Dynamics of Thermalization and Decoherence of a Nanoscale System

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We study the decoherence and thermalization dynamics of a nanoscale system coupled nonperturbatively to a fully quantum-mechanical bath. The system is prepared out of equilibrium in a pure state of the complete system. We propose a random matrix model and show analytically that there are two robust temporal regimes in the approach of the system to equilibrium — an initial Gaussian decay followed by an exponential tail, consistent with numerical results on small interacting lattices [S. Genway, A. F. Ho and D. K. K. Lee, Phys. Rev. Lett. 105 260402 (2010)]. Furthermore, the system decays towards a Gibbs ensemble in accordance with the eigenstate thermalization hypothesis.

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The origin of thermodynamics from a fully quantum-mechanical description has been the subject of much recent research [1]. Emergence of thermal behavior from the unitary evolution of a wavefunction on a generic closed system can be studied using concepts such as the eigenstate thermalization hypothesis [3] (ETH) and canonical typicality [5, 6]. Local or few-body observables in a closed nonintegrable system are expected to ‘thermalize’ at long times [7] in the sense that they converge to a thermal Gibbs distribution. This has been studied with various approaches [8, 12] and for myriad systems [13, 20]. Recent interest has turned to understanding the dynamics of the relaxation to the thermal state [20, 26]. In the canonical model, one considers a composite of system and bath [6] and asks how the system relaxes and decoheres [26, 28] to reach a thermal state at long times.

In previous work [20], we found numerically that a random matrix model provided a generic description of thermalization dynamics for (nonrandom) nanoscale Hubbard clusters. In this Letter, we provide an analytical framework for this random matrix model. We derive [Eqs. (2-4)] the relaxation dynamics of a generic quantum system over the whole temporal range from short to long times. We also confirm that the model produces a thermal state at long times in accordance with ETH. Random matrix methods have been employed to study nanoscale systems coupled to different environments [29, 32]. However, they do not capture the full range of temporal behavior: there is no general account of the Gaussian decay towards thermalization that has been established [14, 20] numerically as a generic feature for the relaxation of local observables in interacting systems.

We focus on a nanoscale system (S) with a discrete energy spectrum embedded in a nonintegrable interacting bath (B) with a quasicontinuous spectrum so that the average bath level spacing $\Delta_B$ is much smaller than the system level spacings. We will examine how the small system thermalizes with the bath via the unitary evolution of the quantum-coherent composite system using a banded coupling model. Previous authors studied a banded coupling [20, 31] but were unable to access the regime where we see Gaussian decay (see below).

The model. — Suppose the system has $N_s$ eigenstates $|s\rangle$ with energies $\varepsilon_s$ and the bath has eigenstates $|b\rangle$ of energies $\epsilon_b$. The Hamiltonian for the composite system is given by $H = H_0 + V$:

$$H = \sum_{sb} E_{sb} |sb\rangle \langle sb| + \sum_{ss'bb'} |sb\rangle \langle sb| V |s'b\rangle \langle s'b|$$

where $|sb\rangle = |s\rangle \otimes |b\rangle$ are product states with energies $E_{sb} = \varepsilon_s + \epsilon_b$ for the decoupled system and bath, and $V$ couples the system and the bath. The coupled system will have an average level spacing of $\Delta = \Delta_B/N_s$. Analogous to the classic random matrix theory of nuclear matter, we model the interacting bath with an energy spectrum that obeys Wigner-Dyson statistics. Note that the randomness does not arise from quenched disorder. We assume that the bath states $|b\rangle$ are random vectors with no special spatial structure, e.g. no spatial localization. This should be valid for generic interacting quantum systems at energies away from strongly correlated states near the bath ground state. The matrix elements of the coupling $V$ in a basis involving these bath states should therefore also be random. We use a banded random matrix of bandwidth $W$ and strength $c$.

More precisely, the matrix element $\langle sb| V |s'b\rangle$ is nonzero only if $|E_{sb} - E_{s'b'}| < W$, and each nonzero element is a Gaussian random variable with zero average and a mean-square value $\langle |\langle sb| V |s'b\rangle|^2 \rangle = c \Delta$. As we see below, this scaling with the level spacing $\Delta$ is consistent with a local coupling between system and bath.

