Determinant Formulas for the Toda and Discrete Toda Equations

Kenji Kajiwara\textsuperscript{a}, Tetsu Masuda\textsuperscript{a}, Masatoshi Noumi\textsuperscript{b}, Yasuhiro Ohta\textsuperscript{c} and Yasuhiko Yamada\textsuperscript{b}

\textsuperscript{a}: Department of Electrical Engineering, Doshisha University, Kyotanabe, Kyoto 610-0321, Japan
\textsuperscript{b}: Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan
\textsuperscript{c}: Department of Applied Mathematics Hiroshima University, Higashi-Hiroshima 739-8527, Japan

Abstract: Determinant formulas for the general solutions of the Toda and discrete Toda equations are presented. Application to the $\tau$ functions for the Painlevé equations is also discussed.

1 Introduction

In the works [1, 3, 4, 5, 6, 7, 8, 18], the determinant formulas for the $\tau$ functions of the Painlevé equations are obtained. These determinant formulas arise as a consequence of the Toda equation which describes Bäcklund (or Schlesinger) transformations of the Painlevé equations [4, 5, 6, 7]. They provide a proof of the miraculous polynomiality of the special polynomials arising as the special solutions of the Painlevé equations [19, 20].

Recently, it is clarified that these determinant formulas can be also applied not only for the special (classical) solutions but also for generic (transcendental) ones [20]. It is natural to expect the existence of such determinant formulas for the general solutions of the Toda equation independent of the Painlevé equations and their (special) solutions.

In this paper, we will give a formula of Hankel type determinant for the solution $\tau_n$ of the Toda equation (viewed as a recurrence relation)

$$\tau_n'' \tau_n - (\tau_n')^2 = \tau_{n-1} \tau_{n+1},$$

for general initial conditions $\tau_0$ and $\tau_1$. In the case of $\tau_0 = 1$, such a formula is known as Darboux’s formula (cf. [4]). As an application of the formula, we will
consider the $\tau$ functions of the Painlevé equations. We also present a similar determinant formula for the discrete Toda equation

$$\rho_{n-1} \rho_{n-1} - (\rho_n^l)^2 = \varepsilon^2 \rho_{n+1}^l \rho_{n-1}^l.$$ 

### 2 Determinant formulas

In this section, we present the determinant formulas for general solutions of Toda and discrete Toda equations.

We consider the following recursion relation of a sequence $\{\tau_n\}_{n \in \mathbb{Z}}$,

$$\tau_n'' - \tau_n^2 = \tau_{n+1} \tau_{n-1} - \psi \varphi \tau_n^2,$$  \hspace{1cm} (1)

with

$$\tau_{-1} = \psi, \quad \tau_0 = 1, \quad \tau_1 = \varphi,$$  \hspace{1cm} (2)

where $\psi$ and $\varphi$ are arbitrary functions, and $'$ denotes a derivation. Equation (1) is called the Toda equation in bilinear form. For given $\psi$ and $\varphi$, $\tau_n$ are uniquely determined as rational functions in them and their derivatives. However, as we shall show below, $\tau_n$ are actually polynomials in $\psi$, $\varphi$ and their derivatives, and moreover, they are expressed in determinantal forms.

**Theorem 2.1** Let $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ be two sequences defined recursively as

$$a_n = a_{n-1}' + \psi \sum_{i+j=n-2, i,j \geq 0} a_i a_j, \quad a_0 = \varphi,$$  \hspace{1cm} (3)

$$b_n = b_{n-1}' + \varphi \sum_{i+j=n-2, i,j \geq 0} b_i b_j, \quad b_0 = \psi.$$  \hspace{1cm} (4)

For any integer $n$, we define $|n| \times |n|$ Hankel determinant $\tau_n$ by

$$\tau_n = \begin{cases} 
\begin{vmatrix}
  a_0 & a_1 & \cdots & a_{n-1} \\
  a_1 & a_2 & \cdots & a_n \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n-1} & a_n & \cdots & a_{2n-2} \\
\end{vmatrix}, & n > 0, \\
1, & n = 0, \\
\begin{vmatrix}
  b_0 & b_1 & \cdots & b_{n-1} \\
  b_1 & b_2 & \cdots & b_n \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n-1} & b_n & \cdots & b_{2n-2} \\
\end{vmatrix}, & n < 0,
\end{cases}$$  \hspace{1cm} (5)

Then, $\tau_n$ satisfies equation (1) with initial condition (2).
Remark 2.2
1. Equation (1) is transformed to the bilinear form of Toda equation in “usual” form,
\[
s_n^2 \sigma_n - \sigma_n^2 = \sigma_{n+1} \sigma_{n-1},
\]
by applying suitable gauge transformation on \(\tau_n\), where \(\sigma_n\) is given by
\[
(\log \sigma_n)' = (\log \tau_n)' + \psi \varphi.
\] (7)
2. Since \(\sigma_n\) involves two arbitrary functions, Theorem 2.1 gives a determinant formula for the general solution of the Toda equation.
3. In the case of \(\varphi = 0\) or \(\psi = 0\), Theorem 2.1 recovers the well-known Darboux formula, namely, the determinant expression for solutions of the so-called Toda molecule equation\([10, 11]\).

