A thermodynamically consistent local master equation

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Master equations under appropriate assumptions are efficient tools for the study of open quantum systems. For many-body systems, subsystems of which locally couple to thermal baths and weakly interact with each other, the local approach provides a more convenient description than the global approach. However, these local master equations are believed to generate inconsistencies with the laws of thermodynamics when inter-subsystem interactions exist. Here we develop an alternative local master equation by applying the secular approximation to modify the inter-subsystem interactions. Our results show that violations of thermodynamic laws will be avoided after correcting inter-subsystem interactions. We finally use our results to investigate a two-qubit heat transfer network, and show the validity of our modified master equation.

I. INTRODUCTION

Theoretical descriptions of open quantum systems is a subject of both fundamental and practical importance. On the one hand, quantum systems can be not completely isolated from the environment; on the other hand, external apparatuses are needed to manipulate and control the quantum system of interest. There have already been various well-established treatments of open quantum systems [1–4], and they have been applied to many fields, such as quantum optics, quantum chemistry, and quantum information [5–10].

For Markovian cases where memory effects of the environment can be neglected, the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation is widely used to describe the dynamics of quantum systems [11–15]. For those quantum systems which are composed of several interacting subsystems, two main approaches – the global approach and the local approach – are developed to describe the evolution of open quantum systems. The global approach is based on eigenstates of the whole Hamiltonian which is composed of the Hamiltonian of each subsystem and the interaction between subsystems. Although it is thermodynamically consistent, those eigenstates and eigenvalues are difficult to get, especially for large-size systems. In contrast, only the eigenstates of each individual subsystem are needed for the local approach, which holds in the weak inter-subsystem coupling regime [23]. However, the local approach was harshly criticized for violating the second law of thermodynamics [31–32]. In order to solve this problem, many researchers have come up with various methods [33–38], among which the most popular one is redefining the heat current [32, 34–36, 38].

Here, we introduce an alternative thermodynamically-consistent local master equation, by modifying the weak inter-subsystem interaction while retaining the GKLS form. The main idea of our work is that we regard the weak inter-subsystem interaction as an external perturbation, and then eliminate some components of the inter-subsystem interaction under the spectral decomposition through the secular approximation [3, 4]. Those redundant components do not contribute to the time evolution of the open quantum system in the weak coupling limit, and the remaining part of the inter-subsystem interaction plays the role of producing transitions in the degenerate eigensubspace of the Hamiltonian $H_s = \sum_{i=1}^{n} H_i$, which is the Hamiltonian of the system without inter-subsystem interactions (see Eq. (1)). In this sense, it is natural to define the local heat current directly using the non-interacting Hamiltonian $H_s$ (see Eq. (12)). Based on this definition of the local heat current, we generally show that our modified local master equation satisfies the first and second laws of thermodynamics at the same time.

This paper is organized as follows. In Sec. II, we present the derivation of the modified local master equation, and discuss its applicable conditions. In Sec. III, we generally prove the thermodynamic consistency of the modified local master equation. In Sec. IV, we consider the two-qubit heat transfer network model and show how our approach is employed to study open quantum systems. Finally, we conclude in Sec. V.

II. THE MODIFIED LOCAL MASTER EQUATION

In this section, we consider the case that a system $\mathcal{S}$ is composed of $n$ subsystems $s_1, s_2, \cdots, s_n$, which weakly interact with each other, and each of which is weakly coupled to a bath $b_i (i = 1, 2, \cdots, n)$. Note that this is a general case for which the local GKLS master equation is suitable. In such a scenario, the total time-independent Hamiltonian can be written as:

$$H = \sum_{i=1}^{n} (H_i + H_{b_i} + \alpha V_{i}) + \alpha H_I,$$

(1)
where $H_i$ is the Hamiltonian of the subsystem $s_i$, $H_b$ is the Hamiltonian of the bath $b_i$, $\alpha V_{ib}$ is the interaction between $s_i$ and $b_i$, and the inter-subsystem interaction is denoted as $\alpha H_I$, which may be composed of two-subsystem interactions, three-subsystem interactions, etc. Here, $\alpha$ is a small dimensionless parameter, which accounts for the weak coupling strength, and we assume that the inter-subsystem interaction and the system-bath coupling are of the same order of magnitude.

