Intermixture of extended edge and localized bulk energy levels in macroscopic Hall systems

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Abstract

We study the spectrum of a random Schrödinger operator for an electron submitted to a magnetic field in a finite but macroscopic two dimensional system of linear dimensions equal to $L$. The $y$ direction is periodic and in the $x$ direction the electron is confined by two smooth increasing boundary potentials. The eigenvalues of the Hamiltonian are classified according to their associated quantum mechanical diamagnetic current in the $y$ direction. Here we look at an interval of energies inside the first Landau band of the random operator for the infinite plane. In this energy interval, with large probability, there exist $\mathcal{O}(L)$ eigenvalues with positive or negative currents of $\mathcal{O}(1)$. Between each of these there exist $\mathcal{O}(L^2)$ eigenvalues with infinitesimal current $\mathcal{O}(e^{-\gamma B (\log L)^2})$. We explain what is the relevance of this analysis of boundary diamagnetic currents to the integer quantum Hall effect.
1 Introduction

In this paper we are concerned about boundary currents in the integer quantum Hall effect, that occurs in disordered electronic systems subject to a uniform magnetic field and confined in a two dimensional interface of an heterojunction [1]. It was recognized by Halperin that boundary diamagnetic equilibrium currents play an important role in understanding the transport properties of such systems [2]. Later on it was realized that there is an intimate connection between these boundary currents and the topological properties of the state in the bulk [3], [4]. Here we will study only diamagnetic currents due to the boundaries, and not those produced by the adiabatic switching of an external infinitesimal electric field (as in linear response theory) which may exist in the bulk. Many features of the integral quantum Hall effect can be described in the framework of one particle random magnetic Schrödinger operators and therefore it is important to understand their spectral properties for finite but macroscopic samples with boundaries. This problem has been approached recently for geometries where only one boundary is present and the operator is defined in a semi-infinite region [5], [6], [7].

Here we will take a finite system: our geometry is that of a cylinder of length and circumference both equal to $L$. There are two boundaries at $x = \pm \frac{L}{2}$ modelled by two smooth confining potentials $U_\ell(x)$ ($\ell$ for left) and $U_r(x)$ ($r$ for right), and we take periodic boundary conditions in the $y$ direction. These potentials vanish for $-\frac{L}{2} \leq x \leq \frac{L}{2}$ and grow fast enough for $|x| \geq \frac{L}{2}$. The Hamiltonian is of the form

$$H_\omega = H_0 + V_\omega + U_\ell + U_r$$

(1.1)

where $H_0$ is the pure Landau Hamiltonian for a uniform field of strength $B$ and $V_\omega$ is a suitable weak random potential produced by impurities with $\sup |V_\omega(x, y)| = V_0 \ll B$ (see section 2 for precise assumptions). Before explaining our results it is useful to describe what is known about the infinite and semi-infinite case.

In the case of the infinite plane $\mathbb{R}^2$ for the Hamiltonian $H_0 + V_\omega$ the spectrum forms “Landau bands” contained in $\bigcup_{\nu \geq 0} \left[ (\nu + \frac{1}{2})B - V_0, (\nu + \frac{1}{2})B + V_0 \right]$. It is proved that the band tails have pure point spectrum corresponding to exponentially localized wavefunctions [8], [9], [10], [11], [12]. There are no rigorous results for energies at the band centers, except for a special model where the impurities are point scatterers [13], [14]. As first shown in [15] these spectral properties of random Schrödinger operators imply that the Hall conductivity – given by the Kubo formula – considered as a function of the filling factor (ratio of electron number and number of flux quanta) has quantized
plateaux at values equal to $\nu e^2/h$ where $\nu$ is the number of filled Landau levels. The presence of the plateaux is a manifestation of Anderson localization while the quantization has a topological origin. The latter was first discovered in particular situations \[16\], and it has been proved for more general models using non commutative geometry \[17\] and the index of Fredholm operators \[18\] (see \[19\] for a review).

In a semi-infinite system where the particle is confined in a half plane with Hamiltonian $H_0 + V_\omega + U_\ell$ (here $(x,y)$ belongs to $\mathbb{R}^2$) the spectrum includes all energies in $[\frac{B}{2}, +\infty]$. The lower edge of the spectrum is between $\frac{B}{2} - V_0$ and $\frac{B}{2}$ and in its vicinity the spectrum is pure point (this follows from techniques in \[11\]). For energies in intervals inside the gaps of the bulk Hamiltonian $H_0 + V_\omega$ the situation is completely different. One can show that the average velocity $(\psi, v_y \psi)$ in the $y$ direction of an assumed eigenstate $\psi$ does not vanish, but since the velocity $v_y$ is the commutator between $y$ and the Hamiltonian, this implies that the eigenstate cannot exist, and that therefore the spectrum is purely continuous \[5\], \[20\]. In fact Mourre theory has been suitably applied to prove that the spectrum is purely absolutely continuous \[6\], \[7\]. These works put on a rigorous basis the expectation that, because of the chiral nature of the boundary currents, the states remain extended in the $y$ direction even in the presence of disorder \[2\]. The same sort of analysis shows that if the $y$ direction is made periodic of length $L$, the same energy intervals have discrete eigenstates which carry a current that is $O(1)$ – say positive – with respect to $L$ \[8\]. Furthermore one can show that the eigenvalue spacing is of order $O(L^{-1})$ \[21\].

The nature of the spectrum for a semi-infinite system for intervals inside the Landau bands of the bulk Hamiltonian $\bigcup_{\nu \geq 0} \left[ (\nu + \frac{1}{2})B - V_0, (\nu + \frac{1}{2})B + V_0 \right]$ has not yet been elucidated.

For the finite system on a cylinder with two boundaries the spectrum consists of finitely degenerate isolated eigenvalues. In \[22\] the results of \[5\], \[6\] for energy intervals inside the gaps of the bulk Hamiltonian are extended to the present two boundary system. The eigenvalues can be classified in two sets distinguished by the sign of their associated current. These currents are uniformly positive or uniformly negative with respect to $L$. For this result to hold it is important to take the circumference and the length of the cylinder both of the order $L$.

In the present work we study the currents of the eigenstates for eigenvalues in the

\footnote{In principle the physical current is $L^{-1}(\psi, v_y \psi)$, but here we will call current the average velocity $(\psi, v_y \psi)$.}
interval \( \Delta \epsilon = \begin{bmatrix} B_2 + \epsilon, \frac{B_2}{2} + V_0 \end{bmatrix} \) where \( \epsilon \) is a small positive number independent of \( L \). We limit ourselves to the first band to keep the discussion simpler. The content of our main result (Theorem 1) is the following. Given \( \epsilon \), for \( L \) large enough there is an ensemble of realizations of the random potential with probability \( 1 - O(L^{-s}) \) for which the eigenvalues of \( H_\omega \) can be classified into three sets that we call \( \Sigma_\ell, \Sigma_r, \) and \( \Sigma_b \). The eigenstates of \( \Sigma_r \) (resp. \( \Sigma_\ell \)) have uniformly positive (resp. negative) currents with respect to \( L \), while those of \( \Sigma_b \) have a current of the order of \( O(e^{-cB(logL)^2}) \). The number of eigenvalues in \( \Sigma_\alpha \) (\( \alpha = \ell, r \)) is \( O(L) \) while that in \( \Sigma_b \) is \( O(L^2) \). This classification of eigenvalues leads to a well defined notion of extended edge and localized bulk states. The edge states are those which belong to \( \Sigma_\alpha \) (\( \alpha = \ell, r \)) and are extended in the sense that they have a current of order \( O(1) \). The bulk states are those which belong to \( \Sigma_b \) and are localized in the sense that their current is infinitesimal. The energy levels of the extended and localized states are intermixed in the same energy interval. See also [23] for a short review on spectral properties of systems defined on a cylinder.

Let us explain the mechanism that is at work. When the random potential is removed \( V_\omega = 0 \) in (1.1) the eigenstates with energies away from \( \frac{B}{2} \) are extended in the \( y \) direction and localized in the \( x \) direction at a finite distance from the boundaries. Their energies form a sequence of “edge levels” and have a spacing of the order of \( O(L^{-1}) \). When the potential of one impurity is added to \( H_0 \) it typically creates a localized bound state with energy between the Landau levels. Suppose now that i) a coupling constant in the impurity potential is fine tuned as a function of \( L \) so that the energy of the impurity level stays at distance greater than \( L^{-p} \) from the edge levels, ii) the position of the impurity is at a distance \( D \) from the boundaries. Then the mixing between the localized bound state and the extended edge states is controlled in second order perturbation theory by the parameter \( L^p e^{-cBD^2} \). Therefore one expects that bound states of impurities that have \( D \gg (\log L)^{1/2} \) are basically unperturbed and have an infinitesimal current. On the other hand bound states coming from impurities with \( D \ll (\log L)^{1/2} \) will mix with edge states. Note that even for impurities with \( D \gg (\log L)^{1/2} \) the coupling constant (equivalently the impurity level) has to be fine tuned as a function of \( L \). Indeed, for a coupling constant with a fixed value the energy of the impurity level is independent of \( L \), and surely for \( L \) large enough the energy difference between the impurity and the edge levels becomes much smaller than \( O(e^{-cBD^2}) \). Remarkably for a random potential the absence of resonance is automatically achieved with large probability and no fine tuning is needed: this is why localized bulk states survive. We have analyzed this
mechanism rigorously for a model (see also [2]) where there are no impurities in a layer of thickness \( \log L \) along the boundary. Then the edge levels are basically non random and the typical spacing between current carrying eigenvalues is easily controlled. Of course it is desirable to allow for impurities close to the boundary but then the edge levels become random and some further analysis is needed. However we expect that the same basic mechanism operates because the typical spacing between edge levels should still be \( O(L^{-1}) \). In connection to the discussion above we mention that for a semi-infinite system the bound state of an impurity at any fixed distance from the boundary turns into a resonance. A similar situation has been analysed in [24].

We note that the spectral region close to \( B \) that is left out in our theorem is precisely the one where resonances between edge and bulk states may occur because edge states become very dense. It is not clear what is the connection with the divergence of the localization length of the infinite system at the band center.

