ADJUNCTION AND INVERSION OF ADJUNCTION

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Abstract. We establish adjunction and inversion of adjunction for log canonical centers of arbitrary codimension in full generality.

§1. Introduction

Throughout this paper, we work over $\mathbb{C}$, the complex number field. We establish the following adjunction and inversion of adjunction for log canonical centers of arbitrary codimension.

Theorem 1.1. Let $X$ be a normal variety, and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $W$ be a log canonical center of $(X, \Delta)$, and let $\nu: Z \to W$ be the normalization of $W$. Then we have the adjunction formula

$$\nu^*(K_X + \Delta) = K_Z + B_Z + M_Z$$

with the following properties:

(A) $(X, \Delta)$ is log canonical in a neighborhood of $W$ if and only if $(Z, B_Z + M_Z)$ is an NQC generalized log canonical pair. Here the abbreviation “NQC” stands for “nef $\mathbb{Q}$-Cartier combinations”.

(B) $(X, \Delta)$ is log canonical in a neighborhood of $W$ and $W$ is a minimal log canonical center of $(X, \Delta)$ if and only if $(Z, B_Z + M_Z)$ is an NQC generalized Kawamata log terminal pair.

For the definition of NQC generalized log canonical pairs and NQC generalized Kawamata log terminal pairs, see [13, §2].

For the formulation of adjunction and inversion of adjunction for log canonical centers of arbitrary codimension in full generality, the notion of b-divisors, which was first introduced by Shokurov, is very useful. In fact, the $\mathbb{R}$-divisors $B_Z$ and $M_Z$ in Theorem 1.1 are the traces of certain $\mathbb{R}$-b-divisors $B$ and $M$ on $Z$, respectively. The precise version of Theorem 1.1 is the following.

Theorem 1.2 (Adjunction and inversion of adjunction). Let $X$ be a normal variety, and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $W$ be a log canonical center of $(X, \Delta)$, and let $\nu: Z \to W$ be the normalization of $W$. Then there exist a b-potentially nef $\mathbb{R}$-b-divisor $M$ and an $\mathbb{R}$-b-divisor $B$ on $Z$ such that $B_Z$ is effective with

$$\nu^*(K_X + \Delta) = K_Z + M_Z + B_Z.$$
More precisely, there exists a projective birational morphism $p: Z' \to Z$ from a smooth quasi-projective variety $Z'$ such that:

1. $M = \overline{M}_{Z'}$ and $M_{Z'}$ is a potentially nef $\mathbb{R}$-divisor on $Z'$.
2. $K + B = K_{Z'} + B_{Z'}$.
3. $\text{Supp} B_{Z'}$ is a simple normal crossing divisor on $Z'$.
4. $\nu \circ p(B_{Z'}^{\geq 1}) = W \cap \text{Nlc}(X, \Delta)$ holds set-theoretically, where $\text{Nlc}(X, \Delta)$ denotes the non-lc locus of $(X, \Delta)$.
5. $\nu \circ p(B_{Z'}^{> 1}) = W \cap (\text{Nlc}(X, \Delta) \cup \bigcup_{W \subseteq W, W \neq W} W')$, where $W'$ runs over log canonical centers of $(X, \Delta)$ which do not contain $W$, holds set-theoretically.

Hence, $(Z, B_Z + M_Z)$ is generalized log canonical, that is, $B_{Z'}^{> 1} = 0$, if and only if $(X, \Delta)$ is log canonical in a neighborhood of $W$. Moreover, $(Z, B_Z + M_Z)$ is generalized Kawamata log terminal, that is, $B_{Z'}^{\geq 1} = 0$, if and only if $(X, \Delta)$ is log canonical in a neighborhood of $W$ and $W$ is a minimal log canonical center of $(X, \Delta)$. We note that $M_{Z'}$ is semi-ample when $\dim W = 1$. We also note that if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then $B$ and $M$ become $\mathbb{Q}$-b-divisors by construction.

In this paper, the $\mathbb{R}$-b-divisors $B$ and $M$ in Theorem 1.2 are defined by using the notion of basic $\mathbb{R}$-slc-trivial fibrations. Here, we explain an alternative definition of $B$ and $M$ for the reader’s convenience. For the details of Definition 1.3, see [11, §5] and [10, Def. 2.1].

**Definition 1.3** (see [11, §5], [10, Def. 2.1], and Remark 6.1). Let $(X, \Delta)$, $W$, and $\nu: Z \to W$ be as in Theorem 1.2. For any higher birational model $\rho: \tilde{Z} \to Z$, we consider all prime divisors $T$ over $X$ such that $a(T, X, \Delta) = -1$ and the center of $T$ on $X$ is $W$. We take a log resolution $f: Y \to X$ of $(X, \Delta)$ so that $T$ is a prime divisor on $Y$ and the induced map $f_T: T \to \tilde{Z}$ is a morphism. We put $\Delta_T = (\Delta_Y - T)|_T$, where $\Delta_Y$ is defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. For any prime divisor $P$ on $\tilde{Z}$, we define a real number $\alpha_{P,T}$ by

$$\alpha_{P,T} = \sup\{\lambda \in \mathbb{R} | (T, \Delta_T + \lambda f_T^* P) \text{ is sub log canonical over the generic point of } P\}.$$

Then the trace $B_{\tilde{Z}}$ of $B$ on $\tilde{Z}$ is defined as

$$B_{\tilde{Z}} = \sum_P (1 - \inf_T \alpha_{P,T}) P,$$

where $P$ runs over prime divisors on $\tilde{Z}$ and $T$ runs over prime divisors over $X$ such that $a(T, X, \Delta) = -1$ and the center of $T$ on $X$ is $W$. When $W$ is a prime divisor on $X$, $T$ is the strict transform of $W$ on $Y$. In this case, we can easily check that $B_{\tilde{Z}} = (f_T)_* \Delta_T$ holds. We consider the $\mathbb{R}$-line bundle $\mathcal{L}$ on $X$ associated with $K_X + \Delta$. We fix an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D_{\tilde{Z}}$ on $\tilde{Z}$ whose associated $\mathbb{R}$-line bundle is $\rho^* \nu^*(\mathcal{L}|_W)$. Then the trace $M_{\tilde{Z}}$ of $M$ on $\tilde{Z}$ is defined as

$$M_{\tilde{Z}} = D_{\tilde{Z}} - K_{\tilde{Z}} - B_{\tilde{Z}}.$$

We simply write

$$\rho^* \nu^*(K_X + \Delta) = K_{\tilde{Z}} + B_{\tilde{Z}} + M_{\tilde{Z}}$$

if there is no danger of confusion (see also Remark 6.1).
As we saw in Definition 1.3, the $\mathbb{R}$-b-divisor $B$ on $Z$ depends only on the singularities of $(X, \Delta)$ near $W$. Conversely, Theorem 1.2(ii)–(v) implies that $B$ remembers properties of the singularities of $(X, \Delta)$ near $W$. If we put $B_Z = B_Z$ and $M_Z = M_Z$, then Theorem 1.1 directly follows from Theorem 1.2. Our new formulation of adjunction and inversion of adjunction includes some classical results as special cases. The following corollary is the case of $\dim W = \dim X - 1$, which recovers the classical adjunction and inversion of adjunction.

**Corollary 1.4 (Classical adjunction and inversion of adjunction).** In Theorem 1.1, we further assume that $\dim W = \dim X - 1$, that is, $W$ is a prime divisor on $X$. Then $M_Z$ and $B_Z$ become zero and Shokurov’s different, respectively. Then (A) recovers Kawakita’s inversion of adjunction on log canonicity. By (B), we have that $(X, \Delta)$ is purely log terminal in a neighborhood of $W$ if and only if $(Z, B_Z)$ is Kawamata log terminal.

We know that we have already had many related results. We only make some remarks on [3], [12].

**Remark 1.5 (Hacon’s inversion of adjunction).** In [12, Th. 1], Hacon treated inversion of adjunction on log canonicity for log canonical centers of arbitrary codimension under the extra assumption that $\Delta$ is a boundary $\mathbb{Q}$-divisor. We note that the $b$-divisor $B(V; X, \Delta)$ in [12] coincides with $B$ in Theorem 1.2 by [10, Th. 1.2]. In [11, Th. 5.4], we proved a generalization of [12, Th. 1]. We note that $B$ in [11, Th. 5.4] coincides with $B$ in Theorem 1.2. Hence, Theorem 1.2 can be seen as a complete generalization of [11, Th. 5.4] and [12, Th. 1].

**Remark 1.6 (Generalized adjunction and inversion of adjunction by Filipazzi).** In [3], Filipazzi established some related results for generalized pairs (see, e.g., [3, Th. 1.6]. Although they are more general than Theorems 1.1 and 1.2 in some sense, they do not include Theorem 1.1.

The main ingredients of Theorem 1.2 are the existence theorem of log canonical modifications established in [11] and the theory of basic slc-trivial fibrations in [6], [7]. Hence, this paper can be seen as a continuation of [7], [11]. Moreover, the theory of partial resolutions of singularities of pairs in [2] is indispensable. We do not use Kawakita’s inversion of adjunction (see [15, Th.]) nor the Kawamata–Viehweg vanishing theorem. If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then Theorem 1.2 easily follows from [6], [7], [11]. Unfortunately, however, the framework of basic slc-trivial fibrations discussed in [6] is not sufficient for our purposes in this paper. Hence, we establish the following result.

**Theorem 1.7 (Corollary 5.2).** Let $f: (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration, and let $B$ and $M$ be the discriminant and moduli $\mathbb{R}$-b-divisors associated with $f: (X, B) \to Y$, respectively. Then we have the following properties:

(i) $K + B$ is $\mathbb{R}$-b-Cartier, where $K$ is the canonical $b$-divisor of $Y$.

(ii) $M$ is $b$-potentially nef, that is, there exists a proper birational morphism $\sigma: Y' \to Y$ from a normal variety $Y'$ such that $M_{Y'}$ is a potentially nef $\mathbb{R}$-divisor on $Y'$ and that $M = M_{Y'}$ holds.

If $f: (X, B) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration, then Theorem 1.7 is nothing but [6, Th. 1.2], which is the main result of [6]. More precisely, we establish the following theorem.

**Theorem 1.8 (see Theorem 5.1).** Let $f: (X, B) \to Y$ be a projective surjective morphism from a simple normal crossing pair $(X, B)$ to a smooth quasi-projective variety $Y$ such that every stratum of $X$ is dominant onto $Y$ and $f_*O_X \cong O_Y$ with:
• $B = B^{≤1}$ holds over the generic point of $Y$,
• there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D$ on $Y$ such that $K_Y + B \sim_{\mathbb{R}} f^*D$ holds, and
• $\text{rank}_f\mathcal{O}_X([-B^{<1}]) = 1$.

We assume that there exists a simple normal crossing divisor $\Sigma$ on $Y$ such that $\text{Supp}D \subset \Sigma$ and that every stratum of $(X, \text{Supp}B)$ is smooth over $Y \setminus \Sigma$. Let $\mathcal{B}$ and $\mathcal{M}$ be the discriminant and moduli $\mathbb{R}$-b-divisors associated with $f : (X, B) \to Y$, respectively. Then:

(i) $K + B = K_Y + B_Y$ holds, where $K$ is the canonical $b$-divisor of $Y$, and
(ii) $\mathcal{M}_Y$ is a potentially nef $\mathbb{R}$-divisor on $Y$ with $\mathcal{M} = \mathcal{M}_Y$.

Note that Theorem 1.8 completely generalizes [14, Lem. 2.8]. By Theorem 1.8, we can use the framework of basic slc-trivial fibrations in [6] for $\mathbb{R}$-divisors. We also note that the main part of this paper is devoted to the proof of Theorem 1.8. In the proof of Theorem 1.2, we naturally construct a basic $\mathbb{R}$-slc-trivial fibration $f : (V, \Delta_V) \to Z$ by taking a suitable resolution of singularities of the pair $(X, \Delta)$. The $\mathbb{R}$-b-divisors $\mathcal{B}$ and $\mathcal{M}$ on $Z$ in Theorem 1.2 are the discriminant and moduli $\mathbb{R}$-b-divisors associated with $f : (V, \Delta_V) \to Z$, respectively.

Conjecture 1.9. In Theorem 1.8, $\mathcal{M}_Y$ is semi-ample.

If Conjecture 1.9 holds true, then $\mathcal{M}$ in Theorem 1.2 is $b$-semi-ample, that is, $\mathcal{M}_{Z'}$ is semi-ample. Note that Conjecture 1.9 follows from [6, Conj. 1.4]. When $\dim Y = 1$, we can easily check that $\mathcal{M}_Y$ is semi-ample by [9, Cor. 1.4]. Unfortunately, however, it is still widely open. In this paper, we prove Conjecture 1.9 for basic slc-trivial fibrations of relative dimension one under some extra assumption (see Theorem 7.2). Then we establish the following theorem.

Theorem 1.10 (see Corollary 7.3). If $W$ is a codimension 2 log canonical center of $(X, \Delta)$ in Theorem 1.2, then $\mathcal{M}$ is $b$-semi-ample.

Theorem 1.10 generalizes Kawamata’s result (see [16, Th. 1]). For the details, see Corollary 7.3.

We briefly look at the organization of this paper. In §2, we recall some basic definitions and results. In §3, we introduce the notion of basic $\mathbb{R}$-slc-trivial fibrations and recall the main result of [6]. In §4, we slightly generalize the main result of [6]. This generalization (see Theorem 4.1) seems to be indispensable in order to treat basic $\mathbb{R}$-slc-trivial fibrations. In §5, we establish a fundamental theorem for basic $\mathbb{R}$-slc-trivial fibrations (see Theorems 1.8 and 5.1). In §6, we prove the main result, that is, adjunction and inversion of adjunction for log canonical centers of arbitrary codimension, in full generality. More precisely, we first establish Theorem 1.2. Then we see that Theorem 1.1 and Corollary 1.4 easily follow from Theorem 1.2. In §7, we treat adjunction and inversion of adjunction for log canonical centers of codimension 2.

