Implementing and an empirical study of rank aggregation approaches based on real world instances

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ABSTRACT
Rank aggregation is an essential approach for aggregating the preferences of multiple agents. One rank aggregation rule of particular interest is the Kemeny rule, which maximises the number of pairwise agreements between the final ranking and the existing rankings, and has an important interpretation as a maximum likelihood estimator. However, Kemeny rankings are NP-hard to compute. This has resulted in the development of various algorithms for computing Kemeny rankings. Fortunately, NP-hardness may not reflect the difficulty of solving problems that arise in practice. As a result, we aim to demonstrate that the Kemeny consensus can be computed efficiently when aggregating different rankings in real case. In this paper, we describe a dynamic programming model for aggregating university rankings. We also provide details on the implementation of the model. Finally, we present results obtained from an empirical comparison of different approach models based on real world and randomly generated problem instances, and show that the dynamic programming approach has comparable performance as other approaches.

Categories and Subject Descriptors
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General Terms
Theory

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1. INTRODUCTION
Rank aggregation has recently been proposed as a useful abstraction that has several applications in the area of both social choice theory and computer science such as metasearch, similarity search, and classification. Rank aggregation concerns how to combine many different independently constructed preferences of rankings on the same sets of alternatives by a number of different agents into a single reasonable ultimate ranking. As a result, it is intended to represent the collective opinion of the agents that constructed these rankings, and we call the collective opinion as “consensus”. Kemeny consensus is one of the most classical and critical considerations in rank aggregation problem for specifying a particular type of collective ranking. Given the best compromise ranking, we can sort the rankings according to their closeness to the collective one.

Unfortunately, the computational drawback of Kemeny rule is that it is known to be NP-hard in the worst case [1, 12, 13]. This results in the development of various algorithms for computing Kemeny rankings. However, NP-hardness may not reflect the difficulty of solving problems that arise in practice. Therefore, in this paper we aim to compute Kemeny consensus for specific data sets such as university rankings, and thus identify which ones are closest to the consensus. More specifically, we aim to prove that Kemeny consensus can be computed efficiently during aggregating rankings in real cases when the rankings are to some extent similar with each other.

Therefore, we focus on modelling different rank aggregation approaches, which are Borda count method, heuristic algorithm, fixed-parameter dynamic programming algorithm respectively. We also emphasise on an experimental study of the developed approaches in order to evaluate their practical performance. We have carried out our research based on two main methodologies. First, it follows an object-oriented programming design method to model the solution. Second, it involves the empirical studies of our models.

The ideal scenario for ranking university rankings is when each agent gives a complete ordering of all the universities in the universe of universities. This, however, is far too unrealistic for two main reasons. The first reason is that the coverage of various university rankings is so different. It is unlikely that all rankings will eventually be capable of ranking the entire collection of universities in the world. Second, agents always add and remove several universities and change their ranking criteria from year to year; this is done to ensure their rankings to be up to date. The issue of efficiency is also a serious bottleneck in performing rank
aggregation for some controversial universities. These approaches must be capable of dealing with any combinations of universities. Therefore, the challenge is to model the approaches that work when there are too large disagreements between university rankings. Finally, in light of the length of rankings, it is implicit that any rank aggregation approach has to be computationally efficient.

In general, the proposed solution to the above considerations is to model these three approaches that can take a collection of rankings on the same sets of universities as input and output corresponding Kemeny ranking and its score, together with the performance analysis on them. In order to analyse the models, a sample of up-to-date university rankings will be collected from real world. Some simulations of university rankings will also be generated randomly to be compared with the real cases to evaluate their successfullness.

After a series of experiments based on the models and specific data sets, we have shown that the fixed-parameter dynamic programming algorithm are capable of finding optimal Kemeny solution exactly, and that it has comparable performance to the other two popular approaches that are Borda count method and heuristic algorithm (local kemenisation) when the parameter “average pairwise Kendall-Tau distance” between all university rankings is not too large.

Our paper is organised as follows. In Section II we describe the theory of rank aggregation and related approaches to it. In Section III we specify how to model the approaches. Then we prove the correctness of our models. In Section IV we perform comparative performance analysis of the three models. Finally, in Section V we conclude and outline directions for future research.

2. BACKGROUND

In this section, we give some basic background of rank aggregation approaches.

2.1 Preliminaries

Rank aggregation is a key method for aggregating the preferences of multiple agents. One rank aggregation rule of particular interest is the Kemeny rule [15], which maximises the number of pairwise agreements between the final ranking and the existing rankings, and has an important interpretation as a maximum likelihood estimator. For the purpose of carrying out the research, we need to have some understandings of some preliminaries of the Kemeny rule. Generally, it can be described as follows.

