Linear and rational factorization of tropical polynomials

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Emory University  
Algebra Seminar  
February 25th, 2020

arXiv:1707.03332v3
Outline

- Background: factorization of tropical polynomials is hard;
- Tools: Cayley trick, signed Minkowski sum of polytopes;
- Main results and algorithms;
- Examples: homogeneous linear polynomials and more.
Tropical algebra

Definition

On the set $\mathbb{R} = \mathbb{R} \cup \{-\infty\}$ we define two commutative binary operations $\oplus$ and $\odot$ as follows: for $a, b \in \mathbb{R}$ and $c \in \mathbb{R}$,

$$a \oplus b = \max(a, b), \quad a \odot b = a + b.$$  

$$c \oplus -\infty = c, \quad c \odot -\infty = -\infty.$$  

The triple $(\mathbb{R}, \oplus, \odot)$ is called the tropical semiring.
Tropical algebra

Definition

On the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ we define two commutative binary operations $\oplus$ and $\odot$ as follows: for $a, b \in \mathbb{R}$ and $c \in \overline{\mathbb{R}}$,

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Remark

We use the so-called max-plus operations for convenience. There is an equivalent way to define the tropical semiring: replace $\max$ by $\min$ and $-\infty$ by $\infty$. 

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Linear and rational factorization of tropical polynomials
Tropical polynomials

**Definition**

A tropical polynomial is a function $f : \mathbb{R}^n \to \mathbb{R}$, such that for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$f(x) = \bigoplus_{a \in A} \left( f_a \odot \bigotimes_{i=1}^{n} x_i^{a_i} \right) = \max_{a \in A} \left( f_a + \sum_{i=1}^{n} a_i x_i \right)$$

where $A \subseteq \mathbb{N}^n$ is finite and $f_a \in \mathbb{R}$ for $a \in A$. 
Tropical polynomials

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\[
  f(x) = \bigoplus_{a \in A} \left( f_a \odot \bigotimes_{i=1}^{n} x_i^{\circ a_i} \right) = \max_{a \in A} \left( f_a + \sum_{i=1}^{n} a_i x_i \right)
\]

where \( A \subseteq \mathbb{N}^n \) is finite and \( f_a \in \mathbb{R} \) for \( a \in A \).

Remark

We ignore the ground field \( K \), and directly take valuations as the coefficients of each term in the polynomial.
Roots of tropical polynomials

Definition

Let $f(x)$ be a tropical polynomial. A point $x \in \mathbb{R}^n$ is a root of $f$ if the maximum is attained at least twice in the evaluation of $f(x)$. 
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Let $f$ be a tropical polynomial. The tropical hypersurface $T(f)$ is the set of all roots of $f$.

**Example**

Suppose $f(x, y) = \max(x, y, 0)$. The graph of $T(f)$ is the right figure.
There are three levels of equivalence between two tropical polynomials $f$ and $g$.

1. $f =_1 g$: $f$ and $g$ have the same terms and coefficients;
2. $f =_2 g$: $f(x) = g(x)$ for all $x \in \mathbb{R}^n$;
3. $f =_3 g$: $T(f) = T(g)$.

Example:

$$\max(2x, 0) = \max(2x, x - 1, 0)$$

for all $x \in \mathbb{R}$. Then

$$\max(2x, 0) = 2\max(2x, x - 1, 0).$$

In addition, they both equal

$$\max(3x + 2, x + 1, 2),$$

as all of them have a unique root $x = 0$. 

In this project, we focus on $=_2$, i.e. the equivalence of polynomial functions.
Equivalence of tropical polynomials

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3. $f =_3 g$: $T(f) = T(g)$.

Example

$max(2x, 0) = max(2x, x - 1, 0)$ for all $x \in \mathbb{R}$. Then $max(2x, 0) =_2 max(2x, x - 1, 0)$. In addition, they both $=_3 max(3x + 2, x + 1, 2)$, as all of them have a unique root $x = 0$.

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Linear and rational factorization of tropical polynomials
Like ordinary polynomials, we also want to write tropical polynomials as (tropical) product of other tropical polynomials.
Fundamental theorem of tropical algebra

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**Theorem (Fundamental theorem of tropical algebra)**

*Every tropical polynomial in one variable with rational coefficients equals to a product of linear tropical polynomials with rational coefficients as functions.*
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*Every tropical polynomial in one variable with rational coefficients equals to a product of linear tropical polynomials with rational coefficients as functions.*

**Example**

\[ f_1(x) = \max(4x, 3x + 2, 2x + 1, -3) =_2 \max(x, 2) + \max(x, -1) + 2 \max(x, -2). \]
Multivariate: factorization is NP-complete

Deciding whether a general tropical polynomial is factorizable is hard.
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**Theorem (Kim-Roush ’05, Grigg ’07)**

*The factorization of multivariate tropical polynomials is NP-complete.*
Newton polytope and regular subdivision

**Definition**

Let $f(x) = \max_{a \in A} (f_a + \sum_{i=1}^{n} a_ix_i)$. The Newton polytope of $f$, denoted by $\text{Newt}(f)$, is the convex hull of \{(a_1, a_2, \ldots, a_n) | a \in A\}.

