“Approximating quantiles in very large datasets”

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Abstract

Very large datasets are often encountered in climatology, either from a multiplicity of observations over time and space or outputs from deterministic models (sometimes in petabytes = 1 million gigabytes). Loading a large data vector and sorting it, is impossible sometimes due to memory limitations or computing power. We show that a proposed algorithm to approximating the median, “the median of the median” performs poorly. Instead we develop an algorithm to approximate quantiles of very large datasets which works by partitioning the data or use existing partitions (possibly of non-equal size). We show the deterministic precision of this algorithm and how it can be adjusted to get customized precisions.

Keywords: quantiles; large datasets; approximation; sorting; algorithm.

1 Introduction

This paper develops an algorithm for approximating the quantiles in petascale (petabyte = one million gigabytes) datasets and uses the “probability loss function” to assess the quality of the approximation. The need for such an approximation does not arise for the sample average, another common data summary. That is because if we break down the data to equal partitions and calculate the mean for every partition, the mean of the obtained means is equal to the total mean. It is also easy to recover the total mean from the means of unequal partitions if their length is known.

However computer memories, several gigabytes (GBs) in size, cannot handle large datasets that can be petabytes (PBs) in size. For example, a laptop with 2 GBs of memory, using the well-known R package, could find the median of a data file of about 150 megabytes (MBs) in size. However, it crashed for files larger than this. Since large datasets are commonly assembled in blocks, say by day or by district, that need not be a serious limitation except insofar as the quantiles computed in that
way cannot be used to find the overall quantile. Nor would it help to sub-sample these blocks, unless these (possibly dependent) sub-samples could be combined into a grand sub-sample whose quantile could be computed. That will not usually be possible in practice. The algorithm proposed here is a “worst-case” algorithm in the sense that no matter how the data are arranged, we will reach the desired precision. This is of course not true if we sample from the data because there is a (perhaps small) probability that the approximation could be poor.

We also address the following question:

**Question**: If we partition the data-file into a number of sub-files and compute the medians of these, is the median of the medians a good approximation to the median of the data-file?

We first show that the median of the medians does not approximate the exact median well in general, even after imposing conditions on the number of partitions or their length. However for our proposed algorithm, we show how the partitioning idea can be employed differently to get good approximations. “Coarsening” is introduced to summarize data vector with the purpose of inferring about the quantiles of the original vector using the summaries. Then the “d-coarsening” quantile algorithm which works by partitioning the data (or use previously defined partitions) to possibly non-equal partitions, summarizing them using coarsening and inferring about the quantiles of the original data vector using the summaries. Then we show the deterministic accuracy of the algorithm in Theorem 6.1. The accuracy is measured in terms of the probability loss function of the original data vector. This is an extension of the work in [Alsabti et al. 1997] to non-equal size partition case. Theorem 6.1 still requires the partition sizes to be divisible by $d$ the coarsening factor. In order to extend the results further to the case where the partitions are not divisible by $d$, we investigate how quantiles of a data vector with missing data or contaminated data relate to the quantiles of the original data in Lemma 6.2 and Lemma 6.3. Also in Lemma 7.1, we show if the quantiles of a coarsened vector are used in place of the quantiles of the original data vector how much accuracy will be lost. Finally we investigate the performance of the algorithm using both simulations and real climate datasets.

We define the loss of estimating/approximating a quantile $q$ by $\hat{q}$ to be the probability that the random variable falls in between the two values. A limited version of this concept only for data vectors can be found in computer science literature, where $\epsilon$-approximations are used to approximate quantiles of large datasets. (See for example [Manku et al. 1998].) However, this concept has not been introduced as
a measure of loss and the definition is limited to data vectors rather than arbitrary distributions.

The traditional definition of quantiles for a random variable $X$ with distribution function $F$,

$$lq_X(p) = \inf \{x | F(x) \geq p \},$$

appears in classic works as Parzen [1979]. We call this the “left quantile function”. In some books (e.g. Rychlik [2001]) the quantile is defined as

$$rq_X(p) = \sup \{x | F(x) \leq p \},$$

this is what we call the “right quantile function”. Also in robustness literature people talk about the upper and lower medians which are a very specific case of these definitions. Hosseini [2009] considers both definitions, explore their relation and show that considering both has several advantages.

**Lemma 1.1:** (Quantile Properties Lemma) Suppose $X$ is a random variable on the probability space $(\Omega, \Sigma, P)$ with distribution function $F$:

a) $F(lq_F(p)) \geq p$.

b) $lq_F(p) \leq rq_F(p)$.

c) $p_1 < p_2 \Rightarrow rq_F(p_1) \leq lq_F(p_2)$.

d) $rq_F(p) = \sup \{x | F(x) \leq p \}$.

e) $P(lq_F(p) < X < rq_F(p)) = 0$. i.e. $F$ is flat in the interval $(lq_F(p), rq_F(p))$.

f) $P(X < rq_F(p)) \leq p$.

g) If $lq_F(p) < rq_F(p)$ then $F(lq_F(p)) = p$ and hence $P(X \geq rq_F(p)) = 1 - p$.

h) $lq_F(1) = -\infty, rq_F(0) = \infty$ and $P(rq_F(0) \leq X \leq lq_F(1)) = 1$.

i) $lq_F(p)$ and $rq_F(p)$ are non-decreasing functions of $p$.

j) If $P(X = x) > 0$ then $lq_F(F(x)) = x$.

k) $x < lq_F(p) \Rightarrow F(x) < p$ and $x > rq_F(p) \Rightarrow F(x) > p$. 
2 Previous work

Finding quantiles and using them to summarize data is of great importance in many fields. One example is the climate studies where we have very large datasets. For example the datasets created by computer climate models are larger than PBs in size. In NCAR (National Center for Atmospheric sciences at Boulder, Colorado), the climate data (outputs of compute models) are saved on several disks. To access different parts of these data a robot needs to change disks form a very large storage space. Another case where we confront large datasets is in dealing with data streams which arise in many different applications such as finance and high–speed networking. For many applications, approximate answers suffice. In computer science, quantiles are important to both data base implementers and data base users. They can also be used by business intelligence applications to drive summary information from huge datasets.

As pointed out in Manku et al. [1998], a good quantile approximation algorithm should

1. not require prior knowledge of the arrival or value distribution of its inputs.
2. provide explicit and tunable approximation guarantees.
3. compute results in a single pass.
4. produce multiple quantiles at no extra cost.
5. use as little memory as possible.
6. be simple to code and understand.

