I showed that how Bose-Einstein condensation (BEC) in a non interacting bosonic system with exponential density of states function yields to a new class of Lerch zeta functions. By looking on the critical temperature, I suggested a possible strategy to prove the "Riemann hypothesis" problem. In a theorem and a lemma I suggested that the classical limit $h \to 0$ of BEC can be used as a tool to find zeros of real part of the Riemann zeta function with complex argument. It reduces the Riemann hypothesis to a softer form. Furthermore I proposed a pair of creation-annihilation operators for BEC phenomena. These set of creation-annihilation operators is defined on a complex Hilbert space. They build a set up to interpret this type of BEC as a creation-annihilation phenomena of the virtual hypothetical particle.

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I. INTRODUCTION

In statistical physics there are many examples of critical phenomena where the system undergoes phase transitions below specific values of temperature/density or other physical quantities. A remarkable example is the BEC were finally demonstrated experimentally in dilute alkali gases by a group of researchers and brings 2001 Nobel Prize in physics [1]. From physical point of view the system in BEC phase exist in the very low temperatures regime $T$. In the bosonic systems, all bosons condense to the ground state with the lowest energy. At this phase the Particles become strongly correlated with a distribution of particles both in coordinate and momentum variables. The correlation between particles happended when their thermal wave-lengths $\lambda$ becomes larger than $\lambda_c$, which is function of the mass of an individual condensate particle $\lambda_c$. The number density $n = \frac{1}{\lambda_c^3} = \frac{N}{V}$ (N is total number of particles and $V$ is volume) and $k_B$ as Boltzmann’s constant $\frac{1}{\kappa_B}$. Later it was discovered that BEC can be emerged in the molecular form in Fermi gas $\sigma$. From the mathematical point of view, the BEC waves are solitons waves with shape invariant forms $\lambda$. The applications of BEC didn’t limit to statistical physics, in relativistic cosmology one can build dark matter model using BEC as well as dark energy. Furthermore one can solve Einstein field equations in general relativity with cylindrical symmtry with BEC matter as energy-momentum source and find out cosmic string solutions $\theta$. In relativistic astrophysics, the general relativistic stars can be formed with BEC matter sources $\phi$. A remarkable discovery was by Horwitz et al., in ref. [3], where they discovered a new relativistic high temperature BEC regime. According to Horwitz et al., the mass spectrum of such a system was bounded.

In this letter I investigated a generalized energy density function for a dilute non interacting bosonic gas in the grand canonical ensemble theory of standard statistical mechanics. I showed that density and pressure can be calculated in the closed forms in terms of a general-ized function, named Lerch function (sometimes called as Hurwitz-Lerch-Phi function). This class of general-ized functions are considered as healthy well possed version of Riemann’s zeta function and have recently reintroduced to the community in a series of works by Lagaria & Li. in refs. [14]-[16]. As far as I know [20], this rich family of special functions don’t have used in physical problems specially when we see they are potentially related to the Riemann hypothesis and can be used to find a solution to this old problem which listed as a “millennium-problem” [21]. As an interesting mathematical physics problem in this letter I explored the role of this Lerch functions in a realistic statistical problem relating to BEC.

II. BEC FOR AN EXPONENTIAL DENSITY OF STATES FUNCTION

The grand partition function for the bosonic gas system given by

$$q = \ln D = \frac{P V}{k_B T} = - \sum \ln (1 - z e^{-\beta \epsilon}) \tag{2.1}$$

