The Story of M*

Paul K. Townsend

DAMTP, Center for Mathematical Sciences,
University of Cambridge,
Wilberforce Road,
Cambridge CB3 0WA, UK

ABSTRACT

The origins of the 11-dimensional supermembrane are recalled, and a curious property is discussed: the field theory limit of a supermembrane in a hyper-Kähler background is a 3-dimensional sigma-model with $N = 4$ supersymmetry, but the higher-order fermion interactions of the supermembrane generically break this to $N = 3$.

They were learning to draw, the doormouse went on, ... and they drew all manner of things—everything that begins with M.

Why with an M? said Alice.

Why not? said the March Hare.

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1 Introduction

Shakespearean actors are traditionally averse to pronouncing the name of the play ‘Macbeth’, preferring to call it ‘the Scottish play’. Presumably it was only distaste for cryptic abbreviations that prevented it from becoming known as the ‘M-play’. M-theory acquired its name from a similar aversion, in this case of string theorists to the word ‘Membrane’. The story of M is thus the story of membranes, supermembranes in particular. The occasion of Stephen Hawking’s 60th birthday is an appropriate one for me to put on record some recollections of this story because Stephen was exceptional in giving his support and encouragement to work on supermembranes during the years in which the ‘M-word’ could not be pronounced. Thank you, Stephen, and happy 60th birthday.

A membrane is of course just a special case of a brane and, as the reader will probably know, M-theory is really an Orwellian democracy in which there are many equal branes but with some being more equal than others. Strings are more equal for all the usual reasons, but membranes are more equal too, for a different set of reasons. In the light-front gauge, membranes are equivalent to the large \( n \) limit of \( SU(n) \) gauge theories, dimensionally reduced to a quantum mechanical model. The M(atrix) model formulation of M-theory could have been, and nearly was, found from the 11-dimensional supermembrane in this way. But this is all well-known, and given my spacetime limitations I prefer to reminisce on the (pre)history of the 11-dimensional supermembrane.

This will be a selective history, chosen to motivate discussion of a surprising, and little-known, fact: the field theory limit of a supermembrane in certain hyper-Kähler backgrounds is a 3-dimensional sigma-model with \( N = 4 \) supersymmetry, but the supermembrane itself generically has only \( N = 3 \) supersymmetry \[5\]. This is a sigma-model analogue of the breaking of \( N = 4 \) to \( N = 3 \) supersymmetry in 3-dimensional gauge theories by the addition of a Chern-Simons term \[6\].

2 The supermembrane

It is well known that string theory arose from attempts to understand the physics of hadrons. What is less well-known is that M-theory has roots in hadron physics too. In 1978, the same year that 11-dimensional supergravity appeared \[7\], a ‘classical’ bag model for hadrons was proposed by Aurilia, Christodoulou and Legovini \[8\]; this was based on the idea that that the closed QCD 4-form \( \text{Tr}(F \wedge F) \) should be replaced, in an effective description of hadrons, by an abelian 4-form field strength \( G = dC \).

\[1\] In the talk I explained how a tubular but axially asymmetric supermembrane, supported against collapse by angular momentum, can be both stable and supersymmetry-preserving \[1\]. This did surprise some members of the audience, although I discovered that the stability issue had been previously adressed in a non-supersymmetric context \[2\]. As a full discussion is available in \[1\] and subsequent papers \[3\], I have chosen to discuss another surprising fact about supermembranes in this write-up.
Hadrons were identified as those regions in which $G$ acquires a non-zero expectation value; these regions would be separated from the vacuum by a membrane coupled to the 3-form potential $C$. I heard about this model from Antonio Aurilia in 1980 and realized that the 4-form field strength $G$ of 11-dimensional supergravity could be similarly used, after reduction on $T^7$, to introduce a positive cosmological constant into $N=8$ $D=4$ supergravity. This supergravity theory had recently been constructed by Cremmer and Julia \cite{9} but they had eliminated the surviving four-dimensional 4-form field strength as if it were a non-dynamical auxiliary field. If one instead uses the field equation of the 3-form potential $C$ then a positive cosmological constant appears as the square of an integration constant $\Sigma$. We enlisted Hermann Nicolai to help construct the new $N=8$ supergravity theory, which turns out to have a positive scalar potential rather than a cosmological constant $\Sigma$. As this potential has no critical points it was unclear what use it might have. We should have continued this research by considering whether a non-vanishing 4-form in $D=4$ could be combined with compactifications on spaces other than $T^7$. We didn’t, but Freund and Rubin did \cite{13} and their demonstration that $D=11$ supergravity could be compactified on a 7-sphere sparked off the revival of interest in Kaluza-Klein theory.

