An initial-boundary value problem of the general three-component nonlinear Schrödinger equation with a $4 \times 4$ Lax pair on a finite interval

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Abstract

We investigate the initial-boundary value problem for the general three-component nonlinear Schrödinger (gtc-NLS) equation with a $4 \times 4$ Lax pair on a finite interval by extending the Fokas unified approach. The solutions of the gtc-NLS equation can be expressed in terms of the solutions of a $4 \times 4$ matrix Riemann-Hilbert (RH) problem formulated in the complex $k$-plane. Moreover, the relevant jump matrices of the RH problem can be explicitly found via the three spectral functions arising from the initial data, the Dirichlet-Neumann boundary data. The global relation is also established to deduce two distinct but equivalent types of representations (i.e., one by using the large $k$ of asymptotics of the eigenfunctions and another one in terms of the Gelfand-Levitan-Marchenko (GLM) method) for the Dirichlet and Neumann boundary value problems. Moreover, the relevant formulae for boundary value problems on the finite interval can reduce to ones on the half-line as the length of the interval approaches to infinity. Finally, we also give the linearizable boundary conditions for the GLM representation.

Keywords: Riemann-Hilbert problem; General three-component nonlinear Schrödinger equation; Initial-boundary value problem; Global relation; Maps between Dirichlet and Neumann problems; Gelfand-Levitan-Marchenko representation

1 Introduction

In the theory of integrable systems, the powerful inverse scattering transform (IST) [1–3] (also called nonlinear Fourier transform) was presented to analytically study the initial value problems of the integrable nonlinear wave equations starting from the spectral analysis of their associated systems of linear eigenvalue equations (also known as the Lax pair [4]). After that, some significant extensions of the IST were gradually developed. For instance, Deift and Zhou [5] developed the IST to present the nonlinear steepest descent method to explicitly explore the long-time asymptotics of the Cauchy problems of (1+1)-dimensional integrable nonlinear evolution equations in terms of RH problems. Fokas [6] extended the idea of the IST to put forward a unified method studying boundary value problems of both linear and integrable nonlinear PDEs with Lax pairs [7–10]. Especially, the Fokas’ method can be used to study integrable nonlinear PDEs in terms of the simultaneous spectral analysis.

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of both parts of the Lax pairs and the global relations among spectral functions. This approach obviously differs from the standard IST in which the spectral analysis of only one part of the Lax pairs was considered [9].

The Fokas’ unified method has been used to explore boundary value problems of some physically significant integrable nonlinear evolution equations (NLEEs) with 2 × 2 Lax pairs on the half-line and the finite interval (e.g., the nonlinear Schrödinger equation [6, 11–14], the sine-Gordon equation [15, 16], the KdV equation [17], the mKdV equation [18–20], the derivative nonlinear Schrödinger equation [21, 22], Ernst equations [23, 24], and etc. [25–31]) and ones with 3 × 3 Lax pairs on the half-line and the finite interval (e.g., [32], the Degasperis-Procesi equation [33], the Sasa-Satsuma equation [34], the coupled nonlinear Schrödinger equations [35–38], and the Ostrovsky-Vakhnenko equation [39]).

To the best of our knowledge, there was no report on the initial-boundary value (IBV) problems of integrable NLEEs with 4 × 4 Lax pairs on the half-line or the finite interval before. The aim of this paper is to develop a methodology for analyzing the IBV problems for integrable NLEEs with 4 × 4 Lax pairs on a finite interval. The extension will contain some novelties from 2 × 2 and 3 × 3 to 4 × 4 matrix Lax pairs, but the two key steps of this method [6–9] keep invariant: (i) Finding an integral representation of the solution in terms of a matrix RH problem formulated in the complex k-plane (k is a spectral parameter of the associated Lax pair). The integral representation in general contains the unknown boundary data such that this expression of the solution is not effective yet; (ii) Applying a global relation to consider the unknown boundary values. The representation of the unknown boundary values in general involves the solution of a nonlinear problem. But, this problem for the linearizable boundary conditions can be ignored since the unknown boundary values can be avoided in terms of only algebraic operations.

In this paper, we will exhibit how steps (i) and (ii) can be actualized for the integrable general three-component nonlinear Schrödinger (gtc-NLS) equation with a 4 × 4 Lax pair [41]

\[
\begin{aligned}
&iq_{tt} + q_{1xx} - 2 \left[ \alpha_{11}|q_1|^2 + \alpha_{22}|q_2|^2 + \alpha_{33}|q_3|^2 + 2 \text{Re} \left( \alpha_{12}\bar{q}_1q_2 + \alpha_{13}\bar{q}_1q_3 + \alpha_{23}\bar{q}_2q_3 \right) \right] q_1 = 0, \\
&iq_{tt} + q_{2xx} - 2 \left[ \alpha_{11}|q_1|^2 + \alpha_{22}|q_2|^2 + \alpha_{33}|q_3|^2 + 2 \text{Re} \left( \alpha_{12}\bar{q}_1q_2 + \alpha_{13}\bar{q}_1q_3 + \alpha_{23}\bar{q}_2q_3 \right) \right] q_2 = 0, \\
&iq_{tt} + q_{3xx} - 2 \left[ \alpha_{11}|q_1|^2 + \alpha_{22}|q_2|^2 + \alpha_{33}|q_3|^2 + 2 \text{Re} \left( \alpha_{12}\bar{q}_1q_2 + \alpha_{13}\bar{q}_1q_3 + \alpha_{23}\bar{q}_2q_3 \right) \right] q_3 = 0,
\end{aligned}
\]  

where the complex-valued vector fields \( q_j = q_j(x,t), j = 1, 2, 3 \) are the sufficiently smooth functions defined in the finite region \( \Omega = \{(x,t)|x \in [0,L], t \in [0,T]\} \), with \( L > 0 \) being the length of the interval and \( T > 0 \) being the fixed finite time, the overbar denotes the complex conjugate, \( \text{Re}(\cdot) \) denotes the real part, and the six coefficients \( \alpha_{ij} \)'s \( (1 \leq i \leq j \leq 3) \) combine a 3 × 3 Hermitian-unitary matrix

\[
\mathcal{M} = \left( \begin{array}{ccc} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \bar{\alpha}_{12} & \alpha_{22} & \alpha_{23} \\ \bar{\alpha}_{13} & \bar{\alpha}_{23} & \alpha_{33} \end{array} \right), \quad \mathcal{M} = \mathcal{M}^\dagger, \quad \mathcal{M}^2 = \mathbb{I}.
\]  

The gtc-NLS equations contain the group velocity dispersion (GVD, i.e., \( q_{jxx} \)), self-phase modulation (SPM, e.g., \( |q_j|^2q_j \)), cross-phase modulation (XPM, e.g., \( |q_j|^2q_s, j \neq s \)), pair-tunneling modulation (PTM, e.g., \( q_j^2\bar{q}_s, j \neq s \)), and three-tunneling modulation (TTM, e.g., \( q_1q_2\bar{q}_3 \)). System (1) admits the distinct cases for the six parameters \( \alpha_{ij}, (1 \leq i \leq j \leq 3) \) such as the three-component focusing NLS equation for \( \alpha_{ij} = -1 \) and \( \alpha_{ij} = 0 \) with \( i < j \), the three-component defocusing NLS equation for \( \alpha_{ij} = 1 \) and \( \alpha_{ij} = 0 \) with \( i < j \), the three-component mixed NLS equation for \( (\alpha_{11} = -1, \alpha_{22} = \alpha_{33} = 1) \) or \( (\alpha_{11} = 1, \alpha_{22} = \alpha_{33} = -1) \) and \( \alpha_{ij} = 0 \) with \( i < j \), and other
general three-component NLS equation. Recently, the three-component defocusing NLS equation with nonzero boundary conditions was studied via the IST [40].

We would like to investigate the gtc-NLS equation (1) with the initial-boundary value problems

Initial conditions: \( q_j(x, t = 0) = q_{0j}(x), \quad j = 1, 2, 3, \)

Dirichlet boundary conditions: \( q_j(x = 0, t) = q_{0j}(t), \quad q_j(x = L, t) = v_{0j}(t), \quad j = 1, 2, 3, \) \( q_{2j}(x = 0, t) = u_{1j}(t), \quad q_{2j}(x = L, t) = v_{1j}(t), \quad j = 1, 2, 3, \) \( q_{3j}(x, t) = 0, \quad j = 1, 2, 3, \) (3)

where the initial data \( q_{0j}(x), (j = 1, 2, 3), \) and Dirichlet and Neumann boundary data \( u_{0j}(t), v_{0j}(t) \) and \( u_{1j}(t), v_{1j}(t), j = 1, 2, 3 \) are sufficiently smooth and compatible at points \((x, t) = (0, 0), (L, 0)\), respectively.

The rest of this paper is organized as follows. In Sec. 2, we investigate the spectral analysis of the associated \( 4 \times 4 \) Lax pair of Eq. (1), such as the eigenfunctions, the jump matrices, and the global relation. Sec. 3 gives the corresponding \( 4 \times 4 \) matrix RH problem by means of the jump matrices obtained in Sec. 2. The global relation is used to establish the map between the Dirichlet and Neumann boundary values in Sec. 4. Particularly, the relevant formulae for boundary value problems on the finite interval can reduce to ones on the half-line as the length of the interval approaches to infinity. In Sec. 5, we present the Gelfand-Levitan-Marchenko (GLM) representation of the eigenfunctions in terms of the global relation. Moreover, we also show that the GLM representation is equivalent to one in Sec. 4. Finally, we also give the linearizable boundary conditions for the GLM representation.

2 The spectral analysis of a \( 4 \times 4 \) Lax pair

2.1. The exact one-form

The gtc-NLS system (1) can be regarded as the compatible condition of a \( 4 \times 4 \) Lax pair [41]

\[
\begin{align*}
\psi_x + ik\sigma_4 \psi &= U(x, t)\psi, \\
\psi_t + 2ik^2\sigma_4 \psi &= V(x, t, k)\psi,
\end{align*}
\]

where \( \psi = \psi(x, t, k) \) is a complex \( 4 \times 4 \) matrix-valued or \( 4 \times 1 \) column vector-valued eigenfunction, \( k \in \mathbb{C} \) is an iso-spectral parameter, \( \sigma_4 = \text{diag}(1, 1, 1, -1) \), and the \( 4 \times 4 \) matrices \( U \) and \( V \) are defined by

\[
U(x, t) = \begin{pmatrix}
0 & 0 & q_1(x, t) & 0 \\
0 & 0 & 0 & q_2(x, t) \\
0 & 0 & 0 & q_3(x, t) \\
p_1(x, t) & p_2(x, t) & p_3(x, t) & 0
\end{pmatrix}, \quad V(x, t, k) = 2kU(x, t) + V_0(x, t), \quad (5)
\]

with \( p_1(x, t) = \alpha_{11}\bar{q}_1 + \alpha_{12}\bar{q}_2 + \alpha_{13}\bar{q}_3, \quad p_2(x, t) = \alpha_{12}\bar{q}_1 + \alpha_{22}\bar{q}_2 + \alpha_{23}\bar{q}_3, \quad p_3(x, t) = \alpha_{13}\bar{q}_1 + \alpha_{23}\bar{q}_2 + \alpha_{33}\bar{q}_3, \) and

\[
V_0(x, t) = -i(U_x + U^2)\sigma_4 = -i \begin{pmatrix}
q_1p_1 & q_1p_2 & q_1p_3 & -q_1x \\
q_2p_1 & q_2p_2 & q_2p_3 & -q_2x \\
q_3p_1 & q_3p_2 & q_3p_3 & -q_3x \\
p_1x & p_2x & p_3x & -(q_1p_1 + q_2p_2 + q_3p_3)
\end{pmatrix}, \quad (6)
\]
Define a new eigenfunction \( \mu(x,t,k) \) by

\[ \mu(x,t,k) = \psi(x,t,k)e^{i(kx+2kt)\sigma_4}, \]

such that the Lax pair (4) becomes the equivalent form for \( \mu(x,t,k) \)

\[
\begin{align*}
\mu_x + ik[\sigma_4,\mu] &= U(x,t)\mu, \\
\mu_t + 2ik[\sigma_4,\mu] &= V(x,t,k)\mu,
\end{align*}
\]

(8)

where \([\sigma_4,\mu] = \sigma_4\mu - \mu\sigma_4\). Let \( \hat{\sigma}_4 \) denote the commutator with respect to \( \sigma_4 \) and the operator acting on a \( 4 \times 4 \) matrix \( X \) by \( \hat{\sigma}_4 X = [\sigma_4, X] \) such that \( e^{\hat{\sigma}_4 X} = e^{\sigma_4 X}e^{-\sigma_4} \), then the Lax pair (8) can be written as a full derivative form

\[
d \left[ e^{i(kx+2kt)\hat{\sigma}_4}\mu(x,t,k) \right] = W(x,t,k),
\]

(9)

where the exact one-form \( W(x,t,k) \) is of the form

\[
W(x,t,k) = e^{i(kx+2kt)\hat{\sigma}_4}[U(x,t)dx + V(x,t,k)dt]\mu(x,t,k).
\]

(10)

2.2. The definition and boundedness of eigenfunctions \( \mu'_j \)'s

For any point \((x,t)\) in the region \( \Omega = \{(x,t)|x \in [0,L], t \in [0,T]\} \), let \( \{\gamma_j\}^4 \) be four contours connecting fours vertexes \((x_1,t_1) = (0,T), (x_2,t_2) = (0,0), (x_3,t_3) = (L,0), (x_4,t_4) = (L,T)\) to \((x,t)\), respectively (see Fig. 1).

Therefore we get the following inequalities on these contours:

\[
\begin{align*}
\gamma_1 : (0,T) &\to (x,t), \quad x - x' \geq 0, \quad t - \tau \leq 0, \\
\gamma_2 : (0,0) &\to (x,t), \quad x - x' \geq 0, \quad t - \tau \geq 0, \\
\gamma_3 : (L,0) &\to (x,t), \quad x - x' \leq 0, \quad t - \tau \geq 0, \\
\gamma_4 : (L,T) &\to (x,t), \quad x - x' \leq 0, \quad t - \tau \leq 0,
\end{align*}
\]

(11)

By means of the Volterra integral equations, it follows from Eqs. (9) and (10) that we introduce the four eigenfunctions \( \{\mu_j\}^4 \) on the four contours \( \{\gamma_j\}^4 \)

\[
\mu_j(x,t,k) = \mathbb{I} + \int_{(x_j,t_j)}^{(x,t)} e^{-i(kx+2kt)\hat{\sigma}_4}W_j(x',\tau,k),
\]

(12)

where \( \mathbb{I} = \text{diag}(1,1,1,1) \), the integral is over a piecewise smooth curve from \((x_j,t_j)\) to \((x,t)\), and \( W_j(x,t,k) \) is given by Eq. (10) with \( \mu(x,t,k) \) replaced by \( \mu_j(x,t,k) \). Since the one-form \( W_j \) are closed, thus \( \mu_j \) are
independent of the path of integration. If we take the paths of integration to be parallel to the $x$ and $t$ axes, then the integral Eq. (12) reduces to

$$
\mu_j(x, t, k) = 1 + \int_{x_j}^{x} e^{-ik(x-x')} \sigma_4(U \mu_j)(x', t, k) dx' + e^{-ik(x-x_j)} \sigma_4 \int_{t_j}^{t} e^{-2ik^2(t-\tau)} \sigma_4(V \mu_j)(x_j, \tau, k) d\tau.
$$

It follows from Eq. (13) that the four columns of the matrix $\mu_j(x, t, k)$ contain the following exponentials

$$
[\mu_j]_s : e^{2ik(x-x') + 4ik^2(t-\tau)}, \quad j = 1, 2, 3, 4; \quad s = 1, 2, 3,
$$

$$
[\mu_j]_4 : e^{-2ik(x-x') - 4ik^2(t-\tau)} e^{-2ik(x-x') - 4ik^2(t-\tau)} e^{-2ik(x-x') - 4ik^2(t-\tau)}, \quad j = 1, 2, 3, 4
$$

To analyze the bounded regions of the eigenfunctions $\{\mu_j\}_4$, we need to use the curve $\{k \in \mathbb{C}|(\text{Re } f(k))(\text{Re } g(k)) = 0, \quad f(k) = ik, \quad g(k) = ik^2\}$ to separate the complex $k$-plane into four regions (see Fig. 2):

$$
D_1 = \{k \in \mathbb{C} | \text{Re } f(k) < 0 \text{ and } \text{Re } g(k) < 0\},
$$

$$
D_2 = \{k \in \mathbb{C} | \text{Re } f(k) < 0 \text{ and } \text{Re } g(k) > 0\},
$$

$$
D_3 = \{k \in \mathbb{C} | \text{Re } f(k) > 0 \text{ and } \text{Re } g(k) < 0\},
$$

$$
D_4 = \{k \in \mathbb{C} | \text{Re } f(k) > 0 \text{ and } \text{Re } g(k) > 0\},
$$

which implies that $D_1$ and $D_3$ ($D_2$ and $D_4$) are symmetric about the origin.

Thus it follows from Eqs. (11), (14) and (15) that the regions, where the different columns of eigenfunctions $\{\mu_j\}_4$ are bounded and analytic in the complex $k$-plane, are presented below:

$$
\begin{align*}
\mu_1 : (f_+ \cap g_+, f_- \cap g_+, f_- \cap g_+, f_+ \cap g_-) =: (D_2, D_2, D_2, D_3), \\
\mu_2 : (f_+ \cap g_-, f_- \cap g_-, f_- \cap g_-, f_+ \cap g_+) =: (D_1, D_1, D_1, D_4), \\
\mu_3 : (f_+ \cap g_-, f_+ \cap g_-, f_+ \cap g_-, f_- \cap g_-) =: (D_3, D_3, D_3, D_2), \\
\mu_4 : (f_+ \cap g_+, f_+ \cap g_+, f_+ \cap g_+, f_- \cap g_-) =: (D_4, D_4, D_4, D_1),
\end{align*}
$$

where $f_+ =: \text{Re } f(k) > 0$, $f_- =: \text{Re } f(k) < 0$, $g_+ =: \text{Re } g(k) > 0$, and $g_- =: \text{Re } g(k) < 0$.

2.3. The definition of the new matrix-valued functions $M_n$’s

To construct the jump matrix in a RH problem, we introduce the solutions $M_n(x, t, k)$ of Eq. (8)

$$
(M_n)_{s,j}(x, t, k) = \delta_{s,j} + \int_{(\gamma^n)_{s,j}} e^{-i(kx+2k^2t)} \sigma_4 W_n(x', \tau, k)_{s,j}, \quad k \in D_n, \quad s, j = 1, 2, 3, 4
$$

![Figure 2: The regions $D_n$ ($n = 1, 2, 3, 4$) separating the complex $k$-plane.](image)
where $W_n(x, t, k)$ is defined by Eq. (10) with $\mu(x, t, k)$ replaced with $M_n(x, t, k)$, and the contours $(\gamma^n)_{sj}$’s are given by

\[
(\gamma^n)_{sj} = \begin{cases} 
\gamma_1, & \text{if } \Re f_s(k) > \Re f_j(k) \text{ and } \Re g_s(k) \leq \Re g_j(k), \\
\gamma_2, & \text{if } \Re f_s(k) > \Re f_j(k) \text{ and } \Re g_s(k) > \Re g_j(k), \\
\gamma_3, & \text{if } \Re f_s(k) \leq \Re f_j(k) \text{ and } \Re g_s(k) \geq \Re g_j(k), \\
\gamma_4, & \text{if } \Re f_s(k) \leq \Re f_j(k) \text{ and } \Re g_s(k) \leq \Re g_j(k),
\end{cases}
\] (18)

for $k \in D_n$, where $f_{1,2,3}(k) = -f_4(k) = -ik, g_{1,2,3}(k) = -g_4(k) = -2ik^2$.

Notice that to distinguish $(\gamma^n)_{sj}$’s to be the contour $\gamma_3$ or $\gamma_4$ for the special cases, $\Re f_s(k) = \Re f_j(k)$ and $\Re g_s(k) = \Re g_j(k)$, we choose them in these cases as $\gamma_3$ (or $\gamma_4$) which must appear in the matrix $\gamma^n$, otherwise, we choose them in all these cases as the same $\gamma_3$ (or $\gamma_4$).

The definition (18) of $(\gamma^n)_{sj}$ implies that $\gamma^n$ ($n = 1, 2, 3, 4$) are explicitly given by

\[
\gamma^1 = \begin{pmatrix} \gamma_4 & \gamma_4 & \gamma_4 & \gamma_2 \\
\gamma_4 & \gamma_4 & \gamma_4 & \gamma_2 \\
\gamma_4 & \gamma_4 & \gamma_4 & \gamma_2 \\
\gamma_4 & \gamma_4 & \gamma_4 & \gamma_4 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix},
\] (19)

\[
\gamma^3 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_1 & \gamma_1 & \gamma_1 & \gamma_3 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \gamma_4 & \gamma_4 & \gamma_4 & \gamma_4 \\
\gamma_4 & \gamma_4 & \gamma_4 & \gamma_4 \\
\gamma_4 & \gamma_4 & \gamma_4 & \gamma_4 \\
\gamma_2 & \gamma_2 & \gamma_2 & \gamma_4 \end{pmatrix},
\]

**Proposition 2.1.** For the matrix-valued functions $M_n(x, t, k)$ ($n = 1, 2, 3, 4$) defined by Eq. (17) for $k \in \mathbb{D}_n$ and $(x, t) \in \Omega$, and any fixed point $(x, t)$, $M_n(x, t, k)$’s are the bounded and analytic functions of $k \in D_n$ away from a possible discrete set of singularity $\{k_j\}$ at which the Fredholm determinants vanish. Moreover, $M_n(x, t, k)$’s admit the bounded and continuous extensions to $\bar{D}_n$ and

\[
M_n(x, t, k) = \mathbb{I} + O\left(\frac{1}{k}\right), \quad k \in D_n, \quad k \rightarrow \infty, \quad n = 1, 2, 3, 4.
\] (20)

**Proof.** Similar to the proof for the $3 \times 3$ Lax pair in [32], we can also proof the boundedness and analyticity of $M_n$. The substitution of \[
\mu(x, t, k) = M_n(x, t, k) = M_n^{(0)}(x, t, k) + \sum_{j=1}^{\infty} \frac{M_n^{(j)}(x, t, k)}{k^j}, \quad k \rightarrow \infty,
\] into the $x$-part of the Lax pair (8) yields Eq. (20). \(\Box\)

The above-defined matrix-valued functions $M_n$’s can be used to formulate a $4 \times 4$ matrix Riemann-Hilbert problem.

### 2.4. The spectral functions and jump matrices

We introduce the spectral functions $S_n(k)$ ($n = 1, 2, 3, 4$) by

\[
S_n(k) = M_n(x = 0, t = 0, k), \quad k \in D_n, \quad n = 1, 2, 3, 4.
\] (21)
Let $M(x, t, k)$ denote the sectionally analytic function on the Riemann $k$-sphere which is equivalent to $M_n(x, t, k)$ for $k \in D_n$. Then $M(x, t, k)$ solves the jump equations

$$M_n(x, t, k) = M_m(x, t, k)J_{mn}(x, t, k), \quad k \in \bar{D}_n \cap \bar{D}_m, \quad n, m = 1, 2, 3, 4, \quad n \neq m,$$

(22)

with the jump matrices $J_{mn}(x, t, k)$ defined by

$$J_{mn}(x, t, k) = e^{-i(kx+2k^2t)\bar{\sigma}_4}(S_{m}^{-1}(k)S_{n}(k)).$$

(23)

2.5. The minors or the transpose of the adjugates of eigenfunctions

To conveniently calculate the spectral functions $S_n(k)$ in the following sections, we need to use the cofactor matrix $X^A$ (or the transpose of the adjugate) of a $4 \times 4$ matrix $X$ defined as

$$\text{adj}(X)^T = X^A = \left(\begin{array}{cccc}
m_{11}(X) & -m_{12}(X) & m_{13}(X) & -m_{14}(X) \\
-m_{21}(X) & m_{22}(X) & -m_{23}(X) & m_{24}(X) \\
m_{31}(X) & -m_{32}(X) & m_{33}(X) & -m_{34}(X) \\
-m_{41}(X) & m_{42}(X) & -m_{43}(X) & m_{44}(X) \end{array}\right),$$

where $m_{ij}(X)$ denotes the $(ij)$th minor of $X$ and $(X^A)^T X = \text{adj}(X)X = \det X$.

It follows from the Lax pair (4) that the eigenfunction $\{\mu^A_j\}_1^4$ of the matrices $\{\mu_j(x, t, k)\}_1^4$ satisfy the Lax equation

$$\left\{\begin{array}{l}
\mu^A_j - ik[\sigma_4, \mu^A] = -U^T(x, t)\mu^A, \\
\mu^A_j - 2ik^2[\sigma_4, \mu^A] = -V^T(x, t, k)\mu^A,
\end{array}\right.$$

(24)

whose solutions can be written as the form

$$\mu^A_j(x, t, k) = \mathbb{I} - \int_{\gamma_j} e^{i(k(x-x')+2k^2(t-\tau))\bar{\sigma}_4} \left[U^T(x', \tau)d\tau + V^T(x', \tau, k)d\tau\right] \mu^A_j(x', \tau, k), \quad j = 1, 2, 3, 4,$$

(25)

in terms of the Volterra integral equations.

It is easy to check that the regions of boundedness of $\mu^A_j$:

$$\left\{\begin{array}{l}
\mu^A_1(x, t, k) \text{ is bounded for } k \in (D_3, D_3, D_3, D_2), \\
\mu^A_2(x, t, k) \text{ is bounded for } k \in (D_4, D_4, D_4, D_1), \\
\mu^A_3(x, t, k) \text{ is bounded for } k \in (D_2, D_2, D_2, D_3), \\
\mu^A_4(x, t, k) \text{ is bounded for } k \in (D_1, D_1, D_1, D_4),
\end{array}\right.$$

which are symmetric ones of $\mu_j$ about the Re$k$-axis (cf. Eq. (16)).

2.6. Symmetries of eigenfunctions

Let

$$\hat{U}(x, t, k) = -ik\sigma_4 + U(x, t), \quad \hat{V}(x, t, k) = -2ik^2\sigma_4 + V(x, t, k).$$

(26)
in the Lax pair (4). Then we have

\[ P\overline{U}(x,t,k)P = -\overline{U}(x,t,k)^T, \quad P\overline{V}(x,t,k)P = -\overline{V}(x,t,k)^T, \]  

(27)

where the symmetric matrix \( P \) is taken as

\[ P = \begin{pmatrix} \alpha_{11} & \bar{\alpha}_{12} & \bar{\alpha}_{13} & 0 \\ \alpha_{12} & \alpha_{22} & \bar{\alpha}_{23} & 0 \\ \bar{\alpha}_{13} & \bar{\alpha}_{23} & \alpha_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad P^2 = I, \quad P = P^\dagger, \]  

(28)

Notice that the symmetric matrix \( P \) used here differs from the diag ones used in \( 3 \times 3 \) Lax pairs [35–38].

Similar to the proof in Ref. [14], based on Eq. (24) and (27) we have the following proposition:

**Proposition 2.2.** The matrix-valued eigenfunctions \( \psi(x,t,k) \) of the Lax pair (4) and \( \mu_j(x,t,k) \) of the Lax pair (8) both possess the same symmetric relations

\[ \psi^{-1}(x,t,k) = P\overline{\psi}(x,t,k)^T P, \quad \mu^{-1}_j(x,t,k) = P\overline{\mu}_j(x,t,k)^T P, \quad j = 1, 2, 3, 4, \]  

(29)

Moreover, In the domains where \( \mu_j \) is bounded, we have

\[ \mu_j(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad j = 1, 2, 3, 4 \]  

(30)

and

\[ \det[\mu_j(x,t,k)] = 1, \quad j = 1, 2, 3, 4 \]  

(31)

since the traces of the matrices \( U(x,t,k) \) and \( V(x,t,k) \) are zero.

### 2.7. The relations between spectral functions and jump matrices \( J_{mn} \)

Since these functions \( \mu_j \) are dependent, thus we can define three \( 4 \times 4 \) matrix-valued functions \( S(k) \), \( s(k) \) and \( S(k) \) between \( \mu_2 \) and \( \mu_j \), \( j = 1, 3, 4 \) in the form (cf. Fig. 3)

\[ \begin{align*}
\mu_1(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{\sigma}_4}S(k), \\
\mu_3(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{\sigma}_4}s(k), \\
\mu_4(x,t,k) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{\sigma}_4}S(k),
\end{align*} \]  

(32)
Evaluating system (32) at \((x, t) = (0, 0)\) and the three equations in system (32) at \((x, t) = (0, T), (L, 0), (L, T)\), respectively, we have

\[
\begin{align*}
S(k) = \mu_1(0, 0, k) &= e^{2ik^2T\bar{\sigma}_4}\mu_2^{-1}(0, T, k), \\
s(k) = \mu_3(0, 0, k) &= e^{ik\bar{\sigma}_4}\mu_2^{-1}(L, 0, k), \\
S_L(k) = \mu_4(0, 0, k) &= e^{i(kL+2k^2T)\bar{\sigma}_4}\mu_2^{-1}(L, T, k),
\end{align*}
\tag{33}
\]

Except for the defined three relations, it follows from Eqs. (33) and (34) that we can find other three relations:

(i) the relation between \(\mu_3(x, t, k)\) and \(\mu_4(x, t, k)\)

\[
\mu_4(x, t, k) = \mu_3(x, t, k)e^{-i[k(x-L)+2k^2(t-T)]\bar{\sigma}_4}\mu_3^{-1}(L, T, k) = \mu_3(x, t, k)e^{-i[k(x-L)+2k^2t]\bar{\sigma}_4}S_L(k),
\]

with

\[
S_L(k) = \mu_4(L, 0, k) = e^{2ik^2T\bar{\sigma}_4}\mu_3^{-1}(L, T, k),
\tag{34}
\]

(ii) the relation between \(\mu_1(x, t, k)\) and \(\mu_4(x, t, k)\)

\[
\mu_3(x, t, k) = \mu_1(x, t, k)e^{-i(kx+2k^2t)\bar{\sigma}_4}\mathcal{S}(k),
\]

with

\[
\mathcal{S}(k) = S^{-1}(k)\mathcal{S}(k),
\tag{35}
\]

and (iii) the relation between \(\mu_1(x, t, k)\) and \(\mu_4(x, t, k)\)

\[
\mu_4(x, t, k) = \mu_1(x, t, k)e^{-i(kx+2k^2t)\bar{\sigma}_4}s_T(k),
\]

with

\[
s_T(k) = S^{-1}(k)\mathcal{S}(k),
\tag{36}
\]

It follows from Eqs. (33) and (34) that we have the relation

\[
S(k) = s(k)e^{ikL\bar{\sigma}_4}S_L(k),
\tag{37}
\]

The map of these relations among \(\mu_j\) is exhibited in Fig. 3.