In the present work we have shown that in quantum Hall samples there exist well defined notions of extended edge states (current of \( O(1) \)) and localized bulk states (infinitesimal current). Instead of classifying the energy levels according to their current one could try to use level statistics. We expect that the localized bulk states have Poissonian statistics whereas the extended edge states should display a level repulsion. In fact such a strong form of level repulsion is proved in [21] for energies in the gap of the bulk Hamiltonian where only extended edge states exist. It is interesting to observe that in the present situation both kind of states have intermixed energy levels. In usual Schrödinger operators (e.g. the Anderson model on a 3D cubic lattice) it is accepted (but not proven) that they are separated by a well defined mobility edge (results in this direction have recently been obtained [25] under a suitable hypothesis). The states at the band edge are localized in the sense that the spectrum is dense pure point for the infinite lattice and has Poisson statistics for the finite system [26]. At the band center the states are believed to be extended in the sense that the spectrum is absolutely continuous for the infinite lattice and has the statistics of the Gaussian Orthogonal Ensemble for the finite lattice.

Others ways of formulating the notion of edge states have been proposed in differents contexts. In [27] the authors consider a clean system with a novel kind of chiral boundary conditions. The Hilbert space then separate in two parts corresponding to edge and bulk states. The bulk states have exactly the Landau energy and the edge
states a linear dispertion relation; the distinction between them being sharp because of the special nature of boundary conditions. It would be interesting to extend this definition to disordered systems. Recently in [28] (see also [29]) another approach has been used in the context of magnetic billiards. The authors study a magnetic billiard with mixed boundary conditions with mixing parameter Λ interpolating between Dirichlet and Neumann boundary conditions. They look at the sensibilit y of the eigenstates and eigenvalues under the variation of Λ and define in this way an edge state as a state that depends strongly on Λ. Let us note that our notion of edge state as well as the other ones all share the feature that an edge state carries a substantial current.

The characterization of the spectrum of (1.1) proposed here also has a direct relevance to the Hall conductivity of the many electron (non interacting) system. In the formulation advocated by Halperin [2] the Hall conductivity is computed as the ratio of the net equilibrium current and the difference of chemical potentials between the two edges. Consider the many fermion state Ψ(µℓ, µr, EF) obtained by filling the levels of Hω (one particle per state) in Σℓ ∩ [B / 2 + ε, µℓ], Σr ∩ [B / 2 + ε, µr] and Σb ∩ [B / 2 + ε, EF] with B / 2 + ε < µℓ < EF < µr < B / 2 + V0. The total current \( I(µℓ, µr, EF) \) of this state – a stationary state of the many particle Hamiltonian – is given by the sum of the individual physical currents of the filled levels (given by \( L^{-1}(ψ, v_ψ) \)). From the estimates (2.16) and (2.18) in Theorem 1

\[
\sum_k J^\ell_k + \sum_k J^r_k + \sum J_β = \sum_k J^\ell_{0k} + \sum_k J^r_{0k} + O(e^{-(logL)^2}L^2) \tag{1.2}
\]

and from (2.11) we get

\[
\frac{1}{L} \sum_k J^r_{0k} = \frac{1}{2\pi} \int_{B / 2 + \varepsilon}^{µr} dE + O(L^{-1}) \tag{1.3}
\]

\[
\frac{1}{L} \sum_k J^\ell_{0k} = \frac{1}{2\pi} \int_{µ\ell}^{B / 2 + \varepsilon} dE + O(L^{-1}) \tag{1.4}
\]

It follows that to leading order

\[
I(µℓ, µr, EF) \simeq \frac{1}{2\pi}(µr - µℓ) \tag{1.5}
\]

In (1.3) the Hall conductance is equal to one (this is because we have considered only the first band). When µℓ and µr vary the density of particles in the state Ψ(µℓ, µr, EF)
does not change since the number of levels in $\Sigma_\alpha (\alpha = \ell, r)$ is of order $O(L)$. However if $E_F$ is increased the particle density (and thus the filling factor) increases since the number of levels in $\Sigma_b$ is of order $O(L^2)$, but the Hall conductance does not change and hence has a plateau. In other words the edge states contribute to the Hall conductance but not to the density of states of the sample in the thermodynamic limit.

In a more complete theory one should also take in account currents possibly flowing in the bulk due to the adiabatic switching of an external electric field, an issue that is beyond the scope of the present analysis. A related problem is the relationship between the conductance in the present picture, defined through (1.5), and the one using Kubo formula (see [30], [31], [32]).

The precise definition of the model and the statement of the main result (Theorem 1) are the subject of the next section.

2 The Structure of the Spectrum

We consider the family of random Hamiltonians (1.1) acting on the Hilbert space $L^2(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}])$ with periodic boundary conditions along $y$, $\psi(x, -\frac{L}{2}) = \psi(x, \frac{L}{2})$. In the Landau gauge the kinetic term of (1.1) is

$$H_0 = \frac{1}{2} p_x^2 + \frac{1}{2} (p_y - Bx)^2$$

and has infinitely degenerate Landau levels $\sigma(H_0) = \{(\nu + \frac{1}{2})B; \nu \in \mathbb{N}\}$. We will make extensive use of explicit point-wise bounds, proved in Appendix A, on the integral kernel of the resolvent $R_0(z) = (z - H_0)^{-1}$ with periodic boundary conditions along $y$.

The confining potentials modelling the two edges at $x = -\frac{L}{2}$ and $x = \frac{L}{2}$ are assumed to be strictly monotonic, differentiable and such that

$$c_1 |x + \frac{L}{2}|^{m_1} \leq U_\ell(x) \leq c_2 |x + \frac{L}{2}|^{m_2} \quad \text{for } x \leq -\frac{L}{2}$$

$$c_1 |x - \frac{L}{2}|^{m_1} \leq U_r(x) \leq c_2 |x - \frac{L}{2}|^{m_2} \quad \text{for } x \geq \frac{L}{2}$$

for some constants $0 < c_1 < c_2$ and $2 \leq m_1 < m_2 < \infty$. Recall that $U_\ell(x) = 0$ for $x \geq \frac{L}{2}$ and $U_r(x) = 0$ for $x \leq \frac{L}{2}$. We could allow steeper confinements but the present polynomial conditions turn out to be technically convenient.

We assume that each impurity is the source of a local potential $V \in C^2$, $0 \leq V(x,y) \leq V_0 < \infty$, $\text{supp } V \subset \mathcal{B}(0, \frac{1}{4})$, and that they are located at the sites of a
finite lattice $\Lambda = \{(n, m) \in \mathbb{Z}^2; n \in [-\frac{L}{2} + \log L, \frac{L}{2} - \log L], m \in [-\frac{L}{2}, \frac{L}{2}]\}$. The random potential $V_\omega$ has the form

$$V_\omega(x, y) = \sum_{(n, m) \in \Lambda} X_{n,m}(\omega)V(x - n, y - m)$$

(2.4)

where the coupling constants $X_{n,m}$ are i.i.d. random variables with common density $h \in C^2([-1, 1])$ that satisfies $\|h\|_{\infty} < \infty$, $\text{supp } h = [-1, 1]$. We will denote by $\mathbb{P}_\Lambda$ the product measure defined on the set of all possible realizations $\omega \in \Omega_\Lambda = [-1, 1]^\Lambda$. Clearly for any realization we have $|V_\omega(x, y)| \leq V_0$. Furthermore it will be assumed that the random potential is weak in the sense that $4V_0 < B$.

We will think of our system as being constituted of three pieces corresponding to the bulk system with the random Hamiltonian

$$H_b = H_0 + V_\omega$$

(2.5)

and the left and right edge systems with non random Hamiltonians

$$H_\alpha = H_0 + U_\alpha, \quad \alpha = \ell, r.$$ 

(2.6)

All the Hamiltonians considered above have periodic boundary conditions along the $y$ direction and are essentially self-adjoint on $C^\infty_0(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}])$. For each realization $\omega$ and size $L$ the spectrum $\sigma(H_\omega)$ of (1.1) (it depends on $L$) consists of isolated eigenvalues of finite multiplicity. In order to state our main result characterizing these eigenvalues we first have to describe the spectra of (2.5) and (2.6).

Let us begin with the edge Hamiltonians (2.6). Here we state their properties without proofs and refer the reader to [5], [20] for more details. Since the edge Hamiltonians $H_\alpha$ commute with $p_y$, they are decomposable into a direct sum

$$H_\alpha = \sum_{k \in \frac{2\pi}{L} \mathbb{Z}} H_\alpha(k) = \sum_{k \in \frac{2\pi}{L} \mathbb{Z}} \left[\frac{1}{2} p_x^2 + \frac{1}{2}(k - Bx)^2 + U_\alpha\right].$$

(2.7)

For each $k$ the one dimensional Hamiltonian $H_\alpha(k)$ has a compact resolvent, thus it has discrete eigenvalues and by standard arguments one can show that they are not degenerate. If the $y$ direction would be infinitely extended, $k$ would vary over the real axis and the eigenvalues of $H_\alpha(k)$ would form spectral branches $\varepsilon_\nu^\alpha(\hat{k})$, $\hat{k} \in \mathbb{R}$ labelled by the Landau level index $\nu$. These spectral branches are strictly monotone, entire functions with the properties $\varepsilon_\nu^\ell(-\infty) = +\infty$, $\varepsilon_\nu^\ell(+\infty) = (\nu + \frac{1}{2})B$ and $\varepsilon_\nu^r(-\infty) = (\nu + \frac{1}{2})B$,
\( \varepsilon_\nu^\prime(+\infty) = +\infty \). Here because of the periodic boundary conditions the set of \( k \) values is discrete so that the spectrum of \( H_\alpha \)

\[
\sigma(H_\alpha) = \left\{ E_{\nu k}^\alpha; \nu \in \mathbb{N}, k \in \frac{2\pi}{L}\mathbb{Z} \right\}
\]

(2.8)

consists of isolated points on the spectral branches \( E_{\nu k}^\alpha = \varepsilon_\nu^\alpha(k), k \in \frac{2\pi}{L}\mathbb{Z} \). The corresponding eigenfunctions \( \psi_{\nu k}^\alpha \) have the form

\[
\psi_{\nu k}^\alpha(x, y) = \frac{1}{\sqrt{L}} e^{\frac{i k y}{m}} \varphi_{\nu k}^\alpha(x)
\]

