Product representation for default bilattices: an application of natural duality theory

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Abstract

Bilattices (that is, sets with two lattice structures) provide an algebraic tool to model simultaneously the validity of, and knowledge about, sentences in an appropriate language. In particular, certain bilattices have been used to model situations in which information is prioritised and so can be viewed hierarchically. These default bilattices are not interlaced: the lattice operations of one lattice structure do not preserve the order of the other one. The well-known product representation theorem for interlaced bilattices does not extend to bilattices which fail to be interlaced and the lack of a product representation has been a handicap to understanding the structure of default bilattices. In this paper we study, from an algebraic perspective, a hierarchy of varieties of default bilattices, allowing for different levels of default. We develop natural dualities for these varieties and thereby obtain a concrete representation for the algebras in each variety. This leads on to a form of product representation that generalises the product representation as this applies to distributive bilattices.

Keywords: bilattice, natural duality, product representation, knowledge order

2010 MSC: Primary: 06D50, Secondary: 08C20, 03G25

1. Introduction

Our objective is to develop a representation theory for classes of algebras which have arisen in the modelling of default logics. Specifically, we consider bilattices which have been used to study logics with prioritised defaults [13]; the simplest and best known of these bilattices was introduced by Ginsberg [10] under the name SEVEN. As we indicate below, such ‘default bilattices’ do not have the interlacing property and so the equational classes they generate fall outside the scope of the Product Representation Theorem, the cornerstone of the structure theory of interlaced bilattices. A novel approach is required in order to develop an analogous structure theory beyond the interlaced setting. This we provide by the application of natural duality theory. In [2], Cabrer and Priestley showed that, for the class DB of distributive bilattices, the product representation can be seen as a consequence of, and very closely allied to, the natural duality for DB presented there. In the present paper we consider an infinite sequence of default bilattices, each having its predecessor as a homomorphic image. We develop natural dualities for the equational classes generated by these bilattices and thereby arrive at a product representation for the members of these classes (Theorem 6.1).

To set the scene we recall the background very briefly. The motivation for Ginsberg’s pioneering paper [10] was his plan to use bilattices as a framework for inference with applications to artificial intelligence and logic programming, in particular for modelling inference in situations where information is incomplete or contradictory. The central idea was to consider sets which carry two lattice orders: ⩽t, interpreted as measuring ‘degree of truth’, and ⩽k, measuring ‘degree of knowledge’. Certain elements of such structures were then treated as distinguished constants, representing degrees of truth or knowledge, t (‘true’), df (‘false by default’), and so on; T and ⊥ are used to denote,
respectively, ‘contradiction’ and ‘no information’. Fig. 1(ii) shows the bilattice \textit{SEVEN} Ginsberg proposed to model this scenario. As is customary in the bilattices literature the two constituent lattices are combined into a single diagram, with knowledge measured vertically and truth horizontally.

The bilattice \textit{SEVEN} may be seen as providing a more refined model of truth and falsity than the best-known bilattice of all, commonly known as \textit{FOUR} and shown in Fig. 1(i). In \textit{FOUR}, the elements \(t\) and \(f\) represent ‘true’ and ‘false’, \(\top\) and \(\bot\) ‘contradiction’ and ‘no information’.

The bilattice \textit{SEVEN} models one level of default. But there are situations in which a hierarchy of degrees of default may be appropriate. Bilattices which model prioritised defaults were discussed by Ginsberg \cite{11} and there is now a range of applications of such structures in artificial intelligence. We note for example the design by Encheva and Tumin \cite{8} of a tutoring feedback system based on a ten-element default bilattice to inform follow-up questions when the initial responses are incomplete or inconsistent. The same ten-element bilattice is employed by Sakama \cite{14}. Prioritised default bilattices have also been applied to visual surveillance by Shet, Harwood and Davis \cite{15}. In Section 2 we introduce an infinite sequence of bilattices \(\mathbf{K}_n\), as a means of modelling prioritised defaults. Here \(\mathbf{K}_0\), \(\mathbf{K}_1\) and \(\mathbf{K}_2\) are the four-, seven- and ten-element bilattices mentioned above (equipped with a negation and appropriate constants). Fig. 2 depicts the knowledge and truth orders of \(\mathbf{K}_n\), for general \(n\).

There is a critical difference between \textit{FOUR} and \textit{SEVEN}. In \textit{FOUR}, each of the four lattice operations distributes over each of the other three; in \textit{SEVEN} (and also in the refinements we consider) this fails. A bilattice with lattice operations \(|\land|,|\lor|\) (with associated order \(\leq_\land\)) and \(|\lor|,|\land|\) (with associated order \(\leq_\lor\)) is \textit{interlaced} if each pair of lattice operations is monotonic with respect to the other order. This holds in \textit{FOUR}. But in \textit{SEVEN} it fails, as is witnessed by the fact that \(df\) \(\leq_\land\) \(t\) and \(df \land \bot = df \land \bot = t \land \bot\). The significance of the interlacing condition is that it is sufficient, and also necessary, for the Product Representation Theorem to be valid: any interlaced bilattice has as its underlying set a product \(L \times L\) of a lattice \(L\) with itself; the lattice operations on the factors determine the bilattice operations; negation sends a pair \((a, b)\) to \((b, a)\). For an account of the theorem and its complicated history, see the recent note by Davey \cite{5}. This note gives a comprehensive list of references both to the theorem itself and to the way in which it is used to study interlaced bilattices.

We conclude this introduction by summarising the content and structure of the paper and highlighting our principal results. We focus on mathematical aspects of default bilattices, rather than logical aspects. We shall consider \(\mathbf{K}_n\) as an algebra of a specified type and investigate the variety \(V_n = \text{HSP}(\mathbf{K}_n)\) generated by \(\mathbf{K}_n\), for an arbitrary value of \(n\). In Section 2 we derive the properties of the algebras \(\mathbf{K}_n\) on which our representation theory will rely.

Our primary tool, as in \cite{2}, will be the theory of natural dualities, for which the text by Clark and Davey \cite{4} serves as the background reference. Here we need the multisorted version of the theory, as it applies to a restricted class of finitely generated lattice-based varieties. Section 2 outlines, as far as possible in black-box style, rudiments of this theory. The framework was first developed more than 25 years ago, but examples of its exploitation are quite scarce. Theorem 4.1 describes our duality for \(V_n\); an instructive new example of the multisorted machinery at work. It also provides us with a springboard to our later results.

In Theorem 4.3 we describe the objects in the category dual to \(V_n\); these multisorted topological structures are such that each sort naturally carries the structure of a Priestley space, and there is a sequence of maps which links each sort to the next. In Section 6 we derive our product representation theorem (Theorem 6.1). The proof makes explicit use of the multisorted structure dual to a given algebra in \(V_n\) to show how the algebra can be obtained from a product
built from a finite sequence of distributive lattices with linking homomorphisms. Section 6 can if desired be studied independently of Section 5. In the latter we connect directly with Priestley duality, relating the natural dual space of an algebra in $V_n$ to the Priestley dual of its (necessarily distributive) knowledge lattice reduct. The paper concludes with a short section devoted to the quasivarieties $\mathcal{ISP}(K_n)$. Here we can employ duality theory in its single-sorted form.

2. A hierarchy of varieties of prioritised default bilattices

The term ‘bilattice’ is not used in a consistent way throughout the extensive bilattice literature. However nowadays it usually refers to a structure which, besides its two lattice structures and, possibly, constants, carries also a negation operation, and we follow this usage. Henceforth a bilattice $B$ will be an algebraic structure $B = (B; \oplus, \ominus, \wedge, \vee, \neg)$ such that the reducts $(B; \ominus, \ominus)$ and $(B; \wedge, \vee)\wedge, \neg)$ are lattices and $\neg$ is a unary operation which preserves the $(\ominus, \ominus)$-order, reverses the $[\wedge, \vee]$-order, and is involutive. We denote by $\leq_i$ the order associated with $(B; \ominus, \ominus)$ and $\leq_i$ the order associated with $(B; \wedge, \vee)$. We shall consider only bilattices in which both of the lattice orders are bounded. The bounds of the knowledge lattice are denoted by $\top$ and $\bot$, and those of the truth lattice by $t$ and $f$.

In an $n$-level prioritised default bilattice, the designated default truth values form two finite sequences $f_0, \ldots, f_n$ and $t_0, \ldots, t_n$, where $f_{i+1}$ and $t_{i+1}$ will be lower in the knowledge order than their respective predecessor default truth values $f_i$ and $t_i$. The connotation is thus that knowledge represented by the truth values at level $k$ has lower priority than that from those at level $k$. In addition, one thinks of $t_{i+1}$ as being ‘less true’ than its predecessor $t_i$, while $f_{i+1}$ is ‘less false’ than $f_i$. That is, $f_{i+1} \leq t_i$ and $f_{i+1} \geq f_i$. Thus we view the truth values hierarchically.

We now describe the $n$-level prioritised default bilattice $K_n$ for $n \geq 0$. The underlying set of this algebra is

$$K_n = \{f_0, \ldots, f_n, t_0, \ldots, t_n, \top_0, \ldots, \top_{n+1}\}.$$ 

We define lattice orders $\leq_k$ and $\leq_i$ on $K_n$ as follows. For $K_0$ and $v \in \{f_0, t_0\}$ we have $\top_1 < v < \top_0$. If $n \geq 1$ and $0 \leq i < j \leq n$,

$$\top_{n+1} < t_v \leq_0 \top_v <_k \top_j <_k v <_i \top_j \quad \text{for } v \in \{f_i, t_j\} \text{ and } v_l \in \{f_i, t_j\},$$

and if $0 \leq i \leq j \leq n$,

$$f_i \leq f_j \leq t_j \leq t_i \quad \text{and } f_i \leq t_{n+1} \leq t_i.$$

These lattice orders are depicted in Hasse diagram style in Fig. 2. We shall, where appropriate, write $\top$ for $\top_0$ and $\bot$ for $\top_{n+1}$.

Negation is defined as follows:

$$\neg t_i = f_i \quad \text{and } \neg f_i = t_i \quad \text{for } 0 \leq i \leq n; \quad \neg a = a \quad \text{if } a \in \{\top_0, \ldots, \top_n\}.$$ 

The elements $t$ and $f$ are inter-definable via $\neg$. Elements of these types contain only information about truth or falsity; they do not contain any information about contradiction or lack of information. We observe that every element of $K_n$ is recursively term-definable from $\top_0$ and $\top_{n+1}$. For $m \in \{0, \ldots, n\}$ we have

$$t_m = \top_m \lor \top_{m+1}, \quad f_m = \top_m \land \top_{n+1} \quad \text{and } \quad \top_{m+1} = (\top_m \lor \top_{m+1}) \land (\top_m \land \top_{m+1}).$$

Note in particular that $f$ and $t$ are $\top \land \bot$ and $\top \lor \bot$ respectively.

We now define the algebra $K_n$ to be $K_n = (K_n; \wedge, \vee, \wedge, \ominus, \neg, \bot, \top)$. It has a term-definable bounded bilattice structure. The cases $n = 0$ and $n = 1$ deserve special mention. The bilattice reducts of $K_0$ and $K_1$ are, up to the labelling of the elements, simply $FOUR$ and $SEVEN$. Moreover, $K_1$ is term-equivalent to the algebra $4 = (\top, 0, \bot, f; \land, \land, \neg, t, t, \top)$ introduced in Section 2; here $\neg$ switches $t$ and $f$, and fixes $\top$ and $\bot$. The algebras $K_0$ are all of the same algebraic type. All have the special property that their knowledge reducts are bounded distributive lattices and the same is true of the algebras in $V_n$ for any $n$. But for $n > 0$ the truth lattice reduct of $K_n$ is not distributive. In what follows we shall take a fixed $n \geq 0$; we include $n = 0$ to emphasise that $K_0$ fits into our general scheme. However our results below give nothing new in this case.
Proposition 2.1. For $m$ such that $0 \leq m \leq n$, let $S_{n,m}$ and $h_{n,m}$ be defined as above. Then the following statements hold.

(i) $S_{n,m} = \{(a, b) \in K_n^2 \mid (h_{n,m}(a), h_{n,m}(b)) \in S_{m,m}\}$.

(ii) $h_{n,m}$ is a surjective homomorphism.

(iii) $S_{n,m}$, with the inherited operations, forms a subalgebra $S_{n,m}$ of $K_n^2$.

(iv) $\mathbb{S}(K_i \times K_j) = \{(h_{i,j}, h_{n,n})(S) \mid S \in \mathbb{S}(K_n^2)\}$, for $i, j \in \{0, \ldots, n\}$.

In our development of the properties of the algebras $K_n$ and the varieties $V_n$ that they generate, we shall occasionally need to draw on basic facts from universal algebra, for example concerning congruences and subdirectly irreducible algebras; [1] provides a good background reference for such material. Our first observation is a triviality: since each element of $K_n$ is a term, $K_n$ has no proper subalgebras. We shall next investigate the subalgebras of $K_n^2$. The characterisation we obtain in Theorem 2 will be of crucial importance for setting up our dualities. As a byproduct, we are able to identify the subdirectly irreducible algebras in the variety $V_n$ and thence obtain a complete description of its lattice of subvarieties.

