A COMPARISON OF MINIMAL SYSTEMS FOR
CONSTRUCTIVE ANALYSIS

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ABSTRACT. We establish a precise relation between the minimal system of
analysis $\mathbf{M}$, a subsystem of the formal axiomatic system of intuitionistic anal-
ysis $\mathbf{FIM}$ of S. C. Kleene, and elementary analysis $\mathbf{EL}$ of A. S. Troelstra, two
weak formal systems of two-sorted intuitionistic arithmetic, both widely used as
basis for (various forms of) constructive analysis. We show that $\mathbf{EL}$ is weaker
than $\mathbf{M}$, by introducing an axiom schema $\text{CF}_d$ asserting that every decidable
(in the sense that it satisfies the law of the excluded middle) predicate of nat-
ural numbers has a characteristic function. As it turns out, $\text{CF}_d$ captures the
essential difference of the two minimal theories. Moreover, we obtain the con-
servativity of $\mathbf{M}$ over first-order intuitionistic arithmetic, and the eliminability
of Church’s $\lambda$ from $\mathbf{EL}$ by modifying the proof of J. Rand Moschovakis of the
 corresponding result for $\mathbf{M}$. We also show that $\mathbf{EL}$ and the formal theory $\mathbf{BIM}$
(\textit{Basic Intuitionistic Mathematics}) of W. Veldman are essentially equivalent,
and we compare some more systems of two-sorted intuitionistic arithmetic.

INTRODUCTION

The investigation in constructive analysis is carried out in a multitude of formal
or informal languages and systems, whose relationships remain in many aspects
unclear. Starting an attempt to elucidate the relations among the various for-
malisms used, we compare particular formal systems of intuitionistic two-sorted
arithmetic that are neutral and formalize the common part of the main varieties of
constructive analysis (corresponding to Brouwer’s intuitionism, Markov’s Russian
recursive mathematics and Bishop’s constructivism), and of classical analysis also.

Despite the different ways in which it can be understood, constructivity affects
the logic inherent in mathematical reasoning, as L. E. J. Brouwer, the founder of
intuitionism, realized and showed, leading to the rejection of the unrestricted use
of the law of the excluded middle. So all the systems that we study are based
on intuitionistic (predicate) logic. In addition, they have the following common
features: they are formulated in two-sorted languages, with variables for natural
numbers and one-place number-theoretic functions (in contrast to set variables
used for the second sort in the classical case, [\textit{Simpson}]), and they have constants
for (different selections of) only primitive recursive functions and functionals, and
equality between natural numbers, with equality between functions defined ex-
tensionally. $\lambda$-abstraction is included in some of them. The function existence
principles assumed are all weak and do not involve real choice.

The differences in the languages as well as the interplay between the possibilities
provided by the languages and the assumed (if any) function existence principles

The content of this paper (except the results of 8.1) is from \textit{VafeiadouPhD}, written under
the supervision of Prof. Joan Rand Moschovakis, to whom the author is deeply grateful for the
invaluable privilege to learn from her, for the encouragement, and for her numerous suggestions
and comments on this work. The author also thanks Iris Loeb for some motivating questions.
do not allow, in most cases, to determine directly how these systems relate to each other. We first obtain a precise relationship between the systems $\mathbf{M}$ and $\mathbf{EL}$, and then apply similar arguments to get comparisons in some other cases.

1. The formal systems $\mathbf{M}$ and $\mathbf{EL}$

1.0.1. Starting with Heyting, intuitionistic logic and arithmetic were formalized as subsystems of the corresponding classical formal theories (see [JRM2009]). On the contrary, Heyting’s formalization of Brouwer’s set theory (the part of intuitionistic mathematics concerning the continuum and the real numbers) failed to allow comparison with classical mathematics. S. C. Kleene and R. E. Vesley ([FIM]) formalized large parts of intuitionistic mathematics, corresponding to mathematical analysis, in the formal system $\mathbf{FIM}$ whose language is suitable for classical analysis also, making such a comparison possible.

The minimal system of analysis $\mathbf{M}$, identified in [JRMPhD], is a subsystem of $\mathbf{FIM}$ consisting of the primitive recursive arithmetical basis of $\mathbf{FIM}$ and a countable function comprehension principle. Within it S. C. Kleene developed formally, with great detail, the theory of recursive partial functionals ([Kleene1969]). The corresponding informal theory $\mathbf{M}$ is used in [JRM2003]. We also note that the system $\mathbf{WKV}$ (Weak Kleene-Vesley) used in the constructive reverse mathematics paper [Loeb] is a “minimalistic” variant of $\mathbf{M}$.

The system of elementary analysis $\mathbf{EL}$ ([Troelstra1973, TvDI]) has been developed mostly by A. S. Troelstra, to serve as a formal basis for intuitionistic analysis. It differs from $\mathbf{M}$ in its arithmetical basis and in the function existence principle it assumes. $\mathbf{EL}$ is used in recent work for the formalization of Bishop’s constructive analysis, especially in relation to the program of Constructive Reverse Mathematics (see for example [Berger] and [Ishihara]).

1.0.2. $\mathbf{M}$ and $\mathbf{EL}$ have many similarities: all the common features mentioned in the introductory paragraph, including $\lambda$-abstraction, and also the possibility of definition by primitive recursion, although with different justification.

Their differences are of two kinds. First, they have differences in their languages. $\mathbf{M}$ has only finitely many function and functional constants, following the paradigm of (the usual presentation of) Peano arithmetic, while $\mathbf{EL}$ has infinitely many function constants (not including the functional constants of $\mathbf{M}$), as it extends (first-order intuitionistic) Heyting arithmetic $\mathbf{HA}$ (as presented in [TvDI]), which contains primitive recursive arithmetic $\mathbf{PRA}$; $\mathbf{EL}$ has also a recursor functional (not included in $\mathbf{M}$). Second, they assume different function existence principles: $\mathbf{M}$ assumes the axiom schema $\mathbf{AC}_{00}$! of countable function comprehension, while $\mathbf{EL}$ assumes the axiom schema $QF-\mathbf{AC}_{00}$ of countable choice for quantifier-free formulas, which is a consequence of $\mathbf{AC}_{00}$!.

The two systems were considered more or less equivalent, but their exact relation was unknown. We have found that $\mathbf{EL}$ is essentially weaker than $\mathbf{M}$, and that their difference is captured by the function existence principle expressed by an axiom schema which we call $\mathbf{CF}_d$, another consequence of $\mathbf{AC}_{00}$!. After identifying $\mathbf{CF}_d$, we show that $\mathbf{EL} + \mathbf{CF}_d$ entails $\mathbf{AC}_{00}$!, and that $\mathbf{EL}$ does not entail $\mathbf{CF}_d$. These results suggest that the formal system $\mathbf{EL} + \mathbf{CF}_d$ is essentially equivalent to $\mathbf{M}$, while $\mathbf{EL}$ is weaker than $\mathbf{M}$. In order to establish these suggested relationships,

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1It is interesting that the formal development of elementary recursion theory in $\mathbf{EL}$ by A. S. Troelstra “relies heavily” on the above mentioned work [Kleene1969], see [Troelstra1973], p. 73.
we have to overcome the differences of the languages. So we extend both systems up to a common language, and we show that the corresponding extensions of $\mathbf{M}$ and $\mathbf{EL} + \mathbf{CF}_d$ are conservative, in fact definitional, and coincide up to trivial notational differences; so we conclude that the suggested relationships hold indeed. We also show that, like $\mathbf{EL}$, $\mathbf{M}$ is conservative over first-order intuitionistic arithmetic and that Church’s $\lambda$ is eliminable from $\mathbf{EL}$, by a slight modification of the corresponding proof for $\mathbf{M}$ given in [JRMPhD].

1.1. Language and underlying logic.

1.1.1. Both systems are based on two-sorted intuitionistic predicate logic with equality, with number and function variables. Their languages $\mathcal{L}(\mathbf{M})$ and $\mathcal{L}(\mathbf{EL})$ have a common part which includes: the logical symbols $\rightarrow$, $\&$, $\lor$, $\neg$, $\forall$, $\exists$, commas and parentheses as punctuation symbols, number variables $x, y, z, \ldots$ intended to range over natural numbers, and function variables $\alpha, \beta, \gamma, \ldots$ intended to range over one-place number-theoretic functions (or choice sequences in the case of intuitionistic analysis). The set of individual constants, predicate and function symbols of each language extends in different ways a common part, and there are common and different formation rules for the terms (type-0 terms, expressions for natural numbers) and the functors (type-1 terms, expressions for one-place number-theoretic functions (or choice sequences)); these will be included in the description of the non-logical part of the formalisms. The number equality predicate symbol $=$ is contained in both languages.

1.1.2. The logical axioms and rules can be introduced in various ways, for example in a natural deduction or Hilbert-type style. We will base our treatment on the formal system of [FIM] which extends that of [IM] (on p. 13 of [FIM] and pp. 82 and 101 of [IM]). The corresponding system of classical logic is obtained by replacing axiom schema $\neg A \rightarrow (A \rightarrow B)$ of “ex falso sequitur quodlibet” by the schema $\neg \neg A \rightarrow A$ of double negation elimination or, equivalently, by adding the schema $A \lor \neg A$ of the excluded middle.

1.1.3. Number equality (between terms) is introduced as a primitive which in $\mathbf{HA}$ and $\mathbf{EL}$ satisfies (the universal closures of) the axiom $\text{REFL} \ x = x$ and the replacement schema $\text{REPL} \ A(x) \ & \ x = y \rightarrow A(y)$, where $A(z)$ is a formula and $x, y$ are distinct number variables free for $z$ in $A(z)$. In the version of first-order arithmetic of [IM] which we call $\mathbf{IA}_0$ and upon which $\mathbf{M}$ is based, as well as in $\mathbf{M}$, these are reduced to a finite number of simple axioms, from which $\text{REFL}$ and the schema corresponding to $\text{REPL}$ are provable.

Equality between functors, in all the systems that we consider, is defined extensionally and is introduced by the abbreviation $u = v \equiv \forall x \ (u)(x) = (v)(x)$, where $u, v$ are functors, $x$ is a number variable not free in $u$ or $v$, $(u)(x)$ and $(v)(x)$ are terms obtained by function application, according to the formation rules given below. Details on the treatment of equality will be given in a later section.

1.2. Underlying arithmetic.

1.2.1. The systems $\mathbf{M}$ and $\mathbf{EL}$ are based on weak systems of two-sorted intuitionistic arithmetic, which we call $\mathbf{IA}_1$ and $\mathbf{HA}_1$, respectively. Both weak theories are based on the two-sorted intuitionistic predicate logic that we described; they extend $\mathbf{IA}_0$ and $\mathbf{HA}$, respectively. The difference between $\mathbf{IA}_0$ and $\mathbf{HA}$ is that the first one has only finitely many primitives, while the second has symbols for
all the primitive recursive functions. But it is well-known (see [IM], §74, for a
detailed proof) that adding function symbols for primitive recursive functions to
IA$_0$ leads to definitional, and so inessential extensions.

1.2.2. IA$_0$ is based on first-order intuitionistic predicate logic (with only number
variables). Besides $=$, its language $L$(IA$_0$) contains the constants 0 (zero), $'$
(successor), $+$ (addition) and $\cdot$ (multiplication).

Terms are defined inductively, as usual: the constant 0 and the number variables
are terms, and if $s$, $t$ are terms, then $(s)'$, $(s) + (t)$, $(s) \cdot (t)$ are terms.

The prime formulas are the equalities between terms: if $s$, $t$ are terms, then
$(s) = (t)$ is a (prime) formula.

The mathematical axioms of IA$_0$ are the axioms for $=$, 0, $'$, $+, \cdot$ (on p. 82 of
[IM]) and the axiom schema of mathematical induction

$$\text{IND } \quad A(0) \land \forall x (A(x) \rightarrow A(x')) \rightarrow A(x).$$

Classical first-order Peano arithmetic PA is IA$_0$ $+$ $\neg
\neg A \rightarrow A$.

1.2.3. Heyting arithmetic HA differs from IA$_0$ in the set of function constants
it contains. Its language $L$(HA) has the constant 0 (zero) and countably infinitely
many constants $h_0, h_1, h_2, \ldots$, including S (successor), for all the primitive
recursive functions, more precisely a function constant for each primitive recursive
description.

The term formation rules are adapted accordingly: the constant 0 and the
number variables are terms, and if $t_1, \ldots, t_k$ are terms and $h$ a $k$-place function
constant, then $h(t_1, \ldots, t_k)$ is a term.

The mathematical axioms of HA, besides the equality axioms given by REFL
and REPL, are the axiom schema of induction IND, the axiom $\neg S(0) = 0$ and
defining axioms for the function constants, which consist of the equations expressing
the corresponding primitive recursive descriptions.

1.2.4. The language $L$(IA$_1$) of IA$_1$ (we note that $L$(IA$_1$) is $L$(M)) extends
$L$(IA$_0$). It has the (finitely many) function and functional constants $f_0, \ldots, f_p$
given in a list below, where each $f_i$ ($i = 0, \ldots, p$) has $k_i$ number arguments and
$l_i$ function arguments. All of them express functions primitive recursive in their
arguments. According to the needs of the development of the theory, a different
selection of which function constants are included in the alphabet may be done, in
agreement with the intuitionistic view that no formal system can exhaust the pos-
sibilities of mathematical activity. The particular formal system that we are con-
sidering, M, contains the 27 function(al)s contained in the list; the system of [FIM]
contains the first 25 of them, the last two have been added in [Kleene1969]. There
are also parentheses serving as constant for function application, and Church’s $\lambda$
for $\lambda$-abstraction.

The terms and functors of IA$_1$ are defined by simultaneous induction: 0 and
the number variables are terms; the function variables and each constant $f_i$ with
$k_i = 1$, $l_i = 0$, are functors; if $t_1, \ldots, t_k$ are terms and $u_1, \ldots, u_l$ functors, then
$f_i(t_1, \ldots, t_k, u_1, \ldots, u_l)$ is a term; if $u$ is a functor and $t$ a term, then $(u)(t)$ is a
term; if $x$ is a number variable and $t$ a term, then $\lambda x(t)$ (we also write $\lambda x.t$) is a
functor.

The prime formulas of IA$_1$ are the equalities $(s) = (t)$ where $s$, $t$ are terms.

