Castelnuovo-Mumford Regularity and Computing the de Rham Cohomology of Smooth Projective Varieties

Peter Scheiblechner*
Department of Mathematics, Purdue University
West Lafayette, IN 47907-2067, USA
pscheibl@math.purdue.edu

Abstract

We describe a parallel polynomial time algorithm for computing the topological Betti numbers of a smooth complex projective variety $X$. It is the first single exponential time algorithm for computing the Betti numbers of a significant class of complex varieties of arbitrary dimension. Our main theoretical result is that the Castelnuovo-Mumford regularity of the sheaf of differential $p$-forms on $X$ is bounded by $p(e m + 1)D$, where $e$, $m$, and $D$ are the maximal codimension, dimension, and degree, respectively, of all irreducible components of $X$. It follows that, for a union $V$ of generic hyperplane sections in $X$, the algebraic de Rham cohomology of $X \setminus V$ is described by differential forms with poles along $V$ of single exponential order. This yields a similar description of the de Rham cohomology of $X$, which allows its efficient computation. Furthermore, we give a parallel polynomial time algorithm for testing whether a projective variety is smooth.

Mathematics Subject Classification (2010) 14Q15, 14Q20, 68W30

1 Introduction

A long standing open problem in algorithmic real algebraic geometry is to construct a single exponential time algorithm for computing the Betti numbers of semialgebraic sets (for an overview see [4]). The best result in this direction is in [3] saying that for fixed $\ell$ one can compute the first $\ell$ Betti numbers of a semialgebraic set in single exponential time.

In the complex setting one approach for computing Betti numbers is to compute the algebraic de Rham cohomology. A result of Grothendieck [27] states

*Partially supported by DFG grant SCHE 1639/1-1.
that the de Rham cohomology of a smooth complex variety is canonically iso-
morphic to the singular cohomology. An algorithm for computing the de Rham
cohomology of the complement of a complex affine variety based on $D$-modules
is given in [43, 56]. This algorithm is used in [57] to compute the de Rham
cohomology of a projective variety. However, these algorithms are not analyzed with regard
to their complexity. Furthermore, they use Gröbner basis computations in a
non-commutative setting, so that a good worst-case complexity is not to be ex-
pected. Indeed, already in the commutative case, computing Gröbner bases is
exponential space complete [37, 36].

A parallel polynomial time algorithm for counting the connected compo-
nents (i.e., computing the zeroth Betti number) of a complex affine variety is given
in [12]. Although this problem can also be solved by applying the corre-
spounding real algorithms, the algorithm of [12] is the first one using the field struc-
ture of $\mathbb{C}$ only. It also extends to counting the irreducible components. In [49]
it is described how one can compute equations for the components in parallel
polynomial time.

Concerning lower bounds it is shown in [48] that it is
PSPACE-hard to com-
pute some fixed Betti number of a complex affine or projective varie-
ty given
over the integers. Note that the varieties constructed in this reduction are
highly singular.

1.1 Main Result

In this paper we describe an algorithm for computing the algebraic de Rham
cohomology of a smooth projective variety running in parallel polynomial time.
It is based on the same techniques as the algorithm in [12] using squarefree
regular chains. Namely, by applying the algorithm of [52, 53] we can con-
struct a linear system of equations describing the ideal (up to a given degree) of an
affine variety. This allows to compute a linear system describing the vanishing
of differential forms of some degree. Given a smooth projective variety $X \subseteq \mathbb{P}^n$
of dimension $m$ and generic hyperplane sections $H_0, \ldots, H_m \subseteq X$ with $H_0 \cap
\cdots \cap H_m = \emptyset$, their complements $U_i = X \setminus H_i$ form an open affine cover of $X$.
Under the additional assumption that the hypersurface $H_0 \cup \cdots \cup H_m$ has normal
crossings, we are able to compute the cohomologies of the affine patches $U_{i_0 \cdots i_q} =
U_{i_0} \cap \cdots \cap U_{i_q}$ and by a Čech process also the cohomology of $X$.

To describe the output of the algorithm explicitly, let $H_i$ be defined by the
linear form $\ell_i$. The cohomology $H^k_{\text{dR}}(X)$ is then represented by a basis consisting
of vectors of rational differential forms $\omega = (\omega_{i_0 \cdots i_q})$ of the form

$$\omega_{i_0 \cdots i_q} = \frac{1}{(\ell_{i_0} \cdots \ell_{i_q}) t} \sum_{0 \leq j_1 < \cdots < j_p \leq n} \omega^{j_1 \cdots j_p}_{i_0 \cdots i_q} dX_{j_1} \wedge \cdots \wedge dX_{j_p}, \quad (1)$$

where $p + q = k$, $t \gg 0$, and the polynomials $\omega^{j_1 \cdots j_p}_{i_0 \cdots i_q} \in \mathbb{C}[X_0, \ldots, X_n]$ are
homogeneous of degree $t(q + 1) - p$.

Furthermore, we show how to test $X$ for smoothness and how to choose the
generic hyperplanes $H_i$ in parallel polynomial time. In summary, we prove the
Theorem 1.1. Given homogeneous polynomials \( f_1, \ldots, f_r \in \mathbb{C}[X_0, \ldots, X_n] \) of degree at most \( d \), one can test whether \( X := Z(f_1, \ldots, f_r) \subseteq \mathbb{P}^n \) is smooth and if so, compute the algebraic de Rham cohomology \( H^*_{\text{dR}}(X) \) in parallel time \( (n \log d)^{O(1)} \) and sequential time \( d^{O(n^2)} \).

As for the necessity of squarefree regular chains in our algorithm we remark that for smooth varieties there are single exponential bounds for the degrees in a Gröbner basis (cf. §1.2). So perhaps one could replace the squarefree regular chains in our approach by Gröbner bases. But we do not know whether such an algorithm would be well-parallelizable.

1.2 Castelnuovo-Mumford Regularity

The main theoretical result of this paper is a bound on the Castelnuovo-Mumford regularity of the sheaf of regular differential \( p \)-forms on a smooth projective variety \( X \). This result allows to bound the degrees of the differential forms one has to deal with in computing the cohomology of \( X \). More precisely, the regularity yields a bound on the order \( t \) in (1), which in turn determines the degree of the coefficients.

The Castelnuovo-Mumford regularity was defined in [40] for sheaves on \( \mathbb{P}^n \). The definition has been modified in [19] to apply to a homogeneous ideal \( I \). This notion was related to computational complexity in [8] by showing that the regularity of \( I \) equals the maximal degree in a reduced Gröbner basis of \( I \) with respect to the degree reverse lexicographic order in generic coordinates. In this respect upper bounds on the regularity of a homogeneous ideal are of particular interest in computational algebraic geometry and commutative algebra. For a general ideal, double exponential upper bounds were shown in [24] and [22]. The famous example of [37] shows that this is essentially best possible. However, there are several results giving better bounds for the regularity in special cases, such as [28, 45, 34, 16, 44, 51, 32, 23]. A nice overview over these kinds of results is given in [8]. This paper also contains a bound on the regularity of the ideal of a smooth variety \( X \), which is asymptotic to the product of the degree and the dimension of \( X \). A more precise bound in terms of the degrees of generators of the ideal of \( X \) is proved in [8]. More generally, the authors prove the vanishing of the higher cohomology of powers of the sufficiently twisted ideal sheaf of \( X \) (cf. Proposition 3.5). Our bound on the regularity of the sheaf of differential forms on \( X \) is deduced from this result. We are not aware of any other bounds on the regularity of a sheaf other than a power of an ideal sheaf.

Acknowledgements

The author is very grateful to Saugata Basu for being his host, many important and interesting discussions, and recommending the book [35]. Without him this work wouldn’t have been possible. The author also thanks Manoj Kummini for
fruitful discussions about the Castelnuovo-Mumford regularity, and Christian Schnell for a discussion about the cohomology of hypersurfaces.

2 Preliminaries

2.1 Basic Notations

Denote by $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$ the projective space over $\mathbb{C}$. A (closed) projective variety $X \subseteq \mathbb{P}^n$ is defined as the zero set

$$X = Z(f_1, \ldots, f_r) := \{ x \in \mathbb{P}^n | f_1(x) = \cdots = f_r(x) = 0 \}$$

of homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[X_0, \ldots, X_n]$. Note that $X$ may be reducible. Occasionally we write $Z_X(f) := X \cap Z(f)$ for $f \in \mathbb{C}[X_0, \ldots, X_n]$. A quasi-projective variety is a difference $X \setminus Y$, where $X$ and $Y$ are closed projective varieties. The term variety will always mean quasi-projective variety. The homogeneous (vanishing) ideal $I(X)$ of the variety $X$ is defined as the ideal generated by the homogeneous polynomials vanishing on $X$. The homogeneous coordinate ring of $X$ is $\mathbb{C}[X] := \mathbb{C}[X_0, \ldots, X_n]/I(X)$. By the (weak) homogeneous Nullstellensatz we have $Z(f_1, \ldots, f_r) = \emptyset$ iff there exists $N \in \mathbb{N}$ with $(X_0, \ldots, X_n)^N \subseteq (f_1, \ldots, f_r)$. Its effective version states that one can choose $N = (n+1)d - n$, when $\deg f_i \leq d$ [33 Théorème 3.3]. According to the affine effective Nullstellensatz, for $f_1, \ldots, f_r \in \mathbb{C}[X_1, \ldots, X_n]$ of degree $\leq d$, we have $Z(f_1, \ldots, f_r) = \emptyset$ iff there exist polynomials $g_1, \ldots, g_r$ with $\deg(g_i f_i) \leq d^n$ such that $1 = \sum_i g_i f_i$ [10, 31, 20, 30]

The dimension $\dim X$ is the Krull dimension of $X$ in the Zariski topology. A variety all of whose irreducible components have the same dimension $m$ is called $(m)$-equidimensional. The local dimension $\dim_x X$ at $x \in X$ is defined as the maximal dimension of all components through $x$. A hypersurface of a variety $X$ is a closed subvariety $V$ with $\dim_x V = \dim_x X - 1$ for all $x \in V$.

