On the global existence of generalized rotational hypersurfaces with prescribed mean curvature in the Euclidean spaces, II

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Abstract

In the previous paper, it has been proved that the generalized rotational hypersurfaces of $O(n-1)$-type and $O(\ell+1) \times O(m+1)$-type, for which the mean curvature is any prescribed continuous function. This paper is a sequel, and a similar existence result is shown for any type.

Keywords: mean curvature, generalized rotational hypersurfaces.

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1 Introduction

Let $H$ be any continuous function on the real line. The purpose of this work is to construct generalized rotational hypersurfaces with any prescribed continuous mean curvature $H$. A generalized rotational hypersurface $M$ in $n$-dimensional Euclidean space $\mathbb{R}^n$, for $n \geq 3$, is defined via a compact Lie group $G$ and its representation to $\mathbb{R}^n$, i.e., $M$ is invariant under an isometric transformation group $(G, \mathbb{R}^n)$ with codimension two principal orbit type. Such transformation groups $(G, \mathbb{R}^n)$ are well known and they were classified by Hsiang [2] into five types:

Type I: $(G, \mathbb{R}^n) = (O(n-1), \mathbb{R}^n)$.

Type II: $(G, \mathbb{R}^n) = (O(\ell+1) \times O(m+1), \mathbb{R}^{\ell+m+2})$.

Type III: $(G, \mathbb{R}^n) = (SO(3), \mathbb{R}^3), (SU(3), \mathbb{R}^8), (Sp(3), \mathbb{R}^{14}), (F_4, \mathbb{R}^{26})$.

Type IV: $(G, \mathbb{R}^n) = (SO(5), \mathbb{R}^{10}), (U(5), \mathbb{R}^{20}), (U(1) \times Spin(10), \mathbb{R}^{32}), (SO(2) \times SO(m), \mathbb{R}^{2m}), (S(U(2) \times U(m)), \mathbb{R}^{4m}), (Sp(2) \times Sp(m), \mathbb{R}^{8m})$.

Type V: $(G, \mathbb{R}^n) = (SO(4), \mathbb{R}^8), (G_2, \mathbb{R}^{14})$. 

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Hypersurfaces of Type I were constructed by Kenmotsu [3] in the case \( n = 3 \) when \( H \) is a continuous function, and by Dorfmeister-Kenmotsu [1] for \( n \geq 4 \), in the case when \( H \) is an analytic function. In the previous paper [4] it is shown the general existence of hypersurfaces of Type I and Type II, with any continuous mean curvature \( H \). In this paper, we get a similar result in all of the five Types I–V.

Let \((x(s), y(s))\) be the generating curve with arclength parameter \( s \):

\[(x'(s))^2 + (y'(s))^2 = 1.\]

For each Type, another equation determines the generalized rotational hypersurfaces with mean curvature \( H \); these can be found in [2]. We shall describe the equations uniformly by introducing the following notation:

\[\mathbf{x}(s) = t(x(s), y(s)), \quad \mathbf{x}'(s) = t(x'(s), y'(s)), \quad \mathbf{x}''(s) = t(-y''(s), x''(s)),\]

and

\[e(\phi) = t(\cos \phi, \sin \phi), \quad e(\phi)\perp = t(-\sin \phi, \cos \phi),\]

**Fact 1.1** For each type of generalized rotational hypersurface, there exist a finite set \( J \), angles \( \phi_j \in (-\frac{\pi}{2}, \frac{\pi}{2}] \), and natural numbers \( n_j \in \mathbb{N} \) for \( j \in J \), such that the equation of the generalized rotational hypersurface can be described by

\[(\mathbf{x}''(s)\perp \cdot \mathbf{x}'(s)) + \sum_{j \in J} n_j e(\phi_j) \cdot \mathbf{x}'(s) = (n - 1)H(s), \quad \| \mathbf{x}'(s) \|^2 = 1,\]

where

\[\sum_{j \in J} n_j = n - 2\]

is the number of principal curvatures, not including the curvature of the generating curve. For each type above, the set \( J \), the angles \( \phi_j \), and the natural numbers \( n_j \) for \( j \in J \) are given as follows:

**Type I:** \( J = \{0\} \), \( \phi_0 = 0 \), \( n_0 = n - 2 \).

**Type II:** \( J = \{0, 1\} \), \( \phi_1 = \frac{\pi}{2} \), \( \phi_0 = 0 \), \( n_0 = m \), \( n_1 = \ell \).

**Type III:** \( J = \{-1, 0, 1\} \) \( \phi_1 = \frac{\pi}{3} \), \( \phi_2 = j\phi_1 \). And \( n_j = 1, 2, 4, \) or 8 for

\((SO(3), \mathbb{R}^5), (SU(3), \mathbb{R}^8), (Sp(3), \mathbb{R}^{14}) \), or \((F_4, \mathbb{R}^{26}) \) respectively.

**Type IV:** \( J = \{-1, 0, 1, 2\} \), \( \phi_1 = \frac{\pi}{4} \), \( \phi_j = j\phi_1 \). And \( n_{\pm 1} = \ell \), \( n_0 = n_2 = k \), where \( (k, \ell) = (2, 2), (5, 4), (9, 6), (m-2, 1), (2m-3, 2), (4m-5, 4) \) for

\((SO(5), \mathbb{R}^{10}), (U(5), \mathbb{R}^{20}), (U(1) \times Sp(10), \mathbb{R}^{32}), (SO(2) \times SO(m), \mathbb{R}^{2m}), (S(U(2) \times U(m)), \mathbb{R}^{4m}), (Sp(2) \times Sp(m), \mathbb{R}^{8m}) \) respectively.

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Type V: \( J = \{-2, -1, 0, 1, 2, 3\}, \phi_i = \frac{x_i}{y_i}, \phi_j = j\phi_1. \) And \( n_j = 1, \) or 2 for \((SO(4), \mathbb{R}^5),\) or \((G_2, \mathbb{R}^{14})\) respectively.

We get these by direct calculations. For example the equation for Type II, see [4], is:

\[
x''(s)y'(s) - y''(s)x'(s) - \frac{\ell y'(s)}{x(s)} + \frac{m x'(s)}{y(s)} = (n - 1)H(s),
\]

together with equation (1). Each term in the left-hand side of the above equation is

\[
x''(s)y'(s) - y''(s)x'(s) = x''(s) \cdot x'(s),
\]

\[
-\frac{\ell y'(s)}{x(s)} = \ell \cdot \frac{(0, 1) \cdot \langle x'(s), y'(s) \rangle}{x(s)} = \frac{n_1 e(\phi_1) \cdot x'(s)}{e(\phi_1) \cdot x(s)},
\]

\[
\frac{m x'(s)}{y(s)} = m \cdot \frac{(1, 0) \cdot \langle x'(s), y'(s) \rangle}{y(s)} = \frac{n_0 e(\phi_0) \cdot x'}{e(\phi_0) \cdot x(s)},
\]

and

\[
\sum_{j \in J} n_j = \ell + m = n - 2.
\]

Our main result is:

**Theorem 1.1** Let \( H \) be a continuous function on \( \mathbb{R}. \) Put

\[
S = \{ x \in \mathbb{R}^2 | e(\phi_j) \cdot x = 0 \text{ for some } j \in J \}.
\]

For any \( x_0 \not\in S \) and \( s_0 \in \mathbb{R}, \) there exists a solution \( x(s) \) to (2) on \( \mathbb{R} \) satisfying \( x(s_0) = x_0. \)

Our equation is singular on the set \( S. \) Since \( x(s_0) \not\in S, \) then it is simple to construct a solution near \( s = s_0, \) and we can extend the solution as long as \( x(s) \not\in S. \) To extend the solution, a problem happens when a solution approaches to \( S \) as \( s \to s_* \) for some \( s_* \in \mathbb{R}. \) It is a non trivial fact that the solution can be extended beyond \( s = s_* \). We shall study the asymptotic behavior of \( x'(s) \) as \( s \to s_* \), and in particular the existence of the limit \( \lim_{s \to s_*} x'(s), \) say \( x'_*. \) Furthermore, we shall construct solutions beyond \( s_* \) with \( x'(s_*) = x'_*. \)

By a formal blow-up argument we can evaluate the limit \( x'_*. \) For simplicity we shall assume \( s_* = 0. \) Let the generating curve be in the sector

\[
(3) \quad S_i = \begin{cases} \{ x \in \mathbb{R}^2 | e(0) \cdot x > 0 \} = \{ (x, y) \in \mathbb{R}^2 | y > 0 \} & \text{for Type I}, \\
\{ x \in \mathbb{R}^2 | e(\phi_i) \cdot x < 0 < e(\phi_i) \cdot x \} & \text{for Types II–V} \end{cases}
\]
for $i = 0$ (Type I), or for some $i \in J \setminus \{ \max J \}$ (Types II–V). There are three cases: (i) $\lim_{s \to 0} x(s) = x_* \neq 0$, $e(\phi_i) \cdot x_* = 0$ (the condition $x_* \neq 0$ can be removed for Type I by a translation); (i)' $\lim_{s \to 0} x(s) = x_* \neq 0$, $e(\phi_{i+1}) \cdot x_* = 0$; and (ii) $\lim_{s \to 0} x(s) = 0$ for Types II–V. Since the argument for (i)' is similar to that for (i), we shall consider cases (i) and (ii) only.

For case (i), we assume that there exists the limit $\lim_{s \to 0} x(s) = x_* \neq 0$ and $e(\phi_i) \cdot x_* = 0$, and (ii) $\lim_{s \to 0} x(s) = 0$ for Types II–V. Since the argument for (i)' is similar to that for (i), we shall consider cases (i) and (ii) only.

For case (i), we assume that there exists the limit $\lim_{s \to +0} x'(s) = e(\theta_*)$ and that $x''(s)$ is bounded. Put

$$x_\lambda(s) = \lambda^{-1}(x(\lambda s) - x_*)$$

for $\lambda > 0$. Then it is easy to see that

$$x''_\lambda(s) \cdot x'_\lambda(s) + \sum_{j \in J} \frac{\lambda n_j e(\phi_j) \cdot x'_\lambda(s)}{e(\phi_j) \cdot (\lambda x_\lambda(s) - x_*)} = (n - 1)\lambda H(\lambda s).$$

Since we have

$$\lim_{\lambda \to +0} x_\lambda(s) = s x'(0) = s e(\theta_*), \quad \lim_{\lambda \to +0} x'_\lambda(s) = x'(0) = e(\theta_*), \quad \lim_{\lambda \to +0} x''_\lambda(s) = 0,$$

we get

$$e(\phi_i) \cdot e(\theta_*) = 0$$

by letting $\lambda \to +0$. Consequently, $\theta_* = \phi_i + \frac{\pi}{2}$. We can obtain a similar result letting $s \to -0$. Thus the generating curve touches perpendicularly the boundary of sector $S_i$.

