The Finsler and non-commutative geometry of point particle in massive gravity background

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Abstract

In this paper, we first investigated collision of two particles in the massive gravity black hole solution and obtained the center-of-mass energy and the effective potential. The corresponding potential plays an important role for describing stability and energy spectrum of the system. In that case, we have some figures. Also, here we obtained the corresponding Lagrangian and calculated some canonical relations and conserved quantities. These quantities helped us to make the Hamiltonian for the massive gravity system without charges. In order to apply the non-commutative geometry to the pointed Hamiltonian we modified such quantity in terms of new variables. The new variables lead us to have new Hamiltonian in form of harmonic oscillator model. And then we applied the non-commutative geometry to the corresponding deformed Hamiltonian and obtained the Lagrangian of massive gravity. Finally, we compared Lagrangian of massive gravity in non-commutative geometry with Finsler geometry.

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1 Introduction

In the introduction, we pay three subjects which are important in the present paper. So first of all, we are going to give some primary reviews on the massive gravity solution. One of the alternatives to explain the present acceleration of the Universe could be the modification of the gravitational law at large distances [1]. This is the main motivation to researcher working in this subject. As we know, the most natural modification that could be considered is to promote gravity from the non-linear theory of a massless spin-2 field. In addition to get a better understanding of GR from a field theoretical point of view, one can investigate the reasons that make the modification of GR so difficult. The most straightforward way to give mass to the graviton is to add to the Einstein-Hilbert action a potential that at the quadratic level produces the mass term for the spin-2 field and that vanishes when one sends the mass parameter to zero. Moreover, the propagation of five and only five degrees of freedom must be guaranteed for the full nonlinear theory. Indeed, adding a mass to the graviton means to add more degrees of freedom in the theory, whose effect is far from trivial. It results in a series of pathologies some of them are not still clear how to avoid. They span from strong classical non-linearities to ghost-like instabilities till very low cutoff for the resulting quantum effective field theory. For massive gravity, we have some problems since the massless limit is non-trivial and it leads to the formulation of the Einstein mechanism. For an introduction to the subject of massive gravity we recommend the following reviews [2-3]. In this paper we take such massive gravity solution and investigate the non-commutative geometry and Finsler geometry for the point particle.

The second section of paper is non-commutative (NC) geometry which has interesting application to physical problems in solid-state and particle physics [4-5]. Such geometry is mainly motivated by the idea of a strong connection of non-commutativity with field and string theories. Also we know that the string theory attempts to unify gravity and quantum mechanics, it ultimately leads to a non-commutative geometry space-time. The phase space of ordinary quantum mechanics is a well-known example of non-commuting space [6]. The momenta of a system in the presence of magnetic field are non-commuting operators as well. The non-commutativity between spatial and time coordinates lead to some problems with unitarity and causality, so usually we just consider spatial non-commutative. Besides, so far quantum theory on the NC space has been extensively studied, the main approach is based on the Weyl-Moyal product, in that case, the usual product replace by ⋆ in the NC space. Therefore, deformation quantization has special significance in the study of physical systems on the NC space. Moreover, the problem of quantum mechanics on NC spaces can be understood in the framework of deformation quantization [7-8]. The NC space from the point of view of deformation quantization for harmonic oscillators have been reported in [9-10]. For this reason we take advantage from information above and deform the Hamiltonian of massive gravity with non-commutative geometry space. In that case, we have some deformation parameter which are non-commutative geometry parameter. Due to small values of deformation parameters, we just keep in our calculation first order of such parameters.

The most important result here is about Finsler geometry, for this reason we give some explanation about such geometry. Generally one can say that the Finsler geometry has a large
potential of applications to physics or other mathematical sciences. As a usual a Finsler metric on $M$ is defined by a function of tangent bundle $TM$. We note that here the Finsler metric are satisfied by the regularity, positive homogeneity and strong convexity [11-13]. In this paper we study some Lagrangian systems and connect the non-commutative geometry (deformation) with Finsler geometry. As we know, one of the applications of Finsler geometry is Lagrangian system, not all systems are satisfied with regularity conditions. It means that Lagrangian is not defined on the whole slit tangent bundle $TM^0 = TM - \{0\}$ but only on a sub-bundle $D(L) \subset TM$ depending to $L$. On the other hand, the gauge theories play an important role to describing some symmetries in different branches of physics. So, the most important Lagrangian system in physics has a gauge theory and does not satisfy the strong convexity. Therefore, we will only consider a weaker regularity condition and the and the following positive homogeneity condition of $L$ as the definition of Finsler metric,

$$L(x, \lambda dx) = \lambda L(x, dx), \quad \forall \lambda > 0.$$  \hspace{1cm} (1)

