Relative entanglement entropy of thermal states of Klein-Gordon and Dirac quantum field theories

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Abstract

An upper bound of the relative entanglement entropy of thermal states at an inverse temperature $\beta$ of linear, massive Klein-Gordon and Dirac quantum field theories across two regions, separated by a nonzero distance $d$ in a Cauchy hypersurface of an ultrastatic spacetime has been computed. This entanglement measure is bounded by a negative constant times $\ln|\tanh(\pi d/2\beta)|$ which signifies power law decay for asymptotic $d$ where the exponent depends on $\beta < \infty$.

1 Introduction

The relative entanglement entropy is a good measure of entanglement for generic states in quantum field theory in Minkowski and curves spacetimes [1]. Hitherto, upper bounds of this entanglement measure have been computed for pure global states [1–3], albeit the measure is well applicable to global mixed states. Thermal states are a class of global mixed states characterized by the Kubo [4]-Martin-Schwinger [5] condition (the mathematical rigour formulation is due to Haag et al. [6]; see also the reviews [7, 8] and the monograph [9] for details), that describe a wide range of phenomena in elementary particle physics, cosmology and condensed matter physics.

In this report, we have considered the Klein-Gordon and the Dirac quantum field theories in a Cauchy hypersurface of an ultrastatic spacetime. By making use of the techniques developed in [1, 2, 10], we implement the required computational modifications to account the mixed nature of thermal states to compute the relative entanglement entropy across two regions of non-vanishing distance $d$ in the Cauchy hypersurface. These modifications result a different bound for the entanglement measure of thermal state compared to the ground state. More precisely, the relative entanglement entropy is bounded by $-\text{cst. } \ln|\tanh(\pi d/2\beta)|$ for both theories which

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implies that this exhibits power law decay with asymptotic $d$, where the exponent depends on the inverse temperature $\beta < \infty$.

Our sign convention for the spacetime metric is mostly minus and the scalar curvature $R$ is positive on the sphere. At first, the Dirac field is presented inasmuch that requires more general strategy than the Klein-Gordon field to prove the results.

2 Relative entanglement entropy of Dirac-Majorana QFT

We begin the section by a minimal review of the quantization of a classical, linear Dirac-Majorana field in a $D \in \{4, 8, 9, 10\}$ mod 8-dimensional\footnote{The restriction of the spacetime dimensions stems from the existence of the time-reversal operator [2, Lem. V.1] required to prove our results and existence of (pseudo-)Majorana spinors [11, 12].} simply-connected ultrastatic spin-spacetime $\mathcal{M} = \mathbb{R} \times \mathcal{C}$ with metric $g = dt^2 - h$ and $H^1(\mathcal{M}) = 0$, where $h$ is the Riemannian metric, independent of $t$ on the on the Riemannian spin-manifold $(\mathcal{C}, h)$ which is assumed to be complete. We assume that $e^0$ be the forward directed time-like normal vector field on the Cauchy hypersurface $\mathcal{C}$ and let $\mathcal{R} := (L^2(\mathcal{M} \upharpoonright \mathcal{C}), \langle \cdot, \cdot \rangle, \Gamma)$ be the space $L^2(\mathcal{M} \upharpoonright \mathcal{C})$ of square-integrable Cauchy data $k$ with respect to the positive definite, non-degenerate hermitian inner product $\langle \cdot, \cdot \rangle$, equipped with an antilinear involution $\Gamma$ [12]. Here $\mathcal{M} \rightarrow \mathcal{M}$ is the spinor bundle whose details is not important for the primary content of this article, and so we resist its exposition and refer [13, 14] and references therein, for instance.

