Non-Linear Operators and Differentiability of Lipschitz Functions

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Received: 27 September 2021 / Accepted: 29 January 2023 / Published online: 7 February 2023
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Abstract
In this work we provide a characterization of distinct types of (linear and non-linear) maps between Banach spaces in terms of the differentiability of certain class of Lipschitz functions. Our results are stated in an abstract bornological and non-linear framework. Restricted to the linear case, we can apply our results to compact, weakly-compact, limited and completely continuous linear operators. Moreover, our results yield a characterization of Gelfand-Phillips spaces and recover some known results about Schur spaces and reflexive spaces concerning the differentiability of real-valued Lipschitz functions.

Keywords Linear and non-linear operators · Differentiability of Lipschitz functions · Bornology · weakly compact operators · Completely continuous operators

Mathematics Subject Classification (2010) Primary: 46A17 · 26A16 · 47B07, Secondary: 47B38 · 49J50

1 Introduction

Differentiability of convex and Lipschitz functions defined on general (infinite dimensional) Banach spaces has attracted the interest of researchers since the last century. Distinct notions of differentiability and their relation with the geometry of the domain space or codomain space is one of the most common topics in the area. A nice survey about

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differentiability of convex functions can be found in [11]. A general approach of differentiability and its applications to renorming can be found in [14]. In particular, the notions of Gâteaux, weak-Hadamard and Fréchet differentiability have been a prolific subject for many authors. For instance, see [5, 8, 11] and reference therein.

In order to motivate this work, let us proceed with the following definition.

**Definition 1.1** Let $X$ be a Banach space. A bounded subset $A$ of $X$ is called limited if for any weak* null sequence $(x_n^*)$, the following limit holds:

$$\lim_{n \to \infty} \sup_{x \in A} |\langle x_n^*, x \rangle| = 0.$$ 

That is, weakly*–null sequences converge uniformly on $A$.

Let $X$ and $Y$ be two Banach spaces. A bounded linear operator $T \in \mathcal{L}(Y, X)$ is called limited if $T(B_Y)$ is a limited set. Further information about limited sets and limited operators can be found in [4, 10, 15] and references therein. In [3, 4], we can find how to characterize limited and compact operators in terms of the differentiability of certain classes of functions. In fact, the main result of [3] reads as follows.

**Theorem 1.2** [3, Theorem 1] Let $X$ and $Y$ be two real Banach spaces, let $\mathcal{U}$ be a nonempty open convex subset of $X$ and let $T \in \mathcal{L}(Y, X)$. Then, $T$ is limited if and only if for every continuous convex function $f : \mathcal{U} \to \mathbb{R}$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f$ is Gâteaux differentiable at $x = Ty \in \mathcal{U}$.

In [4] it is shown that the same statement holds true as well when we replace limited operators by compact operators and convex functions by real-valued Lipschitz functions. This last result exploits the following well-known result: Gâteaux and Hadamard differentiability coincide for Lipschitz functions defined on open subsets of a Banach space. In fact, as we show in Section 5, Theorem 1.2 can also be deduced by using the following fact: Gâteaux and limited differentiability coincide for continuous convex functions (see Proposition 5.3). For definitions see Section 2.

In this paper we generalize Theorem 1.2 in two directions. Indeed, we consider an abstract notion of differentiation (by using bornologies) and we also work with possible non linear operators. In fact, as a consequence of our results we answer [4, Question 5.7], where the authors asked for a possible characterization for weakly-compact linear operators in terms of the differentiability of a given class of functions.

Let us continue with some standard definitions. Let $X$ and $Y$ be two real Banach spaces and let $T \in \mathcal{L}(Y, X)$. We say that $T$ is weakly compact if $T$ sends the closed unit ball of $Y$ onto a relatively weakly compact subset of $X$. We say that $T$ is completely continuous or a Dunford-Pettis operator if, for every weakly convergent sequence $(y_n) \subseteq Y$, the sequence $(Ty_n)$ is norm-convergent in $X$ (see [12, Definition VI.3.2, p. 173]). Equivalently, $T$ is completely continuous if, for every relatively weakly compact subset $K$ of $Y$, we have that $T(K)$ is relatively compact. It is well-known that every compact operator is completely continuous (see [6, p. 143]).

The main result of this manuscript, Theorem 4.5, is an unified statement that characterizes several types of linear and non-linear operators in terms of the "p.h. differentiability" of
certain Lipschitz functions (see Definition 4.3), among which are included weakly-compact, completely continuous, compact and limited linear operators. In order to state the mentioned theorem, we introduce an abstract property for bornologies that we call property (S) in Definition 2.2, which is satisfied by the Hadamard, weak-Hadamard and limited bornology. Also, we use a notion of differentiability (which can be applied to positively homogeneous maps) in the sense of Suchomolinov, see Definition 4.3 (or for instance [24, p. 135] and [22]). In what follow we state two direct consequences of our main result (Theorem 4.5), in which we characterize weakly compact operators and completely continuous operators respectively. Moreover, Theorem 4.5 also implies the characterization established in [4] for compact operators.

**Theorem 1.3** Let $X$ and $Y$ be real Banach spaces and let $T \in L(Y, X)$. Then, $T$ is weakly compact if and only if for every Lipschitz function $f : X \to \mathbb{R}$, $f \circ T$ is Fréchet differentiable at $y$ whenever $f$ is weak-Hadamard differentiable at $Ty$.

**Theorem 1.4** Let $X$ and $Y$ be real Banach spaces and let $T \in L(Y, X)$. Then, $T$ is completely continuous if and only if for every Lipschitz function $f : X \to \mathbb{R}$, $f \circ T$ is weak-Hadamard differentiable at $y$ whenever $f$ is Gâteaux (equivalent to Hadamard) differentiable at $Ty$.

The outline of this paper is as follows: In Section 2 we provide the preliminaries and notation used throughout this manuscript. In Section 3 we present some examples of bornologies that satisfy property (S), see Definition 2.2. Also, for a vector bornology $\beta$ on $X$ which satisfies property (S) and is different from the Fréchet bornology, we construct in Lemma 3.7 a real-valued Lipschitz function $f : X \to \mathbb{R}$ which is $\beta$-differentiable at 0 but not Fréchet-differentiable at 0. In Section 4, we state and prove our main result Theorem 4.5. Moreover, we also prove Theorems 1.3 and 1.4. In Section 5, we present an alternative proof of Theorem 1.2 based in the fact that Gâteaux and limited differentiability coincide for continuous convex functions (see Proposition 5.3). Finally, in Section 6, we present some consequences of our results. More precisely, as a direct consequence of Theorem 4.5 we characterize Gelfand-Phillips spaces, finite dimensional spaces and we recover some well-known characterizations of Schur spaces and reflexive spaces. Also, we present some results on spaceability and an extension of a result of Bourgain and Diestel given in [10].

### 2 Preliminaries and Notation

The notation used through this paper is standard. By $X$, $Y$ and $Z$ we denote real Banach spaces. By $X^*$ we denote the dual space of $X$. For $x \in X$ and $r > 0$ we denote by $B(x, r)$ and $B_X$ the open ball centered at $x$ of radius $r$ and the open unit ball of $X$ respectively. We denote by $L(Y, X)$ the vector space of bounded linear operators from $Y$ to $X$. A set $A \subset X$ is said balanced if $\lambda A \subset A$, for all $|\lambda| \leq 1$. We recall that a bornology $\beta$ on $X$ is a family of nonempty and bounded subsets of $X$ satisfying the following properties:

1. $\bigcup_{A \in \beta} A = X$,
2. for all $A, B \in \beta$, $A \cup B \in \beta$, and
3. for all $A \in \beta$ and $B \subset A$, $B \in \beta$.

If $A \in \beta$, we say that $A$ is a $\beta$-set. For instance, some known bornologies are the ones of Gâteaux, Hadamard, Limited, weak-Hadamard and Fréchet. These bornologies
correspond to the family of finite sets, relatively compact sets, limited sets, relatively weakly-compact sets and bounded sets respectively. For a bornology $\beta$ on $X$ and a linear operator $T : Y \to X$, we say that $T$ is a $\beta$-operator if $T(B(y, r)) \in \beta$, for all $y \in Y$ and all $r > 0$. Since $\beta$ only contains bounded sets, any $\beta$-operator is continuous.

Let $L \geq 0$. We recall that a function $f : X \to Z$ is said $L$-Lipschitz if $\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$ for every $x_1$, $x_2 \in X$. The Lipschitz constant of $f$, denoted by $\text{Lip}(f)$, is the least constant $L \geq 0$ such that $f$ is $L$-Lipschitz. The space of real-valued Lipschitz functions defined on $X$ is denoted by $\text{Lip}(X)$. The Lipschitz constant is a seminorm on $\text{Lip}(X)$. Finally, for a bornology $\beta$ on $X$, we recall that a function $f : X \to Z$ is said $\beta$-differentiable at $x_0 \in X$, with differential $d_\beta f(x_0) \in \mathcal{L}(X, Z)$, if

$$\limsup_{t \to 0^+ \atop u \in A} \frac{\|f(x_0 + tu) - f(x_0)\| - d_\beta f(x_0)(u)}{t} = 0, \text{ for all } A \in \beta.$$

Let us continue with the following definitions which are needed to state our main theorem.

**Definition 2.1** Let $\beta$ be a bornology on $X$. We say that $\beta$ is a vector bornology if:

1. For any $A \in \beta$, $\text{bal}(A) := \{\lambda x : x \in A, \ |\lambda| \leq 1\} \in \beta$.
2. For any $A \in \beta$, $\lambda \in \mathbb{R}$ and $x \in X$, $x + \lambda A \in \beta$,

where $\text{bal}(A)$ stands for the balanced hull of $A$.

