I present a perturbative calculation of the spectrum of the Faddeev-Popov operator in Coulomb gauge in three dimensions, and Landau gauge in two and three dimensions, with an ansatz for the gluon propagator as the non-perturbative input. The results show how the low-lying Faddeev-Popov eigenvalue spectrum is modified as the first Gribov horizon is approached, and how the spectra can differ in Coulomb and Landau gauges.

I. INTRODUCTION

The low-lying spectrum of the Faddeev-Popov (F-P) operator, in Coulomb and covariant gauges, is a probe of the infrared properties of non-abelian gauge theories. Confinement in Coulomb gauge, in particular, is rather directly related to the F-P spectrum. The color Coulomb potential, for example, involves a product of inverse F-P operators, and the Coulombic self-energy of an isolated color charge, which is infrared divergent in a confining theory, depends crucially on the density of low-lying eigenvalues of the F-P operator, as discussed below. The connection to confinement is less apparent in covariant gauges, although the density of near-zero F-P eigenvalues of low-lying eigenvalues of the F-P operator is expected to be relevant to the infrared behavior of the ghost propagator.

Coulomb and Landau gauges are defined on the lattice as the set of elements, on each gauge orbit, for which the quantity

$$R[U] = -\sum_{x}\sum_{\mu=1}^{d}\text{Tr}[U_{\mu}(x)]$$

is stationary with respect to infinitesimal gauge transformations. Here $d$ denotes the number of space dimensions, in Coulomb gauge, and the number of spacetime dimensions in Landau gauge, which is a convention I will adopt from now on. In general, along any gauge orbit, there are many stationary points, known as Gribov copies, and at these points the F-P determinant may be positive or negative. This indefinite sign is closely related to Neuberger’s Theorem [1], which demonstrates that BRST quantization of any non-abelian lattice gauge theory is ill-defined at the non-perturbative level. The picture is that in summing over all copies on a gauge orbit, the copies with a positive F-P determinant are exactly cancelled by the copies with a negative F-P determinant, and the functional integral vanishes. It is for this reason that a constraint of some kind, imposed on the domain of functional integration, is necessary. Ideally the range of functional integration should be a subspace (such as the Fundamental Modular region) containing only a single gauge copy with positive F-P determinant per gauge orbit, but at a minimum the integration range should lie within the Gribov region. This is the region which consists of all gauge copies in which the non-trivial eigenvalues of the F-P operator are all positive; i.e. the Gribov copies which are local minima of $R[U]$. These are, in fact, the configurations obtained by standard lattice gauge-fixing algorithms. The Gribov region is completely bounded, and the Fundamental Modular region is partially bounded, by the first Gribov horizon, where the lowest non-trivial F-P eigenvalue vanishes. It has been argued by Zwanziger [2] that the volume of the Gribov region is concentrated close to the horizon, much as the volume of a sphere in a high dimensional Euclidean space is concentrated near the surface. Since the dimensionality of the space of all lattice configurations is very high indeed, the values of observables obtained at the Gribov horizon should dominate the expectation value. It would then be interesting to understand exactly how proximity to the Gribov horizon affects the behavior of various observables, starting with the spectrum of the F-P operator.

As a step in that direction, this article presents a perturbative calculation of the F-P spectra. Perturbation theory is not necessarily trustworthy when dealing with the low-lying eigenmodes, but something may still be learned from it. In particular, it would be interesting to see whether proximity to the Gribov horizon changes the behavior of the low-lying spectra already at the perturbative level, and whether that behavior is different, for some reason, in Coulomb and Landau gauges. The calculation is carried out for Landau gauge in two and three spacetime dimensions, and Coulomb gauge in three spacetime dimensions, to avoid the complications associated with renormalization in four dimensions. The proximity to the Gribov horizon is controlled by a mass parameter in the transverse gluon propagator, which is where the non-perturbative information enters. I use an ansatz for the gluon propagator, motivated by Gribov’s expression [3], which allows for any desired power behavior in the infrared.

II. F-P EIGENVALUES AND THE COULOMB SELF-ENERGY

The Coulomb potential between a static quark-antiquark pair located at points $x$ and $y$ is given by the expression

$$V_{C}(|x-y|) = -\frac{g^{2}C_{F}}{d_{A}}\left\langle (M^{-1})_{ab}^{al}(-\nabla_{z}^{2})(M^{-1})_{ca}^{l}b_{z}\right\rangle$$

(2.1)

Yang-Mills theory in Coulomb gauge is trivial in two spacetime dimensions if the spacetime manifold is flat and non-compact, and for that reason Coulomb gauge in $D = 2$ will not be considered here. Non-trivial features associated with the Coulomb gauge F-P operator do appear even in two dimensions, if the space direction is compactified to $S^{1}$, and this case has been thoroughly discussed by Reinhardt and Schleifenbaum in ref. [4].
where, $C_r$ is the quadratic Casimir of quarks in color representation $r$, $d_A$ is the dimension of the adjoint representation of the gauge group, and $M$ is the Faddeev-Popov operator, which is

$$M^{ab}_{xy} = \left(-\delta^{ab}\nabla^2 + g f^{abc} A^c(x) \partial_y \right) \delta^3(x-y) \quad (2.2)$$

in the continuum. If $V_C(|x-y|)$ is confining, then this can only be attributed to an infrared singular behavior of $M^{-1}$, which must be related somehow to the low-lying F-P eigenvalue spectrum.

The perturbative evaluation of the F-P spectrum starts with the free-field, $g^2 = 0$ case, on a finite periodic lattice of extension $L$. The eigenmodes of the corresponding F-P operator are simply the plane wave states

$$\phi_{nA}^{(0)} = \frac{1}{\sqrt{V}} e^{ip \cdot x} \chi_A$$

$$\lambda_{nA}^{(0)} = 2 \sum_{\mu} (1 - \cos(p_i))$$

$$p_i = \frac{2\pi n_i}{L}, \quad L/2 < n_i \leq L/2 \quad (2.3)$$

and the $\chi_A$ are some set of orthonormal vectors spanning the $d_A$-dimensional color space. The F-P eigenmodes and eigenvalues \{($\phi_{nA}(x), \lambda_{nA}$)\} at $g^2 > 0$ are also indexed by $(n, A)$, denoted for brevity by $n \equiv (n, A)$, and it is assumed that the eigenmodes and eigenvalues are continuous and differentiable functions of $g^2$, which smoothly approach (2.3) as $g^2 \to 0$. To connect the eigenmode spectrum to the Coulomb self-energy, we begin with the expression

