Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles

M. Palese and E. Winterroth†
Department of Mathematics, University of Torino
Via C. Alberto 10, 10123 Torino, Italy
e-mails: PALESE@DM.UNITO.IT, EKKEHART@DM.UNITO.IT

Abstract

We derive both local and global generalized Bianchi identities for classical Lagrangian field theories on gauge-natural bundles. We show that globally defined generalized Bianchi identities can be found without the a priori introduction of a connection. The proof is based on a global decomposition of the variational Lie derivative of the generalized Euler–Lagrange morphism and the representation of the corresponding generalized Jacobi morphism on gauge-natural bundles. In particular, we show that within a gauge-natural invariant Lagrangian variational principle, the gauge-natural lift of infinitesimal principal automorphisms is not intrinsically arbitrary. As a consequence the existence of canonical global superpotentials for gauge-natural Noether conserved currents is proved without resorting to additional structures.

2000 MSC: 58A20,58A32,58E30,58E40,58J10,58J70.
Key words: jets, gauge-natural bundles, variational principles, generalized Bianchi identities, Jacobi morphisms, invariance and symmetry properties.

1 Introduction

Local generalized Bianchi identities for geometric field theories were introduced [3, 4, 6, 19, 35] to get (after an integration by part procedure) a consistent equation between local divergences within the first variation formula. It was also stressed that in the general theory of relativity these identities coincide with the contracted Bianchi identities for the curvature tensor of the pseudo-Riemannian metric. We recall that in the classical Lagrangian formulation of field theories the description of symmetries amounts to define suitable (vector) densities which generate the conserved currents; in all relevant physical theories this densities are found to be the divergence of skew–symmetric (tensor) densities, which are called superpotentials for the conserved currents. It is also well known that the

* Dedicated to Hartwig in occasion of his first birthday.
† Both of them supported by GNFM of INdAM and University of Torino.
importance of superpotentials relies on the fact that they can be integrated to provide conserved quantities associated with the conserved currents via the Stokes theorem (see e.g. [11] and references quoted therein).

Subsequently, many attempts to “covariantize” such a derivation of Bianchi identities and superpotentials have been made (see e.g. [4, 7, 9, 11, 12, 25, 26] and the wide literature quoted therein) by resorting to background metrics or (fibered) connections used to perform covariant integration by parts to get covariant (variations of) currents and superpotentials. In particular, in [10] such a covariant derivation was implicitly assumed to hold true for any choice of gauge-natural prolongations of principal connections equal to prolongations of principal connections with respect to a linear symmetric connection on the basis manifold in the sense of [27, 28, 33].

In the present paper, we derive both local and global generalized Bianchi identities for classical field theories by resorting to the gauge-natural invariance of the Lagrangian and via the application of the Noether Theorems [39]. In particular we show that invariant generalized Bianchi identities can be found without the a priori introduction of a connection. The proof is based on a global decomposition of the variational Lie derivative of the generalized Euler–Lagrange morphism involving the definition - and its representation - of a new morphism, the generalized gauge-natural Jacobi morphisms. It is in fact known that the second variation of a Lagrangian can be formulated in terms of Lie derivative of the corresponding Euler–Lagrange morphism [13, 14, 17, 20, 40]. As a consequence the existence of canonical, i.e. completely determined by the variational problem and its invariance properties, global superpotentials for gauge-natural Noether conserved currents is proved without resorting to additional structures.

Our general framework is the calculus of variations on finite order jet of fibered bundles. Fibered bundles will be assumed to be gauge-natural bundles (i.e. jet prolongations of fiber bundles associated to some gauge-natural prolongation of a principal bundle $P$ [8, 22, 23, 27, 28, 32]) and variations of sections are (vertical) vector fields given by Lie derivatives of sections with respect to gauge-natural lifts of infinitesimal principal automorphisms (see e.g. [8, 24, 32]).

In this general geometric framework we shall in particular consider finite order variational sequences on gauge-natural bundles. The variational sequence on finite order jet prolongations of fibered manifolds was introduced by Krupka as the quotient of the de Rham sequence of differential forms (defined on the prolongation of the fibered manifold) with respect to a natural exact contact subsequence, chosen in such a way that the generalized Euler–Lagrange and Helmholtz-Sonin mappings can be recognized as some of its quotient mappings [36, 37]. The representation of the quotient sheaves of the variational sequence as sheaves of sections of tensor bundles given in [46] and previous results on variational Lie derivatives and Noether Theorems [10, 15] will be used. Furthermore, we relate the generalized Bianchi morphism to the second variation of the Lagrangian. A very fundamental abstract result due to Kolář concerning global decomposition formulae of vertical morphisms, involved with the integration by parts procedure [21, 29, 30], will be a key tool. In order to apply this results, we stress linearity properties of the Lie derivative operator acting on sections...
of the gauge-natural bundle, which rely on properties of the gauge-natural lift of infinitesimal principal automorphisms. The gauge-natural lift enables one to define the generalized gauge-natural Jacobi morphism (i.e. a generalized Jacobi morphism where the variation vector fields – instead of general deformations – are Lie derivatives of sections of the gauge-natural bundle with respect to gauge-natural lifts of infinitesimal automorphisms of the underlying principal bundle), the kernel of which plays a very fundamental role.

The paper is structured as follows. In Section 2 we state the geometric framework by defining the variational sequence on gauge natural-bundles and by representing the Lie derivative of fibered morphisms on its quotient sheaves; Section 3 is dedicated to the definition and the representation of the generalized gauge-natural Jacobi morphism associated with a generalized gauge-natural Lagrangian. We stress some linearity properties of this morphism as a consequence of the properties of the gauge-natural lift of infinitesimal right-invariant automorphisms of the underlying structure bundle. In Section 4, by resorting to the Second Noether Theorem, we relate the generalized Bianchi identities with the kernel of the gauge-natural Jacobi morphism. We prove that the generalized Bianchi identities hold true globally if and only if the vertical part of jet prolongations of gauge-natural lifts of infinitesimal principal bundle automorphisms is in the kernel of the second variation, i.e. of the generalized gauge-natural Jacobi morphism.

Here, manifolds and maps between manifolds are $C^\infty$. All morphisms of fibered manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified.

2 Variational sequences on gauge-natural bundles

2.1 Jets of fibered manifolds

In this Section we recall some basic facts about jet spaces. We introduce jet spaces of a fibered manifold and the sheaves of forms on the $s$–th order jet space. Moreover, we recall the notion of horizontal and vertical differential [32, 38, 42].

Our framework is a fibered manifold $\pi : Y \to X$, with dim $X = n$ and dim $Y = n + m$.

For $s \geq q \geq 0$ integers we are concerned with the $s$–jet space $J_s Y$ of $s$–jet prolongations of (local) sections of $\pi$; in particular, we set $J_0 Y \equiv Y$. We recall the natural fiberings $\pi^q_s : J_s Y \to J_q Y$, $s \geq q$, $\pi^s_s : J_s Y \to X$, and, among these, the affine fiberings $\pi^q_{s-1}$. We denote with $V Y$ the vector subbundle of the tangent bundle $T Y$ of vectors on $Y$ which are vertical with respect to the fibering $\pi$.

Charts on $Y$ adapted to $\pi$ are denoted by $(x^\sigma, y^i)$. Greek indices $\sigma, \mu, \ldots$ run from 1 to $n$ and they label basis coordinates, while Latin indices $i, j, \ldots$ run from 1 to $m$ and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\sigma, \partial_i)$ and $(d^\sigma, d^i)$ the local basis of vector fields and 1–forms on $Y$ induced
by an adapted chart, respectively. We denote multi–indices of dimension $n$ by boldface Greek letters such as $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \ldots, n$; by an abuse of notation, we denote with $\sigma$ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \sigma$, $\alpha_\mu = 1$, if $\mu = \sigma$. We also set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1! \cdots \alpha_n!$.

The charts induced on $J_s Y$ are denoted by $(x^\alpha, y^i_\alpha)$, with $0 \leq |\alpha| \leq s$; in particular, we set $y^i_0 \equiv y^i$. The local vector fields and forms on $J_s Y$ induced by the above coordinates are denoted by $(\partial^\alpha_i)$ and $(d^\alpha_i)$, respectively.

In the theory of variational sequences a fundamental role is played by the contact maps on jet spaces (see [36, 37, 38, 40]). Namely, for $s \geq 1$, we consider the natural complementary fibered morphisms over $J_s Y \to J_{s-1} Y$

$$\mathcal{D} : J_s Y \times T X \to T J_{s-1} Y, \quad \vartheta : J_s Y \times T J_{s-1} Y \to V J_{s-1} Y,$$

with coordinate expressions, for $0 \leq |\alpha| \leq s - 1$, given by

$$\mathcal{D} = d^\lambda \otimes D_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\lambda+\gamma} \delta^\alpha_j), \vartheta = \partial^\alpha_i \otimes \partial^\alpha_i = (d^\alpha_i - y^j_{\alpha+\lambda} d^\lambda) \otimes \partial^\alpha_i.$$

The morphisms above induce the following natural splitting (and its dual):

$$J_s Y \times J_{s-1} Y \times T^* J_{s-1} Y = \left( J_s Y \times T^* X \right) \oplus C^*_{s-1}[Y], \quad (1)$$

where $C^*_{s-1}[Y] := \text{im} \vartheta^*_s$ and $\vartheta^*_s : J_s Y \times J_{s-1} Y \times V^* J_{s-1} Y \to J_s Y \times J_{s-1} Y \times T^* J_{s-1} Y$.

We have the isomorphism $C^*_{s-1}[Y] \simeq J_s Y \times V^* J_{s-1} Y$. The role of the splitting above will be fundamental in the present paper.

