On the Weyl’s law for discretized elliptic operators

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Abstract

In this paper we give an estimate on the asymptotic behavior of eigenvalues of discretized elliptic boundary values problems. We first prove a simple min-max principle for selfadjoint operators on a Hilbert space. Then we show two sided bounds on the \( k \)-th eigenvalue of the discrete Laplacian by the \( k \)-th eigenvalue of the continuous Laplacian operator under the assumption that the finite element mesh is quasi-uniform. Combining this result with the well-known Weyl’s law, we show that the \( k \)-th eigenvalue of the discretized isotropic elliptic operators, spectrally equivalent to the discretized Laplacian, is \( O(k^{2/d}) \). Finally, we show how these results can be used to obtain an error estimate for finite element approximations of elliptic eigenvalue problems.

1 Introduction

In this work, we focus on the finite element method as discretization of solving linear partial differential equations of the form

\[ Lu = f, \] (1)

with given boundary conditions. As is customary in finite element method, the solution to (1) is approximated by a piece-wise polynomial function (see [1]). The behavior of the eigenvalues of the linear operator \( L \) on the corresponding finite element space when the mesh size becomes small are instrumental in showing optimality of the methods, estimating the error of approximation, construction of efficient solvers for the resulting linear systems, and in all other components of the finite element analysis. Classical references for the finite element approximation of compact eigenvalue problems are the works by Strang and Fix [2] and later the so-called Babuška-Osborn theory [3, 4, 5]. The main result of this theory can be summarized as:

\[ \lambda_k \leq \lambda_{h,k} \leq \lambda_k + C_k \sup_{\|u\|_{E_k} = 1} \inf_{v \in V_h} \|u - v\|_V^2, \] (2)

where \( C_k \) are a constants depending on \( k \) and \( E_k \) denotes the eigenspace associated with \( \lambda_k \), the \( k \)-th eigenvalue of the continuous Laplacian. The proof of (2) found in [3] relies on an induction argument, and does not seem to provide bounds on \( C_k \) which are independent of \( k \). Some refinements on the estimates of the constants \( C_k \) in (2) are given in [4, 5], and, more recently in [6, 7], and also follow from the main result in the present paper. We also mention that bounding the
eigenvalues of the discrete operators plays a crucial role in the convergence analysis of multigrid methods \[8, 9, 10\].

In [11], Hermann Weyl proved the following asymptotic formula for the eigenvalues of the Dirichlet Laplacian in a bounded domain \(\Omega \subset \mathbb{R}^d\):

\[
\lim_{k \to \infty} \frac{\lambda_k}{k^{\frac{2}{d}}} = \frac{(2\pi)^2}{[\omega_d \text{Vol}(\Omega)]^\frac{2}{d}},
\]

where \(\omega_d\) is a volume of the unit ball in \(\mathbb{R}^d\), and \(\text{Vol}(\Omega)\) is the volume of \(\Omega\). This formula was actually conjectured independently by Arnold Sommerfeld [12] and Hendrik Lorentz [13] in 1910 who stated Weyl’s law as a conjecture based on the book of Lord Rayleigh [14]. Recently, V. Ivrii [15] provided comprehensive summary on the research on Weyl’s law since its discovery.

In this paper the main result is a sharp estimate on the asymptotic behavior of eigenvalues for Laplacian operator on the finite element spaces as follows

\[
\lambda_k \leq \lambda_{h,k} \leq C \lambda_k,
\]

with \(C\) independent of \(k, h\). We show such an estimate under the assumption that the finite element mesh is quasi-uniform and in combination with Weyl’s Law for the PDE this result shows that the \(k\)-th smallest eigenvalue of the discretized Laplacian operator is \(O(k^{2/d})\). This leads to the conclusion that the eigenvalues of the finite element discrete operator exhibit the same asymptotic behavior as the eigenvalues of the continuous Laplacian operator.