$$E_{self} = \frac{g^2 C_r}{d_A} \lim_{V \to \infty} \frac{1}{V} \left( \langle (M^{-1})^{ab}_{xy} (-\nabla^2_x) (M^{-1})^{ba}_{yx} \rangle \right) \quad (2.4)$$

and inserting the spectral representation

$$\langle (M^{-1})^{ab}_{xy} \rangle = \sum_n \frac{\phi_n^{a}(x) \phi_n^{b}(y)}{\lambda_n} \quad (2.5)$$

this becomes [5]

$$E_{self} = \frac{g^2 C_r}{d_A} \lim_{V \to \infty} \frac{1}{V} \sum_n \left( \frac{\phi_n^{a} - \nabla^2 \phi_n^{a}}{\lambda_n} \right) \quad (2.6)$$

where $\rho(\lambda)$ is the normalized eigenvalue density

$$\rho(\lambda) = \lim_{V \to \infty} \frac{1}{V} \sum_n \delta(\lambda - \lambda_n) \quad (2.7)$$

$N_c$ is the number of colors, and $V = L^d$. In 2+1 dimensions, the integral in (2.6) is logarithmically divergent at the $\lambda \to 0$ end of the integration even in an abelian theory, and this is because the Coulomb potential in an abelian theory confines with a logarithmically rising potential. The criterion that the infrared divergence in the self-energy is stronger than logarithmic is

$$\lim_{\lambda \to 0} \left( \frac{\rho(\lambda) (\phi_{\lambda} - \nabla^2 \phi_{\lambda})}{\lambda^{1-\epsilon}} \right) > 0 \quad \text{for some } \epsilon > 0 \quad (2.8)$$

This condition involves the near-zero F-P eigenmodes, as well as the eigenvalues. However, assuming that the eigenvalue spectrum is non-degenerate (apart from some rather special cases involving symmetric gauge-field configurations), then at fixed $g^2 > 0$ each $\lambda_n$ is associated with a unique $(n, A)$, which in turn determines $p$. Then $p^2 = p^2(\lambda)$ in the infinite volume limit and

$$\lim_{\lambda \to 0} \left( \frac{\rho(\lambda) p^2(\lambda)}{\lambda^{1-\epsilon}} \right) > 0 \quad \text{for some } \epsilon > 0 \quad (2.9)$$

The proof of this inequality is given in the Appendix. It follows that a sufficient condition for Coulomb confinement is

$$\lim_{\lambda \to 0} \left( \frac{\rho(\lambda) p^2(\lambda)}{\lambda^{1-\epsilon}} \right) > 0$$

III. THE APPROACH TO THE HORIZON

It was stated above that numerical simulations find local minima of $R$, which means, strictly speaking, that all of the eigenvalues of the F-P operator are positive. This statement has to be qualified a little. Even apart from Gribov copies, the Coulomb and Landau gauge conditions do not entirely fix the gauge, because if $U_\mu(x)$ satisfies the gauge condition, so does $G U_\mu(x) G^\dagger$, where $G \in SU(N)$ is any position-independent group element. This is a remnant global gauge symmetry, and it implies that at any stationary point of $R$ there must be flat directions along the gauge orbit corresponding to zero modes of the F-P operator. These are the trivial eigenmodes

$$\phi_{0A}^{a}(x) = \frac{1}{\sqrt{V}} \chi_A^a \quad (3.1)$$

The statement that the F-P determinant is positive in the Gribov region really refers to the determinant of the operator in the subspace orthogonal to these trivial zero modes.

Outside the Gribov region, some of the non-trivial F-P eigenvalues become negative, which means that for configurations which lie exactly on the Gribov horizon there must be at least one non-trivial zero eigenvalue. However, in an infinite volume, the converse is not necessarily true: we cannot deduce, just from the fact that the spectrum of non-trivial eigenvalues begins at zero, that the gauge field lies on the Gribov horizon. Even in an abelian theory, which has no Gribov horizon, the spectrum of the F-P operator $-\nabla^2$ in an infinite volume begins at $\lambda = 0$.

Let us begin with $g^2 = 0$, i.e. a free-field theory, with the eigenvalues and eigenstates shown in (2.3). In this free case
We use $p$ better to replace the integer index the integration is over gauge fields placing a constraint in the functional integral by introducing a $L \rightarrow \infty$ integration is over gauge fields inside the Gribov region, lying a distance and marginally divergent (divergent as log $d$ the Coulomb potential increases logarithmically in $2$ in $s$ dimensions, so the question in $2+1$ dimensions is whether the bov horizon from the outside, the range of negative eigenvalues grows with a non-standard power $\lambda_p \sim p^{2+s}$. For configurations inside the Gribov region ($d_H > 0$) the growth $\lambda_p \sim p^2$ is quadratic. In the Type II scenario, for configurations just outside the Gribov region, the interval of negative eigenvalues does not include $p = 0$, and at the Gribov horizon the non-trivial zero mode is at $|p| > 0$.

we have
\[
\rho(\lambda) \propto \lambda^{(d-2)/2}, \quad (\phi_\lambda) - \nabla^2(\phi_\lambda) = \lambda
\] (3.2)
so that with an ultraviolet regulator, the Coulomb self-energy in $d + 1$ dimensions is finite for all space dimensions $d \geq 3$, and marginally divergent (divergent as $\log(L)$ as extension $L \rightarrow \infty$) at $d = 2$. The latter divergence is expected, since the Coulomb potential increases logarithmically in $2 + 1$ dimensions, so the question in $2+1$ dimensions is whether the condition in eq. (2.12) is satisfied for some $\varepsilon > 0$

Outside the Gribov region, some of the non-trivial F-P eigenvalues become negative, and approaching the first Gribov horizon from the outside, the range of negative eigenvalues should shrink away. Right on the horizon there must exist a non-trivial zero eigenvalue even for a finite spacetime volume. So let us imagine increasing $g^2$ away from zero, and also placing a constraint in the functional integral by introducing a dimensionful parameter $d_H$, and requiring that if $d_H > 0$, the integration is over gauge fields inside the Gribov region, lying a distance $d_H$ from the first Gribov horizon, while if $d_H < 0$, the integration is over gauge fields outside the Gribov region, at a distance $|d_H|$ from the horizon. Then
\[
\langle \lambda_p \rangle = \lambda_p^{(0)} + \langle \Delta \lambda_p \rangle = p^2(1 - F[g, p, d_H])
\] (3.3)
We use $p$ as an index because, in the infinite volume limit, it is better to replace the integer index $n$ by the continuous index $p$.