We will quantize the classical spinors in the selfdual framework by Araki [15] (see also [13, 14]). In this approach a quantum Fermi field $\psi$ defined by a Cauchy hypersurface $\mathcal{C}$ is an algebra of canonical anticommutation relations $\text{CAR}(\mathcal{R})$-valued $\mathbb{C}$-linear distribution $\psi : \mathcal{R} \rightarrow \text{CAR}(\mathcal{R})$. The algebra $\text{CAR}(\mathcal{R})$ is defined as the free unital $\ast$-algebra over $\mathcal{R}$ generated by the symbols $1, \psi(k), \psi(l)^\ast$ modulo the relations

$$\psi(k)^\ast = \psi(\Gamma k), \quad \{\psi(k), \psi(l)^\ast\} = \langle l, k \rangle 1, \quad \forall k, l \in \mathcal{R}. \quad (1)$$

There exists a unique $C^*$-norm [14, 15] on $\text{CAR}(\mathcal{R})$ whose closure $\overline{\text{CAR}(\mathcal{R})}$ defines a $C^*$-algebra. The local algebra of field $\overline{\text{CAR}(\mathcal{R})}$ corresponding to some bounded open region $\mathcal{V} \subset \mathcal{C}$, is by definition, the $C^*$-subalgebra generated by all elements of the form $\psi(k)$ with $\supp(k) \subset \mathcal{V}$.

The local algebra of observables $\mathcal{D}(\mathcal{V})$ is the $C^*$-subalgebra of $\overline{\text{CAR}(\mathcal{R})}$, consisting of only the even elements of $\overline{\text{CAR}(\mathcal{R})}$.

An algebraic thermal state $\Psi$ is a linear functional of $\mathcal{D}(\mathcal{C})$, uniquely characterized by the bounded operator [16, Prop. 1], [15, Thm. 3] (see also, e.g., [9])

$$\Sigma_{\Psi} := \tanh \frac{\beta H}{2}, \quad (2)$$

where $H$ is the Dirac Hamiltonian and $\beta$ is the inverse temperature. Applying the Gelfand-Naimark-Segal (GNS) construction for $\Psi$, we obtain the GNS-triple $(\pi_{\Psi}, \mathfrak{F}_{\Psi}, \Omega_{\Psi})$ where $\Omega_{\Psi} \in \mathbb{C}$ is the GNS-vector and $\pi_{\Psi}$ is a $\ast$-representation of $\mathcal{D}(\mathcal{C})$ on the GNS-Hilbert space $\mathfrak{F}_{\Psi} :=$
\( \mathbb{C} \oplus (\oplus_{n=1}^{\infty} \mathfrak{F}_\Psi) \), where the “1-particle” Hilbert space \( \mathfrak{F}_\Psi \) is constructed from \( \mathfrak{R} \) by dividing out \( \ker(I + \Sigma_\Psi)/2 \), and taking closure with respect to the inner product \([13]\)

\[
([k][l])_\Psi := \frac{1}{2} \langle k, (I + \Sigma_\Psi)l \rangle, \quad \forall k, l \in \mathfrak{R}.
\]  

Employing the GNS-representation, we define the local von Neumann algebra of observables as

\[
\mathfrak{F}_\Psi(\mathcal{Y}) := \pi_\Psi(\mathcal{D}(\mathcal{Y}))''
\]

where “” is the double commutant.

Suppose that \( \mathcal{A}, \mathcal{B} \subset \mathcal{C} \) be any two open sets. We recall that (see, e.g., [1] and references therein for the required background of entanglement in quantum field theoretic setting, and e.g. [9, 17] for the relevant concepts of von Neumann algebra) the relative entropy \( S(\omega, \rho) \) between two global states \( \omega \) and \( \rho \) is defined\(^2\) by the Araki’s formula \([18, 19]\), and the relative entanglement entropy \( E(\omega) \) of a global state \( \omega \) across the bipartite system \((\mathfrak{F}_\Psi(\mathcal{A}), \mathfrak{F}_\Psi(\mathcal{B}))\) of two commuting local von Neumann algebras (in the standard form) \( \mathfrak{F}_\Psi(\mathcal{A}) \) and \( \mathfrak{F}_\Psi(\mathcal{B}) \) is defined by the infimum of \( S(\omega, \rho) \) over the weak *-convex hull of separable states \( \rho \) for all type I von Neumann factors that splits the bipartition \([20]\).

