ON PHASE TRANSITION FOR ONE DIMENSIONAL COUNTABLE STATE 
P-ADIC POTTS MODEL

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Abstract. In the present paper we shall consider countable state $p$-adic Potts model on $\mathbb{Z}_+$. A main aim is to establish the existence of the phase transition for the model. In our study, we essentially use one dimensionality of the model. To show it we reduce the problem, to the investigation of an infinite-dimensional nonlinear equation. We find a condition on weights to show that the derived equation has two solutions, which yields the existence of the phase transition. We prove that measures corresponding to first and second solutions are $p$-adic Gibbs and generalized $p$-adic Gibbs measures, respectively. Note that it turns out that the finding condition does not depend on values of the prime $p$, and therefore, an analogous fact is not true when the number of spins is finite. Note that, in the usual real case, if one considers one dimensional translation-invariant model with nearest neighbor interaction, then such a model does not exhibit a phase transition. Nevertheless, we should stress that in our model there does not occur the strong phase transition, this means that there is only one $p$-adic Gibbs measure. Here we may see some similarity with the real case. Besides, we prove that the $p$-adic Gibbs measure is bounded, and the generalized one is not bounded.

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1. INTRODUCTION

Due to the assumption that $p$-adic numbers provide a more exact and more adequate description of microworld phenomena, starting the 1980s, various models described in the language of $p$-adic analysis have been actively studied [6],[16],[39],[51]. The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [5, 52, 50]. Furthermore, numerous applications of the $p$-adic analysis to mathematical physics have been proposed in [8],[26],[27]. One of the first applications of $p$-adic numbers in quantum physics appeared in the framework of quantum logic in [9]. This model is especially interesting for us because it could not be described by using conventional real valued probability. Besides, it is also known [27, 39, 44, 50] that a number of $p$-adic models in physics cannot be described using ordinary Kolmogorov’s probability theory. New probability models, namely $p$-adic values ones were investigated in [10],[25],[34]. After that in [35] an abstract $p$-adic probability theory was developed by means of the theory of non-Archimedean measures [44],[23]. Using that measure theory in [32],[38] the theory of stochastic processes with values in $p$-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed. In particular, a non-Archimedean analog of the Kolmogorov theorem was proven (see also [18]). Such a result allows us to construct wide classes of stochastic processes using finite dimensional probability distributions1. Therefore, this result gives us a possibility to develop the theory of statistical mechanics in the context of the $p$-adic theory, since it lies on the basis of the theory of probability and stochastic processes. First steps in this theory have been started in [22, 41, 42]. Note that one of the central problems of such a theory is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian, and a description of the set of such measures. In most cases such an analysis depend on a specific properties of Hamiltonian,

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1We point out that stochastic processes on the field $\mathbb{Q}_p$ of $p$-adic numbers with values of real numbers have been studied by many authors, for example, [1, 2, 3, 12, 37, 53]. In those investigations wide classes of Markov processes on $\mathbb{Q}_p$ were constructed and studied. In our case the situation is different, since probability measures take their values in $\mathbb{Q}_p$. This leads our investigation to some difficulties. For example, there is no information about the compactness of $p$-adic values probability measures.
and complete description is often a difficult problem. This problem, in particular, relates to a phase transition of the model (see [19]).

In [29, 30] a notion of ultrametric Markovianity, which describes independence of contributions to random field from different ultrametric balls, has been introduced, and shows that Gaussian random fields on general ultrametric spaces (which were related with hierarchical trees), which were defined as a solution of pseudodifferential stochastic equation (see also [21]), satisfies the Markovianity. In [28, 25] Gaussian $p$-adic valued measure was investigated, and showed that such a measure is not bounded. This phenomena shows the difference between real and $p$-adic valued probability theories. Some applications of the results to replica matrices, related to general ultrametric spaces have been investigated in [31].

The purpose of this paper is devoted to the development of $p$-adic statistical mechanics in $p$-adic probability theory framework. Namely, we study one-dimensional countable state of nearest-neighbor Potts models (see [17, 55]) over $p$-adic filed. We are especially interested in the existence of phase transition for the mentioned model. Here by the phase transition we mean the existence of two different generalized $p$-adic Gibbs measures associated with the model. Note that such measures present more natural concrete examples of $p$-adic Markov processes (see [32], for definitions). It is worth to mention that when the number of states of the model is finite, say $q$, then the corresponding $p$-adic $q$-state Potts models have been studied in [41, 42]. It was established that a strong phase transition occurs if $q$ is divisible by $p$. Here the strong phase transition means the existence of two different $p$-adic Gibbs measures. This shows that the transition depends on the number of spins $q$. Therefore, it is interesting to know the situation in the setting with countable states. In [33] (see also [40]) first steps to investigation of such a countable state $p$-adic Potts model on Cayley tree have been studied. We provided a sufficient condition for the uniqueness of $p$-adic Gibbs measures. Note that such a condition does not depend on the value of $p$.

In the present paper we shall consider countable state $p$-adic Potts model on $\mathbb{Z}_+$. A main aim is to establish the existence of the phase transition for the model. In our study, we essentially use one dimensionality of the model. To show it we reduce the problem, to the investigation of an infinite-dimensional nonlinear equation. We will show that the derived equation has two solutions, which yields the existence of the phase transition. Note that, in the usual real case, if one considers one dimensional translation-invariant model with nearest neighbor interaction, then such a model does not exhibit a phase transition. But in our setting, we are able to produce a model which is translation-invariant and has nearest neighbor interactions, and for such a model we shall prove the existence of the phase transition. Nevertheless, we should stress that in our model there does not occur the strong phase transition, this means that there is only one $p$-adic Gibbs measure. Here we may see some similarity with the real case.

Let us briefly describe the paper. After preliminaries, in section 3 we introduce the model, and define generalized $p$-adic Gibbs measure and $p$-adic Gibbs measure, respectively. Here the provided construction of such measures which depends on a weight $\lambda$. The goal of this investigation is to give a sufficient condition for the existence of two such measures. Note that in comparison to a real case, in a $p$-adic setting, a priori the existence of such kind of measures for the model is not known, since there is not much information on topological properties of the set of all $p$-adic measures defined even on compact spaces. However, in the real case, there is the so called the Dobrushin’s Theorem [13, 14, 19] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians. Using the $p$-adic analog of Kolmogorov’s extension Theorem [32], an investigation of the defined measures is reduced to the examination of an infinite-dimensional nonlinear recursion equation. In next Section 4, we associate a nonlinear operator on the Banach space $c_0$ to the derived recursion equation. We provide a sufficient condition on weight $\lambda$, which ensures the existence of two fixed points of the nonlinear operator. This implies the existence of the phase transition for the model. It turns out that the finding condition does not depend on values of the prime $p$, and therefore, an

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2The classical (real value) counterparts of such models were considered in [55]

3To establish such results we investigated $p$-adic dynamical systems associated with the model. Note that first investigations of non-Archimedean dynamical systems were appeared in [20] (see also [4, 34, 15, 46, 49, 54])
analogous fact is not true when the number of spins is finite. Moreover, we show that the found fixed points define a $p$-adic Gibbs and s generalized $p$-adic Gibbs measures, respectively. Besides, we prove that the $p$-adic Gibbs measure is bounded, and the generalized one is not bounded.

2. Preliminaries

Throughout the paper $p$ will be a fixed prime number greater than 3, i.e. $p \geq 3$. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, $m$ is a positive integer, $(p, n) = 1$, $(p, m) = 1$. The $p$-adic norm of $x$ is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

It satisfies the following strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

this is a non-Archimedean norm.

The completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm is called $p$-adic field and it is denoted by $\mathbb{Q}_p$.

According to non-Archimedeanity of the norm the following statement holds true (see [36]).

**Lemma 2.1.** Let $\{x_n\}$ be a sequence in $\mathbb{Q}_p$. Then $\sum_{k=1}^{\infty} x_k$ converges iff $x_n \to 0$.

Note that any $p$-adic number $x \neq 0$ can be uniquely represented in the form

$$(2.1) \quad x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and $x_j$ are integers, $0 \leq x_j \leq p - 1$, $x_0 > 0$, $j = 0, 1, 2, \ldots$ (see more detail [36]).

In this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called a quadratic residue modulo $p$ if the equation $x^2 \equiv a \pmod{p}$ has a solution $x \in \mathbb{Z}$.

**Lemma 2.2.** [50] In order that the equation

$$x^2 = a, \quad 0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + \ldots), \quad 0 \leq a_j \leq p - 1, \quad a_0 > 0$$

has a solution $x \in \mathbb{Q}_p$, it is necessary and sufficient that the following conditions are fulfilled:

(i) $\gamma(a)$ is even;

(ii) $a_0$ is a quadratic residue modulo $p$ if $p \neq 2$, $a_1 = a_2 = 0$ if $p = 2$.

Denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. Elements of the set $\mathbb{Z}_p$ are called $p$-adic integers.

**Lemma 2.3** (Hensel’s Lemma). [36] Let $P(x)$ be polynomial whose the coefficients are $p$-adic integers. Let $a_0 \in \mathbb{Z}_p$ be a $p$-adic integer such that $P(a_0) \equiv 0 \pmod{p}$ and $P'(a_0) \not\equiv 0 \pmod{p}$. There exists a unique $p$-adic integer $x_0 \in \mathbb{Z}_p$ such that $P(x_0) = 0$ and $|x_0 - a_0|_p \leq 1/p$.

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}, \quad S(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p = r\}.$$

The $p$-adic logarithm is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for $x \in B(1, 1)$; the $p$-adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$. 
Lemma 2.4. [36] Let \( x \in B(0, p^{-1/(p-1)}) \) then we have

\[
|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1 + x)|_p = |x|_p
\]

\[
\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x.
\]

Note the basics of \( p \)-adic analysis, \( p \)-adic mathematical physics are explained in [36, 43, 45].

Let \((X, \mathcal{B})\) be a measurable space, where \( \mathcal{B} \) is an algebra of subsets \( X \). A function \( \mu : \mathcal{B} \to \mathbb{Q}_p \) is said to be a \( p \)-adic measure if for any \( A_1, \ldots, A_n \subset \mathcal{B} \) such that \( A_i \cap A_j = \emptyset \) \((i \neq j)\) the equality holds

\[
\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).
\]

A \( p \)-adic measure is called a probability measure if \( \mu(X) = 1 \). A \( p \)-adic probability measure \( \mu \) is called bounded if \( \sup \{|\mu(A)|_p : A \in \mathcal{B}\} < \infty \). For more detail information about \( p \)-adic measures we refer to [26],[25].

3. \( p \)-adic Potts model and its \( p \)-adic Gibbs measures

In the sequel we will use the notation \( \mathbb{Z}_+ = \{0, 1, 2, \cdots \} \).

Now define the \( p \)-adic Potts model on \( \mathbb{Z}_+ \) with spin values in the set \( \Phi = \{0, 1, 2, \cdots \} \). Note that a configuration \( \sigma \) on \( \mathbb{Z}_+ \) is defined as a function \( x \in \mathbb{Z}_+ \to \sigma(x) \in \Phi \); in a similar manner one defines configurations \( \sigma_n \) and \( \omega(n) \) on \([0, n]\) and \(\{n\}\), respectively. The set of all configurations on \(\mathbb{Z}_+\) (resp. \([0, n]\), \(\{n\}\)) coincides with \(\Omega = \Phi^{\mathbb{Z}_+}\) (resp. \(\Omega_n = \Phi^{[0,n]}\), \(\Omega_{\{n\}} = \Phi\)). One can see that \(\Omega_n = \Omega_{n-1} \times \Phi\).

Using this, for given configurations \(\sigma_{n-1} \in \Omega_{n-1}\) and \(\omega \in \Omega_{\{n\}}\) we define their concatenations by

\[
\sigma_{n-1} \vee \omega = \left\{\left\{\sigma_{n-1}(k), k \in [0, n-1]\right\}, \{\omega\}\right\}.
\]

The Hamiltonian \(H_n : \Omega_n \to \mathbb{Q}_p\) of \( p \)-adic countable state Potts model has the form

\[
H_n(\sigma) = \sum_{k=0}^{n-1} \delta_{\sigma(k), \sigma(k+1)}, \quad n \in \mathbb{N},
\]

here \(\sigma \in \Omega_n\), \(\delta\) is the Kronecker symbol and

\[
|J|_p \leq \frac{1}{p}.
\]

Note that such a condition provides the existence of a \( p \)-adic Gibbs measure (see (3.4)).

Let us construct \( p \)-adic Gibbs measures corresponding to the model.

A given set \(A\) we put \(\mathbb{Q}_p^A = \{\{x_i\}_{i \in A} : x_i \in \mathbb{Q}_p\}\).

Assume that a function \(h : \mathbb{N} \to \mathbb{Q}_p^\Phi\), i.e. \(h_n = \{h_{i,n}\}_{i \in \Phi}, n \in \mathbb{N}\) is given and a non-zero element \(\lambda = \{\lambda(i)\}_{i \in \Phi} \in \mathbb{Q}_p^\Phi\) is fixed such that

\[
|h(n)|_p \to 0 \quad \text{as} \quad n \to \infty
\]

which is called a weight. In what follows, without losing generality we may assume that \(\lambda(0) \neq 0\).

Given \(n = 1, 2, \ldots\) a \( p \)-adic probability measure \(\mu_{h(n)}^{(n)}\) on \(\Omega_n\) is defined by

\[
\mu_{h(n)}^{(n)}(\sigma) = \frac{1}{Z_n^{(h)}} \exp_p\left\{H_n(\sigma)\right\} h_{\sigma(n),n} \prod_{k=0}^{n} \lambda(\sigma(k)),
\]

here, \(\sigma \in \Omega_n\) and \(Z_n^{(h)}\) is the corresponding normalizing factor called a partition function given by

\[
Z_n^{(h)} = \sum_{\sigma \in \Omega_n} \exp_p\left\{H_n(\sigma)\right\} h_{\sigma(n),n} \prod_{k=0}^{n} \lambda(\sigma(k)),
\]

here subscript \(n\) and superscript \((h)\) are accorded to the \(Z\), since it depends on \(n\) and a function \(h\).
Theorem (see [18], [32]) which based on so called compatibility condition \( \mu \) measure is said to be (3.7) i.e. defined on \( N \) \( \sigma \) for any \( p \) even on compact spaces.

\( \) has a special form, i.e. \( p \) \( G \) | if \( a \) phase transition \( p \) such kind of measures and transitions for Ising and Potts models have been studied in [41, 42, 22].

