Delta-nabla optimal control problems

Ewa Girejko¹, Agnieszka B Malinowska¹ and Delfim FM Torres²

Abstract
We present a unified treatment to control problems on an arbitrary time scale by introducing the study of forward-backward optimal control problems. Necessary optimality conditions for delta-nabla isoperimetric problems are proved, and previous results in the literature are obtained as particular cases. As an application of the results of the paper we give necessary and sufficient Pareto optimality conditions for delta-nabla bi-objective optimal control problems.

Keywords
Optimal control, isoperimetric problems, Pareto optimality, time scales

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1. Introduction
In order to deal with nontraditional applications in areas such as medicine, economics, or engineering, where the system dynamics are described on a time scale partly continuous and partly discrete, or to accommodate nonuniform sampled systems, one needs to work with systems defined on a so-called time scale – see, e.g., Atici et al. (2006), Atici and Uysal (2008) and Malinowska and Torres (2010b).

The optimal control theory on time scales was introduced in the beginning of the 21st century in the simpler framework of the calculus of variations, and is now a fertile area of research in control and engineering (Malinowska and Torres, 2010c; Seiffertt et al., 2008). In the literature there are two different approaches to the problems of optimal control on time scales: some authors use the delta calculus (Almeida and Torres, 2009a; Bartosiewicz and Torres, 2008; Bohner, 2004; Bohner et al., 2010; Ferreira and Torres, 2008; Malinowska and Torres, 2009a; Malinowska et al., 2010), while others prefer the nabla methodology (Atici et al., 2006; Atici and Uysal, 2008; Martins and Torres, 2009). In this paper we propose a simple and effective unification of the delta and nabla approaches of optimal control on time scales. More precisely, we consider the problem of minimizing or maximizing a delta-nabla cost integral functional

\[
\mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta\left(t, y^\alpha(t), y^\lambda(t)\right) \Delta t + \gamma_2 \int_a^b L_V\left(t, y^\beta(t), y^\nu(t)\right) \nabla t
\]

subject to given boundary conditions and an isoperimetric constraint of the form

\[
\mathcal{K}(y) = k_1 \int_a^b K_\Delta\left(t, y^\alpha(t), y^\lambda(t)\right) \Delta t + k_2 \int_a^b K_V\left(t, y^\beta(t), y^\nu(t)\right) \nabla t = k.
\]

¹Faculty of Computer Science, Białystok University of Technology, Poland
²Department of Mathematics, University of Aveiro, Portugal

Corresponding author:
Delfim FM Torres, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
Email: delfim@ua.pt
Our main results include Euler–Lagrange necessary optimality type conditions for the delta-nabla isoperimetric problems in equations 1 and 2 (see Section 3.1). Isoperimetric problems have found a broad class of important applications throughout the centuries. Concrete isoperimetric problems in engineering have been investigated by a number of authors (cf. Almeida and Torres (2009b) and Curtis (2004), and references therein. Here, as an application of our results, we obtain the recent results of Almeida and Torres (2009a), Atici et al. (2006), Bohner (2004), and Ferreira and Torres (2010) as straightforward corollaries. In Section 3.2 we consider delta-nabla bi-objective problems. Our more general approach to optimal control in terms of the delta-nabla problem in equations 1 and 2 allows us to obtain necessary and sufficient conditions for Pareto optimality. The results of the paper are illustrated by several examples.

2. Preliminaries

We assume the reader to be familiar with the calculus on time scales. For an introduction to the subject we refer to the seminal papers Aulbach and Hilger (1990) and Hilger (1990), the nice survey Agarwal et al. (2002), and the books Bohner and Peterson (2001), Bohner and Peterson (2003), and Lakshmikantham et al. (1996).

Throughout the whole paper we assume \( \mathbb{T} \) to be a given time scale with \( a, b \in \mathbb{T} \), and we set \( I := [a, b] \cap \mathbb{T} \) for \( [a, b] \subseteq \mathbb{R} \). Moreover, we define \( I_{e} := I^{e} \cap I_{e} \) with the standard notation \( I^{e} = I \setminus (\rho(b), b] \) and \( I_{e} = I \setminus [a, \sigma(a)] \).

We recall some necessary results. If \( y \) is delta differentiable at \( t \in \mathbb{T} \), then \( y^{\Delta}(t) = y(t) + \mu(t)y^{\sigma}(t) \); if \( y \) is nabla differentiable at \( t \), then \( y^{\nabla}(t) = y(t) - \nu(t)y^{\sigma}(t) \). If the functions \( f, g : \mathbb{T} \rightarrow \mathbb{R} \) are delta and nabla differentiable with continuous derivatives, then the following formulas of integration by parts hold:

\[
\begin{align*}
\int_{a}^{b} f^{\sigma}(t)g^{\Delta}(t) \Delta t &= (fg)(b) - \int_{a}^{b} f^{\Delta}(t)g^{\nabla}(t) \Delta t, \\
\int_{a}^{b} f^{\Delta}(t)g^{\Delta}(t) \Delta t &= (fg)(b) - \int_{a}^{b} f^{\Delta}(t)g^{\nabla}(t) \Delta t, \\
\int_{a}^{b} f^{\Delta}(t)g^{\nabla}(t) \nabla t &= (fg)(b) - \int_{a}^{b} f^{\nabla}(t)g^{\Delta}(t) \nabla t, \\
\int_{a}^{b} f^{\nabla}(t)g^{\nabla}(t) \nabla t &= (fg)(b) - \int_{a}^{b} f^{\nabla}(t)g^{\Delta}(t) \nabla t.
\end{align*}
\]

The following fundamental lemma of the calculus of variations on time scales, involving a nabla derivative and a nabla integral, was proved in Martins and Torres (2009).

