Results on the Hilbert coefficients and reduction numbers

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MS received 5 April 2018; revised 26 October 2018; accepted 8 November 2018; published online 26 June 2019

Abstract. Let \((R, m)\) be a \(d\)-dimensional Cohen–Macaulay local ring, \(I\) an \(m\)-primary ideal and \(J\) a minimal reduction of \(I\). In this paper we study the independence of reduction ideals and the behavior of the higher Hilbert coefficients. In addition, we also give some examples.

Keywords. Hilbert coefficient; minimal reduction; associated graded ring.

2010 Mathematics Subject Classification. 13A30, 13D40, 13H10.

1. Introduction

Throughout this paper, we assume that \((R, m)\) is a Cohen–Macaulay local ring of dimension \(d \geq 2\) and the residue class field \(R/m\) is infinite. For an \(R\)-module \(M\), let \(\lambda(M)\) denote the length of \(M\). Let \(I\) be an \(m\)-primary ideal of \(R\). The Hilbert–Samuel function \(H_I(n)\) of \(I\) is defined as \(H_I(n) = \lambda(R/I^n)\). There exists a polynomial \(P_I(x)\) of the form

\[
P_I(x) = e_0 \binom{x + d - 1}{d} - e_1 \binom{x + d - 2}{d - 1} + \cdots + (-1)^d e_d
\]

such that \(P_I(n) = H_I(n)\) for all large \(n\), where \(e_i = e_i(I) \in \mathbb{Z}\) are called the Hilbert coefficients of \(R\) with respect to \(I\).

An ideal \(J \subseteq I\) is called a reduction ideal of \(I\) if \(I^{r+1} = JJ^r\) for some nonnegative integer \(r\) (see [21]). The least such \(r\) is called the reduction number of \(I\) with respect to \(J\) and denoted by \(r_J(I)\). A reduction ideal \(J\) is called a minimal reduction if it does not properly contain a reduction ideal of \(I\). Under our assumption, it is generated by a regular sequence. The reduction number of \(I\) is defined as

\[
r(I) = \min\{r_J(I) : J \text{ is a minimal reduction ideal of } I\}.
\]

The reduction number \(r(I)\) is said to be independent if \(r(I) = r_J(I)\) for all minimal reductions \(J\) of \(I\). Sally in [27] raised the following question: If \((R, m)\) is a \(d\)-dimensional Cohen–Macaulay local ring having an infinite residue field, then is \(r(m)\) independent? A natural extension of this question is to replace \(r(m)\) with \(r(I)\). Let \(G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}\) be the associated graded ring of \(I\). Huckaba [10] and Trung [29] independently proved that if depth \(G(I) \geq d - 1\), then \(r(I)\) is independent (see also [8, 9, 16, 18, 28] and [17]).
Moreover, Wu [30] proved that if depth \( G(I) \geq d - 2 \) and \( r(I) \geq n(I) + d - 1 \), where \( n(I) \) is the postulation number of \( I \), then \( r(I) \) is independent. However, if \( d \geq 2 \) and depth \( G(I) \leq d - 2 \), then \( r(I) \) is not independent, in general. Counter-examples have been obtained in [10,18] and [17].

It is known that \( e_0, e_1 \) and \( e_2 \) are nonnegative integers. Unfortunately, the good behavior of the Hilbert coefficients stops with \( e_2 \). Indeed, Narita [20] showed that it is possible for \( e_3 \) to be negative (see also [2]). However, Itoh [14] proved that if \( I \) is normal ideal; that is \( T^n = I^n \) for all positive integer \( n \), then \( e_3 \) is a nonnegative integer (see also [2]). Puthenpurakal [24] obtained remarkable results about negativity of \( e_3 \).

The main purpose of this paper is to study the independence of reduction number and also the behavior of the higher Hilbert coefficients. In the last section, we collect some examples which disprove a question one can make about the behavior of Hilbert coefficients.

2. Main results

We begin this section by recalling some known definitions, notations and results in [12] and [11]. An element \( x \in I \setminus I^2 \) is said to be superficial for \( I \) if there is an integer \( c \) such that \( (I^{n+1}:x) \cap I^c = I^n \) for all \( n \geq c \). If grade(\( I \)) \( \geq 1 \) and \( x \) is a superficial element, then \( x \) is a regular element of \( R \) and so by Artin–Rees theorem, \( I^{n+1}:x = I^n \) for all \( n \gg 0 \).

