MOVING INTERFACE IN A VISCOUS INCOMPRESSIBLE FLOW
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ABSTRACT. We investigate the sharp interface limit of a diffusive interface system that couples the Allen–Cahn equation with the instationary Stokes system in a bounded domain in $\mathbb{R}^2$. This model is used to describe a propagating front in a viscous incompressible flow with the width of the transition layer being characterized by a small parameter $\varepsilon > 0$. For well-prepared initial data, we show that the solution converges to a limit system that couples the curve-shortening flow and the Stokes system.

Keywords: Allen–Cahn equation, Complex fluid, modulated energy, mean curvature flow, sharp interface limit.

1. Introduction

We consider the Stokes/Allen–Cahn system in a domain $\Omega \subset \mathbb{R}^2$ with smooth boundary:

\begin{align}
\partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - \varepsilon^{-2} W'(c_\varepsilon), \\
\partial_t u_\varepsilon - \Delta u_\varepsilon &= \nabla \pi_\varepsilon - \varepsilon \text{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon), \\
\text{div } u_\varepsilon &= 0.
\end{align}

Here $\varepsilon > 0$ is a small parameter which represents the width of the transition layer. In (1.1a) $W(c)$ is a double equal-well potential taking global minimum zero at $c = \pm 1$. Though our result holds for a general potential, for simplicity we choose $W(c) = \sigma (c^2 - 1)^2$ where $\sigma > 0$ is chosen so that

$$\int_{-1}^1 \sqrt{2W(\lambda)} \, d\lambda = 1.$$ 

An important property of (1.1) for the purpose of this work is the following energy dissipation law:

$$\frac{d}{dt} \int_\Omega \left( \frac{1}{2} |u_\varepsilon|^2 + \frac{\varepsilon}{2} |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) \right) \, dx + \int_\Omega \left( |\nabla u_\varepsilon|^2 + \varepsilon |\dot{c}_\varepsilon|^2 \right) \, dx = 0,$n

where we employed the abbreviation

$$\dot{c}_\varepsilon := \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon.$$ 

So in this work, we shall assume the initial data of (1.1) satisfies

\begin{align}
(c_\varepsilon(x,0), u_\varepsilon(x,0)) &\in C^2(\overline{\Omega}), \\
-1 \leq c_\varepsilon(x,0) &\leq 1, \forall x \in \Omega, \\
\left. \int_\Omega \left( \frac{1}{2} |u_\varepsilon|^2 + \frac{\varepsilon}{2} |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) \right) \, dx \right|_{t=0} &\leq C_{in}
\end{align}

for some $C_{in} > 0$ which is independent of $\varepsilon$. Then for each fixed $\varepsilon > 0$, the global in time classical solution to (1.1) can be established by standard parabolic theory. Another important property of the system (1.1) is a maximum principle estimate to (1.1a):

$$-1 \leq c_\varepsilon(x,t) \leq 1, \quad \forall \varepsilon > 0, x \in \Omega, t \in [0,T].$$

This follows from the observation that the constant functions $\pm 1$ are solutions to (1.1a).
The major contribution of this work is to study the asymptotic behavior of the solution to (1.1) when \( \varepsilon \) tends to 0. Such a limit corresponds to the following system involving a moving front in space-time domain \( \Omega \times [0, T] \)

\[
\Sigma = \bigcup_{t \in [0, T)} \Sigma_t \times \{ t \}
\]  

(1.6)

and unknowns \((u, \pi)\):

\[
\begin{align*}
V &= (H + u) \cdot n \quad \text{on } \Sigma_t, \quad (1.7a) \\
\partial_t u - \Delta u &= \nabla \pi + H^1 \Sigma_t \quad \text{in } \Omega, \quad (1.7b) \\
\text{div } u &= 0 \quad \text{in } \Omega \setminus \Sigma_t, \quad (1.7c) \\
[u] &= 0 \quad \text{on } \Sigma_t. \quad (1.7d)
\end{align*}
\]

In (1.7) \( n, H, V \) are the inner normal vector, the mean curvature vector and the normal velocity of the interface \( \Sigma_t \), respectively. In (1.7d) \([\phi] \) denotes the jump of a function \( \phi \) across \( \Sigma_t \):

\[
[\phi](x) = \lim_{r \downarrow 0} \phi(x + r n(x)) - \phi(x, x - r n(x)) , \quad x \in \Sigma_t.
\]  

(1.8)

In the equation (1.7d), the notation \( H^1 \Sigma_t \) denotes the restriction of 1-dimensional Hausdorff measure to the curve \( \Sigma_t \). More precisely, for any Borel set \( A \subset \mathbb{R}^2 \), \( H^1 \Sigma_t(A) := H^1(A \cap \Sigma_t) \).

Formally a family of solutions of (1.1a) converges to the solution of (1.7a) as \( \varepsilon \) tends to 0. A rigorous proof was given by Abels and the author in [2] using asymptotic expansion and the spectrum analysis of the Allen–Cahn operator [6, 5]. See also a recent work by Abels and Fei [1].

In this work we shall employ the modulated energy method which is also referred to as the relative entropy method. Such a method has been successfully applied to analyze various singular limit problems in continuum mechanics and particle systems. To list a few, we refer the readers to [17] for the scaling limit of particle system, to [13, 14] for the semiclassical limit of Gross–Pitaevskii equation, and to [3, 10] for the hydrodynamic limit of Boltzmann equation. For the applications of such a method to interface problems, we mention [12, 8, 9, 11].

In this work, we shall assume the following regularity of solution to (1.7):

\[
\begin{align*}
\Theta_t(x) : \Omega^+_0 \to \Omega^+_t \text{ is a } C^3 \text{ diffeomorphism on } \overline{\Omega}, \quad \partial_t \Theta \in C([0, T]; C^1(\overline{\Omega})), \quad (1.9a) \\
\Sigma_t = \Theta_t(\Sigma_0) \text{ is a family of smooth simple closed curves with } \Sigma_0 \subset \Omega, \quad (1.9b) \\
\sup_{t \in [0, T]} \left( \| \partial_t \Theta \|_{W_x^1(\Omega)} + \| \Theta \|_{W_x^2(\Omega)} \right) < \infty, \quad (1.9c) \\
u \in W^{1, \infty}([0, T]; W^{1, \infty}(\Omega)) \cap C^0_t C^0_x(\overline{\Omega} \setminus [0, T] \setminus \Sigma) \cap C^0_t C^2_x(\overline{\Omega} \times [0, T] \setminus \Sigma). \quad (1.9d)
\end{align*}
\]

Here \( \Omega^+_t = \Omega_t \) is the (open) domain enclosed by \( \Sigma_t \) and \( \Omega^-_t = \Omega \setminus \overline{\Omega^+_t} \). It is worth mentioning that thanks to (1.7d), the condition (1.7c) can be improved to \( \text{div } u = 0 \) in \( \Omega \) in the sense of distribution. However, in general \( \nabla u \) is not continuous across \( \Sigma_t \) and a possible profile is \( u = (0, |x_1|) \) with \( \Sigma_t = \{ x_1 = 0 \} \). So in (1.9d) \( u \) will not be better than Lipschitz continuous across the interface though it is sufficiently regular in the bulks \( \Omega^+_t \).

Following Jerrard and Smets [12], Fischer and Hensel [8] and Fischer et al. [9], our general strategy is to construct a Lyapunov functional which measures the difference between the solution to the diffusive system (1.1) and that of the sharp interface model (1.7), and then to derive a differential inequality for it. Let \( r = d_\Sigma(x, t) \) be the signed-distance function to the closed curve \( \Sigma_t \). We make the convention that \( d_\Sigma(x, t) \) takes positive values in the domain \( \Omega^+_t = \Omega_t \) enclosed by \( \Sigma_t \) and takes negative values in \( \Omega^-_t = \Omega \setminus \overline{\Omega^+_t} \). We assume the nearest-point projection \( P_\Sigma \) is smooth in \( \Sigma(4\delta) \), the 4\delta tubular neighborhood of \( \Sigma \) for some sufficiently small \( \delta \in (0, 1) \) which only depends on the geometry of \( \Sigma \). If we assume a parametrization \( \varphi_t(s) : S^1 \mapsto \Sigma_t \), then there holds the identity

\[
d_\Sigma(\varphi_t(s) + r n(s, t), t) \equiv r
\]  

(1.10)
with independent variables \((r, s, t)\). Here \(n\) is the inner normal vector to \(\Sigma_t\). Differentiating this identity leads to
\[
\nabla d\Sigma(x, t) = n(s, t), \quad -\partial_t d\Sigma(x, t) = \partial_t \phi_t(s) \cdot n(s, t) =: V(s, t). \tag{1.11}
\]
This extends the inner normal vector and the normal velocity of \(\Sigma_t\) to a neighborhood of it.

