A proposal for implementing an $n$-qubit controlled-rotation gate with three-level superconducting qubit systems in cavity QED

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Received 24 February 2011, in final form 19 April 2011
Published 19 May 2011
Online at stacks.iop.org/JPhysCM/23/225702

Abstract

We present a method for implementing an $n$-qubit controlled-rotation gate with three-level superconducting qubit systems in cavity quantum electrodynamics. The two logical states of a qubit are represented by the two lowest levels of each system while a higher energy level is used for the gate implementation. The method operates essentially by preparing a $W$ state conditioned on the states of the control qubits, creating a single photon in the cavity mode, and then performing an arbitrary rotation on the states of the target qubit with the assistance of the cavity photon. It is interesting to note that the basic operational steps for implementing the proposed gate do not increase with the number of qubits $n$, and the gate operation time decreases as the number of qubits increases. This proposal is quite general, and can be applied to various types of superconducting devices in a cavity or coupled to a resonator.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Multiqubit controlled gates are of importance in constructing quantum computation circuits and quantum information processing. A multiqubit controlled gate can in principle be decomposed into single-qubit and two-qubit gates and thus can be built based on these elementary gates. However, when using the conventional gate-decomposition protocols to construct a multiqubit controlled gate [1–3], the procedure usually becomes complicated as the number of qubits increases. This is because single-qubit and two-qubit gates, required for constructing a multiqubit controlled gate, heavily depend on the number of qubits. Therefore, finding a more efficient way to implement multiqubit controlled gates becomes important.

During the past few years, based on the cavity quantum electrodynamics (QED) technique, several theoretical proposals for implementing an $n$-qubit controlled-phase gate with ion traps, superconducting qubits coupled to a resonator or atoms trapped in a cavity have been presented [4–10]. These previous works opened up a new way for the physical implementation of multiqubit controlled-phase (or controlled-NOT) gates, which play a significant role in quantum information processing, such as quantum algorithms [11, 12] and quantum error correction protocols [13]. On the other hand, experimental realization of a three-qubit controlled-phase gate in NMR quantum systems has been reported [14]. Moreover, a three-qubit quantum gate in trapped ions has been experimentally demonstrated recently [15].

The existing proposals in [4–10] for realizing an $n$-qubit controlled-phase gate cannot be extended to realize an $n$-qubit controlled-rotation gate (denoted as controlled-$R$ gate below). Note that multiqubit controlled-$R$ gates are useful in quantum information processing. For instance, they can be used to construct quantum circuits for: (i) general multiqubit gates [2], (ii) preparation of arbitrary pure quantum states of multiple qubits [16], (iii) transformation of quantum states of multiple qubits [17], and (iv) quantum error correction [18]. In addition, multiqubit controlled-$R$ gates can be applied to...
construct quantum circuits for implementation of quantum algorithms [19] and quantum cloning [20], and so on. In this work, we will focus on how to realize multiqubit controlled-\(R\) gates with superconducting qubit systems. As is well known, superconducting devices have recently appeared to be among the most promising candidates for building quantum information processors, due to their design flexibility, large-scale integration, and compatibility to conventional electronics.

We note that if an \(n\)-qubit controlled-\(R\) gate is constructed by using the conventional gate-decomposition protocols, \(2^n - 3\) two-qubit controlled gates would be needed (for \(n \geq 3\)) [1]. Thus, assuming that realizing any two-qubit controlled gate requires one-step operation only, at least \(2^n - 3\) steps of operations are required, which increase with the number of qubits \(n\) exponentially. For instance, by using the conventional gate-decomposing protocols, 29 basic operational steps are required to implement a five-qubit controlled-\(R\) gate, and 61 basic operational steps are required to realize a six-qubit controlled-\(R\) gate.

In the following, we will propose a way to implement an \(n\)-qubit controlled-\(R\) gate with three-level superconducting qubit systems in cavity QED. The method operates essentially on the basis of this idea: prepare a superposition state used by the following transformation:

\[
|i_1i_2\cdots i_{n-1}i_n\rangle \rightarrow \begin{cases} |i_1i_2\cdots i_{n-1}R(\theta)i_n\rangle, & \text{if } \prod_{k=1}^{n-1} i_k = 1 \\ |i_1i_2\cdots i_{n-1}\rangle, & \text{if } \prod_{k=1}^{n-1} i_k = 0 \end{cases}
\]  

(2)

for all \(i_1, i_2, \ldots, i_n \in \{0, 1\}\). Here, the subscripts 1, 2, \ldots, and \(n - 1\) represent the \(n - 1\) control qubits (1, 2, \ldots, \(n - 1\)) while the subscript \(n\) represents the target qubit \(n\); and \(|i_1i_2\cdots i_{n-1}\rangle\) is the \(n\)-qubit computational basis state. The operator \(R(\theta)\) is described by the following matrix:

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]  

(3)

in a single-qubit computational subspace formed by the two logic states \(|0\rangle = (1, 0)^T\) and \(|1\rangle = (0, 1)^T\) of the target qubit \(n\). It can be seen from equation (2) that if and only if the \(n - 1\) control qubits (1, 2, \ldots, \(n - 1\)) are all in the state \(|1\rangle\), a unitary rotation \(R(\theta)\) is performed on the two logic states \(|0\rangle\) and \(|1\rangle\) of the target qubit \(n\). The definition of the \(n\)-qubit controlled-\(R\) gate here is also shown in figure 1.