We often identify $\mathbb{P}^n \setminus Z(X_i) \simeq \mathbb{C}^n$, $0 \leq i \leq n$, via $(x_0 : \cdots : x_n) \to (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \frac{x_n}{x_i})$, where as usual $\wedge$ denotes omission. Under this identification, a homogeneous polynomial $f \in \mathbb{C}[X_0, X_n]$ corresponds to its dehomogenization $f^i$ by setting $X_i := 1$. We thus get a surjection $^! : \mathbb{C}[X_0, X_n] \to \mathbb{C}[X_0, \ldots, \widehat{X}_i, \ldots, X_n]$, and the image of $I(X)$ under this map is the vanishing ideal of the affine variety $X \setminus Z(X_i)$. Now let $f_1, \ldots, f_r$ be homogeneous polynomials defining the hypersurfaces $H_1, \ldots, H_r$ of $\mathbb{P}^n$. Then we say that the closed variety $X$ is scheme-theoretically cut out by the hypersurfaces $H_1, \ldots, H_r$ iff for each $i$, the dehomogenizations $f_1^i, \ldots, f_r^i$ generate the image of $I(X)$ in $\mathbb{C}[X_0, \ldots, \widehat{X}_i, \ldots, X_n]$.

For a polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ its differential at $x \in \mathbb{C}^n$ is the linear function $d_x f : \mathbb{C}^n \to \mathbb{C}$ defined by $d_x f(v) := \sum_i \frac{\partial f}{\partial x_i}(x) v_i$. The tangent space of the variety $X$ at $x \in X \setminus Z(X_i)$ is defined as the vector subspace

$$T_x X := \{ v \in \mathbb{C}^n | \forall f \in I(X) \; d_x f^i(v) = 0 \} \subseteq \mathbb{C}^n.$$
If $X$ is scheme-theoretically cut out by the hypersurfaces defined by the homogeneous polynomials $f_1, \ldots, f_r$, then $T_x X = \mathcal{Z}(dx_1, \ldots, dx_r)$. We have $\dim T_x X \geq \dim_x X$ for all $x \in X$. We say that $x \in X$ is a smooth point in $X$ iff $\dim T_x X = \dim_x X$. The variety $X$ is smooth iff all of its points are smooth.

The degree $d$ of an irreducible closed variety $X$ of dimension $m$ is defined as the maximal cardinality of $X \cap L$ over all linear subspaces $L \subseteq \mathbb{P}^n$ of dimension $n - m$ [12 §5A]. We define the (cumulative) degree $\deg X$ of a reducible variety $X$ to be the sum of the degrees of all irreducible components of $X$. It follows essentially from Bézout’s Theorem that if $X$ is defined by polynomials of degree $\leq d$, then $\deg X \leq d^n$ [14].

### 2.2 Coherent Sheaves

Let $X$ be a closed variety in $\mathbb{P}^n$. Then every graded $\mathbb{C}[X]$-module $M$ gives rise to a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on $X$ such that, on a principal open set $U = X \setminus \mathcal{Z}(f)$, the sections of $M$ are given by $\Gamma(U, \tilde{M}) = M(f)$, the degree 0 part of the localization of $M$ at $f$. A sheaf $\mathcal{F}$ on $X$ is called coherent iff $\mathcal{F} = \tilde{M}$ with a finitely generated graded $\mathbb{C}[X]$-module $M$.

An important example is of course the structure sheaf $\mathcal{O}_X = \tilde{\mathbb{C}[X]}$. We also define the twisting sheaf $\mathcal{O}_X(k) := \mathbb{C}[X](k)$ for $k \in \mathbb{Z}$, where $\mathbb{C}[X](k) := \mathbb{C}[X]_{k+d}$. Then $\mathcal{O}_X(k)$ is the restriction of $\mathcal{O}_{\mathbb{P}^n}(k)$ to $X$, and we have $\mathcal{O}_X(k) \otimes \mathcal{O}_X(\ell) \cong \mathcal{O}_X(k + \ell)$ [29 II, Proposition 5.12]. The sheaf $\mathcal{O}_X(1)$ is called the very ample line bundle on $X$ determined by the embedding $X \hookrightarrow \mathbb{P}^n$. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on $X$ we define the twisted sheaf $\mathcal{F}(k) := \mathcal{F} \otimes \mathcal{O}_X(k)$ for $k \in \mathbb{Z}$. The ideal sheaf $\mathcal{I}_X$ of $X$ is defined as the kernel of the restriction map $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X$. We have $\mathcal{I}_X = \tilde{I(X)}$, hence the ideal sheaf is coherent.

A particular important example of a coherent sheaf is the sheaf of differential forms which we describe now (cf. §6.1 of [17]). Denote by $\Lambda$ the $\mathbb{C}[X_0, \ldots, X_n]$-module of Kähler differentials $\Omega_{\mathbb{C}[X_0, \ldots, X_n]/\mathbb{C}}$, which is the free module generated by $dX_0, \ldots, dX_n$ [13 Proposition 16.1]. We have the universal derivation $d: \mathbb{C}[X_0, \ldots, X_n] \to \Lambda$ given by $df = \sum_i \frac{\partial f}{\partial X_i} dX_i$. We set $\Lambda^p := \wedge^p \Lambda$ for the $p$-fold exterior power of $\Lambda$. Then $\Lambda^p$ is the free module generated by $dX_{i_1} \wedge \cdots \wedge dX_{i_p}$, $0 \leq i_1 < \cdots < i_p \leq n$. The universal derivation uniquely extends to a derivation $d: \Lambda^p \to \Lambda^{p+1}$ satisfying Leibnitz’ rule and $d \circ d = 0$. This yields the de Rham complex

$$\Lambda^0 = \mathbb{C}[X_0, \ldots, X_n] \xrightarrow{d} \Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \xrightarrow{d} \Lambda^{n+1}$$

The modules $\Lambda^p$ are graded by setting

$$\deg(gX_{i_1} \wedge \cdots \wedge dX_{i_p}) := \deg(g) + p, \quad g \in \mathbb{C}[X_0, \ldots, X_n] \text{ homogeneous}.$$ 

Then $d$ is a map of degree 0. There is another derivation $\Delta: \Lambda^p \to \Lambda^{p-1}$ of degree 0, which can be defined as the contraction with the Euler vector field $\sum_i X_i \frac{\partial}{\partial X_i}$. It is uniquely determined by Leibnitz’ rule and the formula $\Delta(df) = $
deg \cdot f\) for a homogeneous polynomial \(f\), and satisfies \(\Delta(d\alpha) + d(\Delta\alpha) = \deg \alpha \cdot \alpha\) for any homogeneous \(\alpha \in \Lambda^p\).

Now put \(M^p := \ker(\Delta: \Lambda^p \rightarrow \Lambda^{p-1})\). One can define the sheaf of differential \(p\)-forms on \(\mathbb{P}^n\) by setting \(\Omega^p_{\mathbb{P}^n} := \overline{M}^p\). Hence, for a homogeneous polynomial \(f\) of degree \(k\), each differential \(p\)-form on \(\mathbb{P}^n \setminus F(f)\) is of the form

\[
\omega = \frac{\alpha}{f^t} \quad \text{with} \quad \deg \alpha = tk \quad \text{and} \quad \Delta(\alpha) = 0,
\]

where \(\alpha \in \Lambda^p\) is homogeneous and \(t \in \mathbb{N}\). By the usual quotient rule one can extend \(d: \Lambda^p \rightarrow \Lambda^{p+1}\) to localizations. Then one easily checks that \(d(M^p_{(j)}) \subseteq M^{p+1}_{(j)}\) for homogeneous \(f \in \mathbb{C}[X_0, \ldots, X_n]\). This defines the **exterior differential** \(d: \Omega^p_{\mathbb{P}^n} \rightarrow \Omega^{p+1}_{\mathbb{P}^n}\). Now for a smooth subvariety \(X\) of \(\mathbb{P}^n\) we define \(\Omega^p_X\) to be the restriction of the sheaf \(\Omega^p_{\mathbb{P}^n}\) to \(X\). Note that \(\Omega^1_X = \Lambda^1\Omega_X\), where \(\Omega_X = \Omega^1_X\).

A sheaf \(\mathcal{F}\) on \(X\) is said to be **locally free** iff it is locally isomorphic to a direct sum of copies of \(\mathcal{O}_X\). The local rank of \(\mathcal{F}\) is the number of copies of the structure sheaf needed, which is a locally constant function. If \(X\) is smooth and \(m\)-equidimensional, then \(\Omega^p_X\) is locally free of rank \(\binom{m}{p}\).

