AN ALMOST SURE INVARIANCE PRINCIPLE FOR SEVERAL CLASSES OF RANDOM DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we deal with a large class of dynamical systems having a version of the spectral gap property. Our primary class of systems comes from random dynamics, but we also deal with the deterministic case. We show that if a random dynamical system has a fiberwise spectral gap property as well as an exponential decay of correlations in the base, then, developing on Gouëzel’s approach, the system satisfies the almost sure invariance principle. The result is then applied to uniformly expanding random systems like those studied by Denker and Gordin and Mayer, Skorulski, and Urbański.

1. Introduction

The almost sure invariance principle (ASIP) is a powerful statistical property which assures that the trajectories of a process can be approximated in an almost sure manner with the trajectories of a Brownian motion with a negligible error term relative to the length of the trajectories. In particular, the ASIP implies many limit theorems including the law of the iterated logarithm and various versions of the central limit theorem. For more consequences of the ASIP, see [10] and the references therein.

The ASIP was first shown for scalar–valued independent and identically distributed random variables by Strassen in [12, 13] and then for \( \mathbb{R}^d \)-valued observables by Melbourne and Nicol in [9]. In [6], Gouëzel uses spectral properties to show a vector valued ASIP for a wide class of dynamical systems which satisfy the so called “spectral gap” property. Concerning random dynamical systems, the ASIP has been shown for random expanding dynamical systems by Aimino, Nicol, and Vaienti in [1], considering only stationary measures. In the recent paper [5] of Dragičević, Froyland, González–Tokman, and Vaienti for random Lasota–Yorke maps they consider non–stationary fiberwise measures, as we do in this paper, but they prove the ASIP for centered observables. Here we consider general Hölder observables, which are not necessarily centered.

In this paper we build upon Gouëzel’s approach, to present a real valued ASIP for a large class of random dynamical systems with non–stationary fiberwise random measures for which only the central limit theorem and law of the iterated logarithm were previously known. We show that if a random dynamical system has transfer operators which satisfy the spectral gap property as well as a base dynamical system which exhibits an exponential decay of correlations, then the random system satisfies an ASIP. In particular it is the difficulty of dealing with non–centered observables which requires this extra condition on the dynamical system in the base.
In what follows we will show that the uniformly expanding random systems of \cite{8}, and in particular the DG*-systems of \cite{8} based upon the work of Denker–Gordin \cite{4}, are examples of such well-behaved systems for which our theory applies.

2. Preliminaries

2.1. Almost Sure Invariance Principle.

We will consider a real valued stationary stochastic process \((A_n)_{n=0}^{\infty}\) which is bounded in \(L^p\) for some \(p > 2\).

**Definition 2.1.** Suppose \(0 < \lambda \leq 1/2\) and \(\sigma^2 > 0\). We say that the sequence \((A_n)_{n=0}^{\infty}\) satisfies an almost sure invariance principle with error exponent \(\lambda\) and limiting covariance \(\sigma^2\) if there exists a probability space \(\Omega\) and stochastic process \((A'_n)_{n=0}^{\infty}\) and \((B_n)_{n=0}^{\infty}\) on \(\Omega\) such that the following hold:

1. The processes \((A_n)_{n=0}^{\infty}\) and \((A'_n)_{n=0}^{\infty}\) have the same distribution.
2. The random variables \(B_n\) are independent and distributed as \(N(0, \sigma^2)\).
3. Almost surely in \(\Omega\) we have that

\[
\left| \sum_{j=0}^{n-1} A'_j - \sum_{j=0}^{n-1} B_j \right| \leq o(n^\lambda)
\]

when \(n \to \infty\).

As a Brownian motion on the integers corresponds with a sum of i.i.d. Gaussian random variables, this definition can be restated as almost sure approximation by a Brownian motion.

2.2. Random Dynamical Systems.

Suppose \((X, \mathcal{F}, m)\) is a complete (Borel) probability space with metric \(d_X\) and that \(\theta : X \to X\) is an invertible map, often referred to as the base map. We assume that \(\theta\) preserves the measure \(m\), i.e.

\[m \circ \theta^{-1} = m,\]

and that \(\theta\) is ergodic with respect to \(m\). For each \(x \in X\) we associate the metric space \((J_x, \rho_x)\) with each \(x \in X\), and let

\[J := \bigcup_{x \in X} \{x\} \times J_x\]

For ease of exposition we will identify \(J_x\) with \(\{x\} \times J_x\). Further suppose that for each \(x \in X\) there is a continuous map \(T_x : J_x \to J_{\theta(x)}\), and define the associated skew product
map $T : J \longrightarrow J$ by

$$T(x, z) = (\theta(x), T_x(z)).$$

For each $n \geq 0$ we denote

$$T^n_x := T_{\theta^{n-1}(x)} \circ \cdots \circ T_x : J_x \longrightarrow J_{\theta^n(x)}.$$

Similarly, we have

$$T^n(x, z) = (\theta^n(x), T^n_x(z)).$$

Given a continuous function $g : J \longrightarrow \mathbb{R}$, for each $x \in X$ we let

$$g_x := g|_{\{x\} \times J_x} : J_x \longrightarrow \mathbb{R}.$$

For a more thorough treatment of random dynamics see, for example, [2] or [7].

2.3. Random Measures.

Suppose $T : J \longrightarrow J$ is a random dynamical system over the base $(X, \mathcal{F}, m, \theta)$ the as defined above, and let $\mathcal{B} := \mathcal{B}_J$ denote the Borel $\sigma$–algebra on $J$ such that the following hold

(1) The map $T$ and projection function $\pi_X : J \longrightarrow X$, given by $\pi_X(x, y) = x$, are measurable,

(2) for every $A \in \mathcal{B}$, $\pi_X(A) \in \mathcal{F},$

(3) $\mathcal{B}_x := \mathcal{B}|_{J_x}$ is a $\sigma$–algebra on $J_x$.

A measure $\mu$ on $(J, \mathcal{B})$ is said to be random probability measure relative to $m$ if it has marginal $m$, i.e. if

$$\mu \circ \pi_X^{-1} = m.$$

If $(\mu_x)_{x \in X}$ are disintegrations of $\mu$ with respect to the partition $(J_x)_{x \in X}$ of $J$, then these satisfy the following properties:

(1) For every $B \in \mathcal{B}_x$, the map $X \ni x \longmapsto \mu_x(B) \in [0, 1]$ is measurable,

(2) For $m$–a.e. $x \in X$, the map $\mathcal{B}_x \ni B \longmapsto \mu_x(B) \in [0, 1]$ is a Borel probability measure.

We let $\mathcal{P}(J_x)$ denote the space of probability measures on $(J_x, \mathcal{B}_x)$ for each $x \in X$, and by $\mathcal{P}_m(J)$ we denote the space of all random measures on $J$ with marginal $m$. By definition we then have that for $\mu \in \mathcal{P}_m(J)$ and suitable $g : J \longrightarrow \mathbb{R}$

$$\mu(g) = \int_J g d\mu = \int_X \int_{J_x} g_x d\mu_x dm(x).$$

For such a function $g$, we denote the expected value of $g$ with respect to the measure $\mu$ by

$$\mathcal{E}_\mu(g) = \mu(g).$$

For a more detailed discussion of random measures see Crauel’s book [3].
2.4. Continuous and Hölder Potentials.

For each $x \in X$ we let $\mathcal{C}(\mathcal{J}_x)$ be the set of all continuous and bounded functions $g_x : \mathcal{J}_x \rightarrow \mathbb{R}$. Taken together with the sup norm, $\| \cdot \|_{x, \infty}$, over the fiber $\mathcal{J}_x$, $\mathcal{C}(\mathcal{J}_x)$ becomes a (fiberwise) Banach space. We can then consider the global space $\mathcal{C}(\mathcal{J})$ of functions $g : \mathcal{J} \rightarrow \mathbb{R}$ such that for $m$-a.e. $x \in X$ the functions $g_x := g|_{\mathcal{J}_x} \in \mathcal{C}(\mathcal{J}_x)$. Let $\mathcal{C}^0(\mathcal{J})$ and $\mathcal{C}^1(\mathcal{J})$ be subspaces of $\mathcal{C}(\mathcal{J})$ such that for all $g \in \mathcal{C}^0(\mathcal{J})$ the function $x \mapsto \|g_x\|_{x, \infty}$ is $\mathcal{F}$–measurable and $g \in \mathcal{C}^1(\mathcal{J})$ implies that

$$\|g\|_1 := \int_X \|g_x\|_{x, \infty} \, dm(x) < \infty.$$ 

Let $\mathcal{C}^\infty_*(\mathcal{J})$ denote the space of all $\mathcal{B}_\mathcal{J}$–measurable functions $g \in \mathcal{C}(\mathcal{J})$ such that $\sup_{x \in X} \|g_x\|_{x, \infty} < \infty$.

Clearly $\mathcal{C}^\infty_*(\mathcal{J})$ becomes a Banach space when coupled together with the norm $| \cdot |_\infty$ given by

$$|g|_\infty = \sup_{x \in X} \|g_x\|_{x, \infty}, \quad g \in \mathcal{C}^\infty_*(\mathcal{J}).$$

Fix $\alpha \in (0, 1]$. The $\alpha$–variation of a function $g_x \in \mathcal{C}(\mathcal{J}_x)$ is given by

$$v_{x, \alpha}(g_x) = \sup \left\{ \frac{|g_x(y) - g_x(y')|}{\rho_x^\alpha(y, y')} : y, y' \in \mathcal{J}_x, 0 \neq \rho_x(y, y') \leq \eta \right\}.$$ 

We let $\mathcal{H}_\alpha(\mathcal{J}_x)$ denote the collection of all functions $g_x \in \mathcal{C}(\mathcal{J}_x)$ such that $v_{x, \alpha}(g_x) < \infty$. Taken together with the norm given by

$$\| \cdot \|_{x, \alpha} = \| \cdot \|_{x, \infty} + v_{x, \alpha}(\cdot),$$

$\mathcal{H}_\alpha(\mathcal{J}_x)$ becomes a Banach algebra, that is we have

$$\|g_x h_x\|_{x, \alpha} \leq \|g_x\|_{x, \alpha} \cdot \|h_x\|_{x, \alpha}$$

for $g_x, h_x \in \mathcal{H}_\alpha(\mathcal{J}_x)$. We say that a function $g \in \mathcal{C}^1(\mathcal{J})$ is (global) $\alpha$–Hölder continuous over $\mathcal{J}$ if there is a $m$–measurable function

$$H : X \rightarrow [1, \infty), \quad x \mapsto H_x,$$

such that $v_{x, \alpha}(g_x) \leq H_x$ for $m$-a.e. $x \in X$ and $\log H \in L^1(m)$, i.e.

$$\int_X \log H_x \, dm(x) < \infty.$$ 

Let $\mathcal{H}_\alpha(\mathcal{J})$ be the collection of all such functions. For each $1 \leq p < \infty$ we let $\mathcal{H}_\alpha^p(\mathcal{J})$ denote the set of functions $g \in \mathcal{H}_\alpha(\mathcal{J})$ such that $\|g_x\|_{x, \alpha} \in L^p(m)$, that is such that

$$\|g\|_{\alpha, p} := \left( \int_X \|g_x\|_{x, \alpha}^p \, dm(x) \right)^{1/p} < \infty.$$
We then have that $H^\alpha_\alpha(J)$ taken with the norm $||\cdot||_{\alpha,p}$ is a Banach space. Now let $H^\alpha_\alpha(J)$ be the set of all functions $g \in C^\infty_0(J)$ such that 

$$\sup_{x \in X} v_{x,\alpha}(g_x) < \infty.$$ 

For $g \in H^\alpha_\alpha(J)$, we set 

$$V_\alpha(g) = \sup_{x \in X} v_{x,\alpha}(g_x),$$

and then define the norm $|\cdot|_\alpha$ on $H^\alpha_\alpha(J)$ by 

$$|g|_\alpha = |g|_\infty + V_\alpha(g).$$

Then $H^\alpha_\alpha(J)$ is a Banach space when considered with the norm $|\cdot|_\alpha$.

We also consider the set $H^\beta_\beta(X)$, for $\beta \in (0,1]$, of bounded continuous functions $G : X \to \mathbb{R}$ such that the $\beta$-variation of $G$, denoted by $\vartheta_\beta(G)$, is finite, i.e.

$$\vartheta_\beta(G) := \sup \left\{ \frac{|G(x) - G(x')|}{d^\beta_x(x, x')} : x, x' \in X, x \neq x' \right\} < \infty.$$ 

We say that such functions are $\beta$-Hölder continuous on $X$. $H^\beta_\beta(X)$ then becomes a Banach space, and in fact a Banach algebra, when coupled together with the norm 

$$||\cdot||_\beta := ||\cdot||_X + \vartheta_\beta(\cdot),$$

where $||\cdot||_X$ denotes the supremum norm on $X$.

