CO– VERSUS CONTRA V ARIANT FINITENESS
OF CATEGORIES OF REPRESENTATIONS

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Abstract. This article supplements recent work of the authors. (1) A criterion for failure of covariant finiteness of a full subcategory of \( \Lambda\text{-mod} \) is given, where \( \Lambda \) is a finite dimensional algebra. The criterion is applied to the category \( P^\infty(\Lambda\text{-mod}) \) of all finitely generated \( \Lambda \)-modules of finite projective dimension, yielding a negative answer to the question whether \( P^\infty(\Lambda\text{-mod}) \) is always covariantly finite in \( \Lambda\text{-mod} \). Part (2) concerns contravariant finiteness of \( P^\infty(\Lambda\text{-mod}) \). An example is given where this condition fails, the failure being, however, curable via a sequence of one-point extensions. In particular, this example demonstrates that curing failure of contravariant finiteness of \( P^\infty(\Lambda\text{-mod}) \) usually involves a tradeoff with respect to other desirable qualities of the algebra.

1. Introduction and Terminology

Functorial finiteness conditions for certain categories of finitely generated representations of an algebra may have a major impact also on the non-finitely generated representations, as was shown by the authors in [10]. More precisely: Let \( \Lambda \) be an artin algebra, and let \( P^\infty(\Lambda\text{-mod}) \) and \( P^\infty(\Lambda\text{-Mod}) \) be the full subcategories of the categories \( \Lambda\text{-mod} \) of finitely generated left \( \Lambda \)-modules and the full module category \( \Lambda\text{-Mod} \), respectively, consisting of the objects of finite projective dimension in either case. Then contravariant finiteness of \( P^\infty(\Lambda\text{-mod}) \) in \( \Lambda\text{-mod} \) forces arbitrary modules in \( P^\infty(\Lambda\text{-Mod}) \) to be direct limits of objects in \( P^\infty(\Lambda\text{-mod}) \). When combined with a theorem of Auslander and Reiten [1], this entails that

\[
\text{l fin dim } \Lambda = \text{l Fin dim } \Lambda = \sup_{1 \leq i \leq n} \text{p dim } A_i
\]

in this case, where \( A_1, \ldots, A_n \) are the minimal right \( P^\infty(\Lambda\text{-mod}) \)-approximations of the simple left \( \Lambda \)-modules. (Here \( \text{l fin dim } \Lambda \) and \( \text{l Fin dim } \Lambda \) are the suprema of the projective dimensions attained on \( P^\infty(\Lambda\text{-mod}) \) and \( P^\infty(\Lambda\text{-Mod}) \), respectively.)

As a byproduct of the described connections, one obtains that contravariant finiteness of \( P^\infty(\Lambda\text{-mod}) \) implies covariant finiteness of this category in
Indeed, by Crawley-Boevey’s [4, Theorem 4.2], an additive subcategory \( A \) of \( \Lambda\)-mod is covariantly finite if and only if the closure \( \overrightarrow{\mathcal{P}}(A) \) under direct limits is closed under direct products as well. As explained above, contravariant finiteness of \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) implies \( \overrightarrow{\mathcal{P}}(\Lambda\text{-Mod}) = \overrightarrow{\mathcal{P}}^\infty(\Lambda\text{-mod}) \) and \( \text{1Fin dim } \Lambda < \infty \), which guarantees that \( \overrightarrow{\mathcal{P}}^\infty(\Lambda\text{-mod}) \) is closed under direct products.

It is not hard to find examples demonstrating that, in general, contravariant finiteness of \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) in \( \Lambda\text{-mod} \) is properly stronger than covariant finiteness. In fact, the initial example – due to Igusa, Smalø, and Todorov [11] – of a situation where \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) fails to be contravariantly finite already serves to show this. This leaves one wondering whether \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) might always be covariantly finite in \( \Lambda\text{-mod} \). The answer is negative, as we show here, but examples are somewhat harder to come by.

The first part of the present paper is devoted to developing a criterion for failure of covariant finiteness of \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) and to then applying it to a finite dimensional special biserial algebra (for a definition see under ‘Notation and Terminology’ below). We believe that, in the restricted setting of special biserial algebras \( \Lambda \), the conditions of this criterion actually provide an equivalent description for failure of covariant finiteness of \( \mathcal{P}^\infty(\Lambda\text{-mod}) \).

The criterion can be considered as a somewhat weaker twin sibling of the sufficient condition for failure of contravariant finiteness developed by Happel and the first author in [7, Criterion 10]. We remark that, while the concepts of contra- and covariant finiteness are mutually dual, the theories relating them to a prescribed subcategory of \( \Lambda\text{-mod} \) are of course not; in particular, the argument backing up the criterion presented here differs substantially from that used to prove [7, Criterion 10].