Here $z = e^{\beta \epsilon}$ is the fugacity of the gas and is related to the chemical potential $\mu$. $\beta = \frac{1}{k_B T}$ and $z e^{-\beta \epsilon} < 1$, $\forall \epsilon$. Furthermore, for enough large volume $V$, the spectrum of the single-potential state is almost a continuous on $R$, the reason is that the single-potential state of a boson in a cube is proportional to $E_n \propto n^2 V^{-2/3}$ here $n$ denotes a collective set of quantum numbers $(n_x, n_y, n_z)$ and $V$...
denotes the volume. We observe that:
\[
\lim_{V \to \infty} \Delta E_n = \lim_{V \to \infty} (E_{n+1} - E_n) \quad (2.2)
\]
\[
\propto \lim_{V \to \infty} \frac{2n + 1}{V^{2/3}} = 0
\]
as a result, we can use the integral instead of summation:
\[
\sum_{\epsilon} \Rightarrow \int d\epsilon \quad (2.3)
\]

here we consider a bosonic system in which the density of state function \( a(\epsilon) \) in the vicinity of a given energy \( \epsilon \) is given by
\[
a(\epsilon) d\epsilon = \frac{2\pi V}{h^3} (2m)^{3/2} e^{1/2 - \epsilon/c} d\epsilon. \quad (2.4)
\]

Note that by substituting this density function \( (2.3) \) into the \( (2.1) \) we are eventually give a weight to the ground state \( \epsilon = 0 \) incorrectly, because in the quantum mechanical approach, each non-degenerate single particle state in the system has a unity weight, we calculate the \( \epsilon = 0 \) term, we obtain:
\[
P \approx \kappa_c (N - N_0) - \eta c VT^{3/2}. \quad (2.12)
\]

Thus for all \( \epsilon \), we conclude that
\[
\kappa_c = \sqrt{\frac{\epsilon^2 + 1}{2\epsilon}} \eta = 4 (\frac{m \pi k_B}{h^3})^{3/2} \epsilon^{1/2} \epsilon^{3/2} \frac{1}{\sqrt{2\epsilon k_B^3}}.
\]

Furthermore we obtain:
\[
\frac{\partial}{\partial x} \left[ \frac{PV}{kT} \lambda^3 \right] = \lambda^3 \left( \frac{N - N_0}{V} \right) \quad (2.13)
\]

We can find critical temperature \( T_c \) for our system. Let us rewrite equation \( (2.10) \) in the following equivalent form:
\[
\lambda^3 \frac{N_0}{V} = \frac{\lambda^3 \kappa_c (N - N_0)}{V} - \Phi(\frac{3}{2}, z, \tilde{c}). \quad (2.14)
\]

here \( v = \frac{V}{\lambda^3} \) called specific volume. The below figure \( [1] \) shows that for \( z \in [0, 1], \tilde{c} \in \mathbb{R}^+ \), \( \Phi(\frac{3}{2}, z, \tilde{c}) \) is a bound, positive, monotonically increasing single valued function of \( z \).

For small \( z \), we have the power series \( (2.3) \). At \( z = 1 \) and for \( \tilde{c} \in \mathbb{R}^+ \), its value is finite and is called Hurwitz-Zeta function:
\[
\Phi(\frac{3}{2}, 1, \tilde{c}) = \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^2} \leq \zeta(\frac{3}{2}) \approx 2.612. \quad (2.15)
\]

Thus for all \( z, \tilde{c} \in \mathbb{R} \), we conclude that \( \frac{N_0}{V} > 0 \) when the temperature and the specific volume satisfy the following inequality:
\[
\frac{\lambda^3}{v} > \Phi(\frac{3}{2}, 1, \tilde{c}) \quad (2.16)
\]
It implies BEC phenomena, the occupation of ground state with more particles. For a specific volume \( V \), we define "implicitly" a critical temperature \( T_c \):

\[
kT_c = \frac{2\pi\hbar^2}{m\Phi\left(\frac{3}{2}, 1, kT_c\right)^{2/3}}
\]

(2.17)

For \( c kT_c \ll 1 \) we can solve (2.17) and find

\[
c kT_c \approx 0.65 + 0.25m^{-1/2}\sqrt{6.8m - 50.6\hbar^2} \quad (2.18)
\]

for a lower bound on the boson mass as \( m \geq 7.4\hbar^2 \). Such lower bound on the mass of the boson (Higgs) proposed in [23].

