SIMPLICITY CRITERIA FOR GROUPOID $C^*$-ALGEBRAS

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Abstract. We develop a framework suitable for obtaining simplicity criteria for reduced $C^*$-algebras of Hausdorff étale groupoids. This is based on the study of certain non-degenerate $C^*$-subalgebras (in the case of groupoids, the $C^*$-algebra of the interior isotropy bundle), for which one can control (non-unique) state extensions to the ambient $C^*$-algebra. As an application, we give simplicity criteria for reduced crossed products $C_0(Q) times_{red} G$ by discrete groups.

INTRODUCTION

The task of determining which $C^*$-algebras are simple (in the sense of having no proper, non-trivial norm-closed two-sided ideals) has interested operator algebraists for decades. We follow the approach that has focused on classes of $C^*$-algebras defined from topological, combinatorial, or dynamic information. Thus there is a simplicity criterion for graph $C^*$-algebras ([8, Cor. 3.11]), as well as a generalization for $k$-graphs ([7, Prop. 4.8]); there is a simplicity criterion for full étale groupoid $C^*$-algebras ([2, Thm 5.1]); and there is a well-known sufficient condition for a group crossed product $C^*$-algebra to be simple ([1]). In each of these situations, there is a certain $C^*$-subalgebra $B \subset A$, such that simplicity of $A$ is equivalent to the non-existence of non-trivial proper ideals of $B$ which are invariant in some suitable fashion. In a recent series of papers ([11], [12], [3], [4]) there has been a concerted push to understand the uniqueness property for $C^*$-inclusions for graph, $k$-graph, and groupoid $C^*$-algebras. The perspective that has emerged from these investigations shows that the question of uniqueness is closely tied up with the existence of sufficiently many states on $C^*$-subalgebras that extend uniquely to the ambient $C^*$-algebra. In particular, in ([4, Thm. 3.1]) it is proved that, for an étale groupoid with mild additional requirements, a representation of $C^*_{red}(G)$ is faithful if and only if its restriction to $C^*_{red}(IntIso(G))$ is faithful.

In this paper we propose an approach that aims to unify all these results, by identifying such essential $C^*$-inclusions in a manner that circumvents the unique state extension issues that were previously employed. In particular, in Section 3 we extend the abstract uniqueness theorem of [4] by developing the notions of relative dominance (for ideals) and subordination (for families of states), the main result being the Relative Dominance theorem (3.3), which explains how the relative dominance (with respect to a $C^*$-subalgebra $B \subset A$) for ideals $J \triangleleft A$ is equivalent to the existence of certain sets of states on $B$.

The remainder of the paper (of which the first two Sections review background information) is organized as follows. Section 4 contains several general results pertaining to central inclusions $C_0(Q) \subset A$, which are needed in the main application to étale group bundles in the subsequent section. Our analysis pays particular attention to points $q \in Q$ of reduced continuity, relative to a conditional expectation $\mathbb{E} : A \to C_0(Q)$. This methodology is then
applied in Section 5 which specializes on the inclusion $C_0(G^{(0)}) \subset C^*_\text{red}(G)$, which are central for étale group bundles $G$.

In Section 6 we provide a conceptual framework for minimality of $C^*$-inclusions, which is used in Proposition 6.10, where a general simplicity criterion is given. All these results are applied our to groupoid $C^*$-algebras in Section 7 where we characterize simplicity for reduced groupoid $C^*$-algebras in terms of the interior isotropy subgroupoid (Proposition 7.5). The main results in this section are Theorem 7.6 which gives a sufficient condition for simplicity of the reduced $C^*$-algebra of an étale groupoid, along with Corollaries 7.7 and Corollary 7.8 which provide some necessary conditions. Along the way (Theorem 7.10), we also recover the characterization of simplicity for the full groupoid $C^*$-algebra that previously appeared in [2, Thm. 5.1].

The paper concludes with Section 8, where we specialize our preceding results to transformation groupoids, those which correspond to actions of discrete groups on locally compact Hausdorff spaces. We review (a slightly modified, but equivalent) construction of the transformation groupoid $C^*$-algebra $C_0(Q) \rtimes G$. With the help of Theorem 8.6, the simplicity criterion from Corollary 8.7 (which sharpens [13, Thm. 14(1)]) has a nicer formulation than its sibling Theorem 7.6.

In the Appendix we give technical results for embedding the (full) $C^*$-algebra of an open subgroupoid into the full $C^*$-algebra of an étale groupoid.

1. Preliminaries on Essential Inclusions

Notations. Given a $C^*$-algebra $A$, the notation $J \subseteq A$ signifies that $J$ is a closed two-sided ideal in $A$. The instance when $J \subseteq A$ and $J \neq A$ is indicated by the notation $J \lhd A$.

Given $C^*$-algebras $A$, $B$, to any positive linear map $\psi : A \to B$, one associates the following ideals:

- $L_\psi = \{a \in A : \psi(a^*a) = 0\}$, the largest closed left ideal in $A$, on which $\psi$ vanishes;
- $K_\psi = \{a \in A : \psi(xay) = 0, \forall x, y \in A\}$, the largest closed two-sided ideal in $A$, on which $\psi$ vanishes.

(In the scalar case $B = \mathbb{C}$ and $\psi \neq 0$, $K_\psi$ is nothing else but the kernel of the GNS representation $\Gamma_\psi$.

More generally, if $\Psi = \{\psi_i : A \to B_i\}_{i \in I}$ is a collections of positive linear maps between $C^*$-algebras, we let $L_\Psi = \bigcap_i L_{\psi_i}$ and $K_\Psi = \bigcap_i K_{\psi_i}$.

Definition 1.1. A collection $\Psi = \{\psi_i : A \to B_i\}_{i \in I}$, as above, is said to be jointly essentially faithful, if $K_\Psi = \{0\}$. This is a weaker notion than joint (honest) faithfulness, which requires $L_\Psi = \{0\}$. (In the case of single maps, the term “joint” is omitted.)

Notations. For a $C^*$-algebra $A$, we denote its state space by $S(A)$, and we denote its pure state space by $P(A)$.

Definition 1.2. A $C^*$-inclusion $B \subset A$ is said to be non-degenerate, if $B$ contains an approximate unit for $A$ (and consequently, every approximate unit for $B$ is also an approximate unit for $A$). More generally, a $*$-homomorphism $\Phi : B \to A$ is said to be non-degenerate, if $\Phi(B) \subset A$ is a non-degenerate $C^*$-inclusion.

Remark 1.3. Using the Cohen–Hewitt Factorization Theorem, non-degeneracy for a $C^*$-inclusion $B \subset A$ is equivalent to the equality $A = BAB$, i.e. the fact that any $a \in A$ can be written as a product $a = b_1a'b_2$, with $a' \in A$, $b_1, b_2 \in B$. 


With an eye on some further developments in Section 4, the Remark below collects several useful features concerning multipliers.

**Remark 1.4.** A non-degenerate inclusion $B \subset A$ always gives rise to a unital (thus non-degenerate) $C^*$-inclusion

$$M(B) \subset M(A),$$

(1)

by associating to any $m \in M(B)$ (and using Remark 1.3), the left and right multiplication operators $L_m : A \ni a \mapsto (mb_1)a b_2 \in A$ and $R_m : A \ni a \mapsto b_1 (ab_2) m \in A$. (The parentheses surround elements in $B$.) Equivalently, if $(u_\lambda)_\lambda \subset B$ is some approximate unit for $A$, then $L_m a = \lim \lambda \mu u_\lambda (m a u_\mu)$ and $R_m a = \lim \lambda \mu a (u_\lambda m u_\mu)$.

The same argument shows that any non-degenerate $*$-homomorphism $\Phi : B \to A$ extends uniquely to to a unital $*$-homomorphism

$$M\Phi : M(B) \to M(A),$$

which is continuous in the strict topology.

With the help of the inclusion (1), the non-degeneracy of $B \subset A$ yields the following useful identifications

$$M(B) = \{ m \in M(A) : m B \subset B \} = \{ m \in M(A) : B m \subset B \};$$

(2)

$$B = M(B) \cap A.$$  

(3)

The inclusion of the second set in (2) in the third one (and by symmetry, their equality) can justified again using an approximate unit $(u_\lambda)_\lambda \subset B$ for $A$, as follows. If $m \in M(A)$ satisfies $m B \subset B$, then for any $b \in B$ we have $b m = \lim \lambda \mu u_\lambda (m b u_\mu) = \lim \lambda \mu b (u_\lambda m u_\mu) \in B$. The equality (3), now follows immediately from (2), because if $a \in M(B) \cap A$, then $a = \lim \lambda \mu a u_\lambda \in B$.

**Notation.** A non-degenerate $C^*$-inclusion $B \subset A$ yields a restriction map

$$r_{A|B} : S(A) \ni \varphi \mapsto \varphi|_B \in S(B),$$

which (using the Hahn-Banach Theorem) is surjective.

**Definition 1.5.** Given a non-degenerate inclusion $B \subset A$, and some subset $\Phi \subset S(B)$, we call a subset $\Sigma \subset S(A)$ an $A$-lift of $\Phi$, if $r_{A|B}\mid_\Sigma$ is a bijection of $\Sigma$ onto $\Phi$. Equivalently, an $A$-lift is the range of a cross-section $\sigma : \Phi \to S(A)$ of $r_{A|B}$ (i.e. a right inverse of $r_{A|B}$ over $\Phi$), which allows us to enumerate $\Sigma = \{ \sigma(\varphi) \}_{\varphi \in \Phi}$.

In the case when $\Phi \subset P(B)$, we will can an $A$-lift $\Sigma \subset P(A)$ a a pure $A$-lift.

**Remark 1.6.** A standard application of Krein-Milman Theorem shows that any set $\Phi \subset P(B)$ always has pure lifts. Furthermore, if $\Phi \subset P(B)$ has a unique pure $A$-lift $\Sigma$, then $\Sigma$ is in fact the only possible $A$-lift, meaning that every $\varphi \in \Phi$ has a unique, extension to a state $\hat{\varphi} \in S(A)$. In this vein, the “Abstract” Uniqueness Theorem from [4, Thm 3.2] has the following formulation.

**Theorem 1.7.** (cf. [4, Thm. 3.2]) Assume $B \subset A$ is a non-degenerate $C^*$-inclusion, and $\Phi \subset S(B)$ is a collection which has a unique $A$-lift $\Sigma = \{ \sigma(\varphi) \}_{\varphi \in \Phi} \subset S(A)$. If $\Sigma$ is jointly essentially faithful, then:

(E) the only ideal $L \vartriangleleft A$, that satisfies $B \cap L = \{ 0 \}$, is the zero ideal $L = \{ 0 \}$.

A non-degenerate inclusion $B \subset A$ satisfying condition (E) is called essential.

In the spirit of the above theorem, we conclude this section with several technical results, which will be useful to us later, the first of which (stated only as a Remark) is very elementary.
Remark 1.8. For a \(\ast\)-homomorphism between (non-zero) \(C^*\)-algebras \(\pi : A \to B\), the following conditions are equivalent:

(i) \(\pi\) is injective;

(ii) there exists a non-empty set of positive linear maps \(\Sigma = \{B \xrightarrow{\sigma_i} D_i\}_{i \in I}\), such that the set of linear positive maps \(\Sigma^\pi = \{A \xrightarrow{\sigma_{i,\pi}} D_i\}_{i \in I}\) is jointly essentially faithful.

Indeed, “(ii) \(\Rightarrow\) (i)” follows from the observation that \(\text{Ker} \pi \subseteq K_\Theta\), while the implication “(i) \(\Rightarrow\) (ii)” follows by letting \(\Sigma = S(B)\) and using the fact that every state on the \(C^*\)-subalgebra \(\pi(A) \subset B\) can be extended to a state on \(B\), thus (i) implies \(S(B)^\pi \supset S(A)\).

Lemma 1.9. Assume \(A, B\) are \(C^*\)-algebras, \(A_0 \subset A\) is a dense \(\ast\)-subalgebra, and \(\pi_0 : A_0 \to B\) is a \(\ast\)-homomorphism. The following conditions are equivalent.

(i) \(\pi_0\) extends to a (necessarily unique) \(\ast\)-homomorphism \(\pi : A \to B\);

(ii) there exists a jointly faithful set of positive linear maps \(\Sigma = \{B \xrightarrow{\sigma_i} D_i\}_{i \in I}\), such that all maps in the set \(\Sigma^{\pi_0} = \{A_0 \xrightarrow{\sigma_{i,\pi_0}} D_i\}_{i \in I}\) are bounded (in the norm from \(A\)); equivalently, all maps in \(\Sigma^{\pi_0}\) extend to linear positive maps on \(A\).

Proof. The implication “(i) \(\Rightarrow\) (ii)” is obvious, by taking \(\Sigma\) to be the whole state space \(S(B)\) of \(B\). For the implication “(ii) \(\Rightarrow\) (i),” we first observe that, since joint faithfulness is preserved by scaling the \(\sigma_i\)'s, we can assume that \(\sup_{i \in I} \max \{\|\sigma_i\|, \|\sigma_i \circ \pi_0\|\} < \infty\), in which case by considering the faithful positive linear map \(B \ni b \mapsto (\sigma_i(b))_{i \in I} \in \prod_{i \in I} D_i\), we can assume that \(\Sigma\) is a singleton \(\{B \xrightarrow{\sigma} D\}\). Since the desired condition reads \(\|\pi_0(x^*x)\| \leq \|x^*x\|\), \(\forall x \in A_0\), we can assume \(A_0\) and \(A\) are singly generated by a single positive element \(a_0\). In other words, in condition (ii) we can assume both \(A\) and \(B\) are separable (also abelian, if we want). This means that for our singleton set \(\Sigma = \{B \xrightarrow{\sigma} D\}\) we can also assume \(D\) is separable, so it has a faithful state \(\psi\), which will then make \(\psi \circ \sigma : B \to \mathbb{C}\) faithful as well, so in fact we can assume \(D = \mathbb{C}\). With all these reductions in mind, our implication reduces to the following statement: with \(A_0, A, B\) and \(\pi_0\) as above, if \(\sigma\) is a faithful state on \(B\), such that the composition \(\sigma \circ \pi_0 : A_0 \to \mathbb{C}\) is bounded (thus it extends to a positive linear functional \(\phi : A \to \mathbb{C}\)), then \(\|\pi_0(a_0)|| \leq \|a_0\|\), for any element \(a_0 \in A_0\), which is of the form \(a_0 = x^*x\) with \(x \in A_0\). (Thus \(a_0\) is positive in \(A\), and \(\pi_0(a_0)\) is positive in \(B\), as well). However, this implication follows immediately from the well known fact that, whenever \(\sigma : B \to \mathbb{C}\) is a faithful positive linear functional, for any positive element \(b \in B\), one has \(\|b\| = \lim_{n \to \infty} \sigma(b^n)^{1/n}\). Applying this to \(b = \pi_0(a)\) implies \(\|\pi_0(a_0)|| = \lim_{n \to \infty} \phi(a_0^n)^{1/n} \leq \limsup_{n \to \infty} (\|\phi\| \cdot \|a_0\|^n)^{1/n} = \|a_0\|\), and we are done. \(\square\)

Comment. Condition (ii) above cannot be relaxed to essential faithfulness. For example, let \(A = C([0,1])\), let \(A_0 \subset A\) be the \(\ast\)-subalgebra of polynomial functions, and let \(B = M_2\) (the \(2 \times 2\) matrices). The \(\ast\)-homomorphism \(\pi_0 : A_0 \ni f \mapsto \begin{pmatrix} f(1) & 0 \\ 0 & f(2) \end{pmatrix} \in M_2\) is not bounded (in the norm from \(A\)), but the state \(\sigma : M_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \in \mathbb{C}\) is essentially faithful on \(M_2\), and the composition \(\sigma \circ \pi : f \mapsto f(1)\) is bounded (in the norm from \(A\)).

As a combination of the preceding two results, we now have the following embedding criterion.
Proposition 1.10. Assume $A$, $B$ are $C^*$-algebras, $A_0 \subset A$ is a dense $*$-subalgebra, and $\pi_0 : A_0 \to B$ is a $*$-homomorphism. Assume also we have a double $I$-tuple of positive linear maps $\{B \xrightarrow{\psi_i} D_i \leftarrow A\}_{i \in I}$, with the following properties.

(a) the collection $\Sigma = \{B \xrightarrow{\psi_i} D_i\}_{i \in I}$ is jointly faithful (on $B$);
(b) the collection $\Psi = \{A \xleftarrow{\psi_i} D_i\}_{i \in I}$ is jointly essentially faithful (on $A$);
(c) $\sigma_i \circ \pi_0 = \psi_i |_{A_0}$, $\forall i \in I$.

Then $\pi_0$ extends (uniquely) to an injective $*$-homomorphism $\pi : A \to B$. In particular, $\Psi$ is jointly faithful.

Proof. By Lemma 1.9, conditions (a) and (c) imply the fact that $\pi_0$ indeed extends to a (necessarily unique) $*$-homomorphism $\pi : A \to B$. By Remark 1.8, $\pi$ is indeed injective. □

2. Preliminaries on Groupoid $C^*$-algebras

Convention. All groupoids in this paper are assumed to be second countable, locally compact and Hausdorff.

Notations. If $G$ is such a groupoid, then as usual, we denote its unit space by $G^{(0)}$, and we let $r, s : G \to G^{(0)}$ denote the range and source maps; and for each integer $n \geq 2$, the set of composable $n$-tuples is

$$G^{(n)} = \{ (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \prod_1^n G : r(\gamma_j) = s(\gamma_{j-1}), \ 2 \leq j \leq n \}.$$ 

Given some non-empty subset $X \subset G$, we denote the set $\{ \gamma^{-1} : \gamma \in X \}$ simply by $X^{-1}$. Given non-empty subsets $X_1, X_2, \ldots, X_n \subset G$, we define

$$X_1 X_2 \ldots X_n = \{ \gamma_1 \gamma_2 \cdots \gamma_n \mid (\gamma_1, \ldots, \gamma_n) \in G^{(n)} \cap \prod_{j=1}^n X_j \}.$$ 

When one of these sets is a singleton, say $\{ \gamma \}$, we omit the braces; in particular, if $u \in G^{(0)}$, the sets $uG$ and $Gu$ are precisely the range and source fibers $r^{-1}(u)$ and $s^{-1}(u)$, respectively, while $uGu = uG \cap Gu$ is the isotropy group at $u$.

Definition 2.1. A groupoid $G$ is étale, if $r$ and $s$ are local homeomorphisms; equivalently, there exists a basis for the topology on $G$ consisting of open bisections, i.e. open subsets $B \subset G$, for which $r : B \to r(B)$ and $s : B \to s(B)$ are homeomorphisms onto open sets in $G$ (in particular, $G^{(0)}$ is clopen in $G$). As a consequence, for each unit $u \in G^{(0)}$ the sets $uG$ and $Gu$ are discrete in the relative topology; hence compact subsets of $G$ intersect any one of these sets finitely many times.

