ON QUANTUM DE RHAM COHOMOLOGY

HUAI-DONG CAO & JIAN ZHOU

Abstract. We define quantum exterior product $\wedge_h$ and quantum exterior differential $d_h$ on Poisson manifolds, of which symplectic manifolds are an important class of examples. Quantum de Rham cohomology is defined as the cohomology of $d_h$. We also define quantum Dolbeault cohomology. Quantum hard Lefschetz theorem is proved. We also define a version of quantum integral, and prove the quantum Stokes theorem. By the trick of replacing $d$ by $d_h$ and $\wedge$ by $\wedge_h$ in the usual definitions, we define many quantum analogues of important objects in differential geometry, e.g. quantum curvature. The quantum characteristic classes are then studied along the lines of classical Chern-Weil theory, i.e., they can be represented by expressions of quantum curvature. Calculations are done for some examples, which show that quantum de Rham cohomology is different from the quantum cohomology defined using pseudo-holomorphic curves.

Recently, the quantum cohomology rings have generated a lot of researches. Many mathematicians have contributed to this rapidly progressing field of mathematics. We will not described the history here, but refer the interested reader to the original papers and surveys (e.g., [2], [3], [4], [13], [14], [18], [19], [22]–[30], [32]–[40], [42], [43] and the references therein).

The purpose of this paper is to give the construction of another deformation of the de Rham cohomology ring. The existence of a different deformation should not be a surprise, since there is no reason to expect the deformation of the cohomology to be unique. A remarkable feature of our construction is that it follows the traditional construction of the de Rham cohomology. More precisely, we construct a quantum wedge product $\wedge_h$ on exterior forms, and a quantum exterior differential $d_h$, which satisfy the usual property of the calculus of differential forms. This quantum calculus allows us to “deformation quantize” many differential geometric objects, i.e. our quantum objects is a polynomial in an indeterminate $h$, whose zeroth order terms are the classical objects. (In this sense, $h$ should be regarded as the Planck constant.) For example, we will define quantum curvature of an ordinary connection, and define quantum characteristic classes in the same fashion as the classical Chern-Weil theory. Our construction has the following features which are not shared by the quantum cohomology:

1. Quantum de Rham cohomology can be defined for Poisson manifolds, not necessarily compact, or closed.
2. The proof of associative is of elementary nature.
3. It is routine to define quantum Dolbeault cohomology.
4. It is routine to define quantum characteristic class.
5. It is routine to define quantum equivariant de Rham cohomology.
6. The computations for homogeneous examples are elementary.

1Both authors are supported in part by NSF
Our construction is motivated by Moyal-Weyl multiplication and Clifford multiplication. For any finite dimensional vector space $V$ with a basis $\{e_1, \cdots, e_m\}$, let $\{e^1, \cdots, e^m\}$ be the dual basis. Assume that $w = w^{ij} e_i \otimes e_j \in V \otimes V$, then $w$ defines a multiplication $\wedge_w$ on $\Lambda(V^*)$, and a multiplication $*_w$ on $S(V^*)$, such that $e^i \wedge_w e^j = e^i \wedge e^j + w^{ij}$, $e^i *_w e^j = e^i \circ e^j + w^{ij}$. If $w \in S^2(V)$, then $\wedge_w$ is the Clifford multiplication. If $w \in \Lambda^2(V)$ is nondegenerate, $*_w$ is the Moyal-Weyl multiplication. If $w \in \Lambda(V)$, then $\wedge_w$ is what we call a quantum exterior product (or a quantum Clifford multiplication). It is elementary to show that this multiplication is associative. We will use it to obtain a quantum calculus on any Poisson manifold. The main results we obtained in this paper have been announced in [10].

The layout of this paper is clear from the following

\begin{center}
\textbf{CONTENTS}
\end{center}

1. Quantum Exterior Algebra \hfill 2
1.1. Deformation quantization \hfill 2
1.2. Laurent deformation \hfill 3
1.3. Moyal-Weyl quantization \hfill 3
1.4. Quantum exterior algebra \hfill 4
1.5. Complexified quantum exterior algebra \hfill 9
1.6. Multiparameter deformation \hfill 11
2. Quantum de Rham complex \hfill 12
3. Quantum de Rham cohomology \hfill 18
4. Quantum Hard Lefschetz Theorem \hfill 20
5. Quantum Dolbeault cohomology \hfill 26
6. Quantum integral and quantum Stokes Theorem \hfill 27
7. Quantum Chern-Weil theory \hfill 29
7.1. Quantum equivariant de Rham cohomology \hfill 31
8. Computations for some examples \hfill 31
References \hfill 34

\begin{center}
\textbf{Acknowledgement}
The work in this paper was carried out during the second author’s visit at Texas A&M University. He thanks the Department of Mathematics, especially the Geometry-Analysis-Topology group for hospitality and financial support. Both authors want to thank Tony Giaquinto for a wonderful talk on Poisson-Lie groups which jump started our initially unsuccessful approach. We also thank him for providing Vaisman’s book [41], which is a very good reference.

\textbf{Part I. Algebraic Theory}

1. Quantum Exterior Algebra

1.1. Deformation quantization. For more information deformation quantization of algebras, we refer to Donin [11] and the references therein. Let $A$ be an algebra with unit over a field $\mathbf{k}$ of characteristic zero. A deformation quantization of $A$ is an algebra $A_\hbar$ over $\mathbf{k}[\hbar]$ that is isomorphic to $A[\hbar] = A \otimes_{\mathbf{k}} \mathbf{k}[\hbar]$ as a $\mathbf{k}[\hbar]$-module, such that $A_\hbar/\hbar A_\hbar \cong A$. A deformation quantization of an algebra $A$ is uniquely determined by a $\mathbf{k}$-linear map $f : A \otimes_{\mathbf{k}} A \to A[\hbar]$, $f(a, b) = a \cdot b + \sum_{j \geq 0} f_j(a, b) \hbar^j$, for $a, b \in A$, where $a \cdot b$ stands for multiplication in $A$. When $A$ is a $\mathbb{Z}$-graded algebra, a
graded deformation quantization of $A$ is a deformation quantization $A_h \cong A \otimes_k k[h]$, which has the structure of a graded algebra when it is given the induced grading by setting $\text{deg}(h) = 2$.

Assume now $A$ is a graded differential algebra (GDA), i.e., there is a $\mathbb{Z}$-grading on $A$, and a $k$-linear $d : A \to A$ is a derivation of degree $1$ on $A$, such that $d^2 = 0$, and $d(a \cdot b) = (da) \cdot b + (-1)^{|a|}a \cdot (db)$, for homogeneous elements $a, b \in A$, and $|a|$ stands for the degree of $a$. The graded algebra $H^*(A, d) = \text{Ker} d / \text{Im} d$ is called the cohomology of the GDA $(A, d)$. A deformation quantization of a GDA $(A, d)$ is a graded deformation quantization $A_h$, together with a $k[h]$-differential $d_h$ of degree $1$, such that when $A_h / hA_h$ is identified with $A$, the map on $A_h / hA_h$ induced by $d_h$ is identified with $d$. In this paper, we will be concerned with $d_h$ of the form $d_h = d - h\delta$, where $\delta : A \to A$ is a derivation of degree $-1$. We regard the complex $(A[h], d_h)$ as the associated complex of the double complex $(C^{p,q} := h^pA^{q-p}, d, -h\delta)$. There are two spectral sequences associated with this double complex by standard theory (Bott-Tu [8]), one of them has $E_1^{p,q} = h^pH^{q-p}(A, d)$. If this spectral sequence degenerate at $E_1$, we then have

$$Q_hH^n(A) := H^n(A_h, d_h) = \oplus_{p+q=n} h^pH^{q-p}(A, d),$$

for $p, q \geq 0$. It follows then $Q_hH^*(A)$ is a deformation quantization of $H^*(A, d)$ in this case.

1.2. Laurent deformation. Sometimes it is useful to use $k[h, h^{-1}]$ instead of $k[h]$ as the coefficient ring. This will become apparent in our theory in [8]. Let $A$ be an algebra with unit over a field of characteristic zero, a (polynomial) Laurent deformation of $A$ is an algebra over $k[h, h^{-1}]$, which is isomorphic to $A[h, h^{-1}] = A \otimes_k k[h, h^{-1}]$, whose multiplication is determined by a $k$-linear map

$$f : A \otimes_k A \to A[h, h^{-1}]$$

of the following form: $f(a, b) = a \cdot b + \sum_{j \neq 0} f_j(a, b)h^j$, where $a, b \in A$, and $a \cdot b$ stands for multiplication in $A$. We use the following simple construction. Since a deformation quantization $(A_h, \ast_h)$ of an algebra $(A, \cdot)$ is determined by a $k$-linear map $f : A \otimes_k A \to A[h]$ of the form $f(a, b) = a \cdot b + \sum_{j > 0} f_j(a, b)h^j$, it gives rise to a unique Laurent deformation of $A$. We will consider Laurent deformation of a GDA similar to the polynomial deformation case discussed in [1].

1.3. Moyal-Weyl quantization. Reformulation and generalization of Moyal-Weyl quantization for polynomial algebra on a symplectic vector space is presented in this section. It serves as a motivation for our construction below.

Let $V$ be a $k$-vector space. The symmetric tensor algebra $S(V^*)$ of $V^*$ can be regarded as the algebra of $k$-polynomials on $V$. Let $w \in \Lambda^2(V)$. When a basis $\{e_1, \cdots, e_n\}$ is chosen, let $(x^1, \cdots, x^n)$ be the coordinates in this basis, $w = w^{ij}e_i \wedge e_j$. Then the Moyal-Weyl product of two polynomials $u, v$ is given by

$$u \ast_h v = \sum_{n \geq 0} \frac{h^n}{n!} w^{i_1 j_1} \cdots w^{i_n j_n} \frac{\partial^n u}{\partial x^{i_1} \cdots \partial x^{i_n}} \cdot \frac{\partial^n v}{\partial x^{j_1} \cdots \partial x^{j_n}}.$$

It can be formulated without using coordinates. Denote by $T(V^*)$ the tensor algebra of $V^*$, for any $\phi \in \otimes^2 V^*$, define $L_\phi : T(V^*) \otimes T(V^*) \to T(V^*)$ as follows: for $\alpha, \beta \in T(V^*)$, $L_\phi(\alpha \otimes \beta)$ is obtained from $\alpha \otimes \phi \otimes \beta$ by contracting the first factor
in $\phi$ with the last factor in $\alpha$, and contraction the second factor in $\phi$ with the first factor in $\beta$. In coordinates, this is denoted by

$$L_\phi(\alpha \otimes \beta) = \phi^{ij}(\alpha \triangleright e_i) \otimes (e_j \triangleright \beta),$$

where $\triangleright$ is the contraction from the right, $\triangleright$ is the contraction from the left, i.e.,

$$(\alpha \triangleright v)(v_1, \cdots, v_k - 1) = \alpha(v_1, \cdots, v_k - 1, v),$$

$$(v \triangleright \alpha)(v_1, \cdots, v_k - 1) = \alpha(v, v_1, \cdots, v_k - 1),$$

for $v, v_1, \cdots, v_k - 1 \in V$, $\alpha \in T^k(V^*)$. Denote by $m_s : S(V^*) \otimes S(V^*) \rightarrow S(V^*)$ the symmetric product. It is then obvious that Moyal-Weyl product is given by

$$\alpha * h \beta = m_s(\exp(hL_w)(\alpha \otimes \beta)).$$

We give $S_h(V^*) = S(V^*)[\hbar]$ the following $\mathbb{Z}$-grading: elements in $S^p(V^*)$ has degree $p$, and $\hbar$ has degree $2$. If we denote by $S_h^{m}(V^*)$ the space of homogeneous elements of degree $m$, then we have

$$S_h^{[m]}(V^*) * h S_h^{[n]}(V^*) \subset S_h^{[m+n]}(V^*).$$

Since $w$ is anti-symmetric, $u * h v = v * h u$ does not hold in general. It can be checked that $(u * h v) * h w = u * h (v * h w)$. See e.g Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer [3] or the next section. Therefore, $(S_h(V^*), * h)$ is an algebra. It is clear that it is a deformation quantization of the polynomial algebra $S(V^*)$.

1.4. Quantum exterior algebra. Now let $\Lambda(V^*)$ denote the exterior algebra, and $m : \Lambda(V^*) \rightarrow \Lambda(V^*)$ the exterior product. Given any $w = w^{ij} e_i \wedge e_j \in \Lambda^2(V)$, we define the quantum exterior product by $\wedge_h : \Lambda(V^*) \otimes \Lambda(V^*) \rightarrow \Lambda(V^*)[\hbar]$ by

$$\alpha \wedge_h \beta = m(\exp(hL_w)(\alpha \otimes \beta)) = \sum_{n \geq 0} \frac{h^n}{n!} m(L^n_w(\alpha \otimes \beta))$$

$$= \sum_{n \geq 0} \frac{h^n}{n!} w^{i_1j_1} \cdots w^{i_nj_n} (\alpha \triangleright e_{i_1} \triangleright \cdots \triangleright e_{i_n}) \wedge (e_{j_n} \triangleright \cdots \triangleright e_{j_1} \triangleright \beta),$$

for $\alpha, \beta \in \Lambda(V^*)$. (This is evidently independent of the choice of the basis.) Notice that this is just Moyal-Weyl multiplication for exterior algebra. It is defined this way to keep track of the signs. When there is no confusion about $w$, we will simply write $\alpha \wedge_h \beta$ for $\alpha \wedge_h \wedge \beta$. The map $\Lambda(V^*) \otimes \Lambda(V^*) \rightarrow \Lambda(V^*)$ will be denoted by $\wedge_w$. We have

$$\alpha \wedge_h \beta = \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{n(|\alpha|-1)} w^{i_1j_1} \cdots w^{i_nj_n} (e_{i_1} \triangleright \cdots \triangleright e_{i_n} \triangleright \alpha) \wedge (e_{j_n} \triangleright \cdots \triangleright e_{j_1} \triangleright \beta)$$

$$= \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{n(|\alpha|-1)+n(n-1)/2} w^{i_{|I|}j_{|J|}} (\cdot (e_{i_1} \triangleright \cdots \triangleright e_{i_n} \triangleright \alpha) \wedge (e_{j_1} \triangleright \cdots \triangleright e_{j_n} \triangleright \beta).$$

For simplicity of the notations, we will write

$$\alpha \wedge_h \beta = \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{n(|\alpha|-1)+n(n-1)/2} \sum_{|I|=n |J|=n} w^{IJ}(e_I \triangleright \alpha) \wedge (e_J \triangleright \beta).$$
We extend $\wedge_h$ as $k[h]$-module map to $\Lambda_h(V^*) \otimes_{k[h]} \Lambda_h(V^*)$. We give the same
$\mathbb{Z}$-grading on $\Lambda_h(V^*) = \Lambda(V^*)[h]$ as for $S_h(V^*)$, then it is clear that
$$\Lambda_h^{[m]}(V^*) \wedge_h \Lambda_h^{[n]}(V^*) \subset \Lambda_h^{[m+n]}(V^*).$$

**Remark.** To make contact with more familiar objects, we consider the case of
$|\alpha| = 1$, we take $\alpha$ to be an element $v \in V^*$. The bivector $w \in \Lambda^2(V^*)$ defines a
homomorphism $V^* \to V$ by $v \mapsto v_w = w(v, \cdot)$. Then $v \wedge_h \beta = v \wedge \beta + hv \wedge \beta$. This
is the analogue of the Clifford multiplication, which is defined by setting $h = 1,$
and using an element $q \in S^2(V)$ instead of $w$. So it might be more instructive
to call $\wedge_h$ the quantum Clifford multiplication. Of course, when $h = 0,$ both
quantum Clifford multiplication and quantum skew Clifford multiplication gives the
exterior product. This reveals that both of them are deformation quantizations of
the exterior product, the difference being one preserves the super commutativity,
the other destroys it.

**Theorem 1.1.** The quantum exterior product satisfies the following properties

(1) **Supercommutativity** \hspace{2em} $\alpha \wedge_h \beta = (-1)^{|\alpha||\beta|} \beta \wedge_h \alpha$,

(2) **Associativity** \hspace{2em} $(\alpha \wedge_h \beta) \wedge_h \gamma = \alpha \wedge_h (\beta \wedge_h \gamma)$,

for all $\alpha, \beta, \gamma \in \Lambda_h(V^*)$. Therefore, $(\Lambda_h(V^*), \Lambda_h)$ is a deformation quantization of
the exterior algebra $(\Lambda(V^*), \wedge)$.