(2.9)

with \( \varphi_{\nu k}^\alpha \) the normalized eigenfunctions of the one-dimensional Hamiltonian \( H_\alpha(k) \). By definition, the current of the state \( \psi_{\nu k}^\alpha \) in the \( y \) direction is given by the expectation value of the velocity \( v_y = p_y - Bx \),

\[
J_{\nu k}^\alpha = (\psi_{\nu k}^\alpha, v_y \psi_{\nu k}^\alpha) = \int_{\mathbb{R}} |\varphi_{\nu k}^\alpha(x)|^2(k - Bx) \, dx = \partial_k \varepsilon_{\nu}^\alpha(\hat{k}) \bigg|_{\hat{k} = \frac{2\pi m}{L}}
\]

(2.10)

where the last equality follows from the Feynman-Hellman theorem. From (2.10) we notice that for any \( \varepsilon > 0 \), one can find \( j(\varepsilon) > 0 \) and \( L(\varepsilon) \) such that for \( L > L(\varepsilon) \) the states of the two branches \( \nu = 0, \alpha = \ell, r \) with energies \( E_{0 k}^\alpha \geq \frac{1}{2}B + \varepsilon \) satisfy

\[
J_{0 k}^\ell \leq -j(\varepsilon) < 0 \quad J_{0 k}^r \geq j(\varepsilon) > 0 .
\]

(2.11)

In other words the eigenstates of the edge Hamiltonians carry an appreciable current. The spacing of two consecutive eigenvalues greater than \( \frac{1}{2}B + \varepsilon \) satisfies

\[
\left| E_{0 \frac{2\pi (m+1)}{L}}^\alpha - E_{0 \frac{2\pi m}{L}}^\alpha \right| > \frac{j(\varepsilon)}{L} \quad \alpha = \ell, r .
\]

(2.12)

Note that these observations extend to other branches but \( j(\varepsilon) \) and \( L(\varepsilon) \) are not uniform with respect to the index \( \nu \). In the rest of the paper we limit ourselves to \( \nu = 0 \) for simplicity. On the other hand the spacing between the energies of \( \sigma(H_\ell) \) and \( \sigma(H_r) \) is a priori arbitrary. We assume that the confining potentials \( U_\ell \) and \( U_r \) are such that the following hypothesis is fulfilled.

**Hypothesis 1.** Fix any \( \varepsilon > 0 \) and let \( \Delta_\varepsilon = \left[ \frac{1}{2}B + \varepsilon, \frac{1}{2}B + \varepsilon_0 \right] \). There exist \( L(\varepsilon) \) and \( d(\varepsilon) > 0 \) such that for all \( L > L(\varepsilon) \)

\[
\text{dist}(\sigma(H_\ell) \cap \Delta_\varepsilon, \sigma(H_r) \cap \Delta_\varepsilon) \geq \frac{d(\varepsilon)}{L} .
\]

(2.13)
This hypothesis is important because a minimal amount of non-degeneracy between the spectra of the two edge systems is needed in order to control backscattering effects induced by the random potential. Indeed in a system with two boundaries backscattering favors localization and has a tendency to destroy currents. This hypothesis can easily be realized by taking non-symmetric confining potentials $U_\ell$ and $U_r$. In a more realistic model with impurities close to the edges one expects that it is automatically satisfied with a large probability.

Now we describe the spectral properties of the bulk random Hamiltonian (2.5). From the bound (A.3) on the kernel of $R_0(z)$ and the fact that $V_\omega$ is bounded with compact support we can see that $V_\omega$ is relatively compact w.r.t. $H_0$, thus $\sigma_{\text{ess}}(H_b) = \{(\nu + \frac{1}{2})B; \nu \in \mathbb{N}\}$. Since $|V_\omega(x,y)| \leq V_0 < B$ the eigenvalues $E^b_\beta$ of $H_b$ are contained in Landau bands $\bigcup_{\nu \geq 0} \left[ (\nu + \frac{1}{2})B - V_0, (\nu + \frac{1}{2})B + V_0 \right]$. We will assume Hypothesis 2.

Hypothesis 2. Fix any $\varepsilon > 0$. There exist $\mu(\varepsilon)$ a strictly positive constant and $L(\varepsilon)$ such that for all $L > L(\varepsilon)$ one can find a set of realizations of the random potential $\Omega_\Lambda$ with $\mathbb{P}_\Lambda(\Omega_\Lambda) \geq 1 - L^{-\theta}$, $\theta > 0$, with the property that if $\omega \in \Omega_\Lambda$ the eigenstates corresponding to $E^b_\beta \in \sigma(H_b) \cap \Delta_\varepsilon$ satisfy

$$
|\psi^b_\beta(x,\bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L}, \quad |\partial_y \psi^b_\beta(x,\bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L}
$$

for some $\bar{y}_\beta$ depending on $\omega$ and $L$.

Since $V_\omega$ is random we expect that wavefunctions with energies in $\Delta_\varepsilon$ (not too close to the Landau levels where the localization length diverges) are exponentially localized on a scale $O(1)$ with respect to $L$. Inequalities (2.14) are a weaker version of this statement, and have been checked for the special case where the random potential is a sum of rank one perturbations [33] using the methods of Aizenman and Molchanov [34] (see for example [14] where the case of point impurities is treated by these methods). Presumably one could adapt existing techniques for multiplicative potentials to our geometry, to prove hypothesis (H2) at least for energies close to the band tail $\frac{B}{2} + V_0$.

One also expects that $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. The main physical consequence of (H2) (as shown in section 5) is that a state satisfying (2.14) does not carry any appreciable current (contrary to the eigenstates of $H_\alpha$) in the sense that $J^b_\beta = (\psi^b_\beta, v_y \psi^b_\beta) = O(e^{-\mu(\varepsilon)L})$.

We now state our main result.

**Theorem 1.** Fix $\varepsilon > 0$ and assume that (H1) and (H2) are fulfilled. Assume $B > 4V_0$. Let $p \geq 7$ and $s = \min(\theta, p - 6)$. Then there exists a numerical constant $\gamma > 0$ and
an \(L(\varepsilon, p, B, V_0)\) such that for all for all \(L > L(\varepsilon, p, B, V_0)\) one can find a set \(\hat{\Omega}_L\) of realizations of the random potential with \(\mathbb{P}_\Lambda(\hat{\Omega}_L) \geq 1 - 3L^{-s}\) such that for any \(\omega \in \hat{\Omega}_L\), \(\sigma(H_\omega) \cap \Delta_\varepsilon\) is the union of three sets \(\Sigma_\ell \cup \Sigma_0 \cup \Sigma_r\), each depending on \(\omega\) and \(L\), and characterized by the following properties:

a) \(E^\alpha_k \in \Sigma_\alpha (\alpha = \ell, r)\) are a small perturbation of \(E^\alpha_{0k} \in \sigma(H_\alpha) \cap \Delta_\varepsilon\) with

\[
|E^\alpha_k - E^\alpha_{0k}| \leq e^{-\gamma B (\log L)^2}, \quad \alpha = \ell, r.
\] (2.15)

b) For \(E^\alpha_k \in \Sigma_\alpha\) the current \(J^\alpha_k\) of the associated eigenstate satisfies

\[
|J^\alpha_k - J^\alpha_{0k}| \leq e^{-\gamma B (\log L)^2}, \quad \alpha = \ell, r.
\] (2.16)

c) \(\Sigma_b\) contains the same number of energy levels as \(\sigma(H_b) \cap \Delta_\varepsilon\) and

\[
\text{dist}(\Sigma_b, \Sigma_\alpha) \geq L^{-p+1}, \quad \alpha = \ell, r.
\] (2.17)

d) The current associated to each level \(E_\beta \in \Sigma_b\) satisfies

\[
|J_\beta| \leq e^{-\gamma B (\log L)^2}.
\] (2.18)

The proof of the theorem is organized as follows. In section 3 we set up a decoupling scheme by which we express the resolvent of \(H_\omega\) as an approximate sum of those of the edge and bulk systems. Parts a) and c) of Theorem 1 are proven in section 4. First we compute approximations for the spectral projections of \(H_\omega\) in terms of the projectors \(P(E^\alpha_{0k})\) of \(H_\alpha\) and \(P_b(\bar{\Delta})\) of \(H_b\) (Proposition 1). This is done for realizations of the disorder such that the levels of \(H_b\) are not “too close” to those of \(H_\alpha\). We then show that these realizations are typical (have large probability) thanks to a Wegner estimate (Proposition 2). Parts b) and d) are proven in section 5 by estimating currents in term of norms of differences between projectors. The appendices contain some technical estimates.

3 Decoupling of the Bulk and the Edge Systems

The resolvent \(R(z) = (z - H_\omega)^{-1}\) can be expressed, up to a small term, as a sum of the resolvents of the bulk system \(R_b(z) = (z - H_b)^{-1}\) and the two edge systems \(R_\alpha(z) = (z - H_\alpha)^{-1}\) (\(\alpha = \ell, r\)). Here this will be achieved by a decoupling formula
developed in other contexts [35], [36]. We set $D = \log L$ and introduce the characteristic functions

$$
\tilde{J}_i(x) = \chi_{[\frac{1}{4}, \frac{1}{2}]}(x) \quad \tilde{J}_b(x) = \chi_{[\frac{1}{4}, \frac{1}{2} + \frac{4D}{D + 1}]}(x)
$$

$$
\tilde{J}_r(x) = \chi_{[\frac{1}{4}, +\infty]}(x).
$$

(3.1)

We will also use three bounded $C^\infty(\mathbb{R})$ functions $|J_i(x)| \leq 1$, $i \in \mathcal{I} \equiv \{\ell, b, r\}$, with bounded first and second derivatives $sup_x \left| \partial_x^n J_i(x) \right| \leq 2$, $n = 1, 2$, and such that

$$
J_\ell(x) = \begin{cases} 
1 & \text{if } x \leq -\frac{L}{2} + \frac{3D}{4} \\
0 & \text{if } x \geq -\frac{L}{2} + \frac{3D}{4} + 1
\end{cases} \quad J_b(x) = \begin{cases} 
1 & \text{if } x \leq -\frac{L}{2} - \frac{D}{4} \\
0 & \text{if } x \geq -\frac{L}{2} - \frac{D}{4} + 1
\end{cases} \\
J_r(x) = \begin{cases} 
1 & \text{if } x \geq \frac{L}{2} - \frac{3D}{4} - 1 \\
0 & \text{if } x \leq \frac{L}{2} - \frac{3D}{4} - 1
\end{cases}.
$$

(3.2)

For $i \in \mathcal{I}$ we have $H_\omega J_i = H_i J_i$ thus

$$
(z - H_\omega) \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i = \sum_{i \in \mathcal{I}} (z - H_i) J_i R_i(z) \tilde{J}_i = 1 - \mathcal{K}(z)
$$

(3.3)

where

$$
\mathcal{K}(z) = \sum_{i \in \mathcal{I}} K_i(z) = \sum_{i \in \mathcal{I}} \frac{1}{2} \left[ p_x^2, J_i \right] R_i(z) \tilde{J}_i.
$$

(3.4)

To obtain the second equality one commutes $(z - H_i)$ and $J_i$ and then uses the identity $\sum_{i \in \mathcal{I}} J_i \tilde{J}_i = \sum_{i \in \mathcal{I}} \tilde{J}_i = 1$. From (3.3) we deduce the decoupling formula

$$
R(z) = \left( \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) (1 - \mathcal{K}(z))^{-1}.
$$

(3.5)

The main result of this section is an estimate of the operator norm of $\mathcal{K}(z)$. In particular it will assure $\|\mathcal{K}(z)\| < 1$.