The isomorphism classes of locally free sheaves on \(X\) of rank \(k\) are in one-to-one correspondence with those of vector bundles over \(X\) of rank \(k\) [29 II, Ex. 5.18]. A locally free sheaf of rank 1 is called an **invertible sheaf** or **line bundle**. The tensor product \(\mathcal{L} \otimes \mathcal{M}\) of two line bundles \(\mathcal{L}, \mathcal{M}\) is also a line bundle. For any line bundle \(\mathcal{L}\), its **dual** sheaf \(\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)\) is another line bundle satisfying \(\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_X\) [29 II, Proposition 6.12]. In particular, the sheaves \(\mathcal{O}_X(k)\) defined above are line bundles, and we have \(\mathcal{O}_X(k)^\vee = \mathcal{O}_X(-k)\).

### 2.3 Divisors and Line Bundles

Let \(X\) be a smooth variety. A divisor on \(X\) is an element of the free abelian group \(\text{Div} \, X\) generated by the irreducible hypersurfaces of \(X\). This means that each \(D \in \text{Div} \, X\) is a formal linear combination \(D = \sum m_i V_i\), where \(m_i \in \mathbb{Z}\) and \(V_i \subseteq X\) are irreducible hypersurfaces. The **support** of \(D\) is defined as \(\text{supp} D := \bigcup_{m_i \neq 0} V_i\). Each \(D \in \text{Div} \, X\) defines a line bundle \(\mathcal{O}_X(D)\), which is a subsheaf of the sheaf \(\mathcal{K}\) of **total quotient rings** of \(\mathcal{O}_X\) [40 p. 61]. The stalk \(\mathcal{O}_X(D)_x\) is \(\mathcal{O}_{X,x}\), if \(x \notin \text{supp} D\). For \(x \in \text{supp} D\), the stalk is \(\prod_i f_i^{-m_i} \mathcal{O}_{X,x}\), where \(f_i\) is a local equation of \(V_i\) at \(x\). The rule \(D \mapsto \mathcal{O}_X(D)\) maps \(\text{Div} \, X\) bijectively onto the invertible subsheaves of \(\mathcal{K}\), and it maps sums to tensor products [29 II, Proposition 6.13].

Let \(X\) be closed and \(H\) a hypersurface in \(\mathbb{P}^n\) meeting \(X\) properly, i.e., \(H\) does not contain any irreducible component of \(X\), so that \(X \cap H\) is a hypersurface in \(X\). Then the **hyperplane section** \(V := X \cap H\) defines a divisor \(H \cdot X = \sum m_i V_i\), where the \(V_i\) are the irreducible components of \(V\), and \(m_i\) is the **intersection multiplicity** \(i(X, H; V_i)\) between \(X\) and \(H\) along \(V_i\), which can be defined as follows. Choose \(x \in V_i\) and a (reduced) local equation \(f \in \mathcal{O}_{\mathbb{P}^n,x}\) of \(H\). Then \(i(X, H; V_i)\) is the **order of vanishing** \(\text{ord}_{V_i}(f)\) of \(f\) along \(V_i\), i.e., the maximal \(k \in \mathbb{N}\) such that \(f = gh^k\) with some \(g\) in \(\mathcal{O}_{X,x}\), where \(h\) is a local equation of \(V_i\).
at \( x \). Note that \( \mathcal{O}_{X,x} \) is factorial, since \( X \) is smooth. The line bundle \( \mathcal{O}_X(H \cdot X) \) is isomorphic to the very ample line bundle \( \mathcal{O}_X(1) \).

A hypersurface \( V \subseteq X \) is said to have normal crossings iff for each \( x \in V \) contained in \( k \) irreducible components \( V_1, \ldots, V_k \) of \( V \), there exist local equations \( f_i \in \mathcal{O}_{X,x} \) of \( V_i \) around \( x \), such that \( d_x f_1, \ldots, d_x f_k \) are linearly independent in the dual \( (T_x X)^* \). Note that the case \( k = 1 \) implies that all irreducible components of \( V \) are smooth.

### 2.4 Sheaf Cohomology

Let \( \mathcal{F} \) be a coherent sheaf and \( \mathcal{U} := \{ U_i \mid 0 \leq i \leq s \} \) an open cover of the variety \( X \). For \( 0 \leq q \leq s \) and \( 0 \leq i_0 < \cdots < i_q \leq s \) set \( U_{i_0 \cdots i_q} := U_{i_0} \cap \cdots \cap U_{i_q} \).

The Čech complex is defined by \( C^q := C^q(\mathcal{U}, \mathcal{F}) := \bigoplus_{i_0 < \cdots < i_q} \mathcal{F}(U_{i_0 \cdots i_q}) \), with the Čech differential \( \delta : C^q \rightarrow C^{q+1} \) given by

\[
(\delta(\omega))_{i_0 \cdots i_{q+1}} := \sum_{\nu=0}^{q+1} (-1)^\nu \omega_{i_0 \cdots \hat{i}_\nu \cdots i_{q+1}} \mid_{U_{i_0 \cdots i_{q+1}}} \quad \text{for} \quad \omega = (\omega_{i_0 \cdots i_q}) \in C^q. \quad (3)
\]

Then one easily checks that \( \delta \circ \delta = 0 \), hence \((C^\bullet, \delta)\) is indeed a complex. Its cohomology \( H^i(\mathcal{U}, \mathcal{F}) := H^i(C^\bullet, \delta) \) is called the \( i \)-th Čech cohomology of \( \mathcal{F} \) with respect to \( \mathcal{U} \). The \( i \)-th Čech cohomology (or sheaf cohomology) of \( \mathcal{F} \) is defined as the direct limit over all open covers \( \mathcal{U} \) of \( X \), directed by refinements. A sheaf \( \mathcal{F} \) on \( X \) is called acyclic iff \( H^i(X, \mathcal{F}) = 0 \) for all \( i > 0 \).

A cover \( \mathcal{U} \) of \( X \) is called a Leray cover for \( \mathcal{F} \) iff \( \mathcal{F} \) is acyclic on \( U_{i_0 \cdots i_q} \) for all \( i_0 < \cdots < i_q \). Leray’s Theorem states that in this case we have \( H^i(X, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F}) \) \cite[III, Ex. 4.11]{[29]}. Since \( \mathcal{F} \) is a coherent sheaf, this is true for any affine cover \cite[III, Theorem 3.5]{[29]}. It easily follows that for a morphism \( f : X \rightarrow Y \) there is a natural isomorphism \( H^i(X, \mathcal{F}) \cong H^i(Y, f_\ast \mathcal{F}) \), where \( f_\ast \mathcal{F} \) denotes the direct image of the sheaf \( \mathcal{F} \) under \( f \).

### 2.5 Hypercohomology and de Rham Cohomology

The material in this section is explained e.g. in \cite{[26]}. Let \( X \) be a smooth variety and consider a complex of coherent sheaves \((\mathcal{F}, d)\) on \( X \) with \( \mathcal{F}^p = 0 \) for \( p < 0 \). Then, for an open cover \( \mathcal{U} \), the Čech complexes \( C^\bullet(\mathcal{U}, \mathcal{F}^p) \) as defined in \cite{[24]} fit together to the Čech double complex \( C^{\bullet, \bullet}(\mathcal{U}, \mathcal{F}^\bullet) \) by setting

\[
C^{p,q}(\mathcal{U}, \mathcal{F}^\bullet) = \bigoplus_{i_0 < \cdots < i_q} \mathcal{F}^p(U_{i_0 \cdots i_q}) \quad \text{for all} \quad p, q \geq 0.
\]

The two differentials are the one induced by the differential \( d \) of \( \mathcal{F} \) and the Čech differential \( \delta \) defined by \cite{[3]}. Denote by \( \mathbb{H}^\bullet(\mathcal{U}, \mathcal{F}^\bullet) := H^\bullet(\text{tot}^\bullet(C^{\bullet, \bullet})) \) the cohomology of the total complex of \( C^{\bullet, \bullet}(\mathcal{U}, \mathcal{F}^\bullet) \). Then the hypercohomology \( \mathbb{H}^i(X, \mathcal{F}^\bullet) \) of the complex of sheaves \( \mathcal{F}^\bullet \) is defined as the direct limit of \( \mathbb{H}^i(\mathcal{U}, \mathcal{F}^\bullet) \) over all open covers \( \mathcal{U} \) of \( X \), directed by refinement. As for any
double complex [88, §2.4], there are two spectral sequences

\[ \overset{'}{E}^{p,q}_2 = H^q_p(X, \mathcal{F}^\bullet) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet) \]

and

\[ \overset{''}{E}^{p,q}_2 = H^q_p(X, \mathcal{H}^p(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet), \]

where the cohomology sheaf is defined by

\[ H^p\mathcal{F}^\bullet := \ker(d: \mathcal{F}^p \rightarrow \mathcal{F}^{p+1})/\text{im } (d: \mathcal{F}^{p-1} \rightarrow \mathcal{F}^p). \]

The first spectral sequence implies that if all the sheaves \( \mathcal{F}^p \) are acyclic, then \( H\mathcal{F}^\bullet(X) = H^\bullet(\Gamma(X, \mathcal{F}^\bullet)) \) is the cohomology of the complex of global sections. Similarly as for sheaf cohomology we have \( H^i(X, \mathcal{F}^\bullet) \cong H^\bullet(U, \mathcal{F}^\bullet) \), if \( U \) is a Leray cover for all \( \mathcal{F}^p \).

