THE SIZE OF COEFFICIENTS OF CERTAIN POLYNOMIALS RELATED TO THE GOLDBACH CONJECTURE

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ABSTRACT. Recent work of Borwein, Choi, and the second author examined a collection of polynomials closely related to the Goldbach conjecture: the polynomial $F_N$ is divisible by the $N$th cyclotomic polynomial if and only if there is no representation of $N$ as the sum of two odd primes. The coefficients of these polynomials stabilize, as $N$ grows, to a fixed sequence $a(m)$; they derived upper and lower bounds for $a(m)$, and an asymptotic formula for the summatory function $A(M)$ of the sequence, both under the assumption of a famous conjecture of Hardy and Littlewood. In this article we improve these results: we obtain an asymptotic formula for $a(m)$ under the same assumption, and we establish the asymptotic formula for $A(M)$ unconditionally.

1. INTRODUCTION

Let $R(n)$ denote the number of representations of $n$ as the sum of two odd primes. That is, $R(n)$ is the number of ordered pairs $(p, q)$ of odd primes satisfying $p + q = n$. Of course $R(n) = 0$ when $n$ is odd, while the celebrated Goldbach conjecture is equivalent to the statement that $R(n) \geq 1$ for all even integers $n \geq 6$. Subsequently, define

$$a(m) = \sum_{d|m} R(d);$$

these quantities are closely related to a sequence of polynomials, which we describe shortly, that have a surprising connection to the Goldbach conjecture. Also define

$$A(M) = \sum_{m=1}^{M} a(2m) = \sum_{m=1}^{2M} a(m),$$

a summatory function that encodes the average behavior of $a(m)$.

The purpose of this paper is to establish two theorems concerning the sizes of $A(M)$ and $a(m)$ that improve results obtained by Borwein, Choi, and the second author in [1]. The first of these theorems is an asymptotic formula for $A(M)$.

**Theorem 1.** For all $M \geq 3$,

$$A(M) = \frac{\pi^2 M^2}{3 \log^2 M} + O \left( \frac{M^2 \log \log M}{\log^3 M} \right).$$

We emphasize that this theorem is unconditional; by contrast, the authors of [1] established this asymptotic formula without an explicit error term, but only under the assumption of a well-known conjecture on the number of Goldbach representations of an integer $n$:

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Conjecture 2 (Hardy and Littlewood [2]). As \( n \) tends to infinity,
\[
R(2n) \sim 2C_2 \frac{n}{\log^2 n} \prod_{p \mid n} \frac{p - 1}{p - 2},
\]
where \( C_2 \) is the twin primes constant
\[
C_2 = \prod_{p > 2} \left( 1 - \frac{1}{(p - 1)^2} \right).
\]
The authors of [1] do obtain an unconditional lower bound on \( A(M) \), namely
\[
A(M) \geq M \log M + O(M).
\]
This lower bound required the use of a deep result of Montgomery and Vaughan [3] on the exceptional set in the Goldbach conjecture, while our proof of Theorem 1 is elementary, with the deepest ingredient being the prime number theorem. The surprising gap between the asymptotic formula in Theorem 1 and the lower bound (2) can be explained by the fact that the authors of [1] actually prove the much stronger result
\[
2M \sum_{m=1}^{2M} \# \{ d \mid m: R(d) \geq 1 \} \geq M \log M + O(M),
\]
which does indeed imply (2), since the summand on the left-hand side is at most \( \sum_{d \mid m} R(d) = a(m) \).

Our second theorem, an asymptotic formula for \( a(m) \) conditional on the aforementioned conjecture of Hardy and Littlewood, is best stated after defining the following multiplicative function.

**Definition 3.** \( J(m) \) is the multiplicative function given by the following formula: if \( 2^k \parallel m \), then
\[
J(m) = \left( 2 - \frac{1}{2^k} \right) \prod_{p \parallel m \atop p > 2} \left( 1 - \frac{2}{p^{\ell + 1}} \right) \left( 1 - \frac{2}{p} \right)^{-1}.
\]
Here, \( p^\ell \parallel m \) means that \( p^\ell \mid m \) but \( p^{\ell + 1} \nmid m \).

