SOME BERNSTEIN FUNCTIONS AND INTEGRAL REPRESENTATIONS CONCERNING HARMONIC AND GEOMETRIC MEANS

FENG QI, XIAO-JING ZHANG, AND WEN-HUI LI

Abstract. It is general knowledge that the harmonic mean \( H(x, y) = \frac{2}{\frac{1}{x} + \frac{1}{y}} \) and that the geometric mean \( G(x, y) = \sqrt{xy} \), where \( x \) and \( y \) are two positive numbers. In the paper, the authors show by several approaches that the harmonic mean \( H_{x,y}(t) = H(x+t, y+t) \) and the geometric mean \( G_{x,y}(t) = G(x+t, y+t) \) are all Bernstein functions of \( t \in (-\min\{x, y\}, \infty) \) and establish integral representations of the means \( H_{x,y}(t) \) and \( G_{x,y}(t) \).

1. Introduction

1.1. Some definitions. We recall some notions and definitions.

Definition 1.1 ([17, 27]). A function \( f \) is said to be completely monotonic on an interval \( I \subseteq \mathbb{R} \) if \( f \) has derivatives of all orders on \( I \) and
\[
(-1)^n f^{(n)}(t) \geq 0 \tag{1.1}
\]
for all \( t \in I \) and \( n \in \{0\} \cup \mathbb{N} \).

Definition 1.2 ([2]). If \( f^{(k)}(t) \) for some nonnegative integer \( k \) is completely monotonic on an interval \( I \subseteq \mathbb{R} \), but \( f^{(k-1)}(t) \) is not completely monotonic on \( I \), then \( f(t) \) is called a completely monotonic function of \( k \)-th order on an interval \( I \).

Definition 1.3 ([20, 22]). A function \( f \) is said to be logarithmically completely monotonic on an interval \( I \subseteq \mathbb{R} \) if its logarithm \( \ln f \) satisfies
\[
(-1)^k [\ln f(t)]^{(k)} \geq 0 \tag{1.2}
\]
for all \( t \in I \) and \( k \in \mathbb{N} \).

Definition 1.4 ([25, 27]). A function \( f : I \subseteq (-\infty, \infty) \to [0, \infty) \) is called a Bernstein function on \( I \) if \( f(t) \) has derivatives of all orders and \( f'(t) \) is completely monotonic on \( I \).

Definition 1.5 ([25]). A Stieltjes function is a function \( f : (0, \infty) \to [0, \infty) \) which can be written in the form
\[
f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} \, d\mu(s), \tag{1.3}
\]
where \( a, b \) are nonnegative constants and \( \mu \) is a nonnegative measure on \( (0, \infty) \) such that \( \int_0^\infty \frac{1}{1+s} \, d\mu(s) < \infty \).
Definition 1.6 ([9]). Let \( f(x) \) be a nonnegative function and have derivatives of all orders on \((0, \infty)\). A number \( r \in \mathbb{R} \cup \{ \pm \infty \} \) is said to be the completely monotonic degree of \( f(x) \) with respect to \( x \in (0, \infty) \) if \( x^r f(x) \) is a completely monotonic function on \((0, \infty)\) but \( x^{r+\varepsilon} f(x) \) is not for any positive number \( \varepsilon > 0 \).

In what follows, for convenience, we denote the sets of completely monotonic functions on \( I \subseteq \mathbb{R} \), logarithmically completely monotonic functions on \( I \subseteq \mathbb{R} \), Stieltjes functions, and Bernstein functions on \( I \subseteq \mathbb{R} \) by \( C[I], L[I], S, \) and \( B[I] \) respectively.

1.2. Some relationships and a characterization. Now we briefly describe some basic relationships between the above defined classes of functions and list a characterization of Bernstein functions on \((0, \infty)\).

In \([3, 10, 20, 22]\), any logarithmically completely monotonic function on an interval \( I \) was once again proved to be completely monotonic on \( I \). In \([3]\), the set of all Stieltjes functions was proved to be a subset of all logarithmically completely monotonic functions on \((0, \infty)\). See also \([24, \text{Remark 4.8}]\). Conclusively,

\[
S \subset L[(0, \infty)] \subset C[(0, \infty)]. \tag{1.4}
\]

It is obvious that any nonnegative completely monotonic function of first order is a Bernstein function.

The relation between Bernstein functions and logarithmically completely monotonic functions was discovered in \([7, \text{pp. 161–162, Theorem 3}]\) and \([25, \text{p. 45, Proposition 5.17}]\), which reads that the reciprocal of any positive Bernstein function is logarithmically completely monotonic. In other words,

\[
0 < f \in B[I] \implies \frac{1}{f} \in L[I]. \tag{1.5}
\]

A relation between \( S \) and \( B[(0, \infty)] \) was given by \([4, \text{Theorem 5.4}]\) which may be recited as

\[
0 < f \in S \implies \frac{1}{f} \in B[(0, \infty)]. \tag{1.6}
\]

It is easy to see that the degree of any completely monotonic function on \((0, \infty)\) is at least zero. Conversely, if a nonnegative function \( f(x) \) on \((0, \infty)\) has a nonnegative degree \( r \), then it must be a completely monotonic function on \((0, \infty)\). See \([9, \text{p. 9890}]\).

Bernstein functions can be characterized by \([25, \text{p. 15, Theorem 3.2}]\) which states that a function \( f : (0, \infty) \to \mathbb{R} \) is a Bernstein function if and only if it admits the representation

\[
f(x) = a + bx + \int_0^\infty (1 - e^{-xt}) \, d\mu(t), \tag{1.7}
\]

where \( a, b \geq 0 \) and \( \mu \) is a measure on \((0, \infty)\) satisfying \( \int_0^\infty \min\{1, t\} \, d\mu(t) < \infty \).

For information on characterizations of the classes \( C[(0, \infty)] \) and \( L[(0, \infty)] \), please refer to related texts in \([3, 25, 27]\) and references cited therein.

1.3. Some means. We recall from \([26]\) that the extended mean value \( E(r, s; x, y) \) may be defined by

\[
E(r, s; x, y) = \left[ \frac{r(y^s - x^s)}{s(y^r - x^r)} \right]^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0; \tag{1.8}
\]
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\[ E(r, 0; x, y) = \left[ \frac{y^r - x^r}{r(\ln y - \ln x)} \right]^{1/r}, \quad r(x - y) \neq 0; \] (1.9)

\[ E(r, r; x, y) = \frac{1}{e^{1/r}} \left( \frac{x^{r'} - y^{r'}}{y^{r'} - x^{r'}} \right)^{1/(x^{r'} - y^{r'})}, \quad r(x - y) \neq 0; \] (1.10)

\[ E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y; \] (1.11)

\[ E(r, s; x, x) = x, \quad x = x; \]

where \(x, y\) are positive numbers and \(r, s \in \mathbb{R}\). Because this mean was first defined in [26], so it is also called Stolarsky’s mean by a number of mathematicians. Many special means with two positive variables are special cases of \(E\), for example,

\[ E(r, 2r; x, y) = M_r(x, y), \quad \text{(power mean)} \]
\[ E(1, p; x, y) = L_p(x, y), \quad \text{(generalized logarithmic mean)} \]
\[ E(1, 1; x, y) = I(x, y), \quad \text{(exponential mean)} \]
\[ E(1, 2; x, y) = A(x, y), \quad \text{(arithmetic mean)} \]
\[ E(0, 0; x, y) = G(x, y), \quad \text{(geometric mean)} \]
\[ E(-2, -1; x, y) = H(x, y), \quad \text{(harmonic mean)} \]
\[ E(0, 1; x, y) = L(x, y). \quad \text{(logarithmic mean)} \]

For more information on \(E\), please refer to the monograph [6], the papers [11, 12, 19, 13], and a lot of closely-related references therein.