**Lemma 1.** (The nabla Dubois–Reymond lemma – cf. Lemma 14 of Martins and Torres (2009)) Let \( f \in C_{2d}(I, \mathbb{R}) \). If

\[
\int_{a}^{b} f(t)\eta^{\nabla}(t) \nabla t = 0 \quad \text{for all } \eta \in C_{2d}(I, \mathbb{R})
\]

such that \( \eta(a) = \eta(b) = 0 \),

then \( f(t) \equiv c \) for all \( t \in I_{e} \), where \( c \) is a constant.

**Lemma 2.** (The analogous delta version of Lemma 1)

Then \( g(t) \equiv c \) on \( I^{e} \) for some \( c \in \mathbb{R} \).

**Proposition 3.** (cf. Theorems 2.5 and 2.6 of Atici and Guseinov (2002)) (i) If \( f : \mathbb{T} \rightarrow \mathbb{R} \) is delta differentiable on \( \mathbb{T}^{e} \) and \( f^{\Delta} \) is continuous on \( \mathbb{T}^{e} \), then \( f \) is nabla differentiable on \( \mathbb{T}_{e} \) and

\[
(\nabla f)^{\sigma}(t) = \left( f^{\nabla}\right)^{\sigma}(t) \quad \text{for all } t \in \mathbb{T}_{e}.
\]

(ii) If \( f : \mathbb{T} \rightarrow \mathbb{R} \) is nabla differentiable on \( \mathbb{T}_{e} \) and \( f^{\nabla} \) is continuous on \( \mathbb{T}_{e} \), then \( f \) is delta differentiable on \( \mathbb{T}^{e} \) and

\[
\nabla f^{\Delta}(t) = \left( f^{\Delta}\right)^{\sigma}(t) \quad \text{for all } t \in \mathbb{T}^{e}.
\]

**Proposition 4.** (cf. Theorem 2.8 of Atici and Guseinov (2002)) Let \( a, b \in \mathbb{T} \) with \( a < b \) and let \( f \) be a continuous function on \( [a, b] \). Then,

\[
\begin{align*}
\int_{a}^{b} f(t) \Delta t &= \int_{a}^{b} f(t)(b) - \int_{a}^{b} f(t) \Delta t + (b - \rho(b))f^{\sigma}(b), \\
\int_{a}^{b} f(t) \Delta t &= (\sigma(a) - a)f(a) + \int_{a}^{b} f(t) \Delta t, \\
\int_{a}^{b} f(t) \nabla t &= \int_{a}^{b} f(t) \nabla t + (b - \rho(b))f(b), \\
\int_{a}^{b} f(t) \nabla t &= (\sigma(a) - a)f^{\sigma}(a) + \int_{a}^{b} f(t) \nabla t.
\end{align*}
\]

We end our brief review of the calculus on time scales with a relationship between the delta and nabla integrals.
Proposition 5. (cf. Proposition 7 of Gürses et al. (2005)) If function \( f : \mathbb{T} \to \mathbb{R} \) is continuous, then for all \( a, b \in \mathbb{T} \) with \( a < b \) we have
\[
\int_a^b f(t) \Delta t = \int_a^b f^p(t) \Delta t,
\]
and the isoperimetric constraint
\[
\mathcal{K}(y) = k_1 \int_a^b K_\Delta[y](t) \Delta t + k_2 \int_a^b K_\mathcal{V}[y](t) \Delta t = k,
\]
where \( K_\Delta(\cdot, \cdot, \cdot) \) and \( K_\mathcal{V}(\cdot, \cdot, \cdot) \) are given smooth functions from \( \mathbb{T} \times \mathbb{R}^2 \) to \( \mathbb{R} \).

3. Main results

Let \( \mathbb{T} \) be a given time scale with \( a, b \in \mathbb{T}, a < b \), and \( \mathbb{T} \cap (a, b) \neq \emptyset \); let \( L_\Delta(\cdot, \cdot, \cdot) \) and \( L_\mathcal{V}(\cdot, \cdot, \cdot) \) be two given smooth functions from \( \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R} \) and \( \gamma_1, \gamma_2 \in \mathbb{R} \). Our results are trivially generalized for admissible functions \( y : \mathbb{T} \to \mathbb{R}^n \) but for simplicity of presentation we restrict ourselves to the scalar case \( n = 1 \).