This paper is organized as follows. In Section 2 we introduce the basic model problem and some related results on the stability and approximation property of \(L^2\)-projection onto finite element space. In Section 3 we show a generalized min-max principle for selfadjoint operators on separable Hilbert space. This is an important tool, used in the proof of our main result. In Section 4 we recall Weyl’s law for the Laplacian operator on Sobolev spaces and prove the main eigenvalue asymptotic estimate for the discretized Laplacian operator. In Section 5 we prove an error estimate for finite element approximation of 2nd-order elliptic eigenvalue problems.

## 2 Model elliptic PDE operators and finite element discretization

We consider the following boundary value problems

\[
\mathcal{L}u = -\nabla \cdot (\alpha(x)\nabla u) + q(x)u = f, \quad x \in \Omega
\]

where \(\alpha : \Omega \mapsto \mathbb{R}^{d \times d}_{\text{sym}}\) is a matrix valued function taking values in the set of \(d \times d\), symmetric, positive definite matrices.

\[
\alpha_0 ||\xi||^2 \leq \xi^T \alpha(x)\xi \leq \alpha_1 ||\xi||^2, \quad \text{for all } \xi \in \mathbb{R}^d,
\]

for some positive constants \(\alpha_0\) and \(\alpha_1\) and all \(x \in \Omega\). The potential \(q(x)\) is assumed to be bounded and non-negative for almost all \(x \in \Omega\). Here, \(d = 1, 2, 3\) and \(\Omega \subset \mathbb{R}^d\) is a bounded domain with sufficiently smooth (Lipschitz) boundary \(\Gamma = \partial \Omega\).
The variational formulation of (4) is: Find \( u \in V \) such that

\[
a(u, v) = (f, v), \quad \forall v \in V.
\]

(6)

where the bilinear form \( a(\cdot, \cdot) \) and the linear form \( (f, \cdot) \) are defined as

\[
a(u, v) = \int_{\Omega} (\alpha(x) \nabla u) \cdot \nabla v + \int_{\Omega} q u v,
\]

\[
(f, v) = \int_{\Omega} f v.
\]

The Sobolev space \( V \) that can be chosen according to the boundary conditions accompanying the equation (4). For example, in the case of mixed boundary conditions:

\[
u = 0, \quad x \in \Gamma_D,
\]

\[
(\alpha \nabla u) \cdot n = 0, \quad x \in \Gamma_N,
\]

(7)

where \( \Gamma = \Gamma_D \cup \Gamma_N \). The pure Dirichlet problem is when \( \Gamma_D = \Gamma \) while the pure Neumann problem is when \( \Gamma_N = \Gamma \). We then have \( V \) defined as

\[
V = \begin{cases} 
H^1(\Omega) = \{ v \in L^2(\Omega) : \partial_i v \in L^2(\Omega), i = 1 : d \}; \\
H^1_D(\Omega) = \{ v \in H^1(\Omega) : v |_{\Gamma_D} = 0 \}.
\end{cases}
\]

(8)

When we consider a pure Dirichlet problem, \( \Gamma_D = \Gamma \), we denote the space by \( V = H^1_D(\Omega) \). In addition, for pure Neumann boundary conditions, when \( q = 0 \) the condition \( \int_{\Omega} f = 0 \) is usually added to assure uniqueness of the solution to (6). We note that \( a(\cdot, \cdot) \) defines and inner product on \( V \) and

\[
| \cdot |_a^2 := a(u, u),
\]

is a norm (or a seminorm) on \( V \).

One most commonly used model problem is when \( \alpha(x) = I \), and \( q(x) = 0 \) for all \( x \in \Omega \), which corresponds to the Poisson equation

\[
-\Delta u = f.
\]

(9)

This simple problem provides a good representative model for isotropic problems. In the following discussion, we assume \( \alpha(x) = I \) and \( q(x) = 0 \). We further remark that the results carry over to other isotropic elliptic equations by spectral equivalence.

