Optimal unbiased state characterization

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Received 5 October 2012
Accepted for publication 26 October 2012
Published 28 March 2013
Online at stacks.iop.org/PhysScr/T153/014055

Abstract

We propose a general approach to characterize states of a bipartite system composed of a fully controllable and an unaccessible subsystem. The method is based on measuring the interference between states of the uncontrollable subsystem obtained after projecting an appropriately transformed bipartite state on the basis of the accessible subsystem by local operations.

PACS numbers: 03.65.Wj, 03.65.Aa, 42.50.Dv

1. Introduction

The characterization of quantum states is a central problem in quantum information science. Full information about a state with no or very little prior information can be obtained by using quantum tomography (QT) protocols [1–7]. These are usually very resource-demanding tasks, since numerous copies of the original state are required in order to accumulate reliable statistics about the measured probabilities [8–10]. Different strategies for QT have been theoretically proposed and experimentally implemented [11–15]. Nevertheless, even after a large set of measurements the experimental results should still be ‘cleaned’ by applying sophisticated mathematical methods [16–18] in order to ensure the best estimation of the reconstructed state.

On the other hand is situated the quantum state discrimination (QSD) problem [19–21] (for recent reviews see [22–24] and references therein) that is essential in numerous quantum information protocols. In this case the issue is to determine in what state, chosen from a set of known states, the was system prepared. This problem is also far from being simple, especially if one is required to find an optimal way for a conclusive and unambiguous state discrimination. The principal challenge here is to find an experimentally feasible set of positive operator-valued measures (POVMs), the construction of which usually requires us to enlarge the dimension of the quantum system by adding an adequate auxiliary system (ancilla). The ancillary system should be easily accessible and completely manageable so that all desirable operations can be implemented, and in particular, a combination of unitary operations in the Hilbert space of bi-partite (or in general multi-partite) systems plus von Neumann projections allow to carry out QSD protocols.

A general bipartite state (system + ancilla) can be represented in the form of the Schmidt decomposition

$$\left| \Psi \right\rangle = \sum_{l=1}^{d} \sqrt{\lambda_l} \left| \psi_l \right\rangle \left| \alpha_l \right\rangle,$$  \hspace{1cm} (1)

where $\lambda_l$ are positive numbers, $d$ is the so-called Schmidt rank and subscripts $s$ and $a$ are for system and ancilla states correspondingly. Even in the case when we do not have access to the system, i.e. we are not able to apply any operations to the system’s Hilbert space, it is still possible to determine the basic characteristics of the elements of the Schmidt decomposition (1). We can just project out the bipartite state into the ancillary subspace, obtaining a reduced density matrix

$$\rho_a = \text{Tr}_s \left| \Psi \right\rangle \left\langle \Psi \right| = \sum_{k,n} \mu_{kn} \left| u_k \right\rangle \left\langle u_n \right|,$$ \hspace{1cm} (2)

where the coefficients $\mu_{kn}$ depend on the parameters $\lambda_l$ and the scalar products $\lambda_{s,l} = \langle \psi_l | \alpha_l \rangle$. Since it is supposed that the ancilla is completely controllable, the density matrix $\rho_a$ can be reconstructed by applying standard tomographic methods, and thus determining the above-mentioned unknown parameters. Nevertheless, as we mentioned before, this is a very ‘expensive’ way of characterizing the decomposition (1).

In this paper, we propose an alternative way of determining the parameters $\{\lambda_l, \lambda_{s,l}, l, l' = 1, \ldots, d\}$. When the Schmidt rank $d$ is a prime number it is possible to combine the state discrimination protocols and the tomography methods based on the mutually unbiased bases (MUB) [25–30]. The main idea consists in applying specific unitary transformations (exactl those that are used for MUBs generation in prime dimensions) to the ancillary system with a consecutive...
projection on a basis in the Hilbert space of the ancilla. In this way, one can obtain a set of system states such that all the required information about the state (1) can be obtained from the interference picture (mutual projections of the resulting system states). In other words, we can obtain exactly the same information as in tomographic reconstruction of the ancillary density matrix (2), but using less resources.

The paper is organized as follows. In section 2 we describe the general method, applicable in any prime dimension and complex scalar products $\lambda_{i'f}$. In section 3 we provide a method for solving the equations derived in section 2 for the particular case of real $\lambda_{i'f}$. In section 4 we give explicit solutions for the two- and three-dimensional cases.