**Remark.** Similar results hold for $\Lambda_{h,h^{-1}}(V^*)$, and for $\wedge_w$ on $\Lambda(V^*)$.

**Proof of supercommutativity.**
$$\beta \wedge_h \alpha = \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{|\beta|-1+n(n-1)/2} \sum_{|I|=n |J|=n} w^{IJ} (e_I \epsilon \beta) \wedge (e_J \epsilon \alpha)$$
$$= \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{|\beta|-1+n(n-1)/2} (-1)^n \sum_{|I|=n |J|=n} w^{IJ} (-1)^{|\alpha|-1}|\beta|-n) (e_I \epsilon \alpha) \wedge (e_J \epsilon \beta)$$
$$= (-1)^{|\alpha||\beta|} \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{|\alpha|-1+n(n-1)/2} \sum_{|I|=n |J|=n} w^{IJ} (e_I \epsilon \alpha) \wedge (e_J \epsilon \beta)$$
$$= (-1)^{|\alpha||\beta|} \beta \wedge_h \alpha.$$

We will prove the associativity (2) by induction. We say that $A(a,b,c)$ holds,
if (2) holds for all $\alpha, \beta, \gamma \in \Lambda(V^*)$ with $|\alpha| = a, |\beta| = b, |\gamma| = c$. We say that
$A(\leq a,b,c)$ holds, if (2) holds for all $\alpha, \beta, \gamma \in \Lambda(V^*)$ with $|\alpha| \leq a, |\beta| = b, |\gamma| = c$.
Our strategy is as follows. We first prove $A(1,b,c)$ for arbitrary $b$ and $c$, then prove
the general case by induction on $a$. 

Proof of \( A(1, b, c) \). By linearity, we can assume that \( \alpha = e^i \) for some given basis \( \{e_1, \ldots, e_m\} \) of \( V \).

\[
(e^i \land_h \beta) \land_h \gamma = (e^i \land \beta + \hat{h} w^{ik} e_k \vdash \beta) \land_h \gamma
\]

(A) \[= \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{n|\beta| + n(n-1)/2} \sum_{|I|=n \atop |J|=n} w^{IJ} (e_I \vdash (e^i \land \beta)) \land (e_J \vdash \gamma) \]

(B) \[+ \sum_{n \geq 0} \frac{h^{n+1}}{n!} (-1)^{n|\beta| + n(n-1)/2} \sum_{|I|=n \atop |J|=n} w^{ik} w^{IJ} (e_I \vdash e_k \vdash \beta) \land (e_J \vdash \gamma) \]

Now for (B), we use \( e_I \vdash e_k \vdash \beta = (-1)^n e_k \vdash e_I \vdash \beta \). For (A), we use

\[
e_I \vdash (e^i \land \beta) = (-1)^n e^i \land (e_I \vdash \beta) + \sum_{l=1}^n (-1)^{n-l} \delta^i_{l} (e_{i_1} \vdash \cdots \vdash \hat{e}_{i_l} \vdash \cdots \vdash e_{i_n} \vdash \beta). \]

This still holds when \( n = 0 \), if we take the second term on the right to be zero. So (A) is equal to

(A1) \[= \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{n|\beta| + n(n-1)/2} \sum_{|I|=n \atop |J|=n} w^{IJ} (-1)^n e^i \land (e_I \vdash \beta) \land (e_J \vdash \gamma) \]

(A2) \[+ \sum_{n \geq 1} \frac{h^n}{n!} (-1)^{n|\beta| + n(n-1)/2} \sum_{|I|=n \atop |J|=n} w^{IJ} \]
\[\cdot \sum_{1 \leq l \leq n} (-1)^{n-l} \delta^i_{l} (e_{i_1} \vdash \cdots \vdash \hat{e}_{i_l} \vdash \cdots \vdash e_{i_n} \vdash \beta) \land (e_J \vdash \gamma) \]

For (A2), we use the following renaming of the indices: \( I'_i = i_1 \cdots \hat{i_l} \cdots i_n, J'_i = j_1 \cdots \hat{j_l} \cdots j_n \), \( j_l = k \). Then we have

\[
\sum_{|I|=n \atop |J|=n} w^{IJ} \sum_{1 \leq l \leq n} (-1)^{n-l} \delta^i_{l} (e_{i_1} \vdash \cdots \vdash \hat{e}_{i_l} \vdash \cdots \vdash e_{i_n} \vdash \beta) \land (e_J \vdash \gamma) \]

\[= \sum_{l=1}^n \sum_{|I|=n \atop |J|=n} (-1)^{n-l} w^{ik} w^{I'_iJ'_i} (e_{I'_i} \vdash \beta) \land (-1)^{l-1} (e_k \vdash e_{J'_i} \land \gamma) \]

\[= n \sum_{|I|=n \atop |J|=n-1} (-1)^{n-1} w^{ik} w^{IJ} (e_I \vdash \beta) \land (e_k \vdash e_J \land \gamma). \]
So \((\mathbf{A}_2)\) is equal to

\[
\sum_{n \geq 1} \frac{h^n}{n!} (-1)^{n|\beta| + n(n-1)/2} n(n-1)^{n-1} \sum_{|I|=n, |J|=n-1} w^{ik} w^{IJ} (e_l \vdash \beta) \land (e_k \vdash e_J \vdash \gamma)
\]

\[
= \sum_{n \geq 1} \frac{h^n}{(n-1)!} (-1)^{n|\beta| + (n-2)(n-1)/2} \sum_{|I|=n, |J|=n-1} w^{ik} w^{IJ} (e_l \vdash \beta) \land (e_k \vdash e_J \vdash \gamma)
\]

\[
= \sum_{n \geq 0} \frac{h^{n+1}}{n!} (-1)^{(n+1)|\beta| + n(n-1)/2} \sum_{|I|=n, |J|=n} w^{ik} w^{IJ} (e_l \vdash \beta) \land (e_k \vdash e_J \vdash \gamma)
\]

To summarize, we have

\[
(e^l \land_h \beta) \land_h \gamma
\]

\[
(\mathbf{A}_1) \quad = \sum_{n \geq 0} \frac{h^n}{n!} (-1)^{n|\beta| + n(n+1)/2} \sum_{|I|=n} w^{IJ} e^i \land (e_l \vdash \beta) \land (e_J \vdash \gamma)
\]

\[
(\mathbf{A}_2) \quad + \sum_{n \geq 0} \frac{h^{n+1}}{n!} (-1)^{(n+1)|\beta| + n(n-1)/2} \sum_{|I|=n, |J|=n} w^{ik} w^{IJ} (e_l \vdash \beta) \land (e_k \vdash e_J \vdash \gamma)
\]

\[
(\mathbf{B}) \quad + \sum_{n \geq 0} \frac{h^{n+1}}{n!} (-1)^{n|\beta| + n(n+1)/2} \sum_{|I|=n, |J|=n} w^{ik} w^{IJ} (e_k \vdash e_I \vdash \beta) \land (e_J \vdash \gamma)
\]
Therefore, associativity (2) holds for all

$$\eta \in \mathbb{R}$$

It is clear that

$$\phi \in \mathbb{R}$$

Two remarks are in order. First, in the above proof of associativity, we have

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} (-1)^n (|\beta| - 1) + n(n-1)/2 \sum_{|I|=n} w^{it} (\epsilon_I \vdash \beta) \wedge (\epsilon_J \vdash \gamma)$$

It is clear that \((A_1) = (C), (A_2) = (D_2), \) and \((B) = (D_1). \) This completes the proof of \(A(1, b, c). \)

**Proof of \(A(a, b, c). \)** Assume that \(A(\leq a, b, c) \) is proven, we now show how to deduce \(A(1 + a, b, c) \). Without loss of generality, assume that \(\alpha = v \wedge h \eta \), for some \(v \) and \(\eta \) with \(|v| = 1, |\eta| = a. \) By definition, \(v \wedge h \eta = v \wedge \eta - hf(v, \eta) \) for some element \(f(v, \eta) \) with degree \(\leq a - 2. \) Therefore,

$$\alpha \wedge h (\beta \wedge h \gamma) = (v \wedge \eta) \wedge h (\beta \wedge h \gamma)$$

$$= (v \wedge h \eta) \wedge h (\beta \wedge h \gamma) + hf(v, \eta) \wedge h (\beta \wedge h \gamma)$$

$$= v \wedge h (v \wedge h (\beta \wedge h \gamma)) + h(f(v, \eta) \wedge h (\beta \wedge h \gamma) \wedge h \gamma \quad \text{by } A(1, a, b + c) \text{ and } A(\leq a, b, c)$$

$$= v \wedge h ((v \wedge h (\beta \wedge h \gamma)) + h(f(v, \eta) \wedge h (\beta \wedge h \gamma) \wedge h \gamma \quad \text{by } A(a, b, c)$$

$$= (v \wedge h (v \wedge h (\beta \wedge h \gamma)) \wedge h \gamma + h(f(v, \eta) \wedge h (\beta \wedge h \gamma) \wedge h \gamma \quad \text{by } A(1, a + b, c)$$

$$= ((v \wedge h (v \wedge h (\beta \wedge h \gamma)) \wedge h \gamma + h(f(v, \eta) \wedge h (\beta \wedge h \gamma) \wedge h \gamma \quad \text{by } A(1, a, b)$$

$$= (v \wedge h \eta) \wedge h \beta) \wedge h \gamma = (\alpha \wedge h \beta) \wedge h \gamma. \quad \text{by } A(1, a, b)$$

Therefore, associativity \((B)\) holds for all \(\alpha, \beta, \gamma \in \Lambda/V^*. \)

Two remarks are in order. First, in the above proof of associativity, we have never used the anti-symmetric property of \(w. \) Therefore, if we define \(\wedge h \) using any bi-vector \(\phi, \wedge h \) will still be associative. It may not be supercommutative anymore.
Second, the associativity of Moyal-Weyl quantization can be proved in the same fashion. Without those ± signs, it is much simpler. Again we do not use the anti-symmetric property of $w$, so if one defines generalized Moyal-Weyl quantization $*_h$ using any bi-vector $\phi$, $*_h$ is associative. In particular, if $\phi$ is symmetric, $*_h$ defined by $\phi$ is commutative.

1.5. Complexified quantum exterior algebra. In this section, we will consider real vector space $V$ with an almost complex structure $J$, i.e., $J : V \to V$ is a linear transformation such that $J^2 = -Id$. There is an induced linear transformation $\Lambda^2 J : \Lambda^2 (V) \to \Lambda^2 (V)$. For any bi-vector $w \in \Lambda^2 (V)$, we say $J$ preserves $w$ if $\Lambda^2 J (w) = w$. Given any bi-vector $w$ which is preserved by $J$, we can define the quantum exterior product on $\Lambda_h (V^*)$ as in the last section. Now if we tensor everything by $\mathbb{C}$, we get a natural decomposition as follows. There is a linear transformation such that $\Lambda^2 J (w) = w$. Given any bi-vector $w$ which is preserved by $J$, we can define the quantum exterior product on $\Lambda_h (V^*)$ as in the last section. Now if we tensor everything by $\mathbb{C}$, we get a natural decomposition as follows. There are two complex vector spaces $V^{1,0}$ and $V^{0,1}$ with underlying real vector space $V$ by: for $V^{1,0}$, the multiplication by $\sqrt{-1}$ is given by $J$; for $V^{0,1}$, by $-J$. There is a natural identification of complex vector spaces $\mathbb{C}V \cong V^{1,0} \oplus V^{0,1}$ given by $v = \frac{1}{2} (v - \sqrt{-1} J v) + \frac{1}{2} (v + \sqrt{-1} J v)$ for any $v \in V$, and extend it complex linearly to $\mathbb{C}V$. As a consequence, there are decompositions

\[
\mathcal{C}A(V) = \oplus_{p,q} \Lambda^{p,q}(V), \\
\mathcal{C}A(V^*) = \oplus_{p,q} \Lambda^{p,q}(V^*),
\]

where $\Lambda^{p,q}(V) \cong \Lambda^p_\mathbb{C}(V^{1,0}) \otimes \Lambda^q_\mathbb{C}(V^{0,1})$, and $\Lambda^{p,q}(V^*) \cong \Lambda^p_\mathbb{C}((V^{1,0})^*) \otimes \Lambda^q_\mathbb{C}((V^{0,1})^*)$. We give $\mathcal{C}A_h (V^*)$ the following $\mathbb{Z} \times \mathbb{Z}$-bigrading: elements in $\Lambda^{p,q}(V^*)$ has bi-degree $(p, q)$, $h$ has bi-degree $(1, 1)$. Since $w$ is preserved by $J$, it belongs to $\Lambda^{1,1}(V)$ after complexification. Denote by $\Lambda^p_h (V^*)$ the space of homogeneous elements of bi-degree $(p, q)$. When we compute $L$ in $\mathcal{C}A(V^*)$, we can use a complex basis of $\mathbb{C}V$ of the form \{${f_1, \cdots, f_n, f_1^*, \cdots, f_n^*}$\}, where \{${f_1, \cdots, f_n}$\} is a complex basis of $V^{1,0}$, and \{${f_1^*, \cdots, f_n^*}$\} is the complex conjugate basis of $V^{0,1}$. It is then clear from the definition that

\[
\alpha \wedge_h \beta = \sum_{p,q \geq 0} \frac{h^{p+q}}{p!q!} \sum_{|A|=|B|=p \atop |C|=|D|=q} w^{AB} w^{CD} (\alpha \to f_A \to f_C) \wedge (f_{r(D)} \to f_{r(B)} \to \beta),
\]

where if $A = (a_1, \cdots, a_p)$, then $r(A) = (a_p, \cdots, a_1)$, the reverse of $A$. It then follows that

\[
\Lambda^p_h (V^*) \wedge_h \Lambda^q_h (V^*) \subset \Lambda^{p+q+1}_h (V^*).
\]

Now let $\omega$ be a symplectic form on $V$, which is compatible with an almost complex structure on $V$, i.e. rank of $\omega$ is $2n = \dim (V)$, $w(J \cdot, J \cdot) = \omega(\cdot, \cdot)$, and $g(\cdot, \cdot) := \omega(\cdot, J \cdot)$ is a positive definite element of $S^2(V^*)$. For $X, Y \in \mathbb{C}V$, set

\[
H(X, Y) = \frac{1}{\sqrt{-1}} w(X, Y).
\]

Then $H$ is a Hermitian metric on $\mathbb{C}V$, such that $V^{1,0} \perp V^{0,1}$. It induces a Hermitian metric on $\mathcal{C}A(V^*)$, which we will give explicitly in coordinates below. It is possible to find an orthonormal basis of $V$ for $g$ of the form \{${e_1, Je_1, \cdots, e_n, Je_n}$\}. Set

\[
f_a = \frac{1}{2} (e_a - \sqrt{-1} Je_a), \quad \bar{f}_a = \frac{1}{2} (e_a + \sqrt{-1} Je_a),
\]

