The functional integral approach to quantum field theories consists of two basic steps: first the determination of a “physical” Euclidean measure $d\mu$ on the configuration space and second the reconstruction of the quantum theory via an Osterwalder-Schrader procedure. The latter issue has been treated rigorously in several approaches – first by Osterwalder and Schrader [4,5] for scalar fields, by Ashtekar et al. [6] for diffeomorphism invariant theories. However, in contrast to this, the former one kept a problem that has been solved completely only for some examples. Typically, one tried to define this measure $d\mu$ using the action method, this means (up to a normalization factor) simply by

$$d\mu := e^{-S} d\mu_0,$$

where $S$ is the classical action of the theory under consideration and $d\mu_0$ is an appropriate kinematical measure on the configuration space. In this letter we will discuss why just this ansatz can prevent the rigorous description of a wide class of physical theories. More precisely, we present three criteria implying that no function $f$ at all describes such a theory via $d\mu := f d\mu_0$. Our criteria are met, e.g., for the two-dimensional Yang-Mills and other confining theories. Consequently, here the action method fails.

**Framework and Result**

This letter is based on the Ashtekar approach [4,5] to gauge field theories because it is best-suited for solving measure-theoretical problems. Its basic idea goes as follows: The continuum gauge theory is known as soon as its restrictions to all finite floating lattices are known. This means, in particular, that the expectation values of all observables that are sensitive only to the degrees of freedom of a certain lattice can be calculated by the corresponding integration over these finitely many degrees of freedom. Examples for those observables are the Wilson loop variables $\text{tr } h_\beta$ where $\beta$ is some loop in the space or space-time and $h_\beta$ is the holonomy along that loop.

The above idea has been implemented rigorously for compact structure groups $G$ as follows: First the original configuration space of all smooth gauge fields (modulo gauge transforms) has been enlarged by distributional ones. This way the configuration space became compact and could now be regarded as a so-called projective limit of the lattice configuration spaces. These, on the other hand, consist as in ordinary lattice gauge theories of all possible assignments of parallel transports to the edges of the considered floating lattice (again modulo gauge transforms). Since every parallel transport is an element of $G$, the Haar measure on $G$ yields a natural measure for the lattice theories. Now the so-called Ashtekar-Lewandowski measure $d\mu_0$ is just that continuum measure whose restrictions to the lattice theories coincide with these natural lattice Haar measures. It serves as a canonical kinematical measure.

In contrast to the beautiful results in the formulation of quantum geometry within this framework, the progress in the treatment of general gauge theories here is quite small. Only for the two-dimensional Yang-Mills theory the complete quantization program has been performed explicitly [4,5]. However, even there the full measure has not been defined directly via the action method, but using a regularization and a certain limit. This was necessary because no extension of the classical action $S$ to distributive gauge fields is known. Probably neither this does for more complicated models. Therefore we are going to investigate a more fundamental problem: What kind of models at all can be studied via the action method or might it be typical that the action method fails?

For this, we will consider theories satisfying the following, physically rather natural suppositions.

1. Universality of the coupling constant
This encodes the assumption that the interaction between arbitrarily charged, composite particles is determined immediately by the interaction between the elementary particles.

2. Independence principle
This means that certain loops are independent random variables.
3. Geometrical regularity
This signifies that the Wilson-loop expectation value converges to 1 when the loop shrinks, i.e., its holonomy goes to the identity for every smooth gauge field.

Although these criteria are naturally apparent, if all three criteria are met, the continuum measure will be purely singular w.r.t. the Ashtekar-Lewandowski measure. This means, those continuum theories cannot be described by a measure with $\text{d}\mu = e^{-\beta} \text{d}\mu_0$.

Now, we are going to explain our three principles, state precisely their consequences and discuss finally examples.

