On Strong Stability of Explicit Runge–Kutta Methods for Nonlinear Semibounded Operators

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Explicit Runge–Kutta methods are classical and widespread techniques in the numerical solution of ordinary differential equations (ODEs). Considering partial differential equations, spatial semidiscretisations can be used to obtain systems of ODEs that are solved subsequently, resulting in fully discrete schemes. However, certain stability investigations of high-order methods for hyperbolic conservation laws are often conducted only for the semidiscrete versions. Here, strong stability (also known as monotonicity) of explicit Runge–Kutta methods for ODEs with nonlinear and semibounded (also known as dissipative) operators is investigated. Contrary to the linear case, it is proven that many strong stability preserving (SSP) schemes of order two or greater are not strongly stable for general smooth and semibounded nonlinear operators. Additionally, it is shown that there are first order accurate explicit SSP Runge–Kutta methods that are strongly stable (monotone) for semibounded (dissipative) and Lipschitz continuous operators.

Keywords. Runge–Kutta methods · strong stability · monotonicity · strong stability preserving · semibounded · dissipative

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1 Introduction

Considering the numerical solution of (partial) differential equations, stability of the schemes plays an important role. For linear symmetric hyperbolic partial differential equations (PDEs), energy estimates can often be obtained, resulting in both uniqueness of solutions and existence via appropriate approximations, as described in the monographs of Gustafsson, Kreiss and Oliger [18] and Kreiss and Lorenz [30]. Using schemes in the framework of summation by parts operators [31] with simultaneous approximation terms [8, 9], these energy estimates can often be transferred to semidiscrete schemes, as described in the review articles of Fernández, Hicken and Zingg [14] and Svärd and Nordström [49] and references cited therein. While this framework has been developed originally for finite difference schemes, it contains also many other classes of methods such as finite volume [35, 36], discontinuous Galerkin [13, 16], and flux reconstruction schemes [26, 43].

Since numerical methods are used to obtain fully discrete schemes from semidiscrete ones, the preservation of such kind of semidiscrete stability is worth investigating. Strong stability preserving (SSP) methods can be written as convex combinations of explicit Euler steps. Hence, they preserve all convex stability properties of the explicit Euler method, as described in the monograph of S. Gottlieb, Ketcheson and Shu [17] and references cited therein.

However, even for linear ODEs with semibounded operators, the explicit Euler method does not preserve $L^2$ stability in general. Thus, the SSP property cannot be used to obtain strong stability for these schemes. Nevertheless, some well-known high order, explicit SSP Runge–Kutta methods are strongly stable in this general case [42, 51]. While the classical fourth order Runge–Kutta method is not strongly stable after one time step, combining two consecutive time
steps results in a strongly stable scheme for this class of problems [47]. Further results for linear autonomous systems have been obtained by Sun and Shu [48].

Because many hyperbolic conservation laws are nonlinear or semidiscretisations are obtained via nonlinear processes, it is interesting whether explicit SSP Runge–Kutta methods can be strongly stable for general nonlinear ODEs with semibounded operators. In the context of hyperbolic conservation laws, many studies are based on the seminal work of Tadmor [50, 52] concerning entropy stability of semidiscretisations. While there are related studies of explicit Runge–Kutta methods [33], there are no general results on strong stability.

Although strong stability can be considered for general convex functionals, the $L^2$ norm will be used in this article. It is most similar to the linear case and of interest in applications, e.g. in the recent article of Nordström and La Cognata [37]. Moreover, it is a special case of strong stability and it might be expected that there is a larger set of methods that are strongly stable for this convex functional, similar to the case of SSP methods studied by Higuera [22]. Furthermore, the focus is on explicit schemes, since they are widespread, can be implemented easily, are often more efficient if accuracy is a determining factor, and efficient use of parallelism is less expensive than for implicit schemes [28].

In this article, it is proven that many explicit SSP Runge–Kutta methods of order two or greater cannot be strongly stable for nonlinear ODEs with smooth and semibounded operators in general. Some tedious calculations used in these proofs are verified using Mathematica [53] and published online [41]. Moreover, it is shown that first order accurate schemes can be both SSP and strongly stable for semibounded and Lipschitz continuous operators.

To do so, the article is structured as follows. At first, basic definitions such as strong stability and semiboundedness are given in section 2. Additionally, a brief review of Runge–Kutta methods is included to introduce the notation. Afterwards, Runge–Kutta methods of up to three stages are studied in section 3. It is shown that there are no such schemes with order of accuracy of at least two that are strongly stable and SSP. This result is based on the explicit construction of ODEs with nonlinear and semibounded operators and implications of the order conditions.

Since the number of parameters and the complexity of the order conditions increases with the number of stages, the general approach is not really feasible for methods with more stages. Therefore, some known explicit SSP methods with more than three stages are investigated separately in section 4. In particular, it is shown that the families of schemes with optimal SSP coefficient of order two [29, Theorem 9.3] and three [27, Theorem 3] and the ten stage, fourth order method SSPRK(10,4) of Ketcheson [27] are not strongly stable in general.

Thereafter, two well-known and popular SSP methods are studied in more detail in the following sections. While the investigations up to this point are only concerned with strong stability and not with boundedness in general, the popular three-stage method SSPRK(3,3) of Shu and Osher [45] is studied in detail in section 5. It is shown that the norm of the numerical approximation can increase monotonically and without bounds for nonlinear and semibounded operators. Since this article is motivated by applications of SSP methods to semidiscretisations of hyperbolic conservation laws, an energy stable and nonlinear semidiscretisation of the linear transport equation is constructed in section 6. This ODE with semibounded operator is solved numerically with SSPRK(10,4) and it is shown that the norm of the numerical solution increases for a large range of time steps.

Turning to first order schemes in section 7, it is shown that the limitations of high order schemes studied before do not apply. In particular, there are explicit SSP Runge–Kutta methods of first order of accuracy that are strongly stable for semibounded and Lipschitz continuous operators with Lipschitz constant $L$ under a time step restriction $\Delta t \leq \Delta t_{\text{max}}$, where $\Delta t_{\text{max}} \propto L^{-1}$. Finally, the results are summarised and discussed in section 8.
2 Brief Review of Runge Kutta Methods

Consider an ordinary differential equation
\[
\frac{d}{dt} u(t) = g(u(t)), \quad t \in (0, T),
\]
\[
u(0) = u_0, \tag{2.1}
\]
in a real vector space \( \mathcal{H} \) with semi inner product \( \langle \cdot, \cdot \rangle \), inducing the seminorm \( \| \cdot \| \). Typically, \( \mathcal{H} \) can be a Hilbert space. Therefore, \( \| \cdot \| \) will be called norm in the following. However, the property distinguishing a norm from a seminorm will not be used anywhere.

2.1 Strong Stability

For a smooth solution of (2.1), the time derivative of the squared norm is
\[
\frac{d}{dt} \| u(t) \|^2 = 2 \left\langle u(t), \frac{d}{dt} u(t) \right\rangle = 2 \left\langle u(t), g(u(t)) \right\rangle. \tag{2.2}
\]

**Definition 2.1.** A function \( g : \mathcal{H} \to \mathcal{H} \) is semibounded, if
\[
\forall u \in \mathcal{H} : \quad \left\langle u, g(u) \right\rangle \leq 0. \tag{2.3}
\]

**Remark 2.1.** If a complex (semi) inner product space is considered instead of a real one, the real part of the (semi) inner product \( \langle u, g(u) \rangle \) has to be non-positive.

**Remark 2.2.** Sometimes, such operators \( g \) are also called (energy) dissipative. Here, the term semibounded is used instead, in order to emphasise that (energy) conservative operators are included in this definition.

Thus, the (squared) norm of a smooth solution \( u \) of (2.1) is bounded by its initial value if \( g \) is semibounded. However, an approximate solution obtained by a numerical method does not necessarily satisfy this inequality. For example, applying one step of the explicit Euler method to (2.1) yields the new value \( u_+ = u_0 + \Delta t g(u_0) \), satisfying
\[
\| u_+ \|^2 = \| u_0 + \Delta t g(u_0) \|^2 = \| u_0 \|^2 + 2 \Delta t \left\langle u_0, g(u_0) \right\rangle + \Delta t^2 \| g(u_0) \|^2. \tag{2.4}
\]

Thus, for a general semibounded \( g \), the norm of the numerical solution can increase during one time step, e.g. if \( \left\langle u_0, g(u_0) \right\rangle = 0 \). In particular, this happens if \( g(u) = \mathcal{L} u \) where \( \mathcal{L} \) is a linear and skew-symmetric operator.

**Definition 2.2.** A numerical scheme approximating (2.1) during one time step from \( u_0 \approx u(t) \) to \( u_+ \approx u(t + \Delta t) \) with semibounded \( g \) is called strongly stable if \( \| u_+ \|^2 \leq \| u_0 \|^2 \).

**Remark 2.3.** Since this work is motivated by discretisations of PDEs, the term strong stability is used. In the literature on Runge–Kutta methods, such a property is often called monotonicity.

