Elle est à toi cette chanson
Toi l’professeur qui sans façon,
As ouvert ma petite thèse
Quand mon espoir manquait de braise

To the memory of Manuel Bronstein

CYCLOTOMY PRIMALITY PROOFS AND THEIR CERTIFICATES

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Abstract. The first efficient general primality proving method was proposed in the year 1980 by Adleman, Pomerance and Rumely and it used Jacobi sums. The method was further developed by H. W. Lenstra Jr. and more of his students and the resulting primality proving algorithms are often referred to under the generic name of Cyclotomy Primality Proving (CPP). In the present paper we give an overview of the theoretical background and implementation specifics of CPP, such as we understand them in the year 2007.

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1 “Chanson du professeur”, free after G. Brassens

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1. Introduction

Let $n$ be an integer about which one wishes a decision, whether it is prime or not. The decision may be taken by starting from the definition, thus performing trial division by integers $\leq \sqrt{n}$ or is using some related sieve method, when the decision on a larger set of integers is expected. The method is slow for relatively small integers, but may be acceptable in certain contexts. Primality proving becomes a discipline after the realization that rather than the definition, one may test some property or consequence of $n$ being prime, and this can often be done by significantly faster algorithms - basically descending from the exponential to polynomial asymptotic behavior. This way one easily eliminates composites which do not verify the particular property of primes that is tested. The simplest property considered in this context is certainly Fermat’s small theorem: $a^{n-1} \equiv 1 \mod n$ for any $(a,n) = 1$, if $n$ is a prime. Since modular exponentiation is done in polynomial time in $\log(n)$, such a compositeness test is polynomial.

The disadvantage of the above approach is that there are composites which verify the same property; such composites are called Fermat - pseudoprimes base $a$ and there is literature dedicated to these and related pseudoprimes. Stronger statements are obtained when one has sufficient information about the factorization of $n - 1$. For instance, if there is a prime $q|(n-1)$ and $q > \sqrt{n}$, while $(a^{(n-1)/q} - 1, n) = 1$ and $a^{n-1} \equiv 1 \mod n$, then one easily proves that $n$ is prime. Indeed, if $p|n$ is a nontrivial prime factor with $p \leq \sqrt{n}$ – such a prime always exists, if $n$ is composite – then one considers $\hat{a} = a^{(n-1)/q} \mod p \in \mathbb{F}_p$. By hypothesis, $\hat{a} \neq 1$ and $\hat{a}^p = 1$; but then $\hat{a} \in \mathbb{F}_p^\times$ is an element of order $q$ and since $|\mathbb{F}_p^\times| = p - 1$, one should have $q|(p-1) < \sqrt{n}$, which contradicts the choice of $q$. The idea can be refined: $q$ may be replaced by an integer $F|(n-1)$, $F > \sqrt{n}$ which has a known factorization. Based on this factorization and an easy variation of the above argument, one obtains a more general primality test. Note that in these cases a proof of primality (or compositeness) comes along with the result of the algorithm. Tests of this kind can be designed also for small extensions $\mathbb{F}_{p^k} \supset \mathbb{F}_p$, with astute translations of the arithmetic in these extensions, in the case when $\mathbb{F}_p$ is replaced by $\mathbb{Z}/(n \cdot \mathbb{Z})$ and extensions of this ring are used. A general limitation remains the necessity to know some large factored divisors $s|(n^k - 1)$. Tests of this kind are denoted in general by the name of Lucas - Lehmer tests.

The idea of Adleman et. al. in [1] was to bypass the above mentioned restriction, by choosing $k$ so large, that an integer $s > \sqrt{n}$ and which splits completely in small – albeit, not polynomial – prime factors is granted to exist by analytic number theory. The algebraic part consists in a modification of the Lucas - Lehmer setting, which allows more efficient testing. In the original version of [1], the connection to classical test was hard to recognize. This connection was brought to light by H. W. Lenstra Jr. in his presentation of the result of Adleman, Pomerance and Rumely at the Bourbaki Seminar [13].

Let us consider again the Lucas - Lehmer test described above, where $q|(n-1)$ is a prime with $q > \sqrt{n}$. One can assert that this test constructs a primitive $q$-th root of unity modulo $n$, in the sense that $\Phi_q(\alpha) = 0 \mod n$ with $\alpha = a^{(n-1)/q} \mod n$ and $\Phi_q(X)$ the $q$-th cyclotomic polynomial. It is an important remark, that once $\alpha$ was calculated, it suffices to verify $\Phi_q(\alpha) \equiv 0 \mod n$, and this verification is shorter than the original computation. If $q$ is a proved prime, the verification will yield a proof of primality for $n$, which can be quickly verified. This is the core
idea for prime certification: gathering some information during the process of an initial primality proof, which can be used for a quicker a posteriori verification of the proof. Pratt developed this idea in the context of Lucas-Lehmer tests [29].

When replacing, $q$ by some large factored integer $s$ and searching for $s$-th roots of unity $\alpha$ in some extension $A \supset \mathbb{Z}/(n \cdot \mathbb{Z})$, that such roots are zeroes of polynomials over $\mathbb{Z}/(n \cdot \mathbb{Z})$ and this fact yields a common frame for understanding the APR-test and generalized Lucas-Lehmer tests. We present here a slight modification of Lenstra’s Theorem 8 in [18], which is seminal to the approach we take in this paper:

**Theorem 1.** Let $n > 2$ be an integer and $A \supset \mathbb{Z}/(n \cdot \mathbb{Z})$ a commutative ring extension, $s > 1$, $t = \text{ord}_{\alpha}(n)$ and $\alpha \in A^\times$. If the following properties hold

(i) $\Phi_n(\alpha) = 0,$

(ii) $\Psi(X) = \prod_{i=1}^{t} \left(X - \alpha^{n^i}\right) \in \mathbb{Z}/(n \cdot \mathbb{Z})(X),$

then either $n$ is prime or any divisor $r \mid n$ verifies:

\[
(1) \quad r \in \{n^i \text{ rem } s : i = 1, 2, \ldots, t = \text{ord}_s(n)\}.
\]

**Proof.** Suppose that $n$ is not prime and $r \mid n$ is a prime divisor. Then there is a maximal ideal $\mathfrak{M} \supset r \cdot A$ which contains $r$ and $K = A/\mathfrak{M}$ is a finite field while $\tilde{\alpha} = (\alpha \mod \mathfrak{M}) \in K$ verifies $\Phi_n(\tilde{\alpha}) = 0$. If $u = \text{ord}_{\tilde{\alpha}}(r)$, then $\tilde{\alpha}^r = \tilde{\alpha}$ and by galois theory in finite fields, the minimal polynomial of $\tilde{\alpha}$ is

\[
f(X) = \prod_{i=1}^{u} \left(X - \tilde{\alpha}^{i}\right) \in F_r[X].
\]

On the other hand, the polynomial

\[
\hat{\Psi}(X) = \Psi(X) \mod \mathfrak{M} = \prod_{i=1}^{t} \left(X - \tilde{\alpha}^{n^i}\right) \in F_r[X]
\]

has $\tilde{\alpha}$ as zero. By the minimality of $f(X)$, it follows that $f(X)|\hat{\Psi}(X)$ and since $F_r[X]$ has unique factorization, $\tilde{\alpha}^r$ must be a common zero of $f(X)$ and $\hat{\Psi}(X)$. In particular, there is an exponent $j$ such that $\tilde{\alpha}^r = \tilde{\alpha}^{n^j}$ and thus $\tilde{\alpha}^{n^j - r} = 1$. But by (i), $\tilde{\alpha}$ is a primitive $s$-th root of unity, and thus we must have $n^j - r \equiv 0 \mod s$ or $r \in \langle n \mod s \rangle$. This holds for all the prime divisors of $n$ and the more general statement (1) follows by multiplicativity. \hfill $\square$

This Theorem allows a fundamental generalization of the Lucas-Lehmer tests: let $n$ be an integer and suppose that an $s$-th root of unity in the sense of (i) is found in some ring $A \supset \mathbb{Z}/(n \cdot \mathbb{Z})$ and furthermore (ii) holds. If $s > \sqrt{n}$, then, pending upon a test of the fact that all the residues

\[r_i = n^i \text{ rem } s, \quad i = 1, 2, \ldots, t\]

are coprime to $n$, one has a primality proof for $n$. Indeed, if $n$ were composite, then at least one of its prime factors $p \leq \sqrt{n} < s$. But then, the Theorem implies that $p \in \{r_i : i = 1, 2, \ldots, t\}$, which is verified to be false. One should note that prior to Lenstra’s work, Lucas-Lehmer tests in ring extensions of degree $k$ were lacking a transparent criterion for the choice of the size of the completely factored part $s|(n^k - 1)$ required; in particular, the required factored part was often larger than $\sqrt{n}$ even for small values of $k$; it was also not possible to combine informations.
from tests for different values of $k$ \cite{23, 25}. The Theorem 1 solves both questions elegantly.

As we have shown in \cite{23}, the Theorem 1 not only generalizes the notion of Lucas-Lehmer tests and builds a bridge to combining them with the test of Adleman, Pomerance and Rumely, it also indicates a way for a new comprehension of that algorithm. It has become custom to denote the test described in \cite{1} in all its updated variants by Jacobi sum test, while *Cyclotomy Primality Proving* - or CPP - is a word used to cover all variants of tests related to Theorem 1. These may be in Jacobi sum tests, Generalized Lucas Lehmer, combinations thereof or also deterministic variants: we shall indeed see below, that the Jacobi sums test has a probabilistic *Las Vegas* version, which is mostly the version used in implementations, and a computationally more complicated deterministic version. The ideas of CPP were improved by Lenstra et. al. in \cite{20, 19, 12, 10, 23, 24}; their constructive base can be described as building a frame, in which a factor $\Psi(X)/\Phi_s(X) \mod n$ can be constructed for some large $s$ and such that, if $n$ is prime, the factor is irreducible. The computations are performed in *Frobenius rings* extending $\mathbb{Z}/(n \cdot \mathbb{Z})$, which become fields over $\mathbb{Z}/(n \cdot \mathbb{Z})$ if $n$ is prime.

The algorithms of CPP are *de facto* fast, competitive primality proving algorithms, but they have the complexity theoretical intolerable feature of a provable *superpolynomial* run-time

$$O \left( \log(n) \log \log \log(n) \right),$$

which is the expected value for the size of $t$ in \cite{1}. An practical alternative for proving primality on computers is the random polynomial test using the group of points of an elliptic curve over finite fields, originally invented by Goldwasser and Kilian \cite{13}. The test was made practical by a contribution of A. O. L. Atkin \cite{5} and has been implemented at the same time by F. Morain, who maintained and improved \cite{28} a program ECPP \cite{27} since more than a decade.

The purpose of this paper is to give a compact presentation of the theoretical background of the CPP algorithms and an overview of the basic variants. We also present a new method for computing *certificates* of a CPP proof. In the description of algorithms, we follow a balance between efficiency and clarity.

In section two we define the galois, Frobenius and cyclotomic extensions of rings. The last are the algebraic structures in which the various tests are performed. Based on this, we then describe an algorithm for taking roots in cyclotomic ring extensions, which is due to Huang in the field case. Section three gives an overview of Gauss and Jacobi sums over galois rings. We then show the connection to the construction of cyclotomic fields by cyclic field extensions and show that this mechanism is in fact the core idea of the Jacobi sum test. In section four we give some computational criteria which connect this test to the existence and construction of cyclotomic extensions. In section five we introduce the new certification methods and the probabilistic algorithms of CPP are defined in section 6. Finally, in section seven we present the deterministic version of CPP, and show how it could be understood and implemented as a subcase of the general CPP test and section 8 contains observations on the run time and the results from analytic number theory on which the analysis is based.

The ideas of this paper are updated from the thesis \cite{23} and many can be found already in the joint thesis of Bosma and van der Hulst \cite{10} and the seminal papers
of Lenstra. Our perspective of placing Theorem 1 at the center of CPP may be considered as the more personal contribution of this paper. Based on the common structure of Lucas-Lehmer and Jacobi sum tests, such as reflected by Theorem 1, we deduce by analogy to the Pratt certificates for Lucas-Lehmer tests over \( \mathbb{Z}/(n \cdot \mathbb{Z}) \) a certification method for CPP; such a method was not known or predicted to exist previously. The same frame yields also a simple understanding of (a generalized form of) the Berrizbeitia variant \( \mathbb{V} \) of the celebrated polynomial time deterministic test of Agrawal, Kayal and Saxena \([3]\); this is presented in \([6]\) and, independently, by Bernstein in \([8]\).

Finally, the notion of cyclotomic extension of rings can be extended to elliptic extensions of rings - closely connected to the Schoof-Elkies-Atkins algorithm for counting points on elliptic curves over finite fields. Together with the use of dual elliptic primes, some relatives of twin primes in imaginary quadratic extensions of \( \mathbb{Q} \), this leads to a new and very efficient combination of CPP and ECPP (elliptic curve primality proving) algorithms, which is presented in \([20]\). The present paper is herewith both an overview of the recent developments in CPP and a foundation for the description of new results.

1.1. Some notations. Throughout this paper we let \( n > 1 \) be an integer – which can be thought of as a prime candidate. We shall be interested in the ring \( \mathbb{Z}/(n \cdot \mathbb{Z}) \) and its extensions and introduce for simplicity the notation \( \mathcal{N} = \mathbb{Z}/(n \cdot \mathbb{Z}) \). For integers \( s > 0 \) we let \( \Phi_s(X) \in \mathbb{Z}[X] \) be the \( s \)-th cyclotomic polynomial. We shall encounter roots of unity in various rings. For complex roots of unity, we shall write \( \xi_s \in \mathbb{C} \) when \( \Phi(\xi_s) = 0 \); it will be made clear in the context, when a certain complex \( s \)-th root of unity is fixed. If \( G \) is a finite group and \( x \in G \) then \( < x > \) will denote the cyclic group generated by \( x \); e.g. \( < n \) mod \( s > \) is the cycle of \( n \in \mathbb{Z}/(s \cdot \mathbb{Z}) \). We may at times write \( \log_{(k)}(x) \) for the \( k \)-fold iterated logarithm of \( x \). Along with \( n \), we shall often use two parameters \( s, t \) in \( \mathbb{N} \) such that \( t = \text{ord}_s(n) \) or \( s \) is squarefree and \( t = \lambda(s) = \text{lcm}_{q \mid s}(q - 1) \), the product being taken over primes \( q \). In both cases, we consider the following sets related to these parameters:

\[
\mathcal{Q} = \{ q | s : q \text{ prime} \} \quad \text{and} \quad \mathcal{P} = \{ \varphi = (p^k, q) \in \mathbb{N}^2 : p^k || (q - 1), \ q \in \mathcal{Q} \text{ and } p \text{ is prime} \}.
\]

For \( \varphi = (p^k, q) \in \mathcal{P} \), we may use notations like \( p = p(\varphi), k = k(\varphi), \text{ etc.} \), with the obvious signification.

2. Galois extensions of rings and cyclotomy

Let \( A \) be a finite commutative ring and \( \alpha \in \Omega \supset A \) an element which is annihilated by some polynomial from \( A[X] \). Suppose that the powers of \( \alpha \) generate a free module \( R = A[\alpha] \); such modules shall be denoted by simple extensions of \( A \). Alternately, quotient rings of the type \( R = A[X]/(f(X)), \) where \( f(X) \in A[X] \) shall also be called simple extensions. It can be verified that the two types of extensions are equivalent.