If $f : J_s Y \to \mathbb{R}$ is a function, then we set $D_{sf} := D_{sf} f, D_{\alpha+\sigma} f := D_{\alpha} D_{\sigma} f$, where $D_{\sigma}$ is the standard formal derivative. Given a vector field $\Xi : J_s Y \to T J_s Y$, the splitting [11] yields $\Xi \circ \pi^*_{s-1} = \Xi_H + \Xi_V$ where, if $\Xi = \Xi^\gamma \partial_\gamma + \Xi^\alpha \partial^\alpha$, then we have $\Xi_H = \Xi^\gamma D_\gamma$ and $\Xi_V = (\Xi^\alpha - y^j_{\alpha+\gamma} \Xi^\gamma) \partial^\alpha$. We shall call $\Xi_H$ and $\Xi_V$ the horizontal and the vertical part of $\Xi$, respectively.

The splitting [11] induces also a decomposition of the exterior differential on $Y$, $(\pi^*_{s-1})^* \circ d = d_H + d_V$, where $d_H$ and $d_V$ are defined to be the horizontal and vertical differential. The action of $d_H$ and $d_V$ on functions and 1–forms on $J_s Y$ uniquely characterizes $d_H$ and $d_V$ (see, e.g., [12, 40] for more details). A projectable vector field on $Y$ is defined to be a pair $\langle \Xi, \xi \rangle$, where $\Xi : Y \to TY$ and $\xi : X \to TX$ are vector fields and $\Xi$ is a fibered morphism over $\xi$. If there is no danger of confusion, we will denote simply by $\Xi$ a projectable vector field $\langle \Xi, \xi \rangle$. A projectable vector field $\langle \Xi, \xi \rangle$, with coordinate expression $\Xi = \xi^\alpha \partial_\alpha + \xi^i \partial^i$, $\xi = \xi^\alpha \partial_\alpha$, can be conveniently prolonged to a projectable vector field $(j_s \Xi, \xi)$, whose coordinate expression turns out to be

$$j_s \Xi = \xi^\alpha \partial_\alpha + (D_\alpha \xi^i - \sum_{\beta+\gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} D_\beta \xi^\mu y^i_{\gamma+\mu}) \partial_i^\alpha,$$

where $\beta \neq 0$ and $0 \leq |\alpha| \leq s$ (see e.g. [36, 38, 12, 40]); in particular, we have the following expressions $(j_s \Xi)_H = \xi^\alpha D_\sigma, (j_s \Xi)_V = D_\alpha (\Xi_V)^i \partial_i^\alpha$, with
\((\Xi^i) = \xi^i - y^j \xi^\sigma\), for the horizontal and the vertical part of \(j_s \Xi\), respectively. From now on, by an abuse of notation, we will write simply \(j_s \Xi_H\) and \(j_s \Xi_V\). In particular, \(j_s \Xi_V : J_{s+1} Y \times J_s Y \to J_{s+1} Y \times J_s V Y\).

We are interested in the case in which physical fields are assumed to be sections of a fibered bundle and the variations of sections are generated by suitable vector fields. More precisely, fibered bundles will be assumed to be \textit{gauge-natural bundles} and variations of sections are (vertical) vector fields given by Lie derivatives of sections with respect to gauge-natural lifts of infinitesimal principal automorphisms. Such geometric structures have been widely recognized to suitably describe so-called gauge-natural field theories, \textit{i.e.} physical theories in which right-invariant infinitesimal automorphisms of the structure bundle \(P\) uniquely define the transformation laws of the fields themselves (see \textit{e.g.} \cite{[8, 32]}). In the following, we shall develop a suitable geometrical setting which enables us to define and investigate the fundamental concept of conserved quantity in gauge-natural Lagrangian field theories.

2.2 Gauge-natural prolongations

First we shall recall some basic definitions and properties concerning gauge-natural prolongations of (structure) principal bundles (for an extensive exposition see \textit{e.g.} \cite{[22, 32]} and references therein; in the interesting paper \cite{[23]} fundamental reduction theorems for general linear connections on vector bundles are provided in the gauge-natural framework).

Let \(P \to X\) be a principal bundle with structure group \(G\). Let \(r \leq k\) be integers and \(W^{(r,k)} P := J_r P \times L_k (X)\), where \(L_k (X)\) is the bundle of \(k\)-frames in \(X\). \(W^{(r,k)} G := J_r G \circ GL_k (n)\) the semidirect product with respect to the action of \(GL_k (n)\) on \(J_r G\) given by the jet composition and \(GL_k (n)\) is the group of \(k\)-frames in \(\mathbb{R}^n\). Here we denote by \(J_r G\) the space of \((r,n)\)-velocities on \(G\).

Elements of \(W^{(r,k)} P\) are given by \((j^r_{\gamma} \gamma, j^0_k \alpha)\), with \(\gamma : X \to P\) a local section, \(t : \mathbb{R}^n \to X\) locally invertible at zero, with \(t(0) = x, x \in X\). Elements of \(W^{(r,k)} G\) are \((j^r_0 g, j^0_k \alpha)\), where \(g : \mathbb{R}^n \to G\), \(\alpha : \mathbb{R}^n \to \mathbb{R}^n\) locally invertible at zero, with \(\alpha(0) = 0\).

**Remark 1** The bundle \(W^{(r,k)} P\) is a principal bundle over \(X\) with structure group \(W^{(r,k)} G\). The right action of \(W^{(r,k)} G\) on the fibers of \(W^{(r,k)} P\) is defined by the composition of jets (see, \textit{e.g.}, \cite{[22, 32]}).

**Definition 1** The principal bundle \(W^{(r,k)} P\) (resp. the Lie group \(W^{(r,k)} G\)) is said to be the \textit{gauge-natural prolongation of order} \((r,k)\) \textit{of} \(P\) (resp. \textit{of} \(G\)).

**Remark 2** Let \((\Phi, \phi)\) be a principal automorphism of \(P\). It can be prolonged in a natural way to a principal automorphism of \(W^{(r,k)} P\), defined by:

\[W^{(r,k)} (\Phi, \phi) : (j^r_{\gamma} \gamma, j^0_k) \mapsto (j^r_{\phi(\gamma)} (\Phi \circ \gamma \circ \phi^{-1}), j^0_k (\phi \circ t))\]
The induced automorphism $W^{(r,k)}(\Phi, \phi)$ is an equivariant automorphism of $W^{(r,k)}P$ with respect to the action of the structure group $W^{(r,k)}G$. We shall simply denote it by the same symbol $\Phi$, if there is no danger of confusion.

**Definition 2** We define the vector bundle over $X$ of right–invariant infinitesimal automorphisms of $P$ by setting $A := TP/G$.

We also define the vector bundle over $X$ of right invariant infinitesimal automorphisms of $W^{(r,k)}P$ by setting $A^{(r,k)} := TW^{(r,k)}P/W^{(r,k)}G$ ($r \leq k$).

**Remark 3** We have the following projections $A^{(r,k)} \rightarrow A^{(r',k')}$, $r \leq k$, $r' \leq k'$, with $r \geq r'$, $s \geq s'$.

**2.3 Gauge-natural bundles and lifts**

Let $F$ be any manifold and $\zeta : W^{(r,k)}G \times F \rightarrow F$ be a left action of $W^{(r,k)}G$ on $F$. There is a naturally defined right action of $W^{(r,k)}G$ on $W^{(r,k)}P \times F$ so that we can associate in a standard way to $W^{(r,k)}P$ the bundle, on the given basis $X$, $Y_{\zeta} := W^{(r,k)}P \times_\zeta F$.

**Definition 3** We say $(Y_{\zeta}, X, \pi_{\zeta}; F, G)$ to be the gauge-natural bundle of order $(r,k)$ associated to the principal bundle $W^{(r,k)}P$ by means of the left action $\zeta$ of the group $W^{(r,k)}G$.

**Remark 4** A principal automorphism $\Phi$ of $W^{(r,k)}P$ induces an automorphism of the gauge-natural bundle by:

$$\Phi_{\zeta} : Y_{\zeta} \rightarrow Y_{\zeta} : ([j^r_+ \gamma, j^l_0 t], \hat{f})_{\zeta} \mapsto [\Phi(j^r_+ \gamma, j^l_0 t), \hat{f}]_{\zeta},$$

where $\hat{f} \in F$ and $[\cdot, \cdot]_{\zeta}$ is the equivalence class induced by the action $\zeta$.

Denote by $T_X$ and $A^{(r,k)}$ the sheaf of vector fields on $X$ and the sheaf of right invariant vector fields on $W^{(r,k)}P$, respectively. A functorial mapping $\mathcal{G}$ is defined which lifts any right–invariant local automorphism $(\Phi, \phi)$ of the principal bundle $W^{(r,k)}P$ into a unique local automorphism $(\Phi_{\zeta}, \phi)$ of the associated bundle $Y_{\zeta}$. Its infinitesimal version associates to any $\hat{\Xi} \in A^{(r,k)}$, projectable over $\xi \in T_X$, a unique projectable vector field $\hat{\Xi} := \mathcal{G}(\hat{\Xi})$ on $Y_{\zeta}$ in the following way:

$$\mathcal{G} : Y_{\zeta} \times A^{(r,k)} \rightarrow TY_{\zeta} : (y, \hat{\Xi}) \mapsto \hat{\Xi}(y),$$

where, for any $y \in Y_{\zeta}$, one sets: $\hat{\Xi}(y) = \frac{d}{dt}[(\Phi_{\zeta} t)(y)]_{t=0}$, and $\Phi_{\zeta} t$ denotes the (local) flow corresponding to the gauge-natural lift of $\Phi_t$.