A simple algebraic manipulations can be done to show that the quantity \( \frac{N_0}{V} \) is a simple two-part function of \( T_c \):

\[
\frac{N_0}{V} = \begin{cases} 
0 & \text{if } \frac{3}{V} < \Phi\left(\frac{3}{2}, 1, \tilde{c}\right) \\
1 - \left(\frac{3}{V}\right)^\Delta & \text{if } \frac{3}{V} > \Phi\left(\frac{3}{2}, 1, \tilde{c}\right)
\end{cases}
\]

(2.19)

where to find the critical exponent \( \Delta \) we need numerical estimation in equation (2.17).

### III. More on Density of States Function

There is a nice relationship between the exponential density of state function given in (2.3) and momentum of the particle \( \tilde{p} \), is given as the relation between volumes in configuration space \( V q \) and phase space \( V_{q, p} \) as follows,

\[
a(\epsilon) = g \frac{dV_q}{dc} \frac{V_{q, p}}{V_{q, p}}
\]

(3.1)

here \( g \) denotes the number of degrees of freedom for bosonic system (in our case is \( g = 3 \)), by integrating this expression we find an atypical dispersion relation as following:

\[
\epsilon(p) = -c^{-1} \left( 1 + PLog\left[\alpha p^3 + \gamma\right] \right)
\]

(3.2)

here \( p \equiv |\tilde{p}| > 0 \), by \( PLog\left[\epsilon\right] \) we mean the principal solution for \( w \in C \) in equation \( z = we^w \) and \( \omega = \frac{\sqrt{2\epsilon}}{3\epsilon^{1/3}}, \gamma = \frac{\sqrt{2\epsilon}}{2\epsilon^{1/3}}}. \) At low momenta \( p \ll \tilde{p}(= \alpha^{-1/3}) \ll 1 \) we have

\[
\epsilon(p) \approx \epsilon_0 + \gamma p^3 + O(p^4)
\]

(3.3)

For ultra-energetic particles \( p \rightarrow \infty \) we have:

\[
\epsilon(p) \approx -c^{-1} \left[ (\ln p)^3 + ... \right]
\]

(3.4)

A remarkable observation is that for some values of \( \alpha, \gamma \) we can obtain \( \epsilon(p^*) = 0 \), \( p^* \neq 0 \). If we solve equation (3.2) we find:

\[
p^* = \sqrt[3]{\frac{\epsilon \gamma + 1}{-\alpha}}, \quad \alpha < 0, \gamma > 0.
\]

(3.5)

It defines a minimum momentum (effective mass) similar to the [14], where the authors discovered a new relativistic high temperature BEC sytem with an additional mass potential of the ensemble existed. In our system we can demonstrate that there is no physical state with \( p < p^* \) and the system has non zero energy even at zero momentum when \( p = 0 \). The energy of particle will be a monotonically increasing function of \( p \) for all \( p > p^* \). A Theorem can be stated as follows:

**Theorem III.1.** Let \( \epsilon(p) \) be energy spectrum in our atypical system , then \( \epsilon(p) \) is a complex value function for \( p < p^* \).

And a consequence of theorem III.1 is the following corollary:

**Corollary III.1.1.** There’s no physical state with \( p < p^* \).

To prove theorem III.1 let us prove the following lemma:

**Lemma III.2.** Given a physical state with \( p = p^* - \delta \), \( 0 < \delta \ll p^* \), there is a complex number \( W \in C \) such that \( \epsilon(p) = W \).
Proof. To prove, we need to show that the energy spectrum given in equation (3.2) for \( p = p^*(1 - \eta), \) \(|\eta| < 1\) becomes a complex function. Let us first rewrite energy spectrum near \( p^* \) in the following form:

\[
\epsilon(p) = -e^{-1} \left( 1 + P \log \left[ -e^{-1} + \alpha(p^3 - (p^*)^3) \right] \right)
\]