Nevertheless, the explicit Euler method can be strongly stable under stronger assumptions on \( g \). For example, consider the condition
\[
\exists M \in \mathbb{R} \forall u \in \mathcal{H} : \quad \left\langle u, g(u) \right\rangle \leq M \| g(u) \|^2. \tag{2.5}
\]

If \( M < 0 \) in (2.5), the explicit Euler method \( u_+ = u_0 + \Delta t g(u_0) \) is strongly stable under the time step restriction \( \Delta t \in (0, -2M] \), since
\[
\| u_0 + \Delta t g(u_0) \|^2 - \| u_0 \|^2 = 2 \Delta t \left\langle u_0, g(u_0) \right\rangle + \Delta t^2 \| g(u_0) \|^2 \leq (\Delta t + 2M) \Delta t \| g(u_0) \|^2 \leq 0. \tag{2.6}
\]

Such right hand sides with linear \( g \) are called coercive by Levy and Tadmor [32] and Tadmor [51].
Remark 2.4. Instead of the norm $\| \cdot \|$ of the solution, other convex functionals can be considered. For semidiscretisations of hyperbolic conservation laws, some important examples are the $L^1$ norm $\| u(t) \|_1 = \int |u(t,x)| \, dx$, the total variation seminorm $\| u(t) \|_{TV}$, non-negativity (expressed via $-\min_x u(t,x)$), or the total entropy $\int U(u(t,x)) \, dx$, where $U$ is a convex function. In that case, strong stability refers to the monotonicity of that particular convex functional in time.

Since the explicit Euler method is relatively simple, it is desirable to transfer results such as strong stability from that method (which are relatively easy to check) to high order schemes (for which it is considerably more difficult to check these properties). Strong stability preserving numerical schemes are designed to enable exactly this transfer, as described in the monograph of S. Gottlieb, Ketcheson and Shu [17] and references cited therein.

Definition 2.3. A numerical time scheme for (2.1) is called strong stability preserving (SSP) with SSP coefficient $c > 0$ if it is strongly stable under the time step restriction $\Delta t \leq c \Delta t_F$ whenever the explicit Euler method is strongly stable for $\Delta t \leq \Delta t_E$ and any convex functional, i.e. if for all convex functionals $\eta$, $\forall \Delta t \in (0, \Delta t_E]: \eta(u_0 + \Delta t g(u_0)) \leq \eta(u_0)$ implies $\forall \Delta t \in (0, c \Delta t_E]: \eta(u_0) \leq \eta(u_0)$. In particular, if $M \leq 0$, the difference between two solutions $u, v$ of (2.8) with initial conditions $u_0, v_0$ remains bounded. In particular, if $M \leq 0$, the difference between two solutions does not increase, resulting in contractivity. Thus, such a condition yields some important stability/boundedness/robustness properties. Since (2.7) does not restrict the Lipschitz seminorm

$$|f|_{\text{Lip}} := \sup_{u \neq v} \frac{\| f(t,u) - f(t,v) \|}{\| u - v \|}$$

(2.9)
of $f$, results based thereon can be applied to arbitrarily stiff equations. Hence, it is mostly useful for implicit methods. In order to be able to investigate also stability properties of explicit methods, Dahlquist and Jeltsch [11] introduced the condition

$$\exists M \in \mathbb{R} \forall t \in [0,T], u, v \in X: \quad \langle f(t,u) - f(t,v), u - v \rangle \leq M \| f(t,u) - f(t,v) \|^2$$

(2.10)

see also Dekker and Verwer [12, Chapter 6]. If $f$ satisfies (2.10) with $M < 0$, $f$ is Lipschitz continuous in its second argument with $|f|_{\text{Lip}} \leq -M^{-1}$, since

$$-M \| f(t,u) - f(t,v) \|^2 \leq -\langle f(t,u) - f(t,v), u - v \rangle \leq \| f(t,u) - f(t,v) \| \| u - v \|.$$ 

(2.11)

$M < 0$ yields again a contractive ODE and results based on (2.10) can be applied to explicit methods since the Lipschitz constant of $f$ is bounded.

General results on contractivity can be implied by seemingly simpler requirements such as

$$\exists M \in \mathbb{R} \forall t \in [0,T], \hat{u} \in \hat{X}: \quad \langle \dot{f}(t,\hat{u}), \hat{u} \rangle \leq M \| \hat{u} \|^2$$

(2.12)
or

$$\exists M \in \mathbb{R} \forall t \in [0,T], \hat{u} \in \hat{X}: \quad \| \dot{f}(t,\hat{u}) \| \leq M \| f(t,\hat{u}) \|^2.$$ 

(2.13)
A method via the slopes $u$ versions of cf. [6] or [7, Section 357]. In particular, monotonicity/semiboundedness results such as discrete versions of $\|u(t)\| \leq \|u_0\|$ can be transferred directly to contractivity. Therefore, only the former will be studied in this article.

**Remark 2.5.** For linear ODEs with possibly time dependent coefficients, the concepts of contractivity and monotonicity are equivalent. Since many results have been established for contractivity, e.g. by Dahlquist and Jeltsch [11] and Dekker and Verwer [12], they can be transferred directly to monotonicity. In particular, severe limitations of numerical methods result from linearity, e.g. by Dahlquist and Jeltsch [11] and Dekker and Verwer [12], they can be transferred to contractivity. Therefore, only the former will be considered.

**Remark 2.6.** Results for right hand sides $f$ satisfying (2.13) with $M < 0$ have been obtained by Higueras [22], similar to the results for circle contractivity by Dahlquist and Jeltsch [11] and Dekker and Verwer [12]. This is a special case of strong stability preservation and is more directly related to semibounded operators considered in this article. Nevertheless, since numerical methods for hyperbolic conservation laws motivate this study, general SSP methods will be considered.

**Remark 2.7.** For linear and time-independent ODEs (2.1) with semibounded $g$, some strong stability properties have been obtained by Ranocha and Öffner [42] and Sun and Shu [48]. Thus, it is interesting whether similar results can be established under the assumption (2.13) with $M \leq 0$. In order to restrict the stiffness of the ODE (2.1), a Lipschitz condition will be assumed, i.e. $\|g\|_{\text{Lip}} \leq L$. Since there are many negative results even for autonomous problems, (2.1) will be considered instead of (2.8).

### 2.3 Runge–Kutta Methods

A general (explicit or implicit) Runge–Kutta method with $s$ stages can be described by its Butcher tableau [7, 20]

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
\mathbf{b} \\
\end{array}
$$

(2.14)

where $A \in \mathbb{R}^{s \times s}$ and $b, c \in \mathbb{R}^s$. Since (2.1) is an autonomous ODE, there is no explicit dependency on time and one step from $u_0$ to $u_+$ is given by

$$
u_i = u_0 + \Delta t \sum_{j=1}^{s} a_{ij} g(u_j), \quad u_+ = u_0 + \Delta t \sum_{i=1}^{s} b_i g(u_i).

(2.15)
$$

Here, $u_i$ are the stage values of the Runge–Kutta method. It is also possible to express the method via the slopes $k_i = g(u_i)$ [19, Definition II.1.1].

Using the stage values $u_i$ as in (2.15), the change of squared norm (“energy”) is given by

$$
\|u_+\|^2 - \|u_0\|^2 = 2\Delta t \left( u_0, \sum_{i=1}^{s} b_i g(u_i) \right) + (\Delta t)^2 \left\| \sum_{i=1}^{s} b_i g(u_i) \right\|^2

(2.16)
$$

$$
\overset{(2.15)}{=} 2\Delta t \sum_{i=1}^{s} b_i \left( u_i - \Delta t \sum_{j=1}^{s} a_{ij} g(u_j), g(u_i) \right) + (\Delta t)^2 \left\| \sum_{i=1}^{s} b_i g(u_i) \right\|^2
$$

$$
= 2\Delta t \sum_{i=1}^{s} b_i \left( u_i, g(u_i) \right) + (\Delta t)^2 \left( \sum_{i=1}^{s} b_i g(u_i) \right)^2 - 2 \sum_{i,j=1}^{s} b_i a_{ij} \left( g(u_i), g(u_j) \right)
$$

$$
= 2\Delta t \sum_{i=1}^{s} b_i \left( u_i, g(u_i) \right) + (\Delta t)^2 \left\{ \sum_{i,j=1}^{s} \left( b_i b_j - b_i a_{ij} - b_j a_{ji} \right) \left( g(u_i), g(u_j) \right) \right\},
$$

where the symmetry of the scalar product has been used in the last step. Here, the first term on the right hand side is consistent with $\int_{t_0}^{t_0+\Delta t} 2 \left( u(t), g(u(t)) \right) dt$, if the Runge–Kutta method...
is consistent, i.e. \( \sum_{i=1}^{s} b_i = 1 \). Additionally, it can be estimated via the semiboundedness of \( g \) if all \( b_i \) are non-negative.

The second term of order \((\Delta t)^2\) is undesired. Depending on the method (and the stages, of course), it may be positive or negative. However, if it is positive, then a stability error may be introduced.

As a special case, if the method fulfills \( b_i b_j = b_ia_{ij} + b_ja_{ji}, \; i, j \in \{1, \ldots, s\} \), this term vanishes. These methods can conserve quadratic invariants of ordinary differential equations, a topic of geometric numerical integration, see Theorem IV.2.2 of Hairer, Lubich and Wanner [19], originally proved by Cooper [10]. A special kind of these methods are the implicit Gauß methods [19, Section II.1.3].

