On Quantum T-duality in $\sigma$ models

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Abstract

The problem of quantum equivalence between non-linear sigma models related by Abelian or non-Abelian T-duality is studied in perturbation theory. Using the anomalous Ward identity for Weyl symmetry we derive a relation between the Weyl anomaly coefficients of the original and dual theories. The analysis is not restricted to conformally invariant backgrounds. The formalism is applied to the study of two examples. The first is a model based on $SU(2)$ non-Abelian T duality. The second represents a simple realization of Poisson-Lie T duality involving the Drinfeld double based on $SU(2)$. In both cases quantum T duality is established at the 1-loop level.

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1 Introduction

Target space duality (T-duality) symmetries play an important role in modern string theory. A simple and elegant way to introduce T-duality is to identify them as canonical transformations [1] in the σ-model formulation of string theory. The advantage of the canonical transformation approach to T-duality is that it treats various kinds of T-duality transformations (Abelian [2], non-Abelian (NAD) [3, 4] and Poisson-Lie (PL) [5, 6]) on an equal footing. Since the canonical transformation approach is classical, it leaves open the question whether the duality related σ-models give equivalent quantum field theories.

In quantum theory the usual way to argue for the equivalence between the dually related theories is based on gauging the isometries in the functional integral representation. This applies to Abelian and non-Abelian dualities, while it leaves open the problem of Poisson-Lie duality, which does not require any isometry. For Conformal Field Theories (CFT-s) it has been convincingly argued in [7] that the original and dual models are two different functional integral representations of the same CFT. For some special cases (ordinary and gauged Wess-Zumino-Witten-Novikov (WZWN) models) this equivalence has been shown explicitly on the level of current algebras. Nevertheless an example was put forward in [8] where two classically equivalent, non-Abelian duality related conformal invariant sigma models were found to be inequivalent at the one loop level. Subsequently the explanation of this non-equivalence has been given [4] by showing that the gauging procedure is afflicted with anomalies, if the structure constants of the gauge/isometry group have non-vanishing traces.

Conformal invariance is not a prerequisite for the existence of quantum T-duality. Thus it is appropriate to address this problem outside the context of string theory in the general setting of renormalizable sigma models, dropping the requirement of conformal symmetry. Therefore in our study we employ perturbative techniques of quantum field theory, i.e. the loop expansion.

Quantum equivalence of dually related (renormalizable) sigma models is a non-trivial problem already at the conceptual level. Any satisfactory criterion should be based on the comparison of physical quantities as opposed to just considering beta functions. If there are global symmetries in the model then their associated conserved quantities (Noether currents) may be considered physical. The definition of physical quantities, however, is not very clear in diffeomorphic invariant sigma models without a sufficient number of isometries. Bearing this in mind we work in a definite renormalization scheme (background field method with dimensional regularization), hence our results are - at least superficially - dependent on this particular choice. On the other hand the quantities we compare –the Weyl anomaly coefficients– are at least free of field redefinition and gauge transformation ambiguities [4]. They appear naturally and are physical in the sense that
they describe the trace of the world sheet energy momentum tensor.

The study of examples so far has revealed no violations of quantum equivalence at the 1-loop level for Abelian or non-Abelian T-duality (apart from the class of anomalous models of the type of Ref. [8]). A general proof of the one loop equivalence for the Abelian case has been given in [10]. At the two-loop level, the situation is somewhat different. As shown in [11], even for the simplest examples [12] T-duality can be maintained as an exact quantum symmetry only at the price of modifying the classical formulas [2] accompanied by an appropriate change in the renormalization scheme. For the general Abelian case, a form of the 2-loop modifications has been determined in [13] through a consideration of the string effective action; (though it remains to see whether they fit into the conventional field theoretical renormalization scheme).

Based on the study of a number of examples [14], we expect a similar pattern for non-Abelian T-duality. On the other hand, there are too few examples worked out [15] [16] for the most general case of Poisson-Lie T-duality to draw any conclusion at this stage.

The goal of the present paper is twofold: First, we want to establish a general relation between the Weyl-anomaly coefficients (and beta functions) of the original model and those of its dual in a general Abelian or non-Abelian context. The argument is independent of the loop order when applied to the bare quantities, but we consider its implications in terms of renormalized couplings only to 1 loop order. We establish it on general grounds starting from the assumption that the standard gauging procedure is valid in the path integral\footnote{This excludes, in particular, the anomalous cases of \([6], [8]\).}. Our perspective is complementary to that of Ref. [10] where, - in the Abelian case, - such a relation was postulated, and verified by explicit calculation. This is the subject of section 2, where we point out as well that the above relation also implies the 1-loop equivalence of the renormalized partition functions.

In the second half of the paper, we turn to the study of concrete examples. In section 3, we illustrate our general considerations with a non-Abelian example (based on $SU(2)$) with a single spectator field. The beta functions/Weyl anomaly coefficients are explicitly computed, and we verify the validity of the relation mentioned above. In section 4, we analyse a simple example of Poisson-Lie type, involving the Drinfeld double based on $SU(2)$. Two new difficulties appear in the general PL setting: First, no gauging procedure is available to date\footnote{Note that the path integral derivation given by Tyurin and von Unge, [11], is not a conventional gauging.}, thus our starting assumption of Sect. 2 is lacking even a formal derivation in this case. Second, while one can easily show that the isometries responsible for Abelian/non-Abelian duality survive renormalization, it is less obvious that the criterion for PL dualizability does: it is not a priori clear that the Poisson
Lie dualizability of the bare fields is inherited by the renormalized ones. Our general argument of section 2 is thus inapplicable. We show, however, by explicit comparison of the one loop beta functions/Weyl anomaly coefficients of the dual partners that the relation derived in Section 2 does hold for our example. This provides further evidence for the existence of quantum PL T-duality.

2 General considerations

We will consider the most general 2-dimensional $\sigma$-model with background metric $g_{\mu\nu}(X)$, torsion potential $b_{\mu\nu}(X)$ and dilaton field $\phi(X)$. To simplify the notation, sometimes we will use the generic symbol $w$ whose components are the various background fields:

\begin{align}
  w^{(g)}_{\mu\nu} &= g_{\mu\nu}, \\
  w^{(b)}_{\mu\nu} &= b_{\mu\nu}, \\
  w^{(\phi)} &= \phi.
\end{align}

In renormalization theory the above finite fields will play the rôle of the renormalized couplings. The analogous bare (unrenormalized) fields will be denoted by the corresponding capital symbols $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ and collectively denoted by $W$. The Lagrangian is written in terms of the bare fields as

$$
\mathcal{L} = \frac{1}{2} \gamma^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{i}{2} \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)} \Phi.
$$

Here $R^{(2)}$ is the worldsheet curvature and the worldsheet metric $\gamma_{ab}$ can be brought to the conformally flat form

$$
\gamma_{ab}(z) = e^{\sigma(z)} \delta_{ab}.
$$

The partition function of the $\sigma$-model is obtained by functionally integrating over the coordinate fields $X^\mu(z)$.

$$
Z[G, B, \Phi; \sigma] = \int [DX^\mu] e^{-S[G, B, \Phi; \sigma]},
$$

where

$$
S[G, B, \Phi; \sigma] = \frac{1}{2\pi \alpha'} \int d^2z \sqrt{\gamma} \mathcal{L}.
$$

The action (7) is obviously invariant under diffeomorphisms, i.e. transformations of the target space coordinates $X^\mu$:

$$
S[W; \sigma] = S[W^D; \sigma],
$$

where the diffeomorphism $D$ can either be a finite transformation, or, in perturbation theory (using dimensional regularization), a power series in $\alpha'$ with $\epsilon$ poles.
Similarly (7) is invariant under the gauge transformation

\[ S[W; \sigma] = S[W^\Delta; \sigma], \tag{9} \]

where

\[ B^\Delta_{\mu\nu} = B_{\mu\nu} + \partial_\mu \Delta_\nu - \partial_\nu \Delta_\mu \tag{10} \]

with \( \Delta_\mu \) either finite or a perturbative series.