The homological picture available for algebras \( \Lambda \) having the property that \( \mathcal{P}^\infty(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \) naturally raises the question as to how abundant they are. While this condition is known to ‘slice diagonally’ through the standard classes of algebras, the authors conjecture that failure can ‘often’ be fixed in the following sense: Namely, that there exists a sequence \( \Lambda_0 = \Lambda, \Lambda_1, \ldots, \Lambda_m \) of Artin algebras such that each \( \Lambda_i \) is a one-point extension of \( \Lambda_{i-1} \) and \( \mathcal{P}^\infty(\Lambda_{m}\text{-mod}) \) is contravariantly finite in \( \Lambda_m\text{-mod} \). Since in each passage from \( \Lambda_{i-1} \) to \( \Lambda_i \) both the little and big finitistic dimensions increase by at most 1, this might give a handle on the difference \( 1\text{Fin dim } - \text{Fin dim } \) for special classes of algebras. Moreover, the existence of a sequence as above guarantees that \( 1\text{Fin dim } \Lambda < \infty \). In general, this process will force one to leave a given ‘nice’ class of algebras, however. The second part of this article is devoted to the explicit construction of such a sequence \( \Lambda_0 = \Lambda, \Lambda_1, \ldots, \Lambda_m \) such that \( \Lambda \) is a monomial relation algebra while \( \Lambda_m \) cannot be chosen from this class of algebras.

**Terminology and Notation.**

In the following, \( \Lambda \) will be a split finite dimensional algebra over a field \( K \), i.e., \( \Lambda \) will be of the form \( K\Gamma/I \) for some quiver \( \Gamma \) and an admissible
ideal $I$ of the path algebra $K\Gamma$. The Jacobson radical of $\Lambda$ will be denoted by $J$. For us, the primitive idempotents of $\Lambda$ will be those which naturally correspond to the vertices of $\Gamma$; in fact, we will identify a complete set of primitive idempotents of $\Lambda$ with the vertices of the quiver.

According to [2], a full subcategory $\mathcal{A}$ of $\Lambda$-mod is said to be contravariantly finite in $\Lambda$-mod if, for each module $M$ in $\Lambda$-mod, there exists a homomorphism $f : A \to M$ with $A \in \mathcal{A}$ such that the following sequence of contravariant functors is exact:

$$\text{Hom}_\Lambda(-, A) \xrightarrow{\text{Hom}_\Lambda(-, f)} \text{Hom}_\Lambda(-, M) \to 0;$$

in other words, exactness of this sequence means that each $g \in \text{Hom}_\Lambda(B, M)$ with $B \in \mathcal{A}$ factors through $f$. In that case, $A$ is called a right $\mathcal{A}$-approximation of $M$. By [2], the $\mathcal{A}$-approximations of $M$ that have minimal dimension are pairwise isomorphic. This justifies reference to the minimal right $\mathcal{A}$-approximation of $M$, whenever $\mathcal{A}$ is contravariantly finite. The terms covariantly finite and left $\mathcal{A}$-approximation are defined dually.

An algebra $\Lambda$ is said to be special biserial in case it is isomorphic to a path algebra modulo relations, $K\Gamma/I$, with the following properties: Given any vertex $e$ of $\Gamma$, at most two arrows enter $e$ and at most two arrows leave $e$; moreover, for any arrow $\alpha$ in $\Gamma$ there is at most one arrow $\beta$ with $\alpha \beta \notin I$ and at most one arrow $\gamma$ with $\gamma \alpha \notin I$.

Recall, moreover, that given two algebras $\Lambda = K\Gamma/I$ and $\Lambda' = K\Gamma'/I'$, the second is called a one-point extension of the first in case $\Gamma'$ results from $\Gamma$ through addition of a single vertex which is a source of $\Gamma'$ such that $I' \cap K\Gamma = I$. For importance and properties of one-point extensions, we refer to [13].

Given paths $p$ and $q$ of $\Gamma$, we say that $q$ is a subpath of $p$ if $p = p_2qp_1$ in $K\Gamma$ for suitable paths $p_1$ and $p_2$. We call $q$ a right (left) subpath of $p$ in case $p = p_2q$ (respectively, $p = qp_1$) for suitable paths $p_2$ (respectively, $p_1$). The path $p$ is said to start (end) in the arrow $\alpha$ if $\alpha$ is a right (left) subpath of $p$. Furthermore, we call an element $x$ of a left $\Lambda$-module $M$ a top element in case $x \in M \setminus JM$ and $ex = x$ for a primitive idempotent $e$ from our distinguished set.

Finally, we refer the reader to previous work of the authors for their graphing conventions (see, e.g., [7,8,9,10]). The graphs most crucial to the present note are zigzags of the type

$$\begin{array}{cccccc}
e(1) & e(2) & \cdots & e(r) \\
p_1 & q_1 & p_2 & q_2 & \cdots & p_r \\
e(1) & e(2) & e(3) & \cdots & e(r) \end{array}$$
where the $e(i)$ and $\tilde{e}(i)$ denote vertices and, for each $i$, the $p_i$, $q_i$ denote paths of positive length starting in distinct arrows. That a module $M \in \Lambda$-mod has the shown graph relative to a sequence $x_1, \ldots, x_r$ of top elements in particular encodes the following information: $x_i = e(i)x_i$, the $x_i$ are $K$-linearly independent modulo $JM$, each $p_i$ has starting point $e(i)$ and endpoint $\tilde{e}(i)$, each $q_i$ has starting point $e(i)$ and endpoint $e(i + 1)$, and the multiples $q_ix_i = p_{i+1}x_{i+1}$, $1 \leq i \leq r - 1$, are $K$-linearly independent elements of the socle of $M$. The information encoded in the graph guarantees, moreover, that there are no ‘other’ nonzero multiples of the $x_i$ apart from those shown; more precisely, the only paths in $KT$ not annihilating the element $x_i \in M$ are the right subpaths of $p_i$ and $q_i$.