$\text{Newt}(f)$ is a lattice polytope in $\mathbb{R}^n$. It tells us what terms could appear in the polynomial.
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Remark

The regular subdivision of $\text{Newt}(f)$ induced by the weights $f_a$ provides important information of $T(f)$. 
Example: regular subdivision of $\text{Newt}(f_2)$

Let $f_2(x_1, x_2) = \max(2x_1 - 3, 2x_2 - 1, x_1 + x_2, x_1, x_2 + 1, 0)$. Then $\text{Newt}(f_2) = \text{Conv}((0, 0), (1, 0), (0, 1), (0, 2), (1, 1), (2, 0))$. If we choose the weight vector as $w = (-3, -1, 0, 0, 1, 0)$, the regular subdivision of $\text{Newt}(f_2)$ is
Duality between $T(f)$ and the regular subdivision of Newt($f$)

Figure 1: Duality between $T(f_2)$ and $\Delta_{\text{Newt}(f_2)}$
Duality between $T(f)$ and the regular subdivision of $\text{Newt}(f)$

**Proposition**

Let $\Delta_f$ be the regular subdivision of $\text{Newt}(f)$ w.r.t. the vector $f_a$. Then the tropical hypersurface $T(f)$ is the polyhedral complex dual to $\Delta_f$. 
Duality between $T(f)$ and the regular subdivision of Newt($f$)

**Proposition**

Let $\Delta_f$ be the regular subdivision of Newt($f$) w.r.t. the vector $f_a$. Then the tropical hypersurface $T(f)$ is the polyhedral complex dual to $\Delta_f$.

**Remark**

This result tells us that regular subdivision is a useful tool to study tropical polynomials.
Definition

A tropical polynomial $f$ is a unit if the induced regular subdivision on $\text{Newt}(f)$ is trivial. For a set of lattice polytopes $S$ in $\mathbb{R}^n$, an $S$-unit $f$ is a unit such that $\text{Newt}(f)$ is a translation of some polytope in $S$. 

A tropical polynomial $f$ is called $S$-factorizable if it equals to a tropical product of $S$-units. And $f$ is called $S$-rational if there exist a polynomial $g$ and an $S$-factorizable polynomial $h$ such that $f \circ g = h$. $f$ is called strong $S$-rational if in addition $g$ is also $S$-factorizable.
**Definition**

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Cayley trick and Newton polytope

The polyhedral version of the Cayley trick is the following

**Theorem (Sturmfels ’94 (Thm 5.1))**

Let $S$ be a set of polytopes. Then $f$ is $S$-factorizable if and only if $\Delta_f$ is a regular mixed subdivision of $\text{Newt}(f)$ with respect to a sequence of possibly repeated polytopes in $S$. 
Cayley trick and Newton polytope

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**Theorem (Sturmfels ’94 (Thm 5.1))**

Let $S$ be a set of polytopes. Then $f$ is $S$-factorizable if and only if $\Delta_f$ is a regular mixed subdivision of Newt$(f)$ with respect to a sequence of possibly repeated polytopes in $S$.

So we can consider the decomposition of a polytope into Minkowski sums of other polytopes. But this is hard, too.

**Theorem (Gao-Lauder ’01)**

*The decomposition problem of integral polygons is NP-complete.*
There are results that work in special cases:

- Gritzmann-Sturmfels (’93): $d$-dimensional polytopes with up to $n$ vertices;
- Fukuda (’04): zonotopes;
- Fukuda-Weibel (’05): V-polytopes.
In this work, we present a large class of polytopes $S$ such that the set of $S$-factorizable tropical polynomials has unique and local factorization. Here local means that if each cell of $\Delta_f$ is a Minkowski sum of some polytopes in $S$, then $f$ is $S$-factorizable.
Our contribution

In this work, we present a large class of polytopes $S$ such that the set of $S$-factorizable tropical polynomials has unique and local factorization. Here local means that if each cell of $\Delta_f$ is a Minkowski sum of some polytopes in $S$, then $f$ is $S$-factorizable. We also present algorithms to determine whether a tropical polynomials $f$ is $S$-factorizable or $S$-rational. And if the answer is positive, we compute the decomposition (though with exponential time complexity).
Example: $S_{K_3}$-rational but not $S_{K_3}$-factorizable

