Note

EZ gauge is singular at the event horizon

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Abstract

We prove that the EZ gauge of black-hole perturbation theory, introduced by the late Steve Detweiler in his exploration of the physical consequences of the gravitational self-force, is necessarily singular at the event horizon.

Keywords: black hole, perturbation theory, general relativity

1. Introduction and summary

Black-hole perturbation theory, understood here in the specific guise of metric perturbations of the Schwarzschild spacetime, is a mature framework that has found many fruitful applications. The theory originated in the works of Regge and Wheeler [1], Vishveshwara [2], and Zerilli [3], and it was formalized in gauge-invariant formalisms by Moncrief [4], Gerlach and Sengupta [5, 6], Gundlach and Martin-Garcia [7, 8], Sarbach and Tiglio [9], Clarkson and Barrett [10], Nagar and collaborators [11, 12], and Martel and Poisson [13]. The theory was applied to a plethora of phenomena, including the quasinormal modes of vibration of a black hole [14], the gravitational waves produced by a point particle in orbit around a black hole [15, 16], the self-force acting on a small body inspiralling toward a black hole [17, 18], the collision of two black holes in a close-limit approximation [19], and the tidal deformation of black holes [20–22].

The equations of black-hole perturbation theory can be integrated by making use of gauge-invariant master functions, or by selecting a specific gauge to eliminate the redundant coordinate freedom. The most popular choice has been the Regge–Wheeler gauge, introduced in [1], which possesses the virtues of being algebraically simple and unique (except for the monopole and dipole pieces of the perturbation). Another choice, strongly embraced by one of the authors in his exploration of the tidal deformation of black holes, is the light-cone gauge of Preston and Poisson [23]; this gauge assigns a compelling geometrical meaning to the coordinates of the perturbed spacetime, but it is not unique. Another choice of gauge was recently contributed to the vast literature on black-hole perturbation theory. It is known as the EZ (easy) gauge [24], and it was devised (but never published) by the late Steve Detweiler during his exploration of the physical consequences of the gravitational self-force [25].
The choice of gauge is entirely a matter of taste and convenience; metric perturbations related by a gauge transformation are physically equivalent. The choice, however, should be informed by the known properties of the gauge, and our purpose in this paper is to point out that in the EZ gauge, the metric perturbation is necessarily singular on the black-hole horizon. The statement is true for static and time-dependent situations, and is true regardless of the source of the perturbation. This fact does not seem to have been noticed before. The singularity of the EZ gauge at the event horizon does not imply that the gauge is necessarily ‘bad’ or that its use should be discouraged. After all, one can be perfectly comfortable working with the Schwarzschild metric in the standard \((t, r, \theta, \phi)\) coordinates, in spite of the metric singularity at \(r = 2M\); one knows to be careful when investigating processes that occur at or near the horizon. The same lesson applies to the EZ gauge: some care is required if one wishes to adopt it. For example, the EZ gauge can fruitfully be employed to calculate the gravitational waves emitted by a particle in orbit around a black hole, since the metric perturbation in this gauge can readily be related to the gauge-invariant master functions, which have a close relation to the two gravitational-wave polarizations. Such an application was considered in section 10 of [24].

To establish the singular property of the EZ gauge, we begin with the unperturbed Schwarzschild metric presented in \((v, r, \theta, \phi)\) coordinates, with \(v\) denoting the standard advanced-time coordinate. The metric is known to be regular at \(r = 2M\) in these coordinates. We next introduce a metric perturbation formulated in the light-cone gauge [23], which preserves the geometrical meaning of the spacetime coordinates. This property makes it self-evident that in the light-cone gauge, the components of a physically regular metric perturbation will be nonsingular at \(r = 2M\). This perturbation is then transformed to the Regge–Wheeler gauge [1], and shown again to have nonsingular components at the horizon. Finally, the perturbation is transformed to the EZ gauge, and shown to have components that diverge at \(r = 2M\). A physically regular metric perturbation therefore appears to be singular in the EZ gauge. (We note that the proof could avoid the middle step of transforming to the Regge–Wheeler gauge. But we find it useful to establish the regularity of this gauge as a byproduct of the proof.)

