Force acting on a cluster of magnetic nanoparticles in a gradient field: a Langevin dynamics study

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Abstract

Magnetophoretic force acting on a rigid spherical cluster of single-domain nanoparticles in a constant-gradient weak magnetic field is investigated numerically using the Langevin dynamics simulation method. Nanoparticles are randomly and uniformly distributed within the cluster volume. They interact with each other via long-range dipole-dipole interactions. Simulations reveal that if the total amount of particles in the cluster is kept constant, the force decreases with increasing nanoparticle concentration due to the demagnetizing field arising inside the cluster. Numerically obtained force values with great accuracy can be described by the modified mean-field theory, which was previously successfully used for the description of various dipolar media. Within this theory, a new expression is derived, which relates the magnetophoretic mobility of the cluster with the concentration of nanoparticles and their dipolar coupling parameter. The expression shows that if the number of particles in the cluster is fixed, the mobility is a nonmonotonic function of the concentration. The optimal concentration values that maximize the mobility for a given amount of magnetic phase and a given dipolar coupling parameter are determined.

Keywords: magnetophoresis, magnetophoretic mobility, magnetic nanoparticles, magnetic beads, Langevin dynamics, Landau-Lifshitz-Gilbert equation

1. Introduction

Magnetic beads (or microspheres) are composite objects consisting of magnetic nanoparticles embedded in a spherical polymer matrix [1, 2]. Nanoparticles can be homogeneously distributed within the bead volume, placed on its surface or concentrated in its center. Typical sizes of beads are 0.1–10 µm. The most promising applications of beads are in biotechnology and medicine. Among them are magnetic cell separation [3], targeted drug delivery [4] and single-molecule magnetic tweezers [5].

The physical basis for many applications of magnetic beads is the phenomenon of magnetophoresis, i.e. the motion of magnetic particles under the action of nonuniform magnetic field. It is known that the sensitivity of beads to the applied gradient field is among main factors determining their suitability for biomedical purposes [3]. As a result, there are many experimental studies on detailed magnetic characterization of different beads from different manufacturers [6–8]. The present work, on the contrary, uses a simplistic model of the magnetic bead to conduct a numerical and analytical study, which will hopefully provide new qualitative insights into how the magnetophoretic motion of the bead is affected by its size and magnetic content.

2. Model and methods

2.1. Problem formulation

The bead is modeled as a cluster of \( N \) identical spherical magnetic nanoparticles. The diameter \( d \) of particles is small enough (\( \sim 10 \) nm) so that they can be considered as single-domain. Each particle has a magnetic moment \( \mu \), which can rotate freely inside the particle body and has a constant magnitude \( \mu = vM_s \), where \( v = (\pi/6)d^3 \) is the particle volume, \( M_s \) is the saturation magnetization of the particle material. Particles are embedded in a rigid nonmagnetic spherical matrix of diameter \( D \), their positions are fixed. Dipole-dipole interactions between particles are taken into account. The cluster is placed in a constant nonuniform magnetic field with a gradient \( G \). For definiteness, an ideal quadrupole field is considered: \( H = (G_x, -G_y, 0) \) [9]. The schematic sketch of the investigated system is shown in Fig. 1. The primary task of this study is to determine the magnetic force \( F_m \) acting on the cluster due to the field for a given \( R_c \), where \( R_c = (X_c, Y_c, Z_c) \) is the location of the cluster center.