**Theorem 4.** If Conjecture 2 is true, then
\[
a(2m) \sim \frac{2C_2 J(m)m}{\log^2 m}
\]
as \( m \) tends to infinity.

The authors of [1] were able to derive from Conjecture 2 the upper and lower bounds
\[
\frac{2C_2 m}{\log^2 m} \leq a(2m) \leq \frac{4C_2 m}{\log^2 m} \prod_{p \parallel m \atop p > 2} \left( 1 - \frac{2}{p^{\ell + 1}} \right) \left( 1 - \frac{2}{p} \right)^{-1} ;
\]
we are able here to close the small gap between these bounds.

The function \( a(m) \) and the Goldbach conjecture are linked via the sequence of polynomials
\[
F_N(z) = \sum_{k=0}^{N-1} \left( \sum_{n=1}^{N-1} \chi_p(n) z^{kn} \right)^2,
\]
where
\[ \chi_p(n) = \begin{cases} 1, & \text{if } n \text{ is an odd prime}, \\ 0, & \text{otherwise}. \end{cases} \]

For example,
\[
F_{10}(z) = 9 + (z^7 + z^5 + z^3)^2 + (z^{14} + z^{10} + z^6)^2 + (z^{21} + z^{15} + z^9)^2 \\
+ (z^{28} + z^{20} + z^{12})^2 + (z^{35} + z^{25} + z^{15})^2 + (z^{42} + z^{30} + z^{18})^2 \\
+ (z^{49} + z^{35} + z^{21})^2 + (z^{56} + z^{40} + z^{24})^2 + (z^{63} + z^{45} + z^{27})^2.
\]

It is not hard to see that for \( m \geq 1 \), the coefficient of \( z^m \) in \( F_N(z) \) is a nonnegative integer that is at most \( a(m) \), and in fact it equals \( a(m) \) for all \( N \geq m \). For example, when expanded out
\[ F_{10}(z) = 9 + z^6 + 2z^8 + 3z^{10} + \ldots + z^{126}, \]
reflecting the first ten values \( (a(1), \ldots, a(10)) = (0, 0, 0, 0, 1, 0, 2, 0, 3) \).

In other words, the sequence of polynomials \( F_N(z) - F_N(0) \) converges coefficient-wise to the fixed formal power series \( \sum_{m=1}^{\infty} a(m)z^m \).

Letting \( \Phi_k(z) \) denote the \( k \)th cyclotomic polynomial as usual, the authors of [1] show that \( F_{2N}(z) \) is divisible by \( \Phi_{4N}(z) \) for every positive integer \( N \). Experimental evidence suggests:

**Conjecture 5** (Borwein, Choi, and Samuels). *For every integer \( N \geq 3 \), the polynomial \( F_{2N}(z)/\Phi_{4N}(z) \) is irreducible in \( \mathbb{Z}[z] \).*

The relationship between \( F_N \) and the Goldbach conjecture is more than superficial, however, as the following startling theorem displays:

**Theorem 6** (Borwein, Choi, and Samuels). \( \Phi_N(z) \) divides \( F_N(z) \) if and only if there is no representation of \( N \) as the sum of two odd primes. In particular, Conjecture 5 implies the Goldbach conjecture.

### 2. Proofs of our results

We begin by proving Theorem 1, although first we need to devote some time to a technical lemma that counts the number of pairs of primes whose sum lies below a given bound. Afterwards, we derive Theorem 4 from Proposition 8 below.

In order to establish Theorem 1, we must first study the function
\[ Q(x) = \sum_{p+q \leq x} 1, \]

where \( p \) and \( q \) always denote primes in this paper.