ON QUANTUM DE RHAM COHOMOLOGY 9
for \( a = 1, \cdots, n \). It can be easily checked that
\[
\omega_{ab} := \omega(f_a, f_b) = 0, \\
\omega_{\bar{a}\bar{b}} := \omega(f_{\bar{a}}, f_{\bar{b}}) = 0, \\
\omega_{a\bar{b}} = -\omega_{\bar{a}b} = \frac{\sqrt{-1}}{2} \delta_{ab},
\]
for \( a, b = 1, \cdots, n \). Then \( \{\sqrt{2}f_a\} \) is an orthonormal basis of \( V^{1,0} \), and \( \{\sqrt{2}f_{\bar{a}}\} \) is an orthonormal basis of \( V^{0,1} \). Let \( \{\frac{1}{\sqrt{2}} f^a\} \) and \( \{\frac{1}{\sqrt{2}} f^\alpha\} \) be the dual basis for \( (V^{1,0})^* \) and \( (V^{0,1})^* \) respectively. Then in the canonically induced Hermitian metric on \( \Lambda^p,q(V^*) \), \( \{2^{-(p+q)/2} f^a_1 \wedge \cdots \wedge f^a_p \wedge f^b_1 \wedge \cdots \wedge f^b_q, a_1 < \cdots < a_p, b_1 < \cdots < b_q\} \) is an orthonormal basis. The symplectic form \( \omega \in \Lambda^2(V^*) \) determines a unique bivector \( u^\alpha \in \Lambda^2(V) \) in a way similar to raising the index in Riemannian geometry. Let \( \omega^{ij} = \omega^2(e^i, e^j) \), \( \omega_{ij} = \omega(e_i, e_j) \), then the matrix \( (\omega^{ij}) \) is the inverse of the matrix \( (\omega_{ij}) \). After complexification, let \( a, b \) denotes the complex indices, then we have
\[
\omega^{\bar{a}\bar{b}} = \omega_{\bar{a}\bar{b}} = 0, \quad \omega^{a\bar{b}} = -w_{\bar{b}a} = -\frac{2}{\sqrt{-1}} \delta_{ab}.
\]
For any \( \alpha \in \Lambda^p,q(V^*) \), we can write
\[
\alpha = \sum_{|A| = p, |C| = q} \frac{1}{plq!} \alpha_{AC} f^A \wedge f^C,
\]
where \( \alpha_{AC} \) is anti-symmetric in the complex indices \( A \) and \( C \). Then we have
\[
\alpha_{AC} = f_{r(C)} \vdash f_{r(A)} \vdash \alpha.
\]
Therefore, for \( \alpha, \beta \in \Lambda^p,q(V^*) \),
\[
H(\alpha, \beta) = \sum_{|A| = p, |C| = q} \frac{2^{p+q}}{plq!} (f_{r(C)} \vdash f_{r(A)} \vdash \alpha) \cdot (f_{r(C)} \vdash f_{r(A)} \vdash \beta)
\]
\[
= \sum_{|A| = |B| = p, |C| = |D| = q} \frac{2^{p+q}}{plq!} \delta^{AB} \delta^{CD} (f_{r(C)} \vdash f_{r(A)} \vdash \alpha) \cdot (f_{r(D)} \vdash f_{r(B)} \vdash \beta)
\]
\[
= \sum_{|A| = |B| = p, |C| = |D| = q} \frac{(-1)^{p+q}(p+q)!}{p!q!} \omega^{AB} \omega^{CD} (f_{r(C)} \vdash f_{r(A)} \vdash \alpha) \cdot (f_{r(D)} \vdash f_{r(B)} \vdash \beta)
\]
\[
= \frac{(-1)^{p+q}(p+q)!}{2^{p+q}p!q!} \sum_{|A| = |B| = p, |C| = |D| = q} \omega^{AB} \omega^{CD} (\alpha \wedge f_A + f_C) \cdot (f_{r(D)} \vdash f_{r(B)} \vdash \beta)
\]
\[
= \frac{(-1)^{p+q}(p+q)!}{2^{p+q}} (\alpha \wedge \beta)_0,
\]
where \( \alpha \wedge \beta \) is obtained from \( \alpha \wedge_h \beta \) by setting \( h = 1 \). And the subscript 0 means taking the degree 0 zero part. Since for general \( \alpha \in \Lambda^p,q(V^*) \), \( \beta \in \Lambda^r,s(V^*) \),
\( (\alpha \wedge_w \beta)_0 \) can be nonzero only if \( p = r \) and \( q = t \), we see that for any \( \alpha \in \Lambda^{p,q}(V^*) \), \( \beta \in \Lambda^{r,s}(V^*) \), we have
\[
H(\alpha, \beta) = (\sqrt{-1})^{p-q}(1)^{-p+(p+q)(p+q-1)/2} (\alpha \wedge_w \beta)_0
\]
\[
= (\sqrt{-1})^{s-r}(1)^{s+(r+s)(r+s-1)/2} (\alpha \wedge_w \beta)_0.
\]

**Lemma 1.1.** For any three elements \( \alpha \in \Lambda^{p,q}(V^*) \), \( \beta \in \Lambda^{r,s}(V^*) \) and \( \gamma \in \Lambda^{u,v}(V^*) \), we have
\[
H(\alpha \wedge_w \beta, \gamma) = H(\alpha, \beta \wedge_w \gamma).
\]

**Proof.** From (3), we have
\[
\begin{align*}
H(\alpha \wedge_w \beta, \gamma) &= (\sqrt{-1})^{p-q}(1)^{-p+(p+q)(p+q-1)/2} (\alpha \wedge_w \beta)_0 \\
&= (\sqrt{-1})^{s-r}(1)^{s+(r+s)(r+s-1)/2} (\alpha \wedge_w \beta)_0.
\end{align*}
\]
It is nonzero only if \( p + s - u = q + t - v \geq 0 \). Similarly
\[
\begin{align*}
H(\alpha, \beta \wedge_w \gamma) &= (\sqrt{-1})^{p-q}(1)^{-p+(p+q)(p+q-1)/2} (\alpha \wedge_w \beta \wedge_w \gamma)_0 \\
&= (\sqrt{-1})^{1-r}(1)^{1+(1+s)(1+s-1)/2} (\alpha \wedge_w \beta \wedge_w \gamma)_0.
\end{align*}
\]
It is nonzero only if \( u + s - p = v + t - q \geq 0 \). By associativity of \( \wedge_w \), \( H(\alpha \wedge_w \beta, \gamma) \) and \( H(\alpha, \beta \wedge_w \gamma) \) differ only by a constant factor. So we need only to determine this factor when both are nonzero. In this case, we must have \( s = t \). hence \( u - p = v - q \). The factor then is
\[
(\sqrt{-1})^{(u-v)-(p-q)}(1)^{(u-p)+(u+v)(u+v-1)/2-(p+q)(p+q-1)/2}
\]
\[
= (1)^{(u-p)+(u+v+p+q)(u+v-p-q)/2-(u+v-p-q)/2}
\]
\[
= (1)^{(u-p)+(2(u-p)+2(p+q))(u-p)-(u-p)} = 1.
\]

\( \square \)

**1.6. Multiparameter deformation.** Let \( \bar{w} = (w_1, \ldots, w_m) \), \( \bar{h} = (h_1, \ldots, h_m) \), \( \Lambda_{\bar{w}}(V^*) = \Lambda(V^*)[\bar{h}] = \Lambda(V^*)[h_1, \ldots, h_m] \). Define \( \alpha \wedge_{\bar{h}} \beta = m(\exp(h_1 L_{w_1} + \cdots + h_m L_{w_m}) (\alpha \otimes \beta)) \)
\[
= \sum_{n_1, \ldots, n_m} \frac{h_1^{n_1} \cdots h_m^{n_m}}{n_1! \cdots n_m!} \sum_{[J_i] = [n_i]} \prod_{j} w_{\bar{I}}^j \cdot w_{\bar{J}}^j \cdot \alpha \wedge_{\bar{h}_i} e_{\bar{I}} \wedge \cdots \wedge e_{\bar{J}} \wedge \beta.
\]

**Theorem 1.2.** For any \( \bar{w} \) and \( \bar{h} \) as above, \( \alpha \wedge_{\bar{h}} \beta \) satisfies the following properties
\[
\begin{align*}
\alpha \wedge_{\bar{h}} \beta &= (-1)^{|\alpha||\beta|} \alpha \wedge_{\bar{h}} \beta \quad \text{(4)} \\
(\alpha \wedge_{\bar{h}} \beta) \wedge_{\bar{h}} \gamma &= \alpha \wedge_{\bar{h}} (\beta \wedge_{\bar{h}} \gamma) \quad \text{(5)}
\end{align*}
\]
for all \( \alpha, \beta, \gamma \in \Lambda_{\bar{w}}(V^*) \).
Proof. We regard (4) and (5) as polynomial equations in $h_1, \cdots, h_m$. For any values of $h_1, \cdots, h_m$ in $k$, set $w = h_1 w_1 + \cdots + h_m w_m$. Then $\Lambda^k w = \Lambda w$. By Theorem 1.1 and the remark following it, (4) and (5) hold for $\Lambda^k w$. Therefore, they hold as polynomial equations.

Part II. Geometric Applications

2. Quantum de Rham Complex

In this section, we define quantum exterior differential operator on Poisson manifolds. We follow the original route to its discovery, using a Poisson connection, which only exists on regular Poisson manifolds. It has the advantage of making the verification of the desirable properties conceptually simple. Then we will show that it is actually related to some well-known operators which can be defined on any Poisson manifolds. Properties of the quantum exterior differential are then re-established using also proved properties of these operators.

Let $M$ be a smooth manifold, with a fixed bi-vector field $w \in \Gamma(\Lambda^2(TM))$, then $w$ induces quantum exterior product on $\Omega(M)$ by fiberwise quantum exterior product. Suppose now there is a torsionless connection $\nabla$ which preserves the Poisson bi-vector. Such a connection is called regular Poisson manifold. Conversely, any regular Poisson manifold admits a torsionless connection which preserves the Poisson bi-vector. Such a connection is called a Poisson connection. Symplectic manifolds are examples of regular Poisson manifolds. See Vaisman [11] p. 11 and p. 29 for more details.

Let $\{e_i\}$ be a local frame of $TM$ near $x \in M$, and $\{e^i\}$ be the dual frame of $T^*M$. Define $d_h : \Omega(M) \to \Omega(M)[h]$ by

$$d_h \alpha = e^i \wedge_h \nabla_{e_i} \alpha,$$

for $\alpha \in \Omega(M)$. This definition clearly does not depend on the choice of the basis $\{e_i\}$. It follows then that we can use some particularly chosen frame to simplify the calculations. Near each $x \in M$, we will use the normal coordinates with respect to $\nabla$. As in Riemannian geometry, we consider the geodesics through $x$ with respect to $\nabla$, i.e. smooth curves $c : (-1, 1) \to M$, $c(0) = x$ and $\nabla c^{(t)} c'(t) = 0$. Given any basis of $T_x M$, one can use parallel transport along the geodesics starting from $x$ to construct a local frame $\{e_i\}$ near $x$. It then follows that $\nabla_{e_i} e_j = 0$ at $x$. Similarly for the dual frame $\{e^i\}$, if we still use $\nabla$ to denote the induced connection on $T^*M$, we have $\nabla_{e_i} e^j = 0$ at $x$. Since $\nabla$ is torsion free, we also have

$$[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = 0,$$

at $x$. Furthermore, let $w^{jk} = w(e^j, e^k)$, since $\nabla$ preserve $w$, we have

$$\nabla_{e_i} w^{jk} = w(\nabla_{e_i} e^j, e^k) + w(e^j, \nabla_{e_i} e^k),$$

so $\nabla_{e_i} w^{jk} = 0$ at $x$. Given the torsion-free connection on $TM$, let $R_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ be its curvature. It is well-known that Bianchi identity holds for torsionless connections on the tangent bundle, i.e., $R_{X,Y} Z + R_{Y,Z} X + R_{Z,X} Y = 0$ for any vector fields $X, Y, Z$ on $M$. Let $R_{(e_i,e_j)} e_k = R_{i,j,k}^l e_l$, then we have

$$R_{i,j,k}^l + R_{j,k,i}^l + R_{k,i,j}^l = 0,$$
for any $i, j, k$ and $l$. Denote by $\tilde{R}$ the curvature of the induced connection on $T^*M$, let $\tilde{R}_{(e_i,e_j)e^l} = \tilde{R}^l_{ij,k}e^k$. Then it is routine to see that $\tilde{R}^l_{ij,k} = \tilde{R}^l_{ji,k}$. Therefore, we have

$$\tilde{R}^l_{ij,k} + \tilde{R}^l_{jk,i} + \tilde{R}^l_{ki,j} = 0,$$

for any $i, j, k$ and $l$.

**Theorem 2.1.** For any $w \in \Gamma(\Lambda^2(TM))$ and a torsion-free connection $\nabla$ which preserves $w$, $d_h : \Omega(M) \to \Omega(M)[h]$ defined above can be extended to operators

$$d_h : \Omega(M)[h] \to \Omega(M)[h],$$

$$d_h : \Omega(M)[h,h^{-1}] \to \Omega(M)[h,h^{-1}]$$

as derivations, i.e.,

$$d_h(\alpha \wedge h \beta) = (d_h \alpha) \wedge h \beta + (-1)^{|\alpha|} \alpha \wedge h (d_h \beta),$$

for $\alpha, \beta$ both in $\Omega(M)[h]$ or both in $\Omega(M)[h,h^{-1}]$. Furthermore, $d_h^2 = 0$.

**Proof.** We only need to check at each $x \in M$, where we can use the normal coordinates as above. We write $\alpha = \alpha_1 e^i$, $\beta = \beta_1 e^j$. It is clear that $d_h e^j = 0$, $d_h e^j = 0$ at $x$. This implies that at $x$, $d_h \alpha = d\alpha_1 \wedge h e^j$, and $d_h \beta = d\beta_1 \wedge h e^j$.

Since $e^j \wedge h e^j$ is a sum of products of $e^i$’s with $\alpha_1$’s, $\nabla(e^j \wedge h e^j) = 0$, and therefore, $d_h (e^j \wedge h e^j) = 0$ at $x$. By associativity of the quantum multiplication, we have at $x$,

$$d_h(\alpha \wedge h \beta) = d_h(\alpha_1 \beta_1) \wedge h (e^j \wedge h e^j) = d\alpha_1 \wedge h (\beta_1 e^j) + \alpha_1 (d\beta_1 \wedge h e^j)$$

$$= (d\alpha_1 \wedge h (\beta_1 e^j)) + (-1)^{|\alpha_1|} \alpha_1 (d\beta_1 \wedge h e^j)$$

$$= (d_h \alpha) \wedge h (\beta_1 e^j) + (-1)^{|\alpha_1|} (d\beta_1 \wedge h e^j)$$

$$= (d_h \alpha) \wedge h (\beta_1 e^j) + (-1)^{|\alpha_1|} \alpha_1 \wedge h (d_h \beta).$$

This proves (6). Taking $d_h$ on both sides of (6), one sees that

$$d_h^2(\alpha \wedge h \beta) = (d_h^2 \alpha) \wedge h \beta + \alpha \wedge h d_h^2(\beta).$$

Hence to prove $d_h^2 = 0$, it suffices to verify it on $\Omega^0(M)$ and $\Omega^1(M)$. Let $f \in \Omega^0(M)$, then $d_h f = df = (e_j f) e^j$, and at $x$,

$$d_h^2 f = e^j \wedge h \nabla_{e_i}((e_j f) e^j)$$

$$= e^j \wedge h ((e_i e_j f) e^j + (e_j f) \nabla_{e_i} e^j) = e_i e_j f (e^j \wedge h + h w^{ij})$$

$$= \sum_{i<j} [e_i, e_j] f \cdot (e^i \wedge h e^j + h w^{ij}) = 0.$$

For 1-forms, without loss of generality, we can take $\alpha = e^j$. Then $d_h e^j = e^j \wedge h \nabla_{e_j} e^j$. We claim that at $x$,

$$\nabla_{e_i}(e^j \wedge h \nabla_{e_j} e^j) = e^i \wedge h \nabla_{e_i} \nabla_{e_j} e^j.$$

It follows from the claim that

$$d_h^2 e^j = e^i \wedge h \nabla_{e_i}(e^j \wedge h \nabla_{e_j} e^j)$$

$$= e^i \wedge h (e^j \wedge h \nabla_{e_i} \nabla_{e_j} e^j) = \tilde{R}_{ij,k} e^i \wedge h e^j \wedge h e^k$$

$$= 2 \sum_{i<j<k} (\tilde{R}_{ij,k} + \tilde{R}_{jk,i} + \tilde{R}_{ki,j}) e^i \wedge h e^j \wedge h e^k = 0.$$
Now we prove the claim. Let $\nabla_{e_j} e_l = \Gamma^l_{j,k} e^k$, then at $x$, we have

$$
\nabla_{e_i}(e^l \wedge \nabla_{e_j} e^l) = \nabla_{e_i}(e^l \wedge \nabla_{e_j} e^l + h w^{jk} \Gamma^l_{j,k})
$$

$$
= \nabla_{e_i} e^l \wedge \nabla_{e_j} e^l + e^l \wedge \nabla_{e_j} \nabla_{e_i} e^l + h (\nabla_{e_i} w^{jk}) \Gamma^l_{j,k} + h w^{jk} (\nabla_{e_i} \Gamma^l_{j,k})
$$

$$
= e^l \wedge \nabla_{e_i} e^l + h w^{jk} (\nabla_{e_i} \Gamma^l_{j,k})
$$

On the other hand, at $x$, $\nabla_{e_i} \nabla_{e_j} e^l = \nabla_{e_i} (\Gamma^l_{j,k} e^k) = (\nabla_{e_i} \Gamma^l_{j,k}) e^k + \Gamma^l_{j,k} \nabla_{e_i} e^k = (\nabla_{e_i} \Gamma^l_{j,k} e^k).$ And so

$$
e^l \wedge h \nabla_{e_i} \nabla_{e_j} e^l = e^l \wedge \nabla_{e_i} e^l + h w^{jk} (\nabla_{e_i} \Gamma^l_{j,k}).
$$

The claim is proved.

