Accelerating Observers, Area and Entropy

Jarmo Mäkelä

Department of Physics, University of Jyväskylä, PB 35 (YFL), FIN-40351 Jyväskylä, Finland

We consider an explicit example of a process, where the entropy carried by radiation through an accelerating spacelike two-plane is proportional to the decrease in the area of that two-plane even when the two-plane is not a part of any horizon of spacetime. Our results seem to support the view that entropy proportional to area is possessed not only by horizons but by all spacelike two-surfaces of spacetime.

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I. INTRODUCTION

A very interesting discovery was published by Jacobson in 1995. He had found that, in a certain sense, Einstein’s field equation may be viewed as a thermodynamic equation of state of spacetime and matter fields. This discovery was based on the fact that when the Rindler horizon of an accelerating observer emits the so called Unruh radiation, the area of the considered part of the Rindler horizon shrinks. Jacobson observed that if one assumes that the entropy carried away from the considered part of the Rindler horizon is, in natural units, exactly one quarter of the decrease in the area of that part, then Einstein’s field equation follows from the first law of thermodynamics. In other words, it turned out that Einstein’s field equation is just a consequence from the first law of thermodynamics, and an assumption that entropy equal to one quarter of the horizon area is possessed not only by black hole and cosmological horizons, but also by the Rindler horizon. This conclusion supports the view that classical general relativity actually describes the thermodynamic properties of spacetime. If one wants to construct a microscopic theory of gravitation, the correct way to do this may not be to quantize general relativity but to postulate an existence of certain microscopic constituents of spacetime obeying laws which, in the thermodynamic limit, reproduce Einstein’s field equations.

An essential feature in Jacobson’s analysis was that a horizon was looked at: When radiation is emitted by a horizon, the horizon shrinks, and the entropy of the horizon is converted to the entropy of the radiation. It is interesting to investigate, whether entropy would be possessed not only by horizons but by all spacelike two-surfaces of spacetime. Arguments supporting this claim have been given, for instance, in, where it was proposed that every spacelike two-surface of spacetime possesses entropy which, in natural units, is one quarter of the area of that two-surface. If true, this proposal brings great clarity to the concept of gravitational entropy: Entropy is a property of spacetime, and the entropy of a spacelike two-surface is always the same, no matter whether that two-surface is a part of a horizon or not.

To support the view that entropy is possessed not only by the horizons, but by all spacelike two-surfaces of spacetime we consider, in this paper, a slight modification of Jacobson’s analysis. More precisely, we consider a plane parallel to the $xy$-plane, which is accelerating with a constant proper acceleration along the positive $z$-axis. We assume that, from the point of view an observer at rest with respect to the flat Minkowski coordinates $x$, $y$ and $z$, spacetime is filled with electromagnetic radiation in a thermal equilibrium, from which it follows that there is a certain net flow of radiation through the accelerating plane to the negative $z$-direction. We investigate the properties of this radiation,
and it turns out that when radiation propagates through the accelerating plane, the plane shrinks. Using the first law of thermodynamics we shall arrive at an interesting conclusion that the entropy carried by Unruh radiation through the accelerating plane is exactly one half of the decrease in the area of that plane. In other words, it seems that we have found a process where the entropy of a two-plane is converted to the entropy of radiation even when that two-plane is not a part of any horizon of spacetime. This result strongly supports the proposal that not only horizons, but all spacelike two-surfaces of spacetime possess a certain amount of entropy.

This paper is organized as follows: In Section 2 we consider the energy flow and intensity of radiation through an accelerating plane. In Section 3 we investigate the change in the area of the accelerating two-plane when radiation flows through that plane. Finally, in Section 4 we bring to the stage the first law of thermodynamics. The first law of thermodynamics, together with the results of Sections 2 and 3, implies the desired relationship between area and entropy. We end our discussion with concluding remarks. Unless otherwise stated, we shall always use the natural units, where $\hbar = c = G = k_B = 1$.

II. FLOW OF ENERGY

When flat Minkowski spacetime is filled with electromagnetic radiation in thermal equilibrium, there is no net flow of energy in any direction from the point of view of an observer at rest with respect to the flat Minkowski coordinates $t$, $x$, $y$, and $z$. The only non-zero components of the energy momentum stress tensor $T^{\mu\nu}$ of radiation are:

$$T^{00} = \rho, \tag{2.1}$$

where $\rho$ is the energy density of radiation, and

$$T^{11} = T^{22} = T^{33} = p, \tag{2.2}$$

where $p$ is the pressure of the radiation. Because for electromagnetic radiation

$$p = \frac{1}{3} \rho, \tag{2.3}$$

we find that

$$T := T^{00} + T^{11} + T^{22} + T^{33} = 0. \tag{2.4}$$

In other words, the tensor $T^{\mu\nu}$ is trace-free. This result will turn out very useful in our investigations.

