EDGE RINGS SATISFYING SERRE’S CONDITION \((R_1)\)

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Abstract. A combinatorial criterion for the edge ring of a finite connected graph satisfying Serre’s condition \((R_1)\) is studied.

Introduction

The edge polytopes and edge rings of finite connected graphs have been studied from the viewpoints of both combinatorics and computational commutative algebra ([3], [4]). Especially, a combinatorial characterization for the edge ring to be normal is obtained by both [3] and [6] independently. It follows immediately from [2, Theorem 6.4.2] that a normal edge ring is Cohen–Macaulay. However, in general it seems unclear when the edge ring is Cohen–Macaulay. Recall that a noetherian ring is normal if and only if it satisfies Serre’s conditions \((R_1)\) and \((S_2)\). Thus in particular an edge ring satisfying Serre’s condition \((R_1)\) is normal if and only if it is Cohen–Macaulay. In the present paper the problem when a given edge ring satisfies Serre’s condition \((R_1)\) is investigated.

1. Edge rings and edge polytopes of finite connected graphs

First, we recall from [3] the definitions of edge rings and edge polytopes of finite connected graphs. Let \(G\) be a finite connected graph on the vertex set \([d] = \{1, \ldots, d\}\) with \(E(G) = \{e_1, \ldots, e_n\}\) its edge set. We always assume that \(G\) is simple, i.e., \(G\) has no loop and no multiple edge. Let \(e_1, \ldots, e_d\) denote the \(i\)th unit coordinate vectors of \(\mathbb{R}^d\). We associate each edge \(e = \{i, j\} \in E(G)\) with the vector \(\rho(e) = e_i + e_j \in \mathbb{R}^d\). The edge polytope is the convex polytope \(P_G \subset \mathbb{R}^d\) which is the convex hull of the finite set \(\{\rho(e_1), \ldots, \rho(e_n)\}\). Let \(K[t] = K[t_1, \ldots, t_d]\) be the polynomial ring in \(d\) variables over a field \(K\). We associate each edge \(e = \{i, j\} \in E(G)\) with the quadratic monomial \(t^e = t_it_j \in K[t]\). The edge ring is the affine semigroup ring \(K[G] = K[t^{e_1}, \ldots, t^{e_n}]\).

Let, in general, \(P \subset \mathbb{R}^d\) be an integral convex polytope, i.e., a convex polytope all of whose vertices have integer coordinates, which lies on a hyperplane \(H \subset \mathbb{R}^d\) with \(0 \notin H\), where \(0\) is the origin of \(\mathbb{R}^d\). We assume that \(P \subset \mathbb{R}^d_{\geq 0}\), where \(\mathbb{R}^d_{\geq 0}\) is...
the set of nonnegative real numbers. Then for each integer point \( \mathbf{a} = (a_1, \ldots, a_d) \) belonging to \( \mathcal{P} \), we associate the monomial \( t^{a} = t_1^{a_1} \cdots t_d^{a_d} \in K[t] \). The toric ring of \( \mathcal{P} \) is the affine semigroup ring \( K[\mathcal{P}] = K[\{t^a : \mathbf{a} \in \mathcal{P} \cap \mathbb{Z}^d\}] \). Thus in particular the edge ring \( K[G] \) of a finite connected graph \( G \) is the toric ring of the edge polytope \( \mathcal{P}_G \) of \( G \).

We say that an integral convex polytope \( \mathcal{P} \) is normal if its toric ring \( K[\mathcal{P}] \) is normal. It is shown in [3] and [6] that the edge ring of a finite connected graph \( G \) is normal if and only if it is Cohen–Macaulay. In the present paper the problem of when an edge ring satisfies Serre’s condition \((R_1)\) is investigated.

2. When does an edge ring satisfy Serre’s condition \((R_1)\)?

Let \( G \) be a finite connected graph on the vertex set \( [d] = \{1, \ldots, d\} \). If \( G \) is bipartite, then \( K[G] \) is normal and satisfies Serre’s condition \((R_1)\). Thus in what follows we assume that \( G \) is nonbipartite, i.e., \( G \) possesses at least one odd cycle.

