi}_{0,d}(s)} - 1, \\
\check{\varrho}(s) = \frac{1}{2} \left[ \frac{1}{s} - 2r\check{\varphi}_{1,d}(s) + 2r\frac{\dot{\check{\varphi}}_{1,d}(s)}{\check{\varphi}_{0,d}(s)} \check{\varphi}_{0,d}(s) - \frac{\dot{\check{\varphi}}_{1,d}(s)}{s\check{\varphi}_{0,d}(s)} \right].
\]  

(126)

We obtain the steady-state density of domain walls in dimension \( d \) (denoted again by \( \varrho_\infty \), as in equation (73)) by applying the final-value theorem:

\[
\varrho_\infty = \lim_{s \to 0} (s\check{\varrho}(s)) \\
= \frac{1}{2} \lim_{s \to 0} \left( 1 - 2r\check{\varphi}_{1,d}(s) + 2r\frac{\dot{\check{\varphi}}_{1,d}(s)}{\check{\varphi}_{0,d}(s)} \check{\varphi}_{0,d}(s) - \frac{\dot{\check{\varphi}}_{1,d}(s)}{s\check{\varphi}_{0,d}(s)} \right).
\]  

(127)

To obtain a more explicit expression of this density, we need the Fourier–Laplace transform

\( \check{H}(k, s) \), which is the Laplace transform of equation (117). The Laplace transforms \( \check{\varphi}_{0,d}(s) \) and \( \dot{\check{\varphi}}_{1,d}(s) \) are worked out in the appendix (equation (137)) in integral form. The relevant limits are derived in the appendix B. They result in the following expression, which is the solution of Problem (H(\( r \))) in dimension \( d \):

\[
\varrho_\infty = \frac{1}{4\mathcal{I}_d(2rr)} - \frac{rr}{2} - (rr)^2 \left[ \mathcal{I}_d(2rr) + \frac{\mathcal{I}_d'(2rr)}{\mathcal{I}_d(2rr)} \right],
\]  

(128)

where \( \mathcal{I}_d(\epsilon) \) is the function defined in equation (97).

The steady-state density of domain walls is plotted on figure 6 for \( d \leq 5 \). It is an increasing function of the resetting rate, which is intuitive because the resetting processes drive the system away from consensus.

The local behavior of \( \varrho_\infty(\epsilon) \) for \( \epsilon \) close to zero is worked out in appendix C (equations (150), (155), (161) and (166)). These results complement the square-root singularity obtained in equation (75) in dimension one. They yield the behavior of \( \varrho_\infty \) in the limit where the resetting rate \( r \) is small compared to the flipping rate \( r^{-1} \):

\[
\varrho_\infty \sim \frac{1}{4\mathcal{I}_d(2rr)} - \frac{rr}{2} - \frac{(rr)^2}{\mathcal{I}_d(2rr)}.
\]  

31
Figure 6. The steady-state density of domain walls for initial conditions consisting of independent initially undecided voters. There is consensus in the limit of zero resetting rate in dimension $d \leq 2$. The tangent to the graph at $r = 0$ is vertical for $d \leq 4$ (with a square-root singularity if $d = 1$ or $d = 3$, and an inverse-logarithmic singularity if $d = 3$). It has a finite slope for $d \geq 5$, given by the susceptibility to the resetting rate expressed in equation (130).

$$
\varrho_\infty = \begin{cases} 
\sqrt{\frac{\pi}{2}} + o(\sqrt{r\tau}), & (d = 1), \\
-\frac{\pi}{2 \ln(\sqrt{r\tau})} + o\left(\frac{1}{\ln(\sqrt{r\tau})}\right), & (d = 2), \\
\frac{1}{2 \sqrt{2\pi r\tau}} \sqrt{r\tau} + o\left(\sqrt{r\tau}\right), & (d = 3), \\
\frac{1}{2 \sqrt{2\pi r\tau}} - \frac{\pi^2}{12} \int_0^\infty r\tau \ln(r\tau) + o(r\tau \ln(r\tau)), & (d = 4), \\
\frac{1}{2 \sqrt{2\pi r\tau}} - \frac{1}{2} \left[1 + \frac{I_d'(0)}{I_d(0)}\right] r\tau + o(r\tau), & (d \geq 5). 
\end{cases}
$$

In particular, $\varrho_\infty$ goes to zero when the resetting rate goes to zero if and only if $d \leq 2$. This is consistent with the fact that the value of $\varrho_\infty$ in the ordinary voter model is zero (meaning there is consensus in the steady state) if and only if $d \leq 2$. Moreover, the non-zero limit of $\varrho_\infty$ when $r$ goes to zero (in dimension $d \geq 3$) is the value of the steady-state density of domain walls in the ordinary voter model [2].

Moreover, if the dimension in greater than or equal to five, the steady-state density of domain walls has a finite susceptibility to the resetting rate:

$$
\frac{\delta \varrho_\infty}{\delta r} \bigg|_{r=0} = -\frac{\tau}{2} \left(1 + \frac{I_d'(0)}{I_d(0)\tau}\right), \quad (d \geq 5). 
$$

(130)
4. Discussion

In this work we have introduced a toy model of stubborn voters on a hypercubic lattice by resetting the opinion of each of the voters to its initial state at Poisson-distributed times. The extra terms in the flipping rate corresponding to our local resetting prescription induce kinetic equations for the one- and two-point functions. Our derivations are not based on a renewal argument (unlike models of single-particle systems under resetting considered for instance in [11, 35]).