Also the expectation value of $\lambda_{pA}$ can depend on neither the index $A$, since this would violate global color symmetry, nor on the direction of $p$, which would violate rotation invariance.

If $g = 0$ then $F = 0$, but we may speculate on the behavior of $F[g, p, d_H]$ at $g^2 > 0\text{ as } d_H$ varies. Suppose $F$ has the form, near $p = 0$,
\[
F[g, p, d_H] = a[g, d_H] - b[g, d_H]p^s + \text{higher powers of } p
\] (3.4)
and $b[g, d_H]$ is positive for small $|d_H|$. At $d_H > 0$ all non-trivial eigenvalues are positive, so it must be that $a[g, d_H] < 1$ for small $p$. Note that the eigenvalue spectrum in an infinite volume still starts at $\lambda = 0$, even though the configurations are, by definition, off the Gribov horizon. At $d_H < 0$ some eigenvalues are negative, and if those are the eigenvalues near $p = 0$, it means that $a[g, d_H] > 1$. The negative eigenvalues must just disappear at $d_H = 0$, and this is obtained if $a[g, 0] = 1$. exactly. In this last case the subleading power of $p$ in $F[g, p, d_H]$ takes over, and we have
\[
\lambda_p \sim p^{2+s}, \quad \rho(\lambda) \sim \lambda^{(d-2-s)/(2+s)}
\] (3.5)
This a qualitative change in the low-lying F-P spectrum, compared to the behavior inside the Gribov region, and the sufficient condition (2.10) for Coulomb confinement is satisfied if
\[
2s + 2 > d
\] (3.6)
Inside the Gribov region, at $p \to 0$, the spectrum is simply a
rescaling of the zeroth-order spectrum

\[ \lambda_p = (1 - a|g, d_H|)p^2 \]  

(3.7)

and, in the case of Coulomb gauge, the Coulomb self-energy is finite. The conjectured behavior of \( \langle \lambda_p \rangle \) vs. \( p \), for \( d_H \) positive, negative, and zero, is sketched in Fig. 1(a).

But the scenario just outlined is not the only possible behavior near the horizon. Consider, in particular, the case that \( b|g, d_H| \) is negative for small \( |d_H| \). Then we have

\[ \langle \lambda_p \rangle = (1 - a|g, d_H|)p^2 - |b|g, d_H|p^{2+\sigma} + \text{higher powers of } p \]

(3.8)

and it is possible that \( \langle \lambda_p \rangle \) is positive near \( p = 0 \) where the \( p^2 \) term dominates, but negative in some finite region away from \( p = 0 \). The conjectured behavior in this case, for positive, negative, and vanishing \( d_H \), is indicated in Fig. 1(b), and in this case we would still have \( \lambda_p \sim p^3 \) at the horizon, for small \( p^2 \).

Of course, quantization in Coulomb and Landau gauge does not involve setting \( d_H \) to some definite value. What is required, however, is a constraint on the range of functional integration to lie within the first Gribov horizon. If it is true that entropy dominates due to the high dimensionality of the configuration space, and almost all of the volume of the Gribov region is concentrated at the horizon, then only lattice configurations at or very near the horizon will contribute to vacuum expectation values in Coulomb and Landau gauge, just as if the constraint \( d_H = 0 \) were imposed.

IV. PERTURBATIVE EVALUATION OF THE F-P SPECTRUM

The possible spectra shown in Figs. 1(a) and 1(b) are pure speculation at this point, but it is interesting, and somewhat in the spirit of Gribov’s original work [3], to see how far we can go in understanding the F-P spectrum with ordinary perturbation theory.

Let us begin with lattice SU(2) gauge theory in either spacetime dimensions (Coulomb gauge) or \( d \) spacetime dimensions (Landau gauge), starting on a finite \( d \)-dimensional volume \( V \) and taking the infinite volume \( V \to \infty \) and lattice spacing \( a \to 0 \) limits at the end. The F-P operator on the lattice is given by [5]

\[ M^{ab}_{xy} = (K_0)^{ab}_{xy} + (K_1)^{ab}_{xy} + (M_1)^{ab}_{xy} \]

\[ (K_0)^{ab}_{xy} = \delta^{ab} \sum_r \left( 2\delta_{xy} - \delta_{x+r,y} - \delta_{x,y+r} \right) \]

\[ (K_1)^{ab}_{xy} = \frac{1}{2g} \delta^{abc} \sum_r \left[ -A_t^a(x)\delta_{x+r,y} - A_t^a(y)\delta_{x-r,y} \right] \]

\[ (M_1)^{ab}_{xy} = -\delta^{abc} \sum_r \left( \delta_{xy} \left[ (1 - \frac{1}{2}TrU_t(x)) + (1 - \frac{1}{2}TrU_t(x - i)) \right] - \delta_{x+r,y} \right) 
- \delta_{x,y+r}(1 - \frac{1}{2}TrU_t(y)) \]

(4.1)

where

\[ A_t^a = \frac{1}{2ig} Tr[\sigma_a(U_t(x) - U_t^\dagger(x))] \]

(4.2)

The dimensionless lattice coupling \( g_L \) is related to the gauge coupling \( g \) by \( g_L^2 = a^4 - D^2 g^2 \), where \( a \) is the lattice spacing and \( D \) is the spacetime dimension. The eigenvalues and eigenvectors of \( K_0 \) are those shown in eq. (2.3). The operator \( M_1 \) vanishes in the continuum limit, so we will just ignore it in what follows, and treat \( K_1 \) as the only perturbation to \( K_0 \). Lattice Fourier transforms will be defined symmetrically

\[ A_t^a(x) = \frac{1}{\sqrt{V}} \sum_k A_t^a(k) e^{ikx} \]

\[ A_t^a(k) = \frac{1}{\sqrt{V}} \sum_x A_t^a(x) e^{-ikx} \]

(4.3)

The first-order correction to \( \lambda_p^{(0)} \) is

\[ \Delta \lambda_{p,A}^{(1)} = \langle p, A|K_1|p, A \rangle = \sum_{x,y} e^{ip_x A_x^a(x)} e^{ip_y A_x^a(y)} \]

\[ = \frac{1}{V} \sum_{x,y} e^{-ip_x A_x^a(x)} e^{-ip_y A_x^a(y)} \]

\[ = -ig \delta^{abc} \sum_{x,y} \frac{1}{V} \sum_k A_t^a(0) \sin(p_i) \]

