A RELATIONSHIP BETWEEN MULTIPLE CONJUGATION QUANDLE/BIQUANDLE COLORINGS

TOMO MURAO

Abstract. We define a functor $Q$ from the category of multiple conjugation biquandles to that of multiple conjugation quandles. We show that for any multiple conjugation biquandle $X$, there is a one-to-one correspondence between the set of $X$-colorings and that of $Q(X)$-colorings diagrammatically for any handlebody-link and spatial trivalent graph. In particular, we prove that the set of $G$-family of Alexander biquandles colorings is isomorphic to that of $G$-family of Alexander quandles colorings as modules.

1. Introduction

A quandle $[12, 16]$ is an algebraic system whose axioms are derived from the Reidemeister moves on oriented link diagrams, and a biquandle $[2, 3, 15]$ is a generalization of a quandle. The two algebraic systems yield many invariants for not only classical links but also surface links, virtual links and so on. In particular, some invariants obtained from biquandles are stronger than those obtained from quandles for virtual links $[14]$.

On the other hand, as a corollary of $[19]$, it follows that there is a one-to-one correspondence between the set of biquandle colorings and that of quandle colorings for any classical links, where in the proof of the statement, any classical link need to be represented by a closed braid diagram.

Recently, Ishikawa $[10]$ constructed a left adjoint functor $B$ of a functor $Q$ from the category of biquandles to that of quandles which is defined in $[1]$. By using $B$, he proved that we can reconstruct a fundamental biquandle from a fundamental quandle, and there is a one-to-one correspondence between the set of biquandle colorings and that of quandle colorings for any classical and surface links, where in the statement, we can choose any diagram for classical and surface links. Here we note that any left adjoint functor of the functor $Q$ from the category of multiple conjugation biquandles to that of multiple conjugation quandles, which we will define in section 4 in this paper, has not been defined yet. Furthermore, Ishikawa and Tanaka $[11]$ explained the one-to-one correspondence proved in $[10]$ diagrammatically and concretely for classical and surface links.

Ishii, Iwakiri, Jang and Oshiro $[7]$ introduced a $G$-family of quandles to define colorings and invariants for handlebody-links and spatial trivalent graphs. A multiple conjugation quandle (MCQ) is introduced in $[5]$ as a universal symmetric quandle with a partial multiplication to define coloring invariants for handlebody-links, where a partial multiplication is an operation used at trivalent vertices. A $G$-family conjecture...
of biquandles [9] and a multiple conjugation biquandle (MCB) [8, 9] are biquan-
dle versions of those algebraic systems. However, although MCB colorings require
more calculation than MCQ colorings in general, it has not been known whether
an invariant obtained from MCB colorings is more effective than one obtained from
MCQ colorings. In this paper, we partially extend the result in [10, 11] to MCQ and
MCB colorings for handlebody-links and spatial trivalent graphs. Concretely, we
show that for any handlebody-links and spatial trivalent graphs, there is a one-to-
one correspondence between the set of MCB colorings and that of MCQ colorings
diagrammatically (Theorem 4.7). We also show that the set of $G$-family of Ale-
der biquandles colorings is isomorphic to that of $G$-family of Alexander quandles
as modules (Corollary 5.5).

This paper is organized into five sections. In Section 2, we review the defini-
tions of a handlebody-link, a spatial trivalent graph and the Reidemeister moves of
their diagrams. In Section 3, we recall basic notions and facts about quandles and
biquandles. In Section 4, we review the definitions of a multiple conjugation quan-
dle (MCQ) and a multiple conjugation biquandle (MCB) and define a functor $Q$
from the category of MCBs to that of MCQs. Moreover, we introduce colorings for
handlebody-links and spatial trivalent graphs by using an MCQ and an MCB and
show that for any MCB $X$, there is a one-to-one correspondence between the set of
$X$-colorings and that of $Q(X)$-colorings diagrammatically for any handlebody-link
and spatial trivalent graph. In Section 5, we introduce colorings for handlebody-
links and spatial trivalent graphs by using a $G$-family of quandles and a $G$-family
of biquandles. We discuss the similar correspondence between the sets of colorings
by using them.

2. Preliminaries

A handlebody-link is the disjoint union of handlebodies embedded in the 3-sphere
$S^3$ [4]. A handlebody-knot is a handlebody-link with one component. In this paper,
we assume that every component of a handlebody-link is of genus at least 1. An $S^1$-
orientation of a handlebody-link is an orientation of all genus 1 components of the
handlebody-link, where an orientation of a solid torus is an orientation of its core
$S^1$. Two $S^1$-oriented handlebody-links are equivalent if there exists an orientation-
-preserving self-homeomorphism of $S^3$ sending one to the other preserving the $S^1$-
orientation.

A spatial trivalent graph is a finite trivalent graph embedded in $S^3$. In this
paper, a trivalent graph may have a circle component, which has no vertices. A
$Y$-orientation of a spatial trivalent graph is a direction of all edges of the graph
satisfying that every vertex of the graph is both the initial vertex of a directed edge
and the terminal vertex of a directed edge (Figure 1). For a $Y$-oriented spatial
trivalent graph $K$ and an $S^1$-oriented handlebody-link $H$, we say that $K$ represents
$H$ if $H$ is a regular neighborhood of $K$ and the $S^1$-orientation of $H$ agrees with
the $Y$-orientation of $K$. Then any $S^1$-oriented handlebody-link can be represented
by some $Y$-oriented spatial trivalent graph. We define a diagram of an $S^1$-oriented
handlebody-link by a diagram of a $Y$-oriented spatial trivalent graph representing
the handlebody-link. Then the following theorem holds.

**Theorem 2.1** ([6]). For a diagram $D_i$ of an $S^1$-oriented handlebody-link $H_i$ ($i = 1, 2$), $H_1$ and $H_2$ are equivalent if and only if $D_1$ and $D_2$ are related by a finite
sequence of R1–R6 moves depicted in Figure 2 preserving Y-orientations, called the Reidemeister moves.

Here we note that the R1–R5 moves in Figure 2 are the Reidemeister moves for spatial trivalent graphs [13, 20]. Hence we can regard handlebody-links as a quotient structure of spatial trivalent graphs.