Note that the Hadamard, weakly-Hadamard, limited and Fréchet bornologies are clearly vector bornologies. Let $\beta$ be a vector bornology on $X$ and let $T \in \mathcal{L}(Y, X)$. Then, $T$ is a $\beta$-operator if and only if $T(B_Y) \in \beta$. Further, a function $f : X \to Z$ is $\beta$-differentiable at $x_0 \in X$, with differential $d_\beta f(x_0) \in \mathcal{L}(X, Z)$, if

$$\limsup_{t \to 0^+ \atop u \in A} \frac{\|f(x_0 + tu) - f(x_0)\| - d_\beta f(x_0)(u)}{t} = 0, \text{ for all } A \in \beta.$$

We will sometimes simply denote $\beta = G$, $H$, $L$, $wH$ and $F$, for the Gâteaux, Hadamard, limited, weak Hadamard and Fréchet differentiability. The following property, which we will call the property (S), plays a fundamental role in this manuscript.

**Definition 2.2** Let $\beta$ be a vector bornology on $X$. We say that $\beta$ satisfies property (S) if for every bounded set $A \subset X$ such that $A \notin \beta$, there are a sequence $(x_n)_n \subset A$ and $\delta > 0$ such that for any increasing sequence $(n_k) \subset \mathbb{N}$ and for any sequence $(y_k)_k \subset X$ satisfying $\|y_k - x_{n_k}\| \leq \delta$ for all $k \in \mathbb{N}$, the set $\{y_k : k \in \mathbb{N}\} \notin \beta$.

We say that a set $A \subset X$ is semi-balanced if $\lambda A \subset A$, for all $0 \leq \lambda \leq 1$.

**Proposition 2.3** Let $\beta$ be a vector bornology on $X$ satisfying property (S). Let $A \subset X$ be a semi-balanced set such that $A \notin \beta$. Let $(x_n)_n \subset A$ and $\delta > 0$ given by property (S). Then, $(x_n)_n$ does not have accumulation points. Moreover, the sequence $(x_n)_n$ can be chosen such that $\|x_n\| = \|x_1\|$ holds for all $n \in \mathbb{N}$.
Proof Reasoning by contradiction, let us assume that there is a subsequence \((x_{n_k})_k\) of \((x_n)_n\) convergent to some \(\overline{x} \in X\). Up to a subsequence, we can assume that \(\|\overline{x} - x_{n_k}\| < \delta/2\) for all \(k \in \mathbb{N}\). Then, the constant sequence \((y_k)_k \subset X\), defined by \(y_k = \overline{x}\) for all \(k \in \mathbb{N}\), satisfies \(\|y_k - x_{n_k}\| \leq \delta\) for all \(k \in \mathbb{N}\). Thus, by property \((S)\), \(\{y_k : k \in \mathbb{N}\} = \{\overline{x}\} \notin \beta\) which is a contradiction.

In order to prove the second part of Proposition 2.3, we construct a sequence by perturbing a subsequence of \((x_n)_n\). Since \(A\) is a bounded set, up to a subsequence that we still denote by \((x_n)_n\), we can assume that \((\|x_n\|)\) is convergent to some \(\alpha \geq 0\). Thanks to the first part of Proposition 2.3, we have that \(\alpha > 0\). Let us define \(\tilde{\delta} := \min\{\alpha, \delta\}\). Now, we can further assume, up to a subsequence that we still denote by \((x_n)_n\), that \(\|x_n\| \in [\alpha - \tilde{\delta}/4, \alpha + \tilde{\delta}/4]\). For each \(n \in \mathbb{N}\), let us consider

\[
x'_n := \left(\alpha - \frac{\tilde{\delta}}{4}\right) \frac{x_n}{\|x_n\|}.
\]

Notice that \(\|x'_n\| = \alpha - \frac{\tilde{\delta}}{4}\) and that \(\|x_n - x'_n\| \leq \tilde{\delta}/2\) for all \(n \in \mathbb{N}\). Moreover, since \(A\) is a semi-balanced set, we have that \((x'_n)_n \subset A\). Finally, thanks to the triangle inequality, the sequence \((x'_n)_n \subset A\) and \(\delta' := \tilde{\delta}/2 > 0\) can be chosen as the witnesses of property \((S)\) for the set \(A\).

Remark 2.4 In view of Proposition 2.3, if \(\beta\) is a bornology on \(X\) satisfying property \((S)\), then \(\beta\) must contain the relatively compact sets of \(X\).

In the following section we show that the Hadamard, weakly-Hadamard and limited bornologies are vector bornologies that satisfy property \((S)\). Moreover, the Fréchet bornology trivially satisfies property \((S)\).

3 Property \((S)\) and the Construction of a Lipschitz Function

We start this section by showing that some well-known bornologies satisfy the property \((S)\) (see Definition 2.2). In this section \(X\) denotes an infinite dimensional Banach space. Otherwise, the following three proposition are reduced to study the Fréchet bornology, and therefore, there is nothing to prove. Albeit simple, we present the proof of Propositions 3.1 and 3.2.

Proposition 3.1 The Hadamard bornology on \(X\) satisfies property \((S)\).

Proof Let \(A \subset X\) be a bounded non-relatively compact set. Then there exists a sequence \((x_n)_n \subset A\) with no accumulation points. Up to a subsequence, which is still denoted by \((x_n)_n\), there is \(\sigma > 0\) such that \(\|x_n - x_m\| \geq \sigma\) for all \(n \neq m\). Let us fix \(\delta = \sigma/4\). Therefore, for any increasing sequence \((n_k)_k \subset \mathbb{N}\), and any sequence \((y_k)_k \subset X\) satisfying \(\|y_k - x_{n_k}\| \leq \delta\) for all \(k \in \mathbb{N}\), the sequence \((y_k)_k\) has no accumulation points. Thus, the set \(\{y_k : k \in \mathbb{N}\}\) is not relatively compact.

Proposition 3.2 The Limited bornology on \(X\) satisfies property \((S)\).

Proof Let \(A \subset X\) be a bounded non-limited set. Then there is a weak*-null sequence \((x_n^*)_n \subset B_{X^*}\) which does not converge to 0 uniformly on \(A\). Hence, up to a subsequence, which is still denoted by \((x_n^*)_n\), there exist a sequence \((x_n)_n \subset A\) and \(\sigma > 0\) such that
\[|x^*_n(x_n)| \geq \sigma \text{ for all } n \in \mathbb{N}.\] Considering \(\delta = \sigma/2\), we obtain that for any increasing sequence \((n_k)\) in \(\mathbb{N}\) and \((y_k)\) in \(X\) such that \(y_k \in B(x_{n_k}, \delta)\) for all \(k \in \mathbb{N}\), the estimate \(|x^*_{n_k}(y_k)| \geq \sigma/2\) holds true. Therefore, \(\{y_k : k \in \mathbb{N}\}\) is not a limited set.

The following result concerns the weak-Hadamard bornology. It can be found inside of the proof of [9, Theorem 2.1] and we will write it down for the seek of completeness.

**Proposition 3.3** The weak-Hadamard bornology on \(X\) satisfies property \((S)\).

**Proof** Let \(A \subseteq X\) be a bounded non-relatively weakly-compact set. By the Eberlein-Šmulian Theorem, there is a sequence \((x_n)_n \subseteq X\) with no weakly-convergent subsequence. By contradiction, suppose that no subsequence of \((x_n)_n\) satisfies the statement of property \((S)\). Then, there exist an increasing sequence \((n(1,j))_j \subseteq \mathbb{N}\) and a sequence \((z^1_{n(1,j)})_j\) weakly-convergent to \(z\) such that \(z^1_{n(1,j)} \in B(x_{n(1,j)}, 1)\) for all \(j \in \mathbb{N}\). Inductively, for \(k \geq 2\), there exist a subsequence \((n(k, j))_j\) of \((n(k-1, j))_j\) and \((z^k_{n(k, j)})_j\) weakly-convergent sequence to \(z\) such that \(z^k_{n(k, j)} \in B(x_{n(k, j)}, 1/k)\) for all \(j \in \mathbb{N}\). Let us show that the sequence \((z^k)\) is norm convergent. Indeed, let \(k < l\). Recalling that the norm is a weakly-lower semi continuous and that \((n(l, j))_j\) is a subsequence of \((n(k, j))_j\), we obtain

\[
\|z^k - z^l\| \leq \liminf_{j} \|z^k_{n(l,j)} - z^l_{n(l,j)}\| \leq \liminf_{j} \|z^k_{n(l,j)} - x_{n(l,j)}\| + \|x_{n(l,j)} - z^l_{n(l,j)}\| \leq \frac{1}{k} + \frac{1}{l},
\]

proving that \((z^k)\) is a norm-Cauchy sequence. Let us denote by \(z^\infty\) the limit of \((z^k)\). We claim that \((x_{n(k,k)})_k\) weakly-converges to \(z^\infty\). Let \(x^* \in S_X^*\) and \(\varepsilon > 0\). Let \(n_0 \in \mathbb{N}\) such that \(n_0^{-1} \leq \varepsilon/3\). Thus, \(\|z^k - z^\infty\| \leq \varepsilon/3\) for all \(k \geq n_0\). Since the sequence \((z^0_{n(n_0,j)})_j\) is weakly convergent to \(z^0\), there is \(m_0 \in \mathbb{N}\) such that \(|\langle x^*, z^0 - z^0_{n(n_0,j)}\rangle| \leq \varepsilon/3\) for all \(j \geq m_0\). Observe that, for any \(k > n_0\), there is \(j_k \in \mathbb{N}\) such that \(n(k, k) = n(n_0, j_k)\). Hence, for \(k\) large, we have that \(j_k > m_0\) and then

\[
|\langle x^*, z_{n(k,k)} - z^\infty \rangle| \leq |\langle x^*, z_{n(n_0,j_k)} - z^0_{n(n_0,j_k)}\rangle| + |\langle x^*, z^0_{n(n_0,j_k)} - z^\infty\rangle| \leq n_0^{-1} + \frac{\varepsilon}{3} + \|z^0 - z^\infty\| \leq \varepsilon.
\]

Therefore, the sequence \((x_{n(k,k)})_k\) weakly converges to \(z^\infty\). Therefore, the set \(\{x_{n(k,k)} : k \in \mathbb{N}\}\) is weakly-compact, which is a contradiction.