This mapping fulfills the following properties:

1. $\mathcal{G}$ is linear over $id_{Y_{\zeta}}$;
2. we have $T\pi \circ \mathcal{G} = id_{TX} \circ \bar{\pi}^{(r,k)}$, where $\bar{\pi}^{(r,k)}$ is the natural projection $Y_\zeta \times A^{(r,k)} \to TX$;

3. for any pair $(\bar{\Lambda}, \bar{\Xi})$ of vector fields in $A^{(r,k)}$, we have

$$\mathcal{G}([\bar{\Lambda}, \bar{\Xi}]) = [\mathcal{G}(\bar{\Lambda}), \mathcal{G}(\bar{\Xi})];$$

4. we have the coordinate expression of $\mathcal{G}$

$$\mathcal{G} = d^\nu \otimes \partial_\nu + d^\lambda \otimes (Z_\nu^\lambda \partial_\nu) + d^\lambda \otimes (Z_\rho^\lambda \partial_\rho), \quad (4)$$

with $0 < |\nu| < k, 1 < |\lambda| < r$ and $Z_\nu^\lambda, Z_\rho^\lambda \in C^\infty(Y_\zeta)$ are suitable functions which depend on the bundle, precisely on the fibers (see [32]).

**Definition 4** The map $\mathcal{G}$ is called the *gauge-natural lifting functor*. The projectable vector field $(\hat{\Xi}, \xi) \equiv \mathcal{G}((\bar{\Xi}, \xi))$ is called the *gauge-natural lift* of $(\bar{\Xi}, \xi)$ to the bundle $Y_\zeta$.

**2.4 Lie derivative of sections of gauge-natural bundles**

Let $\gamma$ be a (local) section of $Y_\zeta$, $\bar{\Xi} \in A^{(r,k)}$ and $\hat{\Xi}$ its gauge-natural lift. Following [32] we define a (local) section $£_{\bar{\Xi}} \gamma : X \to VY_\zeta$, by setting: $£_{\bar{\Xi}} \gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma$.

**Definition 5** The (local) section $£_{\bar{\Xi}} \gamma$ is called the *generalized Lie derivative* of $\gamma$ along the vector field $\hat{\Xi}$.

**Remark 5** This section is a vertical prolongation of $\gamma$, i.e. it satisfies the property: $\nu_{Y_\zeta} \circ £_{\bar{\Xi}} \gamma = \gamma$, where $\nu_{Y_\zeta}$ is the projection $\nu_{Y_\zeta} : VY_\zeta \to Y_\zeta$. Its coordinate expression is given by $(£_{\bar{\Xi}} \gamma)^i = \xi^\sigma \partial_\sigma \gamma^i - \hat{\Xi}^i(\gamma)$.

**Remark 6** The Lie derivative operator acting on sections of gauge-natural bundles satisfies the following properties:

1. for any vector field $\hat{\Xi} \in A^{(r,k)}$, the mapping $\gamma \mapsto £_{\bar{\Xi}} \gamma$ is a first–order quasilinear differential operator;

2. for any local section $\gamma$ of $Y_\zeta$, the mapping $\hat{\Xi} \mapsto £_{\bar{\Xi}} \gamma$ is a linear differential operator;

3. by using the canonical isomorphism $VJ_1 Y_\zeta \simeq J_1 VY_\zeta$, we have $£_{\bar{\Xi}} [j_1 \gamma] = j_1 [£_{\bar{\Xi}} \gamma]$, for any (local) section $\gamma$ of $Y_\zeta$ and for any (local) vector field $\hat{\Xi} \in A^{(r,k)}$.

We can regard $£_{\bar{\Xi}} : J_1 Y_\zeta \to VY_\zeta$ as a morphism over the basis $X$. In this case it is meaningful to consider the (standard) jet prolongation of $£_{\bar{\Xi}}$, denoted by $j_s £_{\bar{\Xi}} : J_{s+1} Y_\zeta \to VJ_s Y_\zeta$. Furthermore, we have $j_1 £_{\bar{\Xi}}(\gamma) = - £_{j_0 \bar{\Xi}} \gamma$. 

7
4. we can consider $\mathcal{E}$ as a bundle morphism:

$$
\mathcal{E} : J_{s+1}(Y_\zeta \times A^{(r,k)}) \to J_{s+1}Y_\zeta \times VJ_sY_\zeta.
$$

(5)

2.5 Variational sequences

For the sake of simplifying notation, sometimes, we will omit the subscript $\zeta$, so that all our considerations shall refer to $Y$ as a gauge-natural bundle as defined above.

We shall be here concerned with some distinguished sheaves of forms on jet spaces $[36, 37, 42, 46]$. Due to the topological triviality of the fibre of $J_sY \to Y$, we will consider sheaves on $J_sY$ with respect to the topology generated by open sets of the kind $(\pi_s^0)^{-1}(U)$, with $U \subset Y$ open in $Y$.

i. For $s \geq 0$, we consider the standard sheaves $\Lambda^p_s$ of $p$–forms on $J_sY$.

ii. For $0 \leq q \leq s$, we consider the sheaves $H^p_s(s,q)$ and $H^p_s(s,q)$ of horizontal forms, i.e. of local fibered morphisms (following the well known correspondence between forms and fibered morphisms over the basis manifold, see e.g. [32]) over $\pi_s^q$ and $\pi^s$ of the type $\alpha : J_sY \to \mathcal{H}^*_{T_sJ_sY}$ and $\beta : J_sY \to \mathcal{H}^*_{T_sX}$, respectively.

iii. For $0 \leq q < s$, we consider the subsheaf $C^p_s(s,q) \subset H^p_s(s,q)$ of contact forms, i.e. of sections $\alpha \in H^p_{s,q}$ with values into $\mathcal{H}^*(\mathcal{C}^*_q[1])$. We have the distinguished subsheaf $C^p_s \subset C^p_{(s+1),s}$ of local fibered morphisms $\alpha \in C^p_{(s+1),s}$ such that $\alpha = \Lambda^p_s \times \mathcal{H}^*_{T_sJ_sY} \to \mathcal{H}^*_{T_sJ_sY} \to J_{s+1}Y$ which projects down onto $J_sY$.

**Remark 7** Notice that according to [36, 37, 40], the fibered splitting [1] yields the sheaf splitting $H^p_{(s+1),s} = \bigoplus_{t=0}^n C^{p-t}_{(s+1),s} \wedge H^t_{s+1}$, which restricts to the inclusion $\Lambda^p_s \subset \bigoplus_{t=0}^n C^{p-t}_{(s+1),s} \wedge H^t_{s+1}$, where $H^{p,h}_{s+1} := h(\Lambda^p_s)$ for $0 < p \leq n$ and the surjective map $h$ is defined to be the restriction to $\Lambda^p_s$ of the projection of the above splitting onto the non–trivial summand with the highest value of $t$.

The induced sheaf splitting above plays here a fundamental role. We stress again that, for any jet order $s$, it is induced by the natural contact structure on the affine bundle $\pi_{s+1}$. Since a variational problem (described by the corresponding action integral) is insensitive to the addition of any piece containing contact factors, such an affine structure has been pointed out in [36] to be fundamental for the description of the geometric structure of the Calculus of Variations on finite order jets of fibered manifolds. This property reflects on the intrinsic structure of all objects defined and represented in the variational sequence of a given order which we are just going to introduce. In particular this holds true for the generalized Jacobi morphism we will define and represent in Section 3.
We shortly recall now the theory of variational sequences on finite order jet spaces, as it was developed by D. Krupka in [36].

By an abuse of notation, let us denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$ in the standard way. We set $\Theta^* := \ker h + d \ker h$.

In [36] it was proved that the following sequence is an exact resolution of the constant sheaf $R_Y$ over $Y$:

$$0 \longrightarrow R_Y \longrightarrow \Lambda^0 \xrightarrow{\xi_0} \Lambda^1 / \Theta^1 \xrightarrow{\xi_1} \Lambda^2 / \Theta^2 \xrightarrow{\xi_2} \ldots \xrightarrow{\xi_{I-1}} \Lambda^I / \Theta^I \xrightarrow{\xi_I} \Lambda^{I+1} \longrightarrow 0$$

**Definition 6** The above sequence, where the highest integer $I$ depends on the dimension of the fibers of $J_s Y \rightarrow X$ (see, in particular, [36]), is said to be the $s$–th order variational sequence associated with the fibered manifold $Y \rightarrow X$.

For practical purposes, specifically to deal with morphisms which have a well known interpretation within the Calculus of Variations, we shall limit ourselves to consider the truncated variational sequence:

$$0 \longrightarrow R_Y \longrightarrow \mathcal{V}^0 \xrightarrow{\xi_0} \mathcal{V}^1 \xrightarrow{\xi_1} \ldots \xrightarrow{\xi_{n+1}} \mathcal{V}^{n+1} \xrightarrow{\xi_{n+1}(\mathcal{V}^{n+1})} 0,$$

where, following [46], the sheaves $\mathcal{V}^p_s := \mathbb{C}^p_{\gamma} \wedge \mathcal{H}^{n,h}_{s+1} / h(d \ker h)$ with $0 \leq p \leq n + 2$ are suitable representations of the corresponding quotient sheaves in the variational sequence by means of sheaves of sections of vector bundles. We notice that in the following, to avoid confusion, sometimes (when the interpretation could be dubious) we shall denote with a subscript the relevant fibered bundle on which the variational sequence is defined; e.g. in the case above, we would write $(\mathcal{V}^p_s)_Y$.

Let $\alpha \in \mathcal{C}^1_s \wedge \mathcal{H}^{n,h}_{s+1} \subset \mathcal{V}^{n+1}_{s+1}$. Then there is a unique pair of sheaf morphisms

$$E_\alpha \in \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}, \quad F_\alpha \in \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}, \quad (6)$$

such that $(\pi_{s+1})^* \alpha = E_\alpha - F_\alpha$, and $F_\alpha$ is locally of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \mathcal{C}^1_{(2s-1,s-1)} \wedge \mathcal{H}^{n-1}_{2s}$.

