1MP and MP1 inverses and one-sided star orders in a ring with involution

Dragan S. Rakić · Martin Z. Ljubenović

Received: 21 March 2022 / Accepted: 24 October 2022 / Published online: 3 November 2022
© The Author(s) under exclusive licence to The Royal Academy of Sciences, Madrid 2022

Abstract
The classes of 1MP-inverses and MP1-inverses are recently introduced classes of generalized inverses of complex matrices. Actually, they coincide with the classes of \{1, 2, 3\} and \{1, 2, 4\} inverses, respectively. We consider these inverses in the context of a ring with involution and prove that their most important characterizations and properties remain true. We show that the binary relations based on these inverses are in fact the well known left-star and right-star partial orders. We extend these relations to the ring case, connect them with the unified theory of partial order relations based on generalized inverses and provide several properties. Finally, we indicate how these results can be applied to bounded Hilbert space operators.

Keywords One-sided star partial order · 1MP-inverse · \{1, 2, 3\}-Inverse · Ring with involution

Mathematics Subject Classification 16U90 · 06A06 · 15A09

1 Introduction and preliminaries

Let \( A \in \mathbb{C}^{m \times n} \), where \( \mathbb{C}^{m \times n} \) is the set of all \( m \times n \) complex matrices. We denote by \( A^* \), \( \text{Im} A \) and \( \text{Ker} A \) the conjugate transpose, column space and null space of \( A \) respectively. A matrix \( A^- \in \mathbb{C}^{n \times m} \) is called a \( g \)-inverse of \( A \) if the equation \( AA^-A = A \) is satisfied. The set of all \( g \)-inverses of \( A \) is denoted by \( A\{1\} \). If \( AXA = A \) and \( XAX = X \) then \( X \in \mathbb{C}^{n \times m} \) is called a reflexive \( g \)-inverse of \( A \). Recall that the Moore–Penrose inverse of \( A \) is the unique matrix \( A^\dagger \in \mathbb{C}^{n \times m} \) which satisfies the equations

\[
\begin{align*}
(1) \quad AA^\dagger A &= A & (2) \quad A^\dagger AA^\dagger &= A^\dagger & (3) \quad (AA^\dagger)^* &= AA^\dagger & (4) \quad (A^\dagger A)^* &= A^\dagger A.
\end{align*}
\]

The research is financially supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 451-03-68/2022-14/200109 and by the bilateral project between Serbia and Slovenia (Generalized inverses, operator equations and applications, Grant No. 337-00-21/2020-09/32).

✉️ Dragan S. Rakić
rakic.dragan@gmail.com

Martin Z. Ljubenović
martinljubenovic@gmail.com

1 Faculty of Mechanical Engineering, University of Niš, Aleksandra Medvedeva 14, Niš 18000, Serbia
Hernández et al. in [10] considered the following equivalence relation on \( A[1] \). For \( A^-, A^w \in A[1] \),

\[
A^- \sim_I A^w \iff A^- A = A^w A.
\]

It turns out that the simplest representative (relative to the singular value decomposition of \( A \)) of the class \([A^-]_\sim_I\), is given by the matrix \( A^- AA^\dagger \). The authors in [10] defined a new class of \( g \)-inverses of \( A \) as the set of all canonical representatives of the quotient space \( A[1]/_\sim_I \). They analogously defined the dual class as well.

**Definition 1** [10] Let \( A \in \mathbb{C}^{m \times n} \). For each \( A^- \in A[1] \), the matrix \( A^\dagger^- = A^- AA^\dagger \) is called a 1MP-inverse of \( A \) and the set of all 1MP-inverses of \( A \) is denoted by \( A[-\dagger] \). Similarly, the matrix \( A'^\dagger^- = A'^\dagger AA'^- \) is called a MP1-inverse of \( A \) and the set of all MP1-inverses of \( A \) is denoted by \( A[\dagger^-] \).

Among other applications, \( g \)-inverses are used to define nice binary relations. This is the case with 1MP and MP1-inverses as well. For \( A, B \in \mathbb{C}^{m \times n} \) the relation \(<^-\dagger \) was defined in [10] in the following way:

\[
A <^-\dagger B \iff AA^-\dagger = BA^-\dagger \quad \text{and} \quad A^-\dagger A = A^-\dagger B \quad \text{for some} \quad A^-\dagger \in A[-\dagger]. \tag{1}
\]

Similarly,

\[
A <^{\dagger^-} B \iff AA'^\dagger^- = BA'^\dagger^- \quad \text{and} \quad A'^\dagger^- A = A'^\dagger^- B \quad \text{for some} \quad A'^\dagger^- \in A[\dagger^-]. \tag{2}
\]

Our aim is to generalize and investigate the 1MP and MP1 inverses and related binary relations (1) and (2) in the context of an arbitrary ring with involution. We show that these notions are actually \{1, 2, 3\}-inverse, \{1, 2, 4\}-inverse, the left-star order and the right-star order. Although the ring setting is more general than complex matrix setting, we will prove almost all known important properties of these inverses and relations. It shows that the nature of these notions is, to a large degree, purely algebraic.

The theory of generalized inverses and partial orders based on them have recently been extensively studied. The left-star and the right-star order can be defined and studied by the use or without the use of generalized inverses. The former case is typical for the matrix context [1, 3, 6, 17] or the ring context [12, 18, 25], while the latter case is typical for the operator context [4, 5, 7, 19] or the Rickart \(*\)-ring context [11, 13, 15].

Although the results in this paper are mostly algebraic, we will demonstrate in Sect. 6 how they can be applied in the case of bounded Hilbert space operators. We will obtain appropriate space decompositions induced by corresponding idempotents. Furthermore, we will provide in Sect. 2 some new properties in the complex matrix case. Therefore, the presented results can be important not only to the researchers who work on the algebraic properties of generalized inverses and partial orders based on them, but also to the researchers working in the matrix and operator theory.

It is clear that in the proofs in the ring case, we cannot use the standard linear algebra techniques which are dominant in the matrix case. So we use different approaches and methods. One approach is the matrix representation of ring elements with respect to appropriate set of idempotents. This approach is described at the end of this section.

From now on \( R \) denotes a ring with involution \( * \) and the multiplicative identity \( 1 \). Like in the matrix case, if \( a \in R \) and there exist \( x \in R \) such that \( axa = a \) then we say that \( a \) is regular and \( x \) is a \( g \)-inverse of \( a \). If in addition \( xax = x \) then \( x \) is called a reflexive \( g \)-inverse of \( a \). The Moore–Penrose inverse of \( a \) is the element \( a^\dagger \in R \) which satisfies the equations

\[
(1) \ axa = a \quad (2) \ xax = x \quad (3) \ (ax)^* = ax \quad (4) \ (xa)^* = xa.
\]
The Moore–Penrose inverse is unique in the case when it exists. If $x$ satisfies equations $i_1, i_2, \ldots, i_n$ then $x$ is called $\{i_1, i_2, \ldots, i_n\}$-inverse of $a$. By $a\{i_1, i_2, \ldots, i_n\}$ and $R\{i_1, i_2, \ldots, i_n\}$ we denote respectively the set of all $\{i_1, i_2, \ldots, i_n\}$-inverses of $a$ and the set of all elements in $R$ that possess a $\{i_1, i_2, \ldots, i_n\}$-inverse. For short, we will denote by $R^i$ the set of all Moore–Penrose invertible elements in $R$.

This paper is organized as follows.

By the end of this section we will give the basic definitions and set up notation.

In Sect. 2 we will focus on the complex matrix case. We will first note that the set $A\{\dagger\}$ coincides with the set of all reflexive least square $g$-inverses of $A$ and that $A\{\dagger-\}$ is equal to the set of all reflexive minimum norm $g$-inverses of $A$. We will give several other characterizations of these inverses. The most important result in Sect. 2 is Theorem 5 where we show that the relations $<\dagger$ and $<\dagger-$ are actually the well-known left-star partial order ($\ast<\ast$) and right star partial order ($<\ast\ast$), respectively. Recall that these relations were defined by Baksalary and Mitra in [3]:

$$A \ast< B \iff A^* A = A^* B \quad \text{and} \quad \text{Im} A \subseteq \text{Im} B$$
$$A <\ast B \iff AA^* = BA^* \quad \text{and} \quad \text{Im} A^* \subseteq \text{Im} B^*.$$  \hfill (3)

In Sect. 3 we show that the 1MP-inverse can be introduced for elements of an arbitrary ring with involution in the same way as it is done in [10] for complex matrices. We show that, like in the matrix case, the following holds

$$a\{\dagger\} = a\{1, 2, 3\}$$
$$= \{h + (1 - ha)wa\ : \ w \in R\}$$
$$= \{(a^* a)^{-1} a^* : (a^* a)^{-1} \in (a^* a))\{1\}\}$$

where $h$ is a fixed $\{1, 2, 3\}$ inverse of $a$. So, the most important characterizations stay valid in the ring case.