It is also possible to construct a similar formula for the discrete Toda equation. Let \(\Phi^l\) and \(\Psi^l\) be arbitrary functions in \(l\), and \(\{\kappa^l_n\}_{n \in \mathbb{Z}}\) be a sequence defined by
\[
k^{l+1}_n \kappa^{l-1}_n - (1 - \varepsilon^2 \Phi^{l+1} \Psi^{l-1}) (\kappa^n_l)^2 = \varepsilon^2 k^{l+1}_n \kappa^{l-1}_n - 1,
\] (8)
\[
k^{l-1}_0 = \Phi^l, \quad k^l_0 = 1, \quad k^l_1 = \Psi^l,
\] (9)
where \(\varepsilon\) is a parameter corresponding to the lattice interval of \(l\). Equation (8) is called the discrete Toda equation in bilinear form \([12]\). Then we have:

Theorem 2.3 Let \(\{c^l_k\}_{k \in \mathbb{N}}, \{d^l_k\}_{k \in \mathbb{N}}\) be two sequences defined recursively as
\[
c^l_k = \frac{c^l_{k-1} - c^l_{k-1}}{\varepsilon} + \Psi^{l-2} \sum_{i+j=k-2} (c^l_i - \varepsilon c^l_{i+1}) c^l_{j-1}, \quad c^l_0 = \Phi^l,
\] (10)
\[
d^l_k = \frac{d^l_{k-1} - d^l_{k-1}}{\varepsilon} + \Phi^{l+2} \sum_{i+j=k-2} (d^l_i + \varepsilon d^l_{i+1}) d^l_{j-1}, \quad d^l_0 = \Psi^l.
\] (11)

For any integer \(n\), we define \(|n| \times |n|\) Hankel determinant \(\tau_n\) by
\[
\kappa^l_n = \begin{cases} 
\begin{pmatrix}
c^l_0 & c^l_1 & \cdots & c^l_{n-1} \\
c^l_1 & c^l_2 & \cdots & c^l_n \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
c^l_{n-1} & c^l_n & \cdots & c^l_{2n-2}
\end{pmatrix}, & n > 0, \\
1, & n = 0, \\
\begin{pmatrix}
d^l_0 & d^l_1 & \cdots & d^l_{n-1} \\
d^l_1 & d^l_2 & \cdots & d^l_n \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
d^l_{n-1} & d^l_n & \cdots & d^l_{2n-2}
\end{pmatrix}, & n < 0,
\end{cases}
\] (12)

Then, \(\kappa^l_n\) satisfies equation (8) with initial condition (9).
Remark 2.4
1. Equation (8) is transformed to the bilinear form of discrete Toda equation in “usual” form,

\[ \rho_{n+1} \rho_{n-1} - (\rho_n^l)^2 = \varepsilon^2 \rho_{n+1} \rho_{n-1}, \] (13)

by introducing \( \rho_n \) by

\[ \rho_n^l = \frac{1}{\prod_{i=0}^{l} \prod_{j=0}^{l} (1 - \varepsilon^2 \Phi \Psi - 2)} \kappa_n^l. \] (14)

2. Since \( \rho_n \) involves two arbitrary functions, Theorem 2.3 gives a determinant expression for general solution of the discrete Toda equation.

3. In the case of \( \varphi = 0 \) or \( \psi = 0 \), Theorem 2.3 recovers the determinant expression for solution of the so-called discrete Toda molecule equation [13].

4. Discrete Toda equation (8) and its solution (10)-(12) reduce to the Toda equation (1) and its solution (3)-(5) in the limit of \( \varepsilon \to 0 \), respectively.

Let us first prove Theorem 2.1. We consider the case of \( n > 0 \). Let \( D \) be the determinant of an \((n+1) \times (n+1)\) matrix \( X \), and \( D \left( \begin{array}{cccc} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{array} \right) \) the determinant of the matrix obtained from \( X \) by removing the rows with indices \( i_1, \cdots, i_k \) and the columns with indices \( j_1, \cdots, j_k \). Then we have well-known Jacobi’s formula (Lewis Carroll’s formula)

\[ D \left( \begin{array}{c} n \\ n \end{array} \right) D \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) - D \left( \begin{array}{c} n \\ n+1 \end{array} \right) D \left( \begin{array}{c} n+1 \\ n \end{array} \right) = D \cdot D \left( \begin{array}{c} n \\ n \end{array} \right) \left( \begin{array}{c} n+1 \\ n \end{array} \right). \] (15)

We have the following differential formula for \( \tau_n \):

**Lemma 2.5** Putting \( D \equiv \tau_{n+1} \), we have,

\[ D \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) = \tau_n, \quad D \left( \begin{array}{c} n+1 \\ n+1 \end{array} \right) = \tau_{n-1}, \] (16)

\[ D \left( \begin{array}{c} n \\ n+1 \end{array} \right) = D \left( \begin{array}{c} n+1 \\ n \end{array} \right) = \tau'_n, \] (17)

\[ D \left( \begin{array}{c} n \\ n \end{array} \right) = \tau''_n + \varphi \psi \tau_n. \] (18)

