DERIVATIVE EXPANSION OF THE ONE-LOOP EFFECTIVE ACTION IN QED

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The one-loop effective action in QED at zero and finite temperature is obtained by using the worldline approach. The Feynman rules for the perturbative expansion of the action in the number of derivatives are derived. The general structure of the temperature dependent part of the effective action in an arbitrary external inhomogeneous magnetic field is established. The two-derivative term in the effective action for spinor and scalar QED in a static magnetic background at $T \neq 0$ is calculated.

The problem of calculating the effective action in QED is an old one. Its history starts with the well known papers by Heisenberg and Euler and Weisskopf. Later, some results were obtained by Schwinger who, by using the proper time technique, rederived the one-loop effective action for the case of a constant electromagnetic field. Perhaps, the next most natural step in solving the general problem is to take into account the effect of small deviations from a constant configuration of the field. It turns out, however, that the latter is very difficult to realize.

Here we present our recent work that generalizes the previously known results on the effective action in QED. There, in particular, the derivative expansions up to two derivatives of the field strength with respect to space-time coordinates in scalar and spinor QED at zero and at finite temperature are obtained (for some partial results in a non-Abelian gauge theory see [9], [10]).

Our method is heavily based on the very elegant and now widely developed worldline approach to quantum field theory [11], [12], [13], [14], [15]. Here we follow a self-contained approach of Ref. [1], so that even a non-expert in the field should understand all the details.

As is known, the one-loop effective action in QED reduces to computing the fermion determinant

$$W^{(1)}(A) = -i \ln \det (i \hat{D} - m) = -i \frac{1}{2} \ln \det \left( D^2 + \frac{e}{2} \sigma^{\mu\nu} F^{\mu\nu} + m^2 \right)$$

(1)

$$= -i \frac{1}{2} \int d^n x \langle x | tr \ln \left( D^2 + \frac{e}{2} \sigma^{\mu\nu} F^{\mu\nu} + m^2 \right) | x \rangle \equiv \int d^n x \mathcal{L}^{(1)},$$
where
\[ \mathcal{L}^{(1)}(A) = \frac{i}{2} \int_0^\infty d\tau \frac{1}{\tau} e^{-im^2 \tau} tr\langle x| \exp(-i\tau H)|x\rangle. \] (2)

Here \( \hat{D} = \gamma^\mu D_\mu \) and the covariant derivative is \( D_\mu = \partial_\mu + ieA_\mu \). By definition, \( \sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2 \) and \( tr \) refers to the spinor indices of the Dirac matrices \( \gamma_\mu \). States \( |x\rangle \) are the eigenstates of a self-conjugate coordinate operator \( x_\mu \).

Throughout the paper we use the Minkowski metric, i.e., \( \eta_{\mu\nu} = (1, -1, -1) \) or \( \eta_{\mu\nu} = (1, -1, -1, -1) \), depending on the actual space-time dimension. And in both 2 + 1 and 3 + 1 dimensions, we work with the 4 \( \times \) 4 representation of the Dirac \( \gamma \)-matrices.

At finite temperature, the expression analogous to that in Eqs. (1) reads

\[ F^{(1)}(A) = -T \ln \det(i\gamma^\mu D_\mu - m) = -T \frac{1}{2} \ln \det \left( D^\mu D_\mu + e \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} + m^2 \right) = \]

\[ = -T \frac{1}{2} \int_0^\infty dx_0 \int d^3x \langle x|tr\ln \left( D^\mu D_\mu + e \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} + m^2 \right)|x\rangle, \] (3)

so that

\[ \frac{F^{(1)}(A)}{V_3} = -\frac{i}{2} \int_0^\infty d\tau \frac{1}{\tau} e^{-im^2 \tau} tr\langle x| \exp(-i\tau H)|x\rangle. \] (4)

In both cases, \( T = 0 \) and \( T \neq 0 \), the second order differential operator \( H \) is given by

\[ H = D^\mu D_\mu + e \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu}(x). \] (5)