3.1. Delta-nabla isoperimetric problems

We consider the delta-nabla integral functional
\[
\mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta(t, y^\sigma(t), y^\Delta(t)) \Delta t + \gamma_2 \int_a^b L_\mathcal{V}(t, y^\rho(t), y^\mathcal{V}(t)) \Delta t.
\]
For brevity we introduce the operators \([y]\) and \(\{y\}\) defined by
\[
[y](t) = (t, y^\sigma(t), y^\Delta(t)) \quad \text{and} \quad \{y\}(t) = (t, y^\rho(t), y^\mathcal{V}(t)).
\]
Then we can write
\[
\mathcal{L}_\Delta(y) = \int_a^b L_\Delta[y](t) \Delta t,
\]
\[
\mathcal{L}_\mathcal{V}(y) = \int_a^b L_\mathcal{V}[y](t) \Delta t,
\]
\[
\mathcal{L}(y) = \gamma_1 \mathcal{L}_\Delta(y) + \gamma_2 \mathcal{L}_\mathcal{V}(y) = \gamma_1 \int_a^b L_\Delta[y](t) \Delta t + \gamma_2 \int_a^b L_\mathcal{V}[y](t) \Delta t.
\]
Let \( \alpha, \beta, \gamma_1, \gamma_2, k_1, \) and \( k_2 \) be given real numbers. Let us denote by \( C^r_{\Delta}(I, \mathbb{R}) \) the class of functions \( y : I \to \mathbb{R} \) with \((\gamma_1 + k_1) y^\Delta \) continuous on \( I^c \) and \((\gamma_2 + k_2) y^\mathcal{V} \) continuous on \( I_c \). We consider the question of finding \( y \in C^r_{\Delta}(I, \mathbb{R}) \) that is a solution to the problem
\[
\text{extremize } \mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta[y](t) \Delta t + \gamma_2 \int_a^b L_\mathcal{V}[y](t) \Delta t
\]
subject to the boundary conditions
\[
y(a) = \alpha, \quad y(b) = \beta,
\]
and the isoperimetric constraint
\[
\mathcal{K}(y) = k_1 \int_a^b K_\Delta[y](t) \Delta t + k_2 \int_a^b K_\mathcal{V}[y](t) \Delta t = k.
\]
necessarily an extremizer (it is just a candidate to extremizer given by the first-order necessary conditions).

Associated to the problem in equations 8–10 we introduce the following notation:

\[
H_\Delta[y, \lambda](t) := H_\Delta(t, y(t, \lambda), y^\Delta(t, \lambda)) := \gamma_1 L_\Delta[y](t) - k_1 \lambda K[y](t),
\]

\[
H_V[y, \lambda](t) := H_V(t, y(t, \lambda), y^\Delta(t, \lambda)) := \gamma_2 L_V[y](t) - k_2 \lambda K[y](t).
\]

We look for \( H_\Delta \) and \( H_V \) to be functions of four independent variables, and we denote the partial derivatives of \( H_\Delta(\cdot, \cdot, \cdot, \cdot) \) and \( H_V(\cdot, \cdot, \cdot, \cdot) \) with respect to their \( i \)th argument, \( i = 1, 2, 3, 4 \), by \( \partial_i H_\Delta \) and \( \partial_i H_V \), respectively.

**Theorem 9.** (Necessary optimality conditions for normal extremizers of a delta-nabla isoperimetric problem). If \( \tilde{y} \in C^1_\lambda(I, \mathbb{R}) \) is a normal extremizer to the isoperimetric problem in equations 8–10, then there exists \( \lambda \in \mathbb{R} \) such that \( \tilde{y} \) satisfies the following delta-nabla integral equations:

\[
\partial_3 H_\Delta[\tilde{y}, \lambda](t) + \partial_3 H_V[\tilde{y}, \lambda](t)
- \left( \int_a^t \partial_2 H_\Delta[\tilde{y}, \lambda](\tau) \Delta \tau + \int_a^t \partial_2 H_V[\tilde{y}, \lambda](\tau) \Delta \tau \right) = \text{const}
\]

\[
\partial_3 H_\Delta[\tilde{y}, \lambda](t) + \partial_3 H_V[\tilde{y}, \lambda](\sigma(t))
- \left( \int_a^t \partial_2 H_\Delta[\tilde{y}, \lambda](\tau) \Delta \tau + \int_a^t \partial_2 H_V[\tilde{y}, \lambda](\tau) \Delta \tau \right) = \text{const}
\]

where \( H_\Delta \) and \( H_V \) are defined by equation 11.