In this paper, we denote by $A(D)$ and $SA(D)$ the set of all arcs of $D$ and that of all semi-arcs of $D$ respectively, where a semi-arc is a piece of a curve each of whose endpoints is a crossing or a vertex. An orientation of a (semi-)arc of $D$ is also represented by the normal orientation obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. For any $m \in \mathbb{Z}_{\geq 0}$, we put $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. For an $S^1$-oriented handlebody-link $H$, the reverse of $H$, denoted $-H$, is obtained by reversing the orientations of all genus 1 components, and the reflection of $H$, denoted $H^*$, is the image of $H$ under an orientation-reversing self-homeomorphism of $S^3$. A split handlebody-link is a handlebody-link whose exterior is reducible. For any handlebody-links $H_1$ and $H_2$, we denote by $H_1 \sqcup H_2$ the split handlebody-link $H$ such that there exists a 2-sphere in $S^3 - H$ separating $S^3$ into two 3-balls, each of which contains only $H_1$ and $H_2$ respectively. In this paper, we often omit brackets. When we omit brackets, we apply binary operations from left on expressions, except for group operations, which we always apply first. For example, we write $a *_1 b *_2 c d *_3 (e *_4 f *_5 g)$ for $(((a *_1 b) *_2 (cd)) *_3 ((e *_4 f) *_5 g)$ simply, where each $*_i$ is a binary operation, and $c$ and $d$ are elements of the same group.

## 3. Quandles and Biquandles

We recall the definitions of a quandle and a biquandle.
\textbf{Definition 3.1 (12, 16).} A quandle is a non-empty set \( X \) with a binary operation \( \ast : X \times X \rightarrow X \) satisfying the following axioms.

- For any \( x \in X \), \( x \ast x = x \).
- For any \( y \in X \), the map \( S_y : X \rightarrow X \) defined by \( S_y(x) = x \ast y \) is a bijection.
- For any \( x, y, z \in X \), \( (x \ast y) \ast z = (x \ast z) \ast (y \ast z) \).

We define the \textit{type} of a quandle \( X \) by

\[ \text{type } X = \min \{ n \in \mathbb{Z}_{>0} \mid a \ast^n b = a \text{ (for any } a, b \in X) \} \]

where we set \( a \ast^0 b := S_0(a) \) and \( \min \emptyset := \infty \) for any \( i \in \mathbb{Z} \), \( a, b \in X \) and the empty set \( \emptyset \). Any finite quandle is of finite type.

Let \( X \) be an \( R[\pm 1] \)-module, where \( R \) is a commutative ring. For any \( a, b \in X \), we define \( a \ast b = ta + (1 - t)b \). Then \( X \) is a quandle, called an \textit{Alexander quandle}.

\textbf{Definition 3.2 (2, 8, 15).} A biquandle is a non-empty set \( X \) with binary operations \( \ast, \varpi : X \times X \rightarrow X \) satisfying the following axioms.

- For any \( x \in X \), \( x \ast x = x \varpi x \).
- For any \( y \in X \), the map \( S_y : X \rightarrow X \) defined by \( S_y(x) = x \varpi y \) is a bijection.
- For any \( y \in X \), the map \( S_y : X \rightarrow X \) defined by \( S_y(x) = x \varpi y \) is a bijection.
- The map \( S : X \times X \rightarrow X \times X \) defined by \( S(x, y) = (y \varpi x, x \varpi y) \) is a bijection.
- For any \( x, y, z \in X \),
  \[ (x \varpi y) \varpi (z \varpi y) = (x \varpi z) \varpi (y \varpi z), \]
  \[ (x \varpi y) \ast (z \varpi y) = (x \ast z) \varpi (y \varpi z), \]
  \[ (x \varpi y) \varpi (z \varpi y) = (x \varpi z) \varpi (y \varpi z). \]

We note that \((X, \ast)\) is a quandle if and only if \((X, \ast, \varpi)\) is a biquandle with \( x \varpi y = x \). Let \((X, \varpi, \varpi)\) be a biquandle. For any \( i \in \mathbb{Z} \) and \( a, b \in X \), we define \( a \ast^i b := S_i(a) \) and \( a \varpi^i b := S_0(a) \). Then we define two families of binary operations \( \ast^{[n]}, \varpi^{[n]} : X \times X \rightarrow X(n \in \mathbb{Z}) \) by the equalities:

\[ a \varpi^{[0]} b = a, \quad a \varpi^{[1]} b = a \varpi b, \quad a \varpi^{[i+j]} b = (a \varpi^{[i]} b) \varpi^{[j]} (b \varpi^{[i]} b), \]

\[ a \ast^{[0]} b = a, \quad a \ast^{[1]} b = a \ast b, \quad a \ast^{[i+j]} b = (a \ast^{[i]} b) \ast^{[j]} (b \ast^{[i]} b), \]

for any \( i, j \in \mathbb{Z} \). Since \( a = a \ast^{[0]} b = (a \varpi^{[-1]} b) \varpi^{[1]} (b \varpi^{[-1]} b) = (a \varpi^{[-1]} b) \varpi^{[1]} (b \varpi^{[-1]} b) \), we have \( a \varpi^{[-1]} b = a \varpi^{[-1]} (b \varpi^{[-1]} b) \varpi^{[-1]} b = b \), where we note that \( b \varpi^{[-1]} b \) is the unique element satisfying \( b \varpi^{[-1]} b = b \). We define the \textit{type} of a biquandle \( X \) by

\[ \text{type } X = \min \{ n \in \mathbb{Z}_{>0} \mid a \varpi^{[n]} b = a \text{ (for any } a, b \in X) \} \]

where we remind that \( \min \emptyset = \infty \) for the empty set \( \emptyset \). Any finite biquandle is of finite type.

Let \( X \) be an \( R[\pm 1, t \pm 1] \)-module, where \( R \) is a commutative ring. For any \( a, b \in X \), we define \( a \varpi b = ta + (s - t)b \) and \( a \varpi b = sa \). Then \( X \) is a biquandle, called an \textit{Alexander biquandle}. Any Alexander biquandle with \( s = 1 \) coincides with an Alexander quandle. For an Alexander biquandle \( X \), we have \( a \varpi^{[n]} b = t^n a + (s^n - t^n) b \) and \( a \varpi^{[n]} b = s^n a \) for any \( a, b \in X \).

For any biquandle \((X, \ast, \varpi)\), we have a quandle \((X, \ast)\), denoted by \( Q(X) \), by defining \( x \ast y = x \varpi y^{-1} \) for any \( x, y \in X \) [1]. There is a one-to-one correspondence
between the set of $X$-colorings and that of $Q(X)$-colorings for any classical link [14] and surface link [10][11].

For any Alexander biquandle $X$, which is an $R[s^{\pm 1}, t^{\pm 1}]$-module for some commutative ring $R$, $Q(X)$ is the Alexander quandle, which is the $R[(s^{-1}t)^{\pm 1}]$-module. That is, for any $x, y \in Q(X)$, it follows that $x \ast y = s^{-1}tx + (1 - s^{-1}t)y$.

