Operator-Norm Resolvent Asymptotic Analysis of Continuous Media with High-Contrast Inclusions

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Abstract—Using a generalization of the classical notion of Weyl m-function and related formulas for the resolvents of boundary-value problems, we analyze the asymptotic behavior of solutions to a “transmission problem” for a high-contrast inclusion in a continuous medium, for which we prove the operator-norm resolvent convergence to a limit problem of “electrostatic” type. In particular, our results imply the convergence of the spectra of high-contrast problems to the spectrum of the limit operator, with order-sharp convergence estimates. The approach developed in the paper is of a general nature and can thus be successfully applied in the study of other problems of the same type.

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1. INTRODUCTION

Parameter-dependent problems for differential equations have traditionally attracted much interest within applied mathematics, by virtue of their potential for replacing complicated formulations by more straightforward, and often explicitly solvable, ones. This drive has led to a plethora of asymptotic techniques, from perturbation theory to multi-scale analysis, covering a variety of applications to physics, engineering, and materials science. It would be an insurmountable task to give a comprehensive review of the related literature. Notwithstanding the classical status of this subject area, problems that require new ideas continue emerging, often motivated by novel wave phenomena. One of the recent application areas of this kind is provided by composites and structures involving components with highly contrasting material properties (stiffness, density, refractive index). Mathematically, such problems lead to boundary-value formulations for classical operators (such as the Laplace operator), but with parameter-dependent coefficients. For example, problems of this kind have arisen in the study of periodic composite media with high contrast (or “large coupling”) between the material properties of the components, see [1]–[3].

In the present work, we consider a prototype large-coupling transmission problem, posed on a bounded domain \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), see Fig. 1, containing a “low-index” (equivalently, “high propagation speed”) inclusion \( \Omega_- \), located at a positive distance from the boundary \( \partial \Omega \). Mathematically, this is
modelled by a “weighted” Laplacian $-a_\pm \Delta$, where $a_+ = 1$ (the weight on the domain $\Omega_+ := \Omega \setminus \Omega_-)$, and $a_- \equiv a$ (the weight on the domain $\Omega_-$) is assumed to be large, $a_- \gg 1$. This is supplemented by the Neumann boundary condition $\partial u / \partial n = 0$ on the outer boundary $\partial \Omega$, where $n$ is the exterior normal to $\partial \Omega$, and “natural” continuity conditions on the “interface” $\Gamma := \partial \Omega_-$. For each $a$, we consider time-harmonic vibrations of the physical domain represented by $\Omega$, described by the eigenvalue problem for an appropriate operator in $L^2(\Omega)$.

A formal asymptotic argument using expansions in powers of $a^{-1}$ suggests that convergent eigenfunction sequences for the above eigenvalue problems should converge (as $a \to \infty$) to either a constant or a function of the form

$$ v - \frac{1}{|\Omega|} \int_{\Omega_+} v, $$

where $v$ satisfies the spectral boundary-value problem (BVP)

$$ -\Delta v = z \left( v - \frac{1}{|\Omega|} \int_{\Omega_+} v \right) \quad \text{in} \quad \Omega_+, \quad v|_{\Gamma} = 0, \quad \frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0. \quad (1.1) $$

Here the spectral parameter $z$ represents the ratio of the size of the original physical domain to the wavelength in its part represented by $\Omega_+$.

The problem (1.1) is related to the so-called “electrostatic problem” discussed in [4, Lemma 3.4], see also [5] and references therein, namely the eigenvalue problem for the self-adjoint operator $Q$ defined by the quadratic form

$$ q(u, u) = \int_{\Omega_+} \nabla v \cdot \nabla v, \quad u = v + c, \quad v \in H^1_{0, \Gamma} := \{ v \in H^1(\Omega_+), v|_{\Gamma} = 0 \}, \quad c \in \mathbb{C} \quad (1.2) $$

on the Hilbert space $L^2(\Omega_+) \oplus \mathbb{C}$, treated as a subspace of $L^2(\Omega)$.

Indeed (see [4] for details), it is easily seen that the eigenvalue problem $Qu = zu$ is solvable either when $z = 0$, in which case $u = c$, or for $z > 0$ such that the problem (1.1) admits a non-trivial solution. Thus, the formal asymptotic argument suggests that the limiting spectrum is precisely that of the electrostatic problem.

Denote by $A^+_0$ the Laplacian $-\Delta$ on $\Omega_+$, subject to the Dirichlet condition on $\Gamma$ and the Neumann boundary condition on $\partial \Omega$, and write the function $v$ in (1.1) in the form of an eigenfunction series

$$ v = \sum_{j=1}^{\infty} d_j \phi^+_j, $$

where $\lambda^+_j, \phi^+_j, j = 1, 2, \ldots$, are the eigenvalues and the corresponding orthonormal eigenfunctions, respectively, of $A^+_0$. Noticing that the function $\mathbb{1}_+(x) = 1, x \in \Omega_+$, can be written as

$$ \mathbb{1}_+ = \sum_{j=1}^{\infty} \left( \int_{\Omega_+} \phi^+_j \right) \phi^+_j, $$

Fig. 1. Domain with a “stiff” inclusion.
we obtain
\[
\sum_{j=1}^{\infty} d_j \lambda^+_j \phi^+_j = \sum_{j=1}^{\infty} z \left( d_j - \frac{1}{|\Omega|} \int_{\Omega^+} v \int_{\Omega^+} \phi^+_j \right) \phi^+_j,
\]
and, therefore,
\[
v = -\frac{z}{|\Omega|} \left( \int_{\Omega^+} v \sum_{j=1}^{\infty} (\lambda^+_j - z)^{-1} \int_{\Omega^+} \phi^+_j \phi^+_j \right), \tag{1.3}
\]
as long as \( z \neq \lambda^+_j, j = 1, 2, \ldots \). Taking the integral over \( \Omega^+ \) on both sides of (1.3) and assuming that the integral of \( v \) over \( \Omega^+ \) does not vanish yields, by incorporation of \( z = 0 \) into the answer,
\[
z \left[ |\Omega| + z \sum_{j=1}^{\infty} (\lambda^+_j - z)^{-1} \left( \int_{\Omega^+} \phi^+_j \phi^+_j \right) \right] = 0. \tag{1.4}
\]
Thus, the spectrum of the electrostatic problem is the union of two sets:

a) the set of \( z \) solving the equation (1.4);

b) the set of those eigenvalues \( \lambda^+_j \) for which the corresponding eigenfunction \( \phi^+_j \) has the zero mean over \( \Omega^+ \).