We let $\Delta_n$ denote the diagonal subalgebra $\{(a, a) \mid a \in K_n\}$. We now define subsets $S_{n,0}, \ldots, S_{n,n}$ of $K_n^2$ as follows:

$$S_{n,m} = \Delta_n \cup \{(a, b) \mid a, b \leq_k \tau_{m+1} \text{ or } a \leq_k b \leq_k \tau_m\} \quad (\text{for } 0 \leq m \leq n).$$

Figure 2: $K_n$ in its knowledge order (left) and truth order (right); here $0 < i < j < n$

For $0 \leq i < j < n$ we have $S_{n,i} \subsetneq S_{n,j}$.

Later we shall want to view $S_{n,m}$ as a binary relation on $K_n$. It is easily seen that, as such, it is always a quasi-order, and a partial order if and only if $m = n$. The partial orders $S_{m,m}$, for $m \leq n$, appear in our duality theory for $V_n$ in Section 4 and the relations $S_{n,m}$, for $m \leq n$, are employed in the duality for $\mathbb{I}SP(K_n)$ presented in Section 7. By way of illustration, Fig. 3 shows $S_{1,0}$, $S_{1,1}$ and $S_{2,2}$, and also $S_{2,0}$ and $S_{2,1}$.

For $m \leq n$ we let $h_{n,m}: K_n \to K_m$ be given, for $a \in K_n$, by

$$h_{n,m}(a) = \begin{cases} \tau_{m+1} & \text{if } a \leq_k \tau_{m+1}, \\ a & \text{otherwise.} \end{cases}$$

Here $h_{n,n}$ is just the identity map on $K_n$.

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Theorem 2.2. Let $S$ be a subalgebra of $K_n$. Then either $S = K_n$ or there exists $m \in \{0, \ldots, n\}$ such that $S$ is one of the following: $S_{n,m}$, $S_{n,m}, S_{n,m} \cap \mathcal{S}_{n,m}$. (For $r$ a binary relation, $\bar{r}$ denotes its converse.)

Proof. The fact that $K_n$ is generated by $\{\top, \bot\}$ implies that $\Delta_n$ is generated by $\{\top, \bot, (\top, \bot), (\top, \top)\}$. Hence $\Delta_n$ is contained in every subalgebra of $K_n$. Let

$$A = \{\top_{n+1} \cup \{a \mid \exists b \text{ such that } a \equiv_b b \text{ and } (a, b) \in S\}\} \text{ and } B = \{\top_{n+1} \cup \{b \mid \exists a \text{ such that } a \equiv_b b \text{ and } (a, b) \in S\}\}.$$
Claims 1–4 below concern \( A \). These claims, together with corresponding results for \( B \) obtained by swapping the coordinates, will be combined to prove the theorem.

Claim 1: \( \bigoplus A \in A \).

If \( A = \{\tau_{n+1}\} \) the result is trivial. If \( a, b \in A \) are such that \( a \not\equiv_k b \) and \( b \not\equiv_k a \) then there exists \( i \in \{1, \ldots, n\} \) such that \( (a, b) = (t_i, t_i) \). Let \( b, b' \) be such that \( (a, b), (b, b') \in S, b <_k a \) and \( b' <_k b \). Then \( b \oplus b' \equiv_k \tau_{n+1} <_k \tau_i = a \oplus b \). We also have \( (a \oplus b, b \oplus b') = (a, b) \oplus (b, b') \in S \). Hence \( a \oplus b \in A \). Thus \( A \) is a finite set closed under \( \oplus \), and consequently \( \bigoplus A \in A \).

Claim 2: \( \bigoplus A \in \{\tau_0, \ldots, \tau_{n+1}\} \).

Since \( S \) is closed under \( \neg \) and \( \neg \) preserves \( \equiv_k \), we have \( f_i \in A \) if and only if \( t_i \in A \). This, combined with Claim 1, implies that if \( f_i \in A \) or \( t_i \in A \) then \( \tau_i \in A \).

Claim 3: If \( \bigoplus A = \tau_i \) for some \( i \in \{0, \ldots, n\} \), then \((\tau_i, f_i), (\tau_i, t_i), (\tau_i, \tau_{n+1}), (f_i, \tau_{n+1})\) and \((t_i, \tau_{n+1})\) all belong to \( S \).

Let \( b <_k \tau_i \) be such that \( (\tau_i, b) \in S \). If \( b \in \{f_i, t_i\} \), since \( S \) is closed under negation, \((\tau_i, f_i), (\tau_i, t_i) \in S \). If \( b \not\in \{f_i, t_i\} \) then \( b <_k \tau_{n+1} \). Let \( (\tau_i, f_i) = (\tau_i, b) \oplus (f_i, f_i) \in S \) and \((\tau_i, t_i) = (\tau_i, b) \oplus (t_i, t_i) \in S \). We also have \((\tau_i, \tau_{n+1}) = (\tau_i, b) \oplus (t_i, \tau_{n+1}) \in S \). And finally \((f_i, \tau_{n+1}) = (f_i, b) \oplus (f_i, \tau_{n+1}) \in S \).

Claim 4: If \( \bigoplus A = \tau_i \) for some \( i \in \{0, \ldots, n\} \), then \((a, b) \in S \) for each \( a \equiv_k \tau_i \) and \( b \equiv_k \tau_{n+1} \).

By Claim 3, \((f_i, \tau_{n+1}), (t_i, \tau_{n+1}) \in S \) and

\[
(\tau_{n+1}, \tau_{n+1}) = ((\tau_{n+1}, \tau_{n+1}) \lor (f_i, \tau_{n+1})) \land ((\tau_{n+1}, \tau_{n+1}) \land (t_i, \tau_{n+1})) \in S.
\]

Again by Claim 3, \((\tau_i, f_i), (\tau_i, t_i) \in S \), and then \((\tau_i, \tau_{n+1}) = ((\tau_{n+1}, \tau_{n+1}) \lor (\tau_i, f_i)) \land ((\tau_{n+1}, \tau_{n+1}) \land (\tau_i, t_i)) \in S \).

Finally, if \( a \equiv_k \tau_i \) and \( b \equiv_k \tau_{n+1} \), then \((a, b) = ((a, a) \lor (\tau_{n+1}, \tau_{n+1})) \land ((b, b) \land (\tau_{n+1}, \tau_{n+1})) \in S \).

We are now ready to prove the main result. Let \( i, j \in \{0, \ldots, n+1\} \) be such that \( \bigoplus A = \tau_i \) and \( \bigoplus B = \tau_j \). We now have four cases, taking account of how \( i \) and \( j \) are related.

Case 1: Assume \( i = j = 0 \). Then \( S = K^2_n \).

By Claims 3 and 4, \( S_{n,0} \cup \tilde{S}_{n,0} \subseteq S \). Moreover \((t_0, f_0) = (t_i, \tau_i) \lor (t_i, f_i) \in S \) and similarly \((f_0, t_0) \in S \).

Case 2: Assume \( i = j > 0 \). Then \( S = S_{n,i-1} \cap \tilde{S}_{n,i-1} \).

By definition of \( A \) and \( B \), we have \( S \subseteq (1_k \tau_j)^2 \cup \Delta_n \).

By Claims 3 and 4, \( S_{n,i} \cup S_{n,j} \subseteq S \). Moreover \((t_i, f_i) = (t_i, \tau_{n+1}) \lor (\tau_{n+1}, f_i) \in S \) and similarly \((f_i, t_i) \in S \). Thus \( S \subseteq (1_k \tau_j)^2 \cup \Delta_n \).

Case 3: Assume \( 0 \leq j < i \leq n + 1 \). Then \( j = i - 1 \) and \( S = S_{n,j} \).

By Claim 4, \( j = i - 1 \). By definition of \( A \) and \( B \) and Claims 3 and 4, \( S_{n,j} \subseteq S \subseteq (1_k \tau_j)^2 \cup \Delta_n \). But \((1_k \tau_j)^2 \cup \Delta_n \setminus S_{n,j} = \{(f_j, t_j), (t_j, f_j)\} \). If \((f_j, t_j) \in S \), then \((t_j, t_j) = (f_j, t_j) \lor (t_j, t_j) \in S \), and so \( \tau_j \in A \), which contradicts the assumption that \( j < i \). A contradiction is likewise obtained if we assume \((t_j, f_j) \in S \). Thus \( S = S_{n,j} \).

Case 4: Assume \( 0 \leq i < j \leq n + 1 \). Then \( i = j - 1 \) and \( S = S_{n,j} \).

The proof is analogous to that for Case 3.

\[ \square \]

**Corollary 2.3.**

(i) The congruence lattice \( \text{Con}(K_n) \) of \( K_n \) is a chain with \( (n + 2) \) elements.

(ii) The algebra \( K_n \) is subdirectly irreducible.

**Proof.** For \( m \in \{0, \ldots, n\} \), the relation \( S_{n,m} \cap \tilde{S}_{n,m} \) is not a congruence on \( K_n \), as it is not symmetric and so not an equivalence relation. On the other hand each relation \( S_{n,m} \cup \tilde{S}_{n,m} \) is a congruence, and so is \( K_n^2 \). Hence (i) holds. Statement (ii) is an immediate consequence of (i).

\[ \square \]
Let $M$ be a finite algebra. Then $ISP(M)$, the quasivariety generated by $M$, is the class of isomorphic copies of subalgebras of powers of $M$. It is well known (see [2, Proposition 2.3] for a direct proof) that $ISP(4) = HSP(4)$ and that this is the equational class $\mathcal{D}\mathcal{B}$ of distributive bilattices (with bounds). Hence $ISP(K_0) = HSP(K_0)$. On the other hand, $HSP(K_n)$ and $ISP(K_n)$ do not coincide for any $n \geq 1$. We have a homomorphism $h_{n,0}$ from $K_n$ onto $K_0$, and hence $K_0 \in HSP(K_n)$. Suppose for a contradiction that $K_0 \notin ISP(K_n)$. Then, since $K_0$ is not trivial, there exists a homomorphism $u : K_0 \to K_n$. Because $K_n$ has no proper subalgebras, no such map $u$ exists.

Let us fix $n \geq 1$ and consider $V_n = HSP(K_n)$. This variety is lattice-based and hence congruence distributive. Therefore we may appeal to Jónsson’s Lemma (see for example [1, Corollary IV-6.10]) to assert that every subdirectly irreducible algebra in $V_n$ is a homomorphic image of a subalgebra of $K_n$ and hence that $HSP(K_n) = ISP(HSP(K_n))$. Then, because $K_n$ has no proper subalgebras, $HSP(K_n) = ISP(HSP(K_n))$. Corollary [3] showed that every non-trivial congruence in $Con(K_n)$ arises as the kernel of one of the homomorphisms $h_{n,m}$ and, moreover, that a non-trivial algebra in $HSP(K_n)$ is subdirectly irreducible if and only if it is isomorphic to $K_m$ for some $m \in \{0, \ldots, n\}$. We may now record the following proposition.

**Proposition 2.4.** The variety $V_n$ equals $ISP(\{K_m \mid 0 \leq m \leq n\})$. The subvarieties of $V_n$ form an $(n + 2)$-element chain

$HSP(K_0) \subseteq HSP(K_1) \subseteq \cdots \subseteq HSP(K_{n-1}) \subseteq HSP(K_n)$;

here $K_{-1}$ denotes the trivial (one-element) bilattice.

The following lemma is exploited in proving that the dualities we present are optimal, in that the dual category is as simple as possible.

**Lemma 2.5.** Let $m$ and $n$ be such that $0 \leq m \leq n$. Then every homomorphism from $S_{m,m}$ into $K_m$ is the restriction of a projection.

**Proof.** The proof is a special instance of a classic argument from universal algebra, as given, for example, in [9, Theorem 2.5]. It uses the fact that $S_{m,m}$ is lattice-based (and so has a distributive congruence lattice), together with Birkhoff’s Subdirect Product Theorem, to show that any homomorphic image of $S_{m,m}$ is a subdirect product of homomorphic images of $K_n$. Let $g : S_{m,m} \to K_m$ be a homomorphism. Since $K_m$ has no proper subalgebras, $g$ is surjective. Taking account of the fact that $K_m$ is subdirectly irreducible and has no non-identity endomorphisms, the lemma follows easily. □

3. The natural duality framework: multisorted dualities

Assume that we have a quasivariety $\mathcal{A}$ of the form $ISP(M)$, where $M$ is a finite set of finite algebras, later assumed to be lattice-based. When $M$ contains a single algebra $M$, we write $\mathcal{A}$ as $ISP(M)$. We regard $\mathcal{A}$ as a category, in which the morphisms are all homomorphisms. We seek a category $\mathcal{X}$ of topological structures so that there are functors $D : \mathcal{A} \to \mathcal{X}$ and $E : \mathcal{X} \to \mathcal{A}$ setting up a dual equivalence. This will be done in a very specific way, so that $D$ and $E$ are given by appropriately defined hom-functors.