In [17] using that theorem it has been established the existence of the Gibbs measure for the real counterpart of the \( \) \( \hat{\Phi} \) \( \Phi \) \( \lambda \) here and below \( \theta = \exp_p(J) \), a vector \( \hat{h} \) \( \{h_i\}_{i \in \Phi} \) is defined by a vector \( h \) \( \{h_i\}_{i \in \Phi} \) as follows (3.9) \( \hat{h}_i = \frac{h_i \lambda(i)}{h_0 \lambda(0)} \), \( i \in N \)

\( \) In general, à priori the existence of such a kind of measure \( \mu \) is not known, since, there is not much information on topological properties, such as compactness, of the set of all \( p \)-adic measures defined even on compact spaces\(^4\). Therefore, at a moment, we can only use the \( p \)-adic Kolmogorov extension Theorem (see [18], [32]) which based on so called compatibility condition for the measures \( \mu^{(n)}_h \), \( n \geq 1 \), i.e.

\[
\sum_{\omega \in \Phi} \mu^{(n)}(\sigma_{n-1} \lor \omega) = \mu^{(n-1)}(\sigma_{n-1}),
\]

for any \( \sigma_{n-1} \in \Omega_{n-1} \). This condition according to the theorem implies the existence of a unique \( p \)-adic measure \( \mu \) defined on \( \Omega \) with a required condition (3.6). Note that more general theory of \( p \)-adic measures has been developed in [23, 24].

So, if for some function \( h \) the measures \( \mu^{(n)}_h \) satisfy the compatibility condition, then there is a unique \( p \)-adic probability measure, which we denote by \( \mu_h \), since it depends on \( h \). Such a measure \( \mu_h \) is said to be a generalized \( p \)-adic Gibbs measure corresponding to the \( p \)-adic Potts model. By \( GG(H) \) we denote the set of all generalized \( p \)-adic Gibbs measures associated with functions \( h = \{h_n, n \in N\} \). If \( |GG(H)| \geq 2 \) (here \( |A| \) stands for the cardinality of a set \( A \)) i.e. there are at least two different generalized \( p \)-adic Gibbs measures in \( GG(H) \), namely one can find two different functions \( s \) and \( h \) defined on \( N \) such that there exist the corresponding measures \( \mu_s \) and \( \mu_h \), which are different, then we say that a phase transition occurs for the model, otherwise, there is no phase transition. If the function \( h \) has a special form, i.e. \( h = \{\exp_p(\kappa_i, n)\}_{i \in \Phi} \) for some \( \{\kappa_i, n\} \subset Q_p \), then the corresponding measure defined by (3.4) is called \( p \)-adic Gibbs measure. The set of all \( p \)-adic Gibbs measures is denoted by \( G(H) \). If \( |G(H)| \geq 2 \), then we say that for this model there exists a strong phase transition. Note that such kind of measures and transitions for Ising and Potts models have been studied in [41, 42, 22].

Now one can ask for what kind of functions \( h \) the measures \( \mu^{(n)}_h \) defined by (3.4) would satisfy the compatibility condition (3.7). The following theorem gives an answer to this question.

**Theorem 3.1.** [40] The measures \( \mu^{(n)}_h \), \( n = 1, 2, \ldots \) (see (3.4)) satisfy the compatibility condition (3.7) if and only if for any \( n \in N \) the following equation holds:

\[
\hat{h}_{i,n} = \frac{\lambda(i)}{\lambda(0)} F_i(\hat{h}_{i,n+1}; \theta), \quad i \in N
\]

here and below \( \theta = \exp_p(J) \), a vector \( \hat{h} \) \( \{h_i\}_{i \in \Phi} \) is defined by a vector \( h \) \( \{h_i\}_{i \in \Phi} \) as follows (3.9) \( \hat{h}_i = \frac{h_i \lambda(i)}{h_0 \lambda(0)} \), \( i \in N \).

\( \) In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e. Prohorov’s Theorem). When the state space is non-compact, then there is a Dobrushin’s Theorem [13, 14] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians. In [17] using that theorem it has been established the existence of the Gibbs measure for the real counterpart of the studied Potts model. It should be noted that there are even nearest-neighbor models with countable state space for which the Gibbs measure does not exists [48].
and mappings \( F_i : \mathbb{Q}_p^N \times \mathbb{Q}_p \to \mathbb{Q}_p \) are defined by
\[
F_i(x; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{\infty} x_j + 1}{\sum_{j=1}^{\infty} x_j + \theta}, \quad x = \{x_i\}_{i \in \mathbb{N}}, \quad i \in \mathbb{N}.
\]

**Lemma 3.2.** [40] Let \( \{\hat{h}\} \) be a solution of (3.8) such that \( \sum_{j=1}^{\infty} \hat{h}_{j,n} \neq -\theta \) for every \( n \in \mathbb{N} \). Then for every \( n \in \mathbb{N} \) one has
\[
\sum_{j=1}^{\infty} \hat{h}_{j,n} < \infty.
\]

**Remark 3.1.** If every sequence \( \hat{h}_n \) is bounded, then (3.3),(3.9) with Lemma 2.1 imply that the series \( \sum_{j=1}^{\infty} \hat{h}_{j,n} \) is always convergent.

**Observation 3.1.** Here we are going to underline a connection between \( q \)-state Potts model with the defined one. First recall that \( q \)-state Potts model is defined by the same Hamiltonian (3.1), but with the state space \( \Phi_q = \{0, 1, \ldots, q-1\} \). Similarly, one can define \( p \)-adic Gibbs measures for the \( q \)-state Potts model, here instead of the weight \( \{\lambda(i)\} \) we will take a collection \( \{\lambda(0), \lambda(1), \ldots, \lambda(q-1)\} \subset \mathbb{Q}_p \).

Now consider countable Potts model with a weight \( \{\lambda(i)\} \) such that
\[
\lambda(k) = 0 \quad \text{for all } k \geq q, \quad q > 1.
\]

In this case the corresponding Gibbs measures will coincide with those of \( q \)-state Potts model. Indeed, let
\[
\Omega^c = \{\sigma \in \Omega : \exists j \in \mathbb{Z}_+ : \sigma(j) \geq q\}
\]
\[
\Omega^{(q)} = \{\sigma \in \Omega : \sigma(j) \leq q-1 \forall j \in \mathbb{Z}_+\}
\]

It is clear that \( \Omega^{(q)} = \Phi_q^{\mathbb{Z}_+} \). Let \( \mu \) be a Gibbs measure of the countable Potts model with the given weight corresponding to a solution \( h = \{h_i\}_{i \in \Phi} \) of (3.8). Note that here by Gibbs measure we mean genralized \( p \)-adic Gibbs measure. From the definition (3.4) we see that the restriction of \( \mu \) to \( \Omega^c \) is zero, i.e. \( \mu|_{\Omega^c} = 0 \). Moreover, from (3.8) and (3.12) we conclude that \( h_{i,n} = 0 \) for all \( i \geq q \). This means that vectors \( h^{(q)}_{n} = \{h_{i,n}\}_{i \in \Phi_q} \) will be a solution of (3.8) corresponding to the \( q \)-state Potts model. Therefore, the restriction of \( \mu \) to \( \Omega^{(q)} \) coincides with Gibbs measure of \( q \)-state Potts model with a weight \( \{\lambda(0), \lambda(1), \ldots, \lambda(q-1)\} \) corresponding to a solution of \( h^{(q)}_{n} \).

Hence, we conclude that under condition (3.12) all Gibbs measures corresponding to countable Potts model are described by those measures of \( q \)-state Potts model.

Let us recall that a function \( \{h_n\}_{n \in \mathbb{N}} \) is translation-invariant if \( h_n = h_{n+1} := h \) for every \( n \in \mathbb{N} \). It is natural to ask is there a translation invariant solution of (3.8).