Finding quantiles of data vectors and sorting them are parallel problems since once we sort a vector finding any given quantile can be done instantly. A good account of early work in sorting algorithms can be found in Knuth [1973]. Also Munro and Paterson [1980] showed for $P$-pass algorithms (algorithms that scan the data $P$ times) $\Theta(N/P)$ storage locations are necessary and sufficient, where $N$ is the length of the dataset. (See Appendix C for the definitions of complexity functions such as $\Theta$.) It is well-known that the worst-case complexity of sorting is $n \log_2 n + O(1)$ as shown in Manku et al. [1999]. In Paterson [1997], Paterson discusses the progress made in the so-called “selection” problem. He lets $V_k(n)$ be the worst-case minimum number of pairwise comparisons required to find
the $k$-th largest out of $n$ “distinct elements”. In particular $M(n) = V_k(n)$ for $k = \lceil n/2 \rceil$. In Blum and John [1973], it is shown that the lower bound for $V_k(n)$ is $n + \min\{k - 1, n - k\} - 1$, an achieved upper bound by Blum is $5.43n$. Better upper bounds have been achieved through the years. The best upper bound so far is $2.9423N$ and the lower bound is $(2 + \alpha)N$ where $\alpha$ is of order $2^{-40}$.

Yao [1974] shows that finding approximate median needs $\Omega(N)$ comparisons in deterministic algorithms. Using sampling this can be reduced to $O(\frac{1}{\epsilon^2} \log(\delta^{-1}))$ independent of $N$, where $\epsilon$ is the accuracy of the approximation in terms of the “probability loss” in our notation. Munro and Paterson [1980] show that $O(N^{1/p})$ is necessary and sufficient to find an exact $\phi$-quantile in $p$ passes.

Often an exact quantile is not needed. A related problem is finding space-efficient one-pass algorithms to find approximate quantiles. A summary of the work done in this subject and a new method is given in Agrawal and Swami [1995]. Two approximate quantile algorithms using only a constant amount of memory were given in Jain and Chlamtad [1985] and Agrawal and Swami [1995]. No guarantee for the error was given. Alsabti et al. [1997] provide an algorithm and guaranteed error in one pass. This algorithm works by partitioning the data into subsets, summarizing each partition and then finding the final quantiles using the summarized partitions. The algorithm in this chapter is an extension of this algorithm to the case of partitions of unequal length.

3 The median of the medians

A proposed algorithm to approximate the median of a very large data vector partitions the data into subsets of equal length, computes the median for each partition and then computes the median of the medians. For example, suppose $n = lm$ and break the data to $m$ vectors of size $l$. One might conjecture that by picking $l$ or $m$ sufficiently large the median of the medians would ensure close proximity to the exact median. We show by an example that taking $l$ and $m$ very large will not help to get close to the exact median. Let $l = 2b + 1$ and $m = 2a + 1$. 
Table 1: The table of data

| partition number | Partition                      | Median of the partition |
|------------------|--------------------------------|-------------------------|
| 1                | $(1,2,\cdots,b,b+1,10^b,\cdots,10^b)$ | $b+1$                   |
| 2                | $(1,2,\cdots,b,b+1,10^b,\cdots,10^b)$ | $b+1$                   |
| $a+1$            | $(1,2,\cdots,b,b+1,10^b,\cdots,10^b)$ | $10^b$                  |
| $a+2$            | $(10^b,10^b,\cdots,10^b)$          | $10^b$                  |
| $2a+1$           | $(10^b,10^b,\cdots,10^b)$          | $10^b$                  |

Table 1 shows the dataset partitioned into $m = 2a + 1$ vectors of equal length. Every vector is of length $l = 2b + 1$. The first $a + 1$ vectors are identical and $10^b$ is repeated $b$ times in them. The last $a$ vectors are also identical with all components equal to $10^b$. The median of the medians turns out to be $b + 1$. However, the median of the dataset is $10^b$. We show that $b + 1$ is in fact “almost” the first quantile. This is because $(b + 1)$ is smaller than all $10^b$’s. There are $(a + 1)b + a(2b + 1)$ data points equal to $10^b$. Hence $b + 1$ is smaller than this fraction of the data points:

$$\frac{(a + 1)b + a(2b + 1)}{(2a + 1)(2b + 1)} = \frac{2a + 2}{2a + 1} \frac{b}{4b + 2} + \frac{a}{2a + 1} \approx 1 \times \frac{1}{4} + \frac{1}{2} \approx \frac{3}{4}.$$  

With a similar argument, we can show that $b + 1$ is greater than almost a quarter of the data points (the ones equal to $1, 2, \cdots, b$). Hence $b + 1$ is “almost” the first quartile.

One can prove a rigorous version of the following statement.

*The median of the medians is “almost” between the first and the third quartile.*

We only give a heuristic argument for simplicity. To that end, let $n = lm$ and $m = 2a + 1$ and $l = 2b + 1$. Let $M$ be the exact median and $M'$ be the median
of the medians. Order the obtained medians of each partition and denote them by $M_1, \ldots, M_m$. By definition $M' \geq M_j$, $j \leq a$ and $M' \leq M_j$, $j \geq a + 1$. Each $M_j$, $j \leq a$ is less than or equal to $b$ data points in its partition. Hence, we conclude that $M'$ is less than or equal to $ab$ data points. Similarly $M'$ is greater than or equal to $ab$ data points (which are disjoint for the data points used before). But 

$$ab = \frac{ab}{(2a+1)(2b+1)} \approx \frac{1}{4}.$$ 

Hence, $M'$ is greater than or equal to $1/4$ data points and less than or equal to $1/4$ data points.

## 4 Preliminary results

Suppose $y' \in \{y_1, \ldots, y_n\}$, for future reference, we define some additional notations for data vectors.

**Definition 4.1:** The minimal index of $y'$, $m(y')$ and the maximal index of $y'$, $M(y')$ are defined as below:

$$m(y') = \min\{i | y_i = y'\}, \quad M(y') = \max\{i | y_i = y'\}.$$ 

It is easy to see that in $y = \text{sort}(x) = (y_1, \ldots, y_n)$ all the coordinates between $m(y')$ and $M(y')$ are equal to $y'$. Also note that if $y' = z_i$ then $M(y') - m(y') + 1 = m_i$ is the multiplicity of $z_i$. We use the notation $m_x$ and $M_x$ whenever we want to emphasize that they depend on the data vector $x$.

**Lemma 4.1:** Suppose $x = (x_1, \ldots, x_n)$, $y = \text{sort}(x)$ and $z$ a non-decreasing vector of all distinct elements of $x$. Then

a) $m(z_{i+1}) = M(z_i) + 1$, $i = 0, \ldots, r - 1$.

b) Suppose $\phi$ is a bijective increasing transformation over $\mathbb{R}$,

$$m_{\phi(x)}(\phi(z_i)) = m_x(z_i),$$

and

$$M_{\phi(x)}(\phi(z_i)) = M_x(z_i),$$

for $i = 1, \ldots, r$.

**Proof**  a) is straightforward. 

b) Note that

$$m_x(y') = \min\{i | y_i = y'\} = \min\{i | \phi(y_i) = \phi(y')\} = m_{\phi(x)}(\phi(y')).$$
A similar argument works for $M_x$.

