Scaling up Mean Field Games with Online Mirror Descent

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Abstract

We address scaling up equilibrium computation in Mean Field Games (MFGs) using Online Mirror Descent (OMD). We show that continuous-time OMD provably converges to a Nash equilibrium under a natural and well-motivated set of monotonicity assumptions. This theoretical result nicely extends to multi-population games and to settings involving common noise. A thorough experimental investigation on various single and multi-population MFGs shows that OMD outperforms traditional algorithms such as Fictitious Play (FP). We empirically show that OMD scales up and converges significantly faster than FP by solving, for the first time to our knowledge, examples of MFGs with hundreds of billions states. This study establishes the state-of-the-art for learning in large-scale multi-agent and multi-population games.

1. Introduction

Solving decision making problems involving multiple agents has been the topic of intensive research in Artificial Intelligence for decades. It finds applications in a wide variety of domains such as economy (Conitzer & Sandholm, 2011; Othman et al., 2013; Achdou et al., 2014), resource management (Couillet et al., 2012; Freedman et al., 2020), crowd motion modeling (Achdou & Lasry, 2019) among others. Despite the vast literature on Game Theory and numerous fundamental results, application to real-world problems remains a challenge. Recent successes of combining Game Theory and Machine Learning (especially Deep Learning (Goodfellow et al., 2016) and Reinforcement Learning (Sutton & Barto, 2018)) led to solutions for large scale games such as chess (Campbell et al., 2002), Go (Silver et al., 2016; 2017; 2018), Poker (Brown & Sandholm, 2017; 2019; Moravčík et al., 2017) and even complex video games like StarCraft II (Vinyals et al., 2019). Although this allowed for tackling problems involving large states spaces, the number of agents remains still limited and scaling up to large populations of players remains intractable, which prevents a broader real-world impact.

To address this challenge, the Mean Field Game (MFG) theory was introduced in (Lasry & Lions, 2007; Huang et al., 2006b; Huang et al., 2006a) to study a category of games that involves an infinite population of agents. By considering the limit case of a continuous distribution of identical agents (i.e., anonymous and with symmetric interests), the MFG framework allows the learning problem to be reduced to the characterization of the optimal behavior of a single representative agent in its interactions with the full population. Given this asymptotic formulation, traditional solutions to MFGs entail a coupled system of differential equations: one capturing the forward dynamics of the population and a second being the dynamic programming optimality equation of the representative player. Despite important progress in the area, such approaches are based on numerical approximation schemes for partial differential equations (Achdou & Capuzzo-Dolcetta, 2010; Achdou et al., 2012; Carlini & Silva, 2014; 2015; Briceno Arias et al., 2018; Briceno Arias et al., 2019; Achdou & Laurière, 2020) or for stochastic differential equations (Chassagneux et al., 2019; Angiuli et al., 2019), which are not easily scaled to large state spaces. Also, given the sensitivity to limit conditions, only simple configurations of the state space can be considered. So, until recently, we were left with solutions that either scale in terms of the state space dimension (deep RL) or in terms of large populations of agents (MFGs). Moreover, generalizations of the MFG framework to models with multiple populations have been introduced in (Huang et al., 2006a) and have attracted a growing interest (Bensoussan et al., 2013; Carmona & Delarue, 2018a; Bensoussan et al., 2018). Applications include urban settlements (Achdou et al., 2017) and crowd motion (Lachapelle & Wolfram, 2011; Aurell & Djehiche, 2018).

By introducing solutions inspired by game theory (i.e., Fictitious Play (Robinson, 1951; Shapiro, 1958)) into
MFGs (Cardaliaguet & Hadikhanloo, 2017; Elie et al., 2020; Perrin et al., 2020), recent research leverages the generalization capacity of Machine Learning to compute a Nash equilibrium (NE) in large state spaces. Fictitious Play (FP) is a generic algorithm that alternates two steps starting from an arbitrary strategy for the representative player: i) computing the best response of this agent against the rest of the population, ii) compute the mixture of that best response with its previous strategy. Perrin et al. (2020) propose to make use of most recent Reinforcement Learning (RL) methods to learn the best response and solve problems with millions of states with a non-trivial topology. Unfortunately, FP seems hard to scale further for several reasons. First, the computation of the best response remains a hard problem even if RL is promising. Second, its computational efficiency seems very low in practice. Finally, FP requires storing multiple quantities (e.g. averaged policies and induced distributions, etc.) which contributes to cap scalability.

In this context, our first contribution is a new algorithm to compute a NE in lieu of FP, namely Online Mirror Descent (OMD) (Shalev-Shwartz et al., 2011). Inspired by convex optimization and the Mirror Descent algorithm (Nemirovsky & Yudin, 1979), our method doesn’t require the computation of a best response. It rather alternates a step of evaluation of the current strategy with a step of improvement of that strategy. The evaluation is done through the computation of the expected accumulated pay-offs of the strategy over time in the shape of a so-called Q-function. The improvement step reduces to computing the soft-max of the quantity obtained by integrating the Q-functions over iterations (like the MD algorithm suggests). Quantities that need to be stored by OMD (the strategy and the integrated Q-function) are thus limited compared to FP. As a second contribution, we provide a proof of convergence for continuous time OMD to a NE for MFGs under reasonable assumptions (common in the field). These theoretical results extend naturally to multi-population MFGs as well as settings where a noise is commonly shared by all agents. Our third contribution is an extensive empirical evaluation of OMD on different tasks involving single or multiple populations, in the presence of a common noise or not, with non trivial topologies. The scale of the considered problems reaches $10^{11}$ states and trillions of state-action pairs, surpassing by 4 or 5 orders of magnitudes existing results. These experiments demonstrate that OMD’s computational efficiency is much stronger than FP which results in faster convergence.

2. Preliminaries on Mean Field Games

In a Multi-Population Mean Field Game (MP-MFG), an infinite number of players from $N_p$ different populations interact with each other in a temporally and spatially extended game (the case $N_p = 1$ corresponds to a standard MFG). Let $\mathcal{X}$ be the finite discrete state space and $\mathcal{A}$ be the finite discrete action space of the MP-MFG. We denote by $\Delta \mathcal{X}$ and $\Delta \mathcal{A}$ respectively the spaces of probability distributions over states and actions. In this sequential decision problem, a representative player of population $i \in \{1, \ldots, N_p\}$ starts at a state $x^i_0 \in \mathcal{X}$ according to a distribution $\mu^i_0 \in \Delta \mathcal{X}$. We consider a finite time horizon $N > 0$. At each time step $n \in \{0, \ldots, N\}$, the representative player of population $i$ is in state $x^i_n$ and takes an action according to $\pi^i_n(\cdot| x^i_n)$, where $\pi^i_n \in (\Delta \mathcal{A})^{X}$ is a policy. Given this action $a^i_n$, the representative player moves to a next state $x^i_{n+1}$ with probability $p(\cdot|x^i_n, a^i_n)$ and receives a reward $r^i(x^i_n, a^i_n, \mu^1_n, \ldots, \mu^N_n)$, where $\mu^j_n$ is the distribution of the population $j$ at time $n$. Here $p \in (\Delta \mathcal{X})^{X \times \mathcal{A}}$ and $r^i : \mathcal{X} \times \mathcal{A} \times (\Delta \mathcal{X})^{N_p} \to \mathbb{R}$.

Observe that the transition kernel does not depend on the Multi-population distribution as in many classical MFG examples (Lasry & Lions, 2007). For the reader’s convenience, we denote $\pi^i_n = \{\pi^i_n\}_{n \in \{0, \ldots, N\}}$, $\mu^i = \{\mu^i_n\}_{n \in \{0, \ldots, N\}}$, $\pi = \{\pi^i_n\}_{i \in \{1, \ldots, N_p\}}$, $\mu = \{\mu^i_n\}_{i \in \{1, \ldots, N_p\}}$ and $\pi_n = \{\pi^i_n\}_{i \in \{1, \ldots, N_p\}}$ and $\mu_n = \{\mu^i_n\}_{i \in \{1, \ldots, N_p\}}$.