We begin in section 2 with a brief summary of the essential points of black-hole perturbation theory. In section 3 we consider a perturbation presented in the light-cone gauge and construct the transformation that takes it to the Regge–Wheeler gauge; we observe that the perturbation stays regular at \(r = 2M\) after the transformation. In section 4 we construct the transformation to the EZ gauge, and show that the perturbation now diverges at the event horizon. Finally, as a concrete illustration of these properties, we display in section 5 the perturbations that correspond to a black hole deformed by a quadrupolar tidal field.

2. Metric perturbation and gauge transformations

We rely on the summary of black-hole perturbation theory provided by Martel and Poisson [13]. The unperturbed metric \(g_{\alpha\beta}\) is given by the Schwarzschild solution in \((v, r, \theta, \phi)\) coordinates,

\[
g_{\alpha\beta} dx^\alpha dx^\beta = g_{ab} dx^a dx^b + r^2 \Omega_{AB} d\theta^A d\theta^B,
\]

where \(x^a := (v, r), \theta^A := (\theta, \phi), \Omega_{AB} := \text{diag}[1, \sin^2 \theta]\) is the metric on a unit 2-sphere, and

\[
g_{ab} dx^a dx^b = -f dv^2 + 2dv dr
\]

with \(f := 1 - 2M/r; M\) denotes the mass of the black hole. The perturbed metric is \(g_{\alpha\beta} + p_{\alpha\beta}\), with \(p_{\alpha\beta}\) representing the perturbation.

The perturbation is decomposed in spherical harmonics according to
\[ p_{ab} = \sum_{\ell m} h_{ab}^{\ell m} Y_{\ell m}, \]  
\[ p_{AB} = \sum_{\ell m} f_{\ell m} Y_{\ell m}^A + \sum_{\ell m} h_{AB}^{\ell m} X_{\ell m}^A, \]  
\[ p_{AB} = r^2 \sum_{\ell m} \left( K_{\ell m} \Omega_{AB} Y_{\ell m} + G_{\ell m} Y_{\ell m}^{AB} \right) + \sum_{\ell m} h_{AB}^{\ell m} X_{\ell m}^A, \]

where \( Y_{\ell m}^{\ell m}(\theta^A) \) are the usual scalar harmonics, \( Y_{\ell m}^{\ell m} \) and \( Y_{\ell m}^{\ell m} \) are even-parity harmonics, and \( X_{\ell m}^A \) and \( X_{\ell m}^{AB} \) are odd-parity harmonics; definitions are provided in section 3 of [13]. The perturbation fields \( h_{ab}^{\ell m}, f_{\ell m}^{\ell m}, K_{\ell m}^{\ell m}, \) and \( G_{\ell m}^{\ell m} \), all functions of \( x^a \), make up the even-parity sector of the perturbation. The fields \( h_{ab}^{\ell m} \) and \( h_{AB}^{\ell m} \), also functions of \( x^a \), make up the odd-parity sector of the perturbation. In the decomposition we assume that the sums over \( \ell \) begin with \( \ell = 2 \). We exclude the monopole and dipole pieces of the perturbation because both the Regge–Wheeler and EZ gauges are not defined for them; alternative gauge choices must be made.

A gauge transformation is generated by a vector field \( \Xi_a \) that can be decomposed as
\[ \Xi_a = \sum_{\ell m} \xi_{\ell m} Y_{\ell m}^a, \]
\[ \Xi_A = \sum_{\ell m} \xi_{\ell m}^{\ell m,\text{even}} Y_{\ell m}^A + \sum_{\ell m} \xi_{\ell m}^{\ell m,\text{odd}} X_{\ell m}^A. \]

