Semilinear wave equations of derivative type with spatial weights in one space dimension

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Abstract

This paper is devoted to the initial value problems for semilinear wave equations of derivative type with spatial weights in one space dimension. The lifespan estimates of classical solutions are quite different from those for nonlinearity of unknown function itself as the global-in-time existence can be established by spatial decay.

1 Introduction

In this paper, we consider the initial value problems;

\[
\begin{cases}
  u_{tt} - u_{xx} = \frac{|u_t|^p}{(1 + x^2)^{(1+a)/2}} & \text{in } \mathbb{R} \times (0, \infty), \\
  u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R},
\end{cases}
\]

(1.1)

where \( p > 1, a \in \mathbb{R}, f \) and \( g \) are given smooth functions of compact support and a parameter \( \varepsilon > 0 \) is “small enough”. We are interested in the estimate

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of the lifespan $T(\varepsilon)$, the maximal existence time, of classical solutions of (1.1). Our result is the following:

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(p-1)/(1-a)} & \text{for } a < 0, \\ \exp(C\varepsilon^{-(p-1)}) & \text{for } a = 0, \\ \infty & \text{for } a > 0. \end{cases}$$

(1.2)

Here we denote the fact that there are positive constants, $C_1$ and $C_2$, independent of $\varepsilon$ satisfying $A(\varepsilon, C_1) \leq T(\varepsilon) \leq A(\varepsilon, C_2)$ by $T(\varepsilon) \sim A(\varepsilon, C)$. We note that (1.2) is established for classical solutions when $p \geq 2$, while we have to consider $C^1$ solutions of associated integral equations to (1.1) in case of $1 < p < 2$. When $a = -1$, the upper bounds in (1.2) are already obtained by Zhou [11], while the lower bounds are verified only for integer $p$ by general theory which is studied by Li, Yu and Zhou [6, 7]. We see that (1.2) is similar to the one for the time-weighted nonlinear terms of unknown function itself by Kato, Takamura and Wakasa [3] in sense that there is a possibility to obtain the global-in-time existence in spite of one dimension. For such an equation, the lifespan estimates are classified into two cases according to the value of the total integral of the initial speed. But (1.2) has no classification whatever it is. This is due to the fact that Huygens’ principle is always available for the time derivative of the solution of the free wave equation.

In fact, let us compare (1.1) with

$$\begin{cases} u_{tt} - u_{xx} = \frac{|u|^p}{(1 + x^2)^{(1+a)/2}} & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R}. \end{cases}$$

(1.3)

In our previous work [4], it is established that

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(p-1)/(1-a)} & \text{for } a < 0, \\ \phi^{-1}(C\varepsilon^{-(p-1)}) & \text{for } a = 0, \\ C\varepsilon^{-(p-1)} & \text{for } a > 0 \end{cases} \text{ if } \int_{\mathbb{R}} g(x)dx \neq 0,$$

(1.4)

where $\phi^{-1}$ is an inverse function of $\phi$ defined by

$$\phi(s) := s \log(2 + s),$$

and

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-p(p-1)/(1-pa)} & \text{for } a < 0, \\ \psi_p^{-1}(C\varepsilon^{-p(p-1)}) & \text{for } a = 0, \\ C\varepsilon^{-p(p-1)} & \text{for } a > 0 \end{cases} \text{ if } \int_{\mathbb{R}} g(x)dx = 0,$$

(1.5)
where $\psi_p^{-1}$ is an inverse function of $\psi_p$ defined by

$$
\psi_p(s) := s \log^p(2 + s).
$$

We remark that the quantities in all the cases of (1.4) are smaller than those of (1.5). This work in [4] is an extension of the case of $a = -1$ by Zhou [10], and inspired by non-compactly supported case by Suzuki [8], Kubo, Osaka and Yazici [5] and Wakasa [9]. See Introduction in [4] for details.

Finally, we note that our result cannot be valid with our method in this paper if $|u_t|^p$ in (1.1) is replaced with $|u_x|^p$ even in the non-weighted case, $a = -1$, because the blow-up part requires a positiveness of the nonlinear terms in the level of the $x$-derivative of the unknown function. But one can find immediately that it is impossible by the expression of $u_x$ in (2.9). On contrast, it is possible to obtain the same lifespan estimate from below, the existence part, along with our method.