**Definition 7** Let $\gamma \in \Lambda^{n+1}_s$. The morphism $E_{h(\gamma)} \in \mathcal{V}^{n+1}_s$ is called the generalized Euler–Lagrange morphism associated with $\gamma$ and the operator $\xi_s$ is called the generalized Euler–Lagrange operator. Furthermore $p_{h(\gamma)}$ is a generalized momentum associated with $E_{h(\gamma)}$.

Let $\eta \in \mathcal{C}^1_s \wedge \mathcal{C}^1_{(s,0)} \wedge \mathcal{H}^{n,h}_{s+1} \subset \mathcal{V}^{n+2}_{s+1}$, then there is a unique morphism

$$K_\eta \in \mathcal{C}^1_{(2s,s)} \otimes \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}$$

such that, for all $\Xi : Y \rightarrow V Y$, $E_{j_2,\Xi|\eta} = C^1_{(2s,2s)} (j_{2s} \Xi \otimes K_\eta)$, where $C^1_{(2s,2s)}$ stands for tensor contraction on the first factor and $| \eta$ denotes inner product (see [34, 46]). Furthermore, there is a unique pair of sheaf morphisms

$$H_\eta \in \mathcal{C}^1_{(2s,s)} \wedge \mathcal{C}^1_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}, \quad G_\eta \in \mathcal{C}^2_{(2s,0)} \wedge \mathcal{H}^{n,h}_{2s+1}, \quad (7)$$
such that \((\pi_{n+1}^{2s+1})^* \eta = H_\eta - G_\eta\) and \(H_\eta = \frac{1}{2} A(K_\eta)\), where \(A\) stands for antisymmetrisation. Moreover, \(G_\eta\) is locally of the type \(G_\eta = d_H q_\eta\), where \(q_\eta \in C^2_{(2s-1,s-1)} \wedge H^{n-1}_{2s}\), hence \([\eta] = [H_\eta]\). \[\Box\]

**Definition 8** Let \(\gamma \in \Lambda^{n+1}_{s+1}\). The morphism \(H_{hd\gamma} \equiv H_{[e_{n+1}(\gamma)]}\), where square brackets denote equivalence class, is called the *generalized Helmholtz morphism* and the operator \(E_{n+1}\) is called the *generalized Helmholtz operator*.

**Remark 8** A section \(\lambda \in V^n_s\) is just a Lagrangian of order \((s+1)\) of the standard literature.

**Remark 9** It is well known that it is always possible to find global morphisms \(p_{h(\gamma)}\) and \(q_{hd\gamma}\) satisfying decomposition formulae above; however, this possibility depends in general on the choice of a linear symmetric connection on the basis manifold (see [1, 2, 31]). In the present paper, we shall avoid to perform such a choice *a priori*, with the explicit intention of performing an invariant derivation of generalized Bianchi identities, which does not rely on an invariant decomposition involving (local) divergences; that is in fact possible when resorting to the representation of the Second Noether Theorem in the variational sequence, as shown by Theorem 3 below. This is due to the fact that in the quotient sheaves of the variational sequence contact forms and horizontal differentials of contact forms of higher degree are factored out. It is also clear that this will give, at least, prescriptions on the meaningful (within a fully gauge-natural invariant variational problem) possible choices of connections to be used to derive covariantly generalized Bianchi identities in the classical way. \[\Box\]

2.6 Variational Lie derivative

In this Subsection, following essentially [13], we give a representation in the variational sequence of the standard Lie derivative operator acting on fibered morphisms. We consider a projectable vector field \((\Xi, \xi)\) on \(Y\) and take into account the Lie derivative operator \(L_{j_s \Xi}\) with respect to the jet prolongation \(j_s \Xi\) of \(\Xi\). In fact, as well known, such a prolonged vector field preserves the fiberings \(\pi_q^s, \pi_s\); hence it preserves the splitting [14]. Thus we have

\[L_{j_s \Xi} : V^n_s \to V^n_s : [\alpha] \mapsto L_{j_s \Xi}([\alpha]) = [L_j \xi \alpha].\]

**Definition 9** Let \((\Xi, \xi)\) be a projectable vector field. We call the map \(L_{j_s \Xi}\) defined above the *variational Lie derivative*.

Variational Lie derivatives allow us to calculate infinitesimal symmetries of forms in the variational sequence. In particular, we are interested in symmetries of generalized Lagrangians and Euler–Lagrange morphisms which will enable us to represent in this framework Noether Theorems as well as known results stated in the framework of geometric bundles (see e.g. the fundamental papers by Trautman [33, 43, 44]).
Theorem 1 The following two results hold true \cite{15}.

Remark 10 Let \( s \leq q \). Then the inclusions \( \Lambda^p_s \subset \Lambda^p_q \) and \( \Theta^p_s \subset \Theta^p_q \) yield the injective sheaf morphisms (see \cite{36}) \( \chi^q_s : (\Lambda^p_s / \Theta^p_s) \to (\Lambda^p_q / \Theta^p_q) : [a] \mapsto [(\pi^q_s) \circ a] \), hence the inclusions \( \kappa^q_s : \mathcal{V}^p_s \to \mathcal{V}^p_q \) for \( s \leq q \).

The inclusions \( \kappa^q_s \) of the variational sequence of order \( s \) in the variational sequence of order \( q \) give rise to new representations of \( \mathcal{L}_{j,z} \) on \( \mathcal{V}^p_q \). In particular, the following two results hold true \cite{15}.

**Theorem 1** Let \( [\alpha] = h(\alpha) \in \mathcal{V}^n_s \). Then we have locally

\[
\kappa^{2s+1}_s \circ \mathcal{L}_{j,z}(h(\alpha)) = \Xi_V \mathcal{E}_n(h(\alpha)) + dH(j_{2s} \Xi_V \rho_{d\nu} h(\alpha) + \xi | h(\alpha)) .
\]

**Proof.** We have

\[
\kappa^{2s+1}_s \circ \mathcal{L}_{j,z}(h(\alpha)) = h(L_{j,s+1} h(\alpha)) \\
= dH(j_{s+1} \Xi_H | h(\alpha)) + h(j_{s+2} \Xi_V | d\nu h(\alpha)) \\
= dH(\xi | h(\alpha)) + h(j_{2s+1} \Xi_V | (E_{d\nu} h(\alpha) + F_{d\nu} h(\alpha))) .
\]

Since \( F_{d\nu} h(\alpha) = dH \rho_{d\nu} h(\alpha) \) locally, then

\[
\kappa^{2s+1}_s \circ \mathcal{L}_{j,z}(h(\alpha)) = \Xi_V \mathcal{E}_n(h(\alpha)) + dH(j_{2s} \Xi_V \rho_{d\nu} h(\alpha) + \xi | h(\alpha)) .
\]

**QED**

**Theorem 2** Let \( \alpha \in \Lambda^{n+1}_s \). Then we have globally

\[
\kappa^{2s+1}_s \circ \mathcal{L}_{j,z}[\alpha] = \mathcal{E}_n(j_{s+1} \Xi_V | h(\alpha)) + C_1^1(j_s \Xi_V \otimes K_{hda}) .
\]

**Proof.** We have

\[
\kappa^{2s+1}_s \circ \mathcal{L}_{j,z}[\alpha] = \mathcal{E}_n(j_{s+1} \Xi_V | h(\alpha)) + j_{s+1} \Xi_V | d\nu h(\alpha)) \\
= \mathcal{E}_n(j_{s+1} \Xi_V | h(\alpha)) + C_1^1(j_s \Xi_V \otimes K_{hda}) .
\]

**QED**

3 Variations and generalized Jacobi morphisms

To proceed further, we now need to recall some previous results concerning the representation of **generalized Jacobi morphisms** in variational sequences and their relation with the second variation of a generalized Lagrangian (\cite{13} \cite{17} \cite{30}, see also the fundamental paper \cite{20}). In \cite{14} the relation between the classical variations and Lie derivatives of a Lagrangian with respect to (vertical) variation vector fields was worked out and the **variational vertical derivative** was introduced as an operator acting on sections of sheaves of variational sequences, by showing that this operator is in fact a natural transformation, being a functor on the category of variational sequences. We shall here introduce **formal variations** of a morphism as **multiparameter deformations** showing that this is
equivalent to take iterated variational Lie derivatives with respect to (vertical) variation vector fields. Our aim is to relate, on the basis of relations provided by Corollary \[14\] and Proposition \[14\] below, the second variation of the Lagrangian \(\lambda\) to the Lie derivative of the associated Euler–Lagrange morphism and to the generalized Bianchi morphism, defined by Eq. \[24\] in Subsection \[??\] below.

We recall (see \[13, 14, 17\]) a Lemma which relates the \(i\)–th variation with the iterated Lie derivative of the morphism itself. Furthermore, following \[29\], we recall the relation between the variation of the morphism and the vertical exterior differential.

**Definition 10** Let \(\alpha : J_s Y \to \mathbb{R}^n \wedge T^* J_s Y\). Let \(\psi_i^k\), with \(1 \leq k \leq i\), be the flows generated by an \(i\)–tuple \((\Xi_1, \ldots, \Xi_i)\) of (vertical, although actually it is enough that they are projectable) vector fields on \(Y\) and let \(\Gamma_i\) be the \(i\)–th formal variation generated by the \(\Xi_k\)’s (to which we shall refer as variation vector fields) and defined, for each \(y \in Y\), by \(\Gamma_i(t_1, \ldots, t_i)(y) = \psi_i^1 \circ \ldots \circ \psi_i^i(y)\). We define the \(i\)–th formal variation of the morphism \(\alpha\) to be

\[
\delta^i \alpha := \left. \frac{\partial^i}{\partial t_1 \ldots \partial t_i} \right|_{t_1, \ldots, t_i=0} (\alpha \circ j_s \Gamma_i(t_1, \ldots, t_i)(y)). \tag{8}
\]

The following two Lemmas state the relation between the \(i\)–th formal variation of a morphism and its iterated Lie derivative \[13, 14, 17, 20\].