If \( R/m \) is infinite, then a superficial element for \( I \) always exists. A sequence \( x_1, \ldots, x_s \) is called a superficial sequence for \( I \) if \( x_1 \) is superficial for \( I \) and \( x_i \) is superficial for \( I/(x_1, \ldots, x_{i-1}) \) for \( 2 \leq i \leq s \). If \( I \) is an \( m \)-primary ideal and \( J \) is a minimal reduction of \( I \), then there is a superficial sequence \( x_1, \ldots, x_d \) in \( I \) such that \( J = (x_1, \ldots, x_d) \). For any element \( x \in I \), we let \( x^* \) denote the image of \( x \) in \( I/I^2 \). We note that if \( x^* \) is a regular element of \( G(I) \), then \( x \) is a regular element of \( R \) and \( G(I/(x)) = G(I)/(x^*) \).

Huckaba and Marley [12] constructed the complex \( C(x_1, \ldots, x_d, n) \) which has the following form:

\[
0 \rightarrow R/I^{n-d} \rightarrow (R/I^{n-d+1})^d \\
\rightarrow (R/I^{n-d+2})^C \rightarrow \cdots \rightarrow R/I^n \rightarrow 0.
\]

Let \( C_r(n) = C(x_1, x_2, \ldots, x_d, n) \) and \( C_r(n) = C(x_1, x_2, \ldots, x_{d-1}, n) \). For any \( n \), there is an exact sequence of complexes

\[
0 \rightarrow C_r(n) \rightarrow C(n) \rightarrow C_r(n-1)[1] \rightarrow 0,
\]

where \( C_r(n-1)[1] \) is the complex \( C_r(n-1) \) shifted to the left by one degree. Thus, we have the corresponding long exact sequence of homology modules:

\[
\cdots \rightarrow H_l(C_r(n)) \rightarrow H_l(C(n)) \\
\rightarrow H_{l-1}(C_r(n-1)) \xrightarrow{x^d} H_{l-1}(C_r(n)) \rightarrow \cdots.
\]

For \( i \geq 1 \), we define

\[
h_i := \sum_{n=1}^{\infty} \lambda(H_i(C(n)))
\]

and

\[
k_i := \sum_{n=2}^{\infty} (n - 1)\lambda(H_i(C(n))).
\]
By [12, §4], we have
\[
\Delta^d [P_I(n) - H_I(n)] = \lambda(I^n / I^n \cap J) - \sum_{i=1}^d (-1)^i \lambda(H_i(C, n))
\]
\[
= \lambda(I^n / J I^{n-1}) - \sum_{i=2}^d (-1)^i \lambda(H_i(C, n))
\]
and
\[
e_i(I) = \sum_{n=i}^\infty \binom{n-1}{i-1} \Delta^d [P_I(n) - H_I(n)].
\]
Hence by combining the previous two formulas, we have
\[
e_1(I) = \sum_{n=1}^\infty \lambda(I^n / J I^{n-1}) - \sum_{i=2}^d (-1)^i h_i
\]
\[
= \sum_{n=1}^\infty \lambda(I^n / I^n \cap J) + \sum_{i=1}^d (-1)^{i-1} h_i
\]
and
\[
e_2(I) = \sum_{n=2}^\infty (n-1) \lambda(I^n / J I^{n-1}) - \sum_{i=2}^d (-1)^i k_i.
\]
For an ideal \( I \) of \( R \), let \( \bar{I} \) denote the integral closure of \( I \) in \( R \). That is, \( \bar{I} \) is the set of all elements \( x \) in \( R \) satisfying the equation of the form \( x^k + a_1 x^{k-1} + \cdots + a_k = 0 \), where \( a_i \in I^i \) for \( i = 1, 2, \ldots, k \). The ideal \( I \) is integrally closed when \( \bar{I} = I \). Also, the ideal \( I \) is said to be asymptotically normal if there exists an integer \( n_0 \geq 1 \) such that \( I^n \) is integrally closed for all \( n \geq n_0 \). For interesting family of asymptotically normal ideals, see [2, Remark 4.3].