An algebra of the same type as those in $M$ belongs to $\mathcal{A}$ if and only if the sets of homomorphisms $\mathcal{A}(A, M)$, for $M \in M$, jointly separate the elements of $A$; for an explicit statement and proof of this elementary fact from universal algebra, see for example [4, Theorem 1.1.4]. This indicates that the hom-sets $\mathcal{A}(A, M)$ may play a role in a representation theory for $\mathcal{A}$. Indeed, Stone duality for $\mathcal{B}$ (Boolean algebras) and Priestley duality for $\mathcal{D}$ (bounded distributive lattices) can be seen as capitalising on this idea: each of these classes can be represented as the quasivariety generated by an algebra $M$ with universe $\{0, 1\}$. One then builds a dual category $\mathcal{X}$ (of Boolean spaces or of Priestley spaces, as the case may be). There is a natural hom-functor $D : \mathcal{A} \to \mathcal{X}$ which, on objects, assigns to $A$ in $\mathcal{A}$ the hom-set $\mathcal{A}(A, M)$. The objects of $\mathcal{X}$ are obtained by defining an alter ego $M$ for $M$: a discretely topologised structure on the same underlying set $M$. For $\mathcal{B}$, the alter ego $M$ is $\{0, 1\}$ with the discrete topology; for $\mathcal{D}$ it is $\{0, 1\}$, with the partial order $\leq$ for which $0 < 1$, again with the discrete topology. The hom-set $\mathcal{A}(A, M)$ sits inside $M^{A}$, equipped with the product topology, and in the case of $\mathcal{D}$, pointwise lifting of $\leq$. The original algebra $A$ is recaptured as the set of continuous structure-preserving maps from its dual space $D(A)$ into $M$, on which the algebraic operations are defined pointwise from $M$. Readers familiar with the Stone and Priestley dualities formulated in a way different from that we have sketched here can be reassured that passage to the hom-functor approach involves little
more than a simple translation of concepts and notation; for example replacement of prime filters by \{0, 1\}-valued homomorphisms. Details can be found in \cite[Chapter 11]{7} and \cite[Chapter 1]{4}; see also Example 3.2 below.

Our purpose in outlining the hom-functor perspective on the Stone and Priestley dualities has been to provide preliminary motivation for the multisorted dualities we shall employ in this paper. We contend that the ideas involved in setting up the multisorted framework are no more complicated than those in the single-sorted case in which \(\mathcal{M}\) contains one algebra only. Accordingly, we shall pass directly to the general case. An account which parallels that we give below, but confined to the single-sorted setting, and with distributive bilattices in view, can be found in \cite[Section 2]{2} (see also Example 3.3 below).

The theory of multisorted natural dualities is presented, albeit briefly, in \cite[Chapter 7]{4}, and in more detail in the original source \cite[Section 2]{6}. The single-sorted case is much more extensively documented than the multisorted one.

We shall consider an alter ego \(\mathfrak{M}\) of \(\mathcal{A}\) if \(\mathfrak{M}\) is equipped with a set \(S\) of \(\{0, 1, \ldots, n\}\) and is subdirectly irreducible. These assumptions will allow us to work in a more restricted setting than that in \cite{4}. We now need to explain what constitutes an admissible alter ego \(\mathfrak{M}\), how the dual category \(\mathfrak{X}\) of multisorted structures generated by \(\mathfrak{M}\) is constructed and how the associated dual adjunction between \(\mathcal{A}\) and \(\mathfrak{X}\) is set up.

We form multisorted topological \(\mathfrak{M}\)-structures \(X = X_0 \cup \cdots \cup X_n\) where each of the sorts \(X_i\) is a Boolean space, \(X\) is equipped with the disjoint union topology and, regarded as a structure, \(X\) carries relations and operations matching those of \(\mathfrak{M}\). Thus \(X\) is equipped with a set \(\mathfrak{R}\) of relations \(r^X\); if \(r \subseteq M_i \times M_j\), then \(r^X \subseteq X_i \times X_j\); and similarly \(X\) carries a set \(\mathfrak{G}\) of unary operations. Clearly \(\mathfrak{M}\) itself is a structure of this type. Given \(\mathfrak{M}\)-structures \(X\) and \(Y\), a morphism \(\varphi\): \(X \to Y\) is defined to be a continuous map preserving the sorts, so that \(\varphi(X_i) \subseteq Y_i\), and \(\varphi\) preserves the structure. The terms isomorphism, embedding, etc., are defined in the expected way.

We define our dual category \(\mathfrak{X}\) to have as objects those \(\mathfrak{M}\)-structures \(X\) which belong to the class of topological structures which we shall denote by \(\text{ISP}^\uparrow(\mathfrak{M})\). Specifically, \(\mathfrak{X}\) consists of isomorphic copies of closed substructures of powers of \(\mathfrak{M}\). Here powers are formed ‘by sorts’: given a non-empty set \(S\), the underlying set of \(\mathfrak{M}\uparrow\) is the union of disjoint copies of \(M^S\), for \(M \in \mathfrak{M}\), equipped with the disjoint union topology obtained when each \(M^S\) is given the product topology. The structure defined by \(R\) and \(G\) is lifted pointwise to substructures of such powers. The superscript \(\uparrow\) indicates that the empty structure is included in \(\mathfrak{X}\).

We now define hom-functors \(D: \mathcal{A} \to \mathfrak{X}\) and \(E: \mathfrak{X} \to \mathcal{A}\) using \(\mathfrak{M}\) and its alter ego \(\mathfrak{M}\):
a member of $\mathcal{A}$ by virtue of viewing it as a subalgebra of the power $M_i^n \times \cdots \times M_i^n$. The well-definedness of the functors $D$ and $E$ is of central importance to our enterprise. It hinges on the assumption we have made that the relations and operations in the alter ego are algebraic, and that each $M_i$ is finite and carries the discrete topology; cf. [4, Preduality Theorem, 2.5.2]. We can say more (cf. [4, Dual Adjunction Theorem, 2.5.3]): $D$ and $E$ set up a dual adjunction, $(D, E, e, \varepsilon)$ in which the unit and counit maps are evaluation maps, and these evaluations are embeddings.

We say $\mathcal{M}$ yields a multisorted duality if, for each $A \in \mathcal{A}$, the evaluation map $e_A : A \to ED(A)$ is an isomorphism. The duality is full if, for each $X \in \mathcal{X}$, the evaluation map $e_X : X \to DE(X)$ is an isomorphism. Thus a duality provides a concrete representation $ED(A)$ of $A \in \mathcal{A}$. If in addition the duality is full, we also know that every $X \in \mathcal{X}$ arises, up to isomorphism, as a topological structure $D(A)$, for some $A \in \mathcal{A}$.

In practice, fullness of a duality is normally obtained at second hand by showing that the duality is strong. We do not need to use this notion directly; for the formal definition see [4, Chapter 3]. However we do remark that the functors $D$ and $E$ setting up a strong duality have the property that each maps an embedding to a surjection and a surjection to an embedding; this is a very desirable feature of a duality as regards applications.

We record an important fact, true for any multisorted duality, and adding weight to the duality’s claim to be called ‘natural’. In $\mathcal{A} = ISP(M)$, the free algebra $F_A(S)$ on a set $S$ of generators is isomorphic to $E(M^S)$; in particular, $\mathcal{M}$ is the dual space of $F_A(1)$ [4, Lemma 2.2.1 and Section 7.1].

We now state, without further ado, the theorem on which we shall rely, following it with an informal commentary. It is a very restricted form of [4, Theorem 7.1.2] which draws also, mutatis mutandis, on [4, Corollary 3.3.9].

**Theorem 3.1.** (Multisorted NU Strong Duality Theorem, special case) Let $\mathcal{A} = ISP(M)$, where $M = \{M_0, \ldots, M_n\}$ is a set of non-isomorphic subdirectly irreducible algebras of common type having lattice reducts and assume that no $M_i$ has a proper subalgebra. Let $M = (M_0 \cup \cdots \cup M_n; R, G, T)$ where $R = \bigcup \{S(M_i \times M_j) \mid i, j \in \{0, 1, \ldots, n\}\}$, $G = \bigcup \{A(M_i \times M_j) \mid i, j \in \{0, 1, \ldots, n\}\}$, and $T$ is the disjoint union topology obtained from the discrete topology on the sorts $M_i$. Then $\mathcal{M}$ yields a multisorted duality on $\mathcal{A}$ which is strong (and hence full).

It is clear that, from the perspective of universal algebra, the restrictions we have imposed on $\mathcal{M}$ are extremely stringent. However the results of the previous section show that all the assumptions are met when $\mathcal{M} = \{K_0, \ldots, K_n\}$ (for any $n > 0$). We could also take $\mathcal{M} = \{K_n\}$, to obtain a single-sorted duality for the quasivariety $ISP(K_n)$; see Section 7.

The assumption that the algebras in $\mathcal{M}$ be lattice-based comes into play in the following way. Since each $M \in \mathcal{M}$ has a lattice reduct, it has a 3-ary near unanimity term, viz. the lattice median. This ensures, as a consequence of the multisorted version of the NU Duality Theorem [4, Theorem 3.3.8 and Corollary 3.3.9] that the set of all binary relations which are subuniverses of algebras $M_i \times M_j$ (where $i, j$ vary over $\{0, \ldots, n\}$) yields a duality on $\mathcal{A} = ISP(M)$. We stress that a critical part of the conclusion here is dualisability: there exists an alter ego yielding a duality. Moreover we obtain as a bonus a very explicit form of one such alter ego.

Finally we should comment on the claim in the theorem that the duality is strong. A duality can fail to be full if the alter ego is insufficiently rich, so that the dual category $\mathcal{X}$ is too big. In the lattice-based case, adding additional structure to the alter ego in the form of algebraic operations (sometimes partial), one can arrive at a duality which is strong, and hence full. However, under the very restricted conditions imposed on $\mathcal{M}$ in Theorem 3.1 it turns out that unary total operations suffice.

We would like the dualities we present to contain in their alter egos as few relations and operations as possible. Suppose $\mathcal{M}$, as in Theorem 3.1, is an alter ego yielding a duality on a class $\mathcal{A} = ISP(M)$. Then any $A \in \mathcal{A}$ is such that $A \cong ED(A)$; here the structure of $D(A)$ is completely determined by that of $M$, and the elements of $ED(A)$ are the multisorted continuous structure-preserving maps from $D(A)$ into $M$. From this it is clear that, for example, we gain nothing by including in $R$ both a binary relation and its converse. It is also never necessary to include ‘trivial relations’: those which are preserved automatically by $\mathcal{X}$-morphisms. Examples are $M_i^2$ and its diagonal subalgebra, for any $i$. Here we have very simple instances of entailment, sufficient for our immediate needs; see [4, Section 2.4] for further information.

We conclude this summary of facts from natural duality theory by drawing attention to two (single-sorted) dualities which fit into the special framework we have described and can be derived, albeit circuitously, from Theorem 3.1 and the remarks above. First we revisit Priestley duality, which we mentioned briefly at the start of this section. This provides a valuable tool for working with $\mathcal{D}$-based algebras, on which we shall draw heavily in Section 5 recall that the knowledge lattice reduct of each algebra in $\mathcal{V}_n$ belongs to $\mathcal{D}$. 

9
Example 3.2. (Priestley duality) We recall that a Priestley space is a topological structure $(X; \leq, \mathcal{T})$ in which $(X; \mathcal{T})$ is a compact space and $\leq$ is a partial order with the property that, given $x \not\leq y$ in $X$, there exists a $\mathcal{T}$-clopen up-set $U$ such that $x \in U$ and $y \not\in U$. The morphisms in the category $\mathcal{P}$ of Priestley spaces are the continuous order-preserving maps.

There is a dual equivalence between $\mathcal{D}$ and $\mathcal{P}$ constructed as follows. Let $2 = (\{0, 1\}; \land, \lor, 0, 1)$ be the two-element lattice in $\mathcal{D}$ and let its alter ego be $\mathcal{Z} = (\{0, 1\}; \leq, \mathcal{T})$. Then $\mathcal{D} = \mathcal{ISP}(2)$ and $\mathcal{P} = \mathcal{ISP}^+(2)$ and the hom-functors $\mathcal{H} = \mathcal{D}(\cdot, 2)$ and $K = \mathcal{P}(\cdot, 2)$ set up a strong duality between $\mathcal{D}$ and $\mathcal{P}$. This can be seen as a consequence of the single-sorted case of Theorem [3.1] and our comments on entailment. Since we later use Priestley duality in conjunction with a natural duality based on hom-functors $\mathcal{D}$ and $\mathcal{E}$, we adopt non-generic symbols $\mathcal{H}$ and $K$ for the hom-functors between $\mathcal{D}$ and $\mathcal{P}$.

Our second example serves to indicate that, for the base case of our hierarchy of default bilattices, a natural duality has already been worked out. We present this as for $\mathcal{ISP}(K_0)$, recalling that this variety is term-equivalent to the variety $\mathcal{D}\mathcal{B}$ studied in [2].

Example 3.3. (Natural duality for distributive bilattices [2, Theorem 4.2]) There is a dual equivalence between the category $\mathcal{ISP}(K_0)$ of distributive bilattices and the category $\mathcal{P}$ of Priestley spaces constructed as follows. The alter ego $K_0 = (\{T, \bot, t_0, f_0\}; \leq, \mathcal{T})$ for $K_0$ yields a strong (and hence full) duality on $\mathcal{V}_0 = \mathcal{ISP}(K_0)$ and moreover the dual category $\mathcal{ISP}^+(K_0)$ coincides with $\mathcal{P}$. We do not justify here the identification of the dual category, which can be found in [2]. We do, however, draw attention to the occurrence of the knowledge order $\leq_2$ in the alter ego. Because $K_0$ is a distributive bilattice, $\leq_2$ is an algebraic relation.

4. Dualities for varieties of prioritised default bilattices

In this section we present multisorted natural dualities for the varieties $\mathcal{V}_n = \mathcal{ISP}(M_n)$, where $M_n = \{K_0, \ldots, K_n\}$. Then the alter ego

$$\mathcal{M}_n = (K_0 \cup \cdots \cup K_n; R_n, G_n, \mathcal{T}),$$

where $R_n = \{S_{m,n} \mid 0 \leq m \leq n\}$ and $G_n = \{h_{i,j} \mid 1 \leq i \leq n\}$

yields a strong (and hence full) duality on $\mathcal{V}_n$.

