FRACTIONAL VARIATION OF HÖLDERIAN FUNCTIONS

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Abstract. The paper demonstrates the basic properties of the local fractional variation operators (termed fractal variation operators). The action of the operators is demonstrated for local characterization of Hölderian functions. In particular, it is established that a class of such functions exhibits singular behavior under the action of fractal variation operators in infinitesimal limit. The link between the limit of the fractal variation of a function and its derivative is demonstrated. The paper presents a number of examples, including the calculation of the fractional variation of Cauchy sequences leading to the Dirac’s delta-function.

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1. Introduction

Fractional derivatives and fractional calculus have long history since the time of Hôpital and Leibniz [17, 18]. However, only relatively recently fractional calculus has been recognized as a tool for modeling physical and biological problems (see for example [8, 9, 6]). Some classical definitions of fractional derivatives (for example by Riemann and Liouville) are based on extension of the Cauchy integral into non-integer order. However, such derivatives are difficult to compute and their geometric interpretation is unclear because of their non-local character. In particular, there is no relationship between the local geometry of the graph of function and its fractional derivative [1, 3]. A definite disadvantage of the Riemann-Liouville approach is that the fractional derivative of a constant is not zero. This was the starting point for the modified definitions of Caputo [8] and Jumarie [11].

Recently, natural science applications have also inspired the development of local fractional derivatives [14, 2]. The theory of such derivatives is still in its infancy and there are few available results [14, 2, 5, 10]. The starting point of Kolwankar and Gangal [14] was the Riemann-Liouville approach (recent review in [3]). On the other hand, Ben Adda and Cresson [2] introduced from the start a difference operator based definition easily transferable to integer-ordered derivatives. The correspondences between the integral approach and the quotient difference approach have been further investigated by Chen et al. [10] and some of the initial results have been clarified and corrected in [4].
In this paper I present a method based on the so-called fractal variation operators, which can provide a simple way of local characterization of singular and scaling behavior of continuous functions. Fractal variation operators are constructed from power scaling of finite difference operators. Application of this approach can be especially suitable for characterization of Hölderian (especially non-differentiable) functions. One of the main results comprises the calculation of the fractal variation of Cauchy sequences leading to the Dirac’s δ-function.

The manuscript is organized as follows. Section 2 introduces the general definitions and notations. Section 3 introduces Hölderian functions and demonstrates their properties used in further proofs. Section 4 introduces the fractal variation operators. Sections 5 demonstrates some of applications of fractal variation to smooth and singular functions. The fractal variation of a function in the infinitesimal limit corresponds to the definition of local fractional derivative introduced by Ben Adda and Cresson [2, 3, 10]. Notably, these authors define the local factional derivative as

$$\frac{d^\alpha}{dx^\alpha} f(x) := \Gamma(1 + \alpha) \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{\epsilon^\alpha}$$

However, according to the main results in the present work (Theorems 5 and 6) this derivative has very few non-infinitesimal or non-divergent values. Therefore, I prefer the term ”variation” over a derivative. Moreover, such operators can be more useful in finite difference settings, e.g. for numerical applications.

2. GENERAL DEFINITIONS AND NOTATIONS

Along the text I consistently use square brackets for the arguments of operators and round brackets for the arguments of functions. The term function denotes the mapping $$f: \mathbb{R} \mapsto \mathbb{R}$$. The notation $$f(x)$$ is used to refer to the value of the function at the point x. By $$C^0$$ is denoted the class of functions that are continuous and by $$C^n$$ the class of n-times differentiable functions where $$n \in \mathbb{N}$$. Dom[f] denotes the domain of definition of the function $$f(x)$$.

3. HÖLDERIAN FUNCTIONS

**Definition 1.** Let $$\mathbb{H}^\alpha$$ be the class of Hölder $$C^0$$ functions of degree $$\alpha$$, $$\alpha \in (0, 1)$$. That is, $$\forall f(x) \in \mathbb{H}^\alpha$$ there exist two positive constants $$C, \delta \in \mathbb{R}$$ for $$x, y \in \text{Dom}[f]$$ such that for $$|x - y| \leq \delta$$ the following inequality holds

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

Following Mallat and Hwang [15] the definition can be extended to orders greater than one in the following way. Let $$\mathbb{H}^{n+\alpha}$$ be the class of $$C^0$$ double Hölder functions of degree $$n + \alpha$$ for which

$$|f(x) - f(y) - P_n(x - y)| \leq C|x - y|^{n+\alpha}$$
where $P_n$ is a real-valued polynomial of degree $n \in \mathbb{N}$ of the form

$$P_n(z) := \sum_{k=1}^{n} a_k z^k$$

where $P_0(z) = 0$ and $\alpha \in (0, 1)$.

Under this definition we will focus mainly on functions for which $0 < \alpha < 1$. These functions will be further called by the term Hölderian.

**Definition 2.** Let the difference parametrized operators acting on a function $f(x)$ be defined in the following way

$$\begin{align*}
\Delta^+_\epsilon [f](x) &:= f(x + \epsilon) - f(x) \\
\Delta^-_\epsilon [f](x) &:= f(x) - f(x - \epsilon) \\
\Delta^2_\epsilon [f](x) &:= f(x + \epsilon) - 2f(x) + f(x - \epsilon)
\end{align*}$$

where $\epsilon > 0$. The first one we refer to as forward difference operator, the second one we refer to as backward difference operator and the third one as $2^{nd}$ order difference operator.

**Lemma 1 (Difference composition lemma).** The $2^{nd}$ order difference operator is a composition of the backward and forward difference operators.

$$\Delta^2_\epsilon = \Delta^+_\epsilon \circ \Delta^-_\epsilon = \Delta^-_\epsilon \circ \Delta^+_\epsilon = \Delta^+_\epsilon - \Delta^-_\epsilon$$

**Proof.** Let $f(x)$ be an arbitrary function defined in the domain of the application of the operators. Direct calculations shows

$$\Delta^2_\epsilon [f](x) = f(x + \epsilon) - f(x) - (f(x) - f(x - \epsilon)) = \Delta^+_\epsilon [f](x) - \Delta^-_\epsilon [f](x)$$

