Homogeneous symplectic 4-manifolds and finite dimensional Lie algebras of symplectic vector fields on the symplectic 4-space

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Abstract

We classify the finite type (in the sense of E. Cartan theory of prolongations) subalgebras \( h \subset \mathfrak{sp}(V) \), where \( V \) is the symplectic 4-dimensional space, and show that they satisfy \( h^{(k)} = 0 \) for all \( k > 0 \). Using this result, we reduce the problem of classification of graded transitive finite-dimensional Lie algebras \( g \) of symplectic vector fields on \( V \) to the description of graded transitive finite-dimensional subalgebras of the full prolongations \( p_1^{(\infty)} \) and \( p_2^{(\infty)} \), where \( p_1 \) and \( p_2 \) are the maximal parabolic subalgebras of \( \mathfrak{sp}(V) \). We then classify all such \( g \subset p_i^{(\infty)} \), \( i = 1, 2 \), under some assumptions, and describe the associated 4-dimensional homogeneous symplectic manifolds \( (M = G/K, \omega) \). We prove that any reductive homogeneous symplectic manifold (of any dimension) admits an invariant torsion free symplectic connection, i.e., it is a homogeneous Fedosov manifold, and give conditions for the uniqueness of the Fedosov structure. Finally, we show that any nilpotent symplectic Lie group (of any dimension) admits a natural invariant Fedosov structure which is Ricci-flat.

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1 Introduction, statement of the results

Let \((M = G/K, \omega)\) be a homogeneous symplectic manifold of a connected Lie group \(G\). If the group \(G\) is fixed, the description of all homogeneous \(G\)-manifolds reduces to the description of the closed 2-forms \(\Omega \in \Lambda^2_c \mathfrak{g}^*\) on its Lie algebra. In this paper, we consider another problem: to describe the homogeneous symplectic manifolds of a given dimension \(\dim M = 2n\).

We will consider the following approach. The Lie algebra \(\mathfrak{g} = \text{Lie}(G)\), considered as a finite dimensional Lie algebra of symplectic vector fields on the symplectic manifold \((M = G/K, \omega)\), admits a natural filtration

\[
\mathfrak{g} = \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k \supset \{0\},
\]

given by the symplectic vector fields vanishing at a certain order at \(o = eK\). We consider the associated graded Lie algebra

\[
\text{gr}(\mathfrak{g}) = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^k,
\]

it is a transitive graded subalgebra of the Lie algebra \(\mathfrak{sp}(V)^{\infty} = \mathfrak{V} + \mathfrak{sp}(V) + \mathfrak{sp}(V)^{(1)} + \cdots\) of all formal symplectic vector fields, where \(V = \mathfrak{g}^{-1} = T_o M\), \(o = eK\). We note that the Lie algebra \(\mathfrak{g}\) can be realized as a filtered deformation of the graded Lie algebra \(\text{gr}(\mathfrak{g})\) and that \(\mathfrak{g}^1\) is a subspace of the \(j\)-th prolongation \(\mathfrak{h}^{(j)}\) of the linear isotropy subalgebra \(\mathfrak{h} = \mathfrak{g}^0 \subset \mathfrak{sp}(V)\).

The first arising question is to describe all subalgebras \(\mathfrak{h} \subset \mathfrak{sp}(V)\) of finite type, i.e., algebras which have finite-dimensional full prolongation

\[
\mathfrak{h}^{(\infty)} = \mathfrak{V} + \mathfrak{h} + \mathfrak{h}^{(1)} + \cdots + \mathfrak{h}^{(k)}.
\]

For simplicity, consider at first the complex case, where \(\mathfrak{h}\) is a complex subalgebra of the complex symplectic Lie algebra of the symplectic vector space \(V = \mathbb{C}^{2n}\). Then a complex linear Lie algebra \(\mathfrak{h}\) is of finite type if and only if it has no rank one endomorphisms. For \(n = 1\), \(\mathfrak{sp}(V) = \mathfrak{sl}_2(\mathbb{C})\) and (up to a conjugation) the Cartan subalgebra \(\mathfrak{C}\text{diag}(1, -1)\) is the only nonzero subalgebra of finite type. In this paper, we deal with the case \(n = 2\), paving the way to the classification of homogeneous symplectic 4-manifolds and homogeneous Fedosov 4-manifolds.

In \cite{2} and \cite{3} we describe the maximal subalgebras of \(\mathfrak{sp}_2(\mathbb{R})\) and determine those of finite type. We then reduce the classification problem of (maximal) finite type subalgebras of \(\mathfrak{sp}_2(\mathbb{R})\) to the description of finite type subalgebras of maximal parabolic subalgebras \(p_1, p_2\) and the subalgebra \(\mathfrak{s}_1 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)\) preserving an orthogonal decomposition \(V = V_1 + V_2\) of \(V\). Theorem \cite{1} below describes the finite type subalgebras of \(\mathfrak{sp}_2(\mathbb{R})\) and it is a consequence of this analysis, which is carried out in \cite{3}.

Throughout the paper, we shall identify \(\mathfrak{sp}_2(\mathbb{R})\) with the space of quadratic polynomials on \(V^* \cong V = \text{span}(p_1, p_2, q_1, q_2)\).

**Theorem 1** Let \(\mathfrak{h}\) be a subalgebra of \(\mathfrak{sp}_2(\mathbb{R})\). Then \(\mathfrak{h}\) is either of infinite type or it is, up to conjugation, a subalgebra of one of the Lie algebras of the following list:

1. the unitary algebra \(\mathfrak{u}_2\) (maximal compact subalgebra of \(\mathfrak{sp}_2(\mathbb{R})\));

2. the pseudo-unitary algebra \(\mathfrak{u}_{1,1} = \mathfrak{sl}(H) \oplus \mathfrak{so}(E) \subset \mathfrak{sp}(H \otimes E)\), where the symplectic structure on \(V = H \otimes E = \mathbb{R}^2 \otimes \mathbb{R}^2\) is the tensor product \(\Omega = \text{vol}_H \otimes \eta_E\) of the volume form on \(H\) and an Euclidean metric on \(E\);

3. the reductive subalgebra \(Q \vee P \cong \mathfrak{gl}(P)\), where \(P \) and \(Q\) are complementary Lagrangian subspaces;
4. the irreducible singular subalgebra \( \mathfrak{s}_3^2(\mathbb{R}) \) acting on \( V = S^3(\mathbb{R}^2) \);

5. the direct sum \( \mathfrak{so}_2(\mathbb{R}) \oplus \mathbb{R}\text{diag}(1,-1) \) of two Cartan subalgebras of \( \mathfrak{s}_2(\mathbb{R}) \), one compact and one noncompact;

6. the solvable 2-dimensional subalgebra

\[
\mathcal{D}_{4,12} = \text{span}(p_1q_1 + \epsilon p_2^2, p_2q_1),
\]

where \( \epsilon = \pm 1 \);

7. the 1-dimensional subalgebra \( \text{span}(p_2^2 + q_2^2 + \epsilon p_1^2) \), where \( \epsilon = \pm 1 \).

All these subalgebras are maximal finite type subalgebras. In particular a finite type subalgebra of \( \mathfrak{sp}_2(\mathbb{R}) \) has at most dimension 4.

For more details, we refer the reader to Theorems 3.2 and the proof in 3.4 of the maximality of the finite type subalgebras of Theorem 3.1. The upper index in \( \mathfrak{s}_3^2(\mathbb{R}) \) indicates the dimension of the irreducible representation; the notation for the 2-dimensional solvable subalgebra \( \mathcal{D}_{4,12} \) makes contact with the description of conjugacy classes of subalgebras of the similitude algebra in [PWSZ].

A direct but rather surprising corollary of Theorem 3.1 is that all finite type subalgebras of \( \mathfrak{sp}_2(\mathbb{R}) \) have trivial prolongations.

**Corollary 1** Let \( \mathfrak{h} \) be a finite type subalgebra of \( \mathfrak{sp}_2(\mathbb{R}) \). Then the first prolongation \( \mathfrak{h}^{(1)} = 0 \).

Due to this, the finite type subalgebras are a good class of candidates to construct homogeneous Fedosov manifolds. We will expand on this later.

We thank B. Kruglikov, who informed us that there is no upper bound on the dimension of a Lie group \( G \) acting transitively on a symplectic 4-manifold. This is in sharp contrast with the Riemannian case. Let \((M = G/K, \omega)\) be a homogeneous symplectic 4-manifold with associated graded Lie algebra [2]. It is a finite-dimensional subalgebra of the full prolongation \( \mathfrak{h}^{(\infty)} \) of the linear isotropy subalgebra \( \mathfrak{h} = \mathfrak{g}^0 \subset \mathfrak{sp}(V) \). If \( \mathfrak{h} \) has finite type then \( \mathfrak{h}^{(1)} = 0 \) by Corollary 3.1 and therefore \( \text{gr}(\mathfrak{g}) = V + \mathfrak{h} \). If \( \mathfrak{h} = \mathfrak{sp}(V) \) then the first prolongation \( \mathfrak{h}^{(1)} \) is irreducible and, together with \( V + \mathfrak{h} \), it generates the (infinite-dimensional) full prolongation \( \mathfrak{sp}(V)^{(\infty)} \). Hence \( \text{gr}(\mathfrak{g}) = V + \mathfrak{h} \) also in this case. According to the description of maximal subalgebras (Theorem 3.3 and Corollary 3.2), the only maximal infinite type subalgebras of \( \mathfrak{sp}(V) \) are the two maximal parabolic subalgebras \( \mathfrak{p}_1 \) and \( \mathfrak{p}_2 \) and the semisimple subalgebras \( \mathfrak{s}_1 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \) and \( \mathfrak{s}_4 = \mathfrak{s}_2(\mathbb{C}) \).

The classification of all graded transitive finite-dimensional Lie algebras of symplectic vector fields on \( V \) reduces in this way to the description of the graded transitive finite-dimensional subalgebras of the infinite-dimensional Lie algebras \( \mathfrak{p}_1^{(\infty)}, \mathfrak{p}_2^{(\infty)}, \mathfrak{s}_1^{(\infty)}, \mathfrak{s}_4^{(\infty)} \). Furthermore, using Proposition 3.3 and the results of 3.1.1 one can directly see that such subalgebras of \( \mathfrak{s}_1^{(\infty)} \) and \( \mathfrak{s}_4^{(\infty)} \) are either of the form \( \text{gr}(\mathfrak{g}) = V + \mathfrak{h} \) as above or they are subalgebras of \( \mathfrak{p}_1^{(\infty)} \) or \( \mathfrak{p}_2^{(\infty)} \), see 3.3.5 for full details. In other words, we have the following dichotomy result of Theorem 3.2. We recall that a filtered deformation of a graded Lie algebra is called trivial if it is itself graded.

**Theorem 2** Let \((M = G/K, \omega)\) be a 4-dimensional homogeneous symplectic manifold, on which a finite-dimensional connected Lie group \( G \) (not necessarily compact) acts transitively and almost effectively with connected stabilizer \( K \subset G \). Then the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) is a (possibly trivial) filtered deformation of \( \text{gr}(\mathfrak{g}) \) and exactly one of the following two cases occurs:

1. \( \text{gr}(\mathfrak{g}) = V + \mathfrak{h} \) is the affine Lie algebra, where \( \mathfrak{h} \) is any subalgebra of \( \mathfrak{sp}(V) \);
2. $\text{gr}(g) = V + h + g^1 + \cdots + g^k$ is such that $g^1 \neq 0$, where $h$ is an infinite type subalgebra of the maximal parabolic subalgebra $p_1$ or the maximal parabolic subalgebra $p_2$.

If the isotropy representation $j : K \to j(K) = H \subset \text{Sp}_2(\mathbb{R})$ is not infinitesimally exact then there exists an isotropic distribution $\mathcal{D} \subset T\mathbb{M}$ on $\mathbb{M} = G/K$ of $\text{rk} \mathcal{D} = 1$ or 2 left invariant by $G$.

The first alternative of Theorem 2 parallels the situation for Riemannian manifolds. In §4 we focus on the second alternative and use a more suggestive description of the full prolongations $p_1^{(\infty)}$ and $p_2^{(\infty)}$ in order to describe their finite-dimensional subalgebras. Recall that a Lie algebra of vector fields is called primitive if it does not preserve any integrable subdistribution. The notion of a transversally primitive transitive Lie algebra of vector fields is given in §4.1. Under some assumptions, we describe all finite-dimensional transitive and transversally primitive subalgebras of $p_1^{(\infty)}$ and $p_2^{(\infty)}$. They are given in Theorem 10 in §4.2 and in Theorem 11 in §4.3.

We then construct homogeneous symplectic 4-manifolds for any of these Lie algebras and show that most of them do not admit any invariant torsion free symplectic connection.

A Fedosov structure on a symplectic manifold $(\mathbb{M}, \omega)$ is the assignment of a torsion free symplectic connection $\nabla$ [GRS]. The celebrated result of Fedosov gives a deformation quantization canonically defined by the data $(\mathbb{M}, \omega, \nabla)$ [Fed]. Torsion free symplectic connections exist on any symplectic manifold, in other words, a structure of a symplectic manifold can always be extended to a structure of a Fedosov manifold.

The are certain classes of symplectic manifolds for which there is a natural choice of a unique Fedosov structure, for instance those preserving some additional geometric data. These include Levi-Civita connections on Kähler manifolds, symplectic manifolds endowed with a Lagrangian polarization [HS] and symmetric symplectic spaces [Bie]. We also note that there is a notion of a preferred Fedosov structure, which is not imposed by the presence of extra geometric structures (see [BCGRS]). Every compact coadjoint orbit has an invariant preferred Fedosov structure but the connection is not unique in general, see [CGR]. Two important classes of symplectic manifolds with preferred Fedosov structures are the class of Ricci-type manifolds — the Fedosov manifolds $(\mathbb{M}, \omega, \nabla)$ for which the curvature of $\nabla$ is entirely determined by the Ricci tensor $\text{ric}^\nabla$ — and the Ricci flat manifolds, i.e., those satisfying $\text{ric}^\nabla = 0$. The Ricci-type connections are well-understood, both from a local and a global point of view [BCGRS]. Less is known on Ricci-flat Fedosov manifolds, but there is a procedure to construct examples in dimension $\geq 6$ [BCGRS].

In §5.1 we show that any reductive homogeneous symplectic manifold $(\mathbb{M} = G/K, \omega)$ admits an invariant torsion free symplectic connection, in other words, it is a homogeneous Fedosov manifold (see Proposition 17). Corollary 1 then suggests a way to select a class of unique Fedosov structures on reductive homogeneous manifolds. Let $H$ be a subgroup of $\text{Sp}_2(\mathbb{R})$ of finite type (that is, $h \subset \mathfrak{sp}_2(\mathbb{R})$ is included in one of the subalgebras from Theorem 1). We consider the reductive homogeneous symplectic 4-manifolds $(\mathbb{M} = G/K, \omega)$ with isotropy representation $j(K) = H$ and look for invariant torsion free (symplectic) connections compatible with the canonical reduction $\pi : P = G \to \mathbb{M} = G/K$ of the bundle of all symplectic frames on $\mathbb{M}$.

Combining Proposition 17 with Theorem 12 in §5.2 we arrive at the following.

**Theorem 3** Let $(\mathbb{M} = G/K, \omega)$ be a 4-dimensional homogeneous symplectic manifold with a finite type isotropy $j(K) = H \subset \text{Sp}_2(\mathbb{R})$. Then the isotropy representation $j : K \to \text{Sp}_2(\mathbb{R})$ is infinitesimally exact, i.e., it has a discrete kernel and

1. If $\mathbb{M} = G/K$ is reductive, there exists an invariant torsion free symplectic connection;

2. If there exists an invariant torsion free (symplectic) connection which is compatible with the canonical reduction $\pi : P \to \mathbb{M}$, then $\mathbb{M} = G/K$ is reductive. Furthermore, such a connection is unique.
If an invariant torsion free symplectic connection exists then the isotropy representation is exact.

More generally, any torsion free (symplectic) connection on a symplectic 4-dimensional manifold \((M, \omega)\) compatible with a symplectic H-structure \(\pi: P \to M\) whose structure group \(H \subset \text{Sp}(V)\) has finite type is unique. In the case of reductive homogeneous symplectic 4-manifolds, there is a natural choice of \(\pi: P \to M\) and 2. of Theorem\([3]\) can be rephrased by saying that the associated deformation quantization (if it exists) is defined solely in terms of \((M, \omega)\).

Torsion free connections on 4-manifolds whose holonomies exactly realize the representation of \(H = \text{SL}_2(\mathbb{R})\) on \(S^2(\mathbb{R}^2)\) have been considered in great detail in \([Bry]\) and homogeneous Fedosov manifolds (of any dimension) with special symplectic holonomy have been constructed in \([Sch]\).

The classification of reductive homogeneous symplectic 4-manifolds \((M = G/K, \omega)\) such that the isotropy subgroup is a (maximal) finite type subgroup of \(\text{Sp}_2(\mathbb{R})\) — and of their invariant Fedosov structures — will be the content of future work \([AlSa]\).

In \(\S 5.3\) we consider the case of symplectic Lie groups (of any dimension), that is, the homogeneous symplectic manifolds \((M = G/K, \omega)\) with trivial stabilizer \(K = \{1\}\). Using the results of \(\S 5.1-5.2\) and the theory of left-symmetric algebras, we derive the following.