We can motivate this banded coupling model in the context of ultracold atoms in optical lattices. A small cluster of sites (system) is initially isolated from the rest of the lattice (bath) by a high tunneling barrier. The coupling is introduced by lowering this barrier to allow particles to hop between the cluster and the lattice. This particle exchange only couples bath states with an energy
difference of the order of the single-particle bandwidth. This produces a dense banded matrix with bandwidth $W$ (see Fig. 20 of Ref. [7] on the Hubbard model description of this setup). This is the motivation of our banded coupling $V$. Our scaling of the coupling with the level spacing $\Delta$ is also motivated by the local quench in this lattice example: $\text{Tr} V^2 \propto dN J_0^2$ where $N$ is the number of states in the composite system and there are $d$ links with hopping integral $J_0$. Since there are $2NW/\Delta$ nonzero matrix elements, this corresponds [7] to $c \sim dJ_0^2/W$. Unlike in conventional statistical mechanics, we do not assume a weak system-bath coupling so that we can study local observables in a homogeneous optical lattice. Such local measurements are becoming experimentally accessible [33]. Effects of time-reversal symmetry can be studied by trap rotation or artificial gauge fields [34].

**Central Result.**—At time $t = 0$, we prepare the total system in a separable initial state $|\Psi(0)\rangle = |S\rangle \otimes |B\rangle$, for a general system state $|S\rangle$. The bath state $|B\rangle$ is restricted only by the requirement that it should have a small energy uncertainty. This means $|\Psi(0)\rangle$ has significant overlap only with eigenstates of $H$ centered around a total energy $E_0 = \langle \Psi(0) | H | \Psi(0) \rangle$. The system evolves as $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle = \sum_A e^{-iE_At}|A\rangle \langle A|\Psi(0)\rangle$ where $|A\rangle$ are the exact eigenstates of the composite system with energies $E_A$ ($h=1$). We study the reduced density matrix (RDM) obtained by tracing out the bath: $\rho_{ss'}(t) = \sum_{A,B} \langle A| \langle B;[\Psi(0)\rangle |A\rangle |B\rangle \otimes |[\Psi(t)\rangle |B\rangle \rangle |s\rangle \langle s'\rangle$. Our main result is the full temporal evolution of the RDM in the limit of a large bath ($\Delta \ll c, W$), for times $t \ll 1/\Delta$:

$$\rho_{ss}(t) \approx \rho_{ss}(\infty) + [\rho_{ss}(0) - \rho_{ss}(\infty)] e^{-2\Lambda(0,t)},$$

$$\rho_{ss'}(t) \approx \rho_{ss'}(0) e^{-i(\epsilon_s - \epsilon_{s'})t} e^{-2\Lambda(0,t)} \quad (s' \neq s),$$

$$\Lambda(t', t) = \int_{-\infty}^\infty \frac{c(E) R(E)}{E^2} \left( e^{iE't'} - e^{iE(t-t')} \right) dE.$$  (4)

Here, $c(E)$ is the profile for the banded coupling matrix: $c(E) = c$ for $|E| < W$ and zero otherwise. The symmetries of the random matrix model enter via the level repulsion [35], expressed by $R(E) \propto |E|$ or $E^2$ for systems with or without time-reversal symmetry, respectively, for $|E| \lesssim \Delta$, and tending to unity for $|E| \gg \Delta$. Note that the thermalization dynamics discussed below is insensitive to time-reversal symmetry because thermalization occurs over time scales shorter than the time scale $1/\Delta$ over which the system is sensitive to level repulsion.

In this limit of a large bath, we find that the diagonal elements of the RDM decay to reach a steady-state value expected from the Gibbs distribution $\rho_{ss}(\infty) = \nu_0 (E_0 - \epsilon_s) \Delta$ where $\nu_0$ is the bath density of states [36]. Moreover, decoherence has the same dynamics as thermalization, with the off-diagonal elements $\rho_{ss'}(t)$ tending to zero at long times on the same time scales [37].

Most importantly, we establish that the relaxation towards the thermal state has two temporal regimes (as seen in our numerics [20]). The RDM is controlled by $\Lambda(0,t) \approx t^2 \int_{-\infty}^\infty c(E) dE = cWt^2$ for $t \ll W^{-1}$, and $c(E \rightarrow 0) \pi t$ for $W^{-1} \ll t \ll \Delta^{-1}$. So, the RDM has a Gaussian decay with a decay rate of $2\sqrt{cW}$ for $t < W^{-1}$ but has an exponential tail at longer times with decay rate $2\pi c$. For weak coupling ($c \ll W$), the decay is predominantly exponential, as expected from Fermi’s Golden Rule and perturbative Lindblad theory. For stronger coupling [38] ($c \gg W$), the Gaussian form dominates with thermalization completed by the crossover time $W^{-1}$. We stress that the existence of the Gaussian and exponential regimes is robust as our results apply to a general $c(E)$, and the rates are controlled by only two quantities: $\int_{-\infty}^\infty c(E) dE \propto \text{Tr} V^2$ and $c(E \rightarrow 0)$.