Let us introduce a set of appropriate dimensionless parameters that determine the cluster behavior. The field magnitude can be characterized by the so-called Langevin parameter \( \xi = \mu_0 H/k_B T \), where \( \mu_0 \) is the magnetic constant, \( k_B \) is the Boltzmann constant, \( T \) is the system tem-
temperature. The Langevin parameter is the ratio between the magnetic (Zeeman) energy of a particle in the cluster and the thermal energy of the system. Mag-
netite nanoparticles, which are typical in biomedical applications, can be used as an example to estimate these parameters. The saturation magnetization of bulk magnetite is \( M_s \approx 450 \text{ kA} \cdot \text{m}^{-1} \) according to Ref. [11], but the value \( M_s \approx 350 \text{ kA} \cdot \text{m}^{-1} \) is sometimes used for single-domain particles [12, 13]. Thus, for magnetite nanoparticles with \( d = 10 \text{ nm} \), the dipolar coupling parameter at temperature \( T = 300 \text{ K} \) is \( \lambda \approx 1 \); for particles with 15 nm, it is \( \lambda \approx 3–4.5 \). Typical gradient values used in the so-called low gradient magnetic separation are \( G \lesssim 8 \cdot 10^7 \text{ A} \cdot \text{m}^{-2} \) (or \( \leq 100 \text{ T} \cdot \text{m}^{-1} \)) [14]. For magnetite nanoparticles, this corresponds to \( g \sim 10^{-4} \). Sometimes much larger gradients of the order of \( \sim 10^3 \text{ T} \cdot \text{m}^{-1} \) (\( g \sim 10^{-3} \)) are used [15]. As for the field magnitude itself, we will here mainly restrict ourself to values \( \xi \lesssim 1 \). In this range, the magnetic response of a nanoparticle ensemble remains linear.

For 10 nm magnetite particles, \( \xi = 1 \) corresponds to \( H \approx 15 \text{ kA} \cdot \text{m}^{-1} \) (or to \( B \approx 20 \text{ mT} \)). This weak field range is relevant for many biomedical diagnostic systems [16–18]. Besides, the restriction on the field magnitude simplifies the simulation procedure, since now it is possible to neglect magnetic anisotropy of particles. It is known that in the general case magnetic anisotropy can significantly effect the magnetization of nanoparticles distributed in a solid matrix [19]. However, if the distribution of particles’ easy axes is random and uniform and if the applied field is weak, the magnetization does not noticeably depend on anisotropy, it is always close to the magnetization of isotropic particles [12, 20]. Another important parameter is the particle volume fraction (volume concentration) \( \varphi = \nu N/V = N d^3 / D^3 \), where \( V = (\pi / 6) D^3 \) is the cluster volume. The notation \( x^* = x/d \) will be used for the reduced distance.

### 2.2. Langevin dynamics simulations

In order to accurately take into account the combined effect of the applied field, intrachannel interactions and thermal fluctuations on the cluster behavior, the Langevin dynamics method is used. The Langevin equation that describes the magnetodynamics of a single-domain particle is the stochastic Landau-Lifshitz-Gilbert equation [21, 22]. For the \( i \)th particle of the simulated cluster it reads

\[
\frac{d\mu_i}{dt} = -\gamma \left[ \mathbf{H}_i \times \mathbf{M}_i \right] - \frac{\gamma_0}{\mu} \left[ \mathbf{H}_i \times \left[ \mathbf{M}_i \times \mathbf{H}_i \right] \right],
\]

(1)

where \( \gamma = \gamma_0 / (1 + \alpha^2) \), \( \gamma_0 \) is the gyromagnetic ratio (measured in \( \text{m} \cdot \text{A}^{-1} \cdot \text{s}^{-1} \)), \( \alpha \) is the phenomenological dimensionless damping constant, \( \mathbf{H}_i^{\text{tot}} = \mathbf{H}_i^{\text{det}} + \mathbf{H}_i^{\text{fl}} \) is the total deterministic field acting on the particle, it is the sum of the applied field and dipolar fields due to all other particles, \( \mathbf{H}_i^{\text{fl}} \) is the fluctuating thermal field. \( \mathbf{H}_i^{\text{fl}}(t) \) is a Gaussian stochastic process with the following statistical properties:

\[
\langle H_i^{\text{fl}}(t) \rangle = 0,
\]

