AN INVOLUTIVE UPSILON KNOT INVARIANT

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Abstract. Using the theory of involutive Heegaard Floer knot theory developed by Hendricks-Manolescu, we define two involutive analogs of the Upsilon knot concordance invariant of Ozsváth-Stipsicz-Szabó. These involutive invariants are piecewise linear functions defined on the interval [0,2]. Each is a concordance invariant and provides bounds on the three-genus of a knot.

1. Introduction

Heegaard Floer knot theory [7] associates to a knot $K \subset S^3$ a chain complex $\text{CFK}^\infty(K)$. To be more precise,

$$\text{CFK}^\infty(K) = (C, \partial, \text{alg}, \text{Alex}),$$

where $C$ is a graded $\mathbb{Z}/2$–chain complex with boundary map $\partial$ of degree $-1$ and $\text{alg}$ and $\text{Alex}$ are increasing filtrations on $(C, \partial)$. Furthermore, $C$ is a free, finitely generated $\mathbb{Z}/2$–module; the action by $U$ commutes with $\partial$, lowers gradings by 2 and lowers both filtration levels by 1. The construction of $\text{CFK}^\infty(K)$ depends on a series of choices, but any two complexes associated to $K$ are bifiltered chain homotopy equivalent.

Two further structural properties of $\text{CFK}^\infty(K)$ have been discovered: Sarkar [10] described a naturally defined self-chain homotopy equivalence $s$, now called the Sarkar map, and Hendricks and Manolescu [2] used the existence of a skew-bifiltered chain homotopy equivalence, first constructed in [7], $I: (C, \partial) \to (C, \partial)$, to define a family of new invariants called the involutive homology groups, $HFKI^o(K)$, where $o = \infty, +, -, \hat{}$. Here, by skew-bifiltered we mean that $I$ switches algebraic and Alexander filtration levels. That $HFKI^o(K)$ is well-defined depends on the naturality of $I$, which follows from results of Juhasz-Thurston [5]. Hendricks and Manolescu also proved that $I^2$ is bifiltered chain homotopy equivalent to $s$.

The group $HFKI^o(K)$ is the homology of the mapping cone of $I + I$. In Section 2 we will review the construction of this mapping cone, $\text{CFKI}^\infty(K)$. We will also describe a bifiltration on $\text{CFKI}^\infty(K)$. Section 3 describes the computation of $\text{CFKI}^\infty(K)$ for the torus knot $K = T(3,7)$ and presents a generalization that applies to all torus knots or, more generally, to $L$–space knots and their mirror images. In Section 4 we describe how the Upsilon function associated to $\text{CFK}^\infty(K)$, defined in [9], can be extended to give a pair of what we call the involutive Upsilon functions. Section 5 focuses on a single example, the torus knot $T(3,7)$. In Section 6 we prove the concordance invariance of the involutive Upsilon functions. We show in Section 7 that the value of each Upsilon at $t = 0$ is determined by previously defined invariants, $V_0(K)$ and $\overline{V}_0(K)$. Section 8 briefly discusses the three-genus of knots.

1.1. Conventions. Throughout this paper, all complexes are $\mathbb{Z}$–graded chain complexes over the field $\mathbb{Z}/2$, henceforth denoted $\mathbb{F}$. All differentials will lower homological degree by 1. The word grading is synonymous with homological degree.

Recall that if $(I, \leq)$ is a partially ordered set, then an $I$–filtered complex is a complex $C$ together with a collection of subcomplexes $F_i(C) \subset C$, indexed by $i \in I$, with $F_i(C) \subseteq F_j(C)$ whenever $i \leq j$. We will

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always assume that

\[ \bigcap_i \mathcal{F}_i(C) = \{0\} \quad \text{and} \quad \bigcup_i \mathcal{F}_i(C) = C. \]

A homotopy equivalence between $I$–filtered complexes $C \to C'$ is said to be an $I$–filtered homotopy equivalence if all relevant maps (the chain map $C \to C'$, its chain homotopy inverse $C' \to C$, and the chain homotopy) are filtration preserving.

