The Information Criterion GIC of the Trend and Seasonal Adjustment Models

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Abstract

This paper presents an algorithm for computing the GIC and the TIC of the nonstationary state-space models. The gradient and Hessian of the log-likelihood necessary in computing the GIC are obtained by the differential filter that is obtained by extending the Kalman filter. Three examples of the nonstationary time series models, i.e., the trend model, standard seasonal adjustment model and the seasonal adjustment model with stationary AR component are presented to exemplify the specification of structural matrices.

Key words Differential filter, log-likelihood, State-space model, seasonal adjustment model, Kalman filter, gradient, Hessian matrix.

1 Introduction: The Maximum Likelihood Estimation of a State-Space Model

We consider a linear Gaussian state-space model

\[
\begin{align*}
x_n &= F_n(\theta)x_{n-1} + G_n(\theta)v_n \quad (1) \\
y_n &= H_n(\theta)x_n + w_n \quad (2)
\end{align*}
\]

where \(y_n\) is a one-dimensional time series, \(x_n\) is an \(m\)-dimensional state vector, \(v_n\) is a \(k\)-dimensional Gaussian white noise, \(v_n \sim N(0, Q_n(\theta))\), and \(w_n\) is one-dimensional white noise, \(w_n \sim N(0, R_n(\theta))\). \(F_n(\theta), G_n(\theta)\) and \(H_n(\theta)\) are \(m \times m\) matrix, \(m \times k\) matrix and \(m\) vector, respectively. \(\theta\) is the \(p\)-dimensional parameter vector of the state-space model such as the variances of the noise inputs and unknown coefficients in the matrices \(F_n(\theta), G_n(\theta), H_n(\theta), Q_n(\theta)\) and \(R_n(\theta)\). For simplicity of the notation, hereafter, the parameter \(\theta\) and the suffix \(n\) will be omitted.

Various models used in time series analysis, such as the stationary AR and ARMA models, and various nonstationary models including trend model and the seasonal adjustment model, can be treated uniformly within the state-space model framework. Further, many problems of time series analysis, such as prediction, signal extraction, decomposition, parameter estimation and interpolation, can be formulated as the state estimation of a state-space model.

Given the time series \(Y_N \equiv \{y_1, \ldots, y_N\}\) and the state-space model (1) and (2), the one-step-ahead predictor \(x_{n|n-1}\) and the filter \(x_{n|n}\) and their variance covariance matrices \(V_{n|n-1}\) and \(V_{n|n}\) are obtained by the following Kalman filter (Anderson and Moore (2012) and Kitagawa (2020)):

One-step-ahead prediction

\[
\begin{align*}
x_{n|n-1} &= Fx_{n-1|n-1} \\
V_{n|n-1} &= FV_{n-1|n-1}F^T + GQ_nG^T
\end{align*}
\]
Filter

\[ K_n = V_{n|n-1}H^T(HV_{n|n-1}H^T + R)^{-1} \]
\[ x_{n|n} = x_{n|n-1} + K_n(y_n - Hx_{n|n-1}) \]
\[ V_{n|n} = (I - K_nH)V_{n|n-1}. \] (4)

Given the data \( Y_n \), the likelihood of the time series model is defined by

\[ L(\theta) = p(Y_N|\theta) = \prod_{n=1}^{N} g_n(y_n|Y_{n-1}, \theta), \] (5)

where \( g_n(y_n|Y_{n-1}, \theta) \) is the conditional distribution of \( y_n \) given the observation \( Y_{n-1} \) and is a normal distribution given by

\[ g_n(y_n|Y_{n-1}, \theta) = \frac{1}{\sqrt{2\pi r_n}} \exp\left\{ -\frac{\varepsilon_n^2}{2r_n} \right\}, \] (6)

where \( \varepsilon_n \) and \( r_n \) are the one-step-ahead prediction error and its variance defined by

\[ \varepsilon_n = y_n - Hx_{n|n-1} \]
\[ r_n = H_nV_{n|n-1}H_n^T + R. \] (7)

Therefore, the log-likelihood of the state-space model is obtained as

\[ \ell(\theta) = \log L(\theta) = \sum_{n=1}^{N} \log g_n(y_n|Y_{n-1}, \theta) \]
\[ = -\frac{1}{2} \left\{ N \log 2\pi + \sum_{n=1}^{N} \frac{\varepsilon_n^2}{r_n} + \sum_{n=1}^{N} \log r_n \right\}. \] (8)

The maximum likelihood estimates of the parameters of the state-space model can be obtained by maximizing the log-likelihood function. In general, since the log-likelihood function is mostly nonlinear, the maximum likelihood estimates is obtained by using a numerical optimization algorithm based on the quasi-Newton method. According to this method, using the value \( \ell(\theta) \) of the log-likelihood and the first derivative (gradient) \( \partial \ell/\partial \theta \) for a given parameter \( \theta \), the maximizer of \( \ell(\theta) \) is automatically estimated by repeating

\[ \theta_k = \theta_{k-1} + \lambda_k B_{k-1}^{-1} \frac{\partial \ell}{\partial \theta}, \] (9)

where \( \theta_0 \) is an initial estimate of the parameter. The step width \( \lambda_k \) is automatically determined and the inverse matrix \( H_{k-1}^{-1} \) of the Hessian matrix is obtained recursively by the DFP or BFGS algorithms (Fletcher (2013)).