Now we are looking for the translation-invariant solution \( \hat{h} \) of (3.8). Then the equation can be written as follows
\[
\hat{h}_i = \frac{\lambda(i)}{\lambda(0)} \left( \frac{(\theta - 1)\hat{h}_i + \sum_{j=1}^{\infty} \hat{h}_j + 1}{\sum_{j=1}^{\infty} \hat{h}_j + \theta} \right), \quad i \in \mathbb{N}.
\]

Investigating, the derived equation (3.13), in [40] we have proved the following

**Theorem 3.3.** [40] Let \( 0 < |J|_p < p^{-1/(p-1)} \) and for the weight \( \lambda \) the condition
\[
\lambda(0) = 1, \quad \text{and} \quad |\lambda(m)|_p < 1 \quad \forall m \in \mathbb{N}.
\]
be satisfied. Then for one dimensional \( p \)-adic Potts model (3.1) there is a generalized \( p \)-adic Gibbs measure, i.e. \( |\mathcal{G}(H)| \geq 1 \). Moreover, there is a unique \( p \)-adic Gibbs measure, i.e. \( |\mathcal{G}(H)| = 1 \).
Remark 3.2. Under the condition (3.14) from Theorem 3.3 it naturally arises a question: is it possible that $|G\mathcal{G}(H) \setminus \mathcal{G}(H)| \geq 1$. It turns out this situation can occur. Indeed, let us consider

\begin{align*}
\lambda(0) &= 1, \quad \lambda(1) = \lambda(2) = a, \quad \lambda(m)_p = 0 \quad \forall m \geq 3 \\
|a|_p &< 1.
\end{align*}

It is evident that in this case (3.14) is satisfied. Then (3.13) reduces to

\begin{align*}
\hat{h}_1 &= a \left( \frac{\theta \hat{h}_1 + \hat{h}_2 + 1}{\hat{h}_1 + \hat{h}_2 + \theta} \right), \\
\hat{h}_2 &= a \left( \frac{\theta \hat{h}_2 + \hat{h}_1 + 1}{\hat{h}_2 + \hat{h}_1 + \theta} \right).
\end{align*}

From (3.17), (3.18) one gets

\begin{equation}
\frac{\hat{h}_1}{\hat{h}_2} = \left( \frac{\theta \hat{h}_1 + \hat{h}_2 + 1}{\theta \hat{h}_2 + \hat{h}_1 + 1} \right)
\end{equation}

which implies

$$
(\hat{h}_1 - \hat{h}_2)(\hat{h}_1 + \hat{h}_2 + 1) = 0.
$$

This means that either $\hat{h}_1 = \hat{h}_2$ or $\hat{h}_1 = -\hat{h}_2 - 1$.

Now assume that $\hat{h}_1 = -\hat{h}_2 - 1$, and substituting it to (3.17) we immediately find that $\hat{h}_1 = 0$ and $\hat{h}_2 = -1$. From (3.9) and (3.15) one gets that $h_1 = 0$. This means that associated measure (3.4) is a generalized p-adic Gibbs one, i.e. belongs to $G\mathcal{G}(H) \setminus \mathcal{G}(H)$.

Let $\hat{h}_1 = \hat{h}_2$. Then again substituting it to (3.17) and after little algebra one gets

\begin{equation}
Q(\hat{h}_1) = 0
\end{equation}

where $Q(x) = 2x^2 + ((1 - \theta)(1 - a) + 1 - 2a)x - a$. From (3.16) one can see that $|Q(0)|_p = |a|_p < 1$ and $|Q'(0)|_p = 1$. Therefore, thanks to the Hansel Lemma the equation (3.20) has two solutions $\hat{h}_1^{(1)}, \hat{h}_2^{(1)} \in \mathbb{Q}_p$ such that $|\hat{h}_1^{(1)}|_p = |a|_p$, and $|\hat{h}_1^{(2)}|_p = 1$. The measure corresponding to $\hat{h}_1^{(1)}$ due to Theorem 3.3 belongs to $\mathcal{G}(H)$. But from (3.9) and (3.16) one infers that the measure associated with $\hat{h}_1^{(2)}$ belongs to $G\mathcal{G}(H) \setminus \mathcal{G}(H)$.

Hence, for the model (3.1) with a weight (3.15) we have $|G\mathcal{G}(H) \setminus \mathcal{G}(H)| \geq 2$, since $|\mathcal{G}(H)| = 1$.

4. Phase transition

In this section we are going to show that the equation (3.8) has at least two translation-invariant solutions under some conditions.

In this section we will assume the following

\begin{equation}
\lambda(0) = 1, \quad \lambda(1) = \alpha, \quad \text{and} \quad |\lambda(m)|_p < 1 \quad \forall m \geq 2,
\end{equation}

here $\alpha \in \mathbb{Q}_p$ such that

\begin{equation}
|\alpha|_p = 1, \quad |1 - \alpha|_p \leq 1/p.
\end{equation}

It is obvious that in this case (3.14) is not satisfied. Now we are going to find translation invariant solution of (3.8), i.e. $\hat{h}_n = \hat{h}_m$ for all $n, m \in \mathbb{N}$. Therefore, we assume that $\hat{h}_1 = (x_1, \ldots, x_n, \ldots)$. Let us for the sake of shortness, a given sequence $x = \{x_j\}_{j \geq 2}$ we denote

\begin{equation}
X := \sum_{j=2}^{\infty} x_j.
\end{equation}

If $\hat{h}_1$ is a translation invariant solution, then the first equation in (3.13) with (4.1) can be rewritten by

\begin{equation}
x_1 = \alpha \left( \frac{\theta x_1 + X + 1}{x_1 + X + \theta} \right).
\end{equation}
We reduced the last equation to
\[(4.5) \quad P(x_1) = 0,\]
where \(P(x) = x^2 + (X + \theta(1 - \alpha))x - \alpha(X + 1).\)

Direct checking shows that
\[(4.6) \quad P(1) = (1 - \alpha)(X + 1 + \theta), \quad P'(1) = 2 + X + \theta(1 - \alpha).\]

If \(|X + 2|_p = 1\), then (4.2) with the Hensel's Lemma implies that (4.5) has a solution \(x_{+1}\) belonging to \(\mathbb{Q}_p\). Hence, the Vieta Theorem yields that the second solution \(x_{-1}\) of (4.5) also belongs to \(\mathbb{Q}_p\). Note that for both solutions \(x_{\pm1}\) due to Hensel Lemma we have
\[(4.7) \quad |x_{+1} - 1|_p \leq 1/p.\]

Now keeping in mind that \(p \geq 3\), from (4.7) one finds
\[(4.8) \quad |x_{-1} - 1|_p = 1.\]

In the sequel we will need an exact form of these solutions, which can be written as follows
\[(4.9) \quad x_{\pm1} = \frac{(\alpha - 1)\theta - X \pm \sqrt{D_X}}{2},\]
where
\[(4.10) \quad D_X = (X + \theta(1 - \alpha))^2 + 4\alpha(X + 1)
= \theta^2(1 - \alpha)^2 + 2(2X + X\theta + 2)(1 - \alpha) + (X + 2)^2.\]

Note that the existence of the solutions \(x_{\pm1}\) yields the existence \(\sqrt{D_X}\).