(4.4)

Now, according to the above definition of the lattice Fourier transform, the lattice \( A \)-field at zero momentum is

\[ \hat{A}_t^a(0) = \frac{1}{\sqrt{V}} \sum_x A_t^a(x) \]

(4.5)

with \(-2/g < \hat{A}_t^a(x) < 2/g \). Then suppose that the lattice \( A \)-field in Coulomb or Landau gauge has a finite correlation length \( l \). This implies

\[ \sum_x A_t^a(x) \sim \pm \sqrt{\frac{V}{\text{vol}}} l^d \mathcal{A} \]

(4.6)

where \( \mathcal{A} \) is the average value of \( A_t^a \) in a hypercubic region of volume \( l^d \). Then, because of the factor of \( 1/\sqrt{V} \) in (4.4), the first-order correction to \( \lambda_p^{(0)} \) vanishes in the infinite volume limit. Of course, the first-order contribution vanishes even in a finite volume upon taking the expectation value, since \( \langle \hat{A}_t^a(0) \rangle = 0 \).

At second order

\[ \Delta \lambda_{p,A} = \sum_{k,B} |\langle k, B|K_1|p, A \rangle|^2 \]

(4.7)
where

\[
(k, B|K_1| p, A) = \frac{1}{\sqrt{V}} \sum_{x} e^{-i k x} (-A^b_i(x)e^{ip(x+i)})
\]

Then

\[
\Delta \lambda_{p,A} = \frac{1}{2} g^2 \sum_{B} (\lambda^e_{B}l_{e} \lambda^e_{B}l_{e}) \frac{1}{\sqrt{V}} \sum_{k} \frac{1}{\lambda^0_k - \lambda^0_k} \times \sum_{ij} \Delta^2_i (k - p) \Delta^2_j (p - k) \times \left( -e^{i p_a} + e^{-i k_a} \right) \left( -e^{-i p_a} + e^{i k_a} \right)
\]

(4.8)

Inserting these identities into (4.9)

\[
\langle \Delta \lambda_{p}^\prime \rangle = \frac{1}{d^2} \sum_{B} (\lambda^e_{B}l_{e} \lambda^e_{B}l_{e}) \frac{1}{\sqrt{V}} \sum_{k} \frac{1}{\lambda^0_k - \lambda^0_k} \times \sum_{ij} \Delta^2_i (k - p) \Delta^2_j (p - k) \times \left( -e^{i p_a} + e^{-i k_a} \right) \left( -e^{-i p_a} + e^{i k_a} \right)
\]

(4.10)

At this point we can take the continuum limit, and make use of the transversality property \(q_i D_{ij}(q) = 0\) of the gluon propagator, to obtain

\[
\langle \Delta \lambda_{p}^\prime \rangle = g^2 \sum_{B} (\lambda^e_{B}l_{e} \lambda^e_{B}l_{e}) \frac{1}{\sqrt{V}} \sum_{k} \frac{1}{\lambda^0_k - \lambda^0_k} \times \int \frac{d^d k}{(2\pi)^d} \left( p^2 - k^2 \right)^{1/2} p_i D_{ij}(p - k)
\]

(4.15)

The primes, having served their purpose, will now be dropped. It is understood that the unprimed quantities now have their standard engineering dimensions.

Using the competens property

\[
\sum_{B} \lambda^e_{B}l_{e} \lambda^e_{B}l_{e} = \delta^{cf}
\]

(4.16)

we sum over the color indices, which just gives an overall factor of two. The result is

\[
\langle \Delta \lambda_{p} \rangle = -2g^2 \sum_{B} \lambda^e_{B}l_{e} \lambda^e_{B}l_{e} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - p^2} \times \left( p^2 - (p - q)^2 \right)
\]

(4.17)

(note the interchange of \(k^2\) and \(p^2\) in the denominator). Changing variables to \(q = p - k\), and writing

\[
D_{ij}(q) = \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) D(q)
\]

(4.18)

gives

\[
\langle \Delta \lambda_{p} \rangle = -2g^2 \sum_{B} \lambda^e_{B}l_{e} \lambda^e_{B}l_{e} \int \frac{d^d q}{(2\pi)^d} \frac{D(q)}{q^2 - p^2} \times \left( p^2 - (p - q)^2 \right)
\]

(4.19)

We now go to \(d\)-dimensional spherical coordinates

\[
\int d^d q = A_{d-1} \int_{0}^{\infty} dq \ q^{d-1} \int_{0}^{\pi} \sin^{d-2} \theta
\]

(4.20)

where, in Landau gauge, \(D_{ij}(k')\) is the full (i.e. dressed) gluon propagator. In Coulomb gauge it is the spatial Fourier trans-
where
\[ A_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \]  
(4.21) and
\[ R_d = \frac{2A_{d-1}}{(2\pi)^d} = \begin{cases} 1/\pi^2 & d = 2 \\ 1/(2\pi^2) & d = 3 \\ 1/(6\pi^2) & d = 4 \end{cases} \]  
(4.23)

Define
\[ \bar{D}(q) = q^{d-2}D(q) \]  
(4.22)

\[ \langle \Delta \lambda_p \rangle = -g^2 R_d \int_0^\infty dq q^{d-1} \int_0^\pi d\theta \sin^{d-2}\theta (1 - \cos^2\theta) \frac{1}{q^2 - 2pq\cos\theta} D(q)p^2 \]
\[ = -g^2 R_d \int_0^\pi d\theta \sin^{d-2}\theta (1 - \cos^2\theta) \int_0^\infty dq \frac{1}{q - 2p\cos\theta} q^{d-2}D(q)p^2 \]
\[ = -g^2 R_d p^2(I_1 + I_2) \]  
(4.24)

where
\[ I_1 = \int_0^{\pi/2} d\theta \sin^{d-2}\theta (1 - \cos^2\theta) \int_0^\infty dq \frac{1}{q - 2p\cos\theta} \bar{D}(q) \]
\[ I_2 = \int_{\pi/2}^\pi d\theta \sin^{d-2}\theta (1 - \cos^2\theta) \int_0^\infty dq \frac{1}{q - 2p\cos\theta} \bar{D}(q) \]  
(4.25)

Make the change of variables \( \theta \to \pi - \theta \) in \( I_2 \)
\[ I_2 = \int_0^{\pi/2} d\theta \sin^{d-2}\theta (1 - \cos^2\theta) \int_0^\infty dq \frac{1}{q + 2p\cos\theta} \bar{D}(q) \]  
(4.26)

