Remarks on the Hölder-continuity of solutions to parabolic equations with conic singularities

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Abstract

This is a note on [10] and [5]. Using their work line by line, we prove the Hölder-continuity of solutions to linear parabolic equations of mixed type, assuming the coefficient of $\frac{\partial}{\partial t}$ has time-derivative bounded from above. On a Kähler manifold, this Hölder estimate works when the metrics possess conic singularities along a normal crossing divisor.

1 Introduction

Historically, Hölder-continuity of solutions to linear elliptic and parabolic equations (in various cases) has been proved and extensively studied by De Giorgi [4], Nash [13], Moser [12], Krylov-Safonov [9]. Many other experts have contributed to this topic as well. In addition to the above articles, we also refer the interested readers to [1], [5], [6], [10], and the references therein. In [5], Safonov-Ferretti give a unified proof of the Hölder-continuity in both the divergence case and non-divergence case. The key is to establish growth properties for the level sets of the solutions.

In this note, we focus on the work in [5] on divergence-form equations, and the related work in [10]. The operator in equation (4) is exactly the one considered in [5]. The little difference is: $a_0$ in [5] is not allowed to depend on time (line 11 in page 89), but here we allow $a_0$ to depend on time.

The motivation of us is to study the heat equation associated with a Ricci flow. The Ricci flow is a special time-parametrized family of Riemannian metrics $g(t)$. Given a time-family of Riemannian metrics $g(t)$ over a Euclidean ball $B$, the heat equation of this family reads as

$$\frac{\partial u}{\partial t} - \Delta_{g(t)} u = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x_i} (g^{ij} \sqrt{\det g_{ij}} \frac{\partial u}{\partial x_j}) = f, \quad (1)$$

where $x_i$'s are the Euclidean coordinates. To estimate the Hölder norm of $u$, we only care about the $L^\infty$--norm of $f$, though we can assume that everything involved have higher derivatives. Multiplying (1) by $\sqrt{\det g_{ij}}$, we get

$$\sqrt{\det g_{ij}} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} (g^{ij} \sqrt{\det g_{ij}} \frac{\partial u}{\partial x_j}) = F = f \sqrt{\det g_{ij}}. \quad (2)$$
Let \( a_0 = \sqrt{\det g_{ij}} \) and \( a_{ij} = g_{ij} \sqrt{\det g_{ij}} \). (1) is a special case of (4) and equation (D) in page 89 of [5]. Suppose \( \det g_{ij} \) is uniformly bounded, the \( L^\infty \)−norm of \( f \) is equivalent to the \( L^\infty \)−norm of \( F \), thus it makes no difference for the Hölder estimate.

Our main observation (and a one sentence proof of Theorem 1.1) is that when \( a_0 \) depends on time and \( \frac{\partial \log a_0}{\partial t} \) is bounded from above, the general energy estimates are still true (Lemma 4.5). By the proof in [5], these energy estimates imply the main growth theorem (Theorem 5.3) in [5]. Moreover, by an idea in [10], Theorem 5.3 in [5] directly implies the Hölder continuity of solutions, without involving the Harnack inequality in Theorem 1.5 of [5]. We believe these are known by experts. When \( g_t \) is a Ricci flow, the upper bound on \( \frac{\partial \log a_0}{\partial t} \) means

\[
\frac{\partial}{\partial t} dvol_{g_t} \leq K dvol_{g_t} \tag{3}
\]

for some constant \( K > 0 \), where \( dvol_{g_t} \) is the evolving volume form. The \( K \) is actually a lower bound for the scalar curvature of \( g_t \). Fortunately, the scalar curvature is usually bounded from below along Ricci flows without any additional condition, see [2] (page 5) and [8].

The simplest version of our main theorem is stated as follows. Let \( Y = (y, s) \) be a space-time point, and \( C_r(Y) = B_y(r) \times (s-r^2, s) \) be the parabolic cylinder centred at \( Y \) with radius \( r \) \([B_y(r) \text{ is the usual } m\text{-dimensional Euclidean ball}].\)

**Theorem 1.1.** Suppose \( u \in C^\infty[C_r(Y)] \) solves the following equation (or the metric heat equation (1) via the correspondence in (2)) in the classical sense

\[
a_0 \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_j} (a_{ij} \frac{\partial u}{\partial x_i}) = f, \tag{4}
\]

where \( a_0, a_{ij} \ (1 \leq i, j \leq m) \) are space-time smooth functions. Suppose

\[
\frac{1}{K} \leq a_0 \leq K, \quad \frac{\partial \log a_0}{\partial t} \leq K, \quad \frac{I}{K} \leq a_{ij} \leq K I. \tag{5}
\]

Then there exist constants \( \alpha(m, K) \in (0, 1) \) and \( N(m, K) \) such that

\[
r^\alpha \|u\|_{C^\alpha_r(Y)} + \|u\|_{L^\infty[C_r(Y)]} \leq N\left(\frac{|u|_{L^1(C_r(Y))}}{r^{m+2}} + r^{2} |f|_{L^\infty(C_r(Y))}\right).
\]

**Remark 1.2.** The \( \| \cdot \|_\alpha \) is the parabolic Hölder semi-norm of exponent \( \alpha \) (see (4.1) in [11] for definition). **Theorem 1.1** can be generalized to heat equations of Kähler-metrics with conic singularities along normal-crossing divisors (Theorem 2.1). We only prove Theorem 2.1 the proof
of Theorem 1.1 is the same (by discarding the necessary techniques for the conic singularities, see Claim 4.7 for example). We check routinely in Section 5 that the only usage of the equation in proving Theorem 5.8 is the energy estimate (for all test functions, at all levels and in all scales).