**Proof.** Consider a variation of \( \tilde{y} \), say \( \tilde{y} = \hat{y} + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2 \), where \( \eta_i \in C^1_\lambda(I, \mathbb{R}) \) and \( \eta_i(a) = \eta_i(b) = 0 \), \( i \in \{1, 2\} \), and \( \varepsilon_i \) is a sufficiently small parameter \( (\varepsilon_1 \text{ and } \varepsilon_2 \text{ must be such that } \| \tilde{y} - \hat{y} \|_1 < \delta \text{ for some } \delta > 0) \). Here \( \eta_1 \) is an arbitrary fixed function and \( \eta_2 \) is a fixed function that will be chosen later. Define the real function

\[
\tilde{K}(\varepsilon_1, \varepsilon_2) = K(\tilde{y}) = k_1 \int_a^b K_\Delta[\tilde{y}](t) \Delta t + k_2 \int_a^b K_V[\tilde{y}](t) \Delta t - k.
\]

We have

\[
\frac{\partial \tilde{K}}{\partial \varepsilon_2}(0,0) = k_1 \int_a^b \frac{\partial \tilde{K}}{\partial \varepsilon_2}(0,0) = k_1 \int_a^b \frac{\partial \tilde{K}}{\partial \varepsilon_2}(0,0) + k_2 \int_a^b \frac{\partial \tilde{K}}{\partial \varepsilon_2}(0,0) \Delta t.
\]
As \( \tilde{y} \) is a normal extremizer, we conclude by Lemma 1 that there exists \( \eta_2 \) such that \( \frac{\partial \tilde{L}}{\partial \tilde{y}}(0,0) \neq 0 \). Note that the same result can be obtained by transforming the nabla integral in equation 15 to a delta integral by means of equation 7, and then using Lemma 2. Since \( \tilde{K}(0,0) = 0 \), by the implicit function theorem we conclude that there exists a function \( \varepsilon_2 \) defined in the neighborhood of zero such that \( \tilde{K}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0 \), i.e. we may choose a subset of variations \( \tilde{y} \) satisfying the isoperimetric constraint. Let us now consider the real function

\[
\tilde{L}(\varepsilon_1, \varepsilon_2) = \mathcal{L}(\tilde{y}) = \gamma_1 \int_a^b L_\Delta[\tilde{y}](t) \, \Delta t + \gamma_2 \int_a^b L_V[\tilde{y}](t) \, \nabla t.
\]

By hypothesis, \( (0, 0) \) is an extremal of \( \tilde{L} \) subject to the constraint \( \tilde{K} = 0 \) and \( \nabla \tilde{K}(0,0) \neq 0 \). By the Lagrange multiplier rule, there exists some real \( \lambda \) such that \( \nabla \tilde{L}(0,0) - \lambda \nabla \tilde{K}(0,0) = 0 \). Having in mind that \( \eta_1(a) = \eta_1(b) = 0 \), we can write

\[
\begin{align*}
\frac{\partial \tilde{L}}{\partial \varepsilon_1}(0,0) &= \gamma_1 \int_a^b \left( \partial_3 L_\Delta[\tilde{y}](t) - \int_a^t \partial_2 L_\Delta[\tilde{y}](\tau) \, d\tau \right) \eta_1^\Delta(t) \, \Delta t \\
+ &\gamma_2 \int_a^b \left( \partial_3 L_V[\tilde{y}](t) - \int_a^t \partial_2 L_V[\tilde{y}](\tau) \, d\tau \right) \eta_1^V(t) \, \nabla t
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial \tilde{K}}{\partial \varepsilon_1}(0,0) &= k_1 \int_a^b \left( \partial_3 L_\Delta[\tilde{y}](t) - \int_a^t \partial_2 L_\Delta[\tilde{y}](\tau) \, d\tau \right) \eta_1^\Delta(t) \, \Delta t \\
+ &k_2 \int_a^b \left( \partial_3 L_V[\tilde{y}](t) - \int_a^t \partial_2 L_V[\tilde{y}](\tau) \, d\tau \right) \eta_1^V(t) \, \nabla t.
\end{align*}
\]

Transforming the delta integrals in the above equalities to nabla integrals by means of equation 6 and using equation 4, we obtain

\[
\begin{align*}
\frac{\partial \tilde{L}}{\partial \varepsilon_1}(0,0) &= \gamma_1 \int_a^b \left( \gamma_1 m^\Delta(t) + \gamma_2 n(t) \right) \eta_1^V(t) \, \nabla t \\
\frac{\partial \tilde{K}}{\partial \varepsilon_1}(0,0) &= \int_a^b \left( k_1 f^\Delta(t) + k_2 g(t) \right) \eta_1^V(t) \, \nabla t.
\end{align*}
\]

Therefore,

\[
\int_a^b \eta_1^V(t) \left( \gamma_1 m^\Delta(t) + \gamma_2 n(t) - \lambda (k_1 f^\Delta(t) + k_2 g(t)) \right) \, \nabla t = 0.
\]

(18)

Since equation 18 holds for any \( \eta_1 \), by Lemma 1 we have

\[
\gamma_1 m^\Delta(t) + \gamma_2 n(t) - \lambda (k_1 f^\Delta(t) + k_2 g(t)) = c
\]

for some \( c \in \mathbb{R} \) and all \( t \in I_\lambda \). Hence, the condition in equation 12 holds. Equation 12 can also be obtained by transforming nabla integrals to delta integrals by means of equation 7 and then using Lemma 2. Equation 13 can be shown in a totally analogous way.