**Lemma 7.** *Uniformly for \( x \geq 3 \),
\[
Q(x) = \frac{x^2}{2 \log^2 x} + O \left( \frac{x^2 \log \log x}{\log^3 x} \right).
\]

Proof. We begin by writing

\[ Q(x) = \sum_{p \leq x} \pi(x - p) = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \pi(x - p) + O \left( \sum_{p \leq x/\log x} \pi(x - p) + \sum_{x - \sqrt{x} \leq p \leq x} \pi(x - p) \right). \]

Trivially \( \pi(x - p) \leq \pi(x) \leq x \), so

\[ Q(x) = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \pi(x - p) + O \left( \sum_{p \leq x/\log x} \pi(x) + \sum_{x - \sqrt{x} \leq p \leq x} x \right) \]
\[ = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \pi(x - p) + O \left( \pi(x) \pi \left( \frac{x}{\log x} \right) + x \sqrt{x} \right) \]
\[ = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \pi(x - p) + O \left( \frac{x^2}{\log^3 x} \right). \quad (4) \]

In the main term, the prime number theorem gives

\[ \sum_{x/\log x \leq p \leq x - \sqrt{x}} \pi(x - p) = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \left( \text{li}(x - p) + O \left( \frac{x - p}{\log^2 (x - p)} \right) \right) \]

(we could insert a better error term, but it would not improve the final result). Since \( x - p \geq \sqrt{x} \), we have \( \log(x - p) \gg \log x \) and so

\[ = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \text{li}(x - p) + O \left( \sum_{x/\log x \leq p \leq x - \sqrt{x}} \frac{x}{\log^2 x} \right) \]
\[ = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \text{li}(x - p) + O \left( \frac{x}{\log^2 x} \pi(x) \right) \]
\[ = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \text{li}(x - p) + O \left( \frac{x^2}{\log^3 x} \right), \]

which transforms equation (4) into

\[ Q(x) = \sum_{x/\log x \leq p \leq x - \sqrt{x}} \text{li}(x - p) + O \left( \frac{x^2}{\log^3 x} \right). \quad (5) \]

Using partial summation, we have

\[ \sum_{x/\log x \leq p \leq x - \sqrt{x}} \text{li}(x - p) = \int_{x/\log x}^{x - \sqrt{x}} \text{li}(x - t) d\pi(t) \]
\[ = \pi(x - \sqrt{x}) \text{li}(\sqrt{x}) - \pi \left( \frac{x}{\log x} \right) \text{li} \left( x - \frac{x}{\log x} \right) + \int_{x/\log x}^{x - \sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt, \]

[4]
since the $t$-derivative of $\text{li}(x - t)$ is $-1/\log(x - t)$. In other words,

$$\sum_{x/\log x \leq p \leq x - \sqrt{x}} \text{li}(x - p) = O\left(x\sqrt{x} + \pi\left(\frac{x}{\log x}\right) \text{li}(x)\right) + \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt$$

$$= \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt + O\left(\frac{x^2}{\log^3 x}\right),$$

and so equation (5) becomes

$$Q(x) = \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt + O\left(\frac{x^2}{\log^3 x}\right).$$

Using the prime number theorem again, this becomes

$$Q(x) = \int_{x/\log x}^{x-\sqrt{x}} \frac{1}{\log(x - t)} \left(\frac{t}{\log t} + O\left(\frac{t}{\log^2 t}\right)\right) dt + O\left(\frac{x^2}{\log^3 x}\right)$$

$$= \int_{x/\log x}^{x-\sqrt{x}} \frac{t}{(\log t) \log(x - t)} dt + O\left(\int_{x/\log x}^{x-\sqrt{x}}\frac{t}{(\log^2 t) \log(x - t)} dt + \frac{x^2}{\log^3 x}\right). \quad (6)$$

In the error term, again $\log(x - t) \gg \log x$ and $\log^2 t \gg \log^2 x$ due to the endpoints of integration, and so the entire integral is $\ll x^2/\log^3 x$. In the main term, we have

$$\log x \geq \log t \geq \log \frac{x}{\log x} = \log x - \log \log x = (\log x) \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right),$$

and therefore equation (6) becomes

$$Q(x) = \frac{1}{\log x} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right) \int_{x/\log x}^{x-\sqrt{x}} \frac{t}{\log(x - t)} dt + O\left(\frac{x^2}{\log^3 x}\right). \quad (7)$$

