RELATIONS BETWEEN SOME CLASSES OF FUNCTIONS
OF GENERALIZED BOUNDED VARIATION

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Abstract. We prove inclusion relations between generalizing Water-
man’s and generalized Wiener’s classes for functions of two variable.

The notion of function of bounded variation was introduced by C. Jordan [16]. Generalized this notion N. Wiener [30] has considered the class $BV_p$ of functions. L. Young [31] introduced the notion of functions of $\Phi$-variation. In [26] D. Waterman has introduced the following concept of generalized bounded variation

Definition 1. Let $\Lambda = \{\lambda_n : n \geq 1\}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} (1/\lambda_n) = \infty$. A function $f$ is said to be of $\Lambda$-bounded variation ($f \in \Lambda BV$), if for every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$, we have

$$\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty,$$

where $I_n = [a_n, b_n] \subset [0, 1]$ and $f(I_n) = f(b_n) - f(a_n)$.

If $f \in \Lambda BV$, then $\Lambda$-variation of $f$ is defined to be the supremum of such sums, denoted by $V_\Lambda(f)$.

Properties of functions of the class $\Lambda BV$ as well as the convergence and summability properties of their Fourier series have been investigated in [22]-[29].

For everywhere bounded 1-periodic functions, Z. Chanturia [6] has introduced the concept of the modulus of variation.

H. Kita and K. Yoneda [17] studied generalized Wiener classes $BV(p(n) \uparrow p)$. They introduced

Definition 2. Let $f$ be a finite 1-periodic function defined on the interval $(-\infty, +\infty)$. $\Delta$ is said to be a partition with period 1 if there is a set of points

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then the class $\BV Goginava$ 

Belov (see \cite{5}, Z. Chanturia \cite{7}, T. Akhobadze \cite{3} and M. Medvedieva \cite{21},

taken into account in the works of M. Avdispahic \cite{4}, A. Kovcik \cite{19}, A. Belov (see \cite{1}, Z. Chanturia \cite{7}, T. Akhobadze \cite{11} and M. Medvedieva \cite{21},

Goginava \cite{11}, \cite{13}.

We note that if $p(n) = p$ for each natural number, where $1 \leq p < +\infty$, then the class $\BV (p(n) \uparrow p)$ coincides with the Wiener class $V_p$.

Properties of functions of the class $\BV (p(n) \uparrow p)$ as well as the uniform convergence and divergence at point of their Fourier series with respect to trigonometric and Walsh system have been investigated in \cite{9}, \cite{12}, \cite{18}.

Generalizing the class $\BV (p(n) \uparrow p)$ T. Akhobadze (see \cite{11}, \cite{2}) has considered the $\BV (p(n) \uparrow p, \varphi)$ and $BA (p(n) \uparrow p, \varphi)$ classes of functions.

The relation between different classes of generalized bounded variation was taken into account in the works of M. Avdispahic \cite{1}, A. Kovcik \cite{19}, A. Belov (see \cite{1}, Z. Chanturia \cite{7}, T. Akhobadze \cite{11} and M. Medvedieva \cite{21},

Goginava \cite{11}, \cite{13}.

Let $f$ be a real and measurable function of two variable of period 1 with respect to each variable. Given intervals $J_1 = (a, b)$, $J_2 = (c, d)$ and points $x, y$ from $I := [0, 1]$ we denote

$$f(J_1, y) := f(b, y) - f(a, y), \quad f(x, J_2) := f(x, d) - f(x, c)$$

and for the rectangle $A = (a, b) \times (c, d)$, we set

$$f(A) = f(J_1, J_2) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from $I$ ordered in arbitrary way and let $\Omega$ be the set of all such collections $E$.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^\infty$ we denote

$$\LV_1(f) = \sup \sup_{y \in I} \left( \sum_{i} \frac{|f(I_i, y)|}{\lambda_i} \right),$$

$$\LV_2(f) = \sup \sup_{x \in I} \left( \sum_{j} \frac{|f(x, J_j)|}{\lambda_j} \right),$$

$$\LV_{1,2}(f) = \sup_{\{I_i, J_j\} \in \Omega} \left( \sum_{i} \sum_{j} \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j} \right).$$