**Remark.** In the above proof, we use the supercommutativity of quantum exterior product in an essential way. This explains why we cannot define quantum de Rham cohomology using the Riemannian metric.

It is instructive to compare with the classical objects in Riemannian geometry. By Theorem II.5.12 and Lemma II.5.13 in Michelsohn-Lawson [21], when a Riemannian metric $g$ is used, and $h = 1$, then $d_h$ in this context is $d - d^*$, where $d$ is the exterior differential, and $d^*$ its formal adjoint, given by $d^* = - * d*$, where $*: \Lambda(T^*M) \to \Lambda(T^*M)$ is the Hodge star operator defined by the Riemannian metric. Furthermore, if $\nabla$ is the Levi-Civita connection for $g$,

$$
d \alpha = e^l \wedge \nabla_{e_l} \alpha,
$$

$$
d^* \alpha = - \sum_j e_j \top e_j \alpha,
$$

for local orthonormal frame $\{e_1, \cdots, e_n\}$. In Poisson geometry, Koszul [20] introduced a codifferential

$$
\delta : \Lambda^*(T^*M) \to \Lambda^{*-1}(T^*M)
$$

for any Poisson manifold with bi-vector field $w$,

$$
\delta \alpha = w \downarrow (d \alpha) - d(w \downarrow \alpha).
$$

He also proved that $\delta^2 = 0$, $d \delta + \delta d = 0$. (Koszul used letter $\Delta$ for $\delta$.) The complex $(\Omega^*(T^*M), \delta)$ is called the canonical complex of the Poisson manifold, its homology $PH_*(M)$ is called the Poisson homology. When $(M^{2n}, \omega)$ is a symplectic manifold, Brylinski [8] defined an operator $*_w : \Omega^k(M) \to \Omega^{2n-k} (M)$, an analogue of Hodge $*$-star operator. He identified $\delta$ on $\Omega^k(M)$ with $(-1)^{k+1} *_w d w$. A calculation similar to the Riemannian case (see Vaisman [41], Remark 1.16) yields that

$$
(\delta \alpha)_{i_2 \cdots i_k} = -w^{pq} \nabla_q \alpha_{p i_2 \cdots i_k}.
$$

Therefore, for symplectic manifolds, $d_h = d - h \delta$. Vaisman [41] showed that (7) holds for regular Poisson manifolds. In fact, let $\nabla$ be a torsionless connection, which is not required to preserve the Poisson bi-vector field. Then (4.23') in Vaisman [41] gives the following tensorial expression:

$$
(\delta \alpha)_{i_2 \cdots i_k} = w^{pq} \nabla_p \alpha_{q i_2 \cdots i_k} - \frac{1}{2} \sum_{s=2}^k (-1)^s \alpha_{u v i_2 \cdots i_{s-1} i_{s+1} \cdots i_k} \nabla_{i_s} w^{uv}.
$$

When $\nabla$ is a torsionless connection which preserves the Poisson bi-vector $w$, one recovers (7). Therefore, we have
Proposition 2.1. On a regular Poisson manifold \((M, \omega)\), for any Poisson connection \(\nabla\), \(d_h = d - h\delta\). Hence \(d_h\) is independent of the choice of \(\nabla\).

This important result suggests that we should have defined \(d_h = d - h\delta\) for any Poisson manifold, and proved the following stronger version of Theorem 2.3.

Theorem 2.2. For any Poisson manifold \((M, \omega)\), \(d_h = d - h\delta\) satisfies \(d_h^2 = 0\).

\[
(8) \quad d_h(\alpha \wedge_h \beta) = (d_h\alpha) \wedge_h \beta + (-1)^{\vert\alpha\vert} \alpha \wedge_h (d_h\beta),
\]

for \(\alpha, \beta\) both in \(\Omega(M)[h]\) or both in \(\Omega(M)[h, h^{-1}]\).

Proof. Koszul [20] proved that \(\delta^2 = 0\) and \(d\delta + \delta d = 0\), it then follows that \(d_h^2 = 0\). We say that \(D(a, b)\) holds if (8) holds for all \(\alpha, \beta\) with \(\vert\alpha\vert = a, \vert\beta\vert = b\). Our strategy is first prove \(D(1, b)\), then use induction to prove \(D(a, b)\). Recall that \(\delta\alpha = w \wedge d\alpha - d(w \wedge \alpha)\).

Proof of \(D(1, b)\). Let \(\{e^1, \cdots, e^n\}\) be a local frame of \(TM, \beta \in \Omega_h^b(M)\), then we have

\[
d_h(e^i \wedge \beta) = d_h(e^i \wedge \beta + hw^{ij} e_j \wedge \beta) = d_h(e^i \wedge \beta) + hd_h(w^{ij} e_j \wedge \beta).
\]

On the other hand,

\[
d_h(e^i \wedge \beta) = d(e^i \wedge \beta) - h(w \wedge d - dw \wedge)(e^i \wedge \beta)
\]

\[
= de^i \wedge \beta - e^i \wedge d\beta - hw \wedge (de^i \wedge \beta - e^i \wedge d\beta) + hd[e^i \wedge (w \wedge \beta) - w^{ij} e_j \wedge \beta]
\]

\[
= de^i \wedge \beta - e^i \wedge d\beta
\]

\[
- [de^i \wedge (w \wedge \beta) + de^i \wedge (w \wedge \beta) + w^{kl}(e_k \wedge de^i) \wedge (e_l \wedge \beta)]
\]

\[
+ [e^i \wedge (w \wedge d\beta) - w^{ij} e_j \wedge d\beta]
\]

\[
+ [de^i \wedge (w \wedge \beta) - e^i \wedge d(w \wedge \beta)]
\]

\[
- hd[w^{ij}(e_j \wedge \beta)]
\]

\[
= (de^i \wedge \beta - h(e^i \wedge \beta) - hw^{ij}(e_k \wedge de^i) \wedge (e_l \wedge \beta)]
\]

\[
- [e^i \wedge d\beta + hw^{ij} e_j \wedge d\beta - he^i \wedge (w \wedge d\beta) + he^i \wedge d(w \wedge \beta)]
\]

\[
- hd[w^{ij}(e_j \wedge \beta)]
\]

\[
= (de^i \wedge \beta - hde^i \wedge \beta) - (e^i \wedge d\beta - he^i \wedge \delta\beta) - hd[w^{ij}(e_j \wedge \beta)]
\]

\[
= d_h e^i \wedge_h \beta - e^i \wedge_h d_h\beta - h^2[w^{ij}(e_j \wedge \delta\beta)] - hd[w^{ij}(e_j \wedge \beta)]
\]

\[
= d_h e^i \wedge_h \beta - e^i \wedge_h d_h\beta - hd_h[w^{ij}(e_j \wedge \delta\beta)] - h^2[\delta[w^{ij}(e_j \wedge \beta)] - h^2[w^{ij}(e_j \wedge \delta\beta)].
\]

Therefore,

\[
d_h(e^i \wedge_h \beta) = d_h e^i \wedge_h \beta - e^i \wedge_h d_h\beta + h^2[\delta(w^{ij} e_j \wedge \beta) + w^{ij}(e_j \wedge \delta\beta)].
\]
Now,
\[
\begin{align*}
\delta(w^j e_j \vdash \beta) &= w \vdash d(w^j e_j \vdash \beta) - d[w \vdash (w^j e_j \vdash \beta)] \\
&= w \vdash [dw^j \land (e_j \vdash \beta) + w^j d(e_j \vdash \beta)] - d[w^j \vdash (e_j \vdash \beta)] \\
&= [dw^j \land (e_j \vdash \beta) - w^{kl}(e_k \vdash dw^j)(e_l \vdash e_j \vdash \beta)] \\
&\quad + w^j w \vdash d(e_j \vdash \beta) - [dw^j \land (e_j \vdash \beta) + w^j d(e_j \vdash \beta)] \\
&= -w^{kl}(e_k \vdash dw^j)(e_l \vdash e_j \vdash \beta) \\
&\quad + w^j w \vdash d(e_j \vdash \beta) - w^j d(e_j \vdash \beta) \\
&= -w^{kl}(e_k \vdash dw^j)(e_l \vdash e_j \vdash \beta) \\
&\quad + w^j w \vdash (L_{e_j} \beta - e_j \vdash d\beta) - w^j [L_{e_j} (w \vdash \beta) - e_j \vdash d(w \vdash \beta)] \\
&= -w^{kl}(e_k \vdash dw^j)(e_l \vdash e_j \vdash \beta) \\
&\quad + w^j w \vdash L_{e_j} \beta - w^j [L_{e_j} (w \vdash \beta) - w^j e_j \vdash [w \vdash d\beta - d(w \vdash \beta)] \\
&= -w^{kl}(e_k \vdash dw^j)(e_l \vdash e_j \vdash \beta) \\
&\quad + w^j w \vdash L_{e_j} \beta - w^j [L_{e_j} (w \vdash \beta) - w^j e_j \vdash d\beta].
\end{align*}
\]

It is well-known that for \( \alpha \in \Lambda^k(M) \), and smooth vector fields \( X, Y, V_1, \ldots, V_k \),
\[
(L_X \alpha)(V_1, \ldots, V_k) = X \alpha(V_1, \ldots, V_k) - \sum_{j=1}^{k} \alpha(V_1, \ldots, [X, V_j], \ldots, V_k).
\]

Therefore,
\[
\begin{align*}
L_X(Y \vdash \alpha)(V_1, \ldots, V_{k-1}) &= X((Y \vdash \alpha)(V_1, \ldots, V_{k-1})) - \sum_{j=1}^{k-1} (Y \vdash \alpha)(V_1, \ldots, [X, V_j], \ldots, V_{k-1}) \\
&= X \alpha(Y, V_1, \ldots, V_{k-1}) - \sum_{j=1}^{k-1} \alpha(Y, V_1, \ldots, [X, V_j], \ldots, V_{k-1}) \\
&= (L_X \alpha)(Y, V_1, \ldots, V_{k-1}) + \alpha([X, Y], V_1, \ldots, V_{k-1}) \\
&= (Y \vdash L_X \alpha + [X, Y] \vdash \alpha)(V_1, \ldots, V_{k-1}).
\end{align*}
\]

I.e., \( L_X(Y \vdash \alpha) = Y \vdash L_X \alpha + [X, Y] \vdash \alpha \). Since we can assume that \([e_j, e_k] = 0\), we have
\[
L_{e_j}(w \vdash \beta) = L_{e_j} \sum_{k<l} w^{kl} e_k \vdash e_l \vdash \beta = \sum_{k<l} e_j w^{kl} e_k \vdash e_l \vdash \beta + \sum_{k<l} e_j w^{kl} e_k \vdash e_l \vdash L_{e_j} \beta.
\]
So we get

\[ \delta(w^j e_j + \beta) + w^j e_j = -w^{jk}(e_k + d w^j)(e_l + e_j + \beta) + w^j w^l L_{e_j \beta} - w^j L_{e_j}(w + \beta) \]

\[ = -\sum_{k,l} w^{kl} e_k w^j(e_l + e_j + \beta) - \sum_j \sum_{k<\ell} w^j e_j w^{kl}(e_k + e_l + \beta) \]

\[ = -\sum_{k} \sum_{l<j} w^{kl} e_k w^j(e_l + e_j + \beta) - \sum_k \sum_{l>j} w^{kl} e_k w^j(e_l + e_j + \beta) \]

\[ - \sum_j \sum_{k<\ell} w^j e_j w^{kl}(e_k + e_l + \beta). \]

For the first summation, change the indices by \( k \mapsto j, j \mapsto l, l \mapsto k \); for the second summation, change the indices by \( k \mapsto j, j \mapsto k \). Then we get

\[ \delta(w^j e_j + \beta) + w^j e_j = -\sum_j w^{jk} e_j w^l(e_k + e_l + \beta) - \sum_j \sum_{l<k} w^j e_j w^{kl}(e_k + e_l + \beta) \]

\[ = -\sum_{j} \sum_{k<l} (w^{kl} e_k w^j + w^j e_j w^{lk})(e_k + e_l + \beta) = 0. \]

The last equality holds because

\[ w^{kl} e_j w^l + w^j e_j w^{lk} + w^j e_j w^{kl} = 0, \]

which is equivalent to \( w \) be a Poisson bi-vector field (Vaisman [41], (1.5)).

**Proof of \( D(a, b) \)** This is in the same spirit of the proof of \( A(a, b, c) \) in Theorem \ref{thm:main_theorem}.

Assume that \( D(\leq a, b) \) has been proved. Any \( \alpha \in \Omega_h^{[a+1]}(M) \) can be locally written as

\[ \alpha = e^i \wedge \alpha_i \]

for some local frame \( \{e^1, \ldots, e^n\} \) and some \( \alpha_i \in \Omega_h^{[a]}(M) \). Now for each \( i, e^i \wedge \alpha_i = e^i \wedge_h \alpha_i + h f(e^i, \alpha_i) \), for some \( f(e^i, \alpha_i) \in \Omega_h^{[a-1]}(M) \). Then we have

\[ dh(\alpha \wedge_h \beta) \]

\[ = dh[(e^i \wedge_h \alpha_i + h f(e^i, \alpha_i)) \wedge_h \beta] \]

\[ = dh[e^i \wedge_h (\alpha_i \wedge_h \beta) + h f(e^i, \alpha_i) \wedge_h \beta] \]

\[ = dh e^i \wedge_h (\alpha_i \wedge_h \beta) - e^i \wedge_h dh(\alpha_i \wedge_h \beta) \]

by \( D(1, a + b - 1) \)

\[ + h dh f(e^i, \alpha_i) \wedge_h \beta + h(-1)^{|\alpha|-2} f(e^i, \alpha_i) \wedge_h dh \beta \]

by \( D(a - 1, b) \)

\[ = (dh e^i \wedge_h \alpha_i) \wedge_h \beta - e^i \wedge_h (dh \alpha_i \wedge_h \beta + (-1)^{|\alpha|-1} \alpha_i \wedge_h dh \beta) \]

\[ + h dh f(e^i, \alpha_i) \wedge_h \beta + h(-1)^{|\alpha|} f(e^i, \alpha_i) \wedge_h dh \beta \]

\[ = (dh e^i \wedge_h \alpha_i - e^i \wedge_h dh \alpha_i + h dh f(e^i, \alpha_i)) \wedge_h \beta \]

\[ + (-1)^{|\alpha|} (e^i \wedge_h \alpha_i + h f(e^i, \alpha_i)) \wedge_h dh \beta \]

\[ = dh(e^i \wedge \alpha_i) \wedge_h \beta + (-1)^{|\alpha|}(e^i \wedge \alpha_i) \wedge_h dh \beta \]

\[ = dh \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge dh \beta. \]
This completes the proof of Theorem 2.2.

3. Quantum de Rham Cohomology

Definition. For any Poisson manifold \((M, \omega)\), the (polynomial) quantum de Rham cohomology is defined by

\[ Q_h H^*_dR(M) = \ker d_h / \operatorname{Im} d_h, \]

for \(d_h : \Omega(M)[h] \to \Omega(M)[h] \). The Laurent quantum de Rham cohomology is

\[ Q_{h, h^{-1}} H^*_dR(M) = \ker d_h / \operatorname{Im} d_h, \]

for \(d_h : \Omega(M)[h, h^{-1}] \to \Omega(M)[h, h^{-1}] \).

As a consequence of Theorem 2.1 and Theorem 2.2 we have

Theorem 3.1. The quantum de Rham cohomology \( Q_h H^*_dR(M) \) of a Poisson manifold has the following properties:

\[ \alpha \wedge_h \beta = (-1)^{||\alpha|| \beta} \beta \wedge_h \alpha, \]

\[ (\alpha \wedge_h \beta) \wedge_h \gamma = \alpha \wedge_h (\beta \wedge_h \gamma), \]

for \(\alpha, \beta, \gamma \in Q_h H^*_dR(M)\). Similar results hold for Laurent quantum de Rham cohomology.

The goal of this section is to provide a method to compute the quantum de Rham cohomology, and to establish its relationship with the ordinary de Rham cohomology. The primary tool is the spectral sequences associated with any double complex. This approach is motivated by Brylinski’s results [9].