Finally,

$$\int_{x/\log x}^{x-\sqrt{x}} \frac{t}{\log(x - t)} dt = \int_{0}^{x-\sqrt{x}} \frac{t}{\log(x - t)} dt + O\left(\int_{0}^{x/\log x} t dt + \int_{x-\sqrt{x}}^{x-\sqrt{x}} t dt\right)$$

$$= \int_{2}^{x} \frac{x - u}{\log u} du + O\left(\frac{x^2}{\log^2 x}\right)$$

$$= x \text{li}(x) - \int_{2}^{x} \frac{u}{\log u} du + O\left(\frac{x^2}{\log^2 x}\right). \quad (8)$$

By integration by parts, this integral is

$$\int_{2}^{x} \frac{u}{\log u} du = \left. \frac{u^2}{2 \log u}\right|_{2}^{x} + \int_{2}^{x} \frac{1}{2 u \log^2 u} du$$

$$= \left. \frac{x^2}{2 \log x}\right|_{2}^{x} + O\left(\left. \frac{u}{\log^2 u}\right|_{2}^{\sqrt{x}} du + \int_{\sqrt{x}}^{x} \frac{u}{\log^2 u} du\right)$$

$$= \frac{x^2}{2 \log x} + O\left(\sqrt{x} \cdot x + \frac{x}{\log^2 x}\right) = \frac{x^2}{2 \log x} + O\left(\frac{x^2}{\log^2 x}\right).$$

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Therefore equation (8) becomes
\[\int_{x/\log x}^{x-\sqrt{x}} \frac{t}{\log(x-t)} dt = x \log(x) - \frac{x^2}{2\log x} + O\left(\frac{x^2}{\log^2 x}\right) - \frac{x^2}{2\log x} + O\left(\frac{x^2}{\log^2 x}\right)\]
by the fact that \(\log(x) = x/\log x + O(1/x^2)\). Using this in equation (7) finally yields
\[Q(x) = \frac{1}{\log x} \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) \left(\frac{x^2}{2\log x} + O\left(\frac{x^2}{\log^2 x}\right)\right) + O\left(\frac{x^2}{\log^3 x}\right)\]
\[= \frac{x^2}{2\log^2 x} + O\left(\frac{x^2\log\log x}{\log^3 x}\right),\]
as claimed.

Equipped with Lemma 7 we are now prepared to prove Theorem 1.

**Proof of Theorem 1**
Starting with the definitions of \(a(m)\) and \(A(M)\), we have
\[A(M) = \sum_{m=1}^{2M} a(m) = \sum_{m=1}^{2M} \sum_{d|m} R(d) = \sum_{m=1}^{2M} \sum_{d|m} \sum_{p+q=d} 1 = \sum_{p+q\leq 2M} \sum_{1\leq m\leq 2M/2} 1.\]
Writing \(m = (p+q)n\), we obtain
\[A(M) = \sum_{p+q\leq 2M} \sum_{1\leq n\leq 2M/2} 1 = \sum_{1\leq n\leq 2M/2} \sum_{p+q\leq 2M} 1 = \sum_{1\leq n\leq 2M/2} Q\left(\frac{2M}{p+q}\right). \tag{9}\]
The trivial bound \(Q(x) \leq x^2\) allows us to write
\[A(M) = \sum_{1\leq n\leq \log^3 M} Q\left(\frac{2M}{n}\right) + O\left(\sum_{n>\log^3 M} \left(\frac{2M}{n}\right)^2\right) = \sum_{1\leq n\leq \log^3 M} Q\left(\frac{2M}{n}\right) + O\left(\frac{M^2}{\log^3 M}\right),\]
since \(\sum_{n>\log^3 M} n^{-2} \ll 1/\log^3 M\) by comparison with an integral. We use Lemma 7 to get
\[A(M) = \sum_{1\leq n\leq \log^3 M} \left(\frac{(2M/n)^2}{2\log^2 (2M/n)} + O\left(\frac{(2M/n)^2 \log\log(2M/n)}{\log^3 (2M/n)}\right)\right) + O\left(\frac{M^2}{\log^3 M}\right)\]
\[= 2M^2 \sum_{1\leq n\leq \log^3 M} \frac{1}{\log^2 (2M/n)} n^{-2} + O\left(\sum_{1\leq n\leq \log^3 M} \frac{\sqrt{2M} \log\log 2M}{\log^3 2M} \left(\frac{2M}{n}\right)^{3/2} + \frac{M^2}{\log^3 M}\right),\]
since \(\sqrt{x} \log\log x/\log^3 x\) is an (eventually) increasing function of \(x\). By the convergence of \(\sum_n n^{-3/2}\), we obtain
\[A(M) = 2M^2 \sum_{1\leq n\leq \log^3 M} \frac{1}{\log^2 (2M/n)} n^{-2} + O\left(\frac{M^2 \log\log M}{\log^3 M}\right).\]
Finally, we have \(\log(2M/n) = \log M - \log(n/2) = \log M + O(\log\log^3 M) = (\log M)(1 + O(\log\log M/\log M))\) as before. Therefore
\[A(M) = \frac{2M^2}{\log^2 M} \left(1 + O\left(\frac{\log\log M}{\log M}\right)\right) \sum_{1\leq n\leq \log^3 M} \frac{1}{n^2} + O\left(\frac{M^2 \log\log M}{\log^3 M}\right).\]
We conclude that
\[
A(M) = \frac{2M^2}{\log^2 M} \left( 1 + O\left( \frac{\log \log M}{\log M} \right) \right) \left( \zeta(2) + O\left( \frac{1}{\log^3 M} \right) \right) + O\left( \frac{M^2 \log \log M}{\log^3 M} \right) \\
= \frac{\pi^2 M^2}{3 \log^2 M} + O\left( \frac{M^2 \log \log M}{\log^3 M} \right),
\]
as desired. □