It was somehow forgotten, in all the Kaluza-Klein excitement, that the 3-form potential $C$ could couple to membranes (although Bernard Julia was aware of the possibility \cite{14}). I think that the main reason for this collective amnesia was the fact that 11-dimensional supergravity was being promoted as a candidate unified field theory, so the apparent absence of anything to which it could couple was viewed as an advantage. This attitude discouraged thinking about membranes, which didn’t resurface until after the superstring revolution of 1984. Following the construction by Green and Schwarz of a covariant superstring action \cite{15}, it was natural to reconsider the possibility of an 11-dimensional supermembrane. During the summer of 1986, Luca Mezincescu and I attempted to construct a supermembrane generalization of the Green-Schwarz (GS) action but the attempt did not succeed because we were unable to generalize the self-dual worldsheet vector parameter of the GS ‘$\kappa$-symmetry’; this made the two-dimensionality of the string worldsheet seem an essential feature of the construction. In fact, it is not; there is an alternative, but equivalent, form of the $\kappa$-symmetry transformation with a worldsheet scalar parameter. This was found by Hughes, Liu and Polchinski in their construction of an action for a super-3-brane in a $D=6$ Minkowski background \cite{16}; they were motivated by the observation that a vortex of the $D=6$ supersymmetric abelian-Higgs model is a supersymmetry-preserving 3-brane for which the effective action must be of GS-type. I saw this paper the day before I was to travel to Trieste to continue a collaboration with Eric Bergshoeff and Ergin Sezgin, and soon after my arrival we succeeded in constructing an 11-dimensional supermembrane action that is consistent in any background that

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\footnote{This idea occurred independently to Duff and Van Nieuwenhuizen \cite{10} but without the connection to 11-dimensional supergravity.}

\footnote{Stephen Hawking used the idea of a dynamical cosmological constant in his suggestion that the ‘cosmological constant is probably zero’ \cite{12} but it now seems that it probably isn’t zero.}
solves the field equations of 11-dimensional supergravity [17]. For unit tension the action takes the form

$$S = - \int [\text{Vol} \pm \mathcal{C}]$$

(1)

where Vol is the (appropriately defined) induced volume 3-form, and $\mathcal{C}$ is the world-volume 3-form induced by the superspace 3-form potential of 11-dimensional supergravity (of which $\mathcal{C}$ is the bosonic truncation). The choice of relative sign corresponds to the choice between a supermembrane and an anti-supermembrane, or an $M2$-brane and an $\overline{M2}$-brane in modern terminology.

A feature of all GS-type super-brane actions is that the fermions are (apparently) worldvolume scalars. If this were really true then, for example, the GS superstring action could not be equivalent to the worldsheet supersymmetric NSR superstring action. In fact, the GS fermions are not scalars because they are subject to the $\kappa$-symmetry gauge transformation; it is for a similar reason that the 4-vector potential of electrodynamics is not really a 4-vector field. To determine the transformation properties of the GS fermions under any symmetry of the action (which would include spacetime Lorentz transformations for a Minkowski background) one must first fix the $\kappa$-symmetry gauge; the transformation is then a superposition of the ‘naive’ transformation with whatever compensating $\kappa$-symmetry transformation is needed to maintain the gauge choice. The gauge fixing must break the spacetime Lorentz group but can be chosen to preserve the worldvolume Lorentz subgroup, under which the gauge-fixed GS fermions turn out to transform as worldvolume spinors.