We expand the expression (3.7) as power series in \( \eta \),

\[
\epsilon(p) = (-1)^{|B|} \left( -\frac{\sqrt{-6eq^3\alpha}}{c} \eta^{1/2} + O \left( \eta^{3/2} \right) \right)
\] (3.7)

here \( [x] \) gives the greatest integer that is less than or equal to \( x \) and \( B = \frac{1}{2} - \frac{\arg(\eta) + \arg(-3\alpha(p^*)^3)}{2\pi} \). Note that \( \alpha < 0 \) and \( |\eta| < 1 \), consequently \( |\eta| = 0 \) thus we can rewrite the above series as follows:

\[
\epsilon(p) = (-1)^{|B|} \left( -\frac{\sqrt{-6eq^3\alpha}}{c} \eta^{1/2} + O \left( \eta^{3/2} \right) \right)
\] (3.8)

here \( \tilde{B} = \frac{1}{2} - \frac{\arg(3(1+\gamma))}{2\pi} \). Obviously \(-1)^{|\frac{1}{2} - \frac{\arg(3(1+\gamma))}{2\pi}|} = 1, \forall \gamma \in \mathbb{R} \) as a result of \( \eta \to 0^- \), \( \lim_{\eta \to 0^-} \epsilon(p) = ib + O(\eta^{3/2}) \in \mathbb{C} \). It proves our lemma.

IV. ON "RIEMANN HYPOTHESIS" PROBLEM AND CLASSICAL BEC

In this section I will show that how one can find Hurwitz-Zeta function with complex argument \( \Phi(a + ib, 1, \tilde{c}) \) using the following density of states function:

\[
a(\epsilon) = Ae^{-1}e^{-\epsilon} \begin{cases} 
\cos(b \ln \epsilon) \\
\sin(b \ln \epsilon)
\end{cases}
\]

for \( a, b, c \in \mathbb{R} \). Then by taking limit \( c \to 0 \), we lead to Zeta function \( \zeta(a + ib) \). The Riemann hypothesis will translate to a problem of finding specific values for \( T_c \gg 1 \). Let us use density (4.1) to evaluate the grand partition function \( \text{IV} \). It is adequate to rewrite (IV) as follows:

\[
a(\epsilon) = AR \{ e^{-1+ib}e^{-\epsilon} \}, \ A, a, b \in \mathbb{R} \] (4.1)

We can find an \( A \) such that \( \lim_{\epsilon \to 0, a \to \frac{1}{2}, b \to 0} \left[ a(\epsilon) \right] = \frac{2\pi}{h^2} (2m)^{3/2} \), by plugg it into the integral (4.7) we obtain:

\[
\frac{N - N_0}{V} = \frac{A_0}{\lambda^3} \Re \{ \Phi(a + ib, z, \tilde{c}) \}
\]

P \[
\frac{V}{V} = \frac{A_0}{\lambda^3} \int \frac{dz}{z} \Re \{ \Phi(a + ib, z, \tilde{c}) \}.
\] (4.3)

We recall Riemann hypothesis problem here:

**Theorem IV.1. Riemann hypothesis: The zeros of \( \zeta(a + ib) = 0 \) exist only at \( \nu = -n, n \in \mathbb{R} \) \( \vee (\Re \nu = \frac{1}{2}, \nu \in \mathbb{C}) \).**

**Lemma IV.2. In BEC scenario with the critical temperature \( T_c \) given in the equation \( \text{[4.2]} \), does exist a phase**
of classical limit $\hbar \to 0$ (or high critical temperatures $T_c$) bosonic matter with density of states function \([4,0]\) with $a = \frac{1}{2}$.