2. Covariant finiteness of $\mathcal{P}^\infty(\Lambda$-mod)

The main goal of this section will be the development and application of conditions which guarantee that a simple left $\Lambda$-module fails to have a left $\mathcal{P}^\infty(\Lambda$-mod)-approximation. We will start by recalling a result of Auslander and Reiten, including a short alternate proof akin to the arguments of the introduction.

Proposition 1. [1, Proposition 4.2] If $\text{l fin dim } \Lambda \leq 1$, then $\mathcal{P}^\infty(\Lambda$-mod) is covariantly finite in $\Lambda$-mod.

Proof. Suppose $\text{l fin dim } \Lambda \leq 1$. By [4, Theorem 4.2], it suffices to prove that each direct product of objects in $\mathcal{P}^\infty(\Lambda$-mod) belongs to $\mathcal{P}^\infty(\Lambda$-mod). Clearly, each direct product $M$ of objects from $\mathcal{P}^\infty(\Lambda$-mod) has projective dimension $\leq 1$ in $\text{A-Mod}$, and hence [9, Observation 5] shows that $M$ is a direct limit of finitely generated modules of finite projective dimension as required. □

We will see later in this section that the conclusion of Proposition 1 breaks down for algebras of finitistic dimension 2.

Example 2. [11] This is the example exhibited by Igusa, Smalø and Todorov to show that $\mathcal{P}^\infty(\Lambda$-mod) may fail to be contravariantly finite, even in the case of a special biserial algebra $\Lambda$.

Let $\Gamma$ be the quiver

\[
\begin{array}{c}
1 \\
\downarrow \beta \\
2 \\
\end{array}
\]

and $I \subseteq KT$ such that, for $\Lambda = KT/I$, the indecomposable projective left $\Lambda$-modules have the following graphs:
Here $\text{Fin dim } \Lambda = 1$, and hence $\mathcal{P}^\infty(\Lambda\text{-mod})$ is covariantly finite in $\Lambda\text{-mod}$ by Proposition 1. □

In the sequel, $\mathcal{A}$ will denote a full subcategory of $\Lambda\text{-mod}$.

**Criterion 3.** Let $e(1), \ldots, e(r)$ be vertices of $\Gamma$, and let $p_1, \ldots, p_r, q_1, \ldots, q_r$ be $2r$ paths of positive length in $K\Gamma$, none of which is a subpath of any of the others. Moreover, suppose that the following conditions (1) and (2) are satisfied:

1. For each natural number $n$, there exists a module $M_n \in \mathcal{A}$ having graph relative to a suitable sequence of top elements $x_{n,1}, \ldots, x_{n,nr}$ of $M_n$ which are $K$-linearly independent modulo $JM_n$.

2. Each module $A \in \mathcal{A}$ has the following properties:
   (i) $e(1)(\text{Soc } A) \subseteq p_1 A$;
   (ii) $q_i A \cap (\text{Soc } A) \subseteq p_{i+1} A$ for $i < r$, and $q_r A \cap (\text{Soc } A) \subseteq p_1 A$;
   (iii) If $x \in A$ with $p_i x \in \text{Soc } A$, then $q_i x \in \text{Soc } A$. 
Then $S = \Lambda e(1)/Je(1)$ fails to have a left $A$-approximation.

Proof. For $i > r$, let $s(i)$ be the integer in $\{1, \ldots, r\}$ with $i \equiv s(i) \pmod{r}$, and define $p_i := p_{s(i)}$, $q_i := q_{s(i)}$.

We assume that, to the contrary of our claim, $S = \Lambda e(1)/Je(1)$ does have a left $A$-approximation $f : S \to A$ with $A \in \mathcal{A}$. Choose $n > \dim_K A$ and write $x_1, \ldots, x_{nr}$ for the elements $x_{n1}, \ldots, x_{nr}$ of $M_n$. Moreover, let $g : S \to M_n$ be the embedding which sends $e(1) + Je(1)$ to $p_1x_1$, and choose a homomorphism $h : A \to M_n$ with $g = h \circ f$. Since $f(e(1) + Je(1)) \in e(1) \text{Soc } A$, condition 2(i) permits us to pick an element $a_1 \in A$ such that $f(e(1) + Je(1)) = p_1x_1$. In view of the equality $hf(e(1) + Je(1)) = p_1x_1$, we see that $h(a_1) = x_1 + y_1$ with $y_1 \in \sum_{j \not\equiv 1 \pmod{r}} \Lambda x_j + \sum_{j \equiv 1 \pmod{r}} Jx_j$.