**Example 10.** (Normal extremals) (a) Let \( \mathbb{T} = \{1, 3, 4\} \) and consider the problem

\[
\begin{align*}
\text{minimize } & \mathcal{L}(y) = \int_a^4 t(y^\Delta(t))^2 \, \nabla t \\
\text{subject to the constraint } & \mathcal{K}(y) = \int_a^4 t(y^\Delta(t))^2 \, d\tau = \frac{105}{242}.
\end{align*}
\]

Since \( L_V = t(y^\Delta)^2 \) and \( K_\Delta = t(y^\Delta)^2 \), we have

\[
\partial_2 L_V = 0, \quad \partial_3 L_V = 2t y^\Delta, \quad \partial_2 K_\Delta = 0, \quad \partial_3 K_\Delta = 2 t y^\Delta.
\]

Let us assume for the moment that we are in a condition to apply Theorem 9. Applying equation 13 of Theorem 9 we get the following delta-nabla differential equation:

\[
2 \sigma(t) y^\Delta(\sigma(t)) - \lambda 2 t y^\Delta(t) = C, \quad t \in \{1, 3\},
\]

where \( C \in \mathbb{R} \). By equation 5 we can write the above equation in the form

\[
2 \sigma(t) y^\Delta(t) - \lambda 2 t y^\Delta(t) = C, \quad t \in \{1, 3\}.
\]
Since $y^\Delta(1) = (y^3 - y(1))/2 = y^3/2$ and $y^\Delta(3) = y(4) - y(3) = 1 - y(3)$, solving equation 22 subject to the boundary conditions $y(1) = 0$ and $y(4) = 1$ we get

$$
\begin{align*}
\frac{3}{8}(1 - y(3)) - \lambda y(3) &= C, \\
8(1 - y(3)) - 6\lambda(1 - y(3)) &= C,
\end{align*}
$$

what implies

$$
y(t) = \begin{cases} 
0 & \text{if } t = 1, \\
\frac{9}{11} \frac{t}{t+1} & \text{if } t = 3, \\
1 & \text{if } t = 4.
\end{cases}
$$

Substituting equation 23 into equation 21 we obtain

$$\lambda_1 = \frac{11}{13}, \quad \lambda_2 = \frac{131}{127}.$$ Hence, we get two extremals, $y_1$ and $y_2$, corresponding to $\lambda_1$ and $\lambda_2$, respectively:

$$y_1(t) = \begin{cases} 
0 & \text{if } t = 1, \\
\frac{9}{11} \frac{t}{t+1} & \text{if } t = 3, \\
1 & \text{if } t = 4.
\end{cases}$$

$$y_2(t) = \begin{cases} 
0 & \text{if } t = 1, \\
\frac{69}{77} \frac{t}{t+1} & \text{if } t = 3, \\
1 & \text{if } t = 4.
\end{cases}$$

One can easily check that $L(y_1) = \frac{25}{22}$ and $L(y_2) = \frac{1345}{1072}$. We now show that $y_1$ is not an extremal for $K$. Indeed,

$$
\begin{align*}
\partial_1 K_\Delta[y_1(t)] - \int_a^t \partial^\sigma(t) \partial_2 K_\Delta[y_1(t)](\tau) \Delta \tau + \partial_3 K_\Delta[y_1(t)](\sigma(t)) &
\quad - \int_a^{\sigma(t)} \partial_2 K_\Delta[y_1(t)](\tau) \nabla \tau = \partial_3 K_\Delta[y_1(t)] \\
&= 2t\partial_1^\Delta(t) = \begin{cases} 
0 & \text{if } t = 1, \\
\frac{9}{11} \frac{t}{t+1} & \text{if } t = 3, \\
1 & \text{if } t = 4.
\end{cases}
\end{align*}
$$

Thus $y_1$ is a candidate local minimizer to the problem in equations 19–21 .

(b) Let $T = \{1, 3, 4\}$ and consider the problem

$$
\begin{align*}
\text{minimize } & \quad L(y) = \int_1^4 t(y^\Delta)(t)^2 \Delta t \\
\text{subject to the constraint } & \quad y(1) = 0, \quad y(4) = 1
\end{align*}
$$

subject to the constraint

$$
K(y) = \int_1^4 t(y^\nu(t))^2 \Delta t = \frac{25}{22}.
$$

Proceeding analogously as before, we find

$$y_1(t) = \begin{cases} 
0 & \text{if } t = 1, \\
\frac{9}{11} \frac{t}{t+1} & \text{if } t = 3, \\
1 & \text{if } t = 4
\end{cases}$$

as a candidate local minimizer to problem 24–26.

As a particular case of Theorem 9 we obtain the following result.

**Corollary 11.** (Necessary optimality condition for normal extremizers of a delta isoperimetric problem – cf. Theorem 3.4 of Ferreira and Torres (2010)). Suppose that the problem of minimizing

$$J(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t$$

subject to the boundary conditions $y(a) = y_a$, $y(b) = y_b$, and the isoperimetric constraint

$$K(y) = \int_a^b g(t, y^\sigma(t), y^\Delta(t)) \Delta t = l$$

has a local solution at $y$ in the class of functions $y : [a, b] \to \mathbb{R}$ such that $y^A$ exists and is continuous on $[a, b]^*$, and that $y$ is not an extremal for the functional $I_t$. Then, there exists a Lagrange multiplier constant $\lambda$ such that $y$ satisfies

$$
\Delta
\frac{\Delta}{\Delta l} [\partial_1 F(y^\sigma(t), y^\Delta(t))] - \partial_2 F(t, y^\sigma(t), y^\Delta(t)) = 0
$$

for all $t \in [a, b]^*$

with $F(t, x, v) = L(t, x, v) - \lambda g(t, x, v)$.