We also define the position and standardized position of an element of a data vector.

**Definition 4.2:** Let $x = (x_1, \ldots, x_n)$ be a vector and $y = \text{sort}(x) = (y_1, \ldots, y_n)$. Then for $y' \in \{y_1, \ldots, y_n\}$, we define

$$\text{pos}_x(y') = \{m_x(y'), m_x(y') + 1, \ldots, M_x(y')\},$$

where $\text{pos}$ stands for position. Then we define the standardized position of $y'$ to be

$$\text{spos}_x(y') = \left(\frac{m_x(y') - 1}{n}, \frac{M_x(y')}{n}\right).$$

In the following lemma we show that for every $p \in \text{spos}(y')$ (and only $p \in \text{spos}(y')$), we have $rq(p) = lq(p) = y'$. For example if $1/2 \in \text{spos}(y')$ then $y'$ is the (left and right) median.

**Lemma 4.2:** Suppose $x = (x_1, \ldots, x_n)$, $y = \text{sort}(x) = (y_1, \ldots, y_n)$ and $y' \in \{y_1, \ldots, y_n\}$. Then

$$p \in \text{spos}_x(y') \iff lq_x(p) = rq_x(p) = y'.$$

**Proof** Let $z = (z_1, \ldots, z_r)$ be the reduced vector with multiplicities $m_1, \ldots, m_r$. Then $y' = m_i$ for some $i = 1, \ldots, r$.

**case I:** If $i = 2, \ldots, r$, then

$$m(y') = m_1 + \cdots + m_{i-1} + 1,$$

and

$$M(y') = m_1 + \cdots + m_i.$$

**case II:** If $i = 1$, then $m(y') = 1$ and $M(y') = m_1$.

In any of the above cases for $p \in \left(\frac{m(y') - 1}{n}, \frac{M(y')}{n}\right)$ and only $p \in \left(\frac{m(y') - 1}{n}, \frac{M(y')}{n}\right)$,

$$rq_x(p) = lq_x(p) = z_i,$$
by definition.

Now we prove a lemma. It is easy to see that if \( u \in \text{pos}(y') \) then

\[
\left( \frac{u-1}{n}, \frac{u}{n} \right) \subset \text{spos}(y').
\]

We conclude that

\[
\bigcup_{u \in \text{pos}(y')}(\frac{u-1}{n}, \frac{u}{n}) \subset \text{spos}(y').
\]

In fact \( \text{spos}(y') \) can possibly have a few points on the edge of the intervals not in \( \bigcup_{u \in \text{pos}(y')}(\frac{u-1}{n}, \frac{u}{n}) \).

Lemma 4.3: Suppose \( x \) is a data vector of length \( n \) and \( y' \) is an element of this vector. Also assume

\[
y' \geq x_i, \ i \in I, \ y' \leq x_j, \ j \in J,
\]

\[I \cap J = \phi, \ I, J \subset \{1, 2, \ldots, n\} \]

Then there exist a \( p \) in \( (\frac{|I|-1}{n}, 1 - \frac{|J|}{n}) \) that belongs to \( \text{spos}(y') \). In other words \( lq(p) = rq(p) = y' \).

**Proof** From the assumption, we conclude that \( \text{pos}(y') \) includes a number between \( |I| \) and \( n-|J| \). Let us call it \( u_0 \). Hence \( (\frac{u_0-1}{n}, \frac{u_0}{n}) \subset \text{spos}(y') \). Since \( |I| \leq u_0 \leq n-|J| \), we conclude that \( \text{spos}(y') \) intersects with \( \bigcup_{|I| \leq u \leq n-|J|}(\frac{u-1}{n}, \frac{u}{n}) \subset (\frac{|I|-1}{n}, 1 - \frac{|J|}{n}) \).

5 A loss function to assess approximations of quantiles

Our purpose is to find good approximations to the median and other quantiles. We need a method to assess such approximations. We contend that such a method should not depend on the scale of the data. In other words it should be invariant under monotonic transformations. We define a function \( \delta \) that measures a natural “degree
of separation” between data points of a data vector $x$. For the sake of illustration, consider the example $\text{sort}(x) = (1, 2, 3, 4, 4, 4, 5, 6, 6, 7)$. Now suppose, we want to define the degree of separation of 3,4 and 7 in this example. Since 4 comes right after 3, we consider their degree of separation to be zero. There are 3 elements between 4 and 7 so it is appealing to measure their degree of separation as 3 but since the degree of separation should be relative, we can also divide by $n = 11$, the length of the vector, and get: $\delta(4,7) = \frac{3}{11}$. We can generalize this idea to get a definition for all pairs in $\mathbb{R}$. With the same example, suppose we want to compute the degree of separation between 2.5 and 4.5 that are not members of the data vector. Then since there are 5 elements of the data vector between these two values, we define their degree of separation as $\frac{5}{11}$. More formally, we give the following definition.

Definition 5.1: Suppose $x = (x_1, \cdots, x_n)$, a data vector and $z < z'$ let $\Delta_x(z, z') = \{i | z < x_i < z', i = 1, \cdots, n\}$. Then we define

$$\delta_x(z, z') = \frac{|\Delta_x(z, z')|}{n},$$

and $\delta_x(z, z) = 0$, where $|\Delta_x(z, z')|$ is the cardinality of $\Delta_x(z, z')$. We call $\delta_x$ the “degree of separation” (DOS) or the “probability loss function” associated with $x$.

We then have the following lemma about the properties of $\delta$.

Lemma 5.1: The degree of separation $\delta_x$ has the following properties:

a) $\delta_x \geq 0$.

b) $y < y' < y'' \Rightarrow \delta_x(y, y'') \geq \delta_x(y, y')$.

c) $\delta_{\phi(x)}(\phi(z), \phi(z')) = \delta_x(z, z')$ if $\phi$ is a strictly monotonic transformation.

d) $y = \text{sort}(x)$ and $y_i < y_j \Rightarrow \delta_x(y_i, y_j) \leq (j - i - 1)/n$.

Proof

Both a) and b) are straightforward. To show (c), suppose $z < z'$ and $\phi$ is strictly decreasing. (The strictly increasing case is similar.) Then $\phi(z') < \phi(z)$ and hence

$$\Delta_{\phi(x)}(\phi(z), \phi(z')) = \{i | \phi(z') < \phi(x_i) < \phi(z)\} = \{i | z < x_i < z'\} = \Delta_x(z, z').$$
Finally d) is true because \( |\Delta_x(y_i, y_j)| = |\{l|y_i < x_l < y_j, l = 1, \ldots, n\}| \leq j - i - 1. \)

**Remark.** The definition and results above can be applied to random vectors \( S = (X_1, \ldots, X_n) \) as well. In that \( \delta_S(z, z') \) is random.