The definition of the relation $<\dagger$ is algebraic, so we can extend it in the ring context. In Sect. 4, we introduce the left-star order relation $\ast<\ast$ in a ring by analogy with complex matrix case. Then we show that for $a \in R\{1,3\}$ and $b \in R$, $a <\dagger b$ if and only if $a \ast< b$. We will examine this order through the Mitra’s unified theory of partial orders based on generalized inverses, [16] and [20]. In particular, we present in Theorem 13 very useful simultaneous diagonalizations of $a$ and $b$ when $a \ast< b$. This result has interesting interpretation when $a$ and $b$ are bounded Hilbert space operators with closed ranges. Of course, we will prove that $\ast<\ast$ is indeed the partial order relation on $R\{1,3\}$. Moreover, we will show in Theorem 16 that

$$a \ast< b \iff b\{1, 3\} \subseteq a\{1, 3\}$$

which is highly nontrivial result even in the complex matrix case, [3].

In Sect. 5 we present the dual results for MP1-inverse (i.e. $\{1, 2, 4\}$-inverse) and for $<\dagger-$ relation (i.e. right-star order).

In the last Sect. 6, we will give adequate interpretations of the presented results in the case of bounded Hilbert space operators.

We will now give some preliminaries.

For $a \in R$, $Ra = \{xa : x \in R\}$ and $ar = \{ax : x \in R\}$.

An element $e \in R$ is an idempotent (a self-adjoint idempotent) if $e^2 = e$ ($e^2 = e = e^*\ast$).

**Definition 2** Let $p$ and $q$ be idempotents in $R$. We say that an element $a \in R$ is $(p, q)$-invertible if $a \in pRq$ and there exists $a_{p,q}^- \in qRp$ such that

$$aa_{p,q}^- = p \quad \text{and} \quad a_{p,q}^- a = q.$$
In this case we say that \( a_{p,q}^- \) is the \((p, q)\)-inverse of \( a \).

It is easy to see that for \( a \in R \) there exist idempotents \( p \) and \( q \) such that \( a \) is \((p, q)\)-invertible if and only if \( a \) is regular. When it exists, the \((p, q)\)-inverse is unique. If \( a \) is \((p, q)\)-invertible and \( b \in pR \) then the equation \( ax = b \) has the unique solution in the set \( qR \), namely \( x = a_{p,q}^- b \). Similarly, if \( b \in Rq \) then the equation \( xa = b \) has the unique solution in the set \( Rp \) given by \( x = ba_{p,q}^- \).

The idempotents \( e_1, e_2, \ldots, e_n \in R \) are called mutually orthogonal if they are orthogonal in pairs, that is, if \( e_i e_j = 0 \) for \( i \neq j \). If \( e_1, e_2, \ldots, e_n \in R \) are mutually orthogonal idempotents such that

\[
1 = e_1 + e_2 + \cdots + e_n \tag{4}
\]

then the equality (4) is called a decomposition of the identity of the ring \( R \). The decomposition of the identity is orthogonal if \( e_i, i = 1, \ldots, n \) are self-adjoint. The following observation was provided in [21]. Let \( 1 = e_1 + \cdots + e_m \) and \( 1 = f_1 + \cdots + f_n \) be two decompositions of the identity of a ring \( R \). For any \( x \in R \) we have

\[
x = \left( \sum_{i=1}^{m} e_i \right) x \left( \sum_{j=1}^{n} f_j \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} e_i x f_j = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} e \times f
\]

where \( x_{ij} = e_i x f_j \). Note that the representation of an \( x \) as \( x = [x_{ij}]_{e \times f} \) is unique.

If \( y = [y_{ij}]_{e \times f} \) then \( x + y \) can be interpreted as addition of two matrices over \( R \). Let \( 1 = g_1 + \cdots + g_k \) be another decomposition of the identity of \( R \) and let \( z = [z_{jl}]_{f \times g} \), \( z_{jl} = f_j g_l \). As \( f_i f_j = 0 \) for \( i \neq j \), it is easy to see that the product \( xz \) can be calculated as the multiplication of two matrices over \( R \). Also,

\[
x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{m1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{mn}^* \end{bmatrix} \quad \text{f}^* \times \text{e}^*
\]

where the above representation is with respect to decompositions of the identity \( 1 = f_1^* + \cdots + f_n^* \) and \( 1 = e_1^* + \cdots + e_m^* \).

Let \( e, f \in R \) be two idempotents. They induce two decompositions of the identity \( 1 = e + (1 - e) \) and \( 1 = f + (1 - f) \). Then we will write \( x \in R \) in the following way

\[
x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} e \times f
\]

We conclude the section with the following remark. At the period of the review of this paper, Marovt et al. published the paper [14]. In [14], independently of our work, the authors considered the 1MP and MP1 inverses and corresponding partial order relations in an arbitrary ring with involution. Most of the results in [14] differ from our results, but some are the same or similar. Theorem 3.3 in [14] is the same as our Theorem 6. Corollary 3.4, Theorem 4.1 and Theorem 4.5 in [14] are similar to our characterization (13) in Theorem 7, our Theorem 12 and our Theorem 14, respectively. However, the proofs are different. They proved that \( <^\dag \) and \( <^{1\ast} \) are partial orders, but in a completely different way than ours. In contrast to [14], we have proved that the relations \( <^\dag \) and \( <^{1\ast} \) coincide with left-star and right-star orders, respectively.
2 1MP-inverse and $<-\dagger$ partial order in $C^{m \times n}$

Recall that for $A \in C^{m \times n}$ a matrix $A_{\dagger} \in C^{n \times m}$ is a least squares $g$-inverse of $A$ if $A_{\dagger}b$ is a least squares solution of $Ax = b$ for all $b \in C^n$, that is, if for all $b \in C^n$ the $\ell_2$-norm of $Ax - b$ is smallest when $x = A_{\dagger}b$. A matrix $A_{m}^{-} \in C^{n \times m}$ is a minimum norm $g$-inverse of $A$ if $A_{m}^{-}b$ provides a solution with minimum norm of equation $Ax = b$ whenever it is consistent.

The following characterizations of 1MP and MP1 inverses are given in [10].

Theorem 1 ([10, Theorem 3.1]) For $A \in C^{m \times n}$ the following hold

\begin{align*}
A[-\dagger] &= \{X \in C^{n \times m} : AXA = A, XAX = X, (AX)^* = AX\} \\
A[\dagger-] &= \{X \in C^{n \times m} : AXA = A, XAX = X, (XA)^* = XA\}.
\end{align*}

Proof By [17, Theorem 2.5.14], $X$ is a least squares $g$-inverse of $A$ if and only if $AXA = A$ and $(AX)^* = AX$. Similarly, $X$ is a minimum norm $g$-inverse of $A$ if and only if $AXA = A$ and $(XA)^* = XA$, [17, Theorem 2.5.5]. Now the first part of the theorem follows by Theorem 1 while characterizations in (6) follow by [17, Theorem 2.5.19 and Theorem 2.5.9].

We already know by Theorem 1 that $A[-\dagger] = A[1, 2, 3]$. We give some more characterizations of 1MP and MP1-inverses.

Theorem 3 For $A \in C^{m \times n}$ and $X \in C^{n \times m}$ the following conditions are equivalent:

(i) $X$ is a 1MP-inverse of $A$.
(ii) $X \in A[1, 2]$ and Ker $X = \text{Ker} A^*$.  
(iii) $X \in A[1]$ and Ker $A^* \subseteq \text{Ker} X$.
(iv) $X \in A[2]$ and Ker $X \subseteq \text{Ker} A^*$.

Proof (i) $\Rightarrow$ (ii): If $X$ is a 1MP-inverse of $A$ then $AXA = A$, $XAX = X$ and $(AX)^* = AX$. From $A^* = (AXA)^* = A^*AX$ and $X = XAX = X(AX)^* = XX^*A^*$, we conclude that $X = \text{Ker} A^*$.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are trivial.

(iii) $\Rightarrow$ (i): Suppose that $AXA = A$ and Ker $A^* \subseteq \text{Ker} X$. From [23, Lemma 2.1] we know that Ker $A^* \subseteq \text{Ker} X$ implies $X = X(A^*)^\dagger A^* = X(AA^\dagger)^* = XAA^\dagger$. Therefore, $AX = AA^\dagger$ which is self-adjoint. Also, $XAX = XAA^\dagger = X$, so $X \in A[1, 2, 3]$.