A map of complexes of sheaves \( f: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \) is called a quasi-isomorphism iff it induces an isomorphism \( H^\bullet(\mathcal{F}^\bullet) \cong H^\bullet(\mathcal{G}^\bullet) \). By comparing the second spectral sequences of the hypercohomologies of \( \mathcal{F}^\bullet \) and \( \mathcal{G}^\bullet \) it follows that \( f \) induces an isomorphism \( H^\bullet(X, \mathcal{F}^\bullet) \cong H^\bullet(X, \mathcal{G}^\bullet) \).

Now let \( m := \dim X \) and consider the algebraic de Rham complex

\[ \Omega^\bullet_X: 0 \rightarrow O_X \overset{d}{\rightarrow} \Omega^1_X \overset{d}{\rightarrow} \cdots \overset{d}{\rightarrow} \Omega^m_X \rightarrow 0 \]

of regular differential forms on \( X \) together with the exterior differentials. The algebraic de Rham cohomology of \( X \) is defined as the hypercohomology

\[ H^\bullet_{dR}(X) := H^\bullet(X, \Omega^\bullet_X). \]

If \( X \) is affine, then \( H^\bullet_{dR}(X) \) can be computed by taking the cohomology of the de Rham complex \( \Gamma(X, \Omega^\bullet_X) \) of global sections, since all \( \Omega^p_X \) are acyclic.

### 2.6 Computational Model

Our model of computation is that of algebraic circuits over \( \mathbb{C} \), cf. [55, 11]. We set \( \mathbb{C}^\infty := \bigcup_{n \in \mathbb{N}} \mathbb{C}^n \). The size of an algebraic circuit \( C \) is the number of nodes of \( C \), and its depth is the maximal length of a path from an input to an output node. We say that a function \( f: \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty \) can be computed in parallel time \( d(n) \) and sequential time \( s(n) \) iff there exists a polynomial-time uniform family of algebraic circuits \( (C_n)_{n \in \mathbb{N}} \) over \( \mathbb{C} \) of size \( s(n) \) and depth \( d(n) \) such that \( C_n \) computes \( f|_{\mathbb{C}^n} \).

### 2.7 Efficient Parallel Linear Algebra

We use differential forms to reduce our problem to linear algebra, for which efficient parallel algorithms exist. In particular, we need to be able to solve the following problems:

1. Given \( A \in \mathbb{C}^{n \times m} \) and \( b \in \mathbb{C}^n \), decide whether the linear system of equations \( Ax = b \) has a solution and if so, compute one.
2. Compute a basis of the kernel of a matrix \( A \in \mathbb{C}^{n \times m} \).

3. Compute a basis of the image of a matrix \( A \in \mathbb{C}^{n \times m} \).

4. Given a linear subspace \( V \subseteq \mathbb{C}^n \) in terms of a basis, and given linearly independent \( v_1, \ldots, v_i \in V \), extend them to a basis of \( V \).

These problems are easily reduced to inverting a regular square-matrix (thus to computing the characteristic polynomial) and computing the rank of a matrix. For instance, the last problem boils down to rank computations as follows. Let \( b_1, \ldots, b_m \in V \) be the given basis. Set \( B := (v_1, \ldots, v_i) \). For all \( j = 1, 2, \ldots, m \) do: if \( \text{rk} (B, b_j) > \text{rk} B \) then append \( b_j \) to \( B \).

Mulmuley [39] has reduced the problem of computing the rank to the computation of the characteristic polynomial of a matrix. Since we need his construction, we describe it here. Let \( A \in \mathbb{C}^{m \times m'} \) be a matrix. Then

\[
\text{rk} \left( \begin{array}{cc}
0 & A \\
A^T & 0
\end{array} \right) = 2 \text{rk} A,
\]

so we can assume \( m = m' \). Define the diagonal matrix \( X := \text{diag}(1, T, \ldots, T^{m-1}) \) with the additional variable \( T \), and consider the characteristic polynomial \( p_A(Z) \) of \( XA \) over the field \( \mathbb{C}(T) \), \( p_A(Z) := \det(XA - ZI) \). Then the rank of \( A \) equals \( m - s \), where \( s \) is the maximal integer with \( Z^s | p_A(Z) \). We will call \( p_A(Z) \) the Mulmuley polynomial of \( A \).

The characteristic polynomial of an \( m \times m \) matrix can be computed in parallel (sequential) time \( O(\log^2 m) \) \((m^{O(1)}) \) with the algorithm of [7]. If the matrix has polynomial entries of degree \( d \) in \( n \) variables, then the Berkowitz algorithm can be implemented in parallel (sequential) time \( O(n \log m \log(md)) \) \((md)^{O(n)}) \) [47].

3 Castelnuovo-Mumford Regularity

A nice exposition about various versions of Castelnuovo-Mumford regularity and vanishing results is contained in the book [35]. Let \( X \subseteq \mathbb{P}^n \) be a smooth closed subvariety. Recall from [2.2] that \( \mathcal{O}_X(1) \) denotes the very ample line bundle on \( X \) determined by the embedding \( X \hookrightarrow \mathbb{P}^n \), and that for a coherent sheaf \( \mathcal{F} \) on \( X \) we put \( \mathcal{F}(k) := \mathcal{F} \otimes \mathcal{O}_X(k) \). The following definition is due to [40] building on ideas of Castelnuovo.

**Definition 3.1.** The coherent sheaf \( \mathcal{F} \) on \( X \) is called \( k \)-regular iff

\[
H^i(X, \mathcal{F}(k - i)) = 0 \quad \text{for all} \quad i > 0.
\]

The Castelnuovo-Mumford regularity \( \text{reg} (\mathcal{F}) \) of \( \mathcal{F} \) is defined as the infimum over all \( k \in \mathbb{Z} \) such that \( \mathcal{F} \) is \( k \)-regular.

**Remark 3.2.** (i) A fundamental result of [40] is that if \( \mathcal{F} \) is \( k \)-regular, then \( \mathcal{F} \) is \( \ell \)-regular for all \( \ell \geq k \).
Let 
\[ 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \]
be a short exact sequence of coherent sheaves on \( X \). The long exact
cohomology sequence shows that
\[ \text{reg} (\mathcal{H}) \leq \max \{ \text{reg} (\mathcal{F}) - 1, \text{reg} (\mathcal{G}) \} . \]

(iii) Let \( X \subseteq \mathbb{P}^n \) be a subscheme of dimension \( m \) with ideal sheaf \( \mathcal{I} = \mathcal{I}_X \), and
let \( k > 0 \). Then \( \mathcal{I} \) is \( k \)-regular if and only if \( H^i (X, \mathcal{I}(k-i)) = 0 \) for all
\( 0 < i \leq m + 1 \).

**Example 3.3.** (i) Theorem 5.1 in Chapter III of [29] shows \( \text{reg} (\mathcal{O}_{\mathbb{P}^n}) = 0 \).

(ii) The last example together with the exact sequence of [29, II, Theorem 8.13] implies \( \text{reg} (\Omega_{\mathbb{P}^n}) = 2 \).

The aim of this section is to prove the following theorem.

**Theorem 3.4.** Let \( X \neq \mathbb{P}^1 \) be a smooth closed projective variety of dimension \( m \). Let \( D \) be the maximal degree and \( e \) the maximal codimension of all irreducible components of \( X \). Then
\[ \text{reg} (\Omega_X) \leq p(\mathbb{m} + 1)D \quad \text{for} \quad pm > 0, \]
\[ \text{reg} (\mathcal{O}_X) \leq e(D - 1). \]

We will reduce this theorem to the following vanishing result of [8].

**Proposition 3.5.** Let \( \mathcal{I} \) be the ideal sheaf of a smooth irreducible closed variety in \( \mathbb{P}^n \), which is scheme-theoretically cut out by hypersurfaces of degrees at most \( D \). Then
\[ H^i (\mathbb{P}^n, \mathcal{I}^a(k)) = 0 \quad \text{for} \quad a \geq 0, \ i > 0, \ k \geq (a + e - 1)D - n, \]
where \( \mathcal{I}^a \) denotes the \( a \)-th power of the ideal sheaf \( \mathcal{I} \).

**Remark 3.6.** In [8] there is proved a more precise bound in terms of the individual degrees of the hypersurfaces, which we do not need here.

We also use the following result of [41].

**Proposition 3.7.** Each smooth irreducible closed projective variety of degree \( D \) is scheme-theoretically cut out by hypersurfaces of degree \( D \).

Let us first gather some basic properties of regularity. In the following one
can always assume \( X \) to be irreducible. The first two lemmas are a version of a
well known technique to characterize regularity by free resolutions.