2.5. Transfer Operators. 

Given a continuous function $g \in C^1(J)$, for each $n \in \mathbb{N}$ and $x \in X$ we then define the Birkhoff sum $S_{x,n} : C(J_x) \to \mathbb{R}$ by 

$$S_{x,n}g_x := \sum_{j=0}^{n-1} g_{\theta^j(x)} \circ T_{x}^j.$$ 

If there is no confusion about the fiber $J_x$, we will simply write $S_n$. Now given a function $\varphi \in H_\alpha_\alpha(J)$ we define the (Perron–Frobenius) transfer operator $L_{\varphi,x} : C(J_x) \to C(J_{\theta(x)})$ by 

$$L_x(u_x)(w) := L_{\varphi,x}(u_x)(w) := \sum_{z \in T_x^{-1}(w)} u_x(z) e^{\varphi_x(z)}, \quad w \in J_{\theta(x)}.$$ 

For $n \in \mathbb{N}$ the iterates $L^n_x : C(J_x) \to C(J_{\theta^n(x)})$ is given by 

$$L^n_x(u_x) := L_{\theta^{n-1}(x)} \circ \cdots \circ L_x(u_x).$$ 

Inductively one can show that 

$$L^n_x(u_x)(w) = \sum_{z \in T_x^{-n}(w)} u_x(z) e^{S_n\varphi_x(z)}, \quad w \in J_{\theta^n(x)}$$
for each $n \in \mathbb{N}$. Denote by $\mathcal{L}_x^*: \mathcal{C}^*(\mathcal{J}_{\theta(x)}) \to \mathcal{C}^*(\mathcal{J}_x)$, where $\mathcal{C}^*(\mathcal{J}_x)$ is the dual space of $\mathcal{C}(\mathcal{J}_x)$ equipped with the weak* topology. Now suppose there is a random probability measure $\nu$ on $\mathcal{J}$ such that

$$L_x^*\nu_{\theta(x)} = \lambda_x \nu_x \quad \text{for } m - \text{a.e. } x \in X,$$

where

$$\lambda_x := L_x^* (\nu_{\theta(x)})(1) = \nu_{\theta(x)}(L_x 1) = \int_{\mathcal{J}_{\theta(x)}} L_x 1 \, d\nu_{\theta(x)}.$$

We are then able to define the normalized operator $L_{0,x} : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta(x)})$ by

$$L_{0,x}(u_x) := \lambda_x^{-1} L_x(u_x).$$

Clearly we have that the iterates $L_{n,x} : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta^n(x)})$ of the normalized are given by

$$L_{n,x}^*(u_x) = (\lambda_x^n)^{-1} L_{n,x}^*(u_x).$$

where

$$(\lambda_x^n)^{-1} = (\lambda_x)^{-1} \cdots (\lambda_{\theta^{n-1}(x)})^{-1} = \int_{\mathcal{J}_{\theta^n(x)}} L_{n,x}^*(1) \, d\nu_{\theta^n(x)}$$

For each $r \in \mathbb{R}$ we define the perturbed operator given by

$$L_{r,x}(u_x) := L_{0,x}(e^{ir g_x} \cdot u_x)$$

In the sequel, the perturbed operator will be our main technical tool. The following lemma characterizes the iterates of the perturbed operator.

**Lemma 2.2.** For $r_0, \ldots, r_{n-1} \in \mathbb{R}$ we have

$$L_{r_{n-1},\theta^{n-1}(x)} \circ \cdots \circ L_{r_0,x}(u_x) = L_{0,x}^*(e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T_{x}^j} \cdot u_x).$$

**Proof.** For $r_0, r_1 \in \mathbb{R}$ we have

$$L_{r_1,\theta(x)} \left( L_{r_0,x}(u_x) \right) = L_{0,\theta(x)} \left( e^{ir_1 g_{\theta(x)}} \cdot L_{0,x}^*(e^{ir_0 g_x} \cdot u_x) \right)$$

$$= L_{0,\theta(x)} \left( L_{0,x}^* \left( e^{ir_1 g_{\theta(x)} \circ T_x} \cdot e^{ir_0 g_x} \cdot u_x \right) \right)$$

$$= L_{0,x}^* \left( e^{ir_0 g_x + r_1 g_{\theta(x)} \circ T_x} \cdot u_x \right)$$

Inducting on $n \geq 1$ we suppose that for $r_{n-1}, \ldots, r_0 \in \mathbb{R}$ we have

$$L_{r_{n-1},\theta^{n-1}(x)} \circ \cdots \circ L_{r_0,x}(u_x) = L_{0,x}^*(e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T_{x}^j} \cdot u_x),$$

and let $r_n \in \mathbb{R}$. Then

$$L_{r_n,\theta^n(x)} L_{r_{n-1},\theta^{n-1}(x)} \circ \cdots \circ L_{r_0,x}(u_x) = L_{r_n,\theta^n(x)} L_{0,x}^* \left( e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T_{x}^j} \cdot u_x \right)$$

$$= L_{0,\theta^n(x)} \left( e^{i r_n g_{\theta^n(x)}} \cdot L_{0,x}^* \left( e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T_{x}^j} \cdot u_x \right) \right)$$

$$= L_{0,x}^* \left( e^{i \sum_{j=0}^{n} r_j g_{\theta^j(x)} \circ T_{x}^j} \cdot u_x \right).$$

$\square$
In Section 3 we present our a main result, a theorem which establishes an almost sure invariance principle for quite general classes of random dynamical systems. Afterward we provide several classes of examples for which our theorem applies. Throughout our paper, \( C \) will denote some positive constant, which may change from line to line. By \( \mathcal{N}(\mu, \sigma^2) \) we mean the normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

3. ASIP for Random Systems

In this section we give the main result of the paper, which is an adaptation of Theorem 3.5 (Theorem 2.1 of [6]) for random dynamical systems. We will follow the general strategy of Gouëzel’s proof. Our main theorem is the following.

**Theorem 3.1.** Suppose \((X, \mathcal{F}, m, \theta)\) and \(T : \mathcal{J} \to \mathcal{J}\) form a random dynamical system as defined above and suppose that \( \mu \) is a \( T \)-invariant measure on \( \mathcal{J} \). For \( \alpha \in (0, 1) \) and let \( g, \varphi \in \mathcal{H}_\alpha(\mathcal{J}) \). Suppose the transfer operators \( \mathcal{L}_{g,x}, \mathcal{L}_{0,x}, \) and \( \mathcal{L}_{r,x} \) are defined as above and suppose there is \( \varepsilon_0 > 0 \), \( \rho_x \in \mathcal{H}_\alpha(\mathcal{J}_x) \), and \( \nu_x \in \mathcal{P}(\mathcal{J}_x) \) for each \( x \in X \) such that the following hold.

1. There exists \( C > 0 \) such that for each \( x \in X \)
   \[
   \mu_x = \rho_x \nu_x \quad \text{and} \quad \|\rho_x\|_{x, \alpha} \leq C.
   \]
   Moreover, for any \( r_0, \ldots, r_{n-1} \in \mathbb{R} \) with \( |r_j| < \varepsilon_0 \) we have
   \[
   \mathcal{E}_\mu \left( e^{\sum_{j=0}^{n-1} r_j g_{\theta^j}} \right) = \int_X \mathcal{J}_x \mathcal{L}_{r_{n-1}, \theta^{-1}(x)} \circ \cdots \circ \mathcal{L}_{r_0, \theta^{-n}(x)} \rho_x dm(x).
   \]

2. There exists \( C \geq 1 \) such that for all \( n \in \mathbb{N} \), all \( r \in \mathbb{R} \) with \( |r| < \varepsilon_0 \), \( m \)-a.e. \( x \in X \), all \( f_x \in \mathcal{C}(\mathcal{J}_x) \), and all \( h_x \in \mathcal{H}_\alpha(\mathcal{J}_x) \)
   \[
   \|\mathcal{L}_{r,x} f_x\|_{\theta^n(x), \infty} \leq C \|f_x\|_{x, \infty} \quad \text{and} \quad \|\mathcal{L}_{r,x} h_x\|_{\theta^n(x), \alpha} \leq C \|h_x\|_{x, \alpha}.
   \]

3. For each \( x \in X \) there is an operator \( Q_x : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta(x)}) \) defined by
   \[
   Q_x u_x := \int_{\mathcal{J}_x} u_x dm_x \cdot \rho_{\theta(x)},
   \]
   and there are \( C > 0 \) and \( \kappa \in (0, 1) \) such that for \( m \)-a.e. \( x \in X \) and \( u_x \in \mathcal{H}_\alpha(\mathcal{J}_x) \)
   \[
   \|\mathcal{L}_{0,x} u_x - Q_x u_x\|_{\theta^n(x), \infty} \leq C \kappa^n \|u_x\|_{x, \alpha}.
   \]

4. There exist constants \( C > 0 \) and \( \kappa \in (0, 1) \) such that for all \( F \in \mathbb{H}_\beta \), all \( G \in L^1(m) \), and \( n \in \mathbb{N} \) sufficiently large we have that
   \[
   \left|m(G \circ \theta^{-n} \cdot F) - m(G) \cdot m(F)\right| \leq C \kappa^n \|F\|_{\mathbb{H}} \|G\|_{L^1(m)}.
   \]

5. Suppose there exists \( C > 0 \) and \( \beta \in (0, 1] \) such that for each \( n \in \mathbb{N} \) and \( r_0, \ldots, r_{n-1} \in \mathbb{R} \) with \( |r_j| < \varepsilon_0 \) for each \( 1 \leq j \leq n - 1 \), we have that the function
   \[
   X \ni x \mapsto F(x) = \int_{\mathcal{J}_x} \mathcal{L}_{r_{n-1}, \theta^{n-1}(x)} \circ \cdots \circ \mathcal{L}_{r_0 \circ \theta} \rho_x dm_x
   \]
is \(\beta\)-Hölder continuous on \(X\) and furthermore, we have that
\[
\|F\|_H \leq C
\]
independent of the choice of \(n\) and \(r_0, \ldots, r_{n-1}\).

Then either there exists a real number \(\sigma^2 > 0\) such that the stochastic process \(\{g \circ T^n - \mu(g)\}_{n \in \mathbb{N}}\) considered with respect to the measure \(\mu\), satisfies an ASIP with limiting covariance \(\sigma^2\) for any error exponent larger than \(1/4\). Consequently, the sequence
\[
\frac{S_n g - n \cdot \mu(g)}{\sqrt{n}}
\]
converges in probability to \(\mathcal{N}(0, \sigma^2)\) and
\[
\lim_{n \to \infty} \frac{S_n g - n \cdot \mu(g)}{\sqrt{2n \log \log n}} = 1.
\]
Or, if \(\sigma^2 = 0\), then we have that
\[
\sup_{n \in \mathbb{N}} \|S_n g - \mu(g)\|_{L^2(\mu)} < \infty
\]
or equivalently, \(g\) is of the form \(g = k - k \circ T + \mu(g)\) where \(k \in L^2(\mu)\). In addition, almost surely we have
\[
\lim_{n \to \infty} \frac{S_n g}{n^{1/4}} = 0.
\]

**Remark 3.2.** Note that, using induction, we have that the operator
\[
Q^n_x : \mathcal{C}(\mathcal{J}_x) \longrightarrow \mathcal{C}(\mathcal{J}_\theta^n(x))
\]
is defined by
\[
Q^n_x u_x := \int_{\mathcal{J}_x} u_x \, d\nu_x \cdot \rho_{\theta^n(x)}.
\]

**Remark 3.3.** We also note that in [6], Gouëzel refers to hypothesis (i) above as “the characteristic function of the process \(g \circ T^n\) being encoded by the family of operators \(\{L_r\}\)”.

**Remark 3.4.** Assumption (i) essentially says that the dynamical system in the base exhibits an exponential decay of correlations. While this may seem like a strong assumption, it should, in some sense, be expected. Consider a potential \(\varphi\) which is constant on fibers, that is, \(\varphi_x = c_x\) for some constant \(c_x \in \mathbb{R}\), for each \(x \in X\). Such a potential only captures the dynamics of the base map \(\theta\), and, furthermore, the ASIP simply does not hold in general assuming only ergodicity.

Also note that since \(\theta\) is invertible and \(m\)-invariant, we also have that
\[
|m(G \cdot F \circ \theta^n) - m(G) \cdot m(F)| \leq C \kappa^n \|G\|_\mathbb{H} \|F\|_{L^1(m)},
\]
for \(n \in \mathbb{N}\) and \(F, G \in \mathbb{H}_\beta \cap L^1(m)\).

As we will see in the sequel, there are many random systems which satisfy this assumption. In particular, any random system with an expanding base map \(\theta\) and Gibbs measure.
m will satisfy an exponential decay of correlations for H"older continuous observables, see, for example, [11].

In order to prove Theorem 3.1 we will adapt the method of the proof of the following theorem of Gou"ezel:

**Theorem 3.5** (Theorem 2.1, [6]). Let \((A_t)\) be a stochastic process whose characteristic function is encoded by a family of operators \(\mathcal{L}_t : \mathcal{B} \to \mathcal{B}\) (see Remark 3.3) and which is bounded in \(L^p\) for some \(p > 2\). Further assume

\((I0)\) There exists \(u_0 \in \mathcal{B}\) and \(\xi_0 \in \mathcal{B}^*\), the dual of \(\mathcal{B}\), such that for any \(r_0, \ldots, r_{n-1} \in \mathbb{R}^d\) with \(|r_j| \leq \varepsilon_0\),

\[\mathcal{E}(e^{i\sum_{j=0}^{n-1} r_j A_j}) = \langle \xi_0, \mathcal{L}_{r_{n-1}} \circ \cdots \circ \mathcal{L}_{r_0} u_0 \rangle.\]

\((I1)\) One can write \(\mathcal{L}_0 = Q + S\), where \(Q\) is a one-dimensional projection and \(S\) is an operator on \(\mathcal{B}\) with \(SQ = QS = 0\) and \(\|\mathcal{L}_0^n - Q\|_{\mathcal{B} \to \mathcal{B}} \leq C\kappa^n\) for some \(\kappa < 1\).