Although there is no net flow of energy from the point of view of an observer at rest with respect to the coordinates $x$, $y$ and $z$, there is, however, a net flow of energy to the negative $z$-direction from the point of view of an observer in a uniformly accelerating motion along the positive $z$-axis. The world line of such an observer satisfies the equation

$$z^2 - t^2 = \frac{1}{a^2}, \tag{2.5}$$

where $a$ is the proper acceleration of the observer. This world line may be parametrized by the observer’s proper time $\tau$ such that

$$t = \frac{1}{a} \sinh(a\tau), \tag{2.6a}$$

$$z = \frac{1}{a} \cosh(a\tau). \tag{2.6b}$$

The time- and the $z$-coordinates of the accelerating observer are $t'$ and $z'$, respectively, such that for infinitesimal changes of $t'$ and $z'$:

$$dt' = d\tau = \frac{dt - v\,dz}{\sqrt{1 - v^2}}, \tag{2.7a}$$

$$dz' = \frac{dz - v\,dt}{\sqrt{1 - v^2}}. \tag{2.7b}$$
where
\[ v := \tanh(a\tau). \quad (2.8) \]
is the speed of the accelerating observer with respect to the observer at rest. The flow of energy, or intensity of radiation to the negative \( z \)-direction through the \( xy \)-plane at rest with respect to the accelerating observer is
\[ I = -T^{03} = -\frac{\partial x^0}{\partial x^\mu} \frac{\partial x^3}{\partial x^\nu} T^{\mu\nu} = \frac{1}{2} \sinh(2a\tau)(T^{00} + T^{33}), \quad (2.9) \]
and it follows from Eqs. (2.1)-(2.3) that
\[ I = \frac{2}{3} \sinh(2a\tau) \rho. \quad (2.10) \]
Hence we see that if we pick up a region with area \( A \) from the \( xy \)-plane, the energy carried by radiation through that region during an infinitesimal proper time interval \( d\tau \) is
\[ \delta Q = \frac{2}{3} A \rho \sinh(2a\tau) d\tau. \quad (2.11) \]

III. CHANGE OF AREA

In the previous Section we assumed that spacetime is flat, and filled with electromagnetic radiation in thermal equilibrium. Interaction between radiation and the geometry of spacetime, however, makes spacetime curved, and as a result there is a change in the area of the accelerating plane we considered in the previous Section. When we evaluate the change in the area of an accelerating plane in the course of time, we might use, of course, the so called Raychaudhuri equation \[ \text{[4]}, \]
and that was indeed the equation on which Jacobson’s investigations were based. Raychaudhuri equation, however, describes the behavior of geodesics of spacetime, and because an accelerating observer does not move along a spacetime geodesic, it is not quite clear how to apply that equation. Because of the conceptual problems involved in the Raychaudhuri equation when studying the geometric properties of accelerating planes, we shall instead apply the linear field approximation of Einstein’s field equation.

The starting point of the linear field approximation is to write the spacetime metric as:
\[ ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu. \quad (3.1) \]
In this equation, \( x^\mu \)’s are the flat Minkowski coordinates, and \( \eta_{\mu\nu} := \text{diag}(1, -1, -1, -1) \) is the flat Minkowski metric. The field \( h_{\mu\nu} \) represents a small deviation from the flat spacetime geometry. The linear field approximation is therefore particularly useful for the investigation of the small changes in the spacetime geometry. In what follows, we shall assume that at the spacetime point \( P = (t, x, y, z) = (0, 0, 0, 1/a) \), where our accelerating observer is at rest with respect to the radiation, the field \( h_{\mu\nu} \) and its first derivatives vanish. In other words, we shall assume that at the point \( P \) spacetime metric is that of flat spacetime. When we move away from the point \( P \), spacetime metric deviates slightly from its flat spacetime form. We shall also assume that the spacetime metric (3.1) remains unchanged when we interchange the spatial coordinates representing deviations from the point \( P \). In other words, we shall assume that, when looked from the point \( P \), spacetime geometry looks the same in all spatial directions. This requirement is motivated by the fact that the components of the energy momentum stress tensor \( T^{\mu\nu} \) remain unchanged under the interchanges of the spatial coordinates of spacetime. It is reasonable to expect that the same invariance property is possessed also by the spacetime metric itself.