§2. Preliminaries

In this paper, we freely use the standard notation as in [4]–[7]. A scheme means a separated scheme of finite type over $\mathbb{C}$. A variety means an integral scheme, that is, an irreducible and reduced separated scheme of finite type over $\mathbb{C}$. We note that $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational numbers and real numbers, respectively. We also note that $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$ are the sets of positive rational numbers and positive real numbers, respectively. Similarly, $\mathbb{Q}_{≥0}$ denotes the set of nonnegative rational numbers.
Here, we collect some basic definitions for the reader’s convenience. Let us start with the definition of potentially nef divisors.

**Definition 2.1** (Potentially nef divisors; see [6, Def. 2.5]). Let $X$ be a normal variety, and let $D$ be a divisor on $X$. If there exist a completion $X^\dagger$ of $X$, that is, $X^\dagger$ is a complete normal variety and contains $X$ as a dense Zariski open subset, and a nef divisor $D^\dagger$ on $X^\dagger$ such that $D = D^\dagger|_X$, then $D$ is called a potentially nef divisor on $X$. A finite $\mathbb{Q}_{>0}$-linear (resp. $\mathbb{R}_{>0}$-linear) combination of potentially nef divisors is called a potentially nef $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor).

We give two important remarks on potentially nef $\mathbb{R}$-divisors.

**Remark 2.2.** Let $D$ be a nef $\mathbb{R}$-divisor on a smooth projective variety $X$. Then $D$ is not necessarily a potentially nef $\mathbb{R}$-divisor. This means that $D$ is not always a finite $\mathbb{R}_{>0}$-linear combination of nef Cartier divisors on $X$.

**Remark 2.3.** Let $X$ be a normal variety, and let $D$ be a potentially nef $\mathbb{R}$-divisor on $X$. Then $D \cdot C \geq 0$ for every projective curve $C$ on $X$. In particular, $D$ is $\pi$-nef for every proper morphism $\pi: X \to S$ to a scheme $S$.

It is convenient to use b-divisors to explain several results. Here, we do not repeat the definition of b-divisors. For the details, see [6, §2].

**Definition 2.4** (Canonical b-divisors). Let $X$ be a normal variety, and let $\omega$ be a top rational differential form of $X$. Then $(\omega)$ defines a b-divisor $K$. We call $K$ the canonical b-divisor of $X$.

**Definition 2.5** ($\mathbb{R}$-Cartier closures). The $\mathbb{R}$-Cartier closure of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D$ on a normal variety $X$ is the $\mathbb{R}$-b-divisor $D$ with trace $D_Y = f^*D$, where $f: Y \to X$ is a proper birational morphism from a normal variety $Y$.

We use the following definition in order to state our results (see Theorem 1.2).

**Definition 2.6** [6, Def. 2.12]. Let $X$ be a normal variety. An $\mathbb{R}$-b-divisor $D$ of $X$ is b-potentially nef (resp. b-semi-ample) if there exists a proper birational morphism $X' \to X$ from a normal variety $X'$ such that $D = D_{X'}$, that is, $D$ is the $\mathbb{R}$-Cartier closure of $D_{X'}$, and that $D_{X'}$ is potentially nef (resp. semi-ample). An $\mathbb{R}$-b-divisor $D$ of $X$ is $\mathbb{R}$-b-Cartier if $D_{X'}$ is $\mathbb{Q}$-Cartier and $D = D_{X'}$.

For the reader’s convenience, let us recall the definition of singularities of pairs. The following definition is standard and is well known.

**Definition 2.7** (Singularities of pairs). Let $X$ be a variety, and let $E$ be a prime divisor on $Y$ for some birational morphism $f: Y \to X$ from a normal variety $Y$. Then $E$ is called a divisor over $X$. A normal pair $(X, \Delta)$ consists of a normal variety $X$ and an $\mathbb{R}$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $(X, \Delta)$ be a normal pair, and let $f: Y \to X$ be a projective birational morphism from a normal variety $Y$. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$
with
\[ f_* \left( \sum_E a(E, X, \Delta) E \right) = -\Delta, \]
where \( E \) runs over prime divisors on \( Y \). We call \( a(E, X, \Delta) \) the discrepancy of \( E \) with respect to \((X, \Delta)\). Note that we can define the discrepancy \( a(E, X, \Delta) \) for any prime divisor \( E \) over \( X \) by taking a suitable resolution of singularities of \( X \). If \( a(E, X, \Delta) \geq -1 \) (resp. \( > -1 \)) for every prime divisor \( E \) over \( X \), then \((X, \Delta)\) is called sub log canonical (resp. sub Kawamata log terminal). We further assume that \( \Delta \) is effective. Then \((X, \Delta)\) is called log canonical and Kawamata log terminal if it is sub log canonical and sub Kawamata log terminal, respectively. When \( \Delta \) is effective and \( a(E, X, \Delta) > -1 \) holds for every exceptional divisor \( E \) over \( X \), we say that \((X, \Delta)\) is purely log terminal.

Let \((X, \Delta)\) be a log canonical pair. If there exists a projective birational morphism \( f : Y \to X \) from a smooth variety \( Y \) such that the exceptional locus of \( f \) and \( \text{Exc}(f) \cup \text{Supp} f_*^{-1} \Delta \) are simple normal crossing divisors on \( Y \) and that \( a(E, X, \Delta) > -1 \) holds for every \( f \)-exceptional divisor \( E \) on \( Y \), then \((X, \Delta)\) is called divisorial log terminal (dlt). It is well known that if \((X, \Delta)\) is purely log terminal, then it is dlt.

In this paper, the notion of non-lc loci and log canonical centers is indispensable.

**Definition 2.8** (Non-lc loci and log canonical centers). Let \((X, \Delta)\) be a normal pair. If there exist a projective birational morphism \( f : Y \to X \) from a normal variety \( Y \) and a prime divisor \( E \) on \( Y \) such that \((X, \Delta)\) is sub log canonical in a neighborhood of the generic point of \( f(E) \) and that \( a(E, X, \Delta) = -1 \), then \( f(E) \) is called a log canonical center of \((X, \Delta)\).

From now on, we further assume that \( \Delta \) is effective. The non-lc locus of \((X, \Delta)\), denoted by \( \text{Nlc}(X, \Delta) \), is the smallest closed subset \( Z \) of \( X \) such that the complement \((X \setminus Z, \Delta|_{X \setminus Z})\) is log canonical. We can define a natural scheme structure on \( \text{Nlc}(X, \Delta) \) by the non-lc ideal sheaf \( J_{\text{Nlc}}(X, \Delta) \) of \((X, \Delta)\). For the definition of \( J_{\text{Nlc}}(X, \Delta) \), see [4, §7].

We omit the precise definition of NQC generalized log canonical pairs and NQC generalized Kawamata log terminal pairs here since we need it only in Theorem 1.1 and the statement of Theorem 1.2 is sharper than that of Theorem 1.1. For the basic definitions and properties of generalized polarized pairs, we recommend the reader to see [13, §2]. Note that the notion of generalized pairs plays a crucial role in the recent study of higher-dimensional algebraic varieties.

**Definition 2.9.** Let \( X \) be an equidimensional reduced scheme. Note that \( X \) is not necessarily regular in codimension one. Let \( D \) be an \( \mathbb{R} \)-divisor (resp. a \( \mathbb{Q} \)-divisor), that is, \( D \) is a finite formal sum \( \sum_i d_i D_i \), where \( D_i \) is an irreducible reduced closed subscheme of \( X \) of pure codimension one and \( d_i \in \mathbb{R} \) (resp. \( d_i \in \mathbb{Q} \)) for every \( i \) such that \( D_i \neq D_j \) for \( i \neq j \). We put
\[
D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{=1} = \sum_{d_i = 1} D_i, \quad D^{>1} = \sum_{d_i > 1} d_i D_i, \quad \text{and} \quad [D] = \sum_i [d_i] D_i,
\]
where $[d_i]$ is the integer defined by $d_i \leq [d_i] < d_i + 1$. We note that $[D] = -[-D]$ and \{D\} = D - [D].$ Similarly, we put
\[D_{\geq 1} = \sum_{d_i \geq 1} d_i D_i.\]

Let $D$ be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor) as above. We call $D$ a subboundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) if $D = D_{\leq 1}$ holds. When $D$ is effective and $D = D_{\leq 1}$ holds, we call $D$ a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

We further assume that $f : X \to Y$ is a surjective morphism onto a variety $Y$ such that every irreducible component of $X$ is dominant onto $Y$. Then we put
\[D_V = \sum_{f(D_i) \not\subseteq Y} d_i D_i \quad \text{and} \quad D_H = \sum_{f(D_i) = Y} d_i D_i.\]
We call $D_V$ (resp. $D_H$) the vertical part (resp. horizontal part) of $D$ with respect to $f : X \to Y$.

§3. On basic slc-trivial fibrations

Roughly speaking, a basic slc-trivial fibration is a canonical bundle formula for simple normal crossing pairs. It was first introduced in [6] based on [8]. Let us start with the definition of simple normal crossing pairs.

**Definition 3.1 (Simple normal crossing pairs).** A pair $(X, B)$ consists of an equidimensional reduced scheme $X$ and an $\mathbb{R}$-divisor $B$ on $X$. We say that the pair $(X, B)$ is simple normal crossing at a point $x \in X$ if $X$ has a Zariski open neighborhood $U$ of $x$ that can be embedded in a smooth variety $M$, where $M$ has a regular system of parameters $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $x = 0$ in which $U$ is defined by a monomial equation
\[x_1 \cdots x_p = 0\]
and
\[B|_U = \sum_{i=1}^r \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.\]
We say that $(X, B)$ is a simple normal crossing pair if it is simple normal crossing at every point of $X$.

Let $(X, B)$ be a simple normal crossing pair, and let $\nu : X^\nu \to X$ be the normalization. We define $B^\nu$ by $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$, that is, $B^\nu$ is the sum of the inverse images of $B$ and the singular locus of $X$. Then a stratum of $(X, B)$ is an irreducible component of $X$ or the $\nu$-image of some log canonical center of $(X^\nu, B^\nu)$.

Let $(X, B)$ be a simple normal crossing pair, and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of $X$. Then a stratum of $X$ means an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$. It is easy to see that $W$ is a stratum of $X$ if and only if $W$ is a stratum of $(X, 0)$.

We introduce the notion of basic slc-trivial fibrations. In [6], we only treat basic $\mathbb{Q}$-slc-trivial fibrations.
DEFINITION 3.2 (Basic slc-trivial fibrations; see [6, Def. 4.1]). A pre-basic \( \mathbb{Q} \)-slc-trivial (resp. \( \mathbb{R} \)-slc-trivial) fibration \( f : (X,B) \to Y \) consists of a projective surjective morphism \( f : X \to Y \) and a simple normal crossing pair \((X,B)\) satisfying the following properties:

1. \( Y \) is a normal variety.
2. Every stratum of \( X \) is dominant onto \( Y \) and \( f_*O_X \cong O_Y \).
3. \( B \) is a \( \mathbb{Q} \)-divisor (resp. an \( \mathbb{R} \)-divisor) such that \( B = B^{\leq 1} \) holds over the generic point of \( Y \).
4. There exists a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor (resp. an \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor) \( D \) on \( Y \) such that \( K_X + B \sim_\mathbb{Q} f^*D \) (resp. \( K_X + B \sim_\mathbb{R} f^*D \)), that is, \( K_X + B \) is \( \mathbb{Q} \)-linearly (resp. \( \mathbb{R} \)-linearly) equivalent to \( f^*D \).

If a pre-basic \( \mathbb{Q} \)-slc-trivial (resp. \( \mathbb{R} \)-slc-trivial) fibration \( f : (X,B) \to Y \) also satisfies

5. rank \( f_*O_X([-\langle B^{\leq 1}\rangle]) = 1 \),

then it is called a basic \( \mathbb{Q} \)-slc-trivial (resp. \( \mathbb{R} \)-slc-trivial) fibration.

If there is no danger of confusion, we sometimes use (pre-)basic slc-trivial fibrations to denote (pre-)basic \( \mathbb{Q} \)-slc-trivial fibrations or (pre-)basic \( \mathbb{R} \)-slc-trivial fibrations.

REMARK 3.3 (see Remark 4.5). The condition \( f_*O_X \cong O_Y \) in (2) in Definition 3.2 does not play an important role. Moreover, we have to treat the case where \( O_Y \subsetneq f_*O_X \) in this paper. The reader can find that we do not need the condition \( f_*O_X \cong O_Y \) in many places in [6]. Hence, it may be better to remove the condition \( f_*O_X \cong O_Y \) from the definition of pre-basic slc-trivial fibrations (see [6, Def. 4.1] and Definition 3.2). However, we keep it here not to cause unnecessary confusion.

Note that the condition \( f_*O_X \cong O_Y \) always holds for basic slc-trivial fibrations even when we remove it from the definition of pre-basic slc-trivial fibrations. We will see it more precisely. It is sufficient to see that if every stratum of \( X \) is dominant onto \( Y \) with rank \( f_*O_X([-\langle B^{\leq 1}\rangle]) = 1 \), then the natural map \( O_Y \to f_*O_X \) must be an isomorphism. We note that there are natural inclusions

\[
O_Y \hookrightarrow f_*O_X \hookrightarrow f_*O_X([-\langle B^{\leq 1}\rangle])
\]

since \( [-\langle B^{\leq 1}\rangle] \) is effective. Hence, \( O_Y \hookrightarrow f_*O_X \) is an isomorphism over some nonempty Zariski open subset of \( Y \) and rank \( f_*O_X = 1 \) holds. We consider the Stein factorization

\[
f : X \to Z := \text{Spec}_Y f_*O_X \xrightarrow{\alpha} Y
\]

of \( f : X \to Y \). Since every irreducible component of \( X \) is dominant onto \( Y \), \( Z \) is a variety. Moreover, \( \alpha : Z \to Y \) is birational since rank \( f_*O_X = 1 \). By Zariski’s main theorem, \( \alpha : Z \to Y \) is an isomorphism. Hence, the natural map \( O_Y \to f_*O_X \) is an isomorphism.

In order to define discriminant \( \mathbb{R} \)-b-divisors and moduli \( \mathbb{R} \)-b-divisors for basic slc-trivial fibrations, we need the notion of induced (pre-)basic slc-trivial fibrations.