Here, the gradient of the log-likelihood function is usually approximated by numerical difference, such as

\[ \frac{\partial \ell(\theta)}{\partial \theta_j} \approx \frac{\ell(\theta_j + \Delta \theta_j) - \ell(\theta_j \Delta \theta_j)}{2\Delta \theta_j}, \] (10)

where \( \Delta \theta_j \) is defined by \( C|\theta_j| \), for some small \( C \) such as 0.0001. The numerical difference usually yields reasonable approximation to the gradient of the log-likelihood. However, since it requires
2p times of log-likelihood evaluations, the amount of computation becomes considerable if the dimension of the parameters is large. Further, if the maximum likelihood estimates lie very close to the boundary of admissible domain, which sometimes occur in regularization problems, it becomes difficult to obtain the approximation to the gradient of the log-likelihood by the numerical difference.

Analytic derivative of the log-likelihood of time series models were considered by many authors. For example, Kohn and Ansley (1985) gave method for computing likelihood and its derivatives for an ARMA model. Zadrozny (1989) derived analytic derivatives for estimation of linear dynamic models. Kulikova (2009) presented square-root algorithm for the likelihood gradient evaluation to avoid numerical instability of the recursive algorithm for log-likelihood computation. In Kitagawa (2021, 2022), algorithms for computing the gradient and Hessian of the log-likelihood of linear state-space model are given. Details of the implementation of the algorithm for standard seasonal adjustment model, seasonal adjustment model with stationary AR component and ARMA model are given. For each implementation, comparison with a numerical difference method is shown.

In the state-space modeling of time series, the evaluation of the estimated models is important. For that purpose, the information criterion AIC is a standard method for the model evaluation and model comparison. The information criterion GIC was derived as a general model evaluation criterion which can be applied to not only the models whose parameters are estimated by the maximum likelihood method, but also a broad class of estimators defined by statistical functionals, that covers various types of regularization methods and some type of Bayesian models (Konishi and Kitagawa 1996, 2008). Although GIC is theoretically appealing, the difficulty with the GIC in applying the time series model is that it is difficult to compute the GIC since it is necessary to compute the Hessian of the log-likelihood. In this paper, we show the application of the differential filter for computing the GIC of the time series models that can be expressed by a state-space model.

In section 2, we briefly present the differential filter to obtain the gradient and the Hessian of the log-likelihood of the state-space model (Kitagawa (2021)). The formulas for computing the Fisher information matrix and the Hessian of the log-likelihood that are necessary for computing the GIC are also explained in this section. In section 3, three nonstationary time series models, the trend model, the standard seasonal adjustment model and the seasonal adjustment model with stationary AR component are shown to exemplify the model. In the application of the differential filter, it is necessary to specify the first and second order derivative of the structural parameters of the state-space models, i.e., the matrices $F$, $G$, $H$, $Q$ and $R$. In the examples, it will be shown that the most of these terms zero or at least very sparse that makes the computation of the gradient filter rather simple.

2 The Gradient and the Hessian of the log-likelihood

2.1 The gradient of the log-likelihood

The general formula for the differential filter for computing the gradient and the Hessian of the log-likelihood is very complex (Kitagawa 2021, 2022). However, for the time series models considered in section 3, the derivatives of the matrix $F$, $G$, $H$, $Q$ and $R$ satisfy

$$
\frac{\partial^2 F}{\partial \theta \partial \theta^T} = 0, \quad \frac{\partial G}{\partial \theta} = 0, \quad \frac{\partial^2 G}{\partial \theta \partial \theta^T} = 0, \quad \frac{\partial H}{\partial \theta} = 0, \quad \frac{\partial^2 H}{\partial \theta \partial \theta^T} = 0,
$$

where the zero in the right hand side is the zero matrix that makes the algorithm of the differential filter fairly simple. Further, if $\frac{\partial F}{\partial \theta} = 0$ as is the cases for examples shown in subsection 3.1 and
3.2, in the following algorithms, the terms written in red disappear, which make the computation much simpler.

From (8), the gradient of the log-likelihood is obtained by

\[
\frac{\partial \ell(\theta)}{\partial \theta} = -\frac{1}{2} \sum_{n=1}^{N} \left( \frac{1}{r_n} \frac{\partial r_n}{\partial \theta} + 2 \frac{\varepsilon_n}{r_n} \frac{\partial \varepsilon_n}{\partial \theta} - \frac{\varepsilon_n^2}{r_n^2} \frac{\partial r_n}{\partial \theta} \right),
\]

where, from (7), the derivatives of the one-step-ahead prediction \(\varepsilon_n\) and the one-step-ahead prediction error variance \(r_n\) are obtained by

\[
\frac{\partial \varepsilon_n}{\partial \theta} = -H \frac{\partial x_{n|n-1}}{\partial \theta},
\]

\[
\frac{\partial r_n}{\partial \theta} = H \frac{\partial V_{n|n-1}}{\partial \theta} H^T + \frac{\partial R}{\partial \theta}. \tag{13}
\]

To evaluate these quantity, we need the derivative of the one-step-ahead predictor of the state \(\frac{\partial x_{n|n-1}}{\partial \theta}\) and its variance covariance matrix \(\frac{\partial V_{n|n-1}}{\partial \theta}\) which can be obtained recursively in parallel to the Kalman filter algorithm:

**[One-step-ahead-prediction]**

\[
\frac{\partial x_{n|n-1}}{\partial \theta} = F \frac{\partial x_{n-1|n-1}}{\partial \theta} + \frac{\partial F}{\partial \theta} x_{n-1|n-1},
\]

\[
\frac{\partial V_{n|n-1}}{\partial \theta} = F \frac{\partial V_{n-1|n-1}}{\partial \theta} F^T + \frac{\partial F}{\partial \theta} V_{n-1|n-1} F^T + F V_{n-1|n-1} \frac{\partial F^T}{\partial \theta} + G \frac{\partial Q}{\partial \theta} G^T. \tag{14}
\]

**[Filter]**

\[
\frac{\partial K_n}{\partial \theta} = \frac{\partial V_{n|n-1}}{\partial \theta} H^T r_n^{-1} - V_{n|n-1} H^T \frac{\partial r_n}{\partial \theta} r_n^{-2},
\]

\[
\frac{\partial x_n}{\partial \theta} = \frac{\partial x_{n|n-1}}{\partial \theta} + K_n \frac{\partial \varepsilon_n}{\partial \theta} + \frac{\partial K_n}{\partial \theta} \varepsilon_n,
\]

\[
\frac{\partial V_n}{\partial \theta} = \frac{\partial V_{n|n-1}}{\partial \theta} - \frac{\partial K_n}{\partial \theta} H V_{n|n-1} - K_n H \frac{\partial V_{n|n-1}}{\partial \theta}. \tag{15}
\]

### 2.2 Hessian of the Log-likelihood of the State-space Model

To compute the GIC of the model, it is necessary to obtain the Hessian (the second derivative) of the log-likelihood which can also be obtained by a recursive formula, since, from (11), it is given as

\[
\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T} = -\frac{1}{2} \sum_{n=1}^{N} \left\{ \frac{1}{r_n} \left( \frac{\partial^2 r_n}{\partial \theta \partial \theta^T} + 2 \frac{\partial \varepsilon_n}{\partial \theta} \frac{\partial \varepsilon_n}{\partial \theta^T} + 2 \varepsilon_n \frac{\partial^2 \varepsilon_n}{\partial \theta \partial \theta^T} \right) - \frac{1}{r_n^2} \left( \frac{\partial r_n}{\partial \theta} \frac{\partial r_n}{\partial \theta^T} + 2 \varepsilon_n \frac{\partial \varepsilon_n}{\partial \theta} \frac{\partial \varepsilon_n}{\partial \theta^T} + \varepsilon_n^2 \frac{\partial^2 \varepsilon_n}{\partial \theta \partial \theta^T} \right) \right\},
\]

where, from (13), \(\frac{\partial^2 \varepsilon_n}{\partial \theta \partial \theta^T}\) and \(\frac{\partial^2 r_n}{\partial \theta \partial \theta^T}\) are obtained by

\[
\frac{\partial^2 \varepsilon_n}{\partial \theta \partial \theta^T} = -H \frac{\partial^2 x_{n|n-1}}{\partial \theta \partial \theta^T},
\]

\[
\frac{\partial^2 r_n}{\partial \theta \partial \theta^T} = H \frac{\partial^2 V_{n|n-1}}{\partial \theta \partial \theta^T} H^T + \frac{\partial^2 R}{\partial \theta \partial \theta^T}. \tag{16}
\]
Therefore, to evaluate the Hessian, the following computation should be performed along with the recursive formula for the log-likelihood and the gradient of the log-likelihood.

\[
\frac{\partial^2 x_{n|n-1}}{\partial \theta \partial \theta^T} = F \frac{\partial^2 x_{n|n-1}}{\partial \theta \partial \theta^T} + \frac{\partial F}{\partial \theta} \frac{\partial x_{n|n-1}}{\partial \theta^T} + \frac{\partial F}{\partial \theta} \frac{\partial x_{n|n-1}}{\partial \theta^T} + \frac{\partial F}{\partial \theta} \frac{\partial x_{n|n-1}}{\partial \theta^T} + F \frac{\partial V_{n|n-1}}{\partial \theta \partial \theta^T}
\]

\[
\frac{\partial^2 V_{n|n-1}}{\partial \theta \partial \theta^T} = F \frac{\partial^2 V_{n|n-1}}{\partial \theta \partial \theta^T} + \frac{\partial F}{\partial \theta} \frac{\partial V_{n|n-1}}{\partial \theta^T} + \frac{\partial F}{\partial \theta} \frac{\partial V_{n|n-1}}{\partial \theta^T} + \frac{\partial F}{\partial \theta} \frac{\partial V_{n|n-1}}{\partial \theta^T} + F \frac{\partial V_{n|n-1}}{\partial \theta \partial \theta^T}
\]

\[
\frac{\partial^2 K_n}{\partial \theta \partial \theta^T} = \frac{\partial^2 V_{n|n-1}}{\partial \theta \partial \theta^T} H r_n^{-1} - \left( \frac{\partial V_{n|n-1}}{\partial \theta^T} H \frac{\partial r_n}{\partial \theta} + V_{n|n-1} H \frac{\partial^2 r_n}{\partial \theta \partial \theta^T} \right) r_n^{-2}
\]