The above lemma implicitly encourage us to find a classical limit of BEC. Such classical limit well investigated before in literature [24]-[25]. To find the classical analogue of BEC, we need to have a relativistic classical Gibbs ensemble theory. The Gibbs ensembles in relativistic classical and quantum mechanics investigated in [26]. The existence of classical BEC state depends on how to realize classical condensate state as a macroscopic matter wave. The finger prints of quantum matter waves observed in [27] where the authors observed the transition temperature $T_c$ which is lower than the thermodynamic limit. In ref. [27], the author suggested it is possible to have BEC as the coherent dynamics of the system. There is a chance of detecting axions as classical fields for BEC at large scale proposed in [25] where the author suggested that the axions as a motivated dark matter candidate can be considered as a BEC. The classical limit of BEC can be understood when we consider condensate field as a classical field, in analogous to the classical limit of quantum optics. If we have many quanta in each mode, then the fields are treated as the classical fields. As next step we can use the classical fields approximation. We will need two point correlation function. These correlation functions can be depend on the resolution of detectors in an appropriate experimental set up (coarse graining) and we can use the Onsager-Penrose definition of the condensate [28]. I conclude that further studies about classical limits for BEC can be useful to find a general formula for zeros of zeta function on complex plane. My idea here deserved to be considered as a physical proposal to make a connection between a pure mathematical hypothesis and a realistic case of physical situation in nature. Before this calculations done, we were not aware about the connection between zeta function with complex argument and the statistical behavior of a gaseous system. The difficulty was to set up the complex argument in zeta function to the parameters of statistical system. I shopwed that how the real part of the zeta function can be related to the critical temperature and it manifests a nice connection between pure mathematics and applied physics.

V. CREATION-ANNIHILATION OPERATORS FOR BEC

In this section, Im going to study further field theoretic aspects of such atypical condensation phenomena in terms of the creation(raising) and annihilation(lowering) operators pair proposed for Hurwitz-Lerch-Phi function by Lagaria & Li. in refs. [15]-[19]. My focus is to explore the hidden physics of these operators in the mechanism of BEC in our system. In my study I found that the Lerch-zeta function $\Phi(a+ib, z, c)$ plays an essential role. The pair of differential-difference operators $\frac{\partial}{\partial z} z \frac{\partial}{\partial z}$ can be used to define a pair of creation(raising)$D^+_L$ and annihilation(lowering) operators as follows:

$$D^+_L = \frac{\partial}{\partial c}$$

$$D^-_L = \frac{\partial}{\partial \ln z} + c. \quad (5.2)$$

where $[D^+_L, D^-_L] = 1$. Note that these operators are non-self-adjoint linear operators acting in a Hilbert space, a reason is that the following integral identity no longer is valid:

$$\int_{\partial \Omega} \phi^\dagger(z, c) D^+_L \phi(z, c) dz dc = \int_{\partial \Omega} (D^+_L \phi(z, c))^\dagger \phi(z, c) dz dc. \quad (5.3)$$

here $\partial \Omega = [0, 1] \times [0, 1]$. That is a reason that the corresponding eigenvalues of these operators are not real number and their corresponding wave functions can’t build an orthonormal basis for the Hilbert space. which should be
hild for any self sdjoint operator and any function. The  

\[ D_L \equiv D_L^- D_L^+ = \frac{1}{\delta} \frac{\partial^2}{\partial \ln z \partial \ln \tilde{c}} + \frac{\partial}{\partial \ln \tilde{c}}. \]  

(5.4)

here \([D_L, D_L^+] = -D_L^+ [D_L, D_L^-] = D_L^-\). The Hamiltonian’s like operator can be written as \(\tilde{H} = 1 + D_L\). It is illustrative to show that

\[ D_L^+ \Phi(\nu, z, \tilde{c}) = -\nu \Phi(\nu + 1, z, \tilde{c}) \]

\[ D_L^- \Phi(\nu, z, \tilde{c}) = \Phi(\nu - 1, z, \tilde{c}). \]  

(5.5)