**Lemma 3.8.** Let
\[ \mathcal{F}_N \to \mathcal{F}_{N-1} \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0 \]
be an exact sequence of coherent sheaves on \( X \), where \( N + 1 \geq \dim X =: m \). Then
\[ \text{reg} (\mathcal{F}) \leq \max \{ \text{reg} (\mathcal{F}_0), \text{reg} (\mathcal{F}_1) - 1, \ldots, \text{reg} (\mathcal{F}_{m-1}) - m + 1 \}. \]
Proof. This follows easily by chasing through the complex \cite[Prop. B.1.2]{35}, taking into account that \(H^i(X, F) = 0\) for all \(i > m\) and any coherent sheaf \(F\), see \cite[III, Theorem 2.7]{29}. Another proof is given in \cite[Lemma 3.9]{1}.

For a finite dimensional vector space \(V\) and a coherent sheaf \(F\) on \(X\) we denote by \(V \otimes F\) the sheaf \(U \mapsto V \otimes_{\mathbb{C}} F(U)\). If \(v_1, \ldots, v_N\) is a basis of \(V\), then \(V \otimes F = \bigoplus_{i=1}^N v_i \otimes F\). The following lemma is proved as Corollary 3.2 in \cite{1}.

\textbf{Lemma 3.9.} Let \(F\) be a \(k\)-regular coherent sheaf on \(X\). Then there exist finite dimensional vector spaces \(V_i\) and an exact sequence

\[
\cdots \rightarrow V_i \otimes O_X(-k) \rightarrow \cdots \rightarrow V_1 \otimes O_X(-k) \rightarrow V_0 \otimes O_X(-k) \rightarrow F \rightarrow 0,
\]

(5)

where \(R := \text{Reg}(X)\).

Using this we prove a bound on the regularity of tensor products.

\textbf{Proposition 3.10.} Let \(F, G\) be coherent sheaves on \(X\), where \(G\) is locally free, and denote \(m := \dim X\) and \(R := \text{Reg}(X)\) as above. Then

\[\text{reg}(F \otimes G) \leq \text{reg}(F) + \text{reg}(G) + (m-1)(R-1)\]

\textbf{Proof.} The proof parallels the one of the special case \(X = \mathbb{P}^n\) \cite[Proposition 1.8.9]{35}. Let \(k := \text{reg}(F)\), and consider the resolution (5) of \(F\), which exists according to Lemma 3.9. Tensoring with \(G\) yields

\[
\cdots \rightarrow V_i \otimes G(-k) \rightarrow \cdots \rightarrow V_1 \otimes G(-k) \rightarrow V_0 \otimes G(-k) \rightarrow F \otimes G \rightarrow 0.
\]

Since tensoring with a locally free sheaf is an exact functor, this sequence is exact. Furthermore, \(\text{reg}(V_i \otimes G(-k-iR)) \leq k+iR+\text{reg}(G)\), hence \(\text{reg}(F \otimes G) \leq k + \text{reg}(G) + (m-1)(R-1)\) by Lemma 3.9.

\textbf{Corollary 3.11.} Let \(F\) be a locally free sheaf on \(X\). Then for \(p > 0\)

\[\text{reg}(\Lambda^p F) \leq p \cdot \text{reg}(F) + (p-1)(m-1)(R-1)\]

\textbf{Proof.} The same bound for the \(p\)-th tensor power of \(F\) clearly follows from Proposition 3.10. Since the exterior power is a direct summand of the tensor power \cite[III, §7.4]{9}, this implies the claim.

\textbf{Proposition 3.12.} Let \(X\) be a smooth irreducible closed variety of codimension \(e\) and degree \(D > 1\). Then

\[\text{reg}(\Omega_X) \leq (e+1)D - e\]

\textbf{Remark 3.13.} The claim is false for \(D = 1\) as Example 3.3 (ii) shows.
Proof. Denote with \( I \) the ideal sheaf of \( X \). Propositions 3.5 and 3.7 imply together with Part (iii) of Remark 3.2 that \( \text{reg} (I) \leq (a+e-1)D-e+1 \) (note that \( I \) is the ideal sheaf of some subscheme of the same dimension as \( X \)). We will repeatedly apply Part (ii) of Remark 3.2. The exact sequence

\[
0 \rightarrow I \rightarrow \mathcal{O} \rightarrow \mathcal{O}_X \rightarrow 0
\]

and Example 3.3 implies \( \text{reg} (\mathcal{O}_X) \leq \max \{ \text{reg} (I) - 1, 0 \} \leq eD - e \). Furthermore, from the exact sequence

\[
0 \rightarrow I^2 \rightarrow I \rightarrow I/I^2 \rightarrow 0
\]

it follows \( \text{reg} (I/I^2) \leq \max \{ \text{reg} (I), \text{reg} (I^2) - 1 \} \leq (e+1)D - e - 2 \). Finally, the exact conormal sequence \([29, II, \text{Theorem 8.17}]\]

\[
0 \rightarrow I/I^2 \rightarrow \mathcal{O}_X \otimes \Omega_{\mathbb{P}^n} \rightarrow \Omega_X \rightarrow 0
\]

yields \( \text{reg} (\Omega_X) \leq \max \{ \text{reg} (\mathcal{O}_X \otimes \Omega_{\mathbb{P}^n}), \text{reg} (I/I^2) \} \leq (e+1)D - e - 2 \). Here we use that \( \text{Reg} (\mathbb{P}^n) = 1 \), hence \( \text{reg} (\mathcal{O}_X \otimes \Omega_{\mathbb{P}^n}) \leq \text{reg} (\mathcal{O}_X) + 2 \leq eD - e + 2 \) by Proposition 3.10 and Example 3.3.

Proof of Theorem 3.4. The claim is clear for \( D = 1 \). Also, we can assume \( X \) to be irreducible. The claim for \( p = 0 \) follows from the proof of Proposition 3.12. For the case \( pm \geq 1 \), Proposition 3.12 and Corollary 3.11 imply

\[
\text{reg} (\Omega_X^p) \leq p((e+1)D-e)+(p-1)(m-1)(e(D-1)-1) < p((e+1)D-e+(m-1)e(D-1)) < pD(me+1).
\]

4 Cohomology of Hypersurface Complements

4.1 Theory

Let \( X \subseteq \mathbb{P}^n \) be a smooth closed subvariety. Using our result on the Castelnuovo-Mumford regularity of the sheaf of differential forms one can compute the de Rham cohomology of certain hypersurface complements in \( X \) as the cohomology of finite dimensional complexes.

To describe these complexes, let \( H_0, \ldots, H_q \subseteq X \) be hyperplane sections and denote by \( U \) the complement of the hypersurface \( V := \bigcup_{\nu} H_{\nu} \) in \( X \). Assume that \( V \) has normal crossings (see \([2,3]\)). We also consider \( V \) as a divisor \( V = \sum_{\nu} H_{\nu} = \sum_{i} m_i V_i \), where the \( V_i \) are the irreducible components of \( V \) (cf. \([2,3]\)). Then, since \( \mathcal{O}_X (H_{\nu}) \simeq \mathcal{O}_X (1) \), it follows that

\[
\mathcal{O}_X (V) \simeq \bigotimes_{\nu} \mathcal{O}(H_{\nu}) \simeq \mathcal{O}_X (1)^{\otimes (q+1)} \simeq \mathcal{O}_X (q + 1).
\]

Now let \( A = \sum_i a_i V_i \) be any divisor with support in \( V \), and let \( j: U \hookrightarrow X \) be the inclusion. Define the subsheaf \( \Omega_X^p (A) := \Omega_X^p \otimes \mathcal{O}_X (A) \) of \( j_* \Omega_U^p \), which
consists of those rational differential $p$-forms on $X$, which are regular on $U$ and have poles (zeros if $a_i < 0$) of order $|a_i|$ along $V_i$. Define the sheaves
\[ K^p_X(A) := \Omega^p_X(A + pV). \]

Note that $d(K^p_X(A)) \subseteq K^{p+1}_X(A)$, so that $K^*_X(A)$ is in fact a subcomplex of $j_*\Omega^*_U$.

The next lemma is the crucial fact that allows us to compute the algebraic de Rham cohomology of $U$ by a finite dimensional complex. Its proof requires to consider holomorphic differential forms. So let $\Omega^*_U$ denote the complex of holomorphic differential forms on $U$ regarded as a complex manifold, and let $K^*_X(A)$ be the holomorphic version of $K^*_X(A)$. The following lemma is proved analogously to the corresponding statement for the logarithmic complex (cf. [15, 26, 54]). The calculation can be found in [2, Lemma 4.1].

**Lemma 4.1.** Let $A = \sum a_i V_i$ be a divisor with $a_i > 0$ for all $i$, and assume that $V$ has normal crossings. Then the inclusion $K^*_X(A) \hookrightarrow j_*\Omega^*_U$ is a quasi-isomorphism.

The following is the main result of this section and the key for our algorithm.

