In this article a self-contained exposition of proving perturbative renormalizability of a quantum field theory based on an adaption of Wilson’s differential renormalization group equation to perturbation theory is given. The topics treated include the spontaneously broken SU(2) Yang-Mills theory. Although mainly a coherent but selective review, the article contains also some simplifications and extensions with respect to the literature.

In the original version of this review the spontaneously broken Yang-Mills theory, dealt with in Chapter 4, followed [30]. Recently, however, the authors of this article discovered a serious deficiency in their method to restore the Slavnov-Taylor identities (intermediately violated by the regularization), which invalidates their claim. Now in [55] these authors have developed a new approach to accomplish the missing restoration of the Slavnov-Taylor identities. The present revised review concerns solely Chapter 4: the original Sections 4.1-4.3 are essentially unaltered, whereas the former Section 4.4, now obsolete, has been replaced by the new Sections 4.4-4.6, following the recent article [55].
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Chapter 1

Introduction

Dyson’s pioneer work [1] opened the era of a systematic perturbative renormalization theory long ago, and in the late sixties of the last century the rigorous BPHZ-version [2] was accomplished. In place of the momentum space subtractions of BPHZ to circumvent UV-divergences various intermediate regularization schemes were invented: Pauli-Villars regularization [3], analytical regularization [4], dimensional regularization [5]. These different methods, each with its proper merits, are equivalent up to finite counterterms, for a review see e.g. [6]. All of them are based on the analysis of multiple integrals corresponding to individual Feynman diagrams, the combinatorial complexity of which rapidly grows with increasing order of the perturbative expansion. Zimmermann’s famous forest formula [7] provides the clue to disentangle overlapping divergences, organizing the order of subintegrations to be followed. The BPHZ-renormalization, originally developed in case of massive theories, was extended by Lowenstein [8] to cover also zero mass particles. From the point of view of elementary particle physics, renormalization theory culminated in the work of ’t Hooft and Veltman [9], demonstrating the renormalizability of non-Abelian gauge theories.

At the time of these achievements, Wilson’s view [10] of renormalization as a continuous evolution of effective actions – a primarily non-perturbative notion – began to pervade the whole area of quantum field theory and soon proved its fertility. In the domain of rigorous mathematical analysis beyond formal perturbation expansion, the renormalizable UV-asymptotically free Gross-Neveu model in two space-time dimensions has been constructed, [11], [12], by decomposing in the functional integral the full momentum range into a union of discrete, disjoint “slices” and integrating successively the corresponding quantum fluctuations, thereby generating a sequence of effective actions. This slicing can be seen as the equivalent of introducing block-spins in lattices of Statistical Mechanics. Rigorous non-perturbative analysis of the
renormalization flow is the general subject of the lecture notes \cite{13,14} and of the monograph \cite{15} which, among other topics, also treats the problem of summing the formal perturbation series. In these lecture notes and in the monograph references to the original work on non-perturbative renormalization can be found.

In the realm of perturbative renormalization Wilson’s ideas have proved beneficial, too. Gallavotti and Nicolo, \cite{16}, split the propagator of a free scalar field in disjoint momentum slices, i.e. decomposed the field into a sum of independent (generalized) random variables, and developed a tree expansion to perturbative renormalization. Here, due to the slicing, the degrees of freedom are again integrated in finite steps. This method has been applied in the monograph \cite{17} to present a proof of renormalizability of QED which only involves gauge-invariant counterterms. Polchinski \cite{18} realized, that considering renormalization in terms of relevant and irrelevant operators as Wilson, is also effective in perturbation theory, and he gave in the case of the $\Phi_4^4$-theory an inductive self-contained proof of perturbative renormalization, based on Wilson’s renormalization (semi-) group differential equation. His method avoids completely the combinatoric complexity of generating Feynman diagrams and the following cumbersome analysis of Feynman integrals with their overlapping divergences. It rather treats an $n$-point Green function of a given perturbative order as a whole. Due to this fact, his method is particularly transparent.

Polchinski’s approach has proved very stimulating in various directions.

i) In mathematical physics it has been extended to present new proofs of general results in perturbatively renormalized quantum field theory, which are simpler than those achieved before: renormalization of the nonlinear $\sigma$-model \cite{19}, a rigorous version of Polchinski’s argument, together with physical renormalization conditions \cite{20}, renormalization of composite operators and Zimmermann identities \cite{21}, Wilson’s operator product expansion \cite{22}, Symanzik’s improved actions \cite{23,24}, large order bounds \cite{25}, renormalization of massless $\Phi_4^4$-theory \cite{26}, renormalization of QED \cite{27}, decoupling theorems \cite{28}, renormalization with flow equations in Minkowski space \cite{29}, a renormalization of spontaneously broken Yang - Mills theory \cite{30}, temperature independent renormalization of finite temperature field theories \cite{31}. The monograph \cite{32} contains a clear and detailed introduction to Polchinski’s method formulated with Wick-ordered field products \cite{23}, and, in addition, the application of a similar renormalization flow to the Fermi surface problem of condensed matter physics.

ii) In the domain of theoretical physics there is a vast amount of contributions with diverse applications of Polchinski’s approach. Flow equations for vertex functions have been introduced \cite{33} and also employed to investigate pertur-
bative renormalizability of gauge, chiral and supersymmetric theories [34]. In these articles also several explicit one-loop calculations are performed. An effective quantum action principle has been formulated for the renormalization flow breaking gauge invariance [35]. There are also interesting attempts to combine a gauge invariant regularization with a flow equation [36]. Besides aims within perturbation theory, there have been many activities to use truncated versions of flow equations as appropriate non-perturbative approximations in strong interaction physics, more in accord with Wilson’s original goal. In the physically distinguished case of (non-Abelian) gauge theories [37, 38] the effective action is restricted in a local approximation to its relevant part for all values of the flowing scale. As a consequence, the flow equation for the effective action reduces to a system of $r$ ordinary differential equations, $r$ being the number of relevant coefficients appearing. This system is integrated from the UV-scale downward. In these non-perturbative approaches the problem arises to reconcile the truncation with the gauge symmetry. This problem is discussed also in [35]. In a very different field of interest the question of the non-perturbative renormalizability of Quantum Einstein Gravity has been investigated, based on truncated flow equations, [39]. These authors restrict the average effective action to the Hilbert action together with a small number of additional local terms. The flow of the coupling coefficients is studied numerically and the existence of a non-Gaussian fixed point in the ultraviolet found. This result is then interpreted to support the conjecture, that Quantum Einstein Gravity is “asymptotically safe” in Weinberg’s sense. We like to point out that physical applications of flow equations are reviewed in [40], containing an extensive list of references.

The present article is intended to provide a self-contained exposition of perturbative renormalization based on Polchinski’s inductive method, employing the differential renormalization group equation of Wilson. Therefore, emphasis is laid on a coherent presentation of the topics considered. A comprehensive overview of the literature on the subject will not be pursued. The quantum field theories considered are treated in their Euclidean formulation on $d = 4$ dimensional (Euclidean) space-time by means of functional integration. Accordingly, their correlation functions are called Schwinger functions to distinguish them from the Green functions on Minkowski space. In the intermediate steps of the derivations always regularized functionals are used, the controlled removal (within perturbation theory) of this regularization being our main concern. We avoid any manipulation of unregularized “path integrals”.

The plan of this article is as follows. In Chapter 2 Polchinski’s method to prove perturbative renormalizability is elaborated treating the nonsymmetric $\Phi^4$-theory in detail. Besides the system of Schwinger functions of this theory,
the Schwinger functions with one composite field (operator) inserted are also dealt with. The presentation is mainly based on [20, 21] and on some simplifications [41]. Moreover, considering the theory at finite temperature, its temperature independent renormalizability is reviewed, following closely [31].

In chapter 3 two simple cases of the quantum action principle are demonstrated, again treating the nonsymmetric $\Phi^4$-theory: the field equation and the variation of a coupling constant. These applications of the method seem not to have been treated in the literature. Hereafter, somewhat disconnected, flow equations for proper vertex functions are dealt with, [33, 29]. Chapter 4 is devoted to the proof of renormalizability of the physically most important spontaneously broken Yang-Mills theory. Because of the necessity to implement nonlinear field variations, this problem can be regarded as a further instance of the quantum action principle. The initial presentation followed the line of [30]. The authors of this article, however, recently found, that the restoration of the violated Slavnov-Taylor identities claimed there in fact has not been achieved, since Lemma 2 used does not take into account irrelevant boundary terms which are inevitably present in the bare action, thus rendering this Lemma obsolete. Keeping to the general line of the earlier paper the authors have developed in [55] a new approach to cope with the appearance of irrelevant contributions in the violated Slavnov-Taylor identities, which is based on tracing the super-renormalizable couplings in the perturbative expansion. Hence, the revised presentation here keeps with a few adaptions the former Sections 4.1 - 4.3, but replaces the former Section 4.4 by the new Sections 4.4 - 4.6, following [55].
Chapter 2
The Method

2.1 Properties of Gaussian measures

Our point of departure is a Gaussian probability measure $d\mu$ on the space $C(\Omega)$ of continuous real-valued functions on a $d$-dimensional torus $\Omega$. Such a function we identify with a periodic function on $\mathbb{R}^d$, i.e.
$$\phi(x) = \phi(x + nl),$$
where $x \in \mathbb{R}^d$, $n \in \mathbb{Z}^d$, $l = (l_1, \ldots, l_d) \in \mathbb{R}_+^d$ and $nl = n_1l_1 + \cdots + n_d l_d$. A Gaussian measure with mean zero is uniquely defined by its covariance $C(x, y)$,
$$\int d\mu_C(\phi) \phi(x) \phi(y) = C(x, y) = C(y, x). \tag{2.1}$$

The covariance is a positive non-degenerate bilinear form on $C^\infty(\Omega) \times C^\infty(\Omega)$, we assume it to be translation invariant, $C(x, y) = C(x - y)$, too. Moreover, the function $C(x)$ is assumed to have a given number $N \in \mathbb{N}$ of derivatives continuous everywhere on $\Omega$. We list some properties of this Gaussian measure employed in the sequel, proofs can be found e.g. in [12].

- Using the notation
  $$\langle \phi, J \rangle = \int_\Omega dx \phi(x) J(x), \quad \langle J, C J \rangle = \int_\Omega dx \int_\Omega dy J(x) C(x - y) J(y)$$
  where $J \in C^\infty(\Omega)$ is a test function, the generating functional of the correlation functions is given explicitly as
  $$\int d\mu_C(\phi) \ e^{\langle \phi, J \rangle} = e^{\frac{1}{2} \langle J, C J \rangle}. \tag{2.2}$$

- The translation of the Gaussian measure by a function $\varphi \in C^\infty(\Omega)$ results in
  $$d\mu_C(\phi - \varphi) = e^{-\frac{1}{2} \langle \varphi, C^{-1} \varphi \rangle} \ d\mu_C(\phi) \ e^{\langle \phi, C^{-1} \varphi \rangle}. \tag{2.3}$$
• Let $A(\phi)$ denote a polynomial formed of local powers of the field, $\phi(x)^n, n \in \mathbb{N}$, and of its derivatives $(\partial_\mu \phi(x))^m, m \in \mathbb{N}, 2m < N$, at various points $x$. If the covariance $C$ is the sum of two covariances, $C = C_1 + C_2$, then

$$\int d\mu_C(\phi)A(\phi) = \int d\mu_{C_1}(\phi_1) \int d\mu_{C_2}(\phi_2)A(\phi_1 + \phi_2).$$  \hspace{1cm} (2.4)

• Integration by parts of a function $A(\phi)$ as considered in (2.4) yields

$$\int d\mu_C(\phi)\phi(x)A(\phi) = \int d\mu_C(\phi) \int_\Omega dy \frac{\delta}{\delta \phi(y)} A(\phi).$$ \hspace{1cm} (2.5)

• Finally, let the covariance of the Gaussian measure depend differentiably on a parameter,

$$C(x - y) = C_t(x - y), \quad \dot{C}_t \equiv \frac{d}{dt}C_t(x - y).$$

Given again a function $A(\phi)$ as in (2.4), then

$$\frac{d}{dt} \int d\mu_{C_t}(\phi)A(\phi) = \frac{1}{2} \int d\mu_{C_t}(\phi) \left( \frac{\delta}{\delta \phi}, \dot{C}_t \frac{\delta}{\delta \phi} \right) A(\phi).$$ \hspace{1cm} (2.6)

As an example of the class of covariances considered, we present already here the particular covariance which will be mainly used in the flow equations envisaged. The torus $\Omega$ has volume $|\Omega| = l_1 l_2 \cdots l_d$ and a point $x \in \Omega$ has coordinates $-\frac{1}{2} l_i \leq x_i < \frac{1}{2} l_i, \; i = 1, \ldots, d$. Hence, the dual Fourier variables (momentum vectors) $k$ form a discrete set:

$$k = k(n) = \left( \frac{2\pi n_1}{l_1}, \ldots, \frac{2\pi n_d}{l_d} \right), \; n \in \mathbb{Z}^d.$$  

Let $m, \Lambda_0$ be positive constants, $0 < m \ll \Lambda_0$, and the nonnegative parameter $\Lambda$ satisfy $0 \leq \Lambda \leq \Lambda_0$, we define the covariance

$$C^\Lambda,\Lambda_0(x - y) = \frac{1}{|\Omega|} \sum_{n \in \mathbb{Z}^d} e^{ik(x - y)} \left( e^{-\frac{k^2 + m^2}{\Lambda_0^2}} - e^{-\frac{k^2 + m^2}{\Lambda^2}} \right).$$ \hspace{1cm} (2.7)

This covariance obviously has the well-defined infinite volume limit $\Omega \to \mathbb{R}^d$, with $x, y, k \in \mathbb{R}^d$;

$$C^\Lambda,\Lambda_0(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{ik(x - y)} \left( e^{-\frac{k^2 + m^2}{\Lambda_0^2}} - e^{-\frac{k^2 + m^2}{\Lambda^2}} \right).$$ \hspace{1cm} (2.8)
Abusing slightly the notation we did not choose a different symbol for the limit. Later on, however, the case referred to will be clearly stated. Choosing the values Λ₀ = ∞, Λ = 0 the covariances (2.7), (2.8) become the Euclidean propagator of a free real scalar field with mass m on Ω and Rᵈ, respectively. A finite value of Λ₀ generates an UV-cutoff thus regularizing the covariances: they now satisfy the regularity condition assumed for all N. This property is kept introducing the additional term governed by the “flowing” parameter 0 ≤ Λ ≤ Λ₀. Its role is to interpolate differentiably between a vanishing covariance at Λ = Λ₀, corresponding to a δ-measure on the function space, and the free UV-regularized covariance at Λ = 0. As a consequence we remark that the Gaussian measure with covariance (2.7) is supported with probability one on (the nuclear space) C∞(Ω), [43]. Clearly, a modification of the Euclidean propagator showing these properties can be accomplished with a large variety of cutoff functions. In (2.7), (2.8) a factor of the form

\[ R^{Λ,Λ₀}(k) = \sigma_{Λ₀}(k^2) - \sigma_{Λ}(k^2) \]  

has been introduced, with the particular function

\[ \sigma_{Λ}(k^2) = e^{-\frac{k^2+m^2}{Λ^2}}. \]  

We observe, that regularization and interpolation is caused by any positive function σ_{Λ}(k^2) satisfying: i) For fixed Λ it decreases as a function of k², vanishing rapidly for k² > Λ². ii) For fixed k² it increases with Λ from the value zero at Λ = 0 to the value one at Λ = ∞. Later on, our particular choice will prove advantageous.

### 2.2 The flow equation

Perturbative renormalizability of a quantum field theory is based on locality of its action and on power counting. The qualification “perturbative” means expansion of the theory’s Green (or Schwinger) functions as formal power series in the loop parameter \( \sqrt{h} \), and treating them order-by-order, i.e. disregarding questions of convergence. The notions of locality and power counting can be introduced looking at the classical precursor of the quantum field theory to be constructed. There, a local action in d space-time dimensions is the space-time integral of a Lagrangian (density), having the form of a polynomial in the fields entering the theory and their derivatives. The propagators are determined by the free part, which is bilinear in the fields.
For a scalar field and a spin-$\frac{1}{2}$ field this free part is of second and first order in the derivatives, respectively. Defining the canonical (mass) dimension of the corresponding fields to be $\frac{1}{2}(d - 2)$ and $\frac{1}{2}(d - 1)$, respectively, and attributing the mass dimension 1 to each partial derivative, the free part of the Lagrangian has the dimension $d$, the action thus is dimensionless. Vector fields, especially (non-Abelian) gauge fields, pose particular problems to be considered later. Still looking at the classical theory, local interaction terms in the Lagrangian involve by definition more than two fields. Their respective coupling constant has a mass dimension, derived from the mass dimension of the interaction term, the coupling constant of an interaction term of mass dimension $d$ being dimensionless. Any local term entering the Lagrangian is called a relevant operator if it has a mass dimension $\leq d$, but irrelevant, if its mass dimension is greater than $d$. In the physically distinguished case $d = 4$ the central result of perturbative renormalization theory is that UV-finite Green (or Schwinger) functions can be obtained in any order, if the interaction terms have mass dimensions $\leq 4$, by prescribing a finite number of renormalization conditions. This number equals the number of relevant operators forming the full classical Lagrangian.

We consider the quantum field theory of a real scalar field $\phi$ with mass $m$ on four-dimensional Euclidean space-time within the framework of functional integration. The emerging vacuum effects require a finite space-time volume. Therefore we start with a real-valued field $\phi \in C^1(\Omega)$ on a four-dimensional torus $\Omega$. Its bare interaction, labeled by an UV-cutoff $\Lambda_0 \in \mathbb{R}_+$, is chosen as

$$L^{\Lambda_0, \Lambda_0}(\phi) = \int_\Omega dx \left( \frac{f}{3!} \phi^3(x) + \frac{g}{4!} \phi^4(x) \right) + \int_\Omega dx \left( v(\Lambda_0) \phi(x) + \frac{1}{2} a(\Lambda_0) \phi^2(x) + \frac{1}{2} z(\Lambda_0) (\partial_\mu \phi)^2(x) \right. + \left. \frac{1}{3!} b(\Lambda_0) \phi^3(x) + \frac{1}{4!} c(\Lambda_0) \phi^4(x) \right).$$

The first integral has classical roots: its integrand is formed of the field’s self-interaction with real coupling constants $f$ and $g$ having mass dimension equal to one and zero, respectively. The second integral contains the related counterterms, determined according to the following rule. The canonical mass dimension of the field $\phi$ is equal to one. As counterterms in the integrand of the bare interaction have to appear all local terms of mass dimension $\leq 4$ that can be formed of the field and of its derivatives but respecting the (Euclidean) $O(4)$-symmetry. This symmetry is not violated.

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2Stability requires $g$ to be positive, but this property is not felt in a perturbative treatment.
by the intermediate UV-regularization procedure and can thus be main-
tained. In contradistinction to the coupling constants $f, g$ the five coeffi-
cients $v(\Lambda_0), a(\Lambda_0), z(\Lambda_0), b(\Lambda_0), c(\Lambda_0)$ of the counterterms cannot be chosen freely but have to depend on the UV-cutoff $\Lambda_0$. This dependence is dictated by the aim that after functional integration the UV-regularization can be removed, i.e. the limit $\Lambda_0 \to \infty$ can be performed keeping the physical content of the theory finite. As a consequence the coefficients stated above, however, turn out to diverge with $\Lambda_0 \to \infty$. If we restrict the bare inter-
action (2.11) to the case $f = 0, v(\Lambda_0) = b(\Lambda_0) = 0$, it is also invariant under the mirror transformation $\phi(x) \to -\phi(x)$ implying an additional symmetry of the theory.

The regularized quantum field theory on finite volume is defined by the generating functional of its Schwinger functions

$$Z^{\Lambda, \Lambda_0}(J) = \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{-\frac{1}{\hbar} L^{\Lambda_0, \Lambda_0}(\phi) + \frac{1}{\hbar} \langle \phi,J \rangle}$$

(2.12)

with a real source $J \in \mathcal{C}^\infty(\Omega)$, bare interaction (2.11) and a Gaussian measure $d\mu_{\Lambda, \Lambda_0}$ with mean zero and covariance $\hbar C^{\Lambda, \Lambda_0}$, (2.7). The positive parameter $\hbar$ has been introduced with regard to a systematic loop expansion considered later. For fixed $\Omega$ and $\Lambda_0$, and assuming $g + c(\Lambda_0) > 0, z(\Lambda_0) \geq 0$ in the bare interaction, (2.11), the functional integral (2.12) is well-defined. As a functional on $\mathcal{C}^\infty(\Omega)$, the support of the Gaussian measure $d\mu_{\Lambda, \Lambda_0}(\phi)$, the bare interaction is continuous in any Sobolev norm of order $n \geq 1$, and, furthermore, bounded below, $L^{\Lambda_0, \Lambda_0}(\phi) > \kappa$. Hence, with $\Lambda_0$ fixed, we have the uniform bound for $0 \leq \Lambda \leq \Lambda_0$,

$$|Z^{\Lambda, \Lambda_0}(J)| < e^{-\kappa} \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{\frac{1}{\hbar} \langle \phi,J \rangle} \leq e^{-\kappa + \frac{1}{2\hbar} \langle J, C^{0, \Lambda_0} J \rangle}.$$  

(2.13)

From (2.12) one obtains the generating functional $W^{\Lambda, \Lambda_0}(J)$ of the truncated Schwinger functions

$$e^{\frac{1}{\hbar} W^{\Lambda_0, \Lambda_0}(J)} = \frac{Z^{\Lambda, \Lambda_0}(J)}{Z^{\Lambda, \Lambda_0}(0)}$$

(2.14)

which provides the $n$-point functions, $n \in \mathbb{N}$, upon functional derivation:

$$W^{\Lambda, \Lambda_0}_n(x_1, \ldots, x_n) = \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} W^{\Lambda, \Lambda_0}(J)|_{J=0}.$$  

(2.15)

Besides the UV-regularization determined by the cutoff $\Lambda_0$, imperative to have a well-defined functional integral (2.12), an additional flowing cutoff

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3In a representation of these functions in terms of (Feynman) diagrams only connected diagrams appear.
Λ has been built in, suppressing smaller momenta. It is a merely technical device, introduced by Polchinski [18] and inspired by Wilson’s view of renormalization [10]. Decreasing Λ from its maximal value Λ = Λ₀ to its physical value Λ = 0 gradually takes into account the momentum domain, starting at high momenta – in mathematical terms: the parameter Λ interpolates continuously between a δ-measure (i.e. absence of quantum effects) at Λ = Λ₀ and the Gaussian measure dµ₀,Λ₀ on a UV-regularized field, at Λ = 0. Of course, as stressed after eq.(2.10), such an interpolation can also be realized by other cutoff functions than (2.7) used here. In order to make use of the flow parameter Λ it is advantageous to consider the (free propagator-) amputated truncated Schwinger functions with generating functional LΛ,Λ₀(ϕ), ϕ ∈ C∞(Ω), defined as

\[ e^{-\frac{1}{\hbar} (L_{Λ₀} (ϕ) + I_{Λ₀})} = \int dµ_{Λ,Λ₀}(ϕ) e^{-\frac{1}{\hbar} L_{Λ₀,Λ₀}(ϕ + ϕ)}, \quad (2.16) \]

\[ L_{Λ₀}(0) = 0. \quad (2.17) \]

The constant IΛ,Λ₀ is the vacuum part of the theory. Translating on the r.h.s. in (2.16) the source function ϕ to the measure and using (2.3) leads to

\[ e^{-\frac{1}{\hbar} (L_{Λ₀} (ϕ) + I_{Λ₀})} = e^{-\frac{1}{2\hbar} ⟨ϕ, (hC_{Λ,Λ₀})^{-1} ϕ⟩} Z_{Λ,Λ₀}((C_{Λ,Λ₀})^{-1}ϕ) \quad (2.18) \]

relating the generating functionals Z and L. Hereupon, together with the definition (2.14) follows finally

\[ L_{Λ,Λ₀}(ϕ) = \frac{1}{2} ⟨ϕ, (C_{Λ,Λ₀})^{-1} ϕ⟩ - W_{Λ,Λ₀}((C_{Λ,Λ₀})^{-1}ϕ). \quad (2.19) \]

Denoting by ĈΛ,Λ₀ the derivative of the covariance CΛ,Λ₀ with respect to the flow parameter Λ we observe, with Λ₀ kept fixed,

\[ \frac{d}{dΛ} \int dµ_{Λ,Λ₀}(ϕ) e^{-\frac{1}{\hbar} L_{Λ₀,Λ₀}(ϕ + ϕ)} = \frac{\hbar}{2} \int dµ_{Λ,Λ₀}(ϕ) \langle \delta, \dot{C}_{Λ,Λ₀} \delta \rangle e^{-\frac{1}{\hbar} L_{Λ₀,Λ₀}(ϕ + ϕ)} \]

\[ = \frac{\hbar}{2} \langle \delta, \dot{C}_{Λ,Λ₀} \delta \rangle \int dµ_{Λ,Λ₀}(ϕ) e^{-\frac{1}{\hbar} L_{Λ₀,Λ₀}(ϕ + ϕ)} \]

where in the first step (2.6) has been used, whereas the second step follows from the integrand’s particular dependence on the field ϕ. Hence, because of eq. (2.16) we obtain the differential equation

\[ \frac{d}{dΛ} e^{-\frac{1}{\hbar} (L_{Λ₀ (ϕ) + I_{Λ₀})} = \frac{\hbar}{2} \langle \delta, \dot{C}_{Λ,Λ₀} \delta \rangle e^{-\frac{1}{2}(L_{Λ₀ (ϕ) + I_{Λ₀})}. \quad (2.20) \]

The reader notices that the relation (2.6) has been used in the case of a nonpolynomial function. Therefore this extension has to be understood in
terms of a formal power series expansion, i.e. disregarding the question of convergence. Upon explicit differentiation in (2.20) follows the Wilson flow equation

\[
\frac{d}{d\Lambda} \left( L^{\Lambda,A_0}(\varphi) + f^{\Lambda,A_0} \right) = \frac{\hbar}{2} \left( \frac{\delta}{\delta \varphi} C^{\Lambda,A_0} \frac{\delta}{\delta \varphi} L^{\Lambda,A_0}(\varphi) \right) - \frac{1}{2} \frac{\delta}{\delta \varphi} L^{\Lambda,A_0}(\varphi), \frac{\delta}{\delta \varphi} L^{\Lambda,A_0}(\varphi). \tag{2.21}
\]

The form of eq. (2.20) strongly resembles the heat equation. Defining the functional Laplace operator

\[
\Delta^{\Lambda,A_0} = \frac{1}{2} \left( \frac{\delta}{\delta \varphi} C^{\Lambda,A_0} \frac{\delta}{\delta \varphi} \right), \tag{2.22}
\]

the unique solution of the differential equation (2.20), already given in the form (2.16), can also be written as

\[
e^{-\frac{1}{\hbar} \left( L^{\Lambda,A_0}(\varphi) + f^{\Lambda,A_0} \right)} = e^{\hbar \Delta^{\Lambda,A_0}} e^{-\frac{1}{\hbar} L^{\Lambda,A_0}(\varphi)}. \tag{2.23}
\]

Since \(\Delta^{\Lambda,A_0}\) commutes with its derivative \(\dot{\Delta}^{\Lambda,A_0}\) with respect to \(\Lambda\), the r.h.s. of (2.23) satisfies the differential equation. Moreover, the initial condition holds because of \(\Delta^{\Lambda_0,A_0} = 0\) and \(f^{\Lambda_0,A_0} = 0\).

At this point, several remarks concerning the mathematical aspect of the steps performed are in order:

i) Our aim with these preparatory steps is to generate the system of flow equations satisfied by the regularized Schwinger functions of the theory, when considered in the perturbative sense of formal power series. This system then is taken as the starting point for a proof of perturbative renormalizability. As basic “root” acts the UV-regularized finite-volume generating functional (2.12) or one of its direct descendants (2.14), (2.16). Expanding in their respective integrands the exponential function in a power series would provide the standard perturbation expansion in terms of (regularized) Feynman integrals. Bearing in mind our goal stated, we could already view the steps performed in the restricted sense as formal power series.

ii) We mention, that in [32] to begin on safe ground the generating functional of the theory has first been formulated on a finite space-time lattice in order to derive the (perturbative) flow equation - implying a finite-dimensional Gaussian integral -, and the limit to continuous infinite space-time taken afterwards.

iii) Rigorous analysis beyond perturbation theory of flow equations of the Wilson type (2.21) is the subject dealt with in [13], using convergent expansion techniques. Such techniques are developed in the monograph [15].
The flow equation (2.21) for the generating functional $L^{\Lambda, \Lambda_0}(\varphi)$ encodes a system of flow equations for the corresponding $n$-point functions, $n \in \mathbb{N}$, and for the vacuum part $I^{\Lambda, \Lambda_0}$. The flow of the latter is determined considering eq. (2.21) at $\varphi = 0$. From the translation invariance of the theory it follows that the 2-point function is a distribution depending on the difference variable $x - y$ only, (suppressing momentarily the superscript $\Lambda, \Lambda_0$)

$$\delta \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)} L(\varphi) \big|_{\varphi = 0} =: \mathcal{L}_2(x - y),$$

and thus

$$\left< \frac{\delta}{\delta \varphi}, \hat{C} \frac{\delta}{\delta \varphi} \right> L(\varphi) \big|_{\varphi = 0} = \int_{\Omega} dx \int_{\Omega} dy \hat{C}(x - y) \mathcal{L}_2(x - y) = |\Omega| \int_{\Omega} dz \hat{C}(z) \mathcal{L}_2(z).$$

Because of the emerging dependence on the volume $|\Omega|$ the flow equation of the vacuum part cannot be treated in the infinite volume limit. However, due to the covariance (2.8) which corresponds to a massive particle and thus decays exponentially, the flow equation for the $n$-point functions can and in the sequel will be treated in this limit. Hence, at least one functional derivative has to act on the flow equation (2.21).