2. \(N\)-qubit controlled-\(R\) gate

For \(n\) qubits, there are a total of \(2^n\) computational basis states, which form a set of complete orthogonal bases in a \(2^n\)-dimensional Hilbert space of the \(n\) qubits. A quantum controlled-\(R\) gate of \(n\) qubits considered in this paper is defined by the following transformation:

\[
|i_1i_2\cdots i_{n-1}i_n\rangle \rightarrow \begin{cases} |i_1i_2\cdots i_{n-1}R(\theta)i_n\rangle, & \text{if } \prod_{k=1}^{n-1} i_k = 1 \\ |i_1i_2\cdots i_{n-1}\rangle, & \text{if } \prod_{k=1}^{n-1} i_k = 0 \end{cases}
\]  

(2)

for all \(i_1, i_2, \ldots, i_n \in \{0, 1\}\). Here, the subscripts 1, 2, \ldots, and \(n - 1\) represent the \(n - 1\) control qubits (1, 2, \ldots, \(n - 1\)) while the subscript \(n\) represents the target qubit \(n\); and \(|i_1i_2\cdots i_{n-1}\rangle\) is the \(n\)-qubit computational basis state. The operator \(R(\theta)\) is described by the following matrix:

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]  

(3)

in a single-qubit computational subspace formed by the two logic states \(|0\rangle = (1, 0)^T\) and \(|1\rangle = (0, 1)^T\) of the target qubit \(n\). It can be seen from equation (2) that if and only if the \(n - 1\) control qubits (1, 2, \ldots, \(n - 1\)) are all in the state \(|1\rangle\), a unitary rotation \(R(\theta)\) is performed on the two logic states \(|0\rangle\) and \(|1\rangle\) of the target qubit \(n\). The definition of the \(n\)-qubit controlled-\(R\) gate here is also shown in figure 1.

3. Preparation of the \(W\) state conditioned on the states of the controls

In this section, we will discuss how to prepare the \(W\) state given in equation (1), when the \(n - 1\) control qubits are initially in a computational basis state \(|11\cdots 1\rangle\). For the purpose of the gate, the remaining \(2^n-1\) \(1\) computational states of the \(n - 1\) control qubits need not to be affected during the \(W\) state preparation. In this section, we will also give a discussion on how this can be achieved.

The superconducting qubit systems have the three levels shown in figure 2. Note that the three-level structure in figure 2(a) applies to superconducting charge-qubit or flux-qubit systems [23, 24] and the one in figure 2(b) applies to phase-qubit systems [25, 26]. In addition, the three-level structure in figure 2(b) is also available in atoms. In figure 2,
the transition between the two lowest levels $|0\rangle$ and $|1\rangle$ is assumed to: (i) be forbidden due to the selection rules, (ii) be very weak due to the potential barrier between the two lowest levels, or (iii) be highly detuned (decoupled) from the cavity mode during the gate operation. Throughout this paper, the two logic states of a qubit are represented by the two lowest levels $|0\rangle$ and $|1\rangle$.

To simplify our presentation, we will restrict our discussion to the three-level structure in figure 2(a). However, it should be mentioned that the results presented in this section and the method proposed in section 4 for the gate implementation are applicable to the quantum systems with the three-level structure depicted in figure 2(b).

Consider $n-1$ three-level superconducting qubit systems $(1, 2, \ldots, n-1)$ in a single-mode cavity. The $n-1$ systems play the role of controls in realization of the $n$-qubit controlled-$R$ gate, discussed in section 4. Assume that the cavity mode is coupled to the $|1\rangle \leftrightarrow |2\rangle$ transition but does not affect the level $|0\rangle$ (figure 3), which can be achieved by prior adjustment of the level spacings [23–27]. The Hamiltonian is given by (assuming $\hbar = 1$)

$$\hat{H} = \omega_0 \hat{S}_z + \omega_{c} a^+ a + g(a^+ S^- + a S^+),$$

(4)

where $\hat{S}_z = \frac{1}{2} \sum_{j=1}^{n-1} (|2\rangle_j \langle 2| - |1\rangle_j \langle 1|)$, $S^+ = \sum_{j=1}^{n-1} |2\rangle_j \langle 1|$, $S^- = \sum_{j=1}^{n-1} |1\rangle_j \langle 2|$, $a$ and $a^+$ are the photon creation and annihilation operators for the cavity mode, $\omega_{c}$ is the cavity-mode frequency, $\omega_0$ is the $|1\rangle \leftrightarrow |2\rangle$ transition frequency, and $g$ is the coupling constant between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition.