Then, in \( I_1 \), it is useful to rewrite the integral over momenta \( q \)
\[ \int_0^\infty dq \frac{\bar{D}(q)}{q - 2p\cos\theta} = \left\{ \int_0^{2p\cos\theta} dq \frac{\bar{D}(q)}{q - 2p\cos\theta} + \int_{2p\cos\theta}^{4p\cos\theta} dq \frac{\bar{D}(4p\cos\theta - q)}{2p\cos\theta - q} + \int_{4p\cos\theta}^\infty dq \frac{\bar{D}(4p\cos\theta + q)}{2p\cos\theta + q} \right\} \]
\[ = \int_0^{2p\cos\theta} dq \frac{\bar{D}(q)}{q - 2p\cos\theta} + \int_{2p\cos\theta}^{4p\cos\theta} dq \frac{\bar{D}(4p\cos\theta - q)}{2p\cos\theta - q} + \int_0^\infty dq \frac{\bar{D}(4p\cos\theta + q)}{2p\cos\theta + q} \]  
(4.27)

Altogether, we have to second order
\[ \langle \lambda_p \rangle = \lambda_p^{(0)} + \langle \Delta \lambda_p \rangle = p^2 \left( 1 - g^2 R_d I[p,m,\alpha] \right) \]  
(4.28)

where
\[ I[p,m,\alpha] = \int_0^{\pi/2} d\theta \sin^{d-2}\theta (1 - \cos^2\theta) \left\{ \int_0^\infty dq \frac{1}{q + 2p\cos\theta} [\bar{D}(4p\cos\theta + q) + \bar{D}(q)] \right\} \]
\[ + \int_{2p\cos\theta}^{4p\cos\theta} dq \frac{1}{2p\cos\theta - q} [\bar{D}(4p\cos\theta - q) - \bar{D}(q)] \]  
(4.29)

The \( m, \alpha \) in \( I[p,m,\alpha] \) are constants I will use to parametrize the transverse gluon propagator \( D(q) \).

V. AN ANSATZ FOR THE GLUON PROPAGATOR

The gluon propagators \( D_{ij} \) in Coulomb and Landau gauges are transverse with respect to spatial momenta \( q \) in \( d + 1 \) di-
dimensions, and spacetime momenta \( q^2 \) in \( d \) Euclidean dimensions, respectively. Therefore these propagators have the form shown in (4.18). In a free theory

\[
D(q) = \begin{cases} 
\frac{1}{(2q)} & \text{Coulomb gauge} \\
\frac{1}{q^2} & \text{Landau gauge}
\end{cases}
\]

(5.1)

where the propagator in Coulomb gauge is at equal times, with \( q \) the space (rather than spacetime) momentum. The behavior is expected at high momenta, but it is certainly not correct at low momenta, as seen from lattice Monte Carlo simulations. In Landau gauge, the current evidence is that \( D(0) \) is finite and non-zero at \( q = 0 \) in three and four dimensions, while \( D(q) \to 0 \) in two dimensions. In Coulomb gauge it appears that \( D(q) \to 0 \) in four dimensions.

In order to allow for non-singular power behavior in the transverse gluon propagator as \( p \to 0 \), I will adopt the ansatz that

\[
D(q) = \frac{1}{2 \sqrt{q^2 + m^2 + \alpha^2 / q^2}}
\]

(5.2)
in Coulomb gauge, and

\[
D(q) = \frac{1}{q^2 + m^2 + \alpha^2 / q^2}
\]

(5.3)
in Landau gauge. Gribov’s proposal for the gluon propagator in these cases corresponds to \( \alpha = 2 \). The propagators go over to free-field behavior as \( q \to \infty \).

![Image](image_url)

FIG. 2: Equal-times Coulomb-gauge gluon propagator in 2+1 dimensions, at \( \beta = 6 \) and \( L^3 \) lattice volume, for \( L = 24, 32, 50 \).

I am not aware of any lattice Monte Carlo computation of the transverse gluon propagator in Coulomb gauge in \( d = 3 \) dimensions, in position space. In Fig. 2 I show data for \( D(R) \) obtained from the equal times correlator

\[
\langle \text{Tr}[\Lambda_j(x,t)\Lambda_j(y,t)] \rangle
\]

(5.4)
of gluon fields

\[
\Lambda_j(x,t) = \frac{1}{2i}(U_{jp}(x,t) - U_{jp}^+(x,t))
\]

(5.5)
on the lattice. The correlator is calculated via lattice Monte Carlo with an SU(2) Wilson action, on an \( L^3 \) lattice volume at coupling \( \beta = 6 \) and \( L = 24, 32, 50 \), with the equal-times correlator computed after transforming the gauge fields to Coulomb gauge. Note that as the lattice volume increases, the gluon propagator develops a “dip” and actually becomes negative at the larger \( R \) values. This behavior appears to rule out \( \alpha = 0 \), in which the propagator should be everywhere positive. A reliable computation of \( D(q) \) as \( q \to 0 \) will probably require a large-scale lattice calculation, as has been done for the Landau gauge.

VI. RESULTS FOR F-P SPECTRA

In section I introduced a parameter \( d_H \) to control the approach to the first Gribov horizon, and speculated on the low-\( p \) behavior of \( \lambda_p \) as the horizon is approached. In the perturbative calculation, the mass parameter \( m \) in the gluon propagator plays essentially the same role as \( d_H \). Note that in dimensions lower than \( 3+1 \), where \( \lambda(p,m,\alpha) \) is convergent, the coupling \( g^2 \) is dimensionful, and we may as well choose units such that \( g^2 = 1 \). Then

\[
\langle \lambda_p \rangle = p^2 \left( 1 - R_d[p,m,\alpha] \right)
\]

(6.1)

Expanding \( R_d[p,m,\alpha] \) in leading powers of \( p \) near \( p = 0 \), we have

\[
R_d[p,m,\alpha] = a[m,\alpha] - b[m,\alpha]p^4 + \text{higher powers of } p
\]

(6.2)
in which case

\[
\langle \lambda_p \rangle = (1 - a[m,\alpha])p^2 + b[m,\alpha]p^{2+s} + \text{higher powers of } p
\]

(6.3)

Suppose, for a given \( \alpha \), it is possible to find a critical value \( m = m_c \) such that \( a[m_c,\alpha] = 1 \) and \( b[m,\alpha] > 0 \). In that case we have the Type I scenario conjectured in Fig. 1(a) above; i.e.