For each positive integer $n$, our first choice of $S$ is the standard simplex $\text{Conv}(e_i \mid 1 \leq i \leq n)$ and its faces, denoted by $S_{K_n}$. which are the Newton polytopes of the tropical polynomials of the form $\max(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n)$, where each $c_i \in \mathbb{R}$. 
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Example

$\tilde{f}_2(x) = \max(2x_1 - 3, 2x_2 - 1, x_1 + x_2, x_1 + x_3, x_2 + x_3 + 1, 2x_3)$ is not $S_{K_3}$-factorizable, but $\tilde{f}_2$ is $S_{K_3}$-rational. In fact

$$\tilde{f}_2 + \max(x_1 - 1, x_2, x_3) = \max(x_1 - 1, x_2 - 2, x_3)$$

$$+ \max(x_1 - 3, x_2, x_3) + \max(x_1, x_2 + 1, x_3)$$

$$= \max(3x_3, x_2 + 2x_3 + 1, 2x_2 + x_3 + 1, 3x_2 - 1, x_1 + 2x_3, x_1 + x_2 + x_3, x_1 + 2x_2, 2x_1 + x_3 - 1, 2x_1 + x_2 - 1, 3x_1 - 4).$$
Example: $\tilde{f}_2$

Note that $\tilde{f}_2$ is the homogenization of $f_2$. Figure 2 shows the rational factorization:

Figure 2: The rational factorization of $\tilde{f}_2$
Signed Minkowski sum

The $S$-rational examples motivate us to consider the following.

**Definition**

Let $P, Q$ be two nonempty polytopes of $\mathbb{R}^n$. If there exists a nonempty polytope $R \subset \mathbb{R}^n$ such that $P = Q + R$, then we define the **Minkowski difference** $P - Q$ as $R$. 
Signed Minkowski sum

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**Definition**

Let $P_1, P_2, \cdots, P_m$ be non-empty polytopes in $\mathbb{R}^n$, $c_1, c_2, \cdots, c_m \in \mathbb{Z}$ with at least one being positive. If there exists a polytope $P'$ such that

$$\sum_{c_i < 0} (-c_i) P_i + P' = \sum_{c_i > 0} c_i P_i,$$

then the signed Minkowski sum $\sum_{i=1}^m c_i P_i$ is defined to be $P'$. 
**H-representations and b-vectors**

We focus on the $H$-representation of polytopes.

**Definition**

For matrix $H \in \mathbb{Z}^{r \times n}$ whose rows are primitive vectors, and a vector $b \in \mathbb{R}^r$, let $P_{H,b}$ denote the possibly empty polytope given by

$$P_{H,b} = \{ x \in \mathbb{R}^n \mid Hx \leq b \}. $$

Suppose $H = \begin{bmatrix} h_1 & h_2 & \cdots & h_r \end{bmatrix}^T$, where $h_i$ is the $i$-th row vector of $H$. For any polytope $P$, let

$$v(H, P) = \begin{bmatrix} \max_{x \in P} h_1 \cdot x & \max_{x \in P} h_2 \cdot x & \cdots & \max_{x \in P} h_r \cdot x \end{bmatrix}^T. $$

And let $b(H) = \{ b \in \mathbb{R}^r \mid P_{H,b} \neq \emptyset \text{ and } v(H, P_{H,b}) = b \}$. 
Example

Let $H$ be

$$
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
1 & 1
\end{pmatrix}.
$$

And $b = (0, 0, 1)^T$. Then

$$
P(H, b) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}
= \text{Conv}((0, 0), (0, 1), (1, 0)).
$$

And $v(H, P(H, b)) = b$, so $b \in b(H)$. 
**Proposition**

Let $P_i$ be lattice polytopes and $H \in \mathbb{Z}^{r \times n}$ contains all primitive normal vectors of their Minkowski sum $\sum_{i=1}^{m} P_i$. Suppose vectors $b_i$ are such that $P_{H,b_i} = P_i$. For $y^+, y^- \in \mathbb{N}^m$, suppose the signed Minkowski sum $\sum_{i=1}^{m} (y^+_i - y^-_i) P_{H,b_i}$ is well-defined. Then

$$\sum_{i=1}^{m} (y^+_i - y^-_i) P_{H,b_i} = P_{H,\sum_{i=1}^{m} (y^+_i - y^-_i) b_i}.$$
**H**-representable polytopes