By default, filtered complex means $\mathbb{Z}$–filtered and bifiltered means $\mathbb{Z} \times \mathbb{Z}$–filtered. If $(C, \mathcal{F})$ is a filtered complex, we may define the filtration degree of a nonzero $x \in C$ by

\[ \deg_{\mathcal{F}}(x) = \min\{i \mid x \in \mathcal{F}_i(C)\}. \]

By convention, we set $\deg_{\mathcal{F}}(0) = -\infty$. Thus, given the assumptions [1], $\deg_{\mathcal{F}}$ is a well-defined set function $C \to \mathbb{Z} \cup \{-\infty\}$. The function $\deg_{\mathcal{F}}$ satisfies

- (F1) $\deg_{\mathcal{F}}^{-1}(-\infty) = \{0\}$,
- (F2) $\deg_{\mathcal{F}}(x + y) \leq \max\{\deg_{\mathcal{F}}(x), \deg_{\mathcal{F}}(y)\}$, and
- (F3) $\deg_{\mathcal{F}}(\partial(x)) \leq \deg_{\mathcal{F}}(x)$.

Conversely, given a set function with these properties, we can recover the subcomplex $\mathcal{F}_i(C)$ as the linear span of all elements $x \in C$ such that $\deg_{\mathcal{F}}(x) \leq i$. In fact, we can allow more general functions $\deg_{\mathcal{F}}$. If $\deg_{\mathcal{F}} : C \to \mathbb{R}$ is a set function satisfying the properties (F1), (F2), and (F3), then we can define an $\mathbb{R}$–filtration of $C$, which is the associated ordered family of subcomplexes $\mathcal{F}_i(C) \subset C$, indexed by $t \in \mathbb{R}$. In all examples of interest to us, the image of $C \setminus \{0\}$ under $\deg_{\mathcal{F}}$ will be a discrete subset $S \subset \mathbb{R}$ which, as an ordered set, is isomorphic to $\mathbb{Z}$. Thus, every $\mathbb{R}$–filtered complex considered here can also regarded as a $\mathbb{Z}$–filtered complex, though the precise description would require choosing an isomorphism $S \cong \mathbb{Z}$.

**Notation 1.** Henceforth, a filtered complex will mean a pair $(C, \deg_{\mathcal{F}})$, where $C$ is a chain complex with differential $\partial$, and $\deg_{\mathcal{F}}$ is a function $C \to \mathbb{R} \cup \{-\infty\}$ satisfying properties (F1), (F2), and (F3) above. Similarly, a bifiltered complex is a triple $(C, \deg_{\mathcal{F}}, \deg_{\mathcal{G}})$ such that $(C, \deg_{\mathcal{F}})$ and $(C, \deg_{\mathcal{G}})$ are filtered complexes. The reader can verify that a bifiltered complex in this sense corresponds to a filtration indexed by $\mathbb{Z} \times \mathbb{Z}$.

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## 2. Involutive Homology

We begin by reviewing the definition of the mapping cone complex in the context of the chain map $I + I$; we denote this complex $\text{Cone}(C, I + I)$. The underlying graded vector space is $\text{CFK}^\infty(K)[1] \oplus \text{CFK}^\infty(K)$, where $\text{CFK}^\infty(K)[1]$ is the same complex as $\text{CFK}^\infty(K)$ with gradings shifted up by 1. The boundary map is given by

\[
\partial_I = \begin{bmatrix} \partial_C & 0 \\ I + I & \partial_I \end{bmatrix} : \text{CFK}^\infty(K)[1] \oplus \text{CFK}^\infty(K) \rightarrow \text{CFK}^\infty(K)[1] \oplus \text{CFK}^\infty(K).
\]

In other words, the boundary of $(z_1, z_2)$ is $(\partial(z_1), I(z_1) + z_1 + \partial(z_2))$. It is easily checked that $(\partial_I)^2 = 0$.

**Definition 2.** We denote $\text{Cone}(C, I + I)$ by $\text{CFKI}^\infty(K)$.

**Theorem 3.** The homology of $\text{CFKI}^\infty(K)$, $\text{HFKI}^\infty(K)$, is isomorphic as an $\mathbb{F}[U, U^{-1}]$–module to $\mathbb{F}[U, U^{-1}] \oplus \mathbb{F}[U, U^{-1}]$, where $(1, 0)$ has grading 1 and $(0, 1)$ has grading 0. This splitting is natural.