We now recall the construction (cf. [14]) of the $C^*$-algebras associated to an étale groupoid $G$, as above. One starts off by endowing $C_c(G)$ with the following $*$-algebra structure:

$$(f \times g)(\gamma) = \sum_{(\alpha, \beta) \in G^{(0)} : \alpha \beta = \gamma} f(\alpha)g(\beta);$$

$$f^*(\gamma) = \overline{f}(\gamma^{-1}).$$ 

With these definitions, the full $C^*$-norm on $C_c(G)$ is given as

$$\|f\|_{\text{full}} = \sup \{ \|\pi(f)\| : \pi \text{ $*$-representation of } C_c(G) \}, \quad (4)$$
By a deep result – Renault’s Disintegration Theorem ([14] Thm. II.1.21, Corollary II.1.22) – the quantity \( \|f\|_{\text{full}} \) is finite, for each \( f \in C_c(\mathcal{G}) \), and furthermore, satisfies \( \|f\|_{\text{full}} \leq \|f\|_1 \), where

\[
\|f\|_I = \sup_{u \in \mathcal{G}^{(0)}} \max \left\{ \sum_{\gamma \in u} |f(\gamma)|, \sum_{\gamma \in u} |f(\gamma)| \right\}, \quad f \in C_c(\mathcal{G}).
\]

As \( \mathcal{G}^{(0)} \) is clopen in \( \mathcal{G} \), we have an inclusion \( C_c(\mathcal{G}^{(0)}) \subset C_c(\mathcal{G}) \), which turns \( C_c(\mathcal{G}^{(0)}) \) into a \(*\)-subalgebra; however, the \(*\)-algebra operations on \( C_c(\mathcal{G}^{(0)}) \) inherited from \( C_c(\mathcal{G}) \) coincide with the usual (pointwise) operations: \( h^* = \overline{h} \) and \( h \times k = hk \), \( \forall h, k \in C_c(\mathcal{G}^{(0)}) \). In fact, something similar can be said concerning the left and right \( C_c(\mathcal{G}^{(0)}) \)-module structure of \( C_c(\mathcal{G}) \): for all \( f \in C_c(\mathcal{G}), h \in C_c(\mathcal{G}^{(0)}) \) we have

\[
(f \times h)(\gamma) = f(\gamma)h(\sigma(\gamma)); \quad (h \times f)(\gamma) = h(\tau(\gamma))f(\gamma).
\]

When restricted to \( C_c(\mathcal{G}^{(0)}) \), the full \( C^* \)-norm agrees with the usual sup-norm \( \| \cdot \|_\infty \), so by completion, the embedding \( C_c(\mathcal{G}^{(0)}) \subset C_c(\mathcal{G}) \) gives rise to a non-degenerate inclusion \( C_0(\mathcal{G}^{(0)}) \subset C^*(\mathcal{G}) \).

The restriction map \( C_c(\mathcal{G}) \ni f \mapsto f|_{\mathcal{G}^{(0)}} \in C_c(\mathcal{G}^{(0)}) \) (which is contractive in the full norm) gives rise by completion to a conditional expectation of \( C^*(\mathcal{G}) \) onto \( C_0(\mathcal{G}^{(0)}) \), i.e. a norm one linear map \( E : C^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)}) \), such that

- \( E(f) = f, \forall f \in C_0(\mathcal{G}^{(0)}) \);
- \( E(f_1a_{f_2}) = f_1E(a)f_2, \forall a \in C^*(\mathcal{G}), f_1, f_2 \in C_0(\mathcal{G}^{(0)}) \);

The KSGNS representation ([3]) associated with \( E \) is a \(*\)-homomorphism \( \Lambda_E : C^*(\mathcal{G}) \to \mathcal{L}(L^2(C^*(\mathcal{G}), E)) \), where \( L^2(C^*(\mathcal{G}), E) \) is the Hilbert \( C_0(\mathcal{G}^{(0)}) \)-module obtained by completing \( C^*(\mathcal{G}) \) in the norm given by the inner product \( \langle a|b\rangle_E = E(a^*b) \). The kernel of this representation is precisely the ideal \( K_E \) defined in Section 1. The quotient \( C^*(\mathcal{G})/K_E \) is the so-called reduced groupoid \( C^* \)-algebra, denoted by \( C^*_{\text{red}}(\mathcal{G}) \). The ideal \( K_E \) can be described alternatively with the help of the usual GNS representations \( \Gamma_{ev_u \circ E} : C^*(\mathcal{G}) \to \mathcal{B}(L^2(C^*(\mathcal{G}), ev_u \circ E)) \), \( u \in \mathcal{G}^{(0)} \). (The Hilbert space \( L^2(C^*(\mathcal{G}), ev_u \circ E) \) is the completion of \( C_c(\mathcal{G}) \) in the norm given by the inner product \( \langle f|g\rangle_u = (f^* \times g)(u) \).) With these (honest) representations in mind, we have \( K_E = \bigcap_{u \in \mathcal{G}^{(0)}} K_{ev_u \circ E} \).

As was the case with the full groupoid \( C^* \)-algebra, after composing with the quotient map \( \pi_{\text{red}} : C^*(\mathcal{G}) \to C^*_{\text{red}}(\mathcal{G}) \), we still have an embedding \( C_c(\mathcal{G}) \subset C^*_{\text{red}}(\mathcal{G}) \), so we can also view \( C^*_{\text{red}}(\mathcal{G}) \) as the completion of the convolution \( * \)-algebra \( C_c(\mathcal{G}) \) with respect to a (smaller) \( C^* \)-norm, denoted \( \| \cdot \|_{\text{red}} \). Again, when restricted to \( C_c(\mathcal{G}^{(0)}) \), the norm \( \| \cdot \|_{\text{red}} \) agrees with \( \| \cdot \|_{\infty} \), so \( C_0(\mathcal{G}^{(0)}) \) still embeds in \( C^*_{\text{red}}(\mathcal{G}) \), and furthermore, since the natural expectation \( \mathcal{E} \) vanishes on \( K_E \), we will have a reduced version of natural expectation, denoted by \( \mathcal{E}_{\text{red}} : C^*_{\text{red}}(\mathcal{G}) \to C_0(\mathcal{G}^{(0)}) \), which satisfies \( \mathcal{E}_{\text{red}} \circ \pi_{\text{red}} = \mathcal{E} \). Not only we know that \( \mathcal{E}_{\text{red}} \) is essentially faithful (because \( K_E = \ker \pi_{\text{red}} \)), but in fact it is (honestly) faithful on \( C^*_{\text{red}}(\mathcal{G}) \).

3. Relative Dominance

As it turns out, proving that an inclusion \( B \subset A \) is essential is always tied up with some analysis of \( A \)-lifts (i.e. state extensions) for states on \( B \), as Corollary [3.4] below (which also contains some type of converse of Theorem [1.7]) will demonstrate. In preparation for these results, we start off by formulating relative versions for essential faithfulness, as well as property (E) from Theorem [1.7].
Definitions 3.1. Suppose a $C^*$-algebra $A$ is given, along with some ideal $J \triangleleft A$.

(i) A collection $\Psi = \{\psi_i : A \to B_i\}_{i \in I}$ of positive linear maps between $C^*$-algebras is said to be subordinated to $J$, if $K_\Psi \subset J$.

(ii) For a non-degenerate $C^*$-inclusion $B \subset A$, we say that $J$ is dominant relative to $B$, if whenever an ideal $L \triangleleft A$ satisfies $B \cap L \subset B \cap J$, it also satisfies $L \subset J$.

Remark 3.2. When specializing the above Definition to the zero ideal, it is straightforward that

(i) A family $\Psi$ as above is jointly essentially faithful, if and only if $\Psi$ is subordinated to the zero ideal $\{0\}$.

(ii) A a non-degenerate $C^*$-inclusion $B \subset A$ is essential, if and only if the zero ideal $\{0\}$ is dominant relative to $B$.

Theorem 3.3. Suppose that a non-degenerate $C^*$-inclusion $B \subset A$ is given, along with some ideal $J \triangleleft A$. Regard the state space $S(B/B \cap J)$ as a subset of $S(B)$ and the pure state space $P(B/B \cap J)$ as a subset of $P(B)$ (see the Note preceding the proof below). The following are equivalent:

(i) $J$ is dominant relative to $B$;

(ii) there exists $\Phi \subset S(B/B \cap J)$ such that all $A$-lifts of $\Phi$ are subordinated to $J$;

(iii) there exists $\Phi \subset P(B/B \cap J)$ such that all $A$-lifts of $\Phi$ are subordinated to $J$;

(iii') there exists $\Phi \subset P(B/B \cap J)$ such that all pure $A$-lifts of $\Phi$ are subordinated to $J$.

Note. The inclusions mentioned in the statement are simply the compositions with the quotient $*$-homomorphism $\rho : B \to B/B \cap J$, namely the map $S(B/B \cap J) \ni \varphi \mapsto \varphi \circ \rho \in S(B)$ and its restriction to $P(B/B \cap J)$. These inclusions simply identify $S(B/B \cap J)$ with the set $\{\varphi \in S(B) : \varphi|_{B \cap J} = 0\}$ and $P(B/B \cap J)$ with the set $\{\varphi \in P(B) : \varphi|_{B \cap J} = 0\}$.

Proof. (i)$\Rightarrow$(iii). Assume that $J$ is dominant relative to $B$, and let us show that property (iii) holds for $\Phi = P(B/B \cap J)$. Fix for the moment some $A$-lift $\Sigma = \{\sigma(\varphi)\}_{\varphi \in P(B/B \cap J)}$ for $P(B/B \cap J)$ (associated with some cross-section $\sigma : P(B/B \cap J) \to S(A)$), and let us justify the inclusion $K_\Sigma \subset J$. Using the assumption (i), it suffices to prove the inclusion $B \cap K_\Sigma \subset B \cap J$. To this end, fix some $b \in B \cap K_\Sigma$, and let us show that $b \in B \cap J$. By our assumption on $b$, we know that

$$(\sigma(\varphi))(xby) = 0, \forall x, y \in A, \varphi \in P(B/B \cap J).$$

By taking $x$ and $y$ to be terms in an approximate unit for $B$, and using the cross-section condition $\sigma(\varphi)|_B = \varphi$, the above equalities imply

$$\varphi(b) = 0, \forall \varphi \in P(B/B \cap J),$$

which then clearly implies that $b$ indeed belongs to $B \cap J$.

It is obvious that (ii)$\iff$(iii)$\Rightarrow$(iii'), so it stands to prove that [(ii) or (iii')]$\Rightarrow$(i). Start off by assuming the existence of a collection $\Phi \subset S(B/B \cap J)$, which either

(*) satisfies (ii), or

(**) is a subset of $P(B/B \cap J)$ and satisfies (iii'),

let $L \triangleleft A$ be an ideal satisfying $B \cap L \subset B \cap J$, and let us prove the inclusion $L \subset J$.

Consider the quotient $\pi : A \to A/L$ and the non-degenerate $C^*$-subalgebra

$$\pi(B)(\simeq B/B \cap L) \subset A/L.$$
Fix a cross-section $\eta : S(\pi(B)) \to S(A/L)$ for $r_{(A/L)\downarrow \pi(B)}$, which maps $P(\pi(B))$ into $P(A/L)$. (Use Remark 1.6.) In other words, for each state $\varphi \in S(B)$ which vanishes on $B \cap L$, we choose $\eta(\varphi)$ to be an extension of $\varphi$ to a state on $A$, which vanishes on $L$, and furthermore, in case $\varphi$ were pure, $\eta(\varphi)$ is also chosen to be pure.

By the assumption $B \cap L \subset B \cap J$, it follows that $S(B/B \cap J) \subset S(B/B \cap L)$ and $P(B/B \cap J) \subset P(B/B \cap L)$ (that is, if a state $\varphi$ on $B$ vanishes on $B \cap J$, then it also vanishes on $B \cap L$). This means that we can view $\Phi \subset S(B/B \cap L)$ (or, in case (**)), we can view $\Phi \subset P(B/B \cap L)$), thus $\Sigma = \{\eta(\varphi)\}_{\varphi \in \Phi}$ defines an $A$-lift for $\Phi$, which is pure in case (**). Therefore, by our assumptions on $\Phi$, it follows that $\Sigma$ is subordinated to $J$, i.e. $\cap_{\varphi \in \Phi} K_{\eta(\varphi)} \subset J$. However, by our definition of $\eta$, we know that, for each $\varphi \in \Phi$, the state $\eta(\varphi)$ vanishes on $L$. In other words, $L \subset K_{\eta}, \forall \varphi \in \Phi$, which then implies $L \subset \cap_{\varphi \in \Phi} K_{\eta(\varphi)} \subset J$, and we are done. \qed

Note. If some $\Phi \subset S(B/B \cap J)$ satisfies (2), then any larger set $\Psi \supset \Phi$ of states on $B/B \cap J$ will satisfy it as well. Similarly, the class of all subsets $\Phi \subset P(B/B \cap J)$ satisfying (3) or (3') is closed upward.

If in the preceding Theorem we let $J = \{0\}$, we obtain the following generalization of Theorem 1.7.

**Corollary 3.4.** For a non-degenerate inclusion $B \subset A$, the following are equivalent:

(i) $B \subset A$ is essential;

(ii) there exists $\Phi \subset S(B)$ such that all $A$-lifts of $\Phi$ are jointly essentially faithful;

(iii) there exists $\Phi \subset P(B)$ such that all $A$-lifts of $\Phi$ are jointly essentially faithful;

(iii') there exists $\Phi \subset P(B)$ such that all pure $A$-lifts of $\Phi$ are jointly essentially faithful.

4. Central Inclusions

The main applications of Theorem 3.3 and its Corollary 3.4 are Theorem 4.1 and its Corollary 1.2 below, which deal with central inclusions $B \subset A$, i.e. those for which $B$ is contained in the center of $A$, that is, $ba = ab, \forall a \in A, b \in B$. Such inclusions force $B$ to be abelian, so we can identify $B = C_0(Q)$, for some locally compact space $Q$.

Notations. Assume $C_0(Q) \subset A$ is a central non-degenerate inclusion. For any point $q \in Q$, denote by $J_q^{\text{unif}} \subset A$ the ideal generated by $C_0,q(Q) = \{f \in C_0(Q) : f(q) = 0\}$. With the help of $J_q^{\text{unif}}$, we can endow $A$ with the $C^*$-seminorms $p_q^{\text{unif}}, q \in Q$, given by

$$p_q^{\text{unif}}(a) = ||a + J_q^{\text{unif}}||_{A/J_q^{\text{unif}}} = \inf\{||a + x|| : x \in J_q^{\text{unif}}\}$$

$$= \inf\{||fa|| : f \in C_0,q(Q), 0 \leq f \leq 1 = f(q)\}. \quad (7)$$

**Theorem 4.1.** Suppose that $C_0(Q) \subset A$ is a central non-degenerate inclusion, and assume (using the above notation) the set

$$Q^{\text{unif}}_{\text{simple}} = \{q \in Q : A/J_q^{\text{unif}} \text{ is a simple } C^*\text{-algebra}\}$$

is non-empty. Then for any non-empty subset $Q_0 \subset Q^{\text{unif}}_{\text{simple}}$, the ideal $\bigcap_{q \in Q_0} J_q^{\text{unif}}$ is dominant relative to $C_0(Q)$.

**Proof.** For every $q \in Q$, let $ev_q$ denote the evaluation map $C_0(Q) \ni f \mapsto f(q) \in \mathbb{C}$, which defines a pure state on $C_0(Q)$. The main step in our proof is contained in the following
CLAIM. If \( q \in Q_{\text{unif}}^{\text{simple}} \) and \( \psi_q \in S(A) \) is any extension of \( \text{ev}_q \) to a state on \( A \), then \( K_{\psi_q} = J_q^{\text{unif}} \).

First of all, since \( K_{\psi_q} \) is the kernel of a GNS representation (associated to a state on \( A \)), it is trivial that \( K_{\psi_q} \subseteq A \), so by the assumed simplicity of \( A/J_q^{\text{unif}} \), it suffices to prove the inclusion \( J_q^{\text{unif}} \subset K_{\psi_q} \). Secondly, since by construction, the set \( \{f \in C_{0,q}(Q) : 0 \leq f \leq 1\} \) (equipped with the usual order) constitutes an approximate unit for \( J_q^{\text{unif}} \), all we have to justify is the inclusion \( C_{0,q}(Q) \subset K_{\psi_q} \), which amounts to showing that

\[
\psi_q(a^* f^* f a) = 0, \quad \forall f \in C_{0,q}(Q), \, a \in A.
\]

However, since the inclusion \( C_0(Q) \subset A \) is central, for any \( f \in C_{0,q}(Q), \, a \in A \), we have

\[
0 \leq \psi_q(a^* f^* f a) = \psi_q(f^* a^* a f) \leq \psi_q(\|a\|^2 f^* f) = \text{ev}_q(\|a\|^2 f^* f) = \|a\|^2 \cdot |f(q)|^2 = 0,
\]

and (8) follows.

Having proved the Claim, all we have to do is to show that the collection \( \Phi = \{\text{ev}_q\}_{q \in Q_0} \) satisfies condition (ii) from Theorem 3.3 when applied to the ideal \( \bigcap_{q \in Q_0} J_q^{\text{unif}} \). But if we start with a \( A \)-lift \( \Psi = \{\psi_q\}_{q \in Q_0} \) for \( \Phi \) (by way of notation assuming \( \psi_q|_{C_0(Q)} = \text{ev}_q \), \( \forall q \in Q_0 \)), then by the Claim we know that \( K_{\psi_q} = J_q^{\text{unif}}, \, \forall q \in Q_0 \), which clearly implies

\[
K_{\Psi} = \bigcap_{q \in Q_0} K_{\psi_q} = \bigcap_{q \in Q_0} J_q^{\text{unif}}.
\]

\[
\boxed{\text{Corollary 4.2. Suppose } C_0(Q) \subset A \text{ is a central non-degenerate inclusion. If, using the notations from Theorem 4.1, the set } Q_{\text{unif}}^{\text{simple}} \text{ is non-empty, and satisfies }}
\]

\[
\bigcap_{q \in Q_{\text{unif}}^{\text{simple}}} J_q^{\text{unif}} = \{0\},
\]

then the inclusion \( C_0(Q) \subset A \) is essential.

Comment. Central inclusions are special cases of so-called \( C_0(Q) \)-\textit{algebras}, for which the condition that \( C_0(Q) \) is a \( C^* \)-subalgebra of \( A \) is relaxed to the condition that \( C_0(Q) \) is contained in the center \( Z(M(A)) \) of the multiplier algebra \( M(A) \), combined with the (non-degeneracy) condition that an approximate unit for \( C_0(Q) \) converges to \( 1 \in M(A) \) in the strict topology. In this general setting the seminorms \( p_q^{\text{unif}} \) still make sense, when defined by the last equality above; we then can define our ideals by \( J_q^{\text{unif}} = \ker p_q^{\text{unif}} \). Whether we work with arbitrary \( C_0(Q) \)-\textit{algebras}, or just with those special ones considered above, one can show (see e.g. [10, Lemma 1.12], or [17, Appendix C]) that for each \( a \in A \), the map

\[
Q \ni q \longmapsto p_q^{\text{unif}}(a) \in [0, \infty)
\]

is \textit{upper semicontinuous}, in the sense that

\[
\limsup_{q \to q_0} p_q^{\text{unif}}(a) \leq p_{q_0}^{\text{unif}}(a), \quad \forall q_0 \in Q.
\]

(An equivalent characterization of (11) is the fact that all sets \( \{q \in Q : p_q^{\text{unif}}(a) < s\}, \, s > 0, \) are open.) Likewise (using non-degeneracy), it is also pretty clear that

\[
\lim_{q \to \infty} p_q^{\text{unif}}(a) = 0, \quad \forall a \in A.
\]

Regardless of whether (10) is continuous or not, by non-degeneracy we always have

\[
\|a\| = \sup_{q \in Q} p_q^{\text{unif}}(a), \quad \forall a \in A.
\]
As shown below, the seminorms $p_q^{\text{unif}}$ have a certain uniqueness property in conjunction with (13) and (11).