(2)

\[
\langle H_i^{\text{fl}}(t_1) H_j^{\text{fl}}(t_2) \rangle = 2D \delta_{ij} \delta_{kl} \delta(t_1 - t_2),
\]

(3)

\[
D = \frac{\alpha k_B T}{\mu_0 \mu_0 \gamma (1 + \alpha^2)},
\]

(4)

where \( k \) and \( l \) are Cartesian indices, angle brackets denote a mean value. Eq. (1) can be rewritten in the dimensionless form:

\[
\frac{d\mathbf{e}_i}{dt} = -\frac{1}{2\alpha} \left[ \mathbf{e}_i \times \mathbf{e}_i^{\text{det}} \right] - \frac{1}{2} \left[ \mathbf{e}_i \times \left[ \mathbf{e}_i \times \mathbf{e}_i^{\text{det}} \right] \right],
\]

(5)

where the \( \mathbf{e}_i = \mathbf{M}_i / \mu \), \( t^* = t / \tau_D \) is the reduced time, \( \tau_D = \mu_0 H / 2 \alpha \gamma k_B T \) is the characteristic time scale of the rotary diffusion of the magnetic moment (typically, \( \tau_D \sim \).
\(10^{-10} \text{ s} \) \cite{13}], \( \xi_{i}^{\text{det}} = \mu_{0}\mu H_{i}^{\text{tot}}/k_{B}T = \xi_{i}^{\text{det}} + \xi_{i}^{\text{fl}}, \)

\[
\xi_{i}^{\text{det}} = \xi_{i} + \lambda \sum_{j \neq i}^{N} \left[ 3 r_{ij}^{3} (e_{ij} \cdot r_{ij}^{3}) - e_{ij} \right] / r_{ij}^{3}, \quad (6)
\]

\[
\langle \xi_{i}^{\text{fl}}(t) \rangle = 0, \quad (7)
\]

\[
\langle \xi_{i,k}^{\text{fl}}(t^{*}) \rangle = \frac{4\alpha^{2} g^{2}}{1 + \alpha^{2}} \delta_{ij} \delta_{kl} (r_{ij}^{3} - r_{kl}^{3}), \quad (8)
\]

where \( \xi_{i} = (gx_{i}^{*}, -gy_{i}^{*}, 0) \), \( r_{ij}^{3} = r_{ij}^{*} - r_{ij}^{*} \) is the vector between centers of particles \( i \) and \( j \), \( r_{ij}^{*} = (x_{i}^{*}, y_{i}^{*}, z_{i}^{*}) \).

The input parameters of the simulation are \( N, \varphi, \lambda, g \) and \( \xi_{i} \). The latter is the field in the center of the cluster, \( \xi_{c} = g\sqrt{X_{c}^{2} + Y_{c}^{2}} \). In simulations, the cluster is always positioned on the \( Y \) axis: \( X_{c}^{*} = Z_{c}^{*} = 0, Y_{c}^{*} > 0 \). Using \( \xi_{c} \), the cluster position is determined as \( Y_{c}^{*} = \xi_{c}/g \). Using \( N \) and \( \varphi \), the cluster diameter is determined as \( D^{*} = \sqrt{N}/\varphi \). Then the cluster is generated by randomly placing particles within the allowed volume without overlapping. Initial orientations of magnetic moments are chosen at random. After the cluster is generated, the standard Heun scheme \cite{22} is used for numerical integration of Eqs. (5-8).

The damping constant in simulations is \( \alpha = 0.1 \) and the integration time step is \( \Delta t^{*} = 0.002 \). Position and orientation of the cluster as a whole remain fixed during simulations. The following values of input parameters are investigated numerically: \( 0.25 \leq \xi \leq 2, 1 \leq \lambda \leq 7, 0.05 \leq \varphi \leq 0.45, g = 10^{-5}, N = 10^{-2} \cdot 10^{3} \).