(iv) $\Rightarrow$ (i): Suppose that $XAX = X$ and Ker $X \subseteq \text{Ker} A^*$. Again, by [23, Lemma 2.1], we have $A^* = A^*X^\dagger X$, that is, $A = X^\dagger XA$. It follows that $AX = X^\dagger X$ and $AXA = X^\dagger XA = A$, so $X \in A[1, 2, 3]$.

We can prove likewise the dual result for the MP1-inverse.

Theorem 4 For $A \in C^{m \times n}$ and $X \in C^{n \times m}$ the following conditions are equivalent:

(i) $X$ is a MP1-inverse of $A$.
(ii) \( X \in A\{1,2\} \) and \( \text{Im} \ X = \text{Im} \ A^* \).
(iii) \( X \in A\{1\} \) and \( \text{Im} \ X \subseteq \text{Im} \ A^* \).
(iv) \( X \in A\{2\} \) and \( \text{Im} \ A^* \subseteq \text{Im} \ X \).

We now turn to order relations defined by 1MP and MP1 inverses. The next theorem shows that the relations \( <^\dagger \) and \( <^\ast \) defined in (1) and (2) are respectively the left-star and right-star order relations defined in (3).

**Theorem 5** Let \( A, B \in \mathbb{C}^{m \times n} \). Then \( A <^\dagger B \) if and only if \( A * B \). Also, \( A <^\ast B \) if and only if \( A * B \).

**Proof** In [3, Theorem 2.3 and Theorem 2.4] it is proved that \( A * B \) if and only if \( AA_{lr} = BA_{lr} \) and \( A_{lr} A = A_{lr} B \) for some reflexive least square g-inverse of \( A \). In the same theorems, it is proved that \( A * B \) if and only if \( AA_{mr} = BA_{mr} \) and \( A_{mr} A = A_{mr} B \) for some reflexive minimum norm g-inverse of \( A \). Now the proof is a direct consequence of Theorem 2. \( \Box \)

### 3 1MP-inverse in ring

Let \( a \in R \) be regular with g-inverse \( a^- \). Let \( p = aa^- \) and \( q = a^- a \). Then \( p \) and \( q \) are idempotents. Since \( a = paq \), it follows that the matrix representations of \( a \) with respect to \( p \) and \( q \) is

\[
a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q},
\]

where \( a \) is \((p, q)\)-invertible with \( p_{\cdot q} = a^- aa^- \). Thus, if \( a^- \) is a reflexive g-inverse of \( a \) then \( p_{\cdot q} = a^- \). Let \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{q \times p} \in a\{1\} \). We see that \( axa = a \) if and only if \( axa = a \). Multiplying this equation by \( a^- \) from the both sides, we get \( q x_1 p = a^- aa^- \), so \( x_1 = a^- aa^- \). On the other hand, if \( x_1 = a^- aa^- \) then \( axa = a \). Therefore

\[
a\{1\} = \left\{ \begin{bmatrix} a^- aa^- & x_2 \\ x_3 \\ x_4 \end{bmatrix}_{q \times p} : x_2 \in q R(1 - p), x_3 \in (1 - q) R p, x_4 \in (1 - q) R (1 - p) \right\}.
\]

Similarly, we can prove that

\[
a\{1, 2\} = \left\{ \begin{bmatrix} a^- aa^- & x_2 \\ x_3 \\ x_3 a x_2 \end{bmatrix}_{q \times p} : x_2 \in q R(1 - p), x_3 \in (1 - q) R p \right\}.
\]

If \( a \in R^* \) and \( p = aa^\dagger, q = a^\dagger a \) then \( a^\dagger = \begin{bmatrix} a^\dagger \\ 0 \end{bmatrix}_{p \times q} \). We can define a binary relation \( \sim_l \) on \( a\{1\} \) in the same way as it is done for matrices. For \( a^- \), \( a^- \in a\{1\} \),

\[
a^- \sim_l a^- \Leftrightarrow a^- a = a^- a.
\]

The relation \( \sim_l \) is an equivalence relation on \( a\{1\} \). If \( a^- \in a\{1\} \) then by (8),

\[
a^- = \begin{bmatrix} a^\dagger & x_2 \\ x_3 & x_4 \end{bmatrix}_{q \times p}
\]
for some \( x_2 \in qR(1 - p), x_3 \in (1 - q)Rp \) and \( x_4 \in (1 - q)R(1 - p) \). We have

\[
a^{-a} = \begin{bmatrix} q & 0 \\ x_3a & 0 \end{bmatrix}_{q \times q}.
\]

Similarly, for \( a^\perp = \begin{bmatrix} a^\dagger & y_2 \\ y_3 & y_4 \end{bmatrix} \in a[1] \) we have \( a^\perp a = \begin{bmatrix} q & 0 \\ y_3a & 0 \end{bmatrix} \) for some \( y_3 \in (1 - q)Rp \). It follows that \( a^{-} \sim_l a^\perp \) if and only if \( x_3a = y_3a \). Multiplying this by \( a^\dagger \) from the right, we obtain \( x_3p = y_3p \), i.e. \( x_3 = y_3 \). Therefore, \( a^{-} \sim_l a^\perp \) if and only if \( x_3 = y_3 \). It follows that we can choose the element

\[
\begin{bmatrix} a^\dagger & 0 \\ x_3 & 0 \end{bmatrix}_{q \times p}
\]
as the most natural representative of the class \([a^{-}]_{-1}\). This representative is equal to \( a^{-}a a^\dagger \) because

\[
a^{-}a a^\dagger = \begin{bmatrix} a^\dagger & x_2 \\ x_3 & x_4 \end{bmatrix}_{q \times p} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \begin{bmatrix} a^\dagger & 0 \\ 0 & 0 \end{bmatrix}_{q \times p} \begin{bmatrix} a^\dagger a a^\dagger & 0 \\ x_3a a a^\dagger & 0 \end{bmatrix}_{q \times p} = \begin{bmatrix} a^\dagger & 0 \\ x_3 & 0 \end{bmatrix}_{q \times p}.
\]

(10)

We can now extend the definition of 1MP-inverse from the matrix case to the ring case.

**Definition 3** Let \( a \in R^\dagger \) and choose \( a^{-} \in a[1] \). The element

\[
a^{-\dagger} = a^{-}a a^\dagger
\]
is called a 1MP-inverse of \( a \). The set of all 1MP-inverses of \( a \) is denoted by \( a\{-\dagger\} \).

Thus, we have

\[
a\{-\dagger\} = \{a^{-}a a^\dagger : a^{-} \in a[1]\}.
\]

Theorem 1 is also valid in an arbitrary ring with involution.

**Theorem 6** (See [10, Theorem 3.1] for the matrix case.) For \( a \in R^\dagger \) and \( g \in R \) the following statements are equivalent:

(i) \( g \in a\{-\dagger\} \);

(ii) \( g \) is a solution of the system of equations

\[
xax = x, \ ax = aa^\dagger;
\]

(iii) \( g \in a[1, 2, 3] \).

**Proof** (i) \( \Rightarrow \) (ii): If \( g \in a\{-\dagger\} \) then \( g = a^{-}a a^\dagger \) for some \( a^{-} \in a[1] \). We have

\[
 gag = a^{-}a a^\dagger aa^{-}a a^\dagger = a^{-}aa^{-}a a^\dagger = a^{-}a a^\dagger = g
\]

and

\[
 ag = aa^{-}a a^\dagger = aa^\dagger.
\]

(ii) \( \Rightarrow \) (iii): If \( gag = g \) and \( ag = aa^\dagger \) then \( aga = a \) and \( (ag)^* = (aa^\dagger)^* = aa^\dagger = ag \), so \( g \in a[1, 2, 3] \).

(iii) \( \Rightarrow \) (i): Suppose that \( aga = a \), \( gag = g \) and \( (ag)^* = ag \). We obtain

\[
 gaa^\dagger = gagaa^\dagger = (g(ag)^*)(aa^\dagger)^* = g(aa^\dagger ag)(ag)^* = g(ag)^* = gag = g.
\]

Therefore, \( g = a^{-}a a^\dagger \) for \( a^{-} = g \in a[1] \).
Remark 1 When we work with general rings then we don’t have a guarantee that the existence of a \( \{1, 2, 3\} \) inverse of \( a \) implies the existence of \( a^{*} \). Because of that, we will primarily focus on the set \( a \{1, 2, 3\} \) rather than \( a{-}\rangle \), although, by previous theorem, these sets coincide when \( a \) is Moore–Penrose invertible. Note that if \( g \in a \{1, 3\} \) then \( gag \in a \{1, 2, 3\} \). Therefore
\[
R^{(1,2,3)} = R^{(1,3)}.
\]

The characterization similar to characterization (13) in the next theorem was given for complex matrices in [10, Proposition 3.2]. The characterization (14) is a generalization of Theorem 2.