More generally, the \((\Delta t)^2\) term is non-positive if the matrix with entries \((b_i b_j - b_i a_{ij} - b_j a_{ji})_{i,j}\) is negative semidefinite (and \( b_i \geq 0 \), as before), i.e. when the Runge–Kutta method is algebraically stable. Then, the Runge–Kutta method is strongly stable in the \( L^2 \) norm for every time step \( \Delta t > 0 \), i.e. \( B \) stable, cf. [7, section 357]. While there are Runge–Kutta methods with these nice stability properties, these are all implicit.

**Remark 2.8.** Applying explicit methods to (2.1), it can be expected that time step restrictions for strong stability depend on boundedness or Lipschitz constants of \( g \), e.g. \( \Delta t \leq \Delta t_{\text{max}} \propto L^{-1} \). Hence, such restrictions on \( g \) will be used in the following.

The following result will be used in the next sections, cf. [17, Observation 5.2] or [29].

**Lemma 2.1.** Any Runge–Kutta method with positive SSP coefficient \( c > 0 \) has non-negative coefficients and weights, i.e. \( \forall i, j : a_{ij} \geq 0, b_i \geq 0 \).

That the coefficients \( a_{ij}, b_i \) of the schemes are non-negative can also be obtained under other conditions focusing on circle contractivity, cf. [22]. This implies certain restrictions on the possible order of the schemes, cf. [29] or [17, Section 5.1].

### 3 Explicit Methods with Three Stages

In this section, explicit Runge–Kutta methods with three stages are considered. Thus, the corresponding coefficients are

\[
  a_{21}, a_{31}, a_{32}, b_1, b_2, b_3. 
\]

(3.1)

Since the proofs of the negative results obtained in this section are easier if fewer coefficients are considered, third order methods will be investigated at first. Thereafter, schemes of at least second order of accuracy are studied.

The usual conditions for second order accurate Runge–Kutta methods used later are \( \sum_{j=1}^{s} b_j = 1 \) and \( \sum_{j,k=1}^{s} b_j a_{jk} = 1/2 \). Third order schemes have to fulfill \( \sum_{j,k,l=1}^{s} b_j a_{jk} a_{kl} = 1/3 \) and \( \sum_{j,k,l=1}^{s} b_j a_{jk} a_{kl} = 1/6 \) additionally [20, Section II.2].

The basic approach to get negative results can be described as follows. Certain test problems (2.1) using specific semibounded \( g \) are constructed such that the norm of the numerical solution increases during the first time step for each \( \Delta t \in (0, \Delta t_{\text{max}}] \). Then, this result can be transferred to semibounded \( g \) with \( |g|_{\text{Lip}} \leq L \) by considering suitable modifications outside of a bounded region using Kirszbraun’s theorem [44, Theorem 1.31]:

**Theorem 3.1** (Kirszbraun). Suppose \( S \) is a subset of the Hilbert space \( \mathcal{H} \) and \( g : S \rightarrow H \) is Lipschitz continuous. Then, \( g \) can be extended to all of \( \mathcal{H} \) in such a way that the extension satisfies the same Lipschitz condition.

#### 3.1 Third Order Methods

**Theorem 3.2.** There is no explicit Runge–Kutta method that

- is strong stability preserving with positive SSP coefficient,
- is of third order of accuracy & has at most three stages,
• and is strongly stable for (2.1) for all smooth and semibounded \(g\) with \(\|g\|_{\text{Lip}} \leq L\).

To prove Theorem 3.2, the initial value problem (2.1) with

\[
  u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad g(u) = \alpha(u_1 - ru_2) \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

(3.2)

will be used, where \(r\) is a real parameter, \(\alpha > 0\), and \(u_1, u_2\) are real valued functions. Since \(g\) is given by polynomials in \(u\), the squared norm after one step can be calculated explicitly and is a polynomial in the time step \(\Delta t\).

**Lemma 3.1.** Applying an explicit third order Runge–Kutta method with three stages given by the parameters (3.1) to the ODE (2.1) with (3.2) yields \(\|u_+\|^2 - \|u_0\|^2 = \Delta t^4 p(\Delta t)\), where \(p(\Delta t)\) is a polynomial of the form

\[
p(\Delta t) = \frac{\alpha^4}{12} \left(-5 + 7r^2 + (a_{31} + a_{32})(4 - 8r^2)\right) + O(\Delta t),
\]

(3.3)

Lemma 3.1 can be proved by direct but tedious calculations and has been verified using Mathematica [53].

**Lemma 3.2.** An explicit third order Runge–Kutta method with three stages given by the parameters (3.1) that is strongly stable for (2.1) for all smooth and semibounded \(g\) with \(\|g\|_{\text{Lip}} \leq L\) satisfies

\[
  \frac{7}{8} \leq a_{31} + a_{32} \leq \frac{5}{4}.
\]

(3.4)

**Proof.** In order to be strongly stable for the ODE (2.1) with (3.2), the coefficient of the constant term of the polynomial \(p(\Delta t)\) given in Lemma 3.1 has to be non-positive, i.e.

\[
  \frac{12}{\alpha^4} p(0) = -5 + 7r^2 + (a_{31} + a_{32})(4 - 8r^2) \leq 0.
\]

This can be reformulated as

\[
  \begin{align*}
    a_{31} + a_{32} &\leq \frac{5 - 7r^2}{4 - 8r^2}, & \text{if } r^2 < \frac{1}{2} \\
    a_{31} + a_{32} &\geq \frac{5 - 7r^2}{4 - 8r^2}, & \text{if } r^2 > \frac{1}{2}.
  \end{align*}
\]

(3.5)

Basically, the assertion is proved by letting \(r \to 0\) in the first inequality and \(r \to 0\) in the second one. To satisfy the upper bound on the Lipschitz constant, \(\alpha \to 0\) can be coupled with the limiting process on \(r\): In this way, the local Lipschitz constant of \(g\) (3.2) around \(u_0\) can be made arbitrarily small without changing the results of Lemma 3.2. Since only one time step is considered, \(g\) can be modified outside of a suitable neighbourhood of \(u_0\) while keeping the local Lipschitz constant of \(g\) as global Lipschitz constant because of Kirszbraun’s theorem. □

These technical results can be used to prove Theorem 3.2 as follows.

**Proof of Theorem 3.2.** The general solution of the order conditions for third order explicit Runge–Kutta methods with three stages is given by the two parameter family

\[
  a_{21} = a_2, \quad b_1 = 1 + \frac{2 - 3(a_2 + a_3)}{6a_2a_3},
\]

\[
  a_{31} = \frac{3a_2a_3(1 - a_2) - a_2^2}{a_2(2 - 3a_2)}, \quad b_2 = \frac{3a_3 - 2}{6a_2(a_3 - a_2)},
\]

\[
  a_{32} = \frac{a_3(a_3 - a_2)}{a_2(2 - 3a_2)}, \quad b_3 = \frac{2 - 3a_2}{6a_3(a_3 - a_2)}.
\]

(3.6)
where \(\alpha_2, \alpha_3 \neq 0, \alpha_2 \neq \alpha_3, \alpha_2 \neq 2/3\) and the two one parameter families

\[
\begin{align*}
\alpha_2 & = 2/3, \\
\alpha_3 & = 0, \\
\alpha_2 & = 2/3, \\
\alpha_3 & = 2/3,
\end{align*}
\]

where \(\alpha_2 = \alpha_3 = 2/3\) and

\[
\begin{align*}
\alpha_2 & = 2/3, \\
\alpha_3 & = 0, \\
\alpha_2 & = 2/3,
\end{align*}
\]

where \(\alpha_3 = 0\), cf. [40]. Thus, it suffices to check each case separately.

Clearly, then necessary condition \(\alpha_3 + \alpha_2 \geq 7/8\) of Lemma 3.2 is violated for both one parameter families. Thus, it suffices to study the two parameter family (3.6) for all possible cases.

Due to Lemma 2.1, all coefficients \(a_{ij}, b_i\) have to be non-negative for an SSP method. In particular, \(\alpha_2 \geq 0\). Due to Lemma 3.2, \(\alpha_3 = \alpha_3 + \alpha_2 \geq 7/8\), \(\alpha_3 \leq 5/4\).

- \(0 < \alpha_2 < 2/3, 7/8 \leq \alpha_3 \leq 5/4\).
  In this case, \(2 - 3\alpha_2 > 0\) and

\[
\alpha_3 = \frac{1}{\alpha_2(2 - 3\alpha_2)} \left( \alpha_3 \left( \frac{3\alpha_2(1 - \alpha_2)}{4} - \alpha_3 \right) \right) < 0.
\]

Thus, this case is excluded.

- \(2/3 < \alpha_2, 7/8 \leq \alpha_3 \leq 5/4, \alpha_2 \neq \alpha_3\).
  In this case, \(2 - 3\alpha_2 < 0\). Since

\[
\alpha_3 = \frac{\alpha_3}{\alpha_2(2 - 3\alpha_2)} (\alpha_3 - \alpha_2),
\]

the condition \(\alpha_3 \geq 0\) is equivalent to \(\alpha_3 < \alpha_2\). However, due to

\[
b_2 = \frac{3\alpha_3 - 2}{6\alpha_2} \frac{1}{(\alpha_3 - \alpha_2)},
\]

\(b_2 \geq 0\) requires \(\alpha_3 > \alpha_2\), contradicting the requirement for \(\alpha_3 \geq 0\), Hence, this case is also excluded.

