Logarithmically Improved Blow-up Criteria for a Phase Field Navier-Stokes Vesicle-Fluid Interaction Model*

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Abstract

In this paper, we study a hydrodynamical system modeling the deformation of vesicle membrane under external incompressible viscous flow fields. The system is in the Eulerian formulation and is governed by the coupling of the incompressible Navier-Stokes equations with a phase field equation. In the three dimensional case, we establish two logarithmically improved blow-up criteria for local smooth solutions of this system in terms of the vorticity field only in the homogeneous Besov spaces.

Keywords: Phase field; Navier-Stokes equations; fluid vesicle interaction; regularity criterion; Besov space

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1 Introduction

Recently, there have been many numerical and theoretical studies on the configurations and deformations of elastic vesicle membranes under external flow fields \cite{3,4,6,7,8,9,17,19,24,25}. The single component vesicle membranes are possibly the simplest models for the biological cells and molecules and have widely studied in biology, biophysics and bioengineering. Such vesicle membranes can be formed by certain amphiphilic molecules assembled in water to build bilayers, and exhibit a rich set of geometric structures in various mechanical, physical and biological environment \cite{7}. In order to model and understand the formation and dynamics of vesicle membranes and the fluid structure interaction, one approach is to consider equations of elasticity for the vesicle membranes and the Navier-Stokes equations for the fluid. However, the model established in this approach is very difficult to study and numerically simulate due to the fact that the description for elasticity is in Lagrangian coordinate (Hooke’s law) and for fluids is in Eulerian coordinate. To

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overcome this difficulty, in [4, 7], the authors established a phase field Navier-Stokes vesicle fluid interaction model for the vesicle shape dynamics in flow fields via the phase field approach. In this model, the vesicle membrane $\Gamma$ is described by a phase function $\phi$, which is a labeling function defined on computational domain $Q$. The function $\phi$ takes value $+1$ inside of the vesicle membrane and $-1$ outside, with a thin transition layer of width characterized by a small (compared to the vesicle size) positive parameter $\varepsilon$. Obviously, the sharp transition layer of the phase function gives a diffusive interface description of the vesicle membrane $\Gamma$, which is recovered by the zero level set $\{x : \phi(x) = 0\}$. The advantage of introducing such a phase function $\phi$ is to formulate the original Lagrangian description of the membrane evolution in the Eulerian coordinates. On the other hand, the viscous fluid is modeled by the incompressible Navier-Stokes equations with unit density and with an external force defined in terms of $\phi$.

In this paper, we study the three dimensional phase field Navier-Stokes vesicle-fluid interaction model subjecting to the periodic boundary conditions (i.e., in torus $T^3$), which reads as follows:

$$\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla P &= \mu \Delta u + \frac{\delta E(\phi)}{\delta \phi} \nabla \phi & \text{in } Q \times [0, T], \\
\nabla \cdot u &= 0 & \text{in } Q \times [0, T], \\
\partial_t \phi + u \cdot \nabla \phi &= -\gamma \frac{\delta E(\phi)}{\delta \phi} & \text{in } Q \times [0, T]
\end{align*}$$

(1.1) (1.2) (1.3)

with the initial conditions

$$u(x, 0) = u_0(x) \text{ with } \nabla \cdot u_0 = 0, \quad \text{and } \phi(x, 0) = \phi_0(x) \text{ for } x \in Q,$n(1.4)

and the boundary conditions

$$u(x + e_i, t) = u(x, t), \quad \phi(x + e_i, t) = \phi(x, t) \quad \text{for } x \in \partial Q \times [0, T], \quad i = 1, 2, 3,$n(1.5)

where the set of vectors $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ denotes an orthonormal basis of $\mathbb{R}^3$ and $Q$ is the unit square in $\mathbb{R}^3$. Here $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $P = P(x, t) \in \mathbb{R}$ denote the unknown velocity vector field and unknown pressure of the fluid, respectively. $\phi \in \mathbb{R}$ is the phase function of the vesicle membrane $\Gamma$. $E(\phi)$ denotes the physical approximation/regularization of the Helfrich elastic bending energy for the vesicle membrane which is given by (cf. [4, 6, 8, 9])

$$E(\phi) = E_\varepsilon(\phi) + \frac{1}{2} M_1 (A(\phi) - \alpha)^2 + \frac{1}{2} M_2 (B(\phi) - \beta)^2$$

(1.6)

with

$$E_\varepsilon(\phi) = \frac{k}{2\varepsilon} \int_{\Omega} |f(\phi)|^2 dx \text{ and } f(\phi) = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} (\phi^2 - 1) \phi,$n(1.7)

where $\varepsilon$ is a small (compared to the vesicle size) positive parameter that characterizes the thickness of transition layer of the phase function, $M_1$ and $M_2$ are two penalty constants which are introduced in order to enforce the volume

$$A(\phi) = \int_{\Omega} \phi \, dx$$

(1.8)

and the surface area

$$B(\phi) = \int_{Q} \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{4\varepsilon} (\phi^2 - 1)^2 \right) dx$$

(1.9)
of the vesicle conserved (in time), and $\alpha = A(\phi_0)$ and $\beta = B(\phi_0)$ are determined by the initial value of the phase function $\phi_0$. The positive constants $\nu$, $k$, and $\gamma$ denote, respectively, the viscosity of the fluid, the bending modulus of the vesicle, and the mobility coefficient. $\frac{\delta E(\phi)}{\delta \phi}$ is the so-called chemical potential that denotes the variational derivative of $E(\phi)$ in the variable $\phi$. Note that, if we denote
\[
g(\phi) = -\Delta f(\phi) + \frac{1}{\varepsilon^2}(3\phi^2 - 1)f(\phi),
\]
then a direct calculation yields that the variation of the approximate elastic energy is given by (see [4, 6])
\[
\frac{\delta E(\phi)}{\delta \phi} = kg(\phi) + M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi)
\]
\[
= k\varepsilon^2 \phi - \frac{k}{\varepsilon^3}(3\phi^2 - 1)\Delta \phi + \frac{k}{\varepsilon^3}(3\phi^2 - 1)(\phi^2 - 1)\phi
\]
\[
+ M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi).
\]

The system (1.1)–(1.3) describes the dynamic evolution of vesicle membranes immersed in an incompressible, Newtonian fluid, using an energetic variational approach [1, 7] (see [3, 9, 21, 24] for numerical simulations and other studies). Equations (1.1) and (1.2) are the momentum conservation and the mass conservation equations of a viscous fluid with unit density and with an external force caused by the phase field $\phi$. Equation (1.2) is the condition of incompressibility. Equation (1.3) is a relaxed transport equation of $\phi$ with advection by the velocity field $u$. The right-hand side of (1.3) is a regularization term which ensures the consistent dissipation of energy. Roughly speaking, the system (1.1)–(1.3) is governed by the coupling of the hydrodynamic fluid flow and the bending elastic properties of the vesicle membrane. The resulting membrane configuration and the flow field reflect the competition and the coupling of the kinetic energy and membrane elastic energies.

For the system (1.1)–(1.3) subjecting to no-slip boundary condition for the velocity field and Dirichlet boundary condition for the phase function, Du, Li and Liu in [4] obtained that there exists global weak solutions via the modified Galerkin argument, and there holds basic energy inequality
\[
\frac{d}{dt} \left( \frac{1}{2} \|u(\cdot, t)\|_{L^2}^2 + E(\phi(\cdot, t)) \right) + \mu \|\nabla u(\cdot, t)\|_{L^2}^2 + \gamma \frac{\delta E(\phi)}{\delta \phi} \|\delta \phi\|_{L^2}^2 = 0, \quad \forall \ t > 0.
\]
Moreover, the authors also proved that weak solution is unique under an additional regularity assumption $u \in L^8(0, T; L^4(Q))$. Recently, local in time existence and uniqueness of strong solution to the system (1.1)–(1.3) have been established in [19], and under the assumptions that the initial data and the quantity $(|\Omega| + \alpha)^2$ are sufficiently small, the authors proved existence of almost global strong solutions. Note that they have to restrict the working space with proper limited regularity due to some compatibility conditions at the boundary which is required in the fixed point strategy. Very recently, Wu and Xu [25] considered the system (1.1)–(1.3) with initial conditions (1.4) and periodic boundary conditions (1.5) to avoid troubles caused by the boundary terms when performing integration by parts. They proved that, for any given initial data $(u_0, \phi_0) \in$
there exists a positive time $T$ such that the system (1.1)–(1.5) admits a unique smooth solution $(u, \phi)$ satisfying
\[
\begin{aligned}
&u \in C([0, T], H^1_{per}(Q)) \cap L^2(0, T; H^2_{per}(Q)) \cap H^1(0, T; L^2_{per}(Q)), \\
&\phi \in C([0, T], H^4_{per}(Q)) \cap L^2(0, T; H^6_{per}(Q)) \cap H^1(0, T; L^2_{per}(Q)).
\end{aligned}
\]

Moreover, if the viscosity $\mu$ is assumed to be properly large, then the system (1.1)–(1.5) admits a unique global strong solution that is uniformly bounded in $H^1_{per} \times H^4_{per}$ on $[0, \infty)$. However, as for the well-known Navier-Stokes equations, an outstanding open problem is whether or not smooth solution of (1.1)–(1.5) on $[0, T)$ will lead to a singularity at the time $t = T$.

