STRUCTURE THEORY OF NATURALLY REDUCTIVE SPACES

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Abstract. The main result of this paper is that every naturally reductive space can be explicitly constructed from the construction in [23]. This gives us a general formula for any naturally reductive space and from this we prove reducibility and isomorphism criteria.

1. Introduction

Naturally reductive spaces are amongst the simplest of Riemannian homogeneous spaces. The ones which are Riemannian symmetric are of course the most well known. All isotropy irreducible spaces can also be considered to be naturally reductive. However, the class of naturally reductive spaces is much broader and contains many other interesting cases. The holonomy bundle of a naturally reductive connection automatically equips the space with a (non-integrable) $G$-structure, where the naturally reductive connection is also a characteristic connection for the $G$-structure. There are many interesting non-integrable $G$-structures one can obtain in this way such as homogeneous nearly Kähler manifolds (cf. [8, 9]), homogeneous nearly parallel $G_2$-manifolds (cf. [12]), cocalibrated $G_2$-manifolds (cf. [13]), Sasakian $g$-symmetric manifolds (see [21, 5, 6]). Similarly for $Sp(n)Sp(1)$-structures there are the homogeneous $3$-Sasakian manifolds (cf. [7]). In [3] a connection with parallel skew torsion is constructed for any $7$-dimensional $3$-Sasakian structure. Also for $Sp(n)Sp(1)$-structures there are interesting naturally reductive examples. One of these is the quaternionic Heisenberg group, which is discussed in [2]. The naturally reductive connection is here also used to find new examples of generalized Killing spinors. More examples of this phenomena are presented in [4]. Naturally reductive spaces have also been used to find new homogeneous Einstein metrics. The simplest examples are the isotropy irreducible spaces, which are necessarily Einstein. D’Atri and Ziller found many other examples of Einstein metrics on naturally reductive compact Lie groups in [11]. Wang and Ziller classified all normal homogeneous Einstein manifolds $G/H$ with $G$ simple in [28]. These metrics are also naturally reductive. Over the past years there has been an increasing interest in connections with parallel skew torsion because they arise in several fields in theoretical and mathematical physics (e.g. [14] and references therein). The most well known examples of this are naturally reductive connections, which have in particular parallel skew torsion. The simple geometric and algebraic properties of naturally reductive spaces allow one to classify them in small dimensions. This has been done in [26, 18, 19] in dimension 3, 4, 5 and more recently in dimension 6 in [1].

1.1. Results. The most important result in this paper is Theorem 3.18. This states that any naturally reductive space is in a unique way a $(\mathfrak{k}, B)$-extension, defined in [23], of a space with its transvection algebra of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus_{L.a.} \mathbb{R}^n,$$

where $\mathfrak{h} \oplus \mathfrak{m}_0$ is semisimple, $\mathfrak{h}$ is the isotropy algebra and $\oplus_{L.a.}$ denotes the direct sum of Lie algebras. This implies that the discussion in [23] Sec. 2.3] gives an explicit description of all naturally reductive spaces. Remember that $(\mathfrak{g}, B)$-extensions are particular fiber bundles of naturally reductive spaces. More specifically, the fibers are orbits of an abelian group of isometries. This means the fiber distribution is spanned by Killing vectors of constant length, see [21]. Recently in [10] the authors also

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investigate in general connections with parallel skew torsion from a fiber bundle perspective. Their approach however does not cover these fiber bundles. The realization of a naturally reductive space as a \((t, B)\)-extension also allows us to prove whether or not it is isomorphic to another naturally reductive space. This is done in Proposition 4.4. We also provide easy to check criteria for a naturally reductive space to be irreducible. This is done in the combined results of Theorem 2.10, Lemma 2.19 and Proposition 4.7. Surprisingly these last two problems were not touched upon in the literature up to now. It is also nice to note that this approach immediately gives the holonomy algebra of the naturally reductive connection, see Lemma 3.2 and (4.6). This means we always know what the \(G\)-structure is which is induced from the holonomy bundle of the naturally reductive connection.

In a forthcoming paper this theory will be used to give a systematic way to classify naturally reductive spaces and explicitly carry this out up to dimension 8.

2. Preliminaries

The essential structure of a locally homogeneous space is encoded in the infinitesimal model. We now briefly discuss this below.

**Theorem 2.1** (Ambrose-Singer, [4]). A complete simply connected Riemannian manifold \((M, g)\) is a homogeneous Riemannian manifold if and only if there exists a metric connection \(\nabla\) with torsion \(T\) and curvature \(R\) such that
\[
\nabla T = 0 \quad \text{and} \quad \nabla R = 0.
\]

**Remark 2.2.** A Riemannian manifold is locally homogeneous if its pseudogroup of local isometries acts transitively on it. It should be noted that there exist locally homogeneous Riemannian manifolds which are not locally isometric to a globally homogeneous space, see [17]. Of course such manifolds have to be non-complete.

A metric connection satisfying (2.1) is called an Ambrose-Singer connection. The torsion \(T\) and curvature \(R\) of an Ambrose-Singer connection evaluated at a point \(p \in M\) are linear maps
\[
T_p : \Lambda^2 T_p M \to T_p M, \quad R_p : \Lambda^2 T_p M \to \mathfrak{so}(T_p M),
\]
which satisfy
\[
R_p(x, y) \cdot T_p = R_p(x, y) \cdot R_p = 0 \quad (2.3)
\]
\[
\mathfrak{S}^{x, y, z} R_p(x, y) z - T_p(T_p(x, y), z) = 0 \quad (2.4)
\]
\[
\mathfrak{S}^{x, y, z} R_p(T_p(x, y), z) = 0, \quad (2.5)
\]
where \(\mathfrak{S}^{x, y, z}\) denotes the cyclic sum over \(x, y\) and \(z\) and \(\cdot\) denotes the natural action of \(\mathfrak{so}(T_p M)\) on tensors. The first equation encodes that \(T\) and \(R\) are parallel objects for \(\nabla\) and under this condition the first and second Bianchi identity become equations (2.4) and (2.5), respectively. A pair of tensors \((T, R)\), as in (2.2), on a vector space \(\mathfrak{m}\) with a metric \(g\) satisfying (2.3), (2.4) and (2.5) is called an infinitesimal model on \((\mathfrak{m}, g)\). From the infinitesimal model \((T, R)\) of a homogeneous space one can construct a homogeneous space with infinitesimal model \((T, R)\). This construction is known as the Nomizu construction, see [22]. This construction goes as follows. Let
\[
\mathfrak{h} := \{ h \in \mathfrak{so}(\mathfrak{m}) : h \cdot T = 0, \ h \cdot R = 0 \}.
\]
and set
\[
\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}.
\]
On \(\mathfrak{g}\) the following Lie bracket is defined for all \(h, k \in \mathfrak{h}\) and \(x, y \in \mathfrak{m}\):
\[
[h + x, k + y] := [h, k]_{\mathfrak{so}(\mathfrak{m})} - R(x, y) + h(y) - k(x) - T(x, y), \quad (2.7)
\]
where \([-,-]_{\mathfrak{so}(\mathfrak{m})}\) denotes the Lie bracket in \(\mathfrak{so}(\mathfrak{m})\). The bracket from (2.7) satisfies the Jacobi identity if and only if \(R\) and \(T\) satisfy the equations (2.3), (2.4) and (2.5). We will call \(\mathfrak{g}\) the symmetry algebra of the infinitesimal model \((T, R)\). Let \(G\) be the simply connected Lie group with Lie algebra \(\mathfrak{g}\) and
let $H$ be the connected subgroup with Lie algebra $h$. The infinitesimal model is called regular if $H$ is a closed subgroup of $G$. If this is the case, then clearly the canonical connection on $G/H$ has the infinitesimal model $(T, R)$ we started with. In [24 Thm. 5.2] it is proved that every infinitesimal model coming from a globally homogeneous Riemannian manifold is regular.

2.1. Naturally reductive fiber bundles. The important results in this paper revolve around the idea of fiber bundles of naturally reductive spaces. We now discuss the basics of this.

**Definition 2.3.** Let $(g = h \oplus m, g)$ be a Lie algebra together with a subalgebra $h \subset g$, a complement $m$ of $h$ and a metric $g$ on $m$. Suppose $\text{ad}(h)m \subset m$ and for all $x, y, z \in m$ that

$$g([x, y]_m, z) = -g(y, [x, z]_m).$$

Then we call $(g = h \oplus m, g)$ a naturally reductive decomposition with $h$ the isotropy algebra. We will mostly refer to just $g = h \oplus m$ as a naturally reductive decomposition and let the metric be implicit.

The infinitesimal model of the naturally reductive decomposition is defined by

$$T(x, y) := -[x, y]_m, \quad \forall x, y \in m, \quad (2.8)$$

$$R(x, y) := -\text{ad}([x, y]_h) \in \mathfrak{so}(m), \quad \forall x, y \in m, \quad (2.9)$$

where $[x, y]_m$ and $[x, y]_h$ are the $m$- and $h$-component of $[x, y]$, respectively. We call the decomposition an effective naturally reductive decomposition if the restricted adjoint map $\text{ad} : h \to \mathfrak{so}(m)$ is injective.

We will say that $g$ is the transvection algebra of the naturally reductive decomposition $g = h \oplus m$ if the decomposition is effective and $\text{im}(R) = \text{ad}(h) \subset \mathfrak{so}(m)$. Note that [26] implies that $\text{im}(R) \subset \mathfrak{so}(m)$ is a subalgebra and that the transvection algebra is always a Lie subalgebra of the symmetry algebra.

The proof of the following lemma is straightforward and can be found in [20].

**Lemma 2.4.** Let $(T, R)$ and $(T', R')$ be two infinitesimal models on $(m, g)$ and $(m', g')$, respectively. Let $M : m \to m'$ be a linear isometry. The following are equivalent:

i) $M \cdot T = T'$ and $M \cdot R = R'$,

ii) the induced map $M : \text{im}(R) \oplus m \to \text{im}(R') \oplus m'$ is a Lie algebra isomorphism of the transvection algebras.

It is important to recognize fiber bundles on the Lie algebra level. The following lemma and definition deal with this and will be used in the sequel.

**Lemma 2.5.** Let $(g = h \oplus m, g)$ be an effective naturally reductive decomposition. Furthermore, suppose $m = m^+ \oplus m^-$ is an orthogonal decomposition of $h$-modules. Then the following hold:

i) $[m^+, m^-] \subset m$,

ii) $[m^+, m^-] \subset m^-$ if and only if $[m^+, m^+]_m \subset m^+$.

If we assume that $[m^+, m^-] \subset m^-$, then also the following hold:

iii) $b = h \oplus m^+$ is a subalgebra of $g$,

iv) $(g = b \oplus m^-, g|m^- \times m^-)$ is a naturally reductive decomposition.

**Proof.** i) Since $m^+$ and $m^-$ are $h$-invariant we conclude

$$g(R(u, v)x^+, x^-) = 0, \quad \forall x^+ \in m^+, \forall u, v \in m.$$

Combining this with the fact that $R : \Lambda^2 m \to \Lambda^2 m$ is symmetric with respect to the Killing form on $\mathfrak{so}(m) \cong \Lambda^2 m$ it follows that $R(x^+, x^-) = 0$ for all $x^+ \in m^+$. The tensor $R$ is defined by $R(x^+, x^-) = -\text{ad}([x^+, x^-]_h)$. Since we assume our decomposition to be effective $\text{ad}([x^+, x^-]_h) = 0$ implies that $[x^+, x^-]_h = 0$. Hence $[m^+, m^-] \subset m$.

ii) Suppose that $[m^+, m^-] \subset m^-$. If $x^+_1, x^+_2 \in m^+$ and $x^- \in m^-$, then

$$0 = g([x^+_1, x^-], x^+_2) = -g(x^-, [x^+_1, x^+_2]).$$

This implies $[x^+_1, x^+_2]_m \in m^+$. The converse follows from the same equation and i).

iii) From ii) we can easily conclude that $b$ is a subalgebra of $g$. 

For the decomposition $g = b \oplus m^-$ we clearly have $[b, m^-] \subset m^-$ and the decomposition is naturally reductive with respect to the metric $g|_{m^- \times m^-}$. \hfill \qed

**Definition 2.6.** Let $g = h \oplus m$ be a naturally reductive decomposition. Suppose that $[m^+, m^-] \subset m^-$, with the notation from Lemma 2.4. In this case we will call $g = h \oplus m$ the decomposition of the total space of the infinitesimal fiber bundle and the naturally reductive decomposition $g = b \oplus m^-$ with isotropy algebra $b$ the decomposition of the base space. Furthermore, we will call $m^+$ the fiber direction.

If connected subgroup $B \subset G$ with $\text{Lie}(B) = b$ is closed and $G/H$ is globally homogeneous with $H \subset G$ connected, then $G/H \to G/B$ is a homogeneous fiber bundle with $B/H$ as fibers. In general the Lie group $B$ will not be closed. However, the decomposition $g = b \oplus m$ always defines a naturally reductive decomposition and therefore a locally naturally reductive space. This is the reason why we consider infinitesimal fiber bundles.

The following is a basic result on tensors which we use in Lemma 2.8.

**Lemma 2.7.** Let $(V, g)$ be a finite dimensional vector space with a metric $g$. If $\alpha \in \Lambda^2 V \cong \mathfrak{so}(V)$, $\beta \in \Lambda^9 V$ and $e_1, \ldots, e_n$ are orthonormal basis of $V$, then

$$
\sum_{i=1}^{n} (e_i \cdot \alpha) \wedge (e_i \cdot \beta) = \pi^{\wedge q}(\alpha) \beta \equiv \alpha \cdot \beta,
$$

where $\pi$ is the vector representation of $\mathfrak{so}(V)$ and $\pi^{\wedge q}$ is the induced tensor representation on $\Lambda^q V$. Furthermore if $\alpha, \beta \in \Lambda^2 V$, then $\alpha \cdot \beta = [\alpha, \beta]_{\mathfrak{so}(V)}$.

Next we briefly discuss when a Riemannian manifold with a metric connection which has parallel skew torsion can locally be written as a product. It turns out this only depends on the metric and torsion. This result is essential to prove if a space with parallel skew torsion can not be decomposed as a product.

**Lemma 2.8.** Let $(V, g)$ be some vector space with a metric $g$. Let $T \in \Lambda^3 V$ be a 3-form. Let $h \in \mathfrak{so}(V)$ with $h \cdot T = 0$. Suppose that either

i) $T$ has no kernel and $T = T_1 + T_2 \in \Lambda^3 V_1 \oplus \Lambda^3 V_2$, with $V_1 = (V_2)^\perp$ or,

ii) $T$ has a kernel and we set $V_2 = \ker(T)$ and $V_1 = (V_2)^\perp$, so $T = T_1 + T_2 \in \Lambda^3 V_1 \oplus \Lambda^3 V_2$ with $T_2 = 0$.

