Dynamical Analysis of Fractional-Order Hastings-Powell Food Chain Model with Alternative Food

Moh. Nurul Huda¹, Trisilowati², Agus Suryanto²

¹Master Program of Mathematics, University of Brawijaya, Malang, Indonesia
²Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Brawijaya, Malang, Indonesia

Abstract
In this paper, a fractional-order Hastings-Powell food chain model is discussed. It is assumed that the top-predator population is supported by alternative food. Existence and local stability of equilibrium points of fractional-order system are investigated. Numerical simulations are conducted to illustrate analysis results. The analysis results show that alternative food can give a positive impact for top-predator population.

Keywords: Alternative food, Fractional-order, Grunwald-Letnikov approximation, Hastings-Powell model, Stability.

INTRODUCTION
Nowadays, fractional calculus becomes the main focus for the researchers. Many problems of science and engineering can be modeled by using fractional derivatives. The process of developing a differential system of integer order into fractional-order becomes popular in dynamic systems [1]. Basically, a biological mathematical model in predicting the future, not only depends on the current but also the memory or the previous condition. In fractional derivatives, at some certain conditions contain information of previous condition, therefore fractional derivatives can be used to explain more realistic natural phenomena.

Interactions between populations can be described in a food chain model. One of the interactions in the food chain is predation process. Mathematical model used to describe interactions between predator and prey is called the predator-prey model. Furthermore, many interesting phenomena in ecology can be described by mathematical model through predator-prey models such as harvesting in predator population [1], supplying alternative food in a predator population [2], refuging prey population [3], spreading disease in ecosystem [4], and the effect of the present an omnivore [5]. In predator prey model [2], it is assumed that prey populations do not always exist, they also experience migration to find new habitats due to climate change factors and low food reserves.

Therefore, predators need additional food or alternative prey to survive.

In this paper, a food chain model of three-species fractional-order with alternative food is introduced. Examples of three-species ecosystems in this model: vegetation-hare-lynx, mouse-snake-owl and worm-robin-falcon. Moreover, predator-prey food chains have been studied in structured populations in [5,6]. In this paper, Model is a modification of model [7]. Then the conditions of existence and stability of the equilibrium points of the fractional-order are examined in the result and discussion. Numerical simulations are illustrated by the Grunwald-Letnikov approximation [8].

MATERIALS AND METHODS
Model Formulation
In this research, the predatory-prey model [7] is the main object of the study. The model construction is done by modifying the model of Sahoo and Poria [2] by changing the integer order into the fractional-order.

Determination of the Equilibrium Point
In dynamical analysis, the first step is to determine the equilibrium points. The equilibrium point is obtained when the population rate of the system is unchanged or zero. From this condition, the existence properties of each equilibrium points is also obtained.

Stability of the Equilibrium Point
In this paper, the local stability of equilibrium points is analyzed. The discussion of local stability is begun by linearizing the model by using Taylor series. The linearization around its equilibrium...
number of intermediate predators and the top predators can reduce predation rates in intermediate-predators \([2]\), then to give this effect, model (1) can be modified to

\begin{align}
D_t^\alpha X &= R_0 X \left(1 - \frac{X}{K_0}\right) - C_1 A_1 Y X, \\
D_t^\alpha Y &= A_1 X Y - B_2 Y - D_1 Y, \\
D_t^\alpha Z &= C_2 A_2 Z \left(\frac{A Y}{B_2 + \gamma} + (1 - A)\right) - D_2 Z,
\end{align}

with \(0 < \alpha < 1\).