The time evolution of the whole system (including baths) is described by the von Neumann equation, which, in the Schrödinger picture, can be written as

$$\partial_t \rho(t) = -i[H, \rho(t)].$$  

(2)

Note that it is easy to derive the quantum Markovian master equation in the interaction picture, and we transform the problem to the interaction picture through the unitary transformation generated by the non-interacting Hamiltonian $H_0 \equiv \sum_{i=1}^n (H_i + H_b)$. After this transformation, Eq. (2) becomes

$$\partial_t \tilde{\rho}(t) = -i\alpha \left[ \tilde{H}_I(t), \tilde{\rho}(t) \right],$$  

(3)

where operators in the interaction picture are defined as $\tilde{O}(t) = e^{it\hat{H}_b}Oe^{-it\hat{H}_b}$ with $O$ being the operator in the Schrodinger picture.

Following the standard procedure, we integrate Eq. (3) once, and get

$$\tilde{\rho}(t) = \tilde{\rho}(0) - i\alpha \int_0^t dt' \left[ \tilde{H}_I(t'), \tilde{\rho}(t') \right].$$  

(4)

Inserting Eq. (4) into Eq. (3), and keeping terms up to the second order of the small parameter $\alpha$, we can get

$$\partial_t \tilde{\rho}(t) = -i\alpha \left[ \tilde{H}_{int}(t), \tilde{\rho}(0) \right] - \alpha^2 \int_0^t dt' \left[ \tilde{H}_{int}(t'), \left[ \tilde{H}_{int}(t), \tilde{\rho}(t') \right] \right],$$  

(5)

where for brevity, we let $\tilde{H}_{int}(t) \equiv \sum_{i=1}^N \tilde{V}_{ib}(t) + \tilde{H}_I(t)$. Note that now the evolution of the density matrix just depends on the present state $\tilde{\rho}(t)$, and this is the spirit of the Markov approximation. Due to the weak coupling between the system and the bath, the influence of the system on the bath can be neglected. Thus, the state of the total system can be approximately written as $\tilde{\rho}(t) \approx \tilde{\rho}_s(t) \otimes \prod_i \tau_i$, where $\tilde{\rho}_s(t)$ is the reduced density matrix of the system and $\tau_i$ is the state of the bath $b_i$.

Note that the system-bath coupling $V_{ib}$ can be generally written as $\tilde{V}_{ib} = \sum_{\mu} A_{\mu}^b \otimes B_{\mu}^s \tilde{A}_{\mu}^b$, where $A_{\mu}^b$ and $B_{\mu}^s$ are Hermitian operators of the system $i$ and the bath $b_i$, respectively. Taking trace over the baths and assuming that $\text{Tr}(\tau_i B_{\mu}^s) = 0$, one can obtain

$$\partial_t \tilde{\rho}_s(t) = -i\alpha \left[ \tilde{H}_I(t), \tilde{\rho}_s(0) \right] - \alpha^2 \sum_i \int_0^t dt' \text{Tr}_b \left[ \tilde{V}_{ib}(t), \left[ \tilde{V}_{ib}(t'), \tilde{\rho}_s(t') \right] \right],$$  

(6)

where we have used the fact that those cross terms, such as $\text{Tr}_b(\tilde{H}_I(t)\tilde{\rho}(t))$ and $\text{Tr}_b(\tilde{V}_{ib}(t)\tilde{\rho}(t)) (i \neq j)$, vanish due to $\text{Tr}(\tau_i B_{\mu}^s) = 0$. From Eq. (6), one note that by treating the system-bath coupling $(\tilde{V}_{ib})$ and the inter-subsystem interaction $(\tilde{H}_I)$ on the same footing, the system-bath coupling reproduces those terms that one obtained in the traditional derivation of the GKLS master equation [3, 4], while the inter-subsystem interaction generates the second-order time-dependent perturbation theory in quantum mechanics [39].