2. The method

Let us consider the Schmidt decomposition of a bipartite quantum system composed of an unknown and uncontrollable subsystem $\Lambda$ and a controllable subsystem $U$,

$$|\Psi_0\rangle = \sum_{l=0}^{p-1} \sqrt{\lambda_l} |\lambda_l\rangle |u_l\rangle,$$

where $\{|\lambda_l\rangle\}$ and $\{|u_l\rangle\}$ are states of subsystems $\Lambda$ and $U$, respectively, and the Schmidt rank is a prime number $p$. In practice, bipartite states of this form are obtained after applying to a factorized state a combination of local and conditional unitary transformations [31]. The set $\{|u_l\rangle\}$ can be chosen orthonormal, $\langle u_l | u_k \rangle = \delta_{lk}$, leading to the normalization condition on the ‘probabilities’

$$\sum_{l=0}^{p-1} \lambda_l = 1.$$

In order to extract the maximum possible information, i.e. the probabilities $\lambda_l$ and the scalar products $\lambda_{i'f} = \langle \lambda_f | \lambda_i \rangle$, about the state $|\Psi_0\rangle$, we apply to the system $U$ the whole set of Hadamard transformations $H_l$. In prime dimensions, this set consists of $p$ transformations of the form

$$H_l = H_0 D^l = \frac{1}{\sqrt{p}} \sum_{i,k=0}^{p-1} \omega^{-ik |u_l\rangle \langle u_k|},$$

where $\omega = e^{2\pi i/p}$ is the $p$th root of unity, and $e^{-i\phi} = \omega^{2^{-i}}$ for $p > 2$, while $e^{-i\phi} = -1$ for $p = 2$. Here, all the operations in the exponent of $e^{i\phi}$ are modulus $p$. The set of Hadamard matrices (5) is closely related to the standard form of MUB construction [32, 33]. Really, the columns of matrices $H_l$ are elements of bases which are unbiased for different values of the index $s$. In this section we focus on the case $p > 2$. The dimension $p = 2$ will be studied separately in section 4:

$$H_0 = \frac{1}{\sqrt{p}} \sum_{i,k=0}^{p-1} \omega^{-ik} |u_i\rangle \langle u_k|$$

is the finite Fourier transform operator and $D$ is the diagonal operator

$$D = \sum_{i=0}^{p-1} e^{-i\phi |u_i\rangle \langle u_i|},$$

After applying the transformation $I^{(\Lambda)} \otimes H_l^{(U)}$ to the state (3) and projecting over every state $|u_k\rangle$ of the controllable subsystem, we obtain the following normalized $p$ states of the system $\Lambda$:

$$|\psi^l_k\rangle = \frac{1}{\sqrt{N_{ks}}} \sum_{l=0}^{p-1} \omega^{-2^{-l-i_2-k}l} \sqrt{\lambda_l} |\lambda_l\rangle, \quad k = 0, \ldots, p - 1.$$

The normalization factors are given by

$$N_{ks} = 1 + \sum_{l'=0}^{p-1} \omega^{2^{-l'-2^{-l}+k}(l'-l)} x_{l't},$$

where we have introduced $x_{l'f} = \lambda_{l'f} \sqrt{\lambda_f} \lambda_l$; note that these factors automatically fulfill the relations

$$\sum_{k=0}^{p-1} N_{ks} = p$$

for every $s = 0, \ldots, p - 1$.

The state (3) contains $p^2$ unknown parameters: $p$ real (positive) probabilities $\lambda_l$ and $p(p - 1)/2$ complex inner products $\lambda_{i'f}$. These parameters can be determined by measuring projections (obtained from interference experiments) of (8) with $s = 0, \ldots, p - 1$ into a single state $|\psi^l_0\rangle$. Then, we have a set of (complex) equations of the form

$$\langle \psi^l_0 | \psi^l_k \rangle = a_{ks},$$

where the right-hand side is measurable quantities. Explicitly, for a given $s = 1, \ldots, p - 1$ we obtain a set of $p$ equations,

$$a_{ks} \sqrt{N_{00} N_{ks}} = \sum_{l=0}^{p-1} \omega^{-2^{-l-i_2-k}l} \lambda_l + \sum_{l'=0}^{p-1} \omega^{2^{-l'-2^{-l}+k}} x_{l't},$$

while for $s = 0$ one has $p - 1$ equations of the form

$$a_{0s} \sqrt{N_{00} N_{0s}} = \sum_{l=0}^{p-1} \omega^{-kl} \lambda_l + \sum_{l'=0}^{p-1} \omega^{-kl} x_{l't},$$

where $k = 1, \ldots, p - 1$.