**Principle 1: Universality of the Coupling Constant**

We are aiming at the following statement: If the theory considered has a (in a certain sense) universal coupling constant that by itself describes the coupling strength between the elementary (matter) particles of that theory, then $\langle \text{tr } \phi(h_\beta) \rangle$ is determined completely by $\langle \text{tr } h_\beta \rangle$ and the representation $\phi$. Here $\langle f \rangle$ always denotes the physical expectation value of a function $f$.

Let us consider the simplest case of a Yang-Mills theory with structure group $U(1)$. The elementary matter particles are the single-charged particles; the coupling constant be $g = e$. Classically, the interaction, i.e. the potential between a particle and its antiparticle, is obviously proportional to $g^2$. Now we call the coupling constant to be universal if it yields immediately the interaction between arbitrarily charged particles. In particular, for composed particle with charges $n$ and $-n$, resp., it is proportional $(ng)^2$. In general, one assumes that also the Wilson-loop expectation values $\langle h_\beta \rangle$ describe the potential between two oppositely charged static particles $[17][13]$. Namely, if $\beta$ is a rectangular loop running in space between $\vec{x}$ and $\vec{y}$ and in time between 0 and $\Delta t$, then the potential between the elementary particles resting in $\vec{x}$ and $\vec{y}$, resp., is given by

$$V_1(\vec{x} - \vec{y}) = -\lim_{\Delta t \to \infty} \frac{1}{\Delta t} \ln \langle h_\beta \rangle.$$  

A Wilson loop so just carries the interaction between an elementary particle-antiparticle pair; consequently, $n$ loops should yield the interaction between a pair of an $n$-times charged particle and its antiparticle. On the other hand, (by the assumed universality of the coupling constant) the corresponding potential $V_n$ is to be $n^2 V_1$. Hence, we have

$$n^2 V_1(\vec{x} - \vec{y}) = V_n(\vec{x} - \vec{y}) = -\lim_{\Delta t \to \infty} \frac{1}{\Delta t} \ln \langle h_\beta^n \rangle.$$  

Translating these two equations to the level of Wilson-loop expectation values, we get (at least in the limit $\Delta t \to \infty$)

$$\langle h_\beta^n \rangle = \langle h_\beta \rangle^{n^2}. \quad (1)$$  

Indeed, the Wilson-loop expectation values of the $U(1)$ theory for $d = 2$ dimensions in the Ashtekar framework fulfill equation (1) – and namely not only for loops being large w.r.t. the time, but for all loops $[11][13]$. Hence, it is by no means unrealistic to identify the validity of (1) for all loops with the existence of a universal coupling constant.

Let us now turn to gauge theories having general compact structure group $G$. Using the following translation table

| Irreducible representation | $U(1)$ $\rightarrow$ | $G$ |
|---------------------------|----------------------|-----|
| Dimension                 | $n \rightarrow$     | $\phi$ |
| Normalized character      | $g^n \rightarrow \frac{1}{\beta} \text{tr } \phi(g)$ |
| Casimir eigenvalue        | $n^2 \rightarrow \epsilon_\phi$ |

equation (1) becomes

$$\frac{\langle \text{tr } \phi(h_\beta) \rangle}{\epsilon_\phi} = \left( \frac{\langle \text{tr } \phi_1(h_\beta) \rangle}{\epsilon_{\phi_1}} \right)^{\frac{n^2}{\epsilon_\phi}}, \quad (2)$$

where $\phi_1$ denotes some nontrivial representation of $G$, e.g., the standard one of $G \subseteq U(N)$ on $\mathbb{C}^N$. Therefore, we will call a theory having a universal coupling constant if equation (2) is fulfilled for all irreducible representations $\phi$ and all “non-selfoverlapping” loops $\beta$.