**Remark 1.3.** When $\frac{\partial \log a_0}{\partial t}$ is not uniformly bounded from above (while the other conditions in Theorem 1.1 hold true), the above uniform Hölder estimate does not hold in general. We refer the interested readers to the beautiful example constructed by Chen-Safonov (Theorem 4.1 and 4.2 in [3]).

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### 2 The more general version of Theorem 1.1 in Kähler geometry involving conic singularity

In Kähler geometry setting, Theorem 1.1 holds even when the metrics possess conic singularities along analytic hyper surfaces. To state the result, we first give a geometric formulation following [7]. Given a closed Kähler manifold $M$ and a divisor $D = \Sigma_{j=1}^{N} 2\pi (1 - \beta_j)D_j$, where each $D_j$ is an irreducible hyper surface and may have self-intersection, suppose $D$ has (no worse than) normal crossing singularities i.e there is an open cover of $\text{supp}D$ by neighbourhoods $U_i$ such that in each $U_i$, $\text{supp}D \cap U_i = \{z_1 z_2 z_3 \cdots z_k = 0\}$, where $k \leq n$ and $z_1 \cdots z_n$ are holomorphic coordinate functions in $U_i$. A Kähler metric $g$ (defined away from $\text{supp}D$) is said to be a weak-conic metric with quasi-isometric constant $K$, iff it’s Hölder-continuous away from $\text{supp}D$ and in each $U_i$,

$$\frac{g_{\beta}^k}{K} \leq g \leq K g_{\beta}^k \text{ (quasi-isometric)}, \quad g_{\beta}^k = \sum_{j=1}^{k} \frac{\beta_j^2}{|z_j|^{2-2\beta_j}} dz_j \otimes d\bar{z}_j + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j.$$  

(6)

$g_{\beta}^k$ is one of the 2 model metrics on $\mathbb{C}^n$ we work with, and in this local setting we abuse notation by denoting $\text{supp}D$ as $D$.

Similarly, a Kähler metric $g$ is called a $\epsilon$-nearly-conic metric with quasi-isometric constant $K$, iff it’s Hölder-continuous over the whole $M$ (across $\text{supp}D$) and in each $U_i$,

$$\frac{g_{\beta,\epsilon}^k}{K} \leq g \leq K g_{\beta,\epsilon}^k, \quad g_{\beta,\epsilon}^k = \sum_{j=1}^{k} \frac{\beta_j^2}{(|z_j|^2 + \epsilon^2)^{1-\beta_j}} dz_j \otimes d\bar{z}_j + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j, \quad \epsilon > 0.$$  

(7)

We recall the well known intrinsic polar coordinates of $g_{\beta}^k$. Let $\xi_j = r_j e^{-\theta_j}$, $r_j = |z_j|^{\beta_j}$, $1 \leq j \leq k$. In these polar coordinates the model cone $g_{\beta}^k$ is equal to

$$g_{\beta}^k = \sum_{j=1}^{k} (dr_j^2 + \beta_j^2 r_j^2 d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j.$$
and it’s quasi-isometric to the Euclidean metric i.e
\[(\min_j \beta_j^2)g_E \leq g^k_\beta \leq g_E, \ g_E = \sum_{j=1}^{k} (dr_j^2 + r_j^2 d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j. \quad (8)\]

This is important because we want to take advantage of the rescaling and translation invariance of the Euclidean metric.

Similarly, we also have intrinsic polar coordinates for \(g^k_{\beta,\epsilon}\). Let \(s_j\) be the solution to
\[\frac{ds_j}{d\rho_j} = \frac{\beta_j}{(\rho_j^2 + \epsilon^2)^{\frac{\beta_j}{2}}}, \ s_j(0) = 0, \ \rho_j = |z_j|. \quad (9)\]

Then \(\xi_j = s_j e^{\sqrt{2} \theta_j}, \ 1 \leq j \leq k\) defines the polar coordinates of \(g^k_{\beta,\epsilon}\). By Lemma 4.3 in [14], in these coordinates we have
\[g^k_{\beta,\epsilon} = \sum_{j=1}^{k} (ds_j^2 + a_{\beta,\epsilon} s_j^2 d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j, \ \beta_j^2 < a_{\beta,\epsilon} \leq 1. \quad (10)\]

Hence \(g^k_{\beta,\epsilon}\) is also quasi-isometric to the Euclidean metric in its polar coordinate i.e
\[(\min_j \beta_j^2)g_E \leq g^k_{\beta,\epsilon} \leq g_E, \ g_E = \sum_{j=1}^{k} (ds_j^2 + s_j^2 d\theta_j^2) + \sum_{j=k+1}^{n} dz_j \otimes d\bar{z}_j. \quad (11)\]

Unless specified (via a parentheses or a sub-symbol), the constants \(N\) and \(C\) in this article depend on (at most) \(n, K, M, D, \beta_j's, \) and the open cover \(\cup U_i\). They \textbf{don’t depend on} \(\epsilon\). Different \(N's\) could be different. \textbf{The real dimension is} \(m = 2n\) in the Kähler setting.

\textbf{Theorem 2.1. Let} \(\epsilon \in [0, 1]\).