During the game and for a given fixed multi-population distributions sequence $\mu$, a representative player of population $i$ accumulates the following rewards:

$$J^i(\pi^i, \mu) = \mathbb{E}\left[\sum_{n=0}^{N} r^i(x^i_n, a^i_n, \mu_n) \mid x^i_0 \sim \mu^i_0, a^i_0 \sim \pi^i_0(\cdot | x^i_0), x^i_{n+1} \sim p(\cdot | x^i_n, a^i_n)\right].$$

**Backward Equation:** Given a population $i$, a time $n$, a state $x^i$, an action $a^i$, a policy $\pi^i$ and a multi-population distribution sequence $\mu$, we define the Q-function:

$$Q^i_n, \pi^i(\cdot | x^i_n, a^i_n, \mu) = \mathbb{E}\left[\sum_{k=n}^{N} r^i(x^i_k, a^i_k, \mu_k) \mid x^i_n = x^i, a^i_n = a^i, a^i_k \sim \pi^i_k(\cdot | x^i_k), x^i_{k+1} \sim p(\cdot | x^i_k, a^i_k)\right].$$

and the value function:

$$V^i_n, \pi^i(\cdot | x^i_n) = \mathbb{E}\left[\sum_{k=n}^{N} r^i(x^i_k, a^i_k, \mu_k) \mid x^i_n = x^i, a^i_k \sim \pi^i_k(\cdot | x^i_k), x^i_{k+1} \sim p(\cdot | x^i_k, a^i_k)\right].$$

These two quantities can be computed recursively with the following backward equations:

$$Q^i_N, \pi^i(\cdot | x^i, a^i, \mu_N) = r^i(x^i, a^i, \mu_N);$$

$$Q^i_n, \pi^i(\cdot | x^i, a^i) = r^i(x^i, a^i, \mu_{n-1}) + \sum_{x^j \in \mathcal{X}} p(x^j | x^i, a^i) E_{b^i \sim \pi^j(\cdot | x^j)} \left[Q^j_n, \pi^j(\cdot | x^j, b^j) \right];$$

$$V^i_n, \pi^i(\cdot | x^i) = E_{a^i \sim \pi^i(\cdot | x^i)} \left[Q^i_n, \pi^i(\cdot | x^i, a^i) \right].$$
Finally, the sum of rewards is \( J^i(\pi^i, \mu) = \mathbb{E}_{x^i \sim \mu^i_n} [V^i, \pi^i, \mu(x^i)] \).

**Forward Equation:** If all the agents of a population \( i \) follow the policy \( \pi^i \), the distribution of the full population is defined recursively via the following forward equation: for all \( x^i \in \mathcal{X}, \mu^i_n(x^i) = \mu^i_0(x^i) \) and for all \( x^{i+1} \in \mathcal{X} \),

\[
\mu^{i+1}_n(x^{i+1}) = \sum_{(x^i, x^{i+1}) \in \mathcal{X} \times \mathcal{X}} \pi^n_i(a^i|x^i)p(x^{i+1}|x^i, a^i)\mu^{i,n}(x^{i+1})
\]

for \( n \leq N - 1 \). We denote \( \mu^\pi = (\mu^i_\pi)_{i \in \{1, \ldots, N\}} \).

This leads to the following property for the cumulative sum of rewards \( J^i(\pi^i, \mu) = \sum_{n=0}^N \sum_{x^i \in \mathcal{X}} \mu^{i,n}(x^i)\pi^n_i(a^i|x^i)r^i(x^i, a^i, \mu_n) \).

**Best Response and Exploitability:** A best response policy \( \pi^{i,br,\mu}_n \) to a multi-population distribution sequence \( \mu \) verifies the following property \( \max J^i(\pi^i, \mu) = J^i(\pi^{i,br,\mu}_n, \mu) \).

It can be computed recursively by finding the best responding \( Q \)-function \( Q^{i,br,\mu}_{n}(x^i) \):

\[
Q^{i,br,\mu}_n(x^i, a^i) = r^i(x^i, a^i, \mu_N) + \sum_{x^{i+1} \in \mathcal{X}} p(x^{i+1}|x^i, a^i) \max_{b^i} Q^{i,br,\mu}_{n+1}(x^{i+1}, b^i).
\]

Finally, \( \pi^{i,br,\mu}_n(x^i) \in \arg \max Q^{i,br,\mu}_n(x^i) \).

The exploitability measures the distance to an equilibrium and is defined as \( \phi(\pi) = \sum_{i=1}^{N_n} \phi^i(\pi) \) where, for each \( i \),

\[
\phi^i(\pi) = \max_{\mu^i} J^i(\pi^i, \mu^\pi) - J^i(\pi^i, \mu^\pi).
\]

**Monotonicity:** A multi-population game is said to be weakly monotone if for any \( \rho^i_n, \rho^i' \in \Delta(\mathcal{X} \times \mathcal{A}) \) and \( \mu^i_n, \mu^i'_n \in \Delta \mathcal{X} \) such that for all \( i, a^i \), \( \rho^i_n(x^i, a^i) \geq \rho^i_n(x^i, a^i) \) and \( \mu^i_n(x^i, a^i) = \mu^i_0(x^i, a^i) \), we have:

\[
\sum_{(x^i, a^i) \in \mathcal{X} \times \mathcal{A}} (\rho^i_n(x^i, a^i) - \rho^i'_n(x^i, a^i)) (r^i(x^i, a^i, \mu_n) - r^i(x^i, a^i, \mu'_n)) \leq 0.
\]

It is strictly weakly monotone if the inequality is strict whenever \( \rho^i_n \neq \rho^i' \). This condition means that the players are discouraged from taking similar state-action pairs as the rest of the population. Intuitively, it can be interpreted as an aversion to crowded areas.

We have the following consequence, which is enough to derive many properties.

**Lemma 1.** The weak monotonicity property implies that for any \( \pi, \pi' \) with \( \pi \neq \pi' \),

\[
\tilde{M}(\pi, \pi') := \sum_{i=1}^{N_n} [J^i(\pi^i, \mu^\pi) + J^i(\pi^i, \mu^\pi')] - J^i(\pi^i, \mu^\pi) - J^i(\pi^i, \mu^\pi') \leq 0. \quad (2)
\]

Strictly weak monotonicity implies a strict inequality above.

This result is proved in Appendix E.

Moreover, the weak monotonicity condition is met for example in the following classical framework.

**Lemma 2.** Assume the reward is separable, i.e. \( r^i(x^i, a^i, \mu) = \hat{r}^i(x^i, a^i) + \hat{\tilde{r}}^i(x^i, \mu) \) and the following monotonicity condition holds: for all \( \mu \neq \mu', \sum_{i \in \mathcal{X}} \sum_{(x^i)\in\mathcal{X}} (\mu(x^i) - \mu'(x^i))(\hat{r}^i(x^i, \mu) - \hat{\tilde{r}}^i(x^i, \mu')) \leq 0 \) (resp. \( < 0 \)). Then the game is weakly monotone (resp. strictly weakly monotone).

This result is proven in Appendix A.

An example of such a separable and monotone reward can be found in multi-population predator prey models where the reward can be expressed as a network zero-sum game:

\[
r^i(x^i, a^i, \mu) = \hat{r}^i(x^i, a^i) + \hat{\tilde{r}}^i(x^i, \mu) + \sum_{j \neq i} \mu^j(x^j)\hat{\tilde{r}}^{i,j}(x^i)
\]

if \( \forall x^i \in \mathcal{X}, \hat{\tilde{r}}^{i,j}(x^i) = -\hat{r}^{j,i}(x^i) \) and if \( \forall \mu \neq \mu', \forall x^i \in \mathcal{X}, \sum_{x^i \in \mathcal{X}} (\mu^i(x^i) - \mu'^i(x^i))(\hat{r}^{i,i}(x^i) - \hat{\tilde{r}}^{i,i}(x^i)) \leq 0 \) (or with a strict inequality).

**Nash Equilibrium (NE):** A NE is a vector of policies for all populations that has a 0 exploitability. The existence of a NE in MFGs has been studied in many settings (Cardaliaguet, 2012; Bensoussan et al., 2013; Carmona & Delarue, 2018b).

