A Decision Theoretic Approach to A/B Testing

David Goldberg          James E. Johndrow  
            eBay                      Stanford University

October 11, 2017

Abstract

A/B testing is ubiquitous within the machine learning and data science operations of internet companies. Generically, the idea is to perform a statistical test of the hypothesis that a new feature is better than the existing platform—for example, it results in higher revenue. If the p value for the test is below some pre-defined threshold—often, 0.05—the new feature is implemented. The difficulty of choosing an appropriate threshold has been noted before, particularly because dependent tests are often done sequentially, leading some to propose control of the false discovery rate (FDR) rather than use of a single, universal threshold. However, it is still necessary to make an arbitrary choice of the level at which to control FDR. Here we suggest a decision-theoretic approach to determining whether to adopt a new feature, which enables automated selection of an appropriate threshold. Our method has the basic ingredients of any decision-theory problem: a loss function, action space, and a notion of optimality, for which we choose Bayes risk. However, the loss function and the action space differ from the typical choices made in the literature, which has focused on the theory of point estimation. We give some basic results for Bayes-optimal thresholding rules for the feature adoption decision, and give some examples using eBay data. The results suggest that the 0.05 p-value threshold may be too conservative in some settings, but that its widespread use may reflect an ad-hoc means of controlling multiplicity in the common case of repeatedly testing variants of an experiment when the threshold is not reached.

1 Basic A/B Testing Problem

In A/B testing, one has a proposed new version of a software platform and wants to decide whether or not to ship the new version. The classical way of conceiving of this problem is the following. We divide users into two groups: treatment and control. We then roll out the proposed update to the treatment group while leaving the control group with the current version. Using data gathered from this randomized trial, we then ask whether the new version performed “better” with respect to some metric. For the purposes of grounding the discussion, we
assume that the metric is revenue, which at eBay is roughly equivalent to Gross Merchandise Bought (GMB).

The literature on A/B testing has considered several aspects of this problem, ranging from sequential testing issues [6, 3, 2], to study of multi-armed bandits that approximately characterize some applications like search engine optimization and page customization [13, 14, 11], to practical and computational issues [8, 9, 7]. In e-commerce, tests often need to run for several weeks, so it is usually not practical to keep multiple competing versions of the platform active over a period of time in order to pursue an explore and exploit strategy. Accordingly, the traditional A/B testing framework in which a decision is made after every experiment is favored. Sequential testing and multilevel hierarchical dependence structures among experiments are issues in experimentation at eBay, and we return to this in Section 5.

The traditional or generic view is to treat the feature adoption decision like a one-sided hypothesis testing problem. The relevant hypothesis is

\[ H_0 : \text{current version is better.} \]

A very simple setup in which to consider this is to let \( \theta_0 \) be the revenue per user in the control group and \( \theta_1 \) the revenue per user in the treatment group, so that so that \( \Delta = 100(\theta_1 - \theta_0)/\theta_0 \) is the lift, the percentage change in revenue relative to the control group. On observing data

\[ x \sim F(x \mid \theta) \]

from the treatment (\( \theta = \theta_0 \)) and control (\( \theta = \theta_1 \)) groups, we use some procedure, which we leave abstract, to obtain an estimate \( \hat{\Delta} \). We then assume that, at least approximately,

\[ \frac{\hat{\Delta} - \Delta}{\text{SE}(\hat{\Delta})} \sim t_\nu, \]

where \( t_\nu \) is a \( t \) distribution with \( \nu \) degrees of freedom, \( \text{SE}(\hat{\Delta}) \) is the standard error of \( \hat{\Delta} \), and \( \nu \) is known. Now, letting

\[ T(x) = \frac{\hat{\Delta}}{\text{SE}(\hat{\Delta})} \]

we compute the tail probability under repeated sampling

\[ p = \mathbb{P}[T(X) > T(x) \mid H_0], \]

the one-tailed \( p \) value. We then threshold the \( p \) value at some level – typically, 0.05 – and decide to ship if the \( p \) value is smaller than the threshold.