**Proof.** For the mapping cone of any map of complexes $f : C \to D$, there is a long exact sequence

\[ \cdots \rightarrow H_i(C) \xrightarrow{f} H_i(D) \rightarrow H_i(\text{Cone}(C, D, f)) \rightarrow H_{i-1}(C) \xrightarrow{\partial_I} \cdots. \]

The map $(I + I)_*$ is trivial on homology and $H_i(C) \cong H_i(D) \cong \mathbb{F}[U, U^{-1}]$ with the generator 1 of grading 0.

**Definition 4.** $\text{HFKI}^\infty(K) \cong T_o \oplus T_e$, where $T_o$ is the odd tower isomorphic to $\mathbb{F}[U, U^{-1}]$ with all gradings odd, and $T_e$ is similarly the even tower.
2.1. The folded bifiltration. The map $\mathcal{I}$ is skew; it does not preserve the algebraic-Alexander bifiltration. However, there is a pair of filtrations on $\text{CFKI}^\infty(K)$ that are preserved.

**Definition 5.** Suppose $(C, \deg_F, \deg_G)$ is a bifiltered complex. Define a new bifiltered complex $(C, \text{Min}, \text{Max})$, where $\text{Min}(x) = \min\{\deg_F(x), \deg_G(x)\}$ and $\text{Max}(x) = \max\{\deg_F(x), \deg_G(x)\}$. We call the resulting bifiltration the folded, or Min-Max bifiltration on $C$.

We may regard $\text{CFK}^\infty(K)$ as a bifiltered complex with respect to the Min-Max filtration. Note that $\mathcal{I}: \text{CFK}^\infty(K) \to \text{CFK}^\infty(K)$ swaps the alg and Alex filtrations, and hence it preserves the Min-Max filtrations. Thus, $(\text{CFKI}^\infty(K), \text{Min}, \text{Max})$ is a bifiltered complex.

The left diagram in Figure 1 illustrates a model complex for $\text{CFK}^\infty(T(3,7))$; in general, the structure of $\text{CFK}^\infty(T(p,q))$ is determined by the results of [8], which study a more general family of knots, called $L$-space knots. In the diagram, the vertex at $(0,6)$ represents a generator at grading 0. The full complex is constructed from the illustrated model complex by tensoring with $F[U,U^{-1}];$ the $U^k$ translates, if illustrated, would be represented by copies of the finite complex that is drawn, shifted $-k$ units along the main diagonal. It is evident that the only self-chain homotopy equivalence that is also skew is represented by reflection through the diagonal. On the right in Figure 1 the complex $\text{CFK}^\infty(T(3,7))$ is illustrated, where now the bifiltration is given by Min-Max. Figure 2 illustrates the involutive complex $\text{CFKI}^\infty(K)$ for $T(3,7)$ as well as an equivalent reduced complex obtained by bifiltered Gaussian elimination. In the next section we describe the steps in constructing this reduction.

**Figure 1.** $\text{CFK}^\infty(T(3,7))$ and folded $\text{CFK}^\infty(T(3,7))$

3. Reductions

Suppose $(I, \leq)$ is a partially ordered set, and let $(C, \mathcal{F})$ be a filtered complex. We say that $C$ is reduced if each subquotient $\mathcal{F}_{\leq i}(C)/\mathcal{F}_{<i}(C)$ has zero differential. Under mild finiteness assumptions on $I$ (which are satisfied by $I = \mathbb{Z}$ and $I = \mathbb{Z} \times \mathbb{Z}$), any $I$-filtered complex is isomorphic to $C_{\text{red}} \oplus \mathbb{Z}$ where $C_{\text{red}}$ is reduced and $\mathbb{Z}$ is homotopically trivial in the filtered sense. An algorithm for reducing bifiltered complexes is presented in [11]. In this section we will summarize the procedure in the case that the starting complex is the involutive complex associated to a staircase, such as the one illustrated on the left of Figure 1. In this example there are 9 vertices and the steps are $[1, 2, 1, 2, 1, 2, 1, 2, 1]$. The goal is to perform a bifiltered change of basis so that, after removing acyclic summands, the remaining diagram has no arrows within any given square. We first introduce some terminology.