The transformation produces the changes
\[ h_{vv} \to h_{vv} - 2 \partial_v \xi_v + \frac{2M}{r^2} \xi_v + \frac{2M}{r^2} \xi_r, \]
\[ h_{vr} \to h_{vr} - \partial_r \xi_v - \partial_v \xi_r - \frac{2M}{r^2} \xi_r, \]
\[ h_{rr} \to h_{rr} - 2 \partial_r \xi_r, \]
\[ j_v \to j_v - \partial_v \xi_{\text{even}} - \xi_v, \]
\[ j_r \to j_r - \partial_r \xi_{\text{even}} - \xi_r + \frac{2}{r} \xi_{\text{even}}, \]
\[ K \to K - \frac{2f}{r} \xi_r - \frac{2}{r} \xi_r + \ell (\ell + 1) \frac{r^2}{r^2} \xi_{\text{even}}, \]
\[ G \to G - \frac{2}{r^2} \xi_{\text{even}} \]
in the even-parity sector, and
\[ h_v \to h_v - \partial_v \xi_{\text{odd}}, \]
\[ h_r \to h_r - \partial_r \xi_{\text{odd}} + \frac{2}{r} \xi_{\text{odd}}, \]
\[ h_2 \to h_2 - 2 \xi_{\text{odd}} \]
in the odd-parity sector. To unclutter the notation we omit the $\ell m$ labels on the perturbation and gauge fields.

3. From light-cone gauge to Regge–Wheeler gauge

The light-cone gauge of black-hole perturbation theory was introduced by Preston and Poisson [23]. Its formulation begins with the observation that the coordinates $(v, r, \theta, \phi)$ of the background Schwarzschild spacetime possess a clear geometrical meaning. The advanced-time coordinate $v$ is constant on light cones that converge toward the future singularity at $r = 0$, $-\tau$ is an affine-parameter distance on each null generator of these light cones, and $\theta^A = (\theta, \phi)$ is constant on these generators. It is this compelling geometrical meaning that ensures that in these coordinates, the Schwarzschild metric is regular at the event horizon. The argument is simply that since the light cones behave smoothly as they cross $r = 2M$, and since the coordinates are tied to these light cones, the metric will be regular when presented in these coordinates.

The light-cone gauge places conditions on $p_{\alpha\beta}$ that ensure that the geometrical meaning of the coordinates is preserved in the perturbed spacetime. In this way, $v$ continues to label converging light cones (now perturbed), $-\tau$ continues to be an affine parameter on the null generators, and $\theta^A$ continues to be constant on each generator. The preserved geometrical meaning of the coordinates guarantees that in the light-cone gauge, the components of $p_{\alpha\beta}$ in $(v, r, \theta, \phi)$ coordinates will be regular at $r = 2M$. The argument is the same as for the background spacetime, and we therefore take it as self-evident that a perturbation presented in the light-cone gauge will be regular at the black-hole horizon.

The light-cone gauge conditions [23] are

\[ h^{LC}_{vr} = h^{LC}_{rr} = h^{LC}_{rA} = 0, \quad h^{LC}_v = 0. \quad (3.1) \]

The Regge–Wheeler gauge conditions [1] are

\[ f^{RW}_v = f^{RW}_r = G^{RW}_v = 0, \quad h^{RW}_2 = 0. \quad (3.2) \]

When we incorporate these in equation (2.5), we find that the transformation from light-cone gauge to Regge–Wheeler gauge is achieved with

\[ \xi^{LC}_v \rightarrow RW = f^{LC}_v - \frac{1}{2} r^2 \partial_v G^{LC}, \quad (3.3a) \]

\[ \xi^{LC}_r \rightarrow RW = -\frac{1}{2} r^2 \partial_r G^{LC}, \quad (3.3b) \]

\[ \xi^{LC}_{\text{even}} \rightarrow RW = \frac{1}{2} r^2 G^{LC}, \quad (3.3c) \]

in the even-parity sector. In the odd-parity sector we have

\[ \xi^{LC}_{\text{odd}} \rightarrow RW = \frac{1}{2} h^{LC}_2. \quad (3.4) \]

We remark that the gauge vector is uniquely determined (the Regge–Wheeler gauge is unique), and that the operators acting on the light-cone perturbation fields are all smooth at $r = 2M$. The gauge vector is therefore nonsingular at the event horizon.

