Application of refined mixed finite element model for analysis of laminated composite beams

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Abstract. Analysis of laminated composite members is always a critical phenomenon. To address this issue, the authors developed the six nodes two dimensional refined mixed finite element model (r-MFEM) for accurate analysis of laminated composite beam. This model has transverse stress $\sigma_z$ and $\tau_{xz}$ as primary variables along with displacement $u$ and $w$. Equilibrium equations are used to refine the variation of primary variables in the formulation. In the present paper, various laminated composite beams have been analysed by the six nodes two-dimensional r-MFEM element to show the model’s versatility. The results obtained has been compared with the other author’s work.

1. Introduction
Composite beams under cylindrical bending and different load and boundary condition have been solved analytically by the Pagano [1]. However, this solution is for elementary geometries. These simple geometries are far from actual structures, and the process of solution is costly and time-consuming. Various assumptions have been made to ease the analysis of the composite beam, and the beam’s dimension has been reduced. These relevant physical assumptions let the researcher reduce the three-dimensional beam into one dimension. The most straightforward theory came from the assumption that it was equivalent to single-layer theory (ESL). This theory assumes the behaviour of a composite beam as a single layer and analyses it. ESL theory helps in the reduction of complexity and gives a sufficiently justified solution. In ESL, initially, shear deformation was ignored. Later on, the first-order shear deformation theory (FSDT), shear deformation, was considered. However, this theory needs the empirical factor to be multiplied in the solution.

In the analysis of composite laminates, transverse stresses are significant. To analysis transverse stresses precisely, various methods have been proposed. Zig - Zag theories by different scholars like Carrera [2] Kapuria [3] Demasi[4] Eijo [5] Fillipi and Carrera [6] have been developed and tested for various benchmark solution available in the literature. Apart from this, the higher-order theory by Reddy [7] Kant, and Swaminathan [8], and other notable authors have been developed. Later on, Pawar et al. [9] addressed the issue of normal displacement of the beam. They discussed the difference between and bending of the beam and cylindrical bending. There

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is no normal displacement in cylindrical bending, which is there in bending the beam. They
included the warping function, which incorporates the normal deformation, and hence they
claim that it is closer to the realistic behaviour of beam bending. They highlighted this issue in
softcore sandwich beam material.

The mixed finite element method is another dimension to analyse the composite laminates.
However, this dimension requires a more complex mathematical formulation and complex
computation. However, this method allows calculating transverse stresses as primary variables.
Ramtekkar and Desai [10] formulated two and three-dimension elements to analyse the composite
laminated beam and plates. This method is very precise and comparable to the analytical
solution given by Pagano [1]. Compared to other mixed finite element formulations based on the
Hellinger Reissners minimisation principle, Ramtekkar’s formulation is based on the minimum
potential energy principle, which provides numerical stability. As this method could be proved
bit exhaustive, one can combine this with standard finite element using the transition element
developed by Bambole and Desai [11] and applied in different cases by Band and Desai [12].

2. Theoretical formulation
A laminated beam is considered as shown in Fig. 1. The three-dimensional beam is considered
two-dimensional (Fig. 1-C) with the assumption that no variation in the Y direction. Now
two-dimensional beam lying in the XZ plane is being considered for the analysis. A six node
two-dimensional model based on the mixed finite element was developed. Each node of an
element has \( u, w, \sigma_{zz}, \tau_{xz} \) as primary variables. Along with these primary variables, equilibrium
equations are considered in the formulation. Equilibrium equations are used to satisfy the
equilibrium in the formulation. All variations of the different fields in the formulation are not
independent. Variation of stresses has been invoked from the displacement variation by using the
fundamental elasticity relation of kinematics. Using the equation of equilibrium, the variation of
body forces has been invoked from the variation of the displacement field. This constrained
variation satisfies all three (strain-displacement, stress-strain, and equilibrium) elastic relations
in advance.