Theorem 7 Let \( a \in R^{(1,3)} \) and let \( p = ah \) and \( q = ha \) where \( h \) is fixed \( \{1, 2, 3\} \)-inverse of \( a \). Then we have
\[
a\{1, 2, 3\} = a\{1\}aa\{1, 3\} \tag{11}
\]
\[
a\{1, 2, 3\} = \left\{ \begin{bmatrix} h & 0 \\ u & 0 \end{bmatrix} : u \in (1 - q)R_{p} \right\} \tag{12}
\]
\[
= \{ h + (1 - ha)wah : w \in R \} \tag{13}
\]
\[
a\{1, 2, 3\} = \{(a^{*}a)^{-}a^{-} : (a^{*}a)^{-} \in (a^{*}a)\{1\} \}. \tag{14}
\]

Proof Let us prove (11). If \( g \in a \{1, 2, 3\} \) then \( gag \in a\{1\}aa\{1, 3\} \). Conversely, suppose that \( g = a^{-}aa^{(1,3)} \) for some \( a^{-} \in a\{1\} \) and \( a^{(1,3)} \in a \{1, 3\} \). We have
\[
ag\ = \ aa^{-}aa^{(1,3)}a = a
\]
\[
gag\ = \ a^{-}aa^{(1,3)}aa^{-}aa^{(1,3)} = a^{-}aa^{(1,3)} = g
\]
\[
gag\ = \ aa^{-}aa^{(1,3)} = aa^{(1,3)}(aa^{(1,3)})^{-} = (ag)^{*},
\]
so \( g \in a \{1, 2, 3\} \).

Let us prove (12). By (9), \( g \in a \{1, 2\} \) if and only if \( g = \begin{bmatrix} h & x_{2} \\ x_{3} & x_{3}ax_{2} \end{bmatrix}_{q \times p} \). Since
\[
ag = \begin{bmatrix} p & ax_{2} \\ 0 & 0 \end{bmatrix}_{p \times p}
\]
, we obtain that \( ag = (ag)^{*} \) if and only if \( ax_{2} = 0 \) which is equivalent with \( x_{2} = 0 \) because \( x_{2} = qx_{2} = ha x_{2} \). It follows that \( g \in a \{1, 2, 3\} \) if and only if \( g = \begin{bmatrix} h & 0 \\ x_{3} & 0 \end{bmatrix}_{q \times p} \). Note that the characterization (12) is just the matrix record of characterization (13).

Finally, let us prove the equality (14). Let \( g \in a \{1, 2, 3\} \) and set \( s = gg^{*} \). We have \( s \in (a^{*}a)\{1\} \) since
\[
(a^{*}a)gg^{*}a^{*}a = a^{*}(ag)^{*}g^{*}a^{*}a = (agaga)^{*}a = a^{*}a.
\]
Also, \( sa^{*} = gg^{*}a^{*} = g(ag)^{*} = g \), so \( g \) belongs to the right-hand side of (14).

On the other hand, if \( g = (a^{*}a)^{-}a^{-} \) for some \( (a^{*}a)^{-} \in (a^{*}a)\{1\} \) then
\[
ag\ = \ (a^{*}a)^{-}a^{-}a = aha(a^{*}a)^{-}a^{*}a = (ah)s(a(a^{*}a)^{-}a^{*}a
\]
\[
= \ h^{*}a^{*}a(a^{*}a)^{-}a^{*}a = h^{*}a^{*}a = a
\]
\[
gag\ = \ (a^{*}a)^{-}a^{-}a(a^{*}a)^{-}a^{*} = (a^{*}a)^{-}a^{-}a(a^{*}a)^{-}(aha)^{*}
\]
\[
= \ (a^{*}a)^{-}a^{-}a(a^{*}a)^{-}a^{*}ah = (a^{*}a)^{-}a^{*}ah = (a^{*}a)^{-}a^{*} = g
\]
\[
ag\ = \ (a^{*}a)^{-}a^{-}a^{*} = aha(a^{*}a)^{-}(aha)^{*} = (ah)^{*}a(a^{*}a)^{-}a^{*}(ah)^{*}
\]
1MP and MP1 inverses and one-sided star orders...

\[ h^* a^* a (a^* a)^{-1} a^* ah = h^* a^* ah = (ah)^* ah = ah, \]

so \( g \in a \{1, 2, 3\} \). \( \square \)

**4 \(<^{−†}\) order and left-star order in a ring**

The relations \(<^{−†}\) and \(<^{†−}\) can be extended from the complex matrix case (Eqs. (1) and (2)) to a general ring case in a straightforward way. In accordance with Remark 1 we give the following definition.

**Definition 4** Let \( a \in R^{(1,3)} \) and \( b \in R \). We say that \( a \) is lower then or equal to \( b \) with respect to \(<^{−†}\), which is denoted by \( a <^{−†} b \) if there exists \( g \in a \{1, 2, 3\} \) such that

\[ ag = bg \quad \text{and} \quad ga = gb. \]

Recall the definition of the minus partial order.

**Definition 5** [9] Let \( a \in R^{(1)} \) and \( b \in R \). We say that \( a <^{−} b \) if there exists \( a^{−} \in a \{1\} \) such that

\[ aa^{−} = ba^{−} \quad \text{and} \quad a^{−} a = a^{−} b. \]

The relation \(<^{−}\) is a partial order relation on \( R^{(1)} \), [9, Theorem 1].

Note that if \( ag = bg \) and \( ga = gb \) for some \( g \in a \{1, 3\} \) then \( h := gag \in a \{1, 2, 3\} \) and \( ah = bh, ha = hb \). We obtain the following result.

**Proposition 8** Let \( a \in R^{(1,3)} \) and \( b \in R \). Then

\[ a <^{−†} b \iff ag = bg \quad \text{and} \quad ga = gb \text{ for some } g \in a \{1, 3\}. \]

Also,

\[ a <^{−†} b \implies a <^{−} b. \]

It follows that \(<^{−†}\) is reflexive and antisymmetric relation on \( R^{†} \).

We can define left-star partial order in a ring by analogy with the matrix case, see definitions in (3). Some additional explanations will be given in Sect. 6.

**Definition 6** For \( a, b \in R \) we say that \( a < b \) if

\[ a^{*} a = a^{*} b \quad \text{and} \quad aR \subseteq bR. \]

Note that the condition \( aR \subseteq bR \) is equivalent with \( a = bc \) for some \( c \in R \) since \( R \) has the multiplicative identity. Suppose that \( a < b \). Then \( a^{*} a = a^{*} b = b^{*} a \) and \( a = bc \). It is not difficult to show that \( (b-a)^{*} (b-a) = (b-a)^{*} b \) and \( b-a = b(1-c) \in bR \), so \( b-a < b \). Therefore, we obtain the following probably known result:

\[ a < b \iff (b-a) < b. \] (15)

We will prove in the next theorem that, as in the matrix case, the relation \(<^{−†}\) coincides with relation \(*<\).

**Theorem 9** Let \( a \in R^{(1,3)} \) and \( b \in R \). Then

\[ a <^{−†} b \iff a < b. \]
Proof Suppose that \( a <^\dagger b \). There exists \( g \in a\{1, 2, 3\} \) such that \( ag = bg \) and \( ga = gb \). It follows that \( a = aga = bga, \) so \( aR \subseteq bR \). Also

\[
a^*b = (aga)^*b = a^*agb = a^*aga = a^*, \]

and thus \( a *< b \). Suppose now that \( a *< b \), that is, \( a^*a = a^*b \) and \( a = bc \) for some \( c \in R \). Fix \( h \in a\{1, 2, 3\} \) and set \( p = ah \) and \( q = ha \). As before, \( p = p^* \) and

\[
a = \begin{bmatrix} a & 0 & 0 \end{bmatrix}_{p \times q}, \quad h = \begin{bmatrix} h & 0 & 0 \end{bmatrix}_{q \times p}.
\]

Suppose that

\[
b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times q}, \quad c = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{q \times q}.
\]

From

\[
a^*a = \begin{bmatrix} a^*a & 0 \\ 0 & 0 \end{bmatrix}_{q^* \times q} = \begin{bmatrix} a^*b_1 & a^*b_2 \\ 0 & 0 \end{bmatrix}_{q^* \times q} = a^*b
\]

we obtain \( a^*a = a^*b_1 \) and \( a^*b_2 = 0 \). Therefore, \( b_1 = pb_1 = ahb_1 = h^*a^*b_1 = h^*a^*a = pa = a \) and similarly \( b_2 = 0 \). From \( a = bc \) we obtain

\[
\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} = \begin{bmatrix} a & 0 \\ b_3 & b_4 \end{bmatrix}_{p \times q} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{q \times q} = \begin{bmatrix} a^*c_1 & a^*c_2 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{bmatrix}_{p \times q}.
\]