The instantaneous force on a point-like magnetic moment due to external field is \( F_{m} = \mu_{0} (\mu \cdot \nabla) H \) \cite{23}. For a quadrupole field, the force on the ith particle is

\[
F_{m,i} = \mu_{0}\mu G(e_{i,x}, -e_{i,y}, 0). \quad (9)
\]

If the field is large enough, the particle magnetic moment is always aligned with its direction and the magnitude of the force has its maximum value \( \mu_{0}\mu G \). For modeled systems the field magnitude and direction do not significantly change within the cluster. In this case, all particles in the large field are also collinear with each other and the cluster is saturated, its magnetic moment is \( m_{sat} = \mu_{0}N \) and the force is \( F_{sat} = \mu_{0}m_{sat}G \). A normalized magnetic force then can be introduced as \( f_{m} = F_{m}/F_{sat} \). For an arbitrary field magnitude, the net external force is calculated in simulations as

\[
f_{m} = \frac{1}{N} \left( \sum_{i=1}^{N} f_{m,i} \right) = \frac{1}{N} \left( \sum_{i=1}^{N} e_{i,x}, -\sum_{i=1}^{N} e_{i,y}, 0 \right). \quad (10)
\]

To find this average quantity, the sampling of instantaneous force values starts after the time period of \( t = 200\tau_{D} \sim 10^{-8} - 10^{-7} \text{ s} \). In all considered cases, this time is enough for the simulated system to reach equilibrium. Note that in biomedical applications, such as magnetic cell separation, typical velocities of magnetic microparticles are \( 10^{2} - 10^{3} \mu \text{m/s} \) \cite{3}. So, the time it takes a microparticle to travel a distance equal to its diameter (\( 10^{-3} - 10^{-1} \text{ s} \)) is several orders of magnitude larger than the time required to achieve an equilibrium force value. This justifies the neglect of the cluster translational motion in simulations. The same reasoning remains valid even if magnetic anisotropy of particles is taken into account. It is known that anisotropy slows down the relaxation time of magnetic moments, but for 10 nm iron oxide particles this time is still comparable with \( \tau_{D} \) \cite{11}. The situation can be more complicated for other magnetic materials. For example, 10 nm cobalt ferrite particles have the relaxation time \( \gg 1 \text{ s} \) \cite{24}, so the cluster with such particles will presumably remain in a nonequilibrium state during its magnetophoretic motion. This situation is beyond the scope of the present work. For every particular set of input parameters, the force values are averaged not only over simulation time but also over ten independent realizations of the cluster. These realizations differ in positions of particles and initial orientations of magnetic moments. Such averaging can be also considered as an implicit account of the cluster rotation which may arise due to small inhomogeneities in the particle spatial distribution. In practice, force values for different realizations are very close. Error bars presented on the plots below show 95% confidence intervals for calculated averages.

### 2.3. Analytical solution

A much more common approach to the problem at hand is to consider the cluster as a single paramagnetic particle. If the gradient is relatively small \( (gD^{*} \ll 1) \), this particle is homogeneously magnetized. In the weak field range, the magnetization of the cluster is \( M = \chi H \), where \( \chi \) is its initial susceptibility. Let us denote the total magnetic moment of the cluster as \( m = VM \). Then the force on the cluster is

\[
f_{m} = \mu_{0}V(\nabla \cdot \chi_{0} H, 0) = \mu_{0}\chi V(H \cdot \nabla)H. \quad (11)
\]