DEFINITION 3.4 (Induced (pre-)basic slc-trivial fibrations [6, 4.3]). Let \( f : (X,B) \to Y \) be a (pre-)basic slc-trivial fibration, and let \( \sigma : Y' \to Y \) be a generically finite surjective morphism from a normal variety \( Y' \). Then we have an induced (pre-)basic slc-trivial fibration \( f' : (X',B_{X'}) \to Y' \), where \( B_{X'} \) is defined by \( \mu^*(K_X + B) = K_{X'} + B_{X'} \), with the following
commutative diagram:

\[
\begin{array}{ccc}
(X', B_{X'}) & \xrightarrow{\mu} & (X, B) \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{\sigma} & Y,
\end{array}
\]

where \(X'\) coincides with \(X \times_Y Y'\) over a nonempty Zariski open subset of \(Y'\). More precisely, \((X', B_{X'})\) is a simple normal crossing pair with a morphism \(X' \to X \times_Y Y'\) that is an isomorphism over a nonempty Zariski open subset of \(Y'\) such that \(X'\) is projective over \(Y'\) and that every stratum of \(X'\) is dominant onto \(Y'\).

Now, we are ready to define discriminant \(\mathbb{R}\)-b-divisors and moduli \(\mathbb{R}\)-b-divisors for basic slc-trivial fibrations.

**Definition 3.5 (Discriminant and moduli \(\mathbb{R}\)-b-divisors [6, 4.5]).** Let \(f: (X, B) \to Y\) be a (pre-)basic slc-trivial fibration as in Definition 3.2. Let \(P\) be a prime divisor on \(Y\). By shrinking \(Y\) around the generic point of \(P\), we assume that \(P\) is Cartier. We set

\[
b_P = \max \left\{ t \in \mathbb{R} \mid (X', B' + t\nu^*f^*P) \text{ is sub log canonical over the generic point of } P \right\},
\]

where \(\nu: X' \to X\) is the normalization and \(K_{X'} + B' = \nu^*(K_X + B)\), that is, \(B'\) is the sum of the inverse images of \(B\) and the singular locus of \(X\), and set

\[
B_Y = \sum_P (1 - b_P)P,
\]

where \(P\) runs over prime divisors on \(Y\). Then it is easy to see that \(B_Y\) is a well-defined \(\mathbb{R}\)-divisor on \(Y\) and is called the discriminant \(\mathbb{R}\)-divisor of \(f: (X, B) \to Y\). We set

\[
M_Y = D - K_Y - B_Y
\]

and call \(M_Y\) the moduli \(\mathbb{R}\)-divisor of \(f: (X, B) \to Y\). By definition, we have

\[
K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y + M_Y).
\]

Let \(\sigma: Y' \to Y\) be a proper birational morphism from a normal variety \(Y'\), and let \(f': (X', B_{X'}) \to Y'\) be an induced (pre-)basic slc-trivial fibration by \(\sigma: Y' \to Y\). We can define \(B_{Y'}, K_{Y'},\) and \(M_{Y'}\) such that \(\sigma^*D = K_{Y'} + B_{Y'} + M_{Y'}\), \(\sigma_*B_{Y'} = B_Y\), \(\sigma_*K_{Y'} = K_Y\), and \(\sigma_*M_{Y'} = M_Y\). We note that \(B_{Y'}\) is independent of the choice of \((X', B_{X'})\), that is, \(B_{Y'}\) is well defined. Hence, there exist a unique \(\mathbb{R}\)-b-divisor \(B\) such that \(B_{Y'} = B_Y\) for every \(\sigma: Y' \to Y\) and a unique \(\mathbb{R}\)-b-divisor \(M\) such that \(M_{Y'} = M_Y\) for every \(\sigma: Y' \to Y\). Note that \(B\) is called the discriminant \(\mathbb{R}\)-b-divisor and that \(M\) is called the moduli \(\mathbb{R}\)-b-divisor associated with \(f: (X, B) \to Y\). We sometimes simply say that \(M\) is the moduli part of \(f: (X, B) \to Y\).

Let \(g: V \to Y\) be a proper surjective morphism from an equidimensional normal scheme \(V\) onto a normal variety \(Y\) such that every irreducible component of \(V\) is dominant onto \(Y\). Let \(G\) be an \(\mathbb{R}\)-divisor on \(V\) such that \(K_V + G\) is \(\mathbb{R}\)-Cartier. Assume that \((V, G)\) is sub log canonical over the generic point of \(Y\). Let \(\sigma: Y' \to Y\) be a generically finite surjective
morphism from a normal variety $Y'$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
(V', G') & \xrightarrow{\mu} & (V, G) \\
\downarrow{g'} & & \downarrow{g} \\
Y' & \xrightarrow{\sigma} & Y,
\end{array}
$$

where $V'$ is the normalization of the main components of $V \times_Y Y'$ and $G'$ is defined by $K_{V'} + G' = \mu^*(K_V + G)$. Then we can define the discriminant $R$-divisor $B_Y$ on $Y$ and the discriminant $R$-b-divisor $B_Y$ as in Definition 3.5. Let $f: (X, B) \to Y$ be a (pre-)basic slc-trivial fibration, and let $\nu: X' \to X$ be the normalization with $K_{X'} + B' = \nu^*(K_X + B)$. Then the discriminant $R$-b-divisor $B_Y$ associated with $f: (X, B) \to Y$ defined in Definition 3.5 obviously coincides with that of $f \circ \nu: (X', B') \to Y$ by definition.

Let us see the main result of [6].

**Theorem 3.6** [6, Th. 1.2]. Let $f: (X, B) \to Y$ be a basic $\mathbb{Q}$-slc-trivial fibration, and let $B$ and $M$ be the discriminant and moduli $\mathbb{Q}$-b-divisors associated with $f: (X, B) \to Y$, respectively. Then we have the following properties:

(i) $K + B$ is $\mathbb{Q}$-b-Cartier, where $K$ is the canonical b-divisor of $Y$.

(ii) $M$ is b-potentially nef, that is, there exists a proper birational morphism $\sigma: Y' \to Y$ from a normal variety $Y'$ such that $M_{Y'}$ is a potentially nef $\mathbb{Q}$-divisor on $Y'$ and that $M = M_{Y'}$.

The following result was established in [9].

**Theorem 3.7** [9, Cor. 1.4]. In Theorem 3.6, if $Y$ is a curve, then $M_Y$ is semi-ample.

We close this section with important remarks on [6].

**Remark 3.8.** In part (d) in [6, §6], we assume that $\text{Supp} M_Y \subset \text{Supp} \Sigma_Y$. However, this condition is unnecessary. This is because if $P$ is not an irreducible component of $\text{Supp} \Sigma_Y$, then we can always take a prime divisor $Q$ on $V$ such that $\text{mult}_Q(-B_Y + h^*B_Y) = 0$, $h(Q) = P$, and $\text{mult}_Q h^*P = 1$ (see [6, Prop. 6.3(iv)])).

**Remark 3.9.** In [6, 6.1], we assume that $\text{Supp}(B - f^*(B_Y + M_Y))$ is a simple normal crossing divisor on $X$. However, we do not need this assumption. All we need in [6, 6.1] is the fact that the support of $\{\Delta\}$ is a simple normal crossing divisor on $X$. We note that

$$
\text{Supp}\{\Delta\} \subset \text{Supp}(B - f^*(B_Y + M_Y))
$$

always holds since $\Delta = K_{X/Y} + B - f^*(B_Y + M_Y)$.

§4. **Fundamental theorem for basic $\mathbb{Q}$-slc-trivial fibrations**

In this section, we slightly generalize the main theorem of [6] (see Theorem 3.6). The following theorem is the main result of this section.

**Theorem 4.1** (see [6, Th. 1.2]). Let $f: (X, B) \to Y$ be a basic $\mathbb{Q}$-slc-trivial fibration such that $Y$ is a smooth quasi-projective variety. We write $K_X + B \sim_\mathbb{Q} f^*D$. Assume that there exists a simple normal crossing divisor $\Sigma$ on $Y$ such that $\text{Supp} D \subset \Sigma$ and that every stratum of $(X, \text{Supp} B)$ is smooth over $Y \setminus \Sigma$. Then:
(i) \( K + B = K_Y + B_Y \) holds.
(ii) \( M_Y \) is a potentially nef \( \mathbb{Q} \)-divisor on \( Y \) with \( M = M_Y \).

In \( \S 5 \), Theorem 4.1 is generalized for basic \( \mathbb{R} \)-slc-trivial fibrations (see Theorems 1.8 and 5.1). We note that Theorem 4.1 is indispensable for the proof of Theorem 5.1 in \( \S 5 \). For the proof of Theorem 4.1, we prepare a lemma on simultaneous partial resolutions of singularities of pairs. Let us recall the main result of [2].

**Theorem 4.2** [2, Th. 1.4]. Let \( X \) be a reduced scheme, and let \( D \) be a \( \mathbb{Q} \)-divisor on \( X \). Let \( U \) be the largest open subset of \( X \) such that \((U, D|_U)\) is a simple normal crossing pair. Then there is a morphism \( f: \tilde{X} \to X \), which is a composition of blowups, such that:

- the exceptional locus \( \text{Exc}(f) \) is of pure codimension one,
- putting \( \tilde{D} = f_*^{-1}D + \text{Exc}(f) \), then \((\tilde{X}, \tilde{D})\) is a simple normal crossing pair, and
- \( f \) is an isomorphism over \( U \).

**Remark 4.3** (Functoriality; see [2, Rem. 1.5(3)]). By [2, Rem. 1.5(3)], for every reduced scheme \( X \) and a \( \mathbb{Q} \)-divisor \( D \) on \( X \), we may take \( f_X: \tilde{X} \to X \) of Theorem 4.2 satisfying the following functoriality. Suppose that we are given an étale or a smooth morphism \( \phi: X \to Y \) of reduced schemes and \( \mathbb{Q} \)-divisors \( D_X \) and \( D_Y \) on \( X \) and \( Y \), respectively, such that:

- \( \phi^*D_Y = D_X \) and
- the number of irreducible components of \( X \) (resp. \( \text{Supp} D_X \)) at a point \( x \in X \) coincides with that of \( Y \) (resp. \( \text{Supp} D_Y \)) at \( \phi(x) \in Y \) for every \( x \in X \).

Then, the morphisms \( f_X: \tilde{X} \to X \) and \( f_Y: \tilde{Y} \to Y \) as in Theorem 4.2 form the diagram of the fiber product

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi} & \tilde{Y} \\
\downarrow f_X & \square & \downarrow f_Y \\
X & \xrightarrow{\phi} & Y,
\end{array}
\]

that is, \( \tilde{X} = X \times_Y \tilde{Y} \).

The following lemma is a key lemma for the proof of Theorem 4.1.

**Lemma 4.4.** Let \((X, B)\) be a simple normal crossing pair such that \( B \) is a \( \mathbb{Q} \)-divisor. Let \( f: X \to Y \) be a surjective morphism onto a smooth variety \( Y \) such that every stratum of \((X, \text{Supp} B)\) is smooth over \( Y \). We put \( \Delta = K_X + B \) and assume that \( b\Delta \sim 0 \) for some positive integer \( b \). We consider a \( b \)-fold cyclic cover

\[
\pi: \tilde{X} = \text{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{O}_X([i\Delta]) \to X
\]

associated with \( b\Delta \sim 0 \). We put \( K_{\tilde{X}} + B_{\tilde{X}} = \pi^*(K_X + B) \). Let \( \tilde{U} \) be the largest Zariski open subset of \( \tilde{X} \) such that \((\tilde{U}, B_{\tilde{X}}|_{\tilde{U}})\) is a simple normal crossing pair. Then there exists a morphism \( d: V \to \tilde{X} \) given by a composition of blowups such that:

(i) \( d \) is an isomorphism over \( \tilde{U} \),
(ii) \((V, B_V)\) is a simple normal crossing pair, where \( K_V + B_V = d^*(K_{\tilde{X}} + B_{\tilde{X}}) \), and
(iii) every stratum of \((V, \text{Supp} B_V)\) is smooth over \( Y \).
Proof. Let us quickly recall the \( b \)-fold cyclic cover \( \tilde{\pi} : \tilde{X} \to X \). We fix a rational function \( \phi \) on \( X \) such that \( b\Delta = \text{div}(\phi) \). As usual, we can define an \( \mathcal{O}_X \)-algebra structure of \( \bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta \rfloor) \) by \( b\Delta = \text{div}(\phi) \). We note that 
\[
\mathcal{O}_X(\lfloor i\Delta \rfloor) \times \mathcal{O}_X(\lfloor j\Delta \rfloor) \to \mathcal{O}_X(\lfloor (i+j)\Delta \rfloor)
\]
is well defined for \( 0 \leq i, j \leq b-1 \) by \( \lfloor i\Delta \rfloor + \lfloor j\Delta \rfloor \leq \lfloor (i+j)\Delta \rfloor \) and that
\[
\mathcal{O}_X(\lfloor (i+j)\Delta \rfloor) \simeq \mathcal{O}_X(\lfloor (i+j-b)\Delta \rfloor)
\]
for \( i+j \geq b \) defined by the multiplication with \( \phi^{-1} \). We put 
\[
\pi : \tilde{X} = \text{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta \rfloor)
\]
and call it a \( b \)-fold cyclic cover associated with \( b\Delta \sim 0 \). By construction, \( \pi : \tilde{X} \to X \) is \( \acute{e} \text{tale} \) outside \( \text{Supp}\{\Delta\} \). We note that \( \tilde{X} \) is normal over a neighborhood of the generic point of every irreducible component of \( \text{Supp}\{\Delta\} \). We also note that \( (\tilde{X}, B_{\tilde{X}}) \) is simple normal crossing in codimension one. Throughout this proof, we freely use the following commutative diagram:

\[
\begin{array}{ccc}
(X, B) & \xleftarrow{\pi} & (\tilde{X}, B_{\tilde{X}}) \\
\downarrow f & & \downarrow d \\
Y & \xleftarrow{\delta} & (V, B_V)
\end{array}
\]

**Step 1.** Let \( U \) and \( Z \) be affine open neighborhood of \( x \in X \) and \( y = f(x) \in Y \), respectively. Without loss of generality, we may assume that \( U \) is a simple normal crossing divisor on a smooth affine variety \( W \) since \( (X, B) \) is a simple normal crossing pair. By shrinking \( W, U, \) and \( Z \) suitably, we get the following commutative diagram:

\[
\begin{array}{ccc}
U & \xleftarrow{\iota} & W \\
\downarrow f \mid U & & \downarrow p \\
Z
\end{array}
\]

where \( \iota \) is the natural closed embedding \( U \hookrightarrow W \). From now on, we repeatedly shrink \( W, U, \) and \( Z \) suitably without mentioning it explicitly. Since every stratum of \( X \) is smooth over \( Y \), we may assume that \( p \) is a smooth morphism between smooth affine varieties.