Then for both cases $h$ leaves $V_1$ and $V_2$ invariant. In other words

$$
\{ h \in \mathfrak{so}(V) : h \cdot T = 0 \} \cong \{ h_1 \in \mathfrak{so}(V_1) : h_1 \cdot T_1 = 0 \} \oplus \{ h_2 \in \mathfrak{so}(V_2) : h_2 \cdot T_2 = 0 \}.
$$

**Proof.** We view $h$ as a skew-symmetric endomorphism of $V$ and we write $h$ as

$$
h = \begin{pmatrix} A & -B^T \\ B & C \end{pmatrix},
$$

where $A \in \mathfrak{so}(V_1)$, $B \in \text{Lin}(V_1, V_2)$, $C \in \mathfrak{so}(V_2)$. Since the torsion is invariant under $h$ we get

$$
0 = h \cdot T = A \cdot T_1 + B \cdot T_1 - B^T \cdot T_2 + C \cdot T_2.
$$

If any two of these summands are non-zero, then they are linearly independent, since

$$
A \cdot T_1 \in \Lambda^3 V_1, \quad B \cdot T_1 \in \Lambda^2 V_1 \otimes V_2, \quad -B^T T_2 \in V_1 \otimes \Lambda^2 V_2, \quad C \cdot T_2 \in \Lambda^3 V_2.
$$

Hence all terms vanish. We get

$$
0 = B \cdot T_1 = (B - B^T) \cdot T_1 = \sum_i B(e_i) \wedge (e_i \cdot T_1),
$$

where the sum is over an orthonormal basis of $V_1$ and $(B - B^T)$ is considered as a block matrix in $\mathfrak{so}(V)$. For the last equality we used Lemma 2.7. The 2-forms $e_i \cdot T_1$ are all linearly independent, because $T_1$ has no kernel for both case i) and case ii). Since $B(e_i) \in V_2$ and $e_i \cdot T_1 \in \Lambda^2 V_1$ we obtain
the equation $B(e_i) \wedge (e_i, T_1) = 0$ for all $i$. This implies $B(e_i) = 0$ for all $e_i$. We conclude that $B = 0$ and thus $h$ leaves $V_1$ and $V_2$ invariant. \hfill \Box

For this reason we make the following definition.

**Definition 2.9.** Let $(V, g)$ be some vector space with a metric $g$. A 3-form $T \in \Lambda^3 V$ is called reducible if it can be written as $T = T_1 + T_2$ with $T_i \in \Lambda^3 V_i$ for some non-zero $V_1 \subset V$ and $V_2 \subset V$ such that $V_1 \perp V_2$. Otherwise $T$ is called irreducible.

Combining Lemma 2.8 with de Rham’s theorem for Riemannian manifolds we obtain the following.

**Theorem 2.10.** Let $(M, g, \nabla)$ be a complete simply connected manifold with a metric connection $\nabla$ with non-zero parallel skew torsion $T$. Then the following are equivalent

i) $M$ is isometric to a product and $\nabla$ is the product connection:

$$(M, g, \nabla) \cong (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2),$$

where $\nabla_1$ and $\nabla_2$ are connections on $M_1$ and $M_2$, respectively. Both $\nabla_1$ and $\nabla_2$ have parallel skew torsion.

ii) The torsion at some point $x \in M$ is reducible, i.e. $T(x) = T_1(x) + T_2(x) \in \Lambda^3 V_1(x) \oplus \Lambda^3 V_2(x)$, for certain orthogonal subspaces $V_1(x), V_2(x) \subset T_x M$ and $T_1(x) \in \Lambda^3 V_1(x)$.

For naturally reductive spaces this result is already known, see [27]. For naturally reductive spaces a criterion on the transvection algebra is more useful.

**Definition 2.11.** A naturally reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reducible if its torsion, defined by (2.8), is given by Definition 2.11. is isometric to a product and $\nabla$ is the product connection:

$$(M, g, \nabla) \cong (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2),$$

where $\nabla_1$ and $\nabla_2$ are connections on $M_1$ and $M_2$, respectively. Both $\nabla_1$ and $\nabla_2$ have parallel skew torsion.

The following classical result due to Kostant (see also [11]) will prove very useful at several points in this paper.

**Theorem 2.12 (Kostant,[11]).** Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$ be an effective naturally reductive decomposition. Then $\mathfrak{k} := [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \oplus \mathfrak{m}$ is an ideal in $\mathfrak{g}$ and there exists a unique $\text{ad} (\mathfrak{k})$-invariant non-degenerate symmetric bilinear form $\overline{g}$ on $\mathfrak{k}$ such that $\overline{g}_{|_{\mathfrak{m} \times \mathfrak{m}}} = g$ and $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \perp \mathfrak{m}$. Conversely, any ad-$\mathfrak{g}$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $m = \mathfrak{h^⊥}$ and $\overline{g}_{|_{\mathfrak{m} \times \mathfrak{m}}}$ positive definite gives a naturally reductive decomposition.

Our first reducibility criterion is the following.

**Lemma 2.13.** Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a naturally reductive decomposition with $\mathfrak{g}$ its transvection algebra. Let $\overline{g}$ be the unique $\text{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form from Kostant’s theorem, see Theorem 2.12. The reducible decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reducible if and only if there exist two non-trivial orthogonal ideals $\mathfrak{g}_1 \subset \mathfrak{g}$ and $\mathfrak{g}_2 \subset \mathfrak{g}$ with respect to $\overline{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_i \subset \mathfrak{g}_i$ for $i = 1, 2$.

**Proof.** Assume such two ideals exist. Let $\mathfrak{m}_i$ be the orthogonal complement of $\mathfrak{h}_i$ inside $\mathfrak{g}_i$ for $i = 1, 2$. Note that $\mathfrak{m}_i \neq \{0\}$ for $i = 1, 2$, because otherwise $\mathfrak{g}$ is not the transvection algebra. We clearly have $T \in \Lambda^3 \mathfrak{m}_1 \oplus \Lambda^3 \mathfrak{m}_2$, where $T$ is defined by (2.8), and the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reducible, see Definition 2.11.

Conversely suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the transvection algebra of a reducible naturally reductive decomposition, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $\mathfrak{m}_1 \neq \{0\}, \mathfrak{m}_2 \neq \{0\}$, $\mathfrak{m}_1 \perp \mathfrak{m}_2$, and $[\mathfrak{m}_1, \mathfrak{m}_2] = \{0\}$. Then

$$\mathfrak{g} = \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 = ([\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{h}_1} \oplus \mathfrak{m}_1) \oplus ([\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{h}_2} \oplus \mathfrak{m}_2) = (\mathfrak{h}_1 \oplus \mathfrak{m}_1) \oplus (\mathfrak{h}_2 \oplus \mathfrak{m}_2),$$

where $\mathfrak{h}_i := [\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}_i}$. Let $m, m' \in \mathfrak{m}_1$ and $n \in \mathfrak{m}_2$. Then we have

$$[[m, m'], n] = [m, [m', n]] = [m, n, m'] + [m, [m', n]] = 0 + 0 = 0.$$
Since elements of the form \([m, m']_h\) span \(h_1\) it follows that \([h_1, m_2] = \{0\}\). In the same way we get \([h_2, m_1] = \{0\}\). This also implies that \([h_1 \cap h_2, m] = \{0\}\) and because the reductive decomposition is effective we get \(h_1 \cap h_2 = \{0\}\). Let \(h_1 \in h_1\) and \(m_2, m'_2 \in m_2\). Then we have

\[
\mathfrak{g}[h_1, [m_2, m'_2]] = \mathfrak{g}[h_1, [m_2, m'_2]] = \mathfrak{g}[h_1, m_2], m'_2] = 0.
\]

This implies that \(h_1 \perp h_2\) with respect to \(\mathfrak{g}\). We conclude that \(\mathfrak{g} = (h_1 \oplus m_1) \oplus (h_2 \oplus m_2)\) is the direct sum of two ideals in the way required.

2.2. \((\mathfrak{t}, B)\)-extensions. Next we briefly recall how a \((\mathfrak{t}, B)\)-extension is defined in [23]. For a non-zero transvection algebra \(\mathfrak{g} = h \oplus m\) we define a Lie algebra \(\mathfrak{s}(\mathfrak{g})\) by

\[
\mathfrak{s}(\mathfrak{g}) = \{ f \in \text{Der}(\mathfrak{g}) : f(h) = \{0\}, f(m) \subset m, f|_m \in \mathfrak{so}(m)\}.
\]

If \(\mathfrak{g} = \{0\}\), then we define \(\mathfrak{s}(\{0\}) = \mathfrak{so}(\infty)\). For every finite dimensional subalgebra \(\mathfrak{t} \subset \mathfrak{s}(\mathfrak{g})\) with an \(\text{ad}(\mathfrak{t})\)-invariant metric \(B\) on \(\mathfrak{t}\) we can define a Lie algebra structure on

\[
\mathfrak{g}(\mathfrak{t}) := h \oplus \mathfrak{t} \oplus n \oplus m,
\]

where \(n \equiv \mathfrak{t}\) is another copy of \(\mathfrak{t}\). Let \(\varphi : \mathfrak{t} \to \mathfrak{so}(m)\) be the natural Lie algebra representation and let \(\psi : \mathfrak{t} \to \mathfrak{so}(n \oplus m)\) be the Lie algebra representation \(\psi := \text{ad} + \varphi\). Furthermore, let \((T_0, R_0)\) be the infinitesimal model of \(\mathfrak{g} = h \oplus m\). The Lie bracket on \(\mathfrak{g}(\mathfrak{t})\) is defined by:

\[
[h + k, n + m] = \psi(k)(n + m) + h(m), \quad \forall h \in h, \forall k \in \mathfrak{t}, \forall n \in n, \forall m \in m,
\]

\[
[h_1 + k_1, h_2 + k_2] = [h_1, h_2] + [k_1, k_2], \quad \forall h_1, h_2 \in h, \forall k_1, k_2 \in \mathfrak{t},
\]

\[
[x, y] = -R_0(x, y) - R_\mathfrak{t}(x, y) - T(x, y) \quad \forall x, y \in n \oplus m,
\]

where we identified \(\text{im}(R_0)\) with \(h\), and

\[
R_\mathfrak{t}(x, y) = \sum_{i=1}^{l} \psi(k_i)(x, y), k_i, \quad T = T_0 + \sum_{i=1}^{l} \varphi(k_i) \wedge n_i + 2T_n,
\]

and \(T_n(x, y, z) = B([x, y], z)\). Together with the metric \(g := B \oplus g_0\) on \(n \oplus m\) this defines a naturally reductive decomposition with isotropy algebra \(h \oplus \mathfrak{t}\), see [23]. The Lie algebra \(\mathfrak{g}(\mathfrak{t})\) is known as the double extension of \(\mathfrak{g}\) by \(\mathfrak{t}\), see [20]. The naturally reductive infinitesimal model associated to the decomposition \(\mathfrak{g} = h \oplus \mathfrak{t} \oplus n \oplus m\) is \((T, R)\), where \(T\) is given by (2.10) and \(R\) is given by

\[
R = R_0 + R_\mathfrak{t}.
\]

**Definition 2.14.** We call the infinitesimal model \((T, R)\) the \((\mathfrak{t}, B)\)-extension of \((T_0, R_0)\). We also call a naturally reductive decomposition with the infinitesimal model \((T, R)\) the \((\mathfrak{t}, B)\)-extension of the decomposition \(\mathfrak{g} = h \oplus m\).

An important property of the Lie algebra \(\mathfrak{g}(\mathfrak{t})\) is that the diagonal \(a \subset \mathfrak{t} \oplus n\) is an abelian ideal. The spaces studied in Section 3.2 are characterized by such ideals. It is interesting to note that every vector in \(a\) induces a Killing vector field of constant length on the corresponding homogeneous manifold, see [21].

It will be convenient to have the following different formulation of \(\mathfrak{s}(\mathfrak{g})\), which is used in Lemma 3.17

**Lemma 2.15.** Let \(\mathfrak{g} = h \oplus m\) be a naturally reductive decomposition with \(\mathfrak{g} \neq \{0\}\) its transvection algebra. Let \((T_0, R_0)\) be the infinitesimal model of the decomposition. Let \(\mathfrak{so}(m) = \{k \in \mathfrak{so}(m) : [k, \text{ad}(h)]_{\mathfrak{so}(m)} = 0, \forall h \in h\}\). Then the following holds

\[
\mathfrak{s}(\mathfrak{g}) \cong \{ h \in \mathfrak{so}(m) : h \cdot T_0 = 0\}.
\]

**Proof.** For all \(k \in \mathfrak{s}(\mathfrak{g}), h \in h\) and \(m \in m\) we have

\[
k([h, m]) = [k(h), m] + [h, k(m)] = [h, k(m)].
\]
In other words \( \varphi(k) \in \mathfrak{so}_h(m) \). Furthermore, for all \( m_1, m_2 \in \mathfrak{m} \) we have
\[
k(T_0(m_1, m_2)) = -k([m_1, m_2]_m) = -k([m_1, m_2])
\]
\[
= -[k(m_1), m_2]_m - [m_1, k(m_2)]_m = T_0(k(m_1), m_2) + T_0(m_1, k(m_2)).
\]
We conclude that \( \varphi(k) \cdot T_0 = 0 \).