**Definition 6.** A staircase complex $C$ is symmetric if $C \simeq C^{\text{op}}$, where $C^{\text{op}}$ denotes the staircase with bifiltrations swapped. Any symmetric staircase $C$ has steps of size $[a_1, \ldots, a_{2k}]$ with $a_i = a_{2k-i+1}$. A symmetric staircase is inward-pointing if the generator corresponding to the central dot is a cycle, and is outward-pointing otherwise. We call a staircase positive if the first (top) step is to the right, not down, and negative otherwise.
Note that every symmetric staircase has an even number of edges. A positive symmetric staircase is inward-pointing exactly when its length is 0 (mod 4), and a negative symmetric staircase is inward-pointing exactly when its length is 2 (mod 4).

**Note:** Positive torus knots, \( T(p, q) \) with \( p, q > 0 \), have positive symmetric staircase complexes, while their mirror images, \( -T(p, q) \), have negative staircase complexes. More generally, all \( L \)-space knots have positive staircase complexes.

The appropriate change of basis is best described by a schematic, as in Figure 3. The diagram on the left represents the complex \( \text{CFKI}^\infty(T(3, 7)) \). The box labeled \( B \) is the portion of the staircase complex that arises from the portion of the staircase with \( i < j \), and \( B' \) is the portion with \( i > j \). The boxes \( A \) and \( A' \) represent the same complexes, with grading shifted up by one.

A change of basis in which generators from \( A \) are added to corresponding generators of \( A' \), and generators of \( B' \) are added to corresponding generators of \( B \), changes the diagram so that it appears as on the right in Figure 3. The complexes \( A + A' \) and \( \bullet \leftarrow B' \) are each staircase complexes; the one with an even number of vertices is acyclic and the other has homology of rank one. (In the current example, \( A + A' \) and \( B \) both have four vertices, so it is \( \bullet \leftarrow B' \) that is not acyclic.) The complex involving \( A \) and \( B + B' \) is illustrated schematically on the left in Figure 4. Changing basis, adding some of the generators from the top row to those of the bottom row, as indicated in the diagram, offers the reduction as illustrated on the right in Figure 4.

We now see that \( \text{CFKI}^\infty(T(3, 7)) \) is bifiltered chain homotopy equivalent to the complex illustrated in Figure 2. In the diagram, the staircase with four vertices is acyclic, and thus it doesn’t contribute to later computations. Henceforth, we will not include such acyclic pieces in our diagrams.
In considering a general symmetric staircase complex, there are two parity issues: the length of the staircase (mod 4) and whether the staircase is positive or negative. The following theorem summarizes the result in all cases. The proof in each case follows the exact same lines as the computations above.

**Theorem 7.** Let $C$ be a staircase complex $[a_1, a_2, \ldots a_k, a_k, \ldots, a_1]$, let $s = \sum a_i$ and let $d = \sum_{i \text{ odd}, i \leq k} |a_i|$. The involutive complex associated to $C$ is the direct sum of two complexes, one represented by a single vertex $v_0$ on the diagonal and the other a staircase complex $S$.

1. If the staircase $C$ is positive and $k$ is even, then $v_0$ has grading level 1 and is at filtration level $(d, d)$; $S$ is the staircase $[a_1, \ldots, a_k]$ with homology at grading 0, beginning at filtration level $(0, s)$ and ending at $(d, d)$.
2. If the staircase $C$ is positive and $k$ is odd, then $v_0$ has grading level 1 and is at filtration level $(d, d)$; $S$ is the staircase $[a_1, \ldots, a_{k-1}]$ with homology at grading 0, beginning at filtration level $(0, s)$.
3. If the staircase $C$ is negative and $k$ is even, then $v_0$ has grading level 0 and is at filtration level $(-d, -d)$; $S$ is the staircase $[a_1, \ldots, a_k]$ with homology at grading 1, beginning at the filtration level $(-s, 0)$ and ending at $(-d, -d)$.
4. If the staircase $C$ is negative and $k$ is odd, then $v_0$ has grading level 0 and is at filtration level $(-d, -d)$; $S$ is the staircase $[a_1, \ldots, a_{k-1}]$ with homology at grading 1, beginning at filtration level $(-s, 0)$.

Figures 5, 6, 7 and 8 illustrate each case. (In the colored version of this paper, the green, red, gray, and purple dots represent generators of homological degree $-1$, 0, 1 and 2, respectively.)
In summary we have the following general result.