Due to the translation invariance of the theory it is convenient to consider the generating functional $L^{\Lambda, \Lambda_0}(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^4)$, in terms of the Fourier transformed source field $\hat{\varphi}$, the conventions used are

$$\varphi(x) = \int_p e^{ipx} \hat{\varphi}(p), \quad \int_p := \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4},$$

implying for the functional derivative $\delta_{\varphi(x)} := \frac{\delta}{\delta \varphi(x)}$ the transformation

$$\delta_{\varphi(x)} = (2\pi)^4 \int_p e^{-ipx} \delta_{\hat{\varphi}(p)}.$$

From the generating functional $L^{\Lambda, \Lambda_0}(\varphi)$ the correlation functions are obtained by functional derivation, $n \in \mathbb{N}$,

$$(2\pi)^{4(n-1)} \delta_{\hat{\varphi}(p_n)} \cdots \delta_{\hat{\varphi}(p_1)} L^{\Lambda, \Lambda_0}(\varphi) \big|_{\varphi = 0} = \delta(p_1 + \cdots + p_n) \mathcal{L}^{\Lambda, \Lambda_0}_n(p_1, \cdots, p_n).$$

The amputated truncated $n$-point function $\mathcal{L}^{\Lambda, \Lambda_0}_n(p_1, \cdots, p_n)$ is a totally symmetric function of the momenta $p_1, \cdots, p_n$ and, moreover, due to the $\delta$-function, $p_n := -p_1 - \cdots - p_{n-1}$. (In the case where the bare interaction (2.11) shows the mirror symmetry $L^{\Lambda_0, \Lambda_0}(-\phi) = L^{\Lambda_0, \Lambda_0}(\phi)$ all $n$-point functions with $n$ odd vanish.) Observing the definition (2.25) we obtain from
the system of flow equations for the \( n \)-point functions, \( n \in \mathbb{N} \),

\[
\partial_{\lambda} L_n^{\lambda, \Lambda_0} (p_1, \cdots, p_n) = \frac{\hbar}{2} \int \partial_{\lambda} C^{\lambda, \Lambda_0} (k) \cdot L_{n+2}^{\lambda, \Lambda_0} (k, p_1, \cdots, p_n, -k)
\]

\[
- \frac{1}{2} \sum_{r=0}^{n} \sum_{i_1 < \cdots < i_r} L_{r+1}^{\lambda, \Lambda_0} (p_{i_1}, \cdots, p_{i_r}, p) \partial_{\lambda} C^{\lambda, \Lambda_0} (p) \cdot L_{n-r+1}^{\lambda, \Lambda_0} (p, p_{j_1}, \cdots, p_{j_{n-r}})
\]

\( p_1 + \cdots + p_n = 0, \quad -p = p_{i_1} + \cdots + p_{i_r} \) \quad (2.26)

In the quadratic term a given set of momenta \( (p_{i_1}, \cdots, p_{i_r}) \), \( i_1 < \cdots < i_r \), determines (uniquely) the corresponding set \( (p_{j_1}, \cdots, p_{j_{n-r}}) \), \( j_1 < \cdots < j_{n-r} \), such that the union of this pair is the set of momenta \( (p_1, \cdots, p_n) \). Furthermore, the Fourier transform of the covariance \( \tilde{C}^{\lambda, \Lambda_0} (k) \),

\[
\tilde{C}^{\lambda, \Lambda_0} (k) = \frac{1}{k^2 + m^2} \left( e^{-\frac{k^2 + m^2}{\Lambda_0^2}} - e^{-\frac{k^2 + m^2}{\Lambda^2}} \right), \quad (2.27)
\]

is written with a slight abuse of notation omitting the “hat”. In the sequel we shall write the quadratic term appearing in (2.26) more compactly as

\[
- \frac{1}{2} \sum_{r=0}^{n} \sum_{i_1 < \cdots < i_r} \cdots
\]

\[
= - \frac{1}{2} \sum_{n_1, n_2} \left[ L_{n_1+1}^{\lambda, \Lambda_0} (p_1, \cdots, p_{n_1}, p) \partial_{\lambda} C^{\lambda, \Lambda_0} (p) \cdot L_{n_2+1}^{\lambda, \Lambda_0} (p, p_{n_1+1}, \cdots, p_n) \right]_{\text{rsym}}
\]

\( p := -p_1 - \cdots - p_{n_1} = p_{n_1+1} + \cdots + p_n \) \quad (2.28)

where the prime on top of the summation symbol imposes the restriction to \( n_1 + n_2 = n \). Moreover, the symbol “rsym” means summation over those permutations of the momenta \( p_1, \cdots, p_n \), which do not leave invariant the (unordered) subsets \( (p_1, \cdots, p_{n_1}) \) and \( (p_{n_1+1}, \cdots, p_n) \), and, in addition, produce mutually different pairs of (unordered) image subsets. The system of flow equations (2.26) will be treated perturbatively employing a loop expansion of the \( n \)-point functions as formal power series, \( n \in \mathbb{N} \),

\[
L_n^{\lambda, \Lambda_0} (p_1, \cdots, p_n) = \sum_{i=0}^{\infty} \hbar^i L_{i,n}^{\lambda, \Lambda_0} (p_1, \cdots, p_n). \quad (2.29)
\]

Since also flow equations for momentum derivatives of \( n \)-point functions have to be considered, we introduce the shorthand notation

\[
w = (w_{1,1}, \cdots, w_{n-1,4}), \quad w_{i,\mu} \in \mathbb{N}_0, \quad |w| = \sum_{i,\mu} w_{i,\mu}
\]

\[
\partial^w := \prod_{i=1}^{n-1} \prod_{\mu=1}^{4} \left( \frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}}, \quad w! = \prod_{i=1}^{n-1} \prod_{\mu=1}^{4} w_{i,\mu}!. \quad (2.30)
\]
From (2.26) then follows the system of flow equations, \( n \in \mathbb{N}, l \in \mathbb{N}_0 \):

\[
\partial_\Lambda \partial_w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \cdots, p_n) = \frac{1}{2} \int_k \partial_\Lambda C^{\Lambda,\Lambda_0}(k) \cdot \partial_w \mathcal{L}_{l-1,n+2}^{\Lambda,\Lambda_0}(k, p_1, \cdots, p_n, -k)
\]

\[
- \frac{1}{2} \sum_{n_1,n_2} \sum_{l_1,l_2} \sum_{w_1,w_2,w_3} c_{\{w_i\}} \left[ \partial_w \mathcal{L}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(p_1, \cdots, p_{n_1}, p) \cdot \partial_w \mathcal{L}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(-p, p_{n_1+1}, \cdots, p_n) \right]_{\text{rsym}}
\]

\[p = -p_1 - \cdots - p_{n_1} = p_{n_1+1} + \cdots + p_n. \quad (2.31)\]

One should not overlook that the residual symmetrization rsym acts on the momentum \( p \), too. The primes restrict the summations to \( n_1 + n_2 = n, l_1 + l_2 = l, w_1 + w_2 + w_3 = w \), respectively. Moreover, the combinatorial factor \( c_{\{w_i\}} = w!(w_1!w_2!w_3!)^{-1} \) comes from Leibniz’s rule. In the loop order \( l = 0 \), obviously, the first term on the r.h.s. is absent.

### 2.3 Proof of perturbative renormalizability

Perturbative renormalizability of the regularized field theory (2.16) amounts to the following: For given coupling constants \( f, g \) in the bare interaction (2.11) the coefficients \( v(\Lambda_0), a(\Lambda_0), z(\Lambda_0), b(\Lambda_0), c(\Lambda_0) \) of the counterterms can be adjusted within a loop expansion of the theory, i.e.

\[
v(\Lambda_0) = \sum_{l=1}^{\infty} h^l v_l(\Lambda_0), \cdots, c(\Lambda_0) = \cdots \quad (2.32)
\]

in such a way, that all (infinite volume) \( n \)-point functions (2.29) in every loop order \( l \) have finite limits

\[
\lim_{\Lambda_0 \to \infty} \lim_{\Lambda \to 0} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \cdots, p_n), \quad n \in \mathbb{N}, l \in \mathbb{N}_0. \quad (2.33)
\]

These limits emerge directly in the tree order \( l = 0 \), of course. The counterterms are adjusted by requiring that the corresponding \( n \)-point functions at the physical value \( \Lambda = 0 \) of the flow parameter have prescribed values for a chosen set of momenta. Since the theory is massive it is convenient to prescribe these renormalization conditions at vanishing momenta. Hence,
taking into account the Euclidean symmetry we require for all \( l \in \mathbb{N} \):

\[
\mathcal{L}_{l,1}^{0,\Lambda_0} = v_l^R, \quad (2.34)
\]

\[
\mathcal{L}_{l,2}^{0,\Lambda_0}(p, -p) = a_l^R + z_l^R p^2 + \mathcal{O}(p^2)^2, \quad (2.35)
\]

\[
\mathcal{L}_{l,3}^{0,\Lambda_0}(p_1, p_2, p_3) = b_l^R + \mathcal{O}(p_1^2, p_2^2, p_3^2), \quad (2.36)
\]

\[
\mathcal{L}_{l,4}^{0,\Lambda_0}(0, 0, 0, 0) = c_l^R. \quad (2.37)
\]

In each loop order \( l \in \mathbb{N} \) these five real renormalization constants \( v_l^R, \ldots, c_l^R \) can be chosen freely, not depending on \( \Lambda_0 \). Together with the corresponding constants of the tree order \( l = 0 \) they fix the relevant part of the theory completely. A particular (simple) choice would be to set \( v_l^R = a_l^R = z_l^R = b_l^R = c_l^R = 0 \).

The tree order has to be treated first. It is fully determined by the classical part appearing in the bare interaction \( (2.11) \). This classical interaction acts as initial condition at \( \Lambda = \Lambda_0 \) when integrating the flow equations \( (2.31) \) for \( l = 0 \) downwards to smaller values of \( \Lambda \), ascending successively in the number of fields \( n \). The classical interaction contains no terms linear or quadratic in the fields. To bring the system of flow equations to bear, however, at first the crucial properties, \( 0 \leq \Lambda \leq \Lambda_0 \), have to be inferred directly from the representation \( (2.16) \). Hereupon and with the initial condition for \( n = 3 \) follows from \( (2.31) \)

\[
\mathcal{L}_{0,3}^{\Lambda,\Lambda_0}(p_1, p_2, p_3) = f, \quad (2.39)
\]

and then for \( n = 4 \):

\[
\mathcal{L}_{0,4}^{\Lambda,\Lambda_0}(p_1, p_2, p_3, p_4) = g
\]

\[
- f^2 \left( C^{\Lambda,\Lambda_0}(p_1 + p_2) + C^{\Lambda,\Lambda_0}(p_1 + p_3) + C^{\Lambda,\Lambda_0}(p_1 + p_4) \right). \quad (2.40)
\]

Ascending further in the number of fields yields the whole tree order. (For \( n > 4 \) all initial conditions at \( \Lambda = \Lambda_0 \) are equal to zero.)

The first step in proving renormalizability is to establish the

**Proposition 2.1 (Boundedness)**

*For all \( l \in \mathbb{N}, n \in \mathbb{N}, w \) from \( (2.30) \) and for \( 0 \leq \Lambda \leq \Lambda_0 \) holds*

\[
|\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)| \leq (\Lambda + m)^{4-n-|w|} \mathcal{P}_1(\log \frac{\Lambda + m}{m}) \mathcal{P}_2\left( \{ |p_i| \} \right), \quad (2.41)
\]

\(^4\) A weak dependence of these constants on \( \Lambda_0 \) with finite limits when \( \Lambda_0 \to \infty \) could be permitted.
where \( \mathcal{P} \) denotes polynomials having nonnegative coefficients. These coefficients, as well as the degree of the polynomials, depend on \( l, n, w \) but not on \( \{ p_i \}, \Lambda, \Lambda_0 \). For \( l = 0 \) all polynomials \( \mathcal{P}_1 \) reduce to positive constants.

**Remark:** In the following the symbol \( \mathcal{P} \) always denotes a polynomial of this type, possibly a different one each time it appears.

**Proof:** Using again the shorthand (2.30) in the case of one momentum the covariance (2.27) satisfies the bounds, \( 0 \leq \Lambda, |\partial w/\partial \Lambda C^{\Lambda, \Lambda_0} (k)| \leq \Lambda^{-3 -|w|} \mathcal{P} \left( \frac{|k|}{\Lambda} \right) e^{-\frac{k^2+m^2}{\Lambda^2}} \). (2.42)

Here, the polynomials \( \mathcal{P} \) are of respective degree \( |w| \) (and obviously do not depend on \( n, l \)). A weaker version, used too, is

\[
|\partial w/\partial \Lambda C^{\Lambda, \Lambda_0} (k)| \leq (\Lambda + m)^{-3 -|w|} \mathcal{P} \left( \frac{|k|}{\Lambda + m} \right). \tag{2.43}
\]

We first consider the tree order \( l = 0 \). Due to (2.38), (2.39) the bounds (2.41) evidently hold for \( n \leq 3 \). From (2.40) and the very crude bound \( |C^{\Lambda, \Lambda_0} (k)| < 2m^{-2} \) follows the claim for \( (n = 4, w = 0) \). Now in all remaining cases we have \( n + |w| > 4 \). Due to the crucial properties (2.38) they can be treated successively ascending in \( n \), and for given \( n \) the various \( w \) dealt with in arbitrary order, by integrating the respective flow equation (2.31) downwards from the initial point \( \Lambda = \Lambda_0 \). In each such case the initial condition is equal to zero. Then bounds already established together with (2.43) yield

\[
|\partial^w \mathcal{L}_{0,n}^{\Lambda, \Lambda_0} (p_1, \cdots, p_n)| \leq \int_{\Lambda}^{\Lambda_0} d\lambda |\partial \lambda \partial w \mathcal{L}_{0,n}^{\Lambda, \Lambda_0} (p_1, \cdots, p_n)|
\leq \mathcal{P} \left( \left\{ \frac{|p_i|}{\Lambda + m} \right\} \right) \int_{\Lambda}^{\Lambda_0} d\lambda (\lambda + m)^{4-n-|w|-1}
\leq \mathcal{P} \left( \left\{ \frac{|p_i|}{\Lambda + m} \right\} \right) \frac{1}{n + |w|} (\Lambda + m)^{4-n-|w|}. \tag{2.44}
\]

Thus the assertion (2.41) is shown for the tree order.

Given the bounds for \( l = 0 \), those of the higher loop orders can be generated inductively by successive integration of the system of flow equations (2.31): i) Ascending in the loop order \( l \), ii) for fixed \( l \) ascending in \( n \), iii) for fixed \( l, n \) descending with \( w \) down to \( w = 0 \). We observe that in the inductive order adopted the terms on the r.h.s. of a flow equation are always prior to that on the l.h.s. since the linear term has lower loop order and to the quadratic term - because of the key properties (2.38) - only terms of the
same loop order contribute which have a smaller value \( n \). To comply with the growth properties of the bounds (2.41) the integrations are performed as follows:

\( A_1 \) If \( n + |w| > 4 \), the bound decreases with increasing \( \Lambda \). Hence, the flow equation is integrated from the initial point \( \Lambda = \Lambda_0 \) downwards to smaller values of \( \Lambda \) with the initial condition

\[
\partial^w L_{l,n}^{\Lambda_0,\Lambda_0}(p_1, \cdots, p_n) = 0, \quad n + |w| > 4,
\]
as a consequence of the bare interaction (2.11) chosen.

\( A_2 \) In the cases \( n + |w| \leq 4 \) the bounds increase with increasing \( \Lambda \). Therefore, the corresponding flow equations are integrated for a prescribed set of momenta (the renormalization point) with the physical value \( \Lambda = 0 \) as initial point. The respective initial values can be freely chosen order by order, but in accordance with the (Euclidean) symmetry of the theory. As already stated before we choose vanishing momenta as renormalization point, together with the renormalization conditions (2.34 - 2.37) as initial values. Thus, for \( n + |w| \leq 4 \),

\[
\partial^w L_{l,n}^{\Lambda,\Lambda_0}(0, \cdots, 0) = \partial^w L_{l,n}^{0,\Lambda_0}(0, \cdots, 0) + \int_0^\Lambda d\lambda \partial_\lambda \partial^w L_{l,n}^{\Lambda,\Lambda_0}(0, \cdots, 0). \tag{2.46}
\]

(For \( n = 1 \) there is no momentum dependence and \( w = 0 \).) Once a bound has been obtained at the renormalization point, it is extended to general momenta using the Taylor formula

\[
f(p) = f(0) + \sum_{i=1}^n p_i \int_0^1 dt (\partial_i f)(tp) \tag{2.47}
\]

for a differentiable function on \( \mathbb{R}^n \). Applying this formula, the bound of the integrand ( due to the derivative ) yields an additional factor \((\Lambda + m)^{-1}\) which combines with the momentum factor in front to give a new momentum bound of the type considered.

To generate inductively the assertion (2.41) we use it in bounding the r.h.s. of the flow equation (2.31), together with the bounds (2.42) and (2.43) in the linear and in the quadratic term, respectively,

\[
\begin{align*}
&\left| \partial_\Lambda \partial^w L_{l,n}^{\Lambda,\Lambda_0}(p_1, \cdots, p_n) \right| \\
&\leq \int_k e^{-\frac{k^2 + m^2}{\Lambda^2}}(\Lambda + m)^{4-n-2-|w|} P_1(\log \frac{\Lambda + m}{m}) P_2(\frac{\lambda}{\Lambda + m}, \{ \frac{|p_i|}{\Lambda + m} \}) \\
&+ (\Lambda + m)^{4-n-|w|-1} P_3(\log \frac{\Lambda + m}{m}) P_4(\{ \frac{|p_i|}{\Lambda + m} \}).
\end{align*}
\]
The second term on the r.h.s. results from combining a sum of such terms into a single one with new polynomials. In the first term the $k$-integration is performed substituting $k \rightarrow \Lambda k$. The result, easily majorized and combined with the second term yields the bound

$$|\partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)| \leq (\Lambda + m)^{4-n-|w|-1} \mathcal{P}_5(\log \frac{\Lambda + m}{m}) \mathcal{P}_6(\{ \frac{|p_i|}{\Lambda + m} \}).$$

(2.48)

$a_1$) Following the order of the induction stated before, the (irrelevant) cases $n + |w| > 4$ have always to be considered first (for fixed $l, n$). In these cases the bound (2.48) is integrated downwards, observing (2.45), similarly as in the tree order, (2.44). In place of the pure power behaviour, however, we now have

$$\int_0^{\Lambda_0} d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P}_1(\log \frac{\lambda + m}{m}) < (\Lambda + m)^{4-n-|w|} \mathcal{P}_1(\log \frac{\Lambda + m}{m}),$$

with a new polynomial on the r.h.s., see the end of this chapter, section 2.6. Thus, the assertion is established in the cases $n + |w| > 4$.

$a_2$) In the cases $n + |w| \leq 4$ the claim (2.41) has to be deduced from the respective integrated flow equation (2.46) at the renormalization point followed by an extension to general momenta by way of (2.47), proceeding in the order of induction. That is, to start with the particular (momentum independent) case $n = 1$ and continue successively with the cases $(n = 2, |w| = 2), (n = 2, |w| = 1)$, and so on. Converting in the obvious way equation (2.46) into an inequality for absolute values, a bound on the integral is gained using the bound (2.48) at vanishing momenta:

$$\left| \int_0^{\Lambda} d\lambda \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(0, \cdots, 0) \right| \leq \int_0^{\Lambda} d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P}_1(\log \frac{\lambda + m}{m}) \leq (\Lambda + m)^{4-n-|w|} \mathcal{P}_1(\log \frac{\Lambda + m}{m}),$$

where $\mathcal{P}_1$ is a new polynomial, see section 2.6. Hence, the assertion (2.41) is established at the renormalization point. In each case extension to general momenta via (2.47) is guaranteed by bounds established before. This concludes the proof of Proposition 2.1.

The boundedness due to Prop.2.1 would still allow an oscillatory dependence on $\Lambda_0$. Such a (implausible) behaviour is excluded by the

**Proposition 2.2** (Convergence)

For all $l \in \mathbb{N}_0, n \in \mathbb{N}, w$ from (2.30) and for $0 \leq \Lambda \leq \Lambda_0$ holds

$$|\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \cdots, p_n)| \leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} \mathcal{P}_3(\log \frac{\Lambda_0 + m}{m}) \mathcal{P}_4(\{ \frac{|p_i|}{\Lambda + m} \}).$$

(2.49)
Since we need this Proposition for large values of \( \Lambda_0 \) only, we then obviously can write

\[
|\partial_{\Lambda_0} \partial^w L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)| \leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0)^2} \left( \log \frac{\Lambda_0}{m} \right)^{\nu} P_4\left( \frac{|p_i|}{\Lambda_0 + m} \right) \tag{2.50}
\]

with a positive integer \( \nu \) depending on \( l, n, w \). Integration of these bounds with respect to \( \Lambda_0 \) finally shows that for fixed \( \Lambda \) all \( L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n) \) converge to finite limits with \( \Lambda_0 \to \infty \). In particular, one obtains for all \( \Lambda'_0 > \Lambda_0 \):

\[
|L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n) - L_{l,n}^{\Lambda_0,\Lambda_0'}(p_1, \ldots, p_n)| < m^{5-n} \left( \log \frac{\Lambda_0}{m} \right)^{\nu} P_5\left( \frac{|p_i|}{\Lambda + m} \right).
\]

Thus, due to the Cauchy criterion, finite limits \( (2.33) \) exist, i.e. perturbative renormalizability of the theory considered is demonstrated.

**Proof of Proposition 2.2**: We integrate the system of flow equations \( (2.31) \) according to the induction scheme employed before and derive the individual \( n \)-point functions with respect to \( \Lambda_0 \). The r.h.s. of \( (2.31) \) will be denoted by the shorthand \( \partial^w R_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n) \). Due to \( (2.38-2.40) \) the cases \( (l = 0, n + |w| \leq 4) \) evidently satisfy the claim \( (2.49) \).

\( b_1 \) \( n + |w| > 4 \): In these cases, because of the initial condition \( (2.45) \), we have

\[
- \partial_{\Lambda_0} \partial^w L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n) = \int_{\Lambda}^{\Lambda_0} d\lambda \partial^w R_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)
\]

and hence

\[
- \partial_{\Lambda_0} \partial^w L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n) = \partial^w R_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)
+ \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\Lambda_0} \partial^w R_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)	ag{2.51}
\]

To the first term on the r.h.s. only the quadratic part of \( (2.31) \) contributes, cf. \( (2.35) \). It is bounded using Proposition 2.1 and the bound \( (2.43) \) :

\[
|\partial^w R_{l,n}^{\Lambda_0,\Lambda_0}(p_1, \ldots, p_n)| \leq (\Lambda_0 + m)^{3-n-|w|} P_1\left( \log \frac{\Lambda_0 + m}{m} \right) P_2\left( \frac{|p_i|}{\Lambda_0 + m} \right)
\leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} P_1\left( \log \frac{\Lambda_0 + m}{m} \right) P_2\left( \frac{|p_i|}{\Lambda + m} \right),
\]

valid for \( 0 \leq \Lambda \leq \Lambda_0 \), since \( n + |w| > 4 \). The integrand of the second term on the r.h.s. of \( (2.51) \) is the derivative with respect to \( \Lambda_0 \) of the r.h.s. of \( (2.31) \). Observing

\[
\partial_{\Lambda_0} \partial_{\Lambda} C^{\Lambda,\Lambda_0}(k) = 0,
\]

(2.53)
we bound the $\Lambda_0$ - derivative considered using Propositions 2.1 and - in accord with the induction hypothesis - 2.2 together with the bounds (2.42) and (2.43) which are employed in the linear and in the quadratic part, respectively. Proceeding then similarly as in deducing (2.48) yields

$$| \partial_{\Lambda_0} \partial^w R_{t,n}^{\Lambda,\Lambda_0}(p_1, \cdots, p_n) | \leq \frac{(\Lambda + m)^{5-n-\vert w \vert - 1}}{(\Lambda_0 + m)^2} P_t \left( \log \frac{\Lambda_0 + m}{m} \right) P_s \left( \{ \frac{\vert p_i \vert}{\Lambda + m} \} \right).$$

(2.54)

From this follows upon integration, with the bound on the momenta majorized, a bound on the second term on the r.h.s. of (2.51) that has the form (2.52). Therefore, the assertion (2.49) is deduced if $n + \vert w \vert > 4$. Here, the respective flow equations integrated at the renormalization point (2.46) are derived with respect to $\Lambda_0$. They imply, observing that the initial conditions, i.e. the renormalization constants (2.34-2.37) do not depend on $\Lambda_0$, the bound

$$| \partial_{\Lambda_0} \partial^w L_{t,n}^{\Lambda,\Lambda_0}(0, \cdots, 0) | \leq \int_0^\Lambda d\lambda | \partial_{\Lambda_0} \partial^w R_{t,n}^{\Lambda,\Lambda_0}(0, \cdots, 0) | ;$$

(2.55)

where on the r.h.s. the shorthand introduced before has been used. In deducing the bound (2.54) no restriction on $n, w$ entered. Therefore, we can use it in (2.55) also and obtain upon integration

$$| \partial_{\Lambda_0} \partial^w L_{t,n}^{\Lambda,\Lambda_0}(0, \cdots, 0) | \leq \frac{(\Lambda + m)^{5-n-\vert w \vert}}{(\Lambda_0 + m)^2} P_s \left( \log \frac{\Lambda_0 + m}{m} \right).$$

(2.56)

Extension of these bounds to general momenta is again achieved via the Taylor formula (2.47) as in the proof of Proposition 2.1. Thus, Proposition 2.2 is proven.

Remarks: Renormalizability is a consequence of Proposition 2.2 at the value $\Lambda = 0$. From this point of view Proposition 2.1 is of preparatory, technical nature. The bounds established in both Propositions are not optimal but sufficient. Their virtue is to allow a concise and complete proof of renormalizability. These bounds can be refined in various ways. We mention that Kopper and Meunier [44], by sharpening the induction hypothesis with respect to momentum derivatives of $n$-point functions, obtained optimal bounds on the momentum behaviour related to Weinberg’s theorem [45].

The Propositions 2.1 and 2.2 established, it is physically important to notice that they even remain valid, when the original bare interaction (2.11) is extended by appropriately chosen irrelevant terms: It is sufficient to replace
the condition (2.45) by requiring for \( n + |w| > 4 \):

\[
|\partial^w L_{1,n}^{\Lambda_0,\Lambda_0}(p_1, \cdots, p_n)| \leq (\Lambda_0 + m)^{4-n-|w|} \mathcal{P}_1 \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{P}_2 \left( \{ \frac{|p_i|}{\Lambda_0 + m} \} \right),
\]

\[
|\partial_{\Lambda_0} \partial^w L_{1,n}^{\Lambda_0,\Lambda_0}(p_1, \cdots, p_n)| \leq (\Lambda_0 + m)^{3-n-|w|} \mathcal{P}_3 \left( \log \frac{\Lambda_0 + m}{m} \right) \mathcal{P}_4 \left( \{ \frac{|p_i|}{\Lambda_0 + m} \} \right);
\]

(2.57)

evidently, we can also write \( \Lambda_0 \) instead of \( \Lambda_0 + m \) everywhere. One first observes that these bounds agree with Propositions 2.1 and 2.2 considered at \( \Lambda = \Lambda_0 \). Moreover, as bounds on the initial conditions to be added in (2.44), (2.51), respectively, they can be absorbed in the corresponding bounds on the integrals appearing.

### 2.4 Insertion of a composite field

Besides the system of \( n \)-point functions dealt with up to now, \( n \)-point functions with one or more additional composite fields inserted are of considerable physical interest. In particular, the generators of symmetry transformations of a theory appear generally in the form of composite fields. But there are further instances where inserted composite fields – sometimes also called inserted operators – occur. In the sequel we treat the perturbative renormalization of one composite field inserted. Since, by definition, a composite field depends nonlinearly on the basic field (or fields) of the theory considered, new divergences have to be circumvented and hence additional renormalization conditions are required.