According to the definition (13) of \(\mu_j\), Eqs. (33) and (34) imply that

\[
\begin{align*}
s(k) &= 1 - \int_0^L e^{ikx'\bar{\sigma}_4}(U\mu_3)(x', 0, k)dx' = \left[ 1 + \int_0^L e^{ikx'\bar{\sigma}_4}(U\mu_2)(x', 0, k)dx' \right]^{-1}, \\
S(k) &= 1 - \int_0^T e^{2ik^2\bar{\sigma}_4}(V\mu_1)(0, \tau, k)d\tau = \left[ 1 + \int_0^T e^{2ik^2\bar{\sigma}_4}(V\mu_2)(0, \tau, k)d\tau \right]^{-1}, \\
S_L(k) &= 1 - \int_0^T e^{2ik^2\bar{\sigma}_4}(V\mu_4)(L, \tau, k)d\tau = \left[ 1 + \int_0^T e^{2ik^2\bar{\sigma}_4}(V\mu_3)(L, \tau, k)d\tau \right]^{-1}, \\
S(k) &= 1 - \int_0^L e^{ikx'\bar{\sigma}_4}(U\mu_4)(x', 0, k)dx' - e^{ikL\bar{\sigma}_4}\int_0^T e^{2ik^2\bar{\sigma}_4}(V\mu_4)(L, \tau, k)d\tau \\
&= \left[ 1 + e^{2ik^2T\bar{\sigma}_4}\int_0^L e^{ikx'\bar{\sigma}_4}(U\mu_2)(x', T, k)dx' + \int_0^T e^{2ik^2\bar{\sigma}_4}(V\mu_2)(0, \tau, k)d\tau \right]^{-1},
\end{align*}
\tag{38}
\]
which leads to $S(k)$ and $s_T(k)$ in terms of Eqs. (35) and (36), where $\mu_{j_2}(0,t,k)$, $j_2 = 1, 2$, $\mu_{j_3}(L, t, k)$, $j_3 = 3, 4$, $\mu_{j_1}(x, 0, k)$, $j_1 = 2, 3, 4$, $\mu_2(x, T, k)$, $0 < x < L$, $0 < t < T$ are defined by the integral equations

$$\begin{align*}
\mu_1(0, t, k) &= \mathbb{I} + \int_0^t e^{-2 ik^2(t-\tau)\hat{s}_4}(V\mu_1)(0, \tau, k)d\tau, \\
\mu_2(0, t, k) &= \mathbb{I} + \int_0^t e^{-2 ik^2(t-\tau)\hat{s}_4}(V\mu_2)(0, \tau, k)d\tau, \\
\mu_3(L, t, k) &= \mathbb{I} + \int_0^t e^{-2 ik^2(t-\tau)\hat{s}_4}(V\mu_3)(L, \tau, k)d\tau, \\
\mu_4(L, t, k) &= \mathbb{I} + \int_0^t e^{-2 ik^2(t-\tau)\hat{s}_4}(V\mu_4)(L, \tau, k)d\tau, \\
\mu_2(x, 0, k) &= \mathbb{I} + \int_0^x e^{ik\tau\hat{s}_4}(U\mu_2)(x', 0, k)dx', \\
\mu_3(x, 0, k) &= \mathbb{I} + \int_L^x e^{ik\tau\hat{s}_4}(U\mu_3)(x', 0, k)dx', \\
\mu_4(x, 0, k) &= \mathbb{I} + \int_L^x e^{ik\tau\hat{s}_4}(U\mu_4)(x', 0, k)dx' - e^{-ik(x-L)\hat{s}_4}\int_0^T e^{2ik^2\tau\hat{s}_4}(V\mu_4)(L, \tau, k)d\tau, \\
\mu_2(x, T, k) &= \mathbb{I} + \int_0^x e^{-ik(x-x')\hat{s}_4}(U\mu_2)(x', T, k)dx' + e^{-ikx\hat{s}_4}\int_0^T e^{-2ik^2(T-\tau)\hat{s}_4}(V\mu_2)(0, \tau, k)d\tau,
\end{align*}$$

It follows from the properties of $\mu_j$ and $\mu_j^A$ that the functions $\{S(k), s(k), S(k), S_L(k)\}$ and $\{S^A(k), s^A(k), S^A(k), S^A_L(k)\}$ have the following boundedness:

$$\begin{align*}
S(k) &\text{ is bounded for } k \in (D_2 \cup D_4, D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3), \\
s(k) &\text{ is bounded for } k \in (D_3 \cup D_4, D_3 \cup D_4, D_3 \cup D_4, D_1 \cup D_2), \\
S(k) &\text{ is bounded for } k \in (D_4, D_4, D_4, D_1), \\
S_L(k) &\text{ is bounded for } k \in (D_2 \cup D_4, D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3), \\
S^A(k) &\text{ is bounded for } k \in (D_1 \cup D_3, D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4), \\
s^A(k) &\text{ is bounded for } k \in (D_1 \cup D_2, D_1 \cup D_2, D_1 \cup D_2, D_3 \cup D_4), \\
S^A(k) &\text{ is bounded for } k \in (D_2, D_2, D_3), \\
S^A_L(k) &\text{ is bounded for } k \in (D_2 \cup D_4, D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3),
\end{align*}$$

**Proposition 2.3.** The matrix-valued functions $S_n(x, t, k)$ ($n = 1, 2, 3, 4$) defined by

$$M_n(x, t, k) = \mu_2(x, t, k)e^{-i(kx+2k^2t)\hat{s}_4}S_n(k), \quad k \in D_n,$$

(39)

with $M_n$ given by Eq. (17) can be determined by the entries of the data $S(k) = (S_{ij})_{4 \times 4}$, $s(k) = (s_{ij})_{4 \times 4}$, and
\[ S(k) = (s_{ij})_{4 \times 4} \text{ given by Eq. (33) as follows:} \]

\[
S_1(k) = \begin{pmatrix}
    s_{11} & s_{12} & s_{13} & 0 \\
    s_{21} & s_{22} & s_{23} & 0 \\
    s_{31} & s_{32} & s_{33} & 0 \\
    s_{41} & s_{42} & s_{43} & 1/m_{44}(S)
\end{pmatrix}, \quad S_2(k) = \begin{pmatrix}
    s_{11} & s_{12} & s_{13} & S_{14} \\
    s_{21} & s_{22} & s_{23} & S_{24} \\
    s_{31} & s_{32} & s_{33} & S_{34} \\
    s_{41} & s_{42} & s_{43} & S_{44}
\end{pmatrix},
\]

\[
S_3(k) = \begin{pmatrix}
    S_3^{(11)} & S_3^{(12)} & S_3^{(13)} & s_{14} \\
    S_3^{(21)} & S_3^{(22)} & S_3^{(23)} & s_{24} \\
    S_3^{(31)} & S_3^{(32)} & S_3^{(33)} & s_{34} \\
    S_3^{(41)} & S_3^{(42)} & S_3^{(43)} & s_{44}
\end{pmatrix}, \quad S_4(k) = \begin{pmatrix}
    n_{11,44}(S) & n_{12,44}(S) & n_{13,44}(S) & S_{14} \\
    n_{21,44}(S) & n_{22,44}(S) & n_{23,44}(S) & S_{24} \\
    n_{31,44}(S) & n_{32,44}(S) & n_{33,44}(S) & S_{34} \\
    0 & 0 & 0 & S_{44}
\end{pmatrix},
\]

where \( n_{i_1 j_1, i_2 j_2}(X) \) denotes the determinant of the sub-matrix generated by choosing the cross elements of \( i_{1,2} \)th rows and \( j_{1,2} \)th columns of \( X \), and

\[
\begin{align*}
    S_3^{(1)} &= \frac{m_{24}(S)n_{11,24}(s) - m_{34}(S)n_{11,34}(s) + m_{44}(S)n_{11,44}(s)}{(s^T A^T A)_{44}}, \\
    S_3^{(2)} &= \frac{m_{14}(S)n_{21,14}(s) - m_{24}(S)n_{21,24}(s) + m_{44}(S)n_{21,44}(s)}{(s^T A^T A)_{44}}, \\
    S_3^{(3)} &= \frac{m_{14}(S)n_{31,14}(s) - m_{24}(S)n_{31,24}(s) + m_{44}(S)n_{31,44}(s)}{(s^T A^T A)_{44}}, \\
    S_3^{(4)} &= \frac{m_{14}(S)n_{41,14}(s) - m_{24}(S)n_{41,24}(s) + m_{44}(S)n_{41,44}(s)}{(s^T A^T A)_{44}},
\end{align*}
\]

\( l = 1, 2, 3, \)

**Proof.** We introduce the matrix-valued functions \( R_n(k), S_n(k), T_n(k), \) and \( P_n(k), n = 1, 2, 3, 4 \) by \( M_n(x, t, k) \) and \( \mu_j(x, t, k) \)

\[
\begin{align*}
    M_n(x, t, k) &= \mu_1(x, t, k)e^{-i(kx+2k^2t)\xi}R_n(k), \\
    M_n(x, t, k) &= \mu_2(x, t, k)e^{-i(kx+2k^2t)\xi}S_n(k), \\
    M_n(x, t, k) &= \mu_3(x, t, k)e^{-i(kx+2k^2t)\xi}T_n(k), \\
    M_n(x, t, k) &= \mu_4(x, t, k)e^{-i(kx+2k^2t)\xi}P_n(k),
\end{align*}
\]
It follows from Eq. (41) that we have the relations
\[
\begin{align*}
R_n(k) &= e^{2ik^2T^4}M_n(0,T,k), \\
S_n(k) &= M_n(0,0,k), \\
T_n(k) &= e^{ikL^4}M_n(L,0,k), \\
P_n(k) &= e^{i(kL+2k^2T^4)}M_n(L,T,k),
\end{align*}
\] (42)
and
\[
\begin{align*}
S(k) &= \mu_1(0,0,k) = S_n(k)R_n^{-1}(k), \\
s(k) &= \mu_3(0,0,k) = S_n(k)T_n^{-1}(k), \\
S(k) &= \mu_4(0,0,k) = S_n(k)P_n^{-1}(k),
\end{align*}
\] (43)
which can in general obtain the functions \(\{R_n, S_n, T_n, P_n\}\) for the given functions \(\{s(k), S(k), S(k)\}\).

Moreover, we can also determine some entries of \(\{R_n, S_n, T_n, P_n\}\) in terms of Eqs. (17) and (41)
\[
\begin{align*}
(R_n(k))_{ij} &= 0, \quad \text{if } (\gamma^n)_{ij} = \gamma_1, \\
(S_n(k))_{ij} &= 0, \quad \text{if } (\gamma^n)_{ij} = \gamma_2, \\
(T_n(k))_{ij} &= \delta_{ij}, \quad \text{if } (\gamma^n)_{ij} = \gamma_3, \\
(P_n(k))_{ij} &= \delta_{ij}, \quad \text{if } (\gamma^n)_{ij} = \gamma_4.
\end{align*}
\] (44)

Thus it follows from systems (43) and (44) that we can find Eq. (40). □

2.8. The residue conditions for \(M_n\)

Since \(\mu_2(x,t,k)\) is an entire function, it follows from Eq. (39) that \(M_n(x,t,k)\) only have singularities at the points where the \(S_n(k)\)'s have singularities. We find from the expressions of \(S_n(k)\) given by Eq. (40) that the possible singularities of \(M_n\) are as follows:

- \([M]_4\) could admit poles in \(D_1\) at the zeros of \(m_{44}(S)(k)\);
- \([M]_4\) could have poles in \(D_2\) at the zeros of \((S^T s^A)_{44}(k)\);
- \([M]_l, l = 1,2,3\) could be of poles in \(D_3\) at the zeros of \((s^T S^A)_{44}(k)\);
- \([M]_l, l = 1,2,3\) could have poles in \(D_4\) at the zeros of \(S_{44}(k)\).

We introduce the above possible zeros by \(\{k_j\}^N_1\) and suppose that they satisfy the following assumption.

**Assumption 2.4.** We assume that

- \(m_{44}(S)(k)\) has \(n_1\) possible simple zeros in \(D_1\) denoted by \(\{k_j\}^{n_1}_1\);
- \((S^T s^A)_{44}(k)\) has \(n_2 - n_1\) possible simple zeros in \(D_2\) denoted by \(\{k_j\}^{n_2}_{n_1+1}\);
- \((s^T S^A)_{44}(k)\) has \(n_3 - n_2\) possible simple zeros in \(D_3\) denoted by \(\{k_j\}^{n_3}_{n_2+1}\);
• $S_{44}(k)$ has $N - n_3$ possible simple zeros in $D_4$ denoted by $\{k_j\}_{n_3+1}^N$;
and that none of these zeros coincide. Moreover, none of these functions are assumed to have zeros on the boundaries of the $D_n$’s ($n = 1, 2, 3, 4$).

We can deduce the residue conditions at these zeros in the following expressions:

**Proposition 2.5.** Let $\{M_n\}^4_l$ be the eigenfunctions given by Eq. (17) and suppose that the set $\{k_j\}_1^N$ of singularities is as the above-mentioned Assumption 2.4. Then we have the following residue conditions for $M_n$:

$$\text{Res}_{k=j}[M_1]_4 = \frac{n_{12,23}(S(k)) [M_1(k_j)]_1 - n_{11,23}(S(k)) [M_1(k_j)]_2 + n_{11,22}(S(k)) [M_1(k_j)]_3 e^{2\theta(k_j)}}{m_{44}(S(k)) m_{34}(S(k))},$$

for $1 \leq j \leq n_1, k \in D_1$,

$$\text{Res}_{k=j}[M_2]_4 = \frac{[M_2(\theta)]_1 [S_{14}(k_j)] m_{21,43}(S(k)) - s_{24}(k_j) m_{12,23}(S(k)) + S_{44}(k_j) n_{12,23}(S(k))}{(ST^A)_{44}(k_j) m_{34}(S(k)) e^{-2\theta(k_j)}}$$

$$- \frac{[M_2(\theta)]_2 [S_{14}(k_j)] m_{21,43}(S(k)) - S_{24}(k_j) n_{11,43}(S(k)) + S_{44}(k_j) n_{11,23}(S(k))}{(ST^A)_{44}(k_j) m_{34}(S(k)) e^{-2\theta(k_j)}}$$

$$+ \frac{[M_2(\theta)]_3 [S_{14}(k_j)] m_{21,42}(S(k)) - S_{24}(k_j) n_{11,42}(S(k)) + S_{44}(k_j) n_{11,22}(S(k))}{(ST^A)_{44}(k_j) m_{34}(S(k)) e^{-2\theta(k_j)}},$$

for $n_1 + 1 \leq j \leq n_2, k \in D_2$,

$$\text{Res}_{k=j}[M_3]_4 = \frac{n_{41,14}(S(k_j)) m_{41,24}(S(k)) - m_{31,24}(S(k_j)) m_{41,34}(S(k)) + m_{31,34}(S(k_j)) m_{41,3}(S(k))}{(ST^A)_{44}(k_j) s_{44}(S(k)) e^{2\theta(k_j)}}$$

$$\times [M_3(k_j)]_4, \text{ for } n_2 + 1 \leq j \leq n_3, k \in D_3, l = 1, 2, 3,$$

$$\text{Res}_{k=j}[M_4]_4 = \frac{S_{44}(k_j) [M_4(k_j)]_{4} e^{-2\theta(k_j)}}{S_{44}(k_j)}, \text{ for } n_3 + 1 \leq j \leq N, k \in D_4, l = 1, 2, 3,$$

where the overdot stands for the derivative with respect to the parameter $k$ and $\theta = \theta(k) = -i(kx + 2k^2t)$.

**Proof.** It follows from Eqs. (39) and (40) that the four columns of $M_1$ are given by the matrices $\mu_2$ and $S_1(k)$

$$[M_1]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} + [\mu_2]_3 s_{31} + [\mu_2]_4 s_{41} e^{-2\theta},$$

$$[M_1]_2 = [\mu_2]_1 s_{12} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32} + [\mu_2]_4 s_{42} e^{-2\theta},$$

$$[M_1]_3 = [\mu_2]_1 s_{13} + [\mu_2]_2 s_{23} + [\mu_2]_3 s_{33} + [\mu_2]_4 s_{43} e^{-2\theta},$$

$$[M_1]_4 = \frac{[\mu_2]_4}{m_{44}(S)},$$

the four columns of $M_2$ are given by the matrices $\mu_2$ and $S_2(k)$

$$[M_2]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} + [\mu_2]_3 s_{31} + [\mu_2]_4 s_{41} e^{-2\theta},$$

$$[M_2]_2 = [\mu_2]_1 s_{12} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32} + [\mu_2]_4 s_{42} e^{-2\theta},$$

$$[M_2]_3 = [\mu_2]_1 s_{13} + [\mu_2]_2 s_{23} + [\mu_2]_3 s_{33} + [\mu_2]_4 s_{43} e^{-2\theta},$$

$$[M_2]_4 = \frac{[\mu_2]_4 S_{41}}{(ST^A)_{44}} e^{2\theta} + \frac{[\mu_2]_4 S_{24}}{(ST^A)_{44}} e^{-2\theta} + \frac{[\mu_2]_4 S_{34}}{(ST^A)_{44}} e^{2\theta} + \frac{[\mu_2]_4 S_{44}}{(ST^A)_{44}},$$
the four columns of $M_3$ are given by the matrices $\mu_2$ and $S_3(k)$

\[
[M_3]_1 = [\mu_2]_1 s_3^{(11)} + [\mu_2]_2 s_3^{(21)} + [\mu_2]_3 s_3^{(31)} + [\mu_2]_4 s_3^{(41)} e^{-2\theta},
\]

\[
[M_3]_2 = [\mu_2]_1 s_3^{(12)} + [\mu_2]_2 s_3^{(22)} + [\mu_2]_3 s_3^{(32)} + [\mu_2]_4 s_3^{(42)} e^{-2\theta},
\]

\[
[M_3]_3 = [\mu_2]_1 s_3^{(13)} + [\mu_2]_2 s_3^{(23)} + [\mu_2]_3 s_3^{(33)} + [\mu_2]_4 s_3^{(43)} e^{-2\theta},
\]

\[
[M_3]_4 = [\mu_2]_1 s_{14} e^{2\theta} + [\mu_2]_2 s_{24} e^{2\theta} + [\mu_2]_3 s_{34} e^{2\theta} + [\mu_2]_4 s_{44},
\]

(51a)

(51b)

(51c)

(51d)

and the four columns of $M_4$ are given by the matrices $\mu_2$ and $S_4(k)$

\[
[M_4]_1 = [\mu_2]_1 \frac{n_{11,44}(S)}{S_{44}} + [\mu_2]_2 \frac{n_{21,44}(S)}{S_{44}} + [\mu_2]_3 \frac{n_{31,44}(S)}{S_{44}},
\]

\[
[M_4]_2 = [\mu_2]_1 \frac{n_{12,44}(S)}{S_{44}} + [\mu_2]_2 \frac{n_{22,44}(S)}{S_{44}} + [\mu_2]_3 \frac{n_{32,44}(S)}{S_{44}},
\]

\[
[M_4]_3 = [\mu_2]_1 \frac{n_{13,44}(S)}{S_{44}} + [\mu_2]_2 \frac{n_{23,44}(S)}{S_{44}} + [\mu_2]_3 \frac{n_{33,44}(S)}{S_{44}},
\]

\[
[M_4]_4 = [\mu_2]_1 s_{14} e^{2\theta} + [\mu_2]_2 s_{24} e^{2\theta} + [\mu_2]_3 s_{34} e^{2\theta} + [\mu_2]_4 s_{44},
\]

(52a)

(52b)

(52c)

(52d)

For the case that $k_j \in D_1$ is a simple zero of $m_{44}(S)(k)$, it follows from Eqs. (49a)-(49c) that we have $[\mu_2]_j$, $j = 1, 2, 4$ and then substitute them into Eq. (49d) to yield

\[
[M_1]_4 = \frac{n_{12,23}(S)[M_1]_1 - n_{11,23}(S)[M_1]_2 + n_{11,22}(S)[M_1]_3}{m_{31}(S)m_{44}(S)} e^{2\theta} - \frac{[\mu_2]_3}{m_{34}(S)} e^{2\theta},
\]

whose residue at $k_j$ yields Eq. (45) for $k_j \in D_1$, respectively.

Similarly, we solve Eqs. (50a)-(50c) for $[\mu_2]_j$, $j = 1, 2, 4$ and then substitute them into Eq. (50d) to yield

\[
[M_2]_4 = \frac{[M_2]_1 [S_{14} n_{22,44}(s) - S_{24} n_{12,44}(s) + S_{44} n_{12,23}(s)]}{(S^T s^A)_{44} m_{34}(s)} e^{2\theta}
\]

\[- \frac{[M_2]_2 [S_{14} n_{21,44}(s) - S_{24} n_{11,44}(s) + S_{44} n_{11,23}(s)]}{(S^T s^A)_{44} m_{34}(s)} e^{2\theta}
\]

\[+ \frac{[M_2]_3 [S_{14} n_{21,42}(s) - S_{24} n_{11,42}(s) + S_{44} n_{11,22}(s)]}{(S^T s^A)_{44} m_{34}(s)} e^{2\theta} - \frac{[\mu_2]_3}{m_{34}(S)} e^{2\theta},
\]

whose residues at $k_j$ yields Eq. (46) for $k_j \in D_2$, respectively. Similarly, we can show Eq. (47) for $k_j \in D_3$ and Eq. (48) for $k_j \in D_4$ by analyzing Eqs. (51a)-(52d). □

2.9. The global relation

The definitions of the above-mentioned spectral functions $S(k), s(k), S_L(k)$, and $S(k)$ imply that they are dependent. It follows from Eqs. (32) and (34) that

\[
\mu_4(x, t, k) = \mu_2(x, t, k)e^{-i(kx+2k^2t)\sigma_3} s(k)
\]

\[
= \mu_2(x, t, k)e^{-i(kx+2k^2t)\sigma_3} [s(k)e^{ikL\sigma_4} S_L(k)]
\]

\[
= \mu_1(x, t, k)e^{-i(kx+2k^2t)\sigma_3} [S^{-1}(k)s(k)e^{ikL\sigma_4} S_L(k)],
\]

(53)
which leads to the global relation
\[ e(T, k) = \mu_4(0, T, k) = e^{-2ik^2T\sigma_4}[S^{-1}(k)s(k)e^{ikL\sigma_4}S_L(k)], \]
by evaluating Eq. (53) at the point \((x, t) = (0, T)\) and using \(\mu_1(0, T, k) = I\).

3 The 4 \(\times\) 4 matrix Riemann-Hilbert problem

By using the district contours \(\gamma_j (j = 1, 2, 3, 4)\), the integral solutions of the revised Lax pair \((8), \) and \(S_n\) due to \(\{S(k), s(k), S(k), S_L(k)\}\), we have defined the sectionally analytic function \(M_n(x, t, k) (n = 1, 2, 3, 4)\), which solves a 4 \(\times\) 4 matrix Riemann-Hilbert (RH) problem. This RH problem can be formulated on basis of the initial and boundary data of the functions \(q_1(x, t), q_2(x, t)\) and \(q_3(x, t)\). Thus the solution of Eq. (1) for all values of \(x, t\) can be refound by solving the RH problem.

**Theorem 3.1.** Let \((q_1(x, t), q_2(x, t), q_3(x, t))\) be a solution of Eq. (1) in the interval domain \(\Omega = \{(x, t) | x \in [0, L], t \in [0, T]\}\). Then it can be reconstructed from the initial data defined by
\[ q_j(x, t = 0) = q_{0j}(x), \quad j = 1, 2, 3, \]
and Dirichlet and Neumann boundary values defined by

- **Dirichlet boundary data:** \(q_j(x = 0, t) = u_{0j}(t), \quad q_j(x = L, t) = v_{0j}(t), \quad j = 1, 2, 3,\)
- **Neumann boundary data:** \(q_{jx}(x = 0, t) = u_{1j}(t), \quad q_{jx}(x = L, t) = v_{1j}(t), \quad j = 1, 2, 3,\)

We can use the initial and boundary data to define the jump matrices \(J_{nm}(x, t, k), (n, m = 1, ..., 4)\) given by Eq. (23) as well as the spectral functions \(S(k), s(k)\) and \(S(k)\) defined by Eq. (33). Assume that the possible zeros \(\{k_j\}_1^N\) of the functions \(m_{44}(S(k)), (S^T S^A)_{44}(k), (s^T S^A)_{44}(k)\), and \(S_{44}(k)\) are as in Assumption 2.4. Then the solution \((q_1(x, t), q_2(x, t), q_3(x, t))\) of Eq. (1) is given by \(M(x, t, k)\) in the form
\[ q_j(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{j4}, \quad j = 1, 2, 3, \]
where \(M(x, t, k)\) satisfies the following 4 \(\times\) 4 matrix Riemann-Hilbert problem:

- \(M(x, t, k)\) is sectionally meromorphic on the Riemann \(k\)-sphere with jumps across the contours \(\tilde{D}_n \cup \tilde{D}_m, \quad (n, m = 1, ..., 4)\) (see Fig. 2).
- Across the contours \(\tilde{D}_n \cup \tilde{D}_m (n, m = 1, ..., 4), \ M(x, t, k)\) satisfies the jump condition (22).
- The residue conditions of \(M(x, t, k)\) are satisfied in Proposition 2.5.
- \(M(x, t, k) = I + O(1/k)\) as \(k \to \infty\).

**Proof.** System (55) can be deduced from the large \(k\) asymptotics of the eigenfunctions. We can follow the similar one in Refs. [8,14] to show the rest proof of the Theorem. \(\square\)
4 The nonlinearizable boundary conditions

The key difficulty of initial-boundary value problems is to find the boundary values for a well-posed problem. All boundary value conditions are required for the definition of $S(k)$ and $S_L(k)$, and hence for the formulate the RH problem. Our main conclusion exhibits the unknown boundary condition on basis of the prescribed boundary condition and the initial condition in terms of the solution of a system of nonlinear integral equations.

4.1. The generalized global relation

By evaluating Eqs. (53) and (54) at the point $(x,t) = (0,t)$, we have

$$c(t,k) = \mu_2(0,t,k)e^{-2ik^2t_{44}}[s(k)e^{ikL_{44}}S_{L}(k)],$$

which and Eq. (34) lead to

$$c(t,k) = \mu_2(0,t,k)e^{-2ik^2t_{44}}[s(k)e^{ikL_{44}}e^{2ik^2t_{44}}\mu_3^{-1}(L,t,k)],$$

$$= \mu_2(0,t,k)[e^{-2ik^2t_{44}}s(k)][e^{ikL_{44}}\mu_3^{-1}(L,t,k)].$$

Thus, the column vectors $[c(t,k)]_j$, $j = 1, 2, 3$ are analytic and bounded in $D_4$ away from the possible zeros of $S_{44}(k)$ and of order $O(1 + e^{-2ikL_{44}})$ as $k \to \infty$, and the column vector $[c(t,k)]_4$ is analytic and bounded in $D_1$ away from the possible zeros of $m_{44}(S)(k)$ and of order $O(1 + e^{2ikL_{44}})$ as $k \to \infty$. 

16
4.2. Asymptotic behaviors of eigenfunctions

It follows from the Lax pair (8) that the eigenfunctions \( \{\mu_j\} \) possess the following asymptotics as \( k \to \infty \)

\[
\mu_j(x, t, k) = \mathbb{I} + \frac{1}{k^2} \left[ \begin{array}{cc}
\Delta_{11}^{(1)} & \Delta_{12}^{(1)} \\
\Delta_{21}^{(1)} & \Delta_{22}^{(1)} \\
\Delta_{31}^{(1)} & \Delta_{32}^{(1)} \\
\Delta_{41}^{(1)} & \Delta_{42}^{(1)} \\
\end{array} \right] + O\left( \frac{1}{k^3} \right)
\]

where we have introduced the following functions

\[
\Delta_{jl}^{(1)} = \frac{i}{2} q_j p_l dx + \frac{1}{2} (q_j p_l - q_l p_j) dt, \quad j, l = 1, 2, 3,
\]

\[
\Delta_{44}^{(1)} = \frac{i}{2} \sum_{j=1}^{3} q_j p_j dx + \frac{1}{2} \sum_{j=1}^{3} (p_j q_j - p_j q_j) dt,
\]

and

\[
\Delta_{jl}^{(2)} = \frac{i}{2} q_j p_l dx + \frac{1}{2} (q_j p_l - q_l p_j) dt, \quad j, l = 1, 2, 3.
\]

These functions are defined as

\[
\Delta_{jl}^{(2)} = \frac{i}{2} q_j p_l dx + \frac{1}{2} (q_j p_l - q_l p_j) dt, \quad j, l = 1, 2, 3.
\]
and

$$
\begin{align*}
\mu_{j, 4}^{(2)} &= \frac{1}{4} q_{t, x} + \frac{1}{2i} q_{t} \int_{(x, t, j)}^{(x, t)} \Delta_{44}^{(1)} \, \Delta(t), \quad l = 1, 2, 3, \\
\mu_{j, 1}^{(2)} &= \frac{1}{4} p_{t, x} + \frac{i}{2} \sum_{s=1}^{3} p_{s} \int_{(x, t, j)}^{(x, t)} \Delta_{1l}^{(1)} \, \Delta(t), \quad l = 1, 2, 3, \\
\Delta_{4l}^{(2)} &= \left[ \frac{1}{4} q_{s} p_{t, x} + \frac{i}{2} \sum_{s=1}^{3} p_{s} \int_{(x, t, j)}^{(x, t)} \Delta_{nl}^{(1)} \right] \, \Delta(t) \\
&+ \left\{ \frac{1}{4} \left[ q_{s} p_{t, x} + i q_{x} q_{t, x} - i q_{s} p_{t, x} \sum_{j=1}^{3} q_{j} p_{j} \right] + \frac{1}{2} \sum_{n=1}^{3} \left( q_{s} p_{n, x} - q_{s} x p_{n} \right) \int_{(x, t, j)}^{(x, t)} \Delta_{nl}^{(1)} \right\} \, \Delta(t), \quad l = 1, 2, 3, \\
\Delta_{44}^{(2)} &= \left[ \frac{1}{4} \sum_{l=1}^{3} p_{l} q_{t, x} - \frac{i}{2} \sum_{l=1}^{3} p_{l} q_{l} \int_{(x, t, j)}^{(x, t)} \Delta_{44}^{(1)} \right] \, \Delta(t) \\
&+ \left\{ \frac{1}{4} \left[ \sum_{l=1}^{3} (p_{l} q_{t, x} - i p_{t, x} q_{l}) + i \left( \sum_{l=1}^{3} p_{l} q_{l} \right)^{2} \right] + \frac{1}{2} \sum_{l=1}^{3} (p_{l} q_{t, x} - p_{t, x} q_{l}) \int_{(x, t, j)}^{(x, t)} \Delta_{44}^{(1)} \right\} \, \Delta(t),
\end{align*}
$$

The functions \( \{ \mu_{j}^{(i)}(x, t) \}_{i=1, 2} \), \( i = 1, 2 \) are independent of \( k \).

