Strongly Coupled Grand Unification
in Higher Dimensions

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Abstract

We consider the scenario where all the couplings in the theory are strong at the cut-off scale, in the context of higher dimensional grand unified field theories where the unified gauge symmetry is broken by an orbifold compactification. In this scenario, the non-calculable correction to gauge unification from unknown ultraviolet physics is naturally suppressed by the large volume of the extra dimension, and the threshold correction is dominated by a calculable contribution from Kaluza-Klein towers that gives the values for $\sin^2 \theta_w$ and $\alpha_s$ in good agreement with low-energy data. The threshold correction is reliably estimated despite the fact that the theory is strongly coupled at the cut-off scale. A realistic 5d supersymmetric $SU(5)$ model is presented as an example, where rapid $d = 6$ proton decay is avoided by putting the first generation matter in the 5d bulk.
1 Introduction

The unification of the three gauge couplings around $M_U \sim 2 \times 10^{16}$ GeV \[^1\] in the minimal supersymmetric standard model strongly suggests new physics at this energy scale. Conventionally, this new physics has been viewed as 4 dimensional grand unified theories (GUTs) \[^2\], in which all the standard model gauge interactions are unified into a single non-Abelian gauge group and quarks and leptons are unified into smaller numbers of representations under the gauge group. Grand unification in 4 dimensions (4d), however, raises several new questions, including how the GUT gauge symmetry is broken, why the doublet and triplet components of Higgs multiplets split, and why we have not already observed proton decay caused by color triplet Higgsino exchanges \[^3\].

On the other hand, these questions have also been addressed in the context of higher dimensional theories in string theory. In this case, the grand unified group is broken by boundary conditions imposed on the gauge field, and the triplet Higgses are projected out from the zero-mode sector, leaving only the doublet Higgses as massless fields \[^4\], \[^5\]. This is possible because there is no zero-mode gauge symmetry which transforms massless doublet Higgses into massless triplet Higgses. In this framework, however, there is no field theoretic unified symmetry remaining at low energy, so that we have to resort to string threshold calculations to tell whether the three gauge coupling constants are really unified at the string scale \[^4\].

Recently, we have introduced a new framework in which the gauge coupling unification is realized in higher dimensional unified field theories compactified to 4d on orbifolds \[^7\]. Kawamura first suggested a $SU(5)$ GUT in 5d \[^8\], using an $S^1/(Z_2 \times Z_2')$ orbifold earlier introduced in the supersymmetry breaking context \[^9\]. A completely realistic theory was obtained in Ref. \[^7\], where it was shown that a special field theoretic symmetry called restricted gauge symmetry plays a crucial role in this type of theories. This restricted gauge symmetry arises from the fact that there is a moderately large energy interval where the physics is described by higher dimensional grand unified field theories. In the higher dimensional picture it has gauge transformation parameters whose dependence on the extra dimensional coordinates is constrained by orbifold boundary conditions; in the 4d picture it is a symmetry that has different Kaluza-Klein (KK) decompositions for the “unbroken” and “broken” gauge transformations. Using these ideas, various higher dimensional GUT models have been constructed \[^8\], \[^10\]–\[^15\].

In the specific case of 5d $SU(5)$ models in Refs. \[^8\], \[^10\], the 5d $SU(3)_C \times SU(2)_L \times U(1)_Y$ (3-2-1) gauge transformation has a KK decomposition in terms of $\cos[2ny/R]$, while the 5d
\(SU(5)/(SU(3)_C \times SU(2)_L \times U(1)_Y)\) (X-Y) has a decomposition in terms of \(\cos[(2n + 1)y/R]\). Since there is no zero mode for the X-Y gauge transformation, the restricted gauge symmetry does not require that the doublet and triplet Higgses must have the same mass. However, due to higher KK tower gauge transformations, local operators written in the 5d bulk must still preserve the complete \(SU(5)\) symmetry, and all the \(SU(5)\)-breaking local operators must be located on the fixed point where only 3-2-1 gauge symmetry is preserved \[7\]. In particular, \(SU(5)\)-violating effects from unknown ultraviolet physics must appear as boundary operators on this fixed point. This is crucial for guaranteeing the successful gauge coupling unification in this framework. Since the \(SU(5)\)-violating contributions to the gauge couplings which come from the fixed point are suppressed by the volume of the extra dimension compared with the \(SU(5)\)-preserving contribution from the bulk, we can argue that the gauge coupling is (approximately) unified, without invoking any string theory calculation, if the volume of the extra dimension is sufficiently large \[7\]. Then, small deviations from the case of exact unification at a single threshold scale become calculable and improve the agreement between the experimental value and theoretical prediction of \(\sin^2 \theta_w\) \[7, 16\]. The gauge coupling unification in higher dimensional GUTs has been further studied using dimensional deconstruction \[17, 18\] and dimensional regularization \[19\].