As before, we examine the quantum field theory of a real scalar field \( \phi(x) \) with mass \( m \) in four-dimensional Euclidean space-time. Then, a composite field \( Q(x) \) is a local polynomial formed in general of the field \( \phi(x) \) and of its space-time derivatives. It is determined by its classical version \( Q_{\text{class}}(x) \). If we restrict to achieve a renormalized theory with one insertion, \( Q(x) \) has to be chosen as follows: Let \( Q_{\text{class}}(x) \) be a monomial having the canonical mass dimension \( D \), then

\[
Q(x) = Q_{\text{class}}(x) + Q_{c.t.}(x),
\]

(2.58)

where \( Q_{c.t.}(x) \) is a polynomial which is formed of all local terms of canonical mass dimension \( \leq D \). This polynomial \( Q_{c.t.}(x) \) acts as counterterm. If \( Q_{\text{class}}(x) \) shows a symmetry not violated in the intermediate process of regularization this symmetry can be imposed on \( Q_{c.t.}(x) \), too. Since the regularization (2.8) keeps the Euclidean symmetry the counterterms \( Q_{c.t.}(x) \) can be restricted to those showing the same tensor type as \( Q_{\text{class}}(x) \). We
illustrate the notion introduced with the example of a scalar composite field having \( D = 3 \):

\[
Q_{\text{class}}(x) = \frac{1}{3!} \phi(x)^3, \tag{2.59}
\]

\[
Q_{\text{c.f.}}(x) = \frac{1}{3!} r_1(\Lambda_0) \phi(x)^3 - r_2(\Lambda_0) \Delta \phi(x) + \frac{1}{2!} r_3(\Lambda_0) \phi(x)^2 + r_4(\Lambda_0) \phi(x) + r_5(\Lambda_0), \tag{2.60}
\]

with coefficients \( r_i(\Lambda_0) = O(\bar{\hbar}), i = 1, \cdots, 5 \). Our aim here is to show the renormalizability of the theory considered in section 2.3 with one insertion of a scalar composite field \( Q(x) \) of mass dimension \( D \). It will turn out that we can essentially proceed as before, taking minor modifications into account. Hence, we can refrain from repeating definitions and arguments already introduced. In place of the bare interaction (2.11) one starts with a modified one:

\[
\tilde{L}^{\Lambda_0,\Lambda_0}(\varrho; \phi) + \tilde{I}^{\Lambda_0,\Lambda_0}(\varrho) = L^{\Lambda_0,\Lambda_0}(\phi) + \int dx \varrho(x) Q(x), \tag{2.61}
\]

\[
\tilde{L}^{\Lambda_0,\Lambda_0}(\varrho; 0) = 0, \tag{2.64}
\]

where the composite field (2.58), coupled to an external source \( \varrho \in C^\infty(\Omega) \), has been added. Then, as in (2.12), the generating functional of regularized Schwinger functions with insertions \( Q \) is obtained upon functional integration:

\[
\tilde{Z}^{\Lambda_0,\Lambda_0}(\varrho; J) = \int d\mu_{\Lambda_0,\Lambda_0}(\phi) e^{-\frac{1}{\bar{\hbar}}(\tilde{L}^{\Lambda_0,\Lambda_0}(\varrho; \phi) + \tilde{I}^{\Lambda_0,\Lambda_0}(\varrho)) + \frac{1}{\bar{\hbar}}(\phi, J)} \tag{2.62}
\]

Moreover, passing similarly as before to regularized amputated truncated Schwinger functions with insertions, the equations (2.16), (2.17) are replaced by

\[
e^{-\frac{1}{\bar{\hbar}}(\tilde{L}^{\Lambda_0,\Lambda_0}(\varrho; \phi) + \tilde{I}^{\Lambda_0,\Lambda_0}(\varrho))} = \int d\mu_{\Lambda_0,\Lambda_0}(\phi) e^{-\frac{1}{\bar{\hbar}}(L^{\Lambda_0,\Lambda_0}(\varrho; \phi + \varphi) + I^{\Lambda_0,\Lambda_0}(\varrho))}, \tag{2.63}
\]

\[
\tilde{L}^{\Lambda_0,\Lambda_0}(\varrho; 0) = 0. \tag{2.64}
\]

In view of the implicit notation (2.58) we stress that the shift of the field to \( \phi + \varphi \) involves the field dependent insertion \( Q(x) \) in (2.61), too. In exactly the same way as (2.18) was obtained, we find the relation between the generating functionals \( \tilde{L} \) and \( \tilde{Z} \):

\[
e^{-\frac{1}{\bar{\hbar}}(L^{\Lambda_0,\Lambda_0}(\varrho; \phi) + I^{\Lambda_0,\Lambda_0}(\varrho))} = e^{-\frac{1}{\bar{\hbar}}(\phi, (C^{\Lambda_0,\Lambda_0})^{-1}\varphi)} \tilde{Z}^{\Lambda_0,\Lambda_0}(\varrho; (C^{\Lambda_0,\Lambda_0})^{-1}\varphi). \tag{2.65}
\]

\[5\tilde{I}^{\Lambda_0,\Lambda_0}(\varrho) \] is the field independent part that possibly has to enter the modified bare action, as e.g. in (2.60).
Since (2.63) and (2.16) have the same form we obtain the flow equation of the functional $\tilde{L}^{\Lambda_0}(q;\varphi)$ by substituting in the flow equation (2.21):

$$L^{\Lambda_0}(\varphi) \rightarrow \tilde{L}^{\Lambda_0}(q;\varphi), \quad I^{\Lambda_0} \rightarrow \tilde{I}^{\Lambda_0}(q).$$

As a consequence the generating functional of the amputated truncated Schwinger functions with one insertion $Q$,

$$L^{\Lambda_0}(1)(x;\varphi) := \frac{\delta}{\delta q(x)} L^{\Lambda_0}(q;\varphi) |_{q(x) = 0}, \quad (2.66)$$

then satisfies the flow equation

$$\frac{d}{d\Lambda} \left( L^{\Lambda_0}(1)(x;\varphi) + I^{\Lambda_0}(1)(x) \right) = \frac{\hbar}{2} \left( \frac{\delta}{\delta \varphi} \right) C^{\Lambda_0} \frac{\delta}{\delta \varphi} L^{\Lambda_0}(1)(x;\varphi) \quad (2.67)$$

$$- \left( \frac{\delta}{\delta \varphi} L^{\Lambda_0}(\varphi) \right) C^{\Lambda_0} \frac{\delta}{\delta \varphi} L^{\Lambda_0}(1)(x;\varphi),$$

involving the vacuum part with one insertion

$$I^{\Lambda_0}(1)(x) := \frac{\delta}{\delta q(x)} I^{\Lambda_0}(q) |_{q(x) = 0}. \quad (2.68)$$

In deriving the r.h.s. of (2.67) use of the symmetry $\dot{C}^{\Lambda_0}(x-y) = \dot{C}^{\Lambda_0}(y-x)$ has been made. We note that the functional $L^{(1)}$ satisfies a linear equation. Because of the insertion the full flow equation (2.67) can be studied in the infinite volume limit $\Omega \rightarrow \mathbb{R}^4, \varphi \in \mathcal{S}(\mathbb{R}^4)$. The Fourier transform with respect to the insertion is defined as

$$\hat{L}^{\Lambda_0}(1)(q;\varphi) := \int dx e^{iqx} L^{\Lambda_0}(1)(x;\varphi), \quad (2.69)$$

$$\hat{I}^{\Lambda_0}(1)(q) := \int dx e^{iqx} I^{\Lambda_0}(1)(x) = (2\pi)^4 \hat{I}^{\Lambda_0}(q). \quad (2.70)$$

Furthermore, the generating functional is decomposed, observing the conventions (2.24), $n \in \mathbb{N}$,

$$(2\pi)^{4(n-1)} \delta(p_n) \cdots \delta(p_1) \hat{L}^{\Lambda_0}(1)(q;\varphi) |_{\varphi = 0} = \delta(q + p_1 + \cdots + p_n) L^{\Lambda_0}(1)(n)(q; p_1, \cdots, p_n). \quad (2.71)$$

The amputated truncated $n$-point function with one insertion carrying the momentum $q$,

$$L^{\Lambda_0}(1)(n)(q; p_1, \cdots, p_n), \quad (2.71)$$
is at fixed $q$ totally symmetric in the momenta $p_1, \cdots, p_n$. Furthermore, the sum of all momenta has to vanish because of the $\delta$-constraint in (2.71). From (2.67) and by proceeding exactly as before from (2.25) to (2.31), after a loop expansion of the $n$-point functions, $n \in \mathbb{N}$, and of the vacuum part $\Lambda^{A,0} \Lambda_0$, we arrive at the system of flow equations with one insertion:

$$L^{A,\Lambda_0}_{(1),n}(q; p_1, \cdots, p_n) = \sum_{l=0}^{\infty} R^l L^{A,\Lambda_0}_{(1),l,n}(q; p_1, \cdots, p_n),$$  \hspace{1cm} (2.72)

we arrive at the system of flow equations with one insertion:

$$\frac{1}{2} \int k \partial_k C^{A,\Lambda_0}(k) \cdot L^{A,\Lambda_0}_{(1),l-1,2}(0; k, -k)$$

$$- \sum_{l_1,l_2} L^{A,\Lambda_0}_{l_1,l}(0) \partial_k C^{A,\Lambda_0}(0) \cdot L^{A,\Lambda_0}_{(1),l_2,1}(0; 0),$$  \hspace{1cm} (2.73)

$$\frac{1}{2} \int k \partial_k C^{A,\Lambda_0}(k) \cdot \partial^{w} L^{A,\Lambda_0}_{(1),l-1,n+2}(q; k, -k, p_1, \cdots, p_n)$$

$$- \sum_{w_1,w_2,w_3} c_{\{w_i\}} \left[ \partial_{w_1} L^{A,\Lambda_0}_{l_1,n+1}(p_1, \cdots, p_{n+1}, p) \cdot \partial_{w_2} C^{A,\Lambda_0}(p) \right]_{rsym}$$

$$p = -p_1 - \cdots - p_{n+1} = q + p_{n+1} + \cdots + p_n.$$

The notation used above has been introduced in (2.28 - 2.31). Furthermore, the derivations $\partial^{w}$ can be restricted to the momenta $p_1, \cdots, p_n$, because of $q = -p_1 - \cdots - p_n$. The vacuum part does not act back on the functional $L_{(1)}$. We therefore disregard its flow (2.73) and just state that in each loop order the bare parameter $\Lambda^{A,0}_{0,\Lambda_0}$ is determined by a renormalization constant $\Lambda^{A,0}_{0,\Lambda_0}$ prescribed at $\Lambda = 0$.

The task is to show that finite limits

$$\lim_{\Lambda_0 \to \infty} \lim_{\Lambda \to 0} L^{A,\Lambda_0}_{(1),l,n}(q; p_1, \cdots, p_n), \quad n \in \mathbb{N}, \quad l \in \mathbb{N}_0,$$

can be obtained, given the $n$-point functions without insertion which satisfy the Propositions 2.1 and 2.2. This can be achieved following closely the corresponding steps performed before without insertion in section 2.3; the
demonstration here can thus be presented in a concise way. Due to (2.69 - 2.70), (2.66), (2.61), the bare functional is given by

\[ \hat{L}^{\Lambda_0,\Lambda_0}(q; \varphi) + \hat{L}^{\Lambda_0,\Lambda_0}(q) := \int dx e^{ixq} Q(x). \]  

(2.76)

The tree order is completely determined by the classical part of the insertion (2.58). This classical part \( Q_{\text{class}} \) yields the initial condition in integrating the system of flow equations (2.74) for \( l = 0 \) and general momenta from the initial point \( \Lambda = \Lambda_0 \) downwards to smaller values of \( \Lambda \) ascending successively in \( n \). It is not necessary to prescribe the order in which the derivations \( \partial^w \) are treated. Since \( Q_{\text{class}} \) is assumed to be a monomial of canonical mass dimension \( D \) and containing \( n_0 \) field factors, \( 2 \leq n_0 \leq D \), the key properties (2.38) imply for all \( w \):

\[ \partial^w \mathcal{L}^{\Lambda_0,\Lambda_0}_{(1)0,1}(q; p) = 0, \quad \partial^w \mathcal{L}^{\Lambda_0,\Lambda_0}_{(1)0,2}(q; p_1, p_2) = 0. \]  

(2.77)

The nonvanishing tree order starts at \( n = n_0 \) with

\[ \partial^w \mathcal{L}^{\Lambda_0,\Lambda_0}_{(1)0,n_0}(q; \{p_i\}) = \partial^w \mathcal{L}^{\Lambda_0,\Lambda_0}_{(1)0,n_0}(q; \{p_i\}), \]  

(2.78)

i.e. the initial condition. Of course the limits (2.75) exist in the tree order (since no integrations occur). In view of the mass dimension \( D \) of the insertion, the bounds

\[ |\partial^w \mathcal{L}^{\Lambda_0,\Lambda_0}_{(1)0,n}(q; \{p_1, \cdots, p_n\})| \leq (\Lambda + m)^{D-n-|w|} \mathcal{P}\left(\frac{|p_i|}{\Lambda + m}\right) \]  

(2.79)

are established for later use. If \( n_0 < D \), apart from (2.78) there are other relevant instances \( n + |w| \leq D \): they should be evaluated explicitly without using the bounds in the flow equation. (It is instructive to compare the examples \( Q_{\text{class}} = \phi^4 \Delta \phi, \phi^4 \) both having \( D = 4 \).) The irrelevant cases \( n + |w| > D \) in (2.79) are established similarly as (2.44).

For \( l > 0 \) the coefficients of the counterterms inherent in (2.76), extracted via (2.71) - (2.72), have to depend on \( \Lambda_0 \),

\[ \partial^w \mathcal{L}^{\Lambda_0,\Lambda_0}_{(1)l,n}(0; 0, \cdots, 0) = r_{l,n,w}(\Lambda_0), \quad n + |w| \leq D, \quad l > 0. \]  

(2.80)

They are determined by prescribing related renormalization conditions for vanishing flow parameter \( \Lambda \) at a renormalization point which is again chosen at vanishing momenta. Hence, order-by-order, the real constants, \( l > 0 \),

\[ \partial^w \mathcal{L}^{0,\Lambda_0}_{(1)l,n}(0; 0, \cdots, 0) =: r_{l,n,w}^R, \quad n + |w| \leq D, \]  

(2.81)
can be freely chosen, provided they respect the symmetry of $L_{(1)}^{L,Λ_0}$.

**Proposition 2.3** (Boundedness)

Let $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $w$ from (2.30) and $0 \leq Λ \leq Λ_0$, then

$$|∂^w L_{(1)}^{L,Λ_0}(q;p_1,\cdots,p_n)| \leq (Λ + m)^{D-n-|w|} P_1 \left( \frac{Λ + m}{Λ + m} \right).$$

(2.82)

The symbol $P$ denotes polynomials with nonnegative coefficients which depend on $l,n,w$ but not on $\{p_i\},Λ,Λ_0$. For $l = 0$ all polynomials $P_1$ reduce to positive constants.

**Proof**: Due to (2.79), the assertion (2.82) is already shown in the tree order. Given the set of $n$-point functions without insertions satisfying Proposition 2.1, one proceeds for $l > 0$ inductively as in the proof of the latter proposition, i.e. i) ascending in $l$, ii) at fixed $l$ ascending in $n$, iii) at fixed $l,n$ descending in $w$. Inspecting the flow equations (2.74), it is easily seen that for any given $l,n,w$ on the l.h.s. the contributions to the r.h.s. - because of the key properties (2.38) - always precede those on the l.h.s. in the order of induction adopted. Imitating the steps leading to (2.48) provides the bound for $l,n \in \mathbb{N}$,

$$|∂^w L_{(1)}^{L,Λ_0}(q;p_1,\cdots,p_n)| \leq (Λ + m)^{D-n-|w|} P_5 \left( \frac{Λ + m}{Λ + m} \right) P_6 \left( \left\{ \frac{|p_i|}{Λ + m} \right\} \right).$$

(2.83)

$a_1$) In the cases $n + |w| > D$ this bound is integrated downwards from the initial point $Λ = Λ_0$ with vanishing initial conditions, see (2.76). From this follows easily (2.82) for $n + |w| > D$.

$a_2$) If $n + |w| \leq D$, however, the respective flow equation (2.74) has to be integrated upwards at the renormalization point, as in (2.46), employing now the renormalization conditions (2.81). The bound (2.83) then implies the claim (2.82) at vanishing momenta. Again as in section 2.3, extension to general momenta is accomplished appealing to the Taylor formula (2.47). Thus, Proposition 2.3 is proven.

In complete analogy with the steps taken in section 2.3, the proposition just proven prepares the decisive

**Proposition 2.4** (Convergence)

Let $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $w$ from (2.30), $0 \leq Λ \leq Λ_0$, and $Λ_0 > Λ_0$, sufficiently large, then

$$|∂^w L_{(1)}^{L,Λ_0}(q;p_1,\cdots,p_n)| \leq \frac{(Λ + m)^{D+1-n-|w|}}{(Λ_0)^2} \left( \log \frac{Λ_0}{m} \right)^{ν} P_4 \left( \left\{ \frac{|p_i|}{Λ + m} \right\} \right).$$

(2.84)

with a positive integer $ν$ depending on $l,n,w$ only.

**Proof**: One first verifies the assertion directly in the tree order for the relevant
cases \( n + |w| \leq D \). Herewith, the further course of the proof is just a replica of the proof given for Proposition 2.2. As a consequence, in each place the exponent \( 4 + 1 - n - |w| \) appears there, this exponent is changed here into \( D + 1 - n - |w| \), thus proving the assertion \((2.81)\).

Finally, integration of the bound \((2.84)\) of Proposition 2.4 demonstrates, that the renormalized regularized \( n \)-point functions with one insertion have finite limits \((2.75)\).

### 2.5 Finite temperature field theory

There are essentially two formulations of quantum fields at finite temperature: a real-time approach to treat dynamical effects, and an imaginary-time approach to describe equilibrium properties [46]. In this section the problem of renormalization in a temperature independent way is considered. Such a renormalization is required studying the \( T \)- dependence of observables, since then the relation between bare and renormalized coupling constants must not depend on the temperature. Our aim here is to show within the imaginary-time formalism, that a quantum field theory renormalized at \( T = 0 \) stays also renormalized at any \( T > 0 \). For the sake of a succinct presentation the symmetric \( \Phi^4 \)-theory is treated and the generalization to the nonsymmetric theory is stated at the end.

The first steps to be taken do not differ from the zero temperature case: Starting from a finite domain, given by a 4-dimensional torus \( \Omega \), and the Gaussian measure with the regularized covariance \((2.7)\), we obtain Wilson’s flow equation \((2.21)\). Here, the bare interaction \((2.11)\) is restricted to the symmetric theory, as already mentioned, i.e. putting

\[
f = v(\Lambda_0) = b(\Lambda_0) \equiv 0.
\]

Disregarding as before the flow of the vacuum part \( I^{\Lambda_0} \), we imagine at least one functional derivative acting on the flow equation \((2.21)\). Then we can pass to the spatial infinite volume limit, but keeping the periodicity in the imaginary time \( x_4 \) and choosing the period equal to the inverse temperature:

\[
l_4 = \beta \equiv 1/T.
\]

Hence, in this limit the space-time domain is \( \mathbb{R}^3 \times S^1 \) and the theory shows the reduced symmetry \( O(3) \times Z_2 \), as compared to the \( O(4) \)-symmetry at \( T = 0 \). Correspondingly, the dual Fourier variables (momentum vectors) are

\[
\mathbf{p} := (\vec{p}, p_4), \quad \vec{p} \in \mathbb{R}^3, \quad p_4 = 2\pi n T, \quad n \in \mathbb{Z},
\]

and hence we define

\[
\int_{\mathcal{L}} := T \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{(2\pi)^3}.
\]
In the sequel we underline a symbol denoting a quantity at finite temperature or write the T-dependence explicitly. In place of (2.24) the Fourier transform takes the form
\[
\varphi(x) = \int \mathcal{P} e^{i p x} \hat{\varphi}(p), \quad \hat{\varphi}(p) = \int_{\mathbb{R}^3} d^3 x \int_0^\beta d x_4 e^{-i p x} \varphi(x),
\]
(2.88)
implying for a functional derivation:
\[
\delta \varphi(x) = \left(\frac{2\pi}{T}\right)^3 \int \mathcal{P} e^{i p x} \delta \hat{\varphi}(p), \quad \delta \hat{\varphi}(p) = \frac{T}{\left(\frac{2\pi}{T}\right)^3} \int_{\mathbb{R}^3} d^3 x \int_0^\beta d x_4 e^{i p x} \delta \varphi(x).
\]
(2.89)
Furthermore, the regularized covariance (2.27) is restricted to momenta (2.86),
\[
C^{\Lambda,\Lambda_0}(p) = \frac{1}{p^2 + m^2} \left( e^{-\frac{p^2 + m^2}{2\Lambda_0^2}} - e^{-\frac{p^2 + m^2}{2\Lambda_0^2}} \right).
\]
(2.90)
Denoting by \( L^{\Lambda,\Lambda_0}(\varphi; T) \) the generating functional of the amputated truncated Schwinger functions at finite temperature \( T \), we define the \( n \)-point functions, \( n \in \mathbb{N} \), similar to (2.25) as
\[
\left. L^{\Lambda,\Lambda_0}(\varphi; T) \right|_{\varphi \equiv 0} = \delta(p_1 + \cdots + p_n) \delta_{0,(p_1+\cdots+p_n)} \mathcal{L}_n^{\Lambda,\Lambda_0}(p_1, \cdots, p_n; T).
\]
(2.91)
These \( n \)-point functions, after a respective loop expansion in complete analogy to (2.29), then satisfy a system of flow equations obtained from (2.31) by replacing every momentum vector appearing by its underlined analogue, and moreover, restricting the momentum derivatives \( \partial^w \) to spatial momentum components. Employing this system of flow equations, we could prove renormalizability of the theory at finite temperature proceeding similarly as in the case of zero temperature. However, because of the reduced spacetime symmetry, the renormalization conditions (2.34-2.37) for \( l \geq 1 \) would have to be extended by an additional constant:
\[
\begin{align*}
\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(p_i - p; T) &= a_l^R(T) + z_l^{R,1}(T) p^2 + z_l^{R,2}(T) p^4 + O(p^4), \\
\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(0, 0, 0, 0; T) &= c_l^R(T).
\end{align*}
\]
(2.92, 2.93)
The constants for \( n = 1, 3 \) are set equal to zero in the symmetric theory (2.85). Only at \( T = 0 \), the emerging \( O(4) \)-symmetry implies the equality
\footnote{In the symmetric theory the \( n \)-point functions with odd \( n \) vanish.}
\( z_i^{R,1}(0) = z_i^{R,2}(0) \). Our aim is to prove renormalizability in a temperature independent way, i.e. with counterterms that do not depend on the temperature. In this case, the renormalization constants \( a_i^R(T), z_i^{R,1}(T), z_i^{R,2}(T), c_i^R(T) \) cannot be prescribed arbitrarily, since they are related dynamically to the three renormalization constants at \( T = 0 \). Therefore, we follow a different course and study the respective difference of a \( n \)-point function at \( T > 0 \) and at \( T = 0, n \in \mathbb{N} \), with momenta \( \{p\} \equiv (p_1, \ldots, p_n) \) of the form (2.86):

\[
D_{l,n}^{\Lambda,0} (\{p\}) := \mathcal{L}_{l,n}^{\Lambda,0} (\{p\}; T) - \mathcal{L}_{l,n}^{\Lambda,0} (\{p\}).
\]  

(2.94)

These functions are well-defined. From the system of flow equations (2.31) and from its analogue at finite temperature follows the system of flow equations satisfied by the bounds (2.41) and (2.49) satisfied by the \( n \)-point functions at zero temperature, then follows the
Theorem
For \( l, n \in \mathbb{N} \) and for \( 0 \leq \Lambda \leq \Lambda_0 \) holds
\[
|D_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)| \leq (\Lambda + m)^{-s-n} \mathcal{P}_1(\log \frac{\Lambda + m}{m}) \mathcal{P}_2(\{ \frac{|p|}{\Lambda + m} \}) \tag{2.98}
\]
\[
|\partial_\Lambda D_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n)| \leq \frac{(\Lambda + m)^{-s-n}}{(\Lambda_0)^2} \mathcal{P}_3(\log \frac{\Lambda_0}{m}) \mathcal{P}_4(\{ \frac{|p|}{\Lambda + m} \}) \tag{2.99}
\]

The polynomials \( \mathcal{P} \) have positive coefficients, which depend on \( l, n, s, m \) and (smoothly) on \( T \), but not on \( \{ p \}, \Lambda, \Lambda_0 \). The positive integer \( s \) may be chosen arbitrarily.

The \( n \)-point functions at finite temperature \( T \), \( L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n; T) \), when renormalized with the same counterterms as the zero temperature functions, (2.97), satisfy the bounds (2.41) and (2.49) restricted to the case \( w = 0 \) and to momenta (2.86). The coefficients in the polynomials \( \mathcal{P} \) may now depend also (smoothly) on \( T \).

For the proof we refer to [31], pp.396-399, and just indicate that the system of flow equations (2.95) is integrated inductively from the initial point \( \Lambda = \Lambda_0 \) downwards, observing (2.97). The difference of the two terms not involving any function \( D_{l,n}^{\Lambda,\Lambda_0} \), which appears in (2.95), however, is not accessible by induction. It is bounded separately, matching the sharp bound on \( \Lambda \) asserted, by use of the Euler - MacLaurin formula, see e.g. [47].

Due to the Theorem, the \( n \)-point functions of the theory at \( T > 0 \), renormalized at zero temperature, satisfy the bound (2.49) for \( w = 0 \) and momenta (2.86). Hence, they have finite limits
\[
\lim_{\Lambda_0 \to \infty} L_{l,n}^{\Lambda,\Lambda_0}(p_1, \ldots, p_n; T), \quad l, n \in \mathbb{N},
\]
upon removing the UV-cutoff \( \Lambda_0 \).

As already indicated before, a finite theory at given temperature \( T_0 > 0 \) could also be generated imposing renormalization conditions at this temperature. The price to be paid (in the symmetric theory considered) are in each loop order the four constants \( a^R_1(T_0), z^R_i(T_0), e^R_i(T_0), c^R_i(T_0), \) instead of the three constants \( a^R_1, z^R_i, c^R_i \) at zero temperature. However, an arbitrary choice of \( z^R_i(T_0), e^R_i(T_0) \) would not correspond to a theory at zero temperature, which shows the \( O(4) \)- symmetry of Euclidean space-time.

Starting from an \( O(4) \)-invariant theory at zero temperature, the functional
\[
L^{\Lambda,\Lambda_0}(\varphi; T) - L^{\Lambda,\Lambda_0}(\varphi) \tag{2.100}
\]
with initial condition (2.97) has been proven to satisfy the bound (2.99). Hence, the function \( D_{l,2}^{\Lambda,\Lambda_0}(p, -p) \), converging for all \( l \) with \( \Lambda_0 \to \infty \) to a finite
limit, produces a dynamical relation between the renormalization constants $z^{R,1}_i(T_0)$ and $z^{R,2}_i(T_0)$, i.e. fixing one of them determines the other. Thus, a renormalization does not depend on temperature, if this relation is satisfied. It becomes manifest in the equality $z^1_i(\Lambda_0) = z^2_i(\Lambda_0)$ of the corresponding bare parameters.

Concluding we remark that the proof can be easily extended to the non-symmetric $\Phi^4$-theory. In this case, the $n$-point functions with $n$ odd no longer vanish, since the $Z_2$-symmetry is now lacking. Hence, the bare interaction will be of the general form (2.11). Correspondingly, the theory at zero temperature is renormalized by the conditions (2.34-2.37), involving five renormalization constants. Proceeding inductively as before, considering now odd and even values of $n$, establishes the Theorem for the nonsymmetric theory, too.

2.6 Elementary Estimates

Here, rather obvious estimates on some elementary integrals are listed, which we used repeatedly in generating inductive bounds on Schwinger functions.

a1) In the irrelevant cases, the integrals have the form
\[ \int_a^b dx \, x^{-r-1} (\log x)^s, \quad \text{with } 1 \leq a \leq b \text{ and } r \in \mathbb{N}, \; s \in \mathbb{N}_0. \]

Defining correspondingly the function
\[ f_{r,s}(x) := \frac{1}{r} x^{-r} \left( (\log x)^s + \frac{s}{r} (\log x)^{s-1} + \frac{s(s-1)}{r^2} (\log x)^{s-2} + \cdots + \frac{1 \cdot 2 \cdots s}{r^s} \right) \]
we observe $f'_{r,s}(x) = -x^{-r-1}(\log x)^s < 0$ and $f_{r,s}(x) > 0$ for $x > 1$, hence
\[ \int_a^b dx \, x^{-r-1} (\log x)^s = f_{r,s}(a) - f_{r,s}(b) < f_{r,s}(a). \]

a2) The integrals to be bounded in the relevant cases have the form
\[ \int_1^b dx \, x^{-r-1} (\log x)^s, \quad \text{with } 1 \leq b \text{ and } r, s \in \mathbb{N}_0. \]

If $r = 0$, we just integrate. For $r > 0$, defining
\[ g_{r,s}(x) := \frac{1}{r} x^r \left( (\log x)^s - \frac{s}{r} (\log x)^{s-1} + \frac{s(s-1)}{r^2} (\log x)^{s-2} + \cdots + (-)^s \frac{1 \cdot 2 \cdots s}{r^s} \right) \]
we notice $g'_{r,s}(x) = x^{r-1}(\log x)^s$ and hence
\[ \int_1^b dx \, x^{r-1} (\log x)^s = g_{r,s}(b) - g_{r,s}(1) < \frac{1 \cdot 2 \cdots s}{r^s} + |g_{r,s}(b)|. \]
Chapter 3

The Quantum Action Principle

The Green functions of a relativistic quantum field theory depend on the adjustable parameters of this theory and are in general related according to the inherent symmetries of the theory. Clearly, all types of Green functions, whether truncated, amputated, or one-particle-irreducible, show these properties. The quantum action principle deals with the variation of Green functions caused by diverse operations performed:
i) applying the differential operator appearing in the (classical) field equation,
ii) (nonlinear) variations of the fields,
iii) variation of an adjustable parameter of the theory.

The quantum action principle relates each of these different operations on Green functions to the insertion of a corresponding composite field into the Green functions: as a local operator in the first two cases, whereas integrated over space-time in the third. Moreover, in general the local operation has a precursor within classical field theory (e.g. the field equation, the Noether theorem). Then the local composite field to be inserted in the case of a quantum field theory is a sum formed of its classical precursor and of assigned local counterterms, whose canonical mass dimensions are equal to or smaller than the canonical mass dimension ascribed to the term of classical descent. The quantum action principle has been established first by Lam [48] and Lowenstein [49] using the BPHZ-formulation of perturbation theory. This principle is extensively used in the method of algebraic renormalization [50].