\((I2)\) There exists \(C > 0\) such that \(\|\mathcal{L}_0^n\|_{\mathcal{B}} \leq C\) for all \(n \in \mathbb{N}\) and all small enough \(r \in \mathbb{R}^d\).

Then there exists \(a \in \mathbb{R}^d\) and a matrix \(\Sigma^2\) such that \((\sum_{j=0}^{n-1} A_j - na)/\sqrt{n}\) converges to \(\mathcal{N}(0, \Sigma^2)\). Moreover the process \((A_j - a)_{j \in \mathbb{N}}\) satisfies an ASIP with limiting covariance \(\Sigma^2\) for any error exponent larger that \(p/(4p - 4)\).

**Remark 3.6.** We note that we assume in the first half of (3) in the hypotheses of Theorem 3.1 what Gou"ezel proves in his first step. In practice, we will always have such an operator, so our assumption is justified.

However, in order to prove Theorem 3.1 we must invoke Gou"ezel’s Theorem 1.3 and Lemma 2.7 of [6]. First we state the main assumption of [6], which ensures that a given stochastic process is sufficiently close to an independent process for our purposes. We will refer to this assumption as condition (H).

\((H)\) There exists \(\varepsilon_0 > 0\) and constants \(C, c > 0\) such that for any \(n, m, k > 0\), \(b_1 < b_2 < \cdots < b_{n+m+1}\), and \(r_1, \ldots, r_{n+m} \in \mathbb{R}^d\) with \(|r_j| \leq \varepsilon_0\) we have

\[
\left| \mathcal{E} \left( e^{i\sum_{j=1}^{n} t_j (\sum_{\ell=b_j}^{b_j+1-1} A_{\ell}) + i\sum_{j=n+1}^{n+m} t_j (\sum_{\ell=b_j+k}^{b_j+1+k-1} A_{\ell})} \right) \right|
\]

\[
- \mathcal{E} \left( e^{i\sum_{j=1}^{n} t_j (\sum_{\ell=b_j}^{b_j+1-1} A_{\ell})} \right) \cdot \mathcal{E} \left( e^{i\sum_{j=n+1}^{n+m} t_j (\sum_{\ell=b_j+k}^{b_j+1+k-1} A_{\ell})} \right) \leq C \left( 1 + \text{max} |b_{j+1} - b_j| \right)^{C(n+m)} e^{-ck}.
\]

**Theorem 3.7** (Theorem 1.3, [6]). Let \((A_0, A_1, \ldots)\) be a centered \(\mathbb{R}^d\)-valued process, bounded in \(L^p\) for some \(p > 2\), satisfying condition (H). Assume, moreover, that \(\sum |A_{\ell}| < \infty\) and that there exists a matrix \(\Sigma^2\) such that, for any \(\alpha > 0\),

\[
\left| \text{cov} \left( \sum_{\ell=m}^{n+m-1} A_{\ell} \right) - n\Sigma^2 \right| \leq Cn^\alpha,
\]

where \(C\) is a constant depending only on \(\alpha\), \(d\), and \(\mathbb{N}\).
uniformly in $n,m$. The sequence $\sum_{\ell=0}^{n-1} A_\ell / \sqrt{n}$ converges in distribution to $\mathcal{N}(0, \Sigma^2)$. Moreover, the process $(A_0, A_1, \ldots)$ satisfies an ASIP with limiting covariance $\Sigma^2$ for any error exponent $\lambda > p/(4p-4)$.

**Lemma 3.8** (Lemma 2.7, [6]). Let $(A_\ell)$ be a process bounded in $L^p$ for some $p > 2$, satisfying condition $(H)$ such that for any $m \in \mathbb{N}$ there exists a matrix $s_m$ such that uniformly in $\ell, m$ we have

$$|\text{cov}(A_\ell, A_{\ell+m}) - s_m| \leq Ce^{-\delta \ell}.$$ 

Then the series $\Sigma^2 = s_0 + \sum_{m=1}^{\infty} (s_m + s_m^*)$ converges in norm and we have that

$$\left|\text{cov} \left( \sum_{\ell=m}^{\infty} A_\ell \right) - ns^2 \right| \leq C.$$

**Proof (of Theorem 3.1).** First we show that condition $(H)$ holds. Let $b_1 < \cdots < b_{n+m+1}, r_1, \ldots, r_{n+m} \in \mathbb{R}, x \in X$, and $k \in \mathbb{N}$. Letting $y_x = \theta^{-(b_{n+m+1}+1)}(x)$, $w_x = \theta^{-(b_{n+m+1}-b_{n+1})}(x)$, $z_x = \theta^{-(b_{n+m+1}-b_{n+1}+1)}(x)$, and somewhat ignoring the fiberwise subscript notation for the moment, we see

\begin{align*}
\mathcal{E}_\mu \left( e^{\sum_{j=1}^{n} r_j (\sum_{\ell=b_j}^{b_j+1} g \circ f^\ell) + \sum_{j=n+1}^{m+n} r_j (\sum_{\ell=b_j}^{b_j+k} g \circ f^\ell)} \right) \\
= \int_X \int_{J_x} \mathcal{L}_{r_{n+m}}^{b_{n+m+1}-b_{n+m}} \cdots \mathcal{L}_{r_{n+1}}^{b_{n+2}-b_{n+1}} \mathcal{L}_0^k \mathcal{L}_{r_n}^{b_{n+1}-b_n} \cdots \mathcal{L}_{r_1}^{b_2-b_1} \mathcal{L}_0^{b_1} \rho_{y_x} \, d\nu_x \, dm(x) \\
= \int_X \int_{J_x} \mathcal{L}_{r_{n+m}}^{b_{n+m+1}-b_{n+m}} \cdots \mathcal{L}_{r_{n+1}}^{b_{n+2}-b_{n+1}} (\mathcal{L}_0^k - \mathcal{Q}^k) \mathcal{L}_{r_n}^{b_{n+1}-b_n} \cdots \mathcal{L}_{r_1}^{b_2-b_1} \mathcal{L}_0^{b_1} \rho_{y_x} \, d\nu_x \, dm(x) \\
+ \int_X \int_{J_x} \mathcal{L}_{r_{n+m}}^{b_{n+m+1}-b_{n+m}} \cdots \mathcal{L}_{r_{n+1}}^{b_{n+2}-b_{n+1}} \mathcal{Q}^k \mathcal{L}_{r_n}^{b_{n+1}-b_n} \cdots \mathcal{L}_{r_1}^{b_2-b_1} \mathcal{L}_0^{b_1} \rho_{y_x} \, d\nu_x \, dm(x) \\
+ \int_X \left[ \int_{J_x} \mathcal{L}_{r_{n+m}}^{b_{n+m+1}-b_{n+m}} \cdots \mathcal{L}_{r_{n+1}}^{b_{n+2}-b_{n+1}} \rho_{w_x} \, d\nu_x \\
\cdot \int_{J_x} \mathcal{L}_{r_{n+m}}^{b_{n+m+1}-b_{n+m}} \cdots \mathcal{L}_{r_{n+1}}^{b_{n+2}-b_{n+1}} \mathcal{L}_0^{b_1} \rho_{y_x} \, d\nu_x \right] \, dm(x).
\end{align*}

(3.2)

Define the functions $F$ and $G$, which depend on the constants $n, m, r_1, \ldots, r_{n+m}, b_1, \ldots, b_{n+m+1}$, by

$$F(x) = \int_{J_x} \mathcal{L}_{r_{n+m}}^{b_{n+m+1}-b_{n+m}} \cdots \mathcal{L}_{r_{n+1}}^{b_{n+2}-b_{n+1}} \rho_{w_x} \, d\nu_x$$

$$G(x) = \int_{J_{w_x}} \mathcal{L}_{r_{n+m}}^{b_{n+1}-b_n} \cdots \mathcal{L}_{r_1}^{b_2-b_1} \mathcal{L}_0^{b_1} \rho_{\theta^{-b_{n+m+1}}(x)} \, d\nu_{w_x}.$$
Now as \( z_x = \theta^{-(b_{n+m+1}-b_{n+1}+k)}(x) = \theta^{-(b_{n+m+1}-b_{n+1})} \circ \theta^{-k}(x) = w_x \circ \theta^{-k}(x) \) and \( y_x = \theta^{-(b_{n+m+1}+k)}(x) = \theta^{-b_{n+m+1}} \circ \theta^{-k}(x) \) we can rewrite the second term in the above product as
\[
\int_{\mathcal{X}_x} \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \, dv_{z_x} = G \circ \theta^{-k}(x),
\]
and thus the global integral of the products, the last two lines of the above string of equalities beginning with (3.2), becomes
\[
\int_X F(x) \cdot G \circ \theta^{-k}(x) \, dm(x).
\]
By virtue of assumption (1) we then see that
\[
\left| \int_X F(x) \cdot G \circ \theta^{-k}(x) \, dm(x) - \int_X F(x) \, dm(x) \cdot \int_X G(x) \, dm(x) \right| \leq C \kappa^k \|F\|_{\mathcal{H}} \cdot \|G\|_{L^1(m)}.
\]
By assumption (5) we have that \( \|F\|_{\mathcal{H}} \leq C \) and we are able to estimate \( \|G\|_{L^1(m)} \) as follows.
\[
\|G\|_{L^1(m)} = \int_X \left\| \int_{\mathcal{X}_x} \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \, dv_{z_x} \right\| \, dm(x)
\leq \int_X \left\| \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \right\|_{L^1(v_{x,\infty})} \, dm(x)
\leq \int_X \left\| \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \right\|_{w_{x,\infty}} \, dm(x)
\leq C^{n+1} \int_X \|\rho_x\|_{x,\alpha} \, dm(x).
\]
Now we wish to estimate the \( \mathcal{G}_x \) norm of (3.2). In light of assumptions (2) and (3) we see
\[
\left\| \mathcal{L}^{b_{n+m+1}-b_{n+m}} \circ \cdots \circ \mathcal{L}^{b_{n+2}-b_{n+1}} \circ \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \right\|_{x,\infty}
\leq C^m \left\| (\mathcal{L}_0^k - Q^k) \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \right\|_{w_{x,\infty}}
\leq C^{m+1} \kappa^k \left\| \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \right\|_{z_{x,\alpha}}
\leq C^{m+n+1} \kappa^k \|\rho_{y_x}\|_{y_{x,\alpha}}.
\]
Integrating over \( X \) provides the global inequality
\[
\left| \int_X \int_{\mathcal{X}_x} \mathcal{L}^{b_{n+m+1}-b_{n+m}} \circ \cdots \circ \mathcal{L}^{b_{n+2}-b_{n+1}} \circ \mathcal{L}^{b_{n+1}-b_n} \circ \cdots \circ \mathcal{L}^{b_2-b_1} \mathcal{L}^{b_0} \rho_{y_x} \, dv_{z_x} \, dm(x) \right|
\leq C^{m+n+1} \kappa^k \int_X \|\rho_x\|_{x,\alpha} \, dm(x).
\]
Thus combining our estimates we then have
\[
\mathcal{E}_\mu \left( \sum_{j=1}^n r_j \left( \sum_{\ell=b_j}^{b_j+1-1} g_\ell f_\ell \right) + \sum_{j=n+1}^{n+m} r_j \left( \sum_{\ell=b_j}^{b_j+k-1} g_\ell f_\ell \right) \right)
\]
Thus the difference in condition (H) is bounded by \( C^{n+1}k^k \int_X \| \rho_x \|_{x,\alpha} \, dm(x) \) for some \( C > 1 \) and \( 0 < \kappa < 1 \). Setting \( \kappa = e^{-c} \), \( C = 2^{C'} \) then we see that

\[
C^{n+1}k^k \int_X \| \rho_x \|_{x,\alpha} \, dm(x) \leq 2^{C'(n+m+1)} e^{-ck} L \leq L \cdot 2^{C'} (1 + \max |b_j - b_{j-1}|)^{C'(n+m)} e^{-ck},
\]

where \( L \) is some constant such that \( \int_X \| \rho_x \|_{x,\alpha} \, dm(x) \leq L \). This verifies that condition (H) is satisfied.