In view of the important role played by the large volume for the successful prediction of \(\sin^2 \theta_w\), in this paper we study the possibility that the theory has the maximally large volume allowed by strong coupling analysis. We consider the scenario where all the couplings in the theory are strong at the cut-off scale and show that it is consistent with observations. In this paper we restrict our analysis to an order-of-magnitude level, leaving detailed numerical studies for future work. We present a realistic 5d supersymmetric \(SU(5)\) model as an explicit example. The model preserves the successful \(b/\tau\) Yukawa unification and does not have the unwanted \(SU(5)\) mass relations for the first two generations. It also partially explains fermion mass hierarchies due to the configuration of the matter fields in the extra dimension. We find that the observed values of the low-energy gauge couplings are well reproduced if we take the volume of the extra dimension to be large as suggested by the strong coupling analysis. Experimental signatures from \(d = 6\) proton decay are also discussed, and it is shown that the final state generically contains the second or third generation particles. Finally, the values of the cut-off and compactification scales obtained by analyzing gauge couplings give a 4d Planck scale close to the observation, giving a clue of how to solve the conventional problem in string theory of separating the string and the apparent unification scales.
The paper is organized as follows. In the next section, we give a 5d supersymmetric SU(5) model that can accommodate the large volume of the extra dimension without conflicting with the constraint from $d = 6$ proton decay. In section 3, we consider the gauge coupling unification in this model and argue that the model is consistent with low-energy data. The $d = 6$ proton decay and the 4d Planck scale are discussed in section 4. Finally, our conclusions are drawn in section 5.

2 Minimal Model

In this paper, we consider a minimal realization of the scenario where the theory is strongly coupled at the cut-off scale. Thus, we consider a single extra dimension and the smallest grand unified group, SU(5). It should, however, be noted that we present this case as a representative example of more general scenario. We begin with briefly reviewing the bulk structure of 5d supersymmetric SU(5) theories \[8\] [7]. The 5d spacetime is a direct product of 4d Minkowski spacetime $M^4$ and an extra dimension compactified on the $S^1/(Z_2 \times Z'_2)$ orbifold, with coordinates $x^\mu (\mu = 0, 1, 2, 3)$ and $y (= x^5)$, respectively. The $S^1/(Z_2 \times Z'_2)$ orbifold can be viewed as a circle of radius $R$ divided by two $Z_2$ transformations; $Z_2$: $y \to -y$ and $Z'_2$: $y' \to -y'$ where $y' = y - \pi R/2$. Here, $R$ is around the GUT scale, $R \sim M_{\text{GUT}}^{-1}$. The physical space is an interval $y : [0, \pi R/2]$ which has two branes at the two orbifold fixed points at $y = 0$ and $\pi R/2$.

Under the $Z_2 \times Z'_2$ symmetry, a generic 5d bulk field $\phi(x^\mu, y)$ has a definite transformation property

$$\phi(x^\mu, y) \to \phi(x^\mu, -y) = P \phi(x^\mu, y),$$
$$\phi(x^\mu, y') \to \phi(x^\mu, -y') = P' \phi(x^\mu, y'),$$

where the eigenvalues of $P$ and $P'$ must be $\pm 1$. Denoting the field with $(P, P') = (\pm 1, \pm 1)$ by $\phi_{\pm \pm}$, we obtain the following mode expansions [4]:

$$\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{2n} \pi R}} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{R},$$
$$\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n + 1)y}{R},$$
$$\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n + 1)y}{R}. $$
Here, the indices \( mass \) denote the unbroken and broken \( SU(5) \) generators, \( T^a \) and \( T^\bar{a} \), respectively. The \( C \) and \( F \) represent the color triplet and weak doublet components of the Higgs multiplets, respectively: \( H \supset \{ H_C, H_F \} \), \( \bar{H} \supset \{ \bar{H}_C, \bar{H}_F \} \), \( H^c \supset \{ H_C^c, H_F^c \} \), and \( \bar{H}^c \supset \{ \bar{H}_C^c, \bar{H}_F^c \} \). Since only \((+,+)\) fields have zero modes, the massless sector consists of \( N = 1 \) \( SU(3)_C \times SU(2)_L \times U(1)_Y \) vector multiplets \( V^{a(0)} \) with two Higgs doublet chiral superfields \( H^0_F \) and \( \bar{H}^0_F \). The higher modes for the vector multiplets \( V^{a(n)} \) \((n > 0)\) eat \( \Sigma^{a(2n)} \) becoming massive vector multiplets, and similarly for the \( V^{\bar{a}(2n+1)} \) and \( \Sigma^{\bar{a}(2n+1)} \) \((n \geq 0)\). Since the
non-zero modes for the Higgs fields have mass terms of the form $H^{(2n)} F H_{c}^{(2n)}$, $ar{H}^{(2n)} F H_{c}^{(2n)}$, $H_{C}^{(2n+1)} H_{c}^{(2n+1)}$, and $\bar{H}^{(2n+1)} H_{c}^{(2n+1)}$, there is no dimension 5 proton decay from color triplet Higgsino exchange \[7\].