For the Navier-Stokes equations, some results were obtained in early by Prodi \[22\], Serrin \[23\] and Giga \[12\], they proved that if
\[
\int_0^T \|u(\cdot, t)\|_{L^p}^q \, dt < \infty \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} = 1, \ 3 < p \leq \infty,
\]
then the smooth solution $u$ can be extended past the time $T$, while the limit case $p = 3$ was proved by Escauriaza et al. \[10\]. In 1995, Beirão da Veiga \[2\] established similar criterion for the derivative of the solution, i.e., (1.13) can be replaced by the following condition:
\[
\int_0^T \|\nabla u(\cdot, t)\|_{L^p}^q \, dt < \infty \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} = 2, \ 3 < p \leq \infty.
\]

In 1984, Beale, Kato and Majda in their pioneer work \[1\] showed that if the smooth solution $u$ blows up at the time $t = T$, then
\[
\int_0^T \|\omega(\cdot, t)\|_{L^\infty} \, dt = \infty,
\]
where $\omega = \nabla \times u$ is the vorticity of the velocity field. Later, Kozono and Taniuchi \[16\] and Konozo, Ogawa and Taniuchi \[15\] refined the criterion (1.16) to
\[
\int_0^T \|\omega(\cdot, t)\|_{BMO} \, dt = \infty \quad \text{and} \quad \int_0^T \|\omega(\cdot, t)\|_{\dot{B}^{0}_{\infty, \infty}} \, dt = \infty,
\]
respectively, where $BMO$ is the space of Bounded Mean Oscillation and $\dot{B}^{0}_{\infty, \infty}$ is the homogeneous Besov spaces. Recently, Fan et al. \[11\] and Guo and Gala \[13\] improved the above criteria to the following two logarithmic type criteria:
\[
\int_0^T \frac{\|w(\cdot, t)\|_{\dot{B}^{0}_{\infty, \infty}}}{\sqrt{1 + \ln(e + \|w(\cdot, t)\|_{\dot{B}^{0}_{\infty, \infty}})}} \, dt = \infty
\]
and
\[
\int_0^T \frac{\|w(\cdot, t)\|_{\dot{B}^{-1}_{\infty, \infty}}^2}{1 + \ln(e + \|w(\cdot, t)\|_{\dot{B}^{-1}_{\infty, \infty}})} \, dt = \infty.
\]
When the phase function $\phi$ is considered, similar regularity criteria as (1.14) and (1.16) for the system (1.1)–(1.3) have been established in \[25\]. The first author of the present paper in \[26\] obtained that the Beale-Kato-Majda criterion (1.16) still holds for the system (1.1)–(1.5).
Due to the lack of global well-posedness theory of the system (1.1)–(1.5), the investigations of blow-up criteria of local smooth solutions are very important ways to understand the properties of solutions. Motivated by the above results, the purpose of this paper is to establish the blow-up criteria for the system (1.1)–(1.5) in term of the norm of the homogeneous Besov space. The main results of this paper are as follows:

**Theorem 1.1** For \((u_0, \phi_0) \in H^3_{\text{per}}(Q) \times H^6_{\text{per}}(Q)\) with \(\nabla \cdot u_0 = 0\). Let \(T_*\) be the maximal existence time such that the system (1.1)–(1.5) has a unique smooth solution \((u, \phi)\) on \([0, T_*]\). If \(T_* < \infty\), then

\[
\int_0^{T_*} \frac{\|\omega(\cdot, t)\|_{B^{0}_{0, \infty}}^2}{1 + \ln(e + \|\omega(\cdot, t)\|_{B^{0}_{0, \infty}})} dt = \infty, \tag{1.20}
\]

where \(\omega = \nabla \times u\) is the vorticity field. In particular,

\[
\limsup_{t \nearrow T_*} \|\omega(\cdot, t)\|_{B^{0}_{0, \infty}} = \infty.
\]

**Theorem 1.2** For \((u_0, \phi_0) \in H^3_{\text{per}}(Q) \times H^6_{\text{per}}(Q)\) with \(\nabla \cdot u_0 = 0\). Let \(T_*\) be the maximal existence time such that the system (1.1)–(1.5) has a unique smooth solution \((u, \phi)\) on \([0, T_*]\). If \(T_* < \infty\), then

\[
\int_0^{T_*} \frac{\|\omega(\cdot, t)\|_{B^{0}_{0, \infty}}^2}{1 + \ln(e + \|\omega(\cdot, t)\|_{B^{0}_{0, \infty}})} dt = \infty. \tag{1.21}
\]

In particular,

\[
\limsup_{t \nearrow T_*} \|\omega(\cdot, t)\|_{B^{0}_{0, \infty}} = \infty.
\]

**Remark 1.1** Theorems 1.1 and 1.2 are still true, if we replace the vorticity \(\omega\) by \(\nabla u\) in (1.20) and (1.21), due to the boundedness of Riesz transforms in \(\dot{B}^{0}_{\infty, \infty}(Q)\) and \(\dot{B}^{-1}_{\infty, \infty}(Q)\). For the definitions of these spaces, see Section 2.

**Remark 1.2** Since \(L^\infty(Q) \hookrightarrow \dot{B}^{0}_{\infty, \infty}(Q)\), the result (1.20) improves the Beale-Kato-Majda blow-up criterion in [25].

**Remark 1.3** Observe that \(\nabla u \in \dot{B}^{-1}_{\infty, \infty}(Q)\) is equivalent to \(u \in \dot{B}^{0}_{\infty, \infty}(Q)\) and the Sobolev embedding \(L^3_{\text{per}}(Q) \hookrightarrow \dot{B}^{-1}_{\infty, \infty}(Q)\). Therefore, Theorem 1.2 implies that if \(T_* < \infty\), then

\[
(i) \quad \int_0^{T_*} \frac{\|u(\cdot, t)\|_{B^{0}_{0, \infty}}^2}{1 + \ln(e + \|u(\cdot, t)\|_{B^{0}_{0, \infty}})} dt = \infty,
\]

\[
(ii) \quad \int_0^{T_*} \frac{\|\nabla u(\cdot, t)\|_{L^3}^2}{1 + \ln(e + \|\nabla u(\cdot, t)\|_{L^3})} dt = \infty.
\]

The rest of the paper is arranged as follows. In Section 2, we recall the Littlewood-Paley decomposition and definition of the homogeneous Besov spaces, and review some crucial lemmas. In Section 3, we establish the bound of \(\|\nabla \Delta \phi\|_{L^2}\), which enables us to derive some specific higher-order energy estimates. In Section 4, we present the proof of Theorem 1.1. Section 5 is devoted to the proof of Theorem 1.2.
2 Preliminaries

We first recall the Littlewood-Paley decomposition. Let \( \mathcal{S}(\mathbb{R}^3) \) be the Schwartz class of rapidly decreasing function, and \( \mathcal{S}'(\mathbb{R}^3) \) be its dual. Given \( f \in \mathcal{S}(\mathbb{R}^3) \), the Fourier transform of it, \( \mathcal{F}(f) = \hat{f} \), is defined by

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{-ix\cdot\xi} \, dx.
\]

For any given \( g \in \mathcal{S}(\mathbb{R}^3) \), the inverse Fourier transform \( \mathcal{F}^{-1} g = \check{g} \) is defined by

\[
\mathcal{F}^{-1}(g)(x) = \check{g}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} g(\xi) e^{ix\cdot\xi} \, d\xi.
\]

Let \( D_1 = \{ \xi \in \mathbb{R}^3, |\xi| \leq \frac{1}{2} \} \) and \( D_2 = \{ \xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \} \). Choose two non-negative radial functions \( \phi, \psi \in \mathcal{S}(\mathbb{R}^3) \) supported, respectively, in \( D_1 \) and \( D_2 \) such that

\[
\begin{align*}
\psi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) &= 1, \quad \xi \in \mathbb{R}^3, \\
\sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) &= 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.
\end{align*}
\]

Let \( h = \mathcal{F}^{-1} \phi \) and \( \tilde{h} = \mathcal{F}^{-1} \psi \). Then we define the dyadic blocks \( \Delta_j \) and \( S_j \) as follows:

\[
\Delta_j f = \phi(2^{-j} D) f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x - y) \, dy,
\]

\[
S_j f = \psi(2^{-j} D) f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) f(x - y) \, dy,
\]

where \( D = (D_1, D_2, D_3) \) and \( D_j = i^{-1} \partial_{x_j} \) \((i^2 = -1)\). The set \( \{ \Delta_j, S_j \}_{j \in \mathbb{Z}} \) is called the Littlewood-Paley decomposition. Formally, \( \Delta_j = S_j - S_{j-1} \) is a frequency projection to the annulus \( \{ |\xi| \sim 2^j \} \), and \( S_j = \sum_{k \leq -1} \Delta_k \) is a frequency projection to the ball \( \{ |\xi| \leq 2^j \} \). For more details, please refer to [13].

