Chapter 1
Laws relating runs, long runs, and steps in gambler’s ruin, with persistence in two strata

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Abstract Define a certain gambler’s ruin process \( X_j, \ j \geq 0 \), such that the increments \( \varepsilon_j := X_j - X_{j-1} \) take values \( \pm 1 \) and satisfy \( P(\varepsilon_{j+1} = 1|\varepsilon_j = 1, |X_j| = k) = P(\varepsilon_{j+1} = -1|\varepsilon_j = -1, |X_j| = k) = a_k, \) all \( j \geq 1 \), where \( a_k = a \) if \( 0 \leq k \leq f - 1 \), and \( a_k = b \) if \( f \leq k < N \). Here \( 0 < a, b < 1 \) denote persistence parameters and \( f, N \in \mathbb{N} \) with \( f < N \). The process starts at \( X_0 = m \in (-N, N) \) and terminates when \( |X_j| = N \). Denote by \( R'_N, U'_N, \) and \( L'_N \), respectively, the numbers of runs, long runs, and steps in the meander portion of the gambler’s ruin process. Define \( X_N := \left( L'_N - \frac{1-a-b}{(1-a)(1-b)} R'_N - \frac{1}{(1-a)(1-b)} U'_N \right)/N \) and let \( f \sim \eta N \) for some \( 0 < \eta < 1 \). We show \( \lim_{N \to \infty} E\{e^{itX_N}\} = \hat{\varphi}(t) \) exists in an explicit form. In case \( b = 1 - a \) and \( \eta = a \), \( \hat{\varphi}(t) = \sigma^2 t/\{\sinh(\sigma t)|\sigma \cosh(\sigma t) + i(1 - 2a)^2 \sinh(\sigma t)|\} \), for \( \sigma^2 := 1 - 3a^2 + 3a^2 \). If \( b = a \), then \( \hat{\varphi}(t) = At/\sinh(At) \), for \( A := \sqrt{a/(1 - a)} \).

Key words: runs, generating function, excursion, gambler’s ruin, last visit, meander, persistent random walk, generalized Fibonacci polynomial

1.1 Introduction

Define a gambler’s ruin process \( \{X_j, \ j \geq 0\} \), with values in \( \mathbb{Z} \cap [-N, N] \), such that the increments \( \varepsilon_j := X_j - X_{j-1} \) take values \( \pm 1 \) and satisfy \( P(\varepsilon_{j+1} = 1|\varepsilon_j = 1, |X_j| = k) = P(\varepsilon_{j+1} = -1|\varepsilon_j = -1, |X_j| = k) = a_k, \) all \( j \geq 1 \), where \( a_k = a \) if \( 0 \leq k \leq f - 1 \), and \( a_k = b \) if \( f \leq k < N \). Here \( 0 < a, b < 1 \) denote persistence parameters and \( f, N \in \mathbb{N} \) with \( f < N \). The process starts at some fixed level \( m \in (-N, N) \) and terminates at an epoch \( j \) when \( |X_j| = N \). We call the two ranges of values \( |k| \leq f - 1 \) and \( f \leq |k| < N \) as strata for the two
persistence parameter values $a$ and $b$, respectively. In gambling, $X_j$ denotes a fortune after $j$ games on which the gambler makes unit bets. If $a, b > \frac{1}{2}$, then any run of fortune tends to keep going in the same direction. Thus for example a win [loss] resulting in fortune $k$ for some $|k| \leq f-1$ is followed by another win [loss] with probability $a$, whereas a change in fortune occurs with probability $1-a$. We shall take the initial step distribution as $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$. Henceforth we shall simply refer to $\{X_j = X^N_j\}$ as the gambler’s ruin process given by this model, with or without mention of the parameters $a$, $b$, and $f$. Note that $\{X_j\}$ is the classical fair gambler’s ruin process in case $a = b = \frac{1}{2}$, with symmetric boundaries $N$ and $-N$. For the homogeneous case $b = a$, the increments $\{\varepsilon_j, j \geq 0\}$, form a strictly stationary process with zero means, where the correlation between $\varepsilon_j$ and $\varepsilon_{j+1}$ is $2a - 1$. If $b = a$ and also $N = \infty$ then $\{X^\infty_j\}$ becomes a symmetric persistent random walk on $\mathbb{Z}$ (also called a correlated random walk) that is easily seen to be recurrent. Indeed, a run of fortune in one direction followed by a run of fortune in an opposite direction yields a symmetrized version of a geometric random variable with mean zero and finite variance, so recurrence follows by [8], Thm. 8.1. Physical models of persistence often consider the velocity of a particle either staying the same or being changed according to a collision process; in our model the velocity only takes values $\pm 1$. Our introduction of strata corresponds to a change in medium over which the persistence parameter, or likelihood of the velocity staying the same, would deterministically change. See [10] for a random environment context.

We define a nearest neighbor path of length $n$ in $\mathbb{Z}$ to be a sequence $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n$, where $\Gamma_j \in \mathbb{Z}$ and $\delta_j := \Gamma_j - \Gamma_{j-1}$ satisfies $|\delta_j| = 1$ for all $j = 1, \ldots, n$. We also call $n$ the number of steps of $\Gamma$. The graph of the sequence $\Gamma$ over the non-negative $x$-axis may be visualized by connecting successive lattice points $((j-1), \Gamma_{j-1})$ and $(j, \Gamma_j)$ in the plane by straight line segments; we term this connected union of straight line segments the lattice path. We define the number of runs in $\Gamma$ as one plus the number of indices $j$, $1 \leq j \leq n-1$, such that $\delta_{j+1} = -\delta_j$, that is, one more than the number of turns in the lattice path. A run is also called an incline since each run corresponds to an ascent or descent (straight line) of maximal extent along the lattice path; the length of a run is the number of steps in such a maximal ascent or descent. A “long run” is itself a run that consists of at least two steps; in gambling terminology a long run means that the run of fortune does not immediately change direction. A “short run” is on the other hand a run of length exactly one, so every run is either a long run or a short run. An excursion is a nearest neighbor path that starts and ends at $m = 0$, $\Gamma_0 = \Gamma_n = 0$, but for which $\Gamma_j \neq 0$ for $1 \leq j \leq n-1$. A positive excursion is an excursion whose graph lies above the $x$–axis save for its endpoints. For a positive excursion path, the number of runs is just twice the number of peaks, where a peak at lattice point $(j, \Gamma_j)$ corresponds to $\delta_j = 1$ and $\delta_{j+1} = -1$.

In [6], the author establishes in particular a closed form solution of the joint probability generating function of the number of runs and steps for
both the “last visit” and meander portions of the classical fair gambler’s ruin process on the symmetric interval \([-N, N]\). The last visit is defined as

\[
\mathcal{L}_N := \max\{j \geq 0 : X_j = 0\}.
\]  

(1.1)

The meander is the portion of the process that extends from the epoch of the last visit \(\mathcal{L}_N\) to level \(m = 0\) until the gambler’s ruin process terminates. So in the meander, the process never returns to the level \(m = 0\). In [6] it is shown that, for \(b = a = \frac{1}{2}\), if we denote by \(\mathcal{R}_N\) the total number of runs over all excursions of the absolute value process \(\{|X_j|\}\) until the last visit, then \(2\mathcal{R}_N/N^2\) converges in law as \(N \to \infty\), and the limit distribution is the same as the law (weak limit) of \(\lim_{N \to \infty} \mathcal{L}_N/N^2\). Furthermore, with now an order \(N\) scaling, it holds that \((\mathcal{L}_N - 2\mathcal{R}_N)/N\) converges in law. Also, if \(\mathcal{R}'_N\) and \(\mathcal{L}'_N\) denote respectively the number of runs and steps over the meander portion of the process, then it is shown that \((\mathcal{L}'_N - 2\mathcal{R}'_N)/N\) converges in law to a density \(\varphi(x) = (\pi/4)\text{sech}^2(\pi x/2),\) \(-\infty < x < \infty\), with characteristic function \(\int_{-\infty}^{\infty} \varphi(x)e^{i\lambda x}dx = t/\sinh(t)\). We find that the change of measure induced by the parameter \(a\) in the homogeneous case of the persistence model changes this last result qualitatively such that \(\lim_{N \to \infty} E\{e^{(\mathcal{L}'_N - \mathcal{R}'_N)/(1-a))/N}\} = At/\sinh(At)\) with \(A = \sqrt{a/(1-a)}\); see the Remark (1.143) after Corollary 3. Yet in Theorem 2 we extend this type of limit distribution over the meander with scaling of order \(N\) to the full model by utilizing a third counting statistic, the number of long runs \(U'_N\), weaving together \(\mathcal{L}_N\) and \(\mathcal{R}'_N\). There, we find a limit distribution depending on both \(a\) and \(b\) whose characteristic function recovers the prototypical form \(At/\sinh(At)\) when \(b = a\), but which itself is not of this form for \(b \neq a\).

Thus the first motivation of the present paper is to show first how the method of [6] extends to the three statistics, runs, steps, and long (or short) runs, in the homogeneous setting \((b = a)\). As a particular result we find Corollary 2 which connects the present work with a certain combinatorial domain in the study of Dyck paths. Note that the generating function method of [6] which drives the present study depends heavily on a “return to the level 1” type recurrence approach that has been applied extensively in the field of lattice path combinatorics; compare [2]. The second motivation is to extend the persistence model to the case of two distinct strata. This “full model” together with its solution has interesting features, which include: (i) its intrinsic value as physical model, (ii) completely explicit formulae throughout for key polynomials, identities, and generating functions, and (iii) new limiting distributions for a scaling of order \(N\) in both the meander and the last visit portions of the gambler’s ruin; see Sections 1.8 and 1.9.

We now introduce some further definitions to describe our results. For the definitions in this paragraph we assume \(\hat{N} = \infty\), but we work with the general model. Therefore \(f\) is some finite positive integer that separates a stratum of finite range \(|m| < f\) with a stratum of infinite range \(|m| \geq f\). Define the index \(j\), or step, of first return of \(\{X_j\}\) to the origin by \(L := \inf\{j \geq 1 : X_j = 0\}\).
Define the excursion sequence from the origin by \( \Gamma := \{ X_j, j = 0, \ldots, L \} \); again \( L \) is the number of steps of \( \Gamma \). Define the height \( H \) of the excursion \( \Gamma \) as the maximum absolute value of the path over this excursion:

\[
H := \max\{|X_j| : j = 1, \ldots, L\}.
\]  

Also define \( R \) as the number of runs along \( \Gamma \), and further define \( V \) as the number of short runs along \( \Gamma \). Thus officially \( U := R - V \) is the number of long runs along \( \Gamma \). Note that it is possible to have \( H = \infty \), but this does not happen almost surely. One way to see this is to note explicitly, by Proposition 2 and (1.118), that

\[
P(H \geq n) = \frac{ab}{([n - f]a + (f - 1)b - (n - 2)ab)}, \quad n \geq f.
\]

So \( P(H \geq n) \to 0 \), as \( n \to \infty \).

Our first goal is to establish in Proposition 1 for the case \( b = a \), an explicit formula for the conditional joint generating function of the excursion statistics \( R, V, \) and \( L \) given \( \{ H \leq N \} \). Here note that we choose to define our conditional generating function using the number of short runs \( V \), while for certain other purposes the number of long runs \( U \) is a nicer statistic in joint distribution with \( R \) and \( L \). Here we define for the general (full) model:

\[
K_N(r, y, z; a, b) := E_r r^R y^V z^L | X_0 = 0, \ H \leq N \).
\]  

The case of steps alone for simple random walk, that is \( r = y = 1 \) and \( b = a = \frac{1}{2} \) in (1.3), was solved by [1] as noted in [4], p. 327. There the univariate Fibonacci polynomials appear in connection with the derivation of the generating function. The proof of [6] yields an explicit formula for (1.3) in the classical fair gambler’s ruin for the pair of statistics \( R \) and \( L \) (so \( y = 1 \) still, and \( a = b = \frac{1}{2} \)). The proofs of both [6] and the present paper feature bivariate Fibonacci polynomials \( \{ q_n(x, \beta), n \geq 1 \} \) and \( \{ w_n(x, \beta), n \geq 1 \} \), defined as follows.

**Definition 1.** Define sequences \( q_n(x, \beta) \) and \( w_n(x, \beta) \) generated by the following recurrence relation with the specified initial conditions:

\[
q_{n+1} = \beta q_n - xq_{n-1}, \quad q_0 = 0, q_1 = 1, \quad n \geq 1;
\]

\[
w_{n+1} = \beta w_n - xw_{n-1}, \quad w_0 = 1, w_1 = 1, \quad n \geq 1.
\]  

The polynomials \( q_n(x, \beta) \) generalize the univariate Fibonacci polynomials \( F_n(x) = q_n(x, 1) \); also \( w_n(x, 1) = F_{n+1}(x) \). By standard generating function techniques, the fundamental sequences (1.4) have closed formulae given by Lemma 3. Certain linear combinations of these fundamental generalized Fibonacci polynomials, denoted \( q^*_n \) and \( w^*_n \) in (1.9)–(1.11), appear in the solution of the homogeneous case in Proposition 1. We overview (1.9)–(1.11) by noting that the complication in passing from the classical fair gambler’s ruin to the persistence model with two or more statistics defining (1.3), so far in case \( b = a \), is handled by passing the generating function variables and parameter \( a \) into the variables \( x = x_a \) and \( \beta = \beta_a \) of the Fibonacci polynomials.
in a way that is determined by the statistics themselves. A key “inter-

calcing” property of any two term recurrence \(v_{n+1} = \beta v_n - x v_{n-1}, n \geq 1,\) with

coefficients \(\beta\) and \(x\) independent of \(n,\) that we shall exploit, may be written:

\[ v_{n+1}v_{n-1} - v_n^2 = \beta^{-1} x^{n-1} (v_3v_0 - v_2v_1), \quad \beta \neq 0. \quad (1.5) \]

For a proof of (1.5), see (2.7)–(2.8) in [6]. Note that when \(v_0 = 0, v_1 = 1,\) the

polynomials \(v_n = v_n(\beta, -x)\) are called the generalized Fibonacci polynomials

in \(\beta\) and \(-x,\) [9].

We are leading up to a basic feature of our method, which involves condi-
tioning on the event \(\{H = n\};\) see (1.15). We need some additional notation

as follows. For any pair of integers \(m, n \in (-N, N)\) with \(m \neq n\) we define

the following “first passage length” for the process \(\{X_j = X_j^N\}\) that starts

at \(X_0 = m:\)

\[ L'_m,n := \inf\{j \geq 1 : X_j = n \text{ or } |X_j| = N\}. \quad (1.6) \]

Again, for any starting level \(X_0 = m,\) let \(\Gamma'_{m,n} := \{X_j, j = 0, \ldots, L'_m,n\}\)

do notate the ordinary first passage path from level \(m\) to either level \(n\) or to the

boundary of the gambler’s ruin process (if the boundary at levels \(N\) and \(-N\)

appears before level \(n\) appears). Now, for our key definition (1.7), additional

conditions are placed on the first passage path to make it “one-sided” so that

the boundary won’t appear before the intended destination of the path. It is

also for this reason that we put the prime sign in the notation, to remind of

the one-sidedness condition that will be brought to bear whenever we invoke

this notation.

Denote by \(R'_{m,n}\) the number of runs and by \(V'_{m,n}\) the number of short

runs, respectively, along \(\Gamma'_{m,n},\) where \(L'_m,n\) denotes the number of steps along

this path. For \(m < n,\) define \(g_{m,n}(r, y, z; a, b)\) as the following “upward”

conditional joint probability generating function for these counting statistics

given two conditions on the path: (1) the path is a one-sided first passage

path that starts at \(m\) and stays at or above level \(m\) until it reaches level \(n,\)

and (2) the first two steps of this path are both in the positive direction.

\[ g_{m,n}(r, y, z; a, b) := E(r^{R'_{m,n}} y^{V'_{m,n}} z^{L'_m,n} | \varepsilon_1 = \varepsilon_2, X_0 = m, X_j \geq m, j = 0, \ldots, L'_m,n). \quad (1.7) \]

The condition that the first two steps be in the same direction in the definition (1.7)

arises due to the inclusion of the statistic \(V'_{m,n}\) in the analysis. If still \(m < n\) then we also define the analogous “downward” conditional joint

generating function \(g_{n,m}:\)

\[ g_{n,m}(r, y, z; a, b) := E(r^{R'_{n,m}} y^{V'_{n,m}} z^{L'_{n,m}} | \varepsilon_1 = \varepsilon_2, X_0 = n, X_j \leq n, j = 0, \ldots, L'_{n,m}). \quad (1.8) \]

Thus again we are making two conditions in the definition, this time with a

one-sided first passage path starting at level \(n\) and staying at or below the
level \(n\) until level \(m\) is reached, and this time with the first two steps of the path being both downward.

In the homogeneous case of course \(g_{m,n}\) depends only on \(|n-m|\). In this case the two parameter Fibonacci polynomial \(w_n^*\) appears as the denominator of the rational expression for \(g_{0,n}(r, y, z; a, a)\); see (1.12). To formulate this property, define certain functions

\[
\tau_a := 1 + (1 - a)^2 r^2 z^2 y(1 - y), \quad \omega_a := 1 - (1 - a)^2 r^2 y^2 z^2
\]

\[
x_a := a^2 z^2 \tau_a, \quad \beta_a := 1 + z^2 (a^2 - (1 - a)^2 r^2 (y^2 + a^2 (1 - y)^2 z^2)).
\]  (1.9)

In the special case \(y = 1\) and \(b = a = \frac{1}{2}\) we have \(x_a = \frac{1}{4} z^2\), and \(\beta_a = 1 + \frac{1}{4} z^2 (1 - r^2)\) in agreement with the corresponding parameters of [6] for the two variable problem without persistence. The formula (1.12) has a slightly different form than its counterpart in [6] since here we have the added condition on the first two steps of the path under \(g_{m,n}\). However the formula (1.14) for the joint generating function of the meander counting statistics does indeed agree in the special case with [6]. Define also \(q_0^*(a) = q_0^*(r, y, z; a)\) and \(w_0^*(a) = w_0^*(r, y, z; a)\), employing the definitions of (1.9).

\[
q_0^*(a) := -(1 - y)(1 + y + (1 - a)^2 r^2 y^2 z^2(1 - y))/\tau_a^2, \quad q_1^*(r, y, z; a) := y^2;
\]

\[
w_0^*(a) := (1 - (1 - a)^2 r^2 z^2(1 - y)^2)/\tau_a^2, \quad w_1^*(r, y, z; a) := 1.
\]  (1.10)

Finally apply the basic generalized Fibonacci recurrence of Definition (1.4) with \(\beta := \beta_a\) and \(x := x_a\) via the definitions (1.9) to define \(\{q_n^*\}\) and \(\{w_n^*\}\) with initial conditions for these sequences determined by (1.10). Even though the zeroth initial conditions are not polynomial, the sequences are polynomial for all \(n \geq 1\) and may be written of course in terms of the fundamental Fibonacci polynomials of Definition (1.4), as follows:

\[
q_n^*(a) := c_1 q_n(x_a, \beta_a) + c_2 w_n(x_a, \beta_a), \quad c_2 := q_0^*(a), c_1 = y^2 - c_2;
\]

\[
w_n^*(a) := c_1 q_n(x_a, \beta_a) + c_2 w_n(x_a, \beta_a), \quad c_2 := w_0^*(a), c_1 = 1 - c_2.
\]  (1.11)

Then, in the homogeneous case we have:

\[
g_{0,n}(r, y, z; a, a) = C_{n,a} \omega_a r z n r a^{n-2} / w_n^*(r, y, z; a),
\]  (1.12)

for \(C_{n,a} := a^{n-2}(n-(n-1)a)/(2-a)\). See Proposition 5 for the two parameter extension of (1.12). The “numerator” polynomials \(q_n^*\) will appear when we pass to the rational function expression of (1.3) for the homogeneous case in Proposition 1.

We explain the derivation of the expressions \(x_a, \beta_a, w_n^*\) in (1.9)–(1.10). When we set \(b = a\) in (1.28), we obtain a recursive formula for \(g_{n} := g_{0,n}(r, y, z; a, a)\), since \(g_{m,m+i}(r, y, z; a, a)\) is independent of \(m\). We use the algebraic factorization of \(g_{n}\) to find the form of the denominator polynomial \(w_n^*\) that we take to have a unit constant term, with \(w_1^* := 1\). Then, assuming the bivariate Fibonacci paradigm holds, we are led via the interlacing identity
to $x_a$, and so on to $\beta_a$ via the recurrence $w_n^{*} = \beta_a w_{n-1}^* - x_a w_{n-1}$. By backwards iteration we arrive at $w_0^*$. We will verify that indeed this paradigm works when we establish (1.12) by induction, again as a special case of Proposition 5.

By symmetry of the full model, any non-negative path $\Gamma$ started from the origin has the same probability as the corresponding, reflected non-positive path. Therefore if we consider (1.8) with $n = 0$, and with $m = -\ell < 0$, then by one to one correspondence between paths and (1.7)–(1.8), we arrive at $g_0,\ell(r,y,z; a,b)$ for all $\ell \geq 2$. For this reason it will suffice in our study of the meander to consider only a positive meander path.

Now denote by $R', V', L'$ the counting statistics for runs, short runs, and steps, respectively, in the meander portion of the gambler’s ruin process. To account for the condition on the first two steps of the path under $g_0, n$ being the same, we define

$$h_a := (1 + (1 - a)^2 r^2 z^2 y(1 - y))/\omega_a = \tau_a/\omega_a,$$  \hfill (1.13)

where $\tau_a$ and $\omega_a$ are defined by (1.9). By the discussion in Section 1.2 preceding (1.31), the factor $J_a = a(2 - a)h_a/(rz^2)$ is the joint generating function of the numbers of runs, runs of length exactly one, and steps, in an initial part of a positive meander path up until this path reaches the level $m = 3$ for the first time. Therefore, it follows that for $f \geq 3$,

$$E\{r^{R'}_N y^{V'}_N z^{L'}_N\} = a(2 - a)zh_ag_{0,N-1}(r, y, z; a, b).$$  \hfill (1.14)

We denote by $G_n(r, y, z; a, b)$ the conditional joint probability generating function of the excursion statistics $R$, $V$ and $L$ given that the height is $n$:

$$G_n(r, y, z; a, b) := E(r^{R}y^{V}z^{L}|X_0 = 0, H = n), \ n \geq 1.$$  \hfill (1.15)

It is not difficult to recover the conditional generating function $G_n$ of (1.15) from (1.7); see (1.31). It is a basic result of the Fibonacci–based construction that the closed formula for the conditional generating function $K_N(r, y, z; a, a)$ of (1.3) that is written in Proposition 1 for the homogeneous case can be recovered from the formula for $G_n(r, y, z; a, a)$ in (1.15). This is accomplished in Section 1.6 where in fact this recovery property is extended from the homogeneous case to the full model in the guise of Theorem 1.

Our method hinges on establishing a recurrence relation for the sequence $\{g_{m,n}\}$. We accomplish this in Section 1.2 where the doubly-indexed recurrences take final form in (1.28)–(1.29). Let $n > m$. In the formulation of the recurrence for $g_{m,n}$, we must take account of the probability that a one-sided first passage from level $m$ to level $n$ remains at or above the starting level, as defined by $\rho_{m,n}$; we must also define the corresponding probability $\rho_{n,m}$ for a downward transition, as follows.
\[ \rho_{m,n} := P(X_j \geq m, j = 0, \ldots, L'_{m,n}|X_0 = m); \]
\[ \rho_{n,m} := P(X_j \leq n, j = 0, \ldots, L'_{n,m}|X_0 = n). \]  

(1.16)

For \( b = a = \frac{1}{2} \), the probability \( \rho_{0,n} \) is determined by the classical solution of the probability of ruin started from fortune \( n \) on the interval \([0, n+1]\).

For \( b = a \), \( \rho_{m,n} \) depends only on \( k = n - m \) and is determined by \( \rho_{m,m+i} = \frac{1}{2}(i - (i-1)a)^{-1} \). We develop the formula for \( \rho_{m,n} \) in both parameters \( a \) and \( b \) in Proposition 2.

We now state some results for the homogeneous case of our three counting statistics. We prove these results in Sections 1.6–1.7.

**Proposition 1.** Suppose \( b = a \). Then the conditional generating function

\[ K_N(r, y, z; a, a) = \frac{1}{N^2}q_N^a(r, y, z; a)w_N^a(r, y, z; a) \]

where \( q_N^a \) and \( w_N^a \) are defined by (1.9)–(1.11).