**Proposition 3.3.** For any Alexander biquandle $X$ which is of finite type, type $X$ is divisible by type $Q(X)$.

**Proof.** Put $m = \text{type } X$ and $m' = \text{type } Q(X)$. Then it follows that $x^{m'} = t^m x + (s^m - t^m)y = x$ and $x^{m'} = s^m x = x$ for any $x, y \in X$. Hence we have $s^m = t^m = 1$, that is, $x^{s^m} y = s^{-m} t^m x + (1 - s^{-m} t^m) y = x$ for any $x, y \in Q(X)$. Therefore we have $m' \leq m$. We assume that $m = m' l_1 + l_2$ for some $l_1, l_2 \in \mathbb{Z}_{\geq 0}$ such that $0 < l_2 < m'$. Then we have $x^{s^m} y = s^{-l_2} t^{l_2} x + (1 - s^{-l_2} t^{l_2}) y = x$, which contradicts to $m' = \text{type } Q(X)$. Therefore we obtain $m = m' l_1$ for some $l_1 \in \mathbb{Z}_{\geq 0}$. □

Here we see two examples. Let $X$ be the Alexander biquandle $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]/(s - t)$. Then we have type $X = \infty$ and type $Q(X) = 1$. Next, let $X$ be the Alexander biquandle $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]/(s + t, t^4 - 1)$. Then we have type $X = 4$ and type $Q(X) = 2$.

## 4. A Relationship between MCQ/MCB Colorings

In this section, we recall the definitions of a multiple conjugation quandle (MCQ) and a multiple conjugation biquandle (MCB) and define a functor $Q$ from the category of MCBs to that of MCQs. We prove that for any MCB $X$, there is a one-to-one correspondence between the set of $X$-colorings and that of $Q(X)$-colorings for any $S^1$-oriented handlebody-link.

Firstly, we review the definition of a multiple conjugation quandle (MCQ).

**Definition 4.1** ([5]). A multiple conjugation quandle (MCQ) $X$ is the disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a binary operation $\ast : X \times X \to X$ satisfying the following axioms.

- For any $a, b \in G_\lambda$, $a \ast b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x \ast e_\lambda = x$ and $x \ast ab = (x \ast a) \ast b$, where $e_\lambda$ is the identity of $G_\lambda$.
- For any $x, y, z \in X$, $(x \ast y) \ast z = (x \ast z) \ast (y \ast z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $ab \ast x = (a \ast x)(b \ast x)$, where $a \ast x, b \ast x \in G_\mu$ for some $\mu \in \Lambda$.

We remark that an MCQ itself is a quandle. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ and $Y = \bigsqcup_{\mu \in M} G_\mu$ be MCQs. An MCQ homomorphism $\phi : X \to Y$ is a map from $X$ to $Y$ satisfying $\phi(x \ast y) = \phi(x) \ast \phi(y)$ for any $x, y \in X$ and $\phi(ab) = \phi(a) \phi(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. We call a bijective MCQ homomorphism an MCQ isomorphism. $X$ and $Y$ are isomorphic if there exists an MCQ isomorphism from $X$ to $Y$. There is a category MCQ of MCQs, whose objects are MCQs and whose morphisms are MCQ homomorphisms.

Next, we review the definition of a multiple conjugation biquandle (MCB). Let $X$ be the disjoint union of groups $G_\lambda (\lambda \in \Lambda)$. We denote by $G_a$ the group $G_\lambda$ containing $a \in X$. We also denote by $e_\lambda$ the identity of $G_\lambda$. Then the identity of $G_a$ is denoted by $e_a$ for any $a \in X$. 
Definition 4.2 ([8][9]). A multiple conjugation biquandle (MCB) $X$ is the disjoint union of groups $G_{\lambda} (\lambda \in \Lambda)$ with binary operations $\ast, \bar{\ast}, \ast_{\lambda}, \bar{\ast}_{\lambda}, \bar{\ast}_{\lambda}^{-1}$ satisfying the following axioms.

- For any $x, y, z \in X$,
  
  $$(x \ast y) \ast (z \ast y) = (x \ast z) \ast (y \ast z),$$
  $$x \ast (y \ast z) = (x \ast y) \ast (y \ast z),$$
  $$x \ast (y \ast y) = (x \ast z) \ast (y \ast z).$$

- For any $a, x \in X$, $\ast x : G_{a} \to G_{ax}$ and $\bar{\ast} x : G_{a} \to G_{a\bar{x}}$ are group homomorphisms.

- For any $x \in X$ and $a, b \in G_{\lambda}$,
  
  $$x \ast ab = (x \ast a) \ast (b \bar{\ast} a),$$
  $$x \bar{\ast} ab = (x \bar{\ast} a) \bar{\ast} (b \bar{\ast} a),$$
  $$a^{-1} b \bar{\ast} a = ba^{-1} \ast a.$$ 

We remark that an MCB itself is a biquandle. Let $X = \bigsqcup_{\lambda \in \Lambda} G_{\lambda}$ and $Y = \bigsqcup_{\mu \in M} G_{\mu}$ be MCBs. An MCB homomorphism $\phi : X \to Y$ is a map from $X$ to $Y$ satisfying $\phi(x \ast y) = \phi(x) \ast \phi(y)$ and $\phi(x \bar{\ast} y) = \phi(x) \bar{\ast} \phi(y)$ for any $x, y \in X$ and $\phi(ab) = \phi(a)\phi(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_{\lambda}$. We call a bijective MCB homomorphism an MCB isomorphism. $X$ and $Y$ are isomorphic if there exists an MCB isomorphism from $X$ to $Y$. There is a category MCB of MCBs, whose objects are MCBs and whose morphisms are MCB homomorphisms.

Definition 4.3. We define a functor $Q$ from MCB to MCQ by $Q((X, \ast, \bar{\ast})) = (X, \ast)$ with $x \ast y = x \ast_{\lambda} y \bar{\ast}_{\lambda}^{-1} y$ for any MCB $(X, \ast, \bar{\ast})$ and $Q(\phi) = \phi$ for any MCB homomorphism $\phi$.

In the following, we see that the functor $Q$ is well-defined.

Proposition 4.4. The functor $Q : \text{MCB} \to \text{MCQ}$ is well-defined.