The 6-node two-dimensional mixed finite element model shown in Fig. (1-D) has been
developed by considering the displacement \( u(x, z) \) and \( w(x, z) \) having quadratic variation along the
longitudinal x direction and fifth-order variation in transverse z direction. The variation of
displacement field is as shown in Eqn. 1 with \( a \) as a constant and \( g \) as quadratic variation in \( \xi \)
direction. This variation has been used to invoke the variations of other fields.

\[
\sigma_{zz}(x, z) = \sum_{i=1}^{3} g_i a_{0i} + z \sum_{i=1}^{3} g_i a_{1i} + z^2 \sum_{i=1}^{3} g_i a_{2i} + z^3 \sum_{i=1}^{3} g_i a_{3i} + z^4 \sum_{i=1}^{3} g_i a_{4i} + z^5 \sum_{i=1}^{3} g_i a_{5i}
\]  

(1)

Where:

\[
g_1 = \frac{\xi}{2}(\xi - 1); \quad g_2 = (1 - \xi^2); \quad g_3 = \frac{\xi}{2}(1 + \xi); \quad \xi = \frac{1}{L_x}; \]  

(2)

\[
g_1' = \frac{1}{2l}(2\xi - 1); \quad g_2' = \frac{1}{2l}(-2\xi); \quad g_3' = \frac{1}{2l}(2\xi + 1); \]  

(3)

\[
g_1'' = \frac{1}{l^2}; \quad g_2'' = \frac{-2}{l^2}; \quad g_3'' = \frac{1}{l^2}; \]  

(4)

\[
u_1(x, z) = u(x, z); \quad \nu_2(x, z) = w(x, z) \]  

(5)

Constitutive relation for the typical \( i^{th} \) lamina (Fig 1-C) with reference coordinate system can be written as

\[
\{\sigma\} = \{Q\}\{\varepsilon\} \]  

(6)
where:

\[
\{\sigma\} = \begin{bmatrix} \sigma_x & \sigma_z & \tau_{xz} \end{bmatrix}^T; \quad [Q] = \begin{bmatrix} Q_{11} & Q_{13} & 0 \\ Q_{31} & Q_{33} & 0 \\ 0 & 0 & Q_{55} \end{bmatrix} \quad \{\varepsilon\} = \begin{bmatrix} \varepsilon_x & \varepsilon_z & \gamma_{xz} \end{bmatrix}^T \quad (7)
\]

Strain at a point in the lamina and corresponding deformation are the function of assumed displacement field. Linear theory of elasticity gives the general strain-displacement relation as shown in Eqn. 8

\[
\varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_z = \frac{\partial w}{\partial z}; \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (8)
\]

Transverse stresses can be obtained from the constitutive relation Eqn. 6 and strain-displacement relation Eqn. 8 as

\[
\begin{align*}
\sigma_x(x, z) &= Q_{11} \frac{\partial u}{\partial x} + Q_{13} \frac{\partial w}{\partial z} \\
\sigma_z(x, z) &= Q_{13} \frac{\partial u}{\partial x} + Q_{33} \frac{\partial w}{\partial z} \\
\tau_{xz}(x, z) &= Q_{55} \left\{ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right\}
\end{align*} \quad (9)
\]

The body forces can be determined using the equilibrium equation in \(x\) and \(z\) direction can be written as

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + p_x &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + p_z &= 0
\end{align*} \quad (10)
\]

To invoke the variation of stresses \(\sigma_{xx}, \sigma_{zz}, \tau_{xz}\) Eqn. 9 has been used. Variation of body forces has been invoked from the displacement field variation by using the equilibrium equation, strain-displacement relation, and strain displacement relation. After algebraic manipulation, this can be represented in the matrix form as shown in Eqn. 11

\[
\{d\}_{(36\times1)} = [A]_{(36\times36)} \{\alpha\}_{(36\times1)} \quad (11)
\]

where \(\{d\}\) is vector contains nodal primary variables, \([A]\) is the coefficient matrix and \(\{\alpha\}\) is the vector containing constant terms. Nodal primary variables contains body forces \(p_x\) and \(p_z\), along with the displacement \(u, w\) and transverse stresses \(\sigma_{xz}, \tau_{xz}\). On multiplying with \(A^{-1}\) on both side of Eqn. 11 and rearranging the terms

\[
\{\alpha\} = [A]^{-1} \{d\} \quad (12)
\]

where \([A]\) is coefficient matrix which is obtained by putting the coordinate value of node as shown in Fig. 1-D.