**Theorem 4** Any nilpotent symplectic Lie group (of any dimension) admits an invariant torsion free symplectic connection which is Ricci flat.

See Theorem\([13]\) for more details. We recall that the class of nilpotent symplectic Lie groups is particularly large and that they can all be described in terms of double extensions (see \([MR]\)). The explicit classification, up to local equivalence, in dimension \(\leq 6\) can be found in \([Kr, KGM]\).

We remark that structural properties of solvable and nilpotent symplectic Lie groups, as well as cotangent symplectic Lie groups, have been investigated in detail in \([BC]\). In particular, the theory of symplectic reduction w.r.t. isotropic normal subgroups is developed and irreducible symplectic Lie groups are classified.

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## 2 Maximal subalgebras of \(\mathfrak{sp}_2(\mathbb{R})\)

### 2.1 List of maximal subalgebras of \(\mathfrak{sp}_2(\mathbb{R}) \cong \mathfrak{so}(2, 3)\)

Let \((V, \Omega)\) be the 4-dimensional symplectic (real) vector space. The \(\mathfrak{sp}(V)\)-module \(W = \Lambda^2_0(V)\) has an invariant inner product \(\eta = \Omega \wedge \Omega\) of signature \((2, 3)\), which gives an isomorphism \(\mathfrak{sp}_2(\mathbb{R}) = \mathfrak{sp}(V) \cong \mathfrak{so}(W) = \mathfrak{so}(2, 3)\).

According to \([GOVIII]\), the maximal subalgebras of the complexification \(\mathfrak{so}_5(\mathbb{C})\) are exhausted by

1. maximal parabolic subalgebras,
2. the reducible subalgebra \(\mathfrak{so}_4(\mathbb{C})\),
3. a simple irreducible subalgebra \(\mathfrak{sl}_2^5(\mathbb{C})\) defined by the natural action on \(S^4(\mathbb{C}^2)\).

We remark that the reducible subalgebra \(\mathfrak{so}_3(\mathbb{C}) \oplus \mathfrak{so}_2(\mathbb{C})\) has to be removed from the list of \([GOVIII]\) Theorem 3.1, p. 205], since it is contained in the parabolic subalgebra stabilizing an isotropic line, hence it is not maximal. The upper index in \(\mathfrak{sl}_2^5(\mathbb{C})\) indicates the dimension of the irreducible representation.

Since \(\mathfrak{so}(2, 3)\) is the normal real form of \(\mathfrak{so}_5(\mathbb{C})\), there is no big difference between maximal real subalgebras of \(\mathfrak{so}(2, 3)\) and complex subalgebras of \(\mathfrak{so}_5(\mathbb{C})\). It turns out that they are real.
forms of the maximal complex subalgebras listed above or real forms of $\mathfrak{so}_3(\mathbb{C}) \oplus \mathfrak{so}_2(\mathbb{C})$ which do not preserve any isotropic line (see \cite{PSWZ}).

Besides maximal parabolic subalgebras, the real forms of maximal complex subalgebras of $\mathfrak{so}_5(\mathbb{C})$ which belong to $\mathfrak{so}(2, 3) \cong \mathfrak{sp}_2(\mathbb{R})$ give the following list of maximal subalgebras:

(i) $\mathfrak{so}(2, 2) \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$,

(ii) $\mathfrak{so}(1, 3) \cong \mathfrak{sl}_2(\mathbb{C})$,

(iii) $\mathfrak{sl}_2^2(\mathbb{R})$.

In the first case, the symplectic space has an $\Omega$-orthogonal decomposition $V = V_1 + V_2$, where $V_i = \text{span}(p_i, q_i)$ for $i = 1, 2$. We denote the associated semisimple subalgebra by $\mathfrak{s}_1 := \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$ and note that $\mathfrak{sp}_2(\mathbb{R})$ has a symmetric decomposition $\mathfrak{sp}_2(\mathbb{R}) = s_1 + V_1 \vee V_2$ with corresponding pseudo-Riemannian irreducible symmetric space $\text{Sp}_2(\mathbb{R})/\text{SL}_2(\mathbb{R}) \cdot \text{SL}_2(\mathbb{R})$.

The spin representation of $\mathfrak{s}_1 := \mathfrak{so}(1, 3) \cong \mathfrak{sl}_2(\mathbb{C})$ is symplectic real 4-dimensional, and it consists of all symplectic matrices compatible with a complex structure $J$ such that $J^* \Omega = -\Omega$.

The last case gives the singular subalgebra $\mathfrak{s}_5 := \mathfrak{sl}_2(\mathbb{R})$ realised as irreducible subalgebra

$$\mathfrak{s}_5^2(\mathbb{R}) \subset \mathfrak{so}(2, 3) = \mathfrak{so}(W)$$

where $W = S^4(\mathbb{R}^2)$, and as irreducible subalgebra

$$\mathfrak{s}_5^2(\mathbb{R}) \subset \mathfrak{sp}_2(\mathbb{R}) = \mathfrak{sp}(V)$$

where $V = S^3(\mathbb{R}^2)$.

The remaining cases are the reductive maximal subalgebras

(iv) $\mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) \cong \mathfrak{u}_2$,

(v) $\mathfrak{so}(1, 2) \oplus \mathfrak{so}_2(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R})^d \oplus \mathfrak{so}_2(\mathbb{R}) = \mathfrak{u}_{1,1}$.

The subalgebra $\mathfrak{s}_2 := \mathfrak{sp}_2(\mathbb{R}) \cap \text{stab}(J)$ consisting of the symplectic matrices preserving the complex structure $J : p_j \mapsto q_j$, $j = 1, 2$, is the the maximal compact subalgebra $\mathfrak{u}_2$ of $\mathfrak{sp}_2(\mathbb{R})$. We note that the complexification of $\mathfrak{s}_2$ is the complex Lie algebra

$$\mathfrak{s}_2^\mathbb{C} \cong \mathfrak{gl}(V^{10}) = \mathfrak{gl}_2(\mathbb{C})$$

where $V^{10}, V^{01} \subset V^\mathbb{C}$ are $\pm i$ eigenspaces of the complex structure $J$ in $V^\mathbb{C}$. The maximal reductive subalgebra $\mathfrak{s}_3 := \mathfrak{u}_{1,1}$ is similarly described, using a “split” complex structure $J$.

**Remark 1** In case (v), the symplectic space is given by $V = H \otimes E = \mathbb{R}^2 \otimes \mathbb{R}^2$ with symplectic form $\Omega = \text{vol}_H \otimes \eta_E$, where $\text{vol}_H$ is the volume form in $H$ and $\eta_E$ a metric in $E$ of signature $(2, 0)$. In this case $\mathfrak{s}_2(\mathbb{R}) \otimes \text{id} = \mathfrak{s}_2(\mathbb{R})^d$ is the diagonal subalgebra of $\mathfrak{s}_1$.

**Remark 2** If we were to take a metric in $E$ of signature $(1, 1)$, then the associated subalgebra would not be maximal in $\mathfrak{so}(2, 3)$, as it stabilizes an isotropic line.

### 2.2 Type of maximal subalgebras of $\mathfrak{sp}_2(\mathbb{R})$

Let

$$(V, \Omega) = (\mathbb{R}^4, \Omega = -(p_1^* \wedge q_1^* + p_2^* \wedge q_2^*))$$
be the 4-dimensional symplectic vector space with symplectic basis \((p_1, p_2, q_1, q_2)\). Using the symplectic form, we may identify the symplectic Lie algebra \(s = sp(V)\) with the space \(S^2(V)\) of quadratic polynomials on \(V^*\):

\[
uv : w \mapsto \Omega(u, w)v + \Omega(v, w)u ,
\]

for all \(u, v, w \in V\). In particular \(q_1 p_1 : p_1 \mapsto p_1, q_1 \mapsto -q_1\). More generally \(s^{(j)} = S^{j+2}(V)\) and the full algebra of formal symplectic vector fields is the symmetric algebra of \(V\),

\[
s^{(\infty)} = S^+(V) = V + S^2(V) + S^3(V) + \cdots,
\]

modulo constants.

There are two gradings of \(V\) which define two maximal parabolic subalgebras of \(s\). They are described by painted Dynkin diagrams, where the black node corresponds to the simple root of degree one and the white node to the simple root of degree zero.

1. \(\bullet = \circ\) The grading element \(d = \frac{1}{2}(q_1 p_1 + q_2 p_2)\) defines a grading

\[
V = V^{-\frac{1}{2}} + V^{\frac{1}{2}}
\]

of \(V\), where \(V^{-\frac{1}{2}} = Q := \text{span}(q_1, q_2)\), \(V^{\frac{1}{2}} = P := \text{span}(p_1, p_2)\). The induced grading

\[
s = s^{-1} + s^0 + s^1
\]

of \(s\) has corresponding parabolic subalgebra \(p_1 = s^{\geq 0} = Q \vee P + S^2(P) \cong gl(P) + S^2(P)\). Note that the associated flag manifold \(L_G(V) = Sp(V)/GL(P) \cdot S^2(P)\) is the Lagrangian Grassmannian.

2. \(\circ = \bullet\) The grading element \(d = q_1 p_1\) defines a grading

\[
V = V^{-1} + V^0 + V^1
\]

of \(V\), where \(W = V_2 := \text{span}(p_2, q_2)\). It induces the depth 2 grading

\[
s = s^{-2} + s^{-1} + s^0 + s^1 + s^2
\]

of \(s\) with associated parabolic subalgebra \(p_2 = s^{\geq 0} = (S^2(W) + \mathbb{R} q_1) + p_1 W + \mathbb{R} p_1^2\). This is a subalgebra of the Lie algebra of derivations of the 3-dimensional Heisenberg Lie algebra \(\mathfrak{hes}(W) = p_1 W + \mathbb{R} p_1^2\).

**Theorem 5** Any maximal subalgebra of \(s = sp(V)\) is conjugated to one of the seven subalgebras

\[
\begin{align*}
s_1 &= sp(V_1) \oplus sp(V_2) \cong sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R}), \\
s_2 &= sp(V) \cap \text{stab}(J) \cong u_2, \\
s_3 &= sp(V) \cap \text{stab}(\bar{J}) \cong u_{1,1}, \\
s_4 &= sl_2(\mathbb{C}), \\
s_5 &= sl_2(\mathbb{R}), \\
p_1 &= Q \vee P + S^2(P) \cong gl_2(\mathbb{R}) + S^2(\mathbb{R}^2), \\
p_2 &= (S^2(W) + \mathbb{R} p_1 q_1) + p_1 W + \mathbb{R} p_1^2 \cong gl(W) + \mathfrak{hes}(W).
\end{align*}
\]
Corollary 2  All maximal subalgebras of $s = \mathfrak{sp}(V)$ are algebras of infinite type except the Lie algebras of skew-Hermitian matrices $s_2$, $s_3$ and the singular subalgebra $s_5$.

This result follows from the observation that their complexifications have a rank one endomorphism and the following criterion, see [Wil] and also [AS, Lemma 3.4].

Theorem 6

1. A complex linear Lie algebra $\mathfrak{h} \subset \mathfrak{gl}_n(\mathbb{C})$ has infinite type if and only if it has an element of rank one.

2. A real linear Lie algebra $\mathfrak{h} \subset \mathfrak{gl}_n(\mathbb{R})$ has infinite type if and only if its complexification $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{gl}_n(\mathbb{C})$ has infinite type.

The following criteria are for the prolongation to be trivial.

Proposition 1

1. The first prolongation of any compact subalgebra $\mathfrak{h} \subset \mathfrak{so}_n(\mathbb{R})$ is trivial, i.e., $\mathfrak{h}^{(1)} = 0$.

2. Let $\mathfrak{h} \subset \mathfrak{gl}_n(\mathbb{K})$ be a linear Lie algebra over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and

$$\mathfrak{h}^{(0)} = \{(A, \theta A) \mid A \in \mathfrak{h}\} \subset \mathfrak{gl}_n(\mathbb{K}) \oplus \mathfrak{gl}_n(\mathbb{K})$$

the diagonal subalgebra of $\mathfrak{gl}_{2n}(\mathbb{K})$ twisted by a Lie algebra automorphism $\theta : h \to h$ of $\mathfrak{h}$. Then the first prolongation of $\mathfrak{h}^{(0)}$ is trivial.

In particular $s_2 \cong u_2 \subset \mathfrak{so}_4(\mathbb{R})$ satisfies $s_2^{(j)} = 0$ for all $j \geq 1$ and likewise for $s_3 \cong u_{1,1}$.

Proposition 2  The prolongation of $s_5 = \mathfrak{sl}_2^{(1)}(\mathbb{R})$ is trivial, i.e., $s_5^{(j)} = 0$ for all $j \geq 1$.

Proof. The representation of $s_5$ on $S^3(\mathbb{R}^2)$ is irreducible and has finite type. By a classical result of Kobayashi and Nagano, either the first prolongation is trivial or the full prolongation of $s_5$ is a simple Lie algebra [KoNa]. In the latter case, the zero-degree component of the full prolongation must have a non-trivial center, which is not our case. □

It is known that any proper subalgebra of the Lorentz algebra $\mathfrak{so}(1,3)$ preserves a line, either isotropic or not. This implies the following.

Proposition 3  Let $\mathfrak{h}$ be a proper subalgebra of $s_4$. Then either $\mathfrak{h} \subset s_2$ or $\mathfrak{h} \subset p_1$.

2.3 Full prolongation of maximal parabolics $p_1$, $p_2$ and of $s_1$

In view of the results of §2.2, the description of the finite type subalgebras of $\mathfrak{sp}(V)$ reduces to the description of the finite type subalgebras of $s_1$, $p_1$ and $p_2$. We now determine the full prolongation of these Lie algebras. Alternative descriptions in terms of vector fields on the real plane are given in §4.

We depart with $p_1 = Q \vee P + S^2(P)$. Its full prolongation

$$p_1^{(\infty)} = V + p_1 + p_1^{(1)} + \cdots$$

is infinite-dimensional and

$$p_1^{(k)} = \left\{ X \in \mathfrak{sp}(V)^{(k)} \mid \text{ad}_V^k(X) \in p_1 \right\}$$

$$= Q \vee S^{k+1}(P) + S^{k+2}(P),$$

for all $k \geq 1$.  

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Lemma 1 A subalgebra \( \mathfrak{h} \) of \( \mathfrak{p}_1 \) has finite type if and only if \( \mathfrak{a} = \mathfrak{h} \cap S^2(P) \) has finite type.

Proof. Let \( \mathfrak{a}^{(i)} = 0 \) for all \( i \) bigger than or equal a nonnegative integer \( N \) and decompose any element \( X \in \mathfrak{h}^{(N+1)} \) into \( X = X' + X'' \), where

\[
X' \in Q \setminus S^{N+2}(P), \quad X'' \in S^{N+3}(P).
\]

We have

\[
\text{ad}_P X' = \text{ad}_P X \subset h^{(N)} \cap S^{N+2}(P),
\]

that is \( \text{ad}_P X' \subset a^{(N)} = 0 \), whence \( X' = 0 \) and \( X'' \in a^{(N+1)} \). We arrive at \( X = 0 \), so \( \mathfrak{h} \) has finite type, proving an implication. The converse implication is trivial. \( \square \)

The full prolongation

\[
p_2^{(\infty)} = V + p_2 + p_2^{(1)} + \cdots
\]

of \( p_2 \) is also infinite-dimensional and

\[
p_2^{(k)} = \left\{ X \in \mathfrak{sp}(V)^{(k)} \mid \text{ad}_P^k(X) \in p_2 \right\}
= \mathbb{R}p_1^{k+1}q_1 + \sum_{i+j=k+2} S^1(W) \lor p_1^1,
\]

for all \( k \geq 1 \).

Lemma 2 A subalgebra \( \mathfrak{h} \) of \( \mathfrak{p}_2 \) has finite type if and only if the ideal \( \tilde{\mathfrak{h}} = \mathfrak{h} \cap (S^2(W) + p_1 W + \mathbb{R}p_1^2) \) has finite type.

Proof. Let \( \tilde{\mathfrak{h}}^{(i)} = 0 \) for all \( i \) bigger than or equal a nonnegative integer \( N \) and decompose any element \( X \in \tilde{\mathfrak{h}}^{(N+1)} \) into \( X = X' + X'' \), where

\[
X' = \lambda p_1^{N+2}q_1, \quad X'' = \sum_{i+j=N+3} S^1(W) \lor p_1^1,
\]

with \( \lambda \in \mathbb{R} \). Note that \( \text{ad}_P X = -\lambda p_1^{N+2} \in \tilde{\mathfrak{h}}^{(N)} = 0 \), whence \( X' = 0 \) and \( X'' \in \tilde{\mathfrak{h}}^{(N+1)} = 0 \). \( \square \)

The proofs of the following two results are immediate.

Lemma 3 The full prolongation of \( \mathfrak{s}_1 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \) is the Lie algebra direct sum of the full prolongations of \( \mathfrak{sp}(V_1) \) and \( \mathfrak{sp}(V_2) \).

Lemma 4 A subalgebra \( \mathfrak{h} \) of \( \mathfrak{s}_1 \) has finite type if and only if both \( \mathfrak{h} \cap \mathfrak{sp}(V_1) \) and \( \mathfrak{h} \cap \mathfrak{sp}(V_2) \) have finite type.

