Nonperturbative Aspect In $\mathcal{N} = 2$ Supersymmetric Noncommutative Yang-Mills Theory

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Abstract

We investigate asymptotic behaviors of the strong coupling limit in the $\mathcal{N} = 2$ supersymmetric non-commutative Yang-Mills theory. The strong coupling behavior is quite different from the commutative one since the non-commutative dual $U(1)$ theory is asymptotic free, although the monodromy is the same as that of the ordinary theory. Singularities are produced by infinitely heavy monopoles and dyons. Nonperturbative corrections may be determined by holomorphy.
1 S-duality of Non-commutative $U(1)$ gauge theory

In this note we consider the non-perturbative aspect of non-commutative Yang-Mills theory making good use of supersymmetry and duality. We also use perturbative analyses done in Refs.\cite{1, 2, 3}. Strong coupling region can be analyzed by using duality. Holomorphy puts severe constraints on the functional form of the prepotential. We show a strong coupling behavior which is in contrast with the analysis in Ref.\cite{4}. We would like to determine the low energy coupling constant and the $\theta$ parameter.

First, we review the S-duality of non-commutative $U(1)$ gauge theory.\cite{5, 6} In Ref.\cite{5} they give a field redefinition between fields in ordinary theory and in non-commutative theory. Let us suppose the lagrangian of the non-commutative $U(1)$ gauge theory. We denote the field of the non-commutative theory by putting hat like $\hat{A}_\mu$. We perform a field redifinition from the gauge field $\hat{A}_\mu$ to the ordinary one $A_\mu$ according to Ref.\cite{5}. In order to perform S-duality we introduce (dual) auxiliary field $B_\mu$ for imposing the Bianchi identity $dF \equiv 0$. Then, eliminating the field strength $F$ and performing a renormalization transformation from $B_\mu$ back to $\hat{B}_\mu$, we end up with a non-commutative $U(1)$ gauge theory. This is a dual description of the initial non-commutative theory. In this description we find the S-duality relation\cite{5}

$$g_D = 1/g, \quad \theta_{Dij} = -\frac{g^2}{2} \epsilon_{ijkl} \theta_{kl}.$$  \hspace{1cm} (1.1)

This argument holds in order by order of $\theta$, and there is no full order treatment for the non-commutative S-duality. In this paper we assume, or believe, the relevance of this S-duality in quantum field theory.

In Ref.\cite{6} it is conjectured that strongly coupled spatially non-commutative $\mathcal{N} = 4$ Yang-Mills theory is dual to a weakly coupled non-commutative open string (NCOS) theory. One may think that this duality will hold for the $\mathcal{N} = 2$ theory after some modification of the theory. In the NCOS theory the effective Regge slope parameter is given by the non-commutative parameter as $\alpha'_{\text{eff}} = \theta/(2\pi)$. Then, the NCOS theory reduces to a non-commutative Yang-Mills theory when the Yang-Mills non-perturavtive scale $\Lambda$ is smaller than $1/\alpha'_{\text{eff}}$. In this situation the above S-dualities\cite{5, 6} will be essentially the same.
2 The non-commutative Yang-Mills Theory

We consider the $\mathcal{N} = 2$ non-commutative Yang-Mills theory with the gauge group $G = U(2)$, described by a vector multiplet. The $\mathcal{N} = 2$ vector multiplet contains gauge field $A_\mu$, two Weyl fermions $\lambda_\alpha, \psi_\alpha$ and a complex scalar $\phi$, all in the $U(2)$ adjoint representation.

In $\mathcal{N} = 1$ superspace, the lagrangian is

$$g^2 L = \frac{1}{4} \left[ \int d^2 \theta W^\alpha W_\alpha + \text{h.c.} \right] + \int d^4 \theta \Phi e^{-2V} \Phi e^{2V} - \frac{1}{4} F_{mn} F^{mn} - \frac{1}{2} D_m \phi D^m \phi - \frac{1}{2} \left[ \phi, \phi \right] + \frac{1}{\sqrt{2}} \epsilon_{ij} \left( \phi [\lambda^i, \lambda^j] + \text{h.c.} \right),$$

where all the products are non-commutative $\ast$-products, we suppress the trace over the gauge group $U(2)$ and define $\lambda^i = (\lambda, \psi)$, $\lambda \psi \equiv \lambda^\alpha \psi_\alpha$. The $U(2)$ generators are $T^a = \tau^a / 2$ ($a = 1, 2, 3$) and $T^0 = 1/2$, where $\tau^a$ are the Pauli matrices.