The complex \((\Omega(M)[h], d_h)\) can be regarded as a double complex \((C^{p,q}, -h\delta, d)\), where \(C^{p,q} = h^p\Omega^{q-p}(M), p \geq 0\). This is the analogue of Brylinski’s double complex \(C_\ast(M)\) ([8], §1.3). By the standard theory for double complex (Bott-Tu [6], §14), there are two spectral sequences \(E^r, E^s\) abutting to \(H^\ast(\Omega[h], d_h) = Q_h H^*_dR(M)\), with \(E^1_{p,q} = h^p H^q(C^{p,\ast}, d) = h^p H^q_dR(M), (E^1)^{p,q} = h^p H^*(C^{\ast,q}, \delta) = h^p PH_{q-p}(M), p \geq 0\).

Theorem 3.2. For a Poisson manifold with odd Betti numbers all vanishing, the spectral sequence \(E\) degenerate at \(E_1\), i.e. \(d_r = 0\) for all \(r \geq 0\), hence \(Q_h H^*_dR(M)\) is a deformation quantization of \(H^*_dR(M)\).

Proof. This is clear since \(E^1_{p,q} = h^p H^q_dR(M)\) is nontrivial only if \(q - p\) is even. Now for \(r \geq 1\), \(d_r\) maps to \(E^p_{r,q} \coloneq E^{p+r-r+1,q} = \oplus_{p,q} h^p H^q_{dR}(M) = H^*_dR(M) \otimes \mathbb{R}[h].\)

Similarly, \((\Omega(M)[h, h^{-1}], -d - h\delta)\) can be regarded as a double complex \((\tilde{C}^{p,q}, -h\delta, d)\), where \(\tilde{C}^{p,q} = h^p\Omega^{q-p}(M), p, q \in \mathbb{Z}\). This is essentially Brylinski’s double complex \(\tilde{C}^{p,\ast}\), but with a different bi-grading. We get two spectral sequences \(\tilde{E}\) and \(\tilde{E}^\prime\) abutting to \(Q_{h,h^{-1}} H^*_dR(M)\), with \(\tilde{E}^1_{p,q} = h^p H^q_{dR}(M), (\tilde{E}^1)^{p,q} = h^p H^*(C^{\ast,q}, \delta) = h^p PH_{q-p}(M), p, q \in \mathbb{Z}\). The same proof yields

Theorem 3.3. For a Poisson manifold with odd Betti numbers all vanishing, the spectral sequence \(\tilde{E}\) degenerates at \(\tilde{E}_1\), i.e. \(d_r = 0\) for all \(r \geq 0\), hence \(Q_{h,h^{-1}} H^*_dR(M)\) is a Laurent deformation quantization of \(H^*_dR(M)\).
Brylinski proved that on closed Kähler manifold \((M, \omega)\), every de Rham cohomology class has a representative \(\alpha\) such that \(d\alpha = 0, \delta\alpha = 0\). This implies that

**Theorem 1.** For a closed Kähler manifold \(M\), the spectral sequence \(E\) degenerates at \(E_1\), i.e. \(d_r = 0\) for all \(r \geq 0\), hence \(Q_{h,H^*_dR}(M)\) is a deformation quantization of \(H^*_dR(M)\).

We now assume that \((M^{2n}, \omega)\) is a compact symplectic manifold without boundary. Then Corollary 2.2.2 of Brylinski states that \(P(H(M) \cong H^{2n-i}_{dR}(M)\). This is one of the main ingredient in Brylinski’s proof of Theorem 2.3.1, which states that one of the spectral sequences for his double complex \(C^{p,q}_{\text{per}}\) degenerates at \(E_1\).

Therefore, we have

**Theorem 3.4.** For any compact symplectic manifold without boundary, the spectral sequences \(E\) and \(E'\) degenerate at \(E_1\) and \(E'_1\) respectively. Hence \(Q_{h,h^{-1}}H^*_dR(M)\) is a Laurent deformation quantization of \(H^*_dR(M)\).

**Proof.** The degeneracy of \(E'\) is Brylinski’s Theorem 2.3.1. The degeneracy of \(E\) is by the following dimension counting argument. Since \(\dim E^{p,q}_{\infty} \geq \dim E^{p,q}_{1}\) for all \(p, q \in \mathbb{Z}\), with equalities hold for all \(p, q\) if and only if \(E\) degenerates at \(E_1\), we have

\[
\dim Q_{h,h^{-1}}H^m(M) = \sum_{p+q=m} \dim E^{p,q}_{\infty} \\
\geq \sum_{p+q=m} \dim E^{p,q}_{1} = \sum_{p+q=m} \dim H^{q-p}_{dR}(M),
\]

for all \(m \in \mathbb{Z}\), with equalities hold for \(m\) if and only if \(E\) degenerates at \(E_1\). On the other hand, since \(E'\) degenerates at \(E'_1\), we have

\[
\dim Q_{h,h^{-1}}H^m(M) = \sum_{p+q=m} \dim (E'^{p,q}_{\infty}) \\
\geq \sum_{p+q=m} \dim (E'^{p,q}_1) = \sum_{p+q=m} \dim PH_{q-p}(M) \\
= \sum_{p+q=m} \dim H^{2n-(q-p)}_{dR}(M) = \sum_{p+q=m} \dim H^{q-p}_{dR}(M),
\]

for all \(m \in \mathbb{Z}\). \(\square\)

In fact, Brylinski’s proof can be used to give a straightforward proof of the degeneracy of \(E\). It works for any closed Poisson manifold, so we have

**Theorem 3.5.** For any closed symplectic manifold, the spectral sequences \(E\) degenerates at \(E_1\). Hence \(Q_{h,h^{-1}}H^*_dR(M)\) is a Laurent deformation quantization of \(H^*_dR(M)\).

**Remark.** Such results resemble similar results for the double complex and the associated spectral sequence appear in the Cartan model of equivariant cohomology. Such spectral sequences for equivariant cohomology appeared in Kalkman. They were independently discovered by the second author when he prepared for a presentation for a course on Chern-Weil theory by Prof. Lawson in 1992. For a compact symplectic manifold without boundary, Kirwan (Proposition 5.8) proved that the equivariant cohomology of a Hamiltonian action by a compact connected
Lie group $G$ is a free $H^*(BG)$-module generated by $H^*_{dR}(M)$. This result can be interpreted as saying the corresponding spectral sequence for equivariant cohomology degenerates at $E_1$ for any compact symplectic manifold without boundary. It is interesting to find a link between Kirwan’s result with Theorem 3.4 and Theorem 3.5.

Remark. If $M_1$, $M_2$ and $M_1 \times M_2$ all have the property that the (Laurent) quantum de Rham cohomology is isomorphic to the de Rham cohomology tensor with $\mathbb{R}[\hbar] (\mathbb{R}[\hbar, \hbar^{-1}])$, then from Künneth formula for de Rham cohomology, one can deduce that $QhH^*_{dR}(M_1 \times M_2) \cong QhH^*_{dR}(M_1) \otimes QhH^*_{dR}(M_2)$ as graded algebras. Similarly for the Laurent quantum de Rham cohomology. For Künneth formula for quantum cohomology via pseudo-holomorphic curves, cf. Kontsevich-Manin [19] and Tian [39]. It seems plausible to develop a Leray spectral sequence for symplectic fibration for (Laurent) quantum de Rham cohomology.

Brylinski [9] asked the question whether every de Rham class of a closed symplectic can be represented by a form $\alpha$ such that $d\alpha = 0$, $\delta \alpha = 0$. For closed Kähler manifolds, Brylinski [9] showed that $\delta$ is essentially $d^*$ up to the type of the form it acts on. Therefore, By Hodge theory, the answer to the above question for closed Kähler manifolds is yes. It has been answered negatively by Mathieu [31] and Yan [39] negatively for general symplectic manifolds. Nevertheless, Theorem 3.4 implies that on a closed Poisson manifold, any closed $\alpha \in \Omega^k(M)$ can be extended to a $d_h$-closed form $\alpha_h \in \Omega^{[k]}(M)$.

4. Quantum Hard Lefschetz Theorem

The symplectic adjoint of $d_h$ is $\delta = \delta - h^{-1}d = h^{-1}d_h$, hence every quantum de Rham class is represented by a quantum symplectic harmonic form (in the sense that $d_h\alpha = 0$, $\delta_h\alpha = 0$) for a trivial reason. This aspect of quantum Hodge theory has no analogue in the traditional approach to Hodge theory. An important result in Hodge theory on closed Kähler manifolds is the Hard Lefschetz theorem (Griffiths-Harris [15], p. 122) which states that for a closed Kähler manifold $(M, \omega)$ of complex dimension $n$, the map

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism for all $k \leq n$, where $L$ is the map given by wedge product with the Kähler form $\omega$. Furthermore, if one defines the primitive cohomology

$$P^{k-1}(M) = \text{Ker} L^{k+1} : H^{n-K}(M) \to H^{n+k+2}(M),$$

then one has

$$H^m(M) = \oplus_k L^k P^{m-2k}(M),$$
called the Lefschetz decomposition, which is compatible with the Hodge decomposition. This theorem is proved using results concerning finite dimensional representations of $sl(2, \mathbb{C})$, an idea attributed to Chern. This no longer holds for a general symplectic manifold. Mathieu [31] and Yan [39] proved the following theorem by different methods:

**Theorem 4.1.** let $(M^{2n}, \omega)$ be a symplectic manifold of dimension $2n$. Then the following two properties of $M$ are equivalent:

1. Any de Rham cohomology class of $M$ can be represented by a symplectic harmonic differential form.
2. For any $k \leq n$, the map $L^k : H^{n-k}(M) \to H^{n+k}(M)$ is surjective.
Lemma 4.1. Define operators

\[ \text{Lemma 4.2.} \]

\[ \Lambda^{[n+k]}(V^*) \] do not have the same dimension when \( k > 0 \), so we will work with \( \Lambda_{h,h-1}(V^*) \).

Let \( V \) be a 2n-dimensional vector space over a field \( k \) of characteristic zero. \( \omega \in \Lambda^2(V^*) \) a \( k \)-symplectic 2-form. Since \( \omega \) is anti-symmetric, care has to be taken in raising or lowering the indices. Our convention is as follows: \( \omega \) induces an isomorphism \( \sharp : V^* \to V \) by \( \omega(v, \alpha^2) = \alpha(v) \), for \( \alpha \in V^*, v \in V \). Its inverse is denoted by \( \flat : V \to V^* \). Let \( \{e_1, e_2, \ldots, e_{2n-1}, e_{2n}\} \) be a basis of \( V \), \( \omega_{i\ell} = \omega(e_i, e_\ell) \). Let \( (\omega_{kl}) \) be the inverse matrix of \( (\omega_{i\ell}) \), i.e. \( \omega^i_{\ell k} = \delta_i^j \), \( \omega^i_{\ell k} \omega^j_{\ell k} = \delta^j_k \). Then \( (e^\flat)^j = \omega^i_{\ell k} e_i \), \( e^\sharp_i = \omega^i_{\ell k} e^\flat_k \). The musical isomorphism \( \sharp \) induces an isomorphism \( \sharp : \Lambda^2(V^*) \to \Lambda^2(V) \) by

\[ (\phi_1 \wedge \phi_2)^\flat = \phi_1^\sharp \wedge \phi_2^\sharp, \]

for \( \phi_1, \phi_2 \in V^* \). Let \( w = \omega^\flat \in \Lambda^2(V) \). Then we have

\[ w = \frac{1}{2} w_{kl} (e^\flat)^j \wedge (e^\flat)^k = \frac{1}{2} w_{kl} w^{pk} w^{qk} e_p \wedge e_q = \frac{1}{2} \delta_p^q w^{pk} e_p \wedge e_q = \frac{1}{2} w_{pq} e_p \wedge e_q, \]

i.e., \( w_{pq} := w(e^\flat_p, e^\flat_q) = w^{pq} \). Let \( v_w = w^n/n! \). Brylinski \[ \] defined the symplectic star operator

\[ * : \Lambda^k(V^*) \to \Lambda^{2n-k}(V^*) \]

by \( \beta \wedge *\alpha = \lambda^k(w)(\beta, \alpha)v_w \), for all \( \alpha, \beta \in \Lambda^k(V^*) \). He also showed that \( *^2 = 0 \). We define operators \( L, L^*, K, K^* \) and \( A \) as follows:

\[ L(\alpha) = w \wedge \alpha, \quad L^* = -*L^*, \]

\[ K(\alpha) = e^j \wedge (e_j \vdash \alpha), \quad K^* = -*K^*. \]

Lemma 4.1. We have the following identities:

1. \( L^*\alpha = w \vdash \alpha \).
2. \( K(\alpha) = ka, \) if \( \alpha \in \Lambda^k(V^*) \), hence \( K^* = K - 2n, \) \( [K, K^*] = 0 \).
3. \( [L, K] = -2L, \) \( [L, K^*] = -2L^*, \) \( [L^*, K] = 2L^*, \) \( [L^*, K^*] = 2L^* \).
4. \( [L, L^*] = (k - n)\alpha, \) for \( \alpha \in \Lambda^k(V^*) \).

Proof. The first identity has been proved by Yan \[ \] The rest are trivial. \[ \]

Set \( A = -\frac{1}{2}(K + K^*) \), then we have \( A(\alpha) = (n - k)\alpha, \) for \( \alpha \in \Lambda^k(V^*) \). Furthermore,

\[ [L, L^*] = A, \quad [L, A] = 2L, \quad [L^*, A] = -2L^*. \]

This is Corollary 1.6 in Yan \[ \]. Now we define \( L_h : \Lambda_{h,h-1}(V) \to \Lambda_{h,h-1}(V) \) by \( L_h(\alpha) = w \wedge_h \alpha \). We extend \( * \) to \( \Lambda_{h,h-1}(V) \) by setting \( *h = h^{-1}, \) and \( *h^{-1} = h \). Define \( L_h^* = -*L_h, \) then we have

Lemma 4.2. We have

1. \( L_h = L + hK + h^2L^* \).
2. \( L_h^* = L^* + h^{-1}K^* + h^{-2}L \).
The following identities hold:

**Lemma 4.3.** The above discussions actually suggest the following construction. Let \( \text{Lie algebra } g \) such that \( g \) is a representation of the Lie algebra \( \mathfrak{sl}_2 \). It is straightforward to verify that for any constant \( t \), \( \mathfrak{sl}_2 \) is spanned by three vectors \( H, X, Y \) with \( [\cdot, \cdot] : \mathfrak{sl}_2 \to \mathfrak{sl}_2 \) such that

\[
[X, Y] = 0, \quad [X, H] = 2X, \quad [Y, H] = -2Y,
\]

\[
[M^+, M^-] = 0, \quad [X, M^\pm] = [Y, M^\pm] = 0, \quad [H, M^\pm] = \mp 2M^\pm.
\]

Then the linear map defined by \( X \mapsto L_h, Y \mapsto L^*_h, H \mapsto A_h \) is a representation of the Lie algebra \( g' \).

**Proof.** From Lemma 4.2 and \( K^* = K - 2n \), we see that

\[
L^*_h = h^{-2}L_h - 2nh^{-1}.
\]

Therefore \( [L_h, L^*_h] = 0 \). The other identities are trivial.