In addition,

$$\begin{align*}
\Delta^+_\epsilon \circ \Delta^-_\epsilon [f](x) &= \Delta^+_\epsilon (f(x) - f(x - \epsilon)) = \Delta^+_\epsilon [f](x) - \Delta^+_\epsilon [f](x - \epsilon) \\
&= f(x + \epsilon) - f(x) - (f(x) - f(x - \epsilon)) \\
&= \Delta^+_\epsilon [f](x) - \Delta^-_\epsilon [f](x) = f(x + \epsilon) - 2f(x) + f(x - \epsilon)
\end{align*}$$

Finally,

$$\begin{align*}
\Delta^-_\epsilon \circ \Delta^+_\epsilon [f](x) &= \Delta^-_\epsilon (f(x + \epsilon) - f(x)) = \Delta^-_\epsilon [f](x + \epsilon) - \Delta^-_\epsilon [f](x) \\
&= f(x + \epsilon) - f(x) - (f(x + \epsilon) - f(x - \epsilon)) \\
&= f(x + \epsilon) - 2f(x) + f(x - \epsilon)
\end{align*}$$

\[\square\]

**Theorem 1.** Let $f(x) \in \mathbb{H}^{n+\alpha}$ in the interval $[x \ x + \epsilon]$, where $n \in \mathbb{N}$ is a natural number and $\alpha \in (0, 1)$. Then $f(x) \in \mathbb{C}^{\alpha}$ in the interval $[x \ x + \epsilon]$.

**Proof.** Initial case: Let $\epsilon > 0$. For $k = 1$ $P_k(z) = a_1 z$. Then $|\Delta^+_\epsilon [f](x) - a_1\epsilon| \leq C\epsilon^{1+\alpha}$ for arbitrary $x$. We divide both sides by $\epsilon$. Then

$$\left| \frac{\Delta^+_\epsilon [f](x)}{\epsilon} - a_1 \right| \leq C\epsilon^\alpha$$
Therefore, in limit
\[
\lim_{\epsilon \to 0} \frac{\Delta^e_+ [f](x) - a_1}{\epsilon} = 0
\]
Therefore, the right derivative \( f'(x+) \) exists at \( x \).

Similar reasoning can be applied for the opposite case. Then
\[
\lim_{\epsilon \to 0} \frac{\Delta^e_- [f](x) - a_1}{\epsilon} = 0
\]
Therefore, the left derivative \( f'(x-) \) exists at \( x \). Now, if we take the centered difference \( \Delta [f](x) := f(x + \epsilon) - f(x - \epsilon) \), assuming \( \epsilon > 0 \) and repeat the same reasoning we arrive at:
\[
\frac{1}{2} \left| \frac{\Delta [f](x)}{\epsilon} - a_1 \right| \leq 2C \epsilon^{\alpha + 1}
\]
Therefore, the central difference limit exists as well. That is \( f'(x+) = f'(x-) = f'(x) \). Since \( x \) is arbitrary then \( f(x) \in C^1 \).

Inductive step: Let the statement be true for \( n = k \). Moreover, we can identify \( a_1 \) with the first order of the Taylor expansion.

Then for \( k + 1 \) one has:
\[
|\Delta [f](x) - P_{k+1}(\epsilon)| \leq C \epsilon^{k+1+\alpha}
\]
And since we assume that \( f(x) \) is \( k \)-times differentiable
\[
\left| \sum_{i=1}^{k} \frac{1}{i!} f^{(i)}(x) \epsilon^i + R(x, \epsilon) - \sum_{i=1}^{k} a_i \epsilon^i \right| \leq C \epsilon^{k+\alpha+1}
\]
\[
\left| \sum_{i=1}^{k} \left( \frac{1}{i!} f^{(i)}(x) - a_i \right) \epsilon^i + R(x, \epsilon) - a_{k+1} \epsilon^{k+1} \right| \leq C \epsilon^{k+\alpha+1}
\]
\[
\left| R(x, \epsilon) - a_{k+1} \epsilon^{k+1} \right| \leq C \epsilon^{\alpha+1}
\]
\[
\left| \frac{R(x, \epsilon)}{\epsilon^{k+1}} - a_{k+1} \right| \leq C \epsilon^{\alpha}
\]
Therefore, the limit exists and
\[
\lim_{\epsilon \to 0} \frac{R(x, \epsilon)}{\epsilon^{k+1}} = a_{k+1}.
\]
On the other hand, differentiating \( k \) times gives
\[
\frac{d^k}{dx^k} \Delta [f](x) = f^{(k)}(x + \epsilon) = f^{(k)}(x) + \frac{d^k}{dx^k} R(x, \epsilon)
\]
\[
\frac{d^k}{dx^k} R(x, \epsilon) = f^{(k)}(x + \epsilon) - f^{(k)}(x)
\]
\[
(k + 1)! a_{k+1} = f^{(k)}(x + \epsilon) - f^{(k)}(x)
\]
Then the \( k+1 \) derivative of \( f(x) \) exists at \( x \). Therefore, \( f(x) \in C^{k+1} \) and by induction \( f(x) \in C^n \). \( \square \)
Corollary 1. If $|f(x) - f(y) - P_n(x - y)| \leq C'|x - y|^{n+\alpha}$ for $n \in \mathbb{N}$ and $0 < \alpha < 1$ then

$$P_n(x - y) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x)(x - y)^k$$

4. Fractal variation operators

Definition 3. Let the Fractal Variation operators be defined as

$$\upsilon_+^{\beta}[f](x) := \frac{\Delta^+_{\epsilon}[f](x)}{\epsilon^\beta} = \frac{f(x + \epsilon) - f(x)}{\epsilon^\beta} \quad (4.4)$$

and

$$\upsilon_-^{\beta}[f](x) := \frac{\Delta^-_{\epsilon}[f](x)}{\epsilon^\beta} = \frac{f(x) - f(x - \epsilon)}{\epsilon^\beta} \quad (4.5)$$

where $\epsilon > 0$ and $0 < \beta \leq 1$ are real parameters and $f(x)$ is a function.

It is easy to check that the Fractal variation operators are R-linear. This can be formulated in the following Lemma

Lemma 2. Let $K$ and $M$ be real constants. Then

$$\upsilon_+^{\beta}[Kf(x) + Mg(x)] = K\upsilon_+^{\beta}[f](x) + M\upsilon_+^{\beta}[g](x)$$

and

$$\upsilon_-^{\beta}[Kf(x) + Mg(x)] = K\upsilon_-^{\beta}[f](x) + M\upsilon_-^{\beta}[g](x)$$

Definition 4. Let the translation operator acting on the function $f(x)$ be defined as

$$T_{\pm \epsilon}[f](x) := f(x \pm \epsilon)$$

where $\epsilon$ is a real positive parameter.