The mathematical axioms of IA$_1$ are: the axioms of IA$_0$ for $=, 0, ', +, \cdot$, the
axiom schema IND for $L$(IA$_1$), the equations expressing the primitive recursive
definitions of the additional function(al) constants $f_4 \text{ - } f_{26}$ [they are of the following forms, corresponding to explicit definition and definition by primitive recursion:

(1) \[ f_i(y, a, \alpha) = p(y, a, \alpha), \]
(2) \[
\begin{align*}
    f_i(0, a, \alpha) &= q(a, \alpha), \\
    f_i(y', a, \alpha) &= r(y, f_i(y, a, \alpha), a, \alpha), \\
\end{align*}
\]

where $p(y, a, \alpha)$, $q(a, \alpha)$, $r(y, z, a, \alpha)$ are terms containing only the distinct variables shown and only function constants from $f_0, \ldots, f_{i-1}$, and $y, a, \alpha$ are free for $z$ in $r(y, z, a, \alpha)$, the equality axiom for function variables $x = y \rightarrow \alpha(x) = \alpha(y)$, and the axiom schema of $\lambda$-conversion ($\lambda x. t(x))(s) = t(s)$, where $t(x)$ is any term and $s$ is any term free for $x$ in $t(x)$.

We next give the complete list of the function(al) constants of $\mathcal{L}(\mathbf{I}A_1)$ with their defining axioms, where $a$, $b$ are number variables.

- $f_0 \equiv 0$, $k_0 = 0$, $l_0 = 0$
- $f_1 \neg a' = 0$, $a = b \rightarrow a' = b'$, $a' = b' \rightarrow a = b$, $k_1 = 1$, $l_1 = 0$
- $f_2 \mathbb{P} a + 0 = a$, $a + b' = (a + b)'$, $k_2 = 2$, $l_2 = 0$
- $f_3 \mathbb{P} a \cdot 0 = a$, $a \cdot b' = a \cdot b + a$, $k_3 = 3$, $l_3 = 0$
- $f_4 \mathbb{P} a^0 = 1$, $a^b = a^b \cdot a$, $k_4 = 4$, $l_4 = 0$
- $f_5 \mathbb{P} 0! = 1$, $(a')! = (a!) \cdot a'$, $k_5 = 1$, $l_5 = 0$
- $f_6 \mathbb{P} \mathbb{P} x = 0$, $\mathbb{P} a^b = \mathbb{P} (a) \cdot a$, $k_6 = 1$, $l_6 = 0$
- $f_7 \mathbb{P} x = 0$, $\mathbb{P} a0 = a$, $k_7 = 2$, $l_7 = 0$
- $f_8 \mathbb{P} x = 0$, $\mathbb{P} b^a = b^a$, $k_8 = 2$, $l_8 = 0$
- $f_9 \mathbb{P} x = 0$, $\mathbb{P} (a) + b$, $k_9 = 2$, $l_9 = 0$
- $f_{10} \mathbb{P} (a') = 1$, $\mathbb{P} (a') = 0$, $k_{10} = 1$, $l_{10} = 0$
- $f_{11} \mathbb{P} x = 0$, $\mathbb{P} a'(a) = 1$, $k_{11} = 1$, $l_{11} = 0$
- $f_{12} \mathbb{P} x = 0$, $\mathbb{P} (a) + (b)$, $k_{12} = 2$, $l_{12} = 0$
- $f_{13} \mathbb{P} x = 0$, $\mathbb{P} a'(a) = 0$, $k_{13} = 2$, $l_{13} = 0$

where $\mathbb{P} x = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0$, $\mathbb{P} a(\alpha) = 0
\begin{itemize}
  \item $f_{24} \tilde{a}(x) = \prod_{i<x} \rho_i^{a(i)}$, $k_{24} = 1$, $l_{24} = 1$,
  \item $f_{25} a \circ b = \prod_{i<\text{max}(a,b)} P_i^{\text{max}(a_i,b_i)}$, $k_{25} = 2$, $l_{25} = 0$,
  \item $f_{26} \text{ccp}(0) = 1$, $\text{ccp}(y') = \text{ccp}(y) \cdot p_y^{(y',\text{ccp}(y))}$, $k_{26} = 1$, $l_{26} = 0$,
\end{itemize}

where the following abbreviations are used:
\begin{itemize}
  \item $a \triangleq b \equiv a' \cdot b = 0$; $a \leq b \equiv a < b \lor a = b$; $a \mid b \equiv \text{sg}(\text{rm}(b,a)) = 0$;
  \item $\mu y_{<z} R(y) \equiv \sum x_{<z} \prod y_{<x} r(y)$, where $r(y)$ is a term with \vdash R(y) \leftrightarrow r(y) = 0, \vdash r(y) \leq 1$ and $x$ not free in $r(y)$;
  \item $\text{Pr}(a)$ is a prime formula expressing that $a$ is a prime number;
  \item the term $r(y,z)$ in $f_{26}$ is constructed in [Kleene1969], where computation tree numbers are introduced to code partial recursive derivations.
\end{itemize}

**Notation.** (i) In all the systems that we consider, unless otherwise stated, the same abbreviations for $<$ etc. and the same symbols for (the same) functions and functionals as in $\text{IA}_1$. (ii) As in [FIM], we will use the following abbreviation representing a primitive recursive coding of finite sequences of natural numbers: for each $k \geq 0$, $\langle x_0, \ldots, x_k \rangle \equiv p_0^{x_0} \cdot \ldots \cdot p_k^{x_k}$, where $p_i$ is the numeral for the $i$-th prime $p_i$.

1.2.5. The language $\mathcal{L}(\text{HA}_1)$ of $\text{HA}_1$ (note that $\mathcal{L}(\text{HA}_1)$ is $\mathcal{L}(\text{EL})$) extends $\mathcal{L}(\text{HA})$. In addition to the function constants of $\mathcal{L}(\text{HA})$, there are parentheses for function application and Church’s $\lambda$ for $\lambda$-abstraction, as in $\mathcal{L}(\text{IA}_1)$. There is also a functional constant $\text{rec}$ expressing the recursor functional, which corresponds to definition by the schema of primitive recursion.

The terms and functors of $\text{HA}_1$ are defined as in $\text{IA}_1$, with an additional term formation rule: if $t,s$ are terms and $u$ a functor, then $\text{rec}(t,u,s)$ is a term.

The mathematical axioms of $\text{HA}_1$ are: the axioms of $\text{HA}$ with IND and REPL extended to $\mathcal{L}(\text{HA}_1)$, $\lambda$-conversion, and the following axioms for the recursor constant $\text{rec}$:

\[
\text{REC} \left\{ \begin{array}{l}
\text{rec}(t,u,0) = t, \\
\text{rec}(t,u,S(s)) = u(\text{rec}(t,u,s),s),
\end{array} \right.
\]

where $t,s$ are terms and $u$ a functor.

1.3. **Function existence principles.**

1.3.1. The unique existential number quantifier $\exists!y$ is used to express the notion “there exists a unique $y$ such that ...” and it is introduced as an abbreviation:

\[
\exists!y B(y) \equiv \exists y [B(y) \land \forall z (B(z) \to y = z)].
\]

1.3.2. The **minimal system of analysis** $\text{M}$ is the theory $\text{IA}_1 + \text{AC}_00!$, with $\text{AC}_00!$ \vdash \forall x \exists!y A(x,y) \to \exists\alpha \forall x A(x,\alpha(x))$, where $x$ and $\alpha$ are free for $y$ in $A(x,y)$ and $\alpha$ does not occur free in $A(x,y)$.

The schema of unique choice $\text{AC}_00!$ expresses a countable function comprehension principle. Because of the uniqueness condition in the hypothesis, there is no real choice. With classical logic, it is equivalent to $\text{AC}_00$ (just like $\text{AC}_00!$ but without the ! in the hypothesis), expressing a countable numerical choice principle.

2The original formulation of REC uses a pairing function $j$ onto the natural numbers, but as it is remarked in [Troelstra1973], 1.3.9, they “might have used Kleene’s $2^x \cdot 3^y$".
Constructively, as it is shown in [Weinstein] with a highly non-trivial proof, $\text{AC}_{00}$ is weaker than $\text{AC}_{00}$.

1.3.3. **Elementary analysis** $\text{EL}$ is the theory $\text{HA}_1 + \text{QF-AC}_{00}$, with

$$\text{QF-AC}_{00} : \forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)),$$

where $A(x, y)$ is a quantifier-free formula, in which $x$ is free for $y$ and $\alpha$ does not occur.

The schema $\text{QF-AC}_{00}$ expresses a weak principle of countable numerical choice, for quantifier-free formulas. This principle does not involve real choice either, since the quantifier-free formulas are decidable, and in this case, existence entails constructively unique existence (of the least such number). For these basic facts we refer to section 2 below.

2. **Unique existence and decidability**

2.1. In intuitionistic arithmetic unique existence (of a natural number satisfying a predicate) and decidability (of natural number predicates) are closely related. As a consequence, the principles $\text{AC}_{00}$! and $\text{QF-AC}_{00}$ are related in a precise manner, over any reasonable two-sorted intuitionistic arithmetic. We give next some results, most of them well-known, that provide basic facts about the two notions. Using them we will determine how $\text{AC}_{00}$! and $\text{QF-AC}_{00}$ relate to each other. In the following $S$ is any of $\text{IA}_0$, $\text{HA}$, $\text{IA}_1$ or $\text{HA}_1$, and $\vdash$ denotes provability in $S$.

**Lemma 2.1.** $\vdash \forall x \forall y (x = y \lor \neg x = y)$.

**Lemma 2.2.** For any formula $A$ of $S$ built up from the formulas $P_1, \ldots, P_m$ by propositional connectives or bounded number quantifiers,

$$P_1 \lor \neg P_1, \ldots, P_m \lor \neg P_m \vdash A \lor \neg A.$$

**Lemma 2.3.** For any formula $A$ of $S$ which is quantifier-free or has only bounded number quantifiers (and no function quantifiers), $\vdash A \lor \neg A$.

Although the least (natural) number principle fails in intuitionistic arithmetic in general, it holds for number predicates that are assumed decidable, and in this case the least number is unique.

**Lemma 2.4.** In $S$,

$$\vdash \forall y (B(y) \lor \neg B(y)) \rightarrow [\exists y B(y) \rightarrow \exists y (B(y) \land \forall z (z < y \rightarrow \neg B(z)))] .$$

The next lemma asserts that, conversely to the previous, uniqueness entails decidability; it follows from the decidability of number-theoretic equality.

**Lemma 2.5.** $\exists y B(y) \vdash B(y) \lor \neg B(y)$.

The next lemma provides a fact very useful for our purposes.

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3. A constructively equivalent definition of $\exists y B(y)$ is by $\exists y B(y) \land \forall y \forall z (B(y) \land B(z) \rightarrow y = z)$ where the second conjunct expresses “at most one”, for which Bishop constructivists use $\forall y \forall z (y \neq z \rightarrow (\neg B(y) \lor \neg B(z)))$, which is classically but not constructively equivalent: consider the formula $B(x) \equiv (x = 0 \land P) \lor (x = 1 \land \neg P)$, where $P$ is any formula and $x$ any variable not occurring free in $P$ [JRM-GV2012]. However, any of the alternatives could be used in the formulation of $\text{AC}_{00}$!, since under the assumption $\exists x B(x)$ they become constructively equivalent.

4. In the classical case all these are trivialities, as natural number existence always entails unique existence of a least witness and every predicate is decidable.
Lemma 2.6. \(\vdash A \lor \neg A \leftrightarrow \exists y \left[ y \leq 1 \land (y = 0 \leftrightarrow A) \right]\), with \(A\) not containing \(y\) free.

2.1.1. We can now draw a first immediate conclusion about \(\text{AC}_{00}!\) and \(\text{QF-AC}_{00}\).

Proposition 2.7. Over \(\text{IA}_1\) (and \(\text{HA}_1\)) \(\text{AC}_{00}!\) entails \(\text{QF-AC}_{00}\).

Proof. By Lemmas 2.3 and 2.4. \(\square\)

3. A CHARACTERISTIC FUNCTION PRINCIPLE

3.1. The schema \(\text{CF}_d\).

3.1.1. Consider the following schema, which asserts that every decidable predicate of natural numbers has a characteristic function:

\[\text{CF}_d \forall x (B(x) \lor \neg B(x)) \rightarrow \exists \beta \forall x \left[ \beta(x) \leq 1 \land (\beta(x) = 0 \leftrightarrow B(x)) \right],\]

where \(\beta\) does not occur free in \(B(x)\).

Introducing this axiom schema allows to determine the exact relation of \(\text{AC}_{00}!\) and \(\text{QF-AC}_{00}\); and this in its turn will suggest the relation between \(\text{M}\) and \(\text{EL}\).

Proposition 3.1. Over \(\text{IA}_1\) (and \(\text{HA}_1\)), \(\text{AC}_{00}!\) entails \(\text{CF}_d\).

Proof. By Lemma 2.6. \(\square\)

3.1.2. Now we show that the unique choice principle \(\text{AC}_{00}!\) is equivalent to the conjunction of its two consequences \(\text{QF-AC}_{00}\) and \(\text{CF}_d\) over \(\text{IA}_1\) and \(\text{HA}_1\).

Theorem 3.2. Over \(\text{IA}_1\) (and \(\text{HA}_1\)), \(\text{QF-AC}_{00} + \text{CF}_d\) entails \(\text{AC}_{00}!\).

Proof. Assume (a) \(\forall x \exists !y A(x,y)\). By Lemma 2.5 we get \(\forall x \forall y [A(x,y) \lor \neg A(x,y)]\), so, by specializing for \((w)_0, (w)_1\), \(\forall w [A((w)_0, (w)_1) \lor \neg A((w)_0, (w)_1)]\). Applying \(\text{CF}_d\) to this, \(\exists \beta \forall w [\beta(w) \leq 1 \land (\beta(w) = 0 \leftrightarrow A((w)_0, (w)_1))]\), from which (without \(\exists\beta\), towards \(\exists\)-elim., specializing for \((x, y)\) and (a) we get \(\forall x \exists !y \beta((x, y)) = 0\). So by \(\text{QF-AC}_{00}\), \(\exists \alpha \forall x \beta((x, \alpha(x))) = 0\), and finally \(\exists \alpha \forall x A(x, \alpha(x))\). \(\square\)

Corollary 3.3. Over \(\text{IA}_1\) (and \(\text{HA}_1\)), \(\text{AC}_{00}!\) is equivalent to \(\text{QF-AC}_{00} + \text{CF}_d\).