**Lemma 1** Let \(\alpha : J_s Y \to \mathbb{R}^n \wedge T^* J_s Y\) and \(L_j \Xi_k\) be the Lie derivative operator acting on differential fibered morphism.

Let \(\Gamma_i\) be the \(i\)–th formal variation generated by variation vector fields \(\Xi_k\), \(1 \leq k \leq i\) on \(Y\). Then we have

\[
\delta^i \alpha = L_j \Xi_1 \ldots L_j \Xi_i \alpha. \tag{9}
\]

**Lemma 2** Let \(\Xi\) be a variation vector field on \(Y\) and \(\lambda \in \Lambda^n Y\). Then we have \(\delta \lambda = j_s \Xi \, d_V \lambda\) \[22\].

**Remark 11** Owing to the linearity properties of \(d_V \lambda\), we can think of the operator \(\delta\) as a linear morphism with respect to the vector bundle structure \(J_s VY \to X\), so that we can write \(\delta \alpha : J_s Y \to J_s V^* Y \wedge \mathbb{R}^n T^* X\). This property can be obviously iterated for each integer \(i\), so that one can analogously define an \(i\)–linear morphism \(\delta^i\). In particular, we have \(\delta^2 \alpha : J_s Y \times J_s VY \to J_s V^* Y \otimes J_s V^* Y \wedge \mathbb{R}^n T^* X\).

For notational convenience and by an abuse of notation, in the sequel we shall denote with the same symbol an object defined on the vertical prolongation \(VY\) as well as the corresponding one defined on the iterated vertical prolongation \(V(VY)\), whenever there is no danger of confusion.
Corollary 1 Let $\lambda \in (\Lambda^n_0)_Y$. Let $\Xi_1, \Xi_2$ be two variation vector fields on $Y$ generating the formal variation $\Gamma_2$. Then we have

$$\delta^2 \lambda = j_{2s} \Xi_2 | E_n(\delta \lambda) + d_H(j_{2s} \Xi_2 | p_{d_\lambda} \delta \lambda)$$

(10)

$$= \delta(j_{2s} \Xi_1 | E_n(\lambda) + d_H(j_{2s} \Xi_1 | p_{d_\lambda} \lambda))$$

(11)

$$= j_{2s} \Xi_2 | E_n((j_{2s} \Xi_1 | E_n(\lambda)) + d_H(j_{2s+1} \Xi_2 | p_{d_\lambda}(\Xi_1 | E_n(\lambda))) +$$

(12)

$$+ \delta(j_{2s} \Xi_1 | p_{d_\lambda} \lambda)).$$

(13)

Proof. We apply Lemma 2 and decomposition provided by Theorem 1. Furthermore, $d_H \delta = \delta d_H$, which follows directly from the analogous naturality property of the Lie derivative operator.

Remark 12 From the relations above we also infer, of course, that

$$j_{2s} \Xi_2 | E_n((j_{2s} \Xi_1 | E_n(\lambda)) + d_H(j_{2s+1} \Xi_2 | p_{d_\lambda}(\Xi_1 | E_n(\lambda))) =$$

$$= \delta(j_{2s} \Xi_1 | E_n(\lambda)) = \delta^2 \lambda,$$

$$j_{2s} \Xi_2 | E_n((j_{2s} \Xi_1 | E_n(\lambda)) + d_H(\delta(j_{2s} \Xi_1 | p_{d_\lambda} \lambda)) = j_{2s} \Xi_2 | E_n(\delta \lambda).$$

3.1 Variational vertical derivatives and generalized Jacobi morphisms

In this Section we restrict our attention to morphisms which are (identified with) sections of sheaves in the variational sequence. We shall recall some results of ours [13, 14] by defining the $i$-th variational vertical derivative of morphisms.

Let $\alpha \in (\mathcal{V}_n^0)_Y$. We have

$$\delta^i [\alpha] := [\delta^i \alpha] = [L_{\Xi_i} \ldots L_{\Xi_1} \alpha] = L_{\Xi_i} \ldots L_{\Xi_1} [\alpha].$$

Definition 11 We call the operator $\delta^i$ the $i$-th variational vertical derivative.

In [14] the variational vertical derivative was introduced as an operator acting on sections of sheaves of variational sequences, by showing that this operator is in fact a natural transformation, being a functor on the category of variational sequences as it can be summarized by the following commutative diagram.
As a straightforward consequence we have the following characterization of the second variation of a generalized Lagrangian in the variational sequence.

**Proposition 1** Let \( \lambda \in (Y^m)_Y \) and let \( \Xi \) be a variation vector field; then we have

\[
\delta^2 \lambda = [\epsilon_n(j_{2s}\Xi; h\delta\lambda) + C_1(j_{2s}\Xi \otimes K_{h\delta\lambda})].
\]

**Proof.** Since \( \delta \lambda \in (V^{n+1})_Y \) and \( h\delta\lambda \in (V^{n+2})_Y \), the assertion follows by a straightforward application of Theorem 2. Notice that here \( C_1(j_{2s}\Xi \otimes K_{h\delta\lambda}) \): \( \tilde{J}_{2s}Y \times J_{2s}VY \to V^*Y \wedge (\check{n}T^*X) \).

**Remark 13** Let \( l \geq 0 \) and let \( F \) be any vector bundle over \( X \). Let \( \alpha : J_l(Y \times X F) \to \check{n}T^*X \) be a linear morphism with respect to the fibering \( J_lY \times J_lF \to J_lY \) and let \( \check{D}_H \) be the horizontal differential on \( Y \times F \). We can uniquely write \( \alpha \) as

\[
\bar{\alpha} \equiv \alpha : J_lY \to C_0^* [F] \wedge (\check{n}T^*X).
\]

Then \( \check{D}_H \alpha = \check{D}_H \bar{\alpha} \) (this property was pointed out in [16]).

**Lemma 3** Let \( \Xi \) be a variation vector field. Let \( \chi(\lambda, \Xi) := C_1(j_{2s}\Xi \otimes K_{h\delta\lambda}) \equiv E_{j_l, \Xi | h(d\delta\lambda)} \) and let \( \check{D}_H \) be the horizontal differential on \( Y \times VY \). We can see \( \chi(\lambda, \Xi) \) as an extended morphism \( \chi(\lambda, \Xi) : J_{2s}(Y \times VY_X) \to J_{2s}V^*(Y \times VY_X) \) \( V^*Y \wedge (\check{n}T^*X) \) satisfying \( \check{D}_H \chi(\lambda, \Xi) = 0 \).

**Proof.** The morphism \( \chi(\lambda, \Xi) : J_{2s}(Y \times VY_X) \to V^*Y \wedge (\check{n}T^*X) \) is a linear morphism with respect to the projection \( J_{2s}(Y \times VY_X) \to J_{2s}Y \) (see
Remark \[[11]\], then we can apply the Remark above, so that \(\chi(\lambda, \Xi) : J_2V \rightarrow J_2V^*(VY) \otimes V^*Y \land (\wedge^n T^*X) \simeq J_2V^*Y \otimes J_2V^*Y \otimes V^*Y \land (\wedge^n T^*X)\) and again by linearity we get \(\chi(\lambda, \Xi) : J_2s(Y \times VY) \rightarrow J_2sV^*Y \otimes V^*Y \land (\wedge^n T^*X)\).  

The following Lemma is an application of an abstract result, due to Horák and Kolář \[21, 29, 30\], concerning a global decomposition formula for vertical morphisms.

**Lemma 4** Let \(\Xi\) be a variation vector field.  
Let \(\chi(\lambda, \Xi)\) as in the above Lemma. Then we have \((\pi_2^{4s+1})^*\chi(\lambda, \Xi) = E_{\chi(\lambda, \Xi)} + F_{\chi(\lambda, \Xi)}\), where

\[
E_{\chi(\lambda, \Xi)} : J_4s(Y \times VY) \rightarrow C_0^s[Y] \otimes C_0^s[Y] \land (\wedge^n T^*X),
\]

and locally, \(F_{\chi(\lambda, \Xi)} = \hat{D}_HM_{\chi(\lambda, \Xi)}\), with

\[
M_{\chi(\lambda, \Xi)} : J_{4s-1}(Y \times VY) \rightarrow C_{4s-1}^s[Y] \otimes C_0^s[Y] \land (\wedge^n T^*X).
\]

**Proof.** Following e.g. \[29, 30, 40\], the global morphisms \(E_{\chi(\lambda, \Xi)}\) and \(F_{\chi(\lambda, \Xi)}\) can be evaluated by means of a backwards procedure.

**Definition 12** We call the morphism \(J(\lambda, \Xi) := E_{\chi(\lambda, \Xi)}\) the generalized Jacobi morphism associated with the Lagrangian \(\lambda\).