Next, Gouëzel shows that there exists \( a \in \mathbb{R} \) and \( C, \delta > 0 \) such that

\[
|\mathcal{E}_\mu(g \circ T^n) - a| \leq Ce^{-\delta n}.
\]

(3.3)

To accomplish this, he shows that the sequence \( (\mathcal{E}_\mu(g \circ T^n))_{n \geq 0} \) is Cauchy and must have some limit which he denotes by \( a \). However, by the assumed \( T \)-invariance of \( \mu \), we immediately have that \( a \) must be \( \int_J g \, d\mu \) and that the quantity (3.3) must in fact be equal to zero. For the next step in the proof of Theorem 3.1 we claim that for any \( m \in \mathbb{N} \) there is a real number \( s_m \) such that

\[
|\text{cov}(g \circ T^n, g \circ T^{n+m}) - s_m| \leq Ce^{-\delta n}
\]

uniformly in \( n, m \). To show this, we show that \( \text{cov}(g \circ T^n, g \circ T^{n+m}) \) is a Cauchy sequence in \( n \) and therefore must converge to some limit, which we will call \( s_m \). By the \( T \)-invariance of \( \mu \) we have

\[
\text{cov}(g \circ T^n, g \circ T^{n+m}) = \left| \int_J (g \circ T^n)(g \circ T^{n+m}) \, d\mu - \int_J g \circ T^n \, d\mu \cdot \int_J g \circ T^{n+m} \, d\mu \right|
\]

\[
= \left| \int_J (g \cdot g \circ T^n) \circ T^{n+m} \, d\mu - \left( \int_J g \, d\mu \right)^2 \right| \]

\[
= \left| \int_J g \cdot g \circ T^n \, d\mu - \left( \int_J g \, d\mu \right)^2 \right|
\]

\[
= \text{cov}(g, g \circ T^n).
\]

Thus to show that \( \text{cov}(g \circ T^n, g \circ T^{n+m}) \) is Cauchy it suffices to show that \( (\mathcal{E}_\mu(g \cdot g \circ T^n))_{m \geq 0} \) is Cauchy.

\[
\text{cov}(g, g \circ T^n) = \left| \int_J g \cdot g \circ T^n \, d\mu - \int_J g \, d\mu \cdot \int_J g \circ T^n \, d\mu \right|
\]

\[
= \left| \int_X \int_{J_x} g \cdot g^{m(x)} \circ T^n_x \, d\mu_x \, dm(x) - \int_X \int_{J_x} g \cdot \mu_x \, dm(x) \cdot \int_X \int_{J_x} g^{m(x)} \circ T^n_x \, d\mu_x \, dm(x) \right|
\]
Now setting \( \hat{g}_x := g_x - \int_{\mathcal{J}_x} g_x \, d\mu_x \) we can estimate the first summand, \( \Sigma_1 \), as

\[
\Sigma_1 = \left| \int_X \int_{\mathcal{J}_x} g_x \cdot g_{\theta^m(x)} \circ T^m_x \, d\mu_x \, dm(x) - \int_X \left( \int_{\mathcal{J}_x} g_x \, d\mu_x \cdot \int_{\mathcal{J}_x} g_{\theta^m(x)} \circ T^m_x \, d\mu_x \right) \, dm(x) \right|
\]

\[
\leq \int_X \left( \int_{\mathcal{J}_x} g_x \, d\mu_x \right) \cdot \left( \int_{\mathcal{J}_x} g_{\theta^m(x)} \circ T^m_x \, d\mu_x \right) \, dm(x)
\]

\[
= \int_X \left| \int_{\mathcal{J}_x} g_{\theta^m(x)} \circ T^m_x \, d\mu_x \right| \, dm(x)
\]

\[
= \int_X \left| \int_{\mathcal{J}_\theta^m(x)} \mathcal{L}_{0,x}^m \left( \hat{g}_x \rho_x \circ g_{\theta^m(x)} \circ T^m_x \right) \, d\nu_{\theta^m(x)} \right| \, dm(x)
\]

\[
\leq \int_X \left| \int_{\mathcal{J}_\theta^m(x)} g_{\theta^m(x)} \cdot \mathcal{L}_{0,x}^m (\hat{g}_x \rho_x) \right| \, d\nu_{\theta^m(x)} \, dm(x)
\]

\[
= \int_X \left| g_{\theta^m(x)} \cdot \mathcal{L}_{0,x}^m (\hat{g}_x \rho_x) \right| \, dm(x)
\]

\[
\leq \int_X \left| g_{\theta^m(x)} \right| \left| \mathcal{L}_{0,x}^m (\hat{g}_x \rho_x) \right| \, dm(x)
\]

\[
\leq \int_X \left| g_{\theta^m(x)} \right| \left| \mathcal{L}_{0,x}^m (\hat{g}_x \rho_x) \right| \, dm(x)
\]

Now since \( \int_{\mathcal{J}_x} \hat{g}_x \, d\mu_x = \int_{\mathcal{J}_x} \hat{g}_x \rho_x \, d\nu_x = 0 \), assumption (3) gives

\[
\| \mathcal{L}_{0,x}^m (\hat{g}_x \rho_x) \|_{\theta^m(x),\infty} = \left\| \mathcal{L}_{0,x}^m (\hat{g}_x \rho_x) - \int_{\mathcal{J}_x} \hat{g}_x \rho_x \, d\nu_x \cdot \rho_{\theta^m(x)} \right\|_{\theta^m(x),\infty}
\]

\[
\leq C \kappa^m \| \hat{g}_x \rho_x \|_{x,a}.
\]
Using this we can continue to get
\[ \Sigma_1 \leq C^m \int_X \| g_{\theta^m(x)} \|_{\theta^m(x),\alpha} \cdot \| \hat{g}_x \|_{x,\alpha} \, dm(x) \]
\[ \leq C^m \int_X \| g_{\theta^m(x)} \|_{\theta^m(x),\alpha} \cdot \| \hat{g}_x \|_{x,\alpha} \, dm(x). \]

Noting that
\[ \| \hat{g}_x \|_{x,\alpha} \leq \| g_x \|_{x,\alpha} + \left\| \int_{J_x} g_x \, d\mu_x \right\|_{x,\alpha} \leq 2 \| g_x \|_{x,\alpha}, \]
and since \( \| g_x \|_{x,\alpha} \in L^p(m) \) for some \( p > 2 \) we apply the Cauchy-Schwarz inequality to get
\[ \Sigma_1 \leq C^m \left( \int_X \| g_{\theta^m(x)} \|_{\theta^m(x),\alpha}^2 \, dm(x) \right)^{1/2} \left( \int_X \| \hat{g}_x \|_{x,\alpha}^2 \, dm(x) \right)^{1/2} \]
\[ \leq C^m \int_X \| g_x \|_{x,\alpha}^2 \, dm(x) \]
\[ \leq C^m \int_X \| g_x \|_{x,\alpha}^p \, dm(x). \]

Combining these two estimates we have that
\[ \text{cov}(g, g \circ T^m) \leq C^m \left( \int_X \| g_x \|_{x,\alpha}^p \, dm(x) + \| G \|_\mathcal{H} \cdot \| G \|_{L^2(m)} \right), \]
which finishes the claim. To finish the proof of Theorem 3.1 we invoke Lemma 3.8 and Theorem 3.7.

In what follows we present several examples of random dynamical systems for which Theorem 3.1 can be applied. Specifically, we give examples of systems for which assumptions (1)–(3) are known, or can be easily checked, in the literature. We also provide examples of base systems for which assumption (4) is well known. Assumption (5) will need to be checked for most systems as it requires a connection between the random fiber system as well as the system in the base, however, we shall present examples of systems under which this condition is met.

4. Random Distance Expanding Maps

In this section we give an overview of uniformly expanding random systems as they are defined by Mayer, Skorulski, and Urbański in [8]. Suppose \((X, \mathcal{B}, m, \theta)\) is a measure preserving dynamical system with an invertible and ergodic map \( \theta : X \rightarrow X \). For each \( x \in X \) we associate the compact metric space \((J_x, \rho_x)\), which has been normalized in size such that \( \text{diam}_{\rho_x}(J_x) \leq 1 \). Given a \( z \in J_x \) and \( r > 0 \) we denote the ball of radius \( r \) centered at \( z \) in \((J_x, \rho_x)\) by \( B_x(z, r) \). Define the space \( J \) by
\[ J = \bigcup_{x \in X} \{ x \} \times J_x. \]
A map $T : J \longrightarrow J$ is called an expanding random map if the mappings $T_x : J_x \longrightarrow J_{\vartheta(x)}$ are continuous open surjections and if there exists a function $\eta : X \longrightarrow \mathbb{R}$, $x \longmapsto \eta_x$, and $\xi > 0$ such that the following hold:

- **Uniform Openness:** $T_x(B_x(z, \eta_z)) \supseteq B_{\vartheta(x)}(T_x(z, \xi))$ for every $(x, z) \in J$.
- **Measurably Expanding:** There exists a measurable function $\gamma : X \longrightarrow (1, \infty)$, $x \longmapsto \gamma_x$, such that for $m$-a.e. $x \in X$
  
  \[ \vartheta_{\vartheta(x)}(T_x(z_1), T_x(z_2)) \geq \gamma_x \vartheta_x(z_1, z_2) \] 
  
  whenever $\vartheta_x(z_1, z_2) < \eta_x$, $z_1, z_2 \in J_x$.
- **Measurability of the Degree:** The map $x \longmapsto \deg(T_x) := \sup_{y \in J_{\vartheta(x)}} \#T_x^{-1}(\{y\})$ is measurable.
- **Topological Exactness:** There exists a measurable function $x \longmapsto n_\xi(x)$ such that for almost every $x \in X$ and every $z \in J_x$
  
  \[ T_x^{n_\xi(x)}(B_x(z, \xi)) = J_{\vartheta^{n_\xi(x)}(x)} \]

Remark 4.1. Note that the measurably expanding condition implies that $(T_x|_{B_x(z, \eta_z)})$ is injective for each $(x, z) \in J$. Furthermore, given that the spaces $J_x$ are compact, we have that deg$(T_x)$ is finite for each $x \in X$. Considering additionally the uniform openness condition we see that for every $(x, z) \in J$ there exists a unique continuous inverse branch $T_x^{-1} : B_{\vartheta(x)}(T_x(z), \xi) \longrightarrow B_x(z, \eta_z)$ of $T_x$ which sends $T_x(z)$ to $z$.

The map $T$ is called uniformly expanding if it satisfies the following additional properties:

1. $\gamma_* := \inf_{x \in X} \gamma_x > 1$,
2. $\deg(T) := \sup_{x \in X} \deg(T_x) < \infty$,
3. $n_\xi_* := \sup_{x \in X} n_\xi(x) < \infty$.

In addition to the various fiberwise and global Banach spaces defined in Section 2.4, we will find the following definition useful.

First, recall that $u \in \mathcal{H}_\alpha(J)$ provided $u_x \in \mathcal{H}_\alpha(J_x)$ and there is a measurable function $H : X \longrightarrow [1, \infty)$, $x \longmapsto H_x$, such that $\log H \in L^1(m)$ and $v_{x, \alpha}(u_x) \leq H_x$ for $m$-a.e. $x \in X$. For fixed $H$ we let $\mathcal{H}_\alpha(J, H)$ be all the functions $g \in \mathcal{H}_\alpha(J)$ such that $v_{x, \alpha}(g_x) \leq H_x$ for $m$-a.e. $x \in X$ and we say that such a function is $(H, \alpha)$–Hölder continuous over $J$. This allows us to write

\[ \mathcal{H}_\alpha(J) = \bigcup_{H \geq 1} \mathcal{H}_\alpha(J, H). \]

Now for each $H$ and each $\varphi \in \mathcal{H}_\alpha(J, H)$ set

\[ Q_x := Q_x(H) = \sum_{j=1}^{\infty} H_{\theta^{-j}(x)}(\gamma^j_{\theta^{-j}(x)})^{-\alpha}. \]

The following lemma tells us this function is measurable and provides a necessary, though technical, bound.
Lemma 4.2 (Lemma 2.3 of \[8\]). The function $x \mapsto Q_x$ is measurable and $m$-a.e. finite. Moreover, for every $\varphi \in \mathcal{H}_o(J, H)$,
\begin{equation}
|S_n \varphi_x(T_{-n} T_y w_1)) - S_n \varphi_x(T_{-n} T_y w_2))| \leq Q_{\theta^n(x)}(w_1, w_2) \tag{4.2}
\end{equation}
for all $n \geq 1$, a.e. $x \in X$, every $z \in J_x$, and all $w_1, w_2 \in B_{\theta^n(x)}(T^{n}_x(z), \xi)$.

For $\tilde{H} \geq 0$ we denote by $\mathcal{H}_o^{\tilde{H}}(J, \tilde{H})$ to be the space of all functions $\varphi \in \mathcal{H}_o^{\tilde{H}}(J)$ such that
\begin{equation}
\sup_{x \in X} v_{x, \alpha}(\varphi_x) \leq H_x \leq \tilde{H}.
\end{equation}
If $T$ is uniformly expanding then we are able to take $Q_x$ defined in (4.1) as some uniform constant $Q_x$, depending only upon $\varphi$ and $\tilde{H}$ and no longer on $x \in X$, given by
\begin{equation}
Q := Q_{\varphi} := \bar{H} \sum_{j=1}^{\infty} \gamma^{-\alpha j} = \tilde{H} \gamma^{-\alpha}.
\end{equation}
Thus, for uniformly expanding systems, we can rewrite Lemma 4.2 using the following lemma of \[8\].

Lemma 4.3 (Lemma 3.31 of \[8\]). For every $\varphi \in \mathcal{H}_o^{\tilde{H}}(J, \tilde{H})$,
\begin{equation}
|S_n \varphi_x(T_{-n} T_y w_1)) - S_n \varphi_x(T_{-n} T_y w_2))| \leq Q_{\varphi} \theta^n(x)(w_1, w_2) \tag{4.3}
\end{equation}
for all $n \geq 1$, a.e. $x \in X$, every $z \in J_x$, and all $w_1, w_2 \in B_{\theta^n(x)}(T^{n}_x(z), \xi)$.

Now we discuss the established thermodynamic formalism for distance expanding random systems. We begin by defining the transfer operator.