This work was almost completed when the first and second authors were in the master course of Mathematical Institute, Tohoku University and the third author had the second affiliation with Research Alliance Center of Mathematical Sciences, Tohoku University. This paper is organized as follows. In the next section, (1.2) is divided into two theorems, and the preliminaries are introduced. Section 3 is devoted to the proof of the existence part of (1.2). The main strategy is the iteration method in the weighted $L^\infty$ space which is originally introduced by John [1]. In Section 4, we prove a priori estimate. Finally, we prove the blow-up part of (1.2) employing the method by Zhou [11] in Section 5.

# 2 Preliminaries and main results

Throughout of this paper, we assume that the initial data $(f, g) \in C_0^2(\mathbb{R}) \times C_0^1(\mathbb{R})$ satisfies

$$
\text{supp } f, \text{ supp } g \subset \{ x \in \mathbb{R} : |x| \leq R \}, \quad R \geq 1. \quad (2.1)
$$

Let $u$ be a classical solution of (1.1) in the time interval $[0, T]$. Then the support condition of the initial data, (2.1), implies that

$$
\text{supp } u(x, t) \subset \{ (x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R \}. \quad (2.2)
$$

For example, see Appendix in John [2] for this fact.

It is well-known that $u$ satisfies the following integral equation;

$$
u(x, t) = \varepsilon u^0(x, t) + L_a(|u_t|^p)(x, t), \quad (2.3)$$

where $u^0$ is a solution of the free wave equation with the same initial data:

$$u^0(x, t) := \frac{1}{2} \{f(x + t) + f(x - t)\} + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy,$$  \hspace{1cm} (2.4)

and a linear integral operator $L_a$ for a function $v = v(x, t)$ is Duhamel’s term defined by

$$L_a(v)(x, t) := \frac{1}{2} \int_0^t ds \int_{x-t+s}^{x+t-s} \frac{v(y, s)}{(1 + y^2)^{(1+a)/2}} dy.$$ \hspace{1cm} (2.5)

Then, one can apply the time-derivative to (2.3) and (2.4) to obtain

$$u_t(x, t) = \varepsilon u^0_t(x, t) + L'_a(|u_t|^p)(x, t) \hspace{1cm} (2.6)$$

and

$$u^0_t(x, t) = \frac{1}{2} \{f'(x + t) - f'(x - t) + g(x + t) + g(x - t)\}, \hspace{1cm} (2.7)$$

where $L'_a$ for a function $v = v(x, t)$ is defined by

$$L'_a(v)(x, t) := \frac{1}{2} \int_0^t ds \int_{x-t+s}^{x+t-s} \frac{v(x + t - s, s)}{\{1 + (x + t - s)^2\}^{(1+a)/2}} ds$$

$$+ \frac{1}{2} \int_0^t ds \int_{x-t+s}^{x+t-s} \frac{v(x - t + s, s)}{\{1 + (x - t + s)^2\}^{(1+a)/2}} ds.$$ \hspace{1cm} (2.8)

On the other hand, applying the space-derivative to (2.3) and (2.4), we have

$$u_x(x, t) = \varepsilon u^0_x(x, t) + \overline{L'_a}(|u_t|^p)(x, t)$$

and

$$u^0_x(x, t) = \frac{1}{2} \{f'(x + t) + f'(x - t) + g(x + t) - g(x - t)\},$$

where $\overline{L'_a}$ for a function $v = v(x, t)$ is defined by

$$\overline{L'_a}(v)(x, t) := \frac{1}{2} \int_0^t ds \int_{x-t+s}^{x+t-s} \frac{v(x + t - s, s)}{\{1 + (x + t - s)^2\}^{(1+a)/2}} ds$$

$$- \frac{1}{2} \int_0^t ds \int_{x-t+s}^{x+t-s} \frac{v(x - t + s, s)}{\{1 + (x - t + s)^2\}^{(1+a)/2}} ds.$$ \hspace{1cm} (2.9)

Therefore, $u_x$ is expressed by $u_t$. Moreover, one more space-derivative to (2.6) yields that

$$u_{tx}(x, t) = \varepsilon u^0_{tx}(x, t) + p \overline{L'_a}(|u_t|^{p-2}u_tu_{tx})(x, t) - (1 + a)L'_{a+2}(|u_t|^p)(x, t) \hspace{1cm} (2.10)$$
and
\[ u^0_{tx}(x,t) := \frac{1}{2} \left\{ f''(x+t) - f''(x-t) + g'(x+t) + g'(x-t) \right\} . \] (2.11)