Any physical and mathematical Lagrangian systems of finite degree of freedom can be reformulated in Finsler manifolds without changing their physical contents [14-15]. In that case, the action functional is given by the integral of the Finsler metric which is made from the Lagrangian. Then the variational principle becomes geometric and independent of parametrisation, which we will call covariant. From the point of view of a physicist, especially when thinking about the Lagrangian formulation, we are inclined to define a non-linear connection not on a line element space $TM^0$, but directly on the point manifold $M$ [16]. So, in this paper we make the usual Lagrangian from massive gravity metric background and obtain the corresponding Hamiltonian. In such approach, we take some new variable to write the Hamiltonian in form of harmonic oscillator Hamiltonian. This give us motivation to apply non-commutative geometry to the new Hamiltonian and obtain the corresponding Lagrangian. And then, we employ some information from Finsler metric and Lagrangian for the massive gravity system. We show that the corresponding Lagrangian in non-commutative geometry for the massive gravity background completely coincidence with Lagrangian from Finsler geometry with some specification of parameters. Also, we connect $\theta$ and $\beta$ from non-commutative geometry to $\eta$.

### 2 The solution of massive charged black hole

First of all we are going to consider action for a four-dimensional Einstein-Maxwell theory in the framework of massive gravity [17 − 19]. In that case, we need to present its corresponding black hole solution. By using the conventions of [20 − 23], one can write the massive Einstein-Maxwell action consisting of the Ricci scalar, electromagnetic field, cosmological constant term, graviton mass terms and a surface term [24 − 26], which is given by,

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2}(R - 2\Lambda) + \frac{m^2}{\kappa^2}(\alpha_1 u_1 + \alpha_2 u_2) - \frac{1}{16\pi}F^2 \right] + \frac{1}{\kappa^2} \int d^3x \sqrt{-\gamma} K, \quad (2)$$

where

$$u_1 = tr K,$$  \hspace{1cm} (3)

and
\[ u_2 = (tr K)^2 - tr(K^2), \]  
(4)

\[ \alpha_1, \alpha_2 \leq 0, \kappa^2 = 8\pi G, K^{\mu\nu} := \sqrt{g^{\mu\alpha}f_{\alpha\nu}}. \]  
The reference metric without dynamic behavior is chosen as \( f_{\mu\nu} = \text{diag}(0, 0, 1, \sin^2 \theta) \) and \( K \) is the trace of the extrinsic curvature. In order to have some Lagrangian for the massive gravity system, we need the black hole solution of action (1) with gauge field \( A_\mu \) which are given by,

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2, \quad (5) \]

\[ A_\mu = (A_t, 0, 0, 0). \quad (6) \]

With the help of these assumptions and following equations of motion,

\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{l^2}g_{\mu\nu} + m^2\alpha_1(K_{\mu\nu} - tr Kg_{\mu\nu}) + m^2\alpha_2[2(tr K)K_{\mu\nu} - 2K^{\mu}_{\alpha}K^{\nu}_{\alpha}] \]

\[ -m^2\alpha_2 g_{\mu\nu}[(tr K)^2 - tr(K^2)] = 2G(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2), \quad (7) \]

\[ \nabla_\mu F^{\mu\nu} = 0, \quad (8) \]

we can find \( f(r) \) and \( A_t \) as following,

\[ f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + \frac{r^2}{l^2} + m^2\alpha_1 r + 2m^2\alpha_2, \quad (9) \]

\[ A_t = -\frac{Q}{r}, \quad (10) \]

where \( M \) and \( Q \) are black hole mass parameter and total charge. This \( f(r) \) lead us to write the explicit form of Lagrangian for the massive gravity.