The perturbation fields in the Regge–Wheeler gauge are then given by
\[ h_{vv}^{\text{RW}} = h_{vv}^{\text{LC}} - 2 \partial_v \xi_v \rightarrow \text{RW} + \frac{2M}{r^2} \xi_v \rightarrow \text{RW} + \frac{2Mf}{r^2} \xi_v \rightarrow \text{RW}, \]  
\[ h_{vv}^{\text{RW}} = - \partial_v \xi_v \rightarrow \text{RW} + \partial_v \xi_v \rightarrow \text{RW} - \frac{2M}{r^2} \xi_v \rightarrow \text{RW}, \]  
\[ h_{rr}^{\text{RW}} = -2 \partial_r \xi_r \rightarrow \text{RW}, \]  
\[ K^{\text{RW}} = K^{\text{LC}} - \frac{2f}{r} \xi_v \rightarrow \text{RW} - \frac{2}{r} \xi_v \rightarrow \text{RW} + \frac{\ell(\ell + 1)}{r^2} \xi_v \rightarrow \text{even} \]  

in the even-parity sector, and
\[ h_{v}^{\text{RW}} = h_{v}^{\text{LC}} - \partial_{\text{odd}} \xi_v \rightarrow \text{RW}, \]  
\[ h_{r}^{\text{RW}} = - \partial_r \xi_r \rightarrow \text{odd} + \frac{2}{r} \xi_r \rightarrow \text{odd} \]  

in the odd-parity sector. These equations show that when the perturbation fields are regular at \( r = 2M \) in the light-cone gauge, they are also regular in the Regge–Wheeler gauge. A physically regular perturbation is therefore nonsingular on the event horizon when it is presented in Regge–Wheeler gauge.

4. From Regge–Wheeler gauge to EZ gauge

The EZ gauge conditions [24] are
\[ j_r^{\text{EZ}} = K^{\text{EZ}} = G^{\text{EZ}} = 0, \quad h_{2}^{\text{EZ}} = 0, \]  
and the last equation implies that the EZ and Regge–Wheeler gauges coincide in the odd-parity sector. Equations (2.5) reveal that the transformation from Regge–Wheeler gauge to EZ gauge is achieved with
\[ \xi_v \rightarrow \text{EZ} = 0, \]  
\[ \xi_r \rightarrow \text{EZ} = \frac{r}{2f} K^{\text{RW}}, \]  
\[ \xi_{\text{even}} \rightarrow \text{EZ} = 0. \]  

The gauge vector is uniquely determined (the EZ gauge is unique), and the factor of \( f^{-1} = (1 - 2M/r)^{-1} \) in the radial component is the origin of the singularity of the gauge at \( r = 2M \).

The perturbation fields in the EZ gauge are given by
\[ h_{vv}^{\text{EZ}} = h_{vv}^{\text{RW}} + \frac{2Mf}{r^2} \xi_r \rightarrow \text{EZ}, \]  
\[ h_{vr}^{\text{EZ}} = h_{vr}^{\text{RW}} - \partial_v \xi_r \rightarrow \text{EZ} - \frac{2M}{r^2} \xi_r \rightarrow \text{EZ}, \]  
\[ h_{rr}^{\text{EZ}} = h_{rr}^{\text{RW}} - 2 \partial_r \xi_r \rightarrow \text{EZ}, \]
\[ \xi_r^{\text{EZ}} = -\xi_r^{\text{RW}} \rightarrow \text{EZ}, \quad (4.3d) \]

or

\[ h_{vv}^{\text{EZ}} = h_{vv}^{\text{RW}} + \frac{M}{r} k^{\text{RW}}, \quad (4.4a) \]

\[ h_{vv}^{\text{EZ}} = h_{vv}^{\text{RW}} - \frac{1}{f} \left( \frac{1}{2} f \partial_r k^{\text{RW}} + \frac{M}{r} k^{\text{RW}} \right), \quad (4.4b) \]

\[ h_{rr}^{\text{EZ}} = h_{rr}^{\text{RW}} - \partial_r \left( \frac{r k^{\text{RW}}}{f} \right), \quad (4.4c) \]

\[ f_r^{\text{EZ}} = -\frac{r}{f} k^{\text{RW}}, \quad (4.4d) \]

The factors of \( f^{-1} \) and \( f^{-2} \) in the perturbation fields (except for \( h_{vv} \), which is regular) show very clearly that these diverge when \( r = 2M \). A physically regular perturbation therefore appears singular when presented in the EZ gauge. This is our main conclusion.