Then, Theorem 2.1 follows immediately from equation (15) and Lemma 2.5. Therefore, it suffices to prove Lemma 2.5.
Proof of Lemma 2.4. Equation (16) is obvious by definition. To show equation (17), we consider the following equality,

$$D\left(\begin{array}{c} n \\ n+1 \end{array}\right) = D\left(\begin{array}{c} n+1 \\ n \end{array}\right)$$

$$= \left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{array}\right) \left(\begin{array}{cccc} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \cdots & \Delta_{nn} \end{array}\right). \quad (19)$$

Here, for $n \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$, $A \cdot B$ denotes

$$A \cdot B = \sum_{i,j=1}^{n} A_{ij} B_{ij} = \text{Tr} A^t B,$$ \quad (20)

which is the standard scalar product of matrices, and $\Delta_{ij}$ is an $(i,j)$-cofactor of $\tau_n$. The first matrix of equation (19) is rewritten by using the recursion relation (18) as

$$\left(\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{array}\right) + \psi \left(\begin{array}{cccc} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{array}\right) + \left(\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right). \quad (21)$$

Applying scalar product of equation (21) with $(\Delta_{ij})$, we see that the first term of the right hand side of equation (21) gives $\tau_n'$ and second and third terms give no contribution. Thus we have shown that equation (17) holds.
Equation (18) is shown in a similar manner by considering the equality,

\[
D \begin{pmatrix} n \\ n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix} \begin{pmatrix} \Delta'^{1}_{11} & \Delta'^{1}_{12} & \cdots & \Delta'^{1}_{1n} \\ \Delta'^{1}_{21} & \Delta'^{1}_{22} & \cdots & \Delta'^{1}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta'^{1}_{n1} & \Delta'^{1}_{n2} & \cdots & \Delta'^{1}_{nn} \end{pmatrix},
\]

(22)

where \( \Delta'^{ij} \) denotes the \((i, j)\)-cofactor of \( \tau'_n = D \begin{pmatrix} n \\ n+1 \end{pmatrix} \).

The case \( n < 0 \) of Theorem 2.1 is proved in a manner similar to the case of \( n > 0 \), and the case \( n = 0 \) is checked directly. Thus, proof of Theorem 2.1 is completed.

Let us next prove Theorem 2.3. Similarly to the proof of Theorem 2.1, we concentrate on the case of \( n > 0 \). We have the following lemma:

**Lemma 2.6** For \( n \geq 1 \), we have

\[
\varepsilon^{n-1} \kappa^{l+1}_n = \begin{vmatrix} c^l_0 & c^l_1 & \cdots & c^l_{n-2} & C^{l+1}_{0} \\ c^l_1 & c^l_2 & \cdots & c^l_{n-1} & C^{l+1}_{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c^l_{n-1} & c^l_n & \cdots & c^l_{2n-2} & C^{l+1}_{n-1} \end{vmatrix},
\]

(23)

\[
\frac{\varepsilon^{2(n-1)}}{1 - \varepsilon^{2} \Phi^{l+2} \Psi} \kappa^{l+2}_n = \begin{vmatrix} c^l_0 & c^l_1 & \cdots & c^l_{n-2} & C^{l+1}_{0} \\ c^l_1 & c^l_2 & \cdots & c^l_{n-1} & C^{l+1}_{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c^l_{n-2} & c^l_{n-1} & \cdots & c^l_{2n-3} & C^{l+1}_{n-2} \\ C^{l+1}_{0} & C^{l+1}_{1} & \cdots & C^{l+1}_{n-2} & 1 - \varepsilon^{2} \Phi^{l+2} \Psi \end{vmatrix},
\]

(24)

where \( C^l_k \) is defined by

\[
C^l_k = c^l_k + \varepsilon \Psi^{l-2} \sum_{i=1}^{k} c^l_{k-i} c^{l-1}_{i-1}.
\]

(25)
Theorem 2.3 for the case of $n > 0$ is derived as follows. We put

$$D \equiv \frac{\varepsilon^{2n}}{1 - \varepsilon^2 \Phi_{l+2} \Psi^l l_{n+2}} \kappa_{n+2}^{l+1} = \begin{vmatrix}
\varepsilon^l & \cdots & \varepsilon^{l-2} & \varepsilon^{l-1} & C^{l+1}_0 \\
\varepsilon^l & \cdots & \varepsilon^{l-1} & C^{l+1}_n \\
\vdots & \ddots & \vdots & \vdots \\
\varepsilon^{l-2} & \cdots & \varepsilon^{l-4} & \varepsilon^{l-3} & C^{l+1}_{n-2} \\
\varepsilon^{l-1} & \cdots & \varepsilon^{l-3} & \varepsilon^{l-2} & C^{l+1}_{n-1} \\
0 & \cdots & C^{l+1}_{n-2} & C^{l+1}_{n-1} & 1 - \varepsilon^2 \Phi_{l+2} \Psi^l
\end{vmatrix} .$$

(26)

Then we have from Lemma 2.4

$$D \left( \begin{array}{c}
\frac{n}{n} \\
\frac{n}{n+1}
\end{array} \right) = \frac{\varepsilon^{2(n-1)}}{1 - \varepsilon^2 \Phi_{l+2} \Psi^l l_{n} l_{n+2}} \kappa_{n}^{l+1} , \quad D \left( \begin{array}{c}
\frac{n+1}{n+1} \\
\frac{n+1}{n+1}
\end{array} \right) = \kappa_{n}^{l} ,$$