Now, we consider the gauge couplings in our scenario. Here we roughly estimate various quantities at the tree level; more detailed discussions including radiative corrections are given in section 3. Since we require that the theory is strongly coupled at the cut-off scale $M^*$, the gauge kinetic terms are given by

$$S = \int d^4 x \int d^2 \theta \left[ \frac{\eta M_*}{16\pi^3} W^{\alpha} W_{\alpha} + \delta(y) \frac{\eta'}{16\pi^2} W^{\alpha} W_{\alpha} + \delta(y - \frac{\pi}{2} R) \frac{\eta'_i}{16\pi^2} W_i^{\alpha} W_{i\alpha} \right] + \text{h.c.},$$

where we have used naive dimensional analysis (NDA) in higher dimensions. Here, $\eta$, $\eta'$, and $\eta'_i$ are order one coefficients and $i$ runs over $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$. The restricted gauge symmetry requires that the first two terms must preserve the $SU(5)$ symmetry. The last term, however, can have different coefficients for $i = SU(3)_C, SU(2)_L, U(1)_Y$, which encode $SU(5)$-violating effects from unknown physics above the cut-off scale. After integrating over the extra dimension, we obtain the zero-mode gauge couplings at the cut-off scale as

$$\frac{1}{g_i^2} = \frac{\eta M_* R}{16\pi^2} + \frac{\eta'}{16\pi^2} + \frac{\eta'_i}{16\pi^2}.$$  

Since we know that $1/g_i^2 \sim 1$ from the observed values of the low-energy gauge coupling constants, the ratio between the compactification and the cut-off scales must be $M_* R \sim 16\pi^2/g_i^2 = O(10^2 - 10^3)$. We find that the threshold correction from unknown ultraviolet physics above $M_*$ is suppressed by $1/(M_* R) \sim 1/(16\pi^2)$ and thus negligible in the present scenario. Therefore, the threshold correction to $\sin^2 \theta_w$ is dominated by the calculable contribution coming from an energy interval between $1/R$ and $M_*$. 

We next consider the configuration of matter fields. Since $M_* R \gtrsim 100$ corresponds to $1/R \lesssim 10^{15}$ GeV, it requires that the first generation matter must live in the bulk; otherwise $d = 6$ proton decay occurs much faster rate than experimental constraints allow. 

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1 In Ref. 20, a different coefficient of $M_* / 24\pi^3$ was used for the bulk kinetic term, which was derived by considering loop expansions in the non-compactified 5d space. Here we use $M_* / 16\pi^3$ instead, since it correctly reproduces the strong-coupling value for the 4d gauge coupling, $g \simeq 4\pi/(M_* R)^{1/2}$, after integrating out the extra dimension and is more appropriate in the case of the compactified space. The coefficients of brane-localized terms are determined by requiring that all loop expansion parameters are order one in the 4d picture.

2 The actual value of $M_* R$ could be somewhat smaller than the naive estimate given here, due to a group theoretical factor $C$ appearing in loop expansions: $M_* R \sim 16\pi^2/C g_i^2$.

3 The $5^*$ of the first generation may be located in the bulk without conflicting with the bound from proton decay.
other hand, to preserve successful $b/\tau$ unification in supersymmetric GUTs [21], we have to put the third generation matter on the $SU(5)$-preserving brane located at $y = 0$, since if we put quarks and leptons in the bulk there are no $SU(5)$ Yukawa relations \[4\]. These considerations almost fix the location of the matter fields. The remaining choices are only concerning where we put $10$ and $5^*$ of the second generation. Since we do not want the $SU(5)$ relation, $m_s = m_{\mu}$, for the second generation, at least one of $10$ and $5^*$ must be put in the bulk. Thus, we are left with three possibilities: (i) both $10$ and $5^*$ in the bulk, (ii) $10$ in the bulk and $5^*$ on the $y = 0$ brane, (iii) $10$ on the $y = 0$ brane and $5^*$ in the bulk. As we will see later, the second possibility may be preferred in view of quark and lepton mass matrices, especially in view of the large mixing angle between the second and third generation neutrinos observed in the Super-Kamiokande experiment [22]. We therefore take this possibility as an illustrative purpose for the moment. We consider all three possibilities when we discuss $d = 6$ proton decay later [5].

We now explicitly present our model. The gauge and Higgs sectors are as discussed before. For the matter fields, we introduce the third generation matter chiral superfields $T_3(10), F_3(5^*)$ and the second generation one $F_3(5^*)$ on the $y = 0$ brane. In the 5d bulk, we have to introduce six hypermultiplets $\mathcal{T}_2 = \{T_2(10), T_2'(10^*)\}, \mathcal{T}_2' = \{T_2'(10), T_2''(10^*)\}, \mathcal{T}_1 = \{T_1(10), T_1'(10^*)\}, \mathcal{T}_1' = \{T_1'(10), T_1''(10^*)\}, \mathcal{F}_1 = \{F_1(5^*), F_1'(5)\}, \mathcal{F}_1' = \{F_1'(5^*), F_1''(5)\}$, to obtain the correct low-energy matter content. The transformations for these bulk matter fields under $Z_2 \times Z_2'$ are given by $P = (+, +, +, +, +)$ and $P' = (-, -, -, +, +)$ acting on the $5$ for unprimed fields, but for primed fields the $Z_2'$ quantum numbers are assigned to be the opposite of the corresponding unprimed fields \[6\] (for details, see Refs. [11, 12]). Then, the quark and lepton zero modes come from various brane and bulk fields as

\[
\begin{align*}
T_3 & \supset Q_3, U_3, E_3, & F_3 & \supset D_3, L_3, \\
T_2 & \supset U_2, E_2, & T_2' & \supset Q_2, & F_2 & \supset D_2, L_2, \\
T_1 & \supset U_1, E_1, & T_1' & \supset Q_1, & F_1 & \supset L_1, & F_1' & \supset D_1.
\end{align*}
\]

