On subdirect products of type $FP_n$ of limit groups over Droms RAAGs

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Abstract

We generalise some known results for limit groups over free groups and residually free groups to limit groups over Droms RAAGs and residually Droms RAAGs, respectively. We show that limit groups over Droms RAAGs are free-by-(torsion-free nilpotent). We prove that if $S$ is a full subdirect product of type $FP_s(\mathbb{Q})$ of limit groups over Droms RAAGs with trivial center, then the projection of $S$ to the direct product of any $s$ of the limit groups over Droms RAAGs has finite index. Moreover, we compute the growth of homology groups and the volume gradients for limit groups over Droms RAAGs in any dimension and for finitely presented residually Droms RAAGs of type $FP_m$ in dimensions up to $m$. In particular, this gives the values of the analytic $L^2$-Betti numbers of these groups in the respective dimensions.

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1. Introduction

The class of right-angled Artin groups (RAAGs) generalises the class of finitely generated free groups, by allowing relations saying that some of the generators commute. In recent years, RAAGs have been extensively studied due to their rich structure, in particular, due to the variety of their subgroups. For example, it follows from works of I. Agol ([2]) and D. Wise ([28]) that fundamental groups of closed, irreducible, hyperbolic 3-manifolds are virtually subgroups of RAAGs.

In general, not all subgroups of RAAGs are again RAAGs, and Droms RAAGs are precisely those RAAGs with the property that all of their finitely generated subgroups are again RAAGs. In particular, finitely generated free groups are Droms RAAGs. Droms RAAGs can also be described as the ones where the defining graph does not contain induced squares or straight line paths with 3 edges (see [18]). Alternatively, the class of Droms RAAGs can be described as the $\mathbb{Z}*$-closure of $\mathbb{Z}$ (see Section 2 for the definition).
The class of limit groups over Droms RAAGs extends the class of limit groups over free groups. The theory of limit groups over free groups was developed by O. Kharlampovich, A. Miasnikov, under the name of fully residually free groups, and by Z. Sela, as a method for obtaining information on solutions of systems of equations over free groups. The class of limit groups over free groups contains all finitely generated free abelian groups and surface groups of Euler characteristic at most \(-2\). Limit groups over free groups are commutative transitive and, as shown by F. Dahmani ([15]), they satisfy the Howson property.

The study of limit groups over Droms RAAGs was initiated by M. Casals–Ruiz, A. Duncan and I. Kazachkov, in [11, 12], with the aim of studying systems of equations over RAAGs. A group \(\Gamma\) is a limit group over a Droms RAAG if it is precisely a finitely generated group that is fully residually a Droms RAAG, that is, for every finite subset \(S\) of \(\Gamma\) there is a Droms RAAG \(G\) together with a homomorphism of groups \(\varphi : \Gamma \mapsto G\) whose restriction to \(S\) is injective.

Limit groups over Droms RAAGs share many properties with limit groups over free groups, including homological and homotopical finiteness properties. For example, Casals–Ruiz, Duncan and Kazachkov showed in [11] that limit groups over coherent RAAGs are finitely presented and they are subgroups of ICE groups over coherent RAAGs (see Definition 3·8). In particular, this holds for limit groups over Droms RAAGs and implies that limit groups over Droms RAAGs are of type \(FP_\infty\), and that their finitely generated subgroups are again limit groups over Droms RAAGs. In [10] M. Bridson and H. Wilton proved that finitely presentable subgroups of finitely generated residually free groups are separable and that the subgroups of type \(FP_\infty\) are virtual retracts. Though limit groups over Droms RAAGs share many properties with limit groups over free groups, there are properties like commutative transitivity and the Howson property that do not hold for a general Droms RAAG. For example, by [16], the RAAGs that satisfy the Howson property do not hold for a general Droms RAAG. For example, by [16], the RAAGs that satisfy the Howson property are precisely the free products of abelian groups, so in particular, are limit groups over free groups.

In [21] D. Kochloukova showed that limit groups over free groups are free-by-(torsion-free nilpotent). Our first result is a generalisation of this fact for limit groups over Droms RAAGs. The rest of the results in this paper will be heavily dependent on this fact.

**Proposition 3·16.** Every limit group over a Droms RAAG is free-by-(torsion-free nilpotent).

Let us recall some definitions. Let \(S\) be a subgroup of a direct product \(G_1 \times \cdots \times G_n\). We say that \(S\) is full if \(S \cap G_i \neq 1\) for all \(i \in \{1, \ldots, n\}\) and \(S\) is a subdirect product if \(p_i(S) = G_i\) for all \(i \in \{1, \ldots, n\}\), where \(p_i\) is the projection map \(G_1 \times \cdots \times G_n \mapsto G_i\).

G. Baumslag, A. Miasnikov, V. Remeslennikov proved in [4] that every finitely generated residually Droms RAAG is a subgroup of a direct product of finitely many limit groups over Droms RAAGs. The study of subdirect products of free groups was initiated by Baumslag and J. Roseblade in [5]. In a sequence of papers that culminated in [7, 8], Bridson, J. Howie, C. Miller and H. Short studied subgroups of the direct product of limit groups over free groups. One of the results is that if \(\Gamma_1, \ldots, \Gamma_n\) are non-abelian limit groups over free groups and \(S\) is a full subdirect product of \(\Gamma_1 \times \cdots \times \Gamma_n\), then \(S\) is of type \(FP_\infty(\mathbb{Q})\) if and only if \(p_{i,j}(S)\) is of finite index in \(\Gamma_i \times \Gamma_j\) for all \(1 \leq i < j \leq n\), where \(p_{i,j}\) denotes the projection map \(S \mapsto \Gamma_i \times \Gamma_j\). These results were generalised by J. Lopez de Gamiz Zearr in [23] to the class of limit groups over Droms RAAGs (see Section 2·2). In addition, in [21] Kochloukova generalised the previous result concerning limit groups over free groups by showing that if
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S is of type $FP_s$, then it virtually surjects onto $s$ coordinates. However, the converse is still an open problem.

We recall that a group $S$ is of homological type $FP_s(R)$ (or it is $FP_s$ over $R$) for a commutative, associative ring $R$ with identity element if the trivial $RG$-module $R$ has a projective resolution with all modules finitely generated in dimensions up to $s$. A group is of type $FP_s$ if it is $FP_s$ over $\mathbb{Z}$. The aim of Section 4 is to extend the above results to the class of limit groups over Droms RAAGs.

**Theorem 1.1.** Let $G_1, \ldots, G_m$ be limit groups over Droms RAAGs such that each $G_i$ has trivial center and let

$$S < G := G_1 \times \cdots \times G_m$$

be a finitely generated full subdirect product. Suppose further that, for some fixed number $s \in \{2, \ldots, m\}$, for every subgroup $S_0$ of finite index in $S$ the homology group $H_i(S_0, \mathbb{Q})$ is finite dimensional (over $\mathbb{Q}$) for all $i \leq s$. Then for every canonical projection

$$p_{j_1, \ldots, j_s} : S \mapsto G_{j_1} \times \cdots \times G_{j_s}$$

the index of $p_{j_1, \ldots, j_s}(S)$ in $G_{j_1} \times \cdots \times G_{j_s}$ is finite.

In particular, if $S$ is of type $FP_s$ over $\mathbb{Q}$, then for every canonical projection $p_{j_1, \ldots, j_s}$ the index of $p_{j_1, \ldots, j_s}(S)$ in $G_{j_1} \times \cdots \times G_{j_s}$ is finite.

In Section 5 we discuss the growth of homology groups and the volume gradients of limit groups over Droms RAAGs and of finitely presented residually Droms RAAGs. The growth of homology groups is measured by the limit

$$\lim_{n \to \infty} \dim_K H_i(B_n, K)/[G : B_n],$$

whenever this limit exits. Here $K$ is a field and $(B_n)$ is an exhausting normal chain of $G$, that is, a chain $(B_n)$ of normal subgroups of finite index such that $B_{n+1} \subseteq B_n$ and $\bigcap_n B_n = 1$. As every residually finite group has an exhausting normal chain, the same holds for limit groups over coherent RAAGs. Thanks to the approximation theorem of W. Lück we know that if $K$ has characteristic 0, $G$ is residually finite, finitely presented and of type $FP_m$, then the limit exists for $i < m$ and equals the $L^2$-Betti number $\beta_i(G)$. In particular, in this case the limit is independent of the exhausting normal chain.

A group $G$ of homotopical type $F_m$, $\text{vol}_m(G)$ was defined in [9] as the least number of $m$-cells among all classifying spaces $K(G,1)$ with finite $m$-skeleton. For instance, $\text{vol}_1(G)$ equals $d(G)$, where $d(G)$ is the minimal number of generators of $G$. The most studied volume gradient is the 1-dimensional one. That is, the rank gradient of $G$ with respect to $B_n$, which is defined to be $\lim_n d(B_n)/[G : B_n]$ and it is denoted by $\text{RG}(G, (B_n))$. Its study was initiated by M. Lackenby in [22] and a combinatorial approach to the rank gradient was developed by M. Abert, N. Nikolov and A. Jaikin–Zapirain in [1]. In general, the $m$-dimensional volume gradient of $G$ with respect to $(B_n)$ is

$$\lim_{n \to \infty} \text{vol}_m(B_n)/[G : B_n].$$

In [9] Bridson and Kochloukova calculated the above volume and homology type gradients for limit groups over free groups and in the case of residually free groups they found particular filtrations where the homology growth and the rank gradient can be calculated. The
results about homology growth in [9] and in this paper are independent of the characteristic of the field $K$.

In Section 5 we compute the homology growth and the volume gradients for a more general setting, namely, for limit groups over coherent RAAGs. Note that this class of groups is more general than the class of limit groups over Droms RAAGs. We use the approach from [9] and we show that limit groups over coherent RAAGs are slow above dimension 1, hence are $K$-slow above dimension 1. This implies the following results.

**THEOREM 1.2.** Let $G$ be a limit group over a coherent RAAG and let $(B_n)$ be an exhausting normal chain in $G$. Then:

(i) the rank gradient $\text{RG}(G, (B_n)) = \lim_{n \to \infty} \frac{d(B_n)}{[G : B_n]} = -\chi(G)$;

(ii) the deficiency gradient $\text{DG}(G, (B_n)) = \lim_{n \to \infty} \frac{\text{def}(B_n)}{[G : B_n]} = \chi(G)$;

(iii) the $k$-dimensional volume gradient $\lim_{n \to \infty} \frac{\text{vol}_k(B_n)}{[G : B_n]} = 0$ for $k \geq 2$.