The main result of the present paper, which is the norm-resolvent asymptotics for the operator of the BVP introduced above, yields in particular the description (1.4) for the limiting spectrum of the problem, together with an order-sharp estimate on the rate of convergence as \( a \to +\infty \).

Relations similar to (1.4) appear in the analysis of periodic problems with micro-resonances (“meta-materials”), see [3], where they give zeros of the functions describing the dispersion of waves propagating through media modelled by such problems.

The present paper is a development of the recent study [6]–[8], [3], aimed at implementing the ideas of the boundary triples theory as proposed by Ryzhov [9] (in its turn, this analysis heavily draws upon the celebrated Birman-Krein-\( \check{\text{V}} \)išik theory [10]–[13]) in the context of problems of materials science and wave propagation in inhomogeneous media. Our recent papers cited above have shown that the language of boundary triples is particularly fitting for the analysis of composite media. This is due to the fact that the key difficulties in that analysis stem from the presence of interfaces (i.e., boundaries between individual material components) through which an exchange of energy between different components of the medium takes place. We also point out that the papers [14]–[16] further demonstrate the additional convenience of using the boundary triples approach is that the framework of functional models and the approach to scattering theory based on them (see [17], [18]) can be formulated in the most natural terms of Dirichlet-to-Neumann maps pertaining to the interfaces.

An asymptotic analysis of the static (or “equilibrium”) version of the above problem, where \( z = 0 \) and a forcing term is added to the right-hand side, was carried out in [5], in the context of isotropic elasticity (which additionally implies that two material parameters are present, the so-called Lamé coefficients). The authors of [5], using the representation of solutions in terms of boundary layers, prove “strong” resolvent convergence of the original problem to the resolvent version of the “electrostatic” problem (1.1) (albeit framed in the context of linearised elasticity), still with \( z = 0 \). In the work [19], different methods were used to obtain the spectral convergence; however, neither the effective operator of the “limiting” medium nor the norm-resolvent convergence to it were discussed. We argue that the approach we present here allows one to improve such results in two respects: a) the new estimates are of the operator-norm resolvent type, implying, in particular, the control of the convergence of the associated spectra and the exponential groups; b) our estimates are uniform with respect to the spectral parameter and are order-sharp, i.e. the rate of convergence in terms of \( a \to 0 \) cannot be improved further.

We briefly outline the contents of the paper. In Sec. 2, we recall the main points of the abstract construction of [9] and introduce the key tools for our analysis. These include a representation for the resolvents of a class of boundary-value problems in terms of the \( M \)-operator. Using these general
formulas, in Sec. 3 we study the asymptotic behavior of the operators corresponding to transmission problems for two-component media with contrasting material properties, as described above. The asymptotic approximation of the spectra is discussed at the end of the paper, leading to the characterisation (1.4).

2. RYZHOV TRIPLES FOR BOUNDARY-VALUE PROBLEMS

In this section, we follow [9], outlining an operator framework suitable for dealing with boundary-value problems.

2.1. The Boundary Triple Framework

The starting point of our construction is a self-adjoint operator $A_0$ in a separable Hilbert space $H$ with $0 \in \rho(A_0)$, where $\rho(A_0)$, as usual, denotes the resolvent set of $A_0$. Besides $H$, we consider an auxiliary Hilbert space $E$ and a bounded operator $\Pi : E \rightarrow H$ such that

$$\text{dom}(A_0) \cap \text{ran}(\Pi) = \{0\}, \quad \ker(\Pi) = \{0\}.$$ 

Since the kernel of $\Pi$ is trivial, there is a left inverse $\Pi^{-1}$, so that $\Pi^{-1}\Pi = I_E$.

We define

$$\text{dom}(A) := \text{dom}(A_0) + \text{ran}(\Pi), \quad A : A_0^{-1}f + \Pi \phi \mapsto f, \quad f \in H, \quad \phi \in E,$$ 

$$\text{dom}(\Gamma_0) := \text{dom}(A_0) + \text{ran}(\Pi), \quad \Gamma_0 : A_0^{-1}f + \Pi \phi \mapsto \phi, \quad f \in H, \quad \phi \in E,$$ 

where neither $A$ nor $\Gamma_0$ is assumed closed or indeed closable. The operator given in (2.1) is the null extension of $A_0$, while (2.2) is the null extension of $\Pi^{-1}$. Note also that

$$\ker(\Gamma_0) = \text{dom}(A_0).$$

For $z \in \rho(A_0)$, consider the abstract spectral BVP

$$\begin{align*}
Au &= zu, \\
\Gamma_0 u &= \phi, \quad \phi \in E,
\end{align*}$$

(2.3)

where the second equation is understood as a boundary condition. As it is asserted in [9, Theorem 3.1], there is a unique solution $u$ of the BVP (2.3) for any $\phi \in E$. Thus, there is an operator (clearly linear) which assigns to any $\phi \in E$ the solution $u$ of (2.3). This operator is called the solution operator for $A$ and is denoted by $\gamma_z$. An explicit expression for it in terms of $A_0$ and $\Pi$ is

$$\gamma_z : \phi \mapsto (I + z(A_0 - zI)^{-1})\Pi \phi$$

for any $z \in \rho(A_0)$. Note that

$$I + z(A_0 - zI)^{-1} = (I - zA_0^{-1})^{-1},$$

and that (2.2) and (2.4) immediately imply

$$\Gamma_0 \gamma_z = I_E.$$

By (2.4) and a simple calculation, $\gamma(z)$, we have

$$\text{ran} \gamma_z = \ker(A - zI).$$

We note that, since $A$ is not required to be closed, $\text{ran} \gamma_z$ is not necessarily a subspace. This is precisely the kind of situation that commonly occurs in the analysis of BVPs.