In our framework, it is a consequence of the convergence of the fictitious play dynamics in monotone games which will be introduced later and proved in Appendix C.

**Proposition 1 (Existence and uniqueness of Nash).** Any weakly monotone MP-MFG admits a NE. Besides, if the weak monotonicity is strict, the NE is unique.

**Proof.** The existence result follows from Theorem (1), while the uniqueness property is proven in Appendix F. \( \square \)

### 3. Background on Fictitious Play

One can extend the Fictitious Play work of Perrin et al. to a multi-population setting. In the Multi-Population case, the Fictitious play process is defined as follows. Let first picking
1 as an arbitrary but classical reference time. For \( t < 1 \),
we consider a fixed uniform policy for all representative
player \( i \) at all time-step \( n \) denoted \( \pi_{n,t}^{br} \) and inducing a
distribution \( \mu_{n,t}^{br} \). We define \( \forall t \geq 1 \) the distribution \( \mu_{n,t}^{i} \) as:

\[
\forall i, n, \ \mu_{n,t}^{i}(x^{i}) = \frac{1}{t} \int_{s=0}^{t} \mu_{n,s}^{i,br}(x^{i})ds,
\]

where, for all \( t \geq 1 \), \( \mu_{n,t}^{i,br} \) is the distribution of a best
response policy \( \pi_{n,t}^{i} \) to \( \mu_{n,t}^{i} \). The policy \( \pi_{n,t}^{i} \) of the
distribution \( \mu_{n,t}^{i} \) verifies the following equation (see Perrin et al.):

\[
\pi_{n,t}^{i}(a^{i}|x^{i}) = \int_{s=0}^{t} \mu_{n,s}^{i,br}(a^{i}|x^{i})ds
\]

Theorem 1. If a MP-MFG satisfies the weak monotony
assumption, the exploitability is a strong Lyapunov function
of the Fictitious Play dynamical system, \( \forall t \geq 1 \): \( \frac{d}{dt}\phi(\pi^{t}) \leq
\frac{-1}{\rho}\phi(\pi^{t}) \). Hence \( \phi(\pi^{t}) = O(\frac{1}{t}) \).

Proof. This is an extension to multi-population of Theo-
rem 1 of (Perrin et al., 2020). The full proof is in
Appendix C. \( \square \)

4. Online Mirror Descent: algorithm and
convergence result

We now turn to the Online Mirror Descent Algorithm and
introduce a regularizer \( h : \Delta A \to \mathbb{R} \), that is assumed to be
\( \rho \)-strongly convex for some constant \( \rho > 0 \). Furthermore,
we will assume from this point forward that the regularizer
\( h \) is steep, i.e., \( \| \nabla h(\pi) \| \to \infty \) whenever \( \pi \) approaches the
of \( \Delta A \); The classic negentropy regularizer, which results to
replicator dynamics is the prototypical example of this class.
Denote by \( h^{*} : \Delta A \to \mathbb{R} \) its convex conjugate defined by
\( h^{*}(y) = \max_{\pi \in \Delta A} \langle y, \pi \rangle - h(\pi) \). Since \( h \) is differentiable
almost everywhere, we have, for almost every \( y \),

\[
\Gamma(y) := \nabla h^{*}(y) = \arg \max_{\pi \in \Delta A} \langle y, \pi \rangle - h(\pi).
\]

Discrete Time Online Mirror Descent: The OMD
algorithm is implemented as described in Algorithm 1. At
each iteration, the first step consists in computing, for each
population, the evolution of the population’s distribution
by using the current policy, see (1). In the second step,
each population’s policy is updated. This update is done
by first updating the corresponding \( y \) variable and then
obtaining the policy thanks to the function \( \Gamma \). We have for all

Algorithm 1 Online Mirror Descent (OMD)

Input: \( \alpha, y_{n,0}^{i} = 0 \) for all \( i, n; t_{max} \)

repeat

Forward Update: Compute for all \( i, \mu_{n,\pi^{i}}^{i} \)

Backward Update: Compute for all \( i, Q_{n,t}^{i,n,\pi^{i}} \)

Update for all \( i, n, x, a \),

\[
y_{n,t+1}^{i}(x, a) = y_{n,t}^{i}(x, a) + \alpha Q_{n,t}^{i,n,\pi^{i}}(x, a),
\]

\[
\pi_{n,t+1}^{i}(\cdot|x) = \Gamma(y_{n,t+1}^{i}(x, \cdot)).
\]

until \( t = t_{max} \)

We have for all \( t > 0, i \in \{1, \ldots, N_{p}\}, n \in \{0, \ldots, N\}, \)

\[
y_{n,t}^{i}(x^{i}, a^{i}) = \int_{0}^{t} Q_{n,s}^{i,n,\pi^{i}}(x^{i}, a^{i})ds,
\]

\[
\pi_{n,t}^{i}(\cdot|x^{i}) = \Gamma(y_{n,t}^{i}(x^{i}, \cdot)).
\]

From here on, unless otherwise specified, we assume that
the weak monotonicity condition holds and denote by \( \pi^{*} \)
the NE, whose existence follows from Proposition 1. We
let \( y^{i,*} : (x^{i}, a^{i}) \to y^{i,*}(x^{i}, a^{i}) \) be the corresponding dual variable such that \( \pi^{i,*} (\cdot | x^{i}) = \Gamma(y^{i,*}(x^{i}, \cdot)) \) for every \( i \).

Measure of similarity with the NE \( \pi^{*} \): Based on the regularizer
\( h \), we define in the dual space the following measure of
similarity \( H : \mathbb{R}^{[\Delta A]} \to \mathbb{R} \) with the NE \( \pi^{*} \):

\[
H(y) := \sum_{i=1}^{N_{p}} \sum_{n=0}^{N} \sum_{x \in X} \mu_{n,t}^{i,\pi^{*}}(x^{i}) \left[ h^{*}(y_{n,t}^{i}(x^{i}, \cdot)) - h^{*}(y^{i,*}(x^{i}, \cdot)) - \langle \pi_{n,t}^{i,*}, y_{n,t}^{i}(x^{i}, \cdot) - y_{n,t}^{i,*}(x^{i}, \cdot) \rangle \right].
\]

As detailed below, this quantity will be decreasing through
the iterations of CTOMD. Observe that since the regularizer
is steep and thus always maps in the interior of the simplex,
it can also be expressed in terms of Bregman divergence as:

\[
H(y) = \sum_{i=1}^{N_{p}} \sum_{n=0}^{N} \sum_{x \in X} \mu_{n,t}^{i,\pi^{*}}(x^{i}) [D_{h}(\pi_{n,t}^{i,*}(x^{i}, \cdot), \pi_{n}^{i}(x^{i}, \cdot))],
\]
which is always non-negative. Here $D_F$ denotes the Bregman divergence associated with a map $F$ and defined as $D_F(p,q) := F(p) - F(q) - \langle \nabla F(q), p-q \rangle$. In this derivation we have used known relations between Fenchel couplings and Bregman divergences (e.g., Mertikopoulos & Sandholm (2016)) and denoted $\pi_n^t := \Gamma(y_n^t)$. Thus, the similarity measure $H$ can also be expressed in terms of proximity between policies.

We are now in position to characterize the dynamics of the similarity to the Nash mapping via the following lemma, whose proof is provided in Appendix D.

**Lemma 3 (Similarity to Nash dynamics).** In CTOMD, the measure of similarity $H$ to the Nash $\pi^*$ satisfies

$$\frac{d}{dt} H(y_t) = \Delta J(\pi_t, \pi^*) + \tilde{M}(\pi_t, \pi^*)$$

where $\Delta J(\pi_t, \pi^*) := \sum_{n=1}^{N_p} J^i(\pi_t^i, \mu_t^i) - J^i(\pi^i_t, \mu_t^i)$ is always non-positive, and where the weak monotonicity metric $\tilde{M}$ is defined in (2).

**Convergence to the Nash for MP-MFGs:** We now turn to the main theoretical contribution of the paper, by deriving the convergence of CTOMD to the set of NE for MP-MFGs.