**THEOREM 1.3.** Let $K$ be a field, $G$ a limit group over a coherent RAAG and $(B_n)$ an exhausting normal chain in $G$. Then:

(i) $\lim_{n \to \infty} \frac{\dim_K H_i(B_n, K)}{[G : B_n]} = -\chi(G)$;

(ii) $\lim_{n \to \infty} \frac{\dim_K H_j(B_n, K)}{[G : B_n]} = 0$ for all $j \geq 2$.

Using results from [9] we also compute the homology growth up to dimension $m$ and the volume gradients in low dimensions of residually Droms RAAGs of type $\text{FP}_m$ for $m \geq 2$. We cannot apply the same method to the class of residually coherent RAAGs since Theorem 1.1 is a key result in this method and in [14] it is proved that this no longer holds for coherent RAAGs.

**THEOREM 1.4.** Let $m \geq 2$ be an integer, let $G$ be a residually Droms RAAG of type $\text{FP}_m$, and let $\rho$ be the largest integer such that $G$ contains a direct product of $\rho$ non-abelian free groups. Then, there exists an exhausting sequence $(B_n)$ in $G$ so that for all fields $K$:

(i) if $G$ is not of type $\text{FP}_\infty$, then $\lim_{n \to \infty} \frac{\dim H_i(B_n, K)}{[G : B_n]} = 0$ for all $0 \leq i \leq m$;

(ii) if $G$ is of type $\text{FP}_\infty$ then for all $j \geq 1$,

$$\lim_{n \to \infty} \frac{\dim H_\rho(B_n, K)}{[G : B_n]} = (-1)^\rho \chi(G), \quad \lim_{n \to \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = 0 \quad \text{for all} \quad j \neq \rho.$$
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$$\text{RG}(G, (B_n)) = \lim_{n \to \infty} \frac{d(B_n)}{[G : B_n]} = 0.$$  

Furthermore, if $G$ is of type $\text{FP}_3$ but it is not commensurable with a product of two limit groups over Droms RAAGs, $(B_n)$ can be chosen so that the deficiency gradient $DG(G, (B_n)) = 0$.

2. Preliminaries

2.1. Right-angled Artin groups

Let us recall the definition of a right-angled Artin group. Given a finite simplicial graph $X$ with vertex set $V(X)$, the corresponding right-angled Artin group (RAAG), denoted by $G_X$, is given by the following presentation:

$$G_X = \langle V(X) \mid xy = yx \iff x \text{ and } y \text{ are adjacent} \rangle.$$  

Subgroups of RAAGs can be wild, in particular, not all subgroups of RAAGs are themselves RAAGs. RAAGs that have all their finitely generated subgroups again of this type are known as Droms RAAGs. Droms RAAGs can also be described as the ones where the defining graph does not contain induced squares or straight line paths with 3 edges (see [18]).

A group is coherent if all its finitely generated subgroups are also finitely presented. Droms proved that a RAAG is coherent if and only if its underlying graph contains no full subgraph isomorphic to an $n$-cycle, for $n \geq 4$, see [17]. In particular, Droms RAAGs are coherent RAAGs.

**Definition 2.1.** Let $\mathcal{G}$ be a class of groups. The $\mathbb{Z}*$-closure of $\mathcal{G}$, denoted by $\mathbb{Z}*(\mathcal{G})$, is the union of classes $(\mathbb{Z}*(\mathcal{G}))_k$ defined as follows. At level 0, the class $(\mathbb{Z}*(\mathcal{G}))_0$ equals $\mathcal{G}$. A group $G$ lies in $(\mathbb{Z}*(\mathcal{G}))_k$ if and only if

$$G \simeq \mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n),$$

where $m \in \mathbb{N} \cup \{0\}$ and the group $G_i$ lies in $(\mathbb{Z}*(\mathcal{G}))_{k-1}$ for all $i \in \{1, \ldots, n\}$.

The level of $G$, denoted by $l(G)$, is the smallest $k$ for which $G$ belongs to $(\mathbb{Z}*(\mathcal{G}))_k$.

In this terminology, Droms showed in [18] that the class of Droms RAAGs is the $\mathbb{Z}*$-closure of $\mathbb{Z}$. Analogously, it is the $\mathbb{Z}*$-closure of the class of finitely generated free groups. We use the latter when we fix the level of a Droms RAAG.

**Example 2.2.** If $G$ is a Droms RAAG such that $l(G) = 0$, then $G$ is a finite rank free group. If $l(G) = 1$, then

$$G \simeq \mathbb{Z}^m \times F,$$

for some $m \geq 1$, $F$ free of finite rank and $G$ is not $\mathbb{Z}$. If $l(G) = 2$, then

$$G \simeq \mathbb{Z}^m \times ((\mathbb{Z}^{n_1} \times F_{k_1}) \ast \cdots \ast (\mathbb{Z}^{n_l} \times F_{k_l})), $$

for some $m, n_i, k_i \in \mathbb{N} \cup \{0\}$, $\sum_i n_i \geq 1$, $l \geq 2$ and for $i \in \{1, \ldots, l\}$, $F_{k_i}$ free of finite rank $k_i$.  

2. Limit groups over Droms RAAGs

Definition 2.3. A group $\Gamma$ is a limit group over a Droms RAAG if it is precisely a finitely generated group that is fully residually a Droms RAAG, that is, for every finite subset $S$ of $\Gamma$ there is a Droms RAAG $G$ together with a homomorphism of groups $\varphi : \Gamma \mapsto G$ whose restriction to $S$ is injective.

Definition 2.4. If $G$ is a group, a limit group over $G$ is a group $H$ that is finitely generated and fully residually $G$, that is, for any finite set of non-trivial elements $S \subseteq H$ there is a homomorphism $\varphi : H \mapsto G$ which is injective on $S$.

Note that in Remark 2.4 $G$ is a fixed group but in Remark 2.3 $G$ actually depends on $S$.

Lemma 2.5. A group $\Gamma$ is a limit group over a Droms RAAG if and only if there is a Droms RAAG $G$ such that $\Gamma$ is a limit group over $G$.

Proof. It suffices to show that every group $\Gamma$ that satisfies Remark 2.3 is a limit group over $G$ for some appropriate Droms RAAG $G$. By definition, for any finite set of non-trivial elements $S \subseteq \Gamma$ there is a homomorphism $\varphi_S : \Gamma \mapsto G_S$ which is injective on $S$, where $G_S$ is a Droms RAAG. Then by substituting $G_S$ if necessary with $\text{im}(\varphi_S)$ (since $\text{im}(\varphi_S)$ is a Droms RAAG) we can assume that $d(G_S) \leq d(\Gamma)$. But since there are only finitely many Droms RAAGs with at most $d(\Gamma)$ generators, say $D_1, \ldots, D_t$, we conclude that $G_S \in \{D_1, \ldots, D_t\}$ and $G_S$ is a subgroup of the fixed Droms RAAG $G := D_1 \ast \cdots \ast D_t$.

By [4, theorem F1], groups that are residually Droms RAAGs are precisely finitely generated subgroups of a finite direct product of limit groups over Droms RAAGs. In this section we state the basic properties of residually Droms RAAGs and limit groups over Droms RAAGs that we need in later sections.

Classical limit groups, i.e. limit groups over free groups, admit a hierarchical structure. Limit groups over free groups are the finitely generated subgroups of $\omega$- residually free tower groups, known as $\omega$-rft groups (see [20, 25]). An $\omega$-rft space of height $h$ is defined by induction on $h$ and an $\omega$-rft group is the fundamental group of an $\omega$-rft space. A height $0$ space is the wedge of a finite collection of circles, closed hyperbolic surfaces and tori of arbitrary dimension, excluding the closed surface of Euler characteristic $-1$. An $\omega$-rft space $Y_h$ of height $h$ is obtained from an $\omega$-rft space $Y_{h-1}$ of height $h - 1$ by attaching a block of abelian or quadratic type (for details see [6, 25]). The classical height of a limit group (over a free group) $S$ is the minimal height of an $\omega$-rft group that has a subgroup isomorphic to $S$.

Limit groups over Droms RAAGs admit a hierarchical structure that comes from the fact that Droms RAAGs are the direct product of a free abelian group (possibly trivial) and a free product, so we can use the work of Casals–Ruiz and Kazachkov on limit groups over free products (see [13]) to obtain that hierarchy.

Proposition 2.6 ([8, proposition 2.1]). Let $G$ be a Droms RAAG such that $l(G) = 0$ (that is, a finitely generated free group) and let $\Gamma$ be a limit group over $G$ of height $h(\Gamma)$.

If $h(\Gamma) = 0$, then $\Gamma$ equals $M_1 \ast \cdots \ast M_j$ and $M_1, \ldots, M_j$ are free abelian groups or surface groups with Euler characteristic at most $-2$.

If $h(\Gamma) \geq 1$, then $\Gamma$ acts cocompactly on a tree $T$ where the edge stabilisers are trivial or infinite cyclic and the vertex groups are limit groups over $G$ of height at most $h(\Gamma) - 1$. Moreover, at least one of the vertex groups is nonabelian.
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**Proposition 2.7** ([13, theorem 8-2]). Let $G$ be a Droms RAAG with $l(G) \geq 1$ so that $G$ is of the form $\mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n)$ and $l(G_i) \leq l(G) - 1$ for $i \in \{1, \ldots, n\}$ and let $\Gamma$ be a limit group over $G$. Then, $\Gamma$ is $\mathbb{Z}^l \times \Lambda$ where $\Lambda$ is a limit group over $G_1 \ast \cdots \ast G_n$ and if $m = 0$, then $l = 0$. If $h(\Lambda) = 0$, then

$$\Lambda = A_1 \ast \cdots \ast A_j,$$

where for each $t \in \{1, \ldots, j\}$, $A_t$ is a limit group over $G_i$ for some $i \in \{1, \ldots, n\}$.

If $h(\Lambda) \geq 1$, then $\Lambda$ acts cocompactly on a tree $T$ where the edge stabilisers are trivial or infinite cyclic and the vertex groups are limit groups over $G_1 \ast \cdots \ast G_n$ of height at most $h(\Lambda) - 1$. Moreover, at least one of the vertex groups has trivial center.