In the case when the detuning $\Delta_c = \omega_{c} - \omega_0 \gg g\sqrt{n} + 1$ with $\bar{n}$ being the mean photon number of the cavity mode, the Hamiltonian (4) can be rewritten as follows [31]:

$$\hat{H} = \omega_0 \hat{S}_z + \omega_{c} a^+ a - \lambda \sum_{j=1}^{n-1} (|2\rangle_j \langle 2| - |1\rangle_j \langle 1|)a^+ a - \lambda S^+ S^-, \quad \lambda = \frac{g^2}{\Delta_c} \tag{5}$$

where $\lambda = \frac{g^2}{\Delta_c}$. The third term describes the photon-number-dependent Stark shift while the last term describes the dipole coupling among the $n-1$ qubit systems. If the cavity mode is initially in the vacuum state $|0\rangle$, the Hamiltonian (5)
Dicke states properties:

Equation (8) demonstrates that the Dicke state reduces to Figure 4.

basis state $|\text{system} \rangle$ is the eigenstate of the Hamiltonian $H$ 

If the $n-1$ control qubits are initially in the computational basis state $|11\cdots1\rangle$, i.e. the Dicke state $|J, -J\rangle$ with $J = (n-1)/2$, they evolve within the symmetric Dicke subspace spanned by $|J, -J\rangle, |J, -J+1\rangle, \ldots, |J, J\rangle$. Here, the state $|J, -J+k\rangle$ with $k = 0, 1, \ldots, n-1$ is a symmetric Dicke state with $k$ systems being in the state $|2\rangle$ while $n-k-1$ systems being in the state $|1\rangle$. The Dicke state $|J, -J+1\rangle$ is given by

$$|J, -J + k\rangle = \frac{1}{\sqrt{n-1}} \sum P_k |1^{(n-k-1)}2^{2k}\rangle$$

where $P_k$ is the symmetry permutation operator for systems $1, 2, \ldots, n-1$, $\sum P_k |1^{(n-k-1)}2^{2k}\rangle$ denotes the totally symmetric state in which $n-k-1$ of systems $1, 2, \ldots, n-1$ are in the state $|1\rangle$ while the remaining $k$ systems are in the state $|2\rangle$.

One can check that the Hamiltonian $H_0$ has the following properties:

$$H_0|J, -J + k\rangle = \varepsilon_k |J, -J + k\rangle,$$

where

$$\varepsilon_k = \omega_0(-J + k) - k(2J - k + 1)\lambda.$$

Equation (8) demonstrates that the Dicke state $|J, -J + k\rangle$ is the eigenstate of the Hamiltonian $H_0$ with the eigenvalue $\varepsilon_k$. The energy-level spacing between $|J, -J + k\rangle$ and $|J, -J + k + 1\rangle$ is $\varepsilon_{k+1} - \varepsilon_k = \omega_0 - 2(J - k)\lambda$, depending on the excitation number of the state $|J, -J + k\rangle$. It can be seen that the energy-level spacings in the symmetric Dicke subspace are unequal (figure 4(a)). For a detailed discussion see [32].

To prepare the $W$ state of equation (1), we now apply an external driving pulse (with frequency $\omega$) to the $n-1$ systems $(1, 2, \ldots, n-1)$. Suppose that the pulse is coupled to the $|1\rangle \leftrightarrow |2\rangle$ transition but far-off resonant with the transition between any other two levels of each system (figure 3). Thus, the interaction Hamiltonian between the pulse and the $n-1$ systems is given by

$$H_{sp} = \Omega (e^{-i\omega S_z + e^{i\omega S_z})},$$

where $\Omega$ is the Rabi frequency of the pulse. The Hamiltonian for the whole system is

$$\tilde{H} = H_0 + H_{sp}.$$  

Performing the transformation $U = e^{i\omega S_z}$, we obtain the engineered Hamiltonian in the symmetric Dicke subspace

$$H' = U \tilde{H} U^+ - \omega S_z = \sum_{k=0}^{n-1} \delta_k |J, -J + k\rangle\langle J, -J + k|$$

$$+ \sum_{k=0}^{n-2} \Omega_k |J, -J + k + 1\rangle\langle J, -J + k| + \text{H.c.},$$

where

$$\Omega_k = \Omega \sqrt{(2J-k)(k+1)},$$

$$\delta_k = \omega_0(-J + k) - k(2J - k + 1)\lambda - \omega(-J + k).$$

Now assume that the applied pulse is resonant with the transition between the Dicke states $|J, -J\rangle$ and $|J, -J + 1\rangle$ (figure 4(b)). Namely, the pulse frequency $\omega$ is set by $\omega = \omega_0 - 2J\lambda$, i.e. $\Delta_p = \omega_0 - \omega = 2J\lambda$ (figure 3). Discarding the constant energy $-2J^2\lambda$, we have $\delta_0 = \delta_1 = 0$ and $\delta_k = k(k-1)\lambda(k \geq 2)$. Hence the detuning of the pulse frequency with the transition frequency between the two energy levels $|J, -J + 1\rangle$ and $|J, -J + 2\rangle$ is $2\lambda$, (figure 4(b)). Therefore, if we have $\Omega \sqrt{2J-1} \ll 2\lambda$, which is guaranteed by setting

$$\Omega \sqrt{n-1} \ll \lambda,$$
then the transition between the two Dicke states $|J, -J + 1\rangle$ and $|J, -J + 2\rangle$ is negligible due to far-off-resonance with the pulse. As a result, when the systems $(1, 2, \ldots, n - 1)$ are initially in the Dicke state $|J, -J\rangle$ or $|J, -J + 1\rangle$, the Dicke state $|J, -J + 2\rangle$ will not be excited by the pulse and therefore no transition from the state $|J, -J + 2\rangle$ to any one of the Dicke states $|J, -J + 3\rangle, |J, -J + 4\rangle, \ldots, |J, J\rangle$ occurs. The Hamiltonian $\mathcal{H}'$ thus reduces to

$$\mathcal{H}' = \Omega\sqrt{2J}(|J, -J + 1\rangle\langle J, -J| + \text{h.c.}).$$