**Proposition**

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$$\sum_{i=1}^{m} (y^+_i - y^-_i)P_{H,b_i} = P_H, \sum_{i=1}^{m} (y^+_i - y^-_i)b_i.$$ 

**Remark**

This proposition enables us to decompose a polytope into a signed Minkowski sum of polytopes on the level of the $b$-vectors. Then linear algebra can be used.
Basis and unique factorization

Let $S$ be a finite set of lattice polytopes in $\mathbb{R}^n$. Let $H(S) \in \mathbb{Z}^{r \times n}$ be a matrix whose row vectors are all distinct primitive normal vectors of the polytope $\sum_{S \in S} S$, with coordinate-wise lexicographic order. Then $H(S)$ is uniquely defined.

**Definition**

\[
\mathcal{B}(S) = \{ b \in \mathbb{Z}^r \cap b(H(S)) \mid P_{H(S),b} \in S \}, \\
\overline{\mathcal{B}}(S) = \{ b \in \mathbb{Z}^r \cap b(H(S)) \mid \emptyset \neq P_{H(S),b} \subset \mathbb{Z}^n \text{ is a lattice polytope} \}.
\]

$S$ is a basis if $\mathcal{B}(S)$ is a basis over $\mathbb{Z}$ for $\mathbb{Z}\mathcal{B}(S)$. $S$ is a full basis if $\mathcal{B}(S)$ is a basis over $\mathbb{Z}$ for $\overline{\mathcal{B}}(S)$. 
The good polynomials and polytopes of $S$

Let $S$ be a finite set of lattice polytopes.

**Definition**

Let $\mathbb{N}[S]$ be the set of $S$-factorizable polynomials, $\mathbb{E}[S]$ be the set of $S$-rational polynomials and $\mathbb{Z}[S]$ be the set of strong $S$-rational polynomials.
The good polynomials and polytopes of $S$

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**Definition**

Let $\mathbb{N}[S]$ be the set of $S$-factorizable polynomials, $\mathbb{E}[S]$ be the set of $S$-rational polynomials and $\mathbb{Z}[S]$ be the set of strong $S$-rational polynomials.

Let $f$ be a unit tropical polynomial in $\mathbb{R}^n$ and $P = \text{Newt}(f)$. Then

**Proposition**

- $f \in \mathbb{E}[S]$ if and only if $P = P_{H(S),b}$ for some $b \in \overline{B}(S)$
- $f \in \mathbb{N}[S]$ if and only if $P = P_{H(S),b}$ for some $b \in \mathbb{N}B(S)$.
- $f \in \mathbb{Z}[S]$ if and only if $P = P_{H(S),b}$ for some $b \in \overline{B}(S) \cap \mathbb{Z}B(S)$. 
Consider the regular subdivision of the square:

(0, 0) \rightarrow (1, 1)
Example: a basis without local factorization

Consider the regular subdivision of the square:

If both triangles belong to $S$, then $S$ does not have local factorization. This motivates us to exclude such pairs of polytopes in good $S$. 
To define positive basis, we need the following:

**Definition**

A polytope $S$ is canonical if for any proper face $P$, $P \not\subseteq S$. A set of polytopes $S$ is canonical if $S$ is canonical for all $S \in S$. $S$ is hierarchical if for any $S \in S$, all proper faces of $S$ also belong to $S$. 
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**Definition**

A polytope $S$ is canonical if for any proper face $P$, $P \not\subseteq S$. A set of polytopes $S$ is canonical if $S$ is canonical for all $S \in S$. $S$ is hierarchical if for any $S \in S$, all proper faces of $S$ also belong to $S$.

**Definition**

Let $S$ be a hierarchical set of polytopes. An orientation $\tau$ is a map from the row vectors of $H(S)$ to $\{1, -1\}$, such that $\tau(v) = -\tau(-v)$. Let $H^+_\tau = \{v \mid \tau(v) = 1\}$. $S$ is positive with orientation $\tau$ if for each $v \in H^+_\tau$ and $S \in S$, face$_{-v}(S)$ is not a proper face of $S$. 
Example: a non-positive basis

Let

\[ \mathcal{S} = \{ \text{Conv}((0,0), (1,0), (0,1)), \text{Conv}((1,1), (1,0), (0,1)), \text{Conv}((0,0), (1,0)) \}. \]

Then \( \mathcal{S} \) is a basis (note that \( \mathcal{S} \) is not hierarchical). However, \( \mathcal{S} \) is not positive.
Example: a non-positive basis

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\[ \mathcal{S} = \{ \text{Conv}((0,0), (1,0), (0,1)), \text{Conv}((1,1), (1,0), (0,1)), \text{Conv}((0,0), (1,0)) \} \].

