M-furcations in coupled maps

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Abstract

We study the scaling behavior of $M$-furcation ($M = 2, 3, 4, \ldots$) sequences of $M^n$-period ($n = 1, 2, \ldots$) orbits in two coupled one-dimensional (1D) maps. Using a renormalization method, how the scaling behavior depends on $M$ is particularly investigated in the zero-coupling case in which the two 1D maps become uncoupled. The zero-coupling fixed map of the $M$-furcation renormalization transformation is found to have three relevant eigenvalues $\delta$, $\alpha$, and $M$ ($\delta$ and $\alpha$ are the parameter and orbital scaling factors of 1D maps, respectively). Here the second and third ones, $\alpha$ and $M$, called the “coupling eigenvalues”, govern the scaling behavior associated with coupling, while the first one $\delta$ governs the scaling behavior of the nonlinearity parameter like the case of 1D maps. The renormalization results are also confirmed by a direct numerical method.

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I. INTRODUCTION

Universal scaling behaviors of $M$-furcation ($M = 2, 3, 4, \ldots$) sequences of $M^n$-cycles ($n = 1, 2, \ldots$) (i.e., $M^n$-period orbits) have been found in a one-parameter family of one-dimensional (1D) unimodal maps with a quadratic maximum. As an example, consider the logistic map

$$x_{t+1} = f(x_t) = 1 - A x_t^2,$$

where $t$ denotes the discrete time. As the nonlinearity parameter $A$ is increased from 0, a stable fixed point undergoes the cascade of period-doubling bifurcations accumulating at a finite parameter value $A_\infty (= 1.401155 \ldots)$. The bifurcation sequence corresponding to the MSS (Metropolis, Stein, and Stein [1]) sequence $R^{*n}$ (for details of the MSS sequences and the (*)-composition rule, see Refs. [1,2]) exhibits an asymptotic scaling behavior [3].

What happens beyond the bifurcation accumulation point $A_\infty$ is interesting from the viewpoint of chaos. The parameter interval between $A_\infty$ and the final boundary-crisis point $A_c (= 2)$ beyond which no periodic or chaotic attractors can be found within the unimodality interval is called the “chaotic” regime. Within this region, besides the bifurcation sequence, there are many other sequences of periodic orbits exhibiting their own scaling behaviors. In particular, every primary pattern $P$ (that cannot be decomposed using the (*)-operation) leads to an MSS sequence $P^{*n}$. For example, $P = RL$ leads to a trifurcation sequence of $3^n$-cycles, $P = RL^2$ to a tetrafurcation sequence of $4^n$-cycles, and the three different $P = RLR^2$, $RL^2R$, and $RL^3$ to three different period-5$^n$ sequences. Thus there exist infinitely many higher $M$-furcation ($M = 3, 4, \ldots$) sequences inside the chaotic regime. Unlike the bifurcation sequence, stability regions of periodic orbits in the higher $M$-furcation sequences are not adjacent on the parameter axis, because they are born by their own tangent bifurcations. The asymptotic scaling behaviors of these (disconnected) higher $M$-furcation sequences characterized by the parameter and orbital scaling factors, $\delta$ and $\alpha$, vary depending on the primary pattern $P$ [2,4–11].
In this paper we consider two symmetrically coupled 1D maps. This coupled map may help us to understand how coupled oscillators, such as Josephson-junction arrays or chemically reacting cells, exhibit various dynamical behaviors [12–14]. We are interested in the scaling behaviors of $M$-frcations ($M = 2, 3, \ldots$) in the two coupled 1D maps. The bifurcation case ($M = 2$) was previously studied in Refs. [15–20]. Here we extend the results for the bifurcation case to all the other higher multifurcation cases with $M = 3, 4, \ldots$ in the zero-coupling case where the two 1D maps become uncoupled. In Sec. II we investigate the dependence of the scaling behavior on $M$ using a renormalization method. It is found that the zero-coupling fixed point of the $M$-furcation renormalization transformation has three relevant eigenvalues $\delta$, $\alpha$, and $M$. The scaling behavior associated with coupling is governed by two coupling eigenvalues (CE’s) $\alpha$ and $M$, while the scaling behavior of the nonlinearity parameter is also governed by the eigenvalue $\delta$ like the case of 1D maps. As an example, we numerically study the scaling behavior associated with coupling in the trifurcation sequence in Sec. III and confirm the renormalization results. Finally, a summary is given in Sec. IV.