Proof. Let $X = \bigsqcup_{\lambda \in \Lambda} G_{\lambda}$ be an MCB. At first, since $a^{-1} b \bar{\ast} a = ba^{-1} \ast a$ for any $a, b \in G_{\lambda}$, we have $a \ast b = a \ast b \bar{\ast}^{-1} b = b^{-1} ab$. Second, for any $x \in X$ and $a, b \in G_{\lambda}$,

$$x \bar{\ast}^{-1} (a \ast b) \bar{\ast}^{-1} b \ast ab = x \bar{\ast}^{-1} (a \ast b) \bar{\ast}^{-1} b \ast b(a \ast b \bar{\ast}^{-1} b)$$

$$= x \bar{\ast}^{-1} (a \ast b) \bar{\ast}^{-1} b \bar{\ast} b(a \ast b \bar{\ast}^{-1} b)$$

$$= x \bar{\ast}^{-1} (a \ast b) \bar{\ast} (a \ast b)$$

$$= x.$$

Hence we have

$$x \ast ab = x \ast_{\lambda} ab \bar{\ast}_{\lambda}^{-1} ab$$

$$= (x \ast a) \ast (b \bar{\ast} a) \bar{\ast}^{-1} (a \ast b) \bar{\ast}^{-1} b$$

$$= ((x \ast a \bar{\ast}^{-1} a) \bar{\ast} a) \ast (b \bar{\ast} a) \bar{\ast}^{-1} (a \ast b) \bar{\ast}^{-1} b$$

$$= ((x \ast a \bar{\ast}^{-1} a) \ast b) \bar{\ast} (a \ast b) \bar{\ast}^{-1} (a \ast b) \bar{\ast}^{-1} b$$

$$= x \ast a \bar{\ast}^{-1} a \ast b \bar{\ast}^{-1} b$$

$$= (x \ast a) \ast b.$$
Furthermore, we can easily check that \( x * e_\lambda = x \). Third, for any \( x, y, z \in X \), we obtain \( (x * y) * z = (x * z) * (y * z) \) since \( \mathcal{Q}(X) \) is a quandle \([1]\). Finally, for any \( x \in X \) and \( a, b \in G_\lambda \),

\[
ab * x = ab \# x \overline{\tau}^{-1} x \\
= (a \# x \overline{\tau}^{-1} x)(b \# x \overline{\tau}^{-1} x) \\
= (a \# x)(b \# x)
\]

since \( \tau x : G_a \to G_a \overline{\tau} \) and \( \overline{\tau} x : G_a \to G_a \tau \) are group homomorphisms. Therefore \( \mathcal{Q}(X) \) is an MCQ.

On the other hand, for any MCB homomorphism \( \phi : X \to Y \) and \( x, y \in X \), we have

\[
\mathcal{Q}(\phi)(x * y) = \phi(x * y) \\
= \phi(x \# y \overline{\tau}^{-1} y) \\
= \phi(x) \# \phi(y) \overline{\tau}^{-1} \phi(y) \\
= \phi(x) * \phi(y) \\
= \mathcal{Q}(\phi)(x) * \mathcal{Q}(\phi)(y).
\]

Hence \( \mathcal{Q}(\phi) \) is an MCQ homomorphism from \( \mathcal{Q}(X) \) to \( \mathcal{Q}(Y) \). Furthermore it is clear that \( \mathcal{Q}(\text{id}_X) = \text{id}_{\mathcal{Q}(X)} \) and \( \mathcal{Q}(\psi \circ \phi) = \mathcal{Q}(\psi) \circ \mathcal{Q}(\phi) \). This completes the proof. \( \square \)

Let \( X = \bigsqcup_{\lambda \in \Lambda} G_\lambda \) be an MCQ (resp. MCB) and let \( D \) be a diagram of an \( S^1 \)-oriented handlebody-link \( H \). An \( X \)-coloring of \( D \) is a map \( C : A(D) \to X \) (resp. \( S_A(D) \to X \)) satisfying the conditions depicted in Figure 3 (resp. Figure 4) at each crossing and vertex. We denote by \( \text{Col}_X(D) \) the set of all \( X \)-colorings of \( D \).

**Figure 3.** An MCQ-coloring of \( D \).

**Figure 4.** An MCB-coloring of \( D \).
Proposition 4.5 (387). Let \( X = \bigsqcup_{\lambda \in \Lambda} G_{\lambda} \) be an MCQ or MCB and let \( D \) be a diagram of an \( S^3 \)-oriented handlebody-link \( H \). Let \( D' \) be a diagram obtained by applying one of Reidemeister moves to the diagram \( D \) once. For an \( X \)-coloring \( C \) of \( D \), there is a unique \( X \)-coloring \( C' \) of \( D' \) which coincides with \( C \) except near the point where the move is applied.

By this proposition, the cardinality of \( X \)-colorings of \( D \) is an invariant of \( H \).

Let \( D \) and \( D' \) be diagrams of \( S^3 \)-oriented handlebody-links \( H \) and \( H' \) respectively. In the following, we define diagrams \( -D, D^v, D^h, D \sqcup D' \) and \( W(D) \) (see Figure 5). We denote by \( -D \) and \( D^v \) the diagrams of \(-H \) and \( H^* \) obtained from \( D \) by reversing the orientations of all (semi-)arcs and switching all crossings respectively. We can regard that \( D \) is depicted in an \( xy \)-plane. Let \( \iota \) be the involution \( (x,y) \mapsto (-x,y) \). Then we define the diagram \( D^h \) of \( H^* \) by \( D^h = \iota(D) \). We regard \( \iota \) as the map from \( \mathcal{A}(D) \) to \( \mathcal{A}(D^h) \) (or \( \mathcal{SA}(D) \) to \( \mathcal{SA}(D^h) \)).

An \( S^3 \)-oriented handlebody-link diagram in \( S^2 \) is a split diagram if there is a loop in the exterior of the diagram separating \( S^2 \) into two disks each containing part of it. We denote by \( D \sqcup D' \) the split diagram of \( H \sqcup H' \) such that \( D \) and \( D' \) represent \( H \) and \( H' \) respectively. We denote by \( W(D) \) the diagram of the \( S^3 \)-oriented handlebody-link \( H \sqcup -H^* \) obtained from \( D \sqcup -D^v \) by sliding \(-D^v \) under \( D \) and shifting it slightly to the normal orientations of all (semi-)arcs of \( D \).

![Figure 5. Diagrams D, -D, D^v, D^h and W(D).](image)

Let \( X \) be an MCB. We note here that \( \mathcal{SA}(-D) = \mathcal{SA}(D) \). For any \( C \in \mathcal{Col}_X(D) \), we define \( C^* \in \mathcal{Col}_X(-D^h) \) by \( C^* = C \circ \iota \) as shown in Figure 6, where each \( x_i \) is an element of \( X \). We note that the \( X \)-coloring \( C^* \) is shown in Figure 7 at each crossing and vertex. We define \( C \sqcup C^* \in \mathcal{Col}_X(D \sqcup -D^h) \) by \( (C \sqcup C^*)|_{\mathcal{SA}(D)} = C \) and \( (C \sqcup C^*)|_{\mathcal{SA}(-D^h)} = C^* \). We set \( \mathcal{Col}_X^W(D \sqcup -D^h) := \{ C \sqcup C^* \mid C \in \mathcal{Col}_X(D) \} \). We denote by \( \mathcal{Col}_X^W(W(D)) \) the set of \( X \)-colorings of \( W(D) \) satisfying the conditions depicted in Figure 8 at each crossing and vertex.