**Lemma 4.3.** Assume $C_0(Q) \subset A$ is a central non-degenerate inclusion, and $(p_q)_{q \in Q}$ is a family of $C^*$-seminorms on $A$, satisfying

(a) $\|a\| = \sup_{q \in Q} p_q(a), \forall a \in A$;

(b) $p_q(fa) = |f(q)| \cdot p_q(a), \forall a \in A, f \in C_0(Q)$.

If $a \in A$, $q_0 \in Q$ satisfy

$$\lim_{q \to q_0} p_q(a) \leq p_{q_0}(a),$$  \hfill (14)

then $p_{q_0}(a) = p_{q_0}^{\text{unif}}(a)$.

**Proof.** First of all, using the hypotheses (a) and (b), for any $f \in C_0(Q)$ with $0 \leq f \leq 1 = f(q_0)$, we have

$$\|fa\| \geq p_{q_0}(fa) = |f(q_0)| \cdot p_{q_0}(a) = p_{q_0}(a),$$

so by (7), we clearly have the inequality $p_{q_0}^{\text{unif}}(a) \geq p_{q_0}(a)$. For the other inequality, fix for the moment $\varepsilon > 0$ and use (11) to produce some open set $V_\varepsilon \ni q_0$, such that $p_q(a) \leq p_{q_0}(a) + \varepsilon$, $\forall q \in V$. Fix also some $f_\varepsilon \in C_c(V)$ with $0 \leq f_\varepsilon \leq 1 = f_\varepsilon(q_0)$, and observe that, for every $q \in Q$, we have

$$p_q(f_\varepsilon a) = f_\varepsilon(q) \cdot p_q(a) \leq f_\varepsilon(q) (p_{q_0}(a) + \varepsilon) \leq p_{q_0}(a) + \varepsilon,$$

which by (a) implies $\|f_\varepsilon a\| \leq p_{q_0}(a) + \varepsilon$, thus by (7) we obtain $p_{q_0}^{\text{unif}}(a) \leq p_{q_0}(a) + \varepsilon$, and we are done by letting $\varepsilon \to 0$. \qed

When trying to check the hypotheses in Corollary 4.2, a reasonable guess for a sufficient condition would be the density of $Q_\text{simple}^{\text{unif}}$ in $Q$. However, in part due to lack of continuity of some of the maps (10), particularly if one of the inequalities (11) is strict, one cannot expect this to be the case, as illustrated by the following.

**Example 4.4.** Let $A = C([0,2])$, and consider the inclusion $C([0,1]) \subset A$ defined by extending every continuous function $f : [0, 1] \to \mathbb{C}$ to a continuous function on $[0, 2]$, which is constant on $[1, 2]$. Clearly, $J_q^{\text{unif}} = \{ f \in C([0,2]) : f(q) = 0 \}$, for every $q \in [0, 1)$, but $J_1^{\text{unif}} = \{ f \in C([0,2]) : f|_{[1,2]} = 0 \}$, so $Q_\text{simple}^{\text{unif}} = [0, 1)$ is indeed dense in $Q = [0, 1]$, but the ideal $J_{(0,1)}^{\text{unif}} = \{ f \in C([0,2]) : f|_{[0,1]} = 0 \}$ is not $\{0\}$. Coincidentally, the same ideal $J_{(0,1)}^{\text{unif}}$ satisfies $J_{(0,1)}^{\text{unif}} \cap C([0, 1]) = \{0\}$, which also means that $C([0, 1]) \subset A$ is non-essential.

On way to ameliorate the above complication (as shown in Theorem 4.6 below), as well as the failure of continuity for the maps (10), is to use conditional expectations.

**Notations.** Assume $C_0(Q) \subset A$ is a non-degenerate central inclusion, and $E$ is a conditional expectation of $A$ onto $C_0(Q)$. For every $q \in Q$, consider the composition $ev_q \circ E$, which is a state on $A$, and its associated GNS representation $\Gamma_{ev_q \circ E} : A \to \mathcal{B}(L^2(A, ev_q \circ E))$. The associated $C^*$-seminorm $A \ni a \mapsto \|\Gamma_{ev_q \circ E}(a)\| \in [0, \infty)$ will be denoted by $p_q^E$.

**Remark 4.5.** Since $E$ is $C_0(Q)$-linear, for every $q \in Q$ we have

$$ev_q \circ E(fa) = ev_q \circ E(af) = 0, \quad \forall f \in C_0,q(Q), a \in A,$$

thus $J_q^{\text{unif}}$ is contained in the kernel of the GNS representation $\Gamma_{ev_q \circ E}$, which means that

$$p_q^E(a) \leq p_q^{\text{unif}}(a), \quad \forall q \in Q, a \in A.$$  \hfill (15)
Theorem 4.6. Assume $C_0(Q) \subset A$ is a non-degenerate central inclusion, such that $Q^\text{unif}_{\text{simple}}$ is dense in $Q$. If there exists an essentially faithful conditional expectation of $E : A \to C_0(Q)$, then the inclusion $C_0(Q) \subset A$ is essential.

Proof. Consider the ideal

$$H = \bigcap_{q \in Q^\text{unif}_{\text{simple}}} K_{\text{ev}_q \circ E} = \bigcap_{q \in Q^\text{unif}_{\text{simple}}} \ker p_q^E$$

(i.e. the intersection of the kernel of the GNS representations $\Gamma_{\text{ev}_q \circ E}$, $q \in Q^\text{unif}_{\text{simple}}$). On the one hand, by (15), it follows that

$$\bigcap_{q \in Q^\text{unif}_{\text{simple}}} J_q^\text{unif} \subset H,$$

so in order to draw the desired conclusion, by Corollary 4.2 it suffices to show that $H$ is the zero ideal.

On the other hand, by construction, for every $x \in H$, the function $E(x) \in C_0(Q)$ vanishes on $Q^\text{unif}_{\text{simple}}$, which is dense in $Q$, thus by continuity $E(x) = 0$. In other words, $H$ is a closed two-sided ideal contained in $\ker E$, so by the essential faithfulness of $E$, $H$ is indeed zero ideal. □

Remark 4.7. The required essential faithfulness of $E$ should not be regarded as too stringent, since for an essential non-degenerate inclusion $B \subset A$, it follows that all conditional expectations $E : A \to B$ (if any) are essentially faithful.

Remark 4.8. By definition, it is pretty obvious that, for a non-degenerate central inclusion $C_0(Q) \subset A$, the faithfulness of a conditional expectation $E : A \to C_0(Q)$ is equivalent to:

$$\|a\| = \sup_{q \in Q} p_q^E(a), \forall a \in A.$$  \hspace{1cm} (16)

As to the usefulness of conditional expectations in the issues dealing with the map (10), we have the result below, which can be traced to [10, Lemma 1.11]; due to its elementary nature, we include a proof for the benefit of the reader.

Lemma 4.9. Suppose $C_0(Q) \subset A$ is a non-degenerate central inclusion, and $E : A \to C_0(Q)$ is a conditional expectation. For every $a \in A$, the map

$$Q \ni q \mapsto p_q^E(a) \in [0, \infty)$$  \hspace{1cm} (17)

is lower semicontinuous, in the sense that

$$\liminf_{q \to q_0} p_q^E(a) \geq p_{q_0}^E(a), \forall q_0 \in Q.$$  \hspace{1cm} (18)

Proof. An equivalent characterization of (18) is the fact that all sets $D_s = \{q \in Q : p_q^E(a) > s\}$, $s > 0$, are open. Fix $a \in A$, $s > 0$, as well some $q_0 \in D_s$, and let us justify that $q_0$ is also in the interior of $D_s$.

By the definition of the GNS representation $\Gamma_{\text{ev}_q \circ E}$, the condition $p_{q_0}^E(a) > s$ implies the existence of some $x \in A$, such that

$$\text{ev}_{q_0} \circ E(x^* a^* ax) > s^2 \cdot \text{ev}_{q_0} \circ E(x^* x) > 0.$$
Since $E$ is $C_0(Q)$-valued, by continuity, it follows that there exists a whole neighborhood $V$ of $q_0$ in $Q$, such that
\[ ev_q \circ E(x^*a^*ax) > s^2 \cdot ev_q \circ E(x^*x) > 0, \ \forall q \in V, \]
which in turn implies $p^E_q(a) > s, \ \forall q \in V$. \qed

**Proposition 4.10.** Assume $C_0(Q) \subset A$ is a central non-degenerate inclusion and $E : A \to C_0(Q)$ is an essentially faithful conditional expectation. For a point $q_0$, the following conditions are equivalent:

(i) the function (17) is continuous at $q_0$, for each $a \in A$;
(ii) there is some dense subset $A_0 \subset A$, such that the function (17) is continuous at $q_0$, for each $a \in A_0$;
(iii) there is some dense subset $A_0 \subset A$, such that $\limsup_{q \to q_0} p^E_q(a) \leq p^E_{q_0}(a), \ \forall a \in A_0$;
(iv) $p^\text{unif}_{q_0}(a) = p^E_{q_0}(a), \ \forall a \in A$;
(v) there is some dense subset $A_0 \subset A$, such that $p^\text{unif}_{q_0}(a) = p^E_{q_0}(a), \ \forall a \in A_0$.

**Proof.** The equivalences (i) $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) are obvious from (13) and (16).

The implication (ii) $\Rightarrow$ (i) is pretty obvious from the semicontinuity properties (11) and (18). The converse implication (i) $\Rightarrow$ (ii) follows immediately from Lemma 4.3. \qed

**Definition 4.11.** A point $q_0 \in Q$, satisfying the equivalent conditions in the preceding Proposition, will be referred to as a point of reduced continuity relative to $E$. We denote by $Q^E_{\text{r-cont}}$ the set of all such points. On the one hand, using (13), it is pretty obvious that one has the inclusion $Q^\text{unif}_{\text{simple}} \subset Q^E_{\text{r-cont}}$. On the other hand, besides the continuity property (i) above, the maps (10) are also continuous at points in $Q^E_{\text{r-cont}}$.

**Theorem 4.12.** Suppose $C_0(Q) \subset A$ is a non-degenerate central inclusion, and $E : A \to C_0(Q)$ is a conditional expectation. If $A$ is separable and $E$ is essentially faithful, then $Q^E_{\text{r-cont}}$ is a dense $G_\delta$-subset of $Q$.

**Proof.** Fix some dense sequence $\{a_n\}_{n \in \mathbb{N}}$ in $A^+$ (the set of positive elements in $A$), which also satisfies
\[ p^E_q(a_n) > 0, \ \forall q \in Q, \ \forall n \in \mathbb{N}. \quad (19) \]
(This can be achieved with the help of a strictly positive function $f \in C_0(Q)$ and replacing, if necessary, $a_n$ with $a_n + \frac{1}{n}f$. Since $f$ is strictly positive, for each $q \in Q$, it follows that $\Gamma_{ev_q \circ E}(f)$ is an invertible positive operator, namely $f(q)I$.)

Define, for every $r \in (0, 1) \cap \mathbb{Q}$ and every $n \in \mathbb{N}$, the set
\[ F_{n,r} = \{ q \in Q : p^E_q(a_n) \leq r \cdot p^\text{unif}_{q_0}(a_n) \}. \]

By (15) and by density, it is obvious that
\[ Q^E_{\text{r-cont}} = Q \setminus \bigcup_{r,n} F_{n,r}, \]
so the desired conclusion will follow from Baire’s Theorem, once we establish the following:

**Claim.** For each $r \in (0, 1) \cap \mathbb{Q}$ and $n \in \mathbb{N}$, the set $F_{n,r}$ is closed and has empty interior.
The fact that $F_{n,r}$ is closed, is quite clear, since we can write its complement as

$$Q \setminus F_{n,r} = \{ q \in Q : p_q^E(a_n) > r \cdot p_q^{-\text{unif}}(a_n) \} =$$

$$= \bigcup_{t \in (0, \infty)} \{ q \in Q : p_q^E(a_n) > rt \} \cap \{ q \in Q : p_q^{-\text{unif}}(a_n) < t \},$$

with $V_t$'s open by lower semicontinuity of (17), and the $W_t$'s open by upper semicontinuity of (10).

To prove that $F_{n,r}$ has empty interior, argue by contradiction, assuming the existence of some non-identically zero non-negative continuous function $h \in C_0(Q)$ with $\text{supp } h \subset \text{Int}(F_{n,r})$, which gives us a positive element $0 \neq a = ha_n \in A$, such that

(a) $p_q^E(a) = h(q)p_q^E(a_n) \leq r \cdot h(q)p_q^{-\text{unif}}(a_n) = r \cdot p_q^{-\text{unif}}(a)$, \forall $q \in \text{Int}(F_{n,r})$,

(b) $p_q^E(a) = p_q^{-\text{unif}}(a) = 0$, \forall $q \in Q \setminus \text{Int}(F_{n,r})$,

thus yielding

$$0 \neq \|a\| = \sup_{q \in Q} p^E_q(a) \leq r \cdot \sup_{q \in Q} p^{-\text{unif}}_q(a) = r \cdot \|a\|,$$

which is clearly impossible. (The first equality in (20) follows from the essential faithfulness of $E$.)

\[ \square \]

5. Application to étale group bundles

The main example of groupoid $C^*$-algebras that exhibit central inclusions correspond to the so-called étale group bundles (perhaps more suitably referred to as isotropic groupoids), which are those étale groupoids $G$ on which the range and source maps $r$ and $s$ coincide. Of course, for such groupoids, we have $uGu = uG = G\{u, \forall u \in G(0)\}$, so (by our Convention set forth in Section 2) all $uG$'s are countable discrete groups.

**Convention.** In this (and only in this) section, $G$ will be assumed to be an étale group bundle.

**Remark 5.1.** Both inclusions $C_0(G(0)) \subset C^*(G)$ and $C_0(G(0)) \subset C^*_\text{red}(G)$ are non-degenerate and central.

**Notations.** Fix some unit $u \in G(0)$. We denote by $J^\text{full}_u \subset C^*(G)$ the closed two-sided ideal in $C^*(G)$ generated by $C_{0,u}(G(0)) = \{ f \in C_0(G(0)) : f(u) = 0 \}$. (In Section 4 this ideal was denoted by $J^\text{unif}_u$; we use this modified notation in order to distinguish between the two possible central inclusions: $C_0(G(0)) \subset C^*(G)$ and $C_0(G(0)) \subset C^*_\text{red}(G)$.) The ideal $\pi^\text{full}_u(J^\text{full}_u) \subset C^*_\text{red}(G)$ will be denoted by $J^\text{other}_u$. Equivalently, $J^\text{other}_u$ is the closed two-sided ideal in $C^*_\text{red}(G)$ generated by $C_{0,u}(G(0))$. The corresponding $C^*$-semimnorms on $C^*(G)$ and $C^*_\text{red}(G)$ will be denoted by $p^\text{full}_u$ and $p^\text{other}_u$, respectively.

As we have a conditional expectation $E^\text{red}_u : C^*_\text{red}(G) \to C_0(G(0))$, we also have (as defined in Section 4) a $C^*$-semimnorm $p^E_u$ on $C^*_\text{red}(G)$, defined by $p^E_u(p_u) = \|\Gamma_{ev_u \circ E^\text{red}_u}(a)\|$. On the full $C^*$-algebra, the corresponding $C^*$-semimnorm is $p^E_u = p^E_u \circ \pi^\text{full}_u$. In particular, this gives rise to a canonical *-isomorphism $C^*(G)/\ker p^E_u \simeq C^*_\text{red}(G)/\ker p^E_u$.

**Remark 5.2.** For each unit $u \in G(0)$, the quotient $C^*(G)/J^\text{full}_u$ is canonically identified with $C^*(G\{u\})$ – the full group $C^*$-algebra of $G\{u\}$. This identification is constructed in two steps. First, we choose, for any $\gamma \in G\{u\}$, some open bisection $\gamma \in B_\gamma \subset G$ and some $n_\gamma \in C_c(B_\gamma)$, with $n_\gamma(\gamma) = 1$, and let $w_\gamma = n_\gamma(\text{mod } J^\text{full}_u) \in C^*(G)/J^\text{full}_u$. (If $(n', B')$ is another such pair
- i.e. $\gamma \in B' \subset G$ is some bisection, and $n' \in C_c(B')$ is some function with $n'(\gamma) = 1$, then $n' - n_\gamma \in J_u^{\text{full}}$, thus $w_e$ does not depend on the choice of $B_\gamma$ and $n_\gamma$.) It is quite routine to check that $C^*(G)/J_u^{\text{full}}$ is unital, with unit $w_e$ (with $e$ the neutral element in $G_u$), and furthermore, the map

$$G_u \ni \gamma \mapsto w_\gamma \in \mathcal{U}(C^*(G)/J_u^{\text{full}})$$

is a group homomorphism. Next, if $f \in C_c(G)$, then $f - \sum_{\gamma \in G_u \cap \text{supp} f} f(\gamma) n_\gamma \in J_u^{\text{full}}$, so (21) establishes a surjective $*$-homomorphism $\Theta_u^{\text{full}} : C^*(G_u) \to C^*(G)/J_u^{\text{full}}$. Secondly, if we denote by $(v_\gamma)_{\gamma \in G_u}$ the standard unitary generators of $C^*(G_u)$, then the map

$$C_c(G) \ni f \mapsto \sum_{\gamma \in G_u \cap \text{supp} f} f(\gamma) v_\gamma \in C^*(G_u)$$

defines a $*$-representation of $C_c(G)$, which yields a $*$-homomorphism

$$\epsilon_u^{\text{full}} : C^*(G) \to C^*(G_u).$$

The desired conclusion now follows from the pretty obvious observation that $\epsilon_u^{\text{full}} \circ \Theta_u^{\text{full}} = \text{Id}$, which implies that $\Theta_u^{\text{full}}$ is in fact a $*$-isomorphism.

**Remark 5.3.** For each unit $u \in G^{(0)}$, the quotients $C^*(G)/\ker p_u^{\text{red}} \simeq C^*_{\text{red}}(G)/\ker p_u^{\text{red}}$ are canonically identified with $C^*_u(G_u)$ -- the reduced group $C^*$-algebra of $G_u$. First of all, since $\text{ev}_u \circ \mathcal{E}$ vanishes on $J_u^{\text{full}}$, it factors through the quotient $*$-homomorphism $q_u : C^*(G) \to C^*(G)/J_u^{\text{full}}$, so we can write $\text{ev}_u \circ \mathcal{E} = \phi_u \circ q_u$, for some state $\phi_u$ on $C^*(G)/J_u^{\text{full}}$. The desired conclusion now follows from the observation that, if $\Theta_u^{\text{full}} : C^*(G_u) \to C^*(G)/J_u^{\text{full}}$ is the $*$-isomorphism defined in the preceding Remark, then $\phi_u \circ \Theta_u^{\text{full}}$ coincides with the canonical trace $\tau_{G_u}$ on $C^*(G_u)$ (defined on the canonical generators by $\tau_{G_u}(v_\gamma) = \delta_{\epsilon, \gamma}$).