The last ingredient of the proof of Theorem 4.5 is the Lipschitz function constructed in Lemma 3.7 below. A similar construction is used to prove certain properties of non-reflexive spaces, see [9]. In order to continue, let us give some definitions and state some simple results that can be found in [4].

For a set \(A \subseteq X\), \(\text{cone}(A)\) denotes the set \(\{\lambda x : x \in A, \lambda \geq 0\}\).

**Definition 3.4** Let \(X\) be a Banach space, let \((x_n)_n \subseteq X\) such that \(\|x_n\| = \|x_m\|\) for all \(n, m \in \mathbb{N}\) and let \(\sigma \in (0, \|x_1\|)\).

1. We say that \((x_n)_n\) is \(\sigma\)-separated if \(\|x_n - x_m\| \geq \sigma\) for all \(n, m \in \mathbb{N}\), with \(n \neq m\).
2. We say that \((x_n)_n\) is \(\sigma\)-cone separated if the sets \(\{\text{cone}(B(x_n, \sigma)) \setminus \{0\} : n \in \mathbb{N}\}\) are pairwise disjoint.
By definition, a $\sigma$-cone separated sequence is $2\sigma$-separated. Reciprocally, we have the following result.

**Proposition 3.5** [4, Proposition 3.6] Let $(x_n)_n \subset X$ be a $\sigma$-separated sequence. Then $(x_n)_n$ is $\sigma/4$-cone separated.

The core of a set $A \subset X$, denoted by core$(A) \subset X$, is the set defined by

$$\text{core}(A) := \{ x \in A : \forall y \in S_X, \exists t > 0, [x, x + ty) \subset A \}.$$  

Let $f : X \rightarrow \mathbb{R}$ be a function. The level sets of $f$ are denoted by $\{f = r\} := \{ x \in X : f(x) = r \}$, where $r \in \mathbb{R}$.

**Proposition 3.6** [4, Proposition 3.2] Let $f : X \rightarrow \mathbb{R}$ be a function. If $x \in \text{core}(\{f = f(x)\})$, then $f$ is Gâteaux-differentiable at $x$ with Gâteaux-differential equal to $dGf(x) = 0$.

Finally, we proceed with the construction of the mentioned Lipschitz function.

**Lemma 3.7** Let $\beta$ be a vector bornology on $X$, different from the Fréchet bornology, satisfying property (S). Let $A \subset X$ be a nonempty, semi-balanced, bounded set such that $A/\in \beta$. Then, there exist $\sigma > 0$ and a $\sigma$-separated sequence $(x_n)_n \subset A$ such that the Lipschitz function $f : X \rightarrow \mathbb{R}$ defined by

$$f(x) := \text{dist} \left( x, X \setminus \bigcup_{n=1}^{\infty} B \left( \frac{x_n}{n}, \frac{\sigma}{4n} \right) \right), \text{ for all } x \in X,$$

is $\beta$-differentiable at $0$ but not Fréchet-differentiable at $0$.

**Proof** Let $(x_n)_n \subset A$ and let $\delta > 0$ given by property (S). Thanks to Proposition 2.3, we assume that $\|x_n\| = \alpha$, for some $\alpha > 0$, for all $n \in \mathbb{N}$. Since the sequence $(x_n)_n$ do not have accumulation points, up to a subsequence, we can assume that $(x_n)_n$ is a $\sigma$-separated sequence, for some $0 < \sigma \leq \delta$. Let $f : X \rightarrow \mathbb{R}$ be the 1-Lipschitz function on $X$ defined by

$$f(x) := \text{dist} \left( x, X \setminus \bigcup_{n=1}^{\infty} B \left( \frac{x_n}{n}, \frac{\sigma}{4n} \right) \right), \text{ for all } x \in X.$$

By Propositions 3.5 and 3.6, $f$ is Gâteaux-differentiable at $0$, with Gâteaux-differential equal to $dGf(0) = 0$. However, since $nf(x_n/n) = \sigma/4$, $f$ is not Fréchet-differentiable at $0$. Finally, it only remains to prove that $f$ is $\beta$-differentiable at $0$. We proceed by contradiction. Suppose, for some set $W \in \beta$, the ratio of differentiability do not converge uniformly on $W$. That is, there exist a null sequence $(t_k)_k \subset \mathbb{R}^+$, $(w_k)_k \subset W$ and $\varepsilon > 0$ such that:

$$\left| \frac{f(t_k w_k)}{t_k} \right| \geq \varepsilon, \forall n \in \mathbb{N}.$$  

Since $f(t_k w_k) > 0$, there is a sequence $(n_k) \subset \mathbb{N}$ such that $t_k w_k \in B(x_{n_k}/n_k, \sigma/4n_k)$. Due to the fact that $W$ is bounded and that $(t_k w_k)_k$ converges to $0$, we can assume, up to a subsequence, that $(n_k)_k$ is increasing. We have two different cases now. If the sequence $(w_k)_k$ tends to $0$, the set $\{w_k : k \in \mathbb{N}\}$ is relatively compact. However, for Lipschitz functions the quotient of differentiability at a point of Gâteaux differentiability converges uniformly on relatively compact sets. This would imply that $\varepsilon \leq 0$, which is a contradiction.
Thus, the sequence \( (w_k) \) is not a norm-null sequence and then, up to a subsequence, the sequence \( (\|w_k\|) \) converges to some \( v > 0 \). Since \( t_k w_k \in B(x_{n_k}, \sigma/4n_k) \), then \( n_k t_k w_k \in B(x_{n_k}, \sigma/4) \). Therefore,

\[
n_k t_k \in \left[ \frac{\alpha}{\|w_k\|} - \frac{\sigma}{4\|w_k\|}, \frac{\alpha}{\|w_k\|} + \frac{\sigma}{4\|w_k\|} \right].
\]

Thus, the sequence \( (t_k n_k) \) accumulates in \([\alpha \frac{\sigma}{4v}, \alpha \frac{\sigma}{4v}]\). Passing through a subsequence, we assume that \( (t_k n_k) \) converges to some \( \lambda > 0 \). Hence, there is \( K \in \mathbb{N} \) such that \( \lambda w_k \in B(x_{n_k}, \sigma) \), for all \( k \geq K \). This is a contradiction with the property \( (S) \) for the bornology \( \beta \) because \( (\lambda w_k) \subset \lambda W \in \beta \) and \( \|\lambda w_k - x_{n_k}\| \leq \sigma \leq \delta \) for all \( k \geq K \).

Let us end this section with the following result, which can be seen as a natural way to construct vector bornologies that satisfies the property \( (S) \).

**Proposition 3.8** Let \( X \) and \( Z \) be two Banach spaces and let \( f : X \rightarrow Z \) be a Lipschitz function. Assume that \( f \) is Gâteaux differentiable at 0. Then, the family \( \mathcal{F} \) of nonempty bounded sets \( A \subset X \) such that, for every \( a \in X \) and every \( \lambda \in \mathbb{R} \)

\[
\lim_{t \rightarrow 0} \sup_{x \in a + \lambda A} \left\| \frac{f(0 + tx) - f(0)}{t} - d_G f(0)(x) \right\| = 0,
\]

is a vector bornology on \( X \) satisfying the property \( (S) \).