Given a triangulation \( \mathcal{T}_h \) of \( \Omega \), let \( V_h \subset V \) be finite element space consisting of piecewise linear (or higher order) polynomials with respect to this triangulation \( \mathcal{T}_h \). By triangulation here we mean \( d \)-homogenous simplicial complex in \( \mathbb{R}^d \) which covers \( \Omega \). The finite element approximation of the variational problem (6) then is: Find \( u_h \in V_h \) such that

\[
a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h.
\]

(10)

Given a basis \( \{ \phi_i \}_{i=1}^N \) in \( V_h \), we write \( u_h(x) = \sum_{j=1}^N \mu_j \phi_j(x) \) the equation (10) is then equivalent to

\[
\sum_{j=1}^N \mu_j a(\phi_j, \phi_i) = (f, \phi_i), \quad j = 1, 2, \cdots, N,
\]
which is a linear system of equations:

\[ A \mu = b, \quad (A)_{ij} = a(\phi_j, \phi_i), \quad \text{and} \quad (b)_i = (f, \phi_i). \]  

(11)

Here, the matrix \( A \) is known as the stiffness matrix of the nodal basis \( \{\phi_i\}_{i=1}^n \).

With any simplex \( T \in \mathcal{T}_h \), we associate the following geometric characteristics:

\[
\begin{align*}
\bar{h}_T &= \text{diam}(T), \quad h_T = |T|^\frac{1}{2}, \quad h = \max_{T \in \mathcal{T}_h} \bar{h}_T, \\
\bar{h}_T &= 2 \sup \{ r > 0 : B(x, r) \subset T \text{ for } x \in T \}.
\end{align*}
\]

(12)

In the following discussion, we need the definition of quasi-uniform finite element mesh and we recall this next.

**Definition 2.1 (Quasi-uniform mesh)** We say that the mesh \( \mathcal{T}_h \) is quasi-uniform if there exists a constant \( C > 0 \) such that

\[
\max_{T \in \mathcal{T}_h} \frac{h_T}{T} \leq C.
\]

(13)

We assume that we have a projection \( \Pi_h : V \mapsto V_h \), satisfying

\[
|\Pi_h v|_{0,\Omega}^2 \leq c_1 |v|_{0,\Omega}^2, \quad \forall v \in V,
\]

(14)

and

\[
||v - \Pi_h v||_0^2 \leq c_2 h^2 |v|_{0,\Omega}^2, \quad \forall v \in V,
\]

(15)

with \( c_1 \) and \( c_2 \) being constants independent of \( h \) and \( v \).

One example of \( \Pi_h \) is the \( L^2 \) projection \( Q_h : V \mapsto V_h \) defined by

\[
(Q_h v, w)_{L^2(\Omega)} = (v, w)_{L^2(\Omega)}, \quad \forall v \in V, w \in V_h.
\]

(16)

and a proof that \( Q_h \) satisfies (14) and (15) can be found in [16], [17], and [18].

Another example of \( \Pi_h \) is the elliptic projection \( P_h : V \mapsto V_h \) defined as

\[
a(P_h v, w) = a(v, w), \quad \forall v \in V, w \in V_h.
\]

It is well-known that the elliptic projection \( P_h \) satisfies (14) with \( c_1 = 1 \), and, moreover, it also satisfies (15) if the weak solution to the differential equation (4) is \( H^2 \)-regular, namely, there exists a constant \( C_r \) such that

\[
|u|_2^2 \leq C_r ||g||_0^2,
\]

(17)

where \( u \) is the solution of

\[
a(w, u) = (f, w), \quad \forall w \in V.
\]

Next section discusses the min-max principle in (finite dimensional) Hilbert space
3 On the min-max principle

The well known min-max principle for eigenvalues of symmetric matrices was probably first stated and proved in [19]. We state it as the following theorem.

