DYNAMICS OF GENERALISED DERIVATIONS AND ELEMENTARY OPERATORS

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Abstract. We identify concrete examples of hypercyclic generalised deriv-
ations acting on separable ideals of operators and establish some necessary
conditions for their hypercyclicity. We also consider the dynamics of element-
ary operators acting on particular Banach algebras, which reveals surprising
hypercyclic behaviour on the space of bounded linear operators on the Banach
space constructed by Argyros and Haydon.

1. Introduction

For a given Banach space $X$ and its space of bounded linear operators $\mathcal{L}(X)$, the
generalised derivation $\tau_{A,B}: \mathcal{L}(X) \to \mathcal{L}(X)$ is induced by fixed $A, B \in \mathcal{L}(X)$
and defined as

$$\tau_{A,B}(T) = L_A(T) - R_B(T) = AT - TB$$

where $T \in \mathcal{L}(X)$ and $L_A, R_B: \mathcal{L}(X) \to \mathcal{L}(X)$ are, respectively, the left and right
multiplication operators.

Generalised derivations, also known as intertwining operators, have been studied
from many aspects since the initial work by Rosenblum \cite{33}, Lumer and Rosen-
blum \cite{28} and Anderson and Foiaş \cite{2}. In the setting of operator ideals of $\mathcal{L}(X)$
they have been investigated by, amongst others, Maher \cite{29} and Kittaneh \cite{26}. Extensive surveys on this class of operators can be found in \cite{5, 35}, however their
hypercyclic properties remain largely unexplored outside the special case of the com-
mutator operator $L_A - R_A$ \cite{13}.

We recall for a separable Banach space $X$ that $T \in \mathcal{L}(X)$ is hypercyclic if there
exists $x \in X$ such that its orbit under $T$ is dense in $X$, that is

$$\{T^n x : n \geq 0\} = X.$$

Hypercyclicity has been a highly active area of research since the 1980s and com-
prehensive accounts of the topic can be found in \cite{121}.

Hitherto in the setting of separable operator ideals Bonet et al. \cite{6} character-
ised the hypercyclicity of the multipliers $L_A$ and $R_B$ and subsequently Bonilla and
Grosse-Erdmann \cite{12} identified a sufficient condition for when $L_A$ is frequently hy-
percyclic. The next natural question relates to the hypercyclicity of generalised
derivations.

In Section 2 of this paper we introduce our setting and uncover concrete classes
of hypercyclic generalised derivations acting on separable operator ideals. This

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contrasts with the special case of the commutator operator, where the complete answer on the existence of hypercyclic commutator maps remains unclear [18]. Then in Sections 3 and 4 we identify many classes of non-hypercyclic generalised derivations and we extend some observations made in [18] on commutator operators to the generalised derivation case.

In Section 5 we prove that certain Banach algebras do not support hypercyclic elementary operators. This gives the surprising example of a generalised derivation that has different hypercyclic properties on $\mathcal{L}(X_{AH})$ and its non-trivial ideals, where $X_{AH}$ denotes the Banach space constructed by Argyros and Haydon [3]. We note the class of elementary operators has been studied since the 1950s [28] and they have been surveyed in [13, 14, 16, 35].

Finally we mention that hypercyclicity has also been considered on spaces of operators endowed with weaker topologies. Chan [9] characterised the hypercyclicity of the left multiplier $L_A$ on $\mathcal{L}(X)$ under the strong operator topology when $X$ is a separable Hilbert space and subsequently Chan and Taylor [10] characterised it when $X$ is a separable Banach space. This was then extended to the supercyclic case by Montes-Rodríguez and Romero-Moreno [31]. On the space of self-adjoint operators, on a separable Hilbert space, endowed with the topology of uniform convergence on compact sets Petersson [32] gave sufficient conditions for hypercyclicity of the special class of two-sided multipliers $T \mapsto A^*TA$. This was later extended by Gupta and Mundayadan [22] to the supercyclic case.

2. The Setting and Hypercyclic Generalised Derivations

In this section we identify concrete examples of hypercyclic generalised derivations and we demonstrate there is in fact an abundant supply of them. To this end we first recall some basic definitions and establish our setting.

Hypercyclicity requires separability of the underlying space, however for classical Banach spaces $X$ the space $\mathcal{L}(X)$ is non-separable under the operator norm topology. Our approach is to consider generalised derivations acting on separable ideals of $\mathcal{L}(X)$. We say $(J, \| \cdot \|_J)$ is a Banach ideal of $\mathcal{L}(X)$ if

(i) $J \subset \mathcal{L}(X)$ is a linear subspace,
(ii) the norm $\| \cdot \|_J$ is complete in $J$ and $\|S\| \leq \|S\|_J$ for all $S \in J$,
(iii) $BSA \in J$ and $\|BSA\|_J \leq \|B\|_J \|A\| \|S\|_J$ for $A, B \in \mathcal{L}(X)$ and $S \in J$,
(iv) the rank one operators $x^* \otimes x \in J$ and $\|x^* \otimes x\|_J = \|x^*\|\|x\|$ for all $x^* \in X^*$ and $x \in X$.

We recall that a rank one operator $x^* \otimes x \colon X \rightarrow X$ is defined as

$$(x^* \otimes x)(z) = \langle x^*, z \rangle x = x^*(z)x$$

for $x^* \in X^*$, $x \in X$ and any $z \in X$. The space of finite rank operators $\mathcal{F}(X)$ is the linear span of the rank one operators, that is

$$\mathcal{F}(X) = X^* \otimes X = \left\{ \sum_{i=1}^{n} x_i^* \otimes x_i \colon x_i^* \in X^*, x_i \in X \text{ and } n \geq 1 \right\}.$$

For brevity we say a Banach ideal $J$ is admissible when it contains the finite rank operators as a dense subset with respect to the norm $\| \cdot \|_J$, that is

$$\mathcal{F}(X) \| \cdot \|_J = J.$$
Classical examples of separable admissible Banach ideals include the space of nuclear operators \( \mathcal{N}(X, \| \cdot \|_N) \), equipped with the nuclear norm, when the dual \( X^* \) is separable and the space of compact operators \( \mathcal{K}(X) \), under the operator norm, when \( X^* \) is separable and \( X^* \) possesses the approximation property. When \( X \) is a separable Hilbert space the spaces of Schatten \( p \)-class operators \( (C_p, \| \cdot \|_p) \) with the Schatten norm are also important examples of separable admissible Banach ideals. We caution that admissibility does not automatically imply separability of the ideal, for example \( (\mathcal{N}(X, \| \cdot \|_N)) \) is non-separable when \( X^* \) is non-separable.