Proof. The representation of $\mathcal{V}_n$ as $\mathcal{ISP}(M_n)$ was established in Section 2 where we also proved that each $K_m$ is subdirectly irreducible and has no proper subalgebras. Hence the structure $\mathcal{M}_n = (K_0 \cup \cdots \cup K_n; R, G, \mathcal{T})$, where $R = \bigcup \{S(K_i \times K_j) \mid i, j \in \{0, \ldots, n\}\}$ and $G = \bigcup \{V_n(K_i, K_j) \mid i, j \in \{0, \ldots, n\}\}$, yields a strong duality on $\mathcal{V}_n$. The theorem is then a consequence of two claims.

Claim 1: $G_n$ hom-entails $G$ (in the sense of the definition in [4, Section 3.2]).

By Corollary [2.3] $V_n(K_i, K_j) = \{h_{i,j}\}$ if $j \leq i$ and $V_n(K_i, K_j) = \emptyset$ otherwise. If $i = j$ then $h_{i,i}$ is the identity on $K_i$ and it is straightforward to check that $h_{i,j} = h_{i+1,j} \circ \cdots \circ h_{i,j-1}$ when $j < i$.

Claim 2: $R_n$ and $G_n$ entail $R$.

From Proposition [2.1(iv)], $S(K_i \times K_j) = \{(h_{i,j} \times h_{m,n})S) \mid S \in S(K_i^2)\}$. Using the entailment constructs in [4, Section 9.2] (specifically term manipulation and homomorphic relational product), we see that $S(K_i^2)$ and $G$ together entail $R$. By Claim 1, $S(K_i^2)$ and $G_n$ entail $G$. Also, from Proposition 2.1(i) we have $S_{m,n} = \{a, b \mid (h_{m,n}(a), h_{m,n}(b)) \in S_{m,n}\}$. We deduce from Theorem 2.2 and [4, Section 2.4] that $S(K_i^2)$ is entailed by $R_n$ and $G_n$. □
Referring to the proof of Theorem 4.1, let us see what simplification of the duality for \( \mathcal{V}_n \) is achieved by replacing \( G \) and \( R \) by \( G_n \) and \( R_n \). We have \( |G| = \frac{1}{2}(n + 2)(n + 1) \). By contrast, \( |G_n| = n \). Let \( B_1 = \{ j \mid S(K_j^2) \in R \} \). We use Proposition 2.2 to calculate \( |B_2| \) where \( B_2 = R \setminus B_1 \). We obtain \( |B_2| = \sum_{i=1}^{n-1} 2(n - j)(3 + 4) \). Hence \( |R| = \frac{1}{2}(3n + 8)(n + 1) + n(n + 1)(n + 3) = \frac{1}{2}(n + 1)(2n^2 + 9n + 8) \) whereas \( |R_n| = n + 1 \). But can we, using binary relations and unary operations, do any better?

In the setting of Section 3, \( \mathcal{M} = (M_0 \cup \ldots \cup M_n; R, G, T) \) is said to yield an optimal duality on \( \mathcal{A} = \mathcal{ES}(\mathcal{M}) \) if \( \mathcal{M} \) yields a duality on \( \mathcal{A} \) and if the alter ego \( \mathcal{M}' \) obtained by deleting any member of \( R \cup G \) fails to do so. Recalling that each \( r \in R \) is assumed to be algebraic, and so the subuniverse of an algebra \( r \in \mathcal{A} \), we may use \( r \) as a test algebra and seek to show that it is not true that \( r \equiv D'E'(r) \); here \( D' \) and \( E' \) denote the functors associated with the alter ego \( \mathcal{M}' \) obtained by deleting \( r \) from \( \mathcal{M} \). Likewise, we may seek to show an element \( h \in G \) is not redundant by using dom \( h \) as a test algebra. (For a full account of the test algebra strategy, see [4], Section 8.8.1.)

**Theorem 4.2.** The alter ego \( \mathcal{M}'_n = (K_0 \cup \cdots \cup K_n; R_n, G_n, T) \), as defined in Theorem 4.1, yields an optimal duality on \( \mathcal{V}_n \).

**Proof.** Recall that relations and homomorphisms are lifted from the multisorted alter ego pointwise, by sorts. In particular, if \( A \in \mathcal{V}_n \) then, for a binary relation \( r = S_{ij} \), and \( x, y \in \mathcal{D}(A) \),

\[(x, y) \in r^{(A)}(A, K) \iff \forall a \in A((x(a), y(a)) \in S_{ij}).\]

For this to hold, necessarily \( x, y \in \mathcal{V}_n(A, K) \). Hence \( r^{(A)} \) is the empty relation whenever \( \mathcal{V}_n(A, K) = \emptyset \). Likewise, for a homomorphism \( h_{ij-1} \),

\[y = h^{(A)}_{ij-1}(x) \iff \forall a \in A(y(a) = h_{ij-1}(x(a))) \iff y = h_{ij-1} \circ x,\]

and for this to hold it is necessary that \( x \in \mathcal{V}_n(A, K) \) and \( y \in \mathcal{V}_n(A, K) \).

Fix \( m \). We show that if \( S_{nm} \) were deleted from \( R_n \), then the map \( e_{S_{nm}} : S_{nm} \to ED(S_{nm}) \) cannot be surjective. Recall that \( D(S_{nm}) = \bigcup \{ \mathcal{V}_n(S_{nm}, K) \mid j \in \{0, \ldots, n\} \} \). To simplify notation, we shall denote \( D(S_{nm}) \) by \( X \). By Lemma 2.5 \( \mathcal{V}_n(S_{nm}, K_n) = \{ \rho_1, \rho_2 \} \), where \( \rho_1 \) and \( \rho_2 \) are the restrictions of the coordinate projections. Also, \( \mathcal{V}_n(S_{nm}, K_j) = \emptyset \) if \( j > m \). Now consider \( j < m \). Any homomorphism \( g : S_{nm} \to K_n \) is such that \( g(T_{j+1}) = T_{j+1} \) and hence for all \( (a, b) \in K_n \) we have \( g(a, b) = T_{j+1} \). This implies that \( g = h_{m, j} \circ \rho_1 = h_{m, j} \circ \rho_2 \) and hence \( \mathcal{V}_n(S_{nm}, K_j) = \emptyset \). Therefore \( X = \{ \rho_1, \rho_2 \} \setminus \{ \rho_1 \circ \rho_1 \mid 0 \leq j < m \} \). It is easy to check that \( (\rho_1, \rho_1), (\rho_2, \rho_2) \) and \( (\rho_1, \rho_2) \) belong to \( S_{nm} \), whereas \( (\rho_2, \rho_1) \) does not. We now want to construct a map \( \gamma_n : X \to \mathcal{M}_n \) such that \( \gamma_n \) preserves each member of \( (R_n \setminus S_{nm}) \cup G_n \) but does not preserve \( S_{nm} \). We define \( \gamma_n \) by

\[\gamma_n(x) = \begin{cases} I_m & \text{if } x = \rho_1, \\ T_{j+1} & \text{if } x = h_{m, j} \circ \rho_1. \end{cases}\]

This map does not preserve \( S_{nm} \) as \( (\gamma_n(\rho_1), \gamma_n(\rho_2)) = (I_m, T_{j+1}) \notin S_{nm} \). Consequently \( \gamma_n \) cannot be an evaluation map. Now we wish to show that \( \gamma_n \) does preserve the remaining structure in \( \mathcal{M}_n \). First we deal with the relations \( S_{ij} \) for \( j \neq m \). If \( j > m \) the relation \( S_{j,m} \) is empty. Now consider \( j < m \). The only element in \( \mathcal{V}_n(S_{nm}, K_j) \) is \( h_{m, j} \circ \rho_1 \). Since \( S_{j,m} \) is reflexive, \( S_{j,m} = \{ (h_{m, j} \circ \rho_1, h_{m, j} \circ \rho_1) \} \). Hence \( \gamma_n \) preserves \( S_{j,m} \) whenever \( j \neq m \). We claim also that \( \gamma_n \) preserves \( h_{j,i-1} \). If \( i > m \) then \( \mathcal{V}_n(S_{nm}, K_i) = \emptyset \) and \( h_{j,i-1} \) is the empty map and trivially preserved. If \( i \leq m \) then \( \mathcal{A}(S_{nm}, K_{i-1}) = \{ h_{m,i-1} \circ \rho_1 \} \). Thus \( h_{j,i-1} = h_{m,i-1} \circ \rho_1 \). So, for \( i < m \),

\[\gamma_n(h_{j,i-1}(h_{m,j} \circ \rho_1)) = T_{j+1} = h_{j,i-1}(T_{j+1}) = h_{j,i-1}(\gamma_n(h_{m,j} \circ \rho_1)).\]

If \( i = m \) we have \( \gamma_n(h_{m,m-1}(\rho_1)) = T_m = h_{m,m-1}(I_m) = h_{m,m-1}(\gamma_n(\rho_1)) \), and likewise \( \gamma_n(h_{m,m-1}(\rho_2)) = h_{m,m-1}(\gamma_n(\rho_2)) \). We conclude that \( \gamma_n \) preserves \( h_{j,i-1} \) for each \( i \leq m \).

Now we show that, for \( 1 \leq m \leq n \), we cannot remove the homomorphism \( h_{m,m-1} \) from the alter ego. We do this by considering \( Y = \mathcal{D}(K_m) \). We have \( \mathcal{V}_n(K_m, K) = \emptyset \) if \( i > m \) and \( \mathcal{V}_n(K_m, K) = \{ h_{m,j} \} \) if \( i < m \). Therefore if \( i > m \)
then \( h_{i-1}^Y \) is the empty map. If \( i \leq m \) then \( h_{i-1}^Y(h_{m,i}) \) has domain \( K_m \) and is the map \( h_{i-1} \circ h_{m,i} = h_{m,i-1} \). Moreover, \( S_{i,i}^Y = \emptyset \) if \( i > m \) and \( S_{i,i}^Y = \{(h_{m,i}, h_{m,i})\} \) for \( i \leq m \). Define \( \mu_m : Y \to \mathcal{M}_a \) by

\[
\mu_m(h_{m,i}) = \begin{cases} 
\top_{m-1} & \text{if } i = m, \\
\top_{m+1} & \text{if } i < m.
\end{cases}
\]

Trivially, \( \mu_m \) preserves each \( S_{i,j} \). Moreover, if \( i < m \) then

\[
\mu_m(h_{i-1}^Y(h_{m,i})) = \mu_m(h_{m,i-1}) = \top_i = h_{i-1}(\top_{m+1}) = h_{i-1}(\mu_m(h_{m,i})).
\]

that is, \( \mu_m \) respects \( h_{i,j} \) for any \( i < m \). But \( \mu_m \) does not respect \( h_{m,m-1} \):

\[
\mu_m(h_{m,m-1}^Y(h_{m,m})) = \mu_m(h_{m,m-1}) = \top_m \quad \text{and} \quad h_{i-1}(\mu_m(h_{m,m})) = h_{m,m-1}(\top_{m+1}) = \top_{m+1}.
\]

We shall now characterise the objects in our dual category \( \mathbb{P}^+(\mathcal{M}_n) \).

**Theorem 4.3.** Let \( \mathcal{M}_n \) be the alter ego defined in Theorem 4.1. Then a multisorted topological structure

\[
X = (X_0 \cup \ldots \cup X_n; \leq_0, \ldots, \leq_n, g_1, \ldots, g_n, T),
\]

where \( \leq_i \subseteq X_i^2 \) for \( 0 \leq i \leq n \) and \( g_j : X_j \to X_{j-1} \) for \( 1 \leq j \leq n \), belongs to \( \mathbb{P}^+(\mathcal{M}_n) \), if and only if

(i) \( (X_i; \leq_i, T) \) is a Priestley space for \( i \in [0, \ldots, n] \), where \( T \) is the topology induced by \( T \);

(ii) \( g_j : (X_j; T_j) \to (X_{j-1}; T_{j-1}) \) is continuous, for \( i \in [1, \ldots, n] \);

(iii) \( \text{if } x \leq_i y \text{ then } g_i(x) = g_i(y), \text{ for } i \in [1, \ldots, n] \).

**Proof.** Clearly \( S_{m,n} \) is a partial order on \( K_m \). And if \( a, b \in K_m \) with \( 1 \leq m \leq n \) are such that \( a \neq b \) and \( (a, b) \in S_{m,n} \), then \( h_{m,m-1}(a) = h_{m,m-1}(b) = \top_m \). This proves that \( \mathcal{M}_n \) satisfies (i), (ii) and (iii). Since each of (i), (ii) and (iii) are preserved under products and closed substructures, each \( X \in \mathbb{P}^+(\mathcal{M}_n) \) satisfies them.