**Theorem 2 (Convergence of CTOMD).** If a MP-MFG satisfies $\tilde{M}(\pi, \pi^*) < 0$ if $\mu^* \neq \pi^*$ and 0 otherwise, then $(\pi_t)_{t \geq 0}$ generated by CTOMD in (6) converges to the set of Nash equilibria of the game as $t \to +\infty$.

**Proof.** The proof is left in appendix G. \qed

Thanks to Lemma 1 together with Proposition 1, we easily deduce the convergence to the unique NE in some more stringent classes of MP-MFGs.

**Corollary 1 (Convergence of CTOMD for weakly monotone MFG).** For any strictly weakly monotone MP-MFG, $(\pi_t)_{t \geq 0}$ generated by CTOMD in (6) converges to the unique NE of the game, as $t \to +\infty$.

**Corollary 2 (Convergence of CTOMD for multi-population network zero sum MFG).** For any strictly monotone and essentially zero-sum MP-MFG, $(\pi_t)_{t \geq 0}$ generated by CTOMD in (6) converges to the unique NE of the game, as $t \to +\infty$.

It is worth noticing that the argumentation followed in our proof differs from the usual approaches on regret minimization arguments as e.g. in (Zinkevich et al., 2008).

**Restriction to a single population MFG:** Finally, considering the number of populations $N_p$ equals 1, we deduce a convergence result of CTOMD to the NE of single population for strictly weakly monotone MFG.

**Corollary 3 (Convergence of CTOMD for Single Population MFG).** For any single population MFG satisfying the strictly weak monotonicity assumption, $(\pi_t)_{t \geq 0}$ generated by CTOMD given in (6) converges to the unique NE of the game, as $t \to +\infty$.

5. Numerical experiments

We illustrate the theoretical convergence of CTOMD with an extensive empirical evaluation of OMD described in Algorithm 1 within various settings involving single or multiple populations as well as non trivial topologies (videos available here). These settings are typically hardly tractable using classical numerical approximation schemes for partial differential equations. Besides, the scale of the numerical experiments grows up to $10^{12}$ states, establishing a new scalability benchmark in the MFG literature. We emphasize the diversity of tractable environments by considering (randomized MDP) Garnet settings, a twenty-storey high building evacuation, a crowd movement example in the presence of common noise and finally an essentially zero sum multi-population chasing game.

**Experimental setup:** We compare OMD and FP with different learning rates $\alpha$. In discrete-time OMD, $\alpha$ appears in the backward update of $y$: $y_{n,t+1} = y_{n,t}(x,a) + \alpha q_{n}^{i,t} \pi^i_{n,t} (x,a)$, whereas in discrete-time FP, it corresponds to how much we update the average policy with the new best response $\pi_{n,t+1}^i (x,a)$ given by

$$\frac{(1-\alpha_t)\mu_{n,t}^i(x,a)\pi_{n,t}^i(x,a) + \alpha \mu_{n,t}^{br} (x,a)\pi_{n,t}^{br} (x,a)}{(1-\alpha_t)\mu_{n,t}^i(x,a) + \alpha \mu_{n,t}^{br} (x,a)}$$

FP is experimented with decreasing $\alpha_t = \alpha / (2 + t)$ or constant $\alpha_t = \alpha$ learning rate. This latter is referred to hereafter as FP damped, while $\alpha = 1$ corresponds to the fixed point iteration algorithm, i.e. the population applies the last best response policy. The theoretical proof of convergence relies on restrictive conditions which only hold for a small class of games. We provide a thorough evaluation in Table 1 of the complexity of the environments along with the memory required to compute our results. For OMD, we only need to store $y$ of size $|X| \times |A|$ and the distributions, of size $|X|$. For FP, we need to store the last best response, the average policy, the last distribution and the average distribution, requiring a total of $2 \times (|X| \times |A|) + 2 \times |X|$. In all the experiments, $h$ is the entropy: $h = -\sum_{a \in A} \pi(a) \log(\pi(a))$. This implies that $h^*(y) = \log(\sum_{a \in A} \exp(y(a)))$, and we find that $\Gamma$ is a softmax if we take the gradient of $h^*$.

5.1. Garnet

We first evaluate Alg. 1 on a set of randomly generated problems (repeatability of our results for varying sizes).

**Environment:** A garnet is an abstract and randomly generated MDP (Archibald et al., 1995). We adapt this con-
We compare OMD to FP, damped or not. We observe that in varied modeling problem, namely a building evacuation. This case, the number of states influences the convergence rate, consistently provides fast convergence to the Nash. In all results are averaged over 5 randomly generated Garnets.

Numerical results: Fig. 1 (main text) and 6 (Appx. H.1) shows various Garnet experiments. We fix $s_f = 10$, $t = 2000$, $\eta = 1$ and $n_b = 1$ (deterministic dynamics) and vary $n_x \in \{2 \times 10^3; 2 \times 10^4\}$ and $n_a \in \{10, 20\}$. In each case, results are averaged over 5 randomly generated Garnets. We compare OMD to FP, damped or not. We observe that OMD consistently converges faster for the right choice of $\alpha$. $\alpha = 1$ might lead to unstable results while $\alpha = 0.1$ consistently provides fast convergence to the Nash. In all cases, the number of states influences the convergence rate, but much less for OMD.

5.2. Building evacuation

Environment: We now turn to a single-population crowd modeling problem, namely a building evacuation. This kind of problem has been the topic of several works on MFG (see e.g. (Achdou & Laurière, 2015; Achdou & Lasry, 2019) for a single room and (Djehiche et al., 2017) for a multilevel building). The building consists of 20 floors, each of dimension $200 \times 200$. At each floor, two staircases are located at two opposite corners, such as the crowd has to cross the whole floor to take the next staircase. Each agent can remain in place, move in the 4 directions (up, down, right, left) as well as go up or down when on a staircase location. The initial distribution is uniform over all the floors. Each agent of the crowd wants to go downstairs as quickly as possible - as it gets a reward of $10$ at the bottom floor - while favoring social distancing:

$$r(x, a, \mu) = -\eta \log(\mu(x)) + 10 \times \mathbf{1}_{\text{floor}=0}$$

Numerical results: We compute this problem with a horizon of $10000$, so $|\mathcal{X}| = 8^{10}$. We take $\eta = 1$. To ensure that the reward stays bounded, we clip the first part $-\eta \log(\mu(x))$ to $-40$. As expected, we observe in Fig. 2 that the agents go downstairs and do not concentrate on the shortest path but rather spread mildly. OMD converges faster than both FP and FP damped.

5.3. Crowd motion with randomly shifted point of interest

Environment: We consider a second crowd modeling

| Environment     | $|\mathcal{X}|$     | $|\mathcal{X}| \times |A|$     | OMD      | FP       |
|-----------------|---------------------|---------------------|----------|----------|
| Garnet          | $2 \times 10^3 - 2 \times 10^4$ | $2 \times 10^3 - 4 \times 10^4$ | 84Ko - 229Ko | 168Ko - 458Ko |
| Building        | $8 \times 10^7$     | $5.6 \times 10^{11}$ | 0.21To   | 0.42To |
| Common noise    | $2.73 \times 10^{14}$ | $1.092 \times 10^{12}$ | 5.0To    | 10To |
| Multi-Population medium | $5 \times 10^7$   | $2 \times 10^8$   | 0.93Go   | 1.9Go |
| Multi-Population large  | $8 \times 10^8$   | $3.2 \times 10^9$   | 73Go  | 146Go |

Table 1. Number of states, action-states pairs & RAM memory required for the experiments. $|\mathcal{X}| = \text{positions} \times \text{timesteps} \times \text{common noise} \times \text{number of populations}$.