**Proof** Without loss of generality, we assume that \( f(0) = 0 \) and \( d_G f(0) = 0 \). Let us first prove that \( \mathcal{F} \) is a vector bornology. Since \( f \) is Gateaux differentiable at 0, we know that \( \{x\} \in \mathcal{F} \) for all \( x \in X \). Therefore, \( \mathcal{F} \) covers \( X \). The other two properties of bornologies are trivially satisfied by \( \mathcal{F} \) thanks to the algebra of limits. On the other hand, it is clear from the above limit, that if \( a \in X, \lambda \in \mathbb{R} \) and \( A \in \mathcal{F} \), then \( a + \lambda A \in \mathcal{F} \). It remains to show that if \( A \in \mathcal{F} \), then bal(\( A \)) \( \in \mathcal{F} \). Indeed, since \([−1, 1]\) is compact, then for every \( N \in \mathbb{N} \setminus \{0\} \), there exists \( n_N \in \mathbb{N} \) and \( (\beta_i)_{1 \leq i \leq n_N} \subset [−1, 1] \) such that \([−1, 1] = \bigcup_{i=1}^{n_N}[\beta_i - 1/N, \beta_i + 1/N] \). Fix \( a \in X \) and \( \lambda \in \mathbb{R} \),

\[
\sup_{x \in a + \lambda \text{bal}(A)} \left\| \frac{f(tx)}{t} \right\| = \sup_{x \in \text{bal}(A)} \left\| \frac{f(t(a + \lambda x))}{t} \right\|
\]

\[
= \sup_{\beta \in [−1, 1]} \sup_{x \in A} \left\| \frac{f(t(a + \lambda \beta x))}{t} \right\|
\]

\[
= \max_{1 \leq i \leq n_N} \sup_{\beta \in [\beta_i - 1/N, \beta_i + 1/N]} \sup_{x \in A} \left\| \frac{f(t(a + \lambda \beta x))}{t} \right\|
\]

\[
\leq \max_{1 \leq i \leq n_N} \sup_{\beta \in [\beta_i - 1/N, \beta_i + 1/N]} \sup_{x \in A} \left\| \frac{f(t(a + \lambda \beta_i x))}{t} \right\|
\]

\[
+ \max_{1 \leq i \leq n_N} \sup_{\beta \in [\beta_i - 1/N, \beta_i + 1/N]} \sup_{x \in A} \left\| \frac{f(t(a + \lambda \beta_i x))}{t} - \frac{f(t(a + \lambda \beta x))}{t} \right\|
\]
\[
\frac{f(t(a + \lambda \beta_i x))}{t} \leq \max_{1 \leq i \leq n} \sup_{x \in A} \left\| f(t(a + \lambda \beta_i x)) \right\| + \max_{1 \leq i \leq n} \sup_{x \in [\beta_i - 1/N, \beta_i + 1/N]} \sup_{x \in A} \operatorname{Lip}(f) |\lambda||\beta - \beta_i||x|
\]
\[
\leq \max_{1 \leq i \leq n} \sup_{x \in A} \left\| f(t(a + \lambda \beta_i x)) \right\| + \max_{1 \leq i \leq n} \sup_{x \in [\beta_i - 1/N, \beta_i + 1/N]} \sup_{x \in A} \left\| \frac{f(tx)}{t} \right\| + \frac{|\lambda| \operatorname{Lip}(f) \sup_{x \in A} \|x\|}{N}
\]
\[
= \max_{1 \leq i \leq n} \sup_{x \in a + \lambda \beta_i A} \left\| f(tx) \right\| + \frac{|\lambda| \operatorname{Lip}(f) \sup_{x \in A} \|x\|}{N}.
\]
Since \( A \in \mathcal{F} \), tending \( t \to 0 \), we get that, for every \( N \in \mathbb{N} \setminus \{0\} \)
\[
\limsup_{t \to 0} \sup_{x \in a + \lambda \operatorname{bal}(A)} \left\| f(tx) \right\| \leq |\lambda| \operatorname{Lip}(f) \sup_{x \in A} \|x\|/N.
\]
Hence,
\[
\lim_{t \to 0} \sup_{x \in a + \lambda \operatorname{bal}(A)} \left\| f(tx) \right\| = 0.
\]
Thus, \( \operatorname{bal}(A) \in \mathcal{F} \) and finally, we have that \( \mathcal{F} \) is a vector bornology.

Now, let us see that \( \mathcal{F} \) satisfies the property \((S)\). Let \( A \subset X \) be a nonempty bounded set such that \( A \notin \mathcal{F} \). Therefore there are \( a_0 \in X \) and \( \lambda_0 \in \mathbb{R} \setminus \{0\} \), such that
\[
\lim_{t \to 0} \sup_{x \in a_0 + \lambda_0 A} \left\| f(tx) \right\| > 0.
\]
Thus, there are \( \sigma > 0 \), \( (x_n)_n \subset A \) and \( (t_n)_n \subset \mathbb{R} \), convergent to \( 0 \), such that
\[
\left\| f(t_n(a_0 + \lambda_0 x_n)) \right\| / t_n \geq \sigma, \quad \text{for all } n \in \mathbb{N}.
\]
We claim that the sequence \( (x_n)_n \) and \( \delta := \sigma/2|\lambda_0|\operatorname{Lip}(f) \) witness the property \((S)\) for the set \( A \). Indeed, if \( (n_k)_k \subset \mathbb{N} \) is an increasing sequence and \( (y_k)_k \subset X \) is a sequence such that \( \|x_{n_k} - y_k\| \leq \delta \) equivalently, \( \|(a_0 + \lambda_0 x_{n_k}) - (a_0 + \lambda_0 y_k)\| \leq |\lambda_0|\delta \), then
\[
\left\| f(t_{n_k}(a_0 + \lambda_0 y_k)) \right\| \geq \left\| f(t_{n_k}(a_0 + \lambda_0 x_{n_k})) \right\| - \left\| f(t_{n_k}(a_0 + \lambda_0 y_k)) - f(t_{n_k}(a_0 + \lambda_0 x_{n_k})) \right\|
\]
\[
\geq \sigma - \operatorname{Lip}(f) |\lambda_0|\delta t_{n_k} / t_{n_k} \geq \sigma/2.
\]
Therefore \( \{y_k : k \in \mathbb{N}\} \notin \mathcal{F} \). This ends the proof.

### 4 The Main Result: Characterization of \( \beta_Y-\beta_X \)-Operators

In order to prove the main result of this section, we need the following definitions.

**Definition 4.1** Let \( X \) and \( Y \) be two Banach spaces and let \( \beta_X \) and \( \beta_Y \) be vector bornologies on \( X \) and \( Y \) respectively. Let \( T : Y \to X \) be any operator (not necessarily linear). We say that \( T \) is a \( \beta_Y-\beta_X \)-operator, if \( T(\beta_Y) \subset \beta_X \), that is
\[
T(A) \in \beta_X, \quad \text{for all } A \in \beta_Y.
\]
Whenever \( \beta_Y \) is the Fréchet bornology, we simple write that \( T \) is a \( \beta_X \)-operator.
Recall that we denote by $\beta = G, H, L, wH$ and $F$, the Gâteaux, Hadamard, limited, weak Hadamard and Fréchet bornologies respectively. The above definition unifies some classical notions of linear operators, which we recall in the following example.

**Example 4.2** Let $X$ and $Y$ be two Banach spaces. Then.

(i) Linear $F$-$H$-operators or $H$-operators coincide with linear compact operators from $Y$ into $X$.

(ii) Linear $F$-$L$-operators or $L$-operators coincide with linear limited operators from $Y$ into $X$.

(iii) Linear $F$-$wH$-operators or $wH$-operators coincide with linear weakly compact operators from $Y$ into $X$.

(iv) Linear $wH$-$H$-operators coincide with completely continuous linear operators from $Y$ into $X$.

We need the following notion of generalized differential in the spirit of Suchomolinov (see for instance [24, p. 135] and [22]) which is a generalization of $\beta$-differentiability.

**Definition 4.3** Let $X$ and $Y$ be Banach spaces and let $\beta$ be a bornology on $Y$. A map $T : Y \to X$ is said to be $\beta$-positively homogeneous differentiable at $y \in Y$ ($\beta$-p.h. differentiable for short) if there is a continuous positively homogeneous function $R : Y \to X$ such that

$$\lim_{t \to 0^+} \sup_{z \in A} \left\| \frac{T(y + tz) - T(y)}{t} - R(z) \right\| = 0,$$

for all $A \in \beta$.

We denote the $\beta$-p.h. differential of $T$ at $y$, by $R := d_{\beta}^p T(y)$.

The following remark summarizes some simple facts about p.h.-differentiability. In particular, we use the second and fourth points of Remark 4.4 without referring to them.

**Remark 4.4** Let $T : Y \to X$ be an operator (not necessarily linear) and let $\beta$ be a bornology on $Y$. Then:

- If $T$ is $\beta$-p.h. differentiable at $y \in Y$, then the $\beta$-p.h. differential of $T$ at $y$ is unique.

- Definition 4.3 extends the usual notion of $\beta$-differentiability. Indeed, if $T$ is $\beta$-differentiable at some $y \in Y$, then $d_{\beta} T(y) = d_{\beta}^{ph} T(y)$. In particular, if $T$ is a bounded linear operator, then $d_{\beta}^{ph} T(y) = d_{\beta} T(y) = T$, for every $y \in Y$ and any $\beta$ bornology on $Y$.

- If $T$ is a continuous positively homogeneous function, then $T$ is $\beta$-p.h. differentiable at $0$ for any $\beta$ bornology on $Y$ and we have that $d_{\beta}^{ph} T(0) = T$.

- If $T$ is $\beta$-p.h. differentiable at $y \in Y$, then $d_{\beta}^{ph} T(y)$ sends bounded sets into bounded sets.

- If $T : Y \to X$ is $\beta$-p.h. differentiable at $0$ and $d_{\beta}^{ph} T(0) = 0 (= d_{\beta} T(0))$, then $f \circ T$ is $\beta$-differentiable at $0$, for every Lipschitz function $f : X \to \mathbb{R}$ (not necessarily differentiable). Indeed, we have that

$$\lim_{t \to 0^+} \sup_{u \in A} \left| \frac{f \circ T(tu) - f \circ T(0)}{t} \right| \leq \text{Lip}(f) \lim_{t \to 0^+} \sup_{u \in A} \left| \frac{T(tu) - T(0)}{t} \right| = 0,$$

for any $A \in \beta$. 

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Let $X$ and $Y$ be two Banach spaces and let $\beta_X$ and $\beta_Y$ be vector bornologies on $X$ and $Y$ respectively. Then, if $R : Y \to X$ is a continuous positively homogeneous mapping such that there is $A \in \beta_Y$ satisfying $R(A) \notin \beta_X$ (that is, $R$ is not $\beta_Y$-$\beta_X$-operator), then, there exists a semi-balanced set $B \in \beta_Y$ such that $R(B)$ is a semi-balanced, bounded set that does not belong to $\beta_X$. Indeed, thanks to the fact that $R$ is continuous and positively homogeneous, it is enough to consider $B = \text{bal}(A)$, the balanced hull of $A$. Now, we are able to state the main result of this work.