To find a map in the other direction we let \( k \in \mathfrak{so}_h(m) \) with \( k \cdot T_0 = 0 \). We define
\[
\hat{k} : \mathfrak{g} \to \mathfrak{g} ; \quad \hat{k}(h + m) := k(m)
\]
and we show that \( \hat{k} \in \mathfrak{s}(\mathfrak{g}) \). For all \( h, h' \in \mathfrak{h} \) and \( m \in \mathfrak{m} \) we have
\[
\hat{k}([h, h' + m]) = \hat{k}([h, h' + m]_m) = \hat{k}([h, m]) = [h, \hat{k}(m)] = [\hat{k}(h), h' + m] + [h, \hat{k}(h' + m)],
\]
where in the before last equality we used \( k \in \mathfrak{so}_h(m) \). It remains to show that for all \( m_1, m_2 \in \mathfrak{m} \) we have
\[
\hat{k}([m_1, m_2]) = \hat{k}(m_1) + [m_1, \hat{k}(m_2)].
\]
From \( k \cdot T_0 = 0 \) we immediately get
\[
\hat{k}([m_1, m_2]) = \hat{k}([m_1, m_2]_m) = [\hat{k}(m_1), m_2]_m + [m_1, \hat{k}(m_2)]_m.
\]
Furthermore, we have
\[
ad([\hat{k}(m_1), m_2]_h + [m_1, \hat{k}(m_2)]_h) = -R_0(\hat{k}(m_1), m_2) - R_0(m_1, \hat{k}(m_2))
\]
\[
= -R_0(\hat{k}(m_1) \wedge m_2 + m_1 \wedge \hat{k}(m_2))
\]
\[
= -R_0(k \cdot (m_1 \wedge m_2)).
\]
The right-hand-side vanishes precisely when \( k \cdot (m_1 \wedge m_2) \in \text{ad}(\mathfrak{h})^\perp \), where \( \text{ad}(\mathfrak{h})^\perp \) is the orthogonal complement of \( \text{ad}(\mathfrak{h}) \) in \( \mathfrak{so}(\mathfrak{m}) \) with respect to the Killing form \( B_{\mathfrak{so}} \) of \( \mathfrak{so}(\mathfrak{m}) \). Note that Lemma 2.7 gives us \( k \cdot (m_1 \wedge m_2) = [k, m_1 \wedge m_2]_{\mathfrak{so}(\mathfrak{m})} \). For all \( h \in \mathfrak{h} \) we have
\[
B_{\mathfrak{so}}(\text{ad}(h), [k, m_1 \wedge m_2]_{\mathfrak{so}(\mathfrak{m})}) = B_{\mathfrak{so}}([\text{ad}(h), k], m_1 \wedge m_2) = 0.
\]
This implies that \( R_0(k \cdot (m_1 \wedge m_2)) = 0 \) and thus also \( [\hat{k}(m_1), m_2]_h + [m_1, \hat{k}(m_2)]_h = 0 \). From this we now obtain
\[
\hat{k}([m_1, m_2]) = [\hat{k}(m_1), m_2]_m + [m_1, \hat{k}(m_2)]_m = [\hat{k}(m_1), m_2] + [m_1, \hat{k}(m_2)].
\]
Consequently, \( \hat{k} \) defines a derivation of \( \mathfrak{g} \) and \( \hat{k} \in \mathfrak{s}(\mathfrak{g}) \). It is clear that the above two maps are inverse to each other. We conclude that \( \mathfrak{s}(\mathfrak{g}) \cong \{ h \in \mathfrak{so}_h(m) : h \cdot T_0 = 0 \} \).

3. General form of a naturally reductive space

We define two types of naturally reductive spaces:

Type I: The transvection algebra is semisimple.

Type II: The transvection algebra is not semisimple.

First we discuss some basic results for spaces of type I. Most of this section is about describing the spaces of type II. If a Lie algebra is not semisimple, then it contains a non-trivial abelian ideal. This fact will allow us to show that every naturally reductive space of type II is an infinitesimal fiber bundle over another naturally reductive space, see Definition 2.6. In Proposition 3.13 we derive a formula for the infinitesimal model of the total space in terms of the infinitesimal model of the base space and a certain Lie algebra representation. This leads us to the main result: for every naturally reductive space of type II there exists a unique naturally reductive decomposition of the form \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{l}_a. \mathbb{R}^n \), with \( \mathfrak{g} \) as its transvection algebra and \( \mathfrak{h} \oplus \mathfrak{m} \) a semisimple algebra, such that the original infinitesimal model of type II is a \((\mathfrak{t}, B)\)-extension of \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus L_a. \mathbb{R}^n \). Consequently, the construction presented in [23] generates all naturally reductive spaces.
3.1. Type I. In section we will use that there exists for every naturally reductive space a decomposition $g = h \oplus m$ of that space such that the metric on $m$ is induced by an $\text{ad}(g)$-invariant non-degenerate symmetric bilinear form on $g$ for which $h$ and $m$ are perpendicular, see Theorem 2.12. The results below about spaces of type I are quite elementary. The most interesting statement in this section is Lemma 3.3 and the partial duality this induces, see Definition 3.5 and Corollary 3.6.

**Lemma 3.1.** Let $g$ be a compact simple Lie algebra together with a negative multiple of its Killing form as $\text{ad}(g)$-invariant non-degenerate symmetric bilinear form. Any proper subalgebra $h \subset g$ gives a reductive decomposition $g = h \oplus m$, with $m = (h)\perp$. This is either an irreducible naturally reductive decomposition with non-zero torsion or the decomposition of an irreducible symmetric space.

**Proof.** If the torsion is zero, then $g = h \oplus m$ is a decomposition of an irreducible symmetric space. Suppose that the torsion $T$ defined by (2.8) is non-zero and $T \in \Lambda^3m_1 \otimes \Lambda^3m_2$ for some orthogonal decomposition $m = m_1 \oplus m_2$. By Lemma 2.8 the subspace $h \oplus m_1$ defines a non-zero ideal of $g$. Hence it has to be equal to $g$, which means $m_1 = m$. We conclude that $g = h \oplus m$ is irreducible. \hfill $\square$

The next result gives a criterion when $g$ is the transvection algebra of a reductive decomposition $g = h \oplus m$, with $g$ semisimple.

**Lemma 3.2.** Let $g = h \oplus m$ be a naturally reductive decomposition with $g$ semisimple and let $m \perp h$ with respect to some $\text{ad}(g)$-invariant non-degenerate symmetric bilinear form $\overline{\sigma}$ on $g$ such that $\overline{\sigma}_{m \times m} = g$. Let $(T,R)$ be the infinitesimal model defined by (2.8) and (2.9). The following hold:

i) if $[m,m]_h = h$, then $g$ is the transvection algebra of $(T,R)$,

ii) if $g$ is simple, then $[m,m]_h = h$ and by i) the transvection algebra is equal to $g$.

iii) if the reductive decomposition is effective, then $[m,m]_h = h$ and $g$ is the transvection algebra.

**Proof.** i) Let $\text{ad}|_h : h \rightarrow \mathfrak{so}(m)$ denote the restricted adjoint representation. Let $l := \ker(\text{ad}|_h)$. Then $l \subset h$ is an ideal in $g$. This ideal is either semisimple or $\{0\}$. Let $l^\perp = \{g \in g : [g,l] = 0, \forall l \in l\}$ be the complementary ideal. Then $m \subset l^\perp$ and $[m,m] \subset [l^\perp,l^\perp] = l^\perp$. This implies

$$[l,l] = [[m,m]_h,l] = [[m,m],l] = [[m,l],m] + [m,[m,l]] = \{0\}$$

and thus $l \subset h \subset l^\perp$. We conclude that $l = \{0\}$ and $\text{ad}|_h$ is injective. In particular $g$ is the transvection algebra.

ii) Let $t$ be the subalgebra $t := [m,m]_h \oplus m$. By Kostant’s theorem, Theorem 2.12 $t$ is a non-zero ideal in $g$ and thus $t = g$. This gives us $[m,m]_h = h$ and thus by i) the transvection algebra of $(T,R)$ is $g$.

iii) By Kostant’s theorem $[m,m]_h \oplus m$ is an ideal in $g$. Let $h_0$ be a complementary ideal. Since $h_0$ is perpendicular to $m$ with respect to $\overline{\sigma}$ we have $h_0 \subset h$ and $[h_0,m] = \{0\}$. By assumption we obtain $h_0 = \{0\}$ and thus $[m,m]_h = h$. Now i) implies that $g$ is the transvection algebra. \hfill $\square$

The case that $g$ is simple and non-compact is very different from the compact case as the following lemma shows.

**Lemma 3.3.** Let $g$ be a non-compact simple Lie algebra and $g = h \oplus m$ a naturally reductive decomposition. Then $(g,h)$ is a symmetric pair.

**Proof.** By [29] Thm. 12.1.4 we know that any subalgebra $h$ of a reductive Lie algebra $g$ is reductive in $g$ if and only if there is a Cartan involution of $g$ which stabilizes $h$, i.e. $\sigma(h) = h$. Let $\sigma$ be a Cartan involution which stabilizes $h$ and let $h = h^\perp \oplus h^\perp$, with

$$h^\perp = \{h \in h : \sigma(h) = \pm h\}.$$ 

The metric on $m$ is induced from a multiple of the Killing form and $m = h^\perp$. The Killing form is invariant under all automorphisms. This implies that $\sigma$ preserves $m$ as well. Hence we also have $m = m^+ \oplus m^-$ with

$$m^\pm = \{m \in m : \sigma(m) = \pm m\}.$$
Let \( g^- = h^- \oplus m^- \) and \( g^+ = h^+ \oplus m^+ \). Since \( \sigma \) is a Lie algebra automorphism we immediately get
\[
[g^+, g^+] \subset g^+, \quad [g^-, g^+] \subset g^-, \quad \text{and} \quad [g^-, g^-] \subset g^+.
\]
The Killing form is positive definite on \( m^- \) and negative definite on \( m^+ \). This implies that either \( m^+ = \{0\} \) or \( m^- = \{0\} \). Suppose that \( m^+ = \{0\} \). Then we have
\[
[h^-, m] = [h^-, m^+] \subset g^- \cap m = \{0\}.
\]
This implies that \( h^- \subset \ker(\text{ad}_h) \) and by Lemma 3.2 this implies that \( h^- = \{0\} \). In this case we have \( g^- = \{0\} \) and this contradicts the non-compactness of \( g \). Suppose that \( m^+ = \{0\} \). Then we have
\[
[m, m] = [m^-, m^-] \subset g^+ = h^+ \subset h.
\]
This means \((g, h)\) is a symmetric pair.

The above lemma greatly restricts the possible transvection algebras for a type I space. We will now discuss how this allows us to quite easily obtain all type I spaces from the classification of all compact type I spaces.

**Definition 3.4.** A naturally reductive pair \((g, h)\) is a Lie algebra \( g \) together with a subalgebra \( h \subset g \) such that there exists an \( \text{ad}(g) \)-invariant non-degenerate symmetric bilinear form \( \gamma \) for which \( \gamma_{|m \times m} \) is positive definite, where \( m = h^\perp \) and such that \( g \) is the transvection algebra of the corresponding naturally reductive decomposition.

**Definition 3.5.** A naturally reductive pair \((g^*, h^*)\) is a partial dual of a naturally reductive pair \((g, h)\) when \( g^* \) is a real form of \( g \otimes \mathbb{C} \) different from \( g \) and the complexified Lie algebra pairs are isomorphic:
\[
(g \otimes \mathbb{C}, h \otimes \mathbb{C}) \cong (g^* \otimes \mathbb{C}, h^* \otimes \mathbb{C}).
\]

First note that the above definition covers the duality of symmetric pairs, with the exception of the self-dual symmetric pair \((\text{euc}(\mathbb{R}^n), \mathfrak{so}(n))\). We should point out that we are not defining a complete duality for naturally reductive spaces, because it is not a one-to-one correspondence and it is only defined for a very small class of naturally reductive spaces. Also a specific naturally reductive metric does not transfer through the above partial duality.

**Corollary 3.6.** For every non-compact naturally reductive pair \((g, h)\) of type I there exists a partial dual pair \((g^*, h^*)\) for which \( g^* \) is compact.

**Proof.** Let \( g = g_1 \oplus_{L_a} g_2 \) be a direct sum of ideals with \( g_1 \) non-compact and simple and suppose for now that \( g_2 \) is compact. Let \( i = i_1 \oplus i_2 : h \to g_1 \oplus g_2 \) denote the inclusion of the isotropy algebra. Note that \( n := i_1(h)^\perp \subset g_1 \) is non-trivial and contained in \( m = h^\perp \) for every \( \text{ad}(g) \)-invariant non-degenerate symmetric bilinear form. This implies \((g_1, i_1(h))\) defines a naturally reductive pair. From Lemma 3.3 it follows that \((g_1, i_1(h))\) is a non-compact symmetric pair, where \( \mathfrak{t} \subset g_1 \) is the +1 eigenspace of a Cartan involution. We denote the map \( i_1 \) with restricted codomain by \( \varphi : h \to \mathfrak{t} \) and the inclusion of \( \mathfrak{t} \) in \( g_1 \) by \( j : \mathfrak{t} \to g_1 \). We have \( j = j \circ \varphi \). Let \((g_1^*, \mathfrak{t})\) be the dual symmetric pair of \((g_1, \mathfrak{t})\) and \( j^* : \mathfrak{t} \to g_1^* \) the natural inclusion. Let \( h^* := ((j^* \circ \varphi) \oplus i_2)(h) \subset g_1^* \oplus g_2 \). It is clear that \((g^*, h^*)\) defines a dual naturally reductive pair with \( g^* \) compact. If there is more than one non-compact simple factor in \( g \), then we simply apply the above procedure for every factor.

**Remark 3.7.** The process in the above corollary can also be reversed. Let \( g_1 \) be compact semisimple and suppose that \((g_1, i_1(h)) = (g_1, \mathfrak{t})\) is an irreducible compact symmetric pair. Let \((g_1^*, \mathfrak{t})\) be the dual non-compact symmetric pair. Then just as above we obtain a naturally reductive pair \((g^* := g_1^* \oplus g_2, h^*)\).

From Lemma 2.13 and the above corollary we see immediately that a non-compact naturally reductive space of type I is irreducible if and only if its compact dual is irreducible. Dual pairs are algebraically very similar and it is quite easy to obtain all non-compact naturally reductive decompositions from the compact ones because of Lemma 3.3.
3.2. Type II. Now we deal with non-semisimple transvection algebras of naturally reductive spaces.

Lemma 3.8. Let \((g = h \oplus m, g)\) be an effective naturally reductive decomposition. Let \(a \subset g\) be an abelian ideal. Let \(m_a := a \cap m\) and let \(m_0\) be the orthogonal complement of \(m_a\) in \(m\). Let \(a' := (\pi_m|_a)^{-1}(m_0)\), where \(\pi_m\) is the projection onto \(m\) along \(h\) in \(g\). Then the following hold:

i) \([m_a, m] = \{0\}\),

ii) \(g' := h \oplus m_0\) is a subalgebra of \(g\) and a naturally reductive decomposition,

iii) \(a'\) is an abelian ideal of \(g'\) and \(g\) satisfies \(a' \cap m_0 = a' \cap h = \{0\}\).

Proof. i) Since the decomposition \(g = h \oplus m\) is reductive and \(a\) is an ideal we have

\[ [h, m_a] \subset a \cap m = m_a.\]

Hence \(m_a\) and its orthogonal complement \(m_0\) are \(h\)-invariant. Since \(a\) is abelian we have \([m_a, m_a] = \{0\}\). Let \(m \in m_a\) and \(n \in m_0\). Then we can apply Lemma 2.5 to see that \([m, n] \subset m\). Combining this with \(a\) being an ideal gives us \([m, n] \subset a \cap m = m_a\). We obtain \(g([m, n], [m, n]) = g(n, [m, [m, n]]) = 0\) and thus \([m, n] = 0\). We conclude \([m, m_0] = \{0\}\).

ii) We already know that \([h, m_0] \subset m_0\). We just saw that \([m_0, m_a] = \{0\} \subset m_a\). Lemma 2.5 ii) now implies \([m_0, m_0] \subset m_0\). Consequently, \(g' = h \oplus m_0\) is a subalgebra and defines a naturally reductive decomposition with respect to the metric \(g|_{m_0 \times m_0}\).

iii) We know that \(a' \subset g'\) and by ii) \(g' \subset g\) is a subalgebra. Hence \([g', a'] \subset g' \cap a = a'\). This means \(a'\) is an abelian ideal in \(g'\). Clearly \(a'\) is still an abelian ideal in \(g\). Note that \(a' \cap m_0 \subset m_a \cap m_0 = \{0\}\).