**Theorem 8.** The complexes $\text{CFK}^\infty(K)$ and $\text{CFKI}^\infty(K)$ are bifiltered by $\text{Max}$ and $\text{Min}$. The homology group $H_*(\text{CFK}^\infty(K))$ is isomorphic to a tower $T_E$ of even homological grading. The homology group $H_*(\text{CFKI}^\infty(K))$ is isomorphic to a direct sum of towers $T_O$ and $T_E$ of odd and even homological grading, respectively.

**Definition 9.** For each $t \in [0, 2]$, let $\deg_t$ be the function on either $\text{CFK}^\infty(K)$ or $\text{CFKI}^\infty(K)$ given by
\[
\deg_t(x) = \frac{t}{2} \text{Max}(x) + (1 - \frac{t}{2}) \text{Min}(x).
\]
The image of $\deg_t(x)$ as $x$ ranges over all (nonzero) elements of $\text{CFKI}^\infty(K)$ is some discrete subset $S \subset \mathbb{R}$; for each $s$, we have a subcomplex $\text{CFKI}^\infty(K)_{\deg_t \leq s} \subset \text{CFKI}^\infty(K)$ which is the $F$–span of all elements $x$ for which $\deg_t(x) \leq s$. We refer to this as the slope $1 - \frac{t}{2}$ filtration of $\text{CFKI}^\infty(K)$ (or of $\text{CFK}^\infty(K)$). Note also that $\deg_t$ induces filtrations on $\text{HFK}^\infty(K)$ and $\text{HFKI}^\infty(K)$.

Recall that we have a short exact sequence of bifiltered complexes
\[
0 \rightarrow \text{CFK}^\infty(K) \xrightarrow{\sigma} \text{CFKI}^\infty(K) \xrightarrow{\pi} \text{CFK}^\infty(K)[1] \rightarrow 0.
\]
The connecting homomorphism for the associated long exact sequence is induced by the map $\mathcal{I} + I$, and since $\mathcal{I}$ induces the identity map in homology, the connecting map is zero. Thus the long exact sequence in homology splits into a collection of short exact sequences
\[
0 \rightarrow \text{HFK}_i^\infty(K) \xrightarrow{\sigma} \text{HFKI}_i^\infty(K) \xrightarrow{\pi} \text{HFK}_{i-1}^\infty(K) \rightarrow 0
\]
for all $i$. Since $\text{HFK}^\infty(K)$ is supported in even homological degrees, it follows that $\sigma_*$ is an isomorphism $\text{HFK}_0^\infty(K) \rightarrow \text{HFKI}_0^\infty(K)$ and $\pi_*$ is an isomorphism $\text{HFKI}_0^\infty(K) \rightarrow \text{HFK}^\infty(K)$.

Let $c_0 \in \text{HFKI}_0(K)$, $c_1 \in \text{HFKI}_1(K)$, and $d \in \text{HFKI}_0^\infty(K)$ denote generators. It follows that $c_0 = \sigma_*(d)$ and $d = \pi_*(c_1)$. We now define three functions of $t$:
\[
\nu^f_t(K) := \deg_t(d) \\
\nu_t(K) = \deg_t(c_0) \\
\nu_t(K) = \deg_t(c_1)
\]
Here the superscript on $\nu^f_t(K)$ indicates that we are using the folded bifiltration on $\text{HFK}^\infty(K)$. Using these, we define two involutive Upsilon functions and a folded Upsilon function.
Definition 10.
\[ \Upsilon_t(K) = -2\nu_t(K) \quad \Upsilon'_t(K) = -2\nu'_t(K) \quad \Upsilon_t(K) = -2\nu_t(K) \]

Our conventions for “upper bar” and “lower bar” were chosen so that the following inequalities hold:

Proposition 11. \[ \Upsilon_t(K) \leq \Upsilon'_t(K) \leq \Upsilon_t(K). \]

Proof. If \((C, \deg_F)\) and \((D, \deg_F)\) are filtered vector spaces, then any filtration preserving map \(g : C \to D\) satisfies \(\deg_F(g(x)) \leq \deg_F(x)\) by definition. Thus, the inequalities \(\nu_t(K) \leq \nu'_t(K) \leq \nu_t(K)\) follow immediately from the fact that \(c_0 = \sigma_*(d)\) and \(d = \pi_*(c_1)\), discussed in the remarks preceding the proposition. This proves \(\Upsilon_t(K) \leq \Upsilon'_t(K) \leq \Upsilon_t(K)\), as claimed. \(\square\)