II. RENORMALIZATION ANALYSIS

In this section we first introduce two coupled 1D maps and discuss stability of orbits, and then study the scaling behavior of $M$-furcations ($M = 2, 3, \ldots$) in the zero-coupling case using the renormalization method developed in Refs. [15,19]. It is found that there exist three relevant eigenvalues $\delta$, $\alpha$, and $M$. As in the case of 1D maps, the scaling behavior of the nonlinearity parameter is governed by the eigenvalue $\delta$, irrespectively of coupling. However, the scaling behavior associated with coupling depends on the nature of coupling. In a linear-coupling case, in which the coupling function has a leading linear term, it is governed by two CE’s $\alpha$ and $M$, whereas it is governed by only one CE $M$ in the other cases of nonlinear coupling with leading nonlinear terms.

Consider a map $T$ consisting of two symmetrically coupled 1D maps,
\[ T: \begin{cases} x_{t+1} = F(x_t, y_t) = f(x_t) + g(x_t, y_t), \\ y_{t+1} = F(y_t, x_t) = f(y_t) + g(y_t, x_t), \end{cases} \tag{2} \]

where \( f(x) \) is a 1D unimodal map with a quadratic maximum at \( x = 0 \), and \( g(x, y) \) is a coupling function. The uncoupled 1D map \( f \) satisfies a normalization condition

\[ f(0) = 1, \tag{3} \]

and the coupling function \( g \) obeys a condition

\[ g(x, x) = 0 \quad \text{for any } x. \tag{4} \]

The two-coupled map (2) is invariant under the exchange of coordinates such that \( x \leftrightarrow y \). The set of all points which are invariant under the exchange of coordinates forms a symmetry line \( y = x \). An orbit is called an “in-phase” orbit if it lies on the symmetry line, i.e., it satisfies

\[ x_t = y_t \quad \text{for all } t. \tag{5} \]

Otherwise it is called an “out-of-phase” orbit. Here we study only in-phase orbits, which can be easily found from the uncoupled 1D map, \( x_{t+1} = f(x_t) \), because of the condition (4).

Stability of an in-phase orbit with period \( p \) is determined from the Jacobian matrix \( J \) of \( T^p \), which is the \( p \)-product of the Jacobian matrix \( DT \) of \( T \) along the orbit:

\[ J = \prod_{t=1}^{p} DT(x_t, x_t) = \prod_{t=1}^{p} \begin{pmatrix} f'(x_t) - G(x_t) & G(x_t) \\ G(x_t) & f'(x_t) - G(x_t) \end{pmatrix}, \tag{6} \]

where the prime denotes a derivative, and \( G(x) = \partial g(x, y) / \partial y \mid_{y=x} \); hereafter, \( G(x) \) will be referred to as the “reduced coupling function” of \( g(x, y) \). The eigenvalues of \( J \), called the stability multipliers of the orbit, are:

\[ \lambda_1 = \prod_{t=1}^{p} f'(x_t), \quad \lambda_2 = \prod_{t=1}^{p} [f'(x_t) - 2G(x_t)]. \tag{7} \]

Note that the first stability multiplier \( \lambda_1 \) is just that of the uncoupled 1D map and the coupling affects only the second stability multiplier \( \lambda_2 \).
An in-phase orbit is stable only when the moduli of both multipliers are less than or equal to unity, i.e., \(-1 \leq \lambda_i \leq 1\) for \(i = 1, 2\). A tangent (period-doubling) bifurcation occurs when each stability multiplier \(\lambda_i\) increases (decreases) through 1 (−1). Hence the stable region of the in-phase orbit in the parameter plane is bounded by four bifurcation lines associated with tangent and period-doubling bifurcations (i.e., those curves determined by the equations \(\lambda_i = \pm 1\) for \(i = 1, 2\)).

We now consider the \(M\)-furcation renormalization transformation \(N\), which is composed of the \(M\)-times iterating \((T^{(M)})\) and rescaling (\(B\)) operators:

\[
N(T) \equiv BT^{(M)}B^{-1}. \tag{8}
\]

Here the rescaling operator \(B\) is:

\[
B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \tag{9}
\]

because we consider only in-phase orbits.

Applying the renormalization operator \(N\) to the coupled map (2) \(n\) times, we obtain the \(n\)-times renormalized map \(T_n\) of the form,

\[
T_n: \begin{cases} 
  x_{t+1} = F_n(x_t, y_t) = f_n(x_t) + g_n(x_t, y_t), \\
  y_{t+1} = F_n(y_t, x_t) = f_n(y_t) + g_n(y_t, x_t).
\end{cases} \tag{10}
\]

Here \(f_n\) and \(g_n\) are the uncoupled and coupling parts of the \(n\)-times renormalized function \(F_n\), respectively. They satisfy the following recurrence equations:

\[
f_{n+1}(x) = \alpha f_n^{(M)}(\frac{x}{\alpha}), \tag{11}
\]

\[
g_{n+1}(x, y) = \alpha F_n^{(M)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - \alpha f_n^{(M)}(\frac{x}{\alpha}), \tag{12}
\]

where \(f_n^{(M)}(x) = f_n(f_n^{(M-1)}(x))\) and \(F_n^{(M)}(x, y) = F_n(F_n^{(M-1)}(x, y), F_n^{(M-1)}(y, x))\).