**Loss function for distributions**

We define a degree of separation for distributions which corresponds to the notion of “degree of separation” defined for data vectors to measure separation between data points.

**Definition 5.2:** Suppose \( X \) has a distribution function \( F \). Let

\[
\delta_F(z', z) = \delta_F(z, z') = \lim_{u \to z^-} F(u) - F(z') = P(z' < X < z), \quad z > z',
\]

and \( \delta_F(z, z) = 0, \ z \in \mathbb{R} \). We also denote this by \( \delta_X \) whenever a random variable \( X \) with distribution \( F \) is specified. We call \( \delta_X \) the “degree of separation” or the “probability loss function” associated with \( X \).

The following lemma is a straightforward consequence of the definition.

**Lemma 5.2:** Suppose \( x = (x_1, \ldots, x_n) \) is a data vector with the empirical distribution \( F_n \). Then

\[
\delta_{F_n}(z, z') = \delta_x(z, z'), \quad z, z' \in \mathbb{R}.
\]

This lemma implies that to prove a result about the degree of separation of data vectors, it suffices to show the result for the degree of separation of random variables.

**Theorem 5.1:** Let \( X, Y \) be random variables and \( F_X, F_Y \), their corresponding distribution functions.

a) Assume \( Y = \phi(X) \), for a strictly increasing or decreasing function \( \phi : \mathbb{R} \to \mathbb{R} \). Then \( \delta_{F_X}(z, z') = \delta_{F_Y}(\phi(z), \phi(z')) \), \( z < z' \in \mathbb{R} \).

b) \( \delta_{F_X}(z, z') \leq \delta_{F_X}(z, z'') \), \( z \leq z' \leq z'' \).

c) \( \delta_{F_X}(z_1, z_3) \leq \delta_{F_X}(z_1, z_2) + \delta_{F_X}(z_2, z_3) + P(X = z_2) \).

d) Suppose, \( p \in [0, 1] \). Then \( \delta_{F_X}(ql_{F_X}(p), rq_{F_X}(p)) = 0 \).

e) Suppose, \( p_1 < p_2 \in [0, 1] \). Then \( \delta_{F_X}(ql_{F_X}(p_1), rq_{F_X}(p_2)) \leq p_2 - p_1 \).
**Remark.** We may restate Part (c), for data vectors: Suppose $x$ has length $n$ and $z_2$ is of multiplicity $m$, (which can be zero). Then the inequality in (c) is equivalent to $\delta_x(z_1, z_3) \leq \delta_x(z_1, z_2) + \delta_x(z_2, z_3) + m/n.$

**Proof**

a) Note that for a strictly increasing function $\phi$, we have

$$P(z < X < z') = P(\phi(z) < \phi(X) < \phi(z')).$$

Now suppose $\phi$ is strictly decreasing. Then $z < z' \Rightarrow \phi(z') < \phi(z)$. Let $Y = \phi(X)$. Then

$$\delta_X(z, z') = P(z < X < z') = P(\phi(z') < \phi(X) < \phi(z)) = \delta_Y(\phi(z), \phi(z')).$$

b) This is trivial.

c) Consider the case $z_1 < z_2 < z_3$. (The other cases are easier to show.) Then

$$\delta_{F_X}(z_1, z_3) = P(z_1 < X < z_3) = P(z_1 < X < z_2) + P(X = z_2) + P(z_2 < X < z_3)$$

$$= \delta_{F_X}(z_1, z_2) + \delta_{F_X}(z_2, z_3) + P(X = z_2).$$

d) This result is a straightforward consequence of Lemma 1.1 b) and c).

e) This result follows from

$$\delta_{F_X}(lq(p_1), rq(p_2)) = P(lq(p_1) < X < rq(p_2))$$

$$= P(X < rq(p_2)) - P(X \leq lq(p_1)) \leq p_2 - p_1.$$

The last inequality being a result of Lemma 1.1 a) and d).

**Remark:** (e),(b) immediately imply

$$\delta_{F_X}(lq_{F_X}(p_1), lq_{F_X}(p_2)) \leq p_2 - p_1,$$

and

$$\delta_{F_X}(rq_{F_X}(p_1), lq_{F_X}(p_2)) \leq p_2 - p_1.$$

**Remark.** We call part c) of the above theorem the pseudo-triangle inequality.
6 Data coarsening and quantile approximation algorithm

This section introduces an algorithm to approximate quantiles in very large data vectors. As we demonstrated in the previous section the median of medians algorithm is not necessarily a good approximation to the exact median of a data vector even if we have a large number of partitions and large length of the partitions. The algorithm is based on the idea of “data coarsening” which we will discuss shortly. The proposed algorithm can give us approximations to the exact quantile of known precisions in terms of degree of separation. After stating the algorithm, we prove some theorems that give us the precision of the algorithm. The results hold for partitions of non-equal length.

Definition 6.1: Suppose a data vector $x$ of length $n = n_1n_2$ is given, $n_1, n_2 > 1 \in \mathbb{N}$. Also let $\text{sort}(x) = y = (y_1, \cdots, y_n)$. Then the $n_2$–coarsening of $x$, $C_{n_2}(x)$ is defined to be $(y_{n_2}, y_{2n_2}, \cdots, y_{(n_1-1)n_2})$. Note that $C_{n_2}(x)$ has length $n_1 - 1$. Let $p_i = i/n_1, i = 1, 2, \cdots, (n_1 - 1)$. Then $C_{n_2}(x) = (l_{q_x}(p_1), \cdots, l_{q_x}(p_{n_1-1}))$.

We can immediately generalize the coarsening operator. Suppose

$$\text{sort}(x) = (y_1, \cdots, y_n),$$

and $n_2 < n$ is given. Then by The Quotient–Remainder Theorem from elementary number theory, there exist $n_1 \in \mathbb{N} \cup \{0\}$ and $r < n_2$ such that $n = n_1n_2 + r$. Define $C_{n_2}(x) = (y_{n_2}, \cdots, y_{n_2(n_1-1)})$. The expression is similar to before. However, there are $n_2 + r$ elements after $y_{n_2(n_1-1)}$ in the sorted vector $y$. In this sense this coarsening is not fully symmetric. We show that if $n_2$ is small compared to $n$ this lack of symmetry has a small effect on the approximation of quantiles.

Suppose $x$ is a data vector of length $n = \sum_{i=1}^{m} l_i$. We introduce the coarsening algorithm to find approximations to the large data vectors.