For case (ii), we assume the existence of $\lim_{s \to +0} x'(s) = e(\theta_*)$, $\theta_* \neq \phi_j$ for $j \in J$, and the boundedness of $x''(s)$. Putting

$$x_\lambda(s) = \lambda^{-1}(x(\lambda s),$$

we have

$$x''_\lambda(s) \cdot x'_\lambda(s) + \sum_{j \in J} \frac{n_j e(\phi_j) \cdot x'_\lambda(s)}{e(\phi_j) \cdot (\lambda x_\lambda(s) - x_*)} = (n - 1)\lambda H(\lambda s),$$

and

$$\sum_{j \in J} n_j \cot(\theta_* - \phi_j) = 0$$

by $\lambda \to +0$. Put

$$A(\theta) = \sum_{j \in J} n_j \cot(\theta - \phi_j).$$
Since $A(\cdot)$ is monotone decreasing on each interval $(\phi_i, \phi_{i+1})$, and since
\[
\lim_{\theta \to \phi_i+0} A(\theta) = \infty, \quad \lim_{\theta \to \phi_{i+1}-0} A(\theta) = -\infty
\]
there exists a unique $\theta_i$ on each interval $(\phi_i, \phi_{i+1})$ such that $A(\theta_i) = 0$. Thus the generating curve approaches to the origin with angle $\theta_i$.

In the above argument we assume the existence of $\lim_{s \to 0} x'(s)$, the boundedness $x''(s)$ and so on. In the following sections, we shall prove the asymptotic behavior as above without these assumptions, and shall show the existence of solutions of (2) with the initial value
\[
e(\phi_i) \cdot x(0) = 0, \quad x(0) \neq o, \quad x'(0) = e(\phi_i + \frac{\pi}{2})
\]
for all Types I–V, or
\[
x(0) = 0, \quad x'(0) = e(\theta_i)
\]
for Types II–V.

**Remark 1.1** By direct calculation, we get the explicit values of each $\theta_i$:

Type II: $\theta_0 = \arctan \sqrt{\frac{n_0}{n_1}}$

Type III: $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$

Type IV: $\theta_{\pm1} = -\frac{1}{2} \arctan \sqrt{\frac{k}{\ell}}, \theta_0 = \frac{1}{2} \arctan \sqrt{\frac{k}{\ell}}$

Type V: $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$.

For the calculations, see Fact 5.1 in the Appendix.

We shall discuss the following two cases in §§ 3–4 respectively:

Case (i) The asymptotic behavior of $x'(s)$ when $\lim_{s \to 0} x(s) = x_* \neq o$, and the solvability of the initial-value problem (2) and (4), in Propositions 3.1–3.2.

Case (ii) The asymptotic behavior of $x'(s)$ when $\lim_{s \to 0} x(s) = o$, and the solvability of the initial-value problem (2) and (6), in Propositions 4.1–4.2.

Theorem 1.1 follows then from these Propositions. For all types in Case (i) and for Types II–III in Case (ii), the derivations for setting the problem are more complicated but similar to those in [4]. Therefore, we shall present the setting in detail, but some estimates just briefly. For the Types IV–V in the case (ii), we need here one extra procedure in comparison to [4]. We can prove our results without the extra procedure for Types II–III as [4], but this extra procedure is applicable for all types, and in this sense the proof shall be universal.
2 A transformation

The following is a useful transformation. We define the matrix $R(\psi)$ and a vector $u = t(u, v)$ by

$$R(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix}, \quad u = t(u, v) = R(\psi)x.$$

It is easy to see that when $x(s)$ satisfies (2), the new unknown vector function $u(s) = (u(s), v(s))$ satisfies

$$u''(s) \cdot u'(s) + \sum_{j \in J} \frac{n_j e(\phi_j + \psi) \cdot u'(s)}{e(\phi_j + \psi) \cdot u(s)} = (n - 1)H(s), \quad \|u'(s)\|^2 = 1.$$

Assume that the generating curve $x(s)$ is in the sector $S_i$ defined by (3). We transform $x$ to $u$ with $\psi = -\phi_i$, then $u(s)$ is in the sector $S_0$ in $uv$-plane.

3 Case (i)

First we show

**Proposition 3.1** Let the generating curve $x(s)$ be in the sector $S_i$, and assume that

$$\lim_{s \to 0} e(\phi_i) \cdot x(s) = 0, \quad \lim_{s \to 0} x(s) = x_0 \neq 0.$$

Then there exists the limit of $x'(s)$ as $s \to 0$ and

$$\lim_{s \to 0} e(\phi_i) \cdot x'(s) = 0.$$

**Proof.** As stated in § 2, we transform $x(s)$ to $u(s)$ with $\psi = -\phi_i$. Then $u(s) = (u(s), v(s))$ satisfies

$$u''u' - v''u' + \frac{n_i u'}{v} + \sum_{j \neq i} \frac{n_j e(\phi_j - \phi_i) \cdot u'}{e(\phi_j - \phi_i) \cdot u} = (n - 1)H, \quad \|u'\|^2 = 1.$$

The assumption on $\lim_{s \to 0} x(s)$ is written as

$$\lim_{s \to 0} u(s) > 0, \quad \lim_{s \to 0} v(s) = +0$$

in terms of $u(s)$ and $v(s)$. What we want to show is

$$\lim_{s \to 0} u'(s) = 0.$$
Multiplying both sides of the first equation of (7) by \( v^n v' \), and using the second relation, we have
\[
(v^n u')' = \left\{ (n - 1)H - \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot u' \right\} v^n u'.
\]
Taking (8) into account, we get
\[
u' = \frac{1}{v^n(s)} \int_0^s \left\{ (n - 1)H(t) - \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot u'(t) \right\} v^n(t) u'(t) \, dt
\]
by integration of the equation of \( u(s) \).

Now we show \( u'(s) \neq 0 \) near \( s = 0 \). Assume that there exists a sequence \( \{ s_k \} \) such that \( u'(s_k) = 0 \), \( \lim_{k \to \infty} s_k = 0 \). Inserting \( s = s_k \) into \( (u')^2 + (v')^2 \equiv 1 \) and \( u'u' + v'v' \equiv 0 \), we have
\[
u'(s_k) = \pm 1, \quad u''(s_k) = 0.
\]
By evaluating the equation at \( s = s_k \), it follows that
\[
v''(s_k) = \mp \left( 1 + \sum_{j \in J} n_j \right) H(s_k) + \frac{n_i}{v(s_k)} + \sum_{j \neq i} \frac{n_j \cos(\phi_j - \phi_i)}{e(\phi_j - \phi_i) \cdot u(s_k)}
\]
It holds that
\[
\lim_{k \to \infty} e(\phi_j - \phi_i) \cdot u(s_k) = -u(0) \sin(\phi_j - \phi_i) \neq 0 \quad \text{for} \quad j \neq i.
\]
Therefore, we have
\[
\lim_{k \to \infty} v''(s_k) = \lim_{k \to \infty} \frac{n_i}{v(s_k)} = \infty.
\]
Consequently, \( v(s_k) \)'s are always local minimum values for large \( k \). This contradicts the assumption that \( \lim_{s \to 0} v(s) = +0 \).

Since \( v'(s) \neq 0 \) near \( s = 0 \), we can use L’Hospital’s theorem to obtain
\[
\lim_{s \to 0} u'(s) = \lim_{s \to 0} \frac{v'(s)}{v''(s)} = \lim_{s \to 0} \left\{ \left( 1 + \sum_{j \in J} n_j \right) H(s) - \sum_{j \neq i} \frac{n_j e(\phi_j - \phi_i) \cdot u'(s)}{e(\phi_j - \phi_i) \cdot u(s)} \right\} v^{n_i}(s) u'(s)
\]
\[
\quad \quad \quad \quad = \lim_{s \to 0} \left\{ \left( 1 + \sum_{j \in J} n_j \right) H(s) - \sum_{j \neq i} \frac{n_j e(\phi_j - \phi_i) \cdot u'(s)}{e(\phi_j - \phi_i) \cdot u(s)} \right\} \frac{v(s)}{n_i}
\]
\[
\quad \quad \quad \quad = 0.
\]
Here we use
\[ \lim_{s \to 0} \mathbf{e}(\phi_j - \phi_i) \cdot \mathbf{u}(s) = -u(0) \sin(\phi_j - \phi_i) \neq 0 \quad \text{for} \quad j \neq i. \]

Next we prove the converse of Proposition 3.1, i.e., the solvability of (2) and (5). The problem is equivalent to (7) and
\[ u'(0) = 0. \]

Since \( v'(0)^2 = 1 \), the map \( s \mapsto v \) is monotone near \( s = 0 \). Therefore, there exists the inverse function \( s = s(v) \). Put
\[ q = \frac{du}{dv} = \frac{u'}{v'} \]
as a function of \( v \). We divide both sides of the first equation of (7) by \((v')^3\). Using \( \|u'(s)\| \equiv 1 \), we get
\[ \frac{dq}{dv} + \frac{n_i q}{v} = -\frac{n_i q^3}{v} \]
\[ + \sum_{j \neq i} \frac{n_j (q^2 + 1) \{q \cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i)\}}{u(0) + \int_0^v q(\eta) \, d\eta} \sin(\phi_j - \phi_i) - v \cos(\phi_j - \phi_i) \]
\[ + (n - 1) (q^2 + 1)^{\frac{3}{2}} \bar{H}, \]
where
\[ \bar{H} = (\text{sgn } v') H. \]

We multiply both sides by \( v^{n_i} \) and integrate from 0 to \( v \). Since
\[ \lim_{v \to 0} q(v) = \lim_{s \to 0} \frac{u'(s)}{v'} = 0, \]
we obtain
\[ q(v) = \frac{1}{v^{n_i}} \int_0^v \varphi(q)(\eta) \, d\eta, \]
where
\[ \varphi(q)(\eta) = \varphi_1(q)(\eta) + \varphi_2(q)(\eta) + \varphi_3(q)(\eta), \]
\[ \varphi_1(q)(\eta) = -n_i q(q)^3 \eta^{n_i - 1}, \]
\[ \varphi_2(q)(\eta) = \sum_{j \neq i} \left( \frac{q^2 + 1}{u(0) + \int_0^\eta q(\zeta) \, d\zeta} \sin(\phi_j - \phi_i) - \eta \cos(\phi_j - \phi_i) \right) \eta^{n_i}. \]
\[ \varphi_3(q)(\eta) = (n - 1) \left( q(\eta)^2 + 1 \right)^{\frac{3}{2}} \tilde{H}(\eta) \eta^m. \]

Define the Banach space \( X_V \) and its bounded set \( X_{V,M} \) by

\[
X_V = \{ f \in C(0, V) \mid \|f\| < \infty \}, \quad \|f\| = \sup_{v \in (0, V]} \left| \frac{f(v)}{v} \right|, \\
X_{V,M} = \{ f \in X_V \mid \|f\| \leq M \}.
\]

Using the boundedness of \( H \), we can show that if \( M \) is large and if \( V \) is small, the map

\[ \Phi(q)(v) = \frac{1}{v^m} \int_0^v \varphi(q)(\eta) \, d\eta \]

defined on \( X_{V,M} \) into itself and it is contraction. Indeed we have

\[
\left\| \frac{1}{v^m} \int_0^v \phi_1(q)(\eta) \, d\eta \right\| \leq CM^3V^2, \\
\left\| \frac{1}{v^m} \int_0^v \phi_2(q)(\eta) \, d\eta \right\| \leq C \left( 1 + M^3V^2 \right), \\
\left\| \frac{1}{v^m} \int_0^v \phi_3(q)(\eta) \, d\eta \right\| \leq C \left( 1 + M^3V^2 \right)
\]

for \( q \in X_{M,V} \);

\[
\left\| \frac{1}{v^m} \int_0^v (\phi_1(q_1)(\eta) - \phi_1(q_2)(\eta)) \, d\eta \right\| \leq CM^2V^2 \|q_1 - q_2\|, \\
\left\| \frac{1}{v^m} \int_0^v (\phi_2(q_1)(\eta) - \phi_2(q_2)(\eta)) \, d\eta \right\| \leq C \left( M^2V^3 + V + MV^2 + V^2 \right) \|q_1 - q_2\|, \\
\left\| \frac{1}{v^m} \int_0^v (\phi_3(q_1)(\eta) - \phi_3(q_2)(\eta)) \, d\eta \right\| \leq C \left( M^2V^3 + MV^2 \right) \|q_1 - q_2\|
\]

for \( q_1 \in X_{M,V} \) and \( q_2 \in X_{M,V} \). Since these estimates can be obtained in the same way as in [4], we omit details. Hence, there exists the unique fixed point of \( \Phi \) in \( X_{M,V} \), which solves (11). If \( H \) is continuous, then it solves (10) satisfying \( q(0) = 0 \). From this fact we get the solvability of the original problem, and the proof of:

**Proposition 3.2** Let \( H \) be continuous. Then there exists a unique local solution \( x \) to (2) and (5).
4 Case (ii)

Consider the equation for Types II–V.