There is an ideal \( I \subset \mathbb{Z} \) with \( IA = 0 \); the positive generator \( n \) of the annihilator \( I \) is the characteristic of the ring \( A \). We are interested in galois properties of extensions of finite rings. These have been considered systematically for primality by Lenstra in \([18], [20]\). The approach we take here is slightly different and closer to actual computational aspects; the central concept of cyclotomic extensions of rings end up to be identical to the one of Lenstra.
Definition 1. Let \( \mathbf{A} \) be a finite ring of characteristic \( n \) and:

1. Suppose that there is a galois extension of number fields \( \mathbb{L} = \mathbb{K}[X]/(f(X)) \) with \( f(X) \in \mathcal{O}(\mathbb{K})[X] \), and an ideal \( n \subset \mathcal{O}(\mathbb{K}) \) such that \( \mathbf{A} = \mathcal{O}(\mathbb{K})/n \).

2. Let \( \xi = X + (f(X)) \in \mathbb{L} \), \( \tilde{f}(X) = f(X) \mod n \in \mathbf{A}[X] \) and \( \mathbf{R} = (\mathcal{O}(\mathbb{L})/ (n \cdot \mathcal{O}(\mathbb{L}))) = \mathbf{A}[X]/(\tilde{f}(X)) = \mathbf{A}[\rho] \), with \( \rho = \xi \mod (n\mathcal{O}(\mathbb{L})) = X + (f(X)) \).

3. Let \( \mathcal{G} = \text{Gal}(\mathbb{L}/\mathbb{K}) \) and for \( \sigma \in \mathcal{G} \), define \( \hat{\sigma} : \rho \mapsto (\sigma(\xi) \mod (n\mathcal{O}(\mathbb{L}))) \) and \( \hat{\mathcal{G}} = \{ \hat{\sigma} : \sigma \in \mathcal{G} \} \).

4. Suppose that the degree \( d = \deg f \), the discriminant \( \text{disc}(f) \) and the characteristic are coprime:

\[
\text{(4)} \quad (n, \text{disc}(f) \cdot \deg(f)) = 1
\]

If these conditions are fulfilled, then the ring extension \( \mathbf{R} \) is called a galois extension of \( \mathbf{A} \) with group \( \hat{\mathcal{G}} \). Conversely, an extension \( \mathbf{R}/\mathbf{A} \) is galois, if there is a galois extension of number fields \( \mathbb{L}/\mathbb{K} \) from which \( \mathbf{R} \) arises according to 1.-3.

Remark 1. The definition of the galois extension depends in general on the choice of \((\mathbb{K}, \mathbb{L}, n)\) – we may in fact identify \( \mathbf{R} \) to this triple, and unicity of the lift to characteristic zero is not a concern. In fact, considering the case when \( n \) is a prime and \( \mathbf{R} = \mathbb{F}_{n^s} \) is a finite field, it is obvious that the algebra \( \mathbf{R} \) has multiple lifts. We shall in fact use this observation and define also when \( n \) is not known to be prime, some algebras \( \mathbf{R} \) in a simple way, and then construct by operations in \( \mathbf{R} \) additional polynomials that split in \( \mathbf{R} \), leading thus to additional lifts to characteristic zero.

The condition \((4)\) is quite artificial, but harmless in the context of primality testing, where one can think of \( \mathbf{A} \) as \( \mathcal{N} \) or a simple extension thereof: if \( 4 \) fails, one has a non trivial factor of \( n \).

The main property of a galois extension is of course the fact that the base ring is fixed by the galois group:

Fact 1. Let \( \mathbf{R} \supset \mathbf{A} \) be a galois extension of the finite ring \( \mathbf{A} \), let \( \mathbb{L} = \mathbb{K}[X]/(f(X)) \) be the associated extension of number fields and \( \mathcal{G} = \text{Gal}(\mathbb{L}/\mathbb{K}) \) the galois group. Let \( \rho = X + \tilde{f}(X) \in \mathbf{R} \) and suppose that \( \alpha \in \mathbf{R} \) is \( \hat{\mathcal{G}} \)-invariant. Then \( \alpha \in \mathbf{A} \).

Proof. Since \( \mathbf{R} = \pi(\mathcal{O}(\mathbb{L})) \) – where \( \pi \) is the reduction modulo \( n \cdot \mathcal{O}(\mathbb{L}) \) map – is a free \( \mathbf{A} \) - module, we can write \( \alpha = \sum \sigma_{\mathcal{G}} a_{\sigma} \cdot \hat{\sigma}(\rho) \in \mathbf{R} \), with \( a_{\sigma} \in \mathbf{A} \). If \( \alpha \) is \( \hat{\mathcal{G}} \)-invariant and \( d = |\hat{\mathcal{G}}| \in \mathbf{A}^* \), we have:

\[
d \cdot \alpha = \sum_{\sigma, \tau \in \mathcal{G}} \hat{\tau}(\alpha) = \sum_{\sigma, \tau \in \mathcal{G}} a_{\sigma} \cdot \hat{\tau} \circ \hat{\sigma}(\rho) = A \cdot \Theta,
\]

with \( A = \sum_{\sigma \in \mathcal{G}} a_{\sigma} \in \mathbf{A} \) and \( \Theta = \sum_{\sigma \in \mathcal{G}} \hat{\sigma}(\rho) = \pi(\sum_{\sigma \in \mathcal{G}} \sigma(\xi)) = \pi(\text{Tr}_{\mathbb{L}/\mathbb{K}} \xi) \in \mathbf{A} \). It follows that \( \alpha = (A \cdot \Omega) \cdot d^{-1} \in \mathbf{A} \), which completes the proof. \( \square \)

Here are some examples of galois extensions:

Examples 1.  

(a) Let \( \mathbf{A} = \mathcal{N} \), and \( s > 0 \) such that \((n, s \cdot \varphi(s)) = 1\). If \( f(X) = \Phi_s(X), \mathbb{K} = \mathbb{Q} \) and \( \mathbb{L} = \mathbb{Q}(\xi_s) \), then \( \mathbf{R} = \mathbb{Z}[\xi_s]/(n\mathbb{Z}[\xi_s]) \) is a galois extension with group \( \hat{\mathcal{G}} \sim (\mathbb{Z}/s \cdot \mathbb{Z})^* \).
(b) If \( n \) is a prime and \( s, \mathbb{K}, L, \mathbb{R} \) are like in the previous example, let \( H \subset (\mathbb{Z}/s \cdot \mathbb{Z})^* \) be the decomposition group of \( n \) in \( L \) with \( t = |H| \) and suppose that \( (n, s \cdot t) = 1. \) Let \( \mathbb{K}_1 = \mathbb{L}^H \) and \( n \in \mathcal{O}(\mathbb{K}_1) \) an ideal above \( n. \) Then \( \mathcal{O}(\mathbb{K}_1)/n = \mathbb{F}_n \) and there is a polynomial \( f(X) = \prod_{\tau \in H} (X - \tau(\xi)) \) such that \( r = A[X]/(f(X)) \subset \mathbb{R} \) is a galois extension. Of course, \( r = \mathbb{F}_n \) is even a field.

The construction holds for any subgroup \( H \) with \( H \subset K \subset G = \text{Gal}(L/\mathbb{K}), \) yielding a filtration of galois extensions. If \( K = G/H, \) then for any prime ideal with \( n \in \mathfrak{M} \subset \mathbb{Z}[\xi], \) the group \( K \) acts transitively on \( \mathfrak{M} \) and \( (n) = \prod_{\nu \in K} \nu(\mathfrak{M}). \) Furthermore, \( \mathbb{Z}[\xi]/(\nu(\mathfrak{M})) \cong \mathbb{F}_{p^t}. \) There is a canonical decomposition:

\[
\mathbb{R} = \mathbb{Z}[\xi]/(n\mathbb{Z}[\xi]) \cong \prod_{\nu \in H} \mathbb{Z}[\xi]/(\nu(\mathfrak{M})) = \prod_{\nu \in H} \mathbb{F}_{p^t}.
\]

If \( \rho = \zeta_s \mod \mathfrak{M} \in \mathbb{F}_{p^t} \) is fixed, then the image of \( \zeta_s \) in the Chinese Remainder decomposition of \( \mathbb{R} \) above is \( \zeta_s \mod (n) = \left( \rho, \rho^k, \ldots, \rho^{k^{-1}} \right), \)

where \( < k \mod n = K \) and \( \rho^k \equiv \zeta \mod \sigma_{\xi}^{-1}(\mathfrak{M}), \) with obvious meaning of \( \sigma_{\xi}(\xi) = \xi^\xi. \)

(c) Let \( \mathcal{E}(a, b) : Y^2 = X^3 + aX + b \) be the equation of an elliptic curve over some field \( \mathbb{K} \) and suppose that there is an ideal \( n \in \mathcal{O}(\mathbb{K}) \) such that \( \mathcal{O}(\mathbb{K})/n = \mathcal{N}. \)

Let \( \ell \) be a prime and \( \psi_\ell(X) \in \mathcal{O}(\mathbb{K})[X] \) be the \( \ell \)-th division polynomial of \( \mathcal{E}(a, b), \) its reduction being \( \hat{\psi}_\ell(X) \in \mathcal{N}[X]. \) Then \( \mathbb{R} = \mathcal{N}[X]/\left( \hat{\psi}_\ell(X) \right) \) is a galois extension.

The definition of galois extensions is quite general and is not specifically bound to the expectation that \( n \) might be a prime. We specialize below galois extensions to Frobenius extensions, which are related to finite fields.

**Definition 2.** Let \( A \) be a finite commutative ring of characteristic \( n \) and \( \Psi(X) \in A[X] \) a monic polynomial. We say that the simple ring extension \( R = A[X]/(\Psi(X)) \)

is:

- F. a Frobenius extension, if \( \Psi(X^n) \equiv 0 \mod \Psi(X) \) and

  (F1.) There is a set \( S = \{x_1, x_2, \ldots, x_m\} \subset R \) which generates \( R \) as a free \( \mathcal{A} \cdot \) module and such that \( \Psi(x_1) = 0. \)

  (F2.) There is a group \( G \subset \text{Aut}_A (R) \) which fixes \( S \) and such \( g = |G| \in \mathcal{A}^* \).

  (F3.) The traces of \( S \) are \( \text{Tr}(x_i) = \sum_{\sigma \in G} \sigma(x_i) \in A. \)

- SF. a simple Frobenius extension, if it is Frobenius and there is a \( t > 0 \) such that

  \[
  \Psi(X) = \prod_{i=1}^{t} \left( X - \zeta^{n^i} \right), \quad \text{where} \quad \zeta = X + (\Psi(X)) \in \mathbb{R}.
  \]

**Remark 2.** The example (c) is a galois extension which is in general not Frobenius. The other two examples are Frobenius at the same time and the first extension in (b) is simple Frobenius. The property F3. implies that \( A \) is exactly the ring fixed by \( G, \) the proof being similar to the one of Fact [1].

The situation in (b) is crucial for CPP. In fact, in the algorithms we shall investigate integers \( n \) that lead to the decomposition \( \mathbb{Z}(\mathbb{K}_1)/(n\mathbb{Z}(\xi)) \) and show that for such integers the Theorem [2] can be applied.
Next we clarify the notion of primitive root of unity, which has some ambiguity when considering roots of unity over rings. The question is illustrated by the simple example: is \( a = 4 \mod 15 \) a primitive second root of unity modulo 15? One verifies that \( a \in (\mathbb{Z}/15 \cdot \mathbb{Z})^* \) and \( a \neq 1 \) while \( a^2 = 1 \). However, \( a - 1 \notin (\mathbb{Z}/15 \cdot \mathbb{Z})^* \). We shall avoid such occurrences and define

**Definition 3.** Let \( A \) be a commutative ring with 1 and \( s > 1 \) an integer, while \( \Phi_s(X) \in \mathbb{Z}[X] \) is the \( s \)-th cyclotomic polynomial, \( \hat{\Phi}_s(X) \in A[X] \) its image over \( A \). We say that \( \zeta \in A \) is a primitive \( s \)-th root of unity iff \( \hat{\Phi}_s(\zeta) = 0 \).

We leave it as an exercise to the reader to verify that, if \( A \) is finite, then \( \zeta \in A \) is an \( s \)-th primitive root of unity if and only if for all maximal ideals \( \mathfrak{M} \subset A \), the root of unity \( \zeta_{\mathfrak{M}} = (\zeta \mod \mathfrak{M}) \in K = A/\mathfrak{M} \) is a primitive \( s \)-th root of unity in the field \( K \). In particular, \( \zeta - 1 \in A^\times \). Note that \( \zeta_{a} \mod (\mathbb{Z}[\zeta_{a}]) \) in example (b) is a primitive root of unity.

The next step towards the goal of Remark 2 consists in defining the cyclotomic extensions of rings, which are simple Frobenius extensions generated by primitive roots of unity.

**Definition 4.** Let \( n, s \) and \( \mathcal{N} \) be as above and \( \Omega \supset \mathcal{N} \) some ring with \( \zeta \in \Omega \), a primitive \( s \)-th root of unity. We say that

\[
R = \mathcal{N}[\zeta]
\]

is an \( s \)-th cyclotomic extension of the ring \( \mathcal{N} \), if the extension \( R/\mathcal{N} \) is simple Frobenius. In particular, \( R/\mathcal{N} \) has the galois group \( G = \langle \sigma \rangle \) generated by the automorphism with \( \sigma(\zeta) = \zeta^n \) and \( |G| = t = \text{ord}_a(n) \).

We say that \( s \) is the order and \( t \) is the degree of the extension \( R \). Sometimes we shall denote the extension also by the triple \( (\mathbb{R}, \zeta, \sigma) \).

Like for finite fields, a galois extension \( R \supset \mathcal{N} \) can be an \( m \)-th cyclotomic extension of \( \mathcal{N} \) for various values of \( m \). We shall in fact often start with galois extensions \( R \) of degree \( d \) over \( \mathcal{N} \) and then seek \( m \)-th primitive roots of unity in \( R \), for various values \( m|n^d - 1 \) and then prove that these roots together with the galois group generate an \( m \)-th cyclotomic extension. The procedure will be illustrated below, in the results on the Lucas–Lehmer test.

It is also natural to consider subextensions of cyclotomic extensions, i.e. rings of the kind

\[
T = \mathcal{N}[\eta], \quad \text{with} \quad \eta = \sum_{i=1}^{t/u} \sigma^{ui}(\zeta) \in R,
\]

where \( u|t \). Such subextensions are galois (even abelian). They have been considered recently by Lenstra and Pomerance in their version of the AKS algorithm [21]; the term of pseudo-fields was coined in that context.

**Remark 3.** Let \( (R, \zeta, \sigma) \) be some \( s \)-th cyclotomic extension of \( \mathcal{N} \), with \( R = \mathcal{N}[\zeta] \) and \( t = [R : \mathcal{N}] \). Suppose that there is an integer \( u > 1 \), and \( \beta \in R \) with \( \Phi_u(\beta) = 0 \) and such that \( S = \mathcal{N}[\beta] \) is an \( u \)-th cyclotomic extension with automorphism group induced by the restriction of \( \sigma \) to \( S \). We shall say in such a case, by abuse of language, that \( (R, \beta, \sigma) \) is a \( u \)-th cyclotomic extension.