**Definition 3.** We say that the function $f$ has bounded $\Lambda$-variation on $I^2 := [0, 1] \times [0, 1]$ and write $f \in \Lambda BV$, if

$$\LV(f) := \LV_1(f) + \LV_2(f) + \LV_{1,2}(f) < \infty.$$
We say that the function \( f \) has bounded Partial \( \Lambda \)-variation and write \( f \in P\Lambda BV \) if
\[
P\text{AV}(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.
\]
If \( \lambda_n \equiv 1 \) (or if \( 0 < c < \lambda_n < C < \infty \), \( n = 1, 2, \ldots \)) the classes \( ABV \) and \( PABV \) coincide with the Hardy class \( BV \) and \( PBV \) respectively. Hence it is reasonable to assume that \( \lambda_n \to \infty \) and since the intervals in \( E = \{I_i\} \) are ordered arbitrarily, we will suppose, without loss of generality, that the sequence \( \{\lambda_n\} \) is increasing. Thus, in what follows we suppose that

\[
1 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.
\]

In the case when \( \lambda_n = n \), \( n = 1, 2 \ldots \) we say Harmonic Variation instead of \( \Lambda \)-variation and write \( H \) instead of \( \Lambda \) (\( HBV \), \( PHBV \), \( HV \) (\( f \)), etc).

The notion of \( \Lambda \)-variation was introduced by Waterman \[26\] in one dimensional case and Sahakian \[24\] in two dimensional case. The notion of bounded partial variation (class \( PBV \)) was introduced by Goginava \[10\]. These classes of functions of generalized bounded variation play an important role in the theory of Fourier series.

We have proved in \[14\] the following theorem.

**Theorem 1** (Goginava, Sahakian). Let \( \Lambda = \{\lambda_n = n \gamma_n\} \) and \( \gamma_n \geq \gamma_{n+1} > 0, \ n = 1, 2, \ldots \).

1) If
\[
\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,
\]
then \( PABV \subset HBV \).

2) If, in addition, for some \( \delta > 0 \)
\[
\gamma_n = O(\gamma_n^{1+\delta}) \quad \text{as} \quad n \to \infty
\]
and
\[
\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,
\]
then \( PABV \not\subset HBV \).

Dyachenko and Waterman \[8\] introduced another class of functions of generalized bounded variation. Denoting by \( \Gamma \) the set of finite collections of nonoverlapping rectangles \( A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2 \) we define
\[
\Lambda^*V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.
\]

**Definition 4** (Dyachenko, Waterman). Let \( f \) be a real function on \( I^2 \). We say that \( f \in \Lambda^*BV \) if
\[
\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^*V(f) < \infty.
\]
In [15] Goginava and Sahakian introduced a new class of functions of generalized bounded variation and investigated the convergence of Fourier series of function of that classes.

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we denote
\[ \Lambda^#V_1(f) = \sup_{\{y_i\} \subset \mathcal{T}} \sup_{\{I_i\} \in \Omega} \sum_{i} \frac{|f(I_i, y_i)|}{\lambda_i}, \]
\[ \Lambda^#V_2(f) = \sup_{\{x_j\} \subset \mathcal{T}} \sup_{\{J_j\} \in \Omega} \sum_{j} \frac{|f(x_j, J_j)|}{\lambda_j}. \]

**Definition 5** (Goginava, Sahakian). We say that the function $f$ belongs to the class $\Lambda^#BV$, if
\[ \Lambda^#V(f) := \Lambda^#V_1(f) + \Lambda^#V_2(f) < \infty. \]

The following theorem was proved in [15]

**Theorem 2.** a) If
\[ \lim_{n \to \infty} \frac{\lambda_n \log (n+1) \log n}{n} < \infty, \]
then
\[ \Lambda^#BV \subset HBV. \]

b) If $\frac{\lambda_n}{n} \downarrow 0$ and
\[ \lim_{n \to \infty} \frac{\lambda_n \log (n+1)}{n} = +\infty, \]
then
\[ \Lambda^#BV \not\subset HBV. \]

In this paper we introduce new classes of bounded generalized variation.