Since the first generation quarks and leptons which would be unified into a single multiplet

\footnote{Models without the $b/\tau$ unification are obtained if we put the third generation $5^*$ in the bulk.}

\footnote{In these modes, supersymmetry breaking may occur through the mechanism of Ref. [12] that uses small parameters appearing in boundary conditions. In this case, the first possibility of both $10$ and $5^*$ in the bulk is preferred to suppress flavor violating contributions to the first-two generation sfermion masses. The flavor violation would then occur in the processes involving the third generation particles. One way of avoiding all these concerns is to consider 6d models in which gaugino mediation [23] works while suppressing $d = 6$ proton decay [14]. We leave detailed phenomenologies of these models including supersymmetry breaking for future work.}
in the usual 4d GUTs come from different $SU(5)$ multiplets, proton decay from broken gauge boson exchange is absent at the leading order \cite{7}. (This result is also obtained from KK momentum conservation in the fifth dimension.)

The Yukawa couplings are written on the $y = 0$ brane. On this brane, all the operators of the form $[TTH]_{y^2}$ and $[TF\bar{H}]_{y^2}$ are written with the size of their coefficients dictated by NDA in higher dimensions. Here, $T$ and $F$ runs over $\{T_3, T_2, T_1, T_1^\prime\}$ and $\{F_3, F_2, F_1, F_1^\prime\}$, respectively. Similar Yukawa couplings can also be written at $y = \pi R/2$ brane for matter in the bulk. After integrating over $y$, we obtain the Yukawa matrices for low-energy quarks and leptons. At the compactification scale, they take the form

$$
\mathcal{L}_4 \simeq \sqrt{\frac{16\pi^2}{M_4 R}} \begin{pmatrix}
10_1 & 10_2 & 10_3
\end{pmatrix} \begin{pmatrix}
\epsilon^2 & \epsilon & \epsilon \\
\epsilon & \epsilon^2 & \epsilon \\
\epsilon & \epsilon & 1
\end{pmatrix} \begin{pmatrix}
10_1 \\
10_2 \\
10_3
\end{pmatrix} H
$$

$$
+ \sqrt{\frac{16\pi^2}{M_4 R}} \begin{pmatrix}
10_1 & 10_2 & 10_3
\end{pmatrix} \begin{pmatrix}
\epsilon^2 & \epsilon & \epsilon \\
\epsilon & \epsilon^2 & \epsilon \\
\epsilon & \epsilon & 1
\end{pmatrix} \begin{pmatrix}
5_1^* \\
5_2^* \\
5_3^*
\end{pmatrix} \bar{H},
$$

(12)

where $\epsilon \simeq (M_4 R)^{-1/2} \sim 0.1$ and we have omitted order-one coefficients. Low-energy quark and lepton fields are defined as $10_i \equiv \{Q_i, U_i, E_i\}$ and $5_i^* \equiv \{D_i, L_i\}$ ($i = 1, 2, 3$), so that they generically contain fields coming from different hypermultiplets (see Eqs. (9 – 11)). Here, we have normalized these fields canonically in 4d. In the above equation, the matrix elements denoted as $\epsilon$ or $\epsilon^2$ do not respect $SU(5)$ relations, while the ones denoted as 1 must respect $SU(5)$ relations since they entirely come from the Yukawa couplings among the matter fields localized on the $SU(5)$-preserving ($y = 0$) brane. Therefore, the model does not have unwanted $SU(5)$ fermion mass relations for the first two generations, while preserving the $b/\tau$ unification \cite{14}.

Since $\sqrt{16\pi^2/M_4 R} \sim g_i$, the present model predicts $y_t \sim g_i$ at the compactification scale, which is in reasonably good agreement with low-energy data. The over-all mass difference between up- and down-type quarks should be given by $\tan \beta \equiv \langle H_F \rangle / \langle \bar{H}_F \rangle \sim 50$.\footnote{In the case of $5_3^*$ in the bulk, we obtain $\tan \beta \sim (m_t/m_b) \epsilon \sim 5$.} This large value of $\tan \beta$ may also be compatible with the $b/\tau$ Yukawa unification \cite{24}. The above mass matrices roughly explain the observed pattern of quark and lepton masses and mixings; for example, the presence of the mass hierarchy between the first-two generation and the third generation fermions. To reproduce the detailed structure of fermion masses in the first two generations, however, there must be some cancellations among different elements and/or small
numbers in coefficients of order $10^{-1} - 10^{-2}$. It will be interesting to look for the model where more complicated structure gives completely realistic fermion mass matrices \[13\].