Definition 4.4. Fix $\varphi \in \mathcal{H}_o(J)$ and for each $x \in X$ define the operator $L_x := L_{\varphi, x} : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta^n(x)})$ by
\begin{equation}
L_x(u_x)(w) = \sum_{z \in T^{-1}_x(w)} u_x(z)e^{\varphi_x(z)}.
\end{equation}
Clearly, $L_x$ is a positive bounded linear operator with norm bounded by
\begin{equation}
\|L_x\|_{x, \infty} \leq \deg(T_x)e^{\|\varphi\|_{\infty}}.
\end{equation}
Iterating the transfer operator, for each $n \in \mathbb{N}$ we see that $L^n_x : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta^n(x)})$ is given by
\begin{equation}
L^n_x(u_x)(w) = \sum_{z \in T^{-n}_x(w)} u_x(z) e^{S_n \varphi_x(z)}, \quad w \in \mathcal{J}_{\theta^n(x)}(x).
\end{equation}
We can also define the global operator $L : \mathcal{C}(J) \to \mathcal{C}(J)$ by
\begin{equation}
(Lu)_x := L_{\theta^{-1}(x)}u_{\theta^{-1}(x)}.
\end{equation}
Denote by $L^*_x$ the dual operator $L^*_x : \mathcal{C}^*(\mathcal{J}_{\theta^n(x)}) \to \mathcal{C}^*(\mathcal{J}_x)$, where $\mathcal{C}^*(\mathcal{J}_x)$ denotes the dual space of $\mathcal{C}(\mathcal{J}_x)$. 

Theorem 4.5 (Theorem 3.1 (1) of [8]). There exists a unique family of probability measures \( \nu_x \in \mathcal{P}(\mathcal{J}_x) \) such that

\[
\mathcal{L}_x^m \nu_{\theta(x)}(x) = \lambda_x \nu_x \quad \text{for } m - a.e. \ x \in X,
\]

where

\[
\lambda_x := \mathcal{L}_x^*(\nu_{\theta(x)})(1) = \nu_{\theta(x)}(\mathcal{L}_x 1).
\]

Define the normalized operator \( \mathcal{L}_{0,x} : \mathcal{C}(\mathcal{J}_x) \rightarrow \mathcal{C}(\mathcal{J}_0(x)) \) for each \( x \in X \) by

\[
\mathcal{L}_{0,x} = \lambda_x^{-1} \mathcal{L}_x.
\]

where \( \lambda_x \) and \( \nu_x \) are such that \( \mathcal{L}_x^* \nu_{\theta(x)} = \lambda_x \nu_x \), where \( \rho_x \) is such that \( \mathcal{L}_{0,x}(\rho_x) = \rho_{\theta(x)} \). Let \( \mu_x = \rho_x \nu_x \).

For \( p > 2 \) fix a function \( g \in \mathcal{H}^p_\alpha(\mathcal{J}) \). Then for each \( r \in \mathbb{R} \) and \( x \in X \), define the perturbed operator \( \mathcal{L}_{r,x} : \mathcal{C}(\mathcal{J}_x) \rightarrow \mathcal{C}(\mathcal{J}_{\theta(x)}) \) by

\[
\mathcal{L}_{r,x}u_x = \mathcal{L}_{0,x}(e^{irg} \cdot u_x),
\]

and the global perturbed operator by

\[
(\mathcal{L}_r u)_x = \mathcal{L}_{r,\theta^{-1}(x)}u_{\theta^{-1}(x)}.
\]

Remark 4.6. Note that by our assumption that the function \( g : \mathcal{J} \rightarrow \mathbb{R} \) is bounded and Hölder we have that the process \( g \circ T \) is in \( L^p \) for any \( p > 2 \).

Now let \( r_{n-1}, \ldots, r_0 \in \mathbb{R} \). Then

\[
\int_{\mathcal{J}_{\theta^n(x)}} \mathcal{L}_{r_{n-1},\theta^{n-1}(x)} \circ \cdots \circ \mathcal{L}_{r_0,\theta(x)} \nu_{\theta^n(x)} \ d\nu_{\theta^n(x)} = \int_{\mathcal{J}_{\theta^n(x)}} \mathcal{L}_{0,x}^n(e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T^j_x} \cdot \rho_x) \ d\nu_{\theta^n(x)}
\]

\[
= \int_{\mathcal{J}_x} e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T^j_x} \cdot \rho_x \ d\nu_x
\]

\[
= \int_{\mathcal{J}_x} e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T^j_x} \ d\mu_x
\]

Integrating with respect to \( \nu \) then gives

\[
\nu \left( \mathcal{L}_{r_{n-1}} \circ \cdots \circ \mathcal{L}_{r_0}(\rho) \right) = \int_X \int_{\mathcal{J}_{\theta^n(x)}} \mathcal{L}_{r_{n-1},\theta^{n-1}(x)} \circ \cdots \circ \mathcal{L}_{r_0,\theta(x)}(\rho_x) \ d\nu_{\theta^n(x)} \ dm(x)
\]

\[
= \int_X \int_{\mathcal{J}_x} e^{i \sum_{j=0}^{n-1} r_j g_{\theta^j(x)} \circ T^j_x} \ d\mu_x \ dm(x)
\]

\[
= \mathcal{E}_\mu \left( e^{i \sum_{j=0}^{n-1} r_j g \circ T^j} \right).
\]

For fixed \( r \in \mathbb{R} \) and any \( n \in \mathbb{N} \) we see that we can write the iterates of \( \mathcal{L}_r \) as

\[
\mathcal{L}^n_{r,x}(u_x) = \mathcal{L}_{0,x}^n(e^{irS_x g} \cdot u_x).
\]

Now we endeavor to show that for sufficiently small values of \(|r|\), the operator \( \mathcal{L}_r \) is bounded in the \( \alpha \)-norm and the sup norm. We start by applying the following lemma.
Lemma 4.7 (Lemma 3.8 of [8]). For all \( w_1, w_2 \in \mathcal{J}_x \) and \( n \geq 1 \)
\[
\frac{\mathcal{L}^n_{0, \theta^{-n}(x)} \mathbb{1}(w_1)}{\mathcal{L}^n_{0, \theta^{-n}(x)} \mathbb{1}(w_2)} = \frac{\mathcal{L}^n_{\theta^{-n}(x)} \mathbb{1}(w_1)}{\mathcal{L}^n_{\theta^{-n}(x)} \mathbb{1}(w_2)} \leq C_\varphi(x),
\]
where \( C_\varphi \) is given by
\[
C_\varphi(x) := e^{Q_\varphi^S(x)} \deg(T_{\theta^{-j}(x)}^j) \max \left\{ \exp(2 \left\| S_k \varphi_{\theta^{-k}(x)} \right\|_{x, \infty}) : 0 \leq k \leq j \right\} \geq 1.
\]
If in addition we have that \( \varrho_x(w_1, w_2) \leq \xi \), then
\[
\frac{\mathcal{L}^n_{0, \theta^{-n}(x)} \mathbb{1}(w_1)}{\mathcal{L}^n_{0, \theta^{-n}(x)} \mathbb{1}(w_2)} \leq \exp(Q_x \varrho_x^\alpha(w_1, w_2)).
\]
Moreover,
\[
\frac{1}{C_\varphi(x)} \leq \mathcal{L}^n_{0, \theta^{-n}(x)} \mathbb{1}(w) \leq C_\varphi(x)
\]
for every \( w \in \mathcal{J}_x \) and \( n \geq 1 \).

However, we note that for uniformly expanding systems, in view of (4.3), there is some \( C > 1 \) such that for every \( x \in X \) and \( n \in \mathbb{N} \)
\[
(4.5) \quad C^{-1} \leq \mathcal{L}^n_{0, x} \mathbb{1}_x \leq C.
\]
In light of (4.5), then for \( r \in \mathbb{R} \) and \( u \in \mathcal{H}_x^{r, \alpha}(\mathcal{J}) \) we can estimate
\[
\left\| \mathcal{L}^n_{r, x} u_x \right\|_{\theta^n(x), \infty} = \left\| \mathcal{L}^n_{0, x} (e^{i r S_n} u_x) \right\|_{\theta^n(x), \infty} \leq \left\| \mathcal{L}^n_{0, x} (|u_x|) \right\|_{\theta^n(x), \infty}
\]
\[
\leq \left\| u_x \right\|_{x, \infty} \cdot \left\| \mathcal{L}^n_{0, x} \mathbb{1}_x \right\|_{\theta^n(x), \infty} \leq C \left\| u_x \right\|_{x, \infty}.
\]
Now to show that \( \left\| \mathcal{L}^n_{r, x} u_x \right\|_{\theta^n(x), \alpha} \leq C \left\| u_x \right\|_{x, \alpha} \) we must first show that this inequality holds on positive cones of Hölder functions. For \( s \geq 1 \) and \( x \in X \) let
\[
\Lambda_x^s = \{ h_x \in C(\mathcal{J}_x) : h_x \geq 0, v_x(h_x) = 1, \text{ and } h_x(w_1) \leq \exp(s Q_x \varrho_x^\alpha(w_1, w_2)) \cdot h_x(w_2) \}
\]
for all \( w_1, w_2 \in \mathcal{J}_x \) with \( \varrho_x(w_1, w_2) \leq \xi \).

The first of the following two lemmas shows that \( \Lambda_x^s \subseteq \mathcal{H}_x(\mathcal{J}_x) \), and moreover, provides a bound on the variation of such functions, while the second lemma is a sort of converse to the first.

Lemma 4.8 (Lemma 3.11 of [8]). If \( h_x \geq 0 \) and for all \( w_1, w_2 \in \mathcal{J}_x \) with \( \varrho_x(w_1, w_2) \leq \xi \) we have
\[
h_x(w_1) \leq e^{s Q_x e^{\alpha}(w_1, w_2)} h_x(w_2),
\]
then
\[
v_{x, \alpha}(h_x) \leq s Q_x(\exp(s Q_x \xi^\alpha)) \xi^\alpha \left\| h_x \right\|_{x, \infty}.
\]
Lemma 4.9 (Lemma 3.13 of [8]). If $u_x \in \mathcal{H}_\alpha(J_x)$ and $u_x \geq 0$, then the function defined by

$$h_x := \frac{u_x + v_{x,\alpha}(u_x)/Q_x}{v_x(u_x) + v_{x,\alpha}(u_x)/Q_x} \in \Lambda_x^1.$$

The next lemma of [8] establishes the invariance of the cones $\Lambda_x^1$ with respect to the family of normalized operators $L_{0,x}$.

Lemma 4.10 (Lemma 3.14 of [8]). Let $h_x \in \Lambda_x^1$. Then for every $n \geq 1$ and $w_1, w_2 \in J_x$ with $\varrho(w_1, w_2) \leq \xi$ we have

$$\frac{L^n_{0,x}h_x(w_1)}{L^n_{0,x}h_x(w_2)} \leq \exp(sQ_{\varrho^n(x)}\varrho^\alpha(w_1, w_2)).$$

Consequently, $L^n_{0,x}(\Lambda_x^1) \subseteq \Lambda_x^1$ for a.e. $x \in X$ and all $n \geq 1$.

Now we wish to estimate the values $\|L^n_{r,x}u_x\|_{\varrho^n(x),\alpha}$.

Lemma 4.11. There exists $C \geq 1$ such that for all $n \in \mathbb{N}$, all $r \in \mathbb{R}$, m-a.e. $x \in X$, and all $u \in \mathcal{H}_\alpha(J)$ we have

$$\|L^n_{r,x}u_x\|_{\varrho^n(x),\alpha} \leq C \|u_x\|_{x,\alpha}.$$

Proof. In comparing $u_x$ and $h_x$ we note that

$$v_{x,\alpha}(h_x) = \frac{Q_x}{Q_x v_x(u_x) + v_{x,\alpha}(u_x)} v_{x,\alpha}(u_x), \quad \text{and}$$

$$\|h_x\|_{x,\infty} = \left| \frac{Q_x}{Q_x v_x(u_x) + v_{x,\alpha}(u_x)} \right| \|u_x\|_{x,\infty} + \left| \frac{v_{x,\alpha}(u_x)}{Q_x v_x(u_x) + v_{x,\alpha}(u_x)} \right|.$$

Now for $u_x \in \mathcal{H}_\alpha(J_x)$ we can write $u_x = u_x^+ - u_x^-$ where $u_x^+, u_x^-$ are both in $\mathcal{H}_\alpha(J_x)$ and nonnegative. Letting

$$h_x^+ = \frac{u_x^+ + v_{x,\alpha}(u_x^+)/Q_x}{v_x(u_x^+)/v_{x,\alpha}(u_x^+)/Q_x} \quad \text{and} \quad h_x^- = \frac{u_x^- + v_{x,\alpha}(u_x^-)/Q_x}{v_x(u_x^-)/v_{x,\alpha}(u_x^-)/Q_x},$$

then for $h_x^+$ we have the following estimate

$$v_{x,\alpha}(L^n_{0,x}(u_x^+)) = \frac{Q_x v_x(u_x^+)/Q_x + v_{x,\alpha}(u_x^+)/Q_x}{Q_x} \left( v_{x,\alpha}(L^n_{0,x}h_x^+) - \frac{v_{x,\alpha}(u_x^+)/Q_x v_x(u_x^+) v_{x,\alpha}(L^n_{0,x}1_x) \right) \leq Q_x (\exp(Q_x\zeta^\alpha)) \xi^\alpha \frac{Q_x v_x(u_x^+)/Q_x + v_{x,\alpha}(u_x^+)/Q_x}{Q_x} \left( \|L^n_{0,x}h_x^+\|_{\varrho^n(x),\infty} + \|L^n_{0,x}1_x\|_{\varrho^n(x),\infty} \right) \leq \xi^\alpha \exp(Q_x\zeta^\alpha) (Q_x v_x(u_x^+) + v_{x,\alpha}(u_x^+)) \left( \|L^n_{0,x}h_x^+\|_{\varrho^n(x),\infty} + C \right) \leq \max(Q_x, 1) \xi^\alpha Q_x \exp(Q_x\zeta^\alpha) \|L^n_{0,x}u_x^+\|_{\varrho^n(x),\infty} + C \|u_x^+\|_{x,\alpha}.$$. 
Again, we obtain a similar estimate for $v_{\theta^n(x),\alpha}(L^n_{0,x}u^n_x)$. Now if $u_x \in H^\alpha(\mathcal{J})$ then so are $u^n_x, u^\perp_x$ and furthermore $\|u^n_x\|_{x,\infty}, \|u^n_x\|_{x,\infty} \leq \|u^n_x\|_{x,\infty}$ and $v_{x,\alpha}(u^+_x), v_{x,\alpha}(u^\perp_x) \leq v_{x,\alpha}(u_x)$. If $T$ is uniformly expanding then, using (4.3), we are able to simplify the above estimate to

$$v_{\theta^n(x),\alpha}(L^n_{0,x}(u^+_x)) \leq \max(Q_u,1)\xi^n \exp(Q_u\xi^n) \|L^n_{0,x}u^n_x\|_{\theta^n(\mathcal{J}),\infty} + M \|u^n_x\|_{x,\alpha} \leq C \left( \|L^n_{0,x}u^n_x\|_{\theta^n(\mathcal{J}),\infty} + \|u^n_x\|_{x,\alpha} \right).$$