When \( h_{\mu\nu} \) satisfies the so called Hilbert gauge condition
\[ \partial_\mu(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h) = 0, \quad (3.2) \]
the linear field approximation of Einstein’s field equation may be written as:

$$\partial_{\lambda} \partial^{\lambda}(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h) = -16\pi T^{\mu\nu}. \quad (3.3)$$

In these equations, $h := \eta^{\mu\nu} h_{\mu\nu}$, and the components of $T^{\mu\nu}$ are given by Eqs.(2.1)-(2.3). Since the electromagnetic radiation which fills spacetime is assumed to be in thermal equilibrium, and therefore homogeneous, we shall look for a solution, where the diagonal elements of the metric depend on the time $t$ only. It is easy to see that there is just one solution to Eqs.(3.2) and (3.3), which satisfies this requirement, together with the other two requirements given before. The solution in question is the one, where the spacetime metric (3.1) takes the form:

$$ds^2 = (1 - 8\pi \rho t^2) dt^2 - \frac{32}{3} \pi \rho t (x \, dx + y \, dy + \tilde{z} \, d\tilde{z}) \, dt$$
$$- (1 + \frac{8}{3} \pi \rho t^2) (dx^2 + dy^2 + d\tilde{z}^2) - \frac{16}{3} \pi \rho (xy \, dx \, dy + x\tilde{z} \, dx \, d\tilde{z} + y\tilde{z} \, dy \, d\tilde{z}), \quad (3.4)$$

where we have defined:

$$\tilde{z} := z - \frac{1}{a}. \quad (3.5)$$

We are now prepared to calculate the change in the area of a considered region of the accelerating $xy$-plane, from the point of view of an observer at rest with respect to that plane. In general, the area of a region on the $xy$-plane is

$$A = \int_{\mathcal{A}} \sqrt{q} \, dx \, dy, \quad (3.6)$$

where $q$ is the determinant of the metric induced on that plane, and $\mathcal{A}$ is the domain of integration. Change in the area during the course of time is therefore a result from changes both in the metric and the domain of integration. When a plane is in accelerating motion in curved spacetime, the points of the region under consideration move on the plane with respect to the coordinates on that plane (or, rather, the coordinates move with respect to the points), and the geometry of the plane itself changes. Both of these contributions must be taken into account, when we calculate the area change.

To begin with, we investigate the movements of the points on the plane. Consider the point $Q = (x, y)$ on the $xy$-plane, which has the point

$$\mathcal{O} := \left( \frac{1}{a} \sinh(a\tau), 0, 0, \frac{1}{a} \cosh(a\tau) \right) \quad (3.7)$$
of spacetime as its origin. Our aim is to calculate the coordinates $(x', y')$ of $Q$ after the point $\mathcal{O}$ has been transported to the point $\mathcal{O}'$, where $\tau$ has been replaced by $\tau + d\tau$. We shall assume that when $\tau = 0$, the point $Q$ is at rest with respect to the coordinates $x, y$ and $z$. In other words, we shall assume that when $\tau = 0$, the only non-zero component of the four-velocity $u^\mu$ of the point $Q$ is

$$u^0 = 1. \quad (3.8)$$

To calculate the four-velocity of $Q$, when $\tau \neq 0$, we parallel transport the vector $u^\mu$ from the point $(0, x, y, 1/a)$ to the point $(\frac{1}{a} \sinh(a\tau), x, y, \frac{1}{a} \cosh(a\tau))$. In infinitesimal parallel transport from point $x^\mu$ to the point $x^\mu + dx^\mu$ the change experienced by $u^\mu$ is, in general,

$$\delta u^\mu = -\Gamma^\mu_{\alpha\beta} u^\alpha \, dx^\beta, \quad (3.9)$$

where $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol. In the linear field approximation $\Gamma^\mu_{\alpha\beta}$ takes the form:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \eta^{\mu\sigma} (\partial_{\beta} h_{\sigma\alpha} + \partial_{\alpha} h_{\beta\sigma} - \partial_{\sigma} h_{\alpha\beta}). \quad (3.10)$$
Using Eqs. (3.4) and (3.9) we find that at the point \((\frac{1}{a}\sinh(a\tau), x, y, \frac{1}{a}\cosh(a\tau))\) we have:

\[
\begin{align*}
u^1 &= -\frac{16}{3}\pi \rho x \tau, \\
u^2 &= -\frac{16}{3}\pi \rho y \tau,
\end{align*}
\]