In what follows, we consider (abstract) BVPs of the form (2.3) associated with the operator $A$, with variable boundary conditions. To this end, for a self-adjoint operator $\Lambda$ in $E$, define

$$\text{dom}(\Gamma_1) := \text{dom}(A_0) + \Pi \text{ dom}(\Lambda),$$

$$\Gamma_1 : A_0^{-1}f + \Pi \phi \mapsto \Pi^* f + \Lambda \phi, \quad f \in H, \quad \phi \in \text{dom}(\Lambda).$$

\[1\]The function $\gamma$ is sometimes referred to as the $\gamma$-field.
The operator $\Lambda$ can thus be understood as a parameter for the boundary operator $\Gamma_1$.

On the basis of (2.4), one sees from (2.5) (see [9, Eq. 3.7]) that

$$\gamma_z^* = \Gamma_1(A_0 - zI)^{-1}.$$  

Also, according to [9, Theorem 3.2], the following Green’s type identity holds:

$$\langle Au, v \rangle_{\mathcal{H}} - \langle u, Av \rangle_{\mathcal{H}} = \langle \Gamma_1 u, \Gamma_0 v \rangle_{\mathcal{E}} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{\mathcal{E}}, \quad u, v \in \text{dom}(\Gamma_1).$$

The above framework for triples $(A_0, \Lambda, \Pi)$ stems from the Birman-Kreĭn-Višik theory [10]–[13], rather than the theory of boundary triples of the book [20]. We employ it next to introduce the notion of $M$-operator, generalizing the well-known notion of Dirichlet-to-Neumann map in the context of BVPs.

This generalization helps us achieve two goals: on the one hand, it allows us to deal with a transmission (rather than a boundary-value problem), and on the other hand, it produces formulas for the resolvents that we will use in order to obtain operator-norm error estimates in the large-coupling asymptotic regime.

### 2.2. Definition and Properties of the $M$-Operator

Based on the notion of a triple, we now define the mentioned abstract version of the Dirichlet-to-Neumann map.

**Definition 1.** For a given triple $(A_0, \Lambda, \Pi)$, define the operator-valued $M$-function associated with $A_0$ as follows. For any $z \in \rho(A_0)$, the operator $M(z)$ in $\mathcal{E}$ is defined by

$$M(z): \phi \mapsto \Gamma_1 \gamma_z \phi, \quad \phi \in \text{dom}(M(z)) := \text{dom}(\Lambda).$$

A detailed description of how one can state, in the language of boundary triples, the classical boundary-value problems, such as the Dirichlet problem for the Laplace operator on a bounded domain with sufficiently regular boundary, can be found in [9, 16].

Taking into account (2.5), one concludes from Definition 1 that

$$M(z) = \Lambda + z\Pi^*(I - zA_0^{-1})^{-1}\Pi. \quad (2.6)$$

Also, due to the self-adjointness of $\Lambda$, one has

$$M^*(z) = M(\overline{z}).$$

Moreover, it can be checked that $M$ is an unbounded operator-valued Herglotz function, i.e., $M(z) - M(0)$ is analytic and $\exists M(z) \geq 0$ whenever $z \in \mathbb{C}_+$. It is shown in [9, Theorem 3.3(4)] that

$$M(z)\Gamma_0 u_z = \Gamma_1 u_z \quad \text{for all} \quad u_z \in \ker(A - zI) \cap \text{dom}(\Gamma_1).$$

In this work we consider extensions (self-adjoint and non-self-adjoint) of the “minimal” operator

$$\tilde{A} := A_0|_{\ker(\Gamma_1)}, \quad (2.7)$$

that are restrictions of $A$. It is proved in [9, Sec. 5] that $\tilde{A}$ is symmetric with equal deficiency indices. Moreover, [9, Prop. 5.1] asserts that $\tilde{A}$ does not depend on the parameter operator $\Lambda$ are contrary to what could be surmised from (2.7).
2.3. Resolvent Formulas for General Boundary-Value Problems

Still following [9], we suppose \( \alpha \) and \( \beta \) are linear operators in the Hilbert space \( \mathcal{E} \) such that \( \text{dom}(\alpha) \supset \text{dom}(\lambda) \) and \( \beta \) is bounded on \( \mathcal{E} \). Additionally, we assume that \( \alpha + \beta \lambda \) is closable and denote \( \beta := \alpha + \beta \lambda \). Consider the linear set

\[
\mathcal{H}_\beta := \{ A_0^{-1} f + \Pi \phi : f \in \mathcal{H}, \phi \in \text{dom}(\beta) \} \tag{2.8}
\]

Following [9, Lem. 4.1], the identity

\[
(\alpha \Gamma_0 + \beta \Gamma_1)(A_0^{-1} f + \Pi \phi) = \beta \Pi^* f + (\alpha + \beta \lambda) \phi, \quad f \in \mathcal{H}, \, \phi \in \text{dom}(\lambda),
\]

implies that \( \alpha \Gamma_0 + \beta \Gamma_1 \) is well-defined on \( \text{dom}(A_0) + \Pi \text{ dom}(\lambda) \). The assumption that \( \alpha + \beta \lambda \) is closable is used to extend the domain of definition of \( \alpha \Gamma_0 + \beta \Gamma_1 \) to the set (2.8). Moreover, it can be shown that \( \mathcal{H}_\beta \) is a Hilbert space with respect to the norm

\[
\|u\|_\beta^2 := \|f\|_\mathcal{H}^2 + \|\phi\|_\mathcal{E}^2 + \|\beta \phi\|_\mathcal{E}^2, \quad u = A_0^{-1} f + \Pi \phi.
\]

It follows that the constructed extension \( \alpha \Gamma_0 + \beta \Gamma_1 \) is a bounded operator from \( \mathcal{H}_\beta \) to \( \mathcal{E} \).