Figure 1. 5 Garnet sampled with param $n_x = 20000$, $n_a = 10$, $t = 2000$, $s_f = 10$
MFG, extending the Beach Bar problem of (Perrin et al., 2020) in two dimensions. The environment is a 2D torus of dimensions 1000 × 1000, with a point of interest initially located at the center of the square. After 200 timesteps, the point of interest changes location, moving randomly in the direction of one of the corners. This process repeats itself 5 times. This random location change adds common noise to the environment and increases exponentially the number of states. Considering MFG with common noise can be encompassed in our previous study by simply increasing the state space with the common noise and adding time to the reward and the transition kernel. For every random movement, four possible directions are possible, making the total number of states |X| = 2 × 10^8 × \sum_{k=0}^{4} 4^k = 2.73 × 10^{13} states. The reward is: r(x, a, µ) = C × (1 - \frac{∥4∥}{2N_{side}}) - log(µ(x)).

**Numerical results:** We set C = 10. We observe in Fig. 4 that the population is organizing itself with respect to the point of interest and follows it as it randomly moves within the dedicated square region. With common noise we get more than a trillion states, making it hard for FP to succeed. Considering MFG with common noise can be encompassed in our previous study by simply increasing the state space with the common noise and adding time to the reward and the transition kernel. For every movement, four possible directions are possible, making the total number of states |X| = 2 × 10^8 × \sum_{k=0}^{4} 4^k = 2.73 × 10^{13} states. The reward is: r(x, a, µ) = C × (1 - \frac{∥4∥}{2N_{side}}) - log(µ(x)).

**Table 2.** r^{i,j} for three-population.

|     | R | P | S |
|-----|---|---|---|
| R   | 0 | -1| 1 |
| P   | 1 | 0 | -1|
| S   | -1| 1 | 0 |

The reward of population i is monotone (cf. Appx. H.4.1) and follows the definition (3): r^i(x, a, µ^1, ..., µ^N) = - log(µ^i(x)) + \sum_{j \neq i} µ^j(x) r^{j,i}(x). The distributions are initialized either randomly or in different corners. The number of agents of each population is fixed, but the reward encourages the agent to chase the population that it dominates. For example, if an agent is Rock, the second term of the reward is proportional to the amount of Scissors agents µ^S where the Rock agent is located, and inversely proportional to µ^P, the proportion of Paper agents, making the Rock agent to flee from places populated by Paper agents.

**5.4. Multi-population chasing**

**Environment:** We finally look at MP-MFGs, where the populations are chasing each other in a cyclic manner. For the sake of clarity, we explain the reward structure with 3 populations, but more populations are considered in the experiments. With three populations, the game closely relates to the well known Hens-Foxes-Snakes outdoor game for kids. Hens are trying to catch snakes, while snakes are chasing foxes, who are willing to eat hens. It can also be interpreted as a control version of the spatially extended Rock-Paper-Scissors, where patterns of travelling waves appear under certain conditions (Postlethwaite & Rucklidge, 2017). The interplay between nontransitive interactions and biodiversity has been the subject of extensive, mostly experimental, research showing that the setting details critically affect the emergent behavior (Szolnoki et al., 2020).

To ensure r^{i,j} = -r^{j,i} we implement MP-MFGs with the reward structure defined in Table 2 (ex. with 3 populations).

To ensure r^{i,j} = -r^{j,i} we implement MP-MFGs with the reward structure defined in Table 2 (ex. with 3 populations).

**6. Related work**

OMD dynamics have been studied extensively within the field of multi-agent games (Cesa-Bianchi & Lugosi, 2006; Nisan et al., 2007). Leveraging the well known advanta-
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Figure 4. Crowd position at different consecutive dates when the point of interest is randomly shifted to the right by a common noise.

Figure 5. 4-population chasing. Fig 5d: FP (red, $\alpha = 10^{-3}$), FP damped (green, $\alpha = 10^{-5}$) and OMD (blue, $\alpha = 10^{-5}$).

geous regret properties of such dynamics (Srebro et al., 2011), one can prove strong time-average convergence results both in zero-sum games (and network variants thereof) (Freund & Schapire, 1999; Cai et al., 2016) as well as in smooth-games (Roughgarden, 2009). Recently, there has been explicit focus on understanding their day-to-day behavior which has been shown to be non-equilibrating even in standard bilinear zero-sum games (Piliouras & Shamma, 2014; Mertikopoulos et al., 2018). Moreover, even in simple games the behavior of such dynamics can become formally chaotic (Sato et al., 2002; Palaiopanos et al., 2017; Chotibut et al., 2019). Nevertheless, sufficient conditions have been established under which converge to NE is guaranteed even in the sense of the day-to-day behavior (Zhou et al., 2017; Bravo et al., 2018). We find sufficient conditions for convergence in the more demanding setting of MP-MFG.

MP-MFGs have been introduced in (Huang et al., 2006a) and studied from a PDE viewpoint in (Feleqi, 2013; Cirant, 2015; Cirant & Verzini, 2017; Bardi & Cirant, 2018). To the best of our knowledge, our work is the first one to provide a provably converging algorithm.

Related to the question of learning in MFGs, (Yin et al., 2010) studied a MF oscillator game, while (Cardaliaguet & Hadikhanloo, 2017) initiated the study of Fictitious Play in MFGs, which has been further studied in (Hadikhanloo & Silva, 2019). Recently, these ideas have been combined with Reinforcement Learning by Elie et al. (2020); Perrin et al. (2020). These methods allow solving MFGs under a monotonicity assumption, which is at the same time easier to check and less restrictive than the ones used to ensure convergence for fixed point iterations (Guo et al., 2019; Anahtarci et al., 2020) or single-loop fictitious play iterations (Angiuli et al., 2020; Xie et al., 2020). In our work, we also prove convergence under such a weak monotonicity condition, which enables us to cover a large class of MFGs. Furthermore, we consider time-dependent problems (as e.g. in (Mishra et al., 2020)) instead of stationary equilibria. Mirror Descent for MFGs has been introduced in (Hadikhanloo, 2017) for first-order, single-population MFG. Our results cover second order, MP-MFG. Traditional numerical methods for solving MFGs typically rely on a finite difference scheme introduced in (Achdou & Capuzzo-Dolcetta, 2010). This approach can be extended to solve MP-MFG, see (Achdou et al., 2017). However, to the best of our knowledge, there is no general convergence guarantees, nor has it been tested on examples with as many states as we consider. More recently, several numerical methods to solve MFGs based on machine learning tools have been proposed using either an analytical viewpoint (Al-Aradi et al., 2018; Carmona & Laurière, 2019a; Ruthotto et al., 2020; Cao et al.,
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2020; Lin et al., 2020) or a stochastic viewpoint (Fouque & Zhang, 2020; Carmona & Laurière, 2019b; Germain et al., 2019). To the best of our knowledge, these algorithms have not been proved to converge and seem applicable only under rather stringent conditions (on the structure or the regularity of the problem) and do not seem to be directly applicable to complex geometries due to boundary conditions. Last, the question of learning with multiple infinite populations of agents has also been studied recently in (Subramanian et al., 2018). The authors consider several groups where the agents cooperate among each group, which differs from our setting where all the agents compete.

7. Conclusion

We proposed Online Mirror Descent for MP-MFGs. We have proved that under appropriate monotonicity assumptions, OMD converges to a NE. Moreover, we considered multiple experimental benchmarks, some with hundreds of billions states, and compared extensively OMD to FP. OMD scales up remarkably well and consistently converges significantly faster than FP. An interesting direction of future work would be to study the rate of convergence of OMD. We have shown $O(1/t)$ rate for FP in MP-MFG but our technique does not extend to OMD. Empirically, we envision to extend this approach to a model-free setting with function approximation and address even larger problems.
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A. Separability + Monotonicity Imply Weak Monotonicity