Suppose that \(h \in h \cap a\). Then for every \(n \in m_0\) we have \([h, n] \in m_0 \cap a = \{0\}\). If \(m \in m_a\), then \([h, m] = 0\) holds because both \(h\) and \(m\) are in \(a\) and \(a\) is abelian. By assumption the map \(\text{ad} : h \to so(m)\) has trivial kernel. Since \(h \cap a\) is contained in the kernel we conclude that \(h \cap a = \{0\}\). In particular \(h \cap a' = \{0\}\).

From Lemma 3.8 and Theorem 2.10 we immediately obtain that any abelian ideal \(a\) of an irreducible effective naturally reductive decomposition \(h \oplus m\) satisfies \(a \cap m = a \cap h = \{0\}\). In other words \(m_a = \{0\}\) if \(h \oplus m\) is irreducible.

Definition 3.9. Let \(g = h \oplus m\) be an effective naturally reductive decomposition. Let \(a = a' \oplus m_a\) be as in Lemma 3.8. Let \(m^+ := \pi_m(a') \subset m_0, \) where \(\pi_m : g \to m\) is the projection along \(h\) and \(m_0\) is the orthogonal complement of \(m_a\) in \(m\). Let \(m^-\) be the orthogonal complement of \(m^+\) inside \(m_0\). Furthermore, let \(h^+ := \pi_h(a')\), where \(\pi_h : g \to h\) is the projection along \(m\). Note that \(h^+\) is an ideal in \(h\) because \(\pi_h\) is \(h\)-equivariant and \(a\) is an ideal. Let \(h^-\) be a complementary ideal in \(h\), which exists because \(h\) is a reductive Lie algebra. It will be irrelevant which complement we pick. This gives us the following decomposition:

\[ g = h^+ \oplus h^- \oplus m^+ \oplus m^- \oplus m_a.\]

We call this the fiber decomposition with respect to \(a\).

Lemma 3.10. Let the notation be as in Definition 3.9. Then the following hold:

i) the decomposition \(m = m^+ \oplus m^- \oplus m_a\) is \(h\)-invariant,

ii) \([m^+, m^+]_m \subset m^+\),

iii) \([h^-, m^+] = \{0\}\) and \([h^-, h^+] = \{0\}\),

iv) \([a, m^- \oplus m_a] = \{0\}\).

Proof. i) From Lemma 3.8 we know that \(m_a\) and \(m_0\) are \(h\)-invariant. Let \(m \in m^+\) and pick \(h \in h^+\) such that \(h + m \in a\). Then by Lemma 3.8 iii) we have for every \(k \in h\) the following

\[ a' \ni [k, h + m] = [k, h] + [k, m].\]

Hence \([k, m] \in m^+\) and thus \(m^+\) is \(h\)-invariant. The orthogonal complement \(m^-\) in \(m_0\) is automatically also \(h\)-invariant.
Rewriting (3.1) we get

\[\text{Remark 3.11.}\]

invariant, i.e. a submodule, then also \(v\) implies that \(\rho\) is trivial abelian ideal

This implies that \(\rho\) and \(\mathfrak{h}^+\) are both ideals in \(\mathfrak{h}\) we get \([\mathfrak{h}^-, \mathfrak{h}^+] = \{0\}\). Let \(h^- \in \mathfrak{h}^-\). Then

\[a' \ni [h + m, h^-] = [h, h^-] + [m, h^-] = [m, h^-] \in \mathfrak{m}^+.\]

Combining this with \(a' \cap \mathfrak{m}^+ = \{0\}\) we obtain \([\mathfrak{h}^-, \mathfrak{m}^+] = \{0\}\).

Let \(m^- \in \mathfrak{m}^-\). Then

\[a \ni [h + m, m^-] = [h, m^-] + [m, m^-],\]

implies that \([a, \mathfrak{m}^-] \subset a \cap \mathfrak{m}^- = \{0\}\). Since \(a\) is abelian it follows that \([a, \mathfrak{m}_a] = \{0\}\).

In the following we assume we have an abelian ideal \(a \subset \mathfrak{g}\) with \(a \cap \mathfrak{m} = a \cap \mathfrak{h} = \{0\}\). We let

\[\rho: \mathfrak{h}^+ \rightarrow \mathfrak{m}^+,\]

be the linear map defined by the graph \(a \subset \mathfrak{h}^+ \oplus \mathfrak{m}^+\). Let \(k \in \mathfrak{h}^+\) and \(h + m \in a\). Then

\[a \ni [k, h + m] = [k, h] + [k, m].\]

This implies that \(\rho([k, h]) = [k, m] = [k, \rho(h)]\), i.e. the linear map \(\rho: \mathfrak{h}^+ \rightarrow \mathfrak{m}^+\) is an isomorphism of \(\mathfrak{h}^+\)-modules. Let \(h + m, h' + m' \in a\). Then we have

\[0 = [h + m, h' + m'] = [h, h'] + [h, m'] + [m, h'] + [m, m'],\]

or equivalently

\[(3.1) \quad [h, h'] = -[m, m']_h \quad \text{and} \quad [m, m']_m = [h', m] + [m', h] = -2\rho([h, h']).\]

**Remark 3.11.** Rewriting (3.1) we get \([m, m']_m = -2[h, m']\). This implies that if \(v \subset \mathfrak{m}^+\) is \(\mathfrak{h}^+\)-invariant, i.e. a submodule, then also \([v, v]_m \subset v\).

Let \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) be an effective naturally reductive decomposition which has a non-trivial abelian ideal. If we combine Lemma 2.7 with Lemma 3.10 then we obtain an infinitesimal fiber bundle, in the sense of Definition 2.6, for every abelian ideal \(a \subset \mathfrak{g}\).

**Definition 3.12.** Let \((\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)\) be an effective naturally reductive decomposition with a non-trivial abelian ideal \(a \subset \mathfrak{g}\) and with infinitesimal model \((T, R)\). Let \(\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^- \oplus \mathfrak{m}_a\) be the fiber decomposition with respect to \(a\), see Definition 3.9. Let \(\mathfrak{e} := \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{m}_a\). The base space associated to \(a\) is given by the naturally reductive decomposition

\[(\mathfrak{e} \oplus \mathfrak{m}^-, \mathfrak{g})_{\mathfrak{m}^- \times \mathfrak{m}^-},\]

where \(\mathfrak{e}\) is the isotropy algebra. We will denote the infinitesimal model of the base space, defined by (2.8) and (2.9), by \((T_0, R_0)\).

**Notation 1.** Let \(B = \rho^*g|_{\mathfrak{m}^+ \times \mathfrak{m}^+}\) be the pullback metric on \(\mathfrak{h}^+\). This metric is ad(\(\mathfrak{h}^+\))-invariant. We define a 3-form \(T_{h^+}\) on \(\mathfrak{h}^+\) by \(T(h_1, h_2, h_3) := B([h_1, h_2], h_3)\). We define \(T_{m^+} := \rho(T_{h^+})\), where \(\rho\) is the natural extension \(\rho: \Lambda^3\mathfrak{h}^+ \rightarrow \Lambda^3\mathfrak{m}^+\). Let

\[\varphi: \mathfrak{h}^+ \rightarrow \mathfrak{so}(\mathfrak{m}^-) \quad \text{and} \quad \psi: \mathfrak{h}^+ \rightarrow \mathfrak{so}(\mathfrak{m}^+ \oplus \mathfrak{m}^-)\]

denote the restricted adjoint representations in \(\mathfrak{g}\).

Note that \(T_{h^+}\) is invariant under \(\varphi(\mathfrak{h}^+)\). We now derive a formula for the torsion and curvature of a naturally reductive decomposition \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) in terms of \((T_0, R_0)\) and the representations \(\varphi\) and \(\psi\).
Proposition 3.13. Let \( g = h \oplus m \) be an irreducible effective naturally reductive decomposition. Let \( g = h^+ \oplus h^- \oplus m^+ \oplus m^- \) be the fiber bundle decomposition associated with an abelian ideal \( a \subset g \). Its torsion and curvature are given by

\[
T = T_0 + \sum_{i=1}^{l} \varphi(h_i) \otimes m_i + 2T_{m^+} \quad \text{and} \quad R = R_0 + \sum_{i=1}^{l} \psi(h_i) \otimes \psi(h_i),
\]

respectively, where \( m_1, \ldots, m_l \) is an orthonormal basis of \( m^+ \) and \( h_i := \rho^{-1}(m_i) \).

Proof. We know by Lemma 3.10 that \( [m^+, m^-] \subset m^+ \). Thus Lemma 2.5 implies that \( [m^+, m^-] \subset m^- \). These two inclusions tell us that

\[ T \in \Lambda^3 m^+ \oplus \Lambda^2 m^- \oplus m^+ \oplus \Lambda^3 m^- \]

The component in \( \Lambda^3 m^- \) is exactly \( T_0 \) by the definition of \( T_0 \). Let \( h + m \in a \). Then by Lemma 3.10 we have

\[ 0 = [h + m, n] = [h, n] + [m, n] = \varphi(h)(n) + [m, n], \]

for every \( n \in m^- \). This means that \( T(m, n) = -[m, n] = \varphi(h)n \). This proves that the summand in \( \Lambda^3 m^- \oplus m^+ \) is given by \( \sum_{i=1}^{l} \varphi(h_i) \otimes m_i \). From (3.1) we know that \( [m, m'] = -2\rho([h, h']) \). This shows that the summand in \( \Lambda^3 m^+ \) is given by \( 2\rho(T_{h^+}) = 2T_{m^+} \).

The curvature of the base space is by definition given by

\[ R_0(x, y) = -\text{ad}([x, y]_{e}) \in \mathfrak{o}(m^-), \quad \forall x, y \in m^- \]

Let \( x, y, u, v \in m^- \). Then we have

\[
R(x, y, u, v) = g(R(x, y)u, v) = -g([x, y]_{h} - [x, y]_{m^+}, u, v)
\]

\[
= -g(([x, y]_{e} - [x, y]_{m^+}, u, v)
\]

\[
= R_0(x, y, u, v) + g([x, y]_{m^+}, u, v)
\]

\[
= R_0(x, y, u, v) + \sum_{i=1}^{l} -g(g(\psi(h_i)x, y)[m_i, u], v)
\]

\[
= R_0(x, y, u, v) + \sum_{i=1}^{l} g(\psi(h_i)x, y)g(\psi(h_i)u, v)
\]

\[
= (R_0 + \sum_{i=1}^{l} \psi(h_i) \otimes \psi(h_i))(x, y, u, v).
\]

Let \( x, y \in m^+ \) and \( u, v \in m^- \). From (3.11) it follows that

\[
[x, y]_{m} = \sum_{i=1}^{l} g([x, y], m_i) m_i = \sum_{i=1}^{l} g([m_i, x], y)m_i = -2 \sum_{i=1}^{l} g([h_i, x], y)m_i,
\]

and \( [x, y]_{h} = \frac{1}{2}\rho^{-1}([x, y]_{m}) \). Combining these gives

\[
[x, y]_{h} = -\sum_{i=1}^{l} g([h_i, x], y)h_i.
\]

Consequently,

\[
R(x, y, u, v) = -g([x, y]_{h}u, v) = \sum_{i=1}^{l} (\psi(h_i) \otimes \psi(h_i))(x, y, u, v).
\]

From the symmetries of the curvature tensor \( R \) we conclude that

\[
R = R_0 + \sum_{i=1}^{l} \psi(h_i) \otimes \psi(h_i). \]

\( \square \)
In the following lemma we will prove that every effective naturally reductive decomposition admits a maximal abelian ideal. This result will be very useful for the main theorem of this section.

Lemma 3.14. Let \( g = \mathfrak{h} \oplus m \) be an effective naturally reductive decomposition. The sum over all abelian ideals inside \( g \) is again an abelian ideal in \( g \). In other words there always exists a maximal abelian ideal. Every derivation of \( g \) preserves the maximal abelian ideal.

Proof. Let \( a := +_i a_i \) be the sum of all abelian ideals \( a_i \) in \( g \). Then \( a \subset g \) is an ideal. We have to show for all \( x,y \in a \) that \( [x,y] = \sum_i [x_i,y] = 0 \), where \( x = \sum_i x_i \), \( y = \sum_i y_i \), and \( x_i,y_i \in a_i \). In other words the sum of two abelian ideals \( a_i \) and \( a_j \) is an abelian ideal in \( g \). It is clear that \( a_i + a_j \) is an ideal and that \( [a_i,a_j] \subset a_i \cap a_j \). This means that if \( a_{ij} := a_i \cap a_j \) is equal to \( \{0\} \), then \( a_i + a_j \) is also abelian.

Let \( g = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus m^+ \oplus m^- \oplus m_0 \) be the fiber decomposition of \( g \) with respect to \( \alpha \), see Definition 3.9. The intersection \( a_{ij} = a_i \cap a_j \) is an abelian ideal of \( g \) and \( a_{ij} \subset a_i \). Let \( m^+_{ij} \) be the projection of \( a_{ij} \) onto \( m^+ \). Just as in Lemma 3.10 it follows that \( m^+_{ij} \subset m^+ \oplus m_0 \) is \( \mathfrak{h} \)-invariant. Let \( v_i \) be the orthogonal complement of \( m^+_{ij} \) in \( m^+ \oplus m_0 \). Then \( v_i \) is also \( \mathfrak{h} \)-invariant. Remark 3.11 implies \( [v_i,v_i] \subset v_i \) and \( [m^+_{ij},m^+_{ij}] \subset m^+_{ij} \). Therefore, Lemma 2.5(ii) implies that \( [v_i,m^+_{ij}] \subset v_i \cap m_{ij} = \{0\} \).