\textbf{Part I (local estimate): Suppose} \(g_t\) \textbf{is a time-differentiable family of weak-conic Kähler metrics or of \(\epsilon\)-nearly-conic metrics, which is defined over a parabolic cylinder} \(C_r(Y)\) \textbf{in} \(\mathbb{C}^n\) \textbf{under a polar coordinate as below (7) or (9), respectively}. \textbf{Suppose the quasi-isometric constant of} \(g_t\) \textbf{is} \(K\),
\[\frac{\partial}{\partial t} d\text{vol}_t \leq K d\text{vol}_t. \quad (12)\]

and \(u\) \textbf{is a bounded weak solution to}
\[\frac{\partial u}{\partial t} = \Delta_{g_t} u + f \text{ over } C_r(Y). \quad (13)\]

Then there exists \(\alpha(n, \beta, K) \in (0, 1)\) and \(N(n, \beta, K)\) \textbf{such that}
\[r^\alpha |u|_{\alpha, C_r(Y)} + |u|_{L^\infty, C_r(Y)} \leq N \left( |u|_{L^1(C_r(Y))} + r^2 |f|_{L^2(C_r(Y))} \right). \quad (14)\]

\textbf{Part II (global estimate): In the setting of (12) and paragraph above it, suppose all the conditions in part I hold globally on} \(M \times [0, T]\). \textbf{Then for all} \(t_0 \in (0, T)\), \textbf{there exists an} \(\alpha(n, \beta, K)\) \textbf{and} \(C_{t_0}(n, \beta, K)\) \textbf{such that}
\[|u|_{\alpha, M \times [t_0, T]} + |u|_{L^\infty(M \times [t_0, T])} \leq C_{t_0} \left( |u|_{L^1(M \times [0, T])} + |f|_{L^2(M \times [0, T])} \right). \quad (15)\]
Remark 2.2. When the divisor is smooth, a weaker version of this Hölder estimate is in section 4 of [14]. We hope it’s still somewhat valuable to present the proof separately here. The $[u]_α$ is the usual parabolic Hölder semi-norm with respect to $g^k_β (g^k_β,ε)$ [see (8)]. An important point is that Hölder continuity with respect to the distance of $g^k_β (g^k_β,ε)$ is equivalent to Hölder continuity in the usual sense in holomorphic coordinates (apart from a difference of Hölder exponents). We refer interested readers to Lemma 4.4 in [14]. Please see Definition 4.2 for definition of weak solutions (replace $SC_r$ by the underlying domain).

Remark 2.3. Using the Kähler structure, equation (13) can be written as both divergence and non-divergence form. We expect that Theorem 2.1 still holds without condition (12).

3 Proof of the main results assuming Theorem 5.8.

From now on (and in the subsequent sections), we work in the polar coordinates in (8) and (11). In this coordinate, we don’t see the conic singularity (except that the coefficients of the equations and solutions are not defined on $D$). Let $C^0_r$ denote $C_r(y, s – 3r^2)$ (see the paragraph above Theorem 1.1).

Proof. of Theorem 1.1, 2.1 We only prove (part I of) Theorem 2.1 as mentioned at the end of the introduction. Notice that $y$ does not have to be in $suppD$ (as long as integration by parts is true, see proof of Lemma 4.5). By the interior $L^{∞}$-estimate in Proposition 5.3 which holds for every cylinder and every sub-solution, it suffices to show the Hölder norm is bounded by the $L^{∞}$-norm i.e.

$$r^α[u]_{α,C_r^0(Y)} ≤ N(|u|_{L^{∞}|C_r^0(Y)|} + r^2|f|_{L^{∞}|C_r^0(Y)|}).$$

(14)

By Lemma 4.6 in [11], it suffices to show the oscillation decays for every cylinder $C_{2r}$ and every sub-solution $u$ i.e.

$$osc_{C_r} u ≤ (1 – b)osc_{C_{2r}} u + 4r^2|f|_{0,C_{2r}}, \quad b = b(n, β, K) > 0.$$  

(15)

By rescaling and translation invariance, it suffices to assume $r = 1, s = 0$. By adding a constant, it suffices to assume $0 ≤ u ≤ h$, where $h \triangleq osc_{C_1} u$. As in [10], one of the following must hold:

Case 1: $|\{u > \frac{h}{2}\} \cap C^0_1| ≤ \frac{|C^0_1|}{2};$  

Case 2: $|\{u < \frac{h}{2}\} \cap C^0_1| ≤ \frac{|C^0_1|}{2}.$

We only prove (15) in Case 1 in detail. Case 2 is similar by applying the proof in Case 1 to $h – u$. Consider $\bar{u} = u – t|f|_{0,C_{2r}} \quad [t \in (-4, 0)].$ Then

$$\frac{∂\bar{u}}{∂t} – ∆_g \bar{u} ≤ 0, \quad 0 ≤ \bar{u} ≤ h + 4|f|_{0,C_2}.$$  

Moreover, $\bar{u} > \frac{h}{2} + 4|f|_{0,C_2} \Rightarrow u > \frac{h}{2}.$

(16)
Hence the assumption of Case 1 implies

\[ |\{ \bar{u} \geq \frac{h}{2} + 4|f|_{0,C_2} \} \cap C_1^0 | \leq |\{ u > \frac{h}{2} \} \cap C_1^0 | \leq \frac{|C_1^0|}{2}. \]

Then Theorem 5.8 (applied to \( \bar{u} - \frac{h}{2} - 4|f|_{0,C_2} \), (16), and the above inequality imply that there exists \( a(n, \beta, K) > 0 \) such that

\[ \sup_{C_1} (\bar{u} - \frac{h}{2} - 4|f|_{0,C_2}) \leq (1 - a) \sup_{C_2} (\bar{u} - \frac{h}{2} - 4|f|_{0,C_2}) \leq \frac{(1 - a)h}{2}. \]

Then \( osc_{C_1} u \leq \sup_{C_1} \bar{u} \leq (1 - \frac{a}{2})h + 4|f|_{0,C_2} \). The proof of (15) (under the normalization conditions below it) is complete. \( \square \)

4 Energy inequalities

We follow closely the definitions and tricks in [5], the point is that they work equally well in the presence of conic singularity (Definitions 4.1, 4.2, and 4.4). The functions and integrations are all defined away from the divisor \( D \) [see the content below [6]]. If the notation of a function space does not involve \( D \), we mean the space satisfies the indicated asymptotic property (which should be clear from the context). The sets and (slant) cylinders are standard ones minus \( D \). This does not affect any measure theory, integration, or technique in this article, because the space wise co-dimension of \( D \) is 2. For the proof of Theorem 1.1 we don’t have to chop off any singularity.