**Theorem 3.1 (Min-max principle A)** Let $A$ be a $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then

$$\lambda_k = \min_{\dim W = k} \max_{x \in W, x \neq 0} \frac{(Ax, x)}{(x, x)},$$  \hspace{1cm} (18)

and

$$\lambda_k = \max_{\dim W = n-k+1} \min_{x \in W, x \neq 0} \frac{(Ax, x)}{(x, x)}.$$  \hspace{1cm} (19)

We now consider the case when $A : X \mapsto V$ is a selfadjoint operator in a Hilbert space $V$. We assume that the domain of $A$, is $X \subset V$ and $X$ is dense in $V$.

The following lemma is used in the proof of the min-max principle. The result seems obvious (and is obvious in finite dimensional space).

**Lemma 3.2** Let $V$ be a separable Hilbert space with orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ and let for a fixed integer $k \geq 1$,

$$V_{k+} = \text{span}\{\varphi_j\}_{j=k}^\infty = \text{span}\{\varphi_k, \varphi_{k+1}, \ldots\}. \hspace{1cm} (20)$$

If $W \subset V$ is any subspace of dimension $k$, then

$$W \cap V_{k+} \neq \{0\}. \hspace{1cm} (21)$$

**Proof.** Let $\{\psi_j\}_{j=1}^k$ be a basis in $W$ and let $Q : V \mapsto V_{k-}$ be the orthogonal projection on $V_{k-}$ where

$$V_{k-} = \text{span}\{\varphi_j\}_{j=1}^k = \text{span}\{\varphi_1, \ldots, \varphi_k\}.$$ 

Notice that $\dim V_{k-} = \dim W = k$ and that

$$Qv = \sum_{j=1}^k (\varphi_j, v)\varphi_j.$$ 

We consider two cases:

**Case 1:** There exists a $\psi \in W, \psi \neq 0$, such that $Q\psi = 0$;

**Case 2:** For all $\psi \in W, \psi \neq 0$ we have $Q\psi \neq 0$.

In the first case, as $Q\psi = 0$ we obtain that

$$W \ni \psi = (I - Q)\psi \in V_{k+}, \quad \text{and hence}$$

$$W \cap V_{k+} \ni \{\psi\} \neq \{0\},$$

which shows the result of the lemma in the case when the null-space of $Q$ is non-trivial.
Consider now the second case, namely, \( Q \psi \neq 0 \) for all \( \psi \in W \). This implies that the matrix \( C_{ij} = (\varphi_j, \psi_i), i, j = 1, \ldots, k \) is nonsingular. Indeed, if this matrix is singular, so is its transpose. Let \( x \in \mathbb{R}^k \) be such that \( C^T x = 0 \), i.e.

\[
C^T x = 0 = \sum_{j=1}^{k} (\varphi_i, \psi_j) x_j = 0, \quad i = 1, \ldots, k,
\]

Thus, for \( \psi = \sum_{j=1}^{k} x_j \psi_j \) we have \((\varphi_i, \psi) = 0\) for \( i = 1, \ldots, k \) and hence \( Q \psi = 0 \) where \( \psi \in W \) and \( \psi \neq 0 \).

Further, as \( C \) is nonsingular, it follows that there exists \( y \in \mathbb{R}^k \) such that \( Cy = e_k, e_k = (0, \ldots, 0, 1)^T \). Therefore, with

\[
\psi = \sum_{j=1}^{k} y_j \psi_j,
\]

we have

\[
(\varphi_k, \psi) = 1, \quad \text{and} \quad (\varphi_i, \psi) = 0, \quad i = 1, \ldots, (k-1).
\]

Finally, these identities imply that

\[
\psi = \sum_{i=1}^{\infty} (\varphi_i, \psi) \varphi_i = \varphi_k + \sum_{i=1}^{k-1} (\varphi_i, \psi) \varphi_i + \sum_{i=k+1}^{\infty} (\varphi_i, \psi) \varphi_i = \sum_{i=k}^{\infty} (\varphi_i, \psi) \varphi_i.
\]

The right side of the identity above shows that \( \psi \in V_{k^+} \) and by \( (22) \) we have \( \psi \in W \) which completes the proof. \( \square \)

We define the following set of \( k \)-dimensional subspaces:

\[
\mathcal{W}_k = \{ W \subset X \mid \dim W = k \}.
\]

(23)

The theorem which we prove next, shows also a min-max principle \([20, 21]\). Its proof is elementary, and relies on Lemma\[3.2\].

