Abstract

The negative nature of the heat capacity $C_V$ of thermodynamically isolated self-gravitating systems is rediscussed in the framework of a non-extensive kinetic theory. It is found that the dependence of $C_V$ on the non-extensive parameter $q$ gives rise to a negative branch with the critical value corresponding to $q = 5/3$ ($C_V \to -\infty$).

PACS numbers:
Consider a simple analogy between ideal gas and self-gravitating system. The temperature of this system is defined by
\[ \frac{1}{2}mv^2 = \frac{3}{2}k_BT, \]
where \( m \) is the mass of a particle (e.g., a star) and \( k_B \) is the Boltzmann’s constant. If such a system is composed by \( N \) particles, its total kinetic energy is \( K = \frac{3}{2}Nk_BT \) which, according to the virial theorem, is equal to minus its total energy, i.e., \( E = -K \). Therefore, the heat capacity of the system is
\[ C_V = \frac{dE}{dT} = -\frac{3}{2}Nk_B, \]
which is clearly a negative quantity. In other words, this means that by losing energy the system automatically grows hotter, or still, that a self-gravitating system can never be in a true thermodynamical equilibrium state (see [1] and references therein). In actual fact, it is well known that when discussed in the context of the microcanonical ensemble a self-gravitating system interacting via long range forces exhibits a negative heat capacity [2]. The negativeness of the heat capacity has been largely studied in many astrophysical phenomena. For example, Lynden-Bell & Wood [3] used such a concept to explain the gravothermal catastrophe for globular clusters while Beckenstein [4] and Hawking [5] used it in the context of black hole thermodynamics.

On the other hand, a considerable theoretical effort has been concentrated on the study of the statistical description of a large variety of physical systems, resulting in the extension of the Boltzmann-Gibbs’ statistical mechanics. Particular examples are systems endowed with long duration memory, anomalous diffusion, turbulence in pure-electron plasma, self-gravitating systems or, more generally, systems endowed with long range interactions. In order to deal with such problems, Tsallis [6] proposed the following generalization of the Boltzmann-Gibbs (BG) entropy formula
\[ S_q = -k_B \frac{1 - \sum_i p_i^q}{(q - 1)}, \]
where \( p_i \) is the probability of the \( i^{th} \) microstate and \( q \) is a parameter quantifying the degree of non-extensivity. In the limit \( q \to 1 \) the celebrated BG extensive formula, \( S_1 = -k_B \sum_i p_i \ln p_i \), is readily recovered as a particular case.

In the astrophysical context, the first applications of this non-extensive \( q \)-statistics studied stellar polytropes [7] and the peculiar velocity function of galaxy clusters [8]. The Jeans
gravitational instability criterion for a collisionless system was also recently rediscussed in the context of this enlarged formalism \[9\]. In this latter study, it was shown that such systems present instability even for wavelengths of the disturbance smaller than the standard critical Jeans value. Recently, Taruya & Sakagami \[10, 11\] discussed a stability problem in the microcanonical ensemble by extending Padmanabhan’s \[2\] classical analysis of the Antonov instability to the case of polytropic distribution functions. More recently, Chavanis \[12, 13, 14\] considered the Tsallis’ entropy as a particular $H$-function corresponding to isothermal stellar systems and stellar polytropes. In this approach, the maximization of the $H$-function at fixed mass and energy reveals naturally a thermodynamical analogy with the study of the dynamical stability in collisionless stellar systems \[24\].

In this Letter, we discuss the negative nature of the specific capacity $C_V$ of self-gravitating systems in the framework of a kinetic theory resulting from this non-extensive formalism. To do so, we consider self-gravitating systems that can be treated as isolated systems in a first approximation, e.g., globular cluster \[15\] or elliptical galaxies \[16\]. We derive a new analytical expression for this quantity which gives rise to an entire negative branch. We show that for this kind of system the value $q = 5/3$ is an upper limit for the non-extensive parameter. In connection with stellar polytropes, the value of the non-extensive parameter $q = 5/3$ corresponding to a polytropic index $n = 3$ or $n = -1$, depending on the expression between $n$ and $q$.

It has been shown that the kinetic foundations of the Tsallis’ thermostatistic lead to a velocity distribution for free particles given by \[17, 18\]

$$f_0(v) = B_q \left[1 - (1 - q) \frac{mv^2}{2k_B T}\right]^{1/1-q}.$$  \hspace{1cm} (4)

In this velocity $q$-distribution, the $q$-parameter quantifies the nonadditivity property of the associated gas entropy whose the main effect at the level of the distribution is to replace the standard Gaussian form by a power law. Mathematically, this result follows directly from the well known identity $\lim_{w \to 0} (1 + wx)^{1/w} = \exp(x)$ \[19\]. The quantity $B_q$ is a $q$-dependent normalization constant whose expression for $q \leq 1$ and $q \geq 1$ are respectively given by

$$\begin{align*}
B_{q \leq 1} &= \begin{cases} 
  n(1 - q)^{1/2} & (5 - 3q)^{3/2} & \frac{\Gamma(\frac{1}{2} + \frac{1}{q})}{\Gamma\left(\frac{3}{2}\right)} & \left(\frac{m}{2\pi k_B T}\right)^{3/2} \\
  n(q - 1)^{3/2} & & \frac{\Gamma\left(\frac{1}{q - 1} \frac{1}{2}\right)}{\Gamma\left(\frac{1}{q - 1} \frac{1}{2}\right)} & \left(\frac{m}{2\pi k_B T}\right)^{3/2}
\end{cases},
\end{align*}$$  \hspace{1cm} (5)

where $T$ is the temperature and $n$ is the particle number density. This non-extensive velocity
distribution can be derived at least from two different methods: (i) through a simple non-extensive generalization of the Maxwell ansatz, \( f(v) \neq f(v_x)f(v_y)f(v_z) \), which follows from the introduction of statistical correlations between the components of the velocities and (ii) by using a more rigorous treatment based on the non-extensive formulation of the Boltzmann \( H \)-theorem, which requires \( q > 0 \).

