Brownian Intersections, Cover Times and Thick Points via Trees

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Abstract

There is a close connection between intersections of Brownian motion paths and percolation on trees. Recently, ideas from probability on trees were an important component of the multifractal analysis of Brownian occupation measure, in joint work with A. Dembo, J. Rosen and O. Zeitouni. As a consequence, we proved two conjectures about simple random walk in two dimensions: The first, due to Erdős and Taylor (1960), involves the number of visits to the most visited lattice site in the first $n$ steps of the walk. The second, due to Aldous (1989), concerns the number of steps it takes a simple random walk to cover all points of the $n$ by $n$ lattice torus. The goal of the lecture is to relate how methods from probability on trees can be applied to random walks and Brownian motion in Euclidean space.

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1. Introduction

In [18], the author showed that long-range intersection probabilities for random walks, Brownian motion paths and Wiener sausages in Euclidean space, can be estimated up to constant factors by survival probabilities of percolation processes on trees.

More recently, several long-standing problems involving cover times and "thick points" for random walks in two dimensions were solved in joint works [9, 10] of A. Dembo, J. Rosen, O. Zeitouni and the author. These solutions were motivated by powerful analogies with corresponding problems on trees, but these analogies were not discussed explicitly in the research papers cited. The goal of the present note is to describe the tree problems and solutions, that correspond to the problems studied in [9, 10].

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The cover time for a random walk on a finite graph is the number of steps it takes the random walk to visit all vertices. The cover time has been studied intensively by probabilists, combinatorialists, statistical physicists and computer scientists, with a variety of motivations; see, e.g., [7, 16, 2, 8, 17]. The problem of determining the expected cover time $T_n$ for the $n$ by $n$ lattice torus $\mathbb{Z}^2_n$, was posed by Wilf [22] and Aldous [1]. In [9] we proved the following conjecture of Aldous [1].

**Theorem 1** If $T_n$ denotes the time it takes for the simple random walk in $\mathbb{Z}^2_n$ to completely cover $\mathbb{Z}^2_n$, then

$$\lim_{n \to \infty} \frac{T_n}{(n \log n)^2} = \frac{4}{\pi}$$ in probability.  \hspace{1cm} (1.1)

The first step toward proving Theorem 1 was to find a sufficiently robust proof for the asymptotics of the cover time of finite $b$-ary trees. These asymptotics were originally determined by Aldous in [4], but his elegant recursive method was quite sensitive and did not adapt to the approximate tree structure that can be found in Euclidean space. Cover times on trees are discussed in the next section.

Turning to a different but related topic, Erdős and Taylor (1960) posed a problem about simple random walks in $\mathbb{Z}^2$: How many times does the walk revisit the most frequently visited site in the first $n$ steps?

**Theorem 2** (10) Denote by $T_n(x)$ the number of visits of planar simple random walk to $x \in \mathbb{Z}^2$ by time $n$, and let $T^*_n = \max_{x \in \mathbb{Z}^2} T_n(x)$. Then

$$\lim_{n \to \infty} \frac{T^*_n}{(\log n)^2} = \frac{1}{\pi}$$ a.s.

(1.2)

This was conjectured by Erdős and Taylor [11, (3.11)]. After D. Aldous heard one of us describing this result, he pointed us to his cover time conjecture, and this eventually led to Theorem 1. Although the proofs of that theorem and of Theorem 2 differ in important technical points, they follow the same basic pattern:

(i) Formulate a suitable tree-analog and find a “robust” proof.
(ii) Establish a Brownian version using excursion counts.
(iii) Deduce the lattice result via strong approximation a-la [12].

2. Cover times for trees

Let $\Gamma_k$ denote the balanced $b$-ary tree of height $k$, which has $n_k = (b^{k+1} - 1)/(b - 1)$ vertices, and $n_k - 1$ edges.