Keep in mind that $p_1$ equals $p_{r+1} = \cdots = p_{tr+1}$, but is not a subpath of the other $p_i$ or any of the $q_i$. Since $q_1$ is not a subpath of any of the paths $p_1, \ldots, p_r, q_2, \ldots, q_r$ either, we infer that $q_1h(a_1) = q_1x_1$. Due to our choice of $a_1$ such that $p_1a_1 \in \text{Soc } A$, condition 2(iii) guarantees that $q_1a_1 \in \text{Soc } A$ as well, in other words, $q_1a_1 \in q_1A \cap \text{Soc } A$. Next, condition 2(ii) yields $a_2 \in A$ with $p_2a_2 = q_1a_1$. In view of $h(p_2a_2) = q_1h(a_1) = q_1x_1$, we obtain $h(a_2) = x_2 + y_2$ with $y_2 \in \sum_{j \not\equiv 2 \pmod{r}} \Lambda x_j + \sum_{j \equiv 2 \pmod{r}} Jx_j$; indeed this follows at once from the nature of the graph of $M_n$ and the hypothesis that $p_2$ is not a subpath of any of $p_1, p_3, \ldots, p_r, q_1, \ldots, q_r$. Now the non-occurrence of $q_2$ as a subpath of $p_1, \ldots, p_r, q_1, q_3, \ldots, q_r$ allows us to deduce $q_2h(a_2) = q_2x_2$, and since $p_2a_2 = q_1a_1 \in \text{Soc } A$, we observe that also $q_2a_2 \in \text{Soc } A$ by 2(iii). Thus 2(ii) in turn provides us with an element $a_3 \in A$ such that $p_3a_3 = q_2a_2$. As above, we argue that $q_3h(a_3) = q_3x_3$, and proceeding inductively, we thus obtain a sequence $a_1, \ldots, a_{nr}$ of elements in $A$ with the property that $q_ih(a_i) = q_ix_i$ for $1 \leq i \leq nr$. Since the elements $q_1x_1, \ldots, q_{nr}x_{nr}$ of $M_n$ are $K$-linearly independent by hypothesis, so are $a_1, \ldots, a_{nr}$. But this contradicts our choice of $n$ exceeding $\dim_K A$ and proves the criterion. 

As our argument makes clear, the requirement that none of the $p_i, q_i$ is a subpath of any other as called for in the criterion can certainly be weakened. Other than that, the chain reaction exhibited in the proof of Criterion 3 appears prototypical for the failure of a simple module $S$ to have a left $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. In fact, specializing to special biserial algebras, we believe the answer to the following question to be positive.

**Problem 4.** Suppose $S = \Lambda e(1)/Je(1)$ is a simple left module over a finite dimensional special biserial algebra $\Lambda = K\Gamma/I$. If $S$ does not have a left $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation, do there exist paths $p_1, \ldots, p_r, q_1, \ldots, q_r$ in $\text{KT}$ satisfying the conditions (1) and (2) of Criterion 3?

On the other hand, we point out that it is not known whether failure of covariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ in $\Lambda\text{-mod}$ implies that one of the simple left $\Lambda$-modules is devoid of a left $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. This is in contrast with our level of information on contravariant finite-
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ness of \( P^\infty(\Lambda\text{-mod}) \). Indeed, as was shown by Auslander and Reiten in [1], \( P^\infty(\Lambda\text{-mod}) \) is contravariantly finite in \( \Lambda\text{-mod} \) provided that all simple \( \Lambda \)-modules have right \( P^\infty(\Lambda\text{-mod}) \)-approximations. We therefore propose

**Problem 5.** Is \( P^\infty(\Lambda\text{-mod}) \) covariantly finite in \( \Lambda\text{-mod} \) in case each simple left \( \Lambda \)-module has a left \( P^\infty(\Lambda\text{-mod}) \)-approximation?

Criterion 3 is particularly suited to the situation where \( A = P^\infty(\Lambda\text{-mod}) \) and \( \Lambda \) is a finite dimensional special biserial algebra (see §1 for a definition). The representation theory of the finitely generated modules over such algebras is exceptionally well understood (see [3,5,6,12,14]). In fact, the indecomposable objects of \( \Lambda\text{-mod} \) fall into two classes, ‘bands’ and ‘strings’. All we presently need to know about bands is the following: If \( B \in \Lambda\text{-mod} \) is a band, \( e \in \Lambda \) a primitive idempotent and \( x \in e(\text{Soc} B) \), then \( x \in \sum_i u_i B \cap v_i B \) for pairs \((u_i, v_i)\) of paths of positive length ending in \( e \) such that, moreover, \( u_i = \alpha_i u' \) and \( v_i = \beta_i v' \) with distinct arrows \( \alpha_i \) and \( \beta_i \). The strings, on the other hand, are precisely the objects in \( \Lambda\text{-mod} \) having graphs of the form