(5.6)

combining equations \(5.5, 5.6\) we conclude that

\[ D_L \Phi(\nu, z, \tilde{c}) = -\nu \Phi(\nu, z, \tilde{c}). \]  

(5.7)

We can interpret above operator equations as creation and annihilations equations to create and annihilate a hypothetical virtual boson named "Lerchton" be responsible for BEC with exponential density of state function. This pair production phenomena happens at phase space spanned by two coordinates \((\ln z, \tilde{c})\). The creation operator \(D_L^+\) creates a Lerchton at fixed chemical potential and \(D_L^-\) annihilates it but after a trasforming it to the new location, i.e. \(\tilde{c} + 1\) at phase space. We mention here that \(\ln z \sim \mu\) defines an energy scale and \(\tilde{c} \sim \beta \sim E^{-1}\) corresponds to a length scale. The first variable needs an ultraviolet cutoff and the length scale required an infrared cutoff. Using equation \((5.7)\) we are able to interpret \(-\nu\) as the expectation value of the number operator \(D_L\), similar to the number operator in simple harmonic oscillator in ordinary quantum mechanics) at a given bi-partite state defined by \(|\nu, \tilde{c} >\). The Lerch function \(\Phi(\nu, z, \tilde{c}) \equiv< z|\nu, \tilde{c} >\), is the configurational projection of the original ket vector. An alternative form to the equation \((5.7)\) is:

\[ D_L|\nu, \tilde{c} > = -\nu|\nu, \tilde{c} >. \]  

(5.8)

Define \(|\nu, \tilde{c} >\) as the normalized eigenstates of \(D_L\), and let it be understood that the physical states are labeled by the eigenvalue, i.e. \(-\nu \in \mathbb{C}\) is complex definite in general. Note that at negative integer roots of zeta function \(\zeta(\nu = -n) = 0\), the eigenvalue of operator \(D_L\) is positive and it corresponds to a real number of particles. If we consider the quantity

\[ n \equiv< \nu, \tilde{c} |D_L|\nu, \tilde{c} > \]  

(5.9)

It follows that

\[ n = \frac{\zeta(\nu) - 1}{1 - \nu^2} \]  

(5.10)

is real and positive -definite for \(\nu \in \mathbb{R}^+\). We plot number of particles \((5.10)\) in figure 4.

From figure 3, we learn that the number of the particles become infinity when \(\nu \to 1\). It implies an existed BEC condensation at state \(|1, \tilde{c} >\) as the ground state of the system. Further works can be done to build the Fock space corresponds to the multiparticle states.

Just for curiosity I will study coherent states corresponding to the annihilation operator \(D_L^-\), as the solution to the following differential-difference equation:

\[ D_L^- \sum_{\nu} C_{\nu} \Phi(\nu, z, \tilde{c}) = \alpha \sum_{\nu} C_{\nu} \Phi(\nu, z, \tilde{c}). \]  

(5.11)

using equation \((5.6)\) we can tranform it to the following equation:

\[ \sum_{\nu} (C_{\nu+1} - \alpha C_{\nu}) \Phi(\nu, z, \tilde{c}) + \frac{C_0 z}{1 - z} = 0. \]  

(5.12)

Note that here \(\sum_{\nu}\) is a commutative symbol for a more general integral \(\int d\nu\) over full complex plane \(\nu \in [-i\infty, i\infty]\). The equation \((5.12)\) is a differential-difference equation should be solved for \(C_{\nu}\). Note that \(\Phi(\nu, z, \tilde{c})\) are considered as a set of linearly independent functions because they are eigenfunctions for the general operator \(D_L\). A possible ad-hoc solution to equation \((5.12)\) is given for \(C_0 = 0\) and the coherent state corresponds to it is expressed as follows:

\[ |\alpha > = A_0 \sum_{\nu} \alpha^{-\nu-1} \Phi(\nu, z, \tilde{c}). \]  

