There has been some controversies at the large $N$ behaviour of the 2D Yang-Mills and chiral 2D Yang-Mills theories. To be more specific, is there a one parameter family of minima of the free energy in the strong region, or the minimum is unique. We show that there is a missed equation which, added to the known equations, makes the minimum unique.
Recently the large group limit of the 2D Yang–Mills theory (YM$_2$) has become of much interest from different points of view. There is a string formulation of the problem. It has been shown that there is string theory description of pure two dimensional gauge theories with various gauge groups [1-5]. The large-group phase transitions are also of the interesting features of YM$_2$. In ref. 6, the large $N$ behaviour of YM$_2$ for U($N$) gauge group on the sphere has been studied. There, the free energy of YM$_2$ has been obtained for small areas (weak region). Douglas and Kazakov (DK) [6] have studied this theory for arbitrary areas and have shown that the theory has a third order phase transition at some critical area ($A_c$). In their work, a symmetric ansatz for the density function $\rho$ is considered. In a subsequent work, Minahan and Polychronakos (MP)[8] consider an asymmetric ansatz for the density function and introduce a one parameter family of solutions. They introduce a free parameter $Q$ (U(1) charge sector of U($N$)) and express their results in terms of this parameter.

It was shown that the free energy of the complete SU($N$) theory is equal to the $Q = 0$ U($N$) theory in the large area region, where the string expansion is valid [1]. In the words of Crescimanno and Taylor (CT) [10], the results of DK may be the complete description of the phase structure of YM$_2$ and therefore, the DK solution probably represents the extremum of the action with respect to $Q$. To our knowledge, this problem has not been solved so far, and needs more investigations.

Besides this, the chiral version of the large $N$ U($N$) gauge theory on a two-dimensional sphere of area $A$ has been studied in [10]. For small and large areas ($A < A_-$ and $A > A_+$, respectively) a single cut solution has been found. (That is, a solution for the density function, for which the region where this function is less than one, is connected.) For the intermediate region ($A_- < A < A_+$) a two cut solution has been obtained. (That is, a solution for the density function, for which the region where this function is less than one, consists of two disconnected parts.) This again, leads to a one parameter family of solutions for the density function. Although, as noted in [10], the numerical evidence indicates that there is indeed only one solution in the intermediate region. We prove this analytically.

In both cases, the number of equations the authors obtain, is one less than the number of parameters required to determine the density function. In this paper, we show that there is a missed equation in
both cases. This extra equation removes the arbitrariness of the solutions. For the MP case, this extra equation fixes \(Q\) to be zero. This (apparent) insufficiency of the number of equations occurs whenever multi-cut solutions are considered \([11]\).

The partition function of a YM\(_2\) theory on a two dimensional compact Riemann surface of area \(A\) and genus \(g\) is \([12]\).

\[
Z = \sum_r (d_r)^2 \exp \left[ -\frac{A}{2} C_2(r) \right],
\]

where \(C_2(r)\) is the second Casimir of the irreducible representation \(r\) of the gauge group, and \(d_r\) is its dimension. For the gauge group \(U(N)\), the above summation goes over the lengths of the rows of the Young tableau \(\{n_1, n_2, \ldots, n_N\}\) corresponding to the representation \(r\). These should satisfy

\[
n_1 \geq n_2 \geq \cdots \geq n_N.
\]

We also have

\[
C_2(r) = \sum_{i=1}^{N} [(n_i + N - i)^2 - (N - i)^2],
\]

\[
d_r = \prod_{1 \leq i < j \leq N} (1 + \frac{n_i - n_j}{j - i}).
\]

In the large \(N\) limit, one can introduce the continuum variables \(x\) and \(n(x)\) \([6]\):

\[
x := \frac{i}{N}, \quad n(x) := \frac{n_i}{N}.
\]