Further details on Banach ideals and the approximation property can be found for instance in [34].

In the setting of separable operator ideals, Bonet et al. [6] characterised when the multipliers \( L_A \) and \( R_B \) are hypercyclic using tensor techniques developed in [30]. Their main results, expressed in the terminology of Banach ideals, are as follows:

I. \( L_A \) is hypercyclic on \( J \) if and only if \( A \in L(X) \) satisfies the Hypercyclicity Criterion,

II. \( R_B \) is hypercyclic on \( J \) if and only if \( B^* \in L(X^*) \) satisfies the Hypercyclicity Criterion.

The aforementioned Hypercyclicity Criterion is a sufficient condition for hypercyclicity which was initially demonstrated by Kitai [25] and then independently discovered by Gethner and Shapiro [17]. The standard version can be found for instance in [21, Theorem 3.12], however we will use the following equivalent statement which can be found in [21, Proposition 3.20].

We say that \( T \in \mathcal{L}(X) \) satisfies the *Hypercyclicity Criterion* if there exists a dense linear subspace \( X_0 \subset X \), an increasing sequence \( (n_k) \) of positive integers and linear maps \( S_{n_k} : X_0 \to X \), \( k \geq 1 \), such that for any \( x \in X_0 \)

\[
\begin{align*}
(i) & \quad T^{n_k} x \to 0, \\
(ii) & \quad S_{n_k} x \to 0, \\
(iii) & \quad T^{n_k} S_{n_k} x \to x
\end{align*}
\]

as \( k \to \infty \). If \( T \) satisfies the Hypercyclicity Criterion then it is hypercyclic (cf. [4, Theorem 1.6; 21, Theorem 3.12]).

We easily obtain trivial examples of hypercyclic generalised derivations by taking either \( A \equiv 0 \) or \( B \equiv 0 \) in \( \tau_{A,B} \). This is again the case of \( L_A \) and \( R_B \) which was fully characterised in [6].

Non-trivial examples can be induced using weighted backward shifts. Let \( X = c_0 \) or \( \ell^p \) for \( 1 \leq p \leq \infty \), where \( c_0 \) denotes the space of sequences that tend to zero and \( \ell^p \) is the space of \( p \)-summable sequences. The *weighted backward shift* \( B_w \in \mathcal{L}(X) \) is defined as

\[
B_w(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_3 x_3, \ldots)
\]

for \( (x_n) \in X \) and where \( w = (w_n)_{n \geq 2} \in \ell^\infty \) is a bounded weight sequence of nonzero scalars. \( B_w \) is bounded on \( X \) if and only if \( \sup_n |w_n| < \infty \) and when \( w_n = 1 \) for all \( n \) we denote the unweighted backward shift as \( B : X \to X \).

Next we identify a family of hypercyclic generalised derivations on separable admissible Banach ideals. In contrast to the approach taken in [6], where they apply tensor techniques and the hypercyclicity comparison principle to show the
multipliers are hypercyclic, we prove directly that the generalised derivation satisfies the Hypercyclicity Criterion.

**Example 2.1.** Let \( X = \ell^p \) for \( 1 < p < \infty \) and let \( J \subset \mathcal{L}(X) \) be an admissible Banach ideal. We note that \( J \) is separable by the separability of \( X^* \) and we claim there exists a hypercyclic generalised derivation on \( J \).

We denote by \( I \) the identity operator on \( X \) and recall that Salas \[20\] showed the operator \( I + B_w \in \mathcal{L}(X) \) is hypercyclic, while León-Saavedra and Montes-Rodríguez \[21\] proved it satisfies the Hypercyclicity Criterion when \( \sup_{n \geq 1} |w_n| < \infty \).

We thus assume \( \sup_n |w_n| < \infty \). This gives a dense linear subspace \( X_0 \subset X \), a sequence \( (n_k) \) and linear maps \( S_{n_k} : X_0 \to X \) such that \( I + B_w \) satisfies the Hypercyclicity Criterion. We will show the generalised derivation

\[
L_{B_w} - R_{-I} : J \to J
\]

satisfies the Hypercyclicity Criterion for the subspace \( X^* \otimes X_0 \subset J \), the sequence \( (n_k) \) and the linear maps \( L_{S_{n_k}} : X^* \otimes X_0 \to X^* \otimes X \) defined as

\[
L_{S_{n_k}} (x^* \otimes x) = x^* \otimes S_{n_k} (x)
\]

for \( x^* \otimes x \in X^* \otimes X_0 \).

To see that \( X^* \otimes X_0 \) is dense in \( J \), let \( x^* \otimes x \in J \). By assumption \( X_0 \) is dense in \( X \) and hence there exists a sequence \( (x_n) \subset X_0 \) that converges to \( x \in X \) as \( n \to \infty \). We next consider the sequence \( (x^* \otimes x_n) \subset X^* \otimes X_0 \) and notice that

\[
\| x^* \otimes x_n - x^* \otimes x \|_J = \| x^* \otimes (x_n - x) \|_J = \| x^* \| \| x_n - x \| \to 0.
\]

It then follows by linearity and admissibility that \( X^* \otimes X_0 \) is dense in \( J \).