**Theorem 3.3 (Min-max principle B) Let** \( V \) **be a Hilbert space and** \( X \subset V \) **is a dense subset of it. Let us assume that the eigenvectors \( \{ \varphi_j \}_{j=1}^{\infty} \) of a selfadjoint operator \( A : X \mapsto V \) form a complete orthonormal basis for \( V \). Then, we have the following min-max identity:**

\[
\lambda_k = \min_{W \in \mathcal{W}_k} \sup_{v \in W} \frac{(Av, v)}{(v, v)}.
\]

(24)

**Proof.** Let \( \{[\lambda_j, \varphi_j]\}_{j=1}^{\infty} \) be the eigenpairs of \( A \) with \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \) and let \( W_{k^-} = \text{span}\{\varphi_j\}_{j=1}^{k} \). We have that \( W_{k^-} \subset X \), because \( A \varphi_j = \lambda_j \varphi_j, j = 1, \ldots, k \). Next, for any \( v \in W_{k^-} \) we obtain

\[
\frac{(Av, v)}{(v, v)} \leq \frac{\lambda_k(v, v)}{(v, v)} = \lambda_k \Rightarrow \sup_{v \in W_{k^-}} \frac{(Av, v)}{(v, v)} \leq \lambda_k.
\]
Next, let $W \in W_k$ be a space of dimension $k$ and we denote $V_{k+} = \text{span}\{\varphi_j\}_{j=k}^\infty$. By Lemma 3.2 we have that there exists $\psi_W \neq 0$ such that $\psi_W \in V_{k+} \cap W$. Hence,

$$
\frac{(A\psi_W, \psi_W)}{(\psi_W, \psi_W)} = \frac{\sum_{j=k}^{\infty} \lambda_j (\varphi_j, \psi_W)^2}{\sum_{j=k}^{\infty} (\varphi_j, \psi_W)^2} \geq \frac{\lambda_k \sum_{j=k}^{\infty} (\varphi_j, \psi_W)^2}{\sum_{j=k}^{\infty} (\varphi_j, \psi_W)^2} = \lambda_k.
$$

Taking the infimum over all spaces in $W_k$ and we have:

$$
\lambda_k \leq \min_{W_k, \psi_W} \frac{(A\psi_W, \psi_W)}{(\psi_W, \psi_W)} \leq \min_{W_k, \psi_W} \sup_{v \in W_k} \frac{(Av, v)}{(v, v)} \leq \sup_{v \in W_k} \frac{(Av, v)}{(v, v)} \leq \lambda_k.
$$

which completes the proof. \qed

4 Spectral properties of discretized elliptic operators

We now discuss the spectral properties of the operator $L$ given in (4). The following theorem, regarding the eigenfunctions and eigenvalues of $L$ is well-known consequence of the Hilbert-Schmidt theorem for compact operators. Its proof is found, for example, in [22, 23].

**Theorem 4.1** The operator $L$ given in (4) has a complete set of eigenfunctions $(\varphi_k)$ and nonnegative eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$$

such that

$$L\varphi_k = \lambda_k \varphi_k, \quad k = 1, 2, 3 \ldots$$

1. $\lim_{k \to \infty} \lambda_k = \infty$.

2. $(\varphi_i)$ forms an orthonormal basis of $V$ as well as for $L^2(\Omega)$.

We also note that when $q = 0$ and $a(x) = I$ we have

1. For pure Neumann problem, $\lambda_1 = 0$ and $\varphi_1$ is the constant function.

2. For pure Dirichlet problem, $\lambda_1 > 0$ is simple and $\varphi_1$ does not change sign.

For the case of Laplacian operator $L = (-\Delta)$, we have the well-known Weyl’s estimate on the asymptotic behavior of its eigenvalues, as shown in [11, 24, 25].