Again, we obtain a similar estimate for $v_{\theta^n(x),\alpha}(L^n_{0,x}u^n_x)$. Combining (4.8) and (4.5), we see

$$v_{\theta^n(x),\alpha}(L^n_{0,x}(u_x)) \leq v_{\theta^n(x),\alpha}(L^n_{0,x}u^n_x) + v_{\theta^n(x),\alpha}(L^n_{0,x}u^n_x) \leq C \left( \|u^n_x\|_{x,\infty} + \|u^n_x\|_{x,\alpha} + \|u^n_x\|_{x,\infty} + \|u^n_x\|_{x,\alpha} \right) \leq C \|u^n_x\|_{x,\alpha}.$$

Now we are ready to calculate $\|L^n_{r,x}u^n_x\|_{\theta^n(x),\alpha}$. Combining the bounds in (4.6) and (4.9) we see

$$\|L^n_{r,x}u^n_x\|_{\theta^n(x),\alpha} = \|L^n_{0,x}(e^{irS_n g_x} \cdot u^n_x)\|_{\theta^n(x),\alpha} \leq C \|u^n_x e^{irS_n g_x}\|_{x,\alpha}.$$

As $\mathcal{H}^\alpha(\mathcal{J})$ is a Banach algebra we can write the last inequality as

$$\|L^n_{r,x}u^n_x\|_{\theta^n(x),\alpha} \leq C \|u^n_x\|_{x,\alpha} \|e^{irS_n g_x}\|_{x,\alpha}.$$

It suffices to estimate $\|e^{irS_n g_x}\|_{x,\alpha}$, but as $\|e^{irS_n g_x}\|_{x,\infty} = 1$, we need only estimate $v_{x,\alpha}(e^{irS_n g_x})$. Since $|e^{iz_1} - e^{iz_2}| \leq |z_1 - z_2|$, for all $z_1, z_2 \in \mathbb{R}$, we see that $v_{x,\alpha}(e^{irS_n g_x}) \leq v_{x,\alpha}(r S_n g_x)$. Thus it now suffices to estimate $v_{x,\alpha}(r S_n g_x)$. Lemma 4.3 shows that

$$v_{x,\alpha}(e^{irS_n g_x}) \leq v_{x,\alpha}(r S_n g_x) \leq \|r\| Q.$$

In particular, we have the estimate

$$\|L^n_{r,x}u^n_x\|_{\theta^n(x),\alpha} \leq C \|u^n_x\|_{x,\alpha} (1 + \|r\| Q),$$

which finishes the proof. \qed

Now we define the fiberwise operator $Q_x : C(\mathcal{J}_x) \rightarrow C(\mathcal{J}_{\theta(x)})$ by

$$Q_x(u_x) = \int_{\mathcal{J}_x} u_x d\nu_x \cdot \rho_{\theta(x)}.$$

For each $n \in \mathbb{N}$ we can write the iterates $Q^n_x : C(\mathcal{J}_x) \rightarrow C(\mathcal{J}_{\theta^n(x)})$ of $Q_x$ as

$$Q^n_x(u_x) = \int_{\mathcal{J}_x} u_x d\nu_x \cdot \rho_{\theta^n(x)}.$$

Thus we see that for each $k \in \mathbb{N}$ and $x \in X$

$$\left\|((L^k_0 - Q^k)(u))^\lambda\right\|_{x,\infty} = \left\|(L^k_0 - Q^k)(u_{\theta^{-k}(x)})\right\|_{x,\infty}$$

$$= \left\|L^k_0(u_{\theta^{-k}(x)}) - \int_{\mathcal{J}_{\theta^{-k}(x)}} u_{\theta^{-k}(x)} d\mu_{\theta^{-k}(x)} \cdot 1_x\right\|_{x,\infty}.$$
By the proof of Lemma 3.18 of [8], in the case of uniformly expanding random maps, we see that
\[
\|\left(L_0^k - Q^k\right)(u)\|_{x,\infty} \leq \left(\nu_x(u_x) + 2\frac{\nu_{x,\alpha}(u_x)}{Q_x}\right) C\kappa^k \\
\leq \left(\|u_x\|_{x,\infty} + 2\nu_{x,\alpha}(u_x)\right) C\kappa^k \\
\leq 2\|u_x\|_{x,\alpha} C\kappa^k
\]
for some positive constant \(\kappa < 1\).

Thus we have shown that for uniformly expanding systems conditions (1)-(3) of Theorem 3.1 hold. In order for the remaining two conditions to hold we will need to require more structure. In the next section we discuss a class of systems, first described by Denker and Gordin in [4], which fit within the framework of the uniformly expanding systems which we have just discussed. In the same manner as in [8], we shall call refer to these systems as DG and DG*-systems.

5. DG*-Systems

In [4] Denker and Gordin first established the existence and uniqueness of conformal Gibbs measures for DG-systems. Then in [8] Mayer, Skorulski, and Urbański were able to cast these systems as uniformly expanding random systems, which they called DG*-systems, meaning that the full thermodynamic formalism they developed there applies to these DG*-systems. In particular, we see that the results of Section 4 apply, and in particular, the spectral gap property holds for these systems. Thus we have only to check conditions (4) and (5) of Theorem 3.1 hold. However, condition (4) has been shown to hold. In this section we introduce these systems and show that an ASIP holds for such systems. We begin with a definition.

**Definition 5.1.** Suppose \((X_0, d_{X_0})\) and \((Z_0, d_{Z_0})\) are compact metric spaces and that \(\theta_0 : X_0 \to X_0\) and \(T_0 : Z_0 \to Z_0\) are open topologically exact distance expanding mappings in the sense of [11]. Assume that \(T_0\) is a skew product over \(Z_0\), that is, for each \(x \in X_0\) there exists a compact metric space \(J_x\) such that \(Z_0 = \bigcup_{x \in X_0} \{x\} \times J_x\). Further assume that the map \(\theta_0\) is Lipschitz, i.e. there exists \(L > 0\) such that
\[
d_{X_0}(\theta_0(x), \theta_0(x')) \leq L d_{X_0}(x, x')
\]
for all \(x, x' \in X_0\), and that there exists \(\xi, \xi_1 > 0\) such that for all \(x, x' \in X_0\) with \(d_{X_0}(x, x') < \xi_1\) there exist \(y \in J_x\) and \(y' \in J_{x'}\) such that
\[
d_{Z_0}((x, y), (x', y')) < \xi.
\]
Finally we assume that the projection \(\pi : Z_0 \to X_0\) onto the first coordinate given by
\[
\pi(x, y) = x
\]
is an open mapping and that the following diagram commutes:
We call the system \((T_0, Z_0, \theta_0, X_0)\) a DG–system.

In what follows we will assume that there is some metric space \((Y, d_Y)\) such that \(J_x \subseteq Y\) for each \(x \in X\). In this case we see that \(Z_0 \subseteq X \times Y\) and we may take \(d_{Z_0}\) to be the natural product metric given by

\[
d_{Z_0}((x, y), (x', y')) = d_{X_0}(x, x') + d_Y(y, y').
\]

Remark 5.2. Three things to notice about the definition of DG–systems presented above.

- For each \(x \in X_0\) we have that \(T_0(\{x\} \times J_x) \subseteq \{\theta_0(x)\} \times J_{\theta_0(x)}\), giving rise to the map \(T_x : J_x \to J_{\theta_0(x)}\).
- Since \(T_0\) is distance expanding in the sense of \([11]\) the conditions of uniform openness, measurably expanding, measurability of the degree, and topological exactness from the definition of random distance expanding mappings in Section \([11]\) all hold for constants \(\gamma_x \geq \gamma > 1, \deg(T_x) \leq N_1 < \infty\), and \(n_\xi = n_\xi(x)\) independent of \(x\).
- The function \(\theta_0 : X_0 \to X_0\) need not be invertible, meaning that we are not quite able to apply the theory of uniformly expanding random mappings which we described earlier.

In order to rectify the complications involving \(\theta_0\) we define DG*–systems by turning to Rokhlin’s natural extension, i.e. the projective limit, \(\theta : X \to X\) of \(\theta_0 : X_0 \to X_0\).

Definition 5.3. Assume that we are given a DG–system \((T_0, Z_0, \theta_0, X_0)\) as defined above.

We further assume that the space \(X_0\) comes coupled with a Borel probability \(\theta_0\)–invariant ergodic measure \(m_0\) and a Hölder continuous potential \(\varphi : Z_0 \to \mathbb{R}\). Define the space

\[
X = \{(x_n)_{n \leq 0} : \theta_0(x_n) = x_{n+1} \text{ for all } n \leq -1\}
\]

and the map \(\theta : X \to X\) by

\[
\theta((x_n)_{n \leq 0}) = (\theta_0(x_n))_{n \leq 0}.
\]

In this case we have that \(\theta : X \to X\) is invertible and that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\theta} & X \\
\downarrow p & & \downarrow p \\
X_0 & \xrightarrow{\theta_0} & X_0
\end{array}
\]

commutes where \(p : X \to X_0\) is defined by

\[
p((x_n)_{n \leq 0}) = x_0.
\]
Now since $m_0$ is a $\theta_0$–invariant ergodic measure, there exists a unique $\theta$–invariant probability measure $m$ on $X$ such that $m \circ \pi^{-1} = m_0$. Define the set

$$J = \bigcup_{x \in X} \{x\} \times J_{x_0}$$

and the map $T : J \to J$ by

$$T(x, y) = (\theta(x), T_{x_0}(y)).$$

The metrics $d_{X_0}$ and $d_{Z_0}$ extend naturally to metrics $d_X$ and $d_J$ on $X$ and $J$, which are defined by

$$d_X(x, x') = \sum_{n=0}^{\infty} 2^{-n} d_{X_0}(x, x') \circ \theta^{-n}, \quad x, x' \in X$$

and

$$d_J((x, y), (x', y')) = d_Y(y, y') + d_X(x, x'), \quad (x, y), (x', y') \in J$$

respectively. We let $B_J((x, w), r)$ denote the ball of radius $r > 0$ centered at the point $(x, w) \in J$ with respect to the metric $d_J$. In what follows we will assume that $m = m_\psi$ is a $\theta$–invariant Gibbs measure for some continuous Hölder potential $\psi$ on $X$, having nothing to do with our (fiberwise) potential $\varphi$ or density function $\rho$. The system $(T, J, m, \theta, X)$ is then called a DG*–system.

**Remark 5.4.** Note that because DG*–systems are uniformly expanding random systems, we have that there exists $C > 1$ such that

- $C^{-1} \leq \rho_x \leq C$ for all $x \in X$,
- $C^{-1} \leq \mathcal{L}_0 \circ \theta^{-n}(x) \leq C$ for all $x \in X$ and $n \in \mathbb{N}$.

Notice that since the base dynamical system $(X, \theta, m)$ is distance expanding in the sense of [11] and that $m$ is an invariant Gibbs measure, the base system exhibits an exponential decay of correlations for functions $F$ and $G$ so long as one of the two is Hölder continuous on $X$ and the other is integrable with respect to $m$. More precisely, we have the following, which is a consequence of Theorem 5.4.9 of [11].

**Theorem 5.5.** Let $(T, J, m, \theta, X)$ be a DG*–system such that $m$ is an invariant Gibbs measure. Then there exists $C \geq 1$ and $\kappa < 1$ such that for all $F \in \mathbb{H}_\beta$ and all $G \in L^1(m)$ we have

$$\left| \int_X (G \circ \theta^{-n}) \cdot F \, dm - \int_X G \, dm \cdot \int_X F \, dm \right| \leq C \kappa^n \|F\|_{\mathbb{H}} \|G\|_{L^1(m)}$$

In other words, condition (4) of Theorem 3.1 is satisfied for DG*–systems.