Our aim is to demonstrate the parts i) and iii) of the quantum action principle by means of flow equations in the case of the scalar field theory. The particularly interesting part ii) is deferred to a later section, where nonlinear BRST-transformations have to be implemented in showing the renormalizability of a non-Abelian gauge theory.
3.1 Field equation

We consider again the quantum field theory of a real scalar field on four-dimensional Euclidean space-time, which has been treated in the preceding sections. To derive a field equation, we act on the generating functional of its regularized Schwinger functions \(2.12\) as follows:

\[
\hat{h} \int dy (C^{\Lambda,\Lambda_0})^{-1}(x - y) \frac{\delta}{\delta J(y)} Z^{\Lambda,\Lambda_0}(J) = \int dy (C^{\Lambda,\Lambda_0})^{-1}(x - y) \int d\mu_{\Lambda,\Lambda_0}(\phi) \phi(y) e^{-\frac{i}{\hbar} L^{\Lambda_0,\Lambda_0}(\phi) + \frac{i}{\hbar} \langle \phi, J \rangle}.
\]

In presence of the regularization the inverse of the regularized covariance \(2.8\) replaces the differential operator \(-\Delta + m^2\). Integration by parts \(2.5\) on the r.h.s. and recalling that the covariance of the Gaussian measure \(d\mu_{\Lambda,\Lambda_0}(\phi)\) is \(\hat{h}C^{\Lambda,\Lambda_0}\) yields the field equation of the regularized generating functional \(2.12\)

\[
(J(x) - \hat{h} \int dy (C^{\Lambda,\Lambda_0})^{-1}(x - y) \frac{\delta}{\delta J(y)}) Z^{\Lambda,\Lambda_0}(J) = \int d\mu_{\Lambda,\Lambda_0}(\phi) Q(x) e^{-\frac{i}{\hbar} L^{\Lambda_0,\Lambda_0}(\phi) + \frac{i}{\hbar} \langle \phi, J \rangle}.
\]

(3.1)

On the r.h.s. the inserted composite field \(Q(x)\) is given by

\[
Q(x) = \frac{\delta}{\delta \phi(x)} L^{\Lambda_0,\Lambda_0}(\phi).
\]

(3.2)

If we employ the generating functional of regularized Schwinger functions with insertions \(2.61 - 2.62\) we can rewrite the field equation \(3.1\) in the form

\[
J(x) - \hat{h} \int dy (C^{\Lambda,\Lambda_0})^{-1}(x - y) \frac{\delta}{\delta J(y)} Z^{\Lambda,\Lambda_0}(J) = -\hat{h} \frac{\delta}{\delta \phi(x)} \tilde{Z}^{\Lambda,\Lambda_0}(\phi; J)|_{\phi(x)=0}.
\]

(3.3)

Taking into account the relations \(2.18\) and \(2.65\) on the l.h.s. and on the r.h.s. of this equation, respectively, provides the field equation for the generating functional of the amputated truncated Schwinger functions,

\[
\frac{\delta}{\delta \phi(x)} L^{\Lambda_0,\Lambda_0}(\varphi) = L_{(1)}^{\Lambda,\Lambda_0}(x; \varphi) + I_{(1)}^{\Lambda,\Lambda_0}(x).
\]

(3.4)

Hence, in momentum space we have

\[
(2\pi)^4 \frac{\delta}{\delta \varphi(q)} L^{\Lambda,\Lambda_0}(\varphi) = \hat{L}_{(1)}^{\Lambda,\Lambda_0}(q; \varphi) + \hat{I}_{(1)}^{\Lambda,\Lambda_0}(q),
\]

(3.5)
using the conventions (2.24) and (2.69). Our goal is to show within perturbation theory, i.e. in a formal loop expansion, that the field equation (3.5) remains valid taking the limit \( \Lambda = 0, \Lambda_0 \to \infty \). To this end we proceed as follows:

\( \alpha \) In section 2.3 it has been shown that the generating functional \( L^{\Lambda, \Lambda_0}(\varphi) \) of the theory considered (perturbatively) converges to a finite limit with \( \Lambda_0 \to \infty \). The limit theory is determined by the choice of renormalization conditions (2.34 - 2.37) at the renormalization point (chosen at vanishing momenta).

\( \beta \) The generating functional \( L^{\Lambda, \Lambda_0,(1)}(q; \varphi) \) with insertion of one composite field of mass dimension \( D \) and momentum \( q \) has been shown in section 2.4 to have a finite limit with \( \Lambda_0 \to \infty \), too, provided the counterterms of the insertion (2.58) are introduced at first as indeterminate functions of \( \Lambda_0 \) and then determined by a choice of renormalization conditions at the renormalization point. In the case entering the field equation, however, the dependence on \( \Lambda_0 \) of the insertion (3.2) is already given by (2.11), the counterterms of the theory without insertion. In order to maintain in an intermediate stage the freedom in choosing renormalization conditions, we use instead of (3.2) indeterminate counterterms to begin with:

\[
Q(x) = \frac{1}{2!} \phi^2(x) + \frac{g}{3!} \phi^3(x) + v_1(\Lambda_0) + a_1(\Lambda_0) \phi(x) - z_1(\Lambda_0) \Delta \phi(x) + \frac{1}{2!} b_1(\Lambda_0) \phi^2(x) + \frac{1}{3!} c_1(\Lambda_0) \phi^3(x). \tag{3.6}
\]

Then, as has been demonstrated in section 2.4, any choice of admissible renormalization conditions leads to a finite limit of the generating functional \( L^{\Lambda, \Lambda_0,(1)}(q; \varphi) \) in sending \( \Lambda_0 \to \infty \), and thereby a related dependence of the coefficients \( v_1(\Lambda_0), \cdots, c_1(\Lambda_0) \) on \( \Lambda_0 \) arises.

\( \gamma \) We define the functional

\[
\hat{D}^{\Lambda, \Lambda_0}(q; \varphi) := \hat{L}^{\Lambda, \Lambda_0,(1)}(q; \varphi) + \hat{I}^{\Lambda, \Lambda_0,(1)}(q) - (2\pi)^4 \delta \frac{\delta}{\delta \hat{\varphi}(q)} L^{\Lambda, \Lambda_0}(\varphi). \tag{3.7}
\]

If this functional can be forced to vanish at \( \Lambda = 0 \) and for all \( \Lambda_0, \Lambda_0 > \Lambda_0 \), by an appropriate fixed choice of renormalization conditions in \( \beta \), then (3.5) converges to a finite renormalized field equation for \( (\Lambda = 0, \Lambda_0 \to \infty) \).

The functional (3.7) obeys the linear flow equation

\[
\frac{d}{d\Lambda} \hat{D}^{\Lambda, \Lambda_0}(q; \varphi) = \hbar \left( \frac{\delta}{\delta \varphi} \right) \hat{C}^{\Lambda, \Lambda_0} \frac{\delta}{\delta \varphi} \hat{D}^{\Lambda, \Lambda_0}(q; \varphi) - \left( \frac{\delta}{\delta \varphi} L^{\Lambda, \Lambda_0}(\varphi), \hat{C}^{\Lambda, \Lambda_0} \frac{\delta}{\delta \varphi} \hat{D}^{\Lambda, \Lambda_0}(q; \varphi) \right), \tag{3.8}
\]
which follows directly from the flow equations (2.21) and (2.67), performing
a functional derivation of the first and Fourier transforming the latter. To
make use of it we decompose

\[(2\pi)^{-4} \tilde{D}^{\Lambda, \Lambda_0}(q; 0) = \delta(q) D^{\Lambda, \Lambda_0}_0\]

\[(2\pi)^{(n-1)} \delta \varphi(p_n) \cdots \delta \varphi(p_1) \tilde{D}^{\Lambda, \Lambda_0}(q; \varphi)|_{\varphi = 0} = \delta(q + p_1 + \cdots + p_n) D^{\Lambda, \Lambda_0}_n(q; p_1, \cdots, p_n).\]  

From (2.25), (2.69 - 2.70) then results

\[D^{\Lambda, \Lambda_0}_0 = i^{\Lambda, \Lambda_0} - L^{\Lambda, \Lambda_0}(0),\]  

\[D^{\Lambda, \Lambda_0}_n(q; p_1, \cdots, p_n) = -\mathcal{L}^{\Lambda, \Lambda_0}_{(1)n}(q; p_1, \cdots, p_n) - \mathcal{L}^{\Lambda, \Lambda_0}_{n+1}(q; p_1, \cdots, p_n).\]  

We notice that the flow equations (3.18) and (2.67) have the same for-
m. Thus, after a loop expansion, the strict analogue of the system of flow equations
(2.73 - 2.74) is obtained, \(l \in \mathbb{N}_0, n \in \mathbb{N}:\)

\[\partial_\Lambda D^{\Lambda, \Lambda_0}_{l,0} = \frac{1}{2} \int_k \partial_\Lambda C^{\Lambda, \Lambda_0}(k) \cdot D^{\Lambda, \Lambda_0}_{l-1,2}(0; k, -k)\]

\[- \sum_{l_1, l_2} \mathcal{L}^{\Lambda, \Lambda_0}_{l,1}(0) \partial_\Lambda C^{\Lambda, \Lambda_0}(0) \cdot D^{\Lambda, \Lambda_0}_{l,1}(0; 0),\]  

\[\partial_\Lambda \partial_\varphi D^{\Lambda, \Lambda_0}_{l,n}(q; p_1, \cdots, p_n) = \]

\[\frac{1}{2} \sum_{l_1, l_2} \sum_{l_3} C_{l_1, l_2, l_3} \left[ \partial_{w_1} \mathcal{L}^{\Lambda, \Lambda_0}_{l_1, n+1}(p_1, \cdots, p_n, p) \cdot \partial_{w_2} \partial_\varphi C^{\Lambda, \Lambda_0}(p)\right]_{rsym}\]

\[- \partial_{w_2} D^{\Lambda, \Lambda_0}_{l_2, n+1}(q; p, p_{n+1}, \cdots, p_n)\]

\[p = -p_1 - \cdots - p_{n_1} = q + p_{n+1} + \cdots + p_n.\]  

We first treat the tree order. From (2.11), (2.76) follows \(\tilde{D}^{\Lambda_0, \Lambda_0}(q; \varphi)|_{\varphi = 0} = 0.\) Integrating the flow equations with \(l = 0\) and general momenta from the
initial point \(\Lambda = \Lambda_0\) downwards to smaller values of \(\Lambda\), ascending successively
in \(n\), we find due to the properties (2.38) for \(n \in \mathbb{N}, 0 \leq \Lambda \leq \Lambda_0,\)

\[D^{\Lambda, \Lambda_0}_{0,0} = 0, \quad D^{\Lambda, \Lambda_0}_{0,n}(q; p_1, \cdots, p_n) = 0.\]  

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The extension to all loop orders $l$ is achieved by the

**Proposition 3.1**

For all $l \in \mathbb{N}$ and $n + |w| \leq 3$ let

$$D_{t,0}^{0,\Lambda_0} = 0, \quad \partial^w D_{t,n}^{0,\Lambda_0}(0; 0, \cdots, 0) = 0,$$

then for $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $|w| \leq 3$, and $0 \leq \Lambda \leq \Lambda_0$:

$$D_{t,0}^{\Lambda,\Lambda_0} = 0, \quad \partial^w D_{t,n}^{\Lambda,\Lambda_0}(q; p_1, \cdots, p_n) = 0.$$  

(3.16)

**Proof:** In the order $l = 0$ the assertion is already established because of (3.14). We now assume (3.16) to hold for all orders smaller than a fixed order $l$. As a consequence, on the respective r.h.s. of the flow equations (3.12) and (3.13) the first term vanishes and in the second term only the pair $(l_1 = 0, l_2 = l)$ has to be taken into account. Looking first at the vacuum part, we observe that the r.h.s. of (3.12) vanishes due to (2.38). Thus, integration from the initial point $\Lambda = 0$ yields $D_{t,0}^{\Lambda,\Lambda_0} = 0$. To demonstrate the assertion for general $n$ we proceed inductively: ascending in $n$, and for fixed $n$ descending with $w$ from $|w| = 3$. Thus, for each $n$ the irrelevant cases $n + |w| > 3$ always precede the relevant ones, $n + |w| \leq 3$, if present at all. Since

$$\partial^w D_{t,n}^{\Lambda,\Lambda_0}(q; p_1, \cdots, p_n) = 0, \quad n + |w| > 3,$$

(3.17)

the flow equations (3.13) of these cases are integrated from the initial point $\Lambda = \Lambda_0$ downwards. On the other hand, the respective flow equation of the cases $n + |w| \leq 3$ is first integrated at zero momentum from the initial point $\Lambda = 0$ with vanishing initial condition (3.15) and in a now familiar second step the result is extended to general momenta via the Taylor formula (2.47). Following the inductive order stated one notices that for each pair $(n, w)$ occuring, the r.h.s. of (3.13) vanishes due the key properties (2.38) and preceding instances (3.16). Hence, (3.16) also holds for all $n$ in the order $l$ and the proposition is proven.

From $\alpha, \beta$ we know, that letting $\Lambda_0 \to \infty$, each term on the r.h.s. of equation (3.7) converges to a finite limit. Hence, if the renormalization conditions chosen for $L_{(3)}^{\Lambda,\Lambda_0}(q; \varphi)$ are inferred from those of $L^{\Lambda,\Lambda_0}(\varphi)$ to satisfy (3.15), the l.h.s. of (3.7) vanishes for $0 \leq \Lambda \leq \Lambda_0$. Thus, the field equation (3.5) remains valid after removing the cutoffs $\Lambda, \Lambda_0$, written suggestively as

$$(2\pi)^4 \frac{\delta}{\delta \hat{\varphi}(q)} L^{0,\infty}(\varphi) = \hat{L}_{(1)}^{0,\infty}(q; \varphi) + \hat{I}_{(1)}^{0,\infty}(q);$$

(3.18)

in the realm of a formal loop expansion, of course. Considering the relations (3.16) at $\Lambda = \Lambda_0$ reveals that the counterterms entering the insertion (3.6) have to be chosen identical to those of the bare interaction (2.11), $l \in \mathbb{N}$:

$$v_{1,l}(\Lambda_0) = v_l(\Lambda_0), \cdots, c_{1,l}(\Lambda_0) = c_l(\Lambda_0).$$

(3.19)
3.2 Variation of a coupling constant

The renormalized amputated truncated Schwinger functions (2.33) depend on the coupling constants \( f \) and \( g \), which can be freely chosen in the bare interaction (2.11). Our aim is to find a representation for the derivative of these Schwinger functions with respect to \( f \) or \( g \). To this end we start from the defining equation (2.16) of the regularized generating functional. Denoting by \( \kappa \) either \( f \) or \( g \), and defining

\[
W_\kappa(\phi) := \frac{\partial}{\partial \kappa} L^{A_0}(\phi) =: \int_\Omega dx Q_\kappa
\]

where the integrand \( Q_\kappa(x) \) is a composite field and \( W_\kappa(\phi) \) the space-time integral of it, we obtain from deriving (2.16):

\[
\partial_\kappa \left( L^{A_0}(\phi) + I^{A_0} \right) e^{-\frac{1}{\hbar} \left( L^{A_0}(\phi) + I^{A_0} \right)} = \int d\mu_{A_0}(\phi) e^{-\frac{1}{\hbar} L^{A_0}(\phi+\varphi)} W_\kappa(\phi + \varphi).
\]

On the other hand, the functional derivation of eq. (2.63) with respect to \( \varphi(x) \) at \( \varphi(x) = 0 \) yields, observing the shift \( \phi \rightarrow \phi + \varphi \) to be performed in (2.61) and employing the notations (2.66),(2.68):

\[
\left( L_{(1)}^{A_0}(x; \varphi) + I_{(1)}^{A_0}(x) \right) e^{-\frac{1}{\hbar} \left( L^{A_0}(\phi) + I^{A_0} \right)} = \int d\mu_{A_0}(\phi) e^{-\frac{1}{\hbar} L^{A_0}(\phi+\varphi)} Q(x)|_{\phi \rightarrow \phi + \varphi}.
\]

In writing this equation we have already taken account of the identities

\[
L^{A_0}(0; \varphi) = L^{A_0}(\varphi) , \quad I^{A_0}(0) = I^{A_0}.
\]

Choosing in (3.22) the particular composite field \( Q(x) = Q_\kappa(x) \) introduced in (3.20), and integrating over the finite space-time \( \Omega \), implies by comparison with (3.21),

\[
\partial_\kappa L^{A_0}(\varphi) = \int_\Omega dx L_{(1)}^{A_0}(x; \varphi).
\]

We can now pass to the infinite volume limit \( \Omega \rightarrow \mathbb{R}^4, \varphi \in S(\mathbb{R}^4) \). Hence,

\[
\partial_\kappa L^{A_0}(\varphi) = \hat{L}_{(1)}^{A_0}(0; \varphi),
\]

with the Fourier transform (2.69) at vanishing momentum. In the sequel, \( \hat{L}_{(1)}^{A_0}(0; \varphi) \) is always understood as the generating functional with the insertion (3.20).
The task posed is to produce a finite limit of the eq. (3.23) upon removing the cutoffs, i.e. letting $\Lambda = 0, \Lambda_0 \to \infty$. A finite limit of $L^{\Lambda,\Lambda_0}(\varphi)$ has been established in section 2.3. Furthermore, the insertion appearing in (3.23) is a particular instance of the insertion of a composite field $Q(x)$ dealt with in section 2.4. The composite field $Q(\kappa)(x)$ involved here follows from (3.20) and (2.11) as

$$Q(\kappa)(x) = \frac{1}{3!} \delta_{\kappa f} \phi^3(x) + \frac{1}{4!} \delta_{\kappa g} \phi^4(x)$$

$$+ v(\Lambda_0)\phi(x) + \frac{1}{2} a(\Lambda_0) \phi^2(x) + \frac{1}{2} b(\Lambda_0) (\partial_{\mu}\phi)^2(x)$$

$$+ \frac{1}{3!} c(\Lambda_0) \phi^3(x) + \frac{1}{4!} \phi^4(x),$$

where $\delta_{\kappa f}$ is the Kronecker symbol: $\delta_{\kappa f} = 1$, if $\kappa = f$, and $\delta_{\kappa f} = 0$, if $\kappa \neq f$. One should note that this composite field in both cases $\kappa = f$ or $\kappa = g$ has the canonical mass dimension $D = 4$, in contrast to its classical part. The coefficients of the counterterms appearing in (3.24) are the coefficients entering the bare interaction (2.11) derived with respect to $\kappa$. However, since in the process of renormalization the counterterms are determined by the renormalization conditions chosen, we at first treat the counterterms in (3.24) as free functions of $\Lambda_0$, which are then determined by the renormalization conditions prescribed in the case of the insertion. We do not assume the renormalization conditions (2.34-2.37) of $L^{\Lambda,\Lambda_0}(\varphi)$ to depend on $f$ and $g$, hence, their derivative with respect to $\kappa$ vanishes. Requiring (3.23) to be valid at the renormalization point for all values of $\Lambda_0$ then implies, that in all loop orders $l \geq 1$ an $n$-point function of $\hat{L}^{\Lambda,\Lambda_0}(0; \varphi)$ and its momentum derivatives vanish for $\Lambda = 0$ at zero momenta, if $n + |w| \leq 4$. The renormalization conditions fixed, we know from Proposition 2.4, that the functional $\hat{L}^{\Lambda,\Lambda_0}(0; \varphi)$ has a finite limit $\Lambda = 0, \Lambda_0 \to \infty$. To control the renormalization of eq. (4.112) we define

$$D^{\Lambda,\Lambda_0}(\varphi) := \hat{L}^{\Lambda,\Lambda_0}(0; \varphi) - \partial_\kappa L^{\Lambda,\Lambda_0}(\varphi).$$

This functional satisfies, as easily seen, a linear flow equation of the form (3.13); hence, after decomposition, a system of flow equations of the form (3.13) results. (Here, no vacuum part appears.) Comparing the bare interaction (2.11) with the insertion (3.24) we observe that $D^{\Lambda_0,\Lambda_0}$ vanishes in the tree order

$$D^{\Lambda_0,\Lambda_0}|_{l=0} = 0,$$

and its irrelevant part vanishes for $l > 0$,

$$\partial^w D^{\Lambda_0,\Lambda_0}_{l,n}(p_1, \cdots, p_n) = 0, \quad n + |w| > 4.$$
Given these initial conditions we have the

**Proposition 3.2**

Assume for all \( l \in \mathbb{N}, n + |w| \leq 4 \):

\[
\partial^w D^0_{l,n}(0, \cdots, 0) = 0 \quad (3.26)
\]

then follows for \( l \in \mathbb{N}_0, n \in \mathbb{N}, |w| \leq 4 \), and \( 0 \leq \Lambda \leq \Lambda_0 \):

\[
\partial^w D^\Lambda_{\Lambda_0}_{l,n}(p_1, \cdots, p_n) = 0 \quad (3.27)
\]

The proof by induction proceeds exactly as the proof of Proposition 3.1 and is omitted. Proposition 3.2 implies, that equation (3.23) has a finite limit for \( \Lambda = 0, \Lambda_0 \to \infty \),

\[
\partial_{\Lambda} L^0\infty(\varphi) = \hat{L}_1^0\infty(0; \varphi) \quad (3.28)
\]

again to be read in terms of a formal loop expansion. Furthermore, from (4.116) at \( \Lambda = \Lambda_0 \) follows the relation of the counterterms

\[
v_{n,l}(\Lambda_0) = \partial_{\Lambda} v_{n,l}(\Lambda_0), \cdots, c_{n,l}(\Lambda_0) = \partial_{\Lambda} c_{n,l}(\Lambda_0) \quad (3.29)
\]

### 3.3 Flow equations for proper vertex functions

In perturbative renormalization based on the analysis of Feynman integrals, (proper) vertex functions form the building blocks. They are represented by one-particle-irreducible (1PI) Feynman diagrams, see e.g. [51]. Although their generating functional has no representation as a functional integral, flow equations for vertex functions can be derived [33]. Our goal in this section is to deduce in the case of the symmetric \( \Phi^4 \)-theory from Wilson’s differential flow equation (2.21) for the \( L \)-functional the system of flow equations satisfied by the regularized \( n \)-point vertex functions, \( n \in \mathbb{N} \). After that, an inductive proof of renormalizability based on them is outlined.

We start from the regularized generating functional \( W^{\Lambda,\Lambda_0}(J) \) of the truncated Schwinger functions, (2.14), decomposed as

\[
W^{\Lambda,\Lambda_0}(J) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int dx_1 \cdots \int dx_{2n} W^{\Lambda,\Lambda_0}_{2n}(x_1, \cdots, x_{2n}) J(x_1) \cdots J(x_{2n}), \quad (3.30)
\]

according to (2.15). Due to the symmetry \( \phi \to -\phi \) of the theory, all \( n \)-point functions with \( n \) odd vanish identically. Defining the “classical field”

\[
\varphi(x) := \frac{\delta W^{\Lambda,\Lambda_0}(J)}{\delta J(x)}, \quad (3.31)
\]
we then notice, that
\[ \varphi(x)|_{J \equiv 0} = 0, \]
and, moreover, that \( \varphi \) depends on the flow parameter \( \Lambda \) (and on \( \Lambda_0 \)). Since the 2-point function \( W^\Lambda_{\Lambda_0}(\varphi) \) is different from zero, (3.31) can be inverted iteratively as a formal series in \( \varphi(x) \) to yield the source \( J(x) \) in the form
\[ J(x) = J(\varphi(x)) \equiv \sum_{n=0}^\infty \frac{1}{(2n+1)!} \int dx_1 \cdots \int dx_{2n+1} F_{2n+2}(x, x_1, \cdots, x_{2n+1}) \varphi(x_1) \cdots \varphi(x_{2n+1}), \]
where
\[ \int dy F_2(x, y) W^\Lambda_{\Lambda_0}(y, z) = \delta(x - z). \]
The generating functional \( \Gamma^\Lambda_{\Lambda_0}(\varphi) \) of the regularized vertex functions results from the Legendre transformation
\[ \Gamma^\Lambda_{\Lambda_0}(\varphi) := [ - W^\Lambda_{\Lambda_0}(J) + \int dy J(y) \varphi(y) ]_{J = J(\varphi)} \]
implying, due to (3.31),
\[ \frac{\delta \Gamma^\Lambda_{\Lambda_0}(\varphi)}{\delta \varphi(x)} = J(x). \]
The functional \( \Gamma^\Lambda_{\Lambda_0}(\varphi) \) is even under \( \varphi \to -\varphi \) and vanishes at \( \varphi = 0 \). Finally, performing the functional derivation of (3.34) with respect to \( J(y) \) and using again (3.31), provides the crucial functional relation
\[ \int dz \frac{\delta^2 W^\Lambda_{\Lambda_0}(J)}{\delta J(y) \delta J(z)} \frac{\delta^2 \Gamma^\Lambda_{\Lambda_0}(\varphi)}{\delta \varphi(z) \delta \varphi(x)} = \delta(y - x). \]
As an immediate consequence follows
\[ \int dz W^\Lambda_{\Lambda_0}(y, z) \Gamma^\Lambda_{\Lambda_0}(z, x) = \delta(y - x), \]
considering (3.35) at \( \varphi = 0 \), and thus also at \( J = 0 \). In order to obtain the relation between the functionals \( L^\Lambda_{\Lambda_0}(\varphi) \) and \( \Gamma^\Lambda_{\Lambda_0}(\varphi) \), we write (2.19) in the form
\[ W^\Lambda_{\Lambda_0}(J) = - L^\Lambda_{\Lambda_0}(\varphi) + \frac{1}{2} \langle J, C^\Lambda_{\Lambda_0} J \rangle, \]
\[ \varphi(x) = \int dy \, C^{A,A_0}(x-y)J(y). \]  

(3.37)

Deriving (3.36) twice with respect to \( J \) as required in (3.35) one obtains after operating on this equation with \((C^{A,A_0})^{-1}\),

\[
(C^{A,A_0})^{-1}(y-x) = - \int dz \int du \, \frac{\delta^2 L^{A,A_0}(\varphi)}{\delta \varphi(y) \delta \varphi(u)} C^{A,A_0}(u-z) \frac{\delta^2 \Gamma^{A,A_0}(\varphi)}{\delta \varphi(z) \delta \varphi(x)} + \frac{\delta^2 \Gamma^{A,A_0}(\varphi)}{\delta \varphi(y) \delta \varphi(x)}. 
\]

(3.38)

From this analogue of (3.35) follow the relations between the respective n-point functions of \( \Gamma^{A,A_0}(\varphi) \) and \( L^{A,A_0}(\varphi) \) upon repeated functional derivation with respect to \( \varphi \), employing the chain rule together with the relation

\[ \varphi(x) = \int dy \, C^{A,A_0}(x-y) \frac{\delta \Gamma^{A,A_0}(\varphi)}{\delta \varphi(y)}, \]

(3.39)

due to (3.37) and (3.34). With our conventions \((2.24), (2.27)\) for the Fourier transformation, the equations (3.38 - 3.39) appear in momentum space as

\[
(2\pi)^{-4} \frac{\delta(p+q)}{C^{A,A_0}(p)} = -(2\pi)^8 \int_k \frac{\delta^2 L^{A,A_0}(\varphi)}{\delta \tilde{\varphi}(p) \delta \tilde{\varphi}(k)} C^{A,A_0}(k) \frac{\delta^2 \Gamma^{A,A_0}}{\delta \tilde{\varphi}(-k) \delta \tilde{\varphi}(q)} + \frac{\delta^2 \Gamma^{A,A_0}}{\delta \tilde{\varphi}(p) \delta \tilde{\varphi}(q)}, 
\]

(3.40)

\[
\tilde{\varphi}(q) = (2\pi)^4 C^{A,A_0}(q) \frac{\delta \Gamma^{A,A_0}}{\delta \tilde{\varphi}(-q)}. 
\]

(3.41)

In the tree order, \( L^{A,A_0}(\varphi) \) contains no 2-point function, (2.38). Hence, setting in (3.40) \( \varphi = \tilde{\varphi} = 0 \) yields

\[
\frac{\delta^2 \Gamma^{A,A_0}(\varphi)}{\delta \tilde{\varphi}(p) \delta \tilde{\varphi}(q)} \bigg|_{\varphi=0} = (2\pi)^{-4} \frac{\delta(p+q)}{C^{A,A_0}(p)}. 
\]

(3.42)

To deduce the flow equation for the vertex functional, we derive equation (3.33) with respect to the flow parameter \( \Lambda \),

\[
(\partial_\Lambda \Gamma^{A,A_0})(\varphi) + \int dy \, \frac{\delta \Gamma^{A,A_0}}{\delta \varphi(y)} \partial_\Lambda \varphi(y) = - \partial_\Lambda W^{A,A_0}(J) + \int dy \, J(y) \partial_\Lambda \varphi(y). 
\]

Hence, because of (3.34),

\[
(\partial_\Lambda \Gamma^{A,A_0})(\varphi) + \partial_\Lambda W^{A,A_0}(J) = 0. 
\]

(3.43)
Substituting $W^{A,A_0}$ by $L^{A,A_0}$ according to (3.36) then yields

$$
(\partial_\Lambda \Gamma^{A,A_0})(\varphi) - (\partial_\Lambda L^{A,A_0})(\varphi) - \int dz \int dy \frac{\delta L^{A,A_0}}{\delta \varphi(y)} \hat{C}^{A,A_0}(y-z)J(z) \\
+ \frac{1}{2} \langle J, \hat{C}^{A,A_0}J \rangle = 0 ,
$$

(3.44)

$\hat{C}^{A,A_0}$ denoting the derivative of the covariance $C^{A,A_0}$ with respect to $\Lambda$.