We note that

\[
(L_{B_w} - R_{-I}) (x^* \otimes x) = x^* \otimes B_w x + x^* \otimes x = x^* \otimes (I + B_w) x
\]

and moreover \( n_k \)-fold iteration gives

\[
(L_{B_w} - R_{-I})^{n_k} (x^* \otimes x) = x^* \otimes (I + B_w)^{n_k} x.
\]

Next we verify that \( L_{B_w} - R_{-I} \) satisfies the Hypercyclicity Criterion for any rank one operator \( x^* \otimes x \in X^* \otimes X_0 \),

(i) \( \| (L_{B_w} - R_{-I})^{n_k} (x^* \otimes x) \|_J = \| x^* \| \| (I + B_w)^{n_k} x \|_J \to 0 \),

(ii) \( \| L_{S_{n_k}} (x^* \otimes x) \|_J = \| x^* \| \| S_{n_k} (x) \| \to 0 \),

(iii) \( \| (L_{B_w} - R_{-I})^{n_k} L_{S_{n_k}} (x^* \otimes x) - x^* \otimes x \|_J \to 0 \).

Hence \( L_{B_w} - R_{-I} \) satisfies the Hypercyclicity Criterion for rank one operators and by taking linear combinations of rank one operators it further follows that it is satisfied on the dense subspace \( X^* \otimes X_0 \subset J \). Thus we have obtained a hypercyclic generalised derivation on \( J \).

The weighted backward shift from Example 2.1 is in fact a special case of the following class of operators. Following the terminology of \[21\] p. 219, we say that
Let $T \in \mathcal{L}(X)$ be an extended backward shift if

$$\text{span} \left( \bigcup_{j=0}^{\infty} (\ker T^j \cap \text{ran} T^j) \right)$$

is dense in the Banach space $X$.

The following theorem generalises Example 2.1 to extended backward shifts. It also requires the operator $e^T \in \mathcal{L}(X)$ which is defined as

$$e^T = \sum_{j=0}^{\infty} \frac{1}{j!} T^j$$

for $T \in \mathcal{L}(X)$.

**Theorem 2.2.** Let $X$ be a Banach space and let $J \subset \mathcal{L}(X)$ be a separable admissible Banach ideal. If $T \in \mathcal{L}(X)$ is an extended backward shift then the generalised derivations $L_T - R_{-1}$ and $L_T - R_{-j}$ are hypercyclic on $J$, where $T' = \sum_{j=1}^{\infty} \frac{1}{j!} T^j$.

**Proof.** By an unpublished result of Grivaux and Shkarin [20], which can be found in [21, Theorem 8.6], the operators $I + T$ and $e^T$ satisfy the Hypercyclicity Criterion on $X$. (That $e^{B_w}$ satisfies the Hypercyclicity Criterion for the weighted backward shift $B_w$ on $c_0$ or $\ell^p$ for $1 \leq p < \infty$ was originally shown in [15].)

Applying the argument from Example 2.1 to the operator $I + T$, it follows that the generalised derivation $L_T - R_{-1}$ satisfies the Hypercyclicity Criterion on $J$ and is hence hypercyclic.

Next we consider the operator $e^T : X \to X$. Since it satisfies the Hypercyclicity Criterion there exists a dense linear subspace $X_0 \subset X$, an increasing sequence of positive integers $(n_k)$ and linear maps $S_{n_k} : X_0 \to X$, for $k \geq 1$, such that $e^T$ satisfies the Hypercyclicity Criterion.

We will show that the Hypercyclicity Criterion is satisfied by

$$L_{e^T} - R_{-1} : J \to J$$

for the dense subspace $X^* \otimes X_0 \subset J$, the sequence $(n_k)$ and the linear maps $L_{S_{n_k}} : X^* \otimes X_0 \to X^* \otimes X$ defined by $L_{S_{n_k}}(x^* \otimes x) = x^* \otimes S_{n_k}(x)$.

Notice for any rank one operator $x^* \otimes x \in X^* \otimes X_0$ that

$$\|(L_{e^T} - R_{-1})^{n_k} (x^* \otimes x)\|_J = \|x^* \otimes (T' + I)^{n_k} x\|_J = \|x^* \otimes (e^T)^{n_k} x\|_J$$

so condition (i) is satisfied. Conditions (ii) - (iii) follow as in Example 2.1 and arguing as before we have that the generalised derivation $L_{e^T} - R_{-1}$ satisfies the Hypercyclicity Criterion on $J$ and is hence hypercyclic. \qed

Another concrete example covered by Theorem 2.2 is given by the extended backward shift $D_w \in \mathcal{L}(X)$ defined as

$$D_w(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_4 x_4, w_6 x_6, \ldots)$$

for $(x_n) \in X$ and where $X = c_0$ or $\ell^p$ for $1 < p < \infty$. Grivaux [19] showed that $I + D_w \in \mathcal{L}(X)$ satisfies the Hypercyclicity Criterion, so it follows from Theorem 2.2 that $I + D_w$ and $e^{D_w}$ each induce hypercyclic generalised derivations on separable admissible Banach ideals of $\mathcal{L}(X)$. 
Next we uncover another family of hypercyclic generalised derivations, this time induced using forward shifts. For \( X = c_0 \) or \( \ell^p \), for \( 1 \leq p \leq \infty \), the weighted forward shift \( S_w \in \mathcal{L}(X) \) is defined as
\[
S_w(x_1, x_2, x_3, \ldots) = (0, w_1 x_1, w_2 x_2, \ldots)
\]
for \((x_n) \in X\) and where \( w = (w_n)_{n \geq 1} \in \ell^\infty \) is a bounded weight sequence of nonzero scalars. When \( w_n = 1 \) for all \( n \) we denote the unweighted forward shift as \( S : X \to X \).

**Example 2.3.** Let \( X = c_0 \) or \( \ell^p \) for \( 1 < p < \infty \) and \( J \subset \mathcal{L}(X) \) be an admissible Banach ideal. We note the separability of \( X^* \) implies that \( J \) is separable and we consider the weighted shift \( S_{(w_{n+1})} \in \mathcal{L}(X) \) defined as
\[
S_{(w_{n+1})}(x_1, x_2, x_3, \ldots) = (0, w_2 x_1, w_3 x_2, \ldots).
\]
Note that the adjoint \((I + S_{(w_{n+1})})^* = I + B_w \in \mathcal{L}(X^*)\) satisfies the Hypercyclicity Criterion \[27\] for some dense linear subspace \( X_0^* \subset X^* \), a sequence \((n_k)\) and linear maps \( S_{n_k} : X_0^* \to X^* \), for \( k \geq 1 \).