Now, we state the main result of this section in

**Theorem 2.1** Let \( \mathcal{C} \) be a static Cauchy hypersurface in a geodesically complete, simply connected, \( D \in \{4, 8, 9, 10\} \mod 8 \)-dimensional ultrastatic spin-spacetime \( (\mathcal{M} = \mathbb{R} \times \mathcal{C}, g = \text{dt}^2 - h) \) such that \( \inf(m^2 + R(x)/4) > 0 \) holds on \( (\mathcal{C}, h) \), where \( R \) is the scalar curvature of \( (\mathcal{C}, h) \) and \( m \) the mass of the Dirac field. Then the relative entanglement entropy \( E(\Psi) \) for a thermal state \( \Psi \) at an inverse temperature \( \beta \) of the linear Dirac quantum field theory between any open subsets \( \mathcal{A}, \mathcal{B} \subset \mathcal{C} \) such that \( d := \text{dist}(\mathcal{A}, \mathcal{B}) \geq \delta > 0 \), is bounded by

\[
E(\Psi) \leq -\text{cst.} \ln \left| \tanh \frac{\pi d}{2 \beta} \right|,
\]

where \( \text{cst.} \) is a positive constant which depends on \( M := \sqrt{\inf(m^2 + R(x)/4)} \), \( \delta \) and the geometry within a \( \delta \)-neighborhood of \( \mathcal{A} \), and so is independent of \( d \) and \( \beta \).

**Proof** We employ the same strategy as in [2, Sec. V] with the essential replacement of local von Neumann algebra and Fock space for ground state by those, \( \mathfrak{F}_\Psi(\mathcal{Y}) \) and \( \mathfrak{F}_\Psi \) of the thermal state \( \Psi \), respectively. Since, the GNS-vector \( \Omega_\Psi \) is cyclic and separating [21, Thm. 4.8] for \( \mathfrak{F}_\Psi(\mathcal{B}') \) where \( \mathcal{B}' := \mathcal{C} \setminus \mathcal{B} \), we consider the pair \((\mathfrak{F}_\Psi(\mathcal{B}'), \Omega_\Psi)\) and define the map

\[
\Xi^{\mathfrak{F}_\Psi}_{\mathfrak{F}_\Psi} : \mathfrak{F}_\Psi(\mathcal{A}) \to \mathfrak{F}_\Psi, \quad A \mapsto \Xi^{\mathfrak{F}_\Psi}_{\mathfrak{F}_\Psi}(A) := \Delta^{\frac{1}{2}}_{\Psi, \mathfrak{F}_\Psi} A \Omega_\Psi.
\]

As a consequence of Theorems 3 and 4 in [1]:

\[
E(\Psi) \leq \log \min \left( \left\| \Xi^{\mathfrak{F}_\Psi}_{\mathfrak{F}_\Psi} \right\|_1, \left\| \Xi^{\mathfrak{F}_\Psi}_{\mathfrak{F}_\Psi} \right\|_1 \right),
\]

\(^2\)The explicit formulation is not required for the computation in this article.
where the minimum is taken to get a quantity that is symmetric under the exchange of $\mathcal{A}$ with $\mathcal{B}$. Thus we look for an upper bound of the 1-nuclear norms (see, e.g. [10] for mathematical details) of $\Xi^\mathcal{A}$ and $\Xi^\mathcal{B}$. We note that the 1-particle Dirac Hamiltonian commutes with the time-reversal operator [2, Lem. V.1], and so does the operator $\Sigma$ by the spectral theorem. This implies that the 1-particle Hilbert space $\widetilde{\mathcal{H}}$ is invariant under the time-reversal operator and so is the standard real subspace (see, e.g. [22] for details on standard subspace), as the time-reversal operator preserves localization [2, Lem. V.1]. Furthermore, one can employ analogous arguments as in [2, Lem. V.2] to deduce that the 1-particle modular operator commutes with the time reversal operator. Then the arguments (for instance, the doubling prescription, construction of closed complex linear subspaces of $\widetilde{\mathcal{H}}$) in [2, Sec. V] smoothly flow over and we can apply Theorem 3.11 and Theorem 3.5 in [10] with Proposition IV.2 in [2] to deduce that the nuclear norm of $\Xi^\mathcal{A}$ is bounded by the trace-norm of $(P_{\mathcal{B'}} - \Sigma^2_{\mathcal{B},\mathcal{B'}})^{1/4} \uparrow \mathcal{R}(\mathcal{A})$ where $\mathcal{R}(\mathcal{A}) := \mathcal{R} \uparrow \mathcal{A}$, $P_{\mathcal{B'}} : \mathcal{R} \to \mathcal{R}(\mathcal{B'})$ is the projector and $\Sigma_{\mathcal{B},\mathcal{B'}} := \Sigma_{\mathcal{B}} \Sigma_{\mathcal{B'}}$ is the restriction of $\Sigma$ on $\mathcal{R}(\mathcal{B'})$ (cf. [2, Eq. 33]). Altogether, we arrive at