To prove Proposition 1 we use an induction argument that relies on the fact that \( \{q_n^a(a)\} \) and \( \{w_n^a(a)\} \) satisfy the following identity of Lemma 6.

\[ w_n^a(a)q_{n+1}^a(a) - q_n^a(a)w_{n+1}^a(a) = a^2z^2x_a^{n-1}, \quad \text{for all } n \geq 1. \]  

(1.17)

We refer to the left side of (1.17) as an “interlacing bracket”, and in Lemma 6 this bracket is extended to the full model. We are thus able to extend Proposition 1 to the full model in Theorem 1. An application of Theorem 1 is to find limiting distributions for various statistics over the last visit portion of the gambler’s ruin; see Section 1.9. The method for doing this has been shown in [6] for the special case \( b = a = \frac{1}{2} \) and \( y = 1 \).

The unconditional joint generating function of the excursion statistics for \( b = a \), \( K(r, y, z; a) := E\{r^R y^U z^L\} \), follows by taking the limit as \( N \to \infty \) in Proposition 1.

**Corollary 1.** Let \( b = a \) and define \( \alpha_a := -4x_a \sqrt{\beta_a^2 - 4x_a} \) for \( x_a \) and \( \beta_a \) given by (1.9). Then

\[ K(r, y, z; a) := \lim_{N \to \infty} K_N(r, y, z; a, a) = \frac{1 - \frac{1}{2} \beta_a - \frac{1}{2} \alpha_a}{1 - a}. \]  

(1.18)

We prove Corollary 1 in Section 1.7. We give an explicit formula for the joint generating function of the excursion statistics \( R, U, \) and \( L \), by \( K(r, 1/u, z; a) \), in (1.137).

Let \( P_a \) denote the probability for the homogeneous model with persistence parameter \( a \). We obtain the following symmetry for the joint distribution of the excursion statistics.

**Corollary 2.** Let \( b = a \). Then for all \( n \geq 2 \) there holds:
In particular if \( a = \frac{1}{2} \), then

\[
E\{e^{irR}e^{isU}e^{itL}\} - E\{e^{ir(L-R)}e^{isU}e^{itL}\} = \frac{1}{2}e^{2it}(e^{2ir} - 1).
\]

The Corollaries 1–2 extend the known result for the simple symmetric random walk that

\[
P(R = 2k) = P(L - R = 2k), \quad k \geq 2.
\]

Indeed for \( y = 1 \) and \( a = \frac{1}{2} \) the formula (1.18) reduces to

\[
K(r, z) := K(r, 1, z; \frac{1}{2}) = 1 - \frac{1}{4}z^2(1 - r^2) - \sqrt{(1 + \frac{1}{4}z^2(1 - r^2))^2 - z^2},
\]

where \( K(1/r, r) = \frac{5}{4} - \frac{1}{4}r^2 - \frac{1}{4}\sqrt{9 - 10r^2 + r^4} \). We prove Corollary 2 in Section 1.7.1, and there discuss some of its combinatorial features.

There are many calculations used to establish various formulae by the help of certain key definitions, especial Definition (1.45)–(1.46). We reserve the phrase “direct calculation” to mean that computer algebra (Mathematica, [11]) is used to help verify the results. This is especially true of certain calculations in Sections 1.4.2, 1.8, and 1.9. The complication of a second stratumn certainly involves finding the right formulae and then rendering a proof; we often utilize induction based on the proposed formulae. In Section 1.8 we focus on limit laws over the meander with scaling of order \( N \); the main result there is Theorem 2. Its companion, Theorem 3, provides a corresponding limit law over the last visit portion of the gambler’s ruin process.

### 1.2 Recurrence relations for \( \{g_{m,n}\} \)

In this section we establish the general recurrence relations governing the upward and downward one-sided first passage generating functions of (1.7)–(1.8). We first note the initial conditions, as forced by the initial trajectory in either the case of upward or downward paths:

\[
g_{m, m+2}(r, y, z; a, b) := rz^2, \quad g_{m+2, m}(r, y, z; a, b) := rz^2, \quad m \geq 0.
\]

We review the path decomposition of [6] to see how a recursion formula arises. It is convenient to focus on \( g_{n,0} \) with some \( n \geq 3 \). Recall the definition (1.8):

\[
g_{n, 0} = E(r^{R'_{n,0}}y^{V'_{n,0}}z^{L'_{n,0}}|\varepsilon_1 = \varepsilon_2, X_0 = n, X_j \leq n, \quad j = 0, \ldots, L'_{n,0}).
\]

This downward first passage path from \( n \) to \( 0 \) must eventually reach the level \( m = 1 \). So we have an initial factor of \( g_{n,1} \) in a product formula for \( g_{n,0} \). After this initial section, the path may oscillate between levels \( m = 1 \) and \( m = 2 \) for an indefinite period before moving up to level \( m = 3 \) or down to level \( m = 0 \). Thus the initial section is followed by a sequence of steps of the form \((UD)^jUU\) or \((UD)^jD\) where \( U \) and \( D \) stand for one step up or down respectively, and \((UD)^\ell\) is shorthand for \( UDUDD\cdots \) with \( \ell \) repetitions of the
pattern $UD$ for some $\ell \geq 0$. We thus define

$$
\begin{align*}
\omega(a, b) &:= 1 - (1 - a)(1 - b)\pi^2y^2z^2, \quad k(a, b) := (a + b - ab)/\omega(a, b), \\
\tau(a, b) &:= 1 + (1 - a)(1 - b)\pi^2y^2z^2(1 - y); \quad h(a, b) := \tau(a, b)/\omega(a, b),
\end{align*}
$$

(1.21)

where we have suppressed in each case the dependence on the generating function variables $r, y, z$, so for example, $\omega(a, b) = \omega(r, y, z; a, b)$. Here $k(a, b)$ is normalized to unit value when $r = y = z = 1$. It is easy to see that if $f \geq 3$ then the piece of the joint generating function over the indefinite period of oscillations between levels $m = 1$ and $m = 2$, starting and ending at level $m = 1$, where this period is preceded by $DD$ and followed by $UU$, is $k(a, a) = c\sum_{n=0}^{\infty}((1 - a)^2r^2y^2z^2)^f = c/\omega(a, a)$, where $c = a(2 - a)$ will normalize $k(a, a)$. If instead $f = 2$ is the change of stratum parameter then we obtain instead $k(a, b)$ in place of $k(a, a)$ due to the fact that now, with $f = 2$, a change in direction at level $m = 2$ occurs with probability $(1 - b)$ while a change in direction at level $m = 1$ occurs with probability $(1 - a)$. Furthermore $k(a, b)$ is defined to handle all terminating sequences $UD_fD$ starting from level $m = 1$, where again the choice of the pair of parameters $(a, b)$ depends on $f$. To handle the dependence on $f$ we now define

$$
[a, b]_n^\pm := \begin{cases}
(a, a), & \text{if } m \leq f - 1 \\
(a, b), & \text{if } m = f - 1 \\
(b, b), & \text{if } m \geq f
\end{cases}; \quad [a, b]_{n+}^\pm := \begin{cases}
(a, a), & \text{if } n \leq f - 1 \\
(a, b), & \text{if } n = f \\
(b, b), & \text{if } n \geq f + 1
\end{cases}.
$$

(1.22)

Let us suppose that the continuation of the path after the first downward passage to level $m = 1$ takes the form $(UD)^fUU$, since we handle the case of a termination sequence starting from level $m = 1$ of form $(UD)^fD$ via a final factor in our formula for $g_{n,0}$. Now from $UU$ the path makes an upward first passage to level $n$ again, and the pattern “up to level $n$ and down to level 1” repeats for an indefinite number of times, $\ell \geq 0$. To handle the probability associated with the turning of the path downward from a level it will no longer exceed in the future of the path, or in turning from the bottom level $m = 1$ to upwards (in the “return to level 1” of the current example), we define:

$$
\gamma_m := \begin{cases}
1 - a, & \text{if } m \leq f - 1 \\
1 - b, & \text{if } m \geq f
\end{cases}.
$$

(1.23)

By definition (1.16), it now follows that $g_{n,0} = cg_{n,1}\lambda_n\lambda_{n-1} \cdots \lambda_1z\pi[a, b]^\tau_1$, with $\lambda_n = \lambda_{1,n}$ defined as follows:

$$
\begin{align*}
\lambda_{1,n} &:= \sum_{\ell=0}^{\infty}4\gamma_1\gamma_n\rho_{1,n}\rho_{n,1}k[a, b]^\tau_1k[a, b]_n^\tau g_{1,n}g_{n,1}^\ell \\
&= [1 - 4\gamma_1\gamma_n\rho_{1,n}\rho_{n,1}k[a, b]^\tau_1k[a, b]_n^\tau g_{1,n}g_{n,1}]^{-1}.
\end{align*}
$$

(1.24)

Here the factor of 4 arises due to the fact that the stationary probabilities for first step up and down, namely $\pi_+ = \frac{1}{2}$ and $\pi_- = \frac{1}{2}$, get replaced by
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\[ \gamma_1 \text{ and } \gamma_n \text{ respectively in } \rho_{1,n} \text{ and } \rho_{n,1}. \] The successive maximum levels \( n = M_1 \geq M_2 \geq \cdots \geq M_r \) over the whole future of the path, determined in turn from the points of each of its returns to level \( m = 1 \) from the previous such maximum, are the so-called “future maxima” \([6]\) of a downward path from level \( n \geq 2 \) to level \( m = 0 \). We have \( M_r \geq 2 \), and the downward path terminates after the future maximum \( M_r \), that is the path returns to level \( m = 1 \) after this maximum is achieved, and then reaches level \( m = 0 \) after a termination sequence \((UD)^r D\). See Figure 1. The factor \( \lambda_n \) allows for an indefinite number of passages to a future maximum level \( n \), each time only after a return to level 1. Eventually the path will never rise to level \( n \) again; thus the factors \( \lambda_{n-1}, \lambda_{n-2}, \ldots \). The factor \( zh[a, b]_1^{+} \) corresponds to the termination sequence \((UD)^r D\) starting from the last time the path reaches level \( m = 1 \) and never again exceeds level \( m = 2 \). Here, if \( f \geq 3 \), by \([1.22]\), \( h[a, b]^+_f = h(a, a) = 1 + (1 - a)^2 r^2 z^2 y \sum_{i=0}^{\infty} ((1 - a)^2 r^2 y^2 z^2)^i \), which is written by \([1.13]\). Now replace \( m = 1 \) by \( m \geq 1 \) for a final destination level \( m = 1 \), to obtain the following “downward” recurrence relation for any \( m < n - 1 \):

\[ g_{n,m-1} = czh[a, b]_m^+ g_{n,m} \prod_{j=m+2}^{a} \lambda_{m,j}. \] (1.25)

for a normalization constant \( c \) such that \( g_{a,m-1}(1, 1, 1; a, b) = 1 \). Here we officially define \( \lambda_{m,j} = \lambda_{m,j}(r, y, z; a, b) \):

\[ \lambda_{m,j} := [1 - 4 \gamma_m \gamma_j \rho_{m,j}]^{-1} k[a, b]_m^{-1} k[a, b]_m^{+1} g_{m,j} g_{j,m}^{-1}, \quad m + 2 \leq j. \] (1.26)

Note that the product in \([1.25]\) has been written in reverse order to the order we used in the initial downward construction with \( m = 1 \). Each \( \lambda_{j,m} \) in \([1.26]\) is the sum of a geometric sum modeled after \([1.24]\). By symmetric arguments we also obtain the “upward” recursion relation for any \( m < n - 1 \):

\[ g_{m,n+1} = czh[a, b]_n^- g_{m,n} \prod_{j=m+2}^{n-2} [1 - 4 \gamma_n \gamma_j \rho_{n,j}]^{-1} k[a, b]_n^{-1} k[a, b]_n^{+1} g_{n,j} g_{j,n}^{-1} \]

\[ = czh[a, b]_n^- g_{m,n} \prod_{j=m}^{n-2} \lambda_{j,n}. \] (1.27)

where again \( c \) denotes a generic normalization constant. Here the author writes the relation \([1.27]\) by thinking of the level \( m \) being above the level \( n \) in a reversed \( y \)-axis coordinate system, so that the previous argument with a downward path may be applied.

By \([1.25]-[1.27]\) we retrieve “closed” recurrence formulae for \( g_{m,n} \). Indeed, for \( m < n - 2 \), we see for the upward formula \([1.27]\), that the partial product \( \prod_{j=m+1}^{n-2} \lambda_{j,n} \) without the first term \( \lambda_{m,n} \) is of the form \( \prod_{j=m+1}^{n-2} \lambda_{j,n} = c_0 g_{m+1,n+1}/(zh[a, b]_n^- g_{m+1,n}) \), for a normalization constant \( c_0 \). Hence for another normalization constant \( c_1 \), there holds

\[ g_{m,n+1} = c_1 g_{m,n} g_{m+1,n+1}(g_{m+1,n})^{-1} \lambda_{m,n}, \] (1.28)
where factors of $zh[a,b]_{m}^{-}$ have canceled in numerator and denominator. Similarly, by applying (1.25), and by rewriting the product $\prod_{j=m+2}^{n-1}\lambda_{m,j} = c_{0}g_{n-1,m-1}/(zh[a,b]_{m}^{+}g_{n-1,m})$, we obtain for $m < n - 2$ and a normalization constant $c_{2}$,

$$g_{n,m-1} = c_{2}g_{n,m}g_{n-1,m-1}(g_{n-1,m})^{-1}\lambda_{m,n}. \tag{1.29}$$

Observe that the factor $\lambda_{m,n} = [1 - 4\gamma_{m,n}\rho_{m,n}\rho_{n,m}k[a,b]_{m}^{+}k[a,b]_{n}^{-}g_{m,n}g_{n,m}]^{-1}$, with $m < n$, appears exactly the same in both (1.25)–(1.26) and (1.27) and again in (1.28)–(1.29).

![Fig. 1.1 Downward Transition with Future Maxima $M_{1} = 5$, $M_{2} = 4$, and $M_{3} = 2$.](image)

### 1.2.1 Recurrence for $G_{n}$.

We now obtain a companion formula to (1.25)–(1.27) for the conditional joint generating function of the excursion statistics, started from $X_{0} = 0$ and given $H = n$, denoted $G_{n}$ and defined by (1.15). In the definition for $G_{n}$ we note that there is no condition that a path takes first two steps in the same direction. Thus we only have the condition that after the first step from $m = 0$, the path does not return to the $x$-axis until it terminates, but that also the path reaches the specified height, $n$, as a maximum (for a positive path). Now we work in an upright coordinate system. We consider an initial sequence $U(UD)\ell UU$ that brings a positive path for the first time to level $m = 3$ while never returning to level $m = 0$; here $\ell \geq 0$. The probability associated with the initial sequence is $a^{2}(1 - a)^{2\ell}$, so we are really considering a random initial sequence number $\ell \geq 0$ for the initial sequence. We write the joint generating function, $J_{a} = J_{a}(r,y,z)$, for the numbers of runs, runs of length exactly one, and steps, along the initial sequence, but excluding the run of the line of ascent of the final $UU$, as follows:

$$J_{a} = c_{a}z(1 + (1 - a)^{2}r^{2}yz^{2} + (1 - a)^{4}r^{4}y^{3}z^{4} + \cdots) = a(2 - a)z\tau_{a}/\omega_{a}, \tag{1.30}$$
with \( \tau_a \) and \( \omega_a \) defined by (1.9). Now, to make a positive path that starts at level \( m = 0 \) and reaches a level \( n \geq 3 \) for a first time, we start with an initial sequence \( U(UD)^\ell U \). We also consider a path \( \Gamma_m^+ \) that starts at level \( m = 1 \) with \( UU \) and stays at or above level \( m = 1 \) and reaches level \( n \) for a first time. Now any path for \( g_{l,n} \) defined by (1.7) is just such a path \( \Gamma_m^+ \). We link the initial sequence \( U(UD)^\ell U \) and \( \Gamma_m^+ \) together by making them overlap on the \( UU \) on the end of the initial sequence and beginning of \( \Gamma_m^+ \). Then, assuming \( f \geq 3 \), the joint generating function of the number of runs, runs of length exactly one, and steps, until a positive path first reaches level \( n \geq 3 \) is given by \( J_a(r,y,z; \cdot, b) \). We notice by (1.30) that \( J_a(r,y,z; \cdot, b) = a(2 - a)zh_a \) for \( h_a \) defined by (1.13). Hence, with \( G_n \) and defined by (1.15), there holds, for \( f \geq 3 \),

\[
G_n(r,y,z; a,b) = a(2 - a)zh_a g_{n,0} k[a,b]_{a,b}^{-1}, \quad n \geq 3.
\]

The factor \( k[a,b]_{a,b}^{-1} \) in the formula (1.31) takes account of the part of the path from the first maximum at level \( n \) until the first sequence \( DD \) appears after this first maximum.

### 1.3 Formula for \( \rho_{m,n} \).

In this section we establish a formula for \( \{\rho_{m,n}\} \) as defined by (1.16). Note that \( 1 - \rho_{1,N} \) is the probability of ruin for the gambler’s ruin persistence model with two strata on \([0,N]\) in case \( X_0 = 1 \). The novelty of our approach, based on induction, is unnecessary if \( b = a \), since by [5] a difference equation will solve the probability of ruin in this case.

The method we use to establish a formula is based first on the construction of Section 1.2. Indeed, the method in the present section has the same overall structure that is used in Section 1.5 to verify a formula for \( g_{m,n} \). In place of \( \lambda_{m,n} \) of (1.26), here define:

\[
u_{m,j} := [1 - 4\gamma_m \gamma_j \rho_{m,j} \rho_{j,m}]^{-1}, \quad m + 1 \leq j.
\]

In fact, because \( g_{m,n} \) and \( k(a,b) \) are both normalized to take the value 1 at \( r = y = z = 1 \), we have \( u_{m,j} = \lambda_{m,j}(1, 1, 1; a,b) \) when \( m + 2 \leq j \); in (1.32) the definition holds for all \( j \geq m + 1 \).

Let \( m < n \). Similar to the way we developed the formulae (1.25)–(1.27), only without one of the defining conditions of \( g_{m,n} \) specifying that the first two steps of a (one-sided first passage) path be in the same direction, it follows by definition (1.16) that

\[
(i) \quad \rho_{m,n+1} = (1 - \gamma_n)\rho_{m,n} \prod_{j=m}^{n-1} u_{m,j};
\]

\[
(ii) \quad \rho_{n,m-1} = (1 - \gamma_m)\rho_{n,m} \prod_{j=m+1}^{n} u_{m,j}.
\]

(1.33)
The factor \((1 - \gamma_n)\) in \((1.33)\) (ii) gives the probability \((a \text{ or } b)\) of the last step in any one-sided first passage path from level \(m\) to level \(n\); a similar comment applies to \((1.33)\) (ii). By the same method as shown in Section 1.2 to obtain \((1.25)\)–\((1.29)\), we have by \((1.33)\) that

\[
(i) \quad \rho_{m,n+1} = \rho_{m,n}\rho_{m+1,n+1}(\rho_{m+1,n})^{-1}u_{m,n};
\]

\[
(ii) \quad \rho_{n,m-1} = \rho_{n,m}\rho_{n-1,m-1}(\rho_{n-1,m})^{-1}u_{m,n}.
\]

With the help of \((1.34)\), we will now develop a closed formula for \(\rho_{m,n}\). We first make a definition to establish the form of \(\rho_{m,n}\) by which it will be convenient to verify the formula.

**Definition 2.** Let \(\rho_{m,n}\) be given by \((1.10)\). We define a denominator term \(\Pi_{m,n}\) for \(\rho_{m,n}\) as follows; \(m < n\) in all cases:

(I) Upward denominators:

\[
(1) \quad \rho_{m,n} = \frac{1}{2}(b/a)/\Pi_{m,n}, \quad m \leq f - 1; \quad (2) \quad \rho_{m,n} = \frac{1}{2}/\Pi_{m,n}, \quad f \leq m. \quad (1.35)
\]

(II) Downward denominators:

\[
(1) \quad \rho_{n,m} = \frac{1}{2}(b/a)/\Pi_{n,m}, \quad n \leq f - 1; \quad (2) \quad \rho_{n,m} = \frac{1}{2}/\Pi_{n,m}, \quad f \leq n \quad (1.36)
\]

**Proposition 2.** The terms \(\Pi_{m,n}\) determined by Definition \((1.35)\)–\((1.36)\) are given by the following expressions:

I. Between strata formulae:

1. \(\Pi_{f-\ell,f+j} = j + \ell(\frac{b}{a}) - (\ell + j - 1)b, \quad \ell \geq 1, j \geq 0;\)

2. \(\Pi_{f+j,f-\ell} = (j + 1) + (\ell - 1)(\frac{b}{a}) - (\ell + j - 1)b, \quad \ell \geq 1, j \geq 0.\)

II. Within stratum formulae:

1. \(\Pi_{m,m+\ell} = \Pi_{m+\ell,m} = \ell(\frac{b}{a}) - (\ell - 1)b, \quad m < m + \ell \leq f - 1;\)

2. \(\Pi_{m,m+\ell} = \Pi_{m+\ell,m} = \ell - (\ell - 1)b, \quad f \leq m < m + \ell.\)

**Remark 1.** If \(b = a\), we have \(\Pi_{m,m+\ell} = \Pi_{m+\ell,m} = \ell - (\ell - 1)a\) in all cases of Proposition 2 consistent with Definition \((1.35)\)–\((1.36)\) and the result of Section 2.

**Proof (Proposition 2).** By the homogeneous case \((5)\), and Definition \((1.35)\)–\((1.36)\), the within stratum formulae \(\Pi.1–2\) hold in general. The proof of the between strata cases proceeds by induction on \(n - m\), where we always assume \(n > m\). We first verify the cases \(n - m = 2\) for \(\Pi_{m,n}\) and \(\Pi_{n,m}\) in (I); the case \(n - m = 1\) is trivial. We apply \((1.32)\)–\((1.33)\) with \(u_{m,m+1} = [1 - \gamma_m\gamma_{m+1}]^{-1}\). Thus for all \(m\), \(\rho_{m,m+2} = \frac{1}{2}(1 - \gamma_{m+1})/(1 - \gamma_m\gamma_{m+1})\). In particular \(\rho_{f-1,f+1} = \frac{1}{2}b/(1 - (1-a)(1-b)) = \frac{1}{2}(b/a/[1 + \frac{b}{a} - b]).\) This gives the correct form for the denominator in (I) by Definition \((1.35)\) (I)(1). On the other hand, \(\rho_{f,f-2} = \frac{1}{2}(a+b)/(1 - (1-a)(1-b)) = \frac{1}{2}/[1 + \frac{b}{a} - b],\)
again of correct form. Finally, \( \rho_{f-2,f} = \frac{1}{a}/(1 - (1-a)^2) = \frac{1}{a}(b/a)/(2(a/2) - b) \), and \( \rho_{f+1,f} = \frac{1}{a}b_t(f+1) = \frac{1}{a}b/(1 - (1 - b)^2) = \frac{1}{a}/[2 - b] \). Thus all the cases \( n - m = 2 \) have been verified.

Assume by induction that all statements of the proposition hold for \( 2 \leq n - m \leq k \) for some \( k \geq 2 \). We wish to show the following induction step:

Both (i) : \( \Pi_{m,n+1} \), and (ii) : \( \Pi_{n,m-1} \), conform to statements I.1 and I.2, respectively, for all \( m \leq f - 1 \) and \( n \geq f \), with \( n - m = k \).

There are two boundary cases, \( \Pi_{f-k-1,f} \) and \( \Pi_{f+k-1,f} \), that aren’t covered formally by this scheme. However both of these cases actually fall under the within stratum regime. For example, in the calculation of \( \rho_{f-k-1,f} \), the one-sided first passage path from \( f - k - 1 \) to \( f \) never oscillates between levels \( f - 1 \) and \( f \), so the probability \( \rho_{f-k-1,f} \) is governed by a single stratum design. Hence, \( \rho_{f-k-1,f} = \frac{1}{a}/(k + 1 - ka) = \frac{1}{a}(b/a)/[(k + 1)\cdot(a/2) - kb] \), consistent with I.1. For the other boundary case, by similar reasoning, \( \rho_{f+k-1,f} = \frac{1}{a}/(k + 1 - kb) \), consistent with I.2. So the boundary cases have been resolved for all \( k \).

