EXACTNESS FROM PROPER ACTIONS

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Abstract. In this paper we show that if a discrete group $G$ acts properly isometrically on a discrete space $X$ for which the uniform Roe algebra $C^*_u(X)$ is exact then $G$ is an exact group. As a corollary, we note that if the action is cocompact then the following are equivalent: The space $X$ has Yu’s property $A$; $C^*_u(X)$ is exact; $C^*_u(X)$ is nuclear.

Introduction

It is a remarkable result of Yu that any discrete metric space with bounded geometry that satisfies a Følner-type condition, which he called property $A$, also satisfies the coarse Baum-Connes conjecture [7]. The case of a countable discrete group, regarded as a coarse metric space of bounded geometry, was studied by Higson and Roe [2], Guentner and Kaminker [1], and Ozawa [3]. They showed that Yu’s property $A$ is equivalent to exactness of its reduced $C^*$-algebra $C^*_r(G)$ and to nuclearity of its uniform Roe algebra $C^*_u(G)$. Furthermore, Roe showed for a discrete bounded geometry metric space $X$ that if $X$ has property $A$ then $C^*_u(X)$ is nuclear ([4], Proposition 11.41). It is tempting to conjecture therefore that for a discrete metric space of bounded geometry, nuclearity of $C^*_u(X)$ is in fact equivalent to property $A$. This conjecture holds if $X$ admits a free cocompact action by a countable group $G$: since the action is free we may identify $C^*_u(G)$ with a subalgebra of $C^*_u(X)$. If we assume that $C^*_u(X)$ is nuclear, then $C^*_u(G)$ will be exact and $G$ will have property $A$. The fact that $G$ acts cocompactly on $X$ implies that $X$ and $G$ are coarsely equivalent so $X$ also has property $A$.

The conjecture is more delicate than it may appear since while property $A$ is a coarse invariant, the uniform Roe algebra is not. For example, for a finite space $X$ with $n$ points, $C^*_u(X)$ is the algebra of $n \times n$ matrices; however all finite spaces are coarsely equivalent to a point.

In this paper we address the case of the conjecture when the space admits a proper, cocompact isometric action by a countable group $G$. Generalising a theorem in coarse geometry from the case of a free action to a proper action usually requires only a minor adjustment of the argument. This is not the case in our context. The fact that the uniform Roe algebra is not a coarse invariant is manifested in the observation that when the action is not free we no longer have the required embedding of $C^*_u(G)$ into $C^*_u(X)$. The key idea of this paper is to replace the proper action on $X$ by a partially defined free action on an orbit of the original action. In outline we proceed as follows.

Given an orbit $Y$ of the action of $G$ on $X$ and a splitting $\phi$ of the natural surjection $G \to Y$ we identify the algebra $C^*_u(Y)$ with the subalgebra $C^*_u(\phi(Y))$ of $C^*_u(G)$. We construct a free partial action of $G$ on the orbit $Y$ defined in terms of

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the right action of $G$ on itself and the identification of $\phi(Y)$ with $Y$. Finally we may appeal to coarse equivalences to establish the following.

Theorem 1. Let $X$ be a uniformly discrete metric space, and let $G$ be a countable group acting properly by isometries on $X$. If $C^*_u(X)$ is an exact $C^*$-algebra, then $G$ is an exact group.

From this we deduce the following equivalence.

Corollary 2. If $X$ is a uniformly discrete metric space admitting a proper, cocompact isometric action by a countable group $G$ then the following are equivalent.

1. $X$ has property $A$;
2. $C^*_u(X)$ is nuclear;
3. $C^*_u(X)$ is exact; $G$ is exact.

Proof. of Corollary 2
The implication $1 \Rightarrow 2$ was established by Roe in [3, Prop. 11.41]. That 2 implies 3 is a well known result, see, e.g. [6]. Theorem 3 provides the implication $3 \Rightarrow 4$. Finally, as the action of $G$ on $X$ is cocompact, $X$ and $G$ are coarsely equivalent, which gives the implication $4 \Rightarrow 1$. □

BACKGROUND
Throughout the paper we will assume that $G$ is a countable group equipped with the unique (up to coarse equivalence) proper left invariant metric $d_G$. For $R \geq 0$ we will use the notation $B_R(g)$ to denote the closed $R$-ball about $g$ in $G$.

Definition 3. A uniformly discrete metric space $(X, d_X)$ has property $A$ if for all $R, \varepsilon > 0$ there exists a family of finite non-empty subsets $A_x$ of $X \times \mathbb{N}$, indexed by $x$ in $X$, such that

1. for all $x, y$ with $d_X(x, y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$;
2. there exists $S$ such that for all $x$ and $(y, n) \in A_x$ we have $d_X(x, y) \leq S$.