**Proposition 4.1** Let the generating curve \( x(s) \) be in the sector \( S_i \), and assume that
\[
\lim_{s \to \pm 0} x(s) = 0.
\]
Then there exists the limit of \( x'(s) \) as \( s \to \pm 0 \) and
\[
\lim_{s \to \pm 0} x'(s) = \pm e(\theta_i).
\]
Here \( \theta_i \) is the unique angle satisfying
\[
\sum_{j \in J} n_j \cot(\theta_i - \phi_j) = 0, \quad \phi_i < \theta_i < \phi_{i+1}.
\]

This proposition is proved by a series of Lemmas. In what follows, \( x(s) \) satisfies the assumption in Proposition 4.1. For simplicity we consider only the case \( s \to +0 \), and assume that \( x(s) \) is defined on \((0, s_0)\). As in the previous section, we transform \( x \) to \( u \) with \( \psi = -\phi_i \). Then it holds that

\[
(v^n u')' = \left\{ (n-1)H - \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot u'(t) \frac{e(\phi_j - \phi_i)}{e(\phi_j - \phi_i)^\perp \cdot u(t)} \right\} v^n u'.
\]

We integrate this from \( s_0 \) to \( s \in (0, s_0) \), and get

\[
v^n(s) u'(s) = \int_{s_0}^{s} \left\{ (n-1)H(t) - \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot u'(t) \frac{e(\phi_j - \phi_i)}{e(\phi_j - \phi_i)^\perp \cdot u(t)} \right\} v^n(t) u'(t) dt + v^n(s_0) u'(s_0).
\]

Since the left-hand side tends to 0 as \( s \to +0 \), so does the right-hand side. Hence, we get

\[
u'(s) = \frac{1}{v^n(s)} \left[ \int_{s_0}^{s} \left\{ (n-1)H(t) - \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot u'(t) \frac{e(\phi_j - \phi_i)}{e(\phi_j - \phi_i)^\perp \cdot u(t)} \right\} v^n(t) u'(t) dt + v^n(s_0) u'(s_0) \right].
\]

(12)
Next we shall apply L’Hospital’s theorem to this.
Lemma 4.1 If the limit \( u'(s) \) as \( s \to +0 \) exists, then it holds that

\[
\lim_{s \to +0} u'(s) = e(\theta_i - \phi_i).
\]

Proof. Since \( u'(s) \) is a unit vector, so is its limit, say \( e(\psi_i) \). Under the assumption we can apply L’Hospital’s theorem to (12), and get

\[
\begin{align*}
\cos \psi_i &= \frac{1}{n_i} \lim_{s \to +0} \left\{ (n - 1)H(s) - \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot u'(s) \right\} v(s) \\
&= -\frac{1}{n_i} \lim_{s \to +0} \sum_{j \neq i} n_j e(\phi_j - \phi_i) \cdot e(\psi_i) v(s) e(\phi_j - \phi_i) \cdot u(s) \\
&= -\frac{1}{n_i} \lim_{s \to +0} \sum_{j \neq i} n_j \cos(\phi_j - \phi_i - \psi_i) v(s) e(\phi_j - \phi_i) \cdot u(s).
\end{align*}
\]

(13)

Since \( u \in S_0 \) in \( uv \)-plane, it holds that \( \psi_i \in [0, \phi_{i+1} - \phi_i] \).

We will show \( \psi_i \in (0, \phi_{i+1} - \phi_i) \). Assume \( \psi_i = 0 \), and then

\[
e(\phi_j - \phi_i) \cdot e(\psi_i) = \sin(\phi_j - \phi_i) \neq 0
\]

for \( j \neq i \). By use of L’Hopital’s theorem again we have

\[
\lim_{s \to +0} \frac{v(s)}{e(\phi_j - \phi_i) \cdot u(s)} = \lim_{s \to +0} \frac{v'(s)}{e(\phi_j - \phi_i) \cdot u'(s)} = \frac{\sin \psi_i}{\sin(\phi_j - \phi_i)} = 0.
\]

Hence, from (13) it follows that

\[
\cos \psi_i = -\frac{1}{n_i} \times 0 = 0.
\]

This contradicts the assumption \( \psi_i = 0 \). We can show \( \psi_i \neq \phi_{i+1} - \phi_i \) in a similar argument to \( R(-\phi_{i+1})x \).

We have already known \( \psi_i \in (0, \phi_{i+1} - \phi_i) \), and therefore

\[
e(\phi_j - \phi_i) \cdot e(\psi_i) = \sin(\phi_j - \phi_i - \psi_i) \neq 0
\]
for \( j \neq i \). We obtain

\[
\cos \psi_s = -\frac{1}{n_i} \lim_{s \to +0} \sum_{j \neq i} n_j \cos(\phi_j - \phi_i - \psi_s)v(s)
\]

\[
= -\frac{1}{n_i} \lim_{s \to +0} \sum_{j \neq i} n_j \cos(\phi_j - \phi_i - \psi_s)v'(s)
\]

\[
= -\frac{\sin \psi_s}{n_i} \sum_{j \neq i} n_j \cos(\phi_j - \phi_i - \psi_s)
\]

\[
= -\frac{\sin \psi_s}{n_i} \sum_{j \neq i} n_j \cos(\phi_j - \phi_i - \psi_s)
\]

\[
= -\frac{\sin \psi_s}{n_i} \sum_{j \neq i} n_j \cos(\phi_j - \phi_i - \psi_s)
\]

by L’Hospital’s theorem. This shows

\[
\sum_{j \in J} n_j \cot(\psi_s + \phi_i - \phi_j) = 0,
\]

and therefore \( \psi_s = \theta_i - \phi_i \). \( \square \)

**Lemma 4.2** On a neighborhood of \( s \to +0 \), we have

\[
u'(s) > 0, \quad v'(s) > 0, \quad 0 < \frac{v'(s)}{u'(s)} < \tan(\phi_{i+1} - \phi_i).
\]

**Proof.** Since \( \lim_{s \to +0} v(s) = 0 \), and since \( v(s) > 0, v'(s) \neq 0 \) for small \( s > 0 \), we have

\[
v'(s) = e(0) \cdot u'(s) > 0.
\]

Applying a similar argument to \( R(-\phi_{i+1})x \), we have

\[-e(\phi_{i+1} - \phi_i) \cdot u'(s) > 0,
\]

i.e.,

\[
u'(s) \sin(\phi_{i+1} - \phi_i) - v'(s) \cos(\phi_{i+1} - \phi_i) > 0.
\]

Because of \( 0 < \phi_{i+1} - \phi_i \leq \frac{\pi}{2} \), we have \( \sin(\phi_{i+1} - \phi_i) > 0, \cos(\phi_{i+1} - \phi_i) \geq 0 \). Therefore, we obtain

\[
u'(s) > \frac{v'(s) \cos(\phi_{i+1} - \phi_i)}{\sin(\phi_{i+1} - \phi_i)} \geq 0, \quad 0 < \frac{v'(s)}{u'(s)} < \tan(\phi_{i+1} - \phi_i)
\]
**Corollary 4.1** It holds that

\[ 0 \leq \liminf_{s \to +0} \frac{v'(s)}{u'(s)} \leq \liminf_{s \to +0} \frac{v(s)}{u(s)} \leq \limsup_{s \to +0} \frac{v(s)}{u(s)} \leq \limsup_{s \to +0} \frac{v'(s)}{u'(s)} \leq \tan(\phi_{i+1} - \phi_i). \]

**Proof.** It is by virtue of previous lemma and L’Hospital’s theorem for limit superior and limit inferior. \( \square \)

**Lemma 4.3** There exists the limit of \( \frac{u(s)}{\|u(s)\|} \) as \( s \to +0 \).

**Proof.** Put \( w(s) = \frac{v(s)}{u(s)} \), \( z(s) = \frac{v'(s)}{u'(s)} \), \( \liminf_{s \to b-0} w(s) = L \), \( \limsup_{s \to b-0} w(s) = \bar{L} \), and \( L = \tan(\theta_i - \phi_i) \).

Assume \( L \neq \bar{L} \) and \( L < \bar{L} \). And, then, taking into consideration of the shape of the generating curve, there exist sequences \( \{s_j\} \) and \( \{\tilde{s}_j\} \) such that

\[ s_j > \tilde{s}_j > s_{j+1} > \tilde{s}_{j+1}, \quad \lim_{j \to \infty} s_j = +0, \quad \lim_{j \to \infty} \tilde{s}_j = +0, \]

\[ \lim_{j \to \infty} w(s_j) = L, \quad \lim_{j \to \infty} w(\tilde{s}_j) = \bar{L}, \]

the generating curve is tangent to the line \( v = L_j u \) at \( s = s_j \), and \( \lim_{j \to \infty} L_j = \frac{L}{L} \).