**3 Time-like geodesics of the test particles of the massive black hole**

In this section, we show the time-like geodesics of a test particle with charge per unit mass \( e \) around the black hole. In order to investigate test particle on the corresponding black hole we will follow the formalism in Chandrasekhar [27]. Generally, the Lagrangian of the charged test particle in different black hole background is given by,

\[ \mathcal{L} = \frac{1}{2}(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}) + eA_\mu \frac{dx^\mu}{d\tau}. \quad (11) \]

Here the parameter \( \tau \) is the proper time of the charged particle. The metric functions and \( A_\mu \) are given by,

\[ A_t = -\frac{Q}{r}; \quad (12) \]
and
\[ g_{tt} = -f(r), \quad g_{rr} = \frac{1}{f(r)}, \quad g_{\phi\phi} = r^2. \] (13)
So, the most important point about in this paper is the connection between the modified Lagrangian and Finsler Lagrangian in case of massive gravity solution. This leads us to assume some condition and consider the massive gravity solution without charge \( (Q = 0) \). On the other hand, the conservation of angular momentum confines the particle motion to plane. Also we conveniently take the equation plane with \( \theta = \frac{\pi}{2} \), \( d\theta = 0 \) and \( \sin \theta = 1 \). So we consider the black hole metric with the following line element,
\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\phi^2, \] (14)
the Lagrangian of black hole background without charge is given by
\[ \mathcal{L} = \frac{1}{2}(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}). \] (15)
One can define the conjugate momentum \( P_\mu \) to the coordinate \( x^\mu \) as,
\[ P_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x^\mu}} = g_{\mu\nu} \dot{x}^\nu, \] (16)
where
\[ \dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} = u^\mu. \] (17)
Now, we use the equations (15), (14) and (9) in case of \( Q = 0 \) and obtain the corresponding Lagrangian for the massive gravity as,
\[ \mathcal{L} = \frac{1}{2}(-f(r)t^2 + \frac{1}{f(r)}r^2 + r^2\dot{\phi}^2). \] (18)
In terms of the conjugate momenta, the Euler-Lagrange equations can be written as
\[ \frac{d}{d\tau} P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x^\mu}}. \] (19)
The above black hole has two Killing vectors: a time-like Killing vector \( k^\mu = (1, 0, 0, 0) \) and space-like Killing vector \( m^\mu = (0, 0, 0, 1) \). Massive gravity follows time-like geodesics along which two quantities are conserved due to the space-time symmetry represented by the two Killing vectors \( \partial_t \) and \( \partial_\phi \). These constants are given by \( E = -\partial_t U \) and \( L = \partial_\phi U \) where \( U, E \) and \( L \) correspond to the four-velocity of the massive gravity and its energy and angular momentum as observed from infinity, respectively. As we pointed before, for this study, we focus on time-like geodesics in the equatorial plane \( (\theta = \frac{\pi}{2}) \) implying \( U^\theta = 0 \). The radial motion of the massive particle in the metric is obtained by solving the equation \( U.U = -1 \). Hence, the canonical momenta \( P_t \) and \( P_\phi \) are conserved and these constants are labeled as energy per unit mass \( E \) and angular moment per unit mass \( L \). In that case the conserved quantities are given by,
\[ E \equiv -k_\mu u^\mu = -g_{tt} u^t = -P_t, \] (20)
\[ L \equiv m_\mu u^\mu = -g_{\varphi\mu} u^\mu = P_\varphi, \]  
(21)

where \( E \) is the energy at infinity and \( L \) is the angular momentum. In order to obtain \( \dot{t} \) and \( \dot{\varphi} \), we have to consider equations (17) and (18), so we obtain the following equation,

\[ \dot{t} = \frac{E}{f(r)} = u^t, \]  
(22)

\[ \dot{\varphi} = \frac{L}{r^2} = u^\varphi. \]  
(23)

Furthermore, the subject of effective potential and center mass energy play an important role in collision of two particles on the black hole background. In order to have information about such quantities, we employ the following equation,

\[ g_{\mu\nu} u^\mu u^\nu = \kappa, \]  
(24)

where \( \kappa = -1 \) for time-like geodesics. In that case, for obtaining effective potential we use eqs. (19-21) and \( \dot{r}^2 + V_{\text{eff}} = 0 \). So \( V_{\text{eff}} \) can be written by following equation,

\[ V_{\text{eff}} = f(r)(\frac{L^2}{r^2} + 1) - E^2. \]  
(25)

We calculated the roots of \( V_{\text{eff}} \) numerically, in that case we draw the \( V_{\text{eff}} \) with respect to \( r \)

![Graph](image)