Similarly, we have that
\[ u_{tt}(x,t) = \varepsilon u^0_{tt}(x,t) + \frac{|u_t(x,t)|^p}{(1 + x^2)^{(1+a)/2}} + pL_a'(|u_t|^p - 2u_t u_{tx})(x,t) - (1 + a)\frac{L_{a+2}'(|u_t|^p x)(x,t)}{2} \] (2.12)

and
\[ u^0_{tt}(x,t) = \frac{1}{2} \left\{ f''(x+t) + f''(x-t) + g'(x+t) - g'(x-t) \right\} . \]

Therefore, \( u_{tt} \) is expressed by \( u_{tx} \) and \( u_t \), so is \( u_{xx} \) because of
\[ u_{xx}(x,t) = \varepsilon u^0_{xx}(x,t) + p\frac{L_a'(|u_t|^p - 2u_t u_{tx})(x,t)}{2} - (1 + a)\frac{L_{a+2}'(|u_t|^p x)(x,t)}{2} \]

and
\[ u^0_{xx}(x,t) = u^0_{tt}(x,t). \]

First, we note the following fact.

**Proposition 2.1** Assume that \((f,g) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})\). Let \( u_t \) be a \( C^1 \) solution of (2.6). Then,
\[ w(x,t) := \int_0^t u_t(x,s)ds + \varepsilon f(x) \] (2.13)

is a classical solution of (1.1).

**Proof.** It is trivial that \( w \) satisfies the initial condition and
\[ w_t = u_t, \quad w_{tt} = u_{tt}. \] (2.14)

Then, (2.10) yields that
\[ w_x(x,t) = \int_0^t u_{tx}(x,s)ds + \varepsilon f'(x) \]
\[ = \int_0^t \{ pL_a'(|u_t|^p - 2u_t u_{tx})(x,s) - (1 + a)\frac{L_{a+2}'(|u_t|^p x)(x,s)}{2} \} ds \]
\[ + \int_0^t \varepsilon u^0_{tx}(x,s)ds + \varepsilon f'(x) \]
\[ = \frac{L_a'(|u_t|^p)(x,t) + \varepsilon u^0_x(x,t)}{2} \]
because of
\[ pL_a'(u_t^p - 2u_tu_{tx})(x,s) - (1 + a)L_{a+2}'(|u_t|^p)(x,s) = \frac{\partial}{\partial s} L_a(|u_t|^p)(x,s). \]

Therefore we obtain that
\[ w_{xx}(x,t) = \varepsilon u_{xx}^0(x,t) + pL_a'(|u_t|^p - 2u_tu_{tx})(x,t) - (1 + a)L_{a+2}'(|u_t|^p)(x,t) \]
which implies, together with \( 2.12 \) and \( 2.14 \), the desired conclusion,
\[ w_{tt} - w_{xx} = \frac{|u_t|^p}{(1 + x^2)^{(1+a)/2}} = \frac{|w_t|^p}{(1 + x^2)^{(1+a)/2}}. \]

\[ \square \]

Our result in (1.2) is split into the following two theorems.

**Theorem 2.1** Assume (2.1). Then, there exists a positive constant \( \varepsilon_1 = \varepsilon_1(f,g,p,a,R) > 0 \) such that a classical solution \( u \in C^2(\mathbb{R} \times [0,T]) \) of (1.1) for \( p \geq 2 \), or a solution \( u_t \in C(\mathbb{R} \times [0,T]) \) with
\[ \text{supp } u_t(x,t) \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R \} \]
of associated integral equations of (2.6) to (1.1) for \( 1 < p < 2 \), exists as far as \( T \) satisfies
\[ T \leq \begin{cases} c \varepsilon^{-(p-1)/(a-1)} & \text{for } a < 0, \\ \exp(c \varepsilon^{-(p-1)}) & \text{for } a = 0, \\ \exp(c \varepsilon^{-(p-1)}) & \text{for } a > 0, \end{cases} \] (2.15)
where \( 0 < \varepsilon \leq \varepsilon_1 \), \( c \) is a positive constant independent of \( \varepsilon \).