The vacuum is determined by the condition $[\phi, \phi] = 0$. The vacuum expectation value (VEV) $\langle \phi \rangle$ belongs to the Cartan subalgebra $U(1) \times U(1)$ of the $U(2)$. The diagonal part $U(1)_0$ of the $U(2)$ is not broken, since the diagonal part $U(1)_0$ does not act on the VEV $\langle \phi \rangle$. The $SU(2)$ part is broken down to $U(1)_3$, due to the VEV $\langle \phi \rangle$. Then, the gauge symmetry becomes $U(1)_0 \times U(1)_3$ in the low energy region.

The low energy theory is described by a prepotential $F$ which is a holomorphic function as explained in [8]. The low energy effective lagrangian is

$$L_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[ \int d^4 \theta \frac{\partial F}{\partial A^i} A^i + \int d^2 \theta \frac{1}{2} \frac{\partial^2 F}{\partial A^i \partial A^j} W^i W^j \right],$$

where $i, j = 0, 3$. The low energy symmetry $U(1)_0 \times U(1)_3$ does not factor, because of the non-commutativity of the $\ast$-product. The prepotential depends on the components $A^0, A^3$ only through the form $\sqrt{(A^3)^2 + (A^0)^2}$. This is seen easily as follows. The gauge fields of the low energy theory are $A^i = T^i A^i + T^0 A^0$, and the $U(1)_0 \times U(1)_3$ gauge transformations are

$$\delta A_\mu^0 = \partial_\mu A^0 + \frac{1}{2} [A^0_\mu, A^0]_\ast + \frac{1}{2} [A^0_\mu, A^3]_\ast,$$

$$\delta A_\mu^3 = \partial_\mu A^3 + \frac{1}{2} [A^3_\mu, A^3]_\ast + \frac{1}{2} [A^0_\mu, A^0]_\ast.$$  

These two gauge fields are related by this symmetry. These two, and as well as the weak bosons, are put together into the prepotential by virtue of the non-commutative $U(2)$ symmetry. This way of determining a general form of $F$ is essentially described in Ref.[8].
Then, it is enough to consider the $U(1)_3$ part with putting $A^0 = 0$ formally in the effective lagrangian to determine the prepotential. Otherwise we consider the functional form of $F$ in $a = \sqrt{(A^3)^2 + (A^0)^2}$. In the ordinary case the non-perturbative contributions are only due to the anti-self-dual instantons. However, in the non-commutative case, there are also $U(1)$ instantons\cite{10}. These two types of instantons produce non-perturbative corrections, contribute to the prepotential and their roles are very symmetric.

The one-loop contribution to the coupling constant is related to the $U(1)_R$ anomaly by the supersymmetry. The one-loop part of the prepotential contains a logarithmic dependence $\sim a^2 \ln a^2/\Lambda^2$ to reproduce the one-loop $\beta$-function. The $U(1)_R$ phase rotation reveals the anomaly from the prepotential automatically. In this case, the anomaly term takes the $F \wedge F$ form with the $\ast$-product. Then, the $U(1)_R$ is broken to $Z_8$ and for non-zero $\langle \phi \rangle$ the $Z_8$ symmetry is broken to $Z_4$ which acts trivially on the moduli space parameterized by $u$. The discrete symmetry $Z_2 = Z_8/Z_4$ acts on the moduli space by $u \to -u$ as a spontaneously broken symmetry.

3 Asymptotic Behavior

First, let us determine the weak coupling behavior of the non-commutative $U(2)$ theory at large $a$. The non-commutative theory has ultraviolet divergence and its one-loop $\beta$ function has contributions from planar diagrams only\cite{7}, that is the same situation as the commutative theory. The $\beta$-function is $\beta \equiv \mu \frac{d\alpha}{d\mu} = -\frac{g^3}{16\pi^2} 2N_c$ with $N_c = 2$.