The recurrence relations (11) and (12) define a renormalization operator \(R\) of transforming a pair of functions \((f, g)\):
\[
\begin{pmatrix}
    f_{n+1} \\
    g_{n+1}
\end{pmatrix}
= \mathcal{R}
\begin{pmatrix}
    f_n \\
    g_n
\end{pmatrix}.
\] (13)

The renormalization transformation \( \mathcal{R} \) obviously has a fixed point \((f^*, g^*)\) with \(g^*(x, y) = 0\), which satisfies \( \mathcal{R}(f^*, 0) = (f^*, 0) \). Here \( f^* \) is just the 1D fixed function satisfying

\[ f^*(x) = \alpha f^{*(M)}(\frac{x}{\alpha}), \] (14)

where \( \alpha = 1/f^{*(M-1)}(1) \), due to the normalization condition (3), and it has the form,

\[ f^*(x) = 1 + c_1^* x^2 + c_2^* x^4 + \cdots, \] (15)

where \( c_i^* \)'s \((i = 1, 2, \ldots)\) are some constants. The fixed point \((f^*, 0)\) governs the critical behavior near the zero-coupling critical point because the coupling fixed function is identically zero, i.e., \( g^*(x, y) = 0 \). Here we restrict our attention to this zero-coupling case.

Consider an infinitesimal perturbation \((h, \phi)\) to the zero-coupling fixed point \((f^*, 0)\). We then examine the evolution of a pair of functions \((f^* + h, \phi)\) under \( \mathcal{R} \). Linearizing \( \mathcal{R} \) at the zero-coupling fixed point, we obtain a linearized operator \( \mathcal{L} \) of transforming a pair of perturbations \((h, \phi)\):

\[
\begin{pmatrix}
    h_{n+1} \\
    \varphi_{n+1}
\end{pmatrix}
= \mathcal{L}
\begin{pmatrix}
    h_n \\
    \varphi_n
\end{pmatrix}
= \begin{pmatrix}
    \mathcal{L}_u & 0 \\
    0 & \mathcal{L}_c
\end{pmatrix}
\begin{pmatrix}
    h_n \\
    \varphi_n
\end{pmatrix},
\] (16)

where

\[
h_{n+1}(x) = [\mathcal{L}_u h_n](x)
= \alpha \delta f_n^{(M)}(\frac{x}{\alpha}) \equiv \alpha \left[ f_n^{(M)}(\frac{x}{\alpha}) - f_n^{*(M)}(\frac{x}{\alpha}) \right]_{\text{linear}}
= \alpha f^*(f^{*(M-1)}(\frac{x}{\alpha})) \delta f_n^{(M-1)}(\frac{x}{\alpha}) + \alpha h_n(f^{*(M-1)}(\frac{x}{\alpha})),
\] (17)

\[
\varphi_{n+1}(x, y) = [\mathcal{L}_c \varphi_n](x, y)
= \alpha \delta [F_n^{(M)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha})] \equiv \alpha \left[ F_n^{(M)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha}) \right]_{\text{linear}}
= \alpha f^*(f^{*(M-1)}(\frac{x}{\alpha})) \delta [F_n^{(M-1)}(\frac{x}{\alpha}, \frac{y}{\alpha}) - f_n^{(M-1)}(\frac{x}{\alpha})]
+ \alpha \varphi_n(f^{*(M-1)}(\frac{x}{\alpha}), f^{*(M-1)}(\frac{y}{\alpha})).
\] (18)
Here the variations $\delta f_n^{(M)}(x) \alpha$ and $\delta[F_n^{(M)}(\frac{x}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha})]$ are introduced as the linear terms (denoted by $[f_n^{(M)}(\frac{x}{\alpha}) - f^{(M)}(\frac{x}{\alpha})]$linear and $[F_n^{(M)}(\frac{x}{\alpha}) - f^{(M)}(\frac{x}{\alpha})]$linear) in $h$ and $\varphi$ of the deviations of $f_n^{(M)}(\frac{x}{\alpha})$ and $F_n^{(M)}(\frac{x}{\alpha}) - f_n^{(M)}(\frac{x}{\alpha})$ from $f^{(M)}(\frac{x}{\alpha})$ and 0, respectively. A pair of perturbations $(h^*, \varphi^*)$ is then called an eigenperturbation with eigenvalue $\nu$, if it satisfies