**Definition 4.1.** A slant cylinder \( SC_r(y_0, y_1, T_0, T_1) \), which we abbreviate in most the time as \( SC_r \), is the following set

\[ SC_r \triangleq \{ x \mid |x - y(t)| < r, \ T_0 < t \leq T_1 \}, \] (17)

where \( y(t) = y_0 + \frac{(t-T_0)(y_1-y_0)}{T_1-T_0} \). When \( y_1 = y_0 \), \( SC_r \) is just the usual cylinder \( C_r \) defined above Theorem 1.1. We define \( l \triangleq \frac{r(y_1-y_0)}{T_1-T_0} \) as the parabolic slope of \( SC_r \). \( l \) is invariant under

- the usual parabolic rescaling (linear multiplication on \( y_0, y_1, r \) and quadratic multiplication on \( T_0, T_1 \) by the same factor),
- the space-wise translation (on \( y_0, y_1 \) by the same displacement),
- and the time-wise translation (on \( T_0 \) and \( T_1 \) by the same displacement).
**Definition 4.2.** We say $u$ is a weak sub solution to
\[
\frac{\partial u}{\partial t} - \Delta_g u \leq 0,
\] (18)
in a slant cylinder $SC_r$ if

1. $u \in C^{2+\alpha,1+\frac{\alpha}{2}}\{SC_r \setminus D \times [T_0,T_1]\} \cap L^\infty(SC_r)$;

2. Inequality (18) holds over $SC_r \setminus D \times [T_0,T_1]$ in the classical sense.

We call a function $\eta$ (defined in any bounded space-time domain $\Omega \in \mathbb{C}^n \times (-\infty,\infty)$) tame if $\eta \in C^{1,1}\{\Omega \setminus D \times [T_0,T_1]\} \cap L^\infty(\Omega)$ and the following holds.
\[
\frac{\partial \eta}{\partial t} \in L^1(\Omega), \quad \nabla \eta \in L^2(\Omega).
\] (19)

**Remark 4.3.** The $L^\infty(SC_r)$-requirement in Definition 4.2 is crucial, and is the only global condition. It guarantees (18) holds across the singularity in the sense of integration by parts.

**Definition 4.4.** Exactly as in Corollary 2.3 in [5], we define the cutoff function of $u$ as
\[
u = G(u),
\] (20)
where $G$ is a function with one variable such that $G(u) = 0$ when $u \leq \epsilon$, $G(u) = u + G(2\epsilon) - 2\epsilon$ when $u \geq 2\epsilon$, and $G, G', G'' \geq 0$. Consequently, we have
\[
G(2\epsilon) \leq \epsilon \quad \text{and} \quad \max\{u - 2\epsilon,0\} \leq u \leq \max\{u - \epsilon,0\}. \quad (21)
\]

The most important feature of $u_\epsilon$ is that, suppose $u$ is sub solution to (18), so is $u_\epsilon$ i.e
\[
\frac{\partial u_\epsilon}{\partial t} - \Delta_g u_\epsilon \leq 0.
\] (22)

$u_\epsilon$ can be understood as the smoothing of $u^+$ (non-negative part of $u$). We note that in the classical case, $u^+$ is a sub solution (in proper sense) if $u$ is. The above smoothing is point wise, thus works in the presence of conic singularities.

**Lemma 4.5.** Under the same assumptions in Part I of Theorem 2.1 (for any $r$), suppose $u$ is a non-negative weak solution to (18) in the sense of Definition 4.2 in a slant cylinder $SC_r$, $r \leq \frac{1}{100n}$. Then for any non-negative tame function $\eta$ which is compactly supported in $SC_r$ space wisely, we have
\[
\int_{\mathbb{C}^n} w^2 dV_g [\eta_t^2 + \int_{t_1}^{t_2} \int_{\mathbb{C}^n} <\nabla_g u, \nabla_g \eta^2> dV_g ds < \int_{t_1}^{t_2} \int_{\mathbb{C}^n} w^2 \nabla^2_g dV_g ds + K \int_{t_1}^{t_2} \int_{\mathbb{C}^n} w^2 dV_g ds.
\] (23)
Moreover, we have
\[
\int_{C} u^2 \eta^2 dV_g|^{t_2}_{t_1} + \frac{1}{2} \int_{t_1}^{t_2} \int_{C} |\nabla_g u|^2 \eta^2 dV_g ds
\]
(24)

\[
\leq \int_{t_1}^{t_2} \int_{C} u^2 \frac{\partial \eta^2}{\partial t} dV_g ds + (2K + 200) \int_{t_1}^{t_2} \int_{C} u^2 \eta^2 + |\nabla_g \eta|^2 dV_g ds,
\]
and therefore
\[
\int_{\Omega} |\nabla_g u|^2 dV_g ds < +\infty, \text{ for any (parabolic) compact sub-domain } \Omega \text{ of } SC_r.
\]
(25)

**Remark 4.6.** By the same proof, the energy estimate of (4) is similar.

**Proof.** of Lemma 4.5: Let \( r_i \) be the distance function to the smooth hyper-surface \( D_i \). We consider Berdtsson’s cutoff function \( \psi_{i, \epsilon} = \psi(\epsilon \log(-\log r_i)) \), \( \psi \) is the standard cutoff function such that \( \psi(x) \equiv 1 \) when \( x \leq \frac{1}{2} \), and \( \psi(x) \equiv 0 \) when \( x \geq \frac{4}{5} \). Then
\[
\psi_{i, \epsilon} \equiv 0 \text{ when } r_i \leq e^{-e^{4/5} \epsilon}; \ \psi_{i, \epsilon} \equiv 1 \text{ when } r_i \geq e^{-e^{1/2} \epsilon}. \]
(26)

Let \( \psi_\epsilon = \prod_{i=1...n} \psi_{i, \epsilon} \), the following claim is true.