To prove the converse, we shall invoke 4.1 Theorem 1.4.4. Assume that \( X \) satisfies (i), (ii) and (iii) and let \( x, y \in X \) and \( i \in [0, \ldots, n] \) be such that \( x \not\leq_i y \). By (i), since \( (X_i; \leq_i, T) \) is a Priestley space, there exists a clopen up-set \( U_{x,y,i} \) such that \( x \in U_{x,y,i} \) and \( y \notin U_{x,y,i} \). Define \( U_j \subseteq X_j \) by

\[
U_j = \begin{cases} 
U_{x,y,i} & \text{if } j = i, \\
X_j & \text{if } j < i, \\
(g_j \circ \cdots \circ g_{i+1})^{-1}(U_{x,y,i}) & \text{if } j > i.
\end{cases}
\]

By (ii), each \( g_j \) is continuous and, since \( U_{x,y,i} \) is clopen, each \( U_j \) is clopen. By (iii), each \( U_j \) is also an up-set. Define \( f_{x,y,i} : X \to \mathcal{M}_n \) by letting \( f_{x,y,i}(z) = \top_i \) if \( z \in U_j \) and \( f_{x,y,i}(z) = \top_{j+1} \in K_j \) otherwise. Each \( U_j \) is a clopen up-set, so \( f_{x,y,i} \) is order-preserving and continuous sort-wise. Let \( z \in X_j \) with \( 1 \leq j \leq n \). If \( j < i \) then \( f_{x,y,i}(z) = \top_{j+1} \) and \( f_{x,y,i}(g_j(z)) = \top_{j+1} \). Then \( f_{x,y,i}(g_j(z)) = g_j(f_{x,y,i}(z)) \). If \( j > i \) then \( f_{x,y,i}(g_j(z)) = \top_i \) if and only if \( g_j(z) \in U_{j-1} \). That is, \( f_{x,y,i}(g_j(z)) = \top_i \) if and only if \( g_j(z) \in U_{j-1} \). Then \( f_{x,y,i}(g_j(z)) = \top_i \) if and only if \( f_{x,y,i}(z) = \top_i \). We deduce that \( f_{x,y,i}(g_j(z)) = g_j(f_{x,y,i}(z)) \). If \( j = i \), then \( g_j(f_{x,y,i}(z)) = g_j(f_{x,y,i}(z)) = \top_i = f_{x,y,i}(g_j(z)) \). This proves that \( f_{x,y,i} \) is a morphism from \( X \) into \( \mathcal{M}_n \) and that \( (f_{x,y,i}(x), f_{x,y,i}(y)) = (\top_i, \top_{i+1}) \not\in S_{i,j} \). It follows that \( X \in \mathbb{P}^+(\mathcal{M}_n) \). 

**5. Relating the natural duality for \( V_n \) to Priestley duality**

The principal result in this section is Theorem 5.2. It will enable us to give information about free algebras in the varieties \( V_p \) and will later throw light on the product representation we present in Section 6. However in carrying out our analysis we call on recent results from 5.1, and so on aspects of duality theory for \( D \)-based algebras that we have not needed hitherto. Section 6 can if desired be read with almost no reference to this section.

We shall presuppose that the reader has some familiarity with basic facts concerning Priestley duality and its consequences. We recall that we use the non-generic symbols \( H \) and \( K \) for the functors setting up Priestley duality,
retaining $D$ and $E$ for the functors setting up the duality for $V_n$ given in Theorem 4.1. We may identify a lattice $L$ in $D$ with $KH(L)$. Identifying a continuous order-preserving function $x$ from $H(L)$, the Priestley dual space of $L$, into $\mathbb{2}$ with the set $\tau^{-1}(1)$, we may when convenient regard $L$ as the lattice of clopen up-sets of $H(L)$. A full account of Priestley duality and its consequences can be found for example in [7, Chapters 5 and 11], but we warn that the treatment there works with down-sets rather than up-sets.

As we have observed earlier, the algebras in $V_n$ have reducts in the category $D$ of bounded distributive lattices. Formally, there exists a natural forgetful functor $U$ from $\bigcup_{n \geq 0} V_n$ to $D$, sending an algebra $A$ to $(A; \otimes, \oplus, \bot, \top)$ and each morphism to the same map, now regarded as a $D$-morphism. We shall investigate the relationship between the natural duality we have set up for $V_n$ on the one hand and Priestley duality as it applies to the subcategory $U(V_n)$ of $D$ on the other. Of necessity, we work with knowledge lattice reducts since for $n \geq 1$ the truth lattice reduce is not distributive. This means that the treatment below does not align fully with that for $n = 0$ given in [3]. The difference is more notational than real and we can recommend the account given in [2] for the special case as an introduction to ideas we shall use also for general $n$.

Fix $n \geq 1$. We want to know how the multisorted dual space $D(A)$ is related to the Priestley dual space $HU(A)$ of $U(A)$ from which $U(A)$ can be recovered by Priestley duality. For any finitely generated $D$-based variety, and in particular for $V_n$, it is possible to set up an economical natural duality by what is known as the piggybacking method, without recourse to the NU Duality Theorem; see [4, Chapter 7]. Furthermore, this piggyback duality can be related to Priestley duality as it applies to the $D$-reducts, as shown in [3, Section 2]. We opted, however, not to employ this method to set up a natural duality for $V_n$. To have done so would have involved at the outset additional theoretical machinery and would not have yielded a quicker or more informative derivation. But now, with the insights gleaned from the approach we adopted in Sections 2–4, it is profitable to reconcile Theorem 4.1 with results from [3]. This reconciliation elucidates Theorem 5.3 and provides a bridge to the product representation in due course.

In preparation for Theorem 5.2 we need to relate our duality for $V_n$ from Theorem 4.1 to the results of [3, Section 2] as they apply to $A$, where $A = V_n = ISF(M_p)$. The key here—and we cannot emphasise this too strongly—is the relationship between $M_p$, viewed as a member of our natural dual category, and the sets $D(U(K_n), 2)$, for $0 \leq i \leq n$. For each such $i$ let $\omega'_i$ and $\omega'_j$ be the elements of $D(U(K), 2)$ for which $(\omega_i'^{-1})(1) = \top$, and $(\omega_j'^{-1})(1) = \top$; here the up-sets are calculated with respect to the $\leq_k$ order on $K_i$. Let $\Omega_M = \bigcup_{0 \leq i \leq n} \{\omega_i', \omega_j'\}$.

**Lemma 5.1.** Let $\Omega_M$ be as above.

(i) The following separation condition holds: given $m \in \{0, \ldots, n\}$ and $a \neq b \in K_m$, there exists $j \in \{0, \ldots, m\}$ and $\omega \in \{\omega'_i, \omega'_j\}$ such that $\omega(h_m(a)) \neq \omega(h_m(b))$.

(ii) Let $j, m \in \{0, \ldots, n\}$. For $\omega \in \{\omega'_i, \omega'_j\}$ and $\omega' \in \{\omega'_i, \omega'_j\}$, let $R_{\omega, \omega'}$ be the set of binary algebraic relations which are maximal with respect to being contained in $\{(a, b) \in K_j \times K_m \mid \omega(a) \leq \omega'(b)\}$. Then

(a) $R_{\omega, \omega'} = R_{\omega'_i, \omega'_j} = \{s_m, m\}$;

(b) if $j = m$ and $\omega \neq \omega'$ or $j > m$, then $R_{\omega, \omega'} = \emptyset$;

(c) if $j < m$ then $R_{\omega, \omega'} = \{(\text{graph} h_m, i)\}$.

**Proof.** Consider (i). Take $a \neq b$ in $K_m$, and assume without loss of generality that $a <_k b$. We consider two cases. Assume first that there exists $j \in \{0, \ldots, n\}$ with $b \leq_k \tau_j \leq_k a$. Then $\omega'(h_{m,j+1}(a)) = \omega'(h_{m,j+1}(b)) = 1$. On the other hand, $\omega'(h_{m,j}(b)) = 0$ or $\omega'(h_{m,j}(b)) = 0$, or both, must hold. Now assume that $a, b \in \{\tau_j, t_j, f_j, \tau_{j+1}\}$ for some $j$. Then for $\omega = \omega'_i$ or for $\omega = \omega'_j$, we have $\omega(h_m(a)) \neq \omega(h_m(b))$.

We now prove (ii). First take $j = m$ and $\omega \neq \omega'$ and assume that there existed $r \in S(K_m)$ such that $r \in R_{\omega, \omega'}$. Necessarily $r \supseteq \Delta_m$. But $\omega(p) \neq \omega'(p)$ either for $p = t_m$ or for $p = f_m$. Hence $R_{\omega, \omega'} = \emptyset$. Now consider $j > m$ and consider $r$ in $S(K_j \times K_m)$. By Proposition 2.1 iv there exists $s \in S(K_j \times K_m)$ such that $r = \{s, (a, h_{m,b}(b))\} \{a, b \in s\}$. 

13
Hence \((t_r, \tau_{m+1}) \in r\). Then \(\omega'_j(t_r) = 1 \not= 0 = \omega'(\tau_{m+1})\), which implies that \(r \not\in R_{\omega'_j\omega'}\). Similarly, \(\omega'_j(t_i) = 1 \not= 0 = \omega'(\tau_{m+1})\) implies that \(r \not\in R_{\omega'_j\omega'}\), which concludes the proof of (b).

Finally we prove (ii)(c). Assume \(j < m\). Then, for any \(r \in R_{\omega'_j\omega'}\), we have \(r \in \mathcal{S}(K_j \times K_s)\) and hence there exists \(s \in \mathcal{S}(K_j)\) such that \(r = \{ (h_{m,j}(c), b) \mid (c, b) \in s \}\). It is easily seen that \(r \subseteq \{ (a, b) \mid \omega(a) \leq \omega'(b) \}\) when \(s = \Delta_{m,j}\). Hence \(r\) is the convex of the graph of \(h_{m,j}\) and it belongs to \(R_{\omega'_j\omega'}\). If \(s \subseteq S_{m,j} \cap S_{m,j}\), then \(r = (\text{graph} h_{m,j})^\perp\). If \(s \not\subseteq S_{m,j} \cap S_{m,j}\), then \((\tau_j, \tau_{m+1}) \in r\). It follows that \(\omega(\tau_j) = 1 \not= 0 = \omega'(\tau_{m+1})\) and \(r \not\in R_{\omega'_j\omega'}\). \(\square\)

The definition of \(\prec\) in the following theorem may appear complicated, but the intuition behind it is quite simple. We consider the natural dual space of an algebra \(A \in \mathcal{V}_n\). This is a multisorted structure of the type described in Theorem [4,5]. We first ‘double up’ each sort \(X_m\) and give the doubled-up set an order determined by the partial order \(\leq_m\) and the maps \(\omega_m^i\) and \(\omega_m^i\). Then we use the maps \(g_i\) to arrange these sets in layers, in order of increasing \(m\).

**Theorem 5.2.** Let \(A \in \mathcal{V}_n\) and let \(D(A)\) be the structure \(X = (X_0 \cup \ldots \cup X_n; \leq_0, \ldots, \leq_n, g_1, \ldots, g_n, T)\)

Let \(Y = \bigcup \{ X_m \times \{ \omega_m^i, \omega_m^i \} \mid 0 \leq m \leq n \}\). Define a relation \(\prec\) on \(Y\) by

\[(x, \omega) \prec (y, \omega') \iff x \leq y \text{ and } \omega = \omega' \in \{ \omega, \omega' \}; \text{ or} \]

\[x = (g_{i+1} \cdots \circ g_j)(y), \text{ and } \omega \in \{ \omega, \omega' \}, \omega' \in \{ \omega, \omega' \}, \text{ for some } i < j.\]

Then \(Y = (Y; \prec, T)\) is a Priestley space isomorphic to \(HU(A)\).

**Proof.** It is a consequence of Theorems 2.1 and 2.3 and the separation condition established in Lemma 5.1(i) that \(\prec\) is a quasi-order on \(Y\) for which the partially ordered space obtained by quotienting by \(\{ \sim, \succ \}\) is isomorphic to \(HU(A)\).

We now claim that Lemma 5.1(ii) implies that \(\prec\) is a partial order rather than just a quasi-order. Each of the two copies of the sort \(X_m\) carries the pointwise lifting of the partial order \(S_{m,m}\), and no pair of elements, one from each copy, is related by \(\prec\). We view \((X_m \times \{ \omega_m^i \}) \cup (X_m \times \{ \omega_m^i \})\) as constituting the \(m\)-th level of \(HU(A)\), for \(m = 0, \ldots, n\). Lemma 5.1(ii)(b) and (c) tell us that, with respect to \(\prec\), no point at level \(m\) is related to a point at a strictly lower level. \(\square\)

In preparation for analysing the structure of the spaces \(HU(A)\) in particular cases we present some order-theoretic constructions involved in building such spaces. Consider first posets \(S\) and \(T\) and a map \(\varphi: T \to S\). Assume that \(\varphi\) is semi-constant, in the sense that it maps each order component of \(T\) to a singleton (see condition (iii) in Theorem 5.3); any such map is necessarily order-preserving. The restricted linear sum \(S \oplus_T \varphi\) will be the poset obtained by equipping the disjoint union \(S \cup T\) with the relation \(\leq_S \cup \leq_T \cup (\text{graph } \varphi)^\perp\); here \(\leq_S\) and \(\leq_T\) are the partial orders on \(S\) and \(T\), respectively.

Take \(T \xrightarrow{\varphi} S\) as above. We can then form a new poset, which we denote by \(\overline{S} \oplus \overline{T}\) refer to as the doubling of \(S \oplus_T \varphi\). The construction goes as follows Take the disjoint union \(S\) of copies \(S^1\) and \(S^2\) of \(S\) and the disjoint union \(T\) of copies \(T^1\) and \(T^2\) of \(T\). We let \(\varphi\) induce in the obvious way maps \(\varphi^i: T^i \to S^i\) (for \(i, j \in \{1, 2\}\)) and form the restricted linear sums \(T_j \xrightarrow{\varphi^i} S^i\). Pasting the order relations together in the obvious way by taking their union we obtain \(T \xrightarrow{\varphi} S\).