$$\nu \begin{pmatrix} h^* \\ \varphi^* \end{pmatrix} = \mathcal{L} \begin{pmatrix} h^* \\ \varphi^* \end{pmatrix},$$

i.e.,

$$h^*(x) = [\mathcal{L}_u h^*](x),$$

$$\varphi^*(x, y) = [\mathcal{L}_c \varphi^*](x, y).$$

The eigenperturbations of the linear operator $\mathcal{L}$ can be divided into two classes. The first class of eigenperturbations are of the form $(h^*, 0)$. Here $h^*(x)$ is an eigenfunction of the linear “uncoupled operator” $\mathcal{L}_u$ satisfying Eq. (24), which is just the eigenvalue equation in the uncoupled 1D case. It has been found in Refs. [5,6,8] that there exist a unique eigenfunction $h^*(x)$ with (noncoordinate change) relevant eigenvalue $\delta$, associated with scaling of the nonlinearity parameter.

The second class of eigenperturbations have the form $(0, \varphi^*)$, where $\varphi^*(x)$ is an eigenfunction of the linear “coupling operator” $\mathcal{L}_c$ satisfying Eq. (25). However, it is not easy to directly solve the coupling eigenvalue equation (25). We therefore introduce a tractable recurrence equation for a “reduced coupling eigenfunction ” of $\varphi^*(x, y)$, defined by

$$\Phi^*(x) \equiv \frac{\partial \varphi^*(x, y)}{\partial y} \bigg|_{y=x}.$$  

Differentiating Eq. (25) with respect to $y$ and setting $y = x$, we obtain an eigenvalue equation for a reduced linear coupling operator $\tilde{\mathcal{L}}_c$:

$$\nu \Phi^*(x) = [\tilde{\mathcal{L}}_c \Phi^*](x)$$

$$= \delta F_2^{(M-1)}(\frac{x}{\alpha}) = [F_2^{(M-1)}(\frac{x}{\alpha})]_{\text{linear}}$$

$$= f^{*\prime}(f^{*(M-1)}(\frac{x}{\alpha})) \delta F_2^{(M-1)}(\frac{x}{\alpha})$$

$$+ f^{*(M-1)\prime}(\frac{x}{\alpha}) \Phi^*(f^{*(M-1)}(\frac{x}{\alpha})).$$
Here $F(x, y) = f^*(x) + \varphi^*(x, y)$, $F_2^{(M)}(x)$ is a “reduced function” of $F^{(M)}(x, y)$ defined by $F_2^{(M)}(x) \equiv \partial F^{(M)}(x, y)/\partial y|_{y=x}$, and the variation $\delta F_2^{(M)}(\frac{x}{\alpha})$ is also introduced as the linear term (denoted by $[F_2^{(M)}(\frac{x}{\alpha})]_{\text{linear}}$) in $\Phi^*$ of the deviation of $F_2^{(M)}(\frac{x}{\alpha})$ from 0.

In the case $M = 2$, the variation $\delta F_2^{(2)}(\frac{x}{\alpha})$ of Eq. (28) becomes

$$\delta F_2^{(2)}(\frac{x}{\alpha}) = \Phi^*(\frac{x}{\alpha} f''(\frac{x}{\alpha})) + f''(\frac{x}{\alpha}) \Phi^*(f^*(\frac{x}{\alpha})).$$

(30)

Substituting $\delta F_2^{(2)}(\frac{x}{\alpha})$ into Eq. (29), we have $\delta F_2^{(3)}(\frac{x}{\alpha})$ for $M = 3$, which consists of three terms,

$$\delta F_2^{(3)}(\frac{x}{\alpha}) = \Phi^*(\frac{x}{\alpha} f''(\frac{x}{\alpha})) f''(f^*(\frac{x}{\alpha})) + f''(\frac{x}{\alpha}) \Phi^*(f^*(\frac{x}{\alpha})).$$

(31)

Repeating this procedure successively, we obtain $\delta F_2^{(M)}(\frac{x}{\alpha})$ for a general $M$, composed of $M$ terms,

$$\delta F_2^{(M)}(\frac{x}{\alpha}) = \Phi^*(\frac{x}{\alpha} f''(\frac{x}{\alpha})), \ldots$$

$$\delta F_2^{(M)}(\frac{x}{\alpha}) = \Phi^*(\frac{x}{\alpha} f''(\frac{x}{\alpha})), \ldots$$

(32)

where $f^{(0)}(x) = x$.