Cyclotomic extensions do not exist for any pair \( (n, s) \) and their existence is a (sporadic) property of the number \( n \) with respect to \( s \); this fact is used for primality
testing. The following theorem groups a list of equivalent properties of cyclotomic extensions, relating them to Theorem 1 and providing a useful base for algorithmic applications.

**Theorem 2.** Let \( s, n > 1 \) be coprime integers, \( t = \text{ord}_s(n) \) be also coprime to \( n \), and fix \( \xi_s \in \mathbb{C} \). Let \( A \) be the ring of integers in \( \mathbb{Q}[\xi_s]^{< n \mod s> \} \) and consider the polynomial \( \Psi_0(x) = \prod_{i=1}^{n-1} (x - \xi_s^i) \in A[x] \). The following statements are equivalent:

1. An \( s \)-th cyclotomic extension of \( N \) exists.
2. \( r \mid n \iff r < n \mod s > \).
3. There is a surjective ring homomorphism \( \tau_0 : A \rightarrow N \).
4. There is a polynomial \( \Psi(x) = \tau_0(\Psi_0) \in N[x] \) of degree \( t \) with:
   - (i) \( \Psi(x) \mid \Phi_s(x) \)
   - (ii) if \( \zeta = x + (\Psi(x)) \in N[x]/(\Psi(x)) \), then \( \Psi(\zeta^n) = 0 \), for \( i = 1, 2, \ldots, t \).

**Proof.** Suppose that (I) holds and let \( \Psi(X) = \prod_{i=1}^{n-1}(X - \sigma^i(\zeta)) \in N[X] \). The argument used in the proof of Theorem 1 shows that (I) \( \implies \) (II).

Assume that (II) is verified, \( r \mid n \) be some prime factor and let \( \rho \in \mathbb{F}_r \) be a primitive \( s \)-th root of unity. If \( \mathfrak{R} \subset \mathbb{F}_r = \mathbb{Q}(\xi_s)^{< r \mod s> \} \) is some prime ideal above \( r \), then \( \mathfrak{O}(\mathfrak{L}_r) \) mod \( \mathfrak{R} = \mathbb{F}_r \) as follows from the example (c) and relation (6). But since \( r < n \mod s > \) it follows that \( A \subset \mathfrak{O}(\mathfrak{L}_r) \) and there is a fortiori a surjective map \( \tau_r : A \rightarrow \mathbb{F}_r \). By Hensel’s Lemma and the Chinese Remainder Theorem, this map can then be extended to a map \( \tau_0 : A \rightarrow N \), so (II) \( \implies \) (III).

Assume (III) holds and let \( r \mid n \) be a prime. Then \( \tau_0 \) extends by composition with the reduction modulo \( r \) to a map \( \tau_r : A \rightarrow \mathbb{F}_r \). In particular \( \Psi_r(X) = \tau_r(\Psi_0(X)) \in \mathbb{F}_r[X] \) is a polynomial such that \( \Psi_r(X) \mid \Psi_s(X) \) and \( \Psi_r(\zeta^n) = 0 \) if \( \Psi_r(\zeta) = 0 \). Using again Hensel’s Lemma and the Chinese Remainder Theorem, a polynomial \( \Psi(X) \in N(X) \) with the same properties can be constructed and thus (III) \( \implies \) (IV).

Finally, if \( \Psi(x) \in N[x] \) has property (IV), let \( R = N[x]/(\Psi(x)) \) and \( \zeta = x + (\Psi(x)) \); it follows from (i) that \( \zeta \) is a primitive \( s \)-th root of unity. We have to show that \( \sigma : \zeta \mapsto \zeta^n \) is an automorphism of \( R \). By construction, \( \sigma \) permutes the zeroes of \( \Psi \), so \( G < \sigma \) acts transitively on \( S = \{ \zeta, \zeta^n, \ldots, \zeta^{n-1} \} \). This shows that \( R \) is cyclic Frobenius, and since \( \Psi(X) \mid \Phi_s(X) \), it is an \( s \)-th cyclotomic extension of \( N \), so (IV) \( \implies \) (I).

**Remark 4.** It follows from (III), that the extension \( R/N \) is galois in the sense of the Definition 1 and this confirms the fact that its subextensions are galois too.

The relation (ii) is an elementary verifiable condition for the existence of cyclotomic extensions. If \( \lambda(s) = \text{lcm}_q(\varphi(q))^{[n \mod s]} \) is the Carmichael function, while \( \varphi \) is Euler’s totient function, then \( (\mathbb{Z}/s \cdot \mathbb{Z})^* \) contains \( \varphi(s) = \frac{(s)}{\text{ord}_s(n)} \) disjoint cyclic subgroups. The larger \( \varphi(s) \), the more improbable it becomes to find integers \( n \) for which (ii) is verified. This is the core idea of the CPP tests.

The following simple fact has some important implications about the size of cyclotomic extensions.

**Fact 2.** Let \( p \) be a prime, \( n \in \mathbb{N}_{>1}, (n, p) = 1 \) and \( v_p(x) \), \( x \in \mathbb{Z} \) denote the \( p \)-adic valuation. If \( p \) is odd, \( t = \text{ord}_p(n) \) and \( v = v_p(n^t - 1) \), then the order

\[
\text{ord}_p(n) = t \cdot p^u \quad \text{with} \quad u = \max(0, m - v).
\]
For \( p = 2 \) we distinguish the following cases:

1. If \( n \equiv 1 \mod 4 \) and \( v = v_2(n - 1) \), then
   \[
   \text{ord}_{2^{m}}(n) = 2^u \quad \text{with} \quad u = \max(0, m - v),
   \]

2. If \( n \equiv 3 \mod 4 \) and \( v = v_2(n + 1) \), then
   \[
   \text{ord}_{2^{m}}(n^2) = 2^{u+1} \quad \text{with} \quad u = \max(0, m - v).
   \]

\textbf{Proof.} The proof is left as an exercise to the reader, see also [31, Chapter II, § 3].

The remarkable phenomenon above consists in the fact that the order \( \text{ord}_p(n) \) starts from an initial value \( t = \text{ord}_p(n) \) which is constant for \( m \leq v_p(n^t - 1) \) and then increases by factors \( p \), when \( m \) grows. The only exception is the case \( p = 2 \) and \( n \equiv 3 \mod 4 \), when one has to consider \( m \leq v_2(n^2 - 1) \) as starting value. This leads to the following

\textbf{Definition 5.} Let \( n \) be an integer and \( p \) a prime with \( (p, n) = 1 \). We define the saturation index of \( n \) with respect to \( p \) by:

\[
(6) \quad k_n(p) = \begin{cases} 
  v_2(n^2 - 1) & \text{if } p = 2 \text{ and } n \equiv 3 \mod 4, \\
  v_p(n^t - 1) & \text{with } t = \text{ord}_p(n) \text{ otherwise.}
\end{cases}
\]

If \( (R, \sigma, \zeta) \) is a \( p^k \)-th cyclotomic extension of \( \mathcal{N} \) and \( k \geq k_n(p) \), then the extension is saturated. In general, if \( (s, n) = 1 \) and \( (R, \sigma, \zeta) \) is an \( s \)-th cyclotomic extension of \( \mathcal{N} \), we say that the extension is saturated, if \( p|s \Rightarrow p^{k_n(p)}|s \).

For odd \( p \) or \( p = 2 \) and \( n \equiv 1 \mod 4 \), we shall denote by saturated \( p \)-th extension of \( \mathcal{N} \) a galois extension with \( [R : \mathcal{N}] = d \), \( (d, p) = 1 \) or \( d = 2 \), if \( p = 2 \) and \( n \equiv 3 \mod 4 \), and which is a \( q \)-th cyclotomic extension of \( \mathcal{N} \), with \( q = p^{k_n(p)} \).

Note that the term saturated \( p^h \)-th extension, implicitly asserts the fact that \( h \geq k_n(p) \); the definition of a saturated \( p \)-th cyclotomic extension is an exception, since it denotes an extension which not only contains a \( p \)-th root but also a \( p^{k_n(p)} \)-th primitive root of unity. It can happen that a \( p \)-th cyclotomic of \( \mathcal{N} \) exists, but not a saturated one, as illustrated by:

\textbf{Example 1.} Let \( n = 91 = 7 \cdot 13 \). Then (II) implies that a third cyclotomic extension of \( n \) exists, since \( r \in < n \mod 3 > \) for all \( r|n \). However, according to (II) of Theorem 2, this extension is not saturated, since \( n = 1 \mod 9 \) yet \( r = 7|n \) and \( r \notin \{ < n \mod 9 > = < 1 > \} \).

The saturated extensions are characterized by the following property:

\textbf{Theorem 3.} If \( (R, \sigma, \zeta) \) is a saturated \( p^h \)-th extension and \( h > k_n \) then a \( p^h \)-th extension of \( \mathcal{N} \) exists.

\textbf{Proof.} Consider \( h > k_n \), \( R(h) = R[x]/(xp^{(h-k_n)} - \zeta) \) and let \( \zeta(h) \) be the image of \( x \) in \( R(h) \). It is easy to establish by comparing ranks, that \( (R(h), \sigma(h), \zeta(h)) \) is a \( p^h \)-th cyclotomic extension – where \( \sigma(h) \) is the extension of \( \sigma \) to \( R(h) \). \[ \square \]

Theorem 3 motivates the denomination of “saturated”: the existence of a saturated \( p \)-th extension implies existence of cyclotomic extensions of degree equal to any power of \( p \). The Example 1 shows that the existence of a saturated extension is also necessary for this. We shall use for commodity, the term of complete extension for the union of all saturated extensions of orders \( p^h \).
\textbf{Definition 6.} Suppose that a saturated $p^k$-th extension $(R, \sigma, \zeta)$ of $\mathcal{N}$ exists and let:

\begin{equation}
(R_\infty(p), \sigma_\infty(p), \zeta_\infty(p)) = \bigcup_{h=k_s}^{\infty} (R(h), \sigma(h), \zeta(h)).
\end{equation}

$R_\infty$ is called complete $p$-th extension and its existence is granted by the premises and the preceding theorem.

The proving of existence of cyclotomic extensions focuses herewith on proving existence of saturated extensions. The existence of saturated extensions has also implications for the properties of primes $r$ dividing $n$ \cite{12}:

\textbf{Lemma 1} (Cohen and Lenstra, \cite{12}). Suppose that $p$ is a prime with $(p, n) = 1$, for which a saturated $p$-th cyclotomic extensions of $\mathcal{N}$ exists. Then for any $r|n$ there is a $p$-adic integer $l_p(r)$ and, for $p > 2$, a number $u_p(r) \in \mathbb{Z} / ((p-1) \cdot \mathbb{Z})$, such that:

\begin{align}
r &= n^{u_p(r)} \mod p \quad \text{and} \\
r^{p - 1} &= (n^{p - 1})^{l_p(r)} \in \{ 1 + p \cdot \mathbb{Z}_p \} \quad \text{if } p > 2, \\
r &= n^{l_p(r)} \in \{ 1 + 2 \cdot \mathbb{Z}_2 \} \quad \text{if } p = 2.
\end{align}

\textbf{Proof.} Using Theorem 3 the hypothesis implies that $r \in n \mod p^k >$ for all $k \geq 1$ which implies \cite{8}. \qed

\textbf{2.1 Finding Roots in Cyclotomic Extensions.} Consider the following problem: given a finite field $\mathbb{F}_q = \mathbb{F}_p[a]$ with $q = p^k$ a prime power and $r$ a prime with $v_r(q - 1) = a$, and given $x \in \mathbb{F}_q$ with $x^{(q-1)/r} = 1$, find a solution of the equation $y^r = x$ in $\mathbb{F}_q$. The problem has an efficient polynomial time solution, if a $r^a$-th root of unity $\rho \in \mathbb{F}_q$ is known and the algorithm was described by Huang in \cite{14, 15}.

We shall treat here the generalization of the problem to cyclotomic extensions of rings. The basic idea is the same and it is well illustrated by the case $r = 2$ and $q = p \equiv 5 \mod 8$. In this case we let $u = x^{(p-1)/4} = \pm 1$, since $u^2 = x^{(p-1)/2} = 1$ by hypothesis. But $e = (p-1)/4$ is odd and thus $f = (e+1)/2$ is an integer, while $x^{2f} = u \cdot x$. If $\rho^2 = u = \pm 1$ for $\rho \in \mathbb{F}_p$, then a solution of $y^2 = x$ is given by $y = \rho^{-1} \cdot x$. Thus, knowing a $4$-th root of unity, one can find square roots in $\mathbb{F}_p$.

The general case is described in the following:

\textbf{Theorem 4.} Let $p$ be a prime with $(p, n) = 1$ and $(R, \sigma, \zeta)$ a saturated $p$-th cyclotomic extension of $\mathcal{N}$; let $\alpha \in R$ and $l \leq k_p(n)$ be such that

\begin{equation}
\alpha^{N/p^l} = 1
\end{equation}

is satisfied. Then there is a polynomial deterministic algorithm for finding a root $\beta \in R$ of the equation $x^{p^l} = \alpha$.

\textbf{Proof.} Let

\begin{equation}
t = \sfrac{1}{2} < \sigma > = [R : \mathcal{N}], \quad N = n^t - 1, \quad k = v_p(N),
\end{equation}

and let $u$ be given by $N = u \cdot p^k$, so that $(u, p) = 1$ and $k = k_u(p)$. Since $R$ is saturated, $\zeta \in R$ is a $p^k$-th root of unity. If $\alpha$ is a $p^k$-th power in $R$, then

\begin{equation}
\alpha^u = \zeta^{\nu \cdot p^l}, \quad \text{with } \nu \in \mathbb{Z} / (p^k \cdot \mathbb{Z}).
\end{equation}
Note that $\nu \mod p^i$ can be successively computed for $i = 1, 2, \ldots, k - l$ by comparing $\alpha^i p^{k-i-1}$ to powers of $\zeta^{p^{k-i}}$. Given $\nu$, one can define a solution to $x^p = \alpha$ in the following way. Let $u'$ be such that $u \cdot u' = -1 \mod p^k$ and $e = (1 + u \cdot u')/p^l$. Then $\beta = \alpha^e \cdot \zeta^{-u' \nu}$ is such that $\beta^p = \alpha$, which follows from a straightforward computation.

2.2. Finding Roots of Unity and the Lucas – Lehmer Test. The algorithm described above assumes that a saturated root of unity is known in a galois extension of appropriate degree. This can be found naturally by trial and error. Suppose that one wants to construct a saturated $p$–th cyclotomic extension and $t = \text{ord}_p(n)$. The bootstrapping problem that one faces, consists in finding first a galois extension $R/N$ of degree $t$: if such an extension is provided, one seeks a $p$–th power non residue, like one would do if $R$ was a field.