**Brownian Model.**—We use the Dyson Brownian technique [39, 40] which enables us to calculate the ensemble-averaged effects of the random coupling $V$ by building it up as a sum of uncorrelated random perturbations:

$$V \rightarrow V_b(\tau) = \int_0^\tau \xi(\tau') d\tau' \quad \text{with} \quad \tau = 1.$$  (5)

It can be pictured as a random walk in fictitious time $\tau$ in the space of random Hamiltonians. At $\tau = 0$ the system and bath are decoupled. Dyson observed that the $\tau = 1$ case corresponds, after ensemble averaging, to the model defined in [1] with $H = H_0 + V$. More precisely, at each fictitious time step $\delta \tau$, a small perturbation $\xi(\tau) \delta \tau$ is added to the Hamiltonian $H(\tau) = H_0 + V_b(\tau)$ (which has exact eigenstates $|A\rangle$). This perturbation can be written in the basis of these eigenstates as $\langle A|\xi(\tau)\delta \tau|B\rangle = \sqrt{\xi_{AB}} \xi_{AB}$. The banded coupling profile, defined after Eq. [4], is mimicked by $c_{AB} \equiv c(E) \Delta$, with $E = E_A - E_B$. (See the Discussion for the validity of this approach.) Restricting ourselves to time-reversal-invariant systems, we model the randomness by the independent Gaussian random variables $\xi_{AB} (= \xi_{BA})$ with the stochastic properties: $\xi_{AB} = 0$, and $\xi_{AB} \xi_{CD} = (\delta_{AC} \delta_{BD} + \delta_{AD} \delta_{BC}) \delta \tau$. It can be shown [11] from perturbation theory that we have Langevin processes for the eigenstates and eigenenergies:

$$\delta X_A^{sb} = \sum_{B \neq A} \left( \frac{\sqrt{c_{AB}} \xi_{AB}}{E_{AB}} X_b^{sb} - \frac{c_{AB} \delta \tau}{2E_{AB}^2} X_A^{sb} \right),$$  (6)

$$\delta E_A = \sqrt{c_{AA}} \xi_{AA} + \sum_{B \neq A} \frac{c_{AB} \delta \tau}{E_{AB}}.$$  (7)

where $E_{AB} \equiv E_A - E_B$, and the overlap $X_A^{sb}(\tau) = \langle b|A\rangle(\tau)$ is a component of the eigenstate in the decoupled basis. The initial ($\tau = 0$) condition is $X_A^{sb}(0) = \langle b|A\rangle(0)$. ($|A\rangle(0)$ is a product state of system and bath eigenstates.) The perturbations for the overlaps and the energy levels involve independent (off-diagonal and diagonal) elements of $\xi_{AB}$. So, we can replace the sum over energies in [9] with statistical averages over the well-known energy level distribution. Fluctuations should be small owing to the rigidity of the spectrum. The second moment of the overlap, $|X_A^{sb}|^2$, is the ‘local density
of states’ (LDOS) in energy space. Its Brownian motion has been studied \[11\] for an unbounded coupling matrix. We have extended the theory to obtain the fourth moments of the overlap that are needed for the RDM.

**Derivation.** — We will now describe our analytical calculation in more detail. We focus first on the diagonal terms of the overlap of the exact eigenstates and the decoupled product states, \( X_A^b = \langle sb|A \rangle \), which is of random sign over the ensemble of random couplings. It can be shown that

\[
\rho_{ss}(t) = \sum_{AB} \langle A|s_0 b_0 \rangle \langle s_0 b_0 |B \rangle \langle B|sb \rangle \langle sb|A \rangle e^{-iE_{AB}t} \tag{8}
\]

This involves the fourth moments of the overlaps. Let us start with the second moments \( J_{ss}^\beta(\tau) = \sum_{AB} \frac{C_{AB}}{E_{AB}} (J_{AB}^\beta - J_{AB}^{\alpha\beta}) \tag{9} \)

with \( J_{ss}^{\alpha\beta}(0) = \langle sb|A(0) \rangle \delta_{\alpha,ss} \). The sum is in the form of a convolution and so this differential equation can be simplified in the time domain in terms of \( J_s^{\alpha\beta}(t, \tau) = \int J_s^{\alpha\beta}(\tau) e^{-i E_{AB} t} dE_{AB}/\Delta \). It simply becomes \( \partial_t J_s^{\alpha\beta}(t, \tau) = \Lambda(0,t) J_s^{\alpha\beta}(t, \tau) \), and the solution is

\[
J_s^{\alpha\beta}(t, \tau) = \delta_{\alpha,\beta} e^{-i E_{AB} t} e^{-\tau \Lambda(0,0)} \tag{10}
\]

This is nonzero only if \( \alpha = \beta \) because, upon averaging over the random couplings \( V_{Br}(\tau) \), there should be no correlations between different components of \( |A\rangle \) in the decoupled basis. (A test is to consider the terms that survive under an average over random gauge transformations of the set of the basis states \( |sb\rangle \).)