We proceed with our argument for establishing (1.37). By Definition (1.35) (I), we compute \( u_{m,n} \) by (1.32) under the conditions on \( m \) and \( n \) in (1.37) as follows.

\[
\begin{align*}
\Pi_{m,n} = [1 - 4a\gamma_\rho_m \rho_m \rho_n]^{-1} & = \{1 - (1 - a)(1 - b)(b/a)/(\Pi_{m,n}\Pi_{n,m})\}^{-1} \\
& = \Pi_{m,n}\Pi_{n,m}/\{\Pi_{m,n}\Pi_{n,m} - (1 - a)(1 - b)(b/a)\}.
\end{align*}
\]

Now write \( m = f - \ell \) and \( n = f + j \) for some \( \ell \geq 1 \) and \( j \geq 0 \) with \( \ell + j = k \geq 2 \). By the induction hypothesis we can write \( \Pi_{m,n} = j + \ell(\frac{b}{a}) - (\ell + j - 1)b \).

\[
\begin{align*}
\{j + \ell(\frac{b}{a}) - (\ell + j - 1)b\} \{j + 1 + \ell(\frac{b}{a}) - (\ell + j - 1)b\} - \frac{(1-a)(1-b)b}{a} & = \{j + 1 + \ell(\frac{b}{a}) - (\ell + j - 1)b\} \{j + (\ell - 1)(\frac{b}{a}) - (\ell + j - 2)b\} \\
& = \{j + 1 + \ell(\frac{b}{a}) - (\ell + j)b\} \Pi_{m+1,n},
\end{align*}
\]

where at the last step we again applied the induction hypothesis to write \( \Pi_{m+1,n} \). Thus by (1.38), (1.39),

\[
u_{m,n} = \frac{\Pi_{m,n}\Pi_{n,m}}{\{j + 1 + \ell(\frac{b}{a}) - (\ell + j)b\} \Pi_{m+1,n}}.\]

Now rewrite (1.34)(i) by applying Definition (1.35), (1.36) and (1.40) as follows. We have that \( \rho_{m,n+1} \) is given by:
Finally we use that, by the induction hypothesis and the statements of the proposition themselves, we have $\Pi_{m+1,n+1} = \Pi_{n,m}$ for all $n > m$ with $n - m = k$. Therefore, by applying this fact, we find that (1.41) yields

$$\rho_{m,n+1} = \frac{1}{2} \left\{ j + 1 + \ell \left( \frac{b}{a} \right) - (\ell + j) b \right\}.$$  

(1.42)

Thus by (1.42) and Definition (1.37) (I)(1), the induction step (1.37) has been verified for case (i). The argument for the downward case (ii) is wholly similar to the upward case (i), that is, again for $n = f + j$ and $m = f - \ell$ we obtain $\rho_{n,m-1} = \frac{1}{2} \left\{ j + 1 + \ell \left( \frac{b}{a} \right) - (\ell + j) b \right\}$. Therefore the induction step (1.37) has been verified. Thus the proposition is proved. □

1.4 The denominators $\overline{w}_{m,n}$ of $g_{m,n}$.

We recursively define an array of functions $\{\overline{w}_{m,n}(r, y, z; a, b)\}$ such that $\overline{w}_{m,n}$ will turn out to be the denominator polynomial with constant term 1 for the rational expression of $g_{m,n}$. Recall the definitions (1.21)–(1.22). We first define initial cases:

$$\overline{w}_{m,m+2} := \omega[a, b]_{m+}, \quad m \geq 0; \quad \overline{w}_{n,n-2} := \omega[a, b]_{n-}, \quad n - 2 \geq 0; \quad \overline{w}_{m,m+1} = \overline{w}_{m+1,m} = 1, \quad m \geq 0.$$  

(1.43)

For example, if $m \leq f - 2$, then $[a, b]_{m}^+ = (a, a)$, so $\overline{w}_{m,m+2} := \omega(a, a) = 1 - (1 - a)^2 r^2 y^2 z^2$. Note that the initial cases of (1.43) are consistent with Definition (1.35) as illustrated in (1.47). We require a generalization of $\tau_n$, $x_a$, and $\beta_n$ of (1.19) to make our definition of $\{\overline{w}_{m,n}\}$ for two strata, as follows. Define

$$\tau(a, b) := 1 + (1 - a)(1 - b)r^2 z^2 y(1 - y); \quad x(a, b) := b^2 z^2 \tau^2(a, b); \quad \beta(a, b) := 1 + z^2 (b^2 - r^2 ((1 - b)^2 y^2 + b^2 (1 - a)(1 - b)(1 - y)^2 z^2)).$$  

(1.44)

Here we note that $\tau(a, b)$ is symmetric in $a$ and $b$, so $x(h, a) = a^2 z^2 \tau^2(a, b)$. Note further that $\tau(a, b)$ was already defined in (1.21) but we repeat its definition here for convenience. An equivalent formula for $\beta(a, b)$ is in the form of a correction: $\beta(a, b) = \beta_b - (b - a)b^2 (1 - b)r^2 (1 - y)^2 z^4$. We generally write $x_a$ and $\beta_a$ to stand for the equivalent $x(a, a)$ and $\beta(a, a)$, respectively.

We now define the recurrence for the denominators $\{\overline{w}_{m,n}\}$ that will make it clearer how the terms of (1.44) arise. The reason we define the denominators rather than derive them is that the recurrence definition makes the induction proofs to follow easy to resolve. Recall the definition (1.10)–(1.11) for $w_\ast^*(a) :=$
In fact \( w_{r}^{*}(a) = \omega(a,a) \), for \( \omega(a,b) \) of \((1.21)\). Further, in case \( b = a \), we simply have \( \overline{w}_{m,n} = w_{n-m}^{*}(a) \), \( |n-m| \geq 2 \). In the following the terminology “upward” and “downward” refers to the direction of the indices in \( \overline{w}_{m,n} \).

**Definition 3.** Denote \( \overline{w}_{m,n} = \overline{w}_{m,n}(r,y;z,a,b) \).

(I) Define the upward denominator \( \overline{w}_{m,n} \) for all \( n-m \geq 2 \) by:

1. \( \overline{w}_{m,m+\ell} := w_{\ell}^{*}(a) \), \( m < m + \ell \leq f \); \( \overline{w}_{m,m+\ell} := w_{\ell}^{*}(b) \), \( f \leq m < m + \ell; \)
2. \( \overline{w}_{f-\ell,f+1} := \frac{1-b}{1-a} w_{\ell+1}(a) + \frac{b-a}{1-a} w_{\ell}(a) \), \( 1 \leq \ell \leq f; \)
3. \( \overline{w}_{m,f+2} := \beta(a,b) \overline{w}_{m,f+1} - x(a,b) \overline{w}_{m,f}, \) \( m \leq f - 1; \)
4. \( \overline{w}_{m,f+j+1} := \beta_{b} \overline{w}_{m,f+j} - x_{b} \overline{w}_{m,f+j-1}, \) \( m \leq f - 1, j \geq 2. \) \hspace{1cm} (1.45)

(II) Define the downward denominator \( \overline{w}_{n,m} \) for all \( n-m \geq 2 \) by:

1. \( \overline{w}_{m+\ell,m} := w_{\ell}^{*}(a) \), \( m < m + \ell \leq f - 1; \) \( \overline{w}_{m+\ell,m} := w_{\ell}^{*}(b) \), \( f - 1 \leq m; \)
2. \( \overline{w}_{m+j,f-2} := \frac{1-a}{1-b} w_{j+2}(b) + \frac{a-b}{1-b} w_{j+1}(b) \), \( 0 \leq j; \)
3. \( \overline{w}_{n,f-3} := \beta(b,a) \overline{w}_{n,f-2} - x(b,a) \overline{w}_{n,f-1}, \) \( f \leq n; \)
4. \( \overline{w}_{n,f-\ell-2} := \beta_{a} \overline{w}_{n,f-\ell-1} - x_{a} \overline{w}_{n,f-\ell}, \) \( f \leq n, \ell \geq 2. \) \hspace{1cm} (1.46)

Notice that in the downward case (II) of the Definition \((1.45)-(1.46)\), we are effectively reversing the roles of \( a \) and \( b \) from the upward case (I). This definition will be useful to derive the interlacing identity, Proposition 8 as well as a closed form expression for \( \overline{w}_{m,n} \), Proposition 4. This closed form expression will be used in particular to prove a necessary simple identity in Lemma 2 that in turn will be utilized in establishing a closed formula for \( g_{m,n} \) in Proposition 3. For a couple of words of explanation, in Definition \((1.45)-(1.46)\) we write the first step of “crossing over the threshold of the stratum” in either upward or downward directions as a linear combination of two successive homogeneous case solutions. For the next step over the threshold in either direction we use the “mixed” parameters for \( x \) and \( \beta \), and for further steps we use the appropriate homogeneous parameters for \( x \) and \( \beta \). With no crossing over a stratum, the appropriate homogeneous solution is shown. Finally, note the consistency between Definition \((1.45)-(1.46)\) and \((1.43)\). For example, in part (I)(2) of the Definition with \( \ell = 1 \), we find:

\[ \overline{w}_{f-1,f+1} = \frac{1-b}{1-a} w_{2}^{*}(a) + \frac{b-a}{1-a} w_{1}^{*}(a) = \frac{1-b}{1-a} \omega(a,a) + \frac{b-a}{1-a} = \omega(a,b). \] \hspace{1cm} (1.47)

### 1.4.1 Interlacing identity for \( \{\overline{w}_{m,n}\} \)

To establish a formula for \( g_{m,n} \) in Proposition 5 we require a so-called interlacing identity for the denominators \( \{\overline{w}_{m,n}\} \) that is fundamental to the Fibonacci polynomial approach. Define the interlacing braket:
Using the Definition (1.45) and (1.48), that first and then use this as a basis for an induction proof of

But by direct calculation we find

and consider first

It actually suffices to consider only the upward direction for the interlacing bracket \([\overline{w}]_{m,n}\), since by Lemma 2 we find that the natural corresponding downward definition, \([\overline{w}]_{n,m} := \overline{w} \overline{w}_{n-1,m-1} - \overline{w}_{n-1,m} \overline{w}_{n,m}\), \(m \leq n - 2\), satisfies \([\overline{w}]_{m,n} = [\overline{w}]_{m,n}\).

**Proposition 3.** The following identities hold for \([\overline{w}]_{m,n}\) as defined by (1.48):

1. \([\overline{w}]_{f-l, f+j} = a^2 r^2 z^4 (1 - a)(1 - b)x^\ell x^j x^\beta b^{-1}\), \(\ell \geq 2, j \geq 1;\)
2. \([\overline{w}]_{f-l, f} = a^2 r^2 z^4 (1 - a)(1 - b)x^\ell - 2\), \(\ell \geq 2;\)
3. \([\overline{w}]_{f-1, f+j} = b^2 r^2 z^4 (1 - a)(1 - b)x^j - 1\), \(j \geq 1;\)
4. \([\overline{w}]_{m, m+\ell} = a^2 r^2 z^4 (1 - a)^2 x^\ell\), \(m + \ell \leq f - 1;\)
5. \([\overline{w}]_{m, m+\ell} = b^2 r^2 z^4 (1 - b)^2 x^\ell - 2\); \(f \leq m, \ell \geq 2;\)

We first establish a special case of Proposition 3 as follows.

**Lemma 1.** Let \(w^*_n(a) = w^*_n(r, y, z; a)\) be defined by (1.49)–(1.11). Then for all \(n \geq 1\) we have

\[
w^*_n(a)^2 - w^*_{n+1}(a)w^*_{n-1}(a) = a^2 r^2 z^4 (1 - a)^2 x^\beta_a^n - 2.
\]

**Proof.** By the definition of \(w^*_n(a)\) in (1.49)–(1.11), we have that \(w^*_{n+1}(a) = \beta_a w^*_n(a) - x^\beta_a w^*_{n-1}(a)\), for all \(n \geq 1\). Therefore by (1.5) we have:

\[
w^*_n(a)^2 - w^*_{n+1}(a)w^*_{n-1}(a) = -\beta_a^{-1} x^\beta_a^{-1} (w^*_n(a)w^*_n(a) - w^*_n(a)w^*_n(a)). \tag{1.49}
\]

But by direct calculation we find \(w^*_n(a)w^*_n(a) - w^*_n(a)w^*_n(a) = -a^2 r^2 z^4 (1 - a)^2 \beta_a / x^\beta_a\). Hence the lemma follows by substitution of this last formula into (1.49).

**Proof (Proposition 3).** By Definition (1.48) (1)(1) and by Lemma 1 we have that statements 1–5 of the proposition hold. Next fix \(\ell \geq 2\) and consider first the case \(j = 0\) in \(I\) which is just the case of statement 2. We will verify 2 first and then use this as a basis for an induction proof of \(I\). Thus we write, using the Definition (1.45) and (1.48), that \([\overline{w}]_{f-l, f}\) is given by:

\[
w^*_l(a) \{ \frac{1 - b}{a} w^*_l(a) + \frac{b - a}{a} w^*_{l+1}(a) - \{ \frac{1 - b}{a} w^*_l(a) + \frac{b - a}{a} w^*_l(a) \} w^*_l(a). \tag{1.50}
\]

The \(w^*_l(a)w^*_l(a)\) terms cancel in (1.50). Thus we obtain by (1.50) and Lemma 1 that

\[
[\overline{w}]_{f-l, f} = \frac{1 - b}{a} (w^*_l(a) - w^*_{l+1}(a)w^*_l(a)) = a^2(1 - a)(1 - b)r^2 z^4 x^\ell - 2. \tag{1.51}
\]
Thus statement 2 is proved.

We now turn to statement 1. Fix \( \ell \geq 2 \) and let \( j \geq 0 \). Denote \([a, b]_\ell = (a, b)\) and \([a, b]_j = (b, b)\) for \( j \geq 1 \). Thus write, by Definition (1.45) and (1.48),

\[
[w]_{f-\ell, f+j+1} = w_{f-\ell, f+j+1} \{ \beta[a, b]_j w_{f-\ell+1, f+j+1} - x[a, b]_j w_{f-\ell+1, f+j} \}
- \{ \beta[a, b]_j w_{f-\ell, f+j} - x[a, b]_j w_{f-\ell+1, f+j} \} w_{f-\ell+1, f+j+1}.
\] (1.52)

Now the terms of (1.52) involving \( \beta[a, b]_j \) cancel and we obtain from (1.52) and (1.48) that

\[
[w]_{f-\ell, f+j+1} = x[a, b]_j [w]_{f-\ell, f+j+1}.
\] (1.53)

Now put \( j = 0 \) in (1.53) and conclude by (1.51) and (1.53) that statement 1 holds for the initial case \( j = 1 \) for the given fixed \( \ell \geq 2 \). Now for the same fixed index \( \ell \), take statement 1 as an induction hypothesis for induction on \( j \geq 1 \).

We have just established this induction hypothesis for \( j = 1 \). Thus verify by (1.53) again that the induction step holds since \( x[a, b]_j = x(b, b) = x_b \) for all \( j \geq 1 \). Thus statement 1 is proved.

Finally we turn to statement 3. We note that (1.52)–(1.53) continues to hold by Definition (1.45) \( (I)(3) \) with \( \ell = 1 \) as long as \( j \geq 1 \). Now we compute by (1.43)–(1.44), Definition (1.45) \( (I)(3) \), and the interlacing bracket definition (1.48) that, since by (1.47), \( w_{f-1, f+1} = \omega(a, b) \), while by Definition (1.45), \( w_{f, f+2} = w_n^* (b) = \omega(b, b) \),

\[
[w]_{f-1, f+1} = \omega(a, b) \omega(b, b) - \{ \beta(a, b) \omega(a, b) - x(a, b) \cdot 1 \} \cdot 1
\] (1.54)

where at the last step we make a direct calculation based on the definitions in (1.21) and (1.44). Now take statement 3 as an induction hypothesis for induction on \( j \geq 1 \). By (1.54) have established this induction hypothesis for \( j = 1 \). Thus verify by (1.53) with \( \ell = 1 \) and \( j \geq 1 \) that the induction step holds since \( x[a, b]_j = x(b, b) = x_b \) for all \( j \geq 1 \). Thus statement 3 is proved. \( \Box \)

### 1.4.2 Closed formula for \( w_{m,n} \)

As a second consequence of the Definition (1.45) we now establish a closed formula for \( w_{m,n} \), which relies on a closed formulae for \( q_n^* \) and \( w_n^* \) in the definition (1.5)–(1.11). This definition in turn relies on standard closed formulae for \( q_n \) and \( w_n \) defined by (1.4); those formulae are briefly discussed in Section 1.4.3 for easy reference.

To state our formula, Proposition 4, we first must give the idea to develop some parameters. We first consider the “upwards formula” for \( w_{m,n} \). The basic idea is straightforward, since, on the one hand, by Definition (1.45)
(I)(4), given \( m = f - i < f \), \( \bar{m}_{m,f+1} \) and \( \bar{m}_{m,f+2} \) form the initial conditions for a recurrence \( \bar{m}_{m,f+j+1} := \beta_b \bar{m}_{m,f+j} - x_b \bar{m}_{m,f+j-1}, \ j \geq 2 \). For each fixed \( \ell \geq 1 \) we denote the vector of these upward initial conditions across the stratum threshold by By equating the \( 2 \times 1 \) vector \( W(\ell) \). Then we define a \( 2 \times 2 \) matrix \( Q(b) \), and for each \( \ell < f \), a \( 2 \times 1 \) vector \( d = d(\ell) \) by

\[
Q(b) := \begin{bmatrix} q_1^*(b) & w_1^*(b) \\ q_2^*(b) & w_2^*(b) \end{bmatrix}, \quad W(\ell) = \begin{bmatrix} \bar{w}_{f-\ell,f+1} \\ \bar{w}_{f-\ell,f+2} \end{bmatrix} = Q(b) d; \quad d := \begin{bmatrix} d_1(\ell) \\ d_2(\ell) \end{bmatrix}.
\]

(1.55)

Here the entries of the matrix \( Q(b) \) are defined by \( 1.13 \) with \( b \) in place of \( a \). On the other hand, by Definition (1.45) (I)(2)–(3), because we can write each term of the right side of the recurrence \( 1.45 \) (I)(3) in terms of \( w_\ell^*(a) \) and \( w_{\ell+1}^*(a) \) for \( m = f - \ell \), we have that there is a \( 2 \times 2 \) matrix \( B \), independent of \( \ell \), such that

\[
W(\ell) = \begin{bmatrix} \bar{w}_{f-\ell,f+1} \\ \bar{w}_{f-\ell,f+2} \end{bmatrix} = B \begin{bmatrix} w_\ell^*(a) \\ w_{\ell+1}^*(a) \end{bmatrix}.
\]

(1.56)

By equating the two expressions for the vector \( W(\ell) \), we are lead to the \( 2 \times 2 \) matrix \( M := Q(b)^{-1} B \) and hence recover

\[
d(\ell) = \begin{bmatrix} d_1(\ell) \\ d_2(\ell) \end{bmatrix} = M \begin{bmatrix} w_\ell^*(a) \\ w_{\ell+1}^*(a) \end{bmatrix}.
\]

(1.57)

Here it is clear that the entries of the matrix \( M \), which we denote in standard notation as \( M = (\mu_{i,j}), 1 \leq i,j \leq 2 \), do not depend on \( \ell \). In fact we now compute \( B \) and \( M \) explicitly.

First we establish the matrix \( B \). By Definition (1.45) (I)(1)–(3) we have, for \( \ell \geq 1 \), by plugging in \( \bar{w}_{f-\ell,f+1} = \frac{b}{1-a} w_{\ell+1}^*(a) + \frac{b-a}{2} w_\ell^*(a) \) from (1.45) (I)(2) and \( \bar{w}_{f-\ell,f} = w_\ell^*(a) \) from (1.45) (I)(1), that

\[
\bar{w}_{f-\ell,f+2} = \beta(a,b) \left( \frac{1-b}{1-a} w_{\ell+1}^*(a) + \frac{b-a}{2} w_\ell^*(a) \right) - x(a,b) w_\ell^*(a).
\]

(1.58)

Now introduce

\[
k(a,b) := \left( \frac{b-a}{1-a} \right) \beta(a,b) - x(a,b).
\]

(1.59)

Thus by (1.58)–(1.59) we have

\[
\bar{w}_{f-\ell,f+2} = \left( \frac{1-b}{1-a} \right) \beta(a,b) w_{\ell+1}^*(a) + k(a,b) w_\ell^*(a).
\]

(1.60)

Thus by (I)(2) again, (1.60), and the definition (1.56) of \( B \), we have that

\[
B = \begin{bmatrix} \frac{b-a}{1-a} & \frac{1-b}{1-a} \\ k(a,b) & \frac{1-b}{1-a} \beta(a,b) \end{bmatrix}.
\]

(1.61)

Finally we calculate \( M = Q(b)^{-1} B \). Here it turns out that, with \( q_1^*(b) = y^2, \)

\[
w_1^*(b) = 1, \ q_2^*(b) = y^2 + z^2(b^2 - (1-b)^2 r^2 y^4), \text{ and } w_2^*(b) = 1 - (1-b)^2 r^2 y^2 z^2,
\]


we have det \((Q(b)) = -b^2z^2\). The computation of \(M\) then follows by referring to (1.44), (1.59), and (1.61). We first write an auxiliary expression:

\[
\mu_0 := r^2 z^2 (1-y) (b-a + (2-a-b)y + (1-a)^2(1-b)r^2y^2z^2(1-y)).
\]

By direct calculation it follows that:

\[
\begin{align*}
\mu_{1,1} &= -\left(\frac{1-b}{a}\right) (1 + (1-a)\mu_0); \\
\mu_{1,2} &= \left(\frac{1-b}{a}\right) (1 - (1-a)(1-b)r^2z^2(1-y)^2); \\
\mu_{2,1} &= \left(\frac{1-a}{b}\right) (b-a + (1-b)y^2 + (1-a)(1-b)y^2\mu_0); \\
\mu_{2,2} &= \left(\frac{1-a}{b}\right) \left(1 - y^2 + (1-a)(1-b)r^2y^2z^2(1-y)^2\right).
\end{align*}
\]

It is useful to note that at \(y = 1\) we have: \(\mu_0 = 0\), \(\mu_{1,1} = -\frac{1-b}{a}\), \(\mu_{1,2} = \frac{1-a}{b}\), \(\mu_{2,1} = 1\), and \(\mu_{2,2} = 0\). We show a simple calculation to check our result for the case \(b = a\) at \(y = 1\). Indeed in this reduced case we find by (1.55) and (1.57) that \(d_1(\ell) = w_{\ell+1}(a) - w_{\ell}(a)\) and \(d_2(\ell) = w_{\ell}\). Since \(q_{\ell}(a) = 1\) at \(y = 1\), and \(w_{\ell}(a) = 1\), we find that the first entry of \(W(\ell)\), namely \(\varpi_{\ell+1,\ell}\), is given by \(d_1(\ell) + d_2(\ell) = w_{\ell+1}(a)\). This is clearly true since in the homogeneous case \(\varpi_{n,n} = w_{n-n}(a)\) for all \(m \neq n\).

We now state the upward case (1.64) of the formula for \(\varpi_{m,n}\) in the “between strata” context. By this upward formula and the Definition (1.45) (II), we obtain the downward formula in the form of Lemma 2 (1.65). The formula (1.64) is closed via (1.9)–(1.11) and (1.132). The closed formula (1.132) obviously handles the within stratum case of Definition (1.45), parts (I)(1) and (II)(1), as well.

**Proposition 4.** Let \(d_1(\ell)\) and \(d_2(\ell)\) be defined by (1.59), (1.54), (1.61), and (1.62), (1.63). Then

\[
\begin{align*}
\varpi_{\ell+j} &= d_1(\ell)q_{\ell} + d_2(\ell)w_{\ell}(b), \quad \ell \geq 1, j \geq 1.
\end{align*}
\]

**Proof.** Consider (1.64). Fix \(\ell \geq 1\). Now by Definition (1.45) (I)(4), we have: \(\varpi_{\ell+j} = \beta_0\varpi_{\ell+j} - x_0\varpi_{\ell+j-1}, j \geq 2\). But if we denote the right side of (1.64) by \(v_j\), then also \(v_{j+1} = \beta_0v_j - x_0v_{j-1}, j \geq 2\), because by construction each of \(q_{\ell}\) and \(w_{\ell}\) satisfy the same two term recurrence, and the coefficients \(d_1(\ell)\) and \(d_2(\ell)\) are independent of \(j\). Also by hypothesis, for any given \(\ell \geq 1\), (1.64) holds for \(j = 1\) and \(j = 2\), that is, \(v_j = \varpi_{\ell+j}\), \(j = 1, 2\). Hence we have \(v_j = \varpi_{\ell+j}\) for all \(j \geq 1\). Since \(\ell\) was arbitrary the proof of (1.64) is complete. \(\square\)
We now establish the general downward companion to our formula (1.64), that is a consequence of Proposition 4 and Definition (1.45), and that we need to proceed in our proof of a closed formula for \( g_{m,n} \) in Section 1.5.

