Nonlinear multigrid diffusion

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Abstract. In this paper the feasibility of multigrid techniques in the solution of nonlinear heat equations is investigated. The solution of the heat equation with hysteretic diffusivity requires improvements to the presently applied numerical procedures. Not only the diffusivity hysteresis must be modelled but the nonlinear heat equation requires development of a unique solver. In this study the so-called coarse level iterating multigrid method is proposed to reduce the solution time and memory demands. When the exact solution of the problem is sufficiently smooth, which in diffusion processes is a reasonable assumption, this type of problem reduction does not cause considerable deviations in results.

1. Introduction
The analyses of transient thermal responses in composite materials, for instance in chalcogenide glass, could be of great importance in various engineering and scientific fields. These materials have received a lot of attention over the past several years due to their wide range of applications. Experiments prove that thermal diffusivity of composite materials shows hysteretic behavior [1], [2]. For example hysteresis of thermal diffusivity of silica based materials used in coke oven linings is reported in [1]. The thermal conductivity of chalcogenide semiconductor glass has been studied in [3] and hysteresis properties appearing in certain compositions of Se-Te-Cu alloys have been proved. Measured hysteresis values of thermal conductivity under an applied electromagnetic field can be found in [4]. The hysteresis is related to microstructure changes such as phase transitions, micro-cracking induced by thermal expansion anisotropy, interfacial decohesion between particles and so forth.

Several numerical methods have been developed for the analysis of heat transfer problems in solids [5], [6], however solutions of transient problems with variable thermal properties in more than one dimension are very resource demanding and time-consuming, therefore most existing models have been developed under constant coefficient conditions. Efforts to take into account the temperature dependence of thermal diffusivity tend towards polynomial or exponential approximations [7], [8], [9]. Simulation of heat diffusion with hysteresis requires improvements to the presently applied numerical procedures.

The parabolic heat equation with hysteretic diffusivity can only be solved numerically. The finite difference method with high spatial resolution leads to a large system of equations. In the last few years, the multigrid technique has become one of the most popular numerical solvers of heat conduction problems [10], [11], [12], [13], [14]. The multigrid technique can produce very good convergence rates. It can solve a problem with \( N \) unknowns in \( O(N) \) time [15]. However in case of
hysteretic diffusivity, the so-called inner iteration can dramatically increase the computational cost as it should handle the local hysteresis memory at each grid point.

2. Model definition with hysteresis

In this study a 2D transient heat transfer problem is assumed with temperature-dependent thermal diffusivity $\kappa$. (It is assumed that the temperature dependence of the volumetric heat capacity $c_v$ can be neglected and the expression of thermal diffusivity $\kappa = \lambda/c_v$ is valid, where $\lambda$ is the thermal conductivity). The heat equation is written in the form

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[ \kappa(\theta) \frac{\partial \theta}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \kappa(\theta) \frac{\partial \theta}{\partial y} \right] + s(\theta, t),$$

(1)

where $\theta$ is the dimensionless temperature, the thermal diffusivity $\kappa(\theta)$ is temperature dependent, $x, y \in \Omega$, $\Omega = [0, L] \times [0, L]$ are spatial coordinates. The source term $s(\theta, t)$ will be zero in numerical examples below. The initial condition of $\theta(x, y, 0) = \theta_{ini}$, $x, y \in \Omega$ and the first order boundary conditions $\theta(x, y, t) = \theta_{w}$, $x, y \in \partial \Omega$ are assumed. The nonlinearity of the system originates from the composite material properties, since dependence of on temperature is hysteretic. The thermal diffusivity as a function of the temperature is shown in figure 1 and corresponds to the measured data presented in [1]. Santos et al. [1] observed that during a heating cycle, in the range of lower temperatures, there is a sudden change in the thermal diffusivity, associated with the microstructure modification, in the referred case, the phase transition of $\alpha \rightarrow \beta$ cristobalite. The thermal diffusivity upon heating remains relatively unaltered until a higher temperature range, where it presents another steep increase. During cooling a reverse process takes place, but higher diffusivity values can be expected. Furthermore, experiments showed that thermal cycling can reduce the hysteresis of diffusivity in composites.