Model (1) explains that top-predator food sources only depend on intermediate-predators. However, alternative prey (supplementary feeding) for top-predators can reduce predation rates in intermediate-predators \([2]\), then to give this effect, model (1) can be modified to

\begin{align}
D_t^\alpha X &= R_0 X \left(1 - \frac{X}{K_0}\right) - C_1 A_1 Y X, \\
D_t^\alpha Y &= A_1 X Y - B_2 Y - D_1 Y, \\
D_t^\alpha Z &= C_2 A_2 Z \left(\frac{A Y}{B_2 + \gamma} + (1 - A)\right) - D_2 Z,
\end{align}

where \(A\) is a time independent constant to get the alternative resource \((0 < A < 1)\). To make easier the model analysis, variables and some parameter are selected to be

\begin{align}
x &= \frac{X}{K_0}, \\
y &= \frac{Y}{K_0}, \\
z &= \frac{Z}{K_0}.
\end{align}

and the non-dimensional version of model (2) is

\begin{align}
D_t^\alpha x &= x(1 - x) - \frac{a_1 x y}{1 + b_1 x}, \\
D_t^\alpha y &= \frac{a_1 x y}{1 + b_1 x} - \frac{a_2 x y}{1 + b_2 y} - d_1 y, \\
D_t^\alpha z &= \frac{a_2 x z}{1 + b_2 y} + (1 - A) a_2 c z - d_2 z
\end{align}

and initial condition is \(x(0) = x_0, y(0) = y_0, z(0) = z_0\).

Stability of Equilibrium Points
To determine the equilibrium points of differential equation (3), let
\[ D^q_t x = D^q_t y = D^q_t z = 0, \]

then the equilibrium points are

\[ E_1 = (0,0,0), \ E_2 = (1,0,0), \ E_3 = (\bar{x},\bar{y},0), \text{ and } E_4 = (\tilde{x},\tilde{y},\tilde{z}). \]

The Jacobian matrix for system (3) at the equilibrium point \((x^*, y^*, z^*)\) is given by

\[ f(x^*, y^*, z^*) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}, \]

where

\[ a_{11} = 1 - 2x^* - \frac{a_1y^*}{1+b_1x^*}^2, \quad a_{12} = -\frac{a_2z^*}{1+b_2y^*}^2, \]

\[ a_{21} = \frac{a_1y^*}{1+b_1x^*}^2, \quad a_{22} = \frac{a_1x^*}{1+b_1x^*}, \quad a_{23} = \frac{a_2z^*}{1+b_2y^*} \]

and

\[ a_{33} = \frac{a_2z^*}{1+b_2y^*} + ca_2(1-A) - d_2. \]

**Theorem 1.** The equilibrium \(E_1\) of system (3) is always a saddle point.

**Proof.** The Jacobian matrix at \(E_1\) is given by

\[ f(E_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & ca_2(1-A) - d_2 \end{pmatrix}. \]

The eigenvalues of matrix \(f(E_1)\) are obtained by solving the characteristic equation

\[ P(\lambda) = \det(f(E_1) - I\lambda) = (1 - \lambda)(d_1 - \lambda)(ca_2(1-A) - d_2 - \lambda) = 0. \]

The eigenvalues corresponding to the equilibrium \(E_1\) are \(\lambda_1 = 1 > 0, \ \lambda_2 = -d_1, \text{ and } \lambda_3 = ca_2(1-A) - d_2\). Thus \(\arg(\lambda_2) = 0 < \frac{\alpha r}{2}\), \(\arg(\lambda_3) = \pi > \frac{\alpha r}{2}\). Since \(\arg(\lambda_3) = 0 < \frac{\alpha r}{2}\), it follows from convergence of Mittag-Leffler function [9] that the equilibrium \(E_1\) is always a saddle point.

**Theorem 2.** The equilibrium \(E_2\) of system (3) is locally asymptotically stable if \(\frac{a_1}{1+b_1} < d_1\) and \(ca_2(1-A) < d_2\).