Theorem 4.5 Let $X$ and $Y$ be two Banach spaces and let $\beta_X$ and $\beta_Y$ be vector bornologies on $X$ and $Y$ respectively. Assume that $\beta_X$ satisfies property (S) and let $T : Y \to X$ be a $\beta_Y$-p.h. differentiable operator at $y \in Y$. Then the following assertions are equivalent.

(i) $d_{\beta_Y}^{ph} T(y)$ is a $\beta_Y$-$\beta_X$-operator

(ii) for every Lipschitz function $f : X \to \mathbb{R}$, $\beta_X$-p.h. differentiable at $x = Ty$, we have that $f \circ T$ is $\beta_Y$-p.h. differentiable at $y$ and $d_{\beta_Y}^{ph} (f \circ T)(y) = d_{\beta_Y}^{ph} f(x) \circ d_{\beta_Y}^{ph} T(y)$

(iii) for every Lipschitz function $f : X \to \mathbb{R}$, $\beta_X$-differentiable at $x = Ty$ with $d_{\beta_X} f(x) = 0$, we have that $f \circ T$ is $\beta_Y$-differentiable at $y$ and $d_{\beta_Y} (f \circ T)(y) = 0$.

Let us start with the proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3 It is a direct application of Theorem 4.5 where $\beta_X$ is the weakly-Hadamard bornology and $\beta_Y$ is the Fréchet bornology. Indeed, Proposition 3.3 asserts that the weakly-Hadamard bornology satisfies property (S).

Proof of Theorem 1.4 It is a direct application of Theorem 4.5 where $\beta_X$ is the Hadamard bornology and $\beta_Y$ is the weak-Hadamard bornology. Indeed, Proposition 3.1 asserts that the Hadamard bornology satisfies property (S).

Also, as a direct consequence of Theorem 4.5, using the Fréchet bornology on $Y$ and the Hadamard (resp. limited) bornology on $X$, we characterize compact operators (resp. limited operators). Thus, we recover one of the main result of [4].

Proof of Theorem 4.5 (i) $\implies$ (ii). This part is straightforward and it does not require that the bornology $\beta_X$ satisfies property (S). Indeed, let $T : Y \to X$ be a $\beta_Y$-p.h. differentiable at $y$, with $\beta_Y$-p.h. differential equal to $R := d_{\beta_Y}^{ph} T(y)$, which is a $\beta_Y$-$\beta_X$ operator at $y$ and let $f : X \to \mathbb{R}$ be a Lipschitz function $\beta_X$-p.h. differentiable at $x = Ty$, with $p := d_{\beta_X}^{ph} f(x)$. We claim that the $\beta_Y$-p.h. differential of $f \circ T$ at $y$ is equal to $p \circ R$. Indeed, for any $A \in \beta_Y$, we have that for all $t > 0$

$$\sup_{u \in A} \frac{|f \circ T(y + tu) - f \circ T(y)|}{t} \leq \sup_{u \in A} \frac{|f \circ T(y + tu) - f(T(y) + tR(u))|}{t} + \sup_{u \in A} \frac{|f(T(y) + tR(u)) - f(T(y))|}{t} - p(R(u))$$

$$\leq \sup_{u \in A} \text{Lip}(f) \frac{|T(y + tu) - T(y) - tR(u)|}{t} + \sup_{v \in R(A)} \frac{|f(x + tv) - f(x)|}{t} - p(v).$$

Therefore, thanks to the $\beta_Y$-p.h. differentiability of $T$, $R(A) \in \beta_X$ and the $\beta_X$-p.h. differentiability of $f$, sending $t$ to 0 in the above expression, we obtain that $f \circ T$ is $\beta_Y$-p.h.
differentiable at \( y \) with \( \beta_Y \)-p.h. differential equal to \( p \circ R \).

\((ii) \implies (iii)\). This part is trivial. Note that if \( d_{\beta_Y}^{ph}(f \circ T)(y) = 0 \), then \( f \circ T \) is necessarily \( \beta_Y \)-differentiable at \( y \) and \( d_{\beta_Y}(f \circ T)(y) = d_{\beta_Y}^{ph}(f \circ T)(y) = 0 \).

\((iii) \implies (i)\). Redefining \( T \) by \( T := T(\cdot + y) - T(y) \), we assume without loss of generality that \( y = 0 \) and \( T(0) = 0 \). We proceed by contradiction. Assume that \( R := d_{\beta_Y}^{ph}T(0) \) is not a \( \beta_Y \)-\( \beta_X \)-operator. Therefore, there is a balanced set \( A \in \beta_Y \) such that \( R(A) \) is a bounded, semi-balanced set and \( R(A) \not\in \beta_X \). Since \( \beta_X \) satisfies property \((S)\), Lemma 3.7 gives us a \( \sigma > 0 \), a \( \sigma \)-separated sequence \( (x_n)_n \subset R(A) \) and the 1-Lipschitz function \( f : X \to \mathbb{R} \) defined by

\[
f(x) := \text{dist}(x, X \setminus \bigcup_{n=1}^{\infty} B\left(\frac{x_n}{n}, \frac{\sigma}{4n}\right)), \quad \text{for all } x \in X,
\]

which is \( \beta_X \)-differentiable (and thus \( \beta_X \)-p.h. differentiable) at 0 with \( d_{\beta_X}^{ph}f(0) = 0 \). We prove that \( f \circ T \) is not \( \beta_Y \)-differentiable at 0. Indeed, let \( (y_n)_n \subset A \) such that \( R(y_n) = x_n \) for all \( n \in \mathbb{N} \). Finally, notice that (since \( f \circ T(0) = 0 \) and \( f \) is positive)

\[
\left| \frac{f \circ T(\frac{y_n}{n}) - f \circ T(0)}{\frac{1}{n}} \right| = \frac{f \circ T(\frac{y_n}{n}) - f \circ T(0)}{\frac{1}{n}} = f \circ T(\frac{y_n}{n}) - f \circ T(0) - f(\frac{R(y_n)}{n}) + \frac{f(\frac{R(y_n)}{n}) - f \circ T(0)}{\frac{1}{n}} \geq -\text{Lip}(f) \left| \frac{R(y_n)}{n} - \frac{y_n}{n} \right| + nf\left(\frac{x_n}{n}\right) = \sigma - \text{Lip}(f) \left| \frac{y_n}{n} - R(y_n) \right|, \quad \text{for all } n \in \mathbb{N}.
\]

Thanks to our hypothesis (statement \((iii)\)), we have that \( d_{\beta_Y}f \circ T(0) = 0 \). However, sending \( n \) to infinity in the above expression and using the \( \beta_Y \)-p.h. differentiability of \( T \), we get that 0 is not the \( \beta_Y \) differential of \( f \circ T \) at 0, which is a contradiction.

A well-known result asserts that the Fréchet differential of a (not necessarily linear) compact function (resp. completely continuous function) is a linear compact (resp. a linear completely continuous) operator. For more information and details, we refer to [13, 20–23, 25]). The following theorem (which will be used in Corollaries 6.14 and 6.14) is a generalization of these results, which applies to the more general notions of \( \beta \)-p.h. differentiability and \( \beta_Y \)-\( \beta_X \)-operators. We need the following lemma.

**Lemma 4.6** Let \( X \) be a Banach space and let \( \beta \) be a vector bornology on \( X \) satisfying property \((S)\). Let \( A \subset X \) and assume that for every \( \varepsilon > 0 \) there exists \( A_{\varepsilon} \in \beta \) such that \( A \subset A_{\varepsilon} + \varepsilon B_X \). Then \( A \in \beta \).

**Proof** Suppose by contradiction that \( A \not\in \beta \). By property \((S)\), there are a sequence \( (x_n)_n \subset A \) and \( \delta > 0 \) such that for any increasing sequences \( (n_k)_k \subset \mathbb{N} \) and for any sequence \( (y_k)_k \subset X \) satisfying \( \|y_k - x_{n_k}\| \leq \delta \) for all \( k \in \mathbb{N} \), the set \( \{y_k : k \in \mathbb{N}\} \not\in \beta \). By assumption,
with \( \varepsilon = \delta \), \( A \subset A_\delta + \delta B_X \), where \( A_\delta \in \beta \). Thus, for every \( k \in \mathbb{N} \), there exists \( y_k \in A_\delta \) such that \( \| y_k - x_{n_k} \| \leq \delta \). This is a contradiction because \( \{ y_k : k \in \mathbb{N} \} \subset A_\delta \in \beta \).

**Theorem 4.7** Let \( X \) and \( Y \) be two Banach spaces and let \( \beta_X \) and \( \beta_Y \) be vector bornologies on \( X \) and \( Y \) respectively. Assume that \( \beta_X \) satisfies property (S). Let \( T : Y \to X \) be a \( \beta_Y \)-\( \beta_X \)-operator which is \( \beta_Y \)-p.h. differentiable at \( y \in Y \). Then, \( d^{ph}_{\beta_Y} T(y) \) is a \( \beta_Y \)-\( \beta_X \)-operator.