The matrix element \( \langle z|U(\tau)|y\rangle \equiv \langle z| \exp(-i\tau H)|y\rangle \) (which has the interpretation of the evolution operator of a spinning particle), entering the right hand side of Eq. (4), allows a quantum mechanical path integral representation,

\[ tr\langle z|U(\tau)|y\rangle = \frac{1}{N} \int \mathcal{D}[x(t), \psi(t)] \exp \left\{ i \int_0^\tau dt \left[ L_{\text{bos}}(x) + L_{\text{fer}}(\psi, x) \right] \right\}, \] (6)

where \( N \) is a normalization factor, and

\[ L_{\text{bos}}(x) = -\frac{1}{4} \frac{dx_\nu}{dt} \frac{dx^\nu}{dt} - eA_\nu(x) \frac{dx^\nu}{dt}, \] (7)

\[ L_{\text{fer}}(\psi, x) = \frac{i}{2} \psi_\nu \frac{d\psi^\nu}{dt} - ie\psi^\nu \psi^\lambda F_{\nu\lambda}(x). \] (8)
The integration in Eq. (6) goes over trajectories \( x^\mu(t) \) and \( \psi^\mu(t) \) parameterized by \( t \in [0, \tau] \). In addition, the definition of the integration measure assumes the following boundary conditions

\[
x_\nu(0) = y_\nu, \quad \psi(0) = -\psi(\tau) \tag{9}
x_0(\tau) = z_0 \mod(i\beta), \quad x_i(\tau) = z_i \quad (i = 1, 2, 3). \tag{10}
\]

Note that, at zero temperature, \( \mod(i\beta) \) is absent in the boundary conditions.

Since the path integral (in the case of finite temperature) includes the integrations over worldline trajectories with arbitrary integer windings around the compact (imaginary) \( x_0 \)-direction, the expression in Eq. (6) splits into the sum of path integrals labeled by the winding numbers. In case of spinor QED, the weight factors of these separate contributions are given by \((-1)^n\).

Therefore,

\[
\text{tr} (z|U(\tau)|y) = \frac{1}{N} \sum_{n=-\infty}^{\infty} (-1)^n \int \mathcal{D}[x^{(n)}(t), \psi(t)] \\
\times \exp \left\{ i \int_0^\tau dt \left[ L_{bos} \left( x^{(n)}(t) \right) + L_{fer} \left( \psi(t), x^{(n)}(t) \right) \right] \right\}, \tag{11}
\]

where the boundary conditions \( x_\nu^{(n)}(0) = y_\nu \) and \( x_\nu^{(n)}(\tau) = z_\nu + in\beta\eta_\nu0 \) are assumed. Note, that there exists a similar representation for scalar QED as well. In contrast to the case at hand, in scalar QED, the integration over the Grassman field \( \psi(t) \) is absent and all the weight factors are equal to 1. At zero temperature we have only \( n = 0 \) term, and the sum does not appear at all. We will see later, Eqs. (19) and (20), that the same is true if we consider the limit \( T \to 0 \ (\beta \to \infty) \) afterwards, i.e., all the terms with \( n \neq 0 \) go to zero.

Let us consider the effective action in the case of a slightly inhomogeneous static magnetic field. We choose a version of the Fock-Schwinger gauge for the vector potential \( A_\mu(x) \),

\[
A_0(x) = 0, \quad (x_i - y_i) A_i(x) = 0. \tag{12}
\]

The latter leads to the series

\[
A_i(x) = -\frac{1}{2} (x_j - y_j) F_{ji}(y) + \frac{1}{3} (x_j - y_j)(x_l - y_l) \partial_l F_{ji}(y) \\
- \frac{1}{8} (x_j - y_j)(x_l - y_l)(x_k - y_k) \partial_l \partial_k F_{ji}(y) + \ldots \tag{13}
\]
With this choice of gauge, we arrive at a very convenient representation for the diagonal matrix element of the evolution operator,