**Step 2.** Since \( p : W \to Z \) is a smooth morphism, there exists a commutative diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{g} & Z \times \mathbb{C}^n \\
\downarrow p & & \downarrow p_1 \\
Z
\end{array}
\]

where \( g \) is \( \acute{e} \text{tale} \) and \( p_1 \) is the first projection (see, e.g., [1, Chap. VII, Def. 1.1 and Th. 1.8]). By choosing a coordinate system \( (z_1, \ldots, z_n) \) of \( \mathbb{C}^n \) suitably and shrinking \( U \) and \( W \) if necessary, we may further assume that \( U \) is defined by a monomial

\[
x_1 \cdots x_p = 0
\]
on $W$, where $x_i = g^* z_i$ for $1 \leq i \leq p$, and

$$B|_U = \sum_{i=1}^{r} \alpha_i(y_i = 0)|_U \quad \text{with} \quad \alpha_i \in \mathbb{Q}$$

holds, where $y_i = g^* z_{p+i}$ for $1 \leq i \leq r$. Here, we used the hypothesis that every stratum of $(X, \text{Supp} B)$ is smooth over $Y$.

**Step 3.** We put $L = (z_1 \cdots z_p = 0)$ in $\mathbb{C}^n$. Then we have the following commutative diagram.

$$
\begin{array}{ccc}
U & \xrightarrow{g|_U} & Z \times L \\
\downarrow f|_U & & \downarrow p_1 \\
Z & & 
\end{array}
$$

Note that $g|_U$ is étale because it is the base change of $g$ by $L \hookrightarrow \mathbb{C}^n$. We put

$$D = \sum_{i=1}^{r} \alpha_i(z_{p+i} = 0)$$

on $\mathbb{C}^n$. Let $p_2: Z \times \mathbb{C}^n \to \mathbb{C}^n$ be the second projection. Then $B|_U = g^* p_2^* D|_U$ holds.

**Step 4.** Without loss of generality, we may assume that $K_Z \sim 0$ by shrinking $Z$ suitably. Then $K_U \sim 0$ holds. Hence, by using the second projection $p_2: Z \times \mathbb{C}^n \to \mathbb{C}^n$, we have

$$0 \sim b\Delta|_U = b(K_U + B|_U) \sim bg|_U^*(p_2^* D|_{Z \times L}).$$

Since $g|_U$ is étale, we see that all the coefficients of $bp_2^* D|_{Z \times L}$ are integers. Since $p_2$ is the second projection and $D + L$ is a simple normal crossing divisor on $\mathbb{C}^n$, all the coefficients of $bD$ are integers. Therefore, we have $bD \sim 0$. We fix a rational function $\sigma$ on $\mathbb{C}^n$ such that $bD = \text{div}(\sigma)$. We consider the $b$-fold cyclic cover $\alpha: M \to \mathbb{C}^n$ associated with $bD = \text{div}(\sigma)$. We put $N = \alpha^{-1} L$. We define $B_N$ by $K_N + B_N = (\alpha|_N)^* (K_L + D|_L)$ and put $B_{Z \times N} = p_2^* B_N$, where $p_2: Z \times N \to N$ is the second projection. Then we get the following commutative diagram:

$$
\begin{array}{ccc}
U' & \xrightarrow{g'} & Z \times N \\
\downarrow U & \downarrow \text{id} \times (\alpha|_N) & \downarrow \alpha \\
U & \xrightarrow{g|_U} & Z \times L \\
\downarrow f|_U & & \downarrow p_2 \\
Z & & \mathbb{C}^n
\end{array}
$$

where $g': U' \to Z \times N$ is the base change of $g|_U: U \to Z \times L$ by $\text{id} \times (\alpha|_N)$. We put $B_{U'} = g'|^* B_{Z \times N}$. Then $K_{U'} + B_{U'}$ is equal to the pullback of $K_U + B|_U$ to $U'$.

**Step 5.** Since $\alpha: M \to \mathbb{C}^n$ is the $b$-fold cyclic cover associated with $bD = \text{div}(\sigma)$, we see that

$$M = \text{Spec}_{\mathbb{C}^n} \bigoplus_{i=0}^{b-1} \mathcal{O}_{\mathbb{C}^n}(\lfloor iD \rfloor).$$
Since \( p_2 \circ g : W \to Z \times \mathbb{C}^n \to \mathbb{C}^n \) is the composition of an étale morphism and the second projection, we have \( g^*p_2^*[iD]|_U = [ig^*p_2^*D]|_U = [iB]|_U = [iB|_U \), where the last equality follows from that \((U, B|_U)\) is a simple normal crossing pair. Let \( \sigma_U \) be a rational function on \( U \) which is the pullback of \( \sigma \). Then \( bB|_U = \text{div}(\sigma_U) \) because we have \( bD = \text{div}(\sigma) \). By the construction of \( U' \to U \), we see that

\[
U' = \text{Spec}_U \bigoplus_{i=0}^{b-1} \mathcal{O}_U([iB|_U])
\]

and \( U' \to U \) is the \( b \)-fold cyclic cover associated with \( bB|_U = \text{div}(\sigma_U) \).

We recall that \( \Delta = K_X + B \) and \( \bar{X} \to X \) is the \( b \)-fold cyclic cover associated with \( b\Delta = \text{div}(\phi) \). We put \( \phi_U \) as the restriction of \( \phi \) to \( U \). Then, the morphism \( \pi^{-1}(U) \to U \) is the \( b \)-fold cyclic cover associated with \( b\Delta|_U = \text{div}(\phi_U) \). Now, \( \Delta|_U - B|_U \) is a Cartier divisor on \( U \) and \( b(\Delta|_U - B|_U) = \text{div}(\phi_U \cdot \sigma_U^{-1})\). With this relation, we construct a \( b \)-fold cyclic cover \( \tau : \bar{U} \to U \). Then \( \tau \) is étale, \( \tau^*(\Delta|_U - B|_U) \) is Cartier, and \( \tau^*(\Delta|_U - B|_U) \sim 0 \). So there exists a rational function \( \xi \) on \( \bar{U} \) such that \( \xi^b = \tau^*(\phi_U \cdot \sigma_U^{-1}) \), equivalently, \( \tau^*(\Delta|_U - B|_U) = \text{div}(\xi) \).

From this, the \( b \)-fold cyclic cover \( U_2^\dagger \to \bar{U} \) associated with \( b\tau^*(\Delta|_U - B|_U) \) is isomorphic to the \( b \)-fold cyclic cover \( U_2 \to \bar{U} \) associated with \( b\tau^*(\Delta|_U - B|_U) = \text{div}(\tau^*(\phi_U \cdot \xi^{-b})) \).

Since \( \tau : \bar{U} \to U \) is étale, the construction of \( U_2^\dagger \) shows that \( U_2^\dagger \to \bar{U} \) is the base change of \( U' \to U \) by \( \bar{U} \to U \). Similarly, we see that \( U_1^\dagger \to \bar{U} \) is the base change of \( \pi^{-1}(U) \to U \) by \( \bar{U} \to U \).

We put \( a_1 : U_1^\dagger \to \pi^{-1}(U) \) and \( a_2 : U_2^\dagger \to U' \). By construction, \( a_1 \) and \( a_2 \) are étale. We see that the composition \( U_1^\dagger \to \pi^{-1}(U) \to U \) is isomorphic to the composition \( U_2^\dagger \to U' \to U \) by construction. By this isomorphism, we obtain that \( a_1^*(B_{\bar{X}}|_{\pi^{-1}(U)}) \) is isomorphic to \( a_2^*B_{U'} \).

In this way, there exist étale morphisms \( a : U^\dagger \to \pi^{-1}(U) \) and \( a' : U^\dagger \to U' \) over \( \bar{Z} \) such that \( U_1^\dagger \simeq U^\dagger \simeq U_2^\dagger \) with the following commutative diagram:

\[
\begin{array}{ccc}
U^\dagger & \xrightarrow{a} & U^\dagger \\
\downarrow & & \downarrow \\
\pi^{-1}(U) & \xrightarrow{a'} & U' \\
\downarrow & & \downarrow \\
\pi & \xrightarrow{a} & U \\
\end{array}
\]

such that \( a^*(B_{\bar{X}}|_{\pi^{-1}(U)}) = a'^*B_{U'} \).

**Step 6.** We apply [2, Th. 1.4] (see Theorem 4.2) to the pair \((\bar{X}, B_{\bar{X}})\). Then we obtain a morphism \( d : V \to \bar{X} \) given by a composite of blowups satisfying (i) and (ii). Hence, all we have to do is to check that \( d : V \to \bar{X} \) satisfies (iii).

**Step 7.** Recall that \( B_{Z \times N} = p_2^*B_N \), where \( p_2 : Z \times N \to N \), and \( B_{U'} = g'^*B_{Z \times N} \). Recall also the relation \( a^*(B_{\bar{X}}|_{\pi^{-1}(U)}) = a'^*B_{U'} \). We apply [2, Th. 1.4] (see Theorem 4.2) to \( N \) and
$B_N$, and we obtain a morphism $\beta: N' \to N$ given by a composition of blowups. We apply 
[2, Th. 1.4] again to $Z \times N$ and $B_{Z \times N}$. Then we get a morphism $\id_{Z \times \beta}: Z \times N' \to Z \times N$ by the functoriality of [2, Th. 1.4] (see Remark 4.3). We put $\hat{V} = d^{-1}(\pi^{-1}(U)) \subset V$, and we apply [2, Th. 1.4] to the pair of $U^\dagger$ and $a^*(B_{\bar{X}|\pi^{-1}(U)})$, and the pair of $U'$ and $B_U$. Then we obtain morphisms $V^\dagger \to U^\dagger$ and $V' \to U'$.

We check that we may apply the functoriality (see Remark 4.3) to the morphisms $g': U' \to Z \times N, a': U^\dagger \to U'$, and $a: U^\dagger \to \pi^{-1}(U)$ (see the diagram in the next paragraph) and divisors $B_{Z \times N}$ on $Z \times N, B_U$ on $U'$, and $B_{\bar{X}|\pi^{-1}(U)}$ on $\pi^{-1}(U)$ and their pullbacks.

We only check the second condition of Remark 4.3 for schemes because the case of divisors can be proved by the same way. By construction, $g'$ is the base change of $g|_U: U \to Z \times L$ by the morphism $Z \times N \to Z \times L$. Because $Z \times L$ is a simple normal crossing divisor on $Z \times \mathbb{C}^n$, and $g|_U$ is étale, by arguing locally, we see that $g|_U$ satisfies the second condition of Remark 4.3. Then so does $g'$ since $g'$ is constructed by the base change of $g|_U$. Similarly, $a'$ (resp. $a$) is constructed with the base change of $\tau: \overline{U} \to U$ by $U' \to U$ (resp. $\pi^{-1}(U) \to U$), and $U$ is a simple normal crossing divisor on $W$. Thus, the same argument as above implies that $a'$ and $a$ satisfy the second condition of Remark 4.3. Thus, we may apply the functoriality (see Remark 4.3) to the above morphisms and divisors.

Applying the functoriality (see Remark 4.3), we have the following diagram:

$$
\begin{array}{cccccc}
\hat{V} & \leftarrow & V^\dagger & \rightarrow & V' & \rightarrow & Z \times N' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi^{-1}(U) & \leftarrow & a^\dagger & \rightarrow & a' U' & \rightarrow & Z \times N,
\end{array}
$$

where each square is the fiber product. By construction, all the upper horizontal morphisms are étale. Let $B_{V^\dagger}$ (resp. $B_{\hat{V}}$) be the sum of the birational transform of $a^* B_U$ (resp. $B_{\bar{X} | \pi^{-1}(U)}$) and the exceptional locus of $V^\dagger \to U$ (resp. $\hat{V} \to \pi^{-1}(U)$). Then, every stratum of $(V^\dagger, \text{Supp} B_{V^\dagger})$ is smooth over $Z$. Since each irreducible component of $V^\dagger$ is smooth over $Z$ and $V^\dagger \to \hat{V}$ is étale, we see that each irreducible component of $\hat{V}$ is smooth over $Z$. By a similar argument, we see that every stratum of $(\hat{V}, \text{Supp} B_{\hat{V}})$ is smooth over $Z$. This implies that $d: V \to \tilde{X}$ satisfies (iii).

We finish the proof of Lemma 4.4.

Before we start the proof of Theorem 4.1, we make an important remark on [6].

**Remark 4.5** (see Remark 3.3). In Theorem 4.1, we can write

$$K_X + B + \frac{1}{b} \text{div}(\varphi) = f^* D$$

for some positive integer $b$ and a rational function $\varphi \in \Gamma(X, K_X^*)$, where $K_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ and $K_X^*$ denotes the sheaf of invertible elements in $K_X$, such that $b(K_X + B + f^* D) \sim 0$. In general, $b$ is larger than $b(F, B_F)$ in [6, §6]. We take a $b$-fold cyclic cover $\pi: \tilde{X} \to X$ associated with $b\Delta \sim 0$, where $\Delta = K_X + B - f^* D$, as in [6, §6]. Then the general fiber of $h: V \to Y$ is not necessarily connected in [6, §6]. Moreover, $V$
is not necessarily connected. This means that [6, Prop. 6.3(ii)] does not hold true since
the natural map \( O_Y \to h_*O_V \) is not always an isomorphism. Fortunately, the condition
\( h_*O_V \simeq O_Y \) is not necessary for the proof of the other properties of [6, Prop. 6.3]. We note
that the condition \( h_*O_V \simeq O_Y \) is unnecessary in [6, Lem. 7.3 and Th. 8.1]. Hence, it may
be better to remove the condition \( f_*O_X \simeq O_Y \) from part (2) in Definition 3.2.