$d$-Coarsening quantiles algorithm:

1. Partition $x$ into vectors of length $l_1, \cdots, l_m$. (Or use pre-existing partitions, e.g. partitions of data saved in various files on the hard disk of a computer.)

$$x^1 = (x_1, \cdots, x_{l_1}), x^2 = (x_{l_1+1}, \cdots, x_{l_1+l_2}), \cdots, x^m = (x_{\sum_{j=1}^{m-1} l_j+1}, \cdots, x_n)$$
2. Sort each $x_l$, $l = 1, 2, \ldots, m$ and let $y_l = \text{sort}(x_l)$, $l = 1, \ldots, m$:

$$y^1 = (y^1_1, \ldots, y^1_l), \ldots, y^m = (y^m_1, \ldots, y^m_l).$$

3. $d$-Coarsen every vector:

$$(y^1_d, \ldots, y^1_{(c_1-1)d}), \ldots, (y^m_d, \ldots, y^m_{(m-1)d}),$$

and for simplicity drop $d$ and use the notation $w^i_j = y^i_{jd}$.

$$w^1 = (w^1_1, \ldots, w^1_{(c_1-1)}), \ldots, w^m = (w^m_1, \ldots, w^m_{(m-1)}).$$

4. Stack all the above vectors into a single vector and call it $w$. Find $rq_w(p)$ (or $lq_w(p)$) and call it $\mu$. Then $\mu$ is our approximation to $rq_x(p)$ (or $lq_x(p)$).

**Theorem 6.1:** Suppose $x$ is of length $n = \sum_{i=1}^m l_i$, $m \geq 2$ and $l_i = c_id$. Let $C = \sum_{i=1}^m c_i$. Apply the coarsening algorithm to $x$ and find $\mu$ to approximate $rq_x(p)$ (or $lq_x(p)$). Then $\mu$ is a (left and right) quantile in the interval

$$[p - \epsilon, p + \epsilon],$$

where $\epsilon = \frac{m+1}{c-m}$. In other words $\delta_x(\mu, rq_x(p)) \leq \epsilon$ and $\delta_x(\mu, lq_x(p)) \leq \epsilon$. When $l_i = cd$, $i = 1, \ldots, m$, $\epsilon = \frac{m+1}{m-1} \leq \frac{3}{c-1}$.

We need an elementary lemma in the proof of this theorem.

**Lemma 6.1:** (Two interval distance lemma)

Suppose two intervals $I = [a, b]$ and $J = [c, d]$ subsets of $\mathbb{R}$ are given. Then

$$\sup\{|p - q|, p \in I, q \in J\} = \max\{|a - d|, |b - c|\}.$$ 

**Proof** $\sup\{|p - q|, p \in I, q \in J\} \geq \max\{|a - d|, |b - c|\}$ is trivial because $a, b \in I$ and $c, d \in J$.

To show the converse note that $|p - q| = p - q$ or $q - p$, $p \in I, q \in J$. But

$$p - q \leq b - c,$$
and
\[ q - p \leq d - a. \]
Hence
\[ |p - q| \leq \max\{b - c, d - a\} \leq \max\{|b - c|, |a - d|\}. \]
This completes the proof.

**Proof** of Theorem 6.1

Let \( n' = \sum_{i=1}^{m} (c_i - 1) = \sum_{i=1}^{m} c_i - m = C - m \) and \( M_C = \{(i, j) | i = 1, 2, \ldots, m, j = 1, \ldots, c_i - 1\} \), the index set of \( w \). Also let \( c = \max\{c_1, \ldots, c_m\} \).

Suppose, \( \frac{h - 1}{m} \leq p < \frac{h}{m}, \ h = 1, \ldots, n' \). Then since \( \mu = rq_w(p) \), there are disjoint subsets of \( M_C \), \( K \) and \( K' \) such that \( |K| = h, |K'| = n' - h, \mu \geq w^i_j, (i, j) \in K \) and \( \mu \leq w^i_j, (i, j) \in K' \). (This is because if we let \( v = sort(w) \), \( rq_w(p) = v_h \) since \( [n'p] = h - 1 \).) \( K, K' \) are not necessarily unique because of possible repetitions among the \( w^i_t \). Hence we impose another condition on \( K \) and \( K' \). If \( (i, t) \in K \) then \( (i, u) \notin K', \ u < t \). It is always possible to arrange for this condition. For suppose, \( (i, t) \in K \) and \( (i, u) \in K', \ u < t \). Then \( \mu \geq w^i_t \) and \( \mu \leq w^i_u \), hence \( w^i_t \leq w^i_u \). But since \( u < t \) we have \( w^i_t \leq w^i_u \) by the definition of \( w^i \). We conclude that \( w^i_t = w^i_u \). Now we can simply exchange \( (i, t) \) and \( (i, u) \) between \( K \) and \( K' \). If we continue this procedure after finite number of steps we will get \( K \) and \( K' \) with the desired property.

Now define

- \[ K_1 = \{(i, 1) | (i, 1) \in K\}, \]
with \( |K_1| = k_1 \) and
\[ I_1 = \{(i, j) | j \leq d, (i, 1) \in K\}, \]
Then \( |I_1| = k_1d \). Also note that if \( (i, j) \in I_1 \), \( \mu \geq w^i_j \geq y^i_j \).

- Let
\[ K_2 = \{(i, 2) | (i, 2) \in K\}, \]
with \( |K_2| = k_2 \) and
\[ I_2 = \{(i, j) | d < j \leq 2d, (i, 2) \in K\}. \]
Then \( |I_2| = k_2d \). Also note that if \( (i, j) \in I_2 \), \( \mu \geq w^i_j \geq y^i_j \).
Let 

\[ K_t = \{(i, t) | (i, t) \in K\}, \]

with \( |K_t| = k_t \) and

\[ I_t = \{(i, j) | (t - 1)d < j \leq td, (i, t) \in K\}. \]

Then \( |I_t| = k_t d \). Also note that if \((i, j) \in I_t\), \( \mu \geq w^i_t \geq y^j_i \).

Let 

\[ K_0 = \{(i, 0) | (i, 0) \in K\}, \]

with \( |K_0| = k_0 \) and 

\[ I_0 = \{(i, j) | d < j \leq 2d, (i, 2) \in K^\prime\}. \]

Then \( |I_0| = k_0 d \). Also note that if \((i, j) \in I_0\), \( \mu \leq w^i_0 \leq y^j_i \).

Note that \( K = \bigcup_{t=1}^{c} K_t \), \( |K| = k_1 + \cdots + k_{c-1} \). Since the \( K_t \) are disjoint the \( I_t \) are also disjoint. Let \( I = \bigcup_{t=1}^{c} I_t \) then \( |I| = d(k_1 + \cdots + k_{c-1}) = d|K| \). Also note that \((i, j) \in I \Rightarrow \mu \geq y^j_i \).

Similarly define,

- \( K_1' = \{(i, 1) | (i, 1) \in K'\}, |K_1'| = k_1' \),
  and
  
  \[ I_1' = \{(i, j) | d < j \leq 2d, (i, 1) \in K'\}. \]
  Then \( |I_1'| = k_1' d \). Also note that if \((i, j) \in I_1', \mu \leq w^i_1 \leq y^j_i \).

