A NOTE ON FC-NILPOTENCY

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Abstract. The notion of bounded FC-nilpotent group is introduced and it is shown that any such group is nilpotent-by-finite, generalizing a result of Neumann on bounded FC-groups.

1. Introduction

A group in which every element has only finitely many conjugates is called a finite conjugacy group, FC-group for short. Of course, in particular all abelian groups and also all finite groups are FC-groups but there are many more non-trivial examples. The study of this class of groups was initiated by Baer [1] and Neumann [6], and its general theory was strongly developed during the second half of the last century, see [8].

Numerous variations of the notion of FC-group have been considered to study structural properties of infinite groups with some finiteness condition. As a strengthening, Neumann considered FC-groups with a uniform bound on the size of the conjugacy classes, known as bounded FC-groups, and showed that these are finite-by-abelian [6]. On the other hand, Haimo [3] and later Duguid and McLain [2] analyzed FC-nilpotent and FC-solvable groups, which are natural generalizations of nilpotent and solvable groups respectively to the FC context. For instance, a group is FC-solvable of length $n$ if it admits a finite chain of length $n$ of normal subgroups whose factors are FC-groups. Similarly, one can define the notion of FC-nilpotent, see Definition 2.6 for a precise definition. Furthermore, Hickin and Wenzel have shown in [4] that, like for normal nilpotent groups, the product of two normal FC-nilpotent subgroups is normal and again FC-nilpotent.

In this paper we aim to study a suitable version of bounded FC-nilpotency, see again Definition 2.6 and to show that these groups are exactly the nilpotent-by-finite ones. This result generalizes the one of Neumann on bounded FC-groups and another of Duguid and McLain asserting that finitely generated FC-nilpotent groups are nilpotent-by-finite. Furthermore, bounded FC-nilpotent groups appear naturally in the study of groups in model theory such as in $\aleph_0$-categorical groups and groups definable in simple (or even

2000 Mathematics Subject Classification. 20F19, 20F24.

Key words and phrases. nilpotent-by-finite; FC-nilpotent; bounded FC-group.

The second author was partially supported by the project MTM2014-59178-P.
wider families of) first-order theories. For instance, in these cases every definable FC-nilpotent group is indeed bounded. However, such groups are typically not finitely generated and therefore, the aforementioned result of Duguid and McLain cannot a priori be applied to deduce that these groups are nilpotent-by-finite.

The result presented here generalizes some previous cases due to Wagner [9, Proposition 4.4.10] for groups in simple theories, as well as in [5] for groups satisfying a uniform chain condition on centralizers up to bounded index. Our proof involves some machinery on FC-centralizers recently obtained by the first author in [5] using techniques from model theory. Finally, concerning the FC-solvable case note that the situation is more straightforward and an easy argument is given at the end of the paper.

2. Bounded FC-nilpotent groups

Given a group $G$ and a subset $X$ of $G$, we denote by $C_G(X)$ the elements of $G$ that commute with every element in $X$. Moreover, considering a normal subgroup $N$ of $G$, we denote by $C_G(g/N)$ the elements of $G$ in the preimage of $C_{G/N}(gN)$ under the usual projection.

We recall the definition of an FC-centralizer due to Haimo [3] and related notions, which play an essential role along the paper.

**Definition 2.1.** A subgroup $H$ of $G$ is contained up to finite index in another subgroup $K$ if $H \cap K$ has finite index in $H$. We denote this by $H \lesssim K$. Then $H$ and $K$ are commensurable, denoted by $H \sim K$, if $H \lesssim K$ and $K \lesssim H$.

Observe that $\lesssim$ is a transitive relation among subgroups of $G$, and that $\sim$ is an equivalence relation.

**Definition 2.2.** Let $G$ be a group and let $K, H, N$ be subgroups of $G$ with $N$ normalized by $H$. The FC-centralizer of $H$ modulo $N$ in $K$ is defined as

$$FC_K(H/N) = \{ k \in N_K(N) : H \sim C_H(k/N) \}.$$ 

If $N$ is trivial it is omitted.

In other words, the group $FC_K(H/N)$ consists of the elements $k$ in $N_K(N)$ such that $[H : C_H(k/N)]$ is finite, i.e. $k^H/N$ is finite. Moreover, observe that $G$ is an FC-group if $G = FC_G(G)$. As pointed out in the introduction a priori an FC-group may have arbitrarily large conjugacy classes. Those FC-groups in which there is a natural number bounding the size of any conjugacy class are called bounded FC-groups, and are precisely finite-by-abelian groups [6, Theorem 5.1].