How about neutrino masses? Small neutrino masses are obtained by introducing right-handed neutrino fields $N$ through the see-saw mechanism \[25\]. They can be introduced either on the $y = 0$ brane or in the 5d bulk, and have Yukawa couplings of the form $[F N H]_{\theta^2}$ and Majorana masses of the form $[N N]_{\theta^2}$ at the $y = 0$ brane. After integrating out $N$ fields, we obtain the mass matrix for the light neutrinos of the form

$$L_4 \simeq \frac{1}{M_R} \left( \frac{16\pi^2}{M_* R} \right) \begin{pmatrix} 5^*_1 & 5^*_2 & 5^*_3 \end{pmatrix} \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix} \begin{pmatrix} 5^*_1 \\ 5^*_2 \\ 5^*_3 \end{pmatrix} H H,$$  \hspace{0.5cm} (13)

regardless of the configuration of the right-handed neutrino fields. The over-all mass scale $M_R$ is given by right-handed neutrino Majorana masses, which we here assume to be provided by some other physics such as $U(1)_{B-L}$ breaking scale. An interesting point is that the present matter configuration naturally explains the observed large mixing angle between the second and third generation neutrinos, by putting both the second and third generation $5^*$’s on the $y = 0$ brane.

\section{Gauge Coupling Unification}

In this section, we show that the observed values of the low-energy gauge couplings are well reproduced if the volume of the extra dimension is large as is suggested by the strong coupling analysis. We also argue that the situation in the present scenario is better than in usual 4d GUTs, since the masses for the GUT-scale particles are completely determined by KK mode expansions.

Let us first estimate the radiative corrections to the gauge couplings coming from loops of KK towers whose masses lie between $1/R$ and $M_*$. In the 4d picture, the zero-mode gauge couplings $g_i$ at the compactification scale $M_c (= 1/R)$ are given by

$$\frac{1}{g_i^2(M_c)} \simeq \frac{1}{g_0^2(M_*)} - \frac{b}{8\pi^2}(M_* R - 1) + \frac{b'}{8\pi^2} \ln(M_* R),$$  \hspace{0.5cm} (14)

where $b$ and $b'$ are constants of $O(1)$. The second and third terms on the right-hand side represent the pieces which run by power-law and logarithmically. A crucial observation made in Refs. \[4, 13\] is that the coefficient $b$ is necessarily $SU(5)$ symmetric, since the power-law
contributions come from renormalizations of 5d kinetic terms which must be SU(5) symmetric due to the restricted gauge symmetry. The logarithmic contributions come from renormalizations of 4d kinetic terms localized on the branes, and can be different for SU(3)_C, SU(2)_L, and U(1)_Y. Thus, gauge coupling unification is logarithmic even above the compactification scale. This situation is quite different from the power-law unification scenario of Ref. [26].

Since the power-law piece is asymptotically non-free in the present set-up, the ratio between the compactification and cut-off scales could be smaller than the purely classical estimate. This power-law contribution also has a sensitivity to the ultraviolet physics. However, it is expected that this does not change the order of magnitude of the tree-level estimate of M_{SR}, since the theory is strongly coupled only around the cut-off scale and is weakly coupled over a wide energy range from 1/R to M_{SR}. Therefore, we here take M_{SR} \simeq 100 as a representative value.

Note that we have ambiguities coming from \eta's in Eq. (8) in any case, so that the precise value is not very important at this stage.

To calculate the effect of the KK towers on the gauge coupling unification, we consider the one-loop renormalization group equations for the three gauge couplings [7, 16]. Since the SU(5)-violating contribution to the gauge couplings from unknown ultraviolet physics above M_{SR} is suppressed by the large volume, we set the three gauge couplings equal to a unified value g^* at M_{SR}. Then, the equations take the following form:

\begin{align}
\alpha_i^{-1}(m_Z) &= \alpha_*^{-1}(M_{SR}) + \frac{1}{2\pi} \left\{ a_i \ln \frac{m_{SUSY}}{m_Z} + b_i \ln \frac{M_{SR}}{m_Z} 
+ c_i \sum_{n=0}^{N_i} \ln \frac{M_{SR}}{(2n+2)M_c} + d_i \sum_{n=0}^{N_i} \ln \frac{M_{SR}}{(2n+1)M_c} \right\}, \quad (15)
\end{align}

where (a_1, a_2, a_3) = (-5/2, -25/6, -4), (b_1, b_2, b_3) = (33/5, 1, -3), (c_1, c_2, c_3) = (6/5 + n_{5^*} + 3n_{10}, -2 + n_{5^*} + 3n_{10}, -6 + n_{5^*} + 3n_{10}), and (d_1, d_2, d_3) = (-46/5 + n_{5^*} + 3n_{10}, -6 + n_{5^*} + 3n_{10}, -2 + n_{5^*} + 3n_{10}). Here, we have assumed a common mass m_{SUSY} for the superparticles for simplicity, and the sum on n includes all KK modes below M_{SR}, so that (2N_l + 2)M_c \leq M_{SR}; n_{5^*} and n_{10} represent the numbers of generations which are put in the bulk (n_{5^*} = 1 and n_{10} = 2 in the present case). Taking a linear combination of the three equations, we obtain