As in the previous part of the proof, we can easily find that \( c_1 = q, c_2 = 0 \) and \( b_3 + b_4c_3 = 0 \). Note that \( c_3h \in (1 - q)Rp \). Set

\[
g = \begin{bmatrix} h & 0 \\ c_3h & 0 \end{bmatrix}_{q \times p}.
\]

By Theorem 7, we have that \( g \in a\{1, 2, 3\} \). The direct calculation shows that

\[
b = \begin{bmatrix} a & 0 \\ b_3 & b_4 \end{bmatrix}_{p \times q} \begin{bmatrix} h & 0 \\ c_3h & 0 \end{bmatrix}_{q \times p} = \begin{bmatrix} p & 0 \\ b_3h + b_4c_3h & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p \times p} = ag.
\]

Similarly

\[
 ga = gb = \begin{bmatrix} q & 0 \\ c_3 & 0 \end{bmatrix}_{q \times q}.
\]

It follows by definition that \( a <^{-\dagger} b \). \( \square \)

The following observation is well-known:

\[
a^*a = a^*b \iff a^*a = a^*b. \quad (16)
\]

From Theorem 9 it follows that

\[
a <^{-\dagger} b \iff a^*a = a^*b \quad \text{and} \quad aR \subseteq bR.
\]

Remark 2 Since Theorem 9 shows that relations \(<^{-\dagger} \) and \(*< \) are equal on \( R^{(1,3)} \), we will exclusively use the mark \(*< \) for both relations. Of course, when \( a \in R^{(1,3)} \), we are in the position to use both Definition 4 and Definition 6 for \( a *< b \).

\( \square \) Springer
The theory of matrix partial orders based on generalized inverses is well developed, see monograph [17]. The unified theory of these relations was introduced by Mitra in [16]. This unified theory has recently been generalized in an arbitrary ring context by Rakic and Djordjevic in [20]. By the end of this section we will consider our relations through the prism of this unified theory.

First we need to introduce some notions which are given in [16] and [20]. If \( \mathcal{P}(R) \) denotes the power set of \( R \) then a function \( \mathcal{G} : R \rightarrow \mathcal{P}(R) \) is called a \( g \)-map if for every \( a \in R \), \( \mathcal{G}(a) \) is a certain subset of \( a \{1\} \). The set \( \Omega_{\mathcal{G}} = \{ a \in R : \mathcal{G}(a) \neq \emptyset \} \) is called the support of the \( g \)-map \( \mathcal{G} \). We will focus on the \( g \)-map \( \mathcal{G}(a) = a \{1, 2, 3\} \), for which \( \Omega_{\mathcal{G}} = R^{(1,2,3)} = R^{(1,3)} \).

For \( a \in R \) and \( g \)-map \( \mathcal{G} \), the class

\[
\tilde{\mathcal{G}}(a) = \{ g \in R : ga = ha, ag = ah \text{ for some } h \in \mathcal{G}(a) \}
\]

is called the completion of \( \mathcal{G}(a) \). Let us denote by \( \tilde{a}\{i_1, \ldots, i_n\} \) the completion of \( a\{i_1, \ldots, i_n\} \).

If \( \mathcal{G} \) is a \( g \)-map then we say that the pair \( (a, b) \), \( a, b \in R \) satisfies the (T)-condition if \( hah \in \mathcal{G}(a) \) for every \( h \in \mathcal{G}(b) \).

We say that a \( g \)-map \( \mathcal{G} \) is semi-complete if for every \( a \in R \), the pair \( (a, a) \) satisfies the (T)-condition.

**Proposition 10** The \( g \)-map \( \mathcal{G}(a) = a\{1, 2, 3\} \) is semi-complete and its completion is a \( a\{1, 3\} \).

**Proof** Let \( \mathcal{G}(a) = a\{1, 2, 3\} \). If \( g \in \tilde{\mathcal{G}}(a) \) then \( ag = ah \) and \( ga = ha \) for some \( h \in a\{1, 2, 3\} \). From \( ag = ah \) we obtain \( aga = aha = a \) and \( (ag)^* = (ah)^* = ah = ag \), so \( g \in a\{1, 3\} \). Suppose now that \( g \in a\{1, 3\} \). Then \( h := gaa \in a\{1, 2, 3\} \) and \( ah = agag = ag \), \( ha = gaga = ga \), so \( g \in \tilde{\mathcal{G}}(a) \). Since \( h = hah \) for every \( h \in \mathcal{G}(a) \), we conclude that \( \mathcal{G} \) is semi-complete. \( \square \)

In the next theorem, for a fixed \( a \in R^{(1,3)} \), we will characterize all elements which are above \( a \) with respect to \( *<< \) order.

**Theorem 11** For \( a \in R^{(1,3)} \) we have

\[
\{ b \in R : a *<< b \} = \{ a + (1-ag)d(1-ga) : g \in a\{1, 2, 3\}, d \in R \}.
\]

That is, \( a *<< b \) if and only if there exists \( g \in a\{1, 2, 3\} \) such that

\[
b = \begin{bmatrix} a & 0 \\ 0 & v \end{bmatrix}_{p \times q},
\]

for some \( v \in (1-p)R(1-q) \), where \( p = ag \) and \( q = ga \).

**Proof** Since the \( g \)-map \( \mathcal{G}(a) = a\{1, 2, 3\} \) is semi-complete, the proof is a direct consequence of Theorem 9, and Theorem 3.2 and Corollary 3.3 in [20]. \( \square \)

The following result was originally proved for complex matrices in Theorem 2.1 in [1]. The same result for \( <^3 \) order has recently been proved in [10].

**Theorem 12** Let \( a \in R^{(1,3)} \), \( h \in a\{1, 2, 3\} \), and \( p = ah, q = ha \). Then \( a *<< b \) if and only if

\[
b = \begin{bmatrix} a & 0 \\ b_4u & b_4 \end{bmatrix}_{p \times q},
\]

for some \( b_4 \in (1-p)R(1-q) \) and some \( u \in Rq \).
The equalities $a = b$ this by $a < b$). Of course, that decomposition of the identity $1 = 0$ are not included in this theorem (which is specific for $\ast \to \mathbb{C}$.

Proof The most of the theorem directly follows by Theorem 3.1 in [20] where the similar pseudo invertibility of $a$ for some $g$.

The following statements are equivalent

(i) $a \prec b$;
(ii) The equalities

$$1 = p_1 + p_2 + p_3 \quad \text{and} \quad 1 = q_1 + q_2 + q_3$$

are respectively an orthogonal decomposition and a decomposition of the identity of the ring $R$ with respect to which $a$ and $b$ have the following matrix forms:

\[
\begin{align*}
    a &= \begin{bmatrix} a & 0 & 0 \\
                         0 & 0 & 0 \end{bmatrix}_{p \times q}, & b &= \begin{bmatrix} a & 0 & 0 \\
                         0 & 0 & b - a \end{bmatrix}_{p \times q}, & (17)
\end{align*}
\]

where $a$ is $(p_1, q_1)$-invertible with $a_{p_1,q_1} = hah$ and $b - a$ is $(p_2, q_2)$-invertible with $(b - a)_{p_2,q_2} = h - hah$.

Proof The most of the theorem directly follows by Theorem 3.1 in [20] where the similar result is proven for arbitrary semi-complete $g$-map. We will only prove here the results which are not included in this theorem (which is specific for $\ast \to \mathbb{C}$ order). Thus we have to prove that decomposition of the identity $1 = p_1 + p_2 + p_3$ is orthogonal and we have to prove the pseudo invertibility of $a$ and $b - a$. Suppose that $a \prec b$. Then $ag = bg$ and $ga = gb$ for some $g \in a\{1, 2, 3\}$. The decomposition of the identity $1 = p_1 + p_2 + p_3$ is orthogonal because the idempotent $p_1$ is self-adjoint:

$$p_1 = ah = agah = agbh = (bhag)^* = (bhbg)^* = (bg)^* = (ag)^* = ag.$$
By (18), the direct check shows that \( a_{p_1,q_1}^- = q_1 h p_1 = h a h \) and \( (b-a)_{p_2,q_2}^- = q_2 h p_2 = h - h a h \):

\[
\begin{align*}
aha &= agah = ah = p_1 \\
aha &= haga = ha = q_1 \\
(b-a)(h-hah) &= bh - bhah - ah + agah = bh - bhag - ah + agah \\
&= bh - (agbh)^* = bh - (ah)^* = bh - ah = p_2 \\
(h-hah)(b-a) &= hb - ha - hahb + hah = hb - hahb \\
&= hb - hahb = hb - haga = q_2.
\end{align*}
\]

The following characterization is typical for many partial orders based on \( g \)-inverses. The similar result for minus partial order was presented in [22] and for star, sharp, core and dual core partial order in [24].