**Comment.** To summarize the preceding two Remarks, if $u \in G^{(0)}$ is some unit, we have two $*$-isomorphisms $\Theta_u^{\text{full}}, \Theta_u^{\text{red}}$ and two quotient $*$-homomorphisms $\theta_u, \omega_u$

$$C^*(G_u) \xrightarrow{\Theta_u^{\text{full}}} C^*(G)/J_u^{\text{full}} \xrightarrow{\theta_u} C^*_{\text{red}}(G)/J_u \xrightarrow{\omega_u} C^*_{\text{red}}(G)/\ker p_u^{\text{red}} \xleftarrow{\Theta_u^{\text{red}}} C^*(G_u),$$

which yield a commutative diagram

$$\begin{array}{ccc}
C^*(G) & \xrightarrow{\pi_{\text{red}}} & C^*_{\text{red}}(G) \\
\downarrow & & \downarrow \\
C^*(G)/J_u^{\text{full}} & \xrightarrow{\theta_u} & C^*_{\text{red}}(G)/J_u \\
\Theta_u^{\text{full}} & & \Theta_u^{\text{red}}
\end{array}$$

The bottom horizontal arrow designates the standard quotient given by the identification $C^*_{\text{red}}(G_u) = C^*(G_u)/K_{\tau_{G_u}}$. Following the second row, we can split the bottom rectangle into two commutative sub-diagrams (which employ a third $*$-isomorphism $\Theta_u^{\text{other}}$)

$$\begin{array}{ccc}
C^*(G)/J_u^{\text{full}} & \xrightarrow{\theta_u} & C^*_{\text{red}}(G)/J_u \\
\Theta_u^{\text{full}} & & \Theta_u^{\text{other}}
\end{array}$$

$$(C^*(G_u) \xrightarrow{\rho_u} C^*_{\text{other},u}(G_u) \xrightarrow{\kappa_u} C^*_{\text{red}}(G_u)).$$
where the bottom arrows $\rho_u$ and $\kappa_u$ are surjective $*$-homomorphisms, and the intermediary $C^*$-algebra $C^*_{\text{other},u}(G_u)$ is understood as a quotient of $C^*(G_u)/W_{G,u}$ the full group $C^*$-algebra $C^*(G_u)$ by the ideal $W_{G,u} = (\Theta_u^{\text{full}})^{-1}(\ker \theta_u)$, which is contained in $K_{\tau_{G_u}}$.

Equivalently, we have a commutative diagram

$$
\begin{array}{c}
C^*(G) \\ \downarrow \epsilon^\text{full}_u \\
C^*(G_u) \\ \downarrow \rho_u \\
C^*_{\text{other},u}(G_u) \\ \downarrow \epsilon^\text{other}_u \\
C^*_{\text{other},u}(G_u) \\ \downarrow \epsilon^\text{red}_u \\
C^*_{\text{red}}(G_u)
\end{array}
$$

(22)

involving three surjective $*$-homomorphisms $\epsilon^\text{full}_u$, $\epsilon^\text{other}_u$, and $\epsilon^\text{red}_u$, having $\ker \epsilon^\text{full}_u = J^\text{full}_u$, $\ker \epsilon^\text{other}_u = J^\text{other}_u$, and $\ker \epsilon^\text{red}_u = \ker p^\text{red}_u$.

The $C^*$-algebras $C^*(G_u)$, $C^*_{\text{other},u}(G_u)$ and $C^*_{\text{red}}(G_u)$ can all be understood as completions of the group $*$-algebra $C[G_u]$ with respect to the $C^*$-norms $\| \cdot \|_{\text{full}}$, $\| \cdot \|_{\text{other},u}$ and $\| \cdot \|_{\text{red}}$, which satisfy

$$
\| \cdot \|_{\text{full}} \geq \| \cdot \|_{\text{other},u} \geq \| \cdot \|_{\text{red}}.
$$

We caution the reader that, unlike $\| \cdot \|_{\text{full}}$ and $\| \cdot \|_{\text{red}}$, the intermediary $C^*$-norm $\| \cdot \|_{\text{other},u}$ is not intrinsically defined in terms of $G_u$ alone, as it depends on the particular way the group $G_u$ sits in $G$.

Furthermore (as seen in [5, Prop 3]), there are examples available in the literature in which the norms $\| \cdot \|_{\text{other},u}$ and $\| \cdot \|_{\text{red}}$ may fail to coincide.

**Remark 5.4.** As the expectation $E_{\text{red}} : C^*_{\text{red}}(G) \to C_0(G^{(0)})$ is essentially faithful (in fact honestly faithful), using the terminology and the results from Section 4, the inclusion $C_0(G^{(0)}) \subset C^*_{\text{red}}(G)$ exhibits the following properties

A. The set of **continuously reduced units**, described (see Lemma 4.3, Proposition 4.10, and Definition 4.11) in five equivalent ways as

$$
\mathcal{G}_{r-\text{cont}}^{(0)} = \{ u \in G^{(0)} : \lim_{v \to u} p^\text{red}_v(f) = p^\text{red}_u(f), \quad \forall f \in C_c(G) \} =
$$

$$=
\{ u \in G^{(0)} : \limsup_{v \to u} p^\text{red}_v(f) \leq p^\text{red}_u(f), \quad \forall f \in C_c(G) \} =
$$

$$=
\{ u \in G^{(0)} : J^\text{red}_u = \ker p^\text{red}_u \} = \{ u \in G^{(0)} : \kappa_u \text{ is an isomorphism} \} =
$$

$$=
\{ u \in G^{(0)} : \text{the norms } \| \cdot \|_{\text{other},u} \text{ and } \| \cdot \|_{\text{red}} \text{ coincide on } C[G_u] \}
$$

is a dense $G_\delta$ set in $G^{(0)}$.

B. If $G_u$ is **amenable**, then the norms $\| \cdot \|_{\text{full}}$, $\| \cdot \|_{\text{other},u}$ and $\| \cdot \|_{\text{red}}$ all coincide on $C[G_u]$, thus $u \in \mathcal{G}_{r-\text{cont}}^{(0)}$.

C. If the set, defined in two equivalent ways as

$$
\mathcal{X}_G = \{ u \in G^{(0)} : C^*_{\text{other},u}(G_u) \text{ is a simple } C^*-\text{algebra} \} =
$$

$$=
\{ u \in \mathcal{G}_{r-\text{cont}}^{(0)} : G_u \text{ is a } C^*-\text{simple group} \text{ (i.e. } C^*_{\text{red}}(G_u) \text{ is simple}) \}
$$

is dense in $G^{(0)}$, then the inclusion $C_0(G^{(0)}) \subset C^*_{\text{red}}(G)$ is essential.

In connection with Remark 5.4B, but also with an eye on Proposition 5.6 below, we conclude this section with the following construction.
**Definition 5.5.** (Assume still that $\mathcal{G}$ is isotropic.) For any open bisection $\mathcal{B} \subset \mathcal{G}$, consider the isomorphism

$$t_{\mathcal{B}} : C_c(\mathcal{B}) \ni f \mapsto f \circ (r|_{\mathcal{B}})^{-1} \in C_c(r(\mathcal{B})),$$

and let $\mathfrak{N}_c(\mathcal{G})$ denote the two-sided ideal in $C_c(\mathcal{G}, \times)$ generated by

$$\bigcup_{\mathcal{B} \subset \mathcal{G}} \{ f - t_{\mathcal{B}}(f) : f \in C_c(\mathcal{B}) \} \subset C_c(\mathcal{G}).$$

When we view $C_c(\mathcal{G}) \subset C^*(\mathcal{G})$, we denote the norm-closure of $\mathfrak{N}_c(\mathcal{G})$ by $\mathfrak{N}_{\text{full}}(\mathcal{G})$; but when we view $C_c(\mathcal{G}) \subset C^*_\text{red}(\mathcal{G})$, we denote the norm-closure of $\mathfrak{N}_c(\mathcal{G})$ by $\mathfrak{N}_{\text{red}}(\mathcal{G})$. Each one of these ideals – either in $C_c(\mathcal{G})$, or in $C^*(\mathcal{G})$, or in $C^*_\text{red}(\mathcal{G})$ – will be referred to as the augmentation ideal.

**Proposition 5.6.** Assume $\mathcal{G}$ is isotropic.

(i) The augmentation ideal $\mathfrak{N}_c(\mathcal{G})$ (and consequently both $\mathfrak{N}_{\text{full}}(\mathcal{G})$ and $\mathfrak{N}_{\text{red}}(\mathcal{G})$) is non-zero, if and only if $\mathcal{G}u_0 \neq \{e\}$ for some $u_0 \in \mathcal{G}^{(0)}$; equivalently, $\mathcal{G}^{(0)} \subset \mathcal{G}$.

(ii) One has a strict inclusion: $\mathfrak{N}_{\text{full}}(\mathcal{G}) \subset C^*(\mathcal{G})$.

(iii) If there exists $u_0 \in \mathcal{G}^{(0)}$, for which the augmentation homomorphism $\mathbb{C}[\mathcal{G}u_0] \to \mathbb{C}$ (defined by the trivial representation of the group $\mathcal{G}u_0$) is continuous relative to the norm $\| \cdot \|_{\text{other},u_0}$, then one also has a strict inclusion $\mathfrak{N}_{\text{red}}(\mathcal{G}) \subset C^*_\text{red}(\mathcal{G})$.

(A sufficient condition for the hypothesis in statement (iii) to hold true is that $\mathcal{G}u_0$ is amenable, for some $u_0 \in \mathcal{G}^{(0)}$.)

**Proof.** Statement (i) is pretty obvious. As for (ii) and (iii), a unified proof can be provided as follows. Denote by $\mathfrak{N}$ either one of $\mathfrak{N}_{\text{full}}(\mathcal{G})$ or $\mathfrak{N}_{\text{red}}(\mathcal{G})$, denote by $\mathfrak{A}$ either one of $C^*(\mathcal{G})$ – for statement (ii), or $C^*_\text{red}(\mathcal{G})$ – for statement (iii). Fix also some unit $u_0 \in \mathcal{G}^{(0)}$, which either satisfies the hypothesis in (iii), or is arbitrary otherwise, denote by $\mathfrak{A}_{u_0}$ one of $C^*(\mathcal{G}u_0)$ – for (ii), or $C^*_\text{other,}u_0(\mathcal{G}u_0)$ – for (iii), and lastly let $\epsilon_{u_0} : \mathfrak{A} \to \mathfrak{A}_{u_0}$ denote one of the maps from (22). The trivial group representation $\mathcal{G}u_0 \ni \gamma \mapsto 1 \in \mathbb{T}$ induces an augmentation $\ast$-homomorphism $a_{u_0} : \mathfrak{A}_{u_0} \to \mathbb{C}$. (In statement (ii) this is obvious; in statement (iii) this follows from the stated hypothesis.) Now the desired conclusion simply follows from the observation that, for each open bisection $\mathcal{B}$, we have the inclusion

$$\{ f - t_{\mathcal{B}}(f) : f \in C_c(\mathcal{B}) \} \subset \ker(a_{u_0} \circ \epsilon_{u_0}),$$

which in turn implies $\mathfrak{N} \subset \ker(a_{u_0} \circ \epsilon_{u_0})$. □

6. Minimality and Simplicity

In the context of (traditional) dynamical systems, an action $G \curvearrowright X$ of a group $G$ on a locally compact space $X$ is said to be **minimal**, if there do not exist any non-trivial (i.e. non-empty strict subset) open $G$-invariant subsets of $X$. This notion is extended to arbitrary etale groupoids $\mathcal{G}$, by making a similar requirement on the unit space $\mathcal{G}^{(0)}$ (see section 7 for more on this). Minimality is also extended to actions $G \curvearrowright B$ of a group $G$ by automorphisms of a $C^*$-algebra $B$, by requiring the non-existence of non-trivial $G$-invariant ideals $\{0\} \neq J \lhd B$. In what follows (see Definition 6.6.B) we propose yet another generalization of minimality, for non-degenerate of $C^*$-inclusions.

**Notation.** Suppose $B \subset A$ is a $C^*$-inclusion. An element $n \in A$ is called a $B$-**normalizer**, if $nBn^* \cup n^*Bn \subset B$. The collection of all these normalizers is denoted by $\mathcal{N}_A(B)$. 
Remark 6.1. It is fairly obvious that $\mathcal{N}_A(B)$ is norm-closed; furthermore it is also a $*$-monoid, in the sense that

(a) $n \in \mathcal{N}_A(B) \Rightarrow n^* \in \mathcal{N}_A(B)$;
(b) $n_1, n_2 \in \mathcal{N}_A(B) \Rightarrow n_1 n_2 \in \mathcal{N}_A(B)$.

Remark 6.2. In the case of a non-degenerate inclusion $B \subset A$, normalizers are supported in $B$, in the sense that: $nn^*, n^*n \in B, \forall n \in \mathcal{N}(B)$.

Definition 6.3. Suppose $B \subset A$ is a non-degenerate $C^*$-inclusion. Denote the $C^*$-subalgebra of $A$ generated by $\mathcal{N}_A(B)$ by $A^{reg}$. The resulting non-degenerate inclusion $B \subset A^{reg}$ is called the regular part of $B \subset A$.

Comment. The above terminology is consistent with the standard one, by which a non-degenerate inclusion $B \subset A$ is termed regular, if $A^{reg} = A$. Using this language, and the obvious equality

$$\mathcal{N}_{A^{reg}}(B) = \mathcal{N}_A(B).$$

(23)

it follows that $A^{reg}$ is the largest $C^*$-subalgebra of $A$ that makes the inclusion $B \subset A^{reg}$ regular.

By Remark 6.1 it is also fairly obvious that

$$A^{reg} = \overline{\text{span}}\mathcal{N}_A(B).$$

(24)

In order to achieve more flexibility, it is helpful to consider normalizers sitting in multiplier algebras, the properties of which are collected in Proposition 6.4 below.

Proposition 6.4. Assume $B \subset A$ is a non-degenerate $C^*$-inclusion, and let $M(B) \subset M(A)$ be the associated multiplier algebra inclusion (as described in Remark 1.4).

(i) $\mathcal{N}_{M(A)}(B)$ is a norm-closed sub-$*$-semigroup (i.e. a unital $*$-monoid) of $M(A)$; furthermore, if $n \in \mathcal{N}_{M(A)}(B)$ is invertible in $M(A)$, then $n^{-1} \in \mathcal{N}_{M(A)}(B)$.

(ii) $\mathcal{N}_A(B) = \mathcal{N}_{M(A)}(B) \cap A = BN_{M(A)}(B)B = B\mathcal{N}_A(B)B$.

(iii) $\mathcal{N}_{M(A)}(M(B)) = \mathcal{N}_{M(A)}(B)$; in particular, $nn^*, n^*n \in M(B), \forall n \in \mathcal{N}_{M(A)}(B)$.

Proof. Fix an approximate unit $(u_\lambda)_\lambda \subset B$ for $A$, and let us first prove (iii). The inclusion $\mathcal{N}_{M(A)}(M(B)) \subset \mathcal{N}_{M(A)}(B)$ is fairly obvious, by Remark 1.4 because whenever $n \in \mathcal{N}_{M(A)}(M(B))$ and $b \in B$, the elements $nb^n$ and $n^*bn$ clearly belong to both $M(B)$ and $A \unlhd M(A)$, so $nbn^*, n^*bn \in M(B) \cap A = B$. For the reverse inclusion $\mathcal{N}_{M(A)}(B) \subset \mathcal{N}_{M(A)}(M(B))$, fix for the moment some arbitrary elements $n \in \mathcal{N}_{M(A)}(B)$, $x \in M(B)$, and $b \in B$, and notice that, since we have $nxn^*b = \lim_x nxu_\lambda(n^*b)$ (with the element in parenthesis representing an element in $A$), as well as $xu_\lambda \in M(B) \cap B \subset B$, it follows that $n(xu_\lambda)n^*b \in \overline{nBn^*b \subset Bb \subset B}$, so taking the limit, it follows that $nxn^*b \in B$. Since $b \in B$ was arbitrary, we can conclude that the multiplier $m = nxn^* \in M(A)$ satisfies $mB \subset B$, so by Remark 1.4 it follows that $nxn^* \in M(B)$, and likewise, $n^*xn \in M(B)$. Letting $x \in M(B)$ vary, this simply shows that $nM(B)n^* \cup n^*M(B)n \subset M(B)$, so $n \in \mathcal{N}_{M(A)}(M(B))$.

The second statement from (iii), as well as the first statement from (i) are now obvious, by Remark 6.2 applied to the unital (thus non-degenerate) inclusion $M(B) \subset M(A)$. The first statement in (i) is also trivial. As for the second statement from (i), assume $n \in \mathcal{N}_{M(A)}(B)$ is invertible, and let us consider its polar decomposition $n = u(n^*n)^{1/2}$, where $u$ is a unitary in $M(A)$, and $(n^*n)^{1/2}$ is a positive invertible element in $M(B)$. Since $M(B) \subset \mathcal{N}_{M(A)}(M(B)) = \mathcal{N}_{M(A)}(B)$, it follows that $(n^*n)^{-1/2} \in \mathcal{N}_{M(A)}(B)$, so by the semigroup property, the unitary
\( u = n(n^*n)^{-1/2} \) also belongs to \( \mathcal{N}_{M(A)}(B) \), and then again by the \(*\)-semigroup property \( n^{-1} = (n^*n)^{-1/2}u^* \) also belongs to \( \mathcal{N}_{M(A)}(B) \).

(ii) First of all, the equality \( \mathcal{N}_A(B) = \mathcal{N}_{M(A)}(B) \cap A \) is trivial from the definition. Secondly, the inclusion \( B\mathcal{N}_{M(A)}(B)B \subset \mathcal{N}_A(B) \) is fairly obvious, from the inclusion by the \( B\mathcal{N}_{M(A)}(B)B \subset \mathcal{N}_{M(A)}(B) \) (which follows from the semigroup property, as \( B \subset M(B) \subset \mathcal{N}_{M(A)}(B) \)), which combined with the observation that \( BM(A)B \subset AM(A)A \subset A \), implies that \( B\mathcal{N}_{M(A)}(B)B \subset \mathcal{N}_{M(A)}(B) \cap A = \mathcal{N}_A(B) \).