(3.11)

where we have kept the terms linear in \(\tau\) only. When \(\tau\) is changed to \(\tau + d\tau\), the coordinates \((x, y)\) of the point \(Q\) on the \(xy\)-plane transform to

\[
\begin{align*}
x' &= x + u^1 d\tau = (1 - \frac{16}{3}\pi \rho \tau d\tau)x, \\
y' &= y + u^2 d\tau = (1 - \frac{16}{3}\pi \rho \tau d\tau)y.
\end{align*}
\]

(3.12)

In other words, we have just re-scaled the coordinates \(x\) and \(y\) by the factor \((1 - \frac{16}{3}\pi \rho \tau d\tau)\).

It only remains to calculate the change in the metric. The metric induced on the \(xy\)-plane is

\[
edL^2 = (1 + \frac{8}{3}\pi \rho \tau^2)(dx^2 + dy^2) - \frac{16}{3}\pi \rho xy dx dy.
\]

(3.13)

The change experienced by \(\sqrt{q}\) when \(\tau\) is changed to \(\tau + d\tau\) is therefore:

\[
\delta \sqrt{q} = \left(\frac{\partial}{\partial t} \sqrt{q}\right) \frac{dt}{d\tau} d\tau = \frac{16}{3}\pi \rho \tau d\tau,
\]

(3.14)

where we have, again, kept the terms linear in \(\tau\) only. We have also neglected the terms non-linear in \(h_{\mu\nu}\). The relationship between \(t\) and \(\tau\) is given by Eq. (2.6a). Combining Eqs. (3.6), (3.12) and (3.14) we find that during the proper time interval \(d\tau\) the area \(A\) of the region under consideration becomes to

\[
A' = (1 + \frac{16}{3}\pi \rho \tau d\tau)(1 - \frac{16}{3}\pi \rho \tau d\tau)^2 A.
\]

(3.15)

Because \(d\tau\) is infinitesimal, the change experienced by \(A\) is therefore:

\[
dA = -\frac{16}{3}\pi A \rho \tau d\tau.
\]

(3.16)

As one can see, the plane shrinks, when radiation flows through the plane to the negative \(z\)-direction.

### IV. AREA AND ENTROPY

We denoted in Eq. (2.11) the amount of energy carried by radiation through the \(xy\)-plane by \(\delta Q\) for a very good reason: Energy comes through the \(xy\)-plane as heat. According to the first law of thermodynamics the change in the heat of a system may be written as:

\[
\delta Q = dE + p dV,
\]

(4.1)

where \(dE\) is the (infinitesimal) change in the total energy, and \(dV\) in the volume of the system. It is easy to see that both terms on the right hand side of Eq. (4.1) are present on the right hand side of Eq. (2.11): The increase in the total energy in the spatial region ”behind” the accelerating plane from the point of view of the accelerating observer during an infinitesimal proper time interval \(d\tau\) is

\[
dE = \frac{1}{2} A \rho \sinh(2a\tau) d\tau,
\]

(4.2)

and because the increase in the three-volume \(V\) of space ”behind” the accelerating plane is, from the point of view of the accelerating observer,

\[
dV = A \sinh(a\tau) d\tau,
\]

(4.3)
we find, using Eq.(2.3), that the "work term" is

\[ p\,dV = \frac{1}{6}A\rho \sinh(2a\tau)\,d\tau. \] 

(4.4)

When put together, the terms \( dE \) and \( p\,dV \) give the right hand side of Eq.(2.11).

After having convinced ourselves that the right hand side of Eq.(2.11) really gives the heat transported through the accelerating plane from the point of view of the accelerating observer, we may now turn to the relationship between area and entropy. Because between the infinitesimal changes in the heat \( Q \) and the entropy \( S \) of a system there is a relationship:

\[ \delta Q = T\,dS, \] 

(4.5)

where \( T \) is the absolute temperature of the system, and because it follows from Eq.(2.11) that, for very small \( \tau \):

\[ \delta Q = \frac{4}{3}A\rho a\,d\tau, \] 

(4.6)

we find, using Eq.(3.16), that the entropy \( dS \) carried by radiation through the accelerating plane is related to the change \( dA \) in the area of that plane such that:

\[ T\,dS = -\frac{a}{4\pi}dA. \] 