Lemma 4.6. Let \( X \) be an MCB. For the \( X \)-coloring depicted in Figure 9, where \( x_i, x'_i, y_i, y'_i, z_i, z'_i, w_i \) and \( w'_i \) are elements of \( X \) for any \( i \), it follows that \( (x_1, \ldots, x_l) = (x'_1, \ldots, x'_l) \) if and only if \( (y_1, \ldots, y_l) = (y'_1, \ldots, y'_l) \).

Proof. We give the proof by induction on \( l \). When \( l = 1 \), the statement holds immediately. Assume that the statement is proved for \( l - 1 \). Suppose that \( (x_1, \ldots, x_l) = (x'_1, \ldots, x'_l) \)
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Figure 6. Colorings $C$ and $C^*$.

$C \in \text{Col}_X(D)$

$C^* \in \text{Col}_X(-D^h)$

Figure 7. The well-definedness of $C^* \in \text{Col}_X(-D^h)$.

$C \in \text{Col}_X(D)$

$C^* \in \text{Col}_X(-D^h)$

Figure 8. The coloring conditions of $\text{Col}_X^W(W(D))$.

$(x'_1, \ldots, x'_l)$. Then we have $z_i = z'_i$, $y_i = y'_i$ and $w_i = w'_i$ for any $i = 1, \ldots, l - 1$ (see Figure 9). Hence we obtain the $X$-coloring depicted in Figure 10 from the $X$-coloring depicted in Figure 9. Therefore we have $(y_1, \ldots, y_{l-1}) = (y'_1, \ldots, y'_{l-1})$ by the assumption. Consequently, it follows that $(y_1, \ldots, y_l) = (y'_1, \ldots, y'_l)$. In the same way, if we suppose that $(y_1, \ldots, y_l) = (y'_1, \ldots, y'_l)$, then it follows that $(x_1, \ldots, x_l) = (x'_1, \ldots, x'_l)$, where we also have $z_i = z'_i$ and $w_i = w'_i$ for any $i = 1, \ldots, l - 1$. \hfill $\square$
Theorem 4.7. Let $X$ be an MCB and let $D$ be a diagram of an $S^1$-oriented handlebody-link. Then there is a one-to-one correspondence between $\text{Col}_X(D)$ and $\text{Col}_{Q(X)}(D)$.

Proof. By [17], any $S^1$-oriented handlebody-link can be represented by $\text{cl}(\text{braid}^{m_1, \ldots, m_s}(b_0))$, where $b_0$ is a classical $l$-braid diagram and $m_i, n_i \in \mathbb{Z}_{>0}$, and we can deform it into $\text{cl}(b)$, where $b$ is the trivalent braid diagram as shown in Figure 11. Then we may assume that $D$ has the resulting form $\text{cl}(b)$. Here we note that any MCQ(MCB)-coloring of $D$ is determined by the colors of all (semi-)arcs incident to the top endpoints of the trivalent braid diagram $b$.

First, for any $C_1 \in \text{Col}_{Q(X)}(D)$, we denote by $\psi^X_1(C_1)$ the $X$-coloring of $W(D)$ depicted in Figure 12. Then $\psi^X_1$ is a bijective map from $\text{Col}_{Q(X)}(D)$ to $\text{Col}_X(W(D))$. Second, we can deform $W(D)$ into $D \sqcup -D^h$ by Reidemeister moves as shown in Figure 13. By Proposition 4.5 and Lemma 4.6, we obtain a bijective
map $\psi_2^X$ from $\text{Col}_X^W(W(D))$ to $\text{Col}_X^W(D \sqcup -D^h)$ as shown in Figure 13 where $x_i$ and $y_i$ are elements in $X$. Finally, we define a map $\psi_3^X$ from $\text{Col}_X^W(D \sqcup -D^h)$ to $\text{Col}_X(D)$ by $\psi_3^X(C_3 \sqcup C_3^*) = C_3$, which is bijective obviously. Therefore $\psi_3^X \circ \psi_2^X \circ \psi_1^X$ is a bijective map from $\text{Col}_{Q(X)}(D)$ to $\text{Col}_X(D)$.

\[ \text{cl(braid}_{m_1, \ldots, m_s}(b_0)) \]

\[ \text{cl}(b) \]

**Figure 11.** A closed trivalent braid diagram.

\[ (x * y = x * y * y^{-1} y) \]

**Figure 12.** The map $\psi_1^X : \text{Col}_{Q(X)}(D) \rightarrow \text{Col}_X^W(W(D))$. 

Figure 13. The deformation from $W(D)$ to $D \sqcup -D^h$.

5. A RELATIONSHIP BETWEEN $G$-FAMILY OF QUANDLES/BIQUANDLES COLORINGS

A $G$-family of quandles (resp. biquandles), which are algebraic systems whose axioms are motivated from handlebody-knot theory, yields an MCQ (resp. MCB).
In this section, we recall the definitions of a $G$-family of quandles and biquandles and define a map from the set of $G$-families of biquandles to that of $G$-families of quandles. We prove that there is the similar correspondence between $G$-family of quandles and biquandles colorings to the one between MCQ and MCB colorings.

**Definition 5.1 ($[7]$).** Let $G$ be a group with identity element $e$. A $G$-family of quandles is a non-empty set $X$ with a family of binary operations $*^g : X \times X \to X$ ($g \in G$) satisfying the following axioms.

- For any $x \in X$ and $g \in G$, $x *^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x *^{gh} y = (x *^g y) *^h y$ and $x *^e y = x$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$.

Let $R$ be a ring and let $G$ be a group with identity element $e$. Let $X$ be a right $R[G]$-module, where $R[G]$ is the group ring of $G$ over $R$. Then $(X, \{*^g\}_{g \in G})$ is a $G$-family of quandles, called a $G$-family of Alexander quandles, with $x *^g y = xg + ye - g$ $[7]$. Let $(X, \{\times^g\})$ be a quandle and let $m$ be the type of $X$. Then $(X, \{*^g\}_{g \in Z_{km}})$ is a $Z_{km}$-family of quandles for any $k \in Z_{\geq 0}$ $[7]$. In particular, when $X$ is an Alexander quandle, $(X, \{*^i\}_{i \in Z_{km}})$ is called a $Z_{km}$-family of Alexander quandles.