**Lemma 1.** Let $\Re z \in \Delta_z$ such that $\text{dist}(z, \sigma(H_t) \cup \sigma(H_r) \cup \sigma(H_b)) \geq e^{-\frac{B}{\pi^2}(\log L)^2}$. Then for $L$ large enough there exists a constant $C(B, V_0) > 0$ independent of $L$ such that

$$
\|\mathcal{K}(z)\| \leq \epsilon^{-1} C(B, V_0) L e^{-\frac{B}{\pi^2}(\log L)^2}.
$$

(3.6)

**Proof.** Computing the commutator in the definition of $K_i(z)$ and applying the second resolvent formula we have

$$
K_i(z) = \frac{1}{2} (\partial_x^2 J_i) R_i(z) \tilde{J}_i - (\partial_x J_i) \partial_x R_i(z) \tilde{J}_i
$$

$$
= -\frac{1}{2} (\partial_x^2 J_i) R_0(z) \tilde{J}_i - \frac{1}{2} (\partial_x^2 J_i) R_0(z) W_i R_i(z) \tilde{J}_i
$$

$$
- (\partial_x J_i) \partial_x R_0(z) \tilde{J}_i - (\partial_x J_i) \partial_x R_0(z) W_i R_i(z) \tilde{J}_i.
$$

(3.7)
where we have set \( W_\ell = U_\ell, \ W_b = V_\omega \) and \( W_r = U_r \). From the triangle inequality and 

\[ \| K_i(z) \| = \text{dist}(z, \sigma(H_i))^{-1} \]

we obtain

\[
\begin{align*}
\| K_i(z) \| & \leq \frac{1}{2} \| (\partial^2_x J_i) R_0(z) \bar{J}_i \| + \frac{1}{2} \| (\partial^2_x J_i) R_0(z) W_i \| \text{dist}(z, \sigma(H_i))^{-1} \\
& \quad + \| (\partial_x J_i) \partial_x R_0(z) \bar{J}_i \| + \| (\partial_x J_i) \partial_x R_0(z) W_i \| \text{dist}(z, \sigma(H_i))^{-1}.
\end{align*}
\tag{3.8}
\]

To estimate the operator norms on the right hand side it is sufficient to bound them by the Hilbert-Schmidt norms \( \| \cdot \|_2 \). Using bounds \( (3.3) \) on the kernels of \( \partial^2_x R_0(z) \) for \( n = 0, 1 \), and the properties of the functions \( J_i, \bar{J}_i \) we obtain

\[
\begin{align*}
\| (\partial^2_x J_i) \partial_x R_0(z) \bar{J}_i \|_2^2 & = \int_{\text{supp} \partial^2_x J_i} dx |\partial^2_x J_i(x)|^2 \int_{\text{supp} \bar{J}_i} dx' |\partial_x^2 R_0(x, x'; z)|^2 \\
& \leq 4C^2_n(z, B) \int_{\text{supp} \partial^2_x J_i} dx \int_{\text{supp} \bar{J}_i} dx' e^{-\frac{B}{4}(x-x')^2} \\
& \leq 4C^2_n(z, B) e^{-\frac{B}{8}(1+1)^2} \int_{\text{supp} \partial^2_x J_i} dx \int_{\mathbb{R} \times [-\frac{4}{16}, \frac{4}{16}]} dx' e^{-\frac{B}{8}(x-x')^2} \\
& \leq 16 \sqrt{\frac{n}{B}} C^2_n(z, B) L^2 e^{-\frac{B}{128} D^2}.
\end{align*}
\tag{3.9}
\]

For the norms involving the potentials \( W_i \) we obtain in a similar way

\[
\begin{align*}
\| \partial^2_x J_i \partial_x R_0(z) W_i \|_2^2 & = \int_{\text{supp} \partial^2_x J_i} dx |\partial^2_x J_i(x)|^2 \int_{\text{supp} W_i} dx' |\partial_x^2 R_0(x, x'; z)| \| W_i(x') \|^2 \\
& \leq 4C^2_n(z, B) e^{-\frac{B}{128} D^2} \int_{\text{supp} \partial^2_x J_i} dx \int_{\text{supp} W_i} dx' e^{-\frac{B}{8}(x-x')^2} \| W_i(x') \|^2.
\end{align*}
\tag{3.10}
\]

It is clear that since \( V_\omega \) is bounded, and \( U_\ell, U_r \) do not grow faster than polynomials, the double integral in the right hand side of the last inequality is bounded above by \( L^2 \) times a constant depending only on \( B \) and \( V_0 \). From this result, \( (3.8), (3.9) \) and 

\[
\text{dist}(z, \sigma(H_\ell) \cup \sigma(H_r) \cup \sigma(H_b)) \geq e^{-\frac{B}{128}(\log L)^2} \text{ we obtain } (\tilde{C}(B, V_0) a \text{ constant independent of } L)
\]

\[
\| K_i(z) \| \leq \tilde{C}(B, V_0) \varepsilon^{-1} L e^{-\frac{B}{128}(\log L)^2},
\tag{3.11}
\]

where we used the expression for \( C_n(z, B) \) in Appendix A and the fact that \( \Re z \in \Delta_z \). \( \square \)

### 4 Estimates of Eigenprojectors of \( H_\omega \)

In this section we use the decoupling formula \( (3.3) \) to give deterministic estimates for the difference between projectors of \( H_\omega \) and \( H_b, H_\ell \) and \( H_r \). We then combine this
Proposition 1. Assume that (H1) holds. Take \( p \geq 7 \) and any \( e^{-\frac{\rho}{512}(\log L)^2} < \rho < \frac{d(\varepsilon)}{2} L^{-p} \). For \( L > L(\varepsilon) \) let \( \Omega^*_\Lambda \) be the set of realizations of the random potential such that for each \( \omega \in \Omega^*_\Lambda \) \( \text{dist}(\sigma(H_\ell) \cap \Delta_\varepsilon, E_{0k}^\alpha) \geq d(\varepsilon)L^{-p} \) for all \( E_{0k}^\alpha \in \Delta_\varepsilon, \alpha = \ell, r \). Then

i) If \( P(E_{0k}^\alpha) \) is the eigenprojector of \( H_\alpha \) associated to the eigenvalue \( E_{0k}^\alpha \in \Delta_\varepsilon \) and \( P_k^\alpha \) the eigenprojector of \( H_\omega \) for the intervals \( I_k^\alpha = [E_{0k}^\alpha - \rho, E_{0k}^\alpha + \rho] \) we have

\[
\|P_k^\alpha - P(E_{0k}^\alpha)\| \leq \varepsilon^{-1}C'(B, V_0)L e^{-\frac{\rho}{512}(\log L)^2} . \quad (4.1)
\]

ii) Let \( \bar{\Delta} \subset \Delta_\varepsilon \) be an interval such that \( \text{dist}(\bar{\Delta}, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\varepsilon)}{2} L^{-p} \). If \( P_b(\bar{\Delta}) \) is the eigenprojector of \( H_b \) for the interval \( \bar{\Delta} \) and \( P(\Delta) \) the eigenprojector of \( H_\omega \) for the interval \( \Delta \) we have

\[
\|P(\Delta) - P_b(\bar{\Delta})\| \leq \varepsilon^{-3}C'(B, V_0)L^p e^{-\frac{\rho}{512}(\log L)^2} . \quad (4.2)
\]

Proof. We start by proving (4.1) for \( \alpha = r \). The case \( \alpha = \ell \) is identical. From the decoupling formula we have

\[
R(z) - R_r(z) = \left( \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) \left( \sum_{n=1}^{\infty} K(z)^n \right) - (1 - J_r) R_r(z)
- J_r R_r(z)(1 - \tilde{J}_r) + J_r R_r(z) \tilde{J}_r + J_r R_b(z) \tilde{J}_b . \quad (4.3)
\]

Let \( \Gamma \) be a circle of radius \( \rho \) in the complex plane, centered at \( E_{0k}^r \). Because of (H1) and \( \text{dist}(\sigma(H_b) \cap \Delta_\varepsilon, E_{0k}^r) \geq d(\varepsilon)L^{-p} \), \( R_b(z) \) and \( R_\ell(z) \) have no poles in \( \Gamma \). Moreover the only pole of \( R_r(z) \) is precisely \( E_{0k}^r \). Thus integrating (4.3) along the circle \( \Gamma \)

\[
P_k^r - P(E_{0k}^r) = \frac{1}{2\pi i} \oint_{\Gamma} \left( \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) \sum_{n=1}^{\infty} K(z)^n dz
- (1 - J_r) P(E_{0k}^r) - J_r P(E_{0k}^r)(1 - \tilde{J}_r) . \quad (4.4)
\]