(4.7)

It only remains to fix the absolute temperature \( T \) of the electromagnetic radiation. At this point we turn to the Unruh effect. According to that effect an accelerating observer experiences himself to be immersed in a heat bath of thermal particles. The temperature of this heat bath is the Unruh temperature \( T_U \):

\[ T_U := \frac{a}{2\pi}. \] 

(4.8)

Suppose that when \( \tau = 0 \), spacetime is, from the point of view of the accelerating observer, filled with electromagnetic radiation in thermal equilibrium at the Unruh temperature \( T_U \). When \( \tau = 0 \), there is no net flow of heat in any direction in the accelerating observer’s frame of reference. After a very short elapsed proper time \( \tau \), however, there is a net heat flow to the negative \( z \)-direction in the observer’s frame of reference, and the amount of heat transported through the \( xy \)-plane during a proper time interval \( d\tau \) is given by Eq.(4.6). The temperature of this heat flow is still the Unruh temperature \( T_U \), and we may substitute \( T_U \) for \( T \) in Eq.(4.7). When we perform this substitution, we get:

\[ dS = -\frac{1}{2}dA. \] 

(4.9)

In other words, the entropy carried by radiation through the accelerating plane is, in natural units, exactly one half of the decrease in the area of that plane. This is the final result of this paper, and it holds whenever the temperature of the radiation is equal to the Unruh temperature of an observer at rest with respect to the accelerating plane. It is remarkable that we have obtained a simple linear dependence between area and entropy even when our accelerating two-plane is not a part of any horizon of spacetime.

V. CONCLUDING REMARKS

In this paper we have considered an explicit example of a process, where radiation flows through a spacelike two-surface in such a way that the entropy carried through that two-surface is proportional to the decrease in its area even when that two-surface is not a part of any horizon of spacetime. More precisely, we considered the flow of electromagnetic radiation through an accelerating two-plane, and we found that if the temperature of the radiation is
the Unruh temperature of an observer at rest with respect to the accelerating plane, the entropy carried by radiation through that plane is, in natural units, exactly one half of the decrease in the area of that plane. This result bears a close resemblance to the well-known results concerning the radiation emitted by the black hole, cosmological, and Rindler horizons of spacetime. According to those results the entropy carried by radiation out from a horizon is, in natural units, exactly one quarter of the decrease in its area, whereas we found the same relationship—with an important exception that the constant of proportionality is not one quarter but one half—between the entropy flow and the area decrease when the two-surface is not a horizon but an arbitrary accelerating two-plane.

The derivation of our result should be rather uncontroversial. Indeed, we have used just the fundamental results of thermodynamics and classical general relativity, together with the basic properties of Unruh radiation. We first assumed that spacetime is filled with electromagnetic radiation in thermal equilibrium, and then calculated the heat flow through a uniformly accelerating two-plane to the direction opposite to the direction of the motion of that two-plane. When radiation flows through the two-plane, the plane shrinks, and the decrease in its area may be calculated by means of the linear field approximation. Using the first law of thermodynamics one obtains the relationship between the flow of entropy and the decrease in area. Finally, if one substitutes for the temperature the Unruh temperature measured by an observer at rest with respect to the accelerating two-plane, one obtains the core result of this paper.

Our result seems to support the view that entropy is possessed not only by horizons but, in addition, by all accelerating, spacelike two-planes of spacetime. When radiation flows through a two-plane, the plane shrinks, and the entropy of the two-plane is converted to the entropy of radiation. If one accepts the view that entropy proportional to area is possessed by all accelerating two-planes, there is only a small step left to the idea that entropy is possessed by all spacelike two-surfaces of spacetime, no matter whether the two-surface under consideration is a part of any horizon or not.

An enigmatic feature of our analysis is that it suggests that the entropy of an accelerating two-plane is, in natural units, not one quarter as one might expect, but one half of its area. An explanation of this curious result may lie in the fact that the radiation process of a horizon, and the radiation process considered in this paper are completely different: When a horizon radiates, its geometry interacts with the quanta of its radiation by its one side only. When radiation flows through a two-plane, however, the quanta of radiation interact with the geometry of spacetime on the both sides of the two-plane. So it is possible that radiation picks up entropy from the both sides of the two-plane. This may explain why the numerical value given by our analysis for the entropy of an accelerating two-plane is exactly twice the entropy of a spacetime horizon with the same area.

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