We now move on to a proposition from which we will deduce Theorem 4. Define
\[
f(n) = \prod_{\substack{p|n \\ p > 2}} \frac{p - 1}{p - 2}
\]
to be the multiplicative function appearing in Conjecture 2, and note that if \( k \geq 0 \) is the integer such that \( 2^k \parallel m \), then
\[
\sum_{d|m} df(d) = \prod_{p^e|m} \left( 1 + p f(p) + p^2 f(p^2) + \cdots + p^e f(p^e) \right) \\
= \left( 1 + 2^{2k} - 1 \right) \prod_{p^e|m} \left( 1 + \frac{p - 1}{p - 2} \cdot \frac{p^e - 1}{p - 1} \right) \\
= (2^{k+1} - 1) \prod_{p^e|m \atop p > 2} \frac{p^{e+1} - 2}{p - 2} = m J(m)
\]
(10)
by comparison with Definition 3.

**Proposition 8.** Let \( 0 < \varepsilon \leq \frac{1}{2} \) be given. Suppose there exists a positive integer \( n(\varepsilon) \) such that
\[
(1 - \varepsilon) 2C_2 f(n) \frac{n}{\log^2 n} \leq R(2n) \leq (1 + \varepsilon) 2C_2 f(n) \frac{n}{\log^2 n}
\]
(11)
for all \( n > n(\varepsilon) \). Then there exists a constant \( m(\varepsilon) \) such that
\[
(1 - 2\varepsilon) 2C_2 J(m) \frac{m}{\log^2 m} \leq a(2m) \leq (1 + 11\varepsilon) 2C_2 J(m) \frac{m}{\log^2 m}
\]
(12)
for all \( m > m(\varepsilon) \).

It is clear that Theorem 4 follows from Proposition 8 since Conjecture 2 implies that the hypothesis of Proposition 8 holds for every \( \varepsilon > 0 \).

**Proof of Proposition 8.** We shall not keep track explicitly of the necessary value for \( m(\varepsilon) \), instead simply saying “when \( m \) is large enough” (in terms of \( \varepsilon \)) in the appropriate places. We begin by writing
\[
a(2m) = \sum_{c|2m} R(c) = \sum_{d|m} R(2d) = \sum_{d|m \atop d \leq m^{1-\varepsilon}} R(2d) + \sum_{d|m \atop d > m^{1-\varepsilon}} R(2d)
\]
(13)
(where the second equality uses the fact that \( R(c) = 0 \) when \( c \) is odd).
First we establish the upper bound in equation (12). We have $m^{1-\varepsilon} > n(\varepsilon)$ when $m$ is large enough, and so the summands in the second sum on the right-hand side of equation (13) can be bounded above by the upper bound in equation (11). For the first sum on the right-hand side we simply use the trivial bound $R(2n) \leq n$. The result is