**Proof.** The Jacobian matrix of \(E_2\) is given by

\[ f(1,0,0) = \begin{pmatrix} -1 - \frac{a_1}{1+b_1} & 0 \\ 0 & a_1 - d_1 & 0 \\ 0 & 0 & ca_2(1-A) - d_2 \end{pmatrix}. \]

Eigenvalues of matrix \(f(E_2)\) are obtained by solving the characteristic equation

\[ P(\lambda) = \det(f(E_2) - I\lambda) = 0 = (1 - \lambda)(d_1 - \lambda)(ca_2(1-A) - d_2 - \lambda) = 0. \]

The eigenvalues corresponding to the equilibrium \(E_2\) are \(\lambda_1 = 1 < 0, \ \lambda_2 = \frac{a_1}{1+b_1}, \text{ and } \lambda_3 = ca_2(1-A) - d_2\). Thus \(\arg(\lambda_1) = \pi > \frac{\alpha r}{2}\), if \(\frac{a_1}{1+b_1} < d_1\) then \(\arg(\lambda_2) = \pi > \frac{\alpha r}{2}\), if \(d_2 > ca_2(1-A)\) then \(\arg(\lambda_3) = \pi > \frac{\alpha r}{2}\).

It follows from convergence of Mittag-Leffler function [9] that the equilibrium \(E_2\) of system (3) is locally asymptotically stable.

Furthermore, the equilibrium points \(E_3\) and \(E_4\) are discussed as follows. The Jacobian matrix at \(E_3\) is given by

\[ f(E_3) = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}. \]

where

\[ b_{11} = 1 - 2x^* - \frac{a_1y^*}{1+b_1x^*}^2, \quad b_{12} = -\frac{a_2z^*}{1+b_2y^*}^2, \]

\[ b_{21} = \frac{a_1y^*}{1+b_1x^*}, \quad b_{22} = \frac{a_1x^*}{1+b_1x^*} - d_1, \quad b_{23} = \frac{a_2z^*}{1+b_2y^*} \]

and

\[ b_{33} = \frac{a_2z^*}{1+b_2y^*} + ca_2(1-A) - d_2. \]

The eigenvalues of matrix \(f(E_3)\) are \(\lambda_1 = b_{33} = \frac{a_2z^*}{1+b_2y^*} + ca_2(1-A) - d_2\) and the other \(\lambda_2, \lambda_3\) are got by solving the characteristic equation

\[ P(\lambda) = \lambda^2 - \omega_1 \lambda + \omega_2 = 0. \]

where

\[ \omega_1 = 1 - 2x^* - d_1 - \frac{a_1x^*}{1+b_1x^*} - \frac{a_1y^*}{1+b_1x^*}, \]

\[ \omega_2 = \left(1 - 2\bar{x} - \frac{a_1\bar{y}}{1+b_1\bar{x}}\right)\left(\frac{a_1\bar{x}}{1+b_1\bar{x}} - d_1\right) + \frac{a_1\bar{x}\bar{y}}{1+b_1\bar{x}}. \]
The eigenvalues corresponding to the equation 
\[ P(\lambda) = \lambda^2 - 2(\omega_1 + i\psi)\lambda + \frac{1}{2}(\omega_1 - i\psi), \]
where \( \psi = (\omega_1)^2 - 4\omega_2 \). Thus, \( E_3 \) is locally asymptotically stable if it satisfies \( |\arg(\lambda_2)| = \pi > \frac{\pi \alpha}{2} \).

Equation (3) is satisfied if \( |\arg(\lambda_3)| = \pi > \frac{\pi \alpha}{2} \).

If \( \psi = -\omega_1 < 0 \) then \( \lambda_2, \lambda_3 > 0 \) such that \( \pi > \frac{\pi \alpha}{2} \).

If \( \psi > 0, \omega_1 > 0, \omega_2 > 0 \) and \( \sqrt{\omega_1} < |\omega_1| \), then \( \lambda_2, \lambda_3 < 0 \) such that \( \pi > \frac{\pi \alpha}{2} \).