1.4. The arithmetic mean is a Bernstein function. It is easy to see that the arithmetic mean

\[ A_{x,y}(t) = A(x + t, y + t) = A(x, y) + t \]

is a trivial Bernstein function of \(t \in (-\min\{x, y\}, \infty)\) for \(x, y > 0\).

1.5. The exponential mean is a Bernstein function. In [23, p. 116, Remark 6], it was pointed out that,

(1) by standard arguments, it is easy to verify that the reciprocal of the exponential mean

\[ I_{x,y}(t) = I(x + t, y + t) = \frac{1}{e} \left[ \frac{(x + t)^{x+t}}{(y + t)^{y+t}} \right]^{1/(x-y)} \] (1.12)

for \(x, y > 0\) with \(x \neq y\) is a logarithmically completely monotonic function of \(t \in (-\min\{x, y\}, \infty)\);

(2) from the newly-discovered integral representation

\[ I(x, y) = \exp \left( \frac{1}{y-x} \int_x^y \ln u \, du \right), \] (1.13)

it is easy to obtain that the exponential mean \(I_{x,y}(t)\) for \(t > -\min\{x, y\}\) with \(x \neq y\) is also a completely monotonic function of first order (that is, a Bernstein function).
1.6. The logarithmic mean is a Bernstein function. In [18, p. 616, Remark 3.7], the logarithmic mean
\[ L_{x,y}(t) = L(x + t, y + t) \] (1.14)
was proved to be increasing and concave in \( t > -\min\{x, y\} \) for \( x, y > 0 \) with \( x \neq y \).

More strongly, the logarithmic mean \( L_{x,y}(t) \) was proved in [21, Theorem 1] to be a completely monotonic function of first order on \((-\min\{x, y\}, \infty)\) for \( x, y > 0 \) with \( x \neq y \). Therefore, the logarithmic mean \( L_{x,y}(t) \) is a Bernstein function of \( t \in (-\min\{x, y\}, \infty) \).

Remark 1.1. By [7, pp. 161–162, Theorem 3] or [25, p. 45, Proposition 5.17], the logarithmically complete monotonicity of the exponential mean \( I_{x,y}(t) \) and the logarithmic mean \( L_{x,y}(t) \) can be deduced respectively from their common property that they are Bernstein functions.

1.7. Main results. The goals of this paper are to prove that the harmonic mean
\[ H_{x,y}(t) = H(x + t, y + t) = \frac{2}{x+t} + \frac{1}{y+t} \] (1.15)
and the geometric mean
\[ G_{x,y}(t) = G(x + t, y + t) = \sqrt{(x + t)(y + t)} \] (1.16)
are all Bernstein functions of \( t \) on \((-\min\{x, y\}, \infty)\) for \( x, y > 0 \) with \( x \neq y \), and to establish integral representations of \( H_{x,y}(t) \) and \( G_{x,y}(t) \).

2. Lemmas

In order to prove our main results, the following lemmas are needed.

Lemma 2.1. For \( i \in \mathbb{N} \), the \( i \)-th derivatives of the functions
\[ h(t) = \sqrt{1 + \frac{1}{t}}, \] (2.1)
the reciprocal \( \frac{1}{h(t)} \), and
\[ H(t) = h(t) + \frac{1}{h(t)} \] (2.2)
on \((0, \infty)\) may be computed by
\[ h^{(i)}(t) = \frac{(-1)^i}{2^it^{i+1}(1+t)\cdot h(t)} \sum_{k=0}^{i-1} a_{i,k} t^k, \] (2.3)
\[ \left[\frac{1}{h(t)}\right]^{(i)} = \frac{(-1)^{i+1}}{2^it^{i+1}h(t)} \sum_{k=0}^{i-1} b_{i,k} t^k, \] (2.4)
\[ H^{(i)}(t) = \frac{(-1)^i}{2^it^{i+1}(1+t)\cdot h(t)} \sum_{k=0}^{i-1} c_{i,k} t^k, \] (2.5)
where
\[ a_{i,k} = \frac{(i-1)![(2i-2k-1)!!]}{(i-k-1)!(i-k)!k!} 2^k, \] (2.6)
\[ b_{i,k} = \frac{(i-1)!(2i-2k-3)!!}{(i-k-1)!(i-k)!k!} 2^k, \] (2.7)
and

\[ c_{i,k} = \frac{(i-1)! (i+1)! (2i - 2k - 1)!!}{(i-k-1)! (i-k+1)!} 2^k. \quad (2.8) \]

Consequently, the functions \( h(t) \) and \( H(t) \) are completely monotonic on \((0, \infty)\), and the reciprocal \( \frac{1}{h(t)} \) is a Bernstein function on \((0, \infty)\).

**Inductive proof of Lemma 2.1.** A direct calculation yields \( h'(t) = -\frac{1}{2t h(t)} \), which means that

\[ a_{1,0} = 1. \quad (2.9) \]

So, the formulas (2.3) and (2.6) are valid for \( i = 1 \) and \( k = 0 \).

Differentiating on both sides of (2.3) gives

\[
\begin{aligned}
    h^{(i+1)}(t) &= \left[ h^{(i)}(t) \right]' = \frac{(-1)^i}{2^{i+1} t^{i+2} (1+t)^{i+1} h(t)} \sum_{k=0}^{i-1} a_{i,k} t^k \\
    &= \frac{(-1)^i}{2^{i+1} t^{i+2} (1+t)^i h(t)} \sum_{k=0}^{i-1} [1 + 2(i-k) + 2(2i-k)t] a_{i,k} t^k \\
    &= \frac{(-1)^i}{2^{i+1} t^{i+2} (1+t)^i h(t)} \sum_{k=0}^{i} a_{i+1,k} t^k.
\end{aligned}
\]