As in the ordinary voter model, the one-point function can be calculated exactly in any dimension. If the initial condition consists of a single decided voter at the origin, the average opinion state has the same expression as the probability of presence of a diffusive random walker on the lattice, whose position is stochastically reset to the origin. Moreover, for more general initial conditions, the one-point function is the convolution of this particular solution and the initial opinion-state profile.

The evolution equation of the two-point function \( G(x, y, t) = \langle s(x, t) s(y, t) \rangle \) contains terms involving the two-time two-point function \( H(x, y, t) = \langle s(x, t) s(y, 0) \rangle \). This correlator is easily calculated: for any fixed \( y \), it satisfies the same equation as the average opinion state. Introducing a source at the origin to ensure that \( G(x, x, t) = 1 \), we solved the evolution equation of the two-point function in Laplace space, for initial conditions consisting of independent undecided voters. The steady-state density of domain walls \( \varrho_\infty \) follows in closed form in terms of Watson-like integrals. It is a smooth, increasing function of the quantity \( \tau r \) on \([0, \infty[^d\). In dimension \( d \geq 2 \), the density \( \varrho_\infty \) of domain walls in the steady state goes to zero when the resetting rate \( r \) becomes small (compared to the maximum frequency \( \tau^{-1} \) at which opinion states flip in the ordinary voter model). This limit is consistent with the consensus achieved in the ordinary model. There is a square-root singularity at zero resetting rate in dimension one, and an inverse logarithmic singularity at zero resetting rate in dimension two. For \( d \geq 5 \), the function is differentiable at 0.

We may ask how these results could be applied to model real-world systems. The solution of the ordinary voter model was motivated by the kinetics of the dimer-dimer reaction model [1, 2, 36]. In this model, the sites of the lattice are filled with adsorbed monomers of two kinds, say \( A \) and \( B \). Monomers adsorbed on two adjacent sites can react if they are of different kinds. The resulting dimer \( AB \) immediately desorbs and is replaced with an \( A_2 \) or \( B_2 \) dimer with equal probability, which reproduces the dynamics of the voter model (with \( A \) and \( B \) representing the two possible values of the opinion state). In the dimer-dimer reaction model, the resetting prescription is interpreted as follows: the substrate acts like a source of monomers, which replaces (at rate \( r \)) any adsorbed monomer with a monomer of the kind that was adsorbed at the same site in the initial configuration. The dimensions \( d = 1 \) and \( d = 2 \) are of particular interest, as they are the possible dimensions of a substrate: it is difficult to imagine a realization of the dimer-dimer reaction model under resetting in dimension \( d \geq 3 \). The voter model under stochastic resetting is therefore more promising as a model of stylized facts in opinion formation in a social network, where connectivity can be high. Keeping the connectivity fixed throughout the network, one could try to make the model more realistic by allowing the resetting rate to vary across the lattice (for developments on space-dependent resetting rate for Brownian particles, see [37, 38]). Moreover, it would be interesting to adapt the resetting prescription to heterogeneous networks, with uncorrelated distributions of degrees as studied in [39, 40].
Our resetting prescription is local in the sense that the opinion state at different sites are reset at independent times. Local resetting may put the system in a configuration that has never been explored before, on the occasion of the resetting of the opinion state of one voter. Moreover, resetting the opinion state of a single voter does not decrease the number of voters that have never changed their opinion state since the beginning of the process. It would be interesting to explore the decay of the number of persistent spins [41–43] in the voter model under stochastic resetting.

Data availability statement

No new data were created or analysed in this study.

Appendix A. Laplace transforms

A.1. Expressions needed in dimension one

To evaluate the Laplace transform of the modified Bessel functions of the first kind, we used the identity

\[ \tilde{I}_n(s) = \frac{1}{\sqrt{s^2 - 1}} \left( s + \sqrt{s^2 - 1} \right)^{|n|}, \]  

which is the tabulated formula (4.16(1)) in [44]. It is valid for \( s > 1 \), and implies the expression of \( \tilde{I}_n(1 + r\tau) \) as reported in equation (21).

The Laplace transforms of the function denoted by \( \tilde{\varphi}_n \) in equation (67) is calculated by changing the integration variable to \( u := 2\tau^{-1}t \) and using equation (131):

\[ \tilde{\varphi}_n(s) = \int_0^\infty e^{-stu} e^{-2(r+\tau^{-1})u} I_n(2\tau^{-1}u) \, du = \frac{1}{2\tau^{-1}} \int_0^\infty e^{-stu} e^{-(r+\tau^{-1})u} I_n(u) \, du \]
\[ = \frac{\tau}{2} \hat{I}_n \left( r\tau + 1 + \frac{st}{2} \right) \]
\[ = \frac{\tau}{2} \frac{1}{\sqrt{(r\tau + 1 + \frac{st}{2})^2 - 1}} \left( r\tau + 1 + \frac{st}{2} + \sqrt{(r\tau + 1 + \frac{st}{2})^2 - 1} \right)^{|n|}, \quad (n \in \mathbb{N}, s \geq 0). \]  

(132)