If \( T \) is a nonempty subset of \([d]\), then the induced subgraph of \( G \) on \( T \) is denoted by \( G_T \). A nonempty subset \( T \) of \([d]\) is called independent if \( \{i, j\} \notin E(G) \) for all \( i, j \in T \) with \( i \neq j \). If \( T \) is independent and if \( N(G ; T) \) is the set of vertices \( j \in [d] \) with \( \{i, j\} \in E(G) \) for some \( i \in T \), then the bipartite graph induced by \( T \) is defined to be the bipartite graph having the vertex set \( T \cup N(G ; T) \) and consisting of all edges \( \{i, j\} \in E(G) \) with \( i \in T \) and \( j \in N(G ; T) \). We say that a nonempty subset \( T \subset [d] \) is fundamental if

- \( T \) is independent;
- the bipartite graph induced by \( T \) is connected;
- either \( T \cup N(G ; T) = [d] \) or every connected component of the induced subgraph \( G_{[d]\setminus(T \cup N(G ; T))} \) has at least one odd cycle.

Moreover, we call a vertex \( i \in [d] \) regular if every connected component of \( G_{[d]\setminus i} \) has at least one odd cycle. Note that a regular vertex is not the same as a fundamental set with one element.

We are now in the position to state our criterion for an edge ring to satisfy Serre’s condition \((R_1)\).

**Theorem 2.1.** Let \( G \) be a finite connected nonbipartite graph on \([d]\). Then the edge ring \( K[G] \) of \( G \) satisfies Serre’s condition \((R_1)\) if and only if the following conditions are satisfied:

(i) For every regular vertex \( i \in [d] \), the induced subgraph \( G_{[d]\setminus i} \) is connected.

(ii) For every fundamental set \( T \subset [d] \), one has either \( T \cup N(G ; T) = [d] \) or the induced subgraph \( G_{[d]\setminus(T \cup N(G ; T))} \) is connected.

**Example 2.2.** Let \( G \) be the finite connected graph on \( \{1, \ldots, 8\} \) depicted in Figure 1. The graph \( G \) clearly violates the odd cycle condition, hence the edge ring \( K[G] \) is not normal. The only vertices whose removal makes \( G \) disconnected are 3 and 4, but both are not regular. If \( T \subset [8] \) is a set such that \( G_{[d]\setminus(T \cup N(G ; T))} \) is disconnected, then either 3 or 4 are contained in \( T \cup N(G ; T) \). But then \( G_{[d]\setminus(T \cup N(G ; T))} \)
Figure 1. The graph $G$ of Example 1

has only one odd cycle left, so $T$ cannot be fundamental. Hence $K[G]$ satisfies Serre’s condition $(R_1)$. More generally, the same argument shows that the graphs $G_{k+6}$ constructed in [5] satisfy $(R_1)$ if and only if $k \geq 2$.

3. Proof of Theorem 2.1

First recall the description of the facets of $P_G$. To every regular vertex $i$ we associate the linear form $\sigma_i : \mathbb{R}^d \to \mathbb{R}$ which projects onto the $i$th component. Moreover, we set $H_i = \{ x \in \mathbb{R}^d : \sigma_i(x) = 0 \}$ and $F_i = P_G \cap H_i$. Similarly, to every fundamental set $T$ we associate the linear form

$$
\sigma_T : \mathbb{R}^d \ni (x_1, \ldots, x_d) \mapsto \sum_{j \in N(G;T)} x_j - \sum_{i \in T} x_i
$$

and we set $H_T = \{ x \in \mathbb{R}^d : \sigma_T(x) = 0 \}$ and $F_T = P_G \cap H_T$.

**Lemma 3.1** ([3]). The facets of $P_G$ are exactly the sets $F_i$ and $F_T$ for all regular vertices $i$ and all fundamental sets $T$.

A combinatorial condition for a semigroup ring to satisfy Serre’s condition $(R_1)$ is explicitly stated in [7, Theorem 2.7]. In fact, in [7] a characterization of $(R_l)$ for all $l$ is given, but for our purposes we only need the case $l = 1$.