3 Classification of finite type subalgebras of \( \mathfrak{sp}_2(\mathbb{R}) \)

3.1 Finite type subalgebras of \( \mathfrak{s}_1 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \)

Let \( V = V_1 + V_2, V_i = \text{span}(p_i, q_i) \) for \( i = 1, 2 \), be an orthogonal decomposition of \( V \). The problem of description of finite type subalgebras of \( \mathfrak{s}_1 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \cong \mathfrak{s}_2(\mathbb{R}) \oplus \mathfrak{s}_2(\mathbb{R}) \) reduces to the description of the finite type subalgebras of \( \mathfrak{s}_2(\mathbb{R}) \), see Lemmata 3 and 3. We recall that the non-trivial subalgebras of \( \mathfrak{s}_2(\mathbb{R}) \) are, up to conjugation, the diagonal subalgebra \( \mathfrak{r}_{\text{diag}}(1, -1) \), \( \mathfrak{so}_2(\mathbb{R}) \), the Borel subalgebra \( \mathfrak{b}_2 \) (i.e. upper triangular subalgebra of \( \mathfrak{s}_2(\mathbb{R}) \)) and its nilradical \( n_2 \) of strictly upper triangular matrices.
Proposition 4 A non-zero finite type subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ is a Cartan subalgebra, i.e., it is conjugated either to $\mathbb{R}\text{diag}(1,-1)$ or to $\mathfrak{so}_2(\mathbb{R})$.

Proof. The only non-zero subalgebra of the complexification $\mathfrak{sl}_2(\mathbb{C})$ of $\mathfrak{sl}_2(\mathbb{R})$ with no rank one matrices is the Cartan subalgebra. The claim follows from this and Theorem \[6\] \hfill \Box

We get:

Theorem 7 The maximal finite type subalgebras of $\mathfrak{sl}_1(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ are (up to conjugation) the subalgebras

1. $\mathfrak{h}_1 = \mathfrak{t} \oplus \mathfrak{t}'$,
2. $\mathfrak{h}_2 = \mathfrak{sl}_2(\mathbb{R})^d$,
3. $\mathfrak{h}_3 = \mathfrak{sl}_2(\mathbb{R})^d$,

where $\mathfrak{t}$, $\mathfrak{t}'$ are Cartan subalgebras of $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{h}_2$ is the diagonal subalgebra and $\mathfrak{h}_3$ is the diagonal subalgebra twisted by $\theta = \text{Ad}_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$. The first prolongation is trivial in all cases.

To prove the theorem, we first need to recall the description of subalgebras of the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two Lie algebras. This is the content of Goursat’s Lemma, see e.g. [CV].

Lemma 5 (Goursat’s Lemma) Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be Lie algebras. There is a one-to-one correspondence between Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and quintuples $\mathcal{Q}(\mathfrak{h}) = (A, A_0, B, B_0, \theta)$, with $A \subset \mathfrak{g}_1$, $B \subset \mathfrak{g}_2$ Lie subalgebras, $A_0 \subset A$, $B_0 \subset B$ ideals and $\theta: A/A_0 \rightarrow B/B_0$ a Lie algebra isomorphism.

For any subalgebra $\mathfrak{h}$ of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ the associated quintuple $\mathcal{Q}(\mathfrak{h}) = (A, A_0, B, B_0, \theta)$ can be described as follows. Let $\pi_i: \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{g}_i$, $i = 1, 2$, be the natural projections and set

$A := \pi_1(\mathfrak{h}) \subset \mathfrak{g}_1$, \quad $A_0 := \ker(\pi_2|_\mathfrak{h}) \cong \mathfrak{h} \cap \mathfrak{g}_1$,

$B := \pi_2(\mathfrak{h}) \subset \mathfrak{g}_2$, \quad $B_0 := \ker(\pi_1|_\mathfrak{h}) \cong \mathfrak{h} \cap \mathfrak{g}_2$.

It is not hard to see that $A_0$ and $B_0$ can be identified with ideals in $A$ and $B$ respectively. The map $\theta: A/A_0 \rightarrow B/B_0$ is defined as follows. Given $a \in A$, take any $b \in B$ such that $a + b \in \mathfrak{h}$ and set $\theta(a + A_0) := b + B_0$. It is easy to check that this map is well defined and it is a Lie algebra isomorphism. This gives $\mathfrak{h} \mapsto \mathcal{Q}(\mathfrak{h})$.

Conversely, a quintuple $\mathcal{Q} = (A, A_0, B, B_0, \theta)$ as above defines a subalgebra $\mathfrak{h} = \mathcal{H}(\mathcal{Q})$ by

$\mathfrak{h} := \left\{ a + b \in A \oplus B \mid \theta(a + A_0) = b + B_0 \right\}$.

The operation $\mathcal{H}$ and $\mathcal{Q}$ are inverse to each other.

Remark 3 We will consider different subalgebras $\mathfrak{h} = \mathcal{H}(A, A_0, B, B_0, \theta)$ for some fixed $A, A_0$ and $B, B_0$. In this case it is convenient to identify the quotients $A/A_0$ and $B/B_0$ with a fixed abstract Lie algebra $\mathfrak{f}$ and look at $\theta: A/A_0 \rightarrow B/B_0$ as an automorphism of $\mathfrak{f}$.

The conjugacy classes of subalgebras are described in terms of the adjoint action of Lie groups $G_1, G_2$, with Lie algebras $\text{Lie}(G_1) = \mathfrak{g}_1$, $\text{Lie}(G_2) = \mathfrak{g}_2$, on the associated quintuples.

Lemma 6 Two subalgebras $\mathfrak{h}, \mathfrak{h}'$ of $\mathfrak{g}$ with corresponding quintuples $\mathcal{Q}(\mathfrak{h}) = (A, A_0, B, B_0, \theta)$ and $\mathcal{Q}(\mathfrak{h}') = (A', A'_0, B', B'_0, \theta')$ are conjugate if and only if there exists $(\mathfrak{g}_1, \mathfrak{g}_2) \in G_1 \times G_2$ such that $A' = \text{Ad}_{\mathfrak{g}_1}(A), B' = \text{Ad}_{\mathfrak{g}_2}(B), A'_0 = \text{Ad}_{\mathfrak{g}_1}(A_0), B'_0 = \text{Ad}_{\mathfrak{g}_2}(B_0)$ and the diagram
commutes.

We now turn to the proof of Theorem 7. We consider quintuples $Q = (A, A_0, B, B_0, \theta)$, with $A, B$ subalgebras of $\mathfrak{sl}_2(\mathbb{R})$, $A_0, B_0$ ideals of $A$ and $B$, respectively, and $\theta: A/A_0 \to B/B_0$ a Lie algebra isomorphism. By Lemma 4, the associated subalgebra $\mathfrak{h} = \mathfrak{H}(Q)$ of $\mathfrak{sl}_2$ has finite type if and only if $A_0$ and $B_0$ have finite type, i.e., they are Cartan subalgebras of $\mathfrak{sl}_2(\mathbb{R})$ (and in that case $f = 0$) or zero.

Table 1 below describes all the possibilities up to conjugation. Subalgebras $\mathfrak{h} = \mathfrak{H}(Q)$ with trivial $f$ are all contained in the subalgebra $\mathfrak{h}_1$ of Theorem 7. If $f$ is a Cartan subalgebra, $\theta: f \to \hat{f}$ is any non-zero multiple of the identity and the twisted diagonal $\hat{f}_0^d$ is again a subalgebra of $\mathfrak{h}_1$. If $f = n_2, b_2$ or $\mathfrak{sl}_2(\mathbb{R})$ then

$$\mathfrak{h} = \hat{f}_0^d = \{a + b \in f \oplus f | \theta(a) = b\}$$

is the diagonal subalgebra twisted by (the restriction of) the adjoint action $\theta = \text{Ad}_{g_1}$ of a $2 \times 2$-matrix $g_1$ with determinant $\pm 1$. The corresponding subalgebras are conjugated either to subalgebras of $\mathfrak{h}_2$ or to subalgebras of $\mathfrak{h}_3$ as in Theorem 7.

| $(A, A_0)$          | $(B, B_0)$          | $f$                |
|---------------------|---------------------|--------------------|
| $A = A_0 = \{0\}$, $\mathbb{R}\text{diag}(1,-1)$ or $\mathfrak{so}_2(\mathbb{R})$ | $B = B_0 = \{0\}$, $\mathbb{R}\text{diag}(1,-1)$ or $\mathfrak{so}_2(\mathbb{R})$ | $\{0\}$            |
| $A = \mathbb{R}\text{diag}(1,-1)$, $A_0 = \{0\}$ | $B = \mathbb{R}\text{diag}(1,-1)$, $B_0 = \{0\}$ | $\mathbb{R}\text{diag}(1,-1)$ |
| $A = \mathfrak{so}_2(\mathbb{R})$, $A_0 = \{0\}$ | $B = \mathfrak{so}_2(\mathbb{R})$, $B_0 = \{0\}$ | $\mathfrak{so}_2(\mathbb{R})$ |
| $A = n_2$, $A_0 = \{0\}$ | $B = n_2$, $B_0 = \{0\}$ | $n_2$              |
| $A = b_2$, $A_0 = \{0\}$ | $B = b_2$, $B_0 = \{0\}$ | $b_2$              |
| $A = \mathfrak{sl}_2(\mathbb{R})$, $A_0 = \{0\}$ | $B = \mathfrak{sl}_2(\mathbb{R})$, $B_0 = \{0\}$ | $\mathfrak{sl}_2(\mathbb{R})$ |

Table 1: Goursat quintuples for finite type subalgebras of $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{sp}_2(\mathbb{R})$. 
3.2 Finite type subalgebras of $p_1 = QP + S^2(P)$

The description of the finite type subalgebras of $p_1$ reduces to the description of finite type subspaces of $S^2(P)$, see Lemma 1. Note that $S^2(P) = \text{span}(p_1^2, p_1p_2, p_2^2) \subset \text{sp}(V)$ has a rank one endomorphism and it is therefore of infinite type. On the other hand $QP \cong \mathfrak{gl}(P)$ acts diagonally on

$$V = P + Q \cong P + P^*$$

and its first prolongation is trivial, see Proposition 1.

**Lemma 7** The 1-dimensional subspace of $S^2(P)$ generated by $X = x_1p_1^2 + x_2p_1p_2 + x_3p_2^2$ has infinite type if and only if $x_2^2 = 4x_1x_3$.

**Proof.** After complexification, we calculate

$$X : q_1 \mapsto -2x_1p_1 - x_2p_2 \mapsto 0,$$
$$X : q_2 \mapsto -2x_3p_2 - x_2p_1 \mapsto 0,$$

and note that $X$ has rank one if and only if $X(q_1)$ and $X(q_2)$ are proportional. □

Lemma 7 admits a more suggestive interpretation. Under the identification of $\text{SL}(P)$ with the (connected) spin group Spin$^\circ(1, 2)$ in Lorentzian signature, the action on $S^2(P)$ corresponds to the vectorial representation of Spin$^\circ(1, 2)$ on the Lorentzian vector space $\mathbb{R}^{1,2}$ and the elements of infinite type to the lightlike vectors. We shall fix an orthonormal basis $(e_0, e_1, e_2)$ of $S^2(P)$,

$$e_0 = \frac{1}{2}(p_1^2 + p_2^2), \quad e_1 = \frac{1}{2}(p_1^2 - p_2^2), \quad e_2 = p_1p_2,$$  \hspace{1cm} (6)

so that finite type 1-dimensional subspaces of $S^2(P)$ are conjugated either to $\mathbb{R}e_0$ (positive norm) or $\mathbb{R}e_2$ (negative norm). One can directly check that the first prolongation is trivial in both cases, i.e., $(\mathbb{R}e_0)^{(1)} = (\mathbb{R}e_2)^{(1)} = 0$.

**Lemma 8** Any 2-dimensional subspace of $S^2(P)$ has infinite type.

**Proof.** After complexification, it includes a lightlike vector. □

A direct consequence of Lemma 7 and the proof of Lemma 1 is the following a priori estimate of the type of $\mathfrak{h}$:

**Corollary 3** Let $\mathfrak{h} \subset p_1$ be a finite type subalgebra. Then $\dim(\mathfrak{h} \cap S^2(P)) \leq 1$ and $\mathfrak{h}^{(k)} = 0$ for all $k \geq 2$.

To proceed further, we shall use the classification of the subalgebras of $p_1$. Note first that

$$p_1 \cong \mathfrak{so}(1, 2) + \mathbb{R}^{1,2}, \quad QP \cong \mathfrak{so}(1, 2), \quad S^2(P) \cong \mathbb{R}^{1,2},$$

is isomorphic to the similitude algebra of the Lorentzian vector space $\mathbb{R}^{1,2}$. The description of the conjugacy classes of subalgebras of the similitude algebra is rather involved and it can be found in [PWSZ, §4, Tables 2-4]. We recall here only the facts that we need and refer directly to [PWSZ] for more details (see also the overview [Win]).

Let $\mathfrak{h} \subset \mathfrak{so}(1, 2)$ be a subalgebra, $\overline{\mathfrak{h}}$ its projection to $\mathfrak{so}(1, 2) \cong \mathfrak{so}(1, 2)/\mathbb{R}^{1,2}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathbb{R}^{1,2}$. The subalgebra $\mathfrak{h}_s = \overline{\mathfrak{h}} + \mathfrak{a}$ is called the splitting subalgebra associated to $\mathfrak{h}$ and it is an invariant of $\mathfrak{h}$ up to conjugation. All subalgebras conjugated to a subalgebra of the form $\mathfrak{h} = \overline{\mathfrak{h}} + \mathfrak{a}$ are called splitting; otherwise they are nonsplitting.

Nonsplitting subalgebras are all obtained as deformations of splitting subalgebras $\mathfrak{h} = \overline{\mathfrak{h}} + \mathfrak{a}$. More precisely $\overline{\mathfrak{h}}$ is twisted by cocycles $Z^1(\overline{\mathfrak{h}}, \mathfrak{m})$ of the Chevalley-Eilenberg complex of $\overline{\mathfrak{h}}$ with...
values in $m = \mathbb{R}^{1,2}/a$, and coboundaries give subalgebras conjugated under the translation subgroup $\exp(\mathbb{R}^{1,2})$. In other words nonsplitting subalgebras $h$ with underlying associated splitting subalgebra $h_s = \mathfrak{a} + a$ are described by elements of the first Chevalley-Eilenberg cohomology group $H^1(\mathfrak{a}, m)$. We note that
\[
a = h \cap \mathbb{R}^{1,2} = h_s \cap \mathbb{R}^{1,2}
\]
so that $h$ has finite type if and only if $h_s$ has finite type, cf. Lemma 1.

We make contact with the notation of [PWSZ] and introduce the dilation, boost and rotation matrices
\[
F = -\frac{1}{2}(p_1 q_1 + p_2 q_2), \quad K_1 = \frac{1}{2}(p_1 q_1 - p_2 q_2), \\
K_2 = \frac{1}{2}(p_1 q_2 + p_2 q_1), \quad L_3 = -\frac{1}{2}(p_1 q_2 - p_2 q_1),
\]
which together with (5) identify $p_1$ with $\text{sim}(1,2)$. The brackets are as in [PWSZ, p. 951], in particular $[F, e_i] = -e_i$ for $i = 0, 1, 2$. The following result follows from Lemmata 1, 7 and 8 and the classification of the subalgebras of $\text{sim}(1,2)$.

**Proposition 5** The finite type subalgebras $h$ of $p_1$ are, up to conjugation, either contained in $QP \cong \text{co}(1,2)$ or they are one of the subalgebras listed in Table 2.

| Name | Abstract Lie algebra | Generators | Splitting Subalgebra |
|------|----------------------|------------|----------------------|
| $F_{6,5}$ | $\mathbb{R}$ | $e_2$ | |
| $F_{6,6}$ | $\mathbb{R}$ | $e_0$ | |
| $F_{3,5}$ | $\text{so}(1,1) \oplus \mathbb{R}$ | $K_1, e_2$ | |
| $F_{5,3}$ | $\text{so}_2(\mathbb{R}) \oplus \mathbb{R}$ | $L_3, e_0$ | |
| $DF_{6,5}$ | $\mathbb{R} \oplus \mathbb{R}$ | $F, e_2$ | |
| $DF_{6,6}$ | $\mathbb{R} \oplus \mathbb{R}$ | $F, e_0$ | |
| $DF_{5,3}$ | $\text{co}(1,1) \oplus \mathbb{R}$ | $F, K_1, e_2$ | |
| $DF_{5,3}$ | $\text{co}(2) \oplus \mathbb{R}$ | $F, L_3, e_0$ | |
| $F_{3,9}$ | $\mathbb{R}$ | $K_1 + ae e_2$ where $a \neq 0$ | $\mathbb{R}K_1$ |
| $F_{4,7}$ | $\mathbb{R}$ | $K_2 + L_3 + \epsilon(e_0 + e_1)$ where $\epsilon = \pm 1$ | $\mathbb{R}(K_2 + L_3)$ |
| $F_{5,6}$ | $\mathbb{R}$ | $L_3 + ae e_0$ where $a \neq 0$ | $\mathbb{R}L_3$ |
| $D_{4,12}$ | $\mathbb{R} \oplus \mathbb{R}$ | $F - K_1 + \epsilon(e_0 - e_1), K_2 + L_3$ for $\epsilon = \pm 1$ | $D_{4,11}$ with $a = -1$ |
| $D_{4,13}$ | $\mathbb{R} \oplus \mathbb{R}$ | $F + (1/2)K_1, K_2 + L_3 + \epsilon(e_0 + e_1)$ for $\epsilon = \pm 1$ | $D_{4,11}$ with $a = 1/2$ |
| $D_{6,13}$ | $\mathbb{R} \oplus \mathbb{R}$ | $F + aK_1, e_0$ for $a > 0$ | $D_{6,13}$ with $a = +1$ |
| $D_{6,15}$ | $\mathbb{R} \oplus \mathbb{R}$ | $F + aK_1, e_0$ for $a \neq 0$ | $D_{6,21}$ with $a = +1$ |
| $D_{6,22}$ | $\mathbb{R}$ | $F + K_1 + \epsilon(e_0 + e_1)$ for $\epsilon = \pm 1$ | $D_{6,21}$ with $a = +1$ |

Table 2: Finite type subalgebras of $p_1 \subset \text{sp}_2(\mathbb{R})$ not contained in $QP$.