We first extend the normal vector \(n\) of \(\Sigma\) to the whole computational domain \(\Omega\). To this end we introduce the extended inner normal vector
\[
\xi(x, t) = \phi\left(\frac{d\Sigma(x, t)}{\delta}\right) \nabla d\Sigma(x, t), \tag{1.12}
\]
where \(\phi(x) \geq 0\) is an even, smooth function on \(\mathbb{R}\) that decreases for \(x \in [0, 1]\), and satisfies
\[
\begin{align*}
\phi(x) > 0 & \text{ for } |x| < 1, \\
\phi(x) = 0 & \text{ for } |x| \geq 1, \\
1 - 4x^2 & \leq \phi(x) \leq 1 - \frac{1}{2}x^2 \quad \text{ for } |x| \leq 1/2.
\end{align*}
\tag{1.13}
\]

To fulfill these requirements, we can simply choose
\[
\phi(x) = e^{x^2/2} - 1 + 1 \quad \text{ for } |x| < 1 \text{ and } \phi(x) = 0 \text{ for } |x| \geq 1. \tag{1.14}
\]

With these preparations, we define the modulated energy by
\[
E_\varepsilon(t) = \int_\Omega \left( \frac{\varepsilon}{2} \left| \nabla c_\varepsilon \right|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon \right) \, dx, \tag{1.15}
\]
where
\[
\psi_\varepsilon(x, t) = \int_{-1}^{c_\varepsilon(x, t)} \sqrt{2W(\tau)} \, d\tau. \tag{1.16}
\]
As \(|\xi| \leq 1\), the integrand of (1.15) is non-negative by Cauchy–Schwarz’s inequality. Moreover, it follows from (1.14) and (1.5) that
\[
0 \leq \psi_\varepsilon(x, t) \leq 1, \quad \forall \varepsilon > 0, x \in \Omega, t \in [0, T]. \tag{1.17}
\]

We shall also need the following weighted bulk error
\[
A_\varepsilon(t) = \int_\Omega (1_{\Omega_t} - \psi_\varepsilon) \eta(d\Sigma) \, dx, \tag{1.18}
\]
where \(\eta(\cdot)\) denotes the truncation of the identity map defined by
\[
\eta(s) = \begin{cases} 
    s & \text{ when } s \in (-4\delta, 4\delta), \\
    4\delta & \text{ when } s \geq 4\delta, \\
    -4\delta & \text{ when } s \leq -4\delta.
\end{cases} \tag{1.19}
\]

It follows from (1.17) that \(\eta(d\Sigma)(1_{\Omega_t} - \psi_\varepsilon) \geq 0\).

The main result of this work is the following theorem on the growth of \(E_\varepsilon(t)\) and \(A_\varepsilon(t)\):
Theorem 1.1. Assume that the system (1.7) has a classical solution during \([0, T]\) with regularities in (1.9). Moreover, the initial data of (1.7) is smooth, and satisfies (1.4) and

\[
E_\varepsilon(0) + \frac{1}{2}\|u(\cdot, t) - u_\varepsilon(\cdot, t)\|_{L^2}^2 + A_\varepsilon(0) \leq C_0 \varepsilon^2, \quad \forall \varepsilon \in (0, 1),
\]

for some \(C_0 > 0\). Then there exist \(\varepsilon_0, C_T > 0\) depending on \(C_0\) but not on \(\varepsilon\) such that

\[
\max_{t \in [0, T]} \left( E_\varepsilon(t) + \frac{1}{2}\|u(\cdot, t) - u_\varepsilon(\cdot, t)\|_{L^2}^2 + A_\varepsilon(t) \right) \leq C_T \varepsilon^2, \quad \forall \varepsilon \in (0, \varepsilon_0).
\]

The construction of the initial data of (1.7) fulfilling (1.20) is standard. See [15] or [2] for the details.

This work will be organized as follows: in Section 2 we shall setup the notations and establish some technical lemmas. These results will be used in Section 3 to derive a differential inequality of the energy (1.15) for the Allen–Cahn equation with advection term (1.1a). See (3.12) below. In Section 4 we establish the estimate of the difference \(v_\varepsilon := u - u_\varepsilon\) and derive a differential inequality for the weighted bulk error (1.18). Finally we give the proof of the main theorem in Section 5 combining the differential inequalities (3.12), (4.6) (4.11) with the Brezis–Gallouët–Wainger inequality [4, 7] and the method of continuity [16, Section 1.3].

In the sequel, we shall adopt the convention that \(C > 0\) will be a generic constant which depends only on the solution to (1.7) with regularity assumptions (1.9). This constant might change from line to line and will not depend on \(\varepsilon\) or \(t \in [0, T]\).

In order to simplify the presentation, we shall sometimes use the abbreviation \(\int g\) for \(\int_\Omega g \, dx\), especially during the proofs of technical lemmas. Moreover, we shall sometimes abbreviate the estimates like \(X \leq CY\) by \(X \lesssim Y\) for some non-negative quantities \(X, Y\). We shall also use the Frobenius inner product \(A : B := \text{tr} A^T B\) for two square matrices \(A, B\).

2. Preliminaries

The task of this section is to state and prove some technical lemmas (and their corollaries) which will be used in future. These results are originally and essentially due to [9] for the Allen–Cahn equation, though we adapt them to account for the advection terms in (1.1a). The readers are encouraged to read the original work [9]. However, we state the proofs here for the readers’ convenience.

To start with we shall extend the mean curvature vector of \(\Sigma_t\) to the whole domain \(\Omega\) and then derive a transport equation of the limit interface \(\Sigma_t\). Let \(\zeta\) be a cut-off function with

\[
\zeta(\cdot, t) \in C^\infty_{c}\left(\Sigma_t(2\delta)\right), \quad \text{and} \quad \zeta(\cdot, t) = 1 \text{ in } \Sigma_t(\delta).
\]

We define the extended mean curvature vector of \(\Sigma_t\) by

\[
H(x, t) = \kappa \nabla d_\Sigma(x, t) \quad \text{with} \quad \kappa(x, t) = -\Delta d_\Sigma(P_{\Sigma_t}x, t) \zeta(x, t),
\]

where \(P_{\Sigma_t}\) is the projection on \(\Sigma_t\), i.e.

\[
P_{\Sigma_t}(x) := x - \nabla d_\Sigma(x, t) d_\Sigma(x, t).
\]

Note that due to our convention that \(d_\Sigma(x, t)\) takes positive value inside \(\Sigma_t\), the vector \(H\) points to the inside if \(\Sigma_t\) is convex. We define the constant extension of \(u\) off \(\Sigma_t\) by

\[
\tilde{u}(x, t) := u(P_{\Sigma_t}(x, t)) \quad \text{in } \Sigma_t(4\delta).
\]

According to (1.9d) and (1.7d), we deduce that

\[
\tilde{u} \in C^0(\Sigma(4\delta)) \cap C^1(\Sigma(4\delta)).
\]

Moreover, we have the following lemma:

Lemma 2.1. The extended normal vector (1.12) satisfies

\[
\begin{align*}
(\xi \cdot \nabla)H &= 0 \quad \text{in } \Omega, \\
(\xi \cdot \nabla)\tilde{u} &= 0 \quad \text{in } \Omega.
\end{align*}
\]
Moreover, there exists \( C > 0 \) such that
\[
\mathbf{H} \cdot \xi + \text{div} \, \xi = O(d_\Sigma(x,t)) \quad \text{in } \Omega, \tag{2.7}
\]
\[
\partial_t \xi + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) \xi + (\nabla H + \nabla \mathbf{u})^t \xi = O(d_\Sigma(x,t)) \quad \text{in } \Omega, \tag{2.8}
\]
\[
\partial_t |\xi|^2 + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) |\xi|^2 = O(d_\Sigma(x,t)^2) \quad \text{in } \Omega. \tag{2.9}
\]

**Proof.** Concerning (2.6a), since \( \mathbf{H} \) is extended constantly in \( \Sigma_\ell(\delta) \) according to (2.2), we have \( (\xi \cdot \nabla) \mathbf{H} = 0 \) there. Outside \( \Sigma_\ell(\delta) \) we have \( \xi \equiv 0 \). Similarly we obtain (2.6b).

We denote \( \phi_\delta(\tau) = \phi(\xi(\tau)) \) which is an even function. According to (1.12) and the definition of \( \phi \) afterwards, we have \( \phi_\delta'(0) = 0 \). This together with (2.2) leads to
\[
\text{div} \, \xi = |\nabla d_\Sigma|^2 \phi_\delta'(d_\Sigma) + \phi_\delta(d_\Sigma) \Delta d_\Sigma
= O(d_\Sigma) + \phi_\delta(d_\Sigma) \Delta d_\Sigma(P_{\Sigma}(x,t))
= O(d_\Sigma) - \mathbf{H} \cdot \xi
\]
and this leads to (2.7). Regarding (2.8) and (2.9), thanks to (1.12) and (1.14), it suffices to verify these two identities in \( \Sigma_\ell(\delta) \) because \( \xi \) and its derivatives vanish identically outside this region.

To prove (2.8), we first substitute the equations (1.11), (2.4) and (2.2) into (1.7a) and deduce that
\[
\partial_t d_\Sigma + (\mathbf{H} + \mathbf{u}) \cdot \nabla d_\Sigma = 0 \quad \text{in } \Sigma_\ell(\delta). \tag{2.10}
\]

So \( \nabla d_\Sigma \) satisfies the equation
\[
\partial_t \nabla d_\Sigma + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) \nabla d_\Sigma + (\nabla H + \nabla \mathbf{u})^t \nabla d_\Sigma = 0 \quad \text{in } \Sigma_\ell(\delta), \tag{2.11}
\]
and this leads to (2.7). Regarding (2.8) and (2.9), thanks to (1.12) and (1.14), it suffices to verify these two identities in \( \Sigma_\ell(\delta) \) because \( \xi \) and its derivatives vanish identically outside this region.