Let \( a := (\pi_m a_m)^{-1}(v_i) \), where \( \pi_m : g \to m \) is the projection along \( \mathfrak{h} \). Then \( a_i \subset a_i \) and thus Lemma 3.10 implies that \( [a_i,m^+_{ij} \oplus m_0] = \{0\} \). Since \( v_i \) is \( \mathfrak{h} \)-invariant and \( \pi_m \) is \( \mathfrak{h} \)-equivariant we see that also \( a_i \) is \( \mathfrak{h} \)-invariant, i.e. \( [h,a_i] \subset a_i \). Finally, we have \( [a_i,a_j] = \{0\} \). In total this tells us that \( g,a_j = h + a_i + m^+ \oplus m_0,a_j \subset a_i \) and thus that \( a_i \) is an ideal in \( g \). Moreover \( a_i \) is abelian because \( a_i \subset a_i \). We have \( a_i = a_i \oplus a_j \). By construction we have \( a_i \cap a_j = \{0\} \). This implies that \( a_i + a_j \) is again an abelian ideal. Since \( a_{ij} \subset a_j \) we obtain

\[
a_i + a_j = (a_i + a_{ij}) + a_j = a_i + a_j.
\]

We conclude that \( a_i + a_j \) is an abelian ideal and thus also \( a = +_i a_i \) is an abelian ideal. Moreover, \( a \) is maximal in the sense that it contains all other abelian ideals.

The maximal abelian ideal of \( g \) is the sum over all abelian ideals. The image of an abelian ideal under an automorphism is an abelian ideal. Therefore, we see that any automorphism preserves the maximal abelian ideal. This implies that also all derivations preserve the maximal abelian ideal. \( \square \)

Lemma 3.15. Let \( g = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus m^+ \oplus m^- \oplus m_0 \) be an effective naturally reductive decomposition for some abelian ideal \( a \subset g \) with \( a \cap m = \{0\} \). Let \( l := \ker(\alpha) \) and \( l^\perp \) the orthogonal complement in \( \mathfrak{h}^+ \) with respect to \( \rho \). Then we have the following decomposition of ideals

\[
g = (l \oplus \rho(l)) \oplus L.a. \ (l^\perp \oplus \mathfrak{h}^- \oplus \rho(l^\perp) \oplus m^-).
\]

The restricted representation \( \alpha = \text{adj}|_{l^\perp \oplus \mathfrak{h}^-} : l^\perp \oplus \mathfrak{h}^- \to \mathfrak{so}(m^-) \) is faithful.

Proof. Let \( m^+_l := \rho(l) \) and let \( m^-_l := \rho(l^\perp) \) be the orthogonal complement in \( m^+ \). Since \( l \) is an ideal we obtain \( m^+_l \subset m^+ \) is an \( \mathfrak{h}^+ \)-invariant subspace and so is \( m^-_l \). Combining this with Remark 3.11 we see that \( l \oplus m^+_l \) commutes with \( l^\perp \oplus m^-_l \). Let \( n \in m^- \) and \( h + m \in a \) with \( h \in l, m \in m^+_l \). Then by Lemma 3.10 we have

\[
0 = [h,m,n] = [h,n] + [m,n] = [m,n].
\]

Hence \( m^+_l \) also commutes with \( m^- \) and thus it commutes with its orthogonal complement in \( m \). From Lemma 3.10(iii) it follows that \( l \oplus m^+_l \) commutes with \( \mathfrak{h}^- \). Since \( l \oplus m^+_l \) is a subalgebra we obtain it is an ideal and it commutes with \( l^\perp \oplus \mathfrak{h}^- \oplus m^-_l \oplus m^- \). From Proposition 3.13 we can immediately see that \( l^\perp \oplus \mathfrak{h}^- \oplus m^-_l \oplus m^- \) is a subalgebra and thus also an ideal.

Suppose that \( h \in \ker(\alpha) \). For all \( m \in m^+ \) and \( n \in m^- \) we have \( [m,n] \in m^- \) by Lemma 3.10(ii) and Lemma 2.5(ii). Thus

\[
0 = [h,m,n] = [h,m,n] + [m,h,n] = [h,m,n].
\]

We conclude that \( [h,m] \in m^+ \) commutes with \( m^- \). This implies \( \rho^{-1}([h,m]) \in l \). On the other hand \( \rho^{-1}([h,m]) = [h,\rho^{-1}(m)] \in l^\perp \), because \( h \in l^\perp \oplus \mathfrak{h}^- \) and \( \rho^{-1}(m) \in \mathfrak{h}^+ \). We obtain \( \rho^{-1}([h,m]) \in \mathfrak{h}^+ \cap \mathfrak{h}^- \).
Thus \( h, m \) = 0 for all \( m \in m^+ \). In total we have \( h, m = 0 \). This implies \( h = 0 \), because we assumed the reductive decomposition to be effective. We conclude \( \ker(\alpha) = \{0\} \).

By Lemma 2.18 the above Lemma 3.15 implies that for an irreducible naturally reductive decomposition \( g = h \oplus m \) and any abelian ideal \( a \subset g \) there are two possible cases: \( \ker(\varphi) = \{0\} \) or \( m^- = \{0\} \). The case \( m^- = \{0\} \) corresponds to the \((t, B)\)-extensions of a point space.

**Lemma 3.16.** Let \( g = h \oplus m \) be an effective irreducible naturally reductive decomposition with an abelian ideal \( a \subset g \). Let \( g = h^+ \oplus h^- \oplus m^+ \oplus m^- \) be the fiber decomposition associated with \( a \). Let \( h_0 := \pi_h([m^-, m^-]), \) where this time \( \pi_h \) is the projection onto \( h \) along \( a \oplus m^- \) in \( g = h \oplus a \oplus m^- \). Let \( h_0^+ \subset h \) be a complementary ideal of \( h_0 \) in \( h \). Then \( a \oplus h_0 \oplus m^- \) is a subalgebra of \( g \) and

\[
g \cong h_0^+ \ltimes (a \oplus h_0 \oplus m^-).
\]

Moreover, \( a \) is contained in the center of \( a \oplus h_0 \oplus m^- \). If we define a Lie algebra structure on \( g^- := h_0 \oplus m^- \) induced by the quotient \( h_0 \oplus m^- \cong (a \oplus h_0 \oplus m^-)/a \), then \( g^- = h_0 \oplus m^- \) is a naturally reductive decomposition of the base space, with \( g^- \) its transvection algebra.

**Proof.** To see that \( a \oplus h_0 \oplus m^- \) is a subalgebra of \( g \) we first note that \( [h_0, m^-] \subset m^- \) and \( [a, m^-] = \{0\} \), see Lemma 3.10(c). Therefore, the inclusions which we still need to check are:

\[
[m^-, m^-] \subset a \oplus h_0 \oplus m^- \quad \text{and} \quad [a, h_0] \subset a \oplus h_0 \oplus m^-.
\]

Clearly we have \( [m^-, m^-] \subset a \oplus h_0 \oplus m^- \). We know that \( [a, m^-] = \{0\} \) and thus

\[
[a, h_0] = [a, \pi_h([m^-, m^-])] = [a, [m^-, m^-]] = [[a, m^-], m^-] + [m^-, [a, m^-]] = \{0\}.
\]

Thus, \( a \oplus h_0 \oplus m^- \) is a subalgebra and \( a \) is contained in its center. By definition of \( h_0^+ \), we have \( [h_0^+, h_0] = \{0\} \). Furthermore, we know \( [h_0^+, a \oplus m^-] \subset a \oplus m^- \). We conclude that \( g \cong h_0^+ \ltimes (a \oplus h_0 \oplus m^-) \).

We have shown that \( (a \Delta h_0) \Delta m^- \) is a naturally reductive decomposition of the base space. We also know that \( [a, m^-] = \{0\} \). Therefore, the quotient \( h_0 \oplus m^- \) still defines a naturally reductive decomposition of the base space. Moreover, this decomposition is effective by Lemma 3.15 both for the case \( m^- = \{0\} \) and for the case \( \ker(\varphi) = \{0\} \). By definition we have \( [m^-, m^-]_{h_0} = h_0 \) and thus we conclude that \( g^- \) is the transvection algebra of the base space.

**Lemma 3.17.** Let \( g = h^+ \oplus h^- \oplus m^+ \oplus m^- \) be an irreducible naturally reductive decomposition with \( g \) its transvection algebra and with \( \ker(\varphi) = \{0\} \). Let \( g^- \) be the Lie algebra from Lemma 3.10. Then \( h^+ \) can be identified with a subalgebra of \( \mathfrak{s}(g^-) \). Moreover, the maximal abelian ideal of \( g^- \) is preserved by \( h^+ \).

**Proof.** By Lemma 3.10 we know that \( [a, h_0] = \{0\} \) and this implies that \( [h^+, h_0] = \{0\} \). Thus, we obtain

\[
\varphi(h^+) \subset \{ h \in \mathfrak{so}(m^-) : h \cdot T_0 = 0 \}.
\]

Since \( g^- \) is the transvection algebra of \( h_0 \oplus m^- \) it follows by Lemma 2.15 that \( h^+ \) is identified with a subalgebra of \( \mathfrak{s}(g^-) \). By Lemma 3.14 all derivations of \( g^- \) preserve the maximal abelian ideal, so in particular \( h^+ \) preserves it.

Let the notation be as in Lemma 3.16 and let

\[
p : g \rightarrow g/a \cong h_0^+ \ltimes g^-\]

be the quotient map. Now we come to the main result of this section.

**Theorem 3.18.** Let \( (g = h \oplus m, g) \) be an irreducible naturally reductive decomposition with \( g \) its transvection algebra. Let

\[
g = h^+ \oplus h^- \oplus m^+ \oplus m^-\]

be the fiber decomposition with respect to the maximal abelian ideal \( a \). Then the base space associated to \( a \) is isomorphic to the following naturally reductive decomposition

\[
(g^- = (h_0 \oplus m_0) \ltimes L.a, \mathbb{R}^n, g|_{m^- \times m^-}),
\]
where \( h_0 \oplus m_0 \) is semisimple or \{0\}. Moreover, \((g = h \oplus m, g)\) is isomorphic to the \((\varphi(h^+), \rho^*g|_{m^+ \times m^+})\)-extension of \( g^- = h_0 \oplus m^- \).

**Proof.** By assumption our naturally reductive decomposition is irreducible. Therefore, either \( I := \ker(\varphi) = h^+ \) and \( m^- = \{0\} \) or \( I = \{0\} \) holds by Lemma 3.15. In case \( I = h^+ \) we have \( g^- = \{0\} \) and thus the base space is of the required form.

Now we consider the case \( I = \ker(\varphi) = \{0\} \). Let \( g^- = h_0 \oplus m^- \) be the transvection algebra of the base space described in Lemma 3.16. Let \( b \) be the maximal abelian ideal in \( g^- \), which exists by Lemma 3.14. Then \( b \) is also an abelian ideal of \( h_0^\perp \cong g^- \cong g/\alpha \), where \( h_0^\perp \) is defined in Lemma 3.16. By Lemma 3.8 we can decompose \( b \) as

\[
b = b' \oplus m^-_b,
\]

where \( b' \) satisfies \( b' \cap m^- = b' \cap h_0 = \{0\} \). By Lemma 3.17 we know that \( h^+ \subset s(g^-) \) preserves \( b \) and \( m^- \). In particular this tells us that \( m^-_b \) is \( h^+ \)-invariant and thus also the orthogonal complement of \( m^-_b \) in \( m^- \) is \( h^+ \)-invariant. This in turn implies that \( b' \) is \( h^+ \)-invariant. In Lemma 3.17 we saw that \( [h^+, h_0] = \{0\} \). We can write every \( b \in b' \) as \( b = h + m^- \) with \( h \in h_0 \) and \( m^- \in m^- \). If \( d \in h^+ \), then

\[
b' \ni d(h) = d(h) + d(m^-) = d(m^-) \in m^-.
\]

Since \( b' \cap m^- = \{0\} \) we obtain \( d(b') = \{0\} \). Let \( \bar{\alpha} := p^{-1}(b') \), where \( p \) is the map from (3.2). Then \( \bar{\alpha} \) is an ideal in \( g \) and \( \alpha \subset \bar{\alpha} \). Note that \( p(\bar{\alpha}, \bar{\alpha}) = b, \bar{\alpha} = \{0\} \) and thus \( \bar{\alpha}, \bar{\alpha} \subset \alpha \). Let \( \pi_\alpha : g \to m \) be the projection onto \( m \) along \( \bar{\alpha} \). Then \( \pi_{\alpha} \mid_{\bar{\alpha}} \) is injective. This implies that for \( x_1, x_2 \in \bar{\alpha} \) we have

\[
[x_1, x_2] = 0 \iff [x_1, x_2]_m = [x_1, x_2]_{m^+} = 0.
\]

We know that \( x_i = a_i + h_0, i + m^-_i \) with \( h_0, i + m^-_i \in b' \) and \( a_i \in \alpha \) for \( i = 1, 2 \). Let \( m_1, \ldots, m_l \) be an orthonormal basis of \( m^+ \) and \( h_j = \rho^{-1}m_j \).

Then

\[
[x_1, x_2]_{m^+} = [a_1 + h_0, i + m^-_1, a_2 + h_0, i + m^-_2]_{m^+} = [m^-_1, m^-_2]_{m^+} = \sum_{j=1}^l g([h_j, m^-_1, m^-_2]m_j),
\]

where in the second equality we use \( [h_0, i, h_0, j]_{m} = 0 \), \( [h_0, m^-_j]_{m} \in m^- \) and that \( \alpha \) commutes with \( h_0 \oplus m^- \). All the summands vanish by (3.3). We conclude \( [x_1, x_2] = 0 \) and thus \( \bar{\alpha} \) is an abelian ideal. The maximality of \( \alpha \) implies \( \bar{\alpha} = \alpha \). Hence \( b' = \{0\} \). We have

\[
g^- = h_0 \oplus m_0 \oplus m^-_b,
\]

where \( m_0 := (m^-_b)^\perp \subset m^- \). We know from Lemma 3.8(i) that \( [m_0, m^-_b] = \{0\} \). In Lemma 3.16 we saw that \( g^- \) is the transvection algebra of \( g^- = h_0 \oplus m^- \), i.e. \( h_0 = [m^-_b, m^-]_{h_0} \). Thus, we have

\[
[h_0, m^-_b] = [[m^-_b, m^-]_{h_0}, m^-_b] = [[m^-_b, m^-], m^-_b] = [m^-_b, m^-_b].
\]

Hence \( m^-_b \) is in the center of \( g^- \). By Lemma 3.8(ii) we know that \( h_0 \oplus m_0 \) is a subalgebra of \( g^- \). We conclude that

\[
g^- = (h_0 \oplus m_0) \ominus L.a. m^-_b.
\]

The subalgebra \( h_0 \oplus m_0 \) has no non-trivial abelian ideals, since \( b \) is the maximal abelian ideal of \( g^- \). In other words \( h_0 \oplus m_0 \) is semisimple or equal to \( \{0\} \). The infinitesimal model of the \((\varphi(h^+), \rho^*g|_{m^+ \times m^+})\)-extension is identified with the infinitesimal model of \( g = h \oplus m \) through the isometry \( \rho \oplus \text{id} : h^+ \oplus m^- \to m^+ \oplus m^- \). It follows directly from Proposition 3.13 and the equations (2.10) and (2.11) that \( \rho \oplus \text{id} \) is an isomorphism of the infinitesimal models. We conclude that \((g = h \oplus m, g)\) is isomorphic to the \((\varphi(h^+), \rho^*g|_{m^+ \times m^+})\)-extension of \((g^- = h_0 \oplus m^-, g|_{m^- \times m^-})\). \( \square \)

**Definition 3.19.** Let the notation be as in Theorem 3.18. We call the base space associated with the maximal abelian ideal the *canonical base space*. Furthermore, we will call \( m^+ \) the *canonical fiber direction*. 