**Proof.** The result follows from Theorem 9 by considering the particular case $y_1 = k_1 = 1$ and $y_2 = k_2 = 0$. □

One can easily cover abnormal extremizers within our result by introducing an extra multiplier $\lambda_0$. Let

$$
H_\Delta[\tilde{y}, \lambda_0, \lambda](t) := H_\Delta(t, y^\sigma(t), y^\Delta(t), \lambda_0, \lambda) \\
:= \gamma_1 \lambda_0 L[\tilde{y}] \Delta(t) - k_1 \lambda_\Delta[\tilde{y}] \Delta(t) \\
H_\nu[\tilde{y}, \lambda_0, \lambda](t) := H_\nu(t, y^\sigma(t), y^\Delta(t), \lambda_0, \lambda) \\
:= \gamma_2 \lambda_0 L[\tilde{y}] \Delta(t) - k_2 \lambda_\nu[\tilde{y}] \Delta(t)
$$

(27)

**Theorem 12.** (Necessary optimality conditions for normal and abnormal extremizers of a delta-isoperimetric problem). If $\tilde{y} \in C^3\cap(I, \mathbb{R})$ is an extremizer to the isoperimetric problem in equations 8–10, then there exist two constants $\lambda_0$ and $\lambda$, not both zero, such that $\tilde{y}$ satisfies the following delta-nabla integral equations:

$$
\partial_1 H_\Delta[\tilde{y}, \lambda_0, \lambda](\rho(t)) + \partial_3 H_\nu[\tilde{y}, \lambda_0, \lambda](t) \\
\quad - \int_a^{\sigma(t)} \partial_2 H_\Delta[\tilde{y}, \lambda_0, \lambda](\tau) \Delta \tau + \partial_3 H_\nu[\tilde{y}, \lambda_0, \lambda](\sigma(t)) \\
\quad - \int_a^{\sigma(t)} \partial_2 H_\Delta[\tilde{y}, \lambda_0, \lambda](\tau) \Delta \tau = \text{const} \quad \forall t \in I,
$$

where $H_\Delta$ and $H_\nu$ are defined by equation 27.
Proof. Following the proof of Theorem 9, since \((0, 0)\) is an extremal of \(L\) subject to the constraint \(K = 0\), the extended Lagrange multiplier rule (see, for instance, Theorem 4.1.3 of van Brunt (2004)) asserts the existence of reals \(\lambda_0\) and \(\lambda\), not both zero, such that \(\nabla(\lambda_0 L(0, 0) - \lambda K(0, 0)) = 0\). Therefore,

\[
\int_a^b y(t) \left[ \lambda_0 (y_1 m(t) + y_2 n(t)) - \lambda (k_1 f(t) + k_2 g(t)) \right] \Delta t = 0.
\]

(30)

Since equation 30 holds for any \(\eta_1\), by Lemma 1 we have

\[
\lambda_0 (y_1 m(t) + y_2 n(t)) - \lambda (k_1 f(t) + k_2 g(t)) = c
\]

for some \(c \in \mathbb{R}\) and all \(t \in [a, b]\). This establishes equation 28. Equation 29 can be shown using a similar technique.

\[\square\]

Remark 13. If \(\hat{y} \in C^1([a, b])\) is a normal extremizer to the isoperimetric problem in equations 8–10, then we can choose \(\lambda_0 = 1\) in Theorem 9 and obtain Theorem 9. For abnormal extremizers, Theorem 12 holds with \(\lambda_0 = 0\). The condition \((\lambda_0, \lambda) \neq 0\) guarantees that Theorem 12 is a useful necessary optimality condition.

Example 14. (Abnormal extremal) Let \(\mathcal{T} = \{1, 3, 4\}\) and consider the problem

\[
\text{minimize } \mathcal{L}(y) = \int_1^4 t (y^n(t))^2 \Delta t
\]

subject to the constraint

\[
\mathcal{K}(y) = \int_1^4 t (y^n(t))^2 \Delta t = \frac{12}{11}.
\]

Applying equation 28 of Theorem 12 we get the following delta-nabla differential equation:

\[
\lambda_0 2 \rho(t) y^n(t) - \lambda 2 t y^n(t) = C, \quad t \in [3, 4],
\]

where \(C \in \mathbb{R}\). By equation 4 we can write the above equation in the form

\[
\lambda_0 2 \rho(t) y^n(t) - \lambda 2 t y^n(t) = C, \quad t \in [3, 4].
\]

(34)