To analyze the stability of equilibrium point \( E_4 \), first the Jacobian matrix at \( E_4 \) is evaluated by

\[
J(\dot{x}, \dot{y}, \dot{z}) = \begin{pmatrix}
    a_{11} & a_{12} & 0 \\
    a_{21} & a_{22} & a_{23} \\
    0 & a_{32} & a_{33}
\end{pmatrix},
\]

where

\[
a_{11} = 1 - 2\chi - \frac{a_1 \dot{y}}{1 + b_1 \dot{y}^2}, \quad a_{12} = -\frac{a_2 \dot{y}}{1 + b_2 \dot{y}^2},
\]

\[
a_{21} = \frac{a_1 \dot{y}}{1 + b_1 \dot{y}^2}, \quad a_{22} = \frac{a_2 \dot{y}}{1 + b_2 \dot{y}^2} - d_1,
\]

\[
a_{23} = -\frac{a_3 \dot{y}}{1 + b_3 \dot{y}^2}, \quad a_{32} = \frac{a_3 \dot{y}}{1 + b_3 \dot{y}^2}, \quad a_{33} = a_3 + c d_2 (1 - A) - d_2.
\]

Eigenvalues of matrix \( J(E_4) \) are got by solving the characteristic equation

\[ P(\lambda) = \det(J(E_4) - \lambda I) = \lambda^3 + K_1 \lambda^2 + K_2 \lambda + K_3 = 0. \]

where

\[ K_1 = -a_{11} + a_{22} + a_{33}, \quad K_2 = a_{22} a_{33} + a_{12} a_{32} + a_{11} a_{22} - a_{22} a_{12}, \quad K_3 = a_{32} a_{23} a_{11} + a_{12} a_{23} a_{31} + a_{11} a_{22} a_{33}. \]

Let \( D(P) \) is the discriminant of a polynomial \( P(\lambda) \), it can be written

\[
D(P) = \begin{vmatrix}
0 & K_1 & 0 & 0 \\
0 & K_1 & K_2 & K_3 \\
3 & 2K_1 & K_2 & 0 \\
0 & 3 & 2K_1 & K_2 \\
0 & 0 & 3 & 2K_1 & K_2
\end{vmatrix}.
\]

\[
D(P) = 18K_1 K_2 K_3 + (K_1 K_2)^2 - 4K_3 K_1^2 - 4K_2^3 - 27K_3^2.
\]

**Proposition**

Let the equilibrium \( E_4 \) in \( \mathbb{R}^3 \). Then the equilibrium \( E_4 \) of system (3) is asymptotically stable if one of the following conditions [11] are satisfied

1. \( D(P) > 0, \ K_1 > 0, \ K_2 > 0, \) and \( K_3, K_2 > K_3 \).
2. \( D(P) < 0, K_1 > 0, \ K_2 > 0, \ K_3 > 0, \) and \( \alpha < \frac{2}{3} \).

3. \( D(P) < 0, K_1 > 0, \ K_2 > 0, \ K_1 K_2 = K_3, \) and for all \( \alpha \in (0,1) \).

**Numerical Method and Simulations**

Numerical method which is introduced by Grunwald and Letnikov [8] is used to solve nonlinear fractional differential equation [3]. As described in [8,12], by using the Grunwald-Letnikov approximation method, it is obtained the following nonstandard explicit scheme for system [3].

The parameters chosen in the first

\[
x_{n+1} = h^a f(x_n, y_n, z_n) - \sum_{j=0}^{m+1} \binom{a}{j} x_{m+1-j} - \frac{\Gamma(1-\alpha)}{(m+1)h} x_0.
\]

\[
y_{n+1} = h^a g(x_n, y_n, z_n) - \sum_{j=0}^{m+1} \binom{a}{j} y_{m+1-j} - \frac{\Gamma(1-\alpha)}{(m+1)h} y_0.
\]

\[
z_{n+1} = h^a h(x_n, y_n, z_n) - \sum_{j=0}^{m+1} \binom{a}{j} z_{m+1-j} - \frac{\Gamma(1-\alpha)}{(m+1)h} z_0.
\]
condition \((0.5, 1.0, 0.75)\), then the solution convergent to equilibrium point \(E_2(1, 0, 0)\). If the value of \(\alpha\) is approaching to one then the convergence of the rate of change of the three populations is faster and vice versa.