According to [9, Theorem 4.1], if the operator \( \alpha + \beta M(z) \) is boundedly invertible for \( z \in \rho(A_0) \), then, on the one hand, the spectral BVP

\[
\begin{cases}
(A - zI)u = f, \\ (\alpha \Gamma_0 + \beta \Gamma_1)u = \phi, \quad f \in \mathcal{H}, \, \phi \in \mathcal{E}
\end{cases}
\]

has a unique solution \( u \in \mathcal{H}_\beta \), where, as above, \( \alpha \Gamma_0 + \beta \Gamma_1 \) is a bounded operator on \( \mathcal{H}_\beta \). On the other hand, it follows from [9, Theorem 5.1] that the function

\[
(A₀ - zI)^{-1} - (I - zA₀^{-1})^{-1} \Pi [\alpha + \beta M(z)]^{-1} \beta \Pi^* (I - zA₀^{-1})^{-1}
\]

is the resolvent of a closed operator \( A_{\alpha\beta} \) densely defined in \( \mathcal{H} \). Moreover,

\( \tilde{A} \subset A_{\alpha\beta} \subset A, \quad \text{dom}(A_{\alpha\beta}) \subset \{ u \in \mathcal{H}_\beta : (\alpha \Gamma_0 + \beta \Gamma_1)u = 0 \}. \)

3. LARGE-COUPLING ASYMPTOTICS FOR A TRANSMISSION PROBLEM

The aim of this section is to translate a problem familiar to the application-minded reader into the language of boundary triple theory presented above and obtain new results for this problem.

3.1. Problem Formulation

Suppose that \( \Omega \) is a bounded \( C^{1,1} \) domain, and \( \Gamma \subset \partial \Omega \) is a closed \( C^{1,1} \) curve, so that \( \Gamma = \partial \Omega_- \) is the common boundary of domains \( \Omega_+ \) and \( \Omega_- \), where \( \Omega_- \) is strictly contained in \( \Omega \), such that \( \overline{\Omega_+} \cup \overline{\Omega_-} = \overline{\Omega} \), see Fig. 1.

For \( a > 0, z \in \mathbb{C} \) we consider the “transmission” eigenvalue problem (cf. [21])

\[
\begin{align*}
-\Delta u_+ &= zu_+ \quad &\text{in} \quad \Omega_+, \\
- a \Delta u_- &= zu_- \quad &\text{in} \quad \Omega_-,
\end{align*}
\]

\[
\begin{align*}
u_+ &= u_-, \\
\partial u_+ \big{/} \partial n_+ + a \partial u_- \big{/} \partial n_- &= 0 \quad &\text{on} \quad \Gamma,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_+}{\partial n_+} &= 0 \quad &\text{on} \quad \partial \Omega,
\end{align*}
\]

(3.1)

where \( n_\pm \) denotes the exterior normal (defined a.e.) to the corresponding part of the boundary.\(^2\)

The above problem is understood in the strong sense, i.e., \( u_\pm \in H^2(\Omega_\pm) \), the Laplacian differential expression \( \Delta \) is the corresponding combination of second-order weak derivatives, and the boundary values of \( u_\pm \) and their normal derivatives are understood in the sense of traces according to the embeddings of \( H^s(\Omega_\pm) \) into \( H^s(\Gamma), \, H^s(\partial \Omega) \), where \( s = 3/2 \) or \( s = 1/2 \).

\(^2\)The Neumann boundary condition on \( \partial \Omega \) can be replaced by a Robin condition with an arbitrary coupling constant without affecting the analysis of this section.
3.2. A Reformulation in the Language of Triples

In order to make our framework applicable to (3.1), we first consider its weak formulation. We then apply the regularity theory for elliptic BVPs, see, e.g., [21], to show that its solutions are in fact the solutions to (3.1). Indeed, the results of [9], see also (2.9), show that problem (3.1) in the weak formulation is the eigenvalue problem for a self-adjoint operator of the appropriate class \(A_{\alpha\beta}\) introduced in Sec. 2.3, and thus its solutions are in the domain of this operator. Then the result of [21] is used to show that these solutions have higher regularity, as required for the solvability of (3.1) in the strong sense.

We define the Dirichlet-to-Neumann map\(^3\)

\[ \Lambda: \phi \mapsto \frac{\partial u_+}{\partial n_+} - a \frac{\partial u_-}{\partial n_-}, \quad \phi \in H^1(\Gamma), \]

where \(u_\pm\) are the harmonic functions in \(\Omega_\pm\) subject to the above boundary condition on \(\partial \Omega\) and the condition \(u = \phi\) on \(\Gamma\). Clearly \(\Lambda = \Lambda^+ + a \Lambda^-\), where \(\Lambda^+, \Lambda^-\) are the Dirichlet-to-Neumann maps on each side of the interface \(\Gamma\), which are self-adjoint operators in \(L^2(\Gamma)\), with domain \(H^1(\Gamma)\). For sufficiently large values of \(a\), the operator \(\Lambda\) has the same properties as \(\Lambda^+\) and \(\Lambda^-\), see [3, Lem. 2.1].

Translating the spectral BVP (3.1) into the language of the abstract framework developed in Sec. 2.1, we define \(A_0\) as the “Dirichlet decoupling” \(A_0^+ \oplus a A_0^-\), corresponding to the decomposition

\[ L^2(\Omega_+) \oplus L^2(\Omega_-) = L^2(\Omega) =: \mathcal{H}, \]

where \(A_0^+\) is the Laplace operator with Dirichlet condition on \(\Gamma\) and Neumann condition on \(\partial \Omega\), and \(A_0^-\) is the Dirichlet Laplacian on \(\Omega_-\). Furthermore, in the context of the transmission problem (3.1), the boundary space is given by \(L^2(\Gamma) =: \mathcal{E}\) and the abstract operator \(\Pi\) of Sec. 2.1 is simply the Poisson operator of the harmonic lift from \(\mathcal{E}\) to \(\mathcal{H}\) (subject to the Neumann condition on \(\partial \Omega\)), while its left inverse is the trace operator on \(\Gamma\) for functions that are harmonic on \(\Omega_-\) and \(\Omega_+\) and possess square summable boundary values on \(\Gamma\). The operator \(\Gamma_0\) is the null extension of the latter to

\[ (H^2(\Omega_-) \cap H^1_0(\Omega_-)) \oplus (H^2(\Omega_+) \cap H^1_{0,\Gamma}(\Omega_+)) \oplus \Pi L^2(\Gamma), \]

where \(H^1_{0,\Gamma}(\Omega_+\)) consists of functions in \(H^1(\Omega_+)\) with zero trace on \(\Gamma\), cf. (1.2).