**Lemma 4.2 (Weyl’s law)** Assume that $\Omega$ is contented. Then for the homogeneous Dirichlet problem, the eigenvalues of the Laplacian operator satisfy:

$$
\lim_{k \to \infty} \frac{\lambda_k}{k^\frac{d}{2}} = w_\Omega, \quad \text{with} \quad w_\Omega = \frac{(2\pi)^2}{\omega_d \text{Vol}(\Omega)},
$$

where $\omega_d$ is a volume of the unit ball in $\mathbb{R}^d$, and the eigenvalues of the operator $L$ given in (4) satisfy:

$$
(\alpha_0 w_\Omega) k^\frac{d}{2} \leq \lambda_k \leq (\alpha_1 w_\Omega) k^\frac{d}{2}, \quad \forall k \geq 1.
$$
We recall that, by definition, \( \Omega \subset \mathbb{R}^d \) is a **contented domain** if it can be approximated as close as we please by unions of \( d \)-dimensional cubes (see [25, p. 271] for the precise statement of such definition). Since finite element method is often used to discretize problems on Lipschitz polyhedral domains, we note that in [26] it was shown that all Lipschitz polyhedrons are contented domains.

**Remark 4.3** We remark that by the Min-max principle (Theorem 3.1) the asymptotic behavior of \( \{ \lambda_k \} \) is of the same order with respect to \( k \) when \( q(x) \geq 0 \) is bounded coefficient and a matrix valued (symmetric and positive definite) \( \alpha(x) \).

In the following theorem, we extend Weyl’s law stated in Lemma 4.2 to the discretized Laplacian operator defined in (10).

**Theorem 4.4** Let \( V_h \subset H^1_0(\Omega) \) be a family of finite element spaces on a quasi-uniform mesh with \( \dim V_h = N \). Consider the discretized operator of (4)

\[
L_h : V_h \mapsto V_h, \quad (L_h u, v) = a(u, v), \quad \forall u, v \in V_h,
\]

and its eigenvalues:

\[
\lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,N}.
\]

Then, for all \( 1 \leq k \leq N \), there exists a constant \( C_w > 0 \) independent of \( k \) such that we have the following estimates:

\[
\lambda_k \leq \lambda_{h,k} \leq C_w \lambda_k. \tag{27}
\]

and

\[
\gamma_0 k^{2/d} \leq \lambda_{h,k} \leq \gamma_1 k^{2/d}. \tag{28}
\]

**Proof.** For clarity, we present the proof for the Laplacian operator as the proof for the more general case is identical. Using the infinite dimensional version of the min-max principle, Theorem 3.1 for a symmetric bilinear form \( a(\cdot, \cdot) : X \times X \mapsto \mathbb{R} \) with a dense domain \( X \subset V \) and we have

\[
\lambda_k = \inf_{W \subset X} \sup_{\dim W = k, w \in W, w \neq 0} \frac{a(w, w)}{\|w\|_0^2}.
\]

As \( V_h \subset X \subset V \) we have the inequality,

\[
\lambda_{h,k} = \inf_{W \subset V_h} \sup_{\dim W = k, w \in W, w \neq 0} \frac{a(w, w)}{\|w\|_0^2} \geq \inf_{W \subset X} \sup_{\dim W = k, w \in W, w \neq 0} \frac{a(w, w)}{\|w\|_0^2} = \lambda_k,
\]

because the infimum on the left is taken over a smaller collection of spaces. This proves the lower bound in (27).