Given a set of $m$ candidates $C = \{c_1, c_2, ..., c_m\}$, a ranking $\pi$ with respect to $C$ is a permutation (ordering) of all elements of $C$ which represents an agent’s preference on these candidates. For each $c_i \in C(1 \leq i \leq m)$, $\pi(c_i)$ denotes the rank of the element $c_i$ in ranking $\pi$, and for any two elements $c_i, c_j \in C$, $\pi(c_i) > \pi(c_j)$ implies that $c_i$ is ranked higher than $c_j$ by the ranking $\pi$. In other words, we say that candidate $c_j$ has a greater rank than $c_j$ in a ranking $\pi$ if and only if $\pi(c_i) > \pi(c_j)$. There is a collection of $n$ rankings $\pi_1, \pi_2, ..., \pi_n$, which are proposed by a set of agents $A = \{1, 2, ..., n\}$ respectively. A rank aggregation method is used to get a consensus ranking $\pi$ on those $m$ candidates.

Kemeny rule is based on the concept of Kendall-Tau distance [16] between two rankings which counts the total number of pairs of candidates that are assigned to different relative orders in these two rankings. In other words, the Kendall-Tau distance between two rankings $\pi_1$ and $\pi_2$ is defined as:

$$dist_{KT}(\pi_1, \pi_2) = \sum_{1 \leq i < j \leq m} dist_{c}(\pi_1, \pi_2)$$

Kemeny consensus is an optimal ranking $\pi$ with respect to the pre-defined $n$ rankings $\pi_1, \pi_2, ..., \pi_n$, which are provided by those $n$ agents and can minimise the sum of Kendall-Tau distances:

$$SK(\pi, \pi_1, \pi_2, ..., \pi_n) = \sum_{i=1}^{n} dist_{KT}(\pi, \pi_i)$$

As we mentioned in the introductory section, the computational complexity of the problem of finding an optimal Kemeny consensus is NP-hard. $\sum_{i=1}^{n} dist_{KT}(\pi, \pi_i)$ defined above is the score of a ranking $\pi$ with respect to the collection of rankings $\pi_1, \pi_2, ..., \pi_n$. Thus, the score of the optimal Kemeny consensus that minimises those sums of Kendall-Tau distances is denoted as Kemeny score.

2.2 Rank Aggregation Approaches

Ranking procedure for aggregating preferences is an active research area in the computational social choice theory and computer science, and has been widely studied. There are also numerous different rank aggregation approaches that have been proposed in recent years. A good overview of different rank aggregation approaches is given in [18]. In general, there are two main classes of the approaches that are famous and popular: positional approaches such as the Borda count [10] and majority ranking approaches such as Condorcet approaches [8]. The Kemeny rule [15] is another rank aggregation rule, since it has been proposed as a way of looking for a compromise ranking. The Kemeny rule is defined as follows: it produces a ranking that maximises the number of pairwise agreements with the votes, where we have a pairwise agreement whenever the ranking agrees with one of the votes on which a pair of candidates is ranked higher.

Instead greedy heuristic [6] or tractable multi-stage approaches [5] have been developed that combine both positional and majority voting approaches. Previous work [9] [7] have performed computational studies for the efficient computation of a Kemeny consensus, using heuristic approaches. Papers [12] [13] [11] have proposed a new approach to produce good approximation of the optimal Kemeny consensus, and are
quite helpful for us to pursue approximation solutions in our context. Besides, we consider approaches are really vital to our research, being an approach to take advantage of specific aspects of the data that we are hoping will be a feature of our data sets. They prove that a fixed-parameter dynamic programming algorithm could compute the Kemeny score efficiently and exactly whenever the preferences of ranking proposed by any two agents are similar with each other on average. In general, their theoretical results encourage this work for practically relevant, efficiently solvable specific data sets such as university rankings.

In order to model the target fixed-parameter dynamic programming algorithm and to perform experimental studies on it, we have to understand what is parameterised algorithm, and how a dynamic programming algorithm exactly works. According to parameterised algorithmic approaches aims at a multivariate complexity analysis of problems, and this is done by studying relevant problem parameters and their influence on the computational complexity of problems. In our paper, we consider the average pairwise Kendall-Tau distance as the parameterisation. In terms of dynamic programming, it is an approach to solving complex problems by breaking them down into simpler steps. The reason why we choose to model this method is the fact that it takes much less time than naive approaches. In the context of aggregating university rankings, the naive approach takes every permutation of universities in rankings, which makes the problem of finding the optimal one be NP-hard. Thus, we choose the dynamic programming one and hope that it will compute the Kemeny consensus more efficiently in this context.