Let \( b \) be the smallest positive integer such that \( b(K_X + B - f^*D) \sim 0 \). Then we can write
\[
K_X + B + \frac{1}{b} \text{div}(\varphi) = f^*D.
\]
As usual, we consider the \( b \)-fold cyclic cover \( \pi: \bar{X} \to X \) associated with \( b\Delta = \text{div}(\varphi^{-1}) \),
where \( \Delta = K_X + B - f^*D \). Let \( b^2 \) be any positive integer with \( b^2 \geq 2 \). We put \( \varphi^b = \varphi^{b^2} \).
Then we get
\[
K_X + B + \frac{1}{bb^2} \text{div}(\varphi^b) = f^*D.
\]
Let \( \pi^b: X^b \to X \) be the \( bb^2 \)-fold cyclic cover associated with \( bb^2\Delta = \text{div}((\varphi^b)^{-1}) \). We take the
\( H \)-invariant part of \( \pi^b: X^b \to X \), where \( H \) is the subgroup of the Galois group \( \text{Gal}(X^b/X) \simeq \mathbb{Z}/bb^2\mathbb{Z} \) of \( \pi^b: X^b \to X \) corresponding to \( b\mathbb{Z}/bb^2\mathbb{Z} \). Then we can recover \( \pi: \bar{X} \to X \).
Note that \( \pi^b: X^b \to X \) is decomposed into \( b^2 \) components and that each component is isomorphic
to \( \pi: \bar{X} \to X \).

Let us prove Theorem 4.1.

Proof of Theorem 4.1. Here, we only explain how to modify the proof of [6, Th. 1.2] by
using Lemma 4.4.

By taking a completion as in [6, Lem. 4.12], we may further assume that \( Y \) is projective.
By Lemma 4.4, we can construct a commutative diagram (6.4) in [6, §6] satisfying (a)–(g)
such that \( \Sigma_Y = \Sigma \) holds without taking birational modifications of \( Y \). Here, we do not
require the condition \( \text{Supp} M_Y \subset \text{Supp} \Sigma_Y \) in part (d) in [6, §6] (see Remark 3.8). We also
do not require the condition that the general fiber of \( h: V \to Y \) is connected (see Remark 4.5).
The covering arguments and [6, Prop. 6.3] work without any modifications. We note
that \( Y \) is a smooth projective variety. In what follows, we apply the proof of [6, Th. 8.1].
Let \( \gamma: Y' \to Y \) be a projective birational morphism from a normal variety \( Y' \). By replacing
\( Y' \) with a higher model if necessary, we may assume that \( Y' \) is smooth and that \( \gamma^{-1}\Sigma_Y \)
is a simple normal crossing divisor on \( Y' \). With [6, Lem. 7.3], we construct \( \tau: \bar{Y} \to Y \) a
unipotent reduction of the local monodromies around \( \Sigma_Y \). Then the induced fibration over
\( \bar{Y} \) satisfies [6, Prop. 6.3(iv) and (v)]. As in the proof of [6, Th. 8.1], we get a diagram:
\[
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\gamma} & Y' \\
\tau \downarrow & & \downarrow \tau' \\
Y & \xleftarrow{\gamma} & Y'
\end{array}
\]
such that \( \tau' \) is finite and the induced fibration over \( \bar{Y}' \) satisfies [6, Prop. 6.3(iv) and (v)].
By [6, Th. 3.1], we see that \( M_{\bar{Y}'} \) is a nef Cartier divisor and \( \gamma'^*M_{\bar{Y}'} = M_{Y'} \). Moreover, we have
\( \tau^*M_Y = M_{\bar{Y}} \) and \( \tau'^*M_{Y'} = M_{Y'} \) because \( \tau \) and \( \tau' \) are both finite (see [6, Lem. 4.10]).
Thus, we have that \( M_Y \) is a nef \( \mathbb{Q} \)-divisor and \( \gamma^*M_Y = M_{Y'} \). This is Theorem 4.1 (ii).
Theorem 4.1(i) immediately follows from Theorem 4.1(ii). So we are done. ∎
§5. Fundamental theorem for basic $\mathbb{R}$-slc-trivial fibrations

In this section, we establish the following fundamental theorem for basic $\mathbb{R}$-slc-trivial fibrations.

**Theorem 5.1** (see Theorem 1.8). Let $f: (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration such that $Y$ is a smooth quasi-projective variety. We write $K_X + B \sim_{\mathbb{R}} f^* D$. Assume that there exists a simple normal crossing divisor $\Sigma$ on $Y$ such that $\text{Supp} D \subset \Sigma$ and that every stratum of $(X, \text{Supp} B)$ is smooth over $Y \setminus \Sigma$. Then:

(i) $K + B = K_Y + B_Y$ holds, and
(ii) $M_Y$ is a potentially nef $\mathbb{R}$-divisor on $Y$ with $M = \overline{M}_Y$.

By Theorem 5.1, which is obviously a generalization of Theorem 4.1, we can use the theory of basic slc-trivial fibrations in [6], [7] for $\mathbb{R}$-divisors. The following formulation may be useful. Hence, we state it explicitly here for the reader’s convenience. We note that if $f: (X, B) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration, then Corollary 5.2 is nothing but [6, Th. 1.2].

**Corollary 5.2** [6, Th. 1.2]. Let $f: (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration, and let $B$ and $M$ be the discriminant and moduli $\mathbb{R}$-b-divisors associated with $f: (X, B) \to Y$, respectively. Then we have the following properties:

(i) $K + B$ is $\mathbb{R}$-b-Cartier, where $K$ is the canonical b-divisor of $Y$.
(ii) $M$ is b-potentially nef, that is, there exists a proper birational morphism $\sigma: Y' \to Y$ from a normal variety $Y'$ such that $M_{Y'}$ is a potentially nef $\mathbb{R}$-divisor on $Y'$ and that $M = \overline{M}_{Y'}$ holds.

**Remark 5.3** (see [9, Cor. 1.4]). In Theorem 5.1 and Corollary 5.2, we can easily see that $M_Y$ is semi-ample when $Y$ is a curve by Theorem 3.7 and Lemma 5.4.

Let us start with an easy lemma.

**Lemma 5.4.** Let $f: (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration with $K_X + B \sim_{\mathbb{R}} f^* D$. Then there exist a $\mathbb{Q}$-divisor $B_i$ on $X$, a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D_i$ on $Y$, and a positive real number $r_i$ for $1 \leq i \leq k$ such that:

1. $\sum_{i=1}^k r_i = 1$ with $\sum_{i=1}^k r_i B_i = B$ and $\sum_{i=1}^k r_i D_i = D$,
2. $\text{Supp} B = \text{Supp} B_i$, $[B^{>1}] = [B_i^{>1}]$, and $[-(B^{<1})] = [-B_i^{<1}]$ hold for every $i$,
3. if $\text{coeff}_S(B) \in \mathbb{Q}$ for a prime divisor $S$ on $X$, then $\text{coeff}_S(B) = \text{coeff}_S(B_i)$ holds for every $i$,
4. $\text{Supp} D = \text{Supp} D_i$ holds for every $i$,
5. if $\text{coeff}_T(D) \in \mathbb{Q}$ for a prime divisor $T$ on $Y$, then $\text{coeff}_T(D) = \text{coeff}_T(D_i)$ holds for every $i$, and
6. $K_X + B_i \sim_{\mathbb{Q}} f^* D_i$ holds for every $i$.

In particular, $f: (X, B_i) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration with $K_X + B_i \sim_{\mathbb{Q}} f^* D_i$ for every $i$. Moreover, if $t_1, \ldots, t_k$ are real numbers such that $0 \leq t_i \leq 1$ for every $i$ with $\sum_{i=1}^k t_i = 1$, then $f: \left( X, \sum_{i=1}^k t_i B_i \right) \to Y$ is a basic $\mathbb{R}$-slc-trivial fibration with $K_X + \sum_{i=1}^k t_i B_i \sim_{\mathbb{R}} f^* \left( \sum_{i=1}^k t_i D_i \right)$.
Proof. The proof of [6, Lem. 11.1] works with some suitable minor modifications. Therefore, we can take $B_i$, $D_i$, and $r_i$ for $1 \leq i \leq k$ satisfying (1)-(6). By (2), $B_i = B_i^{\leq 1}$ holds over the generic point of $Y$ for every $i$. By (2) again, $\text{rank} f_* \mathcal{O}_X([-1(B_i^{<1})]) = \text{rank} f_* \mathcal{O}_X([-1(\langle B_i^{<1}\rangle)]) = 1$. Hence, $f: (X,B) \to Y$ is a basic $\mathbb{Q}$-scl-trivial fibration with $K_X + B_i \sim_{\mathbb{Q}} f^* D_i$ for every $i$. We put $\tilde{B} = \sum_{i=1}^k t_i B_i$. Then $\tilde{B} = B^{\leq 1}$ holds over the generic point of $Y$ by (2). By (2) again, we see that $[-(\tilde{B}^{<1})] = [-\langle B^{<1}\rangle]$ holds. Therefore, $f: (X,\tilde{B}) \to Y$ is a basic $\mathbb{R}$-scl-trivial fibration.

We also need the following lemma.

Lemma 5.5. Let $f: (X,B) \to Y$ be a basic $\mathbb{R}$-scl-trivial fibration. Let $B$ denote the discriminant $\mathbb{R}$-b-divisor associated with $f: (X,B) \to Y$. Suppose that there are $\mathbb{Q}$-divisors $B_1,\ldots,B_k$ on $X$ and real numbers $r_1,\ldots,r_k$ such that $\sum_{i=1}^k r_i = 1$ and $\sum_{i=1}^k r_i B_i = B$. We put

\[
\mathcal{P} = \left\{ \sum_{i=1}^k t_i B_i \middle| 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^k t_i = 1 \right\}.
\]

Assume that $f: (X,\Delta) \to Y$ has the structure of a basic $\mathbb{R}$-scl-trivial fibration for every $\Delta \in \mathcal{P}$. For $\Delta \in \mathcal{P}$, $B^\Delta$ denotes the discriminant $\mathbb{R}$-b-divisor of the basic $\mathbb{R}$-scl-trivial fibration $f: (X,\Delta) \to Y$. Then, we can find $\Delta_1,\ldots,\Delta_l \in \mathcal{P}$ which are $\mathbb{Q}_{\geq 0}$-linear combinations of $B_1,\ldots,B_k$ and positive real numbers $s_1,\ldots,s_l$ such that:

- $\sum_{j=1}^l s_j = 1$ and $\sum_{j=1}^l s_j \Delta_j = B$, and
- $B_Y = \sum_{j=1}^l s_j B_Y^\Delta_j$.

Here, $B_Y$ (resp. $B_Y^{\Delta_j}$) is the trace of the discriminant $\mathbb{R}$-b-divisor $B$ (resp. $B^\Delta_j$) on $Y$.

Proof. Since $B$ is an $\mathbb{R}$-b-divisor, it is sufficient to prove the lemma for a resolution of $Y' \to Y$ and the induced basic scl-trivial fibrations $f': (X',B_X') \to Y$ and $f': (X',(B_1)_X') \to Y'$. Moreover, by Definition 3.5 and taking the normalization of $X$, we may assume that $X$ is a disjoint union of smooth varieties. Therefore, by replacing $X$, $Y$, $B$, and $B_i$, we may assume that $Y$ is smooth and there are simple normal crossing divisors $\Sigma_X$ on $X$ and $\Sigma_Y$ on $Y$ such that:

- $\text{Supp} B \subset \Sigma_X$ and $\text{Supp} B_i \subset \Sigma_X$ for every $i$,
- $\Sigma_X^v \subset f^{-1} \Sigma_Y \subset \Sigma_X$, where $\Sigma_X^v$ is the vertical part of $\Sigma_X$,
- $f$ is smooth over $Y \setminus \Sigma_Y$, and
- $\Sigma_X$ is relatively simple normal crossing over $Y \setminus \Sigma_Y$.

Then it is clear that $\text{Supp} B_Y \subset \Sigma_Y$ and $\text{Supp} B_Y^{\Delta_j} \subset \Sigma_Y$ for all $\Delta \in \mathcal{P}$. We consider a rational convex polytope

\[
\mathcal{C} = \left\{ v = (v_1,\ldots,v_k) \in [0,1]^k \middle| \sum_{j=1}^k v_j = 1 \right\} \subset [0,1]^k.
\]

Then we may identify $\mathcal{C}$ with $\mathcal{P}$ by putting $\Delta_v = \sum_i v_i B_i \in \mathcal{P}$ for $v = (v_1,\ldots,v_k) \in \mathcal{C}$. We define $v_0 \in \mathcal{C}$ to be the point such that $\Delta_{v_0} = B$.