\[ \text{tr}(y | U(\tau) | y) = \sum_{n=-\infty}^{\infty} (-1)^n \int D[x^{(n)}, \psi] \exp \left[ i \int_0^\tau dt \left( -\frac{1}{4} \frac{dx_0^{(n)}}{dt} \frac{dx_0^{(n)}}{dt} + \frac{e}{2} x_i^{(n)} F_{ij}(y) \frac{dx_j^{(n)}}{dt} + L_{\text{bos}}^{\text{int}}(x_i^{(n)}) \right) \right] \times \exp \left[ i \int_0^\tau dt \left( \frac{i}{2} \frac{d\psi_0}{dt} - \frac{i}{2} \frac{d\psi_i}{dt} - i e \psi_i \psi_j F_{ij}(y) + L_{\text{fer}}^{\text{int}}(x_i^{(n)}, \psi_i) \right) \right] \]

\[ (14) \]

where, as follows from Eqs. (7), (8) and (13), the interacting terms, \( L_{\text{bos}}^{\text{int}}(x) \) and \( L_{\text{fer}}^{\text{int}}(x, \psi) \), containing spatial derivatives of \( F_{ij} \), are given by

\[ L_{\text{bos}}^{\text{int}}(x) = -\frac{e}{3} F_{ij,kl} \frac{dx_i}{dt} \frac{dx_j}{dt} \frac{dx_k}{dt} x_k x_l + \ldots, \]

\[ (15) \]

\[ L_{\text{fer}}^{\text{int}}(x, \psi) = i e F_{ij,k} \psi_i \psi_j x_k - \frac{i e}{2} F_{ij,kl} \psi_i \psi_j x_k x_l + \ldots. \]

\[ (16) \]

The integration variables in Eq. (14) are subject to the following boundary conditions, \( x_0^{(n)}(0) = 0, x_0^{(n)}(\tau) = in\beta \) and \( x_i^{(n)}(\tau) = 0 \) (note that the fields \( x_0^{(n)}(t) \) were preliminary shifted by \( -y_{0,i} \)).

A very nice property of the path integral in Eq. (14) is its factorization into two pieces. One of them contains only the time components of the fields and is, in fact, a Gaussian path integral. The other contains the interacting spatial components of the fields and, what is very important, does not depend on the winding number \( n \). After performing the Gaussian integrations over \( x_0^{(n)}(t) \) and \( \psi_0(t) \), we obtain

\[ \text{tr}(y | U(\tau) | y) = \frac{1}{N} \sum_{n=-\infty}^{\infty} (-1)^n \exp \left( i \frac{n^2 \beta^2}{4\tau} \right) \int D[x_i(t), \psi_i(t)] \exp \left[ i \int_0^\tau dt \left( \frac{i}{2} \frac{d\psi_i}{dt} - i e \psi_i \psi_j F_{ij}(y) + L_{\text{fer}}^{\text{int}}(x_i^{(n)}, \psi_i) \right) \right] \]

\[ (17) \]
This expression, in fact, is one of our most important results here. It states that the temperature dependent part of the evolution operator is exactly factorized from the part depending on an (arbitrary) inhomogeneous magnetic field. The latter, in its turn, puts a strong restriction on the structure of the one-loop finite temperature effective action in QED. Similarly, we obtain the result in scalar QED,

$$tr\langle y|U_{\text{scal}}(\tau)|y\rangle = \frac{1}{N} \sum_{n=-\infty}^{\infty} \exp \left( i \frac{n^2 \beta^2}{4\tau} \right) \int D[x_i(t), \psi_i(t)]$$

$$\times \exp \left[ i \int_{0}^{\tau} dt \left( \frac{1}{4} \frac{dx_i}{dt} \frac{dx_i}{dt} - \frac{e}{2} x_i F_{ij}(y) \frac{dx_j}{dt} + L_{\text{bos}}^{int}(x_i) \right) \right]. \tag{18}$$

As is seen, the path integrals in Eqs. (17) and (18) are almost the same as those at zero temperature. The only difference is due to the additional (Gaussian) integration over $x_0$ at $T = 0$. The latter, however, is irrelevant since it just modifies the overall normalization factor $1/N$.

The further integration (over the spatial components of the fields) can be done only approximately. Remarkably, the result in the case of a constant background field can be obtained exactly from the above expressions. That is due to the fact that the integrals in Eqs. (17) and (18) are Gaussian in that particular case. All the derivatives of an inhomogeneous background field appear in the worldline action only through the interaction terms. Therefore, it is convenient to develop the corresponding perturbative theory and formulate the Feynman rules for calculating the diagrams of interest. Here we briefly outline these rules.