We proceed to estimate the norms of the three contributions on the right hand side of (4.4). The norm of the first term is smaller than

\[
\rho \left( \sum_{i \in \mathcal{I}} \sup_{z \in \Gamma} \| R_i(z) \| \right) \sup_{z \in \Gamma} \| K(z) \| \leq 6\varepsilon^{-1}C(B, V_0) L e^{-\frac{\rho}{512}(\log L)^2} . \quad (4.5)
\]
Indeed, for $i = r$ we have $\sup_{z \in \Gamma} \| R_r(z) \| = \rho^{-1}$ by construction. For $i = \ell, b$ we have $\sup_{z \in \Gamma} \| R_i(z) \| < \frac{2}{d(i)} L_p$. Since $\rho < \frac{d(\ell)}{2} L_p$ we note that in all three cases ($i \in \mathcal{I}$) $\rho \sup_{z \in \Gamma} \| R_i(z) \| \leq 1$. Furthermore, since $\rho > e^{-\frac{d(\ell)}{2} (\log L)^2}$, using Lemma 1 we get (4.3). To estimate the second term in (4.4) we note that by the second resolvent formula

$$\frac{P(E_{0k}^r)}{(z - E_{0k}^r)} = (z - H_0)^{-1} P_r(E_{0k}^r) + (z - H_0)^{-1} U_r \frac{P(E_{0k}^r)}{(z - E_{0k}^r)} .$$

Integrating (4.6) along $\Gamma$ we obtain the identity

$$P(E_{0k}^r) = (E_{0k}^r - H_0)^{-1} U_r P(E_{0k}^r)$$

(4.7)

this implies

$$\| (1 - J_r) P(E_{0k}^r) \| \leq \| (1 - J_r) R_0(E_{0k}^r) U_r \| \leq \| (1 - J_r) R_0(E_{0k}^r) U_r \|_2$$

(4.8)

$$= \left\{ \int dx |1 - J_r(x)|^2 \int dx' R_0(x, x'; E_{0k}^r) U_r(x') \right\}^{1/2}$$

since the distance (in the $x$ direction) between the supports of $1 - J_r$ and $U_r$ is greater than $\frac{D}{2} + 1$ we can proceed in a similar way as in the estimate of (3.10) to obtain

$$\| (1 - J_r) P(E_{0k}^r) \| \leq \epsilon^{-1} \tilde{C}(B) L e^{-\frac{D}{2} \epsilon (\log L)^2}$$

(4.9)

where $\tilde{C}(B)$ is a constant depending only on $B$. For the third term in (4.4) we use the adjoint of (4.7)

$$P(E_{0k}^r) = P(E_{0k}^r) U_r (E_{0k}^r - H_0)^{-1}$$

(4.10)

to get

$$\| J_r P(E_{0k}^r) (1 - \tilde{J}_r) \| \leq \| U_r R_0(E_{0k}^r) (1 - \tilde{J}_r) \|$$

(4.11)

from which we obtain the same bound as in (4.9). Combining this result with (4.4), (4.8), (4.9) we obtain (1.1) in the proposition.

Let us now sketch the proof of (1.2). From the decoupling formula we have

$$R(z) - R_b(z) = \left( \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) \left( \sum_{n=1}^{\infty} K(z)^n \right) - (1 - J_b) R_b(z)$$

$$- J_b R_b(z) (1 - \tilde{J}_b) + J_b R_b(z) \tilde{J}_b + J_r R_r(z) \tilde{J}_r .$$

(4.12)

Given an interval $\Delta \subset \Delta_\epsilon$ such that $\text{dist}(\tilde{\Delta}, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\epsilon)}{2} L^{-p}$, we choose a circle $\Gamma$ in the complex plane with diameter equal to $|\tilde{\Delta}|$. Then if we integrate over $\Gamma$ the last
two terms on the right hand side do not contribute while the second and third ones give 
\((1 - J_b)P_b(\tilde{\Delta})\) and \(J_bP_b(\tilde{\Delta})(1 - J_b)\). Therefore
\[
\|P - P_b(\tilde{\Delta})\| \leq |\tilde{\Delta}| \left( \sum_{i \in I} \sup_{z \in \Gamma} \|R_i(z)\| \right) \frac{\sup_{z \in \Gamma} \|K(z)\|}{1 - \sup_{z \in \Gamma} \|K(z)\|} \\
+ \| (1 - J_b)P_b(\tilde{\Delta}) \| + \| J_bP_b(\tilde{\Delta})(1 - J_b) \|. \tag{4.13}
\]

From Lemma 1, \(|\tilde{\Delta}| < d(\varepsilon)L^{-1}\) and \(\sup_{z \in \Gamma} \|R_i(z)\| < \frac{2}{d(\varepsilon)}L^p\) the first term is bounded above by
\[
12\varepsilon^{-1}C(B, V_0)L^p e^{-\frac{\theta}{8\varepsilon} (\log L)^2}. \tag{4.14}
\]

In order to estimate the second norm in (4.13) we notice that (in the same way as in (4.6), (4.7))
\[
P_b(\tilde{\Delta}) = \sum_{E^b_\beta \in \Delta} R_0(E^b_\beta)V_\omega P_b(E^b_\beta) \tag{4.15}
\]
thus
\[
\| (1 - J_b)P_b(\tilde{\Delta}) \| \leq \sum_{E^b_\beta \in \Delta} \| (1 - J_b)R_0(E^b_\beta)V_\omega \|_2. \tag{4.16}
\]

Each term of the sum can be bounded in a way similar to (3.10), and since the number of terms in the sum is equal to \(\text{Tr} P_b(\tilde{\Delta})\) we get
\[
\| (1 - J_b)P_b(\tilde{\Delta}) \| \leq \varepsilon^{-1}C(B, V_0)L e^{-\frac{\theta}{16\varepsilon} (\log L)^2} \text{Tr} P_b(\tilde{\Delta}) \\
\leq 2\varepsilon^{-3}C(B)^2C(B, V_0)V_0^2L^5 e^{-\frac{\theta}{16\varepsilon} (\log L)^2}. \tag{4.17}
\]

The second inequality follows from Lemma 4 in Appendix B (where we need \(B > 4V_0\)). For \(\|J_bP_b(\tilde{\Delta})(1 - \tilde{J}_b)\|\) one uses the adjoint of identity (4.15) to obtain the same result. The result (4.2) of the proposition then follows by combining (4.13), (4.14) and (4.17). \(\square\)

In appendix B we adapt the method of [10] to our geometry to get the following Wegner estimate.

**Proposition 2.** Let \(B \geq 4V_0\) and \(E \in \Delta_\varepsilon\)
\[
\mathbb{P}_\Lambda (\text{dist}(\sigma(H_b), E) < \delta) \leq 4c(B)\|h\|_\infty \delta e^{-2V_0L^4}. \tag{4.18}
\]
Proof of Theorem 1, part a) and c). Let $\omega \in \Omega''$ where $\Omega''$ is the set given in Proposition 1. Since for $L$ large enough the right hand side of (4.1) is strictly smaller than one the two projectors necessarily have the same dimension. Therefore $\sigma(H_\omega) \cap I_k^\alpha$ contains a unique energy level $E_k^{\alpha}$ for each $I_k^{\alpha}$ of radius $\rho$. In particular by taking the smallest value $\rho = e^{-\frac{B}{512}(\log L)^2}$ we get (2.15). The number of such levels is $O(L)$ since they are in one to one correspondence with the energy levels of $H_\alpha$. The sets $\Sigma_\alpha$ of Theorem 1 are precisely

$$\Sigma_\alpha = \bigcup_k (\sigma(H_\omega) \cap I_k^{\alpha} \cap \Delta_\varepsilon), \quad \alpha = \ell, r.$$  \hfill (4.19)

The set of all other eigenvalues in $\sigma(H_\omega) \cap \Delta_\varepsilon$, defines $\Sigma_b$, and is necessarily contained in intervals $\Delta$ such that $\text{dist}(\Delta, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\varepsilon)}{2}L^{-p}$. In view of (2.15) this implies (2.17). Since the two projectors in (4.2) necessarily have the same dimension, the number of eigenstates in $\Sigma_b$ is the same than that of $\sigma(H_b) \cap \Delta_\varepsilon$. It remains to estimate the probability of the set $\Omega''$. The realizations of the complementary set are such that for at least one $E_{0k}^{\alpha} \in \Delta_\varepsilon$

$$\text{dist}(\sigma(H_b), E_{0k}^{\alpha}) < d(\varepsilon)L^{-p}$$  \hfill (4.20)

but from Proposition 2 this has a probability smaller than

$$4c(B)\|h\|_\infty d(\varepsilon)L^{-p}\varepsilon^{-2}V_0L^4 \cdot O(L)$$  \hfill (4.21)

where $O(L)$ comes from the number of levels in $[\sigma(H_\ell) \cup \sigma(H_r)] \cap \Delta_\varepsilon$. Thus for $L$ large enough

$$\mathbb{P}_\Lambda(\Omega'') \geq 1 - L^{6-p}. \quad (4.22)$$

We recall that $p \geq 7$. 

5 Estimates of Currents

In this section we characterize the eigenvalues of $H_\omega$ in terms of the current carried by the corresponding eigenstates. This will yield parts b) and d) of Theorem 1.

