ON A GENERALIZATION OF THE PLANK PROBLEM

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A strip or a plank $S$ in $\mathbb{R}^n$ is a closed set bounded by two parallel hyperplanes. The distance of these hyperplanes is called the width of $S$. The minimal width of a convex closed set $K$ is the minimal width of a strip containing $K$.

The following theorem was conjectured by A. Tarski in 1932 and proved by T. Bang [2] in 1951:

If a closed convex set $K$ in $\mathbb{R}^n$ is covered by a finite number of strips, then the sum of their widths is greater than or equal to the minimal width of $K$.

This result has recently been generalized to Banach spaces by K. Ball [1].

If $n = 2$ and $K$ is the unit disc then there is an extremally simple proof for the above result.

Assume that the unit disc in $\mathbb{R}^2$ is covered by strips $S_1, \ldots, S_k$ with widths $d_1, \ldots, d_k$. Without loss of generality we can also assume that both bounding lines of the strips intersect the unit circle. Now we consider the unit sphere in $\mathbb{R}^3$ and to each strip $S_i$ in $\mathbb{R}^2$ we construct a three dimensional strip $S_i^*$ which is of width $d_i$ and intersects the $xy$-plane in $S_i$. Since $S_1, \ldots, S_k$ cover the unit disc, hence $S_1^*, \ldots, S_k^*$ cover the unit sphere. The area of the intersection of the unit sphere and the strip $S_i^*$ is $2\pi d_i$ independently of the position of of the $i$-th strip. (This is a well known fact from calculus, already discovered by Archimedes.) Thus the sum of these areas exceeds the area of the unit sphere, i.e.

$$\sum_{i=1}^{k} 2\pi d_i \geq 4\pi \implies \sum_{i=1}^{k} d_i \geq 2,$$

which was to be proved.

We can interpret this proof in the following way: If $S$ is a subset of the disc then we project it up to the sphere, measure the area of the projection and call this number the $\mu$ measure of $S$. Then the $\mu$ measure of a the intersection of a strip and the disc is the width of the strip times $2\pi$. Then the statement is a simple consequence of the subadditivity of $\mu$. 

In what follows, we generalize this idea and extend the result discussed above.

An angular domain in $\mathbb{R}^2$ is a closed convex set $D$ bounded by two halflines. The angle of $D$ is the angle closed by the bounding halflines. The vertex of $D$ is the common endpoint of these two halflines.

**Theorem 1.** Let two concentric circles $k$ and $K$ be given on the plane with radii $r$ and $R$, $r < R$. Assume that the disc bounded by $k$ is covered by angular domains whose vertices are within $K$. Then the sum of the angles of these angular domains is greater than or equal to the view angle of $k$ from an arbitrary point of $K$.

**Remark.** This result was proposed as a problem by the author on the 1985 M. Schweitzer competition (see [3]).

**Proof.** Denote by $O$ the common center of the circles and by $D_1, \ldots, D_k$ the given angular domains with angles $\alpha_1, \ldots, \alpha_k$. An angular domain $D$ will be called regular if the vertex of $D$ is on $K$ and both bounding halflines of $D$ intersect $k$. Without loss of generality, we can assume that $D_1, \ldots, D_k$ are regular domains.

The idea of the proof is the following: We construct a rotation invariant nonnegative measure $\mu$ on the closed disc $T$ bounded by $k$ such that the measure of the intersection of $D$ and $T$ is $\alpha$, where $D$ is an arbitrary regular angular domain with angle $\alpha$. Having such a measure we can give a one line proof for the theorem:

$$\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \mu(D_i \cap T) \geq \mu\left(\bigcup_{i=1}^{k} (D_i \cap T)\right) = \mu(T),$$

and observe that $\mu(T)$ is exactly the view angle of $k$ from any point of $K$.

Now we construct the desired measure. Let $P \in T$ be an arbitrary point, and denote by $\rho$ the distance of $P$ and $O$. Then define

$$F(P) = f(\rho) = \frac{1}{\pi} \cdot \frac{1}{R^2 - \rho^2} \cdot \sqrt{\frac{R^2 - r^2}{r^2 - \rho^2}}.$$

If $S$ is a Lebesgue measurable subset of $T$ then let

$$\mu(S) = \int_S F(P) dP.$$

Obviously $\mu$ is a rotation invariant nonnegative measure on $T$. To prove the key property of $\mu$, let $D$ be a regular angular domain with vertex $A$ and angle $\alpha$. Then we want to show $\mu(D) = \alpha$. Without loss of generality we can assume that one of the bounding halflines of $D$ is tangent to $k$. 