**Lemma 2.** For all \( 1 \leq m < n \), there holds

\[
\overline{w}_{m,n} = \overline{w}_{n-1,m-1}.
\]  

**Proof.** Notice that the lemma holds in the initial cases \( n - m = 1, 2 \) by (1.65). Also, if \( f \leq m < n \) or \( 1 \leq m < n \leq f \) then (1.65) holds by Definition (1.45) (I)(1) and (II)(1). So consider now \( \overline{w}_{f-\ell,f+j} \) for \( 1 \leq \ell < f \) and \( j \geq 1 \). Our method is to prove the statement:

\[
(H)_{\ell,j} : \quad \overline{w}_{f-\ell,f+j} = \overline{w}_{f+j-1,f-\ell-1},
\]  

for both the initial cases \( \ell = 1 \) and \( \ell = 2 \), and all \( j \geq 1 \). We then appeal to a recurrence relation in \( \ell \) to show that \( (H)_{\ell,j} \) holds for all \( \ell \geq 1 \) and all \( j \geq 1 \).

We first establish (1.66) for \( \ell = 1 \) and all \( j \geq 1 \). On the one hand, write \( \overline{w}_{f-1,f+j} \) by (1.64) with \( \ell = 1 \), and on the other hand, write \( \overline{w}_{f+j-1,f-2} \) by Definition (1.45) (II)(2), as follows.

\[
\overline{w}_{f-1,f+j} = d_1(1)q_1^*(b) + d_2(1)w_j^*(b);
\]

\[
\overline{w}_{f+j-1,f-2} = \frac{1-a}{1-b} w_{j+1}^*(b) + \frac{a-b}{1-b} w_j^*(b).
\]  

By (1.57) and (1.62)–(1.63) we explicitly compute

\[
d_1(1) = \mu_{1,1}(a,b)w_1^*(a) + \mu_{1,2}(a,b)w_2^*(a)
\]

and

\[
d_2(1) = \mu_{2,1}(a,b)w_1^*(a) + \mu_{2,2}(a,b)w_2^*(a),
\]

using \( w_1^*(a) = 1 \), and \( w_2^*(a) = \omega(a,a) \), as follows.

\[
d_1(1) = -(1-a)(1-b)r^2z^2; \quad d_2(1) = 1.
\]

Therefore, by substituting (1.68) into (1.67), we find that the two expressions in (1.67) are equal if and only if

\[
(\ast) : \quad -(1-b)^2r^2z^2 q_1^*(b) = w_{j+1}^*(b) - w_j^*(b).
\]

First we see that \((\ast)\) is true at \( j = 1 \), since \( q_1^*(b) = y^2 \), and therefore both sides of \((\ast)\) equal: \( \omega(b,b) - 1 \); see definition (1.21). Now we check \((\ast)\) at \( j = 2 \). We have, by the definitions (1.4) and (1.9)–(1.11), that with either \( \{v_j = w_j^*(b)\} \) or \( \{v_j = q_j^*(b)\} \), the rolling recursion holds:

\[
v_{j+1}(b) = \beta_b v_j - x_b v_{j-1}, \quad j \geq 1.
\]  

Thus by (1.69) with \( j = 2 \), \( w_2^*(b) - w_2^*(b) = (\beta_b - 1)w_2^*(b) - x_b w_2^*(b) = (\beta_b - 1)\omega(b,b) - x_b \). Now find, by the definitions (1.21) and (1.11), that this last expression is in fact equal to: \(- (1-b)^2r^2z^2 q_2^*(b) \) for \( q_2^*(b) = y^2 + z^2 b^2 - (1-b)^2 r^2 y^4 \). Hence \((\ast)\) holds also for \( j = 2 \). But again by (1.69), both sides of \((\ast)\) satisfy the same two-term recurrence. Since we have
determined that \((*)\) holds for the initial cases \(j = 1\) and \(j = 2\) it follows that \((*)\) holds for all \(j \geq 1\). Therefore since we reduced the problem of equality of the two expressions in \((1.67)\) to \((*)\), we have proved that

\[
\overline{w}_{f-1,f+j} = \overline{w}_{f+j-1,f-2}, \quad \text{all } j \geq 1. \tag{1.70}
\]

Next we establish that \((1.66)\) holds with \(\ell = 2\) and all \(j \geq 1\). We proceed similarly as in the previous paragraph. Thus write \(\overline{w}_{f-2,f+j}\) by \((1.64)\) with \(\ell = 2\), and write \(\overline{w}_{f+j-1,f-3}\) by Definition \((1.45)\) (II)(3), as follows.

\[
\overline{w}_{f-2,f+j} = d_1(2)q_j^*(b) + d_2(2)w_j^*(b); \tag{1.71}
\]

\[
\overline{w}_{f+j-1,f-3} = \beta(b,a)\overline{w}_{f+j-1,f-2} - x(b,a)\overline{w}_{f+j-1,f-1}.
\]

Here, in addition we use Definition \((1.45)\) (II)(2) and (II)(1) to rewrite \(\overline{w}_{f+j-1,f-2}\) and \(\overline{w}_{f+j-1,f-1}\) in the second formula of \((1.71)\):

\[
\overline{w}_{f+j-1,f-2} = \frac{1}{2-\beta}w_{j+1}^*(b) + \frac{\beta}{2-\beta}w_j^*(b); \quad \overline{w}_{f+j-1,f-1} = w_j^*(b). \tag{1.72}
\]

Now by \((1.57)\) and \((1.62)-(1.63)\) we explicitly compute \(d_1(2)\) and \(d_2(2)\), with \(w_j^*(a) = \omega(a,a), w_j^*(a) = \beta_a\omega(a,a) - x_a;\)

\[
d_1(2) = -(1-a)(1-b)r^2z\beta(b,a); \quad d_2(2) = \beta(b,a) - x(b,a). \tag{1.73}
\]

To verify that the two expressions in \((1.71)\) are equal, we apply \((1.72)\) and \((1.73)\), to obtain the condition \((**)\): \(-(1-b)^2r^2z^2\beta(b,a)q_j^*(b) = \beta(b,a)\left(w_{j+1}^*(b) - w_j^*(b)\right), \) for all \(j \geq 1\).

But obviously \((**)\) is equivalent to the condition \((*)\) that was verified in the previous paragraph. Hence the two expressions in \((1.71)\) are equal for all \(j \geq 1\), that is \((1.66)\) holds with \(\ell = 2\) and all \(j \geq 1\).

Finally, fix any \(j \geq 1\). We appeal to \((1.57)\) and \((1.64)\) and to Definition \((1.45)\) (II)(4), to obtain, for any \(\ell \geq 3,\)

\[
\overline{w}_{f-\ell,f+j} = \left(\mu_{1,1}w_\ell^*(a) + \mu_{1,2}w_{\ell+1}^*(a)\right)q_j^*(b) + \left(\mu_{2,1}w_\ell^*(a) + \mu_{2,2}w_{\ell+1}^*(a)\right)w_j^*(b); \tag{1.74}
\]

\[
\overline{w}_{f+j-1,f-\ell-1} = \beta_a\overline{w}_{f+j-1,f-\ell} - x_a\overline{w}_{f+j-1,f-\ell+1}.
\]

Here in the first formula of \((1.74)\) we follow the lines of proof in the previous two paragraphs but now write down the formulae for \(d_1(\ell)\) and \(d_2(\ell)\). Now write \(u_\ell := \overline{w}_{f-\ell,f+j}\) for the first line of \((1.74)\). Since with \(j\) fixed, \(u_\ell\) is a linear combination of two successive terms of the sequence \(\{w_j^*(a)\}\), it follows that, by \((1.69)\), \(\{u_\ell, \ell \geq 2\}\) itself satisfies the recursion \((1.69)\) in \(\ell\) (in place of \(j\)), with \(a\) in place of \(b\) and \(u_\ell\) in place of \(v_j\). But now write \(v_\ell := \overline{w}_{f+j-1,f-\ell+1}\), to see explicitly, by the second line of \((1.74)\), that \(\{v_\ell, \ell \geq 2\}\) satisfies the
same recurrence \(1.69\) in \(\ell\) (in place of \(j\)), with \(a\) in place of \(b\) and \(v_\ell\) in place of \(v_j\). Moreover, we proved that \(1.69\) holds for \(\ell = 1\) and \(\ell = 2\), so in particular \((H)_{\ell,j}\) of \(1.66\) holds for the given \(j\), and \(\ell = 1, 2\); that is \(u_1 = v_1\), and \(u_2 = v_2\). Therefore since the initial conditions match for the two matching recurrences for \(\{u_\ell\}\) and \(\{v_\ell\}\), then by \(1.66\) and \(1.74\), \((H)_{\ell,j}\) is proved for all \(\ell \geq 1\) with the given \(j\). Since \(j \geq 1\) was arbitrary, we conclude that \(1.66\) is true in generality for all \(\ell \geq 1\) and all \(j \geq 1\).

\[\square\]

### 1.4.3 The Fibonacci polynomials \(q_n(x, \beta)\) and \(w_n(x, \beta)\).

In this subsection we show closed formulae for \(\{q_n(x, \beta)\}\) and \(\{w_n(x, \beta)\}\) defined by \(1.4\). We also apply these formulae in a special case to evaluate \(w_{m,n}(1, 1, 1; a, b)\).

**Lemma 3.** The generating function of the sequence \(\{q_n, n \geq 0\}\) in Definition \(1.4\) is:

\[q(x, \beta, t) := \sum_{n=0}^{\infty} q_n(x, \beta) t^n = \frac{t}{1 - \beta t + xt^2}.\quad (1.75)\]

**Proof.** The expression on the right side of \(1.75\) follows immediately by applying the recurrence \(1.4\) to the definition of \(q(x, \beta, t)\) and solving algebraically for \(q(x, \beta, t)\).\[\square\]

We have the following closed form expressions for \(q_n(x, \beta)\) and \(w_n(x, \beta)\) of \(1.4\).

**Lemma 4.** Define \(\alpha := \sqrt{\beta^2 - 4x}\). Then, for all \(n \geq 1\), and with \(q_0(x, \beta) = 0\),

\[q_n(x, \beta) = \frac{2^{-n}}{\alpha} ((\beta + \alpha)^n - (\beta - \alpha)^n); \quad w_n(x, \beta) = q_n(x, \beta) - xq_\ell-1(x, \beta).\quad (1.76)\]

**Proof.** Compute that \(\sum_{n=0}^{\infty} t^n \frac{2^{-n}}{\alpha} ((\beta + \alpha)^n - (\beta - \alpha)^n) = q(x, \beta, t)\), as given by \(1.75\). Thus by matching generating functions we have established the formula for \(q_n(x, \beta)\). The formula for \(w_n(x, \beta)\) with \(n \geq 1\) follows because \(q_1(x, \beta) - xq_0(x, \beta) = 1 - 0 = 1 = w_1(x, \beta)\), and \(q_2(x, \beta) - xq_1(x, \beta) = \beta - x = w_2(x, \beta)\). Therefore since \(v_n(x, \beta) := q_n(x, \beta) - xq_{n-1}(x, \beta), n \geq 1,\) satisfies the same two term recurrence in \(1.4\) as both \(q_n(x, \beta)\) and \(w_n(x, \beta), n \geq 1,\) and with the same initial conditions at \(n = 1\) and \(n = 2\) as \(w_n(x, \beta)\), we have \(v_n(x, \beta) = w_n(x, \beta), n \geq 1.\)\[\square\]

As a consequence of Lemma 3 and Proposition 4 we obtain the following evaluation of \(\prod_{m,n}\) at \((r, y, z) = (1, 1, 1)\).

**Lemma 5.** The following identities hold.
1. \( \mathfrak{w}_{f-\ell,f+j}(1,1,1) = a^{\ell-1}b^{j-1}[ja + \ell b - (\ell + j - 1)ab] = a^{\ell-1}b^{j-1}I_{f-\ell,f+j}, \quad \forall \ell \geq 1, \, j \geq 1. \)

2. \( \mathfrak{w}_{f+j,f-\ell}(1,1,1) = a^{\ell-2}b^j[(j+1)a + (\ell - 1)b - (\ell + j - 1)ab] = a^{\ell-2}b^jI_{f+j,f-\ell}, \quad \forall \ell \geq 2, \, j \geq 0. \)

3. \( q^{\ast}_{\ell}(1,1,1) = \ell a^{\ell-1}; \quad w^{\ast}_{\ell}(1,1,1)(a) = a^{\ell-1}[(\ell - 1)a], \quad \forall \ell \geq 1. \)

**Proof.** First note that at \( r = y = z = 1 \) we have \( \beta_a = 2a \) and \( x_a = a^2 \). Thus the parameter \( \alpha \) of Lemma 4 is given by \( \alpha = 0 \). Also by (1.10), \( w^{\ast}_{\ell}(a) = 1, \) and \( q^{\ast}_{\ell}(a) = 0 \). Therefore by Lemma 3, \( q^{\ast}_{\ell}(1,1,1)(a) = \lim_{\alpha \to 0} \frac{2^\alpha}{a} \{ (2\alpha + a^n - (2a - \alpha)^n) \} = \ell a^{\ell-1}. \) Thus, by the second formula of Lemma 1.76, \( w^{\ast}_{\ell}(1,1,1)(a) = \alpha a^{\ell-1} = \alpha^2(\ell - 1)a^{\ell-2} = a^{\ell-1}[(\ell - 1)a] \). So \( \beta \) is proved. Now apply (1.64), also at \( (r,y,z) = (1,1,1) \). By (1.63), we have \( \mu_{1,1} = -\frac{1-b}{1-a}, \) \( \mu_{1,2} = \frac{1-b}{1-a}, \mu_{2,1} = 1, \) and \( \mu_{2,2} = 0 \). Thus by (1.57), \( d_1(\ell) = \frac{1-k}{1-a}[-\ell a^{\ell-1}((\ell - 1)a) + a^{\ell}((\ell - 1)a)] = -((1-a)1-b)\ell a^{\ell-1}, \) and \( d_2(\ell) = w^{\ast}_{\ell}(1,1,1) = a^{\ell-1}[(\ell - 1)a] \). Now plug in \( q^{\ast}_{\ell}(1,1,1)(b) = j_{b}^{j-1} \) and \( w^{\ast}_{\ell}(1,1,1)(b) = b^{j-1}[j - (j-1)b] \), into (1.64) to obtain:

\[ \mathfrak{w}_{f-\ell,f+j}(1,1,1) = a^{\ell-1}b^{j-1}\{-(1-a)(1-b)\ell j + [(\ell - 1)a][j - (j-1)b]\}, \]

which simplifies to 1. The proof of 2 is similar to the proof of 1 from 3. \( \Box \)

### 1.5 Closed formula for \( g_{m,n} \).

The calculation involving the denominators \( \mathfrak{w}_{m,n} \) in the last section culminates in this section with a closed formula for \( g_{m,n} \) developed from the recursions (1.28), (1.29). Recall the definitions (1.42). We also rewrite the definition (1.29) for convenient reference as follows:

\[ \lambda_{m,n} := [1 - 4\gamma_{m,n} \rho_{m,n} \rho_{n,m} k[a,b] k[a,b] g_{m,n} g_{n,m}]^{-1}, \quad m + 2 \leq n. \]  

The reason is that a certain interlacing formula (1.78) for \( \lambda_{m,n} \) plays a fundamental role in our proof of the following.

**Proposition 5.** We have the following formula for \( \{g_{m,n}\} \).

I. The formulae for upward between-strata cases, \( j \geq 1 \):

1. \( g_{f-\ell,f+j} = \frac{\omega(a,b)}{a+b-ab} r^{j+\ell} \tau(a,b) (a \tau(a,a))^{\ell-2}(b \tau(b,b))^{j-1} \left( a I_{f-\ell,f+j} / \mathfrak{w}_{f-\ell,f+j} \right); \)

2. \( g_{f-1,f+j} = \frac{\omega(a,b)}{a+b-ab} r z^{j+1} \tau(a,b) (b \tau(b,b))^{j-1} \left( a I_{f-1,f+j} / \mathfrak{w}_{f-1,f+j} \right); \)

II. The formulae for downward between-strata cases, \( \ell \geq 2 \):

1. \( g_{f+j,f-\ell} = \frac{\omega(b,b)}{2-b} r z^{\ell+j} \tau(a,b) (a \tau(a,a))^{\ell-2}(b \tau(b,b))^{j-1} \left( a I_{f+j,f-\ell} / \mathfrak{w}_{f+j,f-\ell} \right), \quad j \geq 1; \)

2. \( g_{f+j-1,f-\ell} = \frac{\omega(b,b)}{2-b} r z^{\ell+j-1} \tau(a,b) (a \tau(a,a))^{\ell-2}(b \tau(b,b))^{j-1} \left( a I_{f+j-1,f-\ell} / \mathfrak{w}_{f+j-1,f-\ell} \right). \)
2. \( g_{f,f-\ell} = \omega(a,b) r z^\ell [\tau(a,a)]^{\ell-2} (a \Pi_{f,f-\ell}/w_{f,f-\ell}); \)

III. The formulae for within stratum cases:

1. \( g_{m,m+\ell} = g_{m+\ell,m} = \omega(a,a) r z^\ell [\tau(a,a)]^{\ell-2} (\frac{a}{2} \Pi_{m,m+\ell}/w^*_\ell(a)); \) \( m < m + \ell \leq f - 1; \)

1.1 \( g_{f-\ell,f} = \omega(a,a) r z^\ell [\tau(a,a)]^{\ell-2} (\frac{a}{b} \Pi_{f-\ell,f}/w^*_\ell(a)); \)

2. \( g_{m,m+\ell} = g_{m+\ell,m} = \omega(b,b) r z^\ell [\tau(b,b)]^{\ell-2} (\Pi_{m,m+\ell}/w^*_\ell(b)); \) \( f \leq m < m + \ell; \)

2.1 \( g_{f+j,f-1} = \omega(b,b) r z^{j+1}[\tau(b,b)]^{j-1} (\Pi_{f+j,f-1}/w^*_{j+1}(b)), j \geq 1. \)

Furthermore, the following identity holds for all \( n \geq m + 2, \) where \( \lambda_{m,n} \) is defined by (1.77).

\[
\lambda_{m,n} = \frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}{\overline{w}_{m+1,n+1}}. \tag{1.78}
\]

Remark 2. The formula (1.12) holds for the homogeneous case \( b = a \) by Proposition 5.

Proof (Proposition 5). Recall by (1.20) that \( g_{m,m+2} = g_{m+2,m} = r z^2. \) Also recall the definitions of Section 1.2. By (1.28)–(1.29), we must calculate a term \( \lambda_{m,n} \) defined by (1.77). The term \( \lambda_{m,n} \) is the same in both (1.28) and (1.29), so we only consider \( m < n \) in (1.77). The structure of the proof is to first establish (1.78) for \( n = m + 2 \) and to establish all the initial cases of the statements I–III of the proposition. Following this an induction step will be established on the whole for the cases I–III wherein an inductive step for (1.78) shall be the main stepping stone of the proof.

Thus consider first the case \( n = m + 2 \) in (1.77). We note by Definition (1.33)–(1.36) and Proposition 2 that \( 4 \rho_{m,m+2 \rho_{m+2,m}} = \frac{1}{(2-a)^2}, \) for \( m + 2 \leq f - 1, \) while \( 2 \rho_{f-2,f} = \frac{1}{2-a}, 2 \rho_{f-2,f-2} = \frac{a}{a+b-a^2}, \) \( 2 \rho_{f-1,f+1} = \frac{b}{a+b-a^2}, \) and \( 2 \rho_{f+1,f-1} = \frac{1}{2-a}. \) Further \( 4 \rho_{m,m+2 \rho_{m+2,m}} = \frac{1}{(2-b)^2} \) for \( m \geq f. \) Thus by (1.21) and (1.22), we find that in all cases

\[
4 \rho_{m,m+2 \rho_{m+2,m}} k[a,b]^+_m k[b,a]^+_m = \frac{\delta_m}{\omega[a,b]^+_m \omega[a,b]^+_m} \tag{1.79}
\]

for

\[
\delta_m := \begin{cases} a^2, & m \leq f - 2, \\ b^2, & m \geq f - 1 \end{cases}.
\]

Here, by (1.43), we have: \( \omega[a,b]^+_m = \overline{w}_{m,m+2} \) and \( \omega[a,b]^+_m = \overline{w}_{m+2,m}. \) Therefore by the definition of \( \lambda_{m,m+2} \) in (1.77), and by (1.21) and (1.79), that \( (\lambda_{m,m+2})^{-1} \) is given by:

\[
1 - \delta_m \gamma_m \gamma_{m+2} r^2 z^4 \frac{\delta_m}{\overline{w}_{m,m+2} \overline{w}_{m+2,m}} = \frac{\overline{w}_{m,m+2} \overline{w}_{m+2,m}}{\overline{w}_{m,m+2} \overline{w}_{m+2,m} - \delta_m \gamma_m \gamma_{m+2} r^2 z^4}. \tag{1.80}
\]
Now observe from Proposition 3 that for all \( m \geq 0 \), \( \delta_m \gamma_m + 2r^2 z^4 = (\omega)_{m,m+2} \). Hence, by the definition of the interlacing bracket (1.48), and by Lemma 2 we have that the denominator of the right side of (1.80) is equal to

\[
\frac{m_m+2m_{m+2}}{m_m+2m_{m+1,m+3}} = \frac{(\omega)_{m,m+2}}{\omega_{m+m+3}}.
\]  

(1.81)

Hence we conclude by (1.80)–(1.81) and Lemma 2 again that

\[
\lambda_m,m+2 = \frac{m_{m,m+2}+2m_{m+2,m}}{m_{m,m+3}} = \frac{w_{m+m+3}}{\omega_{m_m+3}},
\]  

(1.82)

since \( \omega_{m+1,m+2} = 1 \) by (1.33). We have therefore verified (1.78) for \( n = m+2 \).

It thus follows by (1.27), (1.77), and (1.82) that

\[ g_m,m+3 = c\omega[a,b][m+2m_{m+2}m_{m+2} = c\omega[a,b][m+2r^2 z^4 w_{m,m+3}]. \]

(1.83)

Now take \( m = f = 2, m+2 = f \) in (1.83), and use, by (1.22), that \( h[a,b][m+2 = h[a,b][f = h(a,b), \omega_{m,m+2} = \omega[a,b][m = \omega(a,b), \omega_{m,m+2} = \omega[a,b][m+2 = \omega(a,b) \). Thus by the definition of \( h(a,b) \) in (1.21), we obtain from (1.83) that

\[
g_{f-1,f+1} = c^3 \tau(a,b)[\omega(a,b)[\omega(b,b)[w_{f-1,f+1} = C \omega(a,b) \tau(a,b)[w_{f-1,f+1},
\]  

(1.84)

where \( C \) is determined by Lemma 3 as \( \frac{(2-a)\omega_{f-1,f+1}(1,1,a,b) \omega(1,1,a,a)}{w(1,1,a,a)} = a+2b-2ab = aH_{f-1,f+1} \). Here we have applied Proposition 2 and (1.21). Thus we have shown that formula I.1 holds at the initial case \( \ell = 2, j = 1 \).

Before we can proceed to induction, we must establish the initial cases for \( g_m,n \) and \( g_n,m \) with \( n-m = 3 \) for the other statements of the proposition as well, so that we can work with all cases simultaneously. We next consider I.2 for the case \( j = 2 \), that is \( n-m = 3 \); the case \( j = 1 \) is trivial there because the expression for \( g_{f-1,f+1} \) shown reduces to \( g_{f-1,f+1} = rz^2 \). We apply (1.83) with \( m = f = 1 \) and \( m+3 = f + 2 \). By similar reasoning as shown after (1.83), but now with \( h[a,b][m+2 = h(b,b) \), we obtain,

\[
g_{f-1,f+2} = c^3 \tau(b,b)[\omega(a,b)[\omega(b,b)[w_{f-1,f+2} = C \omega(a,b) \tau(b,b)[w_{f-1,f+2},
\]  

(1.85)

where this time \( C = \frac{(a+b-ab)\omega_{f-1,f+2}(1,1,a,a) \omega(1,1,a,a)}{w(1,1,a,a)} = 2a+b-2ab = aH_{f-1,f+2} \), as required. Next we treat the case \( n-m = 3 \) for III.1. We thus apply (1.83) with \( m+3 \leq f \). Then \( h[a,b][m+2 = h(a,a) \) and \( \omega_{m,m+2} = \omega_{m+1,m+3} = \omega(a,a) \), so by (1.83), in this case

\[
g_m,m+3 = c^3 \tau(a,a)[\omega(a,a)[\omega(a,a)[w_{m,m+3} = C \omega(a,a) \tau(a,a)[w_{m,m+3},
\]  

(1.86)

with \( C = \frac{(2-a)\omega(1,1,a,a)}{a\omega(1,1,a,a)} = 3 - 2a = (a/b)H_{m,m+3} \), as required.
The verification of the formulae for $g_{n,m}$ for $n-m=3$ in the downward cases, II.1–2 and III.2, proceeds similarly as for the upward cases shown above, except now we apply (1.25) in place of (1.27). We demonstrate briefly a proof of the initial case $j=1, \ell = 2$ in II.1. First write (1.25) with $n = m+2$:

$$g_{m+2,m-1} = c_1 h[a, b]^+_m g_{m+2,m+} \lambda_{m,m+2}.$$  

(1.87)

In the present case II.1, we apply (1.87) with $m-1 = f-2$ and $m+2 = f+1$, so $m = f-1$. Therefore $h[a, b]^+_m = h(a, b)$. Also $w_{m,m+2} = \omega[a, b]^+_m = \omega(a, b)$ and $w_{m+1,m+3} = \omega(b, b)$. Thus by (1.82), (1.87), and Lemma 2, we have

$$g_{f+1,f-2} = crz^3 \frac{\tau(a,b)}{\omega(a,b)} w_{m,m+2} w_{m+2,m+1} / \{w_{m,m+3}\} =$$ 

(1.88)

$$crz^3 \frac{\tau(a,b)}{\omega(a,b)} \omega(a, b) \omega(b, b) / \{w_{m+2,m+1}\} = crz^3 \omega(b, b) \tau(a, b) C / w_{f+1,f-2},$$

where $C$ is determined by Lemma 5 as $(2b) w_{f+1,f-2} (1, 1, 1, a, b) = 2a + b - 2ab = a \Pi_{f+1,f-2}$, as required.