(27)

$$D \left( \begin{array}{c}
\frac{n}{n+1} \\
\frac{n+1}{n+1}
\end{array} \right) = D \left( \begin{array}{c}
\frac{n+1}{n} \\
\frac{n+1}{n}
\end{array} \right) = \varepsilon^{n-1} \kappa_{n}^{l+1} , \quad D \left( \begin{array}{c}
\frac{n}{n} \\
\frac{n+1}{n+1}
\end{array} \right) = \kappa_{n}^{l-1} .$$

(28)

Therefore, Jacobi’s identity (15) yields

$$\frac{\varepsilon^{2(n-1)}}{1 - \varepsilon^2 \Phi_{l+2} \Psi^l l_{n} l_{n+2}} \kappa_{n}^{l+1} = \left( \varepsilon^{n-1} \kappa_{n+1}^{l+1} \right)^2 = \frac{\varepsilon^{2n}}{1 - \varepsilon^2 \Phi_{l+2} \Psi^l l_{n+1} l_{n+2}} \kappa_{n+1}^{l+1} \kappa_{n-1}^{l} ,$$

(29)

which is equivalent to the discrete Toda equation (8).

Proof of Lemma 2.4. We rewrite

$$\kappa_{n}^{l+1} = \begin{vmatrix}
\varepsilon^l & \cdots & \varepsilon^{l+1} & \varepsilon^{l+1} & \varepsilon^{l+1} \\
\varepsilon^l & \cdots & \varepsilon^{l+1} & \varepsilon^{l+1} & \varepsilon^{l+1} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\varepsilon^{l-2} & \cdots & \varepsilon^{l-4} & \varepsilon^{l-3} & \varepsilon^{l+1} \\
\varepsilon^{l-1} & \cdots & \varepsilon^{l-3} & \varepsilon^{l-2} & \varepsilon^{l+1} \\
0 & \cdots & \varepsilon^{l+1} & \varepsilon^{l+1} & 1 - \varepsilon^2 \Phi_{l+2} \Psi^l
\end{vmatrix} .$$

by using the recursion relation (10) to obtain eq. (24). We first add $j$-th column multiplied by $\Psi^{-1} c_{n-1-j}$ to $n$-th column for $j = 2, \cdots, n-1$. Next, adding $j$-th column multiplied by $\Psi^{-1} c_{n-2-j}$ to $n$-th column for $j = 1, \cdots, n-2$, we have from eq. (19).
Applying the similar procedure to \((n - 1)\)-th, \(\cdots\), 2nd columns, we obtain,

\[
\kappa_n^{l+1} =
\begin{pmatrix}
\ell_{0}^{l+1} & -\frac{\ell_{1}^{l+1}}{\varepsilon} & \cdots & -\frac{\ell_{n-1}^{l+1}}{\varepsilon} & -\frac{\ell_{n}^{l+1}}{\varepsilon} \\
\ell_{1}^{l+1} & -\frac{\ell_{2}^{l+1}}{\varepsilon} & \cdots & -\frac{\ell_{n}^{l+1}}{\varepsilon} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\ell_{n-2}^{l+1} & \frac{-\varepsilon}{\varepsilon} & \cdots & -\frac{\varepsilon}{\varepsilon} \\
\ell_{n-1}^{l+1} & \frac{-\varepsilon}{\varepsilon} & \cdots & -\frac{\varepsilon}{\varepsilon} 
\end{pmatrix}
\]

Next, we apply the similar procedure in vertical direction. For \(k = 2, \cdots, n\), we add \(j\)-th row multiplied by \(\varepsilon\Psi^{l-1}(\ell_{j-1}^{l+1} - \ell_{j}^{l+1})\) to \(k\)-th row for \(j = 1, \cdots, k - 1\). Then we get

\[
\kappa_n^{l+1} = \varepsilon^{-(n-k)}
\begin{pmatrix}
C_{0}^{l+1} & -\frac{\ell_{1}^{l+1}}{\varepsilon} & \cdots & -\frac{\ell_{n}^{l+1}}{\varepsilon} \\
C_{1}^{l+1} & -\frac{\ell_{2}^{l+1}}{\varepsilon} & \cdots & -\frac{\ell_{n}^{l+1}}{\varepsilon} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n-2}^{l+1} & -\frac{\ell_{n-1}^{l+1}}{\varepsilon} & \cdots & -\frac{\ell_{n}^{l+1}}{\varepsilon} \\
C_{n-1}^{l+1} & -\frac{\ell_{n}^{l+1}}{\varepsilon} & \cdots & -\frac{\ell_{n}^{l+1}}{\varepsilon} 
\end{pmatrix}
\]

where \(C_{k}^{l+1}\) is defined recursively by

\[
C_{k}^{l+1} = C_{k}^{l+1} + \sum_{j=0}^{k-1} \varepsilon\Psi^{l-1}(\ell_{k-1-j}^{l+1} - \ell_{k-j}^{l+1})C_{j}^{l+1}, \quad C_{0}^{l+1} = \ell_{0}^{l+1}, \quad (30)
\]

We can verify that \(C_{k}^{l+1}\) is actually given as eq. (24) by induction. Thus we have proved eq. (23). Since eq. (24) is proved by similar calculation, we omit the detail. This completes the proof of Lemma 2.6.