The last property implies

\[ z(s_j) = L_j \to \bar{L} \quad \text{as} \quad j \to \infty. \]

Put

\[ B_\varepsilon = \{(w, z) \in \mathbb{R}^2 \mid (w - \frac{L}{L})^2 + (z - \frac{L}{L})^2 < \varepsilon^2\}. \]

If \( \varepsilon > 0 \) is sufficiently small, then we may assume that

\[ (w(s_j), z(s_j)) \in B_\varepsilon, \quad (w(\tilde{s}_j), z(\tilde{s}_j)) \in \partial B_\varepsilon. \]

Hence, there exists \( \{\tilde{s}_j\} \) such that

\[ s_j > \tilde{s}_j > \tilde{s}_j, \quad (w(s), z(s)) \in \bar{B}_\varepsilon \quad \text{for} \quad s \in (\tilde{s}_j, s_j), \quad (w(\tilde{s}_j), z(\tilde{s}_j)) \in \partial B_\varepsilon. \]
Now we consider the behavior of \((w(s), z(s))\) on the interval \(I_j = [\hat{s}_j, s_j]\). Then
\[
\frac{1}{2} \frac{d}{ds} (w(s) - L)^2 = (w(s) - L)w'(s) \\
= (w(s) - L) \frac{v'(s)u(s) - v(s)u'(s)}{u(s)^2} \\
= \frac{u'(s)}{u(s)}(w(s) - L)(z(s) - w(s)).
\]
When \(s \in I_j\),
\[
|w(s) - L| \leq C, \\
|z(s) - w(s)| = |z(s) - L - (w(s) - L)| \leq 2\epsilon.
\]
Therefore,
\[
\left| \frac{u'(s)}{u(s)} - \frac{v'(s)}{v(s)} \right| = \left| w(s) - z(s) \right| \frac{u'(s)}{v(s)} \leq 2\epsilon \left| \frac{u'(s)}{v(s)} \right|,
\]
which implies
\[
\frac{u'(s)}{u(s)} = \frac{v'(s) + O(\epsilon)u'(s)}{v(s)}.
\]
Consequently,
\[
\left| \frac{1}{2} \frac{d}{ds} (w(s) - L)^2 \right| = \left| \frac{v'(s) + O(\epsilon)y'(s)}{v(s)} \right| O(\epsilon) = \frac{O(\epsilon)}{v(s)}.
\]
Here we use \(|u'(s)| \leq 1, |v'(s)| \leq 1\). On the other hand
\[
\frac{1}{2} \frac{d}{ds} (z(s) - L)^2 = (z(s) - L)z'(s) \\
= (z(s) - L) \frac{v''(s)u'(s) - v'(s)u''(s)}{(u'(s))^2} \\
= \frac{z(s) - L}{(u'(s))^2} u''(s) \cdot u'(s) \\
= \frac{z(s) - L}{(u'(s))^2} \left\{ -\sum_{j \in J} \frac{n_j}{e(\phi_j - \phi_i) \cdot u(s)} (n - 1)H(s) \right\}.
\]
Define \(\hat{\theta}\) and \(\tilde{\theta}\) by \(w(s) = \tan(\hat{\theta}(s) - \phi_i)\), and \(z(s) = \tan(\tilde{\theta}(s) - \phi_i)\). Then
\[
u(s) = \frac{u(s)}{\cos(\hat{\theta}(s) - \phi_i)} e(\hat{\theta}(s) - \phi_i), \quad u'(s) = \frac{u'(s)}{\cos(\hat{\theta}(s) - \phi_i)} e(\hat{\theta}(s) - \phi_i),
\]
\[
\frac{e(\phi_j - \phi_i) \cdot u'(s)}{e(\phi_j - \phi_i) \cdot u(s)} = \frac{u'(s)w(s) \cos(\hat{\theta}(s) - \phi_i) \cos(\tilde{\theta}(s) - \phi_j) - v(s) \cos(\tilde{\theta}(s) - \phi_i) \sin(\hat{\theta}(s) - \phi_j)}{\cos(\hat{\theta}(s) - \phi_i) \sin(\tilde{\theta}(s) - \phi_j)}.
\]
Define $\theta$ by $L = \tan(\theta - \phi_i)$. When $(w(s), z(s)) \in B_\varepsilon$, we have

$$\hat{\theta}(s) = \theta + O(\varepsilon), \quad \tilde{\theta}(s) = \theta + O(\varepsilon).$$

Hence,

$$\sum_{j \in J} n_j e(\phi_j - \phi_i) \cdot u'(s) = \frac{u'(s) \cos(\tilde{\theta}(s) - \phi_i)}{u(s) \cos(\theta(s) - \phi_i)} \sum_{j \in J} n_j (\cot(\theta - \phi_j) + O(\varepsilon))$$

$$= \frac{u'(s) \cos(\tilde{\theta}(s) - \phi_i)}{u(s) \cos(\theta(s) - \phi_i)} A(\theta) + O(\varepsilon).$$

Our assumption $L < L$ implies $\theta < \theta_i$, and therefore $A(\theta) > A(\theta_i) = 0$. There exists $\lambda \in (0, 1)$ such that

$$0 < w(s) < \frac{(1 + \lambda)L}{2}, \quad 0 < z(s) < \frac{(1 + \lambda)L}{2}, \quad 0 < u'(s) \leq 0, \quad v(s) > 0,$$

$$\cos(\tilde{\theta}(s) - \phi_i)) = \cos(\theta - \phi_i) + O(\varepsilon), \quad \cos(\hat{\theta}(s) - \phi_i)) = \cos(\theta - \phi_i) + O(\varepsilon)$$

hold on $I_j$ for large $j$. Hence, there exists $\delta > 0$ independent of $\varepsilon$ such that

$$\frac{z(s) - L}{(u'(s))^2} \sum_{j \in J} n_j e(\phi_j - \phi_i) \cdot u'(s) \leq \frac{-\delta}{v(s)}.$$

On the interval $I_j$,

$$\frac{1}{(x'(s))^2} = \frac{L^2(1 + o(1))}{(y'(s))^2} = \frac{L^2(1 + o(1))}{1 - (x'(s))^2}.$$

If there exists a sequence $\{\tilde{s}_k\} \subset \bigcup_j I_j$ such that

$$\lim_{k \to \infty} \tilde{s}_k = +0, \quad \lim_{k \to \infty} u'(\tilde{s}_k) = 0,$$

then as $k \to \infty$

$$\infty - \frac{1}{(u'(\tilde{s}_k))^2} = \frac{L^2(1 + o(1))}{1 - (u'(\tilde{s}_k))^2} \to L^2 < L^2.$$

This is a contradiction, therefore we may assume

$$\inf \left\{ (u'(s))^2 \bigg| s \in \bigcup_j I_j \right\} > 0.$$
Hence,

\[ \left| \frac{(n - 1)H(s)(z(s) - L)}{(u'(s))^2} \right| \leq C. \]

Consequently,

\[ \frac{1}{2} \frac{d}{ds} \left\{ (z(s) - L)^2 + (w(s) - L)^2 \right\} \leq -\frac{1}{v(s)} (\delta + O(\epsilon)) + C \]

on \( I_j \). If \( j \) is sufficiently large, then \( v(s) > 0 \) is sufficiently small. Taking \( \epsilon \) small, we have

\[ \frac{1}{2} \frac{d}{ds} \left\{ (z(s) - L)^2 + (w(s) - L)^2 \right\} \leq -\frac{\delta}{2v(s)} < 0 \]

on \( I_j \) for large \( j \). Hence,

\[ (w(\hat{s}_j) - L)^2 + (z(\hat{s}_j) - L)^2 \geq (z(s_j) - L)^2 + (w(s_j) - L)^2 \]

\[ = 2(L_j - L)^2. \]

Taking a suitable subsequence, we have \( (w(\hat{s}_j), z(\hat{s}_j)) \to (\hat{w}, \hat{z}) \), where

\[ (\hat{w}, \hat{z}) \in \partial B_\epsilon \cap \{ (w, z) \in \mathbb{R}^2 \mid (w - L)^2 + (z - L)^2 \geq 2(L - L)^2 \}. \]

This shows that

\[ \hat{w} < L \quad \text{or} \quad \hat{z} < L. \]

This leads to a contradiction. Indeed, if \( \hat{w} < L \), then

\[ \liminf_{s \to +0} w(s) = L > \hat{w} = \lim_{j \to \infty} w(\hat{s}_j) \geq \liminf_{s \to +0} w(s). \]

If \( \hat{z} < L \), then

\[ \liminf_{s \to +0} z(s) = L > \hat{z} = \lim_{j \to \infty} z(\hat{s}_j) \geq \liminf_{s \to +0} z(s). \]

Now we go back to the 5th line of the proof, and \( L = \bar{L} \) or \( L \leq \bar{L} \) has been proved.

Similarly we have \( L = \bar{L} \) or \( \bar{L} \leq L. \)

Combining these, we finally get \( L = \bar{L}. \)

\[ A(\alpha, \beta) = \sum_{j \in J} n_j \frac{\cos(\alpha - \phi_j)}{\sin(\beta - \phi_j)}. \]

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Here we assume $\beta \neq \phi_j$ for all $j \in J$. It follows that
\[
\frac{\partial A}{\partial \alpha} = -\sum_{j \in J} n_j \sin(\alpha - \phi_j) \sin(\beta - \phi_j).
\]
When $\phi_i < \alpha < \phi_{i+1}$ and $\phi_i < \beta < \phi_{i+1}$, it happens that
\[
\text{sgn} \sin(\alpha - \phi_j) = \text{sgn} \sin(\beta - \phi_j).
\]
Therefore, we have
\[
\frac{\partial A}{\partial \alpha} < 0.
\]

**Lemma 4.4** There exists the limit of $u'(s)$ as $s \to +0$ and
\[
\lim_{s \to +0} u'(s) = \lim_{s \to +0} \frac{u(s)}{\|u(s)\|} = e(\theta_i - \phi_i).
\]

**Proof.** It is enough to show
\[
\lim_{s \to +0} \frac{v'(s)}{u'(s)} = \lim_{s \to +0} \frac{v(s)}{u(s)} = \tan(\theta_i - \phi_i).
\]
If $\frac{v'(s)}{u'(s)}$ is monotone near $s = +0$, then there exists $\lim_{s \to +0} \frac{v'(s)}{u'(s)}$.

Otherwise, we put
\[
\liminf_{s \to +0} \frac{v'(s)}{u'(s)} = \tan(\theta' - \phi_i), \quad \limsup_{s \to +0} \frac{v'(s)}{u'(s)} = \tan(\bar{\theta}' - \phi_i).
\]

There exists a sequence $\{s_k\}$ such that $v'(s)u'(s)$ takes a minimum value at $s = s_k$ and
\[
s_k \to 0, \quad \frac{v'(s_k)}{u'(s_k)} \to \tan(\theta' - \phi_i), \quad \left(\frac{v'}{u'}\right)'(s_k) = 0
\]
as $k \to \infty$. From the third relation it follows that $u''(s_k) \cdot u'(s_k) = 0$. By using the equation we have
\[
\|u(s_k)\|(n - 1)H(s_k) = \sum_{j \in J} n_j e(\phi_j - \phi_i) \cdot \frac{u'(s_k)}{\|u(s_k)\|} \to A(\theta', \bar{\theta})
\]
as $k \to \infty$. Because of the boundedness of $H$, it is clear that
\[
\|u(s_k)\|(n - 1)H(s_k) \to 0.
\]
Hence, we have $A(\theta', \bar{\theta}) = 0$. Using a sequence of $s$ where $v'(s)u'(s)$ takes a maximum value, we get $A(\bar{\theta}', \bar{\theta}) = 0$. Combining these, we know $A(\bar{\theta}', \bar{\theta}) = A(\bar{\theta}', \bar{\theta})$. Since $\frac{\partial A}{\partial \alpha} < 0$, we obtain $\theta' = \bar{\theta}$.

Consequently, in any of these cases, $\lim_{s \to +0} \frac{v'(s)}{u'(s)}$ exists. From Lemma 4.1 it follows that the limit value is $\tan(\theta_i - \phi_i)$. Finally we know $\lim_{s \to +0} \frac{v(s)}{u(s)} = \tan(\theta_i - \phi_i)$ by Corollary 4.1.

Thus we have completed the proof of Proposition 4.1.