The associated splitting subalgebras have been illustrated for all the nonsplitting subalgebras and we used the notation $D_{4,11} := \text{span}(F + aK_1, K_2 + L_3)$ and $D_{6,21} := \text{span}(F + aK_1)$.

The maximal finite type subalgebras of $p_1$ are $\text{co}(1,2)$, $DF_{3,5}$, $DF_{5,3}$ and the nonsplitting subalgebras $D_{4,12}$, $D_{4,13}$ and $D_{6,14}$. First note that the commutative radical $\mathbb{R}^{1,2}$ of $p_1$ is an irreducible $\text{co}(1,2)$-module, whence $\text{co}(1,2)$ is a maximal subalgebra. The subalgebras $DF_{3,5}$ and $DF_{5,3}$ are 3-dimensional and cannot be (conjugated in $p_1$) subalgebras of $\text{co}(1,2)$, as they intersect $\mathbb{R}^{1,2}$ nontrivially. Finally, the subalgebras $D_{4,12}$, $D_{4,13}$ and $D_{6,14}$ are all 2-dimensional and clearly not subalgebras of $\text{co}(1,2)$. A straightforward computation shows that they cannot be included in any 3-dimensional finite type subalgebra of $p_1$ either, in particular they are not conjugated to subalgebras of $DF_{3,5}$ and $DF_{5,3}$.

We have proved most of the following.
Theorem 8 The maximal finite type subalgebras of $\mathfrak{p}_1 \cong \mathfrak{co}(1, 2) + \mathbb{R}1.2$ are, up to conjugation, given by

1. $\mathfrak{h} = \mathfrak{co}(1, 2),$

2. the 3-dimensional solvable splitting subalgebras $\mathfrak{h} = \mathfrak{f} + \mathfrak{a}$, where $\mathfrak{a} = \mathbb{R}\mathfrak{e}_0$ or $\mathbb{R}\mathfrak{e}_2$ and $\mathfrak{f} = \text{Nor}_{\mathfrak{co}(1, 2)}(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{co}(1, 2),$

3. the 2-dimensional nonsplitting subalgebras $D_{4,12}, D_{4,13}$ and $D_{6,14}$ illustrated in Table 2.

The first prolongation is trivial in all cases.

Proof. It remains to show the last claim. If $\mathfrak{h} = \mathfrak{co}(1, 2)$ we already know that $\mathfrak{h}^{(1)} = 0$ and the other cases are settled by straightforward computations, which we omit. □

3.3 Finite type subalgebras of $\mathfrak{p}_2 = \mathfrak{gl}(W) + \mathfrak{heis}(W)$

By Lemma 2, the description of the finite type subalgebras of $\mathfrak{p}_2 = \mathfrak{gl}(W) + \mathfrak{heis}(W)$ reduces to the description of the finite type subalgebras of its maximal ideal

$$\tilde{\mathfrak{p}}_2 = \mathfrak{sl}(W) + \mathfrak{heis}(W).$$

We first note that the action of the basic endomorphisms of $\mathfrak{heis}(W) = \text{span}(p_1p_2, p_1q_2, p_1^2)$ on the basic elements of $V$ is given by

$$p_1p_2 : q_1 \mapsto -p_2, \quad q_2 \mapsto -p_1,$$

$$p_1q_2 : q_1 \mapsto -q_2, \quad p_2 \mapsto p_1,$$

$$p_1^2 : q_1 \mapsto -2p_1,$$

and trivial otherwise. This shows the following.

Lemma 9 Any finite type subalgebra of $\mathfrak{heis}(W)$ is conjugated to one of the form $\mathbb{R}p_1w$ for some $w \in W$. We may assume $w = 0$ (trivial subalgebra) or $w = p_2$.

Assume $\tilde{\mathfrak{h}}$ is a finite type subalgebra of $\tilde{\mathfrak{p}}_2$ with nontrivial intersection $\mathfrak{a} = \tilde{\mathfrak{h}} \cap \mathfrak{heis}(W)$ with $\mathfrak{heis}(W)$. Then $\mathfrak{a} = \mathbb{R}p_1p_2$ is an ideal of $\tilde{\mathfrak{h}}$, that is

$$\tilde{\mathfrak{h}} \subset \text{Nor}_{\mathfrak{p}_2}(\mathfrak{a}) = \text{span}(p_2^2, p_2q_2) + \mathbb{R}p_1p_2 + \mathbb{R}p_1^2,$$

where $\text{Nor}_{\mathfrak{p}_2}(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{p}_2$. We note that the normalizer is the semidirect sum of the Borel subalgebra $\mathfrak{b}_2$ of $\mathfrak{sl}(W)$ and a two-dimensional abelian ideal. Hence, we may regard $\mathfrak{h}$ as a (possible trivial) deformation of the corresponding splitting subalgebra $\mathfrak{h}_s = \mathfrak{f} + \mathfrak{a}$, where $\mathfrak{f}$ is the projection of $\mathfrak{h}$ to $\mathfrak{b}_2$ (see [Win]). The list of such splitting subalgebras consists of

$$\mathfrak{b}_2 + \mathbb{R}p_1p_2, \quad n_2 + \mathbb{R}p_1p_2 \quad \text{and} \quad \mathbb{R}\text{diag}(1, -1) + \mathbb{R}p_1p_2,$$

where $n_2 = \mathbb{R}p_2^2$ is the nilradical of $\mathfrak{b}_2$ and $\mathbb{R}\text{diag}(1, -1) = \mathbb{R}p_2q_2$ the diagonal subalgebra.

The proof of the following lemma is straightforward.

Lemma 10 The Chevalley-Eilenberg cohomology group $H^1(\tilde{\mathfrak{h}}, m)$, $m \cong \mathbb{R}p_1^2$, is trivial when $\tilde{\mathfrak{h}}$ is trivial and it coincides with the 1-dimesional space $\text{Hom}(\tilde{\mathfrak{h}}, m)$ when $\tilde{\mathfrak{h}}$ is $n_2$ or $\mathbb{R}\text{diag}(1, -1)$.

The group $H^1(\mathfrak{b}_2, m)$ is 1-dimensional too and it is generated by the cocycle

$$c : \mathfrak{b}_2 \rightarrow m$$

given by $c(p_2^2) = 0$ and $c(p_2q_2) = p_1^2$. 

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It follows that the finite type subalgebras of $\tilde{p}_2$ with nontrivial intersection with $\mathfrak{heis}(W)$ are among the following subalgebras:

(i) $\text{span}(p_2^2, p_2q_2 + c p_1^2, p_1 p_2)$,
(ii) $\text{span}(p_2^2 + c p_1^2, p_1 p_2)$,
(iii) $\text{span}(p_2q_2 + c p_1^2, p_1 p_2)$,
(iv) $\text{span}(p_1 p_2)$,

where $c \in \mathbb{R}$. Using conjugation by the grading element of $p_2$, we may also arrange for $c = 0, \pm 1$.

The action of the basic endomorphisms of $\mathfrak{sl}(W) = \text{span}(p_2^2, p_2q_2, q_2^2)$ on $W$ is

$$p_2^2 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad p_2q_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_2^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

and a finite type subalgebra of $\mathfrak{sl}(W)$ consists only of semisimple elements. This shows that the subalgebra (i) has infinite type and, in a similar way, one sees that the complexification of (ii) has an element of rank one. Using equation (5) with $k = 1$, one finally arrives at the following.

**Proposition 6** The finite type subalgebras of $\tilde{p}_2$ with nontrivial intersection with $\mathfrak{heis}(W)$ are, up to conjugation, the subalgebra (iii) with $c = 0$ or $c = \pm 1$ and the subalgebra (iv). The first prolongation is trivial in all cases.

Assume now that $\tilde{\mathfrak{h}}$ has no intersection with $\mathfrak{heis}(W)$. Let $\pi : \tilde{p}_2 \to \tilde{p}_2/\mathfrak{heis}(W) \cong \mathfrak{sl}(W)$ be the natural projection and set $\overline{\mathfrak{h}} = \pi(\tilde{\mathfrak{h}})$. Clearly $\tilde{\mathfrak{h}} \cong \overline{\mathfrak{h}}$ is isomorphic to a subalgebra of $\mathfrak{sl}(W)$ and we may assume $\overline{\mathfrak{h}}$ to be the whole $\mathfrak{sl}(W)$, $\mathbb{R}\text{diag}(1, -1)$, $\mathfrak{so}_2(\mathbb{R})$, $\mathfrak{b}_2$ or $\mathfrak{n}_2$. However it is not true, in general, that $\tilde{\mathfrak{h}}$ has finite type if and only if $\overline{\mathfrak{h}}$ has finite type.

Write

$$\tilde{\mathfrak{h}} = \{(X, \varphi(X)) \mid X \in \overline{\mathfrak{h}}\},$$

where $\varphi : \overline{\mathfrak{h}} \to \mathfrak{heis}(W)$. Since the Heisenberg algebra is not abelian, the map $\varphi$ has to satisfy a non-linear version of the Chevalley-Eilenberg cocycle condition in order for $\tilde{\mathfrak{h}}$ to be a Lie algebra:

$$\varphi[X, Y] = [\varphi(X), Y] + [X, \varphi(Y)] + [\varphi(X), \varphi(Y)],$$

for all $X, Y \in \overline{\mathfrak{h}}$. It is convenient to decompose $\varphi = c + \psi$ into components

$$c : \overline{\mathfrak{h}} \to p_1 W, \quad \psi : \overline{\mathfrak{h}} \to \mathbb{R} p_1^2,$$

and rewrite (9) as

$$c[X, Y] = [c(X), Y] + [X, c(Y)], \quad \psi[X, Y] = [c(X), c(Y)].$$

We note that $c$ is a cocycle in the Chevalley-Eilenberg complex of $\overline{\mathfrak{h}}$ with values in $p_1 W$ and that the coboundaries correspond to subalgebras conjugated under the analytic subgroup $\exp(p_1 W)$. For any given cocycle, one easily reconstructs the full map $\varphi$ using (10).

**Lemma 11** The Chevalley-Eilenberg cohomology group $H^1(\overline{\mathfrak{h}}, p_1 W)$ is zero, except for $\overline{\mathfrak{h}} = \mathfrak{n}_2$, in which case it is 1-dimensional and generated by the cocycle $c(p_2^2) = p_1 q_2$.

It follows that the finite type subalgebras of $\tilde{p}_2$ with trivial intersection with $\mathfrak{heis}(W)$ are among the following subalgebras:

(v) $\mathfrak{sl}(W)$,
(vi) \( \text{span}(p_2^2, p_2 q_2 + \epsilon p_1^2) \),
(vii) \( \text{span}(p_2 q_2 + \epsilon p_1^2) \),
(viii) \( \text{span}(p_2^2 + q_2^2 + \epsilon p_1^2) \),
(ix) \( \text{span}(p_2^2 + \epsilon p_1^2) \),
(x) \( \text{span}(p_2^2 + \epsilon p_1 q_2) \),

where \( \epsilon = 0 \) or \( \epsilon = \pm 1 \). The last two conjugacy classes correspond to the case \( \mathfrak{h} = \mathfrak{n}_2 \) with the corresponding cocycle either vanishing or, respectively, as in Lemma \( \text{III} \). The proof of the following result is similar to that of Proposition \( 6 \) and we omit it.

**Proposition 7**  The finite type subalgebras of \( \tilde{\mathfrak{p}}_2 \) with trivial intersection with \( \mathfrak{heis}(W) \) are, up to conjugation, the subalgebras (vii)-(viii) with \( \epsilon = 0, \pm 1 \) and the subalgebras (ix)-(x) with \( \epsilon = \pm 1 \). The first prolongation is trivial in all cases.

Let us now sum up the results of Propositions \( 6 \) and \( 7 \). Let \( \mathfrak{h} \) be a finite type subalgebra of \( \mathfrak{p}_2 \) and \( \mathfrak{h} = \mathfrak{h} \cap \tilde{\mathfrak{p}}_2 \) its ideal, which is a subalgebra of \( \tilde{\mathfrak{p}}_2 \) as we just determined. We either have \( \mathfrak{h} = \mathfrak{h} \) or \( \mathfrak{h} \) is a codimension one ideal of \( \mathfrak{h} \). The latter case happens if \( \mathfrak{h} \) is a graded subalgebra of \( \tilde{\mathfrak{p}}_2 \) and \( \mathfrak{h} = \mathfrak{h} + \mathbb{R} p_1 q_1 \), and in few more cases:

\[
\begin{align*}
\text{span}(p_1 p_2, p_1 q_1 + \lambda p_2 q_2), & \quad \text{span}(p_1 p_2, p_1 q_1 + \epsilon p_2^2), \\
\text{span}(p_2^2 + \epsilon p_1 q_2, 3p_1 q_1 + p_2 q_2), & \quad \text{span}(p_2^2 + \epsilon p_1 q_2, 3p_1 q_1 + p_2 q_2),
\end{align*}
\]

where \( \epsilon = \pm 1 \) and \( \lambda \neq 0 \).

**Theorem 9**  The maximal finite type subalgebras of \( \mathfrak{p}_2 \) are, up to conjugation, given by the graded subalgebras

1. \( \text{span}(p_2 q_2, p_1 q_1, p_1 p_2) \),
2. \( \text{span}(p_2^2 + q_2^2, p_1 q_1) \),

and the subalgebras

3. \( \text{span}(p_2 q_2 + \epsilon p_1^2, p_1 p_2) \),
4. \( \text{span}(p_2^2 + q_2^2 + \epsilon p_1^2) \),
5. \( \text{span}(p_2^2 + \epsilon p_1 q_2, p_1 q_1 + p_2 q_2) \),
6. \( \text{span}(p_2^2 + \epsilon p_1 q_2, 3p_1 q_1 + p_2 q_2) \),

where \( \epsilon = \pm 1 \). The first prolongation is trivial in all cases.

**Proof.** The first claim is now immediate and the last follows from a computation. \( \square \)

**Remark 4**  The subalgebras (1), (3) and (6) of Theorem \( 8 \) appeared already in Theorem \( 8 \) and the subalgebra (2) in Theorem \( 7 \). The subalgebra (5) is included in \( \text{DF}_{5,3} \) if \( \epsilon = +1 \) and in \( \text{DF}_{3,5} \) if \( \epsilon = -1 \). (In the second case, it is enough to notice that both vectors \( e_1, e_2 \) in equation (6) have negative norm.)
3.4 Proof of Theorem [\[1\]

In this section we shall complete the proof of Theorem [\[1\]]. Let us first summarize the discussion of \[\[2\]\] and Theorems [\[7\],[\[9\]]] into the following auxiliary result.

**Proposition 8** A finite type subalgebra \(\mathfrak{h}\) of \(\mathfrak{sp}_2(\mathbb{R})\) is, up to conjugation, included in one of the subalgebras of the following list:

1. the unitary algebra \(u_2\);
2. the pseudo-unitary algebra \(u_{1,1}\);
3. the subalgebra \(Q \vee P \cong \mathfrak{gl}(P)\), where \(P\) and \(Q\) are complementary Lagrangian subspaces;
4. the irreducible subalgebra \(\mathfrak{sl}_3^d(\mathbb{R})\) acting on \(V = S^3(\mathbb{R}^2)\);
5. the solvable 3-dimensional subalgebras \(DF_{5,3}\) and \(DF_{3,5}\);
6. the diagonal subalgebra \(\mathfrak{sl}_2(\mathbb{R})^d\) and the twisted diagonal subalgebra \(\mathfrak{sl}_2(\mathbb{R})^d_P\);
7. the direct sum \(\mathfrak{k} \oplus \mathfrak{k}'\) of two Cartan subalgebras of \(\mathfrak{sl}_2(\mathbb{R})\) (i.e., \(\mathbb{R}\text{diag}(1,-1)\) or \(\mathfrak{so}_2(\mathbb{R})\));
8. the solvable 2-dimensional subalgebras \(D_{4,12}\), \(D_{4,13}\) and \(D_{6,14}\);
9. the 1-dimensional subalgebra \(\text{span}(p_2^2 + q_2^2 + \epsilon p_1^2)\), where \(\epsilon = \pm 1\).