2 A Decision Theoretic Perspective

An alternative way to look at A/B testing is as a decision theory problem rather than an inference problem. That is, our primary goal is not to validate
or invalidate the scientific hypothesis in (1), but to maximize revenue for the company. In decision theory, we have an action space $A$ consisting of all of the possible decisions we can make, and a loss function $L(\theta, a)$ which defines what we lose if the true state of nature is $\theta$ and we decide to take action $a \in A$. For A/B testing, the action space only has two elements: “ship” and “don’t ship.”

The obvious loss function is

$$L(\Delta, a) = -a\Delta$$

where $a = 1_{\{\text{ship}\}}$. That is, if we choose to ship and the true lift is positive, then we gain the lift (equivalently, we lose the negative of the lift). Otherwise we lose zero; we just get business as usual GMB. Note that our decision rule $a$ is something that we do upon observing data, so $a = \delta(x)$ is a map from the sample space $X$ into $A$. We emphasize that the loss function in (3) is unusual in the literature, which focuses on loss functions like squared error and continuous action space, and thus the results we derive here are somewhat nonstandard.

The aim of decision theory is to choose an optimal decision rule. The frequentist perspective on decision rule optimality is to compute the expected loss if we use $\delta(X)$ in repeat sampling. This is called the risk

**Definition 1** (risk). The risk function of a decision rule $\delta$ is defined as

$$R(\theta, \delta) = \mathbb{E}_\theta[L(\theta, \delta(X))] = \int L(\theta, \delta(x))dF(x \mid \theta),$$

the expectation of the loss over the sampling distribution of the data conditioned on $\theta$.

The risk conditions on $\theta$, the unknown state of nature. Since $\theta$ is unknown, we seek a decision rule that performs well no matter the true value of $\theta$. There are several ways to formalize this. We focus on the Bayes risk

**Definition 2** (Bayes risk). The Bayes risk of a decision rule $\delta$ is defined as the prior expectation of the risk

$$r(\pi, \delta) = \mathbb{E}_\pi[R(\theta, \delta)] = \mathbb{E}_\pi[\mathbb{E}_{F(x\mid \theta)}[L(\theta, \delta(x))]],$$

where $\mathbb{E}_\pi[f(\theta)] = \int f(\theta)\pi(d\theta)$ is the expectation of $f$ with respect to $\pi$.

A decision rule is considered Bayes optimal if it minimizes the Bayes risk. Thus, Bayes risk deals with the fact that $\theta$ is unknown by weighting the states of nature by our prior beliefs about their plausibility. In the applications that follow, we will take an empirical Bayes approach, where we estimate $\pi$ from the data.

With this basic idea in hand, we can consider the set of all decision rules for A/B testing that correspond to thresholding a $p$-value and derive the risk

---

Throughout we use the standard convention of denoting random variables by upper case Roman letters and their realizations by lower case Roman letters.
function. Suppose the true value of the lift is $\Delta$ and we define $\delta(x) = 1_{\{p(x) < \alpha\}}$.

To simplify calculations, we initially consider a simpler version of the A/B testing problem, where $x$ is a noisy observation of the unknown lift with known variance

$$x \sim \text{No}(\Delta, \sigma^2),$$  \hspace{1cm} (4)

in lieu of the $t$ distribution, which arises when $\sigma^2$ is unknown. In this case

$$T(x) = \frac{x}{\sigma} \sim \text{No} \left( \frac{\Delta}{\sigma}, 1 \right),$$

the $p$ value is

$$p(x) = P[T(X) > T(x) \mid H_0] = 1 - \Phi(x/\sigma),$$

and the decision rule is given by $\delta(x) = 1_{\{p(x) < \alpha\}}$, so we can redefine

$$\delta(x) = 1_{\{\frac{x}{\sigma} > \frac{\beta}{\sigma}\}} = 1_{\{x > \beta\}}$$  \hspace{1cm} (5)

and compute the risk in the Gaussian case as

$$R(\Delta, \delta) = -\Delta \int 1_{\{x > \beta\}} dF(x \mid \Delta)$$

$$= -\Delta \int 1_{\{\frac{x - \Delta}{\sigma} > \frac{\beta - \Delta}{\sigma}\}} dF(x \mid \Delta)$$

$$= -\Delta \Phi \left( \frac{\beta - \Delta}{\sigma} \right)$$

$$= -\Delta \Phi \left( \frac{\Delta - \beta}{\sigma} \right).$$

In fact, if $\mathbb{1}$ had been any location-scale family with a density symmetric about the location,