Definition 4. A kernel $u : X \times X \to \mathbb{R}$ has $(R, \varepsilon)$-variation if $d_X(x, y) \leq R$ implies $|u(x, y) - 1| < \varepsilon$.

Theorem 5 ([5], Proposition 3.2). If $X$ is a bounded geometry discrete metric space, then $X$ has property $A$ if and only if for all $R, \varepsilon > 0$ there exists a positive definite kernel $u$ with $R, \varepsilon$ variation, and such that there exists $S$ for which $d_X(x, y) > S$ implies $u(x, y) = 0$.

Proposition 6. Let $G$ be a countable group acting properly isometrically on a proper metric space $X$. Then for any basepoint $x_0$ in $X$, the map $\psi : g \mapsto gx_0$ is a uniform embedding, and any map $\phi : Gx_0 \to G$ such that $\psi \circ \phi = \text{id}_{Gx_0}$ is a coarse inverse to $\psi$. If moreover $G$ acts cocompactly then $\psi$ is a coarse equivalence between $G$ and $X$.

Proof. First we verify that $\psi$ is a coarse map. For $g, h$ in $G$, we have $d_X(gx_0, hx_0) = d_X(x_0, g^{-1}hx_0)$ as the action is isometric. Given $R$, if $d_G(g, h) \leq R$ then $g^{-1}h$ lies in $B_R(e)$ which, by properness of the metric $d_G$ is finite, so $d_X(x_0, g^{-1}hx_0)$ is bounded by some number $S$. Thus for all $R$ there exists $S$ such that $d_G(g, h) \leq R$ implies $d_X(\psi(g), \psi(h)) \leq S$. Properness of $\psi$ follows from properness of the action, so $\psi$ is a coarse map.
Now let $\phi: Gx_0 \to G$ be a splitting of $\psi$. Then for $x, y$ in $Gx_0$ we have $x = \phi(x)x_0, y = \phi(y)x_0$. If $d_X(x, y) \leq R$ then $d_X(x_0, \phi(x)^{-1}\phi(y)x_0) \leq R$. Properness of the action ensures that there are only finitely many elements $g$ of $G$ with $d_X(x_0, gx_0) \leq R$, so $\phi(x)^{-1}\phi(y)$ lies in $B_S(e)$, for some $S$, i.e. $d_G(\phi(x), \phi(y)) \leq S$. The map $\phi$ is injective, thus it is proper, so $\phi$ is also a coarse map.

The composition $\psi \circ \phi$ is the identity on $Gx_0$ by definition, while $g^{-1}(\phi \circ \psi(g))$ is an element of $G$ fixing $x_0$. The stabiliser of $x_0$ is finite, so $d_G(g, \phi \circ \psi(g))$ is bounded as a function of $g$, i.e. $\phi \circ \psi$ is close to the identity on $G$. Thus $\psi$ is a uniform embedding from $G$ to $X$, and $\phi$ implement a coarse equivalence between $G$ and its image.

If the action is cocompact then for some $R$, the $G$ translations of the $R$-ball about $x_0$ cover $X$, so $Gx_0$ is $R$-dense in $X$. It follows immediately that the inclusion of $Gx_0$ into $X$ is a coarse equivalence, so $\psi: G \to X$ is a coarse equivalence. \hfill $\square$

**Definition 7.** A kernel $k: X \times X \to \mathbb{C}$ has finite propagation if there exists $R \geq 0$ such that $k(x, y) = 0$ for $d(x, y) > R$. The propagation of $k$ is the smallest such $R$.

If $X$ is a proper discrete metric space, and $k$ is a finite propagation kernel then for each $x$ there are only finitely many $y$ with $k(x, y) \neq 0$. Thus $k$ defines a linear map from $l^2(X)$ to itself, $k * v(x) = \sum_{y \in X} k(x, y)v(y)$. These linear maps are also said to have finite propagation. Note that if additionally $X$ has bounded geometry, then every bounded finite propagation kernel gives rise to a bounded operator on $l^2(X)$.

**Definition 8.** The uniform Roe algebra, $C^*_u(X)$, is the $C^*$-algebra completion of the algebra of bounded operators on $l^2(X)$ having finite propagation.

**Definition 9.** A partial translation of $X$ is a subset $t$ of $X \times X$ such that the coordinate projections of $t$ onto $X$ are injective, and such that $d_X(x, y)$ is bounded for $(x, y) \in t$.