In order to investigate the \( q \)-dependence of the specific capacity, we first consider a cloud of ideal gas within the nonrelativistic gravitational context, which is analog to a self-gravitating collisionless gas. The kinetic energy of this system is simply given by \( K = \frac{1}{2} m < v^2 > \), with the statistical content included in the average square velocity of the particles \( < v^2 > \). In this way, the non-extensivity is introduced through a new derivation of expectation value

\[
< v^2 >_q = \frac{\int_{-v_m}^{v_m} f^q v^2 d^3v}{\int_{-v_m}^{v_m} f^q d^3v},
\]

where \( v_m = \left( \frac{2 k_B T}{m(1-q)} \right)^{1/2} \) is a thermal cutoff on the maximum value allowed for the velocity of the particles \( (q < 1) \), whereas for the power law \( q \)-distribution without cutoff \( (q > 1) \), \( v_m \to \infty \). This \( q \)-expectation value can be easily evaluated for \( q \neq 1 \), and the result corresponds to a \( q \)-parameterized family of the square velocity of the particles \( < v^2 >_q \) given by

\[
< v^2 >_q = \frac{6}{5 - 3q} \frac{k_B T}{m}, \text{ for } q < 5/3,
\]

with \( < v^2 > = \frac{2 k_B T}{m} \) being the classical value easily obtained in the limit \( q \to 1 \) (It is worth mentioning that the virial theorem is not modified in this non-extensive formalism).

Now, by substituting the non-extensive \( q \)-expectation value given by Eq. (7) into the heat capacity relation Eq. (2) one finds

\[
C_V = -\frac{3}{5 - 3q} N k_B,
\]

which clearly reduces to the Maxwellian limit \( C_V = -\frac{3}{2} N k_B \) for \( q = 1 \). To better understand the physical implications of the above expression, in Fig. 1 we plot the quantity \( C_V/N k_B \) as a function of the non-extensive parameter \( q \). The graph clearly shows that the specific capacity is a negative quantity for the interval \( q < 5/3 \). For \( q = 5/3 \) (critical value) the specific capacity diverges \( (C_V \to -\infty) \). From this analysis, it is possible to conclude that this particular \( q \)-dependence of \( C_V \) leads to the classical gravothermal catastrophe \( (C_V < 0) \) only for power laws with non-extensive parameter in the range \( 0 < q < 5/3 \).
FIG. 1: The quantity $C_V/Nk_B$ for a self-gravitating collisionless gas obeying the non-extensive Tsallis’ $q$-statistic is shown as a function of the non-extensive parameter $q$. We see that there is a negative branch for the interval $0 < q < 5/3$. For $q = 5/3$ the specific capacity diverges ($C_V \to -\infty$).

At this point we emphasize that the true nature of the non-extensive $q$-parameter appearing in the Tsallis’ statistical framework remains as a completely open question at present. In this way, it is important to show empirical manifestations of these non-extensive effects. In this Letter we have discussed a clear signature of non-extensive effects on some kind of self-gravitating systems. In particular, we have shown that the $q$-dependence of the specific capacity for these systems is physically consistent only for power-laws lying in the interval $0 < q < 5/3$. This particular range of values for the non-extensive parameter $q$ is less restrictive than that one obtained by Abe [22]. We suspect that such a difference occurs because the Abe’s approach is addressed in the context of the canonical ensemble and, as has been shown elsewhere, in this ensemble self-gravitating systems do not present negative specific heat [2]. We also note that, when identified with the polytropic index $n = \frac{3}{2} + \frac{1}{q-1}$, the critical value $q = 5/3$ is identical to the one that gives rise to the gravitational stability/instability transition, i.e., $n = 3$ [12] (or $n = -1$ if $n = \frac{1}{2} + \frac{1}{1-q}$ [11]). In a couple of papers by Taruya & Sakagami [11, 23] it was shown that a thermodynamic instability
appears for values of \( n > 5 \) in systems confined in adiabatic walls (microcanonical ensemble) and for values of \( n > 3 \) in systems surrounded by a thermal bath. In fact, a more consistent study of the gravothermal catastrophe must consider a self-gravitating ideal gas contained in a spherical container. We shall return to this discussion in a more complete analysis involving general non-extensive kinetic theories which will appear in a forthcoming communication.

Acknowledgments: The authors are very grateful to J. A. S. Lima, P. H. Chavanis and A. Taruya for helpful discussions and a critical reading of the manuscript. This work is supported by the project CNPq (62.0053/01-1-PADCT III/Milenio). JSA is also supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq - Brasil).
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