For the following secular approximation, we decompose those operators acting on the Hilbert space of the system $s$ in the eigenbasis of $H_s \equiv \sum_{i=1}^N H_i$. For instance, the decomposition of the inter-subsystem interaction $\tilde{H}_I$ can be expressed as

$$\tilde{H}_I(t) = e^{iH_s t} e^{-iH_s t} = \sum_{\omega} e^{-i\omega t} H_I(\omega),$$  

(7)

where $H_I(\omega) = \sum_{\varepsilon_m, \varepsilon_n} \Pi(\varepsilon_m) H_I \Pi(\varepsilon_n)$ with $\Pi(\varepsilon)$ being the projector on the eigenspace associated to the energy $\varepsilon$ of $H_s$. Substituting those interaction operators of the decomposition form into Eq. (6) and doing the integration once again, one obtains

$$\tilde{\rho}_s(u) = \tilde{\rho}_s(0) - i\alpha \int_0^u dt \sum_{\omega} e^{-i\omega t} \left[ H_I(t), \tilde{\rho}_s(0) \right] - \alpha^2 \sum_{\omega, \omega'} \int_0^u dt \sum_{\varepsilon_m, \varepsilon_n} e^{-i\omega t} \left[ H_I(t), \left[ H_I(t), \tilde{\rho}_s(t) \right] \right] \int_0^t dt' e^{-i\omega' t'}$$

$$+ \alpha^2 \sum_{i=1}^n \int_0^u dt \sum_{\omega, \omega', \mu, \nu} e^{i(\omega - \omega') t} \left[ A_{\mu}^b(\omega) \tilde{\rho}_s(t) A_{\nu}^b(\omega') - A_{\mu}^b(\omega) A_{\nu}^{b\dagger}(\omega') \tilde{\rho}_s(t) \right] \Gamma_{\mu \nu}^{\omega \omega'}(\omega') + h.c.$$  

(8)

where $h.c.$ stands for the Hermitian conjugated expression, $A_{\mu}^b(\omega) = A_{\mu}^b(-\omega)$, and $\Gamma_{\mu \nu}^{\omega \omega'}(\omega')$ is the one-side
Fourier transformation of the correlation function of the bath $b_i$, and is defined as

$$
\Gamma^{\mu\nu}_i(\omega') = \int_0^\infty dt' e^{i\omega't'} \text{Tr} \left[ \tau_i B^{\mu}_i(t') B^{\nu}_i(0) \right],
$$

where we have taken the integral upper limit to infinity based on the fact that the correlation of the bath will decay rapidly \[3\]. In order to find out those dominant terms, following the standard procedure for deriving the GKLS form \[4\], we define rescaled times $\tau = \alpha t, \tau' = \alpha t'$, and $\sigma = \alpha u$. Then, Eq. \((8)\) becomes

$$
\hat{\rho}_{\alpha}^{\sigma} = \hat{\rho}(0) - i \int_0^\sigma d\tau \sum_\omega e^{-i\omega\tau} \left[ H_1(\omega) , \hat{\rho}(0) \right] + \int_0^\sigma d\tau \sum_{\omega,\omega'} e^{-i\omega'\tau} \left[ H_1(\omega), \left[ H_1(\omega'), \hat{\rho}(\frac{\tau}{\alpha}) \right] \right] \int_0^{\tau'} d\tau' e^{-i\omega'\tau'} + \alpha \sum_{i=1}^n \int_0^\sigma d\tau \sum_{\omega,\omega',\mu,\nu} e^{i\omega'\tau} \left[ A^{\mu}_i(\omega) \hat{\rho}(\frac{\tau}{\alpha}) A^{\nu\dagger}_{i'}(\omega') - A^{\mu}_i(\omega) A^{\nu\dagger}_{i'}(\omega') \hat{\rho}(\frac{\tau}{\alpha}) \right] \Gamma^{\mu\nu}_{i'}(\omega') + h.c. \tag{10}
$$

Taking the weak coupling limit $\alpha \to 0$ while keeping $\sigma$ and $\tau$ finite, those terms that satisfy $\omega > O(\alpha), \omega' > O(\omega)$ and $\omega - \omega' > O(\alpha)$, vanish due to the Riemann-Lebesgue lemma \[4\].