### 2.1 Compatibility between duality transformations and the Weyl flow

The way we are going to check quantum T duality in the examples that we will analyze is not by comparison of the partition functions themselves, but rather of the Weyl anomaly coefficients, objects that are closely related to beta functions. They are defined by the anomalous Ward identity

\[ \frac{\delta Y}{\delta \sigma(z)} = \langle T_a^a(z) \rangle = \langle L(\bar{\beta}) \rangle, \tag{11} \]

Here, \( Y[w; \sigma] \) denotes the renormalized partition function, related to the bare one of (6) by

\[ Y[w; \sigma] = Z[W(w); \sigma]. \tag{12} \]

Furthermore, \( T_{ab} \) is the worldsheet energy-momentum tensor. Its anomalous trace is of the form of the original Lagrangian \( L \) where the Weyl anomaly coefficients \( \bar{\beta}(w) \) play the role of the background couplings [9]. The Weyl anomaly coefficients differ from the \( \beta \)-functions of the model only by additive shifts containing the dilaton (see below), but, unlike the \( \beta \)-functions, are free of the ambiguities coming from diffeomorphisms and gauge transformations.

In ref. [10], the following relation connecting the Weyl anomaly coefficients of the original model and its dual was proposed:

\[ \bar{\beta}(w) \frac{\partial \Gamma}{\partial w}(w) = \bar{\beta}(\tilde{w}), \quad \tilde{w} \equiv \Gamma(w), \tag{13} \]

where \( \Gamma(w) \) denotes the duality transformations of the couplings. If \( \hat{T} \) denotes the operator implementing the operation \( w \rightarrow \tilde{w} \) and \( \hat{R} \) the flow operator corresponding to

\[ w \rightarrow w + \lambda \tilde{\beta}(w), \]

(where \( \lambda \) is the infinitesimal flow parameter), then (13) can be interpreted as the commutativity of the “Weyl flow” \( \hat{R} \) with T duality, \( [\hat{T}, \hat{R}] = 0 \). The authors of ref. [10], who postulated (13) somewhat heuristically, proved that it is satisfied at least to 1 loop order for the case of Abelian duality. Furthermore, they noted that it holds exactly and
not only up to diffeomorphisms and gauge transformations, as one may have expected,
but did not provide any explanation for this fact. Our goal in this subsection will be
to derive (13) from a general point of view encompassing Abelian and non-Abelian T
duality alike, and to explain the absence of diffeomorphism and gauge terms. Within
this setting, the function $\Gamma$ takes the general form

$$
\tilde{g}_{\mu\nu} = \Gamma^{(g)}_{\mu\nu}(g, b),
$$

(14)

$$
\tilde{b}_{\mu\nu} = \Gamma^{(b)}_{\mu\nu}(g, b),
$$

(15)

$$
\tilde{\phi} = \phi + S(g, b).
$$

(16)

For Abelian and non-Abelian duality, the set of T-dualizable backgrounds is a linear
space, since the only requirement is the presence of a given set of isometries. The pre-

cence of these isometries is clearly unaffected by adding and rescaling these backgrounds.
Similarly it is obvious from the form of the counterterms (see (38)-(41) below) that if the
renormalized configuration $w$ is dualizable then so is the corresponding bare one, $W$. For
the genuine PL type dualities the situation is more subtle. This is one of the reasons why
we restrict our general considerations to the simpler Abelian and non-Abelian dualities.

The starting point of our argument will be to assume the invariance of the partition
function under T-duality transformation of the bare fields:

$$
Z[W; \sigma] = Z[\tilde{W}; \sigma].
$$

(17)

This is the relation we obtain when we go through the by now standard gauging pro-
dure. Actually, even (17) can in principle be questioned, especially in the dimensional
regularization scheme we will employ in this paper, since the derivation is rather formal.
In particular, we know from refs. [8], [4] that the gauging procedure can be afflicted by
an anomaly. Though such questions go beyond the scope of the present analysis, they
are important and we hope to return to them in future work.

To prepare the stage, let us now consider an infinitesimal local Weyl transformation
$\sigma(z) \rightarrow \sigma(z) + \lambda(z)$. Equation (11) can also be written as

$$
Y[w; \sigma(z) + \lambda(z)] = Y[w + \lambda(z)\overline{\beta}(w); \sigma(z)]
$$

(18)

Actually (18) makes sense for a local (on the worldsheet) $\lambda(z)$ only if we consider a
generalized version of the $\sigma$-model where the background couplings are allowed to depend
on the worldsheet coordinate $z$ as in [3]. We can now translate (18) into the language of
bare quantities using (12):

$$
Y[w; \sigma(z) + \lambda(z)] = Z[W(w); \sigma(z) + \lambda(z)] = Z[W(w + \lambda(z)\overline{\beta}(w)); \sigma(z)]
$$

(19)

$$
= Z[W(w) + \lambda(z)\frac{\partial W}{\partial w}(w)\overline{\beta}(w); \sigma(z)].
$$

(20)
From these last two equations we infer that

\[ Z[W; \sigma(z) + \lambda(z)] = Z[W + \lambda(z)\overline{B}(W); \sigma(z)], \tag{21} \]

where

\[ \overline{B}(W) = \frac{\partial W}{\partial w}(H(W))\overline{\beta}(H(W)). \tag{22} \]

are the bare Weyl anomaly coefficients, and

\[ H(W) \equiv w \tag{23} \]

defines the relation between bare and renormalized couplings. We are now in a position to put our starting equation (17) to work. Applying it to (21), we have

\[ Z[W; \sigma(z) + \lambda(z)] = Z[W + \lambda(z)\overline{B}(W); \sigma(z)] = Z[\Gamma(W + \lambda(z)\overline{B}(W)); \sigma(z)] = Z[\tilde{W} + \lambda(z)\overline{\delta}\Gamma(W); \sigma(z)]. \tag{24} \]

Here we have defined

\[ \lambda\overline{\delta}\Gamma(W) = \Gamma(W + \lambda\overline{B}(W)) - \Gamma(W) + O(\lambda^2). \tag{26} \]

This definition makes sense since the space of dualizable configurations is a linear space, in the context of Abelian or non-Abelian duality, as discussed above. The second of equations (24) deserves further comment. In the literature, the standard gauging procedure which underlies (17) was discussed only for couplings which do not depend explicitly on \( z \), as is the case, of course, for the original and dual sigma models. The replacement \( W \rightarrow W + \lambda(z)\overline{B} \) in (24) requires an extension of this procedure to \( z \)-dependent couplings. This is, however, easily achieved as the only place in the standard procedure where the \( z \) dependence matters are terms where one has to perform partial integrations. Since the only place where a partial integration is necessary is in the Lagrange multiplier term enforcing the flatness of the field strength, we can carry out the functional integration over the gauge fields exactly as in the standard case [3, 17, 18], leading to the same formal result.

Instead of performing the Weyl transformation before duality, we could proceed in the opposite order:

\[ Z[W; \sigma + \lambda] = Z[\tilde{W}; \sigma + \lambda] = Z[\tilde{W} + \lambda\overline{B}(\tilde{W}); \sigma]. \tag{27} \]

Comparing (24) and (27) thus gives

\[ \overline{\delta}\Gamma(W) = \overline{B}(\tilde{W}). \tag{28} \]

Note that in order to arrive at (28), it was crucial to have arbitrary functions \( \lambda(z) \) at our disposal. Had we worked with a simplified version of the above line of reasoning where
λ is a constant parameter, we would still have (28), but only up to diffeomorphisms and gauge transformations. The point is that the partition function (in fact, already the classical action) is not invariant under explicitly $z$-dependent diffeomorphisms and gauge transformations. Our result (28) is formally valid to all orders in perturbation theory, but it is formulated in terms of the bare fields $W$. Its form in the language of renormalized quantities is complicated. However, to lowest (1 loop) order, bare and renormalized Weyl coefficients agree, (as can be seen from (22))

$$\bar{E}(W) = \bar{\beta}(W) + \mathcal{O}(\alpha').$$

(29)

Hence we can write to this order,

$$\delta_{\bar{R}}\Gamma(w) = \bar{\beta}(w)$$

(30)

where $\delta_{\bar{R}}\Gamma(w)$ is defined as the renormalized analog of (26) with $W \rightarrow w$, $\bar{E}(W) \rightarrow \bar{\beta}(w)$. This in turn is nothing else than equation (13) which we wanted to derive.