The two constructions above can unambiguously be extended to the situation in which we start from any finite sequence \(P_0, \ldots, P_n\) of posets and semi-constant maps \(\varphi_i: P_i \to P_{i+1}\). We can first form an iterated restricted linear sum \(P_0 \oplus_{\varphi_0} P_1 \oplus_{\varphi_1} \cdots \oplus_{\varphi_{n-1}} P_{n-1} \oplus_{\varphi_n} P_n\). Pictorially, that is, in terms of a Hasse diagram, we view this poset as having \((n+1)\) layers. The \(m\)-th layer is \(P_m\), and the ordering between the layer \(P_{m+1}\) and the layer \(P_m\) above it is determined by \(\varphi_m\), for \(m \geq 1\). Now we can apply the doubling construction, extended in the obvious way, to obtain a new poset \(\overline{Q_0} \oplus_{\overline{\varphi}_0} \overline{Q_1} \oplus_{\overline{\varphi}_1} \cdots \oplus_{\overline{\varphi}_{n-1}} \overline{Q_{n-1}} \oplus_{\overline{\varphi}_n} \overline{Q_n}\), where \(\overline{Q_i} = \overline{P_i}\), for \(0 \leq i \leq n\) and \(\overline{\varphi_i} = \overline{\varphi_i}\), for \(1 \leq i \leq n\).

We can extend these ideas in the obvious way to the setting of the category \(\mathcal{P}\), replacing posets by Priestley spaces, and requiring the linking maps between them to be continuous as well as semi-constant. Observe that the Priestley space \(T\) in Theorem 5.2 is obtained from the sorts of \(D(A)\) in just the way we have been describing above.

We turn now to examples.
Example 5.3. Take $V_n = ISP(K_0, \ldots, K_n)$ and consider the algebra $K_n$. Up to isomorphism, (the underlying poset of) $HU(K_n) = D(U(K_n), 2)$ is obtained as the doubling of the (restricted) linear sum $\{f_0\} \oplus \{f_1\} \oplus \cdots \oplus \{f_n\}$, and which, suggestively, we label as in Fig. 4. This is exactly what we obtain from Theorem 5.2 if we identify $t_i$ and $f_i$ with the characteristic functions of their up-sets with respect to $\leq_k$ on $K_n$. (Note that $V_n(K_n, K_m) = \{h_{n,m}\}$ for $0 \leq m \leq n$, so that $D(K_n)$ consists of $n + 1$ singletons.) Example 6.2 provides a complementary discussion of this example in terms of our product representation. There we shall consider the representation of $K_n$ and not just of $U(K_n)$.

Example 5.4. (Priestley duals of reducts of free algebras in $V_n$) We recall from Section 3 the fundamental fact that, in a natural duality for a class $A = ISP(M)$ based on an alter ego $M \sim \mathcal{X}$, the free algebra $F_A(S)$ on a non-empty set $S$ of free generators is such that $D(F_A(S)) = M^U$ (up to isomorphism in the topological quasivariety $X = ISP^+(M)$).

Let us apply Theorem 5.2 first to identify $HU(F_{V_1}(1))$ as a poset (its topology is discrete and plays no role). The required poset is obtained by applying doubling to $K_0 \oplus h_{1,0} \oplus K_1 \oplus h_{2,1} \oplus \cdots \oplus h_{n-1,n-2} \oplus K_{n-1} \oplus h_{n,m} \oplus K_m$. Here $K_m$ is equipped with the partial order $S_{m,m}$. With respect to this order it is the disjoint union of an antichain with $3m$ elements and $2^2$, where $2$ denotes the two-element chain. Figure 5 shows the restricted linear sum $K_0 \oplus h_{1,0} \oplus K_1$ and Fig. 6 shows its doubling, $HU(F_{V_1}(1))$. In the figure, points shown by circles belong to level 0 and those by squares belong to level 1.

Now let us describe $HU(F_{V_n}(k))$, where $k$ is finite (we consider the infinite case below). The critical point is that $D(F_{V_n}(k))$ may be identified with $M^U_{k^n}$ with the power being calculated ‘by sorts’. Once this is done, the translation to
the Priestley dual $\text{HU}(F_{V_i}(k))$ proceeds as described in Theorem 5.2. The $m$th-layer of $\text{HU}(F_{V_i}(k))$ is $K_m^k$. This can be obtained by induction on $k$. The linking map from $K_m^k \rightarrow K_{m-1}^k$ is the $k$-fold product map $h_{m,m-1} \times \cdots \times h_{m,1}$. We can then describe $\text{HU}(F_{V_i}(k))$, using doubling, in the same way as for the case $k = 1$.

Our next task is to reveal how to recover $U(A)$ from $\text{HU}(A)$, for $A \in V_n$, taking advantage of the layered structure of this Priestley dual space. For simplicity we shall first confine our remarks to the case $n = 1$ and to finite $A$, so that we are dealing with posets with two layers. We need to describe the up-sets of a poset of the form $Q_0 \oplus_{\psi} Q_1$, obtained by doubling from a poset $P_0 \oplus_{\varphi} P_1$, where $\varphi : P_1 \rightarrow P_0$ is semi-constant.

First consider $U(P)$, the family of up-sets of $P$, where $P = P_0 \oplus_{\psi} P_1$. Every set in $U(P)$ takes the form $(V_0 \cup \varphi^{-1}(V_0)) \cup V_1$, where $V_i$ is an up-set in $P_i$ (for $i = 0, 1$) and, since $\varphi$ is semi-constant, we can choose $V_0$ and $V_1$ such that $\varphi^{-1}(V_0) \cap V_1 = \emptyset$; distinct pairs $(V_0, V_1)$ give rise to distinct up-sets of $P$. Describing $U(P)$ in full can be a complicated task. We note however that we can easily get crude estimates for the cardinality of $U(P)$ when $P_0$ and $P_1$ are finite. We have

$$|U(P_0)| + |U(P_1)| - 1 \leq |U(P)| \leq |U(P_0)| \times |U(P_1)|$$

The upper and lower bounds come from consideration of, respectively, the linear sum $P_0 \oplus P_1$ and the disjoint union $P_0 \cup P_1$.

**Example 5.5.** (The free algebras $F_{V_i}(1)$) We first take $n = 1$. For $i = 0, 1$, we consider $K_i$ equipped with the partial order $S_{ij}$. Each is the Priestley dual of a member $L_0$ of $\mathcal{D}$, for $i = 0, 1$. By elementary Priestley duality, $L_0$ is the linear sum $L_0 = L_0^1 \oplus L_0^2$, that is, a four-element Boolean lattice with new bottom and top elements adjoined. The lattice $L_1$ equals $2^3 \times L_0$. Then $\text{HU}(F_{V_i}(1))$ is a $\mathcal{D}$-sublattice of $L_0^{2^3} \times L_0^{2^3}$.

By calculating the number of up-sets of its Priestley dual, as shown in Fig. 6, we obtain $|F_{V_i}(1)| = 5879$. We can compare this value with our crude upper and lower bounds; $|L_0^{2^3}| + |L_0^{2^3}| - 1 = 2339$ and $|L_0^{2^3} \times L_0^{2^3}| = 82944$. We also draw attention to the difference between the size of $F_{V_i}(1)$ and that of $F_{V_0}(1)$, which is 36 (the Priestley dual is $2^3 \cup 2^3$). Two factors are at work here: the passage to a strictly larger variety and, perhaps more significantly, the weakening of the relation of equivalence between bilattice terms as a result of loss of distributivity.

We can quickly see how $\text{HU}(F_{V_i}(1))$ is obtained order-theoretically from $\text{HU}(F_{V_{i-1}}(1))$, for $n \geq 1$, by adding a new top layer. This gives $|F_{V_i}(1)| \geq |F_{V_{i-1}}(1)| + 36(2^6)^n$. Hence, we can obtain a lower bound for $|F_{V_i}(1)|$ as follows

$$|F_{V_i}(1)| \geq 36 \left(\frac{(2^6)^{n+1} - 1}{2^6 - 1}\right) \geq \frac{1}{2} \left(2^{6^n}\right).$$

So far we have looked at $\text{HU}(A)$ for $A$ a finite algebra in $V_n$ and investigated in particular the lattice $U(F_{V_1}(1))$. We now make some comments applicable to arbitrary algebras. For infinite $A$ the order components within each layer are clopen in $\text{HU}(A)$ and no significant issues arise in passage from the finite to the infinite case. Let $A \in V_n$. Associated with $Y = \text{HU}(A)$ is another Priestley space $Z$ obtained by deleting the order relations between the layers. It is a disjoint union of Priestley spaces $X_i$ ($0 \leq i \leq n$), where each of these is the disjoint union of a Priestley space
is against its negation and vice versa. Thus we obtain a bilattice structure for greater than the corresponding evidence for \( \varphi \) and knowledge order, \( \Box \).

The Product Representation Theorem

Consider \( F_V(\kappa) \), where now \( \kappa \) is infinite. The component layers of \( \mathcal{D}(F_V(\kappa)) \) (for \( 0 \leq m \leq n \)) are the Priestley space powers \( K_m^\kappa \), where \( K_m \) carries the discrete topology; the linking map from \( K_m^\kappa \) to \( K_{m-1}^\kappa \) is determined by its compositions with the coordinate projections; each of these compositions is \( h_{m,m-1} \).

Theorem 5.2 gives us access to a concrete representation of the \( \mathcal{D} \)-reduct \( U(A) \) for \( A \in V_n \), but not in a way which encodes the full bilattice structure. We remedy this omission in Section 6 by presenting our product representation. This will rely on showing how \( U(A) \) regarded as a sublattice of \( L_n^2 \times \cdots \times L_n^2 \) supports operations \( \land, \lor \) and \( \neg \) (the ones suppressed by \( U \)). We thereby arrive at an algebra isomorphic to the original bilattice \( A \).

We were led to consider \( L_n^2 \times \cdots \times L_n^2 \) when we deleted the ordering between successive layers of the Priestley dual space \( HU(A) \). This ordering is derived from (the pointwise lifting to \( D(A) \) of) the maps \( g_i \), for \( 1 \leq i \leq n \) (see Theorem 4.3 as it applies to \( X = D(A) \)). The following elementary lemma reveals the lattice-theoretic content of condition (iii) in that theorem.

**Lemma 5.6.** Let \( L, M \in \mathcal{D} \) and let \( X = H(L) \) and \( Y = H(M) \) be the Priestley dual spaces of \( L \) and \( M \). Let \( f \in \mathcal{D}(L, M) \) and let \( \varphi = H(f) \) be the dual map. Then the following statements are equivalent:

1. \( \varphi : Y \rightarrow X \) is semi-constant;
2. each element of \( f(L) \) has a complement in \( M \).

**Proof.** It will be convenient to identify \( L \) and \( M \) with the clopen up-sets of \( X \) and \( Y \), respectively, and to regard \( f \) as being given by \( f(V) = \varphi^{-1}(V) \), for each clopen up-set \( V \) of \( L \). Re-stated in these terms, (2) becomes the statement that \( \varphi^{-1}(V) \) is a down-set for each clopen up-set \( V \) in \( Y \). Assume that this is false for some \( V \). Then we would be able to find \( p \) and \( q \) in \( Y \) with \( p < q \) but \( \varphi(p) \not\in \varphi^{-1}(V) \) and \( q \in \varphi^{-1}(V) \). But this is incompatible with (1).

Conversely, assume (1) fails. Then there exist \( p \) and \( q \) in \( Y \) with \( p < q \) but \( \varphi(p) \not\in \varphi^{-1}(V) \); here we have used the fact that \( \varphi \) is order-preserving. Since \( X \) is a Priestley space, there exists a clopen up-set \( W \) in \( X \) with \( \varphi(q) \in W \) and \( \varphi(p) \not\in W \). Then \( V = \varphi^{-1}(W) \) is a clopen up-set in \( Y \). But consideration of \( p \) and \( q \) shows that \( V \) cannot be a down-set.

In the lemma the idea of levels of default seems very distant. Interestingly, we shall see shortly that condition (2) emerges in a natural way in the context of reasoning with defaults and helps to motivate the construction underlying our product representation.

6. The Product Representation Theorem

As we stressed at the outset, our objective is to obtain a product-style representation for the members of \( V_n \). Our analysis of \( \mathcal{D} \) reducts in the preceding section gives pointers as to how this might work, but did not give the full-blown representation we seek, because we did not encompass the operations suppressed by the forgetful functor \( U \).

Before introducing the formalism we shall employ, we give some intuition behind the construction of a bilattice from a (not necessarily distributive) lattice (see [3] for background on the construction). Consider a situation in which the lattice \( L \) arises as a lattice of possible evidence (for example collected for a trial). Then, given a certain statement \( s \), we assign to it a pair \((a, b) \in L^2\), where \( a \) denotes the evidence in favour of \( s \) (the positive evidence) and \( b \) the evidence against \( s \) (the negative evidence). Clearly we could have non-empty intersection between the positive and negative evidence for \( s \), depending on the interpretation, and we may also have evidence that is neither for \( s \) nor against it. The product \( L \times L \) admits two orderings. Let \( (a, b) \) and \( (a', b') \) be elements of \( L \times L \). In the knowledge order, \( s \sqsubseteq_k s' \) if \( a \leq b \) and \( b \leq b' \) in \( L \), meaning that both the positive and negative evidence for \( s \) is no greater than the corresponding evidence for \( s' \). In the truth order, \( s \sqsubseteq_s s' \) if \( a \leq b \) and \( b' \leq b \), meaning that the positive evidence for \( s \) is no greater than that for \( s' \), and the opposite holds for the negative evidence. There is also a natural interpretation of negation in this set-up, given by \( \neg(a, b) = (b, a) \), so that whatever evidence is in favour of a statement is against its negation and vice versa. Thus we obtain a bilattice structure \( L \sqcup L \) whose universe is \( L \times L \).