We define the function \( \{ \Psi_{i, j}(t, k) \}_{i, j=1}^{4 \times 4} \) as

$$
\mu_{2}(0, t, k) = (\Psi_{2}(t, k))_{4 \times 4} = \mathbb{I} + \sum_{l=1}^{2} \frac{1}{k^{l}} \left( \begin{array}{cccc}
\Psi_{11}^{(l)}(t) & \Psi_{12}^{(l)}(t) & \Psi_{13}^{(l)}(t) & \Psi_{14}^{(l)}(t) \\
\Psi_{21}^{(l)}(t) & \Psi_{22}^{(l)}(t) & \Psi_{23}^{(l)}(t) & \Psi_{24}^{(l)}(t) \\
\Psi_{31}^{(l)}(t) & \Psi_{32}^{(l)}(t) & \Psi_{33}^{(l)}(t) & \Psi_{34}^{(l)}(t) \\
\Psi_{41}^{(l)}(t) & \Psi_{42}^{(l)}(t) & \Psi_{43}^{(l)}(t) & \Psi_{44}^{(l)}(t)
\end{array} \right) + O\left( \frac{1}{k^{3}} \right), \quad (58)
$$

Based on the asymptotic of Eq. (57) and the boundary data at \( x = 0 \), we find

$$
\begin{align*}
\Psi_{14}^{(1)}(t) &= -\frac{i}{2} u_{01}(t), \quad \Psi_{24}^{(1)}(t) = -\frac{i}{2} u_{02}(t), \quad \Psi_{34}^{(1)}(t) = -\frac{i}{2} u_{03}(t), \\
\Psi_{41}^{(1)}(t) &= \frac{i}{2} \left[ \alpha_{11} \bar{u}_{01}(t) + \alpha_{12} \bar{u}_{02}(t) + \alpha_{13} \bar{u}_{03}(t) \right], \\
\Psi_{42}^{(1)}(t) &= \frac{i}{2} \left[ \alpha_{12} \bar{u}_{01}(t) + \alpha_{22} \bar{u}_{02}(t) + \alpha_{23} \bar{u}_{03}(t) \right], \\
\Psi_{43}^{(1)}(t) &= \frac{i}{2} \left[ \alpha_{13} \bar{u}_{01}(t) + \alpha_{23} \bar{u}_{02}(t) + \alpha_{33} \bar{u}_{03}(t) \right], \\
\Psi_{14}^{(2)}(t) &= \frac{1}{4} u_{11} + \frac{1}{2i} u_{01} \Psi_{14}^{(1)}, \quad \Psi_{24}^{(2)}(t) = \frac{1}{4} u_{12} + \frac{1}{2i} u_{02} \Psi_{14}^{(1)}, \quad \Psi_{34}^{(2)}(t) = \frac{1}{4} u_{13} + \frac{1}{2i} u_{03} \Psi_{14}^{(1)}, \\
\Psi_{44}^{(2)}(t) &= \frac{1}{4} \int_{0}^{t} \left\{ u_{11} \left[ \alpha_{11} \bar{u}_{01}(t) + \alpha_{12} \bar{u}_{02}(t) + \alpha_{13} \bar{u}_{03}(t) \right] + u_{12} \left[ \alpha_{12} \bar{u}_{01}(t) + \alpha_{22} \bar{u}_{02}(t) + \alpha_{23} \bar{u}_{03}(t) \right] \right.
\end{align*}
$$

$$
+ \left. u_{13} \left[ \alpha_{13} \bar{u}_{01}(t) + \alpha_{23} \bar{u}_{02}(t) + \alpha_{33} \bar{u}_{03}(t) \right] - u_{01} \left[ \alpha_{11} \bar{u}_{11}(t) + \alpha_{12} \bar{u}_{12}(t) + \alpha_{13} \bar{u}_{13}(t) \right] \\
- u_{02} \left[ \alpha_{12} \bar{u}_{11}(t) + \alpha_{22} \bar{u}_{12}(t) + \alpha_{23} \bar{u}_{13}(t) \right] - u_{03} \left[ \alpha_{13} \bar{u}_{11}(t) + \alpha_{23} \bar{u}_{12}(t) + \alpha_{33} \bar{u}_{13}(t) \right] \right\} dt,
$$

18
Thus we have the boundary data at $x = 0$:

$$
\begin{align*}
    u_{01}(t) &= 2i\Psi^{(1)}_{14}(t), \quad u_{02}(t) = 2i\Psi^{(1)}_{24}(t), \quad u_{03}(t) = 2i\Psi^{(1)}_{34}(t), \\
    u_{11}(t) &= 4\Psi^{(2)}_{14}(t) + 2iu_{01}(t)\Psi^{(1)}_{44}(t), \\
    u_{12}(t) &= 4\Psi^{(2)}_{24}(t) + 2iu_{02}(t)\Psi^{(1)}_{44}(t), \\
    u_{13}(t) &= 4\Psi^{(2)}_{34}(t) + 2iu_{03}(t)\Psi^{(1)}_{44}(t),
\end{align*}
$$

Similarly, we assume that the asymptotic formula of $\mu_3(L, t, k) = \{\phi_{ij}(t, k)\}_{i, j = 1}^4$ is of the from

$$
\mu_3(L, t, k) = (\phi_{ij}(t, k))_{4 \times 4} = I + \sum_{i=1}^{2} \frac{1}{k^i} \left( \begin{array}{cccc}
\phi^{(i)}_{11}(t) & \phi^{(i)}_{12}(t) & \phi^{(i)}_{13}(t) & \phi^{(i)}_{14}(t) \\
\phi^{(i)}_{21}(t) & \phi^{(i)}_{22}(t) & \phi^{(i)}_{23}(t) & \phi^{(i)}_{24}(t) \\
\phi^{(i)}_{31}(t) & \phi^{(i)}_{32}(t) & \phi^{(i)}_{33}(t) & \phi^{(i)}_{34}(t) \\
\phi^{(i)}_{41}(t) & \phi^{(i)}_{42}(t) & \phi^{(i)}_{43}(t) & \phi^{(i)}_{44}(t)
\end{array} \right) + O\left( \frac{1}{k^3} \right),
$$

By using the asymptotic of Eq. (57) and the boundary data at $x = L$, we find

$$
\begin{align*}
    \phi^{(1)}_{14}(t) &= -\frac{i}{2}v_{01}(t), \quad \phi^{(1)}_{24}(t) = -\frac{i}{2}v_{02}(t), \quad \phi^{(1)}_{34}(t) = -\frac{i}{2}v_{03}(t), \\
    \phi^{(2)}_{11}(t) &= \frac{1}{4}v_{11} + \frac{1}{2i}v_{01}\phi^{(1)}_{44}, \quad \phi^{(2)}_{24} = \frac{1}{4}v_{12} + \frac{1}{2i}v_{02}\phi^{(1)}_{44}, \quad \phi^{(2)}_{34} = \frac{1}{4}v_{12} + \frac{1}{2i}v_{03}\phi^{(1)}_{44}, \\
    \phi^{(4)}_{44} &= \frac{1}{2} \int_0^t \left\{ v_{11} [\alpha_{11}\bar{v}_{01}(t) + \alpha_{12}\bar{v}_{02}(t) + \alpha_{13}\bar{v}_{03}(t)] + v_{12} [\alpha_{12}\bar{v}_{02}(t) + \alpha_{22}\bar{v}_{02}(t) + \alpha_{23}\bar{v}_{03}(t)] \\
    &\quad + v_{13} [\alpha_{13}\bar{v}_{03}(t) + \alpha_{23}\bar{v}_{02}(t) + \alpha_{33}\bar{v}_{03}(t)] \right\} dt,
\end{align*}
$$

which generates the following expressions for the boundary values at $x = L$

$$
\begin{align*}
    v_{01}(t) &= 2i\phi^{(1)}_{14}(t), \quad v_{02}(t) = 2i\phi^{(1)}_{24}(t), \quad v_{03}(t) = 2i\phi^{(1)}_{34}(t), \\
    v_{11}(t) &= 4\phi^{(2)}_{14}(t) + 2iv_{01}(t)\phi^{(1)}_{44}(t), \\
    v_{12}(t) &= 4\phi^{(2)}_{24}(t) + 2iv_{02}(t)\phi^{(1)}_{44}(t), \\
    v_{13}(t) &= 4\phi^{(2)}_{34}(t) + 2iv_{03}(t)\phi^{(1)}_{44}(t).
\end{align*}
$$

For the vanishing initial values, it follows from Eq. (56) that we have the following asymptotic of the global relation $c_{j4}(t, k)$ and $c_{4j}(t, k), j = 1, 2, 3$. 

\[19\]
Proposition 4.1. Let the initial and Dirichlet boundary conditions be compatible at points \( x = 0, L \) (i.e., \( q_0(0) = u_0(0) \) at \( x = 0 \) and \( q_0(L) = v_0(L) \) at \( x = L, \ j = 1, 2, 3 \)). Then, the global relation (56) with the vanishing initial data implies that the large \( k \) behaviors of \( c_{j4}(t, k) \) and \( c_{4j}(t, k), j = 1, 2, 3 \) are of the form

\[
c_{14}(t, k) = \frac{\Psi^{(1)}_{14}}{k} + \frac{\Psi^{(2)}_{14} + \Psi^{(1)}_{44}}{k^2} + O\left(\frac{1}{k^3}\right) \\
- \left\{ \frac{\alpha_{11}\phi^{(1)}_{41} + \alpha_{12}\phi^{(1)}_{42} + \alpha_{13}\phi^{(1)}_{43}}{k} + \frac{1}{k^2} \left[ \alpha_{11}\phi^{(2)}_{41} + \alpha_{12}\phi^{(2)}_{42} + \alpha_{13}\phi^{(2)}_{43} \right] \right\} e^{2ikL}, \ k \to \infty,
\]

\[
c_{24}(t, k) = \frac{\Psi^{(1)}_{24}}{k} + \frac{\Psi^{(2)}_{24} + \Psi^{(1)}_{44}}{k^2} + O\left(\frac{1}{k^3}\right) \\
- \left\{ \frac{\alpha_{12}\phi^{(1)}_{42} + \alpha_{22}\phi^{(1)}_{42} + \alpha_{23}\phi^{(1)}_{43}}{k} + \frac{1}{k^2} \left[ \alpha_{12}\phi^{(2)}_{42} + \alpha_{22}\phi^{(2)}_{42} + \alpha_{23}\phi^{(2)}_{43} \right] \right\} e^{2ikL}, \ k \to \infty,
\]

\[
c_{34}(t, k) = \frac{\Psi^{(1)}_{34}}{k} + \frac{\Psi^{(2)}_{34} + \Psi^{(1)}_{44}}{k^2} + O\left(\frac{1}{k^3}\right) \\
- \left\{ \frac{\alpha_{13}\phi^{(1)}_{41} + \alpha_{23}\phi^{(1)}_{42} + \alpha_{33}\phi^{(1)}_{43}}{k} + \frac{1}{k^2} \left[ \alpha_{13}\phi^{(2)}_{41} + \alpha_{23}\phi^{(2)}_{42} + \alpha_{33}\phi^{(2)}_{43} \right] \right\} e^{2ikL}, \ k \to \infty,
\]

(64a)
\[ c_{41}(t, k) = - \left\{ \frac{\alpha_1 \bar{\phi}^{(1)}_{14} + \alpha_2 \bar{\phi}^{(1)}_{24} + \alpha_3 \bar{\phi}^{(1)}_{34}}{k} + \frac{1}{k^2} \left[ \alpha_{11} \bar{\phi}^{(2)}_{14} + \alpha_{12} \bar{\phi}^{(2)}_{24} + \alpha_{13} \bar{\phi}^{(2)}_{34} \right. \right. \\
+ \Psi_{44}^{(1)} \left( \alpha_{11} \bar{\phi}^{(1)}_{14} + \alpha_{12} \bar{\phi}^{(1)}_{24} + \alpha_{13} \bar{\phi}^{(1)}_{34} \right) \right. + O \left( \frac{1}{k^3} \right) \right\} e^{-2ikL} \\
+ \frac{1}{k} \left\{ (|\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2) \Psi_{41}^{(1)} + (\alpha_{11} \bar{\alpha}_{12} + \alpha_{12} \bar{\alpha}_{13} + \alpha_{13} \bar{\alpha}_{23}) \Psi_{42}^{(1)} \right. \\
+ (\alpha_{11} \bar{\alpha}_{12} + \alpha_{12} \bar{\alpha}_{23} + \alpha_{13} \bar{\alpha}_{13}) \Psi_{43}^{(1)} \right\} + O \left( \frac{1}{k^3} \right), \ k \to \infty, \] 

\[ c_{42}(t, k) = - \left\{ \frac{\bar{\alpha}_1 \bar{\phi}^{(1)}_{14} + \bar{\alpha}_2 \bar{\phi}^{(1)}_{24} + \bar{\alpha}_3 \bar{\phi}^{(1)}_{34}}{k} + \frac{1}{k^2} \left[ \bar{\alpha}_{11} \bar{\phi}^{(2)}_{14} + \bar{\alpha}_{12} \bar{\phi}^{(2)}_{24} + \bar{\alpha}_{13} \bar{\phi}^{(2)}_{34} \right. \right. \\
+ \Psi_{44}^{(1)} \left( \bar{\alpha}_{11} \bar{\phi}^{(1)}_{14} + \bar{\alpha}_{12} \bar{\phi}^{(1)}_{24} + \bar{\alpha}_{13} \bar{\phi}^{(1)}_{34} \right) \right. + O \left( \frac{1}{k^3} \right) \right\} e^{-2ikL} \\
+ \frac{1}{k} \left\{ (|\alpha_{12}|^2 + |\alpha_{22}|^2 + |\alpha_{23}|^2) \Psi_{42}^{(1)} + (\alpha_{11} \bar{\alpha}_{12} + \bar{\alpha}_{12} \bar{\alpha}_{22} + \bar{\alpha}_{13} \bar{\alpha}_{23}) \Psi_{41}^{(1)} \right. \\
+ (\alpha_{11} \bar{\alpha}_{12} + \bar{\alpha}_{12} \bar{\alpha}_{23} + \alpha_{23} \bar{\alpha}_{23}) \Psi_{42}^{(1)} \right\} + O \left( \frac{1}{k^3} \right), \ k \to \infty, \]
The global relation (56) under the vanishing initial data can be simplified as

\[ c_{43}(t, k) = - \left( \frac{\tilde{\alpha}_{13}\phi_{14}^{(1)} + \tilde{\alpha}_{23}\phi_{24}^{(1)} + \alpha_{33}\phi_{34}^{(1)}}{k} + \frac{1}{k^2} \left[ \tilde{\alpha}_{13}\phi_{14}^{(2)} + \tilde{\alpha}_{23}\phi_{24}^{(2)} + \alpha_{33}\phi_{34}^{(2)} \right] \right. \]

\[ + \Psi_{44}^{(1)} \left( \tilde{\alpha}_{13}\phi_{14}^{(1)} + \tilde{\alpha}_{23}\phi_{24}^{(1)} + \alpha_{33}\phi_{34}^{(1)} \right) \left[ \tilde{\alpha}_{13}\phi_{14}^{(2)} + \tilde{\alpha}_{23}\phi_{24}^{(2)} + \alpha_{33}\phi_{34}^{(2)} \right] \right. \]

\[ + \Psi_{44}^{(1)} \left( \tilde{\alpha}_{13}\phi_{14}^{(1)} + \tilde{\alpha}_{23}\phi_{24}^{(1)} + \alpha_{33}\phi_{34}^{(1)} \right) \left[ \tilde{\alpha}_{13}\phi_{14}^{(2)} + \tilde{\alpha}_{23}\phi_{24}^{(2)} + \alpha_{33}\phi_{34}^{(2)} \right] \right. \]

\[ + \frac{1}{k^2} \left( \left( |\alpha_{13}|^2 + |\alpha_{23}|^2 + |\alpha_{33}|^2 \right) \psi_{43}^{(1)} + (\alpha_{11}\tilde{\alpha}_{13} + \tilde{\alpha}_{12}\tilde{\alpha}_{23} + \tilde{\alpha}_{13}\alpha_{33}) \psi_{41}^{(1)} \right) \]

\[ + (\alpha_{11}\tilde{\alpha}_{12} + \tilde{\alpha}_{12}\tilde{\alpha}_{22} + \tilde{\alpha}_{13}\alpha_{23}) \psi_{42}^{(1)} + (\alpha_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33}) \psi_{43}^{(1)} \]

\[ + \Psi_{41}^{(1)} \left[ \alpha_{13} \left( \alpha_{11}\phi_{11}^{(1)} + \alpha_{12}\phi_{12}^{(1)} + \alpha_{13}\phi_{13}^{(1)} \right) + \alpha_{23} \left( \alpha_{11}\phi_{21}^{(1)} + \alpha_{12}\phi_{22}^{(1)} + \alpha_{13}\phi_{23}^{(1)} \right) \right] \]

\[ + \alpha_{33} \left( \alpha_{11}\phi_{31}^{(1)} + \alpha_{12}\phi_{32}^{(1)} + \alpha_{13}\phi_{33}^{(1)} \right) \right. \]

\[ \left. + \Psi_{41}^{(1)} \left[ \alpha_{13} \left( \alpha_{12}\phi_{11}^{(1)} + \alpha_{23}\phi_{11}^{(1)} + \alpha_{23}\phi_{13}^{(1)} \right) \right. \right. \]

\[ \left. \left. + \alpha_{23} \left( \alpha_{12}\phi_{21}^{(1)} + \alpha_{22}\phi_{22}^{(1)} + \alpha_{23}\phi_{23}^{(1)} \right) \right. \right. \]

\[ \left. \left. + \alpha_{33} \left( \alpha_{12}\phi_{31}^{(1)} + \alpha_{22}\phi_{32}^{(1)} + \alpha_{33}\phi_{33}^{(1)} \right) \right] \right. \]

\[ \left. + \psi_{41}^{(1)} \left[ \alpha_{13} \left( \alpha_{13}\phi_{11}^{(1)} + \alpha_{23}\phi_{12}^{(1)} + \alpha_{33}\phi_{13}^{(1)} \right) \right. \right. \]

\[ \left. \left. + \alpha_{23} \left( \alpha_{13}\phi_{21}^{(1)} + \alpha_{23}\phi_{22}^{(1)} + \alpha_{33}\phi_{23}^{(1)} \right) \right. \right. \]

\[ \left. \left. + \alpha_{33} \left( \alpha_{13}\phi_{31}^{(1)} + \alpha_{23}\phi_{32}^{(1)} + \alpha_{33}\phi_{33}^{(1)} \right) \right] \right. \]

\[ + O \left( \frac{1}{k^3} \right), \quad k \to \infty, \quad (67) \]

**Proof.** The global relation (56) under the vanishing initial data can be simplified as

\[ c_{14}(t, k) = \Psi_{14}(t, k)\bar{\phi}_{44}(t, \bar{k}) - e^{2ikL} \left[ \psi_{11}(t, k)(\alpha_{11}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{12}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{13}\tilde{\phi}_{43}(t, \bar{k})) \right. \]

\[ + \psi_{12}(t, k)(\alpha_{12}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{22}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{23}\tilde{\phi}_{43}(t, \bar{k})) \left. \right] \]

\[ + \psi_{13}(t, k)(\alpha_{13}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{32}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{33}\tilde{\phi}_{43}(t, \bar{k})) \right], \quad (68a) \]

\[ c_{24}(t, k) = \Psi_{24}(t, k)\bar{\phi}_{44}(t, \bar{k}) - e^{2ikL} \left[ \psi_{21}(t, k)(\alpha_{11}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{12}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{13}\tilde{\phi}_{43}(t, \bar{k})) \right. \]

\[ + \psi_{22}(t, k)(\alpha_{12}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{22}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{23}\tilde{\phi}_{43}(t, \bar{k})) \left. \right] \]

\[ + \psi_{23}(t, k)(\alpha_{13}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{32}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{33}\tilde{\phi}_{43}(t, \bar{k})) \right], \quad (68b) \]

\[ c_{34}(t, k) = \Psi_{34}(t, k)\bar{\phi}_{44}(t, \bar{k}) - e^{2ikL} \left[ \psi_{31}(t, k)(\alpha_{11}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{12}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{13}\tilde{\phi}_{43}(t, \bar{k})) \right. \]

\[ + \psi_{32}(t, k)(\alpha_{12}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{22}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{23}\tilde{\phi}_{43}(t, \bar{k})) \left. \right] \]

\[ + \psi_{33}(t, k)(\alpha_{13}\tilde{\phi}_{41}(t, \bar{k}) + \tilde{\alpha}_{32}\tilde{\phi}_{42}(t, \bar{k}) + \tilde{\alpha}_{33}\tilde{\phi}_{43}(t, \bar{k})) \right], \quad (68c) \]

where \( \tilde{\phi}_{ij}(t, \bar{k}) = \overline{\phi}_{ij}(t, k). \)

Recalling the time-part of the Lax pair (8)

\[ \mu_t + 2ik^2 [\sigma_4, \mu] = V(x, t, \mu), \quad (69) \]
It follows from the first column of Eq. (69) with $\mu = \mu_2$ that we have

\[
\begin{align*}
\Psi_{11,4}(t, k) &= (2k\alpha_1 + i\alpha_1)\Psi_{41} - i\Psi_{11}(\alpha_{11}|u_{01}|^2 + \alpha_{12}|u_{01}|^2 + \alpha_{13}|u_{01}|^2)
- i\Psi_{21}(\alpha_{12}|u_{01}|^2 + \alpha_{22}|u_{01}|^2 + \alpha_{23}|u_{01}|^2)
- i\Psi_{31}(\alpha_{13}|u_{01}|^2 + \alpha_{32}|u_{01}|^2 + \alpha_{33}|u_{01}|^2), \\
\Psi_{21,4}(t, k) &= (2k\alpha_2 + i\alpha_2)\Psi_{41} - i\Psi_{11}(\alpha_{11}|u_{02}|^2 + \alpha_{12}|u_{02}|^2 + \alpha_{13}|u_{02}|^2)
- i\Psi_{21}(\alpha_{12}|u_{02}|^2 + \alpha_{22}|u_{02}|^2 + \alpha_{23}|u_{02}|^2)
- i\Psi_{31}(\alpha_{13}|u_{02}|^2 + \alpha_{32}|u_{02}|^2 + \alpha_{33}|u_{02}|^2), \\
\Psi_{31,4}(t, k) &= (2k\alpha_3 + i\alpha_3)\Psi_{41} - i\Psi_{11}(\alpha_{11}|u_{03}|^2 + \alpha_{12}|u_{03}|^2 + \alpha_{13}|u_{03}|^2)
- i\Psi_{21}(\alpha_{12}|u_{03}|^2 + \alpha_{22}|u_{03}|^2 + \alpha_{23}|u_{03}|^2)
- i\Psi_{31}(\alpha_{13}|u_{03}|^2 + \alpha_{32}|u_{03}|^2 + \alpha_{33}|u_{03}|^2), \\
\Psi_{41,4}(t, k) &= \Psi_{11}(\alpha_{11}|2k\alpha_{11} - i\alpha_{11}|^2 + \alpha_{12}|2k\alpha_{11} - i\alpha_{11}|^2 + \alpha_{13}|2k\alpha_{11} - i\alpha_{11}|^2)
+ \Psi_{21}(\alpha_{12}|2k\alpha_{12} - i\alpha_{12}|^2 + \alpha_{22}|2k\alpha_{12} - i\alpha_{12}|^2 + \alpha_{23}|2k\alpha_{12} - i\alpha_{12}|^2)
+ \Psi_{31}(\alpha_{13}|2k\alpha_{13} - i\alpha_{13}|^2 + \alpha_{32}|2k\alpha_{13} - i\alpha_{13}|^2 + \alpha_{33}|2k\alpha_{13} - i\alpha_{13}|^2)
+ \Psi_{41}(4k^2 + \alpha_{11}|u_{01}|^2 + \alpha_{12}|u_{02}|^2 + \alpha_{13}|u_{03}|^2) + \alpha_{22}|u_{02}|^2 + \alpha_{23}|u_{03}|^2 + \alpha_{33}|u_{03}|^2),
\end{align*}
\]

The second column of Eq. (69) with $\mu = \mu_2$ yields

\[
\begin{align*}
\Psi_{11,2}(t, k) &= (2k\alpha_1 + i\alpha_1)\Psi_{42} - i\Psi_{11}(\alpha_{11}|u_{01}|^2 + \alpha_{12}|u_{01}|^2 + \alpha_{13}|u_{01}|^2)
- i\Psi_{22}(\alpha_{12}|u_{01}|^2 + \alpha_{22}|u_{02}|^2 + \alpha_{23}|u_{02}|^2)
- i\Psi_{32}(\alpha_{13}|u_{01}|^2 + \alpha_{32}|u_{03}|^2 + \alpha_{33}|u_{03}|^2), \\
\Psi_{22,2}(t, k) &= (2k\alpha_2 + i\alpha_2)\Psi_{42} - i\Psi_{11}(\alpha_{11}|u_{02}|^2 + \alpha_{12}|u_{02}|^2 + \alpha_{13}|u_{02}|^2)
- i\Psi_{22}(\alpha_{12}|u_{02}|^2 + \alpha_{22}|u_{02}|^2 + \alpha_{23}|u_{02}|^2)
- i\Psi_{32}(\alpha_{13}|u_{02}|^2 + \alpha_{32}|u_{02}|^2 + \alpha_{33}|u_{02}|^2), \\
\Psi_{32,2}(t, k) &= (2k\alpha_3 + i\alpha_3)\Psi_{42} - i\Psi_{11}(\alpha_{11}|u_{03}|^2 + \alpha_{12}|u_{03}|^2 + \alpha_{13}|u_{03}|^2)
- i\Psi_{22}(\alpha_{12}|u_{03}|^2 + \alpha_{22}|u_{03}|^2 + \alpha_{23}|u_{03}|^2)
- i\Psi_{32}(\alpha_{13}|u_{03}|^2 + \alpha_{32}|u_{03}|^2 + \alpha_{33}|u_{03}|^2), \\
\Psi_{42,2}(t, k) &= \Psi_{11}(\alpha_{11}|2k\alpha_{11} - i\alpha_{11}|^2 + \alpha_{12}|2k\alpha_{11} - i\alpha_{11}|^2 + \alpha_{13}|2k\alpha_{11} - i\alpha_{11}|^2)
+ \Psi_{22}(\alpha_{12}|2k\alpha_{12} - i\alpha_{12}|^2 + \alpha_{22}|2k\alpha_{12} - i\alpha_{12}|^2 + \alpha_{23}|2k\alpha_{12} - i\alpha_{12}|^2)
+ \Psi_{32}(\alpha_{13}|2k\alpha_{13} - i\alpha_{13}|^2 + \alpha_{32}|2k\alpha_{13} - i\alpha_{13}|^2 + \alpha_{33}|2k\alpha_{13} - i\alpha_{13}|^2)
+ \Psi_{42}(4k^2 + \alpha_{11}|u_{01}|^2 + \alpha_{12}|u_{02}|^2 + \alpha_{13}|u_{03}|^2) + \alpha_{22}|u_{02}|^2 + \alpha_{23}|u_{03}|^2 + \alpha_{33}|u_{03}|^2),
\end{align*}
\]
The third column of Eq. (69) with $\mu = \mu_2$ yields
\[
\begin{align*}
\Psi_{13,\epsilon}(t, k) &= (2k)u_{01} + i\nu_{11})\Psi_{43} - i\Psi_{13}(\alpha_{11}|u_{01}|^2 + \alpha_{12}u_{01}u_{02} + \alpha_{13}u_{01}u_{03}) \\
&\quad - \frac{i}{2}\Psi_{23}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}u_{02} + \alpha_{23}u_{02}u_{03}) \\
&\quad - i\Psi_{33}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}u_{02} + \alpha_{33}u_{01}u_{03}), \\
\Psi_{23,\epsilon}(t, k) &= (2k)u_{02} + i\nu_{12})\Psi_{43} - i\Psi_{13}(\alpha_{11}u_{02}|u_{01}|^2 + \alpha_{12}u_{02}u_{01} + \alpha_{13}u_{02}u_{03}) \\
&\quad - \frac{i}{2}\Psi_{23}(\alpha_{12}u_{02}|u_{01}|^2 + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}u_{03}) \\
&\quad - i\Psi_{33}(\alpha_{13}u_{02}u_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}u_{03}), \\
\Psi_{33,\epsilon}(t, k) &= (2k)u_{03} + i\nu_{13})\Psi_{43} - i\Psi_{13}(\alpha_{11}u_{03}|u_{01}|^2 + \alpha_{12}u_{03}u_{01} + \alpha_{13}u_{03}u_{03}) \\
&\quad - \frac{i}{2}\Psi_{23}(\alpha_{12}u_{03}|u_{01}|^2 + \alpha_{22}u_{03}u_{02} + \alpha_{23}|u_{03}|^2) \\
&\quad - i\Psi_{33}(\alpha_{13}u_{03}u_{01} + \alpha_{23}u_{03}u_{02} + \alpha_{33}|u_{03}|^2),
\end{align*}
\]

The fourth column of Eq. (69) with $\mu = \mu_2$ yields
\[
\begin{align*}
\Psi_{14,\epsilon}(t, k) &= (2k)u_{01} + i\nu_{11})\Psi_{44} - i\Psi_{14}(\alpha_{11}|u_{01}|^2 + \alpha_{12}u_{01}u_{02} + \alpha_{13}u_{01}u_{03}) \\
&\quad - \frac{i}{2}\Psi_{24}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}u_{02} + \alpha_{23}u_{01}u_{03}) \\
&\quad - i\Psi_{34}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}u_{02} + \alpha_{33}u_{01}u_{03}), \\
\Psi_{24,\epsilon}(t, k) &= (2k)u_{02} + i\nu_{12})\Psi_{44} - i\Psi_{14}(\alpha_{11}u_{02}|u_{01}|^2 + \alpha_{12}u_{02}u_{01} + \alpha_{13}u_{02}u_{03}) \\
&\quad - \frac{i}{2}\Psi_{24}(\alpha_{12}u_{02}|u_{01}|^2 + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}u_{03}) \\
&\quad - i\Psi_{34}(\alpha_{13}u_{02}u_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}u_{03}), \\
\Psi_{34,\epsilon}(t, k) &= (2k)u_{03} + i\nu_{13})\Psi_{44} - i\Psi_{14}(\alpha_{11}u_{03}|u_{01}|^2 + \alpha_{12}u_{03}u_{01} + \alpha_{13}|u_{03}|^2) \\
&\quad - \frac{i}{2}\Psi_{24}(\alpha_{12}u_{03}|u_{01}|^2 + \alpha_{22}u_{03}u_{02} + \alpha_{23}|u_{03}|^2) \\
&\quad - i\Psi_{34}(\alpha_{13}u_{03}u_{01} + \alpha_{23}u_{03}u_{02} + \alpha_{33}|u_{03}|^2), \\
\Psi_{44,\epsilon}(t, k) &= \Psi_{14}(\alpha_{11}|u_{01}|^2 + \alpha_{12}u_{01}u_{02} + \alpha_{13}u_{01}u_{03}) \\
&\quad + \frac{i}{2}\Psi_{24}(\alpha_{12}u_{01}u_{02} + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}u_{03}) \\
&\quad + i\Psi_{34}(\alpha_{13}u_{01}u_{03} + \alpha_{23}u_{03}u_{02} + \alpha_{33}|u_{03}|^2),
\end{align*}
\]

Suppose that $\Psi_{j\ell}$'s, $j = 1, 2, 3, 4$ are of the form
\[
\begin{pmatrix}
\Psi_{11} \\ \Psi_{21} \\ \Psi_{31} \\ \Psi_{41}
\end{pmatrix} = \left( a_{10}(t) + \frac{a_{11}(t)}{k} + \frac{a_{12}(t)}{k^2} + \cdots \right) + \left( b_{10}(t) + \frac{b_{11}(t)}{k} + \frac{b_{12}(t)}{k^2} + \cdots \right) e^{i4k^2t},
\]
where the $4 \times 1$ column vector functions $a_{1j}(t), b_{1j}(t) \ (j = 0, 1, \ldots)$ are independent of $k$.