We modify the argument of Example 2.1 to show that
\[
L_I - R_{-S_{(w_{n+1})}} : J \to J
\]
satisfies the Hypercyclicity Criterion for the dense subspace \( X_0^* \otimes X \subset J \), the sequence \((n_k)\) and the linear maps \( R_{S_{n_k}} : X_0^* \otimes X \to X^* \otimes X \) defined as
\[
R_{S_{n_k}}(x^* \otimes x) = S_{n_k}^* (x^*) \otimes x.
\]

For any rank one operator \( x^* \otimes x \in X_0^* \otimes X \) we get that
\[
\left\| \left( L_I - R_{-S_{(w_{n+1})}} \right)^n (x^* \otimes x) \right\|_J = \left\| (I + S_{(w_{n+1})})^* (n_k x^*) \otimes x \right\|_J = \left\| (I + B_w)^{n_k} x^* \otimes x \right\|_J = \left\| (I + B_w)^{n_k} x^* \right\| \to 0.
\]

Using a similar argument to Example 2.1 it follows that conditions \[ii\] and \[iii\] of the Hypercyclicity Criterion are also fulfilled and hence the generalised derivation \( L_I - R_{-S_{(w_{n+1})}} \) is hypercyclic on \( J \).

We briefly note that Example 2.3 can be extended to the following theorem similar to how Theorem 2.2 generalises Example 2.1.

**Theorem 2.4.** Let \( X \) be a Banach space and let \( J \subset \mathcal{L}(X) \) be a separable admissible Banach ideal. If \( T^* \in \mathcal{L}(X^*) \) is an extended backward shift then the generalised derivations \( L_I - R_{-T} \) and \( L_I - R_{T^*} \) are hypercyclic on \( J \), where \( T = \sum_{j=1}^{\infty} \frac{1}{j} T^j \).

To finish this section we recall a result by Ansari and Bernal that states every infinite dimensional separable Banach space \( X \) supports an operator of the form \( I + K \in \mathcal{L}(X) \) that satisfies the Hypercyclicity Criterion, where \( I \in \mathcal{L}(X) \) is the identity operator and \( K \in \mathcal{M}(X) \) is a nuclear operator (cf. [1] Remark 2.12). Using a similar argument to Theorem 2.2 it follows that every separable admissible Banach ideal \( J \subset \mathcal{L}(X) \) supports a hypercyclic generalised derivation of the form \( L_K - R_{-J} \) and furthermore in Example 2.5 we will see the existence of one hypercyclic generalised derivation leads to many more examples. For this we require the notion of quasi-factors and the hypercyclic comparison principle.
For topological spaces $X_0$ and $X$, we say that the map $T: X \to X$ is a \textit{quasi-factor} of $T_0: X_0 \to X_0$ if there exists a continuous map with dense range $\Psi: X_0 \to X$ such that $T\Psi = \Psi T_0$, that is the following diagram commutes.

\[
\begin{array}{ccc}
X_0 & \xrightarrow{T_0} & X_0 \\
\downarrow{\Psi} & & \downarrow{\Psi} \\
X & \xrightarrow{T} & X
\end{array}
\]

When $T_0$ and $T$ are linear and $\Psi$ can be taken as linear we say $T$ is a \textit{linear quasi-factor} of $T_0$.

The \textit{hypercyclic comparison principle} states that hypercyclicity is preserved by quasi-factors and that linear quasi-factors preserve the Hypercyclicity Criterion and supercyclicity (cf. for instance [4, Section 1.1.1]).

**Example 2.5.** Suppose that the generalised derivation $L_A - R_B$ is hypercyclic on a separable Banach ideal $J \subset \mathcal{L}(X)$. We claim its quasi-factors give an abundant supply of hypercyclic generalised derivations.

Let $U \in \mathcal{L}(X)$ be invertible and note that the generalised derivations $L_{U^{-1}AU} - U^{-1}RB: J \to J$ and $L_A - R_{BU^{-1}}: J \to J$ are quasi-factors of $L_A - R_B$ via the following commuting diagrams.

\[
\begin{array}{ccc}
J & \xrightarrow{L_A - RB} & J \\
L_{U^{-1}} \downarrow & & \downarrow L_{U^{-1}} \\
J & \xrightarrow{L_{U^{-1}AU} - RB} & J
\end{array}
\quad
\begin{array}{ccc}
J & \xrightarrow{L_A - RB} & J \\
R_{U^{-1}} \downarrow & & \downarrow R_{U^{-1}} \\
J & \xrightarrow{L_A - R_{BU^{-1}}} & J
\end{array}
\]

Hence it follows by the hypercyclic comparison principle that $L_{U^{-1}AU} - RB$ and $L_A - R_{BU^{-1}}$ are hypercyclic on $J$ and thus we obtain a plentiful supply of hypercyclic generalised derivations.

3. **Classes of Non-Hypercyclic Derivations**

In this section we isolate large classes of operators in the opposite direction that do not induce hypercyclic generalised derivations. We begin by identifying an elementary spectral condition which uses the well known fact that the adjoint of a hypercyclic operator has no eigenvalues (cf. [4 Prop. 1.17]).

**Proposition 3.1.** Let $X$ be a Banach space and $A, B \in \mathcal{L}(X)$. If the point spectra $\sigma_p(A^*) \neq \emptyset$ and $\sigma_p(B) \neq \emptyset$ then the generalised derivation $\tau_{A,B}$ is not hypercyclic on any separable Banach ideal $J \subset \mathcal{L}(X)$.

**Proof.** By assumption we have that $A^*x^* = \alpha x^*$ and $Bx = \beta x$ for some eigenvalues $\alpha, \beta \in \mathbb{C}$ corresponding to nonzero eigenvectors $x \in X$, $x^* \in X^*$. We will show the adjoint $\tau_{A,B}^*: J^* \to J^*$ has a nonempty point spectrum.