Using the fact that $f''(0) = 0$, it can be easily shown that when $x = 0$, the reduced coupling eigenvalue equation (29) becomes

$$\lambda \Phi^*(0) = [\prod_{i=1}^{M-1} f''(f^*(i)(0))][\Phi^*(0)].$$

(33)

Differentiating the 1D fixed-point equation (14) with respect to $x$ and then letting $x \to 0$, we also have

$$\prod_{i=1}^{M-1} f''(f^*(i)(0)) = \lim_{x \to 0} \frac{f''(x)}{f''(\frac{x}{\alpha})} = \alpha.$$  

(34)
Then Eq. (33) reduces to

\[ \lambda \Phi^*(0) = \alpha \Phi^*(0). \]  
(35)

There are two cases. If the coupling eigenfunction \( \varphi^*(x, y) \) has a leading linear term, its reduced coupling eigenfunction \( \Phi^*(x) \) becomes nonzero at \( x = 0 \). For this case \( \Phi^*(0) \neq 0 \), we have the first CE

\[ \nu_1 = \alpha. \]  
(36)

The eigenfunction \( \Phi_1^*(x) \) with CE \( \nu_1 \) is of the form,

\[ \Phi_1^*(x) = 1 + a_1^*x + a_2^*x^2 + \cdots, \]  
(37)

where \( a_i^* \)'s \( (i = 1, 2, \ldots) \) are some constants. For the other case \( \Phi^*(0) = 0 \), it is found that \( f^{*'}(x) \) is an eigenfunction for the reduced coupling eigenvalue equation (29). Since Eq.(32) for the case \( \Phi^*(x) = f^{*'}(x) \) becomes

\[ \delta F_2^{(M)} \left( \frac{x}{\alpha} \right) = M f^{*(M)'} \left( \frac{x}{\alpha} \right), \]  
(38)

the reduced coupling eigenvalue equation (29) reduces to

\[ \nu f^{*'}(x) = M f^{*'}(x). \]  
(39)

We therefore have the second relevant CE

\[ \nu_2 = M, \]  
(40)

with reduced coupling eigenfunction \( \Phi_2^*(x) = f^{*'}(x) \). It is also found that there exists an infinite number of additional (coordinate change) reduced eigenfunctions \( f^{*'}(x) [f^{*l}(x) - x^l] \) with irrelevant CE’s \( \alpha^{-l} \) \( (l = 1, 2, \ldots) \), which are associated with coordinate changes. We conjecture that together with the two (noncoordinate change) relevant CE’s \( (\nu_1 = \alpha, \nu_2 = M) \), they give the whole spectrum of the reduced linear coupling operator \( \tilde{L}_c \) of Eq. (27) and the spectrum is complete.
In order to examine the effect of CE’s on the stability multipliers of periodic orbits in the $M$-furcation sequences, we consider an infinitesimal coupling perturbation $g(x, y) = \varepsilon \varphi(x, y)$ to a critical map at the zero-coupling critical point, in which case the two-coupled map has the form,

$$
T : \begin{cases}
x_{t+1} = F(x_t, y_t) = f_{A_{\infty}^{(M)}}(x_t) + g(x_t, y_t), \\
y_{t+1} = F(y_t, x_t) = f_{A_{\infty}^{(M)}}(y_t) + g(y_t, x_t),
\end{cases}
$$

(41)

where $A_{\infty}^{(M)}$ denotes the accumulation value of the parameter $A$ for the $M$-furcation case, and $\varepsilon$ is an infinitesimal coupling parameter. The map $T$ at $\varepsilon = 0$ is just the zero-coupling critical map consisting of two uncoupled 1D critical maps. It is attracted to the zero-coupling fixed map consisting of two uncoupled 1D fixed maps under iterations of the $M$-furcation renormalization transformation $\mathcal{N}$ of Eq.(8).

The reduced coupling function $G(x)$ of $g(x, y)$ is given by [see Eq. (26)]

$$
G(x) = \varepsilon \Phi(x) \equiv \varepsilon \frac{\partial \varphi(x, y)}{\partial y} \bigg|_{y=x}.
$$

(42)

The $n$th image $\Phi_n$ of $\Phi$ under the reduced linear coupling operator $\tilde{L}_c$ of Eq. (27) is of form,

$$
\Phi_n(x) = [\tilde{L}_c^n \Phi](x) \\ 
\simeq \alpha_1 \nu_1^n \Phi_1^*(x) + \alpha_2 \nu_2^n f'(x) \text{ for large } n,
$$

(43)

because the irrelevant part of $\Phi_n$ becomes negligibly small for large $n$. Here $\alpha_1$ and $\alpha_2$ are some constants.