Fractional variation is translationally invariant. This can be formulated in the following result.

Proposition 1. Fractal variation commutes with the translation operator in the neighborhood of a point if the argument function is defined in the neighborhood.

Let $f(x) \in \mathbb{H}^\alpha$. Then

$$\{\upsilon_+^{\pm \epsilon}T_{\epsilon} - T_{\epsilon}\upsilon_-^{\pm \epsilon}\}[f] = 0[f]$$

The proof follows directly from the commutativity of addition in $\mathbb{R}$.

Remark 1. The following relation can be established from the definitions:

$$\upsilon_+^{\beta}[f](x) = \epsilon^{-\beta} (T_{\epsilon} - I)[f](x) \quad (4.6)$$

$$\upsilon_-^{\beta}[f](x) = -\epsilon^{-\beta} (T_{-\epsilon} - I)[f](x) \quad (4.7)$$

where $I$ is the identity operator. Since for continuous functions the reflection of a translation about the origin is its inverse operation: $T_{-\epsilon} = T_{\epsilon}^{-1}$.

Proposition 2. The right and left fractal variation operators are mapped to each other by translation assuming that the argument function is defined in the domain of the operators.
Proof. Let \( f(x) \) be defined in \([x - \epsilon, x + \epsilon]\). Direct calculation shows that
\[
v_x^{\epsilon^+}[f](x) = T_x \circ v_x^{\epsilon^-}[f](x) = v_x^{\epsilon^-} \circ T_x[f](x) = v_x^{\epsilon^-}[f](x + \epsilon)
\] (4.8)

Theorem 2 (Duality of the limit of variation). If \( f(x) \) is defined in \([x - \epsilon, x + \epsilon]\) and the limit \( \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) \) exists and \( \lim_{\epsilon \to 0} \frac{\Delta^2_x[f](x)}{\epsilon^3} = 0 \) then \( \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) = \lim_{\epsilon \to 0} v_x^{\epsilon^-}[f](x) \).

Proof. The enlargement of the domain follows from Prop. 2. From Lemma 1 it follows directly that
\[
v_x^{\epsilon^+}[f](x) - v_x^{\epsilon^-}[f](x) = \frac{\Delta^2_x[f](x)}{\epsilon^3}
\]
Therefore if \( \lim_{\epsilon \to 0} \frac{\Delta^2_x[f](x)}{\epsilon^3} = 0 \) then \( \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) = \lim_{\epsilon \to 0} v_x^{\epsilon^-}[f](x) \).

Corollary 2. If \( \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) \) is continuous about \( x \) then \( \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) = \lim_{\epsilon \to 0} v_x^{\epsilon^-}[f](x) \). If \( \lim_{\epsilon \to 0} v_x^{\epsilon^-}[f](x) \) is continuous about \( x \) then \( \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) = \lim_{\epsilon \to 0} v_x^{\epsilon^-}[f](x) \).

Proof. The proof follows from Lemma 1. Indeed let \( g(x) = \lim_{\epsilon \to 0} v_x^{\epsilon^+}[f](x) \) be continuous about \( x \). Then \( \lim_{\epsilon \to 0} \Delta^+_x g(x) = 0 \). The other case follows by identical reasoning.

Lemma 3. Let \( f(x) \in \mathbb{H}^\alpha \). Then \( v_x^{\epsilon^+}[f](x) \in \mathbb{H}^{\alpha - \beta} \) and \( v_x^{\epsilon^-}[f](x) \in \mathbb{H}^{\alpha - \beta} \).

Proof. It follows that for the right-hand limit
\[
|f(x + \epsilon) - f(x)| \leq Ce^\alpha \Rightarrow |v_x^{\epsilon^+}[f](x)| = \left| \frac{f(x + \epsilon) - f(x)}{\epsilon^\beta} \right| \leq Ce^{\alpha - \beta}
\]
Analogous calculation can be performed for the right operator.

Theorem 3 (Compound variation theorem). The fractal variation of a compound function \( f(y(x)) \) can be expressed as
\[
v_x^{\epsilon^+}[f(y)](x) = v_y^{\eta^+}[f](y) \left( v_x^{\epsilon^+}[y](x) \right)^\alpha e^{\alpha - \beta}
\] (4.9)
where \( \eta = y(x + \epsilon) - y(x) \neq 0 \), \( \epsilon > 0 \) and \( \alpha > 0 \). The argument function is interpreted as a variable.

Proof. We will prove the general case : Let \( \eta = y(x + \epsilon) - y(x) \neq 0 \). Then
\[
\frac{\Delta_x f(y)}{\epsilon^\beta} = \frac{\Delta_x f(y) \Delta y^\alpha}{(\Delta y)^\alpha} = \frac{\Delta_x f(y)}{(\Delta y)^\alpha} \left( \frac{\Delta y}{\epsilon} \right)^\alpha e^{\alpha - \beta}
\]
where \( \Delta_x f(y) = f(y + \epsilon) - f(y) \). Therefore,
\[
v_x^{\epsilon^+}[f](x) = v_y^{\eta^+}[f](y) \left( v_x^{\epsilon^+}[y](x) \right)^\alpha e^{\alpha - \beta}
\]
So-stated theorem can be specialized in two corollaries that are important for applications.

**Corollary 3** (First differential form). For a compound function \( f(y(x)) \)
\[
v^\epsilon_\beta [f(y)] (x) = v^\eta_\beta [f(y)] v^\epsilon_\beta [y] (x)
\] (4.10)
where \( \eta = y(x + \epsilon) - y(x) \).

**Corollary 4** (Second differential form). For a compound function \( f(y(x)) \):
\[
v^\epsilon_\beta [f(y)] (x) = v^\eta_\beta [f(y)] (v^\epsilon_\beta [y] (x))^\beta
\] (4.11)
where \( \eta = y(x + \epsilon) - y(x) \).

**Remark 2.** These corollaries lead to the following limiting behavior for \( \alpha < 1 \) for a monotonous substituting function \( y(x) \)
\[
\lim_{\epsilon \to 0} v^\epsilon_\beta [f(y)] (x) = f'(y) \lim_{\epsilon \to 0} v^\epsilon_\beta [y] (x)
\]
provided that \( f'(y) \) exists at \( y \) and
\[
\lim_{\epsilon \to 0} v^\epsilon_\beta [f(y)] (x) = (y'(x))^\beta \lim_{\eta \to 0} v^\eta_\beta [f(y)]
\]
provided that \( y'(x) \) exists at \( x \). In both formulas the argument function is interpreted as a variable.