We observe that the notion of height in limit groups over Droms RAAGs given in Proposition 2.7 is a natural generalisation of the notion of height in limit groups over free groups given in Proposition 2.6 except that, for historical reasons, the definition of limit groups over free groups of height 0 includes more than free products of free abelian groups and allows the free factors to be surface groups with Euler characteristic at most $-2$.

In [23], following the paper [8], the results concerning subgroups of direct products of limit groups over free groups were generalised to the case of limit groups over Droms RAAGs.

**Theorem 2.8** ([23, theorem 3-1]). If $\Gamma_1, \ldots, \Gamma_n$ are limit groups over Droms RAAGs and $S$ is a subgroup of $\Gamma_1 \times \cdots \times \Gamma_n$ of type $\text{FP}_n(\mathbb{Q})$, then $S$ is virtually a direct product of limit groups over Droms RAAGs.

**Theorem 2.9** ([23, theorem 8-1]). Let $\Gamma_1, \ldots, \Gamma_n$ be limit groups over Droms RAAGs such that each $\Gamma_i$ has trivial center and let $S < \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated full subdirect product. Then either:

(i) $S$ is of finite index; or

(ii) $S$ is of infinite index and has a finite index subgroup $S_0 < S$ such that $H_j(S_0, \mathbb{Q})$ has infinite dimension for some $j \leq n$.

Another result that will be used later is associated to finitely presented residually Droms RAAGs. In order to state the result, we introduce the following definition: an embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ of a finitely generated group $S$ that is residually a Droms RAAG as a full subdirect product of limit groups over Droms RAAGs is neat if $\Gamma_0$ is abelian (possibly trivial), $S \cap \Gamma_0$ is of finite index in $\Gamma_0$ and $\Gamma_i$ has trivial center for $i \in \{1, \ldots, n\}$.

**Theorem 2.10** ([23, theorem 10-1]). Let $S$ be a finitely generated group that is residually a Droms RAAG. The following are equivalent:

(i) $S$ is finitely presentable;

(ii) $S$ is of type $\text{FP}_2(\mathbb{Q})$;

(iii) $\dim H_2(S_0, \mathbb{Q})$ is finite for all subgroups $S_0 < S$ of finite index;

(iv) there exists a neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ into a product of limit groups over Droms RAAGs such that the image of $S$ under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $0 \leq i < j \leq n$.
for every neat embedding \( S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n \) into a product of limit groups over Droms RAAGs the image of \( S \) under the projection to \( \Gamma_i \times \Gamma_j \) has finite index for \( 0 \leq i < j \leq n \).

3. Limit groups over Droms RAAGs are free-by-(torsion-free nilpotent)

**Definition 3.1.** A graph is called **triangulated** if it contains no induced copy of \( C_n \), for \( n \geq 4 \), where \( C_n \) is the circle with \( n \) vertices.

Recall that a RAAG is coherent precisely when its corresponding graph is triangulated, see [17].

**Theorem 3.2** ([19, theorem 2]). If \( G \) is the RAAG corresponding to the graph \( X \), the commutator subgroup \( G' \) is free if and only if \( X \) is triangulated.

**Lemma 3.3** Coherent RAAGs are free-by-(free abelian).

**Proof.** Let \( G \) be a coherent RAAG corresponding to the graph \( X \). Then, \( G' \) is free, the abelianization is \( \mathbb{Z}^n \) where \( n \) is the number of vertices in the graph \( X \) and there is a short exact sequence

\[
1 \to G' \to G \to \mathbb{Z}^n \to 1.
\]

A filtration \( \{G_i\}_{i \geq 1} \) of normal subgroups of a group \( G \) has property (i) if \( G/G_i \) is torsion-free and \( \bigcap_{i \geq 1} G_i = 1 \), and it has property (ii) if \( \bigcap_{i \geq 1} G_i = 1 \) and for each finitely generated abelian subgroup \( M \) of \( G \) there is \( i = i(M) \) such that \( G_i \cap M = 1 \).

**Lemma 3.4.** Let \( G \) be a group and \( \{G_i\}_{i \geq 1} \) be a filtration of normal subgroups of \( G \). If \( \{G_i\}_{i \geq 1} \) has property (i), it also has property (ii).

**Proof.** Let \( M \) be a finitely generated abelian subgroup of \( G \). Since \( \bigcap_{i \geq 1} G_i = 1 \), there is \( i \) such that \( G_i \cap M \) is not \( M \). The group \( M/(M \cap G_i) \) embeds in \( G/G_i \), so it is non-trivial and torsion-free, that is, a finite rank free abelian group. As \( M \) has finite Hirsch length, we deduce that \( M \cap G_j \) is trivial for sufficiently large \( j \).

**Lemma 3.5.** Let \( H \) be a group, let \( G = \mathbb{Z}^m \times H \) for some \( m \in \mathbb{N} \) and let us denote the projection map \( G \to H \) by \( p \). Suppose that \( \{G_i\}_{i \geq 1} \) is a filtration of normal subgroups of \( G \) with property (ii). Then, \( \{p(G_i)\}_{i \geq 1} \) is a filtration of normal subgroups of \( H \) with property (ii).

**Proof.** Let us denote \( \mathbb{Z}^m \) by \( A \) and let us define \( H_i \) to be \( p(G_i) \). Thus, \( AG_i = AH_i \). Let \( C \) be a finitely generated abelian subgroup of \( H \). Note that the group \( A(H_i \cap C) \) is contained in

\[ AH_i \cap AC = AG_i \cap AC = A(G_i \cap AC). \]

But since the filtration \( \{G_i\} \) has Property (2), there is \( i_1 = i(AC) \) such that \( G_{i_1} \cap AC = 1 \). To sum up, since \( A \cap C \subseteq A \cap H = 1 \), \( A(H_i \cap C) \) being equal to \( A \) implies that \( H_i \cap C = 1 \) for \( i \geq i_1 \).

**Proposition 3.6.** Let \( G \) be a Droms RAAG and \( \{G_i\}_{i \geq 1} \) be a filtration of normal subgroups of \( G \) with property (ii). Then, for sufficiently large \( i_0 \) the group \( G_{i_0} \) is free.
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Proof. Let $G$ be a Droms RAAG of level $l(G)$. If $l(G) = 0$, then $G$ is a free group, so the statement clearly holds. If $l(G) \geq 1$, then $G$ equals $\mathbb{Z}^m \times (K_1 \ast \cdots \ast K_k)$ for some $m \in \mathbb{N} \cup \{0\}$ and $K_1, \ldots, K_k$ Droms RAAGs of level less than $l(G)$. Let us denote $Z$.

Remark. generated subgroup of an ICE group over a Droms RAAG. Moreover, a group is a limit group over a Droms RAAG if and only if it is a finitely generated subgroup of a free group. In conclusion, by taking $s_0$ to be $\max\{s_1, \ldots, s_k\}$, $H_{s_0}$ is a free group.

**Theorem 3.7.** Let $G$ be a Droms RAAG with filtration $\{G_i\}_{i \geq 1}$ of normal subgroups such that $G/G_i$ is torsion-free and $\bigcap_{i \geq 1} G_i = 1$. Then, for sufficiently large $i_0$ the group $G_{i_0}$ is free.

Proof. It follows from Lemma 3.4 and Proposition 3.6.

In order to show that limit groups over Droms RAAGs are free-by-(torsion-free nilpotent), we will show that ICE groups over Droms RAAGs are free-by-(torsion-free nilpotent). A limit group over a Droms RAAG is a finitely generated subgroup of an ICE group over a Droms RAAG, so it will also be free-by-(torsion-free nilpotent).

Let us start recalling the definition of ICE groups over Droms RAAGs given in [11].

**Definition 3.8.** Let $H$ be a group and $Z \subseteq H$ the centraliser of an element. Then, the group $G = H \ast Z (Z \times \mathbb{Z}^n)$ is said to be obtained from $H$ by an extension of a centraliser. An ICE group over $H$ is a group obtained from $H$ by applying finitely many times the extension of a centraliser construction. In the classic limit groups, the standard condition that is asked is that the abelian subgroup $Z$ is maximal in one of the vertex groups and indeed this assures 2-acylindricity of the action. However, if one asks $Z$ to be not just abelian but a centraliser of an element, this maximality condition is satisfied.

**Theorem 3.9 ([11]).** All ICE groups over Droms RAAGs are limit groups over Droms RAAGs. Moreover, a group is a limit group over a Droms RAAG if and only if it is a finitely generated subgroup of an ICE group over a Droms RAAG.

**Remark 3.10.** In [11, lemma 7.5] it is proved that any ICE over a Droms RAAG can be obtained by first extending centralisers of central elements, then extending centralisers of elliptic elements and finally considering extensions of centralisers of hyperbolic elements.

If $G$ is a Droms RAAG such that $l(G) = 0$, then the centraliser of any non-trivial element $t \in G$ is the infinite cyclic group $\langle t_1 \rangle$, where $\langle t \rangle \subseteq \langle t_1 \rangle$ and $[\langle t_1 \rangle : \langle t \rangle]$ is maximal possible.

Now let $G$ be a Droms RAAG of the form $\mathbb{Z}^f \times (G_1 \ast \cdots \ast G_f)$ for $t, f \in \mathbb{N} \cup \{0\}$ such that $l(G) \leq l(G) - 1$ for $i \in \{1, \ldots, f\}$, let $g \in G$ and let us denote the centraliser of $g$ in $G$ by $C_G(g)$.

Firstly, by the result [11, lemma 7.5] mentioned above, we shall start extending centralisers of central elements. If $g \in \mathbb{Z}^f$, then $C_G(g) = G$, so the extension of a centraliser construction gives...
and we again get a Droms RAAG. That is, extending centralisers of central elements in Droms RAAGs leads to another Droms RAAG. Suppose that after extending centralisers of central elements, we have constructed the group \( \mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n) \).