So far, we have \( \mathcal{N}_A(B) = \mathcal{N}_{M(A)}(M(B)) \cap A \supset B\mathcal{N}_{M(A)}(M(B))B \supset B\mathcal{N}_A(B)B \), so all that remains to be justified is the factorization inclusion \( \mathcal{N}_A(B) \subset B\mathcal{N}_A(B) \), for which (by applying adjoints) it suffices to prove the inclusion \( \mathcal{N}_A(B) \subset B\mathcal{N}_A(B) \). Fix \( n \in \mathcal{N}_A(B) \), and consider the elements \( x_n = ((nn^*)^{1/4} + \varepsilon 1)^{-1} \in M(B) \subset \mathcal{N}_{M(A)}(B) \), \( \varepsilon > 0 \), as well as the products \( y_n = x_n \in A \cap \mathcal{N}_{M(A)}(B) = \mathcal{N}_A(B) \). A simple calculation shows that

\[
\|y_n - y_{\delta}\|^2 = \left\|n^* \left( ((nn^*)^{1/4} + \varepsilon 1)^{-1} - ((nn^*)^{1/4} + \delta 1)^{-1} \right) \right\|^2 = \|f_{\varepsilon, \delta}(n^*n)\|,
\]

where \( f_{\varepsilon, \delta}(t) = t \left( \frac{1}{\sqrt[4]{t + \varepsilon}} - \frac{1}{\sqrt[4]{t + \delta}} \right) = (\varepsilon - \delta)^2 \left( \frac{t}{(\sqrt[4]{t + \varepsilon})^2(\sqrt[4]{t + \delta})^2} \right)^2 \). Since \( |f_{\varepsilon, \delta}(t)| \leq 1 \), \( \forall t \geq 0 \), it follows that \( \|y_n - y_{\delta}\| \leq |\varepsilon - \delta| \), so the limit \( y = \lim_{\delta \to 0} y_n \) exists in \( A \), so it defines an element \( y \in \mathcal{N}_A(B) \). By construction, we have \( n = ((nn^*)^{1/4} + \varepsilon 1) y_n \), so taking limit we get \( n = (nn^*)^{1/4} y \in B\mathcal{N}_A(B) \).

We also have the following multiplier version of (23).

**Proposition 6.5.** Suppose \( B \subset A \) is a non-degenerate inclusion, let \( B \subset A^{\text{reg}} \) be its regular part. When one considers the associated multiplier inclusions \( M(B) \subset M(A^{\text{reg}}) \subset M(A) \) (as described in Remark 1.4), the following equality holds:

\[
\mathcal{N}_{M(A^{\text{reg}})}(B) = \mathcal{N}_{M(A)}(B).
\]

**Proof.** The inclusion “\( \subset \)" being trivial, we only need to check the reverse inclusion “\( \supset \)”, which amounts to proving the inclusion \( \mathcal{N}_{M(A)}(B) \subset M(A^{\text{reg}}) \), or equivalently, to showing that \( \mathcal{N}_{M(A)}(B) \cdot A^{\text{reg}} \subset A^{\text{reg}} \). By (25), it suffices to prove the inclusion \( \mathcal{N}_{M(A)}(B) \cdot \mathcal{N}_A(B) \subset \mathcal{N}_A(B) \), which is immediate from Proposition 6.4 and Remark 6.4. \( \square \)

**Definitions 6.6.** Suppose \( B \subset A \) is a non-degenerate \( *\)-inclusion.

A. An ideal \( J \lhd B \) is said to be **fully normalized in \( A \), relative to \( B \)**, if \( \mathcal{N}_{M(A)}(B) \subset \mathcal{N}_{M(A)}(J) \).

B. We declare the inclusion \( B \subset A \) **minimal**, if the only ideals \( J \lhd B \), that are fully normalized in \( A \) relative to \( B \), are the trivial ones: \( J = \{0\} \) and \( J = B \).

**Remark 6.7.** For an ideal \( J \lhd B \), the condition \( \mathcal{N}_{M(A)}(B) \subset \mathcal{N}_{M(A)}(J) \), that appears in the above Definition, is equivalent to the inclusion \( \mathcal{N}_A(B) \subset \mathcal{N}_A(J) \). Indeed, if \( \mathcal{N}_{M(A)}(B) \subset \mathcal{N}_{M(A)}(J) \), then \( \mathcal{N}_A(B) = A \cap \mathcal{N}_{M(A)}(B) \subset A \cap \mathcal{N}_{M(A)}(J) = \mathcal{N}_A(J) \). Conversely, if \( \mathcal{N}_A(B) \subset \mathcal{N}_A(J) \), and \( m \in \mathcal{N}_{M(A)}(B) \), then for every \( x \in J \), we can write \( mxm^* = \lim_{\lambda \to 0}(mu_\lambda)(x(u_\lambda m^*)) \), for some approximate unit \( (u_\lambda)_\lambda \) for \( B \), with both monomials in parentheses being elements in \( B\mathcal{N}_{M(A)}(B) \subset \mathcal{N}_A(B) \subset \mathcal{N}_A(J) \), thus proving that \( mxm^* \in J \) (as well as \( m^*xm \in J \), by same argument applied to \( m^* \)); this argument shows that \( \mathcal{N}_{M(A)}(B) \subset \mathcal{N}_{M(A)}(J) \).

**Remark 6.8.** For any ideal \( L \lhd A \), the intersection \( B \cap L \lhd B \) is fully normalized in \( A \), relative to \( B \). Indeed, for any \( n \in \mathcal{N}_A(B) \) and any \( x \in B \cap L \), the products \( nxn^* \) and \( n^*xn \)
both belong to $B$ (since $n$ normalizes $B$) as well as to $L$ (which is a two-sided ideal), thus proving the inclusion $\mathcal{N}_A(B) \subset \mathcal{N}_A(B \cap L)$.

When verifying (non-)minimality, the following result is particularly useful.

**Lemma 6.9.** Assume $B \subset A$ is a non-degenerate $C^*$-inclusion. If an ideal $J \subset B$ satisfies the condition

(sn) $\mathcal{N}_A(B) \cap \mathcal{N}_A(J)$ generates $A^{\text{reg}}$ as a $C^*$-subalgebra of $A$, then $L = \text{span}\{a_1x_2 : x \in J, a_1, a_2 \in A^{\text{reg}}\}$ is a proper ideal $L \subset A^{\text{reg}}$.

In particular (by the preceding Remark), any ideal $J \subset B$ satisfying condition (sn) is contained in some ideal $J' \subset B$, which is fully normalized in $A$ relative to $B$.

An ideal satisfying condition (sn) is called **sufficiently normalized in $A$, relative to $B$**.

**Proof.** By construction, $L$ is clearly a closed two-sided ideal in $A^{\text{reg}}$, which contains $J$, so it suffices to prove that $L \subsetneq A^{\text{reg}}$. (By non-degeneracy, this will also ensure the strict inclusion $B \cap L \subsetneq B$.)

Fix a state $\varphi \in S(B)$ which vanishes on $J$, let $\psi \in S(A^{\text{reg}})$ be an extension of $\varphi$ to a state on $A^{\text{reg}}$, and let $K_\psi \subset A^{\text{reg}}$ be the kernel of the GNS representation $\Gamma_\psi : A^{\text{reg}} \to \mathcal{B}(L^2(A^{\text{reg}}, \psi))$. Since $K_\psi$ is a proper ideal, the desired conclusion will follow if we prove the inclusion $J \subset K_\psi$ (because this will also force $L \subset K_\psi$). In other words, all we need is to prove that $\Gamma_\psi$ vanishes on $J$, which amounts to showing that

$$\|xa\|_{2,\psi} = 0, \ \forall x \in J, a \in A^{\text{reg}}.$$  \hfill (26)

By Remark 6.1 combined with condition (sn), we know that $A^{\text{reg}} = \text{span}\{\mathcal{N}_A(B) \cap \mathcal{N}_A(J)\}$, so it suffices to verify (26) only for $a \in \mathcal{N}_A(B) \cap \mathcal{N}_A(J)$, which in turn is equivalent to checking that

$$\psi(a^*x^*xa) = 0, \ \forall x \in J, a \in \mathcal{N}_A(B) \cap \mathcal{N}_A(J).$$  \hfill (27)

However, the above equalities are trivial, since $\psi\big|_J = \varphi\big|_J = 0$ and $a^*x^*xa \in a^*Ja \subset J, \ \forall x \in J, a \in \mathcal{N}_A(J)$.

**Proposition 6.10.** Suppose $B \subset A$ is a non-degenerate $C^*$-inclusion, which is regular. Then the following conditions are equivalent.

(i) The ambient $C^*$-algebra $A$ is simple.

(ii) The inclusion $B \subset A$ is essential and minimal.

**Proof.** (i) $\Rightarrow$ (ii). Assume $A$ is simple. The fact that $B \subset A$ is essential is obvious. Minimality is also fairly obvious, for if $J \subset B$ is fully normalized in $A$, relative to $B$, then by Lemma 6.9 there is some (proper) ideal $L \subset A$, such that $J \subset L$, which by simplicity forces $L = \{0\}$, thus $J = \{0\}$ as well.

(ii) $\Rightarrow$ (i). Assume now $B \subset A$ is essential and minimal, and let us prove that $A$ is simple. Fix some proper ideal $L \subset A$, and let us justify that $L$ must be equal to $\{0\}$. Since $B \subset A$ is non-degenerate, the intersection $B \cap L$ is a proper ideal in $B$. As $B \subset A$ is essential, it suffices to prove that $B \cap L = \{0\}$. However, this is immediate from minimality, since by Remark 6.8 $B \cap L \subset B$ is fully normalized in $A$ relative to $B$.

**Example 6.11.** Suppose $\alpha : G \ni g \mapsto \alpha_g \in \text{Aut}(B)$ is an action of a discrete group $G$ by automorphisms on a $C^*$-algebra $B$. The full (resp. reduced) crossed product $C^*$-algebra constructions yield two non-degenerate regular inclusions $B \subset B \rtimes_{\alpha} G$ and $B \subset B \rtimes_{\alpha, \text{red}} G$. If $A$ denotes either one of these $C^*$-algebras, we always have a group homomorphism $G \ni \alpha_g \mapsto $
g ↦→ u_g ∈ \mathcal{U}(M(A))$, and a canonical subset \( \mathcal{K} = \{ bu_g : b ∈ B, g ∈ G \} \subset N_A(B) \), which generates \( A \) as a C*-algebra. Within this framework, the inclusion \( B ⊂ A \) is minimal (in the sense of Definition 6.6.B), if and only if \( \alpha \) is a minimal action, in the sense that \( B \) contains no non-zero ideal \( J < B \) such that \( α_g(J) = J, \forall g ∈ G \).

As Archbold and Spielberg showed ([1]), a sufficient condition for making the inclusion \( B ⊂ B \prec_{α, red} G \) essential is that the action \( \alpha \) is topologically free.

7. Applications to Groupoid C*-Algebras

**Convention.** Besides the standard conditions set forth at the beginning of Section 2, all groupoids mentioned in this section are also assumed to be étale.

**Remark 7.1.** As pointed out in [16], in the case of an étale groupoid \( G \), the non-degenerate inclusions \( C_0(G^{(0)}) \subset C^*(G) \) and \( C_0(G^{(0)}) \subset C_{red}^*(G) \) are regular, since a large supply of normalizers consists of functions supported on bisections. Specifically, if we consider the space

\[
\mathcal{N}(G) = \bigcup_{B ∈ \mathcal{G}} C_c(B) \subset C_c(G),
\]

then all functions in \( \mathcal{N}(G) \) (hereafter referred to as elementary normalizers) normalize \( C_0(G^{(0)}) \) in both \( C^*(G) \) and in \( C_{red}^*(G) \), and furthermore one has \( C_c(G) = \text{span} \mathcal{N}(G) \). (As above, we view \( C_c(G) \) as a dense *-subalgebra in both \( C^*(G) \) and \( C_{red}^*(G) \).) This statement is based on the observation that, whenever \( B_1, B_2 ⊂ G \) are open bisections, and \( n_j ∈ C_c(B_j), j = 1, 2 \), it follows that, for every \( f ∈ C_c(G) \), one has the equality

\[
(n_1 × f × n_2)(γ) = \begin{cases} n_1((|B_1|^{-1}(r(γ)))n_2((|B_2|^{-1}(s(γ))))f\left(\left(|B_1|^{-1}(r(γ))\right)\left(|B_2|^{-1}(s(γ))\right)\right) & \text{if } r(γ) ∈ r(B_1) \text{ and } s(γ) ∈ s(B_2) \\ 0, & \text{otherwise} \end{cases}
\]

(28)

When we specialize (28) to the particular case when \( f ∈ C_c(G^{(0)}) \), and \( n ∈ C_c(B) \) for some open bisection \( B ⊂ G \), it follows that \( n × f × n^* ∈ C_c(G^{(0)}) \), and more explicitly:

\[
(n × f × n^*)(u) = \begin{cases} |n ((|B|^{-1}(u)))|^2 f \left(s\left(|B|^{-1}(u)\right)\right), & \text{if } u ∈ r(B) \\ 0, & \text{otherwise} \end{cases}
\]

(29)

**Remark 7.2.** The formula (29) is also valid for \( f ∈ C_0(G^{(0)}) \). If \( \mathfrak{A} \) denotes any one of \( C^*(G) \) or \( C_{red}^*(G) \), using Lemma 6.9, the following conditions are equivalent.

(i) The inclusion \( C_0(G^{(0)}) \subset \mathfrak{A} \) is minimal, in the sense of Definition 6.6.B.

(ii) The only ideals \( J ≤ C_0(G^{(0)}) \) satisfying \( nJn^* ⊂ J \), \( ∀ n ∈ \mathcal{N}(G) \), are the trivial ideals: \( J = \{ 0 \} \) and \( J = C_0(G^{(0)}) \).

(iii) Every non-empty subset \( \mathcal{X} ⊂ G^{(0)} \) satisfying the inclusion

\[
r\left(s^{-1}(\mathcal{X})\right) ⊂ \mathcal{X}
\]

(30)

is dense in \( G^{(0)} \).

(iii') The only subsets \( \mathcal{X} ⊂ G^{(0)} \), which satisfy (30) and are either open or closed, are the trivial ones: \( \mathcal{X} = \emptyset \) and \( \mathcal{X} = G^{(0)} \).
According to the standard groupoid terminology, these conditions characterize the so-called minimal groupoids. (Whether \( G \) is minimal or not, a set of units satisfying (30) is termed invariant.)

Comment. By Proposition 6.10, minimality of \( G \) is always a necessary condition for the simplicity of either \( C^*(G) \) or \( C^\text{red}_*(G) \). Although it is tempting to try to fit groupoid minimality in the framework provided in Proposition 6.10, this approach will be less fruitful, especially in the reduced case.

Notations. Given an étale groupoid \( G \), we denote its isotropy groupoid, i.e. the set \( \{ \gamma \in G : r(\gamma) = s(\gamma) \} \) by \( \text{Iso}(G) \). Although \( \text{Iso}(G) \) is a nice groupoid in its own right, its interior, hereafter denoted by \( \text{IntIso}(G) \), is more manageable, because the natural inclusion \( C_c(\text{IntIso}(G)) \subset C_c(G) \) gives rise to a natural \( C^* \)-inclusion

\[
i^\text{full} : C^*(\text{IntIso}(G)) \hookrightarrow C^*(G).
\] (31)

(The existence of \( i^\text{full} \) as a \( * \)-homomorphism is obvious, by the definition of \( \| \cdot \|_\text{full} \). We believe the injectivity of \( i^\text{full} \) is well known, but in case it is not, we provide a proof in the Appendix.)

On the other hand, since \( \text{IntIso}(G) \) and \( G \) have the same unit space \( G(0) \), we also have a commutative diagram of conditional expectations

\[
\begin{array}{ccc}
C^*(\text{IntIso}(G)) & \xrightarrow{i^\text{full}} & C^*(G) \\
\uparrow{\pi_{\text{IntIso}(G)}} & & \uparrow{g^G} \\
C_0(\text{IntIso}(G)(0)) & \longrightarrow & C_0(G(0))
\end{array}
\]

which gives rise to another natural \( C^* \)-inclusion

\[
i^\text{red} : C^\text{red}_*(\text{IntIso}(G)) \hookrightarrow C^\text{red}_*(G).
\] (32)

(The injectivity of \( i^\text{red} \) can be justified, for instance, using Proposition 1.10)

Comment. We caution the reader that, for an arbitrary unit \( u \in G(0) \), one might have a strict inclusion of isotropy groups \( \text{IntIso}(G)u \subsetneq \text{Iso}(G)u(= uG/u) \); nevertheless, in this case, \( \text{IntIso}(G)u \) is always a normal subgroup of \( \text{Iso}(G)u \). However, as pointed out in [4], under our overall assumptions on \( G \), the set

\[
G_o(0) = \{ u \in G(0) : \text{IntIso}(G)u = \text{Iso}(G)u \}
\]

is a dense \( G_\delta \) subset of \( G(0) \).

Notations. We adopt the same notations as those from Section 5 for the étale group bundle \( \text{IntIso}(G) \) (with unit space \( G(0) \)), so now, for each unit \( u \in G(0) \), the commutative diagrams (22) become

\[
\begin{array}{ccc}
C^*(\text{IntIso}(G)) & \xrightarrow{\pi_{\text{IntIso}(G)}} & C^*(\text{IntIso}(G)) \\
\uparrow{e^\text{full}_u} & & \uparrow{e^\text{other}_u} \\
C^*(\text{IntIso}(G)u) & \xrightarrow{\rho_u} & C^*_\text{other,u}(\text{IntIso}(G)u)
\end{array}
\]

The three surjective \( * \)-homomorphisms pointing downwards in the above diagram have the following kernels:
Lemma 7.4. Remark

As in Section 5, by construction, we have \( \pi \). By a slight abuse in notation, we will denote \( \text{IntIso}(G)^{(0)}_{\text{r-cont}} \) simply by \( G^{(0)}_r \); in other words,

\[
G^{(0)}_r = \{ u \in G^{(0)} : \ker e^\text{other} = \ker e^\text{red} \}.
\]

Remark 7.3. Using \((28)\), we also have the following properties.

A. When we view \( \mathfrak{N}(G) \) as a subset of \( C^*(G) \), and \( C^*(\text{IntIso}(G)) \) as a \( C^* \)-subalgebra of \( C^*(G) \), all elements in \( \mathfrak{N}(G) \) normalize \( C^*(\text{IntIso}(G)) \), so the inclusion \((31)\) is regular.

B. When we view \( \mathfrak{N}(G) \) as a subset of \( C^*_r(G) \), and \( C^*_r(\text{IntIso}(G)) \) as a \( C^* \)-subalgebra of \( C^*_r(G) \), all elements in \( \mathfrak{N}(G) \) normalize \( C^*_r(\text{IntIso}(G)) \), so the inclusion \((32)\) is also regular.

Lemma 7.4. Assume \( \omega = (\gamma, \mathcal{B}, n) \) is a triple consisting of an element \( \gamma \in G \), an open bisection \( \mathcal{B} \ni \gamma \), and some function \( n \in C_*(\mathcal{B}) \) with \( n(\gamma) = 1 \).