**Theorem 3** (Aldous [1]) Denote by $C_k$ the time it takes for simple random walk in $\Gamma_k$, started at the root, to cover $\Gamma_k$. Then

$$\lim_{k \to \infty} \frac{E C_k}{n_k k^2} = 2 \log(b).$$

(2.1)
Remark The expected hitting time from one vertex to another is bounded by the
commute time, which equals the effective resistance times twice the number of edges
(see, e.g., [5]). Therefore the expected hitting time between two vertices in \( \Gamma_k \) is at
most \( 4kn_k \). From a general result in [3], it follows that also
\[
\lim_{k \to \infty} \frac{C_k}{n_kk^2} = 2 \log(b) \quad \text{in probability.} \tag{2.2}
\]

Proof of theorem Denote by \( C_k^+ \) the time it takes the walk to cover and return
to the root, and by \( R_k \) the number of returns to the root until time \( C_k^+ \). By
the remark preceding the proof, \( \mathbb{E}C_k^+ \to \mathbb{E}C_k \leq 4kn_k \), so to prove the theorem it suffices
to establish that
\[
\lim_{k \to \infty} \frac{\mathbb{E}C_k^+}{n_kk^2} = 2 \log(b). \tag{2.3}
\]
The expected time to return to the root is the reciprocal of the root's stationary
probability \( b/(2n_k - 2) \), so by Wald's lemma
\[
\mathbb{E}(C_k^+) = \frac{2n_k - 2}{b} \mathbb{E}(R_k). \tag{2.4}
\]
Thus the theorem reduces to showing
\[
\lim_{k \to \infty} \frac{\mathbb{E}R_k}{k^2} = b \log(b). \tag{2.5}
\]

We start by reproducing the straightforward proof of the upper bound. Denote
by \( R_v \) the number of returns to the root of \( \Gamma_k \) until the first visit to \( v \), and observe
that \( R_k \) is the maximum of \( R_v \) over all leaves \( v \) at level \( k \). At each visit to the root,
the chance to hit a specific leaf \( v \) before returning to the root is \( 1/bk \), whence
\[
\mathbb{P}[R_v > rbk^2] \leq (1 - \frac{1}{bk})^{rbk^2} \leq e^{-rk}. \tag{2.6}
\]
Summing over all leaves, we infer that
\[
\mathbb{P}[R_k > rbk^2] \leq \min\{1, bke^{-rk}\}. \tag{2.6}
\]
Integrating over \( r > 0 \),
\[
\mathbb{E}[R_k] \leq bk^2(\log b + 1/k). \tag{2.7}
\]
This yields the upper bound in (2.3). To prove a lower bound, Aldous [4] uses
a delicate recursion, and an embedded branching process argument. Here we will
give the shortest argument we know, which only involves an embedded branching
process. Given \( \lambda < \log b \), our next goal is to show that
\[
\mathbb{P}[R_k > \lambda bk^2] \to 1 \quad \text{as } k \to \infty. \tag{2.8}
\]
Let \( T_\lambda \) be the number of steps until the root is visited \( \lambda bk^2 \) times.

Fix \( r \in (\lambda, \log b) \), and choose \( \ell \) large, depending on \( r \). Let \( v \) be a vertex at
level \( k - (j + 1)\ell \) of \( \Gamma_k \), and suppose that \( w \) is a descendant of \( v \) at level \( k - j\ell \).
Observe that the expected number of visits to $v$ by time $T_\lambda$ is $\lambda(b+1)k^2$, and the expected number of excursions between $v$ and $w$ by time $T_\lambda$ is $\lambda k^2/\ell$.