Next we prove the converse of Proposition 4.1, i.e., the solvability of (2) and (3). We define the function $q$ as before. For case (ii),

$$\lim_{v \to 0} q(v) = \cot(\theta_i - \phi_i).$$

Hence, we introduce new unknown functions $r$ and $\rho$ by

$$r(v) = q(v) - \cot(\theta_i - \phi_i), \quad \rho(v) = \frac{1}{v} \int_0^v r(\eta) \, d\eta.$$  

Then our problem is equivalent to

$$v \frac{dr}{dv}(v) + (n - 2)r(v) = -\gamma(n - 2)\rho(v) + \sum_{k=1}^{5} F_k(r, \rho)(v),$$

$$r(0) = \rho(0) = 0,$$

where

$$\gamma = \frac{\pi}{2} J - 1,$$

$$F_1(r, \rho) = F_1(r) = -(n - 2)r^2 \{r + 2 \cot(\theta_i - \phi_i)\} \sin^2(\theta_i - \phi_i),$$

$$F_2(r, \rho) = -\gamma(n - 2)r \rho \{r + 2 \cot(\theta_i - \phi_i)\} \sin^2(\theta_i - \phi_i),$$

$$F_3(r, \rho) = -\rho \left\{r^2 + 2r \cot(\theta_i - \phi_i) + \cosec^2(\theta_i - \phi_i)\right\}$$

$$\times \sum_{j \in J} \frac{n_j \sin^2(\theta_i - \phi_i) \cos(\phi_j - \phi_i) \sin(\phi_j - \phi_i)}{\sin(\theta_i - \phi_j) \{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)\}}$$

$$F_4(r, \rho) = -\rho \left\{r^2 + 2r \cot(\theta_i - \phi_i) + \cosec^2(\theta_i - \phi_i)\right\}$$

$$\times \sum_{j \in J} \frac{n_j \sin^2(\theta_i - \phi_i) \sin^2(\phi_j - \phi_i) \cos(\theta_i - \phi_j)}{\sin^2(\theta_i - \phi_j) \{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)\}},$$

and

$$F_5(r, \rho)(v) = F_5(r)(v) = (n - 1) \left[\{\cot(\theta_i - \phi_i) + r(v)\}^2 + 1\right]^{\frac{3}{2}} \tilde{H}(v)v.$$
Our derivation of (14) is elementary but it needs lengthy calculations, so we present it in the Appendix.

Multiplying both sides of the first equation in (14) by $v^{n-3}$, and integrating from 0 to $v$, we have

$$r(v) = -\frac{\gamma(n-2)}{v^{n-2}} \int_0^v \rho(\eta) \eta^{n-3} d\eta + \frac{1}{v^{n-2}} \int_0^v \sum_{k=1}^5 \psi_k(r, \rho)(\eta) d\eta,$$

where

$$\psi_k(r, \rho)(\eta) = F_k(r, \rho)(\eta)$$.  

Since the function $\rho$ defined by $r$, we can define the map $\Psi$ by

$$\Psi(r)(v) = -\frac{\gamma(n-2)}{v^{n-2}} \int_0^v \rho(\eta) \eta^{n-3} d\eta + \frac{1}{v^{n-2}} \int_0^v \sum_{k=1}^5 \psi_k(r, \rho)(\eta) d\eta.$$  

Taking $M$ large, and $V$ small, we can show this is a contraction map from $X_{V,M}$ into itself for Types II–III. This fact can be proved in the same way as [4]. Indeed, the principal part of $\Psi$ is the map

$$\bar{\Psi} : r \mapsto -\frac{\gamma(n-2)}{v^{n-2}} \int_0^v \rho(\eta) \eta^{n-3} d\eta.$$  

Because

$$\left| -\frac{\gamma(n-2)}{v^{n-1}} \int_0^v (\rho_1(\eta) - \rho_2(\eta)) \eta^{n-3} d\eta \right| \leq \frac{\gamma(n-2)}{2(n-1)} \| r_1 - r_2 \|,$$

The map $\bar{\Psi}$ is contractive when $\gamma \leq 2$, which holds for Types II–III. Since $\Psi$ is a small perturbation of $\bar{\Psi}$, it is also contractive for Types II–III. Testing with the linear functions $r_i = c_i v$, we find that the map $\bar{\Psi}$ is expansive for Types IV–V. This suggests that $\Psi$ may not be contractive for these types, and therefore we must deal with our problem more carefully.

Since

$$\frac{d\rho}{dv} = -\frac{1}{v^2} \int_0^v r \, d\eta + \frac{r}{v} = -\frac{\rho}{v} + \frac{r}{v},$$

we have

$$v \frac{d}{dv} \begin{pmatrix} r \\ \rho \end{pmatrix} + \begin{pmatrix} n-2 & \gamma(n-2) \\ -1 & 1 \end{pmatrix} \begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^5 F_k(r, \rho) \\ 0 \end{pmatrix}.$$  

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Eigenvalues of the matrix in the left-hand side are 

\[ \lambda_\pm = \frac{n - 1 \pm \sqrt{n^2 - 2(2\gamma + 3)n + 8\gamma + 9}}{2}. \]

Since 

\[(2\gamma + 3)^2 - (8\gamma + 9) = 4\gamma(\gamma + 1) > 0,\]

we know \( \lambda_+ \neq \lambda_- \). Therefore, there exists a non-singular matrix \( P \) such that

\[ P^{-1} \begin{pmatrix} n - 2 & \gamma(n - 2) \\ -1 & 1 \end{pmatrix} P = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \]

Put \( P = (p_{ij}) \), \( P^{-1} = (p^{ij}) \). These are matrices with constant entries. Define \( \hat{r} \) and \( \hat{\rho} \) by

\[ \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} = P^{-1} \begin{pmatrix} r \\ \rho \end{pmatrix}. \]

Then the equation can be rewritten as

\[ v \frac{d}{dv} \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} + \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} p^{11} \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho}) \\ p^{21} \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho}) \end{pmatrix}, \quad \hat{r}(0) = \hat{\rho}(0) = 0, \]

where

\[ \hat{F}_k(\hat{r}, \hat{\rho}) = F_k(p_{11} \hat{r} + p_{12} \hat{\rho}, p_{21} \hat{r} + p_{22} \hat{\rho})(= F_k(r, \rho)). \]

Hence, we have

\[ \hat{r}(v) = \frac{p^{11}}{v^{\lambda_+}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\lambda_+ - 1} d\eta, \quad \hat{\rho}(v) = \frac{p^{21}}{v^{\lambda_-}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\lambda_- - 1} d\eta. \]

Considering the Banach spaces \( X_V \) defined earlier in this paper, let \( X_V \times X_V \) be the Banach space with norm

\[ \| (\hat{r}, \hat{\rho}) \|_{X_V \times X_V} = \| \hat{r} \|_{X_V} + \| \hat{\rho} \|_{X_V}. \]

Define the map \( \hat{\Psi} \) by

\[ \hat{\Psi}(\hat{r}, \hat{\rho})(v) = \left( \frac{p^{11}}{v^{\lambda_+}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\lambda_+ - 1} d\eta, \frac{p^{21}}{v^{\lambda_-}} \int_0^v \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\lambda_- - 1} d\eta \right). \]

We will show that if \( M \) is large, and of \( V \) is small, then \( \hat{\Psi} \) is a contraction map from \( X_{V,M} \times X_{V,M} \) into itself.
When $n^2 - 2(2\gamma + 3)n + 8\gamma + 9 < 0$,
\[ \Re\lambda_\pm = \frac{n - 1}{2} > 0. \]

If $n^2 - 2(2\gamma + 3)n + 8\gamma + 9 \geq 0$, then
\[ 0 \leq n^2 - 2(2\gamma + 3)n + 8\gamma + 9 = (n - 1)^2 - 4(\gamma + 1)(n - 2) < (n - 1)^2, \]
and hence
\[ \Re\lambda_\pm = \frac{n - 1 \pm \sqrt{n^2 - 2(2\gamma + 3)n + 8\gamma + 9}}{2} > 0. \]

Therefore, in any of the cases, the integral $\int_0^\eta \eta^{\Re\lambda_\pm + p} d\eta$ converges for $p > -1$, and
\[ \int_0^\eta \eta^{\Re\lambda_\pm + p} d\eta = \frac{v^{\Re\lambda_\pm + p + 1}}{\Re\lambda_\pm + p + 1}. \]

Since both $P$ and $P^{-1}$ are constant matrices, it holds that
\[ \| (r, \rho) \| \leq C \| (\hat{r}, \hat{\rho}) \|, \quad \| (\hat{r}, \hat{\rho}) \| \leq C \| (r, \rho) \|. \]

Let $(\hat{r}, \hat{\rho}) \in X_{V,M} \times X_{V,M}$, and then we have
\[ |\hat{F}_1(\hat{r}, \hat{\rho})(\eta)| = |F_1(r, \rho)(\eta)| \leq C \| r \|^2 \eta^2 (\| r \| \eta + 1) \leq C \| (\hat{r}, \hat{\rho}) \|^2 \eta^2 (\| (\hat{r}, \hat{\rho}) \| \eta + 1) \leq CM^2\eta^2 (M\eta + 1). \]

Therefore, we get
\[ \left| \frac{1}{v^{\Re\lambda_\pm + 1}} \int_0^\eta \hat{F}_1(\hat{r}, \hat{\rho})(\eta) \eta^{\Re\lambda_\pm - 1} d\eta \right| \leq \frac{C}{v^{\Re\lambda_\pm + 1}} \int_0^\eta M^2 \eta^2 (M\eta + 1) \eta^{\Re\lambda_\pm - 1} d\eta \leq \frac{C}{v^{\Re\lambda_\pm + 1}} (M^3 v^{\Re\lambda_\pm + 3} + M^2 v^{\Re\lambda_\pm + 2}) \leq C (M^3 V^2 + M^2 V). \]

We can estimate $\frac{1}{v^{\Re\lambda_\pm + 1}} \int_0^\eta \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\Re\lambda_\pm - 1} d\eta$ for $k = 2, 3, 4, 5$ in a similar manner, and we obtain
\[ \left| \frac{p_{11}^{11}}{v^{\Re\lambda_\pm + 1}} \int_0^\eta \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\Re\lambda_\pm - 1} d\eta \right| \leq C (M^3 V^2 + M^2 V + M^4 V^3 + M^3 V^3 + 1). \]

We can also derive
\[ \left| \frac{p_{21}^{21}}{v^{\Re\lambda_\pm + 1}} \int_0^\eta \sum_{k=1}^5 \hat{F}_k(\hat{r}, \hat{\rho})(\eta) \eta^{\Re\lambda_\pm - 1} d\eta \right| \leq C (M^3 V^2 + M^2 V + M^4 V^3 + M^3 V^3 + 1). \]
in the same manner. From these it follows that
\[
\| \hat{\Psi}(\hat{r}, \hat{\rho}) \| \leq C \left( M^3 V^2 + M^2 V + M^4 V^3 + M^3 V^3 + 1 \right).
\]

Consequently, \( \hat{\Psi} \) is a map from \( X_{V,M} \times X_{V,M} \) into itself provided \( M \) is large and \( V \) is small.