This transformation from spacetime spinor to worldvolume spinor is clearly necessary if any spacetime supersymmetries are to be interpreted as worldvolume supersymmetries after gauge-fixing, but it is not obviously sufficient. In fact, initially it was far from clear that spacetime supersymmetry would imply worldvolume supersymmetry of the supermembrane, partly because the supermembrane has no NSR formulation, and Achúcarro, Gauntlett, Itoh and I went to great lengths to verify it directly [18]; our article was originally entitled Supersymmetry on the brane but we had to change the title to accommodate a referee who insisted that use of the word ‘brane’ would bring the physics community into disrepute.

Nowadays, the connection between spacetime supersymmetry and worldvolume supersymmetry is considered obvious. However, as I hope the following discussion will show, surprises are still possible.

### 3 Backgrounds of reduced holonomy

The $G \wedge G \wedge \mathcal{C}$ term of 11-dimensional supergravity preserves spacetime parity if $\mathcal{C}$ is taken to be parity-odd, and with this parity assignment the coupling of $G$ to

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4Possibly this referee had in mind the earlier use of the word in the 1954 essay *Akquire culture and keep the brane clean* by Nigel Molesworth [19]. As this essay’s subtitle is *How to be Topp in Latin* it is regrettable that it fails to provide the modern translation of *mens sana in corpore sano* which is *clean brane in clean bulk*, otherwise known as the braneworld cosmological principle.
fermion bilinears also preserves parity because of the peculiar way that fermion bilinears behave under parity in odd dimensions. Thus 11-dimensional supergravity preserves parity. It follows that solutions breaking parity must come in parity doublets, each of which will preserve the same fraction of supersymmetry because parity commutes with supersymmetry. We shall be interested in solutions with vanishing $G$, and product 11-metric of the form

$$ds_{11}^2 = ds^2(\mathbb{E}^{(1,2)}) + g_{IJ}(X)dX^IdX^J$$

where $g_{IJ}$ ($I, J = 1, \ldots, 8$) is the metric of some Ricci-flat 8-dimensional manifold $\mathcal{M}_8$, or its orientation reversal $\overline{\mathcal{M}}_8$. Any submanifold with fixed position on $\mathcal{M}_8$ is a minimal surface that we may identify as the Minkowski vacuum of an infinite planar supermembrane. In the gauge in which the worldvolume coordinates $\xi^i$ are identified with coordinates for $\mathbb{E}^{(1,2)}$, the physical bosonic worldvolume fields of the $M^2$-brane are maps $X^I(\xi)$ from the worldvolume to $\mathcal{M}_8$, and the bosonic action is

$$I = -\int d^3\xi \sqrt{-\det(\eta_{ij} + g_{ij})}$$

where $\eta$ is the 2+1 Minkowski metric and

$$g_{ij}(\xi) = \partial_i X^I \partial_j X^J g_{IJ}(X).$$

By convention, we shall take the physical fermion fields of the $M^2$-brane to be in the $(2, 8_s)$ representation and those of the $\overline{M^2}$-brane to be in the $(2, 8_c)$ representation. The spacetime parity transformation that interchanges $\mathcal{M}_8$ with $\overline{\mathcal{M}}_8$ also interchange $M^2$ with $\overline{M^2}$. Thus, an $M^2$-brane in $\mathbb{E}^{(1,2)} \times \mathcal{M}_8$ is equivalent to an $\overline{M^2}$-brane in $\mathbb{E}^{(1,2)} \times \overline{\mathcal{M}}_8$. In the case that $\mathcal{M}_8$ has an orientation reversing isometry we have $\mathcal{M}_8 \cong \overline{\mathcal{M}}_8$, and the $M^2$-brane action will be equivalent to the $\overline{M^2}$-brane action.