The following definition generalizes the notion of bounded FC-group to arbitrary FC-centralizers.
Definition 2.3. Let $G$ be a group and let $H, K, N$ be subgroups of $G$ with $K$ normalized by $H$. We say that $\text{FC}_K(H/N)$ is bounded if there exists a natural number $n$ such that

$$\text{FC}_K(H/N) = \{g \in N_K(N) : |H/C_H(g/N)| \leq n\}.$$ 

FC-centralizers for definable groups and bounded FC-centralizers have been studied by the first author in [5] who has shown that some behaviors of the ordinary centralizers can be generalized to FC-centralizers. This is exemplified in the following two lemmata which can be found as [5, Theorem 2.10] and [5, Theorem 2.18]. For the former we give a proof which is a simple adaptation of the non-definable version to the bounded case.

Fact 2.4 (Symmetry). Let $H$ and $K$ be two subgroups of a group $G$ and $N$ be a subgroup of $G$ that is normalized by $H$ and $K$. Assume further that the FC-centralizer of $\text{FC}_H(K/N)$ is bounded. If $H \preccurlyeq \text{FC}_G(K/N)$, then $K \preccurlyeq \text{FC}_G(H/N)$.

Proof. Assume that $H$ and $\text{FC}_H(K/N)$ be commensurable and let $d$ be the natural number such that for any element $h$ in $\text{FC}_H(K/N)$ we have that

$$|H/C_H(h/N)| \leq d \quad (*) .$$

Now suppose that $K$ and $\text{FC}_K(H/N)$ are not commensurable. In this case, we can choose elements $k_0, \ldots, k_d$ in different cosets of $\text{FC}_K(H/N)$ in $K$. Thus, the index $[H : C_H(k_ik_j^{-1}/N)]$ is infinite. Since no group can be covered by finitely many cosets of subgroups of infinite index by a well-known theorem of Neumann [7], there are infinitely many elements $\{h_i\}_{i \in \mathbb{N}}$ in $H$ such that for all natural numbers $s \neq t$ and $i \neq j \leq d$, we have that $[h_s, h_t^{-1}, k_ik_j^{-1}] \notin N$. Thus, the elements $k_0, \ldots, k_d$ witness that $[K : C_H(h_sh_t^{-1}/N)] > d$. Hence, by $(*)$ none of the elements $h_ish_t^{-1}$ can belong to $\text{FC}_H(K/N)$, and whence $H$ and $\text{FC}_H(K/N)$ are not commensurable, contradicting our initial assumptions. \hfill $\square$

Fact 2.5. Let $G$ be a group and let $H$ and $K$ be two subgroups of $G$ such that $H$ is normalized by $K$. If $H = \text{FC}_H(K)$, $K = \text{FC}_K(H)$, and $\text{FC}_K(H)$ is bounded, then the commutator subgroup $[H, K]$ is finite.

Notice that the latter generalizes the aforementioned result due to Neumann on bounded FC-groups.

Definition 2.6. A group $H$ is FC-nilpotent of class $n$ if there exists a finite chain of normal subgroups

$$\{1\} \leq H_1 \leq H_2 \leq \cdots \leq H_n = H$$

such that $H_{i+1}$ is contained in $\text{FC}_H(H/H_i)$. We say that $H$ is bounded FC-nilpotent if additionally each $\text{FC}_{H_{i+1}}(H/H_i)$ is bounded.
FC-nilpotent groups were introduced by Haimo [3], who studied their basic properties. Duguid and McLain [2] Corollary 1] proved that finitely generated FC-nilpotent groups of class $n$ are (nilpotent of class $n$)-by-finite.

**Remark 2.7.** A nilpotent-by-finite group is bounded FC-nilpotent.

**Proof.** Let $H$ be a nilpotent-by-finite group and choose $N$ a normal nilpotent finite index subgroup of $H$. Let $n$ be the nilpotency class of $N$ and $k$ be the index of $N$ in $H$. It is clear that $Z_{i+1}(N)$ is contained in $FC_H(H/Z_i(N))$ and that

$$[H : C_H(x/Z_i(N))] \leq k$$

for any $x \in Z_{i+1}(N)$. Then

$$\{1\} \leq Z(N) \leq \cdots \leq Z_n(N) \leq H$$

witnesses that $H$ is a bounded FC-nilpotent group. \hfill \Box

The following result yields the equivalence between bounded FC-nilpotent groups and nilpotent-by-finite ones.