This proves Theorem 3.2.

\[\square\]

### 3.2 Schemes of at Least Second Order of Accuracy

A generalisation of Theorem 3.2 is

**Theorem 3.3.** There is no explicit Runge–Kutta method that

- is strong stability preserving with positive SSP coefficient,
- is of at least second order of accuracy & has at most three stages,
• and is strongly stable for (2.1) for all smooth and semibounded \( g \) with \( |\mathcal{S}|_{\text{Lip}} \leq L \).

The basic idea of the proof of Theorem 3.3 is the same as for the proof of Theorem 3.2. However, the technical details are a bit more complicated.

As before, the initial value problem (2.1) with (3.2) will be used, where \( r \) is a real parameter, \( \alpha > 0 \), and \( u_1, u_2 \) are real valued functions.

**Lemma 3.3.** Applying an explicit three stage Runge–Kutta method with at least second order of accuracy given by the parameters (3.1) to the ODE (2.1) with (3.2) yields \( \|u_+\|^2 - \|u_0\|^2 = \Delta t^3 p(\Delta t) \), where \( p(\Delta t) \) is a polynomial of the form

\[
p(\Delta t) = \left(-1 + a_{21} - 2a_{21}a_{31}b_3 + 2a_{31}^2 b_3 + 4a_{31}a_{32}b_3 + 2a_{32}^2 b_3\right) \alpha^3 \\
+ \frac{1}{4} \left(1 + r^2 - 8a_{21}a_{32}b_3(2 - r^2 + a_{31}(-1 + 2r^2) + a_{32}(-1 + 2r^2))\right) \Delta t \alpha^4 + O(\Delta t^2). \quad (3.12)
\]

Since the restrictions imposed by (3.2) do not seem to suffice to prove Theorem 3.3, the initial value problem (2.1) with

\[
u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad g(u) = \alpha(ru_1 - u_2) \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.13)
\]

will be used additionally, where \( r \) is again a real parameter and \( \alpha > 0 \).

**Lemma 3.4.** Applying an explicit three stage Runge–Kutta method with at least second order of accuracy given by the parameters (3.1) to the ODE (2.1) with (3.13) yields \( \|u_+\|^2 - \|u_0\|^2 = \Delta t^3 p(\Delta t) \), where \( p(\Delta t) \) is a polynomial of the form

\[
p(\Delta t) = \left(-1 + a_{21} - 2a_{21}a_{31}b_3 + 2a_{31}^2 b_3 + 4a_{31}a_{32}b_3 + 2a_{32}^2 b_3\right) r^2 \alpha^3 \\
+ \frac{1}{4} r^2 \left(1 + r^2 - 8a_{21}a_{32}b_3(1 - 2r^2 + (a_{31} + a_{32})(-2 + r^2))\right) \Delta t \alpha^4 + O(\Delta t^2). \quad (3.14)
\]

Both Lemma 3.3 and Lemma 3.4 can be proved by direct but tedious calculations and have been verified using Mathematica [53].

These technical results can be used to prove Theorem 3.3 as follows.

**Proof of Theorem 3.3.** For \( s = 3 \) stages, the conditions for an order of accuracy 2 are

\[
\sum_{j=1}^{s} b_j = 1, \quad \sum_{j,k=1}^{s} b_j a_{jk} = \frac{1}{2}. \quad (3.15a)
\]

As in Lemma 3.2, the coefficient of the constant term of the polynomial \( p(\Delta t) \) in Lemma 3.3 has to be non-positive for all \( r \in \mathbb{R} \). Since \( p(0) = (-1 + a_{21} - 2a_{21}a_{31}b_3 + 2a_{31}^2 b_3 + 4a_{31}a_{32}b_3 + 2a_{32}^2 b_3) \alpha^3 \), this implies

\[
-1 + a_{21} - 2a_{21}a_{31}b_3 + 2a_{31}^2 b_3 + 4a_{31}a_{32}b_3 + 2a_{32}^2 b_3 = 0. \quad (3.15b)
\]

Furthermore, Lemma 3.3 (i.e. the right hand side (3.2)) yields the condition

\[
\forall r \in \mathbb{R}: \quad 1 + r^2 - 8a_{21}a_{32}b_3(2 - r^2 + (a_{31} + a_{32})(-1 + 2r^2)) \leq 0. \quad (3.15c)
\]

Similarly, Lemma 3.4 (i.e. the right hand side (3.13)) yields the condition

\[
\forall r \in \mathbb{R}: \quad r^2 \left(1 + r^2 - 8a_{21}a_{32}b_3(1 - 2r^2 + (a_{31} + a_{32})(-2 + r^2))\right) \leq 0. \quad (3.15d)
\]

As in the proof of Lemma 3.2, \( \alpha > 0 \) can be adapted to \( r \) such that the local Lipschitz constant of \( g \) near \( u_0 \) is as small as desired. Outside of such a neighbourhood, \( g \) can be modified to keep this Lipschitz constant (Kirschbraun’s theorem).

Finally, Lemma 2.1 yields the conditions

\[
a_{21} \geq 0, \quad a_{31} \geq 0, \quad a_{32} \geq 0, \quad b_1 \geq 0, \quad b_2 \geq 0, \quad b_3 \geq 0. \quad (3.15e)
\]
Applying the function Reduce of Mathematica [53] to equations (3.15a) to (3.15e) yields the single possibility
\[ a_{21} = \frac{1}{2}, \quad a_{31} = 0, \quad a_{32} = 1, \quad b_{1} = \frac{1}{4}, \quad b_{2} = \frac{1}{2}, \quad b_{3} = \frac{1}{4}. \] (3.16)

This scheme is not strongly stable for the ODE (2.1) with \( g(u) = Lu \), where \( L \) is a general linear and skew-symmetric operator. This can be seen by considering the classical stability region of this scheme. Indeed, the stability function is \( R(z) = \det(1-zA + z1b^T) = 1 + z + \frac{1}{2}z^2 + \frac{1}{8}z^3 \). Considering \( z = y_1 \) with \( y \in \mathbb{R} \) yields
\[ |R(y_1)|^2 = \left| 1 + y_1 - \frac{1}{2}y_2^2 - \frac{1}{8}y_3 \right|^2 = \left( 1 - \frac{1}{2}y_2^2 \right)^2 + \left( y - \frac{1}{8}y_3 \right)^2 = 1 + \frac{1}{64}y^6. \] (3.17)

Hence, \( |R(y_1)| > 1 \) for \( y \neq 0 \) and the scheme (3.16) is not strongly stable in general. This proves Theorem 3.3. \( \square \)

4 Some Known Explicit Methods

Since the general explicit Runge–Kutta with more than three stages has an increased number of coefficients, an approach similar to the one in the previous section is not really feasible. Therefore, some specific methods with up to ten stages will be studied in this section.

As before, the impossibility results will be obtained using some specifically designed test problems. In the following, the ODE (2.1) with
\[ u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad g(u) = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \] (4.1)
will be used, i.e. (3.2) or (3.13) with \( r = 1 \).

4.1 Second Order Methods

The unique second order explicit SSP Runge–Kutta method SSPRK(s,2) with \( s \geq 2 \) stages and optimal (maximal) SSP coefficient is given by the Butcher coefficients [29, Theorem 9.3]
\[ a_{i,j} = \frac{1}{s-1}, \quad b_{i} = \frac{1}{s}, \quad \forall i, j \in \{1, \ldots, s\}, j < i. \] (4.2)

These schemes can be implemented in a low storage form as [27]
\[ u_k = u_{k-1} + \frac{\Delta t}{s-1}g(u_{k-1}), \quad k \in \{1, \ldots, s\}, \]
\[ u_+ = \frac{s-1}{s}u_{s} + \frac{1}{s}u_0. \] (4.3)

**Theorem 4.1.** The second order explicit SSP Runge–Kutta methods SSPRK(s,2), \( s \geq 2 \), of [29, Theorem 9.3] are not strongly stable for the ODE (2.1) for all smooth and semibounded \( g \) with \( \|g\|_{\text{Lip}} \leq L. \)

In order to prove Theorem 4.1, the following technical result will be used.