5. Applications

5.1. Fractal variation of the Hölder exponent.

**Definition 5.** Let the sign operator acting on a function \( f(x) \) be defined as
\[
\text{sign}[f](x) := \begin{cases} +1, & f(x) \geq 0 \\ -1, & f(x) < 0 \end{cases}
\]

In the next section the limit behavior of so-defined operators for the limiting Hölder exponent function will be demonstrated.

**Theorem 4.** Let \( g(x) = |x|^\alpha, \ \alpha > 0 \). Then the limiting behavior of the Fractal variation \( \lim_{\epsilon \to 0} v^\epsilon_\beta [g](x) \) can be summarized in Table 1:

Let \( g(x) = |x|^{-\alpha}, \ \alpha > 0 \). Then the limiting behavior of the Fractal variation \( \lim_{\epsilon \to 0} v^\epsilon_\beta [g](x) \) can be summarized in Table 2:

**Proof.** Let \( g(x) = |x|^\alpha, \ \alpha > 0 \). Obviously, \( g(x) \in \mathbb{H}^\alpha \). For \( x = 0 \)
\[
v^\epsilon_\beta [g](0) = \frac{(0 + \epsilon)^\alpha - 0^\alpha}{\epsilon^\beta} = \epsilon^{\alpha - \beta}
\]
The same equality can be demonstrated for \( v^\epsilon_\beta [g](0) \).
Table 1. Limit behavior of the Fractal variation of $|x|^\alpha$, $\alpha > 0$

| $x = 0$ | $|x| > 0$ |
|---|---|
| $\alpha = \beta$ $\alpha < 1$ | 1 | 0 |
| $\alpha = \beta$ $\alpha > 1$ | 1 | $\infty$ |
| $\alpha > \beta$ $\beta > 1$ | 0 | $\infty$ |
| $\alpha < \beta$ $\beta < 1$ | $\infty$ | 0 |
| $\alpha < 1$ $\beta > 1$ | $\infty$ |
| $\alpha = \beta = 1$ | 1 |
| $\alpha > \beta$ $\beta < 1$ | 0 |
| $\beta = 1$ | $\text{sign}(x)\alpha|x|^{\alpha - 1}$ |

Figure 1. Limit behavior, $v_\beta^{\epsilon}[x^\alpha]$, $x = 0$

Table 2. Limit behavior of the Fractal variation of $|x|^{-\alpha}$, $\alpha > 0$

| $x = 0$ | $|x| > 0$ |
|---|---|
| $\beta < 1$ $-\infty$ | 0 |
| $\beta > 1$ $-\infty$ | 0 |
| $\beta = 1$ $-\text{sign}(x)\alpha|x|^{-\alpha - 1}$ |

Therefore, if $\alpha - \beta > 0$ then $\lim_{\epsilon \to 0} v_\beta^{\epsilon+}[g](0) = 0$. If $\alpha = \beta$ then $\lim_{\epsilon \to 0} v_\beta^{\epsilon+}[g](0) = 1$.

If $\alpha < \beta$ then $\lim_{\epsilon \to 0} v_\beta^{\epsilon+}[g](0) = \infty$. For $x > 0$ we can distinguish 2 cases: when the derivative $g'(x)$ exists and when it does not.

Since the derivative always exists for $x > 0$ then according to Theorem 5

$$\lim_{\epsilon \to 0} v_\beta^{\epsilon+}[g](x) = \lim_{\epsilon \to 0} \epsilon^{1-\beta} \frac{dg}{dx}(x)$$

Therefore, we can distinguish 2 cases. If $\beta = 1$ then

$$\lim_{\epsilon \to 0} v_1^{\epsilon+}[g](x) = \frac{dg}{dx}(x) = \text{sign}(x) \alpha|x|^{\alpha - 1}$$
If \( \beta < 1 \) then \( \lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (x) = 0 \). If \( \beta > 1 \) then \( \lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (x) = \infty \).

If \( \alpha = \beta \) but \( \alpha > 1 \) then we can distinguish 2 cases: for \( x = 0 \)
\[
\lim_{\epsilon \to 0} (x + \epsilon)_{\beta} - x_{\beta} \epsilon = \lim_{\epsilon \to 0} \epsilon_{\beta} = 1
\]
for \( x \neq 0 \) application of L'Hôpital's rule gives
\[
\lim_{\epsilon \to 0} (x + \epsilon)_{\beta} - x_{\beta} \epsilon = \lim_{\epsilon \to 0} \beta (x + \epsilon)^{\beta - 1} = \lim_{\epsilon \to 0} (1 + \frac{x}{\epsilon}) = \infty
\]

Let \( g(x) = |x|^{-\alpha}, \alpha > 0 \). As in the previous example, two cases can be considered here:
For \( x = 0 \),
\[
\nu^\epsilon_\beta [g] (0) = - \lim_{h, \epsilon \to 0} \frac{(h + \epsilon)^{\alpha} - \epsilon^{\alpha}}{\epsilon^{\alpha} h^{\beta} (h + \epsilon)^{\alpha}}
\]
we make the substitution \( h = \epsilon \)
\[
\frac{(h + \epsilon)^{\alpha} - \epsilon^{\alpha}}{\epsilon^{\alpha} h^{\beta} (h + \epsilon)^{\alpha}} = \frac{(\epsilon + \epsilon)^{\alpha} - \epsilon^{\alpha}}{\epsilon^{\alpha} \epsilon^{\beta} (\epsilon + \epsilon)^{\alpha}} = \frac{(2^\alpha - 1)\epsilon^{\alpha}}{2^\alpha \epsilon^{2\alpha + \beta}} = \frac{(2^\alpha - 1)}{2^\alpha \epsilon^{\alpha + \beta}}
\]
Therefore in limit,
\[
\lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (0) = -\infty
\]

For \( |x| > 0 \) since the derivative always exists for \( x > 0 \) according to Theorem 5 the following equality is valid
\[
\lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (x) = \lim_{\epsilon \to 0} \epsilon^{1-\beta} \frac{dg}{dx} (x)
\]
Therefore, if \( \beta = 1 \)
\[
\lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (x) = -\alpha x^{-\alpha-1} \text{sign}(x)
\]
If \( \beta < 1 \) then \( \lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (x) = 0 \). If \( \beta > 1 \) then \( \lim_{\epsilon \to 0} \nu^\epsilon_\beta [g] (x) = -\infty \).