Because

\[
\begin{aligned}
    \sum_{k=0}^{i-1} [1 + 2(i-k) + 2(2i-k)t] a_{i,k} t^k &= \sum_{k=0}^{i-1} [1 + 2(i-k)] a_{i,k} t^k + \sum_{k=0}^{i-1} 2(2i-k) a_{i,k} t^{k+1} \\
    &= \sum_{k=0}^{i-1} [1 + 2(i-k)] a_{i,k} t^k + \sum_{k=1}^{i} 2(2i-k+1) a_{i,k-1} t^k \\
    &= (1 + 2i) a_{i,0} + \sum_{k=1}^{i-1} [(1 + 2(i-k)] a_{i,k} + 2(2i-k+1) a_{i,k-1}] t^k + 2(i+1) a_{i,i-1} t^i,
\end{aligned}
\]

we obtain

\[ a_{i+1,0} = (1 + 2i) a_{i,0}, \quad (2.10) \]

and,

\[ a_{i+1,i} = 2i a_{i,i-1}, \quad (2.11) \]

and, for \( 0 < k < i \),

\[ a_{i+1,k} = [1 + 2(i-k)] a_{i,k} + 2(2i-k+1) a_{i,k-1}. \quad (2.12) \]

Combining (2.9) with (2.10) and (2.11) results in

\[ a_{i,0} = (2i - 1)!! \quad (2.13) \]

and

\[ a_{i,i-1} = 2^{i-1} i!. \quad (2.14) \]

Taking \( k = i - 1 \) in (2.12) and using (2.14) give

\[ a_{i+1,i-1} = 3 a_{i,i-1} + 2(i+2) a_{i,i-2} = 3 \cdot 2^{i-1} i! + 2(i+2) a_{i,i-2}. \quad (2.15) \]

From (2.13), it is easily deduced that \( a_{2,0} = 3 \). Substituting this into (2.15) and recurring repeatedly lead to

\[ a_{i,i-2} = 3(i-1) 2^{i-3} i!. \quad (2.16) \]
Taking \( k = i - 2 \) in (2.12) and using (2.16) show
\[
a_{i+1,i-2} = 5a_{i,i-2} + 2(i + 3)a_{i,i-3} = 15(i - 1)2^{i-3}! + 2(i + 3)a_{i,i-3}.
\] (2.17)

From (2.13), it is readily deduced that \( a_{3,0} = 15 \). Substituting this into (2.17) and recurring repeatedly reveal
\[
a_{i,i-3} = 5(i - 2)(i - 1)2^{i-5}!.
\] (2.18)

Taking \( k = i - 3 \) in (2.12) and using (2.18) show
\[
a_{i+1,i-3} = 7a_{i,i-3} + 2(i + 4)a_{i,i-4} = 35(i - 2)(i - 1)2^{i-5}! + 2(i + 4)a_{i,i-4}.
\] (2.19)

From (2.13), it is immediately obtained that \( a_{4,0} = 105 \). Substituting this into (2.19) and recurring repeatedly yield
\[
a_{i,i-4} = \frac{35}{3}(i - 3)(i - 2)(i - 1)2^{i-8}!.
\] (2.20)

By the same arguments as above, we may obtain
\[
a_{i,i-5} = 21(i - 4)(i - 3)(i - 2)(i - 1)2^{i-11}!.
\] (2.21)

and
\[
a_{i,i-6} = \frac{77}{5}(i - 5)(i - 4)(i - 3)(i - 2)(i - 1)2^{i-13}!.
\] (2.22)

Inductively, we can derive that
\[
a_{i,i-k} = \lambda_{i,i-k} \frac{(i - 1)!}{(i - k)!} 2^{i-k}!.
\] (2.23)

for \( 0 < k < i \}. Specially, we have
\[
\lambda_{i,i-1} = 1, \quad \lambda_{i,i-2} = \frac{3}{2} \quad \lambda_{i,i-3} = \frac{5}{4} \quad \lambda_{i,i-4} = \frac{35}{3 \cdot 2^4} \quad \lambda_{i,i-5} = \frac{21}{2^6}, \quad \lambda_{i,i-6} = \frac{77}{5 \cdot 2^7}.
\] (2.24)

Replacing \( k \) by \( i - \ell \) in (2.23) yields
\[
a_{i,\ell} = \lambda_{i,\ell} \frac{(i - 1)!}{\ell!} 2^{i-\ell}!.
\] (2.25)

for \( 0 < \ell < i \). Substituting (2.25) into (2.12) leads to
\[
[1 + 2(i - \ell)]\lambda_{i,\ell} + \ell(2i - \ell + 1)\lambda_{i,\ell-1} = i(i + 1)\lambda_{i+1,\ell}
\] (2.26)

for \( 0 < \ell < i \). The equality (2.26) is equivalent to
\[
(1 + 2k)\lambda_{i,i-k} + (i - k)(i + k + 1)\lambda_{i,i-k-1} = i(i + 1)\lambda_{i+1,i-k}
\] (2.27)

for \( 0 < k < i \). The quantities in (2.24) implies that \( \lambda_{i,i-k} = \mu_k \), that is, \( \lambda_{i,i-k} \) is independent of \( i \). Then the equality (2.27) may be written as
\[
(1 + 2k)\mu_k = [i(i + 1) - (i - k)(i + k + 1)]\mu_{k+1} = k(1 + k)\mu_{k+1}
\] (2.28)

for \( 0 < k < i \). Recurring (2.28) by \( \mu_1 = \lambda_{i,i-1} = 1 \) reveals
\[
\mu_k = \lambda_{i,i-k} = \frac{(2k-1)!}{(k-1)!k!}
\] (2.29)

for \( 0 < k < i \). As a result, by (2.29), we conclude that
\[
a_{i,i-k} = \frac{(2k-1)!}{(k-1)!k!} \frac{(i - 1)!}{(i - k)!} 2^{i-k}!.
\] (2.30)
for $0 < k < i$. Replacing $k$ by $i - \ell$ in (2.30) shows

$$a_{i,\ell} = \frac{(2i - 2\ell - 1)!}{(i - \ell - 1)!(i - \ell)!} \frac{(i - 1)!}{\ell!},$$

(2.31)

for $0 \leq \ell < i$. It is easy to verify that the sequence (2.31) for $0 \leq \ell \leq i - 1$ meets the recursion formulas (2.10), (2.11), and (2.12). The formulas (2.3) and (2.6) for general terms are thus proved.

It is obvious that $\frac{1}{h(t)} = \frac{1}{t^2 h'(t)}$ which is equivalent to $\frac{1}{h(t)} = -2t^2h'(t)$.