1. \( m < m_c \) and \( a[m,\alpha] > 1 \): The low-lying F-P eigenvalue spectrum has a range of negative eigenvalues, starting at \( p = 0 \). We interpret this to mean that the transverse gluon propagator, which determines the spectrum at second order, is determined by configurations outside the Gribov region.

2. \( m = m_c \) and \( a[m_c,\alpha] = 1 \): The region of negative eigenvalues just disappears, and \( \lambda_p \sim p^{2+s} \). This is the case of particular interest, where the gluon propagator is derived from configurations which mainly lie right on the Gribov horizon.
3. $m > m_c$ and $a[m, \alpha] < 1$. In this case the low-lying spectrum $\lambda_p = (1 - a[m, \alpha]) p^2$ is just a rescaling of the free-field spectrum, and the gluon propagator is derived from configurations inside the Gribov region.

It should be noted at this point that the Type I scenario is in some ways reminiscent of the Dyson-Schwinger approach, and indeed eq. (4.23) resembles the Dyson-Schwinger equation for the ghost propagator in covariant gauges (see, e.g., Fischer [11]). Of course these equations are not the same; (6.1) is an equation for the expectation value of F-P eigenvalues, not the inverse ghost propagator, and it is derived from a perturbative expansion, not the Dyson-Schwinger equations. Nevertheless, the scaling solution [12] is obtained from the Dyson-Schwinger equation by tuning a coupling so that the bare inverse ghost propagator in that equation is exactly cancelled by another term. In the absence of this tuning, the decoupling solution [13] is obtained. Similarly, in our approach, a mass parameter is tuned to exactly cancel the $p^2$ term in the eigenvalue spectrum, resulting in an enhanced density of near-zero eigenmodes. The motivation for the tuning in our case is to study the F-P spectrum at the Gribov horizon, which is only relevant to physics if, in fact, the functional integral over the Gribov region is dominated by horizon configurations.

The Type II scenario is obtained if $b[m, \alpha]$ is negative when $a[m, \alpha] = 1$, in which case there is still a range of negative eigenvalues, so this value of $m$ is not the critical value. The critical value, corresponding to the horizon, is obtained at a value $m = m_c$ where $a[m, \alpha] < 1$, such that the function

$$\langle \lambda_p \rangle = (1 - a[m_c, \alpha]) p^2 - b[m_c, \alpha] p' + c[m_c, \alpha] p'^2$$  \hspace{1cm} (6.4)

approximating $\langle \lambda_p \rangle$ at small $p$ has a zero value, but no negative values, for one choice of $p \neq 0$. In this case the horizon does not alter the power dependence $\lambda_p \sim p^2$ near $p = 0$.

Both the Type I and Type II scenarios assume that

$$a[m, \alpha] = R_d I[p = 0, m, \alpha]$$  \hspace{1cm} (6.5)

is finite. This is not necessarily the case, however, and it is easy to check that $I[0, m, \alpha]$ is divergent at all $\alpha \leq 0$ for Landau gauge in two dimensions and Coulomb gauge in three dimensions, and is divergent for all $\alpha \leq -1$ for Landau gauge in three dimensions. There is no choice of $m$, for those choices of $\alpha$, which completely eliminates negative F-P eigenvalues. This will be referred to as the “no solution” case.

In order to determine which scenario is realized, at each choice of $\alpha$ for which $I[0, m, \alpha]$ is finite, it is necessary to calculate $I[p, m, \alpha]$ numerically. The result, for Coulomb gauge in three dimensions, and Landau gauge in two and three dimensions, is indicated schematically in Fig. 3. To illustrate how these results are obtained, we consider in particular the case of $\alpha = 1$ for Coulomb and Landau gauges in three dimensions (i.e., $d = 2$ for Coulomb, and $d = 3$ for Landau). We begin with Coulomb gauge (Figs. 4(a)). Figure 4(a) shows the low-lying F-P spectrum at $\alpha = 1$ and $m = 0.20 < m_c$, and it is clear that there is a region of negative eigenvalues starting at $p = 0$. As $m$ is increased, the region of negative eigenvalues shrinks in size, until at a critical value $m = m_c(\alpha)$ the interval of negative eigenvalues just vanishes. Figure 4(b) displays the low-lying spectrum just below, at, and just above the critical mass at $\alpha = 1$, which is $m_c = 0.2228$. At $m = m_c$, $\lambda_p$ is proportional to $p^2$ near $p = 0$, with a proportionality constant which is positive or negative, depending on whether $m$ is greater or less than $m_c$. But precisely at $m = m_c$, we find that $\lambda_0 \propto p^{2+\epsilon}$, with $s = s(\alpha) > 0$. Fig. 5 is a log-log plot of $\lambda_p$ vs. $p$ over a large range of $p$, at $\alpha = 1$ and $m_c = 0.223$. For the range $0 < p < 1$, we can determine that $s = 0.53$ in this case, and $\lambda_p \approx 1.21 p^{2.53}$ at small $p$. At around $p \equiv |p| = 1$ (in units $g^2 = 1$), the power behavior shifts to the free case, $\lambda_p = p^2$, and continues that way for all higher $p$, as expected. This is

### Figure 3: Summary of the qualitative behavior of the low-lying F-P spectra, according to 2nd order perturbation theory, for Landau and Coulomb gauges in D=2 and 3 dimensions. The sketch illustrates how the behavior of the F-P spectra depends on the assumed infrared behavior of the gluon propagator, which is parametrized by the exponent $\alpha$.

| Scenario | Landau D=3 | Landau D=2 | Coulomb D=3 |
|----------|------------|------------|-------------|
| Type I   | no solution| no solution| no solution |
| Type II  |            |            |             |

\[
\nu = 0.1, \beta = 0.6\]
an example of the Type I scenario.

The next question is how $m_c$ and $2 + s$ change as $\alpha$ is varied. As already noted, we must choose $\alpha > 0$ to reach the horizon, which means that $D(0) = 0$, and therefore the transverse gluon propagator must vanish at zero momentum for Coulomb gauge in 2+1 dimensions, and for Landau gauge in 1+1 dimensions. As $\alpha \to 0^+$, the increasingly singular behavior of the integrand in $I[p, m, \alpha]$ must be counteracted by an increasingly large value of $m_c$, in order to satisfy $a[m_c, \alpha] = 1$. A plot of $m_c$ vs. $\alpha$ is shown in Fig. 6.