**Theorem 2.2** Assume (2.1) and
\[ \int_{\mathbb{R}} g(x) > 0. \] (2.16)

Then, there exists a positive constant \( \varepsilon_2 = \varepsilon_2(g,p,a,R) > 0 \) such that a solution \( u_t \in C(\mathbb{R} \times [0,T]) \) with
\[ \text{supp } u_t(x,t) \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R \} \]
of associated integral equations (2.6) to (1.1) cannot exist whenever \( T \) satisfies
\[ T \geq \begin{cases} c \varepsilon^{-(p-1)/(a-1)} & \text{for } a < 0, \\ \exp(C \varepsilon^{-(p-1)}) & \text{for } a = 0, \end{cases} \] (2.17)
where \( 0 < \varepsilon \leq \varepsilon_2 \), \( C \) is a positive constant independent of \( \varepsilon \).

The proofs of above theorems are given in following sections.
3 Proof of Theorem 2.1

According to the observations in the previous section, we shall construct a $C^1$ solution of (2.6) when $p \geq 2$ and a continuous solution of (2.6) when $1 < p < 2$.

First, we shall construct a $C^1$ solution for $p \geq 2$. Let $\{U_j(x,t)\}_{j \in \mathbb{N}}$ be a sequence of $C^1(\mathbb{R} \times [0,T])$ defined by

$$U_{j+1} = \varepsilon u^0_t + L'_a(|U_j|^p), \quad U_1 = \varepsilon u^0_t.$$  \hspace{1cm} (3.1)

Then, in view of (2.10), $(U_j)_x$ has to satisfy

$$\begin{cases} (U_{j+1})_x = \varepsilon u^0_{tx} + pL'_a(|U_j|^{p-2}U_j(U_j)_x) - (1 + a)L'_a(|U_j|^p x), \\ (U_1)_x = \varepsilon u^0_{tx}, \end{cases}$$  \hspace{1cm} (3.2)

so that the function space in which $\{U_j\}$ will converge is

$$X := \{U \in C(\mathbb{R} \times [0,T]) : \text{supp } U \subset \{|x| \leq t + R\}\},$$

equipping the norm

$$\|U\|_X := \|U\| + \|U_x\|, \quad \|U\| := \sup_{\mathbb{R} \times [0,T]} |U(x,t)|.$$ 

First we note that supp $U_j \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\}$ implies supp $U_{j+1} \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\}$. It is easy to check this fact by assumption on the initial data (2.1) and the definitions of $L'_a$ in (2.6).

**Proposition 3.1** Let $U \in C(\mathbb{R} \times [0,T])$ and supp $U \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\}$. Then there exists a positive constant $C$ independent of $T$ and $\varepsilon$ such that

$$\|L'_a(|U|^p)\| \leq CE_a(T)\|U\|^p,$$  \hspace{1cm} (3.3)

where

$$E_a(T) := \begin{cases} (T + 2R)^{-a} & \text{if } a < 0, \\ \log(T + 2R) & \text{if } a = 0, \\ 1 & \text{if } a > 0. \end{cases}$$  \hspace{1cm} (3.4)

The proof of Proposition 3.1 is established in the next section. Set

$$M := \|f''\|_{L^\infty(\mathbb{R})} + \|f'''\|_{L^\infty(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})} + \|g'\|_{L^\infty(\mathbb{R})}.$$ 

The convergence of the sequence $\{U_j\}$. 

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First we note that \( \|U_1\| \leq M\varepsilon \) by (2.7). Since (3.1) and (3.3) yield that
\[
\|U_{j+1}\| \leq M\varepsilon + \|L'_a(|U_j|^p)\| \\
\leq M\varepsilon + CE_a(T)\|U_j\|^p,
\]
the boundedness of \( \{U_j\} \);
\[
\|U_j\| \leq 2M\varepsilon \quad (j \in \mathbb{N}) \tag{3.5}
\]
follows from
\[
CE_a(T)(2M\varepsilon)^p \leq M\varepsilon. \tag{3.6}
\]
Assuming (3.6), one can estimate \( U_{j+1} - U_j \) as follows.
\[
\|U_{j+1} - U_j\| \leq \|L'_a(|U_j|^p - |U_{j-1}|^p)\| \\
\leq p\|L'_a((|U_j|^{p-1} + |U_{j-1}|^{p-1})|U_j - U_{j-1}|)\| \\
\leq pCE_a(T)(|U_j|^{p-1} + ||U_{j-1}|^{p-1})\|U_j - U_{j-1}\| \\
\leq pCE_a(T)2(2M\varepsilon)^{p-1}\|U_j - U_{j-1}\|.
\]
Therefore the convergence of \( \{U_j\} \) follows from
\[
\|U_{j+1} - U_j\| \leq \frac{1}{2}\|U_j - U_{j-1}\| \quad (j \geq 2) \tag{3.7}
\]
provided (3.6) and
\[
pCE_a(T)2(2M\varepsilon)^{p-1} \leq \frac{1}{2} \tag{3.8}
\]
are fulfilled.