3. MODELS OF APPROACHES

In order to achieve the aim, we adopt a dynamic programming algorithm in the existing literatures originally for computing Kemeny score. We then modify and extend it to find an exact optimal Kemeny consensus. The reason of choosing this fixed-parameter algorithm with respect to the parameter “average pairwise Kendall-Tau distance” is due to some experimental studies. These studies indicate that the Kemeny consensus is easier to compute when the rankings are close to each other, since we believe that the data sets of university rankings obtained from real world have such characteristics.

Due to the fact that the Borda count method and the heuristic algorithm not only share the same data structures but also they are much simpler to model than the dynamic programming algorithm, we only present the model of the dynamic programming approach in this section. For a detailed description and source code of all three approaches, please see the author’s website.

3.1 Dynamic Programming Approach

The fixed-parameter dynamic programming algorithm is able to compute the optimal Kemeny consensus exactly. The approach is depicted in Figure 2. The approach outputs an optimal Kemeny consensus: for every entry $T(i, u, R_i)$, we additionally store a university $u$ that minimises $T(i - 1, u', (R_i' \cup F(i))(u'))$ in line 11. Then, starting with a minimum entry for position $m - 1$, we reconstruct an optimal Kemeny consensus by iteratively adding the predecessor university.

Although we may use different approaches and algorithmic approaches to realise the aim of the paper, the framework of this program is based on the same modal that could be divided into three core components, which are data representation and preprocessing, processing, and output. Further, there are also some functions for performing sub-tasks within each component, which is able to interact with each other and work as a whole system. The model is depicted in Figure 3.

3.2 Data Structures of the Model

Since the model is modelled with an objected-oriented programming approach, we specify its functionalities by a number of classes and functions. 15 Java classes are initially designed in the model as showed in Table I. For a collection of $m$ universities, we use a list of Java classes

| Class               | Description                                                      |
|---------------------|------------------------------------------------------------------|
| PrefAggregation     | main class realising dynamic programming algorithm               |
| Agent               | institution that proposed university ranking                    |
| University          | individual university in a ranking                               |
| UnivRankPair        | pair of university and its rank that made up of rankings          |
| UnivLeagueTable     | university ranking to be aggregated                               |
| PrefGraph           | adjacency matrix for computing partial Kemeny score              |
| PrefGraphVertex     | vertex of preference graph                                        |
| PrefGraphEdge       | edge of preference graph                                          |
| RSet                | a set $R_i$ of universities can assume $i$ as its rank            |
| ISet                | a set $I_i$ of universities can assume $i$ as its rank            |
| FSet                | a set $F_i$ of universities can assume $i$ as its rank            |
| DefinedPair         | three-dimensional array for dynamic programming table            |
| KemenyScore         | three-dimensional array for dynamic programming table            |
| MajorityGraph       | three-dimensional array for dynamic programming table            |
| MajorityGraphVertex | three-dimensional array for dynamic programming table            |
| MrComparator        | three-dimensional array for dynamic programming table            |
| BordaPair           | three-dimensional array for dynamic programming table            |

Three core components of the model is further described as follows,

- Data representation and preprocessing: to represent and preprocess obtained datasets of rankings;
- Processing: to apply an algorithmic approach to process the representation of the preprocessed datasets;
- Output: to obtain results such as the consensus and best aligned tables after processing.

Figure 3: Components of the model.

Table 1: A list of Java classes

| Class               | Description                                                      |
|---------------------|------------------------------------------------------------------|
| PrefAggregation     | main class realising dynamic programming algorithm               |
| Agent               | institution that proposed university ranking                    |
| University          | individual university in a ranking                               |
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| PrefGraphVertex     | vertex of preference graph                                        |
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| DefinedPair         | three-dimensional array for dynamic programming table            |
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| MajorityGraph       | three-dimensional array for dynamic programming table            |
| MajorityGraphVertex | three-dimensional array for dynamic programming table            |
| MrComparator        | three-dimensional array for dynamic programming table            |
| BordaPair           | three-dimensional array for dynamic programming table            |
The first dimension is rank $R$, the second dimension is every university $u$, and the third dimension is every subset $R'_i \subseteq R_i \setminus \{u\}$.

Computation of the partial Kemeny scores that are obtained by subdividing the overall Kemeny score is a key manipulation of the model. We thus design an class $PrefGraph$ which acts as an intermediate data structure for storing preprocessing outcome of profile of $ulTable$. An object of $PrefGraph$ is actually a weighted directed graph, where each vertex represents one university, and the weight of each edge between vertices represents the number of agents who prefer (rank higher) university represented by start vertex to one represented by end. By this graph, it can be used to compute the partial Kemeny scores rather quickly.