We note that \( J_s^{\alpha\beta}(t, \tau = 1) \) is the Fourier transform of the local density of states \( |X_s^\beta|^{-2} \). From the behavior of \( \Lambda(0,t) \) at short and long times as discussed after [4], we see [22] that the LDOS is a function of \( \omega = E_A - E_{sb} \) which is a Lorentzian of width \( \pi \nu_e \) for \( \omega \ll W \), and is cut off at \( \omega \gg W \) by a Gaussian of width \( \sqrt{2eW} \). The LDOS can also be obtained in large-\( N \) diagrammatics for the random coupling where \( N \) corresponds to the number of bath states. For an unbounded matrix \( (W \to \infty) \), the leading result corresponds to a self-consistent Born approximation, giving the Lorentzian broadening to the LDOS [81]. However, the Gaussian tail for a banded matrix is more difficult to capture in such an approximation.

The result [10] demonstrates analytically ETH [4] which gives a sum rule for the LDOS, i.e. the projection of an eigenstate \( A \) onto a system state \( s \):

\[
\sum_b \langle A|sb \rangle^2 \propto \nu_b (E_A - \varepsilon_s). \tag{11}
\]

This is a sum of the Fourier transform of \( e^{-\tau \Lambda(0,0)} \) at frequencies \( \omega = E_A - \varepsilon_s - \varepsilon_b \) over all \( \varepsilon_b \). Recall that \( e^{-\tau \Lambda(0,0)} \) is mainly Gaussian decay with a rate of \( \sqrt{\pi} eW \) for \( ct \gg W \) and mainly exponential with rate \( \pi c t \) for \( ct \ll W \). Therefore, \( r_1(\omega) \) should be a function centered at \( \omega = 0 \) with width \( \sim \min[ct, \sqrt{\pi} eW] \). Assuming that the bath density of states \( \nu_b \) varies slowly over this width, we find agreement with ETH:

\[
\sum_b \langle A|sb \rangle^2 \simeq \nu_b (E_A - \varepsilon_s) \Delta \int r_1(E_A - \varepsilon_s - \varepsilon_b) d\varepsilon_b = \nu_b (E_A - \varepsilon_s) \Delta \tag{12}
\]
Thus, the right side of (13) becomes

\[
\frac{\Delta}{\pi} \int dt' \int dE \sum_{b} f(E, t) e^{-2\tau \Lambda(0, t') - i E t'} \cos \left[(E_{ss} - \epsilon_b) t'\right]
\]

\[
= \frac{\Delta}{2\pi} \int dt \int dE \int dE' \int dt' \sum_{\eta} \sum_{\pi} r_2(\epsilon - E_{ss} + \eta E) \cdot \left[ (E_{ss} - \epsilon_b) t' \right]
\]

where \( E_{ss} = E_\alpha - \varepsilon_s \) and \( r_2(\omega) \) is the Fourier transform of \( e^{-2\tau \Lambda(0, t)} \) which is peaked at zero with width \( \sim \min[\tau, \sqrt{\pi c}] \). For a smooth \( \nu_b \), \( \nu_b(\epsilon) \simeq \nu_b(\epsilon = E_{ss}) \) for the \( \epsilon \)-range over which \( r_2 \) contributes to the \( \epsilon \)-integration. Then, the right-hand side of (16) becomes

\[
\frac{\partial}{\partial t} + 2\Lambda(0, t) M^\tau(t, \tau) = 2\Lambda(0, t) \nu_b(E_{ss}) \Delta.
\]

From (8), \( \rho_{ss} = M^s,s = s^s b_0 \) for an initial state \( |s_0 b_0\rangle \) of energy \( E_0 = E_{ss} + \varepsilon_s \). The solution at \( \tau = 1 \) for (17) is indeed our result (2) with \( \rho_{ss}(0) = \delta_{ss} \). We can perform an analogous calculation for \( \rho_{ss'}(t) \). The dominant contributions come from terms that are positive definite in the sum over bath states.

\[
\rho_{ss'}(t) \simeq \rho_{ss'}(0) \sum_{AB} |\langle A | s b_0 \rangle|^2 |\langle B | s' b_0 \rangle|^2 e^{-iE_{AB} t} \rho_{ss'}(0) J^s b_0, s' b_0(t) J^{s'} b_0, s b_0(-t).
\]

With (10), this gives our result (3) for decoherence.