We define the linear functional $\varphi \in J^*$ by $\varphi(T) = \langle x^*, Tx \rangle$, where $T \in J$ and we recall that it is bounded since for any $T \in J$

$$|\varphi(T)| = |\langle x^*, Tx \rangle| \leq ||x^*|| \cdot ||x|| \cdot ||T|| \leq ||x^*|| \cdot ||x|| \cdot ||T||.$$
We note for any \( T \in J \) that
\[
\langle \tau_{A,B}^* (\varphi), T \rangle = \langle \varphi, AT - TB \rangle = \varphi(AT) - \varphi(TB)
\]
\[
= \langle A^* x^*, Tx \rangle - \langle x^*, T B x \rangle = \alpha \langle x^*, Tx \rangle - \beta \langle x^*, Tx \rangle
\]
\[
= \alpha \varphi(T) - \beta \varphi(T) = (\alpha - \beta) \langle \varphi, T \rangle.
\]
So \( \alpha - \beta \) is an eigenvalue for \( \tau_{A,B}^* \) and \( \tau_{A,B} \) is not hypercyclic on \( J \).
\( \square \)

We can apply Proposition 3.3 for instance to the concrete generalised derivation \( \tau_{S,B} \in \mathcal{L}(J) \), where \( X = c_0 \) or \( l^p \) for \( 1 < p < \infty \), \( J \subset \mathcal{L}(X) \) is a separable Banach ideal and \( B, S \in \mathcal{L}(X) \) are, respectively, the backward and forward shifts. Recall that the adjoint of the forward shift is \( S^* = B \) and the point spectrum of \( B \) is the open unit disc \( \mathbb{D} \). Hence both \( \sigma_p (B) \) and \( \sigma_p (S^*) \) are nonempty so it follows from Proposition 3.1 that \( \tau_{S,B} \) is not hypercyclic on \( J \).

Next we recall the notion of supercyclicity which is required in the following remark. For a separable Banach space \( X \), we say \( T \in \mathcal{L}(X) \) is supercyclic if there exists \( x \in X \) such that its projective orbit under \( T \) is dense in \( X \), that is
\[
\{ AT^n x : n \geq 0, \lambda \in \mathbb{C} \} = X.
\]
We note that the class of hypercyclic operators is strictly contained in the class of supercyclic operators (cf. [4] Example 1.15).

**Remark 3.2.** We recall that Herrero [24] proved if \( T \in \mathcal{L}(X) \) is supercyclic then either \( \sigma_p (T^*) = \emptyset \) or \( \sigma_p (T^*) = \{ \lambda \} \) for some \( \lambda \neq 0 \) (cf. [4] Proposition 1.26). Hence if either \( A^* \) or \( B \) from Proposition 3.1 has more than one eigenvalue then \( \tau_{A,B} \) is not even supercyclic on \( J \).

We illustrate Remark 3.2 using diagonal operators. Let \( H \) be a separable Hilbert space with orthonormal basis \( (e_j) \) and let \( J \subset \mathcal{L}(H) \) be a separable Banach ideal. The diagonal operators \( D_1, D_2 \in \mathcal{L}(H) \) are defined as
\[
D_1 e_j = \alpha_j e_j, \quad D_2 e_j = \beta_j e_j
\]
where \( (\alpha_j), (\beta_j) \) are bounded sequences of scalars for \( j \geq 1 \). Note for each \( j \geq 1 \) that the complex conjugate \( \overline{\alpha_j} \in \sigma_p (D_1^*) \) and \( \beta_j \in \sigma_p (D_2) \). So it follows from Proposition 3.1 that \( \tau_{D_1,D_2} \) is not hypercyclic and if \( \alpha_i \neq \alpha_j \) and \( \beta_i \neq \beta_j \) for some \( i, j \) then by Remark 3.2 \( \tau_{D_1,D_2} \) is not supercyclic on \( J \).

Next we extend Proposition 3.1 to the more general class of elementary operators. We recall for a Banach space \( X \), the map \( \mathcal{E}_{A,B} : \mathcal{L}(X) \to \mathcal{L}(X) \) is an elementary operator if
\[
\mathcal{E}_{A,B} = \sum_{j=1}^n L_{A_j} R_{B_j}
\]
where \( A = (A_1, \ldots, A_n) \), \( B = (B_1, \ldots, B_n) \in \mathcal{L}(X)^n \) are fixed \( n \)-tuples given by \( A_j \in \mathcal{L}(X) \) and \( B_j \in \mathcal{L}(X) \) for \( j = 1, \ldots, n \).

**Proposition 3.3.** Let \( X \) be a Banach space and \( A, B \in \mathcal{L}(X)^n \) for \( n \geq 1 \). If the operators \( A_j^* \) have eigenvalues sharing a common eigenvector and the operators \( B_j \) have eigenvalues sharing a common eigenvector, for \( 1 \leq j \leq n \), then the elementary operator \( \mathcal{E}_{A,B} \) is not hypercyclic on any separable Banach ideal \( J \subset \mathcal{L}(X) \).

**Proof.** By assumption we have \( A_j^* x^* = \alpha_j x^* \) and \( B_j x = \beta_j x \) for eigenvalues \( \alpha_j, \beta_j \in \mathbb{C} \) corresponding to nonzero eigenvectors \( x \in X \) and \( x^* \in X^* \), for \( 1 \leq j \leq n \).
As in the proof of Proposition 3.1 we define the continuous linear functional \( \varphi \in J^* \) by \( \varphi(T) = \langle x^*, Tx \rangle \), where \( T \in J \). Notice for the adjoint \( \mathcal{E}_{A,B}^* : J^* \to J^* \) and any \( T \in J \) that

\[
\langle \mathcal{E}_{A,B}^*(\varphi), T \rangle = \sum_{j=1}^{n} \varphi(A_j TB_j) = \sum_{j=1}^{n} \langle A_j^* x^*, T B_j x \rangle = \sum_{j=1}^{n} \alpha_j \beta_j \langle \varphi, T \rangle.
\]

Hence \( \sum_{j=1}^{n} \alpha_j \beta_j \) is an eigenvalue of \( \mathcal{E}_{A,B}^* \) and \( \mathcal{E}_{A,B} \) is not hypercyclic on \( J \). \( \square \)