We now proceed by induction on all cases of the proposition at once, where we assume that all statements hold for $g_{m,n}$ and $g_{n,m}$ with $3 \leq n - m \leq k$, for some $k \geq 3$. By the above we have established all the initial cases, $k = 3$, for this hypothesis. We must establish induction steps for all the statements of the proposition. First consider the induction step for I.1. We apply (1.28) and definition (1.77) to write,

$$g_{m,n+1} = c_1 g_{m,n} g_{m+1,n+1} (g_{m+1,n})^{-1} \lambda_{m,n}; \quad n - m = k.$$  

(1.89)

Under I.1 write $m = f - \ell \leq f - 2$ and $n = f + j \geq f + 1$, so we have $k[a, b]^+_m = k(a, a)$ and $k[a, b]^+_n = k(b, b)$. Also, by Proposition 2, 4 $\rho_{m,n} = \rho_{m,n}^{(b/a)} = [a^{(b/a)} H_{m,n} [a^{(b/a)} H_{m,n}]]$. Therefore by definition of $\lambda_{m,n}$ in (1.77), and by our induction hypothesis for the formulae of $g_{m,n}$ and $g_{n,m}$, with $n - m = k,$

$$1 - 1/\lambda_{m,n} = \gamma_m \gamma_n \frac{ab}{[a^{(b/a)} H_{m,n}] [a^{(b/a)} H_{m,n}]} a(2-a) b(2-b) g_{m,n} g_{n,m} =$$

$$= \gamma_m \gamma_n r^2 z^2 (a, b) [a^2 r^2 (a, a)]^{j-2} [b^2 r^2 (b, b)]^{j-1} a^2 b^2 / \{w_{m,n} w_{m,n}\}$$

$$= a^2 \gamma_m \gamma_n r^2 z^2 x_a^{\ell-2} x_b^{j-1} x(a, b) / \{w_{m,n} w_{m+1,n+1}\}.$$  

(1.90)

Here we have used (1.44) to write $x(a, b) = b^2 z^2 r^2 (a, b)$, and also that $x_a = x(a, a)$, and $w_{m,n} = w_{m+1,n+1}$ by Lemma 2. Therefore by (1.90) and the interlacing bracket of Proposition 3 item I, we have

$$\lambda_{m,n} = \frac{w_{m,n} w_{m+1,n+1}}{w_{m,n} w_{m+1,n+1} - a^2 \gamma_m \gamma_n r^2 z^2 x_a^{\ell-2} x_b^{j-1} x(a, b) / \{w_{m,n} w_{m+1,n+1}\}}.$$  

(1.91)
since with \( \ell \geq 2 \) and \( j \geq 1, \gamma_m \gamma_n = (1-a)(1-b) \). Hence we can now substitute this formula (1.91) into (1.89), and apply our induction hypothesis for \( g_{m,n}, g_{m+1,n+1}, \) and \( g_{m+1,n} \), with \( n-m = k \), to obtain via I.1–2,

\[
g_{m,n+1} = c_1 \frac{g_{m,n} g_{m+1,n+1}}{g_{m+1,n}} \frac{w_{m,n} w_{m+1,n+1}}{w_{m,n+1} w_{m+1,n}} = cz [b \tau(b, b)] [g_{m,n} w_{m,n}] / w_{m,n+1}. \tag{1.92}
\]

Here we have used that, since the lower index \( m+1 \) is the same in both numerator and denominator of the ratio \( g_{m+1,n+1} / g_{m,n} \), we obtain by either the induction hypothesis I.1 for \( m+1 \leq f - 2 \), or by II.2 for \( m+1 = f - 1 \), that \( g_{m+1,n+1} / g_{m,n} = cz [b \tau(b, b)] w_{m+1,n} / w_{m,n+1} \). Therefore the second equality in (1.92) follows for some constant \( c \) that may change from line to line. Hence by (1.92), we have

\[
g_{m,n+1} = c \omega(a, a) z^{j+\ell+1} \tau(a, b) [a \tau(a, a)]^{\ell-2} [b \tau(b, b)]^j / w_{m,n+1}. \tag{1.93}
\]

The form (1.93) matches I.1 for \( j+1 \) in place of \( j \) up to the constant \( c \); this constant can easily be checked by Lemma 5 to get the exact form for I.1. Note that the exact form including the constant is necessary for the computation of \( \lambda_{m,n} \) to come out as an “interlacing factor” (1.78). To complete the induction step for I.1, we must still replace \( m \) by \( m-1 \) and \( n \) by \( n-1 \) in the above computation. The argument and parameter values across the stratum remain the same unless \( n = f + 1 \), so we treat that possibility as a special case.

Suppose thus that \( n = f + 1 \) for \( n-m = k, k \geq 3 \), so \( m = f-k+1 \leq f-2 \) under I.1. We wish to write a formula for \( g_{m-1,n} \) for this special case. We apply (1.89) with \( m-1 \) in place of \( m \) and \( n-1 \) in place of \( n \), as follows:

\[
g_{m-1,n} = c_1 g_{m-1,n-1} g_{m,n} (g_{m,n-1})^{-1} \lambda_{m-1,n-1}. \tag{1.94}
\]

Therefore for our choice \( n-1 = f \) we must calculate \( \lambda_{m-1,f} \). To calculate \( \lambda_{m-1,f} \) of (1.77), we make the following steps: we substitute \( k[a,b]_m = k(a,b) \); we apply hypotheses III.1.1 and II.2 in substituting for \( g_{m-1,f} \) and \( g_{f,m-1} \); and we calculate \( 4 \rho_{m-1,f} \rho_{f,m-1} \). Now put \( m-1 = f - (\ell + 1) \), so that \( \ell + 1 = k \) appears in place of \( \ell \) in the formulae III.1(a) and II.2, and also calculate by Definition (1.35)–(1.36) and Proposition 2 that \( 4 \rho_{m-1,f} \rho_{f,m-1} = \frac{a}{(a/b)H_{m-1,f} |aH_{f,m-1}|} \).

Thus

\[
1 - 1/\lambda_{m-1,f} = \gamma_{m-1} f r^2 z^{2\ell+2}[a^2 y^2(a,a)]^{\ell-1} a^2 / \{w_{m-1,f} w_{f,m-1}\}
\]

where at the last step we applied Lemma 2 to rewrite \( w_{f,m-1} = w_{m,f+1} \).

Now, by (1.95), rewrite \( \lambda_{m-1,f} \) by itself:
where at the last step we are able to apply the interlacing bracket of Proposition \[3\] item 2 because still \(\gamma_{m-1}^f = (1-a)(1-b)\). Therefore by (1.91) and (1.96) we have established (1.78) for \(n-m = k\) for all choices of the indices \(m\) and \(n\) under case I.1. The rest of the argument for the induction step works much the same as before. Put \(n = f + 1\) in (1.94) to obtain by (1.96) that

\[
g_{m-1,f+1} = c_1 \frac{g_{m-1,f}g_{m,f+1}}{g_{m,f}} \lambda_{m-1,f} = c_1 \frac{g_{m-1,f}g_{m,f+1}}{g_{m,f}} \frac{\overline{w}_{m-1,f} \overline{w}_{m,f+1}}{\overline{w}_{m-1,f} \overline{w}_{m,f}}.
\]

Note, by our induction hypothesis I.1 and III.1.1, that the ratio \(g_m,f+1/g_{m,f}\) is calculated by \(g_{m,f+1}g_{m,f} = cz\tau(a,b)\overline{w}_{m,f}/\overline{w}_{m,f+1}\), where in fact \(\overline{w}_{m,f} = w_{f+1}^*(a)\). Therefore, by (1.97) and induction hypothesis III.1.1,

\[
g_{m-1,n} = cz\tau(a,b)\{g_{m-1,f} \overline{w}_{m-1,f}\}/\overline{w}_{m-1,n} = c\omega(a,a)r^{\ell+2}\tau(a,b)[\alpha\tau(a,a)]^{\ell-1}/\overline{w}_{m-1,n},
\]

which, up to the constant \(c\), is of the correct form for the induction step in I.1 with \(f + 1\) in place of \(\ell\) and \(j = 1\). Thus by (1.98), and by Lemma \[5\] to evaluate the constant \(c\) therein, the induction step is complete for the special case, and so for I.1 as well.

We show details for the induction step for I.2. We go back to (1.89) where now we set \(m = f - 1\). We also continue to write \(n = f + j\). In the calculation of \(\lambda_{m,n} = \lambda_{f-1,n}\) with \(n-m = k \geq 3\), we have \(k[a,b]_m^+ = k(a,b)\) and \(k[a,b]_m^- = k(b,b)\). Also \(4\rho_{m,n} = [aH_{m,n}][bH_{n,m}]/[aH_{m,n}][bH_{n,m}]\). We apply our induction hypothesis I.2 and III.2.1 to write \(g_{m,n} = g_{f-1,n}\) and \(g_{n,m} = g_{n,f-1}\). Thus by the definition of \(\lambda_{m,n}\) in (1.77),

\[
1 - 1/\lambda_{f-1,n} = \gamma_{f-1}\gamma_n \frac{b}{aH_{m,n}[bH_{n,m}]} \frac{(a+b-ab)b(2-b)}{\omega(a,b)\omega(b,b)} g_{f-1,n}g_{n-1}/\overline{w}_{f-1,n}\overline{w}_{n-1} = \gamma_{f-1}\gamma_n r^{2}\gamma_{n} r^{2}[\overline{w}_{n-1}^{(b)}(b,b)]^{-1} b^{2} / \{\overline{w}_{f-1,n} \overline{w}_{n-1}\}.
\]

By (1.99) and Proposition \[3\] 3, \(1 - 1/\lambda_{f-1,n} = \overline{w}_{f-1,n}/\{\overline{w}_{f-1,n} \overline{w}_{n+1}\}\), where \(\overline{w}_{f-1,n}\) denotes the interlacing bracket, because still \(\gamma_{f-1}\gamma_n = (1-a)(1-b)\) and \(n = f + j\). Hence

\[
\lambda_{f-1,n} = \overline{w}_{f-1,n} \overline{w}_{n+1} / \overline{w}_{f-1,n+1} \overline{w}_{f,n}.
\]
Therefore the argument proceeds as before using \((1.89)\) to write out \(g_{m,n+1} = g_{j-1,n+1}\), where now we focus on the formula for the ratio \(g_{f,n+1}/g_{f,n}\) given by the induction hypothesis \(III.2\) applied to both numerator and denominator of this ratio. Thus \(g_{f,n+1}/g_{f,n} = c z [b r(b, b)] w_{f,n}/w_{f,n+1}\), where \(w_{f,f+j} = w^*_f(b)\). Therefore, since by the induction hypothesis \(I.2\) we have \(g_{j-1,n} w_{f-1,n} = c z (a, b) r z^{j+1} [b r(b, b)]^{j-1}\), then by \((1.89)\) and \((1.100)\) we obtain that \(g_{j-1,n+1}\) takes the correct form \(I.2\) with \(j\) replaced by \(j + 1\).

For the proof of the induction step for the downward between-stratum cases \(II.1–2\) we apply \((1.29)\) to write, in place of the upward recurrence \((1.89)\), a downward recurrence for \(g_{n,m}\):

\[
g_{n,m-1} = c_2 g_{n,m} g_{n-1,m-1} (g_{n-1,m})^{-1} \lambda_{m,n}; \quad n - m = k. \quad (1.101)
\]

Since the structure of the proof of the induction step for \(II.1–2\) is the same as for \(I.1–2\), we omit the details of the downward between-stratum induction steps. We comment that the induction step for the homogeneous cases under \(III.1\) and \(III.2\) are practically immediate from what has gone before. In these induction steps it enough to consider the upward formula by symmetry of the model within a stratum. In fact it is enough to consider \(m = 0\) for the stratum with persistence parameter \(a\), and \(m = f\) for the stratum with persistence parameter \(b\). Therefore, under \(III.1\), \(k[a,b]_0^\pm = k[a,b]_0^\pm = k(a,a)\), and \(4 p_{0,\ell} p_{\ell,0} = \frac{1}{\Pi_{\ell-1}(a+1)} = \frac{1}{\Pi_{\ell+1}(a+1)}\), \(2 \leq \ell \leq f - 1\), while under \(III.2\), \(k[a,b]_f^\pm = k[a,b]_{f-1}^\pm = k(b,b)\) and \(4 p_{f,\ell} p_{\ell,f} = \frac{1}{\Pi_{\ell-1}(b+1)} = \frac{1}{\Pi_{\ell+1}(b+1)}\), \(\ell \geq 2\). The formulae in \(III.1.1\) and \(III.2.1\) are taken for free from the cases \(III.1\) and \(III.2\), respectively. Indeed, consider \(III.1.1\) for example. In the upward formula \((1.27)\) for \(g_{f-t,f}\) with \(n = f - 1\), we see that \(g_{f-t,f} = g_{0,t}\) because all the parameters in \((1.27)\) for \(n = f - 1\) are within the stratum defined by the persistence parameter \(a\). But \(g_{0,t}\), and so \(g_{f-t,f}\), is given by \(III.1\). This concludes our discussion of the induction steps and thus the proof of the proposition.

### 1.6 Proof of Proposition 1: extension to Theorem 1

In the introduction we explained the origin of the definition of \(w^*_n(a)\). We now explain an analogous development for \(q^*_n(a)\). Just as we extended \(\{w^*_n(a)\}\) to the two stratum denominators \(\{w_{m,n}(r, y, z; a, b)\}\) by Definition \((1.15)\), we now attend to a similar issue for \(\{q^*_n(a)\}\). In the case \(b = a\) we obtain a representation \(K_N(r, y, z; a, a) = C r^2 z^2 q_N(a)/w^*_N(a)\) for a normalizing constant \(C\), that is Proposition \(1\). Thus we call \(q^*_n(a)\) a “numerator” polynomial.

In Section \(1.4\) we developed a doubly indexed extension \(\{w_{m,n}\}\) of the denominators \(\{w^*_n(a)\}\) to establish the formula for the generating functions \(\{g_{m,n}\}\) in Proposition \(5\). Our motivation in this section is to obtain a closed formula for the two stratum generating function \(K_N(r, y, z; a, b)\) of
It suffices for this purpose to develop only a singly indexed extension \( \{ \eta_n(x, y; z; a, b) \} \) of the numerators \( \{ q^*_n(a) \} \).

By definitions (1.44) and (1.9)–(1.11), with \( q^*_n := q^*_n(a) = q^*_n(x, y; z; a) \) and \( w^*_n := w^*_n(a) = w^*_n(x, y; z; a) \), we have a common recurrence relation:

\[
q^*_{n+1} = \beta_a q^*_n - x_a q^*_n; \quad w^*_{n+1} = \beta_a w^*_n - x_a w^*_n; \quad \text{for all } n \geq 1,
\]

where \( \beta_a \) and \( x_a \) are defined by (1.9). Likewise, our extension \( \eta_n \) will satisfy an analogous recurrence as \( \overline{w}_{0,n} \) in Definition (1.44), except with \( \eta_n \) we begin the stratum crossing at index \( n = f \) rather than \( n = f + 1 \), as follows. Recall (1.14) for the definitions of \( \beta(a, b) \) and \( x(a, b) \).

**Definition 4.** Define \( \eta_n := \eta_n(x, y; z; a, b) \) for all \( n \geq 1 \) by:

1. \( \eta_n := q^*_n(a), \; 1 \leq n < f; \)
2. \( \eta_f := \frac{1-b}{1-a} q^*_f(a) + \frac{a}{1-a} \eta_{f-1} \); \n3. \( \eta_{f+1} := \beta(a, b) \eta_f - x(a, b) \eta_{f-1} \); \n4. \( \eta_{f+j+1} := \beta(a, b) \eta_{f+j} - x(a, b) \eta_{f+j-1}, \; j \geq 1. \)

To obtain our extension Theorem 1 of Proposition 3 for the persistence model in two strata we require two key ingredients. One is the formula for \( \eta_n \) that we already established in Proposition 5, and the other ingredient is a formula (Lemma 6) for an analogue (1.104) of the interlacing bracket (1.48) that involves both numerators and denominators instead of just denominators. Denote

\[
\overline{w}, \overline{\eta}_n := \overline{w}_{n,0} \eta_n + \overline{w}_{n+1,0} - \overline{w}_{n+1,0}, \; n \geq 1;
\]

We also denote a special case of (1.104) by

\[
[w^*_n(a), q^*_n(a)]_n := w^*_n(a) q^*_n(a) - q^*_n(a) w^*_n(a), \; n \geq 1.
\]

**Lemma 6.** The following identities hold:

1. \( [w^*_n(a), q^*_n(a)]_n = a^2 z^2 x_n^{n-1}, \; n \geq 1; \)
2. \( \overline{w}, \overline{\eta}_{f-1} = \frac{1-b}{1-a} [w^*(a), q^*(a)]_{f-1} = \frac{1-b}{1-a} a^2 z^2 x_{f-2}; \)
3. \( \overline{w}, \overline{\eta}_{f+j-1} = \frac{1-b}{1-a} [w^*(a), q^*(a)]_{f+j} = \frac{1-b}{1-a} a^2 z^2 x_{f-2}^j, \; j \geq 1. \)

**Remark 3.** By (1.44) and Lemma 6, 1 and 3, we obtain

\[
\overline{w}, \overline{\eta}_{f+j-1} = \frac{1-b}{1-a} a^2 z^2 x_{f-2}^j x(a, b), \; \forall \; j \geq 1.
\]

Before we can prove Lemma 6 we write a formula for \( \overline{\eta}_n \) as follows.
Lemma 7. Let the $2 \times 2$ matrix $M = (\mu_{i,j})$ be defined by (1.63). Then, for all $j \geq 1$,
\[ \mathbf{q}_{f+1} = \left[ q^*_f(b) \ w^*_f(b) \right] M \left[ q^*_{f-1}(a) \ w^*_{f-1}(a) \right]. \] (1.107)

Proof (Lemma 7). We follow the proof of Section 1.4.2 to establish (1.107). Define $Q(b)$ as before in (1.55), but write now a revision of (1.55) suitable for computing $\mathbf{q}_n$ by introducing $W_q(f)$ in place of $W(\ell)$ as follows:
\[ W_q(f) := \left[ q^*_f q^*_f + 1 \right] = Q(b) d_q(f); \quad Q(b) := \left[ q^*_1(b) w^*_1(b) \right]. \] (1.108)

We also introduce the matrix $B = B(a,b)$ in the same vein as before in (1.56), so
\[ W_q(f) = B \left[ q^*_{f-1}(a) \right]. \] (1.109)

We do not change the name of $B$, because by Definitions (1.45) and (1.103) the recursions that define $B$ in (1.109) are exactly the same as the ones defining $B$ in (1.56); see (1.58)–(1.60). Thus the matrix $B$ does not change, and is given by (1.61). By equating the two expressions for the vector $W_q(f)$ in (1.108)–(1.109), we are led to the same matrix $M := Q(b)^{-1}B$ as given by (1.63) and hence recover
\[ d_q(f) = M \left[ q^*_{f-1}(a) \right]. \] (1.110)

Now the formula (1.107) follows because by (1.108)–(1.110) we have established the formula for $j = 1, 2$. Therefore by the fact that either side of (1.107), denoted $v_j$, say, satisfies the same recurrence $v_{j+1} = \beta b v_j - x b v_{j-1}$, $j \geq 2$, we have that both sides are equal as stated in the lemma.

Proof (Lemma 6). Note that by Lemma 2 we have $w_{n,0} = w_{1,n+1}$. Further by Proposition 4 we may write
\[ \mathbf{w}_{1,f+j} = \left[ q^*_f(b) \ w^*_f(b) \right] M \left[ w^*_{f-1}(a) \ w^*_{f-1}(a) \right]. \] (1.111)

Thus if we write $n = f + j - 1$ for some $j \geq 1$ then, by (1.111) and (1.107),
\[ \mathbf{w}, \mathbf{q}|_n = \left[ q^*_j \ w^*_j \right] M \times \left[ \begin{array}{c} w^*_{j+1} \ w^*_{j+1} \end{array} \right] M \left[ q^*_{f-1} \ q^*_{f-1} \right] \left[ q^*_{j+1} \ w^*_{j+1} \right] M \left[ w^*_{f-1} \right]. \] (1.112)

where we have abbreviated the notation, so that the polynomials with indices $f-1$ and $f$ depend on the parameter $a$, while the polynomials with indices...
$j$ and $j+1$ depend on the parameter $b$. Now calculate the expression under the curly brackets in (1.112) as follows:

$$[w^*(a), q^*(a)]_{j-1} \left[ \begin{array}{cc} \mu_{1,2} & \mu_{2,2} \\ -\mu_{1,1} & -\mu_{2,1} \end{array} \right] \left[ \begin{array}{c} q_j^*(b) \\ w_j^*(b) \end{array} \right].$$

Here we use the notation $M = (\mu_{i,j})$, so when we plug this last expression for the curly brackets into (1.112) we obtain

$$[w^*(a), q^*(a)]_{j-1} \left[ \begin{array}{cc} 0 & \det(M) \\ \det(M) & 0 \end{array} \right] \left[ \begin{array}{c} q_j^*(b) \\ w_j^*(b) \end{array} \right].$$

(1.113)

By (1.113) we obtain $[w^*(a), q^*(a)]_{j-1} (-\det(M))[w^*(b), q^*(b)]_j$, valid for all $j \geq 1$. Also, we calculate directly from (1.63) that $\det(M) = \frac{1}{1-a}\beta^2(a,b)^2$. So the statement 3 of the lemma is proved.

The formula (1.106) is a consequence of statement 3 that follows from statement 1, which we now prove. Indeed, we calculate by (1.102) that, for all $n \geq 1$,

$$[w^*(a), q^*(a)]_n = w_n^*(\beta_a q_n^* - x_a q_{n-1}^*) - q_n^*(\beta_a w_n^* - x_a w_{n-1}^*),$$

(1.114)

where we have abbreviated $w_n^* = w_n^*(a)$, etc. By direct calculation from the definition (1.10), we find $w_1^*(a) q_2^*(a) - q_1^*(a) w_1^*(a) = 1/\tau_a = a^2 z^2/x_\alpha$, where we refer to (1.9) for the definitions of $\tau_a$ and $x_\alpha$. Therefore, since obviously we may iterate (1.114), we easily obtain statement 1 of the lemma.

Finally, write $[w^*, q^*]_{j-1} = w_{1,j}^* q_j^* - q_{j-1}^* w_{1,j+1}^*$. Thus by Definitions (1.45) and (1.109) we have that

$$[w^*, q^*]_{j-1} = w_{j-1}^* \left( \frac{1}{1-a} q_j^* + \frac{b}{1-a} q_{j-1}^* \right) - q_{j-1}^* \left( \frac{1}{1-a} w_j^* + \frac{b}{1-a} w_{j-1}^* \right),$$

(1.115)

where we have abbreviated $w^* = w^*(a)$ and $q^* = q^*(a)$. Therefore statement 2 of the lemma follows by (1.115) and statement 1.

We now state our generalization of Proposition 4 to the full model with two strata. Then we prove the extended result first in the homogeneous case by itself to clarify the ideas.