The set of equations (12) contains redundant information about the system. In particular, the coefficients of $x_{l'f}$ and $\lambda_l$ in equations (12) labeled by $p - s$ and $p - k$ are complex conjugates to those labeled by $s$ and $k$, leading to the same complex equation when the inner products are real. Thus, the information that can be extracted from equation (12) with $s = (p + 1)/2, \ldots, p - 1$ is the same as the one we can obtain from the first $(p - 1)/2$ sets. The same situation holds for the set corresponding to $s = 0$: only the first $(p - 1)/2$ equations in (13) provide non-redundant information about the system. This leaves us with $p(p - 1)/2 + (p - 1)/2$ complex equations, which together with the normalization condition (4) gives exactly $p^2$ independent real equations.

Equations (12) and (13) are nonlinear, which is a clear drawback in comparison with the standard unbiased tomographic scheme. Nevertheless, in the special case of real scalar products $\lambda_{i'f}$, these sets of equations can be quasilinearized, and thus an analytic solution for an arbitrary dimension $p$ can be found.
3. Real inner products

In the case when \( \lambda_{k/l} \) are real numbers the number of real parameters to be determined is reduced to \( p(p + 1)/2 \).

It is convenient to rewrite equations (12) in the form

\[
\sqrt{N_0} a^2 \sum_{k=0}^{p-1} \omega^{bl} a_{kl} z_{kl} = p y_l, \quad l = 0, \ldots, p - 1, \quad (14)
\]

where we have introduced the new real variables

\[
y_l = \lambda_l + \sum_{l' = 0}^{p-1} \lambda_{l'} y_{l'}/y_l.
\]

and \( z_{kl} = \sqrt{N_0} z_{k-l} = \sqrt{N_{p-k+l-k}} \), for \( s = 1, \ldots, (p - 1)/2 \). The variables \( z_{kl} \) are not independent since relation (10) imposes the following restrictions:

\[
\sum_{l=0}^{p-1} z_{kl}^2 = p,
\]

for any \( s \). The normalization factor \( N_0 \) is related to the new variables through the relation

\[
N_0 = \sum_{l=0}^{p-1} y_l = \frac{1}{p^2} \left[ \sum_{l=0}^{p-1} a_{kl} z_{kl} G(2^{-1}s, k) \right]^2,
\]

where \( G(s, k) = \sum_{l=0}^{p-1} \omega^{sl} \omega^{kl} \) is the Gauss sum. The last equality in (17), obtained by summing up equations in (14), holds for any \( s = 1, \ldots, (p - 1)/2 \) and implies that the measured quantities \( a_{kl} \) are not independent if the scalar products \( \lambda_{k/l} \) are real. Indeed, it is sufficient to measure \( p - 1 \) imaginary parts of \( \langle \psi_i^0 | \psi_i^k \rangle \), for \( s = 1, \ldots, (p - 1)/2 \), and the real part for only a single (arbitrary) value of \( s \). For the complete characterization of the state (3), we need another \( (p - 3)/2 \) measurements from \( s = 0 \).

The condition that the imaginary part of (14) is zero leads to \( p - 1 \) linearly independent equations

\[
\text{Im} \left( \omega^{l-1} \sum_{k=0}^{p-1} \omega^{bl} a_{kl} z_{kl} \right) = 0, \quad l = 0, \ldots, p - 1, \quad (18)
\]

for a given value of \( s = 1, \ldots, (p - 1)/2 \), which together with condition (16) allows us to determine all \( z_{kl} \), for \( s = 1, \ldots, (p - 1)/2 \), and thus all \( N_{kl}, s = 1, \ldots, p - 1 \) since for real \( \lambda_{k/l} \) we have \( N_{kl} = N_{p-k+l-k} \). Having obtained \( z_{kl} \), the variables \( y_l \) are immediately determined from the real part of (14) for any \( s \) and (17).

For \( s = 0 \) equation (14) is reduced to the following form:

\[
\sqrt{N_0} \sum_{k=0}^{p-1} \omega^{bk} a_{k0} z_{k0} = p y_l, \quad l = 0, \ldots, (p - 1)/2, \quad (19)
\]

where \( a_{00} = \langle \psi_0^0 | \psi_0^0 \rangle = 1 \), \( a_{p-10} = a_{10}^* \) and the variables \( z_{k0} \) satisfy the symmetry condition \( z_{p-k+10} = z_{k0} \). Equation (10) now reads

\[
N_{00} + 2 \sum_{k=1}^{p-1} z_{k0}^2 = p, \quad (20)
\]

As above, the variables \( z_{k0}, k = 1, \ldots, (p - 1)/2 \), are obtained from the conditions

\[
\text{Im} \left( \sum_{k=1}^{p-1} \omega^{bl} a_{k0} z_{k0} \right) = 0, \quad l = 1, \ldots, (p - 1)/2, \quad (21)
\]

together with (20).