**Example 1** Let us write explicitly the coordinate expression of \(J(\lambda, \Xi)\). By functoriality of \(\delta\), we have \(h(\delta \lambda) = h(\delta \lambda)\). Let now locally \(\lambda = L\omega\), where \(L\) is a function of \(J_4Y\) and \(\omega\) a volume form on \(X\), then \(d\lambda = \partial_t^\alpha(L)d_\alpha \wedge \omega\) and \(\delta d\lambda = \partial_t^\alpha(\partial_t^\mu L)d_\alpha \wedge \omega\), thus finally \(h(\delta \lambda) = h(\delta \lambda) = \partial_t^\alpha(L)d_\alpha \wedge \omega\). As a consequence we have, with \(0 \leq |\mu|, |\alpha|, |\sigma| \leq 2s + 1\):

\[
\chi(\lambda, \Xi) = D_\sigma \Xi_V \left( \partial_t^\alpha(\partial_t^\mu L) - \sum_{|\alpha| = 0}^{s - |\mu|} (-1)^{|\mu| + |\alpha|} \frac{(\mu + \alpha)!}{\mu! \alpha!} D_\alpha \partial_t^\alpha(\partial_t^\mu L) \right) \partial_t^\sigma \otimes \partial_t^\mu \otimes \partial_t^\mu \wedge \omega =
\]

\[
\hat{\lambda}^\mu_{\sigma j} \partial_t^\sigma \otimes \partial_t^\mu \otimes \partial_t^\mu \wedge \omega;
\]

and by the Lemma above, we get (up to divergencies):

\[
J(\lambda, \Xi) = (-1)^{|\alpha|} D_\alpha \chi_\sigma j_{\sigma j} \partial_t^\sigma \otimes \partial_t^\mu \otimes \partial_t^\mu \wedge \omega. \tag{16}
\]

### 3.2 Generalized gauge-natural Jacobi morphisms

We intend now to specify the just mentioned results and definitions concerning the Jacobi morphism by considering as variation vector fields the vertical parts of prolongations of gauge-natural lifts of infinitesimal principal automorphisms to
the gauge-natural bundle $Y_\zeta$. Owing to linearity properties of the Lie derivative of sections and taking into account the fact that, as we already recalled, $j_\ast \xi_V = -\ell_{j_\ast \xi}$, we can state the following important results.

Recall (see [32], Proposition 15.5) that the jet prolongation of order $s$ of $Y_\zeta$ is a gauge-natural bundle itself associated to some principal prolongation of order $(r+s,k+s)$ of the underlying principal bundle $P$. Let $\hat{\xi} \in A^{(r,k)}$ and $\hat{\xi} := G\hat{\xi}$ associated with the Lagrangian $\lambda$. We consider $j_\ast \hat{\xi}_V$ too, i.e. $j_\ast \xi_\hat{\xi} = \xi_\hat{\xi}$. Let us consider $j_\ast \xi_V$, i.e. the vertical part according to the splitting (1). We shall denote by $j_\ast \hat{\xi}_V$ the induced section of the vector bundle $J^{(s,k)}\xi_\hat{\xi}$ of order $(s,k)$, $J^{(s,k)}\xi_\hat{\xi}$ is a bundle itself associated to some principal prolongation $\xi_\hat{\xi}$, we can state the following important results.

We have the following:

**Proposition 2** The morphism $J_4: Y_\zeta \times X J_4 A^{(r,k)} \rightarrow J_{4k} Y_\zeta$. 

We are now able to provide an important specialization of Theorem 2.

**Proposition 3** Let \( L_j \bar{\Xi} \in (\mathcal{V}_k^{n+1})_Y \). Then we have

\[
\kappa_{s+1}^4 \circ L_j \bar{\Xi} [L_{j+1} \bar{\Xi} \lambda] = E_n(j_s \bar{\Xi} h(L_{j+1} \bar{\Xi} \lambda)) + J(\lambda, \Theta(\bar{\Xi})_V).
\]

**Proof.** By Theorem 2 and the Lemma above we have:

\[
\kappa_{s+1}^4 \circ L_j \bar{\Xi} [L_{j+1} \bar{\Xi} \lambda] = E_n(j_s \bar{\Xi} h(L_{j+1} \bar{\Xi} \lambda)) + (18) + [C_1(j_2 \bar{\Xi} \otimes K h \mathcal{L}_j \bar{\Xi} \lambda)] = E_n(j_s \bar{\Xi} h(L_{j+1} \bar{\Xi} \lambda)) + (19) - E_n(j_s \bar{\Xi} h(\mathcal{L}_{j+1} \bar{\Xi} \lambda)) = E_n(j_s \bar{\Xi} h(L_{j+1} \bar{\Xi} \lambda)) + (20) + J(\lambda, \Theta(\bar{\Xi})_V).
\]

\( \square \)

**Remark 14** Theorem 1 in Subsection 2.6 provides an invariant decomposition, where both pieces are globally defined. However, the second one is only locally a divergence, unless some further geometric structures such as linear symmetric connections on the basis manifold or suitable gauge-natural principal (or prolongations with respect to linear symmetric connections of principal) connections are introduced \([29, 30, 46]\). Proposition 3 above, instead, provides an invariant decomposition into two pieces which are globally defined and no one of them, seen as a section of \((\mathcal{V}_k^{n+1})_Y\), is a (local) divergence. As we shall see, this fact has very important consequences concerning conserved quantities in gauge-natural Lagrangian field theories.

A simple comparison of Remark 12, Proposition 1 and the Proposition above gives us the following.

**Corollary 2** Let \( \delta \lambda \) be the variation of \( \lambda \) with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. We have:

\[
\Theta(\bar{\Xi})_V | E_n(\Theta(\bar{\Xi})_V) | E_n(\lambda) = \delta \lambda = E_n(\Theta(\bar{\Xi})_V) h(d\delta \lambda)) \tag{22}
\]

The reader should notice that, seen as a section of \((\mathcal{V}_k^n)_{Y \times V Y}\), the equivalence class \([E_n(j_s \bar{\Xi} h(\delta \lambda))]\) vanishes being a local divergence of higher degree contact forms. This result can also be compared with \([17]\).

### 4 Noether Theorems and conserved currents for gauge-natural invariant Lagrangians

In the following we assume that the field equations are generated by means of a variational principle from a Lagrangian which is gauge-natural invariant, i.e., invariant with respect to any gauge-natural lift of infinitesimal right invariant vector fields. We consider now a projectable vector field \((\bar{\Xi}, \xi)\) on \(Y_\zeta\) and take into account the Lie derivative with respect to its prolongation \(j_s \bar{\Xi}\).
**Definition 14** Let $(\hat{\xi}, \xi)$ be a projectable vector field on $Y_\zeta$. Let $\lambda \in \mathcal{V}_s^n$ be a generalized Lagrangian. We say $\hat{\xi}$ to be a symmetry of $\lambda$ if $\mathcal{L}_{\hat{j} + 1, \hat{\xi}} \lambda = 0$.

We say $\lambda$ to be a gauge-natural invariant Lagrangian if the gauge-natural lift $(\hat{\xi}, \xi)$ of any vector field $\bar{\xi} \in \mathcal{A}^{(r,k)}$ is a symmetry for $\lambda$, i.e., if $\mathcal{L}_{\bar{j} + 1, \bar{\xi}} \lambda = 0$.

In this case the projectable vector field $\hat{\xi} \equiv \Theta(\bar{\xi})$ is called a gauge-natural symmetry of $\lambda$.

**Remark 15** Due to $\mathcal{E}_n \mathcal{L}_{\bar{j}, \bar{\xi}} = \mathcal{L}_{\bar{j}, \bar{\xi}} \mathcal{E}_n$, a symmetry of a Lagrangian $\lambda$ is also a symmetry of its Euler–Lagrange morphism $E_\lambda$ (but the converse is not true, see e.g. [44]). Symmetries of a Lagrangian $\lambda$ are calculated by means of Noether Theorems, which takes a particularly interesting form in the case of gauge-natural Lagrangians.

**Proposition 4** Let $\lambda \in \mathcal{V}_s^n$ be a gauge-natural Lagrangian and $(\hat{\xi}, \xi)$ a gauge-natural symmetry of $\lambda$. Then we have

$$0 = -\mathcal{L}_{\hat{\xi}} \mathcal{E}_n(\lambda) + d_H(-j_s \mathcal{L}_{\hat{\xi}} p_{dV} \lambda + \xi | \lambda).$$

(23)

Suppose that the section $\sigma$ fulfills the condition $(j_{2s+1} \sigma)^*(-\mathcal{L}_{\hat{\xi}} \mathcal{E}_n(\lambda)) = 0$. Then, the $(n-1)$-form

$$\epsilon = -j_s \mathcal{L}_{\hat{\xi}} p_{dV} \lambda + \xi | \lambda,$$

(24)

fulfills the equation $d((j_{2s} \sigma)^*(\epsilon)) = 0$.

**Remark 16** If $\sigma$ is a critical section for $\mathcal{E}_n(\lambda)$, i.e. $(j_{2s+1} \sigma)^*\mathcal{E}_n(\lambda) = 0$, the above equation admits a physical interpretation as a so-called weak conservation law for the density associated with $\epsilon$.

**Definition 15** Let $\lambda \in \mathcal{V}_s^n$ be a gauge-natural Lagrangian and $\bar{\xi} \in \mathcal{A}^{(r,k)}$. Then the sheaf morphism $\epsilon : J_{2s} Y_\zeta \times V J_{2s} A^{(r,k)} \rightarrow C_0^s[A^{(r,k)}] \land (n-1 \land T^* X)$ is said to be a gauge-natural weakly conserved current.

**Remark 17** In general, this conserved current is not uniquely defined. In fact, it depends on the choice of $p_{dV} \lambda$, which is not unique (see [14] and references quoted therein). Moreover, we could add to the conserved current any form $\mu \in \mathcal{V}_s^{n-1}$ which is variationally closed, i.e. such that $\mathcal{E}_{n-1}(\mu) = 0$ holds. The form $\mu$ is locally of the type $\mu = d_H \gamma$, where $\gamma \in \mathcal{V}_s^{n-2}$.