We claim that all these subalgebras are maximal finite type subalgebras with the exception of

\(DF_{3,5}, \ DF_{5,3}, \ \mathfrak{sl}_2(\mathbb{R})^d, \ \mathfrak{sl}_2(\mathbb{R})^d_P, \ 2\mathbb{R}\text{diag}(1,-1), \ 2\mathfrak{so}_2(\mathbb{R}), \ D_{4,13}\).

Now, the 4-dimensional subalgebras are clearly maximal finite type subalgebras of \(\mathfrak{sp}_2(\mathbb{R})\). The complexification of the singular subalgebra \(\mathfrak{sl}_2^d(\mathbb{R})\) is a maximal subalgebra of \(\mathfrak{sp}_2(\mathbb{C})\) by a classical result of Dynkin [GOVIII Theorem 3.3], hence \(\mathfrak{sl}_2^d(\mathbb{R})\) is a maximal subalgebra of \(\mathfrak{sp}_2(\mathbb{C})\).

We have seen that the 3-dimensional solvable subalgebras \(DF_{5,3}\) and \(DF_{3,5}\) are not conjugated in \(p_1\) to subalgebras of \(\mathfrak{gl}(P)\). However, a simple computation shows that the ring of invariant endomorphisms of \(V\) is two-dimensional, generated in the first case (resp. the second case) by a split complex structure compatible with \(\Omega\) (resp. a paracomplex structure). Hence, a posteriori, \(DF_{5,3} \subset u_{1,1}\) and \(DF_{3,5} \subset \mathfrak{gl}(P)\), up to conjugation in the full symplectic group.

The diagonal subalgebra \(\mathfrak{sl}_2(\mathbb{R})^d\) is also not maximal, as \(\mathfrak{sl}_2(\mathbb{R})^d = u_{1,1} \subset u_{1,1}\). Similarly, the twisted diagonal subalgebra \(\mathfrak{sl}_2(\mathbb{R})^d_P\) preserves a paracomplex structure compatible with \(\Omega\), so that \(\mathfrak{sl}_2(\mathbb{R})^d_P = \mathfrak{sl}_2(\mathbb{R})^d = \mathfrak{gl}(P)\).

The direct sum \(2\mathbb{R}\text{diag}(1,-1)\) of two noncompact Cartan subalgebras of \(\mathfrak{sl}_2(\mathbb{R})\) coincides with the normalizer \(\mathfrak{N}\) of \(\mathfrak{gl}(P)\) of \(a = \mathbb{R}(p_1p_2)\), hence it is contained in \(DF_{3,5} \subset \mathfrak{gl}(P)\). The direct sum \(2\mathfrak{so}_2(\mathbb{R})\) of two compact Cartan subalgebras of \(\mathfrak{sl}_2(\mathbb{R})\) clearly is, up to conjugation, a subalgebra of the maximal compact subalgebra \(u_2\).

On the contrary, the direct sum \(\mathbb{R}\text{diag}(1,-1) \oplus \mathfrak{so}_2(\mathbb{R})\) is a maximal finite type subalgebra. The ring of invariant endomorphisms of \(V\) is 4-dimensional and it is not difficult to see that there is no invariant complex or paracomplex structure compatible with \(\Omega\). Hence, this Cartan subalgebra is not (conjugated to) a subalgebra of a 4-dimensional finite type subalgebra. Furthermore, it is not contained in a 3-dimensional finite type subalgebra either, since \(\mathfrak{sl}_2(\mathbb{R})\) does not have 2-dimensional subalgebras consisting of semisimple elements.

So far we obtained the following list of maximal finite type subalgebras

\[u_2, \ u_{1,1}, \ \mathfrak{gl}(P), \ \mathfrak{sl}_2^d(\mathbb{R}) \text{ and } \mathbb{R}\text{diag}(1,-1) \oplus \mathfrak{so}_2(\mathbb{R}).\]
We note that all such subalgebras are algebraic in the sense that, after complexification, the semisimple and nilpotent components of any element of the subalgebra still lie in the subalgebra. It remains to consider the subalgebras 8-9 of Proposition 8.

We depart with the subalgebras 8 which are not conjugated to a subalgebra from this list. The algebraic closure of subalgebras $D_{4,12} = \text{span}(p_1q_1 + \epsilon p_2^2, p_2q_1)$ and $D_{6,14} = \text{span}(p_2q_2 + \epsilon p_1^2, p_1p_2)$ are given by

$$D_{4,12} = \text{span}(p_1q_1, p_2q_1, p_2^2),$$

$$D_{6,14} = \text{span}(p_2q_2, p_1p_2, p_1^2),$$

and they have infinite type, since they include a rank one element. Therefore $D_{4,12}$ and $D_{6,14}$ are maximal finite type subalgebras. Actually, it is not difficult to see these two subalgebras do coincide, up to conjugation in the full symplectic group.

The derived ideal of the algebra $D_{4,13} = \text{span}(p_1q_1 + 3p_2q_2, p_2q_1 + \epsilon p_1^2)$ consists of a nilpotent endomorphism with Jordan block of order four. Such endomorphism does not exist in the derived ideal of any of the algebras from the list (11), with the exception of $\mathfrak{sl}_2^4(\mathbb{R})$. Indeed, it is easy to see that $D_{4,13}$ is exactly the Borel subalgebra $\mathfrak{h}_2$ of $\mathfrak{sl}_2^4(\mathbb{R})$.

Finally, the algebraic closure of the 1-dimensional subalgebra 9 is $\text{span}(p_2^2 + q_2^2, p_1^2)$, which has infinite type, it is abelian and has a semisimple element generating a compact one-parameter subgroup. One can easily check that the algebras from list (11) and $D_{4,12}$ do not contain subalgebras of this kind. The proof of Theorem 8 is completed.

3.5 Proof of Theorem 2

By the discussion above Theorem 2, we already know that if $\mathfrak{h}$ has finite type or $\mathfrak{h} = \mathfrak{sp}(V)$ then $\text{gr}(\mathfrak{g}) = V + \mathfrak{h}$. Otherwise, $\mathfrak{h}$ is contained in a maximal infinite type subalgebra of $\mathfrak{sp}(V)$, that is a maximal parabolic subalgebra or one of the subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_4$.

If $\mathfrak{h} = \mathfrak{s}_1$ or $\mathfrak{s}_4$ then, after complexification, the first prolongation $\mathfrak{h}^{(1)} = \mathbb{C}^3(\mathbb{C}^2) \oplus \mathbb{C}^3(\mathbb{C}^2)$ and by an argument similar to the case $\mathfrak{h} = \mathfrak{sp}(V)$ we have that $\text{gr}(\mathfrak{g})$ is either infinite-dimensional, which is not possible, or $\text{gr}(\mathfrak{g}) = V + \mathfrak{h}$. Furthermore, if $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{s}_4$ then either $\text{gr}(\mathfrak{g}) = V + \mathfrak{h}$ or $\mathfrak{h}$ is a subalgebra of $\mathfrak{p}_1$ by Proposition 3.

It remains to study the case where $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{s}_1$ which is not of finite type. According to 3.3 we have

$$\mathfrak{h} = \left\{a + b \in A \oplus B \mid \theta(a + A_0) = b + B_0\right\},$$

for some subalgebras $A, B \subset \mathfrak{s}_2(\mathbb{R})$ and ideals $A \subset A_0, B \subset B_0$ such that $A/A_0 \cong B/B_0$. First of all we can assume that $A$ and $B$ are equal to $\mathfrak{s}_2(\mathbb{R})$ or to a Cartan subalgebra of $\mathfrak{s}_2(\mathbb{R})$, otherwise $\mathfrak{h}$ would preserve an isotropic line and therefore be a subalgebra of $\mathfrak{p}_2$. We may also assume that $A_0$ is not contained in a Cartan subalgebra, since $\mathfrak{h}$ has infinite type.

It follows that $A = A_0 = \mathfrak{s}_2(\mathbb{R})$ and $\mathfrak{h}$ is one of the Lie algebras

$$\mathfrak{s}_2(\mathbb{R}) \oplus \mathfrak{s}_2(\mathbb{R}), \quad \mathfrak{s}_2(\mathbb{R}) \oplus \mathbb{R}\text{diag}(1, -1), \quad \mathfrak{s}_2(\mathbb{R}) \oplus \mathfrak{s}_2(\mathbb{R}).$$

Again $\text{gr}(\mathfrak{g}) = V + \mathfrak{h}$, since it is finite-dimensional. We established the first claim of Theorem 2, the last claim is an immediate consequence.

4 Finite-dimensional subalgebras of $\mathfrak{p}_1^{(\infty)}$ and $\mathfrak{p}_2^{(\infty)}$, and their associated homogeneous symplectic 4-manifolds

In this section, we will consider finite-dimensional transitive Lie algebras of symplectic vector fields (on the symplectic 4-dimensional space $V$) which are nonlinear and with the isotropy
subalgebra of infinite type. Our arguments rely on a closer look at the full prolongations of the maximal parabolic subalgebras of $\mathfrak{sp}(V)$.

**Notation:** For any Lie algebra $\mathfrak{g}$ and associative algebra $A$, we will denote by $A \cdot \mathfrak{g}$ the tensor product $A \otimes \mathfrak{g}$ of $A$ and $\mathfrak{g}$ with its natural structure of Lie algebra.

### 4.1 Geometric preliminaries on transitive Lie algebras of vector fields

In this section, we introduce some basic geometric notions on transitive Lie algebras $\mathfrak{g}$ of vector fields on $\mathbb{R}^n$. (In this paper, we will be interested only in $n \leq 4$.) In our conventions, the origin of $\mathbb{R}^n$ is always a regular point for $\mathfrak{g}$.

Let $\mathfrak{g}$ be a transitive Lie algebra of (analytic) vector fields on $\mathbb{R}^n$. The following notions are a direct consequence of classical facts on imprimitive Lie algebras (see, e.g., [SS, §1.5] and [GKO, §2]).

**Definition 1** The Lie algebra $\mathfrak{g}$ is **primitive on an open subset** $U \subset \mathbb{R}^n$ **if there is no foliation of $U$ whose leaves are permuted by the (local) one-parameter subgroups corresponding to elements in $\mathfrak{g}$.** Equivalently, $\mathfrak{g}$ is primitive on $U$ **if there is no non-trivial involutive distribution $\mathcal{D}$ on $U$ left invariant by $\mathfrak{g}$.** Otherwise, $\mathfrak{g}$ is called **imprimitive on $U$**.

**Definition 2** The Lie algebra $\mathfrak{g}$ is called **primitive** if it is primitive in every open subset $U \subset \mathbb{R}^n$. Otherwise, $\mathfrak{g}$ is **imprimitive**.

Let $\mathfrak{g}$ be imprimitive and $x^1, \ldots, x^k, y^1, \ldots, y^{n-k}$ be local coordinates such that the system of imprimitivity corresponding to $\mathcal{D}$ is given by the leaves $y = \text{const}$. Then any vector field $X \in \mathfrak{g}$ is (locally) of the form

$$X = f^1 \frac{\partial}{\partial x^1} + \cdots + f^k \frac{\partial}{\partial x^k} + g^1 \frac{\partial}{\partial y^1} + \cdots + g^{n-k} \frac{\partial}{\partial y^{n-k}},$$

where the $g^i$'s are functions of the $y$'s alone. The set of $X$ with $g^1 = \cdots = g^{n-k} = 0$ forms an ideal $i$ in $\mathfrak{g}$, which consists of the vector fields leaving all the leaves invariant. In general, it is not a transitive Lie algebra of vector fields when acting on a leaf.

**Definition 3** The ideal $i$ is called the **canonical ideal** (of the imprimitive Lie algebra $\mathfrak{g}$, associated to the distribution $\mathcal{D}$).

Since the passage to the quotient space of a manifold under the equivalence relation $\sim$ induced by a foliation commonly entails technical difficulties, the following concepts are only locally defined around the origin.

**Definition 4** Let $\mathfrak{g}$ be a transitive Lie algebra of vector fields on $\mathbb{R}^n$ which is imprimitive on a neighborhood $U$ of the origin. Then we say that $\mathfrak{g}$ is:

1. **transitive on leaves** if the canonical ideal $i$ acting on any leaf $y = \text{const}.$ is a transitive Lie algebra;

2. **primitive on leaves** if it is transitive on leaves and the canonical ideal $i$ acting on any leaf $y = \text{const}.$ is a primitive Lie algebra;

3. **transversally primitive** if the (transitive) Lie algebra $\mathfrak{g}/i$ acting on the local quotient space $\mathbb{R}^{n-k} \cong (U/\sim)$ is a primitive Lie algebra.
4.2 Finite-dimensional subalgebras of $p_2^{(\infty)}$

We recall that the Lie algebra $p_2^{(\infty)}$ of formal symplectic vector fields is graded and with isotropy subalgebra $p_2$. The parabolic $p_2$ is the stabilizer of the line $\mathbb{R}p_1$, hence it leaves invariant also the 3-dimensional orthogonal space $p_1^\perp = \mathbb{R}p_1 + W$, $W = \text{span}(p_2, q_1)$.

It follows that $p_2^{(\infty)}$ is of the “very imprimitive” sort: there exists a flag of involutive distributions

$$0 \subset D_1 \subset D_3 \subset T\mathbb{R}^4,$$

where $\text{rk} D_i = i$ for $i = 1, 3$, which is left invariant by $p_2^{(\infty)}$. The systems of imprimitivity corresponding to these distributions are defined on the entire $\mathbb{R}^4$ and we let $i_1, i_3$ be the associated canonical ideals. Clearly $i_1 \subset i_3$.

Now, the Lie algebra $p_2^{(\infty)}$ is obviously not primitive on the leaves $y = \text{const.}$ of $D_3$ (here and throughout this section, $y$ is a single real coordinate) and it is transversally primitive across these leaves in a trivial way. The action on each leaf of the canonical ideal $i_3$ is transversally primitive (across the leaves of $D_1$), at least formally. To check this, we need the following result, a direct consequence of the natural identification of the algebra of formal symplectic vector fields with the symmetric algebra of $V$ (modulo constants) and equation (5) with $p_1 = y$ and $q_1 = \frac{\partial}{\partial y}$.

**Proposition 9** The full prolongation of $p_2$ is isomorphic, as an abstract Lie algebra, to the semi-direct sum

$$p_2^{(\infty)} = \text{Der}(\mathbb{R}[y]) + i_3 \quad \text{and}$$

$$i_3 = \left( \mathbb{R}[y] \otimes \text{sp}(W)^{(\infty)} + \mathbb{R}_+[y] \right)$$

of the Lie algebra of formal vector fields on the line acting naturally on the canonical ideal $i_3$. The canonical ideal $i_1$ is the center of $i_3$ and it consists of the formal power series in one variable modulo constants:

$$i_1 = \mathbb{R}_+[y] \cong \frac{\mathbb{R}[y]}{\mathbb{R}1}.$$  

Finally

$$[y^iX, y^jY] = \begin{cases} [X, Y] & \text{if } i = j = 0, \\ y^{i+j}[X, Y] + y^i \Omega(X, Y) & \text{if } i + j > 0, \end{cases}$$

for all $X, Y \in \text{sp}(W)^{(\infty)}$, where $\Omega$ is the standard symplectic form at the origin.

**Corollary 4** The Lie algebra $i_3$ acts transversally primitively on each leaf $y = \text{const.}$

**Proof.** The (formal) one-parameter family of transverse actions is given by the natural structure of a Lie algebra of $i_3/i_1 \cong \mathbb{R}[y] \cdot \text{sp}(W)^{(\infty)}$ as $y$-dependent symplectic vector fields on $W$. $\square$

The proof of the following lemma is straightforward.

**Lemma 12** The stability subalgebra $p_2^{>0}$ (resp. the isotropy subalgebra $p_2$) of $p_2^{(\infty)}$ correspond, under the identification (13), to the subalgebras

$$p_2^{>0} = \mathbb{R}_+[y] \frac{\partial}{\partial y} + \left( \mathbb{R}_+[y] \otimes \text{sp}(W)^{(\infty)} + \text{sp}(W)^{>0} + \text{span}(y^2, y^3, y^4, \ldots) \right),$$

$$p_2 = y \frac{\partial}{\partial y} + \left( yW + \text{sp}(W) + \mathbb{R}y^2 \right),$$

where $\text{sp}(W)^{>0}$ is the stability subalgebra of $\text{sp}(W)^{(\infty)}$.  

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We now turn to finite-dimensional transitive subalgebras g of \( p_2^{(\infty)} \). As for the entire Lie algebra, g is not primitive on the leaves of the distribution \( D_3 \) and, as a consequence of the next Proposition 10, it is transversally primitive across these leaves. In this context, there is only one meaningful notion of primitivity: we are led to consider the subalgebra s g whose canonical ideal associated to \( D_3 \) acts transversally primitively (across the leaves of \( D_1 \)) on each leaf \( y = \text{const} \).