By substituting Eq. (74) into Eq. (70) and using the initial conditions

$$a_{10}(0) + b_{10}(0) = (1, 0, 0, 0)^T, \quad a_{11}(0) + b_{11}(0) = (0, 0, 0, 0)^T,$$

we have

$$
\begin{pmatrix}
\Psi_{11} \\
\Psi_{21} \\
\Psi_{31} \\
\Psi_{41}
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{s=1}^{2} \frac{1}{ks} \begin{pmatrix}
\psi_{11}^{(s)} \\
\psi_{21}^{(s)} \\
\psi_{31}^{(s)} \\
\psi_{41}^{(s)}
\end{pmatrix} + O \left( \frac{1}{k^3} \right) + \sum_{s=1}^{2} \frac{1}{2ik} \begin{pmatrix}
0 \\
0 \\
\alpha_{11}\bar{u}_{01}(0) + \alpha_{12}\bar{u}_{02}(0) + \alpha_{13}\bar{u}_{03}(0)
\end{pmatrix}
$$

$$e^{\imath k^2 t}, \quad (75)$$

Similarly, it follows from Eqs. (71)-(73) that we have the asymptotic formulae for $\Psi_{ij}, \ i = 1, 2, 3, 4; \ j = 2, 3, 4$ in the form

$$
\begin{pmatrix}
\Psi_{12} \\
\Psi_{22} \\
\Psi_{32} \\
\Psi_{42}
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{s=1}^{2} \frac{1}{ks} \begin{pmatrix}
\psi_{12}^{(s)} \\
\psi_{22}^{(s)} \\
\psi_{32}^{(s)} \\
\psi_{42}^{(s)}
\end{pmatrix} + O \left( \frac{1}{k^3} \right) + \sum_{s=1}^{2} \frac{1}{2ik} \begin{pmatrix}
0 \\
0 \\
\alpha_{12}\bar{u}_{01}(0) + \alpha_{22}\bar{u}_{02}(0) + \alpha_{23}\bar{u}_{03}(0)
\end{pmatrix}
$$

$$e^{\imath k^2 t}, \quad (76)$$

$$
\begin{pmatrix}
\Psi_{13} \\
\Psi_{23} \\
\Psi_{33} \\
\Psi_{43}
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{s=1}^{2} \frac{1}{ks} \begin{pmatrix}
\psi_{13}^{(s)} \\
\psi_{23}^{(s)} \\
\psi_{33}^{(s)} \\
\psi_{43}^{(s)}
\end{pmatrix} + O \left( \frac{1}{k^3} \right) + \sum_{s=1}^{2} \frac{1}{2ik} \begin{pmatrix}
0 \\
0 \\
\alpha_{13}\bar{u}_{01}(0) + \alpha_{23}\bar{u}_{02}(0) + \alpha_{33}\bar{u}_{03}(0)
\end{pmatrix}
$$

$$e^{\imath k^2 t}, \quad (77)$$

and

$$
\begin{pmatrix}
\Psi_{14} \\
\Psi_{24} \\
\Psi_{34} \\
\Psi_{44}
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{s=1}^{2} \frac{1}{ks} \begin{pmatrix}
\psi_{14}^{(s)} \\
\psi_{24}^{(s)} \\
\psi_{34}^{(s)} \\
\psi_{44}^{(s)}
\end{pmatrix} + O \left( \frac{1}{k^3} \right) + \sum_{s=1}^{2} \frac{i}{2k} \begin{pmatrix}
\bar{u}_{01}(0) \\
\bar{u}_{02}(0) \\
\bar{u}_{03}(0) \\
0
\end{pmatrix}
$$

$$e^{-\imath k^2 t}, \quad (78)$$
Similar to Eqs. (70)-(73) for $\mu_2(0,t,k)$, we also know that the function $\mu(x,t,k) = \mu_3(L,t,k)$ at $x = L$ satisfy the $t$-part of Lax pair (69).

The first column of Eq. (69) with $\mu = \mu_3$ yields

$$
\begin{align*}
\phi_{11,t}(t,k) &= (2k\nu_{01} + iv_{11})\phi_{41} - i\phi_{11}(\alpha_{11}v_{01})^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} - i\phi_{21}(\alpha_{12}v_{01})^2 + \bar{\alpha}_{22}v_{01}\bar{v}_{02} + \bar{\alpha}_{23}v_{01}\bar{v}_{03}, \\
\phi_{21,t}(t,k) &= (2k\nu_{02} + iv_{12})\phi_{41} - i\phi_{11}(\alpha_{11}v_{02})^2 + \bar{\alpha}_{12}v_{02}\bar{v}_{01} + \bar{\alpha}_{13}v_{02}\bar{v}_{03} - i\phi_{21}(\alpha_{12}v_{02})^2 + \bar{\alpha}_{22}v_{02}\bar{v}_{01} + \bar{\alpha}_{23}v_{02}\bar{v}_{03}, \\
\phi_{31,t}(t,k) &= (2k\nu_{03} + iv_{13})\phi_{41} - i\phi_{11}(\alpha_{11}v_{03})^2 + \bar{\alpha}_{12}v_{03}\bar{v}_{01} + \bar{\alpha}_{13}v_{03}\bar{v}_{02} - i\phi_{21}(\alpha_{12}v_{03})^2 + \bar{\alpha}_{22}v_{03}\bar{v}_{01} + \bar{\alpha}_{23}v_{03}\bar{v}_{02}, \\
\phi_{41,t}(t,k) &= \phi_{11}(2k\nu_{01} - iv_{11}) + \bar{\alpha}_{12}(2k\nu_{01} - iv_{12}) + \bar{\alpha}_{13}(2k\nu_{01} - iv_{13}) + \phi_{21}(2k\nu_{02} - iv_{12}) + \bar{\alpha}_{22}(2k\nu_{02} - iv_{12}) + \bar{\alpha}_{23}(2k\nu_{02} - iv_{12}) + \phi_{31}(2k\nu_{03} - iv_{12}) + \alpha_{23}(2k\nu_{03} - iv_{12}) + \phi_{41}(4k^2 + \phi_{01}(\alpha_{11}v_{01})^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} + \phi_{21}(\alpha_{12}v_{02})^2 + \bar{\alpha}_{22}v_{02}\bar{v}_{01} + \bar{\alpha}_{23}v_{02}\bar{v}_{03} + \phi_{31}(\alpha_{12})v_{03})^2 + \bar{\alpha}_{22}v_{03}\bar{v}_{01} + \bar{\alpha}_{23}v_{03}\bar{v}_{02} + \phi_{41}v_{03}^2 + \alpha_{22}v_{02}v_{03} + \alpha_{23}v_{03}v_{02} + \alpha_{33}v_{03}^2, \\
\end{align*}
$$

(79)

The second column of Eq. (69) with $\mu = \mu_3$ yields

$$
\begin{align*}
\phi_{12,t}(t,k) &= (2k\nu_{01} + iv_{11})\phi_{42} - i\phi_{12}(\alpha_{11}v_{01})^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} - i\phi_{22}(\alpha_{12}v_{01})^2 + \bar{\alpha}_{22}v_{01}\bar{v}_{02} + \bar{\alpha}_{23}v_{01}\bar{v}_{03}, \\
\phi_{22,t}(t,k) &= (2k\nu_{02} + iv_{12})\phi_{42} - i\phi_{12}(\alpha_{11}v_{02})^2 + \bar{\alpha}_{12}v_{02}\bar{v}_{01} + \bar{\alpha}_{13}v_{02}\bar{v}_{03} - i\phi_{22}(\alpha_{12}v_{02})^2 + \bar{\alpha}_{22}v_{02}\bar{v}_{01} + \bar{\alpha}_{23}v_{02}\bar{v}_{03}, \\
\phi_{32,t}(t,k) &= (2k\nu_{03} + iv_{13})\phi_{42} - i\phi_{12}(\alpha_{11}v_{03})^2 + \bar{\alpha}_{12}v_{03}\bar{v}_{01} + \bar{\alpha}_{13}v_{03}\bar{v}_{02} - i\phi_{22}(\alpha_{12}v_{03})^2 + \bar{\alpha}_{22}v_{03}\bar{v}_{01} + \bar{\alpha}_{23}v_{03}\bar{v}_{02}, \\
\phi_{42,t}(t,k) &= \phi_{12}(2k\nu_{01} - iv_{11}) + \bar{\alpha}_{12}(2k\nu_{01} - iv_{12}) + \bar{\alpha}_{13}(2k\nu_{01} - iv_{13}) + \phi_{22}(2k\nu_{02} - iv_{12}) + \bar{\alpha}_{22}(2k\nu_{02} - iv_{12}) + \bar{\alpha}_{23}(2k\nu_{02} - iv_{12}) + \phi_{32}(2k\nu_{03} - iv_{12}) + \alpha_{23}(2k\nu_{03} - iv_{12}) + \phi_{42}(4k^2 + \phi_{01}(\alpha_{11}v_{01})^2 + \bar{\alpha}_{12}v_{01}\bar{v}_{02} + \bar{\alpha}_{13}v_{01}\bar{v}_{03} + \phi_{22}(\alpha_{12}v_{02})^2 + \bar{\alpha}_{22}v_{02}\bar{v}_{01} + \bar{\alpha}_{23}v_{02}\bar{v}_{03} + \phi_{32}(\alpha_{12}v_{03})^2 + \bar{\alpha}_{22}v_{03}\bar{v}_{01} + \bar{\alpha}_{23}v_{03}\bar{v}_{02} + \phi_{42}v_{03}^2 + \alpha_{22}v_{02}v_{03} + \alpha_{23}v_{03}v_{02} + \alpha_{33}v_{03}^2, \\
\end{align*}
$$

(80)

26
The third column of Eq. (69) with $\mu = \mu_3$ yields

$$
\begin{align*}
\phi_{13, \ell}(t, k) &= (2k v_{01} + i \nu_{11}) \phi_{43} - i \phi_{13}(\alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12} v_{01} \bar{v}_{02} + \bar{\alpha}_{13} v_{01} \bar{v}_{03}) \\
&\quad - i \phi_{23}(\alpha_{12}|v_{01}|^2 + \alpha_{22} v_{01} \bar{v}_{02} + \alpha_{23} v_{01} \bar{v}_{03}) \\
&\quad - i \phi_{33}(\alpha_{13}|v_{01}|^2 + \alpha_{23} v_{01} \bar{v}_{02} + \alpha_{33} v_{01} \bar{v}_{03}), \\
\phi_{23, \ell}(t, k) &= (2k v_{02} + i \nu_{12}) \phi_{43} - i \phi_{23}(\alpha_{11}|v_{02}|^2 + \bar{\alpha}_{12} v_{02} \bar{v}_{01} + \bar{\alpha}_{13} v_{02} \bar{v}_{03}) \\
&\quad - i \phi_{23}(\alpha_{12}|v_{02}|^2 + \alpha_{22} v_{02} \bar{v}_{01} + \alpha_{23} v_{02} \bar{v}_{03}) \\
&\quad - i \phi_{33}(\alpha_{13}|v_{02}|^2 + \alpha_{23} v_{02} \bar{v}_{01} + \alpha_{33} v_{02} \bar{v}_{03}), \\
\phi_{33, \ell}(t, k) &= (2k v_{03} + i \nu_{13}) \phi_{43} - i \phi_{33}(\alpha_{11}|v_{03}|^2 + \bar{\alpha}_{12} v_{03} \bar{v}_{01} + \bar{\alpha}_{13} v_{03} \bar{v}_{02}) \\
&\quad - i \phi_{33}(\alpha_{12}|v_{03}|^2 + \alpha_{22} v_{03} \bar{v}_{01} + \alpha_{23} v_{03} \bar{v}_{02} + \alpha_{33} |v_{03}|^2), \\
\phi_{43, \ell}(t, k) &= \phi_{13}[\alpha_{11}(2k v_{01} - i \nu_{11}) + \bar{\alpha}_{12}(2k v_{02} - i \nu_{12}) + \bar{\alpha}_{13}(2k v_{03} - i \nu_{13})] \\
&\quad + \phi_{23}[\alpha_{12}(2k v_{01} - i \nu_{11}) + \alpha_{22}(2k v_{02} - i \nu_{12}) + \bar{\alpha}_{23}(2k v_{03} - i \nu_{13})] \\
&\quad + \phi_{33}[\alpha_{13}(2k v_{01} - i \nu_{11}) + \alpha_{23}(2k v_{02} - i \nu_{12}) + \alpha_{33}(2k v_{03} - i \nu_{13})] \\
&\quad + \phi_{43}[4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12} v_{01} \bar{v}_{02} + \bar{\alpha}_{13} v_{01} \bar{v}_{03} + \alpha_{12} v_{01} \bar{v}_{02} + \alpha_{13} v_{01} \bar{v}_{03} + \alpha_{23} v_{01} \bar{v}_{03} + \alpha_{33}|v_{03}|^2],
\end{align*}
$$

The fourth column of Eq. (69) with $\mu = \mu_3$ yields

$$
\begin{align*}
\phi_{14, \ell}(t, k) &= (2k v_{01} + i \nu_{11}) \phi_{44} - i \phi_{14}(4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12} v_{01} \bar{v}_{02} + \bar{\alpha}_{13} v_{01} \bar{v}_{03}) \\
&\quad - i \phi_{24}(\alpha_{12}|v_{01}|^2 + \alpha_{22} v_{01} \bar{v}_{02} + \alpha_{23} v_{01} \bar{v}_{03}) \\
&\quad - i \phi_{34}(\alpha_{13}|v_{01}|^2 + \alpha_{23} v_{01} \bar{v}_{02} + \alpha_{33} v_{01} \bar{v}_{03}), \\
\phi_{24, \ell}(t, k) &= (2k v_{02} + i \nu_{12}) \phi_{44} - i \phi_{24}(4k^2 + \alpha_{11}|v_{02}|^2 + \bar{\alpha}_{12} v_{02} \bar{v}_{01} + \bar{\alpha}_{13} v_{02} \bar{v}_{03}) \\
&\quad - i \phi_{24}(\alpha_{12}|v_{02}|^2 + \alpha_{22} v_{02} \bar{v}_{01} + \alpha_{23} v_{02} \bar{v}_{03}) \\
&\quad - i \phi_{34}(\alpha_{13}|v_{02}|^2 + \alpha_{23} v_{02} \bar{v}_{01} + \alpha_{33} v_{02} \bar{v}_{03}), \\
\phi_{34, \ell}(t, k) &= (2k v_{03} + i \nu_{13}) \phi_{44} - i \phi_{34}(4k^2 + \alpha_{11}|v_{03}|^2 + \bar{\alpha}_{12} v_{03} \bar{v}_{01} + \bar{\alpha}_{13} v_{03} \bar{v}_{02} + \bar{\alpha}_{13}|v_{03}|^2) \\
&\quad - i \phi_{34}(\alpha_{12}|v_{03}|^2 + \alpha_{22} v_{03} \bar{v}_{01} + \alpha_{23} v_{03} \bar{v}_{02} + \alpha_{33} |v_{03}|^2) \\
&\quad - i \phi_{44}(4k^2 + \alpha_{11}|v_{03}|^2 + \alpha_{23} v_{03} \bar{v}_{01} + \alpha_{33}|v_{03}|^2), \\
\phi_{44, \ell}(t, k) &= \phi_{14}[\alpha_{11}(2k v_{01} - i \nu_{11}) + \bar{\alpha}_{12}(2k v_{02} - i \nu_{12}) + \bar{\alpha}_{13}(2k v_{03} - i \nu_{13})] \\
&\quad + \phi_{24}[\alpha_{12}(2k v_{01} - i \nu_{11}) + \alpha_{22}(2k v_{02} - i \nu_{12}) + \bar{\alpha}_{23}(2k v_{03} - i \nu_{13})] \\
&\quad + \phi_{34}[\alpha_{13}(2k v_{01} - i \nu_{11}) + \alpha_{23}(2k v_{02} - i \nu_{12}) + \alpha_{33}(2k v_{03} - i \nu_{13})] \\
&\quad + \phi_{44}[4k^2 + \alpha_{11}|v_{01}|^2 + \bar{\alpha}_{12} v_{01} \bar{v}_{02} + \bar{\alpha}_{13} v_{01} \bar{v}_{03} + \alpha_{12} v_{01} \bar{v}_{02} + \alpha_{13} v_{01} \bar{v}_{03} + \alpha_{23} v_{01} \bar{v}_{03} + \alpha_{33}|v_{03}|^2],
\end{align*}
$$

Similarly, we can also obtain the asymptotic formulae for $\phi_{ij}$, $i, j = 1, 2, 3, 4$. The substitution of these formulae into Eq. (68a) and using the assumption that the initial and boundary data are compatible at $x = 0$ and $x = L$, we find the asymptotic result (64a) of $c_{14}(t, k)$ for $k \to \infty$. Similarly we can also show Eqs. (64b) and (64c) for $c_{24}(t, k)$ and $c_{34}(t, k)$ as $k \to \infty$. 

27
Similarly, we have the global relation (56) under the vanishing initial data as

\[ c_{41}(t, k) = -\Psi_{44}(t, k)(\alpha_{11}\phi_{14} + \alpha_{12}\phi_{24} + \alpha_{13}\phi_{34})e^{-2ikL} \]

\[ + \Psi_{41}(t, k)[\alpha_{11}(\alpha_{11}\phi_{11} + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{12}(\alpha_{11}\phi_{21} + \alpha_{12}\phi_{22} + \alpha_{13}\phi_{23}) + \alpha_{13}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}\phi_{33})] + \Psi_{42}(t, k)[\alpha_{11}(\alpha_{12}\phi_{11} + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{12}(\alpha_{12}\phi_{21} + \alpha_{22}\phi_{22} + \alpha_{23}\phi_{23}) + \alpha_{13}(\alpha_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}\phi_{33})] \]

\[ + \Psi_{43}(t, k)[\alpha_{11}(\alpha_{13}\phi_{11} + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{12}(\alpha_{13}\phi_{21} + \alpha_{23}\phi_{22} + \alpha_{33}\phi_{23}) + \alpha_{13}(\alpha_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}\phi_{33})] \]

\[ (83) \]

\[ c_{42}(t, k) = -\Psi_{44}(t, k)(\alpha_{12}\phi_{14} + \alpha_{22}\phi_{24} + \alpha_{23}\phi_{34})e^{-2ikL} \]

\[ + \Psi_{41}(t, k)[\alpha_{12}(\alpha_{11}\phi_{11} + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{22}(\alpha_{11}\phi_{21} + \alpha_{12}\phi_{22} + \alpha_{13}\phi_{23}) + \alpha_{23}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}\phi_{33})] + \Psi_{42}(t, k)[\alpha_{12}(\alpha_{12}\phi_{11} + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{22}(\alpha_{12}\phi_{21} + \alpha_{22}\phi_{22} + \alpha_{23}\phi_{23}) + \alpha_{23}(\alpha_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}\phi_{33})] \]

\[ + \Psi_{43}(t, k)[\alpha_{12}(\alpha_{13}\phi_{11} + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{22}(\alpha_{13}\phi_{21} + \alpha_{23}\phi_{22} + \alpha_{33}\phi_{23}) + \alpha_{23}(\alpha_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}\phi_{33})] \]

\[ (84) \]

\[ c_{43}(t, k) = -\Psi_{44}(t, k)(\alpha_{13}\phi_{14} + \alpha_{23}\phi_{24} + \alpha_{33}\phi_{34})e^{-2ikL} \]

\[ + \Psi_{41}(t, k)[\alpha_{13}(\alpha_{11}\phi_{11} + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{23}(\alpha_{11}\phi_{21} + \alpha_{12}\phi_{22} + \alpha_{13}\phi_{23}) + \alpha_{33}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}\phi_{33})] + \Psi_{42}(t, k)[\alpha_{13}(\alpha_{12}\phi_{11} + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{23}(\alpha_{12}\phi_{21} + \alpha_{22}\phi_{22} + \alpha_{23}\phi_{23}) + \alpha_{33}(\alpha_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}\phi_{33})] \]

\[ + \Psi_{43}(t, k)[\alpha_{13}(\alpha_{13}\phi_{11} + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{23}(\alpha_{13}\phi_{21} + \alpha_{23}\phi_{22} + \alpha_{33}\phi_{23}) + \alpha_{33}(\alpha_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}\phi_{33})] \]

\[ (85) \]

where \( \bar{\phi}_{ij} = \bar{\phi}_{ij}(t, k) = \phi_{ij}(t, k) \), such that we can show Eqs. (65)-(67) for \( c_{4j}(t, k) \), \( j = 1, 2, 3 \) as \( k \to \infty \). □

4.3. The relation between Dirichlet and Neumann boundary value problems

In what follows we show that the spectral functions \( S(k) \) and \( S_L(k) \) can be expressed in terms of the prescribed Dirichlet and Neumann boundary data and the initial data using the solution of a system of integral equations. Introduce the new notations as

\[ F_{\pm}(t, k) = F(t, k) \pm F(t, -k), \quad \Sigma_{\pm}(k) = e^{2ikL} \pm e^{-2ikL}. \]

The sign \( \partial D_j \), \( j = 1, 2, 3, 4 \) stands for the boundary of the \( j \)th quadrant \( D_j \), oriented so that \( D_j \) lies to the left of \( \partial D_j \). \( \partial D_3^0 \) denotes the boundary contour which has not contain the zeros of \( \Sigma_{-}(k) \) and \( \partial D_3^0 = -\partial D_3^1 \).

**Theorem 4.2.** Let \( q_j(x) = q_x(x, t = 0) = 0, j = 1, 2, 3 \) be the initial data of Eq. (1) on the interval \( x \in [0, L] \) and \( T < \infty \). (i) For the Dirichlet problem, the boundary data \( u_0(t) \) and \( v_0(t) \) \( (j = 1, 2, 3) \) on
the interval \( t \in [0, T] \) are sufficiently smooth and compatible with the initial data \( q_{0j}(x) \), \( j = 1, 2, 3 \) at points \((x_2, t_2) = (0, 0)\) and \((x_3, t_3) = (L, 0)\) respectively, i.e., \( u_{0j}(0) = q_{0j}(0), v_{0j}(0) = q_{0j}(L), j = 1, 2, 3; \) (ii) For the Neumann problem, the boundary data \( u_{1j}(t) \) and \( v_{0j}(t) \) \( j = 1, 2, 3 \) on the interval \( t \in [0, T] \) are sufficiently smooth and compatible with the initial data \( q_{0j}(x) \), \( j = 1, 2, 3 \) at the origin \((x_2, t_2) = (0, 0)\) and \((x_3, t_3) = (L, 0)\), respectively.

For simplicity, let \( n_{33, 44}(\mathbb{S})(k) \) have no zeros in the domain \( D_1 \). Then the spectral functions \( S(k) \) and \( S_L(k) \) are defined by

\[
S(k) = e^{2ik^2T} \begin{bmatrix}
\Psi_{11}(T, k) & \Psi_{21}(T, k) & \Psi_{31}(T, k) & \Psi_{41}(T, k) \\
\Psi_{12}(T, k) & \Psi_{22}(T, k) & \Psi_{32}(T, k) & \Psi_{42}(T, k) \\
\Psi_{13}(T, k) & \Psi_{23}(T, k) & \Psi_{33}(T, k) & \Psi_{43}(T, k) \\
\Psi_{14}(T, k) & \Psi_{24}(T, k) & \Psi_{34}(T, k) & \Psi_{44}(T, k)
\end{bmatrix} P,
\]

\[
S_L(k) = e^{2ik^2T} \begin{bmatrix}
\phi_{11}(T, k) & \phi_{21}(T, k) & \phi_{31}(T, k) & \phi_{41}(T, k) \\
\phi_{12}(T, k) & \phi_{22}(T, k) & \phi_{32}(T, k) & \phi_{42}(T, k) \\
\phi_{13}(T, k) & \phi_{23}(T, k) & \phi_{33}(T, k) & \phi_{43}(T, k) \\
\phi_{14}(T, k) & \phi_{24}(T, k) & \phi_{34}(T, k) & \phi_{44}(T, k)
\end{bmatrix} P,
\]

where the matrix \( P \) is given by Eq. (28), and the complex-valued functions \( \{ \Psi_{ij}(t, k) \}_{i, j = 1}^4 \) have the following system of integral equations

\[
\begin{align*}
\Psi_{11, t}(t, k) &= 1 + \int_0^t \left[ (2ku_{01} + iu_{11})\Psi_{11} - i\Psi_{21}(\alpha_{11}|u_{01}|^2 + \bar{a}_{12}u_{01}\bar{u}_{02} + \bar{a}_{13}u_{01}\bar{u}_{03}) \\
&\quad - i\Psi_{21}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \alpha_{23}u_{01}\bar{u}_{03}) - i\Psi_{31}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03}) \right](t', k)dt', \\
\Psi_{21, t}(t, k) &= \int_0^t \left[ (2ku_{02} + iu_{12})\Psi_{21} - i\Psi_{11}(\alpha_{11}|u_{02}|^2 + \bar{a}_{12}u_{02}\bar{u}_{01} + \bar{a}_{13}u_{02}\bar{u}_{03}) \\
&\quad - i\Psi_{21}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}u_{02}|u_{02}|^2 + \alpha_{23}u_{02}\bar{u}_{03}) - i\Psi_{31}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}u_{02}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03}) \right](t', k)dt', \\
\Psi_{31, t}(t, k) &= \int_0^t \left[ (2ku_{03} + iu_{13})\Psi_{31} - i\Psi_{11}(\alpha_{11}|u_{03}|^2 + \bar{a}_{12}u_{03}\bar{u}_{02} + \bar{a}_{13}u_{03}|u_{03}|^2) \\
&\quad - i\Psi_{21}(\alpha_{12}u_{03}\bar{u}_{02} + \alpha_{22}u_{03}|u_{03}|^2 + \alpha_{23}u_{03}\bar{u}_{03}) - i\Psi_{31}(\alpha_{13}u_{03}\bar{u}_{02} + \alpha_{23}u_{03}|u_{03}|^2 + \alpha_{33}u_{03}|u_{03}|^2) \right](t', k)dt', \\
\Psi_{41, t}(t, k) &= \int_0^t e^{i k^2 (t-t')} \left\{ \Psi_{11}(\alpha_{11}(2k\bar{u}_{01} - iu_{11}) + \bar{a}_{12}(2k\bar{u}_{02} - iu_{12}) + \bar{a}_{13}(2k\bar{u}_{03} - iu_{13})) \\
&\quad + \Psi_{21}(\alpha_{12}(2k\bar{u}_{01} - iu_{11}) + \alpha_{22}(2k\bar{u}_{02} - iu_{12}) + \bar{a}_{23}(2k\bar{u}_{03} - iu_{13})) \\
&\quad + \Psi_{31}(\alpha_{13}(2k\bar{u}_{01} - iu_{11}) + \alpha_{23}(2k\bar{u}_{02} - iu_{12}) + \alpha_{33}(2k\bar{u}_{03} - iu_{13})) \\
&\quad + i\Psi_{41}(\alpha_{11}|u_{01}|^2 + \bar{a}_{12}u_{01}\bar{u}_{02} + \bar{a}_{13}u_{01}\bar{u}_{03} + \alpha_{12}u_{02}\bar{u}_{01}) \\
&\quad + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}\bar{u}_{03} + \alpha_{33}u_{03}|u_{03}|^2 + \alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}|u_{02}|^2 + \alpha_{33}u_{03}|u_{03}|^2 \right\}(t', k)dt',
\end{align*}
\]
\[
\begin{align*}
\Psi_{12,t}(t, k) &= \int_0^t \left[ (2ku_{01} + iu_{11}) \Psi_{42} - i\Psi_{12}(\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03}) \\
- i\Psi_{22}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03}) \right] (t', k) dt', \\
\Psi_{22,t}(t, k) &= 1 + \int_0^t \left[ (2ku_{02} + iu_{12}) \Psi_{42} - i\Psi_{12}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \alpha_{13}u_{02}\bar{u}_{03}) \\
- i\Psi_{22}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03}) \right] (t', k) dt', \\
\Psi_{32,t}(t, k) &= \int_0^t \left[ (2ku_{03} + iu_{13}) \Psi_{42} - i\Psi_{12}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \alpha_{13}|u_{03}|^2) \\
- i\Psi_{22}(\alpha_{12}u_{03}\bar{u}_{01} + \alpha_{22}|u_{03}|^2 + \bar{\alpha}_{23}u_{03}\bar{u}_{03}) - i\Psi_{32}(\alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}|u_{03}|^2 + \alpha_{33}u_{03}\bar{u}_{03}) \right] (t', k) dt', \\
\Psi_{42,t}(t, k) &= \int_0^t e^{i\lambda^2(t-t')} \left\{ \Psi_{12}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{13}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
+ \Psi_{22}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
+ \Psi_{32}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{33}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
+ i\Psi_{42}[\alpha_{11}|u_{01}|^2 + \alpha_{12}u_{01}\bar{u}_{02} + \alpha_{13}u_{01}\bar{u}_{03} + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}\bar{u}_{03} + \alpha_{33}|u_{03}|^2] \right\} (t', k) dt',
\end{align*}
\]