A finite dimensional special biserial algebra \( \Lambda \) for which \( P^\infty(\Lambda\text{-mod}) \) fails to be covariantly finite in \( \Lambda\text{-mod} \): Let \( \Lambda = K\Gamma/I \), where \( \Gamma \) is the quiver

\[
\begin{array}{c}
1 \\
\downarrow \alpha \\
2 \\
\downarrow \beta \\
3 \\
\downarrow \psi \\
5 \\
\downarrow \gamma \\
\downarrow \delta \\
4 \\
\downarrow \epsilon \\
6 \\
\rightarrow 7 \\
\rightarrow 8 \\
\rightarrow \epsilon' \\
\rightarrow \tau \\
\rightarrow \tau' \\
\rightarrow \sigma \
\end{array}
\]

and \( I \) is generated by

\[
\gamma \alpha - \delta \beta, \ \rho \chi - \sigma \psi, \ \gamma \chi, \ \rho \alpha, \ \delta \psi, \ \sigma \beta, \ \epsilon \gamma, \ \epsilon' \epsilon, \ (\epsilon')^2, \ \tau \tau', \ (\tau')^2, \ \tau \sigma.
\]

Then the graphs of the indecomposable projective left \( \Lambda \)-modules are
We will use Criterion 3 to show that $S = \Lambda e_3/J e_3$ does not have a left $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. For that purpose, let $r = 2, e(1) = e_3, e(2) = e_5, p_1 = \beta, \text{ and } p_2 = \chi, q_1 = \alpha, q_2 = \psi$. Then condition 1 of our criterion is satisfied by the modules $M_n$ with graphs

Indeed, one readily verifies that the first syzygy $\Omega^1(M_n)$ of $M_n$ has graph

and the second syzygy $\Omega^2(M_n)$ is a direct sum of modules with graphs

This shows that $p\dim M_n = 2$; in particular, $M_n \in \mathcal{P}^\infty(\Lambda\text{-mod})$.

It is obvious that for every module $A \in \Lambda\text{-mod}$ and every element $x \in A$ the implications $(\beta x \in \text{Soc} A \implies \alpha x \in \text{Soc} A)$ and $(\chi x \in \text{Soc} A \implies \psi x \in \text{Soc} A)$ hold. So condition 2(iii) of Criterion 3 is met as well. To check the remaining conditions under (2), we start by observing that the only nontrivial paths of $K\Gamma$ ending in $e_3$ are the arrows $\beta$ and $\psi$, while...
the only nontrivial paths ending in $e_2$ are the arrows $\alpha$ and $\chi$. Given our comments on bands, this implies that, for each band $B \in \Lambda$-mod, we have $e_3(Soc B) \subseteq \beta B \cap \psi B$ and, in particular, $\psi B \cap (Soc B) \subseteq \beta B$. Analogously, $\alpha B \cap (Soc B) \subseteq e_2(Soc B) \subseteq \chi B$. Hence, we may focus our attention on the situation where $A \in \mathcal{P}^\infty(\Lambda$-mod) is a string with $S_3 \subseteq Soc A$. Note that $A$ cannot be simple, since $p \dim S_3 = \infty$. The only possibility for a copy of $S_3 \subseteq Soc A$ not to belong to $\beta A$ is that of a string $A$ having graph

\[
\begin{array}{c}
\psi \\
\downarrow \\
3 \\
\uparrow \\
2 \\
\vdots
\end{array}
\]

In that case $\Omega^1(A)$ would have a graph of one of the following types

\[
\begin{array}{c}
2 \\
\downarrow \\
6 \quad \text{or} \quad 4 \quad \text{or} \quad 2 \\
\vdots
\end{array}
\]

and $\Omega^2(A)$ would be a copy of $S_4 \oplus S_7$ in the first case, a copy of $S_8$ in the second, and have a direct summand isomorphic to $S_2$ in the third. In all of these cases, we would have $p \dim \Omega^2(A) = \infty$ contradicting our choice of $A$ in $\mathcal{P}^\infty(\Lambda$-mod). This proves $e_3(Soc A) \subseteq \beta A$ and thus 2(i).

For 2(ii) it suffices to observe that any $A \in \mathcal{P}^\infty(\Lambda$-mod) has the stronger property that $e_2(Soc A) \subseteq \chi A$. The argument is analogous to the one we just completed for $e_3$ and $\beta$.

Thus, by Criterion 3, $S = \Lambda e_3/J e_3$ does not have a left $\mathcal{P}^\infty(\Lambda$-mod)-approximation. □

We remark that $\text{Fin dim} \Lambda = 2$ in the preceding example, which shows that Auslander-Reiten’s Proposition 1 cannot be extended to the case of finitistic dimensions exceeding 1.

3. CURING FAILURE OF CONTRAVARIANT FINITENESS OF $\mathcal{P}^\infty(\Lambda$-mod)

We conjecture that, for any monomial relation algebra and for any special biserial algebra $\Lambda$, there exists a sequence of one-point extensions $\Lambda = \Lambda_0, \ldots, \Lambda_m = \Delta$ such that $\mathcal{P}^\infty(\Delta$-mod) is contravariantly finite in $\Delta$-mod.