**Theorem 1.** The conditional generating function (1.3) has the following formula.

$$K_N(r, y, z; a, b) = C_{N,a,b} \frac{r^2 z^2 \mathcal{W}_N}{w_{1,N+1}}, \ N \geq 1,$$
where \( \overline{q}_N \) and \( \overline{\omega}_{1,N+1} \) are given by (1.10) and (1.11) respectively, each with \( j = N - f + 1 \), and with \( M \) defined by (1.63), and where \( C_{N,a,b} = (1-a)(Nf+(f-1)b-(N-1)ab) \).

Proof (Proposition 7 and Theorem 7). For the sake of clarity we first prove Proposition 7 that is we first prove Theorem 7 under the assumption \( b = a \). First we observe, by Remark 2 (1.12), and (1.31), that with \( b = a \) we have that for all \( n \geq 3 \),

\[
G_n(r, y; z; a, a) = a^2(2-a)^2C_{n,a}C_{n-1,a}r^2z^{2n-r_n-4}/\{w_n^*(a)w_{n-1}^*(a)\},
\]

(1.116)

for \( C_{n,a} := \frac{a^{n-2}}{2} (n-(n-1)a) \), where we have written \( h_n k(a, a) \omega_n^2 = a(2-a) \tau_n \) by the definitions (1.9), (1.13), and (1.21). Recall the definition (1.16), and in the homogeneous case \( b = a \) denote \( \rho_n(a) = \rho_i, j \) for \( j - i = n \). Now by Definition (1.35), (1.36) and Proposition 3 we have

\[
\rho_n(a) = \rho_{n-1}(a) = \frac{1}{2n - (n-1)a}, \quad n \geq 1.
\]

Further, in general we have:

\[
P(H = n) = 2a \rho_0n 2 \gamma_n \rho_{n,0}, \quad n \geq 1;
\]

(1.118)

\[
P(H \geq n + 1) = 2a \rho_{n+1}, \quad n \geq 2; \quad P(H = 1) = 2^{\frac{1}{2}}(1-a) = 1 - a.
\]

(1.119)

For brevity, write \( G_n(r, y; z; a, a) = G_n(a) \). Note by direct computation that \( G_1(a) = r^2y^2z^2 \) and \( G_2(a) = r^2z^4k(a, a) \). Therefore by the definitions (1.15) and (1.16) we obtain that \( P(H \leq N)K_N(r, y; z; a, a) \) is written by:

\[
\sum_{n=1}^{N} G_n(a)P(H = n) = (1-a)r^2y^2z^2 + \frac{a(1-a)}{2-a}r^2z^4k(a, a)
\]

(1.119)

+ \sum_{n=3}^{N} a^2(2-a)^2C_{n,a}K_{n-1,a}P(H = n)r^2z^{2n-r_n-4}/\{w_n^*(a)w_{n-1}^*(a)\}.

Now we calculate by (1.116) - (1.118) that \( a^2(2-a)^2C_{n,a}K_{n-1,a}P(H = n) = (1-a)^2a^{2n-2} \). Also, by (1.9) and (1.21), \( a^{2n-2}a^{2n-4} = a^2r^2z^4x_{a}^{n-2} \) while \( k(a, a) = a(2-a)/w_2^*(a) \), since \( \omega(a, a) = w_2^*(a) \). Therefore by (1.119) we have

\[
P(H \leq N)K_N(r, y; z; a, a) = (1-a)^2z^2 + \frac{a^2z^2}{w_2^*(a)} + \sum_{n=3}^{N} \frac{a^2z^2x_{a}^{n-2}}{w_n^*(a)w_{n-1}^*(a)}.
\]

(1.120)

By (1.9) - (1.11) and direct calculation, we have that \( y^2 = q_1^*(a)/w_1^*(a) \) and \( y^2 + a^2z^2/w_2^*(a) = q_2^*(a)/w_2^*(a) \). But, by Lemma 7, 1, we may write
Therefore by (1.111), the sum in (1.120) telescopes such that for all \(N \geq 1\) the right side of (1.120) becomes: \(1 - a r^2 z \gamma_N^2(a)/\gamma_N^2(a)\). Finally, apply (1.118) and Proposition 2 to compute \(P(\mathbf{H} \leq N) = 1 - a \frac{a}{N-(N-1)a} = \frac{(1-a)}{N-(N-1)a}\). Thus after substituting this expression into (1.120), the proof in the case \(b = a\) is complete, so Proposition 1 is proved.

We now indicate the additional steps required to prove the Theorem 1. First, with \(N = f - 1\) in Proposition 1 and by Lemma 6 (1.120) yields:

\[
\sum_{n=1}^{f-1} G_n P(\mathbf{H} = n) = (1 - a)r^2 z^2 \left(y^2 + \sum_{n=2}^{f-1} \frac{w_n^*(a), q_n^*(a)}{w_n^*(a)w_n^*(a)}\right),
\]

(1.122)

where here and in the rest of the proof we abbreviate \(G_n = G_n(r, y, z; a, b)\). Next by (1.122) and Proposition 5 (11.1.1 and II.2,

\[
G_f = a(2 - a)zh_1g_1f(k(a, b)g_{f,0} = c_f r^2 z^2 [a \tau(a, a)]^{2f-4} \frac{w_f z(a)w_{f-1}(a)}{w_f z(a)w_{f-1}(a)}. \]

(1.123)

Also by (1.122) and Proposition 5 (11.1.1 and II.1, for all \(j \geq 1\)

\[
G_{f+j} = a(2 - a)zh_1g_{1,f+j}k(b, b)g_{f+j,0}
= c_{f+j} r^2 z^2 [a \tau(a, a)]^{2f-4} \tau(a, b)^2 [b \tau(b, b)]^{2j-2} / \{w_{f+j} z(a)w_{f+j,0}\}.
\]

(1.124)

In (1.123) and (1.124) the constants \(c_f\) and \(c_{f+1}\), respectively can be determined from Lemma 5 since \(G_n(1, 1, 1) = 1\). Indeed we find in this way, and by Definition (1.35)–(1.36), Proposition 2 and (1.118), that \(c_f = a^2 b^{-1} \Pi_1f \Pi_{f,0}\) and \(P(\mathbf{H} = f) = b(1 - b)/[\Pi_1f \Pi_{f,0}]\), so \(c_f P(\mathbf{H} = f) = a^2(1 - b)\). Also, for \(j \geq 1\), \(c_{f+j} = a^2 b \Pi_1f \Pi_{f+j,0} \) and \(P(\mathbf{H} = f + j) = b(1 - b)/[\Pi_1f \Pi_{f+j,0}]\), so \(c_{f+j} P(\mathbf{H} = f + j) = a^2 b^2(1 - b)\). Thus by (1.123)–(1.124), and since \(w_{f-1}^*(a) = w_1, f\), we have:

\[
G_f P(\mathbf{H} = f) = a^2(1 - b)r^2 z^2 x_a^{f-2}/\{w_{f,1} z(a)w_{f,0}\}
\]

\[
G_{f+j} P(\mathbf{H} = f + j) = a^2(1 - b)r^2 z^2 x_a^{f-2} x(a, b)x_{b}^{j-1}/\{w_{f,1} z(a)w_{f+j,0}\}, \quad j \geq 1.
\]

(1.125)

Now apply (1.122) and (1.123) to obtain that, by Lemma 6 for all \(j \geq 0\) there holds:

\[
\sum_{n=1}^{f+j} G_n P(\mathbf{H} = n) = (1 - a)r^2 z^2 \left(y^2 + \sum_{n=2}^{f+j} \frac{w_n z(a)w_{n-1}(a)}{w_n z(a)w_{n,0}}\right),
\]

(1.126)
where the fraction \( \frac{1-b}{1-a} \) enters to form \([w,q]_{n-1}\) when \( n = f + j \) for \( j \geq 0 \) because we have factored out \((1-a)\) from the entire sum on the right. But by the definition (1.104) and Lemma 2 we have that

\[
[w,q]_{n-1} = \frac{q_n}{w_{1,n+1}} - \frac{q_{n-1}}{w_{1,n}}.
\]

Hence the sum in (1.126) telescopes, and we thereby obtain

\[
K_N(r,y,z; a,b) = P(H \leq N)^{-1} (1-a) \frac{r^2 z^2 q_N}{w_{1,N+1}},\quad N \geq f,
\]

(1.127)

where \( P(H \leq N)^{-1} = (N+1-a) + (f-1)b - (N-1)ab \) by (1.118). Thus by (1.127) the proof is complete. \( \square \)

1.7 Applications of Proposition 1

In this section we prove the corollaries of Proposition 1 announced in the Introduction.

Proof (Corollary 1). We assume \( b = a \). Denote \( K(r,y,z; a) := E\{R_y V_z L\} \) the joint generating function of the counting statistics for runs, runs of length exactly one, and steps, over an excursion of the persistent random walk. Since we have explicitly seen in the proof of Proposition 1 that \( P(H \leq N) = \frac{N(1-a)}{N-(N-1)a} \), we have that the persistent random walk is recurrent: \( \lim_{N \to \infty} P(H \leq N) = 1 \). So we obtain that

\[
K(r,y,z; a) = \lim_{N \to \infty} K_N(r,y,z; a,a) = (1-a) r^2 z^2 \lim_{N \to \infty} \frac{q_N^*}{w_N^*}.
\]

(1.128)

Here and in the rest of the proof we suppress dependence on \( a \) when convenient.

We introduce a substitution variable \( \theta \) to simplify the expression for the quotient \( \frac{q_N^*}{w_N^*} \). Recall the definitions of \( \alpha = \alpha_a, \beta = \beta_a \), and \( x = x_a \) in the statement of Corollary 1 and (1.9). Our substitution is:

\[
\beta := \sqrt{4x \cos \theta}.
\]

(1.129)

Next, since \( \alpha = \sqrt{\beta^2 - 4x} \), we have by (1.129) that

\[
\beta \pm \alpha = \sqrt{4x(\cos \theta \pm i \sin \theta)} = \sqrt{4x} e^{\pm i \theta},
\]

(1.130)

with \( \Im \theta < 0 \) if \( |r| < 1, |y| < 1, z \neq 0 \). The idea of the substitution (1.129)–(1.130) may be found in [3], p. 352. We apply the definitions (1.10)–(1.11) to obtain the following expressions:
Recall that excursion path of $2n$ may view this model in the context of excursion statistics by the way in $R$ distribution of $n$ we obtain the following limit: as the proof of the corollary is complete. 

$\quad$ 

1.7.1 Joint Distribution of $R$, $U$, and $L$. 

Recall that $U = R - V$ is the number of “long” runs (runs of length at least two) in an excursion from the origin. Corollary 2 concerns the joint distribution of $R$, $U$, and $L$ for the homogeneous persistence model. We may view this model in the context of excursion statistics by the way in which excursion paths are weighted relative to one another. Indeed, a specific excursion path of $2n$ steps and $2k$ runs is weighted with the probability $\frac{1}{2} a^{2n-2k}(1-a)^{2k-1}$. In the unweighted case $a = \frac{1}{2}$, it is known that the joint distribution of $(L, R)$ is essentially the same as that of $(L, L - R)$; to see this, set $s = 0$ the joint generating function statement of Corollary 2. This corollary

\begin{align*}
q_n^* &= (q^2 - q_0^2)q_n(x, \beta) + q_0^2w_n(x, \beta); \\
w_n^* &= (1 - w_0^2)q_n(x, \beta) + w_0^2w_n(x, \beta); \quad x = x_0, \beta = \beta_0. \\
\end{align*}

Thus by the closed formulae for the fundamental Fibonacci polynomials $q_n$ and $w_n$ in (1.76), we obtain from (1.131) that

\begin{align*}
q_n^* &= \frac{2q_n}{a} \{q^2((\beta + \alpha)^n - (\beta - \alpha)^n) - 2xq_0^2((\beta + \alpha)^{n-1} - (\beta - \alpha)^{n-1})\}, \\
w_n^* &= \frac{2w_n}{a} \{((\beta + \alpha)^n - (\beta - \alpha)^n) - 2xw_0^2((\beta + \alpha)^{n-1} - (\beta - \alpha)^{n-1})\}. \\
\end{align*}

(1.132)

Now apply the substitution (1.129) - (1.130) in (1.132). Observe that $(\beta + \alpha)^n - (\beta + \alpha)^n = (4x)^{n/2}e^{in\theta}(1 + e^{-2in\theta})$. Also, by our convention for the sign of $\Re\theta$, $1 + e^{-2in\theta} = 1 + o(1)$, as $n \to \infty$. Therefore, using $e^{-i\theta} = (\beta - \alpha)/\sqrt{4x}$, we obtain the following limit: as $n \to \infty$,

$$
\frac{q_n^*}{w_n^*} \to \lim_{n \to \infty} \frac{y^2(1 - e^{-i\theta}) - \sqrt{x}q_0^2e^{-i\theta}(1 - e^{-2i(n-1)\theta})}{1 - q_0^2(\beta - \alpha)/2} = \frac{y^2 - q_0^2(\beta - \alpha)/2}{1 - w_0^2(\beta - \alpha)/2}.
$$

(1.133)

Finally, we simplify the right side of (1.133) by multiplying both numerator and denominator by $1 - w_0(\beta + \alpha)/2$. The new denominator becomes $1 + x_0w_0^2(a)^2 - \beta_0w_0^2(a) = (1 - a)^2r^2z^2/\tau_a^2$. Therefore by bringing the $\tau_a^2$ of this last expression to the numerator we obtain:

$$
\lim_{n \to \infty} (1 - a)^2r^2z^2q_n^*/w_n^* = [y^2 - \beta_0(q_n^* + y^2w_0^*)/2 + x_0w_0^2q_0^*\tau_a^2 + \alpha_0[q_0^* - y^2w_0^2\tau_a^2]/2] = I + \alpha_0II.
$$

(1.134)

By direct calculation we get much simplification of $I$ and $II$ on the right side of (1.134). We find that $I = 1 - \frac{1}{2}\beta_0$, and $II = -\frac{1}{2}$. By (1.128) and (1.134) the proof of the corollary is complete. 

$\square$
yields a symmetry in the joint Taylor expansion of $K(ru, 1/u, 2z; \frac{1}{2})$, where $2z$ is used in place of $z$ to give integer coefficients, and $u$ is the probability generating variable for $U$. Here is a sample from the Taylor expansion of $\frac{1}{2}K(ru, 1/u, 2z; \frac{1}{2})$ about $(r, u, z) = (0, 0, 0)$ at the 16th power of $z$:

$$\{(r^2 + 4r^4 + r^8 + r^{10} + r^{12} + r^{14})u^2 \}
\{(10r^4 + 16r^6 + 18r^8 + 16r^{10} + 10r^{12})u^3 + (10r^4 + 46r^6 + 63r^8 + 46r^{10} + 10r^{12})u^4 \}
\{(36r^6 + 68r^8 + 36r^{10})u^5 + (6r^6 + 23r^8 + 6r^{10})u^6 + 2r^8u^7 \}z^{16}
$$

(1.135)

Via the above symmetry and the near equality of the two joint generating functions in characteristic form stated in the corollary at $a = \frac{1}{2}$, we can obtain (1.19) by a one to one correspondence of paths.

**Proof (Corollary 2).** We first establish the joint generating function identity via a direct calculation. By applying Corollary 1 with $a = \frac{1}{2}$ we obtain

$$K(ru, 1/u, z; \frac{1}{2}) = \frac{1}{16} \left( 16 - 4z^2 + 4r^2z^2 + r^2z^4 - 2r^2uz^4 + r^2u^2z^4 - S \right),
$$

(1.136)

with $S$ given by:

$$S := \sqrt{(4 + 2z + 2rz + r^2z^2 - ru z^2)(4 + 2z - 2rz - r^2z^2 + ru z^2)}
\times \sqrt{(4 - 2z + 2rz - r^2z^2 + ru z^2)(4 - 2z - 2rz + r^2z^2 - ru z^2)}
$$

(1.137)

On the other hand, with the very same main term $S$, we have

$$K(u/r, 1/u, rz; \frac{1}{2}) = \frac{1}{16} \left( 16 + 4z^2 - 4r^2z^2 + r^2z^4 - 2r^2uz^4 + r^2u^2z^4 - S \right)
$$

(1.138)

The two generating functions differ by $K(ru, 1/u, z; \frac{1}{2}) - K(u/r, 1/u, rz; \frac{1}{2}) = \frac{1}{2}z^2(r^2 - 1)$. The difference is mirrored only in the event that $L = 2$, when it happens that $R = 2$ and $U = 0$.

We now have an easy argument for (1.19). Indeed, consider an excursion path $\Gamma$ with $L(\Gamma) = 2n$ and $L(\Gamma) - R(\Gamma) = 2k$. Then $P_n(\Gamma) = \frac{1}{2}a^{2k}(1 - a)^{2n - 2k - 1}$. To see this, simply write $L - R = L - (U + V) = (L - V) - U$. Thus $L(\Gamma) - R(\Gamma)$ is the total length of long runs minus the number of long runs in $\Gamma$, and this gives the number of factors of $a$ in $P_n(\Gamma)$. Now, if $2n \geq 4$, then by the generating function identity of the first part of the proof, there are exactly as many paths $\Gamma$ with the joint information, $L(\Gamma) = 2n$, $L(\Gamma) - R(\Gamma) = 2k$, and $U(\Gamma) = \ell$ as there are paths $\Gamma'$ with $L(\Gamma') = 2n$, $R(\Gamma') = 2k$, and $U(\Gamma') = \ell$. Therefore, since for any such path $\Gamma'$, the probability assigned by the probability measure $P_{1-a}$, defined with persistence parameter $1 - a$, yields $P_{1-a}(\Gamma') = \frac{1}{2}a^{2k}(1 - a)^{2n - 2k}$, we have that (1.19) holds for any fixed $\ell$, for the given $n \geq 2$ and $k \leq n - 1$. □
1.8 Limit distributions over the meander.

In this section we apply our results to obtain the characteristic function of a limiting distribution for a certain linear combination of our counting statistics in the meander portion of the gambler’s ruin, with scaling by the order $N$. For a single counting statistic, such as the total number of steps (or runs alone, or long runs alone, etc.) in either the meander portion or in the last visit portion of the gambler’s ruin, the variable scales to the order of $N^2$; see [6]. We are more interested here in the delicate balance between variables. Let $R_N', V_N', L_N'$, denote respectively the numbers of runs, short runs, and steps, in the meander portion of the gambler’s ruin. If $a = b = \frac{1}{2}$, then the author [6] shows that $(L_N' - 2R_N')/N$ converges in distribution to a density $f(x) = \frac{1}{4} \text{sech}^2(\frac{1}{2}x), -\infty < x < \infty$, with characteristic function $\int_{-\infty}^{\infty} e^{itx} f(x) dx = t/\sinh(t)$. In the homogeneous case the characteristic function $ct/\sinh(ct)$ also appears with order $N$ scaling of certain linear combinations of counting statistics over the meander; see Corollary 3. However, in the case $b \neq a$, and with $f \sim \eta N$ for some $\eta \in (0, 1)$, by incorporating both parameters $a$ and $b$ into a certain linear combination of $R_N', V_N', L_N'$, scaled by order $N$, we obtain convergence in law, but no longer obtain a characteristic function of form $ct/\sinh(ct)$.

Define the following scaled random variable over the meander:

$$X_N := \frac{1}{N} \left( L_N' - \frac{2\sigma_2 - b}{1-a(1-b)} R_N' + \frac{1}{(1-a)(1-b)} V_N' \right). \quad (1.139)$$

**Theorem 2.** Let $f = \eta N$ for some fixed $0 < \eta < 1$. Denote $\kappa_1 := \frac{\eta a^2}{b}$ and $\kappa_2 := \frac{1 - \eta \sigma_2}{1-a}$, with $\sigma_1 = \sqrt{a + b^2 - 2ab}$ and $\sigma_2 = \sqrt{b + a^2 - 2ab}$. Let $X_N$ be defined by (1.139). Then, as $N \to \infty$, then

$$\lim_{N \to \infty} E\{e^{itX_N}\} = \hat{\varphi}(t), \text{ where, } \hat{\varphi}(t) := \frac{(bk_1\sigma_1 + ak_2\sigma_2)t}{a\sigma_1 \cosh(\kappa_1 t) \sinh(\kappa_2 t) + b\sigma_2 \sinh(\kappa_1 t) \cosh(\kappa_2 t) + i(b-a)^2 \sinh(\kappa_1 t) \sinh(\kappa_2 t)} \quad (1.140)$$

Observe that if $b = a$, then in the statement of Theorem 2 we have $\sigma_1 = \sigma_2 = \sqrt{a(1-a)}$ and $\kappa_1 + \kappa_2 = A_1 := \sqrt{a/(1-a)}$. Thus (1.140) holds for $\hat{\varphi}(t) = A_1 t/\sinh(A_1 t)$. But we also have a bivariate result for the homogeneous case as follows. Define

$$Y_{1,N} := \frac{1}{N} \left( R_N' - \frac{1}{(1-a)} V_N' \right); \quad Y_{2,N} := \frac{1}{N} \left( L_N' - \frac{1}{(1-a)} R_N' \right) - Y_{1,N}. \quad (1.141)$$
Corollary 3. Suppose \( b = a \). Then the limiting joint characteristic function of the random variables \( Y_{1,N} \) and \( Y_{2,N} \) is:

\[
\lim_{N \to \infty} E\{e^{itY_{1,N} + itY_{2,N}}\} = \sqrt{(1-a)s^2 + at^2}/\sinh(\sqrt{(1-a)s^2 + at^2}).
\]

(1.142)

Remark 4. Let \( b = a \), and define \( X_{\zeta,N} := \frac{1}{N} \left( \mathcal{L}_N - \frac{1+i\zeta}{(1-a)} R_N + \frac{\zeta}{(1-a)} V_N^t \right) \).

Then by setting \( s = (1 - a - \zeta)t/(1-a) \) in (1.142) we obtain, for all \( \zeta \in \mathbb{R} \),

\[
\lim_{N \to \infty} E\{e^{itX_{\zeta,N}}\} = A\zeta t/\sinh(A\zeta t); \quad A\zeta := \sqrt{[(2\zeta - 1)a + (1 - \zeta)^2]/(1-a)}.
\]

(1.143)

Observe that the case \( \zeta = 0 \) and \( a = \frac{1}{2} \) in (1.143) reduces to the meander limit law of [6]. Furthermore we obtain the case \( b = a \) of Theorem 2 by the conclusion (1.143) with \( \zeta = 1 \). We shall prove the Corollary 3 directly after the proof of Theorem 2. The proof of Theorem 2 is more subtle than that of Corollary 3 though the basic approach is the same for the two proofs.

For the general case of (1.143), one checks that \( \varphi(x) := \int_{-\infty}^{\infty} e^{-ix\xi} \hat{\varphi}(t) dt \) is real since the complex conjugate of \( \varphi(x) \) is equal to itself; observe this by making a change of variables \( t \to -t \) after a direct conjugation of the integral. As a special case of Theorem 2 consider \( b = 1 - a \) and \( \eta = a \). Then \( \kappa_i = \sigma_j = \sigma := \sqrt{1-3a + 3a^2} \), for all \( i, j = 1, 2 \). In this case we have

\[
\hat{\varphi}(t) = \sigma^2 t/ \{ \sinh(\sigma t)[\sigma \cosh(\sigma t) + i(1-2a)^2 \sinh(\sigma t)] \}.
\]

(1.144)

The complex factor of the denominator of \( \hat{\varphi}(t) \) in (1.144) is equal to zero if and only if \( e^{2\sigma t} = -\frac{\sigma + (1-2a)^2}{\sigma + (1-2a)^2} \), where the fraction has modulus one, so we obtain pure imaginary roots, the closest to the origin being \( t = \frac{1}{\pi \sigma} (\pi - \arctan \frac{2(1-2a)^2}{\sigma^2 - (1-2a)^4}) \), with \( \sigma^2 - (1-2a)^4 = a(1-a)(5 - 16a + 16a^2) > 0 \). Thus we can analytically continue \( \hat{\varphi}(t) \) to a suitably chosen ball of positive radius \( c_0 \) about the origin such that \( \sup_{|t| \leq c_0} \| \hat{\varphi}(-t) \| \leq \infty \). It follows by [7], Thm. IX.13, that the inverse Fourier transform \( \varphi(x) \) of \( \hat{\varphi}(t) \) has exponential decay, meaning \( e^{\epsilon |t|} \varphi(t) \) is square integrable for any \( \epsilon < c_0 \). However the inverse Fourier transform, or probability density, \( \varphi(x) \), is not symmetric in \( x \) under (1.143) with \( a \neq \frac{1}{2} \).