![FIG. 1. Comparison of Brownian motion result (solid) with exact diagonalisation (dotted) of 3 random realizations with 2 system states and 7000 bath states. Top: diagonal RDM elements, \( \rho_{ss} \), for initial state \( |s_0 a\rangle \) with state \( a \) near the center of the bath spectrum (high effective temperature). Bottom: off-diagonal RDM elements, \( \text{Re}(\rho_{ss'}) \), for initial state \( (|s_0 a\rangle + |s' a\rangle)/\sqrt{2} \). \( \varepsilon_s - \varepsilon_{s'} = W/4 \), spacing \( \Delta = W/4000 \). Weak coupling (left) shows predominantly exponential decay, while stronger coupling (right) shows the early Gaussian regime.](image)

**Discussion.**— Brownian motion produces a random matrix, \( V_{Br}(\tau = 1) \), that has identical statistical properties to \( V \) only for an unbounded random matrix. For a banded coupling, this is only approximate. This is because the coupling matrix is banded in the eigenstate basis of \( H(\tau) \) at each Brownian step, instead of being banded in the eigenbasis of \( H(\tau = 0) \). For finite \( W \), we can show [45] that \( \mathcal{V}_{Br}(\tau) \) has a broadened profile \( c'(E) \) for its matrix elements with increasing \( \tau \). As discussed after (1), the features of \( c'(E) \) relevant to the physics here are the integrated profile \( \int c'(E) dE \) and small-\( E \) limit of \( c'(E) \). The former gives \( W_{Br}^2 \), which has been fixed at \( W_{Br}^2 \) (implying that the Brownian model reproduces the short-time expansion correctly: \( e^{-iHt} \approx 1 - iHt \) giving \( \rho_{ss0} \approx 1 - 2(\epsilon W)^2 \)). So, the broadening of \( c'(E) \) compared to \( c(E) \) means that \( c'(0) < c(0) \). Thus, we overestimate the exponential decay rate, but this is only significant when \( c \gg W \) so that Gaussian decay dominates and the exponential tail is negligible. As we show in Fig. 1, our analytical results for \( H = H_0 + V_{Br}(\tau = 1) \) agree with the dynamics for \( H = H_0 + V \).

To summarize, we have used a random matrix model to describe the nonequilibrium dynamics of a system coupled to a fully quantum-mechanical bath. In contrast with studies employing an effective scattering approach [40] with a non-Hermitian random Hamiltonian, we study the full Hilbert space of a system with an interacting quantum bath. This provides an analytical demonstration of the eigenstate thermalization hypothesis. (Many previous works provided only numerical support.) We also find that thermalization and decoherence both follow the same dynamical behavior, with Gaussian decay at short times and exponential decay at long times. We should point out that these two regimes have been qualitatively anticipated in works based on semiclassical dynamics of energy wavepackets [47]. Also, a short-time Gaussian regime was found [48] for a global quench that switches on a random two-body interaction among all particles [49]. That Gaussian decay originates from the interactions generating a Gaussian density of states for the total energy spectrum. In contrast, our local quench does not alter drastically the spectrum of the total system, and so we argue that the Gaussian regime in our problem has a completely different physical origin. More recently, Gaussian decay has been found for a small system coupled to a classical bath in a slow local quench [50], with a decay time controlled by the correlation time in the bath. The quench rate can be mimicked in our formalism by the width \( W \). Our model has a short correlation time in the bath. Incorporating bath correlations is the goal of future work.