(i) When viewing \( C_*(\mathcal{B}) \subset \mathfrak{N}(G) \subset \mathcal{N} C^*_r(G) \left( C^*(\text{IntIso}(G)) \right) \subset C^*(G) \), the map

\[
ad_n^\text{full} : C^*(\text{IntIso}(G)) \ni a \mapsto nan^* \in C^*(\text{IntIso}(G))
\]

sends the ideal \( J^\text{full}_{s(\gamma)} \) into the ideal \( J^\text{full}_{r(\gamma)} \). In particular, \((35)\) induces a unique linear map \( a^\omega : C^*(\text{IntIso}(G) s(\gamma)) \rightarrow C^*(\text{IntIso}(G) r(\gamma)) \), which makes a commutative diagram

\[
C^*(\text{IntIso}(G)) \xrightarrow{\text{ad}_n^\text{full}} C^*(\text{IntIso}(G)) \quad \text{and} \quad C^*(\text{IntIso}(G) s(\gamma)) \xrightarrow{\text{ad}_n^\text{full}} C^*(\text{IntIso}(G) r(\gamma)),
\]

(ii) Likewise (when working with the inclusion \( C^*_r(\text{IntIso}(G)) \subset C^*_r(G) \)), the map

\[
ad_n^\text{red} : C^*_r(\text{IntIso}(G)) \ni a \mapsto nan^* \in C^*_r(\text{IntIso}(G))
\]

sends \( J^\text{other}_{s(\gamma)} \) into \( J^\text{other}_{r(\gamma)} \) such that \((35)\) induces a unique linear map \( a^\omega : C^*_{\text{other}, s(\gamma)}(\text{IntIso}(G) s(\gamma)) \rightarrow C^*_{\text{other}, r(\gamma)}(\text{IntIso}(G) r(\gamma)) \), which makes a commutative diagram

\[
C^*_r(\text{IntIso}(G)) \xrightarrow{\text{ad}_n^\text{red}} C^*_r(\text{IntIso}(G)) \quad \text{and} \quad C^*_{\text{other}, s(\gamma)}(\text{IntIso}(G) s(\gamma)) \xrightarrow{\text{ad}_n^\text{red}} C^*_{\text{other}, r(\gamma)}(\text{IntIso}(G) r(\gamma)),
\]

(iii) The map \((35)\) sends the ideal \( \ker p^\text{red}_{s(\gamma)} \) into the ideal \( \ker p^\text{red}_{r(\gamma)} \), so it also induces a unique linear map \( e^\text{red} : C^*_r(\text{IntIso}(G) s(\gamma)) \rightarrow C^*_r(\text{IntIso}(G) r(\gamma)) \),
which makes a commutative diagram

\[
\begin{array}{ccc}
C^*_\text{red}(\text{IntIso}(G)) & \xrightarrow{\text{ad}^\text{red}_n} & C^*_\text{red}(\text{IntIso}(G)) \\
\text{C}^*_\text{red}(\text{IntIso}(G)s(\gamma)) & \xrightarrow{\text{ad}^\text{red}_{\text{C}}}_s & \text{C}^*_\text{red}(\text{IntIso}(G)r(\gamma))
\end{array}
\]

(iv) All maps \( a^\text{full}_\omega, a^\text{other}_\omega, a^\text{red}_\omega \) are *-isomorphisms, with inverses \( a^\text{full}_\omega, a^\text{other}_\omega, a^\text{red}_\omega \) respectively, are the corresponding maps associated with the triple \( \omega^\text{op} = (\gamma^{-1}, B^{-1}, n^*) \).

**Proof.** We prove statements (i) and (ii) simultaneously. First of all, we observe that both \( nn^* \) and \( n^*n \) belong to \( C_0(G^{(0)}) \), which is central in both \( C^*(\text{IntIso}(G)) \) and \( C^*_\text{red}(\text{IntIso}(G)) \). Secondly, if we choose (using approximations of \( \sqrt{k} \)) by polynomials without constant term, and the fact that \( n = \lim_k n(n^*n)^{1/k} \) a sequence of polynomials \( (h_k)_k \subset \mathbb{R}[\mathcal{I}] \), such that

\[
n = \lim_k nn^*nh_k(n^*n),
\]

then for every \( f \in C_{0,s(\gamma)}(G^{(0)}) \), we can write (either in \( C^*(G) \) or in \( C^*_\text{red}(G) \)):

\[
nf = \lim_k nh_k(n^*n)n^*nf = \lim_k (h_k(n^*n)f)n^*n,
\]

with \( h_k(n^*n)f \in C_{0,s(\gamma)}(G^{(0)}) \). So, if \( \mathcal{B} \subset \mathcal{A} \) is either one of the inclusions \( C^*(\text{IntIso}(G)) \subset C^*(G) \), or \( C^*_\text{red}(\text{IntIso}(G)) \subset C^*_\text{red}(G) \), then for every \( f \in C^*_0(s(\gamma))(G^{(0)}) \) and any \( b \in \mathcal{B} \), we can write

\[
nf b n^* = \lim_k (n(h_k(n^*n)f)n^*)(nbn^*),
\]

In particular, if \( \text{ad}_n : \mathcal{B} \to \mathcal{B} \) is the corresponding map (\( \text{ad}^\text{full}_n \) or \( \text{ad}^\text{red}_n \)), then in \( \mathcal{B} \) we have:

\[
\text{ad}_n(fb) = \lim_k \text{ad}_n(h_k(n^*n)f) \text{ad}_n(b),
\]

so the desired conclusion will simply follow for the inclusion

\[
\text{ad}_n \left(C_{0, s(\gamma)}(G^{(0)})\right) \subset C_{0, r(\gamma)}(G^{(0)}),
\]

which is straightforward from (29) (see also Remark 7.2).

Statement (iii) now clearly follows from (37), since by (28) we also know that that

\[
\mathcal{E}_\text{red}(nan^*) = n\mathcal{E}_\text{red}(a)n^*, \quad \forall a \in C^*_\text{red}(G).
\]

Statement (iv) is also pretty clear, since (using the unified notation as above)

\[
\text{ad}_n \circ \text{ad}_n^* = \text{ad}_{nn^*} = \text{multiplication by the central element } (n^*n)^2 \text{ on } \mathcal{B}, \quad \text{and}
\]

\[
\text{ad}_n^* \circ \text{ad}_n = \text{ad}_{n^*n} = \text{multiplication by the central element } (n^*n)^2 \text{ on } \mathcal{B},
\]

and for any one of the *-homomorphisms \( \varepsilon^*_{s(\gamma)} \) and \( \varepsilon^*_{r(\gamma)} \), the elements \( \varepsilon^*_{s(\gamma)}(n^*n) \) and \( \varepsilon^*_{r(\gamma)}(nn^*) \) act as the units in the unital \( C^* \)-algebras \( \varepsilon^*_s(\gamma)(\mathcal{B}) \) and \( \varepsilon^*_r(\gamma)(\mathcal{B}) \).

**Comment.** According to [4, Thm 3.1], the inclusion \( C^*_\text{red}(\text{IntIso}(G)) \subset C^*_\text{red}(G) \) is always essential, so by applying Proposition 6.10 and the other results from Section 6 and we obtain the following simplicity criterion.
Proposition 7.5. The reduced C*-algebra $C^*_r(G)$ is simple, if and only if the inclusion $C^*_r(\text{IntIso}(G)) \subset C^*_r(G)$ is minimal; equivalently, the only ideals $J \leq C^*_r(\text{IntIso}(G))$, satisfying

$$nJn^* \subset J, \ \forall \ n \in \mathfrak{N}(G),$$

are the trivial ideals $J = \{0\}$ and $J = C^*_r(\text{IntIso}(G))$.

We are now in position to formulate one of the main results in this paper.

Theorem 7.6. Assume there is some unit $u_0 \in G^{(0)}_{r\text{-cont}}$, for which the (discrete countable) group $\text{IntIso}(G)u_0$ is $C^*$-simple. Then the following are equivalent:

(i) $G$ is minimal;
(ii) $C^*_r(G)$ is simple.

Proof. By Proposition 6.10, the implication $(ii) \Rightarrow (i)$ is trivial (and holds true without any restrictions).

$(i) \Rightarrow (ii)$. Assume $G$ is minimal, and let us prove the simplicity of $C^*_r(G)$.

With Proposition 7.5 in mind, all we need to prove is that the inclusion $C^*_r(\text{IntIso}(G)) \subset C^*_r(G)$ is minimal. Fix some ideal $J \subset C^*_r(\text{IntIso}(G))$ satisfying (38), and let us prove that $J = \{0\}$. By the minimality assumption on $G$ (cf. Remark 7.2), the set $X = r(Gu_0)$ is dense in $G^{(0)}$. By Lemma 7.4, it follows that $X \subset G^{(0)}_{r\text{-cont}}$, and furthermore, all groups $\text{IntIso}(G)u$, $u \in X$, (whose reduced C*-algebras are *-isomorphic to $C^*_r(\text{IntIso}(G)u_0)$) are $C^*$-simple. By Remark 5.6C, it follows that the inclusion $C_0(G^{(0)}) \subset C^*_r(\text{IntIso}(G))$ is essential, so in order to prove that $J = \{0\}$ it suffices to prove that the ideal $J_0 = J \cap C_0(G^{(0)}) \subset C_0(G^{(0)})$ is the zero ideal $\{0\}$. However, this follows immediately from the minimality of $G$, since we clearly have $nJ_0n^* \subset J_0$, $\forall \ n \in \mathfrak{N}(G)$.

Comment. At this point we are unable to provide any converse statement of Theorem 7.6 (i.e. the fact that its hypothesis is also necessary condition for the simplicity of $C^*_r(G)$). However, based on Proposition 5.6 the following non-simplicity test is available.

Proposition 7.7. If $C^*_r(G)$ is simple, and there exists a unit $u_0 \in G^{(0)}$, such that the augmentation homomorphism $\mathbb{C}[\text{IntIso}(G)u_0] \to \mathbb{C}$ is continuous relative to the norm $\| \cdot \|_{\text{other},u_0}$, then $\text{IntIso}(G)u = \{e\}$, $\forall \ u \in G^{(0)}$.

Proof. First, by the assumption on $u_0$ and by Proposition 5.6, we know that the augmentation ideal $\mathfrak{U}_r(\text{IntIso}(G))$ is a proper ideal in $C^*_r(\text{IntIso}(G))$. Secondly, since we clearly have $n\mathfrak{U}_r(\text{IntIso}(G))n^* \subset \mathfrak{U}_r(\text{IntIso}(G))$, $\forall \ n \in \mathfrak{N}(G)$, by Proposition 7.5, the simplicity of $C^*_r(G)$ forces $\mathfrak{U}_r(\text{IntIso}(G)) = \{0\}$, which in turn by Proposition 5.6 forces $G^{(0)} = \text{IntIso}(G)$.

Corollary 7.8. If $C^*_r(G)$ is simple, and there is a unit $u_0 \in G^{(0)}$, such that $\text{IntIso}(G)u_0$ is amenable, then $\text{IntIso}(G)u = \{e\}$, $\forall \ u \in G^{(0)}$.

Remark 7.9. If the unit $u_0 \in G^{(0)}$ satisfying the hypothesis of Proposition 7.7 is continuously reduced, then $\text{IntIso}(G)u_0$ is necessarily amenable. This is due to the well known fact the augmentation homomorphism on a group algebra $\mathbb{C}[H]$ is $\| \cdot \|_{\text{red}}$-continuous, if and only if $H$ is amenable.

Our methodology also applies to full groupoid C*-algebras, for which we recover the following result from [2].
Corollary 7.10. (cf. [2 Thm. 5.1]) The full C*-algebra $C^*_r(G)$ is simple, if and only if all three conditions below are satisfied

(i) $G$ is minimal;
(ii) $\pi_{\text{red}} : C^*_r(G) \to C^*_{\text{red}}(G)$ is an isomorphism;
(iii) $\text{IntIso}(G)u = \{e\}$, $\forall u \in G^{(0)}$.

Proof. The implication “(i) and (ii) and (iii)” implies “$C^*_r(G)$ simple” is obvious.

Conversely, the implication “$C^*_r(G)$ simple” implies “(i) and (ii)” is also obvious, so we only need to justify the implication “$C^*_r(G)$ simple” implies “(iii)”.

To this end, we argue again as above, by observing that, if $C^*_r(G)$ is simple, then the inclusion $C^*_r(\text{IntIso}(G)) \subset C^*_r(G)$ is minimal, so the full augmentation (proper!) ideal $\mathfrak{N}_{\text{full}}(\text{IntIso}(G)) \triangleleft C^*_r(\text{IntIso}(G))$, which also satisfies

$$n\mathfrak{N}_{\text{full}}(\text{IntIso}(G))n^* \subset \mathfrak{N}_{\text{full}}(\text{IntIso}(G)), \forall n \in \mathfrak{N}(G),$$

must be the zero ideal $\mathfrak{N}_{\text{full}}(\text{IntIso}(G)) = \{0\}$, which again implies $G^{(0)} = \text{IntIso}(G)$. □

Comment. By the density of $G^{(0)}$ in $G^{(0)}$, the condition

$$\text{IntIso}(G)u = \{e\}, \forall u \in G^{(0)}, \tag{39}$$

that appears in Proposition 7.7, as well as in Corollary 7.8 and Theorem 7.10 is equivalent to the condition that the groupoid $G$ is topologically principal, in the sense that the set

$$\{u \in G^{(0)} : \text{Iso}(G)u = \{e\}\}$$

is dense in $G^{(0)}$.

Of course, if some unit $u_0 \in G^{(0)}$ has trivial isotropy $\text{Iso}(G)u_0 = \{e\}$, then it also satisfies the hypothesis of Theorem 7.6 since for any unit $u$, $\text{IntIso}(G)u$ is a normal subgroup of $\text{Iso}(G)u$.

8. Applications to étale transformation groupoids

The results from the preceding section specialize nicely to transformation groupoids, which are étale groupoids constructed as follows. One starts with a (traditional) dynamical system $G \actson Q$, which consists of a countable discrete group $G$ acting by homeomorphisms on a second countable locally compact Hausdorff space $Q$. (For any $g \in G$, the corresponding homeomorphism of $Q$ will be simply denoted by $q \mapsto gq$.) Out of this data, one equips the product space $G = G \times Q$ with the product topology and the groupoid structure obtained by defining the space of composable pairs to be

$$G^{(2)} = \{(g_1, q_1), (g_2, q_2) : g_1, g_2 \in G, q_1, q_2 \in Q, q_1 = g_2q_2\},$$

and the composition operation defined by $(g_1, q_1)(g_2, q_2) = (g_1g_2, q_2)$. The inverse operation is $(g, q) \mapsto (g^{-1}, qg)$, while the unit space $G^{(0)} = \{e\} \times Q$ is of course identified with $Q$. By this identification, the source and range maps $s, r : G \times Q \to Q$ are simply defined as $s(g, q) = q$ and $r(g, q) = gq$. For any element $\gamma = (g, q) \in G$, the set $\{g\} \times Q$ is obviously an open bisection containing $\gamma$.

The full and the reduced C*-algebras of the étale transformation groupoid $G$ are identified with the full and the reduced crossed product C*-algebras $C_0(Q) \rtimes G$ and $C_0(Q) \rtimes_{\text{red}} G$, respectively, which are constructed as follows. First of all, one considers the direct sum $\bigoplus_{g \in G} C_0(Q)$, which consists of all $G$-tuples $(a_g)_{g \in G}$ of functions in $C_0(Q)$, with $a_g = 0$, for all but finitely many $g \in G$, and we endow it with a *-algebra structure defined as follows.
\[(a \times b)_g(q) = \sum_{g_1, g_2 \in G} a_{g_1}(q) b_{g_2}(g_1^{-1} q), \forall (g, q) \in G \times Q, a = (a_g)_{g \in G}, b = (b_g)_{g \in G} \in \bigoplus_{g \in G} C_0(Q); \]
\[\langle a \rangle_{g \in G} = (a_g^{-1}(g^{-1} q), \forall (g, q) \in G \times Q, a = (a_g)_{g \in G} \in \bigoplus_{g \in G} C_0(Q). \]

In order to distinguish between these *-algebra operations and the usual operations in the direct sum *-algebra, we will denote this new *-algebra structure by \(C_0(Q)[G]\). One defines the full \(C^*\)-norm on \(C_0(Q)[G]\) by

\[||a||_{\text{full}} = \sup \{ ||\pi(a)|| : \pi \text{ non-degenerate } *\text{-representation of } C_0(Q)[G] \}. \tag{40}\]

The above definition is slightly different from the standard one found in the literature, which involves the so-called covariant representations of the \(C^*\)-dynamical system \((C_0(Q), G, \lambda)\), where \(\lambda : G \to \text{Aut}(C_0(Q))\) is the action given by \((\lambda_g f)(q) = f(g^{-1} q), g \in G, q \in Q, f \in C_0(Q)\). However, it is fairly easy to show that, for any non-degenerate *-representation \(\pi : C_0(Q)[G] \to B(H)\) there exists a unique unitary representation \(G \ni g \mapsto U^*_g \in \mathcal{U}(H)\), satisfying the identity

\[\pi(a) = \sum_{g \in G} (\pi \circ \chi_a)(a_g) U^*_g, \forall a = (a_g)_{g \in G} \in C_0(Q)[G], \tag{41}\]

where \(\chi_a\) is the *-homomorphism \(C_0(Q) \ni f \mapsto (\delta_{g f})_{g \in G} \in C_0(Q)[G]\).

With this set-up in mind, the full crossed product \(C^*\)-algebra \(C_0(Q) \rtimes G\) is the completion of \(C_0(Q)[G]\) in the \(C^*\)-norm \(\|\|_{\text{full}}\). The *-homomorphism \(\chi_a\) gives rise to a non-degenerate inclusion \(C_0(Q) \hookrightarrow C_0(Q) \rtimes G\); with the help of \([11]\) – applied to the universal representation of \(C_0(Q)[G]\), this gives rise to a group representation \(G \ni g \mapsto u_g \in \mathcal{U}(M(C_0(Q) \rtimes G))\) (the unitary group of the multiplier algebra), which allows us to present the inclusion \(C_0(Q)[G] \subset C_0(Q) \rtimes G\) as

\[C_0(Q) \rtimes G = \overline{\text{span}} \{ f u_g : f \in C_0(Q), g \in G \}. \tag{42}\]

For simplicity, we’ll agree to ignore \(\chi_a\) from our notation (i.e. to replace \(\chi_a(f)\) simply by \(f\), thus viewing \(C_0(Q)\) as a \(C^*\)-subalgebra in \(C_0(Q) \rtimes G\)), so we can simply view

\[C_0(Q) \rtimes G = \overline{\text{span}} \{ f u_g : f \in C_0(Q), g \in G \}, \tag{42}\]

subject to the product and adjoint rules:

\[(f_1 u_{g_1})(f_2 u_{g_2}) = (f_1(\lambda_{g_1} f_2)) u_{g_1 g_2}, \quad f_1, f_2 \in C_0(Q), \quad g_1, g_2 \in G; \tag{43}\]

\[\langle f u_g \rangle^* = (\lambda_{g^{-1}} f) u_{g^{-1}}, \quad f \in C_0(Q), \quad g \in G; \tag{44}\]

(In \([42]\), the linear span – without closure – is \(C_0(Q)[G]\).)