- \( K_2' = \{(i, 2) | (i, 2) \in K'\}, |K_2'| = k_2' \),
  and
  
  \[ I_2' = \{(i, j) | 2d < j \leq 3d, (i, 2) \in K'\}. \]
  Then \( |I_2'| = k_2' d \). Also note that if \((i, j) \in I_2', \mu \leq w^i_2 \leq y^j_i \).
Let
\[ K'_t = \{(i, t) | (i, t) \in K'\}, \quad |K'_t| = k't, \]
and
\[ I'_t = \{(i, j) | td < j \leq (t + 1)d, (i, t) \in K'\}. \]
Then \(|I'_t| = k'_t d\). Also note that if \((i, j) \in I'_t\) then \(\mu \leq w^i_t \leq y^j_i\).

\[ K'_{c-1} = \{(i, (c-1)) | (i, c-1) \in K'\}, \quad |K'_{c-1}| = k'_{c-1}, \]
and
\[ I'_{c-1} = \{(i, j) | j > (c-1)d, (i, c-1) \in K'\}. \]
Then \(|I'_{c-1}| = k'_{c-1} d\). Also note that if \((i, j) \in I'_{c-1}\) then \(\mu \leq w^i_{(c-1)} \leq y^j_i\).

Then \(|I| = |K|d\) and \(|I'| = |K'|d\). We claim that \(I \cap I' = \emptyset\). To see this note that because of how the second components in \(I_t\) and \(I'_t\) are defined, it is only possible that \(I_{t+1} = \{(i, j) | td < j \leq (t + 1)d, (i, t + 1) \in K\}\) and \(I'_t = \{(i, j) | td < j \leq (t + 1)d, (i, t) \in K'\}\) intersect for some \(t = 1, \ldots, c - 2\). But if they intersect then there exist \(i, t\) such that \((i, t + 1) \in K\) and \((i, t) \in K'\) which is against our assumption regarding \(K\) and \(K'\). Hence by Lemma 4.3 \(\mu\) is a quantile between
\[
\left[ \frac{|K|d}{n}, \frac{n - |K'|d}{n} \right] = \left[ \frac{hd}{\sum_{i=1}^{m} c_i d}, \frac{n - (n' - h)d}{\sum_{i=1}^{m} c_i d} \right] = \left[ \frac{h}{C'}, \frac{m + h}{C} \right].
\]
But we know that
\[
p \in \left[ \frac{h - 1}{C - m}, \frac{h}{C - m} \right].
\]
We are dealing with two interval in one of them \(\mu\) is a quantile and the other contains \(p\).

We showed in Lemma 6.1 if two intervals \([a, b]\) and \([c, d]\) are given, the sup distance between two elements of the two intervals is
\[
\max\{|a - d|, |b - c|\}.
\]
Applying this to the above two intervals we get,
\[
\max\{\left\lfloor \frac{m + h}{C} - \frac{h - 1}{C - m} \right\rfloor, \left\lfloor \frac{h - 1}{C - m} - \frac{h}{C} \right\rfloor\},
\]
which is equal to,
\[ \max \left\{ \left| \frac{mC - m^2 - hm + C}{C(C - m)} \right|, \left| \frac{C - hm}{C(C - m)} \right| \right\}. \]

But \( m^2 + hm \leq m^2 + (C - m)m = mC \). Hence
\[ \left| \frac{mC - m^2 - hm + C}{C(C - m)} \right| = \frac{mC - m^2 - hm + C}{C(C - m)} \leq \frac{mC + C}{C(C - m)} = \frac{m + 1}{C - m}. \]

Also
\[ \left| \frac{C - hm}{C(C - m)} \right| \leq \frac{C + mC}{C(C - m)} \leq \frac{m + 1}{C - m}. \]

Hence the max is smaller than \( \epsilon = \frac{m+1}{C-m} \) and we conclude that \( \mu \) is a quantile for \( p' \) which is at most as far as \( \epsilon \) to \( p \).

The case \( l_i = cd \) is easily obtained by replacing \( C = mc \) and noting that \( \frac{m+1}{m-1} \leq 3, \ m \geq 2. \)

In most applications, usually the data partitions are not divisible by \( d \). For example the data might be stored in files of different length with common factors. Another situation involves a very large file that is needed to be read in successive stages because of memory limitations. Suppose that we need a precision \( \epsilon \) (in terms of degree of separation) and based on that we find an appropriate \( c \) and \( m \). Note that \( n \) might not be divisible by \( mc \).

First we prove two lemmas. These lemmas show what happens to the quantiles if we throw away a small portion of the data vector or add some more data to it. The first lemma is for a situation that we have thrown away or ignored a small part of the data. The second lemma is for a situation that a small part of the data are contaminated or includes outliers. In both cases, we show how the quantiles computed in the “imperfect” vectors correspond to the quantiles of the original vector. In both case \( x \) stands for the imperfect vector and \( w \) is the complete/clean data.

**Lemma 6.2:** (Missing data quantile approximation lemma)
Suppose \( x = (x_1, \ldots, x_n) \), \( \text{sort}(x) = (y_1, \ldots, y_n) \) and \( y' = lq_x(p), p \in [0, 1] \). Consider a vector \( x' \) of length \( n' \) and let \( w = \text{stack}(x, x') \). Then \( y' = lq_w(p') \), where \( p' \in [p - \epsilon, p + \epsilon] \) and \( \epsilon = \frac{n'}{n+n'}. \). Similarly if \( y' = rq_x(p) \) and \( p \in [0, 1] \), \( y' = rq_w(p') \), where \( p' \in [p - \epsilon, p + \epsilon] \) and \( \epsilon = \frac{n'}{n+n'}. \)
Proof We prove the result for \( lq_x \) only and a similar argument works for \( rq_x \).