**The convergence of the sequence \( \{(U_j)_x\} \).**

First we note that \( \|(U_1)_x\| \leq M\varepsilon \) by (2.11). Assume that (3.6) and (3.8) are fulfilled. It follows from (3.2) and (3.3) that
\[
\|(U_{j+1})_x\| \leq M\varepsilon + \|L'_a(|U_j|^{p-1})(U_j)_x\| + |1 + a\|L'_{a+1}(|U_j|^p)\| \\
\leq M\varepsilon + CE_a(T)(|U_j|^{p-1})(U_j)_x\| + |1 + a|CE_{a+1}(T)|U_j|^p \\
\leq M\varepsilon + CE_a(T)(2M\varepsilon)^{p-1}|(U_j)_x\| + |1 + a|CE_{a+1}(T)(2M\varepsilon)^p.
\]
Here we have employed the fact that (2.8) yields
\[
|L'_{a+2}(|U_j|^p)(x,t)| \leq \frac{1}{2} \int_0^t \frac{|U_j(x + t - s, s)|^p|x + t - s|}{\{1 + (x + t - s)^2\}^{(3+a)/2}} ds \\
+ \frac{1}{2} \int_0^t |U_j(x - t + s, s)|^p|x - t + s| ds \\
\leq L'_{a+1}(|U_j|^p)(x,t).
\]

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Hence the boundedness of \(\{(U_j)_x\}\);
\[
\| (U_j)_x \| \leq 2M\varepsilon \quad (j \in \mathbb{N})
\] (3.9)
follows from
\[
CE_\alpha(T)(2M\varepsilon)^p + |1 + a|CE_{\alpha+1}(T)(2M\varepsilon)^p \leq M\varepsilon.
\] (3.10)
Assuming (3.10), one can estimate \((U_{j+1})_x - (U_j)_x\) as follows.
\[
\| (U_{j+1})_x - (U_j)_x \| \leq \| L'_\alpha(U_j)^{p-2}U_j(U_j)_x - |U_j-1|^{p-2}U_{j-1}(U_j-1)_x \| \\
+ |1 + a|\| L'_{\alpha+2}((|U_j|^p - |U_j-1|^p)x) \|.
\]
The first term on the right hand side of above inequality is split into three pieces according to
\[
|U_j|^{p-2}U_j(U_j)_x - |U_j-1|^{p-2}U_{j-1}(U_j-1)_x \\
= (|U_j|^p - |U_j-1|^{p-2})U_j(U_j)_x \\
+ |U_j-1|^{p-2}(U_j - U_{j-1})(U_j)_x \\
+ |U_{j-1}|^{p-2}U_{j-1}(U_j)_x - (U_{j-1})_x.
\]
Since
\[
\| |U_j|^p - |U_j-1|^{p-2} \| \\
\leq \begin{cases} 
(p-2)(|U_j|^{p-3} + |U_j-1|^{p-3})|U_j - U_{j-1}| & \text{when } p \geq 3, \\
|U_j - U_{j-1}|^{p-2} & \text{when } 2 < p < 3, \\
0 & \text{when } p = 2,
\end{cases}
\]
the similar manner of handling \(L'_{\alpha+2}\) to above computations leads to
\[
\| (U_{j+1})_x - (U_j)_x \| \\
\leq CE_\alpha(T)\| U_j \|\| (U_j)_x \| \times \\
\times \begin{cases} 
(p-2)(|U_j|^{p-3} + |U_j-1|^{p-3})|U_j - U_{j-1}| & \text{when } p \geq 3, \\
|U_j - U_{j-1}|^{p-2} & \text{when } 2 < p < 3, \\
0 & \text{when } p = 2
\end{cases}
\]
\[
+ CE_\alpha(T)|U_{j-1}|^{p-2}|U_j - U_{j-1}|\| (U_j)_x \| \\
+ CE_\alpha(T)|U_{j-1}|^{p-1}|(U_j)_x - (U_{j-1})_x \| \\
+ |1 + a|CE_{\alpha+1}(T)p(|U_j|^{p-1} + |U_{j-1}|^{p-1})|U_j - U_{j-1}|.
\]
Hence it follows from (3.7) that
\[
\| (U_{j+1})_x - (U_j)_x \| \leq CE_\alpha(T)(2M\varepsilon)^{p-1}\| (U_j)_x - (U_{j-1})_x \| \\
+ \begin{cases} 
O \left( \frac{1}{2^{2(p-2j)}} \right) & \text{when } 2 < p < 3, \\
O \left( \frac{1}{2^j} \right) & \text{otherwise}
\end{cases}
\]
as $j \to \infty$. Here we have employed the fact that $E_{a+1}(T)$ is dominated by $E_a(T)$ with some positive constant. Therefore we obtain the convergence of $\{(U_j)_x\}$ provided