Let us recall some facts and usual notations about cyclotomic fields (see also [32]). The $s$–th cyclotomic field is $L_s = \mathbb{Q}(\zeta_s) = \mathbb{Q}[X]/(\Phi_s(X))$, an abelian extension of degree $\varphi(s)$ with ring of integers $\mathcal{O}(L_s) = \mathbb{Z}[\zeta_s]$ and galois group

$$G_s = \text{Gal}(L_s/\mathbb{Q}) = \{\sigma_a : \zeta_s \mapsto \zeta_s^a, \text{ where } (a, s) = 1\} \cong (\mathbb{Z}/s\mathbb{Z})^\ast.$$  

It is noted in [20] that in fact $\sigma_a = \left(\frac{\zeta_a}{s}\right)$ is in this case the Artin symbol of $a$. We shall adopt the notation of Washington, introduced above. The Theorem of Kronecker - Weber states that all abelian extensions of $\mathbb{Q}$ are subfields of cyclotomic extensions and if $K/\mathbb{Q}$ is an abelian field, then its conductor is by definition the smallest integer $s$ such that $K \subset L_s$, with $L_s$ the $s$–th cyclotomic extension.

The next fact shows where to look for galois extensions of $N$.

**Fact 3.** Let $n > 2$ be an integer and $K/\mathbb{Q}$ be an abelian extension of conductor $s$ such that $(s, n) = 1$ and $\text{Gal}(K/\mathbb{Q}) \cong \langle \text{ord}_n(n) = [K : \mathbb{Q}]$. Then there are $\omega_i \in \mathcal{O}(K)$, $i = 1, 2, \ldots, t$ such that

$$\mathcal{O}(K) = \mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_t] \quad \text{and} \quad R = N[\omega_1, \omega_2, \ldots, \omega_t],$$

where $\omega_i \equiv \omega_i \mod n\mathcal{O}(K)$.

The ring $R = \mathcal{O}(K)/(n \cdot \mathcal{O}(K))$ is a galois extension of $N$ if and only if the ring $\mathcal{O}(K)$ has a normal $\mathbb{Z}$ - base.

If $t = p^k$ is a prime power, this happens in the following cases:

(i) $s$ is prime and $p^k \parallel (s - 1)$.
(ii) $p$ is odd and $s = p^{k+1}$.
(iii) $p = 2$, $k \geq 2$ and $s = 2^{k+2}$.
(iv) $p = 2$ and $k = 1$.

**Proof.** The ring $\mathcal{O}(K)$ is a free $\mathbb{Z}$ - module of rank $t$ and discriminant which divides $s^N$ for some integer $N > 1$, see e.g. [30]. With this, the assertions become simple verifications based upon the definition of $K$ and the one of a galois extension. The assumption that $t = p^k$ can be dropped, by using the linear independence of $s_i$–th cyclotomic extensions ($i = 1, 2$) when $(s_1, s_2) = 1$.  

The following Theorem is useful for constructing roots of unity and the associated cyclotomic extensions, as well as for generalized Lucas-Lehmer tests:

**Theorem 5.** Let $N$ and $K$ be a field of conductor $s$ as described in Fact 3, in particular $R = \mathcal{O}(K)/(n \cdot \mathcal{O}(K))$ is a galois extension of $N$ and with $\text{Gal}(L_s/\mathbb{Q}) = \{\sigma_a : \zeta_s \mapsto \zeta_s^a, \text{ where } (a, s) = 1\} \cong (\mathbb{Z}/s\mathbb{Z})^\ast$. 

\[\zeta_a \equiv \left(\frac{a}{s}\right) \mod n\mathcal{O}(K)\]
<n \mod s >. Suppose that
\begin{equation}
\exists \alpha \in O(L) \text{ such that } \sigma_n(\alpha) = \alpha^n \mod n \cdot O(L).
\end{equation}

For all primes \(q|s\), let \((n^t - 1)/q = \sum_{i=0}^{t-1} c_i(q) \cdot n^i\) and suppose that
\begin{equation}
(\beta(q) - 1) = \prod_{i=0}^{t-1} (\sigma_n^i(\alpha^{c_i(q)})) \mod n \cdot O(\mathbb{K}) \text{ verifies } (\beta(q) - 1) \in R^*.
\end{equation}

Let \(\beta = \prod_{q|s} \beta(q) \mod n \cdot O(\mathbb{K})\) and \(\sigma\) be the automorphism induced by \(\sigma_n\) in \(R\). Then \((R, \sigma, \beta)\) is a saturated \(s\)-th cyclotomic extension of \(N\).

Proof. Let \(\Psi(X) = \prod_{i=0}^{t-1} (X - \sigma_i(\beta))\); then \(\Psi(X) \in \mathcal{N}[X]\) since \(\sigma_n\) generates \(\text{Gal}(\mathbb{K}/\mathbb{Q})\). Furthermore \(\Psi(X)|\Phi_s(X)\) by construction and thus \(\beta\) is a primitive \(s\)-th root of unity. The statement follows from (IV) of Theorem 2. Note that \(\mathcal{N}[\beta(q)]\) are by construction saturated extensions, and thus \(\mathcal{N}[\beta]\) is saturated too.

Remark 5. In practical applications of Theorem 2, \(n > q\). We show that there is a simple expansion of the shape \((n^t - 1)/q = \sum_{i=0}^{t-1} c_i(q) \cdot n^i\) which makes the computation of \(\beta(q)\) in (13) particularly efficient. Let \(n = a \cdot q + b, \) with \(0 \leq b < q\). Then
\begin{equation}
(n^t - 1)/q = ((n^t - b^t) + (b^t - 1))/q = (b^t - 1)/q + a \cdot \sum_{i=0}^{t-1} n^{t-i-1} \cdot b^i.
\end{equation}

This leads to the equation:
\begin{equation}
(n^t - 1)/q = \sum_{i=0}^{t-1} c_i(q) \cdot n^i, \quad \text{with } c_0(q) = (b^t - 1)/q
\end{equation}
\begin{equation*}
\text{and } c_i(q) = a \cdot b^i, \text{ for } i = 1, \ldots, t - 1.
\end{equation*}

Note that \(c_i(q)\) are not necessarily \(< n\), but \(c_i(q)/n\) is at most a small number. The regular shape of the coefficients is however very useful for the simultaneous computation of \(\alpha^n\) and \(\alpha^{c_i(q)}\). Suppose that the cost for the application of one automorphism \(\sigma_n\) is \(c \cdot (\text{multiplication in } R)\) – if no fast polynomial multiplication methods are used, then \(c = 1\). The time needed for the evaluation of \(\beta(q)\) using (14) is bounded by
\begin{equation*}
2 \cdot (t - 1)(\log q + c + 1) + \log n.
\end{equation*}

This method of evaluation is thus for \(t > (\log q)/\log(n/q^2 \cdot 4^{c+1})\) more efficient than when defining \(\beta(q) = \alpha^{(n^t-1)/q}\), and performing the direct exponentiation.

If \(K\) is a field of degree \(p^k\) defined by Fact 3 an \(s\)-th cyclotomic extension can be constructed by using Theorem 5. This is the Lucas-Lehmer approach to constructing cyclotomic extensions. It is obvious that, when the degree of extensions is of importance and the order irrelevant, a minimal \(s\) will be chosen.

Remark 6. The extensions constructed by the Lucas-Lehmer method are saturated. This approach is used in [2] for constructing galois fields. We shall also show that this has useful consequences for combining cyclotomic extensions.
3. Gauss and Jacobi Sums over Cyclotomic Extensions of Rings

Gauss Sums are character sums used in various contexts of mathematics. It will be important for us to note that Gauss sums are Lagrange resolvents encountered when solving the equation $X^s = 1$ with radicals, over $Q$. Or, equivalently, when building the $s$–th cyclotomic field $L_s/Q$ by a succession of prime power galois extensions, see e.g. [17].

Let $n, m > 1$ be integers with $m$ squarefree and let $\lambda(m)$ be the exponent of $(Z/m \cdot Z)^*$, where $\lambda$ is the Carmichael function. In this section, $u = \lambda(m)$ and $f$ will be some divisor of $u$ and we assume that $(n, mu) = 1$. Let $A \supset N$ be a galois extension which contains two primitive roots of unity $\zeta, \rho$ of respective orders $u, m$. A multiplicative character $\chi : (Z/m \cdot Z)^* \to < \zeta >$ is a multiplicative group homomorphism $(Z/m \cdot Z)^* \to < \zeta >$. We denote by $(Z/m \cdot Z)$ the set of multiplicative characters defined on $(Z/m \cdot Z)^*$; the set $(Z/m \cdot Z)$ builds a multiplicative group and the order of $\chi$ is the cardinality of the image $\text{Im}(\chi)$. We shall also denote characters $\chi \in (Z/m \cdot Z)$ by characters modulo $m$.

Let $\chi \in (Z/m \cdot Z)$ and $d$ be a divisor of $m$. If there is a character $\chi' : Z / \left(\frac{m}{d} \cdot Z\right) \to < \zeta >$ such that

$$\chi(x) = \chi'(x \mod (m/d)) \quad \text{for all} \quad x \in (Z/m \cdot Z)^*,$$

then $\chi$ is said to be induced by $\chi'$. A character $\chi : (Z/m \cdot Z)^* \to < \zeta >$ is called primitive if it is induced by no character different from itself; in this case, $m$ is called the conductor of $\chi$. Each character $\chi$ is induced by a unique primitive character $\chi'$ and the conductor of $\chi$ is defined to be equal to the conductor of the primitive character it is induced by. In particular, the principal character $\mathbf{1} : \{1\} \to < 1 >$ is primitive and has conductor 1.

For $\chi \in (Z/m \cdot Z)$, we shall set for ease of notation

$$\chi(x) = 0 \quad \text{for} \quad (x, m) > 1.$$  \hspace{1cm} (15)

The Gauss-Sum of $\chi$ with respect to $x$ is the element of $A$ given by\footnote{We adopt Lang’s sign definition for the character sums}

$$\tau(\chi) = - \sum_{x \in Z/(m \cdot Z)} \chi(x) \cdot \rho^x. \hspace{1cm} (16)$$

The Gauss-Sum depends upon the choice of an element in $< \rho >$ according to:

$$\eta_a(\chi) = \sum_{x \in Z/(m \cdot Z)} \chi(x) \cdot \rho^{ax} = \chi^{-1}(a) \cdot \tau(\chi), \quad \forall a \in (Z/m \cdot Z)^*. \hspace{1cm} (16)$$

Let $\nu \in (Z/m \cdot Z)^*$, $H(\nu) = (Z/m \cdot Z)^*/ < \nu >$ mod $m >$ and $h$ be a coset in $H(\nu)$. The $h$–th Gauss Period with respect to $\nu$ is defined by:

$$\eta_h(\rho, \nu) = \sum_{\mu \in h} \rho^\mu, \quad \forall h \in H(\nu). \hspace{1cm} (17)$$

Let $H(\nu) = \{ \chi \in (Z/m \cdot Z) \mid \chi(\nu) = 1 \}$. $H(\nu)$ is dual to $H(\nu)$ in the sense that characters $\chi \in H(\nu)$ operate on cosets $h \in H(\nu)$. Gauss-Sums and Gauss Periods...
are connected by:
\[(18) \quad \tau(\chi) = - \sum_{h \in H(\nu)} \chi(h) \cdot \eta_h(\rho, \nu)\]
and
\[(19) \quad |H(\nu)| \cdot \eta_h(\rho, \nu) = - \sum_{\chi \in H(\nu)} \chi^{-1}(h) \cdot \tau(\chi), \quad \forall h \in H(\nu).\]

Equation (18) follows by using \(\chi(\nu) = 1\) and regrouping the summation order in (15). Identity (19) is a consequence of the following:

**Fact 4.** If \(G\) is a subgroup of \((\mathbb{Z}/m \cdot \mathbb{Z})\) and \(x \in (\mathbb{Z}/m \cdot \mathbb{Z})^*\), then
\[(20) \quad s(x) = \sum_{\chi \in G} \chi(x) = \begin{cases} 0, & \text{if } \exists \chi \in G \text{ with } \chi(x) \neq 1, \\ |G|, & \text{otherwise}. \end{cases}\]

**Proof.** If \(\chi(\nu) \neq 0\), then \(s(x) \cdot (1 - \chi(\nu)) = 0\) leads to the claimed result. \(\square\)

If \(\chi, \chi' \in (\mathbb{Z}/m \cdot \mathbb{Z})^*\) are primitive, the Jacobi-Sum \(j(\chi, \chi')\) is defined by:
\[(21) \quad j(\chi, \chi') = - \sum_{x \in \mathbb{Z}/(m \cdot \mathbb{Z})} \chi(x) \cdot \chi'(1 - x).\]

Gauss and Jacobi-Sums factor with respect to the ideals \((p^{v(p)} \cdot \mathbb{Z}/(m \cdot \mathbb{Z}))\) of \(\mathbb{Z}/(m \cdot \mathbb{Z})\), where \(p^{v(p)} \parallel m\) and this will allow us to restrict our attention to characters of prime conductor. The factorization is given by the following:

**Fact 5.** Let \(\chi \in (\mathbb{Z}/m \cdot \mathbb{Z})^*\) and \(m = \prod_{p|m} p^{v(p)}\). Then there are characters \(\chi_p \in (\mathbb{Z}/p^{v(p)} \cdot \mathbb{Z})^*\) and Gauss-Sums \(\tau_p(\chi_p)\) such that:
\[(22) \quad \tau(\chi) = \prod_{p|m} \tau_p(\chi_p).\]

If \(\chi', \chi'' \in (\mathbb{Z}/m \cdot \mathbb{Z})^*\), then there are characters \(\chi'_p, \chi''_p \in (\mathbb{Z}/p^{v(p)} \cdot \mathbb{Z})^*\) and Jacobi-Sums \(j_p(\chi'_p, \chi''_p)\) such that:
\[(23) \quad j(\chi', \chi'') = \prod_{p|m} j_p(\chi'_p, \chi''_p).\]

**Proof.** The proof is an exercise in the use of the Chinese Remainder Theorem. \(\square\)

If \(\chi\) is a primitive character, the absolute value of its Gauss-Sum is determined by:
\[(24) \quad |\tau(\chi)| = |\tau(\chi^{-1})| = \chi(-1) \cdot m.\]

Gauss and Jacobi-Sums are connected by:
\[(25) \quad \tau(\chi) \cdot \tau(\chi') = \tau(\chi) \cdot \tau(\chi'), \quad \text{if } \chi, \chi' \text{ are primitive.}\]

Let the \(\chi\) be a character of conductor \(m\) and order \(f\); the multiple Jacobi-Sums \(J(\chi)\) are defined by:
\[(26) \quad \begin{align*}
J_1 &= 1, \\
J_{\nu+1} &= J_\nu \cdot j(\chi, \chi''), \quad \text{for } \nu = 1, 2, \ldots, f-2 \\
J_f &= \chi(-1) \cdot m \cdot J_{f-1}
\end{align*}\]
It is easy to verify by induction that:
\[
J_\nu = \frac{\tau(\chi)^\nu}{\tau(\chi^\nu)}, \quad \text{for } \nu = 1, 2, \ldots, f,
\]
where the sum of the trivial character is set by definition to \(\tau(\chi^f) = 1\). The Chinese Remainder Theorem can be used for expressing the Gauss-Sum of a character \(\chi\) as a product of Gauss-Sums of characters of prime power orders.