**Theorem 2.8.** Any bounded FC-nilpotent group of class $n$ is (nilpotent of class $2n$)-by-finite.

**Proof.** Let $N$ be a bounded FC-nilpotent group of class $n$, and let

$$\{1\} = N_0 \leq N_1 \leq \cdots \leq N_n = N$$

be a series witnessing this. Now, we find recursively on $i \leq n$ a subgroup $F_i$ of finite index in $N$ and a finite chain

$$\{1\} = H^i_0 \leq H^i_1 \leq \cdots \leq H^i_{2i}$$

of normal subgroups of $F_i$ such that $H^i_{j+1}/H^i_j$ is central in $F_i/H^i_j$ for $j < 2i$ and $F_i \cap N_i$ is contained in $H^i_{2i}$. Once this process is concluded, the subgroup $F_n$ is nilpotent of class at most $2n$ and has finite index in $N$.

We start the construction by setting $H^i_0 = \{1\}$ and $F_0 = N$. To continue, let $i > 0$ and assume that we have already defined $H^{i-1}_0, \ldots, H^{i-1}_{2(i-1)}$ and $F_{i-1}$. Now, we work in the quotient $\tilde{N} = F_{i-1}/H^{i-1}_{2(i-1)}$ and consider the subgroup $\tilde{N}_i = N_i \cap F_{i-1}/H^{i-1}_{2(i-1)}$. As $N_{i-1} \cap F_{i-1}$ is contained in $H^{i-1}_{2(i-1)}$ and $N_i = FC_{N_i}(N_i)$ is bounded, we have that $\tilde{N}_i = FC_{\tilde{N}_i}(\tilde{N})$ and that it is bounded. Thus Fact 2.4 yields that $\tilde{N} \sim FC_{\tilde{N}}(\tilde{N}_i)$.

Let $\tilde{N}^*$ be equal to $FC_{\tilde{N}}(\tilde{N}_i)$, a subgroup of finite index in $\tilde{N}$, and let $\tilde{N}^*_i$ be equal to $\tilde{N}_i \cap \tilde{N}^*$. Now, as $\tilde{N}^*_i$ has also finite index in $\tilde{N}_i$, we clearly have that $\tilde{N}^* = FC_{\tilde{N}^*_i}(\tilde{N}^*_i)$ and moreover

$$\tilde{N}^*_i = FC_{\tilde{N}_i}(\tilde{N}) \cap \tilde{N}^* = FC_{\tilde{N}^*_i}(\tilde{N}^*)$$

As the latter remains bounded, Fact 2.5 yields that the group $X$ defined as $[\tilde{N}^*, \tilde{N}^*_i]$ is finite. Note that all considered groups are normal in the ambient
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Hence $X$ is contained in $N^*_i = FC_{N^*_i}(N^*)$ and whence $C_{N^*_i}(X)$ has finite index in $N^*_i$.

Finally, set $F_i$ to be the preimage of $C_{N^*_i}(X)$ in $N$, which as well has finite index in $N$, the group $H_{2i-1}^i$ to be the preimage of $X \cap C_{N^*_i}(X)$ in $N$, and $H_{2i}^i$ to be the preimage of $N^*_i \cap C_{N^*_i}(X)$ in $N$. Moreover, for $j < 2i - 1$, we let $H_j^i$ be $H_{j-1}^i \cap F_i$. Note that by construction, the subgroups $H_j^i$ for $j \leq 2i$ are contained in $F_i$. Then the sequence

$$\{1\} = H_0^i \leq H_1^i \leq \cdots \leq H_{2i}^i$$

together with $F_i$ is as desired. □

**Remark 2.9.** Observe that our proof yields that each quotient $H_{2i+1}^i$ modulo $H_{2i}^i$ is finite. Thus, the group $N$ has a finite index nilpotent subgroup of class $2n$, which admits a series of length $n$ where each factor is (finite central)-by-central.

### 3. Bounded FC-solvability

We say that a group is *bounded FC-solvable* of length $n$ if it is an FC-solvable group of length $n$ in which each factor witnessing the FC-solvability is a bounded FC-group. Thus, by Neumann’s Theorem each of these factors is finite-by-abelian. Moreover, for any finite-by-abelian group $H$, the characteristic subgroup $C_H(H')$ is nilpotent of class two and has finite index. Thus, such a group is nilpotent-by-finite. Using this, we can easily show by induction on the FC-solvability length that a bounded FC-solvable group is solvable-by-finite:

More precisely, let

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd \cdots \lhd G_n = G$$

be a series that witnesses that $G$ is a bounded FC-solvable group. Then, the group $G_{n-1}$ is FC-solvable of smaller length and hence solvable-by-finite. Thus $G$ is a solvable-by-finite-by-abelian group. So, as any finite-by-abelian group is a nilpotent-by-finite group, we can conclude that $G$ is solvable-by-finite.

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