**Claim 4.7.**
\[
\lim_{\epsilon \to 0} \|\nabla E \psi_\epsilon\|_{L^2(B(1/2))} = 0.
\]
(27)

The proof of Claim 4.7 is elementary. We only verify it for \( \frac{\partial \psi_\epsilon}{\partial r_1} \), the other directional derivatives are similar. We compute \( \frac{\partial \psi_\epsilon}{\partial r_1} = -\psi' \frac{\epsilon}{r_1 \log r_1} \prod_{i \neq 1} \psi_{i, \epsilon} \).

Hence in poly-cylindrical coordinates we find
\[
\int_{B(1/2)} \left|\frac{\partial \psi_\epsilon}{\partial r_1}\right|^2 dvol_E \leq C \epsilon^2 \int_0^{\frac{1}{2}} \frac{1}{r_1 (\log r_1)^2} dr_1 \leq C \epsilon^2.
\]

We first prove (24). By definition we have \( \lim_{\epsilon \to 0} \psi_\epsilon = 1 \) everywhere except on \( suppD \). We multiply both hand sides of (18) by \( u \eta \psi_\epsilon^2 \), then integrate by parts and integrate with respect to time, we obtain
\[
\int_{C} u^2 \eta^2 \psi_\epsilon^2 dV_g|^{t_2}_{t_1} + \frac{1}{2} \int_{t_1}^{t_2} \int_{C} |\nabla_g u|^2 \eta^2 \psi_\epsilon^2 dV_g ds
\]
(28)

\[
\leq \frac{1}{2} \int_{t_1}^{t_2} \int_{C} u^2 \eta^2 \psi_\epsilon^2 \frac{dV_g ds}{\partial t} - 2 \int_{t_1}^{t_2} \int_{C} \nabla_g u, \nabla_g \eta > u \eta \psi_\epsilon^2 dV_g ds
\]

\[
+ \frac{1}{2} \int_{t_1}^{t_2} \int_{C} u^2 \frac{\partial \eta^2}{\partial t} \psi_\epsilon^2 dV_g ds - 2 \int_{t_1}^{t_2} \int_{C} \nabla_g u, \nabla_g \psi_\epsilon > u \eta^2 \psi_\epsilon dV_g ds.
\]
Using Cauchy-Schwartz inequality we deduce that
\[
|2 \int_{t_1}^{t_2} \int_{\mathbb{C}^n} < \nabla_g u, \nabla_g \psi_\epsilon > u \eta^2 \psi_\epsilon dV_g ds| \leq \frac{1}{100} \int_{t_1}^{t_2} \int_{\mathbb{C}^n} |\nabla_g u|^2 \psi_\epsilon^2 \eta^2 dV_g ds + 100 \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 |\nabla_g \psi_\epsilon|^2 dV_g ds.
\] (29)

Similarly we have
\[
|2 \int_{t_1}^{t_2} \int_{\mathbb{C}^n} < \nabla_g u, \nabla_g \eta > u \eta^2 \psi_\epsilon^2 dV_g ds| \leq \frac{1}{100} \int_{t_1}^{t_2} \int_{\mathbb{C}^n} |\nabla_g u|^2 \psi_\epsilon^2 \eta^2 dV_g ds + 100 \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \psi_\epsilon^2 |\nabla_g \eta|^2 dV_g ds.
\] (30)

Notice that by (3) we have
\[
\int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 \psi_\epsilon^2 \partial_t ds \leq K \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 \psi_\epsilon^2 dV_g ds.
\] (31)

Then
\[
\frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 \psi_\epsilon^2 dV_g ds + \int_{t_1}^{t_2} \int_{\mathbb{C}^n} |\nabla_g u|^2 \eta^2 \psi_\epsilon^2 dV_g ds \leq K \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 \psi_\epsilon^2 dV_g ds + \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 \psi_\epsilon^2 dV_g ds.
\] (32)

We note that Definition 4.2 requires \( u \in L^\infty \), then (27) implies
\[
\lim_{\epsilon \to 0} 100 \int_{t_1}^{t_2} \int_{\mathbb{C}^n} u^2 \eta^2 |\nabla_g \psi_\epsilon|^2 dV_g ds = 0.
\] (33)

Let \( \epsilon \to 0 \) in (32), the proof of (24) and (25) is complete.

Multiplying both hand sides of (18) by \( \eta \psi_\epsilon \) and integrating by parts over space-time, (23) is proved similarly.

By the same proof as Lemma 4.5 (with Berndtsson’s cutoff function), the Sobolev embedding theorem is true.

**Lemma 4.8. (Sobolev Embedding)** Given a function \( u \in C^1(B \setminus D) \cap L^\infty(B) \), for any cutoff function \( \eta \in C^1_0(B) \), the following holds.
\[
\left( \int_B |\eta u|^{\frac{2n}{2n-1}} dV_E \right)^{\frac{2n-1}{2n}} \leq N(\beta, n) \int_B |\nabla(\eta u)| dV_E.
\]
Proof. It’s true when \( \int_B |\nabla (\eta u)| dV_E = \infty \). When \( \int_B |\nabla (\eta u)| dV_E < \infty \), using \( \psi \), Claim 1.7 and the same proof as in Lemma 1.5 \( \eta u \) belongs to \( W^{1,1}(B) \) in the usual sense. Then it follows from the usual Sobolev-inequality. \( \square \)

Remark 4.9. The \( N(\beta, n) \) above does not depend on the radius or center of the ball. The only place where we use the Sobolev embedding is \( (11) \).