One other comment is in order concerning Theorem 2. We see in the substitution method (1.143) – (1.150) the reasoning behind the specific choice of coefficients that defines \( X_N \). The coefficients are chosen such that the first order term of the Taylor expansions about \( t = 0 \) of the substitutions, \( \cos \theta_1 \) and \( \cos \theta_2 \), composed with the complex exponentials \( (r_N(t), y_N(t), z_N(t)) \) of (1.143), will vanish in (1.150). In detail, with \( U_a(r, y, z) = \cos \theta := \beta_a(r, y, z)/\sqrt{4\pi a(r, y, z)} \), the Taylor expansion of \( U_a(e^{is/N}, e^{iv/N}, e^{it/N}) \) about \((s, v, t) = (0, 0, 0)\) yields an order one term: \( -i(1-a)(t + (1-a)s + (1-a)^2)v \). Thus we solve \( 1 + c_1(1-a) + c_2(1-a)^2 = 0 \), and also \( 1 + c_1(1-b) + c_2(1-b)^2 = 0 \),
to find $c_1 = -(2a-b)c_2$ and $c_2 = 1/[(1-a)(1-b)]$ as the required coefficients of $R_N'$ and $V_N'$, respectively, with a unit coefficient for $L_N'$.

**Proof (Theorem 2).** We fix $t \in \mathbb{R}$. All big oh terms $O(N^{-\nu})$ in the proof will refer to the parameter $N \to \infty$ with implied constants depending only on $a$, $b$, and $t$. The joint generating function of $R_N'$, $V_N'$, and $L_N'$, is given by (1.14). Then, since $(1 + \frac{1}{N})X_{N+1}$ converges in distribution if and only if $\{X_N\}$ does so, it suffices to establish the limit as $N \to \infty$ of the following expression for each fixed $t \in \mathbb{R}$:

$$E\{e^{it(1+\frac{1}{N})X_{N+1}}\} = a(2-a)zh\varrho_0,N(r_N,y_N,z_N; a,b);$$

$$r_N := e^{-it(2-a-b)/((1-a)(1-b))N}, \quad y_N := e^{it/((1-a)(1-b))N}, \quad z_N := e^{it/N}. \quad \text{(1.145)}$$

It is clear that $a(2-a)zh\varrho_0,N(r_N,y_N,z_N) \to 1$ as $N \to \infty$. Therefore we must show that $\lim_{N \to \infty} (g_{0,N}(r_N,y_N,z_N; a,b)) = 1$ in (1.140). We note that all functions of $(r, y, z)$ will be composed with $(r_N, y_N, z_N)$ so that all these compositions will depend on both persistence parameters $a$ and $b$ as well as $t$ and $N$. We apply Proposition 5.1.1 to write

$$g_{0,N} = \frac{\omega_a}{2-a}r_N^z N^\tau (a,b)[a\tau_a]^{-2} [b\tau_b]^{-1} (a\Pi_{0,N}/\varpi_{0,N}) \quad \text{(1.146)}$$

Further, by Proposition 4 we may write

$$\varpi_{0,N} = \left[q_{r-N-f}(b) w_{r-N-f}^*(b)\right] M \left[w_{r}^*(a) w_{r+1}^*(a)\right] = d_1(f) q_{r-N-f}(b) + d_2(f) w_{r-N-f}^*(b). \quad \text{(1.147)}$$

The main work is in calculating an asymptotic expression for $\varpi_{0,N}$ that we will then enter into (1.146) to find the desired limit as $N \to \infty$.

We now make the substitutions and definitions as in (1.129)–(1.130), except now we require a substitution for each stratum:

$$\cos(\theta_1) := \beta_a/\sqrt{4x_a}; \quad \cos(\theta_2) := \beta_b/\sqrt{4x_b};$$

$$\alpha_a := \sqrt{\beta_a^2 - 4x_a}; \quad \alpha_b := \sqrt{\beta_b^2 - 4x_b}, \quad \text{(1.148)}$$

where again all functions on the right sides of these expressions are composed with $(r_N, y_N, z_N)$ of (1.145). Here we refer to (1.39) for definitions. Thus by (1.148),

$$\beta_a \pm \alpha_a = \sqrt{4x_a e^{\pm i\theta_1}}; \quad \beta_b \pm \alpha_b = \sqrt{4x_b e^{\pm i\theta_2}}. \quad \text{(1.149)}$$

Furthermore, by the definition of $(r_N, y_N, z_N)$ in (1.145), the following expansions hold:

$$\cos \theta_1 = 1 + \frac{1}{2} \frac{\sigma_1^2 t^2}{(1-b)^2 N^2} + O(N^{-3}); \quad \cos \theta_2 = 1 + \frac{1}{2} \frac{\sigma_2^2 t^2}{(1-a)^2 N^2} + O(N^{-3}), \quad \text{(1.150)}$$
where $\sigma_1^2$ and $\sigma_2^2$ are as defined in the statement of the theorem. Therefore by (1.150), and by applying the Taylor expansion of $\arccos(u)$ about $u = 1$, we find that $\theta_1$ and $\theta_2$ are both of order $1/N$ as follows:

\[ \theta_1 = i \frac{\sigma_1 t}{(1 - b)N} + O(N^{-3}); \quad \theta_2 = i \frac{\sigma_2 t}{(1 - a)N} + O(N^{-3}). \]  

(1.151)

To apply (1.147) we recall the formulae of (1.132). By (1.9) we shall write $\sqrt{4x_a} = 2az\tau_a$ and $\sqrt{4x_b} = 2bz\tau_b$ when convenient. Thus from (1.132) and (1.148)–(1.149) we have

\[ w_j^+(a) = 2i\alpha_a^{-1}[az\tau_a]^j \{ \sin f\theta_1 - \sqrt{x_a}w_0^+(a)\sin(f - 1)\theta_1 \} \]

(1.152)

\[ w_{j+1}^+(a) = 2i\alpha_a^{-1}[az\tau_a]^j \sqrt{x_a} \{ \sin(f + 1)\theta_1 - \sqrt{x_a}w_0^+(a)\sin f\theta_1 \}, \]

where we have applied the complex exponential formula for the sine. Now write $f_+ := f + 1$, and $f_- := f - 1$. Also denote

\[ A_1 := 2i\alpha_a^{-1}[az\tau_a]^j = (\sin \theta_1)^{-1}[az\tau_a]^j. \]  

(1.153)

since $\alpha_a = i\sqrt{4x_a}\sin \theta_a = 2az\tau_a\sin \theta_1$. By (1.147) and (1.152) we can write expressions for the coefficients $d_1 := d_1(f) = \mu_{1,1}w_j^+(a) + \mu_{1,2}w_{j+1}^+(a)$ and $d_2 := d_2(f) = \mu_{2,1}w_j^+(a) + \mu_{2,2}w_{j+1}^+(a)$ as follows.

\[ \frac{d_1}{A_1} = (\mu_{1,1} - \mu_{1,2}x_a w_0^+(a)) \sin f\theta_1 + \sqrt{x_a} \{ \mu_{1,2} \sin f_+ \theta_1 - \mu_{1,1} w_0^+(a) \sin f_- \theta_1 \} \]

(1.154)

Next we apply the trigonometric identity for the sine of a sum or difference to $\sin f_+ \theta_1 = \sin(f + 1)\theta_1$ and $\sin f_- \theta_1 = \sin(f - 1)\theta_1$ in (1.154). At this point we also introduce some abbreviations to keep the notation a bit compact. Thus write

\[ s_1 := \sin f\theta_1; \quad c_1 := \cos f\theta_1. \]  

(1.155)

We rewrite (1.154), with abbreviation $w_0^+(a)$, by collecting terms with a factor $\sqrt{x_a}$:

\[ d_j/A_1 = (\mu_{1,1} - \mu_{1,2}x_a w_0^+(a))s_1 + \sqrt{x_a} \{ \mu_{1,2}(s_1 \cos \theta_1 + c_1 \sin \theta_1) - \mu_{1,1} w_0^+(s_1 \cos \theta_1 - c_1 \sin \theta_1) \} , \quad j = 1, 2. \]  

(1.156)

We reduce (1.156) a bit more by the following device: we will collect some terms of order $O(1/N^2)$ that will not contribute to the limit in the end. Thus in (1.156), by (1.150)–(1.151), we may treat $\cos \theta_1 = 1 + O(N^{-2})$ and $\sin \theta_1 = \theta_1 + O(N^{-3})$; in addition by (1.155) all other terms are at most $O(1)$. Further we introduce a book-keeping notation for the coefficient $t_j$ of
the variable \( x_j \) in square brackets, within a linear expression \( \sum_j t_j x_j \) in parentheses: \( [x_j] (\sum_j t_j x_j) = t_j \). Thus, after composition of the various functions with \((r_N, y_N, z_N)\), and by direct calculation, we find the following asymptotic expressions for the coefficients of \( s_1 \) and \( c_1 \sin \theta_1 \), respectively, of \( d_1/A_1 \) in (1.156):

\[
[s_1]\left(\frac{d_1}{A_1}\right) = \mu_{1,1} - \mu_{1,2} x a w_0^*(a) + \sqrt{x a}(\mu_{1,2} - \mu_{1,1} w_0^*(a)) \cos \theta_1 \\
= -(1 - a)(1 - b) + 2(1 - ab) \frac{d}{N} + O(N^{-2});
\]

(1.157)

\[
[c_1 \sin \theta_1]\left(\frac{d_1}{A_1}\right) = \sqrt{x a}(\mu_{1,2} + \mu_{1,1} w_0^*(a)) = O(N^{-2}).
\]

(1.158)

By further direct calculations, we find the following asymptotic expressions for the coefficients of \( s_1 \) and \( c_1 \sin \theta_1 \), respectively, of \( d_2/A_1 \) in (1.156):

\[
[s_1]\left(\frac{d_2}{A_1}\right) = \mu_{2,1} - \mu_{2,2} x a w_0^*(a) + \sqrt{x a}(\mu_{2,2} - \mu_{2,1} w_0^*(a)) \cos \theta_1 \\
= 1 - a - \frac{a(b - a) \mu}{4 \sqrt{N}} + O(N^{-2}).
\]

(1.159)

\[
[c_1 \sin \theta_1]\left(\frac{d_2}{A_1}\right) = \sqrt{x a}(\mu_{2,2} + \mu_{2,1} w_0^*(a)) = a + \frac{a(b - a) \mu}{4 \sqrt{N}} + O(N^{-2}).
\]

(1.160)

Therefore by (1.151)–(1.160),

\[
d_1/A_1 = \{-(1 - a)(1 - b) + 2(1 - ab) \frac{d}{N}\} s_1 + O(N^{-2});
\]

\[
d_2/A_1 = \left[1 - a - \frac{a(b - a) \mu}{4 \sqrt{N}}\right] s_1 + a c_1 \sin \theta_1 + O(N^{-2}).
\]

(1.161)

We now find asymptotic expressions for the terms \( q^*_{N,f}(b) \) and \( w^*_{N,f}(b) \) of (1.147). Similar as above, but now with \( b \) in place of \( a \), \( N - f \) in place of \( f \), and using the second substitution \( \theta_2 \) in (1.148)–(1.149). Similar to (1.154) we also introduce

\[
A_2 := (\sin \theta_2)^{-1} [bz \tau_b]^{N-f-1}.
\]

(1.162)

Thus by both lines of (1.132) applied in turn, obtain that

\[
q^*_{N,f}(b) = A_2 \left\{y^2 \sin(N - f) \theta_2 - \sqrt{x a} q_0^*(b) \sin(N - f - 1) \theta_2 \right\},
\]

\[
w^*_{N,f}(b) = A_2 \left\{\sin(N - f) \theta_2 - \sqrt{x a} w_0^*(b) \sin(N - f - 1) \theta_2 \right\}.
\]

(1.163)

Now the expressions in (1.163) are less complicated to deal with than \( d_1 \) and \( d_2 \), but we still expand the second sine in each line using the sine of a difference. Again by (1.151) we have \( \cos \theta_2 = 1 + O(N^{-2}) \) and \( \sin \theta_2 = \theta_2 + O(N^{-3}) \). We again introduce some abbreviations:

\[
s_2 := \sin(N - f) \theta_2; \quad c_2 := \cos(N - f) \theta_2.
\]

(1.164)
We find exact formulae for the coefficients of \( s_2 \) and \( c_2 \) \( \sin \theta_2 \) of \( q_{N-f}^*(b)/A_2 \) and \( w_{N-f}^*(b)/A_2 \), respectively, in (1.163) by using \( \sin(N-f-1)\theta_2 = s_2 \cos \theta_2 - c_2 \sin \theta_2 \):

\[
[s_2] \left( \frac{q_{N-f}^*(b)}{A_2} \right) = y^2 - \sqrt{x_b q_0^*(b)} \cos \theta_2; \quad [c_2 \sin \theta_2] \left( \frac{q_{N-f}^*(b)}{A_2} \right) = \sqrt{x_b q_0^*(b)}; \quad \text{(1.165)}
\]

\[
[s_2] \left( \frac{w_{N-f}^*(b)}{A_2} \right) = 1 - \sqrt{x_b w_0^*(b)} \cos \theta_2; \quad [c_2 \sin \theta_2] \left( \frac{w_{N-f}^*(b)}{A_2} \right) = \sqrt{x_b w_0^*(b)}. \quad \text{(1.166)}
\]

Now by direct calculation, using the exact formulae of (1.165)–(1.166),

\[
[s_2] \left( \frac{q_{N-f}^*(b)}{A_2} \right) = 1 + \frac{2}{1-a} \frac{w}{N} + O(N^{-2}); \quad [c_2 \sin \theta_2] \left( \frac{q_{N-f}^*(b)}{A_2} \right) = O(N^{-1}); \quad \text{(1.167)}
\]

\[
[c_2 \sin \theta_2] \left( \frac{w_{N-f}^*(b)}{A_2} \right) = b + O(N^{-1}). \quad \text{(1.169)}
\]

Since \( \sin \theta_2 = O(N^{-1}) \) by (1.151), we therefore have by (1.163)–(1.169) that

\[
q_{N-f}^*(b) = \left( 1 + \frac{2}{1-a} \frac{w}{N} \right) s_2 + O(N^{-2}); \quad \text{(1.170)}
\]

\[
w_{N-f}^*(b) = \left[ \left( 1 - b - \frac{b(2-a-b) w}{1-a} \right) s_2 + b c_2 \sin \theta_2 \right] + O(N^{-2}). \quad \text{(1.171)}
\]

We are now ready to calculate our asymptotic formula for \( w_{0,N} \). Substitute (1.161) and (1.170) into (1.147) to obtain

\[
\frac{w_{0,N}}{A_1 A_2} = O(N^{-2}) + \left\{ -(1-a)(1-b) + 2(1-ab) \frac{w}{N} \right\} \left( 1 + \frac{2}{1-a} \frac{w}{N} \right) s_1 s_2 + \\
\left[ \left( 1 - a - \frac{a(b-a) w}{1-b} \right) s_2 + a c_1 \sin \theta_1 \right] \left[ \left( 1 - b - \frac{b(2-a-b) w}{1-a} \right) s_2 + b c_2 \sin \theta_2 \right]. \quad \text{(1.171)}
\]

Since by (1.151), \( \sin \theta_1 \) and \( \sin \theta_2 \) are of order 1/\( N \), observe that the sum of the terms of order 1 on the right hand side of (1.171) is exactly zero. Also, since \( \sin \theta_j = \theta_j + O(N^{-3}), \quad j = 1, 2 \), and since \( \theta_j, j = 1, 2 \), are given by (1.151), we substitute these relations into (1.171) and collect the order 1/\( N \) terms to find that:

\[
\frac{w_{0,N}}{A_1 A_2} = \left\{ a \sigma_1 c_1 s_2 + b \sigma_2 s_1 c_2 + (b-a)^2 s_1 s_2 \right\} \frac{it}{N} + O(N^{-2}). \quad \text{(1.172)}
\]

Now plug (1.172) into (1.146), apply Proposition 2.1 to rewrite \( H_{0,N} \), and recall the definitions (1.153) and (1.162) for \( A_j, \ j = 1, 2 \), to find that \( g_{0,N} \) is written.
Finally, to find the limit as \( N \to \infty \) of the expression (1.173), we substitute (1.151) into the definitions (1.155) and (1.164), and also employ \( \sin \theta \sim \theta \) as \( \theta \to 0 \). We note: \( \lim_{N \to \infty} \omega_0 [a(2-a)]^{-1} r_N^2 \tau(a,b) \tau_\alpha^{-1} = 1 \), since \( \omega_0 (1,1,1) = 2a(2-a) \) and \( \tau(a,b)(1,1,1) = \tau_\alpha(1,1,1) = 1 \). Since by assumption \( f \sim \eta \), we have \( [(N-f)a + fb - (N-1)ab] \sim N[(1-\eta)a + \eta b - ab] \), and since by (1.151), \( \theta_1 \theta_2 \sim i^2 \frac{\sigma_2^2}{(a-b)^2} t^2 N^{-2} \), by (1.173) we obtain, as \( N \to \infty \),

\[
g_0, N \sim \frac{i^2 t^2}{N} \frac{\sigma_1 \sigma_2}{(1-a)(1-b)} \frac{(1-\eta)a + \eta b - ab}{[a\sigma_1 c_1 s_2 + b\sigma_2 c_1 c_2 + (b-a)^2 s_1 s_2] \frac{t^2}{N}}. \tag{1.174}
\]

Here we use implicitly that \( \sin(ix) = i \sin(x) \) and \( \cos(ix) = \cosh(x) \), so that by (1.151), (1.155) and (1.164), and by definition of \( \kappa_1 \) and \( \kappa_2 \), \( s_j \sim i \sin(\kappa_j t), j = 1,2 \), and \( c_j \sim \cos(\kappa_j t), j = 1,2 \). Thus by (1.174), wherein the factors of \( N^{-1} \) in numerator and denominator cancel, we obtain, as \( N \to \infty \),

\[
g_0, N \sim \hat{\varphi}(t) \tag{1.175}
\]

for \( \hat{\varphi}(t) \) given by (1.140). This completes the proof of Theorem 2. \( \square \)

**Proof (Corollary 3).** We now assume that \( b = a \) and consider the random variable \( s_{Y_1,N} + t Y_{2,N} \) defined by (1.141) in place of \( t X_N \) in the proof of Theorem 2. By the definition (1.144) we write \( s_{Y_1,N} + t Y_{2,N} = \frac{1}{N} \left[ t L_N^t + \frac{(1-a)s^2 - (2-a)t}{(1-a)} R_N^t + \frac{t}{(1-a)} V_N \right] \). Accordingly, define

\[
r_{s,t,N} := e^{i((1-a)s - (2-a)t)/(1-a)N}), \quad y_{s,t,N} := e^{i(t-s)/(1-a)N}), \quad z_{s,t,N} := e^{t/N}.
\]

We follow the lines of the proof of Theorem 2 making simplifications due to the existence now of only one stratum, while also incorporating both parameters \( s \) and \( t \) into the calculations. It suffices to prove that, for each fixed pair of real numbers \( s, t \in \mathbb{R} \), \( \lim_{N \to \infty} g_0, N(v_{s,t,N}, y_{s,t,N}, z_{s,t,N}; a, a) \) exists and is given by the right side of (1.142). Here, we have by (1.12) that

\[
g_0, N = \frac{\omega_0}{2 - a} r_N^2 [a \tau_a]^{N-2} \left( \frac{Na - (N-1)a}{w_N^t(a)} \right). \tag{1.177}
\]

Define \( \theta = \theta_{s,t,N} \) via \( \cos \theta = \beta_s / \sqrt{4x_a} \), where the functions \( \beta_s \) and \( x_a \) are composed with the complex exponential terms in (1.176). It follows by making a calculation that

\[
\cos \theta = 1 + \frac{1}{2} \frac{(1-a)s^2 + at^2}{N^2} + O(N^{-3}); \quad \theta = \frac{i}{N} \frac{(1-a)s^2 + at^2}{N} + O(N^{-3}). \tag{1.178}
\]
Since the model is homogeneous, we need only apply the first line of (1.152) with \( f := N \) to obtain
\[
\begin{align*}
W^\ast_N(a) &= (\sqrt{x_a \sin \theta})^{-1}[az\tau]N \left\{ \sin N\theta - \sqrt{x_a}w^0_0(a) \sin(N - 1)\theta \right\}.
\end{align*}
\] (1.179)

Expand \( \sin(N - 1)\theta = s \cos \theta - c \sin \theta \), for \( s := \sin N\theta \) and \( c := \cos N\theta \).
Therefore, with \( \Lambda := (\sin \theta)^{-1}[az\tau]N^{-1} \), we have
\[
\begin{align*}
W^\ast_N(a)/\Lambda &= s - \sqrt{x_a}w^0_0(a)(c \sin \theta).
\end{align*}
\] (1.180)

We apply again a direct calculation to expand \( 1 - \sqrt{x_a}w^0_0(a) = 1 - a + O(N^{-1}) \).
In fact this follows easily by noting that, because each variable in (1.176) takes values \( 1 + O(N^{-1}) \) for fixed \( s, t \) as \( N \to \infty \), we have \( \sqrt{x_a} = a + O(N^{-1}) \) and \( w^0_0(a) = 1 + O(N^{-1}) \). But, because we just found that there is no cancellation of the order 1 term in (1.180), we need only explicitly keep order 1 terms in (1.180) and collect other terms of order \( 1/N \) or smaller in a single term \( O(N^{-1}) \).
Therefore by (1.178) and (1.180) we find:
\[
\begin{align*}
W^\ast_N(a)/\Lambda &= (1 - a)s + O(N^{-1}).
\end{align*}
\] (1.181)

Now plug (1.181) into (1.177) to obtain, recalling \( \Lambda := (\sin \theta)^{-1}[az\tau]N^{-1} \), that
\[
\begin{align*}
g_{0,N} = \frac{\omega_a}{a(2 - a)}r\tau^{-1} \frac{(Na - (N - 1)a) \sin \theta}{(1 - a)s + O(N^{-1})}.
\end{align*}
\] (1.182)

Finally apply (1.178) to (1.182) together with our definition \( s := \sin N\theta \) to obtain the conclusion of the corollary. \( \square \)

### 1.9 Limit distributions over the last visit.

For simplicity we assume in this section that \( X_0 = 0 \). Recall the definition of the excursion, \( \Gamma \), just preceding (1.2); since now \( N \) is finite it is possible that the gambler’s ruin process terminates before the process returns to zero.
Define \( M_N \) as the number of consecutive excursions of height at most \( N - 1 \) until there is an excursion of height at least \( N \). So \( M_N \) is the number of returns of the gambler’s ruin process to zero including the last visit to zero, where we recall the definition of the last visit \( L_N \) by (1.1). By the fact that the absolute value process starts afresh at the end of each excursion, we have that \( 1 + M_N \) is a standard geometric random variable (that is the values \( \nu \) of \( M_N \) start from \( \nu = 0 \)) with success probability \( P(H \geq N) \). Thus
\[
\begin{align*}
P(M_N = \nu) &= [P(H < N)]^\nu P(H \geq N), \quad \nu = 0, 1, 2, \ldots.
\end{align*}
\] (1.183)

Let \( L_N, R_N, \) and \( V_N \), respectively, be random variables for the number of steps, runs, and short runs, in an excursion, given that the height of the
excursion is at most \( N - 1 \). We define \( R_N \) and \( V_N \), respectively as the number of runs and short runs of the absolute value process \( \{ |X_j|, j \geq 0 \} \) until the last visit; the number of steps of the absolute value process until the last visit is the same as for the original process, namely \( L_N \). Therefore, in distribution, we may write:

\[
R_N = \sum_{\nu=0}^{M_N} R(\nu), \quad V_N = \sum_{\nu=0}^{M_N} V(\nu), \quad L_N = \sum_{\nu=0}^{M_N} L(\nu),
\]

where \( R(1), R(2), \ldots; V(1), V(2), \ldots; \) and \( L(1), L(2), \ldots, \) respectively, are sequences of independent copies of \( R_N, V_N, \) and \( L_N \). Since the random variables \( R_N, V_N, \) and \( L_N \) already have built into their definitions the condition \( \{ H \leq N - 1 \} \), the probability generating function \( K_N - 1 = E\{ r^{R_N} y^{V_N} z^{L_N} \} \) is calculated by Theorem 1. Thus by (1.183), and by calculating a geometric sum there holds:

\[
E\{ r^{R_N} y^{V_N} z^{L_N} u^{M_N} \} = \sum_{\nu=0}^{\infty} P(\nu) (uK_N - 1)^{\nu} = \frac{P(H \geq N)}{1 - uK_N P(H < N)},
\]

with \( K_N - 1 = K_N - 1 (r, y, z; a, b) \). By analogy with (1.141), but now with last visit statistics in place of the meander statistics, define

\[
Z_{1,N} := \frac{1}{N^2} \left( R_N - \frac{1}{1-a} V_N + a M_N \right);
\]

\[
Z_{2,N} := \frac{1}{N} \left( L_N - \frac{1}{1-a} R_N + \frac{a}{1-a} M_N \right) - Z_{1,N}.
\]