Lemma Let $f(t)$ be integrable in a finite interval $[a, b]$, then

$$
\lim_{x \to \infty} \int_a^b dt e^{ixt} f(t) = 0.
$$

In contrast, those terms that satisfy $\omega \leq O(\alpha), \omega' \leq O(\alpha)$ and $\omega - \omega' \leq O(\alpha)$ will contribute to the time evolution. Note that at this present, it is not enough to obtain the GKLS form. Thus, we introduce further assumptions: $\omega \gg \alpha$ and $\omega - \omega' \gg \alpha$, which, roughly speaking, means that the energy spectrum of $H_s$ is sparse. Therefore, in the first and second line of Eq. \((10)\), only the $\omega = 0$ and $\omega' = 0$ contributions need to be kept, and in the third line of Eq. \((10)\), only the $\omega = \omega'$ contributions need to be kept. Note that this treatment is usually known as the secular approximation \[3, 4\]. Here the secular approximation is used to treat not only the system-bath coupling but also the inter-subsystem interaction. Finally, returning back to the original time variable $(u, t, t')$, and turning to the differential equation in the Schroedinger picture, one obtains

$$
\partial_t \rho_s = -i \left[ H_s + \alpha^2 H_{LS} + \alpha H_1(0), \rho_s \right] + \alpha^2 \sum_{i=1}^n D_i[\rho_s], \tag{11}
$$

where $H_1(\omega = 0) \equiv H_1(0)$, and

$$
H_{LS} = \sum_{i,\omega,\mu,\nu} S^{\mu\nu}_{i}(\omega) A^{\mu}_i(\omega) A^{\nu\dagger}_i(\omega),
$$

$$
D_i[\rho_s] = \sum_{\omega,\mu,\nu} \gamma^{\mu\nu}_{i}(\omega) [A^{\mu}_i(\omega) A^{\nu\dagger}_i(\omega) \rho_s - A^{\mu}_i(\omega) \rho_s A^{\nu\dagger}_i(\omega)]
$$

are the Lamb shift and the dissipation operator of the subsystems $s_i$, respectively. Here, we have let $\Gamma^{\mu\nu}_{i'}(\omega) = \frac{1}{2} \gamma^{\mu\nu}_{i'}(\omega) + i S^{\mu\nu}_{i'}(\omega)$. Eq. \((11)\) is the modified local master equation we derived. One notes that this equation is still in the GKLS form and only the inter-subsystem interaction is modified. In the next section, we will show that this equation not only satisfies the first law of thermodynamics without changing the standard definition of the heat current, but also is consistent with the second law of thermodynamics.