PROPOSITION 2.1

Let \( I \) be an \( m \)-primary integrally closed ideal and \( J \) be a minimal reduction of \( I \). If \( e_2 = \lambda(I^2 / J I) + 1 \), then \( G(I) \) is Cohen–Macaulay, \( e_2 = e_1 - e_0 + \lambda(R/I) \), \( r_J(I) \leq 3 \) and \( r(I) \) are independent for any minimal reduction \( J \) of \( I \).

Proof. By [13, Theorem 12] we have \( e_1 - e_0 + \lambda(R/I) \leq e_2 = \lambda(I^2 / J I) + 1 \). Since \( I \) is integrally closed, we have \( I^2 \cap J = J I \). Suppose \( e_1 - e_0 + \lambda(R/I) \leq \lambda(I^2 / J I) \). Thus \( \lambda(I^2 / J I) + \sum_{n=3}^d \lambda(I^n / I^n \cap J) + \sum_{i=1}^d (-1)^{i-1} h_i \leq \lambda(I^2 / J I) \) and so, by [12, Theorem 3.7], we have \( \sum_{n=3}^d \lambda(I^n / I^n \cap J) = 0 \) and \( \sum_{i=1}^d (-1)^{i-1} h_i = 0 \). Therefore, \( G(I) \) is Cohen–Macaulay and by [17, Lemma 3.2], \( \sum_{i=1}^d (-1)^{i-1} k_i = 0 \). Since \( \sum_{n=3}^d (n-1) \lambda(I^n / I^n \cap J) = 0 \), we have \( e_2 = \lambda(I^2 / J I) \) which is a contradiction, by our hypothesis. Therefore, \( e_1 - e_0 + \lambda(R/I) = \lambda(I^2 / J I) + 1 \) and by [22, Theorem 3.4], \( I^3 = J I^3 \) and so \( r(I) \) is independent for any minimal reduction \( J \) of \( I \). \( \square \)

Corso et al. [2, Remark 3.7] observed that if \( I \) is an integrally closed ideal and \( e_2 = 0, 1, 2 \), then \( G(I) \) is Cohen–Macaulay (see [3–5] and [13]). Also, they observed...
that assumption on the ideal $I$ being integrally closed cannot be weakened, see [2, Example 3.8]. In the following proposition, we prove that if $I$ is an $m$-primary integrally closed ideal and $e_2 = 3$, then depth $G(I) \geq d - 2$ and $r(I)$ is independent.

**PROPOSITION 2.2**

*Let $I$ be an $m$-primary integrally closed ideal and $J$ be a minimal reduction of $I$. If $e_2(I) = 3$, then depth $G(I) \geq d - 2$ and $r(I)$ is independent.*

**Proof.** Let $e_2(I) = 3$. By [13, Theorem 12], we have $e_1 - e_0 + \lambda(R/I) \leq e_2(I) = 3$. If $e_1 - e_0 + \lambda(R/I) \leq 2$, then by [17, Lemma 3.15], depth $G(I) \geq d - 1$ and $r(I)$ is independent. Hence we assume that $e_1 - e_0 + \lambda(R/I) = 3$. In this case, by [22, Corollary 4.7], we have depth $G(I) \geq d - 2$, and so, by [17, Proposition 3.16], $r(I)$ is independent. \(\square\)

Ratliff and Rush [25] introduced the ideal $\bar{I}$, which turns out to be the largest ideal containing $I$ with the same Hilbert coefficients as $I$. In particular, one has the inclusions $I \subseteq \bar{I} \subseteq I$, where equalities hold if $I$ is integrally closed.

**Lemma 2.3.** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension 2. Let $I$ be an $m$-primary ideal and $J$ be a minimal reduction of $I$. If $r_J(I) \leq 2$ and $\bar{I} = I$, then depth $G(I) \geq 1$.

**Proof.** If $r_J(I) \leq 2$, then we have $I^n \cap J = J I^{n-1}$ for all $n \geq 3$. Since $\bar{I} = I$, we have $\bar{I}^2 : x = I$ for any superficial element $x$ in $I$. Hence by [15, Proposition 2.1], we have $\bar{I}^n = I^n$ for any $n \geq 1$ and so, depth $G(I) \geq 1$. \(\square\)

An ideal $I$ is said to be asymptotically normal if there exists an integer $k \geq 1$ such that $I^n$ is integrally closed for all $n \geq k$. In [2, Remark 4.3], there are interesting examples of asymptotically normal ideals that are not normal. The following result improves ([13, Proposition 16]).