For a fixed value \( r > 0 \) of the resetting rate, the function \( \tilde{\varphi}_n(s) \) has a finite limit when \( s \) goes to zero. Hence the limit

\[ \lim_{s \to 0} \tilde{\varphi}_1(s) = \frac{1}{r\tau + 1 + \sqrt{(r\tau + 1)^2 - 1}}, \]  

which is used in equation (74).
The quantity $\Psi_n(s)$ introduced in equation (67) is expressed using the Fourier–Laplace transform of the two-point function $H$, which is obtained from equation (63) by calculating integrals of exponential functions:

$$
\tilde{H}(k,s) = \int_0^\infty dt e^{-st} \left[ \exp \left( \left( \tau^{-1} \cos (ka) - (r+\tau^{-1}) \right) t \right) + \frac{r [\exp \left( \left( \tau^{-1} \cos (ka) - (r+\tau^{-1}) \right) t \right) - 1]}{\tau^{-1} \cos (ka) - (r+\tau^{-1})} \right],
$$

where

$$
\Psi_n(s) = \frac{a}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{s + 2 (r + \tau^{-1} - \tau^{-1} \cos (ka))} \tilde{H}(k,s)
$$

and

$$\begin{align*}
\Psi_n(s) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \frac{\cos (nq)}{s + 2 (r + \tau^{-1} - \tau^{-1} \cos (q))} \left[ \frac{1}{s + r + \tau^{-1} - \tau^{-1} \cos (q)} - \frac{1}{s} \right], \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dq}{s + 2 (r + \tau^{-1} - \tau^{-1} \cos (q))} \left[ \frac{1}{s + r + \tau^{-1} - \tau^{-1} \cos (q)} - \frac{1}{s} \right], \quad (n \in \mathbb{N}, s \geq 0),
\end{align*}
$$

where we have changed the integration variable to $q := ka$ and used parity, since $e^{-inka}$ is multiplied in the integrand by an even function of $k$ (a quantity that depends on $k$ through $\cos(ka)$ only). The last term in the above equation allows to read off the limit

$$
\lim_{s \to 0} (rs \Psi_n(s)) = \frac{r^2 \tau^2}{2\pi} \int_{-\pi}^{\pi} dx \frac{\cos (nx)}{2 (r\tau + 1 - \cos(x))^2}.
$$

This limit is used in the cases $n = 0$ and $n = 1$ in equation (74).

A.2. Expressions needed in dimension $d$

The Laplace transform of the function $\varphi_{n,d}$ defined in equation (123) is obtained in integral form by changing variable to $u = 2(\tau d)^{-1} t$ and inserting the Fourier representation of each of the modified Bessel functions:
\[ \tilde{\varphi}_{n,d}(s) = \int_0^\infty dt e^{-(s+2r+2r^{-1})t} \left[ I_0(2(\tau d)^{-1}t) \right]^{d-1} I_n(2(\tau d)^{-1}t) \]

\[ = \frac{\tau d}{2} \int_0^\infty du e^{-(\frac{s}{2} + \tau r d + d - \sum_{i=1}^{d} \cos(k_i a))} \left[ \prod_{i=1}^{d} \left( \frac{a}{2\pi} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk_i \right] e^{\imath k_i a} \phi \sum_{i=1}^{d} \cos(k_i a) \]

\[ = \left[ \prod_{i=1}^{d} \left( \frac{a}{2\pi} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk_i \right] e^{\imath k_i a} \frac{\tau d}{2} \int_0^\infty du \exp \left( -u \left( \frac{s}{2} + \tau r d + d - \sum_{i=1}^{d} \cos(k_i a) \right) \right) \]

\[ = \left[ \prod_{i=1}^{d} \left( \frac{a}{2\pi} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk_i \right] e^{\imath k_i a} \int_0^\infty dv \exp \left( -v \left( s + 2r + \frac{2}{\tau d} \left( d - \sum_{i=1}^{d} \cos(k_i a) \right) \right) \right) \]

\[ = \left[ \prod_{i=1}^{d} \left( \frac{a}{2\pi} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk_i \right] e^{\imath k_i a} \frac{\cos(qn\lambda)}{s + 2r + \frac{2}{\tau d} \left( d - \sum_{i=1}^{d} \cos(q_i) \right)} \]  

(137)

In the last step we introduced the variables \( q_i = ak_i \) for \( i \) in \([1..d]\) and used parity. The above expression is used in equation (140).

Moreover, we need to take the Laplace transform of the quantity \( \tilde{H}(k,t) \) expressed in equation (117), to calculate the quantity \( \Psi_{n,d}(s) \) defined in equation (123). The operation yields

\[ \Psi_{n,d}(s) = \left[ \prod_{i=1}^{d} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{a}{2\pi} \right) dk_i \right] e^{-\imath k_i a} \frac{1}{s - 2\alpha(k)} \left[ \frac{1}{s - \alpha(k)} + \frac{r}{\alpha(k)} \left( \frac{1}{s - 2\alpha(k)} - \frac{1}{s} \right) \right]. \]  

(138)

**Appendix B. Expression of the steady-state density of domain walls in dimension \( d \)**

Let us work out the limits needed to express the steady-state density of domain walls obtained in equation (127). The last term in equation (138), which is proportional to \( s^{-1} \), yields the limit

\[ \lim_{s \to 0} (2rs\Psi_{n,d}(s)) = 2r^2 \left[ \prod_{i=1}^{d} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{a}{2\pi} \right) dk_i \right] e^{-\imath k_i a} \frac{\cos(qn\lambda)}{2\alpha(k)^2} \]

\[ = (rr)^2 \left[ \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right] \frac{\cos(qn\lambda)}{(rr + \frac{1}{2} \sum_{i=1}^{d} \cos(q_i))}, \]

where we introduced the integration variables \( q_i := ak_i \), \((1 \leq i \leq d)\), and used parity.
Both of the expressions $\tilde{\varphi}_{0,d}(s)$ and $\tilde{\varphi}_{1,d}(s)$ have a finite limit when $s$ goes to zero because the resetting rate is positive. The cases $n = 0$ and $n = 1$ in equation (137) yield

$$
\lim_{s \to 0} \tilde{\varphi}_{0,d}(s) = \tau I_d(2r\tau),
$$

$$
\lim_{s \to 0} \tilde{\varphi}_{1,d}(s) = \tau C_d(2r\tau),
$$

(140)

where the function $I_d$ is the one defined in equation (97), and

$$
C_d(\epsilon) : = \left[ \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right] \frac{\cos(q_d)}{\epsilon + \frac{1}{2} \sum_{i=1}^{d} (1 - \cos(q_i))}.
$$