Therefore, using the formulas (2.3) and (2.6) just verified, we have

$$\left[ \frac{1}{h(t)} \right]^{(i)} = -2\left[ t^2 h'(t) \right]^{(i)}$$

$$= -2 \sum_{\ell=0}^{i} \binom{i}{\ell} (t^2)^{(\ell)} h^{(i-\ell+1)}(t)$$

$$= -2 \left[ \binom{i}{0} t^2 h^{(i+1)}(t) + 2 \binom{i}{1} t h^{(i)}(t) + 2 \binom{i}{2} h^{(i-1)}(t) \right]$$

$$= -2 \left[ \frac{(-1)^{i+1}}{2^{i+1}t^{i+1}} \sum_{k=0}^{i} a_{i+1,k} t^k \right]$$

$$+ \frac{(-1)^{i-1}(i-1)!}{2^{i-1}t^{i-1}} \sum_{k=0}^{i-2} a_{i-1,k} t^k$$

$$= \frac{(-1)^{i+1}}{2^{i+1}(1+t)^i} h(t) \left[ 4i(1+t) \sum_{k=0}^{i-1} a_{i,k} t^k - \sum_{k=0}^{i} a_{i+1,k} t^k \right.$$

$$
- 4(i-1)i(1+t)^2 \sum_{k=0}^{i-2} a_{i-1,k} t^k \right.\left.\right.\right.$$}

$$= \frac{(-1)^{i+1}}{2^{i+1}(1+t)^i} h(t) \left[ 4i a_{i,0} - 4i(i-1)a_{i-1,0} - a_{i+1,0} \right.$$

$$+ [4i(a_{i,1} + a_{i,0}) - 4i(i-1)(a_{i-1,1} + 2a_{i-1,0}) - a_{i+1,1}] t$$

$$+ [4i(a_{i-1,1} + a_{i-1,2}) - 4i(i-1)(a_{i-1,i-3} + 2a_{i-1,i-2}) - a_{i+1,i-1}] t^{i-1}$$

$$+ [4i(a_{i,i-1} - 4i(i-1)a_{i-1,i-2} - a_{i+1,i}) t^i + \sum_{k=2}^{i-2} [4i(a_{i,k} + a_{i,k-1})$$

$$- 4i(i-1)(a_{i-1,k} + 2a_{i-1,k-1} + 2a_{i-1,k-2}) - a_{i+1,k}] t^k \right\}$$

$$= \frac{(-1)^{i+1}}{2^{i+1}(1+t)^i} h(t) \sum_{k=0}^{i-1} \frac{(i-1)!}{(i-k-1)!(i-k)!} a_{i,k} t^k.$$}

Hence, the general formulas (2.4) and (2.7) are obtained.

Adding the two formulas (2.3) and (2.4) yields

$$h^{(i)}(t) + \left[ \frac{1}{h(t)} \right]^{(i)} = \frac{(-1)^{i}}{2^{i+1}(1+t)^i} h(t) \left[ (1+t) \sum_{k=0}^{i-1} a_{i,k} t^k - t \sum_{k=0}^{i-1} b_{i,k} t^k \right]$$
Short proofs of a part of Lemma 2.1. In [25, p. 13, Remark 2.4], it was collected as
□
This implies that the function $H(t)$ is completely monotonic on $(0, \infty)$. The proof of Lemma 2.1 is completed.

**Short proofs of a part of Lemma 2.1.** In [25, p. 13, Remark 2.4], it was collected as an example that the function $\frac{1}{\alpha + t}$ is a Stieltjes function for $\alpha > 0$. The property (iv) in Section 3 of [4] (See also the property (vii) in [16, Theorem 1.3]) reads that if $f \in S$ then $f^\alpha \in S$ for $0 \leq \alpha \leq 1$. Specially for $\alpha = 1$ and $\alpha = \frac{1}{2}$, we have $h_1(t) = \frac{1}{\sqrt{1 + t}} \in S$. The property (i) in Section 3 of [4] (See also the property (i) in [16, Theorem 1.3]) states that if $f \in S \setminus \{0\}$ then $\frac{1}{f(t)} \in S$. Applying this property to $h_1(t)$ brings out

$$h(t) = \frac{1}{h_1(1/t)} \in S \tag{2.32}$$

which means, by the relation from the very ends of the inclusions (1.4), that $h(t) \in C([0, \infty])$ and, by the relation (1.6), that $\frac{1}{h(t)} \in B([0, \infty])$.

In [25, p. 24, Remark 3.11], it was listed as examples that $h_2(t) = t^\beta \in B([0, \infty])$ for $0 < \beta < 1$ and $h_3(t) = \frac{1}{1 + t} \in B([0, \infty])$. The item (iii) of Corollary 3.7 in [25, p. 20] write that if $f_1, f_2 \in B([0, \infty])$ then $f_1 \circ f_2 \in B([0, \infty])$. Applying $f_1$ and $f_2$ respectively to $h_2$ and $h_3$ reveals once again that $\frac{1}{h(t)} = \frac{1}{\sqrt{1 + t}} \in B([0, \infty])$.

Taking $h_3(x) = x + \frac{1}{x}$ and $h_4(t) = \frac{1}{h(t)} = \frac{1}{\sqrt{1 + x}}$. It is easy to see that $h_3 \in C([0, 1])$ and $0 < h_4(t) < 1$. A part of Theorem 3.6 in [25, p. 19] asserts that if $0 < f \in B([0, \infty])$ then $g \circ f \in C([0, \infty])$ for every $g \in C([0, \infty])$. Since $h_4 \in B([0, \infty])$, applying $f$ and $g$ in this assertion respectively to $h_2$ and $h_3$ leads to $H(t) = h(t) + \frac{1}{h(t)} \in C([0, \infty])$. The proof of Lemma 2.1 is completed.

**Lemma 2.2.** For $z \in C \setminus (-\infty, 0]$, the complex functions $h(z)$ and $\frac{1}{h(t)}$ have integral representations

$$h(z) = 1 + \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{u^2 - 1}} \frac{du}{u + z} \tag{2.33}$$

and

$$\frac{1}{h(z)} = 1 - \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{u^2 - 1}} \frac{du}{u + z}. \tag{2.34}$$
Consequently, the functions $h(t)$ and $1 - \frac{1}{h(t)}$ are Stieltjes functions and the complex function $H(z)$ has the integral representation
\begin{equation}
H(z) = 2 + \frac{1}{\pi} \int_{0}^{\infty} \rho(s) e^{-zs} \, ds \tag{2.35}
\end{equation}
for $z \in \mathbb{C} \setminus (-\infty, 0]$, where
\begin{equation}
\rho(s) = \int_{0}^{1/2} q(u) \left[ 1 - e^{-(1-2u)s} \right] e^{-us} \, du = \int_{0}^{1/2} q\left( \frac{1}{2} - u \right) (e^{us} - e^{-us}) e^{-s/2} \, du \tag{2.36}
\end{equation}
is nonnegative on $(0, \infty)$ and
\begin{equation}
q(u) = \sqrt{\frac{1}{u} - 1} - \frac{1}{\sqrt{1/u - 1}} \tag{2.37}
\end{equation}
on $(0, 1)$.