**Proposition 6.** Assume \( b = a \). Define \( Z_{1,N} \) and \( Z_{2,N} \) by (1.186). Then

\[
\lim_{N \to \infty} E\{ e^{i(sZ_{1,N} + tZ_{2,N})} \} = \frac{\tanh(\sqrt{1-a}s^2 + at^2)}{\sqrt{1-a}s^2 + at^2}.
\]

**Proof.** Denote

\[
Z_{s,t,N} := (1 + \frac{1}{N})(sZ_{1,N+1} + tZ_{2,N+1}).
\]

It suffices to show that \( \lim_{N \to \infty} E\{ e^{iZ_{s,t,N}} \} \) exists and is given by the right side of (1.187). By (1.186), we expand \( N Z_{s,t,N} \) as follows:

\[
N Z_{s,t,N} = tL_N + \frac{(1-a)s-(2-a)t}{1-a} R_N + \frac{a-s}{1-a} V_N + \left[ a(s-t) + \frac{a}{1-a} t \right] M_N,
\]

with \( N_+ := N + 1 \). Now define the complex exponentials \( r_N = r_{s,t,N}, y_N = y_{s,t,N} \), and \( z_N = z_{s,t,N} \) by (1.170). Then by (1.186), (1.186) and (1.188), with \( b = a \), we have:
\[ E\{e^{iZ_{s,t,N}}\} = \frac{C_N}{1 - P(H \leq N)e^{i(a(s-t)+\frac{\pi}{2})/N}K_N(r_N,y_N,z_N; a,a)} \]  
(1.190)

with \( C_N := P(H \geq N+1) \). Note that \( C_N \) is of order \( 1/N \) since \( P(H \leq N) = N(1-a) \). Now apply Proposition 1 to write \( P(H \leq N)K_N(r,y,z; a,a) = (1-a)r^2z^2q_N^*(a)/w_N^*(a) \). Let us also denote the centering factor
\[ c(s,t,N) := e^{i(a(s-t)+\frac{\pi}{2})/N} \]  
(1.191)

Therefore by (1.190) we obtain
\[ E\{e^{iZ_{s,t,N}}\} = \frac{C_N w_N^*(a)}{[w_N^*(a)-(1-a)c(s,t,N)r^2z^2q_N^*(a)]}, \]  
(1.192)

with \((r,y,z) = (r_{s,t,N},y_{s,t,N},z_{s,t,N})\). Now as in the proof of Corollary 3 define \( \theta = \theta_{s,t,N} \) by \( \cos \theta = \beta_a/\sqrt{4x_a} \), where the functions \( \beta_a \) and \( x_a \) are composed with the complex exponential terms in (1.176). Thus (1.178) holds. After a bit of simplification, using (1.132) to substitute for \( w_N^*(a) \) and \( q_N^*(a) \), and after a cancellation of a factor \( A = (\sin \theta)^{-1}[ax_\theta]^N \) in numerator and denominator of (1.192), we have that the right side of (1.192) is given by:
\[ \sin N\theta - \sqrt{x_0^*}\sin N_\theta \]
\[ \sin N\theta - \sqrt{x_0^*}\sin N_\theta - (1-a)c(s,t,N)r^2z^2\{y^2\sin N\theta - \sqrt{x_0^*}\sin N_\theta\}, \]  
(1.193)

where we put \( N_- := N-1 \), and also abbreviate \( x = x_a, w_0^* = w_0^*(a) \), and \( q_0^* = q_0^*(a) \). For convenience, write \( s := \sin N\theta \) and \( c := \cos N\theta \). Also expand \( \sin N_- \theta = s \cos \theta - c \sin \theta \). Thus (1.193) becomes
\[ s - \sqrt{x_0^*}(s \cos \theta - c \sin \theta) \]
\[ s - \sqrt{x_0^*}(s \cos \theta - c \sin \theta) - (1-a)c(s,t,N)r^2z^2\{y^2s - \sqrt{x_0^*}(s \cos \theta - c \sin \theta)\} \]  
(1.194)

Now we collect terms involving \( s \) and those involving \( c \) separately. In the numerator, by (1.179)–(1.181), we have
\[ s - \sqrt{x_0^*}(s \cos \theta - c \sin \theta) = (1-a)s + O(N^{-1}). \]  
(1.195)

However, there is a second order cancelation in a portion of the denominator of (1.194) that we find by direct calculation as follows:
\[ s \left( 1 - \sqrt{x_0^*}\cos \theta - (1-a)c(s,t,N)r^2z^2\{y^2 - \sqrt{x_0^*}\cos \theta\} \right) = O(N^{-2}), \]  
(1.196)

where we have used (1.176)–(1.178) to write \( \cos \theta = 1 + O(N^{-2}) \) and applied (1.191). We note that the factor \( c(s,t,N) \) is a centering term, corresponding to the centering terms \( a\mathcal{M}_N \) and \( \frac{q}{\pi}\mathcal{M}_N \) in (1.186), such that the \( O(N^{-1}) \) terms in (1.196) cancel; already the order 1 terms cancel in (1.196) even without the centering term. Thus the main contribution to the denominator of (1.194) is, by (1.196),
\[ \sqrt{X} u_0^* c \sin \theta = c(a + O(N^{-1})) \sin \theta = ac \sin \theta + O(N^{-2}), \quad (1.197) \]

because \( q_0 = O(N^{-1}) \) and so \( \sqrt{X} q_0^* \sin \theta = O(N^{-2}) \). Hence by (1.195) and (1.196), we obtain by (1.190)–(1.193) that

\[ E \{ e^{iZ_{t,N}} \} \sim P(H \geq N + 1) \frac{(1 - a)s + O(N^{-1})}{ac \sin \theta + O(N^{-2})}, \quad \text{as } N \to \infty, \quad (1.198) \]

Here, \( P(H \geq N + 1) \sim \frac{a}{1 - a} N^{-1} \), and by (1.178), \( s \sim i \sinh(\sqrt{(1 - a)s^2 + at^2}) \), \( c \sim \cosh(\sqrt{(1 - a)s^2 + at^2}) \), and \( \sin \theta \sim iN^{-1} \sqrt{(1 - a)s^2 + at^2} \). Thus by these observations and (1.198), we obtain the conclusion of the proposition.

\[ \square \]

We finally state and prove a theorem for the full model that gives a scaling limit of order \( N \) over the last visit portion of the gambler’s ruin. This is a companion result to Theorem 2 as follows. Define the following scaled random variable:

\[ \mathcal{X}_N := \frac{1}{N} \left( \mathcal{L}_N - \frac{2 - a - b}{(1 - a)(1 - b)} \mathcal{R}_N + \frac{1}{(1 - a)(1 - b)} \mathcal{V}_N - \frac{a(b - a)}{(1 - a)(1 - b)} \mathcal{M}_N \right). \quad (1.199) \]

**Theorem 3.** Let \( f \sim \eta N \), as \( N \to \infty \), for some fixed \( 0 < \eta < 1 \). Let \( \mathcal{X}_N \) be defined by (1.199). Let also \( \kappa_j, j = 1, 2 \), and \( \sigma_j, j = 1, 2 \), be as defined in Theorem 2. Let \( \bar{\varphi} \) be defined by (1.140). Then,

\[ \lim_{N \to \infty} E \{ e^{it \mathcal{X}_N} \} = \hat{\varphi}(t)/\bar{\varphi}(t), \quad \text{where} \]

\[ (ab \sigma_1 \sigma_2) \hat{\varphi}(t)^{-1} := ab \sigma_1 \sigma_2 \cosh(\kappa_1 t) \cosh(\kappa_2 t) + a^2 \sigma_1^2 \sinh(\kappa_1 t) \sinh(\kappa_2 t) \]

\[ + i a \sigma_1 (b - a)^2 \cosh(\kappa_1 t) \sinh(\kappa_2 t). \quad (1.200) \]

**Proof.** It suffices to show that \( \lim_{N \to \infty} E \{ e^{it(1 + 1/N) \mathcal{X}_N} \} = \hat{\varphi}(t)/\bar{\varphi}(t) \). We will combine the calculation of \( \mathcal{w}_{0,N} \) of Theorem 2 with the lines of proof of Proposition 6 but with Theorem 1 in place of Proposition 6 in those lines of proof, immediately above. We define \( (r_N, y_N, z_N) \) by (1.145). We define the substitutions \( \theta_1 \) and \( \theta_2 \) by (1.148), so that also (1.149–1.151) hold. By the statement of Theorem 1 of Section 1.6 we must replace the calculation of \( \mathcal{w}_{0,N} \), starting with (1.147) in the proof of Theorem 2 with instead \( \mathcal{w}_{1,N+1} \). However, by (1.111) and (1.117), the difference in the two calculations is simply accounted for by replacing \( f \) by \( f - 1 \) in the calculation of \( \mathcal{w}_{0,N} \). Furthermore, we must now also calculate \( \mathcal{w}_N \) given by (1.107), and, by comparing (1.107) and (1.111), this amounts to replacing \( f \) by \( f - 1 \) and \( (w_j^*(a), w_j+1^*(a)) \) by \( (q_j^*(1/a), q_j^*(1/a)) \) in (1.147), since in (1.107), \( f + j - 1 = N \) is the same as \( j = N - (f - 1) \). Thus, because we simply replace \( f \) by \( f' := f - 1 \), we may use the calculation of \( \mathcal{w}_{0,N} \) in (1.147–1.172) verbatim
in place of the calculation of \( \varpi_{1,N+1} \), and we will do this without changing the names of \( d_j \), \( j = 1, 2 \) and \( A_j \), \( j = 1, 2 \) (see (1.111) and (1.153) and (1.162)). However it turns out that we now need to keep track of the precise order \( N^{-2} \) term in (1.172). To calculate \( \varpi_N \), we use (1.107)–(1.110) together with (1.202), and so follow by analogy the calculation of \( d_1 \) and \( d_2 \) in (1.162)–(1.164); we thus obtain \( d_{q,1} \) and \( d_{q,2} \) of (1.201). We will also have to keep track of the precise \( N^{-2} \) term in \( \varpi_N \). In summary we have that (1.172) holds for \( \varpi_{1,N+1} \) in place of \( \varpi_{0,N} \), where we will explicitly write out the \( O(N^{-2}) \) term below, and where the product \( A_1A_2 \) will appear the same in both \( \varpi_{1,N+1} \) and \( \varpi_N \).

We now proceed to establish the asymptotics for \( d_{q,1}/A_1 \) and \( d_{q,2}/A_1 \), for the vector \( \mathbf{d}_q \) of (1.111) that yields (1.107), so that

\[
d_{q,1} := \mu_{1,1} q_{j}^* + \mu_{1,2} q_{j+1}^*(a); \quad d_{q,2} := \mu_{2,1} q_{j}^* + \mu_{2,2} q_{j+1}^*(a). \tag{1.201}
\]

Here by (1.132) and (1.148)–(1.149), in analogy with (1.152), we have

\[
q_{j}^*(a) = 2i\alpha^{-1}a x_{a} f' \{ y^2 \sin f' \theta_1 - \sqrt{x_{a} q_{0}^*(a) \sin(f' - 1) \theta_1} \}
\]

\[
q_{j+1}^*(a) = 2i\alpha^{-1}a x_{a} f' \sqrt{x_{a}} \{ y^2 \sin(f' + 1) \theta_1 - \sqrt{x_{a} q_{0}^*(a) \sin(f' \theta_1} \}.
\]

Therefore, by (1.201)–(1.202), we have that, with \( A_1 \) defined by (1.153) with \( f' \) in place of \( f \), the term \( d_{q,j}/A_1 \) is given for each \( j = 1, 2 \) by

\[
(y^2 \mu_{j,1} - \mu_{j,2} x_{a} q_{0}^*) \sin f' \theta_1 + \sqrt{x} \{ y^2 \mu_{j,2} \sin(f' + 1) \theta_1 - \mu_{j,1} x_{a} q_{0}^* \sin(f' - 1) \theta_1} \]

(1.203)

where \( x = x_{a} \) and \( q_{0}^* = q_{0}^*(a) \) and all functions are composed with \( (r_N, y_N, z_N) \) of (1.145). We rewrite (1.203) by applying (1.155) exactly as in the proof of Theorem 1, but with \( f' \) in place of \( f \), without changing the names of \( s_1 \) and \( c_1 \), so explicitly \( s_1 = \sin f' \theta_1 \), and \( c_1 = \cos f' \theta_1 \). Thus by (1.203), \( d_{q,j}/A_1 \) is written for each \( j = 1, 2 \) by

\[
d_{q,j}/A_1 = (y^2 \mu_{j,1} - \mu_{j,2} x_{a} q_{0}^*) s_1
\]

\[
+ \sqrt{x} \{ y^2 \mu_{j,2} (s_1 \cos \theta_1 + c_1 \sin \theta_1) - \mu_{j,1} x_{a} q_{0}^* (s_1 \cos \theta_1 - c_1 \sin \theta_1) \}
\]

(1.204)

To guide the asymptotic expansions of the right sides of (1.203) we invoke the proof of Proposition 6. We rewrite (1.192) due to the replacement of Proposition 6 by Theorem 1 and by (1.199), as follows:

\[
E \{ e^{\alpha(1 + 1/N) x_N} \}
\]

\[
= P(H \geq N + 1) \varpi_{1,N+1} / \left[ \varpi_{1,N+1} - (1 - a) e^{-a/(1 - a) N^{1/2} + \alpha(N - 1) N^{-1/2}} \right].
\]

(1.205)
Now it turns out, just as in the proof of Proposition 6, that there is a cancellation in the order of the denominator of (1.205). That is, the leading order of each of $\overline{w}_{1,N+1}/(A_1A_2)$ and $\overline{q}_N/(A_1A_2)$ will be some order 1 trigonometric factor times $it/N$; in fact there holds $(1-a)\overline{q}_N/\overline{w}_{1,N+1} \sim 1$, as $N \to \infty$. Define

$$\Delta_N := \overline{w}_{1,N+1} - (1-a)c(t,N)\mu^2 t^2 \overline{q}_N, \quad c(t,N) := e^{-\frac{a(b-a)}{(1-a)(1-b)}it/N}. \quad (1.206)$$

We will establish that $\Delta_N/(A_1A_2) = O(N^{-2})$, and we will find the exact coefficient of the order $N^{-2}$ term. Note that by (1.118) and Proposition 2 $P(H \geq N+1) \sim C_{a,b}N^{-1}$. Also we already found the exact coefficient for order $N^{-1}$ of $\overline{w}_{1,N+1}/(A_1A_2)$ in (1.172). Therefore by the procedure outlined in this paragraph we will obtain a non-trivial limit for (1.205).

We now reduce both (1.156) and (1.204) by the device of collecting certain lower order terms. The reason we have to treat (1.156) again is due to the cancellation described in the previous paragraph. Indeed we must collect error terms of order $O(N^{-1})$ that will not contribute to the limit in the end and give precise coefficients for the main terms through order $N^{-2}$. Thus we must be more careful with the error terms than in the proof of Theorem 2, wherein we could throw away terms of order $O(N^{-2})$. Now in (1.156) and (1.204), by (1.150) and (1.151), we may still treat $\sin \theta_1 = \theta_1 + O(N^{-3})$, but we must keep explicit the $O(N^{-2})$ term in $\cos \theta_1 = 1 + O(N^{-2})$. For the sake of organization, we follow the same approach as in the proof of Theorem 2 even though there are many terms to display. At the end of the proof we offer an alternative approach which relies solely on machine computation. We first update the asymptotic expressions of (1.157) and (1.159); the asymptotic formulæ (1.158) and (1.160) for the coefficients of $c_1 \sin \theta_1$ stand as previously written since $\sin \theta_1 = O(N^{-1})$. The coefficients of $s_1$ for $d_1/A_1$ and $d_2/A_1$ are expanded as follows, by using the exact formulæ in (1.157) and (1.159):

$$[s_1]\left[\frac{d_1}{A_1}\right] = -(1-a)(1-b) + 2(1-ab)\mu^2 t^2 \frac{\mu^2}{N} + \frac{a^2 - a^3 - 3ab - 9b^2 + 2a^2b^2}{(1-a)(1-b)} \frac{t^2}{N^2} + O(1) \frac{t^2}{N^3}. \quad (1.207)$$

$$[s_1]\left[\frac{d_2}{A_1}\right] = 1 - a - \frac{a(b-a)it}{1-b} - \frac{(-a^2 - 5a^3 + 2a^4 + 8a^5b + 4a^6b + 2ab^2 - 10a^2b^2)}{2(1-a)(1-b)^2} \frac{t^2}{N^2} + O(1) \frac{t^2}{N^3}. \quad (1.208)$$

Here we remind that $d_1$ and $d_2$ appear in Proposition 3 and will be used to compute $\overline{w}_{1,N+1}$ via (1.111) where $d_j = d_j(f') = \mu_{j,1}w_{j'}(a) + \mu_{j,2}w_{j'+1}(a)$, $j = 1, 2$.

We next obtain precise asymptotics for the exact formulæ in (1.204) by using direct calculations:

$$[s_1]\left[\frac{d_j}{A_1}\right] = y^2 \mu_{1,1} - \mu_{1,2}x_aq_0(a) + \sqrt{x_a}(y^2 \mu_{1,2} - \mu_{1,1}q_0(a)) \cos \theta_1

= -(1-b) + \frac{a^2 - abit}{(1-a)} + \frac{(-3a^2 + 2ab + 4a^2b)}{2(1-a)(1-b)^2} \frac{t^2}{N^2} + O(1) \frac{t^2}{N^3}. \quad (1.209)$$
Apply (1.107) and (1.111) to write calculation using the exact formulae in (1.165)–(1.166), we obtain
\[ \Delta s \text{ and take } \min \text{ed by (1.151):} \]

Thus (1.213)–(1.216) extend (1.163)–(1.166) and we thereby extend (1.170).

We must still make (1.170) more precise in the reduction (1.165)–(1.168), with \( f' \) in place of \( f \). Thus without changing the names, define \( A_2 \) by (1.162) and take \( s_2 \) and \( c_2 \) as defined by (1.164), all with \( f' \) in place of \( f \). By direct calculation using the exact formulae in (1.165)–(1.166), we obtain

\[ [s_2] \left( \frac{q_{N-r}^{(b)}}{A_2} \right) = 1 + \frac{2 \sin \theta}{1-a} N - \frac{(2-3b+2ab+b^2) N^2}{2(1-a)^2} + O(1). \]

\[ [c_2 \sin \theta_2] \left( \frac{q_{N-r}^{(b)}}{A_2} \right) = \frac{2b}{1-a} \frac{N}{N^2} + O(1). \]

\[ [s_2] \left( \frac{w_{N-r}^{(b)}}{A_2} \right) = 1 - b - \frac{(2a-b) N}{1-a} + \frac{2b}{1-a} \frac{N}{N^2} + O(1). \]

\[ [c_2 \sin \theta_2] \left( \frac{w_{N-r}^{(b)}}{A_2} \right) = b + \frac{2b}{1-a} \frac{N}{N^2} + O(1). \]

Thus (1.213)–(1.216) extend (1.165)–(1.166) and we thereby extend (1.170). So, via (1.213)–(1.216) we now have asymptotic expansions for \( q_{N-r}^{(b)}(A_2) \) and \( w_{N-r}^{(b)}(b) \) with error term \( O(N^{-3}) \).

We pass now to the calculation of \( \Delta_N/(A_1A_2) \) for \( \Delta_N \) defined by (1.206). Apply (1.107) and (1.111) to write \( \overline{\Pi}_N = d_{q,1} q_{N-r}^{(b)}(b) + d_{q,2} w_{N-r}^{(b)}(b) \) and \( \overline{\Pi}_{1,N+1} = d_1 q_{N-r}^{(b)}(b) + d_2 w_{N-r}^{(b)}(b) \). Expand the factor in front of \( \overline{\Pi}_N \) in the definition (1.206) of \( \Delta_N \):

\[ (1-a)c(t,N) \Delta_N^2 = 1 - a + \frac{2a^2 + ab}{1-b} \frac{N}{N^2} + \frac{(2a^2 + ab)^2}{2(1-a)(1-b)^2} + O(1). \]

Finally, apply (1.158), (1.160), (1.207)–(1.216) and (1.217) to directly calculate \( \Delta_N/(A_1A_2) \), by substituting \( \sin \theta_j = \theta_j + O(N^{-3}), j = 1,2, \) as determined by (1.151):

\[ \frac{\Delta_N}{A_1A_2} = \frac{1}{(1-a)(1-b)} \frac{\overline{\Pi}_N^2}{N^2} \{ -ab \sigma_1 \sigma_2 c_1 c_2 - a \sigma_1 (a-b)^2 c_1 s_2 - a^2 \sigma_1^2 s_1 s_2 \} + O(1). \]

Another way to handle the asymptotic expansion of \( \Delta_N/(A_1A_2) \) by machine calculation is to write out a formula for this expression. Consider \( \overline{\Pi}_N/(A_1A_2) \) for \( \overline{\Pi}_N = d_{q,1} q_{N-r}^{(b)}(b) + d_{q,2} w_{N-r}^{(b)}(b) \). By writing \( \cos \theta_1 = \beta_a/\sqrt{\tan a} = \beta_a/(az \tau_a) \) and \( \cos \theta_2 = \beta_b/(bz \tau_b) \) we can express our formulae (1.163)–(1.166) and (1.204) back in terms of variables \( r, y, z \), with all rational function terms in these variables, except for \( \sin \theta_1 \) and \( \sin \theta_2 \), which
each involve a square root. We treat $s_j$ and $c_j$, $j = 1, 2$, as independent variables. We avoid writing $\sin \theta_j$, $j = 1, 2$ in terms of $(r, y, z)$, because we can simply substitute $\theta_j$ for $\sin \theta_j$ in its asymptotic form given by (1.151), and because the square root will be unwieldy in machine output. So we introduce auxiliary variables to hold the places of $\theta_1$ and $\theta_2$, and simply substitute using (1.151) at the same time that we compose our formula with $(r_N, y_N, z_N)$ of (1.145) to obtain a function of $t$. We also ask the machine for a Taylor series in $t$ about $t = 0$ and to collect the terms of this series according to the order of $N$; the resulting series will be linear in $s_j$ and $c_j$. We apply this approach to the whole formula for $\Delta_N/(\Lambda_1 \Lambda_2)$, including the factor (1.217), and again obtain the result (1.218).

Finally we compute the limit of the ratio (1.205) by the asymptotic relations (1.148), and by (1.172) and (1.218). Thus, because by (1.118) and Proposition 2 we have that

$$P(H \geq N + 1) \sim C_{a,b} N^{-1}$$

for $C_{a,b} = ab/[(1 - \eta)a + \eta b - ab]$, we find:

$$E\left\{e^{it(1+1/N)X_N}\right\} \sim C_{a,b} N^{-1} \left\{\left[ a\sigma_1 c_1 s_2 + b\sigma_2 c_1 c_2 + (b - a)^2 s_1 s_2 \right] t^2 + O(N^{-2}) \right\}$$

$$\div \left\{ \frac{1}{(1-a)(1-b)} \left[ -ab\sigma_1 \sigma_2 c_1 c_2 - a\sigma_1 (a-b)^2 c_1 s_2 + a^2 \sigma_1^2 s_1 s_2 \right] t^2 + \frac{O(1)}{N} \right\}$$

(1.219)

But, as in the proof of Theorem 2 we have $c_j \sim \cosh(\kappa_j t)$, and $s_j \sim i \sinh(\kappa_j t)$, $j = 1, 2$. Therefore, with $\tilde{C}_{a,b} := (1 - a)(1 - b)C_{a,b}$, we obtain that (1.219) has the following limit as $N \to \infty$, where we refer to (1.140) and (1.200) for the definitions of $\tilde{\varphi}(t)$ and $\tilde{\psi}(t)$:

$$\lim_{N \to \infty} E\left\{e^{it(1+1/N)X_N}\right\} = \frac{\tilde{C}_{a,b}}{t} \times \frac{(b\kappa_1 \sigma_2 + a\kappa_2 \sigma_1) t}{\tilde{\varphi}(t)} \times \frac{\tilde{\psi}(t)}{ab\sigma_1 \sigma_2}. \quad (1.220)$$

We have $\tilde{C}_{a,b} = ab\sigma_1 \sigma_2/(a\sigma_1 \kappa_2 + b\sigma_2 \kappa_1)$, so by (1.220) the proof is complete. \hfill \Box

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