Problem (3.1) is then written in the form \(A_{\alpha\beta}u = \alpha \frac{\partial u_+}{\partial n_+} - a \frac{\partial u_-}{\partial n_-}\), equivalently

\[ Au = \alpha \frac{\partial u_+}{\partial n_+} - a \frac{\partial u_-}{\partial n_-}, \quad \Gamma_1 u = 0, \]

where \(A\) is defined by (2.1) and \(\Gamma_1\) is defined by (2.5).

Finally, the operator \(M(z)\) of Definition 1 is the mapping

\[ M(z): \phi \mapsto -\frac{\partial u_+}{\partial n_+} - a \frac{\partial u_-}{\partial n_-}, \quad \phi \in H^1(\Gamma), \]

where

\[
\begin{cases}
-\Delta u_+ = z u_+ & \text{in } \Omega_+, \\
-\alpha \Delta u_- = z u_- & \text{in } \Omega_-, \\
u_+ = u_- = \phi & \text{on } \Gamma, \\
\frac{\partial u_+}{\partial n_+} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and formula (2.6) expresses \(M(z)\) in terms of \(\Lambda\) and the decoupling \(A_0^+ \oplus a A_0^-\). Recall (Sec. 2.2) that \(M(z)\) is an (unbounded) operator-valued Herglotz function, analytic in \(\mathbb{C} \setminus \mathbb{R}\), so that \(M(z) = M^*(\overline{z})\), \(z \in \mathbb{C} \setminus \mathbb{R}\). In addition, for all values of \(z\) except a discrete set of points, \(M(z)\) is invertible and its inverse

\[^3\]For convenience, we define the Dirichlet-to-Neumann map via \(-\partial u/\partial n|_{\partial \Omega}\) instead of the more common \(\partial u/\partial n|_{\partial \Omega}\). As a side note, we mention that this is obviously not the only possible choice for the operator \(\Lambda\). In particular, the trivial option \(\Lambda = 0\) is always possible. Our choice of \(\Lambda\) is motivated by our interest in the analysis of classical boundary conditions.
is compact. Similarly, Definition 1 yields the $M$-operators $M^\pm$ corresponding to the components $\Omega_\pm$, so that
\[ M^+(z): \phi \mapsto -\frac{\partial u_+}{\partial n_+}, \quad M^-(z): \phi \mapsto -a \frac{\partial u_-}{\partial n_-}, \quad \phi \in H^1(\Gamma), \]
where $u_+$ and $u_-$ are as above.

By a similar argument to the one of [9, Theorem 3.3, Theorem 5.1], the following statement holds, cf. [22].

**Theorem 3.1.** The spectrum of (3.1), i.e., the set of values $z$ for which (3.1) has a nonzero solution in the strong sense as described above, coincides with the set of values of $z$ for which the inverse of $\alpha + \beta M(z) \equiv M(z)$ does not admit an analytic continuation.

The representation (2.6) applied to $M^-(z)$ implies that
\[ M^-(z) = a\Lambda^- + z\Pi^*(I - a^{-1}z(A_0^-)^{-1})^{-1}\Pi^- = a\Lambda^- + z\Pi^*\Pi^- + O(a^{-1}), \quad (3.2) \]
where $A_0^-$ is as above the Dirichlet Laplacian on $\Omega_-$, and $\Pi^-$ is the harmonic lift from $\Gamma$ to $\Omega_-$. 

### 3.3. Asymptotic Analysis and the Main Result

In what follows, we analyze the resolvent of the operator $A_\alpha$ of the transmission problem (3.1), which coincides with the resolvent of the operator $A_{0\Gamma}$ in terms of the notation of Sec. 2. In particular, the spectrum of $A_\alpha$ coincides with the spectrum of (3.1). Our approach is based on the use of the Krein formula (2.9) with $\alpha = 0$, $\beta = I$, where for the asymptotic analysis of $M(z)^{-1}$ we employ (3.2) and separate the singular and nonsingular parts of $\Lambda^-$. 

To this end, note first that the spectrum of $\Lambda^-$ consists of the values $\mu$ (“Steklov eigenvalues”) such that the problem
\[
\begin{cases}
\Delta u = 0, & u \in H^2(\Omega_-), \\
\frac{\partial u}{\partial n} = -\mu u & \text{on } \Gamma
\end{cases}
\]
has a non-trivial solution. The least (by absolute value) Steklov eigenvalue is zero, and the associated normalised eigenfunction (“Steklov eigenvector”) is
\[ \psi_* := [\Gamma]^{-1/2}\Pi_\Gamma \in \mathcal{E} \equiv L^2(\Gamma). \]

Introduce the corresponding orthogonal projection $P := \langle \cdot, \psi_* \rangle_{\mathcal{E}}\psi_*$, which is a spectral projection relative to $\Lambda^-$, and decompose the boundary space $\mathcal{E}$:
\[ \mathcal{E} = P\mathcal{E} \oplus P^\perp\mathcal{E}, \quad (3.3) \]
where $P^\perp := I - P$. This yields the following matrix representation for $\Lambda^-$:
\[ \Lambda^- = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^- \end{pmatrix}, \]
where $\Lambda^- := P^\perp\Lambda^-P^\perp$ is treated as a self-adjoint operator in $P^\perp\mathcal{E}$.