Let $(X, \{*^g\}_{g \in G})$ be a $G$-family of quandles. Then $X \times G = \bigsqcup_{x \in X} \{x\} \times G$ is an MCQ with

$$(x, g) * (y, h) := (x *^g y, h^{-1}gh), \quad (x, g)(x, h) := (x, gh)$$

for any $x, y \in X$ and $g, h \in G$ $[5]$. We call it the associated MCQ of $(X, \{*^g\}_{g \in G})$.

**Definition 5.2 ($[8]$ $[9]$).** Let $G$ be a group with identity element $e$. A $G$-family of biquandles is a non-empty set $X$ with two families of binary operations $\bar{*}^g, \bar{\times}^g : X \times X \to X$ ($g \in G$) satisfying the following axioms.

- For any $x \in X$ and $g \in G$,
  $$x \bar{*}^g x = x \bar{\times}^g x.$$
- For any $x, y \in X$ and $g, h \in G$,
  $$x \bar{*}^{gh} y = (x \bar{*}^g y) \bar{\times}^h (y \bar{*}^g y), \quad x \bar{\times}^e y = x,$$
  $$x \bar{\times}^{gh} y = (x \bar{\times}^g y) \bar{*}^h (y \bar{\times}^g y), \quad x \bar{\times}^g y = x.$$
- For any $x, y, z \in X$ and $g, h \in G$,
  $$(x \bar{*}^g y) \bar{\times}^h (z \bar{\times}^g y) = (x \bar{*}^h z) \bar{\times}^{h^{-1}gh} (y \bar{*}^h z),$$
  $$(x \bar{\times}^g y) \bar{*}^h (z \bar{\times}^g y) = (x \bar{\times}^h z) \bar{*}^{h^{-1}gh} (y \bar{\times}^h z),$$
  $$(x \bar{\times}^g y) \bar{\times}^h (z \bar{\times}^g y) = (x \bar{\times}^h z) \bar{\times}^{h^{-1}gh} (y \bar{\times}^h z).$$

Let $R$ be a ring, $G$ be a group with identity element $e$ and let $f : G \to Z(G)$ be a homomorphism, where $Z(G)$ is the center of $G$. Let $X$ be a right $R[G]$-module, where $R[G]$ is the group ring of $G$ over $R$. Then $(X, \{\bar{*}^g\}_{g \in G}, \{\bar{\times}^g\}_{g \in G})$ is a $G$-family of biquandles, called a $G$-family of Alexander biquandles, with $x \bar{*}^g y = xg + y(f(g) - g)$ and $x \bar{\times}^g y = xf(g)$ $[8]$. Let $(X, \{\hat{\times}^g\})$ be a biquandle and let $m$ be the type of $X$. Then $(X, \{\hat{\times}^i\}_{i \in Z_{km}}, \{\bar{\times}^i\}_{i \in Z_{km}})$ is a $Z_{km}$-family of biquandles for any $k \in Z_{\geq 0}$ $[9]$. In particular, when $X$ is an Alexander biquandle, $(X, \{\hat{\times}^i\}_{i \in Z_{km}}, \{\bar{\times}^i\}_{i \in Z_{km}})$ is called a $Z_{km}$-family of Alexander biquandles.
Let \((X, \{*_g\}_{g \in G}, \{\pi_g\}_{g \in G})\) be a \(G\)-family of biquandles. Then \(X \times G = \bigsqcup_{x \in X} \{x\} \times G\) is an MCB with
\[
(x, g) \ast (y, h) := (x \ast y, h^{-1}gh), \quad (x, g)(x, h) := (x, gh),
\]
\[
(x, g) \pi (y, h) := (x \pi^h y, g)
\]
for any \(x, y \in X\) and \(g, h \in G\). We call it the associated MCB of \((X, \{*_g\}_{g \in G}, \{\pi_g\}_{g \in G})\).

Let \((X, \{*_g\}_{g \in G}, \{\pi_g\}_{g \in G})\) be a \(G\)-family of biquandles. For any \(x, y \in X\) and \(g \in G\), it follows that
\[
(x \ast^g y) \ast^{g^{-1}} (y \ast^g y) = x \ast^e y = x
\]
and
\[
x \ast^{g^{-1}} (y \ast^g y) \ast^g y = \{x \ast^{g^{-1}} (y \ast^g y)\} \ast^g \{(y \ast^g y) \ast^{g^{-1}} (y \ast^g y)\}
= x \ast^e (y \ast^g y)
= x.
\]
Hence the map \(*_g y : X \to X\), which sends \(x\) into \(x \ast^g y\), is a bijection and \((*_g y)^{-1}(x) = x \ast^{g^{-1}} (y \ast^g y)\). Similarly, the map \(\pi^g y : X \to X\), which sends \(x\) into \(x \pi^g y\), is a bijection and \((\pi^g y)^{-1}(x) = x \pi^{g^{-1}} (y \pi^g y)\). Then we have the following proposition.

**Proposition 5.3.** Let \((X, \{*_g\}_{g \in G}, \{\pi_g\}_{g \in G})\) be a \(G\)-family of biquandles. Then \((X, \{*_g\}_{g \in G})\) is a \(G\)-family of quandles by defining \(x \ast^g y = (x \ast^g y) \pi^{g^{-1}} (y \pi^g y)\).

**Proof.**

- For any \(x \in X\) and \(g \in G\),
  \[
x \ast^g x = (x \ast^g x) \pi^{g^{-1}} (x \pi^g x) = (x \pi^g x) \pi^{g^{-1}} (x \pi^g x) = x \pi^e x = x.
  \]

- For any \(x, y \in X\), \(g, h \in G\),
  \[
x \ast^{gh} y \pi^g y \pi^h (y \pi^g y)
  = x \ast^{gh} y \ast^{h^{-1}g^{-1}} (y \pi^g y) \pi^g y \pi^h (y \pi^g y)
  = x \ast^{gh} y \ast^{h^{-1}} (y \pi^g y) \pi^{g^{-1}} \{(y \pi^g y) \pi^{h^{-1}} (y \pi^g y)\} \pi^g y \pi^h (y \pi^g y)
  = x \ast^{gh} y \ast^{h^{-1}} (y \pi^g y) \pi^{g^{-1}} (y \pi^g y) \pi^g y \pi^h (y \pi^g y)
  = x \ast^{gh} y \ast^{h^{-1}} (y \pi^g y) \pi^h (y \pi^g y)
  = x \ast^{gh} y \pi^{h^{-1}} \{(y \pi^g y) \pi^h (y \pi^g y)\} \pi^h (y \pi^g y)
  = x \ast^{gh} y \pi^h y.
  \]
On the other hand,
\[
(x \ast^g y) \ast^h y \equiv^g y \ast^h (y \equiv^g y)
\]
\[
= \{ x \ast^g y \ast^h y \ast^{h^{-1}} (y \ast^h y) \equiv^g y \ast^h (y \equiv^g y)
\]
\[
= \{ x \ast^g y \ast^h y \ast^{h^{-1}} (y \ast^h y) \equiv^g y \ast^h (y \equiv^g y)
\]
\[
= (x \ast^g y) \ast^h (y \equiv^g y)
\]
\[
= x \ast^{gh} y.
\]

Therefore we have \( x \ast^{gh} y = (x \ast^g y) \ast^h y \). we can easily check that \( x \ast^g y = x \)
for any \( x, y \in X \).