5. Example: \(T(3,7)\)

In Figure 9 we have redrawn the fully reduced complex \(\text{CFKI}^\infty(T(3,7))\). The figure includes two dashed lines of slope \(-3\) corresponding to filtration levels when \(t = 1/2\). The lower line is the boundary of the region

\[ \frac{t}{2} \min(x) + (1 - \frac{t}{2}) \max(x) \leq \frac{7}{4}. \]

The upper line is the boundary of the region

\[ \frac{t}{2} \min(x) + (1 - \frac{t}{2}) \max(x) \leq \frac{9}{4}. \]

Continuing to work with \(t = 1/2\), the least value of \(s\) so that the region \(\frac{t}{2} \min(x) + (1 - \frac{t}{2}) \max(x) \leq s\) contains a generator of homology at grading 0 is \(s = 3/2\). The least value of \(s\) so that the region \(\frac{t}{2} \min(x) + (1 - \frac{t}{2}) \max(x) \leq s\) contains a generator of homology at grading 1 is \(s = 2\). Thus, \(\Upsilon_{1/2}(K) = -3\) and \(\Upsilon_{1/2}(K) = -4\).

For general \(t\) we have

\[ \Upsilon_t(K) = \begin{cases} -6t, & \text{if } 0 \leq t \leq \frac{2}{3} \\ -4, & \text{if } \frac{2}{3} \leq t \leq 2, \end{cases} \]

\[ \Upsilon_t(K) = -4 \text{ for all } t \in [0, 2]. \]
6. Concurrency Invariance and Knot Inverses.

6.1. Concordance.

Theorem 12. If \( K \) and \( J \) are concordant knots, then \( \Upsilon_t(K) = \Upsilon_t(J) \) and \( \Upsilon_l(K) = \Upsilon_l(J) \).

Proof. A result of Hendricks and Hom [1] states that if \( L \) is a slice knot, then \( \text{CFK}^\infty(L) \) splits as the direct sum of involutive complexes:
\[
\text{CFK}^\infty(L) \cong \mathcal{T} \oplus \mathcal{A},
\]
where \( \mathcal{T} \cong \mathbb{F}[U, U^{-1}] \) and \( \mathcal{A} \) is acyclic. (This result generalizes an analogous result of Hom [4] that holds for noninvolutory complexes. The proof depended on results of Zemke [12] concerning the involutive homology of connected sums of knots.) Thus, we can write
\[
\text{CFK}^\infty(K \# J) \cong \mathcal{T}_1 \oplus \mathcal{A}_1
\]
as involutive complexes, where \( \mathcal{T}_1 \cong \mathbb{F}[U, U^{-1}] \) and \( \mathcal{A}_1 \) is acyclic. It follows that
\[
\text{CFK}^\infty(K \# J) \cong \text{CFK}^\infty(K) \oplus (\mathcal{A}_1 \otimes \text{CFK}^\infty(J))
\]
and according to a connected sum formula for involutive homology given in [3], this is again a direct sum of involutive complexes (but the involution on \( \mathcal{A}_1 \otimes \text{CFK}^\infty(J) \) is not necessarily the tensor product of the involutions; see [12] for details). Since the second summand is acyclic, we write
\[
\text{CFK}^\infty(K \# J) \cong \text{CFK}^\infty(K) \oplus \mathcal{A}_2.
\]
Next, we write
\[
\text{CFK}^\infty(K \# J) \cong \text{CFK}^\infty(K) \otimes \text{CFK}^\infty(-J \# J),
\]
which can be rewritten as
\[
\text{CFK}^\infty(K \# J) \cong \text{CFK}^\infty(K) \otimes (\mathcal{T}_2 \oplus \mathcal{A}_2),
\]
since \( J \# -J \) is slice. As before, \( \mathcal{T}_2 \oplus \mathcal{A}_2 \) is a direct sum of involutive complexes,
\[
\text{CFK}^\infty(K \# J) \cong \text{CFK}^\infty(K) \oplus \mathcal{A}_3.
\]
In summary we have the following decompositions of involutive complexes:
\[
\text{CFK}^\infty(K) \oplus \mathcal{A}_3 \cong \text{CFK}^\infty(K) \oplus \mathcal{A}_2.
\]
The acyclic summands do not affect the value of either \( \Upsilon_t(K) \) or \( \Upsilon_l(K) \), and thus the proof is complete. \( \square \)

7. The Knot Concordance Invariants \( \overline{V}_0(K) \) and \( \underline{V}_0(K) \)

In [2], two knot concordance invariants \( \overline{V}_0(K) \) and \( \underline{V}_0(K) \) are defined. These can be interpreted in terms of \( \Upsilon \).