**Lemma 4.1.** For the ODE (2.1) with parameters (4.1), the stages \( u_k, k \in \{0, \ldots, s\} \), in (4.3) satisfy
\[ u_{k,1} = 1 - \frac{k(k-1)}{2}\left(\frac{\Delta t}{s-1}\right)^2 + \frac{k(k-1)^2}{2}\left(\frac{\Delta t}{s-1}\right)^3 - \frac{(k+1)(k(k-1)(k-2))}{12}\frac{\Delta t}{s-1} + O(\Delta t^5), \]
\[ u_{k,2} = k\left(\frac{\Delta t}{s-1}\right) - \frac{k(k-1)}{2}\left(\frac{\Delta t}{s-1}\right)^2 - \frac{k(k-1)(k-2)}{6}\frac{\Delta t}{s-1}^3 + \frac{(5k-7)k(k-1)(k-2)}{12}\frac{\Delta t}{s-1}^4 + O(\Delta t^5). \] (4.4)
Proof. Since $u_0 = (1, 0)$, the result is true for $k = 0$. Assuming the result holds for $k$, inserting (4.1) into (4.3) proves the result for $k + 1$ and thus for general $k \in \{0, \ldots, s\}$. \qedsymbol

Proof of Theorem 4.1. Consider the ODE (2.1) with parameters (4.1). Using Lemma 4.1,

$$
\|u_+\|^2 - \|u\|_2^2 = \left(\frac{s-1}{s} u_{s,1} + \frac{1}{s}\right)^2 + \left(\frac{s-1}{s} u_{s,2}\right)^2 - 1
$$

$$
= \left(1 - \frac{1}{2} \Delta t^2 + \frac{1}{2} \Delta t^3 - \frac{(s+1)(s-2)}{12(s-1)^2} \Delta t^4\right)^2 - 1 + O(\Delta t^5)
$$

$$
+ \left(\Delta t - \frac{1}{2} \Delta t^2 - \frac{s-2}{6(s-1)} \Delta t^3 + \frac{(5s-7)(s-2)}{12(s-1)^2} \Delta t^4\right)^2 = -\frac{s+1}{6(s-1)^2} \Delta t^4 + O(\Delta t^5).
$$

(4.5)

Since $(s+1)/(6(s-1)^2) > 0$ for $s \geq 2, \|u_+\|^2 > \|u\|_2^2$ for small $\Delta t > 0$ and Theorem 4.1 is proved by applying the same arguments as in the proofs given hitherto to reduce the Lipschitz constant as desired. \qedsymbol

4.2 Third Order Methods

There is also a family of third order SSP methods with optimal SSP coefficient and $s = n^2$ stages for $n \in \mathbb{N}, n \geq 2$ [27, Theorem 3]. This family contains the method SSPrK(4, 3) of Kraaijevanger [29, Theorem 9.5]. The schemes of this family can be implemented in low storage form as [27]

$$
u_0 = \frac{n}{2n-1} u_{(n-1)^2} + \frac{n-1}{2n-1} u_{n^2},$$

$$
u_k = \nu_{k-1} + \frac{\Delta t}{n(n-1)} g(u_{k-1}), \quad k \in \left\{1, \ldots, \frac{n(n+1)}{2}\right\},$$

$$
u_+ = \nu_{n(n-1)}.
$$

Theorem 4.2. The third order explicit SSP Runge–Kutta methods SSPrK(n^2, 3), $n \geq 2$, of [27, Theorem 3] are not strongly stable for the ODE (2.1) for all smooth and semibounded $g$ with $\|s\|_{\text{Lip}} \leq L$.

In order to prove Theorem 4.2, the following technical results will be used.

Lemma 4.2. For the ODE (2.1) with parameters (4.1), the stages $u_k, k \in \left\{0, \ldots, \frac{n(n+1)}{2}\right\}$, in (4.6) satisfy

$$
u_{k,1} = 1 - \frac{k(k-1)}{2n^2(n-1)^2} \Delta t^2 + \frac{k(k-1)^2}{2n^3(n-1)^3} \Delta t^3
$$

$$
- \frac{(k+1)k(k-1)(k-2)}{12n^4(n-1)^4} \Delta t^4 - \frac{(3k-7)(k-1)^2(k-2)}{12n^5(n-1)^5} \Delta t^5 + O(\Delta t^6),
$$

$$
u_{k,2} = \frac{k}{n(n-1)} \Delta t - \frac{k(k-1)}{2n^2(n-1)^2} \Delta t^2 - \frac{k(k-1)(k-2)}{6n^3(n-1)^3} \Delta t^3
$$

$$
+ \frac{(5k-7)(k-1)(k-2)}{12n^4(n-1)^4} \Delta t^4 - \frac{(13k^2 - 41k + 26)(k-1)(k-2)}{60n^5(n-1)^5} \Delta t^5 + O(\Delta t^6).
$$

(4.7)

Proof. Since $u_0 = (1, 0)$, the result is true for $k = 0$. Assuming the result holds for $k$, inserting (4.1) into (4.6) proves the result for $k + 1$ and thus for general $k \in \left\{0, \ldots, \frac{n(n+1)}{2}\right\}$. \qedsymbol

Lemma 4.3. For the ODE (2.1) with parameters (4.1), the stages $v_k, k \in \left\{0, \ldots, \frac{n(n-1)}{2}\right\}$, in (4.6) satisfy

$$
\nu_{k,1} = 1 - \frac{4k^2 + k(-4n^2 + 4n + 4) - n(n^3 - 2n^2 + 3n - 2)}{8(n-1)^2n^2} \Delta t^2
$$

(4.8)
\[+
8k^3 + 4k^2(3n^2 - 3n - 4) + 2k(3n^4 - 6n^3 - n^2 + 4n + 4)
\]
\[+ n(n^5 - 3n^4 + 11n^3 - 17n^2 + 4n + 4)\frac{1}{16(n - 1)^3n^3}\Delta t^3
\]
\[+ (16k^2 + 32k^3(n^2 - n - 1) + 8k^2(3n^4 - 6n^3 - 3n^2 + 6n - 2)
\]
\[+ 8k(n^6 - 3n^5 + 5n^3 + 3n^2 - 6n + 4)
\]
\[+ n(n^7 - 4n^6 + 26n^5 - 64n^4 + 57n^3 - 12n^2 - 20n + 16)\frac{1}{192(n - 1)^4n^4}\Delta t^4
\]
\[+ (96k^5 + 16k^2(15n^2 - 15n - 38) + 16k^3(15n^4 - 30n^3 - 41n^2 + 56n + 86)
\]
\[+ 8k^2(15n^6 - 45n^5 + 27n^4 + 21n^3 - 96n^2 - 114n - 164)
\]
\[+ 2k(15n^8 - 60n^7 + 178n^6 - 324n^5 + 11n^4 + 448n^3 - 284n^2 + 16n + 224)
\]
\[+ n(3n^9 - 15n^8 + 112n^7 - 358n^6 + 247n^5 + 449n^4 - 354n^3 - 428n^2 + 120n + 224)rac{1}{384(n - 1)^5n^5}\Delta t^5 + O(\Delta t^6),
\]

And

\[v_{k,2} = \frac{2}{2(n - 1)n} \Delta t + \frac{4k^2 + k(-4n^2 + 4n + 4) - n(n^3 - 2n^2 + 3n - 2)\Delta t^2}{8(n - 1)^2n^2}
\]
\[- 8k^3 + 12k^2(n^2 - n - 2) + 2k(3n^4 - 6n^3 + 3n^2 + 8)
\]
\[+ n(n^5 - 3n^4 + 9n^3 - 13n^2 - 2n + 8)\frac{1}{48(n - 1)^3n^3}\Delta t^3
\]
\[+ (80k^4 + 32k^3(5n^2 - 5n - 11) + 8k^2(15n^4 - 30n^3 - 27n^2 + 42n + 62)
\]
\[+ 8k(5n^6 - 15n^5 + 30n^4 - 35n^3 + 21n^2 - 6n - 28)
\]
\[+ n(5n^7 - 20n^6 + 106n^5 - 248n^4 + 69n^3 + 252n^2 - 52n - 112)\frac{1}{192(n - 1)^4n^4}\Delta t^4
\]
\[- (416k^5 + 80k^4(13n^2 - 13n - 32) + 80k^3(13n^4 - 26n^3 - 39n^2 + 52n + 70)
\]
\[+ 40k^2(13n^6 - 39n^5 + 15n^4 + 35n^3 + 98n^2 - 122n - 128)
\]
\[+ 2k(65n^8 - 260n^7 + 950n^6 - 1940n^5 + 725n^4 + 1480n^3 - 1500n^2 + 480n + 832)
\]
\[+ n(13n^9 - 65n^8 + 490n^7 - 1570n^6 + 1165n^5 + 1727n^4 - 1508n^3 - 1564n^2
\]
\[+ 480n + 832)\frac{1}{1920(n - 1)^5n^5}\Delta t^5 + O(\Delta t^6).
\]

**Proof.** Using Lemma 4.2, the result can be verified for \(k = 0\). Assuming that the result holds for \(k\), inserting (4.1) into (4.6) proves the result for \(k + 1\) and thus for general \(k \in \left\{0, \ldots, \frac{n(n-1)}{2}\right\}\). \(\square\)

**Proof of Theorem 4.2.** Using Lemma 4.3,

\[\|u_+\|^2 - \|u_0\|^2 = \frac{n^2 - n - 2}{12n^2(n - 1)^2}\Delta t^4 + \frac{n^2 - n + 3}{6n^2(n - 1)^2}\Delta t^5 + O(\Delta t^6), \tag{4.10}
\]

For \(n = 2\), \(\frac{n^2 - n - 2}{12n^2(n - 1)^2} = 0\) and \(\frac{n^2 - n + 3}{6n^2(n - 1)^2} > 0\). For \(n \geq 3\), \(\frac{n^2 - n - 2}{12n^2(n - 1)^2} > 0\). Thus, \(\|u_+\|^2 > \|u_0\|^2\) for small \(\Delta t > 0\) and Theorem 4.2 is proved by applying the same arguments as in the proofs given hitherto to reduce the Lipschitz constant as desired. \(\square\)
4.3 Ten Stage, Fourth Order Method SSPRK(10,4)