Let us now substitute (4.9) into \(F_i\) in (3.10), which has a form
\[(4.11) \quad F_i^{(\pm)}(x; \theta) = \frac{2(\theta - 1)x + (\alpha - 1)\theta + X \pm \sqrt{D_X} + 2}{(\alpha + 1)\theta + X \pm \sqrt{D_X}}, \quad i \geq 2,\]
where \(x = \{x_i\}_{i \geq 2}\).

Note that from (3.9) and (3.3) we see that \(|x_n|_p \to 0\) as \(n \to \infty\). Therefore, it is natural to consider the following space
\[(4.12) \quad c_0 = \{\{x_n\}_{n \geq 2} \subset \mathbb{Q}_p : |x_n|_p \to 0, \quad n \to \infty\}\]
with a norm \(\|x\| = \max_n |x_n|_p\). According to Lemma 2.1 for any \(\{x_n\} \in c_0\) we have \(\sum_{j=2}^{\infty} x_j < \infty\).

Define
\[(4.13) \quad B_r = \{\{x_n\} \in c_0 : \|x\| \leq r\},\]
where \(r \in \{p^k : k \in \mathbb{Z}\}\). It is clear that \(B_{r+}\) is a closed subset of \(c_0\). Now consider the following mapping
\[(4.14) \quad (\mathcal{F}^{(\pm)}(x))_i = \lambda(i)F_i^{(\pm)}(x, \theta), \quad i \geq 2,\]
where \(x = \{x_n\} \in c_0\).

Now our aim is to show the existence of a fixed point of \(\mathcal{F}^{(\pm)}\).

Put
\[(4.15) \quad \delta = \max_{i \geq 2} |\lambda(i)|_p.\]

From (4.1) one immediately finds that \(\delta < 1\).

Note that according to the condition (4.1) from (3.9) we obtain \(|x_n|_p \leq |\lambda(n)|_p, \forall n \geq 2\), which implies that any solution of (3.8) belongs to \(B_\delta\).

**Lemma 4.1.** Let the conditions (4.1),(4.2) be satisfied for \(\lambda\). Then \(\mathcal{F}^{(+)}(B_\delta) \subset B_\delta\).
Proof. Let \( x \in B_\delta \). Then

\[
|\mathcal{X}|_p = \left| \sum_{j=2}^{\infty} x_j \right|_p \leq \|x\| \leq \delta.
\]

Therefore, one has \( |X + 2|_p = 1 \), and according to the above made argument, we infer the existence of \( \sqrt{D_X} \). Now using this fact and (4.15), from (4.10) and (4.2) we conclude that \( \sqrt{D_X} = 2 + \epsilon \) with \( |\epsilon|_p < 1 \), which with (4.2) implies that

\[
\sqrt{D_X} - 2 \leq \frac{1}{p}, \quad |\alpha \theta - \sqrt{D_X}|_p = 1.
\]

Then by means of (4.15), (4.16) and (4.2) we have

\[
|\mathcal{F}(\mathbf{x})|_p = |\lambda(i)|_p \left| \frac{2(\theta - 1)x_i + (\alpha - 1)\theta + X + \sqrt{D_X} + 2}{(\alpha + 1)\theta + X + \sqrt{D_X}} \right|_p
\]

for all \( i \geq 2 \), which implies \( \mathcal{F}(B_\delta) \subset B_\delta \). This completes the proof. \( \square \)

Before going to the main result we need some auxiliary facts.

**Lemma 4.2.** One has

\[
\sqrt{D_X} - \sqrt{D_Y} = \frac{(X - Y)(X + Y + 2\theta(1 - \alpha) + 4\alpha)}{\sqrt{D_X} + \sqrt{D_Y}}.
\]

**Proof.** From (4.10) we immediately find

\[
D_X - D_Y = (X - Y)(X + Y + 2\theta(1 - \alpha) + 4\alpha)
\]

which with

\[
\sqrt{D_X} - \sqrt{D_Y} = \frac{D_X - D_Y}{\sqrt{D_X} + \sqrt{D_Y}}
\]

implies the assertion. \( \square \)

Denote

\[
\xi_X = (\alpha - 1)\theta + X + \sqrt{D_X} + 2.
\]

From the direct calculation we can prove the following

**Lemma 4.3.** One has

\[
\begin{align*}
\xi_X - \xi_Y &= X - Y + \frac{1}{\sqrt{D_X}} - \frac{1}{\sqrt{D_Y}}; \\
Y \xi_X - X \xi_Y &= ((\alpha - 1)\theta + 2)(Y - X) + Y \sqrt{D_X} - X \sqrt{D_Y}; \\
\xi_X \sqrt{D_Y} - \xi_Y \sqrt{D_X} &= ((\alpha - 1)\theta + 2)(\sqrt{D_Y} - \sqrt{D_X}) + X \sqrt{D_Y} - Y \sqrt{D_X}.
\end{align*}
\]

Now we are in a pose to formulate the main estimation.

**Theorem 4.4.** Let the conditions (4.1), (4.2) be satisfied for \( \lambda \). Then one has

\[
\|\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{y})\| \leq \delta|\theta - 1|_p\|\mathbf{x} - \mathbf{y}\|,
\]

for every \( \mathbf{x}, \mathbf{y} \in B_\delta \).
Proof. Let \( x, y \in B_5 \), then from (4.14) we have
\[
|\mathcal{F}^{(+)}(x)_{i} - \mathcal{F}^{(+)}(y)_{i}|_p = |\lambda(i)|_p \left| \frac{2(\theta - 1)x_i + \xi_X}{(\alpha + 1)\theta + X + \sqrt{D_X}} - \frac{2(\theta - 1)y_i + \xi_Y}{(\alpha + 1)\theta + Y + \sqrt{D_Y}} \right|_p
\]
\[
= |\lambda(i)|_p \left| 2(\theta - 1) \left[ x_i((\alpha + 1)\theta + Y + \sqrt{D_Y}) - y_i((\alpha + 1)\theta + X + \sqrt{D_X}) \right] \right|_p
\]
\[
= |\lambda(i)|_p \left| (\alpha + 1)\theta + X + \sqrt{D_X})\xi_X - (\alpha + 1)\theta + X + \sqrt{D_X})\xi_Y \right|_p.
\]

(4.23)

Now step by step, let us estimate I and II.

Put
\[
(4.24)
\Delta = \frac{X + Y + 2\theta(1 - \alpha) + 4\alpha}{\sqrt{D_X} + \sqrt{D_Y}}.
\]

Let us first consider I. Then using Lemma 4.2 one finds
\[
I = (x_i - y_i)((\alpha + 1)\theta + Y + \sqrt{D_Y}) + x_i[Y - X + \sqrt{D_Y} - \sqrt{D_X}]
\]
\[
= (x_i - y_i)((\alpha + 1)\theta + X + \sqrt{D_X}] - x_i(1 + \Delta)(X - Y)
\]
(4.25)

Now turn to II. We easily find that
\[
II = (\alpha + 1)\theta(\xi_X - \xi_Y) + Y\xi_X - X\xi_Y + \xi_X \sqrt{D_Y} - \xi_Y \sqrt{D_X}
\]
according to Lemmas 4.3 and 4.2 one gets
\[
II = 2(1 - \theta)(Y - X) + 2(1 - \theta)(\sqrt{D_Y} - \sqrt{D_X})
\]
(4.26)

Then substituting (4.25) and (4.26) with (4.24) into (4.23) we obtain
\[
|\mathcal{F}^{(+)}(x)_{i} - \mathcal{F}^{(+)}(y)_{i}|_p = |\theta - 1|_p \left| \lambda(i)|_p \left| (x_i - y_i)((\alpha + 1)\theta + X + \sqrt{D_X}) \right|_p
\]
\[
+ (1 - x_i)(1 + \Delta)(X - Y) \right|_p
\]
\[
\leq |\theta - 1|_p |\lambda(i)|_p \max \left\{ |x_i - y_i|_p, 1 + \Delta|_p |X - Y|_p \right\}
\]
\[
\leq |\theta - 1|_p |\lambda(i)|_p \max \left\{ |x_i - y_i|_p \right\}
\]
\[
\leq \delta|\theta - 1|_p |x - y|_p,
\]
(4.27)

here we have used (4.3) and \(|\Delta|_p = 1\).