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[1] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. 83, 863 (2011).
[2] V. I. Yukalov, Laser Phys. Lett. 8, 485 (2011).
[3] M. A. Cazalilla and M. Rigol, New J. Phys. 12, 055006 (2010).
[4] J. M. Deutsch, Phys. Rev. A 43, 2046 (1991); M. Srednicki, Phys. Rev. E 50, 888 (1994); M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
[5] S. Popescu, A. J. Short, and A. Winter, Nature Physics 2, 754 (2006).
[6] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. 96, 050403 (2006). Eur. Phys. J. H 35, 173 (2010).
[7] S. Genway, A. F. Ho, and D. K. K. Lee, Phys. Rev. A 86, 023609 (2012).
[8] H. Tasaki, Phys. Rev. Lett. 80, 1373 (1998).
[9] P. Reimann, Phys. Rev. Lett. 101, 190403 (2008). New J. Phys. 12, 055027 (2010).
[10] J. Gemmer, A. Otte, and G. Mahler, Phys. Rev. Lett. 86, 1927 (2001) G. Mahler, J. Gemmer, and M. Michel, Physica E 29, 53 (2005). J. Gemmer and M. Michel, Europhys. Lett. 73, 1 (2006).
[11] F. G. S. L. Brandão and M. B. Plenio, Nature Physics 4, 873 (2008).
[12] M. Rigol and M. Srednicki, Phys. Rev. Lett. 108, 110601 (2012).
[13] I. Lesanovsky, B. Olmos, and J. P. Garrahan, Phys. Rev. Lett. 105, 100603 (2010).
[14] S. Yuan, M. I. Katsnelson, and H. De Raedt, J. Phys. Soc. Jpn. 78, 044003 (2009). F. Jin, H. De Raedt, S. Yuan, M. I. Katsnelson, S. Miyashita, and K. Michel, J. Phys. Soc. Jpn. 79, 124005 (2010).
[15] B. V. Fine, Phys. Rev. E 80, 051130 (2009) K. Ji and B. V. Fine, Phys. Rev. Lett. 107, 050401 (2011) F. Kolley, O. Bohigas, and B. V. Fine, arXiv:1209.2954 (2012).
[16] M. Kollar, F. A. Wolf, and M. Eckstein, Phys. Rev. B 84, 054304 (2011).
[17] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[18] P. Calabrese, F. H. L. Essler, and M. Fagotti, Phys. Rev. Lett. 106, 227203 (2011).
[19] R. Steinigeweg, J. Herbach, and P. Prelovšek, Phys. Rev. E 87, 012118 (2013).
[20] S. Genway, A. F. Ho, and D. K. K. Lee, Phys. Rev. Lett. 105, 260402 (2010).
[21] C. Ates, J. P. Garrahan, and I. Lesanovsky, Phys. Rev. Lett. 108, 110603 (2012). S. Ji, C. Ates, J. P. Garrahan, and I. Lesanovsky, J. Stat. Mech. 2013, P02005 (2013).
[22] P. A. V. Miranda and G. Mahler, J. Math. Phys. 51, 082107 (2010).
[23] C. Bartsch and J. Gemmer, Phys. Rev. Lett. 102, 110403 (2009).
[24] H. Niemeyer, D. Schmidtko, and J. Gemmer, Europhys. Lett. 101, 10010 (2013).
[25] J. Mossel and J.-S. Caux, New J. Phys. 12, 055028 (2010).
[26] F. H. L. Essler, S. Evangelisti, and M. Fagotti, Phys. Rev. Lett. 109, 247206 (2012).
[27] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003); Nature Physics 5, 181 (2009).
[28] F. M. Cucchietti, D. A. R. Dalvit, J. P. Paz, and W. H. Zurek, Phys. Rev. Lett. 91, 210403 (2003).
[29] P. A. Mello, P. Pereyra, and N. Kumar, J. Stat. Phys. 51, 77 (1988) P. Pereyra, J. Stat. Phys. 65, 773 (1991).
[30] M. Esposito and P. Gaspard, Phys. Rev. E 68, 066113 (2003).
[31] J. L. Lebowitz and L. Pastur, J. Phys. A: Math. Gen. 37, 1517 (2004).
[32] F. David, J. Stat. Mech. 2011, P01001 (2011).
[33] T. Fukuhara, A. Kantiyan, M. Endres, M. Cheneau, P. Schauß, S. Hild, D. Bellem, U. Schollwöck, T. Giamarchi, C. Gross, I. Bloch, and S. Kuhr, Nature Physics 9, 235 (2013).
[34] J. Dalibard, F. Gerbier, G. Juzeliunas, and P. Öhrberg, Rev. Mod. Phys. 83, 1523 (2011).
[35] M. L. Mehta, Random Matrices (Academic Press, 1967).
[36] To be precise, ∆ should be the average level spacing near total energy E\(_0\).
[37] We note that decoherence and thermalization with an interacting bath can have different time scales if there are different couplings for elastic and inelastic processes.
[38] We consider the limit of a large bath where ∆ → 0 and the total bath bandwidth diverges at fixed c and W. This does not cover the scenario (global quench) where c diverges with 1/∆ when the spectrum will be strongly modified.
[39] F. J. Dyson, J. Math. Phys. 3, 1191 (1962); 13, 90 (1972).
[40] J. T. Chalker, I. V. Lerner, and R. A. Smith, Phys. Rev. Lett. 77, 554 (1996).
[41] M. Wilkinson and P. Walker, J. Phys. A: Math. Gen. 28, 6143 (1995).
[42] This has been seen in our numerics [17].
[43] See Section I of Supplemental Material.
[44] See Section II of Supplemental Material.
[45] S. Genway, Thermalisation and Temporal Relaxation in Closed Quantum Systems, Ph.D. thesis, Imperial College London (2011).
[46] F.-M. Dittes, Physics Reports 339, 253 (2000).
[47] We consider the limit of a large bath where ∆ → 0 and the total bath bandwidth diverges at fixed c and W. This does not cover the scenario (global quench) where c diverges with 1/∆ when the spectrum will be strongly modified.
[48] V. V. Flambaum and F. M. Izrailev, Phys. Rev. A 73, 061604 (2006).
[49] D. Cohen, F. M. Izrailev, and T. Kottos, Phys. Rev. Lett. 84, 2052 (2000).
[50] This corresponds to the parameter c in this paper scaling with 1/∆.
[51] L. d’Alessio and A. Polkovnikov, private communication.
SUPPLEMENTAL MATERIAL

I. Diagonal elements of the reduced density matrix

We consider here the fourth moments of the overlaps $X_{AB}(\tau) \equiv \langle sb|A(\tau)\rangle$ needed for the evaluation of the reduced density matrix (RDM):

$$
M_{AB}^{\alpha\beta}(\tau) = X_{A}^{\alpha}X_{B}^{\beta}, \quad N_{AB}^{\alpha\beta}(\tau) = \frac{1}{2} \left[ X_{A}^{\alpha}X_{A}^{\beta}X_{B}^{\beta} + (A \leftrightarrow B) \right]
$$

where $|\alpha\rangle = |ra\rangle$ and $|\beta\rangle = |sb\rangle$ correspond to system-bath product states.