There is an alternative way to arrive at equation (3.44), starting from (3.33) but treating $\varphi$ as $\Lambda$-independent and thus $J$ to depend on $\Lambda$ according to (3.31). Deriving (3.33) with respect to $\Lambda$ then reads

$$
\partial_\Lambda \Gamma^{A,A_0}(\varphi) = \left[ - (\partial_\Lambda W^{A,A_0})(J) - \int dy \frac{\delta W^{A,A_0}}{\delta J(y)} \partial_\Lambda J(y) \right. \\
+ \left. \int dy \partial_\Lambda J(y) \varphi(y) \right]_{J=\hat{J}(\varphi)} = - (\partial_\Lambda W^{A,A_0})(J) \big|_{J=\hat{J}(\varphi)} .
$$

(3.45)

The substitution of $W^{A,A_0}$ by $L^{A,A_0}$ due to (3.36)-(3.37) again provides (3.44). Given (3.44) the flow equation (2.21) with its vacuum part subtracted can be taken into account, leading to

$$
(\partial_\Lambda \Gamma^{A,A_0})(\varphi) + \frac{1}{2} \left( \frac{\delta L^{A,A_0}}{\delta \varphi} - J, \hat{C}^{A,A_0} \left( \frac{\delta L^{A,A_0}}{\delta \varphi} - J \right) \right) \\
= \frac{\hbar}{2} \left( \left[ \frac{\delta}{\delta \varphi}, \hat{C}^{A,A_0} \frac{\delta}{\delta \varphi} \right] L^{A,A_0}(\varphi) - \left. \left( \frac{\delta}{\delta \varphi}, \hat{C}^{A,A_0} \frac{\delta}{\delta \varphi} \right) L^{A,A_0}(\varphi) \right|_{\varphi=0} \right) .
$$

In the second term on the l.h.s. we use the relation

$$
\int dx \left( C^{A,A_0} \right)^{-1}(z-x) \varphi(x) = - \frac{\delta L^{A,A_0}}{\delta \varphi(z)} + J(z) ,
$$

(3.46)

resulting from (3.31) together with (3.39), and note $(\partial_\Lambda C)^{-1} + C \partial_\Lambda C^{-1} = 0$. Hence, the flow equation for the vertex functional turns out as

$$
(\partial_\Lambda \Gamma^{A,A_0})(\varphi) - \frac{1}{2} \langle \varphi, (\partial_\Lambda (C^{A,A_0})^{-1}) \varphi \rangle = \frac{\hbar}{2} \left[ \left( \frac{\delta}{\delta \varphi}, \partial_\Lambda C^{A,A_0} \frac{\delta}{\delta \varphi} \right) \hat{\Gamma}^{A,A_0}(\varphi) \right] ,
$$

(3.47)

with the r.h.s. defined by

$$
\frac{\delta^2 \hat{\Gamma}^{A,A_0}(\varphi)}{\delta \varphi(x) \delta \varphi(y)} := \frac{\delta^2 L^{A,A_0}(\varphi)}{\delta \varphi(x) \delta \varphi(y)} \bigg|_{\varphi=C^{A,A_0},J(\varphi)} - \frac{\delta^2 L^{A,A_0}(\varphi)}{\delta \varphi(x) \delta \varphi(y)} \bigg|_{\varphi=0} .
$$

(3.48)
Looking at this definition of the functional $\tilde{\Gamma}^{A,A_0}(\varphi)$ we notice, that its 2-point function vanishes, and furthermore, that its higher $n$-point functions, $n = 4, 6, 8, \ldots$, emerge from the first term on the r.h.s.. These latter are recursively determined by the functional equations (3.38) or (3.40) (which could also be obtained via (3.46) and (3.34)), by performing successively two, four, six, \ldots functional derivations with respect to $\varphi$. The r.h.s. of the flow equation (3.47) can also be given another form, expressing (3.48) first in terms of the functional $W^{A,A_0}(J)$ by way of (3.36), (3.37) and then using the functional relation (3.35),

$$\frac{\hbar}{2} \left( \frac{\delta}{\delta \varphi}, (\partial_A C^{A,A_0}) \frac{\delta}{\delta \varphi} \right) \tilde{\Gamma}^{A,A_0}(\varphi) = \frac{\hbar}{2} \int dx \int dy \partial_A (C^{A,A_0})^{-1}(y - x) \left( \frac{\delta^2 W^{A,A_0}(J)}{\delta J(x) \delta J(y)} - \frac{\delta^2 W^{A,A_0}(J)}{\delta J(x) \delta J(y)} \bigg|_{J = 0} \right)$$

This form is (also) met in the literature [38, 35], and the flow equation (3.47) called there “exact renormalization group”. Similar to (2.25) regarding the functional $L^{A,A_0}(\varphi)$ we define the $n$-point functions, $n \in 2\mathbb{N}$, of the functional $\Gamma^{A,A_0}(\varphi)$ in momentum space as

$$(2\pi)^{4(n-1)} \frac{\delta}{\delta \varphi(p_1)} \cdots \frac{\delta}{\delta \varphi(p_n)} \Gamma^{A,A_0}(\varphi) \bigg|_{\varphi = 0} = \delta(p_1 + \cdots + p_n) \Gamma_n^{A,A_0}(p_1, \ldots, p_n),$$

and analogously in the case of the functional $\tilde{\Gamma}^{A,A_0}(\varphi)$. Performing in addition a respective loop expansion, $n \in 2\mathbb{N}$,

$$\Gamma_n^{A,A_0}(p_1, \ldots, p_n) = \sum_{l=0}^{\infty} \hbar^l \Gamma_{l,n}^{A,A_0}(p_1, \ldots, p_n),$$

and for the functions $\tilde{\Gamma}_n^{A,A_0}(p_1, \ldots, p_n)$ alike, the flow equation (3.47) is finally converted into the system of flow equations, satisfied by the $n$-point functions, $n \in 2\mathbb{N}, l \in \mathbb{N}$,

$$\partial_A \Gamma_{l,n}^{A,A_0}(p_1, \ldots, p_n) = \frac{1}{2} \int_k \partial_A C^{A,A_0}(k) \cdot \tilde{\Gamma}_{l-1,n+2}^{A,A_0}(k, -k, p_1, \ldots, p_n).$$

In contrast to the system of flow equations (2.31) satisfied by the amputated truncated Schwinger functions, here the r.h.s. is in total of lower loop order,
but there is no closed form for it. As explained before, it has to be determined recursively via (3.40), treated in a loop expansion and using (3.42). It then emerges in the form

\[
\tilde{\Gamma}^{\Lambda, \Lambda_0}_{l, n+2}(k, -k, p_1, \ldots, p_n) = \Gamma^{\Lambda, \Lambda_0}_{l, n+2}(k, -k, p_1, \ldots, p_n)
\]

\[
- \sum_{r \geq 2} \sum_{\{n_i\}, \{l_i\}} \sigma \left[ \Gamma^{\Lambda, \Lambda_0}_{l_1, n_1+1}(\ldots) \right] C^{\Lambda, \Lambda_0} \Gamma^{\Lambda, \Lambda_0}_{l_2, n_2+2} \ldots \\
\ldots \sum_{r \geq 2} \sum_{\{n_i\}, \{l_i\}} \sigma \left[ \Gamma^{\Lambda, \Lambda_0}_{l_{r-1}, n_{r-1}+2}(\ldots) \right] C^{\Lambda, \Lambda_0} \Gamma^{\Lambda, \Lambda_0}_{l_r, n_r+1}(-k, \ldots). \tag{3.53}
\]

The prime restricts summation to \(l_1 + l_2 + \cdots + l_r = l\) and \(n_1 + n_2 + \cdots + n_r = n + 2\), in addition, 2-point functions in the tree order are excluded as factors. The momentum assignment has been suppressed, it goes without saying that the sum inherits from the l.h.s. the complete symmetry in the momenta \(p_1, \ldots, p_n\). Moreover, there is a sign factor \(\sigma\) depending on \(\{n_i\}\) and \(\{l_i\}\). The form of (3.53) is easily understood when represented by Feynman diagrams: To the first term (on the r.h.s.) correspond 1PI - diagrams, whereas to the sum correspond chains of 1PI - diagrams, minimally connected by single lines and thus not of 1PI - type. These chains are closed to 1PI - diagrams by the contraction involved in the flow equation.

The system of flow equations (3.52) can alternatively be employed to prove the renormalizability of the theory considered. In the tree order, only the 2-point function (3.42) and the 4-point function \(\Gamma^{\Lambda, \Lambda_0}_{0, 4}(p_1, \ldots, p_4) = g\) are different from zero. The latter is easily obtained via (3.40, 3.42) from (2.40) observing \(f = 0\) there. In each loop order \(l \geq 1\), the three counterterms

\[
\Gamma_{l, 2}^{\Lambda, \Lambda_0}(p, -p) = a_{l}(\Lambda_0) + z_{l}(\Lambda_0) p^2, \quad \Gamma_{l, 4}^{\Lambda, \Lambda_0}(p_1, \ldots, p_4) = c_{l}(\Lambda_0) \tag{3.54}
\]

form the respective bare action, determined in the end by the renormalization conditions, \(l \geq 1\),

\[
\Gamma_{l, 2}^{0, \Lambda_0}(p, -p) = \xi_{l}^{R} + \xi_{l}^{R} p^2 + \mathcal{O}((p^2)^2), \quad \Gamma_{l, 4}^{0, \Lambda_0}(0, 0, 0, 0) = \xi_{l}^{R} \tag{3.55}
\]

The renormalization constants \(a_{l}, z_{l}, c_{l}\) can be freely chosen. To prove renormalizability, we also have to make use of momentum derivatives of the flow equations (3.52), i.e. acting on them with \(\partial^w\), (2.30). Then, the proof by induction follows step-by-step the proof given in section 2.3 considering amputated truncated Schwinger functions. It will therefore not be repeated. As result, the analogue of the Propositions 2.1 and 2.2 is established, where the function \(\xi_{l, n}^{\Lambda, \Lambda_0}\) appearing there is now replaced by the function \(\Gamma_{l, n}^{\Lambda, \Lambda_0}\)
Chapter 4

Spontaneously Broken SU(2) Yang-Mills Theory

Attempting to prove renormalizability of a non-Abelian gauge theory via flow equations, following the path taken before in the case of a scalar field theory, one finds oneself confronted with a serious obstacle to be surmounted. By their definition the Schwinger functions of a non-Abelian gauge theory are not gauge invariant individually, the local gauge invariance of the theory, however, compels them to satisfy the system of Slavnov-Taylor identities \[52\]. These identities are inevitably violated, if one employs in the intermediate regularization procedure a momentum cutoff. Moreover, the Slavnov-Taylor identities are generated by nonlinear transformations of the fields – the BRS-transformations \[53\] – which, being composite fields, have to be renormalized, too. In the sequel we follow the general line of \[55\], hereafter referred to as “II”. For the sake of readability, however, a coherent detailed argumentation is kept up. As concerns a number of proofs and technical derivations, we refer to the original article. After presenting in Section 4.1 the classical action of the theory considered, as a first step we disregard in Section 4.2 the Slavnov-Taylor identities and establish for an arbitrary set of renormalization conditions at a physical renormalization point a finite UV-behaviour of the Schwinger functions without and with the insertion of one BRS-variation. The procedure is essentially the same as in the case of the scalar theory. Having thus established a family of finite theories, in Section 4.3 the violation of the Slavnov-Taylor identities of the amputated truncated Schwinger functions at the physical value \(\Lambda = 0\) of the flow parameter and fixed \(\Lambda_0\) is worked out, as well as the BRS-variation of the bare action. Moreover, the corresponding violated Slavnov-Taylor identities in terms of proper vertex functions are also deduced, as this formulation appears more accessible to analysis. Inspection of the relevant part of these identities reveals, that in
this part irrelevant contributions from the vertex functions with insertion of a BRS-variation appear. To overcome this obstruction, in Section 4.4 a particular mass expansion is developed. It results from scaling the super-renormalizable 3-point couplings appearing at tree level, i.e. in the classical action, and traces the effect of these couplings in the perturbative expansion. On account of the corresponding adapted renormalization scheme in the relevant part of the violated Slavnov-Taylor identities now no longer irrelevant terms contribute, but solely renormalization constants. In Section 4.5 the equation of motion of the antighost is considered. Finally, in Section 4.6 the restoration of the Slavnov-Taylor identities is dealt with. Requiring the relevant part of the violated Slavnov-Taylor identities to vanish amounts to satisfy 53 equations containing the 37+7 relevant parameters of the respective generating functionals of proper vertex functions without and with insertion of a BRS-variation. Taking also into account the field equation of the antighost, it is shown that this can be accomplished extracting a particular set of 7 relevant parameters as renormalization constants to be chosen freely, in terms of which the remaining ones are uniquely determined in satisfying these 53 equations. Given a vanishing relevant part of the violated Slavnov-Taylor identities then implies that their irrelevant part has a vanishing UV-limit, thus achieving the Slavnov-Taylor identities in this limit.

4.1 The classical action

We begin collecting some basic properties of the classical Euclidean SU(2) Yang-Mills-Higgs model on four-dimensional Euclidean space-time, following closely the monograph of Faddeev and Slavnov [54]. This model involves the real Yang-Mills field \( \{ A_\mu^a \} \) and the complex scalar doublet \( \{ \phi_\alpha \} \) assumed to be smooth functions which fall-off rapidly. The classical action has the form

\[
S_{\text{inv}} = \int dx \left\{ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} \left( \nabla_\mu \phi \right)^* \nabla_\mu \phi + \lambda \left( \phi^* \phi - \rho^2 \right)^2 \right\}, \tag{4.1}
\]

with the curvature tensor

\[
F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g \epsilon^{abc} A_\mu^b(x) A_\nu^c(x) \tag{4.2}
\]

and the covariant derivative

\[
\nabla_\mu = \partial_\mu + \frac{1}{2} \sigma^a A_\mu^a(x) \tag{4.3}
\]

acting on the SU(2)-spinor \( \phi \). The parameters \( g, \lambda, \rho \) are real positive, \( \epsilon^{abc} \) is totally skew symmetric, \( \epsilon^{123} = +1 \), and \( \{ \sigma^a \} \) are the standard Pauli
matrices. For simplicity the wave function normalizations of the fields are chosen equal to one. The action (4.1) is invariant under local gauge transformations of the fields

$$\frac{1}{2i} \sigma^a A^a_\mu(x) \rightarrow u(x) \frac{1}{2i} \sigma^a A^a_\mu(x) u^*(x) + g^{-1} u(x) \partial_\mu u^*(x),$$

$$\phi(x) \rightarrow u(x) \phi(x),$$

(4.4)

with \( u : \mathbf{R}^4 \rightarrow \text{SU}(2) \), smooth. A stable ground state of the action (4.1) implies spontaneous symmetry breaking, taken into account by reparametrizing the complex scalar doublet as

$$\phi(x) = \begin{pmatrix} B^0(x) + iB^1(x) \\ \rho(x) + h(x) - iB^3(x) \end{pmatrix},$$

(4.5)

where \( \{ B^a(x) \}_{a=1,2,3} \) is a real triplet and \( h(x) \) the real Higgs field. Moreover, in place of the parameters \( \rho, \lambda \) the masses

$$m = \frac{1}{2} g \rho, \quad M = \left( 8 \lambda \rho^2 \right)^{\frac{1}{2}}$$

(4.6)

are used. Since we aim at a quantized theory pure gauge degrees of freedom have to be eliminated. We choose the ’t Hooft gauge fixing

$$S_{g.f.} = \frac{1}{2\alpha} \int dx (\partial_\mu A^a_\mu - \alpha m B^a)^2,$$

(4.7)

with \( \alpha \in \mathbf{R}_+ \), implying complete spontaneous symmetry breaking. With regard to functional integration this condition is implemented by introducing anticommuting Faddeev-Popov ghost and antighost fields \( \{ c_a \}_{a=1,2,3} \) and \( \{ \bar{c}_a \}_{a=1,2,3} \), respectively, and forming with these six independent scalar fields the additional interaction term

$$S_{gh} = - \int dx c^a \left\{ (-\partial_\mu \partial_\mu + \alpha m^2) \delta^{ab} + \frac{1}{2} \alpha gm h \delta^{ab} \\
+ \frac{1}{2} \alpha gm \epsilon^{abc} B^c - g \partial_\mu \epsilon^{abc} A^c_\mu \right\} \bar{c}^b. $$

(4.8)

Hence, the total “classical action” is

$$S_{BRS} = S_{\text{inv}} + S_{g.f.} + S_{gh},$$

(4.9)

---

\[1\] The general \( \alpha \)-gauge would lead to mixed propagators, in the Lorentz gauge the fields \( \{ B^a \} \) would be massless.
which we decompose as

$$S_{\text{BRS}} = \int dx \{ \mathcal{L}_{\text{quad}}(x) + \mathcal{L}_{\text{int}}(x) \} \quad (4.10)$$

into its quadratic part, with $\Delta \equiv \partial_{\mu} \partial_{\mu}$,

$$\mathcal{L}_{\text{quad}} = \frac{1}{4} (\partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a)^2 + \frac{1}{2\alpha} (\partial_{\mu} A_{\mu}^a)^2 + \frac{1}{2} m^2 A_{\mu}^a A_{\mu}^a$$

$$+ \frac{1}{2} h(-\Delta + M^2)h + \frac{1}{2} B^a(-\Delta + \alpha m^2)B^a$$

$$- \bar{c}^a(-\Delta + \alpha m^2)c^a, \quad (4.11)$$

and into its interaction part

$$\mathcal{L}_{\text{int}} = g\epsilon^{abc}(\partial_{\mu} A_{\nu}^a)A_{\mu}^bA_{\nu}^c + \frac{1}{4} g^2 (\epsilon^{abc}A_{\mu}^bA_{\mu}^c)^2$$

$$+ \frac{1}{2} g \left\{ (\partial_{\mu} h)A_{\nu}^a B^a - h A_{\nu}^a \partial_{\mu} B^a - \epsilon^{abc}A_{\mu}^a(\partial_{\mu} B^b)B^c \right\}$$

$$+ \frac{1}{8} g A_{\mu}^a A_{\mu}^a \left\{ 4m h + g(h^2 + B^a B^a) \right\}$$

$$+ \frac{1}{4} g M^2 h(h^2 + B^a B^a) + \frac{1}{32} g^2 \left( \frac{M}{m} \right)^2 (h^2 + B^a B^a)^2$$

$$- \frac{1}{2} \alpha g m c^a \left\{ h \delta^{ab} + \epsilon^{acb} B^c \right\} c^b$$

$$- g\epsilon^{abc}(\partial_{\mu} c^a)A_{\mu}^c c^b. \quad (4.12)$$

In (4.11) we recognize the important properties that all fields are massive and that no coupling term $A_{\mu}^a \partial_{\mu} B^a$ appears.

The classical action $S_{\text{BRS}}, (4.10)$, shows the following symmetries:

i) Euclidean invariance: $S_{\text{BRS}}$ is an $O(4)$-scalar.

ii) Rigid SO(3)-isosymmetry: The fields $\{A_{\mu}^a\}, \{B^a\}, \{c^a\}, \{\bar{c}^a\}$ are isovectors and $h$ an isoscalar; $S_{\text{BRS}}$ is invariant under spacetime independent SO(3)-transformations.

iii) BRS-invariance: Introducing the classical composite fields

$$\psi_{\mu}^a(x) = \left\{ \partial_{\mu} \delta^a + g \epsilon^{arb}A_{\mu}^r(x) \right\} c^b(x),$$

$$\psi(x) = -\frac{1}{2} g B^a(x)c^a(x),$$

$$\psi^a(x) = \left\{ (m + \frac{1}{2}gh(x))\delta^{ab} + \frac{1}{2} g \epsilon^{arb}B^r(x) \right\} c^b(x),$$

$$\Omega^a(x) = \frac{1}{2} g \epsilon^{apq}c^p(x)c^q(x), \quad (4.13)$$
the BRS-transformations of the basic fields are defined as

\[
\begin{align*}
A^a_\mu(x) &\rightarrow A^a_\mu(x) - \psi^a_\mu(x)\epsilon, \\
h(x) &\rightarrow h(x) - \psi(x)\epsilon, \\
B^a(x) &\rightarrow B^a(x) - \psi^a(x)\epsilon, \\
c^a(x) &\rightarrow c^a(x) - \Omega^a(x)\epsilon, \\
\bar{c}^a(x) &\rightarrow \bar{c}^a(x) - \frac{1}{\alpha}(\partial_\nu A^a_\nu(x) - \alpha m B^a(x))\epsilon.
\end{align*}
\] (4.14)

In these transformations \(\epsilon\) is a spacetime independent Grassmann element that commutes with the fields \(\{A^a_\mu, h, B^a\}\) but anticommutes with the (anti-) ghosts \(\{c^a, \bar{c}^a\}\). To show the BRS-invariance of the total classical action (4.9) one first observes that the composite classical fields (4.13) are themselves invariant under the BRS-transformations (4.14). Moreover, we can write (4.8) in the form

\[
S_{gh} = -\int dx \bar{c}^a \{-\partial_\mu \psi^a_\mu + \alpha m \psi^a\}. 
\] (4.15)

Using these properties the BRS-invariance of the classical action (4.9) follows upon direct verification.

It is convenient to add to the classical action (4.9) source terms both for the fields and the composite fields introduced, defining the extended action

\[
S_c = S_{\text{BRS}} + \int dx \{\gamma^a_{\mu} \psi^a_\mu + \gamma \psi + \gamma^a \psi^a + \omega^a \Omega^a\} \\
- \int dx \{j^a_\mu A^a_\mu + sh + b^a B^a + \bar{\eta}^a c^a + \bar{c}^a \eta^a\}. 
\] (4.16)

The sources \(\gamma^a_{\mu}, \gamma, \gamma^a\) all have canonical dimension 2, ghost number -1 and are Grassmann elements, whereas \(\omega^a\) has canonical dimension 2 and ghost number -2; the sources \(\eta^a\) and \(\bar{\eta}^a\) have ghost number +1 and -1, respectively, and are Grassmann elements. Employing the BRS-operator \(\mathcal{D}\), defined by

\[
\mathcal{D} = \int dx \left\{j^a_\mu \delta \gamma^a_{\mu} + s \frac{\delta}{\delta \gamma} + b^a \frac{\delta}{\delta \gamma^a} + \bar{\eta}^a \frac{\delta}{\delta \omega^a} + \eta^a \left(\frac{1}{\alpha} \partial_\nu \frac{\delta}{\delta j^a_\nu} - m \frac{\delta}{\delta b^a}\right)\right\}, 
\] (4.17)

the BRS-transformation of the extended action \(S_c\), (4.16), can be written as

\[
S_c \rightarrow S_c + \mathcal{D} S_c \epsilon 
\] (4.18)

Of course, \(\epsilon\) also anticommutes with the sources of Grassmannian type.
4.2 Flow equations: renormalizability
without Slavnov-Taylor identities

In view of the various fields present, it is convenient to introduce a short collective notation for the fields and sources. We denote:
i) the bosonic fields and the corresponding sources, respectively, by
\[ \varphi_\tau = (A_\mu^a, h, B_a) \quad J_\tau = (j_\mu^a, s, b^a), \] (4.19)

ii) all fields and their respective sources by
\[ \Phi = (\varphi_\tau, c^a, \bar{c}^a), \quad K = (J_\tau, \bar{\eta}^a, \eta^a), \] (4.20)

iii) the insertions and their sources
\[ \psi_\tau = (\psi_\mu^a, \psi, \psi^a), \quad \gamma_\tau = (\gamma_\mu^a, \gamma, \gamma^a), \quad \xi = (\gamma_\tau, \omega^a). \] (4.21)

The quadratic part of \( S_{\text{BRS}} \), (4.10), defines the inverses of the various unregularized free propagators. We start from the theory defined on finite volume, as described in Section 2.1. With the notation introduced there, we have
\[
\int dx L \quad \equiv \quad Q(\Phi) = \frac{1}{2} \langle A_\mu^a, (C^{-1})_{\mu\nu} A_\nu^a \rangle + \frac{1}{2} \langle h, C^{-1} h \rangle + \frac{1}{2} \langle B^a, S^{-1} B^a \rangle - \langle \bar{c}^a, S^{-1} c^a \rangle,
\] (4.22)

where the Fourier transforms of these free propagators, (compare (2.7) with \( \Lambda = 0, \Lambda_0 = \infty \)) turn out to be
\[
C(k) = \frac{1}{k^2 + M^2}, \quad S(k) = \frac{1}{k^2 + \alpha m^2}, \\
C_{\mu\nu}(k) = \frac{1}{k^2 + m^2} \left( \delta_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 + \alpha m^2} \right).
\] (4.23)

Again the notation has been abused omitting the “hat”. Furthermore, we shall use \( C(k) \) as a collective symbol for these propagators. A Gaussian product measure, the covariances of which are a regularized version of the propagators (4.23), forms the basis to quantize the theory by functional integration. Although gauge symmetry is violated by any momentum cutoff one should try to reduce the bothersome consequences as far as possible. Instead of the simple form (2.10) we choose the cutoff function
\[
\sigma_\Lambda(k^2) = \exp \left( -\frac{(k^2 + m^2)(k^2 + \alpha m^2)(k^2 + M^2)(k^2 + 1)^2}{\Lambda^{10}} \right).
\] (4.24)
It is positive, invertible and analytic as the former, but satisfies in addition
\[
\frac{d}{dk^2} \sigma_\Lambda(k^2) |_{k^2=0} = 0. \quad (4.25)
\]
This property is the raison d’être for the particular choice (4.24), turning out helpful in the later analysis of the relevant part of the STI. Employing this cutoff function we define the regularized propagators,
\[
0 \leq \Lambda \leq \Lambda_0 < \infty,
\]

\[
C^{\Lambda,\Lambda_0}(k) \equiv C(k)\sigma_{\Lambda,\Lambda_0}(k^2) := C(k) (\sigma_{\Lambda_0}(k^2) - \sigma_\Lambda(k^2)). \quad (4.26)
\]

They satisfy the bounds
\[
\left| \prod_{i=1}^{\left|\mathcal{W}\right|} \frac{\partial}{\partial k_{\mu_i}} \right| \partial_\Lambda C^{\Lambda,\Lambda_0}(k) \leq \begin{cases} 
  c_{\left|\mathcal{W}\right|} \sigma_2 \Lambda(k^2) & \text{for} \quad 0 \leq \Lambda \leq m, \\
  \Lambda^{-3-\left|\mathcal{W}\right|} P_{\left|\mathcal{W}\right|}(\frac{|k|}{\Lambda}) \sigma_\Lambda(k^2) & \text{for} \quad \Lambda > m.
\end{cases} \quad (4.27)
\]
with polynomials $P_{\left|\mathcal{W}\right|}$ having positive coefficients. These coefficients, as well as the constants $c_{\left|\mathcal{W}\right|}$, only depend on $\alpha, m, M, \left|\mathcal{W}\right|$. Considering $\sigma_\Lambda(k^2)$, (4.24), as a function of $(\Lambda, k^2)$, it cannot be extended continuously to $(0, 0)$. We set $\sigma_0(0) := \lim_{k^2 \to 0} \sigma_0(k^2) = 0$, and hence $\sigma_{\Lambda_0}(0) = \sigma_\Lambda(0) = 1$.

Writing
\[
\langle \Phi, K \rangle := \int dx \left( \sum_\tau \varphi_\tau(x) J_\tau(x) + \bar{c}^a(x) \eta^a(x) + \bar{\eta}^a(x) c^a(x) \right), \quad (4.28)
\]
the characteristic functional of the Gaussian product measure with covariances $\hbar C^{\Lambda,\Lambda_0}$, (4.26), (4.23), is given by
\[
\int d\mu_{\Lambda,\Lambda_0}(\Phi) e^{\frac{i}{\hbar} \langle \Phi, K \rangle} = e^{\frac{i}{\hbar} P(K)}, \quad (4.29)
\]

\[
P(K) = \frac{1}{2} \langle j^a_\mu, C^{\Lambda,\Lambda_0} j^a_\nu \rangle + \frac{1}{2} \langle s, C^{\Lambda,\Lambda_0} s \rangle + \frac{1}{2} \langle b^a, S^{\Lambda,\Lambda_0} b^a \rangle - \langle \bar{\eta}^a, S^{\Lambda,\Lambda_0} \eta^a \rangle. \quad (4.30)
\]
The free propagators (4.23) reveal the mass dimensions of the corresponding quantum fields: each of the fields has a mass dimension equal to one, attributing equal values to the ghost and antighost field.

To promote the classical model to a quantum field theory we consider the generating functional $L^{\Lambda,\Lambda_0}(\Phi)$ of the amputated truncated Schwinger functions. It unfolds according to the integrated flow equation, cf. (2.23),
\[
e^{-\frac{i}{\hbar} \langle L^{\Lambda,\Lambda_0}(\Phi) + I^{\Lambda,\Lambda_0} \rangle} = e^{\hbar \Delta_{\Lambda,\Lambda_0}} e^{-\frac{i}{\hbar} L^{\Lambda_0,\Lambda_0}(\Phi)} \quad (4.31)
\]

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from the bare functional \( L_{\Lambda_0}(\Phi) \), which forms its initial value at \( \Lambda = \Lambda_0 \). The functional Laplace operator appearing has the form

\[
\Delta_{\Lambda, \Lambda_0} = \frac{1}{2} \left( \frac{\delta}{\delta A_\mu} C^{\Lambda_0}_{\mu\nu} \frac{\delta}{\delta A_\nu} \right) + \frac{1}{2} \left( \frac{\delta}{\delta h} C^{\Lambda_0} \frac{\delta}{\delta h} \right) + \frac{1}{2} \left( \frac{\delta}{\delta B^a} S^{\Lambda_0} \frac{\delta}{\delta B^a} \right) + \left( \frac{\delta}{\delta c^a} S^{\Lambda_0} \frac{\delta}{\delta c^a} \right). \tag{4.32}
\]

Since the local gauge symmetry is violated by the regularization, the bare functional

\[
L_{\Lambda_0}(\Phi) = \int dx L_{\text{int}}(x) + L_{c.t.}(\Phi) \tag{4.33}
\]

has at first to be chosen sufficiently general in order to allow the restoration of the Slavnov-Taylor identities at the end. Therefore, we add as counterterms to the given interaction part \((4.12)\) of classical origin all local terms of mass dimension \( \leq 4 \), which are permitted by the unbroken global symmetries, i.e. Euclidean \( O(4) \)- invariance and \( SO(3) \)- isosymmetry. There are 37 such terms, by definition all of order \( O(\bar{h}) \). The bare functional is presented in Appendix A.