$$CE_a(T)(2M\varepsilon)^{p-1} \leq \frac{1}{2}. \quad (3.11)$$

**Continuation of the proof.**

It is easy to find a positive constant $C_0$ independent of $\varepsilon$ and $T$ such that all the conditions, (3.6), (3.8), (3.10), (3.11), on the convergence of $\{U_j\}$ in the closed subspace of $X$ satisfying $\|U\|, \|U_x\| \leq 2M\varepsilon$ follows from

$$C_0\varepsilon^{p-1}E_a(T) \leq 1.$$

Therefore we obtain Theorem 2.1 for $p \geq 2$.

For $1 < p < 2$, $X$ and $M$ in the proof for $p \geq 2$ above are replaced with $Y$ and $N$ respectively, where

$$Y := \{U \in C(\mathbb{R} \times [0, T]): \text{supp } U \subset \{|x| \leq t + R\}\}$$

equipping

$$\|U\|_Y := \|U\| = \sup_{\mathbb{R} \times [0, T]} |U(x, t)|$$

and

$$N := \|f\|_{L^\infty(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})},$$

respectively. It is trivial that the convergence of $\{U_j\}$ in the closed subspace $Y$ satisfying $\|U\| \leq 2N\varepsilon$ follows from (3.6) and (3.8), so that the proof of Theorem 2.1 is completed now by taking $\varepsilon$ small enough. \hfill \Box

## 4 Proof of Proposition 3.1

In this section, we prove a priori estimate (3.3). Recall the definition of $L_a'$ in (2.8). From now on, a positive constant $C$ independent of $T$ and $\varepsilon$ may change from line to line. Since

$$\frac{1}{2}(1 + |x|) \leq (1 + x^2)^{1/2} \leq (1 + |x|),$$

we have that

$$|L_a'(|U|^p)(x, t)| \leq C\|U\|^p \{I_+(x, t) + I_-(x, t)\},$$

where the integrals $I_+$ and $I_-$ are defined by

$$I_\pm(x, t) := \int_0^t \frac{\chi_{\pm}(x, t; s)}{(1 + |t - s \pm x|)^{1+a}} ds$$

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and the characteristic functions $\chi_+$ and $\chi_-$ are defined by

$$
\chi_{\pm}(x,t; s) := \chi\{s;|t-s\pm x|\leq s+R\} = \begin{cases} 
1 & \text{when } s \text{ satisfies } |t-s \pm x| \leq s + R, \\
0 & \text{otherwise,}
\end{cases}
$$

respectively. First we note that it is sufficient to estimate $I_{\pm}$ for $x \geq 0$ due to its symmetry,

$$
I_+(x,t) = I_-(x,t).
$$

Hence it follows from $0 \leq x \leq t + R$ as well as

$$
|t - s + x| \leq s + R \quad \text{and} \quad 0 \leq s \leq t
$$

that

$$
\frac{t + x - R}{2} \leq s \leq t,
$$

so that

$$
I_+(x,t) \leq \int_{(t+x-R)/2}^{t} \frac{1}{(1 + t - s + x)^{1+a}} ds.
$$

When $a < 0$, we have

$$
I_+(x,t) \leq C \left(1 + t + x - \frac{t + x - R}{2}\right)^{-a} \leq C(T + 2R)^{-a}.
$$