**Proposition 3.2** ([7]). Let $M$ be an affine monoid, $K$ a field and $K[M]$ its semigroup ring. Then $K[M]$ satisfies Serre’s condition $(R_1)$ if and only if every facet $F$ of $M$ satisfies the following two conditions:

(i) There exists $x \in M$ such that $\sigma_F(x) = 1$, where $\sigma_F$ is a support form of $F$ taking integer values on $gp(M)$.

(ii) $gp(M \cap F) = gp(M) \cap H$, where $H$ is the supporting hyperplane of $F$.

Here $gp(M)$ denotes the additive group generated by $M$.

We apply Proposition 3.2 to the affine monoid

$$
M_G = \mathbb{N}(P_G \cap \mathbb{Z}^d)
$$

generated by the integer points in $P_G$. Note that the support hyperplanes $H_i$ and $H_T$ of $P_G$ are also the support hyperplanes of $M_G$. We start proving Theorem 2.1 by the following.

**Lemma 3.3.** Let $G$ be a finite connected nonbipartite graph on the vertex set $[d]$. Then the facets of $M_G$ satisfy the first condition of Proposition 3.2.
Proof. First, let \( i \in [d] \) be a regular vertex. Since \( G \) is connected, there exists an edge \( e = \{i, j\} \in E(G) \) to another vertex \( j \). Then \( \sigma_{i}(\rho(e)) = 1 \).

Second, let \( T \subset [d] \) be a fundamental set. If \( T \cup N(G; T) \subset [d] \), then there exists an edge \( e = \{i, j\} \in E(G) \) such that \( i \in N(G; T) \) and \( j \in [d] \setminus (T \cup N(G; T)) \). This edge satisfies \( \sigma_{T}(\rho(e)) = 1 \). If instead \( T \cup N(G; T) = [d] \), then every edge of \( G \) has either both endpoints in \( N(G; T) \), or one in \( N(G; T) \) and one in \( T \). Hence \( \sigma_{T}(e) \in \{0, 2\} \) for every edge \( e \) of \( G \). It then follows that \( \frac{1}{2}\sigma_{T} \) satisfies the condition of Proposition 3.2. \qed

To check the second condition of Proposition 3.2, we need to compute the lattice generated by \( M_{G} \). The following Lemma 3.4 appears in [3, p. 426] without an explicit proof. However, for the sake of completeness, we give its detailed proof.

**Lemma 3.4.** Let \( G \) be a finite connected nonbipartite graph on the vertex set \([d]\). Then the lattice \( gp(M_{G}) \) is the set of all integer vectors in \( \mathbb{Z}^{d} \) with an even coordinate sum.

**Proof.** Since every generator of \( M_{G} \) has an even coordinate sum, it follows that the lattice \( gp(M_{G}) \) is contained in the set of all integer vectors in \( \mathbb{Z}^{d} \) with an even coordinate sum.

To prove the converse, assume the edges \( e_{1}, \ldots, e_{\ell} \) form an odd cycle of \( G \) and let \( i \) be the common vertex of \( e_{1} \) and \( e_{\ell} \). Then

\[
2e_{i} = \sum_{j=1}^{\ell} (-1)^{j+1} \rho(e_{j}) \in gp(M_{G}).
\]

Now consider a spanning tree \( G' \) of \( G \). The set \( \{ \rho(e) \mid e \in E(G') \} \) together with \( 2e_{i} \) forms a \( \mathbb{Z} \)-basis for the space of all integer vectors in \( \mathbb{Z}^{d} \) with an even coordinate sum. \qed

Now, we can prove two propositions which complete our proof of Theorem 2.1.

**Proposition 3.5.** Let \( G \) be a finite connected nonbipartite graph on the vertex set \([d]\) and let \( i \in [d] \) be a regular vertex of \( G \). Then \( \mathcal{F}_{i} \) satisfies the second condition of Proposition 3.2 if and only if \( G_{[d]\setminus i} \) is connected.