Remark 3.20. The partial duality of pairs from Definition 3.3 also takes a very simple form for spaces of type II. If two spaces of type II are partial dual to each other, then it easily follows from Theorem 3.15 that the canonical base spaces also define partial dual pairs. This means that for every naturally reductive pair of type II there exists a partial dual pair for which the semisimple part of the canonical base space is compact.

Remark 3.21. In [20] the authors proved that the class of Lie algebras which admit an invariant non-degenerate symmetric bilinear form on it is the smallest class which contains the simple and abelian Lie algebras and which is stable under direct sums and double extensions. Theorem 3.15 is similar in the sense that every irreducible infinitesimal model is obtained as a \((\mathfrak{k}, B)\)-extension of an naturally reductive infinitesimal model which has a reductive transvection algebra. The biggest difference is that we do not obtain any new spaces by repeated \((\mathfrak{k}, B)\)-extensions. Therefore, the formula in [23, Sec. 2.3] directly describes all naturally reductive spaces.

4. ISOMORPHISM AND IRREDUCIBILITY CRITERIA

With the knowledge that any naturally reductive decomposition is a particular \((\mathfrak{k}, B)\)-extension we prove in this section relatively easy to check criteria for two naturally reductive spaces to be locally isomorphic. It is also important to known when a naturally reductive space is irreducible. Therefore, we give a necessary and sufficient condition for a \((\mathfrak{k}, B)\)-extension to be irreducible in Proposition 4.7.

First we will investigate under which conditions the canonical base space of a \((\mathfrak{g}, \mathfrak{h})\)-extension is isomorphic. It is also important to known when a naturally reductive space is irreducible. Therefore, we will prove in this section relatively easy to check criteria for two naturally reductive spaces to be locally isomorphic.

Let \((\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)\) be a naturally reductive decomposition of the form

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus \mathfrak{n} \oplus \mathbb{R}^n,
\]

with \(\mathfrak{g}\) its transvection algebra and \(\mathfrak{h} \oplus \mathfrak{m}_0\) a semisimple Lie algebra. Let \(\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathbb{R}^n\), where \(\mathfrak{g}_1, \ldots, \mathfrak{g}_k\) are simple ideals of \(\mathfrak{g}\). Furthermore, let \(\mathfrak{t} \subset \mathfrak{s}(\mathfrak{g})\) and \(B\) some \(\text{ad}(\mathfrak{t})\)-invariant inner product on \(\mathfrak{t}\). Let \((T, R)\) be the infinitesimal model of the \((\mathfrak{t}, B)\)-extension. The transvection algebra of \((T, R)\) is given by

\[
\mathfrak{f} := \text{im}(R) \oplus \mathfrak{n} \oplus \mathbb{R}^n,
\]

with the Lie bracket defined by \((2.7)\). Let \(\mathfrak{d} \subset \mathfrak{f}\) be the maximal abelian ideal. We will prove when \(\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) = \mathfrak{n}\), i.e. when the base space \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) is equal to the canonical base space of the \((\mathfrak{t}, B)\)-extension.

Remark 4.1. The map \(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{t})} : \text{ad}(\mathfrak{h} \oplus \mathfrak{t}) \rightarrow \text{ad}(\mathfrak{h} \oplus \mathfrak{t})\) is symmetric with respect to the Killing form of \(\mathfrak{s}(\mathfrak{n} \oplus \mathfrak{m})\), denoted by \(B_{\mathfrak{s}(\mathfrak{n} \oplus \mathfrak{m})}\), and is given by

\[
R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{t})} = R_0 + \sum_{i=1}^l \psi(k_i) \otimes \psi(k_i),
\]

where \(k_1, k_2, \ldots, k_l\) is an orthonormal basis of \(\mathfrak{t}\) with respect to \(B\). This means we have an orthogonal direct sum

\[
\text{ad}(\mathfrak{h} \oplus \mathfrak{t}) = \ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{t})}) \oplus \text{im}(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{t})}).
\]

Notation 2. In this section we will denote \(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{t})}\) simply by \(R\) and \(R_\psi = \sum_{i=1}^l \psi(k_i) \otimes \psi(k_i)\). Furthermore, the center of a Lie algebra \(\mathfrak{g}\) will be denoted by \(\mathcal{Z}(\mathfrak{g})\) and the semisimple part of a reductive Lie algebra \(\mathfrak{g}\) will be denoted by \(\mathfrak{g}^{ss}\). Let \(B_{A^2}\) denote the metric on \(\mathfrak{s}(\mathfrak{m})\) defined by \(B_{A^2}(x, y) = -\frac{1}{4} \text{tr}(xy)\). Note that \(B_{A^2}\) is a multiple of \(B_{\mathfrak{s}(\mathfrak{m})}\).

We recall some definitions from [23].
**Proposition 4.4.** Let \((g = \mathfrak{h} \oplus \mathfrak{m}, g)\) be a as in (4.1). Let \(\mathfrak{t} \subset \mathfrak{s}(g)\) be a Lie subalgebra and let \(B\) be an \(\text{ad}(\mathfrak{t})\)-invariant inner product on \(\mathfrak{t}\). Then we define \(\varphi_1 : \mathfrak{t} \rightarrow \mathfrak{so}(m_0)\) and \(\varphi_2 : \mathfrak{t} \rightarrow \mathfrak{so}(\mathbb{R}^n)\) to be the restricted representations of \(\mathfrak{t}\) and

\[
\mathfrak{e}_1 := \ker(\varphi_2), \quad \mathfrak{e}_3 := \ker(\varphi_1), \quad \mathfrak{e}_2 := (\mathfrak{e}_1 + \mathfrak{e}_2)^\perp \subset \mathfrak{t},
\]

where the orthogonal complement is taken with respect to \(B\). Furthermore, recall that \(\mathfrak{s}(\mathfrak{h} \oplus m_0) \cong \mathcal{Z}(\mathfrak{h}) \oplus \mathfrak{p}\), where \(\mathcal{Z}(\mathfrak{h}) \subset \mathfrak{h}\) is the center of \(\mathfrak{h}\) and \(\mathfrak{p} := \{m \in m_0 : [h, m] = 0, \forall h \in \mathfrak{h}\}\). In this way we identify \(\mathfrak{e}_1 \oplus \mathfrak{e}_2 \subset \text{Aut}(\mathfrak{h} \oplus m_0)\) with inner derivations: \(b_1 \oplus b_2 \subset \mathcal{Z}(\mathfrak{h}) \oplus \mathfrak{p} \subset \mathfrak{h} \oplus m_0\).

**Lemma 4.3.** Let \(g = \mathfrak{h} \oplus \mathfrak{m}\) be a naturally reductive decomposition with \(g\) its transvection algebra as in (4.1). Let \(\mathfrak{t} \subset \mathfrak{s}(g)\) and let \(B\) be an \(\text{ad}(\mathfrak{t})\)-invariant inner product on \(\mathfrak{t}\). Let \((T, R)\) be the infinitesimal model of the \((\mathfrak{t}, B)\)-extension. Then

\[
\text{ad}(\mathfrak{h}^{ss}) \oplus \text{ad}(\mathfrak{t}^{ss}) = \text{ad}(\mathfrak{h}^{ss} \oplus \mathfrak{t}^{ss}) \subset \text{im}(R) \quad \text{and} \quad \ker(R) \subset \text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{m})).
\]

Moreover, if \(\mathfrak{e}_1 = \{0\}\), then \(\ker(R) = \{0\}\).

**Proof.** Note that \(\mathfrak{h} = \mathfrak{h}^{ss} \oplus \mathfrak{a}_{L.a.} \mathcal{Z}(\mathfrak{h})\). If \(h_1, h_2 \in \mathfrak{h}^{ss}\) and \(k \in \mathfrak{k}\), then

\[
B_{\mathfrak{h}^{ss}}([h_1, h_2], \psi(k)) = B_{\mathfrak{h}^{ss}}([h_1, k], \psi(h_2)) + B_{\mathfrak{h}^{ss}}([h_2, k], \psi(h_1)) = 0.
\]

The Lie algebra \(\mathfrak{h}^{ss}\) is semisimple, so \([\mathfrak{h}^{ss}, \mathfrak{h}^{ss}] = \mathfrak{h}^{ss}\). Therefore, for all \(h \in \mathfrak{h}^{ss}\) and \(k \in \mathfrak{k}\) we obtain

\[
B_{\mathfrak{h}^{ss}}(h, \psi(k)) = 0. \quad \text{For all } h \in \mathfrak{h}^{ss} \text{ this implies }
\]

\[
R_\psi(\text{ad}(h)) = \sum_{i=1}^l B_{\mathfrak{h}^{ss}}(\text{ad}(h), \psi(k_i)) \psi(k_i) = 0,
\]

where \(R_\psi\) is as defined in (4.23). Thus \(R(\text{ad}(h)) = R_0(\text{ad}(h)) \neq 0\). By assumption we have \(R_0(\text{ad}(h)) = \text{ad}(\mathfrak{h})\). Hence, \(R_0 : \text{ad}(h) \rightarrow \text{ad}(h)\) is a Lie algebra isomorphism. This implies that \(R(\text{ad}(\mathfrak{h}^{ss})) = R_0(\text{ad}(\mathfrak{h}^{ss})) = \text{ad}(\mathfrak{h}^{ss})\). Similarly we prove that \(R(\psi(\mathfrak{t}^{ss})) = \psi(\mathfrak{t}^{ss})\). Consequently, \(\ker(R) \subset \text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{m}))\), because \(\ker(R) \perp \text{im}(R)\) and \(\text{im}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{m}))\) is the orthogonal complement of \(\text{ad}(\mathfrak{h}^{ss} \oplus \mathfrak{t}^{ss})\) in \(\text{ad}(\mathfrak{h} \oplus \mathfrak{t})\).

If \(\mathfrak{e}_1 = \{0\}\), then \(R_0(\text{ad}(h)) \cap R_\psi(\psi(\mathfrak{t})) = \{0\}\). Therefore, \(\ker(R) = \{0\}\), because \(R_0 : \text{ad}(h) \rightarrow \text{ad}(h)\) and \(R_\psi : \psi(\mathfrak{t}) \rightarrow \psi(\mathfrak{t})\) are both injective.

Since \(\text{ad}(\mathfrak{h} \oplus \mathfrak{t}) \subset \{h \in \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m}) : h \cdot T = 0, h \cdot R = 0\}\) we get a Lie algebra homomorphism

\[
q : \mathfrak{g}(\mathfrak{t}) \longrightarrow \text{ad}(\mathfrak{h} \oplus \mathfrak{t}) \oplus \mathfrak{n} \oplus \mathfrak{m}; \quad h + k + n + m \mapsto \text{ad}(h + k) + n + m,
\]

where \(\text{ad}(\mathfrak{h} \oplus \mathfrak{t}) \oplus \mathfrak{n} \oplus \mathfrak{m}\) is a subalgebra of the symmetry algebra defined in (2.6). Note that \(\mathfrak{f}\) is a ideal of \(\text{ad}(\mathfrak{h} \oplus \mathfrak{t}) \oplus \mathfrak{n} \oplus \mathfrak{m}\). Let \(\mathfrak{a} \subset \mathfrak{t} \oplus \mathfrak{n}\) be the diagonal subspace. It is easy to see that \(\mathfrak{a}\) is an abelian ideal. Furthermore, let \(p : \mathfrak{g}(\mathfrak{t}) \rightarrow \mathfrak{g}(\mathfrak{t})/\mathfrak{a}\) be the quotient Lie algebra homomorphism. We summarize this in the following diagram:

\[
\begin{array}{ccc}
\mathfrak{g}(\mathfrak{t}) & \xrightarrow{q} & \mathfrak{g}(\mathfrak{t})/\mathfrak{a} \\
\mathfrak{f} \hookrightarrow & & \mathfrak{f} \\
\text{ad}(\mathfrak{h} \oplus \mathfrak{t}) \oplus \mathfrak{n} \oplus \mathfrak{m} & \xrightarrow{p} & \mathfrak{g}(\mathfrak{t})/\mathfrak{a} \cong \mathfrak{k} \times \mathfrak{g}.
\end{array}
\]

The following proposition proves when the canonical base space of a \((\mathfrak{t}, B)\)-extension of \(g = \mathfrak{h} \oplus \mathfrak{m}\) is again equal to \(g = \mathfrak{h} \oplus \mathfrak{m}\). In the following the diagonal in \(\mathfrak{t}^{ss} \oplus \mathfrak{n}^{ss}\) is denoted by \(\mathfrak{a}^{ss}\).

**Proposition 4.4.** Let \(g = \mathfrak{h} \oplus \mathfrak{m}\) be as in (4.1) and let \(\mathfrak{f}\) be the transvection algebra of a \((\mathfrak{t}, B)\)-extension as in (4.2) with maximal abelian ideal \(\mathfrak{d} \subset \mathfrak{f}\). The following are equivalent

\[
\pi_{n \oplus m}(\mathfrak{d}) = \mathfrak{n} \iff \begin{cases} (i) \quad \pi_m(\mathcal{Z}(\mathfrak{b}_1)) = \{0\} \quad \text{and}, \\ (ii) \quad \ker(R) = \{0\}, \end{cases}
\]

where \(\pi_{n \oplus m}\) and \(\pi_m\) denote the projections onto \(n \oplus m\) and \(m\), respectively.
Proof. Suppose that $\pi_{n_{\otimes m}}(0) \subseteq n$. Let $n \in \pi_{n_{\otimes m}}(0) \cap n$ and $n \neq 0$. Let $k \in k$ be the element corresponding to $n$. From Lemma 1.3 we know that $\psi(k) \subset \text{im}(R)$, thus $q(a^*) \subset f$. Note that $a^* \subset g(k)$ is an abelian ideal. Thus, the subalgebra $q(a^*)$ is also an abelian ideal in $ad(h \oplus k) \oplus m$, because $q$ is a surjective Lie algebra homomorphism. Therefore, $q(a^*) \subset d$ and we obtain $n^* \subset \pi_{n_{\otimes m}}(0)$. This implies $k \in Z(k)$. Suppose that $\psi(k) \in \text{im}(R)$. It is easy to see that $k + n \in \text{Z}(g(k))$. The homomorphism $q$ is surjective and thus $q(k + n) = \psi(k) + n \in Z(f)$ and 

$$\text{span}\{\psi(k) + n\} \oplus d \subset f$$

is an abelian ideal. This contradicts the maximality of $d$. We conclude that $\psi(k) \notin \text{im}(R)$ and thus that $\ker(R) \neq \{0\}$. We have shown that $(ii)$ does not hold. Now we can assume that $n \subset \pi_{n_{\otimes m}}(d)$. 