In our line of thinking, the $[\hat{T}, \hat{R}] = 0$ relation has an obvious interpretation: It expresses the fact that the form of the duality transformation does not depend on the configuration of the worldsheet metric $\sigma(z)$. In our examples, we will verify explicitly that equation (13) holds, providing a check of duality on the level of the Weyl anomaly coefficients.

### 2.2 Existence of the dual model

On the level of renormalized couplings, the naive equivalent of (17) would be

$$Y[w; \sigma] = Y[\tilde{w}; \sigma].$$

(31)

However, it is well known that this is not correct. For Abelian dualities (31) holds at the 1-loop level [10], but it is already violated at 2-loop [11]. Based on these experiences it is natural to postulate that instead of (31) we have

$$Y[w; \sigma] = Y[\tau(w); \sigma],$$

(32)

where $\tau(w)$ is the quantum corrected duality transformation [12, 11, 13]

$$\tau(w) = \tilde{w} + \mathcal{O}\left(\alpha'\right).$$

(33)

\footnote{In fact, a slightly more precise version of it. Since the domain of definition of the dual mapping $\Gamma$ is not the full coupling space, but only its subspace under a certain set of isometries, the gradient appearing in (13) is not well-defined. However, the specific combination $\bar{\beta}(w)\frac{\partial}{\partial w}$ is, because it represents a directional derivative in a tangential direction - cf. (24).}
The situation is actually even more subtle since it turns out that true quantum equivalence requires in addition to the quantum correction (33) also a change in the renormalization scheme [11]. We will not discuss these subtleties here since we mainly concentrate on the leading 1-loop results where they can be neglected.

Using definition (12) and assuming (17) we have

\[ Y[\tau(w); \sigma] = Z[W(\tau(w)); \sigma] \] (34)

and

\[ Y[w; \sigma] = Z[W(w); \sigma] = Z[\tilde{W}(w); \sigma]. \] (35)

Quantum equivalence of the model and its dual thus follows if

\[ W(\tau(w)) \approx \tilde{W}(w), \] (36)

where \( \approx \) means that the equation holds up to diffeomorphisms and gauge transformations. This can formally be solved using (23):

\[ \tau(w) = H(\tilde{W}^{(D, \Delta)}(w)), \] (37)

but it is not clear if the diffeomorphism \( D \) and the gauge transformation \( \Delta \) (both possibly infinite) can always be chosen such that this solution is finite. In other words, the question is whether the renormalization of the original theory, transported by duality to its dual formulation, will give rise to a consistent definition of renormalized couplings of the dual theory. We will now show explicitly to 1 loop order that this is the case, provided (13) holds. Let us first discuss briefly the explicit form of the 1 loop counterterms (and hence the beta functions) in dimensional regularization. Writing

\[ W = W(w) = w + \frac{\alpha'}{\epsilon} \rho(w) + \mathcal{O}\left((\alpha')^2\right), \] (38)

where

\[ \rho^{(g)}_{\mu\nu} = \hat{R}_{\mu\nu}(g, b), \] (39)

\[ \rho^{(b)}_{\mu\nu} = \hat{R}_{[\mu|\nu]}(g, b), \] (40)

\[ \rho^{(\phi)} = -\frac{1}{2} D^\mu \partial_\mu \phi - \frac{1}{24} b^2 \] (41)

with \( \hat{R}_{\mu\nu} \) being the generalized Ricci tensor (built from the torsion containing connection) and \( b^2 = b_{\mu\nu\rho} b^{\mu\nu\rho} \) the scalar square of the torsion. The formal inverse of (38) is

\[ w = H(W) = W - \frac{\alpha'}{\epsilon} \rho(W) + \mathcal{O}\left((\alpha')^2\right). \] (42)
For the Weyl anomaly coefficients, one obtains
\begin{equation}
\beta^{(g)}_{\mu\nu} = \beta^{(g)}_{\mu\nu} + 2D_\mu \partial_\nu \phi + \mathcal{O}(\alpha'), \tag{43}
\end{equation}
\begin{equation}
\beta^{(b)}_{\mu\nu} = \beta^{(b)}_{\mu\nu} + b_{\mu\nu} \phi + \mathcal{O}(\alpha'), \tag{44}
\end{equation}
\begin{equation}
\beta^{(\phi)} = \beta^{(\phi)} + \partial^\mu \phi \partial_\mu \phi + \mathcal{O}(\alpha'), \tag{45}
\end{equation}
where
\begin{equation}
b_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} + \text{cyclic}. \tag{46}
\end{equation}

Now to lowest order in $\alpha'$, we must have $\tau(w) = \tilde{w} \equiv \Gamma(w)$, and therefore (36) becomes at this level
\begin{equation}
\tilde{w} + \frac{\alpha'}{\epsilon} \rho(\tilde{w}) \approx \Gamma\left(w + \frac{\alpha'}{\epsilon} \rho(w)\right) = \tilde{w} + \frac{\alpha'}{\epsilon} \partial \Gamma \frac{w}{\rho(w)}. \tag{47}
\end{equation}
Comparing it to (13) and using (43-45) together with the fact that
\begin{equation}
\rho^{(g)}_{\mu\nu} = \bar{R}_{(\mu\nu)} = \beta^{(g)}_{\mu\nu}, \tag{48}
\end{equation}
\begin{equation}
\rho^{(b)}_{\mu\nu} = \bar{R}_{[\mu\nu]} = \beta^{(b)}_{\mu\nu}, \tag{49}
\end{equation}
\begin{equation}
\rho^{(\phi)} = -\frac{1}{2} D^\mu \partial_\mu \phi - \frac{1}{24} b^2 = \beta^{(\phi)} \tag{50}
\end{equation}
we see that (47) is satisfied up to an infinitesimal diffeomorphism with parameter
\begin{equation}
\xi^\mu = \frac{\alpha'}{\epsilon} \tilde{g}^{\mu\nu} \partial_\nu S, \tag{51}
\end{equation}
where $S = \tilde{\phi} - \phi$ is the dilaton shift.

### 3 The non-Abelian dual of an SU(2) model

Let us now apply the above formalism to specific examples. The first model we consider is defined by the action
\begin{equation}
S = \int d^2 \sigma \left\{ f(x) \partial_\mu x \partial^\mu x + h(x) \text{Tr} \left[ (g^{-1} \partial_\mu g) \left( g^{-1} \partial^\mu g \right) \right] \right\}, \tag{52}
\end{equation}
where $g \in \text{SU}(2)$, $f(x)$ and $h(x)$ are arbitrary functions. In Eq. (52) $x(\sigma)$ is a spectator field coupled to the principal $\text{SU}(2)$ sigma model and the form of the Lagrangian follows from the global $\text{SU}(2) \times \text{SU}(2)$ symmetry. The renormalisation properties of the model (52) are rich enough to test to the full our findings of the previous sections.

Using the Euler angles $(y, z, w)$, the parametrisation of the SU(2) group element is
\begin{equation}
g = \exp(iy\tau_3) \exp(iz\tau_1) \exp(iw\tau_3), \tag{53}
\end{equation}
with \( \tau_a = \sigma_a/2 \) and the \( \sigma_a \) are the usual Pauli matrices. The target spacetime metric is then given by

\[
G_{ij} = \begin{pmatrix}
    f & 0 \\
    0 & h_{ab}
\end{pmatrix}, \quad \text{where} \quad g_{ab} = \begin{pmatrix}
    1 & 0 & \cos(z) \\
    0 & 1 & 0 \\
    \cos(z) & 0 & 1
\end{pmatrix}, \quad (54)
\]

where \( i, j, \ldots = 1, \ldots, 4 \) label the coordinates \( (x, y, z, w) \) and \( a, b, \ldots = 2, 3, 4 \) label the group manifold coordinates \( (y, z, w) \). We note that for \( f = 1 \) and \( h = x^2 \), the metric \( G_{ij} \) is that of the four sphere \( S^4 \) and the second term of the action coincides with the \( SU(2) \) principal chiral sigma model when \( h = 1 \).