Finally, having obtained all \( N_{kl} \) and \( y_l \) we can invert equations (9) and (15) to determine the physical parameters \( \lambda_i \) and \( \lambda_{k/l} \). It should be stressed here that since \( N_{kl} \) are not linearly independent for a fixed value of the index \( s = 0, \ldots, p - 1 \), as it follows from equations (18), (21), there are \( (p - 1)^2/2 - 1 \) linearly independent \( N_{kl} \) and \( p \) linearly independent \( y_l \), so that the normalization condition (4) should be added in order to be able to reconstruct \( \lambda_i \) and \( \lambda_{k/l} \).

In the next section, we show how this approach works in the particular cases of \( p = 2 \) and 3.

4. Examples

4.1. Dimension two

In the case of \( p = 2 \) we have \( \omega = -1 \) and \( \omega^{-1} \rightarrow -i \), so that equations (12) read as

\[
\sqrt{N_0} a_{01} + i y_1 = 0, \quad (22)
\]

\[
\sqrt{N_0} a_{11} - i y_1 = 0, \quad (23)
\]

where \( y_0 = \lambda_0 + x_0, \quad y_1 = \lambda_1 + x_1, \) and

\[
\begin{align*}
N_{00} &= 1 + 2x_R, \quad N_{01} = 1 - 2x_1, \quad N_{11} = 1 + 2x_1;
\end{align*}
\]

here we have introduced \( x_0 = x_R + i x_1 \). Observe that similarly to the case of \( p > 2 \), equations (22) and (23) provide the same information, so that we pick, for instance, equation (22). From (13) we obtain a single relation

\[
\sqrt{N_0} a_{10} = y_0 - y_1, \quad (24)
\]

with

\[
N_{01} = 1 - 2x_R,
\]

where again only the imaginary part will be considered. Introducing the imaginary and real parts of the measurement as \( a_{10} = a_{00} + i \beta_0 \), and \( a_{01} = a_{10}^* = a_{11} + i \beta_1 \), we arrive at three real equations

\[
\begin{align*}
\sqrt{1 + 2x_R}(1 - 2x_1) a_1 &= \lambda_0 + x_R - x_1, \quad (25) \\
(1 + 2x_R)(1 - 2x_1) \beta_1 &= \lambda_1 + x_R - x_1, \quad (26) \\
\sqrt{1 - 4x^2_R} \beta_0 &= -2x_1, \quad (27)
\end{align*}
\]

where the first two equations correspond to the real and the imaginary part of (22), while the last one is the imaginary part of (24). Equations (25)–(27) together with the normalization condition \( \lambda_0 + \lambda_1 = 1 \) allow us to determine four real parameters, \( \lambda_{0,1} \) and \( x_{R,1} = \sqrt{\text{Re(Im)} \lambda_{0,1}} \).
4.2. Dimension three

Here we discuss the case of \( p = 3 \) where the scalar products between any pair of states \( |\lambda_i\rangle \) are real.

Equations (14) for \( s = 1 \) take the form

\[
\sqrt{N_{00}} (z_{01} a_{01} + z_{11} a_{11} + z_{21} a_{21}) = 3y_0, \tag{28}
\]

\[
\sqrt{N_{00}} (\omega^2 z_{01} a_{01} + z_{11} a_{11} + \omega z_{21} a_{21}) = 3y_1, \tag{29}
\]

\[
\sqrt{N_{00}} (\omega^2 z_{01} a_{01} + \omega z_{11} a_{11} + z_{21} a_{21}) = 3y_2, \tag{30}
\]

where \( \omega = e^{2\pi i/3} \),

\[
z_{11}^2 = N_{11} = 1 + 2x_{01} - x_{02} - x_{21},
\]

\[
z_{21}^2 = N_{21} = 1 - x_{01} + 2x_{02} - x_{21},
\]

and \( z_{01} = N_{01} = 3 - N_{11} - N_{21} \).