**Corollary 3** Let $\epsilon : J_{2s} Y_\zeta \times V J_{2s} A^{(r,k)} \rightarrow C_0^s[A^{(r,k)}] \land (n-1 \land T^* X)$ be a conserved current. As an immediate consequence of Remark 15 we can regard $\epsilon$ as the equivalent morphism $\epsilon \equiv \tau : J_{2s} Y_\zeta \times V J_{2s} A^{(r,k)} \rightarrow C_{2s}^s[A^{(r,k)}] \otimes C_0^s[A^{(r,k)}] \land (n-1 \land T^* X)$. 

18
Remark 18 Let $\eta \in V^0_{n+1}$ and let $\Xi$ be a symmetry of $\eta$. Then, as a special case of Theorem 2 we have $0 = \mathcal{E}_n(\Xi V | \eta) + [C^1_1(J_{2s+1}\Xi V \otimes K_{bdn})]$ Suppose that $K_{bdn} = 0$; then we have $\mathcal{E}_n(\Xi V | \eta) = 0$. This implies that $\Xi V | \eta$ is variationally trivial, i.e. it is locally of the type $\Xi = d_{\eta} H$, where $H \in \Lambda^1_{n-1}$.

Suppose that the section $\sigma : X \to Y_\zeta$ fulfills $(j_{2s+1}\sigma)^* (\Xi V | \eta) = 0$. Then we have $d((j_2\sigma)^* \mu) = 0$ so that, as in the case of symmetries of Lagrangians, $\mu$ is a conserved current along $\sigma$.

As in the case of Lagrangians, a conserved current for an Euler–Lagrange type morphism is not uniquely defined. In fact, we could add to $\Xi V | \eta$ any variationally trivial Lagrangian, obtaining different conserved currents. Moreover, such conserved currents are defined up to variationally trivial $(n-1)$–forms. □

4.1 The Bianchi morphism

Let $\lambda$ be a gauge-natural Lagrangian. By the linearity of $L$ we have

$$\omega(\lambda, \Theta(\Xi)_V) = \mathcal{E}_n(\lambda) : J_{2s} Y_\zeta \to C^*_2 [A^{(r,k)}] \otimes C^n_0 [A^{(r,k)}] \wedge (\wedge^n T^* X)$$

We have $D_H \omega(\lambda, \Theta(\Xi)_V) = 0$. We can regard $\omega(\lambda, \Theta(\Xi)_V)$ as the extended morphism $\omega(\lambda, \Theta(\Xi)_V) : J_{2s} Y_\zeta \times X J_{2s} A^{(r,k)} \to C^*_2 [A^{(r,k)}] \otimes C^n_0 [A^{(r,k)}] \wedge (\wedge^n T^* X)$. Thus we can state the following.

Lemma 6 Let $\omega(\lambda, \Theta(\Xi)_V)$ be as in the above Remark. Then we have globally

$$(\sigma_{2s+1})^* \omega(\lambda, \Theta(\Xi)_V) = \beta(\lambda, \Theta(\Xi)_V) + F_{\omega(\lambda, \Theta(\Xi)_V)}$$

where

$$\beta(\lambda, \Theta(\Xi)_V) = E_{\omega(\lambda, \Theta(\Xi)_V)} : J_{4s} Y_\zeta \times X J_{4s} A^{(r,k)} \to C^*_2 [A^{(r,k)}] \otimes C^n_0 [A^{(r,k)}] \wedge (\wedge^n T^* X)$$

19
and locally, $F_{\omega(\mathcal{E}(\xi)V)} = D_H M_{\omega(\mathcal{E}(\xi)V)}$, with

$$
M_{\omega(\mathcal{E}(\xi)V)} : J_{4s-1}Y_\xi \times V J_{4s-1}A^{(r,k)} \to X \rightleftharpoons C_{2s}^+ [A^{(r,k)}] \otimes C_{2s-1}^+ [A^{(r,k)}] \otimes C_0^+ [A^{(r,k)}] \wedge (\wedge^{n-1} T^* X).
$$

In particular, we get the following local decomposition of $\omega(\mathcal{E}(\xi)V)$:

$$
\omega(\mathcal{E}(\xi)V) = \beta(\mathcal{E}(\xi)V) + D_H \tilde{e}(\mathcal{E}(\xi)V), \tag{29}
$$

\textbf{Proof.} We take into account that $D_H \omega(\mathcal{E}(\xi)V)$ is obviously vanishing, then the result is a straightforward consequence of Lemma 4. \qed

\textbf{Definition 16} We call the global morphism $\beta(\mathcal{E}(\xi)V) := E_{\omega(\mathcal{E}(\xi)V)}$ the generalized Bianchi morphism associated with the Lagrangian $\lambda$.

\textbf{Remark 20} For any $(\mathcal{E}, \xi) \in A^{(r,k)}$, as a consequence of the gauge-natural invariance of the Lagrangian, the morphism $\beta(\mathcal{E}(\xi)V) \equiv E_{\omega(\mathcal{E}(\xi)V)}$ is locally identically vanishing. We stress that these are just local generalized Bianchi identities. In particular, we have $\omega(\mathcal{E}(\xi)V) = D_H \tilde{e}(\mathcal{E}(\xi)V)$ locally $[3, 5, 6, 19, 35]$.

\textbf{Definition 17} The form $\tilde{e}(\mathcal{E}(\xi)V)$ is called a local reduced current.

\section{4.2 Global generalized Bianchi identities}

We are now able to state our main result providing necessary and sufficient conditions on the gauge-natural lift of infinitesimal right-invariant automorphisms of the principal bundle $P$ in order to get globally defined generalized Bianchi identities. Let $\mathcal{R} := Ker J(\mathcal{E}(\xi)V)$ be the kernel of the generalized gauge-natural morphism $J(\mathcal{E}(\xi)V)$. As a consequence of the considerations above, we have the following important result.

\textbf{Theorem 3} The generalized Bianchi morphism is globally vanishing if and only if $\delta^2_0 \lambda \equiv J(\mathcal{E}(\xi)V) = 0$, i.e. if and only if $\mathcal{E}(\xi)V \in \mathcal{R}$.

\textbf{Proof.} By Corollary 2 we get

$$
\mathcal{E}(\xi)V \mid \beta(\mathcal{E}(\xi)V) = \delta^2_0 \lambda \equiv J(\mathcal{E}(\xi)V).
$$

Now, if $\mathcal{E}(\xi)V \in \mathcal{R}$ then $\beta(\mathcal{E}(\xi)V) \equiv 0$, which are global generalized Bianchi identities. Vice versa, if $\mathcal{E}(\xi)V$ is such that $\beta(\mathcal{E}(\xi)V) = 0$, then $J(\mathcal{E}(\xi)V) = 0$ and $\mathcal{E}(\xi)V \in \mathcal{R}$. Notice that $\mathcal{E}(\xi)V \mid \beta(\mathcal{E}(\xi)V)$ is nothing but the Hessian morphism associated with $\lambda$ (see [17]). \qed

20
Remark 21 We recall that given a vector field $j_s \hat{\Xi} : J_s Y_\zeta \to TJ_s Y_\zeta$, the splitting yields $j_s \hat{\Xi} \circ \pi_s^{s+1} = j_s \hat{\Xi}_H + j_s \hat{\Xi}_V$ where, if $j_s \hat{\Xi} = \hat{\Xi}^\gamma \partial_\gamma + \hat{\Xi}_\alpha^i \partial_\alpha^i$, then we have $j_s \hat{\Xi}_H = \hat{\Xi}^\gamma D_\gamma$ and $j_s \hat{\Xi}_V = D_\alpha (\hat{\Xi}^i - y_i^\gamma \hat{\Xi}^\gamma) \partial_\alpha^i$. Analogous considerations hold true of course also for the unique corresponding invariant vector field $j_s \bar{\Xi}$ on $W^{(r,k)} P$. In particular, the condition $j_s \bar{\Xi}_V \in \mathcal{K}$ implies, of course, that the components $\bar{\Xi}_\alpha^i$ and $\bar{\Xi}^\gamma$ are not independent, but they are related in such a way that $j_s \bar{\Xi}_V$ must be a solution of generalized gauge-natural Jacobi equations for the Lagrangian $\lambda$ (see coordinate expression (16)). The geometric interpretation of this condition will be the subject of a forthcoming paper [41]. Our results are quite evidently related to the theory of $G$-reductive Lie derivatives developed in [18]. It is in fact our opinion that the Kosmann lift (the kind of gauge-natural lift used to correctly define the Lie derivative of spinors, in [18] interpreted as a special kind of reductive lift) can be recognized as a kind of gauge-natural Jacobi vector field. Even more, we believe that the kernel of the generalized gauge-natural Jacobi morphism induces a canonical reductive pair on $W^{(r,k)} P$. We also remark that, for each $\Xi \in \mathcal{A}^{(r,k)}$ such that $\bar{\Xi}_V \in \mathcal{R}$, we have $\mathcal{L}_{j_s \bar{\Xi}_V} \omega(\lambda, \mathcal{R}) = 0$. Here it is enough to stress that, within a gauge-natural invariant Lagrangian variational principle the gauge-natural lift of infinitesimal principal automorphism is not intrinsically arbitrary. It would be also interesting to compare such results with reduction theorems stated in [23].

In the following we shall refer to canonical globally defined objects (such as currents or corresponding superpotentials) by their explicit dependence on $\mathcal{R}$.

Corollary 4 Let $\lambda \in \mathcal{V}_n^s$ be a gauge-natural Lagrangian and $j_s \hat{\Xi}_V \in \mathcal{R}$ a gauge-natural symmetry of $\lambda$. Being $\beta(\lambda, \mathcal{R}) \equiv 0$, we have, globally, $\omega(\lambda, \mathcal{R}) = D_H \epsilon(\lambda, \mathcal{R})$, then the following holds:

$$D_H(\epsilon(\lambda, \mathcal{R}) - \bar{\epsilon}(\lambda, \mathcal{R})) = 0.$$  \hfill (30)

Eq. (30) is referred as a gauge-natural ‘strong’ conservation law for the global density $\epsilon(\lambda, \mathcal{R}) - \bar{\epsilon}(\lambda, \mathcal{R})$.