To illustrate Proposition 3.3 let \( X = c_0 \) or \( \ell^p \) for \( 1 < p < \infty \). We consider the separable Banach ideal \( J \subset \mathcal{L}(X) \) and the elementary operator

\[
\mathcal{E}_{U,V} : J \to J
\]

where \( U = (S, I, S^2), V = (I, B, B^2) \in \mathcal{L}(X)^3 \) and \( B, S \in \mathcal{L}(X) \) are the backward and forward shift operators. A common eigenvector for the 3-tuple \( V \) is easily obtained, we choose \( \beta \in \mathbb{D} \) which gives \( x = (1, \beta, \beta^2, \ldots) \in X \) such that \( Bx = \beta x \). Hence \( 1, \beta, \beta^2 \) are, respectively, eigenvalues for \( I, B, B^2 \). Similarly we choose \( \alpha \in \mathbb{D} \) which gives the eigenvector \( x^* = (1, \alpha, \alpha^2, \ldots) \in X^* \) such that \( \alpha, 1, \alpha^2 \) are the corresponding eigenvalues for \( S^*, I, S^* \). So it follows from Proposition 3.3 that \( \mathcal{E}_{U,V} \) is not hypercyclic on \( J \).

Next we extend some observations on commutator maps induced by Riesz operators made in [18] to the generalised derivation case. We recall that \( T \in \mathcal{L}(X) \) is a Riesz operator if its essential spectrum \( \sigma_e(T) = \{0\} \) and they are never hypercyclic [21, p. 160]. The spectrum of \( T \) is \( \sigma(T) = \{0\} \cup \{\lambda_n : n \geq 1\} \), where \( \{\lambda_n : n \geq 1\} \) is a discrete, at most countable (possibly empty) set containing nonzero eigenvalues of finite multiplicity.

The spectrum of the generalised derivation \( \tau_{A,B} \) acting on \( \mathcal{L}(X) \) was initially identified by Lumer and Rosenblum [28]. Their formula extends to \( \tau_{A,B} \) restricted to any Banach ideal \( J \subset \mathcal{L}(X) \) and satisfies

\[
\sigma_J(\tau_{A,B}) = \sigma(A) - \sigma(B)
\]

where \( \sigma_J(\tau_{A,B}) \) denotes the spectrum of \( \tau_{A,B} : J \to J \). Further details can be found, for instance, in the survey [35, Theorem 3.12].

We also recall Kitai’s [25] spectral condition that every connected component of the spectrum of a hypercyclic operator intersects the unit circle (cf. [4, Theorem 1.18]).

**Theorem 3.4.** Let \( X \) be a Banach space and \( J \subset \mathcal{L}(X) \) be a separable Banach ideal. If \( A, B \in \mathcal{L}(X) \) are Riesz operators then the induced generalised derivation \( \tau_{A,B} : J \to J \) is not hypercyclic.

**Proof.** The spectrum of \( \tau_{A,B} \) on \( J \) is given by

\[
\sigma_J(\tau_{A,B}) = \sigma(A) - \sigma(B) = \{\alpha_m - \beta_n : \alpha_m \in \sigma(A), \beta_n \in \sigma(B), m, n \geq 0\}
\]

where for convenience we set \( \alpha_0 = 0 = \beta_0 \).

Note that \( \sigma_J(\tau_{A,B}) \) is a closed and compact set which is at most countable. So it is a discrete set containing the singleton \( \{0\} \) as a connected component and it follows from the spectral condition of Kitai that \( \tau_{A,B} \) is not hypercyclic on \( J \). \( \square \)
We remark that compact operators are an important class of Riesz operators and hence it follows by Theorem 3.4 that $\tau_{A,B}$ is not hypercyclic on any separable Banach ideal $J \subset \mathcal{L}(X)$ for compact $A, B \in \mathcal{K}(X)$.

4. Normal Derivations

The Hilbert space setting provides interesting families of non-hypercyclic operators and in this section we extend some observations made in [18] to the generalised derivation case. Here $H$ denotes an infinite dimensional Hilbert space over the complex field.

Generalised derivations induced by normal operators and acting on $\mathcal{L}(H)$, or normal derivations, were first studied by Anderson [1] and Anderson and Foiaş [2]. The Banach ideal case was then investigated by Maher [29] and Kittaneh [26].

We recall that $T \in \mathcal{L}(H)$ is positive, denoted $T \geq 0$, if the inner product $\langle Tx, x \rangle \geq 0$ for all $x \in H$. We say $T \in \mathcal{L}(H)$ is hyponormal if $T^*T - TT^* \geq 0$ or equivalently if $\|Tx\| \geq \|T^*x\|$ for all $x \in H$. The class of hyponormal operators contains some well known classes of operators such as the subnormal, normal and self-adjoint operators [23]. Kitai [25] showed hyponormal operators are never hypercyclic and Bourdon [8] proved that they are never even supercyclic.

We consider generalised derivations induced by hyponormal operators acting on the space of Hilbert-Schmidt operators $C_2$, which is complete in the Hilbert-Schmidt norm $\|T\|_2^2 = \text{tr}(T^*T) = \sum_j (T^*Te_j, e_j) = \sum_j (Te_j, Te_j) = \sum_j \|Te_j\|^2$ where $T \in C_2$, $\text{tr}(T)$ denotes the trace of $T$ and $(e_j)$ is any orthonormal basis of $H$. It is a Hilbert space with the corresponding inner product $\langle S, T \rangle = \text{tr}(T^*S)$.

Further details on Hilbert-Schmidt operators can be found in [12].

We note that a characterisation of hyponormal generalised derivations is stated, without proof, in [11, p. 50]. In the proof of Theorem 4.1 below we recall the part we need for the convenience of the reader.

**Theorem 4.1.** Let $A, B \in \mathcal{L}(H)$ be such that $A$ and $B^\ast$ are hyponormal. Then the generalised derivation $\tau_{A,B} : C_2 \to C_2$ is not supercyclic.

**Proof.** We first observe that

\[
\tau_{A,B}^* \tau_{A,B} - \tau_{A,B} \tau_{A,B}^* = (L_{A^\ast} - R_{B^\ast})(L_A - R_B) - (L_A - R_B)(L_{A^\ast} - R_{B^\ast}) = L_{A^\ast A} - L_A \cdot R_B - R_B \cdot L_A + R_{BB^\ast}
\]

\[
- L_{AA^\ast} + L_A R_{B^\ast} + R_B L_{A^\ast} - R_{B^\ast B} = L_{A^\ast A - AA^\ast} + R_{BB^\ast - B^\ast B}
\]

where above we have used the facts that $LSR_T = R_TLS$ and $R_SR_T = R_STS$ for any $S, T \in \mathcal{L}(H)$.