In this study we assumed that the temperature dependent diffusivity hysteresis could be modeled with a temperature dependent hysteresis operator [16], [17], namely, with an appropriate parameterized Preisach-type hysteresis model [18].

![Figure 1. Hysteresis of thermal diffusivity with experimentally observed damping effects of thermal cycling [1].](image-url)
Repeated heating-cooling cycles can decrease the magnitude of the thermal diffusivity and decrease the area of the hysteresis loops as well. This can be introduced into the model through a cycle factor, \( v \). The numerical implementation of the Preisach scalar model is based on the Preisach triangle of elementary hysteresis relays [18]. The diffusivity at grid points \( \kappa(\theta^{i+1}) \) is calculated by a recursive formula provided by discretization of the Preisach operator [17]

\[
\kappa(\theta^{i+1}) = H(\theta^{i+1}) = \kappa(\theta^i) \pm \left( b^i T_p \frac{1}{m} \sum_{j} \sum_{i} \mu(\alpha_i, \beta_j) \right), \quad ij = m, (i,j) \in P,
\]

where \( T_p \) is the area of the domain \( P \) over which in the last input step the hysteresis relays have changed their state and \( m \) is the number of relays in the domain \( P \), \( \nu \) is the heating-cooling cycle number. The sign of the second part of the right-hand side depends on the sign of the temperature change. The normalized distribution function \( \mu(\alpha, \beta) = 1/ N_{2D} (m_1, m_2, \sigma_1^2, \sigma_2^2) \) is discretized on a 500\( \times \)500 grid, \( N_{2D} \) represents the normalized density function of 2D Gaussian distribution with mean values \( m_1 \), \( m_2 \) and variances \( \sigma_1^2 \), \( \sigma_2^2 \).

The hysteresis cycles in figure 2 can be provided by the following parameters: \( m_1 = m_2 = 0.525 \), \( \sigma_1 = \sigma_2 = 0.18 \), in first thermal cycle \( \kappa_0^1 = 28 \cdot 10^{-8} \text{ m}^2 / \text{s} \), \( b^1 = 1 \), in the second cycle \( \kappa_0^2 = 27 \cdot 10^{-8} \text{ m}^2 / \text{s} \), \( b^2 = 0.57 \) and in the third and all higher cycles \( \kappa_0^k = 26 \cdot 10^{-8} \text{ m}^2 / \text{s} \), \( b^k = 0.43 \), \( \nu = 3, 4, \ldots \).

### 3. Discretization of the nonlinear model

The solution domain \( \Omega \) has been discretized by a Cartesian mesh with a uniform mesh size \( h = L/n \). Time differentiation is discretized by the Crank-Nicholson second-order method [15]. The right-hand side of (1) is discretized in two steps. First the thermal flux differences are calculated, considering cell interfaces at the distance of \( \pm h/2 \) from grid points. For example between points \((i, j)\) and \((i+1, j)\), the assumed flux is \( J_{i+1/2, j} = \kappa_{i+1/2, j} (\theta_{i+1/2, j} - \theta_j) / h \), where \( \kappa_{i+1/2, j} \) is a space averaged diffusivity, \( \kappa_{i+1/2, j} = 0.5 (\kappa_{ij} + \kappa_{i,j+1}) \).

Depending on the time averaging of the diffusivity, various approximations to (1) can be obtained. The Crank-Nicholson method with space averaged, time-instantaneous diffusivity has the form of

\[
\frac{\theta_{i+1/2, j}^{k+1} - \theta_{i+1/2, j}^k}{\Delta t} = \frac{1}{h} \left( \sigma (\Delta J_{i+1/2, j}^{k+1} + \Delta J_{i+1/2, j}^{k+1/2}) + (1 - \sigma) (\Delta J_{i+1/2, j}^k + \Delta J_{i+1/2, j}^k) \right) + s^{k+1/2},
\]

where the first superscript \( k \) refers to the time step and the second superscript \( l \) refers to the iteration cycle. The weighting factor is \( \sigma = 0.5 \) for the Crank-Nicholson method, \( J \) is the heat flux and \( \Delta J \) denotes the flux differences. Each flux difference has been computed in the same manner, for example

\[
\Delta J_{i+1/2, j}^{k+1} = J_{i+1/2, j}^{k+1} - J_{i-1/2, j}^{k+1}.
\]

The thermal fluxes in the iteration step \( l + 1 \) and at time step \( k + 1 \) are determined by the diffusivity hysteresis model \( \kappa_{ij}^{k+1/2} = H(\theta_{ij}^{k+1/2}) \). The hysteretic diffusivity \( \kappa_{ij} = H(\theta_{ij}^l) \) is not involved in the inner iteration marked by \( l \).