It is straightforward to verify that for any constant \( t \), the \( k \)-vector space \( g_t \) spanned by \( H, X, Y \) with \( [\cdot, \cdot] : \Lambda^2(g_t) \to g_t \) such that

\[
[X, Y]_t = tH, \quad [X, H]_t = 2X, \quad [Y, H]_t = -2Y,
\]

is a Lie algebra. Over the complex field, it is easy to see that for any \( t \neq 0 \), \( (g_t, [\cdot, \cdot]_t) \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \). When \( t = 0 \), it gives us the Lie algebra \( g \) in Lemma 4.3. The above discussions actually suggest the following construction. Let \( \phi : \mathfrak{sl}(2, k) \to \text{End}(W) \) be any representation of \( \mathfrak{sl}(2, k) \) (over \( k \)). As a \( k \)-vector space, \( \mathfrak{sl}(2, k) \) is spanned by three vectors \( H, X, Y \), such that

\[
[X, Y] = H, \quad [X, H] = 2X, \quad [Y, H] = -2Y.
\]
Define the following operators on \( W \otimes_k k[h, h^{-1}] \):

\[
L_h(\pm, p) = \phi(X) \pm h\phi(H) + h^2\phi(Y) + ph,
L_h^*(\pm, q) = \phi(Y) \pm h^{-1}\phi(H) + h^{-2}\phi(X) + qh,
\]

and \( A_h(r) \) is defined to be \( \phi(H) + r \) on \( W \), and \( A_h(h)(h) = 2h, A_h(r)(h^{-1}) = -2h \). Then the linear map given by \( X \mapsto L_h(\pm, p), Y \mapsto L_h^*(\pm, q), H \mapsto A_h(r) \) is a representation of \( g \). If we also send \( M^+ \) to \( h \), and \( M^- \) to \( h^{-1} \), then we get a representation of \( g' \). In particular, if \( W = \Lambda(V^*) \), \( L = \phi(X), L^* = \phi(Y) \), \( A = \phi(H) \), then

\[
L_h = L_h(-, n), \quad L_h^* = L_h^*(-, -n), \quad A_h = A_h(0).
\]

To get the analogue of Hard Lefschetz Theorem, we will not use the representation theory for \( g \) or \( g' \). Instead, there is a simpler algebra: let \( M_n = h^{-1}L_h, M_n^* = hL_h^* \), where \( 2n = \dim V \). Then \( M_n^* = M_n - 2n \). Furthermore, \( M, M^* \) and \( A_h \) commute with each other. Since multiplications by \( h \) and \( h^{-1} \) are isomorphisms which commutes with \( M_n \) and \( M_n^* \), it suffices to examine the representation of this commutative Lie algebra on \( \Omega^{[0]}_{h,h^{-1}}(V^*_n) \) and \( \Omega^{[1]}_{h,h^{-1}}(V^*_n) \). So we only need to find the eigenvalues of \( M_n \) on these spaces. We will need the following easy lemma in linear algebra:

**Lemma 4.4.** Let \( \{M_n\} \) be a sequence of square matrices with coefficient in \( k \) obtained in the following way:

\[
M_{n+1} = \begin{pmatrix} M_n & -I \\ I & M_n + 2I \end{pmatrix}
\]

for \( n \geq 1 \). where \( I \) is the identity matrix of the same size as \( M_n \).

(a) For any \( \lambda \in k \), and \( n \geq 1 \), we have

\[
det(M_{n+1} + \lambda I) = \det[M_n + (\lambda + 1)I]^2.
\]

Therefore, the eigenvalues of \( M_{n+1} \) can be obtained by adding 1 to that of \( M_n \), with the multiplicities doubled.

(b) For any \( \lambda \in k \), \( n \geq 0 \), \( \det(M_{n+1} + \lambda I) = [\det(M_1 + (\lambda + n)I)]^{2^n} \). Therefore, the eigenvalues of \( M_{n+1} \) can be obtained by adding \( n \) to that of \( M_1 \), with \( 2^n \) times the multiplicities. If particular, if \( \det(M_1 + nI) \neq 0 \) for \( n \geq 0 \), then \( \det M_{n+1} \neq 0 \).

**Proof.** (a) We use the standard trick of making one block of the matrix zero. Notice that if \( M_1 \) is a \( m \times m \) matrix, then the size of \( M_n \) is \( m2^{n-1} \times m2^{n-1} \).

\[
det(M_{n+1} + \lambda I) = det \begin{pmatrix} M_n + \lambda I & -I \\ I & M_n + (\lambda + 2)I \end{pmatrix}
= det \begin{pmatrix} (M_n + \lambda I) & -I \\ I & M_n + (\lambda + 2)I \end{pmatrix}
= det \begin{pmatrix} 0 & -I \\ (M_n + (\lambda + 1)I) & M_n + 2I \end{pmatrix}
= det[M_n + (\lambda + 1)I]^2.
\]

(b) An easy consequence of (a) by induction. \( \Box \)
Remark. It is clear that similar results hold for the sequence of matrices defined by

\[
M_{n+1} = \begin{pmatrix} M_n & I \\ -I & M_n - 2I \end{pmatrix}
\]

for \( n \geq 1 \). For such a sequence, we have \( \det(M_{n+1} + \lambda I) = \det(M_1 + (\lambda - n)I)^2 \), for any \( \lambda \in \mathbb{k}, n \geq 0 \).

Lemma 4.5. The eigenvalues of \( M_1 \) on \( \Lambda_{h,h-1}^0(V^*) \) are \( 1 \pm \frac{\sqrt{n}}{2} \), on \( \Lambda_{h,h-1}^1(V^*) \), there is only one eigenvalue \( 1 \) with multiplicity 2. For any \( n > 0 \), and any \( 2(n+1) \)-dimensional symplectic vector space \( V_{n+1} \), the eigenvalues of the operator \( M \) on both \( \Lambda_{h,h-1}^0(V^*) \) and \( \Lambda_{h,h-1}^1(V^*) \) are \( n - \frac{\sqrt{n}}{2} \), \( n \), and \( n + \frac{\sqrt{n}}{2} \).

Proof. We will express the operator \( M \) as a matrix in a suitable basis. Let \( \{ e^1, e^2, \ldots, e^{2n+1}, e^{2n+2} \} \) be a symplectonormal basis of \( V_{n+1} \), let \( V_n \) be the span of the first \( 2n \) base vectors. Then \( \{ h^{-k}e^{i_1} \wedge \cdots \wedge e^{i_{2k}} : k \geq 0, i_1 < \cdots < i_{2k} \} \) is a basis of \( \Lambda_{h,h-1}^0(V_n^*) \), and \( \{ h^{-k}e^{i_1} \wedge \cdots \wedge e^{i_{2k+1}} : k \geq 0, i_1 < \cdots < i_{2k+1} \} \) is a basis of \( \Lambda_{h,h-1}^1(V_n^*) \). Let \( M^0_n \) and \( M^1_n \) be the matrices of \( M \) for \( V_n \) in these two bases. Now for \( V_{n+1} \),

\[
\begin{align*}
&h^{-k}e^{i_1} \wedge \cdots \wedge e^{i_{2k}}, \\
h^{-(k+1)}e^{2n+1} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2k+1}}, \\
h^{-k}e^{2n+1} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2k}}, \\
h^{-(k+1)}e^{2n+2} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2k+1}}, \\
k \geq 0, i_1 < \cdots < i_{2k}, \text{ form a basis for } \Lambda_{h,h-1}^0(V_n^*).
\end{align*}
\]

Similarly,

\[
\begin{align*}
h^{-k}e^{i_1} \wedge \cdots \wedge e^{i_{2k+1}}, \\
h^{-(k+1)}e^{2n+1} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2k+1}}, \\
h^{-k}e^{2n+2} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2k}}, \\
h^{-(k+1)}e^{2n+2} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2k+1}}, \\
k \geq 0, i_1 < \cdots < i_{2k+1}, \text{ form a basis for } \Lambda_{h,h-1}^1(V_n^*).
\end{align*}
\]

Let \( M^0_{n+1} \) and \( M^1_{n+1} \) be the matrices of \( M \) in these bases. It is straightforward to verify that

\[
M^0_{n+1} = \begin{pmatrix} M^0_n & -I & 0 & 0 \\ I & M^0_n + 2I & 0 & 0 \\ 0 & 0 & M^1_n + I & 0 \\ 0 & 0 & 0 & M^1_n + I \end{pmatrix},
\]

\[
M^1_{n+1} = \begin{pmatrix} M^1_n & -I & 0 & 0 \\ I & M^1_n + 2I & 0 & 0 \\ 0 & 0 & M^0_n + I & 0 \\ 0 & 0 & 0 & M^0_n + I \end{pmatrix}.
\]

In fact, for any \( \alpha \in \Lambda_{h,h-1}(V_n^*) \), we have

\[
M_{n+1}(\alpha) = M_{n}(\alpha) + \alpha,
M_{n+1}(h^{-1}e^{2n+1} \wedge e^{2n+2} \wedge \alpha) = -\alpha + h^{-1}e^{2n+1} \wedge e^{2n+2} \wedge M_{n}(\alpha) + 2h^{-1}e^{2n+1} \wedge e^{2n+2} \alpha,
M_{n+1}(e^{2n+1} \wedge \alpha) = M_{n}(e^{2n+1} \wedge \alpha) + e^{2n+1} \wedge \alpha,
M_{n+1}(e^{2n+2} \wedge \alpha) = M_{n}(e^{2n+2} \wedge \alpha) + e^{2n+2} \wedge \alpha.
\]

Furthermore, we have

\[
M^0_1 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad M^1_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
We then inductively work out the eigenvalues of $M^0_{n+1}$ and $M^1_{n+1}$ with the help of Lemma 4.4. The eigenvalues of $M^0_1$ are $1 \pm \frac{2}{\sqrt{5}}$, $M^1_1$ has eigenvalue 1 with multiplicity 2. $M^0_2$ and $M^2_2$ both have eigenvalues $2 \pm \frac{2}{\sqrt{5}}$ and 2. For $n > 2$, we obtain the eigenvalues of $M_{n+1}$ by adding 1 to that of $M_n$.

As a consequence, we have the following algebraic version of Quantum Hard Lefschetz Theorem

**Theorem 4.2.** For a symplectic vector space $V$, the operators $L_h$ and $L_h^*$ are isomorphisms. Furthermore, $\Lambda_{h,h^{-1}}(V^*)$ decomposes into one dimensional eigen spaces of $h^{-1}L_h$ (or $hL_h^*$) with nonzero eigenvalues.

**Proof.** Recall that $L_h = hM_n$, $L_h^* = h^{-1}M_n^*$, and $M_n^* = M_n - 2n$.

**Remark.** By the same method, it is easy to find the values of $p$ and $q$ such that the operators $L_h(\pm p)$ and $L_h^*(\pm q)$ defined in (10) for $W = \Lambda(V^*)$ are isomorphisms.

On $\Lambda_h(V^*)$, we do not have such rich structures. It is easy to see that $\Lambda^{[n-k]}(V^*)$ and $\Lambda^{[n+k]}(V^*)$ do not have the same dimension when $k > 0$.

Now let $(M^{2n}, \omega)$ be a $2n$-dimensional symplectic manifold. Then $L_h, L_h^*, A_h$ can be defined on $\Omega_{h,h^{-1}}(M)$ by fiberwise actions.

**Lemma 4.6.** On a symplectic manifold $(M, \omega)$, we have

$$[L_h, d_h] = 0, \quad [L_h^*, d_h] = 0, \quad [A_h, d_h] = -d_h.$$

**Proof.** Since $\delta w = w \vdash dw - d(w \vdash w) = 0$, we have $d_h w = (d - h\delta)(w) = 0$. Therefore, for any $\alpha \in \Omega_{h,h^{-1}}(M)$, we have

$$[L_h, d_h] \alpha = w \wedge_h d_h \alpha - d_h (w \wedge_h \alpha) = -d_h w \wedge_h \alpha = 0.$$

The second identity follows from the first and (9). The third identity is trivial.

**Theorem 4.3.** On a symplectic manifold $(M^{2n}, \omega)$, $Q_{h,h^{-1}}H_R^\ast(M)$ is a representation of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$.

**Proof.** If $\alpha \in \Omega_{h,h^{-1}}(M)$, such that $d_h \alpha = 0$, then by Lemma 4.6

$$d_h(L_h \alpha) = L_h(d_h \alpha) = 0,$$

$$d_h(L^*_h \alpha) = L^*_h(d_h \alpha) = 0,$$

$$d_h(A_h \alpha) = A_h(d_h \alpha) + d_h \alpha = 0.$$

I.e., the action of $\mathfrak{g}$ maps $d_h$-closed forms to $d_h$-closed forms. Similarly, for any $\beta \in \Omega_{h,h^{-1}}(M),$

$$L_h(d_h \beta) = d_h(L_h \beta) = 0,$$

$$L_h^*(d_h \beta) = d_h(L_h^* \beta) = 0,$$

$$A_h(d_h \beta) = d_h(A_h \beta - \beta) = 0.$$

I.e., the action of $\mathfrak{g}$ maps $d_h$-coboundaries to $d_h$-coboundaries. Therefore, the action of $\mathfrak{g}$ goes down to an action on the cohomology.

As a consequence of Theorem 4.2, we have
Theorem 4.4. (Quantum Hard Lefschetz Theorem) For any symplectic manifold \((M^{2n}, \omega)\), its Laurent quantum de Rham cohomology \(Q_{h^1, h^{-1}} H^*_{dR}(M)\) decomposes into one-dimensional eigenspaces of the operator \(\hbar^{-1} L_h\) (or \(\hbar L^*_{h}\)) with nonzero eigenvalues. In particular, \(L^*_h\) are isomorphisms.

5. Quantum Dolbeault Cohomology

Let \((M, \omega)\) be a Poisson manifold, which admits an almost complex structure \(J\) which preserves \(\omega\). Assume that there is a torsionless connection \(\nabla\) on \(TM\), such that \(\nabla \omega = 0, \nabla J = 0\), then \((M, \omega)\) is regular Poisson, and \(J\) is integrable. (As an example, consider a Kähler manifold with its Levi-Civita connection.) Complexify \(d_h : \Omega_h(M) \to \Omega_h(M)\), we get a decomposition
\[
\mathfrak{C} \Omega_h(M) = \oplus_{p,q} \Omega_{h}^{p,q}(M),
\]
and correspondingly \(d_h = \partial_h + \overline{\partial}_h\), where
\[
\partial_h \alpha = (e^1)^{0,0} \wedge \nabla \overline{e}^{0,0} \alpha, \\
\overline{\partial}_h \alpha = (e^1)^{0,1} \wedge \nabla \overline{e}^{0,1} \alpha,
\]
for any \(\alpha \in \mathfrak{C} \Omega_h(M)\). It is clear that
\[
\partial_h \Omega_{h}^{p,q}(M) \subset \Omega_{h}^{p+1,q}(M), \quad \overline{\partial}_h \Omega_{h}^{p,q}(M) \subset \Omega_{h}^{p,q+1}(M).
\]
Now \(0 = d^2_h = \partial^2_h + (\partial_h \overline{\partial}_h + \overline{\partial}_h \partial_h) + \overline{\partial}_h \partial_h\), since they have bi-degrees \((2, 0)\), \((1, 1)\) and \((0, 2)\) respectively, we have
\[
(11) \quad \partial^2_h = 0, \quad \partial_h \overline{\partial}_h + \overline{\partial}_h \partial_h = 0, \quad \overline{\partial}_h \partial_h = 0.
\]
Similar to \(\S 2\), the use of the connection is only an expedient way of definition. On a complex manifold \((M, J)\) with a Poisson structure \(w\), not necessarily regular, such that \(J\) preserves \(w\), define \(\delta^{-1,0} : \Omega^{p,q}(M) \to \Omega^{p-1,q}(M)\) and \(\delta^{0,-1} : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)\) by
\[
\delta^{0,-1} \alpha = w \wedge (\partial \alpha) - \partial (w \wedge \alpha), \\
\delta^{-1,0} \alpha = w \wedge (\overline{\partial} \alpha) - \overline{\partial} (w \wedge \alpha),
\]
for \(\alpha \in \Omega^{p,q}(M)\). It is easy to see that for regular Poisson manifolds, \(\partial_h = \partial - h \delta^{0,-1}\), and \(\overline{\partial}_h = \overline{\partial} - h \delta^{-1,0}\). So we will take these as the definitions for \(\partial_h\) and \(\overline{\partial}_h\) on general Poisson manifolds. It is clear that \((11)\) still holds. We call
\[
Q_h H^{p,q}(M) = H(\Omega_{h}^{p,q}(M), \overline{\partial}_h)
\]
the quantum Dolbeault cohomology. We can get the quantum version of the usual Frölicher spectral sequence as follows. From \((11)\), we get a double complex \((\Omega_{\hbar}^{p,q}(M), \partial_h, \overline{\partial}_h)\), whose associated complex is \((\mathfrak{C} \Omega_{\hbar}^{p,q}(M), d_h)\), one of the standard spectral sequences \(\mathfrak{C} Q_h H(M)\) has \(E_1^{p,q} = Q_h H^{p,q}(M)\). Now \(0 = \overline{\partial}_h^2 = \overline{\partial}^2 - h(\overline{\partial} \delta^{-1,0} + \delta^{-1,0} \overline{\partial}) + h^2 (\delta^{-1,0})^2\), so
\[
\overline{\partial}_h^2 = 0, \quad \overline{\partial}_h \delta^{-1,0} + \delta^{-1,0} \overline{\partial}_h = 0, \quad (\delta^{-1,0})^2 = 0.
\]
So we get a double complex \((C^{p,q} = h^n \Omega^{p-m,n}(M), \partial_h, -h \delta^{-1,0})\). It has two associated spectral sequences abutting to \(Q_h H^{p,q}(M)\). Taking cohomology in \(\overline{\partial}_h\) first, we get a spectral sequence with \(E_1^{m,n} = h^m H^{p-m,n}(M)\). Similar to Theorem 3.2 and Theorem 3.3, we get
Theorem 5.1. When $M$ is a compact Kähler manifold with vanishing odd Betti numbers, the spectral sequence of $(C^{m,n} = h^mΩ^{p-m,n}(M), δ, -hδ^{-1,0})$ with $E_1^{p,n} = H^p_{DR}(M)$ degenerate at $E_1$. Hence $Q_hH^{p,q}(M) = \oplus_{k=0}^\infty h^kH^{p-k,q-k}(M)$.