(15)

It is straightforward to show from equation (15) that the states $|J, -J\rangle$ and $|J, -J + 1\rangle$ evolve as follows:

$$|J, -J\rangle \rightarrow \cos(\sqrt{2J}\Omega t)|J, -J\rangle - i\sin(\sqrt{2J}\Omega t)|J, -J + 1\rangle,$$

$$|J, -J + 1\rangle \rightarrow \cos(\sqrt{2J}\Omega t)|J, -J + 1\rangle - i\sin(\sqrt{2J}\Omega t)|J, -J\rangle.$$

(16)

(17)

Based on equations (1) and (7), it can be seen that the Dicke state $|J, -J + 1\rangle$ is the W state defined in equation (1).

From equation (16), it can be seen that when the control qubits $(1, 2, \ldots, n - 1)$ are initially in the state $|J, -J\rangle$ (i.e. the computational state $|1\cdots1\rangle$), the W state state $|J, -J + 1\rangle$ is prepared through a transformation $|J, -J\rangle \rightarrow -i|J, -J + 1\rangle$ after a pulse duration $t = \pi/(2\sqrt{2J}\Omega)$.

For the gate implementation below, we will need to transform the W state $|J, -J + 1\rangle$ back to the state $|J, -J\rangle$. Equation (17) shows that this can be achieved by applying the same pulse to the systems $(1, 2, \ldots, n - 1)$ for a time $t = \pi/(2\sqrt{2J}\Omega)$, via the transformation $|J, -J + 1\rangle \rightarrow -i|J, -J\rangle$.

The remaining $2^{n-1}$ computational basis states $|i_1i_2\cdots i_{n-1}\rangle$ of the $n - 1$ control qubit systems can be classified into a set of Dicke states $|n - 1/2, -(n - 1/2)\rangle$ with $l = 1, 2, \ldots, n - 1$. For instance, the $(n - 1)$-qubit computational basis states $|1\cdots1\rangle$ and $|1\cdots0\rangle$ are written in term of the Dicke states $|1, -1, -(2 - 1)/2\rangle$ (for $l = 1$) and $|1, -J + 1, -(J - 1)/2\rangle$ (for $l = 2$), respectively. To see how the set of Dicke states $|l, -l, -(l - 1)/2\rangle$ remains unaffected during the W state preparation, let us focus on a Dicke state $|J, -J + 2, -(J - 2)/2\rangle$ ($l \neq 0$), and discuss how to make this Dicke state unaffected by the pulse.

The level spacing between the Dicke states $|J, -J + 1/2, -(J - 1)/2\rangle$ and $|J, -J + 1, -(J - 1)/2\rangle + 1\rangle$ is given by

$$\tilde{\varepsilon}_1 - \tilde{\varepsilon}_0 = \omega_0 - (2J - 1)\lambda.$$

(18)

Therefore, the detuning of the pulse frequency from the transition frequency between the two Dicke states $|J, -J + 1/2, -(J - 1/2)\rangle$ and $|J, -J + 1, -(J - 1)/2\rangle + 1\rangle$ would be $\omega - (\tilde{\varepsilon}_1 - \tilde{\varepsilon}_0) = -\lambda$. The pulse Rabi frequency between the two Dicke states $|J, -J + 1/2, -(J - 1/2)\rangle$ and $|J, -J + 1, -(J - 1)/2\rangle + 1\rangle$ is $\Omega\sqrt{2J} - \lambda$, which can be seen from the expression of $\Omega_k$ in equations (13) (for the present case, $k = 0$ and $J$ is replaced by $J - 1/2$). If the large detuning $\Omega\sqrt{2J} - \lambda \ll \lambda$ is met, the transition between the two Dicke states $|J, -J + 1/2, -(J - 1/2)\rangle$ and $|J, -J + 1, -(J - 1)/2\rangle + 1\rangle$ can be neglected due to far-off-resonance with the pulse. Thus, the pulse does not excite the Dicke state $|J - 1/2, -(J - 1)/2\rangle + 1\rangle$ when the control qubits $(1, 2, \ldots, n - 1)$ are initially in the state $|J - 1/2, -(J - 1)/2\rangle$.

Note that for any $l \in \{1, 2, \ldots, n - 1\}$, we have $\Omega\sqrt{2J} - \lambda \ll \lambda$ when $\Omega(n - 1) \ll \lambda$. Therefore, as long as the condition in equation (14) is satisfied, the Dicke state $|J - 1/2, -(J - 1)/2\rangle$ with any given $l = 1, 2, \ldots$ or $n - 1$ will not be affected by the pulse, i.e. the remaining $2^{n-1}$ computational basis states of the control qubit systems $(1, 2, \ldots, n - 1)$ are unchanged during the pulse.