Next we recall the definition of homogeneous Besov spaces. Let \( \mathcal{S}'_h(\mathbb{R}^3) \) be the space of temperate distributions \( f \) such that

\[
\lim_{j \to \infty} S_j f = 0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^3).
\]

For \( s \in \mathbb{R} \) and \( (p, q) \in [1, \infty] \times [1, \infty] \), the homogeneous Besov space \( \dot{B}^s_{p,q}(\mathbb{R}^3) \) is defined by

\[
\dot{B}^s_{p,q}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'_h(\mathbb{R}^3) : \| f \|_{\dot{B}^s_{p,q}} < \infty \right\},
\]

where

\[
\| f \|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j \in \mathbb{Z}} 2^{jsq} \| \Delta_j f \|_{L^p}^q)^{1/q} & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{L^p} & \text{for } q = \infty.
\end{cases}
\]

It is well-known that if either \( s < \frac{3}{p} \) or \( s = \frac{3}{p} \) and \( q = 1 \), then \( \dot{B}^s_{p,q}(\mathbb{R}^3), \| \cdot \|_{\dot{B}^s_{p,q}} \) is a Banach space. In particular, when \( p = q = 2 \), we get the homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^3) = \dot{B}^s_{2,2}(\mathbb{R}^3) \) which
is endowed the equivalent norm \( \|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_{L^2} \). The notation \( H^s(\mathbb{R}^3) \) is the standard inhomogeneous Sobolev spaces which is endowed the standard norm \( \|f\|_{H^s} = \|(-\Delta)^{s/2} f\|_{L^2} + \|f\|_{L^2} \).

We also need to introduce some well-established functional settings for periodic problems: For \( 1 \leq r \leq \infty \), we denote by
\[
L^r_{\text{per}}(Q) := \{ u \in L^r(\mathbb{R}^3) \mid u(x + e_i) = u(x) \}
\]
edowed the usual norm \( \| \cdot \|_{L^r} \). For an integer \( m > 0 \), we denote by
\[
H^m_{\text{per}}(Q) := \{ u \in H^m(\mathbb{R}^3) \mid u(x + e_i) = u(x) \}
\]
edowed with the usual norm \( \|u\|_{H^m} \). For \( s \in \mathbb{R} \) and \( (p,q) \in [1,\infty] \times [1,\infty] \), we denote by
\[
\dot{B}^s_{p,q}(Q) = \left\{ u \in \dot{B}^s_{p,q}(\mathbb{R}^3) : u(x + e_i) = u(x) \right\}
\]
associated with the norm \( \| \cdot \|_{\dot{B}^s_{p,q}} \).

Before ending this section, we state some well-known inequalities. The first one comes from [14]:

**Lemma 2.1** ([14]) For \( s > 1 \), we have
\[
\|\nabla^s (fg) - f\nabla^s g\|_{L^p} \leq C \left( \|\nabla f\|_{L^{p_1}} \|\nabla^{s-1} g\|_{L^{q_1}} + \|\nabla^s f\|_{L^{p_2}} \|g\|_{L^{q_2}} \right)
\]
with \( 1 < p, q_1, p_2 < \infty \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} \).

The second one can be found in [15] and the proof follows from the Littlewood-Paley decomposition.

**Lemma 2.2** ([15]) For all \( f \in H^{s-1}(\mathbb{R}^3) \) with \( s > \frac{5}{2} \), we have
\[
\|f\|_{L^\infty} \leq C \left( 1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \ln^{1/2}(e + \|f\|_{H^{s-1}}) \right).
\]

The last one comes from [20], see also [13].

**Lemma 2.3** ([20] [13]) For all \( f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \), we have
\[
\|f\|_{L^4} \leq C \|f\|^{1/2}_{\dot{B}^{-1/2}_{\infty,\infty}} \|f\|^{1/2}_{\dot{H}^1}.
\]

### 3 The bound of \( \|\nabla \Delta \phi\|_{L^2} \)

By the basic energy estimate (1.12), we can easily get the following uniform estimates (cf. [1] [25]):
\[
\|u(\cdot,t)\|_{L^2} + \|\phi(\cdot,t)\|_{H^2} \leq C \quad \text{for all} \quad t \geq 0,
\]
\[
\int_0^{+\infty} \left( \mu \|\nabla u(\cdot,t)\|_{L^2}^2 + \gamma \|\frac{\delta E}{\delta \phi}(\cdot,t)\|_{L^2}^2 \right) dt \leq C,
\]
where \( C \) is a constant depending only on \( \|u_0\|_{L^2}, \|\phi_0\|_{H^2} \) and coefficients of the system except the viscosity \( \mu \).
Lemma 3.1 Assume that \((u_0, \phi_0) \in H^3_{\text{per}}(Q) \times H^6_{\text{per}}(Q)\) with \(\nabla \cdot u_0 = 0\). For any smooth solution \((u, \phi)\) to the system (1.1)–(1.4), we have

\[
\sup_{0 \leq t \leq T} \|\nabla \Delta \phi(\cdot, t)\|_{L^2} \leq C
\]  \tag{3.3}

for any \(0 < T < \infty\), where \(C\) is a constant depending only on \(\|u_0\|_{H^1}, \|\phi_0\|_{H^3}, T\) and coefficients of the system.

Proof. Taking \(\Delta\) on (3.3), multiplying the resultant by \(-\Delta^2 \phi\), and integrating over \(Q\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \Delta \phi\|^2_{L^2} \leq -\int_Q \nabla \cdot (u \cdot \nabla \phi) \cdot \nabla \Delta^2 \phi dx - \gamma \int_Q \nabla \frac{\delta E}{\delta \phi} \cdot \nabla \Delta^2 \phi dx \equiv I_1 + I_2. \tag{3.4}
\]

For \(I_1\), by using the interpolation inequality \(\|\nabla^2 \phi\|^2_{L^2} \leq C \|\nabla^2 \phi\|_{L^2} \|\nabla \Delta \phi\|_{L^2}\), we can infer from (3.1) that

\[
I_1 \leq \frac{k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\nabla u \cdot \nabla \phi\|^2_{L^2} + C \|u \cdot \nabla^2 \phi\|^2_{L^2}
\]

\[
\leq \frac{k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\nabla u\|^2_{L^2} \|\nabla \phi\|^2_{L^\infty} + \|u\|^2_{L^2} \|\nabla^2 \phi\|^2_{L^\infty}
\]

\[
\leq \frac{k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\nabla u\|^2_{L^2} (\|\nabla \Delta \phi\|^2_{L^2} + 1) + C \|\nabla^2 \phi\|^2_{L^2} \|\nabla \Delta \phi\|_{L^2}
\]

\[
\leq \frac{k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\nabla u\|^2_{L^2} (\|\nabla \Delta \phi\|^2_{L^2} + 1). \tag{3.5}
\]

For \(I_2\), since \(A(\phi)\) and \(B(\phi)\) are functions depending only on time, by (1.1), we obtain

\[
I_2 = -\gamma \int_Q \nabla \left[ k g(\phi) + M_1 (A(\phi) - \alpha) + M_2 (B(\phi) - \beta) f(\phi) \right] \cdot \nabla \Delta^2 \phi dx
\]

\[
= k \gamma \int_Q \nabla \Delta f(\phi) \cdot \nabla \Delta^2 \phi dx - \frac{k \gamma \varepsilon}{\varepsilon^2} \int_Q \nabla [(3 \phi^2 - 1) f(\phi)] \cdot \nabla \Delta^2 \phi dx
\]

\[
- M_2 \gamma (B(\phi) - \beta) \int_Q \nabla f(\phi) \cdot \nabla \Delta^2 \phi dx
\]

\[
:= I_{21} + I_{22} + I_{23}. \tag{3.6}
\]

Note that \(f(\phi) = -\varepsilon \Delta \phi + \frac{4}{\varepsilon^2} (\phi^2 - 1)\phi\), by (3.1), we can estimate \(I_{2i} \) \((i = 1, 2, 3)\) as follows:

\[
I_{21} = -k \varepsilon \|\nabla \Delta^2 \phi\|^2_{L^2} + \frac{k \gamma \varepsilon}{\varepsilon^2} \int_Q \nabla \Delta (\phi^3 - \phi) \cdot \nabla \Delta^2 \phi dx
\]

\[
\leq -\frac{7k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\nabla \Delta (\phi^3 - \phi)\|^2_{L^2}
\]

\[
\leq -\frac{7k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\phi\|^2_{L^4} \|\nabla \Delta \phi\|^2_{L^2} + \|\phi\|^2_{L^4} \|\nabla \phi\|^2_{L^6} \|\Delta \phi\|^2_{L^3}
\]

\[
+ \|\nabla \phi\|^2_{L^6} + \|\nabla \Delta \phi\|^2_{L^2}
\]

\[
\leq -\frac{7k \gamma \varepsilon}{8} \|\nabla \Delta^2 \phi\|^2_{L^2} + C \|\nabla \Delta \phi\|^2_{L^2} + 1; \tag{3.7}
\]

\[
I_{22} = -\frac{6k \gamma \varepsilon}{\varepsilon^2} \int_Q \phi \nabla f(\phi) \cdot \nabla \Delta^2 \phi dx - \frac{k \gamma \varepsilon}{\varepsilon^2} \int_Q (3 \phi^2 - 1) \nabla f(\phi) \cdot \nabla \Delta^2 \phi dx
\]
\[
\frac{k\varepsilon\gamma}{8} \| \nabla^2 \phi \|_{L^2}^2 + C \left( \| \phi \nabla \phi f(\phi) \|_{L^2}^2 + \| (3\phi^2 - 1) \nabla f(\phi) \|_{L^2}^2 \right) \\
\leq \frac{k\varepsilon\gamma}{8} \| \nabla^2 \phi \|_{L^2}^2 + C \left( \| \phi \nabla \phi \Delta \phi \|_{L^2}^2 + \| \phi^2 - 1 \|_{L^2}^2 \right) \\
+ \| (3\phi^2 - 1) \nabla \Delta \phi \|_{L^2}^2 + \| (3\phi^2 - 1) \nabla (\phi^3 - \phi) \|_{L^2}^2
\]