\[
\begin{align*}
\Psi_{13,t}(t, k) &= \int_0^t \left[ (2ku_{01} + iu_{11}) \Psi_{43} - i\Psi_{13}(\alpha_{11}|u_{01}|^2 + \bar{\alpha}_{12}u_{01}\bar{u}_{02} + \bar{\alpha}_{13}u_{01}\bar{u}_{03}) \\
- i\Psi_{23}(\alpha_{12}|u_{01}|^2 + \alpha_{22}u_{01}\bar{u}_{02} + \bar{\alpha}_{23}u_{01}\bar{u}_{03}) - i\Psi_{33}(\alpha_{13}|u_{01}|^2 + \alpha_{23}u_{01}\bar{u}_{02} + \alpha_{33}u_{01}\bar{u}_{03}) \right] (t', k) dt', \\
\Psi_{23,t}(t, k) &= \int_0^t \left[ (2ku_{02} + iu_{12}) \Psi_{43} - i\Psi_{13}(\alpha_{11}u_{02}\bar{u}_{01} + \bar{\alpha}_{12}|u_{02}|^2 + \alpha_{13}u_{02}\bar{u}_{03}) \\
- i\Psi_{23}(\alpha_{12}u_{02}\bar{u}_{01} + \alpha_{22}|u_{02}|^2 + \bar{\alpha}_{23}u_{02}\bar{u}_{03}) - i\Psi_{33}(\alpha_{13}u_{02}\bar{u}_{01} + \alpha_{23}|u_{02}|^2 + \alpha_{33}u_{02}\bar{u}_{03}) \right] (t', k) dt', \\
\Psi_{33,t}(t, k) &= 1 + \int_0^t \left[ (2ku_{03} + iu_{13}) \Psi_{43} - i\Psi_{13}(\alpha_{11}u_{03}\bar{u}_{01} + \bar{\alpha}_{12}u_{03}\bar{u}_{02} + \alpha_{13}|u_{03}|^2) \\
- i\Psi_{23}(\alpha_{12}u_{03}\bar{u}_{01} + \alpha_{22}u_{03}\bar{u}_{02} + \bar{\alpha}_{23}|u_{03}|^2) - i\Psi_{33}(\alpha_{13}u_{03}\bar{u}_{01} + \alpha_{23}u_{03}\bar{u}_{02} + \alpha_{33}|u_{03}|^2) \right] (t', k) dt', \\
\Psi_{43,t}(t, k) &= \int_0^t e^{i\lambda^2(t-t')} \left\{ \Psi_{13}[\alpha_{11}(2k\bar{u}_{01} - i\bar{u}_{11}) + \bar{\alpha}_{12}(2k\bar{u}_{02} - i\bar{u}_{12}) + \alpha_{13}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
+ \Psi_{23}[\alpha_{12}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{22}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{23}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
+ \Psi_{33}[\alpha_{13}(2k\bar{u}_{01} - i\bar{u}_{11}) + \alpha_{23}(2k\bar{u}_{02} - i\bar{u}_{12}) + \bar{\alpha}_{33}(2k\bar{u}_{03} - i\bar{u}_{13})] \\
+ i\Psi_{43}[\alpha_{11}|u_{01}|^2 + \alpha_{12}u_{01}\bar{u}_{02} + \alpha_{13}u_{01}\bar{u}_{03} + \alpha_{22}|u_{02}|^2 + \alpha_{23}u_{02}\bar{u}_{03} + \alpha_{33}|u_{03}|^2] \right\} (t', k) dt',
\end{align*}
\]
and

\[
\Psi_{14,t}(t, k) = \int_0^t e^{-4ik(t-t')} [(2k u_{01} + i u_{11}) \Psi_{44} - i \Psi_{14}(a_{11}|u_{01}|^2 + \bar{a}_{12}u_{01}\bar{u}_{02} + \bar{a}_{13}u_{01}\bar{u}_{03}) - i \Psi_{24}(a_{12}|u_{01}|^2 + \bar{a}_{22}u_{02}\bar{u}_{02} + \bar{a}_{23}u_{02}\bar{u}_{03}) - i \Psi_{34}(a_{13}|u_{01}|^2 + \bar{a}_{23}u_{03}\bar{u}_{02} + \bar{a}_{33}u_{03}\bar{u}_{03})](t', k) dt',
\]

\[
\Psi_{24,t}(t, k) = \int_0^t e^{-4ik(t-t')} [(2k u_{02} + i u_{12}) \Psi_{44} - i \Psi_{14}(a_{11}|u_{02}|^2 + \bar{a}_{12}u_{02}\bar{u}_{02} + \bar{a}_{13}u_{02}\bar{u}_{03}) - i \Psi_{24}(a_{12}|u_{02}|^2 + \bar{a}_{22}u_{02}\bar{u}_{02} + \bar{a}_{23}u_{02}\bar{u}_{03}) - i \Psi_{34}(a_{13}|u_{02}|^2 + \bar{a}_{23}u_{03}\bar{u}_{02} + \bar{a}_{33}u_{03}\bar{u}_{03})](t', k) dt',
\]

\[
\Psi_{34,t}(t, k) = \int_0^t e^{-4ik(t-t')} [(2k u_{03} + i u_{13}) \Psi_{44} - i \Psi_{14}(a_{11}|u_{03}|^2 + \bar{a}_{12}u_{03}\bar{u}_{02} + \bar{a}_{13}u_{03}\bar{u}_{03}) - i \Psi_{24}(a_{12}|u_{03}|^2 + \bar{a}_{22}u_{02}\bar{u}_{02} + \bar{a}_{23}u_{02}\bar{u}_{03}) - i \Psi_{34}(a_{13}|u_{03}|^2 + \bar{a}_{23}u_{03}\bar{u}_{02} + \bar{a}_{33}u_{03}\bar{u}_{03})](t', k) dt',
\]

\[
\Psi_{44,t}(t, k) = 1 + \int_0^t \{ \Psi_{14}(a_{11}(2k u_{01} - i \bar{u}_{11}) + \bar{a}_{12}(2k u_{02} - i \bar{u}_{12}) + \bar{a}_{13}(2k u_{03} - i \bar{u}_{13})) + \Psi_{24}(a_{12}(2k u_{02} - i \bar{u}_{12}) + \bar{a}_{22}(2k u_{02} - i \bar{u}_{12}) + \bar{a}_{23}(2k u_{03} - i \bar{u}_{13})) + \Psi_{34}(a_{13}(2k u_{03} - i \bar{u}_{13}) + \bar{a}_{23}(2k u_{03} - i \bar{u}_{13}) + \bar{a}_{33}(2k u_{03} - i \bar{u}_{13})) + \Psi_{44}(a_{11}|u_{01}|^2 + \bar{a}_{12}u_{01}\bar{u}_{02} + \bar{a}_{13}u_{01}\bar{u}_{03} + \bar{a}_{12}u_{02}\bar{u}_{01} + \bar{a}_{12}u_{02}\bar{u}_{01} + \bar{a}_{23}u_{03}\bar{u}_{02} + \bar{a}_{23}u_{03}\bar{u}_{02} + \bar{a}_{33}u_{03}\bar{u}_{03})](t', k) dt',
\]

(92)

The functions \(\{\phi_{ij}(t, k)\}_{i,j=1}^4\) are of the same integral equations (89)-(92) by replacing the functions \(\{u_{ij}, v_{ij}\}\) with \(\{v_{ij}, v_{ij}\}\), \(j = 1, 2, 3\), that is, \(\phi_{ij}(t, k) = \Phi_{ij}(t, k)|_{\{u_{ij}(t) = v_{ij}(t), u_{ij}(t) = v_{ij}(t)\}}\) \(i, j = 1, 2, 3, 4; l = 1, 2, 3\)

(i) For the given Dirichlet problem, the unknown Neumann boundary data \(\{u_{ij}(t)\}_{j=1}^3\) and \(\{v_{ij}(t)\}_{j=1}^3\), \(0 < t < T\) can be given by

\[
u_{11}(t) = \int_{\partial D^+} \left[ \frac{2\Sigma}{i\pi \Sigma} (k \Psi_{14} + i u_{01}) + u_{01}(2\Psi_{44} - \bar{\phi}_{44}) \right] dk + \int_{\partial D^-} \left[ \frac{4}{i\pi \Sigma} (a_{11}(-ik \bar{\phi}_{41} + \alpha_{11}v_{01} + \alpha_{12}v_{02} + \alpha_{13}v_{03}) + \bar{a}_{12}(-ik \bar{\phi}_{42} + \bar{a}_{12}v_{01} + \alpha_{22}v_{02} + \alpha_{23}v_{03}) + \bar{a}_{13}(-ik \bar{\phi}_{43} + \bar{a}_{13}v_{01} + \alpha_{23}v_{02} + \alpha_{33}v_{03}) \right] dk + \int_{\partial D^+} \left[ \frac{4k}{i\pi \Sigma} \Phi_{14}(\bar{\phi}_{44} - 1)e^{-2ikL} - (\Psi_{11} - 1)(a_{11}\bar{\phi}_{41} + \alpha_{12}\bar{\phi}_{42} + \bar{a}_{13}\bar{\phi}_{43}) - \Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{a}_{23}\bar{\phi}_{43}) - \Psi_{13}(\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{43} + \alpha_{33}\bar{\phi}_{43}) \right] dk,
\]
\[
u_{12}(t) = \int_{\partial D_2^0} \left[ \frac{2\Sigma^+}{i\pi \Sigma_-} (k\Psi_{24-} + iv_{02}) + u_{02} (2\Psi_{44-} - \bar{\phi}_{44-}) \right] dk
+ \int_{\partial D_2^0} \frac{4}{i\pi \Sigma_-} \left[ \alpha_{12} \left( -ik\bar{\phi}_{41-} + \alpha_{11}v_{01} + \alpha_{12}v_{02} + \alpha_{13}v_{03} \right)
+ \alpha_{22} \left( -ik\bar{\phi}_{42-} + \bar{\alpha}_{12}v_{01} + \alpha_{22}v_{02} + \alpha_{23}v_{03} \right)
+ \alpha_{23} \left( -ik\bar{\phi}_{43-} + \bar{\alpha}_{13}v_{01} + \alpha_{23}v_{02} + \alpha_{33}v_{03} \right) \right] dk
+ \int_{\partial D_2^0} \frac{4k}{i\pi \Sigma_-} \left[ \Psi_{24}(\bar{\phi}_{44-} - 1)e^{-2ikL} - \Psi_{21}(\alpha_{11}\bar{\phi}_{41} + \alpha_{12}\bar{\phi}_{42} + \alpha_{13}\bar{\phi}_{43})
- (\Psi_{22} - 1)(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) - \Psi_{23}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43}) \right] dk,
\]
(94)

\[
u_{13}(t) = \int_{\partial D_3^0} \left[ \frac{2\Sigma^+}{i\pi \Sigma_-} (k\Psi_{34-} + iv_{03}) + u_{03} (2\Psi_{44-} - \bar{\phi}_{44-}) \right] dk
+ \int_{\partial D_3^0} \frac{4}{i\pi \Sigma_-} \left[ \alpha_{13} \left( -ik\bar{\phi}_{41-} + \alpha_{11}v_{01} + \alpha_{12}v_{02} + \alpha_{13}v_{03} \right)
+ \alpha_{23} \left( -ik\bar{\phi}_{43-} + \alpha_{13}v_{01} + \alpha_{23}v_{02} + \alpha_{33}v_{03} \right) \right] dk
+ \int_{\partial D_3^0} \frac{4k}{i\pi \Sigma_-} \left[ \Psi_{34}(\bar{\phi}_{44-} - 1)e^{-2ikL} - \Psi_{31}(\alpha_{11}\bar{\phi}_{41} + \alpha_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})
- \Psi_{32}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \bar{\alpha}_{23}\bar{\phi}_{43}) - (\Psi_{33} - 1)(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43}) \right] dk,
\]
(95)

and
\[
v_{1j}(t) = 4\phi_{j4}^{(2)} + \frac{2}{\pi} \int_{\partial D_j^0} v_{0j} \phi_{44-} dk, \quad j = 1, 2, 3,
\]
(96)

where
\[
\begin{align*}
\left( \phi_{14}^{(2)} \right) = & \frac{1}{4\pi} \int_{\partial D_4^0} \left[ \frac{2\Sigma^+}{\Sigma_-} \begin{pmatrix} k\phi_{44-} + iv_{01} \\ k\phi_{24-} + iv_{02} \end{pmatrix} - \Psi_{44}^{(1)} \begin{pmatrix} v_{01} \\ v_{02} \end{pmatrix} \right] dk + \frac{M^T}{2i\pi} \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix},
\end{align*}
\]
\[ I_1(t) = \int_{\partial D^3} \left\{ \frac{2}{\Sigma} \left[ (\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2) (k \bar{\Psi}_{41}^- + i(\bar{\alpha}_{11}u_{01} + \alpha_{12}u_{02} + \alpha_{13}u_{03})) \right. \right. \\
+ (\alpha_{11}\bar{\alpha}_{12} + \alpha_{12}\bar{\alpha}_{22} + \alpha_{13}\alpha_{23}) (k \bar{\Psi}_{42}^- + i(\bar{\alpha}_{12}u_{01} + \alpha_{22}u_{02} + \alpha_{23}u_{03})) \\
+ (\alpha_{11}\bar{\alpha}_{13} + \alpha_{12}\bar{\alpha}_{23} + \alpha_{13}\alpha_{33}) (k \bar{\Psi}_{43}^- + i(\bar{\alpha}_{13}u_{01} + \bar{\alpha}_{23}u_{02} + \alpha_{33}u_{03})) \left. \right\} dk \\
+ \int_{\partial D^3} \left\{ \frac{2k}{\Sigma} \left\{ (1 - \bar{\Psi}_{44}^+)(\alpha_{11}\phi_{14} + \bar{\alpha}_{12}\phi_{24} + \bar{\alpha}_{13}\phi_{34})e^{2ikL} \right. \right. \\
+ \bar{\Psi}_{41}[\alpha_{11}(\alpha_{11}(\phi_{11} - 1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \bar{\alpha}_{12}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22} - 1) + \alpha_{13}\phi_{23}) \right. \\
+ \bar{\alpha}_{13}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33} - 1))] + \bar{\Psi}_{42}[\alpha_{11}(\bar{\alpha}_{12}(\phi_{11} - 1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \right. \\
+ \bar{\alpha}_{12}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22} - 1) + \alpha_{23}\phi_{23}) + \bar{\alpha}_{13}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33} - 1))] \\
+ \bar{\Psi}_{43}[\alpha_{11}(\bar{\alpha}_{13}(\phi_{11} - 1) + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \bar{\alpha}_{12}(\bar{\alpha}_{13}\phi_{21} + \alpha_{23}(\phi_{22} - 1) + \alpha_{33}\phi_{23}) \right. \\
+ \bar{\alpha}_{13}(\bar{\alpha}_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}(\phi_{33} - 1))] \left. \right\}_- dk, \] (97)

\[ I_2(t) = \int_{\partial D^3} \left\{ \frac{2}{\Sigma} \left[ (\alpha_{12}^2 + |\alpha_{22}|^2 + |\alpha_{23}|^2) (k \bar{\Psi}_{42}^- + i(\bar{\alpha}_{12}u_{01} + \alpha_{22}u_{02} + \alpha_{23}u_{03})) \right. \right. \\
+ (\alpha_{12}\bar{\alpha}_{13} + \alpha_{22}\bar{\alpha}_{23} + \alpha_{23}\alpha_{33}) (k \bar{\Psi}_{43}^- + i(\bar{\alpha}_{13}u_{01} + \bar{\alpha}_{23}u_{02} + \alpha_{33}u_{03})) \left. \right\} dk \\
+ \int_{\partial D^3} \left\{ \frac{2k}{\Sigma} \left\{ (1 - \bar{\Psi}_{44}^+)(\alpha_{12}\phi_{14} + \bar{\alpha}_{22}\phi_{24} + \bar{\alpha}_{23}\phi_{34})e^{2ikL} \right. \right. \\
+ \bar{\Psi}_{41}[\alpha_{12}(\alpha_{11}(\phi_{11} - 1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{22}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22} - 1) + \alpha_{13}\phi_{23}) \right. \\
+ \alpha_{23}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33} - 1))] + \bar{\Psi}_{42}[\alpha_{12}(\bar{\alpha}_{12}(\phi_{11} - 1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) \right. \\
+ \alpha_{22}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22} - 1) + \alpha_{23}\phi_{23}) + \alpha_{23}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33} - 1))] \\
+ \bar{\Psi}_{43}[\alpha_{12}(\bar{\alpha}_{13}(\phi_{11} - 1) + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{22}(\bar{\alpha}_{13}\phi_{21} + \alpha_{23}(\phi_{22} - 1) + \alpha_{33}\phi_{23}) \right. \\
+ \alpha_{23}(\bar{\alpha}_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}(\phi_{33} - 1))] \left. \right\}_- dk, \] (98)
\[ I_3(t) = \int_{\partial D_3^0} \left\{ \frac{2}{\pi \Sigma} \left[ (\alpha_{12}^2 + \alpha_{22}^2 + |\alpha_{23}|^2) (k \Phi_{43}^- + i(\bar{\alpha}_{13} u_{01} + \bar{\alpha}_{23} u_{02} + \alpha_{33} u_{03})) \\
+ (\alpha_{11} \alpha_{13} + \alpha_{12} \alpha_{23} + \bar{\alpha}_{13} \alpha_{23}) (k \Phi_{41}^- + i(\bar{\alpha}_{11} u_{01} + \alpha_{12} u_{02} + \alpha_{13} u_{03})) \\
+ (\alpha_{13} \bar{\alpha}_{12} + \alpha_{22} \alpha_{23} + \bar{\alpha}_{13} \alpha_{23}) (k \Phi_{42}^- + i(\bar{\alpha}_{12} u_{01} + \alpha_{22} u_{02} + \alpha_{23} u_{03})) \right]\} dk \\
+ \int_{\partial D_3^0} \frac{2k}{\pi \Sigma} \left\{ (1 - \Phi_{44}) (\alpha_{13} \phi_{14} + \alpha_{23} \phi_{24} + \alpha_{33} \phi_{34}) e^{2ikL} \\
+ \Phi_{41} (\alpha_{13} (\alpha_{11} \phi_{11} - 1) + \alpha_{12} \phi_{12} + \alpha_{13} \phi_{13}) + \alpha_{23} (\alpha_{11} \phi_{21} + \alpha_{12} (\phi_{22} - 1) + \alpha_{13} \phi_{23}) \\
+ \alpha_{33} (\alpha_{11} \phi_{31} + \alpha_{12} \phi_{32} + \alpha_{13} (\phi_{33} - 1)) + \Phi_{42} (\alpha_{13} (\bar{\alpha}_{12} \phi_{11} - 1) + \alpha_{22} \phi_{12} + \alpha_{23} \phi_{13}) \\
+ \alpha_{23} (\bar{\alpha}_{12} \phi_{21} + \alpha_{22} (\phi_{22} - 1) + \alpha_{23} \phi_{23}) + \alpha_{33} (\bar{\alpha}_{12} \phi_{31} + \alpha_{22} \phi_{32} + \alpha_{23} (\phi_{33} - 1)) \\
+ \Phi_{43} (\alpha_{13} (\alpha_{13} \phi_{11} - 1) + \alpha_{23} \phi_{12} + \alpha_{33} \phi_{13}) + \alpha_{23} (\alpha_{13} \phi_{21} + \bar{\alpha}_{23} (\phi_{22} - 1) + \alpha_{33} \phi_{23}) \\
+ \alpha_{33} (\bar{\alpha}_{13} \phi_{31} + \alpha_{23} \phi_{32} + \alpha_{33} (\phi_{33} - 1)) \right\}^- dk, \tag{99} \] 

(ii) For the known Neumann problem, the unknown Dirichlet boundary data \(\{u_{0j}(t)\}_{j=1}^3\) and \(\{v_{0j}(t)\}_{j=1}^3\), \(0 < t < T\) can be given by

\[ u_{01}(t) = \int_{\partial D_3^0} \frac{1}{\pi \Sigma} \left[ \Sigma_+ \Phi_{14}^- + 2(\alpha_{11} \bar{\phi}_{41} + \alpha_{12} \bar{\phi}_{42} + \bar{\alpha}_{13} \bar{\phi}_{43}) \right] dk \]

\[ + \int_{\partial D_3^0} \frac{2}{\pi \Sigma} \left\{ \Phi_{14}^- (\bar{\phi}_{44} - 1) e^{-2ikL} - [(\Phi_{11} - 1) (\alpha_{11} \bar{\phi}_{41} + \alpha_{12} \bar{\phi}_{42} + \bar{\alpha}_{13} \bar{\phi}_{43}) \\
+ \Phi_{12} (\alpha_{12} \bar{\phi}_{41} + \alpha_{22} \bar{\phi}_{42} + \bar{\alpha}_{23} \bar{\phi}_{43}) + \Phi_{13} (\alpha_{13} \bar{\phi}_{41} + \alpha_{23} \bar{\phi}_{42} + \alpha_{33} \bar{\phi}_{43}) \right] \}^+ dk, \tag{100a} \]

\[ u_{02}(t) = \int_{\partial D_3^0} \frac{1}{\pi \Sigma} \left[ \Sigma_+ \Phi_{24}^- + 2(\alpha_{12} \bar{\phi}_{41} + \alpha_{22} \bar{\phi}_{42} + \bar{\alpha}_{23} \bar{\phi}_{43}) \right] dk \]

\[ + \int_{\partial D_3^0} \frac{2}{\pi \Sigma} \left\{ \Phi_{24}^- (\bar{\phi}_{44} - 1) e^{-2ikL} - [(\Phi_{21} (\alpha_{11} \bar{\phi}_{41} + \alpha_{12} \bar{\phi}_{42} + \bar{\alpha}_{13} \bar{\phi}_{43}) \\
+ (\Phi_{22} - 1) (\alpha_{12} \bar{\phi}_{41} + \alpha_{22} \bar{\phi}_{42} + \bar{\alpha}_{23} \bar{\phi}_{43}) + \Phi_{23} (\alpha_{13} \bar{\phi}_{41} + \alpha_{23} \bar{\phi}_{42} + \alpha_{33} \bar{\phi}_{43}) \right] \}^+ dk, \tag{100b} \]

\[ u_{03}(t) = \int_{\partial D_3^0} \frac{1}{\pi \Sigma} \left[ \Sigma_+ \Phi_{34}^- + 2(\alpha_{13} \bar{\phi}_{41} + \alpha_{23} \bar{\phi}_{42} + \alpha_{33} \bar{\phi}_{43}) \right] dk \]

\[ + \int_{\partial D_3^0} \frac{2}{\pi \Sigma} \left\{ \Phi_{34}^- (\bar{\phi}_{44} - 1) e^{-2ikL} - [(\Phi_{31} (\alpha_{11} \bar{\phi}_{41} + \alpha_{12} \bar{\phi}_{42} + \bar{\alpha}_{13} \bar{\phi}_{43}) \\
+ \Phi_{32} (\alpha_{12} \bar{\phi}_{41} + \alpha_{22} \bar{\phi}_{42} + \bar{\alpha}_{23} \bar{\phi}_{43}) + (\Phi_{33} - 1) (\alpha_{13} \bar{\phi}_{41} + \alpha_{23} \bar{\phi}_{42} + \alpha_{33} \bar{\phi}_{43}) \right] \}^+ dk, \tag{100c} \]

and

\[ v_{01}(t) = 2i\phi_{14}^{(1)}, \quad v_{02}(t) = 2i\phi_{24}^{(1)}, \quad v_{03}(t) = 2i\phi_{34}^{(1)}, \tag{101} \]
where

\[
\begin{pmatrix}
\phi_{14}^{(1)} \\
\phi_{24}^{(1)} \\
\phi_{34}^{(1)}
\end{pmatrix}
= -\frac{1}{2i\pi} \int_{\partial D_3^0} \frac{\Sigma_+}{\Sigma_-} \begin{pmatrix}
\phi_{14'} \\
\phi_{24'} \\
\phi_{34'}
\end{pmatrix} dk
+ \frac{\mathcal{M}^T}{2i\pi} \begin{pmatrix}
J_1(t) \\
J_2(t) \\
J_3(t)
\end{pmatrix},
\]

with

\[
J_1(t) = \int_{\partial D_3^0} \frac{2}{\Sigma_-} \left\{ \left[ (\alpha_{11}^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2) \bar{\Psi}_{41} + (\alpha_{11}\bar{\alpha}_{12} + \bar{\alpha}_{12}\alpha_{22} + \bar{\alpha}_{13}\alpha_{32}) \bar{\Psi}_{22} \right.ight.
\]
\[
\left. + (\alpha_{11}\bar{\alpha}_{13} + \bar{\alpha}_{12}\bar{\alpha}_{23} + \bar{\alpha}_{13}\alpha_{33}) \bar{\Psi}_{43} \right] dk
+ \int_{\partial D_3^0} \frac{2}{\Sigma_-} \left\{ (1 - \bar{\Psi}_{44})(\alpha_{11}\phi_{14} + \bar{\alpha}_{12}\phi_{24} + \bar{\alpha}_{13}\phi_{34})e^{2ikL} 
\right.
\]
\[
\left. + \bar{\Psi}_{41}(\alpha_{11}(\alpha_{11}(\phi_{11} - 1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \bar{\alpha}_{12}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22} - 1) + \alpha_{13}\phi_{23})
+ \bar{\alpha}_{13}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33} - 1)))
+ \bar{\Psi}_{42}(\alpha_{11}(\bar{\alpha}_{12}(\phi_{11} - 1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13})
+ \alpha_{12}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22} - 1) + \alpha_{23}\phi_{23}) + \bar{\alpha}_{13}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33} - 1)))
\right.
\]
\[
\left. + \bar{\Psi}_{43}(\alpha_{11}(\bar{\alpha}_{13}(\phi_{11} - 1) + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{12}(\bar{\alpha}_{13}\phi_{21} + \alpha_{23}(\phi_{22} - 1) + \alpha_{33}\phi_{23})
+ \alpha_{13}(\bar{\alpha}_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}(\phi_{33} - 1))) \right\} + dk,
\]

\[
J_2(t) = \int_{\partial D_3^0} \frac{2}{\Sigma_-} \left\{ \left[ (\alpha_{12}^2 + |\alpha_{22}|^2 + |\alpha_{23}|^2) \bar{\Psi}_{42} + (\alpha_{11}\alpha_{12} + \bar{\alpha}_{21}\alpha_{22} + \bar{\alpha}_{23}\alpha_{32}) \bar{\Psi}_{41} \right.ight.
\]
\[
\left. + (\alpha_{12}\bar{\alpha}_{13} + \alpha_{22}\bar{\alpha}_{23} + \alpha_{23}\alpha_{33}) \bar{\Psi}_{43} \right] dk
+ \int_{\partial D_3^0} \frac{2}{\Sigma_-} \left\{ (1 - \bar{\Psi}_{44})(\alpha_{12}\phi_{14} + \alpha_{22}\phi_{24} + \bar{\alpha}_{23}\phi_{34})e^{2ikL} 
\right.
\]
\[
\left. + \bar{\Psi}_{41}(\alpha_{12}(\alpha_{11}(\phi_{11} - 1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{22}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22} - 1) + \alpha_{13}\phi_{23})
+ \alpha_{23}(\alpha_{11}\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33} - 1)))
+ \bar{\Psi}_{42}(\alpha_{12}(\bar{\alpha}_{12}(\phi_{11} - 1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13})
+ \alpha_{22}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22} - 1) + \alpha_{23}\phi_{23}) + \alpha_{23}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33} - 1)))
\right.
\]
\[
\left. + \bar{\Psi}_{43}(\alpha_{12}(\bar{\alpha}_{13}(\phi_{11} - 1) + \alpha_{23}\phi_{12} + \alpha_{33}\phi_{13}) + \alpha_{22}(\bar{\alpha}_{13}\phi_{21} + \alpha_{23}(\phi_{22} - 1) + \alpha_{33}\phi_{23})
+ \alpha_{23}(\bar{\alpha}_{13}\phi_{31} + \alpha_{23}\phi_{32} + \alpha_{33}(\phi_{33} - 1))) \right\} + dk,
\]
\[ J_3(t) = \int_{\partial D_3} \frac{2}{\Sigma_+} \left[ (\alpha_{12}^2 + \alpha_{22}^2 + |\alpha_{23}|^2)\Psi_{43} + (\alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{23} + \bar{\alpha}_{13}\alpha_{23})\Psi_{41} + (\alpha_{13}\bar{\alpha}_{12} + \alpha_{22}\alpha_{23} + \alpha_{13}\alpha_{23})\Psi_{42} \right] dk \]

\[ + \int_{\partial D_3} \frac{2}{\Sigma_+} \left\{ (1 - \Psi_{44})(\alpha_{13}\phi_{14} + \alpha_{23}\phi_{24} + \alpha_{33}\phi_{34})e^{2ikL} \right\} dk \]

\[ + \Psi_{41}[\alpha_{13}(\alpha_{11}(\phi_{11} - 1) + \alpha_{12}\phi_{12} + \alpha_{13}\phi_{13}) + \alpha_{23}(\alpha_{11}\phi_{21} + \alpha_{12}(\phi_{22} - 1) + \alpha_{13}\phi_{23}) + \alpha_{33}(\alpha_{11}(\phi_{31} + \alpha_{12}\phi_{32} + \alpha_{13}(\phi_{33} - 1))] \]

\[ + \Psi_{42}[\alpha_{13}(\bar{\alpha}_{12}(\phi_{11} - 1) + \alpha_{22}\phi_{12} + \alpha_{23}\phi_{13}) + \alpha_{23}(\bar{\alpha}_{12}\phi_{21} + \alpha_{22}(\phi_{22} - 1) + \alpha_{23}\phi_{23}) + \alpha_{33}(\bar{\alpha}_{12}\phi_{31} + \alpha_{22}\phi_{32} + \alpha_{23}(\phi_{33} - 1))] \]

\[ + \Psi_{43}[\alpha_{13}(\bar{\alpha}_{13}(\phi_{11} - 1) + \alpha_{23}\phi_{12} + \alpha_{33}(\phi_{13}) + \alpha_{23}(\bar{\alpha}_{13}\phi_{21} + \alpha_{23}(\phi_{22} - 1) + \alpha_{33}\phi_{23}) + \alpha_{33}(\bar{\alpha}_{13}\phi_{31} + \alpha_{33}(\phi_{33} - 1))] \] +dk,

where \( \Psi_{14} = \Psi_{14}(t,k) \), \( \phi_{44} = \phi_{44}(t,k) = \bar{\phi}_{44}(t,k) \) and other functions have the similar expressions.