To prove (2.9), we first substitute the equations (1.11), (2.4) and (2.2) into (1.7a) and deduce that
\[
\partial_t |\xi|^2 + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) |\xi|^2 = 0. \tag{2.12}
\]

Combining (2.12) with (2.11) leads to
\[
\partial_t \xi + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) \xi + (\nabla H + \nabla \mathbf{u})^t \xi = 0. \tag{2.13}
\]

By (2.4) and (2.5) we have for some non-negative function \( \omega(t) \in L_\ell^\infty \) that
\[
|\mathbf{u}(x,t) - \mathbf{u}(x,t)| \leq \omega(t)|d_\Sigma(x,t)|, \tag{2.14}
\]
which leads to (2.8). To prove (2.9), we test (2.13) by \( \xi \) and obtain
\[
\partial_t |\xi|^2 + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) |\xi|^2 = - (\xi \cdot \nabla - \xi \cdot \nabla)\mathbf{u} \cdot \xi. \tag{2.15}
\]

The right hand side of the above equation vanishes in \( \Sigma_\ell(\delta) \) due to (2.6a) and (2.6b), and vanishes outside \( \Sigma_\ell(\delta) \) because \( \xi \) does. Finally we obtain by (2.14) that
\[
\partial_t |\xi|^2 + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) |\xi|^2 = |\mathbf{u} - \mathbf{u}| \cdot \nabla |\xi|^2 = O(d_\Sigma(x,t)^2), \tag{2.16}
\]
and this finishes the proof of (2.9).

It is obvious that the integrand of the energy (1.13) is non-negative. Moreover, we have the following lemma:

**Lemma 2.2.** There exists a universal constant \( C < \infty \) which is independent of \( t \in (0,T) \) and \( \varepsilon \) such that the following estimates hold for every \( t \in (0,T) \):
\[
\int_\Omega \left( \sqrt{\varepsilon} |\nabla c_\varepsilon| - \varepsilon^{-\frac{1}{2}} \sqrt{2W(c_\varepsilon)} \right)^2 \, dx \leq E_\varepsilon(t), \tag{2.17a}
\]
\[
\int_\Omega \left( \frac{\varepsilon}{2} |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) + |\nabla \psi_\varepsilon| \right) (1 - \xi \cdot \mathbf{n}) \, dx \leq 4E_\varepsilon(t), \tag{2.17b}
\]
\[
\int_\Omega \left( \frac{\varepsilon}{2} |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) + |\nabla \psi_\varepsilon| \right) \min (d_\Sigma^2, 1) \, dx \leq CE_\varepsilon(t). \tag{2.17c}
\]
Proof. Since $|\xi| \leq 1$ by (1.12), we can write the integrand of $E$ (1.15) as the sum of two non-negative terms:

$$E_\varepsilon(t) = \int_\Omega \left( \frac{\varepsilon}{2} |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} |W(c_\varepsilon) - |\nabla \psi_\varepsilon|| \right) + \int_\Omega (1 - \xi \cdot n_\varepsilon)|\nabla \psi_\varepsilon| \cdot \varepsilon \Delta c_\varepsilon - \frac{1}{\varepsilon} W'(c_\varepsilon) \right) |\nabla c_\varepsilon|.$$

(2.18)

Completing the square leads to (2.17a). Using (2.18) and $0 \leq 1 - \xi \cdot n_\varepsilon \leq 2$ yields (2.17b). Finally, by (1.13) and $\delta \in (0, 1)$ we have

$$1 - \xi \cdot n_\varepsilon \geq 1 - \phi \left( \frac{d_\varepsilon}{\delta} \right) \geq \min \left( \frac{d_\varepsilon^2}{\delta^2}, 1 \right) \geq \min (d_\varepsilon^2, 1).$$

(2.19)

This together with (2.17b) implies (2.17c). □

Lemma 2.3. Let $v = H$ or $\hat{u}$, then there exists a universal constant $C < \infty$ which is independent of $t \in (0, T)$ and $\varepsilon$ such that the following estimates hold for every $t \in (0, T)$:

$$\int_\Omega \nabla v \cdot n_\varepsilon \leq C E_\varepsilon.$$  

(2.20)

Proof. We denote the integral on the left-hand side of (2.20) by $J_\varepsilon^1$. Using $n_\varepsilon = n_\varepsilon - \xi + \xi$ and $\nabla v \cdot n_\varepsilon \leq (\xi \cdot \nabla) v \cdot n_\varepsilon$, we can estimate $J_\varepsilon^1$ by

$$J_\varepsilon^1 \leq \int_\Omega \nabla v \cdot n_\varepsilon \leq \int_\Omega (\xi \cdot \nabla) v \cdot n_\varepsilon \leq \int_\Omega \xi \cdot \nabla \psi_\varepsilon \leq C E_\varepsilon.$$  

(2.21)

Finally applying the Cauchy–Schwarz inequality and then (2.17a) and (2.17b), we obtain (2.20). □

We introduce the phase-field analogues of the inner normal vector and the mean curvature vector, respectively

$$n_\varepsilon := \frac{\nabla c_\varepsilon}{|\nabla c_\varepsilon|} = \frac{\nabla \psi_\varepsilon}{|\nabla \psi_\varepsilon|}, \quad H_\varepsilon = - \left( \varepsilon \Delta c_\varepsilon - \frac{1}{\varepsilon} W'(c_\varepsilon) \right) \frac{\nabla c_\varepsilon}{|\nabla c_\varepsilon|}.$$  

(2.22)

Combining (2.22) and (1.3), we can write (1.1a) as

$$\varepsilon \dot{c}_\varepsilon = - H_\varepsilon \cdot n_\varepsilon.$$  

(2.23)

Lemma 2.4. We have the following identities:

$$\int_\Omega \nabla v \cdot n_\varepsilon \leq \int_\Omega (\div \xi) n_\varepsilon \cdot H |\nabla \psi_\varepsilon| dx - \int_\Omega (\div H) n_\varepsilon \cdot \xi |\nabla \psi_\varepsilon| dx,$$

(2.24)

and

$$0 = \int_\Omega (\div \xi) \hat{u} \cdot n_\varepsilon |\nabla \psi_\varepsilon| dx - \int_\Omega (\div \hat{u}) n_\varepsilon |\nabla \psi_\varepsilon| dx - \int_\Omega (\hat{u} \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| dx.$$  

(2.25)

Proof. We first note that for any matrix-valued function $A = (a_{ij})$, we define its divergence by $
abla A := (\sum_j \partial_j a_{ij})_{1 \leq i \leq 3}$. To prove (2.24), using symmetry and integration by parts, we can calculate

$$0 = - \int \div \left( \div (H \otimes \xi - \xi \otimes H) \right) \psi_\varepsilon.$$
In view of (2.22), we can verify the identity

\[ H \]

Testing the identity (2.29) by \( \hat{u} \) and \( \xi \) are both smooth, by symmetry, \( \text{div} \text{div}(\hat{u} \otimes \xi - \xi \otimes \hat{u}) = 0 \). This combined with the divergence theorem implies

\[
0 = -\int \text{div} \left( \text{div} (\hat{u} \otimes \xi - \xi \otimes \hat{u}) \right) \psi_{\varepsilon} \\
= \int \text{div} (\hat{u} \otimes \xi - \xi \otimes \hat{u}) |\nabla \psi_{\varepsilon}| \\
= \int (\text{div} \xi) \hat{u} \cdot n_{\varepsilon} |\nabla \psi_{\varepsilon}| - (\text{div} \hat{u}) \xi \cdot n_{\varepsilon} |\nabla \psi_{\varepsilon}| - (\hat{u} \cdot \nabla) \xi \cdot n_{\varepsilon} |\nabla \psi_{\varepsilon}| \\
+ \int (\xi \cdot \nabla) \hat{u} \cdot n_{\varepsilon} |\nabla \psi_{\varepsilon}|.
\]

The last integral vanishes due to (2.6a), i.e. \( (\xi \cdot \nabla) \hat{u} = 0 \) in \( \Omega \), and this proves (2.25). \( \square \)

**Lemma 2.5.** The following identity holds

\[
\int_{\Omega} \nabla H : (\xi \otimes n_{\varepsilon}) |\nabla \psi_{\varepsilon}| \, dx - \int_{\Omega} (\text{div} H) \xi \cdot \nabla \psi_{\varepsilon} \, dx \\
= \int_{\Omega} \nabla H : (\xi - n_{\varepsilon}) \otimes n_{\varepsilon} |\nabla \psi_{\varepsilon}| \, dx + \int H_{\varepsilon} \cdot H |\nabla c_{\varepsilon}| \, dx \\
+ \int_{\Omega} (\text{div} H) \left( \frac{\varepsilon}{2} |\nabla c_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(c_{\varepsilon}) - \xi \cdot \nabla \psi_{\varepsilon} \right) \, dx \\
- \int_{\Omega} (\varepsilon |\nabla c_{\varepsilon}|^2 - |\nabla \psi_{\varepsilon}|) n_{\varepsilon} \otimes n_{\varepsilon} : \nabla H \, dx
\]

(2.26)

**Proof of Lemma 2.5.** The left-hand side of the equation (2.26) can be written as

\[
\int \nabla H : (\xi \otimes n_{\varepsilon}) |\nabla \psi_{\varepsilon}| - \int (\text{div} H) \xi \cdot \nabla \psi_{\varepsilon} \\
= \int \nabla H : (\xi - n_{\varepsilon}) \otimes n_{\varepsilon} |\nabla \psi_{\varepsilon}| + \int \nabla H : (n_{\varepsilon} \otimes n_{\varepsilon}) |\nabla \psi_{\varepsilon}| - \int (\text{div} H) \xi \cdot \nabla \psi_{\varepsilon}.
\]

(2.27)

To treat the second integral on the right-hand side of the equation (2.26), we introduce the energy stress tensor \( T_{\varepsilon} \)

\[
T_{\varepsilon} = \left( \frac{\varepsilon}{2} |\nabla c_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(c_{\varepsilon}) \right) I_2 - \varepsilon \nabla c_{\varepsilon} \otimes \nabla c_{\varepsilon}.
\]

(2.28)