Fix a prime divisor $Q$ on $Y$ which is a component of $\Sigma_Y$. We shrink $Y$ near the generic point of $Q$ so that all components of $f^* Q$ dominate $Q$. We can write $f^* Q = \sum_i m_i P_i$, where...
where $P_i$ are components of $\Sigma_X$ such that $f(P_i) = Q$, and $m_{P_i} = \text{coeff}_{P_i}(f^*Q)$. We fix a component $P_{(B,Q)}$ of $f^*Q$ such that

$$
\frac{1 - \text{coeff}_{P_{(B,Q)}}(B)}{m_{P_{(B,Q)}}} = \min_{P_i} \left\{ \frac{1 - \text{coeff}_{P_i}(B)}{m_{P_i}} \right\}.
$$

Note that $\frac{1 - \text{coeff}_{P_{(B,Q)}}(B)}{m_{P_{(B,Q)}}}$ is the log canonical threshold of $(X,B)$ with respect to $f^*Q$ over the generic point of $Q$ because $(X,B + \mu f^*Q)$ is sub log canonical over the generic point of $Q$ if and only if $\text{coeff}_{P_i}(B) + \mu m_{P_i} \leq 1$ for all $P_i$. For every component $P_i$ of $f^*Q$, we can define a function

$$
H^{(P_i)}(v) := \frac{1 - \text{coeff}_{P_{(B,Q)}}(\Delta v)}{m_{P_{(B,Q)}}} - \frac{1 - \text{coeff}_{P_i}(\Delta v)}{m_{P_i}}
$$

and the half-space

$$
H^{(P_i)}_{\leq 0} := \{ v \in \mathcal{C} \mid H^{(P_i)}(v) \leq 0 \}.
$$

It is easy to check that $H^{(P_i)}$ are rational affine functions and the half spaces $H^{(P_i)}_{\leq 0}$ contain $v_0$ since $v_0$ is the point such that $\Delta v_0 = B$. Therefore, the set

$$
\mathcal{C}_Q := \mathcal{C} \cap \left( \bigcap_{P_i} H^{(P_i)}_{\leq 0} \right)
$$

is a rational polytope in $\mathcal{C}$ containing $v_0$, where $P_i$ runs over components of $f^*Q$. We put

$$
t(\Delta v, Q) := 1 - \text{coeff}_{Q}(B^\Delta v) = \sup \{ \mu \in \mathbb{R} \mid (X, \Delta v + \mu f^*Q) \text{ is sub log canonical over the generic point of } Q \}.
$$

Then, by the definitions of $H^{(P_i)}_{\leq 0}$, every $v \in \mathcal{C}_Q$ satisfies

$$
t(\Delta v, Q) = \min_{P_i} \left\{ \frac{1 - \text{coeff}_{P_i}(\Delta v)}{m_{P_i}} \right\} = \frac{1 - \text{coeff}_{P_{(B,Q)}}(\Delta v)}{m_{P_{(B,Q)}}}.
$$

Here, to prove the first equality, we used the fact that $(X, \Delta v + \mu f^*Q)$ is sub log canonical over the generic point of $Q$ if and only if $\text{coeff}_{P_i}(\Delta v) + \mu m_{P_i} \leq 1$ for all $P_i$.

Finally, we define

$$
\mathcal{C} := \bigcap_Q \mathcal{C}_Q,
$$

where $Q$ runs over all irreducible components of $\Sigma_Y$. It is easy to see that $\mathcal{C}$ is a rational polytope in $\mathcal{C}$ and $\mathcal{C}$ contains $v_0$. Thus, we can find rational points $v_1, \ldots, v_l$ and positive real numbers $s_1, \ldots, s_l$ such that $\sum_{j=1}^l s_j = 1$ and $\sum_{j=1}^l s_j v_j = v_0$. We put $\Delta_j = \Delta v_j$ for each $1 \leq j \leq l$. Then $B = \Delta v_0 = \sum_{j=1}^l s_j \Delta j$. For every component $Q$ of $\Sigma_Y$, the equation
(1) implies that
\[ t(B, Q) = \frac{1 - \text{coeff}_{P(B, Q)}(B)}{m_{P(B, Q)}} \quad \text{(see (1))} \]
\[ = \frac{1 - \text{coeff}_{P(B, Q)}(\sum_{j=1}^{l} s_j \Delta_j)}{m_{P(B, Q)}} \quad \text{for } B = \sum_{j=1}^{l} s_j \Delta_j \]
\[ = \sum_{j=1}^{l} s_j \cdot \frac{1 - \text{coeff}_{P(B, Q)}(\Delta_j)}{m_{P(B, Q)}} \quad \text{(sum from 1 to } l) \]
\[ = \sum_{j=1}^{l} s_j \cdot t(\Delta_j, Q). \quad \text{(see (1)).} \]

Since \( t(\Delta_j, Q) = 1 - \text{coeff}_{Q}(B_\Delta^y) \) for every \( 1 \leq j \leq l \) and every irreducible component \( Q \) of \( \Sigma_Y \), we see that \( B_Y = \sum_{j=1}^{l} s_j B_\Delta^y \).

We are ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** Fix an arbitrary projective birational morphism \( \sigma: Y' \to Y \) from a normal quasi-projective variety \( Y' \), and let
\[ f': (X', B_{X'}) \mathrel{\xrightarrow{\sigma}} (X, B) \]
\[ f' \downarrow \quad f \downarrow \quad \sigma \downarrow \quad Y' \quad f \quad Y \]
be the induced basic \( \mathbb{R} \)-slc-trivial fibration (see Definition 3.4). It is sufficient to show that
\( \sigma^*(K_Y + B_Y) = K_{Y'} + B_{Y'} \) and \( M_Y \) is a potentially nef \( \mathbb{R} \)-divisor on \( Y \) with \( \sigma^*M_Y = M_{Y'} \).

We pick \( \mathbb{Q} \)-divisors \( B_1, \ldots, B_k \) on \( X \), \( \mathbb{Q} \)-divisors \( D_1, \ldots, D_k \) on \( Y \) and positive real numbers \( r_1, \ldots, r_k \) as in Lemma 5.4. Then, the following properties hold:
- \( \sum_{i=1}^{k} r_i = 1 \) with \( \sum_{i=1}^{k} r_i B_i = B \) and \( \sum_{i=1}^{k} r_i D_i = D \).
- \( \text{Supp} B = \text{Supp} B_i \) and \( \text{Supp} D = \text{Supp} D_i \) hold for every \( i \), and
- \( K_X + B_i \sim_{Q} f^*D_i \) holds for every \( i \).

We put \( D'_i = \sigma^*D_i \) and we define \( B'_i \) by \( K_{X'} + B'_i = \mu^*(K_X + B_i) \) for any \( 1 \leq i \leq k \). Then \( f': (X', B'_i) \to Y' \) are basic \( Q \)-slc-trivial fibrations with \( K_{X'} + B'_i \sim_{Q} f^*D'_i \). As in Lemma 5.5, we put

\[ \mathcal{P}' = \left\{ \sum_{i=1}^{k} t_i B'_i \bigg| 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^{k} t_i = 1 \right\}. \]

We may assume that \( f': (X', \Delta) \to Y' \) is a basic \( \mathbb{R} \)-slc-trivial fibration for every \( \Delta \in \mathcal{P}' \). We define \( \mathcal{P}'_Q \) as

\[ \mathcal{P}'_Q := \left\{ \sum_{i=1}^{k} t_i B'_i \bigg| t_i \in \mathbb{Q} \text{ and } 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^{k} t_i = 1 \right\}. \]

Note that \( B_{X'} \in \mathcal{P}' \).
Here, we used \( \mu_* B'_i = B_i \), we have \( \mu_* \Delta = \sum_{i=1}^k t_i B_i \) such that \( t_i \in \mathbb{Q} \). Therefore, the morphism \( f : (X, \mu_* \Delta) \to Y \) is a basic \( \mathbb{Q} \)-slc-trivial fibration such that \( K_X + \mu_* \Delta \sim_{\mathbb{Q}} f^*(\sum_{i=1}^k t_i D_i) \). Let \( B^\Delta \) and \( M^\Delta \) be the discriminant \( \mathbb{Q} \)-b-divisor and the moduli \( \mathbb{Q} \)-b-divisor of the basic \( \mathbb{Q} \)-slc-trivial fibration \( f : (X, \mu_* \Delta) \to Y \), respectively. Because we have \( \text{Supp} \left( \sum_{i=1}^k t_i D_i \right) \subset \text{Supp} B \) and \( \text{Supp} \mu_* \Delta \subset \text{Supp} B \), we may apply Theorem 4.1. Therefore, for every \( \Delta \in \mathcal{P}'_\mathbb{Q} \), it follows that \( \sigma^*(K_Y + B_Y^\Delta) = K_{Y'} + B_{Y'}^\Delta \) and \( M_Y^\Delta \) is a potentially nef \( \mathbb{Q} \)-divisor on \( Y \) with \( \sigma^* M_Y^\Delta = M_{Y'}^\Delta \). It also follows from the second paragraph that \( f' : (X', \Delta) \to Y' \) is the basic \( \mathbb{Q} \)-slc-trivial fibration induced from \( f : (X, \mu_* \Delta) \to Y \) such that \( K_{X'} + \Delta \sim_{\mathbb{Q}} f'^*(\sum_{i=1}^k t_i D_i) \). It is because \( K_{X'} + \Delta = \mu^*(K_X + \mu_* \Delta) \) by construction.

We apply Lemma 5.5 to \( f' : (X', B_{X'}) \to Y' \) and \( \mathcal{P}' \). Then, we can find \( \Delta_1, \ldots, \Delta_l \in \mathcal{P}'_\mathbb{Q} \) and positive real numbers \( s_1, \ldots, s_l \) such that:

\[
\begin{align*}
\sum_{j=1}^l s_j &= 1 \quad \text{and} \quad \sum_{j=1}^l s_j \Delta_j = B_{X'}, \\
B_{Y'} &= \sum_{j=1}^l s_j B_{Y'}^\Delta.
\end{align*}
\]

Since \( B \) and \( B_{Y'}^\Delta \) are \( \mathbb{R} \)-b-divisors, we have \( B_Y = \sum_{j=1}^l s_j B_{Y'}^\Delta \). Then

\[
\sigma^*(K_Y + B_Y) = \sigma^* \left( K_Y + \sum_{j=1}^l s_j B_{Y'}^\Delta \right) = \sum_{j=1}^l s_j \sigma^*(K_{Y'} + B_{Y'}^\Delta)
\]

\[
= \sum_{j=1}^l s_j (K_{Y'} + B_{Y'}^\Delta) = K_{Y'} + \sum_{j=1}^l s_j B_{Y'}^\Delta = K_{Y'} + B_{Y'}.
\]

Therefore, we have \( \sigma^*(K_Y + B_Y) = K_{Y'} + B_{Y'} \). Recalling that \( \sigma : Y' \to Y \) is an arbitrary projective birational morphism, we see that part (i) of Theorem 5.1 holds, that is,

\[ K + B = K_{Y'} + B_{Y'}. \]

As in the third paragraph, for each \( j \), we define \( D_{Y'}^\Delta \) to be the \( \mathbb{Q} \)-divisor on \( Y' \) associated with the basic \( \mathbb{Q} \)-slc-trivial fibration \( f' : (X', \Delta_j) \to Y' \). Note that \( K_{X'} + \Delta_j \sim_{\mathbb{Q}} f'^* D_{Y'}^\Delta \) for all \( j \). Since \( \sum_{j=1}^l s_j = 1 \) and \( \sum_{j=1}^l s_j \Delta_j = B_{X'} \), we have \( \sigma^* D = \sum_{j=1}^l s_j D_{Y'}^\Delta \). By the relation \( B_{Y'} = \sum_{j=1}^l s_j B_{Y'}^\Delta \) and the definition of the moduli \( \mathbb{R} \)-b-divisors (see Definition 3.5), we have

\[
M_{Y'} = \sum_{j=1}^l s_j M_{Y'}^\Delta \quad \text{and} \quad M_Y = \sum_{j=1}^l s_j M_Y^\Delta.
\]

Then \( M_Y \) is a potentially nef \( \mathbb{R} \)-divisor on \( Y \) and

\[
\sigma^* M_Y = \sigma^* \left( \sum_{j=1}^l s_j M_Y^\Delta \right) = \sum_{j=1}^l s_j M_{Y'}^\Delta = M_{Y'}.
\]

Here, we used \( \sigma^* M_Y^\Delta = M_{Y'}^\Delta \) for every \( j \), which follows from the third paragraph. We complete the proof.

The following result is essentially obtained in the proof of Theorem 5.1. We explicitly state it here for future use.
Theorem 5.6. Let \( f : (X, B) \to Y \) be a basic \( \mathbb{R} \)-slc-trivial fibration with \( K_X + B \sim_{\mathbb{Q}} f^*D \). Then there are \( \mathbb{Q} \)-divisors \( B_1, \ldots, B_l \) on \( X \), \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisors \( D_1, \ldots, D_l \) on \( Y \), and positive real numbers \( r_1, \ldots, r_l \) satisfying the following properties:

- \( \sum_{j=1}^l r_j = 1 \) with \( \sum_{j=1}^l r_j B_j = B \) and \( \sum_{j=1}^l r_j D_j = D \).
- \( \text{Supp } B = \text{Supp } B_j \), \( [B^>1] = [B_j^>1] \), and \( -(B^<1)] = -(B_j^<1] \) hold for every \( j \).
- if \( \text{coeff}_S(B) \in \mathbb{Q} \) for a prime divisor \( S \) on \( X \), then \( \text{coeff}_S(B) = \text{coeff}_S(B_j) \) holds for every \( j \).
- \( \text{Supp } D = \text{Supp } D_j \) holds for every \( j \).
- if \( \text{coeff}_T(D) \in \mathbb{Q} \) for a prime divisor \( T \) on \( Y \), then \( \text{coeff}_T(D) = \text{coeff}_T(D_j) \) holds for every \( j \).
- \( K_X + B_j \sim_{\mathbb{Q}} f^*D_j \) holds for every \( j \).
- \( B_j = \sum_{i=1}^l r_j B_j^i \) as \( \mathbb{Q} \)-divisors, where \( B \) (resp. \( B_j \)) is the discriminant \( \mathbb{R} \)-b-divisor (resp. the discriminant \( \mathbb{Q} \)-b-divisor) of \( f : (X, B) \to Y \) (resp. \( f : (X, B_j) \to Y \)), and
- \( M = \sum_{i=1}^l r_j B_j^i \) as \( \mathbb{Q} \)-divisors, where \( M \) (resp. \( M_j \)) is the moduli \( \mathbb{R} \)-b-divisor (the moduli \( \mathbb{Q} \)-b-divisor) associated with \( f : (X, B) \to Y \) (resp. \( f : (X, B_j) \to Y \)).

Sketch of Proof. It can be proved by Theorem 5.1 and Lemmas 5.4 and 5.5. We only outline the proof.