In Figure 2, some parameters are set as \(a_1 = 3, a_2 = 0.1, b_1 = 2.5, b_2 = 2, d_1 = 0.6, d_2 = 0.3, c = 0.45, A = 0.6\) and \(h = 0.1\). According Matignon’s condition [13], stability of the equilibrium \(E_3\) is stabilized by \(\alpha^* = 0.94\). The initial condition of Figure 2 is \((1, 0.8, 0.5)\) and the solution is stable at point \((0.461, 0.414, 0)\) for \(\alpha = 0.9\), and it is unstable for \(\alpha = 0.95\). From this simulation, it can be concluded that the stability of the equilibrium point of the fractional-order model depends on the parameter of \(\alpha\) if \(\alpha^* > \alpha\) then the equilibrium \(E_3\) is stable. Conversely if \(\alpha^* < \alpha\) then the equilibrium \(E_3\) is unstable.

In Figure 3, some parameters are set as \(a_1 = 3, a_2 = 0.1, b_1 = 2.5, b_2 = 2, d_1 = 0.6, d_2 = 0.02, c = 0.45, A = 0.6\) and \(h = 0.2\). With initial conditions \((1, 0.5, 1.5)\) and different value of \(A = 1, A = 0.9, \) and \(A = 0.8\), then the solution convergent to \((0.6, 0.333, 2), (0.71, 0.26, 2.89)\) and \((0.812, 0.189, 3.518)\) respectively. These simulations explain that, if there is no alternative food \((A = 1)\), the number of top-predator population decreases compared with the presence of alternative food \((0 < A < 1)\). When the value of \(A = 1\) indicates that top-predator population doesn’t perform activities to search for additional food and the food source of the top-predator population depends only on the intermediate-predator population. However, in this case the three populations still survive in a ecosystem. On the other hand, if top-predator eat only alternative food \((A = 0)\) then the top-
interaction with harvesting. J. Appl. Math. Model. 37. 8946-8956.

[2] Sahoo, B., S. Poria. 2014. Effects of supplying alternative food in a predator–prey model with harvesting. Appl. Math. Comput. 234. 150-166.

[3] Ilmiyah, N.N., Trisilowati, A.R. Alghofari. 2014. Dynamical analysis of a harvested predator-prey model with ratio-dependent response function and prey refuge. Appl. Math. Sci. 8.5027-5037.

[4] Trisdiani, P.I., Trisilowati, A. Suryanto. 2014. Dynamics of harvested predator-prey system with disease in predator and prey in refuge. Int. J. Ecol. Econ. Stat. 33. 47-57.

[5] Perc., M. A. Szolnoki, G. Szabó. 2007. Cyclic interactions with alliance-specific heterogeneous invasion rates. Phys. Rev. E. 75. 052-102.

[6] Perc., M, A. Szolnoki. 2007. Noise-guided evolution within cyclical interactions. New J. Phys. 9. 267.13.

[7] Matouk, A.E., A.A. Elsadany, E. Ahmed, H.N. Agiza. 2015. Dynamical behavior of fractional-order Hastings–Powell food chain model and its discretization. J. Commun Nonlinear Sci. Numer. Simulat. 27. 153-167.

[8] Scherer, R., S.L. Kalla, Y. Tang, J. Huang. 2011. The Grünwald–Letnikov method for fractional differential equations. Comput. Math. Appl. 62. 902-917.

[9] Petras, I. 2011. Fractional-Order Nonlinear systems. Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg. Beijing.

[10] Hastings, A., T. Powell. 1991. Chaos in three-species food chain. J. Ecol. 72. 896–903.

[11] Ahmed, E., A.M.A. El-Sayed, H.AA. El-Saka. 2006. On some Routh-Hurwitz conditions for Fractional-order differential equations and their applications in Lorenz, Rossler, Chua and Chen Systems. Phys. Lett. A. 358. 1–4.

[12] Arenas, A.J., G.G. Parra, B.M Chen-Charpentier. 2016. Construction of non-standard finite difference schemes for the SI and SIR epidemic models of fractional-order. Math. Comput. Simulat. 121. 48-63.

[13] Matignon D. 1996. Stability results for fractional differential equations with applications to control Processing. Proceedings of Computational Engineering in Systems and Application Multi-Conference, Vol. 2: IMACS, IEEE-SMC. Lille, France. 963–968.