The decomposition of the generating functional \( L^{\Lambda, \Lambda_0}(\Phi) \) is written employing a multiindex \( n \), the components of which denote the number of each source field species appearing:

\[
n = (n_A, n_h, n_B, n_c, n_e), \quad |n| = n_A + n_h + n_B + n_c + n_e. \tag{4.34}
\]

Moreover, we consider the functional within a formal loop expansion, hence

\[
L^{\Lambda, \Lambda_0}(\Phi) = \sum_{|n|=3}^{\infty} L_{l=0,n}(\Phi) + \sum_{l=1}^{\infty} h^l \sum_{|n|=1}^{\infty} L_{l,n}(\Phi). \tag{4.35}
\]

Disregarding the vacuum part, we can study the flow of the \( n \)-point functions in the infinite volume limit \( \Omega \to \mathbb{R}^4, \Phi \in \mathcal{S}(\mathbb{R}^4) \). With our conventions \((2.24)\) of the Fourier transformation, the momentum representation of the \( n \)-point function with multiindex \( n \), \((4.34)\), at loop order \( l \) is obtained as an \( |n| \)-fold functional derivative

\[
(2\pi)^4 |n|^{-1} \delta_{\Phi(p)} L_{l,n}^{\Lambda, \Lambda_0}(p_1, \cdots, p_{|n|}). \tag{4.36}
\]

To avoid clumsiness, the notation does not reveal how the momenta are assigned to the multiindex \( n \), and in addition, it suppresses the \( O(4) \)- and \( SO(3) \)- tensor structure. From the definition \((4.36)\) of the \( n \)-point function follows that it is completely symmetric (antisymmetric) upon permuting the
variables belonging to each of the bosonic (fermionic) species occurring. Proceeding exactly as in the case of the scalar field, the flow equation (4.31) is converted into a system of flow equations relating the $n$-point functions. It looks like (2.31), where $n$ is now a multiindex and the residual symmetrization has to be extended to a corresponding antisymmetrization in case of the (anti)ghost fields. The system is integrated in the familiar way. At first the tree order $l = 0$ has to be gained, fully determined by the classical descendant (4.12) appearing in the initial condition (4.33) at $\Lambda = \Lambda_0$. Given the tree order $l = 0$, the inductive integration ascends in the loop order $l$, for fixed $l$ ascends in $|n|$, and for fixed $l, n$ descends in $w$ from $|w| = 4$ to $w = 0$, with initial conditions as follows:

A1) For $|n| + |w| > 4$ at $\Lambda = \Lambda_0$,

$$\partial^\mu L^{\Lambda_0, \Lambda_0}(p_1, \cdots, p_{|n|}) = 0,$$

(4.37)
due to the choice of the bare functional (4.33).

A2) For the (relevant) cases $|n| + |w| \leq 4$, renormalization conditions at the physical value $\Lambda = 0$ and a chosen renormalization point are freely prescribed order-by-order, subject only to the unbroken $O(4)$- and $SO(3)$-symmetries. These conditions determine the 37 local counterterms entering the bare functional (4.33). For simplicity we choose, as before, vanishing momenta as renormalization point.

Repeating exactly the steps that in the case of the scalar field led to the Propositions 2.1 and 2.2, one establishes in the present case analogous bounds, just reading now $n$ as a multiindex. As a consequence of these bounds a finite theory results in the limit $\Lambda_0 \to \infty$, however, not yet the gauge theory looked for! The problem still to be solved is to select renormalization conditions $A_2$) such that the $n$-point functions in the limit $\Lambda = 0, \Lambda_0 \to \infty$ satisfy the Slavnov-Taylor identities.

As worked out in the next section, to establish the Slavnov-Taylor identities necessitates to consider Schwinger functions with a composite field inserted. There will appear two kinds of such insertions: the composite BRS-fields forming local insertions, and a space-time integrated insertion describing the intermediate violation of the Slavnov-Taylor identities. The classical composite BRS-fields (4.13) all have mass dimension 2 and transform as vector-isovector, scalar-isoscalar, scalar-isovector and scalar-isovector, respectively. Moreover, the first three have ghost number 1, whereas the last one has ghost number 2. Thus, adding counterterms according to
the rules formulated in Section 2.4, we introduce the bare composite fields

\[ \psi^a_{\mu}(x) = R_1^0 \partial_{\mu} e^0(x) + R_2^0 g e^{arb} A^r_{\mu}(x)c^b(x), \]
\[ \psi(x) = -R_3^0 g \frac{1}{2} B^a(x) e^0(x), \]
\[ \psi^a(x) = R_4^0 m e^a(x) + R_5^0 g \frac{1}{2} h(x) c^a(x) + R_6^0 \frac{g}{2} e^{arb} (x)c^b(x), \]
\[ \Omega^a(x) = R_7^0 \frac{g}{2} e^{a|p} c^0(x)c^q(x), \]

keeping the notation introduced for the classical terms and using it henceforth exclusively according to \(4.38\). We set

\[ R_i^0 = 1 + O(\bar{h}), \]

thus viewing the counterterms again as formal power series in \(\bar{h}\); the tree order \(\bar{h}_0\) provides the classical terms \(4.13\). The reader notices that there is no insertion attributed to the linear variation of the antighost field. It will be seen that the Slavnov-Taylor identities can be established generating this variation by functional derivation with respect to the sources of the fields involved. We shall have to deal with Schwinger functions with one insertion. Similarly as in Section 2.4, the bare interaction \(4.33\) is modified adding the composite fields \(4.38\) coupled to corresponding sources, introduced in \(4.16\),

\[ \mathcal{L}_{\Lambda_0, \Lambda^0} := L^{\Lambda_0, \Lambda^0} + L^{\Lambda_0, \Lambda^0}(\xi), \]

\[ L^{\Lambda_0, \Lambda^0}(\xi) = \int dx \{ \gamma^a_{\mu}(x)\psi^a_{\mu}(x) + \gamma(x)\psi(x) + \gamma^a(x)\psi^a(x) + \omega^a(x)\Omega^a(x) \}. \]

Then, from the corresponding generating functional of the regularized amputated truncated Schwinger functions with one insertion \(\psi(x)\),

\[ L_{\gamma_0}^{\Lambda_0, \Lambda^0}(x; \Phi) := \delta \frac{\delta}{\delta \gamma(\xi)} \mathcal{L}_{\Lambda_0, \Lambda^0}\big|_{\xi=0}, \quad \hat{L}_{\gamma_0}^{\Lambda_0, \Lambda^0}(q; \Phi) = \int dx e^{iqx} L_{\gamma_0}^{\Lambda_0, \Lambda^0}(x; \Phi) \]

with analogous expressions for the other insertions, after a loop expansion, follows for the \(n\)-point functions with one insertion \(\psi\),

\[ \delta(q + p_1 + \cdots + p_{|n|}) \mathcal{L}_{\gamma_0}^{\Lambda_0, \Lambda^0}(q; p_1, \cdots, p_{|n|}) := (2\pi)^{4(|n|-1)} \delta^{n}_{\Phi(p)} \hat{L}_{\gamma_0}^{\Lambda_0, \Lambda^0}(q; \Phi)|_{\Phi=0}, \]

a system of flow equations. From each of these systems the renormalizability of the amputated truncated Schwinger functions with one insertion can be deduced inductively in the familiar way. We denote by \(\xi\) any of the labels \(\gamma^a_{\mu}, \gamma, \gamma^a, \omega^a\). First, the tree order \(l = 0\) is obtained from its initial condition
at \( \Lambda = \Lambda_0 \). For \( l \geq 1 \) the initial conditions are:

\begin{equation}
\partial^w L_{\xi; l, n}^{\Lambda_0, \Lambda_0}(q; p_1, \ldots, p_{|n|}) = 0. \tag{4.44}
\end{equation}

\( B_1 \) If \(|n| + |w| > 2\) at \( \Lambda = \Lambda_0 \),

\begin{equation}
\partial^w L_{\xi; l, n}^{\Lambda_0, \Lambda_0}(q; p_1, \ldots, p_{|n|}) = 0.
\end{equation}

\( B_2 \) If \(|n| + |w| \leq 2\) at \( \Lambda = 0 \) and at vanishing momenta (the renormalization point) the initial condition can be fixed freely in each loop order, provided the Euclidean symmetry and the isosymmetry are respected. In total, there are 7 such renormalization conditions which then determine the 7 parameters \( R_i^0 \) entering the bare insertions (4.38). Given the bounds of the case without insertion one deduces inductively the analogues of the Propositions 2.3 and 2.4, with \( n \) now a multiindex and \( D = 2 \). Hence, we have boundedness and convergence of the amputated truncated Schwinger functions with the insertion of one BRS-variation.

The intermediate violation of the Slavnov-Taylor identities, as will be derived in the following section, leads to a bare space-time integrated insertion of the form

\begin{equation}
L_{1}^{\Lambda_0, \Lambda_0}(\Phi) = \int dx N(x), \tag{4.45}
\end{equation}

\begin{equation}
N(x) = Q(x) + Q'(x; (\Lambda_0)^{-1}). \tag{4.46}
\end{equation}

Here \( Q(x) \) is a local polynomial in the fields and their derivatives, having canonical mass dimension \( D = 5 \), whereas \( Q'(x; (\Lambda_0)^{-1}) \) is nonpolynomial in the field derivatives but with powers \( (\Lambda_0)^{-1} \) as coefficients such that it becomes irrelevant. The individual terms composing \( N(x) \) involve at most five fields and have ghost number equal to one. We have to control \( L_{1}^{\Lambda_0, \Lambda_0}(\Phi) \), the \( L \)-functional with one (bare) insertion (4.45). Hence, in analogy to the local case, cf. (2.61), a modified bare action

\begin{equation}
L_{1}^{\Lambda_0, \Lambda_0}(\Phi) + \chi L_{1}^{\Lambda_0, \Lambda_0}(\Phi) \tag{4.47}
\end{equation}

is introduced as initial condition in the (integrated form of the) flow equation

\begin{equation}
e^{-\frac{1}{\hbar}(L_{1}^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0})} = e^{\bar{\chi} L_{1}^{\Lambda_0, \Lambda_0}} e^{-\frac{1}{\hbar}(L_{1}^{\Lambda_0, \Lambda_0} + \chi L_{1}^{\Lambda_0, \Lambda_0})}. \tag{4.48}
\end{equation}

Herefrom results the generating functional of the (regularized) amputated truncated Schwinger functions with one insertion (4.45) as

\begin{equation}
L_{1}^{\Lambda, \Lambda_0}(\Phi) = \frac{\partial}{\partial \chi} L_{1}^{\Lambda_0, \Lambda_0}(\Phi)|_{\chi = 0}. \tag{4.49}
\end{equation}

It satisfies a linear differential flow equation which is easily obtained relating it to the case of a bare local insertion, cf. (2.61-2.69);

\begin{equation}
\int dx \, q(x) N(x)
\end{equation}

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and observing
\[ \frac{\partial}{\partial \chi} L^\Lambda_{\chi,0}(\Phi)|_{\chi=0} = \int dx \frac{\delta}{\delta \phi(x)} \tilde{L}^\Lambda_{\chi,0}(\phi; \Phi)|_{\phi=0} = \int dx L^\Lambda_{(1)}(x; \Phi) = \tilde{L}^\Lambda_{(1)}(0; \Phi) \]

Hence, the differential flow equation satisfied by the functional \( L^\Lambda_{\chi,0}(\Phi) \) is the space-time integrated analogue of \( (2.67) \). Performing a loop expansion, the amputated truncated \( n \)-point functions with one insertion \( (4.45) \), \( n \) a multiindex \( (4.34) \),
\[ \delta(p_1 + \cdots + p_{|n|}) \mathcal{L}^\Lambda_{1:1,0}(p_1, \cdots, p_{|n|}) := \left(2\pi\right)^{|n|-1} \delta_n^{\Phi(p)} \tilde{L}^\Lambda_{1:1,0} (\Phi)|_{\phi=0} \]
then satisfy a system of flow equations similar to the case of the local BRS-insertions, letting there the momentum take the value \( q = 0 \). As a consequence, we obtain analogous bounds, but observing in the present case the dimension \( D = 5 \). The irrelevant part appearing in the bare insertion \( (4.45) - (4.46) \) satisfies the required bounds to be admitted, cf. \( (2.57) \).

4.3 Violated Slavnov-Taylor identities

The Schwinger functions of the spontaneously broken Yang-Mills theory should be uniquely determined by its free physical parameters \( g, \lambda, m \) and the gauge fixing parameter \( \alpha \), once the normalization of the fields has been fixed. This uniqueness - as well as the physical gauge invariance - is accomplished by requiring the Schwinger functions to satisfy the Slavnov-Taylor identities. These identities, however, are inevitably violated by the intermediate regularization in momentum space. Our ultimate goal is to show, that by a proper choice of the renormalization conditions the Slavnov-Taylor identities emerge upon removing the regularization. To this end we first examine the violation of the Slavnov-Taylor identities produced by the UV-cutoff \( \Lambda_0 \). Our starting point is the generating functional of the regularized Schwinger functions, here considered at the physical value \( \Lambda = 0 \) of the flow parameter,
\[ Z^{0,\Lambda_0}(K) = \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{1}{\kappa}L^{0,\Lambda_0} + \frac{1}{\kappa}(\Phi,K)}. \]
The Gaussian measure \( d\mu_{0,\Lambda_0}(\Phi) \) corresponds to the quadratic form \( \frac{1}{\kappa}Q^{0,\Lambda_0}(\Phi) \), cf. \( (4.22), (4.26) \), to wit:
\[ Q^{0,\Lambda_0}(\Phi) = \frac{1}{2} \langle A^a_\mu, (C^{0,\Lambda_0})^{-1}_{\mu \nu} A^a_\nu \rangle + \frac{1}{2} \langle h, (C^{0,\Lambda_0})^{-1} h \rangle \]
\[ + \frac{1}{2} \langle B^a, (S^{0,\Lambda_0})^{-1} B^a \rangle - \langle \bar{c}^a, (S^{0,\Lambda_0})^{-1} c^a \rangle. \]
(4.51)
Defining *regularized* BRS-variations \((4.14), (4.38)\) of the fields by
\[
\delta_{\text{BRS}} \varphi_\tau(x) = -\left(\sigma_{0,\Lambda_0} \psi_\tau\right)(x) \, \epsilon,
\delta_{\text{BRS}} \epsilon^a(x) = -\left(\sigma_{0,\Lambda_0} \Omega^a\right)(x) \, \epsilon,
\delta_{\text{BRS}} \bar{c}^a(x) = -\left(\sigma_{0,\Lambda_0} \left(\frac{1}{\alpha} \partial_\nu A_\nu^a - m B^a\right)\right)(x) \, \epsilon,
\]
the BRS-variation of the Gaussian measure follows as
\[
d\mu_{0,\Lambda_0}(\Phi) \mapsto d\mu_{0,\Lambda_0}(\Phi) \left(1 - \frac{1}{\hbar} \delta_{\text{BRS}} Q^{0,\Lambda_0}(\Phi)\right).
\]
Written more explicitly,
\[
\delta_{\text{BRS}} Q^{0,\Lambda_0}(\Phi) = \left(-\sum_\tau \langle \varphi_\tau, \left(C_\tau^{0,\Lambda_0}\right)^{-1} \sigma_{0,\Lambda_0} \psi_\tau \rangle + \langle \bar{c}^a, \left(S^{0,\Lambda_0}\right)^{-1} \sigma_{0,\Lambda_0} \Omega^a \rangle \right)
- \left(-\frac{1}{\alpha} \partial_\nu A_\nu^a - m B^a, \sigma_{0,\Lambda_0} \left(S^{0,\Lambda_0}\right)^{-1} \epsilon^a\right) \epsilon,
\]
it reveals that \(\sigma_{0,\Lambda_0}\) just cancels its inverse appearing in the inverted propagators, and as a consequence, the BRS-variation of the Gaussian measure has mass dimension \(D=5\). The essential reason for using regularized BRS-variations \((4.52)\) is to assure this property. From the requirement, that the regularized generating functional \(Z_{0,\Lambda_0}(K)\) be invariant under the BRS-variations \((4.52)\), result the violated Slavnov-Taylor identities \([2]\)
\[
0 = \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{1}{\hbar} L^{0,\Lambda_0} + \frac{1}{\pi} \langle \Phi, K \rangle} \left(\delta_{\text{BRS}} \langle \Phi, K \rangle - \delta_{\text{BRS}} \left(Q^{0,\Lambda_0} + L^{0,\Lambda_0}\right)\right).
\]
This equation can be rewritten, introducing modified generating functionals: i) With the modified bare interaction \((4.40)\) we define
\[
\tilde{Z}^{0,\Lambda_0}(K, \xi) := \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{1}{\hbar} L^{0,\Lambda_0} + \frac{1}{\pi} \langle \Phi, K \rangle},
\]
in combination with a regularized version of the BRS-operator \((4.17)\),
\[
D_{\Lambda_0} = \sum_\tau \langle J_\tau, \sigma_{0,\Lambda_0} \frac{\delta}{\delta \gamma_\tau} \rangle + \langle \bar{\eta}^a, \sigma_{0,\Lambda_0} \frac{\delta}{\delta \omega^a} \rangle + \langle \frac{1}{\alpha} \partial_\nu \frac{\delta}{\delta j_\nu^a} - m B^a, \sigma_{0,\Lambda_0} \eta^a \rangle.
\]
ii) In addition, we treat the BRS-variation of the bare action,
\[
L^{0,\Lambda_0}_1 \epsilon := -\delta_{\text{BRS}} \left(Q^{0,\Lambda_0} + L^{\Lambda_0,\Lambda_0}\right).
\]
\[\text{As long as the vacuum part is involved, one has to stay in finite volume}\]
as a space-time integrated insertion with ghost number 1. Because of the regularizing factor \( \sigma_0, \Lambda_0 \), cf. (4.52), the integrand is not a polynomial in the fields and their derivatives. With \( \chi \in \mathbb{R} \), we then define
\[
Z_{\chi}^{0,\Lambda_0}(K) := \int d\mu_{0,\Lambda_0}(\Phi) e^{-\frac{i}{\hbar}(L_{\Lambda_0} + \chi L_{1,\Lambda_0} + \frac{1}{2}\Phi K)}.
\] (4.59)
Due to these definitions, the violated Slavnov-Taylor identities (4.55) can be written in the form
\[
\mathcal{D}_{\Lambda_0} \tilde{Z}^{0,\Lambda_0}(K, \xi)|_{\xi=0} = \frac{d}{d\chi} Z_{\chi}^{0,\Lambda_0}(K)|_{\chi=0}.
\] (4.60)
From the modified functionals (4.56) and (4.59) follow, cf. (2.65), the generating functionals of the corresponding amputated truncated Schwinger functions
\[
\tilde{Z}^{0,\Lambda_0}(K, \xi) = e^{\frac{i}{\hbar} P(K)} e^{-\frac{i}{\hbar}(L_{\Lambda_0}(\psi, c, \bar{c}; \xi) + I_{0,\Lambda_0})},
\] (4.61)
\[
Z_{\chi}^{0,\Lambda_0}(K) = e^{\frac{i}{\hbar} P(K)} e^{-\frac{i}{\hbar}(L_{\Lambda_0}(\psi, c, \bar{c}) + I_{0,\Lambda_0})},
\] (4.62)
where the variables of the \( Z \)- and the \( L \)-functional are related as
\[
\varphi_\tau(x) = \int dy C^{0,\Lambda_0}_\tau(x-y) J_\tau(y),
\]
\[
c^a(x) = -\int dy S^{0,\Lambda_0}(x-y) \eta^a(y),
\]
\[
\bar{c}^a(x) = -\int dy S^{0,\Lambda_0}(x-y) \bar{\eta}^a(y).
\] (4.63)
Furthermore, \( P(K), (4.30) \), has to be taken here at \( \Lambda = 0 \). We observe, that the vacuum part \( I_{0,\Lambda_0} \) present without insertions appears, since both insertions have positive ghost number. To have a less cumbersome notation in the rest of this section, we abbreviate
\[
L \equiv L^{0,\Lambda_0}, \quad L^{0,\Lambda_0}|_{\xi=0} = L_{\chi}^{0,\Lambda_0}|_{\chi=0}, \quad L^0 \equiv L_{\Lambda_0,\Lambda_0},
\]
\[
L_1 \equiv L_1^{0,\Lambda_0} = \frac{d}{d\chi} L_{\chi}^{0,\Lambda_0}|_{\chi=0}, \quad L_1^0 \equiv L_1^{\Lambda_0,\Lambda_0},
\]
\[
L_\gamma \equiv L_{\gamma}^{0,\Lambda_0}(x; \Phi), \quad L_\gamma^0 \equiv L_{\gamma}^{\Lambda_0,\Lambda_0}(x; \Phi) \left( = \frac{\delta}{\delta \gamma(x)} L_{\Lambda_0,\Lambda_0}(\xi)|_{\xi=0} \right),
\] (4.64)
see (4.40-4.42). Moreover, we denote the inverted unregularized propagators by
\[
D_\tau \equiv \left( (-\Delta + m^2) \delta_{\mu,\nu} - \frac{1}{\alpha} \partial_\mu \partial_\nu, -\Delta + M^2, -\Delta + \alpha m^2 \equiv D \right).
\] (4.65)
From (4.60) we derive via (4.61 - 4.63), employing the previous abbreviations, the violated Slavnov-Taylor identities of the amputated truncated Schwinger functions:

\[
\langle c^a, D \left( \frac{1}{\alpha} \partial_\nu A^a_\nu - m B^a \right) \rangle - \langle c^a, \sigma_{0,\Lambda_0} \left( \partial_\nu \delta L_{\delta A^a_\nu} - m \delta L_{\delta B^a} \right) \rangle + \sum_\tau \langle \phi_\tau, D_\tau L_{\gamma_\tau} \rangle - \langle \bar{c}^a, DL_\omega \rangle = L_1.
\] (4.66)

As will turn out, we also need the explicit form of \( L_1 \), (4.58), i.e. the BRS-variation of the bare action. From its definition (4.58) follows directly, using (4.40 - 4.41),

\[
L_1 = \langle c^a, D \left( \frac{1}{\alpha} \partial_\nu A^a_\nu - m B^a \right) \rangle - \langle \delta L_{0,\sigma_0,\Lambda_0} \left( \frac{1}{\alpha} \partial_\nu A^a_\nu - m B^a \right) \rangle + \sum_\tau \langle \phi_\tau, D_\tau L_{0,\gamma_\tau} \rangle - \langle \bar{c}^a, DL_{0,\omega} \rangle.
\] (4.67)

Moreover, to restore the Slavnov-Taylor identities we shall rely on proper vertex functions, too. Therefore, the violated form in terms of these functions is derived here, too. In the following, all functionals appearing should carry the superscript \( 0, \Lambda_0 \) which is omitted, cf. (4.64). Considering the generating functional of the truncated Schwinger functions

\[
e^{\frac{1}{\hbar} \tilde{W}(K,\xi)} = \frac{\tilde{Z}(K,\xi)}{\tilde{Z}(0,0)},
\] (4.68)

it follows from (4.60), together with (4.61 - 4.62) and using notation defined in (4.64), that

\[
D_{\Lambda_0} \tilde{W}(K,\xi)|_{\xi=0} = -L_1(\phi_\tau, c^a, \bar{c}^a),
\] (4.69)

with arguments according to (4.64). Because of the inherent symmetries, the functional \( L \), and hence also \( \tilde{W} \), contain only one 1-point function, which we force to vanish by the renormalization condition

\[
\left. \frac{\delta L}{\delta h(x)} \right|_{\Phi=0} = 0, \quad \rightarrow \left. \frac{\delta \tilde{L}}{\delta h(x)} \right|_{\Phi=0} = 0.
\] (4.70)

A Legendre transformation yields the (modified) generating functional of the proper vertex functions,

\[
\tilde{\Gamma}(\phi_\tau, c^a, \bar{c}^a; \xi) + \tilde{W}(J_\tau, \eta^a, \bar{\eta}^a; \xi) = \int dx \left( \sum_\tau \phi_\tau J_\tau + \bar{\eta}^a \bar{c}^a + \bar{\bar{c}}^a \eta^a \right),
\] (4.71)
with variables related by

\begin{align}
\varphi_r(x) &= \frac{\delta \tilde{W}}{\delta J_r(x)}, & J_r(x) &= \frac{\delta \tilde{\Gamma}}{\delta \varphi_r(x)}, \\
\xi^a(x) &= \frac{\delta \tilde{W}}{\delta \bar{\eta}^a(x)}, & \bar{\eta}^a(x) &= -\frac{\delta \tilde{\Gamma}}{\delta \xi^a(x)}, \\
\bar{\xi}^a(x) &= -\frac{\delta \tilde{W}}{\delta \eta^a(x)}, & \eta^a(x) &= \frac{\delta \bar{\Gamma}}{\delta \bar{\xi}^a(x)}.
\end{align}

Since \( \tilde{W} \) does not contain 1-point functions, because of (4.70), but begins with 2-point functions, the equations on the left in (4.72) imply, that the variables \( \varphi_r, \xi^a, \bar{\xi}^a \) of \( \tilde{\Gamma} \) vanish, if the variables \( J_r, \eta^a, \bar{\eta}^a \) are equal to zero. Inverting these equations provides \( J_r, \eta^a, \bar{\eta}^a \) as respective functions of \( \varphi_r, \xi^a, \bar{\xi}^a \), to be used in the definition (4.71) of \( \tilde{\Gamma} \). It follows, that there is no 1-point proper vertex function, i.e.

\[ \frac{\delta \tilde{\Gamma}}{\delta \bar{h}(x)} \bigg|_{\Phi = 0} = 0. \]

From the functional derivation of (4.71) with respect to the source \( \gamma(x) \) at fixed \( \Phi \),

\[ \frac{\delta \tilde{\Gamma}}{\delta \gamma(x)} \bigg|_{\Phi} + \frac{\delta \tilde{W}}{\delta \gamma(x)} \bigg|_{K} \]

\[ + \int dy \left( \sum_r \frac{\delta \tilde{W}}{\delta J_r(y)} \frac{\delta J_r(y)}{\delta \gamma(x)} + \frac{\delta \eta^a(y)}{\delta \gamma(x)} \frac{\delta \tilde{W}}{\delta \bar{\eta}^a(y)} + \frac{\delta \eta^a(y)}{\delta \gamma(x)} \frac{\delta \bar{\Gamma}}{\delta \xi^a(y)} \right) \]

\[ = \int dy \left( \sum_r \varphi_r(y) \frac{\delta J_r(y)}{\delta \gamma(x)} + \frac{\delta \bar{\eta}^a(y)}{\delta \gamma(x)} \xi^a(y) - \frac{\delta \eta^a(y)}{\delta \gamma(x)} \bar{\xi}^a(y) \right), \]

we infer, because of (1.72),

\[ \frac{\delta \tilde{\Gamma}}{\delta \gamma(x)} \bigg|_{\Phi} = -\frac{\delta \tilde{W}}{\delta \gamma(x)} \bigg|_{K}, \]

and similar relations for the derivatives with respect to the sources \( \gamma^\mu, \gamma^a \) and \( \omega \). These relations are employed at \( \xi = 0 \). Using a notation in accord with (4.64),

\[ \Gamma \equiv \tilde{\Gamma}^{0,A_0} \bigg|_{\xi = 0}, \quad \Gamma_{\gamma_r}(x) \equiv \frac{\delta \tilde{\Gamma}^{0,A_0}}{\delta \gamma_r(x)} \bigg|_{\xi = 0}, \]

(4.75)
the violated Slavnov-Taylor identities for proper vertex functions emerge from (4.67) via (4.72), (4.74) as
\[ \sum_{\tau} \langle \delta \Gamma_{\rho_0,\Lambda_0} \gamma_\tau \rangle - \langle \delta \Gamma_{\rho_0,\Lambda_0,\omega_0} \rangle - \frac{1}{\alpha} \partial_\nu \bar{A}_\nu - mB_2 \sigma_{0,\Lambda_0} \frac{\delta \Gamma}{\delta \tilde{c}_a} \rangle = \Gamma_1(\varphi_\tau, c_a, \bar{c}_a), \tag{4.76} \]
with
\[ \Gamma_1(\varphi_\tau, c_a, \bar{c}_a) = L_1(\varphi_\tau, c_a, \bar{c}_a). \tag{4.77} \]
In (4.77) the variables are related, suppressing the superscript \(0,\Lambda_0\) of the propagators, as
\[ \varphi_\tau(x) = \int dy C_\tau(x-y) \frac{\delta \Gamma}{\delta \varphi(y)}, \tag{4.78} \]
\[ c_a(x) = - \int dy S(x-y) \frac{\delta \Gamma}{\delta c_a(y)}; \quad \bar{c}_a(x) = \int dy \frac{\delta \Gamma}{\delta \bar{c}_a(y)} S(y-x). \]
Comparing (4.67) with (4.76) we observe, that \(L_1^0\) and \(\Gamma_1\) have the same form! The apparently additional terms in \(L_1^0\) result from the quadratic part of the classical action, which by definition is excluded from the bare interaction \(L^0\), but is contained in \(\Gamma\).

### 4.4 Mass scaling of super-renormalizable couplings

The systems of amputated truncated Schwinger functions and of proper vertex functions are equivalent formulations of the theory. To restore the Slavnov-Taylor identities, however, still to be accomplished, analysing their relevant part in terms of the proper vertex functions turns out to be simpler. A necessary condition to achieve the Slavnov-Taylor identities then is a vanishing relevant part of the violating functional \(\Gamma_{1,0,\Lambda_0}^0,\Lambda_0\) (4.76). The means available to generate this property is the freedom in choosing the relevant terms appearing in the functionals \(\Gamma_{1,0,\Lambda_0}^{0,\Lambda_0}\) and \(\Gamma_{0,\Lambda_0}^{0,\Lambda_0}\), \(\gamma = \gamma_\tau, \omega_0\), i.e. the renormalization conditions of the functionals \(\Gamma_{0,\Lambda_0}^{0,\Lambda_0}\) and \(\Gamma_{0,\Lambda_0}^{0,\Lambda_0}\). Inspecting (4.76), however, reveals an obstacle in bringing this freedom of choice to bear. In order to exhaust all relevant terms of (4.76), up to 5 field- and momentum derivatives have to be applied, according to the dimension 5 of the insertion defining \(\Gamma_{1,0,\Lambda_0}^{0,\Lambda_0}\). Due to the property (4.25) we observe that the momentum derivatives of the cutoff function \(\sigma_{0,\Lambda_0}(k^2) = \sigma_{\Lambda_0}(k^2)\) do not contribute to these terms. Thus, the field- and momentum derivatives in
question generate from \((4.76)\) terms with \(d_1\) such derivatives applied to the factors of the form \(\delta \Gamma / \delta \varphi\) and \(d_2\) such derivatives applied to factors of the form \(\Gamma_\gamma, \partial A^a\), or \(mB^a\), with \(d_1 + d_2 \leq 5\). Applying \(d_2 \geq 3\) derivatives on the functionals \(\Gamma_{0,0}\), however, generates irrelevant contributions, since the respective insertions of these functional are of dimension 2. It appears, that the elimination of these unknown irrelevant terms can only be accomplished making the relevant terms from \(\Gamma_{l,n}\), which form products with them, disappear. One notices, however, that already in the tree order \(\Gamma_{0,0}\) there are terms preventing this procedure, to wit, the nonvanishing super-renormalizable three-point couplings and the mass terms of the two-point functions (see Appendix A).