We write the operator $M(z)$ as a block-operator matrix relative to the decomposition (3.3), followed by an application of the Schur–Frobenius inversion formula, see [23, Theorem 2.3.3]. To this end, notice that, for all $\psi \in \text{dom } \Lambda$, we have $P\psi \in \text{dom } \Lambda$ and, therefore, $P^\perp\Lambda P$ is well defined on $\text{dom } \Lambda$. Similarly, $P^\perp\psi = \psi - P\psi \in \text{dom } \Lambda$, and $P\Lambda P^\perp$ is also well defined. Furthermore, by the self-adjointness of $\Lambda$, we can write
\[ P\Lambda P^\perp\psi = \langle P^\perp\psi, \Lambda\psi_* \rangle \psi_*, \]
and, therefore,
\[ \|P\Lambda P^\perp\|_{\mathcal{E} \to \mathcal{E}} \leq \|\Lambda\psi_*\|_{\mathcal{E}}. \]
It follows that $PAP^{-1}$ is extendable to a bounded mapping on $P^\perp \mathcal{E}$. A similar calculation applied to $P^\perp \Lambda P$ and $PAP$ shows that these are extendable to bounded mappings on $P\mathcal{E}$. Therefore, for each $z \in \rho(A_0^+) \cap \rho(A_0^-)$, the operator $M(z)$ admits the representation

$$M(z) = \begin{pmatrix} A & B \\ E & D \end{pmatrix}, \quad A, B, E \text{ bounded.}$$

In order to evaluate $M(z)^{-1}$, we use the Schur-Frobenius inversion formula \cite{24}, \cite[Theorem 2.3.3]{23}

$$\begin{pmatrix} A & B \\ E & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BD^{-1}EA^{-1} & -A^{-1}BD^{-1} \\ -S^{-1}EA^{-1} & S^{-1} \end{pmatrix}, \quad S := D - EA^{-1}B. \quad (3.4)$$

Employing the fact that $S^{-1} = (I - D^{-1}EA^{-1}B)^{-1}D^{-1}$,

where $\|D^{-1}\| \leq C a^{-1}$, and, therefore, $S$ is boundedly invertible with a uniformly small bound, we obtain (see \cite{23} for details)

$$M(z)^{-1} = \begin{pmatrix} A & B \\ E & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(a^{-1}). \quad (3.5)$$

**Theorem 3.2.** Fix $\sigma > 0$ and a compact set $K \subset \mathbb{C}$, and denote $K_\sigma := \{z \in K : \text{dist}(z, \mathbb{R}) \geq \sigma\}$. There exist $C, a_0 > 0$ such that, for all $z \in K_\sigma$, $a \geq a_0$,

$$\|(A_a - z)^{-1} - (A_{P^\perp, P} - z)^{-1}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C a^{-1}.$$  

**Proof.** We use formula (2.9) with $\alpha = 0$, $\beta = 1$ for the resolvent $(A_a - z)^{-1}$ and with $\alpha = P^\perp$, $\beta = P$ for $(A_{P^\perp, P} - z)^{-1}$. In the former case, we use (3.5) and in the latter case, we write

$$(P^\perp + PM(z))^{-1}P = P(PM(z)P)^{-1}P,$$

by the Schur-Frobenius inversion formula \cite[Sec. 1.6]{23}, see (3.4).\footnote{We note that $P^\perp + PM(z)$ is triangular $(A = PM(z)P, B = PM(z)P^\perp, E = 0, D = I$ in (3.4)) with respect to the decomposition $\mathcal{E} = P\mathcal{E} \oplus P^\perp \mathcal{E}$.} The claim follows by comparing the obtained expressions for the two resolvents. \hfill \square

We now rewrite the result of Theorem 3.2 in block-matrix form relative to the decomposition

$$\mathcal{H} = P_- \mathcal{H} \oplus P_+ \mathcal{H} = L^2(\Omega_-) \oplus L^2(\Omega_+),$$

where $P_-, P_+$ are orthogonal projections from $L^2(\Omega)$ to $L^2(\Omega_-), L^2(\Omega_+)$, respectively. This allows us to express the asymptotics of $(A_a - z)^{-1}$ in terms of the generalized resolvent (see \cite{25}, \cite{26}) $R_a(z) := P_+(A_a - z)^{-1}P_+$, analyzed next.

**Theorem 3.3.** The following relation holds:

$$R_a(z) = (A_0^+ - z)^{-1} - \gamma_z^+(M^+(z) + M^-(z))^{-1}(\gamma_z^+)^*,$$

where $\gamma_z^+: \phi \mapsto u$ is the solution operator of the BVP

$$\begin{cases}
-\Delta u - zu = 0, \quad u \in \text{dom } A_0^+ + \text{ran } \Pi_+, \\
u|_\Gamma = \phi, \\
\frac{\partial u}{\partial n_+} = 0 \quad \text{on } \partial \Omega.
\end{cases} \quad (3.6)$$

**Proof.** By the definition of $M^+$, $M^-$ and a direct application of (2.9) as in [3, Lemma 3.2], the assertion follows. \hfill \square
A comparison of the latter result with (2.9) taking into account (2.4) yields:

**Corollary 3.4.** The generalized resolvent \( R_\alpha(z) \) is the solution operator for the BVP

\[
\begin{align*}
-\Delta u - zu &= f, & f \in L^2(\Omega_+), \\
\Gamma^+_1 u &= -M^-(z)\Gamma^+_0 u, \\
\frac{\partial u}{\partial n_+} &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Theorem 3.2 now implies an operator-norm asymptotics for the generalized resolvents \( R_\alpha \) as \( a \to \infty \).

**Theorem 3.5.** For all \( z \in K_\sigma \) the operator \( R_\alpha(z) \) admits the asymptotics

\[
R_\alpha(z) = R_{\text{eff}}(z) + O(a^{-1}),
\]

as \( a \to \infty \), in the operator-norm topology, where \( R_{\text{eff}}(z) \) is the solution operator for the BVP

\[
\begin{align*}
-\Delta u - zu &= f, & f \in L^2(\Omega_+), \\
\alpha(z)\Gamma^+_0 u + \beta\Gamma^+_1 u &= 0, \\
\frac{\partial u}{\partial n_+} &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

with \( \alpha(z) = P^\perp + PM^-(z)P \) and \( \beta = P \).