Just as non-degenerate *-representations of \(C_0(Q)[G]\) are in bijective correspondence to those of \(C_0(Q) \rtimes G\), the same can be said about non-degenerate *-homomorphisms (in the sense of Definition [12]) a *-homomorphism \(\Psi : C_0(Q)[G] \to A\) is non-degenerate, if \(\Psi|_{C_0(Q)} : C_0(Q) \to A\) is such.) In particular, using Remark [14] every non-degenerate *-homomorphism \(\Phi : C_0(Q) \rtimes G \to A\) yields a group homomorphism \(G \ni g \mapsto u_g^\Phi = M\Phi(u_g) \in \mathcal{U}(M(A))\), so that

\[\Phi(f u_g) = \Phi(f) u_g^\Phi, \quad \forall f \in C_0(Q), \quad g \in G. \tag{45}\]
Since \((C_0(Q)[G], \| \cdot \|_{\text{full}})\) contains as a dense \(*\)-subalgebra  
\[ C_c(Q)[G] = \{ a = (a_g)_{g \in G} \in C_0(Q)[G], \ a_g \in C_c(Q), \ \forall \ g \in G \}, \]
the non-degenerate \(*\)-representations of (or \(*\)-homomorphisms defined on) \(C_0(Q)[G]\) are in a bijective correspondence (by restriction) to the non-degenerate \(*\)-representations of (or \(*\)-homomorphisms defined on) \(C_c(Q)[G]\); in other words, we can also view the full crossed product \(C_0(Q) \rtimes G\) as the completion of \(C_c(Q)[G]\) with respect to the \(C^*\)-norm  
\[
\| a \|_{\text{full}} = \sup \{ \| \pi(a) \| : \pi \text{ non-degenerate } *\text{-representation of } C_c(Q)[G] \}. 
\tag{46}
\]
The point here is the fact that the \(*\)-algebra \(C_c(Q)[G]\) is \(*\)-isomorphic to the convolution \(*\)-algebra \(C_c(G)\) of our transformation groupoid \(G\). Explicitly, this \(*\)-isomorphism \(\gamma : C_c(G) \to C_c(Q)[G]\) assigns to every function \(f \in C_c(G \times Q)\) the \(G\)-tuple \(a = (a_g)_{g \in G} \in C_c(Q)[G]\) given by \(a_g(q) = f(g, g^{-1}q)\). By completion – using the equalities (40) and (46), this gives rise to a \(*\)-isomorphism \(\gamma_{\text{full}} : C^*(G) \to C_0(Q) \rtimes G\).

The linear surjection  
\[
C_0(Q)[G] \ni (a_g)_{g \in G} \mapsto a_e \in C_0(Q) \tag{47}
\]
is \(\| \cdot \|_{\text{full}}\)-contractive, thus it gives rise to conditional expectation \(E^\times\) of \(C_0(Q) \rtimes G\) onto \(C_0(Q)\), which acts on the generators as  
\[
E^\times(fu_g) = \delta_{ge}f, \quad f \in C_0(Q), \quad g \in G.
\]

Using the identification between \(Q\) and the unit space \(\mathcal{G}^{(0)}\) of our transformation groupoid \(\mathcal{G}\), we have a commutative diagram  
\[
\begin{array}{ccc}
C^*(\mathcal{G}) & \xrightarrow{\gamma_{\text{full}}} & C_0(Q) \rtimes G \\
\downarrow{\pi^\times} & & \downarrow{E^\times} \\
C_0(\mathcal{G}^{(0)}) & \xrightarrow{\gamma_{\text{red}}} & C_0(Q)
\end{array}
\]
When we apply the KSGNS construction, the above diagram will yield a \(*\)-isomorphism  
\(\gamma_{\text{red}} : C^*_\text{red}(\mathcal{G}) \to C_0(Q) \rtimes_\text{red} G\) between the reduced \(C^*\)-algebra of \(\mathcal{G}\) and the reduced crossed product \(C^*_\text{red}\)-algebra. Equivalently, one can view \(C_0(Q) \rtimes_\text{red} G\) as the completion of \(C_c(Q)[G]\) with respect with the unique \(*\)-isomorphism \(\pi_{\text{red}} : (C_0(Q)[G], \| \cdot \|_{\text{red}}) \to (C_c(Q)[G], \| \cdot \|_{\text{red}})\) isometric; one then has two commutative diagrams  
\[
\begin{array}{ccc}
C^*(\mathcal{G}) & \xrightarrow{\gamma_{\text{full}}} & C_0(Q) \rtimes G \\
\downarrow{\pi^\times_{\text{red}}} & & \downarrow{\pi^\times_{\text{red}}} \\
C^*_\text{red}(\mathcal{G}) & \xrightarrow{\gamma_{\text{red}}} & C_0(Q) \rtimes_\text{red} G \\
\downarrow{\pi^\times_{\text{red}}} & & \downarrow{E^\times_{\text{red}}} \\
C_0(\mathcal{G}^{(0)}) & \xrightarrow{\gamma_{\text{red}}} & C_0(Q)
\end{array}
\]
where \(\pi^\times_{\text{red}} : C_0(Q) \rtimes G \to C_0(Q) \rtimes_\text{red} G\) is the quotient \(*\)-homomorphism, and \(E^\times_{\text{red}} : C_0(Q) \rtimes_\text{red} G \to C_0(Q)\) is the (unique) conditional expectation arising from (47) (which is also \(\| \cdot \|_{\text{red}}\)-contractive), that yields the factorization \(E^\times = E^\times_{\text{red}} \circ \pi^\times_{\text{red}}\). (Again, with the help of the \(*\)-homomorphism \(\chi_{e}^\text{red} := \pi^\times_{\text{red}} \circ \chi_{e} : C_0(Q) \hookrightarrow C_0(Q) \rtimes_\text{red} G\), we view \(C_0(Q)\) as a non-degenerate \(C^*\)-subalgebra of \(C_0(Q) \rtimes_\text{red} G\), and omit \(\chi_{e}^\text{red}\) from our notation.)
Lastly, by applying the construction of the unitaries satisfying (15) to the surjective (thus non-degenerate) \( \ast \)-homomorphism \( \pi^\ast_{\text{red}} : C_0(Q) \times G \to C_0(Q) \times_{\text{red}} G \), we also obtain a group homomorphism \( G \ni g \mapsto u^\ast_{g} \in \mathcal{U}(M(C_0(Q) \times_{\text{red}} G)) \) satisfying
\[
\pi^\ast_{\text{red}}(fu_g) = fu^\ast_{g}, \quad \forall f \in C_0(Q), \ g \in G.
\]

**Remark 8.1.** A point \((g, q) \in G \times Q = \mathcal{G}\) belongs to the isotropy groupoid \(\text{Iso}(\mathcal{G})\), if and only if \(gq = q\); in other words, \(g\) belongs to the stabilizer group \(G_q = \{ g \in G : gq = q \}\). Furthermore, using the fact that \(G\) comes equipped with the discrete topology, \((g, q)\) belongs to the interior \(\text{IntIso}(\mathcal{G})\), if and only if \(g \in G_q\), for all \(q\) in some neighborhood of \(q\). In other words, if we define, for any open set \(V \subset Q\), the set stablizer group
\[
G_V = \{ g \in G : gq = q, \ \forall q \in V \},
\]
then, when we identify \(Q\) with the unit space \(\mathcal{G}^{(0)}\) of our transformation groupoid \(\mathcal{G}\), for any \(q \in Q\), the group \(\text{IntIso}(\mathcal{G})q\) is equal to the interior stabilizer group
\[
G_q^\circ = \bigcup_{\substack{V \text{ open} \\ V \ni q}} G_V.
\]

**Comment.** When translating the groupoid dictionary to dynamical systems, the following slightly different terminology is employed.

(a) The transformation groupoid \(\mathcal{G} = G \times Q\) is minimal (as in Remark 7.2), if and only \(G \curvearrowright Q\) is a minimal action, in the sense that the only open/closed \(G\)-invariant subsets \(S \subset Q\) are \(S = \emptyset, Q\).

(b) The transformation groupoid \(\mathcal{G} = G \times Q\) is topologically principal (as in condition (iii) from Corollary 7.10), if and only the action is \(G \curvearrowright Q\) is a topologically free, in the sense that the set \(\{ q \in Q : G_q = \{ e \} \}\) is dense in \(Q\)

As pointed out in the Comment that followed Remark 7.2 minimality for the action \(G \curvearrowright Q\) is always a necessary condition for the simplicity of either \(C_0(Q) \rtimes G\) or \(C_0(Q) \rtimes_{\text{red}} G\).

When we specialize Corollary 7.10 to the transformation groupoid, and use the above dictionary, one recovers the following well-known result of Kawamura and Tomyama.

**Corollary 8.2.** (cf. [1] and [6] Thm. 4.4.) The full crossed product \(\ast\)-algebra \(C_0(Q) \rtimes G\) is simple, if and only if all three conditions below are satisfied

(i) the action \(G \curvearrowright Q\) is minimal;

(ii) \(\pi^\ast_{\text{red}} : C_0(Q) \rtimes G \to C_0(Q) \rtimes_{\text{red}} G\) is an isomorphism;

(iii) the action \(\mathcal{G} \curvearrowright Q\) is topologically free.

Concerning the simplicity of the reduced crossed product, in preparation for our adaptation of the main results from the previous section (Theorem 7.8 and Proposition 7.9), we begin by clarifying the status of the open subgroupoid \(\text{IntIso}(\mathcal{G}) \subset \mathcal{G}\).

**Notation.** For each \(g \in G\), denote the fixed point set \(\{ q \in Q : gq = q \}\) by \(Q^g\).

**Remark 8.3.** Using the \(\ast\)-isomorphism \(\Upsilon : C_c(\mathcal{G}) \to C_c(Q)[G]\), and viewing \(C_c(Q)[G]\) as either a \(\ast\)-subalgebra in \(C_0(Q) \rtimes G\), or in \(C_0(Q) \rtimes_{\text{red}} G\), the \(\ast\)-subalgebra \(C_c(\text{IntIso}(\mathcal{G})) \subset\)
$C_c(G)$ gets identified with either one of the $*$-subalgebras (using the convention $C_c(\emptyset) = \{0\}$):

$$\mathcal{A} = \sum_{g \in G} C_c(\text{Int}(Q^g)) u_g \subset C_0(Q) \rtimes G,$$

$$\mathcal{A}_{\text{red}} = \pi_{\text{red}}^{\infty}(\mathcal{A}) = \sum_{g \in G} C_c(\text{Int}(Q^g)) u_g^{\text{red}} \subset C_0(Q) \rtimes_{\text{red}} G.$$

The sums defining these $*$-subalgebras are direct sums. The fact that $\mathcal{A}$ is a $*$-subalgebra can also be seen (without any reference to the $*$-homomorphism $\Upsilon$) from the following easy observations:

(i) if $f \in C_c(\text{Int}(Q^g))$, then $f u_g = u_g f$;

(ii) if $f_j \in C_c(\text{Int}(Q^{g_j}))$, $j = 1, 2$, then $(f_1 u_{g_1}) (f_2 u_{g_2}) = u_{g_1} (f_1 f_2) u_{g_2} = (f_1 f_2) u_{g_1 g_2}$, where $f_1 f_2 \in C_c(\text{Int}(Q^{g_1} \cap \text{Int}(Q^{g_2})) \subset C_c(\text{Int}(Q^{g_1 g_2})).$

Using either one of the $*$-isomorphisms $C^* (G) \xrightarrow{\Upsilon_{\text{full}}} C_0(Q) \rtimes G$, or $C^*_{\text{red}}(G) \xrightarrow{\Upsilon_{\text{red}}} C_0(Q) \rtimes_{\text{red}} G$, we can identify $C^* (\text{IntIso}(G))$ with the closure $\mathcal{A} \subset C_0(Q) \rtimes G$, and $C^*_{\text{red}}(\text{IntIso}(G))$ with the closure $\mathcal{A}_{\text{red}} \subset C_0(Q) \rtimes_{\text{red}} G$.

**Remark 8.4.** Fix for the moment $q \in Q$ (viewed as a unit in $\text{IntIso}(G)^{(0)}$). Following the treatment from Section 5.6, we have a $C^*$-seminorm $p^\text{full}_q$ on $C^* (\text{IntIso}(G))$ and two $C^*$-seminorms $p^\text{other}_q, p^\text{red}_q$ on $C^*_{\text{red}}(\text{IntIso}(G))$. Using the central inclusions

$$C^* (\text{IntIso}(G)) \ni C_0(Q) \subset C^*_{\text{red}}(\text{IntIso}(G)),$$

the seminorms $p^\text{full}_q$ and $p^\text{other}_q$ are defined as

$$b \mapsto \inf \{ \| fb \| : f \in C_{0,q}(Q), 0 \leq f \leq 1 = f(q) \},$$

where $b$ belongs to either $C^* (\text{IntIso}(G))$, or to $C^*_{\text{red}}(\text{IntIso}(G))$. The seminorm $p^\text{red}_q$ is defined using the GNS representation $\Gamma_{ev_q \circ E_{\text{red}}}$ of $C^*_{\text{red}}(\text{IntIso}(G))$ associated with the state $ev_q \circ E_{\text{red}}$.

When transferring this set-up to $\mathcal{A} \cong C^* (\text{IntIso}(G))$ and $\mathcal{A}_{\text{red}} \cong C^*_{\text{red}}(\text{IntIso}(G))$, we are now dealing with seminorms which we denote $p^\text{full}_q$ (on $\mathcal{A}$), $p^\text{other}_q$ and $p^\text{red}_q$ (on $\mathcal{A}_{\text{red}}$). As in the preceding paragraph, the seminorms $p^\text{full}_q$ and $p^\text{other}_q$ are given by

$$p^\text{full}_q (a) = \inf \{ \| fa \| : f \in C_{0,q}(Q), 0 \leq f \leq 1 = f(q) \}, \quad a \in \mathcal{A},$$

$$p^\text{other}_q (a) = \inf \{ \| fa \|_{\text{red}} : f \in C_{0,q}(Q), 0 \leq f \leq 1 = f(q) \}, \quad a \in \mathcal{A}_{\text{red}}, \quad (49)$$

while the seminorm $p^\text{red}_q$ is given by the GNS representation $\Gamma_{ev_q \circ E_{\text{red}}} \big|_{\mathcal{A}_{\text{red}}}$, associated with the state $ev_q \circ E_{\text{red}}$.

Alternatively, these seminorms can be realized as follows. Start off with the group algebra $\mathbb{C}[G^g] = \mathbb{C}[\text{IntIso}(G)q]$, denote its canonical unitary generators by $\{x_g\}_{g \in G^g}$, and consider the $*$-homomorphisms (defined for sums indexed by finite sets $F \subset G$)

$$\epsilon_q : \mathcal{A} \ni \sum_{g \in F} f_g u_g \longmapsto \sum_{g \in F} f_g(q) x_g \in \mathbb{C}[G^g],$$

$$\epsilon_q' : \mathcal{A}_{\text{red}} \ni \sum_{g \in F} f_g u_g^{\text{red}} \longmapsto \sum_{g \in F} f_g(q) x_g \in \mathbb{C}[G^g].$$

On the group algebra $\mathbb{C}[G^g]$ we now have two $C^*$-norms $\| \cdot \|_{\text{full}}$ (the full $C^*$-norm) and $\| \cdot \|_{\text{other},q} \leq \| \cdot \|_{\text{full}}$, which by completion allow us to extend the above $*$-homomorphisms to
two surjective \ast\hbox{-}homomorphisms, \(e^\text{full}_q : \mathcal{A} \to C^*(G_q^0)\) and \(e^\text{other}_q : \mathcal{A}_{\text{red}} \to C^\ast_{\text{other}, q}(G_q^0)\), so that
\[
\begin{align*}
p^\text{full}_{q,x}(a) &= \|e^\text{full}_q(a)\|_{\text{full}}, \quad \forall a \in \mathcal{A}; \\
p^\text{other}_{q,x}(a) &= \|e^\text{other}_q(a)\|_{\text{other}, q}, \quad \forall a \in \mathcal{A}_{\text{red}}.
\end{align*}
\]
(51)
(52)

Also, if we equip \(\mathbb{C}[G_q^0]\) with the reduced \(C^\ast\hbox{-}\)norm \(\|\cdot\|_{\text{red}} \leq \|\cdot\|_{\text{other}, q}\) by completion, \(e^\text{full}_q\) also gives rise to a surjective \ast\hbox{-}homomorphism \(e^\text{red}_q : \mathcal{A}_{\text{red}} \to C^\ast_{\text{red}}(G_q^0)\), which allows us to realize
\[
P^\text{red}_{q,x}(a) = \|e^\text{red}_q(a)\|_{\text{red}}, \quad \forall a \in \mathcal{A}_{\text{red}}.
\]
(53)

Using (13), we have
\[
\begin{align*}
\|a\| &= \sup_{q \in Q} p^\text{full}_{q,x}(a) = \sup_{q \in Q} \|e^\text{full}_q(a)\|_{\text{full}}, \quad a \in \mathcal{A}; \\
\|a\|_{\text{red}} &= \sup_{q \in Q} p^\text{other}_{q,x}(a) = \sup_{q \in Q} \|e^\text{other}_q(a)\|_{\text{other}, q}, \quad a \in \mathcal{A}_{\text{red}}.
\end{align*}
\]
(54)
(55)

Using the faithfulness of \(E^\times_{\text{red}}\), we also have:
\[
\|a\|_{\text{red}} = \sup_{q \in Q} p^\text{red}_{q,x}(a) = \sup_{q \in Q} \|e^\text{full}_q(a)\|_{\text{red}}, \quad a \in \mathcal{A}_{\text{red}}.
\]
(56)

**Theorem 8.5.** Let \(\{v^q_g\}_{g \in G} \subset C^\ast(G)\) and \(\{v^\text{red}_g\}_{g \in G} \subset C^\ast_{\text{red}}(G)\) denote the standard unitary generators if the full, or reduced group \(C^\ast\hbox{-}\)algebras, respectively. The linear maps (defined on sums indexed by finite subsets \(F \subset G\))
\[
\begin{align*}
\Delta : \mathcal{A} &\ni \sum_{g \in F} f u_g \longmapsto \sum_{g \in F} f \otimes v_g \in C_0(Q) \otimes C^\ast(G), \\
\Delta' : \mathcal{A}_{\text{red}} &\ni \sum_{g \in F} f u^\text{red}_g \longmapsto \sum_{g \in F} f \otimes v^\text{red}_g \in C_0(Q) \otimes C^\ast_{\text{red}}(G)
\end{align*}
\]
(57)
(58)

extend to injective \ast\hbox{-}homomorphisms \(\Delta^\text{full} : \mathcal{A} \to C_0(Q) \otimes C^\ast(G)\) and \(\Delta^\text{red} : \mathcal{A}_{\text{red}} \to C_0(Q) \otimes C^\ast_{\text{red}}(G)\).

(In (57) and (58), the symbol \(\otimes\) denoted the maximal tensor product. Of course, since \(C_0(Q)\) is nuclear, the maximal and minimal tensor products involved here coincide.)