Substituting \(t = 3\) and \(t = 4\) into equation 34 we obtain

\[
\begin{cases}
\lambda_0 y(3) - 3 \lambda y(3) = C \\
6 \lambda_0 (1 - y(3) - 8 \lambda (1 - y(3)) = C.
\end{cases}
\]

If we put \(\lambda_0 = 1\), then the above system of equations has no solutions. Therefore, we fix \(\lambda_0 = 0\). In this case we obtain

\[
y_0(t) = \begin{cases}
0 & \text{if } t = 1 \\
\frac{2}{11} & \text{if } t = 3 \\
1 & \text{if } t = 4
\end{cases}
\]

as a candidate local minimizer to the problem in equations 31–33. Observe that \(y_0\) is an extremal for \(K\). Indeed,

\[
\frac{\partial K}{\partial y}(y_0) = 0.
\]

As a particular case of Theorem 12 we obtain the main result of Almeida and Torres (2009a).

Corollary 15. (Necessary optimality condition for normal and abnormal extremizers of a nabla isoperimetric problem – cf. Theorem 2 of Almeida and Torres (2009a)). Let \(\mathcal{T}\) be a time scale, \(a, b \in \mathcal{T}\) with \(a < b\). If \(\hat{y}\) is a local minimizer or maximizer to the problem

\[
\text{extimize } \int_a^b f(t, y^n(t), y^\Delta(t)) \Delta t
\]

\[
\int_a^b g(t, y^n(t), y^\Delta(t)) \Delta t = \Lambda
\]

\[
y(a) = \alpha, \quad y(b) = \beta
\]

in the class of functions \(y : [a, b] \to \mathbb{R}\) such that \(y^\Delta\) exists and is continuous on \([a, b]_\mathcal{V}\), then there exist two constants \(\lambda_0\) and \(\lambda\), not both zero, such that

\[
\frac{\nabla}{\Delta t} \left[ \frac{\partial G(t, y^n(t), y^\Delta(t))}{\partial t} \right] - \frac{\partial^2 G(t, y^n(t), y^\Delta(t))}{\partial t^2} = 0
\]

for all \(t \in [a, b]_\mathcal{V}\), where \(G(t, x, v) = \lambda_0 f(t, x, v) - \lambda g(t, x, v)\).

Proof. The result follows from Theorem 12 by considering the particular case \(\gamma_1 = k_1 = 0\) and \(\gamma_2 = k_2 = 1\).

Other interesting corollaries are easily obtained from Theorem 12:

Corollary 16. (The delta-nabla Euler–Lagrange equations on time scales (Girejko et al., 2010)). If \(\hat{y} \in C^1([a, b])\) is a local extremizer to the problem

\[
\text{extimize } \mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta(y(t)) \Delta t + \gamma_2 \int_a^b L_\nabla(y(t)) \Delta t
\]

\[
y(a) = \alpha, \quad y(b) = \beta
\]

\[y \in C^1([a, b], \mathbb{R}),
\]

\[\square\]
then $\hat{y}$ satisfies the following delta-nabla integral equations:

$$
\gamma_1 \left( \partial_1 L_\Delta[\hat{y}](\rho(t)) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau) \Delta \tau \right) \\
+ \gamma_2 \left( \partial_1 L_V[\hat{y}](t) - \int_a^t \partial_2 L_V[\hat{y}](\tau) \nabla \tau \right) = \text{const}
$$

for all $t \in I_\varepsilon$.

**Proof.** The result follows from Theorem 12 by considering the particular case $k_1 = k_2 = k = 0$, for which the isoperimetric constraint in equation 10 is trivially satisfied.

### 3.2. Delta-nabla bi-objective problems

We are now interested in studying the following bi-objective problem:

$$
\text{minimize } F(y) = \begin{bmatrix} L_\Delta(y) \\ L_V(y) \end{bmatrix}
$$

with

$$
L_\Delta(y) = \int_a^b L_\Delta(t, y^\alpha(t), y^\beta(t)) \Delta t = \int_a^b L_\Delta(y(t)) \Delta t,
$$

$$
L_V(y) = \int_a^b L_V(t, y^\alpha(t), y^\beta(t)) \nabla t = \int_a^b L_V(y(t)) \nabla t,
$$

and $y \in C^1_b(I, \mathbb{R})$, $y(a) = \alpha$, $y(b) = \beta$, $t \in I$. A solution to this vector optimization problem is understood in the Pareto sense.

**Definition 17.** (Locally Pareto optimal solution). A function $\hat{y} \in C^1_b(I, \mathbb{R})$ is called a local Pareto optimal solution if there exists $\alpha \delta > 0$ such that for each $\delta > 0$ there does not exist $y \in C^1_b(I, \mathbb{R})$ with $\|y - \hat{y}\|_{1, \infty} < \delta$ and

$$
L_\Delta(y) \leq L_\Delta(\hat{y}) \quad \text{and} \quad L_V(y) \leq L_V(\hat{y}),
$$

where at least one of the above inequalities is strict.