**Theorem 4.2.** Let $X$ be a smooth closed projective variety of dimension $m$. Let $D$ be the maximal degree and $e$ the maximal codimension of all irreducible components of $X$. Let $H_0, \ldots, H_q$ be hyperplane sections of $X$ such that $V = H_0 \cup \ldots \cup H_q$ has normal crossings, and denote $U := X \setminus V$. For $s \in \mathbb{N}$ set $K^*_s := \Gamma(X, K^*_X(sV))$. Then we have
\[ H^i_{dR}(U) \simeq H^i(K^*_s) \quad \text{for} \quad s \geq m(em + 1)D. \]

**Proof.** It follows from Lemma 4.1 that
\[ \mathbb{H}^i(X^an, K^*_X(sV)) \simeq \mathbb{H}^i(X^an, j_*\Omega^*_U). \quad (7) \]

But by [2, 2] we have $H^i(X^an, j_*\Omega^*_U) = H^i(U^an, \Omega^p_{U^an}) = 0$ for $i > 0$. Hence the right hypercohomology in (7) is $H^*_{dR}(U^an) = H^*_{dR}(U)$.

On the other hand, since $K^*_X(sV)$ is a coherent sheaf on $X$, by GAGA [50] the left hypercohomology in (7) can be replaced by $\mathbb{H}^i(X, K^*_X(sV))$. Furthermore, Theorem 3.4 implies $s \geq \text{reg} (\Omega^p_X)$ for all $0 \leq p \leq m$, thus
\[ H^i(X, \Omega^p_X((s+p)V)) = H^i(X, \Omega^p_X((s+p)(q+1))) = 0 \quad \text{for all} \quad i > 0 \]
(use (6)). Hence, the left side of (7) is $H^i(K^*_s)$ as claimed.

**4.2 Computation**

We adopt the notations and assumptions of the last section. We choose $s$ according to Theorem 4.2 and set $K^* := K^*_s$. In this section we describe this finite dimensional complex more explicitly and show how to compute its cohomology.
Let $H_\nu = \mathbb{Z}_X(\ell_\nu)$ with a linear form $\ell_\nu$, $0 \leq \nu \leq q$. We can assume w.l.o.g. that $\ell_0 = X_0$. Set $f := X_0\ell_1 \cdots \ell_q$ and $U := X \setminus \mathcal{Z}(f)$. Recall from [22] that each $p$-form on $\mathbb{P}^n \setminus \mathcal{Z}(f)$ is given by
\[
\omega = \frac{\alpha}{f^t} \quad \text{with} \quad \deg \alpha = t(q + 1) \quad \text{and} \quad \Delta(\alpha) = 0,
\]
where $\alpha$ is a homogeneous $p$-form on $\mathbb{C}^{n+1}$ and $\Delta$ denotes contraction with the Euler vector field. Furthermore, by Exercise 6.1.7 of [17], each regular differential form on $U$ is the restriction of a form given by (8). In particular, each element of $K^p$ is of that form with $t := s + p$, which we assume henceforth. Denote by $\Omega^p$ the space of $p$-forms $\omega$ as in (8) with fixed $t$.

We identify $\mathbb{C}^n \cong \{X_0 \neq 0\} \subseteq \mathbb{P}^n$ and set $X^0 := X \setminus \mathcal{Z}(X_0)$. As with polynomials one can dehomogenize a homogeneous differential form $\alpha$ on $\mathbb{C}^{n+1}$ by setting $X_0 = 1$ and $dX_0 = 0$ to get a form $\alpha^0$ on $\mathbb{C}^n$. Hence for $\omega \in \Omega^p$ one gets a regular form $\omega^0$ on $\mathbb{C}^n \setminus \mathcal{Z}(f^0)$. Its restriction defines a regular form on the dense open subset $U = X^0 \setminus \mathcal{Z}(f^0)$ of $X^0$.

We use the algorithm of Szántó to compute a decomposition $I := I(X^0) = \bigcap_j I_j$, where each $I_j$ is the saturated ideal of a squarefree regular chain $G_j$. Note that $\mathcal{Z}(I_j)$ is equidimensional. We will construct for all $j$ a linear system of equations describing the identity $\omega = 0$ on $\mathcal{Z}(I_j)$ for $\omega \in \Omega^p$. For simplicity we assume that $I$ is represented by a single $G = \{g_1, \ldots, g_r\}$. In the general case one only has to combine all the linear systems to one large system.

Let $k \in \mathbb{N}$. In [12] we have constructed a linear system of equations
\[
\text{prem}_k(f, G) = 0
\]
in the coefficients of $f \in \mathbb{C}[X_1, \ldots, X_n]$, whose solution space is $I_{\leq k}$, the set of polynomials of degree $\leq k$ vanishing on $X^0$.

Szántó’s algorithm also yields a polynomial $h$ which is a non-zerodivisor mod $I$, and such that the module of differentials on $X^0 \setminus \mathcal{Z}(h)$ is the free module generated by $m$ of the $dX_j$, where $m = \dim X^0$. More precisely, let $X_1, \ldots, X_m$ denote the free variables, and $Y_1, \ldots, Y_e$ the dependent variables, where $m + e = n$. For a polynomial $F \in \mathbb{C}[X_1, \ldots, X_m, Y_1, \ldots, Y_e]$ we denote $\overline{F} := F \mod I \in \mathbb{C}[X^0]$. Then by [12, Proposition 3.13] we have $\Omega_{\mathbb{C}[X^0]/\mathbb{C}} = \sum_{i=1}^m \mathbb{C}[X^0]h_i \overline{dX_i}$. Furthermore, for all $F \in \mathbb{C}[X_1, \ldots, X_m, Y_1, \ldots, Y_e]$
\[
d\overline{F} = \sum_{i=1}^m \left( \frac{\partial F}{\partial X_i} \overline{dY} - \frac{\partial F}{\partial Y} \left( \frac{\partial g}{\partial Y} \right)^{-1} \frac{\partial g}{\partial X_i} \right) \overline{dX_i},
\]
where $g := (g_1, \ldots, g_r)^T$. Note that $h$ is a multiple of $\det(\frac{\partial g}{\partial Y})$, so that the entries of $(\frac{\partial g}{\partial Y})^{-1}$ lie in $h^{-1}\mathbb{C}[X_1, \ldots, X_n]$. Using (10) for the coordinates $Y_j$, one can write the restriction of a form $\omega \in \Omega^p$ to $U \setminus \mathcal{Z}(h)$ in terms of the free generators of $\Omega^p_{\mathbb{C}(U)h/\mathbb{C}}$, which are $d\overline{X_i} \wedge \cdots \wedge d\overline{X_p}$, where $1 \leq i_1 < \cdots < i_p \leq m$. It follows that $\omega = \omega_h$, where
\[
\omega_h = \sum_{1 \leq i_1 < \cdots < i_p \leq m} (\omega_h)_{i_1 \cdots i_p} d\overline{X_{i_1}} \wedge \cdots \wedge d\overline{X_{i_p}} \in \Omega^p_{\mathbb{C}[X^0]/\mathbb{C}}.$
Then, since $U \setminus Z(h)$ is dense in $U$ and in $X^0$, we have

$$
\omega = 0 \text{ on } U \iff \omega_h = 0 \text{ on } U \setminus Z(h)
\iff \forall i_1 < \cdots < i_p: (\omega_h)_{i_1 \cdots i_p} = 0 \text{ on } U \setminus Z(h)
\iff \forall i_1 < \cdots < i_p: (\omega_h)_{i_1 \cdots i_p} \in I_{\leq k}
$$

(11)

for sufficiently large $k$.

Now we compute the cohomology of $K^\bullet$. First note that the contraction with the Euler vector field $\Delta$ (cf. §2.2) can be easily computed, so that we can compute a basis for $\Omega^p$. Consider the commutative diagram

\[
\begin{array}{ccc}
\Omega^p & \xrightarrow{d} & \Omega^{p+1} \\
\pi \downarrow & & \pi \downarrow \\
K^p & \xrightarrow{d} & K^{p+1},
\end{array}
\]

where $\pi$ is the restriction of forms to $U$. Let $N^p := \ker(\pi: \Omega^p \to K^p)$. According to (11) and (11), $N^p$ is the solution set of a linear system of equations. Since $\pi$ is surjective, we have $K^p \cong \Omega^p/N^p$, thus $K^p$ can be identified with any complementary subspace of $N^p$ in $\Omega^p$. So we compute a basis of $N^p$ and extend it to a basis of $\Omega^p$ to get a basis of $K^p$ via this identification. The differential $d: K^p \to K^{p+1}$ is just the restriction of the differential $d: \Omega^p \to \Omega^{p+1}$, which we can evaluate efficiently. Hence we can compute the matrix of $d: K^p \to K^{p+1}$ with respect to the computed bases of $K^\bullet$. By computing kernel and image of this matrix and taking their quotient we get the cohomology of $K^\bullet$.

**Proposition 4.3.** Under the notations and assumptions of Theorem 4.2, let $X \subseteq \mathbb{P}^n$ be given by equations of degree $\leq d$. Then one can compute the cohomology $H^\bullet_{\text{dR}}(U)$ in parallel time $(d \log n)^{O(1)}$ and sequential time $d^{O(n^4)}$.