To describe the data structures to be used by the model, some further definitions and notations are employed. More specifically, there are 3 important sets of universities are defined: $R_i$, $I(i)$ and $F(i)$. For any rank $i$ from 0 to $m - 1$, $R_i$ denotes the set of all possible universities that can take this rank, that is, $R_i = \{u \in U \mid \tau_{ave}(u) - d < i < \tau_{ave}(u) + d\}$; $I(i)$ denotes the set of universities that could be “inserted” at rank $i$, that is $I(i) = \{u \in U \mid u \in R_i \land u \notin R_{i-1}\}$; $F(i)$ denotes the set of universities that must be “forgotten” at least at this rank, that is $F(i) = \{u \in U \mid u \in R_{i-1} \land u \notin R_i\}$. All these sets could be represented by a $<University> ArrayList$, thus we then encapsulate these $ArrayList$ into three separate classes: $RSet$, $ISet$, and $FSet$.

We use the term “three dimensional dynamic programming table” to describe an abstract mechanism where we save information (the integer value) that we can later retrieve. The values depend upon the “minimum partial Kemeny score” over all possible orders of the universities of $R'_i$ given $u$ taking rank $i$ and all universities of $R'_i$ taking ranks below $i$. Thus, it is represented by a three-dimensional array $T$, where the first dimension is rank $i$, the second dimension is every university $u$ can assume $i$, and the third dimension is every university subset $R'_i \subseteq R_i \setminus \{u\}$.

We gives a typical example of a three dimensional array structure with 24 elements and different ranks in Figure 4. Each element in the illustration shows the index values that access it. For example, you can access the third subset of the first university of the second rank of the three-dimensional array by specifying indexes (1, 0, 2).

### 3.3 Implementation of the Model

#### 3.3.1 $generatePrefGraph()$

it is used to generate a preference graph given a collection of university rankings. As we previously stated, an object of the abstract data type $PrefGraph$ is a directed weighted graph, and it is represented by an adjacency matrix in the programs. We attempt to use a simple example to describe the preference graph and its representation, and they are also showed in Figure 5.

Considering three university rankings expressed by the profile $((0, A), (1, B), (2, C)), ((0, A), (1, C), (2, B)), ((0, B), (1, A), (2, C))$, this means that: the first agent ranks university $A$ better than university $B$ than university $C$, the second agent ranks $A$ better than $C$ than $B$, the third agent ranks $C$ better than $B$ than $A$. 

```plaintext
Initialise:
1 for i = 0, ..., m - 1
2 for all u ∈ R_i
3 for all R'_i \subseteq R_i \setminus \{u\}
4 T(i, u, R'_i) = +∞
5 for all u ∈ R_0
6 T(0, u, \emptyset) = pK(u, U \setminus \{u\})
Update:
7 for i = 0, ..., m - 1
8 for all u ∈ R_i
9 for all R'_i \subseteq R_i \setminus \{u\}
10 if | R'_i \cup \bigcup_{j=0}^{m-1} F(j) | = i - 1 and T(i - 1, u', (R'_i \cup F(j)) \setminus \{u'\}) is defined
11 then T(i, u, R'_i) = \min_{u' \in R'_i \cup F(j)} T(i - 1, u', (R'_i \cup F(j)) \setminus \{u'\}) + pK(u, (R_i \cup \bigcup_{j=1}^{m-1} I(j)) \setminus \{u\}) and storing u'
12 for i = m, ..., 1
13 if T(i - 1, u', (R'_i \setminus \{u'\}) is defined
14 add u' that minimises T(i - 1, u, R'_i) to the optimal Kemeny consensus at rank i - 1
Output:
15 the optimal Kemeny consensus and its K-score = \min_{u \in R_{m-1}} T(m - 1, u, R_{m-1} \setminus \{u\})
```

Figure 2: Fixed-parameter dynamic programming approach. The input is a collection of university rankings for $U$ given $A$, and for every $0 \leq i < m$, the set $R_i$ of universities that can assume rank $i$ in an optimal Kemeny consensus. The output is the optimal Kemeny consensus and its Kemeny score.

Figure 4: A three dimensional array with 24 elements.
3.3.2 computeAveRank()

It is used to return the average rank of each university given a collection of university rankings. Let the rank of a university \( u \) in a university ranking proposed by a gent \( a \), denoted by \( r_a(u) \), be the number of universities that are better than \( u \) in \( a \). That is, the topmost and best university in \( a \) has rank 0 and the bottommost has rank \( m - 1 \). For a collection of university rankings for \( U \) given \( A \) and university \( u \in U \), the average rank \( r_{ave}(u) \) of \( u \) is defined as:

\[
r_{ave}(u) = \frac{1}{n} \cdot \sum_{a \in A} r_a(u)
\]

With regards to this function, the average ranks of all universities can be computed in \( O(|A| \cdot m) \) time by iterating once over every university ranking and adding the rank (an integer value) of every university to a counter variable for this university.