**Proof.** On the one hand, by Theorem 3.2, the resolvent \( (A_\alpha - z)^{-1} \) is \( O(a^{-1}) \)-close to

\[
(A_{P^\perp} - z)^{-1} = (A_0 - z)^{-1} - \gamma_\alpha(P^\perp + PM(z))^{-1}P\gamma_\alpha^*,
\]

and, therefore,

\[
R_\alpha(z) = P_+(A_0 - z)^{-1}P_+ - P_+\gamma_\alpha(P^\perp + PM(z))^{-1}P\gamma_\alpha^* + O(a^{-1})
= (A_0^+ - z)^{-1} - \gamma_\alpha^+(P^\perp + PM(z))^{-1}P(\gamma_\alpha^*)^* + O(a^{-1})
= (A_0^+ - z)^{-1} - \gamma_\alpha^*P(PM(z)P)^{-1}P(\gamma_\alpha^*)^* + O(a^{-1}).
\]

On the other hand, using (2.9) and applying the inversion formula (3.4), we obtain

\[
R_{\text{eff}}(z) = (A_0^+ - z)^{-1} - \gamma_\alpha^*(P^\perp + PM^-(z)P + PM^+(z))^{-1}P(\gamma_\alpha^*)^*
= (A_0^+ - z)^{-1} - \gamma_\alpha^*P(PM^+(z)P + PM^-(z)P)^{-1}P(\gamma_\alpha^*)^*.
\]

Comparing the right-hand sides of (3.8) and (3.9) completes the proof.

Theorem 3.5 can be further clarified by considering the “truncated” boundary space \( \tilde{\mathcal{E}} := PE \). Introduce the truncated harmonic lift by \( \tilde{\Pi}_+ := \Pi_+|_{\tilde{\mathcal{E}}} \) and Dirichlet-to-Neumann map \( \tilde{\Lambda}^+ := PA^+|_{\tilde{\mathcal{E}}} \).

**Theorem 3.6.** Denote

\[
\tilde{M}^+(z) := PM^+(z)|_{\tilde{\mathcal{E}}} = \tilde{\Lambda}^+ + z\tilde{\Pi}_+(1 - z(A_0^+)^{-1})^{-1}\tilde{\Pi}_+.
\]

The formula

\[
R_{\text{eff}}(z) = (A_0^+ - z)^{-1} - \gamma_\alpha^+(\tilde{M}^+(z) + PM^-(z)P)^{-1}(\gamma_\alpha^*)^*,
\]

holds, where \( \gamma_\alpha^+ : \phi \mapsto u_\phi \) is the solution operator of the problem

\[
\begin{align*}
-\Delta u_\phi - zu_\phi &= 0, & u_\phi \in \text{dom } A_0^+ + \text{ran } \tilde{\Pi}_+, \\
\Gamma^+_0 u_\phi &= \phi, & \phi \in \tilde{\mathcal{E}}.
\end{align*}
\]

\(^5\text{In what follows, we consistently supply the (finite-dimensional) “truncated” spaces and operators pertaining to them by the breve overscript.}\)
Note that here $PM^-(z)P$, as opposed to $\tilde{M}^+(z)$, is attributed the meaning of the operator defining nonlocal boundary conditions in the BVP.

**Proof.** By the definition of $\gamma^+_z$, one has

$$\gamma^+_z = (I - z(A^+_0)^{-1})^{-1}\Pi_+,$$

and, therefore, $\gamma^+_z |_{\mathcal{E}}$. It follows that (3.9) is equivalent to (3.10), whence the assertion of Theorem follows.

**Corollary 3.7.** The operator $R_{\text{eff}}(z)$ is the solution operator of the problem

$$
\begin{aligned}
-\Delta u - zu &= f, \quad f \in L^2(\Omega_+), \quad u \in \text{dom} A^+_0 + \text{ran} \Pi_+, \\
PT_1u &= -P M^-(z)\Pi_0^+ u.
\end{aligned}
$$

Equipped with Theorems 3.5 and 3.6, we obtain a more convenient representation for the asymptotics of $(A_a - z)^{-1}$ of Theorem 3.2.

**Theorem 3.8.** Denote $\mathcal{R}_z := \Gamma_0^+|_{\text{ran} \gamma^+_z P}$, where $z \in \mathbb{C}_\pm$. For the resolvent $(A_a - z)^{-1}$,

$$(A_a - z)^{-1} = \begin{pmatrix}
R_{\text{eff}}(z) & (\mathcal{R}_z[R_{\text{eff}}(z)] - (A^+_0 - z)^{-1}]^{*}\Pi_+^*
\\
\Pi_-\mathcal{R}_z[R_{\text{eff}}(z) - (A^+_0 - z)^{-1}] & \Pi_-\mathcal{R}_z[R_{\text{eff}}(z) - (A^+_0 - z)^{-1}]^{*}\Pi_+^* + O(a^{-1}),
\end{pmatrix}
$$

where the operator matrix is written with respect to the decomposition

$$L^2(\Omega) = L^2(\Omega_+) \oplus L^2(\Omega_-).$$

**Proof.** First, we note that since $\text{ran}(\gamma^+_z P)$ is one-dimensional, the operator $\mathcal{R}_z$ is well defined as the bounded linear operator from $\text{ran}(\gamma^+_z P)$ to $\mathcal{E}$, where the former is equipped with the standard norm of $L^2(\Omega_+)$. We proceed by representing the operator $(A_{P_1 -} - z)^{-1}$, see Theorem 3.2, in block-operator matrix form relative to the orthogonal decomposition $\mathcal{H} = L^2(\Omega_+) \oplus L^2(\Omega_-)$. The upper left matrix entry $P_+(A_{P_1 -} - z)^{-1}P_+$, according to Theorem 3.5, is $O(a^{-1})$-close to $R_{\text{eff}}(z)$. Next we consider the lower left matrix entry:

$$P_-(A_{P_1 -} - z)^{-1}P_+ = -\gamma^-_z P(PM^+(z)P + PM^-(z)P)^{-1}P(\gamma^+_z)^*$$

$$= -\gamma^-_z \Gamma_0^+ \gamma^+_z P(PM^+(z)P + PM^-(z)P)^{-1}P(\gamma^+_z)^*$$

$$= \gamma^-_z \Gamma_0^+ [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}] = \gamma^-_z \mathcal{R}_z[R_{\text{eff}}(z) - (A^+_0 - z)^{-1}],$$

where $\gamma^-_z : \phi \mapsto u_\phi$ is the solution operator of the BVP (cf. (3.6))

$$\begin{cases}
-a\Delta u_\phi - zu_\phi = 0, \quad u_\phi \in \text{dom} A_0^- + \text{ran} \Pi_-, \\
\Gamma_0^- u_\phi = \phi.
\end{cases}
$$