Using
\[
\| (r_1, \rho_1) - (r_2, \rho_2) \| \leq C \| (\hat{r}_1, \hat{\rho}_1) - (\hat{r}_2, \hat{\rho}_2) \|,
\]
\[
\| (\hat{r}_1, \hat{\rho}_1) - (\hat{r}_2, \hat{\rho}_2) \| \leq C \| (r_1, \rho_1) - (r_2, \rho_2) \|,
\]
we can get
\[
\| \hat{\Psi}(\hat{r}_1, \hat{\rho}_1) - \hat{\Psi}(\hat{r}_2, \hat{\rho}_2) \| \leq C \left( M^2 V^2 + MV + M^4 V^4 + M^2 V^3 + V \right) \| (\hat{r}_1, \hat{\rho}_1) - (\hat{r}_2, \hat{\rho}_2) \|.
\]

Indeed, from
\[
\left| \hat{F}_1(\hat{r}_1, \hat{\rho}_1) - \hat{F}_1(\hat{r}_2, \hat{\rho}_2) \right| = |F_1(r_1) - F_1(r_2)|
\]
\[
= |-(n - 2)(r_1 - r_2) \left\{ r_1^2 + r_1 r_2 + r_2 + 2(r_1 + r_2) \cot(\theta_i - \phi_i) \right\} \sin^2(\theta_i - \phi_i) |
\]
\[
\leq C \| r_1 - r_2 \| \eta \left( M^2 \eta^2 + M \eta \right)
\]
\[
\leq C \| \hat{r}_1 - \hat{r}_2 \| \eta \left( M^2 \eta^2 + M \eta \right)
\]

it follows that
\[
\left| \frac{p^{11}}{v^{\lambda + 1}} \int_0^v \left( \hat{F}_1(r_1, \rho_1)(\eta) - \hat{F}_1(r_2, \rho_2)(\eta) \right) \eta^{\lambda - 1} d\eta \right|
\]
\[
\leq C \| \hat{r}_1 - \hat{r}_2 \| \int_0^v \left( M^2 \eta^3 + M \eta^2 \right) \eta^{\lambda - 1} d\eta
\]
\[
\leq C \| \hat{r}_1 - \hat{r}_2 \| \left( M v^{\lambda + 3} + M v^{\lambda + 2} \right)
\]
\[
\leq C \left( M^2 V^2 + MV \right) \| \hat{r}_1 - \hat{r}_2 \|.
\]

Similarly we have
\[
\left| \frac{p^{11}}{v^{\lambda + 1}} \int_0^v \left( \hat{F}_2(r_1, \rho_1)(\eta) - \hat{F}_2(r_2, \rho_2)(\eta) \right) \eta^{\lambda - 1} d\eta \right|
\]
\[
\leq C \left( M^2 V^2 + MV \right) \| \hat{r}_1 - \hat{r}_2 \|,
\]

\[
\left| \frac{p^{11}}{v^{\lambda + 1}} \int_0^v \left( \hat{F}_k(r_1, \rho_1)(\eta) - \hat{F}_k(r_2, \rho_2)(\eta) \right) \eta^{\lambda - 1} d\eta \right|
\]
\[
\leq C \left( M^4 V^4 + MV \right) \left( \| \hat{r}_1 - \hat{r}_2 \| + \| \hat{\rho}_1 - \hat{\rho}_2 \| \right)
\]
for \( k = 3, 4, \)
\[
\left| \frac{p^{11}}{v^{\lambda + 1}} \int_0^v \left( \hat{F}_5(r_1, \rho_1)(\eta) - \hat{F}_5(r_2, \rho_2)(\eta) \right) \eta^{\lambda - 1} d\eta \right|
\]
\[
\leq C \left( M^2 V^3 + V \right) \| \hat{r}_1 - \hat{r}_2 \|.
\]
Therefore, it holds that

\[
\left\| \frac{p^{11}}{v^{N+}} \int_0^v \sum_{k=1}^5 \left( \hat{F}_k(\hat{r}_1, \hat{\rho}_1)(\eta) - \hat{F}_k(\hat{r}_2, \hat{\rho}_2)(\eta) \right) \eta^{N+1} d\eta \right\|_{X_V} \leq C \left( M^2V^2 + MV + M^4V^4 + M^2V^3 + V \right) \| (\hat{r}_1, \hat{\rho}_1) - (\hat{r}_2, \hat{\rho}_2) \|.
\]

We can also derive

\[
\left\| \frac{p^{21}}{v^{N-}} \int_0^v \sum_{k=1}^5 \left( \hat{F}_k(\hat{r}_1, \hat{\rho}_1)(\eta) - \hat{F}_k(\hat{r}_2, \hat{\rho}_2)(\eta) \right) \eta^{N-1} d\eta \right\|_{X_V} \leq C \left( M^2V^2 + MV + M^4V^4 + M^2V^3 + V \right) \| (\hat{r}_1, \hat{\rho}_1) - (\hat{r}_2, \hat{\rho}_2) \|
\]

in the same manner.

Consequently, the map \( \hat{\Psi} \) is contraction if \( V \) is sufficiently small. The unique fixed point is a local solution to (14).

Because eigenvalues \( \lambda_{\pm} \) and matrix \( P \) are not necessarily real, our solution might not be real-valued. Therefore, we must confirm that our \( r \) and \( \rho \) are real-valued. Putting \( r_I = \Re r \) and \( \rho_I = \Re \rho \), we want to show \( r_I = \rho_I \equiv 0 \). It is easy to see that \( r_I \) satisfies

\[
v \frac{d}{dv} r_I + (n - 2)r_I + \gamma(n - 2)\rho_I = \sum_{k=1}^5 \Re F_k(r, \rho).
\]

Multiplying both sides by \( 2r_I \), we have

\[
\frac{d}{dv} \left[ v \left\{ r_I^2 + \gamma(n - 2)\rho_I^2 \right\} \right] + (2n - 5)r_I^2 + \gamma(n - 2)\rho_I^2 = 2r_I \sum_{k=1}^5 \Re \hat{F}_k(\hat{r}, \hat{\rho}).
\]

We shall use the following lemma first and give its proof immediately after.

**Lemma 4.5** Assume that \( V \) is sufficiently small. There exists a positive constant \( C \) depending on \( M \) and \( V \) such that

\[
|\Re \hat{F}_k(\hat{r}, \hat{\rho})| \leq C v \left( |r_I| + |\rho_I| \right)
\]

Using the lemma, \( n - 2 > 0 \), and \( \gamma > 0 \), we have

\[
v \left\{ r_I^2 + \gamma(n - 2)\rho_I^2 \right\} \leq C \int_0^v \eta \left\{ r_I^2 + \gamma(n - 2)\rho_I^2 \right\} d\eta.
\]

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From Gronwall’s lemma it follows that

\[ r_i \equiv 0, \quad \rho_i \equiv 0. \]

**Proof of Lemme 4.5** for \( \mathfrak{F}_1, \ldots, \mathfrak{F}_4 \). Put \( \Re r = r_R, \Re \rho = \rho_R \). Since \( r_R, r_i, \rho_R, \rho_i \in X_{V,M} \), these moduli are dominated by \( Cv \). Therefore, we have

\[
|\mathfrak{F}_1(\hat{r}, \hat{\rho})| = |\mathfrak{F}_1(r)| \leq C \left( |\Im (r^3)| + |\Im (r^2)| \right)
\]

\[
\leq C \left( |r_R r_i| + |r_R^2| + |r_R r_i| \right)
\]

\[
\leq C (v^2 + v) |r_i|,
\]

\[
|\mathfrak{F}_2(\hat{r}, \hat{\rho})| = |\mathfrak{F}_2(r, \rho)| \leq C \left( |\Im (r^2 \rho)| + |\Im (r \rho)| \right)
\]

\[
\leq C \left( r^2 |r_i| + |r_R \rho_R| |r_i| + |\rho_R^2| + |r_R | |\rho_i| \right)
\]

\[
\leq C (v^2 + v) (|r_i| + |\rho_i|).
\]

\( \mathfrak{F}_3 \) is estimated as follows:

\[
|\mathfrak{F}_3(\hat{r}, \hat{\rho})| = |\mathfrak{F}_3(r, \rho)|
\]

\[
\leq |\Im \left[ r \rho \left\{ r^2 + 2r \cot(\theta_i - \phi_i) + \csc^2(\theta_i - \phi_i) \right\} \right] |
\]

\[
\leq C \left( |\Im (r^3 \rho)| + |\Im (r^2 \rho)| + |\Im (r \rho)| \right)
\]

\[
\leq C v (|r_i| + |\rho_i|).
\]

We can estimate each term as follows:

\[
|\Im \left[ r \rho \left\{ r^2 + 2r \cot(\theta_i - \phi_i) + \csc^2(\theta_i - \phi_i) \right\} \right] |
\]

\[
\leq C \left( |\Im (r^3 \rho)| + |\Im (r^2 \rho)| + |\Im (r \rho)| \right)
\]

\[
\leq C v (|r_i| + |\rho_i|),
\]

\[
|\Im \left[ \sum_{j \in J} \frac{n_j \sin^2(\theta_i - \phi_i) \cos(\phi_j - \phi_i) \sin(\phi_j - \phi_i)}{\sin(\theta_i - \phi_j) \{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)\}} \right] |
\]

\[
\leq C,
\]

\[
|\Im \left[ \sum_{j \in J} \frac{n_j \sin^2(\theta_i - \phi_i) \cos(\phi_j - \phi_i) \sin(\phi_j - \phi_i)}{\sin(\theta_i - \phi_j) \{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)\}} \right] |
\]

\[
\leq C v \sum_{j \in J} \left| \sin(\theta_i - \phi_j) \{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)\} \right|^2
\]

\[
\leq C |\rho_i|.
\]
Similarly it holds for $\Im \hat{F}_4$ that
\[
\Im \hat{F}_4(r, \rho) = \Im F_4(r, \rho) \\
\leq \Im \left[ \rho^2 \left\{ r^2 + 2r \cot(\theta_i - \phi_i) + \csc^2(\theta_i - \phi_i) \right\} \right] \\
\times \left| \Re \sum_{j \in J} \sin^2(\theta_i - \phi_j) \left\{ \sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \right\} \right| \\
+ \left| \Re \rho^2 \left\{ r^2 + 2r \cot(\theta_i - \phi_i) + \csc^2(\theta_i - \phi_i) \right\} \right| \\
\times \left| \Im \sum_{j \in J} \sin^2(\theta_i - \phi_j) \left\{ \sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \right\} \right| \\
\leq C \left( |\Im (\rho^2 r^2)| + |\Im (\rho^2 r)| + \Im (\rho^2) \right) + C \nu |\rho_i| \\
\leq C \nu (|r_i| + |\rho_i|).
\]

We need the following lemma for the estimate of $\Im \hat{F}_5$.

**Lemma 4.6** Let $B_R$ be the closed disc in $\mathbb{C}$ with center $O$ and with radius $R$. Assume that the function $f$ is analytic on $B_R$, and that $f|_{B_R \cap \mathbb{R}}$ is real-valued. Then there exists $C > 0$ such that

\[
|\Im f(r)| \leq C |r_i|
\]
holds for $r \in B_{R/2}$.