Consequently, from (4.27) we get the required inequality. \( \square \)

Now let us turn to \( \mathcal{F}^{(-)} \). This case is a little bit tricky. Therefore, impose some extra conditions. Namely, we assume
\[
(4.28) \quad |\alpha - 1|_p \leq \frac{1}{p^2},
\]
\[
(4.29) \quad |\theta - 1|_p = \frac{1}{p}.
\]

Lemma 4.5. Let the conditions (4.1),(4.2),(4.28),(4.29) be satisfied. Then \( \mathcal{F}^{(-)}(B_{p^{-1}\delta}) \subset B_{p^{-1}\delta} \).

Proof. Let \( x \in B_{p^{-1}\delta} \). Then from (4.1),(4.15) we have \(|X|_p \leq 1/p^2\). Using this with (4.28), from (4.10) one finds that \( \sqrt{D_X} = 2 + \epsilon_1 \), where \(|\epsilon_1|_p \leq 1/p^2\). This with (4.9) yields that
\[
(4.30) \quad |x_{-1} + 1|_p \leq \frac{1}{p^2}.
\]
Whence with (4.29) one finds
\[
|x_{-1} + 1 + X + \theta|_p = |x_{-1} + 1 + X + \theta - 1|_p = |\theta - 1|_p.
\]
Consequently, using (4.31), (4.30) and (4.29) we have
\[
|\langle \mathcal{F}(x) \rangle_{i}|_p = |\lambda(i)|_p \left| \frac{(\theta - 1)x_i + x_{-1} + X + 1}{x_{-1} + X + \theta} \right|_p \\
\leq \frac{1}{p}|\lambda(i)|_p \leq p^{-1}\delta
\]
for all \(i \geq 2\), which implies the assertion. \(\square\)

By the same argument of the proof of Theorem 4.4 one can prove

**Theorem 4.6.** Let the conditions (4.1), (4.2), (4.28), (4.29) be satisfied. Then one has
\[
\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq \delta \|x - y\|,
\]
for every \(x, y \in B_{p^{-1}\delta}\).

Now we are ready to formulate our main result.

**Theorem 4.7.** Let the conditions (4.1), (4.2), (4.28), (4.29) be satisfied. Then a phase transition occurs for the countable state \(p\)-adic Potts model (3.1).

**Proof.** From the conditions (4.1), (4.2), (4.28), (4.29) we infer that statements of both Theorems 4.4 and 4.6 are valid. Noting that \(\delta < 1\) with Theorem 4.4 (resp. Theorem 4.6) we can apply the fixed point theorem to \(\mathcal{F}(x)\) (resp. \(\mathcal{F}(x)\)), which means that the existence of a unique fixed point \(x_+ = \{x_{+,i}\} \in B_{\delta}\) (resp. \(x_- = \{x_{-,i}\} \in B_{p^{-1}\delta}\)). Hence, equation (3.8) has at least two translation-invariant solutions \((x_{+,1}, x_+\)\) and \((x_{-,1}, x_-\)). These solutions according to Theorem 3.1 define \(\mu_+\) and \(\mu_-\) generalized \(p\)-adic Gibbs measures, respectively. To show that such measures are different, it is enough to establish that \(x_+\) and \(x_-\) are different. Therefore, using (4.31) one finds
\[
|x_{+,i} - x_{-,i}|_p \leq |\langle \mathcal{F}(x) \rangle_{i} - \langle \mathcal{F}(y) \rangle_{i}|_p \\
\leq |\lambda(i)|_p \left| \frac{(\theta - 1)x_{+,i} + x_{+,1} + X + 1}{x_{+,1} + X + \theta} - \frac{(\theta - 1)x_{-,i} + x_{-,1} + X + 1}{x_{-,1} + X + \theta} \right|_p \\
\leq |\lambda(i)|_p \left| \frac{\theta - 1}{x_{+,1} + X + \theta} \right|_p |x_{+,1} - x_{-,1}|_p \\
\leq |\lambda(i)|_p \left| \frac{\theta - 1}{p|x_{+,1} - x_{-,1}|_p} \right| |x_{+,1} - x_{-,1}|_p \\
\leq |\lambda(i)|_p \left| \frac{\theta - 1}{p} \right| |x_{+,1} - x_{-,1}|_p.
\]
From \(|x_{+,1} - x_{-,1}|_p = |\sqrt{D_{x}}|_p = 1\) we conclude that \(\|x_+ - x_-\| = \delta\), which means the measures \(\mu_+\) and \(\mu_-\) are different. \(\square\)

**Remark 4.1.** Note that if the conditions (4.1), (4.2) are not satisfied, then it may exist only one generalized translation-invariant \(p\)-adic Gibbs measure. Indeed, consider weights defined by (3.15) with
\[
|a|_p = 1, \quad |2a - 1|_p \leq 1/p,
\]
\[
\sqrt{a} \text{ does not exist in } \mathbb{Q}_p.
\]

From Remark 3.2 we already knew that (3.13) has a solution \(\hat{h}_1 = 0\) and \(\hat{h}_2 = -1\) which defines a generalized translation-invariant \(p\)-adic Gibbs measure.

Now we show that equation (3.20) does not have any solution belonging to \(\mathbb{Q}_p\). Indeed, one can compute that its discriminant has a form
\[
D = ((1 - \theta)(1 - a) + 1 - 2a)^2 + 8a
\]
due to \( |1 - \theta|_p \leq 1/p \) and (4.33) one finds that \( |D|_p = |a|_p \). Hence, the assumption (4.34) with Lemma 2.2 implies that \( \sqrt{D} \) does not exist in \( \mathbb{Q}_p \). This means there is no solution of (3.20) belonging to \( \mathbb{Q}_p \).

So, we have two different generalized \( p \)-adic Gibbs measures. It is natural to ask: which of them would be a \( p \)-adic Gibbs measure?

Now recall that a translation-invariant generalized \( p \)-adic measure associated with \( h = \{h_i\} \in \mathbb{Q}_p^\Phi \) would be \( p \)-adic Gibbs one, if there is a sequence \( \{\kappa_i\} \in \mathbb{Q}_p^\Phi \) such that the equality \( h_i = \exp_p \kappa_i \) holds for all \( i \in \Phi \).