**Proof by Cauchy integral formula.** By standard arguments, we immediately obtain that
\begin{align*}
\lim_{z \to 0} [zh(z)] &= \lim_{z \to 0} \sqrt{z^2 + z} = \sqrt{\lim_{z \to 0} (z^2 + z)} = 0, \tag{2.38} \\
\lim_{z \to 0} \frac{z}{h(z)} &= \lim_{z \to 0} \frac{z}{\sqrt{1 + z} - 1} = \sqrt{\lim_{z \to 0} \frac{z}{1 + z}} = 0, \tag{2.39} \\
\lim_{z \to \infty} \sqrt{1 + \frac{1}{z}} &= \sqrt{1 + \lim_{z \to \infty} \frac{1}{z}} = 1, \tag{2.40} \\
\lim_{z \to \infty} \frac{1}{\sqrt{1 + \frac{1}{z}}} &= \lim_{z \to \infty} \frac{1}{\sqrt{1 + \frac{1}{z}}} = 1, \tag{2.41} \\
\frac{h(z)}{h(z)} &= \frac{1}{h(z)}. \tag{2.42}
\end{align*}

For $t \in (0, \infty)$ and $\varepsilon > 0$, we have
\begin{align*}
h(-t + i\varepsilon) &= \sqrt{1 + \frac{1}{-t + i\varepsilon}} = \sqrt{1 + \frac{-t - i\varepsilon}{t^2 + \varepsilon^2}} = \exp \left\{ \frac{1}{2} \ln \left( 1 + \frac{-t - i\varepsilon}{t^2 + \varepsilon^2} \right) \right\} \tag{2.43} \\
&= \exp \left\{ \frac{1}{2} \left[ \ln \left( \frac{t^2 + \varepsilon^2 - t - i\varepsilon}{t^2 + \varepsilon^2} \right) + i \arg \left( \frac{t^2 + \varepsilon^2 - t - i\varepsilon}{t^2 + \varepsilon^2} \right) \right] \right\} \\
&= \exp \left\{ \frac{1}{2} \left[ \ln p(t, \varepsilon) + i \arctan \frac{\varepsilon}{t^2 + \varepsilon^2} \right] \right\}, \quad t^2 + \varepsilon^2 - t > 0, \\
&= \exp \left\{ \frac{1}{2} \left[ \ln p(t, \varepsilon) + i \left( \arctan \frac{\varepsilon}{t^2 + \varepsilon^2} - \frac{\pi}{2} \right) \right] \right\}, \quad t^2 + \varepsilon^2 - t < 0, \\
&= \exp \left\{ \frac{1}{2} \left[ \ln \frac{\varepsilon}{t^2 + \varepsilon^2} - i \frac{\pi}{2} \right] \right\}, \quad t^2 + \varepsilon^2 - t = 0,
\end{align*}
where
\begin{equation}
p(t, \varepsilon) = \sqrt{\left( \frac{t^2 + \varepsilon^2 - t}{t^2 + \varepsilon^2} \right)^2 + \left( \frac{\varepsilon}{t^2 + \varepsilon^2} \right)^2}. \tag{2.44}
\end{equation}
Hence,
\[
\Im(-t + i\varepsilon) = \begin{cases} 
\exp\left[\frac{1}{2}\ln p(t, \varepsilon) \sin\left(\frac{1}{2}\arctan\frac{\varepsilon}{t^2 + \varepsilon^2}\right)\right], & t^2 + \varepsilon^2 - t > 0; \\
\exp\left[\frac{1}{2}\ln p(t, \varepsilon) \sin\left(\frac{1}{2}\arctan\frac{\varepsilon}{t^2 + \varepsilon^2} - \frac{\pi}{2}\right)\right], & t^2 + \varepsilon^2 - t < 0; \\
-\exp\left[\frac{1}{2}\ln \frac{\varepsilon}{t^2 + \varepsilon^2}\right] \sin\left(\frac{\pi}{4}\right), & t^2 + \varepsilon^2 - t = 0.
\end{cases}
\]

Accordingly,
\[
\lim_{\varepsilon \to 0^+} \Im(-t + i\varepsilon) = \begin{cases} 
-\sqrt{1 - t}, & 0 < t < 1; \\
\infty, & t = 1; \\
0, & t > 1.
\end{cases}
\] (2.44)

Similarly, for \(t \in (0, \infty)\) and \(\varepsilon > 0\), we have
\[
\frac{1}{h(-t + i\varepsilon)} = \exp\left[\frac{1}{2}\ln\left(1 + \frac{-t - i\varepsilon}{t^2 + \varepsilon^2}\right)\right]
\]
\[
= \begin{cases} 
\exp\left\{-\frac{1}{2}\ln p(t, \varepsilon) + i\arctan\frac{\varepsilon}{t^2 + \varepsilon^2}\right\}, & t^2 + \varepsilon^2 - t > 0; \\
\exp\left\{-\frac{1}{2}\ln p(t, \varepsilon) + i\left(\arctan\frac{\varepsilon}{t^2 + \varepsilon^2} - \pi\right)\right\}, & t^2 + \varepsilon^2 - t < 0; \\
\exp\left\{-\frac{1}{2}\ln \frac{\varepsilon}{t^2 + \varepsilon^2} - i\frac{\pi}{2}\right\}, & t^2 + \varepsilon^2 - t = 0.
\end{cases}
\]

Therefore,
\[
\Im\left[\frac{1}{h(-t + i\varepsilon)}\right] = \begin{cases} 
-\exp\left[\frac{1}{2}\ln p(t, \varepsilon) \sin\left(\frac{1}{2}\arctan\frac{\varepsilon}{t^2 + \varepsilon^2}\right)\right], & t^2 + \varepsilon^2 - t > 0; \\
-\exp\left[\frac{1}{2}\ln p(t, \varepsilon) \sin\left(\frac{1}{2}\arctan\frac{\varepsilon}{t^2 + \varepsilon^2} - \frac{\pi}{2}\right)\right], & t^2 + \varepsilon^2 - t < 0; \\
\exp\left[\frac{1}{2}\ln \frac{\varepsilon}{t^2 + \varepsilon^2}\right] \sin\left(\frac{\pi}{4}\right), & t^2 + \varepsilon^2 - t = 0.
\end{cases}
\]

Consequently,
\[
\lim_{\varepsilon \to 0^+} \Im\left[\frac{1}{h(-t + i\varepsilon)}\right] = \begin{cases} 
\sqrt{\frac{t}{1-t}}, & 0 < t < 1; \\
\infty, & t = 1; \\
0, & t > 1.
\end{cases}
\] (2.45)

Let \(D\) be a bounded domain with piecewise smooth boundary. The famous Cauchy integral formula (See [8, p. 113]) reads that if \(f(z)\) is analytic on \(D\), and \(f(z)\) extends smoothly to the boundary of \(D\), then
\[
f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(w)}{w - z} dw, \quad z \in D.
\] (2.46)

For any fixed point \(z \in \mathbb{C} \setminus (-\infty, 0]\), choose \(0 < \varepsilon < 1\) and \(r > 0\) such that \(0 < \varepsilon < |z| < r\), and consider the positively oriented contour \(C(\varepsilon, r)\) in \(\mathbb{C} \setminus (-\infty, 0]\) consisting of the half circle \(z = \varepsilon e^{i\theta}\) for \(\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) and the half lines \(z = x \pm i\varepsilon\)
for \( x \leq 0 \) until they cut the circle \(|z| = r\), which close the contour at the points \(-r(\varepsilon) \pm i\varepsilon\), where \( 0 < r(\varepsilon) \rightarrow r \) as \( \varepsilon \rightarrow 0 \). See Figure 1.

\[ z = \varepsilon e^{i\theta} \]