(5.13)

We stress here that the above coherent state isn’t normalized, but we can make it normalized if we opt

\[ A_0 = \left( \sum_{\nu = -i\infty}^{i\infty} \frac{\zeta(\nu) - 1}{1 - \nu^2} \alpha^{2(\nu - 1)} \right)^{-1/2} \]  

(5.14)

Where we taking into the account all values of \(\nu \in \mathbb{C}\).
A. Finding minimum of \( \nu \) via variational method

The eigenvalue-eigenfunction operator equation given in (5.7) provides an easy way to find minimum of \( \nu \) through a useful variational method. It is needed we introduce a suitable functional on the phase space \((\ln z, \tilde{c})\) in the form that the resulted Euler-Lagrange equation gives us the linear second order partial differential equation proposed in The eigenvalue-eigenfunction operator introduced given in (5.7). A possible functional to be minimize for a trial function \( \psi \) is given as following:

\[
\Re \nu_0 = -\Re \text{Minimize} \left[ \int_{\partial \mathcal{D}} \bar{\psi}(z, \tilde{c}) D_L^+ D_L^- \psi(z, \tilde{c}) dz d\tilde{c} \right. \\
\left. - \int_{\partial \mathcal{D}} |\psi(z, \tilde{c})|^2 dz d\tilde{c} \right]
\tag{5.15}
\]

It defines a single-valued analytic functional to the region \( \partial \mathcal{D} \) and the trial function \( \psi \) should satisfy the essential boundary condition:

\[
\psi(z, 0) = \begin{cases} 
0 & \text{if } z \to 0 \\
1 & \text{if } z \to 1 
\end{cases}
\tag{5.16}
\]

A possible trial function can be \( \psi(z, \tilde{c}) = (1 + \tilde{c})z \) we evaluate (5.15) we obtain \( \Re \nu_0 = -1 \).

VI. SUMMARY

The special functions of mathematical physics play essential roles to build new physics and develop new mathematical tools in modern physics. The Hurwitz-Lerch-Phi function proposed as an attempt to generalize Riemann zeta function to complex plane as well as break its non simplicity to write it as a solution for any linear partial differential equation, the last was Riemann’s attempt for zeta function. The Riemann zeta functions appeared as key functions in study statistical mechanics on a non interacting Boson dilute gas in grand canonical ensemble. So far in my knowledge there isn’t any physical system with Hurwitz-Lerch-Phi functions as their key functions. As a pioneering work I demonstrated that statistical mechanics for an exponential density of states functions leads to Hurwitz-Lerch-Phi function. If we set the Lerch parameter to zero, it was showed that the statistical expressions for number density (specific volume) and pressure can be represented as real part of the Hurwitz-Lerch-Phi function. A remarkable observation was the Bosonic system doesn’t exist for momentum below a specific momentum \( p^* \), where below this value the energy spectrum becomes complex and it will break the reality of our system. This statement proved in details by considering the left branch of the energy spectrum function. Furthermore I showed that there is a real valued density of states function with a supremum value over whole energy ranges. I proposed a softer theorem in support of the idea that the roots of real part of the zeta function in the half-plane of the complex argument can be obtained in general closed form from investigating classical limits of condensation phenomena. I mention here that such classical limits can be understood by two ways: one is by passing to \( \hbar \to 0 \) of the system and secondly by considering very hot (high) temperature bath in condensed phase. A field theoretical point of view introduced to interpret the Lerch functions as normalized state for an irregular type of number operator in Fock space. I demonstrated that the number of particles in a given state can be a positive number on the half complex plane of the zeta function parameter \( \nu \). It provides a systematic basis to study multiparticle states for such BEC system from field theory point of view. In the forthcoming paper I will study more field theoretical aspects of this system. The aim will be to prove Riemann hypothesis in the softer form that proposed in this paper.

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[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995); C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet,
Phys. Rev. Lett. 75, 1687 (1995); K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).