**Proof.** We can show that Eqs. (87) and (88) hold by means of Eqs. (33) and (34) with replacing \( T \) by \( t \), that is, \( S(k) = e^{-2ik^2t\alpha_4}\mu_2^{-1}(0, t, k) \) and \( S_L(k) = e^{-2ik^2t\alpha_4}\mu_3^{-1}(L, t, k) \) and the symmetry relation (29). Moreover, Eqs. (89)-(92) for \( \Psi_{ij}(t,k) \), \( i,j = 1,2,3,4 \) can be obtained by using the Volteral integral equations of \( \mu_2(0,t,k) \). Similarly, the expressions of \( \phi_{ij}(t,k) \), \( i,j = 1,2,3,4 \) can be found by means of the Volteral integral equations of \( \mu_3(L,t,k) \).

In what follows we show Eqs. (93)-(101), that is the map between Dirichlet and Neumann boundary conditions.

(i) The Cauchy’s theorem is employed to study Eq. (58) to generate

\[ i\pi \Psi_{14}^{(1)}(t) = - \left( \int_{\partial D_2} + \int_{\partial D_1} \right) [\Psi_{44}(t,k) - 1]dk = \left( \int_{\partial D_1} + \int_{\partial D_3} \right) [\Psi_{44}(t,k) - 1]dk \]

\[ = \int_{\partial D_3} [\Psi_{44}(t,k) - 1]dk - \int_{\partial D_3} [\Psi_{44}(t,-k) - 1]dk = \int_{\partial D_3} \Psi_{44}^-(t,k)dk, \]

and

\[ i\pi \Psi_{14}^{(2)}(t) = \left( \int_{\partial D_1} + \int_{\partial D_3} \right) \left[ k\Psi_{14}(t,k) + \frac{i}{2}u_{01}(t) \right] dk, = \int_{\partial D_3} \left[ k\Psi_{14}(t,k) + \frac{i}{2}u_{01}(t) \right] _- dk, \]

\[ = \int_{\partial D_3} \sum_{\Sigma_+} (k\Psi_{14}^- + iu_{01})dk + C_1(t), \]

where we have introduced the function \( C_1(t) \) as

\[ C_1(t) = - \int_{\partial D_3} \left\{ \frac{2e^{-2ikL}}{\Sigma_-} \left[ k\Psi_{14}(t,k) + \frac{i}{2}u_{01}(t) \right] _- \right\} dk, \]

36
Thus substituting Eqs. (105) and (109) into the third one of system (60), we can get Eq. (93). Similarly, we can apply the Cauchy’s theorem and asymptotic (64a) to Eq. (107), we find that

\[
C_1(t) = -\int_{\partial D_3^{(2)}} \left\{ \frac{2e^{-2ikL}}{\Sigma^2} \left[ k\Psi_{14}(t, k) + \frac{i}{2}v_{01}(t) \right] \right\}_- \, dk
\]

\[
= \int_{\partial D_3^{(2)}} \left\{ \frac{2e^{-2ikL}}{\Sigma^2} \left[ -k\bar{c}_{14} + \Psi_{14}^{(1)} + \frac{\Psi_{14}^{(1)}\bar{\phi}_{14}^{(1)}}{k} - (\alpha_{11}\phi_{41}^{(1)} + \bar{\alpha}_{12}\bar{\phi}_{42}^{(1)} + \bar{\alpha}_{13}\bar{\phi}_{43}^{(1)})e^{2ikL} \right] \right\}_- \, dk
\]

\[
- \int_{\partial D_3^{(2)}} \left\{ \frac{2e^{-2ikL}}{\Sigma^2} \left[ \frac{\Psi_{14}^{(1)}\bar{\phi}_{44}^{(1)}}{k} + \left( \alpha_{11}(k\phi_{41} - \bar{\phi}_{41}^{(1)}) + \bar{\alpha}_{12}(k\bar{\phi}_{42} - \bar{\phi}_{42}^{(1)}) \right) + \bar{\alpha}_{13}(k\bar{\phi}_{43} - \bar{\phi}_{43}^{(1)})e^{2ikL} \right] \right\}_- \, dk
\]

\[
+ \int_{\partial D_3^{(2)}} \left\{ \frac{2ke^{-2ikL}}{\Sigma^2} \left[ \Psi_{14}(\bar{\phi}_{44} - 1) - \left[ (\Psi_{11} - 1)(\alpha_{11}\phi_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) + \Psi_{12}(\alpha_{12}\phi_{41} + \alpha_{22}\bar{\phi}_{42} + \alpha_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\phi_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43}) \right] e^{2ikL} \right] \right\}_- \, dk;
\]

By applying the Cauchy’s theorem and asymptotic (64a) to Eq. (107), we find that \( C_1(t) \) can reduce to

\[
C_1(t) = -i\pi\Psi_{14}^{(2)} - \int_{\partial D_3^{(2)}} \left\{ \frac{i}{2}u_{01}\bar{\phi}_{44} - \frac{2i}{\Sigma} \left[ \alpha_{11}(-ik\phi_{41} + \alpha_{11}v_{01} + \bar{\alpha}_{12}v_{02} + \alpha_{13}v_{03}) \right. \right.
\]

\[
\left. + \bar{\alpha}_{12}(-ik\bar{\phi}_{42} + \alpha_{12}v_{01} + \alpha_{22}v_{02} + \alpha_{23}v_{03}) + \bar{\alpha}_{13}(-ik\bar{\phi}_{43} + \bar{\alpha}_{13}v_{01} + \bar{\alpha}_{23}v_{02} + \alpha_{33}v_{03}) \right] \right\}_- \, dk
\]

\[
+ \int_{\partial D_3^{(2)}} \frac{2k}{\Sigma} \left[ \Psi_{14}(\bar{\phi}_{44} - 1)e^{-2ikL} - (\Psi_{11} - 1)(\alpha_{11}\phi_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \right.
\]

\[
\left. - \Psi_{12}(\alpha_{12}\phi_{41} + \alpha_{22}\bar{\phi}_{42} + \alpha_{23}\bar{\phi}_{43}) - \Psi_{13}(\alpha_{13}\phi_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43}) \right] \right\}_- \, dk;
\]

It follows from Eqs. (106) and (108) that we have

\[
2i\pi\Psi_{14}^{(2)}(t) = \int_{\partial D_3^{(2)}} \left[ \frac{\Sigma}{\Sigma^2}(k\Psi_{14} + iv_{01}) - \frac{i}{2}u_{01}\bar{\phi}_{44} \right] \, dk
\]

\[
+ \int_{\partial D_3^{(2)}} \frac{2i}{\Sigma} \left[ \alpha_{11}(-ik\phi_{41} + \alpha_{11}v_{01} + \bar{\alpha}_{12}v_{02} + \alpha_{13}v_{03}) \right. \right.
\]

\[
\left. + \bar{\alpha}_{12}(-ik\bar{\phi}_{42} + \alpha_{12}v_{01} + \alpha_{22}v_{02} + \alpha_{23}v_{03}) \right. \left. + \bar{\alpha}_{13}(-ik\bar{\phi}_{43} + \bar{\alpha}_{13}v_{01} + \bar{\alpha}_{23}v_{02} + \alpha_{33}v_{03}) \right] \right\}_- \, dk
\]

\[
+ \int_{\partial D_3^{(2)}} \frac{2k}{\Sigma} \left[ \Psi_{14}(\bar{\phi}_{44} - 1)e^{-2ikL} - (\Psi_{11} - 1)(\alpha_{11}\phi_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) \right. \right.
\]

\[
\left. - \Psi_{12}(\alpha_{12}\phi_{41} + \alpha_{22}\bar{\phi}_{42} + \alpha_{23}\bar{\phi}_{43}) - \Psi_{13}(\alpha_{13}\phi_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43}) \right] \right\}_- \, dk;
\]

Thus substituting Eqs. (105) and (109) into the third one of system (60), we can get Eq. (93). Similarly, we can also show Eqs. (94) and (95).

To use Eq. (63) to show Eq. (96) for \( v_{11}(t) \) we need to find these functions \( \phi_{14}^{(1)}(t, k) \) and \( \phi_{14}^{(2)}(t, k) \). Applying
the Cauchy’s theorem to Eq. (61), we have
\[ i\pi[\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)] = \int_{\partial D^0} \left[ \alpha_{11}(k\phi_{14}(t,k) - \phi_{14}(t,k)) + \alpha_{12}(k\phi_{24}(t,k) - \phi_{24}(t,k)) + \alpha_{13}(k\phi_{34}(t,k) - \phi_{34}(t,k)) \right]_0^k dx, \]
\[ = \int_{\partial D^0} \left\{ \alpha_{11} \left[ k\phi_{14}(t,k) - \phi_{14}(t,k) - \frac{2e^{2ikL}}{\Sigma} \right] \right\}_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \alpha_{12} \left[ k\phi_{24}(t,k) - \phi_{24}(t,k) - \frac{2e^{2ikL}}{\Sigma} \right] \right\}_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \alpha_{13} \left[ k\phi_{34}(t,k) - \phi_{34}(t,k) - \frac{2e^{2ikL}}{\Sigma} \right] \right\}_0^k dx + C_2(t), \]
where we have introduced the function \( C_2(t) \) as
\[ C_2(t) = \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(k\phi_{14}(t,k) - \phi_{14}(t,k)) + \alpha_{12}(k\phi_{24}(t,k) - \phi_{24}(t,k)) + \alpha_{13}(k\phi_{34}(t,k) - \phi_{34}(t,k)) \right] \right\}_0^k dx, \]
We use the global relation (83) to further reduce \( C_2(t) \) in the form
\[ C_2(t) = \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)) \right]_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)) \right]_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)) \right]_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)) \right]_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)) \right]_0^k dx, \]
\[ + \int_{\partial D^0} \left\{ \frac{2e^{2ikL}}{\Sigma} \left[ \alpha_{11}(\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)) \right]_0^k dx, \]
We need to further reduce \( C_2(t) \) by using the asymptotic (65) and the Cauchy’s theorem such that we have
\[ C_2(t) = -i\pi[\alpha_{11}(t) + \alpha_{12}(t) + \alpha_{13}(t)] - \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
\[ + \int_{\partial D^0} \frac{1}{2} \left[ \frac{1}{\Sigma} \right]_0^k dx, \]
38
It follows from Eqs. (110) and (112) that we have

\[
2i\pi \left[ \alpha_{11}\phi_{14}^{(2)} + \bar{\alpha}_{12}\phi_{24}^{(2)} + \alpha_{13}\phi_{34}^{(2)} \right] = -\int_{\partial D_3} \frac{\Sigma_+(k)}{\Sigma_-(k)} [\alpha_{11}(k\phi_{14} + iv_{01}) + \bar{\alpha}_{12}(k\phi_{24} + iv_{02}) + \alpha_{13}(k\phi_{34} + iv_{03})] dk
\]

\[
-\int_{\partial D_3} \frac{i}{2} \Psi_{14}^{(1)}(\alpha_{11}v_{01} + \bar{\alpha}_{12}v_{02} + \alpha_{13}v_{03}) dk + I_1(t),
\]

where \( I(t) \) is given by Eq. (97). Similarly, we can also the expressions of \( i\pi \left[ \alpha_{12}\phi_{14}^{(2)} + \alpha_{22}\phi_{24}^{(2)} + \bar{\alpha}_{23}\phi_{34}^{(2)} \right] \) and \( 2i\pi \left[ \alpha_{13}\phi_{14}^{(2)} + \alpha_{23}\phi_{24}^{(2)} + \alpha_{33}\phi_{34}^{(2)} \right] \) such that we can show that Eq. (96) holds.

(ii) We now deduce the Dirichlet boundary value problems (100a)-(100c) at \( x = 0 \) from the Neumann boundary value problems. It follows from the first one of Eq. (60) that \( u_{01}(t) \) can be expressed by means of \( \Psi_{14}^{(1)} \). Applying the Cauchy’s theorem to Eq. (58) yields

\[
i\pi \Psi_{14}^{(1)}(t) = \left( \int_{\partial D_1} + \int_{\partial D_3} \right) \Psi_{14}(t,k) dk = \int_{\partial D_3} \Psi_{14}-(t,k) dk
\]

\[
= \int_{\partial D_3} \left[ \Psi_{14}-(t,k) + \frac{2}{\Sigma_-(k)}(e^{-2ikL}\Psi_{14})_+ \right] dk + C_3(t)
\]

\[
= \int_{\partial D_3} \frac{\Sigma_+(k)}{\Sigma_-(k)} \Psi_{14}+ dk + C_3(t),
\]

where \( C_3(t) \) is defined by

\[
C_3(t) = -\int_{\partial D_3} \frac{2}{\Sigma_-(k)}(e^{-2ikL}\Psi_{14})_+ dk,
\]

By applying the global relation (68a), the Cauchy’s theorem and asymptotics (64a) to Eq. (115), we find

\[
C_3(t) = -\int_{\partial D_3} \frac{2}{\Sigma_-(k)}(e^{-2ikL}\Psi_{14})_+ dk
\]

\[
= \int_{\partial D_3} \frac{2}{\Sigma_-(k)} [-c_{14}e^{-2ikL} - (\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})]_+ dk
\]

\[
+ \int_{\partial D_3} \frac{2}{\Sigma_-(k)} \left[ \Psi_{14}(\bar{\phi}_{44} - 1)e^{-2ikL} - [(\Psi_{11} - 1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) + \Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \alpha_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})]_+ dk,
\]

\[
= -i\pi \Psi_{14}^{(1)} - \int_{\partial D_3} \frac{2}{\Sigma_-(k)}(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43})_+ dk
\]

\[
+ \int_{\partial D_3} \frac{2}{\Sigma_-(k)} \left[ \Psi_{14}(\bar{\phi}_{44} - 1)e^{-2ikL} - [(\Psi_{11} - 1)(\alpha_{11}\bar{\phi}_{41} + \bar{\alpha}_{12}\bar{\phi}_{42} + \bar{\alpha}_{13}\bar{\phi}_{43}) + \Psi_{12}(\alpha_{12}\bar{\phi}_{41} + \alpha_{22}\bar{\phi}_{42} + \alpha_{23}\bar{\phi}_{43}) + \Psi_{13}(\alpha_{13}\bar{\phi}_{41} + \alpha_{23}\bar{\phi}_{42} + \alpha_{33}\bar{\phi}_{43})]_+ dk,
\]

39
Eqs. (114) and (116) imply that

\[
2\pi\Psi_1^{(1)}(t) = \int_{\partial D_0^+} \left[ \sum_{+}(k) \Psi_{14+} - \frac{2}{\sum_-}(\alpha_{11}\tilde{\phi}_{41} + \tilde{\phi}_{42} + \tilde{\alpha}_{13}\phi_{43}) \right] dk
\]

\[
+ \int_{\partial D_0^+} \frac{2}{\sum_-} \left\{ \Psi_{14}(\tilde{\phi}_{44} - 1) e^{-2\pi kl} - \left[ (\Psi_1 - 1)(\alpha_{11}\tilde{\phi}_{41} + \tilde{\alpha}_{12}\phi_{42} + \tilde{\alpha}_{13}\phi_{43}) \right] + \Psi_{12}(\alpha_{12}\tilde{\phi}_{41} + \alpha_{22}\phi_{42} + \alpha_{23}\phi_{43}) + \Psi_{13}(\alpha_{13}\tilde{\phi}_{41} + \alpha_{32}\phi_{42} + \alpha_{33}\phi_{43}) \right\} dk,
\]

(117)

Thus, substituting Eq. (117) into the first one of Eq. (60) yields Eq. (100a). Similarly, by applying the expressions of \(\Psi_2^{(1)}(t)\) and \(\Psi_4^{(1)}(t)\) to the second one of Eq. (60), we can find Eqs. (100b) and (100c).

Similarly we also can show that the Dirichlet boundary value problems (101) at \(x = L\) hold from the Neumann boundary value problems. □

4.3. The effective characterizations

Substituting the perturbed expressions for eigenfunctions and initial boundary conditions

\[
\begin{align*}
\Psi_{ij}(t, k) &= \Psi^{[0]}_{ij}(t, k) + \epsilon \Psi^{[1]}_{ij}(t, k) + \epsilon^2 \Psi^{[2]}_{ij}(t, k) + \cdots, \quad i, j = 1, 2, 3, 4, \\
\phi_{ij}(t, k) &= \phi^{[0]}_{ij}(t, k) + \epsilon \phi^{[1]}_{ij}(t, k) + \epsilon^2 \phi^{[2]}_{ij}(t, k) + \cdots, \quad i, j = 1, 2, 3, 4, \\
u_{sij}(t) &= \epsilon^s u^{[1]}_{sij}(t) + \epsilon^s u^{[2]}_{sij}(t) + \cdots, \quad s = 0, 1; j = 1, 2, 3, \\
v_{sij}(t) &= \epsilon^s v^{[1]}_{sij}(t) + \epsilon^s v^{[2]}_{sij}(t) + \cdots, \quad s = 0, 1; j = 1, 2, 3,
\end{align*}
\]

(118)

into Eqs. (70)-(73), where \(\epsilon > 0\) is a small parameter, we have these terms of \(O(1)\), and \(O(\epsilon)\) as

\[
O(1): \begin{cases} 
\Psi^{[0]}_{jj} = 1, & j = 1, 2, 3, 4, \\
\Psi^{[0]}_{ij} = 0, & i, j = 1, 2, 3, 4, i \neq j,
\end{cases}
\]

(119)

\[
O(\epsilon): \begin{cases} 
\Psi^{[1]}_{ss} = \Psi^{[1]}_{44} = 0, & j, s, = 1, 2, 3, \\
\Psi^{[1]}_{sj} = \int_0^t e^{-4\pi k(t-t')} \left( 2k\bar{u}_{0j} + i\bar{u}_{1j} \right) (t')dt', \quad j = 1, 2, 3, \\
\Psi^{[1]}_{4j} = \int_0^t e^{-4\pi k(t-t')} \left[ \alpha_{11} \left( 2k\bar{u}_{01} - i\bar{u}_{11} \right) + \alpha_{12} \left( 2k\bar{u}_{02} - i\bar{u}_{12} \right) + \alpha_{13} \left( 2k\bar{u}_{03} - i\bar{u}_{13} \right) \right] (t')dt', \\
\Psi^{[1]}_{41} = \int_0^t e^{-4\pi k(t-t')} \left[ \alpha_{11} \left( 2k\bar{u}_{01} - i\bar{u}_{11} \right) + \alpha_{12} \left( 2k\bar{u}_{02} - i\bar{u}_{12} \right) + \alpha_{23} \left( 2k\bar{u}_{03} - i\bar{u}_{13} \right) \right] (t')dt', \\
\Psi^{[1]}_{42} = \int_0^t e^{-4\pi k(t-t')} \left[ \alpha_{11} \left( 2k\bar{u}_{01} - i\bar{u}_{11} \right) + \alpha_{12} \left( 2k\bar{u}_{02} - i\bar{u}_{12} \right) + \alpha_{23} \left( 2k\bar{u}_{03} - i\bar{u}_{13} \right) \right] (t')dt', \\
\Psi^{[1]}_{43} = \int_0^t e^{-4\pi k(t-t')} \left[ \alpha_{11} \left( 2k\bar{u}_{01} - i\bar{u}_{11} \right) + \alpha_{12} \left( 2k\bar{u}_{02} - i\bar{u}_{12} \right) + \alpha_{33} \left( 2k\bar{u}_{03} - i\bar{u}_{13} \right) \right] (t')dt', \end{cases}
\]

(120)

Similarly, we can also obtain the analogous expressions for \(\phi^{[l]}_{ij} \) and \(v^{[l]}_{ij} \) for \(l = 0, 1\) by means of the boundary values at \(x = L\), that is, \(\{v^{[l]}_{ij}\}_i, l = 0, 1; j = 1, 2, 3; l = 0, 1\).
If we assume that $m_{44}(S)$ has no zeros, then we expand Eqs. (93)-(96) to have

$$
u_{11}^{[n]}(t) = \int_{\partial D_3} \left\{ \frac{2\Sigma^+}{i\pi\Sigma^-} (k\Psi_{14}^{[n]} + i\nu_{01}^{[n]}) dk + \frac{4}{\pi\Sigma^-} \left[ \alpha_{11} \left( -ik\phi_{41}^{[n]} + \alpha_{11}\nu_{01}^{[n]} + \alpha_{12}\nu_{02}^{[n]} + \alpha_{13}\nu_{03}^{[n]} \right) \right. \\
+ \tilde{\alpha}_{12} \left( -ik\phi_{42}^{[n]} + \tilde{\alpha}_{12}\nu_{01}^{[n]} + \alpha_{22}\nu_{02}^{[n]} + \alpha_{23}\nu_{03}^{[n]} \right) \\
+ \tilde{\alpha}_{13} \left( -ik\phi_{43}^{[n]} + \tilde{\alpha}_{13}\nu_{01}^{[n]} + \alpha_{23}\nu_{02}^{[n]} + \alpha_{33}\nu_{03}^{[n]} \right) \right\} \ dt + \text{lower order terms},
$$

(121a)

$$
u_{12}^{[n]}(t) = \int_{\partial D_3} \left\{ \frac{2\Sigma^+}{i\pi\Sigma^-} (k\Psi_{24}^{[n]} + i\nu_{02}^{[n]}) dk + \frac{4}{\pi\Sigma^-} \left[ \alpha_{12} \left( -ik\phi_{41}^{[n]} + \alpha_{11}\nu_{01}^{[n]} + \alpha_{12}\nu_{02}^{[n]} + \alpha_{13}\nu_{03}^{[n]} \right) \right. \\
+ \tilde{\alpha}_{22} \left( -ik\phi_{42}^{[n]} + \tilde{\alpha}_{12}\nu_{01}^{[n]} + \alpha_{22}\nu_{02}^{[n]} + \alpha_{23}\nu_{03}^{[n]} \right) \\
+ \tilde{\alpha}_{23} \left( -ik\phi_{43}^{[n]} + \tilde{\alpha}_{13}\nu_{01}^{[n]} + \alpha_{23}\nu_{02}^{[n]} + \alpha_{33}\nu_{03}^{[n]} \right) \right\} \ dt + \text{lower order terms},
$$

(121b)

$$
u_{13}^{[n]}(t) = \int_{\partial D_3} \left\{ \frac{2\Sigma^+}{i\pi\Sigma^-} (k\Psi_{34}^{[n]} + i\nu_{03}^{[n]}) dk + \frac{4}{\pi\Sigma^-} \left[ \alpha_{13} \left( -ik\phi_{41}^{[n]} + \alpha_{11}\nu_{01}^{[n]} + \alpha_{12}\nu_{02}^{[n]} + \alpha_{13}\nu_{03}^{[n]} \right) \right. \\
+ \tilde{\alpha}_{23} \left( -ik\phi_{42}^{[n]} + \tilde{\alpha}_{12}\nu_{01}^{[n]} + \alpha_{22}\nu_{02}^{[n]} + \alpha_{23}\nu_{03}^{[n]} \right) \\
+ \tilde{\alpha}_{33} \left( -ik\phi_{43}^{[n]} + \tilde{\alpha}_{13}\nu_{01}^{[n]} + \alpha_{23}\nu_{02}^{[n]} + \alpha_{33}\nu_{03}^{[n]} \right) \right\} \ dt + \text{lower order terms},
$$

(121c)

where ‘lower order terms’ stands for the result involving known terms of lower order.

The terms of $O(e^n)$ in Eqs. (89)-(92) and the similar equations for $\phi_{ij}$ yield

$$
\left\{ \Psi_{14}^{[n]}(t, \bar{k}) = \int_0^t e^{-4ik^2(t-\tau') \left( 2\bm{k}\cdot\nu_{0j}^{[n]} + i\nu_{1j}^{[n]} \right)} (\tau') d\tau' + \text{lower order terms,} \quad j = 1, 2, 3, \\
\phi_{41}^{[n]}(t, \bar{k}) = \int_0^t e^{-4ik^2(t-\tau') \left( \alpha_{11}\nu_{01}^{[n]} + i\nu_{11}^{[n]} + \alpha_{12}\nu_{02}^{[n]} + i\nu_{12}^{[n]} + \alpha_{13}\nu_{03}^{[n]} + i\nu_{13}^{[n]} \right)} (\tau') d\tau' + \text{lower order terms},

(122)
$$

$$
\phi_{42}^{[n]}(t, \bar{k}) = \int_0^t e^{-4ik^2(t-\tau') \left( \tilde{\alpha}_{12}\nu_{01}^{[n]} + \tilde{\alpha}_{12}\nu_{01}^{[n]} + \alpha_{22}\nu_{02}^{[n]} + i\nu_{12}^{[n]} + \alpha_{23}\nu_{03}^{[n]} + i\nu_{13}^{[n]} \right)} (\tau') d\tau' + \text{lower order terms},
$$

$$
\phi_{43}^{[n]}(t, \bar{k}) = \int_0^t e^{-4ik^2(t-\tau') \left( \tilde{\alpha}_{13}\nu_{01}^{[n]} + \tilde{\alpha}_{13}\nu_{01}^{[n]} + \alpha_{23}\nu_{02}^{[n]} + i\nu_{13}^{[n]} \right)} (\tau') d\tau' + \text{lower order terms},
$$

+lower order terms,
which leads to

\[ \Psi_{j4-}(t, k) = 4k \int_0^t e^{-4ik^2(t-t')} u_{0j}^{[n]}(t') dt' + \text{lower order terms}, \quad j = 1, 2, 3, \]

\[ \phi_{41-}(t, k) = 4k \int_0^t e^{-4ik^2(t-t')} \left( \alpha_{11} v_{10}^{[n]} + \alpha_{12} v_{12}^{[n]} + \alpha_{13} v_{13}^{[n]} \right)(t') dt' + \text{lower order terms}, \]

\[ \phi_{42-}(t, k) = 4k \int_0^t e^{-4ik^2(t-t')} \left( \alpha_{12} v_{10}^{[n]} + \alpha_{22} v_{20}^{[n]} + \alpha_{23} v_{23}^{[n]} \right)(t') dt' + \text{lower order terms}, \]

\[ \phi_{43-}(t, k) = 4k \int_0^t e^{-4ik^2(t-t')} \left( \alpha_{13} v_{10}^{[n]} + \alpha_{23} v_{20}^{[n]} + \alpha_{33} v_{30}^{[n]} \right)(t') dt' + \text{lower order terms}, \]  

(123)

It follows from system (123) that \( \Psi_{1j-}^{[n]} \) and \( \phi_{4j-}^{[n]} \), \( j = 1, 2, 3 \) can be generated at each step from the known Dirichlet boundary data \( u_{0j}^{[n]} \) and \( v_{0j}^{[n]} \) such that we know that the Neumann boundary data \( u_{1j}^{[n]} \) can then be given by Eqs. (121a)-(121c). Similarly, we also show that the Neumann boundary data \( v_{1j}^{[n]} \) can then be determined by the known Dirichlet boundary data \( u_{0j}^{[n]} \) and \( v_{0j}^{[n]} \).

Similarly, the substitution of Eq. (118) into Eqs. (100a) and (100b) yields the terms of \( O(e^n) \) as

\[ u_{01}^{[n]}(t) = \int_{\partial D_0^0} \left[ \frac{\Sigma_+}{\pi \Sigma_-} \Psi_{14+}^{[n]} - \frac{2}{\pi \Sigma_-} \left( \alpha_{11} \tilde{v}_{11}^{[n]} + \tilde{\alpha}_{12} \tilde{v}_{12}^{[n]} + \tilde{\alpha}_{13} \tilde{v}_{13}^{[n]} \right) \right] dk + \text{lower order terms}, \quad (124a) \]

\[ u_{02}^{[n]}(t) = \int_{\partial D_0^0} \left[ \frac{\Sigma_+}{\pi \Sigma_-} \Psi_{24+}^{[n]} - \frac{2}{\pi \Sigma_-} \left( \alpha_{12} \tilde{v}_{12}^{[n]} + \tilde{\alpha}_{22} \tilde{v}_{22}^{[n]} + \tilde{\alpha}_{23} \tilde{v}_{23}^{[n]} \right) \right] dk + \text{lower order terms}, \quad (124b) \]

\[ u_{03}^{[n]}(t) = \int_{\partial D_0^0} \left[ \frac{\Sigma_+}{\pi \Sigma_-} \Psi_{34+}^{[n]} - \frac{2}{\pi \Sigma_-} \left( \alpha_{13} \tilde{v}_{13}^{[n]} + \tilde{\alpha}_{23} \tilde{v}_{23}^{[n]} + \tilde{\alpha}_{33} \tilde{v}_{33}^{[n]} \right) \right] dk + \text{lower order terms}, \quad (124c) \]

Eq. (122) implies that

\[ \Psi_{j4+}^{[n]}(t, k) = 2i \int_0^t e^{-4ik^2(t-t')} u_{1j}^{[n]}(t') dt' + \text{lower order terms}, \quad j = 1, 2, 3, \]

\[ \tilde{\phi}_{41+}^{[n]}(t, k) = \int_0^t e^{-4ik^2(t-t')} \left( \alpha_{11} \tilde{v}_{11}^{[n]} + \alpha_{12} \tilde{v}_{12}^{[n]} + \alpha_{13} \tilde{v}_{13}^{[n]} \right)(t') dt' + \text{lower order terms}, \]

\[ \tilde{\phi}_{42+}^{[n]}(t, k) = \int_0^t e^{-4ik^2(t-t')} \left( \alpha_{12} \tilde{v}_{12}^{[n]} + \alpha_{22} \tilde{v}_{22}^{[n]} + \alpha_{23} \tilde{v}_{23}^{[n]} \right)(t') dt' + \text{lower order terms}, \]

\[ \tilde{\phi}_{43+}^{[n]}(t, k) = \int_0^t e^{-4ik^2(t-t')} \left( \alpha_{13} \tilde{v}_{13}^{[n]} + \alpha_{23} \tilde{v}_{23}^{[n]} + \alpha_{33} \tilde{v}_{33}^{[n]} \right)(t') dt' + \text{lower order terms}, \]  

(125)

It follows from system (125) that \( \Psi_{j4+}^{[n]} \) and \( \phi_{4j+}^{[n]} \), \( j = 1, 2, 3 \) can be generated at each step from the known Neumann boundary data \( u_{1j}^{[n]} \) and \( v_{1j}^{[n]} \) such that we know that the Dirichlet boundary data \( u_{0j}^{[n]} \) can then be given by Eqs. (124a)-(124c). Similarly, we also show that the Dirichlet boundary data \( v_{0j}^{[n]} \) can then be determined by the known Neumann boundary data \( u_{1j}^{[n]} \) and \( v_{1j}^{[n]} \).
4.4. The large $L$ limit from the interval to the half-line

The formulae for the initial and boundary value conditions $u_{0j}(t)$ and $u_{1j}(t), j = 1, 2, 3$ of Theorem 4.2 in the limit $L \to \infty$ can reduce to the corresponding ones on the half-line. Since when $L \to \infty$,

$$v_{0j} \to 0, \quad v_{1j} \to 0, \quad j = 1, 2, 3, \quad \phi_{ij} \to \delta_{ij}, \quad \frac{\Sigma_{+}(k)}{\Sigma_{-}(k)} \to 1 \quad \text{as} \quad k \to \infty \quad \text{in} \quad D_3, \quad (126)$$

Thus, according to Eq. (126), the $L \to \infty$ limits of Eqs. (93), (94), (100a), and (100b) yield the unknown Neumann boundary data

$$u_{1j}(t) = \frac{2}{\pi} \int_{\partial D_3^0} u_{0j}(\Psi_{44} - 1) - i k \Psi_{j4} -] dk, \quad j = 1, 2, 3, \quad (127)$$

for the given Dirichlet boundary problem, and the unknown Dirichlet boundary data

$$u_{0j}(t) = \frac{1}{\pi} \int_{\partial D_3^0} \Psi_{j4} + dk, \quad j = 1, 2, 3, \quad (128)$$

for the given Neumann boundary problem.