4. Implementation of an $n$-qubit controlled-$R$ gate in cavity QED

Let us now consider $n$ superconducting qubit systems $(1, 2, \ldots, n)$ placed in a cavity or coupled to a resonator. Each system has the three-level configuration. Initially, the transition between any two levels of each system is highly detuned (decoupled) from the cavity mode (figures 5(a) and (a’)), which can be achieved via prior adjustment of the level spacings. In addition, assume that the cavity mode is initially in the vacuum state $|0\rangle$.

The procedure for implementing the $n$-qubit controlled-$R$ gate is listed as follows:

**Step (i).** Leave the level structure of system $n$ unchanged (figure 5(b’)) while adjusting the level spacings of systems $(1, 2, \ldots, n - 1)$ such that the $|1\rangle \leftrightarrow |2\rangle$ transition of each of systems $(1, 2, \ldots, n - 1)$ is non-resonantly coupled to the cavity mode, with a detuning $\Delta_c$ (figure 5(b)). Then, apply a pulse to systems $(1, 2, \ldots, n - 1)$ for a duration $t_1 = \pi/(2\sqrt{2J}\Omega)$ (figure 5(b)). As discussed in section 3, when the systems $(1, 2, \ldots, n - 1)$ are initially in the computational basis state $|1\cdots1\rangle$, the W state $|J, -J + 1\rangle$ is created after the pulse, via a transformation $|J, -J\rangle \leftrightarrow -i|J, -J + 1\rangle$. Note that the cavity mode remains in the vacuum state during the operation of this step, as shown in section 3.

**Step (ii).** Leave the level structure of system $n$ unchanged (figure 5(c’)) while adjusting the level spacings of systems $(1, 2, \ldots, n - 1)$ such that the $|1\rangle \leftrightarrow |2\rangle$ transition of each of systems $(1, 2, \ldots, n - 1)$ is resonantly coupled to the cavity mode for an interaction time $t_2 = (\pi/2)/(\sqrt{2J}g)$ (figure 5(c)). The Hamiltonian describing this step is given by (in the interaction picture)

$$H_t = g'aS^+ + g'a^+S^-.$$  

(19)

Here and below, $g'$ is the resonantly coupling constant between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition. It can be found that under this Hamiltonian, the time evolution for the state $|J, -J + 1\rangle \otimes |0\rangle_c$ of the systems $(1, 2, \ldots, n - 1)$ and the cavity mode is described by

$$|J, -J + 1\rangle \otimes |0\rangle_c \rightarrow \cos(\sqrt{2J}g't)|J, -J + 1\rangle \otimes |0\rangle_c - i\sin(\sqrt{2J}g't)|J, -J\rangle \otimes |1\rangle_c,$$

(20)

which shows that when systems $(1, 2, \ldots, n - 1)$ are initially in the W state $|J, -J + 1\rangle$, a single photon is created in the cavity mode after an interaction time $t_2$ given above, through a transformation $|J, -J + 1\rangle \otimes |0\rangle_c \rightarrow -i|J, -J\rangle \otimes |1\rangle_c$. Note that the operation time $t_2$ $(\propto 1/\sqrt{n - 1})$ decreases as the number $n - 1$ of the qubit systems increases.
structures for systems \(1, 2, \ldots, n - 1\) back to the original situation (figure 5(d)) such that the cavity mode does not couple to the systems \(1, 2, \ldots, n - 1\). Meanwhile, adjust the level spacings of system \(n\) so that the \(|1\rangle \leftrightarrow |2\rangle\) transition of this system is resonantly coupled to the cavity mode for an interaction time \(t_1\) (figure 5(d')). The Hamiltonian describing this step of operation is given by

\[
H_t = \hbar(g'a^\dagger|1\rangle\langle 2| + \text{h.c}).
\]  

The time evolution of the state \(|1\rangle_n|1\rangle_c\) is described by

\[
|1\rangle_n|1\rangle_c \rightarrow \cos(g't)|1\rangle_n|1\rangle_c - i\sin(g't)|2\rangle_n|0\rangle_c. 
\]  

It can be seen from equation (22) that after an interaction time \(t_2 = \pi/(2g')\), the state \(|1\rangle_n|1\rangle_c\) changes to \(-i|2\rangle_n|0\rangle_c\). Note that the state \(|0\rangle_n|1\rangle_c\) remains unchanged since the state \(|0\rangle_n\) is not coupled to the cavity mode. Here and below, the subscript \(n\) represents the system \(n\).

Step (iv). Leave the level structure of systems \(1, 2, \ldots, n - 1\) unchanged (figure 5(e)) while adjust the level spacings of system \(n\) so that the \(|0\rangle \leftrightarrow |2\rangle\) transition of system \(n\) is resonantly coupled to the cavity mode for an interaction time \(t_3\) (figure 5(e')). The Hamiltonian describing this step is

\[
H_t = \hbar(g''a^\dagger|0\rangle\langle 2| + \text{h.c}).
\]  

Here and below, \(g''\) is the resonantly coupling constant between the cavity mode and the \(|0\rangle \leftrightarrow |2\rangle\) transition. According to this Hamiltonian, one can easily find that after an interaction time

\[
t_4 = \theta/g'',
\]

the transformations \(|0\rangle_n|1\rangle_c \rightarrow \cos \theta|0\rangle_n|1\rangle_c - i \sin \theta|2\rangle_n|0\rangle_c\) and \(|2\rangle_n|0\rangle_c \rightarrow -i \sin \theta|0\rangle_n|1\rangle_c + \cos \theta|2\rangle_n|0\rangle_c\) are obtained.