It is natural to consider evidence being accumulated in an iterative fashion, with pre-existing evidence taken into account by default, and to be seen as having priority. Such earlier evidence, encoded by a lattice \( L' \), might have
come from statistics, previous trials, or from other sources. So new information, captured by a lattice \( L \), might be of a kind different from that encoded by \( L' \) but should be assumed to be somehow connected to it. The connection between \( L \) and \( L' \) can be modelled by a homomorphism from \( L \) into \( L' \), meaning that the evidence encoded in \( a \in L \) has precedence over any information that is contained in \( h(a) \). For example, if \( a \) and \( b \) in \( L \) represent the positive and negative evidence for a certain statement, then any default information (positive or negative) that is less informative than \( h(a \lor b) \) does not add truly new information. Therefore, if the evidence about a statement is encoded by \((a, b) \in L \times L\) and the default information that we get about the same statement is encoded by \((c, d) \in L' \times L'\), then \( h(a \lor b) \leq c \) and \( h(a \lor b) \leq d \). In this fashion the evidence overrides any positive or negative default information that we have about certain statements.

We shall now present our product representation. We begin by setting up an equivalence between \( \mathbb{I} \), as described in Theorem 4.3, and another category related to \( V_n \). This equivalence implicitly subsuces parts of the ‘doubling’ framework presented in Section 4 and provides a convenient formalism for developing our theory in an algebraic setting.

For \( n \geq 0 \), we define an \( n \)-default sequence to be a sequence

\[
\Xi \equiv L_0 \xrightarrow{h_1} L_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} L_{n-1} \xrightarrow{h_n} L_n,
\]

where \( L_i \in \mathcal{D} \), for \( 0 \leq i \leq n \), and \( h_j \in \mathcal{D}(L_{j-1}, L_j) \), for \( 1 \leq j \leq n \), are such that each \( c \in h_j(L_{j-1}) \) has a complement \( c' \) in \( L_j \). The \( n \)-default sequences support a natural categorical structure. More precisely: an \( n \)-default morphism \( f \) from \( \Xi \) to \( \Xi' \) is an \((n + 1)\)-tuple of \( \mathcal{D} \)-morphisms \((f_0, \ldots, f_n)\) such that the diagram in Fig. 7 commutes. It is easy to see that the class \( \mathcal{D}S_n \) of \( n \)-default sequences with \( n \)-default morphisms is indeed a category.

![Figure 7: n-default morphism](image)

By Theorem 4.3, a simple extension of the functors \( H \) and \( K \) determines a dual equivalence between \( \mathbb{I} \), and \( \mathcal{D}S_n \). This is set up by the functors \( H_\Xi : \mathcal{D}S_n \rightarrow \mathbb{I} \) and \( K_\Xi : \mathbb{I} \rightarrow \mathcal{D}S_n \). The functor \( H_\Xi \) is defined as follows:

- on objects: \( H_\Xi(L_0) \xrightarrow{h_1} H_\Xi(L_1) \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} H_\Xi(L_{n-1}) \xrightarrow{h_n} H_\Xi(L_n) \), and \( H_\Xi(h_i) = (X_i; \leq_i, \cup_i) \) for \( i \in \{0, \ldots, n\} \) then
  \[
  H_\Xi(\Xi) = (X_0 \cup \ldots \cup X_n; \leq_0, \ldots, \leq_n, H(h_1), \ldots, H(h_n), \cup),
  \]

- where \( \cup \) is the disjoint union topology;

- on morphisms: \( H_\Xi(f_0, \ldots, f_n) = H(f_0) \cup \cdots \cup H(f_n) \).

In the other direction, we define \( K_\Xi \) as follows:

- on objects: \( X = (X_0 \cup \cdots \cup X_n; \leq_0, \ldots, \leq_n, g_1, \ldots, g_n, \cup) \) and \( E_i = K(X_i; \leq_i, \cup_i) \) for \( i \in \{0, \ldots, n\} \) then
  \[
  K_\Xi(E) = E_0 \xrightarrow{K(g_0)} E_1 \xrightarrow{K(g_1)} \cdots \xrightarrow{K(g_{n-1})} E_{n-1} \xrightarrow{K(g_n)} E_n
  \]

- on morphisms: \( K_\Xi(u) = (K(u | x_0), \ldots, K(u | x_n)) \), where \( u | x_i : X_i \rightarrow K_i \) is the restriction of \( u \) to \( X_i \) for \( i \in \{0, \ldots, n\} \).

The discussion above paves the way to our construction of a \emph{product default bilattice}. Given an \( n \)-default sequence \( \Xi \equiv L_0 \xrightarrow{h_1} \cdots \xrightarrow{h_n} L_n \), we shall define a default bilattice \( \Xi \oplus \Xi \) isomorphic in \( V_n \) to \( E \circ H_\Xi(\Xi) \) whose universe \( A \) is included in \( L_0^2 \times \cdots \times L_n^2 \). If \( a \in L_0^2 \times \cdots \times L_n^2 \), then \( (a_{i,1}, a_{i,2}) \in L_i^2 \) denotes the pair formed from the \((2i + 1)\)- and \((2i + 2)\)-coordinates of \( a \). We define

\[
A = \{ a \in L_0^2 \times \cdots \times L_n^2 \mid h_i(a_{i-1,1} \lor a_{i-1,2}) \leq a_{i,1}, a_{i,2} \text{ for } 1 \leq i \leq n \}.
\]
In terms of coordinates, the negation operation is given by
\[ \neg a = a \wedge \top, \quad \neg a = a \vee \top \quad \text{for } a \in \mathcal{L}_i. \]

We define \( \iota : A \to \mathbf{E} \circ H_n(\mathcal{E}) \) recursively in the following way. Fix \( a \in A \) and let \( z \in H_n(\mathcal{L}_i) \).

**Case 1:** Assume that either \( i = 0 \) or that \( i > 0 \) and \( \iota(a)(z \circ h_1) = \top_i \). Then we define
\[
\iota(a)(z) = \begin{cases} 
\top_i & \text{if } z(a_{i,1}) = z(a_{i,1}) = 1, \\
\bot_i & \text{if } z(a_{i,1}) = 1 \text{ and } z(a_{i,1}) = 0, \\
\iota & \text{if } z(a_{i,1}) = 0 \text{ and } z(a_{i,1}) = 1, \\
\top_{i+1} & \text{if } z(a_{i,1}) = z(a_{i,1}) = 0.
\end{cases}
\]

**Case 2:** Assume that \( i > 0 \) and \( \iota(a)(z \circ h_1) \neq \top_i \). Then we define \( \iota(a)(z) = \iota(a)(z \circ h_1) \).
Since each \( x \in H_n(\mathcal{L}_i) \) is continuous and order-preserving, so is \( \iota(a) \). It is routine to check that \( \iota(a) \circ H_n(h_1) = h_{i-1} \circ \iota(a) \).
Hence \( \iota \) is well defined. Furthermore, \( \iota \) is bijective and its inverse is given as follows. Let \( f = f_0 \cup \cdots \cup f_n \) belong to \( \mathbf{E} \circ H_n(\mathcal{E}) \) and \( a_{i,1} \) and \( a_{i,0} \) be the unique elements of \( \mathcal{L}_i \) determined by the clopen up-sets \( f_i^{-1}(\top_i) \) and \( f_i^{-1}(\bot_i) \), respectively. Then \( \iota^{-1}(f) = ((a_{i,0}, a_{i,1}), \ldots, (a_{i,0}, a_{i,1})) \).

We shall now use \( \iota \) to define the bilattice operations
\[
\mathcal{E} \circ \mathcal{E} = (A; \otimes, \wedge, \vee, \neg, \bot, \top)
\]
in such a way that \( \mathcal{E} \circ \mathcal{E} \approx \mathbf{E} \circ H_n(\mathcal{E}) \). This is done simply by arranging that, for each \( a, b \in A \),
\[
a \star b = \iota^{-1}(\iota(a) \star \iota(b)) \quad \text{for } \star \in \{ \vee, \wedge, \otimes, \neg \}; \quad \neg a = \iota^{-1}(\neg \iota(a)); \quad \top = \iota^{-1}(\top); \quad \bot = \iota^{-1}(\bot).
\]

In what follows we present an alternative description of the operations in \( \mathcal{E} \circ \mathcal{E} \) in terms of the coordinates of the elements of \( A \) involved and the homomorphisms \( h_i \) of the sequence \( \mathcal{E} \). This description is intrinsic to \( \mathcal{E} \) and does not refer to the map \( \iota \). First we let
\[
(a \otimes b)_{i,1} = a_{i,1} \wedge b_{i,1}, \quad (a \otimes b)_{i,1} = a_{i,1} \vee b_{i,1}, \quad (a \otimes b)_{i,1} = a_{i,1} \wedge b_{i,1}, \quad (a \otimes b)_{i,1} = a_{i,1} \vee b_{i,1}, \quad (a \vee b)_{i,1} = a_{i,1} \otimes b_{i,1}, \quad (a \vee b)_{i,1} = a_{i,1} \otimes b_{i,1}.
\]

Inductively, if we have specified \( (a \star b)_{i-1} \), then in the \( i \)th pair of coordinates, we take account of the previously encoded information to obtain \( (a \star b)_i \). We recall that the definition of a default sequence assumes that, for \( i > 0 \), the homomorphism \( h_i : L_{i-1} \to L_i \) is such that each element \( c \) of \( h_i(L_{i-1}) \) has a complement \( c' \) in \( L_i \). We let
\[
(a \otimes b)_{i,1} = a_{i,1} \wedge b_{i,1}, \quad (a \otimes b)_{i,1} = a_{i,1} \wedge b_{i,1}, \quad (a \wedge b)_{i,1} = a_{i,1} \vee b_{i,1}, \quad (a \wedge b)_{i,1} = a_{i,1} \vee b_{i,1}, \quad (a \wedge b)_{i,1} = a_{i,1} \wedge b_{i,1}, \quad (a \wedge b)_{i,1} = a_{i,1} \wedge b_{i,1}.
\]

In terms of coordinates, the negation operation is given by
\[
\neg a_{i,1} = a_{i,1}, \quad \neg a_{i,1} = a_{i,1} \quad \text{for } i \in \{0, \ldots, n\};
\]

\[ \mathcal{V}_n = \text{ISP}(M_n) \xrightarrow{D} \text{ISP}^1(M_n) \xrightarrow{K_n} \mathbf{DS}_n \]
and the constants by \( T = (1, \ldots, 1) \) and \( \bot = (0, \ldots, 0) \).

We demonstrate for \( \oplus \) and \( \lor \) that our alternative specifications fit with the definitions in terms of \( \iota \) that we gave initially. The remaining operations are handled similarly. Given \( a, b \in A \) and \( x_0 \in H(L_0) \),

\[
x_0(\iota^{-1}(a \oplus b), x_0) = 1 \iff (a \oplus b)(x_0) \in [t_0, t_0] \iff \iota(a)(x_0) \in [t_0, t_0] \lor \iota(b)(x_0) \in [t_0, t_0] \iff x_0(a_0, x_0, b_0, x_0) = 1 \iff x_0(a_0, x_0) = 1 \lor x_0(b_0, x_0) = 1;
\]

likewise one can show

\[
x_0(\iota^{-1}(a \lor b), x_0) = 1 \iff (a \lor b)(x_0) \in [t_0, t_0] \iff x_0(a_0, x_0) = 1 \lor x_0(b_0, x_0) = 1;
\]

\[
x_0(\iota^{-1}(a \lor b), x_0) = 1 \iff (a \lor b)(x_0) \in [t_0, t_0] \iff x_0(a_0, x_0) = 1 \lor x_0(b_0, x_0) = 1;\]

Now consider \( i \geq 1 \). We give the first calculation in full detail so as to bring out clearly how the definition of the universe \( A \) comes into play. The second and subsequent calculations are more abbreviated but involve the same ideas. Let \( x_i \in H(L_i) \). Then

\[
x_i(\iota^{-1}(a \oplus b), x_i) = 1 \iff (a \oplus b)(x_i) \in [t_i, t_i],
\]

\[
x_i(\iota^{-1}(a \lor b), x_i) = 1 \iff (a \lor b)(x_i) \in [t_i, t_i],
\]

\[
x_i(\iota^{-1}(a \lor b), x_i) = 1 \iff (a \lor b)(x_i) \in [t_i, t_i].
\]

We now consider \( \lor \). In the last step of the first calculation we use the fact that \( x_i : L_i \to 2 \) is a lattice homomorphism.