Then \( \mathcal{S} \) is a basis (note that \( \mathcal{S} \) is not hierarchical). However, \( \mathcal{S} \) is not positive.

Since \( \text{Conv}((1,0), (0,1)) \) is a facet for some polytope in \( \mathcal{S} \), the vectors \((1,1)\) and \((-1,-1)\) are row vectors of \( H(\mathcal{S}) \). Then for any orientation \( \tau \), if \( \tau((1,1)) = 1 \), then \( \text{face}_{(-1,-1)}(\text{Conv}((1,1), (1,0), (0,1))) = \text{Conv}((1,0), (0,1)) \) is a proper face of \( \text{Conv}((1,1), (1,0), (0,1)) \); the other case is similar.
Example: a positive basis

Let $S$ consist of the polytope $\text{Conv}((0, 0), (1, 0), (0, 1))$ and its proper faces. Then $S$ is a positive basis.
Example: a positive basis

Let $S$ consist of the polytope $\text{Conv}((0,0), (1,0), (0,1))$ and its proper faces. Then $S$ is a positive basis. If we homogenize, this $S$ becomes the family $S_{K_3}$. 
Main Theorems

Theorem (L.-Tran ’17+)

*If $S$ is a positive basis, then $\mathbb{N}[S]$ has unique and local factorizations.*
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Theorem (L.-Tran ’17+)

If \( S \) is a positive basis, then \( \mathbb{N}[S] \) has unique and local factorizations.

Theorem (L.-Tran ’17+)

If \( S \) is a positive basis, then \( \mathbb{Z}[S] \) has unique and local factorizations. In addition, \( \mathbb{Z}[S] = \mathbb{E}[S] \) if and only if \( S \) is a full positive basis. In this case, \( f \in \mathbb{Z}[S] = \mathbb{E}[S] \) if and only if the edges of cells in \( \Delta_f \) are parallel to edges in \( S \).
Given a tropical polynomial $f$ and a finite set $S$ of lattice polytopes. We have algorithms for each of the following purposes:

1. Decide whether $S$ is a positive basis.
2. Given a positive basis $S$, decide whether $f \in \mathbb{Z}[S] \setminus \mathbb{N}[S]$, $f \in \mathbb{N}[S]$, or neither.
3. If $f \in \mathbb{N}[S]$, obtain the unique factorization for $f$.
4. If $f \in \mathbb{Z}[S] \setminus \mathbb{N}[S]$, obtain a $g \in \mathbb{N}[S]$ such that $f \odot g \in \mathbb{N}[S]$. 
The family $S_{K_n}$ of simplices is important because the factorization into linear polynomials has applications in economics and combinatorics.
$S_{Kn}$ revisited

The family $S_{Kn}$ of simplices is important because the factorization into linear polynomials has applications in economics and combinatorics.

**Proposition**

$S_{Kn}$ is a full positive basis.
A formula using generalized permutohedra

If $f$ is strong $S_{K_n}$-rational, then each cell $C'$ in $\Delta f$ is a generalized permutohedra. So there exist $(z_I)_{I \subseteq [n]} \in \mathbb{R}^{2^n}$ such that

$$C' = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \geq z_I \quad \forall I \subseteq [n]\}.$$
A formula using generalized permutohedra

If $f$ is strong $S_{K_n}$-rational, then each cell $C'$ in $\Delta_f$ is a generalized permutohedra. So there exist $(z_I)_{I \subseteq [n]} \in \mathbb{R}^{2n}$ such that

$$C' = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \geq z_I \quad \forall I \subseteq [n]\}.$$ 

The following result gives a Möbius inversion formula to decompose $C'$.

**Theorem (Ardila-Benedetti-Doker '10)**

$$C = \sum_{I \subseteq [n]} y_I \cdot \text{Conv}(\{e_i \mid i \in I\}),$$

where $y_I = \sum_{J \subseteq I} (-1)^{|I|-|J|} z_J$. 
Factors of a $S_{K_3}$-factorizable $h_1$

Let $n = 3$ and consider the finite set $S_{K_3}$. Let

$$h_1(x_1, x_2, x_3) = \max(x_2 + 2x_3 + 4, 2x_2 + x_3 + 6, 3x_2 + 7,$$

$$x_1 + 2x_3 + 5, x_1 + x_2 + x_3 + 7, x_1 + 2x_2 + 8,$$

$$2x_1 + x_3 + 7, 2x_1 + x_2 + 8, 3x_1 + 5).$$
Factors of a $S_{K_3}$-factorizable $h_1$