From (3.7)-(3.9), we get

\[
\text{The estimate (3.12) with (3.2) imply (3.3) immediately. We complete the proof of Lemma 3.1.}
\]

\[
I_{23} = M_2\varepsilon\gamma (B(\phi) - \beta) \int \nabla \Delta \phi \cdot \nabla^2 \phi dx - \frac{M_2\gamma (B(\phi) - \beta)}{\varepsilon} \int \nabla (\phi^3 - \phi) \cdot \nabla^2 \phi dx
\]

\[
\leq \frac{k\varepsilon\gamma}{8} \| \nabla^2 \phi \|_{L^2}^2 + C (\| \phi \|_{L^2} + \| \nabla (\phi^3 - \phi) \|_{L^2}^2)
\]

\[
\leq \frac{k\varepsilon\gamma}{8} \| \nabla^2 \phi \|_{L^2}^2 + C \left( \| \phi \|_{L^2}^2 + \| \phi^2 - 1 \|_{L^2}^2 + 1 \right) (\| \nabla \Delta \phi \|_{L^2}^2 + \| \phi^2 - 1 \|_{L^\infty}^2 \| \nabla \phi \|_{L^2}^2)
\]

\[
\leq \frac{k\varepsilon\gamma}{8} \| \nabla^2 \phi \|_{L^2}^2 + C (\| \nabla \Delta \phi \|_{L^2}^2 + 1).
\]  

Combining the above estimates (3.5) and (3.10), we obtain

\[
\frac{d}{dt} \| \nabla \phi(t) \|_{L^2}^2 + k\varepsilon\gamma \| \nabla^2 \phi \|_{L^2}^2 \leq C \left( \| \nabla u(t) \|_{L^2}^2 + 1 \right) (\| \nabla \Delta \phi \|_{L^2}^2 + 1).
\]  

The Gronwall’s equality yields that

\[
\| \nabla \phi(t) \|_{L^2}^2 \leq \| \nabla \phi(0) \|_{L^2}^2 \exp \left( C \int_0^t (\| \nabla u(\tau) \|_{L^2}^2 + 1) d\tau \right).
\]  

The estimate (3.12) with (3.2) imply (3.3) immediately. We complete the proof of Lemma 3.1. \(\square\)

By (3.1) and (3.3), for any \(0 < T < \infty\), we obtain

\[
\sup_{0 \leq t \leq T} \| \phi(\cdot, t) \|_{H^3} \leq C.
\]

By the Sobolev embedding \(H^2_{\text{per}}(Q) \hookrightarrow L^\infty_{\text{per}}(Q)\), (3.13) yields that

\[
\sup_{0 \leq t \leq T} \| \nabla \phi \|_{L^\infty} \leq C.
\]

This result will be used frequently in the proofs of Theorems 1.1 and 1.2.
4 The proof of Theorem 1.1

We argue Theorem 1.1 by contradiction. Assume that the result (1.20) is not true, which means that there exists a constant $M > 0$ such that
\[ \int_0^{T_*} \frac{\| \omega(\cdot, t) \|_{B^0_{\infty, \infty}}}{\sqrt{1 + \ln(e + \| \omega(\cdot, t) \|_{B^0_{\infty, \infty}})}} \, dt \leq M. \] (4.1)

Under the condition (4.1), if we can prove that
\[ \limsup_{t \to T_*} (\| u(\cdot, t) \|_{H^3} + \| \phi(\cdot, t) \|_{H^6}) \leq C \] (4.2)
holds for some constant $C$ depending only on $u_0, \phi_0, M, T_*$ and coefficients of the system (1.1)–(1.5), then we can extend the solution $(u, \phi)$ beyond the time $t = T_*$, which leads to the contradiction. Therefore, it suffices to show that under the condition (4.1), we get (4.2).

Taking the curl on (1.1), we obtain
\[ \partial_t \omega - \mu \Delta \omega + u \cdot \nabla \omega = \omega \cdot \nabla u + \nabla \times (\frac{\delta E}{\delta \phi} \nabla \phi). \] (4.3)

Multiplying (4.3) by $\omega$ and integrating over $Q$, we have
\[ \frac{1}{2} \frac{d}{dt} \| \omega \|_{L^2}^2 + \mu \| \nabla \omega \|_{L^2}^2 = \int_Q w \cdot \nabla u \cdot \omega \, dx - \int_Q \frac{\delta E}{\delta \phi} \nabla \phi \cdot \nabla \times \omega \, dx, \] (4.4)
where we have used the fact $\int_Q u \cdot \nabla \omega \cdot \omega \, dx = 0$ due to $\nabla \cdot u = 0$. Since the Riesz operators are bounded in $L^2$ and $\nabla u = (-\Delta)^{-1} \nabla (\nabla \times \omega)$, we have $\| \nabla u \|_{L^2} \leq C \| \omega \|_{L^2}$. This implies that
\[ \left| \int_Q w \cdot \nabla u \cdot \omega \, dx \right| \leq C \| \omega \|_{L^\infty} \| \nabla u \|_{L^2} \| \omega \|_{L^2} \leq C \| \omega \|_{L^\infty} \| \omega \|_{L^2}^2. \] (4.5)

Applying Young’s inequality and (3.14), we have
\[ \left| \int_Q \frac{\delta E}{\delta \phi} \nabla \phi \cdot \nabla \times \omega \, dx \right| \leq \frac{\mu}{4} \| \nabla \omega \|_{L^2}^2 + C \| \frac{\delta E}{\delta \phi} \nabla \phi \|_{L^2}^2 \]
\[ \leq \frac{\mu}{4} \| \nabla \omega \|_{L^2}^2 + C \| \frac{\delta E}{\delta \phi} \|_{L^2}^2 \| \nabla \phi \|_{L^\infty}^2 \]
\[ \leq \frac{\mu}{4} \| \nabla \omega \|_{L^2}^2 + C \| \frac{\delta E}{\delta \phi} \|_{L^2}^2. \] (4.6)

Taking (4.5) and (4.6) into (4.4), we obtain
\[ \frac{d}{dt} \| \omega \|_{L^2}^2 + \frac{3\mu}{2} \| \nabla \omega \|_{L^2}^2 \leq C \left( \| \omega \|_{L^\infty} + 1 \right) \left( \| \omega \|_{L^2}^2 + \| \frac{\delta E}{\delta \phi} \|_{L^2}^2 \right). \] (4.7)

On the other hand, after integration by parts, we obtain from (1.11) that
\[ \frac{1}{2} \frac{d}{dt} \| \frac{\delta E}{\delta \phi} \|_{L^2}^2 = \int_Q \frac{\partial}{\partial t} \frac{\delta E}{\delta \phi} : \frac{\delta E}{\delta \phi} \, dx \]
\[
\begin{align*}
\int_Q \frac{\partial}{\partial t} [kg(\phi) + M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi)] \cdot \frac{\delta E}{\delta \phi} d\mathbf{x} \\
- k \int_Q \frac{\partial}{\partial \mathbf{t}} \mathbf{\Delta} f(\phi) \cdot \frac{\delta E}{\delta \phi} d\mathbf{x} + \frac{k}{\varepsilon^2} \int_Q \frac{\partial}{\partial \mathbf{t}} [(3\phi^2 - 1)f(\phi)] \cdot \frac{\delta E}{\delta \phi} d\mathbf{x} + M_1 \frac{d}{dt} A(\phi) \int_Q \frac{\delta E}{\delta \phi} d\mathbf{x} \\
+ M_2 \frac{d}{dt} B(\phi) \int_Q f(\phi) \cdot \frac{\delta E}{\delta \phi} d\mathbf{x} + M_2 (B(\phi) - \beta) \int_Q \frac{\partial}{\partial \mathbf{t}} f(\phi) \cdot \frac{\delta E}{\delta \phi} d\mathbf{x} \\
:= J_1 + J_2 + J_3 + J_4 + J_5. \tag{4.8}
\end{align*}
\]

