The minimum forcing number of perfect matchings in the hypercube

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Abstract Let $M$ be a perfect matching in a graph. A subset $S$ of $M$ is said to be a forcing set of $M$, if $M$ is the only perfect matching in the graph that contains $S$. The minimum size of a forcing set of $M$ is called the forcing number of $M$. Pachter and Kim [Discrete Math. 190 (1998) 287–294] conjectured that the forcing number of every perfect matching in the $n$-dimensional hypercube is at least $2^{n-2}$, for all $n \geq 2$. Riddle [Discrete Math. 245 (2002) 283-292] proved this for even $n$. We show that the conjecture holds for all $n \geq 2$. The proof is based on simple linear algebra.

1 Introduction

Let $M$ be a perfect matching in a graph $G$. A subset $S$ of edges in $M$ is said to be a forcing set of $M$, if $M$ is the only perfect matching in $G$ that contains $S$. The minimum size of a forcing set of $M$ is called the forcing number of $M$.

Forcing sets and forcing numbers of perfect matchings were first studied by Harary et al. [3], and have applications in chemistry [2]. Pachter and Kim [4] found tight upper and lower bounds on the forcing number of perfect matchings in a 2-dimensional square grid. They also conjectured that every perfect matching in the $n$-dimensional hypercube has forcing number at least $2^{n-2}$, for all $n \geq 2$. This was proved by Riddle [5] for even $n$. Adams et al. [1] showed that for all $n \geq 5$ and integers $i$ in the interval $[2^{n-2}, 2^{n-2} + 2^{n-5}]$,
there exists a perfect matching in the $n$-dimensional hypercube with forcing number $i$.

In this note, we complete the proof of Pachter and Kim’s conjecture, and show that it holds for all $n \geq 2$. Our proof is much simpler than the one in [5], and uses only basic linear algebra. We actually prove a slightly stronger statement. If $S$ is a forcing set of a perfect matching $M$ in a graph, then the graph obtained by deleting all endvertices of edges in $S$ has a unique perfect matching $M \setminus S$. Thus an upper bound on the order of an induced subgraph with a unique perfect matching directly gives a lower bound on the forcing number of every perfect matching in the graph. We show that any induced subgraph of the $n$-dimensional hypercube, with a unique perfect matching, has at most $2^{n-1}$ vertices, for $n \geq 2$. This implies the lower bound on the forcing number. The technique used is quite general, and can be applied to any bipartite graph.

2 Hypercube

We describe our technique in general, and then apply it to the specific case of the hypercube. Let $G$ be a bipartite graph, with a bipartition $X, Y$ of the vertex set. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. The weighted bipartite adjacency matrix of $G$ is an $n \times m$ matrix $W$, with $W_{ij} = 0$ if $x_i$ is not adjacent to $y_j$, and $W_{ij} = w$ otherwise. Here, $w$ is an indeterminate used to indicate the presence of an edge in the graph.

Let $H$ be an induced subgraph of $G$ with a unique perfect matching. The rows and columns of $W$ that correspond to vertices in $H$ form a square sub-matrix $W_H$ of $W$. Since $H$ has a unique perfect matching, the determinant of $W_H$ must be $\pm w^{\vert H \vert / 2}$, since only one term in the expansion of the determinant is non-zero. This implies that if we replace each occurrence of $w$ in $W$ by an arbitrary non-zero element from some field $F$, the resulting matrix must have rank at least $\vert H \vert / 2$ over $F$. Thus, if we choose a field $F$, and appropriate non-zero values from the field as the weights of the edges, then twice the rank of the resulting bipartite adjacency matrix is an upper bound on the order of any induced subgraph with a unique perfect matching. We now apply this technique to the hypercube.