Using a change of variable

\[
\phi(x) := -n(x) + x - 1,
\]

the partition function takes a simpler form:

\[
Z = \int D\phi(x) e^{S[\phi(x)]},
\]

where

\[
S[\phi(x)] = N^2 \left[ \int_0^1 dx \int_0^1 dy \log |\phi(x) - \phi(y)| - \frac{A}{2} \int_0^1 dx \phi^2(x) \right],
\]

apart from an unimportant constant. The inequality \([3]\), expressed in terms of \(\phi\), is

\[
\frac{\phi(x) - \phi(y)}{x - y} \geq 1.
\]
One can introduce the variable $\rho[\phi(x)]$ as

$$\rho[\phi(x)] := \frac{dx}{d\phi}. \quad (9)$$

Then

$$\int_{d=\phi(0)}^{a=\phi(1)} \rho(z) \, dz = 1, \quad (10)$$

and, form (8),

$$\rho(z) \leq 1. \quad (11)$$

Forgetting about the constraint (11), the saddle point equation for the action (5) is

$$\frac{A z}{2} = P \int d \lambda \frac{\rho(\lambda)}{z - \lambda}, \quad (12)$$

where $P$ indicates the principal value of the integral. The solution of the above integral equation, along with (10), is the well known semicircle law [6]:

$$\rho(z) = \frac{A}{2\pi} \sqrt{\frac{4}{A} - z^2}, \quad (13)$$

with

$$a = -d = \sqrt{\frac{4}{A}}. \quad (14)$$

The free energy is $F(A) = -\frac{1}{N^2} \log Z$, and its derivative with respect to the area $A$ is

$$F'(A) = \frac{1}{2} \int_{0}^{1} dx \, \phi^2(x) = \int_{-a}^{a} dz \, z^2 \rho(z) = \frac{1}{2A}. \quad (15)$$

But the equation (13) is valid only if $\rho(z) \leq 1$, and this holds for $A \leq A_c = \pi^2$. For $A > A_c$, there is a region where $\rho(z) > 1$, and one should search for another solution, which does not violate (11).

DK have considered a symmetric ansatz for $\rho(z)$. They take $\rho(z)$ to be equal to one in the region $[-b, b]$, and less than one in the region $(-a, -b) \cup (b, a)$, where $0 < b < a$. They obtain the derivative of the free energy and show that the theory has a third order phase transition. MP have considered a more general asymmetric ansatz in which $\rho(z)$ is equal to one in the region $[c, b]$, and less than one in the region $L := (d, c) \cup (b, a)$, where $d < c < b < a$. They obtain three equations, whereas there are four
parameters. They claim that a one parameter family of solutions exists, which is expressed in terms of $Q$, the $U(1)$ charge sector of $U(N)$ theory. We show that an equation is missed, and the solution is unique.

To see this, let’s go back to the action (7), write the saddle point equation with respect to $\phi(x)$ for $A > A_c$, and express the result in terms of $\rho(z)$. We obtain

$$\frac{A z}{2} = P \int_a^a d\lambda \frac{\rho_s(\lambda)}{z - \lambda} + \log \frac{z - c}{z - b}, \quad (16)$$

where

$$\rho_s(z) = \begin{cases} \tilde{\rho}_s(z), & z \in L \\ 1, & z \in [c, b] \end{cases} \quad (17)$$

Now define the functions

$$H_s(z) := \int_a^a d\lambda \frac{\rho_s(\lambda)}{z - \lambda}$$

$$\tilde{H}_s(z) := \int_{L} d\lambda \frac{\tilde{\rho}_s(\lambda)}{z - \lambda} \quad (18)$$

The function $H_s(z)$ has a cut $[d, a]$ and,

$$H_s(z \pm i\epsilon) = \mp i\pi \rho_s(z) + \text{(some continuous function)}, \quad z \in [d, a]$$

$\tilde{H}_s(z)$ has two cuts $[d, c]$ and $[b, a]$. The solution for $\tilde{H}_s(z)$, and then $H_s(z)$ is (18),

$$H_s(z) = \log \frac{z - c}{z - b} + \sqrt{(z - a)(z - b)(z - c)(z - d)} \int_{c_L} \frac{A \frac{\lambda}{2} - \log \frac{\lambda - c}{\lambda - b}}{(z - \lambda)\sqrt{(\lambda - a)(\lambda - b)(\lambda - c)(\lambda - d)}} d\lambda \quad (19)$$

where $c_L$ is the contour encircling the cuts $[d, c]$ and $[b, a]$, leaving the point $z$ out. One can deform the contour $c_L$ to a contour consisting of three parts: $c_L'$, which is a contour encircling $[c, b]$; $c_z$, a contour encircling the pole $\lambda = z$; and $c_\infty$, a contour at the infinity. In this way one arrives at

$$H_s(z) = \frac{A z}{2} - \sqrt{(z - a)(z - b)(z - c)(z - d)} \int_c^b \frac{d\lambda}{(z - \lambda)\sqrt{(\lambda - a)(\lambda - b)(\lambda - c)(\lambda - d)}} \quad (20)$$