For a system prepared in the initial state $\alpha$, the quantities needed for the diagonal elements of the RDM, $\rho_{ss}(\tau)$, are $M_{AA}^{ss}(\tau) = \sum_b M_{AA}^{ss,b}(\tau)$ and $N_{AA}^{ss}(\tau) = \sum_b N_{AA}^{ss,b}(\tau)$.

Using the Langevin equations for the overlaps and then averaging over the ensemble, we obtain two coupled equations of motion for the Brownian motion:

$$
\frac{\partial M_{AB}^{\alpha\beta}}{\partial \tau} = \sum_{D \neq A} \left[ \frac{c_{AD}}{E_{AD}^2} (M_{DB}^{\alpha\beta} - M_{AB}^{\alpha\beta}) + (A \leftrightarrow B) \right] - 2(1 - \delta_{AB}) \frac{c_{AB}}{E_{AB}^2} (M_{AB}^{\alpha\beta} + N_{AB}^{\alpha\beta}) + 2\delta_{AB} \sum_{D \neq A} \frac{c_{AD}}{E_{AD}^2} (M_{AD}^{\alpha\beta} + N_{AD}^{\alpha\beta})
$$

$$
\frac{\partial N_{AB}^{\alpha\beta}}{\partial \tau} = \sum_{D \neq A} \left[ \frac{c_{AD}}{E_{AD}^2} (N_{DB}^{\alpha\beta} - N_{AB}^{\alpha\beta}) + (A \leftrightarrow B) \right] - 4(1 - \delta_{AB}) \frac{c_{AB}}{E_{AB}^2} M_{AB}^{\alpha\beta} + 4\delta_{AB} \sum_{D \neq A} \frac{c_{AD}}{E_{AD}^2} N_{AD}^{\alpha\beta},
$$

where $M_{AB}^{ss}(0) = \delta_{ss}\delta_{AB}\delta_{A(0)}$ and $N_{AB}^{ss}(0) = (\delta_{A(0)}\delta_{B(0)} + \delta_{A,B(0)}\delta_{B,A(0)})/2$ where $A(0)$ and $B(0)$ are the decoupled states at $\tau = 0$.

We are interested in the limit of $\Delta \to 0$ and so we will take the continuum limit $\sum_{D} \to \int dE_D/\Delta$. Since $c_{AB}/E_{AB}^2$ is a function of only the energy difference $E_{AD} = E_{A} - E_{B}$, we note that the terms involving sums over the exact eigenstates $D$ are in the form of convolutions. Thus, the equation can be simplified in the time domain. Let us define $\mathcal{M}^{\alpha\beta}(t, \tau) = \int \mathcal{M}_{AB}^{\alpha\beta}e^{-iE_{AB}t}dE_AdE_B/\Delta^2$ and similarly for $\mathcal{N}^{\alpha\beta}$. The equations of motion become:

$$
\left[ \frac{\partial}{\partial \tau} + 2\Lambda(0, t) \right] \mathcal{M}^{\alpha\beta}(t, \tau) = 2\Delta \int \frac{dt'}{2\pi} \Lambda(t', t)[\mathcal{M}^{\alpha\beta}(t', \tau) + \mathcal{N}^{\alpha\beta}(t', \tau)],
$$

$$
\left[ \frac{\partial}{\partial \tau} + 2\Lambda(0, t) \right] \mathcal{N}^{\alpha\beta}(t, \tau) = 4\Delta \int \frac{dt'}{2\pi} \Lambda(t', t)\mathcal{M}^{\alpha\beta}(t', \tau),
$$

where $\Lambda(t', t)$ is defined by equation (4) in the main text. The initial conditions at $\tau = 0$ become $M^{\alpha\beta}(t, 0) = \delta_{\alpha\beta}$ and $N^{\alpha\beta}(t, 0) = \cos[(E_A - E_B)t]$.