The limiting behavior is represented graphically in Figs. 1 and 2.
5.2. **Fractal variation of smooth functions.** We will prove a general theorem allowing one to compute the limit of the fractal variation for smooth functions.

**Theorem 5** (Limit of Fractal Variation about a point). Let \( f(x) \in C^1 \). Then

\[
\lim_{\epsilon \to 0} v_{\epsilon \beta}^+[f](x) = \frac{1}{\beta} \lim_{\epsilon \to 0} \epsilon^{1-\beta} f'(x + \epsilon)
\]

(5.13)

\[
\lim_{\epsilon \to 0} v_{\epsilon \beta}^-[f](x) = \frac{1}{\beta} \lim_{\epsilon \to 0} \epsilon^{1-\beta} f'(x - \epsilon)
\]

(5.14)

**Proof.** We will prove the forward case. The backward case is similar. The proof follows by direct application of L'Hôpital's rule in evaluating the limit and treating \( \epsilon \) as an independent variable while fixing \( x \).

\[
\lim_{\epsilon \to 0} v_{\epsilon \beta}^+[f](x) = \frac{f(x + \epsilon) - f(x)}{\epsilon^\beta} = \frac{(f(x + \epsilon) - f(x))'}{(\epsilon^\beta)'} = \frac{1}{\beta} \lim_{\epsilon \to 0} \epsilon^{1-\beta} f'(x + \epsilon)
\]

☐

**Corollary 5** (Vanishing variation theorem). Let \( f(x) \in C^1 \) about \( x \) and \( 0 < \beta < 1 \) then

\[
\lim_{\epsilon \to 0} v_{\epsilon \beta}^+[f](x_0) = 0
\]

**Proof.** The proof follows directly from the definition.

\[
\lim_{\epsilon \to 0} v_{\epsilon \beta}^+[g](x_0) = \lim_{\epsilon \to 0} \epsilon^{1-\beta} \frac{dg}{dx}(x_0) = \lim_{\epsilon \to 0} \epsilon^{1-\beta} g'(x_0) = 0
\]

☐

This result corresponds with the result obtained in [10].

5.3. **Fractal Variation of functions with singular derivatives.** Closely related to the results of the Vanishing Variation Theorem is the next Theorem. First let’s introduce the concept of the critical exponent acting around a singularity of a function.

**Definition 6.** Let \( g(x) \) be a continuous function having a singularity at \( x_s \).

Then the left critical exponent \( \alpha \) be the minimal exponent in the power term for which the quantity \( h^\alpha T_{-h}[g](x) \) is finite at \( x_s \). Or formally,

\[
\mathcal{P}_+[g](x = x_s) := \alpha \left\{ 0 < \inf_{\alpha} \lim_{h \to 0} h^\alpha T_{-h}[g](x) < \infty \right\}
\]

where \( h > 0 \). Let the right critical exponent \( \alpha \) be the minimal exponent in the power term for which

\[
\mathcal{P}_-[g](x = x_s) := \alpha \left\{ 0 < \inf_{\alpha} \lim_{h \to 0} h^\alpha T_{h}[g](x) < \infty \right\}
\]

When \( g(x) \) is bounded \( \mathcal{P}[g](x) \) will be assumed 0.
**Theorem 6** (Singular derivative variation theorem). Let \( f(x) \in \mathbb{C}^n \) has a singular derivative at \( x_s \) \( f'(x) \) be such that \( \mathcal{P}_+[f'](x_s) = \alpha \). Then in limit

- for \( \beta < |1 - \alpha| \): \( \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) = 0 \)
- for \( \beta = |1 - \alpha| \): \( \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) \) is finite
- for \( \beta > |1 - \alpha| \): \( \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) \) is unbounded.

assuming always \( \beta \in [0, 1) \). For \( x \neq x_s \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) = 0 \). The result can be illustrated from Figs. 1 and 2.

**Proof.** Let the derivative \( f'(x) \equiv g(x) \) be such that \( \mathcal{P}_+[g](x_s) = \alpha \) and \( g(x) \) is defined in the neighborhood of \( x_s \). That is let \( K' < \lim_{h \to 0} h^{\alpha} \mathcal{T}_h[g](x) < K \) for a finite \( K \). Consider also the Taylor expansion of \( f \) about \( x_s - h \):

\[
v^{+\beta}_f [f] (x_s - h) = \frac{f(x_s - h + \epsilon) - f(x_s - h)}{\epsilon^\beta} = g(x_s - h) \epsilon^{1-\beta} + \gamma \epsilon^{1-\beta}
\]

where \( \gamma = o(\epsilon) \) and \( h > 0 \).

Then in \( \epsilon \)-limit

\[
\lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s - h) = \lim_{\epsilon \to 0} \left( g(x_s - h) \epsilon^{1-\beta} + \gamma \epsilon^{1-\beta} \right) = \lim_{\epsilon \to 0} \frac{h^\alpha g(x_s - h) \epsilon^{1-\beta}}{h^\alpha} = g(x_s - h) \epsilon^{1-\beta} = K' \epsilon^{1-\beta} h^{-\alpha} < g(x_s - h) \epsilon^{1-\beta} < K \epsilon^{1-\beta} h^{-\alpha}
\]

We make the substitution \( h = \epsilon \) and we arrive at

\[
\lim_{\epsilon \to 0} K' \epsilon^{1-\alpha-\beta} < \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) < \lim_{\epsilon \to 0} K \epsilon^{1-\alpha-\beta}
\]

Therefore, the (value of the) limit depends on the sign of \( 1 - \alpha - \beta \).