**Proof.** First of all, \(\Delta\) and \(\Delta'\) are clearly \ast\hbox{-}homomorphisms.

Second, consider, for each \(q \in Q\) the injective \ast\hbox{-}homomorphisms \(\iota^\text{full}_q : C^\ast(G_q^0) \to C^\ast(G)\) and \(\iota^\text{other}_q : C^\ast_{\text{other}, q}(G_q^0) \to C^\ast_{\text{other}}(G)\) arising from the inclusion \(\mathbb{C}[G_q^0] \subset \mathbb{C}[G]\) (which in turn comes from the inclusion \(G_q^0 \subset G\)). Also, denote by \(\text{ev}^\text{full}_q : C_0(Q) \otimes C^\ast(G) \to C^\ast(G)\) and \(\text{ev}^\text{red}_q : C_0(Q) \otimes C^\ast_{\text{red}}(G) \to C^\ast_{\text{red}}(G)\) the evaluation maps. The desired conclusion now follows from Proposition 1.10 applied

(a) to the \ast\hbox{-}homomorphism \(\Delta\) and the families \(\Sigma^\text{full}_q = \{C_0(Q) \otimes C^\ast(G) \xrightarrow{\text{ev}^\text{full}_q} C^\ast(G)\}_{q \in Q}\), \[\Psi^\text{full}_q = \{\mathcal{A} \xrightarrow{\text{ev}^\text{full}_q} \mathcal{A} \xrightarrow{\iota^\text{full}_q} C^\ast(G)\}_{q \in Q},\]
and likewise,

(b) to the \ast\hbox{-}homomorphism \(\Delta'\) and the families \(\Sigma^\text{red}_q = \{C_0(Q) \otimes C^\ast_{\text{red}}(G) \xrightarrow{\text{ev}^\text{red}_q} C^\ast_{\text{red}}(G)\}_{q \in Q}\), \[\Psi^\text{red}_q = \{\mathcal{A}_{\text{red}} \xrightarrow{\text{ev}^\text{red}_q} \mathcal{A}_{\text{red}} \xrightarrow{\iota^\text{other}_q} C^\ast_{\text{other}}(G)\}_{q \in Q}.
\]

Indeed, on the one hand, it is pretty evident that, for a fixed \(q \in Q\), the \ast\hbox{-}homomorphisms \(\text{ev}^\text{full}_q\) and \(\iota^\text{full}_q \circ \epsilon^\text{full}_q\) act the same way on monomials of the form \(f u_g \in \mathcal{A}\), and likewise, \(\text{ev}^\text{red}_q\)
and \( \iota_q^\text{red} \circ \epsilon_q^\text{red} \) act the same way on \( f u_q^\text{red} \in A_q^\text{red} \). (The monomial referred to in each instance involves some \( g \in G \) and \( f \in C_c(\text{Int}(Q^g)) \).) One the other hand, by (54) and (56), both \( \Psi^\text{full} \) and \( \Psi^\text{red} \) are jointly faithful, while the joint faithfulness of \( \Sigma^\text{full} \) and \( \Sigma^\text{red} \) is obvious.

**Theorem 8.6.** The isotropic groupoid \( \text{IntIso}(G) \) associated to the étale transformation groupoid \( G = G \times Q \) exhibits the following continuity properties:

1. For each \( a \in C^*(\text{IntIso}(G)) \), the map \( Q \ni q \mapsto p_q^\text{full}(a) \in [0, \infty) \) is continuous.
2. For each \( a \in C_q^\text{red}(\text{IntIso}(G)) \), the map \( Q \ni q \mapsto p_q^\text{red}(a) \in [0, \infty) \) is continuous. In particular, every unit \( q \in Q \) is continuously reduced, i.e. the \( C^* \)-seminorms \( p_q^\text{other} \) and \( p_q^\text{red} \) coincide on \( C_q^\text{erd}(\text{IntIso}(G)) \), thus the \( C^* \)-norms \( \| \cdot \|_{\text{other}, q} \) and \( \| \cdot \|_{\text{red}} \) coincide on \( C[\text{IntIso}(G)q] = C[G_q] \).

**Proof.** The continuity statements from parts (i) and (ii) follow from the following elementary facts.

**Fact 1.** For any \( C^* \)-algebra \( B \), the evaluation maps \( \text{ev}_q : C_0(Q) \otimes B \to B \), \( q \in Q \) give rise, for each \( a \in C_0(Q) \otimes B \), to a continuous map \( Q \ni q \mapsto \| \text{ev}_q(a) \| \in [0, \infty) \), which satisfies \( \sup_{q \in Q} \| \text{ev}_q(a) \| = \| a \| \).

**Fact 2.** If \( C_0(Q) \subset A \) is a central non-degenerate inclusion, and \( \Delta : A \to C_0(Q) \otimes B \) is an injective \( * \)-homomorphism, which is \( C_0(Q) \)-linear (i.e. \( \Delta(fa) = (f \otimes 1)\Delta(a) \), \( \forall f \in C_0(Q) \), \( a \in A \)), then

\[
p_q^{\text{unif}}(a) = \|(\text{ev}_q \circ \Delta)(a)\|, \quad \forall a \in A, \ q \in Q.
\]

In particular, the map \( Q \ni q \mapsto p_q^{\text{unif}}(a) \in [0, \infty) \) is continuous, for each \( a \in A \).

As for the second statement from (ii), the equality \( p_q^{\text{other}} = p_q^{\text{red}} \) follows from Fact 2, applied to the inclusion \( C_q^{\text{red}}(\text{IntIso}(G)) \hookrightarrow C_0(Q) \otimes C_q^{\text{red}}(G) \) obtained from Theorem 8.5 (by composing \( \Delta_{\text{red}} \) with the isomorphism \( C_q^{\text{red}}(\text{IntIso}(G)) \xrightarrow{\iota_q^{\text{red}}} A_q^{\text{red}} \)), which implies that the group algebra inclusion \( C[G_q] \subset C[G] \) extends to a \( C^* \)-algebra inclusion \( j_q : C_{\text{other}, q}(G_q) \hookrightarrow C_{\text{red}}(G) \). (Since we always have an inclusion \( C_{\text{red}}(G_q) \hookrightarrow C_{\text{red}}(G) \), and \( j_q \) factors through the quotient \( * \)-homomorphism \( \kappa_q : C_{\text{other}, q}(G_q) \to C_{\text{red}}(G_q) \), the injectivity of \( j_q \) indeed forces \( \kappa_q \) to be isometric.

Since all units are in transformation groupoids are continuously reduced, when we adapt Theorem 7.6 to this setting, we obtain the following.

**Corollary 8.7.** If there is some \( q_0 \in Q \), such that the group \( G_{q_0} \) is a \( C^* \)-simple, then the following conditions are equivalent

1. the action \( G \acts Q \) is minimal;
2. the reduced crossed product \( C_0(Q) \rtimes_{\text{red}} G \) is a simple \( C^* \)-algebra.

**Comment.** The above result offers a slight improvement to a Theorem of Ozawa ([13, Thm. 14(1)]), in which our hypothesis – \( C^* \)-simplicity of some interior stabilizer \( G_{q_0} \) – is replaced by a stronger condition: the \( C^* \)-simplicity of some full stabilizer group \( G_q \). (After all, for any \( q \in Q \), we know that \( G_q \) is a normal subgroup of \( G_q \); also it is well known that normal subgroups of \( C^* \)-simple groups are also \( C^* \)-simple.)

Once again, since all units are continuously reduced, when adapted to crossed products, the hypothesis from Proposition 7.7 is indistinguishable to the one from Corollary 7.8 so
our non-simplicity criterion for crossed products is stated as follows (compare to [13 Thm. 14(2)])

**Corollary 8.8.** If the reduced crossed product $C_0(Q) \rtimes_{\text{red}} G$ is a simple $C^*$-algebra, and there is some $q_0 \in Q$, such that the group $G_{q_0}^\circ$ is amenable, then $G_q = \{e\}, \forall q \in Q$, i.e. the action $G \curvearrowright Q$ is topologically free.

**APPENDIX: ON THE INCLUSION $C^*(\text{IntIso}(G)) \subset C^*(G)$**

As in Section 7 we fix an étale Hausdorff, second countable groupoid $G$, so that its full $C^*$-algebra is the completion of $(C_c(G), \times, \star)$ in the full norm $\|\|_\text{full}$ defined in (4).

In our analysis of the isotropy groupoid, we will use the well known identification between the following three spaces, associated with a discrete group $H$:

(i) the space of positive linear functionals on the full group $C^*$-algebra $C^*(H)$;

(ii) the space of positive linear functionals on the group algebra $\phi : C[H] \to \mathbb{C}$, i.e. those satisfying $\phi(f^* \times f) \geq 0, \forall f \in C[H]$;

(iii) the space of positive definite functions on $H$, i.e. those functions $\theta : H \to \mathbb{C}$, with the property that: for any finite set $\{h_1, \ldots, h_n\} \subset H$, the matrix $[\theta(h_i^{-1} h_j)]_{i,j=1}^n \in M_n$ is positive.

This correspondence assigns to every positive definite $\theta$ the positive linear functional $\mathbb{C}[H] \ni f \mapsto \sum_{h \in H} \theta(h) f(h) \in \mathbb{C}$, which turns out to be $\|\|_\text{full}$-bounded, thus it extends to a unique positive linear functional on $C^*(H)$.

Another well know elementary fact, is that, whenever $H_0 \subset H$ is a subgroup, and $\theta_0$ is a positive definite function on $H_0$, the function $\theta : H \to \mathbb{C}$ given by

$$\theta(h) = \begin{cases} \theta_0(h), & \text{if } h \in H_0 \\ 0, & \text{otherwise} \end{cases}$$

is again a positive definite function on $H$. Using this, one obtains the well known fact that the canonical inclusion $\mathbb{C}[H_0] \subset \mathbb{C}[H]$ gives rise to two $C^*$-inclusions of group $C^*$-algebras $C^*(H_0) \subset C^*(H)$ and $C^*_{\text{red}}(H_0) \subset C^*_{\text{red}}(H)$. (These can also be justified quickly using Proposition [1.10])

The next two results are well known, but due to their elementary nature, we supply them with proofs.

**Lemma A.1.** (a variation of [15 Prop.4.2]) If $\phi : C_c(G) \to \mathbb{C}$ is a linear functional, which is positive, in the sense that

$$\phi(f^* \times f) \geq 0, \forall f \in C_c(G), \quad (59)$$

then $\phi$ also satisfies the inequalities

$$\phi(f^* \times h^* \times h \times f) \leq \|h\|_{\text{full}}^2 \cdot \phi(f^* \times f), \forall f, h \in C_c(G). \quad (60)$$

In particular, if $\phi|_{C_c(G^{(0)})}$ is $\|\|_\infty$-bounded, then $\phi$ is $\|\|_{\text{full}}$-continuous, thus it extends to a positive linear functional on $C^*(G)$.

**Proof.** The key step is the following slightly weaker version of (60):

**Claim.** For every $h \in C_c(G)$, there exists some constant $C_h \geq 0$, such that

$$\phi(f^* \times h^* \times h \times f) \leq C_h \cdot \phi(f^* \times f), \forall f, h \in C_c(G). \quad (61)$$
Using the fact that $C_c(\mathcal{G})$ is spanned by functions supported on open bisections, and Cauchy-Bunyakovsky-Schwarz – which implies

$$\phi(f^* \times \left( \sum_{j=1}^{k} h_j^* \times \left( \sum_{j=1}^{k} h_j \right) \times f \right) \leq k^2 \sum_{j=1}^{k} \phi(f^* \times h_j^* \times h_j \times f),$$

it suffices to prove the Claim under the additional assumption that $h \in C_c(\mathcal{B})$, for some open bisection $\mathcal{B} \subset \mathcal{G}$. But in this case, $h^* \times h$ belongs to $C_c(\mathcal{G}^{(0)})$, so by replacing $h$ with $(h^* \times h)^{1/2}$ we can in fact assume that $h \in C_c(\mathcal{G}^{(0)})$. But now if we take a function $k \in C_c(\mathcal{G}^{(0)})$, such that $0 \leq k \leq 1$ and $k|_{(\text{supp } f) \cup \text{supp } h} = 1$, then $k \times f = f$, so the function $\|h\|^2_k h^* \times k - h^* \times h \in C_c(\mathcal{G}^{(0)})$ is positive, so if we take $g$ to be its square root, we have the identity $h^* \times h + g^* \times g = \|h\|^2_k h^* \times k$, which yields:

$$f^* \times h^* \times h \times f + f^* \times g^* \times g \times f = \|h\|^2_k f^* \times f,$$

which by positivity implies (61) with $C_h = \|h\|^2_k$.

Having justified the Claim, we can conclude that, by considering the separate completion $\mathcal{H}$ of $C_c(\mathcal{G})$ in the seminorm given by the inner product $\langle f | g \rangle_\phi = \phi(f^* \times g)$, the left multiplication operators $L_f : C_c(\mathcal{G}) \ni g \mapsto f \times g \in C_c(\mathcal{G})$ give rise to a $*$-representation $\pi : C_c(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$, so by the definition of $\| \cdot \|$, full we must have $\|L_f\| \leq \|f\|_{\text{full}}$. □

**Lemma A.2.** Given a unit $u \in \mathcal{G}^{(0)}$ and a positive definite function $\theta$ on the isotropy group $u \mathcal{G}u$, the linear map $\eta_\theta : C_c(\mathcal{G}) \ni f \mapsto \sum_{\gamma \in u \mathcal{G}u} \theta(\gamma) f(\gamma) \in \mathbb{C}$ is positive, in the sense of (59). (For each $f \in C_c(\mathcal{G})$, the sum that defines $\eta_\theta(f)$ has only finitely many non-zero terms.)

**Proof.** Let $\{v_\gamma\}_{\gamma \in u \mathcal{G}u}$ be the standard unitary generators of the group algebra $\mathbb{C}[u \mathcal{G}u]$ (so that we can view $\mathbb{C}[u \mathcal{G}u] = \text{span}\{v_\gamma\}_{\gamma \in u \mathcal{G}u}$), and let $\omega$ be the positive linear map on $\mathbb{C}[u \mathcal{G}u]$ associated with $\theta$, i.e. $\omega(\sum_{\gamma \in F} \zeta_\gamma v_\gamma) = \sum_{\gamma \in F} \theta(\gamma) \zeta_\gamma$ (for all finite sets $F \subset u \mathcal{G}u$).

Fix now some $f \in C_c(\mathcal{G})$, and let us justify the inequality

$$\eta_\theta(f^* \times f) \geq 0. \tag{62}$$

Consider the finite set $X = \text{supp } f \cap \mathcal{G}u$, and assume $X \neq \emptyset$ (otherwise, $\eta_\theta(f^* \times f) = 0$, and there is nothing to prove). Let us list the finite set of units $r(X) = \{u_1, \ldots, u_k\}$, and fix, for each $j$, an element $\gamma_j \in X$, such that $r(\gamma_j) = u_j$, along with an open bisection $\mathcal{B}_j \ni \gamma_j$, and some normalizer $n_j \in C_c(\mathcal{B}_j)$ with $n_j(\gamma_j) = 1$. By suitably shrinking these bisections, we can assume that the sets $r(\mathcal{B}_j)$, $j = 1, \ldots, k$ are disjoint.

Using (29), it is easy to see that

$$(f^* \times f)(\gamma) = \sum_{j=1}^{k} (f^* \times n_j^* \times n_j \times f)(\gamma), \quad \forall \gamma \in u \mathcal{G}u,$$

so it suffices to prove (62) with each of $n_j \times f$’s in place of $f$; in other words, it suffices to prove (62) for a function $f \in C_c(\mathcal{G})$ which has $\mathcal{G}u \cap \text{supp } f \subset u \mathcal{G}u$. However, this special case is trivial, since for any such $f$, when we consider the element $\hat{f} = \sum_{\gamma \in \text{supp } f \cap u \mathcal{G}u} f(\gamma) v_\gamma \in \mathbb{C}[u \mathcal{G}u]$, we have $\eta_\theta(f^* \times f) = \omega(\hat{f}^* \times \hat{f}) \geq 0$. □

**Theorem A.3.** Assume $\mathcal{Y} \subset \text{Iso}(\mathcal{G})$ is an open subgroupoid. Then the $*$-algebra inclusion $C_c(\mathcal{Y}) \subset C_c(\mathcal{G})$
Since $\eta (63)$, we have a family $\Sigma = \{ \eta \}$ then use Lemma A.2 to produce a linear positive map $C$. □

Remark 1.8. A subgroup in $\mathfrak{G}$ is associated to $\theta$ where $\mathfrak{G}$.

In order to distinguish between the full norms on $C^*(\mathcal{Y})$ and $C^*(\mathcal{G})$, we will denote them by $\| \cdot \|_{C^*(\mathcal{Y})}$ and $\| \cdot \|_{C^*(\mathcal{G})}$, respectively.

As pointed out in Section 3, for each unit $u \in \mathcal{Y}^{(0)}$, we have a surjective $\ast$-homomorphism $\rho_u : C^*(\mathcal{Y}) \to C^*(\mathcal{Y} u)$ (from the full groupoid $C^*$-algebra to the full group $C^*$-algebra), which when restricted to $C_c(\mathcal{Y})$, acts as:

$$\rho_u : C_c(\mathcal{Y}) \ni f \mapsto f|_{\mathcal{Y} u} \in \mathbb{C}[\mathcal{Y} u].$$

Using (63), we also know that

$$\| a \|_{C^*(\mathcal{Y})} = \sup_{u \in \mathcal{Y}^{(0)}} \| \rho_u(a) \|_{C^*(\mathcal{Y} u)}, \quad \forall a \in C^*(\mathcal{Y})$$

(64)

where $\| \cdot \|_{C^*(\mathcal{Y} u)}$ denotes the norm in the full group $C^*$-algebra of $\mathcal{Y} u$.

Consider now the set

$$\Xi = \{(u, \theta) : u \in \mathcal{Y}^{(0)}, \theta \text{ positive definite function on the group } \mathcal{Y} u\}.$$ Fix for the moment $(u, \theta) \in \Xi$. Let $\varepsilon_{\theta} : C^*(\mathcal{Y} u) \to \mathbb{C}$ be the positive linear functional associated to $\theta$, and let $\psi_{(u, \theta)} = \varepsilon_{\theta} \circ \rho_u$. As $\theta$ is a positive definite function on $\mathcal{Y} u$, which is a subgroup in $u\mathfrak{G} u$, extend it (as zero outside $\mathcal{Y} u$) to a positive definite function $\theta'$ on $u\mathfrak{G} u$, then use Lemma A.2 to produce a linear positive map

$$\eta_{\theta'} : C_c(\mathcal{G}) \ni f \mapsto \sum_{\gamma \in u\mathfrak{G} u} \theta'(\gamma) f(\gamma) = \sum_{\gamma \in \mathcal{Y} u} \theta(\gamma) f(\gamma) \in \mathbb{C}.$$ Since $\eta_{\theta'}$ acts on $C_c(\mathcal{G}^{(0)})$ as a multiple ($\theta(e)$ or zero) of the evaluation map $f \mapsto f(u)$, by Lemma A.1 $\eta_{\theta'}$ extends to a positive linear functional, hereafter denoted by $\phi_{(u, \theta)}$ on $C^*(\mathcal{G})$.

Now, when we consider the canonical $\ast$-homomorphism $\pi : C^*(\mathcal{Y}) \to C^*(\mathcal{G})$ arising from (63), we have a family $\Sigma = \{ \phi_{(u, \theta)} \}_{(u, \theta) \in \Xi}$ of positive linear maps on $C^*(\mathcal{G})$, for which the associated family

$$\Sigma^\pi = \{ \phi_{(u, \theta)} \circ \pi \}_{(u, \theta) \in \Xi} = \{ \psi_{(u, \theta)} \}_{(u, \theta) \in \Xi} = \{ \psi \circ \rho_u : u \in \mathcal{Y}^{(0)}, \psi \text{ linear positive on } C^*(\mathcal{Y} u)\}$$
is clearly jointly faithful on $C^*(\mathcal{Y})$, by (64), so the desired injectivity of $\pi$ follows from Remark 1.8. □

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