\[
\frac{\partial^2 x_{n|n}}{\partial \theta \partial \theta^T} = \frac{\partial^2 x_{n|n-1}}{\partial \theta \partial \theta^T} + \frac{\partial K_n}{\partial \theta} \frac{\partial x_n}{\partial \theta^T} + \frac{\partial K_n}{\partial \theta} \frac{\partial x_n}{\partial \theta^T} + \frac{\partial K_n}{\partial \theta} \frac{\partial x_n}{\partial \theta^T} + \frac{\partial^2 K_n}{\partial \theta \partial \theta^T} \frac{\partial x_n}{\partial \theta^T}
\]

\[
\frac{\partial^2 V_{n|n}}{\partial \theta \partial \theta^T} = \frac{\partial^2 V_{n|n-1}}{\partial \theta \partial \theta^T} - \frac{\partial^2 K_n}{\partial \theta \partial \theta^T} H V_{n|n-1} - \frac{\partial K_n}{\partial \theta} H \frac{\partial V_{n|n-1}}{\partial \theta} - \frac{\partial K_n}{\partial \theta} H \frac{\partial V_{n|n-1}}{\partial \theta} - \frac{\partial K_n}{\partial \theta} H \frac{\partial V_{n|n-1}}{\partial \theta}
\]

2.3 Information Criteria GIC for the State-Space Model

The information criterion GIC \(^{10}\) for the state-space model is given by

\[
\text{GIC} = -2 \log L(\hat{\theta}) + 2 \text{tr} \left( I(\hat{\theta}) J(\hat{\theta})^{-1} \right)
\]

where \(\hat{\theta}\) is the maximum likelihood estimate of the parameter \(\theta\), and \(I(\hat{\theta})\) and \(J(\hat{\theta})\) are the Fisher information and negative of the Hessian defined by

\[
I(\hat{\theta}) = \frac{1}{N} \sum_{j=1}^{N} \frac{\partial \ell(\hat{\theta})}{\partial \theta} \frac{\partial \ell(\hat{\theta})}{\partial \theta^T} \bigg|_{\theta = \hat{\theta}}
\]

\[
J(\hat{\theta}) = -\frac{1}{N} \sum_{j=1}^{N} \frac{\partial^2 \ell(\hat{\theta})}{\partial \theta \partial \theta^T} \bigg|_{\theta = \hat{\theta}}
\]

These term can be obtained as the byproduct of the differential filter. Note that if the parameters are estimated by the maximum likelihood method, GIC is identical to the TIC \(^{16}\) \(^{18}\).

3 Examples

In order to implement the differential filter, it is necessary to to specify the first and the second derivatives of \(F, G, H, Q\) and \(R\) along with the original state-space model. In this section, we shall consider three typical cases. The first two examples are the trend model and the standard seasonal
adjustment model, for which three matrices (or vector), $F$, $G$ and $H$ do not contain unknown parameters and thus the derivatives of these matrices becomes 0. This makes the algorithm for the gradient and the Hessian of the log-likelihood considerably simple. The third example is the seasonal adjustment model with AR component. For this model, the matrix $F$ depends on the unknown AR coefficients, but the derivative of $F$ is very simple and very sparse.

3.1 Trend model

This is a typical example of the case where only the noise covariances $Q$ and $R$ depend on the unknown parameter $\theta$. Consider a trend model

$$y_n = T_n + w_n,$$

where $T_n$ is the trend component that typically follow the following model

$$(1 - B)^k T_n = v_n,$$  

where $B$ is the back-shift operator satisfying $B T_n = T_{n-1}$, $v_n$ and $w_n$ are assumed to be Gaussian white noise with variances $\tau^2$ and $\sigma^2$, respectively (Kitagawa and Gersch (1984,1996) and Kitagawa (2020)). Note that for $k = 1$ and $k = 2$, the model (21) becomes $T_n = T_{n-1} + v_n$ and $T_n = 2T_{n-1} - T_{n-2} + v_n$, respectively.

This trend model can be expressed in the state-space model form as

$$x_n = F x_{n-1} + G v_n$$
$$y_n = H x_n + w_n,$$

with $v_n \sim N(0, Q)$ and $w_n \sim N(0, R)$ and the state vector $x_n$ and the matrices $F$, $G$, $H$, $Q$ and $R$ are defined by

$$x_n = T_n, \quad F = 1, \quad G = 1, \quad H = 1, \quad Q = \tau^2, \quad R = \sigma^2,$$

for $k = 1$ and

$$x_n = \begin{bmatrix} T_n \\ T_{n-1} \end{bmatrix}, \quad F = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad Q = \tau^2, \quad R = \sigma^2,$$

for $k = 2$, respectively.

In this state-space representation, the parameter is $\tau^2$, and the $F$, $G$ and $H$ do not depend on the parameter. Therefore, we have $\frac{\partial F}{\partial \theta} = \frac{\partial G}{\partial \theta} = \frac{\partial H}{\partial \theta} = 0$ and $\frac{\partial^2 F}{\partial \theta \partial \theta^T} = \frac{\partial^2 G}{\partial \theta \partial \theta^T} = \frac{\partial^2 H}{\partial \theta \partial \theta^T} = 0$.