**Proof.** It remains to analyze the algorithm described above. Let $\delta$ denote the maximal degree of the polynomials in the squarefree regular chain $G$. Then the system (11) has asymptotic size $O((nk\delta)^n)$ and can be computed in parallel time $(n \log(k\delta))^{O(1)}$ and sequential time $(k\delta)^{O(n^4)}$ [12]. Since the numerator of each $\omega \in \Omega^p$ has degree $(s+p)(q+1)$, the dimension of $\Omega^p$ is $\binom{n+1}{s+p} \binom{s+p+1}{s+1} = O(s^n n^{O(n)})$. Furthermore, for $\omega \in \Omega^p$, the degree of the coefficients of $\omega_h$ is bounded by $(s+p)(q+1) + p(e+1)\delta$, hence we must choose the $k$ in (11) of that order. Thus, $N^p$ is described by a linear system of equations of size $O((s+n)n + n^2\delta)^n \delta^{(s+n)n} \leq n^{O(n)} s^n \delta^{(s+n)n}$. Now let $X$ be given by equations of degree $d$. According to Theorem 4.2 we have to choose $s$ of order $n^3 \deg X \leq n^3 d^n$. Furthermore, by [3] we have $\delta = d^{O(n^3)}$. Hence the size of this system is $d^{O(n^4)}$. The algorithms of [2,4] imply the claimed bounds. \[\square\]
5 Patching Cohomologies

Let $X$ be a smooth closed projective variety of dimension $m$. Our aim is to compute the de Rham cohomology of $X$ by way of an open affine cover.

So let $H_0, \ldots, H_m \subseteq X$ be hyperplane sections with $H_0 \cap \cdots \cap H_m = \emptyset$ and set $U_i := X \setminus H_i$. Then $\mathcal{U} := \{U_i \mid 0 \leq i \leq m\}$ is an open affine cover of $X$.

Consider the Čech double complex $C^{\bullet,\bullet} := C^{\bullet,\bullet}(\mathcal{U}, \Omega^\bullet_X)$ as defined in §2.2. Recall that with $U_{i_0 \cdots i_q} := U_{i_0} \cap \cdots \cap U_{i_q}$ we have $C^{p,q}(\mathcal{U}, \Omega^\bullet_X) = \bigoplus_{i_0 < \cdots < i_q} \Omega^p_X(U_{i_0 \cdots i_q})$. Since $\mathcal{U}$ is a Leray cover for all the sheaves $\Omega^p_X$, we have

**Lemma 5.1.** $H^\bullet_{dR}(X) \simeq H^\bullet(\det (C^{\bullet,\bullet}))$, where $\det (C^{\bullet,\bullet})$ denotes the total complex associated to $C^{\bullet,\bullet}$.

To compute this cohomology, we replace the infinite dimensional double complex $C^{\bullet,\bullet}$ by a finite dimensional one, which is built from the complex of the last section for each $U_{i_0 \cdots i_q}$. More precisely, let $e, D$ have the meanings of Theorem 4.2, and choose $s \geq m(em + 1)D$. For a hypersurface $V$ in $X$ we denote $K^p(V) := \Gamma(X, \Omega^p_X((s + p)V))$. This corresponds to the complex $K^\bullet_r$ from Theorem 4.2. Now we define the double complex

$$K^{p,q} := \bigoplus_{i_0 < \cdots < i_q} K^p(H_{i_0} \cup \cdots \cup H_{i_q})$$

together with the differential $\delta: K^{p,q} \to K^{p+1,q}$, which is the restriction of the Čech differential $\partial_K$, and the exterior differential $d: K^{p,q} \to K^{p-1,q}$. Then $K^{\bullet,\bullet}$ is a subcomplex of $C^{\bullet,\bullet}$.

**Lemma 5.2.** We have $H^\bullet_{dR}(X) \simeq H^\bullet(\det (K^{\bullet,\bullet}))$.

**Proof.** Clearly, the inclusion $K^{\bullet,\bullet} \hookrightarrow C^{\bullet,\bullet}$ induces a morphism of spectral sequences $''E_r(K^{\bullet,\bullet}) \to ''E_r(C^{\bullet,\bullet})$ between the second spectral sequences of these double complexes. Theorem 4.2 implies that this is an isomorphism

$$''E^p_{1,q}(K^{\bullet,\bullet}) \simeq \bigoplus_{i_0 < \cdots < i_q} H^p_{dR}(U_{i_0 \cdots i_q}) = ''E^p_{1,q}(C^{\bullet,\bullet})$$

on the first level of these spectral sequences. According to [38, Theorem 3.5], this induces an isomorphism on their $\infty$-terms and, since the corresponding filtrations are bounded, also on the cohomologies of the total complexes, so $H^\bullet(\det (K^{\bullet,\bullet})) \simeq H^\bullet(\det (C^{\bullet,\bullet}))$. Together with Lemma 5.1 this completes the proof.

**Proposition 5.3.** Assume that one is given homogeneous polynomials of degree at most $d$ defining the smooth variety $X \subseteq \mathbb{P}^n$, and linear forms defining the hyperplane sections $H_0, \ldots, H_m$ such that $\bigcup H_i$ has normal crossings. Then one can compute $H^\bullet_{dR}(X)$ in parallel time $(d \log n)^{O(1)}$ and sequential time $d^{O(n^4)}$. 

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Proof. By Lemma 5.2 one has to compute the cohomology of the total complex $T^k := \text{tot}^k(K^\bullet) = \bigoplus_{p+q=k} K^{p,q}$ with the differential

$$d_T : T^k \to T^{k+1}, \quad (\omega_{p,q})_{p+q=k} \mapsto (d\omega_{p-1,q} + (-1)^p \delta \omega_{p,q-1})_{p+q=k+1}.$$ 

As in §4.2 one can compute bases for $K^{p,q}$ and hence for $T^\bullet$ within the claimed bounds. Since the differential $d_T$ is easily computable and the vector spaces have dimension $d^{O(n^4)}$, the cohomology of this complex can be computed using the algorithms of §2.7. 

6 Testing Smoothness

In this section we describe how one can test in parallel polynomial time whether a closed projective variety $X$ is smooth.

Crucial is Proposition 3.7 implying that if $X$ is smooth, then $X$ is scheme-theoretically cut out by hypersurfaces of degree $\leq D = \deg X$. Using the linear system of equations (9) one can compute a vector space basis $f_1, \ldots, f_N$ of $I_{\leq D}$, where $I = I(X)$. Let $U_i$ be the open subset $X \setminus Z(X_i)$ for $0 \leq i \leq n$. Then (cf. §2.1) the tangent space of $X$ at each $x \in U_i$ is

$$T_x X = Z(d_x f_1, \ldots, d_x f_N) \subseteq \mathbb{C}^n \cong \mathbb{P}^n \setminus Z(X_i).$$

Hence, if we assume $X$ to be $m$-equidimensional and denote with $L_x^i$ the linear subspace $Z(d_x f_1, \ldots, d_x f_N)$, we have

$$X \text{ smooth } \iff \bigwedge_i \forall x \in U_i \dim L_x^i = m,$$

which is the Jacobian criterion. Indeed, if $X$ is not smooth at $x \in U_i$, then $\dim L_x^i \geq \dim T_x X > m$, since in general $T_x X \subseteq L_x^i$.

Now our algorithm reads as follows.

Algorithm Smoothness Test

input $X$ given by homogeneous polynomials of degree $\leq d$.

1. Compute the equidimensional decomposition $X = Z_0 \cup \cdots \cup Z_m$, where $Z_m$ is either empty or $m$-equidimensional.
2. if $Z_n \neq \emptyset$ then output “Yes”.
3. for $0 \leq m < m' < n$ do if $Z_m \cap Z_{m'} \neq \emptyset$ then output “No”.
4. Set $D := d^n$.
5. Compute a basis $f_1, \ldots, f_N$ of $I_{\leq D}$, where $I = I(X)$.
6. for $0 \leq m < n$ do
7. for $0 \leq i \leq n$ do

8. Compute the matrix $A := \left(\frac{\partial f^i_v}{\partial x_k}\right)_{i,v}$, where $f^i_v$ is the dehomogenization of $f_v$ with respect to $X_i$.

9. Compute the Mulmuley polynomial $p(Z)$ of $A$ (see [27]), which lies in $\mathbb{C}[X_0, \ldots, \hat{X}_i, \ldots, X_n, T, Z]$. Write $p(Z) = p_0 + p_1 Z + \cdots + p_K Z^K$, and let $F_1, \ldots, F_L \in \mathbb{C}[X_0, \ldots, \hat{X}_i, \ldots, X_n]$ be the coefficients of all $T^k$ in $p_0, \ldots, p_m$.

10. if $Z_m \cap Z(F_1, \ldots, F_L) \cap \{X_i \neq 0\} \neq \emptyset$ then output “No”.

11. output “Yes”.

**Proposition 6.1.** The algorithm Smoothness Test is correct and can be implemented in parallel time $(n \log d)\mathcal{O}(1)$ and sequential time $d\mathcal{O}(n^4)$.

**Proof.** Correctness: If $X$ is smooth, then it clearly passes the test in step 3. By [12] we have $\dim L_x^i = m$ for all $m, i$ and all $x \in Z_m$. Denote by $A_x$ the matrix $A$ evaluated at $x$, and similarly for $p(Z)$. Then $L_x^i = \ker A_x$, hence

$$\dim L_x^i > m \iff Z^{m+1} | p_x(Z) \iff F_1(x) = \cdots = F_L(x) = 0.$$ 

If $X$ is not smooth, then it doesn’t pass the test in step 3 or some $Z_m$ is not smooth. In the latter case at some point $x \in Z_m \cap U_i$ we will have $\dim L_x^i > m$.