The stability multipliers $\lambda_{1,n}$ and $\lambda_{2,n}$ of the $M^n$-cycle of the map $T$ of Eq. (41) are the same as those of the fixed point of the $n$-times renormalized map $\mathcal{N}^n(T)$ [19], which are given by

$$
\lambda_{1,n} = f'_n(\hat{x}_n), \quad \lambda_{2,n} = f'_n(\hat{x}_n) - 2G_n(\hat{x}_n).
$$

(44)

Here $f_n$ is the uncoupled part of the $n$th image of $(f_{A_{\infty}^{(M)}}, g)$ under the renormalization transformation $\mathcal{R}$, $G_n(x)$ is the reduced coupling function of the coupling part $g_n(x, y)$ of
the \( n \)th image, and \( \hat{x}_n \) is just the fixed point of \( f_n(x) \) [i.e., \( \hat{x}_n = f_n(\hat{x}_n) \)] and converges to the fixed point \( x^* \) of the 1D fixed map \( f^*(x) \) as \( n \to \infty \). In the critical case (\( \varepsilon = 0 \)), \( \lambda_{2,n} \) is equal to \( \lambda_{1,n} \) and they converge to the 1D critical stability multiplier \( \lambda^* = f^{**}(x^*) \). Since \( G_n(x) \simeq [\hat{L}_c^n G](x) = \varepsilon \Phi_n(x) \) for infinitesimally small \( \varepsilon \), \( \lambda_{2,n} \) has the form

\[
\lambda_{2,n} \simeq \lambda_{1,n} - 2\varepsilon \Phi_n \\
\simeq \lambda^* + \varepsilon [e_1 \nu_1^n + e_2 \nu_2^n] \text{ for large } n, \quad (45)
\]

where \( e_1 = -2\alpha_1 \Phi_1^*(x^*) \) and \( e_2 = -2\alpha_2 f^{**}(x^*) \). Hence the slope \( S_n \) of \( \lambda_{2,n} \) at the zero-coupling point (\( \varepsilon = 0 \)) is

\[
S_n \equiv \frac{\partial \lambda_{2,n}}{\partial \varepsilon} \bigg|_{\varepsilon=0} \simeq e_1 \nu_1^n + e_2 \nu_2^n \text{ for large } n. \quad (46)
\]

Here the coefficients \( e_1 \) and \( e_2 \) depend on the initial reduced function \( \Phi(x) \), because the constants \( \alpha_1 \) and \( \alpha_2 \) are determined only by \( \Phi(x) \). Note that the magnitude of slope \( S_n \) increases with \( n \), unless both coefficients \( e_1 \) and \( e_2 \) are zero.

We choose monomials \( x^l \) (\( l = 0, 1, 2, \ldots \)) as initial reduced functions \( \Phi(x) \), because any smooth function \( \Phi(x) \) can be represented as a linear combination of monomials by a Taylor series. Expressing \( \Phi(x) = x^l \) as a linear combination of eigenfunctions of \( \hat{L}_c \), we have

\[
\Phi(x) = x^l = \alpha_1 \Phi_1^*(x) + \alpha_2 f^{**}(x) \\
+ \sum_{l=1}^{\infty} \beta_l f^{**}(x)[f^{*l}(x) - x^l], \quad (47)
\]

where \( \alpha_1 \) is nonzero only for \( l = 0 \), and hence zero for \( l \geq 1 \), and all \( \beta_l \)'s are irrelevant components. Therefore the slope \( S_n \) for large \( n \) becomes

\[
S_n \simeq \begin{cases} 
  e_1 \alpha^n + e_2 M^n \text{ for } l = 0, \\
  e_2 M^n \text{ for } l \geq 1.
\end{cases} \quad (48)
\]

There are two kinds of coupling. In the case of a linear coupling, in which the coupling function \( \varphi(x, y) \) has a leading linear term, its reduced coupling function \( \Phi(x) \) has a leading constant term. However, for any other nonlinear-coupling case, in which the coupling
function has a leading nonlinear term, its reduced coupling function contains no constant term. Hence it follows from Eq. (48) that the growth of \( S_n \) for large \( n \) is governed by the two relevant CEs \( \nu_1 = \alpha \) and \( \nu_2 = M \) for the linear-coupling case \( (l = 0) \), but by only the second relevant CE \( \nu_2 = M \) for the other nonlinear-coupling cases \( (l \geq 1) \).

**III. NUMERICAL ANALYSIS**

Taking the trifurcation case with \( M = 3 \) as an example, we numerically study the scaling behavior associated with coupling in the two coupled 1D maps (41) with \( f(x) = 1 - A x^2 \) and \( \varphi(x, y) = \frac{1}{m}(y^m - x^m) \) \( (m = 1, 2, \ldots) \), and confirm the renormalization results. In this trifurcation case, we follow the 3\(^m\)-cycles up to level \( n = 9 \) and obtain the slopes of Eq. (46) at the zero-coupling critical point \( (A_\infty^{(3)}, 0) \) \( (A_\infty^{(3)} = 1.786 440 255 563 639 354 534 447 \ldots) \) when the reduced coupling function \( \Phi(x) \) is a monomial \( x^l \) \( (l = 0, 1, \ldots) \).