The power behavior $\lambda_p = b p^{2+s}$ in the low-lying spectrum is crucial for Coulomb confinement, and the exponent $2 + s$ vs. $\alpha$, obtained at $m = m_c$ is shown in Fig. 7(a). In 2+1 dimensions the condition for Coulomb confinement (beyond the marginal divergence of the free theory) is that $s > 0$, which is seen to hold throughout the range shown.

We also see that there is a sudden jump in $s$ from roughly $s = 1$ to $s = 2$ at $\alpha = 2$. This is where the transition from Type I to Type II behavior takes place. As $\alpha \to 2$ the coefficient $b[m_c, \alpha]$ approaches zero (cf. Fig. 7(b)) and then changes sign. Exactly at $\alpha = 2$, where $b[m_c, \alpha] = 0$, the term which has the next higher power in $p$ takes over, accounting for the sudden jump in $s$.

Landau gauge in three dimensions, at $\alpha = 1$, furnishes an example of the Type II scenario. The F-P spectrum at small $p$ is shown in Fig. 8 for the mass parameter above ($m = 0.087$), below ($m = 0.086$), and equal $m = m_c = 0.08644$ to the critical value.

FIG. 4: F-P spectra at $\alpha = 1$. (a) $m = 0.20 < m_c$. There is an interval of negative eigenvalues in the region $0 < p < 0.009$. (b) $\lambda_p$ at low $p$, for $m = 0.20 < m_c$, $m = m_c = 0.2228$, and $m = 0.25 > m_c$.

FIG. 5: Log-log plot of the spectrum of the Fadeev-Popov operator, for $\alpha = 1$ at the critical $m_c = 0.223$. A best fit at $p \ll 1$ yields $\lambda_p = 1.21 p^{2.53}$.

FIG. 6: Critical value $m_c$ for the mass parameter in the transverse gluon propagator, vs. the power $\alpha$.

VII. CONCLUSIONS

If the integration over gauge fields is dominated by configurations on or near the first Gribov horizon, then the low-
FIG. 7: (a) Exponent $2+s$ vs. $\alpha$; and (b) coefficient $b$ vs $\alpha$; for the power-law behavior $\lambda_p = bp^{2+s}$ at the critical mass parameter $m = m_c(\alpha)$, for the Coulomb gauge F-P spectrum in 2+1 dimensions. The sudden rise to $2+s = 4$ at $\alpha = 2$ is correlated with $b \to 0$.

FIG. 8: The low-lying F-P eigenvalue spectra near the Gribov horizon, for Landau gauge in D=3 spacetime dimensions and $\alpha = 1$, according to second-order perturbation theory. This is an example of the Type II scenario.

The most non-trivial F-P eigenvalue must be very close or equal to zero, even in a finite spacetime volume. The main finding of the perturbative treatment presented here is that if there is, in fact, a non-trivial zero mode, and the F-P eigenvalues are labeled by the lattice momenta, then this non-trivial zero mode may occur at either zero momentum (Type I scenario) or non-zero momenta (Type II scenario), depending on the infrared behavior of the gluon propagator. While the spectrum of F-P eigenvalues does not translate directly into a prediction for the behavior of the ghost propagator (because the momentum behavior of the F-P eigenmodes must also be taken into account), it is natural to conjecture that the Type I scenario is associated with an infrared singular ghost dressing function, as in Coulomb gauge, while the Type II scenario corresponds to a finite ghost dressing function, as appears to be the case in Landau gauge. This would most likely be the case if $|\phi_{pA}(k)|^2$ is narrowly peaked around $k = p$, where $\phi_{pA}(k)$ is the Fourier transform of an F-P eigenmode $\phi_{pA}(x)$ with a low-lying eigenvalue $\lambda_{pA}$.

Since the FP spectra at the Gribov horizon have been derived here from ordinary 2nd order perturbation theory (plus an ansatz for the gluon propagator), there is obviously a question of whether perturbation theory can be trusted in this context. In $D = 3$ spacetime dimensions the coupling $g^2$ has units of mass, so the expansion parameter at $p \to 0$ will be $g^2/m$, while the expansion parameter at large $p$ will be $g^2/|p|$. The perturbative calculation of the FP eigenvalue spectrum at $p \to 0$ should therefore be trustworthy for large $m/g^2$. Unfortunately, we have seen that the critical mass parameter $m_c$ corresponding to the Gribov horizon is actually rather small, in units of $g^2$, with, e.g., $m_c/g^2 = 0.223$ in Coulomb gauge, and $m_c/g^2 = 0.0864$ in Landau gauge in three spacetime dimensions and $\alpha = 1$. Of course, the perturbative expansion may also involve some numerical factors, and without calculating to higher orders, or estimating the radius of convergence in some way, it is difficult to judge the accuracy of the second-order term in the series. But there is no particular reason for confidence in the second-order results at $m = m_c$ at the quantitative level. It was argued however in section III on rather general grounds, that it is natural to expect either Type I or Type II behavior of the Faddeev-Popov spectrum at the Gribov horizon. The perturbative calculation, at this stage, simply provides a concrete illustration in support of this rather general qualitative argument.

Acknowledgments

This research was supported in part by the U.S. Department of Energy under Grant No. DE-FG03-92ER40711.
In order to derive the inequality \( \text{(2.9)} \) stated in section II, we begin with the following.

**Theorem.** Let \( Q \) be any Hermitian operator with a discrete set of eigenstates \( \{|n\}\}, \) whose corresponding eigenvalue spectrum \( \{q_n\} \) is bounded from below, and ordered such that \( q_n \leq q_{n+1} \). Here the index \( n \) runs from \( 1 \) up to the dimension of the Hilbert space \( N_H \) (which need not be finite). Let \( \{|\phi_n\}\} \) be any other complete set of orthonormal states spanning the same Hilbert space as the \( \{|n\}\} \). Then, for any \( N \leq N_H \),

\[
\sum_{n=1}^{N} \langle \phi_n | Q | \phi_n \rangle \geq \sum_{n=1}^{N} q_n \tag{A.1}
\]

This is a fairly trivial generalization of the inequality underlying the Rayleigh-Ritz variational method (the case \( N = 1 \)), and the proof goes as follows: Define

\[
T \equiv \sum_{n=1}^{N} \langle \phi_n | Q | \phi_n \rangle = \sum_{n=1}^{N} \sum_{k} \sum_{m} \langle \phi_n | k \rangle \langle k | Q | m \rangle \langle m | \phi_n \rangle = \sum_{m} q_m P_N(m) \tag{A.2}
\]

where

\[
P_N(m) = \sum_{n=1}^{N} \langle m | \phi_n \rangle \langle \phi_n | m \rangle \tag{A.3}
\]