Proof of Lemma 2. Let us assume that the reward is separable \( r^i(x^i, a^i, \mu) = \bar{r}^i(x^i, \mu) \) and that it follows the monotonicity condition: \( \forall \mu \neq \mu', \sum_{x \in X} (\mu^i(x^i) - \mu'^i(x^i)) (\bar{r}^i(x^i, \mu) - \bar{r}^i(x^i, \mu')) \leq 0 \). Then, we have:

\[
\begin{align*}
&\sum_{i=1}^{N_p} \left[ J^i(\pi, \mu^\pi) - J^i(\pi', \mu^\pi) - J^i(\pi, \mu^\pi') + J^i(\pi', \mu^\pi') \right] \\
= &\sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{(x^i, a^i) \in X \times A} \left[ \mu^i_n \pi^i_n(x^i|a^i) r^i(x^i, a^i, \mu^\pi_n) - \mu^i_n \pi^i_n(x^i|a^i) r^i(x^i, a^i, \mu^\pi'_n) \right. \\
&\left. - \mu^i_n \pi^i_n(x^i|a^i) r^i(x^i, a^i, \mu^\pi'_n) + \mu^i_n \pi^i_n(x^i|a^i) r^i(x^i, a^i, \mu^\pi'_n) \right] \\
= &\sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{(x^i, a^i) \in X \times A} \left( \mu^i_n \pi^i_n(x^i|a^i) - \mu^i_n \pi^i_n(x^i|a^i) \right) \left( r^i(x^i, a^i, \mu^\pi_n) - r^i(x^i, a^i, \mu^\pi'_n) \right) \\
= &\sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x \in X} \left( \mu^i_n \pi^i_n(x^i) - \mu^i_n \pi^i_n(x^i) \right) \left( \bar{r}^i(x^i, \mu^\pi_n) - \bar{r}^i(x^i, \mu^\pi'_n) \right) \leq 0.
\end{align*}
\]

With a similar proof, we obtain the corresponding property with strict inequality.

B. Multi-Population Reward

Let us suppose:

\[
r^i(x^i, a^i, \mu) = \bar{r}^i(x^i, a^i) + \bar{r}^i(x^i, \mu) + \sum_{j \neq i} \mu^j(x^j) \bar{r}^{i,j}(x^i)
\]

With \( \forall x \in X, \bar{r}^{i,j}(x) = -\bar{r}^{j,i}(x) \) and if \( \forall \mu \neq \mu', \forall x, \sum_{x \in X} \left( \mu^i(x^i) - \mu'^i(x^i) \right) \left( \bar{r}^i(x^i, \mu^i) - \bar{r}^i(x^i, \mu'^i) \right) \leq 0 \).

\[
\begin{align*}
&\sum_{i} \sum_{x \in X} \left( \mu^i(x^i) - \mu'^i(x^i) \right) \left( \bar{r}^i(x^i, \mu) - \bar{r}^i(x^i, \mu') \right) \\
= &\sum_{i} \sum_{x \in X} \left( \mu^i(x^i) - \mu'^i(x^i) \right) \left( \bar{r}^i(x^i, \mu) + \sum_{j \neq i} \mu^j(x^j) \bar{r}^{i,j}(x^i) - \bar{r}^i(x^i, \mu^i) - \sum_{j \neq i} \mu^j(x^j) \bar{r}^{i,j}(x^i) \right) \\
= &\sum_{i} \sum_{x \in X} \left( \mu^i(x^i) - \mu'^i(x^i) \right) \left( \bar{r}^i(x^i, \mu) - \bar{r}^i(x^i, \mu^i) \right) + \sum_{j \neq i} \sum_{x \in X} \left( \mu^j(x^j) - \mu'^j(x^j) \right) \left( \mu^i(x^i) - \mu'^i(x^i) \right) \bar{r}^{i,j}(x^i) \\
\leq &0 \quad \text{(since } \bar{r}^{i,j}(x) = -\bar{r}^{j,i}(x) \text{)}
\end{align*}
\]

\( \leq 0 \)
C. Fictitious Play

In this section, we prove that under the weak monotonicity condition, the Fictitious Play process converges to a NE. First, we prove the following property, which stems from the weak monotonicity.

**Property 1.** Let \( f \) be a smooth enough function and let assume that the ODE \( \dot{\rho} = f(\rho) \) (with \( \dot{\rho} = \frac{d}{dt}\rho \)) has a solution \( (\rho^i_t(x))_{i \geq 0, x \in X} \). If the game is weakly monotone, then:

\[
\sum_{i=1}^{N_p} \sum_{x^i, a^i \in X \times A} \langle \nabla \rho r^i(x^i, a^i, \rho), \dot{\rho} \rangle \rho^i(x^i, a^i) \leq 0.
\]

**Proof.** The monotonicity condition implies that, for all \( \tau \geq 0 \), we have:

\[
\sum_{i=1}^{N_p} \sum_{x^i, a^i \in X \times A} (\rho^i_t(x^i, a^i) - \rho^i_{t+\tau}(x^i, a^i)) (r^i(x^i, a^i, \rho_t) - r^i(x^i, a^i, \rho_{t+\tau})) \leq 0.
\]

Thus:

\[
\sum_{i=1}^{N_p} \sum_{x^i, a^i \in X \times A} \frac{\rho^i_t(x^i, a^i) - \rho^i_{t+\tau}(x^i, a^i)}{\tau} r^i(x^i, a^i, \rho_t) - r^i(x^i, a^i, \rho_{t+\tau}) \leq 0.
\]

The result follows when \( \tau \to 0 \). \( \square \)

In the space of distributions over state actions, the Fictitious Play process can be expressed as follows. First, we start with a distribution \( \rho_{n,t}^i \) following the balance equation on the state action distributions:

\[
\sum_{a^i \in A} \rho_{n-1,t}^i(a^i, x^i) = \sum_{x^i, a^i \in X \times A} p(x^i | x^i, a^i) \rho_{n,t}^i(x^i, a^i).
\]

And for \( t < 1 \), the policy \( \pi_{n,t}^i(a^i | x^i) = \frac{\rho_{n,t}^i(x^i, a^i)}{\sum_{a^i \in A} \rho_{n,t}^i(x^i, a^i)} \) is the uniform policy whenever \( \sum_{a^i \in A} \rho_{n,t}^i(x^i, a^i) > 0 \).

A best response state action distribution to \( \rho \) is written \( \rho_{n,t}^{i,br}(x^i, a^i) \) (which will be assumed to be equal to \( \rho_t \) for \( t < 1 \)) and finally the FP process on the state action distribution is written as for all \( t \geq 1 \):

\[
\rho_{n,t}^i(x^i, a^i) = \frac{1}{t} \int_0^t \rho_{n,s}^{i,br}(x^i, a^i) ds.
\]

The exploitability can then be written as:

\[
\phi(t) = \max_{\rho} \left[ \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i, a^i \in X \times A} \rho_n^i(x^i, a^i) r^i(x^i, a^i, \rho_n, t) - \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i, a^i \in X \times A} \rho_n^i(x^i, a^i) r^i(x^i, a^i, \rho_n, t) \right]
\]

**Property 2.** We have that \( \frac{d}{dt} \rho_{n,t}^i(x^i, a^i) = \frac{1}{t} \left[ \rho_{n,s}^{i,br}(x^i, a^i) - \rho_{n,s}^i(x^i, a^i) \right] \) by taking the derivative of \( \rho_{n,s}^i(x^i, a^i) = \frac{1}{t} \int_0^t \rho_{n,s}^{i,br}(x^i, a^i) ds \) on both sides.