In view of (2.22), we can verify the identity

\[
\nabla \cdot T_{\varepsilon} = -\varepsilon \Delta c_{\varepsilon} \nabla c_{\varepsilon} + \frac{1}{\varepsilon} W'(c_{\varepsilon}) \nabla c_{\varepsilon} = H_{\varepsilon} |\nabla c_{\varepsilon}|.
\]

(2.29)

Testing the identity (2.29) by \( H \), and then integrating by parts, we obtain

\[
0 = \int H_{\varepsilon} \cdot H |\nabla c_{\varepsilon}| + \int \nabla H : T_{\varepsilon} \\
= \int H_{\varepsilon} \cdot H |\nabla c_{\varepsilon}| + \int (\text{div} H) \left( \frac{\varepsilon}{2} |\nabla c_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(c_{\varepsilon}) \right) - \int \varepsilon \nabla c_{\varepsilon} \otimes \nabla c_{\varepsilon} : \nabla H \\
= \int H_{\varepsilon} \cdot H |\nabla c_{\varepsilon}| + \int (\text{div} H) \left( \frac{\varepsilon}{2} |\nabla c_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(c_{\varepsilon}) \right)
\]

(2.26)
\[
- \int (\varepsilon |\nabla c_\varepsilon|^2 - |\nabla \psi_\varepsilon|) \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon : \nabla H - \int \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon : \nabla H|\nabla \psi_\varepsilon|. \tag{2.30}
\]

So adding (2.30) into (2.27) leads to (2.26).

\[\square\]

**Corollary 2.6.** There exists a universal constant \( C > 0 \) which is independent of \( t \in (0, T) \) and \( \varepsilon \) such that the following estimates hold for every \( t \in (0, T) \):

\[
\int_{\Omega} \nabla H : (\mathbf{\xi} \otimes \mathbf{n}_\varepsilon)|\nabla \psi_\varepsilon| \, dx \leq C\varepsilon + \int_{\Omega} (\text{div} \, \mathbf{H}) \mathbf{\xi} \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx + \int_{\Omega} \mathbf{H}_\varepsilon \cdot \nabla |\nabla c_\varepsilon| \, dx, \tag{2.31a}
\]

\[
\int_{\Omega} \nabla \hat{u} : (\mathbf{\xi} \otimes \mathbf{n}_\varepsilon)|\nabla \psi_\varepsilon| \, dx \leq C\varepsilon + \int_{\Omega} (\text{div} \, \hat{u}) \mathbf{\xi} \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx + \int_{\Omega} \mathbf{H}_\varepsilon \cdot \hat{u} |\nabla c_\varepsilon| \, dx. \tag{2.31b}
\]

**Proof.** Using \((\mathbf{\xi} \cdot \nabla) \mathbf{H} = 0 \) in \( \Omega \) (see (2.6a)), we can write the first integral on the right-hand side of the equation (2.26) as

\[
\int \nabla H : (\mathbf{\xi} - \mathbf{n}_\varepsilon) \otimes \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| = \int \nabla H : (\mathbf{\xi} - \mathbf{n}_\varepsilon) \otimes (\mathbf{n}_\varepsilon - \mathbf{\xi}) |\nabla \psi_\varepsilon|, \tag{2.32}
\]

and this integral can be estimated by (2.17b). The third integral on the right-hand side of (2.26) can be controlled using (1.15). The integrand of the last line can be written as

\[
\nabla H : \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon \left( |\nabla \psi_\varepsilon| - \varepsilon |\nabla c_\varepsilon|^2 \right),
\]

and is estimated in Lemma 2.3. So we prove (2.31a).

Observe that the identity (2.26) still stands if we replace \( \mathbf{H} \) by \( \hat{u} \). Actually it suffices to test the identity (2.29) by \( \hat{u} \) instead. This combined with (2.6b) leads to (2.31b) in the same way. \[\square\]

### 3. Deriving a differential inequality

The aim of this section is to derive a differential inequality of (1.16) for the Allen–Cahn equation with advection term (1.1a). We start with the derivation of the energy dissipation law (1.2). We first square both sides of the equation (1.1a), integrate by parts, and use (1.1c):

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) \right) \, dx + \int_{\Omega} \varepsilon |\dot{c}_\varepsilon|^2 \, dx = \varepsilon \int_{\Omega} \mathbf{u}_\varepsilon \cdot \nabla c_\varepsilon \Delta c_\varepsilon \, dx. \tag{3.1}
\]

On the other hand, testing (1.1b) by \( \mathbf{u}_\varepsilon \) and integrating by parts yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathbf{u}_\varepsilon|^2 \, dx + \int_{\Omega} |\nabla \mathbf{u}_\varepsilon|^2 \, dx = -\varepsilon \int_{\Omega} \mathbf{u}_\varepsilon \cdot \nabla c_\varepsilon \Delta c_\varepsilon \, dx. \tag{3.2}
\]

Adding this equation into (3.1) leads to (1.2). Recall (1.15). Using (3.1), (2.22) and integration by parts, we obtain

\[
\frac{d}{dt} \mathcal{E}_\varepsilon(t) + \int \varepsilon |\dot{c}_\varepsilon|^2 = \int -\partial_t \mathbf{\xi} \cdot \nabla \psi_\varepsilon + (\text{div} \, \mathbf{\xi}) \partial_t \psi_\varepsilon - \mathbf{u}_\varepsilon \cdot \mathbf{H}_\varepsilon |\nabla c_\varepsilon|. \tag{3.3}
\]

Recall (2.22), we have \( \mathbf{n}_\varepsilon = \frac{\nabla \psi_\varepsilon}{|\nabla \psi_\varepsilon|} \). Now use (2.8) to replace \( \partial_t \mathbf{\xi} \) in (3.3) above and integrate by parts

\[
\int -\partial_t \mathbf{\xi} \cdot \nabla \psi_\varepsilon = \int (\mathbf{H} + \mathbf{u}) \cdot \nabla \mathbf{\xi} \cdot \nabla \psi_\varepsilon + (\nabla \mathbf{H} + \nabla \hat{u})^t \mathbf{\xi} \cdot \nabla \psi_\varepsilon
\]

\[
- \int \left( \partial_t \mathbf{\xi} + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) \mathbf{\xi} + (\nabla \mathbf{H} + \nabla \hat{u})^t \mathbf{\xi} \right) \cdot \nabla \psi_\varepsilon
\]

\[
= \int \mathbf{n}_\varepsilon \cdot (\mathbf{H} \cdot \nabla) \mathbf{\xi} |\nabla \psi_\varepsilon| + (\mathbf{u} \cdot \nabla) \mathbf{\xi} \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| + (\mathbf{n}_\varepsilon \cdot \nabla) \mathbf{H} \cdot \mathbf{u} \cdot \mathbf{\xi} |\nabla \psi_\varepsilon| + (\mathbf{n}_\varepsilon \cdot \nabla) (\mathbf{H} + \hat{u}) \cdot \mathbf{\xi} |\nabla \psi_\varepsilon| \tag{3.4a}
\]

\[
- \int \left( \partial_t \mathbf{\xi} + ((\mathbf{H} + \mathbf{u}) \cdot \nabla) \mathbf{\xi} + (\nabla \mathbf{H} + \nabla \hat{u})^t \mathbf{\xi} \right) \cdot (\mathbf{n}_\varepsilon - \mathbf{\xi}) |\nabla \psi_\varepsilon| \tag{3.4b}
\]
To proceed we need (2.31a), i.e.

\[ - \int \left( \partial_t \xi + ((H + u) \cdot \nabla) \xi \right) \cdot \xi |\nabla \psi_e| + \int (\nabla H + \nabla \hat{u}) : (\xi \otimes \xi) |\nabla \psi_e|. \quad (3.4c) \]

By (2.6), the second integral in (3.4c) vanishes. By (2.8) and (2.9), we obtain

\[ (3.4b) + (3.4c) \leq \int \left( |n_e - \xi|^2 + \min(1, d_{\xi}^2) \right) |\nabla \psi_e| \lesssim E_e(t). \quad (3.5) \]

To handle the first term in (3.4a), we need (2.24), i.e.

\[ \int n_e \cdot (H \cdot \nabla) \xi |\nabla \psi_e| = \int (\text{div} \xi) n_e \cdot H |\nabla \psi_e| - \int (\text{div} H) n_e \cdot \xi |\nabla \psi_e|. \]

So altogether, we can update (3.4) and obtain

\[ \int -\partial_t \xi \cdot \nabla \psi_e \leq \int (\text{div} \xi) n_e \cdot H |\nabla \psi_e| - (\text{div} H) n_e \cdot \xi |\nabla \psi_e| \]
\[ + \int (u \cdot \nabla) \xi \cdot n_e |\nabla \psi_e| + (n_e \cdot \nabla) (H + \hat{u}) \cdot \xi |\nabla \psi_e| + C\mathcal{E}_e(t). \quad (3.6) \]

Substituting (3.6) into (3.3) yields

\[ \frac{d}{dt} \mathcal{E}_e(t) - C\mathcal{E}_e(t) \leq \int -\varepsilon |\hat{c}_e|^2 + (\text{div} \xi) \partial_t \psi_e + \int (\text{div} \xi) n_e \cdot H |\nabla \psi_e| - (\text{div} H) n_e \cdot \xi |\nabla \psi_e| \]
\[ + \int (u \cdot \nabla) \xi \cdot n_e |\nabla \psi_e| + (n_e \cdot \nabla) (H + \hat{u}) \cdot \xi |\nabla \psi_e| - u_e \cdot H_e |\nabla c_e|. \quad (3.7) \]

To proceed we need (2.31a), i.e.