Let us find the corresponding sequence \( \{\kappa_i\} \) for \( (x_{+1}, x_+) \). From (3.9) we have

\[
\exp_p(\kappa_i - \kappa_0) = x_{+i}, \quad i \in \mathbb{N}
\]

Since \( (x_{\pm 1}, x_\pm) \) is a fixed point of (3.8), therefore from (4.5) one gets

\[
(4.35) \quad \exp_p(\kappa_1 - \kappa_0) = \frac{\theta x_{+1} + X_+ + 1}{x_{+1} + X_+ + \theta}
\]

and

\[
(4.36) \quad \exp_p(\kappa_i - \kappa_0) = \frac{(\theta - 1)x_{+i} + x_{+1} + X_+ + 1}{x_{+1} + X_+ + \theta}, \quad i \geq 2,
\]

where as before

\[
(4.37) \quad X_\pm = \sum_{j=2}^{\infty} x_{\pm j}.
\]

By means of (4.7) and Lemma 4.1 one gets

\[
\frac{\theta x_{+1} + X_+ + 1}{x_{+1} + X_+ + \theta} \Big|_p = 1,
\]

\[
\frac{\theta x_{+1} + X + 1}{x_{+1} + X + \theta} - 1 \Big|_p = |\theta - 1|_p |x_{+1} - 1|_p < 1/p,
\]

\[
\frac{(\theta - 1)x_{+i} + x_{+1} + X_+ + 1}{x_{+1} + X_+ + \theta} \Big|_p = 1,
\]

\[
\frac{(\theta - 1)x_{+i} + x_{+1} + X_+ + 1}{x_{+1} + X_+ + \theta} - 1 \Big|_p = |\theta - 1|_p |x_{+i} - 1|_p = |\theta - 1|_p \leq 1/p,
\]

which, thanks to Lemma 2.2, allow us to take \( \log_p \) from both sides of (4.35) and (4.36). Hence putting \( \kappa_0 = 0 \), we able to find \( \{\kappa_i\} \). This means that \( \mu_+ \) is a \( p \)-adic Gibbs measure.

Now turn to \( (x_{-1}, x_-) \). From (4.29),(4.31) and (4.8) we have

\[
\left| \frac{(\theta - 1)x_{-j} + x_{-1} + X_- + 1}{x_{-1} + X_- + \theta} \right|_p \leq 1/p.
\]

This due to Lemma 2.2 implies that \( x_- \) can not be represented as \( \exp_p \kappa_i \). Therefore, \( \mu_- \) is a strictly generalized \( p \)-adic Gibbs measure.

So, we have

**Theorem 4.8.** Assume all the conditions of Theorem 4.7 are satisfied. Then \( \mu_+ \) is a \( p \)-adic Gibbs measure, but \( \mu_- \) is a generalized \( p \)-adic Gibbs measure.

In Theorem 3.3 we have provided a sufficient condition on uniqueness of the \( p \)-adic Gibbs measure. But nevertheless, it is interesting to know whether the measure \( \mu_+ \) is a unique \( p \)-adic Gibbs measure.

**Theorem 4.9.** Let the conditions (4.1),(4.2) be satisfied for \( \lambda \). Then the measure \( \mu_+ \) is a unique \( p \)-adic Gibbs measure for \( p \)-adic Potts model (3.1), i.e. \( |\mathcal{G}(H)| = 1 \).
Corollary 4.10. Assume that the conditions of Theorem 4.7 are satisfied. Then

Proof. Let \( \mathfrak{h} = \{ \hat{h}_{i,n}, \hat{h}_n \} \) be any solution of (3.8), where \( \hat{h}_n = \{ \hat{h}_{i,n} \}_{i \geq 2} \) and \( \hat{h}_{i,n} = \lambda(i) \exp(\kappa_{i,n}) \), \( i \in \mathbb{N} \). We will show that such a solution coincides with \( (x_{+1}, x_+) \). Indeed, it is clear that \( \hat{h}_{i,n} |_p = 1 \) and \( \| \hat{h}_n \| = \delta \).

Let us fix \( n \in \mathbb{N} \) and consider the difference

\[
|\hat{h}_{1,n} - x_{+1}|_p = |a| \left| \frac{\theta \hat{h}_{1,n+1} + H_{n+1} + 1}{\hat{h}_{1,n+1} + H_{n+1} + \theta} - \frac{\theta x_{+1} + X_+ + 1}{x_{+1} + X_+ + \theta} \right|_p
\]

(4.38)

\[
\leq |\theta - 1|_p (|\hat{h}_{1,n+1} - 1| (X_+ - H_{n+1}) + (\hat{h}_{1,n+1} - x_{+1}) (\theta + 1 + H_{n+1})|_p
\]

(4.39)

Here we have used that

\[
H_{n+1} := \sum_{k=2}^{\infty} \hat{h}_{k,n+1}
\]

and \( |H_{n+1}|_p \leq \| \hat{h}_{n+1} \| < 1 \).

Similarly reasoning as in the proof of Theorem 4.4 one gets

\[
|\hat{h}_{i,n} - x_{+1}|_p = |\lambda(i)|_p |F_i(\hat{h}_{i,n+1}, \hat{h}_n; \theta) - F_i(x_{+1}, x_+; \theta)|_p
\]

(4.40)

\[
\leq |\lambda(i)|_p |\theta - 1|_p \max \{|\hat{h}_{i,n+1} - x_{+1}|_p, \| \hat{h}_n + x_+ \| \}
\]

From (4.38),(4.39) we obtain

\[
\max \{|\hat{h}_{1,n} - x_{+1}|_p, \| \hat{h}_n - x_+ \| \} \leq |\theta - 1|_p \max \{|\hat{h}_{1,n+1} - x_{+1}|_p, \| \hat{h}_n + x_+ \| \}
\]

Now take an arbitrary \( \epsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that \( |\theta - 1|_{n_0}^p < \epsilon \). Then iterating (4.40) \( n_0 \) times, one gets

\[
\max \{|\hat{h}_{1,n} - x_{+1}|_p, \| \hat{h}_n - x_+ \| \} \leq |\theta - 1|_{n_0}^p < \epsilon
\]

Due to the arbitrariness of \( \epsilon \) we have \( h_{1,n} = x_{+1} \) and \( \hat{h}_n = x_+ \) for every \( n \in \mathbb{N} \).

Remark 4.2. Theproved Theorem 4.9 indicates that the condition (3.14) was a sufficient for the uniqueness of the \( p \)-adic Gibbs measure.

Now let us turn to \( \mu_- \). Take any solution \( \mathfrak{h}_- \) of (3.8) of the form \( \mathfrak{h}_- = \{ x_{-1}, \hat{h}_n^{(-)} \} \), where \( \hat{h}_n^{(-)} \in \mathbb{B}_{p^{-1}\delta} \). As a consequence of Theorem 4.6 one can formulate the following

Corollary 4.10. Assume that the conditions of Theorem 4.7 are satisfied. Then \( \mathfrak{h}_- \) coincides with \( (x_{-1}, x_-) \).

Proof. Now fix any vertex \( n \in \mathbb{N} \) and \( i \geq 2 \). From (4.32) one gets

\[
|\hat{h}_{i,n}^{(-)} - x_{-1}|_p = |\lambda(i)|_p \left| F_i^{(-)}(\hat{h}_{i,n+1}^{(-)}, \theta) - F_i^{(-)}(x_{-1}, \theta) \right|_p
\]

(4.41)

\[
\leq \delta \| \hat{h}_{n+1}^{(-)} - x_- \|
\]

Take an arbitrary \( \epsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that \( \delta^{n_0} < \epsilon \). Then (4.41) implies that

\[
\| \hat{h}_n^{(-)} - x_- \| \leq \delta \| \hat{h}_{n+1}^{(-)} - x_- \| \leq \cdots \leq \delta^{n_0} \| \hat{h}_{n+n_0}^{(-)} - x_- \| < \epsilon.
\]

Hence, from the arbitrariness of \( \epsilon \) we obtain \( \hat{h}_n^{(-)} = x_- \) for every \( n \in \mathbb{N} \). This proves the assertion.