When $a = 0$, we have

$$
I_+(x,t) \leq \log \frac{1 + t + x - (t + x - R)/2}{1 + x} \leq \log(T + 2R).
$$

When $a > 0$, we have

$$
I_+(x,t) \leq C(1 + x)^{-a} \leq C.
$$

Therefore we obtain

$$
I_+ \leq CE_a(T) \quad \text{in } \mathbb{R} \times [0,T].
$$

On the other hand, the estimate for $I_-$ is divided into two cases. If $t - x \geq 0$, then $|t - s - x| \leq s + R$ yields that

$$
I_-(x,t) \leq \int_{(t-x-R)/2}^{t-x} \frac{1}{(1 + t - s - x)^{1+a}} ds + \int_{t-x}^{t} \frac{1}{(1 + t + s + x)^{1+a}} ds.
$$
follows. When $a < 0$, we have
\[ I_-(x,t) \leq C(1 + t - x - (t - x - R)/2)^{-a} + C \leq C(T + 2R)^{-a}. \]

When $a = 0$, we have
\[ I_-(x,t) = \log(1 + t - x - (t - x - R)/2) + \log(1 + x) \leq 2\log(T + 2R). \]

When $a > 0$, we have
\[ I_-(x,t) \leq C. \]

Therefore we obtain
\[ I_- \leq CE_a(T) \quad \text{in } \mathbb{R} \times [0,T] \cap \{t - x \geq 0\}. \]

If $(-R \leq t - x \leq 0, |t - s - x| \leq s + R$ yields that
\[ I_-(x,t) \leq \int_0^t \frac{1}{(1 + s - t + x)^{1+a}} ds. \]

When $a < 0$, we have
\[ I_-(x,t) \leq C(1 + x)^{-a} \leq C(T + 2R)^{-a}. \]

When $a = 0$, we have
\[ I_-(x,t) \leq \log \frac{1 + x}{1 - t + x} \leq \log(T + 2R). \]

When $a > 0$, we have
\[ I_-(x,t) \leq C(1 - t + x)^{-a} \leq C. \]

Therefore we obtain
\[ I_- \leq CE_a(T) \quad \text{in } \mathbb{R} \times [0,T] \cap \{-R \leq t - x \leq 0\}. \]

Summing up all the estimates for $I_+$ and $I_-$, we have
\[ |L'_a(|U|^p)| \leq C||U||^pE_a(T) \quad \text{in } \mathbb{R} \times [0,T]. \]

This completes the proof of Proposition 3.1. \qed
5 Proof of Theorem 2.2

In this section, a positive constant $C$ independent of $T$ and $\varepsilon$ may change from line to line. Let $U \in C(\mathbb{R} \times [0,T])$ with

$$\text{supp } U(x,t) \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\} \tag{5.1}$$

be a solution of the integral equation (2.6), namely

$$U = \varepsilon u^0 + L_a(|U|^p).$$

Then it is easy to see by simple integration that

$$V(x,t) := \int_0^t U(x,s)ds + \varepsilon f(x)$$

satisfies a integral equation,

$$V = \varepsilon u^0 + L_a(|V|^p).$$

Set $t = x + R$, $x \geq R$. Then, inverting the order of the $(y,s)$-integral and diminishing its domain, we have that

$$L_a(|V|^p)(x,t) \geq C \int_R^x \frac{1}{(1+y)^{1+a}}dy \int_{y-R}^{y+R} |V_t(y,s)|^p ds.$$  

Hence Hölder’s inequality yields that

$$L_a(|V|^p)(x,t) \geq C \int_R^x \frac{1}{(1+y)^{1+a}}dy \left| \int_{y-R}^{y+R} V_t(y,s)ds \right|^p$$

which implies that, due to (5.1),

$$L_a(|V|^p)(x,t) \geq C \int_R^x \frac{|V(y,y+R)|^p}{(1+y)^{1+a}}dy.$$  

On the other hand, it follows from the assumption on the support of the data that

$$u^0(x,x + R) = \frac{1}{2} \int_{\mathbb{R}} g(x)dx =: G > 0 \quad \text{for } x \geq R.$$  

Hence $V$ satisfies

$$V(x,x + R) > G\varepsilon + C \int_R^x \frac{|V(y,y+R)|^p}{(1+y)^{1+a}}dy \quad \text{for } x \geq R. \tag{5.2}$$
We note that the equality in the inequality above can be removed without loss of generality by taking slightly smaller \( G \).