**Theorem 18.** (Necessity). If $\hat{y}$ is a local Pareto optimal solution to the bi-objective problem in equation 36, then $\hat{y}$ is a minimizer to the isoperimetric problems

$$
\text{minimize } L_\Delta(y) \text{ subject to } L_V(y) = L_V(\hat{y})
$$

and

$$
\text{minimize } L_V(y) \text{ subject to } L_\Delta(y) = L_\Delta(\hat{y})
$$

simultaneously.

**Proof.** A proof can be done similarly to the proof of Theorem 3.8 in Malinowska and Torres (2009b). \(\boxempty\)

**Example 19.** Let us consider $I = \{1, 3, 4\}$ and the bi-objective optimization problem in equation 36 with

$$
L_\Delta(y) = \int_1^4 t(y^\alpha(t))^2 \Delta t,
$$

$$
L_V(y) = \int_1^4 t(y^\beta(t))^2 \nabla t.
$$

We pose the question of finding local Pareto optimal solutions to equation 37 under the boundary conditions

$$
y(1) = 0, \quad y(4) = 1.
$$

Let us consider the following function:

$$
\hat{y}(t) = \begin{cases} 
0 & \text{if } t = 1 \\
\frac{9}{11} & \text{if } t = 3 \\
1 & \text{if } t = 4.
\end{cases}
$$

As is shown in Example 10, $\hat{y}$ is, simultaneously, a candidate minimizer to the problem

$$
\text{minimize } L_V(y) = \int_1^4 t(y^\beta(t))^2 \nabla t
$$

subject to

$$
L_\Delta(y) = \int_1^4 t(y^\alpha(t))^2 \Delta t = \frac{105}{242},
$$

and

$$
\text{minimize } L_\Delta(y) = \int_1^4 t(y^\alpha(t))^2 \Delta t
$$

subject to

$$
y(1) = 0, \quad y(4) = 1.
$$
subject to
\[ L_v(y) = \int_1^d t \left( y^V(t) \right)^2 \, dt \leq \frac{25}{22} \]

According to Theorem 18, the function \( y \) is a candidate Pareto optimal solution to the bi-objective problem in equations 37–38.

Theorem 18 shows that necessary optimality conditions to isoperimetric problems (see Section 3.1) are also necessary for local Pareto optimality of a bi-objective variational problem on time scales. Indeed, the functional in equation 8 in the particular cases when \( \gamma_1 = 1 \) and \( \gamma_2 = 0 \) or \( \gamma_1 = 0 \) and \( \gamma_2 = 1 \) is reduced either to \( L(y) = L_\Delta(y) \) or to \( L(y) = L_v(y) \), respectively.

The next theorem asserts that sufficient conditions of optimality for scalar optimal control problems are also sufficient conditions for Pareto optimality.

Theorem 20. ( Sufficiency.) A local minimizer \( \hat{y} \) to the functional \( \gamma L_\Delta(y) + (1 - \gamma) L_v(y) \) with \( \gamma \in (0, 1) \) is a local Pareto optimal solution to the bi-objective problem in equation 36.

Proof. A proof can be done similarly to the proof of Theorem 3.7 in Malinowska and Torres (2009b). □

Example 21.

a. Let us consider \( \mathbb{T} = \{1, 3, 4\} \) and the bi-objective optimization problem in equation 36 defined by
\[ L_\Delta(y) = \int_1^d t \left( y^\Delta(t) \right)^2 \, dt, \]
\[ L_v(y) = \int_1^d t \left( y^V(t) \right)^2 \, dt \]
subject to
\[ y(1) = 0, \quad y(4) = 1. \] 

By Theorem 20 we can find Pareto optimal solutions to this problem by considering the family of problems
\[ \min \gamma L_\Delta(y) + (1 - \gamma) L_v(y) \]
\[ y(1) = 0, \quad y(4) = 1, \]

where \( \gamma \in (0, 1) \). Using the condition in equation 35 of Corollary 16 we get the following equation:
\[ y \int_0^\rho(t) y^\sigma(t) \, d\tau + (1 - \gamma) \int_0^1 (y^\rho(t) - 3) \, d\tau = c \quad \forall t \in \{1, 2\} \]

for some \( c \in \mathbb{R} \). Substituting \( t = 1 \) and \( t = 2 \) into equation 44 we obtain
\[ y \int_0^\rho(t) y^\sigma(t) \, d\tau + (1 - \gamma) \int_0^1 (y^\rho(t) - 3) \, d\tau = c, \]
\[ y \int_0^\rho(t) y^\sigma(t) \, d\tau + (1 - \gamma) \int_0^1 (y^\rho(t) - 3) \, d\tau = c, \]

and from this we have \( y(1) = 3 - 3\gamma, \gamma \in (0, 1) \). Since \( L_\Delta(\cdot, \cdot, \cdot) \) and \( L_v(\cdot, \cdot, \cdot) \) are jointly convex with respect to the second and third arguments for any \( t \in \mathbb{T} \), the local Pareto optimal solutions to the problem in equations 42–43 are
\[ y(t) = \begin{cases} 0, & \text{if } t = 0 \\ k, & \text{if } t = 1, \quad k \in (0, 3) \\ 0, & \text{if } t = 2. \end{cases} \]

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