Secondly, we extend centralisers of elliptic elements, so we assume that \( g \in \mathbb{Z}^m g_0 \), \( g_0 \neq 1 \) and \( g_0 \) is an elliptic element in \( G_1 \ast \cdots \ast G_n \). Then, \( g_0 \in \bigcup_{1 \leq i \leq n} G_i^G \), \( C_G(g) = C_G(g_0) \) and we can assume without loss of generality that \( g = g_0 \). Since extensions of centralizers are conjugate invariant, we can assume that \( g \in G_i \) for some \( i \in \{1, \ldots, n\} \). Thus,

\[
C_G(g) = \mathbb{Z}^m \times C_{G_i}(g).
\]

Therefore,

\[
G \ast_{C_G(g)} \left( C_G(g) \times \mathbb{Z}^k \right) \simeq \mathbb{Z}^m \times \left( G_1 \ast \cdots \ast G_{i-1} \ast \left( G_i \ast_{C_{G_i}(g)} (\mathbb{Z}^k \times C_{G_i}(g)) \right) \ast G_{i+1} \ast \cdots \ast G_n \right),
\]

so we get a group of the form \( H := \mathbb{Z}^m \times (H_1 \ast \cdots \ast H_n) \). Elliptic elements of \( H \) lie again in \( \mathbb{Z}^m g_0 \) with \( g_0 \neq 1 \) and \( g_0 \) elliptic in \( H_1 \ast \cdots \ast H_n \). Extending centralisers of elliptic elements in groups of this form thus ends up giving again a group of the form \( \mathbb{Z}^l \times (K_1 \ast \cdots \ast K_v) \).

Thirdly, we shall extend the centraliser of a hyperbolic element in a group of the form \( G := \mathbb{Z}^l \times (K_1 \ast \cdots \ast K_v) \). Hence, assume that \( g \in \mathbb{Z}^l g_0 \), \( g_0 \neq 1 \) and \( g_0 \) is a hyperbolic element in \( K_1 \ast \cdots \ast K_v \). Then, \( g_0 \in (K_1 \ast \cdots \ast K_v) \setminus \left( \bigcup_{1 \leq i \leq v} K^G_i \right) \), \( C_K(g) = C_K(g_0) \) and we can assume without loss of generality that \( g = g_0 \). Hence, \( C_K(g) = \mathbb{Z}^l \times \langle g \rangle \cong \mathbb{Z}^{l+1} \), where \( \langle g \rangle \subseteq \langle g_0 \rangle \) and \( \{(g_1) : \langle g \rangle \} \) is maximal possible, and

\[
K \ast_{C_K(g)} \left( C_K(g) \times \mathbb{Z}^k \right) \simeq \left( \mathbb{Z}^l \times (K_1 \ast \cdots \ast K_v) \right) \ast_{\mathbb{Z}^l \times \langle g \rangle} \left( \mathbb{Z}^l \times \langle g \rangle \times \mathbb{Z}^k \right) \simeq \mathbb{Z}^l \times \left( (K_1 \ast \cdots \ast K_v) \ast_{\mathbb{Z}^l} \mathbb{Z}^{k+1} \right).
\]

In this case, we obtain a group of the form \( H := \mathbb{Z}^l \times (A \ast_{B} C) \). Hyperbolic elements of this group are elements \( g \in \mathbb{Z}^l g_0 \) with \( g_0 \) hyperbolic in \( A \ast_{B} C \), that is, \( g_0 \in (A \ast_{B} C) \setminus (A^H \cup C^H) \) and extending centralizers of hyperbolic elements in groups of this form yields a group with the same structure.

Thus, the class of ICE groups over Droms RAAGs can be given the following hierarchical structure.

**Definition 3.11.** Assume that \( G \) is a Droms RAAG of level 0, that is, \( G \) is a free group. An ICE group over \( G \) of altitude 0 is precisely \( G \). An ICE group over \( G \) of altitude \( k \geq 1 \) is an amalgamated free product over \( \mathbb{Z}^{n_0} \) of an ICE group over \( G \) of altitude \( \leq k - 1 \) and a free abelian group \( \mathbb{Z}^{n_0} \times \mathbb{Z}^{m_0} \), where \( \mathbb{Z}^{n_0} \) embeds in \( \mathbb{Z}^{n_0} \times \mathbb{Z}^{m_0} \) naturally.

Assume that \( G \) is a Droms RAAG of level \( l \geq 1 \), that is, \( G = \mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n) \) where each \( G_i \) is a Droms RAAG of level less than \( l \). An ICE group over \( G \) of altitude 0 is of the form \( \mathbb{Z}^{m'} \times (H_1 \ast \cdots \ast H_l) \) where \( m' \geq m \) and \( H_i \) is an ICE group over \( G_i \) for \( i \in \{1, \ldots, n\} \).

An ICE group over \( G \) of altitude \( k \geq 1 \) is an amalgamated free product over \( \mathbb{Z}^{n_0} \) of an ICE group over \( G \) of altitude \( \leq k - 1 \) and a free abelian group \( \mathbb{Z}^{n_0} \times \mathbb{Z}^{m_0} \), where \( \mathbb{Z}^{n_0} \) embeds in \( \mathbb{Z}^{n_0} \times \mathbb{Z}^{m_0} \) naturally.
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Remark 3.12. The term used in the literature for the altitude of ICE groups over free groups is *level*. However, since level has been used when defining Droms RAAGs hierarchically, we have decided to use *altitude* for ICE groups over Droms RAAGs.

**Theorem 3.13.** Let $G$ be a Droms RAAG and let $K$ be an ICE group over $G$. Let $\{K_i\}_{i \geq 1}$ be a filtration of normal subgroups of $K$ with property (ii). Then, for sufficiently large $i_0$ the group $K_{i_0}$ is free.

**Proof.** We prove it by induction on the level $l(G)$ of $G$ as a Droms RAAG. Suppose that $l(G) = 0$. Then, if $K$ has altitude 0, $K$ is precisely $G$ which is a free group, and the result is obvious. If $K$ has altitude $\geq 1$, then $K$ is of the form

$$H \ast_{\mathbb{Z}^{n_0}} (\mathbb{Z}^{m_0} \times \mathbb{Z}^{n_0}),$$

and $H$ is an ICE group over $G$ of smaller altitude than $K$. Since $K_i$ is a normal subgroup of $K$, $K_i$ inherits a graph of groups decomposition where the vertex groups are of the form

$$(K_i \cap H)^{g_{a_j}} \text{ or } (K_i \cap (\mathbb{Z}^{m_0} \times \mathbb{Z}^{n_0}))^{g_{a_j}}, \text{ for } g_{a_j} \in K,$$

and the edge groups are of the form

$$(K_i \cap \mathbb{Z}^{n_0})^{g_{b_j}} \text{ for } g_{b_j} \in K.$$

Note that $\{K_i \cap H\}_{i \geq 1}$ is a filtration of $H$ with property (ii), so by inductive hypothesis, there is $i_1$ such that $K_{i_1} \cap H$ is free. In addition, the filtration $\{K_i\}_{i \geq 1}$ has property (ii), so there is $i_2 = i(\mathbb{Z}^{m_0} \times \mathbb{Z}^{n_0})$ such that $K_{i_2} \cap (\mathbb{Z}^{m_0} \times \mathbb{Z}^{n_0})$ is trivial. In particular, $K_{i_2} \cap \mathbb{Z}^{m_0}$ is also trivial. Therefore, taking $i_0$ to be $\max\{i_1, i_2\}$, $K_{i_0}$ is free.

Now suppose that $l(G) \geq 1$. Then, $G$ is of the form

$$\mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n),$$

for some $m \in \mathbb{N} \cup \{0\}$ and $G_i$ is a Droms RAAG with $l(G_i) \leq l(G) - 1$ for each $i \in \{1, \ldots, n\}$. If $K$ has altitude 0 as an ICE group over $G$, then

$$K = \mathbb{Z}^{m'} \times (H_1 \ast \cdots \ast H_n),$$

where $m' \geq m$ and $H_i$ is an ICE group over $G_i$ for $i \in \{1, \ldots, n\}$. Let us denote the projection map $K \mapsto H_1 \ast \cdots \ast H_n$ by $p$.

By hypothesis, there is $i_1 = i(\mathbb{Z}^{m'})$ such that $\mathbb{Z}^{m'} \cap K_{i_1} = 1$. In addition, by Lemma 3.5, $\{N_i\}_{i \geq 1}$ is a filtration of normal subgroups of $H_1 \ast \cdots \ast H_n$ with property (ii), where $N_i = p(K_i)$. As a consequence, $N_i \simeq K_i$ for $i \geq i_1$.

Note that the group $N_i$ is a free product of conjugates of $N_i \cap H_j$ for $j \in \{1, \ldots, n\}$ and a free group. For $j \in \{1, \ldots, n\}$, $\{N_i \cap H_j\}_{i \geq 1}$ is a filtration of normal subgroups of $H_j$ with property (ii), so by inductive hypothesis, there is $r_j$ such that $N_{i_j} \cap H_j$ is free. In conclusion, taking $i_0$ to be $\max\{i_1, r_1, \ldots, r_n\}$, $N_{i_0}$ is a free group.

Finally, suppose that $K$ is an ICE group over $G$ of altitude $k \geq 1$. Then, $K$ is an amalgamated free product over $\mathbb{Z}^{n_0}$ of an ICE group over $G$ of altitude $\leq k - 1$ and a free abelian group $\mathbb{Z}^{m_0} \times \mathbb{Z}^{n_0}$. This case may be treated as the case when $K$ is an ICE group of altitude greater than 0 over a free group.
Recall that $\gamma_i(G)$ denotes the lower central series of a group $G$. That is,

$$
\gamma_1(G) = G \quad \text{and} \quad \gamma_{i+1}(G) = [G, \gamma_i(G)].
$$

We denote by $\text{tor}(G)$ the set of torsion elements of $G$. In the case when $G$ is finitely generated nilpotent, then $\langle \text{tor}(G) \rangle$ is the maximal finite subgroup of $G$.

**Theorem 3.14.** Let $G$ be a Droms RAAG, let $K$ be an ICE group over $G$ and define $K_i$ to be $K_i/\gamma_i(K) = \langle \text{tor}(K/\gamma_i(K)) \rangle$. Then, $K_{i+1} < K_i$, $K_i$ is normal in $K$, $K/K_i$ is torsion-free nilpotent and $\bigcap_{i \geq 1} K_i = 1$.