The following theorem addresses condition (5) of Theorem 3.1 and is due to Denker and Gordin.
Theorem 5.6 (Theorem 2.10 of [4]). Let \((T_0, Z_0, \theta_0, X_0)\) be a DG–system with random Gibbs measures \((\nu_x)_{x \in X_0}\). If \(f : Z_0 \rightarrow \mathbb{R}\) is \((D, \alpha)\)–Hölder continuous, for any \(\alpha \in (0, 1]\) and \(D > 0\), then there exists \(\beta_\alpha \in (0, 1]\) and \(C_\alpha > 0\) such that the function

\[
x \mapsto \int_{\mathcal{J}_x} f_x(z) \, d\nu_x(z)
\]

is \((C_\alpha D, \beta_\alpha)\)–Hölder continuous on \(X_0\). In particular the function

\[
x \mapsto \lambda_x
\]

is \(\beta_\alpha\)–Hölder continuous on \(X_0\).

Remark 5.7. Note that the same theorem applies for DG*–systems. Theorem 8.12 of [8] reproves the special case of the previous theorem for the function \(x \mapsto \lambda_x\) in the case of DG*–systems.

We shall now prove the following.

Proposition 5.8. Suppose that \(u \in \mathcal{H}_\tau(\mathcal{J})\), for \(\tau \in (0, 1]\), and that there is \(C > 1\) such that

\[
C^{-1} \leq u_x \leq C
\]

for all \(x \in X\). Given \(g \in \mathcal{H}_\alpha^p(\mathcal{J})\) and \(\varphi \in \mathcal{H}_\alpha(\mathcal{J})\), define the transfer operators as before. Then there exists \(\beta \in (0, 1]\), depending only on \(\alpha\) and \(\tau\), such that for each \(n \in \mathbb{N}\) and each \(r_0, \ldots, r_{n-1} \in \mathbb{R}\) with each \(|r_j| < \varepsilon_0\), for some \(\varepsilon_0 > 0\), the function

\[
x \mapsto \int_{\mathcal{J}_x} \mathcal{L}_{r_{n-1},\theta^{-1}(z)} \circ \cdots \circ \mathcal{L}_{r_0,\theta^{-n}(z)}(u_{\theta^{-n}(x)}) \, d\nu_x
\]

is in \(\mathbb{H}_\beta\). Moreover, we have that

\[
\left\| \int_{\mathcal{J}_x} \mathcal{L}_{r_{n-1},\theta^{-1}(z)} \circ \cdots \circ \mathcal{L}_{r_0,\theta^{-n}(z)}(u_{\theta^{-n}(x)}) \, d\nu_x \right\|_H \leq C
\]

independent of the choice of \(n\) or the \(r_j\).

Proof of Proposition 5.8. Fix \(n \in \mathbb{N}\) and let \((x, w) \in \mathcal{J}\). Then for each \(z \in T_x^{-n}(w)\) there is a unique continuous inverse branch

\[
T_{\theta^{-n}(x),z} : B_{\mathcal{J}}((x, w), \xi) \rightarrow B_{\mathcal{J}}((\theta^{-n}(x), z), \xi)
\]

which sends the point \((x, w)\) to \((\theta^{-n}(x), z)\). Similarly, for \((x', w') \in B_{\mathcal{J}}((x, w), \xi)\), there is a unique continuous inverse branch

\[
T_{\theta^{-n}(x'),z'} : B_{\mathcal{J}}((x, w), \xi) \rightarrow B_{\mathcal{J}}((\theta^{-n}(x), z), \xi)
\]

which sends the point \((x', w')\) to \((\theta^{-n}(x'), z') \in B_{\mathcal{J}}((\theta^{-n}(x), z), \xi)\). Thus for each \(z \in T_x^{-n}(w)\) there is a unique and bijectively defined \(z' \in T_x^{-n}(w')\) such that \(z' \in B_{\mathcal{J}}((\theta^{-n}(x), z), \xi)\).
Consequently, we may re-index the sum in the definition of the transfer operator allowing us to write
\[
\mathcal{L}^n_{x'}(u_{\theta^{-n}(x')})(w') = \sum_{z' \in T^n_{x'}(w')} e^{S_n\varphi_{\theta^{-n}(x')}(z')} u_{\theta^{-n}(x')}(z')
\]
\[
= \sum_{z \in T^n_x(w)} e^{S_n\varphi_{\theta^{-n}(x')}(z')} u_{\theta^{-n}(x')}(z').
\]
(5.1)

For \( \zeta \in (0, 1] \) let \( \beta_\zeta \) be the number coming from Theorem 5.6. In order to prove Proposition 5.8 we wish to employ Theorem 5.6 thus it suffices to show that
\[\mathcal{L}_{r_{n-1}} \circ \cdots \circ \mathcal{L}_{r_0}(u) \in \mathcal{H}_\zeta(\mathcal{J}).\]
for some \( \zeta \in (0, 1] \). To that end, we will first prove two lemmas, however we begin with an observation concerning the following definition. For each \( n \in \mathbb{N} \) and \( r_0, \ldots, r_{n-1} \in \mathbb{R} \), set
\[
\mathcal{S}_n h_x(z) := \mathcal{S}_{n,r_0,\ldots,r_{n-1}} h_x(z) := \sum_{j=0}^{n-1} r_j \cdot h_{\theta^j(x)} \circ T^j_x(z)
\]
for \( h : \mathcal{J} \rightarrow \mathbb{R} \). Now for \( h \in \mathcal{H}_\zeta(\mathcal{J}) \), for any \( \zeta \in (0, 1] \), we have that
\[
|h(T^k(\theta^{-n}(x)), z)) - h(T^k(\theta^{-n}(x'), z'))| \leq C d^h_{\zeta}(T^k(\theta^{-n}(x), z), T^k(\theta^{-n}(x'), z'))
\]
\[
\leq C d^h_{\zeta}((x, w), (x', w')) \cdot \gamma^{-\zeta(n-k)}.
\]
Hence we have
\[
\left| \mathcal{S}_n h_{\theta^{-n}(x)}(z) - \mathcal{S}_n h_{\theta^{-n}(x')}(z') \right| \leq C d^h_{\zeta}((x, w), (x', w')) \cdot \sum_{k=0}^{n-1} \gamma^{-\zeta(n-k)}
\]
(5.2)
and similarly we have
\[
\left| \mathcal{S}_n h_{\theta^{-n}(x)}(z) - \mathcal{S}_n h_{\theta^{-n}(x')}(z') \right| \leq \frac{C}{1 - \gamma^{-\zeta}} \cdot d^h_{\zeta}((x, w), (x', w')).
\]
(5.3)
We now wish to prove the first of two lemmas.

**Lemma 5.9.** The function \( \mathcal{L}^n \mathbb{1} \) is \( \alpha \)-Hölder on \( \mathcal{J} \).

**Proof.** To see this we calculate
\[
\left| \mathcal{L}^n_{\theta^{-n}(x)} \mathbb{1}_{\theta^{-n}(x)}(w) - \mathcal{L}^n_{\theta^{-n}(x')} \mathbb{1}_{\theta^{-n}(x')}(w') \right| = \left| \sum_{z \in T^n_x} e^{S_n\varphi_{\theta^{-n}(x)}(z)} - \sum_{z' \in T^n_{x'}} e^{S_n\varphi_{\theta^{-n}(x')}(z')} \right|
\]
\[
\leq \sum_{z \in T^n_x} \left| e^{S_n\varphi_{\theta^{-n}(x)}(z)} - e^{S_n\varphi_{\theta^{-n}(x')}(z')} \right|
\]
\[
= \sum_{z \in T^n_x} \left( \left| e^{S_n\varphi_{\theta^{-n}(x')}(z')} \right| \left| e^{S_n\varphi_{\theta^{-n}(x)}(z)} - e^{S_n\varphi_{\theta^{-n}(x')}(z')} - 1 \right| \right)
\]

where we have used the fact that \( e^z - 1 \) is \( \alpha \)-Hölder for any \( \alpha > 0 \).
\[ \begin{align*}
&\leq C \cdot \sum_{z \in T_x^{-n}(w)} \left( \left| e^{S_n \varphi_{\theta^{-n}(x)}(z')} \right| \left| S_n \varphi_{\theta^{-n}(x)}(z) - S_n \varphi_{\theta^{-n}(x')}(z') \right| \right) \\
&\leq C \cdot \mathcal{L}_{\theta^{-n}(x')}^{n} 1_{\theta^{-n}(x')} (w') \cdot d^\alpha_T ((x, w), (x', w')) \\
&\leq C \cdot d^\alpha_T ((x, w), (x', w')).
\end{align*} \]

This finishes the proof. \( \square \)

Using the previous lemma we can now show the same for the normalized operator.

**Lemma 5.10.** The function \( \mathcal{L}_{\theta}^{n} 1 \) is \( \kappa \)-Hölder on \( J \), where \( \kappa := \min \{ \alpha, \beta_\alpha \} \).

**Proof.** Note that Theorem 5.6 shows that the function

\[ x \mapsto \lambda_x \]

is \( \beta_\alpha \)-Hölder on \( X \). Thus to see the claim we consider the following calculation.

\[ \begin{align*}
&\left| \mathcal{L}_{0, \theta^{-n}(x)}^{n} 1_{\theta^{-n}(x)} (w) - \mathcal{L}_{0, \theta^{-n}(x')}^{n} 1_{\theta^{-n}(x')} (w') \right| \\
&\leq \left| \left( \lambda_\theta^{n, \theta^{-n}(x)} \right)^{-1} \sum_{z \in T_x^{-n}(w)} e^{S_n \varphi_{\theta^{-n}(x)}(z)} - \left( \lambda_\theta^{n, \theta^{-n}(x')} \right)^{-1} \sum_{z \in T_x^{-n}(w)} e^{S_n \varphi_{\theta^{-n}(x)}(z)} \right| \\
&\quad + \left| \left( \lambda_\theta^{n, \theta^{-n}(x')} \right)^{-1} \sum_{z \in T_x^{-n}(w)} e^{S_n \varphi_{\theta^{-n}(x)}(z)} - \left( \lambda_\theta^{n, \theta^{-n}(x')} \right)^{-1} \sum_{z' \in T_x^{-n}(w')} e^{S_n \varphi_{\theta^{-n}(x')} (z')} \right| \\
&\leq \sum_{z \in T_x^{-n}(w)} \left| e^{S_n \varphi_{\theta^{-n}(x)}(z)} \left| \left( \lambda_\theta^{n, \theta^{-n}(x)} \right)^{-1} - \left( \lambda_\theta^{n, \theta^{-n}(x')} \right)^{-1} \right| \\
&\quad + \left| \lambda_\theta^{n, \theta^{-n}(x')}^{-1} \left| \sum_{z \in T_x^{-n}(w)} e^{S_n \varphi_{\theta^{-n}(x)}(z)} - \sum_{z' \in T_x^{-n}(w')} e^{S_n \varphi_{\theta^{-n}(x')} (z')} \right| \\
&\leq \left| \lambda_\theta^{n, \theta^{-n}(x')} \right|^{-1} \sum_{z \in T_x^{-n}(w)} \left| e^{S_n \varphi_{\theta^{-n}(x)}(z)} \left| 1 - \frac{\lambda_\theta^{n, \theta^{-n}(x)}}{\lambda_\theta^{n, \theta^{-n}(x')}} \right| \\
&\quad + \left| \lambda_\theta^{n, \theta^{-n}(x')} \right|^{-1} \left| \mathcal{L}_{0, \theta^{-n}(x')}^{n} 1_{\theta^{-n}(x')} (w') - \mathcal{L}_{0, \theta^{-n}(x)}^{n} 1_{\theta^{-n}(x)} (w) \right| \\
&\leq C \cdot \left| \lambda_\theta^{n, \theta^{-n}(x')} \right|^{-1} \mathcal{L}_{\theta^{-n}(x')}^{n} 1_{\theta^{-n}(x')} (w') \cdot d^\alpha_T ((x, w), (x', w')).
\end{align*} \]
\[ \leq C \cdot \left| \lambda_{\theta}^{n}(x, x') \right|^{-1} \mathcal{L}_{\theta}^{n}(x, x') + C \cdot \mathcal{L}_{0, \theta}^{n}(x, x') \cdot d_{\beta_{\alpha}}(x, x') \]
\[ = C \cdot \mathcal{L}_{0, \theta}^{n}(x, x') \cdot d_{\beta_{\alpha}}(x, x') \]
\[ = C \cdot \mathcal{L}_{0, \theta}^{n}(x, x') \cdot d_{\beta_{\alpha}}(x, x') \]
\[ \leq C \cdot d_{\beta_{\alpha}}(x, x'), \]

where \( \kappa := \min \{ \alpha, \beta_{\alpha} \} \). The proof is now complete. \( \square \)

We now wish to show the function

\[ x \mapsto \int_{J_{x}} \mathcal{L}_{r_{n-1}, \theta-1}(x) \cdots \mathcal{L}_{r_{0}, \theta-n}(x)(u_{\theta-n}(x)) \, d\nu_{x} \]

is Hölder continuous on \( X \) by showing the integrand is Hölder continuous on \( J \) and then applying the Denker–Gordin Hölder continuity theorem (Theorem 5.6).

\[ \mathcal{L}_{r_{n-1}, \theta-1}(x) \cdots \mathcal{L}_{r_{0}, \theta-n}(x)(u_{\theta-n}(x)) - \mathcal{L}_{r_{n-1}, \theta-1}(x') \cdots \mathcal{L}_{r_{0}, \theta-n}(x')(u_{\theta-n}(x')) \]
\[ = \left( \lambda_{\theta}^{n}(x) \right)^{-1} \sum_{z \in T_{x}^{n}(w)} e^{S_{n}g_{\theta-n}(z)} e^{S_{n}g_{\theta-n}(z)} u_{\theta-n}(x)(z) \]
\[ - \left( \lambda_{\theta}^{n}(x') \right)^{-1} \sum_{z' \in T_{x'}^{n}(w')} e^{S_{n}g_{\theta-n}(z')} e^{S_{n}g_{\theta-n}(z')} u_{\theta-n}(x')(z') \]