- If \( \alpha + \beta < 1 \) then \( \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) = 0 \).
- If \( \alpha + \beta = 1 \) then \( \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) = \pm K \) depending of the sign of \( g(x_s-) \).
- If \( \alpha + \beta > 1 \) then \( \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s-) = \pm \infty \) depending of the sign of \( g(x_s-) \).

\[ \square \]

**Theorem 7.** Let \( f(x) \in \mathbb{C}^n \) is such that \( \mathcal{P}_-[f'](x_s) = \alpha \). Then in limit

- for \( \beta < |1 - \alpha| \): \( \lim_{\epsilon \to 0} v^{-\beta}_f [f] (x_s+) = 0 \)
- for \( \beta = |1 - \alpha| \): \( \lim_{\epsilon \to 0} v^{-\beta}_f [f] (x_s+) \) is finite
- for \( \beta > |1 - \alpha| \): \( \lim_{\epsilon \to 0} v^{-\beta}_f [f] (x_s+) \) is unbounded.

assuming always \( \beta \in [0, 1) \). The proof is analogous to the proof of Theorem 6.

**For** \( x \neq x_s \lim_{\epsilon \to 0} v^{+\beta}_f [f] (x_s+) = 0 \).

**Definition 7.** Let \( u(x) \) be the function with the following definition:

\[
u(x) := \lim_{n \to 0} u_n (x)
\]
defined as the limiting sequence of

\[ u_n(x) := \begin{cases} 
0, & x < 0 \\
1, & x < \epsilon_n \\
0, & x \geq \epsilon_n 
\end{cases} \]

for \( \epsilon_n = \frac{\nu}{2n} \) where \( 0 < \nu < 2 \) is an arbitrary small number, \( n \in \mathbb{N} \).

**Example 1.** The result can easily be extended for scaled and translated versions of \( x^\alpha : f(x) = (sx - x_0)^\alpha \) because of the commutativity of translation and the homogeneity property. Then

\[ \lim_{\epsilon \to 0} \nu^{\alpha + [|x - x_0|^\alpha]} = u(x - x_0) \]  \hspace{1cm} (5.15)

for \( 1 > \alpha > 0 \).

### 5.4. Fractal variation of functions with singularities.

The behavior of the fractal variation is non trivial when the argument function can become singular. Notably, it can be demonstrated that the fractal variation preserves singularities in limit.

**5.4.1. Fractal variation of Delta function sequences.** The difficulty in the standard treatment of the Delta "function"s properties come from the fact that the "function" has to assume unlimited growth at the origin and therefore is discontinuous. This makes the notion difficult to handle in the standard framework of analysis (recent review on historical developments in [12]). With standard analytic arguments the notion of Delta "function" can be treated in Swartz’s distribution theory. In contrast, using Non Standard reasoning [19], the Delta "function" can be handled using more algebraical approach.

In the following we give elementary treatment of some of the properties of the Dirac’s Delta "function". We will always assume that the value at the origin can be defined in some sense.

In the standard analysis setting, the next results have to be interpreted in distributional sense.

**Theorem 8** (Fractal variation of the \( \delta \)-function). Let \( \delta(x) \) be the Dirac’s delta function/distribution

\[ \delta(x) := \lim_{n \to \infty} \delta_n(x) \]

defined as the limiting sequence of functions-prototypes (i.e. "predistributions" in the language of distribution theory)

\[ \delta_n(x) := \frac{1}{s_n} \psi \left( \frac{x}{s_n} \right) \]

having the following properties:

- \( \int_{-\infty}^{\infty} \psi(x) \, dx = 1 \)
- \( \psi(x) = \psi(-x) \)
\[ \frac{1}{2\epsilon_n} \]

\[ \frac{1}{\epsilon_n} \]

**Figure 3.** Graph of two Delta sequences

The graph of the triangular sequence is rescaled by factor 2.

- positive at the origin
- monotonously decreasing from the origin towards ±∞ as \( \sim \frac{1}{x^2} \);

that are parametrized by the Cauchy sequence \( \lim_{n \to \infty} s_n = 0 \) with \( s_1 \leq 1 \) and \( s_n > 0, n \in \mathbb{N} \).

Then formally

\[
\lim_{\epsilon \to 0} \psi_{\beta}^+ \left[ \delta \right] (x) = \frac{\delta(x+) - \delta(x-)}{|x|^\beta} \tag{5.16}
\]

for \( \beta \leq 1 \). In the equation the RHS should be interpreted as limits: \( \delta(x\pm) = \lim_{\epsilon \to 0} \delta(x \pm \frac{\epsilon}{2}) \) and \( \frac{1}{x} = \lim_{\epsilon \to 0} \min \left( \frac{1}{x-\epsilon}, \frac{1}{x+\epsilon} \right) \)

Due to the symmetry about the origin it can be demonstrated that \( \psi'(x) = -\psi'(-x), \psi''(x) = \psi''(-x) \) etc. From these properties it follows in particular that \( \psi'(0) = 0 \) and \( \psi''(0) = 0 \).

We are going to prove two lemmas considering the limiting cases.

**Lemma 4** (Fractal variation of the pulsed \( \delta \)-sequence). Let \( \delta_n(x) \) be a rectangular sequence

\[
\delta_n(x) := \begin{cases} 0, & |x| \geq \epsilon_n \\ \frac{1}{2\epsilon_n}, & |x| < \epsilon_n \end{cases}
\]

where \( \epsilon_n = \frac{\nu}{2n}, 0 < \nu < 2 \) is an arbitrary small number, \( n \in \mathbb{N} \).

Then for \( \beta \leq 1 \)

\[
\psi_{\beta}^{\epsilon_n+} \left[ \delta \right] (x) = \begin{cases} 0, & |x| \geq \epsilon_n \\ -\frac{\text{sign}(x)}{2\epsilon_n^{\beta+1}}, & \epsilon_{n-1} \leq |x| < \epsilon_n \\ 0, & |x| < \epsilon_{n-1} \end{cases}
\]

**Proof.** For \( |x| > \epsilon_n \) we have

\[
\frac{\delta_n(x + \epsilon_n) - \delta_n(x)}{\epsilon_n^\beta} = \frac{\delta_n(x + \epsilon_n)}{\epsilon_n^\beta} = \frac{0}{\epsilon_n^\beta} = 0
\]
For \( x = 0 \) and step \( \epsilon_n \) we have
\[
\frac{\delta_n (\epsilon_n + 0) - \delta_n (0)}{\epsilon_n^\beta} = \frac{1}{2\epsilon_n^\beta} \left( 0 - \frac{1}{\epsilon_n} \right) = -\frac{1}{2\epsilon_n^{\beta+1}}
\]
On the other hand
\[
- \frac{\delta_n (0 - \epsilon_n) - \delta_n (0)}{\epsilon_n^\beta} = -\frac{1}{\epsilon_n^\beta} \left( 0 - \frac{1}{\epsilon_n} \right) = \frac{1}{2\epsilon_n^{\beta+1}}
\]
And
\[
\frac{\delta_n (0 + \epsilon_n - 1) - \delta_n (0)}{\epsilon_n^\beta} = \frac{1}{\epsilon_n^\beta} \left( \frac{1}{\epsilon_n} - \frac{1}{\epsilon_n} \right) = 0
\]
\[\square\]