\[ \int \nabla H : (\xi \otimes n_e) |\nabla \psi_e| \leq C\mathcal{E}_e + \int (\text{div} H) \xi \cdot n_e |\nabla \psi_e| + \int H_e \cdot H |\nabla c_e|. \]

Adding this inequality to (3.7) yields

\[ \frac{d}{dt} \mathcal{E}_e(t) - C\mathcal{E}_e \leq \int -\varepsilon |\hat{c}_e|^2 + (\text{div} \xi) \partial_t \psi_e + (\text{div} \xi) n_e \cdot H |\nabla \psi_e| \]
\[ + \int (u \cdot \nabla) \xi \cdot n_e |\nabla \psi_e| + \nabla \hat{u} : \xi \otimes n_e |\nabla \psi_e| \]
\[ + \int H_e \cdot H |\nabla c_e| - u_e \cdot H_e |\nabla c_e|. \quad (3.8) \]

Using (2.23) we can write (3.5) by

\[ \frac{d}{dt} \mathcal{E}_e - C\mathcal{E}_e(t) \leq \int -\varepsilon |\hat{c}_e|^2 + (\text{div} \xi) \hat{c}_e \sqrt{2W(c_e)} + H_e \cdot H |\nabla c_e| + (\text{div} \xi) n_e \cdot H |\nabla \psi_e| \quad (3.9a) \]
\[ + \int (u \cdot \nabla) \xi \cdot n_e |\nabla \psi_e| + \nabla \hat{u} : \xi \otimes n_e |\nabla \psi_e| \quad (3.9b) \]
\[ + \int -u_e \cdot H_e |\nabla c_e| - u_e \cdot \nabla c_e \sqrt{2W(c_e)} (\text{div} \xi) \quad (3.9c) \]

The following lemma gives the estimate of (3.9a).

**Lemma 3.1.** There exists a universal constant \( C > 0 \) which is independent of \( t \in (0, T) \) and \( \varepsilon \) such that the following estimates hold for every \( t \in (0, T) \):

\[ (3.9a) \leq C\mathcal{E}_e(t) - \frac{1}{2\varepsilon} \int \left( |\varepsilon \hat{c}_e - (\text{div} \xi) \sqrt{2W(c_e)}|^2 + |H_e - \varepsilon |\nabla c_e|H|^2 \right) \quad (3.10) \]
Proof. The integrands in (3.9a) can be arranged by

\[-\varepsilon|\dot{c}_\varepsilon|^2 + (\text{div } \xi)\sqrt{2W(c_\varepsilon)}\dot{c}_\varepsilon + (\text{div } \xi)H \cdot \nabla \psi_\varepsilon + H_\varepsilon \cdot H|\nabla c_\varepsilon|
\]

\[= -\frac{1}{2\varepsilon} \left( |\varepsilon \dot{c}_\varepsilon|^2 - 2(\varepsilon \dot{c}_\varepsilon) (\text{div } \xi)\sqrt{2W(c_\varepsilon)} + (\text{div } \xi)^2|\sqrt{2W(c_\varepsilon)}|^2 \right)
\]
\[- \frac{1}{2\varepsilon}|\varepsilon \dot{c}_\varepsilon|^2 + \frac{1}{2\varepsilon}(\text{div } \xi)^2|\sqrt{2W(c_\varepsilon)}|^2 + (\text{div } \xi)H \cdot \nabla \psi_\varepsilon
\]
\[- \frac{1}{2\varepsilon} \left( |H_\varepsilon|^2 - 2\varepsilon|\nabla c_\varepsilon|H_\varepsilon \cdot H + \varepsilon^2|\nabla c_\varepsilon|^2|H|^2 \right) + \frac{1}{2\varepsilon} \left( |H_\varepsilon|^2 + \varepsilon^2|\nabla c_\varepsilon|^2|H|^2 \right)
\]
\[= -\frac{1}{2\varepsilon} |\varepsilon \dot{c}_\varepsilon - (\text{div } \xi)\sqrt{2W(c_\varepsilon)}|^2 - \frac{1}{2\varepsilon} |H_\varepsilon - \varepsilon|\nabla c_\varepsilon|H|^2
\]
\[+ \frac{1}{2\varepsilon} ( (\text{div } \xi)^2|\sqrt{2W(c_\varepsilon)}|^2 + 2\varepsilon(\text{div } \xi)\nabla \psi_\varepsilon \cdot H + \varepsilon^2|\nabla c_\varepsilon|^2|H|^2 ) \quad (3.11)
\]

The last line on the right-hand side of (3.11) is a square term. To control its integral, we use triangle inequality

\[
\int \frac{1}{2\varepsilon} \left( (\text{div } \xi)^2|\sqrt{2W(c_\varepsilon)}|^2 + 2\varepsilon(\text{div } \xi)\nabla \psi_\varepsilon \cdot H + \varepsilon^2|\nabla c_\varepsilon|^2|H|^2 \right)
\]
\[= \frac{1}{2} \int \left| \frac{1}{\sqrt{\varepsilon}}(\text{div } \xi)\sqrt{2W(c_\varepsilon)}n_\varepsilon + \sqrt{\varepsilon}|\nabla c_\varepsilon|H \right|^2 dx
\]
\[\leq \int \left| (\text{div } \xi) \left( \sqrt{\varepsilon}|\nabla c_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(c_\varepsilon)} \right) n_\varepsilon \right|^2 dx
\]
\[+ \int \left| (\text{div } \xi)\sqrt{\varepsilon}|\nabla c_\varepsilon|(n_\varepsilon - \xi) \right|^2 dx
\]
\[+ \int \left| (H + (\text{div } \xi)\xi)\sqrt{\varepsilon}|\nabla c_\varepsilon| \right|^2 dx.
\]

The first integral on the right-hand side of the above inequality is controlled by (2.17a). Due to the elementary inequality $|\xi - n_\varepsilon|^2 \leq 2(1 - n_\varepsilon \cdot \xi)$, the second integral is controlled by (2.17b). The third integral can be treated using the relation $H = (H \cdot \xi)\xi + O(\varepsilon^2)$ and (2.7). So it can be controlled by (2.17c). Altogether, we obtain (3.10).

To simplify the presentation, we use the abbreviation $v_\varepsilon = u - u_\varepsilon$ in the sequel.

**Proposition 3.2.** There exists a universal constant $C > 0$ which is independent of $t \in (0,T)$ and $\varepsilon$ such that the following estimates hold for every $t \in (0,T)$:

\[
\frac{d}{dt} \mathcal{E}_\varepsilon(t) + \frac{1}{4\varepsilon} \int \left[ |\varepsilon \dot{c}_\varepsilon - (\text{div } \xi)\sqrt{2W(c_\varepsilon)}|^2 + |H_\varepsilon - \varepsilon|\nabla c_\varepsilon|H|^2 \right] dx
\]

\[\leq C\|v_\varepsilon\|_{L^\infty}^2 \mathcal{E}_\varepsilon + C \int (|\xi|^2 + |\nabla \psi_\varepsilon|) dx. \quad (3.12)
\]

**Proof.** It suffices to estimate the terms on the right-hand side of (3.9). The terms in (3.9a) is estimated in (3.10). The terms in (3.9c) can be written by

\[
- u_\varepsilon \cdot H_\varepsilon |\nabla c_\varepsilon| - u_\varepsilon \cdot \nabla c_\varepsilon \sqrt{2W(c_\varepsilon)}(\text{div } \xi)
\]
\[\geq \frac{\varepsilon}{2} u_\varepsilon \cdot n_\varepsilon \left( \varepsilon \dot{c}_\varepsilon - (\text{div } \xi)\sqrt{2W(c_\varepsilon)} \right) |\nabla c_\varepsilon|. \quad (3.13)
\]

So we can estimate the sum of (3.9b) and (3.9c) as

\[
(3.9b) + (3.9c) = \int u_\varepsilon \cdot n_\varepsilon \left( \varepsilon \dot{c}_\varepsilon - (\text{div } \xi)\sqrt{2W(c_\varepsilon)} \right) |\nabla c_\varepsilon|
\]
\[+ \int (u \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| + \nabla \hat{u} \cdot \xi \otimes n_\varepsilon |\nabla \psi_\varepsilon|
\]
Lemma 3.3. There exists a universal constant $C > 0$ which is independent of $t \in (0, T)$ and $\varepsilon$ such that the following estimates hold for every $t \in (0, T)$:

$$
(B2) + (B3) \leq C \mathcal{E}_\varepsilon + \frac{1}{16\varepsilon} \int \left( \varepsilon \dot{c}_\varepsilon - (\text{div} \, \xi) \sqrt{2W(c_\varepsilon)} \right)^2 dx.
$$

Proof. Recall that

$$
(B2) + (B3) = \int (u \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| + (n_\varepsilon \cdot \nabla) \tilde{u} \cdot \xi |\nabla \psi_\varepsilon| - (\text{div} \, \xi) u \cdot n_\varepsilon |\nabla \psi_\varepsilon| - u \cdot H_\varepsilon |\nabla c_\varepsilon|.
$$

Recall also (2.31b) and (2.25):

$$
\int_\Omega \nabla \tilde{u} : (\xi \otimes n_\varepsilon) |\nabla \psi_\varepsilon| \leq C \mathcal{E}_\varepsilon + \int (\text{div} \, \tilde{u}) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| + \int H_\varepsilon \cdot \tilde{u} |\nabla c_\varepsilon|,
$$

$$
0 = \int_\Omega (\text{div} \, \xi) \tilde{u} \cdot n_\varepsilon |\nabla \psi_\varepsilon| - \int_\Omega (\text{div} \, \tilde{u}) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| - \int_\Omega (\tilde{u} \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon|.
$$