In actual likelihood maximization, since there are positivity constraints, $\tau^2 > 0$, and $\sigma^2 > 0$, it is frequently used log-transformations,

$$\theta_1 = \log(\tau^2), \quad \theta_2 = \log(\sigma^2)$$

and maximize the log-likelihood with respect to this transformed parameter $\theta = (\theta_1, \theta_2)^T$. In this case,

$$\frac{\partial Q}{\partial \theta_1} = \frac{\partial^2 Q}{\partial \theta_1 \partial \theta_1} = \begin{bmatrix} \tau^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial Q}{\partial \theta_2} = \frac{\partial^2 Q}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 Q}{\partial \theta_2 \partial \theta_1} = \frac{\partial^2 Q}{\partial \theta_2 \partial \theta_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{\partial R}{\partial \theta_1} = \frac{\partial^2 R}{\partial \theta_1 \partial \theta_1} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial R}{\partial \theta_2} = \frac{\partial^2 R}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 R}{\partial \theta_2 \partial \theta_1} = \frac{\partial^2 R}{\partial \theta_2 \partial \theta_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (27)$$
Since log-transformation is a monotone increasing function, we can get the same parameter by solving this modified optimization problem.

For Whard (whole sale hardware) data (Kitagawa (2020)), $N = 155$, the parameter $\theta = (\log \tau^2, \log \sigma^2)^T$ of the trend model with $m_1 = 1$ was estimated using the initial values $\tau_0^2 = 0.1 \times 10^{-3}$ and $\sigma_0^2 = 0.2 \times 10^{-3}$. By a numerical optimization procedure, the maximum likelihood estimates of the parameters are obtained as $\hat{\tau}^2 = 0.687264 \times 10^{-3}$ and $\hat{\sigma}^2 = 0.131613 \times 10^{-3}$. In this case, the bias correction term of the GIC, $I(\hat{\theta})J(\hat{\theta})^{-1}$ is evaluated as 1.4547. Note that since this model contains two parameters, the bias correction terms is 2.

Table I shows the log-likelihoods, the gradients, the Hessians and the observation noise variances of the initial and the final estimates. The log-likelihood of the model with these initial and final estimates are $\ell(\theta) = -307.6616$ and $-317.9243$, respectively.

Table II shows the results for the second order trend model. In this case, the final estimate obtained by the numerical optimization procedure depends on the initial estimate and two cases are shown in the table. If the initial estimate is set to $\tau_0^2 = 10^{-4}$ and $\sigma_0^2 = 2 \times 10^{-4}$, the final estimate is $\hat{\tau}^2 = 1.9222 \times 10^{-4}$ and $\hat{\sigma}^2 = 3.4960 \times 10^{-4}$ with the log-likelihood value $\ell(\hat{\theta}) = 293.0193$. On the other hand, if we set the initial estimate as $\theta_0 = 2 \times 10^{-7}$ and $\sigma_0^2 = 2 \times 10^{-4}$, the final estimate becomes $\hat{\theta} = 1.16423 \times 10^{-6}$ and $\hat{\sigma}^2 = 1.11047 \times 10^{-1}$ with $\ell(\hat{\theta}) = 278.6631$. Comparing the log-likelihood values, $\hat{\tau}^2 = 1.9222 \times 10^{-4}$ and $\hat{\sigma}^2 = 3.4960 \times 10^{-4}$ are the maximum likelihood estimate of the second order trend model. The bias correction term $b(\text{GIC})$ is evaluated as 1.9115 and 2.8151, respectively. Note that the bias correction term $b(\text{GIC})$ takes different values depending on the estimated parameters. This is because the curvature of the log-likelihood function is different depending on the pair of parameter values.

### 3.2 The standard seasonal adjustment model

As the second example, we consider a standard seasonal adjustment model

$$y_n = T_n + S_n + w_n,$$  \hspace{1cm} (28)

where $T_n$ and $S_n$ are the trend component and the seasonal component that typically follow the following model

$$T_n = 2T_{n-1} - T_{n-2} + u_n,$$  $S_n = -(S_{n-1} + \cdots + S_{n-p+1}) + v_n.$$ \hspace{1cm} (29)

| \hline
| $(\tau^2, \sigma^2)$ | Initial Model | MLE |
|---|---|---|
| $[0.1 \times 10^{-3}, 0.2 \times 10^{-3}]$ | $[6.87264 \times 10^{-4}, 1.31613 \times 10^{-4}]$ |
| $\theta$ | $[-9.21034, -8.51719]$ | $[-7.28279, -8.93564]$ |
| $\ell(\theta)$ | $253.1868$ | $317.5342$ |
| $\frac{\partial \ell(\theta)}{\partial \theta}$ | $[72.417363, 59.11161]$ | $[0.1347 \times 10^{-8}, -1.2742 \times 10^{-8}]$ |
| $\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T}$ | $[35.77478, 62.19858]$ | $[45.69891, 12.22819]$ |
| $b(\text{GIC})$ | $-2.4551$ | $1.4547$ |
are defined by model form as and \( \sigma \). Theree noise terms, \( u \), \( v \) and \( w \) are assumed to be Gaussian white noise with variances \( \tau_1^2, \tau_2^2 \) and \( \sigma^2 \), respectively (Kitagawa and Gersch 1984,1996 and Kitagawa (2020)).