3.2. Classical models for weak theories of two-sorted arithmetic.

3.2.1. Let \(T\) be the formal theory \(\text{IA}_1 + \text{QF-AC}_{00}\). \(T\) can be extended to a corresponding classical theory \(T^0\), by replacing the axiom schema \(\neg A \rightarrow (A \rightarrow B)\) by \(\neg \neg A \rightarrow A\). We will use \(T^0\) to see that \(T\) does not prove \(\text{CF}_d\) by showing that \(T^0\) has a classical model in which \(\text{CF}_d\) fails.

Theorem 3.4. (a) \(\text{IA}_1 + \text{QF-AC}_{00}\) does not prove \(\text{CF}_d\).

(b) \(\text{EL}\) does not prove \(\text{CF}_d\).

Proof. (a) Let \(T\) and \(T^0\) be as in the discussion above. We consider the structure \(\mathcal{GR}\) for the language of \(T^0\) consisting of the sets and functions given by (i)-(iii):

(i) The set \(\mathbb{N}\) of the natural numbers, that serves as the universe of the first sort, over which the number variables range.

(ii) The subset \(\mathcal{GR}^2\) of the set of all functions from \(\mathbb{N}\) to \(\mathbb{N}\) consisting of all the general recursive functions from \(\mathbb{N}\) to \(\mathbb{N}\), which serves as the universe of the second sort, over which the function variables range.

The structure that we consider is characterized by the choice of the universe of the second sort, so we use the same name for both.
(iii) The function(al)s \( f_0, \ldots, f_p \) that correspond to the function(al) constants \( f_0, \ldots, f_p \): each \( f_i \), \( i = 0, \ldots, p \), is the primitive recursive function(al) obtained by the primitive recursive description expressed by the defining axioms of \( f_i \).

The interpretation of a term or functor under an assignment into \( \mathcal{G} \mathcal{R} \) and the notions of satisfaction and truth are as usual. In particular, the interpretation \( u^{\mathcal{G} \mathcal{R}} \) in \( \mathcal{G} \mathcal{R} \) of a functor \( u \) of the form \( \lambda x.t \) where \( t \) is a term, under an assignment \( v \), is given by

\[
(\lambda x.t)^{\mathcal{G} \mathcal{R}} = \lambda n.v(x|n)(t),
\]

where \( v(x|n) \) is the extension to all terms and functors of the assignment which assigns the natural number \( n \) to \( x \) and agrees with \( v \) on all other variables, the \( \lambda \) in the interpretation is the usual (informal) Church’s \( \lambda \), and \( n \) ranges over \( \mathbb{N} \). Function application (represented by parentheses) is interpreted accordingly. It is straightforward that \( \mathcal{G} \mathcal{R} \) is a model of \( \text{IA}_1 \).

QF-\( AC_0 \) holds in \( \mathcal{G} \mathcal{R} \): using the following fact (shown in \cite{FIM}, pp. 27-31) and the least number operator, we obtain the function asserted to exist by QF-\( AC_0 \).

**Fact.** For any formula \( Q \) which is quantifier-free (or has only bounded number quantifiers), we can construct a term \( q \), with the same free variables as \( Q \), such that \( \vdash q \leq 1 \) and \( \vdash Q \leftrightarrow q = 0 \). The construction of \( q \) and the proofs are done in \( \text{IA}_1 \).

It is easy to see that \( \text{CF}_d \) does not hold in \( \mathcal{G} \mathcal{R} \), since the law of the excluded middle holds in \( \mathcal{G} \mathcal{R} \) while e.g. the predicate \( \exists y T(x, x, y) \), where \( T(x, y, z) \leftrightarrow z \) is the code of the computation of the value of the partial recursive function with Gödel number \( x \) at the argument \( y \) (the Kleene \( T \)-predicate), does not have a general recursive characteristic function.

(b) The argument is similar to the one for (a). We only have to consider now \( \mathcal{G} \mathcal{R} \) as a structure with infinitely many functions, corresponding to the function constants for number-theoretic functions of \( \text{EL} \), and the recursor functional (which, we note, is itself a primitive recursive functional).

**Remark.** It is well-known that in the presence of \( \text{AC}_0 ! \) Church’s Thesis in the form \( \forall \alpha \exists x \forall y \exists z (T(x, y, z) \& U(z) = \alpha(y)) \), where \( T(x, y, z) \) is the Kleene \( T \)-predicate and \( U \) the result-extracting function, contradicts classical logic. The previous theorem makes it clear that this is due to \( \text{CF}_d \).

**Corollary 3.5.** (a) \( \text{IA}_1 + \text{QF-AC}_0 \) is a proper subtheory of \( \text{M} \).

(b) \( \text{EL} \) is a proper subtheory of \( \text{EL} + \text{AC}_0 ! = \text{EL} + \text{CF}_d \).

3.2.2. By interpreting the function variables as varying over all primitive recursive functions of one number variable we obtain, as in the previous theorem, a classical model for \( \text{HA}_1 \) in which QF-\( AC_0 \) does not hold, as it guarantees closure under the notion “general recursive in”. Similarly for \( \text{IA}_1 \).

**Theorem 3.6.** (a) \( \text{HA}_1 \) does not prove QF-\( AC_0 \).

(b) \( \text{IA}_1 \) does not prove QF-\( AC_0 \).

**Proof.** By using for example the primitive recursive characteristic function of the Kleene \( T \)-predicate and the general but not primitive recursive Ackermann function.

4. Introduction of a Recursor in \( \text{M} \)

To show that \( \text{M} \) and \( \text{EL} + \text{CF}_d \) are essentially equivalent we will find a common conservative, in fact definitional, extension of both. In order to obtain it, we add
one by one the missing constants of each system, and show that the corresponding extension is definitional. In this way we reach conservative extensions of the two systems in the same language, which are identical (except for trivial notational differences). Our treatment is based on [IM, §74, where the one-sorted first-order case of definitional extensions is covered, and on [JRMPhD], where the method is applied for a result in the two-sorted case.

The first step is to add a recursor constant to \( M \). The notions of conservative and of definitional extension will be our main tool. We first give the definitions that we will use (see also [Troelstra1973]) and then make some useful observations concerning equality and replacement.

4.1. **Conservative and definitional extensions.**

4.1.1. **Definition.** Let \( S_1, S_2 \) be formal systems based on (many-sorted) intuitionistic predicate logic with equality, and let the language \( \mathcal{L}(S_2) \) of \( S_2 \) extend the language \( \mathcal{L}(S_1) \) of \( S_1 \), and the theorems of \( S_2 \) contain the theorems of \( S_1 \). \( S_2 \) is a *conservative extension* of \( S_1 \) if the theorems of \( S_2 \) that are formulas of \( S_1 \) are exactly the theorems of \( S_1 \).

**Definition.** Let \( S_1, S_2 \) be formal systems with \( \mathcal{L}(S_1) \) contained in \( \mathcal{L}(S_2) \). \( S_2 \) is a *definitional extension* of \( S_1 \) if there exists an effective mapping (or translation) \( ' \) which, to each formula \( E \) of \( S_2 \), assigns a formula \( E' \) of \( S_1 \) such that:

I. \( E' \equiv E \), for \( E \) a formula of \( \mathcal{L}(S_1) \).
II. \( \vdash_{S_2} E' \leftrightarrow E \).
III. If \( \Gamma \vdash_{S_2} E \), then \( \Gamma' \vdash_{S_1} E' \).
IV. \( ' \) commutes with the logical operations (of \( S_1 \)).

If the addition of a symbol gives a definitional extension, the symbol is called eliminable (from the extended to the original system); conditions I - IV are called elimination relations; and we say that the symbol is added definitionally.

A definitional extension is obviously conservative, and moreover every theorem of the extended system is equivalent (in the extended system), by a translation, to one of the original. So it is an inessential extension.

4.2. **On equality and replacement.**

4.2.1. **Many-sorted intuitionistic predicate logic with equality.** The systems that we are studying are based on many-sorted (and specifically two-sorted) intuitionistic predicate logic with equality, so the following axiom and axiom schema should be satisfied for each sort \( i \) (we refer to [Troelstra1973]).

\[
\begin{align*}
\text{REFL}^i & \quad x^i = x^i, \\
\text{REPL}^i & \quad x^i = y^i \rightarrow (A(x) \rightarrow A(y)), \quad \text{with } x, y \text{ free for } z^i \text{ in } A(z).
\end{align*}
\]

4.2.2. **Treatment of equality in the systems under study.** In all the systems that we consider only number equality is given as a primitive; function equality is defined extensionally by the abbreviation \( u = v \equiv \forall x \, u(x) = v(x) \).

By EQ we denote all the axioms \( \text{REFL}^i \) and \( \text{REPL}^i \), \( i = 0, 1 \). It is possible ([IM, §73]), as in the case of \( \text{IA}_0 \) and \( M \), to reduce the axioms of EQ to simpler (and in some cases only finitely many) axioms, as follows (we refer to systems with only function(al) constants; the case of predicate constants is treated similarly):

(A) By equality axioms for the binary predicate symbol \( = \) are meant the axioms \( x = x \), and \( x = y \rightarrow (x = z \rightarrow y = z) \).
(B) By equality axioms for a function(al) symbol $f$ of $k$ number and $l$ function arguments are meant the $k$ formulas:

$$x = y \rightarrow f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k, \alpha_1, \ldots, \alpha_l) = f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k, \alpha_1, \ldots, \alpha_l), \quad i = 1, \ldots, k,$$

and the corresponding $l$ formulas for function variables.

(C) The axioms EQ of a two-sorted formal system with type-0 equality as a primitive and type-1 equality defined as above, and with only function(al) constants, are provable from the following instances or consequences of them:

1. The equality axioms for $=$.
2. The equality axioms for the function(al) constants of its alphabet.
3. The equality axiom for function variables $x = y \rightarrow \alpha(x) = \alpha(y)$.

Thanks to the fact that the function(al) constants of $M$ (and $IA_1$) are introduced successively via the primitive recursive description of the corresponding functions, the equality axioms for these are provable in $M$; the proofs are by use of IND (see Lemma 5.1 on p. 20 of [FIM]). In the case of $EL$ (and $HA_1$), the axioms by REFL$^0$, REPL$^0$ are all introduced from the beginning, but it is easy to see that, in this case too, it suffices to include (C) 1, 3.

4.2.3. The replacement theorem. Lemma 4.2, p. 16 of [FIM], gives the replacement theorem for $M$. Since the proviso of the lemma is satisfied in the case of $EL$ as a consequence of the preceding paragraph, Lemma 4.2 of [FIM] provides the replacement theorem for $EL$. The same holds for the system that we will obtain by adding a recursor to $M$.

We note that the replacement theorem requires the equality axioms only for the function symbols that have the specified occurrence to be replaced within their scope, in the formula in which the replacement takes place.

4.3. Introducing a recursor in $M$.

4.3.1. We will add now to $M$ a recursor functional and prove that the resulting extension is definitional. Let $S_1$ be the minimal system of analysis $M$ and $S_2$ the system $M + \text{Rec}$, obtained by adding to $M$ the functional constant $\text{rec}$ together with the corresponding term formation rule “if $t$, $s$ are terms and $u$ a functor, then $\text{rec}(t, u, s)$ is a term”, and the following axiom Rec defining it:

$$\text{Rec} \quad A(x, \alpha, y, \text{rec}(x, \alpha, y)),$$

where $A(x, \alpha, y, w)$ is the formula

$$\exists \beta \left[ \beta(0) = x \land \forall z \beta(z') = \alpha(\langle \beta(z), z \rangle) \land \beta(y) = w \right].$$

The new constant rec represents then the recursor functional, which corresponds to definition by the schema of primitive recursion.

REMARK. We could have introduced the new functional constant $\text{rec}$ in $M$ by the pair of equations REC that define it in $EL$ and consider it as the $f_{27}$, extending the list of constants of $M$ (using the second of the forms of the definitions of the constants $f_i$). We have not adopted this choice, because the presence of rec would make redundant many of the constants of the list and because we consider this addition temporary, only for the purpose of comparison.

Lemma 5.3(b) of [FIM] (stated below) provides definition by primitive recursion in $M$, so we based our definition directly on it. We followed [IM], §74, in introducing a new function symbol by a formula for which the formalism proves
that it has a functional character. We note also that some of the formal systems that we will consider (BIM, WKV, H) have definition by primitive recursion as an axiom or axiom schema, in forms very similar to Lemma 5.3(b) of [FIM]. As we will see the two ways of introducing the new constant are equivalent.

4.3.2. Interderivability of Rec and REC. We can easily see that $S_2 = M + \text{Rec}$ is equivalent with $S'_2 = M + \text{REC}$, in the sense that every instance of REC is provable in $S_2$ and vice versa. In fact, we can show that REC and Rec are interderivable over IA

Notation. (a) In the following, by $\vdash_1$ and $\vdash_2$ we denote provability in $S_1$ and $S_2$, respectively.

(b) The unique existential function quantifier is introduced as an abbreviation by

$$\exists! \beta C(\beta) \equiv \exists \beta \left[ C(\beta) \land \forall \gamma (C(\gamma) \to \beta = \gamma) \right].$$

Remark 1. With the help of the unique existential function quantifier, we can formulate compactly the following version of $\text{AC}_{[0]}!$:

$$\forall x \exists y A(x, y) \to \exists! \alpha \forall x A(x, \alpha(x)).$$

Although this schema is apparently stronger than $\text{AC}_{[0]}!$, it is easily shown that it is a consequence of it, hence equivalent (over two-sorted intuitionistic logic with equality). We will use this version in some proofs.

The following lemma ([FIM], p. 39) is proved in M and justifies “definition by primitive recursion” in this formal theory.

Lemma 5.3(b) ([FIM]). Let $y, z$ be distinct number variables, and $\alpha$ a function variable. Let $q, r(y, z)$ be terms not containing $\alpha$ free, with $\alpha$ and $y$ free for $z$ in $r(y, z)$. Then

$$\vdash \exists! \alpha \left[ \alpha(0) = q \land \forall z \alpha(z) = r(y, \alpha(y)) \right].$$

The two next lemmas are easy consequences of Lemma 5.3(b) ([FIM]).

Lemma 4.1. $\vdash_1 \exists! \beta \left[ \beta(0) = x \land \forall z \beta(z') = \alpha(\langle \beta(z), z \rangle) \right].$

Lemma 4.2. $\vdash_1 \forall y \exists! m A(x, \alpha, y, m)$.