Adding the above two equations into (3.18) and then rearrange the terms:

$$
(B2) + (B3) \leq C \mathcal{E}_\varepsilon + \int (u \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| - (\text{div} \, \xi) u \cdot n_\varepsilon |\nabla \psi_\varepsilon| - u \cdot H_\varepsilon |\nabla c_\varepsilon|
$$

$$
+ \int (\text{div} \, \tilde{u}) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| + H_\varepsilon \cdot \tilde{u} |\nabla c_\varepsilon|
$$

$$
+ \int (\text{div} \, \xi) \tilde{u} \cdot n_\varepsilon |\nabla \psi_\varepsilon| - (\text{div} \, \tilde{u}) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| - (\tilde{u} \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon|
$$

$$
= C \mathcal{E}_\varepsilon + \int ((u - \tilde{u}) \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| + (\text{div} \, \xi) (\tilde{u} - u) \cdot n_\varepsilon |\nabla \psi_\varepsilon| + (\tilde{u} - u) \cdot H_\varepsilon |\nabla c_\varepsilon|
$$

$$
\leq C \mathcal{E}_\varepsilon + \int ((u - \tilde{u}) \cdot \nabla) \xi \cdot n_\varepsilon |\nabla \psi_\varepsilon| + \int |\nabla c_\varepsilon| (\tilde{u} - u) \cdot \left( \sqrt{2W(c_\varepsilon)} (\text{div} \, \xi) n_\varepsilon + H_\varepsilon \right).
$$

(3.19)
To control the first integral in the last line of (3.19), we first deduce from (1.12) and (2.14) that
\[
\left| (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla |\xi|^2 \right| \leq C \min(\delta^2, 1).
\] (3.20)
On the other hand, by (2.23) we can replace \( H_\varepsilon \) in the second integral by \(-c_\varepsilon n_\varepsilon\). Altogether, we obtain
\[
(B2) + (B3) \leq C\varepsilon + \int (|\xi - n_\varepsilon|^2 + \min(\delta^2, 1)) |\nabla \psi_\varepsilon|
\]
\[
+ \int \varepsilon^\theta |\nabla c_\varepsilon| (\tilde{\mathbf{u}} - \mathbf{u}) \cdot n_\varepsilon \varepsilon^{-\frac{\theta}{2}} \left( \sqrt{2W(c_\varepsilon)(\text{div } \xi)} - \varepsilon \dot{c}_\varepsilon \right).
\] (3.21)
Finally applying the Cauchy–Schwarz inequality and using (2.17) yields (3.17).

4. Estimates of the velocity fields

We test (1.7b) by \( \mathbf{v}_\varepsilon = \mathbf{u} - \mathbf{u}_\varepsilon \) and integrate by parts
\[
\int \partial_t \mathbf{u} \cdot \mathbf{v}_\varepsilon + \int \nabla \mathbf{u} : \nabla \mathbf{v}_\varepsilon = \int_{\Sigma_t} \mathbf{H} \cdot \mathbf{v}_\varepsilon + \int \nabla \pi \cdot \mathbf{v}_\varepsilon.
\]
The last integral above vanishes due to (1.9d) and (1.7d):
\[
\int \nabla \pi \cdot \mathbf{v}_\varepsilon = \int \nabla \pi \cdot \mathbf{u} = \sum_{\pm} \int_{\Omega_t^\pm} \nabla \pi \cdot \mathbf{u} = - \sum_{\pm} \int_{\partial \Omega_t^\pm} \pi \mathbf{u} \cdot n_{\partial \Omega_t^\pm} = 0
\] (4.1)
with \( n_{\partial \Omega_t^\pm} \) being the inward normal of \( \Omega_t^\pm \). So we obtain
\[
\int \partial_t \mathbf{u} \cdot \mathbf{v}_\varepsilon + \int \nabla \mathbf{u} : \nabla \mathbf{v}_\varepsilon = \int_{\Sigma_t} \mathbf{H} \cdot \mathbf{v}_\varepsilon.
\] (4.2)
By (2.22), we can write (1.1b) as
\[
\partial_t \mathbf{u}_\varepsilon - \Delta \mathbf{u}_\varepsilon = \nabla \pi_\varepsilon + \mathbf{H}_\varepsilon |\nabla c_\varepsilon|.
\] (4.3)
As the calculation in (4.1) is also valid when \( \pi \) is replaced by \( \pi_\varepsilon \), we have \( \int \nabla \pi_\varepsilon \cdot \mathbf{v}_\varepsilon = 0 \). Now we test (4.3) by \( \mathbf{v}_\varepsilon \), and integrate by parts
\[
\int \partial_t \mathbf{u}_\varepsilon \cdot \mathbf{v}_\varepsilon + \int \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v}_\varepsilon = \int \mathbf{v}_\varepsilon \cdot \mathbf{H}_\varepsilon |\nabla c_\varepsilon|.
\]
Combining the above equation with (1.2), and using the divergence theorem, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int |\mathbf{v}_\varepsilon|^2 + \int |\nabla \mathbf{v}_\varepsilon|^2 = \int_{\Sigma_t} \mathbf{H} \cdot \mathbf{v}_\varepsilon - \int \mathbf{v}_\varepsilon \cdot \mathbf{H}_\varepsilon |\nabla c_\varepsilon|.
\] (4.4)
To treat the second integral on the right-hand side of (4.4) we need \( \mathbf{H} = \kappa \mathbf{n} \) with \( \kappa \) being defined by (2.2). This combined with the divergence theorem yields
\[
\int_{\Sigma_t} \mathbf{H} \cdot \mathbf{v}_\varepsilon = - \int_{\Omega_t} \text{div } \left( \mathbf{v}_\varepsilon \kappa \right) = - \int_{\Omega_t} \mathbf{v}_\varepsilon \cdot \nabla \kappa
\]
\[
= - \int (1_{\Omega_t} - \psi_\varepsilon) \mathbf{v}_\varepsilon \cdot \nabla \kappa - \sum_{\pm} \int_{\Omega_t^\pm} \psi_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla \kappa.
\]
Note that according to the last assumption in (1.9d), \( \nabla \mathbf{u} \) might have a jump across \( \Sigma_t \). However using (1.7c) and (1.7d) we can calculate as in (4.1) and obtain
\[
\int_{\Sigma_t} \mathbf{H} \cdot \mathbf{v}_\varepsilon = - \int (1_{\Omega_t} - \psi_\varepsilon) \mathbf{v}_\varepsilon \cdot \nabla \kappa
\]
\[
+ \int \kappa \mathbf{v}_\varepsilon \cdot (\mathbf{n}_\varepsilon - \xi) |\nabla \psi_\varepsilon| + \int \mathbf{v}_\varepsilon \cdot \mathbf{\xi} \kappa |\nabla \psi_\varepsilon|.
\]
Submitting this formula into (4.4) yields
\[
\frac{1}{2} \frac{d}{dt} \int |v_\varepsilon|^2 + \int |\nabla v_\varepsilon|^2 \\
= - \int (1_{\Omega_t} - \psi_\varepsilon) v_\varepsilon \cdot \nabla \kappa \\
+ \int \kappa v_\varepsilon \cdot (n_\varepsilon - \xi) |\nabla \psi_\varepsilon| \\
+ \int v_\varepsilon \cdot (\xi \kappa |\nabla \psi_\varepsilon| - H_\varepsilon |\nabla c_\varepsilon|).
\] (4.5a)

**Proposition 4.1.** There exists a universal constant \(C > 0\) which is independent of \(t \in (0, T)\) and \(\varepsilon\) such that the following estimates hold for every \(t \in (0, T)\):
\[
\frac{1}{2} \frac{d}{dt} \int |v_\varepsilon|^2 + \int |\nabla v_\varepsilon|^2 \\
\leq C \varepsilon (1 + \|v_\varepsilon\|^2_{L^\infty}) + C \left( \varepsilon \|v_\varepsilon\|^2_{L^2} + A_\varepsilon \right) \\
+ \frac{1}{16 \varepsilon} \int \left( \varepsilon \hat{c}_\varepsilon - (\text{div} \, \xi) \sqrt{2W(c_\varepsilon^\varepsilon)} \right)^2.
\] (4.6)

**Proof.** It suffices to show the terms (4.5a), (4.5b) and (4.5c) above are estimated by
\[
\int (1_{\Omega_t} - \psi_\varepsilon) v_\varepsilon \cdot \nabla \kappa \, dx,
\] (4.7a)

\[
\int \left( \varepsilon \hat{c}_\varepsilon - (\text{div} \, \xi) \sqrt{2W(c_\varepsilon^\varepsilon)} \right)^2 + \frac{1}{8} \|\nabla v_\varepsilon\|^2_{L^2} \\
+ \|v_\varepsilon\|^2_{L^\infty} C + C \left( \|v_\varepsilon\|^2_{L^2} + \varepsilon \right).
\] (4.7c)

To simplify the presentation, we denote \(\chi_\varepsilon = 1_{\Omega_t} - \psi_\varepsilon\). By (1.17), we have \(\|\chi_\varepsilon\|_{L^\infty} \leq 1\). To estimate the integral in (4.5a), we shall need the following inequality which follows from Fubini’s theorem
\[
\left( \int_0^\tau g(r) \, dr \right)^2 \leq 2 \|g\|_{L^\infty(0, \tau)} \int_0^\tau g(r) r \, dr, \quad \forall g(r) \geq 0, \tau \in \mathbb{R}^+.
\] (4.8)

We shall follow the proof of [2, Lemma 5.1]. Since \(\kappa\) is supported in a neighborhood of \(\Sigma_t\) (recall (2.2)), we have
\[
\int (1_{\Omega_t} - \psi_\varepsilon) v_\varepsilon \cdot \nabla \kappa \, dx \\
\lesssim \int_{\Sigma_t} \int_{-2\delta}^{2\delta} |\chi_\varepsilon \nabla \kappa \cdot v_\varepsilon (y + nr)| \, dr dH^1(y) \\
\lesssim \int_{\Sigma_t} \sup_{|r| \leq 2\delta} |v_\varepsilon (y + nr)| \left( \int_{-2\delta}^{2\delta} |\chi_\varepsilon| (y + nr) \, dr \right) \, dH^1(y) \\
\lesssim \int_{\Sigma_t} \|v_\varepsilon (y + nr)\|^2_{L^2(-2\delta, 2\delta)} \|v_\varepsilon (y + nr)\|^2_{H^1(-2\delta, 2\delta)} \sqrt{\int_{-2\delta}^{2\delta} |r| |\chi_\varepsilon| (y + nr) \, dr} \, dH^1(y). \tag{4.9}
\]

Note that in the last step, we employed interpolation inequality in dimension 1, (1.5) and (4.8). So by Hölder’s inequality and (4.9) we obtain (4.7a).