Here, we end this section with some intuitive notes about the modified master equation so as to help readers have a clear understanding. 1) In order to apply the secular approximation, we decompose operators in the basis of $H_s$. This makes the sparse spectrum condition ($\omega \gg \alpha$ and $\omega - \omega' \gg \alpha$) stricter, because intuitively, the spectrum of $H_s$ will be quite dense for large $n$ (the number of subsystems) even though the spectrum of each subsystem is sparse. However, actually the decomposition basis will reduce to the eigenbasis of the Hamiltonian of several subsystems, and in this case the corresponding spectrum will be sparse. For example, the decomposition basis of $A^{\mu}_i$ coming form the system-bath coupling will reduce to the eigenbasis of $H_i$. Therefore, from this perspective, we know that the sparse condition can be fulfilled easily. 2) In order to eliminate those terms that satisfy $\omega > O(\alpha), \omega' > O(\alpha)$ and $\omega - \omega' > O(\alpha)$ in Eq. \((10)\), we take the weak coupling limit $\alpha \to 0$ while keep $\sigma$ finite. The finite $\sigma$ implies that $\sigma < O(1/\alpha)$, as when $\sigma$ exceeds $O(1/\alpha)$, the Riemann-Lebesgue lemma can not be applied in the weak coupling limit anymore (as the integration interval is not finite when we take $\alpha \to 0$). Therefore, this implication sets the working time scale of Eq. \((11)\), that is $\tau_{\sigma} < t < O(1/\alpha^2)$, where the lower bound $\tau_{\sigma}$ which is the correlation time of the bath, is provided by the Markov approximation. Note that in the weak coupling limit, there exists a large scope of application for our modified master equation. One should also note that the traditional GKLS master equation has a working time scale as well, which is also generated by the secular approxi-
mation and the Markov approximation. 3) In Eq. (11), we express the contribution of $H_I$ through the commutator $[\alpha H_I(0), \rho_s]$. Following the procedure in Eqs. (12) and up to $\alpha^2$, it is equivalent with the contribution of $H_I$ in Eq. (10) (modified by the secular approximation). 4) Note that in the weak coupling limit, the order of magnitude of the energy shift resulted from the Lamb shift $\alpha^2 H_{LS}$ is $\mathcal{O}(\alpha^2)$, while the sparse spectrum condition requires that the energy gap $\omega$ should satisfy $\omega \gg \alpha$. Therefore, the contribution of the Lamb shift to the energy level can be safely ignored for those cases to which our modified local master equation is applicable.

### III. THERMODYNAMICAL CONSISTENCY

In this section, we will show that our local master equation is consistent with the first and second law of thermodynamics at the same time. In the following discussions, we ignore the Lamb shift as mentioned at the end of Sec. II.

As discussed in Sec. II, the weak inter-subsystem interaction can be treated as an external perturbation, which produces transitions in the degenerate eigensubspace of $H_s$ after doing the secular approximation. In this sense, the local heat current should come from the transition between energy levels of $H_s$. Thus, it is natural to define the local heat current flowing from bath $b_i$ to the system as

$$\dot{Q}_i = \alpha^2 \text{Tr} \left( H_s D_i [\rho_s(t)] \right).$$

Note that this definition is also used in Ref. [34]. Since the transition in the degenerate eigensubspace does not cost extra energy, for a system without external power, the dynamical version of the first law of thermodynamics can be expressed as

$$\dot{E}_s = \sum_{i=1}^{n} \dot{Q}_i,$$

where

$$\dot{E}_s = \frac{d}{dt} \text{Tr}[H_s \rho_s(t)]$$

is the change rate of the internal energy. We now show that Eq. (13) (the first law of thermodynamics) holds for our modified local master equation.

Using the local master equation Eq. (11) and ignoring the Lamb shift, the left-hand side of Eq. (13) will be

$$\dot{E}_s = \text{Tr} [H_s \partial_t \rho_s]$$

$$= \text{Tr} \left( -i H_s [H_s + \alpha H_I(0), \rho_s] + \alpha^2 \sum_{i=1}^{n} \text{Tr} (H_s D_i [\rho_s]) \right)$$

$$= \sum_{i=1}^{n} \dot{Q}_i,$$

which proves Eq. (13). Note that in the third equality, we have used the equality $\text{Tr} (H_s [H_s + \alpha H_I(0), \rho_s]) = 0$.

For the entire setup that contains the baths and the subsystems, the second law of thermodynamics can be described by the irreversible entropy production:

$$\frac{dS}{dt} - \sum_{i=1}^{n} \beta_i \dot{Q}_i \geq 0,$$

where $\beta_i$ denotes the inverse temperature of the bath $b_i$, and $S = -\text{Tr}(\rho_s \ln \rho_s)$ is the entropy of the system $S$. In the following, we will show how to produce Eq. (16) from our modified local master equation. The derivations are similar with those in Ref. [36].