A partial translation can be viewed as a partially defined map from $X$ to $X$ which is close to the identity (where defined). Therefore a partial translation gives rise to a partial isometry of $l^2(X)$ which has finite propagation and hence is an element of $C^*_u(X)$. By definition, the partial isometry is defined by $t(\delta_y) = \delta_x$ if $t(y) = x$, and $t(\delta_y) = 0$ if $t(y)$ is undefined.

**Proof of Theorem 1**

Fix a basepoint $x_0$ in $X$, and let $Y$ be the orbit of $x_0$ under the action of $G$. Given a finite propagation operator on $l^2(Y)$ we can extend it to $l^2(X) = l^2(Y) \oplus l^2(X \setminus Y)$ by defining it to be zero on $l^2(X \setminus Y)$. Thus the algebra $C^*_u(Y)$ is a subalgebra of $C^*_u(X)$, hence exactness for $C^*_u(X)$ implies exactness for $C^*_u(Y)$.

For each $y \in Y$, pick an element $\phi(y)$ of $G$ such that $\phi(y)x_0 = y$. We construct a partially defined action of the group $G$ on the space $Y$ as follows. For $g \in G, y \in Y$ we define $g \circ y = x$ if and only if $\phi(y)g^{-1} = \phi(x)$. The element $g \circ y$ in $Y$ is uniquely determined if it exists, since then $g \circ y = \phi(y)g^{-1}x_0$. However it will be undefined if $\phi(y)g^{-1}$ is not in the image of $\phi$. Note that $g \circ$, viewed as a partially defined map from $X$ to $X$ is a partial translation, since

$$d_G(\phi(y), \phi(g \circ y)) = d_G(\phi(y), \phi(y)g^{-1}) = d_G(e, g^{-1})$$
which is independent of $y$, and $\phi$ is a coarse equivalence by Proposition\(\text{[1]}\). Note that
in the case where $G$ acts freely, the map $\phi$ is uniquely determined and is a bijection
between $Y$ and $G$. Using this to identify $Y$ with $G$, the action $g \circ \phi$ is identified with
the action of $G$ on itself by right multiplication by $g^{-1}$.

Given $R, \varepsilon$ we will produce a positive kernel $u$ on $Y$ with $(R, \varepsilon)$-variation, and
satisfying the hypothesis that $u(x, y)$ vanishes for $x, y$ far apart. Let $E_R$ be the set
of elements of $G$ of the form $\phi(x)^{-1}\phi(y)$, $x, y$ in $Y$ with $d_X(x, y) \leq R$. As $\phi$ is
a coarse map, there exists $S$ such that if $d_X(x, y) \leq R$ then $d_G(\phi(x), \phi(y)) \leq S$, so
$E_R$ is contained in the ball $B_S(e)$ in $G$, in particular it is finite. Elements of $G$ act
as partial translations on $Y$ via $\circ$, hence we can identify $E_R$ with a finite subset of
$C_u^*(Y)$.

Using the characterisation of exactness (\text{[8]} Lemma 2), there exists a completely
positive finite rank map $\theta: C_u^*(Y) \to B(l^2(Y))$ such that:

1. $\theta$ has the form $\theta(.) = \sum_{i=1}^d (\delta_{a_i}, \delta_{b_i})T_i$, for some $a_i, b_i$ in $Y$, and $T_i$ in
   $B(l^2(Y))$, and
2. for all $g$ in $E_R$ we have $\|\theta(g) - g\| < \varepsilon$.

Let $F_R$ denote the set $\{a_i, b_i : i = 1, \ldots, d\}$, and note that for any partial translation
t such that the image of $F_R$ under $t$ does not meet $F_R$, we have $\theta(t) = 0$. We now define

$$u(x, y) = \langle \delta_x, \theta(\phi(x)^{-1}\phi(y))\delta_y \rangle$$

for $x, y$ in $Y$, where the elements $\phi(x)^{-1}\phi(y)$ of $G$ are regarded as elements of $C_u^*(Y)$ as above.

First we will verify positivity of $u$: this is not immediate because as an operator on
$l^2(X)$, $\phi(x)^{-1}\phi(y)$ is not necessarily the composition of the operators corresponding
to $\phi(x)^{-1}$ and $\phi(y)$. This is because $(\phi(x)^{-1}\phi(y)) \circ y'$ may be defined even when
$\phi(x)^{-1} \circ (\phi(y) \circ y')$ is not.