Another approximation scheme uses time-averaged temperatures to determine the diffusivity hysteresis. In this case the discretized form of (1) is
\[ \frac{\theta^{k+1/l+1}_j - \theta^k_j}{\Delta t} = \frac{1}{h} \left( \sigma (\Delta I^{k+1/l+1}_{i,j} + \Delta I^{k+1/l+1}_{i,j+1}) + (1 - \sigma) (\Delta I^{k+1/l+1}_{i,j}, \Delta I^{k+1/l+1}_{i,j+1}) \right) + s^{k+1.5}_j. \]  

(5)

Grid points at times \( k \) and \( k+1 \) have the same values of diffusivity, \( \kappa^{k+1/l+1}_j = H(0.5(\theta^{k+1/l}_j, \theta^k_j)) \). Diffusivity at cell interfaces is interpolated linearly by

\[ \kappa^{k+1/l+1}_{i,j} = 0.5(\kappa^{k+1/l+1}_{i,j+1} + \kappa^{k+1/l+1}_{i+1,j}). \]  

(6)

The thermal flux across the cell interfaces at time \( k+1 \) and in iteration cycle \( l+1 \) is calculated according to

\[ J^{k+1/l+1}_{i,j} = \kappa^{k+1/l+1}_{i,j} (\theta^{k+1/l+1}_{i,j+1} - \theta^k_j)/h. \]  

(7)

At time \( k \) the thermal flux is \( J^{k+1/l+1}_{i+1,j} = \kappa^{k+1/l+1}_{i,j+1} (\theta^{k+1/l+1}_{i+1,j} - \theta^k_j)/h \). All the other thermal fluxes can be derived in the same manner. Substituting thermal fluxes (7) into (5) produces the so-called time-centered discretization scheme

\[ (1 + p) \theta^{k+1/l+1}_j - \frac{\Delta t}{2h^2} (\kappa^{k+1/l+1}_{i,j} 2 \theta^{k+1/l+1}_{i,j+1} + \kappa^{k+1/l+1}_{i,j+1} 2 \theta^k_j) = \frac{\Delta t}{2h^2} (\kappa^{k+1/l+1}_{i+1,j} 2 \theta^{k+1/l+1}_{i,j+1} + \kappa^{k+1/l+1}_{i,j+1} 2 \theta^k_j) + s^{k+1.5} \Delta t, \]  

(8)

where \( \kappa^{k+1/l+1}_{i,j} 2 \theta^{k+1/l+1}_{i,j+1} \) is an abbreviation of \( \kappa^{k+1/l+1}_{i+1,j} 2 \theta^{k+1/l+1}_{i,j+1} + \kappa^{k+1/l+1}_{i,j+1} 2 \theta^k_j \) and

\[ p = \frac{\Delta t}{2h^2} (\kappa^{k+1/l+1}_{i+1,j} 2 \theta^{k+1/l+1}_{i,j+1} + \kappa^{k+1/l+1}_{i,j+1} 2 \theta^k_j). \]  

(9)

Assembling (8) for each interior node yields a system of equations

\[ A^{k+1/l+1}_h \theta^{k+1/l+1}_k = B^{k+1/l+1}_h \theta^k_k + S^{k+1.5/l}_h, \]  

(10)

where \( A^{k+1/l+1}_h \) is the matrix of the coefficients on the left-hand side of (8), \( \theta^{k+1/l+1}_k \) is the vector of unknown temperatures, \( B^{k+1/l+1}_h \) is the matrix of coefficients on the right-hand side, \( \theta^k_k \) is the solution temperature vector at time step \( k \), \( S^{k+1.5/l}_h \) is the vector of the source terms multiplied by \( \Delta t \). Subscript \( h \) refers to the level of the spatial resolution.

The semi-implicit finite difference method (8) resembles the 2D variation of the DuFort-Frankel scheme, which is consistent with the original PDE only if \( \Delta t \kappa_{max}/h^2 << 1 \), where \( \kappa_{max} = \max(\kappa_j) \).  