3.3.3 computeAveKTdistance()

It is used to return the average pairwise Kendall-Tau distance between a given collection of university rankings. For a collection of university rankings for \( U \) given \( A \), the average Kendall-Tau distance \( d_{ave}(u) \) is defined as:

\[
d_{ave} = \frac{2}{n(n - 1)} \cdot \sum_{a, a' \in A, a \neq a'} dist_{KT}(a, a')
\]

where for each pair of university ranking \( a \) and \( a' \), the Kendall-Tau distance (KT-dist) between universities \( u \) and \( u' \) is defined as:

\[
dist_{KT}(a, a') = \sum_{u, u' \in U} d_{a, a'}(u, u')
\]

where the sum is taken over all unordered pairs \( u, u' \) of universities, and \( d_{a, a'}(u, u') \) is 0 if \( a \) and \( a' \) rank \( u \) and \( u' \) in the same order, and 1 otherwise.

Furthermore, the value of \( d_{ave} \) may be a non-integer. However, for the purpose of the whole dynamic programming algorithm, this method is supposed to return a \(< int >\) value, thus \( d \) is defined as: \( d = \lceil d_{ave} \rceil \)

3.3.4 computeSubsets()

It is used to return all subsets of a given set of universities. Given an arraylist (set) of length \( k \) number of distinct elements (universities), this method will eventually generate \( 2^k \) number of arraylists (subsets) including the empty set \( \emptyset \). The result will be stored in a set of sets, which actually acts as a two-dimensional table whose entries are \(< University >\) objects.

For example, given a set consisting of 3 universities A, B and C, that is: \{A, B, C\}, as the parameter, invoking of this method will return \( 2^3 = 8 \) subsets of set \{A, B, C\}, that are: \( \emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \text{ and } \{A, B, C\} \).

3.3.5 computePKScore()

It is used to compute partial Kemeny score given an university, a subset of universities excluded this university, and a collection of university rankings. As for the dynamic programming, it is necessary to subdivide the overall Kemeny score into partial Kemeny scores. More precisely, for an university \( u \) and a subset \( R \) of universities with \( u \notin R \), we define:

\[
pK(u, R) = \sum_{u' \in R} \sum_{a \in A} d_{a}^R(u, u')
\]

where for \( u \notin R \) and \( u' \in R \) we have \( d_{a}^R(u, u') = 0 \) if in the university ranking proposed by \( a \) we have \( u > u' \), and \( d_{a}^R(u, u') = 1 \), otherwise.

Intuitively, the partial Kemeny score denotes the score that is induced by university \( u \) and the university subset \( R \) if the universities of \( R \) have greater ranks than \( u \) in an optimal Kemeny consensus.

3.3.6 findBestTable()

It is used to return which university rankings are closest to the Kemeny consensus given a collection of university rankings and their Kemeny consensus. The score of a university ranking \( t \) with respect to a collection of university rankings for \( U \) given \( A \) is denoted as:

\[
SK(t, a_1, a_2, ..., a_n) = \sum_{a \in A} dist_{KT}(t, a)
\]

We already know that a university ranking \( t \) with the minimum score is called an optimal Kemeny consensus of \( (U, A) \), thus we aim to identify which university rankings given by agents are closest to the consensus. This function is independent and used as the last step.

3.3.7 computeKemenyConsensus()

It is used to acts as an entrance function of the dynamic programming algorithm and other implemented approaches, and all algorithmic processing tasks are embedded within it. If using the dynamic programming, two more functions need to be specified in the implementation, which are setUnion() and setElimination(). Both functions are used a number of times by this function. They are similar to the function computeSubsets since they both actually manipulate on arraylists (sets).

The function setUnion() is used to find an union set of two given sets. For example, if we have two sets of universities \{A\} and \{B, C\} as parameters of setUnion(), it eventually returns the set of universities \{A, B, C\}.
The function method setElimination() is used to find a set that is same as the first set after eliminating the second set. For example, if we have two sets of universities \( \{A, B, C\} \) and \( \{C\} \) as parameters of setElimination(), it eventually returns the set of universities \( \{A, B\} \).