To show the upper bound, we first consider \( k = N \). Since the finite element mesh is quasi-uniform, by the inverse inequality, we have

\[
a(v, v) \leq h^{-2} \|v\|_0^2, \quad \forall v \in V_h.
\]

As \( 1 \leq k \leq N \), the mesh is quasi-uniform, and we have

\[
\lambda_k \leq \lambda_{h,k} \leq C_w \lambda_k.
\]

\[
\gamma \leq \gamma_0 k^{2/d} \leq \gamma_1 k^{2/d}.
\]
Therefore
\[ \lambda_{h,N} = \max_{v \in \mathcal{V}_h} \frac{a(v, v)}{\|v\|_0^2} \lesssim h^{-2} \]

Clearly, for all \( k \) such that \( \lambda_k \geq \frac{1}{2c_2h^2} \), where \( c_2 \) is the constant from (15), we have
\[ \lambda_{h,k} \leq \lambda_{h,N} \lesssim h^{-2} \leq 2c_2\lambda_k. \]

Next, we consider the case for \( k \) such that \( \lambda_k < \frac{1}{2c_2h^2} \).

Let \( W_k \subset \mathcal{V} \) be the space spanned by the first \( k \) eigenvectors of \(-\Delta\), namely,
\[ W_k = \text{span}\{\phi_j\}_{j=1}^k. \]

Since \( \frac{|w|^2}{\|w\|_0^2} \leq \lambda_k \) for all \( w \in W_k \), from (15) we have
\[ \| (I - \Pi_h)w \|_0^2 \leq c_2h^2\|w\|_0^2 \leq c_2h^2\lambda_k\|w\|_0^2 \leq \frac{1}{2}\|w\|_0^2. \] (31)

This implies
\[ \|\Pi_h w\|_0 \geq \|w\|_0 - \| (I - \Pi_h)w \|_0 \geq \left(1 - \frac{\sqrt{2}}{2}\right)\|w\|_0, \quad \forall w \in W_k. \] (32)

The above inequality implies that if \( w \in W_k \) is such that \( \Pi_h w = 0 \), then \( w = 0 \). This implies that \( \{\Pi_h \phi_j\}_{j=1}^k \) are linearly independent. We further denote
\[ W_{h,k} := \Pi_h W_k = \text{span}\{\Pi_h \phi_j\}_{j=1}^k \subset \mathcal{V}_h. \]

We now use (14) and (32) and we have
\[ \lambda_{h,k} \leq \sup_{v = \Pi_h w \in W_{h,k}, w \neq 0} \frac{|v|^2_0}{\|v\|_0^2} = \sup_{w \in W_k, w \neq 0} \frac{\|\Pi_h w\|_0^2}{\|\Pi_h w\|_0^2} \leq \sup_{w \in W_k, w \neq 0} \frac{c_1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2 \|w\|_0^2} \lambda_k = \frac{c_1}{\left(1 - \frac{\sqrt{2}}{2}\right)^2} \lambda_k. \]

This completes the proof. \( \square \)

5 An error estimate

In this section, we provide an error estimate for the finite element approximations of eigenvalue problems. Before we state the main result of this section let us comment on similar estimates found in the literature. The classical work [2] provides the estimate
\[ 0 < \frac{\lambda_{k,h} - \lambda_k}{\lambda_k} \leq h^D \lambda_k^D, \] (33)
where $D$ is the polynomial degree of the finite element space. More recently, Zhang \[7\] combines (33) with Babuška-Osborn theory and shows that

$$0 < \frac{\lambda_{k,h} - \lambda_k}{\lambda_k} \approx h^{2D} \lambda_k^D.$$  \hfill (34)

Furthermore, the results in \[6\], when combined with (33) and (34) imply that

$$0 < \lambda_{k,h} - \lambda_k \leq \sup_{u \in W_k, \|u\|_0 = 1} \inf_{v \in V_h} \|u - v\|^2_V.$$  \hfill (35)

The main result of this section can be stated as the following theorem and provides an improved estimate for small $k$.

**Theorem 5.1** \textbf{For all $k$ satisfying (30) with $\Pi_h = Q_h$, the following error estimates hold:}

$$0 \leq \sqrt{\lambda_{k,h}} - \sqrt{\lambda_k} \leq C \sup_{w \in W_k, \|w\|_0 = 1} \|I - Q_h w\|_a.$$  \hfill (36)

Here $C = 1 + \frac{c_1}{2}$ with $c_1$ being the constants in (14), which is independent of $k$ and $h$.