At this point, our conviction is based mainly on a long list of examples. One of our most interesting examples shows that, in general, one cannot expect $\Delta$ to retain the ‘good’ properties of $\Lambda$ in this process, in other words, a ‘cure’ for failure of contravariant finiteness of $\mathcal{P}^\infty(\Lambda$-mod) by successive one-point extensions, will usually involve a trade-off. Here we will construct an example of a monomial relation algebra $\Lambda$, together with a sequence $\Delta = \Lambda_0, \Lambda_1, \Lambda_2 = \Delta$ as above such that $\Delta$ cannot be chosen within the class
of monomial relation algebras. Simultaneously, this example will illustrate the potential intricacy of the structure of the minimal right \( P^\infty (\Delta\text{-mod}) \)-approximations of the simple modules.

Example 7. Let \( \Lambda = K\Gamma / I \), where \( \Gamma \) is the quiver

\[
\begin{array}{cccccccc}
1 &  &  &  &  &  &  &  & 2 \\
\alpha & \beta_1 & \gamma_1 & \delta & \rho & \sigma & \gamma_2 & \delta & 3 \\
2 &  &  &  &  &  &  &  & 4 \\
\end{array}
\]

and the ideal \( I \subseteq K\Gamma \) is generated by

\[
\delta \alpha, \ \epsilon \alpha, \ \epsilon \beta_i \ (i = 1, 2), \ \delta \gamma_i \ (i = 1, 2), \ \rho \delta, \ \sigma \epsilon, \ \rho \beta_2, \ \sigma \gamma_2.
\]

This yields indecomposable projective left \( \Lambda \)-modules with graphs

\[
\begin{array}{cccccccc}
1 &  &  &  &  &  &  &  & 2 \\
\alpha & \delta & \epsilon & \rho & \sigma & \beta_1 & \beta_2 & \gamma_1 & \gamma_2 \\
2 &  & 3 & 4 & 3 & 4 & 2 & 2 & 2 \\
3 &  & 3 & 4 & 3 & 4 & 4 & 4 & 1
\end{array}
\]

To see that \( P^\infty (\Lambda\text{-mod}) \) fails to be contravariantly finite in \( \Lambda\text{-mod} \), more precisely, that \( \Lambda e_1 / Je_1 \) fails to have a right \( P^\infty (\Lambda\text{-mod}) \)-approximation, we exhibit a \( P^\infty (\Lambda\text{-mod}) \)-phantom of infinite \( K \)-dimension for \( \Lambda e_1 / Je_1 \) (see [7, Definition 5 and Theorem 9]). The routine check that the following module \( H \) is indeed such a phantom is left to the reader:

\[
H = \lim_{\rightarrow} H_n
\]

where \( H_n = (\Lambda z \oplus \bigoplus_{i=1}^{n} \Lambda x_i \oplus \bigoplus_{i=1}^{n} \Lambda y_i) / U_n \),

with \( z = e_1, x_{2m-1} = e_5, x_{2m} = e_6, y_{2m-1} = e_7, y_{2m} = e_8 \) for \( m \geq 1 \), and

\[
U_n = \Lambda (\alpha z - \beta_1 x_1 - \gamma_1 y_1) + \sum_{i=1}^{n-1} \Lambda (\tilde{\gamma}_i y_i - \tilde{\beta}_{i+1} x_{i+1} - \tilde{\gamma}_{i+1} y_{i+1}),
\]

with \( \tilde{\gamma}_i \) equal to \( \gamma_1 \) or \( \gamma_2 \), depending on whether \( i \) is odd or even, and \( \tilde{\beta}_i \) equal to \( \beta_1 \) or \( \beta_2 \), depending on whether \( i \) is odd or even. The modules \( H_n \) can be pictured via graphs of the form
relative to the top elements $z, x_1, \ldots, x_n, y_1, \ldots, y_n$, where we extend our graphing conventions as follows: The dotted loop around the vertices labeled ‘2’ which represent $\alpha z, \beta_1 x_1, \gamma_1 y_1$ indicates that any two of the three listed vectors are $K$-linearly independent while $\dim_K(K\alpha z + K\beta_1 x_1 + K\gamma_1 y_1) = 2$. The same holds for the additional triples $\tilde{\gamma}_i y_i, \tilde{\beta}_i x_i, \tilde{\gamma}_{i+1} y_{i+1}$ surrounded by loops. Note that $\Omega^1(H_n) \cong (\Lambda e_2)^n$ for $n \in \mathbb{N}$; in particular, $H_n \in \mathcal{P}^\infty(\Lambda\text{-mod})$ for $n \in \mathbb{N}$.

Let $\Gamma_1$ be the quiver obtained from $\Gamma$ by adding a single vertex, labeled 9, and two arrows leaving 9, namely $\chi_1 : 9 \to 5$ and $\chi_2 : 9 \to 6$. Moreover, let $\Lambda_1 = K\Gamma_1/I_1$, where the ideal $I_1 \subseteq K\Gamma_1$ is generated by $I$ and the relation $\beta_1 \chi_1 - \beta_2 \chi_2$.