To avoid solutions that do not satisfy the physical nature of the modeled system, the upper limit to the time step can be evaluated from the inequality \( \Delta t \leq h^2/(2(1 - \sigma)\kappa_{max}) \).

4. The coarse level iterating multigrid solver

The nonlinearity of \( \kappa \) in method (8) needs iteration at every time step. The Newton iteration is very expensive in CPU time because of the large set of unknown variables [15]. Hysteresis-like temperature dependence of the diffusivity increases dramatically the computational cost as it requires to manage the local memory at each grid point (10). Treatment of this memory is very resource and time consuming. The finer the resolution, the more hysteresis memory has to be treated. Using coarser resolution decreases the computational costs, but produces a coarser solution as well. In this study we propose a coarse level iterating multigrid algorithm, in which the fine resolution of the temperature
field is preserved, but the hysteresis of diffusivity is reduced to a coarser grid. Thus, this multigrid algorithm works on several resolutions [16], [19].

In a general case, the multigrid solver assumes an initial guess on the fine resolution for the diffusivity \( k_h^{k+1/0} = H(\theta_h^k) \) and for the temperature field \( \theta_h^{k+1,0} = \theta_h^k \). In every iteration cycle the diffusivity coefficients have to be re-calculated for all fine grid points. To reduce the solution time, before the iteration, problem (10) should be reduced to the next coarser grid, with mesh size \( H = 2h \), in the form of

\[
A_H^{k+1/4} \theta_H^{k+1/4} = B_H^{k+1/4} \theta_H^k + S_H^{k+1/2},
\]

where the initial temperature \( \theta_H^{k+1,0} \) of time step \( k+1 \) has been determined by the restriction of the fine grid solution at time step \( k \), \( \theta_H^{k+1,0} = R \theta_h^k \). The initial diffusion coefficient field has been defined with the hysteresis operator as \( k_h^0 = H(\theta_h^0) \). The source term \( S_H^{k+1/2} \) has been restricted to \( S_H^{k+1/2} \), or \( S_H^{k+1/2} \) could be calculated directly as well. The temperature field approximation for time step \( k+1 \) and mesh size \( H \) could be evaluated by iteration (11) with the new values of diffusivity \( k_H^{k+1/4} = H(0.5(\theta_H^{k+1/4} + \theta_H^{k+1,0})) \), until the absolute difference decreases below a preliminarily defined \( \varepsilon \) limit, \( \| \theta_H^{k+1,1} - \theta_H^{k+1,0} \| < \varepsilon \). After this so-called inner-iteration, the diffusivity and the temperature are interpolated to the fine grid. By denoting \( P \) as the prolongation operator and \( I_H \) as the interpolation operator, the fine grid diffusivity is \( k_h^{k+1/2} = S_{GS} P k_h^{k+1/2,1} \), the initial temperature is \( \theta_h^{k+1,0} = I_H \theta_H^{k+1/4} \). The prolonged diffusivity needs smoothing, \( S_{GS} \) denotes the Gauss-Seidel smoothing operator [19][20]. The interpolated temperature could be seen as a good initial solution to the grid level \( h \). To evaluate the fine grid solution \( \theta_h^{k+1} \), multigrid V-cycles are applied, until the convergence criterion is reached. During the fine grid V-cycles the diffusivity field remains unaltered. The graphical scheme of the iteration method can be seen in figure 2.

\[ \]

**Figure 2.** Schematic representation of the coarse level iterating multigrid method.

This proposed coarse level iterating hierarchical algorithm consists of two types of multigrid methods. The inner-iteration is on the coarser grid, the solver is the so-called full multigrid algorithm. An advantage of this so called Full Multigrid (FMG) method is the exact initial solutions on the coarsest grid, and therefore a good convergence. The nonlinear iteration has a reduced set of variables and the amount of hysteresis memory data also decreases on the coarser grid. The stencils of \( A_H^{k+1/4} \)
and \( B^{k+1,j+1}_H \) are changed point-by-point, depending on the diffusivity. The stencils of \( A^{k+1,j+1}_H \) and \( B^{k+1,j+1}_H \) are