For completeness we display the transformation that takes any old gauge to the EZ gauge. In the even-parity sector the gauge vector is given by

\[ \xi_v^{\rightarrow \text{EZ}} = \xi_v^{\text{old}} - \frac{1}{2} r^2 \partial_t G^{\text{old}}, \quad (4.5a) \]

\[ \xi_r^{\rightarrow \text{EZ}} = \frac{r}{2f} \left[ K^{\text{old}} - \frac{2}{r} h_r^{\text{old}} + r \partial_r G^{\text{old}} + \frac{1}{2} \ell (\ell + 1) G^{\text{old}} \right], \quad (4.5b) \]

\[ \xi_{\text{even}}^{\rightarrow \text{EZ}} = \frac{1}{2} r^2 G^{\text{old}}, \quad (4.5c) \]

and it reduces to equation (4.2) when the old gauge is the Regge–Wheeler gauge. The factor of \( f^{-1} \) in \( \xi_r^{\rightarrow \text{EZ}} \) continues to imply that the EZ gauge is singular at \( r = 2M \). The gauge vector is given by

\[ \xi_{\text{odd}}^{\rightarrow \text{EZ}} = \frac{1}{2} h_2^{\text{old}}, \quad (4.6) \]

in the odd-parity sector.

### 5. Tidally deformed black hole

To provide a concrete illustration of the results obtained in the preceding sections, we consider a black hole perturbed by remote bodies that exert tidal forces. This is a well-studied problem [20–22], and for the most part we shall simply import results from the literature.

At the leading, quadrupole (\( \ell = 2 \)) order, the tidal environment is characterized by the symmetric-tracefree Cartesian tensor \( \mathcal{E}_{ab} \), which is assumed to vary slowly with respect to time—the tidal perturbation is formally taken to be time-independent. The spherical-harmonic decomposition of the perturbation is aided by the identity

\[ \mathcal{E}_{ab} \Omega^a \Omega^b = \sum_{m=-2}^{2} \mathcal{E}_m \nu^{2,m}, \quad (5.1) \]
where $\Omega^a := [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$. The identity maps the five independent components of $E_{ab}$ to the five harmonic coefficients $E_m$. The perturbation associated with $E_{ab}$ is entirely in the even-parity sector.

In the light-cone gauge, the nonvanishing perturbation fields are given by [22]

$$h_{vv}^{LC} = -r^2 f^2 E_m,$$

$$j_v^{LC} = -\frac{1}{3} r^3 f E_m,$$

$$G_v^{LC} = -\frac{1}{3} r^2 \left(1 - \frac{2M^2}{r^2}\right) E_m.$$  

In the Regge–Wheeler gauge they are [26]

$$h_{vv}^{RW} = -r^2 f^2 E_m,$$

$$h_{vv}^{RW} = r^2 f E_m,$$

$$h_{rr}^{RW} = -2r^2 E_m,$$

$$K^{RW} = -r^2 \left(1 - \frac{2M^2}{r^2}\right) E_m.$$  

And in the EZ gauge they are given by

$$h_{vv}^{EZ} = -r^2 \left(1 - \frac{M}{r}\right) \left(1 - \frac{2M}{r} + \frac{2M^2}{r^2}\right) E_m,$$

$$h_{rr}^{EZ} = \frac{r^2}{f} \left(1 - \frac{M}{r}\right) \left(1 - \frac{2M}{r} + \frac{2M^2}{r^2}\right) E_m,$$

$$h_{rr}^{EZ} = \frac{r^2}{f^2} \left(1 - \frac{10M^2}{r^2} + \frac{8M^3}{r^3}\right) E_m,$$

$$f_r^{EZ} = \frac{r^3}{2f} \left(1 - \frac{2M^2}{r^2}\right) E_m.$$  

It is manifest that while the perturbation fields are regular at $r = 2M$ in the light-cone and Regge–Wheeler gauges, they are singular in the EZ gauge.

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