Next let $\Gamma_2$ be the quiver obtained from $\Gamma_1$ by adding a single vertex, 10, and two arrows leaving 10, namely $\psi_1 : 10 \to 7$ and $\psi_2 : 10 \to 8$. Now $\Delta = \Lambda_2 = K\Gamma_2/I_2$, where $I_2$ is generated by $I_1$ and the relation $\gamma_1 \psi_1 - \gamma_2 \psi_2 \in K\Gamma_2$.

Clearly, $\Lambda_1$ is a one-point extension of $\Lambda = \Lambda_0$, and $\Delta = \Lambda_2$ is a one-point extension of $\Lambda_1$. The ‘new’ indecomposable projective left $\Delta$-modules have graphs

Note that $\Delta e_i = \Lambda e_i$ for $1 \leq i \leq 8$.

One can verify that $\mathcal{P}^\infty(\Delta\text{-mod})$ is contravariantly finite in $\Delta\text{-mod}$ by exhibiting right $\mathcal{P}^\infty(\Delta\text{-mod})$-approximations of the simple left $\Delta$-modules $S_i = \Delta e_i/J(\Delta)e_i$ for $1 \leq i \leq 10$. It is comparatively easy to see that the following are the (minimal) right $\mathcal{P}^\infty(\Delta\text{-mod})$-approximations of $S_2, \ldots, S_{10}$: Namely, $\Delta e_i$ for $i = 2, 3, 4,$ and
for $i = 5, 6, \ldots, 10$, respectively. We will sketch an argument backing the claim that the canonical epimorphism $f : A_1 \rightarrow S_1$ with

$$A_1 = (\Delta e_1 \oplus \Delta e_9 \oplus \Delta e_{10})/\Delta(\alpha, \beta_1 \chi_1, \gamma_1 \psi_1)$$

is a right $\mathcal{P}^\infty(\Delta\text{-mod})$-approximation of $S_1$. To buttress intuition, start by noting that $A_1$ has graph

with the above convention for loops. Observe, moreover, that $\Omega^1_{\Delta}(A_1) \cong \Delta e_2$, whence $A_1 \in \mathcal{P}^\infty(\Delta\text{-mod})$.

We will sketch an argument showing that each epimorphism $g : M \rightarrow S_1$ with $M \in \mathcal{P}^\infty(\Delta\text{-mod})$ factors through $f$. It is clearly harmless to assume that $M$ is indecomposable. Moreover, it suffices to consider the case where $\ker(g)$ does not contain any nonzero submodules in $\mathcal{P}^\infty(\Delta\text{-mod})$; indeed, given $U \subseteq M$ with $U \in \mathcal{P}^\infty(\Delta\text{-mod})$, it is enough to factor the induced map $\overline{g} : M/U \rightarrow S_1$ through $f$. In particular, this means that $M$ does not contain any submodules isomorphic to $\Delta e_i$, with $i \geq 2$, nor any submodules with graphs of type
As a consequence, we can zero in on the structure of $M$ as follows: Let $\mathcal{A}$ be the class of $\Delta$-modules isomorphic to $\Delta e_1$, $B$ the class of modules isomorphic to one of the $\Delta$-modules with graphs

and, finally, $C$ the class of those $\Delta$-modules which have one of the graphs

Our assumptions on $M$ guarantee that, up to isomorphism, $M = X/Y \in P^\infty(\Delta\text{-mod})$, where

$$X = \bigoplus_{1 \leq i \leq m_1} A_i \oplus \bigoplus_{1 \leq i \leq m_2} B_i \oplus \bigoplus_{1 \leq i \leq m_3} C_i$$

with $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$, $C_i \in \mathcal{C}$ and $Y \subseteq \text{Soc } X \cong S_{m_1+m_2+m_3}^2$.

Let $a_i$, $b_i$, $c_i$ be top elements of $A_i$, $B_i$, $C_i$, respectively, and let $a'_i = \alpha a_i$, $b'_i = (\beta_1 + \beta_2 + \beta_1 \chi_1) b_i$, $c'_i = (\gamma_1 + \gamma_2 + \gamma_1 \psi_1) c_i$. Moreover, let

be the obvious projective cover of $X/Y$ mapping $\overline{a}_i$, $\overline{b}_i$, $\overline{c}_i$ to $a_i$, $b_i$, $c_i$ respectively. Then

$$\text{Soc } X = \bigoplus_{1 \leq i \leq m_1} \Delta a'_i \oplus \bigoplus_{1 \leq i \leq m_2} \Delta b'_i \oplus \bigoplus_{1 \leq i \leq m_3} \Delta c'_i,$$

and we can therefore write $Y$ in the form $\bigoplus_{1 \leq h \leq t} \Delta y_h$ with