As usual, let us combine the coupling constant and theta parameter in the form $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. The low energy values of $\tau$ are related to the prepotential. The effective coupling, which we denote as $\tau(a)$, is parameterized by $a$ and is given by $\tau(a) = \frac{\partial a}{\partial a} = \frac{\partial^2 F}{\partial^2 a}$. We integrate the above formula obtaining

$$ a_D = \frac{2i}{\pi} (a \ln a + a) + \cdots, \quad a = \sqrt{2u} + \cdots, \quad (3.1) $$

at $u \sim \infty$. Let us circle around the infinity $u \to u e^{2\pi i}$ ($u \sim \infty$), then we obtain the same monodromy

$$ M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \quad (3.2) $$

as that of the ordinary $SU(2)$ Yang-Mills theory. This result is obvious since only planar diagrams contribute to the $\beta$-function in both cases.
Next, let us show the strong coupling behavior of the theory. Take the S-duality transformation to the $U(1)_0 \times U(1)_3$ non-commutative gauge theory. The S-dual of the low energy theory will be the $U(1) \times U(1)$ gauge theory with one hypermultiplet. In the classical and $\theta \to 0$ limit this hypermultiplet is the ’tHooft-Polyakov monopole. The classical monopole solution is derived in order by order of $\theta$ in Refs.\[11, 12\]. The perturbative behavior of the dual coupling constant $\tau_D$ is determined by this dual non-commutative gauge theory. Calculating the $\beta$-function of the non-commutative $U(1)$ theory with one hypermultiplet, we have

$$\tau_D = \frac{i\beta_0}{2\pi} \ln a_D + \cdots = -\frac{\partial a}{\partial a_D},$$

(3.3)

where $\beta_0 = 2$ is a coefficient of the one-loop $\beta$-function. This is integrated to be

$$a = \frac{i\beta_0}{2\pi}(a_D \ln a_D - a_D) + \cdots.$$

(3.4)

Since the behavior of $a_D$ is not known, we assume the form

$$a_D = c_0 (u - 1)^k$$

(3.5)

with some constants $c_0$ and $k$. We can determine the value of $k$ by a $Z_2$ symmetry of the moduli space. Thus, the monodromy around $u - 1 \to (u - 1)e^{2\pi i}$ is

$$M_{+1} = e^{2\pi ik} \begin{pmatrix} 1 & 0 \\ k\beta_0 & 1 \end{pmatrix}.$$

(3.6)

Since the monodromies must obey $M_{+1}M_{-1} = M_\infty$, we obtain

$$M_{-1} = e^{-2\pi ik} \begin{pmatrix} -1 & 2 \\ k\beta_0 & -(2k\beta_0 + 1) \end{pmatrix}.$$

(3.7)

Now, let us determine the value of $k$. Electric/magnetic charge $(n_m, n_e)$ transforms under a monodromy transformation by $(n_m, n_e) \to M^{-1}(n_m, n_e)$. Since the charges must be integer, $M^{-1}$ should be a integer-valued matrix. Then, $k$ is a integer or half-integer. Next, we expect a $Z_2$ symmetry which interchanges the singularities $u = \pm 1$. This symmetry implies a similarity transformation between the two monodromies as $M_{-1} = AM_{+1}A^{-1}$ with some matrix $A$. This similarity condition determines $k$ to be

$$k = -1.$$

(3.8)

Thus, we determine the three monodromies

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_{+1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix},$$

(3.9)
which turn out to be exactly the same as those of the ordinary SU(2) Yang-Mills theory. We should notice that, although we obtain the same monodromies, singular behaviors of the non-commutative theory are completely different. At \( u \sim 1 \) the VEV \( a_D \) diverges as \( a_D \sim 1/(u-1) \) which means that monopoles get infinitely heavy mass. Asymptotic freedom and \( a_D \sim \infty \) means that the dual theory is in a weakly coupled phase.

4 Seiberg-Witten differential

Supersymmetry implies that the section \((a, a_D)\) is a holomorphic function of \( u \). Even in the case of the concerned non-commutative theory, we cannot help supposing that the section may be given in terms of a elliptic curve. The curve should have singularities at \( u = \infty, \pm 1 \). Thus, one of the candidates will be a torus of the form

\[
y^2 = (x^2 - 1)(x - u). \tag{4.1}
\]

This is exactly the same as that in Ref.\[9\]. Then, the significant difference from the ordinary theory should be implemented in the Seiberg-Witten differential one-form \( \lambda \). Now, the section is given by

\[
a_D = \oint_{\beta} \lambda, \quad a = \oint_{\alpha} \lambda. \tag{4.2}
\]

In what follows we would like to find the condition which should be satisfied by the one-form \( \lambda \) for the non-commutative theory.