Strictly speaking, the above expressions must contain the quantity \( \Delta \chi = \chi - \chi_{s} \), where \( \chi_{s} \) is the susceptibility of the surrounding medium \cite{3}, but the latter is assumed to be negligible. For a quadrupole field, this force can be rewritten in the normalized form

\[
f_{m} = \frac{\mu_{0}V\chi G_{2}^{2}}{F_{sat}} \left( x_{c}, y_{c}, 0 \right) = \frac{\mu_{0}V\chi G_{2}^{2} \xi d}{F_{sat} g} \left( \frac{x_{c}^{*}}{\sqrt{x_{c}^{*2} + y_{c}^{*2}}}, \frac{y_{c}^{*}}{\sqrt{x_{c}^{*2} + y_{c}^{*2}}}, 0 \right) = \chi \frac{\xi d}{3N} \left( \frac{x_{c}^{*}}{\sqrt{x_{c}^{*2} + y_{c}^{*2}}}, \frac{y_{c}^{*}}{\sqrt{x_{c}^{*2} + y_{c}^{*2}}}, 0 \right), \quad (12)
\]

where \( \chi_{L} = \mu_{0}^{2}N/3k_{B}TV = 8\varphi \) is the so-called Langevin susceptibility, which describes the initial magnetic response of an ideal paramagnetic gas \cite{20}. The only unknown quantity in Eq. (12) is the initial susceptibility \( \chi \). To estimate it, we will use the so-called modified mean-field the-
ory (MMFT). This approach was first proposed for the description of static magnetic properties of concentrated ferrofluids [25, 26]. MMFT also showed its effectiveness in the description of other media containing magnetic nanoparticles, such as ferrogels [27] and magnetic emulsions [28]. It was shown in Ref. [29] that, at least in the limited parameter range, MMFT is able to describe the initial susceptibility of rigid quasi-spherical clusters similar to those investigated here. Main MMFT results, which are relevant for this works, are as follows. Consider an elongated cylindrical sample homogeneously filled with magnetic nanoparticles. According to MMFT, if a weak uniform magnetic field is applied along the main axis of the cylinder, then the ratio between the sample magnetization and the field is given by the susceptibility

$$\chi_0 = \chi_L(1 + \chi_L/3). \quad (13)$$

For a spherical sample, the situation becomes more complicated. Now $\chi_0$ describes the relation between the magnetization and the magnetic field inside the sample $H_{int}$, i.e. $M = \chi_0 H_{int}$. The internal field does not coincide with the applied one, the difference between two fields is called the demagnetizing field. It is created by the surface divergence of the sample’s own magnetization [30]. For a sphere, the relation between applied and internal fields is $H = H_{int} + M/3$ (for a cylinder, $H = H_{int}$ if $H$ is directed along the main axis). Thus, the final formula for the susceptibility of the spherical nanoparticle sample, according to MMFT, is

$$\chi = \frac{M}{H} = \frac{\chi_0}{1 + \chi_0/3} = \chi_L \frac{1 + \chi_L/3}{1 + \chi_L/3 + \chi_L^2/9}. \quad (14)$$

Eqs. (12) and (14) completely determine the magnetic force acting on the cluster. Their applicability range is to be tested via numerical simulations.

3. Results and discussion

3.1. Magnetic force

One the simulation results is that the force acting on the cluster positioned on the $Y$ axis is directed predominantly along this axis, the average $x$-component of the force is zero within the error bar. Fig. 2 illustrates dependencies of the magnetic force magnitude on the magnetic field intensity $\xi_c$. It is seen that Eqs. (12) and (14) accurately describe simulation results in the weak field limit. With increasing field, the growth of the force slows down due to the fact that the cluster magnetization curve is nonlinear – its magnetic moment cannot be larger than $m_{sat}$ and the force cannot be larger than the corresponding value $F_{sat}$. It is noteworthy that the field range, where the linear response assumption is valid, increases with increasing particle concentration. In Fig. 2a, which corresponds to $\varphi = 0.05$, nonlinearity becomes noticeable already at $\xi \approx 1$, but in Fig. 2b ($\varphi = 0.4$) the linear law Eq. (12) is valid up to $\xi = 2$. Fig. 2 gives simulation results for clusters of different sizes, $N = 100$ and $N = 1000$. Simulation points for two cases are very close and this is an encouraging result. Due to limited computational resources, we only investigate clusters with $N \sim 10^3$, which at the lowest considered concentrations $\varphi \sim 0.1$ have a diameter of a few tenths of a micron. But the weak dependency of the cluster reduced properties on its size indicates that obtained
results should remain relevant for larger structures with \( D \approx 1–10 \mu m \).