Since the antisymmetric tensor field and the dilaton are both absent from the model \( (52) \) the Weyl anomaly coefficients \( \bar{\beta}_{ij} \) are identical to the beta functions, \( \beta_{ij} \), which are given as:

\[
\bar{\beta}_{ij} = \beta_{ij} = R_{ij}. \quad (55)
\]

The non-vanishing components of the Ricci tensor are

\[
R_{ab} = (\delta h) g_{ab} \\
\delta h = \left\{ \frac{1}{2} - \frac{1}{2} f^{-1} \left( h'' - \frac{1}{2} f^{-1} f' h' \right) - \frac{1}{4} (fh)^{-1} (h')^2 \right\} \\
R_{11} = \delta f \\
\delta f = \frac{3}{4} \left\{ \left( h^{-1} h' \right)^2 - 2h^{-1} h'' + (fh)^{-1} f' h' \right\}, \quad (56)
\]

where the prime denotes the derivative with respect to \( x \).

The non-Abelian dual of the original theory, \( (52) \), is constructed by exploiting the presence of the symmetry \( g \rightarrow LgR \), where \( L \) and \( R \) are constant elements of \( SU(2) \). The classical non-Abelian dual with respect to the ‘left’ \( SU(2) \) symmetry of our model can be written (in light-cone coordinates) as:

\[
\tilde{S} = \int d^2 \sigma \left\{ f(x) \partial_+ x \partial_- x + \tilde{M}_{ab} \partial_+ \chi^a \partial_- \chi^b \right\}, \quad (57)
\]

where \( \chi_a \) is the auxiliary field introduced in the gauging precedure and \( \partial_x = \partial_\tau \pm \partial_r \). The matrix \( \tilde{M}_{ab} \) is the inverse of

\[
M^{ab} = h \delta^{ab} + f_c^{ab} \chi^c. \quad (58)
\]

The dual theory still has a ‘right’ \( SU(2) \) symmetry. To make this manifest we introduce spherical coordinates

\[
\chi^a = r n^a, \quad n^a n^a = 1, \quad n^a = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad (59)
\]
in terms of which \( \tilde{S} \) takes the simple form

\[
\tilde{S} = \int d^2\sigma \left\{ f \partial_\mu x \partial^\mu x + \frac{1}{h} \partial_\mu r \partial^\mu r + \frac{hr^2}{D} [\partial_\mu \theta \partial^\mu \theta + \sin^2 \theta \partial_\mu \varphi \partial^\mu \varphi] + 2 \epsilon^{\mu \nu} \frac{r^3}{D} \sin \theta \partial_\mu \theta \partial_\nu \varphi \right\}
\]

(60)

where \( D = h^2 + r^2 \). To get the one loop beta functions of (60) we have to compute the generalised Ricci tensor, \( \tilde{R}_{ij} \) where \( i, j, \ldots \) label the coordinates \((x, r, \theta, \varphi)\). The non-vanishing components of the generalised Ricci tensor are as follows

\[
\begin{align*}
\tilde{R}_{12} &= \tilde{R}_{21} = -\frac{1}{D^2 h} r h' (-r^2 + 3h^2) \\
\tilde{R}_{11} &= \frac{1}{4D^2 h^2 f} (-r^2 + 3h^2)(-h^3 h' f' + 2h^3 h'' f - 3h' h^2 f + 2r^2 h^4 f) \\
&\quad - r^2 h' h f' + 2r^2 h^2 f \\
\tilde{R}_{22} &= \frac{1}{4D^3 h^3 f^2} (-r^4 h' h + 2r^4 h'' h f - 2r^4 f^2 h + 2r^2 h' h^3 + 4r^2 h'' h f) \\
&\quad - 12r^2 h^3 f^2 - 6r^2 h'^2 f - 2h' h^5 + 6h f^2 \\
\tilde{R}_{33} &= \frac{1}{4D^3 h^2 f} r^2 (r^4 h' f h + 2r^4 h'' f + 2r^4 f^2 h - 2r^4 h'' h f + 4r^2 h f) \\
&\quad - 5h'^2 h^4 + 6h h^5 f - h' h^5 \\
\tilde{R}_{34} &= -\tilde{R}_{43} = \frac{r^2 \sin^2 \theta}{2D^3 f^2} (-r^2 h h' f - r^2 h'^2 f + 2r^2 h h'' f - 5h'^2 h^2 + 2h'^2 f) \\
&\quad + 4h^3 f^2 - h^3 h' \\
\tilde{R}_{44} &= \frac{r^2 \sin^2 \theta}{4D^3 h f} (r^4 h' f h + 2r^4 h'' f + 2r^4 f^2 h - 2r^4 h'' h f) \\
&\quad + 4r^2 h'^2 f - 5h'^2 h^4 + 6h^5 f^2 + 2h'' h^5 f - h' h^5)
\end{align*}
\]

(61)

The part of the dual action involving \((r, \theta, \varphi)\), when \( h = 1 \), does indeed correspond to the dual of the principal model and if in addition \( f = 1 \), the corresponding components of \( \tilde{R}_{ij} \) coincide also.

It is worth pointing out that the one loop counterterms do not all have the same form as those present in the classical dual action (60) (e.g. the presence of counterterms of the form \( \tilde{R}_{12} \partial_\mu x \partial^\mu r \)). This is to be contrasted with the original theory where all the counterterms are of the same form as those present in the classical Lagrangian (52). Consequently in the dual model one has to perform a nonlinear renormalization of the \( r \) field.

\footnote{In our conventions the torsion is given by \( H_{ijk} = \partial_i B_{jk} + \partial_j B_{ik} + \partial_k B_{ij} \) and the generalised Riemann tensor is given by \( \tilde{R}^i_{jk} = \partial_j \Omega^i_{kl} + \Omega^i_{j} \Omega^l_{kl} - (k \leftrightarrow l) \), where \( \Omega^i_{jk} = \left\{ \begin{array}{cl} i \left( j \right) \end{array} \right\} + H^i_{jk} \) is the generalised connection. Finally, the generalised Ricci tensor is given by \( \tilde{R}_{jk} = \tilde{R}^r_{ijk} \).}
Our goal now is to show that the Weyl anomaly coefficients of the original and of the dual theories are related through the main equation of the paper. Equations (13) are now explicitly written as

\[ \tilde{\beta}_{(ij)} \equiv \tilde{\beta}_{(ij)} + 2 \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\Phi} = \frac{\partial \tilde{G}_{ij}}{\partial G_{rs}} \tilde{\beta}_{(rs)} + \frac{\partial \tilde{G}_{ij}}{\partial B_{rs}} \tilde{\beta}_{[rs]} \]

\[ \tilde{\beta}_{[ij]} \equiv \tilde{\beta}_{[ij]} + \tilde{G}^{kl} \partial_k \tilde{\Phi} \tilde{H}_{lij} = \frac{\partial \tilde{B}_{ij}}{\partial G_{rs}} \tilde{\beta}_{(rs)} + \frac{\partial \tilde{B}_{ij}}{\partial B_{rs}} \tilde{\beta}_{[rs]} \]  

(62)