Let $f$ be a function defined on $\mathbb{R}^2$ with 1-periodic relative to each variable. $\Delta_1$ and $\Delta_2$ is said to be a partitions with period 1, if
\[ \Delta_i : \ldots < t_{i-1}^{(i)} < t_0^{(i)} < t_1^{(i)} < \ldots < t^{(i)}_{m_i} < t_{m_i+1}^{(i)} < \ldots, \quad i = 1, 2 \]
satisfies $t_{k+m_i}^{(i)} = t_k^{(i)} + 1 \text{ for } k = 0, \pm 1, \pm 2, \ldots$, where $m_i, i = 1, 2$ are a positive integers.

**Definition 6.** Let $p(n)$ be an increasing sequence such that $1 \leq p(n) \uparrow p$, $n \to \infty$, where $1 \leq p \leq +\infty$. We say that a function $f$ belongs to the class $BV^#(p(n) \uparrow p)$ if
\[ V_1^#(f, p(n) \uparrow p) := \sup_{\{y_i\} \subset \mathcal{T}} \sup_{n \geq 1} \sup_{\Delta_1} \left\{ \left( \sum_{i=1}^{m_i} |f(I_i, y_i)|^{p(n)} \right)^{1/p(n)} : \inf_i |I_i| \geq \frac{1}{2^n} \right\} < +\infty, \]
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and

\[ V_2^\# (f, p(n) \uparrow p) := \sup_{\{x_j\} \subset I} \sup_{n \geq 1} \Delta \left\{ \left( \sum_{j=1}^{m_2} |f(x_j, J_j)|^{p(n)} \right)^{1/p(n)} : \inf_j |J_j| \geq \frac{1}{2^n} \right\} < +\infty, \]

where

\[ I_i := (t^{(1)}_{i-1}, t^{(1)}_i), J_j := (t^{(2)}_{j-1}, t^{(2)}_j). \]

C(\(I^2\)) and B(\(I^2\)) are the spaces of continuous and bounded functions given on \(I^2\), respectively.

In this paper we prove inclusion relations between \(\Lambda^\# BV\) and \(BV^\# (p(n) \uparrow \infty)\) classes. In particular, the following are true

**Theorem 3.** \(\Lambda^\# BV \subset BV^\# (p(n) \uparrow \infty)\) if and only if

\[ \lim_{n \to \infty} \sup_{1 \leq m \leq 2^n} m^{1/p(n)} \sum_{i=1}^{m} (1/\lambda_i) < \infty. \]  

**Theorem 4.** Let \(\sum_{n=1}^{\infty} (1/\lambda_n) = +\infty\). Then there exists a function \(f \in BV^\# (p(n) \uparrow \infty) \cap C(\{I^2\})\) such that \(f \notin \Lambda BV^\#\).

**Corollary 1.** \(BV^\# (p(n) \uparrow \infty) \subset \Lambda^\# BV\) if and only if \(\Lambda^\# BV = B(\{I^2\})\).

**Proof of Theorem 3.** Let us take an arbitrary \(f \in \Lambda^\# BV\). Follow of the method of the paper Kuprikov in [20], we can prove that the following estimations hold:

\[ \left( \sum_{k=1}^{m_1} |f(I_k, y_k)|^{p(n)} \right)^{1/p(n)} \leq \Lambda^\# V_1 (f) \sup_{1 \leq m \leq 2^n} m^{1/p(n)} \sum_{i=1}^{m} (1/\lambda_i) < \infty \]

and

\[ \left( \sum_{k=1}^{m_2} |f(x_k, J_k)|^{p(n)} \right)^{1/p(n)} \leq \Lambda^\# V_2 (f) \sup_{1 \leq m \leq 2^n} m^{1/p(n)} \sum_{i=1}^{m} (1/\lambda_i) < \infty. \]

Therefore, \(f \in \Lambda^\# BV (p(n) \uparrow \infty)\).

Next, we suppose that the condition (7) does not hold. As an example we construct function from \(\Lambda^\# BV\) which is not in \(BV^\# (p(n) \uparrow \infty)\).