Simulations also show that the force acting on an \( N \)-particle cluster is smaller for more concentrated clusters. It is clearly seen in Fig. 3. In the limit \( \varphi \to 0 \), the normalized force is \( f_m = \xi_c/3 \). With increasing \( \varphi \), the force starts a nonlinear decline, which is more pronounced at larger coupling parameters \( \lambda \). At \( \lambda = 7 \) and \( \varphi = 0.45 \), the force drops by almost an order of magnitude. MMFT accurately describes simulation results for all considered values of \( \varphi \) and \( \lambda \) and can be used to analyze the observed behavior. The total magnetic moment of the cluster and the magnetic force Eq. (11) are proportional to the quantity \( V = (\chi/\varphi)V_m \), where \( V_m = vN \) is the total amount of magnetic material in the cluster. In the case when \( V \) is fixed, the force is controlled by the susceptibility \( \chi \), which is a measure of the magnetic response per unit volume. But if \( V_m \) is fixed (this is the case in simulations), the force is determined by \( \chi/\varphi \), which is a measure of the magnetic response per particle. If intrachannel interactions between particles are neglected (the Langevin approximation), the susceptibility is given by the Langevin value \( \chi = \chi_L \), which grows linearly with the concentration \( \varphi \). As a result, for a given \( V_m \), the quantity \( \chi/\varphi \) and hence the force do not depend on the particle concentration. The force always equals to the zero-concentration value \( F_{m0} = (\xi_c/3)F_{sat} = (8\varphi V_m/\mu_0G H_c) \). Eq. (13) goes beyond the Langevin approximation and takes into account the fact that dipole-dipole interactions between an arbitrary particle and its local surroundings, on average, help the particle to align with the field. \( \chi_0 \) grows quadratically with the concentration and \( \chi_0/\varphi \) grows linearly. Eq. (14) additionally takes into account the demagnetizing field, which is the long-range effect of dipole-dipole interactions. This field, in accordance with its name, weakens the response of an arbitrary particle to the applied field. The demagnetizing field is proportional to \( \chi \), which grows slower than linearly with \( \varphi \) and bounded from above by the value \( \chi = 3 \) (see Fig. 4 below). Consequently, at fixed \( V_m \) and large \( \varphi \), the quantity \( \chi/\varphi \) and hence the force decrease hyperbolically with the concentration, \( F_m = (\xi_c/8\lambda \varphi)F_{sat} = (3V_m/\varphi)\mu_0G H_c \).

### 3.2. Magnetophoretic mobility

The analytical model based on MMFT shows very good agreement with the simulation results in a wide range of interaction parameters \( \varphi \) and \( \lambda \). Potentially, the model can be used for an accurate description of the cluster magnetophoresis at temporal and spatial scales that are not easily accessible via direct nanoscale simulations. For example, the model can be used to obtain a universal expression for the so-called magnetophoretic mobility. It is known that magnetic microparticles moving in a viscous nonmagnetic liquid with time attain a constant velocity \( u \), which is determined by the balance between the magnetic force \( F_m \) and the drag force \( F_d \) [3]. The latter for spherical particles with low Reynolds numbers is given by the Stokes's law:

\[
F_d = -3\pi \eta D u, \quad (15)
\]