The above discussion proves Theorem 2.3 for the case of \(n > 0\). The case of \(n < 0\) is proved similarly, and the case of \(n = 0\) is checked immediately. Thus we have proved Theorem 2.3.
3 Application for Painlevé equations

It is established by K. Okamoto that the \( \tau \) functions of the Painlevé equations \( P_{\text{II}}, \ldots, P_{\text{VI}} \) satisfy the Toda equation. Recall that each of the Painlevé equations \( P_J \) (\( J = \text{II}, \ldots, \text{VI} \)) can be written as a Hamiltonian system

\[
\delta q = \frac{\partial H}{\partial p}, \quad \delta p = -\frac{\partial H}{\partial q},
\]

(31)

where \( H \) is a certain polynomial in \( q, p \), and the derivation \( \delta \) is given by \( \delta = \partial_t \) for \( P_{\text{II}}, P_{\text{IV}} \), \( \delta = t\partial_t \) for \( P_{\text{III}}, P_{\text{V}} \), and \( \delta = t(t-1)\partial_t \) for \( P_{\text{VI}} \), respectively. (The explicit formula for \( H \) will be given below.) The \( \tau \) function, which we denote \( \sigma \), is defined up to constant factor as

\[
\delta (\log \sigma) = H.
\]

(32)

By using a Bäcklund transformation (Schlesinger transformation) \( T \) which acts as a translation on the parameter space, one can define a sequence of functions \( \sigma_n \) by choosing an appropriate normalization factor for \( T_n \) (\( n \in \mathbb{Z} \)).

**Theorem 3.1 (Okamoto [4])** The sequence of \( \tau \) functions \( \sigma_n \) for the Painlevé equations \( P_J \) satisfies the Toda equation

\[
(\delta^2 \sigma_n)\sigma_n - (\delta \sigma_n)^2 = \sigma_{n+1}\sigma_{n-1}.
\]

(33)

We will review the derivation of the Toda equation in the next section for completeness. Let us put

\[
\sigma_n = \sigma_0 \left( \frac{\sigma_1}{\sigma_0} \right)^n T_n,
\]

(34)

then the function \( T_n \) satisfy the recurrence relation

\[
T_{n+1}T_{n-1} - (\delta^2 T_n)T_n - (\delta T_n)^2 + (n\delta v + u)T_n^2,
\]

(35)

with initial condition \( T_0 = T_1 = 1 \). Here \( u = \delta^2 \log \sigma_0 \), \( v = \delta \log \frac{\sigma_1}{\sigma_0} \). The following has been conjectured by H. Umemura. (see [20] for example)

**Theorem 3.2** Let \( (q, p) \) be a solution for \( P_J \) and define functions \( T_n = T_n(q, p, t) \) through the recurrence relation (35). Then the function \( T_n \) is polynomial in \( q, p \).

**Proof.** We will show for \( n \geq 0 \). The case \( n \leq 0 \) is similar. Note that the functions \( \tau_n = \sigma_n/\sigma_0 \) (\( n \in \mathbb{Z} \)) satisfy the equation

\[
\tau_n'' \tau_n - \tau_n'^2 = \tau_{n+1}\tau_{n-1} - \psi \varphi \tau_n^2,
\]

(36)

with

\[
\psi = \frac{\sigma_1 - \sigma_0}{\sigma_0}, \quad \varphi = \frac{\sigma_1}{\sigma_0},
\]

(37)
From the determinant formula (Theorem 2.1), the solution of the Toda equation (33) is given by
\[ \sigma_n = \sigma_0 \tau_n = \sigma_0 \det(a_{i+j})_{0 \leq i,j \leq n-1}. \] (38)

We introduce \( g_i \) (\( i \in \mathbb{N} \)) by setting \( a_i = \sigma_1 g_i / \sigma_0 = \varphi g_i \), so that
\[ \sigma_n = \sigma_0 \left( \frac{\sigma_1}{\sigma_0} \right)^n \det(g_{i+j})_{0 \leq i,j \leq n-1}. \] (39)

Putting \( u = \delta^2 \log \sigma_0 = \psi \varphi \) and \( v = \delta \log \varphi = \delta \log \sigma_0 / \sigma_0 \), we can rewrite the recurrence formula for \( a_n \) to that of \( g_n \):
\[ g_0 = 1, \quad \text{and} \quad g_n = \delta g_{n-1} + v g_{n-1} + u \sum_{i,j \geq 0} g_i g_j, \quad (n \geq 1). \] (40)

For the Painlevé \( \tau \) functions, \( u, v \) (these are essentially \( H \) and \( Y \) given in the next section) and their \( \delta \)-derivatives are all polynomials in \( q, p \). Hence the coefficients \( g_k \) and their determinants \( T_n \) are also polynomials in \( q, p \).

4 Derivation of the Toda equations

In this section we will review the derivation of the Toda equation following the work by Okamoto [4, 5, 6, 7].