**Remark 7.** If \(m = q\) is a prime, \(\nu \in (\mathbb{Z}/q\mathbb{Z})^*\) and \(t\) and \(f\) are such that \(t = \text{ord}_q(\nu)\) and \(f = \varphi(q)/t\), then \(H(\nu) = (\mathbb{Z}/q\mathbb{Z})^*/<\nu \mod q>\) is a cyclic group isomorphic to \(\{g^{t^i} | i = 1, 2, \ldots, f\}\), where \(g\) is a generator of \((\mathbb{Z}/q\mathbb{Z})^*\). With \(\rho\) a primitive \(q\)-th root of unity, the relations (17) – (19) can be rewritten explicitly as:

\[
\eta_j(\rho, \nu) = \sum_{i=1}^t \rho^{g^{t^i} \cdot j^i} \quad \text{for } j = 1, 2, \ldots, f.
\]

\[
\tau(\chi) = \sum_{j=1}^f \eta_j(\rho, \nu) \cdot \chi(g^{t^i}j^i), \quad \forall \chi \in H(\nu)\dagger.
\]

\[
f \cdot \eta_j(\rho, \nu) = \sum_{j=1}^f \chi^{-1}(g^{t^i}j^i) \cdot \tau(\chi^j) \quad \text{for } j = 1, 2, \ldots, f.
\]

It follows from (27), that the Gauss-Sums \(\tau(\chi), \chi \in H(\nu)\dagger\) are Lagrange resolvents for the Gauss Periods \(\eta_j(\rho, \nu)\). In this context, one can interpret (18) as a generalization of Lagrange resolvents to abelian extensions. We shall see in the next chapter, that Gauss Periods generate intermediate extensions in cyclotomic fields. The Gauss-Sums can be used to calculate the periods and thus to generate intermediate cyclotomic fields.

Gauss sums can be defined for primitive characters of prime power conductors; the properties arising in this context have been investigated in [24] but are not of interest in our present context. This explains the choice of \(s\) as being squarefree in the definitions above.

In the case when \(n = r\) is a prime and \(A\) is a field of characteristic \(r\), the action of the Frobenius upon Gauss sums induces some formulae which are specific for character sums over finite fields. Let \(\chi\) be a primitive character of conductor \(m\) and order \(f\); \(\zeta, \rho \in A\) are primitive roots of unity, with respective orders \(f\) and \(m\). We investigate the action of the automorphism \(\phi_r : x \mapsto x^r\) of \(A\) upon \(\tau(\chi)\):

\[
\tau(\chi)^r = \sum_x \left(\chi(x) \cdot \rho^x\right)^r = \sum_x \chi^r(x) \cdot \rho^{rx}.
\]

By using (14) we have:

\[
\tau(\chi)^r = \chi^{-r}(r) \cdot \tau(\chi^r),
\]

and iterating (31) we get:

\[
\tau(\chi)^{rk} = \chi^{-rk}(r) \cdot \tau(\chi^{rk}), \quad \text{for } k \geq 1.
\]

If \(r^t = 1 \mod f\), then

\[
\tau(\chi)^{r^{t-1}} = \chi^{-t}(r).
\]
The relations (31) and (33) are central in primality testing. It will be important to have efficient computing methods for powers of Gauss and Jacoby sums, if they are to be used in practical algorithms.

4. Further Criteria for Existence of Cyclotomic Extensions

The condition (III) in Theorem 2 is central for primality proving and motivates the interest in proving the existence of cyclotomic extensions. One way of doing this is shown in Theorem 5 and it generalizes the classical Lucas – Lehmer tests.

The resulting conditions indicate the direction for the Jacobi sum test. Before stating them, let us introduce some notations. Let $n,s,t$ be like in Theorem 2 and $\xi, \xi_s \in \mathbb{C}$ be fixed; furthermore, we assume that there exists a saturated $t$–th cyclotomic extension $R \supset \mathcal{N}$ and $\zeta \in R$ is a primitive $t$–th root of unity.

We shall write like previously $\chi \in \mathcal{H}$ for the characters with image in $R$ while $\frac{\mathbb{Z}/s \cdot \mathbb{Z}}{s} = \{ \chi : (\mathbb{Z}/s \cdot \mathbb{Z})^* \to \xi >, \chi \text{ multiplicative } \}$. For $a \in (\mathbb{Z}/s \cdot \mathbb{Z})^*$ we let $H(a) = (\mathbb{Z}/s \cdot \mathbb{Z})^*/<a \mod s>$ and

$$H(a) = \{ \chi \in (\mathbb{Z}/s \cdot \mathbb{Z}) : \chi(a) = 1 \} \subset (\mathbb{Z}/s \cdot \mathbb{Z})$$

be its dual. The set $H(a)^T \subset (\mathbb{Z}/s \cdot \mathbb{Z})^T$ is defined by analogy. Then,

**Theorem 6.** The following statement is equivalent to (I) – (IV) of Theorem 2.

(V) If the Gauss sums $\tau(\chi)$ are defined for $\chi \in (\mathbb{Z}/s \cdot \mathbb{Z})^T$ with respect to $\xi_s$, then:

$$\chi \in H(n)^\perp \iff \exists \text{ a homomorphism } \vartheta : \mathbb{Z}[\xi, \tau(\chi)] \to R.$$

**Proof.** Suppose that (III) holds, thus a map $\tau_0 : A = \mathcal{O}(\{\xi_s\}^{<n \mod s>}) \to \mathcal{N}$ exists. In particular, it follows that the Gauss periods $\eta_h(\xi_s, n) = \sum_{\mu \in \mathbb{H}} \xi_{s}^{\mu}$ with $h \in H(n)$ are mapped to $\mathcal{N}$. Let $\vartheta$ be the lift of $\tau_0$ with $\vartheta(\xi_s) = \zeta \in R$. If $\tau(\chi)$ are Gauss sums with respect to $\xi_s$ and $\chi \in H(n)^\perp$, then we gather from (18) that $\vartheta(\tau(\chi)) \in R$, which proves that (III) $\Rightarrow$ (V).

Suppose now that (V) holds and let $B \subset \mathbb{Z}[\xi_s, \xi]$ be the ring generated by $\xi_s$ and the Gauss sums $\tau(\chi)$, $\chi \in H(n)^\perp$, while $\vartheta : B \to R$ is such that $\vartheta(\tau(\chi)) \in R$. Using (19) we see that $\vartheta$ maps the Gauss periods $\eta_h$ to $R$, and if $\sigma$ generates the Galois group of $R/\mathcal{N}$ acting on $\zeta$, then $\vartheta(\eta_h)$ are $\sigma$ invariant, so $\vartheta(\eta_h) \in \mathcal{N}$. Using reduction modulo primes $r|n$ and arguments from the proof of Theorem 2 we deduce that $r \in < n \mod s >$ and thus (V) $\Rightarrow$ (II), which completes the proof. \[ \square \]

Note that since only characters $\chi \in H(n)^\perp$ are considered, the condition (V) is a slight improvement of the one used in the initial form of the Jacobi sum test [1], and which involved all characters in $(\mathbb{Z}/s \cdot \mathbb{Z})^\perp$.

**Lemma 2.** Let $p, q$ be primes not dividing $n$, with $p^k \parallel (q - 1)$ and $(R, \sigma, \zeta)$ be a saturated $p$–th cyclotomic extension of $\mathcal{N}$. Let $\chi \in (\mathbb{Z}/q \cdot \mathbb{Z})$ be a character of order $p^k$ and $\alpha, \beta \in R$ be given by:

$$\alpha = J_p(\chi) \quad \text{and} \quad \beta = J_\nu(\chi), \quad \text{where } \nu = n \mod p^k.$$
Let \( l = \lfloor n/p^k \rfloor \) and suppose that
\[
\alpha^l \cdot \beta = \eta^{-n} \quad \text{holds for some } \eta \in \mathbb{C}.
\]
Then \( \eta = \chi(n) \) and \( \chi(r) = \chi(n)^{\rho_p(r)}, \forall r \mid n \), with \( \rho_p(r) \) defined in Lemma 4.

**Proof.** Let \( \mathbf{R}' = \mathbf{R}[X]/\langle \Phi_q(X) \rangle \) and define \( \zeta_q = X + \Phi_q(X) \in \mathbf{R}' = \mathbf{R}[X]/\langle \Phi_q(X) \rangle \); one proves that \( \mathbf{R}' \) is a galois extension of \( \mathbf{R} \) and also of \( \mathcal{N} \). We then define the Gauss sum \( \tau(\chi) \) with respect to \( \zeta_q \) and claim that the identities on multiple Jacobi sums hold for this sum; this is a simple verification and is left to the reader. The actual identities are meaningful in the ring \( \mathbf{R} \), but we need \( \mathbf{R}' \) for introducing the Gauss sums. By the definition of \( \alpha, \beta \) and \( l \), (35) is equivalent to
\[
\tau(\chi)^n = \eta^{-n} \cdot \sigma(\tau(\chi)).
\]
Raising (36) to the power \( n \) repeatedly, we find:
\[
\tau(\chi)^{n^i} = \eta^{-i \cdot n} \cdot \sigma^i(\tau(\chi)) \quad \forall i \geq 1
\]
and, with \( i = p^k \cdot (p - 1) \) and \( N = n^i \),
\[
\tau(\chi)^{N - 1} = 1.
\]
If \( r \mid n \) is a prime and \( \mathfrak{R} \subset \mathbf{R}' \) a maximal ideal through \( r \), then by (38)
\[
\tau(\chi)^r = \chi(r)^{-r} \cdot (\tau(\chi)) \mod \mathfrak{R}.
\]
From the existence of the saturated \( p \)-th extension \( \mathbf{R} \) we gather, by Fact 1, that there are two integers \( l_p(r), u_p(r) \) verifying (3). With these, we let \( m \in \mathbb{N} \) be such that \( m = l_p(r) \mod p^k \) and \( m = u_p(r) \mod (p - 1) \), so that \( \sigma^m(\chi) = \chi^r \) and
\[
v_p(r - n^m) = v_p(m \cdot (r/n^m - 1)) \geq v_p(N - 1)
\]
We let \( i = m \) in (37), use \( \sigma^m(\tau(\chi)) = \tau(\chi^r) \) and divide by (39). This is allowed, since \( \tau(\chi) \cdot \tau(\chi^{-1}) = \pm q \) and \( (q, n) = 1 \); the result is:
\[
\tau(\chi)^{n - m} = (\chi(r) \cdot \eta^{-m})^r \mod \mathfrak{R}.
\]
Let \( u \) be the largest divisor of \( (N - 1) \) which is coprime to \( p \). From (38), (40) and by raising (11) to the power \( u \), we get:
\[
1 = (\chi(r) \cdot \eta^{-m})^{r - u} \mod \mathfrak{R}.
\]
Now \( \rho = \chi(r) \cdot \eta^{-m} \in \mathbf{R} \) is a primitive root of unity of some order \( p^v \) and such that \( \rho \equiv 1 \mod \mathfrak{R} \). We claim that \( v = 0 \) and \( p = 1 \); if this was not the case, then
\[
p^v = \prod_{\rho = 1}^{p^v - 1}(1 - \rho) = \frac{X^{p^v - 1}}{X - 1} \bigg|_{X = 1}
\]
and since \( \rho \equiv 1 \mod \mathfrak{R} \), we should have a fortiori \( p^v \equiv 0 \mod \mathfrak{R} \) which contradicts \( (p, r) = 1 \). So \( p = 1 \) and thus \( \chi(r) = \eta^m = \eta^{\rho_p(r)} \).
This holds for all primes \( r \mid n \) and, by multiplicativity, for all divisors \( r' \mid n \). In particular, since \( l_p(n) = 1 \), it follows that \( \eta = \chi(n) \).

**Remark 8.** The equivalent relations (35) and (36) are reminiscent of the identity (37) holding in finite fields. The statement of the Lemma holds a fortiori when replacing (35) by
\[
\alpha^{(n^p - 1)/p^k} = \chi^{-t_p(n)}, \quad \text{with } \ t_p = \text{ord}_{p^k}(n),
\]
which is the analog of (33) and is obtained by iteration of (37). Here \( \alpha = J_p(\chi) \) like in the hypothesis above.
The Lemma \[2\] indicates the steps for proving the existence of \(s\)-th cyclotomic extensions with Jacobi sums. This is the corner stone of the Jacobi sum test:

**Corollary 1.** Suppose that \(s\) is square-free, \(t = \text{ord}_s(n)\) and \(R\) is a saturated \(t\)-th extension of \(N\) with \(\zeta \in R, \Phi_t(\zeta) = 0\). We let \((\mathbb{Z}/s \cdot \mathbb{Z})^\sim\) be the set of characters of conductor \(s\) with images in \(<\zeta>\), the sets \(\mathcal{P}, \mathcal{Q}\) be given by (3) and

\[
(44) \quad \mathcal{C} = \left\{ \chi_\varphi \in (\mathbb{Z}/s \cdot \mathbb{Z})^\sim : \varphi \in \mathcal{Q}, \chi \text{ has conductor } q \text{ and order } p^k \right\}.
\]

Suppose that

\[
(45) \quad \tau(\chi_\varphi)^{n-\sigma} \in <\chi_\varphi(n)>, \quad \forall \varphi \in \mathcal{Q}.
\]

or, alternately, for all \(\varphi \in \mathcal{Q}\) one has:

\[
(46) \quad \alpha_{\varphi}^{p^k \varphi} = \chi_\varphi(n)^{t_\varphi}, \quad \text{with } t_\varphi = \text{ord}_s(n) \text{ and } \alpha_{\varphi} = \tau(\chi_\varphi)^{p^k}.
\]

Then an \(s\)-th cyclotomic extension of \(N\) exists.

**Proof.** Using Lemma \[2\] respectively (43), we deduce from (44) or (46) that \(\chi(r) = \chi(n^{L(r)})\) for all the characters \(\chi \in (\mathbb{Z}/s \cdot \mathbb{Z})^\sim\). Let \(L(r) \equiv l_p(r) \mod p^k\) for all \(p^k \parallel t\); then we have a fortiori \(\chi(r) = \chi(n^{L(r)})\) for all \(\chi \in (\mathbb{Z}/s \cdot \mathbb{Z})^\sim\) and by duality, \(r \equiv n^{L(r)} \mod s\). This holds for all \(r|n\) which implies (II) and the fact that an \(s\)-th cyclotomic extension of \(N\) exists. \(\square\)

The conditions for existence of \(s\)-th cyclotomic extensions, which are based on Gauss sums, require \(s\) to be squarefree. This is not the case for the Lucas–Lehmer test in Theorem \[5\]. We wish to combine the information about extensions proved by the two methods. This happens to be quite easy, since the extensions proved by means of Theorem \[3\] are saturated and thus (8) holds by Lemma \[4\]. We group these observations in

**Fact 6.** Let \((s_1, s_2) = 1\) with \(s_2\) squarefree, \(s = s_1 \cdot s_2\) and \(t_i = \text{ord}_s(n), i = 1, 2, t = \text{ord}_s(n)\). Suppose that \((R, \sigma, \zeta)\) is a saturated \(t\)-th cyclotomic extension of \(N\) and \(t_1||[R:N]\). Furthermore there is a \(\beta \in R\) with \(\Phi_{s_i}(\beta) = 0\), and \(\beta^p = \sigma(\beta)\), such that \((R, \beta, \sigma)\) is saturated as a \(s_1\)-th extension. If the conditions of Corollary \[7\] apply for \(s = s_2\), then an \(s\)-th cyclotomic extension of \(N\) exists.