5 Proof of Theorem 5.8 by energy inequalities

5.1 Growth Lemma

Proposition 5.1. (Growth Lemma) Suppose \( u \) is a weak sub solution to \( (18) \) in a cylinder \( C_{2r}(Y) \). Then there exists a \( \mu_2(n, \beta, K) > 0 \) such that

\[
\frac{|\{ u > 0 \} \cap C_{2r}(Y)|}{|C_{2r}(Y)|} \leq \mu_2 \implies \sup_{C_r} u \leq \frac{1}{2} \sup_{C_{2r}} u^+. \tag{34}
\]

Proof of Proposition 5.1: The proof is formally the same as Lemma 4.1 in [5]. Since condition (3) is involved, we still give a detailed proof for the reader’s convenience. The point is to show that we don’t need more on the equation than the energy estimates of sub-solutions (Lemma 4.5 and the proof of it). The constants \( N \) in this proof only depend on \( n, \beta, K \).

By rescaling invariance of the sub-equation \( (18) \), it suffices to assume \( r = 1 \) and \( \sup_{C_1} u = 1 \). We let \( \mu_2 \) be small enough. It suffices to prove that for all \( Z \notin D \) and \( Z \in C_1(Y) = C_1 \), under the condition

\[
\frac{|\{ u > 0 \} \cap C_1(Z)|}{|C_1(Z)|} \leq 2^{2n+2} \frac{|\{ u > 0 \} \cap C_2(Z)|}{|C_2(Z)|} \leq 2^{2n+2} \mu_2 \overset{=}{=} \mu_1, \tag{35}
\]
the following estimate holds

\[
u(Z) \leq \frac{1}{2}. \tag{36}
\]

We only need to apply Lemma 5.2 ( (3.8) in page 33 of [5]). Using exactly the induction argument from the last line of page 99 to line 16 of page 100 in [5] (only involving Lemma 5.2), we deduce for any integer \( j \geq 0 \), for some \( N(n, \beta, K) \), the following estimate holds when \( N \mu_1^{2n+2} < \frac{1}{2} \).

\[
|\{ u > \frac{1}{2} - \rho \} \cap C_{\rho}(Z)| \leq \mu_1 \rho^{2n+2} |C_{\rho}(Z)|, \quad \rho = 2^{-j}. \tag{37}
\]

Since \( Z \notin D \), (37) directly implies that \( u(Z) \leq \frac{1}{2} \). Were this not true, \( u(Z) > \frac{1}{2} \) implies that there exists dyadic \( \rho_0 \) small enough such that \( C_{\rho_0}(Z) \) does not touch the singularity \( D \), and \( u > \frac{1}{2} \) over \( C_{\rho_0}(Z) \). This contradicts (37). \( \square \)
Lemma 5.2. Under the same setting as in Proposition 5.1 and its proof above, for any constant $A \geq 0$, we have

$$\int_{C_T^1(Z)} (u - A)_+ dV_E ds \leq \frac{N}{\rho} |\{u > A\} \cap C_\rho(Z)|^{1 + \frac{1}{2n+2}}.$$  

Proof. of Lemma 5.2. By linearity and rescaling invariance of the sub equation (18), without loss of generality we can assume $A = 0$ and $\rho = 1$ (note $u \leq 1$). Denote the set $\{(u > 0) \cap C_1(Z)\}$ as $E_u$, and the space-wise set $\{x| (x, t) \in (u > 0) \cap C_1(Z)\}$ as $Q(t)$. Hence $|E_u| = \int_0^1 |Q(t)| dt$. We need to prove

$$\int_{C_T^1(Z)} u_+ dV_E ds \leq N|E_u|^{1 + \frac{1}{2n+2}} \tag{38}$$

To show (38) is true, it suffices to show that for any $\epsilon$ small enough, $u_\epsilon$ satisfies

$$\int_{C_T^1(Z)} u_\epsilon dV_E ds \leq N|E_{u_\epsilon}|^{1 + \frac{1}{2n+2}}. \tag{39}$$

The advantage of $u_\epsilon$ is that it’s supported in $Q(t)$, and $0 \leq u_\epsilon \leq 1$. Then integration by parts implies the energy estimates in Lemma 4.5 holds true near the parabolic boundary, Hölder’s inequality and Lemma 4.5 imply

$$\int_B |\eta u_\epsilon dV_E| t \leq |Q(t)|^\frac{1}{2} (\int_B |\eta^2 u_\epsilon^2 dV_E| t \leq NE_{u_\epsilon}^\frac{1}{2} |Q(t)|^{\frac{1}{2}}. \tag{40}$$

We also have the following bootstrapping estimate on the same term.

$$\int_B |\eta u_\epsilon dV_E| \leq (\int_B |\eta u_\epsilon|^{\frac{2n}{2n-1}} dV_E)^{\frac{2n-1}{2n}} |Q(t)|^{\frac{1}{2n}} \leq N(\int_B |\nabla(\eta u_\epsilon)| dV_E)|Q(t)|^{\frac{1}{2n}} \leq N(\int_B |\nabla(\eta u_\epsilon)|^2 dV_E)^{\frac{1}{2}} |Q(t)|^{\frac{1}{2n} + \frac{1}{4}} \text{ [since supp}\nabla(\eta u_\epsilon) \subset \{u > 0\} \cap B]. \tag{41}$$