Since

\[ \lim_{n \to \infty} \sup_{1 \leq m \leq 2^n} m^{1/p(n)} \sum_{j=1}^{m} (1/\lambda_j) = +\infty, \]
there exists a sequence of integers \( \{ n'_k : k \geq 1 \} \) such that

\[
\lim_{k \to \infty} \frac{m(n'_k)^{1/p(n'_k)}}{\sum_{j=1}^{m(n'_k)} (1/\lambda_j)} = +\infty,
\]

where

\[
\sup_{1 \leq m \leq 2^n} \frac{m^{1/p(n)}}{\sum_{j=1}^{m (n)} (1/\lambda_j)} = \frac{m(n)^{1/p(n)}}{\sum_{j=1}^{m(n)} (1/\lambda_j)}.
\]

We choose a monotone increasing sequence of positive integers \( \{ n_k : k \geq 1 \} \subset \{ n'_k : k \geq 1 \} \) such that

\[
\frac{m(n_k)^{1/p(n_k)}}{\sum_{j=1}^{m(n_k)} (1/\lambda_j)} \geq 4^k,
\]

\[
p(n_k) \geq n_{k-1}.
\]

\[
n_k > 3n_{k-1} + 1 \quad \text{for all } k \geq 2.
\]

From (8) and (10) it is evident that \( 2^{2n_{k-1}} < m(n_k) \leq 2^{n_k} \).

Two cases are possible:

a) Let there exists a monotone sequence of positive integers \( \{ s_k : k \geq 1 \} \subset \{ n_k : k \geq 1 \} \) such that

\[
2^{2s_{k-1}} < m(s_k) \leq 2^{s_k - s_{k-1} - 1}.
\]

Consider the function \( f_k \) defined by

\[
f_k(x) = \begin{cases} 
  h_k(2^{s_k}x - 2j + 1), & x \in [(2j - 1)/2^{s_k}, 2j/2^{s_k}) \\
  -h_k(2^{s_k}x - 2j - 1), & x \in [2j/2^{s_k}, (2j + 1)/2^{s_k}) \\
  0, & \text{otherwise}
\end{cases}
\]

where

\[
h_k = \left( \frac{1}{2^k \sum_{j=1}^{m(s_k)} (1/\lambda_j)} \right)^{1/2}.
\]

Let

\[
f(x, y) = \sum_{k=2}^{\infty} f_k(x) f_k(y),
\]

where

\[
f(x + l, y + s) = f(x, y), \quad l, s = 0, \pm 1, \pm 2, \ldots.
\]
First we prove that $f \in \Lambda^\#BV$. For every choice of nonoverlapping intervals $\{I_n : n \geq 1\}$, we get

$$
\Lambda^\#V_1(f; p(n) \uparrow \infty) \leq \sum_{j=1}^{\infty} \frac{|f(I_j, y_j)|}{\lambda_j}
$$

$$
\leq 4 \sum_{i=1}^{\infty} h^2_i \sum_{j=1}^{m(s_i)} \frac{1}{\lambda_j} = 4 \sum_{i=1}^{\infty} \frac{1}{2^j} = 4.
$$

Analogously, we can prove that

$$
\Lambda^\#V_2(f; p(n) \uparrow \infty) \leq 4.
$$

Next, we shall prove that $f \notin BV^\# (p(n) \uparrow \infty)$. By [11], [12] and from the construction of the function we get

$$
V_1(f; p(n) \uparrow \infty)
\geq \left\{ \sum_{j=m(s_k-1)}^{m(s_k)-1} \left| f\left(\frac{2j-1}{2^{s_k}}, \frac{2j}{2^{s_k}}\right) - f\left(\frac{2j}{2^{s_k}}, \frac{2j}{2^{s_k}}\right)\right|^\frac{1}{p(s_k)} \right\}
$$

$$
= \left\{ \sum_{j=m(s_k-1)}^{m(s_k)-1} \left( f_k\left(\frac{2j-1}{2^{s_k}}\right) - f_k\left(\frac{2j}{2^{s_k}}\right)\right)^\frac{1}{p(s_k)} \right\}
$$

$$
= h^2_k (m(s_k) - m(s_{k-1}))^{\frac{1}{p(s_k)}}
\geq c \frac{m(s_k)}{m(s_{k-1})} \geq c 2^k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.
$$