Let $n = 3$ and consider the finite set $S_{K_3}$. Let

$$h_1(x_1, x_2, x_3) = \max(x_2 + 2x_3 + 4, 2x_2 + x_3 + 6, 3x_2 + 7, x_1 + 2x_3 + 5, x_1 + x_2 + x_3 + 7, x_1 + 2x_2 + 8, 2x_1 + x_3 + 7, 2x_1 + x_2 + 8, 3x_1 + 5).$$

$\Delta_{h_1}$ has 5 maximal cells:

1. $\text{Conv}(\{(2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1)\})$,
2. $\text{Conv}(\{(1, 2, 0), (1, 1, 1), (0, 3, 0), (0, 2, 1)\})$,
3. $\text{Conv}(\{(1, 1, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2)\})$,
4. $\text{Conv}(\{(1, 1, 1), (1, 0, 2), (2, 0, 1)\})$,
5. $\text{Conv}(\{(2, 1, 0), (2, 0, 1), (3, 0, 0)\})$. 

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Next we write the 5 maximal cells as Minkowski sums of polytopes in $S_{K^3}$. Both methods work here. Let’s decompose $C_1 = \text{Conv}((2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1))$ using the Möbius inversion formula. $C_1$ is a generalized permutohedron with parameters:

| z | $\emptyset$ | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{1, 2, 3\} |
|---|--------|--------|--------|--------|--------|--------|--------|--------|
| $z$ | 0 | 1 | 0 | 0 | 2 | 1 | 1 | 3 |
signed Minkowski sum - Möbius inversion

Next we write the 5 maximal cells as Minkowski sums of polytopes in $S_{K_3}$. Both methods work here. Let’s decompose $C_1 = \text{Conv}(\{(2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1)\})$ using the Möbius inversion formula. $C_1$ is a generalized permutohedron with parameters:

\[
\begin{array}{cccccccc}
\emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\
 z & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 3 \\
\end{array}
\]

Then the coefficients $y_I = \sum_{J \subseteq I} (-1)^{|I| - |J|} z_J$ are:

\[
\begin{array}{cccccccc}
\emptyset & \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \\
y & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

Thus $C_1 = S_{\{1\}} + S_{\{1,2\}} + S_{\{2,3\}}$. 
signed Minkowski sum - $b$-vector

$H(S_{K_3})$ is the transpose of (up to row permutation):

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & -1
\end{bmatrix}.
$$
signed Minkowski sum - $b$-vector

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\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & -1
\end{bmatrix}.
$$

The vectors $v(H(S_{K_3}), P)$ for the simplices $P \in S_{K_3}$ are

$$
\begin{bmatrix}
\{1,2,3\}: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1,2\}:  & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\{1,3\}:  & 1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
\{2,3\}:  & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
\{1\}: & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\
\{2\}:  & 1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\
\{3\}:  & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & -1
\end{bmatrix}.
$$

This $7 \times 14$ matrix has full rank.
Take $C_2 = \text{Conv}(\{(1, 2, 0), (1, 1, 1), (0, 3, 0), (0, 2, 1)\})$. Then

$$v(H, C_2) = (3, 3, 2, 3, 1, 3, 1, -3, -2, 0, -2, 0, -1, 0).$$
signed Minkowski sum - $C_2$

Take $C_2 = \text{Conv} \left( \{(1, 2, 0), (1, 1, 1), (0, 3, 0), (0, 2, 1)\} \right)$. Then

$$v(H, C_2) = (3, 3, 2, 3, 1, 3, 1, -3, -2, 0, -2, 0, -1, 0).$$

It turns out that $v(H, C_2)$ belongs to the span of the previous matrix and it is the sum of the 2-nd, 4-th, and 6-th rows. Thus

$$C_2 = S_{\{2\}} + S_{\{1,2\}} + S_{\{2,3\}}.$$
Factors of a $S_{K_3}$-factorizable $h_1$

So we write the 5 maximal cells as Minkowski sums of polytopes in $S_{K_3}$ respectively:

$$S\{1\} + S\{1,2\} + S\{2,3\}, S\{2\} + S\{1,2\} + S\{2,3\},$$
$$S\{3\} + S\{1,2\} + S\{2,3\}, S\{1\} + S\{3\} + S\{1,2,3\}, 2S\{1\} + S\{1,2,3\}.$$
Factors of a $S_{K_3}$-factorizable $h_1$

So we write the 5 maximal cells as Minkowski sums of polytopes in $S_{K_3}$ respectively:

\[ S\{1\} + S\{1,2\} + S\{2,3\}, S\{2\} + S\{1,2\} + S\{2,3\}, \]
\[ S\{3\} + S\{1,2\} + S\{2,3\}, S\{1\} + S\{3\} + S\{1,2,3\}, 2S\{1\} + S\{1,2,3\}. \]