One can also define Laurent quantum Dolbeault cohomology $Q_{h,h^{-1}}H^{p,q}(M)$, and consider the corresponding spectral sequences.

Theorem 5.2. For a closed Kähler manifold $M$, we have

\[ \mathbb{C}Q_{h,h^{-1}}H^n_{DR}(M) = \oplus_{p+q=n} Q_{h,h^{-1}}H^{p,q}(M), \]
\[ \mathbb{C}Q_{h,h^{-1}}H^p\,(M) = \oplus_{k=0}^\infty h^kQ_{h,h^{-1}}H^{-k,q-k}(M). \]

Proof. By Theorem 3.4,

\[ Q_{h,h^{-1}}H^n_{DR}(M) = \oplus_{k=\mathbb{Z}} h^k\mathbb{C}H^n_{DR}(M). \]

By Hodge theorem, $\mathbb{C}H^{n-2k}(M) \cong \oplus_{p+q=n} H^{p-1,q-k}(M)$. So we have

\[ (12) \quad \dim Q_{h,h^{-1}}H^n_{DR}(M) = \sum_{k=\mathbb{Z}} \sum_{p+q=n} \dim H^{p-1,q-k}(M), \]

where all the dimensions are dimensions as complex vector spaces, It is a sum of finitely many finite numbers. Now there is a spectral sequence abutting to $Q_{h,h^{-1}}H^n_{DR}(M)$ with $E_1^{p,q} = Q_{h,h^{-1}}H^{p,q}(M)$. So we have

\[ (13) \quad \dim Q_{h,h^{-1}}H^n_{DR}(M) \leq \sum_{p+q=n} \dim Q_{h,h^{-1}}H^{p,q}(M), \]

equality holds iff the spectral sequence degenerates at $E_1$. Similarly, there is a spectral sequence abutting to $Q_{h,h^{-1}}H^p\,(M)$ with $E_1^{k,l} = h^kH^{p-k,l}(M)$. Therefore,

\[ (14) \quad \dim Q_{h,h^{-1}}H^{p,q}(M) \leq \sum_{k+l=q} \dim H^{p-k,l}(M) = \sum_{k=\mathbb{Z}} \dim H^{p-k,q-k}(M). \]

Equality holds iff the spectral sequence degenerates at $E_1$. Combining (13) with (14), one gets

\[ \dim Q_{h,h^{-1}}H^n_{DR}(M) \leq \sum_{k=\mathbb{Z}} \sum_{p+q=n} \dim H^{p-k,q-k}(M), \]

with equality iff both $E$ and $\tilde{E}$ degenerate at $E_1$. Comparing with (12), one sees that all the relevant spectral sequences degenerate at $E_1$. This completes the proof. \qed

6. Quantum integral and quantum Stokes Theorem

Let $(M, \omega)$ be a closed $2n$-dimensional symplectic manifold. Define an integral $\int_h : \Omega(M) \to \mathbb{R}[h]$ as follows. For any $\alpha \in \Omega^j(M)$, if $j$ is odd, set $\int_h \alpha = 0$; if $j = 2n - 2k$ for some integer $k$, set

\[ \int_h \alpha = \int_M \alpha \wedge \frac{\omega^k}{k!}. \]

Extend $\int_h$ to $\Omega(M)$ as a $\mathbb{R}[h]$-module map. We call $\int_h$ the quantum integral. Straightforward calculations yield the following
Lemma 6.1. For $\alpha, \beta \in \Omega(M)$, we have

$$w \vdash (\alpha \wedge \beta) = (w \vdash \alpha) \wedge (w \vdash \beta) + 2w^{ij}(e_i \vdash \alpha) \wedge (e_j \vdash \beta) + \alpha \wedge (w \vdash \beta).$$

Lemma 6.2. (i) We have

$$w \vdash \omega_k^{k+1} = (n + k)\frac{\omega^k}{k!}.$$  

(ii) For $\beta \in \Omega^p(M)$, we have

$$w^{ij}(e_i \vdash \beta) \wedge (e_j \vdash \omega_{k+1}^{k+1} \frac{(k+1)!}{(k+1)!}) = (-1)^{p-1}p\beta \wedge \omega_k^k.$$  

Theorem 6.1. (Quantum Stokes Theorem) We have $\int_h d\alpha = 0$, $\int_h h\delta\alpha = 0$, and therefore $\int_h d_h\alpha = 0$.

Proof. We can assume that $\alpha$ has odd degree, write $\alpha = \sum_{k=0}^n \frac{k^k}{k!}\alpha_{2n-1-2k}$, where $\deg(\alpha_{2n-1-2k}) = 2n - 2k - 1$. Then we have

$$\int_h d\alpha = \int_M \sum_{k=0}^n d\alpha_{2n-1-2k} \wedge \frac{\omega^k}{k!} = \int_M \sum_{k=0}^n d(\alpha_{2n-1-2k} \wedge \frac{\omega^k}{k!}) = 0.$$  

Recall that $\delta\alpha = w \vdash d\alpha - d(w \vdash \alpha)$, therefore,

$$\int_h h\delta\alpha = \int_h w \vdash d\alpha = \int_M \sum_{k=0}^n (w \vdash d\alpha_{2n-1-2k}) \wedge \frac{\omega^k}{(k+1)!}$$

$$\quad = \int_M \sum_{k=0}^n w \vdash (d\alpha_{2n-1-2k} \wedge \frac{\omega^k}{(k+1)!})$$

$$\quad - \int_M \sum_{k=0}^n d\alpha_{2n-1-2k} \wedge (w \vdash \frac{\omega^k}{(k+1)!})$$

$$\quad - 2\int_M w^{ij}(e_i \vdash d\alpha_{2n-1-2k}) \wedge (e_j \vdash \frac{\omega^k}{(k+1)!}).$$

The first term vanishes since $d\alpha_{2n-1-2k} \wedge \omega^k$ has degree $2n + 2 > \dim(M)$. By Lemma 6.2,

$$\int_h h\delta\alpha$$

$$= -\sum_{k=0}^n (n + k) \int_M d\alpha_{2n-1-2k} \wedge \frac{\omega^k}{k!} + 2\sum_{k=0}^n (2n - 2k) \int_M d\alpha_{2n-1-2k} \wedge \frac{\omega^k}{k!} = 0.$$  

$\square$
7. Quantum Chern-Weil Theory

The classical constructions in Chern-Weil theory of representing characteristic classes of a vector bundle over a smooth manifold by curvature expressions can be generalized in the context of quantum de Rham cohomology. As usual, the case of a complex line bundle is very simple. We will go over it first to illustrate the generalized in the context of quantum de Rham cohomology. As usual, the case of a vector bundle over a smooth manifold by curvature expressions can be classes of a vector bundle over a smoothly manifold by curvature expressions can be

\[
\text{On Quantum De Rham Cohomology}
\]

\[
\begin{align*}
\text{Lemma 7.1.} & \quad \text{The definition of } d_h \text{ is independent of the choice of the local frames.}
\end{align*}
\]
Proof. If $s'$ is another local frame such that $s' = s \cdot G$, $\alpha = s' \otimes \phi'$, and $\nabla s' = s' \otimes \theta'$. Then $
abla' = G^{-1} \nabla \theta G + G^{-1} dG$. Hence,

$$
\nabla' = s \otimes (\theta + G^{-1} dG) + d_h (s) (\theta). 
$$

Alternatively, let $\{e_i\}$ be a local frame of $TM$ near $x \in M$, and $\{e_i'\}$ be the dual frame of $T^*M$. Then

$$
d_{\nabla'^E} e_i = \theta \wedge e_i',
$$

where $\nabla'$ is the connection on $\Lambda(T^*M) \otimes E$ induced by the admissible connection $\nabla$ on $TM$ and $\nabla_E$ on $E$. From the definition and Theorem 2.1, it is routine to verify the following

**Lemma 7.2.** The quantum covariant derivative is a $\Omega^k_h(M)$-module derivation of degree 1, i.e.,

$$
d_{\nabla'^E} (\Phi \wedge h \alpha) = (d_{\nabla'^E} \Phi) \wedge h \alpha + (-1)^{\deg \Phi} \Phi \wedge h (d_h \alpha),
$$

where $\Phi \in \Omega^k_h(E)$, $\alpha \in \Omega^*_h(M)$.

Notice that $\Omega^*_h(E)$ is also a right $\Omega^*_h(\text{End}(E))$-module.

**Theorem/Definition 7.1.** There is an element $R^E_h \in \Omega^2_h(\text{End}(E))$, such that for each $k \geq 0$, $(d_{\nabla'^E}^2) \Phi$ on $\Omega^k_h(M)$ is given by $(d_{\nabla'^E}^2) \Phi = \Phi \wedge h R^E_h$, for any $\Phi \in \Omega^*_h(E)$. $R^E_h$ is called the quantum curvature of $\nabla^E$.

**Proof.** We use the local frame $s$ and local connection 1-form $\theta$ as above. Then by Theorem 2.1

$$
(d_{\nabla'^E}^2) \Phi = d_{\nabla'^E} (d_{\nabla'^E} \Phi) = s \otimes (\theta \wedge h \phi + d_h \phi)
$$

For a different local frame $s' = sG$ with $\nabla_E s' = s' \otimes \theta'$. A calculation as in the ordinary case shows that

$$
d_h \theta' + \theta' \wedge h \theta' = G^{-1} (d_h \theta + \theta \wedge h \theta) G.
$$

This shows that $\Theta^*_h := d_h \theta + \theta \wedge h \theta$ in different frames patches up to give us an element $R^E_h$ in $\Omega^2_h(\text{End}(E))$.

For $(n \times n)$-matrix valued differential forms $\alpha = (\alpha_{ij})$ and $\beta = (\beta_{ij})$, define

$$
[\alpha \wedge h \beta]_{ij} = \sum_k (\alpha_{ik} \wedge h \beta_{kj} - \beta_{ik} \wedge h \alpha_{kj}).
$$

In a local frame $s$, we have

$$
d_{\nabla'^E}^2 s = d_{\nabla'^E} (d_{\nabla'^E} s) = s \otimes (d_h \Theta^*_h + \theta \wedge h \Theta_h),
$$

$$(d_{\nabla'^E} s) \wedge h \Theta^*_h = (d_{\nabla'^E} s) \wedge h s (\Theta_h \wedge h \theta).
$$
Since \(d_h^{\nabla_E}(d_h^{\nabla_E})^2 = (d_h^{\nabla_E})^2d_h^{\nabla_E}\), we get
\begin{equation}
(15)
\quad d_h \Theta_h = [\Theta_h \wedge_h \theta].
\end{equation}

If \(p\) is a polynomial on the space of \(n \times n\)-matrices, such that \(p(G^{-1}AG) = p(A)\), for any invertible \(n \times n\)-matrix \(G\), then \(p(\Theta^a)\) for different frames patch up to a well-defined element \(p(R^E) \in \Omega^*(M)[h]\). Similar to the ordinary Chern-Weil theory, it is easy to see that \(d_h p(R^E) = 0\). So it defines a class in \(Q_h H_{dR}^*(M)\). The usual construction of transgression operator carries over to show that this class is independent of the choice of the connection \(\nabla^E\). In this way, one can define quantum Chern classes, quantum Euler class etc. We will call them quantum characteristic classes. It is clear that we can repeat the same story in Laurent case. Notice that in Atiyah-Singer index theorem, the index of an elliptic operator on a closed manifold is expressed as the integral of a power series of the curvature. If we use quantum curvature and quantum exterior product in the power series, we then get a power series in \(h\), whose 0-th order term yields the ordinary index.

7.1. Quantum equivariant de Rham cohomology. Let \((M, \omega)\) be a Poisson manifold, which admits an action by a compact connected Lie group \(G\), such that the \(G\)-action preserves the Poisson bi-vector field \(\omega\). Let \(g\) be the Lie algebra of \(G\), \(\{\xi_a\}\) a basis of \(g\), denote by \(\iota_a\) the contraction by the vector field generated by the one parameter group corresponding to \(\xi_a\), and \(L_a\) the Lie derivative by the same vector field. Imitating the Cartan model for equivariant cohomology, we consider the operator \(D_{hG} = d_h + \Theta^a \iota_a = d - h\delta + \Theta^a \iota_a\) acting on \((S(g^*) \otimes \Omega(M))^G[h]\). It is well-known that \(d + \Theta^a \iota_a\) maps \((S(g^*) \otimes \Omega(M))^G\) to itself. Since the \(G\)-action preserves \(w\), it is easy to check that \(\delta\) also preserves \((S(g^*) \otimes \Omega(M))^G\). Therefore, \(D_{hG}\) is an operator from \((S(g^*) \otimes \Omega(M))^G[h]\) to itself. Now on \((S(g^*) \otimes \Omega(M))^G[h]\), we have
\begin{align*}
D_{hG}^2 &= d_h^2 + (\Theta^a \iota_a)^2 + \Theta^a (d \iota_a + \iota_a d) - h\Theta^a(\delta \iota_a + \iota_a \delta) \\
&= -h\Theta^a(\delta \iota_a + \iota_a \delta).
\end{align*}

Since \(\delta = \iota_a \omega - d \iota_a \omega\), we have
\begin{align*}
\delta \iota_a + \iota_a \delta &= \iota_a \omega d \iota_a - d \iota_a \omega + \iota_a \omega d - \iota_a \omega d \\
&= \iota_a \omega dt_a - d \iota_a \omega t_a + \iota_a \omega d - \iota_a \omega d \\
&= \iota_a \omega \Lambda - \Lambda \omega t_a = -\iota_a \omega t_a = 0.
\end{align*}

Hence, \(D_{hG}^2 = 0\). We call the cohomology
\[Q_h H^*_G(M) := H^*((S(g^*) \otimes \Omega(M))^G[h], D_{hG})\]
the quantum equivariant de Rham cohomology. Similar definitions can be made using Laurent deformation. We will study quantum equivariant de Rham cohomology in a forthcoming paper.

8. Computations for some examples

The quantum Chern-Weil theory and Theorem [3.4] provide us with tools to compute the quantum de Rham cohomology rings of some important examples of symplectic manifolds such as projective spaces, complex Grassmannians and flag manifolds.

Example. (Complex projective space) For any symplectic form on \(\mathbb{C}P^n\), \(H^*_{dR}(\mathbb{C}P^n)\) is the ring \(\mathbb{R}[\omega]/(\omega^{n+1} = 0)\). By Theorem [3.4], \(Q_h H^*_{dR}(\mathbb{C}P^n) = H^*_{dR}(\mathbb{C}P^n) \otimes \mathbb{R}[h]\).
So we need to compute \( \omega^k \wedge_h \omega^l \). It is clear from the definition that it is a linear combination of \( \omega^{k+1}, \omega^2, \ldots, \omega^{k-l} \), with coefficients polynomials of \( \hbar \). This can be done inductively as follows: we first compute \( \omega \wedge \omega^k \), then by induction compute \( (\omega)^k \wedge \omega \wedge_h \omega^k \) (\( k \) times), then for \( k \geq 2 \), reduce the computation of \( \omega^k \wedge_h \omega^l \) to the first computations. In fact, by Darboux Theorem, locally we write \( \omega = e^1 \wedge e^2 + \cdots + e^{2n-1} \wedge e^{2n} \). By results in \([4]\), we have
\[
(\omega)_{h}^{k+1} = \omega \wedge_h (\omega)^k = a_{k}^{(k)}(h) \omega^{k+1} + \cdots + a_{0}^{(k)}(h) \omega^k.
\]

Without loss of generality, we can assume that \( \omega \) is the trivial rank \( n \). Then, it is possible to check that
\[
(\omega)^k \wedge \omega = n + 1 \left( \begin{array}{c} n + 1 \\ k \end{array} \right) \omega^k.
\]

This can be written as a linear combination of \( \omega^1, \ldots, \omega^{k+1} \) by induction hypothesis and \([4]\). The result of such recursive procedures is very complicated. But it gives us the full information about how the ring structure is deformed.