**Lemma 5** (Fractal variation of the triangular \( \delta \)-sequence). Let \( \delta_n^T(x) \) be defined as a sequence of triangular functions
\[
\delta_n^T(x) := \begin{cases} 
0, & |x| \geq \epsilon_n \\
\frac{1}{\epsilon_n} - \frac{x}{\epsilon_n^\beta}, & 0 \leq x < \epsilon_n \\
\frac{1}{\epsilon_n} + \frac{x}{\epsilon_n^\beta}, & 0 \geq x > -\epsilon_n 
\end{cases}
\]
where \( \epsilon_n = \frac{v}{2^n}, \ 0 < v < 2 \) is an arbitrary small number, \( n \in \mathbb{N} \).
Then for \( \beta \leq 1 \)
\[
v_{n+1}^\epsilon \delta_n^T(x) = \begin{cases} 
0, & |x| \geq \epsilon_n \\
n\frac{\text{sign}(x)}{\epsilon_n^{\beta+1}}, & \epsilon_{n-1} \leq |x| < \epsilon_n \\
0, & |x| < \epsilon_{n-1} 
\end{cases}
\]
**Proof.** For \( |x| \geq \epsilon_n \) we have
\[
\frac{\delta_n^T(x + \epsilon_n - 1) - \delta_n^T(x)}{\epsilon_n^\beta} = \frac{\delta_n^T(x + \epsilon_n)}{\epsilon_n^\beta} = 0 = 0
\]
For \( x = \epsilon_n \) we take the step \( \epsilon_n \). Then we have
\[
\frac{\delta_n^T(\epsilon_n) - \delta_n^T(0)}{\epsilon_n^\beta} = \frac{1}{\epsilon_n^\beta} \left( 0 - \frac{1}{\epsilon_n} \right) = -\frac{1}{\epsilon_n^{\beta+1}}
\]
For \( x = -\epsilon_n \) we take the step \( \epsilon_n \). Then we have
\[
\frac{\delta_n^T(0) - \delta_n^T(-\epsilon_n)}{\epsilon_n^\beta} = \frac{1}{\epsilon_n^\beta} \left( \frac{1}{\epsilon_n} - 0 \right) = \frac{1}{\epsilon_n^{\beta+1}}
\]
And
\[
\frac{\delta_n^T(-\epsilon_{n-1} + \epsilon_n) - \delta_n^T(\epsilon_{n-1})}{\epsilon_n^\beta} = 0
\]
\[\square\]
From these two lemmas it follows that any scale-dependent parametric smooth function constrained between the rectangular pulse and the triangular pulse at some scales will also exhibit this scaling limit behavior.
Central argument for the subsequent presentation will be the differentiating property of the $\delta$-function:

$$\int_{-\infty}^{\infty} \delta'(x)f(x)\,dx = \delta(x)f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)f'(x)\,dx = -f'(0)$$

For the prototype function by applying integration by parts we have

$$\int_{-\infty}^{\infty} \psi'(x)f(x)\,dx = \psi(x)f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(x)f'(x)\,dx$$

Then by mapping to scale

$$\int_{-\infty}^{\infty} \frac{1}{s_n} \frac{d}{dx} \left[ \psi \left( \frac{x}{s_n} \right) \right] f(x)\,dx = \frac{1}{s_n} \psi \left( \frac{x}{s_n} \right) f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{s_n} \psi \left( \frac{x}{s_n} \right) f'(x)\,dx$$

Therefore,

$$\lim_{n \to \infty} \frac{1}{s_n} \frac{d}{dx} \psi \left( \frac{x}{s_n} \right) = \delta'(x)$$

Further, in order to have vanishing of the first term in the RHS for an arbitrary $L_1$ function we would require that $\lim_{s_n \to 0} \frac{1}{s_n} \psi' \left( \frac{x}{s_n} \right) = 0$ for $x$ not equal to an extremum point $x_m$.

In order to study the behavior at the extrema more easily we will rewrite the (shifted) prototype derivative as the limit of the symmetric central difference

$$\frac{d}{dx} \psi \left( \frac{x}{s_n} - \frac{1}{2} \right) = \lim_{n \to \infty} \frac{\psi \left( \frac{x}{s_n} + \frac{1}{2} \right) - \psi \left( \frac{x}{s_n} - \frac{1}{2} \right)}{s_n}$$

If follows that the last expression will have extrema tending towards $x_m = \pm \frac{s_n}{2}$, since by assumption the prototype has a maximum at the origin and for non-extremal point the function vanishes in scale limit. The extrema of the nominator of the RHS are given by

$$- \frac{x}{s_n} \left[ \psi' \left( \frac{x}{s_n} + \frac{1}{2} \right) - \psi' \left( \frac{x}{s_n} - \frac{1}{2} \right) \right] = 0$$

Therefore, at $x = \pm \frac{s_n}{2}$ (LHS) (0 for RHS respectively) for $s_n \neq 0$ we have the RHS equal to 0. Similar arguments starting from the backward difference can handle the case for $x = -\frac{s_n}{2}$. After substitution $x = ks_n$ we would have

$$\frac{d}{dx} \psi \left( \frac{x}{s_n} - \frac{1}{2} \right) = \lim_{n \to \infty} \frac{k \left( \psi' \left( k + \frac{1}{2} \right) - \psi' \left( k - \frac{1}{2} \right) \right)}{s_n}$$

Therefore the LHS will scale like $\frac{1}{s_n}$ which can be mapped to the scaling of Lemma's 4 and 5.