Lemma 4.3. $\vdash_2 \text{rec}(x, \alpha, y) = z \leftrightarrow A(x, \alpha, y, z)$.

Proof. The formula (a) $A(x, \alpha, y, \text{rec}(x, \alpha, y))$ is an axiom of $S_2$.

(i) Assume $\text{rec}(x, \alpha, y) = z$. From this, (a) and the replacement property of equality (which requires only the predicate calculus with the equality axioms for $\equiv$ and the function symbols of $A(x, \alpha, y, z)$) we get $A(x, \alpha, y, z)$.

(ii) Assuming $A(x, \alpha, y, z)$, from (a) with Lemma 4.2 we obtain $\text{rec}(x, \alpha, y) = z$.

Remark 2. The equality axioms for rec become now provable from the above lemmas, or alternatively from the (equivalent) definition of rec by REC, by the method of [FIM], Lemma 5.1 (see 4.2.2, on the treatment of equality).

Notation. (i) Let $t$ be a term. Let all the free number variable$\text{6}$ of $t$ be among $x_0, \ldots, x_k$, and let $w$ be a number variable not occurring in $t$. We will write $t^w$ for the result of replacing in $t$, for each $i = 0, \ldots, k$, each free occurrence of $x_i$ by

\[6\text{Note that the } \lambda\text{-prefixes } \lambda x, \text{ where } x \text{ is any number variable, bind number variables.}\]
an occurrence of the term \((w)\). The same notation will be used for functors too. Since the exponential will not appear in the proofs, there is no chance of confusion by the use of this notation.

(ii) By the notation \(A(t)\) we represent as usual the result of substituting a term \(t\) for all the free occurrences of \(x\) in a formula \(A(x)\), and we tacitly assume that, if needed, some bound variables are renamed, so that the substitution becomes free. Similarly for functors, and also for many “arguments”.

**Lemma 4.4.** Let \(t, s\) be terms and \(u\) a functor of \(S_1\). Let \(x_0, \ldots, x_k\) include all the number variables occurring free in \(t, u\) or \(s\), let \(w\) and \(v\) be distinct number variables not occurring in \(t, u, s\) and \(\gamma\) a function variable free for \(v\) in \(A(t^w, u^w, s^w, v)\), not occurring free in \(A(t^w, u^w, s^w, v)\). Then

\[
\vdash_1 \exists! \forall v A(t^w, u^w, s^w, \gamma(w)).
\]

**Proof.** Let \(x, y, z\) be distinct number variables different from \(w, x_0, \ldots, x_k\) and \(\alpha\) a function variable not occurring free in \(t\), free for \(\delta\) in \(A(x, \delta, y, z)\). By Lemma 4.2 we get \(\vdash_1 \exists! \forall x \forall y \exists! z A(x, \alpha, y, z)\), so, after specializing for \(t, u, s\) with the corresponding \(\forall\)-eliminations, we get \(\vdash_1 \forall x_0 \ldots \forall x_k \exists! z A(t, u, s, z)\). From this, after specializing for each \(i = 0, \ldots, k\) for \((w)_i\), we get \(\vdash_1 \forall w \exists! z A(t^w, u^w, s^w, z)\), and by AC\(0\)! with Remark 1 of 4.3.3 we get \(\vdash_1 \exists! \forall v A(t^w, u^w, s^w, \gamma(w))\). \(\square\)

**Notation.** (i) We use the notation \(E[a]\) to indicate some specified occurrences of a term or functor \(a\) in an expression \(E\). We will also use similarly \(E[a_1, \ldots, a_k]\) for \(k > 0\) to indicate some specified occurrences of \(k\) terms or functors. This notation may leave some ambiguity regarding the indicated occurrences, but in each case we will explain its use.

(ii) We use \(\alpha(x_0, \ldots, x_k)\) as an abbreviation for \(\alpha((x_0, \ldots, x_k))\).

**Lemma 4.5.** Let \(t, s\) be terms and \(u\) a functor of \(S_2\), let \(x_0, \ldots, x_k\) be all the number variables occurring free in \(t, u\) or \(s\), and let \(w\) be a number variable not occurring in \(A(t, u, s, v)\). Let \(E[\gamma(x_0, \ldots, x_k)]\) be a formula of \(S_2\) in which \(\gamma(x_0, \ldots, x_k)\) is not within the scope of some function quantifier \(\forall \alpha\) or \(\exists \alpha\), where \(\alpha\) is a function variable occurring free in \(t, u\) or \(s\) or \(\alpha\) is \(\gamma\) and \(\gamma\) is new for \(E[\text{rec}(t, u, s)]\), is free for \(v\) in \(A(t^w, u^w, s^w, v)\), and does not occur free in \(A(t^w, u^w, s^w, v)\), and where \(E[\text{rec}(t, u, s)]\) is obtained by replacing in \(E[\gamma(x_0, \ldots, x_k)]\) each of the (specified) occurrences of \(\gamma(x_0, \ldots, x_k)\) by an occurrence of \(\text{rec}(t, u, s)\). Then

\[
\vdash_2 E[\text{rec}(t, u, s)] \leftrightarrow \exists! \forall v A(t^w, u^w, s^w, \gamma(w)) \land E[\gamma(x_0, \ldots, x_k)].
\]

**Proof.** (i) Assume (a) \(E[\text{rec}(t, u, s)]\). We have (b) \(\vdash_2 \exists! \forall v A(t^w, u^w, s^w, \gamma(w))\) by Lemma 4.4, so we assume (c) \(\forall v A(t^w, u^w, s^w, \gamma(w))\) by Lemma 4.3 we have \(\forall w \text{rec}(t^w, u^w, s^w) = \gamma(w)\), so by specializing for \((x_0, \ldots, x_k)\) we get

\[
\forall x_0 \ldots \forall x_k \text{rec}(t, u, s) = \gamma(x_0, \ldots, x_k).
\]

Since the occurrences of \(\gamma(x_0, \ldots, x_k)\) are not within the scope of some function quantifier \(\forall \alpha\) or \(\exists \alpha\) where \(\alpha\) occurs free in \(s, t, u\) or \(\alpha\) is \(\gamma\), by the replacement theorem, from (a) we get (e) \(E[\gamma(x_0, \ldots, x_k)]\), so with (c) and \&-introduction and then \&\&-introduction and \&\&\gamma-elimination discharging (c), after \&\&-introduction we get the “\&\&” case from (a).

(ii) Assume (a) \(\forall w A(t^w, u^w, s^w, \gamma(w))\) and (b) \(E[\gamma(x_0, \ldots, x_k)]\). By (a), specializing for \((x_0, \ldots, x_k)\), we get \(A(t, u, s, \gamma(x_0, \ldots, x_k))\), and by Lemma 4.3 we get (c) \(\forall x_0 \ldots \forall x_k \text{rec}(t, u, s) = \gamma(x_0, \ldots, x_k)\). By the conditions on the bindings
due to function quantifiers, the replacement theorem applies and from (b), (c) we get $E[\text{rec}(t,u,s)]$, and after $\exists \gamma$-elimination discharging (a) and (b), and with $\rightarrow$-introduction, we get the “$\leftarrow$” case. $\square$.

**Terminology.** A term of the form $\text{rec}(t,u,s)$ is called a rec-term. A term in which the constant rec does not occur is called a rec-less term. A term $\text{rec}(t,u,s)$ where rec does not occur in $t,u,s$ is called a rec-flat term. An occurrence of the constant rec in a formal expression is called a rec-occurrence.

**Lemma 4.6.** To each formula $E$ of $S_2$ there can be correlated a formula $E'$ of $S_1$, called the principal rec-less transform of $E$, in such a way that the elimination relations I, II hold, no free variables are introduced or removed, and the logical operators of the two-sorted predicate logic are preserved (elimination relation IV).

Proof. The definition of $E'$ is done by induction on the number $q$ of occurrences of the logical operators in $E$. The basis of the induction consists in giving the definition for $E$ prime; this is done by induction on the number $q$ of occurrences of rec-terms in $E$.

Case $E$ is rec-less: then $E'$ shall be $E$.

Case $E$ has $q > 0$ rec-occurrences: let $\text{rec}(t,u,s)$ be the first (the leftmost) occurrence of a rec-less term, so that $E \equiv E[\text{rec}(t,u,s)]$, and let $x_0, \ldots, x_k$ be all the free number variables of $t, u, s$ a number variable and $\gamma$ a function variable new for both $E \equiv E[\text{rec}(t,u,s)]$ and $A(t,u,s,v)$. Then we define

$$E' \equiv \exists \gamma[\forall w A(t^w,u^w,s^w,\gamma(w)) \& [E[\gamma(x_0,\ldots,x_k)]]']$$

where $E[\gamma(x_0,\ldots,x_k)]$ is the result of replacing in $E$ the specified (first) occurrence of $\text{rec}(t,u,s)$ by an occurrence of $\gamma(x_0,\ldots,x_k)$. Then $E[\gamma(x_0,\ldots,x_k)]$ is prime and contains $q - 1$ occurrences of rec-terms, and $w$ is the only number variable free in $A(t^w,u^w,s^w,\gamma(w))$. About the choice of the bound variables $\gamma$ and $w$ and the possibly necessary changes of the bound variables of $A(x,\alpha,y,v)$ to make the substitutions of $t^w, u^w, s^w$ free, all permissible choices lead to congruent formulas.

The condition that the logical operators are preserved, i.e. $(\neg A)' \equiv \neg (A')$, $(A \circ B)' \equiv A' \circ B'$, for $\circ \equiv \rightarrow, \&, \vee$, and $(Q\alpha A(x))' \equiv Q\alpha (A(x))'$, for $Q \equiv \forall, \exists$, $(Q\alpha A(\alpha))' \equiv Q\alpha (A(\alpha))'$ for $Q \equiv \forall, \exists$, determines in a unique way the definition of $'$ for all formulas of $S_2$.

We immediately see that elimination relations I and IV hold. Elimination relation II is now proved easily by induction on the number of logical operators in $E$. The basis of the induction is the case of $E$ prime, and is proved (easily) by induction on the number of rec-occurrences in $E$, using Lemma 4.5 and replacement. $\square$

We still need to prove elimination relation III, so we have to show: if $\Gamma \vdash_{S_2} E$, then $\Gamma' \vdash_{S_1} E'$. The proof depends on a sequence of lemmas that follow. We will use the version with function variables of Lemma 25 [IM], p. 408. This lemma provides useful facts, most of them consequences of unique existence, for number variables, in intuitionistic predicate logic with equality. Corresponding results have been obtained by S. C. Kleene for function variables, in the two-sorted case (in a manuscript mentioned in [JRMPhD]); we can use this version thanks to Lemma 4.4. The versions of [IM] *181 - *190 with a function variable instead of $v$ are mentioned as *181F - *188F, *189F, *190F, and there are also cases *189F, *190F, *189f, *190f, with variables whose sorts are obvious from the notation. We will state the needed cases in the places we use them.
Lemma 4.7. Let \(\text{rec}(t, u, s)\) be any specified occurrence of a rec-plain term in a prime formula \(E\) of \(S_2\), so that we have \(E \equiv E[\text{rec}(t, u, s)]\). Then

\[
\vdash_1 E' \iff \exists \gamma [\forall w A(t^w, u^w, s^w, \gamma(w)) \land [E[\gamma(x_0, \ldots, x_\kappa)]']',
\]

where the conditions on the variables are as in the definition of \(E\).

Proof. The proof is by induction on the number \(q\) of occurrences of rec-terms in the prime formula \(E\). We will use the functional version of \(78\), [IM], p. 162,

\[
\vdash_{190} F' \vdash \exists \alpha \exists \beta D(\alpha, \beta) \leftrightarrow \exists \beta \exists \alpha D(\alpha, \beta),
\]

and the following case of the functional version of Lemma 25 of [IM],

\[
\vdash_{78} F' \vdash \exists \beta [F(\beta) \land \exists \alpha D(\alpha, \beta)] \leftrightarrow \exists \alpha \exists \beta [F(\beta) \land D(\alpha, \beta)],
\]

where \(\alpha\) does not occur free in \(F(\beta)\).

Cases \(q = 0\) or \(q = 1\) are trivial. For \(q > 1\), assume that Lemma 4.7 holds for prime formulas having \(q - 1\) rec-occurrences, and let \(E\) be a prime formula with \(q\) rec-occurrences. Let \(\text{rec}(t, u, s)\) be a specified occurrence of a rec-plain term in \(E\), so \(E \equiv E[\text{rec}(t, u, s)]\). If \(\text{rec}(t, u, s)\) is the first occurrence of a rec-plain term in \(E\), then (a) holds by the definition of \(E\). If \(\text{rec}(t, u, s)\) is not the first occurrence of a rec-plain term, then let \(\text{rec}(t_1, u_1, s_1)\) be the first such, so that \(E \equiv E[\text{rec}(t_1, u_1, s_1), \text{rec}(t, u, s)]\). Then, by the definition of \(E\), we have

\[
\vdash_1 E' \iff \exists \gamma [\forall w A(t^w_1, u^w_1, s^w_1, \gamma(w)) \land [E[\gamma(y_0, \ldots, y_1), \text{rec}(t, u, s)]']',
\]

where \(y_0, \ldots, y_1\) are the free number variables of \(t_1, u_1, s_1\) and \(\gamma, w\) as in the definition of \(E\). By the inductive hypothesis, since \(E[\gamma(y_0, \ldots, y_1), \text{rec}(t, u, s)]\) has \(q - 1\) rec-occurrences, with the replacement theorem, from (b) we get

\[
\vdash_1 E' \iff \exists \gamma [\forall w A(t^w_1, u^w_1, s^w_1, \gamma(w)) \land \exists \delta [\forall w A(t^w, u^w, s^w, \delta(w)) \land [E[\gamma(y_0, \ldots, y_1), \delta(x_0, \ldots, x_\kappa)]']',
\]

where \(x_0, \ldots, x_\kappa\) are the free number variables of \(t, u, s\). And by (c), \(\vdash_{190} F', \vdash_{78} F'\),

\[
\vdash_1 E' \iff \exists \delta [\forall w A(t^w, u^w, s^w, \delta(w)) \land \exists \gamma [\forall w A(t^w_1, u^w_1, s^w_1, \gamma(w)) \land [E[\gamma(y_0, \ldots, y_1), \delta(x_0, \ldots, x_\kappa)]']'].
\]