The explicit strong-stability preserving method SSPRK(10,4) of Ketcheson [27] is given by the Butcher tableau

\[
\begin{array}{cccccccccccc}
0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/10 & 1/10 & 1/10 & 1/10 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/10 & 1/10 & 1/10 & 1/10 \\
1/3 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/10 & 1/10 & 1/10 & 1/10 \\
1/2 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/10 & 1/10 & 1/10 & 1/10 \\
2/3 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/10 & 1/10 & 1/10 & 1/10 \\
1/3 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/10 & 1/10 & 1/10 & 1/10 \\
1/2 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/10 & 1/10 & 1/10 & 1/10 \\
2/3 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/10 & 1/10 & 1/10 & 1/10 \\
5/6 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/10 & 1/10 & 1/10 & 1/10 \\
1 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/10 & 1/10 & 1/10 & 1/10 \\
\end{array}
\]

which is not sparse in the sense of many zeros, but “data sparse” in the sense of a clear structure with few different values of the entries. This results in a sparse Shu-Osher form and the low-storage implementation

\[
\begin{align*}
\begin{array}{c}
0 := u_0, \\
u_1 := u_{i-1} + \frac{\Delta t}{6} g(u_{i-1}), \quad i \in \{2, 3, 4, 5\}, \\
u_6 := \frac{3}{5} u_0 + \frac{2}{5} \left( u_5 + \frac{\Delta t}{6} g(u_5) \right), \\
u_i := u_{i-1} + \frac{\Delta t}{6} g(u_{i-1}), \quad i \in \{7, 8, 9, 10\}, \\
u_+ := \frac{1}{25} u_0 + \frac{9}{25} \left( u_5 + \frac{\Delta t}{6} g(u_5) \right) + \frac{3}{5} \left( u_{10} + \frac{\Delta t}{6} g(u_{10}) \right).
\end{array}
\end{align*}
\]

\[u_1 := u_0, \quad u_2 := u_1 + \Delta t g(u_1), \quad u_3 := \frac{3}{4} u_0 + \frac{1}{4} \left( u_2 + \Delta t g(u_2) \right) = u_0 + \frac{1}{4} \Delta t g(u_1) + \frac{1}{4} \Delta t g(u_2), \quad u_+ := \frac{1}{3} u_0 + \frac{2}{3} \left( u_3 + \Delta t g(u_3) \right) = u_0 + \frac{1}{6} \Delta t g(u_1) + \frac{1}{6} \Delta t g(u_2) + \frac{2}{3} \Delta t g(u_3).
\]

Theorem 4.3. The ten stage, fourth order, explicit strong stability preserving method SSPRK(10,4) of Ketcheson [27] is not strongly stable for the ODE (2.1) for all smooth and semibounded \( g \) with \( \| g \|_{\text{Lip}} \leq L \).

Proof. Consider the ODE (2.1) with parameters (4.1). A lengthy calculation that has been verified using Mathematica [53] shows that \( \| u_+ \|^2 - \| u_0 \|^2 = \Delta t^6 p(\Delta t) \), where \( p(\Delta t) \) is a polynomial in \( \Delta t \) with

\[p(\Delta t) = \frac{23}{3240} - \frac{1}{240} \Delta t + \frac{161}{29160} \Delta t^2 + O(\Delta t^3).
\]

Hence, there is some \( \tau > 0 \) such that \( \| u_+ \|^2 - \| u_0 \|^2 > 0 \) for all \( \Delta t \in (0, \tau) \). Thus, SSPRK(10,4) is not strongly stable for this test problem. The same arguments as in the proofs given hitherto can be used to reduce the Lipschitz constant as desired. \( \square \)

5 Three Stage, Third Order Method SSPRK(3,3)

The third-order explicit strong stability preserving Runge–Kutta method SSPRK(3,3) with three stages given by Shu and Osher [45] is determined by the Butcher tableau

\[
\begin{array}{cccc}
0 & 1 & 1/2 & 1/4 \\
1 & 1/2 & 1/4 & 1/4 \\
\end{array}
\]

and can be represented using the Shu-Osher form

\[
\begin{align*}
\begin{array}{c}
0 := u_0, \\
u_1 := u_1 + \Delta t g(u_1), \\
u_2 := \frac{3}{4} u_0 + \frac{1}{4} \left( u_2 + \Delta t g(u_2) \right) = u_0 + \frac{1}{4} \Delta t g(u_1) + \frac{1}{4} \Delta t g(u_2), \\
u_+ := \frac{1}{3} u_0 + \frac{2}{3} \left( u_3 + \Delta t g(u_3) \right) = u_0 + \frac{1}{6} \Delta t g(u_1) + \frac{1}{6} \Delta t g(u_2) + \frac{2}{3} \Delta t g(u_3).
\end{array}
\end{align*}
\]
Theorem 3.2 implies that SSPRK(3,3) is not strongly stable for general semibounded \( g \). Since this method is often used, it is considered for further stability investigations in this section. In particular, the following properties will be studied.

- The classical fourth order, four stage explicit Runge–Kutta method RK(4,4) is not strongly stable for general semibounded and linear \( g \). However, the method given by two consecutive steps of RK(4,4) is strongly stable for such \( g \), cf. [47]. Thus, it is of interest whether something similar is true for SSPRK(3,3) for general nonlinear semibounded \( g \).

- Even if SSPRK(3,3) is not strongly stable after a finite number of steps, the increase of the norm might still be bounded, cf. [23–25] for investigations of such a property when the explicit Euler method is assumed to be strongly stable. Although boundedness and monotoncity are equivalent in this context for a large class of Runge–Kutta methods [25], this result cannot be applied here directly. Hence, it is of interest to study the behaviour of SSPRK(3,3) for semibounded \( g \).

Of course, both properties are related in some way. In particular, it will be proven that there are semibounded \( g \) such that the norm of a numerical solution obtained using SSPRK(3,3) is monotonically increasing and unbounded. This implies that SSPRK(3,3) cannot be strongly stable after any finite number of steps.

**Theorem 5.1.** There are smooth and semibounded \( g \) with \( \| g \|_\text{Lip} \leq L \) and the property that the application of the three stage, third order explicit strong stability preserving Runge–Kutta method SSPRK(3,3) of [45] to the ODE (2.1) yields a numerical solution \( u_{\text{num}} \) such that the sequence \( \| u_{\text{num}}(n\Delta t) \|^2 \) is bounded from above by \( 3\alpha \) in \( \mathbb{R}^2 \setminus B_1(0) \).

As the proofs of the previous results, this one is based on the explicit construction of carefully designed test problems. Here, the ODE (2.1) with

\[
 u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\alpha}{\| u \|^2} \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix},
\]

will be considered for \( \alpha > 0 \). Since the norm of the numerical solution will be shown to increase monotonically and the norm of a smooth solutions remains constant, the function \( g \) could be modified to remove the singularity at zero and keep the Lipschitz constant as small as desired for a suitable choice of \( \alpha > 0 \). Indeed, for \( \| u \|, \| v \| \geq 1 \),

\[
 \| g(u) - g(v) \| 
\leq \alpha \frac{\| v \| - \| u \|}{\| u \| + \| v \|} + \| v \| - \| u \| \frac{\| v \|}{\| u \|} + \| v \| \frac{\| v \|}{\| u \|} - \| u \| \frac{\| v \|}{\| v \|}
\leq \frac{3\alpha}{\| u \| + \| v \|} \| u - v \|,
\]

showing that the Lipschitz constant of \( g \) is bounded from above by \( 3\alpha \) in \( \mathbb{R}^2 \setminus B_1(0) \).

**Proof of Theorem 5.1.** Consider one step of SSPRK(3,3) from \( u_0 = (u_{0,1}, u_{0,2}) \) to \( u_+ = (u_{+,1}, u_{+,2}) \). A lengthy calculation that has been verified using Mathematica [53] yields

\[
 \| u_+ \|^2 - \| u_0 \|^2 = f_{\alpha\Delta t}(\| u_0 \|^2),
\]

\[ f_{\Delta t}(x) = \Delta t^4 + \frac{\Delta t^4 + 96\Delta t^2 x^2 + 240x^4}{36x(\Delta t^2 + x^2)(\Delta t^4 + 12\Delta t^2 x^2 + 16x^4)}, \]

for \( \Delta t > 0 \). In particular, the squared norms of the numerical solution \( \| u_{\text{num}}(n\Delta t) \|^2 \) are given recursively by

\[
 \| u_{\text{num}}((n + 1)\Delta t) \|^2 = \| u_{\text{num}}(n\Delta t) \|^2 + f_{\alpha\Delta t}(\| u_{\text{num}}(n\Delta t) \|^2).
\]

Since \( f_{\Delta t}(x) > 0 \) for every \( \Delta t > 0 \) and \( x > 0 \), they increase monotonically with \( n \) if, e.g. \( \| u_0 \| = 1 \). If they were bounded, the sequence of squared norms would have a (positive) limit \( x \) satisfying \( f_{\alpha\Delta t}(x) = 0 \) which is impossible for \( \alpha\Delta t > 0 \).
Numerical solutions of (5.3) with $\alpha = 1$ and initial condition $u_0 = (1, 0)$ have been computed using SSPRK(3,3) implemented in DifferentialEquations.jl [39] in Julia v1.1.0 [4] using using floating point numbers with extended precision (BigFloat with setprecision(500)). As visualised in Figure 1, the energy increases monotonically for every time step $\Delta t > 0$, in accordance with Theorem 5.1.