### 3.4 Correctness of the Model

Concerning the dynamic programming, we test all possible universities for every rank \( 0 \leq i \leq length - 1 \). Thus, having chosen an university \( u \) for rank \( i \), the remaining universities that could assume \( i \) must either has smaller rank or bigger than \( i \) in an optimal Kemeny consensus. To ensure the correctness of the implementation, we show that the implementation satisfies the following two conditions:

First, the capability of the implementation to find an optimal Kemeny solution is ensured. From the specification, we know that the Kemeny score can be decomposed into partial Kemeny scores. Therefore, we can show that the approach considers a decomposition that results in the final optimal Kemeny consensus. For every rank, the algorithm tests every university in \( R_i \). Based on the definition of the set \( R_i \) of universities for rank \( i \), one of these universities must be the correct university \( u \) for this rank. Furthermore, for \( u \) we are able to find that the algorithm tests a sufficient number of possibilities to partition all remaining universities \( U \setminus \{u\} \) such that they either be left or right of rank \( i \). More precisely, every university from \( U \setminus \{u\} \) must be in exactly one of the following three subsets:

- The set \( F \) of universities that have already been forgotten, that is, \( F = \bigcup_{j=0}^{i} F(j) \). We realise it in the implementation as the following:

\[
\text{ArrayList<University> } F_j = \text{newArrayList<University> } (); \text{ for(int } i = 1; i < \text{length}; i++) \text{ F}_j = F.getF(0); \text{ for(int } j = 1; j <= i; j++) \text{ F}_j = \text{setUnion(F}_j, F.getF(j));
\]

- The set of universities that can assume rank \( i \), that is, \( R_i \setminus \{u\} \). This is realised by the following lines of codes:

\[
\text{for(int } p = 0; p < R.getR(i).size(); p++) \text{ ArrayList<University> } u = \text{transferToSet(R.getR(i).get(p)); ArrayList<University> } \text{ newList = setElimination(R.getR(i), u);}
\]

- The set \( I \) of universities that are not inserted yet, that is, \( I = \bigcup_{m=1}^{\text{length} - 1} I(j) \). We realise it in the implementation by the following codes:

\[
\text{ArrayList<University> } I_j = \text{newArrayList<University> } (); \text{ I}_j = I.getI(length - 1); \text{ for(int } h = \text{length} - 2; h > i; h--) \text{ I}_j = \text{setUnion(I}_j, I.getI(h));
\]

Second, all entries in the three dimensional table are well defined and its value is computed correctly. For any entry \( T(i, u, R_i) \), in terms of rank \( i \) there must be exactly \( i - 1 \) number of universities that have ranks smaller than \( i \). This requirement is guaranteed by the following lines of codes from 4 to 5, and from 7 to 8 in Figure 6.

![Figure 6: Well formedness of three dimensional table.](image)

4. **EMPIRICAL EVALUATION**

In this section, we present a critical appreciation of the strengths and limitations of our models of the approaches. To the best of knowledge, there is no existing experimental study of the fixed-parameter algorithmic approaches and there is no analysis of the performance of such approach and some other approaches. As a novelty, we aim to compare the actual effectiveness and efficiency of these approaches in real case. As a result, we analyse their performance by observing whether it accords with the theoretical prediction and make some comparisons with each other based on their models.

Our experiment aims to compare the rank aggregation approaches and algorithms in the context of specific data sets such as university rankings. Comparative analysis of three approaches has been performed. These approaches are Borda count method, heuristic algorithm (local Kemenenisation), and dynamic programming algorithm respectively. We perform comparative analysis is to estimate the effectiveness and efficiency of different approaches for aggregating university rankings. The aim is to prove that Kemeny consensus can be efficiently computed by the fixed parameter dynamic programming algorithm when aggregating rankings in real case and the rankings are similar with each other.

#### 4.1 Data Sets

We will use three kinds of data sets of university rankings obtained from both reality and simulation, which are denoted by, \textsc{same}, \textsc{diff}, and \textsc{random} respectively. First, \textsc{same} provides five university rankings, and they are obtained from five consecutive years of rankings that were published by the “same” organisation in real case. In our case, we choose one of the most authoritative and popular media in the world, which is The Times, and we also select five years of its UK university rankings from 2008 to 2012. Second, \textsc{diff} offers the recent rankings proposed by five “different” organisations. More precisely, these UK university rankings come from the medias and organisations, which are the Times UK University Ranking, the Guardian UK
University Ranking, the Independent UK University Ranking, the QS World University Rankings, and the UK section of the Shanghai Jiaotong University Academic Ranking of World Universities.