From the physical point of view such an assumption has a very interesting consequence: If a theory describes confinement (in the sense of an area law) between the elementary particles, all other charged particle-antiparticle pairs are confined as well. In the case of QCD this just explains why only particles containing exclusively of baryons and mesons are freely observable; they are simply those particles whose total color charge $\sqrt{c_\phi}$ equals zero, i.e. whose quark product state transforms according the trivial $SU(3)$ representation. We remark that this discussion is not new because already about twenty years ago Yang-Mills theories with non-elementary charges has been considered (cf., e.g., $[13]$) and it has been shown that there occurs an area law as well. However, there one started with the action $\frac{1}{\beta} (\phi(F), \phi(F))$ specially tailored to those charges, such that a comparison between differently charged particles is possible within one model – in contrast to our description.

Finally, we note that just the universality of the coupling constant might be a desirable property of unified theories.

**Principle 2: Independence Principle**

It is well-known that non-overlapping loops yield independent random variables in the two-dimensional Yang-Mills theory. This means, for all finite sets $\beta_1, \ldots, \beta_n$ of
such loops and for all representations $\phi_1, \ldots, \phi_n$ of the structure group $G$ we have

$$\langle \text{tr} \phi_1(h_{\beta_1}) \cdots \text{tr} \phi_n(h_{\beta_n}) \rangle = \langle \text{tr} \phi_1(h_{\beta_1}) \rangle \cdots \langle \text{tr} \phi_n(h_{\beta_n}) \rangle.$$

(3)

However, to demand equation (3) being satisfied for general theories is too restrictive physically because then every quantum state will be ultralocal and the Hamiltonian vanishes [12]. For our purposes it is completely sufficient to demand that there is a sufficiently large number of “small” independent loops. Of course, non-overlapping loops remain natural candidates for this although their precise definition is worth discussing – in particular from dimension 3 on. As a minimal version one could view a set of loops as non-overlapping if there is a surface in the space-time such that these loops form a set of non-overlapping loops. However, this condition seems to be too restrictive. Perhaps one could resort to the knot theoretical interpretation and the readability. Thus, we are given a theory that has a universal coupling constant and that is geometrically regular w.r.t. the area $|G_\beta|$ enclosed by the loop $\beta$. Therefore we will call a theory geometrically regular if there is a nonnegative real function $\sigma(\beta)$ such that first

$$\frac{d_\phi - \langle \text{tr} \phi(h_{\beta}) \rangle}{\sigma(\beta)}$$

is bounded as a function of $\beta$ and second $\sigma$ goes to 0 for shrinking $\beta$. Examples of conceivable functions $\sigma(\beta)$ are the area $|G_\beta|$ enclosed by $\beta$ or the length $L(\beta)$ of $\beta$.

We remark, that we will only need the validity of equation (4) for the case that $\phi$ is the representation having the smallest nonzero Casimir eigenvalue.

**Implications of these Principles**

1. If a theory obeys the principles 1 and 2, then all lattice measures are absolutely continuous w.r.t. to the lattice Haar measure.

2. If a theory obeys the principles 1, 2 and 3, then the continuum measure is purely singular w.r.t. to the Ashtekar-Lewandowski measure. This means it cannot be gained by the action method. Additionally, the measure is concentrated near non-generic strata, i.e. certain singular gauge fields.

The proofs are quite technical and will therefore be contained in a subsequent, detailed paper [3]. They use chiefly Fourier analysis on compact Lie groups.

We note that as already indicated several times the assumptions of the theorem above can be weakened drastically, but we skipped this here in favour of the physical interpretation and the readability.

**Examples**

Two-dimensional Yang-Mills Theory ($\mathbb{R}^2$)

The Wilson-loop expectation values of the Yang-Mills theory on the space-time $\mathbb{R}^2$ are completely known within the Ashtekar approach [16]

$$\langle \text{tr} \phi(h_{\beta}) \rangle = d_\phi e^{-\frac{1}{4} k^2 c_\phi |G_\beta|}.$$

(5)

Thus, we are given a theory that has a universal coupling constant and that is geometrically regular w.r.t. the area as indicated above. Moreover, it has been shown that non-overlapping loops are indeed independent. Consequently, the continuum measure is purely singular w.r.t. the Ashtekar-Lewandowski measure $d\mu_0$.