Inserting the above form of $H_s(z)$ in (18), and expanding both sides of (18) for large $z$, one can obtain the following equations from the coefficients of $z$, $z^0$, and $z^{-1}$, respectively.

$$\frac{A}{2} - \int_c^b \frac{d\lambda}{\sqrt{(a - \lambda)(b - \lambda)(\lambda - c)(\lambda - d)}} = 0, \quad (21)$$

$$\frac{a + b + c + d}{2} \int_c^b \frac{d\lambda}{\sqrt{(a - \lambda)(b - \lambda)(\lambda - c)(\lambda - d)}} - \int_c^b \frac{\lambda d\lambda}{\sqrt{(a - \lambda)(b - \lambda)(\lambda - c)(\lambda - d)}} = 0, \quad (22)$$

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\[
\begin{align*}
&\left\{-\frac{a(b+c+d) + b(c+d) + cd}{4} + \frac{a^2 + b^2 + c^2 + d^2}{8}\right\} + \int_c^b \frac{d\lambda}{\sqrt{(a - \lambda)(b - \lambda)(\lambda - c)(\lambda - d)}} \\
&+ \frac{a+b+c+d}{2} \int_c^b \frac{\lambda d\lambda}{\sqrt{(a - \lambda)(b - \lambda)(\lambda - c)(\lambda - d)}} - \int_c^b \int_c^b \lambda^2 d\lambda \\
&= \int_d^a \rho(\lambda)d\lambda = 1. \hspace{1cm} (23)
\end{align*}
\]

The coefficient of \(z^{-2}\) of the right hand side of (19) is \(\int \lambda \rho_s(\lambda) d\lambda\), which is the charge defined in [8].

It is obvious that the three equations (21-23) are not adequate for determining \((a, b, c, d)\). MP used the charge \(Q\) to obtain the remained parameter. As we claimed, there exists another equation which must be considered. Consider the action (5) with the constraint (10), in term of \(\rho_s(z)\)

\[
\tilde{S} = -N^2 \left\{-\int_d^a dz \int_d^a dw \rho_s(z) \rho_s(w) \log(z - w) + \frac{A}{2} \int_d^a z^2 \rho_s(z) dz + \lambda \left[ \int_d^a \rho_s(z) dz - 1 \right] \right\}, \hspace{1cm} (24)
\]

where \(\lambda\) is a Lagrange multiplier. Variating \(\tilde{S}\) with respect to \(\rho_s(z)\) leads to

\[
-\frac{1}{N^2} \frac{\delta \tilde{S}}{\delta \rho_s(z)} = -2 \int_d^a dw \rho_s(w) \log(z - w) + \frac{Az^2}{2} + \lambda. \hspace{1cm} (25)
\]

We must impose

\[
\frac{\delta \tilde{S}}{\delta \rho_s(z)} = 0, \quad z \in L. \hspace{1cm} (26)
\]

Differentiating (26) with respect to \(z\) gives

\[
P \int_d^a dw \frac{\rho_s(w)}{z - w} = \frac{Az}{2}, \quad z \in L \hspace{1cm} (27)
\]

which is the same as (12). But (27) has more information than (27). In fact, from (26) we also have

\[
2 \int_d^a dw \rho_s(w) \log \frac{b - w}{c - w} + \frac{A}{2} \left( b^2 - c^2 \right) = 0, \hspace{1cm} (28)
\]

which is the difference of eq. (24) at \(z = b\) and \(z = c\). Eq. (28) can be written as

\[
\int_c^b dz \left[ P \int_d^a \frac{dw \rho_s(w)}{z - w} - \frac{Az}{2} \right] = 0, \hspace{1cm} (29)
\]

or

\[
P \int_c^b dz \int_d^b \frac{dw \sqrt{(a - z)(b - z)(z - c)(z - d)}}{(z - w) \sqrt{(a - w)(b - w)(w - c)(w - d)}} = 0. \hspace{1cm} (30)
\]

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This can be written as,

\[ P \int_c^b \! dz \int_c^b \! dw \frac{dz}{z-w} + \int_c^b \! dz \int_c^b \! dw \sqrt{\frac{(z-a)(z-b)(z-c)(z-d)}{(w-a)(w-b)(w-c)(w-d)}} - \frac{1}{z-w} = 0. \]  \hspace{1cm} (31)