**Proposition 14** (See Theorem 4.1 in [1] for the similar result in the matrix case.) Let \( a \in R^{(1,3)} \) and \( b \in R \). Then \( a \prec b \) if and only if there exist a self-adjoint idempotent \( p \) and an idempotent \( q \) such that \( a = pb = bq \).

**Proof** If \( a \prec b \) then \( ag = bg \) and \( ga = gb \) for some \( g \in a\{1,2,3\} \). If \( p = ag \) and \( q = ga \) then \( p \) is self-adjoint idempotent, \( q \) is idempotent and \( a = aga = agb = pb \) and \( a = bga = qa \). If \( a = pb = bq \) where \( p \) is self-adjoint idempotent and \( q \) is idempotent then \( a = pa, aR \subseteq bR \) and \( a^*b = (pa)^*b = a^*pb = a^*a \).

In the study of partial orders based on generalized inverses, it is usual to examine its relationships with inclusions of appropriate subsets of \( g \)-inverses. This is already done for the minus, star, sharp, core and dual core partial orders in [22] and [24]. The next two theorems show that the left-star partial order is not exception in this respect.

**Theorem 15** Let \( a, b \in R^{(1,3)} \) and suppose that \( a \prec b \). Then

(i) The pair \( (a, b) \) satisfies the (T)-condition, that is,

\[
hah \in a\{1,2,3\}, \forall h \in b\{1,2,3\}.
\]

(ii) \( b\{1,3\} \subseteq a\{1,3\} \).

**Proof** Suppose that \( a \prec b \). Since the \( g \)-map \( G(a) = a\{1,2,3\} \) is semi-complete and since \( b\{1,3\} \) is the completion of \( b\{1,2,3\} \), it follows by Corollary 3.6 in [20] that the conditions (i) and (ii) are equivalent. Therefore, it is enough to prove only the inclusion (ii). There exists \( g \in a\{1,2,3\} \) such that \( ag = bg \) and \( ga = gb \). Let \( h \in b\{1,3\} \). Then we have

\[
aha = agah = agbh = (bhag)^* = (bbhg)^* = (bg)^* = (ag)^* = ag \\
aha = aga = a,
\]

so \( h \in a\{1,3\} \).

The following observation follows by (15) and Theorem 15. Suppose that \( a, b \in R^{(1,3)} \) and \( a \prec b \). Then for every \( h \in b\{1,2,3\} \)

\[
h(b - a)h = h - hah = (b - a)_{p_2,q_2}^- \in (b - a)\{1,2,3\},
\]
where \( p_2 \) and \( q_2 \) are as in Theorem 13. In particular, we conclude that \( b - a \in R^{(1,3)} \).

From the unified theory of \( g \)-based partial orders it follows that any of the conditions (i) or (ii) from previous Theorem 15 is sufficient to conclude that \( * \prec \) is a partial order relation on \( R^{(1,3)} \), see Corollary 3.6 in [20]. For the reader’s convenience we will postpone this conclusion because it directly follows by the following important result.

Theorem 16 is originally proved by Baksalary and Mitra in [3]. Another proof of the same result can be found in Theorem 6.5.17 in [17]. Our proof is in the spirit of that proof. The starting point of the proof in [17] is the singular value decomposition of the matrix \( A \).

Theorem 16 Let \( a, b \in R^{(1,3)} \). Then

\[
\quad a * < b \iff b \{ 1, 3 \} \subseteq a \{ 1, 3 \}.
\]

Proof We have proved the "if" part in Theorem 15. Suppose that \( b \{ 1, 3 \} \subseteq a \{ 1, 3 \} \). Fix an \( h \in b \{ 1, 2, 3 \} \) and let \( p = bh = (bh)^* \) and \( q = hb \). Suppose that \( a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \). Like in Theorem 7, we can show that \( g \in b \{ 1, 3 \} \) if and only if

\[
g = \begin{bmatrix} h & 0 \\ x_3 & x_4 \end{bmatrix}_{q \times p}
\]

for some \( x_3 \in (1 - q)Rp \) and \( x_4 \in (1 - q)R(1 - p) \). Since \( b \{ 1, 3 \} \subseteq a \{ 1, 3 \} \), we have that \( ag = a \) and \( (ag)^* = ag \) for every \( x_3 \) and \( x_4 \). If we take \( x_3 = 0 \) and \( x_4 = 0 \) then the condition \( (ag)^* = ag \) gives \( (a_1h)^* = a_1h \) and \( a_3h = 0 \). Thus, \( a_3 = a_3q = 0 \). The condition \( ag = a \) gives

\[
\begin{bmatrix} a_1ha_1 & a_1ha_2 \\ 0 & 0 \end{bmatrix}_{p \times q} = \begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix}_{p \times q}.
\]

Therefore,

\[
a_4 = 0, \quad a_1ha_1 = a_1 \quad \text{and} \quad a_1ha_2 = a_2. \quad (19)
\]

For an arbitrary \( g \in b \{ 1, 3 \} \) the conditions \( ag = (ag)^* \) and \( ag = a \) provide

\[
\begin{bmatrix} a_1h + a_2x_3 & a_2x_4 \\ 0 & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} a_1h + (a_2x_3)^* & 0 \\ (a_2x_4)^* & 0 \end{bmatrix}_{p \times p}
\]

and

\[
\begin{bmatrix} a_1ha_1 + a_2x_3a_1 & a_1ha_2 + a_2x_3a_2 \\ 0 & 0 \end{bmatrix}_{p \times q} = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p \times q}.
\]

Therefore,

\[
a_2x_4 = 0, \quad (a_2x_3)^* = a_2x_3, \quad a_2x_3a_1 = 0, \quad a_2x_3a_2 = 0
\]

for every \( x_3 \in (1 - q)Rp \) and for every \( x_4 \in (1 - q)R(1 - p) \). It follows that

\[
a_2x_3 = a_1ha_2x_3 = (a_2x_3a_1h)^* = 0.
\]

Take \( x_3 = (1 - q)p \) and \( x_4 = (1 - q)(1 - p) \) to conclude that

\[
a_2 = a_2(1 - q) = a_2(1 - q)p + a_2(1 - q)(1 - p) = a_2x_3 + a_2x_4 = 0.
\]
We have proved that
\[ a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q}, \]
so \( a = a_1 \in pRq \) and \( aha = a \), \((ah)^* = ah\). Hence, \( a = pa = bha \), so \( aR \subseteq bR \). Also,
\[ a^*b = (aha)^*b = a^*ahb = a^*aq = a^*a. \]
It follows that \( a^*b \) and the proof is complete. \( \Box \)

**Corollary 17** The relation \( \ast < \), that is, the relation \( <_{\ast}^\dagger \), is a partial order relation on \( R(1,3) \).

**Proof** We have already establish that the relations \( \ast < \) is reflexive and antisymmetric. The transitivity of \( \ast < \) follows by Theorem 16, since the subset relation is transitive. \( \Box \)

**5 MP1-inverse, \( <_{\ast}^\dagger \) order and the right-star order**

In this section we will present dual results concerning the MP1-inverse, the relation \( <_{\ast}^\dagger \) and the right-star order. All results can be proved in a very similar (analogous) way as its duals. Because of that, we will omit the proofs.

Similarly as in the case of 1MP-inverse, the set of MP1-inverses can be introduced as the set of all canonical representatives of the quotient space \( a\{1\}/\sim_r \), where the relation \( \sim_r \) is defined on \( a\{1\} \) by \( a^- \sim_r a^= \) if \( aa^- = aa^=, a^-, a^= \in a\{1\} \). It turns out that we arrive to the following definition.

**Definition 7** Let \( a \in R^\dagger \) and choose \( a^- \in a\{1\} \). The element
\[ a^- = a^\dagger aa^- \]
is called a MP1-inverse of \( a \). The set of all MP1-inverses of \( a \) is denoted by \( a\{\dagger^-\} \).

Note that \( R^{(1,2,4)} = R^{(1,4)} \).