Notice that \((\tilde{G}_{ij}, \tilde{B}_{ij})\) depend only on the two functions \(f\) and \(h\). Therefore, using the fact that \(h = \frac{1}{3} g^{ab} G_{ab}\) and \(f = G_{11}\) (that is \(\frac{\partial}{\partial G_{11}} = \frac{\partial}{\partial f}\) and \(\frac{\partial}{\partial G_{ab}} = \frac{1}{3} g^{ab} \frac{\partial}{\partial h}\), where \(a, b = 2, 3, 4\)), one can write these equations as

\[ \tilde{\beta}_{(ij)} = \tilde{\beta}_{(ij)} + 2 \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\Phi} = \delta \Gamma_{(ij)} \]

\[ \tilde{\beta}_{[ij]} = \tilde{\beta}_{[ij]} + \tilde{G}^{kl} \partial_k \tilde{\Phi} \tilde{H}_{lij} = \delta \Gamma_{[ij]} \]  

(63)

where the non-vanishing components of the matrix \(\delta \Gamma_{ij}\) are given by

\[ \delta \Gamma_{11} = \frac{\partial \tilde{G}_{11}}{\partial f} \delta f \]

\[ \delta \Gamma_{(ab)} = \frac{\partial \tilde{G}_{ab}}{\partial h} \delta h \]

\[ \delta \Gamma_{[ab]} = \frac{\partial \tilde{B}_{ab}}{\partial h} \delta h \]  

(64)

In finding \(\delta \Gamma_{ij}\), we have made use of the explicit forms of \(\tilde{\beta}_{rs}\) in (56) and of \((\tilde{G}_{ij}, \tilde{B}_{ij})\) as given in (60).

To illustrate the self-consistency of our results we are now going to use the above equations to determine the dilaton shift \(\tilde{\Phi}\) and verify that it does coincide with the result \(\tilde{\Phi} - \Phi \sim \log \det M\).

Writing \(\tilde{\Phi}\) in the general form

\[ \tilde{\Phi} = -\frac{1}{2} \ln [N(x, r, \theta, \varphi)] \]  

(65)

then the equations corresponding to \(\delta \Gamma_{[13]}\) and \(\delta \Gamma_{[14]}\) lead respectively to

\[ \frac{\partial N}{\partial \theta} = 0 \quad , \quad \frac{\partial N}{\partial \varphi} = 0 \]  

(66)

Hence \(N\) can only be function of \(x\) and \(r\), that is \(N = N(x, r)\).

The rest of the equations emerging from considering the other components of \(\delta \Gamma_{ij}\), namely \(\delta \Gamma_{(11)}, \delta \Gamma_{(12)}, \delta \Gamma_{(22)}, \delta \Gamma_{(33)}, \delta \Gamma_{(44)}\) and \(\delta \Gamma_{[34]}\) lead respectively to the following
six algebro-differential equations

\[ 0 = (2r^4h''hf - r^4h'f'h - 2r^4h'^2f + 8r^2h''h^3f - 4r^2h'f'h^3 - 3h'f'h^5) \]
\[ + 6h''h^5f - 6h'^2f h^4)N^2 - h^2D^2(-f' \frac{\partial N}{\partial x} + 2f \frac{\partial^2 N}{\partial x^2})N + 2h^2f D^2(\frac{\partial N}{\partial x})^2 \]
\[ 0 = -2h'(-r^2 + 3h^2)N^2 - 2hD^2N \frac{\partial^2 N}{\partial x \partial r} + D^2(2h \frac{\partial N}{\partial x} - Nh') \frac{\partial N}{\partial r} \]
\[ 0 = +2fh^3D^2(\frac{\partial N}{\partial r})^2 + (-4r^2h^3f + 4fh^5 - r^4h'^2 - 4h'^2h^2r^2 - 3h'^2h^4)N^2 \]
\[ - hD^2(2fh^2 \frac{\partial^2 N}{\partial r^2} - h' \frac{\partial N}{\partial x})N \]
\[ 0 = r^2(r^4h'^2 + 2h^2h'^2 + 4fh^5 - 3h'^2h^4)N + r^2h'h(2r - 2)D \frac{\partial N}{\partial x} \]
\[ - 2fh^5D \frac{\partial N}{\partial r} \]
\[ 0 = -r(r^4h'^2 + 2h^2h'^2 + 4fh^5 - 3h'^2h^4)N - r(h^2 - 2)D \frac{\partial N}{\partial x} \]
\[ + 2fh^5D \frac{\partial N}{\partial r} \]
\[ 0 = -2r(3h^2 + r^2)(hf - h'^2)Nu \delta - 2rh'hD \frac{\partial N}{\partial x} + h(f(3h^2 + r^2))D \] (67)

In these equations, we notice that \( \frac{\partial^2 N}{\partial x^2} \) comes only from the \( \delta \Gamma_{(11)} \) equation. Therefore we consider this equation on its own. On the other hand, the last five equations can be treated as a system of equations in the ‘independent’ variables \((N, \frac{\partial N}{\partial x}, \frac{\partial N}{\partial r}, \frac{\partial^2 N}{\partial x \partial r}, \frac{\partial^2 N}{\partial r^2})\). The solution to this system is given by surprisingly simple expressions

\[ \frac{\partial N}{\partial r} = \frac{2N}{D}, \quad \frac{\partial^2 N}{\partial r^2} = \frac{2N}{D}, \quad \frac{\partial N}{\partial x} = \frac{Nh'(3h^2 + r^2)}{hD}, \quad \frac{\partial^2 N}{\partial x \partial r} = \frac{2rNh'}{hD}. \] (68)

These differential equations, in turn, have a unique solution given by

\[ N(x, r) = ahD \] (69)

with \( D = h^2 + r^2 \) and \( a \) a constant. This solution does also satisfy the \( \delta \Gamma_{(11)} \) equation.

We note that

\[ N(x, r) = a \det(M), \] (70)

where \( M \) is the matrix defined in (58). This shows that the dilaton shift \( \tilde{\Phi} \), determined by our conditions to ensure one loop quantum equivalence of the dually related models coincides (up to the irrelevant constant \( a \)) with the result of Ref. [3].
4 Poisson Lie duality at one loop

Let us now investigate the question of quantum T-duality for models related by Poisson-Lie duality. As the validity of the argument leading to (13) is not immediately obvious in the Poisson-Lie case, we will follow a slightly different path as compared to the previous example. We will first establish that the condition of Poisson-Lie dualizability is compatible with the 1 loop RG flow, at least in an infinitesimal neighborhood of the starting point. We then proceed to show that (36) can be satisfied for a suitable choice of diffeomorphism and gauge transformation. As we have seen in section 2.2, this equation expresses quantum T duality, and is equivalent (at least to 1 loop) to (13).

To investigate the interplay between renormalization and (genuine) Poisson Lie duality we consider two mutually dual sigma models defined on a specific six dimensional Drinfeld double. The double we use is the same that appeared in refs [15], [16]. In [15] a pair of PL duality related WZWN models were considered on it, while in [16] the canonical transformations connecting the PL duality related family of sigma models were worked out. In [16] the question of quantum equivalence of these sigma models was also investigated in a particular limit - when the target space of the sigma models reduces to two dimensions. Here we consider another particular example, when the target spaces of the PL dual models are three dimensional and investigate whether the (one loop) renormalization and the duality transformations commute.