In the second calculation we need additionally the fact that, for a complemented element \( c \) in \( L_i \), we have \( x_i(c) = 0 \) if and only if \( x_i(c^*) = 1 \). We have

\[
x_i(\iota^{-1}(a \lor b), x_i) = 1 \iff (a \lor b)(x_i) \in [t_i, t_i],
\]

\[
x_i(\iota^{-1}(a \lor b), x_i) = 1 \iff (a \lor b)(x_i) \in [t_i, t_i].
\]

20
\[ \iff a(x_i) \in \{ t \}_{j \in \mathbb{N}} \cup \{ \top \}_{j \in \mathbb{N}} \text{ or } b(x_i) \in \{ t \}_{j \in \mathbb{N}} \cup \{ \top \}_{j \in \mathbb{N}} \text{ or } \{ \neg(a(x_i)), \neg(b(x_i)) \} \leq \{ \top, f \}, \]
\[ \iff a(x_i \circ h_j) \in \{ t \}_{j \in \mathbb{N}} \cup \{ \top \}_{j \in \mathbb{N}} \text{ or } b(x_j \circ h_i) \in \{ t \}_{j \in \mathbb{N}} \cup \{ \top \}_{j \in \mathbb{N}} \]
\[ \text{or } (x_i(a_{j,t}) = 1 \text{ and } x_i(b_{j,t}) = 1) \]
\[ \iff x_i(h_j((a \lor b)_{i-1,t})) = 1 \text{ or } x_i(h_j((a \lor b)_{i-1,t})) = 1 \text{ or } x_i(a_{j,t} \land b_{j,t}) = 1. \]

Very little work is needed to complete the proof of the following theorem.

**Theorem 6.1. (Product Representation Theorem)** For each default bilattice \( B \in \mathcal{V}_n \) there exists an \( n \)-default sequence \( \Xi \) such that \( B \cong \Xi \circ \Xi \).

**Proof.** It suffices to consider \( \Xi = (K_n \circ D)(B) \) and to observe that, by definition of \( \Xi \circ \Xi \) and the equivalence between \( \mathcal{HSP}(K_n) \) and \( \mathcal{DS}_n \),
\[ B \cong (E \circ H_n)((K_n \circ D)(B)) = (E \circ H_n)(\Xi) \cong \Xi \circ \Xi. \]

The operations in a lattice determine, and are determined by, the underlying order. This leads us to enquire how the knowledge and truth orders of \( \Xi \circ \Xi \) can be characterised. Our definitions of \( \odot \) and of \( \odot \) imply immediately that these operations on the subset \( A \) coincide with the coordinatewise-defined lattice operations on \( (L_0^2 \times \cdots \times L_0^2) \). As a consequence, the knowledge order \( \leq_k \) of the bilattice \( \Xi \circ \Xi \) is the inherited coordinatewise order. (An application of Theorem 5.2 provides an alternative proof of this statement.) One may then ask whether the truth order \( \leq_t \) can likewise be described in terms of coordinates. Certainly, for \( a, b \in A \), we have \( a \leq_t b \) if and only if \( a \vee b = b \) (or equivalently if and only if \( a \land b = a \)). The fact that \( a \vee b = c^{-1}(a \lor b) \lor b \) implies that \( a \leq_t b \) if and only if \( (a \lor b, \lor b) \) implies \( a \leq_t b \). However, an intrinsic description of \( \leq_t \) is not easy to formulate in general.

We end this section with a simple example, spelled out in detail. We include a recursive description of the truth order, capitalising on the fact that each lattice in our default sequence has only two elements.

**Example 6.2.** Let \( n \geq 1 \) and \( \Xi_n \) be the \( n \)-default sequence:

\[ \Xi_n = 2 \xrightarrow{id} 2 \xrightarrow{id} \cdots \xrightarrow{id} 2 \xrightarrow{id} 2, \]

where \( h_i = \text{id} : 2 \to 2 \), the identity map, for each \( i \). Here
\[ A_n = \{ a \in 2^{2(n+1)} \mid h_i(a_{i-1,1} \lor a_{i-1,1}) \leq a_{i,t}, a_{i,t} \text{ for } 1 \leq i \leq n \}\]
\[ = \{ a \in 2^{2(n+1)} \mid \max(a_{i-1,1}, a_{i-1,1}) \leq \min(a_{i,t}, a_{i,t}) \text{ for } 1 \leq i \leq n \}. \]

No restrictions are imposed on \( a_{0,t} \) or \( a_{0,\ell} \), so there are four choices for this pair. However, if \( 1 \in \{ a_{1,1}, a_{1,\ell} \} \) then \( a_{1,1} = 1 = a_{1,\ell} \) for all \( j > i \). It follows that \( A_n \) has \( 4 + 3n \) elements.

This calculation strongly suggests that \( \Xi_n \circ \Xi_n \) is isomorphic to \( K_n \) (note also Example 5.3). We would like to show that the map \( \Phi_n \) defined by
\[ \Phi_n(a) = \begin{cases} 
\top_i & \text{if } 1 = a_{i,t} = a_{i,\ell}, \text{ and } a_{j,\ell} = a_{j,t} = 0 \text{ for each } j < i, \\
\top_j & \text{if } 1 = a_{i,t} \text{ and } 0 = a_{i,\ell}, \\
\top_j & \text{if } 1 = a_{i,\ell} \text{ and } 0 = a_{i,t}, \\
\top_{n+1} & \text{if } a_{i,\ell} = a_{i,t} = 0 \text{ for } 0 \leq j \leq n 
\end{cases} \]
is an isomorphism of default bilattices. Certainly \( \Phi_n \) is surjective. Since its domain and range have the same cardinality, it is also injective. We want to show that \( \Phi_n \) preserves all the bilattice operations. This is trivial for \( \perp, \top \) and \( \neg \) and routine for \( \odot \) and \( \odot \). We now apply our general formulae to calculate \( (a \land b)_{i,v} \) for \( a, b \in A_n \), where \( v \in \{ t, \ell \} \).

Since \( (a \land b)_{i,v} \in \{ 0, 1 \} \), it will suffice to give the conditions under which the value is 1. Note first that \( (a \land b)_{0,1} = 1 \) if and only if \( a_{0,1} = b_{0,1} = 1 \) and that \( (a \land b)_{0,\ell} = 1 \) if and only if \( a_{0,\ell} = 1 \) or \( b_{0,\ell} = 1 \). For \( i > 1 \),
\[ (a \land b)_{i,t} = 1 \iff (a_{1,t} = 1 \text{ and } b_{1,t} = 1) \text{ or } ((a \land b)_{i-1,t} = 1 \text{ or } (a \land b)_{i-1,\ell} = 1); \]
\[ (a \land b)_{i,\ell} = 1 \iff (a_{1,\ell} = 1 \text{ and } a_{i-1,1} = 0) \text{ or } (b_{1,\ell} = 1 \text{ and } b_{i-1,1} = 0) \text{ or } ((a \land b)_{i-1,\ell} = 1 \text{ or } (a \land b)_{i-1,t} = 1). \]
Note that the recursive components of these definitions reflect the restriction imposed by \( a \land b \) belonging to \( \Xi_n \otimes \Xi_n \). Similar considerations apply to \( \lor \). It can now be verified that \( \Phi_n \) preserves \( \land \) and \( \lor \).

We already know that \( \Xi_n \otimes \Xi_n \) inherits its knowledge order coordinatewise from \( 2^{2n+1} \). To access the truth order, we use the fact that \( a \leq b \) if and only if \( a_{i+1} = (a \land b)_{i+1} \) and \( b_{i+1} = (a \lor b)_{i+1} \) for \( 0 \leq i \leq n \). By treating odd and even coordinates in this way we are able to save work by exploiting the parallels between the definitions of \( \land \) and \( \lor \).

Certainly \( a_{i+1} = (a \land b)_{i+1} \) if and only if \( a_{i+1} \leq a_{i+1} \) and \( a_{i+1} = (a \land b)_{i+1} \). Hence in identifying when \( a \leq b \) it will be sufficient to restrict attention to pairs \( a \) and \( b \) whose 0, \( \nu \)-coordinates are related in this way. Consider \( i = 1 \). Taking into account the fact that \( a \) and \( b \) are members of \( A \), we see easily that

\[
a_{1,1} = (a \land b)_{1,1} \iff a_{0,1} = 1 \text{ or } a_{0,1} = 1 \text{ or } a_{1,1} \leq b_{1,1}
\]

and, likewise,

\[
b_{1,1} = (a \lor b)_{1,1} \iff b_{0,1} = 1 \text{ or } b_{0,1} = 1 \text{ or } a_{1,1} \geq b_{1,1}.
\]

We now proceed by recursion, in the following way. We let \( a_{0,i} \in \{0, 1\}^{2n+1} \) be the projection of \( a \) onto its first \( 2(i+1) \) coordinates. We have \( a \leq b \) if and only if \( a_{i+1} = (a \land b)_{i+1} \) and \( b_{i+1} = (a \lor b)_{i+1} \) for each \( i \in [1, \ldots, n] \) and for this to hold we must have in particular \( a_{0,1} = (a_{0,0}, a_{0,1}, \ldots, a_{1,1}, a_{1,1}) \leq (b_{0,0}, b_{0,1}, \ldots, b_{1,1}, b_{1,1}) = b_{0,1} \). Assume by induction that we already know necessary and sufficient conditions for \( a_{0,i-1} \leq b_{0,i-1} \). Now \( (a \land b)_{i+1} = (a_{i+1} \land b_{i+1}) \lor b_{i+1}((a \land b)_{i+1} \land b_{i+1}) = (a_{i+1} \land b_{i+1}) \lor a_{i+1} \land b_{i+1} \). And from here we obtain \( a_{i+1} = (a \land b)_{i+1} \) and only if \( a_{i+1} = 1 \). A similar argument applies to \( b_{i+1} \). From this we can deduce that the truth order is that induced by the lexicographic order on the power \( (2 \times 2^n)^{n+1} \) of the poset \( 2 \times 2^n \); here \( 2^n \) denotes the order dual of \( 2 \) that is \( \{0, 1\} \) with strict order in which \( 1 < 0 \). Thus \( 2 \times 2^n \) is just \( \text{FOUR} \) in its truth order (recall Fig. 1). By way of illustration, Fig. 9 shows the knowledge and truth orders on \( \Xi_n \otimes \Xi_n \) for \( n = 1 \). In it we have labelled the elements with four-element binary strings \( a_{0,i}a_{1,i}a_{0,i}a_{1,i} \).

![Figure 9: The knowledge order (left) and truth order (right) of \( \Xi_1 \otimes \Xi_1 \)](image)

Our product representation applied to the default sequence \( \Xi_n \otimes \Xi_n \) associated with the algebra \( K_n \) tells us how \( K_n \) inherits its knowledge order and its truth order from a full power. The latter is a power of what is known as the twist structure based on \( 2 \) (see for example [12] for a discussion of this notion). We stress that the lexicographic ordering on powers arises only because the default sequences we have been considering are so very simple. (Of course, too, the truth operations are not the join and meet inherited from \( 2 \times 2^n \) with the lexicographic order.) Returning full circle to our starting point in Section 3 we can see how the particular default bilattices \( \text{SEVEN}, \text{TEN}, \ldots \) relate to twist structures in a way which generalises the construction of \( \text{FOUR} \) as a twist structure.

7. Dualities for quasivarieties of prioritised default bilattices

In this final section we turn our attention from the variety \( V_n = \text{HSP}(K_n) \) to the quasivariety \( Q_n = \text{ISP}(K_n) \), for \( n \geq 1 \). We cannot expect there to be a product representation entirely within the quasvariety because our work in the
Acknowledgements

The first author was supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Program (ref. 299401-FP7-PEOPLE-2011-IEF). The second author was supported by the Claude Leon Foundation during the writing of this paper. He acknowledges also the support of the Rhodes Trust during the period in which he was studying for his DPhil degree at the University of Oxford, under the supervision of the third author: during this period he undertook the initial research from which the paper subsequently evolved.

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Theorem 7.1. Consider \( \mathcal{Q}_n \). Then \( K_n = (K_n; S_{n,0}, \ldots, S_{n,n}, \mathcal{T}) \) yields a strong (and hence full) duality on \( \mathcal{Q}_n \). Moreover, this duality is optimal.

In outline, the proof of Theorem 7.1 proceeds as follows. Theorem 2.2 combined with very simple entailment arguments, leads to the alter ego we present. Besides the entailment constructs of converse and trivial relations we need also intersection \[ \mathcal{S}_{K_n, \mathcal{T}} \] of \( \mathcal{S}_{K_n, \mathcal{T}} \). The proof of optimality is similar to the first part of the proof of Theorem 4.3: we exploit the Separation Theorem for Topological Quasivarieties \[ 4, \text{Theorem 1.4.4}. \]

Theorem 7.2. Let \( X = (X; \leq_0, \leq_1, \ldots, \leq_n, \mathcal{T}) \) be a structured topological space. Then \( X \in \mathcal{S}_{K_n}^\mathcal{T} \) if and only if

(i) \( (X; \leq_0, \mathcal{T}) \) is a Priestley space;
(ii) \( \leq_i \) is a Priestley quasi-order extending \( \leq_0 \), for \( i \in \{1, \ldots, n\} \);
(iii) \( \leq_0 \leq \leq_1 \leq \cdots \leq \leq_n \);
(iv) \( \geq_i \leq \geq_j, \) for \( 0 \leq i < j \leq n \).

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Acknowledgements

The first author was supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Program (ref. 299401-FP7-PEOPLE-2011-IEF). The second author was supported by the Claude Leon Foundation during the writing of this paper. He acknowledges also the support of the Rhodes Trust during the period in which he was studying for his DPhil degree at the University of Oxford, under the supervision of the third author: during this period he undertook the initial research from which the paper subsequently evolved.

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