$$E(\Psi) \leq \text{cst. } \left\|(P_{\mathcal{B'}} - \Sigma^2_{\mathcal{B},\mathcal{B'}})^{1/4} \uparrow \mathcal{R}(\mathcal{A}) \right\|_1.$$  

(8)

Thus, our task boils down to the

**Proposition 2.1** *In the preceding notations, on a complete Riemannian spin-manifold $(\mathcal{C}, h)$, we have for any $\delta > 0$*

$$\left\|(P_{\mathcal{B'}} - \Sigma^2_{\mathcal{B},\mathcal{B'}})^{1/4} \uparrow \mathcal{R}(\mathcal{A}) \right\|_1 \leq -\text{cst. } \ln \tanh \frac{\pi d}{2 \beta}, \quad \forall d \geq \delta,$$  

(9)

*where cst. is a positive constant independent of $d$ and $\beta$ but may depend on $\delta$ and $M$.*

**Proof:** Complying to the Prop. V.1 in [2], we note that $0 < \Sigma^2 < I$ in contrast to the pure states (where it is an involution). Let $P_{\mathcal{A'}}$ be the restriction map $\mathcal{R} \to \mathcal{R}(\mathcal{A'})$ for $\mathcal{A'} = \mathcal{A}, \mathcal{B}, \mathcal{B'}$ and compute using the properties of projectors

$$P_{\mathcal{A}}(P_{\mathcal{B'}} - \Sigma^2_{\mathcal{B},\mathcal{B'}})P_{\mathcal{A}} = P_{\mathcal{A}}(I - \Sigma \Sigma_{\mathcal{B}})P_{\mathcal{A}}$$

$$\leq P_{\mathcal{A}}(I - \Sigma \Sigma_{\mathcal{B}})P_{\mathcal{A}} + P_{\mathcal{A}} \Sigma^2 P_{\mathcal{A}}$$

$$= P_{\mathcal{A}} + \left| P_{\mathcal{A}} \Sigma P_{\mathcal{A}} \right|^2$$

$$\leq (P_{\mathcal{A}} + |P_{\mathcal{A}} \Sigma P_{\mathcal{A}}|^2)^2.$$  

(10)

Employing the definition of projection operator and operator monotone property of the square root one can show that (cf. [2, Eqs. 37 and 38])

$$\left| (P_{\mathcal{B'}} - \Sigma^2_{\mathcal{B},\mathcal{B'}})^{1/4} P_{\mathcal{A}} \right| \leq \sqrt{P_{\mathcal{A}}(P_{\mathcal{B'}} - \Sigma^2_{\mathcal{B},\mathcal{B'}})P_{\mathcal{A}}} \leq \tilde{\chi} P_{\mathcal{A}} + |P_{\mathcal{B}}(I - \tilde{\chi}) \Sigma \tilde{\chi} P_{\mathcal{A}}|,$$  