For each maximal cell $C$, we find a unique homogeneous linear function $l_C(x_1, x_2, x_3)$ such that for $a \in C$,
\[ l_C(a_1, a_2, a_3) = -(h_1)_a. \] This is called the Legendre transform.
Legendre transform of the 5 maximal cells

The coefficients of the linear functions are

\((8/3, \ 8/3, \ 5/3), (10/3, \ 7/3, \ 4/3), (11/3, \ 8/3, \ 2/3),\n(3, \ 3, \ 1), (5/3, \ 14/3, \ 11/3)\).
Legendre transform of the 5 maximal cells

The coefficients of the linear functions are

\((8/3, 8/3, 5/3), (10/3, 7/3, 4/3), (11/3, 8/3, 2/3), (3, 3, 1), (5/3, 14/3, 11/3)\).

The algorithm always chooses a polytope from \(S_{K_3}\) with largest dimension that appears in the Minkowski sums, and fix a maximal cell \(\sigma\). In this case, we choose \(S_{\{1,2,3\}}\) and \(\sigma = \text{Conv}(\{(1, 1, 1), (1, 0, 2), (2, 0, 1)\})\). Then we get a linear factor from the Legendre transform

\[ \max(x_1 + 3, x_2 + 3, x_3 + 1). \]
The remaining factors

Now there are two more linear factors. The important thing is to determine what are the contributions of the first factor in the Minkowski sums. For another maximal cell \( \eta \), we find all vertices \( v \in S_{\{1,2,3\}} \) such that \( l_{\sigma}(v) - l_{\eta}(v) \) is maximal. Write the convex hull of these vertices as a Minkowski sum of polytopes in \( S_{K_3} \), which is the contribution of the first factor, and we want to delete them.
The remaining factors

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The five polytopes are

$$S_{\{1,2\}}, S_{\{2\}}, S_{\{3\}}, S_{\{1,2,3\}}, S_{\{1\}}.$$
The remaining Minkowski sums

Now the Minkowski sums become

1. \( S\{1\} + S\{1,2\} + S\{2,3\} \),
2. \( S\{2\} + S\{1,2\} + S\{2,3\} \),
3. \( S\{3\} + S\{1,2\} + S\{2,3\} \),
4. \( S\{1\} + S\{3\} + S\{1,2,3\} \),
5. \( S\{1\} + S\{1\} + S\{1,2,3\} \).
The remaining Minkowski sums

Now the Minkowski sums become

1. \( S_1 + S_{1,2} + S_{2,3} \),
2. \( S_2 + S_{1,2} + S_{2,3} \),
3. \( S_3 + S_{1,2} + S_{2,3} \),
4. \( S_1 + S_3 + S_{1,2,3} \),
5. \( S_1 + S_1 + S_{1,2,3} \).

Repeat the procedure, we can find the other two linear factors of \( h_1 \):

\[ \max(x_1, x_2 + 3, x_3 + 2), \max(x_1 + 1, x_2). \]
$	ilde{f}_2$ revisited

Recall $S_{K3}$ consists of polytopes $\text{Conv}((1,0,0),(0,1,0),(0,0,1))$ and its faces. For convenience denote these polytopes as $S_{\{1,2,3\}}, S_{\{1,2\}}, S_{\{3\}}$, etc. Note that $\Delta_{\text{Newt}(\tilde{f}_2)}$ has four 2-dimensional cells: $\text{Conv}((2,0,0),(1,0,1),(1,1,0))$ and other two symmetric ones, plus $\text{Conv}((1,0,1),(1,1,0),(0,1,1))$. 
Recall $S_{K_3}$ consists of polytopes $\text{Conv}((1, 0, 0), (0, 1, 0), (0, 0, 1))$ and its faces. For convenience denote these polytopes as $S\{1,2,3\}, S\{1,2\}, S\{3\}$, etc. Note that $\Delta_{\text{Newt}(\tilde{f}_2)}$ has four 2-dimensional cells: $\text{Conv}((2, 0, 0), (1, 0, 1), (1, 1, 0))$ and other two symmetric ones, plus $\text{Conv}((1, 0, 1), (1, 1, 0), (0, 1, 1))$.

$\text{Conv}((2, 0, 0), (1, 0, 1), (1, 1, 0)) = S\{1,2,3\} + (1, 0, 0)$, but

$\text{Conv}((1, 0, 1), (1, 1, 0), (0, 1, 1)) = S\{1,2\} + S\{1,3\} + S\{2,3\} - S\{1,2,3\}$.