To tackle this problem requires to trace in the perturbative expansion the effects of the super-renormalizable three-point couplings appearing. To this end, in the tree-level part \((4.12)\) of the interaction \((4.33)\) the mass parameters entering the three-point couplings, as well as in the BRS-insertions \((4.38)\) the mass parameter appearing there in \(\psi^\alpha(x)\), are scaled by a common factor \(\lambda > 0\):

\[
m \to \lambda m, \quad M \to \lambda M.
\]

It is important to notice that the mass parameters present in the regularized propagators appearing in the flow equations are not scaled. All amputated truncated Schwinger functions, considered first, will then depend smoothly on \(\lambda\), and are expanded as

\[
L^{A,0}_{l,n}(\lambda; \vec{p}) = \sum_{\nu=0}^{\infty} (m \lambda)^\nu L^{(\nu),A,0}_{l,n}(\vec{p}), \quad \vec{p} = (p_1, \ldots, p_{|n|}),
\]

\[
L^{A,0}_{\gamma;l,n}(\lambda; q; \vec{p}) = \sum_{\nu=0}^{\infty} (m \lambda)^\nu L^{(\nu),A,0}_{\gamma;l,n}(q; \vec{p}),
\]

with finite sums in suitable renormalization schemes.

In the Renormalization scheme adopted here relevant terms satisfy

i) \(|n| + |w| + \nu \leq 4\) in case of the functional \(L^{A,0}\),

ii) \(|n| + |w| + \nu \leq 2\) in case of the functionals \(L^{A,0}_\gamma, \gamma = \gamma_\tau, \omega\),

in accord with the bounds stated below.

At tree level holds (there are no functions with \(|n| \leq 2\) at \(l = 0\))

\[
(\partial^w L^{(\nu),A,0}_{0,n})(\vec{0}) = 0, \quad \text{if} \quad |n| + |w| + \nu < 4.
\]

For \(l \geq 1\) renormalization conditions on the relevant terms are imposed as follows:

\[
(\partial^w L^{(\nu),0,A,0}_{l,n})(\vec{0}) \equiv 0, \quad \text{if} \quad |n| + |w| + \nu < 4,
\]
whereas if \(|n| + |w| + \nu = 4\), on the r.h.s. a free constant \(r(\nu, l, n)\) can be chosen.

Correspondingly, in the case of a BRS-insertion, at the tree level holds

\[
(\partial^\nu L_{\gamma; 0, n}^{(\nu), \Lambda, \Lambda_0})(0; \vec{0}) = 0, \quad \text{if } |n| + |w| + \nu < 2,
\]

and renormalization conditions are imposed as

\[
(\partial^\nu L_{\gamma; l, n}^{(\nu), 0, \Lambda_0})(0; \vec{0}) = 0, \quad \text{if } |n| + |w| + \nu < 2,
\]

but if \(|n| + |w| + \nu = 2\), on the r.h.s. again a free constant can be chosen.

According to the expansions (4.80) and (4.81) the flow equations of the form (2.31) and (2.74) have to be adjusted attributing a superscript \((\nu)\) to the \(n\)-point functions \(L_{\gamma}^{\Lambda, \Lambda_0, \Lambda_0 l, n}\) and \(L_{\gamma}^{\Lambda, \Lambda_0, \gamma; l, n}\) and in the quadratic term to sum \(\nu_1 + \nu_2 = \nu\), in complete analogy to the loop index \(l\). Employing these extended flow equations the following bounds can be deduced,

**Proposition 1, II**

Let \(l \in \mathbb{N}_0\) and \(0 \leq \Lambda \leq \Lambda_0\), then

\[
|\partial^\nu L_{l, n}^{(\nu), \Lambda, \Lambda_0}(\vec{p})| \leq (\Lambda + m)^4 |n| |w| \nu P_1(\log \frac{\Lambda + m}{m}) P_2(\frac{|\vec{p}|}{\Lambda + m}),
\]

(4.86)

\[
|\partial^\nu L_{\gamma; l, n}^{(\nu), \Lambda, \Lambda_0}(q, \vec{p})| \leq (\Lambda + m)^2 |n| |w| \nu P_1(\log \frac{\Lambda + m}{m}) P_2(\frac{|q, \vec{p}|}{\Lambda + m}).
\]

(4.87)

In these bounds \(P_i, i = 1, 2\), denote (each time they appear possibly new) polynomials with nonnegative coefficients independent of \(\Lambda, \Lambda_0, \vec{p}, q, m\). The coefficients may depend on \(n, l, w\), and the other free parameters of the theory \(\alpha, M/m, g\).

These bounds are uniform in \(\Lambda_0\). The proof in II follows closely the line of proof presented before in the case of a scalar field.

The system of flow equations satisfied by the proper vertex functions \(\Gamma_{l, n}^{\Lambda, \Lambda_0}\) is a direct extension of (3.52) to various types of fields now present, as before in the case of the Schwinger functions. The functions \(\tilde{\Gamma}_{l, n}^{\Lambda, \Lambda_0}\) appearing on the r.h.s. of the flow equation (3.52), denoted in II, eq.(84), by \(L_{l, n}^{\Lambda, \Lambda_0}\), have to be determined recursively according to their definition (3.48) from the crucial relation (3.40) extended to various types of fields, providing finally the functions \(\tilde{\Gamma}_{l, n}^{\Lambda, \Lambda_0}\) in the form (3.53). In addition, as in the case of Schwinger functions, flow equations for proper vertex functions with a BRS-insertion, \(\Gamma_{\gamma; l, n}^{\Lambda, \Lambda_0}\), have to be considered, too, see II, eq.(86). The r.h.s. of these flow equations has again to be determined recursively as described above regarding proper vertex functions without insertion. Having obtained this way the functions entering the r.h.s. of the flow equations in the form
the mass scaling (4.79) in the tree-level interaction and the insertion can be performed, leading to the expansions

\[ \Gamma_{\Lambda,\Lambda_0}^{\Lambda,\Lambda_0}(\lambda; \vec{p}) = \sum_{\nu=0}^{\infty} (m\lambda)^\nu \Gamma_{\Lambda,\Lambda_0}^{(\nu),\Lambda,\Lambda_0}(\vec{p}) , \quad \vec{p} = (p_1, \ldots, p_{|n|}) , \quad (4.88) \]

\[ \Gamma_{\gamma;l,n}^{\Lambda,\Lambda_0}(\lambda; q; \vec{p}) = \sum_{\nu=0}^{\infty} (m\lambda)^\nu \Gamma_{\gamma;l,n}^{(\nu),\Lambda,\Lambda_0}(q; \vec{p}) . \quad (4.89) \]

Considering the tree level \( l = 0 \) first, the scaling (4.79) leads to

i) in case of (4.88),

\[ \frac{\partial^w}{\partial \vec{w}} \Gamma_{0,n}^{(\nu),0,\Lambda_0}(0; \vec{0}) = 0 , \quad |n| = 3 , \quad |w| + \nu \neq 1 , \quad (4.90) \]

ii) the masses of the two-point functions, fixed by the regularized propagators, are not scaled, and there is no \(|n| = 1\) content,

iii) in case of (4.89),

\[ \frac{\partial^w}{\partial \vec{w}} \Gamma_{\gamma;0,n}^{(\nu),0,\Lambda_0}(0; \vec{0}) = 0 , \quad |n| + |w| + \nu < 2 . \quad (4.91) \]

Implementing the expansions (4.88) and (4.89) in the flow equations of \( \Gamma_{\Lambda,\Lambda_0}^{\Lambda,\Lambda_0} \) and \( \Gamma_{\gamma;l,n}^{\Lambda,\Lambda_0} \), respectively, a superscript \((\nu)\) with corresponding values has to be attached to the various \( n \)-point functions involved there. Employing these flow equations and proceeding inductively as in the case of the Schwinger functions, renormalizability of the proper vertex functions may be deduced. The relevant terms are subject to the renormalization conditions as follows, where \( l \geq 1 \):

\[ \frac{\partial^w}{\partial \vec{w}} \Gamma_{l,n}^{(\nu),0,\Lambda_0}(0; \vec{0}) \bigg|_{\nu=0} = 0 , \quad |n| + |w| + \nu < 4 , \quad (4.92) \]

whereas if \(|n| + |w| + \nu = 4\), on the r.h.s. a nonvanishing constant can be chosen.

In the case of a BRS-insertion

\[ \frac{\partial^w}{\partial \vec{w}} \Gamma_{\gamma;l,n}^{(\nu),0,\Lambda_0}(0; \vec{0}) \bigg|_{\nu=0} = 0 , \quad |n| + |w| + \nu < 2 , \quad (4.93) \]

whereas if \(|n| + |w| + \nu = 2\), again a nonvanishing constant on the r.h.s. may be imposed.

Proceeding inductively as indicated provides the bounds:

**Proposition 2, II**

\[ \left| \frac{\partial^w}{\partial \vec{w}} \Gamma_{l,n}^{(\nu),\Lambda,\Lambda_0}(\vec{p}) \right| \leq (\Lambda + m)^{|n| - |w| - \nu} \mathcal{P}_1 \left( \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left( \frac{|\vec{p}|}{\Lambda + m} \right) , \quad (4.94) \]

\((l, |n|) \neq (0, 2)\)
\[ | \partial^{\nu} \Gamma^{(\nu),\Lambda,\Lambda_0}_{\gamma_{\tau},l,n}(q; \vec{p}) | \leq (\Lambda + m)^{2-|n|-|w|-\nu} \mathcal{P}_1(\log \frac{\Lambda + m}{m}) \mathcal{P}_2(\frac{|q, \vec{p}|}{\Lambda + m}) , \quad (4.95) \]

The notations are those from (4.86), (4.87).

Using the expansions (4.88) and (4.89) of the \( n \)-point functions of \( \Gamma_{1,l,n}^{\Lambda,\Lambda_0} \), \( \Lambda_0 \) and \( \Gamma^{\Lambda,\Lambda_0}_{\gamma_{\tau},l,n} \), respectively, which record the number of inherent super-renormalizable vertices, allows now to proceed as envisaged at the beginning of this section in making vanish the relevant part of \( \Gamma_{1,0,\Lambda_0} \). The bounds (4.94) and (4.95) show that the degree of divergence decreases with this number.

The expansions of the \( n \)-point functions of the functionals \( \Gamma_{1,0,\Lambda_0} \), \( \Lambda_0 \) and \( L_{1,0,\Lambda_0} \), resulting from the mass scaling (4.79),

\[ L_{1;1,n}^{\Lambda,\Lambda_0}(\lambda; \vec{p}) = \sum_{\nu=0}^{\infty} (m\lambda)^{\nu} L_{1;1,n}^{(\nu),\Lambda,\Lambda_0}(\vec{p}) , \quad \vec{p} = (p_1, \cdots, p_{|n|}) , \quad (4.96) \]

\[ \Gamma_{1;1,n}^{\Lambda,\Lambda_0}(\lambda; \vec{p}) = \sum_{\nu=0}^{\infty} (m\lambda)^{\nu} \Gamma_{1;1,n}^{(\nu),\Lambda,\Lambda_0}(\vec{p}) , \quad (4.97) \]

form the starting point of the demonstration. Moreover, in a consistent mass expansion of the violated Slavnov-Taylor identities (4.76), conform with the treatment of the BRS-insertions, the mass scaling (4.79) has to be performed in the BRS-variation \( \frac{1}{\alpha} (\partial_{\nu} A_{\nu}^{\sigma}(x) - \alpha m B^{\sigma}(x)) \) of the antighost, too. The aim envisaged is to determine the relevant part of the functional \( \Gamma_{1,0,\Lambda_0} \), given by the values \( \partial^{\nu} \Gamma^{(\nu),0,\Lambda_0}_{1,l,n}(0) \), \( |n| + |w| + \nu \leq 5 \), via (4.76). It is important to observe that irrelevant contributions only emerge from the functionals containing a BRS-insertion. Upon requiring the vertex functions entering (4.76) to satisfy the boundary conditions, \( l \in \mathbb{N}_0 \),

\[ \partial^{\nu} \Gamma^{(\nu),0,\Lambda_0}_{1,l,n}(0) \, \big|_{\nu=0}^{1} = 0 , \quad \text{if} \quad |n| + |w| + \nu < 4 , \quad (4.98) \]

then annihilates the irrelevant contributions generated from the functionals \( \Gamma^{(\nu),0,\Lambda_0}_{\gamma_{\tau},l,n} \), \( \Gamma^{(\nu),0,\Lambda_0}_{\omega} \) by multiplication and only contributions of these functionals with \( |n_2| + |w_2| + \nu_2 \leq 2 \) field-, momentum- and mass derivatives, i.e. relevant terms, do appear. The boundary conditions (4.98) are satisfied for \( l \geq 1 \) due to the renormalization conditions (4.92), and in the tree order, if \( |n| = 3 \), (4.94).

At this stage one has to remember that the mass scaling is only performed with regard to the boundary terms appearing in the the flow equations, but not touching the free propagators. In (4.76), however, the inverted free propagators \( \Gamma^{0,\Lambda_0}_{0,0} \), \( |n| = 2 \), appear as boundary terms at \( \Lambda = 0 \) for the functionals \( \Gamma^{\Lambda,\Lambda_0}_{1,l,n} \), and their masses then are scaled (4.79), satisfying (4.98) in this case.
too. One should notice, that in the approach followed the flow equations and the violated Slavnov-Taylor identities have been derived before the mass expansion, and only afterwards the mass expansion is applied to all boundary terms, and the bounds on the vertex functions have been verified inductively.

Because of the renormalization conditions (4.92) imposed on (a subset of) the relevant terms of the vertex functions, in all two-point functions the leading terms of the mass expansion are fixed to zero, i.e. with the notation of Appendix A,

\[ \delta m_{(\nu)}^2 = 0, \quad \Sigma^{cc(\nu)}(0) = 0, \quad \Sigma^{BB(\nu)}(0) = 0, \quad \Sigma^{hh(\nu)}(0) = 0 \quad \text{for} \quad \nu \leq 1, \]

and also

\[ \Sigma^{AB(\nu)}(0) = 0 \quad \text{for} \quad \nu = 0; \quad \kappa^{(\nu)} = 0 \quad \text{for} \quad \nu \leq 2. \quad (4.100) \]

The respective relevant parts of vertex functions with a BRS-insertion are listed in Appendix B. The corresponding restricted set of renormalization conditions (4.93) is automatically fulfilled.

The functionals \( L^{\Lambda,\Lambda_0}, \Gamma^{\Lambda,\Lambda_0} \) serve to control the violation of the Slavnov-Taylor identities. In contrast to the functionals \( \Gamma^{\Lambda,\Lambda_0} \) and \( \Gamma^{\Lambda,\Lambda_0}_{\gamma} \), these functionals contain irrelevant boundary terms at \( \Lambda = \Lambda_0 \), resulting from the presence of the factors \( \sigma_{0,\Lambda_0} \).\footnote{Due to the property (4.25) these factors do not affect the relevant part.}

The bound on the functional \( \Gamma^{\Lambda_0,\Lambda_0} \) as stated below in (4.104) does not follow from the choice of standard renormalization conditions for insertions. Rather it is \textit{assumed} here that the relevant part of this functional at the physical value \( \Lambda = 0 \) of the flow parameter does vanish, \( l \in \mathbb{N}_0 \),

\[ |\partial^w \Gamma_{1;1,n}^{(\nu),\Lambda_0,\Lambda_0}(\vec{0})| \leq (\Lambda_0 + m)^{5-|n|-|w|-\nu} \left( \log \frac{\Lambda_0}{m} \right)^r \mathcal{P} \left( \left| \frac{\vec{p}}{\Lambda_0} \right| \right), \quad (4.101) \]

which holds trivially, unless \( 2 \leq |n| \leq 5 \). Using the relation

\[ \Gamma_{1;1,n}^{\Lambda_0,\Lambda_0}(p_1, \cdots, p_{|n|}) = \mathcal{L}_{1;1,n}^{\Lambda_0,\Lambda_0}(p_1, \cdots, p_{|n|}), \quad (4.102) \]

this assertion can be established from (4.67) employing there the previous bounds on \( \partial^w \mathcal{L}_{1,n}^{(\nu),\Lambda,\Lambda_0} \), (4.36), and on \( \partial^w \mathcal{L}_{1;1,n}^{(\nu),\Lambda,\Lambda_0} \), (4.37), at the value \( \Lambda = \Lambda_0 \), together with a bound on \( \partial^w \sigma_{0,\Lambda_0}(k^2) \) easily obtained from (4.24). The bound on the functional \( \Gamma_{1;1,n}^{\Lambda,\Lambda_0} \) as stated below in (4.104) does not follow from the choice of standard renormalization conditions for insertions. Rather it is \textit{assumed} here that the relevant part of this functional at the physical value \( \Lambda = 0 \) of the flow parameter does vanish, \( l \in \mathbb{N}_0 \),

\[ (\partial^w \Gamma_{1;1,n}^{(\nu),0,\Lambda_0})(\vec{0}) = 0, \quad |n| + |w| + \nu \leq 5. \quad (4.103) \]
In Section 4.6 it will be shown that this assumption can be accomplished via the violated Slavnov-Taylor identities \((4.76)\) choosing for the functionals entering the l.h.s. suitable renormalization conditions within the class \((4.92), (4.93)\) considered.

Taking this assumption for granted the irrelevant part of the functional \(\Gamma_{1,0}^0\) then vanishes upon shifting the UV- cutoff to infinity, owing to the

**Proposition 3. II**

Given \((4.103)\), then for \(l \in \mathbb{N}_0, |n| \geq 2\) and \(0 \leq \Lambda \leq \Lambda_0\),

\[
|\partial^{\nu} \Gamma_{1;l,n}^{(\nu),0,0} (\vec{p})| \leq \frac{1}{\Lambda_0} (\Lambda + m)^{5+1-|n|-|w|+\nu} \left( \log \frac{\Lambda_0}{m} \right)^r P \left( \frac{|\vec{p}|}{\Lambda + m} \right). \tag{4.104}
\]

with a positive integer \(r\) depending on \(n, l, w\), and a polynomial \(P\) as in \((4.86), (4.87)\).

In the proof given in II the system of flow equations for the vertex functions \(\Gamma_{1;l,n}^{(\nu),0,0}\) is integrated inductively. The flow equations in the case of an integrated insertion occurring here coincide with those in the case of a local insertion at insertion momentum zero, cf. \((4.49)-(4.50)\). Given the condition \((4.103)\), from the bound \((4.104)\) then follows that the Slavnov-Taylor identities are restored in the limit \(\Lambda_0 \to \infty\).

### 4.5 Equation of motion of the antighost

The renormalization of a non-Abelian gauge theory using a gauge invariant renormalization scheme is generally based on the Slavnov-Taylor identities, complemented by the equation of motion of the antighost \([54],[51]\). Approaching renormalization via flow equations this equation is derived from the regularized functional integral representation of the generating functional and its renormalization is considered in conjunction with the restoration of the Slavnov-Taylor identities. In Section 3.1 the equation of motion in connection with renormalization within the theory of a scalar field has been treated, in the present theory the field equation of the antighost is obtained as the analogue of \((3.4)\),

\[
\frac{\delta L^{\Lambda,0}(\Phi)}{\delta \zeta^a(x)} = L^{\Lambda,0}_\zeta(x; \Phi), \tag{4.105}
\]

which emerges from extending the original bare interaction \(L^{\Lambda,0}_0(\Phi)\) by the insertion, cf. \((3.2)\),

\[
L^{\Lambda,0}_0(\zeta; \Phi) = \int dx \, \zeta^a(x) \frac{\delta L^{\Lambda,0}_0(\Phi)}{\delta \zeta^a(x)}, \tag{4.106}
\]

69
where the source $\zeta^a(x)$ is now a Grassmann element carrying ghost number $-1$. The BRS-invariant classical action (4.9) satisfies the classical field equation

$$\delta/\delta\bar{c}^a(x)S_{BRS} = \partial_\mu \psi^a_\mu(x) - \alpha m \psi^a(x),$$

where the form (4.15) of $S_{gh}$ has been used. The aim is to show that at the physical value $\Lambda = 0$ of the flow parameter the field equation

$$\delta L^{0,\Lambda_0}(\Phi)/\delta\bar{c}^a(x) = \partial_\mu L^{0,\Lambda_0}_\gamma(x;\Phi)|_{\text{mod}} - \alpha m L^{0,\Lambda_0}_\gamma(x;\Phi)|_{\text{mod}}, \quad (4.107)$$

still holds in the renormalized theory, following at the tree level from the classical action. The label "mod" prescribes in the bare insertions (4.38) to replace

$$R_i^0 \rightarrow \tilde{R}_i^0 = O(\bar{h}) \quad \text{for} \quad i = 1, 4$$

since the respective tree order does not appear on the l.h.s. The field equation (4.107) can be rewritten in terms of proper vertex functions using the conventions introduced,

$$\left(2\pi\right)^4 \frac{\delta \Gamma^{0,\Lambda_0}(\Phi)}{\delta\bar{c}^a(q)} = - \frac{q^2 + \alpha m^2}{\sigma_{0,\Lambda_0}(q^2)} \xi^a(-q)$$

$$- iq_\mu \Gamma^{0,\Lambda_0}_\gamma(q;\Phi)|_{\text{mod}} - \alpha m \Gamma^{0,\Lambda_0}_\gamma(q;\Phi)|_{\text{mod}}. \quad (4.108)$$

The first term on the r.h.s. is the tree level 2-point function. Restricting (4.108) to its relevant part, $\sigma_{0,\Lambda_0}(q^2)$ is replaced by $\sigma_{0,\Lambda_0}(0) = 1$ due to (4.25), the first term then provides the tree order of $R_1$ and $R_4$ excluded in the insertions as indicated by the label mod, cf. (4.107).

Taken for granted in the present case the outcome of Section 3.1, the field equation (4.108) holds in the limit $\Lambda_0 \rightarrow \infty$, if its relevant part is fulfilled.

### 4.6 Restoration of the Slavnov-Taylor identities

To achieve the Slavnov-Taylor identities it is necessary, and due to Proposition 3 also sufficient, that the relevant part of the functional $\Gamma^{0,\Lambda_0}_1$ vanishes.

Enforcing this behaviour in accord with the violated Slavnov-Taylor identities (4.76), the conditions (4.103) amount to satisfy the 53 equations listed in the Appendix C. These equations are satisfied in the tree order. Since the analysis to be performed for $l > 0$, although rather technical, is a crucial step in the method developed, it appears meaningful to present it verbatim as given in II.

Noticing that the normalization constants of the BRS-insertions behave as $R_i = 1 + O(\bar{h}), \quad i = 1, \cdots, 7$, we first analyse the equations IX to XXIX,
but take already into account the equations $VII_d$, $VIII_c$, the latter ones providing
\[ r^{{hBA}}_2 = r^{{cA}}_2 = 0. \] (4.109)

In proceeding we use conditions determined before, if needed. From $XIV_b$, $XIV_c$, $XV_2$, $XXIII$ directly follow
\[ r^{A\bar{a}c} = r^{B\bar{a}c} = r^{A\bar{A}BB} = 0, \] (4.110)
and then, from $XIV_{a+c}$, $XVII_b$, $XVIII_c$, $XXVIII$, $XXIX$,
\[ r^{AAAA} = r^{h\bar{A}c} = r^{c\bar{c}c} = r^{h\bar{B}c} = r^{BB\bar{c}} = 0. \] (4.111)

$XVI_a$, $XVIII_a$, and $XV_{2a}$ combined with $XVI_b$, respectively, require
\[ R_2 = R_5 = (R_2)^2. \] (4.112)

$XIV_c$:
\[ 2F^{AAAA} R_1 = F^{AAA} g R_2 \] (4.113)

$XI$:
\[ F^{ccB(1)} R_5 = F^{c\bar{c}h(1)} R_2. \] (4.114)

From $X$, $XX$, $XIX$, $IX$ follow for the self-coupling of the scalar field
\[ 8 F^{BBB} R_4 = F^{BBh(1)} g R_3, \] (4.115)
\[ 4 F^{BBhh} R_4 = F^{BBh(1)} g R_5, \] (4.116)
\[ 8 F^{hhhh} R_4 R_3 = F^{BBh(1)} g (R_5)^2, \] (4.117)
\[ F^{hh(1)} R_3 = F^{BBh(1)} R_5, \] (4.118)
and from $XVI_b$, $XVII_a$, $XXI$, $XIII_2$ for the scalar-vector coupling
\[ 2 F^{BBA} R_5 = - F^{hBA} R_2, \] (4.119)
\[ 4 F^{AAhh} R_3 = F^{hBA} g R_5, \] (4.120)
\[ 4 F^{AABB} R_1 = F^{hBA} g R_3, \] (4.121)
\[ F^{Ahh(1)} R_3 = F^{hBA} R_4. \] (4.122)

One easily verifies that the remaining equations of $IX$ to $XXIX$ are satisfied due to these conditions (4.109)-(4.112).

At this stage, all those relevant couplings with $|n| = 3, 4$ not appearing already in the tree order are required to vanish: (4.109)-(4.111). All other couplings involving four fields are determined by particular couplings with $|n| = 3$: (4.113), (4.115)-(4.117), (4.120), (4.121). In addition, there are 4
conditions relating couplings with \(|n| = 3\): \((4.114), (4.118), (4.119)\) and \((4.122)\). Moreover, the normalization constants of the BRS-insertions are required to satisfy the three conditions \((4.112)\).

There are still 18 − 2 equations among \(I\) to \(VIII\) to be considered. They contain the relevant parameters of \(\Gamma^{0,\Lambda\nu}\) with \(|n| = 1, 2, 3\), except \(F^{hhh}\), together with the normalization constants of the BRS-insertions. Since two of these parameters have been fixed before, \((4.109)\), there remain 26 to be dealt with. \((F^{hhh}\) will then be determined by \((4.118)\).\) These parameters in addition have to obey the conditions derived before: We first observe that the condition \((4.122)\) is identical to equation \(VIB\). There remain the 5 conditions to be satisfied: 3 conditions \((4.112)\), together with \((4.114), (4.119)\). All these conditions generate 4 linear relations among the equations still to be considered: denoting by \(\{X\}\) the content of the bracket \(\{\cdots\}\) appearing in equation \(X\), we find

\[
0 = \alpha^{-1}\{VIIIb\} + gR_2\{I_b\} + R_1(\{IIIa\} + \{III_b\}), \quad (4.123)
\]
\[
0 = gR_2\{IIa\} - \{VIIIb\} + R_1\{IV_b\} - 2R_4\{V\}, \quad (4.124)
\]
\[
0 = R_2\{IVa\} - R_3(\{VI_a\} - \{VI_b\}), \quad (4.125)
\]
\[
0 = R_2\{V\} - R_3\{VIIc\}. \quad (4.126)
\]

Hence, the 26 parameters in question are constrained by 16 + 5 − 4 = 17 equations. As renormalization conditions we then fix \(\kappa^{(3)} = 0\) and let

\[
\Sigma_{\text{trans}}, \Sigma_{\text{long}}, \Sigma^{AB(1)}, \Sigma^{cc}, \Sigma^{BB}, F^{AAA}, F^{BBh(1)}, R_3
\]

be chosen freely. These parameters correspond to the number of wave function renormalizations (including one for the BRS sector) and coupling constant renormalizations of the theory. Thus, there are 26 − 9 parameters left, together with 17 equations. These parameters are now determined successively in terms of \((4.127)\) and possibly parameters determined before in proceeding. We list them in this order, writing in bracket the particular equation fulfilled:

\[
R_1(I_b), R_4(II_b), R_2(III_b) \rightarrow R_6, R_7, R_5 \quad \text{due to } (4.112),
\]
\[
F^{ccA}(IIIa), F^{BBA}(V) \rightarrow F^{hBA} \quad \text{due to } (4.119),
\]
\[
F^{Ahb(1)}(VI_b), F^{ccB(1)}(IV_a) \rightarrow F^{ccb(1)} \quad \text{due to } (4.114),
\]
\[
\Sigma^{cc(2)}(VIIa), \Sigma^{BB(2)}(IIa), \delta m^2(2)(Ia), \Sigma^{hh(2)}(VIIa), \Sigma^{hh}(VII_{b+c}). \quad (4.128)
\]

Now all parameters are determined, without using the equations \(IV_b, VI_a, VII, VIII_b\). These equations, however, are satisfied because of the relations
Finally, the relevant couplings with \(|n| = 4\) as well as \(F_{hh(1)}^{hh}\) then are explicitly given by (4.113), (4.115)-(4.118), (4.120) and (4.121).