We observe that there are two different types of local interactions in Eqs. (15) and (16). The first (bosonic) type contains only the bosonic fields, $x_{\mu}(t)$. The corresponding vertices are shown in Figure 1. The other type involves both the boson, $x_{\mu}(t)$, and the spinor fields, $\psi_{\mu}(t)$. These latter produce the vertices given in Figure 2. Regarding the notation, the integers in the vertices denote the number of derivatives of the electromagnetic field with respect to space-time coordinates. Some legs in the diagrams are marked by circles and bullets. The circles correspond to legs related to the first Lorentz index ($\nu$) of the tensor weight, $F_{\nu\lambda\mu_1...\mu_n}$, assigned to the fermion vertices. The bullets, on the other hand, mark those legs which contain the derivatives with respect to the proper time. The latter act on the boson propagators attached to the corresponding legs.

The rest of the Feynman rules are easy to derive as well. In calculation, one has to use the appropriate propagators of $x_{\mu}(t)$ and $\psi_{\mu}(t)$ fields for
Figure 1: Diagrammatic notations for the boson interaction vertices. The curly brackets denote symmetrization of the type: $F_{\nu(\lambda, \mu_1 \ldots \mu_n)} = F_{\nu\lambda, \mu_1 \ldots \mu_n} + F_{\nu\mu_1, \lambda \ldots \mu_n} + \ldots + F_{\nu\mu_n, \mu_1 \ldots \lambda}$.

Connecting the boson (solid) and the fermion (dashed) legs, respectively.

A somewhat disappointing feature of the theory is an infinite number of local interactions (see Eqs. (15) and (16)). Fortunately, while working at any finite order of the perturbative theory, one requires only a finite number of those interactions.

Omitting the intermediate calculations, let us give the final results for the first non-trivial term in the derivative expansion of the effective action (free energy) in QED. Assuming that the background magnetic field is directed along the third spatial axis, we get the following expression for the two-derivative part of the expansion,

$$
\frac{F^1_{\text{der}}(B)}{V_3} = \frac{e^2}{(8\pi^2)\beta |eB|} \int_0^\infty \frac{d\omega}{\omega} e^{-m^2 \omega/|eB|} \frac{d^3}{d\omega^3} (\omega \coth \omega) \times \sum_{n=-\infty}^{\infty} (-1)^n \exp \left( -\frac{n^2 \beta^2 |eB|}{4\omega} \right),
$$

where $(\partial_\perp B)^2 \equiv (\partial_1 B)^2 + (\partial_2 B)^2$. The result in Eq. (19) differs from the analogous expression at $T = 0$ (see Ref. [4]) only by the last factor containing the sum over the winding numbers. Again, as all the other results, this expression
allows a straightforward generalization to 2+1 dimensional case as well as to scalar QED. For example, in 3+1 dimensional scalar QED, the result reads

$$f_{\text{der}, \text{scal}}(B) = \frac{e^2}{2(8\pi)^2|eB|} \int_0^\infty \frac{d\omega}{\omega} e^{-m^2\omega/|eB|} \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega^2}{\sinh \omega} \right)$$

$$\times \sum_{n=-\infty}^\infty \exp \left( -\frac{n^2\beta^2|eB|}{4\omega} \right).$$

In conclusion, here we derived the one-loop zero and finite temperature effective potential (free energy) in spinor and scalar QED using the worldline formulation of quantum field theory. The only difference (at one loop) between zero and finite temperature cases appears in applying the boundary conditions to the saddle point trajectory. By making use of this method, we were able to establish the general structure of the temperature dependent part of the effective action in QED in an arbitrary external inhomogeneous magnetic field. Also, we established the Feynman rules for calculating the perturbative expansion of the effective action (free energy) in the number of derivatives of the background field with respect to space-time coordinates. The explicit result for the first non-trivial term in the expansion (containing two derivatives of the magnetic field) is presented.
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