[3] E. A. Cornell and C. E. Wieman, Rev. Mod. Phys. 74, 875 (2002); W. Ketterle, Rev. Mod. Phys. 74, 1131 (2002); R. A. Duine and H. T. C. Stoof, Phys. Rep. 396, 115 (2004).

[4] Q. Chen, J. Stajic, S. Tan and K. Levin, Phys. Rep. 412, 1 (2005).

[5] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Clarendon Press, Oxford, 2003).

[6] C. J. Pethick and H. Smith, Bose-Einstein Condensation in Dilute Gases (Cambridge University Press, 2008).

[7] A. Griffin, T. Nikuni, and E. Zaremba, Bose-condensed Gases at Finite Temperatures (Cambridge University Press, 2009).

[8] M. Greiner, C. A. Regal, and D. S. Jin, Nature 426, 537-540 (2003).

[9] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, Sci. 287, 97-101 (2000)

[10] D. Bettoni, M. Colombo and S. Liberati, JCAP 1402, 004 (2014) doi:10.1088/1475-7516/2014/02/004 arXiv:1310.3753 [astro-ph.CO].

[11] S. Das and R. K. Bhaduri, Class. Quant. Grav. 32, no. 10, 105003 (2015) doi:10.1088/0264-9381/32/10/105003 arXiv:1411.0753 [gr-qc].

[12] T. Harko and M. J. Lake, Phys. Rev. D 91, 045012 (2015) doi:10.1103/PhysRevD.91.045012 arXiv:1410.6899 [gr-qc].

[13] P. H. Chavanis and T. Harko, Phys. Rev. D 86, 064011 (2012) doi:10.1103/PhysRevD.86.064011 arXiv:1108.3986 [astro-ph.SR].

[14] L. Buravovsky, L. P. Horwitz and W. C. Schieve, Phys. Rev. D 54, 4029 (1996) doi:10.1103/PhysRevD.54.4029 hep-th/9604039.

[15] Jeffrey C. Lagarias, W.-C. Winnie Li, Forum Math. 24 (2012), no. 1, 1-48. arXiv:1005.4712 [math.NT].

[16] Jeffrey C. Lagarias, W.-C. Winnie Li, Forum Math. 24 (2012), no. 1, 49-84. arXiv:1005.4967 [math.NT].

[17] Jeffrey C. Lagarias, W.-C. Winnie Li, Research in the Mathematical Sciences (2016) 3:3 arXiv:1511.08116 [math.NT].

[18] Jeffrey C. Lagarias, W.-C. Winnie Li, Research in the Mathematical Sciences (2016) 3:3 arXiv:1511.0815 [math.NT].

[19] See talk delivered by Prof. Jeffrey C. Lagaria at International Conference in Number Theory and Physics :https://www.youtube.com/watch?v=EbkOTJSGT0Q

[20] See Clay Mathematics Institute website http://www.claymath.org/millennium-problems/riemann-hypothesis

[21] R. S. Willey, Phys. Lett. B 381, 255 (1996) Erratum: [Phys. Lett. B 381, 255 (1996)] doi:10.1016/0370-2693(96)00690-9. 10.1016/0370-2693(96)00321-8 hep-ph/9512226.

[22] J. A. de Reyna, R. P. Brent, J. van de Lune, arXiv:1112.4910.

[23] K. Stalinas, arXiv:cond-mat/0001347.

[24] S. Davidson, Astropart. Phys. 65, 101 (2015) , arXiv:1405.1139 [hep-ph].

[25] L. P. Horwitz, W. C. Schieve and C. Piron, Annals Phys. 137, 306 (1981) [Annals Israel Phys. Soc. 2, 924 (1978)].

[26] W. Ketterle , N. J. van Druten, Phys. Rev. A 54, 656(1996).

[27] O. Penrose , L. Onsager, Phys. Rev. 104, 576(1956).