For any $T \in C_2$ notice that

\[
\langle L_{A^\ast A - AA^\ast} (T), T \rangle = \langle A^\ast AT - AA^\ast T, T \rangle = \langle AT, AT \rangle - \langle A^\ast T, A^\ast T \rangle = \|AT\|^2 - \|A^\ast T\|^2.
\]
Since $A$ is hyponormal we know that $\|AT e_n\| \geq \|A^* T e_n\|$ for all $n \geq 1$ and any orthonormal basis $(e_n)$ of $H$. Hence
$$\|AT\|^2 = \sum_n \|AT e_n\|^2 \geq \sum_n \|A^* T e_n\|^2 = \|A^* T\|^2$$
and hence $\|AT\|^2 - \|A^* T\|^2 \geq 0$ for all $T \in C_2$.

Similarly for any $T \in C_2$
$$\langle RBB^* - B^* B (T), T \rangle = \langle B^* T, B^* T \rangle - \langle BT, BT \rangle = \|B^* T\|^2 - \|BT\|^2$$
and it follows from the hyponormality of $B^*$ that $\|B^* T\|^2 - \|BT\|^2 \geq 0$.

Hence $\tau_A^* B \tau_A B - \tau_A B \tau_A^* B \geq 0$ so $\tau_{A,B}$ is hyponormal and cannot be supercyclic on $C_2$. □

Next we extend Theorem 4.1 using the hypercyclic comparison principle.

**Corollary 4.2.** Let $A, B \in L(H)$ be such that $A$ and $B^*$ are hyponormal and let $J$ be an admissible Banach ideal contained in $C_2$. Then the induced generalised derivation $\tau_{A,B}$ is never supercyclic on $J$.

**Proof.** The finite rank operators are contained in $J$ and they form a dense subset of $C_2$. This gives a natural linear inclusion, with dense range, $\Psi: J \rightarrow C_2$. Hence $\tau_{A,B}: C_2 \rightarrow C_2$ is a linear quasi-factor of $\tau_{A,B}: J \rightarrow J$ via the following commuting diagram.

\[
\begin{array}{ccc}
J & \xrightarrow{\tau_{A,B}} & J \\
\Psi \downarrow & & \downarrow \Psi \\
C_2 & \xrightarrow{\tau_{A,B}} & C_2
\end{array}
\]

If $\tau_{A,B}$ was supercyclic on $J$ then it would follow by the comparison principle that it is supercyclic on $C_2$. However we know from Theorem 4.1 that $\tau_{A,B}$ is not supercyclic on $C_2$ and hence $\tau_{A,B}$ is not supercyclic on $J$. □

5. **Elementary Operators on the Argyros-Haydon Space**

Argyros and Haydon resolved the famous *scalar-plus-compact* problem with the construction of the extreme Banach space $X_{AH}$ [3]. While this type of space is rare, $X_{AH}$ is relatively nice and possesses many remarkable properties. In this section we reveal some surprising differences in the hypercyclic behaviour of elementary operators acting on $\mathcal{L}(X_{AH})$ and $\mathcal{K}(X_{AH})$.

We first recall some properties of $X_{AH}$ relevant to our discussion. The space $X_{AH}$ has a Schauder basis and every $T \in \mathcal{L}(X_{AH})$ is of the form
$$T = \lambda I + K$$
where $\lambda \in \mathbb{C}$ and $K \in \mathcal{K}(X_{AH})$ is a compact operator.

The existence of a Schauder basis implies that $X_{AH}$ possesses the approximation property and it is shown in [3] that the dual $X_{AH}^*$ is isomorphic to the sequence space $\ell^1$ and is therefore separable. Hence the space of compact operators $\mathcal{K}(X_{AH})$ is a separable admissible Banach ideal under the operator norm topology and it further follows that $\mathcal{L}(X_{AH}) = \mathbb{C} \cdot I + \mathcal{K}(X_{AH})$ is separable under the operator norm topology.

The separability of $\mathcal{L}(X_{AH})$ naturally leads to the question of whether it supports hypercyclic elementary operators. In [21] p. 300 an argument is partially
outlined showing that the left multiplier is not hypercyclic on \( \mathcal{L}(X_{AH}) \). Here we establish a somewhat more general observation for elementary operators acting on particular Banach algebras and we then apply it to \( \mathcal{L}(X_{AH}) \).

For a Banach algebra \( \mathcal{A} \), the elementary operator \( E_{a,b} : \mathcal{A} \to \mathcal{A} \) is given by

\[
E_{a,b} = \sum_{j=1}^{n} L_{a_j} R_{b_j}
\]

where \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \in \mathcal{A}^n \), \( n \geq 1 \) and we define \( L_{a_j}(s) = a_j s \), \( R_{b_j}(s) = s b_j \) for any \( s \in \mathcal{A} \) and \( j = 1, \ldots, n \).

We further recall that a multiplicative linear functional \( \varphi : \mathcal{A} \to \mathbb{C} \) is a nonzero linear functional such that \( \varphi(ab) = \varphi(a)\varphi(b) \) for all \( a, b \in \mathcal{A} \) and it is well known that they are always continuous.

**Theorem 5.1.** Let \( \mathcal{A} \) be a Banach algebra which admits a non-trivial multiplicative linear functional \( \varphi \in \mathcal{A}^* \). Then the elementary operator \( E_{a,b} : \mathcal{A} \to \mathcal{A} \) is not hypercyclic.

**Proof.** We will prove that \( E_{a,b} \) is not hypercyclic on \( \mathcal{A} \) by showing that \( \varphi \) is an eigenvector of the adjoint \( E_{a,b}^* : \mathcal{A}^* \to \mathcal{A}^* \).