Let \( z = \text{sort}(w) \) then \( lq_z = lq_w \). For \( p = 1 \) the result is easy to see. Otherwise, \( \frac{1}{n} \leq p < \frac{i+1}{n} \) for some \( i = 0, \ldots, n-1 \). But then \( y' = lq_z(p) = y_i \). In the new vector \( z \) since we have added \( n^* \) elements \( y' = z_j \) for some \( j, i \leq j < i + n^* \). Hence \( y' = lq_z(\frac{j}{n+n^*}) \). From \( np - 1 < i \leq np \), we conclude

\[
\frac{np - 1}{n + n^*} < \frac{i}{n + n^*} < \frac{j}{n + n^*} < \frac{i + n^*}{n + n^*} < \frac{np + n^*}{n + n^*}.
\]

Hence,

\[
\frac{n^*(1-p) - 1}{n + n^*} < \frac{j}{n + n^*} - p < \frac{n^*(1-p)}{n + n^*} \Rightarrow
\]

\[
|\frac{j}{n + n^*} - p| < \max\{|\frac{n^*(1-p) - 1}{n + n^*}|, |\frac{n^*(1-p)}{n + n^*}|\}.
\]

But \( |\frac{n^*(1-p)}{n + n^*}| \leq \frac{n^*}{n+n^*} \) and \( |\frac{n^*(1-p) - 1}{n + n^*}| \leq \max\{\frac{n^*-1}{n+n^*}, \frac{1}{n+n^*}\} \) since \( p \) ranges in \([0, 1] \). We conclude that that

\[
|\frac{j}{n + n^*} - p| < \frac{n^*}{n + n^*}.
\]

Lemma 6.3: (Contaminated data quantile approximation lemma)

Suppose \( x = (x_1, \ldots, x_n) \), \( \text{sort}(x) = (y_1, \ldots, y_n) \) and \( y' = lq_x(p), p \in [0, 1] \). Consider the vector \( w = (x_1, x_2, \ldots, x_{n-n^*}) \) then \( y' = lq_w(p') \), where \( p' \in [p - \epsilon, p + \epsilon] \) and \( \epsilon = \frac{n^*}{n-n^*} \).

Similarly if \( y' = rq_x(p) \) and \( p \in [0, 1] \), \( y' = rq_w(p') \), where \( p' \in [p - \epsilon, p + \epsilon] \) and \( \epsilon = \frac{n^*}{n-n^*} \).

Proof We only show the case for \( lq_x \) and a similar argument works for \( rq_x \).

Let \( z = \text{sort}(w) \). Then \( lq_z = lq_w \). If \( p = 1 \) the result is easy to see. Otherwise, \( \frac{1}{n} \leq p < \frac{i+1}{n} \) for some \( i = 0, \ldots, n-1 \). But then \( y' = lq_z(p) = y_i \). In the new vector \( z \) since we have removed \( n^* \) elements \( y' = z_j \) for some \( j, i - n^* \leq j \leq i \). Hence \( y' = lq_z(\frac{j}{n-n^*}) \). From \( np - 1 < i \leq np \), we conclude \( np - n^* < j \leq np \Rightarrow np - n^* \leq j \leq np \). Hence

\[
\frac{-n^* + n^* p}{n-n^*} \leq \frac{j}{n-n^*} - p \leq \frac{n^* p}{n-n^*} \Rightarrow
\]

\[
|\frac{j}{n-n^*} - p| \leq \frac{n^*}{n-n^*}.
\]
In the case that the partitions are not divisible by $d$, we can use the same algorithm with generalized coarsening. The error will increase obviously and the next two lemmas say by how much.

**Lemma 6.4:** Suppose $x$ has length $n = lm + r$, $0 \leq r < l$ and $m = cd$. To find $lq_x(p)$, apply the algorithm in the previous theorems to a sub-vector of $x$ of length $lm$. Then the obtained quantile is a quantile for a number in $[p - \epsilon, p + \epsilon]$, where

$$
\epsilon = \frac{m+1}{m-1} \frac{1}{c-1} + \frac{r}{lm+r}.
$$

**Proof** The result is a straightforward consequence of the Theorem 6.1 and the Lemma 6.2.

**Lemma 6.5:** Suppose $x$ has length $n = \sum_{i=1}^{m} l_i$ and $l_i = cd_i + r_i$, $r_i < d$. Let $R = \sum_{i=1}^{m} r_i$. Then apply the algorithm above to $x$ to find $lq_x(p)$, using the generalized coarsening. The obtained quantile is a quantile for a number in $[p - \epsilon, p + \epsilon]$ where

$$
\epsilon = m \frac{1}{C-m} + \frac{R}{R+Cd}.
$$

**Proof** Let $l'_i = cd_i$. Consider $x'$ a sub-vector of $x$ consisting of

$$
(y_1^1, \cdots, y_{l'_1}^1), (y_1^2, \cdots, y_{l'_2}^2), \cdots, (y_1^m, \cdots, y_{l'_m}^m).
$$

Then $x'$ has length $\sum_{i=1}^{m} l'_i$. By Lemma 6.2 $p$-th quantile found by the algorithm is a quantile in $[p - \epsilon_1, p + \epsilon_1]$, $\epsilon_1 = \frac{m+1}{C-m}$ for $x'$. $x$ has $R = \sum_{i=1}^{m} r_i$ elements more than $x'$. Hence the obtained quantile is a quantile for $x$ for a number in $[p - \epsilon, p + \epsilon]$, $\epsilon = \epsilon_1 + \frac{R}{R+Cd}$.

### 7 Applications and computations

Suppose a data vector $x$ has length $n$. To find the quantiles of this vector, we only need to sort $x$. Since then for any $p \in (0,1)$, we can find the first $h$ such that $p \geq h/n$. Note that

$$
\text{sort}(x) = (lq_x(1/n), lq_x(2/n), \cdots, lq_x(1)) = (rq_x(0), rq_x(1/n), \cdots, rq_x\left(\frac{n-1}{n}\right)).
$$
We only focus on left quantiles here. Similar arguments hold for the right quantile. Obviously, the longer the vector $x$, the finer the resulting quantiles are. Now imagine that we are given a very long data vector which cannot even be loaded on the computer memory. Firstly, sorting this data is a challenge and secondly, reporting the whole sorted vector is not feasible. Assume that we are given the sorted data vector so that we do not need to sort it. What would be an appropriate summary to report as the quantiles? As we noted also the sorted vector itself although appropriate, maybe of such length as to make further computation and file transfer impossible. The natural alternative would be to coarsen the data vector and report the resulting coarsened vector. To be more precise, suppose, $\text{length}(x) = n = n_1n_2$ and $y = \text{sort}(x) = (y_1, \cdots, y_n)$. Then we can report

$$y' = C_{n_2}(y) = (y_{n_2}, \cdots, y_{(n-1)n_2}).$$

This corresponds to

$$(\text{lq}_{y'}(1/n_2), \cdots, \text{lq}_{y'}(1)).$$

How much will be lost by this coarsening? Suppose, we require the left quantile corresponding to $(h-1)/n < p \leq h/n$, $h = 1, \cdots, n$. Then $x$ would give us $y_h$. But since $(h-1)/n < p \leq h/n$

$$np < h \leq np + 1.$$  

Also suppose for some $h' = 1, \cdots, n_1$,

$$(h' - 1)/(n_1 - 1) < p \leq (h')/(n_1 - 1) \Rightarrow (h' - 1) < p(n_1 - 1) \leq h'$$

$$\Rightarrow (n_1 - 1)p \leq h' < p(n_1 - 1) + 1.$$  

Then

$$(h-1)(n_1 - 1)/n < h' < h(n_1 - 1)/n + 1,$$

and

$$(h-1)(n_1 - 1)n_2/n < h'n_2 < h(n_1 - 1)n_2/n + n_2. \quad (1)$$

Using the coarsened vector, we would report $y_{h'(n_2)}$ as the approximated quantile for $p$. The degree of separation between this element and the exact quantile using Equation (1) is less than or equal to

$$\max\left\{\frac{|h - (h-1)(n_1 - 1)n_2/n|}{n}, \frac{|h(n_1 - 1)n_2/n + n_2 - h|}{n}\right\}.$$
This equals
\[ \max\{|-\frac{hn_2 - n_1 n_2 + n_2}{n^2}|, \frac{-hn_2 + n n_2}{n^2}|\}. \]