Observe that

\[
0 \leq P_N(m) \leq P_{N_H}(m) = 1 \tag{A.4}
\]

and

\[
\sum_{m=1}^{N_H} P_N(m) = N \tag{A.5}
\]

Since \( P_N(m) \leq 1 \), and with regard to the constraint \( \text{(A.5)} \), the smallest possible value of \( T \) is obviously obtained for

\[
P_N(m) = \begin{cases} 
1 & m \leq N \\
0 & m > N
\end{cases} \tag{A.6}
\]

Substituting this optimal choice into the last line of \( \text{(A.2)} \), we find that

\[
T \geq T_{\text{min}} = \sum_{m=1}^{N} q_m \tag{A.7}
\]

and the inequality stated in the theorem is established. From this theorem it follows that

\[
\sum_{n}^{(N)} \langle \phi_n | - \nabla^2 | \phi_n \rangle \geq \sum_{n}^{(N)} \lambda_n^{(0)} \tag{A.8}
\]

where now the \( \{|\phi_n\}\} \) are the eigenstates of the F-P operator, and where we have defined

\[
\sum_{n}^{(N)} \equiv \sum_{n \neq a} \theta(N^2 - n \cdot n) \tag{A.9}
\]

Now let

\[
\rho_N(\lambda) = \frac{1}{N_{\text{color}} V} \sum_{n}^{(N)} \delta(\lambda - \lambda_n) \tag{A.10}
\]

Assuming non-degeneracy, each \( \lambda \) determines \( n, A \) uniquely, \( n \) determines \( \lambda_n^{(0)} \), and \( p_n^2 = \lambda_n^{(0)} \) in the large-volume limit. In this limit we can therefore we can express \( \text{(A.8)} \) as

\[
\int d\lambda \, \rho_N(\lambda) (\phi_k | - \nabla^2 | \phi_k) \geq \int d\lambda \, \rho_N(\lambda) p_k^2(\lambda) \tag{A.11}
\]

At small \( \lambda \) there is some leading power behavior

\[
\langle \rho_N(\lambda) (\phi_k | - \nabla^2 | \phi_k) \rangle \approx a\lambda^c \tag{A.12}
\]

Then, since \( \text{(A.11)} \) must be true for any mode cutoff \( N \), no matter how small, it follows that either \( r < q \), or \( r = q \) and \( a > b \). Either way,

\[
\lim_{\lambda \to 0} \left( \frac{\rho_N(\lambda) (\phi_k | - \nabla^2 | \phi_k)}{\lambda^{1-\epsilon}} \right) \geq \lim_{\lambda \to 0} \left( \frac{\rho_N(\lambda) p_k^2(\lambda)}{\lambda^{1-\epsilon}} \right) \tag{A.13}
\]

Finally, given that all the near-zero modes are included in the sum \( \text{(A.9)} \), we have

\[
\rho_N(\lambda) = \rho(\lambda) \quad \text{as} \quad \lambda \to 0 \tag{A.14}
\]

and the inequality \( \text{(A.13)} \) becomes

\[
\lim_{\lambda \to 0} \left( \frac{\rho(\lambda) (\phi_k | - \nabla^2 | \phi_k)}{\lambda^{1-\epsilon}} \right) \geq \lim_{\lambda \to 0} \left( \frac{\rho(\lambda) p_k^2(\lambda)}{\lambda^{1-\epsilon}} \right) \tag{A.15}
\]

This establishes eq. \( \text{(A.9)} \).

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[1] H. Neuberger, Phys. Lett. B183, 337 (1987).
[2] D. Zwanziger, Nucl. Phys. B 518, 237 (1998).
[3] H. Reinhardt and W. Schleifenbaum, Annals Phys. 324, 735 (2009) [arXiv:0809.1764 [hep-th]].
[4] V. N. Gribov, Nucl. Phys. B 139 (1978) 1.
[5] J. Greensite, Š. Olejnık, and D. Zwanziger, JHEP 0505, 070 (2005) [arXiv:hep-lat/0407032].
[6] D. Zwanziger, Nucl. Phys. B 364, 127 (1991).
[7] A. Cucchieri and T. Mendes, PoS LAT2007, 297 (2007) [arXiv:0710.0412 [hep-lat]].
   A. Cucchieri and T. Mendes, Phys. Rev. D 78, 094503 (2008) [arXiv:0804.2371 [hep-lat]].
   Phys. Rev. Lett. 100, 241601 (2008) [arXiv:0712.3517 [hep-lat]].
[8] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker and A. Sternbeck, Phys. Lett. B 676, 69 (2009) [arXiv:0901.0736 [hep-lat]].
[9] A. Maas, Phys. Rev. D 75, 116004 (2007) [arXiv:0704.0722 [hep-lat]].
[10] M. Quandt, G. Burgio, S. Chimchinda and H. Reinhardt, PoS CONFINEMENT8, 066 (2008) [arXiv:0812.3842 [hep-th]].
    A. Cucchieri and D. Zwanziger, Nucl. Phys. Proc. Suppl. 119, 727 (2003) [arXiv:hep-lat/0209068].
    K. Langfeld and L. Moyaerts, Phys. Rev. D 70, 074507 (2004) [arXiv:hep-lat/0406024].
[11] C. S. Fischer, J. Phys. G 32 (2006) R253 [arXiv:hep-ph/0605173].
[12] L. von Smekal, R. Alkofer and A. Hauck, Phys. Rev. Lett. 79 (1997) 3591, [hep-ph/9705242]; Annals Phys. 267 (1998) 1 [Erratum-ibid. 269 (1998) 182], [hep-ph/9707327].
    C. S. Fischer, R. Alkofer and H. Reinhardt, Phys. Rev. D 65 (2002) 094008, [hep-ph/0202195].
    C. S. Fischer and R. Alkofer, Phys. Lett. B 536 (2002) 177, [hep-ph/0202202].
[13] Ph. Boucaud, J. P. Leroy, A. L. Yaouanc, J. Micheli, O. Pene and J. Rodriguez-Quintero, JHEP 0806 (2008) 012 [arXiv:0801.2721 [hep-ph]].
    A. C. Aguilar, D. Binosi and J. Papavassiliou, Phys. Rev. D 78 (2008) 025010 [arXiv:0802.1870 [hep-ph]].
    D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel and H. Verschelde, Phys. Rev. D 78 (2008) 065047 [arXiv:0806.4348 [hep-th]].