where \( \eta \) is the viscosity of the suspending liquid. In the general case, the drag force should contain the hydrodynamic diameter of the cluster \( D_h \), which can be larger than \( D \) if the cluster core is covered by some nonmagnetic shell.
The mobility under these optimal conditions is \( \mathcal{M}_{\text{opt}} \simeq 4\lambda^{2/3} \mathcal{M}_{\text{ref}}(V_m) \). The observed nonmonotonic dependency can be explained as follows. In the limit \( \varphi \to 0 \), the parameter \( (\chi/\varphi)V_m \), which controls the magnetic force, has a definite finite value \( 8\lambda V_m \). On the contrary, the cluster diameter and hence the friction coefficient are infinitely large, so the mobility in this limit is zero. With increasing concentration, the friction coefficient decreases as \( \sim 1/\varphi^{1/3} \) and the mobility initially increases as \( \mathcal{M} = 8\lambda\varphi^{1/3} \mathcal{M}_{\text{ref}}(V_m) \). But at \( \varphi > \varphi_{\text{opt}} \), the magnetic force decrease becomes hyperbolic (due to strong demagnetizing fields) and dominates over the drag decrease. As a result, the mobility eventually falls down as \( \mathcal{M} = (3/\varphi^{2/3}) \mathcal{M}_{\text{ref}}(V_m) \).

4. Conclusions

In this work, the force acting on a polymer magnetic bead in a constant-gradient field is calculated by means of the Langevin dynamics method. The bead is modeled as a spherical rigid cluster of randomly distributed single-domain particles. The magnitude of the applied field is typically small enough so that the cluster magnetization remains a linear function of the field. It is demonstrated that if the total number of particles in the cluster is fixed, the increase in the particle concentration leads to the nonlinear decrease in the force magnitude. The reason for this is the demagnetizing field inside the cluster, which weakens the response of an arbitrary particle to the applied field and hence decreases the cluster net average magnetic moment. It is also shown that the cluster can be successfully represented as a single paramagnetic particle whose magnetization obeys MMFT. The theory describes numerically obtained force values with great accuracy in a broad range of simulation parameters. Within MMFT, a new universal formula is obtained for the magnetophoretic mobility of an isolated cluster moving in a viscous nonmagnetic liquid. The formula shows that for a given number of particles and a given dipolar coupling parameter there is an optimal concentration value (and hence an optimal diameter) for which the mobility is maximal. Below this value, the mobility becomes smaller due to the increase of amount of magnetic material \( V_m \) is fixed. The quantity \( \mathcal{M}_{\text{ref}}(V_m) = (6V_m/\pi)^{2/3}/18\eta \) may be chosen as a reference mobility value for this case. For \( N = 10^6, d = 10 \text{ nm} \) and \( \eta = 10^{-3} \text{ Pa·s} \), this value is \( \mathcal{M}_{\text{ref}} = 1.2 \times 10^{-11} \text{ m}^3/(\text{T·A·s})^{-1} \). Concentration dependencies of \( \mathcal{M}/\mathcal{M}_{\text{ref}}(V_m) \) at different \( \lambda \) are given in Fig. 5. It is seen that dependencies are nonmonotonic, for every \( \lambda \) there is an optimal concentration value at which the mobility is maximal. The corresponding value of \( \chi_L \) can be found by solving \( (\partial \mathcal{M}/\partial \chi_L)_\lambda \) for \( \varphi = 0 \). It gives the optimal value of the Langevin susceptibility \( \chi_{L,\text{opt}} \simeq 1.9813 \). Then the optimal volume fraction and diameter are

\[
\varphi_{\text{opt}} = \frac{1.9813}{8\lambda}, \quad D_{\text{opt}} = \left[ \frac{6V_m}{\pi \varphi_{\text{opt}}} \right]^{1/3}.
\]
the cluster friction coefficient; above this value, the mobility becomes smaller due to the discussed decrease of the magnetic force.

In future, we hope to investigate a more general problem when nanoparticles do not occupy the whole bead, but instead distributed only in an outer spherical shell surrounding a nonmagnetic polymer core.

5. Acknowledgments

The work was supported by Russian Science Foundation (project No. 17-72-10033).

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