Therefore, we get $f \notin BV^\#(p(n) \uparrow \infty)$.

b) Let

$$
2^{n_k-n_{k-1}-1} < m(n_k) \leq 2^{n_k} \quad \text{for all} \quad k > k_0.
$$

Consider the function $g_k$ defined by

$$
g_k(x) = \begin{cases} 
  d_k(2^{n_k}x - 2j + 1), & x \in [(2j-1)/2^{n_k}, 2j/2^{n_k}) \\
  -d_k(2^{n_k}x - 2j - 1), & x \in [2j/2^{n_k}, (2j + 1)/2^{n_k}) \\
  0, & \text{otherwise}
\end{cases}
$$

where

$$
d_k = \left( \frac{1}{2^k \sum_{j=1}^{m(n_k)} (1/\lambda_j)} \right)^{1/2}.
$$
Let

\[ g(x, y) = \sum_{k=k_0+2}^{\infty} g_k(x) g_k(y), \]

where

\[ g(x + l, y + s) = g(x, y), \quad l, s = 0, \pm 1, \pm 2, \ldots. \]

For every choice of nonoverlapping intervals \( \{I_n : n \geq 1\} \) we get

\[
\sum_{j=1}^{\infty} \frac{|f(I_j, y_j)|}{\lambda_j} \leq 4 \sum_{i=k_0+1}^{\infty} d_i^2 \sum_{j=1}^{2^{n_i}-n_{i-1}-1} \frac{1}{\lambda_j} \leq 4 \sum_{i=k_0+1}^{\infty} d_i^2 \sum_{j=1}^{m(n_i)} \frac{1}{\lambda_j} < \infty.
\]

Analogously, we can prove that

\[
\sum_{j=1}^{\infty} \frac{|f(x_j, J_j)|}{\lambda_j} < \infty.
\]

Hence we have \( g \in \Lambda^\# BV. \)
Next we shall prove that \( g \notin BV^\# (p(n) \uparrow \infty) \). By (8), (10), (11) and from the construction of the function we get
\[
V_1^\# \left( g; p(n) \uparrow \infty \right) \geq \left\{ \begin{array}{c}
\sum_{j=2n_{k-1}}^{2n_k-1} \left| g \left( \frac{2j-1}{2n_k}, \frac{2j}{2n_k} \right) - g \left( \frac{2j}{2n_k}, \frac{2j}{2n_k} \right) \right|^{1/p(n_k)} \\
\sum_{j=2n_{k-1}}^{2n_k-1} \left| g_k \left( \frac{2j-1}{2n_k} \right) - g_k \left( \frac{2j}{2n_k} \right) \right|^{1/p(n_k)} 
\end{array} \right. 
\]
\[
= d_k^2 \left( 2^{n_k-n_{k-1}} - 2^{n_{k-1}-n_{k-2}} \right)^{1/p(n_k)} 
\]
\[
\geq \frac{1}{4} d_k^2 2^{(n_k-n_{k-1})/p(n_k)} 
\]
\[
\geq c 2^{n_k/p(n_k)} \sum_{j=1}^{m(n_k)} (1/\lambda_j) 
\]
\[
\geq c m \left( n_k \right)^{1/p(n_k)} \sum_{j=1}^{n_k} (1/\lambda_j) 
\]
\[
\geq c 2^{n_k} \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty. 
\]
Therefore, we get \( g \notin BV^\# (p(n) \uparrow \infty) \) and the proof of Theorem 1 is complete.