Figure 1: The variation \( V_{\text{eff}} \) with respect to \( r \), where \( G = \frac{1}{8}, M = 12.25, Q = 0, E = 7.7, L = 11.5, m = 4.46, l = 1, \alpha_1 = -1, \alpha_2 = -0.5 \)

As we know there are several paper about center-of-mass black hole. For example, Banados, Silk and West (BSW) [28] have shown that free particles falling from rest at infinity outside a Kerr black holes may collide with arbitrarily high center-of-mass (CM) energy and hence the maximally rotating black hole might be regarded as a Planck-energy-scale collided. They proposed that this might lead to signals from ultra high energy collisions such as dark matter.
The center-of-mass energy and effective potential in above results give us opportunity to investigate the relation between Finsler and non-commutative geometry about Lagrangian of
two particle on the massive gravity background. So, first in next section try to understand non-commutative geometry of collision two particles in massive gravity background.

4 Non-commutative geometry on Lagrangian of massive gravity

In this section, we take the Lagrangian of massive gravity in equation (18). In that case, the canonical relations help to obtain the corresponding Hamiltonian. In order to apply non-commutative geometry to the pointed Hamiltonian, we need to write the Hamiltonian in terms of new variable. In that case, the new Hamiltonian corresponding to the Lagrangian of massive gravity will be a form of simple harmonic oscillator. So, this form of Hamiltonian leads us to apply non-commutative geometry. First we try to choose following variables

\begin{align*}
x_1 &= \sqrt{f(r)} \cosh t, \quad x_2 = r \sinh \varphi, \quad x_3 + y_3 = \sqrt{2r} + \sqrt{f(r)},\nonumber\ny_1 &= \sqrt{f(r)} \sinh t, \quad y_2 = r \cosh \varphi, \quad x_3 - y_3 = \sqrt{2r} - \sqrt{f(r)},
\end{align*}

and

\begin{align*}
x_4 &= \frac{1}{\sqrt{f(r)}} \sinh r, \quad x_5 + y_5 = \frac{1}{\sqrt{f(r)}} + r,\nonumber\ny_4 &= \frac{1}{\sqrt{f(r)}} \cosh r, \quad x_5 - y_5 = \frac{1}{\sqrt{f(r)}} - r.
\end{align*}

The Hamiltonian is,

\begin{equation}
H = \frac{1}{2} \sum_{i=1}^{5} ((P_{x_i}^2 - P_{y_i}^2) + \omega_i^2 (x_i^2 - y_i^2)),
\end{equation}

where

\begin{align*}
P_{x_i} &= \frac{\partial L}{\partial \dot{x}_i} = \dot{x}_i,\nonumber\P_{y_i} &= \frac{\partial L}{\partial \dot{y}_i} = -\dot{y}_i.
\end{align*}
and

$$\omega_i^2 = -1. \quad (33)$$

It is obvious that the Hamiltonian eq. (31) has an oscillator form that is useful for gravitational theories. Before applying non-commutative geometry to the Hamiltonian (31), one can need to review some stuff of such geometry.

As we know in commutative case we have usual Poisson brackets which are given by,

$$\{x_i, x_j\} = 0, \quad \{P_{x_i}, P_{x_j}\} = 0, \quad \{x_i, P_{x_j}\} = \delta_{ij}, \quad (34)$$

where $$x_i(i = 1, 2)$$ and $$P_{x_i}(i = 1, 2)$$. To compare non-commutative and commutative phase space we need to explain some approaches of non-commutative. In such approach, quantum effects can be dissolved by the Moyal brackets $$\{f, g\}_\alpha = f \star_\alpha g - g \star_\alpha f$$ which is based on the Moyal product as,

$$\left( f \star_\alpha g \right)(x) = \exp\left( \frac{1}{2} \lambda^{ab} \partial_a \partial_b \right) f(x_1)g(x_2) \bigg|_{x_1 = x_2 = x}. \quad (35)$$

After corresponding calculations, we find the algebra of variables as,

$$\{x_i, x_j\}_\alpha = \theta_{ij}, \quad \{x_i, P_{x_j}\}_\alpha = \delta_{ij} + \sigma_{ij}, \quad \{P_{x_i}, P_{x_j}\} = \beta_{ij}. \quad (36)$$