Furthermore, if \(s_i\) are any coprime integers such that saturated \(s_i\)-th cyclotomic extensions of \(N\) exist and \(s = \prod_i s_i\), then a saturated \(s\)-th extension exists.

**Proof.** Let \(r|n\) be a prime. The proof of Corollary \[10\] and the fact that the \(s_1\)-th extension is saturated imply, by means of Lemma \[12\] that \(\chi(r) = \chi(n^{L(r)})\) for all characters \(\chi \in (\mathbb{Z}/s \cdot \mathbb{Z})^\sim\) with \(L(r) \equiv l_p(r) \mod p^k\) and all \(p|t\). The statement about combinations of saturated extensions is a direct consequence of Lemma \[11\]. \(\square\)

### 5. Certification

Certificates for primality proofs are data collected during the performance of the test of primality for a given number \(n\). The certificate allows to perform a verification of the primality of \(n\) in (sensibly) less time than it took to collect the data. A recursive Pratt certificate \[29\] is the following: suppose that \(n = aF + 1\) is a prime and \(\prod_{i=1}^k P_i^{e_i} = F > \sqrt{n}\), with \(q_i = P_i^{e_i}\) being prime powers. Furthermore,
suppose that \( b_i \in \mathbb{Z} \) are such that \( \Phi_p(b_i) \equiv 0 \mod n \), or \( b_i \equiv c_i^{(n-1)/q_i} \mod n \), while \( (c_i^{(n-1)/p_i} - 1, n) = 1 \) and \( c_i^{(n-1)/p_i} \equiv 1 \mod n \). A certificate \( C(n) \) is defined recursively by

\[
C(n) = \{ b_i : i = 1, 2, \ldots, k \} \bigcup (\bigcup_{i=1}^k C(p_i))
\]

with \( C(p_i) \) being certificates for \( p_i \). If \( b_i \) are computed by trial and error, using the \( c_i \) above, the time for building a certificate is larger than the one required for its verification. This example suggests a generalization to CPP. We may mention that it was believed until recently that certification was an advantage of ECPP and not achievable for CPP. It is not the case, as we show here.

The relation 46 shows that if \( \chi(n) = 1 \) and an \( s \)-th cyclotomic extension does exist, then not only \( \tau(\chi) \in \mathbb{R} \) as follows from (IV), but it also can be explicitly computed in \( \mathbb{R} \) by means of the Theorem 4. This would provide for a certificate which can be verified by exponentiations with exponent \( p^k \) in \( \mathbb{R} \); however the list \( C \) contains also characters which do not vanish at \( n \). In such cases, one first modifies \( \alpha_\varphi \) accordingly before taking a \( p^k \)-th root.

The resulting criteria are given in

**Theorem 7.** Let \( s \) be squarefree, \( t = \text{ord}_s(n) \) and \( \mathbb{R} = \mathbb{N}[\zeta] \) be a saturated \( t \)-th cyclotomic extension. Let \( Q, P, C \) be defined in 3, 44 and suppose that for all \( \varphi \in Q \) there is a \( \beta_\varphi \in \mathbb{R} \) such that, for \( i \in \mathbb{Z} \) and \( \varphi, t = \text{ord}_p(n) \):

\[
\beta_\varphi^k = \chi_\varphi(n) \frac{\zeta^k}{\zeta^i-1} \cdot \alpha_\varphi, \quad \text{with} \quad t = \text{ord}_p(n) \quad \text{and} \quad \alpha_\varphi = \tau(\chi_\varphi)^{p^k}.
\]

Then an \( s \)-th cyclotomic extension of \( N \) exists.

**Proof.** If \( n \) is prime, then

\[
\left( \alpha_\varphi \cdot \chi_\varphi(n) \frac{\zeta^k}{\zeta^i-1} \right)^{\frac{n^k-1}{p^k-1}} = 1
\]

as a consequence of 33 and the expression \( \chi_\varphi(n) \frac{\zeta^k}{\zeta^i-1} \cdot \alpha_\varphi \) is in this case a \( p^k \)-th power in \( \mathbb{R} \), as follows from Theorem 4. The existence of \( \beta_\varphi \) is a necessary condition for primality and thus consistent with our purpose.

Since \( \mathbb{R} \) is saturated, \( S = \mathbb{R}[X]/(X^t - \zeta) \) is a \( t^2 \)-th cyclotomic extension and in particular galois with group of order \( t \cdot \text{ord}_t(n) \). We claim that \( S \) contains a primitive \( s \)-th root of unity \( \omega \) upon which \( \varphi \) acts making \( (S, \omega, \varphi) \) into an \( s \)-th cyclotomic extension in the sense of Remark 3. Our proof relies upon Theorem 4.

We first prove an auxiliary fact about saturation. Let \( \varphi \in Q \), let

\[
\delta_\varphi = \chi_\varphi(n)^{-\frac{i}{n^t-1}}
\]

and \( u = k_p(n) \) be the saturation exponent of \( p \) with respect to \( n \). Then there is an integer \( 0 \leq v < p^u \) such that

\[
\frac{t}{n^t-1} = \frac{v}{p^u} + m, \quad \text{with} \quad m \in \mathbb{Z},
\]

and hence

\[
(48) \quad \delta_\varphi \times \chi_\varphi(n)^{\frac{iv}{p^u}} \in \mathbb{R} \quad \text{and} \quad \delta_\varphi \in S.
\]

---

2We suppress here, for typographic reasons, writing out the explicite dependency on \( \varphi \).
Indeed, let \( t' = \text{ord}_p(n) \) so that \( v_p(n^{t'} - 1) = u \) and suppose \( \frac{t'}{n^{t'}} = \frac{u}{p^u} \mod \mathbb{Z} \).

The assertion follows for \( \varphi = (p^k, q) \) with \( k \leq u \); if \( k = u + j \), then by the definition of saturation, \( t = \text{ord}_{p^{u+j}}(n) = p^j \cdot t' \). Since

\[
\frac{n^{p^j \cdot t'} - 1}{p^j (n^{t'} - 1)} \equiv 1 \mod p^u,
\]

as shown by a short calculation, it follows that

\[
\frac{t}{n^{t'} - 1} = \frac{t'}{n^{t'} - 1} \times \left( \frac{p^j}{n^{p^j \cdot t'} - 1} \right) \equiv \frac{u}{p^u} \mod \mathbb{Z},
\]

thus proving the claim. Note that \( \gamma_\varphi = \beta_\varphi \cdot \delta_\varphi \) is a solution of \( X^{p^k} = \alpha_\varphi = \tau(\chi_\varphi)^{p^k} \).

Let as usual \( \xi_1, \xi_2 \in \mathbb{C} \) be fixed and \( \psi \in H^T(n) \) be a character of order \( p^u \) with image in \( \langle \xi_1, \xi_2 \rangle \), satisfying \( \psi(n) = 1 \) and let \( \chi \in (\mathbb{Z}/s \mathbb{Z})^\times \) be the image of \( \psi \) by \( \theta : \xi_i \mapsto \zeta_i \). We want to show that \( \theta \) can be extended to \( \tau(\psi) \). This is done as follows: \( a(\chi) = (\tau(\psi))^{p^k} \in \mathbb{Z}[\xi_i] \) so we can set \( \alpha(\chi) = \theta(a(\psi)) \in \mathbb{R} \) and then \( \mathbb{Z} \xi_i, \mathbb{Z} \psi(\mathbb{Z}) \subseteq \mathbb{Z}[\xi_i, X]/(X^{p^k} - a(\psi)) \). The map \( \theta \) extends to \( \tau(\psi) \) if we can show that the equation \( \tau^{p^k} = \theta(a(\chi)) \) has a solution in \( \mathbb{R} \). Furthermore, if this holds for any \( \varphi \in \mathbb{Q} \), we conclude that for each \( \psi \in H^T(n) \), the Gauss sum \( \tau(\psi) \) maps to \( \mathbb{R} \) and the claim then follows from (V). Now if \( \psi \in H^T(n) \) is a character of order \( m \), it can be decomposed in a product of characters \( \psi = \prod_{p^k \mid m} \psi_p \) of characters of prime power orders \( p^k \mid m \). The Gauss sum \( \tau(\psi) = J(\psi) \times \prod_{p^k \mid m} \tau(\psi_m) \) where we assumed that \( \theta(\tau(\psi_m)) \in \mathbb{R} \) and \( J(\psi) \) is a product of Jacobi sums which also maps to \( \mathbb{R} \).

Suppose that the prime decomposition of \( s \) is \( s = \prod_{i=1}^n q_i \) and define the factor characters \( \chi_i(x) = \chi(x \mod q_i) \); the decomposition formula (22) implies that \( \tau(\chi) = \prod_{i=1}^n \tau(\chi_i) \). By definition of \( P \), there are pairs \( \varphi_i = (p^{k_i}, q_i) \in P \) such that \( \chi_i = \chi_{\varphi_i} \). Using (19), we have \( (\tau(\chi_i))^{p^{k_i}} = (\beta_i \cdot \delta_i)^{p^{k_i}} \in \mathbb{R} \) and \( \beta_i = \beta_{\varphi_i} \), etc. Note that we have to raise to the power \( p^{k_i} \) in the previous formula, in order to consider elements which are defined in \( \mathbb{R} \); an alternative solution would be a formal adjunction of an \( s \)-th root of unity to \( \mathbb{R} \). The hypothesis \( \chi(n) = 1 \) and relation (18) imply that

\[
\tau(\chi)^{p^u} = \left( \prod_{i=1}^d \beta_i \right)^{p^u} \times \prod_{i=1}^d \chi_i(n)^{m_i \cdot p^u} = \left( \prod_{i=1}^d \beta_i \right)^{p^u} \cdot \chi(n)^{p^u} \cdot \prod_{i=1}^d \chi_i(n)^{m_i \cdot p^u} = \beta^{p^u},
\]

with \( m_i = \frac{t_i}{n^{t_i} - 1} - \frac{u}{p^u} \) and \( \beta = \prod_{i=1}^d \beta_i \cdot \chi_i(n)^{m_i} \in \mathbb{R} \). This shows that \( \theta(\tau(\psi)) \in \mathbb{R} \) as claimed, and completes the proof for odd \( p \) or \( p = 2 \) and \( n \equiv 1 \mod 4 \). If \( n \equiv 3 \mod 4 \) and \( p = 2 \), the saturation context is different. The proof uses an appropriate variant of (18) and shall be skipped here.

It is useful to note, that (14) substantially accelerates the evaluation of (16), making it comparable to the one of (15). As a consequence, computing a certificate requires no substantial additional work compared to the classical Jacobi sum test.

### 5.1. Computation of Jacobi Sums and their Certification

We are interested in the computation of Jacobi sums \( j(\chi, \chi^a) \), where \( \chi = \chi_\varphi \) is a character of prime
conductor $q$ and prime power order $p^k|(q - 1)$. For these sums, the absolute value is

$$j(\chi, \chi^a) \times \overline{j(\chi, \chi^a)} = q. \quad (49)$$

Since the conductor $q$ of Jacobi sums in CPP has superpolynomial size, their computation is a critical step which deserves some attention. From the theoretical point of view, the recent random polynomial algorithm of Ajtai, Kumar and D. Sivakumar [4] for finding shortest vectors in lattices solves the concrete problem in polynomial, and in fact linear time and space. Indeed, as we detail below, Jacobi sums of characters of order $P$ are shortest vectors in certain well rounded lattices, i.e. lattices with a base of vectors of equal length. In a lattice of dimension $P$, the algorithm [4] takes $O(2^P)$ space and time, and since in the context of CPP, the size $P = O(\log_2(P))$, it follows that Jacobi sums can be computed in random linear time.

In practice, the dimensions of lattices are quite small and in view both of constants and implementation complexity of the shortest vector algorithm, it is useful to discuss some simpler practical methods too.

For moderate values of $q$, possibly $q < 10^{14}$, the direct computation based on the definition (21) is adequate and fast. The bottleneck is the necessity to store a table of discrete logarithms modulo $q$. This can simply be avoided, by performing the computation of Gauss periods in $\mathbb{C}$, then computing Gauss and Jacobi sums in $\mathbb{C}$ too; finally, from the conjugates of a Jacobi sum, one recovers its coefficients as an algebraic integer. The method is straightforward and was implemented in the Master Thesis [22].

For larger conductors, it is preferable to use methods of lattice reduction. These have been investigated in [11], [23], [32] and are based on the following observation. Let $\mathfrak{Q} \subset \mathbb{Z}[\xi_p^k]$ be a prime ideal above $q$; note that the choice of $p$ implies that $q$ splits completely and $\mathfrak{Q}$ has inertial degree one. Let $G = \text{Gal}(\mathbb{Q}(\xi_p^k)/\mathbb{Q})$ and $I = \mathbb{Z}[G]$ be the Stickelberger ideal. There is an element

$$\theta = \sum_{(c,p) = 1; \ 0 < c < p^k} \left[ \frac{ac}{p^k} \right] \cdot \sigma_c^{-1} \in I \quad (50)$$

such that

$$\langle j(\chi, \chi^a) \rangle = \mathfrak{Q}^\theta,$$

for some $\sigma \in G$. The ideal $\mathfrak{Q}^\theta$ can be represented by a $\mathbb{Z}$ - base, being a free $\mathbb{Z}$ - module of rank $\varphi(p^k)$. As such, it is a lattice and it follows from (49) that $\sigma^{-1}(j(\chi, \chi^a)) \in \mathfrak{Q}^\theta$ is a shortest vector of this lattice, with respect to the embedding (Gauss) norm $\| x \| = \sum_{\sigma \in G} |\sigma(x)|^2$.

This opens the road for applications of methods of lattice reduction. Without entering in details, which can be found in the references, we mention that lattice reduction allows use of large conductors, but the growth of the order – which controls the dimension of the lattice – is critical. Indeed, the problem of finding the shortest vector in a lattice of dimension $d$ with initial base of vectors bounded by $q$ has complexity $O(d^d \cdot \log(q)^{O(1)})$. In practice, due in part to the particularity that the lattices to consider (generated by Jacobi sums) have a basis of shortest vectors – they are well rounded – the computations are quite efficient, and shortest vectors are frequently found directly by LLL, for character orders up to at least $P \sim 125$ [22].
A more efficient LLL based approach which works for small class numbers of the cyclotomic field $\mathbb{Q}(\zeta_p)$ follows the method used by Buhler and Koblitz in [11]: Let $\mathfrak{Q} \subset \mathbb{Q}(\zeta_p)$ be an ideal above the conductor $q$. If $h$ is the class number, then find by LLL a generator of $\mathfrak{Q}^h$ and compute Jacobi sum powers $j(\chi, \chi')^h$ by use of Stickelberger elements. If the generator of $\mathfrak{Q}^h$ is found correctly by LLL, then this method uses only one LLL computation for a given conductor and order.

Finally, the implementations of PARI for computing the structure of class and unit groups of number fields turned out to be very efficient in computing Jacobi sums too. The bottleneck there is the space requirement, since finding generators of principal ideals is based on building up all the information on class and unit groups. Here, we use the fact that multiple Jacobi sums have to be computed in the same field, so the field construction which is slower, happens only once.