Say that $w$ is “special” if the number of excursions from $v$ to $w$ by time $T_\lambda$ is at most $r\ell j^2$. Note that vertices close to the root (i.e., at level $k-j\ell$ where $r\ell^2 j^2 > \lambda k^2$) are special with high probability, because $r > \lambda$. If $k > (j+2)\ell$, then every visit to $v$ is equally likely to start an excursion to $w$ as to the ancestor of $v$ at distance $\ell$ from $v$. Thus, if $v$ is special then $w$ is special with probability at least $P[X < r\ell j^2]$, where $X$ has binomial law with parameters $r\ell(j^2 + (j+1)^2)$ and $1/2$. By the central limit theorem, as $j$ grows, $P[X < r\ell j^2] \to P[Z > (2r\ell)^{1/2}]$, where $Z$ is standard normal. Since $r < \log b$, we find that $P(Z > (2r\ell)^{1/2}) > b^{-\ell}$, if $\ell$ is large enough. Therefore, special vertices considered at jumps of $2\ell$ levels (to ensure the required independence) dominate a supercritical branching process; the survival probability tends to 1 as $k \to \infty$, because vertices near the root are almost guaranteed to be special. This establishes (2.8). It follows that $E(R_k) > \lambda b k^2$ for large $k$, and since $\lambda < \log b$ is arbitrary, this completes the proof of (2.5) and the theorem.

Remark The argument above is quite robust: it readily extends to family trees of Galton watson trees with mean offspring $b > 1$. With a little more work, using the notion of quasi-Bernoulli percolation (see [13] or [19]), it can be extended to the first $k$ levels of any tree $\Gamma$ that has growth and branching number both equal to $b > 1$. The most robust argument, the truncated second moment method used in [9], is too technical to include here.

3. From trees to Euclidean space

The following “dictionary” was offered in [18] to illustrate the reduction of certain intersection problems from Euclidean space to trees:

| Problem in Euclidean space | Corresponding problem on trees |
|---------------------------|--------------------------------|
| How many (independent) Brownian paths in $\mathbb{R}^d$ can intersect? | Which branching processes can have an infinite line of descent? |
| What is the probability that several random walk paths, started at random in a cube of side-length $2^k$, will intersect? | What is the probability that a branching process survives for at least $k$ generations? |
| Which sets in $\mathbb{R}^3$ contain double points of Brownian motion? | Which trees percolate at a fixed threshold $p$? |
| What is the Hausdorff dimension of the intersection of a fixed set in $\mathbb{R}^d$ with one or two Brownian paths? | What is the dimension of a percolation cluster on a general tree? |

The Brownian analogs of Theorems 1 and 2, respectively, are given below. Throughout, denote by $D(x, \epsilon)$ the disk of radius $\epsilon$ centered at $x$.

**Theorem 4** ([9]) *For Brownian motion $w_T(\cdot)$ in the two-dimensional torus $\mathbb{T}^2$,*
consider the hitting time of a disk,
\[ T(x, \epsilon) = \inf\{t > 0 \mid X_t \in D(x, \epsilon)\}, \]
and the \( \epsilon \)-covering time,
\[ C_\epsilon = \sup_{x \in \mathbb{T}^2} T(x, \epsilon) \]
which is the amount of time needed for the Wiener sausage of radius \( \epsilon \) to completely cover \( \mathbb{T}^2 \). Then
\[ \lim_{\epsilon \to 0} \frac{C_\epsilon}{(\log \epsilon)^2} = \frac{2}{\pi} \quad \text{a.s.} \quad (3.1) \]

**Theorem 5 ([10])** Denote by \( \mu_w \) the occupation measure for a planar Brownian motion \( w(\cdot) \) run for unit time. Then
\[ \lim_{\epsilon \to 0} \sup_{x \in \mathbb{R}^2} \frac{\mu_w(D(x, \epsilon))}{\epsilon^2 \left(\log \frac{1}{\epsilon}\right)^2} = 2, \quad \text{a.s.} \quad (3.2) \]
(This was conjectured by Perkins and Taylor [20].)

The basic approach used to prove these results, which goes back to Ray, [21], is to control occupation times using excursions between concentric discs. The approximate tree structure that is (implicitly) used arises by considering discs of the same radius \( r \) around different centers and varying \( r \); for fixed centers \( x, y \), and “most” radii \( r \) (on a logarithmic scale) the discs \( D(x, r) \) and \( D(y, r) \) are either well-separated (if \( r << |x - y| \)) or almost coincide (if \( r >> |x - y| \)).

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