One of the main results is the following.

**Theorem 2.4.** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension 3. Let $I$ be an $m$-primary ideal and $J$ be a minimal reduction of $I$. Assume that $I$ is an asymptotically normal ideal and $\bar{I} = I$. Then $P_I(n) = H_I(n)$ for $n = 1, 2$ if and only if $r_J(I) \leq 2$.

**Proof.** Let $x$ be a superficial element of $I$ and set $A = R/(x)$, $B = IA$ and $C = JA$. Then $\dim A = 2$. Since $I$ is an asymptotically normal ideal, by [23, Corollary 7.11], $\bar{B} = B$. If $r_J(I) \leq 2$, then $r_C(B) \leq 2$. Therefore, by Lemma 1.3, depth $G(B) \geq 1$ and so by [12, Lemma 2.2], depth $G(I) \geq 2$. Hence by [19, Theorem 2], we have $r_J(I) = n(I) + 3$. Thus $n(I) \leq -1$ and $P_I(n) = H_I(n)$ for all $n \geq 0$. Conversely, if $P_I(n) = H_I(n)$ for $n = 1, 2$, then by [30, Lemma 2.3], we have $P_B(n) = H_B(n)$ for $n = 1, 2$. So by [13, Proposition 16], $r_C(B) \leq 2$ and by Lemma 1.3, depth $G(B) \geq 1$. Therefore, depth $G(I) \geq 2$ and $r_J(I) \leq 2$. \(\square\)

Let $\mathcal{R}(I)$ be the Rees-algebra of $I$ and $E$ be an $\mathcal{R}(I)$-module. Then in the following theorem, we set $H^i(E)$ to be the $i$-th local cohomology module of $E$ with respect to the maximal homogeneous ideal $M = (m, I_I)$ of $\mathcal{R}(I)$ as the support.
The second main result with application is the following theorem.

**Theorem 2.5.** Let \((R, \mathfrak{m})\) be a Cohen–Macaulay local ring of dimension 4. Let \(I\) be an \(\mathfrak{m}\)-primary ideal and \(J\) be a minimal reduction of \(I\). If \(I\) is an asymptotically normal ideal and \(r_J(I) \leq 3\), then \(e_4(I) \leq 0\).

**Proof.** By [8, Lemma 2.4], for \(n \gg 0\), we have

\[
H^i(G(I^n))_j = 0 \quad \text{for } j \geq 1 \text{ and } i = 0, 1, 2, 3, 4
\]

and since \(r_J(I) \leq 3\), we get by [29, Proposition 3.2] and [8, Lemma 2.4],

\[
H^4(G(I^n))_0 = 0.
\]

Let \(q\) be the integer such that \(I^q\) is normal. By [23, Theorem 7.3], depth \(G(I^n) \geq 2\) for \(n \gg 0\), and so, \(H^i(G(I^n)) = 0\) for \(n = 0, 1\). Thus

\[
a_2(I^n) < a_3(I^n) = 0,
\]

and therefore \(a_2(I^n) \leq -1\) and \(H^2(G(I^n))_0 = 0\).

Set \(h_i = \lambda(H^i(G(I^n)))\) for \(i = 0, 1, 2, 3, 4\). Then

\[
h_0 = h_1 = h_2 = h_4 = 0.
\]

Set \(K := I^n\) for \(n \gg 0\) and let

\[
P_K(z) = c_0 \left(\frac{z + 3}{3}\right) - c_1 \left(\frac{z + 2}{2}\right) + c_2 \left(\frac{z + 1}{1}\right) - c_3
\]

be the Hilbert polynomial of \(G(K)\), i.e.

\[
P_K(i) = \lambda(K^i/K^{i+1}) \quad \text{for } i \gg 0.
\]

By Grothendieck–Serre formula, we get

\[
H_K(i) - P_K(i) = \sum_{s=0}^{4} (-1)^s \lambda(H^s(G(K)))_{i}.
\]

Set \(i = 0\). We get

\[
\lambda \left(\frac{R}{I^n}\right) - [c_0 - c_1 + c_2 - c_3] = h_0 - h_1 + h_2 - h_3 + h_4 = -h_3.
\]

Let \(\varphi_I(n)\) be the Hilbert–Samuel polynomial of \(I\), i.e.