The vertex set of the $n$-dimensional hypercube $Q_n$ is $\{0, 1\}^n$, the set of all sequences of length $n$ in which each element is either 0 or 1. Two sequences are adjacent if they differ in exactly one position. Let $E_n$ denote the subset of
these sequences with an even number of 1 elements, and \(O_n\) the subset with an odd number of 1 elements. The hypercube is a bipartite graph with \(E_n; O_n\) the two parts of the bipartition. Construct the weighted bipartite adjacency matrix \(W_n\) of \(Q_n\) with rows indexed by elements of \(E_n\) in lexicographic order, and columns indexed by elements of \(O_n\) in lexicographic order. Then \(W_n\) is a symmetric matrix with a simple recursive structure. \(W_1\) is the 1 \(\times\) 1 matrix with \(w\) as the only entry and

\[
W_{n+1} = \begin{bmatrix}
W_n & wI_{2^{n-1}} \\
wI_{2^{n-1}} & W_n
\end{bmatrix}.
\]

Here, \(I_k\) denotes the \(k \times k\) identity matrix.

We use the recursive structure of \(W_n\) to assign non-zero values from the field \(Z_3\) of integers modulo 3, to the \(w\) elements in \(W_n\), so that the rank of the resulting matrix is \(2^{n-2}\) over \(Z_3\).

**Lemma 1** There exists a nonsingular matrix \(A_n\), obtained by assigning non-zero values from the field \(Z_3\) to the \(w\) elements in \(W_n\), such that \(A_n^{-1}\) can also be obtained by assigning non-zero values to the \(w\) elements in \(W_n\).

**Proof:** We construct the matrix inductively. For \(n = 1\), let \(A_1\) be the 1 \(\times\) 1 matrix containing the entry 1. Clearly, \(A_1^{-1} = A_1\) satisfies the required property. Let

\[
A_{n+1} = \begin{bmatrix}
2A_n & I_{2^{n-1}} \\
I_{2^{n-1}} & A_n^{-1}
\end{bmatrix}.
\]

Then it is easy to verify that

\[
A_{n+1}^{-1} = \begin{bmatrix}
A_n^{-1} & 2I_{2^{n-1}} \\
2I_{2^{n-1}} & 2A_n
\end{bmatrix}.
\]

By induction, and using the recursive structure of \(W_{n+1}\), we conclude that both \(A_{n+1}\) and \(A_{n+1}^{-1}\) can be obtained from \(W_{n+1}\) by assigning non-zero values to the \(w\) elements in \(W_{n+1}\). \(\square\)

**Theorem 1** Any induced subgraph of \(Q_n\) that has a unique perfect matching contains at most \(2^{n-1}\) vertices, for all \(n \geq 2\).
Proof: Let $A_{n-1}$ be the matrix over $\mathbb{Z}_3$ satisfying the properties in Lemma 1. Let

$$B_n = \begin{bmatrix} A_{n-1} & I_{2^{n-2}} \\ I_{2^{n-2}} & A_{n-1} \end{bmatrix}.$$

Now, it is easy to see that $B_n$ has rank $2^{n-2}$ over $\mathbb{Z}_3$, and is obtained from $W_n$ by replacing all occurrences of $w$ in $W_n$ by non-zero values from $\mathbb{Z}_3$. □

Note that for even $n$, we get a simpler proof by considering the field $\mathbb{Z}_2$. In this case, if we replace all occurrences of $w$ in $W_n$ by 1, we get a matrix of rank $2^{n-2}$ over $\mathbb{Z}_2$. However, for odd $n$, this gives a nonsingular matrix which is its own inverse. Riddle’s proof for even $n$ is purely combinatorial. It would be interesting to see if the proof for odd $n$ can also be made purely combinatorial. It would also be interesting to see if this technique can be used to find tight bounds on the forcing numbers of perfect matchings in other bipartite graphs.

References

[1] P. Adams, M. Mahdian, E. S. Mahmoodian, On the forced matching number of bipartite graphs, Discrete Math. 281 (2004) 1–12.

[2] Z. Che, Z. Chen, Forcing on perfect matchings—a survey, MATCH Commun. Math. Comput. Chem. 66 (2011) 93–136.

[3] F. Harary, D. J. Klein, T. P. Živković, Graphical properties of polyhexes: perfect matching vector and forcing, J. Math. Chem. 6 (1991) 295–306.

[4] L. Pachter, P. Kim, Forcing matchings on square grids, Discrete Math. 190 (1998) 287–294.

[5] M. E. Riddle, The minimum forcing number for the torus and hypercube, Discrete Math. 245 (2002) 283–292.