4.1 The second Painlevé equation: \( P_{II} \)

The Hamiltonian is
\[ H = \frac{1}{2} p^2 - (q^2 + \frac{1}{2} t)p - \alpha_1 q. \] (41)

The equation for \( y = q \) is the Painlevé equation \( P_{II} \) given as follows:
\[ y'' = 2y^3 + ty + a, \] (42)

with \( a = \alpha_1 - \frac{1}{2} \).

The Bäcklund transformations are given by
\[
\begin{array}{c|cccc}
\hline
x & \alpha_0 & \alpha_1 + 2\alpha_0 & \alpha_1 & \alpha_0 + 2\alpha_1 & \alpha_1 & \alpha_0 \\
\hline
s_0(x) & -\alpha_0 & \alpha_1 & p + \frac{4\alpha_0^2}{f} & 2\alpha_0^2 & \frac{q + \alpha_0}{p} & q + \alpha_0 \\
s_1(x) & \alpha_0 + 2\alpha_1 & -\alpha_1 & p & -f & \frac{q + \alpha_1}{p} & -q \\
\pi(x) & \alpha_1 & \alpha_0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\] (43)

where \( \alpha_0 = 1 - \alpha_1 \) and \( f = p - 2q^2 - t \).

Let \( T = \pi s_1 \) be the translation which acts on the parameter as \( T(\alpha_0, \alpha_1) = (\alpha_0 + 1, \alpha_1 - 1) \), then we have
\[ T(H) - H = Y = q, \quad \partial_t(H) = -\frac{p}{2}, \] (44)
\[\partial_t (\log \partial_t H) = T(H) - 2H + T^{-1}(H).\]  
(45)

Hence we have the Toda equation:

\[\partial_t^2 \log \sigma_n = c(n) \frac{\sigma_{n-1} \sigma_{n+1}}{\sigma_n^2},\]
(46)

where \(c(n)\) is a non-zero constant. This can be transformed to the standard Toda equation by changing the normalization as

\[\sigma_n \to C_n \sigma_n.\]
(47)

### 4.2 The third Painlevé equation: \(P_{III}\)

The Hamiltonian is

\[H_{III} = q^2 p^2 - (q^2 + v_1 q - t)p - \frac{1}{2}(v_1 + v_2)q.\]
(48)

The equation for \(y = q/s\) \((t = s^2)\) is given by the third Painleve equation \(P_{III}\)

\[\frac{d^2 y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s} ay^2 + b + cy^3 + \frac{d}{y},\]
(49)

with

\[a = -4v_2, \quad b = 4(v_1 + 1), \quad c = 4, \quad d = -4.\]
(50)

The Bäcklund transformations are

\[
\begin{array}{cccccc}
 x & v_1 & v_2 & p & q & t \\
 s_0(x) & -1 - v_2 & -1 - v_1 & \frac{q}{4}(p - 1) - \frac{1}{2}(v_1 - v_2) + 1 & -\frac{1}{2} & t \\
 s_1(x) & v_2 & v_1 & p & q + \frac{1}{4}(v_2 - v_1) & t \\
 s_2(x) & v_1 & -v_2 & 1 - p & -q & -t \\
\end{array}
\]
(51)

Let \(T = s_0s_2s_1s_2\) be the translation which acts on the parameter as \(T(v_1, v_2) = (v_1 + 1, v_2 + 1)\), then

\[T(H) - H = Y = q(1 - p), \quad \partial_t(H) = p.\]
(52)

and

\[\delta \log(\partial_t H) = T(H) - 2H + T^{-1}(H).\]
(53)

Hence we have the Toda equation

\[\partial_t \delta \log \sigma_n = c(n) \frac{\sigma_{n-1} \sigma_{n+1}}{\sigma_n^2},\]
(54)

where \(c(n)\) is a non-zero constant. This equation can be transformed to the standard Toda equation by the change of the normalization as

\[\sigma_n \to C_n t^{n^2/2} \sigma_n.\]
(55)
4.3 The fourth Painlevé equation : \( P_{IV} \)

The Hamiltonian is

\[
H_{IV} = (p - q - 2t)pq - 2\alpha_1 p - 2\alpha_2 q.
\]  (56)

Equation for \( y = q \) is given by

\[
y'' = \frac{y'^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - a)y + \frac{b}{y},
\]  (57)

with

\[
a = \alpha_0 - \alpha_2, \quad b = -2\alpha_1^2.
\]  (58)

The Bäcklund transformations are

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & \alpha_0 & \alpha_1 & \alpha_2 & p & q \\
\hline
s_0(x) & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 + \alpha_0 & p + \frac{2\alpha_1}{q} & q + \frac{2\alpha_2}{p} \\
\hline
s_1(x) & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & p - \frac{2\alpha_1}{q} & q \\
\hline
s_2(x) & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & p & q + \frac{2\alpha_2}{p} \\
\hline
\pi(x) & \alpha_1 & \alpha_2 & \alpha_0 & -f & -p \\
\hline
\end{array}
\]  (59)

where \( \alpha_0 = 1 - \alpha_1 - \alpha_2 \) and \( f = p - q - 2t \).