Since the computations in $\mathbb{C}$ and the LLL based method are not guaranteed to yield Jacobi sums - the first due to rounding errors, the second due to the shortest vector problem - it is therefore interesting that one can certify very easily that the value of a Jacobi sum is correct, using the very formulae displayed above. This comes both as a verification and as part of a certificate for ultimate verifications of a primality proof. More precisely, we have:

**Lemma 3.** Let $\varphi = (p^k, q)$ with $p^k | (q - 1)$ and $p, q$ being primes. Let $0 < a < p^k$ be an integer and $\xi = \xi_{p^k} \in \mathbb{C}$ be fixed. If $\alpha \in \mathbb{Z}[\xi]$, there is a deterministic algorithm which verifies whether $\alpha = j(\chi, \chi^a)$ for some character $\chi \in (\mathbb{Z}/q \cdot \mathbb{Z})^\times$ of conductor $q$ and order $p^k$. The verification is done in $O(p^{2k} \cdot \log(q))$ binary operations.

**Proof.** A first condition which must be fulfilled by a Jacobi sum is the local $p$-adic norming condition $j \equiv \pm 1 \mod (1 - \xi)^2$, see e.g. [16], and this fixes the choice of a root of unity factor $\xi$. Thus one starts by verifying that

$$\alpha \times \overline{\alpha} = q, \quad \text{and} \quad \alpha \equiv \pm 1 \mod (1 - \xi)^2,$$

in $O(p^{k} \cdot \log(q))$ operations – note that the coefficients of a Jacobi sums have size $\sim \sqrt{q}$ and thus a multiplication of two Jacobi sums has the complexity above.

Since $q \equiv 1 \mod p^k$ there is a $c \in \mathbb{Z}$ with $\Phi_{p^k}(c) \equiv 0 \mod q$ and thus $\mathfrak{Q} = (\xi - c, q)$ is a prime ideal above $q$. Next one computes $\beta = (\xi - c)^\theta \in \mathbb{Z}[\xi]$ with $\theta \in I$ defined by (50). This is done in $O(p^{2k} \cdot \log(q))$ operations. Finally, one checks if there is a $\sigma \in G$ such that $\sigma(\beta) \equiv 0 \mod \alpha$. If yes, then $\alpha$ is a Jacobi sum and $\alpha = j(\chi, \chi^a)$ for some character of order $p^k$ and conductor $q$, otherwise the claim is false. \hfill \Box

### 6. Algorithms

The previous sections provide the theoretical foundation for the CPP primality proving algorithms. These consist of three steps, which are partially interdependent. Like usual, we denote by $n$ a number to be proved prime and $Q, P, C$ are defined by (3) and (44), respectively. The main steps of the algorithms are the following:

**A. Work Extensions:** Select two parameters $s, t$ such that $t = \text{ord}_s(n)$ and build a saturated $t-$th extension $R/\mathcal{N}$ – e.g. by using the Lucas – Lehmer method of Theorem 5

---

3The sign is always positive, if one adopts Lang’s definition of Gauss sum, with a minus sign.
B. Parameters: Let \( s'|(n^k - 1) \) be a totally factored part\(^4\) with \((s, s') = 1\), let \( s_1 \) be the order of a saturated \( s' \)-th extension – thus \( s_1 = \prod_{q'|s'} q^{k_p(n)} \) and \( S = s \cdot s_1 \). Verify \( S > \sqrt{n} \). An optimization cycle can lead back to \( A \). At the end, optimal values of \( S, s, s' \) and \( t \) are chosen and the fixed conditions \( S > \sqrt{n} \) and \( t = ord_s(n) \) hold.

C. Test part:

C1. Prove the existence of a saturated \( s' \)-th cyclotomic extension in \( \mathbb{R} \), by using the Theorem 5. This is the Lucas – Lehmer part of the test, and it can be void.

C2. Build the sets \( \mathcal{P}, \mathcal{C} \), with respect to the current value of \( s \) and verify (35) for all characters \( \chi \in \mathcal{C} \). This is the Jacobi sum part of the test.

C2’. Alternately, if a certificate is required along with the test, after building the list \( \mathcal{C} \), one finds \( \beta_0 \in \mathbb{R} \) verifying (47).

C3. Perform the final trial division, verifying that (1) yields no nontrivial factors of \( n \).

Unless \( n \) has some special form, so that many prime factors of \( F_k = n^k - 1 \) are known for small \( k|t \), the parameter \( s' \) is either set to 1 and thus neglected, or gained by investing some time in the factorization of the same \( F_k \). An important observation, which does not influence the asymptotic behavior of the algorithms but generates a useful speed up, consists of the fact that one can verify (35) simultaneously for a set of characters of mutually coprime orders.

**Definition 7.** We define an amalgam as a subset \( \mathcal{A} \subset \mathbb{Q} \) such that \( \{ p(\varphi) : \varphi \in \mathcal{A} \} \) are pairwise coprime. If \( \varphi = (q^k, q) \) and \( t(\varphi) = \text{ord}_{p^k}(n) \), then an amalgam \( \mathcal{A} \) is rooted, if there is a \( \varphi_0 \in \mathcal{A} \) such that \( t(\varphi)|t(\varphi_0) \) for all \( \varphi \in \mathcal{A} \).

The relevance of amalgams is provided by the following:

**Theorem 8.** Let \( \mathcal{A} \) be an amalgam,

\[
f = f(\mathcal{A}) = \prod_{\varphi \in \mathcal{A}} p^k(\varphi), \quad f' = \text{rad } f = \prod_{\varphi \in \mathcal{A}} p(\varphi), \quad t = \text{ord}_{p^k}(n),
\]

and \( (\mathbb{R}, \sigma, \zeta) \) a saturated \( f \)-th cyclotomic extension of \( N \), the roots \( \{ \zeta_p : p = p(\varphi), \varphi \in \mathcal{A} \} \subset \mathbb{R} \) being all saturated of orders \( p(\varphi) \). For \( \varphi \in \mathcal{A} \), let:

\[
\begin{align*}
\alpha(\varphi) &= J_{p^k(\varphi)}(\chi_{\varphi}) \quad \text{and} \\
\beta(\varphi) &= J_{\nu(\varphi)}(\chi_{\varphi}), \quad \text{where } \nu(\varphi) = n \text{ rem } p^k.
\end{align*}
\]

Let \( n = f \cdot l + \nu \) with \( 0 \leq \nu < f \) and \( \nu = p^{k(\varphi)} \cdot \lambda(\varphi) + \nu(\varphi) \), for \( \varphi \in \mathcal{A} \). Define \( \alpha \) and \( \beta \) by

\[
\begin{align*}
\alpha &= \prod_{\varphi \in \mathcal{A}} \alpha(\varphi)^{\nu(\varphi) + p^{k(\varphi)}} \in \mathbb{R} \quad \text{and} \\
\beta &= \prod_{\varphi \in \mathcal{A}} \alpha(\varphi)^{\lambda(\varphi)} \cdot \beta(\varphi) \in \mathbb{R}.
\end{align*}
\]

Suppose there is an \( \eta \in < \zeta_f > \) such that

\[
\alpha^l \cdot \beta = \eta^{-n}
\]

Then \( \chi(\varphi)(r) = \chi(\varphi)(n)^{l(\varphi)}, \ \forall r \ | n \) and \( \varphi \in \mathcal{A} \). Furthermore \( \eta = \prod_{\varphi \in \mathcal{A}} \chi(\varphi)(n). \)

\(^4\) It is assumed that \( s' \) is built up from primes \( q'|s' \) such that the orders \( t(q') = \text{ord}_{p^k}(n)|t \) are small.
Proof. The proof is similar to the one of the Lemma 2. We shall describe the general ideas and refer the reader to [23] for the complete proof. One first adds formal $q(\wp)$-th roots of unity to $R$ in order to define some Gauss sums which verify, by definition of $\alpha$ and $\beta$ and (54):

$$\prod_\wp \tau(\chi_\wp)^n = \eta(\wp)^{-n} \cdot \prod_\wp \tau(\sigma(\wp)). \quad (55)$$

Then one decomposes $\eta$ in a product of $p$-th power roots of unity and raising (55) repeatedly to the $n$-th power, obtains:

$$\prod_\wp \tau(\chi_\wp)^{n^h} = \prod_\wp \eta(\wp)^{-h \cdot n^h} \cdot \sigma^h(\tau(\chi_\wp)). \quad \forall h \geq 1. \quad (56)$$

Inserting $h = tf$, one has:

$$\prod_\wp \tau(\chi_\wp)^{n^{tf} - 1} = 1. \quad (57)$$

Let $r|n$ be a prime and $\mathfrak{R} \supset (r)$ a maximal ideal. By analogous steps to the proof of Lemma 2, one eventually shows that:

$$\prod_\iota \left( \frac{\chi_\wp(r)}{\eta^{m^i}} \right)^{ru} \equiv 1 \mod \mathfrak{R}. \quad (58)$$

Since $(ru, f) = 1$, we get $\prod_\wp \left( \frac{\chi_\wp(r)}{\eta^{m^i}} \right) = 1$. This product of roots of unity of coprime order can only be 1 if all factors are 1 and thus:

$$\chi_\wp(r) = \eta^{m^i}.$$

The rest of the statement follows by multiplicativity and using $l_{p(\wp)}(n) = 1. \quad \square$

7. Deterministic primality test

The Corollary 1, Theorem 5 and the certification - theorem 7 are used as bases for an explicit primality test, which proceeds by providing a proof of existence of an $s$-th cyclotomic extension of $N$ for some $s > \sqrt{n}$ such that $t = \text{ord}_s(n)$ is small, de facto $O\left( \log(n)^{\log_2(3)}(n) \right)$.

In all cases, the existence of saturated $p$-th extensions is required for all $p|t$. Such an extension or a proof of compositeness for $n$ can be gained in polynomial time, if one assumes the existence of some $p$-power non residues of small height [7] – existence which follows from the GRH. The versions of CPP based on this assumption are thus probabilistic Las Vegas algorithms; they shall be described with algorithmic details in a separate paper dealing with implementations.

The use of GRH is in the case of CPP explicite, in the sense that the failure to find the required non residues in the expected range together with an a posteriori proof of primality for $n$, which can be gained with a variety of methods, would yield a counterexample to the generalized Riemann hypothesis.

It is however of a certain theoretical interest, that one can prove also a deterministic version of the Jacobi sum test, one thus that does not relay upon the existence of saturated extensions. This version was proposed by Adleman, Pomerance and Rumely in [1] and adapted by Lenstra in his exposition [18]. Both sources present the deterministic algorithm as one which is independent of the Las Vegas variant of the Jacobi sum test, and are based on computation in exceedingly large extensions.
We give here an improved and simplified version, based on the ideas in [15]. Certainly, the question about the interest for this variant after the AKS test [3] must be addressed. In fact, provided the highly improbable event occurs, that the Las Vegas version is not sufficient, then the deterministic version of CPP may still be more efficient than AKS for larger numbers; this is due both to asymptotic behavior and mostly to space requirements which are very high for AKS. We add the theory for the deterministic variant here, for the sake of completeness.

Let thus, as usual, \( n \) be an integer to be tested for primality and \( s, t \) integers with \( t = \text{ord}_a(n) \) and set

\[
P = \{ p : t : p \text{ is a prime such that } p \mid \frac{q-1}{\text{ord}_a(p)} \text{ for some } q \text{ and no saturated } p-\text{th extension of } N \text{ is known } \}.
\]

Since the Jacobi sum method can be used for actually constructing \( p-\text{th} \) extensions, it follows that in the cases of interest for the deterministic version, the valuation \( v_p(n^{p-1} - 1) \geq 1 \) for odd \( p \) or \( n \equiv 1 \bmod 4 \) and \( v_p(n^{2} - 1) > 3 \) otherwise.

The deterministic test described in [15] generalizes the idea of the Rabin-Miller test. It gives an alternative version of [8] in the \( p \)-adic numbers \( \mathbb{Z}_p \). This leads to proving that the divisors \( r|n \) also lay in some cycles generated by a number \( \nu \bmod t \), which can be explicitly constructed: the structure of the criterium is similar to (II) in Theorem 2 replacing \( n \) by \( \nu \). Since in general, \( \nu \neq n \), the approach paradoxically suggests that no \( s-\text{th} \) cyclotomic extensions exist, according to Theorem 2.

We consider in depth the case when \( p \in P \) is odd or \( n \equiv 1 \bmod 4 \). The saturation index is in these cases \( k_p(n) = v_p(n^{p-1} - 1) \) and we shall assume that

\[
k_p(n) = \kappa + 1 > 1.
\]

For such \( p \) we let \( Q_p = \{ \varphi = (p^{k}(q), q) : q|s : p^{k}(q) \parallel (q - 1) \} \) and define \( k_m = \max_{\varphi \in Q_p} (k(q)) \). With this we fix \( \zeta = \zeta_{p^{k_m}} \) and for \( l < k_m \) we shall assume the compatibility conditions \( \xi^{p^{l+1}} = \zeta^{p^{l+1}} \). For a \( q|s \) let \( \xi = \xi_q \) be a root of unity, \( \Pi \subset G = \text{Gal}(\mathbb{Q}(\xi_q)/\mathbb{Q}) \) be the maximal \( p \)-group, \( H = G/\Pi \) and \( \eta_q = \sum_{\sigma \in H} \sigma(\xi) \).

We shall consider the rings

\[
R = \mathbb{Z}[\zeta]/(n \cdot \mathbb{Z}[\zeta]) \quad \text{and} \quad Q = \mathbb{Z}[\zeta, \eta]/(n \cdot \mathbb{Z}[\zeta, \eta_q]).
\]

Let \( \varphi = (p^{k}(q), q) \in Q_p \); \( n^{\varphi(p^{k}(q))} = u(q) \cdot p^{r+k(q)} \) with \( (u(q), p) = 1 \), and fix a character \( \chi = \chi_p : (\mathbb{Z}/q \cdot \mathbb{Z})^* \rightarrow \langle \zeta \rangle \). We assume that

\[
(\tau(\chi))^{n-\sigma_n} = \omega(\chi)^{-n} \in \zeta > 0
\]

(59)

where \( \sigma_n \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) : \zeta \rightarrow \zeta^n \), holds in \( Q \); a fortiori, if \( K = \mathbb{Q}(\zeta, \xi) : \mathbb{Q} \), we have \( (\tau(\chi))^{\frac{1}{n-\sigma_n}} \in \zeta > 0 \). Let \( \lambda_q(\chi) = \tau(\chi)u(q) \in Q \) and \( p = \begin{cases} p & \text{if } p \text{ is odd} \\ 4 & \text{otherwise} \end{cases} \).

With this, we define

\[
J_\chi \subset 1 + p\mathbb{Z}_p = \{ a \in 1 + p\mathbb{Z}_p : \lambda(\chi)^{a-\sigma_n} \in \zeta > 0 \}.
\]

By (39), we have \( n^{p-1} \in J_\chi \) and thus \( J_\chi \) is a non empty subgroup of \( U_1 = 1 + p\mathbb{Z}_p \). The structure of \( U_1 \) implies that \( J_\chi = (1 + p)^{\zeta_\pi} \) for a given, yet to determine, positive integer \( j \). By analogy to the Rabin-Miller test, we let \( a_i = 1 + p^i \in \mathbb{Z}_p \), build the sequence

\[
x_i(q) = (\lambda_q(\chi))^{a_i - \sigma_n}, \quad i = 0, 1, \ldots, k(q) + \kappa,
\]

and consider the following conditions:
D1 The halting condition \( x_{h(q)+\kappa} \in< \zeta > \) holds. This is the condition \([33]\) and is related to \([34]\).