at the point $Q$. (In the general case $D$ can be obtained as the difference of two such angular domains.) Denote by $\varepsilon$ the angle $OAQ < \pi$ and by $d$ the (signed) distance of the other bounding halfline from $O$. (This distance is positive if $O$ is outside $D$, and negative if $O$ is inside $D$.) Then $d = R \sin(\varepsilon - \alpha)$. Using successive integration, we obtain

$$\mu(D) = \int_D F(P) dP = \int_d \int_{-\arccos(d/\rho)}^{\arccos(d/\rho)} f(\rho) \rho d\varphi d\rho$$

$$= \int_d^{2f(\rho)\rho \arccos(d/\rho)} d\rho$$

$$= \int_{R \sin(\varepsilon - \alpha)}^{2f(\rho)\rho \arccos\left(\frac{R \sin(\varepsilon - \alpha)}{\rho}\right)} d\rho.$$ 

Thus we have to show that

$$\int_{R \sin(\varepsilon - \alpha)}^{r} 2f(\rho)\rho \arccos\left(\frac{R \sin(\varepsilon - \alpha)}{\rho}\right) d\rho = \alpha$$

for all $0 \leq \alpha \leq 2\varepsilon$. Substituting the new variable $t = R \sin(\varepsilon - \alpha)$ this reduces to

$$\int_{t}^{r} 2f(\rho)\rho \arccos(t/\rho) d\rho = \varepsilon - \arcsin(t/R),$$

for $-r \leq t \leq r$. This latter equation is obviously valid for $t = r$, thus it suffices to show that the derivatives of both sides with respect to $t$ are identical, i.e.

$$\int_{t}^{r} \frac{2f(\rho)\rho}{\sqrt{\rho^2 - t^2}} d\rho = \frac{1}{\sqrt{R^2 - t^2}}, \quad -r < t < r.$$ 

However

$$\int_{t}^{r} \frac{2f(\rho)\rho}{\sqrt{\rho^2 - t^2}} d\rho = \int_{t}^{r} \frac{2}{\pi} \cdot \frac{\rho}{R^2 - \rho^2} \cdot \sqrt{\frac{R^2 - r^2}{(r^2 - \rho^2)(\rho^2 - t^2)}} d\rho$$

$$= \left[ \frac{2}{\pi} \cdot \frac{1}{\sqrt{R^2 - t^2}} \cdot \arctan \frac{\sqrt{R^2 - r^2}}{\sqrt{R^2 - t^2}} \cdot \frac{\rho^2 - t^2}{r^2 - \rho^2} \right]_{\rho=r}^{\rho=t}$$

$$= \frac{1}{\sqrt{R^2 - t^2}}.$$
Thus the proof is complete.

**Remark.** When \( n = 2 \) and \( K \) is the unit disc, then the statement of the plank problem can easily be derived from our theorem. Denote the unit disc by \( T \) and assume that it is covered by strips \( S_1, \ldots, S_k \) (whose bounding lines intersect \( T \)). Take a concentric circle \( K \) with radius \( R \), where \( R \) is sufficiently large. Assume that the two bounding lines of \( S_i \) intersect \( K \) in \( A_i, B_i \) and in \( C_i, D_i \). We choose the notation such that \( S_i \) is covered by the two regular angular domains \( A_iB_iD_i < \) and \( B_iD_iC_i < \). Denote by \( \alpha_i' \) and \( \alpha_i'' \) their angle and by \( d_i \) the width of \( S_i \). Then we have

\[
\frac{d_i}{2R - 2} \geq \tan \alpha_i' \geq \alpha_i', \quad \frac{d_i}{2R - 2} \geq \tan \alpha_i'' \geq \alpha_i''.
\]

Thus the theorem yields

\[
2 \sum_{i=1}^{k} \frac{d_i}{2R - 2} \geq \sum_{i=1}^{k} (\alpha_i' + \alpha_i'') \geq 2\varepsilon \geq 2 \sin \varepsilon \geq \frac{2}{R}.
\]

Now taking the limit \( R \to \infty \) we obtain the statement.

**References**

[1] K. Ball, The plank problem in general normed spaces, manuscript

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[3] B. Brindza—Zs. Páles, Jelentés az 1985. évi Schweitzer Miklás emlékverseny-ről (Report on the 1985 M. Schweitzer memorial competition, in Hungarian), Matematikai Lapok 13(1982-1986), 149-169.

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