Transformations on the classical phase space variables are expressed as,

$$\hat{x}_i = x_i + \frac{\theta}{2} P_{y_i}, \quad \hat{y}_i = y_i - \frac{\theta}{2} P_{x_i}, \quad \hat{P}_{x_i} = P_{x_i} - \frac{\beta}{2} y_i, \quad \hat{P}_{y_i} = P_{y_i} + \frac{\beta}{2} x_i. \quad (37)$$

The deformed algebra for new variables are given by,

$$\{\hat{y}, \hat{x}\}_\theta = \theta, \quad \{\hat{x}, \hat{P}_{x}\} = \{\hat{y}, \hat{P}_{y}\} = 1 + \sigma, \quad \{\hat{P}_{y}, \hat{P}_{x}\} = \beta, \quad (38)$$

where $$\sigma = \frac{\beta \theta}{2}$$. In order to construct the deformed Hamiltonian of massive gravity, we borrow the form of Hamiltonian from eq. (31) with new variables in eq. (37). Hence, the form of Hamiltonian in the deformed analysis is found as,

$$\hat{H} = \frac{1}{2} \sum_{i=1}^{5} ((P_{x_i}^2 - P_{y_i}^2) - \gamma_i^2 (y_i P_{x_i} + x_i P_{y_i}) + \tilde{\omega}_i^2 (x_i^2 - y_i^2)), \quad (39)$$

where

$$\tilde{\omega}_i^2 = \frac{\omega_i^2 - \frac{\beta^2}{4}}{1 - \omega_i^2 \frac{\beta^2}{4}}, \quad \gamma_i^2 = \frac{\beta - \omega_i^2 \theta}{1 - \omega_i^2 \frac{\beta^2}{4}}. \quad (40)$$

The above deformed Hamiltonian lead us to have a new Lagrangian which is deformed form of original Lagrangian of massive gravity. Now, for the corresponding model we arrange the deformed Lagrangian as,

$$\hat{L} = \frac{1}{2} \sum_{i=1}^{5} ((P_{x_i}^2 - P_{y_i}^2) - \gamma_i^2 (y_i P_{x_i} + x_i P_{y_i}) - \tilde{\omega}_i^2 (x_i^2 - y_i^2)), \quad (41)$$
where
\[ \dot{\omega}_i^2 = \frac{\omega_i^2 + \frac{\beta^2}{4}}{1 + \omega_i^2 \frac{\beta^2}{4}}, \quad \dot{\gamma}_i^2 = \frac{\beta + \omega_i^2 \theta}{1 + \omega_i^2 \frac{\beta^2}{4}}. \] (42)

We use the equation (32), then the deformed Lagrangian can be written as following,
\[ \hat{\mathcal{L}} = \frac{1}{2} (-f(r)\dot{t}^2 + \frac{1}{f(r)} \dot{r}^2 + r^2 \dot{\phi}^2 - (\beta - \theta)(-f(r)\dot{t} + r^2 \dot{\phi}) - (\sqrt{2f(r)} + \frac{1}{f(r)} - \frac{\sqrt{2}}{2} \frac{\dot{f}(r)r}{\sqrt{f(r)}} - \frac{\dot{f}(r)r}{2f(r)\sqrt{f(r)}})\dot{r}), \] (43)

where
\[ \dot{f}(r) = \frac{d}{dr} f(r). \] (44)

In the next section, we shall find a counterpart for the above lagrangian in the context of Finsler geometry.

5 The Lagrangian of massive gravity in Finsler geometry

Now we can concentrate our attention on Finsler geometry issues and also we take advantage of Lagrangian in massive gravity system. At the first, we are going to review some properties in Finsler geometry. So, here we note that, a Finsler space \( F^n = (M, F) \) is an \( n \)-dimensional manifold \( M \) equipped with a Finsler metric \( F : TM \to \mathbb{R} \), \( (x, y) \mapsto F(x, y) \), \( x \in M, y \in TM \), with following conditions. First condition is regularity where \( F \) is a \( C^\infty \) function on \( TM \setminus \{0\} \), second condition is positive homogeneity (of degree 1) where \( F(x, \lambda Y) = \lambda F(x, y) \), \( \lambda \in \mathbb{R} \) and third condition is strong convexity where \( g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y_i \partial y_j}(x, y) \) is positively defined.