By (40), (41), Lemma 4.5 and Fubini-Theorem [with the help of (25)],

$$\int_{-1}^0 \int_B |\eta u_\epsilon dV_E| ds = \int_{-1}^0 (\int_B |\eta u_\epsilon dV_E|)^{\frac{1}{n+1}} (\int_B |\eta u_\epsilon dV_E|)^{\frac{n}{n+1}} ds \leq NE_{u_\epsilon}^{\frac{1}{2n+2}} \int_{-1}^0 |Q(t)|^{\frac{n+2}{2n+2}} (\int_B |\nabla(\eta u_\epsilon)|^2 dV_E)^{\frac{n}{2n+2}} dt \tag{42}$$

$$\leq N|E_{u_\epsilon}|^{\frac{1}{2n+2}} (\int_{-1}^0 |Q(t)| dt)^{\frac{n+2}{2n+2}} (\int_{-1}^0 (\int_B |\nabla(\eta u_\epsilon)|^2 dV_E dt)^{\frac{n}{2n+2}} \leq N|E_{u_\epsilon}|^{1 + \frac{1}{2n+2}}$$

Since $\eta \equiv 1$ over $C_T^1(Z)$, the proof is complete. As we’ve seen, nothing in this proof involves more than Lemma 4.5 on the sub-solutions.
Proposition 5.3. Suppose $u$ is a weak sub solution to (18) in a cylinder $C_r(Y), y \notin D$. Then

$$u(Y) \leq \frac{N}{|C_r|} \left( \int_{C_r(Y)} u_+ dV_E \right).$$

(43)

Proof. of Proposition 5.3 The proof is exactly as of Theorem 3.4 in [5]. The only thing worth mentioning is that we should deal with the singularity $D$. In [5], they consider the maximal point of $d^\gamma u$, where $\gamma = \frac{2n+2}{p}$ and $d$ is the parabolic distance to the the parabolic boundary of $C_r(Y)$. However, when singularity is present, $d^\gamma u$ might not attain maximum away from $D$. To overcome this, we simply assume $u(Y) > 0$, and use the fact that there exist an almost maximal point away from $D$. Namely, there exist $X_0 = (x_0, t_0)$ such that

$$d^\gamma(X_0)u(X_0) \geq M, \quad M \triangleq \sup_{C_r} d^\gamma u$$

(44)

(we can assume $M > 0$ with out loss of generality). Then the rest of the proof is line by line as from line 13 to line -3 in Page 101 of [5], except the $\mu_1$ in line 19 should correspond to $\beta_1 = 2^{-\gamma-2}$, because we have an additional $\frac{1}{2}$ in (44).

Remark 5.4. As mentioned in Remark 3.5 in [5], this proof does NOT involve explicitly the sub equation (18). Instead, it only requires the Growth Lemma (5.1). Thus the conditions in (12) is not involved explicitly in this proof.

Theorem 5.5. (Slant Cylinder Theorem) Suppose $u$ is a weak sub solution to (18) in a slant cylinder $SC_r$. Suppose $u \leq 0$ in $B_r \times \{T_0\}$. Then

$$u(Y) \leq (1 - \lambda) \sup_{SC_r} u^+, \quad \lambda \in (0, 1) \text{ depends on } n, \beta, \frac{r|y_1 - y_0|}{T_1 - T_0} (|l|), T_1 - T_0, \text{ and } K.$$ 

(45)

Proof of Theorem 5.5: The first paragraph in the proof of Proposition 5.1 also applies here. By translation and rescaling (see Definition 4.1), without changing the parabolic slope, we can transform $SC_r$ to a slant cylinder $SC_1$ with $r = 1$, $T_0 = 0$, $T_1 = T$, $y_0 = \{0\}$, and $y_1 = y$. We then pull back $u$ and the matrix of the metric $g$ on $SC_r$ to $"u"$ (by abuse of notation) and $\hat{g}$ on $SC_1$. Thus, $u$ satisfies in $SC_1$ the following.

$$\frac{\partial u}{\partial t} - \Delta \hat{g} u \leq 0 \text{ in the sense of Definition 4.2 and }$$

$$\frac{g_{Euc}}{K} \leq \hat{g} \leq K g_{Euc} \text{ in } SC_1.$$ 

(46)

(47)
It suffices to prove (45) for \( u \). By rescaling, we can assume \( u \leq 1 \) and \( \sup_{SC_1} u = 1 \). Then \( 0 \leq u_\epsilon \leq 1 - \epsilon \) and \( \sup_{SC_1} u_\epsilon \geq 1 - 3\epsilon \). It suffices to derive an estimate for for \( v = -\log(1 - u_\epsilon) \) which is independent of \( \epsilon \). Since \( u_\epsilon \) satisfies (46), \( v \) satisfies

\[
\frac{\partial v}{\partial t} - \Delta \tilde{g} v \leq -|\nabla \tilde{g} v|^2
\]  

(48)
in the sense of Definition 4.2. Let \( \eta \) be the standard cut-off function in the Euclidean unit ball \( B(1) \) which only depends on \(|x|^2\). By (proof of) Lemma 4.5 [replace the 0 on the right hand side of (18) by \(-|\nabla \tilde{g} v|^2\)], using \( u_\epsilon \geq 0 \), \( u_\epsilon |_{t=0} = 0 \), by abuse of notation with Lemma 4.5, we consider \( \eta = \eta[x-y(t)] \) and obtain (similarly to (23))

\[
\int_{C_n} v\eta^2 dV_g \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{C_n} <\nabla g v, \nabla g \eta^2 > dV_g \, ds + \int_{t_1}^{t_2} \int_{C_n} |\nabla g u|^2 \eta^2 dV_g \, ds \\
\leq \int_{t_1}^{t_2} \int_{C_n} v \frac{\partial \eta^2}{\partial t} dV_g \, ds + K \int_{t_1}^{t_2} \int_{C_n} v \eta^2 dV_g \, ds. 
\]  

(49)