Now let us consider more concrete examples.

Example 4.1. Assume that \( \{ \hat{h}_m \} \) is a solution of (3.13) defined by

\[
\hat{h}_m = p^{m+1} - \frac{J^{m-1}}{(m-1)!}, \quad m \in \mathbb{N},
\]

(4.42)
where \(|J|_p = 1/p\). Then one can see
\[
\sum_{m=1}^{\infty} \hat{h}_m = \frac{p^2}{1 - p} - \theta,
\]
here as before \(\theta = \exp_p(J)\). Now substituting (4.42) to (3.13) we obtain the corresponding \(\lambda\) by
\[
\frac{\lambda(m)}{\lambda(0)} = \frac{J^{m-1}}{(m-1)!} \left( \frac{p^2}{(p-1)(\theta - 1)(\frac{J^{m-1}}{(m-1)!}) - 1 + p^2} \right).
\]
Put \(\lambda(0) = 1\). Then it is clear that \(\lambda(1) = 1\). The equality \(|\theta - 1|_p = |J|_p = 1/p\) implies that
\[
\left| \frac{p^2}{(p-1)(\theta - 1)(\frac{J^{m-1}}{(m-1)!}) - 1 + p^2} \right|_p = \frac{1}{p^2|\theta - 1|_p} = \frac{1}{p}.
\]
This with \(|J^{m-1}/(m-1)!|_p < 1\) yields that the conditions of Theorem 4.7 are satisfied. Now keeping in mind that \(\hat{h}_m\) is a solution of (3.13) and by means of (3.9),(4.44) we find that
\[
\left| \frac{h_m}{h_0} \right|_p = \left| \frac{(p-1)(\theta - 1)(\frac{J^{m-1}}{(m-1)!}) - 1 + p^2}{p^2} \right|_p = p
\]
for every \(m \geq 2\). Hence, (4.45) shows that \(\left| \frac{h_m}{h_0} \right|_p > 1\) which means that the equality \(h_m = \exp_p(\kappa_m)\) impossible for any \(\kappa_m\). This implies that the corresponding measure belongs to \(G(H) \setminus G^*(H)\). From Corollary 4.10 we infer that the constructed generalized \(p\)-adic Gibbs measure is \(\mu_-\). Thanks to Theorem 4.9 for the weights (4.43) there is also a unique \(p\)-adic Gibbs measure.

**Example 4.2.** Now suppose that \(\{\hat{h}_m\}\) is a solution of (3.13) defined by \(\hat{h}_m = \frac{J^{m-1}}{(m-1)!}, \ m \in \mathbb{N}\), with \(|J|_p \leq 1/p\). Then \(\sum_{m=1}^{\infty} \hat{h}_m = \theta\). By the same argument used in Example 4.1 one can define \(\lambda\), for which the conditions (4.1),(4.2) are satisfied as well. In this case one can show that \(\left| \frac{h_m}{h_0} \right|_p = 1\). Hence, according to Theorem 4.9 the corresponding measure is a unique \(p\)-adic Gibbs measure.

Now let us turn to the boundedness of the measures \(\mu_+\) and \(\mu_-\). We need the following

**Lemma 4.11.** Let \(h\) be a translation-invariant solution of (3.8), and \(\mu_h\) be the associated Gibbs measure. Then for the corresponding partition function \(Z^{(b)}_n\) (see (3.5)) the following equality holds
\[
Z^{(b)}_{n+1} = A_h Z^{(b)}_n,
\]
where
\[
A_h = \lambda(0) \left( \theta + \sum_{j=1}^{\infty} \hat{h}_j \right).
\]

**Proof.** From (3.8) we conclude that there is a constant \(A_h \in \mathbb{Q}_p\) such that
\[
\sum_{j \in \Phi} \exp_p \{J \delta_{ij}\} h_j \lambda(j) = A_h h_i
\]
for any \(i \in \Phi\).
On the other hand, using (3.4) and (4.48) we have

\[ 1 = \sum_{\sigma \in \Omega_n, \omega \in \Phi} \mu_h^{(n+1)}(\sigma \lor \omega) \]

\[ = \sum_{\sigma \in \Omega_n, \omega \in \Phi} \frac{1}{Z_n^{(b)}} \exp_p \{ H(\sigma \lor \omega) \} h_{\omega} \prod_{k=0}^{n} \lambda(\sigma(k)) \lambda(\omega) \]

\[ = \sum_{\sigma \in \Omega_n, \omega \in \Phi} \frac{1}{Z_n^{(b)}} \exp_p \{ H(\sigma) \} \prod_{k=0}^{n} \lambda(\sigma(k)) \sum_{j \in \Phi} \exp_p \{ J_\delta_{\sigma(n),j} \} h_{j} \lambda(j) \]

\[ = \frac{A_b}{Z_{n+1}^{(h)}} \sum_{\sigma \in \Omega_n} \exp_p \{ H(\sigma) \} h_{\sigma(n)} \prod_{k=0}^{n} \lambda(\sigma(k)) \]

which implies (4.46). From (4.48) we may easily find (4.47).

Now we are ready to formulate a result.

**Theorem 4.12.** Assume all the conditions of Theorem 4.7 are satisfied. Then the p-adic Gibbs measure \( \mu_+ \) is bounded, but the generalized p-adic Gibbs measure \( \mu_- \) is not bounded.

**Proof.** Let us first consider \( \mu_+ \). Take any \( \sigma \in \Omega_n \). Then from (3.4) with (4.46), (4.47) one gets

\[ |\mu_+(\sigma)|_p = \frac{1}{|Z_n^{(+)}|_p} \left| \exp_p \{ H(\sigma) \} x_{+,\sigma(n)} \prod_{k=0}^{n-1} \lambda(\sigma(k)) \right|_p \]

\[ = \frac{1}{|x_{+,1} + X_+ + \theta |^{n-1} Z_1^{(+)}|_p} |x_{+,\sigma(n)}|_p \prod_{k=0}^{n-1} |\lambda(\sigma(k))|_p \]

\[ \leq \frac{1}{|Z_1^{(+)}}_p|, \]

this means that \( \mu_+ \) is bounded.

Now consider \( \mu_- \). Let us take

\[ \sigma_n^{(1)} = \{ \sigma(k) = 1, k \in [0, n] \}. \]

Then analogously as above with (4.1), (4.31) and (4.29) we find

\[ |\mu_-(\sigma_n^{(1)})|_p = \frac{1}{|Z_n^{(-)}|_p} \left| \exp_p \{ H(\sigma_n^{(1)}) \} x_{-,1} \prod_{k=0}^{n-1} \lambda(1) \right|_p \]

\[ = \frac{|x_{-,1} + X_- + \theta |^{n-1} Z_1^{(-)}|_p}{|x_{-,1} + X_- + \theta |^{n-1} Z_1^{(-)}|_p} \]

\[ = \frac{1}{|\theta - 1|^{n-1} Z_1^{(-)}|_p} \]

\[ = \frac{p^{n-1}}{|Z_1^{(-)}|_p}, \]

which means that \( \mu_- \) is not bounded. \( \square \)

**Remark 4.3.** In [41] we have proved that at \( p = 3 \) there is two p-adic Gibbs measures for that 3-state Potts model, i.e. \( |G(H)| \geq 2 \). Hence a strong phase transition occurs. There, it was shown that those p-adic Gibbs measures were unbounded. Hence, Theorem 4.12 shows the difference between the finite state Potts models, since there the p-adic Gibbs measures are unbounded when the phase transition occurs.
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