\[
\varphi_I(z) = e_0(I) \left(\frac{z + 3}{4}\right) - e_1(I) \left(\frac{z + 2}{3}\right)
\]

\[
+ e_2(I) \left(\frac{z + 1}{2}\right) - e_3(I) \left(\frac{z}{1}\right) + e_4(I).
\]
Write

\[ \varphi_K(z) = c_0 \left( \frac{z + 3}{4} \right) - c_1 \left( \frac{z + 2}{3} \right) + c_2 \left( \frac{z + 1}{2} \right) - c_3 \left( \frac{z}{1} \right) + c_4. \]

Clearly \( \varphi_K(z) = \varphi_I(nz) \). In particular, \( c_4 = e_4(I) \). Also notice that

\[ \varphi_K(1) = c_0 - c_1 + c_2 - c_3 + c_4 = \varphi_I(n) = \lambda \left( R/I^n \right). \]

So we get \( c_4 = -h_3 \). Thus \( e_4 = -h_3 \). Thus \( e_4 \leq 0 \). \( \square \)

For any ideal \( I \), the set of ideals \( (I^{n+1} : I^n) \) forms an ascending chain. Let \( \tilde{T} \) denote the union of these ideals. Ratliff and Rush [25] showed that \( \tilde{T} \) is the largest ideal containing \( I \) which has the same Hilbert polynomial as \( I \). In the following proposition, we use the notations \( B_I(x, R) = \bigoplus_{n=0}^{\infty} \frac{I^{n+1}x}{I^n} \) and \( L_I(R) = \bigoplus_{n=0}^{\infty} R/I^{n+1} \).

**PROPOSITION 2.6**

Let \( (R, m) \) be a Cohen–Macaulay local ring of dimension 3. Let \( I \) be an \( m \)-primary ideal and \( J \) be a minimal reduction of \( I \). If \( e_2(I) \leq 1 \), then \( e_3(I) \leq 0 \).

**Proof.** By [24, Proposition 6.4] we can assume that \( e_2(I) = 1 \). Let \( x \) be a superficial element of \( I \). Set \( A = R/(x) \), \( B = IA \) and \( C = JA \). Then \( \dim A = 2 \), \( e_i(I) = e_i(B) \) for \( i = 0, 1, 2 \) and \( e_2(B) = 1 \). By [12, Corollary 4.13], we have

\[ e_2(B) = e_2(B) = \sum_{n \geq 1} n \lambda(B^{n+1}/C B^n) = n. \]

Therefore, we have

\[ \sum_{n \geq 2} \lambda(B^{n+1}/C B^n) = 0. \]

Thus by [7, Proposition 1.9], we have

\[ \tilde{e}_3(B) = \sum_{n \geq 2} \binom{n}{2} \lambda(B^{n+1}/C B^n) = 0. \]

By using [23, §6.3], we have the following exact sequence:

\[ 0 \to B^I(x, R) \to H^0(L^I(R))(-1) \to H^0(L^I(R)) \to H^0(L^I(A)). \]

Therefore, we obtain

\[ b := \lambda(B^I(x, R)) \leq \lambda(H^0(L^I(A))) =: r. \]

By using [24, §1.5], we get that \( e_3(B) = e_3(B) - r \) and by [26, Proposition 1.2], \( e_3(I) = e_3(B) + b \). Then \( e_3(I) \leq 0 \). \( \square \)
As an application of Proposition 1.6, we give the following example. The computation of examples is performed by using Macaulay2 [6] and CoCoA [1]. For the convenience of the reader, we calculate Hilbert series and Hilbert polynomial of the following example.

**Example 2.7.** Let \( R = k[x, y, z]_{(x,y,z)} \), where \( k \) is a field, and let \( I = (x^3, y^3, x^2y + z^3, xz^2, y^2z + x^2z) \). Then depth \( G(I) = 0 \) and we have the following Hilbert series:

\[
P_I(t) = \frac{16 + 5t + 5t^2 - 5t^3 + 6t^4 + 10t^5 - 13t^6 + 2t^7 + t^8}{(1 - t)^3}.
\]

and the Hilbert polynomial

\[
P_I(n) = 27 \left( \frac{n + 2}{3} \right) - 18 \left( \frac{n + 1}{2} \right) + \left( \frac{n}{1} \right) + 15.
\]

Hence \( e_2(I) = 1 \) and \( e_3(I) \leq 0 \).