**Proof of Theorem 4.** We choose a monotone increasing sequence of positive integers \( \{l_k : k \geq 1 \} \) such that \( l_1 = 1 \) and
\[
(13) \quad p \left( l_{k-1} \right) \geq \ln k \quad \text{for all} \quad k \geq 2. 
\]
Set \( (k = 1, 2, \ldots) \)
\[
r_k (x) = \begin{cases} 
2^{k+1} c_k \left( x - 1/2^{k} \right), & \text{if} \ 1/2^k \leq x \leq 3/2^{k+1} \\
-2^{k+1} c_k \left( x - 1/2^{k-1} \right), & \text{if} \ 3/2^{k+1} \leq x \leq 1/2^{k-1} \\
0, & \text{otherwise} 
\end{cases} 
\]
where
\[
c_k = \left( \sum_{j=1}^{k} \frac{1}{\lambda_j} \right)^{-1/4} 
\]
and
\[
r (x, y) = \sum_{k=1}^{\infty} r_k (x) r_k (y), 
\]
where
\[
r (x + l, y + s) = r (x, y) \quad l, s = 0, \pm 1, \pm 2, \ldots. 
\]
It is easy to show that function $r \in C(I^2)$.

First we show that $r \in BV^#(p(n) \uparrow \infty)$. Let $\{I_i\}$ be an arbitrary partition of the interval $I$ such that $\inf_i |I_i| \geq 1/2^l$. For this fixed $l$, we can choose integers $l_{k-1}$ and $l_k$ for which $l_{k-1} \leq l < l_k$ holds. Then it follows that $p(l_{k-1}) \leq p(l) \leq p(l_k)$ and $1/2^{l_{k-1}} < 1/2^l \leq 1/2^{l_k-1}$.

By (13) and from the construction of the function $r$ we obtain

$$
\left\{ \sum_{j=1}^{m} |r(I_i, y_i)|^{p(l)} \right\}^{1/p(l)} = \left\{ \sum_{j=1}^{k} \left( \sum_{i : 2^{-l_j} \leq y_i < 2^{-l_j+1}} |r(I_i, y_i)|^{p(l)} \right) \right\}^{1/p(l)} \\
\leq \left\{ \sum_{j=1}^{k} \left( \sum_{i : 2^{-l_j} \leq y_i < 2^{-l_j+1}} |r(I_i, y_i)|^{p(l)} \right) \right\}^{1/p(l)} \\
\leq \left\{ \sum_{j=1}^{k} \left( \sum_{i : 2^{-l_j} \leq y_i < 2^{-l_j+1}} |r(I_i, y_i)| \right) \right\}^{1/p(l)} \\
\leq \left\{ \sum_{j=1}^{k} \left( \frac{3}{2^{l_j+1}} \right)^{p(l)} \right\}^{1/p(l)} \\
\leq \left\{ \sum_{j=1}^{k} \left( 2c_j^2 \right)^{p(l)} \right\}^{1/p(l)} \leq 2k^{1/p(l_{k-1})} \leq 4k^{1/\ln k} = 4e.
$$

Therefore $r \in BV^#(p(n) \uparrow \infty)$ holds.
Finally, we prove that \( r / \not \in \Lambda BV^\# \). Since \( c_n \downarrow 0 \), we get
\[
\sum_{j=1}^{k} \left| \frac{r \left(1/2^{l_j}, 3/2^{l_j}+1\right) - r \left(3/2^{l_j}+1, 3/2^{l_j}+1\right)}{\lambda_j} \right| \\
\begin{align*}
&= \sum_{j=1}^{k} \left| \left( r_j \left(1/2^{l_j}\right) - r_j \left(3/2^{l_j}+1\right) \right) \right| \frac{1}{\lambda_j} \\
&= \sum_{j=1}^{k} \frac{c_j^2}{\lambda_j} \geq \frac{k}{\lambda_j} \sum_{j=1}^{k} \frac{1}{\lambda_j} \\
&= \left( \frac{k}{\sum_{j=1}^{k} \lambda_j} \right)^{1/2} \to \infty \text{ as } k \to \infty.
\end{align*}
\]
Therefore, we get \( r / \not \in \Lambda BV^\# \) and the proof of Theorem 4 is complete. \( \square \)

Since \( \Lambda BV^\# = B \left( I^2 \right) \) if and only if \( \sum_{j=1}^{\infty} (1/\lambda_j) < \infty \) the validity of Corollary 4 follows from Theorem 4.

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