By the above mentioned Cauchy integral formula, we have

\[
h(z) = \frac{1}{2\pi i} \oint_{C(\varepsilon, r)} \frac{h(w)}{w-z} \, dw
= \frac{1}{2\pi i} \left[ \int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta} h(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta} - z} \, d\theta + \int_{-r(\varepsilon)}^{0} \frac{h(x + i\varepsilon)}{x + i\varepsilon - z} \, dx \right. \\
\left. + \int_{0}^{-r(\varepsilon)} \frac{h(x - i\varepsilon)}{x - i\varepsilon - z} \, dx + \int_{\arg[-r(\varepsilon) - i\varepsilon]}^{\arg[-r(\varepsilon) + i\varepsilon]} \frac{ire^{i\theta} h(re^{i\theta})}{re^{i\theta} - z} \, d\theta \right].
\] (2.47)

By the limit (2.38), it follows that

\[
\lim_{\varepsilon \to 0^+} \int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta} h(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta} - z} \, d\theta = 0.
\] (2.48)

In virtue of the limit (2.40), it can be derived that

\[
\lim_{\varepsilon \to 0^+} \int_{\arg[-r(\varepsilon) - i\varepsilon]}^{\arg[-r(\varepsilon) + i\varepsilon]} \frac{ire^{i\theta} h(re^{i\theta})}{re^{i\theta} - z} \, d\theta = \lim_{r \to \infty} \int_{-\pi}^{\pi} \frac{ire^{i\theta} h(re^{i\theta})}{re^{i\theta} - z} \, d\theta = 2\pi i.
\] (2.49)

Making use of the limits (2.42) and (2.44) yields that

\[
\int_{-r(\varepsilon)}^{0} \frac{h(x + i\varepsilon)}{x + i\varepsilon - z} \, dx + \int_{0}^{-r(\varepsilon)} \frac{h(x - i\varepsilon)}{x - i\varepsilon - z} \, dx = \int_{-r(\varepsilon)}^{0} \left[ \frac{h(x + i\varepsilon)}{x + i\varepsilon - z} - \frac{h(x - i\varepsilon)}{x - i\varepsilon - z} \right] \, dx
\]
By virtue of the limit (2.41), it may be deduced that

\[
\int_{-r(\varepsilon)}^{0} \frac{(x - i\varepsilon - z)h(x + i\varepsilon) - (x + i\varepsilon - z)h(x - i\varepsilon)}{(x + i\varepsilon - z)(x - i\varepsilon - z)} \, dx
\]

Employing the limits (2.43) and (2.45) yields that

\[
\int_{-r(\varepsilon)}^{0} \frac{(x - z)[h(x + i\varepsilon) - h(x - i\varepsilon)] - i\varepsilon[h(x - i\varepsilon) + h(x + i\varepsilon)]}{(x + i\varepsilon - z)(x - i\varepsilon - z)} \, dx
\]

as \(\varepsilon \to 0^+\) and \(r \to \infty\). Substituting equations (2.48), (2.49), and (2.50) into (2.47) and simplifying produce the integral representation (2.33).

Similarly, by the above mentioned Cauchy integral formula, we have

\[
\text{as } \varepsilon \to 0^+ \text{ and } r \to \infty. \quad \text{Substituting equations (2.48), (2.49), and (2.50) into (2.47) and simplifying produce the integral representation (2.33).}
\]

Similarly, by the above mentioned Cauchy integral formula, we have

\[
\frac{1}{h(z)} = \frac{1}{2\pi i} \int_{C(\varepsilon,r)} \frac{1/h(w)}{w - z} \, dw
\]

From the limit (2.39), it follows that

\[
\lim_{\varepsilon \to 0^+} \int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta} [1/h(e^{i\theta})]}{\varepsilon e^{i\theta} - z} \, d\theta = 0.
\]

By virtue of the limit (2.41), it may be deduced that

\[
\lim_{\varepsilon \to 0^+} \int_{\arg[-r(\varepsilon) - i\varepsilon]}^{\arg[-r(\varepsilon) + i\varepsilon]} \frac{i\varepsilon e^{i\theta} [1/h(e^{i\theta})]}{r e^{i\theta} - z} \, d\theta = 2\pi i.
\]

Employing the limits (2.43) and (2.45) yields that

\[
\int_{-r(\varepsilon)}^{0} \frac{1/h(x + i\varepsilon)}{x + i\varepsilon - z} \, dx + \int_{0}^{-r(\varepsilon)} \frac{1/h(x - i\varepsilon)}{x - i\varepsilon - z} \, dx
\]

\[
\quad \to 2i \int_{0}^{r} \frac{(x - z)\Im[1/h(x + i\varepsilon)] - \varepsilon\Re[1/h(x + i\varepsilon)]}{(x + i\varepsilon - z)(x - i\varepsilon - z)} \, dx
\]

\[
\quad \to 2i \int_{-r}^{0} \lim_{\varepsilon \to 0^+} \Im[1/h(x + i\varepsilon)] \, dx \quad \text{as } \varepsilon \to 0^+
\]

\[
\quad \to -2i \int_{0}^{\infty} \lim_{\varepsilon \to 0^+} \Im[h(-t + i\varepsilon)] \, dt \quad \text{as } r \to \infty
\]
\[ \alpha = 1 \] yields \[ 1 \] results in

The property (x) in [16, Theorem 1.3] util 

Substituting equations (2.52), (2.53), and (2.54) into (2.51) and simplifying produce the integral representation (2.34).

Adding (2.33) and (2.34) leads to

Substituting equations (2.52), (2.53), and (2.54) into (2.51) and simplifying produce the integral representation (2.34).

The proof of Lemma 2.2 is thus completed. \[ \square \]

The proof of Lemma 2.2 is thus completed. \[ \square \]

Proof by Stieltjes-Perron inversion formula. The property (x) in [16, Theorem 1.3] formulates that if \( f \in \mathcal{S} \) then \( f^\alpha (0^+) - f^\alpha (\frac{1}{2}) \in \mathcal{S} \) for \( 0 \leq \alpha \leq 1 \). Since \( h(t) \in \mathcal{S} \), see (2.32), and, by the property (i) in [16, Theorem 1.3], \( \frac{h(1/t)}{t} \in \mathcal{S} \), replacing \( f \) by \( \frac{h(1/t)}{t} \), making use of the easy fact that \( f(0^+) = \lim_{t \to 0^+} f(t) = 1 \), and letting \( \alpha = 1 \) yield \( 1 - \frac{1}{t} \in \mathcal{S} \).