(141)

The density of domain walls in the steady state in dimension $d$ follows from equation (127) as

$$
\varphi_\infty = \frac{1}{2} \left[ 1 - (r\tau)^2 \left( \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right) \frac{\cos(q_1)}{(r\tau + \frac{1}{2} \sum_{i=1}^{d} (1 - \cos(q_i))}^2 \right] - \frac{C_d(2r\tau)}{I_d(2r\tau)} \left( 1 - (r\tau)^2 \left[ \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right] \frac{1}{(r\tau + \frac{1}{2} \sum_{i=1}^{d} (1 - \cos(q_i))}^2 \right].
$$

(142)

The remaining integral terms in the above expression are related to the derivatives of the functions $I_d$ and $C_d$ defined in equations (97) and (141):

$$
I_d'(\epsilon) = - \left[ \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right] \frac{1}{\left[ \epsilon + \frac{1}{2} \sum_{i=1}^{d} (1 - \cos(q_i)) \right]^2},
$$

$$
C_d'(\epsilon) = - \left[ \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right] \frac{\cos(q_d)}{\left[ \epsilon + \frac{1}{2} \sum_{i=1}^{d} (1 - \cos(q_i)) \right]^2}.
$$

(143)

Substituting into equation (142) yields

$$
\varphi_\infty = \frac{1}{2} \left[ 1 + 4(r\tau)^2 C_d'(2r\tau) - \frac{C_d(2r\tau)}{I_d(2r\tau)} \left( 1 + 4(r\tau)^2 I_d'(2r\tau) \right) \right].
$$

(144)

Moreover, we can easily express $C_d(\epsilon)$ in terms of $I_d(\epsilon)$, because the denominator in the integrand is symmetric in the variables $q_1, \ldots, q_d$. Indeed, for any positive $\epsilon$,

$$
1 = \left( \prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi} \right) \frac{\epsilon + 2 - \frac{1}{2} \sum_{j=1}^{d} \cos(q_j)}{\epsilon + 2 - \frac{1}{2} \sum_{j=1}^{d} \cos(q_j)}
$$

$$
= (\epsilon + 2) I_d(\epsilon) - d \times \frac{2}{d} C_d(\epsilon),
$$

(145)

hence

$$
C_d(\epsilon) = \left( \frac{\epsilon}{2} + 1 \right) I_d(\epsilon) - \frac{1}{2}.
$$

(146)
Differentiating equation (146) yields
\[ \mathcal{C}'_d(\epsilon) = \left( \frac{\epsilon}{2} + 1 \right) \mathcal{I}'_d(\epsilon) + \frac{1}{2} \mathcal{I}_d(\epsilon). \] \hfill (147)

Substituting equations (146) and (147) into equation (144) yields equation (128) reported in the main text.

Appendix C. Local behaviour of the density of domain walls at small resetting rate

The quantity \( \mathcal{I}_d(\epsilon) \) is defined by an integral in equation (97). If \( \epsilon = 0 \), the denominator of the integrand takes the value 0 at the origin (at \( \mathbf{q} = \mathbf{0} \)). However, on a \((d - 1)\)-dimensional sphere of small radius \( \rho \), the integrand is equivalent to \( \rho^{-2} \), because
\[ \frac{1}{d} \sum_{i=1}^{d} (1 - \cos(q_i)) \rho^{d-3} \sum_{i=1}^{d} q_i^2 = \rho^d \sum_{i=1}^{d} q_i^2. \] \hfill (148)

Moreover, the integration measure contributes a factor of \( S_{d-1} \rho^{d-2} \, d\rho \) (where \( S_{d-1} \) denotes the measure of the \((d - 1)\)-dimensional unit sphere of equation \( \sum_{i=1}^{d} q_i^2 = 1 \)). The quantity \( \mathcal{I}_d(\epsilon) \) therefore goes to infinity when \( \epsilon \) goes to zero if \( d \leq 2 \) (and it goes to a finite limit denoted by \( \mathcal{I}_d(0) \) if \( d \geq 3 \)).

On a sphere of radius \( \rho \) centered at the origin, the integrand in the expression of \( \mathcal{I}'_d(\epsilon) \) (see equation (143)) is equivalent to \( \rho^{-4} \) when the radius \( \rho = ||\mathbf{q}|| \) goes to zero. Hence \( \mathcal{I}'_d(\epsilon) \) has a finite limit when \( \epsilon \) goes to zero (and \( \mathcal{I}_d \) is differentiable at zero) if \( d \geq 5 \). Moreover, \( \mathcal{I}'_d(\epsilon) \) goes to infinity when \( \epsilon \) goes to zero if \( d \leq 4 \). These observations allow to work out equivalents of the quantity \( \varphi_\infty \) at low resetting rate.