\[
A^{k+1,j+1}_H = \begin{bmatrix}
0 & -\beta k^{-1/2,j+1} & 0 \\
-\beta k^{-1/2,j+1} & (1+p)_{jH} & -\beta k^{1/2,j+1} \\
0 & -\beta k^{1/2,j+1} & 0 
\end{bmatrix},
\]

and

\[
B^{k+1,j+1}_H = \begin{bmatrix}
0 & -\beta k^{1/2,j+1} & 0 \\
-\beta k^{1/2,j+1} & (1-p)_{jH} & -\beta k^{-1/2,j+1} \\
0 & -\beta k^{-1/2,j+1} & 0 
\end{bmatrix},
\]

respectively, where \( \beta = \Delta t / 2H^2 \). In the proposed algorithm the diffusivity fields \( k \), the temperature fields \( \theta \) and the source terms \( S \) are stored at every grid level. The FMG determines the solutions at each grid level, so the approximated truncation error can be the stop criterion of the FMG cycles. The hysteresis memory is treated and stored on the coarser grid level.

To refine the temperature field after the inner iteration, some simple multigrid V-cycles run on the finest grid. The temperature field is iterated on the coarser grid because of nonlinearity, and after interpolation it is iterated on the fine grid to increase the accuracy. The diffusivity is iterated on the coarser grid as well, but diffusivity values remain unchanged on the fine grid. The high level V-cycles (MG-V) coarsening technique and stencil determination are similar to the techniques used in FMG, but the temperature field is calculated only on the finest grid. On the coarse levels MG-V works with the corrections of the temperatures. The V-cycles iteration could be stopped when the residual norm has been reduced below the \( \eta \) times of the initial norm, i.e.

\[
\| f_h^{k+1} \| < \eta \| A^{k+1,j+1}_h \theta^{k+1,j+1}_H - B^{k+1,j+1}_h \theta^k_h - S^{k+1}_h \|, \quad \eta < 1.
\]

The coarsest problem has been solved in both multigrid algorithms by the successive over-relaxations method (SOR). The schematic flowchart of the proposed coarse level iterating multigrid algorithm is shown in figure 3.

5. Numerical results

In this section the proposed method and algorithm are analysed. In all the test problems the source term is considered to be zero. The test parameters are \( L = 0.5 \text{ m}, \quad \kappa_b = 2.5 \cdot 10^{-7} \text{ m}^2/\text{s}, \quad \Delta t = 250 \text{ s} \), the finest level consists of \( 33 \times 33 \) mesh points. In all tested FMG and also in MG-V-cycles, full weighting restriction and bilinear interpolation have been applied, except for the diffusivity prolongation at the end of the coarse level iteration. SOR was the coarsest grid solver. The number of pre- and post-relaxation is equal to 3, the coarsest grid resolution has \( 5 \times 5 \) points, for nonlinear iteration \( \varepsilon = 1 \cdot 10^{-4} \) and the initial norm reduction in MG-V is \( \eta = 0.1 \).

5.1. Verifying the coarse level iterating algorithm

The finite difference methods (3) and (5) are compared without reduction of the diffusivity hysteresis, i.e. the diffusivity has been iterated in both cases at the finest resolution. Initial conditions are \( \theta^0 = 1 \) with constant boundary conditions \( \theta(x,y,0) = 0, \quad x, y \in \partial \Omega \). Figure 4a shows the norm of differences between the solutions obtained by two distinct difference schemes versus time. The highest norm value is less than \( 2.5 \cdot 10^{-3} \).
Comparing the numbers of iteration cycles, it can be stated that the proposed scheme (5) generally needs two iterations, however formulation (3) at larger temperature differences needs considerably more iteration cycles [16], [17][19]. When approaching the steady-state, the cycle number could also decrease. In the proposed algorithm, after the coarse level iteration, the diffusivity remains unchanged. The crucial question is how the diffusivity should be prolongated to the finest grid. When dealing with two-dimensional cases the bilinear interpolation is the most popular technique. Another possibility is the simple prolongation. According to the required rapid iteration, higher order interpolation is not suitable. Figure 4b and figure 4c show the norms of differences between the fine grid iteration and

Figure 3. Flow chart of the proposed coarse level iterating multigrid algorithm.
coarse level iteration with two distinct diffusivity projections. The deviation in the norm with the prolongation algorithm (curve c) all the time remains beneath the norm of bilinear interpolation (curve b). Comparing the results, it can be concluded, that the algorithm with prolongation approximates the results obtained by using the fine grid iteration algorithm better, therefore we propose that the temperature field can be interpolated but the diffusivity field should be prolonged to the finest grid.