The renormalization result implies that the growth of the slopes \( S_n \) is governed by one CE \( \nu_2 = 3 \) for the nonlinear-coupling cases with \( l \geq 1 \), i.e., the sequence of \( S_n \) obeys a one-term scaling law asymptotically:

\[
S_n = d_1 r_1^n,
\]

where \( d_1 \) is some constant. We therefore define the growth rate of the slopes as follows:

\[
r_{1,n} \equiv \frac{S_{n+1}}{S_n}.
\]

Then it will converge to a constant \( r_1 \) as \( n \to \infty \). As an example consider the case \( \Phi(x) = x \). Figure [1] shows three plots of \( \lambda_{2,n}(A_\infty^{(3)}, \varepsilon) \) versus \( \varepsilon \) for \( n = 5, 6, \) and 7. For \( \varepsilon = 0 \), \( \lambda_{2,n} \) converges to the 1D critical stability multiplier \( \lambda^* \) \( (= 1.872 705 929 \ldots) \) as \( n \to \infty \). However, when \( \varepsilon \) is nonzero it diverges as \( n \to \infty \), i.e., its slope \( S_n \) at the zero-coupling critical point diverges as \( n \to \infty \). The sequence \( \{r_{1,n}\} \) of the growth rate of \( S_n \) is shown in the second column of Table [1]. Note that it converges fast to \( r_1 = \nu_2 = 3 \). We have also studied two other nonlinear-coupling cases with \( l = 2, 3 \) and found that the sequences of \( r_{1,n} \) also converge fast to \( r_1 = \nu_2 = 3 \).
However, in a linear-coupling case with $l = 0$, two relevant CE’s $\nu_1 = \alpha (= -9.277 \, 341 \ldots)$ and $\nu_2 = 3$ govern the growth of the slopes $S_n$. We therefore extend the simple one-term scaling law (19) to a two-term scaling law:

$$S_n = d_1 r_1^n + d_2 r_2^n \quad \text{for large } n,$$

(51)

where $d_1$ and $d_2$ are some constants, and $|r_1| > |r_2|$. This is a kind of multiple scaling law [21,22]. The equation (51) gives

$$S_{n+2} = q_1 S_{n+1} - q_2 S_n,$$

(52)

where $q_1 = r_1 + r_2$ and $q_2 = r_1 r_2$. Then $r_1$ and $r_2$ are solutions of the following quadratic equation,

$$r^2 - q_1 r + q_2 = 0.$$

(53)

To evaluate $r_1$ and $r_2$, we first obtain $q_1$ and $q_2$ from $S_n$’s using Eq. (52):

$$q_1 = \frac{S_{n+1} S_n - S_{n+2} S_{n-1}}{S_n^2 - S_{n+1} S_{n-1}}, \quad q_2 = \frac{S_{n+1}^2 - S_n S_{n+2}}{S_n^2 - S_{n+1} S_{n-1}}.$$

(54)

Note that Eqs. (51)-(54) are valid for large $n$. In fact, the values of $q_i$’s and $r_i$’s ($i = 1, 2$) depend on the level $n$. Thus we denote the values of $q_i$’s in Eq. (54) explicitly by $q_{i,n-1}$’s, and the values of $r_i$’s obtained from Eq. (53) are also denoted by $r_{i,n-1}$’s. Then each of them converges to a constant as $n \to \infty$:

$$\lim_{n \to \infty} q_{i,n} = q_i, \quad \lim_{n \to \infty} r_{i,n} = r_i, \quad i = 1, 2.$$

(55)

When $\Phi(x) = 1$, plots of $\lambda_{2,n}(A_{\infty}^{(3)},\varepsilon)$ versus $\varepsilon$ for $n = 2, 3,$ and 4 are shown in Fig. 2. The slopes $S_n$ at $(A_{\infty}^{(3)},0)$ obeys well the two-term scaling law (51). Sequences $\{r_{1,n}\}$ and $\{r_{2,n}\}$ are shown in the third and fourth columns of Table 1. Note that they converge fast to $r_1 = \nu_1 = \alpha$ and $r_2 = \nu_2 = 3$, respectively.
IV. SUMMARY

The scaling behavior of $M$-furcations is studied in two symmetrically coupled 1D maps. Using a renormalization method, the dependence of the scaling behavior on $M$ is particularly investigated in the zero-coupling case. It is found that the zero-coupling fixed map of the $M$-furcation renormalization operator has three relevant eigenvalues $\delta$, $\alpha$, and $M$. As in the case of 1D maps, the eigenvalue $\delta$ governs the scaling behavior of the nonlinearity parameter, irrespectively of coupling. However, the scaling behavior associated with coupling depends on the nature of coupling. In a linear-coupling case, it is governed by two CE's $\alpha$ and $M$, whereas it is governed by only one CE $M$ in the case of a nonlinear-coupling case. Taking the trifurcation case as an example, we also study the coupling effect on the second stability multipliers of $3^n$-cycles by a direct numerical method and confirm the renormalization results.