Concerning (4.5b), we employ the Cauchy–Schwarz inequality and (2.17b)
\[
\int |v_\varepsilon|^2 |\nabla \psi_\varepsilon| \sqrt{\int |n_\varepsilon - \xi|^2 |\nabla \psi_\varepsilon|} \lesssim \sqrt{\int |v_\varepsilon|^2 |\nabla \psi_\varepsilon|} + \varepsilon. \tag{4.10}
\]
The integral in the last step can be estimated by

\[
\int |v_{\varepsilon}|^2 |\nabla \psi_{\varepsilon}| = \int |v_{\varepsilon}|^2 (1 - n_{\varepsilon} \cdot \xi) |\nabla \psi_{\varepsilon}| + \int |v_{\varepsilon}|^2 |\nabla \psi_{\varepsilon} \cdot \xi| \leq 4 \|v_{\varepsilon}\|^2_{L^\infty} E_{\varepsilon} - \int |v_{\varepsilon}|^2 \psi_{\varepsilon}.
\]

The above two estimates together with (1.17) imply (4.7b). To estimate (4.5c), we first deduce from (2.7), (1.12) and (2.2) that

\[
\kappa + \text{div} \xi = (1 - \phi \left( \frac{d\psi}{\delta} \right)) \kappa + O(d_{\Sigma}) \quad \text{(1.13)}
\]

This combined with (2.17b), (2.17c) and Cauchy–Schwarz’s inequality yields

\[
\text{(4.5c)} = \int v_{\varepsilon} \cdot \left( \kappa \sqrt{2W(c_{\varepsilon})} \xi - H_{\varepsilon} \right) |\nabla c_{\varepsilon}|
\]

\[
= - \int v_{\varepsilon} \cdot \left( (\text{div} \xi) \sqrt{2W(c_{\varepsilon})} \xi + H_{\varepsilon} \right) |\nabla c_{\varepsilon}| + \int v_{\varepsilon} \cdot \xi \left( \kappa + \text{div} \xi \right) \sqrt{2W(c_{\varepsilon})} |\nabla c_{\varepsilon}|
\]

\[
\leq - \int v_{\varepsilon} \cdot \left( (\text{div} \xi) \sqrt{2W(c_{\varepsilon})} n_{\varepsilon} + H_{\varepsilon} \right) |\nabla c_{\varepsilon}|
\]

\[
+ C \int |v_{\varepsilon}|^2 + C \int \left( \min \left( 1, d_{\Sigma}^2 \right) + |n_{\varepsilon} - \xi|^2 \right) |\nabla c_{\varepsilon}|^2
\]

\[
\leq C E_{\varepsilon} + \frac{1}{16\varepsilon} \int \left( \varepsilon c_{\varepsilon} - (\text{div} \xi) \sqrt{2W(c_{\varepsilon})} \right)^2 + \int |v_{\varepsilon}|^2 |\nabla c_{\varepsilon}|^2 + C \int |v_{\varepsilon}|^2.
\]

The second integral in the last line is estimated in (3.16), and (4.7c) is proved.

In order to apply the Grönwall’s inequality to (4.6) and (3.12), we need to derive a differential inequality of (1.18). See [9] for the corresponding result for the Allen–Cahn equation.

**Lemma 4.2.** There exists a universal constant $C > 0$ which is independent of $t \in [0, T]$ so that

\[
\frac{d}{dt} A_{\varepsilon}(t) \leq C \|v_{\varepsilon}\|_{L^2}^2 E_{\varepsilon}(t) + C \left( E_{\varepsilon}(t) + \|v_{\varepsilon}\|_{L^2}^2 + A_{\varepsilon}(t) \right)
\]

\[
+ \frac{1}{8} \|\nabla v_{\varepsilon}\|^2_{L^2} + \frac{1}{16\varepsilon} \int \left| H_{\varepsilon} - \varepsilon |\nabla c_{\varepsilon}| H \right|^2.
\]

**Proof.** Using (2.22) and (2.23), we can write

\[
\partial_t \psi_{\varepsilon} = \sqrt{2W(c_{\varepsilon})} \partial_t c_{\varepsilon} = -\varepsilon^{-1} \sqrt{2W(c_{\varepsilon})} H_{\varepsilon} \cdot n_{\varepsilon} - u_{\varepsilon} \cdot \nabla \psi_{\varepsilon}
\]

By (1.19), we deduce that $1_{\Omega_t} \eta(d_{\Sigma})$ is a Lipschitz function. So we have

\[
\frac{d}{dt} \int (1_{\Omega_t} - \psi_{\varepsilon}) \eta(d_{\Sigma})
\]

\[
= \int \varepsilon^{-1} \sqrt{2W(c_{\varepsilon})} H_{\varepsilon} \cdot n_{\varepsilon} \eta(d_{\Sigma}) + \int u_{\varepsilon} \cdot \nabla \psi_{\varepsilon} \eta(d_{\Sigma}) + \int (1_{\Omega_t} - \psi_{\varepsilon}) \eta'(d_{\Sigma}) \partial_t d_{\Sigma}
\]

\[
= \int \varepsilon^{-1} \sqrt{2W(c_{\varepsilon})} H_{\varepsilon} \cdot n_{\varepsilon} \eta(d_{\Sigma}) + \int u_{\varepsilon} \cdot \nabla \psi_{\varepsilon} \eta(d_{\Sigma}) - \int (1_{\Omega_t} - \psi_{\varepsilon}) \left( H + \tilde{u} \right) \cdot \nabla \eta(d_{\Sigma})
\]

\[
+ \int (1_{\Omega_t} - \psi_{\varepsilon}) \eta'(d_{\Sigma}) \left( \partial_t d_{\Sigma} + H \cdot \nabla d_{\Sigma} + \tilde{u} \cdot \nabla d_{\Sigma} \right)
\]

\[
= \int (\varepsilon^{-1} \sqrt{2W(c_{\varepsilon})} H_{\varepsilon} \cdot n_{\varepsilon} - \nabla \psi_{\varepsilon} \cdot H) \eta(d_{\Sigma})
\]

\[
+ \int (1_{\Omega_t} - \psi_{\varepsilon}) \eta(d_{\Sigma}) \left( \text{div } H + \text{div } \tilde{u} \right)
\]

\[
+ \int (u_{\varepsilon} - \tilde{u}) \cdot \nabla \psi_{\varepsilon} \eta(d_{\Sigma})
\]

(4.13a) (4.13b) (4.13c)
implies

Applying the Cauchy–Schwarz inequality to the last integral in the last line yields (4.11).

We can estimate (4.13a) by
\[ \int (u_\varepsilon - \tilde{u}) \cdot \nabla \psi_\varepsilon \eta (d_\Sigma) \]
\[ = \int (v_\varepsilon \cdot \nabla \psi_\varepsilon + \tilde{u} - u_\varepsilon) \cdot \nabla \psi_\varepsilon \eta (d_\Sigma) \]
\[ \leq C \mathcal{E}_\varepsilon + \int |v_\varepsilon|^2 ((|\nabla \psi_\varepsilon| - \nabla \psi_\varepsilon \cdot \xi) + \int |v_\varepsilon|^2 (\nabla \psi_\varepsilon \cdot \xi) \]
\[ \leq C \mathcal{E}_\varepsilon + C \mathcal{E}_\varepsilon \leq \int \text{div} \left( |\nabla \psi_\varepsilon|^2 \xi \right) \psi_\varepsilon. \]

Applying the Cauchy–Schwarz inequality to the last integral in the last line yields
\[ \leq C \mathcal{E}_\varepsilon + \int |v_\varepsilon|^2 \mathcal{E}_\varepsilon + \frac{1}{8} \|
abla v_\varepsilon\|_{L^2}^2 + C \left( \mathcal{E}_\varepsilon^2 + \mathcal{E}_\varepsilon \right). \]

Substituting the above inequalities into (4.13) and using the Cauchy–Schwarz inequality we obtain (4.11).

In order to apply Grönnwall’s inequality to (4.6), we need to estimate \( \|v_\varepsilon\|_{L^2_t L^\infty_x} \). In (4.6) we have an estimate of \( \|v_\varepsilon\|_{L^2_t H^2_x} \), and this is almost sufficient for \( d = 2 \) with the aid of the Brezis–Gallouët–Wainger inequality:

**Lemma 4.3** (Brezis–Gallouët–Wainger inequality [4] [7]). There exists \( C > 0 \) only depending on \( \Omega \subset \mathbb{R}^2 \) and \( p > 2 \) such that
\[ \|f\|_{L^\infty(\Omega)} \leq C \|f\|_{H^1(\Omega)} \left[ 1 + \log^\frac{1}{2} \left( 1 + \frac{\|f\|_{W^{1,p}(\Omega)}}{\|f\|_{H^1(\Omega)}} \right) \right], \quad \forall f \in \|f\|_{W^{1,p}(\Omega)}. \]

To apply the Brezis–Gallouët–Wainger inequality, we shall need the growth estimate of \( \|u_\varepsilon\|_{L^\infty_t W^{1,p}_x} \), which is given below. Note that the rate \( \varepsilon^{-6} \) in (4.16) below is far from being optimal.