Let \( T = \pi s_2 s_1 \) be the translation which acts on the parameter as \( T(\alpha_0, \alpha_1, \alpha_2) = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2) \), then

\[
T(H) - H = Y = 2p, \quad \partial_t(H) = -2pq.
\]  (60)

and

\[
\delta \log(\partial_t H + 4\alpha_1) = T(H) - 2H + T^{-1}(H).
\]  (61)

Hence we have the Toda equation

\[
\partial_t \delta \log \sigma_n + 4(\alpha_1 - n) = c(n) \frac{\sigma_{n-1}\sigma_{n+1}}{\sigma_n^2},
\]  (62)

where \( c(n) \) is a non-zero constant. The change of normalization is given by

\[
\sigma_n \rightarrow C_n e^{2(\alpha_1-n)t^2} \sigma_n.
\]  (63)

4.4 The fifth Painlevé equation : \( P_{V} \)

The Hamiltonian is

\[
H_{V} = p(p + t)q(q - 1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q - 1).
\]  (64)

Equation for \( y = 1 - 1/q \) is given by

\[
y'' = \left(\frac{1}{2y} + \frac{1}{y - 1}\right)(y')^2 - \frac{y'}{t^2} + \frac{(y - 1)^2}{t} (ay + \frac{b}{y}) + \frac{y}{t} + d \frac{y(y + 1)}{y - 1},
\]  (65)
where
\[
a = \frac{\alpha_1^2}{2}, \quad b = -\frac{\alpha_3^2}{2}, \quad c = \alpha_0 - \alpha_2, \quad d = -\frac{1}{2}
\]  
(66)

The Bäcklund transformations are
\[
x = \frac{\alpha_0}{x} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_0 \quad p = \frac{\alpha_1}{q} \quad q = \frac{\alpha_1}{p + q}
\]
(67)

where \(\alpha_0 = 1 - \alpha_1 - \alpha_2 - \alpha_3\).

Let \(T = \pi s_3 s_2 s_1\) be the translation which acts on the parameters as \(T(\alpha_0, \ldots, \alpha_3) = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2, \alpha_3)\), then
\[
T(H) - H = Y = p(q - 1), \quad \partial_t(H) = pq(q - 1) + \alpha_2q,
\]
(68)

and
\[
\delta \log(\partial H + \alpha_1) = T(H) - 2H + \dagger(\dagger(H)).
\]
(69)

Hence, we obtain the Toda equation
\[
\partial_t \delta \log \sigma_n + (\alpha_1 - n) = c(n) \frac{\sigma_{n-1} \sigma_{n+1}}{\sigma_n^2},
\]
(70)

where \(c(n)\) is a non-zero constant. The change of the normalization is given by
\[
\sigma_n \to C_n \frac{t^{n^2/2} e^{(\alpha_1 - n)t}}{\sigma_n}.
\]
(71)

### 4.5 The sixth Painlevé equation \(P_{VI}\)

The Hamiltonian is
\[
H = q(q-1)(q-t)p^2 - \alpha_4(q-1)(q-t) + \alpha_3 q(q-t) + (\alpha_0 - 1)q(q-1) \left| p + \alpha_2(\alpha_1 + \alpha_2)(q-t) \right|.
\]
(72)

The equation for \(y = q\) is given by
\[
y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - t} \right)y' - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{y - t} \right)y' + \frac{y(y - 1)(y - t)}{t^2(t - 1)^2} \left[ a + b \frac{t}{y^2} + c \frac{t - 1}{(y - 1)^2} + d \frac{t(t - 1)}{(y - t)^2} \right],
\]
with
\[
a = \alpha_1^2 / 2, \quad b = -\alpha_4^2 / 2, \quad c = \alpha_3^2 / 2, \quad d = -(\alpha_0^2 - 1)/2.
\]
(73)
\( (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1) \). The Bäcklund transformations are as follows

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
x & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & p & q \\
\hline
s_0(x) & -\alpha_0 & \alpha_1 & \alpha_2 + \alpha_0 & \alpha_3 & \alpha_4 & p - \frac{\alpha_4}{q-1} & q \\
\hline
s_1(x) & \alpha_0 & -\alpha_1 & \alpha_2 + \alpha_1 & \alpha_3 & \alpha_4 & p & \frac{q}{q-1} \\
\hline
s_2(x) & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & \alpha_3 + \alpha_2 & \alpha_4 + \alpha_2 & p & \frac{q}{q-1} \\
\hline
s_3(x) & \alpha_0 & \alpha_1 & \alpha_2 + \alpha_3 & -\alpha_3 & \alpha_4 & p \rightarrow p - \frac{\alpha_1}{q-1} & q \\
\hline
s_4(x) & \alpha_0 & \alpha_1 & \alpha_2 + \alpha_4 & \alpha_3 & -\alpha_4 & p \rightarrow p - \frac{\alpha_2}{q-1} & q \\
\hline
r(x) & \alpha_1 & \alpha_0 & \alpha_2 & \alpha_3 & \alpha_4 & \frac{q(t-1)^2 + \alpha_2(q-t)}{q-1} & q \rightarrow \frac{(q-1)t}{q-1} \\
\hline
\end{array}
\]  

(74)