For a Stieltjes function \( f \) given by (1.3), by the Stieltjes-Perron inversion formula in [14, p. 591], we can determine the scalars \( a = \lim_{x \to 0^+} [xf(x)] \) and \( b = \lim_{x \to \infty} f(x) \) and the measure

\[ \mu(s) = -\frac{1}{\pi} \lim_{t \to 0^+} \Im \int_{-\infty}^{-s} f(u + ti) \, du, \] (2.55)
as done in [3, 15]. Specially, for the function \( h(x) \), since 
\[ a = \lim_{x \to 0^+} x h(x) = 0 \]
and 
\[ b = \lim_{x \to \infty} h(x) = 1, \]
we have
\[ h(z) = 1 + \int_0^\infty \frac{d\Phi(u)}{u + z} \quad (2.56) \]
for \(|\arg z| < \pi\), where
\[ \Phi(u) = \frac{1}{\tau} \lim_{s \to 0^+} \int_u^\infty \sqrt{1 - \frac{1}{\tau - is}} \, d\tau = -\frac{1}{\pi} \int_u^\infty \sqrt{\frac{1}{\tau} - 1} \, d\tau \]
when \( 0 < \tau < 1 \) and \( \Phi(u) = 0 \) when \( \tau > 1 \) because taking \( s \to 0^+ \) we obtain
\[ \sqrt{1 - \frac{1}{\tau - is}} \to 0 \] when \( 0 < \tau < 1 \) and \( \Phi(u) = 0 \) when \( u > 1 \). Substituting \( \Phi(u) \) in the representation (2.56) results in the formula (2.33).

The rest is the same as in the first proof. Lemma 2.2 is proved once again. □

3. The harmonic mean is a Bernstein function

Our results on the harmonic mean \( H_{x,y}(t) \) may be stated as the theorem below.

**Theorem 3.1.** The harmonic mean \( H_{x,y}(t) \) defined by (1.15) is a Bernstein function of \( t \) on \((-\min\{x, y\}, \infty)\) for \( x, y > 0 \) with \( x \neq y \) and has the integral representation
\[ H_{x,y}(t) = H(x, y) + t + \frac{(x - y)^2}{4} \int_0^\infty \frac{1}{1 - e^{-(x+y)u}} \, du. \quad (3.1) \]
Consequently,
\[ H(x, y) = A(x, y) - \frac{(x - y)^2}{2} \int_0^\infty \frac{e^{-(x+y)u}}{u} \, du \quad (3.2) \]
\[ H(s, y + s) = s + \frac{y^2}{4} \int_0^\infty \frac{1}{1 - e^{-su}} \, du, \quad s > 0. \quad (3.3) \]

**Proof.** The harmonic mean \( H_{x,y}(t) \) meets
\[ H'_{x,y}(t) = \frac{2[x^2 + y^2 + 2(x + y)t + 2t^2]}{(x + y + 2t)^2} = 1 + \frac{(x - y)^2}{(x + y + 2t)^2} > 1. \quad (3.4) \]
It is obvious that the derivative \( H'_{x,y}(t) \) is completely monotonic with respect to \( t \). As a result, the harmonic mean \( H_{x,y}(t) \) is a Bernstein function of \( t \) on \((-\min\{x, y\}, \infty)\) for \( x, y > 0 \) with \( x \neq y \).

In [1, p. 255, 6.1.1], it was listed that, for \( \Re z > 0 \) and \( \Re k > 0 \), the classical Euler gamma function
\[ \Gamma(z) = k^z \int_0^\infty t^{z-1} e^{-kt} \, dt. \quad (3.5) \]
This formula can be rearranged as
\[
\frac{1}{z} = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1} e^{-zt} \, dt
\]
(3.6)
for \(\Re z > 0\) and \(\Re w > 0\). Combining (3.6) with (3.4) yields
\[
H'_{x,y}(t) = 1 + (x - y)^2 \int_0^\infty u e^{-(x+y+2t)u} \, du,
\]
(3.7)
and so, by integrating with respect to \(t \in (0,s)\) on both sides of (3.7), the formula (3.1) follows.

Letting \(s \to \infty\) on both sides of (3.1) and using the limit \(\lim_{s \to \infty} [H_{x,y}(s) - s] = A(x,y)\) generate the formula (3.2).

Taking \(x \to 0^+\) in (3.1) produces (3.3). Theorem 3.1 is thus proved. \(\square\)

**Remark 3.1.** By [7, pp. 161–162, Theorem 3] or [25, p. 45, Proposition 5.17], it can be derived that the reciprocal of the harmonic mean \(H_{x,y}(t)\), that is, the function \(\frac{1}{A(t, x+t, t, y+y)}\), is logarithmically completely monotonic.

This logarithmically complete monotonicity can also be proved by considering
\[
[\ln H_{x,y}(t)]' = \frac{x^2 + y^2 + (x + y)t + 2t^2}{(x + t)(y + t)(x + y + 2t)} = \frac{1}{2} \left( \frac{1}{x + t} + \frac{1}{y + t} \right) \left[ 1 + \frac{(x - y)^2}{(x + y + 2t)^2} \right]
\]
and that the product and sum of finitely many completely monotonic functions are also completely monotonic functions.

Moreover, from (3.4), it follows readily that \(H_{x,y}(t) - t\) is an increasing function in \(t \in (- \min \{x,y\}, \infty)\) for \(x, y > 0\) with \(x \neq y\).

4. The geometric mean is a Bernstein function

Our results on the geometric mean \(G_{x,y}(t)\) can be summarized as two theorems.

**Theorem 4.1.** Let \(x, y > 0\) with \(x \neq y\). Then the geometric mean \(G_{x,y}(t)\) defined by (1.16) is a Bernstein function of \(t\) on \((- \min \{x,y\}, \infty)\).

We supply three proofs of Theorems 4.1.

**First proof.** By a direct differentiation, we have
\[
G'_{x,y}(t) = \sqrt{\frac{x + t}{y + t} \frac{x + y + 2t}{2(x + t)}}.
\]
Taking the logarithm on both sides of the above equality creates
\[
\ln G'_{x,y}(t) = \frac{1}{2} \ln \frac{x + t}{y + t} + \ln \frac{x + y + 2t}{2(x + t)},
\]
(4.1)
In [1, p. 230, 5.1.32], it was collected that for \(a > 0\) and \(b > 0\),
\[
\ln \frac{b}{a} = \int_0^\infty e^{-au} - e^{-bu} \, du.
\]
(4.2)
Using this formula in (4.1) leads to
\[
\ln G'_{x,y}(t) = \int_0^\infty \frac{e^{-(x+t)v} + e^{-(y+t)v} - 2e^{-v[(x+t)+(y+t)]/2}}{2v} \, dv.
\]
Since the function \(e^{-t} \) is convex on \(\mathbb{R}\), we have
\[
e^{-(x+t)v} + e^{-(y+t)v} - 2e^{-v[(x+t)+(y+t)]/2} \geq 0.
\]
Therefore, we have
\[
[\ln G'_{x,y}(t)]^{(k)} = \frac{(-1)^k}{2} \int_0^\infty \{e^{-(x+t)u} + e^{-(y+t)u} - 2e^{-u[(x+t)+(y+t)]/2}\} u^{k-1} \, d\nu.
\]
This means that the derivative \( G'_{x,y}(t) \) is logarithmically completely monotonic, and so it is also completely monotonic. As a result, the geometric mean \( G_{x,y}(t) \) is a Bernstein function. □