**Proof.** By construction, $K_i/\gamma_i(K)$ is a characteristic subgroup of $K/\gamma_i(K)$, so $K_i$ is normal in $K$ and $K_{i+1} < K_i$. It remains to show that $\bigcap_{i} K_i = 1$. Suppose that $k \in (\bigcap_{i} K_i) \setminus \{1\}$. Since $K$ is a limit group over $G$, there is a homomorphism $\varphi : K \to G$ such that $\varphi(k) \neq 1$. Let $G_0 = \text{im}(\varphi)$, so $G_0$ is a Droms RAAG. By [26, theorem 6.4], we have that $G_0/\gamma_i(G_0)$ is torsion-free, hence $\varphi(K_i) = \varphi(\gamma_i(K))$. Then $\varphi(k) \in \bigcap_{i} \varphi(K_i) = \bigcap_{i} \varphi(\gamma_i(K)) \subseteq \bigcap_{i} \gamma_i(G) = 1$, a contradiction.

**Corollary 3.15** Every ICE group over a Droms RAAG is free-by-(torsion-free nilpotent).

**Proof.** It follows from Lemma 3.4, Theorem 3.13 and Theorem 3.14.

**Proposition 3.16.** Every limit group over a Droms RAAG is free-by-(torsion-free nilpotent).

**Proof.** It follows from Theorem 3.9 and Corollary 3.15.

4. Corollaries on subdirect products of limit groups over Droms RAAGs

In this section we aim to prove Theorem 1.1. Theorem 1.1 is a generalisation of [21, theorem 11].

**Theorem 4.1** ([21, theorem 11]). Let $G_1, \ldots, G_m$ be non-abelian limit groups over free groups and let

$$
S < G := G_1 \times \cdots \times G_m
$$

be a finitely generated subdirect product intersecting each factor $G_i$ non-trivially. Suppose further that, for some fixed natural number $s \in \{2, \ldots, m\}$, for every subgroup $S_0$ of finite index in $S$ the homology group $H_i(S_0, \mathbb{Q})$ is finite dimensional (over $\mathbb{Q}$) for all $i \leq s$. Then for every canonical projection

$$
p_{j_1, \ldots, j_s} : S \to G_{j_1} \times \cdots \times G_{j_s}
$$

the index of $p_{j_1, \ldots, j_s}(S)$ in $G_{j_1} \times \cdots \times G_{j_s}$ is finite.

We observe that in [21] the domain of $p_{j_1, \ldots, j_s}$ is defined to be $G$ and not as we stated it above, but in [7, 8] it is $S$. It looks more convenient to stick to the latter.

The starting point in the proof of Theorem 1.1 is that the proof of Theorem 4.1 applies in bigger generality. We explain this in the following result.
THEOREM 4.2. Let \( 2 \leq s \leq m \) be integers and \( S < G_1 \times \cdots \times G_m \) be a finitely generated subdirect product such that:

(i) there exist normal free subgroups \( L_i \) in \( G_i \) with \( Q_i := G_i/L_i \) nilpotent;
(ii) each \( G_i \) is finitely presented;
(iii) for every \( 1 \leq j_1 < \cdots < j_s \leq m \) if \( M_{j_1,\ldots,j_s} \) is a subgroup of infinite index in the group \( G_{j_1} \times \cdots \times G_{j_s} \), then there exists \( i \leq s \) and a subgroup \( M_{j_1,\ldots,j_s}^i \) of finite index in \( M_{j_1,\ldots,j_s} \) such that \( H_i(M_{j_1,\ldots,j_s}^i, \mathbb{Q}) \) is infinite dimensional over \( \mathbb{Q} \);
(iv) for every \( 1 \leq j_1 < \cdots < j_s \leq m \) and every subgroup \( H_{j_i} \) of finite index in \( G_{j_i} \) we have that if a subdirect product \( M_{j_1,\ldots,j_s} \subseteq H_{j_1} \times \cdots \times H_{j_s} \) is finitely presented, then there is a subgroup \( K_{j_i} \) of finite index in \( H_{j_i} \) and \( N_{j_i} \), a normal subgroup of \( H_{j_i} \) such that \( K_{j_i}/N_{j_i} \) is nilpotent and \( N_{j_1} \times \cdots \times N_{j_s} \subseteq M_{j_1,\ldots,j_s} \);
(v) \( S \) virtually surjects on pairs;
(vi) \( L := L_1 \times \cdots \times L_m \subseteq S \);
(vii) for every subgroup \( S_0 \) of finite index in \( S \), the homology group \( H_i(S_0, \mathbb{Q}) \) is finite dimensional (over \( \mathbb{Q} \)) for all \( i \leq s \).

Then for every canonical projection

\[
p_{j_1,\ldots,j_s} : S \mapsto G_{j_1} \times \cdots \times G_{j_s},
\]
the index of \( p_{j_1,\ldots,j_s}(S) \) in \( G_{j_1} \times \cdots \times G_{j_s} \) is finite.

Proof. We will prove the result by induction on \( s \). For \( s = 2 \), this is condition (v). Suppose that \( s \geq 3 \). We divide the proof in several steps.

(a) Set \( Q := S/L < Q_1 \times \cdots \times Q_m \). Consider the Lyndon–Hochschild–Serre spectral sequence

\[
E^2_{i,j} = H_i(Q, H_j(L, \mathbb{Q}))
\]
that converges to \( H_{i+j}(S, \mathbb{Q}) \).

Note that since each \( L_j \) is a free group, for every \( t \leq m \) we have that

\[
H_j(L_t, \mathbb{Q}) \simeq \bigoplus_{1 \leq j_1 < j_2 < \cdots < j_t \leq m} H_1(L_{j_1}, \mathbb{Q}) \otimes \cdots \otimes H_1(L_{j_t}, \mathbb{Q}),
\]
where each summand is \( \mathbb{Q} \)-invariant, where the \( \mathbb{Q} \)-action is induced by conjugation. Thus,

\[
E^2_{0,s} = H_0(Q, H_s(L, \mathbb{Q})) \simeq \bigoplus_{1 \leq j_1 < j_2 < \cdots < j_s \leq m} \left( H_1(L_{j_1}, \mathbb{Q}) \otimes \cdots \otimes H_1(L_{j_s}, \mathbb{Q}) \otimes \mathbb{Q} \mathbb{Q} \right).
\]

By inductive hypothesis, \( S \) virtually surjects onto \( s - 1 \) factors. This implies that \( H_j(L, \mathbb{Q}) \) is a finitely generated \( \mathbb{Z} \mathbb{Q} \)-module for \( j \leq s - 1 \), and hence,

\[
E^2_{i,j} \text{ is finite dimensional over } \mathbb{Q} \text{ for every } j \leq s - 1.
\]

For \( i \geq 2 \), we have that \( s + 1 - i \leq s - 1 \), so by (2), \( E^2_{i,s+1-i} \) is finite dimensional over \( \mathbb{Q} \). Hence, for all differential maps \( d^j_{i,s+1-i} : E^1_{i,s+1-i} \mapsto E^1_{0,s} \), the subgroup \( \text{im} \left( d^j_{i,s+1-i} \right) \) is
finite dimensional over $\mathbb{Q}$. Thus,

$$E_{0,s}^{i+1} = \ker \left( d^i_{0,s} \right) / \text{im} \left( d^i_{s+1-i} \right) = E_{0,s}^i / \text{im} \left( d^i_{s+1-i} \right)$$

is finite dimensional over $\mathbb{Q}$ if and only if $E_{0,s}^{1}$ is finite dimensional over $\mathbb{Q}$. This implies that $E_{0,s}^{2}$ is finite dimensional over $\mathbb{Q}$ if and only if $E_{0,s}^{\infty}$ is finite dimensional over $\mathbb{Q}$. Combining this with the convergence of the spectral sequence and (2) we deduce that $H_s(S, \mathbb{Q})$ is finite dimensional over $\mathbb{Q}$ if and only if

$$E_{0,s}^{2} = H_0(Q, H_s(L, \mathbb{Q}))$$

is finite dimensional over $\mathbb{Q}$.

The condition that $H_s(S, \mathbb{Q})$ is finite dimensional over $\mathbb{Q}$ implies that for each $1 \leq j_1 < j_2 < \cdots < j_s \leq m,$

$$\dim_{\mathbb{Q}} \left( H_1(L_{j_1}, \mathbb{Q}) \otimes \cdots \otimes H_1(L_{j_s}, \mathbb{Q}) \otimes_{\mathbb{Q} \mathbb{Q}} \mathbb{Q} \right)$$

is finite dimensional over $\mathbb{Q}$ if and only if $E_{0,s}^{1}$ is finite dimensional over $\mathbb{Q}$. Thus, by the convergence of the spectral sequence, we deduce that $\dim_{\mathbb{Q}} H_i(S, \mathbb{Q}) < \infty$ for $i \leq s$.

(c) Now we consider $\widetilde{S}$ a subgroup of finite index in $p_{j_1, \ldots, j_s}(S)$ that contains $\widetilde{L} := L_{j_1} \times \cdots \times L_{j_s}$ and set $\widetilde{Q} := \widetilde{S} / \widetilde{L}$. Then, the Lyndon–Hochschild–Serre spectral sequence

$$E_{ij}^2 = H_i \left( \widetilde{Q}, H_j \left( \widetilde{L}, \mathbb{Q} \right) \right)$$

converges to $H_{i+j}(\widetilde{S}, \mathbb{Q})$. By (1) and (3) we deduce that $\dim_{\mathbb{Q}} E_{ij}^2 < \infty$ for $j \leq s-1$ and $\dim_{\mathbb{Q}} E_{0,s}^2 < \infty$. Then, by the convergence of the spectral sequence, we deduce that $\dim_{\mathbb{Q}} H_i(\widetilde{S}, \mathbb{Q}) < \infty$ for $i \leq s$.

By condition (vii), $H_i(S_0, \mathbb{Q})$ is finite dimensional (over $\mathbb{Q}$) for all $i \leq s$. Applying Step 2 for $S$ substituted with $S_0$, $G_j$ substituted with $p_j(S_0)$ and $L_j$ substituted with $\gamma_i(p_j(S_0))$, we deduce that for every subgroup $B$ of finite index in $p_{j_1, \ldots, j_s}(S_0) = S_0 \cap (K_{j_1} \times \cdots \times K_{j_s})$ such that $B$ contains $\gamma_i(p_j(S_0))$ we have that

$$\dim_{\mathbb{Q}} H_i(\widetilde{Q}, \mathbb{Q}) < \infty \quad \text{for} \quad i \leq s.$$

We apply this for $B = p_{j_1, \ldots, j_s}(S_0) = S_0 \cap (K_{j_1} \times \cdots \times K_{j_s})$ and note that $B$ is a normal subgroup of finite index in $S_0$. Hence the Lyndon–Hochschild–Serre spectral sequence
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$H_i(\widetilde{S}_0/B, H_j(B, \mathbb{Q}))$ that converges to $H_{i+j}(\widetilde{S}_0, \mathbb{Q})$ implies that

$$\dim_{\mathbb{Q}} H_i(\widetilde{S}_0, \mathbb{Q}) < \infty$$

for $i \leq s$.