**Proof.** We have

\[
f(r) = \sum_{k=0}^{\infty} c_k r^k \quad \text{for} \quad r \in B_R, \quad \text{and} \quad \sum_{k=0}^{\infty} |c_k| R^k < \infty.
\]

Since $f|_{B_R \cap \mathbb{R}}$ is real-valued, so are $c_k$'s. Therefore, we get

\[
\Im f(r) = \sum_{k=1}^{\infty} c_k \Im (r^k).
\]

Furthermore, we have

\[
|\Im (r^k)| = |\Im (r_R + \sqrt{-1} r_i)^k| \leq \sum_{\ell=1}^{k} \ell C_\ell |r_R^{k-\ell} r_i^\ell| \leq \sum_{\ell=0}^{k} \ell C_\ell |r|^{k-1} |r_i| = \frac{(2 |r|)^k}{R} |r_i|.
\]

Consequently,

\[
\left| \sum_{k=1}^{\infty} c_k \Im (r^k) \right| \leq \frac{|r_i|}{R} \sum_{k=1}^{\infty} c_k (2 |r|)^k,
\]
and the right-hand side converges for $r \in B_{R/2}$. \qed
Proof of Lemma 4.5 for $\Im \hat{F}_5$.

The function $f(r) = \left[\{\cot(\theta_i - \phi_i) + r\}^2 + 1\right]^\frac{3}{2}$ satisfies the assumption of Lemma 4.6. Taking $V$ small, we may assume $|r_i(v)| \leq \frac{R}{2}$. Hence, we have

$$\left|\Im \left[\{\cot(\theta_i - \phi_i) + r(v)\}^2 + 1\right]^\frac{3}{2}\right| \leq C|r_i|.$$ 

Consequently, it holds that

$$|\Im \hat{F}_5(\hat{r}, \hat{\rho})| = |\Im F_5(r)| \leq C_v\left|\Im \left[\{\cot(\theta_i - \phi_i) + r(v)\}^2 + 1\right]^\frac{3}{2}\right| \leq C_v|r_i|.$$ 

Proposition 4.2 Let $H$ be continuous. Then there exists a unique local solution $x$ to (2) and (6).

5 Appendix

5.1 The values of $\theta_i$

Fact 5.1 If $A(\theta_i) = 0$, $\phi_i < \theta_i < \phi_{i+1}$, then

Type II: $\theta_0 = \arctan\sqrt{\frac{n_0}{n_1}}$,

Type III: $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$,

Type IV: $\theta_{i+1} = -\frac{1}{2}\arctan\sqrt{\frac{k}{\pi}}$, $\theta_0 = \frac{1}{2}\arctan\sqrt{\frac{k}{\pi}}$,

Type V: $\theta_i = \frac{1}{2}(\phi_i + \phi_{i+1})$.

Proof. For Type II: Since $J = \{0, 1\}$, $\phi_j = \frac{j}{2}\pi$, we have

$$0 = \sum_{j \in J} n_j \cot(\theta_0 - \phi_j) = n_0 \cot \theta_0 + n_1 \cot \left(\theta_0 - \frac{\pi}{2}\right) = n_0 \cot \theta_0 - n_1 \tan \theta_0.$$ 

Combining this with $\theta_0 \in (0, \frac{\pi}{2})$, we get the assertion.

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Next we deal with Types III–V. When \( \pm j \in J \), we have \( \phi_{-j} = -\phi_j \), \( n_{-j} = n_j \). Therefore, it holds that

\[
\cot(\theta_i - \phi_{-j}) + \cot(\theta_i - \phi_j) = \frac{\sin 2\theta_i}{\sin^2 \theta_i \cos^2 \phi_j - \cos^2 \theta_i \sin^2 \phi_j}.
\]

Furthermore, \( -\max J \notin J \), \( n_0 = n_{\max J} \), \( \phi_0 = 0 \) and \( \phi_{\max J} = \frac{\pi}{2} \) for Types IV and V. For these cases

\[
\cot(\theta_i - \phi_0) + \cot(\theta_i - \phi_{\max J}) = 2 \cot 2\theta_i.
\]

Using these relations we have

\[
0 = \sum_{j \in J} n_j \cot(\theta_i - \phi_j) = n_0 \sum_{j=-1}^{1} \cot \left( \frac{i}{j} \pi \right)
\]

\[
= n_0 \left[ \cot \left( \theta_i + \frac{\pi}{3} \right) + \cot \left( \theta_i - \frac{\pi}{3} \right) \right] + \cot \theta_i
\]

\[
= \frac{3n_0 \cos \theta_i (2 \sin \theta_i - 1) (2 \sin \theta_i + 1)}{\sin \theta_i (\sin^2 \theta_i - 3 \cos^2 \theta_i)}
\]

for Type III;

\[
0 = \sum_{j \in J} n_j \cot(\theta_i - \phi_j)
\]

\[
= \ell \left\{ \cot \left( \theta_i + \frac{\pi}{4} \right) + \cot \left( \theta_i - \frac{\pi}{4} \right) \right\} + k (\cot \theta_i - \tan \theta_i)
\]

\[
= -2\ell \tan 2\theta_i + 2k \cot 2\theta_i
\]

for Type IV;

\[
0 = \sum_{j \in J} n_j \cot(\theta_i - \phi_j) = n_0 \sum_{j=-2}^{3} \cot \left( \frac{i}{j} \pi \right)
\]

\[
= n_0 \left[ \left\{ \cot \left( \theta_i + \frac{\pi}{3} \right) + \cot \left( \theta_i - \frac{\pi}{3} \right) \right\} + \left\{ \cot \left( \theta_i + \frac{\pi}{6} \right) + \cot \left( \theta_i - \frac{\pi}{6} \right) \right\} \right]
\]

\[
+ \left\{ \cot \theta_i + \cot \left( \theta_i - \frac{\pi}{2} \right) \right\} \right]
\]

\[
= \frac{6n_0 \cos 2\theta_i (1 - 2 \sin 2\theta_i) (1 + 2 \sin 2\theta_i)}{\sin 2\theta_i (\sin^2 \theta_i - 3 \cos^2 \theta_i) (3 \sin^2 \theta_i - \cos^2 \theta_i)}
\]

for Type V. Taking \( \theta_i \in (\phi_i, \phi_{i+1}) \) into consideration, we have the assertion. \( \Box \)

**Remark 5.1** The result \( \theta_i = \frac{1}{2}(\phi_i + \phi_{i+1}) \) for Types III and V is by virtue of the symmetry \( n_j \equiv n_0 \).
5.2 The derivation of (14)

We insert \( r = q - \cot(\theta_i - \phi_i) \) into (10) with \( u(0) = 0 \). It is trivial that

\[
\frac{dq}{dv} = \frac{dr}{dv}
\]

\[(n - 1) (q^2 + 1)^\frac{3}{2} \dot{H} = (n - 1) \left[ (\cot(\theta_i - \phi_i) + r)^2 + 1 \right]^{\frac{3}{2}} \dot{H} = \frac{F_3(r)}{v}.
\]

The summation of remainder terms is

\[- \frac{n_i (q^2 + 1) q}{v} + \sum_{j \neq i} n_j (q^2 + 1) \{ q \cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i) \} \sum_{j \in J} \frac{n_j \{ q \cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i) \}}{\sin(\phi_j - \phi_i)} \int_0^v q(\eta) \, d\eta - v \cos(\phi_j - \phi_i) \]

\[= (q^2 + 1) \sum_{j \in J} n_j \{ q \cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i) \} \sum_{j \in J} \frac{n_j \{ q \cos(\phi_j - \phi_i) + \sin(\phi_j - \phi_i) \}}{\sin(\phi_j - \phi_i)} \int_0^v q(\eta) \, d\eta - v \cos(\phi_j - \phi_i) \]

\[= - \{ \cot^2(\theta_i - \phi_i) + 2r \cot(\theta_i - \phi_i) + r^2 + 1 \} \sum_{j \in J} n_j \{ \cos(\theta_i - \phi_j) + r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \} \sum_{j \in J} \frac{n_j \{ \cos(\theta_i - \phi_j) + r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \}}{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)} \]

\[= - \{ r^2 + 2r \cot(\theta_i - \phi_i) + \cosec^2(\theta_i - \phi_i) \} \sum_{j \in J} \frac{n_j \{ \cos(\theta_i - \phi_j) + r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \}}{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)} - \cot(\theta_i - \phi_i) \}

\[\times \sum_{j \in J} \frac{n_j \{ \cos(\theta_i - \phi_j) + r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \}}{\sin(\theta_i - \phi_j)} \sum_{j \in J} \frac{n_j \{ \cos(\theta_i - \phi_j) + r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \}}{\sin(\theta_i - \phi_j)} \]

\[= - \{ r^2 + 2r \cot(\theta_i - \phi_i) + \cosec^2(\theta_i - \phi_i) \} \sum_{j \in J} \frac{n_j \sin(\theta_i - \phi_i) \{ r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_j) + \rho \sin(\phi_j - \phi_i) \cos(\theta_i - \phi_j) \}}{v \sin(\theta_i - \phi_j) \{ \sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \}} \sum_{j \in J} \frac{n_j \{ \cos(\theta_i - \phi_j) + r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \}}{\sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i)} \].
Extracting the linear parts with respect to \( r \) and \( \rho \), we get

\[
\left\{ r^2 + 2r \cot(\theta_i - \phi_i) + \cosec^2(\theta_i - \phi_i) \right\} - \left\{ \sum_{j \in J} n_j \sin(\theta_i - \phi_i) \{ r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_j) + \rho \sin(\phi_j - \phi_i) \cos(\theta_i - \phi_j) \} \right\} \\
\times \frac{v \sin(\theta_i - \phi_j) \{ \sin(\theta_i - \phi_j) - \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_j) \}}{v \sin(\theta_i - \phi_j) \sin(\theta_i - \phi_j)}
\]

\[
= \frac{\left\{ \sum_{j \in J} n_j \sin(\theta_i - \phi_i) \{ r \cos(\phi_j - \phi_i) \sin(\theta_i - \phi_j) + \rho \sin(\phi_j - \phi_i) \cos(\theta_i - \phi_j) \} \right\} \sin(\theta_i - \phi_j)}{\sin(\theta_i - \phi_j)}
\]

\[
= -r \sum_{j \in J} \frac{n_j \rho \sin(\phi_j - \phi_i) \cos(\theta_i - \phi_j)}{v \sin(\theta_i - \phi_j) \sin(\theta_i - \phi_j)}
- r \left\{ \sum_{j \in J} n_j \sin(\theta_i - \phi_i) \cos(\phi_j - \phi_i) \cos(\theta_i - \phi_j) \right\} \\
+ \left\{ \sum_{j \in J} n_j \rho \sin(\phi_j - \phi_i) \sin(\theta_i - \phi_i) \sin(\theta_i - \phi_j) \right\} \\
+ \left\{ \sum_{j \in J} n_j \rho \sin^2(\theta_i - \phi_i) \sin(\phi_j - \phi_i) \cos(\theta_i - \phi_j) \right\}
\]