But
\[ \left| -\frac{hn_2 - n_1 n_2 + n_2}{n^2} \right| = \frac{n_2(n_1 + n - 1)}{n^2} < \frac{n_2(n_1 + n)}{n^2} = \frac{1}{n} + \frac{n_2}{n}, \]
and
\[ \left| -\frac{hn_2 + n n_2}{n^2} \right| < \frac{n_2}{n}. \]

Hence the degree of separation is less than $1/n + 1/n_1$. We have proved the following lemma.

**Lemma 7.1**: Suppose $x$ is a data vector of the length $n = n_1 n_2$ and $y = \text{sort}(x), \ y' = C_{n_2}(y)$. Then if we use the quantiles of $y'$ in place of $x$, the accuracy lost in terms of the probability loss of $x$ ($\delta_x$) is less than $1/n + 1/n_1$.

The algorithm proposes that instead of sorting the whole vector and then coarsening it, coarsen partitions of the data. The accuracy of the quantiles obtained in this way is given in the theorems of the previous section. This allows us to load the data into the memory in stages and avoid program failure due to the length of the data vector. We are also interested in the performance of the method in terms of speed, and do a simulation study using the “R” package (a well-known software for statistical analysis) to assess this. In order to see theoretical results regarding the complexity of the special case of the algorithm for equal partitions see [Alsabti et al. 1997](#). For the simulation study, we create a vector, $x$, of length $n = 10^7$. We apply the algorithm for $m = 1000, c = 20, d = 500$. We create this vector in a loop of length 1000. During each iteration of the loop, we generate a random mean for a normal distribution by first sampling from $N(0, 100)$. Then we sample 10,000 points from a normal distribution with this mean and standard deviation 1. We compare two scenarios:

1. Start by a NULL vector $x$ and in each iteration add the full generated vector of length 10000 to $x$. After the loop has completed its run, sort the data vector which now has length $10^7$ by the command sort in R and use this to find the quantiles.
2. Start with a NULL vector $w$. During each iteration after generating the random vector, $d$-coarsen the data by $d = 500$. (Hence $m = 1000, c = 20$.) In order to do that computing, first apply the sort command to the data and then simply $d$-coarsen the resulting sorted vector. During each iteration, add the coarsened vector to $w$. After all the iterations, sort $w$ and use it to approximate quantiles.

**Remark.** The first part corresponds to the straightforward quantiles’ calculation and the second corresponds to our algorithm. Note that in the real examples instead of the loop, we could have a list of 1000 data files and still this example serves as a way of comparing the straightforward method and our algorithm.

**Remark.** Note that if we wanted to create an even longer vector say of length $10^{10}$ then the first method would not even complete because the computer would run out of memory in saving the whole vector $x$.

**Remark.** The final stage of the algorithm can use the fact that $w$ is built of ordered vectors to make the algorithm even faster. We will leave that a problem to be investigated in the future.

We have repeated the same procedure for $n = 2 \times 10^7, m = 1000, d = 500$ and $n = 10^8, m = 1000, d = 500$. The results of the simulation are given in Table 2 in which “DOS” stands for the degree of separation between the exact median and the approximated median. The “DOS bound” bounds the degree of separation obtained by the theorems in the previous section. For $n = 10^7, n = 2 \times 10^7$ significant time accrue by using the algorithm. For a vector of length $10^8$, R crashed when we tried to sort the original vector and only the algorithm could provide results. For all cases the exact and approximated quantiles are close. In fact the dos is significantly smaller than the dos bound. This is because this is a “worst-case” bound. The exact and approximated quantiles for $n = 10^7$ are plotted in Figure 1.
Fig. 1: Comparing the approximated quantiles to the exact quantiles $N = 10^7$. The circles are the exact quantiles and the + are the corresponding approximated quantiles.

| Length       | $n = 10^7$ | $n = 2 \times 10^7$ | $n = 10^8$ |
|--------------|------------|---------------------|------------|
| Exact median | 1.847120   | 1.857168            | NA         |
| Algorithm median | 1.866882   | 1.846463            | 1.846027   |
| DOS          | 0.00012    | $-6.475 \times 10^{-5}$ | NA         |
| DOS bound    | 0.05268421 | 0.02566667          | 0.005030151|
| Time for exact median | 186 sec   | 461 s               | NA         |
| Time for the algorithm | 6 sec     | 18 s                | 98 s       |

Tab. 2: Comparing the exact method with the proposed algorithm in R run on a laptop with 512 MB memory and a processor 1500 MHZ, $m = 1000, d = 500$. “DOS” stands for degree of separation in the original vector. “DOS bound” is the theoretical degree of separation obtained by Theorem 6.1.

Next, we apply the algorithm on a real dataset. The dataset includes the daily maximum temperature for 25 stations over Alberta during the period 1940–2004. We focus on the 95th percentile. The results are given in Table 3. The
Fig. 2: Comparing the approximated quantiles to the exact quantiles for $MT$ (daily maximum temperature) over 25 stations in Alberta 1940–2004. The circles are the exact quantiles and the + the approximated quantiles.
algorithm finds the percentile more quickly but the time difference is not as large as the simulation. This is because most of the time of the algorithm and the exact computation is spent on reading the files from the hard drive. The dos bound is about 0.01 (on the 0–1 probability scale). The true degree of separation is about 0.001. The estimated quantiles and the exact quantiles are plotted in Figure 2. Notice that the exact and approximated values match except at the very beginning (very close to zero) and end (when it is close to 1), where we see that the circles (corresponding to exact quantiles) and the +s (corresponding to the approximated quantiles) do not completely match. This difference is at most 0.01 in terms of dos in any case.

|                      |            |
|----------------------|------------|
| Exact 95th percentile| 27°C       |
| Algorithm 95th percentile| 26.7°C   |
| DOS                  | 0.001278726|
| DOS bound            | 0.01052189 |
| time for exact median| 8 min 6 sec|
| time for the algorithm| 7 min 29 sec|

Tab. 3: Comparing the exact method with the proposed algorithm in R (run on a laptop with 512 MB memory and processor 1500 MHZ) to compute the quantiles of $MT$ (daily maximum temperature) over 25 stations with data from 1940 to 2004.

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