In the following calculations, Taylor expansions and equivalents are meant in the limit \( \tau \tau \ll 1 \). Repeated use is made of the change of integration variables from \( q_i \) to \( y_i = (2d\tau\tau)^{-\frac{1}{2}} q_i \), for \( i \) in \([1,d] \). It is motivated by the following Taylor expansion of the denominator, in the integrand involved in the definition of the function \( \mathcal{I}_d(2\tau\tau) \):
\begin{align*}
2\tau\tau + \frac{2}{d} \sum_{i=1}^{d} (1 - \cos(q_i)) &= 2\tau\tau + \frac{2}{d} \sum_{i=1}^{d} (2d\tau\tau)^{-\frac{1}{2}} \frac{y_i^2}{2} + o(\tau\tau) \sim (2\tau\tau) \left( 1 + \sum_{i=1}^{d} y_i^2 \right). \hfill (149)
\end{align*}

C.1. Dimension \( d \geq 5 \)

As the function \( \mathcal{I}_d \) is differentiable at 0, we obtain the following Taylor expansion of equation (128) at order 1 in \( \tau \tau \ll 1 \):
\begin{align*}
\varphi_\infty &= \frac{1}{4[\mathcal{I}_d(0) + 2\tau\tau \mathcal{I}'_d(0) + o(\tau\tau)]} \frac{\tau\tau}{2} + o(\tau\tau) \\
&= \frac{1}{4\mathcal{I}_d(0)} \left[ 1 - 2\tau\tau \frac{\mathcal{I}'_d(0)}{\mathcal{I}_d(0)} + o(\tau\tau) \right] \frac{\tau\tau}{2} + o(\tau\tau) \\
&= \frac{1}{4\mathcal{I}_d(0)} - \frac{1}{2} \left[ 1 + \frac{\mathcal{I}'_d(0)}{\mathcal{I}_d(0)} \right] \tau\tau + o(\tau\tau). \hfill (150)
\end{align*}
C.2. Dimension $d = 4$

The function $\mathcal{I}_d$ is not differentiable at 0. The change of variables

$$y_i := \frac{q_i}{\sqrt{8 \tau \times \pi}}, \quad (1 \leq i \leq d), \quad (151)$$

yields

$$\mathcal{I}_d(2\tau r) - \mathcal{I}_d(0) = \left(\prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi}\right) \frac{2\tau r}{\left[2\tau r + \frac{2}{3} \sum_{j=1}^{d} (1 - \cos(q_j))\right]^{3/2}}$$

$$\sim \frac{2(2\tau d)^{2/3}}{(2\tau)^{2}} \left(\frac{4}{\pi} \int_{-\pi}^{\pi} \frac{1}{y^2} \frac{d\rho}{\sqrt{1 + \rho^2}}\right)$$

$$\sim -\frac{2(8\tau)^{2/3}}{(2\tau)^{2}} \frac{S_3}{(2\pi)^{2}} \int_{0}^{\pi} \frac{\rho^3 d\rho}{(1 + \rho^2)^{3/2}} = O(\ln(\tau)),$$

(152)

where $S_3 = 2\pi^2$ is the area of the three-sphere. The quantity $\mathcal{I}_d'(2\tau r)$ diverges logarithmically when $\tau r$ goes to zero, indeed

$$\mathcal{I}_d'(2\tau r) = -\left(\prod_{i=1}^{d} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi}\right) \frac{1}{\left[2\tau r + \frac{2}{3} \sum_{j=1}^{d} (1 - \cos(q_j))\right]^{2/3}}$$

$$\sim \frac{(8\tau)^{2/3}}{(2\tau)^{2}} \frac{S_3}{(2\pi)^{2}} \int_{0}^{\pi} \frac{\rho^3 d\rho}{(1 + \rho^2)^{3/2}} = O(\ln(\tau)).$$

Moreover, $\mathcal{I}_d(0)$ is finite. Hence all the terms in the expression of $\varrho_{\infty}$ apart from $(4\mathcal{I}_d(2\tau r))^{-1}$ are negligible compared to $\tau r \ln(\tau r)$:

$$(\tau r)^2 \frac{\mathcal{I}_d'(2\tau r)}{\mathcal{I}_d(0)^2} = o(\tau r \ln(\tau r)),$$

$$\tau r = o(\tau r \ln(\tau r)),$$

$$\tau r \mathcal{I}_d(2\tau r) = o(\tau r \ln(\tau r)).$$

(154)

Combining with the equivalent of $\mathcal{I}_d(2\tau r) - \mathcal{I}_d(0)$ derived in equation (152), we obtain the following expansion of the density of domain walls when $\tau r \ll 1$:

$$\varrho_{\infty} = \frac{1}{4 \left[ \mathcal{I}_d(0) + \frac{2}{\pi} \tau r \ln(\tau r) + o(\tau r \ln(\tau r)) \right]} + o(\tau r \ln(\tau r))$$

$$= \frac{1}{4\mathcal{I}_d(0)} - \frac{1}{2\pi^2 \mathcal{I}_d(0)^2} \tau r \ln(\tau r) + o(\tau r \ln(\tau r))$$

$$\sim 0.4036 - 0.1320 \tau r \ln(\tau r) + o(\tau r \ln(\tau r)),$$

(155)
C.3. Dimension $d = 3$

The function $I_3$ is not differentiable at 0. The change of variables

$$y_i := \frac{q_i}{\sqrt{6 \times r\tau}}, \quad (1 \leq i \leq 3),$$

(156)

yields

$$I_3(2r\tau) - I_3(0) = \frac{2r\tau}{\prod_{i=1}^{3} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi}} \left( \frac{2r\tau}{2r\tau + \frac{3}{2} \sum_{j=1}^{3} (1 - \cos(q_j))} \right) \left( \frac{1}{1 + \sum_{j=1}^{3} y_j^2} \right)^{\frac{d}{2}} \left( \frac{1}{\sum_{k=1}^{3} y_k^2} \right),$$