**Proof of Theorem 1, part b).** Let $E_k^{\alpha} \in \Sigma_\alpha$. The associated current is by definition

$$J_k^{\alpha} = \text{Tr} \, v_y P_k^{\alpha}$$  \hfill (5.1)
and will be compared to that of $\psi_{0k}^\alpha$

$$J_{0k}^\alpha = \text{Tr} v_y P(E_{0k}^\alpha). \quad (5.2)$$

The difference between these two currents will be estimated by $\|P_k^\alpha - P(E_{0k}^\alpha)\|$. First we observe that $v_y P_k^\alpha$ is trace class. Indeed, $v_y P_k^\alpha = v_y P_k^\alpha P_k^\alpha$ with $v_y P_k^\alpha$ bounded and $\|P_k^\alpha\|_1 = \text{Tr} P_k^\alpha = 1$

$$\|v_y P_k^\alpha\|_1^2 \leq \|v_y P_k^\alpha\|^2 \leq \|P_k^\alpha v_y^2 P_k^\alpha\| \leq 2\|P_k^\alpha (H_\omega - V_\omega) P_k^\alpha\| \leq 2E_k^\alpha + V_0 \quad (5.3)$$

to get the second inequality one has simply added positive terms to $v_y^2$. Similarly

$$\|v_y P(E_{0k}^\alpha)\|^2_1 \leq \|v_y P(E_{0k}^\alpha)\|^2 \leq \|P(E_{0k}^\alpha) v_y^2 P(E_{0k}^\alpha)\| \leq 2\|P(E_{0k}^\alpha) H_\alpha P(E_{0k}^\alpha)\| \leq 2E_{0k}^\alpha. \quad (5.4)$$

The identity

$$P_k^\alpha - P(E_{0k}^\alpha) = [P_k^\alpha - P(E_{0k}^\alpha)]^2 + [P_k^\alpha - P(E_{0k}^\alpha)] P(E_{0k}^\alpha) + P(E_{0k}^\alpha) [P_k^\alpha - P(E_{0k}^\alpha)] \quad (5.5)$$

implies

$$|J_k^\alpha - J_{0k}^\alpha| = |\text{Tr} v_y [P_k^\alpha - P(E_{0k}^\alpha)]| \leq \left|\text{Tr} v_y [P_k^\alpha - P(E_{0k}^\alpha)]^2\right|$$

$$+ \left|\text{Tr} v_y [P_k^\alpha - P(E_{0k}^\alpha)] P(E_{0k}^\alpha)\right|$$

$$+ \left|\text{Tr} v_y P(E_{0k}^\alpha) [P_k^\alpha - P(E_{0k}^\alpha)]\right|. \quad (5.6)$$

From (5.6), (5.3) and (5.4) we get

$$|J_k^\alpha - J_{0k}^\alpha| \leq 2 \left(\|v_y P_k^\alpha\|_1 + \|v_y P(E_{0k}^\alpha)\|_1\right) \|P_k^\alpha - P(E_{0k}^\alpha)\|$$

$$\leq 2 \left((B + 3V_0)^{1/2} + (B + 2V_0)^{1/2}\right) \|P_k^\alpha - P(E_{0k}^\alpha)\|. \quad (5.7)$$

Combining this last inequality with (4.1) we get the result (2.16) of Theorem 1. \qed

In order to prove part d) of Theorem 1 we need the following lemma.

**Lemma 2.** Fix $\omega \in \Omega_\Lambda'$ the set of realizations in (H2). Let $\psi_1^b, \psi_2^b$ be two eigenstates of $H_b$ with eigenvalues $E_1^b$ and $E_2^b$. Then

$$|\langle \psi_1^b, v_y \psi_2^b \rangle| \leq 2|E_1^b - E_2^b| L + e^{-\frac{\mu(b)}{4}} L. \quad (5.8)$$
For \( \psi_b^1 = \psi_b^2, E_b^1 = E_b^2 \) this shows that eigenstates of \( H_b \) do not carry any appreciable current. The main idea of the proof sketched below is that \( v_y \) is equal to the commutator \([-iy, H_b]\) up to a small boundary term.

**Proof.** The wavefunctions \( \psi_b^1 \) and \( \psi_b^2 \) are defined on \( \mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}] \), are periodic along \( y \) and are twice differentiable in \( y \). Here we will work with periodized versions of these functions where the \( y \) direction is infinite (but we keep the same notation). This allows us to shift integrals over \( y \) from \([−\frac{L}{2}, \frac{L}{2}]\) to \([\bar{y}_2, \bar{y}_2 + L]\). We have

\[
(\psi_b^1, v_y \psi_b^2) = \int_{\mathbb{R}} dx \int_{\bar{y}_2}^{\bar{y}_2+L} dy [\psi_b^1(x)]^*(-i\partial_y - Bx)\psi_b^2(x) .
\] (5.9)

An integration by parts yields

\[
i(\psi_b^1, v_y \psi_b^2) = \frac{1}{2} \int_{\mathbb{R}} dx \int_{\bar{y}_2}^{\bar{y}_2+L} dy [\psi_b^1(x)]^*y(-i\partial_y - Bx)^2 \psi_b^2(x)
- \frac{1}{2} \int_{\mathbb{R}} dx \int_{\bar{y}_2}^{\bar{y}_2+L} dy [(-i\partial_y - Bx)^2 \psi_b^1(x)]^* y \psi_b^2(x) + \mathcal{B}
\] (5.10)

where \( \mathcal{B} \) is a boundary term given by

\[
\mathcal{B} = i \frac{L}{2} \int_{\mathbb{R}} dx [(-i\partial_y - Bx) \psi_b^1(x, \bar{y}_2)]^* \psi_b^2(x, \bar{y}_2)
+ [\psi_b^1(x, \bar{y}_2)]^*(-i\partial_y - Bx) \psi_b^2(x, \bar{y}_2) .
\] (5.11)

We can add a periodized version of \( V_\omega \) and \( \frac{1}{2}p_x^2 \) to the kinetic energy operator in both terms on the right hand side of (5.10) and use that \( \psi_b^1 \) and \( \psi_b^2 \) are eigenfunctions of \( H_b \) to obtain

\[
i(\psi_b^1, v_y \psi_b^2) = (E_b^2 - E_b^1) \int_{\mathbb{R}} dx \int_{\bar{y}_2}^{\bar{y}_2+L} dy y [\psi_b^1(x)]^* \psi_b^2(x) + \mathcal{B} .
\] (5.12)

From \( |y| \leq |\bar{y}_2| + L \leq 2L \) and the Schwarz inequality we obtain

\[
|\langle \psi_b^1, v_y \psi_b^2 \rangle| \leq 2L|E_b^2 - E_b^1| + |\mathcal{B}| .
\] (5.13)

With the help of (C.6), (C.7) in Appendix C we get

\[
|\mathcal{B}| \leq e^{-\mu \Omega L}.
\] (5.14)

this concludes the proof of (5.8). \( \square \)
Proof of Theorem 1, part d). Let $\Delta$ an interval like in part $ii)$ of Proposition 4. We consider the maximal set of intervals $F_k \subset \Delta$ such that $|F_k| = e^{-\frac{B}{1024}(\log L)^2}$ and $\text{dist}(F_k, F_\lambda) \geq 4e^{-\frac{B}{1024}(\log L)^2}$, $k \neq \lambda$. Since the number of gaps between the $F_k$ in $\Delta$ is less than $e^{\frac{B}{1024}(\log L)^2} |\Delta|$ and $|\Delta| < \frac{d(\varepsilon)}{L}$, it follows from Proposition 2 that

$$P(\Omega_{\Lambda''}) = P(\omega \in \Omega_{\Lambda} : \sigma(H_0) \cap \Delta \subset \bigcup_k F_k) \geq 1 - 16c(B)\|h\|_\infty \varepsilon^{-2}V_0L^4e^{-\frac{B}{1024}(\log L)^2} e^{\frac{B}{1024}(\log L)^2} \frac{d(\varepsilon)}{L} = 1 - 16c(B)\|h\|_\infty \varepsilon^{-2}V_0d(\varepsilon)L^3e^{-\frac{B}{1024}(\log L)^2}.$$  

(5.15)

Now suppose that $\psi_\beta$ is an eigenstate of $H_0$ corresponding to $E_\beta \in \Delta$. For a given $\omega \in \Omega_{\Lambda''}$ one can show that $E_\beta$ is necessarily included in one of the fattened intervals $\bar{F}_k \equiv F_k + e^{-\frac{B}{1024}(\log L)^2}$. In order to check this it is sufficient to adapt the estimates (4.13) to (4.17) to the difference of projectors $\|P(\bar{F}_k) - P_b(\bar{F}_k)\|$. The main point is to check that with our choice of intervals one is allowed to replace the circle $\bar{\Gamma}$ by circles $\bar{\Gamma}_k$ centered at the midpoint of $F_k$ and of diameter $e^{-\frac{B}{1024}(\log L)^2} + 2e^{-\frac{B}{1024}(\log L)^2}$. We do not give the details here. One finds

$$\|P(\bar{F}_k) - P_b(\bar{F}_k)\| \leq \varepsilon^{-3}C''(B, V_0)L e^{-\frac{B}{1024}(\log L)^2}.$$  

(5.16)

Therefore $P(\bar{F}_k)\psi_\beta = \psi_\beta$ for some $k$ and we have

$$J_\beta = (\psi_\beta, v_\gamma \psi_\beta) = (\psi_\beta, v_\gamma P(\bar{F}_k)\psi_\beta) = (P_b(\bar{F}_k)\psi_\beta, v_\gamma P_b(\bar{F}_k)\psi_\beta) = (P_b(\bar{F}_k)\psi_\beta, v_\gamma P_b(\bar{F}_k)\psi_\beta) + ([P(\bar{F}_k) - P_b(\bar{F}_k)]\psi_\beta, v_\gamma P_b(\bar{F}_k)\psi_\beta) + (\psi_\beta, v_\gamma [P(\bar{F}_k) - P_b(\bar{F}_k)]\psi_\beta).$$  

(5.17)

To estimate the first term on the right hand side of (5.17) we use the spectral decomposition in terms of eigenstates of $H_0$,

$$P_b(\bar{F}_k)\psi_\beta = \sum_{E_\tau \in \bar{F}_k} (\psi^b_\tau, \psi_\beta) \psi^b_\tau.$$  

(5.18)

We have

$$(P_b(\bar{F}_k)\psi_\beta, v_\gamma P_b(\bar{F}_k)\psi_\beta) = \sum_{E_\tau, E_\sigma \in \bar{F}_k} (\psi_\beta, \psi^b_\tau)(\psi^b_\sigma, \psi_\beta)(\psi^b_\tau, v_\gamma \psi^b_\sigma).$$  

(5.19)

From Lemma 2 and Lemma 4 in Appendix B we get

$$\|(P_b(\bar{F}_k)\psi_\beta, v_\gamma P_b(\bar{F}_k)\psi_\beta)\| \leq (\text{Tr} P_b(\bar{F}_k))^2 4L e^{-\frac{B}{1024}(\log L)^2}$$

$$\leq 16c(B)^4 \varepsilon^{-4} V_0^4 L^9 e^{-\frac{B}{1024}(\log L)^2}.$$  

(5.20)
The second term on the right hand side of (5.17) is estimated by the Schwarz inequality
\[
(P(\tilde{F}_k) - P_b(\tilde{F}_k)\psi_\beta, v_y P_b(\tilde{F}_k)\psi_\beta)^2 \leq \|v_y P_b(\tilde{F}_k)\psi_\beta\|^2 \|P(\tilde{F}_k) - P_b(\tilde{F}_k)\|^2
\]
\[
\leq 2(P_b(\tilde{F}_k)\psi_\beta, (H_b - V_\omega) P_b(\tilde{F}_k)\psi_\beta) \|P(\tilde{F}_k) - P_b(\tilde{F}_k)\|^2
\]
\[
\leq (B + 3V_0) \|P(\tilde{F}_k) - P_b(\tilde{F}_k)\|^2 .
\] (5.21)

The third term is treated in a similar way
\[
(\psi_\beta, v_y[P(\tilde{F}_k) - P_b(\tilde{F}_k)]\psi_\beta)^2 \leq \|v_y\psi_\beta\|^2 \|P(\tilde{F}_k) - P_b(\tilde{F}_k)\|^2
\]
\[
\leq 2(\psi_\beta, (H_\omega - V_\omega)\psi_\beta) \|P(\tilde{F}_k) - P_b(\tilde{F}_k)\|^2
\]
\[
\leq (B + 3V_0) \|P(\tilde{F}_k) - P_b(\tilde{F}_k)\|^2 .
\] (5.22)

The last estimate (2.18) of Theorem 1 then follows from (5.16), (5.20), (5.21) and (5.22).

\[\square\]

Remark. The set $\hat{\Omega}_A$ in Theorem 1 may be taken equal to $\Omega_A' \cap \Omega_A'' \cap \Omega_A'''$. This set has a probability larger than $1 - 3L^{-s}$ with $s = \min(\theta, p - 6)$.