Noticing from (1.3) that
\[
\begin{align*}
\| \frac{\partial \phi}{\partial t} \|_{L^2} & \leq C \left( \| u \cdot \nabla \phi \|_{L^2} + \| \frac{\delta E}{\delta \phi} \|_{L^2} \right) \\
& \leq C \left( \| u \|_{L^2} \| \nabla \phi \|_{L^\infty} + \| \frac{\delta E}{\delta \phi} \|_{L^2} \right) \\
& \leq C \left( \| \frac{\delta E}{\delta \phi} \|_{L^2} + 1 \right),
\end{align*}
\]

\[
\begin{align*}
\| \nabla \frac{\partial \phi}{\partial t} \|_{L^2} & \leq C \left( \| \nabla u \cdot \nabla \phi \|_{L^2} + \| u \cdot \nabla^2 \phi \|_{L^2} + \| \nabla \frac{\delta E}{\delta \phi} \|_{L^2} \right) \\
& \leq C \left( \| \nabla u \|_{L^2} \| \nabla \phi \|_{L^\infty} + \| u \|_{L^3} \| \nabla^2 \phi \|_{L^6} + \| \nabla \frac{\delta E}{\delta \phi} \|_{L^2} \right) \\
& \leq C \left( \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2} + \| \nabla u \|_{L^2} + \| \frac{\delta E}{\delta \phi} \|_{L^2} + 1 \right),
\end{align*}
\]

\[
\begin{align*}
\| \frac{\partial f(\phi)}{\partial t} \|_{L^2} & \leq C \left( \| \frac{\Delta \phi}{\partial \mathbf{t}} \|_{L^2} + \| \frac{\partial}{\partial \mathbf{t}} (\phi^3 - \phi) \|_{L^2} \right) \\
& \leq C \left( \| \Delta u \cdot \nabla \phi \|_{L^2} + \| \nabla u \cdot \nabla^2 \phi \|_{L^2} + \| u \cdot \nabla \phi \|_{L^2} + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2} + \| \frac{\partial}{\partial \mathbf{t}} (\phi^3 - \phi) \|_{L^2} \right) \\
& \leq C \left( \| \nabla \phi \|_{L^\infty} \| \Delta u \|_{L^2} + \| \nabla u \|_{L^3} \| \nabla^2 \phi \|_{L^6} + \| u \|_{L^\infty} \| \nabla \phi \|_{L^2} + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2} + \| \frac{\partial}{\partial \mathbf{t}} \phi \|_{L^2} \right) \\
& \leq C \left( \| \Delta u \|_{L^2} + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2} + \| \frac{\delta E}{\delta \phi} \|_{L^2} + 1 \right).
\end{align*}
\]

Then we can estimate \( J_i \) (\( i = 1, 2, 3, 4, 5 \)) as follows: For \( J_1 \), we can further split it into the following two terms:
\[
J_1 = k\varepsilon \int_Q \frac{\partial \phi}{\partial \mathbf{t}} \cdot \frac{\Delta \delta E}{\delta \phi} d\mathbf{x} - \frac{k}{\varepsilon} \int_Q \frac{\partial}{\partial \mathbf{t}} \Delta (\phi^3 - \phi) \cdot \frac{\delta E}{\delta \phi} d\mathbf{x} := J_{11} + J_{12}. \tag{4.9}
\]

By using Leibniz's rule, (1.3) yields that
\[
\begin{align*}
J_{11} & = -k\varepsilon \gamma \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 - k\varepsilon \int_Q \Delta (u \cdot \nabla \phi) \cdot \Delta \frac{\delta E}{\delta \phi} d\mathbf{x} \\
& \leq -\frac{9k\varepsilon \gamma}{10} \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \| \Delta (u \cdot \nabla \phi) \|_{L^2}^2 \\
& \leq -\frac{9k\varepsilon \gamma}{10} \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \Delta u \cdot \nabla \phi \|_{L^2}^2 + 2 \| \nabla u \cdot \nabla^2 \phi \|_{L^2}^2 + \| u \cdot \nabla \Delta \phi \|_{L^2}^2 \right)
\end{align*}
\]
Similarly, we can estimate
\[ J_1 \leq -\frac{4k\varepsilon\gamma}{5} \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla E\|_{L^2}^2 + 1). \] (4.10)

Hence, we infer from (4.10) and (4.11) that
\[ J_1 \leq -\frac{4k\varepsilon\gamma}{5} \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla E\|_{L^2}^2 + 1). \] (4.12)

Similarly, we can estimate \( J_2, J_3, J_4 \) and \( J_5 \) as follows:
\[
J_2 = \frac{6k}{\varepsilon^2} \int_Q \phi f(\phi) \frac{\partial \phi}{\partial t} \cdot \frac{\delta E}{\delta \phi} dx + \frac{k}{\varepsilon^2} \int_Q (3\phi^2 - 1) \frac{\partial f(\phi)}{\partial t} \frac{\delta E}{\delta \phi} dx
\]
\[ = \frac{6k}{\varepsilon} \int_Q \phi \Delta \frac{\delta \phi}{\delta \phi} \frac{\delta E}{\delta \phi} dx + \frac{6k}{\varepsilon^3} \int_Q \phi^2 (\phi^2 - 1) \frac{\partial \phi}{\partial t} \frac{\delta E}{\delta \phi} dx + \frac{k}{\varepsilon^2} \int_Q (3\phi^2 - 1) \frac{\partial f(\phi)}{\partial t} \frac{\delta E}{\delta \phi} dx
\]
\[ \leq C(\|\phi\|_{L^\infty} \|\Delta \phi\|_{L^2} + \|\phi\|_{L^\infty}^2 \|\phi^2 - 1\|_{L^\infty} \|\frac{\partial \phi}{\partial t}\|_{L^2} + \|\phi^2 - 1\|_{L^\infty} \|\frac{\partial f(\phi)}{\partial t}\|_{L^2}) \|\frac{\delta E}{\delta \phi}\|_{L^2}
\]
\[ \leq C(\|\frac{\partial \phi}{\partial t}\|_{L^2} + \|\nabla \phi\|_{L^2} + \|\nabla \phi\|_{L^2} + \|\nabla \phi\|_{L^2} + \|\frac{\delta E}{\delta \phi}\|_{L^2} + 1) \|\frac{\delta E}{\delta \phi}\|_{L^2}
\]
\[ \leq k\varepsilon\gamma \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2 + 1), \] (4.13)

\[
J_3 \leq C\left| \frac{dA(\phi)}{dt} \right| \int_Q \frac{\delta E}{\delta \phi} dx \leq C \int_Q \frac{\partial \phi}{\partial t} \|\frac{\partial \phi}{\delta \phi}\|_{L^2} \leq C \left( \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 \right), \] (4.14)

\[
J_4 \leq C \left| \frac{dB(\phi)}{dt} \right| \|f(\phi)\|_{L^2} \|\frac{\delta E}{\delta \phi}\|_{L^2}
\]
\[ \leq C(\|\nabla \phi\|_{L^2} \|\nabla \phi\|_{L^2} + \|\phi^2 - 1\|_{L^\infty} \|\frac{\partial \phi}{\partial t}\|_{L^2}) \|f(\phi)\|_{L^2} \|\frac{\delta E}{\delta \phi}\|_{L^2}
\]
\[ \leq C(\|\Delta \frac{\delta E}{\delta \phi}\|_{L^2} + \|\nabla u\|_{L^2} + \|\frac{\delta E}{\delta \phi}\|_{L^2} + 1) \|\frac{\delta E}{\delta \phi}\|_{L^2}
\]
\[ \leq \frac{k\varepsilon\gamma}{10} \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1), \] (4.15)
\[ J_5 \leq C|B(\phi) - \beta\|\frac{\partial f(\phi)}{\partial t}\|_{L^2} \|\frac{\delta E}{\delta \phi}\|_{L^2} \]
\[ \leq C \left( \|\nabla \phi\|_{L^2}^2 + \|\phi^2 - 1\|_{L^2}^2 \right) \left( \|\frac{\partial f(\phi)}{\partial t}\|_{L^2} \|\frac{\delta E}{\delta \phi}\|_{L^2} \right) \]
\[ \leq C \left( \|\Delta u\|_{L^2} + \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2} + \|\nabla u\|_{L^2} + \|\frac{\delta E}{\delta \phi}\|_{L^2} + 1 \right) \|\frac{\delta E}{\delta \phi}\|_{L^2} \]
\[ \leq \frac{k_2 \gamma}{10} \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 + C \|\Delta u\|_{L^2}^2 + C \left( \|\nabla u\|_{L^2}^2 + \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]  
(4.16)

Taking (4.12) – (4.16) into (4.8), by using the fact that \( \|\Delta u\|_{L^2} \leq C \|\nabla w\|_{L^2} \), we conclude that
\[ \frac{d}{dt} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 \right) + \mu \|\nabla w\|_{L^2}^2 + k_2 \gamma \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 \leq \tilde{C} \|\nabla w\|_{L^2}^2 + C \left( \|w\|_{L^2}^2 + \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) , \]  
(4.17)

where \( \tilde{C} \) is a constant depending only on \( \|u_0\|_{H^1} \), \( \|\phi_0\|_{H^3} \), \( T \), and coefficients of the system due to the estimate (2.3).