\begin{align}
(5\alpha_1^{-1} - 3\alpha_2^{-1} - 2\alpha_3^{-1})(m_Z) = \frac{1}{2\pi} \left\{ 8 \ln \frac{m_{SUSY}}{m_Z} + 36 \ln \frac{(2N_l + 2)M_c}{m_Z} - 24 \sum_{n=0}^{N_l} \ln \frac{(2n+2)}{(2n+1)} \right\}, \quad (16)
\end{align}

where we have set M_{SR} = (2N_l + 2)M_c. Note that n_{5^*} and n_{10} drop out from this equation, since a combination of bulk hypermultiplets whose massless modes give a complete SU(5) representa-
tation has $SU(5)$ symmetric matter content at each KK mass level. Since the corresponding linear combination in the usual 4d minimal supersymmetric $SU(5)$ GUT takes the form

$$(5\alpha_1^{-1} - 3\alpha_2^{-1} - 2\alpha_3^{-1})(m_Z) = \frac{1}{2\pi} \left\{ 8 \ln \frac{m_{\text{SUSY}}}{m_Z} + 36 \ln \frac{M_U}{m_Z} \right\},$$

where $M_U = (M_Z^2 M_V)^{1/3}$ [27], we find the following correspondence between the two theories:

$$\ln \frac{M_c}{m_Z} = \ln \frac{M_U}{m_Z} + \frac{2}{3} \sum_{n=0}^{N_l} \ln \frac{2n+2}{2n+1} - \ln(2N_l + 2),$$

as far as the running of the gauge couplings is concerned. An important point here is that this KK contribution improves the agreement between the experimental value and theoretical prediction of $\sin^2 \theta_w$ and $\alpha_s$ [7, 16]. This is because $b_i'$s in Eq. (14) are given by $b_i' = b_i - c_i/2$, and are not equal to the low-energy $\beta$-function coefficients, $b_i$, plus some universal pieces. At the leading order, the contributions from KK towers to $\sin^2 \theta_w$ and $\alpha_s$ are given by $\Delta_{\sin^2 \theta_w} \simeq -(1/5\pi)\alpha \ln(M_* R)$ and $\Delta_{\alpha_s} \simeq -(3/7\pi)\alpha_s^2 \ln(M_* R)$, respectively, which well reproduce experimental values with $M_* R \simeq 10^2 - 10^3$. A more detailed analysis including the next to leading order effect has been given in Ref. [19], where it was shown that if $M_* R \simeq 100$ the KK contribution would indeed give the right values for $\sin^2 \theta_w$ and $\alpha_s$ in a reasonable range of $m_{\text{SUSY}}$.

Using the experimental values of the gauge couplings, we obtain $1 \times 10^{16} \text{ GeV} \lesssim M_U \lesssim 3 \times 10^{16} \text{ GeV}$, and this translates into the range of $M_c$ for a given $N_l$. Taking $M_* R \simeq 100$, we find that the compactification scale must be in the range

$$5 \times 10^{14} \text{ GeV} \lesssim M_c \lesssim 2 \times 10^{15} \text{ GeV},$$

which is considerably lower than the usual 4d unification scale $M_U \simeq 2 \times 10^{16} \text{ GeV}$. Since the mass for the broken gauge bosons is given by $1/R$, it induces the $d = 6$ proton decay at a rate contradicting the bound from Super-Kamiokande [28], if quarks and leptons are localized on the $SU(5)$-preserving brane. In fact, this constraint was used in Ref. [19] to conclude that strict NDA assumption does not work, and the contribution from unknown ultraviolet physics is needed to obtain the right values for $\sin^2 \theta_w$ and $\alpha_s$. In other words, the contribution from KK towers alone is insufficient to explain the small difference of $\sin^2 \theta_w$ between the experiment and naive 4d GUT prediction, since $M_* R$ must be smaller than $\sim 10$ from the proton decay.
constraint. In the present case, however, the first generation matter lives in the bulk so that the constraint from \( d = 6 \) proton decay is evaded even if the compactification scale is low. This allows us to consider larger values for \( M_s R \); that is, the scenario where the theory is strongly coupled at the cut-off scale. Then, the calculable contribution from KK towers could completely explain the small discrepancy of \( \sin^2 \theta_w \) between the experimental and theoretical values that was present in the case of the minimal 4d GUT with a single threshold. Note that the non-calculable contribution from unknown physics above \( M_s \) is expected to be small in this case through NDA in higher dimensions.

We here consider uncertainties for the present analysis. Since the theory is assumed to be strongly coupled at the cut-off scale, higher order effects could be important around that energy scale. However, the logarithmic contribution from KK towers discussed above comes from entire energy range from \( 1/R \) to \( M_s \), and the theory is weakly coupled in most of this energy region. Actually, various interactions quickly become weak below \( M_s \), suppressed by powers of \( (E/M_s) \) at energy scale \( E \). This is because in the 5d picture the couplings in the theory have negative mass dimensions, and in the 4d picture the number of KK states circulating in the loop decreases with decreasing energies so that loop expansion parameters in the theory (‘t Hooft couplings) become small by powers of \( (E/M_s) \). Therefore, we expect that the leading log calculation of the threshold correction to \( \sin^2 \theta_w \) is reliable at least at the order of magnitude level, although the precise coefficients may receive corrections from this higher order effect. To be more precise, the difference of the gauge couplings runs logarithmically in all energy regions between \( 1/R \) and \( M_s \), and the one-loop estimates are reliable only when the renormalization scale is at least a factor of a few smaller than \( M_s \); higher loop effects would equally be important around the cut-off scale. This would give \( O(10\%) \) uncertainties in the calculations of the threshold corrections of \( \sin^2 \theta_w \) and \( \alpha_s \). A similar size of uncertainties is also expected from tree-level \( SU(5) \)-breaking boundary operators. We emphasize that the uncertainties are for the threshold corrections and are not \( O(10\%) \) uncertainties for the values of \( \sin^2 \theta_w \) and \( \alpha_s \) themselves.