(157)

The quantity $I_3(0)$ is known explicitly as one of the Watson integrals [45–47]:

$$I_3(0) = \frac{\sqrt{6}}{64\pi^3} \Gamma \left( \frac{1}{24} \right) \Gamma \left( \frac{5}{24} \right) \Gamma \left( \frac{7}{24} \right) \Gamma \left( \frac{11}{24} \right) \simeq 0.7582.$$  

(158)

The same change of variables as above yields

$$I_3'(2r\tau) = -\frac{\prod_{i=1}^{3} \int_{-\pi}^{\pi} \frac{dq_i}{2\pi}}{\prod_{i=1}^{3} \int_{-\pi}^{\pi} \frac{dy_i}{2\pi}} \left( \frac{1}{2r\tau + \frac{3}{2} \sum_{j=1}^{3} (1 - \cos(q_j))} \right)^{\frac{d}{2}} \left( \frac{1}{1 + \sum_{j=1}^{3} y_j^2} \right)^{\frac{d}{2}} \left( \frac{1}{\sum_{k=1}^{3} y_k^2} \right)^{\frac{d}{2}}$$

(159)

$$\sim \frac{(6r\tau)^d}{(2\pi)^d (2r\tau)^d} \times 4\pi \int_{0}^{\infty} \frac{d\rho}{1 + \rho^2} \left( \frac{\rho^2}{(1 + \rho^2)^{\frac{d}{2}}} \right)^d = O \left( (r\tau)^{-\frac{d}{2}} \right).$$

Hence the following terms in the expression of $\varrho_\infty$ in equation (128) are negligible compared to $\sqrt{\tau}$ at low resetting rate:

$$r\tau = o \left( \sqrt{\tau} \right),$$

$$(r\tau)^2 I_3(r\tau) = O \left( (r\tau)^2 \right) = o \left( \sqrt{\tau} \right),$$

(160)

$$(r\tau)^2 I_3'(2r\tau) = O \left( (r\tau)^2 \right) = o \left( \sqrt{\tau} \right).$$
Combining with the expansion in equation (157) yields
\[
\varrho_\infty = \frac{1}{4} \mathcal{I}_3(0) - \frac{1}{\sqrt{2\pi}} \sqrt{\tau} + o\left(\sqrt{\tau}\right)
\]
\[
= \frac{1}{4\mathcal{I}_3(0)} + \frac{6^3}{32\pi \mathcal{I}_3(0)} \sqrt{\tau} + o\left(\sqrt{\tau}\right) + o\left(\sqrt{\tau}\right)
\]
\[
\simeq 0.3297 + 0.25431 \sqrt{\tau} + o\left(\sqrt{\tau}\right).
\]

C.4. Dimension d = 2

The quantity \( \mathcal{I}_2(2\tau) \) goes logarithmically to infinity when \( r \) goes to 0. Indeed, changing variables to \( y_i = q_i/\sqrt{4\tau} \) (for \( i \) in \( \{1, 2\} \)) yields:
\[
\mathcal{I}_2(2\tau) = \int_{-\pi}^{\pi} \frac{dq_1}{2\pi} \int_{-\pi}^{\pi} \frac{dq_2}{2\pi} \left( 2\tau + 2 - \cos(q_1) - \cos(q_2) \right)^{-1}
\]
\[
\simeq \frac{4\tau}{2\pi} \int_{-\pi}^{\pi} \frac{dy_1}{2\pi} \int_{-\pi}^{\pi} \frac{dy_2}{2\pi} \left( \tau + 1 + y_1^2 + y_2^2 \right)^{-1} \]
\[
\simeq \frac{4\tau}{(2\pi)^2} \times 2\pi \int_{0}^{\pi} \rho d\rho \left( 1 + \rho^2 \right)^{-1} \]
\[
= \frac{\pi}{2\ln(\tau)}.
\]

Moreover,
\[
\mathcal{I}_2^i(2\tau) = \int_{-\pi}^{\pi} \frac{dq_1}{2\pi} \int_{-\pi}^{\pi} \frac{dq_2}{2\pi} \left( 2\tau + 2 - \cos(q_1) - \cos(q_2) \right)^{-1}
\]
\[
\simeq \frac{4\tau}{(2\tau)^2} \int_{-\pi}^{\pi} \frac{dy_1}{2\pi} \int_{-\pi}^{\pi} \frac{dy_2}{2\pi} \left( \tau + 1 + y_1^2 + y_2^2 \right)^{-1} \]
\[
\simeq \frac{4\tau}{(2\pi)^2(2\tau)^2} \times 2\pi \int_{0}^{\pi} \rho d\rho \left( 1 + \rho^2 \right)^{-1} \]
\[
= O(\tau^{-1}).
\]