The operations for the last three steps are the reverse operations of steps (i), (ii) and (iii) above, and are described below.

Step (v). Leave the level structure of systems \(1, 2, \ldots, n - 1\) unchanged (figure 5(d)) while adjusting the level spacings of system \(n\) such that the \(|1\rangle \leftrightarrow |2\rangle\) transition of system \(n\) is resonant with the cavity mode for an interaction time \(t_5\) (figure 5(d')). The Hamiltonian describing this step is the one in equation (21). The time evolution of the state \(|2\rangle_n|0\rangle_c\) is described by

\[
|2\rangle_n|0\rangle_c \rightarrow \cos(g't)|2\rangle_n|0\rangle_c - i\sin(g't)|1\rangle_n|1\rangle_c. 
\]  

Thus, after an interaction time \(t_5 = 3\pi/(2g')\), the state \(|2\rangle_n|0\rangle_c\) becomes \(i|1\rangle_n|1\rangle_c\). Note that the state \(|0\rangle_n|1\rangle_c\) remains unchanged during this step of the operation.

Step (vi). Adjust the level spacings of system \(n\) such that the cavity mode does not couple to this system (figure 5(c)). Meanwhile, adjust the level spacings of systems \(1, 2, \ldots, n - 1\) such that the \(|1\rangle \rightarrow |2\rangle\) transition of each of systems \(1, 2, \ldots, n - 1\) is resonant with the cavity mode for an interaction time \(t_6\) (figure 5(c')). The Hamiltonian describing this step is the one in (19), from which one can easily find that the time evolution for the state \(|J, -J\rangle \otimes |1\rangle_c\) of the systems \((1, 2, \ldots, n - 1)\) and the cavity mode is described by

\[
|J, -J\rangle \otimes |1\rangle_c \rightarrow \cos((\sqrt{2}J/g')t)|J, -J\rangle \otimes |1\rangle_c,
\]

\[-i\sin((\sqrt{2}J/g')t)|J, -J + 1\rangle \otimes |0\rangle_c.\]  

Figure 5. The level structures of the systems \((1, 2, \ldots, n)\) during gate preparation. Figures on the left side represent the level structures for systems \((1, 2, \ldots, n - 1)\), while figures on the right size represent the level structures of system \(n\). Here, \(g\) is the non-resonantly coupling constant between the cavity mode and the \(|1\rangle \leftrightarrow |2\rangle\) transition, \(g'\) is the resonantly coupling constant between the cavity mode and the \(|1\rangle \leftrightarrow |2\rangle\) transition, and \(g''\) is the resonantly coupling constant between the cavity mode and the \(|0\rangle \leftrightarrow |2\rangle\) transition. In addition, the transition between any two levels linked by a dashed line is highly detuned (decoupled) from the cavity mode and/or the pulse.
It can be seen from equation (25) that the operation of this step results in the transformed state \([J, -J] \otimes |1\rangle_c \rightarrow -i|J, -J + 1\rangle \otimes |0\rangle_c\) for \(t_6 = \pi/(2\sqrt{2}J')\).

**Step (vii).** Leave the level structure of system \(n\) unchanged (figure 5(b)) while adjusting the level spacings of systems \((1, 2, \ldots, n - 1)\) such that the \(|1\rangle \leftrightarrow |2\rangle\) transition of each of systems \((1, 2, \ldots, n - 1)\) is non-resonantly coupled to the cavity mode, with a detuning \(\Delta_n\) (figure 5(b)). Then, apply a pulse to systems \((1, 2, \ldots, n - 1)\) for a duration \(t_7 = \pi/(2\sqrt{2}J')\) (figure 5(b)). As discussed in section 3, the operation of this step leads to the transformation \([J, -J + 1] \rightarrow -i|J, -J\rangle\).

Note that after the last step of the operation, we will need to leave the level structure of system \(n\) unchanged (figure 5(a)) while adjusting the level spacings of systems \((1, 2, \ldots, n - 1)\) back to the original situation as shown in figure 5(a), such that the cavity mode does not couple to each system after the above manipulation.

The states of the whole system after each step of the above operations are summarized below:

\[
\begin{align*}
&|11 \cdots 1 \rangle |0\rangle \otimes |0\rangle_c, \\
&\text{Step}(i) \quad -i|J, -J + 1\rangle |0\rangle \otimes |0\rangle_c, \\
&\text{Step}(ii) \quad -|J, -J\rangle |0\rangle \otimes |1\rangle_c, \\
&\text{Step}(iii) \quad -|J, -J\rangle |0\rangle \otimes |1\rangle_c, \\
&\text{Step}(iv) \quad -|J, -J\rangle (\cos \theta |0\rangle + \sin \theta |1\rangle), \\
&\text{Step}(v) \quad -|J, -J\rangle (\cos \theta |0\rangle + \sin \theta |1\rangle), \\
&\text{Step}(vi) \quad |J, -J + 1\rangle (\cos \theta |0\rangle + \sin \theta |1\rangle), \\
&\text{Step}(vii) \quad |11 \cdots 1\rangle (\cos \theta |0\rangle + \sin \theta |1\rangle), \\
&\text{Step}(i) \quad -i|J, -J + 1\rangle |1\rangle \otimes |0\rangle_c, \\
&\text{Step}(ii) \quad -|J, -J\rangle |1\rangle \otimes |1\rangle_c, \\
&\text{Step}(iii) \quad -|J, -J\rangle |2\rangle \otimes |0\rangle_c, \\
&\text{Step}(iv) \quad |J, -J\rangle (\sin \theta |0\rangle + i \cos \theta |2\rangle), \\
&\text{Step}(v) \quad -|J, -J\rangle (-\sin \theta |0\rangle + \cos \theta |1\rangle), \\
&\text{Step}(vi) \quad |J, -J + 1\rangle (-\sin \theta |0\rangle + \cos \theta |1\rangle), \\
&\text{Step}(vii) \quad |11 \cdots 1\rangle (-\sin \theta |0\rangle + \cos \theta |1\rangle),
\end{align*}
\]

(26) For all \(i_1, i_2, \ldots, i_{n-1} \in \{0, 1\}\) and \(\prod_{k=1}^{n-1} i_k = 0\) remain unchanged during the entire operation. This is because: (a) during the operation of step (i), the states \(|i_1i_2 \cdots i_{n-1}\rangle\) of systems \((1, 2, \ldots, n - 1)\) were not affected by the applied pulse, as discussed in section 3; and (b) no photon was emitted to the cavity during the operation of step (ii), when systems \((1, 2, \ldots, n - 1)\) are in any one of the states \(|i_1i_2 \cdots i_{n-1}\rangle\). Hence, it can be concluded from equations (26) and (27) that the transformation (2), i.e. the \(n\)-qubit controlled-R gate, was implemented with \(n\) systems (i.e. the \(n - 1\) control systems \((1, 2, \ldots, n - 1)\) and the target system \(n\)) after the above process.

The systems not involved in each step of the operations above need to be decoupled from the cavity field and/or the pulse. This requirement can be achieved by adjusting the level spacings (e.g. this can be done for superconducting devices as discussed in section 1).

The detunings \(\Delta_n\) and \(\Delta_c\) are set to be identical for each of the systems \((1, 2, \ldots, n - 1)\) in steps (i) and (vii), and systems \((1, 2, \ldots, n - 1)\) are brought to resonance with the cavity mode in steps (ii) and (vi). Therefore, the level spacings for systems \((1, 2, \ldots, n - 1)\) can be synchronously adjusted by changing the common external parameters of the qubit systems during the entire operation. In addition, the cavity mode is virtually excited during the operation of steps (i) and (vii). Thus, decoherence caused by the cavity decay for these two steps is greatly reduced.

From the description given above, it can be seen that the level \(|0\rangle\) of each of qubit system \((1, 2, \ldots, n - 1)\) is not affected during the entire operation, because the cavity mode was set to be highly detuned (decoupled) from the \(|0\rangle \leftrightarrow |1\rangle\) transition and the \(|0\rangle \leftrightarrow |2\rangle\) transition. Thus, neither the level spacing between the two levels \(|0\rangle |0\rangle\) nor the level spacing between the two levels \(|0\rangle |1\rangle\) are required to be identical for each of the qubit systems \((1, 2, \ldots, n - 1)\). However, as shown above, the level spacing between the two levels \(|1\rangle |0\rangle\) and \(|2\rangle |0\rangle\) needs to be identical for each of the qubit systems \((1, 2, \ldots, n - 1)\). Note that for superconducting qubit systems, it is difficult to have the level spacing between any two levels the same for each qubit system, but it is easy to make the level spacing between two particular levels (i.e. levels \(|1\rangle\) and \(|2\rangle\) for the present case) identical by adjusting device parameters or varying external parameters [33].

Finally, it should be mentioned that non-uniformity of the device parameters for qubit systems \((1, 2, \ldots, n - 1)\) may cause the coupling strength \(g\) or \(g'\) (i.e. the coupling strength between the cavity mode and the \(|1\rangle \leftrightarrow |2\rangle\) transition) not to be the same for each of the qubit systems \((1, 2, \ldots, n - 1)\). However, it is noted that for a superconducting qubit system, the qubit–cavity coupling strength is adjustable by varying the position of the qubit system in the cavity. Thus, by having the qubit systems \((1, 2, \ldots, n - 1)\) located at appropriate positions in the cavity, one can make the coupling strength \(g\) or \(g'\) identical for each of the qubit systems \((1, 2, \ldots, n - 1)\).

5. Possible experimental realization

In this section, we discuss possible experimental implementations. For the method to work:
(a) During the operation of step (i) or step (vii), the occupation probability $p_1$ of the Dicke state $|J, -J + 1\rangle \leftrightarrow |J, -J + 2\rangle$ due to the $|J, -J + 1\rangle \leftrightarrow |J, -J + 2\rangle$ transition induced by the pulse, and the occupation probability $p_2$ of the Dicke state $|J - 1/2, -(J - 1/2) + 1\rangle$ due to the $|J - 1/2, -(J - 1/2)\rangle \rightarrow |J - 1/2, -(J - 1/2) + 1\rangle$ transition induced by the pulse, given by \[ p_1 \approx \frac{\Omega^2}{\Omega^2 + \lambda^2/[2(n-2)]}, \] \[ p_2 \approx \frac{\Omega^2}{\Omega^2 + (\Omega^2 + \lambda^2)/[4(n-l-1)]}, \] need to be negligibly small in order to reduce the gate error. 