We then find that the observed values for the gauge coupling constants are well reproduced by taking \( M_s R = O(10^2 - 10^3) \). That is, we can explain the difference of \( \sin^2 \theta_w \) (and \( \alpha_s \))

\[ 7 \text{ The constraint from } d = 6 \text{ proton decay was also used in Ref. }[18] \text{ to conclude that } N_l (\simeq M_s R) \text{ must be smaller than } \sim 20 \text{ and that the calculable contribution from KK towers cannot explain the discrepancy of the gauge coupling values between the experiment and the theoretical prediction of 4d GUT. Ref. }[17] \text{ also argues that } N_l \text{ must be smaller than } \sim 25 \text{ using } \alpha N_l \lesssim 1 \text{ to estimate the strong coupling bound, while we here use } \alpha N_l/4\pi \lesssim 1 \text{ to estimate it.} \]
between the experimental value and the theoretical prediction obtained by assuming the exact gauge unification at a single threshold. We note that the situation is better in the present scenario than in usual 4d GUTs. Let us consider, for example, predicting \( \alpha_s \) from the observed values of \( \sin^2 \theta_w \) and \( e \). It is known that if we calculate \( \alpha_s \) without including any threshold correction, we obtain a somewhat larger value \( \alpha_s|_{\text{th}} \simeq 0.130 \) than the experimentally measured value \( \alpha_s|_{\text{ex}} \simeq 0.118 \pm 0.002 \). Thus, we have to explain the difference \( \alpha_s|_{\text{ex}} - \alpha_s|_{\text{th}} \simeq -0.012 \pm 0.002 \) by the GUT-scale threshold correction \( \Delta_{\text{gut}}^{\alpha_s} \). (Here we ignore the weak-scale threshold corrections, which typically give \( |\Delta_{\text{weak}}^{\alpha_s}| \lesssim 0.004 \).) In usual 4d GUTs, the size of the GUT-scale threshold correction is given by \( |\Delta_{\text{gut}}^{\alpha_s}| \lesssim 0.02 \), but we cannot predict the value of \( \Delta_{\text{gut}}^{\alpha_s} \) in general since it strongly depends on the mass spectrum of the GUT scale particles.

On the other hand, in the present case, we completely know the pattern of the GUT-scale particle masses, so that we can calculate the threshold correction, \( \Delta_{\text{gut}}^{\alpha_s} \), for a given value of \( M_sR \). It is given by \( \Delta_{\text{gut}}^{\alpha_s} \simeq -\left(3/7 \pi\right) \alpha_s^2 \ln(M_sR) \). Numerically, we find \( \Delta_{\text{gut}}^{\alpha_s} \simeq -0.009 \pm 0.002 \) (\( \Delta_{\text{gut}}^{\alpha_s} \simeq -0.013 \pm 0.003 \)) if \( M_sR = 100 \) (\( M_sR = 1000 \)), where the errors represent the \( O(10\%) \) uncertainties discussed before. We find that the observed value of \( \alpha_s \) is well reproduced with the values of \( M_sR \) suggested by the NDA analysis.

Of course, we cannot prove that these values of \( M_sR \) exactly give a truly strongly coupled theory at the cut-off scale (all \( \eta \)'s equal to 1), since there are many uncertainties in estimating the over-all value for the gauge coupling (but not the differences between the three couplings) at the cut-off scale. For example, the contribution from \( SU(5) \) symmetric power-law running (scheme dependence, in other words) could change the value. However, within the uncertainties in estimating various quantities, we can say that the scenario where the theory is (moderately) strongly coupled at the cut-off scale is consistent with low-energy observations. It is particularly interesting that the \( M_sR \) value giving the desired low-energy gauge coupling values is consistent with the requirement that the theory is strongly coupled at the cut-off scale.