By the inductive hypothesis again, we have

\[
\vdash_1 [E[\text{rec}(t_1, u_1, s_1), \delta(x_0, \ldots, x_\kappa)]]' \iff \exists \gamma [\forall w A(t^w_1, u^w_1, s^w_1, \gamma(w)) \land [E[\gamma(y_0, \ldots, y_1), \delta(x_0, \ldots, x_\kappa)]']'.
\]

The left part of the \(\iff\) in (e) is just the result of replacing the specified occurrence \(\text{rec}(t, u, s)\) by an occurrence of \(\delta(x_0, \ldots, x_\kappa)\) in \(E\). So, by (d), (e) and the replacement theorem we get (a congruent of) (a). \(\square\)

Lemma 4.8. Let \(E \equiv E[\text{rec}(t, u, s)]\) be any formula of \(S_2\) such that \(\text{rec}(t, u, s)\) is a specified occurrence of a rec-plain term in \(E\) not in the scope of any quantifier binding a free variable of \(\text{rec}(t, u, s)\). Then, if \(x_0, \ldots, x_\kappa\) are the free number variables of \(t, u, s\) and the conditions on \(\gamma, w\) are as in the definition of \(E\),

\[
\vdash_1 E' \iff \exists \gamma [\forall w A(t^w, u^w, s^w, \gamma(w)) \land [E[\gamma(x_0, \ldots, x_\kappa)]']'.
\]
Proof. The proof is by induction on the complexity of the formula E. For E prime apply Lemma 4.7. For E composite, let r be the term \( \gamma(x_0, \ldots, x_k) \). Then \( E[r] \) will be of one of the forms \( \neg B[r], B[r] \land C, B \land C[r], B[r] \lor C, B \lor C[r], B \rightarrow C, B \rightarrow C[r], \forall x B(x)[r], \exists x B(x)[r], \forall \alpha B(\alpha)[r], \exists \alpha B(\alpha)[r] \). Thanks to Lemma 4.4 and the conditions on the variables in the present lemma and in the definition of \( \gamma' \), we can apply for each form the corresponding case of the functional version of Lemma 25 of [IM]. In the case that we treat here we will use

\[ \forall \alpha B(\alpha) \equiv \forall \alpha B(\alpha) \left[ \text{rec}(t, u, s) \right], \]

where \( \text{rec}(t, u, s) \) is the occurrence to be eliminated, so \( \alpha \) does not occur free in \( t, u, s \). From the definition of \( \gamma' \) with the inductive hypothesis

\[ \vdash_1 E' \leftrightarrow \forall \alpha \exists \gamma [ \forall w A(t^w, u^w, s^w, \gamma(w)) \land [B(\alpha) [\gamma(x_0, \ldots, x_k)]]'] \].

Since \( \alpha \) is not free in \( \forall w A(t^w, u^w, s^w, \gamma(w)) \), by \( \forall \exists \gamma' \) with Lemma 4.4

\[ \vdash_1 E' \leftrightarrow \exists \gamma [ \forall w A(t^w, u^w, s^w, \gamma(w)) \land \forall \alpha [B(\alpha) [\gamma(x_0, \ldots, x_k)]]'] \].

Lemma 4.9. Let \( E \) be a formula of \( S_2 \) in which the \( \text{rec}-\text{plain} \) term \( \text{rec}(t, u, s) \) has some occurrences such that no (number or function) variable free in \( t, u \) or \( s \) becomes bound in \( E \) by a universal or existential quantifier. Let \( E[\gamma(x_0, \ldots, x_k)] \) be the result of replacing in \( E \) one or more specified occurrences of \( \text{rec}(t, u, s) \) by \( \gamma(x_0, \ldots, x_k) \), where \( x_0, \ldots, x_k \) are all the free number variables of \( t, u, s \) and \( \gamma, w \) as in the definition of \( \gamma' \). Then

\[ \vdash_1 E' \leftrightarrow \exists \gamma [ \forall w A(t^w, u^w, s^w, \gamma(w)) \land [E[\gamma(x_0, \ldots, x_k)]]'] \].

Proof. The proof is by induction on the number of occurrences of \( \text{rec}(t, u, s) \) that are replaced (by \( \gamma(x_0, \ldots, x_k) \)). For \( q=1 \) the lemma follows by Lemma 4.8. For the inductive step, let \( E \equiv E[\text{rec}(t, u, s)] \) indicate \( q \) specified occurrences of \( \text{rec}(t, u, s) \) in \( E \), and let \( E[\gamma(x_0, \ldots, x_k)] \) be the result of replacing these \( q \) specified occurrences by \( \gamma(x_0, \ldots, x_k) \). By Lemma 4.8 we have

\[ \vdash_1 E' \leftrightarrow \exists \gamma [ \forall w A(t^w, u^w, s^w, \gamma(w)) \land [E[\gamma(x_0, \ldots, x_k), \text{rec}(t, u, s)]]'] \],

where \( E[\gamma(x_0, \ldots, x_k), \text{rec}(t, u, s)] \) is the result of replacing in \( E \) the first of the \( q \) specified occurrences of \( \text{rec}(t, u, s) \) by \( \gamma(x_0, \ldots, x_k) \). By the inductive hypothesis with the replacement theorem,

\[ \vdash_1 E' \leftrightarrow \exists \gamma [ \forall w A(t^w, u^w, s^w, \gamma(w)) \land \exists \delta [ \forall w A(t^w, u^w, s^w, \delta(w)) \land [E[\gamma(x_0, \ldots, x_k), \delta(x_0, \ldots, x_k)]]'] \],

where \( E[\gamma(x_0, \ldots, x_k), \delta(x_0, \ldots, x_k)] \) is the result of replacing the other \( q-1 \) specified occurrences of \( \text{rec}(t, u, s) \) in \( E[\gamma(x_0, \ldots, x_k), \text{rec}(t, u, s)] \) by \( \delta(x_0, \ldots, x_k) \). By Lemma 4.4 and the following case of the functional version of Lemma 25, [IM],

\[ \forall \exists \gamma' F(\beta) \vdash \exists \alpha [F(\alpha) \land C(\alpha, \alpha)] \leftrightarrow \exists \alpha [F(\alpha) \land \exists \beta [F(\beta) \land C(\alpha, \beta)]] \],

where \( \alpha \) does not occur free in \( F(\beta) \) and is free for \( \beta \) in \( F(\beta) \) and in \( C(\alpha, \beta) \),

\[ \vdash_1 E' \leftrightarrow \exists \gamma [ \forall w A(t^w, u^w, s^w, \gamma(w)) \land [E[\gamma(x_0, \ldots, x_k), \gamma(x_0, \ldots, x_k)]]'] \],

and, since \( E[\gamma(x_0, \ldots, x_k), \gamma(x_0, \ldots, x_k)] \) is just \( E[\gamma(x_0, \ldots, x_k)] \), we get the result. \( \square \)
Lemma 4.10. If $E$ is any axiom of $S_2$, then $\vdash E'$.

Proof. (i) If $E$ is an axiom of $S_2$ by an axiom schema of the propositional logic, then $E'$ is equivalent in $S_1$ to an axiom of $S_1$ by the same axiom schema. This follows by the fact that, by its definition, the translation $'$ preserves the logical operators, together with the fact that the translations of different instances of the same formula may differ only in their bound variables, so they are congruent, hence equivalent. So by the replacement theorem for equivalence, $\vdash E'$.

(ii) The logical axioms for the quantifiers need a different treatment. We give the proof for axiom schema 10N, and the other cases follow similarly.

Axiom schema 10N: $E \equiv \forall x B(x) \rightarrow B(r)$, where $r$ is a term of $S_2$ free for $x$ in $B(x)$. Then $E' \equiv \forall x [B(x)]' \rightarrow [B(r)]'$.

Ia. If $x$ has no free occurrences in $B(x)$, or $r$ is rec-less and $x$ does not occur free in any rec-occurrence, we simply choose the same bound variables at corresponding steps in the elimination processes in $B(x)$ and $B(r)$, so the resulting formula $\forall x C(x) \rightarrow C(r)$ is congruent to $E'$ and is an axiom of $S_1$ by 10N, so $\vdash E'$.

Ib. If $x$ occurs free in some of the rec-occurrences of $B(x)$ and $r$ is rec-less, we show first, by induction on the number $g$ of logical operators in $B(x)$,

$$(A) \vdash \vdash [B(x)]'(x/r) \leftrightarrow [B(r)]' .$$

The case of $B(x)$ prime is proved by induction on the number $q$ of rec-occurrences in $B(x)$ as follows. For $q > 0$, consider the first occurrence of a rec-plain term in $B(x)$. If $x$ does not occur free in it, then (A) follows from the inductive hypothesis and the definition of $'$. Otherwise, let $rec(t(x), u(x), s(x))$ with $x$ occurring free in $t(x)$, $u(x)$ or $s(x)$ be the considered occurrence, so $B(x) \equiv B(x)[rec(t(x), u(x), s(x))]$, and let $x_0, \ldots , x_k, x$ be all the free number variables of $t(x)$, $u(x)$, $s(x)$. Then, by the definition of $'$ we get

$$(a) \vdash [B(x)]'(x/r) \leftrightarrow \exists \gamma [\forall w A(t(x)^w, u(x)^w, s(x)^w, \gamma(w)) \& [B(x) [\gamma(x_0, \ldots , x_k, x)]]'(x/r)].$$

$$\forall w A(t(x)^w, u(x)^w, s(x)^w, \gamma(w)) \& [B(x) [\gamma(x_0, \ldots , x_k, x)]]'(x/r).$$

From (a) by $\forall$-introduction and elimination, since $w$ is the only number variable free in the formula $A(t(x)^w, u(x)^w, s(x)^w, \gamma(w))$, we have

$$(a1) \vdash [B(x)]'(x/r) \leftrightarrow \exists \gamma [\forall w A(t(x)^w, u(x)^w, s(x)^w, \gamma(w)) \& [B(x) [\gamma(x_0, \ldots , x_k, x)]]'(x/r)].$$

Assume (a2) $[B(x)]'(x/r)$. We will get $[B(r)]'$. By (a1) and (a2) we may assume

$$(a3) \forall w A(t(x)^w, u(x)^w, s(x)^w, \gamma(w)) \& [B(x) [\gamma(x_0, \ldots , x_k, x)]]'(x/r).$$

By the inductive hypothesis,

$$(a4) \vdash [B(x) [\gamma(x_0, \ldots , x_k, x)]]'(x/r) \leftrightarrow [B(r) [\gamma(x_0, \ldots , x_k, r(x,z))]]'.$$
By Lemma 4.4 we may assume (a5) \( \forall w A(t(r)^w, u(r)^w, s(r)^w, \delta(w)) \). Now, specializing for \( q_0 \equiv \langle x_0, \ldots, x_k, r(x), z \rangle \) and \( q_1 \equiv \langle x_0, \ldots, x_k, x, z \rangle \) from the first conjunct of (a3) and from (a5) respectively, since \( t(x)^{q_0} \) is just \( t(r)^{q_1} \), and similarly for \( u(x), s(x) \), using Lemma 4.2 \( \vdash \exists y \exists m A(x, \alpha, y, m) \) and then \( \forall \)-intros. we get

\[
(c) \quad \forall x_0 \ldots \forall x_k \forall x \forall z \gamma(x_0, \ldots, x_k, r(x), z) = \delta(x_0, \ldots, x_k, x, z).
\]

We will use now the following

**FACT.** For any formula \( D[x] \) of \( S_2 \) with a specified occurrence of \( x \) indicated, if \( s, t \) are rec-less terms, \( x_0, \ldots, x_k \) include all the free number variables of \( s \) and \( t \), and no free function variable of \( s \) or \( t \) becomes bound in the corresponding replacements, then

\[
\forall x_0 \ldots \forall x_k s = t \vdash \_\_ \exists [D[s]]' \leftrightarrow [D[t]]'.
\]

The proof is by induction on the number of logical operators in \( D[x] \). The basis (case of prime formulas) is obtained by induction on the number \( q \) of rec-occurrences in the formula, by use of the replacement theorem for \( S_1 \).

By the above fact and (c), we get now

\[
(d) \quad [B(r) [\delta(x_0, \ldots, x_k, x, z)]'] \leftrightarrow [B(r) [\gamma(x_0, \ldots, x_k, r(x), z)]]'.
\]

From the second conjunct of (a3) with (a4) and with (d) we get

\[
(e) \quad [B(r) [\delta(x_0, \ldots, x_k, x, z)]]'.
\]

Now from (a5) and (e) with \( \exists \delta \)-introds. we get

\[
[B(r) [\gamma(x_0, \ldots, x_k, r(x), z)]]' \equiv [B(r)'].
\]

The other direction of the equivalence of (A) is obtained similarly and the case of composite formulas (inductive step for \( g > 0 \)) follows easily. From (A) follows immediately that \( E' \equiv \forall x \exists [B(x)]' \to [B(r)]' \) is a congruent of an axiom of \( S_1 \) by the same axiom schema, so \( \vdash E' \).

II. If \( r \) has some rec-occurrences, the result is obtained by an induction on the number \( q \) of these occurrences. If \( r \) contains \( q \) (\( q > 0 \)) rec-occurrences, we consider the first rec-plain occurrence in \( r \), say \( rec(t, u, s) \), so

\[
(a) \quad E \equiv E(r [\text{rec}(t, u, s)]) \equiv \forall x B(x) \to B(r [\text{rec}(t, u, s)]).
\]

By Lemma 4.9,

\[
(b) \quad \vdash E' \leftrightarrow \exists \gamma [\forall w A(t^w, u^w, s^w, \gamma(w)) \& [E(r [\gamma(x_0, \ldots, x_k)])]']',
\]

with \( \gamma \) new for \( E \) and \( x_0, \ldots, x_k, w \) as usual. By the inductive hypothesis,

\[
(c) \quad \vdash \_\_ \forall x B(x) \to B(r [\gamma(x_0, \ldots, x_k)])'.
\]

But \( E(r [\gamma(x_0, \ldots, x_k)]) \) is just \( \forall x B(x) \to B(r [\gamma(x_0, \ldots, x_k)]) \), so by Lemma 4.4 with \( ^*182^F \), from (b) and the universal closure of (c) we get \( \vdash E' \).