Finally, RANDOM denotes the data sets of five university rankings, in which all universities and their ranks are generated and arranged “randomly” during the runtime of each execution of corresponding model in the experiments. For the purpose of our analysis, we have chosen a set of the same 40 universities. Thus, the full list in each university ranking consists of exactly the same 40 universities in the SAME, DIFF, and RANDOM data sets. Further, we evaluate different rank aggregation approaches in terms of their performance. We consider subsets of the data sets, that are, SAME\textsubscript{m}, DIFF\textsubscript{m}, and RANDOM\textsubscript{m}, where m is the length of each partial ranking.

4.2 Experimental Setup

After collecting the data sets of all three kinds of university ranking, each method has been evaluated in terms of its effectiveness, i.e., its average Kemeny score of the Kemeny solution was computed, and its efficiency, i.e., its average running time was also computed. We consider the execution results of SAME\textsubscript{m}, DIFF\textsubscript{m}, and RANDOM\textsubscript{m}.

Our approach to examining the effectiveness and efficiency of different approaches is as follows. Each experiment is applied to the partial data set SAME\textsubscript{m}, DIFF\textsubscript{m}, and RANDOM\textsubscript{m} individually. For each data set, each data point is the average of 20 trials. Each trial is performed as follows: for every possible m ∈ {6, 8, 10, 12}, randomly select a subset U that contains the number m universities from 40 universities in the full set, and then to form a new university ranking made up of this subset of same m of universities. Therefore, for m = 6, we have: SAME\textsubscript{6}, DIFF\textsubscript{6}, and RANDOM\textsubscript{6}, and for m = 8, we have: SAME\textsubscript{8}, DIFF\textsubscript{8}, and RANDOM\textsubscript{8}, and so on.

4.3 Preliminary Results

There are four essential aspects that we investigate in the experiment, which are the ranking length, average pairwise Kendall-Tau distance, Kemeny score of consensus ranking, and algorithmic computational complexity. Therefore, for partial data sets with different ranking length, we investigate three parameters (outputs) from three perspectives of measurement criteria, that are, minimum value, maximum value, and average value. The preliminary results of our experiments are reported in Table II to IV as follows.

| Table 2: Kendall-Tau Distance of Different Approaches |
|-------------------------------------------------------|
| **Data Sets** | **Borda count method** | **Heuristic algorithm** | **Dyn. pro. algorithm** |
| **Min** | **Max** | **Ave** | **Min** | **Max** | **Ave** | **Min** | **Max** | **Ave** |
| SAME\textsubscript{6} | 2.0 | 8.0 | 3.6 | 2.0 | 8.0 | 3.6 | 2.0 | 8.0 | 3.6 |
| SAME\textsubscript{8} | 3.0 | 10.0 | 5.6 | 3.0 | 10.0 | 5.6 | 3.0 | 10.0 | 5.6 |

In Table II, the labels “Min.”, “Max.”, and “Ave.” in the first row indicates the minimum, maximum, and average pairwise Kendall-Tau distance respectively. In Table III, the labels “Min.”, “Max.”, and “Ave.” in the first row indicates the minimum, maximum, and average Kemeny score of consensus ranking respectively. In Table IV, the labels “Min.”, “Max.”, and “Ave.” in the first row indicates the minimum, maximum, and average total running time of corresponding algorithm respectively. The last column shows the average result of the 20 trials for each partial data set.

As we discussed in previous sections, we already know that dynamic programming algorithm can output Kemeny consensus exactly. Therefore, its effectiveness for finding optimal Kemeny ranking should be the best, and its precision is 100%. In Figure 5(a), we can see that the heuristic algorithm achieves a very good approximation of the exact solution, because its Kemeny score is really close to the dynamic programming algorithm. The largest Kemeny score indicates the least precision of result consensus, thus the Borda count method has the worst effectiveness among all.
three approaches.

Figure 8: Average running time of the Borda count method.

Figure 9: Average running time of the heuristic algorithm.

In Figure 5(b) and 6(a), it is not hard to see that the total running time $t$ of both Borda count method and heuristic algorithm is almost linear to the ranking length $m$ under the experiments on three different kinds of data sets of university rankings $SAME_m$, $DIFF_m$, and $RANDOM_m$, where $m \in \{6, 8, 10, 12\}$. It indicates that the experimental computational complexity for both approaches is constant no matter how many universities each university ranking consists of. Thus, we could test much larger $m$ (e.g., 50, 100, etc) for both approaches, although that $m$ cannot be tested on dynamic programming algorithm due to its memory consumption problem in dynamic programming table construction.