We note that the properties of Theorem 5.6 except the last two properties correspond to
parts (1)–(6) of Lemma 5.4, respectively. By Lemma 5.4, we can find \( \mathbb{Q} \)-divisors \( \tilde{B}_1, \ldots, \tilde{B}_k \) on \( X \), \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisors \( \tilde{D}_1, \ldots, \tilde{D}_k \) on \( Y \), and positive real numbers \( \tilde{r}_1, \ldots, \tilde{r}_k \) satisfying parts (1)–(6) of Lemma 5.4. Then \( \tilde{B}_i, \tilde{D}_i, \) and \( \tilde{r}_i \) satisfy all the properties of Theorem 5.6 except the last two properties. More specifically, \( \tilde{B}_i, \tilde{D}_i, \) and \( \tilde{r}_i \) satisfy:

- \( \sum_{i=1}^k \tilde{r}_i = 1 \) with \( \sum_{i=1}^k \tilde{r}_i \tilde{B}_i = B \) and \( \sum_{i=1}^k \tilde{r}_i \tilde{D}_i = D \) (see part (1) of Lemma 5.4),
- \( \text{Supp } B = \text{Supp } \tilde{B}_i \) and \( \text{Supp } D = \text{Supp } \tilde{D}_i \) hold for every \( i \), and
- \( K_X + \tilde{B}_i \sim_{\mathbb{Q}} f^* \tilde{D}_i \) holds for every \( i \) (see part (6) of Lemma 5.4),

and parts (2)–(5) in Lemma 5.4. We take a smooth higher model \( \sigma : Y' \to Y \) so that the induced basic \( \mathbb{R} \)-slc-trivial fibration \( f' : (X', B') \to Y' \) satisfies the property that there exists a simple normal crossing divisor \( \Sigma' \) on \( Y' \) such that \( \text{Supp } \sigma^*D \subset \Sigma' \) and that every stratum of \( (X', \text{Supp } B') \) is smooth over \( Y' \setminus \Sigma' \). The morphism \( X' \to X \) is denoted by \( \mu \). For each \( 1 \leq i \leq k \), let \( B_i' \) be a \( \mathbb{Q} \)-divisor on \( X' \) defined by \( K_{X'} + B_i' = \mu^*(K_X + \tilde{B}_i) \). Note that \( K_{X'} + B_i' \sim_{\mathbb{Q}} f'^* \sigma^* \tilde{D}_i \). We may assume that \( \text{Supp } \sigma^* \tilde{D}_i \subset \Sigma' \) and that every stratum of \( (X', \text{Supp } B_i') \) is smooth over \( Y' \setminus \Sigma' \) for every \( i \) by taking \( \sigma : Y' \to Y \) suitably. We define

\[
P = \left\{ \sum_{i=1}^k t_i B_i' \mid 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^k t_i = 1 \right\}.
\]

By Lemma 5.5, we can find \( B_1', \ldots, B_l' \in P \), which are \( \mathbb{Q} \geq 0 \)-linear combinations of \( \tilde{B}_1', \ldots, \tilde{B}_l' \) and positive real numbers \( r_1, \ldots, r_l \) such that:

- \( \sum_{j=1}^l r_j = 1 \) and \( \sum_{j=1}^l r_j B_j' = B' \), and
- \( B_{Y'} = \sum_{j=1}^l r_j B_{Y'j} \).

Here, \( B_j \) is the discriminant \( \mathbb{Q} \)-b-divisor associated with \( f' : (X', B_j') \to Y' \). By Theorem 5.1, we have \( K + B = K_{Y'} + B_{Y'} \) and \( K + B_j = K_{Y'} + B_{Y'} \). We put \( B_j = \mu_* B_j' \) for each \( 1 \leq j \leq l \). Then we can find \( \mathbb{Q} \)-divisors \( D_1, \ldots, D_l \) on \( Y \) such that \( K_X + B_j \sim_{\mathbb{Q}} f^* D_j \) and
\[
\sum_{j=1}^{l} r_j D_j = D. \] By construction, we can easily see that \(B_1, \ldots, B_l, D_1, \ldots, D_l\), and \(r_1, \ldots, r_l\) constructed above satisfy the desired properties.

\section{Proof of Theorem 1.2}

In this section, we prove Theorem 1.2, which is the main result of this paper. Then we treat Theorem 1.1 and Corollary 1.4. We note that we freely use the framework of quasi-log schemes in the proof of Theorem 1.2. For the details of quasi-log schemes, see [5, Chap. 6]. Let us start with the proof of Theorem 1.2.

\textit{Proof of Theorem 1.2.} From Step 1 to Step 3, we define a natural quasi-log scheme structure on \(Z\). This part is essentially contained in [5, Chap. 6] and [7].

\textbf{Step 1.} In this step, we give a natural quasi-log scheme structure on \(W' := W \cup \text{Nlc}(X, \Delta)\). This step is essentially the adjunction for quasi-log schemes (see [5, Th. 6.3.5(i)]).

We put \(W' := W \cup \text{Nlc}(X, \Delta)\) as above. We sketch how to define a natural quasi-log scheme structure on \(W'\). Let \(f : Y \to X\) be a projective birational morphism from a smooth quasi-projective variety \(Y\) such that \(K_Y + \Delta_Y = f^*(K_X + \Delta)\) and that \(\text{Supp}\Delta_Y\) is a simple normal crossing divisor on \(Y\). By taking some more blowups, we may assume that the union of all log canonical centers of \((Y, \Delta_Y)\) mapped to \(W'\) by \(f\), which is denoted by \(V'\), is a union of some irreducible components of \(\Delta_Y^{-1}\). As usual, we put \(A = [-\Delta_Y^{-1}]\) and \(N = [\Delta_Y^{>1}]\) and consider the following short exact sequence:

\[
0 \to \mathcal{O}_Y(A - N - V') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{V'}(A - N) \to 0.
\]

By taking \(R^1f_*\), we obtain

\[
0 \to f_*\mathcal{O}_Y(A - N - V') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{V'}(A - N) \to \cdots.
\]

The connecting homomorphism \(\delta\) is zero since no associated prime of \(R^1f_*\mathcal{O}_Y(A - N - V')\) is contained in \(W' = f(V')\) (see [4, Th. 6.3 (i)] and [5, Th. 5.6.2(i)]). Hence, we have

\[
0 \to f_*\mathcal{O}_Y(A - N - V') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{V'}(A - N) \to 0.
\]

Note that \(J_{\text{Nlc}}(X, \Delta) = f_*\mathcal{O}_Y(A - N)\) by definition. We put \(\mathcal{I}_{W'} = f_*\mathcal{O}_Y(A - N - V')\) and \(\mathcal{I}_{W', \infty} = f_*\mathcal{O}_{V'}(A - N)\). We define \(\Delta_{V'}\) by \((K_Y + \Delta_Y)|_{V'} = K_{V'} + \Delta_{V'}\). Then

\[
(W', (K_X + \Delta)|_{W'}, f : (V', \Delta_{V'}) \to W')
\]

is a quasi-log scheme. By construction, \(\text{Nlc}(W', (K_X + \Delta)|_{W'}) = \text{Nlc}(X, \Delta)\) holds. By construction again, a subset \(C \subset X\) is a qlc stratum of \([W', (K_X + \Delta)|_{W'}]\) if and only if \(C\) is a log canonical center of \((X, \Delta)\) included in \(W\). We note that the above construction is independent of the choice of \(f : Y \to X\) by [5, Prop. 6.3.1].

\textbf{Step 2.} In this step, we give a natural quasi-log scheme structure on \([W, (K_X + \Delta)]_W\]. This step is essentially [7, Lem. 4.19].

In Step 1, we may further assume that the union of all strata of \((V', \Delta_{V'})\) mapped to \(W \cap \text{Nlc}(X, \Delta)\) is also a union of some irreducible components of \(V'\). Let \(\tilde{V}\) be the union of the irreducible components of \(V'\) mapped to \(W\) by \(f\). We put \(\Delta_{\tilde{V}}\) by \((K_Y + \Delta_Y)|_{\tilde{V}} =
Then, by the proof of [7, Lems. 4.18 and 4.19],

\[ (W, (K_X + \Delta) |_W, f : (\hat{V}, \Delta_{\hat{V}}) \to W) \]

is a quasi-log scheme. By [7, Lem. 4.19], we obtain that \( \mathcal{I}_{W_{\to \infty}} = \mathcal{I}_{W_{\to \infty}'} \) holds and that a subset \( C \subset X \) is a qlc stratum of \( [W, (K_X + \Delta) |_W] \) if and only if \( C' \) is a qlc stratum of \( [W, (K_X + \Delta) |_W] \). Hence, \( W \cap \text{Nlc}(X, \Delta) = W_{\to \infty} \) and

\[ W \cap \left( \text{Nlc}(X, \Delta) \cup \bigcup_{W \subset W'} W' \right) = \text{Nqklt}(W, (K_X + \Delta) |_W) \]

hold set-theoretically, where \( W' \) runs over log canonical centers of \( (X, \Delta) \) which do not contain \( W \).

**Step 3.** In this step, we give a natural quasi-log scheme structure on \( Z \). This step is nothing but [7, Th. 1.9].

In Step 2, we may further assume that the union of all strata of \( (\hat{V}, \Delta_{\hat{V}}) \) mapped to \( \text{Nqklt}(W, (K_X + \Delta) |_W) \) is a union of some irreducible components of \( \hat{V} \). Let \( V \) be the union of the irreducible components of \( \hat{V} \) which are dominant onto \( W \). Then, by the proof of [7, Th. 1.9], \( f : V \to W \) factors through \( Z \) and

\[ (Z, \nu^*(K_X + \Delta), f : (V, \Delta_V) \to Z) \]

becomes a quasi-log scheme, where \( \Delta_V \) is defined by \( (K_Y + \Delta_Y) |_V = K_Y + \Delta_V \). By construction, we have \( \nu_* \mathcal{I}_{\text{Nqklt}(Z, \nu^*(K_X + \Delta))} = \mathcal{I}_{\text{Nqklt}(W, (K_X + \Delta) |_W)} \). Hence,

\[ \text{Nqklt}(Z, \nu^*(K_X + \Delta)) = \nu^{-1} \text{Nqklt}(W, (K_X + \Delta) |_W) \]

holds.

**Step 4.** Then \( f : (V, \Delta_V) \to Z \) is a basic \( \mathbb{R} \)-slc-trivial fibration. Hence, we can apply Corollary 5.2 and Remark 5.3 to \( f : (V, \Delta_V) \to Z \). We note that \( f : (V, \Delta_V) \to Z \) is a basic \( \mathbb{Q} \)-slc-trivial fibration when \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. In that case, Theorem 3.6 with Theorem 3.7 is sufficient.

**Step 5.** By [7, Th. 7.1] and Steps 1–3 in its proof, we can construct a projective birational morphism \( p : Z' \to Z \) from a smooth quasi-projective variety \( Z' \) satisfying (i)–(v). We note that we can directly apply Step 3 in the proof of [7, Th. 7.1] to basic \( \mathbb{R} \)-slc-trivial fibrations by Corollary 5.2. We also note that \( B \) is a well-defined \( \mathbb{R} \)-b-divisor on \( Z \) and is independent of \( f : Y \to X \) (see [11, Lem. 5.1] and [10, Th. 1.2]).

**Step 6.** (see [11, Th. 5.4]). In this final step, we prove (iv). This step is essentially [11, Th. 5.4]. We explain it here for the reader’s convenience.

Without loss of generality, we may assume that \( X \) is affine by taking a finite affine open cover of \( X \). Let \( g_{dlt} : X_{dlt} \to X \) be a good dlt blowup of \( (X, \Delta) \) such that \( K_{X_{dlt}} + \Delta_{X_{dlt}} = g_{dlt}^*(K_X + \Delta) \) (see [11, Lem. 3.5]). We may assume that there is an irreducible component \( S \) of \( \Delta_{X_{dlt}}^{\geq 1} \) with \( g_{dlt}(S) = W \). We put

\[ D = \Delta_{X_{dlt}}^{\geq 1} - \text{Supp} \Delta_{X_{dlt}}^{\geq 1} = \Delta_{X_{dlt}}^{\geq 1} - \text{Supp} \Delta_{X_{dlt}}^{\geq 1} \]

Then \( -D \) is semi-ample over \( X \) and \( \text{Supp} D = \text{Nlc}(X_{dlt}, \Delta_{X_{dlt}}) \) holds set-theoretically (see [11, Lem. 3.5]). By taking the contraction morphism \( \varphi : X_{dlt} \to X_{lc} \) associated with \( -D \)
over $X$, we get a log canonical modification $g_{lc}: X_{lc} \to X$ with $K_{X_{lc}} + \Delta_{X_{lc}} = g_{lc}^{-1}(K_X + \Delta)$ (see [11, Th. 1.3]).

$$
\begin{array}{c}
X_{dlt} \\
\downarrow \varphi \\
X_{lc} \\
\downarrow g_{dlt} \\
X \\
\downarrow g_{lc}
\end{array}
$$

We put $D' = \varphi_* D$. Then $-D'$ is ample over $X$, and

$$g_{lc}^{-1} \text{Nlc}(X, \Delta) = \text{Nlc}(X_{lc}, \Delta_{X_{lc}}) = \text{Supp } D'
$$

holds set-theoretically. We note that

$$\text{Nlc}(X_{dlt}, \Delta_{X_{dlt}}) = \varphi^{-1} \text{Nlc}(X_{lc}, \Delta_{X_{lc}}) = g_{dlt}^{-1} \text{Nlc}(X, \Delta)
$$

holds set-theoretically. Let $\tilde{W}$ be the strict transform of $W$ on $X_{lc}$. Let $\tilde{\nu}: \tilde{Z} \to \tilde{W}$ be the normalization. Then we can easily see that

$$\text{Supp } B^1 Z' = \tilde{\nu}^* D' = \tilde{\nu}^{-1} \left( \text{Nlc}(X_{lc}, \Delta_{X_{lc}}) \cap \tilde{W} \right) = (g_{lc} \circ \tilde{\nu})^{-1} (\text{Nlc}(X, \Delta) \cap W)
$$

holds set-theoretically. We note that $B^1 = 0$ over $X \setminus \text{Nlc}(X, \Delta)$ by construction. Hence, we obtain $\nu \circ p(B^1 Z') = W \cap \text{Nlc}(X, \Delta)$ set-theoretically.

We finish the proof of Theorem 1.2.

Finally, we prove Theorem 1.1 and Corollary 1.4.