We assume, for simplicity, that our finite-dimensional and transitive subalgebra \( g \subset p_2^{(\infty)} \) is consistent with the decomposition (13), i.e., it is of the form

\[
g = a + \tilde{g} + \xi
\]

with

\[
a = \text{Der}(\mathbb{R}[[y]]) \cap g, \quad \tilde{g} = \left( \mathbb{R}[[y]] \otimes \mathfrak{sp}(W)^{(\infty)} \right) \cap g, \quad \xi = \mathbb{R}_+[y] \cap g.
\]  

We remark that \( \tilde{g} \) is not a subalgebra of \( g \) in general, as \([\tilde{g}, \tilde{g}] \subset \tilde{g} + \xi \) due to (14). Nonetheless, we shall often identify \( \tilde{g} \) with the quotient of \( \tilde{g} + \xi \) by its central ideal \( \xi \), the Lie brackets being (14) with the symplectic form at the origin set to zero, i.e., the Lie brackets of \( \mathbb{R}_+[y] \cdot \mathfrak{sp}(W)^{(\infty)} \).

Transitivity of \( g \) reads as the transitivity of \( a \) and \( \tilde{g} \) (in the sense that the natural projection of \( \tilde{g} \) to \( W \) is surjective) and the fact that \( \xi \) has a formal power series of the form \( y + h.o.t. \).

**Definition 5** We say that a subalgebra \( g \) of \( p_2^{(\infty)} \) is splitting if:

1. it is consistent with the decomposition (13) of \( p_2^{(\infty)} \), i.e., it is of the form (16);
2. if a vector field \( X = \sum_{i=0}^{+\infty} y^i X_i, X_i \in \mathfrak{sp}(W)^{(\infty)} \) for all \( i \geq 0 \), is in \( \tilde{g} \), then its component \( X_0 \) is in \( \tilde{g} \) too.

The classification of general (nonsplitting) subalgebras requires studying different deformation and extension problems; in this paper, we will consider only the splitting case. We depart with the following.

**Proposition 10** (see, e.g., [Dra]) The following list exhausts, up to isomorphism, the finite-dimensional transitive subalgebras \( a \) of the Lie algebra of vector fields on the real line:

1. \( a_1 = \text{span}(\frac{\partial}{\partial y}) \),
2. \( a_2 = \text{aff}(\mathbb{R}) = \text{span}(\frac{\partial}{\partial y}, y \frac{\partial}{\partial y}) \),
3. \( a_3 = \text{proj}(\mathbb{R}) = \text{span}(\frac{\partial}{\partial y}, y \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial y}) \).

We recall that an isomorphism of Lie algebras of vector fields on the line is by definition the map induced by an analytic coordinate change \( \phi : \mathbb{R}[[y]] \to \mathbb{R}[[y]] \) at the origin. In particular, any such isomorphism acts on \( p_2^{(\infty)} \) too, in a way compatible with the decomposition into sum of three Lie algebras. Hence, without any loss of generality, we may take the component \( a \) of the Lie algebra (16) as in Proposition 10 from now on, in particular \( \frac{\partial}{\partial y} \in a \).

**Lemma 13**

1. Any non-zero finite-dimensional subspace \( \xi \), invariant w.r.t. \( a_2 \) has the form

\[
\xi = p_+^k = \text{span}(y, y^2, \ldots, y^k).
\]

The same holds for \( a_1 \) if, in addition, \( \xi \) consists of polynomials;
2. There is no finite-dimensional subspace ξ, which is invariant w.r.t. a₃.

Proof.

1. Using $y\frac{\partial}{\partial y} \in a₂$ and Vandermonde determinants, one sees that any subspace ξ, which is $a₂$-invariant and does not consist just of polynomials is necessarily infinite-dimensional. The rest follows from $\frac{\partial}{\partial y} \in a$.

2. Clear. □

Lemma 13 implies that $a = a₃$ is not permissible as a subalgebra of a splitting finite-dimensional transitive $g = a + \tilde{g} + \xi$. In view of this lemma, we will also enforce one growth condition on $g$, namely, we will assume from now on that $g$ is polynomial in $y$.

Let $pₖ \subset \mathbb{R}[y]$ be the space of all polynomials of degree $\leq k$. Since $\tilde{g}$ is finite-dimensional and polynomial in $y$, there is a nonnegative integer $N$ such that $\tilde{g} \subset p^N \otimes sp(W)^{∞}$. We assume that $N$ is the smallest integer with this property and decompose any $X \in \tilde{g}$ into

$$X = \sum_{i=0}^{N} y^i X_i ,$$

where $X_i \in sp(W)^{∞}$ for all $i = 0, \ldots, N$. We also set $\mathfrak{g} = \tilde{g} \cap sp(W)^{∞}$ and

$$\tilde{g}^i = \{ Y \in sp(W)^{∞} \mid \text{there exists } X \in \tilde{g} \text{ with } X_i = Y \}$$

for all $i = 0, \ldots, N$. Clearly $\mathfrak{g} \subset \tilde{g}^0$ and, by the second condition of Definition 5, $\mathfrak{g} = \tilde{g}^0$.

Proposition 11

1. The space $\tilde{g}^i$ is contained in $\tilde{g}^{i-1}$ and it is an ideal of $\mathfrak{g}$, for all $i > 0$;

2. If $\mathfrak{g}$ has no non-zero nilpotent ideals then $\tilde{g} = \mathfrak{g}$.

Proof. The first claim follows by repeatedly applying $\frac{\partial}{\partial y} \in a$ to elements of $\tilde{g}$. It remains to show that $\tilde{g} \subset \mathfrak{g}$ if $\mathfrak{g}$ has no non-zero nilpotent ideals. This follows immediately, as the ideal $\tilde{g}^N$ of $\tilde{g}$ is non-zero and abelian if $N > 0$. □

The classification of the finite-dimensional Lie algebras of vector fields in the real plane can be found in [GKO, Table 1] (the notation therein is $p = \frac{\partial}{\partial x}$ and $q = \frac{\partial}{\partial y}$; caution: the origin is not a regular point for the Lie algebras (2) and (17)-(19), a translation of coordinates has to be performed to match our present set-up).

There are transitive Lie algebras of symplectic vector fields, both primitive and not primitive. The primitive ones are the unimodular affine Lie algebra $sl_2(\mathbb{R}) + \mathbb{R}^2$ (case (5) in [GKO, Table 1]), its subalgebra $\mathbb{R} + \mathbb{R}^2$ of Euclidean motions (case (1) for $\alpha = 0$) and the two filtered deformations $sl_2(\mathbb{R})$ and $sp_3(\mathbb{R})$ of the latter given by the Lie algebra of infinitesimal automorphisms (2) of the hyperbolic plane and (3) of the Euclidean 2-sphere.

The imprimitive ones are the Lie algebras (17)-(18), the Lie algebra (24) and its subalgebra (12) for $\alpha = -1$ and, finally, the Lie algebra (22) for an appropriate choice of the defining functions $\eta_i(x) - \text{e.g. } \eta_1(x) = x^{i-1}$ for all $i = 1, \ldots, r$.

In all cases, there is a natural decomposition $\mathfrak{g} = s + n$.
into the semidirect sum of a subalgebra $s$ and a (possibly trivial) abelian ideal $n$. The latter coincides with the maximal nilpotent ideal of $\mathfrak{g}$, unless $\mathfrak{g}$ is (24) or (22) with all polynomial defining functions as above.

We now state the main result of this section. Recall the definition of a transversally primitive Lie algebra across the flag of distributions (12) on $\mathbb{R}^4$, cf. Definition 4 and the discussion after Lemma 12.

**Theorem 10** Let $\mathfrak{g} = s + n$ be one of the four finite-dimensional primitive Lie algebras of symplectic vector fields on the real plane, as included in [GKO, Table 1]. Then:

1. If $\mathfrak{g} = s$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of infinitesimal automorphisms of the hyperbolic plane or the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ of infinitesimal automorphisms of the Euclidean 2-sphere, then
   $$\mathfrak{g} = \text{aff}(\mathbb{R}) + \mathfrak{g} + P^k$$
   (19)
   is a finite-dimensional transitive and transversally primitive Lie algebra of symplectic vector fields on the 4-dimensional space. Here
   $$\text{aff}(\mathbb{R}) = \text{span} \left( \frac{\partial}{\partial y}, y \frac{\partial}{\partial y} \right)$$
   and
   $$P^k = \text{span} \left( y, y^2, \ldots, y^k \right),$$
   (20)
   for some integer $k \geq 1$. The associated stability subalgebra $\mathfrak{k}$ and isotropy algebra $\mathfrak{h}$ are given by
   $$\mathfrak{h} = \left\{ \begin{array}{ll}
   \mathbb{R} y \frac{\partial}{\partial y} + \mathfrak{g} & \text{if } k > 1 \\
   \mathbb{R} y \frac{\partial}{\partial y} + \mathfrak{g} + \mathfrak{h} & \text{if } k = 1
   \end{array} \right. $$
   (21)
   where $\mathfrak{k}$ is the stability subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \cong \mathfrak{so}_2(\mathbb{R})$ the linear isotropy algebra of $\mathfrak{g}$.

2. If $\mathfrak{g} = s + n$ is the unimodular affine Lie algebra $\mathfrak{sl}_2(\mathbb{R}) + \mathbb{R}^2$ or its subalgebra $\mathbb{R} + \mathbb{R}^2$ of Euclidean motions, then the nilradical $n = \mathbb{R}^2$ of $\mathfrak{g}$ is abelian and
   $$\mathfrak{g} = \text{aff}(\mathbb{R}) + (s + P^N \otimes n) + P^k$$
   (22)
   is a finite-dimensional transitive and transversally primitive Lie algebra of symplectic vector fields on the 4-dimensional space, where the notation is as in (21) and
   $$P^N = \text{span} \left( 1, y, y^2, \ldots, y^N \right)$$
   for some nonnegative integer $N$ with $2N \leq k$. The stability subalgebra $\mathfrak{k}$ is given by
   $$\mathfrak{k} = \left\{ \begin{array}{ll}
   \mathbb{R} y \frac{\partial}{\partial y} + \left( s + P^N \otimes n + \text{span} (y^2, \ldots, y^k) \right) & \text{if } k > 1 \text{ and } N > 0 \\
   \mathbb{R} y \frac{\partial}{\partial y} + \left( s + \text{span} (y^2, \ldots, y^k) \right) & \text{if } k > 1 \text{ and } N = 0 \\
   \mathbb{R} y \frac{\partial}{\partial y} + s & \text{if } k = 1
   \end{array} \right.$$ 
   (23)
and the isotropy algebra $\mathfrak{h}$ by

\[
\mathfrak{h} = \begin{cases} 
\mathcal{R} \frac{\partial}{\partial y} + \left( s + yn + \mathcal{R} y^2 \right) & \text{if } k > 1 \text{ and } N > 0 \\
\mathcal{R} \frac{\partial}{\partial y} + \left( s + \mathcal{R} y^2 \right) & \text{if } k > 1 \text{ and } N = 0 \\
\mathcal{R} \frac{\partial}{\partial y} + s & \text{if } k = 1 
\end{cases}
\] (24)

The Lie algebras (19) and (22) just described are subalgebras of $\mathfrak{p}_2^{(\infty)}$, the explicit expression of their Lie brackets is as in Proposition 2.

Conversely:

3. Any finite-dimensional transitive and transversally primitive subalgebra $\mathfrak{g}$ of $\mathfrak{p}_2^{(\infty)}$ that is splitting and polynomial in $y$ is a subalgebra

$$\mathfrak{g} = a + \tilde{\mathfrak{g}} + \xi$$ (25)

of (19) or (22). More precisely, $a = \text{span} \left( \frac{\partial}{\partial y} \right)$ or $a = \text{aff}(R)$, $\tilde{\mathfrak{g}} \subset s + P^N \otimes n$ for one of the finite-dimensional primitive Lie algebras $\mathfrak{g} = s + n$ of symplectic vector fields on the real plane and $\xi = P_k^k$ for some $k \geq 1$.

4. For all Lie algebras (25) of symplectic vector fields, there exists a choice of a connected Lie group $G$ with Lie algebra $\text{Lie}(G) = \mathfrak{g}$ so that the analytic subgroup $K$ of $G$ with Lie algebra $\text{Lie}(K) = \mathfrak{k} = \mathfrak{g} \cap \mathfrak{p}_2^{\geq 0}$ is closed and

$$M = (G/K, \omega)$$

is a homogeneous symplectic 4-manifold. If $k > 2$, then $M = G/K$ does not admit any invariant linear connection, in particular it is not reductive and it is not a homogeneous Fedosov manifold.

Proof. The first two claims follow from direct computations using the Lie brackets described in Proposition 9, the fact that the given Lie algebras (19) and (22) are transitive and transversally primitive is also immediate. We now show the converse implication.

Let $\mathfrak{g} = a + \tilde{\mathfrak{g}} + \xi$, be a splitting subalgebra of $\mathfrak{p}_2^{(\infty)}$ with the desired properties. By transitivity of $\mathfrak{g}$ and the fact that $\mathfrak{g}$ is polynomial in $y$ we have $a \subset \text{aff}(R)$ and $\xi = P_k^k$ for some $k \geq 1$, see Proposition 10 and Lemma 13. We now focus on $\tilde{\mathfrak{g}}$.

First $\mathfrak{g} = \tilde{\mathfrak{g}} \cap \mathfrak{sp}(W)^{(\infty)}$ is a transitive Lie algebra of symplectic vector fields on the real plane, since the natural projection of $\tilde{\mathfrak{g}}$ to $W$ is surjective and the component $X_0$ of any element $X = \sum_{i=0}^{\infty} y^i X_i$ of $\tilde{\mathfrak{g}}$ is again in $\tilde{\mathfrak{g}}$. By assumption, $\mathfrak{g}$ is transversally primitive on each leaf $y = \text{const}$. and so, in particular, on the leaf $y = 0$. This says that $\mathfrak{g}$ is primitive too, hence one of the four finite-dimensional primitive Lie algebras of symplectic vector fields on the real plane.

If $\mathfrak{g}$ is simple, then $\tilde{\mathfrak{g}} = \mathfrak{g}$ by Proposition 11 and $\mathfrak{g}$ is a subalgebra of (19). Otherwise $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) + \mathbb{R}^2$ or $\mathfrak{g} = \mathbb{R} + \mathbb{R}^2$ and we now deal with the two cases separately.

Recall first that

$$\tilde{\mathfrak{g}} \subset \mathfrak{p}_2^{\geq 0} \otimes \mathfrak{sp}(W)^{(\infty)},$$

for some nonnegative integer $N$. The case $N = 0$ is trivial, as it amounts to $\tilde{\mathfrak{g}} = \mathfrak{g}$. If $N > 0$, then by Proposition 11 and the fact that $\tilde{\mathfrak{g}}^N$ is a non-zero abelian ideal of $\mathfrak{g}$, we have

$$\tilde{\mathfrak{g}}^i = \begin{cases} 
\mathfrak{g} & \text{for all } 0 \leq i \leq M \\
n & \text{for all } M + 1 \leq i \leq N \\
0 & \text{for all } N < i 
\end{cases}$$
for some nonnegative integer $M < N$. We want to show $M = 0$, so that $\mathfrak{g} \subseteq \mathfrak{s} + \mathbb{P}^N \otimes \mathfrak{n}$.

(i) Let $\mathfrak{g} = \mathfrak{s} + \mathfrak{n} = \mathfrak{s}l_2(\mathbb{R}) + \mathbb{R}^2$ and assume $M > 0$. We take elements

$$X = \sum_{i=0}^{N} y^i X_i \quad \text{and} \quad Y = \sum_{i=0}^{N} y^i Y_i$$

of $\tilde{\mathfrak{g}}$ such that $X_M, Y_M \in \mathfrak{s}$. Then the component of the bracket

$$[X, Y]_{2M} \equiv [X_M, Y_M] \mod \mathfrak{n},$$

since $\mathfrak{n}$ is an ideal. On the other hand $2M > M$, whence $[X, Y]_{2M} \in \mathfrak{n}$, which is not possible for all $X_M, Y_M \in \mathfrak{s}$.

(ii) Let $\mathfrak{g} = \mathfrak{s} + \mathfrak{n} = \mathbb{R} + \mathbb{R}^2$, and let $J$ be the complex structure generating $\mathfrak{s}$. Assume $M > 0$ and take $X$ and $Y$ in $\tilde{\mathfrak{g}}$ such that $X_M = J$ and $Y_N$ is a nonzero element of $\mathfrak{n}$. By possibly replacing $Y$ with $[J, Y]$, we may assume that all components of $Y$ are in $\mathfrak{n}$. Hence

$$0 = [X, Y]_{N+M} = [X_M, Y_N] = [J, Y_N] \neq 0,$$

since $N + M > N$ and $\mathfrak{n}$ is an abelian ideal, an absurd.

This proves claim 3. of the theorem.

We now turn to the existence of homogeneous symplectic 4-manifolds. This amounts to show that the putative stability subgroup is closed. First we note that it is sufficient to establish claim 4. for the “maximal” Lie algebras (19) and (22) (for the nonmaximal Lie algebras, one simply considers the orbits under smaller group actions).

Let $G$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $K$ the analytic subgroup of $G$ with $\text{Lie}(K) = \mathfrak{k} = \mathfrak{g} \cap \mathfrak{p}_{>0}^2$. Let $\mathfrak{k}^{\text{Mal}}$ be the topological closure of $K$ in $G$ and $\text{Lie}(\mathfrak{k}^{\text{Mal}}) = \mathfrak{k}^{\text{Mal}}$ the corresponding subalgebra of $\mathfrak{g}$. It is called the Malcev closure of $\mathfrak{k}$ and it satisfies the following fundamental properties (see [GOVI]):

- $\mathfrak{k}^{\text{Mal}} \supset \mathfrak{k}$,
- $[\mathfrak{k}^{\text{Mal}}, \mathfrak{k}^{\text{Mal}}] = [\mathfrak{k}, \mathfrak{k}]$.