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REFERENCES

[1] N. Metropolis, M. L. Stein, and P. R. Stein, J. Comb. Theor. 15, 25 (1973).

[2] B. Derrida, A. Gervois, and Y. Pomeau, J. Phys. A 12, 269 (1979).

[3] M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978); 21, 669 (1979).

[4] B. Hu and I. I. Satija, Phys. Lett. A 98, 143 (1983).

[5] J.-P Eckmann, H. Epstein, and P. Wittwer, Commun. Math. Phys. 93, 495 (1984).

[6] W.-Z. Zeng and B.-L. Hao, Commun. in Theor. Phys. 3, 283 (1984).

[7] P. R. Hauser, Constantino, Tsallis, and E. M. F. Curado, Phys. Rev. A 30, 2074 (1984).

[8] S.-J. Chang and J. McCown, Phys. Rev. A 31, 3791 (1985).

[9] R. Delbourgo, W. Hart, and B. G. Kenny, Phys. Rev. A 31, 514 (1985).

[10] R. Delbourgo and B. G. Kenny, Phys. Rev. A 33, 3292 (1986).

[11] V. Urumov and L. Kocarev, Phys. Lett. A 144, 220 (1990).

[12] K. Kaneko, Prog. Theor. Phys. 68, 1427 (1983).

[13] J.-M. Yuan, M. Tung, D. H. Feng, and L. M. Narducci, Phys. Rev. A 28, 1662 (1983).

[14] T. Hogg and B. A. Huberman, Phys. Rev. A 29, 275 (1984).

[15] S. Kuznetsov, Radiophys. Quantum Electron. 28, 681 (1985).

[16] H. Kook, F. H. Ling, and G. Schmidt, Phys. Rev. A 43, 2700 (1991).

[17] S.-Y. Kim and H. Kook, Phys. Rev. A 46, R4467 (1992).

[18] S.-Y. Kim and H. Kook, Phys. Lett. A 178, 258 (1993).

[19] S.-Y. Kim and H. Kook, Phys. Rev. E 48, 785 (1993).

[20] S.-Y. Kim, Phys. Rev. E 49, 1745 (1994).
[21] J.-m. Mao and B. Hu, J. Stat. Phys. 46, 111 (1987); Int. J. Mod. Phys. B 2, 65 (1988).

[22] C. Reick, Phys. Rev. A 45, 777 (1992).
TABLE I. In a nonlinear-coupling case $\Phi(x) = x$, a sequence $\{r_{1,n}\}$ for a one-term scaling law is shown in the second column, and in the linear coupling case $\Phi(x) = 1$, two sequences $\{r_{1,n}\}$ and $\{r_{2,n}\}$ for a two-term scaling law are shown in the third and fourth columns, respectively.

| n | $r_{1,n}$ | $r_{1,n}$ | $r_{2,n}$ |
|---|---|---|---|
| 1 | 2.997 929 154 | -9.276 543 16 | 2.005 8 |
| 2 | 3.000 141 141 | -9.277 415 78 | 2.692 5 |
| 3 | 2.999 990 417 | -9.277 335 54 | 2.927 8 |
| 4 | 3.000 000 651 | -9.277 341 50 | 2.984 5 |
| 5 | 2.999 999 956 | -9.277 341 09 | 2.996 7 |
| 6 | 3.000 000 003 | -9.277 341 12 | 2.999 3 |
| 7 | 3.000 000 000 |             |             |
| 8 | 3.000 000 000 |             |             |
FIGURES

FIG. 1. Plots of the second stability multipliers $\lambda_{2,n}(A_{\infty}^{(3)}, \varepsilon)$ versus $\varepsilon$ for $n = 5, 6, 7$ in a non-linear-coupling case $\Phi(x) = x$.

FIG. 2. Plots of the second stability multipliers $\lambda_{2,n}(A_{\infty}^{(3)}, \varepsilon)$ versus $\varepsilon$ for $n = 2, 3, 4$ in the linear-coupling case $\Phi(x) = 1$.