## A Resolvent of the Landau Hamiltonian

The kernel $R_0(x, x'; z)$ of the resolvent $R_0(z) = (z - H_0)^{-1}$ with periodic boundary conditions along $y$ can be expressed in term of the kernel $R_0^\infty(x, x'; z)$ of the resolvent of the two dimensional Landau Hamiltonian defined on the whole plane $\mathbb{R}^2$. Since the spectrum and the eigenfunctions of $H_0$ are exactly known, by writing down the spectral decomposition of $R_0(x, x'; z)$ and applying the Poisson summation formula we get for $z \in \rho(H_0)$
\[
R_0(x, x'; z) = \sum_{m \in \mathbb{Z}} R_0^\infty(xy - mL, x'y'; z) .
\] (A.1)

The formula for $R_0^\infty(x, x'; z)$ is (see for example [14])
\[
R_0^\infty(x, x'; z) = \frac{B}{2\pi} \Gamma(\alpha_z)U \left(\alpha_z, 1; \frac{B}{2}|x - x'|^2\right) e^{-\frac{B}{2}|x - x'|^2} M(x, x')
\] (A.2)

where $\alpha_z = (\frac{1}{2} - \frac{z}{B})$ and
\[
M(x, x') = \exp \left(\frac{i}{2}B(x + x')(y - y')\right)
\] (A.3)
is the phase factor in the Landau gauge. In (A.2) the Landau levels appear as simple poles of the Euler \( \Gamma \) function and \( U(-\lambda, b; \rho) \) is the logarithmic solution of the Kummer equation (see eqns. (13.1.1) and (13.1.6) of [37])

\[
\rho \frac{d^2 U}{d\rho^2} + (b - \rho) \frac{dU}{d\rho} + \lambda \rho = 0 .
\]

(A.4)

Lemma 3. If \(|\text{Im} z| \leq 1, \text{Re} z \in ]\frac{1}{2} B, \frac{3}{2} B[\) and \(\frac{B}{2} |x - x'|^2 > 1\) then, for \(L\) large enough, there exists \(C_n(z, B), n = 0, 1\) independent of \(L\) such that

\[
|\partial_x^n R_0(x, x'; z)| \leq C_n(z, B) e^{-\frac{B}{8} (x-x')^2}
\]

(A.5)

where \(C_n(z, B) = C_n B^{1+n} \text{dist}(z, \sigma(H_0))^{-1}\) with \(C_n\) a numerical positive constant.

For our purposes we need only decay in the \(x\) direction as provided by the lemma but in fact there is also a Gaussian decay in the \(y\) direction as long as \(|y - y'|| < \frac{1}{2}\). One can also prove similar estimates when \(\text{Re} z\) is between higher Landau levels but the constant is not uniform with respect to \(\nu\). Finally we point out that this estimate does not hold for \(\frac{B}{2} |x - x'|^2 < 1\) because of the logarithmic singularity in the Kummer function for \(\rho \to 0\) (see also Appendix C).

Proof. The proof relies on the estimate (6.10) of [14] which we state here for convenience. For \(\lambda = x + iy, N - 1 < x < N \ (N \geq 1), b \in \mathbb{N}\) and \(\rho > 1\)

\[
|U(-\lambda, b; \rho)| \leq 2^{b+N-1} \rho^x (b + N + |y|)^N \frac{\Gamma(-x)}{\Gamma(-\lambda)} + e^{-(\rho-2)(\rho + 1 + |y|)N} \frac{(b + N)!}{\Gamma(N - \lambda)} .
\]

(A.6)

Using this estimate for \(N = 1, |y| < 1\) and \(b = n\) together with \(\Gamma(1-\lambda) = -\lambda \Gamma(-\lambda)\) we have \((C''_n\) a numerical constant)

\[
|\Gamma(-\lambda)||U(-\lambda, n + 1; \rho)| \leq C''_n \rho \left\{ \Gamma(-x) + |\lambda|^{-1} \right\} .
\]

(A.7)

From (A.7) if \(|\text{Im} z| \leq 1, \text{Re} z \in ]\frac{1}{2} B, \frac{3}{2} B[\) and \(\frac{B}{2} |x - x'|^2 > 1\) we deduce the estimate \((C''_n\) a numerical constant)

\[
|\Gamma(\alpha_x)U\left(\alpha_x, n + 1; \frac{B}{2} |x - x'|^2\right)| \leq BC''_n \text{dist}(z, \sigma(H_0))^{-1} |x - x'|^2 .
\]

(A.8)

From (A.8) for \(n = 0\) and (A.1) we get

\[
|R_0(x, x'; z)| \leq 2BC''_0 \text{dist}(z, \sigma(H_0))^{-1} e^{-\frac{B}{8} (x-x')^2} \sum_{m \in \mathbb{Z}} e^{-\frac{B}{8} (y-y'-mL)^2}
\]

(A.9)
since $|y - y'| < L$ the last sum can be bounded by a constant, which yields \((A.3)\) for $n = 0$.

To estimate the first derivative it is convenient to use the relation \([37]\)
\[
\frac{dU(-\lambda, 1; \rho)}{d\rho} = U(-\lambda, 1; \rho) - U(-\lambda, 2; \rho)
\]
which yields
\[
\partial_x R_0^\infty(x, x'; z) = \frac{B}{2} [(x - x') + i(y - y')] R_0^\infty(x, x'; z)
\]
\[- B(x - x') \frac{B}{2\pi} \Gamma(\alpha_z) U \left( \alpha_z, 2; \frac{B}{2}|x - x'|^2 \right) e^{-\frac{B}{2}|x - x'|^2} M(x, x').
\]
Using \((A.8)\) to bound the two terms on the right hand side of \((A.11)\) we get
\[
|\partial_x R_0^\infty(x, x'; z)| \leq B \frac{B}{2} C_1^\alpha \text{dist}(z, \sigma(H_0))^{-1} e^{-\frac{B}{2}(x - x')^2 + (y - y' - mL)^2} \] (A.12)
the result \((A.3)\) for $n = 1$ then follows from \((A.12)\) and \((A.1)\).

\[\square\]

**B  Bounds on the Number of Eigenvalues in Small Intervals**

We first prove a deterministic Lemma on the maximal number of eigenvalues of $H_b$ belonging to energy intervals $I$ contained in $\Delta_\varepsilon$. Then we sketch the proof of Proposition 2. The ideas in this appendix come from the method used by Combes and Hislop to obtain the Wegner estimate which gives the expected number of eigenvalues in $I$. Since Lemma 4 does not appear in \([10]\) and we need to adapt the technique to our geometry we give some details below.

We begin with some preliminary observations on the kernel $P_0(x, x')$ of the projector onto the first Landau level with periodic boundary conditions along $y$. Using the spectral decomposition and the Poisson summation formula one gets
\[
P_0(x, x') = \sum_{m \in \mathbb{Z}} P_0^\infty(x, x' - mL)
\]
where
\[
P_0^\infty(x, x') = \frac{B}{2\pi} e^{-\frac{B}{2}|x - x'|^2} e^{i\frac{B}{2}(x + x')(y - y}')
\] (B.2)
is the projector on the first Landau level for the infinite plane. The above formula can also be obtained by computing the residues of the poles of the $\Gamma$ function. We observe
that $V_i^{1/2}P_0V_j^{1/2}$ is trace class. Indeed it is the product of two Hilbert-Schmidt operators $V_i^{1/2}P_0$ and $P_0V_j^{1/2}$ and from the expression of the kernel \((\ref{B.1})\) it is easily seen that \((c(B)\) a constant independent of $L$)

$$
\|V_i^{1/2}P_0V_j^{1/2}\|_1 \leq \|V_i^{1/2}P_0\|_2\|P_0V_j^{1/2}\|_2 \leq c(B)V_0 . \tag{B.3}
$$

**Lemma 4.** Let $I$ be any interval contained in $\Delta_\epsilon$ and $P_b(I)$ the eigenprojector associated to $H_b$. Then

$$
\text{Tr} \ P_b(I) \leq 2\varepsilon^{-2}c(B)^2V_0^2L^4 . \tag{B.4}
$$

**Proof.** Let $Q_0 = 1 - P_0$ and $E$ the middle point of $I$. Using $Q_0(H_0 - E)Q_0 \geq 0$ and $Q_0R_0(E)Q_0 \leq (B - V_0)^{-1}Q_0$ we can write

$$
P_b(I)Q_0P_b(I) = P_b(I)Q_0(H_0 - E)^{1/2}R_0(E)(H_0 - E)^{1/2}Q_0P_b(I) \tag{B.5}
$$

$$
\leq (B - V_0)^{-1}P_b(I)(H_0 - E)Q_0P_b(I) \\
\leq (B - V_0)^{-1}[P_b(I)(H_b - E)Q_0P_b(I) - P_b(I)V_\omega Q_0P_b(I)]
$$

and thus from $\|P_b(I)(H_b - E)\| \leq \frac{|I|}{2}$, we get

$$
\|P_b(I)Q_0P_b(I)\| \leq (B - V_0)^{-1} \left( \frac{|I|}{2} + V_0 \right) \leq \frac{3V_0}{2(B - V_0)} \leq \frac{1}{2} . \tag{B.6}
$$