Now we employ the comparison argument with a solution of the related ordinary differential equation. Let \( W \) be a solution of

\[
W(x) = G\varepsilon + C \int_{R}^{x} \frac{|W(y)|^p}{(1 + y)^{1+a}} dy \quad \text{for } x \geq R. \tag{5.3}
\]

Then we have

\[
V(x, x + R) > W(x) \quad \text{for } x \geq R. \tag{5.4}
\]

Because \( V(R, 2R) > W(R) \) and the continuity of \( V, W \) yield that (5.4) holds in the neighborhood of \( x = R \). If there exists a point

\[
x_0 \defeq \inf\{x \geq R : V(x, x + R) = W(x)\},
\]

we immediately reach to a contradiction,

\[
0 = V(x_0, x_0 + R) - W(x_0) = C \int_{R}^{x_0} \frac{|V(y, y + R)|^p - |W(y)|^p}{(1 + y)^{1+a}} dy > 0.
\]

Hence (5.4) is true and implies that the existence time of \( V(x, x + R) \) is less than the blow-up time of \( W(x) \).

Therefore the conclusion of Theorem 2.2 follows by solving the initial value problem for ordinary differential equations,

\[
\begin{cases}
W' = \frac{C|W|^p}{(1 + x)^{1+a}} & \text{in } [R, \infty), \\
W(R) = G\varepsilon,
\end{cases}
\]

which is equivalent to (5.3). In fact, the blow-up time \( X \) of \( W \) has to satisfy

\[
(G\varepsilon)^{1-p} = \begin{cases}
\frac{p - 1}{-a} C \left\{(1 + X)^{-a} - (1 + R)^{-a}\right\} & \text{for } a < 0, \\
(p - 1)C \log \frac{1 + X}{1 + R} & \text{for } a = 0.
\end{cases}
\]

Since (5.4) implies that the blow-up time \( T \) of \( W \) has to satisfy the inequality \( T \geq X + R \), we have the blow-up condition (2.17) by taking \( \varepsilon \) small enough. To see this, if \( a < 0 \), one can estimate \( X + R \) as

\[
X + R = \left\{ \frac{-a}{(p - 1)C} (G\varepsilon)^{1-p} + (1 + R)^{-a} \right\}^{1/(-a)} - 1 + R
\]

\[
\leq \left\{ \frac{-a}{(p - 1)C} (G\varepsilon)^{1-p} + 2(1 + R)^{-a} \right\}^{1/(-a)}.
\]
Therefore $\varepsilon_2$ in Theorem 2.2 should be defined by
\[
\frac{-a}{(p-1)C} (G\varepsilon_2)^{1-p} = 2(1 + R)^{-a}
\]
because it makes
\[
X + R \leq \left\{ \frac{2(-a)G^{1-p}}{(p-1)C} \right\}^{1/(-a)} \varepsilon^{-(p-1)/(-a)} \text{ for } 0 < \varepsilon \leq \varepsilon_2.
\]
Similarly, if $a = 0$, one can estimate $X + R$ as
\[
X + R = \exp \left\{ \frac{(G\varepsilon)^{1-p}}{(p-1)C} + \log(1 + R) \right\} - 1 + R
\leq \exp \left\{ \frac{(G\varepsilon)^{-p}}{(p-1)C} + 2 \log(1 + R) \right\}.
\]
Therefore $\varepsilon_2$ in Theorem 2.2 should be defined by
\[
\frac{(G\varepsilon_2)^{1-p}}{(p-1)C} = 2 \log(1 + R)
\]
because it makes
\[
X + R \leq \exp \left\{ \frac{-2G^{1-p}}{(p-1)C} \varepsilon^{-(p-1)} \right\} \text{ for } 0 < \varepsilon \leq \varepsilon_2.
\]
The proof of Theorem 2.2 is now completed. \hfill \Box

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