(b) According to the discussion in section 3, the following conditions need to be satisfied:

\[ g \ll \Delta_c, \quad \omega \sqrt{n-1} \ll g^2/\Delta_c, \] \[ \Delta_p = (n-1)g^2/\Delta_c. \] \[ (30) \]

Note that these conditions can in principle be achieved because: (i) the Rabi frequency $\Omega$ can be adjusted by changing the intensity of the pulse, (ii) the detuning $\Delta_c$ can be adjusted by changing the $|1\rangle \leftrightarrow |2\rangle$ transition frequency $\omega_c$, and (iii) the detuning $\Delta_p$ can be adjusted by varying the pulse frequency $\omega_c$.

(c) The total operation time is given by

\[ \tau = \pi/(\Omega\sqrt{n-1}) + \pi/(g'\sqrt{n-1}) + 2\pi/g' + \theta + \gamma + 8\tau_0, \] \[ (31) \]

which shows that for a given $\Omega\sqrt{n-1}$, $\tau$ decreases as the number of qubits $n$ increases. Here, $\tau_0$ is the typical time required to adjust the level spacings during each step. $\tau$ should be much shorter than the energy relaxation time $\gamma_2^{-1}$ and dephasing time $\gamma_2^{z^*}$ of the level $|2\rangle$, such that decoherence, caused by spontaneous decay and dephasing of the qubit systems, is negligible during the operation. Also, $\tau$ needs to be much shorter than the lifetime of the cavity photon, which is given by $\kappa^{-1} = Q/(2\pi\nu_c^L)$, such that the decay of the cavity photon can be neglected during the operation. Here, $Q$ is the (loaded) quality factor of the cavity and $\nu_c^L$ is the cavity field frequency. To obtain these requirements, one can design the qubit systems to have a sufficiently long energy relaxation time and dephasing time, such that $\tau \ll \gamma_2^{-1}, \gamma_2^{z^*}$, and choose a high-$Q$ cavity such that $\tau \ll \kappa^{-1}$.

For the sake of definitiveness, let us consider the experimental possibility of realizing a six-qubit controlled-Hadamard gate (i.e. the controlled-$R$ gate for $\theta = \pi/4$), using six identical superconducting qubit systems coupled to a resonator (figure 6(a)). Each qubit system could be a superconducting charge-qubit system (figure 6(b)), flux-qubit system (figure 6(c)), or flux-biased phase-qubit system (figure 6(d)). As a rough estimate, assume $g/2\pi \sim 220$ MHz, which could be reached for a superconducting qubit system coupled to a one-dimensional standing-wave CPW (coplanar waveguide) transmission resonator [35]. With the choice of $g', g'' \sim g$, $\Delta_x \sim 10g$, $\Omega/2\pi \sim 1.1$ MHz (i.e. $\lambda/\Omega \sim 20$), and $\tau_0 \sim 1$ ns, one has $\tau \sim 0.2$ $\mu$s, much shorter than $\min[\gamma_2^{-1}, \gamma_2^{z^*}] \sim 1$ $\mu$s [25, 36]. In addition, consider a resonator with frequency $\nu_c \sim 3$ GHz (e.g. [37]) and $Q \sim 5 \times 10^4$, we have $\kappa^{-1} \sim 2.7$ $\mu$s, which is much longer than the operation time $\tau$ here. Note that superconducting coplanar waveguide resonators with a quality factor $Q > 10^6$ have been experimentally demonstrated [38].

For the choice of $\Delta_c \sim 10g$ and $\lambda/\Omega \sim 20$ here, we have $p_1 \sim 0.02$ and $p_2 \lesssim 0.04$, which can be further reduced by increasing the ratio $\Delta_c/g$ and $\lambda/\Omega$. How well this gate would work needs to be further investigated for each particular experimental setup or implementation. However, we note that this requires a rather lengthy and complex analysis, which is beyond the scope of this theoretical work.

6. Conclusion

In summary, we have proposed a method for implementing an $n$-qubit controlled-rotation gate with three-level superconducting qubit systems in cavity QED. This proposal requires seven steps of operation only, which is independent of the number of qubits $n$. In contrast, when the proposed gate is constructed by using the conventional gate-decomposing protocols, the basic operational steps increase exponentially with the number of qubits $n$. Thus, when the number of qubits $n$ is large, the gate operation is significantly simplified by using the present proposal. In addition, as shown above, the gate operation time
for this proposal decreases as the number of qubits increases. This proposal is quite general, and can be applied to various types of superconducting devices in a cavity or coupled to a resonator.

Acknowledgments

CPY is grateful to Shi-Biao Zheng for very useful comments. This work is supported in part by the National Natural Science Foundation of China under grant no. 11074062, the Zhejiang Natural Science Foundation under grant no. Y6100098, funds from Hangzhou Normal University, and the Open Fund from the SKLPS of ECNU.

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