**Proof** Let \( A \in \beta_Y \). From the definition of the \( \beta_Y \)-p.h. differentiability of \( T \) at \( y \in Y \), we have that for every \( \varepsilon > 0 \), there exists \( \delta_\varepsilon > 0 \) such that
\[
\sup_{z \in A} \| \delta_\varepsilon^{-1}(T(y + \delta_\varepsilon z) - T(y)) - d^{ph}_{\beta_Y} T(y)(z) \| \leq \varepsilon.
\]
This shows that, for every \( \varepsilon > 0 \), there exists \( \delta_\varepsilon > 0 \) such that
\[
d^{ph}_{\beta_Y} T(y)(A) \subset \delta_\varepsilon^{-1}(T(y + \delta_\varepsilon A) - T(y)) + \varepsilon B_X.
\]
Since \( T \) is \( \beta_Y \)-\( \beta_X \)-operator and \( \beta_X \) is a vector bornology, it follows that
\[
\delta_\varepsilon^{-1}(T(y + \delta_\varepsilon A) - T(y)) \in \beta_X.
\]
Thus, using Lemma 4.6, we get that \( d^{ph}_{\beta_Y} T(y)(A) \in \beta_X \). Hence, \( d^{ph}_{\beta_Y} T(y) \) is a \( \beta_Y \)-\( \beta_X \)-operator.

## 5 Alternative Proof of Theorem 1.2

In this section we provide an alternative proof of Theorem 1.2, which was initially given in [3]. Our proof is based on the fact that Gâteaux-differentiability and limited-differentiability coincide for continuous convex functions defined on an open convex subset of a Banach space (Proposition 5.3). We point out that this is not the case for general Lipschitz functions defined on open subsets of Banach spaces which are not Gelfand-Phillips (see Definition 6.2 and Corollary 6.5). In what follows, we state two useful results that can be found in [11].

**Proposition 5.1** [11, Proposition 8.1.1] Let \( \beta \) be a bornology on \( X \). Let \( (x^*_n)_n \subset X^* \) be a bounded sequence. Let \( f : X \to \mathbb{R} \) be the convex function defined by:
\[
f(x) = \sup_n \left\{ 0, x^*_n(x) - \frac{1}{n} \right\}.
\]
Then, \( f \) is \( \beta \)-differentiable at \( 0 \) if and only if \( (x^*_n)_n \to_{\tau_\beta} 0 \), where \( \tau_\beta \) denotes the topology on \( X^* \) of the uniform convergence on \( \beta \)-sets.

**Theorem 5.2** [11, Theorem 8.1.3] Let \( X \) be a Banach space and let \( \beta_1 \) and \( \beta_2 \) be two bornologies on \( X \) such that \( \beta_1 \subseteq \beta_2 \). Then, the following assertions are equivalent.

(a) \( \tau_{\beta_1} \) and \( \tau_{\beta_2} \) agree sequentially in \( X^* \).

(b) \( \beta_1 \)-differentiability and \( \beta_2 \)-differentiability coincide for continuous convex functions.

We know that Gâteaux differentiability and Hadamard differentiability coincide for real-valued Lipschitz functions defined on any Banach space. The following proposition can be seen as the analogous of this result for convex functions.

**Proposition 5.3** Gâteaux differentiability and limited differentiability coincide for continuous convex functions defined on open convex subsets of a Banach space.
Proof Let $X$ be a Banach space. We use Theorem 5.2 with the bornologies $\beta_1 = \text{Gâteaux}$ and $\beta_2 = \text{limited}$. Let $(x_n^*) \subset X^*$ be a sequence $\tau_{\beta_1}$-convergent to 0, i.e., $(x_n^*)$ is a weak$^*$-null sequence. Let $A \subseteq X$ be limited set on $X$. By definition of limited set, we have that
\[
\lim_{n \to \infty} \sup_{x \in A} |x_n^*(x)| = 0.
\]
Since $A$ is an arbitrary limited subset of $X$, we have that $(x_n^*)$ converges to 0 in $\tau_{\beta_2}$. Applying Theorem 5.2 we obtain the desired result.

In [3], the proof of Theorem 1.2 follows a completely different approach compared with our proof of Theorem 4.5. Let us see how Theorem 1.2 can be deduced from Proposition 5.3.

Alternative proof of Theorem 1.2 Thanks to Proposition 5.3, the necessity part of Theorem 1.2 is straightforward. In fact, it is analogous to the necessity part of Theorem 4.5. Conversely, we proceed by contradiction. Let $T \in \mathcal{L}(Y, X)$ be a non-limited operator. Then, there exist a weak$^*$-null sequence $(x_n^*) \subset X^*$ and a sequence $(y_n) \subset BY$ such that $x_n^*(Ty_n) \geq 2$. Let us consider the function $f : X \to \mathbb{R}$ defined by:
\[
f(x) = \max \left\{ 0, \sup \left\{ x_n^*(x) - \frac{1}{n} \right\} \right\}, \quad \text{for all } x \in X.
\]
Thanks to Proposition 5.1, $f$ is Gâteaux-differentiable at 0. Since $f$ is a positive function and $f(0) = 0$, we know that $d_G f(0) = 0$. Thus, the only candidate to Fréchet-differential of $f \circ T$ at 0 is also the functional 0. However, the computation
\[
nf \circ T \left( \frac{yn}{n} \right) \geq n \left( \frac{2}{n} - \frac{1}{n} \right) = 1, \quad \text{for all } n \in \mathbb{N},
\]
shows that $f \circ T$ is not Fréchet-differentiable at 0.

6 Consequences of Theorem 4.5

In this section we provide some applications of our main result.

6.1 Characterization of a $\beta_2$-$\beta_1$-space

Let us start with the following definitions.

Definition 6.1 Let $X$ be a Banach space and $\beta_1$, $\beta_2$ be two vector bornologies on $X$. We say that $X$ is a $\beta_2$-$\beta_1$-space if $\beta_2 \subset \beta_1$ or equivalently, if the identity $Id : X \to X$ is a $\beta_2$-$\beta_1$-operator. Whenever $\beta_2 = F$, we simple say that $X$ is a $\beta_1$-space, which is equivalent in this case to the fact that the closed unit ball $\overline{B}_X \in \beta_1$.

Definition 6.2 A Banach space $X$ is called a Gelfand-Phillips space if all limited sets in $X$ are relatively norm-compact.

As a simple fact, a Banach space $X$ is Gelfand-Phillips if and only if every limited operator with range in $X$ is compact. Further information of Gelfand-Phillips spaces can be found in [15]. The following proposition summarizes some well-known results.

Proposition 6.3 Let $X$ be a Banach space. Then
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1. $X$ is a $H$-space if and only if $X$ is finite dimensional (this is the Riesz theorem).
2. $X$ is a $wH$-space if and only if $X$ is reflexive (this is the James theorem).
3. $X$ is a $L$-space if and only if $X$ is finite dimensional (this is the Josefson-Nissenzweig theorem).
4. $X$ is a $wH$-$H$-space if and only if $X$ has the Schur property (easy to see, using the Eberlein-Šmulian Theorem).
5. $X$ is $L$-$H$-space if and only if $X$ is a Gelfand-Phillips space (simply by coincidence of definitions).

Thanks to Proposition 6.3 (1) and (3), a Banach space $X$ is a $L$-space if and only if it is a $H$-space. The following characterization of a $\beta_Y$-$\beta_X$-space is an easy consequence of Theorem 4.5.

**Proposition 6.4** Let $X$ be a Banach space and let $\beta_1$ and $\beta_2$ be two vector bornologies on $X$ such that $\beta_1$ satisfies property (S). Then, $X$ is a $\beta_2$-$\beta_1$-space (i.e. $\beta_2 \subset \beta_1$) if and only if for every real-valued Lipschitz function $f$ on $X$, $f$ is $\beta_2$-differentiable at some point whenever it is $\beta_1$-differentiable at this point.

**Proof** It is enough to apply Theorem 4.5 to the identity operator on $X$. 

We end this subsection by stating some corollaries of Propositions 6.3 and 6.4 which are characterizations of Gelfand-Phillips spaces, finite dimensional spaces, reflexive spaces and Schur spaces in terms of the differentiability of Lipschitz functions. Up to the best of our knowledge, Corollaries 6.5 and 6.6 are new results while Corollaries 6.7 and 6.8 are known results in [9].

**Corollary 6.5** Let $X$ be a Banach space. Then, $X$ is a Gelfand-Phillips space if and only if Gâteaux-differentiability (equivalently Hadamard-differentiability) and limited-differentiability coincide for real-valued Lipschitz functions on $X$.

It is a well-known result that a Banach space $X$ is finite dimensional if and only if Gâteaux-differentiability and Fréchet differentiability coincide for real-valued Lipschitz functions. On the other hand, Corollary 6.5 shows that outside Gelfand-Phillips spaces, Gâteaux-differentiability and limited-differentiability do not coincide in general for real-valued Lipschitz functions on $X$. We give in the following corollary a new characterization of finite dimensional spaces.

**Corollary 6.6** Let $X$ be a Banach space. Then, $X$ is finite dimensional if and only if limited-differentiability and Fréchet-differentiability coincide for real-valued Lipschitz functions on $X$.

We recover the following two known results.

**Corollary 6.7** (see [9, Theorem 1.4]) A Banach space $X$ is a reflexive space if and only if weak-Hadamard differentiability and Fréchet differentiability coincide for real-valued Lipschitz functions.

