NONLINEAR SECOND ORDER OSCILLATORS OFF RESONANCE AT CERTAIN FUNCTIONAL SPACES

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Abstract. We will deal with the existence of odd and $T$-periodic solutions of the scalar equation

\begin{equation}
\label{eq:1}
u'' + g(u) = k(t),
\end{equation}

where $g : \mathbb{R} \to \mathbb{R}$ is an odd function and $k$ is an odd and $T$-periodic function of mean zero. By putting (1) in means of

\[ Lu = Nu, \]

where $Lu = u''$ is the linear part and $Nu = k - g(u)$ is the nonlinear part, generally if one denotes by $P_T$ the continuous $T$-periodic functions and by $Q_T$ the continuous $T$-periodic functions of mean zero, then

\[ L : P_T \cap C^2 \to P_T \]

and one have that $\text{Ker}(L) = \mathbb{R}$ and $\text{Rank}(L) = Q_T$, which is clearly a resonant problem. We will consider the space of odd and $T$-periodic functions where one can avoid resonance. In this space we state two results of existence, one including a priori bounds and one of uniqueness. This results generalize the results obtained by Hamel in \cite{2} on the periodic problem for the forced pendulum equation.

1. Observations at resonace

We denote $\mathcal{H}$ the set of continuous, odd and $T$-periodic functions.

\[ \mathcal{H} := \{ u \in P_T | u(t) = -u(-t) \ \forall \ t \in [0, T] \} \]

Observation 1. $\mathcal{H}$ is a complete normed space.

Lemma 1. Let $u \in P_T$ be an odd function, then $\overline{u} := \frac{1}{T} \int_0^T u \, dt = 0$

Proof. Let

\begin{align*}
\Omega^+ & := \{ t \in [-T/2, T/2] : u(t) > 0 \}, \\
\Omega^- & := \{ t \in [-T/2, T/2] : u(t) < 0 \}.
\end{align*}

If $t \in \Omega^+ \Rightarrow u(t) > 0 \Rightarrow -u(-t) > 0 \Rightarrow u(-t) < 0 \Rightarrow -t \in \Omega^-$, therefore $-\Omega^+ \subseteq \Omega^-$. In a similar way one proves that $-\Omega^- \subseteq \Omega^+$ and consequently

\begin{equation}
\label{eq:2}
\Omega^+ = -\Omega^-.
\end{equation}
From (2), it follows that
\[ u = \frac{1}{T} \left( \int_{\Omega^+} u(t) \, dt + \int_{\Omega^-} u(t) \, dt \right) = \frac{1}{T} \left( \int_{\Omega^+} u(-t) \, dt + \int_{\Omega^-} u(t) \, dt \right). \]

Finally, the odd property of \( u \) in (3) implies
\[ u = \frac{1}{T} \left( -\int_{\Omega^-} u(t) \, dt + \int_{\Omega^-} u(t) \, dt \right) = 0. \]

**Observation 2.** \( \mathcal{H} \subset Q_T \).

**Lemma 2.** \( L|_{\mathcal{H}} : dom(L) \cap \mathcal{H} \rightarrow \mathcal{H} \) is invertible.

From the above observation it is clear that \( L(\mathcal{H}) \subset P_T \), therefore it is necessary to prove that \( Lu \) is an odd function for all odd functions \( u \).

**Proof.** For \( u' \), we have
\[ u'(t) = \lim_{h \to 0} \frac{u(t) - u(t + h)}{h} = \lim_{h \to 0} \frac{u(-t) - u(-t + h)}{-h} = u'(-t), \]

it follows that \( u' \) is an even function. Using (4) in \( u'' \) it follows that
\[ u''(t) = \lim_{h \to 0} \frac{u'(t) - u'(t + h)}{h} = \lim_{h \to 0} \frac{u'(-t) - u'(-t + h)}{-h} = -u''(-t). \]

This proves that \( L|_{\mathcal{H}} : dom(L) \cap \mathcal{H} \rightarrow \mathcal{H} \) is well defined. It is well known from [3] the existence of an integral operator \( S : Q_T \rightarrow P_T \), a right inverse of \( L \) such that
\[ \|S(f)\|_\infty \leq \frac{T^2}{2} \|f\|_\infty \]

and \( S(\mathcal{H}) \subseteq dom(L) \cap \mathcal{H} \), for it is sufficient to observe that \( Ker(L|_{\mathcal{H}}) = \{0\} \). □

Lemma 2 tells us that (1) is a non-resonant problem in \( \mathcal{H} \). Naturally, the question arises to answer when does \( Nu \in \mathcal{H} \), in order to express any solution of (1) as a classic fixed point problem of the form
\[ u = L^{-1} Nu. \]

From now on, let us put \( L := L|_{\mathcal{H}} \).