The Lie algebra of the double is generated by $J_a$ and $T^a$ ($a = 1, 2, 3$), satisfying

$$[J_a, J_b] = \epsilon_{abc}J_c, \quad [T^a, T^b] = f^{ab}_{\ c}T^c,$$

$$[J_a, T^b] = f^{bc}_{\ a}J_c - \epsilon_{abc}T^c,$$

where the only non vanishing commutator among the $T^a$-s are $[T^i, T^3] = cT^i$ for $i = 1, 2$ with $c$ being a constant. (If $c \neq 0$ then it could be absorbed into $T^3$, but we keep it, as then setting $c = 0$ should lead back to the case of ordinary non-Abelian duality). The $T^a$ generators may be thought of as describing two translations and a dilatation. It is straightforward to determine the various matrices of the Klimčík – Severa construction using e.g. the $6 \times 6$ representation of the generators following from Eq.(71). Parametrizing the elements of the $SU(2)$ subgroup as:

$$g = \exp(J_3 f) \exp(J_2 t) \exp(J_3 p),$$

the $b_{ab}(g)$, $d_{ab}(g)$ matrices are defined as:

$$g^{-1}T^a g = b_{ab}(g)J_b + d_{ab}(g)T^b,$$

while the third matrix of this construction, $g^{-1}J_a g = a_{ab}(g)J_b$, is related to $d_{ab}$ by $d^c_{ab}(g) = a^{-1}_{ab}(g)$. This can be derived from the consistency of the invariant scalar product on the
double:

\[ < J^a, J^b > = 0, ~ < T^a_a, T^b_b > = 0, ~ < J^a_a, T^b_b > = \delta^b_a, \quad (74) \]

From the explicit form of these matrices we find the \( \Pi \) matrix as:

\[
\Pi = b(g)a^{-1}(g) = \begin{bmatrix}
0 & -c \cos(t) + c & \sin(f) c \sin(t) \\
c \cos(t) - c & 0 & -\cos(f) c \sin(t) \\
-\sin(f) c \sin(t) & \cos(f) c \sin(t) & 0
\end{bmatrix}. \quad (75)
\]

The Lagrangian of the original model is given by

\[
\mathcal{L} = (E + \Pi)^{-1} \partial_+ g g^{-1} \partial_- g g^{-1}
\]

where \( E \) is an arbitrary constant matrix and \((\partial_+ g g^{-1})_i\) are the right invariant one forms: \((\partial_+ g g^{-1})_i = \text{Tr}(\partial_+ g g^{-1} J_i)\). Various choices of \( E \) give different (dualizable) \( \sigma \) models living on the same double, and the expression in Eq.(74) is the most general solution of the dualizability conditions. Since both \((\partial_\pm g g^{-1})_i\) and \( \Pi \) depend only on the derivatives of the \( p \) field but not on \( p \) itself, \( \mathcal{L} \) is invariant under the \( p \) translations for any choice of \( E \). The simplest choice of the original model is to set \( E \equiv \text{Id} \), and in this case we find \((t_\pm \equiv \partial_\pm t \text{ etc.})\)

\[
\mathcal{L} = \frac{16}{-1 + 2 c^2 \cos(t)} \left[ -t_+ t_- + p_+ c_2 f_+ - f_+ c_2 f_- - p_- p_+ - p_+ c_2 p_- \\
- f_+ f_- + f_+ c_2 p_- - c(-t_- f_+ + t_+ p_- - t_- p_+ + t_+ f_-) \sin(t) \\
+ \left(2 p_+ c_2 p_- + 2 f_+ c_2 f_- - 2 p_+ c_2 f_- - 2 f_+ c_2 p_- - p_+ f_- - f_+ p_- \right) \cos(t) \\
- c^2 (-f_+ + p_+) \cos(t) \right]. \quad (77)
\]

Note that \( \mathcal{L} \) is also invariant under the \( f \) translations. Writing

\[
\mathcal{L} = g_{\mu\nu} \partial_\mu \xi^\mu \partial_\nu \xi^\nu + \epsilon_{ij} \partial_\mu \xi^\mu \partial_\nu \xi^\nu
\]

\((\xi^\mu = t, f, p)\) effectively means that we identify the metric as the coefficients of \((\xi_+^\mu \xi_-^\nu + \xi_-^\mu \xi_+^\nu)/2\) and the torsion potential, \( B_{\mu\nu} \), as the coefficients of \((\xi_+^\mu \xi_-^\nu - \xi_-^\mu \xi_+^\nu)/2\) in Eq.(74).

It turns out that this \( B_{\mu\nu} \) is a pure gauge, as it has vanishing torsion. Therefore the one loop counterterm, \( \Sigma_1 \), is determined by the (ordinary) Ricci tensor, \( R_{\mu\nu} \), which is symmetric in \( \mu, \nu \).

Since Eq.(74) (with an arbitrary \( E \)) is the most general solution of the dualizability condition we can go on with this construction if the change in the background caused by renormalization can be absorbed by changing the \( E \) matrix. Thus the – one loop – compatibility between Poisson-Lie dualizability and renormalization requires

\[
\alpha' R_{\mu\nu} \partial_+ \xi^\mu \partial_- \xi^\nu = \alpha' \frac{\partial \mathcal{L}}{\partial n_k} Y_k(n), \quad (79)
\]
where the \( n_k (k = 1, \ldots, 9) \) parametrize the \( E = E(n_k) \) matrix (e.g. \( n_k \) are its elements) and change as \( n_k \to n_k + \alpha' Y_k(n) \) under renormalization. In our case \( E(n_k)|_{n=0} = \text{Id} \) thus the right hand side of Eq. (79) becomes

\[
- [(\text{Id} + \Pi)^{-1}\partial E(n)_k]|_{n=0}(\text{Id} + \Pi)^{-1}]_{ij}(\partial_+ g^{-1})_i(\partial_- g^{-1})_j
\]

(80)

where \( S \equiv \frac{\partial E(n)_k}{\partial n_k}|_{n=0} \) is a completely arbitrary \( 3 \times 3 \) matrix. Thus in our case the compatibility between the renormalization flow and the dualizability condition depends on whether we can choose the (a priori completely arbitrary) elements of \( S \) in such a way that

\[
R_{\mu\nu} \partial_+ \xi^\mu \partial_- \xi^\nu = -[(\text{Id} + \Pi)^{-1}S(\text{Id} + \Pi)^{-1}]_{ij}(\partial_+ g^{-1})_i(\partial_- g^{-1})_j
\]

(81)

holds, where the left hand side is given by the specific \( R_{\mu\nu} \) following from Eq. (77). More precisely the symmetric part of \( M_{\mu\nu} \) must coincide with \( R_{\mu\nu} \), while for its antisymmetric part it is enough to require a vanishing torsion:

\[
\partial_p(M_{[12]}) + \partial_t(M_{[23]}) + \partial_f(M_{[31]}) = 0.
\]

(82)

These requirements lead only to \( 6 + 1 = 7 \) equations for the 9 unknown elements of \( S \). Writing

\[
S = \begin{pmatrix}
  m_1 & m_2 & m_3 \\
  m_4 & m_5 & m_6 \\
  m_7 & m_8 & m_9
\end{pmatrix}
\]

(83)

Maple found from these seven equations that all the other \( m_k \) can be determined in terms of \( m_2 \) and \( m_4 \) if they are related as \( m_4 = -m_2 \). However, looking at the explicit solution we deduce that the \( m_k \) obtained this way are constant (i.e. are independent of \( t, f \) and \( p \)) only for \( m_2 = c \), and for these values \( S \) takes the form:

\[
S = \begin{pmatrix}
  \frac{1}{2} & c & 0 \\
  -c & \frac{1}{2} & 0 \\
  0 & 0 & 2c^2 + \frac{1}{2}
\end{pmatrix}
\]

(84)

Thus – for the simplest original model at least – the dualizability condition seems to be satisfied along the renormalization flow, and this allows us to construct the righthand side of (36). The validity of (36) then means that the dualization of the model with bare couplings of the form (38) will lead to a dual theory which is finite in the limit \( \epsilon \to 0 \), for a suitable choice of the diffeomorphism and gauge transformation implicit in (36). In order to see this, we will study the renormalization of the dual model and check that the required renormalizations of the dual couplings are exactly those induced from the original model by the duality transformation.

\[\text{Since } t, f \text{ and } p \text{ have a geometrical meaning (i.e. they are angular variables on } S^3 \text{) we do not expect any field renormalizations (reparametrizations) for them, that’s why these terms are missing in Eq. (79).}\]
According to the Klimčík-Severa construction the dual Lagrangian is again of the same form as in (70):

$$\tilde{L} = (\tilde{E} + \tilde{\Pi})^{-1} (\partial_+ \tilde{g} \tilde{g}^{-1})_i (\partial_- \tilde{g} \tilde{g}^{-1})_j$$  \hspace{1cm} (85)

The constant matrix $\tilde{E}$ is related to $E$ by $E \tilde{E} = \text{Id}$, and this should be true even when $E$ undergoes a renormalization as above:

$$\tilde{E} \equiv \text{Id} + \alpha' \tilde{S} = E^{-1} = \left(\text{Id} + \alpha' S\right)^{-1} = \text{Id} - \alpha' S + \ldots . \hspace{1cm} (86)$$

On the other hand, $\tilde{S}$ will be determined independently by the analog of the above renormalization analysis for the dual model, and thus (86) is a nontrivial condition.