**Corollary 6.8** (see [9, Theorem 4.1]) A Banach space $X$ has the Schur property if and only if Gâteaux differentiability and weak Hadamard differentiability coincide for real-valued Lipschitz functions.
6.2 Spaceability and Differentiability in the Space of Lipschitz Functions

Let $\beta_X$ and $\beta_Y$ be vector bornologies on $X$ and $Y$ respectively such that $\beta_X$ satisfies property $(S)$. Let $T \in \mathcal{L}(X, Y)$ be a non-$\beta_Y$-$\beta_X$-operator. Then, the set of Lipschitz functions, $\beta_X$ differentiable at 0 such that $f \circ T$ is not $\beta_Y$-differentiable at 0, denoted by $\mathcal{F}$, is dense in the space of Lipschitz, $\beta_X$-differentiable functions at 0 (for the topology generated by the Lipschitz seminorm). In what follows, we want to measure the size of $\mathcal{F}$ in an algebraic sense. To do so, we need the notions of lineability and spaceability: Let $\alpha$ be a cardinal number. A set $A \subset X$ is said $\alpha$-lineable if $A \cup \{0\}$ contains a subspace of dimension $\alpha$. A set $A \subset X$ is said $\alpha$-spaceable, if $A \cup \{0\}$ contains a closed subspace of dimension $\alpha$. The following corollary states that the set $\mathcal{F}$ is $\varepsilon$-spaceable, meaning that it contains an isometric copy of a Banach space of dimension of the continuum. More on lineability and spaceability can be found in [1, 2, 16] and references therein. The following result extends [4, Corollary 3.9].

**Corollary 6.9** Let $X$ and $Y$ be two Banach spaces. Let $\beta_X$ and $\beta_Y$ be a vector bornologies on $X$ and $Y$ respectively such that $\beta_X$ satisfies property $(S)$. Let $T : Y \to X$ be a continuous operator, $\beta_Y$-p.h. differentiable at 0 such that $T(0) = 0$ and $d_{\beta_Y}^{ph} T(0)$ is not a $\beta_Y$-$\beta_X$-operator. Then, the set of real-valued Lipschitz functions defined on $X$ which are $\beta_X$-differentiable at 0, with $\beta_X$-differential equal to 0, but $f \circ T$ is not $\beta_Y$-differentiable at 0, contains a subspace isometric to $\ell^\infty(\mathbb{N})$, up to the function 0 (whenever it is regarded as a subspace of $\text{Lip}(X)$).

**Proof** Let us fix $R = d_{\beta_Y}^{ph} T(0)$, which is a continuous positively homogeneous function from $Y$ to $X$. Let $A \subset Y$ be a bounded semi-balanced set such that $A \in \beta_Y$ but the bounded, semi-balanced set $R(A) \notin \beta_X$. Let $\sigma > 0$ and let $(x_n) \subset R(A)$ be a $\sigma$-separated sequence given by Lemma 3.7. Let $(y_n) \subset A$ such that $R(y_n) = x_n$ for all $n \in \mathbb{N}$. For each $p \in \mathbb{N}$ prime number define the sets $B_{p,n} = B(\frac{x}{p^r}, \frac{\sigma}{p^{r+1}})$ for every $n \in \mathbb{N}$, and $B_p := \bigcup B_{p,n}$. Note that, the family of sets $\{B_{p,n} : p, n \in \mathbb{N}\}$ is pairwise disjoint. As in Lemma 3.7, for each $p \in \mathbb{N}$, we define $f_p : X \to \mathbb{R}$ by

$$f_p(x) = \text{dist}(x, X \setminus B_p),$$

which is 1-Lipschitz, $\beta_X$-differentiable at 0, with $d_{\beta_X} f_p(0) = 0$. Moreover, as in the proof of Theorem 4.5, $f_p \circ T$ is not $\beta_Y$-differentiable at 0. Let $(p_i)_i$ be an enumeration of the prime numbers. Note that, for every $p_i$, $\{x \in X : f_{p_i}(x) > 0\} = B_{p_i}$. Therefore, $(f_{p_i})_i \subset \text{Lip}(X)$ is a sequence of linearly independent functions. Moreover, if $\mu \in \ell^\infty(\mathbb{N})$, the function

$$f_\mu(x) := \sum_{i=1}^\infty \mu_i f_{p_i}(x),$$

is well defined (because for each $x \in X$ there is at most one non-zero term in the series), $f(0) = 0$ and is $\|\mu\|_\infty$-Lipschitz. On the other hand, since $f_\mu = \mu_i f_{p_i}$ on supp$(f_{p_i})$, we have that Lip$(f_\mu) \geq \sup\{|\mu_i| : i \in \mathbb{N}\}$. That is, the linear operator $L : \ell^\infty(\mathbb{N}) \to \text{Lip}(X)$ given by $L\mu = f_\mu$ is an isometry onto its image. Since $(x_n)_n$ is a $\sigma$-separated sequence, Proposition 3.5 implies that $0 \in \text{core}(X \setminus \cup_i B_{p_i})$ and, thanks to Proposition 3.6, $f_\mu$ is Gâteaux-differentiable at 0 with $d_G f_\mu(0) = 0$. Moreover, following the lines of the proof of Lemma 3.7, we show that $f_\mu$ is, in fact, $\beta_X$-differentiable at 0. On the other hand, if $\mu \in \ell^\infty(\mathbb{N})$ and $\mu \neq 0$, $f_\mu \circ T$ is not $\beta_Y$-differentiable at 0. Indeed, observe first that the only candidate for $\beta_Y$-(p.h.) differential of $f_\mu \circ T$ at 0 is the functional 0 (since $T$ is
continuous, for any \( u \in X \), there is a null-sequence \((t_n)\in\mathbb{R}_+\) such that \( f_{\mu} \circ T(t_n u) = 0 \).

Let \( k \in \mathbb{N} \) such that \( \mu_k \neq 0 \). Since \( A \in \beta_Y \) and recalling that \( R = d^{n_k}_{\beta_Y} T(0) \), for any \( n \in \mathbb{N} \) we have that
\[
\frac{f_{\mu} \circ T(y_{p_k}^n / p_k^n) - f_{\mu} \circ T(0)}{1/p_k^n} = \frac{f_{\mu} \circ T(y_{p_k}^n / p_k^n) - f_{\mu} \circ R(y_{p_k}^n / p_k^n)}{1/p_k^n} + \frac{f_{\mu} \circ R(y_{p_k}^n / p_k^n)}{1/p_k^n}
\]
\[
\geq -\text{Lip}(f_{\mu}) \left\| \frac{T(y_{p_k}^n / p_k^n) - R(y_{p_k}^n / p_k^n)}{1/p_k^n} \right\| + p_k^n f_{\mu}(x_{p_k}^n / p_k^n)
\]
\[
= \frac{\sigma}{4} - \text{Lip}(f_{\mu}) \left\| \frac{T(y_{p_k}^n / p_k^n) - R(y_{p_k}^n / p_k^n)}{1/p_k^n} \right\|.
\]

Therefore, since \( R \) is the \( \beta_Y \)-p.h. differential of \( T \) and \((y_{p_k}^n)\in A \in \beta_Y \), by sending \( n \) to infinity we obtain that the last expression does not converge to 0. Hence, 0 is not the \( \beta_Y \)-differential of \( f_{\mu} \circ T \) at 0. Thus, \( f_{\mu} \circ T \) is not \( \beta_Y \) differentiable at 0.

6.3 Non-linear Strict Cosingularity and Almost Weakly Compactness

In this last subsection we give an extension of the so-called Bourgain-Diestel Theorem, see [10] or [4]. Recall from [10] (resp. from [17]) that a bounded linear operator \( T : Y \to X \) between Banach spaces \( Y \) and \( X \) is called strictly cosingular (resp. almost weakly compact) if the only Banach spaces \( E \) for which one can find a bounded linear operator \( q : X \to E \) for which \( q \circ T \) is surjective, are finite dimensional (resp. are reflexive spaces). The strict cosingularity and almost weak compactness, are generalizations of the notions of compact and weakly compact operators respectively. A generalization of (linear) strict cosingularity can be found in [18].

In the following definitions we state a non-linear extension of the notion of strict cosingularity and almost weak compactness. Recall the notion of co-Lipschitz mapping, which was introduced in [7, 19]: a mapping \( L : X \to E \) is called co-Lipschitz if there is \( c > 0 \) such that
\[
L(x) + cr B_E \subset L(x + r B_X), \text{ for all } x \in X, \ r > 0.
\]

Note that, thanks to the Open-mapping theorem, a bounded linear operator \( L : X \to E \) is co-Lipschitz if and only if it is surjective.

**Definition 6.10** Let \( X \) and \( Y \) be two Banach spaces. An operator (not necessarily linear) \( T : Y \to X \) is said to be strictly cosingular (resp. strongly strictly cosingular) if the following property holds: for every Banach space \( E \), if there exists a bounded linear operator \( q : X \to E \) (resp. a Lipschitz map \( q : X \to E \), limited differentiable at 0 with \( q(0) = 0 \)), such \( q \circ T \) is co-Lipschitz, then \( E \) is a finite dimensional.

**Definition 6.11** Let \( X \) and \( Y \) be two Banach spaces. An operator (not necessarily linear) \( T : Y \to X \) is said to be almost weakly compact (resp. strongly almost weakly compact) if the following property holds: for every Banach space \( E \), if there exists a bounded linear operator \( q : X \to E \) (resp. a Lipschitz map \( q : X \to E \), weak Hadamard differentiable at 0 with \( q(0) = 0 \)), such \( q \circ T \) is co-Lipschitz, then \( E \) is a reflexive space.
Clearly, the strong strict cosingularity (resp. the strong almost weakly compactness) implies the strict cosingularity (resp. the almost weakly compactness). Bourgain and Diestel proved in [10] that every limited bounded linear operator (i.e. linear $L$-operator) is strict cosingular. Theorem 6.12 gives an extension to the Bourgain-Diestel result, where the operator $T$ is not assumed to be necessarily linear but only its (generalized positively homogeneous) differential at 0 is supposed to be a limited operator. Moreover, Theorem 6.12 gives the strong strict cosingularity. A similar result concerning weak Hadamard bornologies and strong almost weakly compactness is given in Theorem 6.13.