These coupled equations of motion are linear equations in the moments. So, setting $\beta = (sb)$ and summing over $b$, we see that $\mathcal{M}^{ss}(t, \tau)$ and $\mathcal{N}^{ss}(t, \tau)$ [defined in the main text above equation (13)] obey the same coupled set of equations as $\mathcal{M}^{ss}(t, \tau)$ and $\mathcal{N}^{ss}(t, \tau)$. Thus, we find the equations of motion in fictitious time $\tau$ for $M$ and $N$ as given in equations (13) and (14) of the main text. The initial conditions are $\mathcal{M}^{ss}(t, 0) = \delta_{ss}$ and $\mathcal{N}^{ss}(t, 0) = \sum_b \cos[(E_A - E_{sb})t]$.

II. Off-diagonal elements of the reduced density matrix

Now, let us turn to the dynamics of decoherence. To observe decoherence, we prepare the subsystem in an entangled state and then study the off-diagonal elements of the RDM. Suppose we start with a single bath state $a$, then $|\Psi(t = 0)\rangle = \sum_s c_s|sa\rangle$. Then, the off-diagonal elements of the RDM are given by $s \neq s'$

$$
\rho_{ssl}(t) = \sum_{ABb} \sum_{rr'} \langle A|ra\rangle \rho_{rr'}(0) \langle r'a|B\rangle \langle B|sb\rangle \langle sb|A\rangle e^{-iE_{AB}t}, \quad \text{with} \quad \rho_{rr'}(0) = c_r^*c_{r'},
$$

The terms which survive averaging over the disorder should be the terms which are invariant under random gauge transformation on the basis states.

Nonzero contributions come from the terms with $s = r$, $s' = r'$ and $b = a$. These terms give

$$
\rho_{ss}(t) = \rho_{ss}(0) \sum_{AB} X_{A}^{sa}X_{A}^{sb}X_{B}^{sa}X_{B}^{sb} e^{-iE_{AB}t} = \rho_{ss}(0) N_{sa,s'b}(t)
$$

(24)
Following an analogous analysis of the coupled equations (13) and (14) of the main text, we expect that \( \mathcal{N} \) is decoupled from \( \mathcal{M} \) in the thermodynamic limit \( (\Delta \to 0) \) so that (22) simplifies to \([\partial_\tau + 2\Lambda(0,t)]N^{s\alpha,s'\alpha}(t,\tau) = 0\). The solution is simply:

\[
\rho_{1ss'}(t,\tau) = e^{-2\tau\Lambda(0,t)}e^{i(\varepsilon_s - \varepsilon_{s'})t}\rho_{ss'}(0). \tag{25}
\]

where \( \varepsilon_s \) is the eigenenergy for subsystem state \( s \). This is the result given in equation (18) of the main text.

Another possible contribution to \( \rho_{ss'} \) comes from setting \( s = r' \), \( s' = r \) and \( a = b \) in (23). This gives

\[
\rho_{2ss'}(t) = \rho_{s's}(0) \sum_{AB} \overline{X}_{s'a}^A X_{s'a}^A \overline{X}_{s'a}^B X_{s'a}^B e^{-iE_{AB}t} = \rho_{s's}(0)M^{s\alpha,s'\alpha}(t). \tag{26}
\]

\( M^{s\alpha,s'\alpha}(t) \) obeys (22) and is coupled to \( N^{s\alpha,s'\alpha}(t) \). For \( s \neq s' \), the initial conditions in fictitious time are \( M^{s\alpha,s'\alpha}(t,\tau = 0) = 0 \) and \( N^{s\alpha,s'\alpha}(t,\tau = 0) = \cos[(\varepsilon_s - \varepsilon_{s'})t] \). Since \( \mathcal{N} \) is not of order \( 1/\Delta \) (unlike in our calculation for the diagonal RDM elements), this does not contribute to the differential equation for \( \mathcal{M} \) in the thermodynamic limit. This means that (21) simplifies to \([\partial_\tau + 2\Lambda(0,t)]M^{s\alpha,s'\alpha}(t,\tau) = 0\). With the initial condition that it is zero at \( \tau = 0 \), this means that \( \rho_{2ss'} = 0 \) in the thermodynamic limit and does not contribute to the off-diagonal elements of the RDM.

Note that the Langevin equations can be used to derive a Fokker-Planck equation for the joint distribution \( P(\{X\},\tau) \) for the overlaps:

\[
\frac{\partial P}{\partial \tau} = \sum_{\alpha\beta AB} \frac{c_{AB}}{2E_{AB}} \left[ \delta_{\alpha\beta} \frac{\partial(X_{A}^\alpha P)}{\partial X_{A}^\alpha} + \frac{\partial^2(X_{B}^\beta X_{A}^\beta P)}{\partial X_{A}^\alpha \partial X_{A}^\alpha} - \frac{\partial^2(X_{B}^\beta X_{A}^\beta P)}{\partial X_{A}^\beta \partial X_{A}^\beta} \right]. \tag{27}
\]

All the results for the moments of the overlaps can be derived from this Fokker-Planck equation.