(11)

where we have used (10) and $\tilde{\chi}, \chi, \tilde{\chi}$ be the multiplication operators by the smooth functions $\tilde{\chi}, \chi, \tilde{\chi}$, respectively. These smooth functions are defined as follows [2]. We introduce two
intermediate regions between $\mathcal{A}$ and $\mathcal{B}$, called $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ such that $\mathcal{A} \subset \hat{\mathcal{A}} \subset \hat{\mathcal{B}} \subset \mathcal{B}$. Then the smooth functions $\hat{\chi}, \hat{\chi}, \chi$ are defined by: (i) $\text{supp}(\hat{\chi}) \subset \hat{\mathcal{A}}$ and $\hat{\chi} \equiv 1$ on $\hat{\mathcal{A}}$; (ii) $\text{supp}(\hat{\chi}) \subset \mathcal{B}$ and $\hat{\chi} \equiv 1$ on $\hat{\mathcal{B}}$, $1 - \hat{\chi} \equiv 1$ on $\hat{\mathcal{B}}$; (iii) The distance $\text{dist}(\text{supp}(\hat{\chi}), \text{supp}(1 - \hat{\chi})) = d - \varepsilon$, where $d := \text{dist}(\mathcal{A}, \mathcal{B})$ and $\varepsilon > 0$ is thought of as small; (iv) $\chi$ is a function of compact support such that $\chi \equiv 1$ on $\text{supp}(\hat{\chi})$, $\chi \subset \text{supp}(\hat{\chi})$ and $\text{dist}(\text{supp}(\hat{\chi}), \text{supp}(1 - \hat{\chi})) = d - 2\varepsilon$ (see Fig. 1 in [2] for a schematic visualization).

We utilize the supports of the smooth functions $\hat{\chi}, \chi, \hat{\chi}$ and properties of the trace-norm and operator norm to deduce

\[
\left\| (\hat{\mathcal{B}} - \Sigma^2_{\hat{\mathcal{B}}}) \hat{\mathcal{A}}_\chi \right\|_\infty \leq \left\| \hat{\mathcal{A}}_\chi \right\|_\infty + \left\| \hat{\mathcal{B}}^{\frac{1}{2}}(1 - \hat{\chi})\Sigma_\chi \hat{\mathcal{A}}_\chi \right\|_\infty \leq \left\| \hat{\mathcal{A}}_\chi \right\|_\infty + \left\| (1 - \hat{\chi})\Sigma_\chi \hat{\mathcal{A}}_\chi \right\| \left\| \hat{\mathcal{A}}_\chi \right\|_\infty. \tag{12}
\]

Finally we estimate the operator norm of $(1 - \hat{\chi})\Sigma_\chi \hat{\mathcal{A}}_\chi$ using (2) and by exploiting the finite propagation speed of the spinor wave operator, $\partial_t^2 + L, L := -\Delta_{\gamma\mathcal{E}} + \kappa /4 + m^2 - M^2$ following the original idea due to [23, Prop. 1.1] and subsequently generalized in [2, Appendix]:

\[
\left\| (1 - \hat{\chi})\Sigma_\chi \hat{\mathcal{A}}_\chi \right\| = \left\| (1 - \hat{\chi}) \frac{\beta H}{2} X \right\|
\leq \left\| (1 - \hat{\chi}) \frac{\beta |H|}{2} \right\|
= \frac{1}{\beta} \left\| (1 - \hat{\chi}) \int_0^\infty \tanh(s) \sin(s \sqrt{L}) \, ds \right\|
\leq \frac{1}{\beta} \left\| (1 - \hat{\chi}) \right\|_\infty \| X \|_\infty \int_{d-2\varepsilon}^\infty \left| \tanh(s) \right| \, ds
= \frac{1}{\beta} \left| I - \hat{\chi} \right|_\infty \| X \|_\infty \int_{d-2\varepsilon}^\infty \text{csch} \left( \frac{\pi s}{2 \beta} \right) \, ds
= \frac{1}{\beta} \left| I - \hat{\chi} \right|_\infty \| X \|_\infty \ln \left| \tanh \left( \frac{\pi d - 2\varepsilon}{2} \right) \right|. \tag{13}
\]

Here we have used $H \leq |H|$, monotonicity of $\tanh$, and Schrödinger [24]-Lechnerowaicz [25] formula to deduce $H^*H = L$ in the intermediate steps. The Fourier sine transformation $\tanh(s)$ of $\tanh(\beta H/2)$ is with prefactor $1/\pi$ in our convention and $\| \cdot \|_\infty$ is the suprimum norm.