Around a singularity any curve looks like a cylinder of the form

\[
X_z : x^2 + y^2 = z \tag{4.3}
\]

up to changing variables where the parameter \( z \) is a function of \( u \). In the following explanation we denote a vanishing cycle by \( \alpha' \) and other non-vanishing cycle by \( \beta' \). Let us consider \( \alpha' \) winding on the above cylinder and \( \beta' \) intersecting with \( \alpha' \). We choose a particular intersection of the cycles \( \alpha \) and \( \beta \). When the moduli parameter \( z \) circles around the origin \( z \rightarrow ze^{2\pi i} \), we have the Picard-Lefshetz formula

\[
\alpha' \rightarrow \alpha', \quad \beta' \rightarrow \beta' - \alpha', \tag{4.4}
\]

which means the curve is twisted one time around the \( A_1 \) singularity.

Here we briefly describe the above result. The parametrization \( r \equiv x + iy \) allows us to find a identification between \( X_z \) and a cylinder \( \mathbb{C}^* \). The Milnor fibers \( M_{\pm} \) are
defined as a upper part and lower part of the region satisfied by \(|x|^2 + |y|^2 > \rho\) for some \(\rho > |z|^2\). The allowed regions of \(r\) are \(0 < |r|^2 < R_+ \equiv \sqrt{\rho - \sqrt{\rho^2 - |z|^2}}\) for \(M_-\) and \(\sqrt{\rho + \sqrt{\rho^2 - |z|^2}} \equiv R_- < |r|^2 < \infty\) for \(M_+\), respectively. Then, the mappings, \(C^* \rightarrow M_{\pm}\), are \(r = R_+ \exp(u + i\theta)\) and \(r = R_- \exp(-u - i\theta)\), respectively for \(M_+\) and \(M_-\), where \(u > 0, \theta \in [0, 2\pi]\). Now, let us circle around the origin \(z \rightarrow ze^{2\pi i}\). We find \(r \rightarrow re^{\pi i}\), and then \(M_{\pm}\) rotates by \(\pm \pi\) in opposite direction with each other. This result leads to the Picard-Lefshetz formula (4.4).

Let us adopt this formula to the curve (4.1). At \(u \sim \infty\) the cycle \(\alpha' = \alpha\) is vanishing. Near the singularity the curve takes the form (4.3) with \(z = u^{-2}\) up to changing the variables as \(x \rightarrow ux, y \rightarrow u^{3/2}y\). Thus, when we circle around the infinity \(u \sim \infty\), the curve is twisted \(-2\) times around the singularity. So, we have

\[
\alpha \rightarrow \alpha, \quad \beta \rightarrow \beta - 2\alpha.
\]  

In order to reproduce the monodromy \(M_\infty\) the Siberg-Witten one-form has to transform under this encircling as \(\lambda \rightarrow -\lambda\). This means that \(\lambda\) has a square root singularity at \(u \sim \infty\). In the infinity \(u \sim \infty\) the \(\alpha\)-cycle shrinks (seen from the eyes after changing variables), while the section \(a\) diverges as \(a \sim \sqrt{2u}\). Then, \(\lambda\) have to diverge as \(\lambda \sim \sqrt{u}\). This singular behavior is the same as that of the ordinary case of the Seiberg-Witten differential \(\lambda = \sqrt{\frac{u-x}{1-x}}dx\), where \(x \sim O(1)\).

In the case \(u \sim 1\) the cycle \(\alpha' = \beta\) vanishes, and the section \(a_D\) diverges as \(a_D \sim 1/(u - 1)\) since \(k = -1\). We find \(z = -(u - 1)^2/2\) and then

\[
\beta \rightarrow \beta, \quad \alpha \rightarrow \alpha - 2\beta,
\]  

where we take account of the effect of flipping the intersection between \(\alpha'\) and \(\beta'\). Comparing this with the monodromy matrix \(M_{+1}\) in eqs. (3.9), we obtain \(\lambda \rightarrow \lambda\) for \(u - 1 \rightarrow (u - 1)e^{2\pi i}\). Then, the one-form \(\lambda\) have to behave as \(\lambda \sim 1/(u - 1)^2\) without fractional power.

In the ordinary case \(\lambda\) has a singularity only at \(u \sim \infty\), whereas in the present case \(\lambda\) has three singularities at \(u \sim \infty, \pm 1\). We can learn from these situations that when a perturbative picture appears with asymptotic freedom around a point of the moduli space, the Seiberg-Witten one-form \(\lambda\) becomes singular at that point.

Up to now we do not have found a way for completely determine the form of \(\lambda\) yet.

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