We have not yet implemented the field equation of the antighost (4.103). Performing the mass scaling as before and then extracting the local content \(|n| + |w| + \nu \leq 4\) leads to the relations

\[
1 + \dot{\Sigma}_{cc} = R_1, \quad (4.129)
\]

\[
\alpha + \Sigma_{cc}^{(2)} = \alpha R_4, \quad (4.130)
\]

\[
F_{1ccA}^{\bar{c}} = gR_2, \quad (4.131)
\]

\[
F_{ccB(1)}^{\bar{c}} = \frac{\alpha}{2} gR_6, \quad (4.132)
\]

\[
F_{cch(1)}^{\bar{c}} = -\frac{\alpha}{2} gR_5. \quad (4.133)
\]

Fixing now the hitherto free renormalization constant \(\Sigma_{long}\) at the particular value \(\Sigma_{long} = 0\), we claim these relations to be satisfied: (4.129) and (4.131) follow at once from \(I_b\) and \(III_{a+b}\), respectively; (4.132) follows from \(2\{IV_a\} - \{IV_b\}\), due to (4.131) and (4.112); and herefrom follow (4.133) due to (4.114), and (4.130) because of \(VIII_a\), thus establishing the claim.

Given these additional relations (4.129)-(4.133) we can adjust the procedure (4.128) choosing now a reduced set of free renormalization conditions (4.127) in which \(\Sigma_{long}\) is excluded. Proceeding similarly as before we find

\[
I_b: \quad \Sigma_{long} = 0, \quad II_a: \quad \Sigma_{BB(2)}^{B(2)} = 0, \quad (4.134)
\]

\[
III_b: \quad gR_2 = -2F_{aaa}^{AA} \frac{1 + \dot{\Sigma}_{cc}}{1 + \Sigma_{trans}} \rightarrow R_6, R_7, R_5 \quad (4.135)
\]

due to (4.112),

\[
II_b: \quad R_4 = \frac{1 + \dot{\Sigma}_{cc}}{1 + \Sigma_{BB}} (1 + \Sigma_{AB}^{(1)}), \quad (4.136)
\]

\[
I_a: \quad 1 + \delta m^2(2) = \frac{1}{1 + \Sigma_{BB}} (1 + \Sigma_{AB}^{(1)})^2, \quad (4.137)
\]

\[
V: \quad 2F_{BBA}^{BB} = F_{aaa}^{AA} \frac{1 + \dot{\Sigma}_{BB}}{1 + \Sigma_{trans}} \rightarrow F_{hBA}^{hBA} \rightarrow F_{Ah(1)}^{Ah(1)} \quad (4.138)
\]

due to (4.119), (4.122),

\[
VII_a: \quad \left(\frac{M}{m}\right)^2 + \Sigma_{hh}^{(2)} = \frac{4}{g} F_{BBh(1)}^{BBh(1)} \frac{R_4}{R_3}, \quad (4.139)
\]

\[
VII_{b+c}: \quad 1 + \dot{\Sigma}_{hh} = (1 + \dot{\Sigma}_{BB}) \frac{R_5}{R_3}. \quad (4.140)
\]
This list closes the analysis of (4.103) quoted from II.

In retrospect the method followed first dealt with the renormalization of the functional $\Gamma^{0,\Lambda_0}$ and of the accompanying functionals with a BRS-insertion $\Gamma^{0,\Lambda_0}_\gamma$, $\Gamma^{0,\Lambda_0}_\omega$, disregarding the Slavnov-Taylor identities. In these functionals there appear $37 + 7$ relevant parameters. Requiring the absence of tadpoles ($\kappa = 0$) and fixing $\Sigma_{\text{long}} = 0$ because of the field equation of the antighost, it is then shown that the set (4.127) without $\Sigma_{\text{long}}$ can be used as renormalization constants to be chosen freely: given these renormalization constants the remaining relevant parameters are determined uniquely upon requiring that the relevant part (4.103) of the functional $\Gamma^{0,\Lambda_0}$ does vanish via the violated Slavnov-Taylor identities (4.76). According to Proposition 3, the irrelevant part of the $\Gamma^{0,\Lambda_0}$ then vanishes in the limit $\Lambda_0 \to \infty$, too. Thus, within perturbation theory, the functionals $\Gamma^{0,\infty}$ and $\Gamma^{0,\infty}_\gamma$, $\Gamma^{0,\infty}_\omega$ are finite and satisfy the Slavnov-Taylor identities, i.e. equation (4.76) for $\Lambda_0 \to \infty$ with the r.h.s. vanishing.

A The relevant part of $\Gamma$

The bare functional $L^{\Lambda_0,\Lambda_0}$ and the relevant part of the generating functional $\Gamma^{0,\Lambda_0}$ for the proper vertex functions have the same general form. We present the latter and give the tree order of both explicitly. The cutoff symbols $0, \Lambda_0$ are suppressed. We write

$$\Gamma(A, h, B, \bar{c}, c) = \sum_{|n| = 1}^{4} \Gamma_{|n|} + \Gamma_{(|n|>4)},$$

$|n|$ counting the number of fields, and extract its relevant part, i.e. its local field content with mass dimension not greater than four. Generally we will not underline the field variable symbols in the Appendices, though of course all arguments in the $\Gamma$-functional should appear underlined. The modification to obtain the bare functional $L^{\Lambda_0,\Lambda_0}$ is stated at the end.

1) One-point function:

$$\Gamma_1 = \kappa \hat{h}(0).$$

2) Two-point functions:
\[
\Gamma_2 = \int_p \left\{ \frac{1}{2} A^a_\mu(p) A^a_\nu(-p) \Gamma_{\mu\nu}^{AA}(p) + \frac{1}{2} h(p) h(-p) \Gamma^{hh}(p) \\
+ \frac{1}{2} B^a_\mu(p) B^a_\nu(-p) \Gamma_{\mu\nu}^{BB}(p) - \bar{c}^a(p) c^a(-p) \Gamma_{\mu}^{\bar{c}c}(p) \\
+ A^a_\mu(p) B^a_\nu(-p) \Gamma_{\mu\nu}^{AB}(p) \right\},
\]

\[
\Gamma_{\mu\nu}^{AA}(p) = \delta_{\mu\nu}(m^2 + \delta m^2) + (p^2 \delta_{\mu\nu} - p_\mu p_\nu)(1 + \Sigma_{\text{trans}}(p^2)) \\
+ \frac{1}{\Lambda}(p_\mu p_\nu(1 + \Sigma_{\text{long}}(p^2))),
\]

\[
\Gamma^{hh}(p) = p^2 + M^2 + \Sigma^{hh}(p^2), \quad \Gamma^{BB}(p) = p^2 + \alpha m^2 + \Sigma^{BB}(p^2), \\
\Gamma^{\bar{c}c}(p) = p^2 + \alpha m^2 + \Sigma^{\bar{c}c}(p^2), \quad \Gamma^{AB}(p) = i p_\mu \Sigma^{AB}(p^2).
\]

Besides the unregularized tree order explicitly stated, there emerge 10 relevant parameters from the various self-energies:

\[
\delta m^2, \Sigma_{\text{trans}}(0), \Sigma_{\text{long}}(0), \Sigma^{hh}(0), \Sigma^{BB}(0), \Sigma^{\bar{c}c}(0), \Sigma^{\bar{c}c}(0), \Sigma^{AB}(0)
\]

where the notation \( \Sigma(0) \equiv (\partial_{p^2} \Sigma)(0) \) has been used. We note, that in transforming the regularized \( L \)-functional into the corresponding \( \Gamma \)-functional the inverse of the regularized propagators \( \Gamma \) become the 2-point functions of the latter in the tree order \( l = 0 \). The factor \((\sigma_{AA}(p^2))^{-1}\) thus appearing, however, does not contribute to the relevant part due to the property \( [1,25] \).

3) Three-point functions:

Only the relevant part is given explicitly: \( r \in \mathcal{O}(\hbar) \) denotes a relevant parameter which vanishes in the tree order, otherwise a relevant parameter is denoted by \( F \). Moreover, we indicate an irrelevant part by a symbol \( \mathcal{O}_n \), \( n \in \mathbb{N} \), indicating that this part vanishes as an \( n \)-th power of the momentum in the limit when all momenta tend to zero homogeneously.

\[
\Gamma_3 = \int_p \int_q \left\{ \epsilon^{rst} A_{\mu}^r(p) A_{\nu}^s(q) A_{\lambda}^t(-p - q) \Gamma_{\mu\nu\lambda}^{AAA}(p, q) \\
+ A_{\mu}^r(p) A_{\nu}^s(q) h(-p - q) \Gamma_{\mu\nu}^{Ahh}(p, q) \\
+ \epsilon^{rst} B_{\nu}^r(p) B_{\lambda}^s(q) A_{\mu}^t(-p - q) \Gamma_{\mu\nu\lambda}^{BBB}(p, q) \\
+ h(p) B_{\nu}^r(q) A_{\mu}^t(-p - q) \Gamma_{\mu\nu}^{Bhh}(p, q) + \epsilon^{rst} \bar{c}^r(p) c^s(q) A_{\mu}^t(-p - q) \Gamma_{\mu\nu}^{\bar{c}cA}(p, q) \\
+ B_{\nu}^r(q) h(-p - q) \Gamma_{\mu\nu}^{BBh}(p, q) + h(p) h(q) h(-p - q) \Gamma_{\mu\nu}^{hh}(p, q) \\
+ \bar{c}^r(p) c^s(q) h(-p - q) \Gamma_{\mu\nu}^{\bar{c}c}(p, q) + \epsilon^{rst} \bar{c}^r(p) c^s(q) B_{\mu}^t(-p - q) \Gamma_{\mu\nu}^{\bar{c}cB}(p, q) \right\},
\]
The 3-point functions explicitly.

With parameters

\[ \Gamma \]

Hence, in total \( \Gamma \) involves \( 1 + 10 + 11 + 15 = 37 \) relevant parameters.

4) Four-point functions:

With parameters \( r \) and \( F \) defined as before

\[ \Gamma_{4\text{rel}} = \int_k \int_p \int_q \{ e^{abc} e^{ars} A^b_\mu(k) A^r_\mu(p) A^s_\mu(q) A^c_\mu(-k - p - q) F_{1}\text{AAAA} \]

\[ + A^b_\mu(k) A^r_\mu(p) A^s_\mu(q) A^c_\mu(-k - p - q) r_2^{AAB} \]

\[ + A^b_\mu(k) A^r_\mu(p) \bar{c}^s_\mu(q) \bar{c}^c_\mu(-k - p - q) (\delta^{ab} \delta^{rs} r_1^{AAP} + \delta^{as} \delta^{bc} r_2^{AAP}) \]

\[ + B^c_\mu(k) B^r_\mu(p) \bar{c}^s_\mu(q) \bar{c}^c_\mu(-k - p - q) (\delta^{ab} \delta^{rs} r_1^{ABB} + \delta^{cs} \delta^{bc} r_2^{ABB}) \]

\[ + h(k) h(p) h(q) h(-k - p - q) F_{1}\text{hhh} \]

\[ + B^c_\mu(k) B^r_\mu(p) h(q) h(-k - p - q) F_{1}\text{Bhh} \]

\[ + B^c_\mu(k) B^r_\mu(p) B^s_\mu(q) B^c_\mu(-k - p - q) F_{1}\text{BBB} \]

\[ + A^b_\mu(k) A^r_\mu(p) h(q) h(-k - p - q) F_{1}\text{Ahb} \]

\[ + h(k) h(p) \bar{c}^s_\mu(q) \bar{c}^c_\mu(-k - p - q) r_1^{Bhc} \]

\[ + \bar{c}^s_\mu(k) \bar{c}^c_\mu(p) \bar{c}^c_\mu(-k - p - q) r_1^{Bhc} \]

\[ + e^{s\alpha}(\delta^{ab} \delta^{rs} r_1^{AAP} + \delta^{as} \delta^{bc} r_2^{AAP}) \}

\[ F_{1}\text{AAAA} = \frac{1}{8} g^2 + r_1^{AAB} \]

\[ F_{1}\text{ABB} = \frac{1}{8} g^2 + r_1^{AAB} \]

\[ F_{1}\text{BBB} = \frac{1}{8} g^2 + r_1^{Bhh} \]

Hence, in total \( \Gamma \) involves \( 1 + 10 + 11 + 15 = 37 \) relevant parameters.

After deleting in the two-point functions the contributions of the order \( l = 0 \), i.e. keeping only the 10 parameters which appear in the various self-energies, we have the form of the bare functional \( L^{A_0 A_0} \), and its order \( l = 0 \) also given explicitly.

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The relevant part of the BRS-insertions

We also have to consider the vertex functions (4.74-4.75) with one operator insertion, generated by the BRS-variations. These insertions have mass dimension \( D = 2 \). Performing the Fourier-transform

\[
\hat{\Gamma}_0^{\Lambda_0}(q) = \int dx \ e^{i q x} \Gamma_0^{\Lambda_0}(x)
\]

and similarly in the other cases, we list the respective relevant part of these four vertex functions with one insertion, suppressing the superscript \( 0, \Lambda_0 \):

\[
\begin{align*}
\hat{\Gamma}_\gamma^a(q)|_{\text{rel}} &= -i q_\mu c^a(-q)R_1 + e^{arb} \int_k A_\mu^r(k)c^b(-q-k)gR_2, \\
\hat{\Gamma}_\gamma(q)|_{\text{rel}} &= \int_k B^r(k)c^r(-q-k)(-\frac{1}{2}gR_3), \\
\hat{\Gamma}_\gamma^a(q)|_{\text{rel}} &= mc^a(-q)R_4 \\
&\quad + \int_k h(k)c^a(-q-k)\frac{1}{2}gR_5 + e^{arb} \int_k B^r(k)c^b(-q-k)\frac{1}{2}gR_6, \\
\hat{\Gamma}_\omega^a(q)|_{\text{rel}} &= e^{ars} \int_k c^r(k)c^s(-q-k)\frac{1}{2}gR_7.
\end{align*}
\]

There appear 7 relevant parameters

\[
R_i = 1 + r_i, \quad r_i = \mathcal{O}(\hbar), \quad i = 1, \ldots, 7.
\]

All the other 2-point functions, and the higher ones, of course, are of irrelevant type.

The relevant part of \( \Gamma_1 \)

As a consequence of the expansion in the mass parameters the conditions following from the fact that the relevant part of the functional \( \Gamma_1 \) should vanish

\[
\Gamma_1(\mathcal{A}, \mathcal{B}, \bar{\mathcal{B}}, \bar{\mathcal{C}})|_{\text{dim}\leq 5} = 0.
\]

can be reordered according to the value of \( \nu \) which appears. We get contributions for \( 0 \leq \nu \leq 3 \). The value of \( \nu \) in the various relevant couplings is indicated as a superscript in parentheses \( \text{if } \nu > 0 \). We explicitly indicate the momentum and the power of \( m \) in front of each STI. The power of \( m \) indicates the value of \( \nu \) in the corresponding contribution to \( \Gamma_1 \).

Two fields
\[ I) \; \delta_{A_{i}(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

a) \[ 0 \overset{!}{=} m^{2}q_{\mu}\left\{ -(1 + \delta m^{2}_{(2)}) R_{1} + \sum^{AB(1)} R_{4} + 1 + \frac{1}{\alpha} \sum^{cc(2)} \right\}, \]

b) \[ 0 \overset{!}{=} q^{2}q_{\mu}\left\{ -\frac{1}{\alpha}(1 + \sum_{\text{long}}) R_{1} + \frac{1}{\alpha}(1 + \sum^{cc}) \right\}. \]

\[ II) \; \delta_{B^{*}(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

a) \[ 0 \overset{!}{=} m^{2}\left\{ (\alpha + \sum^{BB(2)} R_{4} - (\alpha + \sum^{cc(2)}) - \frac{3}{2} \sum^{cc(3)} R_{3} \right\}, \]

b) \[ 0 \overset{!}{=} m q^{2}\left\{ -\sum^{AB(1)} R_{1} + (1 + \sum^{BB}) R_{4} - (1 + \sum^{cc}) \right\}. \]

Three fields

\[ III) \; \delta_{A_{i}(p)}\delta_{A_{j}(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

a) \[ 0 \overset{!}{=} (p_{\mu}p_{\nu} - q_{\mu}q_{\nu})\left\{ -2F^{AAA}R_{1} + \frac{1}{\alpha} (F^{ccA} - r_{2}^{ccA}) + \frac{1}{\alpha}(1 + \sum_{\text{long}}) - (1 + \sum_{\text{trans}}) \right\} g R_{2}, \]

b) \[ 0 \overset{!}{=} (p^{2} - q^{2})\delta_{\mu\nu}\left\{ 2F^{AAA}R_{1} + (1 + \sum_{\text{trans}}) g R_{2} \right\}, \]

\[ IV) \; \delta_{A_{i}(p)}\delta_{B^{*}(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

a) \[ 0 \overset{!}{=} m p_{\mu}\left\{ 2F^{BBB}R_{4} + \frac{1}{2} g \sum^{AB(1)} R_{6} + \frac{1}{\alpha} F^{ccB(1)} - r_{2}^{ccA} \right\}, \]

b) \[ 0 \overset{!}{=} m q_{\mu}\left\{ g \sum^{AB(1)} R_{2} + 4F^{BBB}R_{4} + (F^{ccA} - r_{2}^{ccA}) \right\}, \]

\[ V) \; \delta_{B^{*}(p)}\delta_{B^{*}(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

\[ 0 \overset{!}{=} (p^{2} - q^{2})\left\{ 2R_{1}F^{BBB} + (1 + \sum^{BB}) \frac{g}{2} R_{6} \right\}, \]

\[ VI) \; \delta_{A_{i}(p)}\delta_{h(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

a) \[ 0 \overset{!}{=} m p_{\mu}\left\{ -2R_{1}F^{AAh(1)} + R_{4}(F^{hBA} - r_{2}^{hBA}) + \sum^{AB(1)} \frac{1}{2} g R_{5} - \frac{1}{\alpha} F^{cch(1)} \right\}, \]

b) \[ 0 \overset{!}{=} m q_{\mu}\left\{ -2R_{1}F^{AAh(1)} + 2R_{4}F^{hBA} \right\}, \]

\[ VII) \; \delta_{h(p)}\delta_{B^{*}(q)}\delta_{e_{c}(k)}\Gamma_{1}|0 \]

a) \[ 0 \overset{!}{=} m^{2}\left\{ \left( \frac{M^{2}}{m^{2}} + \sum^{hh(2)} \right)(-\frac{1}{2} g R_{3}) + 2F^{BBh(1)}R_{4} + F^{cch(1)} + (\alpha + \sum^{BB(2)})\frac{1}{2} g R_{5} \right\}, \]

b) \[ 0 \overset{!}{=} p^{2}\left\{ F^{hBA}R_{1} - (1 + \sum^{hh})\frac{1}{2} g R_{3} \right\}, \]
c) \(0 \overset{\text{I}}{=} q^2 \left\{ - F_1^{hBA} R_1 + (1 + \sum^{BB} \frac{1}{2}) g R_5 \right\},
\)

d) \(0 \overset{\text{I}}{=} k^2 \left\{ r_2^{hBA} R_1 \right\},
\)

VIII) \(\delta_{\nu(q)} \delta_{\rho(p)} \delta_{\tau(k)} \Gamma_1|_0\)

a) \(0 \overset{\text{I}}{=} m^2 \left\{ 2 F^{\bar{\nu}\bar{c}B(1)} R_4 - (\alpha + \sum^{\bar{\nu}\bar{c}(2)}) g R_7 \right\},
\)

b) \(0 \overset{\text{I}}{=} k^2 \left\{ F_1^{\bar{v}\bar{c}A} R_4 - r_2^{\bar{v}\bar{c}A} R_1 - (1 + \sum^{\bar{v}\bar{c}}) g R_7 \right\},
\)

c) \(0 \overset{\text{I}}{=} (p^2 + q^2) \left\{ r_2^{\bar{v}\bar{c}A} R_1 \right\}.
\)

Four fields

IX) \(\delta_{h(q)} \delta_{h(p)} \delta_{B^1(k)} \delta_{\nu(l)} \Gamma_1|_0\)

\(0 \overset{\text{I}}{=} m \left\{ 6 F^{hBhh(1)} (-\frac{1}{2} g R_3) + 4 F^{BBhh} R_4 + 2 F^{BBh, (1)} g R_5 + 2 r^{hB\bar{c}c} \right\}.
\)

X) \(\delta_{B^1(k)} \delta_{B^2(p)} \delta_{B^2(q)} \delta_{\nu(l)} \Gamma_1|_0\)

\(0 \overset{\text{I}}{=} m \left\{ - F^{BBh, (1)} g R_3 + 8 F^{BBBB} R_4 + (2 r_1^{BB\bar{c}c} + r_2^{BB\bar{c}c}) \right\}.
\)

XI) \(\delta_{h(l)} \delta_{c(k)} \delta_{c(p)} \delta_{c(q)} \Gamma_1|_0\)

\(0 \overset{\text{I}}{=} m \left\{ 2 r_1^{hB\bar{c}c} R_4 + F^{\bar{c}cB(1)} g R_5 + F^{\bar{c}c, (1)} g R_7 \right\}.
\)

XII) \(\delta_{c(k)} \delta_{c(l)} \delta_{c(p)} \delta_{c(q)} \Gamma_1|_0\)

\(0 \overset{\text{I}}{=} m \left\{ F^{\bar{c}c, (1)} (-\frac{1}{2} g R_3) + (2 r_1^{BB\bar{c}c} - r_2^{BB\bar{c}c}) R_4 + F^{\bar{c}cB(1)} (\frac{1}{2} g R_6 - g R_7) + 2 r^{c\bar{c}\bar{c}c} \right\}.
\)

XIII) \(\delta_{A_{1}^{\nu}(k)} \delta_{A_{2}^{\nu}(q)} \delta_{B^1(l)} \delta_{c(l)} \Gamma_1|_0\)

\(0 \overset{\text{I}}{=} 2 r_2^{AABB} R_4 + r_2^{AA\bar{c}c}.
\)

XIII) \(\delta_{A_{1}^{\nu}(k)} \delta_{A_{1}^{\nu}(p)} \delta_{B^2(q)} \delta_{c(l)} \Gamma_1|_0\)

\(0 \overset{\text{I}}{=} m \left\{ - F^{AA\bar{c}h(1)} g R_3 + 4 F_1^{AABB} R_4 + 2 r_1^{AA\bar{c}c} \right\}.
\)

XIV) \(\delta_{A_{1}^{\nu}(p)} \delta_{A_{1}^{\nu}(q)} \delta_{A_{2}^{\nu}(k)} \delta_{c(l)} \Gamma_1|_0\)

a) \(0 \overset{\text{I}}{=} 2 \delta_{\nu \mu} l_\nu \left\{ 4 (F_1^{AAAA} + r_2^{AAAA}) R_1 + 2 F^{AAAA} g R_2 + \frac{1}{\alpha} r_1^{AA\bar{c}c} \right\},
\)

b) \(0 \overset{\text{I}}{=} \delta_{\mu \nu} (p_\rho + q_\rho) \left\{ \frac{2}{\alpha} r_1^{AA\bar{c}c} \right\}.
\)
\( c) \quad 0 \overset{1}{=} (\delta_{\mu\nu} I_{\mu} + \delta_{\nu\mu} I_{\nu}) \left\{ -4F_1^{AAAA} R_1 - 2F^{AAAA} gR_2 \right\}, \\
\( d) \quad 0 \overset{1}{=} (\delta_{\mu\nu} P_{\nu} + \delta_{\nu\mu} q_{\mu}) \{ 0 \}, \\
\( e) \quad 0 \overset{1}{=} (\delta_{\mu\nu} q_{\nu} + \delta_{\nu\mu} P_{\mu}) \left\{ -\frac{1}{\alpha} r^{AAcc} \right\}. \\
XV) \quad \delta^{B_1(p)} B_1^{(q)} A_1^{(k)} \delta^{\ell_1(l)} \Gamma_1 | 0 \\
\( a) \quad 0 \overset{1}{=} l_{\mu} \left\{ 4F_1^{AABB} R_1 + 2F^{BBBA} gR_6 \right\}, \\
\( b) \quad 0 \overset{1}{=} k_{\mu} \left\{ r_1^{BBcc} \right\}, \\
XVII) \quad \delta^{B_2(p)} B_2^{(q)} \delta^{\ell_2(l)} \delta^{\ell_1(l)} \Gamma_1 | 0 \\
\( a) \quad 0 \overset{1}{=} p_{\mu} \left\{ -2r_2^{AAAA} R_1 + 2F^{BBBA} gR_2 + F_1^{hBA} gR_3 \right\}, \\
\( b) \quad 0 \overset{1}{=} q_{\mu} \left\{ -2r_2^{AAAA} R_1 - 2F^{BBBA} gR_2 + 2F^{BBBA} gR_6 \right\}, \\
\( c) \quad 0 \overset{1}{=} k_{\mu} \left\{ -2r_2^{AAAA} R_1 + F_1^{hBA} gR_3 + r_2^{hBA} \overline{g} R_3 + F^{BBBA} gR_6 - \frac{1}{\alpha} r_2^{BBcc} \right\}, \\
XVIII) \quad \delta^{A_2^{(p)}} \delta^{A_1^{(k)}} \delta^{\ell_2(l)} \delta^{\ell_1(l)} \Gamma_1 | 0 \\
\( a) \quad 0 \overset{1}{=} l_{\mu} \left\{ 4F_1^{AAhh} R_1 - F_1^{hBA} gR_3 \right\}, \\
\( b) \quad 0 \overset{1}{=} k_{\mu} \left\{ r_2^{hBA} \overline{g} R_6 + F^{BBBA} gR_5 - \frac{1}{\alpha} r_2^{BBcc} \right\}, \\
XV) \quad \delta^{B_1(p)} \delta^{B_1^{(q)}} \delta^{A_1^{(k)}} \delta^{\ell_1(l)} \Gamma_1 | 0 \\
\( a) \quad 0 \overset{1}{=} p_{\mu} \left\{ F_1^{hBA} g(R_6 - R_2) - r_2^{hBA} gR_2 \right\}, \\
\( b) \quad 0 \overset{1}{=} q_{\mu} \left\{ F_1^{hBA} gR_2 - r_2^{hBA} gR_2 + 2F^{BBBA} gR_5 \right\}, \\
\( c) \quad 0 \overset{1}{=} k_{\mu} \left\{ F_1^{hBA} gR_6 - r_2^{hBA} \overline{g} R_6 + F^{BBBA} gR_5 - \frac{1}{\alpha} r_2^{BBcc} \right\}, \\
XVII) \quad \delta^{B_2(p)} \delta^{B_2^{(q)}} \delta^{A_1^{(k)}} \delta^{\ell_1(l)} \Gamma_1 | 0 \\
\( a) \quad 0 \overset{1}{=} l_{\mu} \left\{ F_1^{\overline{c}A} g(R_2 - R_7) + \frac{1}{\overline{c}} r^{\overline{c}cc} \right\}, \\
\( b) \quad 0 \overset{1}{=} p_{\mu} \left\{ 2r_1^{AA\overline{c}} R_1 + r_1^{\overline{c}A} g(R_2 - R_7) + \frac{1}{\overline{c}} r^{\overline{c}cc} \right\}, \\
\( c) \quad 0 \overset{1}{=} q_{\mu} \left\{ -r_2^{AA\overline{c}} R_1 - r_2^{\overline{c}A} gR_7 + \frac{1}{\overline{c}} r^{\overline{c}cc} \right\}. \\
\}

Five fields

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XIX) \[ \delta_{h(p)} \delta_{h(q)} \delta_{h(k)} \delta_{B^1(t)} \delta_{c^1(l)} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} -2 F^{hhhh} R_3 + F^{hhBB} R_5. \]

XX) \[ \delta_{h(p)} \delta_{B^1(q)} \delta_{B^1(k)} \delta_{c^2(l)} \delta_{c^2(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} -2 F^{BBhh} R_3 + 2 F^{BBBB} R_5. \]

XXI) \[ \delta_{A^2_2(k)} \delta_{A^2_2(p)} \delta_{c^2(q)} \delta_{c^2(l)} \delta_{c^2(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} -F^{AAhh} R_3 + F_1^{AAAB} R_5. \]

XXII) \[ \delta_{A^2_2(k)} \delta_{B^1(q)} \delta_{c^3(l)} \delta_{A^2_2(q)} \delta_{B^2(l)} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} r_2^{ABB} (R_6 - 2 R_2). \]

XXIII) \[ \delta_{A^2_2(k)} \delta_{B^1(q)} \delta_{A^2_2(p)} \delta_{c^3(l)} \delta_{c^3(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} r_2^{ABB} R_5. \]

XXIV) \[ \delta_{A^2_2(k)} \delta_{A^2_2(p)} \delta_{c^2(q)} \delta_{c^3(l)} \delta_{c^3(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} r_2^{AACC} R_2 + r_1^{AACC} R_7. \]

XXV) \[ \delta_{A^2_2(k)} \delta_{c^3(q)} \delta_{A^2_2(p)} \delta_{c^3(l)} \delta_{c^3(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} r_2^{AACC} (3 R_2 - R_7). \]

XXVI) \[ \delta_{B^1(q)} \delta_{B^1(q)} \delta_{c^2(l)} \delta_{c^2(l)} \delta_{c^2(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} r_2^{BBcc} (R_6 - R_7) - r_1^{BBcc} R_7. \]

XXVII) \[ \delta_{B^1(q)} \delta_{c^3(k)} \delta_{B^2(q)} \delta_{c^3(l)} \delta_{c^3(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} -r_1^{BBcc} R_3 + r_2^{BBcc} (3 R_6 - 2 R_7). \]

XXVIII) \[ \delta_{h(p)} \delta_{h(q)} \delta_{c^2(l)} \delta_{c^2(l)} \delta_{c^2(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} r_1^{BBcc} R_5 + r_1^{hhhcc} R_7. \]

XXIX) \[ \delta_{h(p)} \delta_{B^1(q)} \delta_{c^2(l)} \delta_{A^2_2(k)} \delta_{c^2(l')} \Gamma_1 \big|_0 \]
\[ 0 \overset{!}{=} 2 r_1^{BBcc} R_3 - 2 r_1^{BBcc} R_5 + r_2^{BBcc} R_5 + r_1^{BBcc} (- R_6 + 2 R_7). \]
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