Note that the field content of the gauge-fixed supermembrane is bose-fermi balanced, as would be required for worldvolume supersymmetry. Whether the supermembrane is worldvolume supersymmetric will depend on the choice of $\mathcal{M}_8$. This follows from the fact that (super)symmetries of the supermembrane action arise from (super)isometries of the background that leave invariant the superspace 4-form field strength. In particular, for bosonic backgrounds of the type under consideration, supersymmetries arise from Killing superfields whose spinor component is a Killing spinor of $M$, and these exist only if $\mathcal{M}_8$ has special holonomy.
Let $H \subset SO(8)$ be the holonomy group. The number $N$ of linearly-realized supersymmetries of the supermembrane is the number of singlets in the decomposition of the spinor representation $8_s$ of $SO(8)$ into irreps of $H$. The number $N'$ of non-linearly realized supersymmetries is the number of singlets of the $8_c$ representation of $SO(8)$ in its decomposition into irreps of $H$. For the anti-supermembrane the numbers $N$ and $N'$ are interchanged. The groups $H$ for which $N > 0$ fall into one of two nested sequences. One sequence is

$$G_2 \supset SU(3) \supset SU(2).$$

The corresponding types of 8-manifold, and the values of $N$ and $N'$ are given in Table 1 (where $CY_n$ is a 2n-dimensional Calabi-Yau $n$-fold and $HK_{4n}$ is a hyper-Kähler manifold of quaternionic dimension $n$).

| $H$       | $M_8$     | $N$ | $N'$ |
|-----------|-----------|-----|-----|
| $G_2$     | $M_7 \times E^4$ | 1   | 1   |
| $SU(3)$   | $CY_3 \times E^2$ | 2   | 2   |
| $SU(2)$   | $HK_4 \times E^4$ | 4   | 4   |

In each of these cases the 8-manifold $M_8$ takes the form $M_8 = M_{8-k} \times \mathbb{R}^k$ ($k = 1, 2, 4$) for some irreducible (8-k)-dimensional manifold $M_{8-k}$. Such 8-manifolds have an orientation-reversing isometry, so the M2-brane in these backgrounds is equivalent to the M2-brane. In fact, they are identical because an anti-membrane can be obtained from a membrane by a rotation in some $E^3$ subspace of $E^{2+k}$. The reason that the M2 and $\overline{M2}$ actions can be identical is that their $\kappa$-symmetry transformations differ and this difference can compensate for the different sign in (1).

Note that fixing the position in $M_{8-k}$ yields a supermembrane in a Minkowski spacetime of dimension $D = 4, 5$ or $7$, according to whether $k = 1, 2$ or $4$, respectively; as it happens, these are precisely the other dimensions for which the supermembrane action is classically consistent \cite{17}, so the existence of these lower-dimensional supermembrane actions is explained by the existence of the 11-dimensional supermembrane.

The other sequence of holonomy groups is

$$Spin(7) \supset SU(4) \supset Sp_2 \supset Sp_1 \times Sp_1.$$

The corresponding types of 8-manifold, and the values of $N$ and $N'$ are given in Table 2. In each of these cases there are no non-linearly-realized supersymmetries, so replacing the M2-brane by the $\overline{M2}$-brane breaks all supersymmetries. Equivalently, replacing the 8-manifold $M_8$ by its orientation reversal $\overline{M_8}$ breaks all $N$ supersymmetries of the $M2$-brane action$^5$.

$^5$Note that the background solutions preserve $N$ supersymmetries irrespective of the orientation.
Table 2:

| H       | $\mathcal{M}_8$ | $N$ | $N'$ |
|---------|-----------------|-----|------|
| $Spin(7)$ | $Spin(7)$       | 1   | 0    |
| SU(4)   | $CY_4$          | 2   | 0    |
| $Sp_2$  | $HK_4$          | 3   | 0    |
| $Sp_1 \times Sp_1$ | $HK_4 \times HK_4$ | 4   | 0    |

4 The sigma model limit

The action (3) can be expanded as a power series in $\partial X$. Discarding a constant and terms with more than two derivatives we arrive at the field theory action

$$S = -\frac{1}{2} \int d^3 \xi \sqrt{-\det \eta_{ij} \partial_i X^I \partial_j X^J g_{IJ}(X)}.$$ (7)