$$a(2m) \leq \sum_{d|m} \frac{d}{\log^2 d} + \sum_{d|m} \frac{(1 + \varepsilon)2C_2 f(d)}{\log^2 d} \sum_{d|m} d \leq \sum_{d|m} \frac{m^{1-\varepsilon} + (1 + \varepsilon)2C_2}{(1 - \varepsilon)^2 \log^2 m} \leq \sum_{d|m} \frac{m^{1-\varepsilon} \tau(m) + \frac{1}{(1 - \varepsilon)^2} 2C_2 mJ(m)}{\log^2 m},$$

using the identity (10), where $\tau(m)$ denotes the number of divisors of $m$. It is well known that $\tau(m) \ll m^{\varepsilon/3}$, and so the first term is less than $\varepsilon m/\log^2 m$ when $m$ is large enough. Also $(1 + \varepsilon)/(1 - \varepsilon)^2 \leq 1 + 10\varepsilon$ for $0 < \varepsilon \leq 1/2$. Therefore

$$a(2m) \leq \varepsilon \frac{m}{\log^2 m} + (1 + 10\varepsilon) \frac{2C_2 mJ(m)}{\log^2 m} \leq (1 + 11\varepsilon)2C_2 J(m) \frac{m}{\log^2 m}$$

when $m$ is large enough, since $J(m) \geq 1$ for all positive integers $m$ and $2C_2 > 1$. This establishes the upper bound in equation (12).

A similar method addresses the lower bound in equation (12). Since $m^{1-\varepsilon} > n(\varepsilon)$ when $m$ is large enough, the summands in the second sum on the right-hand side of equation (13) can be bounded below by the lower bound in equation (11); the first sum on the right-hand side is nonnegative, and so we can simply delete it. We obtain the lower bound

$$a(2m) \geq \sum_{d|m} \frac{(1 + \varepsilon)2C_2 f(d)}{\log^2 d} \sum_{d|m} \frac{d}{\log^2 d} \geq (1 - \varepsilon) \frac{2C_2}{\log^2 m} \sum_{d|m} \frac{df(d)}{d} = (1 - \varepsilon) \frac{2C_2}{\log^2 m} (mJ(m) - \sum_{d|m} \frac{df(d)}{d}),$$

again using the identity (10). This last sum is bounded above by

$$\sum_{d|m} \frac{df(d)}{d} \leq \sum_{d|m} \left(\frac{m^{1-\varepsilon}}{d}\right)^{1+\varepsilon/2} \leq m^{1-\varepsilon/2} \sum_{d|m} \prod_{p | d} \frac{1}{p^{\varepsilon/2}(p - 2)}.$$

There are only finitely many primes $p$ for which $(p - 1)/p^{\varepsilon/2}(p - 2)$ exceeds 1, and so the inner product on the right-hand side is uniformly bounded by some constant $C(\varepsilon)$. Therefore

$$\sum_{d|m} \frac{df(d)}{d} \leq C(\varepsilon) m^{1-\varepsilon/2} \sum_{d|m} 1 = C(\varepsilon) m^{1-\varepsilon/2} \tau(m),$$

and so

$$a(2m) \geq \frac{1}{2} \sum_{d|m} \frac{df(d)}{d} \geq \frac{1}{2} C(\varepsilon) m^{1-\varepsilon/2} \tau(m),$$

which establishes the lower bound in equation (12).
which as above is less than $\varepsilon m$ for $m$ large enough. Therefore equation (14) becomes

\[ a(m) \geq (1 - \varepsilon) \frac{2C_2}{\log^2 m} (mJ(m) - \varepsilon m) \geq (1 - 2\varepsilon) \frac{2C_2 J(m)}{\log^2 m} \frac{m}{m} \]

when $m$ is large enough, again since $J(m) \geq 1$ always. This establishes the lower bound in equation (12).

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