5 The GLM representation and equivalence

In this section we deduce the eigenfunctions $\Psi(t, k)$ and $\phi(t, k)$ in terms of the Gel’fand-Levitan-Marchenko (GLM) approach [25–28]. Moreover, the global relation can be used to find the unknown Neumann (Dirichlet) boundary values from the given Dirichlet (Neumann) boundary values by means of the GLM representations. Moreover, the GLM representations are shown to be equivalent to the ones obtained in Sec. 4. Finally, the linearizable boundary conditions are presented for the GLM representations.

5.1. The GLM representation

**Proposition 5.1.** The eigenfunctions $\Psi(t, k)$ and $\phi(t, k)$ possess the GLM representation

$$\Psi(t, k) = I + \int_{-t}^{t} \left[ L(t, s) + \left( k + \frac{i}{2} U^{(0)} \sigma_4 \right) G(t, s) \right] e^{-2ik(s-t)\sigma_4} ds, \quad (129a)$$

$$\phi(t, k) = I + \int_{-t}^{t} \left[ L(t, s) + \left( k + \frac{i}{2} U^{(L)} \sigma_4 \right) G(t, s) \right] e^{-2ik(s-t)\sigma_4} ds, \quad (129b)$$

where the $4 \times 4$ matrix-valued functions $L(t, s) = (L_{ij})_{4 \times 4}$ and $G(t, s) = (G_{ij})_{4 \times 4}, -t \leq s \leq t$ satisfy a Goursat system

$$\begin{cases}
L_t(t, s) + \sigma_4 L_s(t, s) \sigma_4 = i\sigma_4 U_x^{(0)} L(t, s) + \frac{1}{2} \left[ (U^{(0)})^3 + i\hat{U}^{(0)} \sigma_4 + [U_x^{(0)}, U^{(0)}] \right] G(t, s), \\
G_t(t, s) + \sigma_4 G_s(t, s) \sigma_4 = 2U^{(0)} L(t, s) + i\sigma_4 U_x^{(0)} G(t, s),
\end{cases} \quad (130)$$

43
with the initial conditions

\[
\begin{aligned}
L_{ij}(t, -t) &= L_{44}(t, -t) = 0, \quad l, j = 1, 2, 3, \\
G_{ij}(t, -t) &= G_{44}(t, -t) = 0, \quad l, j = 1, 2, 3, \\
G_{14}(t, t) &= u_0(t), \quad G_{24}(t, t) = u_{02}(t), \quad G_{34}(t, t) = u_{03}(t), \\
G_{41}(t, t) &= \alpha_{11}\bar{u}_{01}(t) + \bar{\alpha}_{12}\bar{u}_{02}(t) + \bar{\alpha}_{13}\bar{u}_{03}(t), \\
G_{42}(t, t) &= \alpha_{12}\bar{u}_{01}(t) + \alpha_{22}\bar{u}_{02}(t) + \bar{\alpha}_{23}\bar{u}_{03}(t), \\
G_{43}(t, t) &= \alpha_{13}\bar{u}_{01}(t) + \alpha_{23}\bar{u}_{02}(t) + \bar{\alpha}_{33}\bar{u}_{03}(t), \\
L_{14}(t, t) &= \frac{i}{2}u_{11}(t), \quad L_{24}(t, t) = \frac{i}{2}u_{12}(t), \quad L_{34}(t, t) = \frac{i}{2}u_{13}(t), \\
L_{41}(t, t) &= -\frac{i}{2}(\alpha_{11}\bar{u}_{11}(t) + \bar{\alpha}_{12}\bar{u}_{12}(t) + \bar{\alpha}_{13}\bar{u}_{13}(t)), \\
L_{42}(t, t) &= -\frac{i}{2}(\alpha_{12}\bar{u}_{11}(t) + \alpha_{22}\bar{u}_{12}(t) + \bar{\alpha}_{23}\bar{u}_{13}(t)), \\
L_{43}(t, t) &= -\frac{i}{2}(\alpha_{13}\bar{u}_{11}(t) + \alpha_{23}\bar{u}_{12}(t) + \alpha_{33}\bar{u}_{13}(t)),
\end{aligned}
\]

(131)

\[U(0) = \begin{pmatrix}
0 & 0 & 0 & u_{01}(t) \\
0 & 0 & 0 & u_{02}(t) \\
0 & 0 & 0 & u_{03}(t) \\
p_{01}(t) & p_{02}(t) & p_{03}(t) & 0
\end{pmatrix}, \quad U_x(0) = \begin{pmatrix}
0 & 0 & 0 & u_{11}(t) \\
0 & 0 & 0 & u_{12}(t) \\
0 & 0 & 0 & u_{13}(t) \\
p_{11}(t) & p_{12}(t) & p_{13}(t) & 0
\end{pmatrix}, \quad (132)
\]

with

\[
p_{01} = \alpha_{11}\bar{u}_{01} + \bar{\alpha}_{12}\bar{u}_{02} + \bar{\alpha}_{13}\bar{u}_{03}, \quad p_{02} = \alpha_{12}\bar{u}_{01} + \alpha_{22}\bar{u}_{02} + \bar{\alpha}_{23}\bar{u}_{03}, \quad p_{03} = \alpha_{13}\bar{u}_{01} + \alpha_{23}\bar{u}_{02} + \alpha_{33}\bar{u}_{03},
\]

\[
p_{11} = \alpha_{11}\bar{u}_{11} + \bar{\alpha}_{12}\bar{u}_{12} + \bar{\alpha}_{13}\bar{u}_{13}, \quad p_{12} = \alpha_{12}\bar{u}_{11} + \alpha_{22}\bar{u}_{12} + \bar{\alpha}_{23}\bar{u}_{13}, \quad p_{13} = \alpha_{13}\bar{u}_{11} + \alpha_{23}\bar{u}_{12} + \alpha_{33}\bar{u}_{13},
\]

Similarly, \( \mathcal{L}(t, s), \mathcal{G}(t, s) \) satisfy the similar Eqs. (130) and (131) with \( u_{0j} \rightarrow v_{0j}, \ u_{1j} \rightarrow v_{1j}, \ U(0) \rightarrow U^{(L)} = U^{(0)} \big|_{u_{0j} \rightarrow v_{0j}}, \ U_x^{(0)} \rightarrow U_x^{(L)} = U_x^{(0)} \big|_{u_{1j} \rightarrow v_{1j}}. \)

**Proof.** We assume that the function

\[
\psi(t, k) = e^{-2ik^2\sigma_4} + \int_{-t}^{t} \left[ L_0(t, s) + kG(t, s) \right] e^{-2ik^2\sigma_4} ds, \quad (133)
\]

satisfies the time-part of Lax pair (4) with the boundary data \( \psi(0, k) = 1 \) at \( x = 0 \), where \( L_0(t, s) \) and \( G(t, s) \) are the unknown \( 4 \times 4 \) matrix-valued functions. We substitute Eq. (133) into the time-part of Lax pair (4) with the boundary data (3) and use the identity

\[
\int_{-t}^{t} F(t, s) e^{-2ik^2\sigma_4} ds = \frac{i}{2\kappa^2} \left[ F(t, t) e^{-2ik^2\sigma_4} - F(t, -t) e^{2ik^2\sigma_4} - \int_{-t}^{t} F_s(t, s) e^{-2ik^2\sigma_4} ds \right] \sigma_4, \quad (134)
\]
where the function $F(t, s)$ is a $4 \times 4$ matrix-valued function. As a consequence, we find

\[
\begin{align*}
L_0(t, -t) + \sigma_4 L_0(t, -t) \sigma_4 &= -iU^{(0)} G(t, -t) \sigma_4, \\
G(t, -t) + \sigma_4 G(t, -t) \sigma_4 &= 0, \\
L_0(t, t) - \sigma_4 L_0(t) \sigma_4 &= iU^{(0)} G(t, t) \sigma_4 + V_0^{(0)}, \\
G(t, t) - \sigma_4 G(t, t) \sigma_4 &= 2U^{(0)}, \\
L_0(t, s) + \sigma_4 L_0(t, s) \sigma_4 &= -iU^{(0)} G_s(t, s) \sigma_4 + V_0^{(0)} L_0(t, s), \\
G_s(t, s) + \sigma_4 G_s(t, s) \sigma_4 &= 2U^{(0)} L_0(t, s) + V_0^{(0)} G(t, s),
\end{align*}
\]

(135)

where $U^{(0)}$ is given by Eq. (132) and

\[
V_0^{(0)} = -i(U_x^{(0)} + U^{(0)}^2) \sigma_4 = -i \begin{pmatrix}
\frac{u_{01p01}}{u_{03p01}} & \frac{u_{01p02}}{u_{03p02}} & \frac{u_{01p03}}{u_{03p03}} & -u_{11} \\
\frac{u_{02p01}}{u_{03p01}} & \frac{u_{02p02}}{u_{03p02}} & \frac{u_{02p03}}{u_{03p03}} & -u_{21} \\
\frac{u_{03p01}}{u_{03p01}} & \frac{u_{03p02}}{u_{03p02}} & \frac{u_{03p03}}{u_{03p03}} & -u_{31} \\
p_{11} & p_{21} & p_{31} & -(u_{01p01} + u_{02p02} + u_{03p03})
\end{pmatrix},
\]

To reduce system (135) we further introduce the new matrix $L(t, s)$ by

\[
L(t, s) = L_0(t, s) - \frac{i}{2} U^{(0)} \sigma_4 G(t, s),
\]

(136)
such that the first four equations of system (135) become

\[
\begin{align*}
L(t, -t) + \sigma_4 L(t, -t) \sigma_4 &= 0, \\
G(t, -t) + \sigma_4 G(t, -t) \sigma_4 &= 0, \\
L(t, t) - \sigma_4 L(t) \sigma_4 &= V_0^{(0)}, \\
G(t, t) - \sigma_4 G(t, t) \sigma_4 &= 2U^{(0)},
\end{align*}
\]

which leads to Eq. (131), and from the last two equations of system (135) we have Eq. (130). By means of transformation (7), that is, $\mu_2(0, t, k) = \Psi(t, k) = \psi(t, k)e^{2ikx^2t},$ we know that $\Psi(t, k)$ is given by Eq. (129a). Similarly, we can also show that Eq. (129b) holds. $\Box$

For convenience, we rewrite a $4 \times 4$ matrix $C = (C_{ij})_{4 \times 4}$ as

\[
C = (C_{ij})_{4 \times 4} = \begin{pmatrix}
\tilde{C}_{3 \times 3} & \tilde{C}_{j4} \\
\tilde{C}_{4j} & \tilde{C}_{44}
\end{pmatrix}, \quad \tilde{C}_{3 \times 3} = (C_{ij})_{3 \times 3}, \quad \tilde{C}_{j4} = (C_{14}, C_{24}, C_{34})^T, \quad \tilde{C}_{4j} = (C_{41}, C_{42}, C_{43}),
\]

The Dirichlet and Neumann boundary values at $x = 0, L$ are simply written as

\[
\begin{align*}
u_j(t) &= (u_{j1}(t), u_{j2}(t), u_{j3}(t)), & v_j(t) &= (v_{j1}(t), v_{j2}(t), v_{j3}(t)), & j = 1, 2, 3, \\
w_{j0}(t) &= (p_{j1}(t), p_{j2}(t), p_{j3}(t)), & w_{jL}(t) &= (p_{j1}(t), p_{j2}(t), p_{j3}(t))|_{u_{j1} \to v_{j1}, s = 0, j = 1, 2, 3},
\end{align*}
\]

(137)

For a matrix-valued function $F(t, s)$, we introduce the $\hat{F}(t, k)$ by

\[
\hat{F}(t, k) = \int_{-t}^{t} F(t, s)e^{2ik^2(s-t)} ds,
\]

45
Thus, the GLM expressions (129a) and (129b) of \( \{ \Psi_{ij}, \phi_{ij} \} \) can be rewritten as

\[
\begin{align*}
\Psi_{3 \times 3}(t, k) &= \mathbb{I} + \hat{L}_{3 \times 3} - \frac{i}{2} \bar{v}_0^T(t) \hat{G}_{44} + k \hat{G}_{3 \times 3}, \\
\tilde{\Psi}_{j4}(t, k) &= \hat{L}_{j4} - \frac{i}{2} \bar{u}_0^T(t) \hat{G}_{44} + k \hat{G}_{j4}, \quad j = 1, 2, 3, \\
\tilde{\Psi}_{4j}(t, k) &= L_{4j} + \frac{i}{2} \bar{u}_0(t) \hat{M}_{G_{3 \times 3}}^{\wedge} + k \hat{G}_{44}, \quad j = 1, 2, 3, \\
\tilde{\Psi}_{44}(t, k) &= 1 + \hat{L}_{44} + \frac{i}{2} \bar{u}_0(t) \hat{M}_{G_{j4}}^{\wedge} + k \hat{G}_{44},
\end{align*}
\]  
(138a)

\[
\begin{align*}
\phi_{3 \times 3}(t, k) &= \mathbb{I} + \hat{L}_{3 \times 3} - \frac{i}{2} \bar{v}_0^T(t) \hat{G}_{j4} + k \hat{G}_{3 \times 3}, \\
\tilde{\phi}_{j4}(t, k) &= \hat{L}_{j4} - \frac{i}{2} \bar{v}_0^T(t) \hat{G}_{j4} + k \hat{G}_{j4}, \quad j = 1, 2, 3, \\
\tilde{\phi}_{4j}(t, k) &= \hat{L}_{4j} + \frac{i}{2} \bar{v}_0(t) \hat{M}_{G_{j4}}^{\wedge} + k \hat{G}_{44}, \quad j = 1, 2, 3, \\
\tilde{\phi}_{44}(t, k) &= 1 + \hat{L}_{44} + \frac{i}{2} \bar{v}_0(t) \hat{M}_{G_{j4}}^{\wedge} + k \hat{G}_{44},
\end{align*}
\]  
(138b)

For the given Eqs. (138a) and (138b) we have the following proposition.

**Proposition 5.2.**

\[
\begin{align*}
\lim_{t' \to t} \int_{\partial D^0} \frac{k e^{4ik^2(t-t')}}{\Sigma_-} \left( \tilde{F}_{j4} e^{-2ikL} \right) dk &= \int_{\partial D^0} \left[ \frac{i k}{2} \bar{u}_0^T \left( \hat{G}_{44} - \bar{G}_{44} \right) + \frac{k}{\Sigma_-} \left( \tilde{F}_{44} e^{-2ikL} \right) \right] dk, \\
\lim_{t' \to t} \int_{\partial D^0} \frac{k e^{4ik^2(t-t')}}{\Sigma_-} \tilde{F}_{4j} dk &= \int_{\partial D^0} \left[ \frac{i k^2}{2} \bar{M}^T \bar{v}_0 \left( \hat{G}_{j4} - \bar{G}_{j4} \right) + \frac{k}{\Sigma_-} \tilde{F}_{4j} \right] dk, \\
\lim_{t' \to t} \int_{\partial D^0} \frac{e^{4ik^2(t-t')}}{\Sigma_-} \tilde{F}_{4j+} dk &= \int_{\partial D^0} \left( \tilde{F}_{4j+} \right) dk,
\end{align*}
\]  
(139a, 139b, 139c)

where the vector-valued functions \( \tilde{F}_{j4}(t, k) \) and \( \tilde{F}_{4j}(t, k) \) \( (j = 1, 2, 3) \) are defined by

\[
\begin{align*}
\tilde{F}_{j4}(t, k) &= -\frac{i}{2} \bar{u}_0^T \hat{G}_{44} + \frac{i}{2} \bar{M}^T \hat{G}_{3 \times 3} \bar{v}_0 e^{2ikL} \\
&\quad + \left( \hat{L}_{j4} - \frac{i}{2} \bar{v}_0^T \hat{G}_{44} + k \hat{G}_{j4} \right) \left( \hat{G}_{44} - \frac{i}{2} \bar{G}_{j4} \bar{v}_0^T + k \hat{G}_{44} \right) \\
&\quad - \left( \hat{L}_{3 \times 3} - \frac{i}{2} \bar{u}_0^T \hat{G}_{4j} + k \hat{G}_{3 \times 3} \right) \bar{M}^T \left( \bar{G}_{j4} - \frac{i}{2} \bar{G}_{3 \times 3} \bar{v}_0^T + k \hat{G}_{4j} \right) e^{2ikL},
\end{align*}
\]  
(140)

\[
\begin{align*}
\tilde{F}_{4j}(t, k) &= -\frac{i}{2} \bar{v}_0^T \hat{G}_{44} + \bar{v}_0^T \hat{G}_{3 \times 3} \bar{v}_0 e^{2ikL} \\
&\quad - \bar{M}^T \left( \hat{L}_{3 \times 3} - \frac{i}{2} \bar{v}_0^T \hat{G}_{4j} + k \hat{G}_{3 \times 3} \right) \bar{M}^T \left( \bar{G}_{j4} - \frac{i}{2} \bar{G}_{3 \times 3} \bar{v}_0^T + k \hat{G}_{4j} \right) e^{2ikL},
\end{align*}
\]  
(141)
Thus we take the limit $t'$ to $t$ and integrate along $\partial D^1_t$ with respect to $dk$ to yield

$$
\int_{\partial D^1_t} k^2 e^{ik^2(t-t')} \left( \hat{F}_{34} e^{-2ikL} \right)_{tk} dk = \int_{\partial D^1_t} \frac{ik}{2} e^{ik^2(t-t')} u_0^T \tilde{\hat{G}}_{44} dk - \int_{\partial D^1_t} k^3 e^{ik^2(t-t')} \tilde{\hat{G}}_{34} \tilde{\hat{G}}_{44} dk
$$

Similarly, we multiply Eq. (140) by $k$ and integrate along $\partial D^1_t$ with respect to $dk$ to yield

$$
\int_{\partial D^1_t} k e^{ik^2(t-t')} \left( \hat{F}_{34} e^{-2ikL} \right)_{tk} dk = \int_{\partial D^1_t} \frac{ik}{2} e^{ik^2(t-t')} u_0^T \tilde{\hat{G}}_{44} dk - \int_{\partial D^1_t} k^3 e^{ik^2(t-t')} \tilde{\hat{G}}_{34} \tilde{\hat{G}}_{44} dk
$$

To further analyse the above equation, the following identities are introduced

$$
\int_{\partial D^1_t} k e^{ik^2(t-t')} \hat{F}(t, k) dk = \begin{cases} \frac{\pi}{2} F(t, 2t' - t), & 0 < t' < t, \\ \frac{\pi}{4} F(t, t), & 0 < t' = t, \end{cases}
$$

which also holds for the cases that $\frac{k^2}{\Sigma_t}$ is taken place by $\frac{k^2}{\Sigma_t}$ or $k^2$.

It follows from the first integral on the RHS of Eq. (142) and Eq. (143) that we have

$$
\lim_{t' \to t} \int_{\partial D^1_t} \frac{ik}{2} e^{ik^2(t-t')} u_0^T \tilde{\hat{G}}_{44} dk = \lim_{t' \to t} \int_{\partial D^1_t} \frac{ik}{2} u_0^T \tilde{\hat{G}}_{44} dk = \int_{\partial D^1_t} \frac{ik}{2} u_0^T \tilde{\hat{G}}_{44} dk = \frac{\pi}{8} u_0^T \tilde{\hat{G}}_{44} (t, t),
$$

$$
\lim_{t' \to t} \int_{\partial D^1_t} \frac{ik}{2} e^{ik^2(t-t')} u_0^T \tilde{\hat{G}}_{44} dk = \int_{\partial D^1_t} \frac{ik}{2} u_0^T \tilde{\hat{G}}_{44} dk = \frac{\pi}{8} u_0^T \tilde{\hat{G}}_{44} (t, t),
$$

Therefore, we know that the first integral on the RHS of Eq. (142) yields the following two terms

$$
\lim_{t' \to t} \int _{\partial D^1_t} \frac{ik}{2} e^{ik^2(t-t')} u_0^T \tilde{\hat{G}}_{44} dk = \int_{\partial D^1_t} \frac{ik}{2} u_0^T \tilde{\hat{G}}_{44} dk |_{(145a)} + \int_{\partial D^1_t} \frac{ik}{2} u_0^T \tilde{\hat{G}}_{44} dk |_{(145b)},
$$

Nowadays we study the second integral on the RHS of Eq. (142). It follows from the second integral on the RHS of Eq. (142) and Eq. (144) that we have

$$
- \int_{\partial D^1_t} k^3 e^{ik^2(t-t')} \tilde{\hat{G}}_{34} \tilde{\hat{G}}_{44} dk = -2 \int_{\partial D^1_t} k^3 \left[ \int _{0}^{t} e^{ik^2(t-t')} \tilde{\hat{G}}_{34} \tilde{\hat{G}}_{44} d\tau \right] dk
$$

Thus we take the limit $t' \to t$ of Eq. (147) to have

$$
- \lim_{t' \to t} \int_{\partial D^1_t} k^3 e^{ik^2(t-t')} \tilde{\hat{G}}_{34} \tilde{\hat{G}}_{44} dk = - \int_{\partial D^1_t} k^3 \tilde{\hat{G}}_{34} \tilde{\hat{G}}_{44} dk + \int_{\partial D^1_t} \frac{k}{2} u_0^T \tilde{\hat{G}}_{44} dk
$$

47
Finally, following the proof in Ref. [31] we can show the limits $t' \to t$ of the rest three integrals (i.e., the third, fourth and fifth integrals) of Eq. (142) can be deduced by simply making the limit $t' \to t$ inside the every integral, that is, no additional terms arise in these integrals. For example,

$$
\lim_{t' \to t} \int_{\partial D^j_t'} k e^{i k^2 (t-t')} \left( \tilde{L}_{j4} - \frac{i}{2} u_0^T \tilde{G}_{j4} \right) \left( \tilde{L}_{44} - \frac{i}{2} \tilde{G}_{j4}^T M v_0^T \right) dk
$$

$$
= \int_{\partial D^j_t} k \left( \tilde{L}_{j4} - \frac{i}{2} u_0^T \tilde{G}_{j4} \right) \left( \tilde{L}_{44} - \frac{i}{2} \tilde{G}_{j4}^T M v_0^T \right) dk.
$$

Thus we complete the proof of Eq. (139a). Similarly, we can show that Eqs. (139b), (139c) and (139d) also hold. □

**Theorem 5.3.** Let $q_0(x) = q_j(x, t = 0) = 0$, $j = 1, 2, 3$ be the initial data of Eq. (1) on the interval $x \in [0, L]$ and $T < \infty$. For the Dirichlet problem, the boundary data $u_{0j}(t)$ and $v_{0j}(t)$ ($j = 1, 2, 3$) on the interval $t \in [0, T)$ are sufficiently smooth and compatible with the initial data $q_0(x)$ ($j = 1, 2, 3$) at the points $(x_2, t_2) = (0, 0)$ and $(x_3, t_3) = (L, 0)$, respectively. For the Neumann problem, the boundary data $u_{1j}(t)$ and $v_{1j}(t)$ ($j = 1, 2, 3$) on the interval $t \in [0, T)$ are sufficiently smooth and compatible with the initial data $q_0(x)$ ($j = 1, 2, 3$) at the points $(x_2, t_2) = (0, 0)$ and $(x_3, t_3) = (L, 0)$, respectively. For simplicity, let $v_{33,44}(S)(k)$ have no zeros in the domain $D_1$. Then the spectral functions $S(k)$ and $S_L(k)$ are defined by Eqs. (87) and (88) with $\Psi(t, k)$ and $\phi(t, k)$ given by Eq. (129a) and (129b).

(i) For the given Dirichlet boundary values $u_0(t)$ and $v_0(t)$, the unknown Neumann boundary values $u_1(t)$ and $v_1(t)$ are given by

$$
u_1^j(t) = \frac{4}{i \pi} \int_{\partial D^j_t} \left\{ \frac{\Sigma^+}{\Sigma^-} \left[ k^2 \tilde{G}_{j4}(t, t) + \frac{i}{2} u_0^T(t) \right] - \frac{2 M^T}{\Sigma^-} \left[ k^2 \tilde{G}_{4j}^T(t, t) + \frac{i}{2} M v_0^T(t) \right] \right\} dk,
$$

$$
u_1^j(t) = \frac{4}{i \pi} \int_{\partial D^j_t} \left\{ \frac{\Sigma^+}{\Sigma^-} \left[ k \tilde{G}_{j4}^T(t, t) + \frac{i}{2} u_0^T(t) \right] + \frac{2 M^T}{\Sigma^-} \left[ k \tilde{G}_{4j}(t, t) + \frac{i}{2} M v_0^T(t) \right] \right\} dk,
$$

(ii) For the given Neumann boundary values $u_1(t)$ and $v_1(t)$, the unknown Dirichlet boundary values $u_0(t)$ and $v_0(t)$ are given by

$$
u_0^j(t) = \frac{2}{\pi} \int_{\partial D^j_t} \left\{ \frac{\Sigma^+}{\Sigma^-} \left[ \tilde{L}_{j4} - \frac{2 M^T}{\Sigma^-} \tilde{L}_{4j}^T + \frac{1}{\Sigma^-} \left( \tilde{F}_{j4} e^{-2 i k L} \right) \right] \right\} dk,
$$

$$
u_0^j(t) = \frac{2}{\pi} \int_{\partial D^j_t} \left\{ \frac{2 M^T}{\Sigma^-} \tilde{L}_{j4}^T - \frac{1}{\Sigma^-} \tilde{L}_{4j} + \frac{M^T}{\Sigma^-} \tilde{F}_{4j}^T \right\} dk,
$$

where $\tilde{F}_{j4}(t, k)$ and $\tilde{F}_{4j}(t, k)$ are defined by Eqs. (140) and (141).

**Proof.** By means of the global relation (56) and Proposition 5.1, we can show that the spectral functions $S(k)$ and $S_L(k)$ are defined by Eqs. (87) and (88) with $\Psi(t, k)$ and $\phi(t, k)$ given by Eq. (129a) and (129b).
(i) we firstly consider the Dirichlet problem. It follows from the global relation (56) with the vanishing initial data
\[ c(t, k) = \mu_2(0, t, k)e^{ikL\xi}L^{-1}(L, t, k), \tag{150} \]
that we find
\[ \tilde{c}_{j4}(t, k) = -\tilde{\Psi}_{j4}(t, k)e^{2ikL} + \tilde{\Psi}_{j4}(t, k), \tag{151a} \]
\[ \tilde{c}_{4j}(t, k) = \tilde{\Psi}_{4j}MT\tilde{\phi}_{j3}(t, k)M^T - \tilde{\Psi}_{4j}MTe^{-2ikL}, \tag{151b} \]
Substituting Eqs. (138a) and (138b) into Eq. (151a) yields
\[ \mathcal{M}^T\tilde{\mathcal{L}}_{j4}e^{2ikL} - \tilde{L}_{j4} = k\tilde{G}_{j4} + k\mathcal{M}^T\tilde{\mathcal{G}}_{4j}e^{2ikL} + \tilde{F}_{j4}(t, k) - \tilde{c}_{j4}(t, k), \tag{152} \]
where \( \tilde{F}_{j4}(t, k) \) is given by Eq. (140). Eq. (152) with \( k \to -k \) further yields
\[ \mathcal{M}^T\tilde{\mathcal{L}}_{j4}e^{-2ikL} - \tilde{L}_{j4} = -k\tilde{G}_{j4} + k\mathcal{M}^T\tilde{\mathcal{G}}_{4j}e^{-2ikL} + \tilde{F}_{j4}(t, -k) - \tilde{c}_{j4}(t, -k), \tag{153} \]
It follows from Eqs. (152) and (153) that we get
\[ \hat{L}_{j4} = \frac{k\Sigma+\tilde{G}_{j4} - 2k\Sigma-\mathcal{M}^T\tilde{\mathcal{G}}_{4j}}{\Sigma-}\left[\tilde{F}_{j4}(t, k)e^{-2ikL}\right]_-. \tag{154} \]
We multiply Eq. (154) by \( ke^{4ik^2(t-t')} \) with \( 0 < t' < t \) and integrate them along \( \partial D_1^0 \) with respect to \( dk \), respectively to yield
\[ \int_{\partial D_1^0} ke^{4ik^2(t-t')}\hat{L}_{j4}dk = \int_{\partial D_1^0} e^{4ik^2(t-t')}\frac{k^2\Sigma+\tilde{G}_{j4}dk}{\Sigma-} - \int_{\partial D_1^0} e^{4ik^2(t-t')}\frac{2k^2}{\Sigma-}\mathcal{M}^T\tilde{\mathcal{G}}_{4j}dk \]
\[ + \int_{\partial D_1^0} \frac{ke^{4ik^2(t-t')}}{\Sigma-}[\tilde{F}_{j4}(t, k)e^{-2ikL}]_-dk, \tag{155} \]
where we have used
\[ \int_{\partial D_1^0} ke^{4ik^2(t-t')}\tilde{c}_{j4}(t, k)dk = \int_{\partial D_1^0} ke^{4ik^2(t-t')}\tilde{c}_{j4}(t, k)e^{-2ikL}\_dk = 0 \]
in terms of their analytical properties in \( D_1^0 \).