**Proof.** Let

\[
P_I(t) = f(t)/(1 - t)^3
\]

be the Hilbert series of the ideal \( I \) and

\[
f(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + a_7t^7 + a_8t^8.
\]

Then

\[
a_0 = \lambda(R/I), \quad a_i = \lambda(I^i/I^{i+1}) - \sum_{n=0}^{i-1} \left( \frac{d + n}{d - 1} \right) a_{i-1-n},
\]

where \( i \) is a non-negative integer.

Therefore, by using Macaulay2, we have \( \lambda(R/I) = 16, \lambda(I/I^2) = 53, \lambda(I^2/I^3) = 116, \lambda(I^3/I^4) = 200, \lambda(I^4/I^5) = 311, \lambda(I^5/I^6) = 459, \lambda(I^6/I^7) = 631, \lambda(I^7/I^8) = 829, \lambda(I^8/I^9) = 1054, \lambda(I^9/I^{10}) = 1306, \lambda(I^{10}/I^{11}) = 1585 \) and so on. Hence we can obtain the following

\[
a_0 = \lambda(R/I) = 16,
\]

\[
a_1 = \lambda(I/I^2) - 3\lambda(R/I) = 5,
\]

\[
a_2 = \lambda(I^2/I^3) - 6\lambda(I/I^2) - 3\lambda(R/I) = 5,
\]

and also by the above formula, we have \( a_3 = -5, a_4 = 6, a_5 = 10, a_6 = -13, a_7 = 2, a_8 = 1, a_9 = 0, a_{10} = 0 \).

For computing of the Hilbert polynomial, we have \( e_0 = f(1) = 27, e_1 = f'(1)/1! = 18, e_2 = f''(1)/2! = 1, e_3(I) = f^{(3)}(1)/3! = -15 \). This completes the proof. \( \square \)

**Theorem 2.8.** Let \((R, m)\) be a Cohen–Macaulay local ring of dimension 3, \(I\) an \(m\)-primary integrally closed ideal and \(J\) a minimal reduction of \(I\). If \(e_1(I) - e_0(I) + \lambda(R/I) = e_2(I)\), then \(e_3(I) \leq 0\).
Proof. By [13, Lemma 11], there exists a superficial element $x$ of $I$ so that $I/(x)$ is an integrally closed ideal of $A = R/(x)$. Set $B = IA$ and $C = JA$. Then $\dim A = 2$, $e_i(I) = e_i(B)$ for $i = 0, 1, 2$ and $e_1(B) - e_0(B) + \lambda(A/B) = e_2(B)$. By [12, Corollary 4.13], we have

$$e_1(B) - e_0(B) + \lambda(A/B) = e_1(B) - e_0(B) + \lambda(A/B) = \sum_{n \geq 1} \lambda(B^{n+1}/C \tilde{B}^{n})$$

and

$$e_2(B) = e_2(B) = \sum_{n \geq 1} n\lambda(B^{n+1}/C \tilde{B}^{n}).$$

Therefore,

$$\sum_{n \geq 2} \lambda(B^{n+1}/C \tilde{B}^{n}) = 0$$

and so,

$$e_3(B) = \sum_{n \geq 2} \left( \frac{n}{2} \right) \lambda(B^{n+1}/C \tilde{B}^{n}) = 0.$$

By using [23, §6.3], we have the following exact sequence:

$$0 \to B^I(x, R) \to H^0(L^I(R))(-1) \to H^0(L^I(R)) \to H^0(L^I(A)).$$

Hence

$$b := \lambda(B^I(x, R)) \leq \lambda(H^0(L^I(A))) =: r.$$

By using [24, §1.5], we get that $e_3(B) = e_3(B) - r$ and so by [26, Proposition 1.2], $e_3(I) = e_3(B) + b$. Thus $e_3(I) \leq 0$. $\square$

As an application of Theorem 1.8, we give the following example.