Parametrizing the elements of the dual group as

$$\tilde{g} = \exp(XT^1) \exp(YT^2) \exp(ZT^3), \hspace{1cm} (87)$$

we find the dual equivalent of the $\Pi$ matrix in the form:

$$\tilde{\Pi} = \begin{bmatrix} 0 & \frac{X^2 e^2 + Y^2 e^2 + e^{-2} Z e^{-1}}{2 e} & Y \\ -\frac{X^2 e^2 + Y^2 e^2 + e^{-2} Z e^{-1}}{2 e} & 0 & -X \\ -Y & X & 0 \end{bmatrix}. \hspace{1cm} (88)$$

(Note that for $c \to 0$ $\tilde{\Pi}$ reduces to what we already know from the non-Abelian dual of the principal model). Using Eq.(87) the coefficients of the right invariant one form $\partial_+ \tilde{g} \tilde{g}^{-1} = (\partial_+ \tilde{g} \tilde{g}^{-1})_i T^i$ are given by very simple expressions:

$$(\partial_+ \tilde{g} \tilde{g}^{-1})_1 = X_\pm + cXZ_\pm, \quad (\partial_+ \tilde{g} \tilde{g}^{-1})_2 = Y_\pm + cYZ_\pm, \quad (\partial_+ \tilde{g} \tilde{g}^{-1})_3 = Z_\pm. \hspace{1cm} (89)$$

The dual Lagrangian, following from the dual version of Eq.(76) (with $\tilde{E} = \text{Id}$), becomes easily tractable if we use cylindrical coordinates:

$$X = r \cos(\alpha), \quad Y = r \sin(\alpha), \quad Z = z. \hspace{1cm} (90)$$

Indeed introducing the notation $\chi^1 = r$, $\chi^2 = \alpha$, $\chi^3 = z$, and defining

$$D = c^4 r^4 + 2c^2 r^2 \left(1 + e^{-2cz}\right) + 4c^2 + \left(1 - e^{-2cz}\right)^2, \hspace{1cm} (91)$$

together with

$$M = r^2 e^2 + 2e^2 + 1 - e^{-2cz}, \hspace{1cm} (92)$$

the Lagrangian, $\tilde{\mathcal{L}} = \tilde{g}_{\mu\nu} \partial_\mu \chi^\nu \partial_\nu \chi^\nu + \tilde{b}_{\mu\nu} \epsilon_{ij} \partial_\mu \chi^i \partial_\nu \chi^j$, assumes the form:

$$\tilde{\mathcal{L}} = \frac{1}{D} \left[4c^2(1 + r^2) \partial_t r^2 + 4c^2 r^2 \partial_\alpha^2 + \left(4c^2(e^{-2cz} - c^2) + M^2\right) \partial_z^2 + 4crM \partial_r \partial^i z + 2 \epsilon_{ij} (-2cr \left(-1 + e^{-2cz} + r^2 e^2\right) \partial^r \partial^i \alpha + 2c^2 r^2 \left(1 + e^{-2cz} + r^2 e^2\right) \partial^i \alpha \partial^j z) \right]. \hspace{1cm} (93)$$
Note that this Lagrangian is independent of $\alpha$. The metric appearing here has only four non vanishing components while the antisymmetric tensor field, $b_{\mu\nu}$, has two non vanishing components. The one loop counterterm is determined by the generalized Ricci tensor. The explicit form of $\tilde{R}_{\mu\nu}$ reveals that $\tilde{R}_{12} = -\tilde{R}_{21}$, $\tilde{R}_{23} = -\tilde{R}_{32}$, $\tilde{R}_{13} = \tilde{R}_{31}$, i.e. the one loop counterterm contains no new derivative couplings that are not present in $\tilde{L}$.

In the dual model we have to check whether the dual version of the renormalizability condition, Eq.(73),

$$\alpha' \tilde{R}_{\mu\nu} \partial_+ \chi^\mu \partial_- \chi^\nu = \alpha' \frac{\partial \tilde{L}}{\partial \tilde{m}_k} \tilde{Y}_k(\tilde{n}) + \alpha' \frac{\delta \tilde{L}}{\delta \chi^\mu} \chi_1^\mu(\chi^\nu),$$

is satisfied by the $\tilde{E}$ in Eq.(84). In this equation $\chi_1^\mu(\chi^\nu)$ denote the potential reparametrizations (which we expect, since in the dual of the principal model they were also present), and as usual, equality is required modulo the gauge transformations on the antisymmetric parts.

From the (anti)symmetry properties of $\tilde{R}_{\mu\nu}$ – by the same arguments as in the paper on NAD, Balázs et al. [14], – we conclude that the reparametrizations must have the form:

$$r_0 \to r + \frac{\alpha'}{\epsilon} F(r, z), \quad z_0 \to z + \frac{\alpha'}{\epsilon} G(r, z), \quad \alpha_0 \to \alpha + \frac{\alpha'}{\epsilon} y_2 \alpha,$$

with $y_2$ being an $r$ and $z$ independent constant. Therefore the last term in Eq.(94) is independent of the $\alpha$ field. The first term in Eq.(94) is computed in the same way as in the case of the original model, Eq.(80): denoting $\frac{\partial E(\tilde{n})}{\partial \tilde{m}_k} \tilde{Y}_k(\tilde{n})|_{\tilde{n}=0} \equiv \tilde{S}$, it has the form:

$$-\left[ (\text{Id} + \tilde{\Pi})^{-1} \tilde{S}(\text{Id} + \tilde{\Pi})^{-1} \right]_{ij} (\partial_+ \tilde{g} \tilde{g}^{-1})_i (\partial_- \tilde{g} \tilde{g}^{-1})_j \equiv \tilde{M}_{\mu\nu} \partial_+ \chi^\mu \partial_- \chi^\nu.$$  

A hopeful sign of consistency comes from the observation that $\tilde{M}_{\mu\nu}$ becomes $\alpha$ independent (which we need as the other two pieces in Eq.(94) are also $\alpha$ independent), if $\tilde{S}$ has the form:

$$\tilde{S} = \begin{pmatrix} \tilde{m}_1 & \tilde{m}_2 & 0 \\ -\tilde{m}_2 & \tilde{m}_1 & 0 \\ 0 & 0 & \tilde{m}_9 \end{pmatrix},$$

with arbitrary $\tilde{m}_1$, $\tilde{m}_2$ and $\tilde{m}_9$. Furthermore this choice of $\tilde{S}$ yields an $\tilde{M}$ that also satisfies $\tilde{M}_{12} = -\tilde{M}_{21}$, $\tilde{M}_{32} = -\tilde{M}_{23}$, $\tilde{M}_{13} = \tilde{M}_{31}$. Therefore we use this $\tilde{M}_{\mu\nu}$ in Eq.(94).