Here in the second equality we use the fact that $\Gamma_0^+ \gamma^+_z = I$, and in the third equality we use (2.9), also see (3.9). Passing over to the top-right entry, we write

$$P_+(A_{P_1 -} - z)^{-1}P_- = -\gamma^+_z P(PM^+(z)P + PM^-(z)P)^{-1}P(\gamma^-_z)^*$$

$$= (\mathcal{R}_z[R_{\text{eff}}(z) - (A^+_0 - z)^{-1}]^{*}(\gamma^-_z)^*$$

$$= (\mathcal{R}_z[R_{\text{eff}}(z) - (A^+_0 - z)^{-1}]^{*}\Pi_+^*(1 - a^{-1}z(A^-_0)^{-1})^{-1},$$

and the claim pertaining to the above-mentioned entry follows by a virtually unchanged argument. Finally, for the bottom-right entry, we have

$$P_-(A_{P_1 -} - z)^{-1}P_- = (aA_0^- - z)^{-1} + \gamma^-_z \mathcal{R}_z[R_{\text{eff}}(z) - (A^+_0 - z)^{-1}]^{*}(\gamma^-_z)^*,$$

which completes the proof if we take into account the equality (cf. (*))

$$\gamma^-_z = (I - a^{-1}z(A^-_0)^{-1})^{-1}\Pi_- = \Pi_- + O(a^{-1}).$$

□
The resolvent Theorem 3.9. and hence the following statement holds. As a result, we have

\[ RM^{-1}(z)P = PA^{-1}P + zPI\Pi_\ast \Pi_\ast P + O(a^{-1}) = z\tilde{\Pi}^\ast \Pi_\ast + O(a^{-1}), \quad \Pi_\ast := \Pi_\ast |_{\tilde{E}}. \]

where the operator matrix is written with respect to the decomposition of the operator \( A \).

The spectra of the operators Corollary 3.11. immediately yields (see, e.g., [30]) the following corollary. Note that the resolvent of Remark 1.

We note that the generalized resolvent \( \tilde{R}_{\text{eff}}(z) \) is thus the solution operator of a spectral BVP of a special class. Namely, the dependence of its boundary condition on the spectral parameter is linear with respect to the latter. Such spectral BVPs were considered, e.g., in [27].
An explicit representation for the spectrum of $\mathcal{A}_{\text{eff}}$, i.e., the set of $z \in \mathbb{C}$ for which the problem

$$
\mathcal{A}_{\text{eff}} \left( \begin{array}{c}
u \\ \eta \end{array} \right) = z \left( \begin{array}{c} u_+ \\ \eta \end{array} \right)
$$

has a nontrivial solution $(u_+, \eta)^\top$, can be obtained as follows. We represent $u_+ \in H^2(\Omega_+)$ in the form $u_+ = v + c$, where $c \in \mathbb{C}$ is related to $\eta$ by the formula $c \sqrt{\Omega_-} = \eta$, cf. (3.12), and $v$ solves the problem $-\Delta v = z(v + c)$ subject to the Neumann boundary condition $\left( \partial v / \partial n_+ \right)|_{\partial \Omega} = 0$ and the Dirichlet boundary condition $v|_{\Gamma} = 0$, or equivalently $v = z(c(A_0^+ - z)^{-1} \mathbb{1}_{\Omega_+})$, where $\mathbb{1}_{\Omega_+}$ is the unity function on $\Omega_+$. Therefore, in terms of the pair $(v, c)^\top$, the eigenvalue problem (3.13) has the form:

$$
\begin{pmatrix}
-\Delta v \\
\frac{1}{\sqrt{|\Omega_-|}} \int_{\Gamma} \frac{\partial v}{\partial n_+}
\end{pmatrix} = z \begin{pmatrix} v + c \\ c \sqrt{|\Omega_-|} \end{pmatrix},
$$

and so its solvability is easily seen to be equivalent to the boundary equation appearing in the second component of (3.14):

$$
\frac{1}{\sqrt{|\Omega_-|}} \int_{\Gamma} \frac{\partial v}{\partial n_+} = z c \sqrt{|\Omega_-|}.
$$

Suppose first that $c \neq 0$. Denoting, as in the Introduction, by $\lambda_j^+, j = 1, 2, \ldots$, and $\phi_j^+, j = 1, 2, \ldots$, the eigenvalues and the corresponding normalised eigenfunctions of the operator $A_0^+$, relation (3.15) reduces to $c = -|\Omega|^{-1} \int_{\Omega_+} v$ by using Green’s formula and the equality $-\Delta v = z(v + c)$. Therefore, based on the calculation contained in the Introduction, the solvability of (3.13) turns out to be equivalent to (1.4). Alternatively, if $c = 0$, which corresponds to the case when $\eta = 0$, the function $u_+$ is clearly an eigenfunction of $A_0^+$, and it can easily be shown to have zero mean\(^7\) over $\Omega_+$.

**Remark 2.** It has been conjectured in [32] (and established rigorously in the case of the Dirichlet Laplace operator in two dimensions in [33]) that the eigenfunctions of a BVP on a sufficiently regular bounded domain $\Omega$ can be “nearly extended” (see [33] for rigorous details) to generalized eigenfunctions of the whole-space problem, corresponding to the same eigenvalue. This can be interpreted as an effect of “transparency” of the domain $\Omega$ to the waves of certain wavelengths. The set of values described by (1.4) can therefore be interpreted as the set of wavelengths at which a similar effect of transparency occurs for problems (3.1), as $a \to \infty$, in other words for problems with low-index dielectric inclusions. A rigorous proof of this observation requires the development of scattering theory for high-contrast transparent obstacles, and it will be addressed in a separate publication.

**Remark 3.** The spectrum of $A_0^+$, contrary to what would seem from Theorem 3.9, in the generic case does not enter the spectrum of (3.13). The mechanism for this is described in [34].

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\(^7\)This situation was shown in [31] to be nongeneric, at least for the case of simply connected domains and the Dirichlet condition on the boundary.
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