### 2. Existence of Solutions in \( \mathcal{H} \) in the Sublinear Case

Since odd functions form are closed under composition we get
\[ \overline{g(u)} = 0, \]

and even more
\[ Nu = k - g(u) \in \mathcal{H} \]

if \( g \) is an odd function. The latter discussion lead us to our first result.
Theorem 1. Let \( g : \mathbb{R} \to \mathbb{R} \) an odd sublinear function and \( k \in Q_T \) an odd function, then equation
\[
u'' + g(u) = k(t)\]
has an odd and \( T \)-periodic solution. Even more, the set of solutions is bounded.

Proof. Based on the discussion of the first section and the latter remark it is clear that
\[
u = k - g(u) \in \mathcal{H},\]
and that \( u \) is a solution of (1) if and only if \( u \) is a fixed point of
\[
K(u) = L^{-1}Nu.
\]
From Schäfer’s theorem it suffices to show that the set
\[
\Sigma := \{ u \in \mathcal{H} : u = \lambda L^{-1}Nu, \lambda \in (0,1] \}
\]
is bounded. Let us suppose the opposite and let us take \((u_n) \subset \Sigma\) and \((\lambda_n) \subset (0,1]\) such that
\[
u_n = \lambda_nL^{-1}Nu_n
\]
and
\[
\|u_n\|_\infty \to \infty \quad \text{as} \quad n \to \infty.
\]
The sublinearity of \( g \) implies that, for all \( \epsilon > 0 \) there exists \( M := M(\epsilon) \) such that
\[
g(t) < M + \epsilon t.
\]
For \( n \in \mathbb{N} \) it follows that
\[
\|u_n\|_\infty \leq \lambda_n\|L^{-1}\|_\mathcal{L}\|k - g(u_n)\|_\infty
\]
\[
\leq \lambda_n\|L^{-1}\|_\mathcal{L}(\|k\|_\infty + M(\epsilon) + \epsilon\|u_n\|_\infty)
\]
\[
\leq \lambda_n\epsilon\|L^{-1}\|_\mathcal{L}\|u_n\|_\infty + C(\epsilon)
\]
\((C(\epsilon) > 0)\).

Consequently,
\[
\infty = \lim_{n \to \infty} \|u_n\|_\infty \leq \lim_{n \to \infty} \frac{C(\epsilon)}{1 - \lambda_n\epsilon\|L^{-1}\|_\mathcal{L}},
\]
which in any way means that
\[
\lim_{n \to \infty} \lambda_n = \frac{1}{\epsilon\|L^{-1}\|_\mathcal{L}}.
\]
By putting \( \epsilon < \|L^{-1}\|^{-1} \) we get
\[
\lim_{n \to \infty} \lambda_n > 1
\]
which contradicts the fact that \((\lambda_n) \subset (0,1]\). □
3. Uniqueness of solutions in $H$ under convexity conditions

Before stating the main result let us remember that we first considered $L$ as an operator with domain in $P_T$ and therefore $\|S\|_L = \|L^{-1}\|_L$ depends only on $T$.

**Theorem 2.** If $g \in C^1(\mathbb{R})$ is an odd function such that $\|g'\|_\infty < 2/T^2$ and $k \in Q_T$ is an odd function then there exists a unique solution in $H$ of equation

$$u'' + g(u) = k(t).$$

Which between lines tells us that the uniqueness depends on the period $T$.

**Proof.** As in Theorem 1, we look for a fixed point of the equation

$$K(u) = L^{-1}Nu \quad (u \in H).$$

The fact that $\|g'\|_\infty < 2/T^2$, implies the existence of $\lambda \in (0, 1)$ such that

$$\|L^{-1}\|_L |g(x) - g(y)| < \lambda |x - y| \quad \text{for all } x, y \in \mathbb{R}. $$

Then, if $u, v \in H$ we obtain

$$\|K(u) - K(v)\|_\infty \leq \|L^{-1}\|_L \|Nu - Nv\|_\infty$$

$$= \|L^{-1}\|_L \|g(v) - g(u)\|_\infty$$

$$\leq \lambda \|v - u\|_\infty = \lambda \|u - v\|_\infty.$$ 

Banach’s fixed point theorem guarantees the existence and uniqueness of a solution of (1) in $H$. \hfill $\Box$

**References**

[1] P. Amster and M. Clapp, *Periodic solutions of resonant systems with rapidly rotating nonlinearities*, 2010.

[2] H. G., *Über erzwungene Schwingungen bei endlichen Amplituden*, Math. Ann., 86 (1922), 1-13.

[3] A. Lazer, *On Schauder’s fixed point theorem and forced second-order nonlinear oscillations*, J. Math. Anal. Appl. 21 (1968), 421-425.