Based on these conditions given by Eqs. (143) and (144), Eq. (155) can become
\[ \frac{\pi}{2}\hat{L}_{j4}(t, 2t' - t) = 2\int_{\partial D_1^0} \frac{k^2\Sigma+}{\Sigma-}\left[ \int_0^{t'} e^{4ik^2(t-t')}\tilde{G}_{j4}(t, 2\tau - t)\ d\tau - \frac{\tilde{G}_{j4}(t, 2t' - t)}{4ik^2} \right]dk \]
\[ -4\int_{\partial D_1^0} \frac{k^2\mathcal{M}^T}{\Sigma-}\left[ \int_0^{t'} e^{4ik^2(t-t')}\tilde{\mathcal{G}}_{4j}(t, 2\tau - t)\ d\tau - \frac{\tilde{\mathcal{G}}_{4j}(t, 2t' - t)}{4ik^2} \right]dk \]
\[ + \int_{\partial D_1^0} \frac{k}{\Sigma-} e^{4ik^2(t-t')}[\tilde{F}_{j4}(t, k)e^{-2ikL}]_-dk, \tag{156} \]
We choose the limit \( t' \to t \) of Eq. (156) with the initial data (131) and Proposition 5.2 to find
\[
\frac{\pi}{2} \tilde{L}_{j4}(t, t) = 2 \lim_{t' \to t} \int_{\partial D_0^0} \frac{k^2 \Sigma^+}{\Sigma^-} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{G}_{j4}(t, 2\tau - t) d\tau - \frac{\tilde{G}_{j4}(t, 2t' - t)}{4ik^2} \right] dk
\]
\[
-4 \lim_{t' \to t} \int_{\partial D_0^0} \frac{k^2 M^T}{\Sigma^-} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{G}_{j4}(t, 2\tau - t) d\tau - \frac{\tilde{G}_{j4}(t, 2t' - t)}{4ik^2} \right] dk
\]
\[
+ \lim_{t' \to t} \int_{\partial D_0^0} \frac{k}{\Sigma^-} e^{4ik^2(t-t')} \tilde{F}_{j4}(t, k) e^{-2ikL} dk
\]
\[
= \int_{\partial D_0^0} \left\{ \frac{\Sigma^+}{\Sigma^-} \left[ k^2 \tilde{\xi}_{j4}(t, t) + \frac{i}{2} \tilde{G}_{j4}(t, t) \right] - \frac{2M^T}{\Sigma^-} \left[ k^2 \tilde{G}_{j4}^T(t, t) + \frac{i}{2} \tilde{G}_{j4}(t, t) \right] \right. \\
\left. + \frac{i k}{2} u_T^0 \left( \tilde{G}_{j4} - \tilde{g}_{j4} \right) + \frac{k}{\Sigma^-} \left( \tilde{F}_{j4} e^{-2ikL} \right) \right\} dk, \tag{157}
\]

Since the initial data (131) are of the form
\[
\tilde{L}_{j4}(t, t) = \frac{i}{2} u_T^0 (t) = \frac{i}{2} (u_{11}(t), u_{12}(t), u_{13}(t))^T, \tag{158}
\]
then we know that Eq. (148a) holds by means of Eqs. (157) and (158).

To show Eq. (148b) we rewrite Eq. (151b) in the form
\[
\tilde{c}_{j4}^T(t, \bar{k}) = M^T \tilde{\phi}_{3 	imes 3} M^T \tilde{\Psi}_{j4}^T(t, \bar{k}) - M^T \tilde{\phi}_{j4} \tilde{\psi}_{j4}^T(t, \bar{k}) e^{2ikL}, \tag{159}
\]

We substitute Eqs. (138a) and (138b) into Eq. (159) to have
\[
- \tilde{L}_{j4}^T + M^T \tilde{L}_{j4} e^{2ikL} = k \tilde{G}_{j4}^T - kM^T \tilde{G}_{j4} e^{2ikL} + \tilde{F}_{j4}(t, k) - \tilde{c}_{j4}(t, \bar{k}), \tag{160}
\]

where \( \tilde{F}_{j4}(t, k) \) is given by Eq. (141). Eq. (160) with \( k \to -k \) yields
\[
- \tilde{L}_{j4}^T + M^T \tilde{L}_{j4} e^{-2ikL} = -k \tilde{G}_{j4}^T + kM^T \tilde{G}_{j4} e^{-2ikL} + \tilde{F}_{j4}(t, -k) - \tilde{c}_{j4}(t, \bar{k}), \tag{161}
\]

It follows from Eqs. (160) and (161) that we have
\[
M^T \tilde{L}_{j4} = \frac{2k}{\Sigma^-} \tilde{G}_{j4}^T - \frac{k \Sigma^+}{\Sigma^-} M^T \tilde{G}_{j4} + \frac{1}{\Sigma^-} [\tilde{F}_{j4}(t, k) - \tilde{c}_{j4}(t, \bar{k})], \tag{162}
\]

We multiply Eq. (162) by \( ke^{4ik^2(t-t')} \) with \( 0 < t' < t \), integrate them along \( \partial D_0^0 \) with respect to \( dk \), and use these conditions given by Eqs. (143) and (144) to yield
\[
\frac{\pi}{2} M^T \tilde{L}_{j4}(t, 2t' - t) = -2 \int_{\partial D_0^0} \frac{k^2 \Sigma^+}{\Sigma^-} M^T \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{G}_{j4}(t, 2\tau - t) d\tau - \frac{\tilde{G}_{j4}(t, 2t' - t)}{4ik^2} \right] dk
\]
\[
+ 4 \int_{\partial D_0^0} \frac{k^2}{\Sigma^-} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{G}_{j4}^T(t, 2\tau - t) d\tau - \frac{\tilde{G}_{j4}^T(t, 2t' - t)}{4ik^2} \right] dk \tag{163}
\]
\[
+ \int_{\partial D_0^0} \frac{k}{\Sigma^-} e^{4ik^2(t-t')} \tilde{F}_{j4}(t, k) dk,
\]

\]

50
where we have used the relation
\[
\int_{\partial D_1^0} \frac{k}{\Sigma} e^{4ik^2(t-t')} \bar{c}^T_{4j} - (t, \bar{k}) dk = 0
\]
due to the analytical property of the integrand in \(D_1^0\).

We consider the limit \(t' \to t\) of Eq. (163) with the initial data (131) and Proposition 5.2 to have
\[
\frac{\pi}{2} M^T \tilde{L}_{j4}(t, t) = \int_{\partial D_1^0} \left\{ -\frac{\Sigma^+}{\Sigma^-} M^T \left[ k^2 \tilde{G}_{j4}(t, t) + \frac{i}{2} \tilde{G}_{j4}(t, t) \right] + \frac{2}{\Sigma} \left[ k^2 \tilde{G}^T_{4j}(t, t) + \frac{i}{2} \bar{G}^T_{4j}(t, t) \right] \right\} dt,
\]
(164)

Since the initial conditions are of the form
\[
\tilde{L}_{j4}(t, t) = \frac{i}{2} u^T(t) = \frac{i}{2} (v_{11}(t), v_{12}(t), v_{13}(t))^T,
\]
then we have Eq. (148b) by combining Eqs. (164) and (165).

(ii) We now turn to consider the Neumann problem. It follows from Eqs (152), (153), (160) and (161) that we have
\[
\hat{G}_{j4} = \frac{1}{k \Sigma^-} \left\{ \frac{1}{2} \left[ \tilde{L}_{j4}(t, t) + \frac{i}{2} \tilde{G}_{j4}(t, t) \right] + \frac{1}{2} \left[ \tilde{G}^T_{4j}(t, t) + \frac{i}{2} \bar{G}^T_{4j}(t, t) \right] \right\},
\]
(166a)
\[
\hat{G}_{j4} = \frac{1}{k \Sigma^-} \left\{ \frac{1}{2} \left[ \tilde{L}_{j4}(t, t) + \frac{i}{2} \tilde{G}_{j4}(t, t) \right] + \frac{1}{2} \left[ \tilde{G}^T_{4j}(t, t) + \frac{i}{2} \bar{G}^T_{4j}(t, t) \right] \right\},
\]
(166b)

We multiply Eqs. (166a) and (166b) by \(ke^{4ik^2(t-t')}\) with \(0 < t' < t\), integrate them along \(\partial D_1^0\) with respect to \(dk\), and use the conditions given by Eqs. (143) and (144) to yield
\[
\frac{\pi}{2} \hat{G}_{j4}(t, 2t' - t) = \int_{\partial D_1^0} \frac{2 \Sigma^+}{\Sigma^-} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{L}_{j4}(t, 2\tau - t) d\tau - \frac{\tilde{L}_{j4}(t, 2t' - t)}{4ik^2} \right] d\tau
\]
\[
- \int_{\partial D_1^0} \frac{4 M^T}{\Sigma} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{L}^T_{4j}(t, 2\tau - t) d\tau - \frac{\tilde{L}^T_{4j}(t, 2t' - t)}{4ik^2} \right] d\tau
\]
\[
+ \int_{\partial D_1^0} \frac{e^{4ik^2(t-t')}}{\Sigma^-} (\tilde{L}_{j4} e^{-2ikL}) d\tau,
\]
(167a)
\[
\frac{\pi}{2} \hat{G}_{j4}(t, 2t' - t) = \int_{\partial D_1^0} \frac{4 M^T}{\Sigma} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{L}^T_{4j}(t, 2\tau - t) d\tau - \frac{\tilde{L}^T_{4j}(t, 2t' - t)}{4ik^2} \right] d\tau
\]
\[
- \int_{\partial D_1^0} \frac{2}{\Sigma} \left[ \int_0^{t'} e^{4ik^2(t-t')} \tilde{L}_{j4}(t, 2\tau - t) d\tau - \frac{\tilde{L}_{j4}(t, 2t' - t)}{4ik^2} \right] d\tau
\]
\[
+ \int_{\partial D_1^0} \frac{M^T}{\Sigma^-} e^{4ik^2(t-t')} \tilde{L}_{j4} d\tau,
\]
(167b)
where we have used the analytical property of the matrix-valued functions
\[
\int_{\partial D_1^0} \frac{1}{\Sigma^-} e^{4ik^2(t-t')} (\tilde{L}_{j4}(t, k) e^{-2ikL}) d\tau = \int_{\partial D_1^0} \frac{1}{\Sigma^-} e^{4ik^2(t-t')} \tilde{L}^T_{4j} (t, \bar{k}) dq = 0.
\]
Since the initial conditions are of the form
\[
\hat{G}_{j4}(t, t) = u_0^T(t) = (u_{01}(t), u_{02}(t), u_{03}(t))^T, \quad \hat{G}_{j4}(t, t) = v_0^T(t) = (v_{01}(t), v_{02}(t), v_{03}(t))^T, \]
then we have Eqs. (149a) and (149b) by using Eqs. (168a) and (168b). This completes the proof of the Theorem.

\[\square\]

5.2. Equivalence of the two distinct representations

We now show that the above-mentioned GLM representation for the Dirichlet and Neumann boundary data in Theorem 5.3 is equivalent to one in Theorem 4.2.

**Case i. From the Dirichlet boundary conditions to the Neumann boundary ones**

It follows from Eqs. (138a) and (138b) that we obtain
\[
\hat{G}_{j4} = \frac{1}{2k} \tilde{\Psi}_{j4} - \hat{G}_{j4} = \frac{1}{2k} \tilde{\phi}_{j4} - \hat{G}_{44} = \frac{1}{2k} \tilde{\Psi}_{44} - \hat{G}_{44} = \frac{1}{2k} \tilde{\phi}_{44} - \hat{G}_{44},
\]

Substituting Eqs. (140) and (170) into Eq. (148a) yields
\[
u_0^T(t) = \frac{4}{i\pi} \int_{\partial D_0^1} \left\{ \frac{\Sigma^+}{\Sigma^-} \left[ k^2 \hat{G}_{j4}(t, t) + \frac{i}{2} u_0^T(t) \right] - \frac{2M^T}{\Sigma^-} \left[ \tilde{k}^2 \hat{G}_{j4}(t, t) + \frac{i}{2} \tilde{M} \nu_0^T(t) \right] \\
+ ik u_0^T \hat{G}_{j4} + \frac{k}{2} u_0^T \tilde{G}_{44} + \frac{k}{\Sigma} \left[ \tilde{\Psi}_{j4}(\tilde{\phi}_{44} - 1) e^{-2ikL} - (\tilde{\phi}_{j4} - \hat{G}_{44}) \right] \right\} dk
\]
\[
= \frac{4k}{i\pi \Sigma^-} \left\{ \frac{\Sigma^+}{\Sigma^-} \left[ k \hat{G}_{44} + \frac{i}{2} u_0^T(t) \right] + \frac{4iM^T}{\pi \Sigma^-} \left[ \tilde{k} \hat{G}_{44} - \tilde{M} \nu_0^T(t) \right] \right\} \int_{\partial D_0^1} \left\{ \frac{4k}{i\pi \Sigma^-} \right\} dk; \tag{171}
\]

Since the integrand in Eq. (171) is an odd function about \(k\), which makes sure that the contour \(\partial D_0^1\) can be replaced by \(\partial D_0^3\), thus we can find the same Neumann boundary data \(u_{1j}(t)\) (\(j = 1, 2, 3\)) at \(x = 0\) given by Eqs. (93)-(95) from Eq. (171). Similarly, we can also find the Neumann boundary data \(v_{1j}(t)\) (\(j = 1, 2, 3\)) at \(x = L\) given by Eq. (96) from Eq. (148b).

**Case ii. From the Neumann boundary conditions to the Dirichlet boundary ones**

Eqs. (138a) and (138b) imply that
\[
\hat{L}_{j4} = \frac{1}{2} \tilde{\Psi}_{j4+k} + \frac{i}{2} u_0^T \hat{G}_{44}, \quad \hat{L}_{4j} = \frac{1}{2} \tilde{k} \hat{G}_{44+k} + \frac{i}{2} \tilde{M} \nu_0^T, \tag{172}
\]

52
The substitution of Eqs. (172) and (140) into Eq. (149a) yields

\[
u_0^T(t) = \frac{2}{\pi} \int_{\partial D^T_1} \left[ \frac{\Sigma_j \tilde{\Psi}_{j4} - 2M^T \tilde{\Phi}_{j4}}{\Sigma_j \tilde{L}_{j4}} + \frac{1}{\Sigma_j} \left( \tilde{F}_{j4} e^{-2ikL} \right) \right] dk
\]

\[
= \int_{\partial D^T_1} \left\{ \frac{\Sigma_j \tilde{\Psi}_{j4} - 2M^T \tilde{\Phi}_{j4}}{\pi \Sigma_j} + \frac{2}{\pi \Sigma_j} \left[ \tilde{\Psi}_{j4}(\phi_{44}(t, \tilde{k}) - 1) e^{-2ikL} - (\tilde{\Psi}_{33} - i)M^T \tilde{\Phi}_{j4} \right] \right\} dk,
\]

(173)

Since the integrand in Eq. (173) is an odd function about \( k \), which makes sure that the contour \( \partial D^T_1 \) can be replaced by \( \partial D^T_3 \), thus Eq. (173) yields the Dirichlet boundary values \( u_{0j}(t) , j = 1, 2, 3 \) again. Similarly, we can also deduce the Dirichlet boundary values \( v_{0j}(t), j = 1, 2, 3 \) from Eq. (149b).

5.3. Linearizable boundary conditions for the GLM representation

In what follows we further explore the linearizable boundary conditions for the GLM representation given in Theorem 5.3.

Proposition 5.4. Let \( q_j(x, t = 0) = q_{0j}(x), j = 1, 2, 3 \) be the initial conditions of the gtc-NLS equation (1) on the interval \( x \in [0, L] \), and one of the following boundary conditions, either

(i) the Dirichlet boundary conditions at \( x = 0, L, q_j(x = 0, t) = u_{0j}(t) = 0 \) and \( q_j(x = L, t) = v_{0j}(t) = 0, j = 1, 2, 3 \),

or

(ii) the Robin boundary conditions \( x = 0, L, q_{0j}(x = 0, t) = \chi q_j(x = 0, t) = u_{0j}(t) - \chi u_{0j}(t) = 0, j = 1, 2, 3 \) and \( q_{0j}(x = L, t) - \vartheta q_j(x = L, t) = v_{0j}(t) - \vartheta v_{0j}(t) = 0, j = 1, 2 \), where \( \chi \) and \( \vartheta \) are both real parameters.

Then the eigenfunctions \( \Psi(t, k) \) and \( \phi(t, k) \) can be expressed as

(i)

\[
\Psi(t, k) = \mathbb{I} + \begin{pmatrix} \tilde{L}_{33} & \tilde{L}_{j4} \\ \tilde{L}_{4j} & \tilde{L}_{44} \end{pmatrix},
\]

\[
\phi(t, k) = \mathbb{I} + \begin{pmatrix} \tilde{L}_{33} & \tilde{L}_{j4} \\ \tilde{L}_{4j} & \tilde{L}_{44} \end{pmatrix},
\]

(174a)

(174b)

where the \( 4 \times 4 \) matrix-valued function \( L(t, s) = (L_{ij})_{4 \times 4} \) satisfies a reduced Goursat system

\[
\begin{cases}
\tilde{L}_{33} \partial t + \tilde{L}_{33} \partial s = \mathbb{I} \tilde{u}_1^T(t) \tilde{L}_{4j}, \\
\tilde{L}_{j4} \partial t - \tilde{L}_{j4} \partial s = \mathbb{I} \tilde{u}_1^T(t) \tilde{L}_{44}, \quad j = 1, 2, 3, \\
\tilde{L}_{4j} \partial t - \tilde{L}_{4j} \partial s = -i \tilde{u}_1(t) \mathcal{M} \tilde{L}_{33}, \quad j = 1, 2, 3, \\
\tilde{L}_{44} \partial t + \tilde{L}_{44} \partial s = -i \tilde{u}_1(t) \mathcal{M} \tilde{L}_{4j},
\end{cases}
\]

(175)

with the initial data (cf. Eq. (131))

\[
\tilde{L}_{33}(t, -t) = 0_{3 \times 3}, \quad \tilde{L}_{44}(t, -t) = 0, \quad \tilde{L}_{j4}(t, t) = \frac{i}{2} \tilde{u}_1^T(t), \quad \tilde{L}_{4j}(t, t) = -\frac{i}{2} \tilde{u}_1(t) \mathcal{M},
\]

(176)
Similarly, the $4 \times 4$ matrix-valued function $\mathcal{L}(t, s) = (\mathcal{L}_{ij})_{4 \times 4}$ satisfies the analogous system (175) with $u_1(t)$ replaced by $v_1(t)$.

(ii)  

$$
\begin{align*}
\Psi(t, k) &= \mathbb{I} + \left( \hat{L}_{3 \times 3} \quad \hat{L}_{j4} \right) + \begin{pmatrix}
-\frac{i}{2} u_0^T(t) \hat{G}_{4j} & k \hat{G}_{j4} \\
k \hat{G}_{4j} & i \frac{1}{2} \bar{u}_0(t) \mathcal{M} \hat{G}_{j4}
\end{pmatrix}, \\
\phi(t, k) &= \mathbb{I} + \left( \hat{L}_{3 \times 3} \quad \hat{L}_{j4} \right) + \begin{pmatrix}
-\frac{i}{2} v_0^T(t) \hat{G}_{4j} & k \hat{G}_{j4} \\
k \hat{G}_{4j} & i \frac{1}{2} \bar{v}_0(t) \mathcal{M} \hat{G}_{j4}
\end{pmatrix},
\end{align*}
$$

where the $4 \times 4$ matrix-valued functions $L(t, s) = (L_{ij})_{4 \times 4}$ and $G(t, s) = (G_{ij})_{4 \times 4}$ satisfy the reduced nonlinear Goursat system

$$
\begin{align*}
\left\{ 
\begin{array}{l}
\hat{L}_{3 \times 3} + \bar{L}_{3 \times 3} = i \chi u_0^T(t) \bar{L}_{4j} + \frac{1}{2} \left[ i u_0^T(t) - u_0^T(t) u_0(t) \mathcal{M} u_0^T(t) \right] \hat{G}_{4j}, \\
\bar{L}_{44} + \bar{L}_{44} = -i \bar{u}_0(t) \mathcal{M} \bar{L}_{j4} - \frac{1}{2} \left[ i \bar{u}_0(t) \mathcal{M} + \bar{u}_0(t) \mathcal{M} u_0^T(t) \bar{u}_0(t) \mathcal{M} \right] \hat{G}_{j4}, \\
\hat{L}_{j4} - \bar{L}_{j4} = i \chi u_0^T(t) \hat{L}_{44}, \\
\bar{L}_{4j} - \bar{L}_{4j} = -i \bar{u}_0(t) \mathcal{M} \bar{L}_{3 \times 3}, \\
\hat{G}_{j4} - \bar{G}_{j4} = 2 u_0^T(t) \hat{L}_{44}, \\
\bar{G}_{j4} - \bar{G}_{j4} = 2 \bar{u}_0(t) \mathcal{M} \bar{L}_{3 \times 3},
\end{array}
\right.
\end{align*}
$$

with the initial data (cf. Eq. (131))

$$
\begin{align*}
\left\{ 
\begin{array}{l}
\bar{L}_{3 \times 3}(t, -t) = 0_{3 \times 3}, \\
\hat{L}_{44}(t, -t) = 0, \\
\bar{L}_{j4}(t, t) = \frac{i}{2} \chi u_0^T(t), \\
\hat{L}_{4j}(t, t) = -\frac{i}{2} \chi \bar{u}_0(t) \mathcal{M}, \\
\bar{G}_{j4}(t, t) = u_0^T(t), \\
\hat{G}_{4j}(t, t) = \bar{u}_0(t) \mathcal{M},
\end{array}
\right.
\end{align*}
$$

Similarly, the $4 \times 4$ matrix-valued functions $\mathcal{L}(t, s) = (\mathcal{L}_{ij})_{4 \times 4}$ and $\mathcal{G}(t, s) = (\mathcal{G}_{ij})_{4 \times 4}$ satisfy the similar system (178) with $u_0(t)$ and $\chi$ replaced by $v_0(t)$ and $\vartheta$, respectively.

**Proof.** Let us show that the linearizable boundary data correspond to the special cases of Proposition 5.1.

Case (i) The Dirichlet zero boundary data $q_j(x = 0, t) = u_{0j}(t) = 0$. It follows from the second one of system
(130) that $\tilde{G}_{ij}(t, s)$ satisfy

$$
\begin{align*}
\tilde{G}_{3\times 3t} + \tilde{G}_{3\times 3s} &= iu_1^T(t)\tilde{G}_{4j}, \\
\tilde{G}_{j4t} - \tilde{G}_{j4s} &= iu_1^T(t)\tilde{G}_{44}, \\
\tilde{G}_{4jt} - \tilde{G}_{4js} &= -i\tilde{u}_1(t)M\tilde{G}_{3\times 3}, \\
\tilde{G}_{44t} + \tilde{G}_{44s} &= -i\tilde{u}_1(t)M\tilde{G}_{j4},
\end{align*}
$$

(180)

with the initial data (cf. Eq. (131))

$$
\begin{align*}
\tilde{G}_{3\times 3}(t, -t) &= 0_{3\times 3}, & \tilde{G}_{44}(t, -t) &= 0, & \tilde{G}_{j4}(t, t) &= 0_{j4}, & \tilde{G}_{4j}(t, t) &= 0_{4j},
\end{align*}
$$

(181)

Thus the unique solution of Eq. (180) is trivial, that is, $\tilde{G}_{3\times 3}(t, s) = 0, \tilde{G}_{4j}(t, s) = 0, \tilde{G}_{j4}(t, s) = 0, \tilde{G}_{44}(t, s) = 0$ such that Eq. (129a) reduces to Eq. (174a) and the condition (130) with (131) becomes (175) with (176). Similarly, for the Dirichlet zero boundary data $q_j(x = L, t) = v_0(t) = 0, j = 1, 2, 3$, we can also show Eq. (174b).

(ii) Consider the Robin boundary data $q_j\dot{x}(x = 0, t) - \chi q_j(x = 0, t) = u_{1j}(t) - \chi u_0(t) = 0, (j = 1, 2, 3)$, that is, the Dirichlet and Neumann boundary data have the linear relation

$$
u_1(t) = \chi u_0(t).
$$

(182)

We introduce a $4 \times 4$ matrix

$$
Q(t, s) = 2L(t, s) - i\chi \sigma_3 G(t, s)
$$

by the linear combinations of $L$ and $G$ such that we have

$$
\begin{align*}
\dot{Q}_{3\times 3}(t, s) &= 2\dot{L}_{3\times 3}(t, s) - i\chi \dot{G}_{3\times 3}(t, s), \\
\dot{Q}_{j4}(t, s) &= 2\dot{L}_{j4}(t, s) - i\chi \dot{G}_{j4}(t, s), \\
\dot{Q}_{4j}(t, s) &= 2\dot{L}_{4j}(t, s) + i\chi \dot{G}_{4j}(t, s), \\
\dot{Q}_{44}(t, s) &= 2\dot{L}_{44}(t, s) + i\chi \dot{G}_{44}(t, s),
\end{align*}
$$

(183)

It follows from Eq. (130) and (183) with Eq. (182) that $\dot{Q}_{ij}(t, s), \tilde{Q}_{ij}(t, s), i, j = 1, 2$ satisfy

$$
\begin{align*}
\dot{Q}_{3\times 3t} + \dot{Q}_{3\times 3s} &= [i\dot{u}_0^T(t) - u_0^T(t)\tilde{u}_0(t)M\dot{u}_0^T(t) + \chi^2 \dot{u}_0^T(t)] \tilde{G}_{4j}, \\
\dot{Q}_{j4t} - \dot{Q}_{j4s} &= [i\dot{u}_0^T(t) - u_0^T(t)\tilde{u}_0(t)M\dot{u}_0^T(t) + \chi^2 \dot{u}_0^T(t)] \tilde{G}_{44}, \\
\dot{Q}_{4jt} - \dot{Q}_{4js} &= [-\tilde{u}_0(t)M\dot{u}_0^T(t)\tilde{u}_0(t)M - i\tilde{u}_0(t)M + \chi^2 \tilde{u}_0(t)M] \tilde{G}_{3\times 3}, \\
\dot{Q}_{44t} + \dot{Q}_{44s} &= [-\tilde{u}_0(t)M\dot{u}_0^T(t)\tilde{u}_0(t)M - i\tilde{u}_0(t)M + \chi^2 \tilde{u}_0(t)M] \tilde{G}_{j4}, \\
\tilde{G}_{3\times 3t} + \tilde{G}_{3\times 3s} &= u_0^T(t)\dot{Q}_{4j}, \\
\tilde{G}_{j4t} - \tilde{G}_{j4s} &= u_0^T(t)\dot{Q}_{44}, \\
\tilde{G}_{4jt} - \tilde{G}_{4js} &= \tilde{u}_0(t)M\dot{Q}_{3\times 3}, \\
\tilde{G}_{44t} + \tilde{G}_{44s} &= \tilde{u}_0(t)M\dot{Q}_{j4},
\end{align*}
$$

(184)

55
with the initial data (cf. Eq. (131))

\[
\begin{align*}
\tilde{G}_{3 \times 3}(t, -t) &= 0_{3 \times 3}, \quad \tilde{G}_{44}(t, -t) = 0, \quad \tilde{G}_{j4}(t, t) = u^T_0(t), \quad \tilde{G}_{4j}(t, t) = \bar{u}_0(t)M, \\
\tilde{Q}_{3 \times 3}(t, -t) &= 0_{3 \times 3}, \quad \tilde{Q}_{44}(t, -t) = 0, \quad \tilde{Q}_{j4}(t, t) = 0_{4j}, \quad \tilde{Q}_{4j}(t, t) = 0_{4j},
\end{align*}
\]

(185)

Thus the unique solution of Eq. (184) is trivial, that is, \(\tilde{Q}_{j4}(t, s) = \tilde{Q}_{4j}(t, s) = \tilde{G}_{3 \times 3}(t, s) = \tilde{G}_{44}(t, s) = 0\) such that Eq. (129a) reduces to Eq. (177a) and the condition (130) with Eq. (131) becomes Eq. (178) with Eq. (179).

Similarly, for the Robin boundary data \(q_j x(x = L, t) - \partial q_j(x = L, t) = v_1(t) - \partial v_0(t) = 0, j = 1, 2, 3\), that is, \(v_1(t) = \partial v_0(t)\), we can also show Eq. (177b). \(\square\)

Based on the Theorem 5.3 and Proposition 5.4, we have the following Proposition.

**Proposition 5.5** For the linearizable Dirichlet boundary data \(u_0(t) = v_0(t) = 0\), we have the Neumann boundary data \(u_1(t)\) and \(v_1(t)\):

\[
u^T_1(t) = \frac{4i}{\pi} \int_{\partial D^0_1} k \tilde{\Psi}_{j4}(\tilde{\phi}_{44} - \mathbb{I})dk, \quad v^T_1(t) = \frac{4i}{\pi} \int_{\partial D^0_1} k \tilde{\phi}_{j4}(\tilde{\phi}_{44} - \mathbb{I})dk,
\]

(186)

where

\[
\begin{align*}
\tilde{\Psi}_{j4} + 4ik^2\tilde{\Psi}_{j4} &= iu^T_1(t)(\tilde{\Psi}_{44} + \mathbb{I}), \\
\tilde{\Psi}_{44} &= -i\bar{u}_1(t)M\tilde{\psi}_{j4}, \\
\tilde{\phi}_{j4} + 4ik^2\tilde{\phi}_{j4} &= iv^T_1(t)(\tilde{\phi}_{44} + \mathbb{I}), \\
\tilde{\phi}_{44} &= -i\bar{v}_1(t)M\tilde{\phi}_{j4}.
\end{align*}
\]

**Remark 5.6.** The analogous analysis of the Fokas unified method will use also to explore the IBV problems for other integrable nonlinear evolution PDEs with \(4 \times 4\) Lax pairs both on the the half-line and the finite interval, such as the three-component derivative-NLS equation and the three-component higher-order NLS equation, which will be considered in other papers.

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