Notice for any \( s \in \mathcal{A} \) that

\[
\langle E_{a,b}^*(\varphi), s \rangle = \langle \varphi, E_{a,b}(s) \rangle = \varphi \left( \sum_{j=1}^{n} a_j s b_j \right)
\]

\[
= \sum_{j=1}^{n} \varphi(a_j)\varphi(s)\varphi(b_j) = \varphi(s) \sum_{j=1}^{n} \varphi(a_j)\varphi(b_j)
\]

\[
= \left( \sum_{j=1}^{n} \varphi(a_j)\varphi(b_j) \right) \langle \varphi, s \rangle,
\]

where the step on line (5.2) follows from the linearity and multiplicativity of \( \varphi \). Hence \( \varphi \) is an eigenvector of \( E_{a,b}^* \) corresponding to the eigenvalue \( \sum_{j=1}^{n} \varphi(a_j)\varphi(b_j) \) and it follows that \( E_{a,b} \) is not hypercyclic on \( \mathcal{A} \). \( \square \)

When the Banach algebra \( \mathcal{A} \) contains a unit element we note that Saldivia [37, Theorem 5.3] showed the left and right multipliers are not topologically transitive on \( \mathcal{A} \). We recall that \( T : \mathcal{A} \to \mathcal{A} \) is a left multiplier if \( T(ab) = T(a)b \) and a right multiplier if \( T(ab) = aT(b) \) for all \( a, b \in \mathcal{A} \). When \( \mathcal{A} \) is an infinite dimensional and separable Banach algebra the Birkhoff Transitivity Theorem (cf. [4, Theorem 1.2]) implies that topological transitivity is equivalent to hypercyclicity. So when \( \mathcal{A} \) contains a unit element this gives an alternative proof that \( L_a \) and \( R_a \) are not hypercyclic on \( \mathcal{A} \) for any \( a \in \mathcal{A} \).

Next we apply Theorem 5.1 to the Banach algebra \( \mathcal{L}(X_{AH}) \).

**Corollary 5.2.** No elementary operator is hypercyclic on \( \mathcal{L}(X_{AH}) \).

**Proof.** We define the linear functional \( \varphi : \mathcal{L}(X_{AH}) \to \mathbb{C} \) by

\[
\varphi(\lambda I + K) = \lambda
\]

where \( \lambda \in \mathbb{C} \) and \( K \in \mathcal{K}(X_{AH}) \). This is well defined since it follows from line (5.1) that the representation of any operator in \( \mathcal{L}(X_{AH}) \) is unique.
Further notice $\varphi$ is a multiplicative functional since for any $T = \lambda_0 I + K_0$ and $S = \lambda I + K \in \mathscr{L}(X_{AH})$ we have that

$$\varphi(TS) = \varphi(\lambda_0 \lambda I + \lambda K + \lambda K_0 + K_0) = \lambda_0 \lambda$$

and

$$\varphi(T) \varphi(S) = \varphi(\lambda_0 I + K_0) \varphi(\lambda I + K) = \lambda_0 \lambda.$$

So it follows by Theorem 5.1 that elementary operators are not hypercyclic on $\mathscr{L}(X_{AH})$. \hfill $\square$

The argument from Corollary 5.2 does not apply to $\mathscr{K}(X_{AH})$ since $\varphi|_{\mathscr{K}(X_{AH})} \equiv 0$. However in the next example the subtle nature of the hypercyclicity of elementary operators is revealed when we see that the space $\mathscr{K}(X_{AH})$ supports hypercyclic generalised derivations.

**Example 5.3.** We claim there exists a hypercyclic generalised derivation on the separable admissible Banach ideal $\mathscr{K}(X_{AH}) \subset \mathscr{L}(X_{AH})$. By a result of Ansari and Bernal [21, Theorem 8.9] every infinite dimensional separable Banach space supports an operator that satisfies the Hypercyclicity Criterion. So in particular there exists some $T = \lambda I + K \in \mathscr{L}(X_{AH})$, for $\lambda \in \mathbb{C}$ and $K \in \mathscr{K}(X_{AH})$, such that $T$ satisfies the Hypercyclicity Criterion.

By using an argument similar to Example 2.1, it follows that the generalised derivation $L_K - R_{-\lambda I} : \mathscr{K}(X_{AH}) \to \mathscr{K}(X_{AH})$ is hypercyclic. \hfill $\square$

Hence the seemingly minor difference of one dimension between $\mathscr{K}(X_{AH})$ and $\mathscr{L}(X_{AH})$ completely alters the hypercyclicity of the generalised derivation $L_K - R_{-\lambda I}$. The delicate nature of this question is further illustrated below where we show that $\mathscr{K}(X_{AH})$ does not support any hypercyclic commutator maps.

**Proposition 5.4.** Let $A \in \mathscr{L}(X_{AH})$. Then the commutator map $\Delta_A = L_A - R_A$ is not hypercyclic on $\mathscr{K}(X_{AH})$.

**Proof.** The operator $A \in \mathscr{L}(X_{AH})$ has the form $A = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and $K \in \mathscr{K}(X_{AH})$. So $\Delta_A : \mathscr{K}(X_{AH}) \to \mathscr{K}(X_{AH})$ is given by

$$\Delta_A = L_{\lambda I + K} - R_{\lambda I + K} = L_K - R_K = \Delta_K.$$

Since $\mathscr{K}(X_{AH})$ is a separable Banach ideal and $K$ is compact it follows from Theorem 3.4 that $\Delta_A$ is not hypercyclic on $\mathscr{K}(X_{AH})$. \hfill $\square$

We remark that the argument from Proposition 5.4 can also be used to prove directly that no commutator operator is hypercyclic on $\mathscr{L}(X_{AH})$.

### 6. Further Questions

Some natural questions arising from this paper include the following.

1. Do reasonable sufficient conditions exist on the pair $(A, B)$ that induce hypercyclic $\tau_{A,B}$ on separable Banach ideals?

2. Corollary 5.2 and Example 5.3 contrast the hypercyclic properties of particular generalised derivations on different spaces. Furthermore in [18] it is shown for particular Banach spaces $X$ that $\mathscr{N}(X)$ does not support a hypercyclic commutator operator. Do there exist further examples of contrasting hypercyclic behaviour of elementary operators on different Banach ideals?
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