Now we are going to introduce the general form of metric in Finsler geometry which is given by following,
\[ F(x, y) = \alpha a(x, y) + \eta b(x, y) + \gamma \frac{a^2(x, y)}{b(x, y)} + \Xi(x, y), \] (45)

where \( a(x, y) = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric, \( a^{ij} \) is the inverse matrix of \( a_{ij} \) and \( b(x, y) \) is a differential one-form on \( M \) with \( \|b(x, y)\|_a := \sqrt{a^{ij}b_i b_j} < 1 \). Here, we used Einstein notation and such equation (44), defines a Finslerian metric. Also one can say that \( \Xi(x, y) \) is a Finsler fundamental function on \( TM \) and \( \alpha, \eta, \gamma \in \mathcal{F}(M) \). The first two terms of \( F \) in above metric determine a Randers metric. The Finsler metric (Lagrangian) corresponding to massive gravity solution without charge will be as,
\[
F(x, y) = \alpha \sqrt{-f(r)\dot{t}^2 + \frac{1}{f(r)} \dot{r}^2 + r^2 \dot{\phi}^2 + \eta(-f(r)\dot{t} + r^2 \dot{\phi}) + (\sqrt{2f(r)} + \frac{1}{f(r)} - \frac{\sqrt{2}}{2} \frac{\dot{f}(r)r}{\sqrt{f(r)}} - \frac{\dot{f}(r)r}{2f(r)\sqrt{f(r)}})\dot{r}).
\] (46)
Now, we compare such metric to equation (44) one can suppose that $\Xi = \gamma = 0$. The most important result here is the comparison of obtained results of Lagrangian between Finsler and non-commutative geometry. In order to compare such results in equations (43) and (44) we have to apply following conditions to equation (44),

$$r^2 \ll V_{\text{eff}}, \quad \alpha = \sqrt{\frac{l^2}{r^2} - \frac{E^2}{f(r)}}.$$  \hspace{1cm} (47)

Also, we note that the system with mentioned conditions has a stability as $\omega_{\varphi}^2 \gg 1$. According to the mentioned conditions $F(x, y)$ can be written by,

$$F(x, y) \simeq (-f(r)t^2 + \frac{1}{2f(r)} \dot{r}^2 + r^2 \dot{\varphi}^2) + \eta(-f(r)t + r^2 \dot{\varphi} + (\sqrt{2f(r)} + \frac{1}{f(r)} - \frac{1}{\sqrt{f(r)}} - \frac{\sqrt{2}}{2} \frac{\dot{f}(r)}{\sqrt{f(r)}} \frac{\dot{f}(r)}{2f(r)\sqrt{f(r)}} $$

$$\dot{r}).$$  \hspace{1cm} (48)

In the next section, we shall present a conceptional comparison between two geometries.

6 Conclusion

In this paper we first investigated two particle collision in the massive gravity black hole solution and obtained the center-of-mass energy. In that case we achieved the effective potential which is important for the stability and energy spectrum of system. Also, here we obtained the corresponding Lagrangian and calculated some canonical relations and conserved quantities. These quantities help us to make the Hamiltonian for the massive gravity system without charges. In order to apply the non-commutative geometry to the pointed Hamiltonian we modified such quantity in terms of new variables. The new variables lead us to have new Hamiltonian in form of harmonic oscillator model. And then we applied the non-commutative geometry to the corresponding deformed Hamiltonian and obtained the Lagrangian of massive gravity. Finally, by comparing Lagrangian of massive gravity in non-commutative and Finsler geometries (equations (43) and (48)), it is found that they can be the same if $\eta = \frac{\theta - \beta}{1 - \frac{\theta^2}{\beta}}$ for $\theta^2 = 0$, it means that this similarity lead us to consider both geometries be equivalent under some considerations. Another proof for this result can be found in the physical concept of the mentioned condition. As we know, Finsler geometry is related to the momentum component in addition to position (eq.(37)) and it can be traced in the mentioned condition as the dependence of $\eta$ to non-commutative parameters $\theta$ and $\beta$. When we compare the corresponding Lagrangian in massive gravity without charge for non-commutative and Finsler geometry the effective potential must be zero. In that case, the kinetic energy of particle near the black hole is smaller than its effective potential. If we account the rotation to the black hole the center-of-mass energy be large. For this reasons, it may be interesting to add some charge and rotation to the massive gravity solution and investigate the relation between two geometries. Also here one can check the mentioned approaches to different black hole with charge and without charges. It will be also interesting to see the effect of electromagnetic field on the above equivalency, we see in future work.
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