Global displacement vector \(\{U\}\) can be obtained from the multiplication of \([X]\) and \(\{\alpha\}\).

\[
\{U\}_{(2\times1)} = [X]_{(2\times36)} \{\alpha\}_{(36\times1)} \quad (13)
\]

Substituting value of \(\{\alpha\}\) from Eqn. 12 in Eqn. 13

\[
\{U\} = [X][A]^{-1} \{d\} \quad (14)
\]
From Eqn. 14 it could be concluded that the portion that is relating the global primary variables to nodal primary variables could be treated as interpolation function \([N]\).

\[
[N] = [X][A]^{-1}
\]  

(15)

After deriving the interpolation function from the above equation, the strain matrix could be obtained by differentiating the interpolation matrix as shown in Eqn. 16.

\[
[B] = [L][X][A]^{-1}
\]  

(16)

Where:

\[
[L] = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z}
\end{bmatrix}^T
\]

\[
[X] = \begin{bmatrix}
g_1 & g_2 & \cdots & z^5 g_3 & 0 & 0 & \cdots & 0
\end{bmatrix}_{2 \times 3}
\]

After deriving these equations, for beginning of finite element formulation variational formulation is being used. In this formulation Minimum total potential energy function is used. Functional for Total energy formulation could be referred from Eqn. 17:

\[
\Pi = \frac{1}{2} \int_V \{\epsilon\}^T \{\sigma\} dV - \int_V \{q\}^T \{p_b\} dV - \int_{\Sigma} \{q\}^T \{p_t\} ds
\]  

(17)

where \(\{p_b\}\) is body force vector per unit volume and \(\{p_t\}\) is traction load per unit area acting on any surface of element. Here \(\sum\)' is a surface element subjected to the traction forces. The strain vector \(\{\epsilon\}\) and the stress vector \(\{\sigma\}\) can be expressed as:

\[
\{\epsilon\} = [B] \{q\}
\]  

(18)

\[
\{\sigma\} = [Q] [B] \{d\}
\]  

(19)

By minimisation of total potential energy functional Eqn. 17 element property matrix \([K]^e\) and element influence vector \(\{f\}^e\) can be obtained as

\[
[K]^e = \int_{-L_x}^{L_x} \int_{-L_y}^{L_y} \int_{-L_z}^{L_z} [B]^T [D] [B] dx \; dy \; dz
\]  

(20)

\[
\{f\}^e = \int_{-L_x}^{L_x} \int_{-L_y}^{L_y} \int_{-L_z}^{L_z} [N]^T \{p_b\} dx \; dy \; dz + \int_{\Sigma} [N]^T \{p_t\} ds
\]  

(21)

This element level stiffness matrix could be seen in following way

\[
\begin{bmatrix}
[K]_{pp}^e & [K]_{pb}^e \\
[K]_{bp}^e & [K]_{bb}^e
\end{bmatrix}
\begin{bmatrix}
\{d\}^e_p \\
\{d\}^e_b
\end{bmatrix}
= 
\begin{bmatrix}
\{f\}^e_p \\
\{f\}^e_b
\end{bmatrix}
\]  

(22)

Where \([K]_{pp}^e\) = Stiffness matrix elements due to primary variables only, \([K]_{bb}^e\) is stiffness matrix elements due to body force components \([K]_{pb}^e\) & \([K]_{bp}^e\) are the cross components.

Performing the static condensation on the Eqn. 22 to take the effect of body force into the primary variables. After performing the Static condensation refined element level property matrix \([K]^{e*}\) and refined element influence vector \(\{f\}^{e*}\) which is now refined element level stiffness matrix.

\[
[K]_{bp}^e \{d\}^e_p + [K]_{bb}^e \{d\}^e_b = \{f\}^e_b
\]  

(23)
where \( \{ f \}^e_b \) is component of force vector due to body force which will be known, it this situation we consider it zero.

\[
[K]^e_{bp}\{d\}^e_p + [K]^e_{bb}\{d\}^e_b = \{ f \}^e_p
\]

\[
\{d\}^e_b = -[K]^e_{bb}^{-1}[K]^e_{bp}\{d\}^e_p + [K]^e_{bb}^{-1}\{ f \}^e_p
\]

\[
[K]^e_{pp}\{d\}^e_p + [K]^e_{bp}\{d\}^e_b = \{ f \}^e_p - [K]^e_{bb}^{-1}\{ f \}^e_p
\]

\[
([K]^e_{pp} - [K]^e_{bp}[K]^e_{bb}^{-1}[K]^e_{bp})\{d\}^e_p = \{ f \}^e_p - [K]^e_{bb}^{-1}\{ f \}^e_p
\]

\[
[K]^e_{pp}\{d\}^e_p = \{ f \}^e_p
\]

where \([K]^e\) is refined element level stiffness matrix and \(\{ f \}^e\) is refined force vector which will be used for the further finite element formulation.