**Proof.** We denote the connected components of \( G_{[d]\setminus i} \) with \( G'_{j} \). Then it is easy to see that \( M_{G_{[d]\setminus i}} = \bigoplus_{j} M_{G'_{j}} \); and hence \( gp(M_{G} \cap \mathcal{F}_{i}) = gp(M_{G_{[d]\setminus i}}) = \bigoplus_{j} gp(M_{G'_{j}}) \). Since every \( G'_{j} \) is connected and contains an odd cycle, we can use Lemma 3.4 to describe \( gp(M_{G'_{j}}) \). If \( G_{[d]\setminus i} \) is connected, then \( gp(G_{[d]\setminus i}) \) and \( gp(M_{G}) \cap \mathcal{H}_{i} \) are both the set of integer vectors in \( \mathbb{Z}^{d} \) with even coordinate sum and ith coordinate equal to zero; thus these sets coincide.

We consider the case that \( G_{[d]\setminus i} \) has at least two different connected components \( G'_{1}, G'_{2} \). Then we can choose a vector \( x \in \mathbb{Z}^{d} \) such that (i) its coordinate sum is even, (ii) \( \sigma_{i}(x) = 0 \), and (iii) the restricted coordinate sum over the vertices in \( G'_{1} \) is odd. This \( x \) is contained in \( gp(M_{G}) \cap \mathcal{H}_{i} \) but not in \( gp(G_{[d]\setminus i}) \); thus \( \mathcal{F}_{i} \) violates the condition. \qed

**Proposition 3.6.** Let \( G \) be a finite connected nonbipartite graph on the vertex set \([d]\) and let \( T \subset [d] \) be a fundamental set of \( G \). Then \( \mathcal{F}_{T} \) satisfies the second condition of Proposition 3.2 if and only if one has either \( T \cup N(G; T) = [d] \) or the induced subgraph \( G_{[d]\setminus (T \cup N(G; T))} \) is connected.
Proof. Again, we denote the connected components of $G_{[d]} \setminus (T \cup N(G; T))$ with $G'_j$.

We claim that
\begin{equation}
gp(M_G \cap \mathcal{F}_T) = \bigoplus_j gp(M_{G'_j}) \oplus \{ x \in \mathbb{Z}^d \mid \text{supp}(x) \subset T \cup N(G; T), \sigma_T(x) = 0 \}.
\end{equation}

Here, supp(.) denotes the support of a vector. The sum is direct because the supports of the summands are disjoint. $M_G \cap \mathcal{F}_T$ (and thus $gp(M_G \cap \mathcal{F}_T)$) is generated by the set $\{ \rho(e) \mid e \in E(G), \sigma_T(\rho(e)) = 0 \}$. For an edge $e \in E(G)$, it holds that $\sigma_T(\rho(e)) = 0$ if and only if either both endpoints lie in $T \cup N(G; T)$ or both are not contained in this set. Thus, a set of generators on the left side of (1) is contained in the right side of the equation, and hence one inclusion follows.

Furthermore $\bigoplus_j gp(M_{G'_j}) \subset gp(M_G \cap \mathcal{F}_T)$. Thus it remains to show that
\begin{equation}
\{ x \in \mathbb{Z}^d \mid \text{supp}(x) \subset T \cup N(G; T), \sigma_T(x) = 0 \} \subset gp(M_G \cap \mathcal{F}_T).
\end{equation}

For this we consider a spanning tree of the induced bipartite graph on $T \cup N(G; T)$. Its edges form a $\mathbb{Z}$-basis for the left set, hence it is contained in $gp(M_G \cap \mathcal{F}_T)$. Next, we note that
\begin{equation}
gp(M_G) \cap \mathcal{H}_T = \left\{ x \in \mathbb{Z}^d \mid \text{supp}(x) \cap (T \cup N(G; T)) = \emptyset, \sum x_i \text{ even} \right\} \oplus \left\{ x \in \mathbb{Z}^d \mid \text{supp}(x) \subset (T \cup N(G; T)), \sigma_T(x) = 0 \right\}.
\end{equation}

Now the reasoning is completely analogous to the proof of Proposition 3.5. \qed

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