Suppose 

$$\text{ad}(h' + k') + m \in d, \quad \text{with} \quad m \in m \setminus \{0\}.$$ 

We will use the diagram (1.5) to transfer the abelian ideal $d \subset f$ to $g$ and conclude that $\pi_{m}(\text{Z}(b_1)) \neq \{0\}$. By Lemma 3.17 we know that $d$ is also preserved by all derivations of $f$. As pointed out above, $f \subset \text{ad}(h \oplus k) \oplus n \oplus m$ is an ideal. It follows that $d$ is also an abelian ideal in $\text{ad}(h \oplus k) \oplus n \oplus m$. Note that $\ker(q) \subset \text{Z}(h \oplus k)$ and $\ker(q)$ commutes with $n \oplus m$, thus $\ker(q) \subset \text{Z}(g(h))$. The subspace $q^{-1}(d)$ is a 2-step nilpotent ideal in $g(h)$ with $q^{-1}(d)$ contained in its center. Therefore, the subalgebra $d := q(q^{-1}(d))$ is a 2-step nilpotent ideal in $f \times g$. The reductive decomposition $\text{ad}(h \oplus k) \oplus n \oplus m$ is effective. Thus, we know that $q(a) + d$ is an abelian ideal in $\text{ad}(h \oplus k) \oplus n \oplus m$, see Lemma 3.14 Let $\text{ad}(u) + n \in d$ with $u \in h \oplus k$ and $n \in n$. Let $k \in k$ such that $k + n \in a$. We have $\text{ad}(u - k) = \text{ad}(u) + n - (\text{ad}(k) + n) \in q(a) + d$. From Lemma 3.8 $(ii)$ we obtain $\text{ad}(u) = \text{ad}(k)$ and thus $q(a) \subset d$. In particular, for every $k + n \in a$ we have 

$$0 = [\text{ad}(h' + k') + m, \text{ad}(k) + n] = [\text{ad}(k'), \text{ad}(k) + n],$$

where we used Lemma 3.10. This implies that $k' \subset \text{Z}(g)$ Let 

$$\tilde{d} := p(h' + k' + m) = k' + h' + m = k' + g_1 + \cdots + g_k + x \in \tilde{d},$$

where $x \in R^n$ and $g_i \in g_i$ with $g_i$ a simple ideal of $g$ for $i = 1, \ldots, k$. Consider 

$$[\tilde{d}, g_i] \in \tilde{d} \cap g_i,$$

for $i = 1, \ldots, k$. If $[\tilde{d}, g_i] \neq \{0\}$, then this implies that $g_i \subset \tilde{d}$, because $g_i$ is simple and $\tilde{d}$ is an ideal. This is not possible because $\tilde{d}$ is 2-step nilpotent and $g_i$ is simple. We conclude that $[\tilde{d}, g_i] = \{0\}$. Suppose that $y \in R^n$ and $(k', y) = z \neq 0$. Then $\tilde{d}, y) = z \in \tilde{d} \cap R^n$. Moreover, $w := (k', z) \in \tilde{d} \cap R^n$ and $g(w, y) = g(k', z) = -g(z, z) \neq 0$. In particular $w \neq 0$. We already saw that $q(a) \subset d$. Therefore, $q^{-1}(\tilde{d}) = \text{q}^{-1}(d)$ and $q(p^{-1}(\tilde{d})) = d$. It follows that $z, w \in d \subset f$. If we take the Lie bracket of $z$ and $w$ in $f$, we obtain 

$$[z, w] = \sum_{i=1}^{l} g([k_i, z], w)\text{ad}(n_i + k_i) = \sum_{i=1}^{l} g([k_i, z], [k', z])\text{ad}(n_i + k_i) \neq 0,$$

where $k_1, \ldots, k_l$ is an orthonormal basis of $k$ with respect to $B$ and $n_i$ is the corresponding basis of $n$. This contradicts the fact that $\tilde{d}$ is abelian. We conclude that $(k', y) = 0$ for all $y \in R^n$. In other words $k' \subset \text{Der}(g)$ and we showed $k' \subset \text{Der}(g_1 \oplus \cdots \oplus g_k) \subset \text{Der}(g)$ and $k' = -\text{ad}(g_1 + \cdots + g_k) = -\text{ad}(h' + m)$. Hence we see that $0 \neq k' \subset \text{Z}(g)$ and $\pi_{m}(h' + m) = m \neq 0$. Remember that we defined $b_1 \subset \text{g}_1 \oplus \cdots \oplus \text{g}_k$ by $\text{ad}(b_1) = \text{e}_i$ in Definition 4.2. This proves that $(i)$ does not hold. 

For the converse, suppose $\pi_{n_{\otimes m}}(0) = n$. Let $n \in n$ and $\omega \in \text{im}(R)$ such that $\omega + n \in d$. Let $k \in k$ be the corresponding element of $n$. Note that Lemma 3.10$(iv)$ and Remark 3.11 imply $\omega = \psi(k)$. Thus for all $k \in k$ we get $\psi(k) \in \text{im}(R)$ and if $\omega' \in \ker(R)$, then $B_{ax}(\omega', \psi(k)) = 0$. It follows that 

$$0 = R(\omega') = R_0(\omega') + R_\psi(\omega') = R_0(\omega') + \sum_{i=1}^{l} B_{ax}(\omega', \psi(k_i))\psi(k_i) = R_0(\omega'),$$
where \(k_1, \ldots, k_l\) is an orthonormal basis of \(\mathfrak{k}\). This implies \(\omega' \perp \mathfrak{ad}(\mathfrak{h})\), because \(\text{im}(R_0) = \mathfrak{ad}(\mathfrak{h})\) and \(R_0\) is symmetric with respect to \(B_{\mathfrak{g}_2}\). We have \(\omega' \perp \mathfrak{ad}(\mathfrak{h} \oplus \mathfrak{k})\) and thus \(\omega' = 0\). We conclude that \(\ker(R) = \{0\}\).

Finally, we still need to show that if \(\ker(R) = \{0\}\) and \(\pi_m(Z(\mathfrak{b}_1)) \neq \{0\}\), then \(\pi_{m \oplus} (\mathfrak{d}) \neq \mathfrak{n}\). Let \(b = h + m \in Z(\mathfrak{b}_1) \subset \mathfrak{h} \oplus m\) with \(m \neq 0\). Let \(n \in \mathfrak{n}\) and \(k \in \mathfrak{k}\) be the elements corresponding to \(b\). Since \(\ker(R) = \{0\}\) we know that \(\psi(k) \in \mathfrak{im}(R)\) and \(\mathfrak{ad}(h) \in \mathfrak{im}(R)\). We easily see that \(-\psi(k) + \mathfrak{ad}(h) + m \in Z(\mathfrak{f})\) and thus in particular that \(-\psi(k) + \mathfrak{ad}(h) + m \in \mathfrak{d}\). We have \(0 \neq m \in \pi_{m \oplus} (\mathfrak{d})\) and \(m \notin \mathfrak{n}\) and thus \(\pi_{m \oplus} (\mathfrak{d}) \neq \mathfrak{n}\). \(\square\)

From the above lemma we see that if \(\pi_{m \oplus} (\mathfrak{d}) = \mathfrak{n}\), then \(\mathfrak{im}(R) = \mathfrak{ad}(\mathfrak{h} \oplus \mathfrak{k})\) and \(\mathfrak{ad}(Z(\mathfrak{f}_1)) \subset \mathfrak{ad}(\mathfrak{h})\). More precisely, we obtain
\[
\mathfrak{im}(R) = \mathfrak{ad}(\mathfrak{h} \oplus \mathfrak{k}) = \mathfrak{ad}(\mathfrak{h}) \oplus \psi(\mathfrak{t}_1 \oplus \mathfrak{t}_2 \oplus \mathfrak{t}_3).
\]

Note that his is a formula for the holonomy algebra of the naturally reductive connection of a \((\mathfrak{k}, \mathfrak{B})\)-extension.

**Remark 4.5.** In [23 Sec. 2.3] an explicit description of \((\mathfrak{k}, \mathfrak{B})\)-extensions of spaces as in \([4, 11]\) is given under the additional assumption that \(\mathfrak{b}_i \subset Z(\mathfrak{h}) \oplus \mathfrak{p}\) splits as \(\mathfrak{b}_i = \mathfrak{b}_{1,i} \oplus \mathfrak{b}_{2,i}\) with \(\mathfrak{b}_{1,i} \subset Z(\mathfrak{h})\) and \(\mathfrak{b}_{2,i} \subset \mathfrak{p}\) for \(i = 1, 2\). Note that Proposition 4.4 condition (i) implies this assumption. Consequently, this together with Theorem 3.15 implies that [23 Sec. 2.3] describes really all naturally reductive spaces.

Next we give a criterion when two \((\mathfrak{k}, \mathfrak{B})\)-extensions are isomorphic.

**Proposition 4.6.** For \(i = 1, 2\) let \(\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i\) be naturally reductive decompositions with \(\mathfrak{g}_i\) their transvection algebras and with \(\mathfrak{g}_i\) of the form
\[
\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i \oplus \mathfrak{L}_a \mathbb{R}^{n_i},
\]
where \(\mathfrak{h}_i \oplus \mathfrak{m}_i\) is semisimple or \(\{0\}\). Let \((\mathfrak{T}_i, \mathfrak{R}_i)\) be the infinitesimal model of \(\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i\) for \(i = 1, 2\). Furthermore, let \(\mathfrak{f}_i = \mathfrak{r}_i \oplus \mathfrak{n}_i \oplus \mathfrak{m}_i\) be the transvection algebra of the \((\mathfrak{f}_i, \mathfrak{B}_i)\)-extension of \((\mathfrak{T}_i, \mathfrak{R}_i)\), where \(\mathfrak{r}_i\) is the isotropy algebra. Suppose \(\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i\) is the canonical base space of the \((\mathfrak{f}_i, \mathfrak{B}_i)\)-extension for \(i = 1, 2\) and that the \((\mathfrak{f}_1, \mathfrak{B}_1)\)-extension and \((\mathfrak{f}_2, \mathfrak{B}_2)\)-extension are isomorphic. Then there is a Lie algebra isomorphism \(\tau : \mathfrak{g}_1 \to \mathfrak{g}_2\). Furthermore, \(\tau(\mathfrak{h}_1) = \mathfrak{h}_2\), \(\tau|_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{m}_2\) is an isometry and \(\tau_* : \mathfrak{t}_1 \to \mathfrak{t}_2\) is an isometry, where \(\tau_* : \text{Der}(\mathfrak{g}_1) \to \text{Der}(\mathfrak{g}_2)\) is the induced map on derivations.

**Proof.** From Lemma 2.4 we obtain a Lie algebra isomorphism
\[
\sigma : \mathfrak{f}_1 \to \mathfrak{f}_2,
\]
such that \(\sigma(\mathfrak{r}_1) = \mathfrak{r}_2\) and \(\sigma\) preserves the unique bilinear form from Kostant’s theorem, see Theorem 24.12. The maximal abelian ideal \(\mathfrak{a}_1\) of \(\mathfrak{f}_1\) is bijectively mapped to the maximal abelian ideal \(\mathfrak{a}_2\) of \(\mathfrak{f}_2\) by \(\sigma\). This implies that \(\sigma(\mathfrak{h}_1) = \mathfrak{h}_2\) and thus we obtain \(\sigma(\mathfrak{m}_1) = \mathfrak{m}_2\), because \(\sigma|_{\mathfrak{h}_1 \oplus \mathfrak{m}_1} : \mathfrak{h}_1 \oplus \mathfrak{m}_1 \to \mathfrak{h}_2 \oplus \mathfrak{m}_2\) is an isometry. For all \(x, y \in \mathfrak{m}_1\) we obtain
\[
\sigma(T_1(x, y)) = -\sigma([x, y]_{\mathfrak{m}_1}) = -[\sigma(x), \sigma(y)]_{\mathfrak{m}_2} = T_2(\sigma(x), \sigma(y))
\]
and
\[
\sigma(R_1(x, y)) = -\sigma(\mathfrak{ad}(x, y))_{\mathfrak{m}_2} = -\mathfrak{ad}(\sigma(x), \sigma(y))_{\mathfrak{m}_2} = R_2(\sigma(x), \sigma(y)),
\]
where \(\sigma\) also denotes the linear map \(\mathfrak{L}^2\mathfrak{m}_1 \to \mathfrak{L}^2\mathfrak{m}_2\) induced by \(\sigma|_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{m}_2\). By Lemma 2.4 the isometry \(\sigma|_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{m}_2\) induces a Lie algebra isomorphism \(\tau : \mathfrak{g}_1 \to \mathfrak{g}_2\), which satisfies \(\tau(\mathfrak{h}_1) = \mathfrak{h}_2\) and \(\tau|_{\mathfrak{m}_1} = \sigma|_{\mathfrak{m}_1}\) is an isometry. Recall from Lemma 2.15 that \(\tau|\mathfrak{g}_1 = \{x \in \text{so}_h(\mathfrak{m}_1) : h \cdot T_1 = 0\}\). Under this identification \(\tau_* : \mathfrak{f}_1 \to \mathfrak{f}_2\) is given by \(\tau_*|_{\mathfrak{t}_1} : \mathfrak{t}_1 \to \mathfrak{t}_2\) is given by the isometry \(\sigma|_{\mathfrak{h}} : \mathfrak{h}_1 \to \mathfrak{h}_2\). Therefore, \(\tau_*|_{\mathfrak{t}_1} : \mathfrak{t}_1 \to \mathfrak{t}_2\) is given by the isometry \(\sigma|_{\mathfrak{h}} : \mathfrak{h}_1 \to \mathfrak{h}_2\). \(\square\)
This proposition also implies that the canonical base space is unique for every space. It can be quite non-trivial to see whether two infinitesimal models \((T_1, R_2)\) and \((T_2, R_3)\) on \((m, g)\) are equivalent.

We can view the canonical base space as an invariant of the infinitesimal model. For a base space \(g = h \oplus m_0 \oplus L.a. \mathbb{R}^n\) it is also quite tractable to decide when two algebras \(\mathfrak{t}_1, \mathfrak{t}_2 \subset s(g)\) are conjugate to each other and thus to decide if two naturally reductive spaces are isomorphic.