### 3 Relative entanglement entropy of Klein-Gordon QFT

In this section, we present a minimal review of the quantization of a classical, real-linear Klein-Gordon field on any finite-dimensional $D \geq 3$, ultrastatic spacetime $(\mathcal{M}, g) = \mathbb{R} \times \mathcal{E}, dt^2 - h)$ and refer [14] and the expository articles [7, 8] for details. We let $(\mathcal{S} := C_c(\mathcal{E}, \mathbb{R}) \oplus C_c(\mathcal{E}, \mathbb{R}), \sigma)$ be the space $C_c(\mathcal{E}, \mathbb{R}) \oplus C_c(\mathcal{E}, \mathbb{R})$ of Cauchy data of the linear Klein-Gordon equation equipped with sympletic form $\sigma$. 

5
The algebraic Weyl algebra $\mathcal{W}(s)$ over the symplectic space $(s, \sigma)$ is generated by the symbols $W(0) := 1$ and $W(F)$, modulo the relations [26, 27] (see also [14, A.2] and the expository articles [7, 8]):

$$W(f)^* = W(-f), \quad W(f)W(g) = \exp\left(-\frac{i}{2}\sigma(f, g)\right)W(f + g), \quad \forall f, g \in s. \quad (14)$$

We turn the $*$-algebra $\mathcal{W}(s)$ into a $C^*$-algebra $\mathcal{W}^*(s)$, called the Weyl algebra, by taking the completion with respect to the topology induced by the unique $C^*$-norm on $\mathcal{W}(s)$.

A thermal state $\Phi$ is a $\mathbb{R}$-linear functional of $\mathcal{W}(s)$, given by [27] (see also the expository articles [7, 8]):

$$\Phi(W(f)) := \exp\left(-\frac{\langle f, f \rangle_\Phi}{2}\right), \quad \forall f \in s, \quad (15)$$

where $\langle \cdot, \cdot \rangle_\Phi$ is a real inner product on $s$ satisfying

$$\frac{1}{2} |\sigma(f, g)| \leq \sqrt{\langle f, f \rangle_\Phi} \sqrt{\langle g, g \rangle_\Phi}, \quad \forall f, g \in s. \quad (16)$$

This results a real Hilbert space $(s, \langle \cdot, \cdot \rangle_\Phi)$ (after taking the Hilbert space completion in the norm induced by $\langle \cdot, \cdot \rangle_\Phi$). It is convenient to introduce the complexification $\mathcal{E}$ of $s$ and the extension (with the convention of being antilinear in the first argument) $\langle \cdot \rangle_\Phi$ of $\langle \cdot, \cdot \rangle_\Phi$ to describe the GNS representation of $\Phi$. Then we have the complex Hilbert space (after taking completion as before) $(\mathcal{E}, \langle \cdot, \cdot \rangle_\Phi)$ and the Riesz representation theorem implies that there exists a unique, bounded, selfadjoint operator $\Sigma$ on $\mathcal{E}$ saturating (16): $i\sigma(f, g)/2 = \langle f | \Sigma g \rangle_\Phi$. Applying the GNS construction, we obtain the GNS triplet $(\pi_\Phi, \Phi_\pi, \Omega_\Phi)$, where $\Omega_\Phi \in \mathbb{C}$ is the GNS-vector and $\pi_\Phi$ is the $*$-representation of $\mathcal{W}(\mathcal{E})$ by bounded operators on the GNS-space $\Phi_\pi$ obtained from the symmetrized tensor product of the 1-particle Hilbert space $\Phi_\Phi$, defined by factoring out $\mathcal{E}$ by $\ker(I + \Sigma)/2$ and taking the completion with respect to the GNS inner product

$$\langle [F] | [G] \rangle_\Phi := \frac{1}{2} \langle F | (I + \Sigma)G \rangle_\Phi, \quad \forall F, G \in \mathcal{E}. \quad (17)$$