Hence the following terms in \( \varrho_\infty \) are negligible compared to \( o\left(\ln(\tau)\right)^{-1}\) at low resetting rate:
\[
\tau = o\left(\ln(\tau)\right)^{-1},
\]
\[
(\tau^{-2}) \mathcal{I}_2(\tau) = O\left(\tau^{-2} \ln(\tau)\right) = o(\tau) = o\left(\sqrt{\tau}\right) = o\left(\ln(\tau)\right)^{-1},
\]
\[
(\tau^{-2}) \mathcal{I}_2^i(2\tau) = O\left(\tau \left(\ln(\tau)\right)^{-1}\right) = o\left(\left(\ln(\tau)\right)^{-1}\right).
\]
Hence the density $\varrho_{\infty}$ is equivalent to $(4I_2(2r\tau))^{-1}$ at small resetting rate:

$$\varrho_{\infty} \underset{r\tau \ll 1}{\sim} \frac{\pi}{2 \ln(r\tau)}. \quad (166)$$

### C.5. Dimension $d = 1$

Starting from the expansion

$$\frac{1}{r\tau + 1 + \sqrt{(r\tau + 1)^2 - 1}} = \frac{1}{r\tau + 1 + \sqrt{r\tau \sqrt{2 + r\tau}}} = 1 - \sqrt{2r\tau} + o(r\tau), \quad (167)$$

we obtain the equivalent at low resetting rate of the limit worked out in equation (133)

$$1 - \frac{\tilde{\varphi}_1(0)}{\varphi_0(0)} = 2\sqrt{r\tau} + o(r\tau). \quad (168)$$

For completeness, let us work out the same equivalent starting from the general expression of the steady-state density of domain walls in terms of the function $I_1$ in equation (128). To ensure consistency with equation (74), the expression $(4I_1(2r\tau))^{-1}$ should behave in the same way as the quantity $\frac{1}{2} \left(1 - \frac{\tilde{\varphi}_1(0)}{\varphi_0(0)}\right)$ in the regime $r\tau \ll 1$. The quantity $I_1(\epsilon)$ goes to infinity when $\epsilon$ goes to 0. Changing the integration variable to $y := x/(\sqrt{2r\tau})$ yields

$$I_1(2r\tau) = \sqrt{2r\tau} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{1}{2r\tau(1 + y^2)(1 + o(1))} \sim \frac{1}{2\sqrt{2r\tau}}. \quad (169)$$

Hence

$$\frac{1}{4I_1(2r\tau)} \underset{r\tau \ll 1}{\sim} \frac{\sqrt{2r\tau}}{2}, \quad (170)$$

which is consistent with equation (168).

On the other hand, the integral terms in equation (74) diverge when the resetting rate goes to zero. We can work out an equivalent of these integral terms as follows:

$$\lim_{s \to 0} \left[ -2sx\Psi_1(s) + 2sx\tilde{\varphi}_1(s)\Psi_0(s) \right] \underset{r\tau \ll 1}{\sim} \frac{(r\tau)^2}{2\pi} \int_{-\pi}^{\pi} dx \frac{1 - \cos(x)}{(1 - \cos(x) + r\tau)^2}. \quad (171)$$

$$\frac{(r\tau)^2}{\pi} \int_{-\pi}^{\pi} dx \frac{1 - \cos(x)}{(1 - \cos(x) + r\tau)^2} = \frac{(r\tau)^2}{2\pi} \int_{-\pi}^{\pi} dx \frac{2\sin^2\left(\frac{x}{2}\right)}{(2\sin^2\left(\frac{\sqrt{r\tau}}{2}\right) + r\tau)^2}$$

$$= \frac{(r\tau)^2}{\pi} \sqrt{2r\tau} \int_{-\pi}^{\pi} \frac{dy}{\sqrt{2\pi}} \frac{2\sin^2\left(\frac{\sqrt{r\tau}}{2}\right)}{(2\sin^2\left(\frac{\sqrt{r\tau}}{2}\right) + r\tau)^2} \quad (172)$$

$$= \frac{(r\tau)^2}{\sqrt{2r\tau}} \int_{-\infty}^{\infty} dy \frac{y^2}{(y^2 + 1)^2(1 + o(1))} = O(r\tau)^{\frac{3}{2}}.$$


where we have changed the integration variable $y := x/\sqrt{2r\tau}$. The contribution of the integral terms to the density $\varrho_\infty$ at low resetting rate is therefore subdominant compared to $\sqrt{r\tau}$, which yields the equivalent displayed in equation (75).