### 3. Numerical application

Various problem with different end condition and lay-ups has been solved using the refined mixed finite element model in this paper. These problems are listed below:

(i) Two layer unsymmetric laminated beam having lamina scheme of \((0^\circ/90^\circ)\) with clamped end condition from table 2 and material set 1 from table 1

(ii) Three layer symmetric laminated beam having lamina scheme of \((0^\circ/90^\circ/0^\circ)\) with clamped end condition from table 2 and material set 1 from table 1

Domain of the beam under consideration is \(0 \leq x \leq \alpha; \left( \frac{b}{2} \leq y \leq \frac{b}{2} \right)\) and \(\frac{h}{2} \leq z \leq \frac{h}{2}\). Details of this beam under laminates under consideration is shown in part (C) of the fig 1. Detailed diagram of this beam subjected to sinusoidal loading and uniformly distributed load is given in part (A) and (B) of the fig 1. Six node plane stress element with node number is shown in part (D) of the fig 1. Numerical result presented in this work are normalised as shown in Eq 30:

\[
\bar{\sigma}_x = \frac{b}{q_0} \sigma_x \left( \frac{L}{2}, \pm \frac{h}{2} \right), \quad \bar{\sigma}_z = \frac{b}{q_0} \sigma_z \left( \frac{L}{2}, z \right), \quad \bar{\tau}_{zx} = \frac{b}{q_0} \tau_{zx}(0, z), \quad \bar{u} = \frac{bE_2}{q_0h} u(0, z), \quad \bar{w} = \frac{100bh^3E_2}{q_0L^4} \left( \frac{L}{2} \right)^2 z
\]

The bar over the variables shows the normalized value. Two different types of loading like sinusoidal and uniformly distributed load has been used as shown in Eq 31 and Eq ?? respectively. Different boundary condition like simply supported and clamped condition (Table 2) is also used in the numerical problems. Two different load has been and different support condition has been used to show case the ability of formulation to handle different condition very easily and also to increase the existing database.

\[
q(x) = q_0 \sin \frac{\pi x}{a}
\]

### 4. Results and Discussion

In the numerical example, clamped composite unsymmetric and symmetric laminates with lamina scheme of \((0^\circ/90^\circ)\) and \((0^\circ/90^\circ/0^\circ)\) respectively analysed for sinusoidal loading condition. Fig 2 and fig 3 shows the through-thickness variation of different parameters and compared with Kant [13] and Ramtekar and Desai [10]. Transverse displacement \(\bar{u}\) and inplane stress \(\bar{\sigma}_x\) shows agreement up to some extent. Transverse shear stress \(\tau_{xz}\) seems more physically intuitive in the case of present formulation as compared with Kant [13]. Only a finite element scheme is
Figure 1. Details of sandwich beam with six node element (A) simply supported sandwich beam subjected to sinusoidal load in positive Z direction (B) simply supported beam subjected to uniformly distributed load in positive Z direction (C) plane stress sandwich beam simply supported subjected to sinusoidal load with primary variables (D) six two-dimensional node element with node number used in the formulation.

Table 1. Material properties.