The first term is equal to zero. Now, we define the parameters \( \bar{x} := \frac{b+c}{2} \) and \( \xi =: \frac{b-c}{2} \), and expand (31) in terms of \( \xi \) (note that \( \xi = 0 \) at \( A = A_c \)). This yields

\[ \frac{\pi^2 \xi^2}{4} (a + d - 2\bar{x}) \left[ 1 + \xi^2 \frac{8(\bar{x} - d)^2 + 8(a - \bar{x})^2 + 3(a - d)^2}{32(a - \bar{x})^2(\bar{x} - d)^2} + \cdots \right] = 0. \]  \hspace{1cm} (32)

The term in the brackets is obviously positive. So \( a + d - 2\bar{x} \) should be zero, which means that, around the critical area, \( a \) and \( d \) are symmetric with respect to \( \bar{x} \). This is the fourth equation, and now, the parameters \( a, b, c, \) and \( d \) can be determined uniquely.

There is also a simple and somehow straightforward way to deduce \( Q = 0 \). Consider the action (7), shifting the solution \( \phi \) by a constant \( c \) results

\[ S(c) = -N^2 \left\{ -\int_0^1 \! dx \int_0^1 \! dy \log |\phi(x) - \phi(y)| + \frac{A}{2} \int_0^1 \! dx (\phi + c)^2 \right\}. \]  \hspace{1cm} (33)

Now, as \( \phi \) is an extermum of \( S \), the derivative of \( S \) with respect to \( c \) if \( \phi \) should be zero:

\[ \frac{dS}{dc} \bigg|_{c=0} = 0. \]  \hspace{1cm} (34)

So,

\[ \int_0^1 \! \phi(x) dx = 0. \]  \hspace{1cm} (35)

This is nothing but the U(1) charge \( Q \):

\[ Q = \int z \rho(z) dz = \int_0^1 \! \phi(x) dx = 0. \]  \hspace{1cm} (36)

MP has shown that this theory is equivalent to a system of \( N \) nonrelativistic fermions living on a circle and at the critical area \( A_c \) fermion condensation occurs \[8\]. The partition function for the fermionic theory is

\[ Z = \sum_{p_i} \prod_{i>j} (p_i - p_j)^2 \exp(-\frac{LT}{2} \sum p_i^2), \]  \hspace{1cm} (37)

where \( p_i \) is the momentum of the \( i \)th fermion, and \( L \) and \( T \) are the parameters of the two dimensional space-time cylinder on which the electrons live. Comparing (37) with (1), where \( C_2(r) \) and \( d_r \) are given
by (3), it is seen that the U(1) charge \( Q \) is \( \frac{\sum p_i}{N^2} \), or the center of mass momentum divided by \( N^2 \).

The above reasoning leading to \( Q = 0 \) holds true for \( \frac{\sum p_i}{N^2} = 0 \).

As a second example, let’s consider the chiral YM\(_2\) theory. The chiral U(\( N \)) gauge theory at large \( N \) has been studied in [10]. For small and large areas (\( A < A_- \) and \( A > A_+ \), respectively) a single cut solution has been found. In the intermediate region \( A_- < A < A_+ \), the number of equations are insufficient: there are three equations while there are four unknown parameters. As in the previous case, there is also a missed equation. As discussed in [10], consider the ansatz

\[
\rho(z) = \begin{cases} 
1, & z \in R := [c, b] \cup [a, 1/2] \\
\tilde{\rho}(z), & z \in [d, c] \cup [b, a].
\end{cases}
\] (38)

The integral equation for the saddle point is

\[
\frac{Az}{2} + \log \frac{(z - \frac{b}{2})(z - b)}{(z - a)(z - c)} = P \int_R d\lambda \frac{\rho(\lambda)}{z - \lambda}.
\] (39)

Using the same argument leading to (29), one obtains

\[
\int_c^b dz \left[ \frac{Az}{2} + \log \frac{(z - \frac{b}{2})(z - b)}{(z - a)(z - c)} - P \int_R d\lambda \frac{\rho(\lambda)}{z - \lambda} \right] = 0.
\] (40)

This equation, added to the three equations obtained in [10], gives a complete set of equations. So the solution is, again, unique.

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