We will now show that $\mathfrak{k}^{\text{Mal}} = \mathfrak{k}$ in our case. If $\mathfrak{g} = \mathfrak{s}$ is simple, then

$$[\mathfrak{k}, \mathfrak{k}] = \begin{cases} \text{span} \{ y^2, \ldots, y^k \} & \text{if } k > 1 \\ 0 & \text{if } k = 1 \end{cases}$$

as the stability subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is 1-dimensional compact. The subalgebra $\mathfrak{k}$ has codimension four in $\mathfrak{g}$ and

$$\mathfrak{g}/\mathfrak{k} \cong \text{span} \left( \frac{\partial}{\partial y}, y \right) + \mathfrak{g}/\mathfrak{k}$$

as vector spaces. Let $X \in \mathfrak{k}^{\text{Mal}}$ be of the form

$$X = \lambda \frac{\partial}{\partial y} + \mu y + Z,$$
where $Z \in \mathfrak{g}/\mathfrak{t}$. Then the bracket

$$[X, y \frac{\partial}{\partial y}] = \lambda \frac{\partial}{\partial y} - \mu y$$

belongs to $[\mathfrak{k}, \mathfrak{k}]$, since $y \frac{\partial}{\partial y} \in \mathfrak{k}$, but this is possible only if $\lambda = \mu = 0$. Now, the hyperbolic plane and the Euclidean 2-sphere are irreducible Riemannian symmetric spaces and the stability subalgebra $\mathfrak{t} \cong \mathfrak{so}_2(\mathbb{R})$ is a maximal subalgebra of $\mathfrak{g}$. Hence either $\mathfrak{t}^{\text{Mal}} = \mathfrak{t}$ or

$$\mathfrak{t}^{\text{Mal}} = \begin{cases} \mathbb{R} y \frac{\partial}{\partial y} + \mathfrak{g} + \text{span}(y^2, \ldots, y^k) & \text{if } k > 1 \\ \mathbb{R} y \frac{\partial}{\partial y} + \mathfrak{g} & \text{if } k = 1 \end{cases}$$

but in the second case $[\mathfrak{k}, \mathfrak{k}] \supset \mathfrak{g}$, which is not possible. Hence $\mathfrak{t}^{\text{Mal}} = \mathfrak{t}$ and $K = K^{\text{Mal}}$ since both Lie groups are connected. The case $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$ not simple is similar and we omit the proof.

An immediate consequence of Theorem 10 is the following.

**Corollary 5** There exist finite-dimensional transitive subalgebras $\mathfrak{g}$ of $\mathfrak{p}_2(\infty)$ of any positive order and with associated isotropy algebra of infinite type.

**Proof.** The Lie algebras (19) and (22) with any $k > 1$ do the job. It is not difficult to see that the isotropy subalgebra includes a rank one element and it is therefore of infinite type. $\square$

### 4.3 Finite-dimensional subalgebras of $\mathfrak{p}_1(\infty)$

The isotropy algebra $\mathfrak{p}_1$ of $\mathfrak{p}_1(\infty)$ is the stabilizer of the Lagrangian plane $P = \text{span}(\mathfrak{p}_1, \mathfrak{p}_2)$. Hence, there exists an involutive Lagrangian distribution

$$0 \subset D \subset T\mathbb{R}^4$$

of $\text{rk} D = 2$ which is left invariant by $\mathfrak{p}_1(\infty)$. The system of imprimitivity of this distribution is defined on the entire $\mathbb{R}^4$ and we denote by $i$ the associated canonical ideal (see §4.1).

The following is an alternative characterization of $\mathfrak{p}_1(\infty)$.

**Proposition 12** The full prolongation of $\mathfrak{p}_1$ is isomorphic, as an abstract Lie algebra, to the semi-direct sum

$$\mathfrak{p}_1(\infty) = \text{Der}(\mathbb{R}[[x, y]]) + i \quad \text{and} \quad i = \mathbb{R}_+[[x, y]] \cong \frac{\mathbb{R}[[x, y]]}{\mathbb{R}1}$$

of the Lie algebra of formal vector fields on the plane acting naturally on the canonical ideal. The latter is abelian and identifiable with the formal power series in two variables, modulo constants.

The stability subalgebra $\mathfrak{p}_1^{\geq 0}$ (resp. the isotropy subalgebra $\mathfrak{p}_1$) of $\mathfrak{p}_1(\infty)$ correspond, under this identification, to the subalgebras

$$\mathfrak{p}_1^{\geq 0} = \text{Der}^{\geq 0}(\mathbb{R}[[x, y]]) + \text{span}(x^2, xy, y^2, \ldots),$$

$$\mathfrak{p}_1 = \mathfrak{gl}_2(\mathbb{R}) + \text{span}(x^2, xy, y^2),$$

where $\text{Der}^{\geq 0}(\mathbb{R}[[x, y]])$ is the stability subalgebra of $\text{Der}(\mathbb{R}[[x, y]])$.  

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In particular the Lie algebra \( p_1^{(\infty)} \) acts in a transversally primitive fashion across the leaves of \( \mathcal{D} \) (cf. Definition \[\text{[4]}\] in our case, this means that the quotient \( g/\mathfrak{i} \) acts primitively on the plane).

Let \( g \) be a finite-dimensional subalgebra of \( p_1^{(\infty)} \) consistent with the above decomposition, that is
\[
g = \tilde{g} + \xi,
\]
with
\[
\tilde{g} = \text{Der}(\mathbb{R}[x, y]) \cap g, \quad \xi = \mathbb{R}_+[x, y] \cap g. \tag{31}
\]
We call any such \( g \) splitting. The classification of nonsplitting subalgebras of \( p_1^{(\infty)} \) requires the study of a deformation problem; here, we will consider only splitting subalgebras.

**Proposition 13** Let \( \tilde{g} \) be any transitive Lie algebra of vector fields on the real plane and \( \xi \), a subspace of \( \mathbb{R}_+[x, y] \) which is \( \tilde{g} \)-invariant and such that \( x + \text{h.o.t.} \) and \( y + \text{h.o.t.} \) are in \( \xi \). Then
\[
g = \tilde{g} + \xi,
\]
is a transitive splitting subalgebra of \( p_1^{(\infty)} \). Conversely, any transitive splitting subalgebra of \( p_1^{(\infty)} \) is obtained in this way. The Lie algebra \( g \) acts in a transversally primitive way across the leaves of \( \mathcal{D} \) if and only if \( \tilde{g} \) acts primitively on the plane.

The list of finite-dimensional primitive Lie algebras of vector fields on the real plane is in \[\text{[GKO, Table 1, I]}\]. (We emphasize once again that in \[\text{[GKO]}\] the origin is not a regular point for the Lie algebra of infinitesimal automorphisms of the hyperbolic plane). They all consist of polynomial vector fields. For simplicity, we also assume that \( \xi \) consists of polynomials.

**Lemma 14**

1. If \( \tilde{g} \) is the Lie algebra of (unimodular) affine infinitesimal transformations of the plane, then any nontrivial finite-dimensional polynomial \( \tilde{g} \)-module \( \xi \), has the form
\[
\xi = \mathbb{P}^k = \text{span} (x, y, x^2, xy, y^2, \ldots, x^k, \ldots, y^k);
\]

2. Let \( \tilde{g} \) be the Lie algebra \( \mathfrak{sl}_2(\mathbb{R}) \) of infinitesimal automorphisms of the hyperbolic plane, \( \mathfrak{so}_3(\mathbb{R}) \) of the Euclidean 2-sphere, \( \mathfrak{sl}_3(\mathbb{R}) \) of the projective plane and \( \mathfrak{so}(1, 3) \) of the conformal 2-sphere. Then any finite-dimensional polynomial \( \tilde{g} \)-module \( \xi \), is trivial, i.e., \( \xi = 0 \).

**Proof.** The first claim is immediate and we omit the proof. Note also that it is sufficient to establish the second claim for the Lie algebras \( \tilde{g} = \mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}_3(\mathbb{R}) \) and \( \mathfrak{sl}_3(\mathbb{R}) \).

Let \( \xi \neq 0 \) and take a nonconstant polynomial \( f \) of maximum degree in \( \xi \), say \( \deg(f) = k \), which we decompose into the sum \( f = \sum_{i=1}^{k} f_i \) of (nonconstant) homogeneous polynomials. If \( \tilde{g} = \mathfrak{sl}_2(\mathbb{R}) \) or \( \mathfrak{so}_3(\mathbb{R}) \), we may act on such polynomial with a vector field of the form
\[
X = (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} + \text{t.o.t.}
\]
The resulting component of highest degree is given by
\[
(x^2 - y^2) \frac{\partial}{\partial x} (f_k) + 2xy \frac{\partial}{\partial y} (f_k)
\]
and its vanishing implies \( f_k = 0 \), an absurd. The case \( \tilde{g} = \mathfrak{sl}_3(\mathbb{R}) \) is similar. \( \Box \)

It remains to give \( \xi \), for the Lie algebra of infinitesimal conformal automorphisms of the plane and a 1-parameter family of deformations of the subalgebra of Euclidean motions. We set
\[
E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad J = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},
\]
so that
(i) \( \widetilde{g} = \text{conf}(\mathbb{R}^2) = \text{span}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, E, J) \),

(ii) \( \widetilde{g} = \text{cuc}_\alpha(\mathbb{R}^2) = \text{span}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, J_\alpha = \alpha E - J) \),

for any \( \alpha \geq 0 \). It is clear that the action of \( C = \text{span}(E, J) \) on the complexified formal power series is diagonal, with 1-dimensional eigenspaces

\[
p^{k,\ell} = \text{span}(\frac{\partial}{\partial z} \cdot \frac{k+\ell}{2}, \frac{k-\ell}{2}) \text{, } E|_{p^{k,\ell}} = k\text{Id} \text{ and } J|_{p^{k,\ell}} = i\ell\text{Id} .
\]

Here we are assuming \( \ell = k, k - 2, \ldots, 2 - k, -k \), otherwise we set \( p^{k,\ell} = 0 \). In this way, the space of complexified formal power series inherits a \( \mathbb{Z} \)-bigrading of the form \( \mathbb{C}[x, y] = \bigoplus_{k,\ell} p^{k,\ell} \).

Consider the “upward triangle” with bigraded nodes the 1-dimensional eigenspaces and top node given by \( p^{k_o,\ell_o} \), pictorially represented by

\[
\begin{align*}
\partial_{z} & \quad p^{k_o,\ell_o} \\
\partial_{\bar{z}} & \quad p^{k_o-1,\ell_o+1} \\
\partial_{z} & \quad p^{k_o-1,\ell_o-1} \\
p^{k_o-2,\ell_o+2} & \quad p^{k_o-2,\ell_o} \\
p^{k_o-2,\ell_o-2} & \quad 0
\end{align*}
\]

The partial derivatives

\[
\frac{\partial}{\partial z} : p^{k,\ell} \rightarrow p^{k-1,\ell-1} , \quad \frac{\partial}{\partial \bar{z}} : p^{k,\ell} \rightarrow p^{k-1,\ell+1} ,
\]

sends nodes into nodes, moving downwards right and left, respectively. We also note that \( E \) (resp. \( J \)) acts with the same eigenvalue on all the nodes on a same vertical (resp. horizontal) level. We denote the direct sum of all the nodes \( p^{k,\ell} \) with \( k > 0 \) included in the triangle with top node \( p^{k_o,\ell_o} \) by \( W^{k_o,\ell_o} \), it is a (complex) module for \( \widetilde{g} = \text{conf}(\mathbb{R}^2) \).

The modules \( W^{k,\ell} \) just constructed are all inequivalent. This implies the following.

**Proposition 14**

1. Any nontrivial finite-dimensional complex polynomial \( \text{conf}(\mathbb{R}^2) \)-module \( \Xi \) is the sum (not necessarily direct) of a finite number of modules \( W^{k,\ell} \). It is invariant by conjugation if and only if it includes both modules \( W^{k,\pm \ell} \) or none of them;

2. Any nontrivial finite-dimensional (real) polynomial \( \text{conf}(\mathbb{R}^2) \)-module \( \xi \) is the real form of a unique complex module \( \Xi \) invariant by conjugation.
Note that $J_\alpha$ acts on $P^{k, \ell}$ with eigenvalue $\alpha k - i\ell$. The modules $P^{k, \ell}$ are still all inequivalent under this action, unless $\alpha = 0$ so that $J_\alpha = -J$ and $P^{k, \ell} \cong P^{k', \ell'}$. In any case, Proposition 14 carries over with minor modifications and it is not difficult to see that any $\text{euc}_\alpha(R^2)$-module is always included in a $\text{conf}(R^2)$-module as we just described.

We have proved most of the following.

**Theorem 11** Let $\tilde{g}$ be one of the following finite-dimensional primitive Lie algebras of vector fields on the real plane:

1. the Lie algebra $\tilde{g} = \mathfrak{sl}_2(R) + R^2$ of unimodular affine infinitesimal transformations;
2. the Lie algebra $\tilde{g} = \mathfrak{gl}_2(R) + R^2$ of affine infinitesimal transformations;
3. the Lie algebra $\tilde{g} = \text{conf}(R^2)$ of infinitesimal conformal automorphisms of the plane;
4. a 1-parameter family $\tilde{g} = \text{euc}_\alpha(R^2)$ of deformations of the subalgebra of Euclidean motions.

Then the semi-direct sum

$$\mathfrak{g} = \tilde{g} + \xi$$

is a finite-dimensional transitive and transversally primitive subalgebra of $\mathfrak{p}_1^{(\infty)}$, where in the first two cases $\xi = P^k_+$, $k \geq 1$, is the space of nonconstant polynomials in two variables of degree at most $k$ and in the last two cases $\xi$ is a sum of “triangle” modules $W^{k, \ell}$ as in Proposition 14.

The stability subalgebra $\mathfrak{k}$ and the isotropy algebra $\mathfrak{h}$ are given by

$$\mathfrak{k} = \tilde{\mathfrak{k}} + (\xi \cap \text{span}(x^2, xy, y^2, \ldots)),$$

$$\mathfrak{h} = \tilde{\mathfrak{h}} + (\xi \cap \text{span}(x^2, xy, y^2)),$$

where $\tilde{\mathfrak{k}}$ is the stability subalgebra of $\tilde{g}$.

Let $\tilde{G} = \tilde{K} \ltimes R^2$ be the connected Lie group of affine transformations of $R^2$, where $\text{Lie}(\tilde{G}) = \tilde{g}$ and $\text{Lie}(\tilde{K}) = \tilde{\mathfrak{k}}$. Then $K = \tilde{K} \ltimes (\xi \cap \text{span}(x^2, xy, y^2, \ldots))$ is a closed subgroup of $G = \tilde{G} \ltimes \xi$, and $M = (G/K, \omega)$ a homogeneous symplectic 4-manifold. If $k > 2$, then $M = G/K$ is not reductive and it is not a homogeneous Fedosov manifold.

Conversely, any finite-dimensional transitive and transversally primitive subalgebra $\mathfrak{g}$ of $\mathfrak{p}_1^{(\infty)}$ that is splitting and polynomial in $x$ and $y$ is a subalgebra of some of the Lie algebras $\tilde{g}$.

Actually, it coincides with one of them, unless $\tilde{g} = \text{euc}_\alpha(R^2)$ and $\alpha = 0$, in which case $\xi$ may be properly contained in a sum of “triangle” modules.

**Proof.** It remains to note that the isotropy representation is not exact if $k > 2$ and hence $M = G/K$ does not admit any invariant linear connection. $\square$

**Corollary 6** There exist finite-dimensional transitive subalgebras $\mathfrak{g}$ of $\mathfrak{p}_1^{(\infty)}$ of any positive order and with associated isotropy algebra of infinite type.

### 5 Homogeneous torsion free symplectic connections (Fedosov structures)

#### 5.1 Existence of homogeneous Fedosov structures

It is a well known fact that on a $2n$-dimensional symplectic manifold $(M, \omega)$ there always exists a linear connection $\nabla$ which is torsion free and preserves $\omega$. In other words, the triple $(M, \omega, \nabla)$
is a Fedosov manifold, in the terminology of [GRS]. The usual way to see this is to take trivial local connections in Darboux coordinates and globally glue them using a partition of unity. The following alternative argument is borrowed from [BCGRS].