Let \( T = s_0s_2s_3s_4s_2s_0r \) be the translation which acts on the parameters as

\[ T(\alpha_0, \ldots, \alpha_4) = (\alpha_0 - 1, \alpha_1 + 1, \alpha_2, \alpha_3, \alpha_4), \]

then we have

\[ T(H) - H = Y = pq(1-q) + \alpha_2(t-q), \quad \partial_t H = p^2q(1-q) - \alpha_2(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4)pq - \alpha_4p, \]

(75)

and

\[ \delta \log(\partial_t H + (\alpha_1 + \alpha_2)(1 - \alpha_0)) = T(H) - 2H + T^{-1}(H). \]

(76)

The Toda equation is given as

\[ \partial_t \delta \log \sigma_n + (\alpha_1 + \alpha_2 + n)(1 - \alpha_0 + n) = c(n) \frac{\sigma_{n-1}\sigma_{n+1}}{\sigma_n^2}, \]

(77)

where \( c(n) \) is a non-zero constant. The change of the normalization is given by

\[ \sigma_n \rightarrow C_n (t(t-1))^{(\alpha_1 + \alpha_2 + n)(1-\alpha_0 + n)} \sigma_n. \]

(78)

5 Conclusion

In this paper, we have presented determinant formulas for the general solutions of both Toda and discrete Toda equations. We have applied the results to the Painlevé equations, and obtained a new direct proof for the polynomiality of the solutions of Painlevé equations.

Acknowledgment. We wish to thank Professor H. Umemura for stimulating discussions.

References

[1] K. Kajiwara and Y. Ohta, “Determinant Structure of the Rational Solutions for the Painlevé IV Equation”, J. Phys. A 31(1998) 2431–2446.
[2] K. Kajiwara, K. Yamamoto and Y. Ohta, “Rational Solutions for the Discrete Painlevé II Equation”, Phys. Lett. A232(1997), 189-199.

[3] K. Kajiwara and Y. Ohta, “Determinant Structure of the Rational Solutions for the Painlevé II Equation”, J. Math. Phys. 37 (1996), 4693–4704.

[4] K. Okamoto, “Studies on the Painlevé Equations I, Sixth Painlevé equation P_{VI}”, Annali di Matematica pura ed applicata CXLVI (1987), 337-381.

[5] K. Okamoto, “Studies on the Painlevé Equations II, Fifth Painlevé Equation P_{V}”, Japan J. Math. 13 (1987),47-76.

[6] K. Okamoto, “Studies on the Painlevé Equations III, Second and Fourth Painlevé Equations, P_{II} and P_{IV}”, Math. Ann. 275(1986), 222-254.

[7] K. Okamoto, “Studies on the Painlevé Equations IV, Third Painlevé Equation P_{III}”, Funkcial. Ekvac. 30 (1987), 305-332.

[8] K. Kajiwara and T. Masuda, “On the Umemura Polynomials for the Painlevé III equation”, solv-int/9903013.

[9] K. Kajiwara and T. Masuda, “A Generalization of Determinant Formulas for the Solutions of Painlevé II and XXXIV equations”, J. Phys. A: Math. Gen. 32(1999), 3763-3778.

[10] A.N. Leznov and M.V. Saveliev, ” Theory of Group Representations and Integration of Nonlinear Systems $X_{a,z \bar{z}} = \exp(kX)_a$”, Physica 3D(1981) 62-72.

[11] R. Hirota, Y. Ohta and J. Satsuma, “Wronskian Structures of Solutions for Soliton Equations”, Prog. Theor. Phys. Suppl. 94(1988), 59-72.

[12] R. Hirota, M. Ito and F. Kako, “Two-Dimensional Toda Lattice Equations”, Prog. Theor. Phys. Suppl. 94(1988), 42-58.

[13] R. Hirota, “Discrete Two-Dimensional Toda Molecule Equation”, J. Phys. Soc. Jpn. 56(1987), 4285-4288.

[14] A. Nakamura, ”Bilinear Structure of the Painlevé II Equation and its Solutions for Half Integer Constants”, J. Phys. Soc. Jpn. 61(1992), 3007–3008.

[15] A. P. Veselov and A. B. Shabat, ”Dressing Chains and the Spectral Theory of the Schrödinger Operator”, Funct. Anal. Appl. 27(1993)1-21.

[16] V. E. Adler, ”Nonlinear chains and Painlevé equations”, Physica D73 (1994) 335-351.

[17] M. Noumi and Y. Yamada, “Symmetries in the Fourth Painlevé Equations and Okamoto Polynomials”, Nagoya Math. J. 153(1999), 53–86.
[18] M. Noumi and Y. Yamada, "Unemura Polynomials for Painlevé V Equation", Phys. Lett. A247 (1998) 65–69.

[19] H. Umemura, “Special Polynomials associated with the Painlevé Equations I”, to appear in the proceedings of the Workshop on "Painlevé Transcendents", CRM, Montreal, Canada, 1996.

[20] M. Noumi, S. Okada, K. Okamoto and H. Umemura, “Special Polynomials associated with the Painlevé Equations II”, in Integrable Systems and Algebraic Geometry, Proceedings of the Taniguchi Symposium 1997, 349–372, World Scientific, 1998.