In the last inequality we have assumed that $B \geq 4V_0$. Using $\text{Tr} \ P_b(I) = \text{Tr} \ P_b(I)P_0P_b(I) + \text{Tr} \ P_b(I)Q_0P_b(I)$, $\text{Tr} \ P_b(I)Q_0P_b(I) \leq \|P_b(I)Q_0P_b(I)\| \text{Tr} \ P_b(I)$, and \((\ref{B.3})\) we obtain

$$
\text{Tr} \ P_b(I) \leq 2 \text{Tr} \ P_b(I)P_0P_b(I) = 2 \text{Tr} \ P_0P_b(I)P_0 . \tag{B.7}
$$

Now, from

$$
\text{dist}(I, \frac{B}{2})^2P_b(I)^2 \leq \left( P_b(I)(H_b - \frac{B}{2})P_b(I) \right)^2 \tag{B.8}
$$

it follows that

$$
\text{Tr} \ P_0P_b(I)P_0 \leq \varepsilon^{-2} \text{Tr} \left( P_0P_b(I)(H_b - \frac{B}{2})P_b(I)(H_b - \frac{B}{2})P_b(I)P_0 \right) \\
= \varepsilon^{-2} \text{Tr} \left( P_0V_\omega P_b(I)V_\omega P_0 \right) \leq \varepsilon^{-2}\|P_0V_\omega\|_2\|V_\omega P_0\|_2 \tag{B.9}
$$

each Hilbert-Schmidt norm in \((\ref{B.9})\) is bounded by $c(B)V_0L^2$. This observation together with \((\ref{B.7})\) gives the result of the lemma. \qed

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Let us now sketch the proof of Proposition 2.

**Proof of Proposition 2.** Let \( E \in \Delta_\varepsilon \) and \( I = [E - \delta, E + \delta] \) for \( \delta \) small enough (we require that \( I \) is contained in \( \Delta_\varepsilon \)). By the Chebyshev inequality we have

\[
P_\Lambda (\text{dist}(\sigma(H_s), E) < \delta) = P_\Lambda (\text{Tr} P_b(I) \geq 1) \leq E_\Lambda (\text{Tr} P_b(I))
\]  

(B.10)

where \( E_\Lambda \) is the expectation with respect to the random variables in \( \Lambda \). To estimate it we use an intermediate inequality of the previous proof

\[
E_\Lambda (\text{Tr} P_b(I)) \leq 2\varepsilon^{-2} E_\Lambda (\text{Tr} P_0 V_\omega P_b(I) V_\omega P_0).
\]

(B.11)

Writing \( V_{\omega,\Lambda} = \sum_{i \in \Lambda} X_i(\omega)V_i \)

\[
\text{Tr} P_0 V_\omega P_b(I) V_\omega P_0 = \sum_{i,j \in \Lambda^2} X_i(\omega)X_j(\omega) \text{Tr} P_0 V_i P_b(I) V_j P_0
\]

(B.12)

\[
= \sum_{i,j \in \Lambda^2} X_i(\omega)X_j(\omega) \text{Tr} V_j^{1/2} P_0 V_i^{1/2} P_b(I) V_j^{1/2}.
\]

Since \( V_j^{1/2} P_0 V_i^{1/2} \) is trace class we can introduce the singular value decomposition

\[
V_j^{1/2} P_0 V_i^{1/2} = \sum_{n=0}^{\infty} \mu_n(\psi_n, \cdot) \phi_n
\]

(B.13)

where \( \sum_{n=0}^{\infty} \mu_n = \| V_j^{1/2} P_0 V_i^{1/2} \|_1 \). Then

\[
\text{Tr} V_j^{1/2} P_0 V_i^{1/2} P_b(I) V_j^{1/2} = \sum_{n=0}^{\infty} \mu_n(\phi_n, V_j^{1/2} P_b(I) V_j^{1/2} \psi_n)
\]

\[
\leq \sum_{n=0}^{\infty} \mu_n(\phi_n, V_i^{1/2} P_b(I) V_i^{1/2} \phi_n)^{1/2}(\psi_n, V_j^{1/2} P_b(I) V_j^{1/2} \psi_n)^{1/2}
\]

\[
\leq \frac{1}{2} \sum_{n=0}^{\infty} \mu_n \left\{ (\phi_n, V_i^{1/2} P_b(I) V_i^{1/2} \phi_n) + (\psi_n, V_j^{1/2} P_b(I) V_j^{1/2} \psi_n) \right\}.
\]

(B.14)

An application of the spectral averaging theorem of [10] shows that

\[
E_\Lambda ((\psi_n, V_j^{1/2} P_b(I) V_j^{1/2} \psi_n)) \leq \| h \|_\infty 2\delta
\]

(B.15)

as well as for the term with \( i \) replacing \( j \) and \( \phi_n \) replacing \( \psi_n \). Combining (B.11), (B.14), (B.13) and (B.12) we get

\[
E_\Lambda (\text{Tr} P_b(I)) \leq 4\| h \|_\infty \delta \varepsilon^{-2} \sum_{i,j \in \Lambda^2} \| V_j^{1/2} P_0 V_i^{1/2} \|_1 \leq 4\| h \|_\infty \delta \varepsilon^{-2} c(B) V_0 L^4.
\]

(B.16)
C Estimate on the Eigenfunction of $H_b$

In this section we prove Gaussian decay of the eigenfunction $\psi^b_\beta$ and its $y-$derivative outside the support of the random potential $V_\omega$. From the eigenvalue equation $(H_0 + V_\omega)\psi^b_\beta = E^b_\beta \psi^b_\beta$ we get

$$\psi^b_\beta = R_0(E^b_\beta) V_\omega \psi^b_\beta. \quad (C.1)$$

Thus

$$|\psi^b_\beta(x)| \leq \int_{\mathbb{R} \times I_p} |R_0(x, x'; E^b_\beta) V_\omega(x') \psi^b_\beta(x')| \, dx'$$

$$\leq V_0 \left\{ \int_{\text{supp} V_\omega} |R_0(x, x'; E^b_\beta)|^2 \, dx' \right\}^{1/2}, \quad (C.2)$$

and

$$|\partial_y \psi^b_\beta(x)| \leq V_0 \sup_x |\psi^b_\beta(x)| \int_{\text{supp} V_\omega} |\partial_y R_0(x, x'; E^b_\beta)| \, dx'. \quad (C.3)$$

We need bounds on the integral kernel $R_0$ and its $y-$derivative to get an estimate of the eigenfunctions and their $y-$derivative. From [14] we have ($E \in \Delta_x$)

$$|R_0^\infty(x, x'; E)| \leq C(B) |\Gamma(\alpha_E)| e^{-\frac{B}{8} |x-x'|^2} \times \begin{cases} 1 & \text{if } \frac{B}{2} |x-x'|^2 > 1 \\ 1 + |\ln(\frac{B}{2} |x-x'|^2)| & \text{if } \frac{B}{2} |x-x'|^2 \leq 1. \end{cases} \quad (C.4)$$

Calculating the $y-$derivative thanks to (A.10), and using bounds (6.16) of [14] we have

$$|\partial_y R_0^\infty(x, x'; E)| \leq C'(B) |\Gamma(\alpha_E)| e^{-\frac{B}{8} |x-x'|^2} \times \begin{cases} 1 + |x| & \text{if } \frac{B}{2} |x-x'|^2 > 1 \\ (1 + |\ln(\frac{B}{2} |x-x'|^2)|) (1 + |x| + |x-x'|^{-1}) & \text{if } \frac{B}{2} |x-x'|^2 \leq 1. \end{cases} \quad (C.5)$$

With the help of (C.4) and (C.3) we can see that for $L$ large enough

$$|\psi^b_\beta(x)| \leq C(B) \varepsilon^{-1} V_0 L \times \begin{cases} e^{-\frac{\beta}{8} (x - \frac{L}{2} + \log L)^2} & \text{if } x \not\in \left[ -\frac{L}{2}, \frac{L}{2} \right] \\ \ln(BL^2) & \text{if } x \in \left[ -\frac{L}{2}, \frac{L}{2} \right]. \end{cases} \quad (C.6)$$

and

$$|\partial_y \psi^b_\beta(x)| \leq C'(B) \varepsilon^{-2} V_0^2 L^2 \times \begin{cases} e^{-\frac{\beta}{8} (x - \frac{L}{2} + \log L)^2} (1 + |x|) & \text{if } x \not\in \left[ -\frac{L}{2}, \frac{L}{2} \right] \\ L \ln(BL^2)^2 (1 + |x|) & \text{if } x \in \left[ -\frac{L}{2}, \frac{L}{2} \right]. \end{cases} \quad (C.7)$$
Indeed, for $|m| > 1 \frac{B}{2} [(x - x')^2 + (y - y' - mL)^2] > 1$ thus we have

$$|R_0(x, x'; E^b_\beta)| \leq \tilde{C}(B) \varepsilon^{-1} e^{-\frac{B}{2}(x-x')^2} + \sum_{|m| \leq 1} |R_0^\infty(x, y, x' y' - mL; E^b_\beta)|.$$  \hspace{1cm} (C.8)

If $x \notin [-\frac{L}{2}, \frac{L}{2}]$ since $x' \in \text{supp} V_\omega$ the terms $|m| \leq 1$ have also a Gaussian bound and

$$|R_0(x, x'; E^b_\beta)| \leq \tilde{C}'(B) \varepsilon^{-1} e^{-\frac{B}{2}(x-x')^2}.$$  \hspace{1cm} (C.9)

Replacing this bound in \hspace{1cm} (C.2) \hspace{1cm} we get the Gaussian decay in \hspace{1cm} (C.6). \hspace{1cm} On the other hand if $x \in [-\frac{L}{2}, \frac{L}{2}]$ we can use the logarithmic bounds for the terms $|m| \leq 1$ and we remark they are integrable and bounded by $L^2 \ln(BL^2)$. The same arguments hold for the $y-$derivative.

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