For each $y$ in $Y$, we define a bounded linear map $s_y$ from $l^2(Y)$ to $l^2(G)$ as follows. Let
$s_y(\delta_{y'}) = \delta_y$ where $g = \phi(y')\phi(y)^{-1}$ in $G$. Then its adjoint $s_y^*$ satisfies
$s_y^*(\delta_y) = \delta_y'$ if there exists $x'$ with $\phi(x')\phi(x)^{-1} = g$, (such an $x'$ must be unique by
injectivity of $\phi$) and is zero otherwise. Thus for $x, y, y'$ in $Y$, the vector $s_y^*s_y(\delta_{y'})$
is $\delta_{y'}$ for some $x' \in Y$ if we have

$$\phi(y')\phi(y)^{-1} = \phi(x')\phi(x)^{-1},$$

and it is zero otherwise. Note that we can rewrite this as $\phi(y')(\phi(x)^{-1}\phi(y))^{-1} =
\phi(x')$ i.e. $x' = (\phi(x)^{-1}\phi(y)) \circ y'$. We conclude that

$$s_y^*s_y(\delta_{y'}) = \delta_{x'} = (\phi(x)^{-1}\phi(y))(\delta_{y'}) \quad \text{if } x' = (\phi(x)^{-1}\phi(y)) \circ y',$$

$$s_y^*s_y(\delta_{y'}) = 0 = (\phi(x)^{-1}\phi(y))(\delta_{y'}) \quad \text{if } (\phi(x)^{-1}\phi(y)) \circ y' \text{ is undefined.}$$

Hence $s_y^*s_y$ is $\phi(x)^{-1}\phi(y)$ as an operator on $l^2(Y)$. The operators $(\phi(x)^{-1}\phi(y)) =
(s_y^*s_y)$ therefore form a positive matrix over $Y$, so positivity of $u$ follows from
complete positivity of $\theta$.

We will now show that $u$ has $(R, \varepsilon)$-variation. For $x, y$ with $d_X(x, y) \leq R$ we
have $\phi(x)^{-1}\phi(y)$ in $E_R$, hence it follows from (\text{[2]} above that

$$\|\phi(x)^{-1}\phi(y) - \theta(\phi(x)^{-1}\phi(y))\| < \varepsilon.$$
image of \( \phi \), so \((\phi(x)^{-1}\phi(y)) \circ y \) is defined and equals \( x \). Thus \( \phi(x)^{-1}\phi(y)(\delta_0) = \delta_x \), so \( \langle \delta_x, \phi(x)^{-1}\phi(y)(\delta_y) \rangle = 1 \). Hence
\[
|1 - u(x, y)| = |\langle \delta_x, (\phi(x)^{-1}\phi(y) - \theta(\phi(x)^{-1}\phi(y))) (\delta_y) \rangle| < \varepsilon.
\]
We conclude that \( u \) has \((R, \varepsilon)\) variation as required.

Finally we will show that \( u(x, y) \) vanishes for \( d_X(x, y) \) sufficiently large. As \( \phi \) is a uniform embedding, for all \( R' \) there exists \( S' \) such that if \( d_G(\phi(x), \phi(y)) \leq R' \) then \( d_X(x, y) \leq S' \). As \( F_R \) is a finite subset of \( Y \) we note that \( \{ \phi(x')^{-1}\phi(y') : x', y' \in F_R \} \) is a finite subset of \( G \), and choose \( R' \) such that this is contained in the ball \( B_R(e) \) in \( G \). Now if \( x, y \in Y \) and \( x', y' \in F_R \) with \( (\phi(x)^{-1}\phi(y)) \circ y' = x' \) then \( \phi(y')\phi(x)^{-1}\phi(y))^{-1} = \phi(x') \) so \( \phi(x')^{-1}\phi(y') = \phi(x)^{-1}\phi(y) \). Hence
\[
d_G(e, \phi(x)^{-1}\phi(y)) = d_G(e, \phi(x')^{-1}\phi(y')) \leq R',
\]
i.e. \( d_G(\phi(x), \phi(y)) \leq R' \), so \( d_X(x, y) \leq S' \). Thus if \( d_X(x, y) > S' \) then for \( y' \) in \( F_R \), \( \phi(y') \circ y' \) cannot be an element of \( F_R \). It follows that if \( d_X(x, y) > S' \) then \( \langle \delta_x, (\phi(x)^{-1}\phi(y))(\delta_y) \rangle \) vanishes. Hence \( u(x, y) \) also vanishes as required.

We have shown that for all \( R, \varepsilon \) there exists a positive kernel on \( Y \) with \((R, \varepsilon)\) variation, and which vanishes for \( x, y \) far apart. It follows by Theorem 3 that \( Y \) has property A. As \( G \) is coarsely equivalent to \( Y \), it too has property A, so by Ozawa’s result \( G \) is exact.

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