### 4 Other Issues

In the model discussed in the previous sections, \( d = 6 \) proton decay occurs through mixings between the first and heavier generations occurring at the coupling to the heavy broken gauge
bosons Thus, their rates are suppressed by mixing angles that are expected to have similar order of magnitudes to the corresponding CKM angles. In the present case where $10_2$ resides in the bulk and $5_2^*$ lives on the brane, the dominant decay mode is $K^+\nu_\mu$ or $\mu^+\pi^0$. However, their amplitudes receive suppression of order $V_{ub}V_{cb}V_{e3}$ and $V_{ub}^2V_{e2}$, respectively, which are $10^{-5} - 10^{-6}$. Therefore, the lifetime is roughly $10^{40}$ years, and there would be little hope for detection in the near future. In the case where both $10_2$ and $5_2^*$ are in the bulk, the dominant decay mode is $K^+\nu_\tau$, whose amplitude also receives suppression of order $V_{ub}V_{cb}V_{e3}\sim 10^{-5}$. On the other hand, if $10_2$ is on the brane and $5_2^*$ is in the bulk, the $p\rightarrow K^+\nu_\tau$ and $p\rightarrow \mu^+K^0$ decays could occur with only $V_{us}V_{e3}\sim V_{us}^2\sim 10^{-2}$ suppression in their amplitudes, that is, with the lifetime of $10^{33} - 10^{35}$ years. Incidentally, if $M_*R$ is somewhat larger than 100, the proton lifetime becomes shorter. In the case of $M_*R\simeq 500$ ($M_*R\simeq 1000$), for example, the lifetime becomes factor 70 (450) shorter compared with the case of $M_*R\simeq 100$. Thus, in the case where $10_2$ is on the brane, it is probable that strange $d=6$ proton decay, involving the second generation particles in the final state, could be discovered in future experiments.

Finally, we estimate 4d reduced Planck scale $M_P$ assuming that the strength of the gravitational interaction is also dictated by the NDA analysis. Since the theory is strongly coupled at $M_*$, the kinetic term for the graviton is given by $S = \int d^4x dy (M_*^{2}/16\pi^3)R$, where $R$ is the Ricci scalar. Thus, after integrating $y$, $M_P$ is given by $M_P^2 = M_*^3R/(16\pi^2)$. Substituting the value obtained in Eq. (19) with $M_*R\simeq 100$, we obtain $M_P \simeq 10^{17}$ GeV. This is substantially higher than the 4d unification scale $M_U \simeq 2\times 10^{16}$ GeV, but still somewhat lower than the observed value $M_P \simeq 2\times 10^{18}$ GeV. To reproduce the observed value, we need either an $O(10)$ coefficient, $M_*R \gtrsim 1000$, or $n$ extra dimensions with radius $R \simeq O(10^{2/n})$ in which (only) gravity propagates. However, it is true that $M_P$ is an order of magnitude separated from the apparent unification scale $M_U$ by the presence of the large extra dimension necessary to break the GUT symmetry. The precise estimate is also dependent on the number of extra dimensions, gauge group and matter content, which we here took those of the minimal 5d $SU(5)$ model as a representative case. Thus, we expect that the existence of this type of dimension may provide a general way of separating the two scales in string theory. 

\footnote{Similar situations are also discussed in the context of dimensionally deconstructed models [17].}
5 Conclusions

In this paper, we have explored the scenario where all the couplings in the theory are strong at the cut-off scale, in the context of higher dimensional grand unified field theories. This provides a calculable framework for gauge coupling unification in higher dimensions. The non-calculable effect from unknown ultraviolet physics is suppressed by assuming that all the operators in the theory scale according to naive dimensional analysis in higher dimensions \cite{7}. Then, the threshold correction to $\sin^2 \theta_w$ dominantly comes from the calculable contribution from KK towers, giving the values for $\sin^2 \theta_w$ and $\alpha_s$ in good agreement with low-energy data. Although the theory is strongly coupled at the cut-off scale $M_*$, it quickly becomes weakly coupled below $M_*$, allowing reliable estimates of threshold corrections to the gauge coupling unification. A crucial point is that we can have large values of $M_* R$ without conflicting with the constraint from proton decay by putting the first generation matter in the bulk. This enables us to consider the strong coupling scenario, in contrast with the previous work \cite{19} where it was concluded that $M_* R$ must be smaller than $\sim 10$ due to the proton decay constraint and thus the calculable contribution from KK towers cannot fully explain the low-energy data.

We have shown that the ansatz where all the coupling constants are dictated by naive dimensional analysis in higher dimensions is consistent with low-energy observations. We have presented a completely realistic 5d supersymmetric $SU(5)$ model as an explicit example. This suggests that the higher dimensional grand unified theory is a low-energy effective theory of some more fundamental theory that is strongly coupled at the scale $M_*$. In the present scenario, the observed weakness of various couplings is attributed to the presence of a moderately large extra dimension(s). The hierarchy among various couplings arise from different numbers of dimensions in which various fields propagate. The presence of this large dimension(s) is required to solve the problems in conventional GUTs, such as doublet-triplet splitting and $d = 5$ proton decay problems, by extra dimensional mechanisms while preserving successful prediction of $\sin^2 \theta_w$ \cite{7, 8}. Therefore, in this framework, solving the many conventional problems in GUTs is transformed to finding a single mechanism of naturally getting such a large extra dimension(s) with the radius of order $10^2 - 10^3$ in units of the fundamental scale. It would be interesting to consider a mechanism of generating this type of large extra dimension(s) in the context of string theory.

Note added

While this work is being completed, we received Ref. \cite{31} where it is hoped that $SU(5)$-
breaking boundary operators may not exist when the gauge group is broken only by orbifold reflections (not translations). Even then, however, there are fixed points which do not preserve full $SU(5)$ symmetry. Thus, $SU(5)$-violating local operators can be written on these points, since they are not prohibited by the restricted gauge symmetry.

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