This seasonal adjustment model with two component models can be expressed in state-space model form as

\[
\begin{align*}
x_n &= Fx_{n-1} + Gv_n \\
y_n &= Hx_n + w_n
\end{align*}
\]  

(30)

with \( v_n \sim N(0, Q) \) and \( w_n \sim N(0, R) \) and the state vector \( x_n \) and the matrices \( F, G, H, Q \) and \( R \) are defined by

\[
x_n = \begin{bmatrix} T_n \\ T_{n-1} \\ S_n \\ S_{n-1} \\ \vdots \\ S_{n-p+2} \end{bmatrix}, \quad F = \begin{bmatrix} 2 & -1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & -1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

(31)

\[
H = \begin{bmatrix} 1 & 0 & 1 & \cdots & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2 \end{bmatrix}, \quad R = \sigma^2
\]

(32)

In this case, the parameter is \( \theta = (\tau_1^2, \tau_2^2, \sigma^2)^T \), and the \( F, G, H \) and \( R \) do not depend on the parameter. In actual likelihood maximization, since there are positivity constrains, \( \tau_1^2 > 0, \tau_2^2 > 0 \)

Table 2: Comparison of numerical difference and gradient methods for the second order trend model.

| \( (\tau^2, \sigma^2) \) | Initial model | Optimized model |
|--------------------------|--------------|----------------|
| \( 0.1 \times 10^{-3}, 0.2 \times 10^{-3} \) | [1.9222 \times 10^{-4}, 3.4960 \times 10^{-4}] | |
| \( \theta \) | [-9.21034, -8.51719] | [-8.55687, -7.95871] |
| \( \ell(\theta) \) | 276.6621 | 293.0193 |
| \( \frac{\partial \ell(\theta)}{\partial \theta} \) | [20.50334, 40.91088] | [0.8543 \times 10^{-8}, 0.3189 \times 10^{-8}] |
| \( \frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T} \) | [20.09159, 24.63536] | [14.27777, 10.60807] |
| \( b(GIC) \) | 5.4927 | 1.9115 |
| \( (\tau^2, \sigma^2) \) | [0.2 \times 10^{-6}, 0.2 \times 10^{-3}] | [1.16423 \times 10^{-6}, 1.11047 \times 10^{-4}] |
| \( \theta \) | [-15.42495, -8.51719] | [-8.55687, -7.95871] |
| \( \ell(\theta) \) | 37.3252 | 278.6631 |
| \( \frac{\partial \ell(\theta)}{\partial \theta} \) | [20.46039, 328.96519] | [0.01318 \times 10^{-7}, -1.24767 \times 10^{-7}] |
| \( \frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T} \) | [4.54019, 20.31043] | [1.78678, 2.66212] |
| \( b(GIC) \) | 37.3252 | 2.8151 |
and $\sigma^2 > 0$, we use the log-transformation,

$$\theta_1 = \log(\tau_1^2), \quad \theta_2 = \log(\tau_2^2), \quad \theta_3 = \log(\sigma^2).$$

(33)

In this case,

$$\frac{\partial Q}{\partial \theta_1} = \frac{\partial^2 Q}{\partial \theta_1 \partial \theta_1} = \begin{bmatrix} \tau_1^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial Q}{\partial \theta_2} = \frac{\partial^2 Q}{\partial \theta_2 \partial \theta_2} = \begin{bmatrix} 0 & 0 \\ 0 & \tau_2^2 \end{bmatrix}, \quad \frac{\partial Q}{\partial \theta_3} = \frac{\partial^2 Q}{\partial \theta_3 \partial \theta_2} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\frac{\partial^2 Q}{\partial \theta_i \partial \theta_j} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (i \neq j), \quad \frac{\partial R}{\partial \theta_1} = \frac{\partial R}{\partial \theta_2} = 0, \quad \frac{\partial R}{\partial \theta_3} = \sigma^2,$$

(34)

$$\frac{\partial^2 R}{\partial \theta_3 \partial \theta_3} = \sigma^2, \quad \frac{\partial^2 R}{\partial \theta_i \partial \theta_j} = 0 \quad \text{(unless } i = j = 3).$$

Further, since $F, G$ and $H$ do not depend on $\theta$, $\frac{\partial F}{\partial \theta} = \frac{\partial^2 F}{\partial \theta \partial \theta^T} = 0$, $\frac{\partial G}{\partial \theta} = \frac{\partial^2 G}{\partial \theta \partial \theta^T} = 0$ and $\frac{\partial H}{\partial \theta} = \frac{\partial^2 H}{\partial \theta \partial \theta^T} = 0$ hold, where 0 indicates a zero matrix with appropriate size.

For Whard data, the standard seasonal adjustment model with $m_1 = 2, m_2 = 1$ is estimated using the initial estimates of parameters, $\theta = (\log \tau_1^2, \log \tau_2^2, \log \sigma^2) = (-9.21034, -10.81978, -8.5179)^T$. The log-likelihood of the model with these initial parameters is $\ell(\theta) = -346.5115$ and the Gradient obtained by the differential filter are shown in the Table 3.

The maximum likelihood estimate of the parameter vector is $\hat{\theta} = (-12.10001, -10.04570, -9.85025)$ with maximum log-likelihood $\ell(\hat{\theta}) = 384.9600$. The gradient and the Hessian matrix of this model are shown in table. The bias correction term of this estimated standard seasonal adjustment model is 3.9558.