Figure 1: Evolution of the energy of numerical solutions computed using SSPRK(3,3) with different time steps $\Delta t$.

**Remark 5.1.** In applications, only finite final times $T > 0$ are relevant. Hence, it can be interesting whether a bound of the form

$$\forall n \in \mathbb{N}, n\Delta t \leq T: \quad \|u(n\Delta t)\| \leq c_T$$

holds for some constant $c_T > 0$ depending on $T$. For explicit schemes, a useful bound seems to require an additional restriction of the time step $\Delta t$ because of stability issues. Choosing $0 < \Delta t < \Delta t_{\text{max}}$ small enough, such a bound will trivially hold if the scheme converges. However, it does not seem to be trivial to guarantee good estimates of $c_T$ and $\Delta t_{\text{max}}$ in general.

### 6 Ten Stage, Fourth Order Method SSPRK(10,4) and the Transport Equation

In this section, the method SSPRK(10,4) of [27] described in section 4.3 will be used to integrate a semidiscretisation of a hyperbolic conservation law in time. It will be demonstrated numerically that the energy (squared norm) of the solution increases for a wide range of positive time steps.

Consider the linear advection equation with periodic boundary conditions

$$\partial_t u(t, x) + \partial_x u(t, x) = 0, \quad t \in (0, T), x \in (x_L, x_R),$$

$$u(0, x) = u_0(x), \quad x \in (x_L, x_R),$$

$$u(t, x_L) = u(t, x_R), \quad t \in (0, T),$$

and the initial condition $u_0(x) = -\sin(\pi x)$ in the domain $(x_L, x_R) = (-1, 1)$. Using the $L^2$ entropy $\mathcal{U}(u) = \frac{1}{2} u^2$, the entropy flux is $F(u) = \frac{1}{2} u^2$ and the flux potential is $\psi(u) = \frac{1}{2} u^2$. Smooth solutions fulfil $\|u(t)\|_2^2 = \|u_0\|_2^2$ and the entropy inequality

$$\partial_t u(t, x)^2 + \partial_x u(t, x)^2 \leq 0$$

yields $\|u(t)\|_2^2 \leq \|u_0\|_2^2$ for general solutions, cf. [50, 52].

Recently, Abgrall [1] proposed a general method to make numerical schemes entropy conservative/stable. He described this approach using residual distribution schemes and explains
how some other frameworks can be recast in this way, see also [2, 3]. For nodal discontinuous Galerkin (DG) schemes in one space dimension, this approach will be adapted in the following.

Consider a general polynomial collocation approach using \( p + 1 \) nodes and polynomials of degree \( \leq p \) in an element \([x_{i-1}, x_i]\). Besides the choice of the nodes, the main ingredients are

- a mass matrix \( M \), approximating the \( L^2 \) scalar product via \( \int_{x_{i-1}}^{x_i} u(x)v(x) \, dx = \langle u, v \rangle_{L^2} \approx \langle u, v \rangle_{M} = u^T M v \).
- a derivative matrix \( D \), approximating the derivative \( \partial_x u \approx D u \).
- a restriction operator \( R \), performing interpolation to the boundary nodes \( x_{i-1}, x_i \) via \( R u = (u_L, u_R)^T \).
- a diagonal boundary matrix \( B = \text{diag} (-1, 1) \), giving the difference of boundary values as in the fundamental theorem of calculus.

If the mass matrix is exact for polynomials of degree \( \leq 2p - 1 \), the summation by parts property
\[
MD + DT M = R^T B R \tag{6.3}
\]
will be satisfied, cf. [13, 16, 21]. In that case, semidiscrete stability can be proven in many cases, as described in the review articles of Fernández, Hicken and Zingg [14] and Svärd and Nordström [49] and references cited therein.

A nodal DG semidiscretisation of the advection equation (6.1) will be performed as follows. At first, the domain \((x_L, x_R)\) is divided uniformly into \( N \) non-overlapping elements. Each element is mapped via an affine-linear mapping to the reference element \((-1, 1)\) and all computations are performed there. On each element, the semidiscretisation is
\[
\partial_t u + D u = -M^{-1} R^T B \left( f_{\text{num}} - R u \right), \tag{6.4}
\]
where the numerical flux will be the central flux \( f_{\text{num}}(u_-, u_+) = \frac{u_- + u_+}{2} \). This flux is entropy conservative for the \( L^2 \) entropy \( U(u) = \frac{u^2}{2} \). Thus, if SBP operators are used, e.g. via bases on Gauss or Lobatto Legendre nodes, the resulting semidiscretisation is entropy conservative. Indeed, using SBP operators yields
\[
\frac{u^T M \partial_t u}{2} = -\frac{u^T M D u}{2} - \frac{u^T R^T B f_{\text{num}}}{2} = \frac{u^T D u}{2} - \frac{u^T R^T B f_{\text{num}}}{2}.
\tag{6.5}
\]

where the SBP property (6.3) has been used in the second line. Writing the element index as an upper index and suppressing the index \(i\) for the \(i\)-th element, this can be rewritten as

\[
\frac{u^T M \partial_t u}{2} = -\frac{u^T R^T B f_{\text{num}}}{2} = \frac{u^T R^T B f_{\text{num}}}{2},
\tag{6.6}
\]

where \( F_{\text{num}} = (u_- + u_+)/2 \cdot f_{\text{num}}(u_-, u_+) - (\psi(u_-) + \psi(u_+))/2 \) is the entropy flux of Tadmor [52]. The basic idea of Abgrall [1] is to enforce (6.6) for any semidiscretisation via the
addition of a correction term $r$ on the left hand side of (6.4) that is consistent with zero and does not violate the conservation relation (using $D_1 = 0$)

$$1^T M \partial_t u = -1^T M D u - 1^T R^T B \left( f^{\text{num}} - R u \right) = -1^T R^T B f^{\text{num}}. \quad (6.7)$$

He proposes a correction term of the form

$$r = \alpha \left( u - \frac{1^T M u}{1^T M 1} \right), \quad \alpha = \frac{\mathcal{E}}{u^T M u - \frac{1^T M u^2}{1^T M 1}} \quad (6.8)$$

$$\mathcal{E} = 1^T R^T B f^{\text{num}} - u^T M D u - u^T R^T B \left( f^{\text{num}} - R u \right).$$

Indeed, the conservation relation (6.7) is left unchanged, since

$$1^T M t = \alpha \left( 1^T M u - \frac{1^T M u}{1^T M 1} \right) = 0. \quad (6.9)$$

Moreover, the entropy rate satisfies the desired equation (6.6), since

$$u^T M \partial_t u = -u^T M r - u^T M D u - u^T R^T B \left( f^{\text{num}} - R u \right) =$$

$$= -\alpha \left( u^T M u - \frac{1^T M u}{1^T M 1} \right) - u^T M D u - u^T R^T B \left( f^{\text{num}} - R u \right)$$

$$= -1^T R^T B f^{\text{num}} \quad (6.10)$$

If the denominator of $\alpha$ in (6.8) is zero, the numerical solution is constant in the element because of the Cauchy Schwarz inequality (since $1$ and $u$ are linearly dependent in that case). Then, the DG scheme reduces to a finite volume scheme using the numerical flux $f^{\text{num}}$ and is therefore entropy conservative/stable depending on $f^{\text{num}}$.

In the following, some numerical experiments will be conducted using nodal DG methods on equidistant nodes including the boundaries and diagonal mass matrices using the weights of the closed Newton Cotes quadrature formula (which are positive in the cases considered below). The domain is divided into $N = 16$ elements and polynomials of degree $\leq p = 3$ are applied. The initial condition is advanced one time step $\Delta t$ using SSPRK(10,4) of [27].

These methods have been implemented in Julia v0.6.4 [4] using floating point numbers with extended precision (BigFloat with setprecision(5000)). Using 500 different values of $\Delta t$ (with uniformly distributed logarithms), the discrete energy errors after one time step $\|u^{\text{num}}(\Delta t)\|_M - \|u_0\|_M$ (computed via the mass matrix $M$) are shown in Figure 2. As can be seen there, the discrete energy increases for every choice of $\Delta t > 0$. Moreover, the increase of the energy scales as $O(\Delta t^5)$, as expected for one time step of a fourth order Runge–Kutta method.

To sum up, a semidiscrete DG scheme with insufficient quadrature strength is made semidiscretely energy conservative following the approach of Abgrall [1]. Integrating the resulting ordinary differential equation in time with SSPRK(10,4) results in monotonically increasing energies, even for ridiculously small time steps $\Delta t > 0$.