Finally, we take the derivative of the exploitability and get:
Online Mirror Descent for Mean Field Games

\[
\frac{d}{dt} \phi(t) = \frac{d}{dt} \max_\rho \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \rho_n^i(x^i, a^i) r^i(x^i, a^i, \rho_{n,t}) \right] - \frac{d}{dt} \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \rho_n^i(x^i, a^i) r^i(x^i, a^i, \rho_{n,t}) \right]
\]

\[
= \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i, a^i \in X \times A} \rho_{n,t}^{i,b}(x^i, a^i) \frac{dt}{dt} \left( r^i(x^i, a^i, \rho_{n,t}) \right) \right]
- \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i, a^i \in X \times A} \left( \rho_{n,t}^i(x^i, a^i) \frac{dt}{dt} \left( r^i(x^i, a^i, \rho_{n,t}) \right) + r^i(x^i, a^i, \rho_{n,t}) \frac{dt}{dt} \left( \rho_{n,t}^i(x^i, a^i) \right) \right) \right]
\]

\[
= \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i, a^i \in X \times A} \left( \rho_{n,t}^{i,b}(x^i, a^i) - \rho_{n,t}^i(x^i, a^i) \right) \frac{dt}{dt} \left( r^i(x^i, a^i, \rho_{n,t}) \right) \right]
- \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i, a^i \in X \times A} r^i(x^i, a^i, \rho_{n,t}) \frac{dt}{dt} \left( \rho_{n,t}^i(x^i, a^i) \right) \right]
\]

\[
= \frac{t}{N_p} \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i, a^i \in X \times A} \left[ \frac{dt}{dt} \left( \rho_{n,t}^i(x^i, a^i) \right) \right] \langle \nabla_{\rho} r^i(x^i, a^i, \rho_{n,t}), \delta \rho_{n,t} \rangle \leq 0
\]

\[
- \frac{1}{t} \left[ \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i, a^i \in X \times A} \left( \rho_{n,t}^{i,b}(x^i, a^i) - \rho_{n,t}^i(x^i, a^i) \right) r^i(x^i, a^i, \rho_{n,t}) \right]
\]

\[
\leq - \frac{1}{t} \phi(t).
\]
D. Online Mirror Descent Dynamics

Proof of Lemma 3. The Continuous Time Online Mirror Descent (CTOMD) algorithm is defined as: for all $t > 0, i \in \{1, \ldots, N_p\}, n \in \{0, \ldots, N\}$,
\[
y_{n,t}^i(x^i, a^i) = \int_0^t Q_n^{i,\pi^*_n,\mu^*_n}(x^i, a^i) ds,
\]
\[
\pi_{n,t}^i(\cdot|x^i) = \Gamma(y_{n,t}^i(x^i, \cdot)).
\]

\[
\frac{d}{dt} H(y_t) = \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) \left[ h_n^{i,\pi_n^*}(y_{n,t}^i(x^i, \cdot)) - h_n^{i,*}(y_{n,t}^i(x^i, \cdot)) - \langle \pi_{n,t}^i, y_{n,t}^i(x^i, \cdot) - y_{n,t}^{i,s}(x^i, \cdot) \rangle \right]
\]
\[
= \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) \left[ h_n^{i,\pi_n^*}(y_{n,t}^i(x^i, \cdot)) - h_n^{i,*}(y_{n,t}^i(x^i, \cdot)) - \langle \pi_{n,t}^i, y_{n,t}^i(x^i, \cdot) - y_{n,t}^{i,s}(x^i, \cdot) \rangle \right]
\]
\[
= \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) \left[ \pi_n^{i,s}(\cdot|x^i) - \pi_n^{i,*}(\cdot|x^i), Q_n^{i,\mu_n^*}(x^i, \cdot) \right]
\]
\[
= \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) \left[ V_n^{i,\pi_n^*,\mu_n^*}(x^i) - \langle \pi_n^{i,s}(\cdot|x^i), Q_n^{i,\mu_n^*}(x^i, \cdot) \rangle \right]
\]
\[
= \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) \left[ \sum_{x^i \in X} V_n^{i,\pi_n^*,\mu_n^*}(x^i) - \pi_n^{i,s}(\cdot|x^i), r^i(x^i, \cdot, \mu_n^*) \right.
\]
\[
- \left[ \sum_{x^i \in X} \pi_n^{i,*}(x^i) \pi_n^{i,s}(x^i, \cdot|x^i) p(x^i|x^i, a^i) \right]
\]
\[
= \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) V_n^{i,\pi_n^*,\mu_n^*}(x^i) - \sum_{n=0}^N \sum_{x^i \in X} \pi_n^{i,s}(x^i, \cdot|x^i) p(x^i|x^i, a^i)
\]

\[
= \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) V_n^{i,\pi_n^*,\mu_n^*}(x^i) - \sum_{n=0}^N \sum_{x^i \in X} V_n^{i,\pi_n^*,\mu_n^*}(x^i) \mu_{n+1}^{i,\pi_n^*}(x^i)
\]
\[
- \sum_{i=1}^{N_p} \sum_{n=0}^N \sum_{x^i \in X} \mu_n^{i,\pi_n^*}(x^i) \langle \pi_n^{i,s}(\cdot|x^i), r^i(x^i, \cdot, \mu_n^*) \rangle
\]
\[
= J(\pi_t, \pi^*) - J(\pi^{i,s}, \mu^*)
\]
\[
= \Delta J(\pi_t, \pi^*) + \tilde{M}(\pi_t, \pi^*).
\]
E. Weak monotonicity implies $\dot{M} \leq 0$

Proof of Lemma 1. Consider two policies $\pi, \pi'$. Denote by $\mu = \mu^\pi, \mu' = \mu^\pi'$ respectively the induced distribution sequences. Let $\rho, \rho'$ be the associated joint distribution sequences:

$$\rho_i^n(x^i, a^i) = \mu_i^n(x^i)\pi_i^n(a^i|x^i)$$

and likewise for $\rho'$. By the weak monotonicity, we have:

$$0 \geq \sum_i \sum_{(x^i, a^i) \in X \times A} (\rho_i^n(x^i, a^i) - \rho'_i^n(x^i, a^i))(r^i(x^i, a^i, \mu_n) - r^i(x^i, a^i, \mu'_n)) = \Delta J(\pi, \pi') + \Delta J(\pi', \pi), \quad (8)$$

with

$$\Delta J(\pi, \pi') = \sum_i \sum_{(x^i, a^i) \in X \times A} (\rho_i^n(x^i, a^i) - \rho'_i^n(x^i, a^i))r^i(x^i, a^i, \mu_n),$$

and

$$\Delta J(\pi', \pi) = \sum_i \sum_{(x^i, a^i) \in X \times A} (\rho'_i^n(x^i, a^i) - \rho_i^n(x^i, a^i))r^i(x^i, a^i, \mu'_n).$$

From here, we deduce (2). Similarly, the strictly weak monotonicity implies a strict inequality in (2).

\hfill $\Box$

F. Strictly weak monotonicity implies uniqueness

Proof of Lemma 1. Consider a strictly weakly monotone game. For the sake of contradiction, assume that there exist two different Nash equilibria, say $\pi, \pi'$. Proceeding as in the proof of Lemma 1, we obtain (8) with a strict inequality.

Note that $\Delta J(\pi, \pi')$ corresponds to the difference between the reward of a typical player following $\pi$ when the population follows $\pi$ and the reward of a typical player following $\pi'$ when the population still follows $\pi$, and vice versa for $\Delta J(\pi', \pi)$. Moreover, $\pi, \pi'$ are Nash equilibria, so we deduce that these two terms are non-negative, which yields a contradiction with (8).

\hfill $\Box$
G. Online Mirror descent convergence

Proof of Theorem 2. Let $\Xi$ be defined as:

$$
\Xi(\pi^*, \pi) := \sum_{i=1}^{N_p} \sum_{n=0}^{N} \sum_{x^i \in X} \mu_n^i(x^i)(D_h(\pi^*_n(x^i, \cdot), \pi_n^i(x^i, \cdot))).
$$

We pick $\pi \in \Delta A$. If $\Delta J(\pi, \pi^*) + \tilde{M}(\pi, \pi^*) = 0$ then, $\tilde{M}(\pi, \pi^*) = 0$ and we can deduce that $\mu^* = \mu^{\pi^*}$. This implies that $\pi$ is a Nash as $\pi$ and $\pi^*$ share the same distribution and thus the reward of a best response against $\pi$ or $\pi^*$ will be the same.

Let us suppose now that $\pi$ is a Nash and $\Delta J(\pi, \pi^*) + \tilde{M}(\pi, \pi^*) < 0$, then $\sum_{i=1}^{N_p} J_i(\pi^i, \mu^*) - J_i(\pi^{i*, \mu^*}) < 0$ meaning that there exists an $i$ such that $J_i(\pi^i, \mu^*) - J_i(\pi^{i*, \mu^*}) < 0$. But as $\pi$ is a Nash, for all $\pi'$, $i$, we have $J_i(\pi^i, \mu^*) - J_i(\pi^{i', \mu^*}) \geq 0$ which is a contradiction.