The singularity can also be interpreted physically: If one calculated the expectation values of $\text{tr} \phi(h_{\beta})$ w.r.t. $d\mu_0$, one would get 0 for all nontrivial $\phi$ and 1 in the trivial case. By means of equation (4) we see that the Ashtekar-Lewandowski measure is simply the naive strong-coupling limit $g \to \infty$ of the Yang-Mills measure. But, physically it should be clear that the cases of finite
and of infinite coupling are significantly different. The singularity encodes just this difference.

Two-dimensional Yang-Mills Theory (general)

There are also striking hints that the same results are valid for the other Yang-Mills theories on two-dimensional spaces as well. A more detailed analysis \[8\] analysis shows that our three criteria need only be met for appropriate “small” homotopically trivial loops. But just this has been shown by Sengupta \[13\]. He could prove on the classical level that in certain graphs the lattice measures are given by heat-kernel measures as in the \(\mathbb{R}^2\)-case. It can be expected that these results can be transferred to the Ashtekar approach as for \(\mathbb{R}^2\) because holonomies outside a graph have been unimportant for the continuum limit in \(\mathbb{R}^2\). In contrast to this, calculations of Aroca and Kubyshin \[1\] indicate for compact space-time that the area of the complement of a graph influences the expectation values by its finiteness. Hence, the universality of the coupling constant is given only approximately. However, the interpretation of our principles have to be handled with care at least for compact space-times: A limit \(\Delta t \to \infty\) is hard to define.

Nevertheless, in general one can expect singular continuum measures, hence a failure of the action method for \(d = 2\).

Theories Showing Confinement

Strictly speaking, the only theory that is proven to fulfill all three criteria is the two-dimensional Yang-Mills theory. However, the geometrical regularity is given for every theory with an area law \(\langle \text{tr } \phi(\mathcal{A}_\beta) \rangle = d_\phi \ e^{-\text{const}\left|G_\beta\right|}\) or a length law \(\langle \text{tr } \phi(\mathcal{A}_\beta) \rangle = d_\phi \ e^{-\text{const}L(\beta)}\). The former one is regarded as an indicator for confinement, and the latter one for deconfinement. Since among our three criteria just the geometrical regularity is the most important one for the singularity of the continuum measure, one could expect for both classes of theories that the action method fails. However, we have to mention that both the deconfinement and the confinement criterion need the corresponding laws for loops that are large in the time direction, but we actually need loops of small size to prove the singularity of the measure. Both requirements can be matched together only in the area-law case: Here one can still generate loops with small area by choosing very narrow loops that are large w.r.t. the time which is impossible in the length-law case. Therefore, up to now, we can only claim that the appearance of an area law is a convincing indicator for a singular continuum measure.

Conclusions

Despite to the mentioned difficulties, the singularity of the full interaction measure \(d\mu\) can be viewed as a typical property of the continuum. Hence, in particular, regular continuum limit and action method exclude each other: Assuming regularity the definition of the interaction measure via \(d\mu := e^{-S} \ d\mu_0\) is impossible. For all that it is mostly tried to get \(d\mu\) this way. Maybe that just this sticking to the action method is a deeper reason for the problems with the continuum limit or quantizations occurring permanently up to now. The desired absolute continuity seems to be a deceptfully simple tool, since it hides important physical phenomena. But, the singularity of a measure per se is completely harmless. In fact, strictly speaking, the measure is no physically relevant quantity; only expectation values are detectable. So far it is to be evaluate absolutely positive that the interaction measure \(d\mu\) has not been used in our principles, but rather some of its expectation values. It has been completely sufficient to know that \(d\mu\) does exist at all for extracting properties of \(d\mu\) from our physical principles in a mathematically rigorrous way. Thus, a measure is only the mathematical arena where anything happens. To know it might be superflious from the physical point of view; however, one must be able to rely on it.

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