### 7 First Order Schemes

In contrast to the negative results of the previous sections for explicit methods of at least second order of accuracy, there are first order schemes that are strongly stable. To prove this, it suffices
to consider schemes with two stages, i.e.

\begin{align}
   u_1 &= u_0, \\
   u_2 &= u_0 + a_2 \Delta t g(u_1), \\
   u_+ &= u_0 + b_1 \Delta t g(u_1) + b_2 \Delta t g(u_2).
\end{align}

(7.1)

**Theorem 7.1.** There are first order accurate explicit Runge–Kutta method with two stages (7.1) that are

- strong stability preserving

- and strongly stable for the ODE (2.1) with semibounded and Lipschitz continuous \( g \) with \( |g|_{\text{Lip}} \leq L \) under a time step constraint \( 0 < \Delta t \leq \Delta t_{\text{max}} \propto L^{-1} \).

**Proof.** Inserting (7.1) and using \( u_1 = u_0 \) yields

\begin{align}
   \|u_+\|^2 - \|u_0\|^2 &= 2\Delta t \langle u_0, b_1 g(u_1) + b_2 g(u_2) \rangle \\
   &\quad + \Delta t^2 \left( b_1^2 \|g(u_1)\|^2 + 2b_1b_2 \langle g(u_1), g(u_2) \rangle + b_2^2 \|g(u_2)\|^2 \right) \\
   &= 2b_1\Delta t \langle u_0, g(u_0) \rangle + 2b_1\Delta t \langle u_2 - a_2 \Delta t g(u_0), g(u_2) \rangle \\
   &\quad + \Delta t^2 \left( b_1^2 \|g(u_0)\|^2 + 2b_1b_2 \langle g(u_0), g(u_2) \rangle + b_2^2 \|g(u_2)\|^2 \right) \\
   &= 2b_1\Delta t \langle u_0, g(u_0) \rangle + 2b_2\Delta t \langle u_2, g(u_2) \rangle \\
   &\quad + \Delta t^2 \left( b_1^2 \|g(u_0)\|^2 + 2(b_1b_2 - b_2a_2) \langle g(u_0), g(u_2) \rangle + b_2^2 \|g(u_2)\|^2 \right).
\end{align}

(7.2)

Since \( g \) is semibounded, the inner products can be estimated as \( \langle u_i, g(u_i) \rangle \leq 0 \). Thus, the terms proportional to \( \Delta t \) can be estimated if \( b_1, b_2 \geq 0 \). Since the conditions for first order are \( b_1 + b_2 = 1 \), this becomes

\begin{align}
   0 &\leq b_2 \leq 1, \quad b_1 = 1 - b_2.
\end{align}

(7.3)
The terms multiplied by $\Delta t^2$ satisfy

\[
\begin{align*}
& b_1^2 \|g(u_0)\|^2 + 2(b_1 b_2 - b_2 a_{21}) \langle g(u_0), g(u_2) \rangle + b_2^2 \|g(u_2)\|^2 \\
= & \ b_1^2 \|g(u_0)\|^2 + 2(b_1 b_2 - b_2 a_{21}) \langle g(u_0), g(u_0) + g(u_2) - g(u_0) \rangle + b_2^2 \|g(u_0) + g(u_2) - g(u_0)\|^2 \\
= & \ (b_1^2 + 2(b_1 b_2 - b_2 a_{21}) + b_2^2) \|g(u_0)\|^2 \\
& + 2(b_1 b_2 - b_2 a_{21} + b_2^2) \langle g(u_0), g(u_2) - g(u_0) \rangle + b_2^2 \|g(u_2) - g(u_0)\|^2 \\
\leq & \ (b_1 + b_2)^2 - 2b_2 a_{21}) \|g(u_0)\|^2 \\
& + 2(b_1 b_2 - b_2 a_{21} + b_2^2) \|g(u_0)\| \|g(u_2) - g(u_0)\| + b_2^2 \|g(u_2) - g(u_0)\|^2 \\
\leq & \ (b_1 + b_2)^2 - 2b_2 a_{21}) \|g(u_0)\|^2 \\
& + 2(b_1 b_2 - b_2 a_{21} + b_2^2) |a_{21}| L \Delta t \|g(u_0)\|^2 + b_2^2 a_{21} L^2 \Delta t^2 \|g(u_0)\|^2.
\end{align*}
\] (7.4)

Inserting the order condition (7.3), the last expression can be written as

\[
\cdots \leq \ (1 - 2b_2 a_{21}) \|g(u_0)\|^2 + 2|1 - a_{21}| |b_2 a_{21}| L \Delta t \|g(u_0)\|^2 + |b_2 a_{21}|^2 L^2 \Delta t^2 \|g(u_0)\|^2.
\] (7.5)

If $g(u_0) = 0$, then $u_+ = u_2 = u_1 = u_0$ and strong stability is obvious. Otherwise, the term without $\Delta t$ is negative if

\[
1 - 2b_2 a_{21} < 0.
\] (7.6)

In that case, strong stability is achieved for sufficiently small $\Delta t$ (and the natural assumption $b_2 a_{21} L \neq 0$), since

\[
(1 - 2b_2 a_{21}) + 2|1 - a_{21}| |b_2 a_{21}| L \Delta t + |b_2 a_{21}|^2 L^2 \Delta t^2 \leq 0
\] (7.7)

for

\[
\Delta t \leq \sqrt{(1 - a_{21})^2 - (1 - 2b_2 a_{21}) - |1 - a_{21}|}{|b_2 a_{21}| L}.
\] (7.8)

Thus, there are strongly stable schemes.

Choosing for example

\[
b_1 = b_2 = \frac{1}{2}, \quad a_{21} = \frac{3}{2}.
\] (7.9)

the new value $u_+$ can be written as

\[
\begin{align*}
u_+ & = u_0 + \frac{1}{2} \Delta t g(u_0) + \frac{1}{2} \Delta t g(u_2) \\
& = \frac{3}{4} u_0 + \frac{1}{2} \Delta t g(u_0) + \frac{1}{4} \left( u_2 - \frac{3}{2} \Delta t g(u_0) \right) + \frac{1}{2} \Delta t g(u_2) \\
& = \frac{3}{4} \left( u_0 + \frac{1}{6} \Delta t g(u_0) \right) + \frac{1}{4} \left( u_2 + 2 \Delta t g(u_2) \right).
\end{align*}
\] (7.10)

Since $u_2 = u_0 + \frac{3}{2} \Delta t g(u_0)$, $u_+$ is a convex combination of explicit Euler steps with positive step sizes, the resulting scheme is strong stability preserving. □

\textbf{Remark 7.1.} Of course, the scheme constructed in the proof of Theorem 7.1 and the derived lower bound on the SSP coefficient are not optimal. However, since such first order schemes are not really relevant in practice, no attempt to optimise them has been made. △

\textbf{Remark 7.2.} Generalising the approach used in proof of Theorem 7.1, it can be expected that there are strongly stable schemes of first order if more stages are used. △

\textbf{Remark 7.3.} If a non-autonomous problem (2.8) is considered, the proof of Theorem 7.1 requires Lipschitz continuity in $(t, u)$ instead of continuity in $t$ and Lipschitz continuity in $u$ as required for the Picard-Lindelöf theorem, since $f(t_2, u_2) - f(t_0, u_0)$ has to be estimated. △

\textbf{Remark 7.4.} Since higher-order schemes satisfy $\|u_+\|^2 - \|u_0\|^2 = O(\Delta t^p)$ for $p > 2$ if $g$ is smooth, it does not seem to be (easily) possible to get similar estimate for higher order schemes using only Lipschitz continuity of $g$. △

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8 Summary and Discussion

In this article, strong stability of explicit SSP Runge–Kutta methods has been investigated. Many well-known and widespread high order schemes are not strongly stable for ODEs with general nonlinear, smooth, and semibounded operators with bounded Lipschitz constant, cf. Theorems 3.3, 4.1, 4.2, and 4.3. Moreover, it has been proven that the norms of the numerical solutions can even increase monotonically and without bounds for the popular three stage, third order method SSPRK(3,3) of [45], cf. Theorem 5.1. Additionally, it has been shown in section 6 that the ten stage, fourth order method SSPRK(10,4) of [27] can result in increasing norms of the solution for an energy stable and nonlinear semidiscretisation of a hyperbolic conservation law. Finally, it has been proven that such restrictions do not apply to first order Runge–Kutta methods, cf. Theorem 7.1. In particular, there are strongly stable SSP methods, even for nonlinear and semibounded operators that are Lipschitz continuous. In that case, strong stability can be guaranteed under a time step restriction \( \Delta t \leq \Delta t_{\text{max}} \), where \( \Delta t_{\text{max}} \) is proportional to the inverse of the Lipschitz constant of the right hand side.

It is well-known that implicit Runge–Kutta methods can have more favourable stability properties than explicit ones. In particular, there are strongly stable methods for general semibounded operators [7, sections 357–359]. Furthermore, summation by parts operators can be used to construct schemes with these properties [5, 34, 38], resulting e.g. in energy stable schemes for nonlinear equations [37]. This is also related to space-time discontinuous Galerkin schemes, where entropy stability can be obtained [15].

In this light, it seems interesting to investigate whether there are general possibilities to obtain strong stability of explicit Runge–Kutta methods by approximating the original problem, e.g. by adding sufficient artificial dissipation, cf. [54]. In the light of the current results, it might be conjectured that such a dissipative mechanism might be necessary to obtain strong stability with explicit methods. If this is possible, it is interesting to compare such schemes with fully implicit ones.

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