But this combined with condition (iii) and the fact that $\widetilde{S}_0$ is an arbitrary subgroup of finite index in $p_{j_1, \ldots, j_s}(S)$ implies that $p_{j_1, \ldots, j_s}(S)$ has finite index in $G_{j_1} \times \cdots \times G_{j_s}$.

Let $n \geq 0$, $R$ be an associative ring with identity and let $M$ be a (right) $R$-module. Then $M$ is of (homological) type $\text{FP}_n$ if there is a projective resolution

$$\mathcal{P} : \cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that each projective $R$-module $P_i$ is finitely generated for $0 \leq i \leq n$. A group $G$ is of type $\text{FP}_n$ over a commutative ring $S$ if the trivial $S G$-module $S$ is of type $\text{FP}_n$ over the group algebra $S G$.

**Lemma 4.3** ([21, lemma 6]). Let $Q_1, \ldots, Q_i$ be finitely generated nilpotent groups and $V_j$ be a finitely generated $Q_i Q_j$-module such that $V_j$ contains a cyclic non-zero free $Q_i Q_j$-submodule $W_j$ for $1 \leq j \leq i$. Suppose that $\widetilde{Q}$ is a subgroup of $Q := Q_1 \times \cdots \times Q_i$ such that $V_1 \otimes_{Q_1} \cdots \otimes_{Q_i} V_i$ is finitely generated as a $\widetilde{Q}$-$Q$-module. Then, $\widetilde{Q}$ has finite index in $Q$.

**Proposition 4.4** ([21, proposition 7]). Let $G$ be a group of negative Euler characteristic $\chi(G)$ such that the trivial $Q G$-module $Q$ has a free resolution with finitely generated modules and finite length. Then, for any normal subgroup $M$ of $G$ such that $Q := G/M$ is torsion-free nilpotent and $M$ is free, the $Q Q$-module $V = M/[M, M] \otimes_{\mathbb{Q}} Q$ has a non-zero free $Q Q$-submodule, where $Q$ acts on $M = M/[M, M]$ via conjugation.

**Theorem 4.5.** Let $S < G_1 \times \cdots \times G_m$ be a finitely generated subdirect product such that:

(i) for each $1 \leq i \leq m$ the trivial $Q G_i$-module $Q$ has a free resolution of finite length with finitely generated modules;

(ii) for each $1 \leq i \leq m$ there is a normal free subgroup $L_i$ of $G_i$ such that $G_i/L_i$ is torsion-free nilpotent;

(iii) $L_1 \times \cdots \times L_m \subseteq S$;

(iv) $S$ is of type $\text{FP}_s$ over $Q$;

(v) $\chi(G_i) < 0$.

Then, for every canonical projection $p_{j_1, \ldots, j_s} : S \to G_{j_1} \times \cdots \times G_{j_s}$ the index of $p_{j_1, \ldots, j_s}(S)$ in $G_{j_1} \times \cdots \times G_{j_s}$ is finite.

**Proof.** Note that since $G_i$ is of type $FP_{\infty}$ over $Q$, it is $FP_1$ over $Q$ and so $G_i$ is finitely generated. By Proposition 4.4, $V_i := (L_i/[L_i, L_i]) \otimes_{\mathbb{Q}} Q$ has a non-zero free $Q Q_i$-submodule, where $Q_i := G_i/L_i$ acts via conjugation.

Let

$$\mathcal{F} : \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to Q \to 0$$

be a free resolution of the trivial $\mathbb{Q} S$-module $\mathbb{Q}$ with $F_i$ finitely generated for $i \leq s$. We define $L$ to be $L_1 \times \cdots \times L_m$ and since each $L_i$ is free, by the Künneth formula,

$$H_s(L, \mathbb{Q}) = \bigoplus_{1 \leq j_1 < \cdots < j_s \leq m} L_{j_1} / [L_{j_1}, L_{j_2}] \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L_{j_s} / [L_{j_s}, L_{j_1}] \otimes_{\mathbb{Z}} \mathbb{Q} \simeq$$

$$\bigoplus_{1 \leq j_1 < \cdots < j_s \leq m} V_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_{j_s}.$$  

Note that

$$H_s(L, \mathbb{Q}) \simeq H_s(\mathcal{F} \otimes_{\mathbb{Q} L} \mathbb{Q}) = \ker (d_s) / \text{im}(d_{s+1}),$$

where $d_j : F_j \otimes_{\mathbb{Q} L} \mathbb{Q} \to F_{j-1} \otimes_{\mathbb{Q} L} \mathbb{Q}$ is the differential of $\mathcal{F} \otimes_{\mathbb{Q} L} \mathbb{Q}$. Since $F_s \otimes_{\mathbb{Q} L} \mathbb{Q}$ is a finitely generated $\mathbb{Q}\hat{Q}$-module, where $\mathbb{Q}\hat{Q} := S/L$ is a finitely generated nilpotent group and $\mathbb{Q}\hat{Q}$ is a Noetherian ring, we deduce that ker $(d_s)$ is a finitely generated $\mathbb{Q}\hat{Q}$-module. In particular, $H_s(L, \mathbb{Q})$ and $V_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_{j_s}$ are finitely generated $\mathbb{Q}\hat{Q}$-modules. Note that the action of $\hat{Q}$ on $V_{j_1} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} V_{j_s}$ factors through $\hat{Q} := p_{j_1,\ldots,j_s}(S)/L_{j_1} \times \cdots \times L_{j_s}$. Then, by Lemma 4.3, $\hat{Q}$ has finite index in $Q_{j_1} \times \cdots \times Q_{j_s}$. This is equivalent to $p_{j_1,\ldots,j_s}(S)$ having finite index in $G_{j_1} \times \cdots \times G_{j_s}$.

The next step is to prove that limit groups over Droms RAAGs satisfy the conditions of Theorem 4.5.

**Lemma 4.6.** Let $G$ be a Droms RAAG and $\Gamma$ a limit group over $G$. Then, $\chi(\Gamma) \leq 0$. Furthermore, $\chi(\Gamma) = 0$ if and only if $\Gamma$ has non-trivial center. The latter happens precisely when $\Gamma = \mathbb{Z}^{l_0} \times \Lambda_0$ for some $l_0 \geq 1$ and with $\Lambda_0$ a centerless subgroup of $\Gamma$.

**Proof.** Let us prove it by induction on the level $l(G)$ of $G$. If $l(G) = 0$, then $G$ is a free group, so the result follows from [21, lemma 5].

Now assume that $l(G) \geq 1$. Then, $G$ equals

$$\mathbb{Z}^m \times (G_1 \ast \cdots \ast G_n),$$

where $m \in \mathbb{N} \cup \{0\}$ and $G_i$ is a Droms RAAG such that $l(G_i) \leq l(G) - 1$ for $i \in \{1, \ldots, n\}$. From Proposition 2.7 we get that $\Gamma$ is of the form $\mathbb{Z}^l \times \Lambda$ where $\Lambda$ is a limit group over $G_1 \ast \cdots \ast G_n$ and if $m = 0$, then $l = 0$. Therefore,

$$\chi(\Gamma) = \chi(\mathbb{Z}^l) \chi(\Lambda),$$

so if $l \geq 1$, then $\chi(\Gamma) = 0$. Let us compute $\chi(\Lambda)$. If the height of $\Lambda$ is 0, i.e. $h(\Lambda) = 0$, then

$$\Lambda = A_1 \ast \cdots \ast A_j,$$

where for each $t \in \{1, \ldots, j\}$ $A_t$ is a limit group over $G_i$ for some $i \in \{1, \ldots, n\}$. Hence,

$$\chi(\Lambda) = \sum_{t \in \{1, \ldots, j\}} \chi(A_t) - (j - 1),$$

so applying the inductive hypothesis, we get that $\chi(\Lambda) \leq 1 - j$. If $j \geq 2$, then $\chi(\Lambda) < 0$. If $j = 1$, $\chi(\Lambda) = \chi(A_1)$ and $A_1$ is a limit group over $G_i$. Thus, by induction, $\chi(A_1) \leq 0$ and $\chi(A_1) = 0$ if and only if $A_1$ has non-trivial center. In this case the center $Z(A_1)$ is a direct factor of $A_1$. 
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If $h(\Lambda) \geq 1$, then $\Lambda$ acts cocompactly on a tree $T$ where the edge stabilisers are cyclic and the vertex groups are limit groups over $G_1 \ast \cdots \ast G_n$ of height at most $h(\Lambda) - 1$. Moreover, at least one vertex group $H_{V_0}$ has trivial center and so by inductive hypothesis, $\chi(H_{V_0}) < 0$. If $X$ is the quotient graph $T/\Lambda$,

$$\chi(\Lambda) = \sum_{v \in V(X)} \chi(H_v) - \sum_{e \in E(X)} \chi(H_e) \leq \sum_{v \in V(X)} \chi(H_v) \leq \chi(H_{V_0}) < 0.$$ 

**Lemma 4.7.** Let $\Gamma$ be a limit group over a Droms RAAG. Then, the trivial $\mathbb{Q}\Gamma$-module $\mathbb{Q}$ has a free resolution with finitely generated modules and of finite length.

**Proof.** Limit groups over Droms RAAGs are of type $\text{FP}_\infty$ over $\mathbb{Z}$ (and so over $\mathbb{Q}$) and of finite cohomological dimension.