Theorem 13.
\[
\overline{V}_0(K) = -\frac{1}{2} \Upsilon_2(K),
\]
\[
\underline{V}_0(K) = -\frac{1}{2} \Upsilon_2(K).
\]

Proof. Both \( \overline{V}_0(K) \) and \( \underline{V}_0(K) \) are defined in terms of the involutive correction terms for large surgery on \( K \): \( \overline{d}(S^3_0(K), [0]) \) and \( \underline{d}(S^3_0(K), [0]) \). These are computed in terms of the maximal gradings of even and odd non-torsion elements in the homology of \( \text{CFK}^\infty(K)_{\text{Max}(i,j) \leq 0} \). More precisely, they are minus one half these gradings.

Suppose the maximal grading of a (non-torsion) class in \( \text{CFK}^\infty(K)_{\text{Max}(i,j) \leq 0} \) of even grading is \( a \). Then it follows from Formula 1 of [2] that \( \overline{V}_0(K) = -a/2 \). As a consequence, the involutive complex \( \text{CFK}^\infty(K)_{\text{Max}(i,j) \leq s} \) contains a non-torsion class of grading 0 if and only if \( s \geq -a/2 \). Thus \( \overline{V}_2(K) = -a/2 \). It now follows that \( \Upsilon_2(K) = a \), as desired. A similar argument works for the lower invariants. \( \square \)
8. Three-genus

The three-genus bounds that arises from \( \Upsilon_K \) and \( \Upsilon'_K \) are almost immediate, following in the same way as the lower bounds on \( g_4(K) \) coming from \( \Upsilon_K \): 
\[
g_4(K) \geq \Upsilon'_K(t) \quad \text{for all } t \in [0, 2] \text{ at which } \Upsilon_K(t) \text{ is nonsingular.}
\]

The proof of this inequality uses only the fact that \( \text{CFK}^\infty(K) \) is chain homotopy equivalent to complex for which all filtration levels satisfy \[ \| \text{alg} - \text{Alex} \| \leq g_3(K) \] (see [9], or the expository account in [6]). The same constraint holds for the involutive complex with the Max-Min filtration, so the same proof applies.

References

[1] K. Hendricks and J. Hom, A note on knot concordance and involutive knot Floer homology, arxiv.org/abs/1708.06389
[2] K. Hendricks and C. Manolescu, Involutive Heegaard Floer homology, Duke Math. J. 166 (2017), 1211–1299.
[3] K. Hendricks, C. Manolescu and I. Zemke, A connected sum formula of involutive Heegaard Floer homology, arxiv.org/abs/1607.07499
[4] J. Hom, A note on the concordance invariants epsilon and upsilon, Proc. Amer. Math. Soc. 144 (2016), 897–902.
[5] A. Juhász and D. Thurston, Naturality and mapping class groups in Heegaard Floer homology, arxiv.org/abs/1210.4999.
[6] C. Livingston, Notes on the knot concordance invariant Upsilon, Algebr. Geom. Topol. 17 (2017) 111–130.
[7] P. Ozsváth and Z. Szabó, Holomorphic disks and knot invariants, Adv. Math. 186 (2004), 58–116.
[8] P. Ozsváth and Z. Szabó, On knot Floer homology and lens space surgeries, Topology 44 (2005), 1281–1300.
[9] P. Ozsváth, A. Stipsicz, and Z. Szabó, Concordance homomorphisms from knot Floer homology, Adv. Math. 315 (2017), 366–426.
[10] S. Sarkar, Moving basepoints and the induced automorphisms of link Floer homology, Algebr. Geom. Topol. 15 (2015), 2479–2515.
[11] J. Rasmussen, Floer homology and knot complements, arxiv.org/abs/math/0306378.
[12] I. Zemke, Connected sums and involutive knot Floer homology, arxiv.org/abs/1705.01117.

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