Following the procedure of section 3, we express \( z_{11} \) and \( z_{21} \) as functions of \( z_{01} \) from the imaginary part of equations (29), (30) and then using (10) one obtains \( z_{01} \) in terms of the measured quantities,

\[
z_{01} = \frac{\sqrt{3}(\alpha_1 \alpha_2 - 3\beta_1 \beta_2)}{R}, \tag{31}
\]

\[
z_{11} = \frac{\sqrt{3}\alpha_0 (\alpha_2 - \sqrt{3}\beta_2)}{R}, \tag{32}
\]

\[
z_{21} = \frac{\sqrt{3}\alpha_0 (\alpha_1 - \sqrt{3}\beta_1)}{R}, \tag{33}
\]

where

\[
R = \left( (\alpha_1 \alpha_2 - 3\beta_1 \beta_2)^2 + \alpha_0^2 \left( (\alpha_1 - \sqrt{3}\beta_1)^2 + (\alpha_2 - \sqrt{3}\beta_2)^2 \right) \right)^{1/2},
\]

and we have introduced \( a_{ij} = \alpha_j + i\beta_j, \) \( j = 0, 1, 2 \). It is worth noting that \( \beta_0 \) does not appear in the above expressions. Now, following the general procedure, we obtain

\[
N_{00} = \frac{1}{3} \left[ 2z_{01} z_{11} + \left( \sqrt{3}\alpha_1 - \beta_1 \right) z_{11} + \left( \sqrt{3}\alpha_2 - \beta_2 \right) z_{21} \right]^2, \tag{34}
\]

where \( z_{j1} \) are given in (31)–(33).

Using (31)–(33) and (34) it is straightforward to express \( y_1 \) as a function of \( z_{j1} \) and \( N_{j1}, N_{00} \) form the real part of equations (28)–(30). It is worth noting here that \( N_{00} \) can be found directly from \( 2N_{10} = 3 - N_{00} \). Now we can completely characterize the initial state by the probabilities

\[
\lambda_0 = y_0 + \frac{1}{3} (N_{01} - N_{00}),
\]

\[
\lambda_1 = y_1 + \frac{1}{3} (N_{21} - N_{00}),
\]

\[
\lambda_2 = y_2 + \frac{1}{3} (N_{11} - N_{00}),
\]

and the inner products \( \lambda_{ij} = x_{ij}/\sqrt{\lambda_i \lambda_j} \), where

\[
x_{01} = \frac{1}{6} (N_{00} + N_{11} - N_{01} - N_{21}),
\]

\[
x_{02} = \frac{1}{6} (N_{00} + N_{21} - N_{01} - N_{11}),
\]

\[
x_{12} = \frac{1}{6} (N_{00} + N_{01} - N_{11} - N_{21}).
\]

5. Conclusions

We proposed a general characterization of bipartite states of the form (1) in the case where one of the subsystems is not accessible by unitary transformations. We have shown that such a characterization can be done without applying the complete tomographic procedure but by only measuring the interference between states of the uncontrollable subsystem obtained after projecting an appropriately transformed bipartite state on the basis of the accessible (by local operations) subsystem.

In this approach we can reduce the number of quantum resources required and can also avoid processing a vast amount of statistical information obtained usually from tomographic data. In fact, we need only \( p^2 - 1 \) copies of the initial state required for producing the \((p^2 - 1)/2\) projections to state \( |\psi_{01}^j\rangle\) that we need for characterizing the desired state. In this sense the characterization we have proposed is optimal since we require the same number of measurements (and not only setups, as in the MUB tomography scheme) as the number of unknown parameters.

In the present approach, we have used the properties of the Hadamard transformations in the case of prime dimension, which allows us to approach the analytical solution of the general problem. These Hadamard transformations are exactly those that generate a complete set of MUB. Obviously, in prime power dimensions our methods can be applied almost literally. When the dimension is not a prime power, and the whole set of Hadamard matrices is unknown, this method of quantum state characterization will still work: we need only \( p^2 - 1 \) equations (excluding the normalization condition) to determine all the parameters of the state (1). These equations can be obtained by applying a set of appropriately chosen unitary transformations. Although it would not generally be possible to establish an analytical procedure similar to that described in section 3, a numerical solution to the resulting system of equations can still be found. Such equations can be obtained by applying a set of appropriately chosen unitary transformations. Although, in general it would not be possible to establish an analytical procedure as described in section 3, a numerical solution can still be found.

Finally, we would like to mention the recent paper [34], where a tomographic reconstruction of a single qubit by applying methods of QSD was proposed.

Acknowledgments

This work was partially supported by grant no. 106525 of the CONACyT (Mexico).
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