**Proof of Theorem 1.1.** Here, we use the same notation as in Theorem 1.2. We put $B_Z = B_Z$ and $M_Z = M_Z$ in Theorem 1.2. We note that $M_{Z'}$ is a finite $\mathbb{R}_{>0}$-linear combination of potentially nef Cartier divisors on $Z'$ with $p_* M_{Z'} = M_Z$. Hence, the desired statement follows from Theorem 1.2. \[ \square \]

**Proof of Corollary 1.4.** By the definition of $B$ in Theorem 1.2 (see the proof of Theorem 1.2 and Definition 1.3), we can easily check that $B_Z$ is nothing but Shokurov’s different (see [4, §14]) and $\nu^*(K_X + \Delta) = K_Z + B_Z$ holds, where $\nu: Z \to W$ is the normalization of $W$. In particular, we have $M_Z = 0$. By part (A) in Theorem 1.1, we obtain that $(X, \Delta)$ is log canonical in a neighborhood of $W$ if and only if $(Z, B_Z)$ is log canonical in the usual sense. It recovers Kawakita’s inversion of adjunction (see [15, Th.]). By part (B), we see that $(Z, B_Z)$ is Kawamata log terminal if and only if $(X, \Delta)$ is log canonical in a neighborhood of $W$ and $W$ is a minimal log canonical center of $(X, \Delta)$ (see [4, Th. 9.1] and [5, Th. 6.3.11]). Note that $(X, \Delta)$ is purely log terminal in a neighborhood of $W$ if and only if $(X, \Delta)$ is log canonical in a neighborhood of $W$ and $W$ is a minimal log canonical center of $(X, \Delta)$. \[ \square \]

We close this section with the following remark, which summarizes the construction of the $\mathbb{R}$-b-divisors $B$ and $M$ on $Z$.

**Remark 6.1.** Let $X$ be a normal variety, and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $W$ be a log canonical center of $(X, \Delta)$, and let $\nu: Z \to W$ be the normalization of $W$.

We take a log resolution $f: Y \to X$ of $(X, \Delta)$ which is a sufficiently high birational model. We define $\Delta_Y$ by $K_Y + \Delta_Y = f^*(K_X + \Delta)$, and let $V$ be the union of the irreducible components of $\Delta_Y^{-1}$ which map onto $W$. Let $\Delta_V$ be an $\mathbb{R}$-divisor on $V$ defined by $K_V + \Delta_V =$
(K_Y + \Delta_Y)|_V$, then we get the morphism $f: (V, \Delta_V) \to Z$, which has the structure of a basic $\mathbb{R}$-scl-trivial fibration. Then $\mathcal{B}$ and $\mathcal{M}$ are defined to be the discriminant $\mathbb{R}$-b-divisor and the moduli $\mathbb{R}$-b-divisor as in Definition 3.5. By construction, we can easily check that the construction in the proof of Theorem 1.2 and the one in Definition 1.3 define the same $\mathbb{R}$-b-divisor $\mathcal{B}$ on $Z$ (see [11, Lem. 5.1] and [10, Th. 1.2]). Precisely speaking, when $\dim W \leq \dim X - 2$, we consider the $\mathbb{R}$-line bundle $\mathcal{L}$ on $X$ associated with $K_X + \Delta$. We fix an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D$ on $Z$ whose associated $\mathbb{R}$-line bundle is the pullback of $\mathcal{L}$. Then we put $M = D - K - B$, where $D$ is the $\mathbb{R}$-Cartier closure of $D$ and $K$ is the canonical $b$-divisor of $Z$.

§7. Adjoining for codimension 2 log canonical centers

In this final section, we first discuss basic scl-trivial fibrations under some extra assumption and then prove adjoining for codimension 2 log canonical centers.

**Theorem 7.1.** Let $f: (X, B) \to Y$ be a basic $\mathbb{R}$-scl-trivial fibration. Assume that there exists a stratum $S$ of $(X, B)$ such that the induced morphism $S \to Y$ is generically finite and surjective. Then there exists a proper birational morphism $p: Y' \to Y$ from a smooth quasi-projective variety $Y'$ such that $M = \overline{M_{Y'}}$ with $\mathcal{M}_{Y'} \sim_{\mathbb{R}} 0$. In particular, $M$ is $b$-semiample.

**Proof.** By Theorem 5.6, we may assume that $f: (X, B) \to Y$ is a basic $\mathbb{Q}$-scl-trivial fibration. Let $\nu: X' \to X$ be the normalization. We define a $\mathbb{Q}$-divisor $B'$ on $X'$ by $K_{X'} + B' = \nu^*(K_X + B)$. Note that after the reduction we may find a log canonical center $S$ of $(X', B')$ such that the induced morphism $S \to Y$ is generically finite and surjective. By [6, Lem. 4.12], we may further assume that $Y$ is a complete variety. By replacing $Y$ with a smooth higher birational model and $f: (X, B) \to Y$ with the induced basic $\mathbb{Q}$-scl-trivial fibration, we may assume that $Y$ is a smooth projective variety, $M = \overline{M_Y}$, and $\mathcal{M}_Y$ is nef. The induced morphism $S \to Y$ is denoted by $f_S$. We define a $\mathbb{Q}$-divisor $B_S$ on $S$ by $K_S + B_S = (K_{X'} + B')|_S$.

From now on, we show that $-\mathcal{M}_Y$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor. We consider the divisor $\nu^* f_\ast M_Y \sim_{\mathbb{Q}} K_{X'} + B' - \nu^* f_\ast (K_Y + B_Y)$. By restricting it to $S$, we get the relation $f_S^\ast M_Y \sim_{\mathbb{Q}} K_S + B_S - f_S^\ast (K_Y + B_Y)$. Let $g: S \to T$ be the Stein factorization of $f_S$. The finite morphism $T \to Y$ is denoted by $f_T$. We put $B_T = g_\ast B_S$. Then the relation $K_S + B_S = g^\ast (K_T + B_T)$ holds because $K_S + B_S$ is $\mathbb{Q}$-linearly trivial over $Y$. We also have the relation

$$f_T^\ast M_Y \sim_{\mathbb{Q}} K_T + B_T - f_T^\ast (K_Y + B_Y).$$

To show that $-\mathcal{M}_Y$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor, it is sufficient to prove that $-(K_T + B_T - f_T^\ast (K_Y + B_Y))$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.

By the definition of the discriminant $\mathbb{Q}$-b-divisor (see Definition 3.5), for every prime divisor $P$ on $Y$, we have $\text{coeff}_P (B_Y) = 1 - b_P$, where $b_P$ is the log canonical threshold of $(X'', B'')$ with respect to $\nu' f' P$ over the generic point of $P$. Since $f_T$ is finite, we may write $f_T^\ast P = \sum Q_i m_i Q_i$, where $Q_i$ runs over prime divisors on $T$ with $f_T(Q_i) = P$ and $m_i$ is the multiplicity of $Q_i$ with respect to $f_T$. By the ramification formula, over a neighborhood of
the generic point of \( P \), we may write 
\[
f_T^*(K_Y + B_Y) = f_T^*(K_Y + (1 - b_P)P) = K_T - \sum_{Q_i}(m_i - 1)Q_i + (1 - b_P)\sum m_i Q_i = K_T + \sum_{Q_i}(1 - m_i b_P)Q_i.
\]
We define \( E := \sum_{Q_i}(\text{coeff}_{Q_i}(B_T) - (1 - m_i b_P))Q_i \). Then, over a neighborhood of the generic point of \( P \), we have 
\[
f_T^*M_Y \sim Q K_T + B_T - f_T^*(K_Y + B_Y) = \sum_{Q_i}(\text{coeff}_{Q_i}(B_T) - (1 - m_i b_P))Q_i = E.
\]
On the other hand, by the definition of \( b_P \) (see Definition 3.5) and the fact that \( S \) is a log canonical center of \( (X^\nu, B^\nu) \), the pair \((S, B_S + b_P f_S^*P)\) is sub canonical over the generic point of \( P \). Since \( g : S \to T \) is birational and \( K_S + B_S = g^*(K_T + B_T) \), the pair \((T, B_T + b_P f_T^*P)\) is sub log canonical over the generic point of \( P \). This shows \( \text{coeff}_{Q_i}(B_T) + m_i b_P \leq 1 \) for all \( Q_i \) such that \( f_T(Q_i) = P \). Thus, \(-E\) is effective. Hence, \(-M_Y \sim Q\)-linearly equivalent to an effective \( Q\)-divisor.

Finally, since \( M_Y \) is nef, we see that \( M_Y \sim Q \). \( \square \)

We prove the b-semi-ampleness of \( M \) for basic slc-trivial fibrations of relative dimension one under some extra assumption.

**Theorem 7.2.** Let \( f : (X,B) \to Y \) be a basic \( \mathbb{R}\)-slc-trivial fibration with \( \dim X = \dim Y + 1 \) such that the horizontal part \( B^h \) of \( B \) is effective. Then the moduli \( \mathbb{R}\)-b-divisor \( M \) is b-semi-ample.

**Proof.** By Theorem 5.6, we may assume that \( f : (X,B) \to Y \) is a basic \( Q\)-slc-trivial fibration. By [6, Lem. 4.12], we may further assume that \( Y \) is a complete variety. When \( X \) is reducible, by the definition of basic slc-trivial fibrations (see Definition 3.2), there is a stratum \( S \) of \( X \) such that the morphism \( S \to Y \) is generically finite and surjective since \( \dim X = \dim Y + 1 \). Thus, we can apply Theorem 7.1. By Theorem 7.1, the moduli \( Q\)-b-divisor \( M \) is b-semi-ample when \( X \) is reducible. So we may assume that \( X \) is irreducible. Let \( F \) be a general fiber of \( f \). Then \( B|_F \geq 0 \) by the assumption \( B^h \geq 0 \). If \((F, B|_F)\) is not Kawamata log terminal, then there is a log canonical center \( S' \) of \((X,B)\), that is, \( S' \) is a stratum of \((X,B)\), such that the morphism \( S' \to Y \) is generically finite and surjective. As in the reducible case, by applying Theorem 7.1, we see that the moduli \( Q\)-b-divisor \( M \) is b-semi-ample. If \((F, B|_F)\) is Kawamata log terminal, then the morphism \( f : (X,B) \to Y \) satisfies [17, Assump. 7.11]. Therefore, by [17, Th. 8.1], the moduli \( Q\)-b-divisor \( M \) is b-semi-ample. In this way, in any case, the moduli \( Q\)-b-divisor \( M \) is b-semi-ample. \( \square \)

By combining Theorem 7.2 with the proof of Theorem 1.2, we obtain the following result, which generalizes Kawamata’s theorem (see [16, Th. 1]).

**Corollary 7.3 (Adjunction and inversion of adjunction in codimension 2).** Under the same notation as in Theorem 1.2, we further assume that \( \dim W = \dim X - 2 \). Then \( M \) is b-semi-ample. Equivalently, \( M_{Z'} \) is semi-ample. In particular, there exists an effective \( \mathbb{R}\)-divisor \( \Delta_Z \) on \( Z \) such that:
\begin{itemize}
  \item $\nu^*(K_X + \Delta) \sim_{\mathbb{R}} K_Z + \Delta_Z$, \\
  \item $(Z, \Delta_Z)$ is log canonical if and only if $(X, \Delta)$ is log canonical near $W$, and \\
  \item $(Z, \Delta_Z)$ is Kawamata log terminal if and only if $(X, \Delta)$ is log canonical near $W$ and $W$ is a minimal log canonical center of $(X, \Delta)$.
\end{itemize}

When $K_X + \Delta$ is $\mathbb{Q}$-Cartier, we further make $\Delta_Z$ an effective $\mathbb{Q}$-divisor on $Z$ such that $\nu^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Z + \Delta_Z$ in the above statement.

\textbf{Proof.} We use the same notation as in Theorem 1.2. Note that $W$ is a codimension 2 log canonical center of $(X, \Delta)$ by assumption. Let $f: Y \to X$ be a projective birational morphism from a smooth quasi-projective variety $Y$ such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$ and that $\text{Supp}\Delta_Y$ is a simple normal crossing divisor on $Y$. Without loss of generality, we may assume that $f^{-1}(W)$ is a simple normal crossing divisor on $Y$ such that $f^{-1}(W) = \sum_i E_i$ is the irreducible decomposition. We put

$$E = \sum_{a(E_i, X, \Delta) = -1} E_i.$$ 

We define $\Delta_E$ by $K_E + \Delta_E = (K_Y + \Delta_Y)|_E$. In this situation, we can check that $\Delta_E$ is effective over the generic point of $W$. Indeed, if $X$ is a surface, then we can check this fact by using the minimal resolution. In the general case, by shrinking $X$ and cutting $X$ by general hyperplanes, we can reduce the problem to the case where $X$ is a surface.

Let $Z$ be the normalization of $W$. By the same arguments as in Steps 1–3 in the proof of Theorem 1.2, we can construct a basic $\mathbb{R}$-slc-trivial fibration $f: (V, \Delta_V) \to Z$. Then $\text{dim} V = \text{dim} Z + 1$ because $\text{dim} V = \text{dim} X - 1$ and $W$ is a codimension 2 log canonical center of $(X, \Delta)$. Furthermore, by the discussion in the first paragraph, we see that the horizontal part $\Delta_V^h$ of $\Delta_V$ with respect to $f: V \to Z$ is effective. By the same arguments as in Steps 4–6 in the proof of Theorem 1.2, we get a projective birational morphism $p: Z' \to Z$ from a smooth quasi-projective variety $Z'$ satisfying parts (i)–(v) of Theorem 1.2. Moreover, by Theorem 7.2, $\mathbf{M}$ is b-semi-ample, that is, $\mathbf{M}_{Z'}$ is semi-ample.

Let $N \sim_{\mathbb{R}} \mathbf{M}_{Z'}$ be a general effective $\mathbb{R}$-divisor such that $N$ and $\mathbf{B}_{Z'}$ have no common components, $\text{Supp}(N + \mathbf{B}_{Z'})$ is a simple normal crossing divisor on $Z'$, and all the coefficients of $N$ are less than 1. We put $\Delta_Z = p_* N + \mathbf{B}_{Z}$. Then, it is easy to see that $\Delta_Z$ satisfies the desired three conditions of Corollary 7.3. By the above construction, we can make $\Delta_Z$ an effective $\mathbb{Q}$-divisor such that $K_Z + \Delta_Z \sim_{\mathbb{Q}} \nu^*(K_X + \Delta)$ when $K_X + \Delta$ is $\mathbb{Q}$-Cartier. So we are done.

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