**Theorem 18** Let \( a \in R^{(1,4)} \) and let \( p = ah, q = ha \) where \( h \) is fixed \( \{1, 2, 4\} \)-inverse of \( a \). Then we have
\[
a\{1, 2, 4\} = a\{1, 4\}aa\{1\} = \left\{ \begin{bmatrix} h & u \\ 0 & 0 \end{bmatrix}_{q \times p} : u \in qR(1-p) \right\} = \{h + haw(1-ah) : w \in R\} = \{a^*(aa^*)^- : (aa^*)^- \in (aa^*)\{1\} \}.\]

If \( a \in R^\dagger \) then
\[ a\{\dagger^-\} = a\{1, 2, 4\} = \{x \in R : xax = x, xa = a^\dagger a\}. \]

The following observation is evident
\[ g \in a\{1, 2, 3\} \iff g^* \in a^*\{1, 2, 4\}. \tag{20} \]

The definitions of the \( <_{\ast}^\dagger \) relation and the right-star order are in an analogy with the definitions of the \( <_{\ast}^\dagger \) relation and the left-star order, respectively.
Definition 8 For \(a, b \in R\) we say that \(a\) is lower then or equal to \(b\) with respect to \(\prec^\dagger\), which is denoted by \(a \prec^\dagger - b\) if there exists \(g \in a\{1, 2, 4\}\) such that
\[
ag = bg \quad \text{and} \quad ga = gb.
\]

Proposition 19 Let \(a, b \in R\). Then
\[
a \prec^\dagger - b \iff ag = bg \quad \text{and} \quad ga = gb \quad \text{for some} \quad g \in a\{1, 4\}.
\]

Also,
\[
a \prec^\dagger - b \Rightarrow a \prec - b.
\]

Definition 9 We say that \(a \prec^* - b\) if
\[
aa^* = ba^* \quad \text{and} \quad Ra \subseteq Rb.
\]

The relations \(\prec^\dagger\) and \(\prec^*\) coincides on \(R^{(1,4)}\).

Theorem 20 Let \(a \in R^{(1,4)}\) and \(b \in R\). Then
\[
a \prec^\dagger - b \iff a \prec^* - b.
\]

From Theorem 20 and (20) we obtain the following well-know characterization
\[
a^* < b \iff a^* <^* b^*.
\]

Suppose that \(a \in R^\dagger\). From
\[
aa^* = ba^* \iff aa^\dagger = ba^\dagger
\]
and Theorem 20 it follows that
\[
a \prec^\dagger - b \iff aa^\dagger = ba^\dagger \quad \text{and} \quad Ra \subseteq Rb.
\]

Like for the \(\prec^*\) order, the semi-completeness given in the next proposition allow us to transfer some results of the unified theory to the \(\prec^*\) order relation.

Proposition 21 The g-map \(G(a) = a\{1, 2, 4\}\) is semi-complete and its completion is \(a\{1, 4\}\).

The next two theorems characterize all elements which are greater then \(a \in R^{(1,4)}\) with respect to \(\prec^*\) order.

Theorem 22 For \(a \in R^{(1,4)}\) we have
\[
\{b \in R : a \prec^* b\} = \{a + (1 - ag)d(1 - ga) : g \in a\{1, 2, 4\}, \ d \in R\}.
\]
That is, \(a \prec^* b\) if and only if there exists \(g \in a\{1, 2, 4\}\) such that
\[
b = \begin{bmatrix} a & 0 \\ 0 & v \end{bmatrix}_{p \times q},
\]
for some \(v \in (1 - p)R(1 - q)\), where \(p = ag\) and \(q = ga\).

Theorem 23 Let \(a \in R^{(1,4)}\), \(h \in a\{1, 2, 4\}\), and \(p = ah, q = ha\). Then \(a \prec^* b\) if and only if
\[
b = \begin{bmatrix} a & ub_4 \\ 0 & b_4 \end{bmatrix}_{p \times q},
\]
for some \(b_4 \in (1 - p)R(1 - q)\) and \(u \in pR\).
The following canonical matrix representations is characteristic for every $g$-based semi-complete relation.

**Theorem 24** Let $a, b \in R^{(1,4)}$. Fix $h \in b \{1, 2, 4\}$ and set
\[
p_1 = ah, \quad p_2 = (b - a)h, \quad p_3 = 1 - bh
\]
\[
q_1 = ha, \quad q_2 = h(b - a), \quad q_3 = 1 - hb.
\]

The following statements are equivalent

(i) $a <\ast b$;

(ii) The equalities
\[
1 = p_1 + p_2 + p_3 \quad \text{and} \quad 1 = q_1 + q_2 + q_3
\]
are respectively a decomposition and an orthogonal decomposition of the identity of the ring $R$ with respect to which $a$ and $b$ have the following matrix forms:

\[
a = \begin{bmatrix}
a & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_{p \times q}, \quad b = \begin{bmatrix}
a & 0 & 0 \\
0 & b - a & 0 \\
0 & 0 & 0
\end{bmatrix}_{p \times q}, \quad (21)
\]

where $a$ is $(p_1, q_1)$-invertible with $a_{p_1,q_1}^{-1} = hah$ and $b - a$ is $(p_2, q_2)$-invertible with $(b - a)_{p_2,q_2}^{-1} = h - hah$.

**Proposition 25** (See Theorem 4.1 in [1] for the similar result in the matrix case.) Let $a \in R^{(1,4)}$ and $b \in R$. Then $a <\ast b$ if and only if there exist an idempotent $p$ and a self-adjoint idempotent $q$ such that $a = pb = bq$.

The relationships with inclusions of appropriate subsets of $g$-inverses is presented in the next theorem.

**Theorem 26** Let $a, b \in R^{(1,4)}$. Then
\[
a <\ast b \Leftrightarrow b\{1, 4\} \subseteq a\{1, 4\}.
\]

Also, if $a <\ast b$ then the pair $(a, b)$ satisfies the $(T)$-condition, that is,
\[
hah \in a\{1, 2, 4\}, \quad \forall h \in b\{1, 2, 4\}.
\]

As a direct consequence of previous theorem, we conclude that $<\ast$ is transitive. We already know that it is reflexive and antisymmetric, so it is a partial order relation.

**Corollary 27** The relation $<\ast$, that is, the relation $<\dagger$, is a partial order relation on $R^{(1,4)}$.

It is natural to see the connection between one-sided star orders and star order $<\ast$ which is defined by Drazin in [8]:
\[
a <\ast b \Leftrightarrow aa^* = ba^* \quad \text{and} \quad a^*a = a^*b.
\]

Recall that $<\ast$ is a partial order relation in arbitrary semigroup with proper involution, [8].

The star order has recently been examined in an arbitrary ring with involution in [24]. It was shown that the relation $<\ast$ is a partial order relation on $R^\dagger$. The following is one of the basic characterization of the star order when $a, b \in R^\dagger$, see Theorem 2.6 in [24]
\[
a <\ast b \Leftrightarrow a = pb = br \quad \text{for some self-adjoint idempotents} \quad p \quad \text{and} \quad r. \quad (22)
\]

From Propositions 14 and 25 and characterization (22) we obtain the following expected result, which is known in the matrix case (see Theorem 2.1. in [3]).
Proposition 28 Let $a \in R^\dagger$ and $b \in R$. Then

$$a <^* b \iff a *< b \quad \text{and} \quad a <*> b.$$ 

6 Concluding remarks

We have considered several things in this paper. After examining the 1MP and MP1 inverses and associated partial orders $<^-=^-$ and $<^+=^+$ in the complex matrix case, we introduced and studied these notions in the context of an arbitrary ring with involution. Beside, we introduced the left-star $*<$ and right-star $<*$ partial orders by analogy with matrix case and showed that these orders coincide on $R^{(1,3)}$ with $<^-^-$ and $<^+^+$, respectively. After that we investigated the left-star order through the prism of unified theory of partial orders based on generalized inverses. The dual results of $\{1, 2, 4\}$-inverses, $<^-^-$ order and the right-star order are also presented.

It is worth to say that the orders $*<$ and $<$ are also defined and studied in the context of Rickart $*$-rings, see [11, 13, 15]. The two generalizations are in the complete agreement because one can show that the left-star order in the sense of Definition 6 coincide with the left-star order given in Definition 10 in [15] when $a \in R^{(1,3)}$. This fact can be proved using characterization given in Proposition 14. But each of these two cases has its own specificity and its own proving techniques. Let us also mention the closely related diamond partial order which is defined for matrices in [2] and also studied in rings in [12].