![Figure 4](image_url)

**Figure 4.** Norm tendencies differences a) between schemes (3) and (5), b) between CLI-MG by bilinear interpolation and fine level iteration, c) CLI-MG by prolongation and fine level iteration, d) CLI-MG with $\Delta t = 500s$ and $\Delta t = 50s$.

The correctness of the time-centered diffusivity has been tested by employing different time steps using the same resolutions in space. The reference time step was a tenth of the original time step. figure 4d shows the norm of differences between two steps in time. Apart from the first few steps, there is no difference between the two calculations, so time averaging is suitable, as long as regularity and consistency are ensured.

Comparing the solution time, we measured twenty time steps. Assuming that the solution time for the coarse level iteration is one unit, the solution time with fine grid iteration is four units. Analysing the memory usage, the memory requirement of the multigrid part of the two methods are roughly the same. The discrete Preisach algorithm in the proposed form has one large distribution matrix. The memory character of the Preisach model requires to store the previous input values of all grid points in a matrix of size $(n+1) \times (n+1)$ and the return points of the characteristic of $\theta_i(k\Delta t)$ at each grid point, which means that about 30 values have to be stored and treated per cycle and per grid point. If the Preisach algorithm works on the next coarser grid, the memory demand can be calculated in the same way, replacing $n$ by $N = n/2$. The saved memory size if $n >> 1$ is about $\approx 23n^2$ data. With large problems, significant differences in memory usage can be accompanied by reduced CPU-time as well [17][19].

5.2. **Hysteresis induced heating-cooling asymmetry**

To investigate the results of simulations using the proposed algorithm, the heating-cooling thermal cycles with constant and temperature dependent hysteretic thermal diffusivity are compared. Numerical simulations represent well the thermal cycling asymmetry induced by hysteresis of diffusivity. In the first test the thermal diffusivity changes on the main hysteresis curves. It can be reached by monotone heating or cooling of the boundaries. In the heating process the initial and
boundary conditions are \( \theta(x,y,0) = 0 \) at \( t = 0 \), and \( \theta(x,y,t) = 1 \), \( x, y \in \partial \Omega \) at \( t > 0 \). The initial and boundary conditions in the cooling process are \( \theta(x,y,0) = 1 \) at \( t = 0 \); \( \theta(x,y,t) = 0 \), \( x, y \in \partial \Omega \) and \( t > 0 \). In the tests presented, the constant diffusivity is considered to be the base diffusivity of the hysteretic diffusion \( \kappa_0 \). Temperature fields and diffusivity fields at four different time steps are shown in figure 5. Temperature transients at the middle of the domain are shown in figure 6.

**Figure 5.** Heating and cooling a rectangular 2D sample with constant and with hysteretic diffusivity.

Heating-cooling transients with constant thermal diffusivity are symmetrical. In hysteretic diffusion, the heating-cooling transients differ from each other. The thermal process speeds up with increased diffusivity. The cooling process begins at higher diffusivity [17]. Differences between constant and hysteretic diffusion processes are expressed by the Euclidean norm. Figure 7 shows the norm of differences for heating and cooling processes. The norm first increases in time and after reaching a maximum slowly decreases. In the transient time domain, the difference between cooling processes is higher than the difference between heating processes. The cooling process with hysteretic diffusivity approaches the steady-state more rapidly.

5.3. **Periodically changing boundary temperatures**

The hysteresis minor loops have been avoided using the presented monotone heating or cooling models. However with periodically heating-cooling of the boundaries, it has to be assumed, that diffusivity could vary along minor loops. In this test, a rectangular domain with sinusoidal varying temperatures at two neighboring boundaries, zero temperature boundary conditions on the other two sides of the domain are considered. The transient thermal characteristic of a material with diffusivity hysteresis is compared to the characteristic of a material with constant diffusivity. Hundred time steps have been evaluated. The temperature fields and diffusivities at time steps \( k = 10, 20, 50, 100 \) can be seen in figure 8. In figure 8a-d the linear, in figure 8e-h the hysteresis heat diffusion time series and in figure 8i-l the appropriate diffusivity fields can be seen. Diffusivity minor loops are shown in figure 9.
Figure 6. Temperature transients in the middle of the domain with hysteretic and with constant diffusivity.