**Second proof.** It is clear that the geometric mean \( G_{x,y}(t) \) satisfies
\[
G'_{x,y}(t) = \frac{1}{2} \left( \frac{x + t}{y + t} + \frac{y + t}{x + t} \right) = \frac{1}{2} \left( \sqrt{u} + \frac{1}{\sqrt{u}} \right) \triangleq f(u) \tag{4.3}
\]
and
\[
[\ln G_{x,y}(t)]' = \frac{1}{2} \left( \frac{1}{x + t} + \frac{1}{y + t} \right), \tag{4.4}
\]
where
\[
u \triangleq u_{x,y}(t) = \frac{x + t}{y + t} = 1 + \frac{x - y}{y + t}. \tag{4.5}
\]
If \( 0 < x < y \), then \( 0 < u_{x,y}(t) < 1 \) for \( t \in (-x, \infty) \) and \( u_{x,y}(t) = \frac{y - x}{y + t} \) is completely monotonic in \( t \in (-x, \infty) \). On the other hand, the function \( f(u) \) is positive and
\[
\begin{align*}
f^{(i)}(u) &= \frac{1}{2} \left[ (-1)^{i-1} \frac{(2i-3)!!}{2^i} u^{-(2i-1)/2} + (-1)^i \frac{(2i-1)!!}{2^i} u^{-(2i+1)/2} \right] \\
&= \frac{(-1)^i (2i-3)!!}{2^{i+1}} \frac{1}{u^{(2i-1)/2}} \left( \frac{2i-1}{u} - 1 \right)
\end{align*}
\]
for \( i \in \mathbb{N} \), which implies that the function \( f(u) \) is completely monotonic on \((0,1)\).

A ready modification of a conclusion in [5, p. 83] yields the following conclusion: If \( g \) and \( h' \) are completely monotonic functions such that \( g(h(x)) \) is defined on an interval \( I \), then \( x \mapsto g(h(x)) \) is also completely monotonic on \( I \); So, when \( y > x > 0 \), the derivative \( G'_{x,y}(t) \) is completely monotonic and the geometric mean \( G_{x,y}(t) \) is a Bernstein function. Consequently, considering the symmetric property \( G_{x,y}(t) = G_{y,x}(t) \), it is easily obtained that the geometric mean \( G_{x,y}(t) \) for \( t \in (-\min\{x,y\}, \infty) \) with \( x \neq y \) is a Bernstein function. □

**Remark 4.1.** From the equality in (4.3), it is easy to derive that the function \( G_{x,y}(t) - t \) is increasing in \( t \in (-\min\{x,y\}, \infty) \) for \( x, y > 0 \) with \( x \neq y \).

From (4.4), it is immediate to deduce that the reciprocal of the geometric mean \( G_{x,y}(t) \) is a logarithmically completely monotonic function of \( t \in (-\min\{x,y\}, \infty) \) for \( x, y > 0 \) with \( x \neq y \).

**Third proof.** By (4.3) and (4.5), it follows that
\[
G'_{x,y}(t) = \frac{1}{2} \left[ h \left( \frac{y + t}{x - y} \right) + \frac{1}{h \left( \frac{y + t}{x - y} \right)} \right] = \frac{1}{2} H \left( \frac{y + t}{x - y} \right) \tag{4.6}
\]
and
\[
[G'_{x,y}(t)]^{(i)} = \frac{1}{2(x - y)^{i}} H^{(i)} \left( \frac{y + t}{x - y} \right)
\]
for \( i \in \{0\} \cup \mathbb{N} \). By the formula (2.5) in Lemma 2.1, we have
\[ [G_{x,y}'(t)]^{(i)} = \frac{(-1)^i}{2^{i+1}(y+t)^{i+1}(x+t)^i h\left(\frac{y+t}{x-t}\right)} \times \sum_{k=0}^{i-1} \frac{(i-1)!(i+1)!(2i-2k-1)!!}{(i-k-1)!(i-k+1)!} y^k \left(\frac{y+t}{x-y}\right)^k, \]

which means that, when \( x > y \), the derivative \( G_{x,y}'(t) \) is completely monotonic. Since \( G_{x,y}(t) = G_{y,x}(t) \), when \( x < y \), the derivative \( G_{y,x}'(t) \) is also completely monotonic. This implies that the geometric mean \( G_{x,y}(t) \) is a Bernstein function of \( t \in (-\min\{x,y\}, \infty) \). \( \square \)

**Theorem 4.2.** For \( x > y > 0 \) and \( z \in \mathbb{C} \setminus (-\infty, -y] \), the geometric mean \( G_{x,y}(z) \) has the integral representation

\[
G_{x,y}(z) = G(x,y) + z + \frac{x-y}{2\pi} \int_0^\infty \frac{\rho((x-y)s)}{s} e^{-ys} \left(1 - e^{-s^2}\right) ds, \tag{4.7}
\]

where the function \( \rho \) is defined by (2.36). Consequently, the geometric mean \( G_{x,y}(t) \) is a Bernstein function of \( t \) on \((-\min\{x,y\}, \infty)\). 

**Proof.** For \( x > y > 0 \) and \( z \in \mathbb{C} \setminus (-\infty, -y] \), making use of

\[
G_{x,y}'(z) = \frac{1}{2} \left[ h\left(\frac{y+z}{x-y}\right) + \frac{1}{h\left(\frac{z+y}{x-y}\right)} \right] = \frac{1}{2} H\left(\frac{y+z}{x-y}\right)
\]

and (2.35) gives

\[
G_{x,y}'(z) = 1 + \frac{1}{2\pi} \int_0^\infty \rho(s) \exp\left(-\frac{y+z}{x-y}s\right) ds.
\]

Integrating with respect to \( z \) from 0 to \( w \) on both sides of the above equation and interchanging the order of integrals yield

\[
G_{x,y}(w) - G_{x,y}(0) = w + \frac{x-y}{2\pi} \int_0^\infty \frac{\rho(s)}{s} \exp\left(-\frac{ys}{x-y}\right) \left[1 - \exp\left(-\frac{sw}{x-y}\right)\right] ds
\]

\[
= w + \frac{x-y}{2\pi} \int_0^\infty \frac{\rho((x-y)s)}{s} e^{-ys} \left(1 - e^{-ws}\right) ds.
\]

Since \( G_{x,y}(0) = G(x,y) \), the integral representation (4.7) is readily deduced.

By the characterization expressed by (1.7) and the integral representation (4.7) applied to \( z = t \in (-\min\{x,y\}, \infty) \), it is immediate to see that the geometric mean \( G_{x,y}(t) \) is a Bernstein function of \( t \) on \((-\min\{x,y\}, \infty)\). \( \square \)

**Remark 4.2.** Taking \( z \to \infty \) in (4.7) and using \( \lim_{z \to \infty} [G_{x,y}(z) - z] = A(x,y) \) yield

\[
A(x,y) = G(x,y) + \frac{x-y}{2\pi} \int_0^\infty \frac{\rho((x-y)s)}{s} e^{-ws} ds \geq G(x,y). \tag{4.8}
\]

The equality in (4.8) is valid if and only if \( x = y \). This gives a new proof of the fundamental and well known AG mean inequality.
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(Qi) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.wordpress.com

(Zhang) Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
E-mail address: xiao.jing.zhang@qq.com

(Li) Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
E-mail address: wen.hui.li@foxmail.com