Take $\nabla^o$ any torsion free linear connection (e.g., the Levi-Civita connection associated to a Riemannian metric on $M$). The covariant derivative $\nabla^o \omega$ is a section of $T^*M \otimes \Lambda^2 T^*M$ and, since $\omega$ is closed, we have

$$\mathcal{S}_{XYZ} \left\{ \nabla^o_X \omega(Y,Z) \right\} = 0 ,$$

where $\mathcal{S}_{XYZ}$ is the cyclic sum over the vector field $X,Y,Z$ on $M$. We define the section $N$ of $T^*M \otimes T^*M \otimes TM$ by $\omega(N(X,Y),Z) = \nabla^o_X \omega(Y,Z)$ and a new connection by

$$\nabla_X Y = \nabla^o_X Y + \frac{1}{3} N(X,Y) + \frac{1}{3} N(Y,X) .$$

**Proposition 15** [BCGRS] The connection $\nabla$ is torsion free and preserves $\omega$.

**Proof.** It is clear that $\nabla$ is torsion free. We then compute

$$\nabla_X \omega(Y,Z) = X(\omega(Y,Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z)$$

$$= \nabla^o_X \omega(Y,Z) - \frac{1}{3} \omega(N(X,Y), Z) - \frac{1}{3} \omega(N(Y,X), Z)$$

$$- \frac{1}{3} \omega(Y, N(X,Z)) - \frac{1}{3} \omega(Y, N(Z,X))$$

$$= \nabla^o_X \omega(Y,Z) - \frac{1}{3} \omega(N(X,Y), Z) - \frac{1}{3} \omega(N(Y,X), Z)$$

$$- \frac{1}{3} \omega(N(X,Y), Z) - \frac{1}{3} \omega(N(Z,Y), X)$$

$$= \left( 1 - \frac{1}{3} - \frac{1}{3} \right) \omega(N(X,Y), Z) + \frac{1}{3} \omega(N(X,Z), Y) = 0 .$$

Let $(M, \omega)$ be an almost symplectic manifold with an action of a Lie group $G$ preserving $\omega$. It is a classical result of I. Vaisman that if $(M, \omega)$ has a $G$-invariant linear connection then it also admits a $G$-invariant linear connection preserving $\omega$ (see e.g. [GRS] Corollary 1.3)).

If $d\omega = 0$, Vaisman’s result can be strengthened as follows.

**Proposition 16** Let $(M, \omega)$ be a symplectic manifold with an action of a Lie group $G$ which preserves $\omega$. Assume there exists a $G$-invariant linear connection on $M$. Then $M$ has a torsion free $G$-invariant connection preserving $\omega$.

**Proof.** Let $\nabla$ be a $G$-invariant linear connection on $M$. Symmetrisation (in the sense of Gelfand et al.) of this connection yields another connection

$$\nabla^o_X Y = \frac{1}{2} (\nabla_X Y + \nabla_Y X + [X,Y])$$

which is $G$-invariant and torsion free. The corresponding $\omega$-preserving torsion free connection from Proposition 15 is $G$-invariant.

Let $(M = G/K, \omega)$ be a homogeneous symplectic manifold, on which a Lie group $G$ (not necessarily compact) acts effectively. Assume for simplicity that $G$ and $K$ are connected. We recall that the homogeneous manifold $M = G/K$ is called reductive if the Lie algebra $\mathfrak{g}$ of $G$ may be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ of the Lie algebra $\mathfrak{k}$ of $K$ and a $\mathfrak{k}$-invariant subspace $\mathfrak{m}$.

**Proposition 17** Any reductive homogeneous symplectic manifold $(M = G/K, \omega)$ admits a torsion free $G$-invariant connection which preserves $\omega$, i.e., it is a homogeneous Fedosov manifold.

**Proof.** Any such manifold admits a $G$-invariant linear connection (for instance, the canonical connection [KoNo]) and Proposition 15 applies.
5.2 Uniqueness of homogeneous Fedosov structures

Let \((M = G/K, \omega)\) be a 2n-dimensional homogeneous symplectic manifold. For any \(g \in G\), we denote by \(L_g\) the corresponding left action on \(M\). Let

\[ \pi: O_\omega(M) \to M \] (34)

be the symplectic frame bundle of \(M\), i.e., the \(\text{Sp}_n(\mathbb{R})\)-bundle of linear frames \(u = (e_i)\) of the tangent spaces of \(M\) which are adapted to the symplectic form.

Given a symplectic frame \(u_o = (e_i) \in O_\omega(M)\) at \(o = eK\), we let

\[ P = G \cdot u_o = \{ g \cdot u_o := (L_g e_i) | g \in G \} \subset O_\omega(M) \]

be the \(G\)-orbit of \(u_o\). One gets in this way a natural reduction \(\pi: P \to M\) of (34), usually called the \textit{homogeneous} \(H\)-\textit{structure associated with} \(M = G/K\).

If \(K_1\) is the kernel of the linear isotropy representation

\[ j: K \to \text{Sp}(V), \quad V = T_o M, \quad k \mapsto L_{k|o} \]

then the map \(g \mapsto g \cdot u_o\) determines a natural bundle isomorphism between

\[ \pi: G/K_1 \to M = G/K \] (35)

and the \(H\)-structure \(\pi: P \to M\), in such a way that the structure group \(H = j(K) \subset \text{Sp}(V)\) has finite type. We assume a fixed choice of \(u_o\) and tacitly use it to identify \(\pi: P \to M\) with (35).

We shall now restrict to the 4-dimensional case. The following result is a straightforward consequence of Corollary [11].

\textbf{Lemma 15} Let \((M = G/K, \omega)\) be a 4-dimensional homogeneous symplectic manifold and assume the isotropy \(H = j(K) \subset \text{Sp}(V)\) has finite type. Then \(K_1\) is discrete.

Our result deals with the uniqueness of homogeneous Fedosov structures compatible with extra geometric data.

\textbf{Theorem 12} Let \((M = G/K, \omega)\) be a homogeneous symplectic 4-manifold with finite type isotropy \(H \subset \text{Sp}(V)\). Assume there exists a torsion free \(G\)-invariant connection \(\nabla\) which is, in addition, compatible with the associated homogeneous \(H\)-structure \(\pi: P \to M\) (in particular, \(\nabla\) preserves the symplectic form \(\omega\)). Then \(K_1 = \{1\}\) and

1. \(M = G/K\) is reductive;
2. the connection \(\nabla\) is unique.

\textit{Proof.} An affine transformation of \(M\) is the identity transformation if it leaves one linear frame fixed. This says \(K_1 = \{1\}\) and that \(\pi: P \to M\) is the \(H\)-principal bundle \(\pi: G \to G/H\). It is a well known fact that a \(G\)-invariant connection on \(\pi: G \to G/H\) exists if and only if the homogeneous manifold \(M = G/H\) is reductive (see, e.g., [CS, Corollary 1.4.6]). This proves (1).

An invariant connection on \(\pi: G \to G/H\) is completely determined by the associated Nomizu map, an \(H\)-equivariant map \(L: m \to \mathfrak{h}\), and it is torsion free if \(L(x)y - L(y)x - \pi_m [x,y] = 0\), where \(\pi_m\) is the projection of \(g\) on \(m\) relative to the decomposition \(g = \mathfrak{h} + m\) (see e.g., [KoNo]). The difference of two Nomizu maps is an element of \(\mathfrak{h}^{(1)}\), which is zero by Corollary [11]. \qed
5.3 Applications to symplectic Lie groups

We conclude with an application of the results of §5.1 and §5.2 to *symplectic Lie groups*, that is, the homogeneous symplectic manifolds \((M = G/K, \omega)\) with trivial stabilizer \(K = \{1\}\). It is clear that the existence of \(\omega\) is equivalent to \(g\) being a *symplectic Lie algebra*, i.e., a Lie algebra endowed with a nondegenerate scalar 2-cocyle.

Let \((G, \omega)\) be a symplectic Lie group, of any (even) dimension. It is known that \(G\) admits an invariant flat and torsion free connection \(\nabla^\omega\) [MR]. At the Lie algebra level, it can be described by

\[
\omega(\nabla^\omega_x y, z) = -\omega(y, [x, z]),
\]

where \(x, y, z \in g\) (we identify \(g\) with the Lie algebra of left-invariant vector fields on \(G\)). It is customary to introduce the product on \(g\) given by covariant differentiation

\[
xy := \nabla^\omega_x y
\]

and recast the flatness and zero-torsion conditions as

\[
(xy)z - x(yz) = (yx)z - y(xz),
\]

\[
[x, y] = xy - yx.
\]

A vector space \(A\) endowed with a bilinear product \(xy\) satisfying the first equation in (38), or equivalently for which the associator is symmetric in the first two variables, is called a *left-symmetric algebra*. Any left-symmetric algebra \(A\) is Lie-admissible, that is, the commutator of the left-symmetric product is a Lie bracket. If this Lie bracket coincides with the Lie bracket of a Lie algebra structure \(g\) given a priori on \(A\), the left-symmetric product is called *compatible* (with \(g\)). The construction of [MR] may be then summarized as follows.

**Proposition 18** Any symplectic Lie algebra \(g\) admits a compatible structure \((37)\) of a left-symmetric algebra.

For more details on left-symmetric algebras, we refer the reader to the survey article [Bu] and the references therein.

We collect a number of trace identities useful for our purposes. We denote left (resp. right) multiplication by \(x \in g\) in the left-symmetric algebra by \(L_x : y \mapsto xy\) (resp. \(R_x : y \mapsto yx\)). We let \((e_i)\) be a fixed symplectic basis of \(g\) and \((e^i)\) the dual basis of \(g\) (i.e., \(\omega(e_i,e^l) = \delta^l_i\)).

**Proposition 19** Let \(g\) be a symplectic Lie algebra, with compatible left-symmetric structure. Then for all \(x, y, z \in g\) we have:

1. \(\omega(xy, z) + \omega(zy, x) = 0\),
2. \(\mathfrak{S}_{xyz}\{\omega(xy, z)\} = 0\),
3. \(\text{tr } (R_x \circ R_y) = \text{tr } (R_{xy}) = 2\text{tr } (L_{xy})\),
4. \(\text{tr } (R_x \circ R_y) = 2\text{tr } (R_y \circ L_x) = 2\text{tr } (R_x \circ L_y)\).

where \(\mathfrak{S}_{xyz}\) is the cyclic sum over \(x, y, z\).

**Proof.** The first property is immediate and the second is equivalent to the closure of \(\omega\), since \(\nabla^\omega\) is torsion free. We now use the Einstein summation convention and compute

\[
\text{tr } (R_x) = \omega(e_i, e^i) = -\omega(e^i e_i, x) - \omega(xe^i, e_i)
\]

\[
= \omega(e_i, e_i, x) + \omega(xe_i, e^i) = -\omega(x, e_i e^i) + \text{tr } (L_x)
\]

\[
= -\frac{1}{2}\omega(x, [e_i, e^i]) + \text{tr } (L_x) = \frac{1}{2}\omega(e_i x, e^i) + \text{tr } (L_x)
\]

\[
= \frac{1}{2}\text{tr } (R_x) + \text{tr } (L_x),
\]

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whence $\text{tr} (R_x) = 2\text{tr} (L_x)$, for all $x \in \mathfrak{g}$; similarly
\[ \text{tr} (R_x \circ R_y) = \omega((e_i x)y, e^i) \]
\[ = \omega(e_i(xy), e^i) + \omega((xe_i)y, e^i) - \omega(x(e_i y), e^i) \]
\[ = \text{tr} (R_{xy}) + \text{tr} (R_y \circ L_x) - \text{tr} (L_x \circ R_y) \]
\[ = \text{tr} (R_{xy}). \tag{40} \]
Finally:
\[ \text{tr} (R_x \circ R_y) = \omega((e_i x)y, e^i) \]
\[ = -\omega(e^i(e_i x), y) - \omega(y e^i, e_i x) \]
\[ = \omega(y(e_i x), e^i) + \omega(e_i x, ye^i) \]
\[ = \text{tr} (L_y \circ R_x) - \omega((ye^i)x, e_i) \]
\[ = \text{tr} (L_y \circ R_x) + \text{tr} (R_x \circ L_y), \tag{41} \]
which readily implies our last claim. \hfill \square

The bilinear forms $\text{tr} (L_{xy})$ and $\text{tr} (R_{xy})$ on a left-symmetric algebra have been extensively studied in the context of convex homogeneous cones \cite{Min}, respectively, in connection with complete left-symmetric algebras \cite{He, Sc}. To state the main result of this section, we need one last preliminary fact. Left multiplication in the left-symmetric algebra yields a representation $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ and we call the corresponding $\mathfrak{g}$-equivariant symmetric bilinear form
\[ \kappa(x, y) := \text{tr} (L_x \circ L_y) \]
the left trace form. We are not aware of any simple identity which relates the left trace form with the Killing form of the associated Lie algebra or with (any of) the bilinear forms considered in Proposition \cite{19}. Nonetheless, the following holds.

**Proposition 20** Let $\mathfrak{g}$ be a symplectic Lie algebra, with the compatible left-symmetric structure. If the left trace form $\kappa = 0$ then $\mathfrak{g}$ is solvable and, conversely, if $\mathfrak{g}$ is nilpotent then $\kappa = 0$.

**Proof.** Iterating \cite{36} we have
\[ \omega(y, \text{ad}^k(x)(z)) = (-1)^k \omega(L^k_x(y), z), \]
for all positive integers $k$, where $x, y, z \in \mathfrak{g}$. In particular an element $x \in \mathfrak{g}$ is ad-nilpotent if and only if the operator $L_x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Note also that the kernel of the left regular representation $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ coincides with the center $\mathfrak{z}$ of $\mathfrak{g}$.

If $\kappa = 0$, then we may apply Cartan’s criterion to the matrix Lie algebra $L(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{z}$, and infer that the derived algebra of $\mathfrak{g}/\mathfrak{z}$ is nilpotent, whence $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is nilpotent then the Lie algebra $L(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ consists of nilpotent operators and by Engel’s theorem it can be represented by upper triangular matrices. Hence $\kappa = 0$. \hfill \square

Let now $(G, \omega)$ be a symplectic Lie group and $\nabla$ the torsion free symplectic connection associated to the left-symmetric product \cite{37} on $\mathfrak{g}$ via Proposition \cite{15}. The section $N$ of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ satisfies
\[ \omega(N(x, y), z) = \nabla_x^\omega(y, z) \]
\[ = x(\omega(y, z)) - \omega(xy, z) - \omega(y, xz) \]
\[ = -\omega(xy, z) + \omega([x, y], z) \]
\[ = -\omega(yx, z), \]
whence $N(x, y) = -yx$ and

$$\nabla xy = \nabla y x + \frac{1}{3} N(x, y) + \frac{1}{3} N(y, x)$$

$$= \frac{2}{3} xy - \frac{1}{3} y x ,$$

for all $x, y \in g$.

**Proposition 21** The curvature of $\nabla$ is given by

$$R(x, y) = - \frac{1}{9} [R_x, R_y] - \frac{2}{9} L_{[x,y]} + \frac{1}{9} R_{[x,y]} ,$$

for all $x, y \in g$.

**Proof.** We compute

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$$

$$= \nabla_x (\frac{2}{3} y z - \frac{1}{3} z y) - \nabla_y (\frac{2}{3} x z - \frac{1}{3} z x) - \nabla_{xy} z + \nabla_{yx} z$$

$$= \frac{4}{9} (x y z - y x z) + \frac{2}{9} ((x z y) - y (x z)) - \frac{2}{9} ((y z x - y z x)) - \frac{1}{9} (z x) y + \frac{1}{9} (z y) x$$

$$- \frac{2}{3} (x y z + 2 (y x) z + \frac{1}{3} z (x y) - \frac{1}{3} z (y x))$$

$$= \frac{1}{9} (z x) y - \frac{1}{9} (z y) x - \frac{2}{9} [x, y] z + \frac{1}{9} [z, x, y] ,$$

where in the last step we repeatedly used the left-symmetric property of the product. \(\square\)

We now state the main result of this section.

**Theorem 13** Let $(G, \omega)$ be a symplectic Lie group (of any dimension) with Lie algebra $g$ and

$$\nabla x y = \frac{2}{3} xy - \frac{1}{3} y x ,$$

$x, y \in g$, the symplectic torsion free connection naturally associated to the compatible structure of left-symmetric algebra on $g$ (cf. [57]). Then the Ricci tensor of $\nabla$ is given by

$$\text{ric}^\nabla (x, y) = \frac{1}{9} \left( \text{tr} (L_{xy}) + \text{tr} (L_x \circ L_y) \right) .$$

If $(G, \omega)$ is a nilpotent symplectic Lie group then $\text{ric}^\nabla = 0$.

**Proof.** We fix a symplectic basis $(e_i)$ of $g$ with dual basis $(e^i)$, use the Einstein summation convention and compute

$$9 \text{ric}^\nabla (x, y) = 9 \text{tr} \left\{ z \mapsto R(x, z) y \right\}$$

$$= 9 \omega (R(x, e_i) y, e^i)$$

$$= + \omega([(y x) e_i, e^i]) - \omega((y e_i) x, e^i) - 2 \omega((x e_i) y, e^i)$$

$$+ 2 \omega((e_i x) y, e^i) + \omega(y(x e_i), e^i) - \omega(y(e_i x), e^i)$$

$$= \text{tr} (L_{xy}) - \text{tr} (R_{x} \circ L_{y}) - 2 \text{tr} (R_{y} \circ L_{x})$$

$$+ 2 \text{tr} (R_{y} \circ R_{x}) + \text{tr} (L_{y} \circ L_{x}) - \text{tr} (L_{y} \circ R_{x}) .$$

Using Proposition [19] one readily gets the formula for the Ricci tensor. If $G$ is nilpotent, then each $L_x : g \to g$ is nilpotent too (see the proof of Proposition [20]) and $\text{ric}^\nabla = 0$ follows from $\text{tr} (L_{xy}) = 0$ and Proposition [20]. \(\square\)
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