Hence, if $\Delta J(\pi, \pi^*) + \tilde{M}(\pi, \pi^*) < 0$ then $\pi$ is not a Nash.

This proves that the Bregman divergence $\min_{\pi^* \in \text{Nash}} \Xi(\pi^*, \cdot)$ is a strict Lyapunov function of the CTOMD system. Hereby, $\pi_t$ converges to the set of Nash equilibria.

Related to the Hypothesis in Theorem 2, we can show the following:

Lemma 4. If a MP-MFG satisfies $\tilde{M}(\pi, \pi') < 0$ if $\mu^* \neq \mu'^*$ and 0 otherwise, then there is at most one Nash equilibrium distribution.

Note that uniqueness of the equilibrium distribution does not imply uniqueness of the equilibrium policy. This implication holds however under extra assumptions (e.g., some kind of strict convexity of the cost function).

Proof. Consider a MP-MFG satisfying the assumption. Consider two Nash equilibria, say $\pi$, $\pi'$. For the sake of contradiction, assume that they generate two different distributions $\mu^*$, $\mu^{\pi'}$. We have:

$$
0 > \tilde{M}(\pi, \pi')
= \sum_{i=1}^{N_p} \left[ J_i(\pi^i, \mu^*) + J_i(\pi^{i'}, \mu^{\pi'}) - J_i(\pi^{i*, \mu^*}) - J_i(\pi^{i*, \mu^{\pi'}}) \right]
= \sum_{i=1}^{N_p} \left[ J_i(\pi^i, \mu^*) - J_i(\pi^{i'}, \mu^{\pi'}) \right] + \sum_{i=1}^{N_p} \left[ J_i(\pi^{i*, \mu^*}) - J_i(\pi^{i*, \mu^{\pi'}}) \right]
$$

where both terms are non-negative because $\pi$ and $\pi'$ are Nash equilibria. Hence we must have $\mu^* = \mu^{\pi'}$. 

\qed
H. Numerical Experiments

H.1. Garnet

Figure 6. Garnet Experiments performances
H.2. Building experiment

H.2.1. Building experiment performances

The full building evacuation dynamics over the 20 floors is presented in Figure 8 below.

Figure 7. Building Experiment performances

H.2.2. Building experiment solution

The full building evacuation dynamics over the 20 floors is presented in Figure 8 below.
H.3. Crowd motion with randomly shifted point of interest

In this section, we discuss how to extend our results to the case of multi-population MFGs with common noise. In the example of Section 5.3, the common noise corresponds to the geographical shifts of the point of interest.

The action space and the state space are the same but the dynamics and the reward are affected by a common noise sequence \( \{\xi_n\}_{0 \leq n \leq N} \). We denote \( \Xi_n := \{\xi_k\}_{0 \leq k < n} = \Xi_{n-1} - \xi_{n-1} \) the concatenation of the sequence \( \Xi_{n-1} \) and the new noise \( \xi_{n-1} \). By convention, we denote by \( \Xi_0 \) the empty sequence \( \{\} \). \( |\Xi_n| = n \) represents the total length of the sequence. The distribution of \( \xi_n \) given the past sequence \( \Xi_n \) is denoted by \( P(\cdot|\Xi_n) \). Here, \( \xi \) plays the role of a source of randomness which affects both the reward \( r(x, a, \mu, \xi) \) and the probability transition function \( p(x'|x, a, \xi) \). It appears on top of the idiosyncratic randomness affecting each player. Policies and population distributions are now functions of the common noise and denoted respectively by \( \pi_n^i(a|x, \Xi) \) and \( \mu_n^i(x|\Xi) \) for population \( i \). We will sometimes simply write \( \pi_n^i|\Xi (a|x) \) and \( \mu_n^i|\Xi (x) \). Notice that the common noise is shared by all populations (we could also, with a slight modification, consider noises which are common to players of a given population and not shared with other populations).

For each population, the evolution of the distribution is conditioned on the realization of the common noise. It satisfies the forward equation: for all \( x^i \in \mathcal{X} \), \( \mu_{0|\Xi_0}^i(x) = \mu_0^i(x) \) and for all \( x'^i \in \mathcal{X} \),

\[
\mu_{n+1|\Xi_n}^i(x'^i) = \sum_{(x^i, a^i) \in \mathcal{X} \times A} \pi_n^i(a^i|x^i, \Xi_n) p(x'^i|x^i, a^i, \xi_n) \mu_n^i|\Xi_n (x^i)
\]

for \( n \leq N - 1 \). We denote \( \mu^\Xi = (\mu_{i|\Xi}^i)_{i \in \{1, \ldots, N\}} \).

The expected total reward for a representative player of population \( i \) using policy \( \pi^i \) and facing the crowd behavior given by \( \mu \) is:

\[
J^i(\pi^i, \mu) = \mathbb{E} \left[ \sum_{n=0}^{N} r^i(x_n^i, a_n^i, \mu_n|\Xi_n, \xi_n) \mid x_0^i \sim \mu_0^i, a_0^i \sim \pi_0^i(\cdot|\xi_n), x_{n+1}^i \sim p(\cdot|x_n^i, a_n^i, \xi_n), \xi_n \sim P(\cdot|\Xi_n) \right].
\]

**Continuous time Online Mirror Descent for MP-MFGs with common noise:**

In this setting, the Continuous Time Online Mirror Descent (CTOMD) algorithm is defined as: for all \( i \in \{1, \ldots, N_p\}, n \in \{0, \ldots, N\}, y_n^{i, 0} = 0 \), and for all \( t \in \mathbb{R}_+ \),

\[
y_{n,t}^i(x^i, a^i|\Xi_n) = \int_0^t Q_{\Xi_n}^i(x^i, a^i|\Xi_n) ds, \quad (9)
\]

\[
\pi_{n,t}^i(x^i, \Xi_n) = \Gamma(y_{n,t}^i(x^i, |\Xi_n)). \quad (10)
\]

Our theoretical results naturally extend to this setting by following similar arguments as the ones in (Perrin et al., 2020).
Online Mirror Descent for Mean Field Games

Figure 8. Building Experiment solution (ground floor on the upper left corner)
Online Mirror Descent for Mean Field Games

H.4. Multi-population

H.4.1. Monotony of the multi-population reward

We prove rigorously that the MP-MFG reward is monotone. As \( \tilde{r}(x, a) = 0 \), the separability condition is trivially verified. Furthermore, we have:

\[
\sum_i \sum_{x \in X} (\mu^i(x) - \mu'^i(x))(\hat{r}^i(x, \mu) - \hat{r}^i(x, \mu')) = \sum_i \sum_{x \in X} (\mu^i(x) - \mu'^i(x))(-\log(\mu^i(x)) + \sum_{j \neq i} \mu^j(x)\tilde{r}^{j,i}(x) + \log(\mu^i(x)) - \sum_{j \neq i} \mu^j(x)\tilde{r}^{j,i}(x))
\]

\[
= \sum_i \sum_{x \in X} (\mu^i(x) - \mu'^i(x))(-\log(\mu^i(x)) + \log(\mu'^i(x)) + \sum_{j \neq i} \mu^j(x)\tilde{r}^{j,i}(x) - \sum_{j \neq i} \mu^j(x)\tilde{r}^{j,i}(x));
\]

where we have:

- (1) \( \leq 0 \) because \( \forall x, \forall i, (\mu^i(x) - \mu'^i(x))(-\log(\mu^i(x)) + \log(\mu'^i(x))) \leq 0 \) as \( \log \) is an increasing function;
- (2) = 0 because \( \forall i, \forall j, i \neq j, \tilde{r}^{j,i}(x) = -\tilde{r}^{i,j}(x) \).

Thus,

\[
\sum_i \sum_{x \in X} (\mu^i(x) - \mu'^i(x))(\hat{r}^i(x, \mu) - \hat{r}^i(x, \mu')) \leq 0.
\]

H.4.2. Multi-population performances

The performances of Fictitious Play and Online Mirror Descent for the multi-population chasing Mean Field Game with different field topologies and initial distribution are presented in Figure 9.
Figure 9. Multi-population experiments, performances with different topologies