**Theorem 6.12** Let $X$ and $Y$ be two Banach spaces. Let $T : Y \to X$ be a (not necessarily linear) operator $F$-p.h. differentiable at 0 such that $T(0) = 0$ and $d^p_F T(0)$ is a limited operator. Then, $T$ is strong strictly cosingular.

**Theorem 6.13** Let $X$ and $Y$ be two Banach spaces. Let $T : Y \to X$ be a (not necessarily linear) operator $F$-p.h. differentiable at 0 such that $T(0) = 0$ and $d^p_F T(0)$ is a weakly compact operator. Then, $T$ is strong almost weakly compact.

The proofs of Theorems 6.12 and 6.13 are given as an immediate consequence of Lemma 6.18 at the end of the paper. Using the above theorems, together with Theorem 4.7, we get the following corollaries.

**Corollary 6.14** Let $X$ and $Y$ be two Banach spaces. Let $T : Y \to X$ be a (not necessarily linear) operator $F$-p.h. differentiable at 0 such that $T(0) = 0$. Suppose that $T$ is a $L$-operator. Then, $T$ is strong strictly cosingular.

**Corollary 6.15** Let $X$ and $Y$ be two Banach spaces. Let $T : Y \to X$ be a (not necessarily linear) operator $F$-p.h. differentiable at 0 such that $T(0) = 0$. Suppose that $T$ is a $wH$-operator. Then, $T$ is strong almost weakly compact.

In order to state Lemma 6.18 in all its generality, let us continue with the following definition which generalizes the notion of co-Lipschitz map.

**Definition 6.16** Let $X$ and $E$ be two Banach spaces and $\beta_X, \beta_E$ be two bornologies on $X$ and $E$ respectively. Let $q : X \to E$ be a mapping. We say that $q$ is $\beta_E-\beta_X$-co-Lipschitz, if for every $A \in \beta_E$ and every $\varepsilon > 0$, there exists $B \in \beta_X$ such that

$$\limsup_{t \to 0^+} \sup_{h \in A} \operatorname{dist}(h, \frac{q(tB)}{t}) := \limsup_{t \to 0^+} \sup_{h \in A, x \in B} \left\| \frac{q(tx)}{t} - h \right\| \leq \varepsilon.$$ 

Whenever $\beta_E$ is the Fréchet bornology, we simple say that $q$ is $\beta_X$-co-Lipschitz.

**Remark 6.17** Let $X$ and $E$ be two Banach spaces and $\beta_X, \beta_E$ be two bornologies on $X$ and $E$ respectively. Let $q : X \to E$ be a mapping. Suppose that for every $A \in \beta_E$ and every $\varepsilon > 0$, there exists $B \in \beta_X$ such that

$$rA \subset q(rB) + \varepsilon rB_Z, \text{ for all } r > 0,$$

then, $q$ is $\beta_E-\beta_X$-co-Lipschitz. Notice also that, if $q : X \to E$ is a co-Lipschitz map and $q(0) = 0$, then $q$ is $F-F$-co-Lipschitz.
Lemma 6.18 Let $X$, $Y$ and $E$ be Banach spaces, let $\beta_X$ and $\beta_Y$ be vector bornologies on $X$ and $Y$ respectively and $\beta_E$ and $\beta'_E$ be two vector bornologies on $E$. Assume that $\beta_X$ and $\beta_E$ satisfy property (S). Let $T : Y \to X$ be a $\beta_Y$-p.h. differentiable at 0 mapping such that $T(0) = 0$ and $d^\beta_T(0)$ is a $\beta_E$-operator. Suppose that there exists a Lipschitz mapping $q : X \to E$, such that $q(0) = 0$, $q$ is $\beta_X$-p.h. differentiable at 0, $d^\beta_T q(0)$ is a $\beta_X$-$\beta_E$-operator and $q \circ T$ is $\beta'_E$-$\beta_Y$-co-Lipschitz. Then, $E$ is a $\beta'_E$-$\beta_E$-space.

Proof Suppose that $E$ is not a $\beta'_E$-$\beta_E$-space, then by Proposition 6.4, there exists a positive Lipschitz function $f : E \to \mathbb{R}$ such that $f$ is $\beta_E$-differentiable at 0 but not $\beta'_E$-differentiable at 0, with $f(0) = 0$ and $d\beta_E f(0) = 0$.

Since $f$ is not $\beta'_E$-differentiable at 0, then there exist $A \in \beta'_E$ and $\varepsilon > 0$ such that

$$\forall \delta > 0, \exists t \in (0, \delta) \text{ such that } \sup_{h \in A} \frac{f(th)}{t} \geq \varepsilon. \quad (1)$$

Since $q \circ T$ is $\beta'_E$-$\beta_Y$-co-Lipschitz, there exists $B \in \beta_Y$ such that

$$\lim_{t \to 0^+} \sup_{h \in A} \sup_{y \in B} \inf_{t \to 0^+} \frac{q \circ T(ty)}{t} - h \leq \frac{\varepsilon}{2 \text{Lip}(f)}. \quad (2)$$

On the other hand, since $f$ is Lipschitz, we have for all $y \in B$, for all $h \in A$ and for all $t > 0$

$$t^{-1} f \circ q \circ T(ty) \geq t^{-1} f(th) - \text{Lip}(f) \frac{q \circ T(ty)}{t} - h. \quad (3)$$

Since $f$ is a positive function and $f(q(T(0))) = 0$, the only candidate for $\beta_Y$-differentiable of $f \circ q \circ T$ at 0 is $0$. Thanks to (1) and (3), for any $\delta > 0$, there is $t \in (0, \delta)$ such that

$$\sup_{y \in B} t^{-1} f \circ q \circ T(ty) \geq \sup_{h \in A} \left( t^{-1} f(th) - \text{Lip}(f) \inf_{y \in B} \frac{q \circ T(ty)}{t} - h \right)$$

$$\geq \sup_{h \in A} t^{-1} f(th) - \text{Lip}(f) \sup_{h \in A} \inf_{y \in B} \frac{q \circ T(ty)}{t} - h$$

$$\geq \varepsilon - \text{Lip}(f) \sup_{h \in A} \inf_{y \in B} \frac{q \circ T(ty)}{t} - h.$$

Therefore, combining the last inequality and (2), it follows that

$$\lim_{t \to 0^+} \sup_{y \in B} t^{-1} f \circ q \circ T(ty) \geq \varepsilon - \text{Lip}(f) \lim_{t \to 0^+} \sup_{h \in A} \inf_{y \in B} \frac{q \circ T(ty)}{t} - h \geq \varepsilon / 2.$$

This shows that the $\beta_Y$-differential of $f \circ q \circ T$ at 0 cannot be 0. Hence, $f \circ q \circ T$ is not $\beta_Y$-differentiable at 0.

On the other hand, using Theorem 4.5, $f \circ q$ is $\beta_X$-differentiable at 0 = $q(0)$, with $d_{\beta_X} f \circ q(0) = 0$ since $d^\beta_T q(0)$ is a $\beta_X$-$\beta_E$-operator, $d_{\beta_E} f(0) = 0$. Now, since $f \circ q : X \to \mathbb{R}$ is Lipschitz, $\beta_X$-differentiable at 0 = $T(0)$ and $f \circ q \circ T$ is not $\beta_Y$-differentiable at 0, we get from Theorem 4.5 that $d^\beta_T T(0)$ is not a $\beta_Y$-$\beta_X$-operator, which is a contradiction. \qed

Now, we give the proofs of Theorems 6.12 and 6.13.

Proof of Theorem 6.12 Let $q : X \to E$ be a Lipschitz map such that $q(0) = 0$, $q$ is $L$-differentiable at 0 and $q \circ T$ is co-Lipschitz. We apply Lemma 6.18 to the operator $T$, with $\beta_Y = \beta'_E = F$ and $\beta_X = \beta_E = L$ (Fréchet and limited bornology respectively), using
Remark 6.17 and the observation that $d_{L}q(0)$, as a bounded linear operator, is always a $L$-$L$-operator. We get that $E$ is an $L$-space, that is a finite dimensional space by Proposition 6.3. Thus, $T$ is a strong strictly cosingular operator. 

**Proof of Theorem 6.13** Let $q : X \rightarrow E$ be a Lipschitz map such that $q(0) = 0$, $q$ is $wH$-differentiable at 0 and $q \circ T$ is co-Lipschitz. We apply Lemma 6.18 to the operator $T$, with $\beta_Y = \beta'_E = F$ and $\beta_X = \beta_E = wH$ (Fréchet and weak Hadamard bornology respectively), using Remark 6.17 and the observation that $d_{wH}q(0)$, as a bounded linear operator, is always a $wH$-$wH$-operator. We get that $E$ is a $wH$-space, that is a reflexive space by Proposition 6.3. Thus, $T$ is a strong almost weakly compact operator.

Acknowledgements This research has been conducted within the FP2M federation (CNRS FR 2036) and SAMM Laboratory of the University Paris Panthéon-Sorbonne. The second author was partially supported by ANID-PFCHA/Doctorado Nacional/2018-21181905 and by CMM (UMI CNRS 2807), Basal grant: AFB170001.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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