Set
\[ \eta = \frac{\mu}{2C} . \]

Then multiplying (4.17) by \( \eta \), adding (4.17) together, applying Lemma 2.2 with \( s = 3 \), we obtain
\[ \frac{d}{dt} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 \right) + \mu \|\nabla w\|_{L^2}^2 + k_2 \gamma \|\Delta \frac{\delta E}{\delta \phi}\|_{L^2}^2 \]
\[ \leq C \left( \|w\|_{L^\infty}^2 + 1 \right) \left( \|w\|_{L^2}^2 + \gamma \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]
\[ \leq C \left( 1 + \|w\|_{\dot{B}^0_{\infty, \infty}} \sqrt{1 + \ln(1 + \|w\|_{\dot{B}^0_{\infty, \infty}})} \right) \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]
\[ \leq C \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) + C \frac{\|w\|_{\dot{B}^0_{\infty, \infty}}}{\sqrt{1 + \ln(1 + \|w\|_{\dot{B}^0_{\infty, \infty}})}} \sqrt{1 + \ln(1 + \|w\|_{\dot{B}^0_{\infty, \infty}})} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]
\[ \times \sqrt{\ln(1 + \|w\|_{H^2})} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]
\[ \leq C \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]
\[ + C \frac{\|w\|_{\dot{B}^0_{\infty, \infty}}}{\sqrt{1 + \ln(e + \|w\|_{\dot{B}^0_{\infty, \infty}})}} \ln(1 + \|\nabla^3 u\|_{L^2}) \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) , \]  
(4.18)

where \( C \) is a constant which may depend on \( \eta \).

By the condition (4.20), one concludes that for any small constant \( \sigma > 0 \), there exists \( T_0 < T \) such that
\[ \int_{T_0}^{T} \frac{\|w\|_{\dot{B}^0_{\infty, \infty}}}{\sqrt{1 + \ln(e + \|w\|_{\dot{B}^0_{\infty, \infty}})}} dt < \sigma . \]
(4.19)

For any \( T_0 < t \leq T \), we set
\[ h(t) := \sup_{T_0 \leq \tau \leq t} \left( \|\nabla \Delta u(\tau)\|_{L^2}^2 + \eta \|\Delta \frac{\delta E}{\delta \phi}(\tau)\|_{L^2}^2 \right) . \]
(4.20)
where $\hat{\eta}$ is a determined constant which specified later. Applying Gronwall’s inequality to (4.18) in the time interval $[T_0, t]$, one has

$$
\|w(t)\|^2_{L^2} + \eta T_0^{\frac{1}{2}} \frac{\delta E}{\delta \phi}(t) \|_{L^2}^2 \\
\leq C_0 \exp \left( \int_{T_0}^t C \ln(e + h(t)) \int_{T_0}^t \|\omega\|_{B_{\infty, \infty}} \, dt \right) \\
\leq C_0 \exp \left( C(T - T_0) + C \sigma \ln(e + h(t)) \right) \\
\leq C_0 (e + h(t))^{2C},
$$

(4.21)

where $C_0 = \|w(T_0)\|^2_{L^2} + \eta \|\frac{\delta E}{\delta \phi}(T_0)\|^2_{L^2}$ is a positive constant depending on $T_0$.

Now we are in a position to derive higher order energy estimates of the solution. Taking $\nabla \Delta$ on (1.11), multiplying $\nabla \Delta u$ and integrating over $Q$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\nabla \Delta u\|^2_{L^2} + \mu \|\Delta^2 u\|^2_{L^2} = -\int_Q \nabla \Delta(u \cdot \nabla u) \cdot \nabla \Delta u \, dx + \int_Q \nabla \Delta \left( \frac{\delta E}{\delta \phi} \nabla \phi \right) \cdot \nabla \Delta u \, dx \\
:= \tilde{I}_1 + \tilde{I}_2.
$$

(4.22)

Since $\nabla \cdot u = 0$, $\tilde{I}_1$ can be rewritten as

$$
\tilde{I}_1 = -\int_Q [\nabla \Delta(u \cdot \nabla u) - u \cdot \nabla \nabla \Delta u] \cdot \nabla \Delta u \, dx.
$$

By using Lemma 2.1, we can estimate $\tilde{I}_1$ as follows:

$$
\tilde{I}_1 \leq C \|\nabla \Delta(u \cdot \nabla u) - u \cdot \nabla \nabla \Delta u\|_{L^{4/3}} \|\nabla \Delta u\|_{L^4} \leq C \|\nabla u\|_{L^2} \|\nabla \Delta u\|_{L^2}^2 \\
\leq C \|\nabla u\|^2_{L^2} \|\Delta^2 u\|^2_{L^2} + C \|\nabla u\|^4_{L^2} \\
\leq \frac{\mu}{8} \|\Delta^2 u\|^2_{L^2} + C \|\nabla u\|^4_{L^2},
$$

(4.23)

where we have used the Gagliardo-Nirenberg inequality:

$$
\|\nabla \Delta u\|_{L^4} \leq C \|\nabla u\|_{L^2}^{1/12} \|\Delta^2 u\|_{L^2}^{11/12}.
$$

For $\tilde{I}_2$, after integration by parts, by using (3.3) and (3.14), we obtain

$$
\tilde{I}_2 = -\int_Q \Delta \left( \frac{\delta E}{\delta \phi} \nabla \phi \right) \Delta^2 u \, dx \\
\leq \frac{\mu}{8} \|\Delta^2 u\|^2_{L^2} + C \|\Delta \left( \frac{\delta E}{\delta \phi} \nabla \phi \right)\|_{L^2}^2 \\
\leq \frac{\mu}{8} \|\Delta^2 u\|^2_{L^2} + C \left( \|\Delta \frac{\delta E}{\delta \phi} \nabla \phi\|^2_{L^2} + 2 \|\nabla \frac{\delta E}{\delta \phi} \nabla^2 \phi\|_{L^2}^2 + \|\frac{\delta E}{\delta \phi} \nabla \Delta \phi\|_{L^2}^2 \right) \\
\leq \frac{\mu}{8} \|\Delta^2 u\|^2_{L^2} + C \left( \|\Delta \frac{\delta E}{\delta \phi}\|^2_{L^2} \|\nabla \phi\|^2_{L^\infty} + \|\nabla \frac{\delta E}{\delta \phi} \nabla^2 \phi\|^2_{L^2} + \|\nabla \frac{\delta E}{\delta \phi} \nabla \Delta \phi\|^2_{L^2} \right) \\
\leq \frac{\mu}{8} \|\Delta^2 u\|^2_{L^2} + C \left( \|\Delta \frac{\delta E}{\delta \phi}\|^2_{L^2} + \|\frac{\delta E}{\delta \phi}\|^2_{L^2} + 1 \right).
$$

(4.24)
Combining (4.22)–(4.24), we deduce that
\[
\frac{d}{dt} \| \nabla \Delta u \|^2_{L^2} + \frac{3 \mu}{2} \| \Delta^2 u \|^2_{L^2} \leq C \left( \| u \|^4_{L^4} + \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + \| \frac{\delta E}{\delta \phi} \|^2_{L^2} + 1 \right). \tag{4.25}
\]

To obtain the desired estimates for \( \phi \), we start from (1.11) that
\[
\frac{1}{2} \frac{d}{dt} \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} = \int_Q \frac{\partial}{\partial t} \Delta \frac{\delta E}{\delta \phi} \cdot \Delta \frac{\delta E}{\delta \phi} \, dx
\]
\[
= \int_Q \frac{\partial}{\partial t} \Delta [kg(\phi) + M_1(A(\phi) - \alpha) + M_2(B(\phi) - \beta)f(\phi)] \cdot \Delta \frac{\delta E}{\delta \phi} \, dx
\]
\[
= \int_Q \frac{\partial}{\partial t} \Delta [kg(\phi) + M_2(B(\phi) - \beta)f(\phi)] \cdot \Delta \frac{\delta E}{\delta \phi} \, dx
\]
\[
= \int_Q \frac{\partial}{\partial t} (kg(\phi) + M_2(B(\phi) - \beta)f(\phi)) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]
\[
= -k \int_Q \frac{\partial}{\partial t} \Delta f(\phi) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx + \frac{k}{\varepsilon^2} \int_Q \frac{\partial}{\partial t} [3(\phi^2 - 1)f(\phi)] \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]
\[
+ M_2 \frac{d}{dt} B(\phi) \int_Q f(\phi) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx + M_2 (B(\phi) - \beta) \int_Q \frac{\partial}{\partial t} f(\phi) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]
\[
:= \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4. \tag{4.26}
\]

Let us estimate the terms \( \tilde{J}_i \) (\( i = 1, 2, 3, 4 \)) one by one. For \( \tilde{J}_1 \), we divide it into the following two parts:
\[
\tilde{J}_1 = k \varepsilon \int_Q \Delta^2 \frac{\partial \phi}{\partial t} \cdot \Delta \frac{\delta E}{\delta \phi} \, dx - \frac{k}{\varepsilon} \int_Q \frac{\partial}{\partial t} \Delta (\phi^3 - \phi) \cdot \Delta \frac{\delta E}{\delta \phi} \, dx := \tilde{J}_{11} + \tilde{J}_{12}. \tag{4.27}
\]