We can now state the following fundamental result about the existence and globality of gauge-natural superpotentials in the framework of variational sequences.

Theorem 4 Let $\lambda \in \mathcal{V}_n^s$ be a gauge-natural Lagrangian and $(j_s \hat{\Xi}, \xi)$ a gauge-natural symmetry of $\lambda$. Then there exists a global sheaf morphism $\nu(\lambda, \mathcal{R}) \in \mathcal{V}_{n-2}^{s-1} Y_\times \times \mathcal{R}$ such that

$$D_H \nu(\lambda, \mathcal{R}) = \epsilon(\lambda, \mathcal{R}) - \bar{\epsilon}(\lambda, \mathcal{R}).$$

Definition 18 We define the sheaf morphism $\nu(\lambda, \mathcal{R})$ to be a canonical gauge-natural superpotential associated with $\lambda$.

Acknowledgments. The authors wish to thank Prof. I. Kolár for many interesting and useful discussions.
References

[1] R. J. Alonso: Decomposition of higher order tangent fields and calculus of variations. Proc. Diff. Geom. Appl. (Brno, 1998), 451–460, Masaryk Univ., Brno, 1999.

[2] R. J. Alonso: $D$-modules, contact valued calculus and Poincaré–Cartan form. Czech. Math. J. 49(124) (3) (1999) 585–606.

[3] J.L. Anderson, P.G. Bergmann: Constraints in Covariant Field Theories, Phys. Rev. 83 (5) (1951) 1018–1025.

[4] G. Allemandi, L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri: Nöther Conserved Quantities and Entropy in General Relativity, in: Recent Developments in General Relativity; Genoa 2000; R. Cianci et al. eds.; Springer Italia, Milano (2001) 75–92.

[5] P.G. Bergman: Non-Linear Field Theories, Phys. Rev. 75 (4) (1949) 680–685.

[6] P.G. Bergman: Conservation Laws in General relativity as the Generators of Coordinate Transformations, Phys. Rev. 112 (1) (1958) 287–289.

[7] P.T. Chrusciel: On the relation between the Einstein and the Komar expressions for the energy of the gravitational field, Ann. Inst. Henri Poincaré 42 (3) (1985) 267–282.

[8] D.J. Eck: Gauge-natural bundles and generalized gauge theories, Mem. Amer. Math. Soc. 247 (1981) 1–48.

[9] L. Fatibene, M. Francaviglia, M. Raiteri: Gauge natural field theories and applications to conservation laws. Proc. VIII Conf. Diff. Geom. and Appl.; O. Kowalski et al. eds.; Silesian University at Opava, Opava (Czech Republic) 2001, 401–413.

[10] L. Fatibene, M. Francaviglia, M. Palese: Conservation laws and variational sequences in gauge-natural theories, Math. Proc. Camb. Phil. Soc. 130 (2001) 555–569.

[11] M. Ferraris, M. Francaviglia: The Lagrangian Approach to Conserved Quantities in General Relativity, in: Mechanics, Analysis and Geometry: 200 Years after Lagrange; M. Francaviglia ed.; Elsevier Science Publishers B. V. (Amsterdam, 1991), 451–488.

[12] M. Ferraris, M. Francaviglia, M. Raiteri: Conserved Quantities from the Equations of Motion (with applications to natural and gauge natural theories of gravitation) Class. Quant. Grav. 20 (2003) 4043–4066.

[13] M. Francaviglia, M. Palese: Second Order Variations in Variational Sequences, Steps in differential geometry (Debrecen, 2000) Inst. Math. Inform. Debrecen, Hungary (2001) 119–130.

[14] M. Francaviglia, M. Palese: Generalized Jacobi morphisms in variational sequences, in Proc. XXI Winter School Geometry and Physics, Srni 2001 Rend. Circ. Matem. di Palermo. Serie II, Suppl. 69 (2002) 195–208.

[15] M. Francaviglia, M. Palese, R. Vitolo: Symmetries in Finite Order Variational Sequences, Czech. Math. J. 52(127) (2002) 197–213.

[16] M. Francaviglia, M. Palese, R. Vitolo: Superpotentials in variational sequences, Proc. VII Conf. Diff. Geom. and Appl., Satellite Conf. of ICM in Berlin (Brno 1998); I. Kolář et al. eds.; Masaryk University in Brno (Czech Republic) 1999, 469–480.

[17] M. Francaviglia, M. Palese, R. Vitolo: The Hessian and Jacobi Morphisms for Higher Order Calculus of Variations, Diff. Geom. Appl. 22 (1) (2005) 105–120.
[18] M. Godina, P. Matteucci: Reductive G-structures and Lie derivatives, *J. Geom. Phys.* 47 (1) (2003) 66–86.

[19] J.N. Goldberg: Conservation laws in general relativity. *Phys. Rev.* (2) 111 (1958) 315–320.

[20] H. Goldschmidt, S. Sternberg: The Hamilton–Cartan Formalism in the Calculus of Variations, *Ann. Inst. Fourier, Grenoble* 23 (1) (1973) 203–267.

[21] M. Horáček, I. Kolář: On the Higher Order Poincaré–Cartan Forms, *Czech. Math. J.*, 33 (108) (1983) 467–475.

[22] J. Janýška: Natural and gauge-Natural Operators on the Space of Linear Connections on a Vector Bundle, *Proc. Diff. Geom. and its Appl.* (Brno, 1989); J. Janýška, D. Krupka eds.; World Scientific (Singapore, 1990) 58–68.

[23] J. Janýška: Reduction theorems for general linear connections, *Differential Geom. Appl.* 20 (2004), no. 2, 177–196.

[24] J. Janýška, M. Modugno: Infinitesimal natural and gauge-natural lifts, *Differential Geom. Appl.* 2 (2) (1992) 99–121.

[25] B. Julia, S. Silva: Currents and superpotentials in classical gauge theories. II. Global Aspects and the example of affine gravity, *Class. Quant. Grav.*, 17 (22) (2000) 4733–4743.

[26] J. Katz: A note on Komar’s anomalous factor, *Classical Quantum Gravity* 2 (3) (1985) 423–425.

[27] I. Kolář: On some operations with connections, *Math. Nachr.*, 69 (1975) 297–306.

[28] I. Kolář: Prolongations of generalized connections, *Coll. Math. Soc. János Bolyai,* (Differential Geometry, Budapest, 1979) 31 (1979) 317–325.

[29] I. Kolář: A Geometrical Version of the Higher Order Hamilton Formalism in Fibred Manifolds, *J. Geom. Phys.*, 1 (2) (1984) 127–137.

[30] I. Kolář: Some Geometric Aspects of the Higher Order Variational Calculus, *Geom. Meth. in Phys., Proc. Diff. Geom. and its Appl.,* (Nové Město na Moravě, 1983); D. Krupka ed.; J. E. Purkyně University (Brno, 1984) 155–166.

[31] I. Kolář: Natural operators related with the variational calculus. Proc. Diff. Geom. Appl. (Opava, 1992), 461–472, *Math. Publ.* 1 Silesian Univ. Opava, Opava, 1993.

[32] I. Kolář, P.W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, (Springer–Verlag, N.Y., 1993).

[33] I. Kolář, G. Vírsk: Connections in first principal prolongations, in Proc. XVI Winter School Geometry and Physics, Srni 1995 *Rend. Circ. Matem. di Palermo. Serie II, Suppl.* 43 (1995) 163–171.

[34] I. Kolář, R. Vitolo: On the Helmholtz operator for Euler morphisms, *Math. Proc. Cambridge Phil. Soc.*, 135 (2) (2003) 277–290.

[35] A. Komar: Covariant Conservation Laws in General Relativity, *Phys. Rev.* 113 (1959) (3) 934–936.

[36] D. Krupka: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. and its Appl.* (Brno, 1989); J. Janýška, D. Krupka eds.; World Scientific (Singapore, 1990) 236–254.
[37] D. Krupka: Topics in the Calculus of Variations: Finite Order Variational Sequences; O. Kowalski and D. Krupka eds., Proc. Diff. Geom. and its Appl. (Opava, 1992), Math. Publ. 1, Silesian Univ. Opava, Opava, 1993, 473–495.

[38] L. Mangiarotti, M. Modugno: Fibered Spaces, Jet Spaces and Connections for Field Theories, in Proc. Int. Meet. on Geom. and Phys.; M. Modugno ed.; Pitagora Editrice (Bologna, 1983) 135–165.

[39] E. Nöther: Invariante Variationsprobleme, Nachr. Ges. Wiss. Gött., Math. Phys. Kl. II (1918) 235–257.

[40] M. Palese: Geometric Foundations of the Calculus of Variations, Variational Sequences, Symmetries and Jacobi Morphisms. Ph.D. Thesis, University of Torino (2000).

[41] M. Palese, E. Winterroth: in preparation.

[42] D.J. Saunders: The Geometry of Jet Bundles, Cambridge Univ. Press (Cambridge, 1989).

[43] A. Trautman: Conservation laws in general relativity. in Gravitation: An introduction to current research pp. 169–198 Wiley, New York (1962).

[44] A. Trautman: Noether equations and conservation laws, Comm. Math. Phys. 6 (1967) 248–261.

[45] A. Trautman: A metaphysical remark on variational principles, Acta Phys. Pol. B XX (1996) 1–9.

[46] R. Vitolo: Finite Order Lagrangian Bicomplexes, Math. Proc. Camb. Phil. Soc. 125 (1) (1999) 321–333.