We are mainly interested in irreducible naturally reductive spaces. This is now investigated for type II. Suppose that \(g = h \oplus m\) is a naturally reductive decomposition of type II with \(g\) its transvection algebra. Furthermore, suppose that the naturally reductive decomposition is reducible, i.e.

\[
g = (h_1 \oplus m_1) \oplus L.a. (h_2 \oplus m_2),
\]

see Lemma 2.13. Let \(a \subset g\) be the maximal abelian ideal. Let \(\pi_i : g \to g_i := h_i \oplus m_i\) be the projection for \(i = 1, 2\). Now \(\pi_i(a) \subset g_i\) is an abelian ideal in \(g_i\). Hence \(\pi_1(a) \oplus \pi_2(a)\) is also an abelian ideal of \(g\). We have \(a \subset \pi_1(a) \oplus \pi_2(a)\) and \(a\) is maximal, thus \(a = \pi_1(a) \oplus \pi_2(a)\). This means that if a reductive decomposition of type II is reducible, then we also obtain a decomposition \(g^\perp = g_1^\perp \oplus g_2^\perp\), where \(g^\perp\) is the transvection algebra of the canonical base space as obtained in Lemma 3.10. Moreover, the Lie algebra \(h^\perp\) splits as an orthogonal direct sum \(h^\perp = h_1^\perp \oplus h_2^\perp\), with \(h_i^\perp \subset s(g_i^\perp)\). Note that it is also possible that \(g_1^\perp = \{0\}\) or \(g_2^\perp = \{0\}\). Conversely, if we start with a base space

\[
(g^\perp = g_1^\perp \oplus g_2^\perp, h_1 \oplus h_2, g),
\]

with \(h_i \subset g_i^\perp\) and \(g_i^\perp\) ideals of \(g^\perp\), a Lie algebra \(\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2\) with \(\mathfrak{t}_i \subset s(g_i^\perp)\) and \(\mathfrak{t}_1 \perp \mathfrak{t}_2\) with respect to \(B\), then the \((\mathfrak{t}, B)\)-extension is clearly always reducible. The above discussion also implies that if a type II space admits a partial dual pair, then it is reducible if and only if its partial dual pair is reducible.

We would also like to have a criterion when a \((\mathfrak{t}, B)\)-extension of a naturally reductive decomposition \(g = h \oplus m \oplus L.a. \mathbb{R}^n\) is irreducible. The following proposition will give us such a criterion. For this it is good to remember that the algebra \(s(h \oplus m \oplus L.a. \mathbb{R}^n) \cong \mathcal{Z}(h) \oplus p \oplus so(\mathbb{R}^n)\), where \(p = \{m \in m : [h, m] = 0, \forall h \in h\}\).

**Proposition 4.7.** Let \(g = h \oplus m \oplus L.a. \mathbb{R}^n\) be an effective naturally reductive decomposition, with \(g\) its transvection algebra and \(h \oplus m\) semisimple. Furthermore, let \(\mathfrak{t} \subset s(g)\) and let \(B\) be some \(ad(\mathfrak{t})\)-invariant inner product on \(\mathfrak{t}\). Consider the following decomposition

\[
(4.7) \quad g = (h_1 \oplus m_1) \oplus L.a. \cdots \oplus L.a. (h_p \oplus m_p) \oplus L.a. m_{p+1} \oplus L.a. \cdots \oplus L.a. m_{p+q},
\]

where \(h_i \oplus m_i\) is an irreducible naturally reductive decomposition with \(h_i \subset h\) and \(m_i \subset m\) for \(i = 1, \ldots, p\) and \(m_{p+j} \subset \mathbb{R}^n\) is an irreducible \(\mathfrak{t}\)-module for \(j = 1, \ldots, q\). We choose the \(m_1, \ldots, m_{p+q}\) mutually orthogonal. Suppose that \(g = h \oplus m \oplus L.a. \mathbb{R}^n\) is the canonical base space of the \((\mathfrak{t}, B)\)-extension. The \((\mathfrak{t}, B)\)-extension is reducible if and only if there exists a non-trivial partition:

\[
\{m_1, \ldots, m_p, m_{p+1}, \ldots, m_{p+q}\} = W' \cup W'', \quad W' \cap W'' = \emptyset,
\]

and an orthogonal decomposition of ideals \(\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{t}''\) with respect to \(B\) such that \(\mathfrak{t}'\) acts trivially on all elements of \(W''\) and \(\mathfrak{t}''\) acts trivially on all elements of \(W'\).

**Proof.** If such a partition exists, then it is clear from the formula of the \((\mathfrak{t}, B)\)-extension and Theorem 2.10 that the \((\mathfrak{t}, B)\)-extension is reducible.

For the converse we suppose that the \((\mathfrak{t}, B)\)-extension is reducible. Let \(v := \{v \in m \oplus \mathbb{R}^n : \varphi(k)v = 0, \forall k \in \mathfrak{t}\}\). Suppose that \(m_i \subset v\) for some \(i = 1, \ldots, p\). Then we can define a partition by \(W' := \{m_i\}, W'' := \{m_1, \ldots, m_i, \ldots, m_{p+q}\}\) and define \(\mathfrak{t}' := \{0\}\) and \(\mathfrak{t}'' := \mathfrak{t}\). From now on we can assume that no \(m_i\) contained in \(v\). Let \(j\) be the transvection algebra of the \((\mathfrak{t}, B)\)-extension \((T, R)\). If the \((\mathfrak{t}, B)\)-extension is reducible, then by Lemma 2.13 there exist two orthogonal ideals \(f_1 \subset f\) and \(f_2 \subset j\) with respect to the unique bilinear form from Kostant’s theorem, such that \(j = f_1 \oplus f_2\) and \(\text{im}(R) = f_1 \oplus f_2\) with \(f_i \subset f_i\). Let \(a\) be the maximal abelian ideal. Let \(\pi_i : f \to f_i\) be the projections for \(i = 1, 2\). Now \(\pi_1(a) \subset f_1\) is an abelian ideal in \(f_1\). Hence also \(\pi_1(a) \oplus \pi_2(a)\) is an abelian ideal of \(f\). Since \(a \subset \pi_1(a) \oplus \pi_2(a)\) and \(a\) is maximal we obtain \(a = \pi_1(a) \oplus \pi_2(a)\). Hence \(n = n' \oplus n''\) with \(n' \subset f_1\)
and $\mathfrak{n}'' \subset \mathfrak{f}_2$. In particular this implies that $\mathfrak{n}' \perp \mathfrak{n}''$. Let $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{t}''$ be the corresponding orthogonal decomposition of $\mathfrak{t}$. We will now show for all $\mathfrak{m}_i$ that either $\mathfrak{m}_i \subset \mathfrak{f}_1$ or $\mathfrak{m}_i \subset \mathfrak{f}_2$. Since there is no $\mathfrak{m}_i$ contained in $\mathfrak{v}$ we have

$$\mathbb{R}^n = [\mathfrak{t}, \mathbb{R}^n] = [\mathfrak{t}', \mathbb{R}^n] + [\mathfrak{t}'', \mathbb{R}^n].$$

Note that $[\mathfrak{t}', \mathbb{R}^n] \subset \mathfrak{f}_1$ and $[\mathfrak{t}'', \mathbb{R}^n] \subset \mathfrak{f}_2$, hence $\mathbb{R}^n = [\mathfrak{t}', \mathbb{R}^n] \oplus [\mathfrak{t}'', \mathbb{R}^n]$. This implies that $\mathfrak{m}_{p+j}$ is contained in either $\mathfrak{f}_1$ or $\mathfrak{f}_2$ for all $j = 1, \ldots, q$. We consider the case that $\mathfrak{h}_1 \oplus \mathfrak{m}_i$ is not a reductive decomposition of an irreducible symmetric space. Note that $[\mathfrak{t}, \mathfrak{m}_i] \neq \{0\}$, because we assumed that $\mathfrak{m}_i$ is not contained in $\mathfrak{v}$. Suppose that $v \in [\mathfrak{t}', \mathfrak{m}_i]$ for some $v \neq 0$ and $1 \leq i \leq p$. Then $v \in \mathfrak{f}_1 \cap \mathfrak{m}_i$. Define $V_0 := \{v\}$ and $V_j := \text{span}(V_{j-1}, [V_{j-1}, \mathfrak{m}_i]_{\mathfrak{m}_i})$ for $j \geq 1$. By assumption $\mathfrak{h}_1 \oplus \mathfrak{m}_i$ is an irreducible decomposition. It is easy to see that this implies there exists a $p \in \mathbb{N}$ for which $V_p = \mathfrak{m}_i$. Since $\mathfrak{f}_1$ is an ideal we conclude that $\mathfrak{m}_i \subset \mathfrak{f}_1$. Similarly with $\mathfrak{t}'$ replaced by $\mathfrak{t}''$ and $\mathfrak{f}_1$ replaced by $\mathfrak{f}_2$. If $\mathfrak{h}_1 \oplus \mathfrak{m}_i$ defines an irreducible symmetric space, then $\mathfrak{s}(\mathfrak{h}_1 \oplus \mathfrak{m}_i) = \mathcal{Z}(\mathfrak{h}_1)$. If $\mathcal{Z}(\mathfrak{h}_1) = \{0\}$, then $\mathfrak{m}_i \subset \mathfrak{v}$ and this we assumed not to be the case. The irreducible symmetric spaces for which $\mathcal{Z}(\mathfrak{h}_1) \neq 0$ are exactly the irreducible hermitian symmetric spaces and $\mathcal{Z}(\mathfrak{h}_1)$ is then 1-dimensional. 

Theorem 3.18 together with the results in [23] give us an explicit construction for any naturally reductive space. Furthermore, we showed in this section that this general formula of a naturally reductive space allows us to decide when two naturally reductive spaces are isomorphic or whether one naturally reductive space is irreducible. In a forthcoming paper we will illustrate the use of these results by classifying all naturally reductive spaces up to dimension 8.

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References

[1] Ilka Agricola, Ana Cristina Ferreira, and Thomas Friedrich. The classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$. Differential Geom. Appl., 39:59–92, 2015. URL: http://dx.doi.org/10.1016/j.difgeo.2014.11.005

[2] Ilka Agricola, Ana Cristina Ferreira, and Reinier Storm. Quaternionic Heisenberg groups as naturally reductive homogeneous spaces. Int. J. Geom. Methods Mod. Phys., 12(8):1560007, 10, 2015. URL: http://dx.doi.org/10.1142/S0219887815600075

[3] Ilka Agricola and Thomas Friedrich. 3-Sasakian manifolds in dimension seven, their spinors and $G_2$-structures. J. Geom. Phys., 60(2):326–332, 2010. URL: https://doi.org/10.1016/j.geomphys.2009.10.003

[4] W. Ambrose and I. M. Singer. On homogeneous Riemannian manifolds. Duke Math. J., 25:647–669, 1958.

[5] David Blair and Lieven Vanhecke. New characterizations of $\phi$-symmetric spaces. Kodai Math. J., 10(1):102–107, 1987. URL: http://dx.doi.org/10.2996/kmj/1138037365

[6] David Blair and Lieven Vanhecke. Symmetries and $\phi$-symmetric spaces. Tohoku Math. J. (2), 39(3):373–383, 1987. URL: http://dx.doi.org/10.2748/tmj/1178228284

[7] Charles P. Boyer, Krzysztof Galicki, and Benjamin M. Mann. The geometry and topology of 3-Sasakian manifolds. J. Reine Angew. Math., 455:183–220, 1994.

[8] Jean-Baptiste Butruille. Classification des variétés approximativement kählériennes homogènes. Ann. Global Anal. Geom., 27(3):201–225, 2005. URL: https://doi.org/10.1007/s10455-005-1581-x

[9] Jean-Baptiste Butruille. Homogeneous nearly Kähler manifolds. In Handbook of pseudo-Riemannian geometry and supersymmetry, volume 16 of IRMA Lect. Math. Theor. Phys., pages 399–423. Eur. Math. Soc., Zürich, 2010. URL: https://doi.org/10.4171/079-1/11

[10] R. Cleyton, A. Moroianu, and U. Semmelmann. Metric connections with parallel skew-symmetric torsion. ArXiv e-prints, June 2018. arXiv:1807.00191
[11] J. E. D’Atri and W. Ziller. Naturally reductive metrics and Einstein metrics on compact Lie groups. *Mem. Amer. Math. Soc.*, 18(215):iii+72, 1979.

[12] Th. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann. On nearly parallel $G_2$-structures. *J. Geom. Phys.*, 23(3-4):259–286, 1997. URL: http://dx.doi.org/10.1016/S0393-0440(97)80004-6

[13] Thomas Friedrich. $G_2$-manifolds with parallel characteristic torsion. *Differential Geom. Appl.*, 25(6):632–648, 2007. URL: http://dx.doi.org/10.1016/j.difgeo.2007.06.010

[14] Thomas Friedrich and Stefan Ivanov. Parallel spinors and connections with skew-symmetric torsion in string theory. *Asian J. Math.*, 6(2):303–335, 2002.

[15] Sigurður Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original. URL: http://dx.doi.org/10.1090/gsm/034

[16] O. Kowalski. Counterexample to the “second Singer’s theorem”. *Ann. Global Anal. Geom.*, 8(2):211–214, 1990. URL: http://dx.doi.org/10.1007/BF00128004

[17] O. Kowalski and L. Vanhecke. Four-dimensional naturally reductive homogeneous spaces. *Rend. Sem. Mat. Univ. Politec. Torino*, (Special Issue):223–232 (1984), 1983. Conference on differential geometry on homogeneous spaces (Turin, 1983).

[18] Kazumi Nomizu. Invariant affine connections on homogeneous spaces. *Amer. J. Math.*, 76:33–65, 1954.

[19] Toshio Takahashi. Sasakian $\phi$-symmetric spaces. *Tohoku Math. J.*, (2), 29(1):91–113, 1977.

[20] F. Tricerri and Lieven Vanhecke. Homogeneous structures on Riemannian manifolds, volume 83 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1983. URL: http://dx.doi.org/10.1017/CBO9781107325531

[21] Kazumi Tsukada. Totally geodesic hypersurfaces of naturally reductive homogeneous spaces. *Osaka J. Math.*, 33(3):697–707, 1996. URL: http://projecteuclid.org/euclid.ojm/1200787097

[22] McKenzie Y. Wang and Wolfgang Ziller. On normal homogeneous Einstein manifolds. *Ann. Sci. École Norm. Sup. (4)*, 18(4):563–633, 1985. URL: http://www.numdam.org/item?id=ASENS_1985_4_18_4_563_0

[23] Joseph A. Wolf. *Spaces of constant curvature*. AMS Chelsea Publishing, Providence, RI, sixth edition, 2011.

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