We first estimate the term \( \int_{t_1}^{t_2} \int_{C_n} v \frac{\partial \eta^2}{\partial t} dV_g \, ds \). It’s the same as in [5]. We note that \( |\frac{\partial \eta^2}{\partial t}| \leq |l| |\nabla E \eta^2| \) (Definition 4.1). Then

\[
\left| \int_{t_1}^{t_2} \int_{C_n} v \frac{\partial \eta^2}{\partial t} dV_g \, ds \right| \leq |l| K 2^n \int_{t_1}^{t_2} \int_{C_n} v |\nabla E \eta^2| dV_E. 
\]  

(50)

Using line 14 to line 23 in page 103 of [5], we obtain

\[
\int_{C_n} v |\nabla E \eta^2| dV_E \leq N \int_{C_n} (|v| + |\nabla_E v|) \eta^2 dV_E. 
\]  

(51)

Then Cauchy-Schwartz inequality and the quasi-isometric condition (47) imply

\[
\left| \int_{t_1}^{t_2} \int_{C_n} v \frac{\partial \eta^2}{\partial t} dV_g \, ds \right| \leq \frac{1}{100} \int_{t_1}^{t_2} \int_{C_n} |\nabla g v|^2 \eta^2 dV_g \, ds + N \int_{t_1}^{t_2} \int_{C_n} v \eta^2 dV_g \, ds. 
\]  

(52)

By the same reason we have

\[
\left| \int_{t_1}^{t_2} \int_{C_n} <\nabla g v, \nabla g \eta^2 > dV_g \, ds \right| \leq \frac{1}{100} \int_{t_1}^{t_2} \int_{C_n} |\nabla g v|^2 \eta^2 dV_g \, ds + N. 
\]  

(53)

Then (49), (52), and (53) imply

\[
\int_{C_n} v \eta^2 dV_g \bigg|_{t_1}^{t_2} \leq N + N \int_{t_1}^{t_2} \int_{C_n} v \eta^2 dV_g \, ds. 
\]  

(54)
Denote \( \int_{C_n} v \eta^2 dV \[mid C = I(t) \), since \( I(0) = 0 \), \( (54) \) implies \( I(t) \) satisfy the assumption in Lemma 5.6. Hence Lemma 5.6 implies \( I(t) \leq N \) for all \( t \in [0, T] \).

Then Proposition (5.3) implies \( v(Y) \leq N \). Hence for some \( \lambda \) (as in Theorem 5.5) which is independent of \( \epsilon \), \( u(\epsilon) \leq 1 - 2\lambda \leq (1 - \lambda) \sup_{SC_1} u \) when \( \epsilon \) is small enough. Let \( \epsilon \to 0 \), the proof of (45) is complete. Again, nothing in this proof involves more than the energy estimates of the subsolutions.

**Lemma 5.6.** Suppose \( I(t) \), \( t \in [T_0, T_1] \) is an everywhere defined \( L^\infty \) function. Suppose \( I(t) \geq 0 \) for all \( t \), \( I(T_0) = 0 \), and

\[
I(t) \leq I(t_1) + N_1 \int_{t_1}^{t_2} I(s) ds + N_2, \quad \text{for all } t_1, t_2 \text{ and } t \in [t_1, t_2]. \tag{55}
\]

Then there exists \( N \) depending on \( N_1, N_2 \), and \( T_1 - T_0 \) such that \( I(t) \leq N \).

**Proof.** Choose \( a \) such that \( a \leq \frac{1}{100N_1} \) and \( \frac{T_1 - T_0}{a} = k_0 \) is an integer. Then for \( k \leq k_0 - 1 \), we deduce \( \max_{ka \leq t \leq (k+1)a} I(t) \leq \frac{1}{2} \max_{ka \leq t \leq (k+1)a} I(t) + N_2 + I(ka) \), then

\[
\max_{ka \leq t \leq (k+1)a} I(t) \leq 2N_2 + 2I(ka). \tag{56}
\]

Since \( I(T_0) = 0 \), the proof is complete by induction. \( \square \)

The short proof of Theorem 4.2 in [5] (only involving Theorem 5.5) gives

**Proposition 5.7.** Suppose \( u \) is a weak sub solution to \((18)\) in a cylinder \( C_r(Y) \). Suppose \( u \leq 0 \) on \( B_{\rho}(z) \times \{ \tau \} \), where \( s - r^2 \leq \tau < s - r^2 - \rho^2 \). Then

\[
\sup_{C_{2r}} u \leq (1 - \lambda) \sup_{C_r} u, \quad \text{where } \lambda \in (0, 1) \text{ depends on } n, \beta, \frac{\rho}{r}, K. \tag{57}
\]

**Theorem 5.8.** (Main Growth Theorem) Suppose \( u \) is a weak sub solution to \((18)\) in a cylinder \( C_{2r}(Y) \). Suppose

\[
\frac{|\{ u > 0 \} \cap C_r(y, s - 3r^2)|}{|C_r(y, s - 3r^2)|} \leq \frac{1}{2}. \tag{58}
\]

Then

\[
\sup_{C_r} u \leq (1 - \lambda) \sup_{C_{2r}} u, \quad \text{where } \lambda \in (0, 1) \text{ depends on } n, \beta, K. \tag{59}
\]

**Proof of Theorem 5.8:** Instead of directly quoting the work in [3], we would like to make the crucial point:

Except measure theory which does not involve the sub equation \((18)\), the proof of Theorem 5.3 in [3] only depends on the fact that Proposition 5.7 (Theorem 3.3 in [3]) and 5.7 (Theorem 4.2 in [3]) hold true for any sub solution (with suitable conditions on initial value or level sets) in any scale.
Actually both propositions are applied in case (a) in page 109 of [5]. Thus, using Proposition 5.1 (in the position of Theorem 3.3 in [5]) and 5.7 (in the position of Theorem 4.2 in [5]), the proof of Theorem 5.3 in [5] goes through for Theorem 5.8.

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