Since $x$ will tend to zero, if we take the LHS and using the definition of the $\delta$-"function", it is justified to write

$$\lim_{s_n \to 0} \psi' \left( \frac{x}{s_n} \right) = \delta(x+) - \delta(x-)$$

where $\delta(x \pm)$ is interpreted as the limit $\lim_{\epsilon \to 0} \delta(x \pm \frac{\epsilon}{2})$. 
Remark 3. In a frequently used another notation \[7\] the derivative of the \(\delta\)-“function” is frequently written as \(x\delta'(x) = -\delta(x)\), which would imply

\[
\delta'(x) = -\frac{\text{sign}(x)}{|x|}\delta(x), \ x \neq 0
\]

but from the expression for the central difference this can be interpreted also as

\[
-\lim_{\varepsilon \to 0} \frac{\delta(x + \frac{\varepsilon}{2}) - \delta(x - \frac{\varepsilon}{2})}{\max(x,\varepsilon)}
\]

for a positive \(\varepsilon\) which for \(x = 0\) will yield

\[
\frac{\delta(-\frac{\varepsilon}{2}) - \delta(\frac{\varepsilon}{2})}{\varepsilon}
\]

which from negative approach will be positive and from the positive approach will be negative.

5.4.2. Scale relative \(s/\varepsilon\)-limit procedure. Let \(f(x) \in \mathbb{C}^\infty\) be a function of a Delta sequence. That is

\[
f(x) = \frac{1}{s} \psi\left(\frac{x}{s}\right)
\]

where the index has been omitted for clarity. \(f(x)\) can be expanded in a 4\(^{th}\) order Taylor series about the origin as

\[
f(x) = \frac{1}{s} \left( \psi(0) + \frac{1}{2s^2} \psi''(0) x^2 + \frac{1}{24s^4} \psi^{iv}(0) x^4 + o\left(\frac{x^6}{s^6}\right) \right)
\]

and

\[
f'(x) = \frac{1}{s} \left( \frac{1}{s^2} \psi''(0) x + \frac{1}{6s^4} \psi^{iv}(0) x^3 + o\left(\frac{x^5}{s^5}\right) \right)
\]

From the area condition it follows that \(\lim_{x \to \pm\infty} \psi(x) = 0\). Since for large \(x\) \(\psi(x)\) must decay very fast towards 0 the signs of its derivatives must alternate in order to have cancellation. Let’s denote \(\psi''(0) = b\) and \(\psi^{iv}(0) = c\). Then by assumption \(b < 0\). Then it follows that about the origin the extrema of \(f'(x)\) can be found by solving for \(x\)

\[
\frac{b x}{s^3} + \frac{c x^3}{6 s^5} = 0
\]

which has solutions

\[
x_m = \pm \sqrt{-\frac{2 b}{c s}}
\]

Since \(b\) and \(c\) have different signs the roots \(x_m\) are all real.

\[
f'(x_m) = \frac{2 b}{3 s^2} \sqrt{-\frac{2 b}{c}}
\]
On the other hand, if we consider the 4th order Taylor expansion, the fractal variation at the positive extremum is

\[ v^e_\beta [f] (x_m) = -\frac{e^{1-\beta}}{s^2 24\sqrt{2}} \left( 16 \left( -\frac{b}{c} \right)^3 c + 48 b \sqrt{-\frac{b}{c}} + 8 \sqrt{-\frac{b}{c}} \left( \frac{\epsilon}{s} \right)^2 + \sqrt{2} c \left( \frac{\epsilon}{s} \right)^3 \right) \]

Thereafter, the behavior of the extremum is dictated by two exponents: \( \frac{e^{1-\beta}}{s^2} \) and \( \frac{\epsilon}{s} \). Let’s make the substitution \( s = k \epsilon^p \). Then

\[ v^e_\beta [f] (x_m) = -\frac{2\sqrt{2} b \sqrt{-\frac{b}{c}} e^{-2p-\beta+1}}{3 k^2} - \frac{\sqrt{-\frac{b}{c}} e^{-4p-\beta+3} c}{3 \sqrt{2} k^4} - \frac{c e^{-5p-\beta+4}}{24 k^5} \]

Then if we wish to retain the appearance of singularity at the extremum while decreasing scale we would have

\[ p > \frac{1-\beta}{2} \cap p > \frac{3-\beta}{4} \cap p > \frac{4-\beta}{5} \]

Therefore, only \( p > \frac{1-\beta}{2} \) is an admissible exponent in the substitution \( s = \epsilon^p \).

This amounts to a scale-relative \( s/\epsilon \)-limit procedure.

In particular, if \( p = 1 \) and \( k = 1 \) then

\[ v^e_\beta [f] (x_m) = -\frac{16 \left( -\frac{b}{c} \right)^3 + 8 \sqrt{-\frac{b}{c}} + \sqrt{2}}{24 \sqrt{2} \epsilon^{1+\beta}} c + 48 b \sqrt{-\frac{b}{c}} \]

which can be fit between the bonds established in Lemma’s 4 and 5.

**Remark 4.** There is a critical value of the order \( \beta = \frac{1}{4} \) for which separation of limits can not be used and power-law substitution can not be applied. The result is demonstrated in Fig. 4. Only in the limit \( \beta = 1 \), which corresponds to usual differentiation the \( \epsilon \) and \( s \)-limiting procedures seem to be unrelated.

This principle corresponds with the theory of scale relativity where the scale \( \epsilon \)-resolution is related to the temporal \( t \)-resolution as argued heuristically by Nottale [16]. In other words, not all trajectories towards singularity in the resolution space are admissible if the separation of limits is used.

**Corollary 6.** Let

\[ \delta_\beta (x) := \frac{\delta (x+) - \delta (x-)}{|x|^\beta} \]

Formally we can write the following equation for the limit of the fractal variation in distributional sense:

\[ \lim_{\epsilon \to 0} v^e_\beta [f] (x) = -\int_A^B \delta_\beta (x - z) f(z) \, dz \quad (5.18) \]

for the exponent \( \beta \leq 1 \) and \( 0 \in [A, B] \).
The primitive function maps to $u(x)$ along the critical line $1 - 2p - b = 0$.

**Proof.** From the last Theorem for a positive $\epsilon$ we have

$$\int_A^B \delta_\beta(x-z) f(z) \, dz = \lim_{\epsilon \to 0} \int_A^B \frac{\delta(x-z + \frac{\epsilon}{2}) - \delta(x-z - \frac{\epsilon}{2})}{|x-z|^{\beta}} f(z) \, dz =$$

$$- \lim_{\epsilon \to 0} \frac{f(x + \frac{\epsilon}{2}) - f(x - \frac{\epsilon}{2})}{\epsilon^{\beta}} = - \lim_{\epsilon \to 0} \nu_{\beta}^+ [f] \left( x - \frac{\epsilon}{2} \right)$$

\[ \square \]

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