The inspiration for most of the results and proofs in this paper come from the operator theory. For instance, the results that we develop can find nice applications in $*$-algebras of bounded linear operators, in theory. For instance, the results that we develop can find nice applications in *-algebras, or in Rickart $*$-rings. We will conclude this paper by indicating how the presented results can be applied in the case of Hilbert space operators.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $B(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Let $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$. Like in the matrix case, for $A \in B(\mathcal{H}, \mathcal{K})$, we use $A^*$, Im $A$ and Ker $A$ to denote respectively the Hilbert adjoint of $A$, the range and the null-space of $A$. As usual, $I$ stands for the identity operator. We write $\mathcal{H}_1 \perp \mathcal{H}_2$ for the direct sum of subspaces and we write $\mathcal{H}_1 \oplus \mathcal{H}_2$ for the orthogonal direct sum. The different generalized inverses of $A$ are defined by the same equations as in the matrix (ring) case. Recall that $A \in B(\mathcal{H}, \mathcal{K})$ has a $g$-inverse if and only if it has the Moore–Penrose inverse if and only if Im $A$ is closed in $\mathcal{K}$.

Let $A, B \in B(\mathcal{H}, \mathcal{K})$ be operators with closed ranges. We can define relations $<^-=^-$ and $*<$ like in the matrix case, using definitions (1) and (3), respectively. By Lemma 2.1 in [23] we know that Im $A \subseteq$ Im $B$ if and only if $A = BC$ for some $C \in B(\mathcal{H})$. Because of that, from Theorem 9, it follows that the relations defined by (1) and by (3) are the same in the operator case as well. From the proof of Theorem 9 one can see that the technical obstacle that $B(\mathcal{H}, \mathcal{K})$ is not a ring has no effect on the validity of this claim.

We will now give the interpretation of Theorem 13. The interpretation of other results can be achieved by similar reasoning. Let us follow the notation as in Theorem 13 and its proof. By (18), we have

$$\text{Im } P_1 = \text{Im } (AH) = \text{Im } (AG) = \text{Im } A. \quad (23)$$

Recall that $A *< B$ implies $B - A *< B$ by (15). Now, by the same argument as in (23), we conclude that Im $P_2 = \text{Im } (B - A)$. From the same reason as in the proof of Theorem 3, we have that Ker $H = \text{Ker } B^*$. Therefore,

$$\text{Im } P_3 = \text{Im } (I - BH) = \text{Ker } (BH) = \text{Ker } H = \text{Ker } B^*.$$
In [21], the connection of the decomposition of the identity $I$ of ring $B(\mathcal{K})$ and topological direct sum was presented in detail. From that connection, it follows that $\mathcal{K} = \text{Im} \, P_1 \oplus \text{Im} \, P_2 \oplus \text{Ker} \, P_3$, so

$$\mathcal{K} = \text{Im} \, A \oplus \text{Im} \, (B - A) \oplus \text{Ker} \, B^*,$$

where the above decomposition is an orthogonal direct sum decomposition of $\mathcal{K}$. Finally, note that $\text{Im} \, Q_3 = \text{Im} \, (I - HB) = \text{Ker} \, (HB) = \text{Ker} \, B$. We are now in a position to restate Theorem 13 in the operator case.

**Theorem 29** Let $A, B \in B(\mathcal{H}, \mathcal{K})$ be operators with closed ranges. Then $A \preceq B$ if and only if the following hold

(i) There exist closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \text{Ker} \, B$;

(ii) $\mathcal{K} = \text{Im} \, A \oplus \text{Im} \, (B - A) \oplus \text{Ker} \, B^*$;

(iii) $A, B : \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \text{Ker} \, B \to \text{Im} \, A \oplus \text{Im} \, (B - A) \oplus \text{Ker} \, B^*$ have matrix representations

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $A_1 \in B(\mathcal{H}_1, \text{Im} \, A)$ and $B_1 \in B(\mathcal{H}_2, \text{Im} \, (B - A))$ are invertible operators.

**Acknowledgements** The authors gratefully acknowledge the referee for the detailed and helpful comments and suggestions that improved the quality of the paper.

**References**

1. Baksalary, J.K., Baksalary, O.M., Liu, X.: Further properties of the star, left-star, right-star, and minus partial orderings. Linear Algebra Appl. 375, 83–94 (2003)
2. Baksalary, J.K., Hauke, J.: A further algebraic version of Cochran’s theorem and matrix partial orderings. Linear Algebra Appl. 127, 157–169 (1990)
3. Baksalary, J.K., Misra, S.K.: Left-star and right-star partial orderings. Linear Algebra Appl. 149, 73–89 (1991)
4. Cirulis, J.: One-sided star partial orders for bounded linear operators. Oper. Matrix 9(4), 891–905 (2015)
5. Dolinar, G., Guterman, A., Marovt, J.: Monotone transformations on $B(\mathcal{H})$ with respect to the left-star and the right-star partial order. Math. Inequal. Appl. 17, 573–589 (2014)
6. Dolinar, G., Halicioglu, S., Harmanci, A., Kuzma, B., Marovt, J., Ungor, B.: Preservers of the left-star and right-star partial orders. Linear Algebra Appl. 587, 70–91 (2020). https://www.sciencedirect.com/science/article/pii/S0024379519304720
7. Dolinar, G., Marovt, J.: On a generalized concept of order relations on B(H). Math. Slovaca 68(1), 33–40 (2018). https://doi.org/10.1515/ms-2017-0077
8. Drazin, M.P.: Natural structures on semigroups with involution. Bull. Am. Math. Soc. 84, 139–141 (1978)
9. Hartwig, R.E.: How to partially order regular elements. Math. Japon. 25(1), 1–13 (1980)
10. Hernández, M.V., Lattanzi, M.B., Thome, N.: From projectors to 1MP and MP1 generalized inverses and their induced partial orders. RACSAM 115, 148 (2021). https://doi.org/10.1007/s13398-021-01090-8
11. Krémer, I.: Left-star order structure of Rickart *-rings. Linear Multilinear Algebra 64(3), 341–352 (2016)
12. Lebtahi, L., Patrício, P., Thome, N.: The diamond partial order in rings. Linear Multilinear Algebra 62(3), 386–395 (2014). https://doi.org/10.1080/03081087.2013.779272
13. Marovt, J.: On partial orders in Rickart rings. Linear Multilinear Algebra 63(9), 1707–1723 (2015). https://doi.org/10.1080/03081087.2014.972314
14. Marovt, J., Mosić, D., Cremer, I.: On some generalized inverses and partial orders in *-rings. J. Algebra Appl. (2022) [online ready]. https://doi.org/10.1142/S0219498823502560
15. Marovt, J., Rakić, D.S., Djordjević, D.S.: Star, left-star, and right-star partial orders in Rickart *-rings. Linear Multilinear Algebra 63(2), 343–365 (2015). https://doi.org/10.1080/03081087.2013.866670
16. Misra, S.K.: Matrix partial orders through generalized inverses: unified theory. Linear Algebra Appl. 148, 237–263 (1991)
17. Mitra, S.K., Bhimasankaram, P., Malik, S.B.: Matrix Partial Orders, Shorted Operators and Applications. World Scientific, Singapore (2010)
18. Mosić, D.: One-sided core partial orders on a ring with involution. RACSAM 112, 1367–1379 (2018). https://doi.org/10.1007/s13398-017-0433-4
19. Mosić, D., Cvetković-Ilić, D.S.: Some orders for operators on Hilbert spaces. Appl. Math. Comput. 275, 229–237 (2016). https://doi.org/10.1016/j.amc.2015.11.059
20. Rakić, D.S., Djordjević, D.S.: Partial orders in rings based on generalized inverses—unified theory. Linear Algebra Appl. 471, 203–223 (2015). https://doi.org/10.1016/j.laa.2015.01.004
21. Rakić, D.S., Djordjević, D.S.: A note on topological direct sum of subspaces. Funct. Anal. Approx. Comput. 10(1), 9–20 (2018). http://operator.pmf.ni.ac.rs/www/pmf/publikacije/faac/2018/FAAC-10-1/ faac-10-1-contents.htm
22. Rakić, D.S.: Decomposition of a ring induced by minus partial order. Electron. J. Linear Algebra 23, 1040–1059 (2012). https://doi.org/10.13001/1081-3810.1573
23. Rakić, D.S., Djordjević, D.S.: Space pre-order and minus partial order for operators on Banach spaces. Aequat. Math. 85, 429–448 (2013). https://doi.org/10.1007/s00010-012-0133-2
24. Rakić, D.S., Djordjević, D.S.: Star, sharp, core and dual core partial order in rings with involution. Appl. Math. Comput. 259, 800–818 (2015). https://doi.org/10.1016/j.amc.2015.02.062
25. Wang, L., Chen, J.: Further results on partial ordering and the generalized inverses. Linear Multilinear Algebra 63(12), 2419–2429 (2015). https://doi.org/10.1080/03081087.2015.1016885

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.