D2 For \( j(q) = \min\{ i : x_i = 1 \}, x_{j-1} \in< \zeta >\), if \( j > 0 \).

D3 If \( \exists l \geq 0 : x_l \notin< \zeta >\), then for the maximal such \( l,\)
\[
(x_l - \xi_l^{k(q)}), \ n) = 1; \ h = 1, 2, \ldots, p^{k(q)}.
\]

The first two conditions are Rabin-Miller; by the third, the value of \( j(q) \) in the definition of \( J_X \) is the one determined in D2. This provides the information that will be used for combining tests.

We now show how this functions. Let \( \omega \in \mathbb{Q}_p \) and suppose \( r^{p-1} \notin J_X \). Then, with \( j = j(\omega) \) given by D2, there is an \( m \in \mathbb{Z}_p^* \) with \( 1 + p^{j-1} = r^{(p-1)m} \) and the relation \([32]\) implies:
\[
x_{j-1} = \tau(\chi)^{(p-1)m} = \chi(r)^{-m} \mod r\mathbb{R}.
\]

This contradicts condition D3 and thus:
\[
r^{p-1} \in J_X = (1 + p^j)^{m}.\]

We shall define \( j_p = \max\{ j(\omega) : \omega \in \mathbb{Q}_p \} \) and choose some \( q(p)\)s such that \( q \) gives rise to the maximal value of \( j \), so there is a \( \omega = (p^\alpha, q(p)) \in \mathbb{Q}_p \) with \( j(q) = j_p \). The condition D3 applied to this particular choice of \( \omega \) – which we shall also refer to as maximal pair \( \omega \in \mathbb{Q}_p \) – implies:
\[
r^{p-1} = (1 + p^{j_p})^{m}, \ \forall \ r | n, \ \text{with some} \ \mu_p(r) \in \mathbb{Z}_p^*.
\]

Of course, \((n^{p-1})^p = (1 + p^{j_p})^{m}_{p^p} \). If \( j_p = \kappa + 1 \), then \( r^{p-1} \in (n^{p-1})^{p} \) for all \( r | n \) and consequently, the condition \([35]\) is fulfilled. A saturated \( p \)-th extension exists – albeit, could not be constructed by the trial and error method of Theorem \([5]\). We deduce from \((61)\) a condition which is similar to the one in Lemma \([2]\).

**Lemma 4.** Notations being like above, we assume that \( n \equiv 1 \mod 4 \) if \( p = 2 \in P \). Suppose that the existence of \( \mu_p(r) \) in \((61)\) is proved by verifying D3 for a maximal \( \omega \in \mathbb{Q}_p \) for all \( p \in P \) and that for all \( \omega = (p^{k(q)}, q) \in \mathbb{Q}_p \), letting \( \chi = \chi_\omega : (\mathbb{Z}/q \cdot \mathbb{Z})^* \in< \zeta >\), the condition \([56]\) is verified. Then there is a character:
\[
\chi: J_X \rightarrow <\zeta > \ \text{with} \ \hat{\chi}(r) = \chi(r) \ \forall \ r | n.
\]

In particular, \( \hat{\chi}(n) = \chi(n) \).

**Proof.** We may assume that \( J_X = (a)^{\hat{\omega}} \) with \( a = (1 + p^j) \) and \( j \leq j_p \). Let us define \( \eta \in< \zeta > \) by the relation \( \eta^{-\omega}a = \lambda(\chi)^{a-\sigma_a} \in< \zeta >\) and fix the character \( \hat{\chi}: J_X \rightarrow< \zeta > \) by \( \hat{\chi}(a) = \eta \). If \( r | n \) is a prime, by \([32]\),
\[
(\tau(\chi)^{p-1})^{r^{p-1}} = \chi(r)^{-1} \mod r\mathbb{R},
\]
while setting \( r^{p-1} = (1 + p^j)^{m} = a^{\mu}'\), with the obvious definition of \( \mu' \) in dependence of \( \mu_p(r) \), yields
\[
(\tau(\chi)^{p-1})^{r^{p-1}} = \lambda(\chi)^{a^{\mu}'-\sigma_{a^{\mu}'}} = \lambda(\chi)^{(a-\sigma_a)(\sum_{i=0}^{\mu'-1} a^{\mu'-1} a^\sigma)} = (\eta^{-\omega}a^{\mu'}\mu'^{-1})^{a^{\mu'-1}}.
\]
Comparing the last two identities, we find:
\[
\chi(r)^{-1} \mod r\mathbb{R} = \eta^{-\mu'}ur{n-1} \mod r\mathbb{R}.
\]
From $(ur^{p−1}, p) = 1$ and Lemma 2 we have
\[ \chi(r)^{p−1} = r^{\nu'} = \tilde{\chi}(a)^{\mu'} = \tilde{\chi} \left( a^{\mu'} \right) = \tilde{\chi} \left( r^{p−1} \right). \]
Since $p − 1 \in \mathbb{Z}_p^*$, we also have $\chi(r) = \tilde{\chi}(r)$ and, by multiplicativity, $\tilde{\chi}(n) = \chi(n)$, which completes the proof. \hfill \Box

**Remark 9.** It is of practical relevance, to note that all computations can in fact be performed in the rings $\mathbb{R} = \mathbb{Z}[\zeta]/(n\mathbb{Z}[\zeta])$, by using multiple Jacobi sums. This is clear for the verification of \[(\mathbb{M}). \] In order to determine the value of $j$ in $D2$, one has to compute $(r(\chi)^{\mu−\sigma_n})$ for $a_i = 1 + p^j$, and this computation can also be completed in $\mathbb{R}$, by definition of the multiple Jacobi sum $J_\alpha(\chi)$.

Let us introduce the notation $\pi_2(p) = \{ q/s : \exists \varphi = (p^{k(q)}, q) \in \mathbb{Q}_p \}$ and $\pi_2(P) = \bigcup_{p \in P} \pi_2(p)$. We have the following deterministic test variant:

**Corollary 2.** Let the notations be like above and suppose that if $2 \in P$ then $n \equiv 1 \mod 4$. Suppose that for all $p \not\in P$ and $\varphi \in \mathbb{Q}$ with $\varphi = (p^{k}, q)$, the relation \[(\mathbb{M})\] holds and that the existence of the characters $\tilde{\chi}$ in Lemma 64 has been proved for all $\chi = \chi_\varphi, \varphi \in \mathbb{Q}_p$ and $p \in P$. For all $q/s$, let $\nu(q)$ be defined by
\[ \chi(\nu(q)) = \begin{cases} 
\chi(n) & \text{if } q \not\in \pi_2(P) \\tilde{\chi}_\varphi(1 + p^{j\mu}) & \text{for all } p \in P \text{ with } q \in \pi_2(p), \varphi = (p^{j\mu}, q) \in \mathbb{Q}.
\end{cases} \]

Let $\nu \in (\mathbb{Z}/s \cdot \mathbb{Z})^*$ be defined with the Chinese Remainder Theorem, by the congruences $\nu \equiv \nu(q) \mod q$ for all $q/s$. Then all divisors $r|n$ verify $r \equiv \nu \mod s$.

**Proof.** Let $r|n$ and $\chi_\nu$ be a character, with $\varphi = (p^{k}, q)$; if $p \not\in P$, then $\chi_\nu(\nu) = \chi_\nu(n)$ and $\chi(r) = \chi(\nu)^{\nu(r)}$, as a consequence of Corollary 1. If $\varphi \in \bigcup_{p \in P} \mathbb{Q}_p$, then the proof of Lemma 4 implies that $\chi(r) = \chi(\nu)^{\nu(r)}$. By choosing
\[ m \equiv \begin{cases} 
\mu_p(r) \mod p^{\nu_p(t)} & \text{if } p \in P \\
\nu_p(r) \mod p^{\nu_p(t)} & \text{otherwise,}
\end{cases} \]
we find that $\chi(r) = \chi(\nu)^m$ for all characters $\chi \in (\mathbb{Z}/s \cdot \mathbb{Z})^\top$. By duality it follows that $r \equiv \nu^m \mod s$ as claimed. \hfill \Box

We shall sketch now the case $p = 2$ and $n \equiv 3 \mod 4$. As suggested by saturation, we consider here $n^2 − 1$ instead of $n − 1 = n^{p−1} − 1$ and note that $\mathbb{Z}_p^* = \mathbb{Z}_2^* \times 5^{\mathbb{Z}_2}$ is not cyclic any more. For all $q$, one defines like before the characters $\chi = \chi_\varphi$ and determines $J_\chi \subset \mathbb{Z}_2^*$. If $J_\chi \neq \{ n^2 \}$, then $n$ is composite, while for the remaining cases one can define characters $\tilde{\chi}$ and show eventually that an $s$–th cyclotomic extension of $\mathcal{N}$ exists. There are some technical obstructions \[(\mathbb{M}), \] resulting from the fact that in a first step, only $\tilde{\chi}^2$ is naturally defined and $\tilde{\chi}$ having a power of 2, there is an ambiguity in its definition. The condition D3 has to be modified and the ambiguity is removed by considering a $\rho \in (\mathbb{Z}/s \cdot \mathbb{Z})^*$ with $\rho^2 = 1$ and showing that the possible divisors $\nu|n$ belong this time to the set $\{ \rho^{k}, \rho^{k} \mod s : k = 1, 2, \ldots, t \}$, with $\nu$ defined like in the Corollary. We refer to \[(\mathbb{M}), \] for details.
8. Asymptotics and run times

In this section we evaluate the asymptotic expected run-time of the cyclotomy test. We shall use, for ease of notation, the symbol $P$ for the set of all rational primes. The following theorem is well-known in the context of primality tests [1], [2].

**Theorem 9** (Prachar, Odlyzko, Pomerance). There exists an effectively computable positive constant $c$ such that $\forall \ n > e^c, \ \exists t > 0$ satisfying

$$t < (\log n)^{c \cdot \log(3)(n)} \quad \text{and} \quad f(t)^2 = \left( \prod_{\{g \in P, (g-1)|t\}} q \right)^2 > n. \quad (63)$$

Heuristics indicate that the expected value of $c > \log(e)/\log(4)$ and the Theorem shows that one can choose, $(t, s = f(t))$ in the given range, and then the existence of an $s$-th cyclotomic extension can be proved in time polynomial in $t$. The claim follows from (II) of Theorem 2. More precisely, if the existence of an $s$-th cyclotomic extension is proved by \((65)\), then this relation should be proved for all pairs $
abla = (p^k, q) \in \mathbb{Q}$, as defined in Corollary 1. The verification of \((65)\) for one fixed $
abla$ takes $O^\sim (p^k \cdot \log(n)^2)$ binary operations – with the standard $O^\sim$ notation, in which factors that are polynomial in $\log(p), \log_2(n)$ are neglected. We would wish to deduce some upper bounds on $p^k, q$ and $\nabla \mathbb{Q}$ using the above Theorem. From the prime number Theorem, if $1 < c$ is such that $\pi(X) < c \cdot \frac{X}{\log(X)}$ for all $X > e^c$, we have the estimate

$$\prod_{p^f < c \cdot \log(X)} p^f > X^{1/2},$$

for all $X > e^c$, where $p^f$ are prime powers. Conversely, if $g(X) = \prod_{p^f < c \cdot \log(X)} p^f$ and $h(Y) = \min\{X : g(X) > Y^{1/2}\}$, the estimate implies:

$$h(Y) < c \log(Y) \quad \text{and} \quad \pi(h(Y)) < c^2 \frac{\log(Y)}{\log_2(Y)}, \quad \forall \ Y > 9. \quad (64)$$

From this and $q < t$ we deduce that $d(s) < \frac{\log(n)}{\log_2(n)}$, where $d(s)$ – the number of factors of $s = f(t)$ – is equal to the number of distinct primes $q$ in the list of pairs $\mathbb{Q}$. We shall assume here that it is possible to build $t = \prod_{p^k < B} p^k$ as the product of the first prime powers such that $f(t) > \sqrt{n}$. This is a hypothesis and not a consequence of Theorem 9. If this holds, it follows from (64) that for $\nabla = (p^k, q) \in \mathbb{Q}$ we have $p^k < c^2 \log_2(n)$. Altogether,

$$\nabla \mathbb{Q} < c^3 \log(n), \quad p^k < c^2 \cdot \log_2(n). \quad (65)$$

We have the following

**Fact 7.** Let $n, s$ be coprime integers with $n > s > \sqrt{n}$ squarefree. There is a probabilistic Las Vegas algorithm which requires $O^\sim (\log(n)^3)$ binary operations for proving the existence of an $s$-th cyclotomic extension. The algorithm generates a certificate for the existence of such extension and the certificate can be verified, together with the validity of the Jacobi sums, in $O^\sim (\log(n)^3)$ binary operations.

**Proof.** The proof follows directly from (65) and the description of the algorithm in Section 6. Building up the saturated working extensions for all primes $p|t$ takes
\( \mathcal{O}^\sim (\log(n)^2) \) operations and in the certificate generation phase, one has to perform an exponentiation with exponents \( O(n) \) in extensions of small degree \( (O(\log_2(n))) \), for each of \( \varphi \in \mathbb{Q} \): this leads to the claimed run time \( \mathcal{O}^\sim (\log(n) \times \log(n)^2) \). The certification requires merely exponents of size \( O(\log_2(n)) \), which explains the verification time, given the fact that certification of Jacobi sums is negligible by Lemma 3.

The operations using superpolynomial time in the CPP primality proofs are quite elementary: they are the computation of \( (2 \cdot \# \mathcal{Q}) \sim \log(n) \) multiple Jacobi sums and the test that \( n \mod (n^k \text{ rem } s) \neq 0 \) for \( k = 2, 3, \ldots, t - 1 \). Both operations take \( \mathcal{O}^\sim (t \log(n)) \) binary operations, and only the final test is specific for \( n \); the Jacobi sums can be reutilized for numerous test and it is conceivable to store large tables of precomputed sums. Although \( t \) and \( \log(n) \) are of different orders of magnitude, we specified the explicit factor \( \log(n) \) for obvious reasons: the exponent of \( \log(n) \) in the upper bound for \( t \) diverges so slowly, that it is indicative to know by what polynomial factor \( t \) is multiplied.

**Remark 10.** We only estimated the certificates for the existence of \( s \)-th cyclotomic extensions. The existence of such an extension does not grant primality, and one still has to perform the final trial divisions, requiring a superpolynomial amount of operations, and for which we did not provide any possible certification. The interest of CPP certification would be thus rather theoretical, without a method to circumvent completely.

Such a method is described in [26], in connection with dual elliptic primes and a new algorithm which intimately combines CPP with ECPP. This combination yields a random cubic time primality test with certificates that can be verified in quadratic time, being thus the fastest general primality test up to date. Like the Atkin version of ECPP, the run time estimates are based on some heuristics.

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