In light of (5.1), we may rewrite the last equality from above as

\[ \mathcal{L}_{r_{n-1}, \theta-1}(x) \cdots \mathcal{L}_{r_{0}, \theta-n}(x)(u_{\theta-n}(x)) - \mathcal{L}_{r_{n-1}, \theta-1}(x') \cdots \mathcal{L}_{r_{0}, \theta-n}(x')(u_{\theta-n}(x')) \]
\[ = \left( \lambda_{\theta}^{n}(x) \right)^{-1} \sum_{z \in T_{x}^{n}(w)} e^{S_{n}g_{\theta-n}(z)} e^{S_{n}g_{\theta-n}(z)} u_{\theta-n}(x)(z) \]
\[ - \left( \lambda_{\theta}^{n}(x') \right)^{-1} \sum_{z \in T_{x}^{n}(w)} e^{S_{n}g_{\theta-n}(z)} e^{S_{n}g_{\theta-n}(z')} u_{\theta-n}(x')(z') \]

We can then split this difference in to the sum of four differences in the standard way, which we call \( (\Delta_{1}), \ldots, (\Delta_{4}) \), that is

\[ \left( \lambda_{\theta}^{n}(x) \right)^{-1} \sum_{z \in T_{x}^{n}(w)} e^{S_{n}g_{\theta-n}(z)} e^{S_{n}g_{\theta-n}(z)} u_{\theta-n}(x)(z) \]
\[ - \left( \lambda_{\theta}^{n}(x') \right)^{-1} \sum_{z \in T_{x}^{n}(w)} e^{S_{n}g_{\theta-n}(z')} e^{S_{n}g_{\theta-n}(z')} u_{\theta-n}(x')(z') \]
We now estimate each of the previous differences \((\Delta_1) - (\Delta_4)\), beginning with \((\Delta_1)\). Theorem 5.6 and Lemma 5.9 allows us to write

\[
(\Delta_1) = \left| \sum_{z \in T_x^{-n}(w)} e^{S_n \varphi_{\theta, n}(x)} e^{iS_n g_{\theta, n}(x)} u_{\theta, n}(x)(z) \right| \left| (\lambda_{\theta, n}(x))^{-1} - (\lambda_{\theta, n}(x'))^{-1} \right|
\]

\[
= \left| \lambda_{\theta, n}(x) \right| \left| L_{0, \theta, n}(x) \right| \left| (S_n g_{\theta, n}(x)) u_{\theta, n}(x)(w) \right| \left| (\lambda_{\theta, n}(x))^{-1} - (\lambda_{\theta, n}(x'))^{-1} \right|
\]

\[
\leq L_{0, \theta, n}(x) \left| u_{\theta, n}(x) \right| \left| (\lambda_{\theta, n}(x))^{-1} - (\lambda_{\theta, n}(x'))^{-1} \right|
\]

\[
\leq C \cdot |u_x(w)| \cdot \left| \lambda_{\theta, n}(x) \right|^{-1} \left| \lambda_{\theta, n}(x) - \lambda_{\theta, n}(x') \right|
\]

\[
\leq C \cdot \lambda_{\theta, n}(x)^{-1} \int_{J_x} L_{0, \theta, n}(x) 1_{\theta, n}(x) d\nu_x - \int_{J_{x'}} L_{0, \theta, n}(x') 1_{\theta, n}(x') d\nu_{x'}
\]
Using (5.2), the difference \( (\Delta_2) \) can be estimated as

\[
(\Delta_2) = \left| \left( \lambda_{\theta-n(x')}^n \right)^{-1} \sum_{z \in T_x^n(w)} e^{S_n g_{\theta-n(x)}(z)} u_{\theta-n(x)}(z) \left( e^{S_n \varphi_{\theta-n(x')}(z)} - e^{S_n \varphi_{\theta-n(x')}(z')} \right) \right|
\]

\[
\leq \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} \left| u_{\theta-n(x)}(z) \right| \left| e^{S_n \varphi_{\theta-n(x)}(z)} - e^{S_n \varphi_{\theta-n(x')}(z')} \right| \leq C \cdot \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} e^{S_n \varphi_{\theta-n(x)}(z')} \left| e^{S_n \varphi_{\theta-n(x)}(z)} - e^{S_n \varphi_{\theta-n(x')}(z')} \right| - 1
\]

\[
\leq C \cdot \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} e^{S_n \varphi_{\theta-n(x)}(z')} \left| S_n \varphi_{\theta-n(x)}(z) - S_n \varphi_{\theta-n(x')}(z') \right| \leq C \cdot d_{\mathcal{J}}^\alpha((x, w), (x', w')) \cdot \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} e^{S_n \varphi_{\theta-n(x')}(z')}
\]

Similarly, using (5.3), the difference \( (\Delta_3) \) can be estimated as

\[
(\Delta_3) = \left| \left( \lambda_{\theta-n(x')}^n \right)^{-1} \sum_{z \in T_x^n(w)} u_{\theta-n(x)}(z) \cdot e^{S_n \varphi_{\theta-n(x')}(z')} \left( e^{S_n g_{\theta-n(x)}(z)} - e^{S_n g_{\theta-n(x')}(z')} \right) \right|
\]

\[
\leq \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} u_{\theta-n(x)}(z) \cdot e^{S_n \varphi_{\theta-n(x')}(z')} \left| e^{S_n g_{\theta-n(x)}(z)} - e^{S_n g_{\theta-n(x')}(z')} \right| \leq C \cdot \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} e^{S_n \varphi_{\theta-n(x)}(z')} \left| S_n g_{\theta-n(x)}(z) - S_n g_{\theta-n(x')}(z') \right| \leq C \cdot d_{\mathcal{J}}^\alpha((x, w), (x', w')) \cdot \left| \lambda_{\theta-n(x')}^n \right|^{-1} \sum_{z \in T_x^n(w)} e^{S_n \varphi_{\theta-n(x')}(z')}
\]

Given that \( u \in \mathcal{H}_T(\mathcal{J}) \), the final difference \( (\Delta_4) \) can be estimated as

\[
(\Delta_4) = \left| \left( \lambda_{\theta-n(x')}^n \right)^{-1} \sum_{z \in T_x^n(w)} e^{S_n \varphi_{\theta-n(x')}(z')} e^{S_n g_{\theta-n(x')}(z')} \right| \left( u_{\theta-n(x)}(z) - u_{\theta-n(x')}(z') \right)
\]
Theorem 5.6, would mean that each of the functions $x \mapsto \lambda_{\theta-n(x')}^{-1} \, \sum_{z \in T_{x}^{-n}(w)} e^{Sn_{\theta-n(x')}}(z') \cdot \left| u_{\theta-n(x)}(z) - u_{\theta-n(x')}^{n}(z') \right|

\leq C \cdot \mathcal{D}_{\mathcal{F}}((x, w), (x', w')) \cdot \lambda_{\theta-n(x')}^{-1} \, \sum_{z \in T_{x}^{-n}(w)} e^{Sn_{\theta-n(x')}}(z')

= C \cdot \mathcal{D}_{\mathcal{F}}((x, w), (x', w')) \cdot \mathcal{L}_{0, \theta-n(x')}(\mathcal{L}_{\theta-n(x')})(w')

\leq C \cdot \mathcal{D}_{\mathcal{F}}((x, w), (x', w'))

All together this gives that

$$\left| \mathcal{L}_{r_{n-1}, \theta-1(x)} \circ \cdots \circ \mathcal{L}_{r_{0}, \theta-n(x)}(u_{\theta-n(x)})(w) - \mathcal{L}_{r_{n-1}, \theta-1(x') \circ \cdots \circ \mathcal{L}_{r_{0}, \theta-n(x')}(u_{\theta-n(x')})(w') \right|

\leq \Delta_{1} + \Delta_{2} + \Delta_{3} + \Delta_{4}

\leq C \cdot \mathcal{D}_{\mathcal{F}}((x, w), (x', w')) + C \cdot \mathcal{D}_{\mathcal{F}}((x, w), (x', w'))

\leq C \cdot \mathcal{D}_{\mathcal{F}}((x, w), (x', w'))

where

$$\zeta = \min \{ \alpha, \beta_{\alpha}, \tau \}.$$

Upon application of Theorem 5.6, we see that the function

$$x \mapsto \int_{\mathcal{F}} \mathcal{L}_{r_{n-1}, \theta-1(x)} \circ \cdots \circ \mathcal{L}_{r_{0}, \theta-n(x)}(u_{\theta-n(x)}) \, d\nu_{x}$$

is $\beta_{\zeta}$-Hölder continuous on $X$ with uniformly bounded $\mathbb{H}_{\beta_{\zeta}}$ norm. The proof of Proposition 5.8 is now complete.

Remark 5.11. The proof of Proposition 5.8 gives more. We have actually shown that the functions $\mathcal{L}_{x}^{n}(u_{x}), \mathcal{L}_{0,x}^{n}(u_{x}),$ and $\mathcal{L}_{r,x}^{n}(u_{x})$ are each Hölder continuous on $\mathcal{F}$, which applying Theorem 5.6 would mean that each of the functions $x \mapsto \mathcal{L}_{x}^{n}(u_{x}), x \mapsto \mathcal{L}_{0,x}^{n}(u_{x}),$ and $x \mapsto \mathcal{L}_{r,x}^{n}(u_{x})$ are each Hölder continuous on $X$.

Remark 5.12. Considering Remark 5.4, we see that Proposition 5.8 provided that we know that the function $\rho$ is Hölder on $\mathcal{F}$, implies that the function

$$x \mapsto \int_{\mathcal{F}} \mathcal{L}_{r_{n-1}, \theta-1(x)} \circ \cdots \circ \mathcal{L}_{r_{0}, \theta-n(x)}(\rho_{\theta-n(x)}) \, d\nu_{x}$$

is Hölder continuous on $X$ with uniformly bounded $\mathbb{H}$ norm, which satisfies condition 5 of Theorem 3.1.

In order to show the function $\rho$ is indeed Hölder continuous over $\mathcal{F}$ we will first need the following result while follows from the proof of Lemma 3.8 of [8].

Lemma 5.13. For each $x \in X$ the sequence

$$\rho_{x,n} := \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}_{0, \theta-k(x)}^{k} \mathcal{L}_{\theta-k(x)}$$
is equicontinuous, i.e. there exists a subsequence $n_j$ such that

$$\rho_{x,n_j} \longrightarrow \rho_x$$

uniformly.

Thus, in light of the previous remark, the following lemma establishes that the hypotheses of Theorem 3.1 hold for $DG^s$–systems.

**Lemma 5.14.** The function $\rho$ is $\kappa$–Hölder on $X$, where $\kappa = \min\{\alpha, \beta, \gamma\}$.

**Proof.** To see this we appeal to the Hölder continuity of the function $L^k_{0,\theta^n(x)}1_{\theta^n(x)}$ which we just showed in Lemma 5.10. Since we have that the functions

$$\rho_{x,n} := \frac{1}{n} \sum_{k=0}^{n-1} L^k_{0,\theta^{-k}(x)}1_{\theta^{-k}(x)}$$

converge to $\rho_x$ for each $x \in X$ and from the proof of Lemma 5.10, we also have that for $(x, w), (x', w') \in J$ and each $n \in \mathbb{N}$

$$|\rho_{x,n}(w) - \rho_{x',n}(w')| = \left| \frac{1}{n} \sum_{k=0}^{n-1} L^k_{0,\theta^{-k}(x)}1_{\theta^{-k}(x)} - \frac{1}{n} \sum_{k=0}^{n-1} L^k_{0,\theta^{-k}(x')}1_{\theta^{-k}(x')} \right| \leq C \cdot d^\kappa_J((x, w), (x', w')).$$

Thus we have that

$$|\rho_x(w) - \rho_{x'}(w')| \leq C \cdot d^\kappa_J((x, w), (x', w')).$$

In particular, we see that for $z \in T_x^{-n}(w)$ and $z' \in T_{x'}^{-n}(w')$

$$|\rho_{\theta^{-n}(x)}(z) - \rho_{\theta^{-n}(x')}(z')| \leq C \cdot \gamma^{-\kappa n} \cdot d^\kappa_J((x, w), (x', w')).$$

This finishes the proof. \qed

We have finally shown that the hypotheses of Theorem 3.1 hold for $DG^s$–systems, and we now have the following.

**Theorem 5.15.** Let $T : J \longrightarrow J$ be a $DG^s$–system and $g \in \mathcal{H}^s_\alpha(J)$. Then either there exists a number $\sigma^2 > 0$ such that the process $\{g \circ T^n - \mu(g)\}_{n \in \mathbb{N}}$ satisfies an ASIP with limiting covariance $\sigma^2$ for any error exponent larger that $1/4$, or, if $\sigma^2 = 0$, then we have that

$$\sup_{n \in \mathbb{N}} \|S_n g - \mu(g)\|_{L^2(\mu)} < \infty.$$

**Acknowledgments**

The author would like to thank Mariusz Urbański for the many thoughtful discussions which inspired this work.
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