### 3.3 Seasonal adjustment model with stationary AR component

The third example is a seasonal adjustment model with stationary AR component

$$y_n = T_n + S_n + p_n + w_n,$$

(35)

| $(\tau_1^2, \tau_2^2, \sigma^2)$ | Initial model | Optimal model |
|----------------------------------|---------------|---------------|
| $\theta$ | $[-9.21034, -10.81978, -8.5179]$ | $[-12.10001, -10.04570, -9.85025]$ |
| $\ell(\theta)$ | $346.5115$ | $384.9600$ |
| $\frac{\partial \ell(\theta)}{\partial \theta}$ | $[-18.12229, -4.82792, -17.81465]$ | $[-0.47689 \times 10^{-6}, -1.31732 \times 10^{-6}, 0.0856 \times 10^{-6}]$ |
| $\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \theta^T}$ | $[5.77960, -0.06331, -1.89624]$ | $[8.66117, 1.12346, 3.41325]$ |
| $b(GIC)$ | $1.1946$ | $3.9558$ |
where $T_n$ and $S_n$ are the trend component and the seasonal component introduced in the previous subsection and $p_n$ is an AR component with AR order $m_3$ defined by

$$p_n = \sum_{j=1}^{m_3} a_j p_{n-j} + v_n^{(t)}.$$  \hfill (36)

Here $v_n^{(t)}$ is a Gaussian white noise with variance $\tau_3^2$. The model contains $4 + m_3$ parameters and the parameter vector is given by $\theta = (\theta_1, \ldots, \theta_{4+m_3}) \equiv \tau_1^2, \tau_2^2, \tau_3^2, \sigma^2 a_1, \cdots, a_{m_3})^T$.

The matrices $F$, $G$, $H$, $Q$ and $R$ are defined by

$$x_n = \begin{bmatrix} T_n \\ T_{n-1} \\ \vdots \\ S_{n-p+1} \\ p_{n-1} \\ p_{n-2} \\ \vdots \\ p_{n-m_3} \end{bmatrix}, \quad F = \begin{bmatrix} 2 & -1 & 1 & 1 & \cdots & -1 & 1 \\ -1 & -1 & \cdots & 1 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\ a_1 & a_2 & \cdots & a_{m_3} & 1 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \hfill (37)

$$H = \begin{bmatrix} \tau_1^2 & 0 & 0 \\ 0 & \tau_2^2 & 0 \\ 0 & 0 & \tau_3^2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \hfill (38)

In this case,

$$\frac{\partial Q}{\partial \theta_1} = \frac{\partial^2 Q}{\partial \theta_1^2} = \begin{bmatrix} \tau_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial Q}{\partial \theta_2} = \frac{\partial^2 Q}{\partial \theta_2^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tau_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial Q}{\partial \theta_3} = \frac{\partial^2 Q}{\partial \theta_3^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau_3^2 \end{bmatrix}, \hfill (39)

\frac{\partial^2 Q}{\partial \theta_i \partial \theta_j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{for } i \neq j

\frac{\partial R}{\partial \theta_1} = \frac{\partial R}{\partial \theta_2} = 0, \quad \frac{\partial R}{\partial \theta_3} = \frac{\partial^2 R}{\partial \theta_3^2} = \sigma^2, \quad \frac{\partial^2 R}{\partial \theta_i \partial \theta_j} = 0 \quad \text{(unless } i, j = 3)\hfill (39)

\left( \frac{\partial F}{\partial \theta_k} \right)_{pq} = \begin{cases} 1 & \text{if } k = 4 + i, p = 4, q = k, (i = 1, \ldots, m_3). \\ 0 & \text{otherwise} \end{cases}

\left( \frac{\partial^2 F}{\partial \theta_j \partial \theta_k} \right)_{pq} = 0

where $\left( \frac{\partial F}{\partial \theta_k} \right)_{pq}$ and $\left( \frac{\partial^2 F}{\partial \theta_j \partial \theta_k} \right)_{pq}$ denote the $(p, q)$ components of the matrices $\left( \frac{\partial F}{\partial \theta_k} \right)$ and $\left( \frac{\partial^2 F}{\partial \theta_j \partial \theta_k} \right)$, respectively.
For $m_1 = 2$, $m_2 = 1$ and $m_3 = 2$, the matrices $\frac{\partial F}{\partial \theta_j}, (j = 5, 6)$ are given by

$$
\frac{\partial F}{\partial \theta_5} = \begin{bmatrix}
0 & \ldots & 0 & 0 & 0 \\
\vdots & & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 1 \\
0 & \ldots & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & 0
\end{bmatrix}, \quad \frac{\partial F}{\partial \theta_6} = \begin{bmatrix}
0 & \ldots & 0 & 0 & 0 \\
\vdots & & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 1 \\
0 & \ldots & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & 0
\end{bmatrix}.
$$

(40)

Table 4: The gradient vectors and the Hessian matrix of the two TSAR models with $M_3 = 1$ obtained by the proposed method.