Example 2.9. Let $R = k[x, y, z, u, v, w]_{(x, y, z, u, v, w)}$, where $k$ is a field. Let $Q = (z^2, zu, zv, uv, u^3 - yz, v^3 - xz)$, $m = (x, y, z, u, v, w)$ and $S = R/Q$. Then $S$ is a 3-dimensional Cohen–Macaulay local ring and $m$ is a maximal ideal (integrally closed $m$-primary ideal) of $S$. Then depth $G(I) = 1$ and we have the following Hilbert series:

$$P_I(t) = \frac{1 + 3t + 3t^3 - t^4}{(1 - t)^3},$$

and the Hilbert polynomial

$$P_I(n) = 6 \left( \binom{n + 2}{3} \right) - 8 \left( \binom{n + 1}{2} \right) + 3 \left( \binom{n}{1} \right) + 1.$$

Thus $e_1 - e_0 + \lambda(R/I) = e_2$ and $e_3 \leq 0$. 
3. Examples

Marley [19] proved that if \((R, m)\) is a \(d\)-dimensional Cohen–Macaulay local ring and \(I\) an \(m\)-primary ideal such that \(\text{depth } G(I) \geq d - 1\), then all Hilbert coefficients \(e_0(I), e_1(I), \ldots, e_d(I)\) are non-negative integers.

Thus it is natural to ask if \(d - 3 \leq \text{depth } G(I) = t \leq d - 2\) whenever \(d \geq 3\), then is \(e_i(I) \geq 0\) for \(i = t, t + 1, \ldots, d\) in the following examples, we show that the question is negative.

**Example 3.1.** Let \(R = k[x, y, z, u]_{(x,y,z,u)}\) where \(k\) is a field, and \(I = (x^3, y^3, z^3, u^3, xy^2, yz^2, zu^2, xyz, xyu)\). Then we have depth \(G(I) = 2\), Hilbert series

\[
P_I(t) = \frac{33 + 19t + 21t^2 + 7t^3 + 5t^4 - 3t^5 - t^6}{(1 - t)^4},
\]

and the Hilbert polynomial

\[
P_I(n) = 81 \left( \frac{n + 3}{4} \right) - 81 \left( \frac{n + 2}{3} \right) + 27 \left( \frac{n + 1}{2} \right) + 23 \left( \frac{n}{1} \right) - 50.
\]

Hence \(e_0(I) = 81, e_1(I) = 81, e_2(I) = 27, e_3(I) = -23\) and \(e_4(I) = -50\).

In the following example, we show that if \((R, m)\) is a 5-dimensional Cohen–Macaulay local ring and \(I\) an \(m\)-primary ideal and \(\text{depth } G(I) = 3\), then \(e_i(I) < 0\) for \(i = 3, 4, 5\).

**Example 3.2.** Let \(R = k[x, y, z, u, v]_{(x,y,z,u,v)}\), where \(k\) is a field and \(I = (x^3, y^3, z^3, u^2, v, xy^2, yz^2, xyz, xyu)\). Then we have depth \(G(I) = 3\), the Hilbert series

\[
P_I(t) = \frac{28 + 11t + 10t^2 + 5t^3 + t^4 - t^5}{(1 - t)^5},
\]

and the Hilbert polynomial

\[
P_I(n) = 54 \left( \frac{n + 4}{5} \right) - 45 \left( \frac{n + 3}{4} \right) + 21 \left( \frac{n + 2}{3} \right)
\]

\[+ \left( \frac{n + 1}{2} \right) - 4 \left( \frac{n}{1} \right) + 1.
\]

Therefore, \(e_0(I) = 54, e_1(I) = 45, e_2(I) = 21, e_3(I) = -1, e_4(I) = -4\) and \(e_5(I) = -1\).

In the following example, we show that if \((R, m)\) is a 4-dimensional Cohen–Macaulay local ring and \(I\) an \(m\)-primary ideal, depth \(G(I) = 2\) and \(e_3(I) > 0\), then \(e_4(I) < 0\).

**Example 3.3.** Let \(R = k[x, y, z, u]_{(x,y,z,u)}\), where \(k\) is a field and \(I = (x^4, y^4, z^4, u^4, x^2 y^2, y^2z^2, z^2u^2, xyz, xyu)\). Then we have depth \(G(I) = 2\), the Hilbert series

\[
P_I(t) = \frac{81 + 58t + 31t^2 + 7t^3 - t^4}{(1 - t)^4},
\]
and the Hilbert polynomial

$$P_I(n) = 176 \left( \frac{n + 3}{4} \right) - 137 \left( \frac{n + 2}{3} \right) + 46 \left( \frac{n + 1}{2} \right) - 3 \left( \frac{n}{1} \right) - 1.$$ 

So we have $e_0(I) = 176$, $e_1(I) = 137$, $e_2(I) = 46$, $e_3(I) = 3$ and $e_4(I) = -1$.