(iii) *The axiom schema of induction.* \( \equiv A(0) \& \forall x A(x) \to A(x') \to A(x) \). Then \( E' \equiv [A(0)]' \& \forall x([A(x)]' \to [A(x')]') \to [A(x')]' \), and the result follows by the arguments of (ii), Ia and Ib.

(iv) *The axiom schema of \( \lambda \)-conversion.* \( \equiv (\lambda x.r(x))(p) = r(p) \), where \( r(x), p \) are terms of \( S_2 \) and \( p \) is free for \( x \) in \( r(x) \).

If \( r(x), p \) are rec-less, then \( E' \) is \( E \) and is an axiom of \( S_1 \) by the same schema.

If there are \( q \) (\( q > 0 \)) rec-occurrences in total in \( r(x) \) and \( p \), we consider the first occurrence of a rec-plain term in \( r(x) \) (in case that \( r(x) \) is rec-less, then consider...
the first such in p), say rec(t, u, s), with free number variables the \(x_0, \ldots, x_k\), and \(\gamma, \omega\) new. Then
\[ E \equiv E[rec(t, u, s)] \equiv (\lambda x.r(x)[rec(t, u, s)])(p) = (r(x)[rec(t, u, s)])(x/p). \]

By Lemma 4.9,
(a) \( \vdash_1 E' \leftrightarrow \exists \gamma [\forall w A(t^w, u^w, s^w, \gamma(w)) \land \left[ (\lambda x.r(x)[\gamma(x_0, \ldots, x_k)])(p) = (r(x)[\gamma(x_0, \ldots, x_k)])(x/p) \right]' \].

The instance of \(\lambda\)-conversion shown in (a) \((x\) may or may not be one of \(x_0, \ldots, x_k\), but in both cases \(p\) is free for \(x\) in \(r(x)[\gamma(x_0, \ldots, x_k)]\) has \(q - 1\) rec-occurrences in the terms involved, so the induction hypothesis applies and we obtain
(b) \( \vdash_1 [(\lambda x.r(x)[\gamma(x_0, \ldots, x_k)])](p) = (r(x)[\gamma(x_0, \ldots, x_k)])(x/p)' \).

By Lemma 4.4 with \(^*182^F\), we get now from (a) and (b) that \( \vdash_1 E' \). In case that the rec-occurrence to be eliminated is in the term \(p\), the argument is similar.

(v) The axiom schema \(\text{AC}_{00}!\).

\[ E \equiv \forall x \exists y \left[ A(x, y) \land \forall z (A(x, z) \rightarrow y = z) \right] \rightarrow \exists \alpha \forall x A(x, \alpha(x)). \]

By the arguments of (ii), Ia and Ib, \(E'\) is (congruent to) an axiom of \(S_1\) by the same axiom schema, so \( \vdash_1 E' \).

(vi) The axiom Rec \(A(x, \alpha, y, \text{rec}(x, \alpha, y))\).

\[ E \equiv \exists \beta [\beta(0) = x \land \forall z \beta(z') = \alpha(\langle \beta(z), z\rangle) \land \beta(y) = \text{rec}(x, \alpha, y)], \]

and so we have
\[ E' \equiv \exists \beta [\beta(0) = x \land \forall z \beta(z') = \alpha(\langle \beta(z), z\rangle) \land [\beta(y) = \text{rec}(x, \alpha, y)]'. \]

We have already shown
(a) \( \vdash_1 [\beta(y) = \text{rec}(x, \alpha, y)]' \leftrightarrow \exists \gamma [\forall w A(x^w, \alpha^w, y^w, \gamma(w)) \land \beta(y) = \gamma(x, y)] \).

By Lemma 4.1 we may assume
(b) \( \beta(0) = x \land \forall z \beta(z') = \alpha(\langle \beta(z), z\rangle) \land \forall z \delta(z') = \alpha(\langle \delta(z), z\rangle) \rightarrow \beta = \delta \).

By Lemma 4.4 we may assume (c) \(\forall w A(x^w, \alpha^w, y^w, \gamma(w))\), and from this by specializing for \(x, y\) we may also assume
(c1) \(\delta(0) = x \land \forall z \delta(z') = \alpha(\langle \delta(z), z\rangle) \land \delta(y) = \gamma(x, y)\).

By (b) and (c1) we get \(\beta = \delta\), and then also (d) \(\beta(y) = \gamma(x, y)\). After \(\exists \delta\)-elimination discharging (c1), we get from (c), (d),
(e) \(\forall w A(x^w, \alpha^w, y^w, \gamma(w)) \land \beta(y) = \gamma(x, y)\).

And after \(\exists \gamma\)-introduction, \(\exists \gamma\)-elimination discharging (c), from (e), (b) we get
(f) \(\beta(0) = x \land \forall z \beta(z') = \alpha(\langle \beta(z), z\rangle) \land \exists \gamma [\forall w A(x^w, \alpha^w, y^w, \gamma(w)) \land \beta(y) = \gamma(x, y)] \).

After \(\exists \beta\)-introduction, \(\exists \beta\)-elimination discharging (b), we get \(E'\) in \(S_1\) (using (a) and the replacement theorem for equivalence).

(vii) The remaining axioms are finitely many axioms not containing the constant rec, and they are also axioms of \(S_1\). \(\Box\)
Lemma 4.11. If $E$ is an immediate consequence of $F$ (and $G$) in $S_2$, then $E'$ is an immediate consequence of $F'$ (and $G'$) in $S_1$.

Proof. Since by the definition of the translation $'$ the logical operators are preserved and since, by Lemma 4.6, no free variables are introduced or removed, and also since congruent formulas are equivalent, it follows that to each instance of a rule of $S_2$ corresponds an instance of the same rule in $S_1$. □

We conclude that if $\Gamma \vdash_2 E$, where $\Gamma$ is a list of formulas and $E$ is a formula of $S_2$ and no variables are varied in the deduction, then $\Gamma' \vdash_1 E'$. So elimination relation III is satisfied.

5. Comparison of $M$ and $EL$

5.1. Introduction of the other function(al) constants.

5.1.1. Having the recursor constant rec in the formalism, it is immediate that any constant for a function with a primitive recursive description in which are used only functions with names already in the symbolism can be added definitionally. The needed translation is trivial, it amounts just to the replacement of each occurrence of the new constant by the corresponding (longer) term provided by the formalism. More concretely, constants for all the primitive recursive functions can be added definitionally, successively according to their primitive recursive descriptions, as follows:

- For the initial functions and for functions defined by composition from functions for which we already have constants, it is very easy to find terms expressing them.
- For the case of definition by primitive recursion we use the constant rec: for example if $f(x,0) = g(x)$ and $f(x,y + 1) = h(f(y),y,x)$, we introduce a new constant $f_j$ by $f_j(x,y) = \text{rec}(g(x), \lambda z.h((z)_0, (z)_1, x), y)$, if we have already in our symbolism constants $g$ and $h$ for $g$ and $h$, respectively.

We note that in this way, not only functions, but also functionals can be added in a formalism having a recursor, like $EL$ or $M + \text{Rec}$. We also note that the equality axioms for the new constants become provable.

5.2. Comparison of $M$ and $EL$.

5.2.1. Let $M^+$ be obtained by adding to $M + \text{Rec}$ all the (infinitely many) function constants of $HA$, with their defining axioms, extending also all axiom schemata to the new language.

Let $EL^+$ be obtained by adding to $EL$ all the (finitely many) functional constants of $M$, with their defining axioms, extending also all axiom schemata to the new language.

We see that the languages of the extended systems $M^+$ and $EL^+$ coincide (with trivial differences). Using the relations between the function existence principles that we have obtained as well as the equivalence of the different definitions of the recursor constant, we arrive at the following:

$$EL^+ + \text{CF}_d = HA_1 + \text{fin.list(M)} + \text{QF-AC}_{00} + \text{CF}_d = IA_1 + \text{Rec} + \text{inf.list(HA)} + \text{AC}_{001} = M^+.$$

Theorem 5.1. $EL^+ + \text{CF}_d$ is a conservative (in fact definition) extension of $M$.

Proof. It suffices to observe that every proof in $EL^+ + \text{CF}_d$ is done in a finite subsystem of it, so in a definitional extension of $M$. □
Theorem 5.2. $M^+$ is a conservative (in fact definitional) extension of $EL + CF_d$.

Corollary 5.3. The systems $M^+$ and $EL^+ + CF_d$ essentially coincide, so $M$ and $EL + CF_d$ are essentially equivalent, in the sense that they have a common conservative extension obtained by definitional extensions.

In [Troelstra1974] p. 585, a result of N. Goodman is mentioned and used, stating that $EL_1$ is conservative over $HA$, where $EL_1$ is $EL + AC_01$, where $AC_01$ (which entails $AC_00$!) is the countable choice schema assumed by $FIM$. With our previous results, we obtain the following.

Theorem 5.4. $EL + CF_d$ is a conservative extension of $HA$.

Theorem 5.5. $M^+$ is a conservative extension of $HA$.

Theorem 5.6. $M$ is a conservative extension of $IA_0$.

6. Elimination of the symbol λ from $EL$ and $IA_1 + QF-AC_{00}$

6.1. From [JRMPhD] it is known that $λ$ can be eliminated from the formal systems that S. C. Kleene set up to formalize parts of intuitionistic analysis, including $M$. The proof uses $AC_00$! so, by the previous results, it is not valid for the cases of $EL$ and $IA_1 + QF-AC_{00}$. Here we modify a part of it, and obtain the corresponding results.

Let $EL - λ$ be obtained from $EL$ by omitting the symbol $λ$, the corresponding functor formation rule and the axiom schema of $λ$-conversion. As axioms for the constant rec we include in both systems the following version, equivalent by logic to the one used in the definition of $EL$, with terms and a functor in the places of the number variables and the function variable, respectively:

$$\text{REC} \begin{cases} \text{rec}(x, α, 0) &= x, \\ \text{rec}(x, α, S(y)) &= α(\text{rec}(x, α, y), y). \end{cases}$$

Both systems include $QF-AC_{00}$. Instead of it we choose to include, in both systems, the following term-version of it:

$$QF_{t-AC_{00}} \quad ∀x∃y t((x, y)) = 0 \rightarrow ∃α∀x t((x, α(x))) = 0,$$

where $x$ is free for $y$ in $t((x, y))$ and $α$ does not occur in $t((x, y))$. Similarly, we consider $IA_1 + QF-AC_{00}$ (its version with $QF_{t-AC_{00}}$) and the corresponding $IA_1 + QF-AC_{00} - λ$.

In the following $S_2$ is $EL$ or $IA_1 + QF-AC_{00}$ and $S_1$ is $S_2 - λ$. By $\vdash_1, \vdash_2$ we denote provability in $S_1, S_2$, respectively.

Proposition. Over $HA_1$, $IA_1$ and the corresponding systems without $λ$, $QF-AC_{00}$ is interderivable with $QF_{t-AC_{00}}$.

Proof. Observe that $QF_{t-AC_{00}}$ is a special case of $QF-AC_{00}$. We obtain $QF-AC_{00}$ from $QF_{t-AC_{00}}$ as follows. In all the systems mentioned, for any quantifier-free formula $A(x, y)$, we can find a term $q(x, y)$ with the same free variables as $A(x, y)$, such that $\vdash A(x, y) \leftrightarrow q(x, y) = 0$. We get easily the result if we consider the term $t(w) ≡ q((w)₀, (w)₁)$, for which we obtain $\vdash A(x, y) \leftrightarrow t((x, y)) = 0$. □

---

Motivated by Iris Loeb’s question whether this method of elimination (that we initially applied) for $EL$ would have worked for $M$, we considered also the case of $IA_1 + QF-AC_{00}$.
6.2. To obtain \( \lambda \)-eliminability in [JRMPhD] a translation \( \prime \) is defined and then it is shown that the elimination relations are satisfied. We give the definition for prime formulas; for composite, the translation \( \prime \) is defined so that the logical operations are preserved.

**Remark on terminology and notation.** We use terminology and notation similar to those used in the case of rec. In particular, if we consider a specified occurrence of a term or functor \( R \) in an expression \( E \), we write \( E[R] \) to indicate this occurrence, and when this occurrence is replaced by a function variable, say \( \alpha \), we write \( E[\alpha] \) to denote the result of this replacement.

**Definition.** If \( P \) is any prime formula of \( S_2 \) with no \( \lambda \)'s, then \( P' \) is \( P \). Otherwise, if \( \lambda x.s(x) \) is the first (free) \( \lambda \)-occurrence in \( P \), in which case we use the notation \( P[\lambda x.s(x)] \), then

\[
P' \equiv \exists \alpha [\forall x [s(x) = \alpha(x)]' \& [P[\alpha]]'],
\]

where \( \alpha \) is a function variable which does not occur in \( P \), and \( P[\alpha] \) is obtained from \( P \) by replacing the occurrence \( \lambda x.s(x) \) by an occurrence of \( \alpha \).

The only point of the proof in [JRMPhD] where \( \text{AC}_{00} \) is used (except the treatment of \( \text{AC}_{00} \) itself) is in order to obtain \( \vdash \exists \alpha \forall x [t(x) = \alpha(x)]' \). Here we present some lemmas by which we obtain this result using only \( \text{QF-AC}_{00} \), so, after treating the axioms not included in the systems of [JRMPhD], we obtain \( \lambda \)-eliminability for \( \text{EL} \) and \( \text{IA}_1 + \text{QF-AC}_{00} \).

**Remark.** Free substitution of a free (number) variable by a \( \lambda \)-less term commutes with the translation \( \prime \) (it follows by induction on the number of logical operators of the considered formula, where the case of prime formulas follows by induction on the number of \( \lambda \)-occurrences). From this we also obtain:

(a) Consider \( E \equiv \forall x \exists t(x) = A(t) \), where \( A(t) \) is any formula of \( S_2 \) (with \( t \) free for \( x \) in \( A(t) \)), an axiom of \( S_2 \) by schema 10N. If \( t \) has no \( \lambda \)'s, then \( \vdash E' \).

(b) If \( s, t \) are \( \lambda \)-less terms free for \( z \) in a formula \( B(z) \) of \( S_2 \), then

\[
s = t \vdash [B(s)]' \leftrightarrow [B(t)]'.
\]

Corresponding results are obtained for function variables and \( \lambda \)-less functors (and in particular for the \( \forall \alpha \)-elimination schema 10F).