\* For any \( x, y, z \in X, g, h \in G \) and \( \alpha = (y \ast^h z) \equiv^{h^{-1}} (z \ast^h z), \)
\[
(x \ast^g y) \ast^h z \equiv^{h^{-1}} g \ast^h \ast^{h^{-1}} g h g (z \ast^h z)
\]
\[
= \{ ((x \ast^g y) \equiv^{h^{-1}} (y \ast^g y) \ast^h z \equiv^{h^{-1}} (z \ast^h z)) \equiv^{h^{-1}} g h g (z \ast^h z) \}
\]
\[
= \{ (x \ast^g y) \equiv^{h^{-1}} (y \ast^g y) \ast^h z \equiv^{h^{-1}} (z \ast^h z) \}
\]
\[
= (x \ast^g y) \ast^h (y \ast^g y)
\]
\[
= x \ast^{gh} y.
\]

On the other hand,
\[
(x \ast^h z) \ast^{h^{-1}} g h g (y \ast^h z) \ast^{h^{-1}} g h (z \ast^h z)
\]
\[
= ((x \ast^h z) \ast^{h^{-1}} (z \ast^h z)) \ast^{h^{-1}} g h g ((y \ast^h z) \ast^{h^{-1}} (z \ast^h z)) \ast^{h^{-1}} g h (z \ast^h z)
\]
\[
= ((x \ast^h z) \ast^{h^{-1}} (z \ast^h z)) \ast^{h^{-1}} g h g (\alpha \equiv^{h^{-1}} g h g (z \ast^h z))
\]
\[
= (x \ast^h z) \ast^{h^{-1}} g h g (z \ast^h z)
\]
\[
= (x \ast^g y) \ast^h (z \ast^g y).
\]

Therefore we have \( x \ast^g y \ast^h z = (x \ast^h z) \ast^{h^{-1}} g h (y \ast^h z) \).

\[\blacksquare\]

By Proposition 5.3, for any \( G \)-family of biquandles \( (X, \{\ast^g\}_{g \in G}, \{\equiv^g\}_{g \in G}) \), we
have a \( G \)-family of quandles \( (X, \{\ast^g\}_{g \in G}) \), denoted by \( \mathcal{Q}_G(X) \), by defining \( x \ast^g y = (x \ast^g y) \equiv^{g^{-1}} (y \ast^g y) \). Then \( \mathcal{Q}_G \) is a map from the set of \( G \)-families of biquandles
of \( G \)-families of quandles. In particular, let \( (X, \{\ast^g\}_{g \in G}, \{\equiv^g\}_{g \in G}) \) be a
\( G \)-family of Alexander biquandles, where \( X \) is a right \( R[G] \)-module for some ring \( R \)
and group \( G \) with a homomorphism \( \phi : G \to Z(G) \). Then \( \mathcal{Q}_G(X) \) is a \( G \)-family of
Alexander quandles with the action \( xg := xg\phi(g) \) since for any \( x, y \in X \) and \( g \in G \),
we have \( x \ast^g y = (x \ast^g y) \equiv^{g^{-1}} (y \ast^g y) = xg\phi(g) + y(\epsilon - g\phi(g)) \).
For any $G$-family of biquandles $(X, \{x^g\}_{g \in G}, \{\bar{x}^g\}_{g \in G})$ and its associated MCB $X \times G$, the MCQ $Q(X \times G)$ coincides with the associated MCQ $Q_G(X) \times G$ of the $G$-family of quandles $Q_G(X)$ with $(x, g) \ast (y, h) = ((x \bar{x}^h y)^{h^{-1}} (y \bar{x}^h y), h^{-1}gh)$.

Let $G$ be a group and let $D$ be a diagram of an $S^1$-oriented handlebody-link $H$. A $G$-flow of $D$ is a map $\phi : \mathcal{A}(D) \to G$ satisfying the conditions depicted in Figure 14 at each crossing and vertex. In this paper, to avoid confusion, we often represent an element of $G$ with an underline. We denote by $(D, \phi)$, which is called a $G$-flowed diagram of $H$, a diagram $D$ given a $G$-flow $\phi$, and by $\text{Flow}(D; G)$ the set of all $G$-flows of $D$. We can identify a $G$-flow $\phi$ with a homomorphism from the fundamental group $\pi_1(S^3 - H)$ to $G$.

![Figure 14. A $G$-flow of $D$.](image)

Let $G$ be a group and let $D$ be a diagram of an $S^1$-oriented handlebody-link $H$. Let $D'$ be a diagram obtained by applying one of Reidemeister moves to the diagram $D$ once. For any $G$-flow $\phi$ of $D$, there is a unique $G$-flow $\phi'$ of $D'$ which coincides with $\phi$ except near the point where the move is applied. Therefore the cardinality of the set of $G$-flows of $D$, denoted by $\#\text{Flow}(D; G)$, is an invariant of $H$. We call the $G$-flow $\phi'$ the associated $G$-flow of $\phi$ and the $G$-flowed diagram $(D', \phi')$ the associated $G$-flowed diagram of $(D, \phi)$.

Let $X$ be a $G$-family of quandles (resp. biquandles) and let $(D, \phi)$ be a $G$-flowed diagram of an $S^1$-oriented handlebody-link. An $X$-coloring of $(D, \phi)$ is a map $C : \mathcal{A}(D, \phi) \to X$ (resp. $\mathcal{SA}(D, \phi) \to X$ ) satisfying the conditions depicted in Figure 15 (resp. Figure 16) at each crossing and vertex. We denote by $\text{Col}_X(D, \phi)$ the set of all $X$-colorings of $(D, \phi)$. We note that when $X$ is a $G$-family of Alexander (bi)quandles that is a right $R[G]$-module for some ring $R$, the set $\text{Col}_X(D, \phi)$ is a right $R$-module with the action $(C \cdot r)(\alpha) := C(\alpha)r$ and the addition $(C + C')(\alpha) := C(\alpha) + C'(\alpha)$ for any $C, C' \in \text{Col}_X(D, \phi)$, $\alpha \in \mathcal{A}(D, \phi)$ (or $\alpha \in \mathcal{SA}(D, \phi)$) and $r \in R$.