In Figure 6(b), the total running time $t$ on data set $RANDOM_m$ is increasing significantly if $m$ continues to become larger from 6 to 12. Because we know that $RANDOM_m$ has average pairwise Kendall-Tau distance $d$ much larger than $SAME_m$ and $DIFF_m$, in from Table II to Table IV, it indicates that its time complexity is exponential to its ranking length $m$. Further, to see whether this NP-hardness could be overcome in some circumstances, we look into the $SAME_m$ and $DIFF_m$ cases applied to dynamic programming algorithm. As we can see, the performance of $SAME_m$ is much better than $DIFF_m$, and it has also clearly decreased the running time than $RANDOM_m$.

Figure 10: Average running time of the dynamic programming algorithm.

As a result, our experimental results proved that finding Kemeny consensus is fixed-parameter tractable with respect to the parameter “average pairwise Kendall-Tau distance $d$” ($d_{SAME_m} < d_{DIFF_m} < d_{RANDOM_m}$). Further, it is easy to see that to a great extent the efficiency of dynamic programming algorithm also relies on the ranking length. To conclude, we say computing optimal Kemeny consensus can work effectively and efficiently for data set $SAME$ if there is not too many controversial universities in all rankings. Further, we still need to improve the performance of the algorithm to make it be more realistic on data set $DIFF$.

4.4 Discussion

Thanks to the dynamic programming algorithm, finding an exact optimal Kemeny consensus is the main strength of the method. Besides, there are theoretical and practical advantages with respect to the approaches and their models. The theoretical advantage is that: if the average pairwise Kendall-Tau distance $d$ between a number of $m$ university rankings is relatively not too large (e.g. say: $d \leq m/2$), which means university rankings are similar each other, we could then overcome the NP-hard computational complexity and compute the Kemeny ranking effectively and efficiently no matter how many university are there in a ranking.

The practical advantage is that: given a profile of a number $n$ of university rankings and each one containing a full list of the same $C$ universities, we could choose any integer number $m$ that is not larger than $C$, that is $m \leq C$, and then select any $m$ number of universities among the full list of $C$ universities in order to aggregate the new partial university rankings formed by only those $m$ universities.

Unfortunately, if $d$ is too large, the developed algorithm will be inefficient. Since the set $R_i$ of universities that can take rank $i$ probably contains all universities in the ranking, this will increase the workload of data processing and operation significantly. However, we could still compute the exact Kemeny optimal solution efficiently. The solution may be improving the computation power or changing the computation environment from single computing to distributed or parallel computing.

In addition, whenever $d \geq m$, and $m$ is too large at the
same time, the developed program will terminate its running unexpectedly and give a prompt of exception that is as: "java.lang.OutOfMemoryError: Java heap space at ThreeDTable.<init>". It indicates that there are no enough memory spaces for the computer system to establish the three dynamic programming table. Therefore, it is impossible to allocate resources for the program if this number is too large. This is due to the size of the dynamic programming table $T$.

Concerning the size, there are $m$ ranks (universities) in a partial ranking, and any rank can be assumed by at most $4d$ universities. The number of considered subsets is bounded from above by $2^{4d}$. Hence, the size of the table $T$ is $O(2^{4d} \cdot d \cdot m)$. However, if $d \geq m$, the size of the table $T$ then become $O(2^{m^2} \cdot m^2)$, which is extremely large. For instance, the computer we used has a 2GB of physical memory. When $m = 21$ and $d \geq 21$, it will need $2^{21} \cdot 21^2 \cdot 2 = 2.0 \times 10^9$ Bytes = 2GB of space for storing the dynamic programming table, which necessarily exceeds the available memory of the machine.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we have studied the rank aggregation problem based on Kemeny rule. We have modelled fixed-parameter dynamic programming algorithm, Borda count approach, and a heuristic algorithm based on local Kemenisation approach. Moreover, we performed an empirical comparison of three models. We find that the dynamic programming approach has its own strengths for finding optimal Kemeny consensus for the specific data sets in real case. In addition, we show that, if the average pairwise Kendall-Tau distance is not too large, it has comparable performance as the Borda approach and the heuristic algorithm. This is especially important in the context of aggregating a collections of similar rankings. It is worth noting that its practical performance relies on the memory consumption of its three-dimensional dynamic programming table.

There are some possible directions for future research. First, we can extend our cases to the rank aggregation problem with university rankings that may have ties or that may be incomplete. Concerning the number of university rankings, if the number is even, there may be a tie between these ranking [14] [17]. It is worth to investigate the case that a university in a ranking may not appear in another one. Second, we can extend our cases to university rankings that may have weights. We measure that different ranking may have weights. We measure that different ranking may have weights. We measure that different ranking may have weights. We measure that different ranking may have weights. We measure that different ranking may have weights. We measure that different ranking may have weights. We measure that different ranking may have weights.

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