**Proof.** If $k$ satisfies (30), by (31)

$$\|I - Q_h w\|_0^2 \leq \frac{1}{2}, \quad \forall w \in W_k, \|w\|_0 = 1.$$  \hfill (37)

Since by (32) the image $Q_h W_k$ is a $k$-dimensional space, it follows that

$$\lambda_{k,h} \leq \sup_{w \in W_k, \|w\|_0 = 1} \frac{a(Q_h w, Q_h w)}{\|Q_h w\|_0^2} = \sup_{w \in W_k, \|w\|_0 = 1} \frac{a(Q_h w, Q_h w)}{1 - \|I - Q_h w\|_0^2} \leq \sup_{w \in W_k, \|w\|_0 = 1} a(Q_h w, Q_h w) (1 + 2\|I - Q_h w\|_0^2)$$  \hfill (38)

Then

$$\sqrt{\lambda_{k,h}} \leq \sup_{w \in W_k} |Q_h w|_a \sqrt{1 + 2\|I - Q_h w\|_0^2} \leq \sup_{w \in W_k} |Q_h w|_a (1 + \|I - Q_h w\|_0^2)$$  \hfill (39)

Here we have used the following inequalities:

$$\sqrt{1 + 2x} \leq 1 + x \leq \frac{1}{1 - x} \leq 1 + 2x, \quad x \in [0, 1/2].$$

From (15), we have

$$\|I - Q_h w\|_0^2 = \|I - Q_h (I - Q_h) w\|_0^2 \leq c_2 h^2 \|I - Q_h w\|_0^2.$$
Combining with (37), we obtain

\[
\| (I - Q_h)w \|_0^2 \leq \frac{1}{\sqrt{2}} \| (I - Q_h)w \|_0 \leq \sqrt{\frac{c_2}{2}} |(I - Q_h)w \|_a. \tag{40}
\]

Since for all \( w \in W_k, \| w \|_0 = 1 \), we have

\[
\lambda_k = \sup_{v \in W_k, \| w \|_0 = 1} a(v, v) \geq a(w, w),
\]

and hence we have that

\[
\sqrt{\lambda_k} \geq |w|_a.
\]

By (14) and (30) for all \( k \in W_k, \) and \( w \) with \( \| w \|_0 = 1 \) we obtain

\[
|Q_hw|_a \leq c_1 |w|_a \leq c_1 \sqrt{\lambda_k} \leq \frac{c_1}{\sqrt{2c_2h}}. \tag{41}
\]

From (39), (40) and (41), it then follows that

\[
\sqrt{\lambda_{h,k}} \leq \sup_{w \in W_k, \| w \|_0 = 1} \left( |Q_hw|_a + \frac{c_1}{2} \| (I - Q_h)w \|_a \right).
\]

This leads to

\[
\sqrt{\lambda_{h,k}} - \sqrt{\lambda_k} \leq \sup_{w \in W_k, \| w \|_0 = 1} \left( |Q_hw|_a - |w|_a + \frac{c_1}{2} |(I - Q_h)w|_a \right)
- \sup_{w \in W_k, \| w \|_0 = 1} \left( |(I - Q_h)w|_a + \frac{c_1}{2} |(I - Q_h)w|_a \right)
\leq \sup_{w \in W_k, \| w \|_0 = 1} \left( |Q_hw|_a - |w|_a + \frac{c_1}{2} |(I - Q_h)w|_a \right)
\leq \sup_{w \in W_k, \| w \|_0 = 1} \left( |(I - Q_h)w|_a + \frac{c_1}{2} |(I - Q_h)w|_a \right)
= \left( 1 + \frac{c_1}{2} \right) \sup_{w \in W_k, \| w \|_0 = 1} \| (I - Q_h)w \|_a.
\]

This completes the proof. \( \Box \)

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