**Lemma 6.1.** Let \( r(x) \) be any term of \( S_2 \) with \( \alpha \) not free in it. Then

\[
\vdash \exists \alpha \forall x [r(x) = \alpha(x)]'.
\]

**Proof.** By induction on the number \( q \) of \( \lambda \)'s in the term \( r(x) \).

For \( q = 0 \), since \( \vdash \forall x \exists \alpha r(x) = w \), just apply \( \text{QF-AC}_{00} \).

Let \( r(x) \) have \( q > 0 \) \( \lambda \)'s, and let \( \lambda z.s(z,x) \) be the first \( \lambda \)-occurrence in \( r(x) \), so that \( r(x) \equiv r(x)[\lambda z.s(z,x)] \). We have to show

(a) \( \vdash \exists \alpha \forall x \exists \beta [\forall z [s(z,x) = \beta(z)]' \& [r(x)[\beta] = \alpha(x)]'] \).

We may assume (b) \( \forall z [s(z,x) = \beta(z)]' \) and (c) \( \forall x [r(x)[\beta] = \alpha(x)]' \), since we have both (d) \( \vdash \exists \beta \forall z [s(z,x) = \beta(z)]' \) and (e) \( \vdash \exists \alpha \forall x [r(x)[\beta] = \alpha(x)]' \) by the inductive hypothesis. From (b) and (c) with \( \forall x \)-elim., \&-introd. and \( \exists \beta \)-introd. we get

(f) \( \exists \beta [\forall z [s(z,x) = \beta(z)]' \& [r(x)[\beta] = \alpha(x)]'] \).

and with \( \exists \beta \)-elim. disch. (b) (from (d)), and then \( \forall x \)-introd. (\( x \) is not free in (e)) and \( \exists \alpha \)-introd., \( \exists \alpha \)-elim. disch. (c) (from (e)), we get (a). \( \square \)
Lemma 6.2. Let $t$ be any term of $S_2$ with $z$ not free in it. Then

$$\vdash_1 \exists! z \, [t = z]' .$$

Proof. By induction on the number $q$ of $\lambda$’s in $t$.

For $q = 0$ the lemma follows by $\vdash_1 \exists! z \, t = z$.

If $t$ has $q > 0$ $\lambda$’s and $\lambda x.s(x)$ is the first $\lambda$-occurrence in $t$, so that $t \equiv t[\lambda x.s(x)]$, we have $[t = z]' \equiv \exists \alpha \, [\forall x \, s(x) = \alpha(x)]' \land [t[\alpha] = z]'$. We may assume

(a) $\forall x \, [s(x) = \alpha(x)]'$ and (b) $[t[\alpha] = z]'$,

since by Lemma 6.1, we have (c) $\vdash_1 \exists! \alpha \forall x \, [s(x) = \alpha(x)]'$ and by the inductive hypothesis (d) $\vdash_1 \exists! z \, [t[\alpha] = z]'$. By (a), (b) with $\exists!\alpha$-introd. we get (e) $[t = z]'$.

For the uniqueness, we assume

(f) $[t = y]' \equiv \exists \beta \, [\forall x \, s(x) = \beta(x)]' \land [t[\beta] = y]'$,

so we may also assume

(g1) $\forall x \, [s(x) = \beta(x)]'$ and (g2) $[t[\beta] = y]'$.

From (a) and (g1) we get $\alpha = \beta$ as follows: from (a) and (g1), by $\forall$-elims, we get

(h1) $[s(x) = \alpha(x)]'$ and (h2) $[s(x) = \beta(x)]'$,

respectively. By the inductive hypothesis, $\vdash_1 \exists! z \, [s(x) = z]'$, so by (h1), (h2) and the Remark we get $\alpha(x) = \beta(x)$, so $\forall x \alpha(x) = \beta(x)$ (x is not free in (a), (g1)), so (i) $\alpha = \beta$. By the Remark, from (g2), (i), we get (j) $[t[\alpha] = y]'$, and then from (b), (j) and (d) we get $y = z$. So after $\exists!\beta$-elim. disch. (g1), (g2), with $\rightarrow$-introd. disch. (f), and with $\forall y$-introd. and (e), we get

$$[t = z]' \land \forall y ([t = y]' \rightarrow y = z),$$

and with $\exists y$-introd., $\exists! y$-elim. disch. (b), $\exists!\alpha$-elim. disch. (a) we complete the proof. \hfill $\square$

Lemma 6.3. Let $t(x)$ be any term of $S_2$ with $\alpha$ not free in it. Then

$$\vdash_1 \exists!\alpha \forall x \, [t(x) = \alpha(x)]'. $$

Proof. By Lemma 6.1, $\vdash_1 \exists! \alpha \forall x \, [t(x) = \alpha(x)]'$, so we assume (a) $\forall x \, [t(x) = \alpha(x)]'$ and, for the uniqueness, (b) $\forall x \, [t(x) = \beta(x)]'$, from which by $\forall$-elims we get (c) $[t(x) = \alpha(x)]'$ and (d) $[t(x) = \beta(x)]'$. By Lemma 6.2 with (c), (d) and the Remark, we get $\alpha(x) = \beta(x)$, so $\forall x \alpha(x) = \beta(x)$ (from (a), (b)), so (e) $\alpha = \beta$. So from (a), (b), (e) with $\rightarrow$-introd. disch. (b), we get

$$\forall x \, [t(x) = \alpha(x)]' \land \forall \beta \, [\forall x \, [t(x) = \beta(x)]' \rightarrow \alpha = \beta].$$

So after $\exists!\alpha$-introd. and $\exists!\beta$-elim. disch. (a) we complete the proof. \hfill $\square$

For the treatment of the axioms not included in the systems of \textbf{JRMPhD}, the case of the axioms REC is trivial, since there are no $\lambda$-occurrences in them, and the case of the axiom schema $QF_{t,AC_{00}}$ is obtained as follows.

Lemma 6.4. Let

$$E \equiv \forall x \exists y \, t(\langle x, y \rangle) = 0 \rightarrow \exists \alpha \forall x \, t(\langle x, \alpha(x) \rangle) = 0$$

where $x$ is free for $y$ in $t(\langle x, y \rangle)$ and $\alpha$ does not occur in $t(\langle x, y \rangle)$ be an instance of $QF_{t,AC_{00}}$ in $S_2$. Then $\vdash_1 E'$. 
Proof. We have that

\[ E' \equiv \forall x \exists y \left[ t(\langle x, y \rangle) = 0 \right]' \rightarrow \exists \alpha \forall x \left[ t(\langle x, \alpha(x) \rangle) = 0 \right]' .\]

By Lemma 6.1, \( (a) \vdash \exists \exists \forall w \left[ t(w) = \beta(w) \right]' \), so we assume \( (a) \exists \forall w \left[ t(w) = \beta(w) \right]' \) from which we get by \( \forall \)-elim. with the REMARK

\[ (a2) \quad \left[ t(\langle x, y \rangle) = \beta(\langle x, y \rangle) \right]' .\]

We assume now \( (b) \forall x \exists y \left[ t(\langle x, y \rangle) = 0 \right]' \), so we get \( (c) \exists y \left[ t(\langle x, y \rangle) = 0 \right]' \) and we may assume \( (c1) \left[ t(\langle x, y \rangle) = 0 \right]' \). By \( (c1), (a2) \), the REMARK and Lemma 6.2, we have \( \beta(\langle x, y \rangle) = 0 \) so we get \( (d) \exists y \beta(\langle x, y \rangle) = 0 \) and discharge \( (c1) \) with \( \exists \)-elim. (from \( (c) \)). With \( \forall x \)-introd. to \( (d) \) \( x \) not free in \( (a1), (b) \), we get

\[ (e) \forall x \exists y \beta(\langle x, y \rangle) = 0. \]

Now by applying QF-AC\(_{00} \) to \( (e) \) we get

\[ (f) \exists \alpha \forall x \beta(\langle x, \alpha(x) \rangle) = 0. \]

We assume now \( (g) \forall x \beta(\langle x, \alpha(x) \rangle) = 0 \) from which we get \( (g1) \beta(\langle x, \alpha(x) \rangle) = 0 \).

With \( \forall w \)-elim. from \( (a1) \) and the REMARK we get \( \left[ t(\langle x, \alpha(x) \rangle) = \beta(\langle x, \alpha(x) \rangle) \right]' \), so by \( (g1) \) and the REMARK after \( \forall \)-introd. \( x \) not free in \( (a1), (b), (g) \), we have

\[ (h) \forall x \left[ t(\langle x, \alpha(x) \rangle) = 0 \right]' , \]

and finally

\[ (i) \exists \alpha \forall x \left[ t(\langle x, \alpha(x) \rangle) = 0 \right]' , \]

with \( \exists \alpha \)-introd. and \( \exists \alpha \)-elim. disch. \( (g) \) (from \( (f) \)). After completing the \( \exists \beta \)-elim. disch. \( (a1) \) (from \( (a) \)), with \( \rightarrow \)-introd. disch. \( (b) \) we get \( \vdash (E') \). □

Theorem 6.5. (a) EL is a definitional extension of EL − \( \lambda \).

(b) IA\(_1\) + QF-AC\(_{00} \) is a definitional extension of IA\(_1\) + QF-AC\(_{00} \) − \( \lambda \).

7. Comparison of M with BIM, H and WKV

7.1. The formal system BIM.

7.1.1. The formal system BIM of Basic Intuitionistic Mathematics (described variously in [Veldman1], [Veldman2], [Veldman3]) has been introduced by W. Veldman to serve as a formal basis for intuitionistic mathematics. This system has a very small collection of primitives: the most recent version in [Veldman3] includes constants for 0, for the unary function with constant value 0, for the identity function, for the successor, and constants J, K, L, for a binary pairing function (which is taken to be onto the natural numbers) with the corresponding projection functions. Besides axioms concerning equality, the non-logical axioms comprise the axiom schema of mathematical induction, finitely many axioms on the function constants, presented as a single axiom, and axioms guaranteeing the closure of the set of (one-place number-theoretic) functions under the operations of composition, pairing and primitive recursion, again presented as conjuncts of a single axiom. Constants for primitive recursive functions and relations can be added conservatively. There is no \( \lambda \)-abstraction. The following axiom of unbounded search is also included, as the last conjunct of the axiom on the closure of the set of functions:

\[ \forall \alpha \left[ \forall m \exists n \alpha(m, n) = 0 \rightarrow \exists \gamma \forall m \left[ \alpha(m, \gamma(m)) = 0 \land \forall n < \gamma(m) \alpha(m, n) \neq 0 \right] \right] , \]

where \( \alpha(m, n) \) abbreviates \( \alpha(J(m, n)) \) which guarantees closure under the unbounded least number operator and through its consequence

\[ \forall \alpha \left[ \forall m \exists n \alpha(m, n) = 0 \rightarrow \exists \gamma \forall m \alpha(m, \gamma(m)) = 0 \right] \]
(called the minimal axiom of countable choice) expresses, again in the form of
a single axiom, countable numerical choice for a special case of quantifier-free
formulas.

**Proposition 7.1.** Over BIM, the schema $\text{CF}_d$ entails $\text{AC}_{00}!$.

**Proof.** The proof is similar to that of Theorem 3.2 since BIM proves the decid-
ability of number equality (by the axiom schema of induction). We only have to
add (conservatively) addition $+$ to BIM so that $<$ can be expressed efficiently,
and use the constants $J, K, L$ instead of the pairing and projection functions of
$\text{IA}_1$ and the axiom of unbounded search instead of QF-$\text{AC}_{00}$. □

In fact, over BIM without the axiom of unbounded search, the conjunction of
the axiom of unbounded search with the schema $\text{CF}_d$ is equivalent to $\text{AC}_{00}!$. The
one direction is given by the preceding proposition, and for the converse, $\text{AC}_{00}!$
entails $\text{CF}_d$ like in Proposition 3.1, and it is easy to see that $\text{AC}_{00}!$ entails the
axiom of unbounded search adapting Proposition 2.7, (again by adding addition
$+$ to BIM so that $<$ can be expressed efficiently).

7.1.2. The result of [JRMPhD] on $\lambda$-eliminability applies to any formal system of
the sort that we are studying which has at least function constants $0, \cdot, +, \cdot, \exp,$
and assumes $\text{AC}_{00}!$. The system BIM can be extended definitionally to a system
$\text{BIM}'$, so that it includes all these constants. Consequently, the $\lambda$ symbol with
the related formation rule and axiom schema can be added definitionally to the
system $\text{BIM}' + \text{CF}_d$. Consider now the system obtained by the addition of $\lambda,$
$\text{BIM}' + \text{CF}_d + \lambda$. In this system, thanks to the presence of $\lambda$, we can easily obtain
Lemma 5.3(b) of [FIM] from the following **Axiom of Primitive Recursion** of BIM:

$$\forall \alpha \forall \beta \exists \gamma \forall m \forall n \left[ \gamma(m, 0) = \alpha(m) \land \gamma(m, S(n)) = \beta(m, n, \gamma(m, n)) \right].$$

By the same method that the recursor constant $\text{rec}$ is added to M, all the addi-
tional function and functional constants of M can be added definitionally, succes-
sively according to their primitive recursive descriptions, to this extended system.
Let $\text{BIM}^+$ be the result of adding to BIM $\lambda$ and all the additional constants
of M, with their formation rules and axioms. Let $\text{M}^\dagger$ be the trivial extension of
M obtained by adding the constants $J, K, L$ (note that there exists a primitive
recursive pairing function which is onto the natural numbers and has primitive
recursive inverses, and of course all three can be added definitionally to M, to
serve as interpretations for $J, K, L$). Then $\text{BIM}^+ + \text{CF}_d$ and $\text{M}^\dagger$ coincide up to
trivial differences, and we have the following:

**Theorem 7.2.** $\text{M}^\dagger$ is a definition extension of $\text{BIM} + \text{CF}_d$.

We conclude that $\text{BIM} + \text{CF}_d$ and M are essentially equivalent, and also
essentially equivalent with $\text{EL} + \text{CF}_d$. It is remarkable that, in developing the
theory within $\text{BIM}$, W. Veldman defines a set of natural numbers to be decidable
if it has a characteristic function, as this means that $\text{CF}_d$ is implicitly assumed.

**Theorem 7.3.** BIM does not prove $\text{CF}_d$.

**Proof.** This is obtained as in the case of EL, Thm. 3.4, interpreting the function
variables as varying over the general recursive functions of one number variable. □
7.2. The formal system H. In [Howard-Kreisel], the formal system of elementary intuitionistic analysis $H$ is used. $H$ differs from BIM essentially only in that it does not assume the axiom of unbounded search (some primitive recursive functions and relations are introduced in the system from the beginning, but this is an inessential difference).