![Figure 15. A $G$-family of quandles coloring of $(D, \phi)$.](image)

**Proposition 5.4 ([7, 9]).** Let $X$ be a $G$-family of (bi)quandles and let $(D, \phi)$ be a $G$-flowed diagram of an $S^1$-oriented handlebody-link. Let $(D', \phi')$ be the associated...
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G-flowed diagram of $(D, \phi)$. For any $X$-coloring $C$ of $(D, \phi)$, there is a unique $X$-coloring $C'$ of $(D', \phi')$ which coincides with $C$ except near the point where the move is applied.

By this proposition, we have $\#\text{Col}_X(D, \phi) = \#\text{Col}_X(D', \phi')$.

Let $X$ be a $G$-family of quandles (resp. biquandles), $X \times G$ the associated MCQ (resp. MCB) of $X$ and let $\text{pr}_G$ and $\text{pr}_X$ be the natural projections from $X \times G$ to $G$ and from $X \times G$ to $X$ respectively.

For any $\phi \in \text{Flow}(D; G)$, we define $\text{Col}_{X \times G}^\phi(D) := \{ C \in \text{Col}_{X \times G}(D) \mid \text{pr}_G \circ C = \phi \}$, where for any $a \in \text{SA}(D)$ and $\tilde{\alpha} \in \text{A}(D)$ satisfying $\alpha \subset \tilde{\alpha}$, we put $\phi(\alpha) := \phi(\tilde{\alpha})$ when $X$ is a $G$-family of biquandles. Then we can identify $\text{Col}_{X \times G}^\phi(D)$ with $\text{Col}_X(D, \phi)$, that is, for any $C \in \text{Col}_{X \times G}^\phi(D)$, the map $\text{pr}_G \circ C$ corresponds to the $G$-flow of $\phi$, and the map $\text{pr}_X \circ C$ corresponds to the $X$-coloring of $(D, \phi)$. Therefore $\text{Col}_{X \times G}^\phi(D)$ is also a right $R$-module in the same way as $\text{Col}_X(D, \phi)$. Then we obtain the following corollary by Theorem 4.7.

**Corollary 5.5.** Let $X$ be a $G$-family of biquandles and let $(D, \phi)$ be a $G$-flowed diagram of an $S^1$-oriented handlebody-link. Then there is a one-to-one correspondence between $\text{Col}_X(D, \phi)$ and $\text{Col}_{Q_G(X)}(D, \phi)$. In particular, when $X$ is a $G$-family of Alexander biquandles, $\text{Col}_X(D, \phi)$ is isomorphic to $\text{Col}_{Q_G(X)}(D, \phi)$ as right $R$-modules.

**Proof.** We remind that we can identify $\text{Col}_{X \times G}^\phi(D)$ with $\text{Col}_X(D, \phi)$ and $\text{Col}_{Q_G(X) \times G}^\phi(D)$ with $\text{Col}_{Q_G(X)}(D, \phi)$, and we note that $\text{Col}_{X \times G}(D) = \bigsqcup_{\phi' \in \text{Flow}(D; G)} \text{Col}_{X \times G}^{\phi'}(D)$ and $\text{Col}_{Q_G(X) \times G}(D) = \text{Col}_{Q_G(X) \times G}(D) = \bigsqcup_{\phi' \in \text{Flow}(D; G)} \text{Col}_{Q_G(X) \times G}^{\phi'}(D)$. By the proof of Theorem 4.7, the map $\Psi^{X \times G} := \psi_3^{X \times G} \circ \psi_2^{X \times G} \circ \psi_1^{X \times G}$ is a bijective map from $\text{Col}_{Q_G(X) \times G}(D)$ to $\text{Col}_{X \times G}(D)$, and $\Psi^{X \times G}(\text{Col}_{Q_G(X) \times G}^{\phi'}(D)) \subset \text{Col}_{X \times G}(D)$ for any $\phi' \in \text{Flow}(D; G)$ (see Figures 12 and 13). Hence $\Psi^{X \times G}|_{\text{Col}_{Q_G(X) \times G}^{\phi'}(D)}$ is a bijective map from $\text{Col}_{Q_G(X) \times G}^{\phi'}(D)$ to $\text{Col}_{X \times G}(D)$. Next, suppose that $X$ is a $G$-family of Alexander biquandles. Then $\psi_1^{X \times G}$ and $\psi_3^{X \times G}$ preserve module structures clearly. Furthermore $\psi_2^{X \times G}$ also preserves module structures since in Lemma 1.6 each $y_i$ can be represented by using each $x_i$ and the operations $*$ and $\tau$. Therefore $\Psi^{X \times G}|_{\text{Col}_{Q_G(X) \times G}^{\phi'}(D)}$ is an isomorphism of right $R$-modules.

Finally, we see an example. Let $H_n$ be the handlebody-knot represented by the $Z_8$-flowed diagram $(D_n, \phi_n)$ depicted in Figure 17 for any $n \in \mathbb{Z}_{>0}$. Let $s = t + 1 \in \mathbb{Z}$.
and let $f(t) = t^2 + t + 2 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_3$-family of Alexander biquandles and a field. By [18 Example 7.3], it follows that $\dim \text{Col}_X(D_n, \phi_n) = n$ as vector spaces over $X$, and the assignment of elements $x_1, \ldots, x_n$ of $X$ to each semi-arc of $(D_n, \phi_n)$ as shown in Figure 17 corresponds to a basis of $\text{Col}_X(D_n, \phi_n)$. By Proposition 5.3, $QG(X)$ is a $\mathbb{Z}_8$-family of Alexander quandles with $x^s y = s^{-1} t^i x + (1 - s^{-1} t^i) y = t^i x + (1 - t^i) y$ for any $i \in \mathbb{Z}_8$. By Corollary 5.5, we have $\dim \text{Col}_{QG(X)}(D_n, \phi_n) = n$ as vector spaces over $X$, and the assignment of elements $x_1, \ldots, x_n$ of $X$ to each arc of $(D_n, \phi_n)$ as shown in Figure 17 corresponds to a basis of $\text{Col}_{QG(X)}(D_n, \phi_n)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure17.png}
\caption{A $\mathbb{Z}_8$-flowed diagram $(D_n, \phi_n)$ of $H_n$.}
\end{figure}

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