For \( \tilde{J}_{11} \), by using Leibniz’s rule, we deduce from (1.3) that
\[
\tilde{J}_{11} = -k \varepsilon \gamma \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} - k \varepsilon \int_Q \Delta^2 (u \cdot \nabla \phi) \cdot \Delta \frac{\delta E}{\delta \phi} \, dx
\]
\[
\leq -\frac{15 k \varepsilon \gamma}{16} \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + C \| \Delta^2 (u \cdot \nabla \phi) \|^2_{L^2}
\]
\[
\leq -\frac{15 k \varepsilon \gamma}{16} \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + C \left( \| \Delta^2 u \cdot \nabla \phi \|^2_{L^2} + 4 \| \nabla \Delta u \cdot \nabla^2 \phi \|^2_{L^2}
\right.
\]
\[
+ 6 \| \Delta u \cdot \nabla \phi \|^2_{L^2} + 4 \| \nabla u \cdot \nabla^2 \Delta \phi \|^2_{L^2} + \| u \cdot \nabla^2 \Delta \phi \|^2_{L^2}
\left.
\right)
\]
\[
\leq -\frac{15 k \varepsilon \gamma}{16} \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + C \left( \| \nabla \phi \|^2_{L^2} \| \Delta^2 u \|^2_{L^2} + \| \nabla \Delta u \|^2_{L^2} \| \nabla^2 \phi \|^2_{L^2}
\right.
\]
\[
+ \| \Delta u \|^2_{L^2} \| \nabla \phi \|^2_{L^2} + \| \nabla u \|^2_{L^2} \| \Delta^2 \phi \|^2_{L^2} + \| u \|^2_{L^2} \| \nabla^5 \phi \|^2_{L^2}
\left.
\right)
\]
\[
\leq -\frac{15 k \varepsilon \gamma}{16} \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + C \left( \| \Delta^2 u \|^2_{L^2} + C \left[ \| \frac{\delta E}{\delta \phi} \|^2_{L^2} + 1 \right] \| \nabla \Delta u \|^2_{L^2}
\right.
\]
\[
+ \| \nabla u \|^2_{L^2} + 1 \right) \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + 1 \right)
\left.
\right)
\]
\[
\leq -\frac{15 k \varepsilon \gamma}{16} \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + C \left( \| \nabla u \|^2_{L^2} + \| \frac{\delta E}{\delta \phi} \|^2_{L^2} + 1 \right) \left( \| \nabla \Delta u \|^2_{L^2} + \| \Delta \frac{\delta E}{\delta \phi} \|^2_{L^2} + 1 \right). \tag{4.28}
\]
where we have used the facts \( \| \Delta^2 \phi \|_{L^2}^2 \leq C(\| \Delta \phi \|_{L^2}^2 + 1) \) and \( \| \nabla^5 \phi \|_{L^6}^2 \leq C\| \nabla^6 \phi \|_{L^2}^2 \leq C(\| \Delta \phi \|_{L^2}^2 + 1) \). For \( J_{12} \), it clear that

\[
J_{12} = -\frac{k}{\epsilon} \int_Q \frac{\partial}{\partial t} \Delta(\phi^3 - \phi) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx - \frac{6k}{\epsilon} \int_Q \frac{\partial}{\partial t} (\Delta \phi^3 \frac{\delta E}{\delta \phi}) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]

\[
- \frac{3k}{\epsilon} \int_Q \frac{\partial}{\partial t} (\phi^2 \Delta \phi) \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx + \frac{k}{\epsilon} \int_Q \Delta \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]

\[
= -\frac{12k}{\epsilon} \int_Q \phi \nabla \phi \nabla \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx - \frac{6k}{\epsilon} \int_Q |\nabla \phi|^2 \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]

\[
- \frac{6k}{\epsilon} \int_Q \phi \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx - \frac{3k}{\epsilon} \int_Q \phi \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx
\]

\[
+ \frac{k}{\epsilon} \int_Q \Delta \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} \, dx := \sum_{i=1}^5 J_{12i}.
\]

Similarly, we can estimate the terms \( J_{12i} \) \((i = 1, 2, 3, 4, 5)\) as follows:

\[
J_{121} \leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
\leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
J_{122} \leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
\leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
J_{123} \leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
\leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
J_{124} \leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]

\[
\leq \frac{k}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C\| \nabla \phi \|_{L^6}^2 \| \nabla \frac{\partial \phi}{\partial t} \|_{L^2}^2
\]
Putting estimates (4.30)–(4.34) together, we obtain from (4.29) that
\[
\| \Delta u \|_{L^2}^2 \leq C k \varepsilon \gamma + 6 \| \Delta u \|_{L^2}^2 + C \left( \| \Delta u \|_{L^2}^2 + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right),
\] (4.33)

\[
\hat{J}_{125} \leq \frac{k \varepsilon \gamma}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \| \Delta \frac{\partial \phi}{\partial t} \|_{L^2}^2
\leq \frac{k \varepsilon \gamma}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \Delta u \|_{L^2}^2 + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right).
\] (4.34)

Putting estimates (4.30)–(4.34) together, we obtain from (4.29) that
\[
\hat{J}_{12} \leq \frac{5k \varepsilon \gamma}{16} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \Delta u \|_{L^2}^2 + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right).
\] (4.35)

Since \( \| \Delta u \|_{L^2}^2 \leq C (\| \nabla \Delta u \|_{L^2}^2 + 1) \), we obtain from (4.35), (4.27) and (4.28) that
\[
\hat{J}_1 \leq \frac{5k \varepsilon \gamma}{8} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \| \Delta u \|_{L^2}^2
\quad + C \left( \| \nabla u \|_{L^2}^2 + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right) \left( \| \nabla \Delta u \|_{L^2}^2 + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right).
\] (4.36)

The estimates of \( \hat{J}_i \) \((i = 2, 3, 4, 5)\) can be proceeded as that of \( J_i \) \((i = 2, 3, 4)\), thus we have

\[
\hat{J}_2 = \frac{6k \varepsilon}{\varepsilon^2} \int Q f(\phi) \frac{\partial \phi}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} dx + \frac{k}{\varepsilon^2} \int Q (3 \phi^2 - 1) \frac{\partial f(\phi)}{\partial t} \cdot \Delta^2 \frac{\delta E}{\delta \phi} dx
\leq \frac{6k \varepsilon \gamma}{8} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \phi \|_{L^\infty}^2 \| \Delta \phi \|_{L^2}^2 \| \frac{\partial \phi}{\partial t} \|_{L^2}^2 + \| \phi \|_{L^4}^2 \| \phi \|_{L^2}^2 - 1 \right) \| \frac{\partial \phi}{\partial t} \|_{L^2}^2
\quad + 3 \| \phi \|_{L^2}^2 - 1 \| \frac{\partial \phi}{\partial t} \|_{L^2}^2 \| \frac{\partial f(\phi)}{\partial t} \|_{L^2}^2 \|_{L^2}^2 \|_{L^2}^2 \|_{L^2}^2 \|_{L^2}^2 \|_{L^2}^2 \|_{L^2}^2 \|_{L^2}^2
\leq \frac{6k \varepsilon \gamma}{8} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \frac{\partial \phi}{\partial t} \|_{L^2}^2 + \| \Delta \frac{\partial \phi}{\partial t} \|_{L^2}^2 + \| \frac{\partial \phi}{\partial t} \|_{L^2}^2 \| \frac{\partial \phi}{\partial t} \|_{L^2}^2 + \| \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right),
\] (4.37)

\[
\hat{J}_3 \leq C \frac{dB(\phi)}{dt} \| \| f(\phi) \|_{L^2}^2 \| \frac{\delta E}{\delta \phi} \|_{L^2}^2 \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2
\leq \frac{k \varepsilon \gamma}{8} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \nabla \phi \|_{L^2}^2 \| \Delta \frac{\partial \phi}{\partial t} \|_{L^2}^2 + \| \phi \|_{L^2}^2 \| \frac{\partial \phi}{\partial t} \|_{L^2}^2 \right) \| f(\phi) \|_{L^2}^2
\leq \frac{k \varepsilon \gamma}{8} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \nabla u \|_{L^2}^2 + \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2 + \| \frac{\delta E}{\delta \phi} \|_{L^2}^2 + 1 \right),
\] (4.38)

\[
\hat{J}_4 \leq C |B(\phi) - \beta| \| \frac{\partial f(\phi)}{\partial t} \|_{L^2}^2 \| \Delta \frac{\delta E}{\delta \phi} \|_{L^2}^2
\leq \frac{k \varepsilon \gamma}{8} \| \Delta^2 \frac{\delta E}{\delta \phi} \|_{L^2}^2 + C \left( \| \nabla \phi \|_{L^2}^2 + \| \phi \|_{L^2}^2 - 1 \right) \| \frac{\partial f(\phi)}{\partial t} \|_{L^2}^2.
\]
\[
\leq \frac{k\varepsilon\gamma}{8} \|\Delta^2 \delta E \|_{L^2}^2 + C \left( \|\nabla \Delta \phi \|_{L^2}^2 + \|\frac{\partial \phi}{\partial t} \|_{L^2}^2 \right)
\]
\[
\leq \frac{k\varepsilon\gamma}{8} \|\Delta^2 \delta E \|_{L^2}^2 + C \left( \|\nabla \Delta \phi \|_{L^2}^2 + \|\frac{\partial \phi}{\partial t} \|_{L^2}^2 + 1 \right). \quad (4.39)
\]

Taking (4.36)–(4.39) into (4.26), we conclude that
\[
\frac{d}{dt} \|\Delta \delta E \delta \phi \|_{L^2}^2 + \frac{k\varepsilon\gamma}{2} \|\Delta^2 \delta E \|_{L^2}^2 \leq \hat{C} \|\Delta^2 u \|_{L^2}^2
\]
\[
+ C \left( \|\nabla \Delta u \|_{L^2}^2 + \|\Delta \delta E \delta \phi \|_{L^2}^2 + 1 \right) \left( \|\nabla \Delta u \|_{L^2}^2 + \|\Delta \delta E \delta \phi \|_{L^2}^2 + 1 \right), \quad (4.40)
\]
where \(\hat{C}\) is a constant depending only on \(\|u_0\|_{H^1}, \|\phi_0\|_{H^3}, T^*, \) and coefficients of the system due to the estimate (3.3).