In the following example, we show that if $(R, m)$ is a 4-dimensional Cohen–Macaulay local ring and $I$ an $m$-primary ideal, depth $G(I) = 1$, then $e_4(I) > 0$ but $e_3(I) < 0$.

Example 3.4. Let $R = k[x, y, z, u]_{(x, y, z, u)}$, where $k$ is a field and $I = (x^3, y^3, z^3, u^3, xy^2, xz^2, xu^2, xy, xyz, xyu)$. Then we have depth $G(I) = 1$, the Hilbert series

$$P_I(t) = \frac{37 + 14t + 17t^2 + 15t^3 + 6t^4 - 12t^5 + 4t^6}{(1 - t)^4}$$ 

and the Hilbert polynomial

$$P_I(n) = 81 \left( \frac{n + 3}{4} \right) - 81 \left( \frac{n + 2}{3} \right) + 38 \left( \frac{n + 1}{2} \right) + \left( \frac{n}{1} \right) + 6.$$ 

Thus $e_0(I) = 81$, $e_1(I) = 81$, $e_2(I) = 38$, $e_3(I) = -1$ and $e_4(I) = 6$.

In the following example, $(R, m)$ is a 5-dimensional Cohen–Macaulay local ring, $I$ an $m$-primary ideal, depth $G(I) = 3$ and $e_4(I) < 0$, but $e_5(I) \geq 0$.

Example 3.5. Let $R = k[x, y, z, u, v]_{(x, y, z, u, v)}$, where $k$ is a field and $I = (x^4, y^4, z^4, u^4, v, x^2y^2, y^2z^2, z^2u^2, xyz, xyu)$. Then depth $G(I) = 3$, the Hilbert series

$$P_I(t) = \frac{81 + 58t + 31t^2 + 7t^3 - t^4}{(1 - t)^5}$$ 

and the Hilbert polynomial

$$P_I(n) = 176 \left( \frac{n + 4}{5} \right) - 137 \left( \frac{n + 3}{4} \right) + 46 \left( \frac{n + 2}{3} \right) - 3 \left( \frac{n + 1}{2} \right) + \left( \frac{n}{1} \right).$$

Thus $e_0(I) = 176$, $e_1(I) = 137$, $e_2(I) = 46$, $e_3(I) = 3$, $e_4(I) = -1$ and $e_5(I) = 0$.

In the following example, we prove that if $(R, m)$ is a $d$-dimensional Cohen–Macaulay local ring and $I$ an $m$-primary ideal, then all Hilbert coefficients are positive, however, depth $G(I) \leq d - 2$.

Example 3.6. Let $R = k[x, y, z]_{(x, y, z)}$, where $k$ is a field, and $I = (x^4, y^4, z^4, x^3y, y^3z, xyz)$. Then we have depth $G(I) = 1$, the Hilbert series

$$P_I(t) = \frac{31 + 9t + 7t^2 + t^3}{(1 - t)^3}$$
and the Hilbert polynomial
\[ P_I(n) = 48 \binom{n + 2}{3} - 26 \binom{n + 1}{2} + 10 \binom{n}{1} - 1. \]

Thus all Hilbert coefficients are positive.

In the following example, we show that depth \( G(I) = 0 \) but all Hilbert coefficients are non negative.

**Example 3.7.** Let \( R = k[x, y, z]_{(x, y, z)} \), where \( k \) is a field and \( I = (x^4, y^4, z^4, x^3y, xy^3, y^3z, yz^3) \). Then depth \( G(I) = 0 \), the Hilbert series
\[ P_I(t) = \frac{30 + 12t + 22t^2 + 8t^3 - 2t^4 - 12t^5 + 6t^6}{(1 - t)^3} \]

and the Hilbert polynomial
\[ P_I(n) = 64 \binom{n + 2}{3} - 48 \binom{n + 1}{2} + 4 \binom{n}{1}. \]

Hence all Hilbert coefficients are non negative.

**Acknowledgements**

The authors would like to thank the referee for a careful reading of the manuscript and for providing helpful suggestions.

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Communicating Editor: B Sury