This is a D=3 sigma-model with the 8-manifold $\mathcal{M}_8$ as its target space. If the supermembrane preserved $N$ supersymmetries then an analogous expansion yields a supersymmetric sigma-model with at least $N$ supersymmetries. In most cases one can easily see that it can have no more than $N$ supersymmetries because of the constraints imposed on the target space of a sigma-model by extended supersymmetry; specifically, a supersymmetric D=3 sigma model with an irreducible target space has $N = 2$ supersymmetry if the target space is Kähler and $N = 4$ if it is hyper-Kähler [22]. For example, because $Spin(7)$ manifolds are not Kähler we know that the sigma-model obtained from the supermembrane action can have at most $N = 1$ supersymmetry. The same is true for the $G_2$ case, although the conclusion is less immediate in this case because the 8-manifold is not irreducible. Note that for the $N = 2$ case of either Table the target space is Kähler, as consistency requires, but not hyper-Kähler, so the sigma-model has $N = 2$ supersymmetry. Similarly, for both $N = 4$ cases the target space is hyper-Kähler, as required for consistency. This leaves only the case of $Sp_2$ holonomy of Table 2 to consider, and here we find a surprise. As one sees from Table 2, the gauge-fixed supermembrane action has only $N = 3$ supersymmetry but, as its target space is hyper-Kähler, the sigma model obtained from the field theory limit has $N = 4$ supersymmetry. Thus, in this one case, the low-energy sigma-model has more supersymmetries than the supermembrane action from which it was derived!

From the sigma-model perspective, the supermembrane just adds higher-dimension terms to the action. An interaction term that breaks $N = 4$ supersymmetry to $N = 3$ must also break worldvolume parity. Majorana mass terms break parity in three dimensions [23] and although the supermembrane has no mass terms it does have a mass parameter, determined by the membrane tension. Higher dimension fermion of $\mathcal{M}_8$; it is the only the rigid worldvolume supersymmetries on the supermembrane that are broken when $\mathcal{M}_8$ is replaced by $\overline{\mathcal{M}_8}$. This is in contrast to the related phenomenon of supergravity solutions with non-zero $G$ that are supersymmetric for one orientation but non-supersymmetric for the other orientation [21].
interactions in the supermembrane must involve this parameter and so may break parity. For example, if \( \psi \) is the 8-plet of real two-component \( Sl(2;\mathbb{R}) \) spinor fields then a term of the form
\[
(\bar{\psi}\psi)(\bar{\psi}\gamma \cdot \partial \psi)
\]
breaks parity for the same reason that Majorana mass terms break parity. The supersymmetric completion of this term will not include any purely bosonic term, consistent with parity preservation of the bosonic truncation of the supermembrane, and it will not survive in the field theory limit, consistent with parity preservation of the supersymmetric sigma model. Moreover, it can occur only when there are no non-linearly realized supersymmetries, and hence must be absent in the cases of Table 1. Thus, a term of the above type is a candidate for a parity-violating interaction that will break \( N = 4 \) to \( N = 3 \) supersymmetry when the hyper-Kähler target space has \( Sp_2 \) holonomy, although it must be absent if the holonomy is contained in the \( Sp_1 \times Sp_1 \) subgroup of \( Sp_2 \).

There is a gauge theory precedent for all this. The addition of a Chern-Simons to a 3-dimensional gauge theory with \( N = 4 \) supersymmetry can preserve at most \( N = 3 \) supersymmetry, in which case its supersymmetric completion will include parity-violating fermion mass terms \[3\]. In fact, this phenomenon is an M-theory dual of the one discussed here, at least for the class of toric hyper-Kähler 8-manifolds, because the M2-brane in such a background is dual to a D3-brane suspended between \((p,q)\)-fivebranes and the effective field theory on the intersection is precisely a 3-dimensional \( N = 3 \) gauge theory with a Chern-Simons term \[3, 24\].

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