**Proof of Theorem 1.1.** By [23, theorem 6-2], $G_i/(S \cap G_i)$ is virtually nilpotent for every $i \in \{1, \ldots, m\}$. By substituting $G_i$ and $S$ with subgroups of finite index if necessary we can assume that $G_i/(S \cap G_i)$ is torsion-free nilpotent. By Proposition 3-16, there is a normal free subgroup $M_i$ de $G_i$ such that $G_i/M_i$ is torsion-free nilpotent, hence for $L_i = S \cap M_i$ we have that $L_i$ is a normal subgroup of $G_i$ such that $L_i$ is free, $G_i/L_i$ is torsion-free nilpotent and $L_1 \times \cdots \times L_m \subseteq S$. Note that conditions 1 and 6 from Theorem 4-2 hold. The other conditions from Theorem 4-2 hold when each $G_i$ is a limit group over a Droms RAAG with trivial center: condition (ii) is [11, Corollary 7-8], condition (iii) is Theorem 2-9, condition (v) is Theorem 2-10.

Observe that a finite index subgroup of a limit group over a Droms RAAG is a limit group over a Droms RAAG. Hence condition (iv) follows from [23, theorem 6-2]. Finally, condition (vii) is assumed in the statement.

5. On the $L^2$-Betti numbers and volume gradients of limit groups over Droms RAAGs and their subdirect products

The aim of this section is to study the growth of homology groups and the volume gradients for limit groups over Droms RAAGs and for finitely presented residually Droms RAAGs, following the paper [9]. Some of the results concerning limit groups over Droms RAAGs hold in a more general setting, more precisely, they also hold for limit groups over coherent RAAGs. Thus, these results (see Theorem 1-2 and Theorem 1-3) will be stated for limit groups over coherent RAAGs. However, the results for finitely presented residually Droms RAAGs make use of Theorem 2-10 and in [14] it was shown that this no longer holds for coherent RAAGs. Thus, Theorem 1-4 and Theorem 1-6 are stated just for residually Droms RAAGs.

In order to study the homology growth and volume gradients, we work with exhausting normal chains: a chain $(B_n)$ of normal subgroups of finite index such that $B_{n+1} \subseteq B_n$ and $\bigcap B_n = 1$. Note that if a group is residually finite, then it has an exhausting normal chain. In particular, limit groups over coherent RAAGs have exhausting normal chains.

A group $G$ is of homotopical type $F_m$ if there is a classifying space $K(G,1)$ with finite $m$-skeleton. Given a group $G$ of homotopical type $F_m$, $\text{vol}_m(G)$ is defined to be the least number of $m$-cells among all classifying spaces $K(G,1)$ with finite $m$-skeleton. For instance, $\text{vol}_1(G)$ equals $d(G)$, where $d(G)$ is the minimal number of generators of $G$.
One of the aims of this section is to prove Theorem 1.2 and Theorem 1.3. These two results are proved in [9] for limit groups over free groups via more technical results that make use of slowness of limit groups over free groups (see Section 5.1 for the definition). In [9] it is shown that limit groups over free groups are slow above dimension 1 and hence are $K$-slow above dimension 1 for any field $K$. Thus, the key point is to show that limit groups over coherent RAAGs are also slow above dimension 1 (see Section 5.2).

**Theorem 5.1** ([9, theorem D]). If a residually finite group $G$ of type $F$ is slow above dimension 1, then with respect to every exhausting normal chain $(B_n)$:

(i) Rank gradient:

$$RG(G, (B_n)) = \lim_{n \to \infty} \frac{d(B_n)}{[G : B_n]} = -\chi(G);$$

(ii) Deficiency gradient:

$$DG(G, (B_n)) = \lim_{n \to \infty} \frac{\text{def}(B_n)}{[G : B_n]} = \chi(G).$$

**Lemma 5.2** ([9, lemma 5.2]). Let $K$ be a field and let $G$ be a residually finite group of type $F$ with an exhausting normal chain $(B_n)$. If $G$ is $K$-slow above dimension 1, then

$$\lim_{n \to \infty} \frac{\dim H_1(B_n, K)}{[G : B_n]} = -\chi(G).$$

We state other results from [9] that will be important for us, as we will show that residually Droms RAAGs that are of type $FP_m$ for some $m \geq 2$ satisfy virtually the assumptions of Theorem 5.3.

**Theorem 5.3** ([9, theorem F]). Let $G \subseteq G_1 \times \cdots \times G_k$ be a subdirect product of residually finite groups of type $F$, each of which contains a normal free subgroup $F_i < G_i$ such that $G_i/F_i$ is torsion-free and nilpotent. Assume $F_i \subseteq G \cap G_i$. Let $m < k$ be an integer, let $K$ be a field, and suppose that each $G_i$ is $K$-slow above dimension 1.

If the projection of $G$ to each $m$-tuple of factors $G_{j_1} \times \cdots \times G_{j_m}$ is of finite index, then there exists an exhausting normal chain $(B_n)$ in $G$ so that for $0 \leq j \leq m$,

$$\lim_{n \to \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = 0.$$

**Theorem 5.4** ([9, theorem G]). Every finitely presented residually free group $G$ that is not a limit group over a free group admits an exhausting normal chain $(B_n)$ with respect to which the rank gradient

$$RG(G, (B_n)) = \lim_{n \to \infty} \frac{d(B_n)}{[G : B_n]} = 0.$$

Furthermore, if $G$ is of type $FP_3$ but it is not commensurable with a product of two limit groups over free groups, $(B_n)$ can be chosen so that the deficiency gradient $DG(G, (B_n)) = 0$. 


5.1. Preliminaries on groups that are K-slow and slow above dimension 1

Let us start recalling the definitions from [9] about slowness and K-slowness.

**Definition 5.5.** Let $G$ be a group. A sequence of non-negative integers $(r_j)_{j \geq 0}$ is a *volume vector* for $G$ if there is a classifying space $K(G,1)$ that, for all $j \in \mathbb{N}$, has exactly $r_j$ $j$-cells.

**Definition 5.6.** A group $G$ of homotopical type $F$ is *slow above dimension 1* if it is residually finite and for every exhausting normal chain $(B_n)$, there exist volume vectors $(r_j(B_n))_j$ for $B_n$ with finitely many non-zero entries, so that

$$\lim_{n \to \infty} \frac{r_j(B_n)}{[G:B_n]} = 0,$$

for all $j \geq 2$. $G$ is *slow* if it satisfies the additional requirement that the limit exists and is zero for $j = 1$ as well.

**Example 5.7** ([9, examples 4.4]). Finitely generated torsion-free nilpotent groups are slow. The trivial group is slow. Free groups are slow above dimension 1. Surface groups are slow above dimension 1.

**Proposition 5.8** ([9, proposition 4.5]). If a residually finite group $G$ is the fundamental group of a finite graph of groups where all of the edge groups are slow and all of the vertex groups are slow above dimension 1, then $G$ is slow above dimension 1.

**Definition 5.9.** Let $K$ be a field and let $G$ be a residually finite group. $G$ is *K-slow above dimension 1* if for every exhausting normal chain $(B_n)$, we have

$$\lim_{n \to \infty} \frac{\dim_K H_j(B_n, K)}{[G:B_n]} = 0,$$

for all $j \geq 2$.

$G$ is *K-slow* if it satisfies the additional requirement that the limit exists and is zero for $j = 1$ as well.

It follows directly by the definitions that if a group $G$ is slow above dimension 1 (respectively, slow), then it is K-slow above dimension 1 (respectively, K-slow).

**Proposition 5.10** ([9, proposition 5.3]). Let $K$ be a field. If a residually finite group $G$ is the fundamental group of a finite graph of groups where all of the edge groups are $K$-slow and all of the vertex groups are $K$-slow above dimension 1, then $G$ is $K$-slow above dimension 1.

5.2. Limit groups over coherent RAAGs are slow above dimension 1

**Lemma 5.11** ([11, corollary 9.7]). Limit groups over coherent RAAGs are CAT(0).

In particular, since limit groups over coherent RAAGs are torsion-free, they are of type $F$.

**Lemma 5.12.** Coherent RAAGs are slow above dimension 1. In particular, Droms RAAGs are slow above dimension 1.
Proof. In [17] Droms proved that if $G_X$ is a coherent RAAG, then $G_X$ splits as a finite graph of groups where all the vertex groups are free abelian. Thus, by Proposition 5.8, coherents RAAGs are slow above dimension 1.

The aim of this section is to show that limit groups over coherent RAAGs are slow above dimension 1. For that, we will use the work on [11, 12] about limit groups over coherent RAAGs.

Let us recall some definitions that are needed to understand the results below. If $G_X$ is a RAAG, the elements of $X$ are called the canonical generators of $G_X$. A non-exceptional surface is a surface which is not a non-orientable surface of genus 1, 2 or 3. We can use the following proposition to define graph towers over coherent RAAGs.

**Proposition 5.13** ([11, lemma 7.3]). Let $G$ be a coherent right-angled Artin group. A graph tower over $G$ of height 0 is a coherent RAAG $H$ which is obtained from $G$ by extending centralizers of canonical generators of $G$.

A graph tower over $G$ of height $\geq 1$ can be obtained as a free product with amalgamation, where the edge group is a free abelian group, one of the vertex groups is a graph tower over $G$ of lower height and the other vertex group is either free abelian, or the direct product of a free abelian group and the fundamental group of a non-exceptional surface.

**Theorem 5.14** ([12, theorem 8.1]). Let $\Gamma$ be a limit group over a RAAG $G$. Then, $\Gamma$ is a subgroup of a graph tower over $G$.

**Lemma 5.15** [27]. Let $1 \to C \to D \to E \to 1$ be a short exact sequence of groups of type $F$. Suppose that there are classifying spaces $K(C,1)$ and $K(E,1)$ with $\alpha_t(C)$ and $\alpha_t(E)$ $t$-cells, respectively. Then, there is a $K(D,1)$ complex with $\alpha_i(D)$ $i$-cells such that

$$\alpha_i(D) = \sum_{0 \leq t \leq i} \alpha_t(C)\alpha_{i-t}(E).$$

We now prove some results that will be used in order to show that limit groups over coherent RAAGs are slow above dimension 1.

**Lemma 5.16.** Suppose that $G$ is a group of homotopical type $F$, $H$ is a normal subgroup of finite index in $G$ and $(r_j(G))_j$ is a volume vector for $G$. Then, $([G:H]r_j(G))_j$ is a volume vector for $H$.