So this cell is the reason that $\tilde{f}_2$ is not $S_{K_3}$-factorizable. And if $\text{Newt}(g) = S\{1,2,3\}$, $g$ may suffice. To find the coefficients of $g$ we need the Legendre transform and the steps are similar to the previous example.
Another full positive basis $S_2$

This example comes from tropical plane curves of degree 2. Let $S_2$ consists of the following ten polytopes $P_1, \ldots, P_{10}$ (and their faces).

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
Another full positive basis $S_2$

This example comes from tropical plane curves of degree 2. Let $S_2$ consists of the following ten polytopes $P_1, \ldots, P_{10}$ (and their faces).

$H(S_2)$ is the following $14 \times 3$ matrix:

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 2 & -2 & 1 & -1 \\
0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 2 & -2 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}.
$$
Example: \( \Delta_{\text{Newt}}(f_3) \)

Let \( f_3(x_1, x_2, x_3) = \max(2x_1 + 2x_2, x_1 + 3x_2 - 2, x_1 + x_2 + 2x_3 - 3, 3x_1 + x_3 - 1, x_1 + 2x_2 + x_3 - 4, 4x_1 - 3) \). Then \( \text{Newt}(f_3) \) is \( \text{Conv}((2, 2, 0), (1, 3, 0), (1, 1, 2), (3, 0, 1), (1, 2, 1), (4, 0, 0)) \) and \( \Delta_{\text{Newt}}(f_3) \) consists of three triangles \( C_1, C_2, C_3 \):

\[
\begin{align*}
C_1 & \quad \text{Triangle } C_1 \\
C_2 & \quad \text{Triangle } C_2 \\
C_3 & \quad \text{Triangle } C_3
\end{align*}
\]

Figure 3: The projection of \( \Delta_{\text{Newt}}(f_3) \) onto coordinates \( x_1 \) and \( x_2 \).
Computing $g_3$

Using the $b$-vectors, we can write

\[ C_1 = P_1 - P_2 + P_3 - P_4 + P_{10} + \text{Conv}((1, 0, 1)), \]
\[ C_2 = -P_1 + 2P_4 + P_7 + \text{Conv}((1, 1, -2)), \]
\[ C_3 = -P_3 + 2P_4 + P_9 + \text{Conv}((2, 0, -2)). \]

Then $\text{Newt}(g)$ should be at least the Minkowski sum

$P_1 + P_2 + P_3 + P_4$. 

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\[ C_3 = -P_3 + 2P_4 + P_9 + \text{Conv}((2, 0, -2)). \]

Then $\text{Newt}(g)$ should be at least the Minkowski sum $P_1 + P_2 + P_3 + P_4$.

In fact, the following polynomial works:

\[
g_3(x_1, x_2, x_3) = \max(2x_1, 2x_3 - 10/3, x_2 + x_3 - 2) \]
\[ + \max(x_1 + x_3, 2x_3 - 5/3, x_2 + x_3 - 1/3) \]
\[ + \max(2x_3, 2x_2 - 1, x_1 + x_3 - 2) + \max(2x_1, 2x_3 - 5, x_1 + x_2 - 2). \]
Computing $h_3$

We take the product $f_3 \odot g_3$ and apply the algorithms again, we get

$$h_3(x_1, x_2, x_3) = x_1 - 3x_3 + 2 \max\left(2x_3 - \frac{5}{2}, x_1 + x_3, x_2 + x_3 - 2\right)$$

$$+ \max\left(2x_3 - \frac{8}{3}, x_1 + x_3 - 1, 2x_2\right) + \max(x_2 + x_3 - 2, 2x_1)$$

$$+ 2 \max\left(2x_3, x_1 + x_3 - 2, x_2 + x_3 - 1/2\right)$$

$$+ \max\left(2x_3 - \frac{10}{3}, x_1 + x_2 - \frac{1}{3}, 2x_1\right).$$
The subdivision of $\text{Newt}(h_3)$

Figure 4: Two ways to decompose $\Delta_{h_3}$: by writing $h_3$ as a product of $S_2$-units, or by writing $h_3 = f_3 \odot g_3$. 
Conjecture

Let $E$ be a finite set of primitive lattice edges in $\mathbb{R}^n$. Then there exists a full positive basis $S$ such that $E$ corresponds to the 1-dimensional polytopes in $S$. 

This conjecture implies the following:

Conjecture

Any tropical polynomial is $S$-rational for some full positive basis $S$. 

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This conjecture implies the following:

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Any tropical polynomial is $S$-rational for some full positive basis $S$. 
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The End

Thank you!