It is straightforward to obtain the symmetric part of Eq.(94), i.e. the equalities of the coefficients of $(\partial r)^2$, $(\partial r)^2$, $(\partial z)^2$ and $\partial_1 r \partial^2 z$ on the two sides. Denoting by $(LS)_{\mu\nu}$, $(LA)_{\mu\nu}$ the symmetric and antisymmetric coefficients of the field derivatives in $\frac{\delta \tilde{L}}{\delta \chi^\mu} \chi_1^\mu(\chi^\nu)$:

$$\frac{\delta \tilde{L}}{\delta \chi^\mu} \chi_1^\mu(\chi^\nu) = (LS)_{\mu\nu} \partial_1 \chi^\mu \partial^2 \chi^\nu + (LA)_{\mu\nu} \epsilon_{ij} \partial^i \chi^\mu \partial^j \chi^\nu,$$  

(98)
and introducing the notation:

\[ \partial_r F = FR, \quad \partial_z F = FZ, \quad \partial_r G = GR, \quad \partial_z G = GZ, \]

the symmetric part of Eq.(94) can be written as:

\[ (LS)_{\mu\nu}(F, G, FR, FZ, GR, GZ) = \tilde{R}_{\mu\nu} - \tilde{M}_{\mu\nu}, \quad (\mu \nu = 11, 22, 33, 13). \]

(Note that the left hand side depends linearly on \( F, G, FR, FZ, GR, \) and \( GZ \)). We handle the antisymmetric part of Eq.(94) in the following way: we define \( \Delta^{12} \) and \( \Delta^{23} \) as the coefficients of \( \epsilon_{ij} \partial^i r \partial^j \alpha \) (respectively \( \epsilon_{ij} \partial^i \alpha \partial^j z \)) in the difference between the left hand sides and the right hand sides of Eq.(94). Then the requirement that the two sides of Eq.(94) be equal up to a gauge transformation can be written as:

\[ \partial_z (\Delta^{12}) + \partial_r (\Delta^{23}) = 0. \]

Naively this equation contains second derivatives of \( F(r, z) \) and \( G(r, z) \), however it is easy to show, that the antisymmetry of \( \tilde{b}_{\mu\nu} \) guarantees that all second derivatives of \( F \) and \( G \) cancel from Eq.(101). Therefore Eq.(100) and Eq.(101) give five (linear) equations for the six unknowns \( F, G, FR, FZ, GR \) and \( GZ \). The great surprise is that \( F \) and \( G \) can be expressed completely algebraically from these equations, while \( FR, GR \) and \( FZ \) are given by three (inhomogeneous) linear expressions in \( GZ \). Renormalizability of the dual model amounts to the problem of choosing the \( y_2, \tilde{m}_1, \tilde{m}_2 \) and \( \tilde{m}_9 \) parameters in such a way (if there is any), that the \( r \) and \( z \) derivatives of the algebraically determined \( F \) and \( G \) functions satisfy the aforementioned three linear equations between \( FR, GR, FZ \) and \( GZ \). The claim is that

\[ y_2 = 0, \quad \tilde{S} = - \begin{pmatrix} \frac{1}{2} & c & 0 \\ -c & \frac{1}{2} & 0 \\ 0 & 0 & 2c^2 + \frac{1}{2} \end{pmatrix}, \]

is the only such choice. In this case, \( F \) and \( G \) simplify to

\[ F = \frac{c^2 r}{D} \left( P^2 + 4P - 1 - 4c^2 + 2r^2c^2P + c^4 r^4 \right), \]

\[ G = -\frac{c}{D} \left( P^2 - 1 - 4c^2 + 4r^2c^2 - 2r^2c^2P + c^4 r^4 \right), \]

where \( P = e^{-2cz} \).

Thus \( \tilde{S} = -S \) as required by (86), and we conclude that quantum T duality holds for our PL example. So we see that \( - \) at least for \( E = \text{Id} \) – the infinitesimal renormalization flow and the PL duality transformations commute in the same way as for the Abelian

\[ ^6 \text{Note that } FR, FZ, GR, GZ \text{ are single variables and should not be read as products.} \]
and non-Abelian cases. The $F$ and $G$ reparametrizations appearing in Eq. (103-104) have some interesting properties. For $c \to 0$ they give

$$F \to \frac{r}{1 + r^2 + z^2}, \quad G \to \frac{z}{1 + r^2 + z^2},$$

which is: (I) identical to what was found for the NAD of the principal model (of course without the rescaling by the coupling constant), (II) identical to what is obtained from the ‘dilaton shift’ for the $c \equiv 0$ (NAD) case (see Eq. (51)). Tyurin and Unge, [6], gave an expression for the (dual) dilaton shift in the general case; for our model ($c \neq 0$) this dilaton shift is proportional to $\ln D$. Note, however, that the appropriate gradient of this, $\frac{1}{2}\tilde{g}^{ij}\partial_j \ln D$, gives expressions, which are different from those in Eq. (103-104).

Nevertheless these $F$ and $G$ can also be written as the components of a gradient, namely as $\frac{1}{2}\tilde{g}^{ij}\partial_j \ln(De^{2cz})$. This may be interpreted in two ways: either that for PL duality the relation between the reparametrization and the dilaton shift is different from the one found for the Abelian and non-Abelian dualities, or that we are working in a coordinate system, which is not the ‘natural’ one (i.e. where the connection between the dilaton shift and the reparametrizations is the simple gradient form). Also an interesting problem is that our $\tilde{f}^{ab}_c \equiv f^{ab}_c$ have non vanishing traces $f^{31}_1 = f^{32}_2 \neq 0$ – when Tyurin and Unge found that quantum equivalence is broken already in the conformal case – yet we found this equivalence in the form of the commutative nature of the (one loop) renormalization flow and the PL duality transformations – at least at a certain point of the modulus space described by $E = Id$.

### 5 Conclusions and discussion

In this paper we studied the question of quantum equivalence among dually related sigma models in perturbation theory. Using the anomalous Ward-identity for Weyl symmetry, we derived the Haagensen-Olsen [10] relation between the Weyl-anomaly coefficients (and beta functions) of these models and their duals in a general context of Abelian or non-Abelian dualities. We obtained this relation (to 1-loop order in terms of renormalized quantities) from the assumption that the standard gauging procedure is valid for the bare fields. We pointed out that it also implies the quantum equivalence of the dual models in the sense that duality induces the correct renormalization of the dual model from the renormalization of the original one.

This simple criterion - which we believe to hold even in the PL context - certainly provides a convenient way to check quantum T duality at an elementary level, as we illustrated through the study of a non-Abelian example (based on $SU(2)$) with a single spectator field.

On the other hand, it is clear that a comprehensive analysis must go further than
this, for several reasons: As T duality should be checked for physical quantities and not just for the partition function, sources should be introduced. Second, the question of the existence of a consistent regularization compatible with duality should be addressed; dimensional regularization, while formally preserving duality, has well-known problems with the antisymmetric tensor coupling (though this should be irrelevant at the one-loop level we have been considering). Finally, an extension of our general argument to all orders in loops seems desirable. The reason why we restricted our attention to the 1-loop order is that the 2 (and higher) loop problems are not only technically more involved, but also conceptionally different from the 1-loop case because of the quantum modifications of both the transformation formulae and the renormalization ‘scheme’ [1]-[4].

At the two-loop level, up to now the situation is clear only in the Abelian case [13] where the corrections to the classical duality transformation have been obtained in general. The derivation of these corrections for the non-Abelian case would be interesting, since no special non-Abelian example with two loop equivalence is known at present.

Moreover much work remains to be done in the larger context of PL duality. Our general considerations do not immediately apply for the case of genuine Poisson-Lie dualities, due to the problem that PL dualizability - not being related to any isometry in general - cannot as easily be seen to survive renormalization as in the traditional Abelian and non-Abelian cases. For the simple PL example we studied, however, it turned out that dualizability is indeed compatible with renormalization at 1 loop, and quantum equivalence could be established on the same level as for the non-Abelian example.

This finding, when taken together with the one loop equivalence found for sigma models with two dimensional target spaces [16], and the quantum equivalence of the special WZNW models described in Ref. [15], gives some support to the expectation that quantum PL T-duality exists in a form similar to the Abelian and non-Abelian cases. For PL dualities, however, even the question of one-loop equivalence has not been settled. The conformal example of Ref. [15] and the non conformally invariant ones of Ref. [16] and of section 4 are defined on a Drinfeld double which does not satisfy the conditions which, according to Ref. [8], guarantee that the dual theory is conformally invariant if the original one is. Obviously more work is needed to clarify this point.

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