Set
\[
\eta = \frac{\mu}{2C}.
\]

Multiplying (4.40) by \(\eta\) and adding the resultant to (4.25), we obtain
\[
\frac{d}{dt} \left( \|\nabla \Delta u \|_{L^2}^2 + \|\Delta \delta E \delta \phi \|_{L^2}^2 + 1 \right) \left( \|\nabla \Delta u \|_{L^2}^2 + \|\Delta \delta E \delta \phi \|_{L^2}^2 + 1 \right)
\]
\[
\leq C \left( \|\nabla u \|_{L^2}^2 + \|\delta E \delta \phi \|_{L^2}^2 + 1 \right) \left( \|\nabla \Delta u \|_{L^2}^2 + \|\Delta \delta E \delta \phi \|_{L^2}^2 + 1 \right). \quad (4.41)
\]

It follows from (4.21) that
\[
\frac{d}{dt} (e + h(t)) \leq C \left( \|\nabla u \|_{L^2}^2 + \|\delta E \delta \phi \|_{L^2}^2 + 1 \right) \left( (e + h(t))^14C^\sigma + h(t) + 1 \right). \quad (4.42)
\]

Choosing \(\sigma\) small enough such that \(14C^\sigma \leq 1\), we get
\[
\frac{d}{dt} (e + h(t)) \leq C \left( \|\nabla u \|_{L^2}^2 + \|\delta E \delta \phi \|_{L^2}^2 + 1 \right) (e + h(t)). \quad (4.43)
\]

Applying Gronwall’s inequality leads to
\[
h(t) \leq (e + h(T_0)) \exp \left( C \int_{T_0}^{t} \left( \|\nabla u \|_{L^2}^2 + \|\delta E \delta \phi \|_{L^2}^2 + 1 \right) d\tau \right). \quad (4.44)
\]

This combines with the basic energy (1.12) yield the boundness of \(h(t)\) on the time interval \([T_0, T]\).

Since it is easy to verify that
\[
\|\delta \phi \|_{H^6}^2 \leq C(\|\Delta \delta E \delta \phi \|_{L^2}^2 + 1),
\]
we finally obtain from (4.41) that
\[
\sup_{T_0 \leq t \leq T} \left( \|u(\cdot, t)\|_{H^3} + \|\phi(\cdot, t)\|_{H^6} \right) \leq C.
\]
This completes the proof of Theorem 1.1.
5 The proof of Theorem 1.2

Similarly we prove Theorem 1.2 by contradiction. It suffices to prove that if

\[ \int_0^{T_*} \frac{\|\omega(\cdot, t)\|_{B_{\infty, \infty}^{1, 1}}^2}{1 + \ln(e + \|\omega(\cdot, t)\|_{B_{\infty, \infty}^{1, 1}})} dt \leq M < \infty, \]  

(5.1)

then

\[ \limsup_{t \to T_*} (\|u(\cdot, t)\|_{H^3} + \|\phi(\cdot, t)\|_{H^6}) \leq C \]

(5.2)

for some constant depending only on \( u_0, \phi_0, M, T_* \) and coefficients of the system (1.1)–(1.5).

Multiplying (5.3) by \( \omega \) and integrating over \( Q \), we have

\[ \frac{1}{2} \frac{d}{dt}\|\omega\|_{L^2}^2 + \mu \|\nabla \omega\|_{L^2}^2 = \int_Q w \cdot \nabla u \cdot \omega dx - \int_Q \frac{\delta E}{\delta \phi} \nabla \phi \cdot \nabla \omega dx, \]

(5.3)

Since \( \|\nabla u\|_{L^2} \leq C \|\omega\|_{L^2} \), by using Lemma 2.3 we obtain

\[ \left| \int_Q w \cdot \nabla u \cdot \omega dx \right| \leq C \|\omega\|_{L^2}^2 \|\nabla u\|_{L^2} \leq C \|\omega\|_{L^2}^2 \|\omega\|_{L^2} \]

\[ \leq C \|w\|_{B_{\infty, \infty}^{1, 1}} \|\nabla w\|_{L^2} \]

\[ \leq \frac{\mu}{8} \|\nabla w\|_{L^2}^2 + C \|w\|_{B_{\infty, \infty}^{1, 1}} \|w\|_{L^2}^2. \]

(5.4)

The second term on the right-hand side of (5.3) can be estimated the same as (4.6):

\[ \left| - \int_Q \frac{\delta E}{\delta \phi} \nabla \phi \cdot \nabla \omega dx \right| \leq \frac{\mu}{8} \|\nabla w\|_{L^2}^2 + C \|\frac{\delta E}{\delta \phi}\|_{L^2}^2. \]

(5.5)

Taking (5.4) and (5.5) into (5.3), we obtain

\[ \frac{d}{dt}\|w\|_{L^2}^2 + \frac{3\mu}{2} \|\nabla w\|_{L^2}^2 \leq C \left( \|w\|_{B_{\infty, \infty}^{1, 1}}^2 + 1 \right) \left( \|w\|_{L^2}^2 + \frac{\delta E}{\delta \phi} \right)^2. \]

(5.6)

The estimate for \( \frac{\delta E}{\delta \phi} \) can be proceeded the same as that in the proof of Theorem 1.1, thus we also get (4.17). Multiplying (4.17) by \( \eta \) and adding (5.6) together, we obtain

\[ \frac{d}{dt} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 \right) + \mu \|\nabla w\|_{L^2}^2 + k \xi \gamma \eta \|\Delta w\|_{L^2}^2 \]

\[ \leq C \left( \|w\|_{B_{\infty, \infty}^{1, 1}}^2 + 1 \right) \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]

\[ \leq C \left( \|w\|_{B_{\infty, \infty}^{1, 1}}^2 + 1 \right) \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \]

\[ = C \frac{\|w\|_{B_{\infty, \infty}^{1, 1}}^2 + 1}{1 + \ln(e + \|w\|_{B_{\infty, \infty}^{1, 1}})} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \left[ 1 + \ln(e + \|\nabla \Delta u\|_{L^2}) \right] \]

\[ \leq C \frac{\|w\|_{B_{\infty, \infty}^{1, 1}}^2 + 1}{1 + \ln(e + \|w\|_{B_{\infty, \infty}^{1, 1}})} \left( \|w\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}\|_{L^2}^2 + 1 \right) \left[ 1 + \ln(e + \|\nabla \Delta u\|_{L^2}) \right], \]

(5.7)
where $C$ is a constant which may depend on $\eta$.

By the condition (1.21), one concludes that for any small constant $\sigma > 0$, there exists $T_0 < T$ such that
\[ \int_{T_0}^{T} \frac{\|\omega\|_{B_{\infty,1}^{-1}}^2 + 1}{1 + \ln(\sigma + \|\omega\|_{B_{\infty,1}^{-1}})} \, dt < \sigma. \] (5.8)

For any $T_0 < t \leq T$, we set
\[ h(t) := \sup_{T_0 \leq \tau \leq t} \left( \|\nabla \Delta u(\tau)\|_{L^2}^2 + \hat{\eta} \|\Delta \frac{\delta E}{\delta \phi}(\tau)\|_{L^2}^2 \right), \] (5.9)
where $\hat{\eta}$ is a determined constant which specified later. Applying Gronwall’s inequality to (5.7) in the time interval $[T_0, t]$, one has
\[ \|\nabla u(t)\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}(t)\|_{L^2}^2 \leq C_0 \exp \left( C(1 + \ln(e + h(t))) \int_{T_0}^{t} \frac{\|\omega\|_{B_{\infty,1}^{-1}}^2 + 1}{1 + \ln(e + \|\omega\|_{B_{\infty,1}^{-1}})} \, d\tau \right) \leq C_0 \exp \left( C\sigma(1 + \ln(e + h(t))) \right) \leq C_0 (e + h(t))^{2C\sigma}, \] (5.10)

where $C_0 = \|\nabla u(T_0)\|_{L^2}^2 + \eta \|\frac{\delta E}{\delta \phi}(T_0)\|_{L^2}^2$ is a positive constant depending on $T_0$.

The derivations of higher derivative estimates are analogously the proof of Theorem 1.1, thus we safely omit it. This completes the proof of Theorem 1.2.

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