By using \( \sum_{j \in J} n_j \cot(\theta_i - \phi_j) = 0 \) again, the coefficient of the linear terms are simplified as follows:

\[
\sum_{j \in J} \frac{n_j \cos(\phi_j - \phi_i)}{\sin(\theta_i - \phi_i) \sin(\theta_i - \phi_j)} = \sum_{j \in J} \frac{n_j \cos \{(\phi_j - \theta_i) + (\theta_i - \phi_i)\}}{\sin(\theta_i - \phi_i) \sin(\theta_i - \phi_j)}
= \sum_{j \in J} n_j \frac{\cos(\phi_j - \theta_i) \cos(\theta_i - \phi_i) - \sin(\phi_j - \theta_i) \sin(\theta_i - \phi_i)}{\sin(\theta_i - \phi_i) \sin(\theta_i - \phi_j)}
= \cot(\theta_i - \phi_i) \sum_{j \in J} n_j \cot(\theta_i - \phi_j) + \sum_{j \in J} n_j \sin(\theta_i - \phi_i) \sin(\theta_i - \phi_j)
= \cot(\theta_i - \phi_i) \sum_{j \in J} n_j \cot(\theta_i - \phi_j) + \sum_{j \in J} n_j = \sum_{j \in J} n_j = n - 2.
\]
\[
\sum_{j \in J} n_j \sin(\phi_j - \phi_i) \cos(\theta_i - \phi_j) = \sum_{j \in J} n_j \sin \{ (\phi_j - \theta_i) + (\theta_i - \phi_j) \} \cos(\theta_i - \phi_j) \\
= \sum_{j \in J} n_j \sin(\phi_j - \theta_i) \cos(\theta_i - \phi_j) + \cos(\phi_j - \theta_i) \sin(\theta_i - \phi_j) \cos(\theta_i - \phi_j) \\
= -\cot(\theta_i - \phi_i) \sum_{j \in J} n_j \cot(\theta_i - \phi_j) + \sum_{j \in J} n_j \cot^2(\theta_i - \phi_j) \\
= \sum_{j \in J} n_j \cot^2(\theta_i - \phi_j).
\]

Consequently, we get (14) if

\[
\sum_{j \in J} n_j \cot^2(\theta_i - \phi_j) = \gamma(n - 2),
\]

which we shall prove next. To do this, we use Fact 5.1.

**Type II:** Since \(J = \{0, 1\}\), \(\gamma = \#J - 1 = 1\), \(\phi_j = \frac{j}{2} \pi\), and \(\tan^2 \theta_0 = \frac{n_0}{n_1}\), we get

\[
\sum_{j \in J} n_j \cot^2(\theta_i - \phi_j) = n_0 \cot^2 \theta_0 + n_1 \cot^2 \left( \theta_0 - \frac{\pi}{2} \right) = n_0 \cot^2 \theta_0 + n_1 \tan^2 \theta_0 \\
= n_0 \frac{n_1}{n_0} + n_1 \frac{n_0}{n_1} = n_1 + n_0 = \sum_{j \in J} n_j = \gamma(n - 2).
\]

As in the proof of Fact 5.1, we have the following for Types III–V. When \(\pm j \in J\), we have \(\phi_{-j} = -\phi_j\), \(n_{-j} = n_j\). Therefore, it holds that

\[
\cot^2(\theta_i - \phi_{-j}) + \cot^2(\theta_i - \phi_j) = \frac{2 \left( \sin^2 \theta_i \cos^2 \phi_j + \sin^2 \phi_j \cos^2 \phi_j \right)}{\sin^2 \theta_i \cos^2 \phi_j - \cos^2 \phi_j \sin^2 \phi_j}.
\]

Furthermore \(- \max J \not\in J\), \(n_0 = n_{\max J}\), \(\phi_0 = 0\) and \(\phi_{\max J} = \frac{\pi}{2}\) for Types IV and V. For these cases

\[
\cot^2(\theta_i - \phi_0) + \cot^2(\theta_i - \phi_{\max J}) = \frac{2(2 - \sin^2 2\theta_i)}{\sin^2 2\theta_i}.
\]

**Type III:** Since \(J = \{-1, 0, 1\}\), \(\gamma = \#J - 1 = 2\), \(\phi_j = \frac{j}{3} \pi\), \(n_j \equiv n_0\), and
\[
\sin^2 \theta_i = \frac{1}{4}, \quad \cos^2 \theta_i = \frac{3}{4}, \quad \text{for all } i \in J, \text{ we have}
\]
\[
\sum_{j \in J} n_j \cot^2 \theta_i - \phi_j = n_0 \left\{ \cot^2 \theta_i - \phi_{-1} + \cot^2 \theta_i - \phi_1 + \cot^2 \theta_i \right\}
\]
\[
= n_0 \left\{ \frac{2 \left( \sin^2 \theta_i \cos^2 \theta_i + \sin^2 \phi_1 \cos^2 \phi_1 \right)}{\left( \sin^2 \theta_i \cos^2 \phi_1 - \cos^2 \theta_i \sin^2 \phi_1 \right)^2 + \sin^2 \theta_i} \right\}
\]
\[
= n_0 \left\{ \frac{2 \left( \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} \right)}{\left( \frac{3}{4} \cdot \frac{3}{4} - \frac{3}{4} \cdot \frac{3}{4} \right)^2 + \frac{3}{4}} \right\}
\]
\[
= n_0 \left( \frac{3}{4} + 3 \right) = 6n_0 = 2 \sum_{j \in J} n_j = \gamma(n - 2).
\]

**Type IV:** Since \( J = \{-1, 0, 1, 2\}, \gamma = 4J - 1 = 3, \phi_1 = \frac{4}{7} \pi, \ n_{-1} = n_1 = \ell, \ n_0 = n_2 = k, \) and
\[
\sin^2 2\theta_i = \frac{k}{k + \ell}, \quad \cos^2 2\theta_i = \frac{\ell}{k + \ell}
\]
for all \( i \in J, \) we have
\[
\sum_{j \in J} n_j \cot^2 \theta_i - \phi_j
\]
\[
= \ell \left\{ \cot^2 \left( \theta_i + \frac{\pi}{4} \right) + \cot^2 \left( \theta_i - \frac{\pi}{4} \right) \right\} + k \left\{ \cot^2 \theta_i + \cot^2 \left( \theta_i - \frac{\pi}{2} \right) \right\}
\]
\[
= \frac{2\ell \left( \sin^2 \theta_i \cos^2 \theta_i + \sin^2 \frac{\pi}{4} \cos^2 \frac{\pi}{4} \right)}{\left( \sin^2 \theta_i \cos^2 \theta_i - \cos^2 \theta_i \sin^2 \theta_i \right)^2 + \sin^2 \theta_i} + \frac{2k \left( 2 - \sin^2 2\theta_i \right)}{\sin^2 2\theta_i}
\]
\[
= \frac{\frac{1}{4} \left( \sin^2 \theta_i \cos^2 \theta_i + \frac{1}{4} \right)}{\cos^2 2\theta_i} + \frac{2k \left( 1 + \cos^2 2\theta_i \right)}{\sin^2 2\theta_i}
\]
\[
= 2(k + \ell + k) + 2(k + \ell + \ell) = 6(k + \ell) = 3 \sum_{j \in J} n_j = \gamma(n - 2).
\]

**Type V:** Since \( J = \{-2, -1, 0, 1, 2, 3\}, \) and \( n_j \equiv n_0, \) we have
\[
\sum_{j \in J} n_j \cot^2 \theta_i - \phi_j
\]
\[
= n_0 \left\{ \sum_{j=1}^{2} \left\{ \cot^2 \left( \theta_i - \phi_j \right) + \cot^2 \theta_i - \phi_j \right\} + \cot^2 \theta_i + \cot^2 \left( \theta_i - \phi_3 \right) \right\}
\]
\[
= n_0 \left\{ \sum_{j=1}^{2} 2 \left( \sin^2 \theta_i \cos^2 \theta_i + \sin^2 \phi_j \cos^2 \phi_j \right) + \frac{2 \left( 2 - \sin^2 2\theta_i \right)}{\sin^2 2\theta_i} \right\}.
\]
When $i = -2, 1$, it holds that
\[ \sin^2 \theta_i = \cos^2 \theta_i = \frac{1}{2}, \quad \sin^2 2\theta_i = 1. \]

Using $\gamma = \#J - 1 = 5$ and $\phi_j = \frac{j}{6}\pi$, we get
\[
\sum_{j \in J} n_j \cot^2(\theta_i - \phi_i) = n_0 \left\{ \sum_{j=1}^{2} \frac{2 \left( \frac{1}{4} + \sin^2 \phi_j \cos^2 \phi_j \right) + 2(2 - 1)}{\cos^2 \frac{2\phi_j}{3}} \right\}
\]
\[
= n_0 \left\{ \sum_{j=1}^{2} \frac{2 \left( 2 + \sin^2 2\phi_j \right)}{\cos^2 \frac{2\phi_j}{3}} + 2 \right\}
\]
\[
= n_0 \left\{ \frac{2 \left( 1 + \sin^2 \frac{\pi}{3} \right)}{\cos^2 \frac{\pi}{3}} + \frac{2 \left( 1 + \sin^2 \frac{2\pi}{3} \right)}{\cos^2 \frac{2\pi}{3}} + 2 \right\}
\]
\[
= n_0 \left\{ \frac{2 \left( 1 + \frac{3}{4} \right)}{\frac{1}{4}} + \frac{2 \left( 1 + \frac{3}{4} \right)}{\frac{1}{4}} + 2 \right\}
\]
\[
= 30n_0 = 5 \sum_{j \in J} n_j = \gamma(n - 2).
\]

When $i = -1, 0, 2$, it holds that
\[ \sin^2 2\theta_i = \frac{1}{4}, \quad \sin^2 \theta_i = \frac{2 - \sqrt{3}}{4}, \quad \cos^2 \theta_i = \frac{2 + \sqrt{3}}{4}. \]

Therefore, we obtain
\[
\sum_{j \in J} n_j \cot^2(\theta_i - \phi_j)
\]
\[
= n_0 \left[ \sum_{j=1}^{2} \frac{2 \left( \frac{2-\sqrt{3}}{4} \cdot \frac{2+\sqrt{3}}{4} + \sin^2 \phi_j \cos^2 \phi_j \right)}{\cos^2 \phi_j - \sqrt{3}} \right] + \frac{2(2 - \frac{1}{4})}{\frac{1}{4}}
\]
\[
= n_0 \left\{ \sum_{j=1}^{2} \frac{2 \left( 1 + 4 \sin^2 2\phi_j \right)}{\cos 2\phi_j - \sqrt{3}} + \frac{2 \cdot \frac{7}{4}}{\frac{1}{4}} \right\}
\]
\[
= n_0 \left\{ \frac{2 \left( 1 + 4 \sin^2 \frac{\pi}{3} \right)}{\cos \frac{\pi}{3} - \sqrt{3}} + \left( 1 - \frac{\sqrt{3}}{4} \right)^2 + \frac{2 \left( 1 + 4 \sin^2 \frac{2\pi}{3} \right)}{\cos \frac{2\pi}{3} - \sqrt{3}} \right\}
\]
\[
= n_0 \left\{ \frac{2(1+3)}{4} + \frac{2(1+3)}{4} \right\}
\]
\[
= n_0 \left[ 4 \left\{ \frac{2 + \sqrt{3}}{4} + \frac{2 - \sqrt{3}}{4} \right\} + 14 \right]
\]
\[
= 30n_0 = 5 \sum_{j \in J} n_j = \gamma(n - 2).
\]
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