Figure 7. Norm of differences between hysteretic and normal heat diffusion a) heating and b) cooling along the main hysteresis curves.

The periodical changing of boundary temperatures can be seen in figure 10. From determining the norm of differences of the nonlinear and the linear temperature fields in time, it can be concluded, that the norm is increasing rapidly when the boundary temperatures are increasing and is decreasing slowly when the boundary temperatures are decreasing. The characteristic of the curve approaches a quasi-static state. The asymmetry in the norm cycles represents well the nonlinear effects of the hysteresis.

5.4. Nonlinear diffusion with inhomogeneous initial conditions

Special problems arise from inhomogeneous initial conditions. In these cases the initial diffusivity field depends on how the initial temperature distribution has been reached. A hypothetical initial temperature distribution can be seen in figure 11a. To have a correct solution, the initial diffusivity field has to be determined as well. If the initial diffusivity field can be calculated on the main hysteresis curves, different initial diffusivity fields can be found on the increasing and decreasing main hysteresis curves, see figure 11b and figure 11c, respectively. Considering zero temperature boundary
conditions, each initial diffusivity field induces different transients in the cooling process that can be seen in figure 12 for constant diffusivity (figure 12a-c), for hysteretic diffusivities starting from the increasing curve (figure 12d-f) and for hysteretic diffusivities starting from the decreasing curve (figure 12g-i) at three time steps, \( k=6,20,40 \). The temperature fields are homogenized faster in the initially high temperature domain with nonlinear diffusivity than with constant low values of diffusivity. After \( k = 40 \) time steps, the temperature fields are well smoothed in all cases, but the effect of initial conditions has been retained in the diffusivity fields, see figure 13a and figure 13b.

![Temperature fields with linear, nonlinear heat diffusion and diffusivity at four different time steps.](image)

**Figure 8.** Temperature fields with a)-d) linear, e)-h) nonlinear heat diffusion and i)-l) diffusivity at four different time steps.
Figure 9. Minor loops of diffusivity near the boundaries.

Figure 10. Top: Norm of differences between linear and nonlinear heat diffusions. Bottom: Periodically changing boundary temperatures.

Figure 11. a) Hypothetical initial temperature field, b) initial diffusivity field determined on the main increasing curve and c) initial diffusivity field determined on the main decreasing curve.
Figure 12. Temperature fields at three different time steps, top to bottom, $k = 6, 20, 40$, with constant diffusivity (a-c), with initial diffusivity of figure 11b (d-f), and with initial diffusivity of figure 11c (g-i).

Figure 13. Diffusivity fields after $k = 40$ time steps.

6. Conclusion
The experimentally proved hysteresis of diffusivity for composite materials is not yet incorporated in numerical models of systems applied in common engineering applications. Hysteresis of diffusivity can be approximated from a macroscopic point of view by a hysteresis operator. This operator
describes the temperature dependence of diffusivity. Numerical solution of nonlinear heat equations in more than one dimension is very demanding in computational resources and time consuming, even more so with hysteresis memory incorporated at every discretization unit. This study has proved that the execution time and the memory demand can be reduced by the proposed coarse level iterating multigrid technique. Even with the lower computational costs, good convergence properties have been achieved by applying the full multigrid method in the inner iteration. When the exact solution of the problem is sufficiently smooth, which in diffusion processes is a reasonable assumption, this type of problem reduction does not cause considerable deviations in the results.

The presented transient heat diffusion examples indicate that due to the hysteretic character of diffusivity, heat transfer processes become non-symmetric which can be different from processes with constant diffusivity. The transient process depends not only on the initial and boundary conditions but on the prehistory of the sample, i.e. the correct solution requires in addition to the initial distribution of temperature the initial diffusivity distribution and the prehistory of diffusivity at the initial moment. We can conclude that for the proper analysis of heat transfer processes over a broad temperature range inside composite materials, the hysteretic temperature dependence of the diffusivity should not be neglected.

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