**Proposition 4.4.** There exists some \( p > 2 \) and a universal constant \( C_p > 0 \) which is independent of \( T > 0 \) and \( \varepsilon > 0 \) so that
\[ \|u_\varepsilon\|_{L^\infty_t W^{1,p}_x} \leq C_p \varepsilon^{-6}. \]

**Proof.** We first deduce from (1.2), (1.4c) and (2.23) that \( \|H_\varepsilon\|_{L^2_t L^2_x} \lesssim \varepsilon^{-\frac{1}{2}} \). This together with (2.22) implies
\[ \|\delta_\varepsilon\|_{L^2_t L^2_x} \lesssim \varepsilon^{-\frac{1}{2}}, \quad \|\Delta c_\varepsilon\|_{L^2_t L^2_x} \lesssim \varepsilon^{-2}. \]
Let $q < 2$ be the dual index of $p > 2$, i.e. $1/p + 1/q = 1$, and $q_* = \frac{2q}{2-q}$. Then (4.17) combined with interpolation inequality implies
\begin{equation}
\|(\partial_t c_\varepsilon, \nabla^2 c_\varepsilon)\|_{L^2_t L^q_x} + \|\nabla c_\varepsilon\|_{L^q_t L^q_x} \lesssim \varepsilon^{-2}.
\end{equation}
(4.18)

Now we can apply the $W^{2,q}$-estimate of the instationary Stokes system to (4.13)
\begin{equation}
\|u_\varepsilon\|_{L^2_t W^{2,q}_x} + \|\partial_t u_\varepsilon\|_{L^2_t L^q_x} \lesssim \|\nabla c_\varepsilon\|_{H^2_x} + \|H_\varepsilon\|_{L^2_t L^q_x} \lesssim \varepsilon^{-\frac{2}{3}}.
\end{equation}
(4.19)

Now we choose $p$ to be slightly larger than 2 in order to estimate $u_\varepsilon \cdot \nabla c_\varepsilon$,
\begin{equation}
\|u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^2_t L^p_x} \lesssim \|u_\varepsilon\|_{L^4_t L^2_x} \|\nabla c_\varepsilon\|_{L^4_t L^2_x} \lesssim \varepsilon^{-\frac{2}{5}}.
\end{equation}
(4.20)

So we can improve the estimate of (1.1a) from (4.18) to
\begin{equation}
\|(\partial_t c_\varepsilon, \nabla c_\varepsilon, \nabla^2 c_\varepsilon)\|_{L^2_t L^q_x} \lesssim \varepsilon^{-\frac{2}{5}}.
\end{equation}
(4.21)

This combined with interpolation inequality yields $\|\nabla c_\varepsilon\|_{L^6_t L^\infty_x} \lesssim \varepsilon^{-\frac{7}{2}}$. So we can apply the $W^{2,p}$-estimate to (1.1b)
\begin{equation}
\|u_\varepsilon\|_{L^2_t W^{2,p}_x} + \|\partial_t u_\varepsilon\|_{L^2_t L^p_x} \lesssim \varepsilon \|\Delta c_\varepsilon \nabla c_\varepsilon\|_{L^2_t L^p_x} \lesssim \varepsilon^{-\frac{7}{2}} \|\nabla c_\varepsilon\|_{L^6_t L^\infty_x} \lesssim \varepsilon^{-6}.
\end{equation}
(4.22)

Now (4.22) together with interpolation inequality yields (4.16).

\section{5. Proof of the Main Theorem}

We first recall our convention that $C > 0$ is a generic constant that only depend on the solution of (1.1) with regularity (1.9). Now we combine (3.12), (4.16) and (4.11):
\begin{equation}
\frac{d}{dt} \left( \mathcal{E}_\varepsilon(t) + \frac{1}{2} \|v_\varepsilon\|_{L^2_x}^2(t) + \mathcal{A}_\varepsilon(t) \right) + \frac{1}{2} \|\nabla v_\varepsilon\|_{L^2_x}^2(t)
\end{equation}
\begin{equation}
+ \frac{1}{8\varepsilon} \int \left[ (|\varepsilon c_\varepsilon - (\text{div } \xi)\sqrt{2W(c_\varepsilon)})^2 + |H_\varepsilon - \varepsilon|\nabla c_\varepsilon|H_\varepsilon|^2 \right]
\end{equation}
\begin{equation}
\lesssim (1 + \|v_\varepsilon\|_{L^\infty_x}) \mathcal{E}_\varepsilon(t) + (\mathcal{E}_\varepsilon(t) + \|v_\varepsilon\|_{L^2_x}^2(t) + \mathcal{A}_\varepsilon(t)).
\end{equation}
(5.1)

Applying the Brezis–Gallouët–Wainger inequality (4.17) with $f = |v_\varepsilon| + \varepsilon^3$ leads to
\begin{equation}
\|v_\varepsilon\|_{L^2_t L^\infty_x}^2 \lesssim \|v_\varepsilon| + \varepsilon^3\|_{L^2_t L^2_x}^2
\end{equation}
\begin{equation}
\lesssim \|v_\varepsilon| + \varepsilon^3\|_{L^2_t H^1_x}^2 \left(1 + \log \left(1 + \sup_{t \in [0,T]} \frac{\|v_\varepsilon| + \varepsilon^3\|_{W^{1,p}_x}}{\|v_\varepsilon| + \varepsilon^3\|_{H^1_x}} \right) \right)
\end{equation}
\begin{equation}
\lesssim \left(\varepsilon^6 + \|v_\varepsilon\|_{L^2_t H^1_x}^2 \right) \left(1 + \log \left(C + \varepsilon^{-3}\|v_\varepsilon\|_{L^\infty_t W^{1,p}_x} \right) \right).
\end{equation}
(5.2)

This combined with (4.16) yields
\begin{equation}
\|v_\varepsilon\|_{L^2_t L^\infty_x}^2 \lesssim \left(\varepsilon^6 + \|v_\varepsilon\|_{L^2_t H^1_x}^2 \right) \left(1 + \log \left(C + 1 + \varepsilon^{-9} \right) \right).
\end{equation}
(5.2)

In order to apply (5.2) and Grönwall’s inequality to (5.1), we shall employ the method of continuity. For each $\tau \in [0,T]$, we denote
\begin{equation}
\Lambda(\varepsilon, \tau) := \sup_{t \in [0,\tau]} \left( \mathcal{E}_\varepsilon(t) + \frac{1}{2} \|v_\varepsilon\|_{L^2_x}^2(t) + \mathcal{A}_\varepsilon(t) \right) + \frac{1}{2} \int_0^\tau \|\nabla v_\varepsilon\|_{L^2_x}^2.
\end{equation}
(5.3)
By (1.20) we have $\Lambda(\varepsilon, 0) \leq C_0\varepsilon^2$. We make the hypothesis
\[
\text{hypothesis: } \Lambda(\varepsilon, \tau) \leq C_\tau\varepsilon^2, \quad \forall \varepsilon \in (0, \varepsilon_0)
\] (5.4)
for some $C_\tau, \varepsilon_0 > 0$ to be determined later on. We can first apply Grönwall’s inequality $\dagger$ to (5.1), and then use (5.2)
\[
\sup_{t \in [0, \tau]} \Lambda(\varepsilon, t) \leq \Lambda(\varepsilon, 0) \exp \left( C + C \int_0^\tau \|v_\varepsilon\|_{L^\infty} \right) \leq C_0\varepsilon^2 \exp \left( C \left( 1 + [\varepsilon^6 + C_\tau\varepsilon^2] \log(C + 1 + \varepsilon^{-9}) \right) \right) .
\] (5.5)

For a generic constant $\tilde{c} = \tilde{c}(C) > 0$ depending only on $C$, we have
\[
\exp \left( C \left( 1 + [\varepsilon^6 + C_\tau\varepsilon^2] \log(C + 1 + \varepsilon^{-9}) \right) \right) \leq \exp \left( \tilde{c} + C_\tau\varepsilon \right), \quad \forall \varepsilon \in (0, 1).
\] (5.6)

Then with the choices
\[
C_\tau := 2e^{2\tilde{c}}C_0 \text{ and } \varepsilon_0 := \frac{1}{C_\tau},
\] (5.7)
we deduce from (5.4), (5.5) and (5.6) that
\[
\sup_{t \in [0, \tau]} \Lambda(\varepsilon, t) \leq \exp \left( \tilde{c}(1 + C_\tau\varepsilon) \right) C_0\varepsilon^2 \leq \frac{C_\tau}{2}\varepsilon^2, \quad \forall \varepsilon \in (0, \varepsilon_0).
\] (5.8)

Combining this with (5.4) and the method of continuity implies that, with the choices of constants in (5.7), the hypothesis (5.4) holds for any $\tau \in [0, T]$. This implies (1.21), and the proof of the main theorem is finished.

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$\dagger$ Let $a(t), b(t)$ be two non-negative functions such that $a'(t) + b(t) \leq C(t)a(t)$ for some $C(t) \geq 0$. Then
\[
a(t) + \int_0^t b(\tau) d\tau \leq e^{\int_0^t C(s) ds} a(0).
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