$$y_h = \sum_{1 \leq i \leq m_1} k_{hi} a'_i + \sum_{1 \leq i \leq m_2} l_{hi} b'_i + \sum_{1 \leq i \leq m_3} m_{hi} c'_i,$$
where \( k_{hi}, l_{hi}, m_{hi} \in K \) such that \( \Delta y_h \cong \Delta e_2 = \Lambda e_2 \) for each \( h \). Let \( k_h = (k_{hi})_{1 \leq i \leq m_1} \in K^{m_1}, \ l_h = (l_{hi})_{1 \leq i \leq m_2} \in K^{m_2}, \) and \( m_h = (m_{hi})_{1 \leq i \leq m_3} \) in \( K^{m_3} \). Then the vectors \( k_h, l_h, m_h \) are \( K \)-linearly independent; indeed if we had \( \sum_{1 \leq h \leq t} d_h l_h = 0 \) with \( d_h \in K \) not all zero, we would obtain a top element \( z \) of \( \Omega_1(\Delta(M)) \) with the property that \( \delta z = 0 \), namely \( z = \sum_{1 \leq h \leq t} d_h z_h \), where

\[
  z_h = \sum_{1 \leq i \leq m_1} k_{hi} \alpha_i + \sum_{1 \leq i \leq m_2} l_{hi} (\beta_1 + \beta_2 + \beta_1 \chi_1) \tilde{\tau}_i + \sum_{1 \leq i \leq m_3} m_{hi} (\gamma_1 + \gamma_2 + \gamma_1 \psi_1) \tau_i
\]

in \( \Omega_1(\Delta(M)) \). This would place a direct summand isomorphic to \( S_3 \) into \( \Omega_1(\Delta(M)) \), which – in view of \( \text{p dim}_\Delta \Delta e_3/J(\Delta e_3) = \text{p dim} \Lambda e_3/J e_3 = \infty \) – is incompatible with \( \text{p dim}_\Delta(M) < \infty \). Similarly \( m_1, \ldots, m_t \) in \( K^{m_3} \) are linearly independent, since otherwise we would obtain a direct summand isomorphic to \( S_4 \) in \( \Omega_1(\Delta(M)) \). If we set \( \tilde{g}(a_i) = (r_i e_1, 0, 0) \in A_1 \), where \( f(a_i) = \tau_i e_1 \) with \( r_i \in K \), the above independence information allows us to extend the assignment \( \tilde{g} \) to a homomorphism \( \tilde{g} : M \to A \). Any such homomorphism clearly satisfies \( f \circ \tilde{g} = g \).

Now let \( \Lambda = R_0, R_1, \ldots, R_m = R \) be successive one-point extensions such that \( R \) is a monomial relation algebra. We leave the justification of our claim that \( P\infty(R\text{-mod}) \) fails to be contravariantly finite in \( R\text{-mod} \) as an exercise, but provide hints of the underlying ideas.

1.) Due to the fact that \( R \) is a monomial relation algebra obtained from \( \Lambda \) via one-point extensions, the following holds: Whenever a left \( R \)-module \( M \) has a submodule \( N \) with graph

\[
  \begin{array}{c}
    e(1) & \quad & e(2) \\
    2 & \quad & 2 \\
  \end{array}
\]

or

\[
  \begin{array}{c}
    e(1) & \quad & e(2) & \quad & e(3) \\
    2 & \quad & 2 & \quad & 2 \\
  \end{array}
\]

where \( e(1), e(2), e(3) \) are distinct vertices in \( \{e_1, e_5, e_6, e_7, e_8\} \), the first syzygy \( \Omega_1(M) \) of \( M \) has a top element \( x \) of type \( e_2 \); moreover, if \( e(1), e(2), e(3) \) belong to \( \{e_1, e_5, e_6\} \), we can choose \( x \) such that \( Rx \) has graph

\[
  \begin{array}{c}
    2 \\
    \bullet \\
    2 \\
  \end{array}
\]

or

\[
  \begin{array}{c}
    3 \\
    \bullet \\
  \end{array}
\]

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and if $e(1), e(2), e(3) \in \{e_1, e_7, e_8\}$, we can choose $x$ such that $Rx$ has graph

2
\hspace{1cm} \bullet
\hspace{1cm} 2

or

4

In each of these two situations, $\text{pdim}_R(M) = \infty$.

2.) If there were a $\mathcal{P}^\infty(R\text{-mod})$-approximation $B_1$ of $Re_1/J(R)e_1 = \Lambda e_1/Je_1 = S_1$, then all homomorphisms in $\text{Hom}_R(H_n, S_1) = \text{Hom}_\Lambda(H_n, S_1)$ with $H_n$ as above would factor through $B_1$, because $H_n \in \mathcal{P}^\infty(R\text{-mod})$. Using the first part, one would deduce the existence of a submodule of $B_1$ with graph

1\hspace{0.5cm}5\hspace{0.5cm}7\hspace{0.5cm}6\hspace{0.5cm}8
\hspace{1cm}2\hspace{1cm}2\hspace{1cm}2\hspace{1cm}2

and then proceed to show that $H = \lim H_n$ would still be a $\mathcal{P}^\infty(R\text{-mod})$-phantom for $S_1$.

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