**ORCID iD**

Pascal Grange https://orcid.org/0000-0002-9621-1990

**References**

[1] Krapivsky P L 1992 Kinetics of monomer-monomer surface catalytic reactions Phys. Rev. A 45 1067
[2] Frachebourg L and Krapivsky P L 1996 Exact results for kinetics of catalytic reactions Phys. Rev. E 53 R3009
[3] Liggett T M and Liggett T M 1985 *Interacting Particle Systems* vol 2 (Springer)
[4] Redner S 2019 Reality-inspired voter models: a mini-review C. R. Physique 20 275–92
[5] Krapivsky P L, Redner S and Ben-Naim E 2010 *A Kinetic View of Statistical Physics* (Cambridge University Press)
[6] Miron A and Reuveni S 2021 Diffusion with local resetting and exclusion Phys. Rev. Res. 3 L012023
[7] Evans M R and Majumdar S N 2011 Diffusion with stochastic resetting Phys. Rev. Lett. 106 160601
[8] Evans M R and Majumdar S N 2011 Diffusion with optimal resetting J. Phys. A: Math. Theor. 44 435001
[9] Pal A 2015 Diffusion in a potential landscape with stochastic resetting Phys. Rev. E 91 012113
[10] Kusmierz L, Majumdar S N, Sabhapandit S and Schehr G 2014 First order transition for the optimal search time of levy flights with resetting Phys. Rev. Lett. 113 220602
[11] Evans M R and Majumdar S N 2018 Run and tumble particle under resetting: a renewal approach J. Phys. A: Math. Theor. 51 475003
[12] Evans M R and Majumdar S N 2018 Effects of refractory period on stochastic resetting J. Phys. A: Math. Theor. 52 01LT01
[13] Kumar V, Sadekar O and Basu U 2020 Active brownian motion in two dimensions under stochastic resetting Phys. Rev. E 102 052129
[14] Evans M R, Majumdar S N and Schehr G 2020 Stochastic resetting and applications J. Phys. A: Math. Theor. 53 193001
[15] Gupta S and Jayannavar A M 2022 Stochastic resetting: a (very) brief review Front. Phys. 10 789097
[16] Grange P 2021 Aggregation with constant kernel under stochastic resetting J. Phys. A: Math. Theor. 54 294001
[17] Pelizzola A, Pretti M and Zamparo M 2021 Simple exclusion processes with local resetting Europhys. Lett. 133 60003
[18] Magoni M, Majumdar S N and Schehr G 2020 Ising model with stochastic resetting Phys. Rev. Res. 2 033182
[19] Aron C and Kulkarni M 2020 Nonanalytic nonequilibrium field theory: Stochastic reheating of the ising model Phys. Rev. Res. 2 043390
[20] Basu U, Kundu A and Pal A 2019 Symmetric exclusion process under stochastic resetting Phys. Rev. E 100 032136
[21] Sadekar O and Basu U 2020 Zero-current nonequilibrium state in symmetric exclusion process with dichotomous stochastic resetting J. Stat. Mech. 073209
[22] Karthika S and Nagar A 2020 Totally asymmetric simple exclusion process with resetting J. Phys. A: Math. Theor. 53 115003
[23] Quetzalcoatl Toledo-Marin J, Boyer D and Sevilla F J 2019 Predator-prey dynamics: chasing by stochastic resetting (arXiv:1912.2019)
[24] Evans M R, Majumdar S N and Schehr G 2022 An exactly solvable predator prey model with resetting J. Phys. A: Math. Theor. 55 274005
[25] Mercado-Vásquez G and Boyer D 2018 Lotka–Volterra systems with stochastic resetting J. Phys. A: Math. Theor. 51 405601
[26] Gupta S, Majumdar S N and Schehr G 2014 Fluctuating interfaces subject to stochastic resetting Phys. Rev. Lett. 112 220601
[27] Gupta S and Nagar A 2016 Resetting of fluctuating interfaces at power-law times J. Phys. A: Math. Theor. 49 445001
[28] Sarkar M and Gupta S 2022 Synchronization in the Kuramoto model in presence of stochastic resetting Chaos 32 073109
[29] Durang X, Henkel M and Park H 2014 The statistical mechanics of the coagulation–diffusion process with a stochastic reset J. Phys. A: Math. Theor. 47 045002
[30] Grange P 2020 Entropy barriers and accelerated relaxation under resetting J. Phys. A: Math. Theor. 53 375002
[31] Grange P 2019 Steady states in a non-conserving zero-range process with extensive rates as a model for the balance of selection and mutation J. Phys. A: Math. Theor. 52 365601
[32] Grange P 2020 Non-conserving zero-range processes with extensive rates under resetting J. Phys. Commun. 4 045006
[33] Nagar A and Gupta S 2023 Stochastic resetting in interacting particle systems: a review J. Phys. A: Math. Theor. 56 283001
[34] Evans M R and Majumdar S N 2014 Diffusion with resetting in arbitrary spatial dimension J. Phys. A: Math. Theor. 47 285001
[35] Masó-Puigdellosas A, Campos D and Méndez V 2022 Conditioned backward and forward times of diffusion with stochastic resetting: a renewal theory approach Phys. Rev. E 106 034126
[36] Evans J W and Ray T 1993 Kinetics of the monomer-monomer surface reaction model Phys. Rev. E 47 1018
[37] Roldán E and Gupta S 2017 Path-integral formalism for stochastic resetting: Exactly solved examples and shortcuts to confinement Phys. Rev. E 96 022130
[38] Pinsky R G 2020 Diffusive search with spatially dependent resetting Stoch. Process. Their Appl. 130 2954–73
[39] Sood V and Redner S 2005 Voter model on heterogeneous graphs Phys. Rev. Lett. 94 178701
[40] Sood V, Antal T and Redner S 2008 Voter models on heterogeneous networks Phys. Rev. E 77 041121
[41] Derrida B, Hakim V and Pasquier V 1995 Exact first-passage exponents of 1d domain growth: relation to a reaction-diffusion model Phys. Rev. Lett. 75 751
[42] Derrida B, Hakim V and Pasquier V 1996 Exact exponent for the number of persistent spins in the zero-temperature dynamics of the one-dimensional potts model J. Stat. Phys. 85 763–97
[43] Ben-Naim E, Frachebourg L and Krapivsky P L 1996 Coarsening and persistence in the voter model Phys. Rev. E 53 3078
[44] Abramowitz M, Stegun I A and Romer R H 1988 Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables
[45] Watson G N 1939 Three triple integrals Q. J. Math. os-10 266–76
[46] Glasser M L and Zucker I J 1977 Extended Watson integrals for the cubic lattices Proc. Natl Acad. Sci. 74 1800–1
[47] Zucker I 2011 70+ years of the Watson integrals J. Stat. Phys. 145 591–612