Like $IA_1$ and $HA_1$, $H$ has a classical model consisting of the primitive recursive functions. Essentially $H$ is a proper subtheory of BIM.

7.3. The formal system WKV. In [Loeb] Iris Loeb presents and uses the formal system $WKV$ (for “Weak Kleene-Vesley”). The constants include 0, $S$, $+$, $\cdot$, =, $j$ (for a pairing function which is onto) and $j_1$, $j_2$ (projections). This system has $\lambda$-abstraction. Among the axioms assumed, there is an axiom schema of primitive recursion in the version

$$\exists \beta \left[ \beta(0) = t \land \forall y \beta(S(y)) = r(j(y, \beta(y))) \right],$$

where $t$ is a term and $r$ a functor, and $AC_{00}$! We observe that the assumed version of the axiom schema of primitive recursion is very similar to Lemma 5.3(b) of [FIM]; in fact, as it is easily seen, these schemas are equivalent (modulo the pairing) over both $WKV$ and $M$. Using the method of the addition of the recursor constant $rec$ to $M$, all constants of $M$ can, successively according to their primitive recursive descriptions, be added definitionally to $WKV$. So, if we consider the trivial extension $M^j$ of $M$ by the pairing and projections of $WKV$ with their axioms, we conclude that the two systems are essentially equivalent.

**Theorem 7.4.** $M^j$ is a definitional extension of $WKV$.

8. Concluding observations

8.1. More on BIM and $H$. The following clarify, among other things, the relationship between $EL$ and BIM.

**Observation 1.** In the presence of enough function constants in BIM without the axiom of unbounded search and in $H$, the single minimal axiom of countable choice is equivalent to QF-AC$_{00}$. To see this, consider the description of BIM in [Veldman3]. What follows holds also for $H$. As it is noted, thanks to the presence of the pairing function, the binary, ternary etc. functions can be treated as unary. So we can consider (definitional) extensions of BIM by only unary function constants, expressing primitive recursive functions, introduced as functors in the formalism (see below the treatment of the recursor $R$); then, since the only non-unary function constant is the binary constant $J$ for the pairing function, for which we get from the axioms (*) $\forall \alpha \forall \beta \exists \gamma \forall n \gamma(n) = J(\alpha(n), \beta(n))$, we have:

(a) For every term $t$ with free number variables say for simplicity the $x$, $y$, we can prove $\exists \alpha(\alpha(x, y) = t(x, y))$ (for $t(x) \equiv x$ consider $\alpha$ with $\alpha(x) = J(K(x), L(x))$, justified by (*) and the axioms for $J$, $K$, $L$, or the identity function, absent from the earlier versions of the system).

If an extension as above contains a sufficient supply of primitive recursive functions, essentially like in [FIM], pp. 27, 30, we have:

(b) For any quantifier-free formula $A(x, y, z)$ with free number variables the indicated three (for example), we can prove $\exists \gamma \forall x \forall y \forall z (A(x, y, z) \leftrightarrow \gamma(x, y, z) = 0)$.

Moreover:

(c) By formal induction on $z$, we can prove $\forall z \exists \beta \forall n \beta(n) = z$. 
(d) Given a (number) \( z \) and a (function) \( \gamma \), we may assume, by (c), \( \forall n \beta(n) = z \) and define \( \delta(w) = \gamma(K(w), L(w), z) = \gamma(K(w), L(w), \beta(w)) \), using (\(*\)).

Now let \( \forall x \exists y A(x, y, z) \) be the hypothesis of an instance of QF-AC\(_{00}\). By the above facts we get a \( \delta \) such that \( A(x, y, z) \leftrightarrow \delta(x, y) = 0 \), then get \( \forall x \exists y A(x, y) = 0 \), and applying the minimal axiom of countable choice to this get \( \exists \alpha \forall x A(x, \alpha(x)) = 0 \) and finally \( \exists \alpha \forall x A(x, \alpha(x), z) \).

Observation 2. We can add definitionally to \( H \) and BIM (and in their extensions as above, but in this case the previous additions become redundant) a recursor constant: we add a constant \( R \) together with the formation rule “if \( t \) is a term and \( u \) a functor, then \( R(t, u) \) is a functor” and defining axiom \( A(x, \alpha, R(x, \alpha)) \), where \( A(x, \alpha, \beta) \) is the formula

\[
\forall y \forall z [\beta(y, 0) = x \& \forall z \beta(y, S(z)) = \alpha(y, z, \beta(y, z))].
\]

We note that facts (a) - (d) hold in the present extension too. Since we have \( \vdash \exists! \beta A(x, \alpha, \beta) \), we can show that this extension is definitional, using the translation ’ defined as follows, with terminology, notation and conditions on the variables as in the previous cases, and with use of functional versions of results from \[IM\], see discussion before Lemma 4.7. For each prime formula \( E \) without \( R \), \( E' \) is \( E \). Otherwise, if \( R(t, u) \) is the first occurrence of an \( R \)-plain functor in \( E \), then

\( E' \equiv \exists \beta [A(t, u, \beta) \& [E(\beta)']] \).

Observation 3. In Lemmas 6.1 - 6.3 (on \( \lambda \)-eliminability) there is just one use of QF-AC\(_{00}\), for the basis of the inductive argument of Lemma 6.1. By the facts (a), (d) above, this use can be avoided. So Lemmas 6.1 - 6.3 are valid for suitable extensions (which may include the recursor \( R \)) of BIM and H, allowing to add \( \lambda \) definitionally.

We have now the following conclusions (note that \( R \) can be introduced in HA\(_1\) and EL by explicit definition).

**Theorem 8.1.** The systems \( H \) and HA\(_1\) have a common conservative extension obtained by definitional extensions, so they are essentially equivalent.

**Theorem 8.2.** The systems BIM and EL have a common conservative extension obtained by definitional extensions, so they are essentially equivalent.

8.2. Additional results. In proving that the recursor constant rec can be added definitionally to \( M \), AC\(_{00}\)! has been used in two cases (except from the point where it is shown that the translation of AC\(_{00}\)! itself is a theorem of \( M \)): in the proof of Lemma 5.3(b) of [FIM], and in the proof of Lemma 4.4. We will prove these two lemmas by using only QF-AC\(_{00}\). In this way, together with the observation that QF-AC\(_{00}\) is equivalent over \( IA_1 \) with the single axiom

\[
\forall \alpha [\forall x \exists y \alpha(\langle x, y \rangle) = 0 \rightarrow \exists \gamma \forall x \alpha(\langle x, \gamma(x) \rangle) = 0],
\]

we obtain the fact that rec can be added definitionally to \( IA_1 + QF-AC_{00} \).

**Lemma 8.3.** In \( IA_1 + QF-AC_{00} \),

\[
\vdash \exists \beta [\beta(0) = x \& \forall z \beta(z') = \alpha(\langle \beta(z), z \rangle)].
\]

\(^9\)For the treatment of definitional extensions by the introduction of functors in the formalism, see [Kreisel-Troelstra]. Note that the system called EL (and its subsystem EL\(_0\)) in that work, assumes AC\(_{00}\)!.
Proof. We slightly modify the proof of Lemma 5.3(b) of [FIM] as follows. Let
\[ P(x, \alpha, y, v) \equiv (v)_0 = x \& \forall i < y (v)_i = \alpha((v)_i, i)) \]
By formal induction (IND) on \( y \) we show first (a) \( \vdash \forall y \exists v P(x, \alpha, y, v) \) as follows:

**BASIS.** We get \( \exists v P(x, \alpha, 0, v) \) by “setting” \( v = p^0_0 \).

**IND. STEP.** Assuming \( P(x, \alpha, y, w) \) and “setting” \( v = \Pi_i \geq y p^i_0 \ast p_{y'}^{\alpha((w)_y, y)} \), since then
\[ \forall i < y (v)_i = (w)_i \& (v)_{y'} = \alpha((w)_y, y) \]
we get \( \exists v P(x, \alpha, y', v) \) from the inductive hypothesis.

Using \( \Pi_{i \leq y} p^i_0 \ast p_{y'}^{\alpha((w)_y, y)} \ast p_{y'}^{\alpha((w)_y, y)} \), we can apply QF-AC\(_{00} \) to (a) and get
\[ (b) \quad \exists \gamma \forall y P(x, \alpha, y, \gamma(y)) \]
Assume now (c) \( \forall y P(x, \alpha, y, \gamma(y)) \). We “define” now
\[ \beta = \lambda y. \gamma(y) \]
(justified by Lemma 5.3(a) of [FIM]). Then we can show
\[ (d) \quad \exists \beta[\beta(0) = x \& \forall z \beta(z') = \alpha(\langle \beta(z), z \rangle)] \]
(for the second conjunct in (d), specialize from (c) for \( z' \) and \( i = z (z < z') \), and get \( \gamma(z')_{z'} = \alpha(\gamma(z'), z) \)). But \( \gamma(z')_{z} = \gamma(z)_{z} \) (by formal induction (IND) on \( z \)), using (c), we can show \( \forall z \forall i \leq z \gamma(z')_{i} = \gamma(z)_{i} \), so \( \beta(z') = \alpha(\langle \beta(z), z \rangle) \). \( \square \)

**Lemma 8.4.** In \( \text{IA}_1 + \text{QF-AC}_{00} \),
\[ \vdash \exists ! \beta[\beta(0) = x \& \forall z \beta(z') = \alpha(\langle \beta(z), z \rangle)] . \]
In the next lemma the notation and abbreviations are as in Lemma 4.4.

**Lemma 8.5.** Let \( t, s \) be terms and \( u \) a functor of \( \text{IA}_1 \). Let \( x_0, \ldots, x_k \) include all the number variables occurring free in \( t, u \) or \( s \), let \( w \) and \( v \) be distinct number variables not occurring in \( t, u, s, \) and \( \gamma \) a function variable free for \( v \) in \( A(t^w, u^w, s^w, v) \), not occurring free in \( A(t^w, u^w, s^w, v) \). Then in \( \text{IA}_1 + \text{QF-AC}_{00} \)
\[ \vdash \exists ! \gamma \forall w A(t^w, u^w, s^w, \gamma(w)) . \]

**Proof.** We want to show
\[ \vdash \exists ! \gamma \forall w \exists \beta[\beta(0) = t^w \& \forall z \beta(z') = (u^w)((\langle \beta(z), z \rangle) \& \beta(s^w) = \gamma(w))] . \]
From (a) in the proof of Lemma 8.3 we have \( \vdash \forall w \exists ! v P(t^w, u^w, s^w, v) \). Applying QF-AC\(_{00} \) we get (a) \( \vdash \exists ! \delta \forall w P(t^w, u^w, s^w, \delta(w)) \). Assume
\[ (a1) \quad \forall w \big[(\delta(w))_0 = t^w \& \forall i < s^w (\delta(w))_{i'} = (u^w)((\langle (\delta(w))_{i}, i \rangle)) \big] , \]
and let (\( * \)) \( \gamma = \lambda w. (\delta(w))_{s^w} \). By (a1), we get
\[ (a2) \quad (\delta(w))_0 = t^w \& \forall i < s^w (\delta(w))_{i'} = (u^w)((\langle (\delta(w))_{i}, i \rangle)) . \]
From Lemma 8.4,
\[ (b) \quad \vdash \forall w \exists ! \beta[\beta(0) = t^w \& \forall z \beta(z') = (u^w)((\langle \beta(z), z \rangle))] , \]
so we may assume
\[ (b1) \quad \beta(0) = t^w \& \forall z \beta(z') = (u^w)((\langle \beta(z), z \rangle)) \]
Then we get (c) \( \beta(s^w) = (\delta(w))_{s^w} \) (to get (c) we prove \( \forall i \leq s^w \beta(i) = (\delta(w))_i \) by formal induction (IND) on \( i \)), so (d) \( \gamma(w) = \beta(s^w) \). By \( \rightarrow - \text{elim. disch.} \) (b1) with
∀β- and ∀w-intros. we get (e) ∀w∀β [(b1) → γ(w) = β(s^w)]. From (b) and (e) we get
(f) ∀w∃β [β(0) = t^w & ∀z β(z') = (u^w)((β(z), z)) & β(s^w) = γ(w)].
For the uniqueness, assume now
(g) ∀w∃β [β(0) = t^w & ∀z β(z') = (u^w)((β(z), z)) & β(s^w) = ε(w)].
Then from (e) and (f) we get easily (h) ∀w ε(w) = γ(w). So from (g) we get
(i) ε = γ,
and then ∀ε ((g) → (i)) (disch. (g) before ∀-introd.). So with (f) and ∃γ-introd. we get the lemma, after completing the ∃γ, ∃δ-elims. disch. (∗) and (a1), respectively.

Corollary 8.6. The system IA_1 + QF-AC_{00} + Rec is a definitional extension of IA_1 + QF-AC_{00}.

8.3. Obtained relations. Below we collect the main results following from the arguments we presented. We note that II(b) justifies completely the observation of J. Rand Moschovakis ([JRM2014], Thm. 1) that BIM, EL and IA_1 + QF-AC_{00} can be used interchangeably as a basis for intuitionistic reverse analysis.

I. In relation to the small classical models that the weak constructive systems we studied admit:
(a) The systems H, IA_1, HA_1, IA_1 + Rec have a classical model consisting of primitive recursive functions.
(b) BIM and all the systems resulting from adding QF-AC_{00} to IA_1, HA_1, IA_1 + Rec have a classical model consisting of general recursive functions.
(c) Adding CF_d to the systems of (b) gives stronger systems, which do not admit small classical models consisting of recursive functions.

II. The systems of each group are proof-theoretically essentially equivalent:
(a) IA_1 + Rec, HA_1, H.
(b) IA_1 + QF-AC_{00}, EL, BIM, H + Unbounded Search.
(c) M, WKV, EL + CF_d, BIM + CF_d, H + AC_{00}!

III. We observe also the following.
(a) Adding QF-AC_{00} to any of H, IA_1, HA_1 gives stronger systems.
(b) Adding QF-AC_{00} to any of H + CF_d, IA_1 + CF_d, HA_1 + CF_d gives stronger systems (J. Rand Moschovakis, [JRM-GV2012]).
(c) Adding CF_d to any of H, IA_1, HA_1, EL, BIM gives stronger systems.

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