Analysis: All the algorithms we use are well-parallelizable. We therefore state only the sequential time bounds. The equidimensional decomposition in step 1 can be done in time $d\mathcal{O}(n^2)$ with the algorithm of [23]. For each $m$, this algorithm returns $d\mathcal{O}(n^2)$ polynomials of degree bounded by $\deg Z_m = \mathcal{O}(d^m)$ whose zero set is $Z_m$. Testing feasibility of a system of $r$ homogeneous equations of degree $\bar{d}$ can be done in time $r(n \bar{d})\mathcal{O}(n)$ using the effective homogeneous Nullstellensatz. Hence step 3 takes time $d\mathcal{O}(n^2)$. Szántó’s algorithm in step 5 runs in time $d\mathcal{O}(n^4)$, and clearly $N = \mathcal{O}(D^{n+1}) = d\mathcal{O}(n^2)$. Furthermore, the computation of the Mulmuley polynomial in step 9 can be done in time $d\mathcal{O}(n^5)$ by [27], and we have $L = \mathcal{O}(N^2m) = d\mathcal{O}(n^5)$ and $\deg F_i \leq ND = d\mathcal{O}(n^5)$. Thus step 10 takes time $d\mathcal{O}(n^5)$ by the affine effective Nullstellensatz.

## 7 Finding Generic Hyperplanes

The algorithm for computing the cohomology of $X$ described in [12] and [14] depends on a choice of sufficiently generic hyperplane sections $H_v$ of $X$. More precisely, it is required that $V := H_0 \cup \cdots \cup H_m$ is a hypersurface with normal crossings in $X$, where $m = \dim X$. Note that as a consequence $H_{i_0} \cup \cdots \cup H_{i_q}$ has normal crossings for each tuple $i_0 < \cdots < i_q$. Here we describe how to find sufficiently generic hyperplanes deterministically in parallel polynomial time.

Throughout this section we assume $X$ to be smooth, and let us first assume that $X$ is $m$-equidimensional. We will formulate a sufficient condition for normal
crossings in terms of transversality. Recall that a linear subspace $L \subseteq \mathbb{P}^n$ is called transversal to $X$ in $x \in X \cap L$, written $X \cap_x L$, iff $\dim(T_xX \cap T_xL) = \dim T_xX + \dim T_xL - n$. Now let the hypersurfaces $H_0, \ldots, H_m$ be given by the linear forms $\ell_0, \ldots, \ell_m \in \mathbb{C}[X_0, \ldots, X_n]$. Denote $L_{i_0 \cdots i_q} := Z(\ell_{i_0}, \ldots, \ell_{i_q})$ for all $0 \leq q \leq m$ and all $0 \leq i_0 < \cdots < i_q \leq m$.

**Lemma 7.1.** If $\ell_0, \ldots, \ell_m$ are linearly independent and for all $0 \leq q \leq m$ and all $0 \leq i_0 < \cdots < i_q \leq m$ we have

$$\forall x \in X \cap L_{i_0 \cdots i_q} \cap_x L_{i_0 \cdots i_q},$$

then $V \subseteq X$ is a hypersurface with normal crossings.

**Proof.** Suppose that the condition (13) holds. First note by choosing $q = 0$ that $L_i = \mathbb{Z}(\ell_i)$ is transversal to $X$ at all $x$, thus $V$ is indeed a hypersurface. Furthermore, $H_i = X \cap L_i$ is smooth in $x$, so that $x$ lies in only one irreducible component of $H_i$, and $\ell_i \in \mathcal{O}_{X,x}$ is a local equation of that component. By transversality we have $\dim(T_xX \cap T_xL_{i_0 \cdots i_q}) = m - q - 1$. But $T_xX \cap T_xL_{i_0 \cdots i_q}$ is the kernel of the linear map $\varphi := (d_x\ell_{i_0}, \ldots, d_x\ell_{i_q}) : T_xX \rightarrow \mathbb{C}^{q+1}$, which thus must be surjective. Hence $d_x\ell_{i_0}, \ldots, d_x\ell_{i_q}$ are linearly independent on $T_xX$, which proves the claim.

In order to work with condition (13) algorithmically, we introduce some notation. Set $I := I(X)$ and $D := \deg X$. Recall from (6) that if $f_1, \ldots, f_N$ is a vector space basis of $I_{\leq D}$, then

$$T_xX = \mathbb{Z}(d_xf_1^i, \ldots, d_xf_N^i) \subseteq \mathbb{C}^n$$

for all $x \in U_i = X \setminus \mathbb{Z}(X_i)$ and $0 \leq i \leq n$. For each tuple $i_0 < \cdots < i_q$ and each $i$ we define the matrix

$$A_{i_0 \cdots i_q}^i := \begin{pmatrix}
    d_xf_1^i \\
    \vdots \\
    d_xf_N^i \\
    d_x\ell_{i_0}^i \\
    \vdots \\
    d_x\ell_{i_q}^i
\end{pmatrix} \in \mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]^{(N+q+1) \times n}. \quad (14)$$

Then the kernel of $A_{i_0 \cdots i_q}^i$ is the kernel of $\varphi$ of the proof of Lemma 7.1. Assume that $\ell_0, \ldots, \ell_m$ are linearly independent. Then condition (13) is equivalent to the statement that the nullity of $A_{i_0 \cdots i_q}^i$ is $m - q - 1$, its minimal possible value, at each point $x \in U_i \cap L_{i_0 \cdots i_q}$. Note that this condition also implies the linear independence. Now let $p(Z)$ be the Mulmuley polynomial of $A_{i_0 \cdots i_q}^i$, which lies in $\mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n, T, Z]$. Let $F_1, \ldots, F_L \in \mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]$ be the coefficients of all $T^k$ in the coefficient of $Z^{m-q}$ in $p(Z)$. Then a sufficient condition for (13) is

$$\bigwedge_{i} U_i \cap L_{i_0 \cdots i_q} \cap Z(F_1, \ldots, F_L) \neq \emptyset. \quad (15)$$
Using this formula we can prove

**Proposition 7.2.** Given polynomials of degree \( \leq d \) defining a smooth subvariety \( X \subseteq \mathbb{P}^n \) of dimension \( m \), one can compute in parallel time \( (n \log d)O(1) \) and sequential time \( dO(n^2) \) linear forms \( \ell_0, \ldots, \ell_m \) such that \( V = \bigcup_j H_j \) is a hypersurface with normal crossings, where \( H_j = Z_X(\ell_j) \).

**Proof.** First we set \( D := d^n \) and compute a basis \( f_1, \ldots, f_N \) of \( I_D \). Then we compute the equidimensional components \( Z_m \) of \( X \). We find the linear forms \( \ell_0, \ldots, \ell_m \) successively, one at a time. So assume that \( \ell_0, \ldots, \ell_{j-1} \) have been already found, and take \( \ell_j = \alpha_0 X_0 + \cdots + \alpha_n X_n \) with indeterminate coefficients \( \alpha = (\alpha_0, \ldots, \alpha_n) \). Now consider the conjunction of the conditions \( 15 \) for all \( m \leq \dim X \) and \( i_0 < \cdots < i_q = j \), which is a first order formula with free variables \( \alpha \). Note that here one has to take \( U_i = Z_m \cap \{ X_i \neq 0 \} \). By quantifier elimination compute an equivalent quantifier-free formula \( \Phi(\alpha) \) in disjunctive normal form. Let \( G_1, \ldots, G_M \) be all polynomials accruing \( \Phi(\alpha) \). Since \( \Phi(\alpha) \) is satisfied for generic \( \alpha \), it is easy to see that \( G_\nu(\alpha) \neq 0 \) for all \( \nu \) implies \( \Phi(\alpha) \) \( [13] \) proof of Theorem 3.8). Let \( \delta \) be the maximal degree of the \( G_\nu \). Now take a set \( S \subseteq \mathbb{C} \) of cardinality \( > M n \delta \) and test for all \( b \in S \) in parallel, whether \( P_\nu(b) := G_\nu(1, b, \ldots, b^n) \neq 0 \) for all \( 1 \leq \nu \leq M \). Since \( P_\nu \) is a univariate polynomial of degree \( \leq n \delta \), there must exist a successful \( b \). Then we can take \( \ell_j = X_0 + b X_1 + \cdots + b^n X_n \).

Analysis: The computation of \( f_1, \ldots, f_N \), of the equidimensional decomposition, and of the Mulmuley polynomial can be done within the claimed time bounds. Recall that \( N, L \), the degrees of the defining equations for \( Z_m \), as well as \( \deg F_i \) are of order \( dO(n^2) \). Condition \( 15 \) is a universal first order formula with \( O(n) \) free and bounded variables, and \( dO(n^2) \) atomic formulas involving polynomials of degree \( dO(n^2) \). According to \( 21 \), one can eliminate the universal quantifier and hence compute the polynomials \( G_\nu \) in parallel time \( (n \log d)O(1) \) and sequential time \( dO(n^2) \). Furthermore, \( M \) and \( \delta \) are also bounded by \( dO(n^2) \). Hence the cardinality of the set \( S \) is \( dO(n^2) \) and our claim follows. \( \square \)

Theorem 1.1 follows from the Propositions 6.1, 7.2, and 5.3

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