On the other hand, there is simple ways to give presentations of the deformation quantization. Let \( \omega_h = e^1 \wedge e^2 + \cdots + e^{2n-1} \wedge e^{2n} = \omega - \hbar \omega e^{2n} \). Since \( \wedge_h \) is super-commutative, if we expand \( (\omega_h)^{n+1}_h \) as a sum of terms of the form \( (e^{2n_1} \wedge_h e^{2n_2}) \wedge_h \cdots \wedge_h (e^{2n_{n+1}} \wedge_h e^{2n_{n+1}}) \), it is clear that
\[
(\omega)^{n+1} = 0.
\]

This does not imply that we get a trivial deformation, since \( (\cdot)^{n+1} \) is given by the deformed multiplication. Equivalently,
\[
(\omega)^{n+1} = \sum_{k=0}^{n} (-1)^{n-k} h^{n+1-k} \left( \begin{array}{c} n + 1 \\ k \end{array} \right) (\omega)_h^k.
\]

Let \( \nu \) be the tautological line bundle over \( \mathbb{C}P_n \), and \( Q = \mathbb{C}P_n^i / \nu \), where \( \mathbb{C}P_n^i \) is the trivial rank \( n + 1 \) bundle. Then from the exact sequence \( 0 \to \nu \to \mathbb{C}P_n^{i+1} \to Q \to 0 \), we get \( c(\nu)_{h} h \wedge c(Q)_{h} = c(\mathbb{C}P_n^{i+1})_{h} = 1 \). Therefore,
\[
c(Q)_{h} = 1/c(\nu)_{h} = \sum_{j=0}^{i} (-c_1(\nu)_h)^j_{h}.
\]

Without loss of generality, we can assume that \( [\omega] = c_1(\nu) \). It is easy to check that \( c_1(\nu)_h = -(\omega - \lambda h) \) for some constant \( \lambda \). Since \( Q \) has rank \( n \), \( \omega + \lambda h \wedge_h h^{n+1} = c_n(Q)_{h} = 0 \). It follows from \([\mathbb{C}7]\) that \( \lambda = n \). On the other hand, if \( \omega \) is the Kähler form for Fubuni-Study metric, then it is possible to check that \( c_1(\nu)_h = -(\omega - nh) \).

This then yields \([\mathbb{C}7]\).

This example illustrates the complexity in the calculation of quantum multiplications in quantum de Rham cohomology.

**Example.** (Complex Grassmannian) The same method can be used for complex Grassmannian \( G_{r,n}(\mathbb{C}) \). Let \( \nu \) be the tautological vector bundle, and \( Q = \mathbb{C}P^n / \nu \).
Let $c_j = c_j(\nu)$, and $s_j = c_j(Q)$. Then from the exact sequence $0 \to \nu \to \bigoplus^n \to Q \to 0$, we get $c(\nu) \wedge c(Q) = c(\bigoplus^n) = 1$, i.e
\[(18)\quad s_j = -s_{j-1}c_1 - \cdots - s_1c_{j-1} - c_j,
\]
for $j \geq 1$. Since $Q$ has rank $n - r$, we must have $s_j = 0$ for $j > n - r$. In fact, the de Rham cohomology ring of $G_{r,n}(\mathbb{C})$ is given by (e.g. Fulton \[12\], Ex. 14.6.6)
\[\mathbb{R}[c_1, \ldots, c_r]/(s_{n-r+1}, \ldots, s_n),\]
where $s_j$'s are given by (18). Since $H^*_{dR}(G_{k,n}(\mathbb{C})$ is one-dimensional, given any symplectic structure $\omega$ on $G_{k,n}(\mathbb{C})$, we may assume without loss of generality that $[\omega] = -c_1$. Let $c_{j,h} = c_j(\nu)_h$, $s_{j,h} = c_j(Q)_h$. Then using quantum Chern classes, we get $s_{j,h} = 0$, for $j = n - r + 1, \ldots, n$, where $s_{j,h}$'s are given by
\[(18')\quad s_{j,h} = -s_{j-1,h}c_1,h - \cdots - s_{1,h}c_j,h - c_j,h.
\]
Therefore, we need to compute $c_{j,h}$ and their multiplications.

**Example.** (Complex flag manifold) Let $F_{n+1}$ denote the manifold of complete flags $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$, where each $V_j$ is a subspace of dimension $j$, for $j = 1, \ldots, n$. There are tautological line bundles $L_j$ on $F_{n+1}$, whose fiber at a flag $V_1 \subset \cdots \subset V_{n-1} \subset V_n = V_j/V_{j-1}$. Then it is clear that
\[L_1 \oplus \cdots \oplus L_n = \bigoplus^n.\]

This is a special case of splitting principle (see e.g. Bott-Tu \[3\], §21, especially the proof of Proposition 21.15), which states that for any complex vector bundle $E$ of rank $n$, if $\pi: F(E) \to M$ is the flag bundle associated with $E$, then $\pi^*E = L_1 \oplus \cdots \oplus L_n$, where $L_j = V_j(E)/V_{j-1}(E)$. Therefore, we have
\[(19)\quad c(L_1) \cdots c(L_n) = 1.
\]
Let $x_j = c_1(L_j)$, $\sigma_j = \sigma_j(x_1, \ldots, x_n)$ the $j$-th elementary symmetric polynomial in $x_1, \ldots, x_n$. Then (18) is equivalent to $\sigma_j = 0$, $j = 1, \ldots, n$. Now we set $x_{j,h} = c_1(L_j)_h$, $\sigma_{j,h} = \sigma_j(x_{j,h}, \ldots, x_{n,h})$, the $j$-th elementary symmetric polynomial computed by quantum multiplications. Then from quantum Chern-Weil theory, we have $\sigma_{j,h} = 0$, $j = 1, \ldots, n$. As in the projective space case, this does not help much in writing down the quantum multiplications in quantum de Rham cohomology.

We identify $F_{n+1}$ with $X = U(n)/T^n$, where $T^n$ is the diagonal subgroup. Let $x_0$ denote the point in $X$ which corresponds to $T^n$. Then we can identify $T_{x_0}X$ with
\[n = \{(a_{ij}) \in M(n \times n, \mathbb{C}) : a_{jj} = 0, j = 1, \ldots, n, a_{ij} = -\bar{a}_{ji}\}.
\]
Since $H^*_d(U(n)/T^n) \cong H^*(\Omega(U(n)/T^n)^{U(n)}, d)$, every de Rham class can be represented by a $U(n)$-invariant closed form. In particular, by choosing a $U(n)$-invariant connection, the first Chern class of the line bundle $L_j$, can be represented by a $U(n)$-invariant closed form $\alpha_j$, for each $j = 1, \ldots, n$. The calculations of quantum de Rham cohomology for $U(n)$-invariant symplectic structures on $U(n)/T^n$ will be indicated later.

**Example.** (Generalized flag manifolds) We identify $F_{n+1}$ with $X = U(n)/T^n$, where $T^n$ is the diagonal subgroup. This reveals the fact that complex flag manifold is a special example of an important class of Fano manifolds used in Borel-Weil-Bott theory. In general, let $G^C$ be a semisimple Lie group over $\mathbb{C}$, and $B$ be a Borel subgroup of $G^C$, then $G^C/B$ is a projective variety (see Borel \[3\]). In fact, let
$G$ be the maximal compact subgroup of $G^C$, and $T$ the maximal torus of $G$, then $G/B \cong G/T$. Use a $G$-invariant inner product, e.g. the negative of Killing form on $\mathfrak{g}$, one gets a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}.$$ 

one can identify $T_{x_0}(G/T)$ with $\mathfrak{n}$, where $x_0$ is the point in $X = G/T$ corresponding to $T$. It is straightforward to find compatible almost complex structure $J$ and symplectic structure $\omega$ on $T_{x_0}X$ which are invariant under the action of the Weyl group $W$. Use the translation by action of $G$ to extend $J$ and $\omega$ to $G/T$, we then get a homogeneous Kähler manifold, which is Fano. Each weight $\lambda \in \mathfrak{t}^*$ of $G$ determines a representation $\phi_\lambda : T \to S^1$. Since $\pi : G \to G/T$ is a principal $T$-bundle, we then get an associated line bundle $L_\lambda$ from $\phi_\lambda$. Let $\lambda_1, \cdots, \lambda_k$ be the set of simple roots of $G$, and $x_j = c_1(L_{\lambda_j})$. If $p \in I(G) = S(T^*G)$ is an invariant polynomial, then by the isomorphism $S(\mathfrak{g}^*)G \cong S(T^*)^W$, we can identify $p$ with a polynomial on $\mathfrak{t}$, which is invariant under the action of the Weyl group. It turn out that $p(x_1, \cdots, x_k) = 0$, if $p(0) = 0$. Let $I_+(G)$ be the ideal generated by $p \in S(T^*)^T$ such that $p(0) = 0$. Borel used the degeneracy of the Leray spectral sequence at $E_2$ of the fibration $G/T \to BG \to BT$ to show that the cohomology of $X$ is isomorphic to

$$S(T^*)/I_+(G).$$

When $G = U(n)$, $T = T^n$ the diagonal subgroup, $G/T$ is diffeomorphic to the complex flag manifold, one recovers the result of the last example.

To compute the quantum de Rham cohomology, we use the fact $H^*_{dR}(G/T) = H^*(\Omega(G/T)^G, d)$, which implies that de Rham classes of $G/T$ can be represented by $G$-invariant forms on $G/T$. Such forms are determined by their values at $x_0$. Since $T_{x_0}(G/T) \cong \mathfrak{n}$, $\Omega(G/T)^G \cong \Lambda(\mathfrak{n}^*)^T$. For any element $\xi \in \mathfrak{g}$, let $X_\xi$ be the fundamental vector field of $\xi$ on $X = G/T$. For any weight $\lambda$, define a $G$-invariant 2-form $\omega_\lambda$ by setting

$$\omega_\lambda(X_\xi, X_\eta) = \lambda(\xi, \eta),$$

at $x_0$, for two fundamental vector fields $X_\xi$ and $X_\eta$. Following Lemma 8.67 and Lemma 8.68 in Besse, it is easy to see that $\omega_\lambda$ is closed. Let $\lambda_1, \cdots, \lambda_r$ be the simple roots of $\mathfrak{g}$, then they form a basis of $\mathfrak{t}^*$. Borel’s result indicates that $\omega_{\lambda_1}, \cdots, \omega_{\lambda_r}$ generates $H^*_{dR}(G/T)$. We will consider $G$-invariant symplectic forms which will be given below.

**Example.** (Coadjoint orbits) The coadjoint orbits of a compact connect Lie group $G$ are parameterized by $\mathfrak{t}^*/W$, or equivalently, a closed Weyl chamber $\overline{C}$. For any $\lambda \in \overline{C}$, let $O_\lambda$ denote the orbit of $\lambda$. For $\lambda$ in the interior of $\overline{C}$, $O_\lambda$ is diffeomorphic to $G/T$; for $\lambda$ on the wall of the Weyl chamber, there is a fibration of $G/T$ over $O_\lambda$ (Besse, Proposition 8.116). Besse, §8H shows complex flag manifolds, partial flag manifolds, Grassmannians, etc., are all examples of coadjoint orbits. It can be shown that $\omega_\lambda$ is a $G$-invariant symplectic forms on $O_\lambda$. It is called Kirillov-Kostant-Souriau form. It is straightforward to explicitly write down the quantum exterior multiplication on $H^*_{dR}(O_\lambda, \omega_\lambda)$, since we can do it on the tangent space of one point.

**References**

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization*, *Annals of Physics* 111 (1978), 61-110 and 111-152.
[2] K. Behrend, Gromov-Witten invariants in algebraic geometry, Invent. Math. 127 (1997), no. 3, 601–617.
[3] K. Behrend, B. Fantechi, The intrinsic normal cone, Invent. Math. 128 (1997), no. 1, 45–88.
[4] K. Behrend, Y. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Math. J. 85 (1996), 1-60.
[5] A. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10. Springer-Verlag, Berlin-New York, 1987.
[6] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. (2) 57, (1953), 115–207.
[7] A. Borel, Linear algebraic groups. Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.
[8] R. Bott, L. W. Tu, Differential forms in algebraic topology. Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982.
[9] J.-L. Brylinski, A differential complex for Poisson manifolds, J. Differential Geom. 28 (1988), no. 1, 93–114.
[10] H.-D. Cao, J. Zhou, On quantum de Rham cohomology theory (announcement), preprint, DG/9804145.
[11] J. Donin, On the quantization of Poisson brackets, Adv. Math. 127 (1997), no. 1, 73–93.
[12] W. Fulton, Intersection Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 2. Springer-Verlag, Berlin-New York, 1984.
[13] K. Fukaya, K. Ono, Arnold conjecture and Gromov-Witten invariants, preprint, 1996.
[14] A. Givental, B. Kim, Bumsig Quantum cohomology of flag manifolds and Toda lattices. Comm. Math. Phys. 168 (1995), no. 3, 609–641.
[15] P. Griffiths, J. Harris, Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.
[16] J. Kalkman, BRST model for equivariant cohomology and representatives for the equivariant Thom class, Comm. Math. Phys. 153 (1993), no. 3, 447–463.
[17] F.C. Kirwan, Cohomology of quotients in symplectic and algebraic geometry. Mathematical Notes, 31. Princeton University Press, Princeton, N.J., 1984.
[18] M. Kontsevich, Y. Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, Comm. Math. Phys. 164 (1994), no. 3, 525–562.
[19] M. Kontsevich, Y. Manin, Quantum cohomology of a product, with an appendix by R. Kaufmann, Invent. Math. 124 (1996), no. 1-3, 313–339.
[20] J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, The mathematical heritage of lie Cartan (Lyon, 1984). Astérisque 1985, Numero Hors Serie, 257–271.
[21] H.B. Lawson, Jr., M.-L. Michelsohn, Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.
[22] W. Lerche, C. Vafa, N. Warner, Chiral rings in N = 2 superconformal theories, Nuclear Phys. B 324 (1989), no. 2, 427–474.
[23] J. Li, G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, J. Amer. Math. Soc. 11 (1998), no. 1, 119–174.
[24] J. Li, G. Tian, Virtual moduli cycles and GW-invariant of general symplectic manifolds, preprint 1996.
[25] J. Li, G. Tian, Comparison of the algebraic and the symplectic Gromov-Witten invariants, preprint, alg-geom/9712033.
[26] G. Liu, Associativity of quantum multiplication, preprint, 1994.
[27] G. Liu, G. Tian, Floer homology and Arnold conjecture, preprint, 1997.
[28] G. Liu, G. Tian, On the equivalence of multiplicative structures in Floer and quantum cohomology, in preparation.
[29] G. Liu, G. Tian, Weinstein Conjecture and GW Invariants, preprint, alg-geom/9712021.
[30] P. Lu, A rigorous definition of fiberwise quantum cohomology and equivariant quantum cohomology, preprint, 1995.
[31] O. Mathieu, Harmonic cohomology classes of symplectic manifolds, Comment. Math. Helv. 70 (1995), no. 1, 1–9.
[32] D. McDuff, D. Salamon, J-holomorphic curves and quantum cohomology. University Lecture Series, 6. American Mathematical Society, Providence, RI, 1994.
[33] Y. Ruan, *Symplectic topology on algebraic 3-folds*, J. Differential Geom. **39** (1994), no. 1, 215–227.
[34] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. J. **83** (1996), no. 2, 461–500.
[35] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, preprint, 1996.
[36] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), no. 2, 259–367.
[37] Y. Ruan, G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997), no. 3, 455–516.
[38] B. Siebert, *Gromov-Witten invariants for general symplectic manifolds*, preprint, 1996.
[39] G. Tian, *Quantum cohomology and its associativity*, in Current developments in mathematics, 1995 (Cambridge, MA), 361–401, Internat. Press, Cambridge, MA, 1994.
[40] C. Vafa, *Topological mirrors and quantum rings*, in Essays on mirror manifolds, 96–119, Internat. Press, Hong Kong, 1992.
[41] I. Vaisman, *Lectures on the geometry of Poisson manifolds*. Progress in Mathematics, 118. Birkhuser Verlag, Basel, 1994.
[42] E. Witten, *Topological sigma models*, Comm. Math. Phys. **118** (1988), no. 3, 411–449.
[43] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, Nuclear Phys. B **340** (1990), no. 2-3, 281–332.
[44] D. Yan, *Hodge structure on symplectic manifolds*, Adv. Math. **120** (1996), no. 1, 143–154.