Employing the GNS-representation, we define the local von Neumann algebra of observables as

$$\mathcal{B}_\Phi(\mathcal{V}) := \pi_\Phi(\mathcal{W}(\mathcal{E}))'' = \pi_\Phi(\mathcal{W}(\mathcal{E}))(\mathcal{V}), \quad \forall F \in \mathcal{E} | \text{supp}(F) \subset \mathcal{V}, \quad (18)$$

where $''$ is the double commutant, as before.

Entanglement and its relative entropy of a state of the algebra $\mathcal{B}_\Phi(\mathcal{C})$ is defined exactly the same way as those for the fermionic case $\mathcal{F}_\Phi(\mathcal{C})$. Without repetition, we state the main result of this section as

**Theorem 3.1** Let $\mathcal{C}$ be a static Cauchy hypersurface in a geodesically complete, finite $D \geq 3$-dimensional ultrastatic spacetime $(\mathcal{M} = \mathbb{R} \times \mathcal{C}, g = dt^2 - h)$. Then, for a thermal state $\Phi$ at an inverse temperature $\beta$ of the linear Klein-Gordon quantum field theory of mass $m$, the relative
entanglement entropy $E(\Phi)$ between any open subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ such that $d := \text{dist}(\mathcal{A}, \mathcal{B}) \geq \delta > 0$, is bounded by

$$E(\Phi) \leq -\text{cst.} \ln \left| \tanh \frac{\pi d}{2 \beta} \right| . \quad (19)$$

Here the positive constant $\text{cst.}$ is only $m, \delta$ and the geometry within a $\delta$-neighborhood of $\mathcal{A}$ dependant, thus is independent of $d$ and $\beta$.

**Proof** The proof proceeds in parallel to that of our Theorem 2.1, when we note that $\Omega_{\Phi}$ is cyclic and separating [21, Thm. 4.8] for $\mathcal{R}_\Phi(\mathcal{B}')$ and $\Sigma := \tanh(\beta K/2)$ where $K := \sqrt{-\Delta + m^2}$ [28, Thm. 3.2] (see also, e.g., [9, Exam 5.3.2]). So we consider the modular operator $\Delta_{\mathcal{A}, \mathcal{B}'}$ for the pair $(\mathcal{R}_\Phi(\mathcal{B}'), \Omega_{\Phi})$ and define the map

$$\Xi_\Phi^\mathcal{A} : \mathcal{H}_\Phi(\mathcal{A}) \to \mathfrak{H}_\Phi, \quad A \mapsto \Xi_\Phi^\mathcal{A}(A) := \Delta_\Phi^{1/2} A \Omega_{\Phi}. \quad (20)$$

As before, we are forced to estimate the 1-nuclear norm of this operator. We can now implement the same strategy as in [2, Sec. V]. We defy the details as the steps will be quite analogous to those of Theorem 2.1. It is well-known that the time-reversal operator commutes with the Klein-Gordon Hamiltonian and due to its involutive nature, no doubling procedure is required. Then one can utilize it to construct the closed complex linear subspaces of $\mathfrak{H}_\Phi$ and apply Propositions 5.2 and 5.3 with Theorems 3.11 and 3.5 in [10] to deduce that the nuclear norm of $\Xi_\Phi^\mathcal{A}$ is bounded by the trace-norm of $(P_{\mathcal{B}'} - \Sigma_{\mathcal{B}'}^2)^{1/4} \uparrow \Xi(\mathcal{A})$ where $\Xi(\mathcal{A}) := \Xi \uparrow \mathcal{A}$. The trace-norm estimation then follows from the positivity of the Klein-Gordon Hamiltonian $K$ (in contrast to the Dirac Hamiltonian) and exploiting the finite propagation speed of wave operator [23, Prop. 1.1], as successfully contrived in [10, Prop. 4.3].

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