According to the definition of the entropy $S$ and the local master equation, Eq. (11), the time derivative of $S$ can be expressed as

$$\frac{dS}{dt} = -\text{Tr} (\mathcal{L}[\rho_s] \ln \rho_s),$$

where we have introduced the Liouvillian superoperator $\mathcal{L}$, whose definition is

$$\mathcal{L}[\rho_s] = -i [H_s + \alpha H_I(0), \rho_s] + \alpha^2 \sum_{i=1}^{n} D_i [\rho_s].$$

We further introduce a partial superoperator $\mathcal{L}'$, which is defined as

$$\mathcal{L}'[\rho_s] = -i [H_s, \rho_s] + \alpha^2 \sum_{i=1}^{n} D_i [\rho_s].$$

Note that $\tau_s = \prod_i \exp (-\beta_i H_i) / Z$ with $Z$ being the partition function and $H_{LS}$ being the Lamb shift of the subsystem $s_i$, is the steady state of the partial superoperator $\mathcal{L}'$, that is $\mathcal{L}'[\tau_s] = 0$. Applying the Spohn’s inequality [41] to the partial superoperator $\mathcal{L}'$, one can get

$$-\text{Tr}(\mathcal{L}'[\rho_s] \ln \rho_s) \geq -\text{Tr}(\mathcal{L}'[\rho_s] \ln \tau_s).$$

Inserting the explicit expression of $\tau_s$ into the right-hand
side of Eq. (20), one has

\[ -\text{Tr}(\mathcal{L}'[\rho_s] \ln \tau_s) = \alpha^2 \sum_{i=1}^{n} \beta_{i} \text{Tr} (H_s D_{i}[\rho_s]) = \sum_{i=1}^{n} \beta_{i} \dot{Q}_i. \] (21)

The left-hand side of Eq. (20) can be rewritten as

\[ -\text{Tr}(\mathcal{L}'[\rho_s] \ln \rho_s) = -\text{Tr}(\mathcal{L}'[\rho_s] \ln \rho_s) + i \text{Tr} \{ [\alpha H_I(0), \rho_s] \ln \rho_s \} = -\text{Tr}(\mathcal{L}[\rho_s] \ln \rho_s) = \frac{dS}{dt}, \] (22)

where in the second equality, we have used the identity \( \text{Tr} \{ [H_I(0), \rho_s] \ln \rho_s \} = 0 \).

Combining Eqs. (20)-(22), the second law of thermodynamics, Eq. (10) can be obtained. And thus, we recover the thermodynamical consistency in our local master equation.

IV. THE TWO-QUBIT HEAT TRANSFER MODEL

As an example of our modified local master equation, in this section, we consider a two-qubit heat transfer network. This model is composed of two qubits, each of which is coupled to a bosonic bath with temperature \( T_i \) \((i = 1, 2)\). The Hamiltonian of this model reads

\[ H = H_s + H_I + \sum_{i=1}^{2} (H_{b_i} + V_{b_i}), \] (23)

where \( H_s = (E_1 \sigma_1^+ + E_2 \sigma_2^+ + \sigma_2^-) / 2 \) is the bare Hamiltonian of the two qubits, \( H_I = \alpha \sigma_1^+ \sigma_2^- \) is the weak inter-subsystem interaction, \( H_{b_i} = \int_{0}^{\infty} d\omega \omega a_i^{\dagger}(\omega)a_i(\omega) \) is the Hamiltonian of the bath \( b_i \), and \( V_{b_i} = \int_{0}^{\infty} d\omega \alpha h_i(\omega) \sigma_1^+ [a_i^{\dagger}(\omega) + a_i(\omega)] \) is the coupling between qubit \( i \) and bath \( b_i \). \( \omega \) is the energy of the bosonic modes in the baths, \( \alpha h_i(\omega) \) is the coupling function, and \( a_i(\omega) \) denotes the bosonic annihilation operator.