|                | Maximum likelihood model |                |                |                |                |
|----------------|--------------------------|----------------|----------------|----------------|----------------|
| $\tau_1^2$   | $5.3908 \times 10^{-14}$ | $\tau_2^2$    | $5.4129 \times 10^{-3}$ | $\tau_3^2$ | $6.9080 \times 10^{-3}$ | $\sigma^2$ | $3.8147 \times 10^{-8}$ | $a_1^1$ |
| $\theta$     | -30.55150                | -9.82413       | -9.58025       | -17.0818      | 99.9900       |
| $\frac{\partial \ell (\theta)}{\partial \theta}$ | -0.00000         | -0.00000       | -0.00112       | -0.00178      | 0.00087       |
|              | 0.00000                  | -0.00000       | 0.00000        | -0.00000      | 0.00000       |
|              | -0.00000                 | 30.99869       | 7.66932        | 0.01036       | -0.07670      |
| $\frac{\partial^2 \ell (\theta)}{\partial \theta \partial \theta'}$ | 0.00000         | 7.66932        | 25.22303       | 0.00551       | -0.24763      |
|              | -0.00000                 | 0.01036        | 0.00551        | 0.00178       | -0.00006      |
|              | 0.00000                  | -0.07670       | -0.24763       | -0.00066      | 0.07058       |

|                | Local MLE model           |                |                |                |                |
| $\tau_1^2$   | $5.4741 \times 10^{-6}$  | $\tau_2^2$    | $4.3834 \times 10^{-5}$ | $\tau_3^2$ | $2.2564 \times 10^{-23}$ | $\sigma^2$ | $5.2261 \times 10^{-5}$ | $a_1^1$ |
| $\theta$     | -12.11548                | -10.03510      | -54.44829      | -9.85926      | 96.81629      |
| $\frac{\partial \ell (\theta)}{\partial \theta}$ | 0.00000          | 0.00000        | 0.00000        | 0.00000       | 0.00000       |
|              | 8.79305                  | 1.09359        | 0.00000        | 3.43159       | -0.00443      |
|              | 1.09359                  | 19.24777       | 0.00000        | 11.29246      | 0.00427       |
| $\frac{\partial^2 \ell (\theta)}{\partial \theta \partial \theta'}$ | 0.00000         | 0.00000        | -0.00000       | 0.00000       | 0.00000       |
|              | 3.43159                  | 11.29246       | 0.00000        | 11.86720      | 0.00053       |
|              | -0.00443                 | 0.00427        | 0.00000        | 0.00053       | 0.00108       |

Table 4 shows the gradients and the Hessians of the TSAR model with the first order AR component $m_3 = 1$. Two models with the maximum likelihood estimates and the local maximum likelihood estimates (the second best model) are shown.

The maximum likelihood estimates of the parameter the AR coefficient is $a_1^1 = 0.9999$ that show the AR process has of the model has almost unit root. The log-likelihood of the maximum likelihood model is 392.1341. Instead, the variances of the trend component and observation noise are very small. The log-likelihood of the model is 385.2710. On the other hand, the variance of the AR coefficient of the second best model is almost zero. The bias correction terms are $b(GIC) = 1.7655$ and 4.2173, respectively.

Table 5 shows the gradients and the Hessians of the TSAR model with the second order AR component $m_3 = 2$. In this case the observation noise variance is almost zero. The log-likelihood and the bias correction term are $\ell (\hat{\theta}) = 393.0525$ and $b(GIC) = 4.1985$, respectively.

Table 6 shows the log-likelihood, the number of parameters, bias correction term of BIC, AIC and GIC of various state-space models, such as the trend model with order 1 and 2 ($m_1 = 1$ or 2,
Table 5: The gradient vectors and the Hessian matrix of TSAR model with \( m_3 = 2 \) obtained by the proposed method.

| \( \tau_1^2 \) | \( \tau_2^2 \) | \( \tau_3^3 \) | \( \sigma^2 \) | \( a_1^2 \) | \( a_2^2 \) |
|----------------|----------------|----------------|-------------|--------------|--------------|
| 2.1040 \times 10^{-10} | 6.2681 \times 10^{-5} | 3.4119 \times 10^{-3} | 8.7565 \times 10^{-2t} | 1.36666 | -0.37573 |

Optimized model:

\[ \tau_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \sigma^2, \quad a_1^2, \quad a_2^2 \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

\[ \begin{pmatrix} 0.00000 \\ 0.00000 \\ -0.00000 \\ -0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix} \]

Table 6: Log-likelihoods and bias correction terms of AIC and GIC for the seasonal adjustment model with AR components.

| \( m_1 \) | \( m_2 \) | \( m_3 \) | log-likelihood | \( b_{AIC} \) | \( b_{GIC} \) | \( AIC \) | \( GIC \) |
|---------|---------|---------|---------------|-------------|-------------|--------|--------|
| 1       | 0       | 0       | 319.5067      | 2           | 1.4669      | -635.0134 | -636.0796 |
| 2       | 0       | 1       | 296.7171      | 2           | 1.9232      | -589.4342 | -589.5878 |
| 2       | 1       | 0       | 384.9600      | 3           | 3.9558      | -763.9201 | -762.0084 |
| 2       | 1       | 1       | 392.1015      | 5           | 1.8232      | -774.2030 | -780.5566 |
| 2       | 1       | 2       | 393.0525      | 6           | 4.1998      | -774.1050 | -777.7054 |
| 2       | 1       | 3       | 393.1091      | 7           | 4.4603      | -772.2182 | -777.2976 |

\( m_2 = 1, m_3 = 0 \), the seasonal adjustment model with AR order \( m_3 = 0, 1, 2, 3 \). It can be seen that the \( b_{AIC} \) and \( b_{GIC} \) are considerably different, in this case the both criteria select the same model \( m_1 = 2, m_2 = 1 \) and \( m_3 = 1 \).

4 Summary

The gradient and Hessian of the log-likelihood of linear state-space model are given. Details of the implementation of the algorithm for standard seasonal adjustment model, seasonal adjustment model with stationary AR component and ARMA model are given. For each implementation, comparison with a numerical difference method is shown.

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