| S.No | Source     | Elastic Properties                  |
|------|------------|-------------------------------------|
| 1    | Pagano [1] | $E_1 = 172.4$ GPa, $E_2 = 6.89$ GPa, $E_3 = E_2$  
       |            | $G_{12} = G_{13} = 3.45$ GPa, $G_{23} = 1.378$ GPa  
       |            | $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$ |
Figure 2. Distribution of normalised (a) in-plane stress $\sigma_x$ (b) transverse displacement $W$ (c) transverse shear stress $\tau_{xz}$ (d) transverse normal stress $\sigma_z$ in the thickness direction for asymmetric double layer laminated beam with lamina scheme of $(0^\circ/90^\circ)$ which is clamped at both ends, material properties given by Pagano [1] and subjected to sinusoidal load and compared with the available solution given by Kant [13]
Figure 3. Distribution of normalised (a) in-plane stress $\overline{\sigma}_x$ (b) transverse displacement $\overline{W}$ (c) transverse shear stress $\overline{\tau}_{xz}$ (d) transverse normal stress $\overline{\sigma}_z$ in the thickness direction for symmetric triple layer laminated beam with lamina scheme of $(0^\circ/90^\circ/0^\circ)$ which is clamped at both ends, material properties given by Pagano [1] and subjected to sinusoidal load and compared with the available solution given by Kant [13] and Ramtekkar and Desai [10]
Table 2. Boundary condition

| Description     | Location | Degree of Freedom |
|-----------------|----------|-------------------|
| Beam under      | $X = 0$  | $u$ 0 0 - -       |
| clamped         | $X = a/2$| $0$ - 0 - -       |
| supports        | $Z = +h/2$| - - 0 $\hat{q}(X)$|
| at both ends    | $Z = -h/2$| - - 0            |

Table 3. Tripple layer composite laminated beam with lamina scheme $0^\circ/90^\circ/0^\circ$ clamped and subjected to sinusoidal loading conditions.

| AR  | Source  | $0^\circ/90^\circ$ unsymmetric laminate | $0^\circ/90^\circ/0^\circ$ symmetric laminate |
|-----|---------|----------------------------------------|---------------------------------------------|
|     |         | $\bar{\sigma}_x$ | $\bar{\sigma}_y$ | $\bar{\tau}_{xy}$ | $\bar{\tau}_{xz}$ | $\bar{\tau}_{yz}$ | $\bar{\sigma}_x$ | $\bar{\sigma}_y$ | $\bar{\tau}_{xy}$ | $\bar{\tau}_{xz}$ | $\bar{\tau}_{yz}$ |
|     |         | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ | $\left(\frac{a}{b}\right)$ |
| 4   | Present | 0.1194 | -0.8235 | 1.8736 | 2.6453 | 0.6645 | -0.6057 | 1.6106 | 2.1000 |
|     | Kant[13]| 0.1241 | -0.8041 | 1.4232 | 2.7930 | 0.5511 | -0.5166 | 0.8833 | 2.0886 |
| 10  | Present | 0.0783 | -0.6588 | 0.4499 | 0.9039 | 0.3327 | -0.3300 | 0.4133 | 0.4905 |
|     | Kant[13]| 0.0909 | -0.6697 | 2.8418 | 1.0623 | 0.3320 | -3.004 | 1.5079 | 0.4898 |
| 20  | Present | 0.0708 | -0.6275 | 0.5289 | 0.6467 | 0.2552 | -0.2527 | 0.4525 | 0.2113 |
|     | Kant[13]| 0.0739 | -0.5956 | 2.6678 | 0.6792 | 0.2428 | -0.2424 | 0.1562 | 0.2007 |
| 50  | Present | 0.0687 | -0.6290 | 0.9102 | 0.5709 | 0.2318 | -0.2320 | 0.5793 | 0.1259 |
|     | Kant[13]| 0.0649 | -0.5746 | 2.3424 | 2.7930 | 0.2065 | -0.2064 | 1.6142 | 0.1083 |

used, and no empirical correction factor is used, and the result shows that this formulation is free from any shear correction factor. Fixed end beam posses a very high-stress gradient at the end of the beam and stress-free top and bottom layer, which can be achieved by putting the boundary condition in the formulation.

5. Conclusion
In this paper, the authors applied the refined six nodes two-dimensional mixed finite element-based element developed to analyse the different numerical examples. This theory is based on the mixed finite element as, along with displacement, transverse stresses are also considered primary variables. To develop the element level matrices and solve the problem principle of minimum potential energy is used. In this formulation, the equilibrium equation is explicitly satisfied at the node points. Static condensation is used to refine the other primary variable variation.

The important observation and features of the theory are given below:

(i) Even in the case of sharp gradient due to fixed end condition formulation captures accurate results.

(ii) This theory avoids the prior assumptions of deformation.

(iii) This theory does not need the shear correction factor.

(iv) This formulation allows the different stress boundary conditions.
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