For the case that \( E_1 = E_2 \equiv E, \{ |10\rangle, |01\rangle \} \) in the basis of \( \sigma^z \) is a degenerate eigensubspace of the Hamiltonian \( H_s \). Thus, the inter-subsystem interaction \( H_I \) will be modified to be

\[ H_I(0) = \alpha (\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+), \] (24)

where \( \sigma^\pm = (\sigma^x \pm i \sigma^y)/2 \).

Here, we study the behavior of the steady state in this model through our modified local master equation. Note that in the steady state, Eq. (11) becomes

\[ 0 = -i [H_s + H_I(0), \rho_{ss}] + \alpha^2 \sum_{i=1}^{2} D_{i}[\rho_{ss}], \] (25)

where \( \rho_{ss} \) is the density matrix of the steady state, the Lamb shift has been ignored, the condition that \( \text{Tr}(\tau B_0^2) = 0 \) is satisfied, and the dissipation operator reads

\[ D_{i}[\rho_{ss}] = 2\pi \hbar_i^2(E_i) \{ n_i(E_i) + 1 \} \left[ \sigma_1^+ \rho_{ss} \sigma_1^- - \frac{1}{2} \{ \sigma_1^+, \sigma_1^- \}, \rho_{ss} \right] + 2\pi \hbar_i^2(E_i) n_i(E_i) \left[ \sigma_i^+ \rho_{ss} \sigma_i^- - \frac{1}{2} \{ \sigma_i^+, \sigma_i^- \}, \rho_{ss} \right], \] (26)

where \( n_i(\omega) = 1/(e^{\beta_i \omega} - 1) \) is the Bose-Einstein distribution of the bosonic bath.

Combining Eqs. (25) and (26), the steady state can be obtained. And then using Eq. (12), one can get the heat current flowing from the bath \( b_1 \) to the qubits, that is

\[ \dot{Q}_1 = \frac{e^{\beta_1 E} - e^{\beta_2 E}}{(e^{\beta_1 E} + 1)(e^{\beta_2 E} + 1)} F, \] (27)

where \( F \) is a positive function. The heat current flowing from bath \( b_2 \) to the qubits can be derived in the similar way and can be expressed as \( \dot{Q}_2 = -\dot{Q}_1 \). Since \( \dot{Q}_1 + \dot{Q}_2 = 0 \) and the change rate of the internal energy is zero, it is obvious that the first law of thermodynamics is satisfied. One also finds that when \( \beta_1 < \beta_2 \), \( \dot{Q}_1 > 0 \) and \( \dot{Q}_2 < 0 \), which means that the heat current flows from the hot bath to the system, and then to the cold bath. This process conforms to the second law of thermodynamics.

Note that for the case that \( E_1 \neq E_2 \), the inter-subsystem interaction dose not contribute to the master equation, and the local master equation Eq. (11) reduces to two separated local master equations – one for the qubit 1 and the bath \( b_1 \) and one for the qubit 2 and the bath \( b_2 \). In the steady state, \( \dot{Q}_1 = \dot{Q}_2 = 0 \), which means that there is no heat current flowing between these two baths. This case is also consistent with the second law of thermodynamics, as now these two baths are disconnected.

In summary, taking the two-qubit heat transfer model as an example, we have shown that how to employ our modified local master equation and that the first and second law of thermodynamics can be satisfied at the same time.

V. CONCLUSION

In conclusion, in this work, we have presented an approach to derive an alternative thermodynamically-consistent local master equation, which works when the
spectrum condition and the time scale condition are satisfied (see Sec. III). Based on a concise and natural definition of the heat current, we generally prove that the modified local master equation fulfills the first and second law of thermodynamics at the same time. This good property suggests that our modified local master equation may be a more suitable method to deal with weak-interacting systems, especially when thermodynamic properties of open quantum systems are considered.

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