**Proof.** Let $Y$ be a classifying space $K(G,1)$ that, for all $j \in \mathbb{N} \cup \{0\}$, has exactly $r_j(G)$ $j$-cells. Then, if we denote by $\tilde{Y}$ the universal cover of $Y$, $\tilde{Y}$ is contractible and $Y = \tilde{Y}/G$. Therefore, $\tilde{Y}/H$ is a classifying space for $H$ with exactly $[G:H]r_j(G)$ $j$-cells.

**Lemma 5.17.** Let $G$ be a group of homotopical type $F$ and $H$ a residually finite group where there is a short exact sequence

$$1 \to \mathbb{Z}^n \to H \to G \to 1.$$

(a) If $n \geq 1$, then $H$ is slow.

(b) If $n \geq 0$ and $G$ is slow above dimension 1, then $H$ is slow above dimension 1.
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Proof. Let us denote $\mathbb{Z}^n$ by $A$.

(a) Let $(B_i)$ be a chain of finite index normal subgroups in $H$ with $\bigcap B_i = 1$. We need to show that for each $i$ there is a $K(B_i, 1)$ complex with $r_j(B_i)$ $j$-cells such that for each $j \geq 0$,

$$\lim_{i \to \infty} \frac{r_j(B_i)}{|H : B_i|} = 0.$$  

For each $i$, the short exact sequence from the statement induces a short exact sequence

$$1 \to A \cap B_i \to B_i \to p(B_i) \to 1.$$  

Let us show that

$$[H : B_i] = [A : A \cap B_i][G : p(B_i)].$$  

Indeed, note that $[H : B_i] = [H : AB_i][AB_i : B_i]$. Firstly, $[AB_i : B_i]$ equals $[A : A \cap B_i]$. Secondly, $[H : AB_i]$ equals $[H/A : AB_i/A]$, and $H/A \simeq G$ and $AB_i/A \simeq p(B_i)$. Therefore, $[H : AB_i] = [G : p(B_i)]$.

Let $\alpha_j(G)$ be the number of $j$-cells in a fixed $K(G,1)$ complex. By Lemma 5.16, there is a $K(p(B_i), 1)$ complex with $\alpha_j(p(B_i))$ $j$-cells such that

$$\alpha_j(p(B_i)) = [G : p(B_i)]\alpha_j(G).$$  

Since $A \cap B_i$ has finite index in $A$, there is a $K(A \cap B_i, 1)$ complex with $\binom{n}{j}$ $j$-cells. By Lemma 5.15, there is a $K(B_i, 1)$ complex with $\alpha_j(B_i)$ $j$-cells such that

$$\alpha_j(B_i) = \sum_{0 \leq a \leq j} \binom{n}{j - a} \alpha_a(p(B_i)) = \sum_{0 \leq a \leq j} \binom{n}{j - a} [G : p(B_i)]\alpha_a(G),$$  

and we set $r_j(B_i) = \alpha_j(B_i)$. Then,

$$\lim_{i \to \infty} \frac{r_j(B_i)}{|H : B_i|} = \lim_{i \to \infty} \frac{[G : p(B_i)]\sum_{0 \leq a \leq j} \binom{n}{j - a} \alpha_a(G)}{[G : p(B_i)][A : B_i \cap A]} = \sum_{0 \leq a \leq j} \binom{n}{j - a} \alpha_a(G) \lim_{i \to \infty} \frac{1}{[A : B_i \cap A]} = 0.$$  

(b) If $n \geq 1$, then we apply a). If $n = 0$, by assumption $G$ is slow above dimension 1.

**Theorem 5.18.** Limit groups over coherent RAAGs are slow above dimension 1.

Proof. Let $G$ be a coherent RAAG and let $\Gamma$ be a limit group over $G$. Then, by Theorem 5.14, $\Gamma$ is a subgroup of a graph tower over $G$, say $L$. Let us prove by induction on the height of $L$ that $\Gamma$ is slow above dimension 1 (see Proposition 5.13).

If $L$ has height 0, $L$ is a coherent RAAG. Therefore, $L$ is the fundamental group of a graph of groups where the vertex groups are free abelian. Thus, $\Gamma$ also admits a decomposition as a graph of groups where the vertex groups are free abelian, so by Proposition 5.8 we get that $\Gamma$ is slow above dimension 1.

Now suppose that the height of $L$ is greater than 0. Then, $L$ is a free product with amalgamation, where the edge group is a free abelian group, one of the vertex groups is a graph tower over $G$ of lower height and the other vertex group is either free abelian, or the direct
product of a free abelian group and the fundamental group of a non-exceptional surface. Then, \( \Gamma \) admits a decomposition as a graph of groups where the edge groups are free abelian (including the possibility that some are trivial), and the vertex groups are either subgroups of graph towers over \( G \) of lower height (and by induction, those vertex groups are slow above dimension 1) or free abelian groups or subgroups of the direct product of a free abelian group and the fundamental group of a non-exceptional surface. It suffices to show that finitely generated subgroups of the direct product of a free abelian group and the fundamental group of a non-exceptional surface are slow above dimension 1. Then, by Proposition 5·8 we obtain that \( \Gamma \) is slow above dimension 1.

If \( H \) is a finitely generated subgroup of \( \mathbb{Z}^m \times G_0 \) where \( m \in \mathbb{N} \cup \{0\} \) and \( G_0 \) is the fundamental group of a non-exceptional surface, then there is a short exact sequence

\[
1 \to A \to H \to N \to 1,
\]

where \( A \) is a free abelian group and \( N \) is a finitely generated subgroup of \( G_0 \). In particular, \( N \) is either a surface group or a free group, so \( N \) is slow above dimension 1. Then, by Lemma 5·17, \( H \) is slow above dimension 1.

Remark 5·19. Observe that by Theorem 5·18 coherent RAAGs are slow above dimension 1. It is not true that every RAAG is slow above dimension 1, as this would imply that it is \( K \)-slow above dimension 1 for every field \( K \). Indeed, by [3, theorem 1], if \( G \) is a right-angled Artin group with underlying flag complex \( L \), for any exhausting normal chain \( (B_n) \) in \( G \),

\[
\lim_{n \to \infty} \frac{\dim_K H_j(B_n, K)}{[G : B_n]} = \tilde{b}_{j-1}(L, K),
\]

where \( \tilde{b}_{j-1}(L, K) \) denotes the reduced Betti number of \( L \) with coefficients in \( K \).

Proof of Theorem 1·2. Parts (i) and (ii) follow from Theorem 5·1 and Theorem 5·18. Part (iii) follows from Theorem 5·18.

Proof of Theorem 1·3. It follows from Lemma 5·2 and the fact that Theorem 5·18 implies that every limit group over a coherent RAAG is \( K \)-slow above dimension 1.

Proof of Theorem 1·4. The proof is an adaptation of the proof from [9] to the setting of residually Droms RAAGs. Thus, we just give a sketch of the proof.

From Theorem 2·10 we get that \( G \) is a full subdirect product of limit groups over Droms RAAGs \( G_0 \times G_1 \times \cdots \times G_r \) such that \( G_0 \) is abelian (possibly trivial), \( G \cap G_0 \) has finite index in \( G_0 \) and \( G_i \) has trivial center for \( i \in \{1, 2, \ldots, r\} \). By Proposition 3·16, \( G_i \) is free-by-(torsion-free nilpotent).

Suppose that \( G_0 \) is trivial. Then, part (i) from Theorem 1·4 follows from Theorem 5·3 and Theorem 1·1. Indeed, by Proposition 3·16, each \( G_i \) has a free normal subgroup \( N_i \) such that \( G_i/N_i \) is torsion-free nilpotent. By [23, theorem 6·2], \( G_i/(G \cap G_i) \) is virtually nilpotent, hence \( F_i = G \cap N_i \) is a free normal subgroup of \( G_i \) such that \( G_i/F_i \) is virtually nilpotent. Hence there is a subgroup of finite index \( \tilde{G}_i \) in \( G_i \) that contains \( F_i \) such that \( \tilde{G}_i/F_i \) is torsion-free nilpotent. Then we can apply Theorem 5·3 for \( \tilde{G} := G \cap (\tilde{G}_1 \times \cdots \times \tilde{G}_r) < \tilde{G}_1 \times \cdots \times \tilde{G}_r \). This guarantees an exhausting normal chain \( (B_n) \) of \( \tilde{G} \) that in general might not be a normal chain in \( G \) but it is still an exhausting chain in \( G \) such that for \( 0 \leq j \leq m \),

\[
\lim_{n \to \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = 0.
\]
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If $G_0$ is non-trivial, $G \cap G_0$ is non-trivial, free abelian and central, so part (i) from Theorem 1.4 follows from [9, lemma 7.2].

It remains to consider part (ii) from Theorem 1.4. From Theorem 2.8, $G$ has a subgroup of finite index $H = H_1 \times \cdots \times H_{r_0}$ where each $H_i$ is a limit group over Droms RAAGs. Following the proof of [9, theorem E], we can choose $(B_i)$ an exhausting normal chain in $H$ such that $B_i = (B_i \cap H_1) \times \cdots \times (B_i \cap H_{r_0})$. Then, as in [9], we can deduce that for $j \geq 1$,

$$
\lim_{n \to \infty} \frac{\dim H_j(B_n, K)}{[G : B_n]} = (-1)^{r_0} \delta_j r_0 \chi(G),
$$

where $\delta_{j, r_0}$ is the Kronecker symbol.

A limit group over a Droms RAAG does not contain a direct product of two or more non-abelian free groups, since limit groups over Droms RAAGs are coherent (see [11, corollary 7.8]) and the direct product of two non-abelian free groups is not coherent. In addition, every limit group over a Droms RAAG that has trivial center contains a non-abelian free group (see, for instance, [23, property 2-10]), so $r_0 = \rho$ unless one or more $H_i$ has non-trivial center. If some $H_i$ has non-trivial center, $\chi(H_i) = 0$ and so $\chi(H) = 0$ and $\chi(G) = 0$.

Proof of Theorem 1.6. In Theorem 5.4 Bridson and Kochloukova proved a low dimensional homotopical version of Theorem 1.4 for limit groups over free groups. The proof of Theorem 5.4 relies on some general results about residually finite groups, see [9, lemma 8.1, lemma 8.2], and some specific results for limit groups over free groups. The properties of limit groups over free groups that are used in the proof of Theorem 5.4 still hold for the class of limit groups over Droms RAAGs and correspond to Theorem 1.1 and Theorem 2.10, so the same proof can be used in order to show Theorem 1.6.

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