$$x \sim F(x; \Delta, \sigma)$$  \hspace{1cm} (6)

we would have obtained

$$R(\Delta, \delta) = -\Delta F \left( \frac{\Delta - \beta}{\sigma} \right),$$  \hspace{1cm} (7)

with $F$ the CDF of the member of the location-scale family with location 0 and scale 1. Moreover, if we had replaced $\frac{x}{\sigma}$ with the statistic

$$T(x) = \frac{\hat{\Delta}}{\text{SE}(\hat{\Delta})},$$
we would still have obtained this representation, since $T(x)$ has a $t$, which a location-scale family with a density that is symmetric about the location. We will therefore mainly consider the case where $F(x \mid \theta)$ is Gaussian, with the understanding that the approach extends to other location-scale families.

Having defined the risk function, we consider the Bayes risk. Suppose the lifts are exchangeable realizations of a random variable, so that

$$\Delta_i \overset{iid}{\sim} \pi(\Delta_i; \eta),$$

where $\eta$ are the prior hyperparameters. For example, $\pi$ could be a normal distribution with parameters $\eta = (\mu, \tau^2)$. A general expression for the Bayes risk of any thresholding decision rule is

$$r(\pi, \delta) = \mathbb{E}_{\pi}[\mathbb{E}_{F(x|\Delta)}[-\delta(x)\Delta]]$$

$$= -\mathbb{E}_{\pi}[\Delta \mathbb{E}_{F(x|\Delta)}[\mathbb{1}_{\{x>\beta\}}]]$$

$$= -\mathbb{E}_{\pi}[\Delta \mathbb{P}_{\pi}[X > \beta]]$$

$$= -\mathbb{E}_{\pi}[\Delta (1 - F(\beta \mid \Delta))],$$

In the sequel, we estimate $\pi$ from eBay data and obtain some explicit expressions for $r(\pi, \delta)$.

### 3 Bayes Risk of Thresholding Rules

We now return to the class of decision rules in (5) that thresholds at $\beta$ a statistic that is distributed according to a location-scale family, with risk given in (7). If we knew $F$ in (6) and $\pi$ in (8), we could optimize the Bayes risk over $\beta$ to determine the Bayes-optimal strategy for deciding whether to ship a proposed update to the platform. In many applications, we are willing to assume that $F$ is $t$ or normal. This is particularly true in A/B testing applications in industry, where $x$ is typically the (normalized) difference of means from two quite large populations. At eBay, a somewhat more sophisticated procedure is used to estimate the lift, but the estimator is then assumed to be approximately Gaussian for hypothesis testing purposes. We make the same assumption here. We then model the true lifts as student $t$ with unknown location $\mu$, scale $\tau$, and degrees of freedom $\nu$ so that

$$x_i \mid \Delta_i, \sigma_i^2 \sim \text{No}(\Delta_i, \sigma_i^2),$$

$$\Delta_i \overset{iid}{\sim} t_\nu(\mu, \tau)$$

is a hierarchical Bayesian specification of the process generating the data, where we have selected $t_\nu$ for $\pi$. Here $\sigma_i^2$ is the estimated standard error of $x_i$, which we take to be known, and $t_\nu(\mu, \tau)$ denotes a three-parameter student $t$ distribution with density

$$p(\Delta; \mu, \tau, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi \nu \tau^2}} \left(1 + \frac{1}{\nu} \left(\frac{x - \mu}{\tau}\right)^2\right)^{-\frac{\nu+1}{2}},$$
Figure 1: Left: histogram of historical lifts. Right: histogram of historical standard errors.

We fit the model in (9) using Markov chain Monte Carlo (MCMC) implemented in the Stan environment with the rstan package for R. The data \((x_i, \sigma_i)\) for \(i = 1, \ldots, n\) are historical lift estimates and corresponding standard errors from A/B tests performed at eBay during the year 2016. Histograms of the \(x_i\) and \(\sigma_i\) are shown in the top and center panels of Figure 1. The \(x_i\) are centered near zero and the distribution is apparently symmetric, but the tails are considerably heavier than Gaussian. The distribution of standard errors has a mode of approximately 0.3, with some values as large as 2.

For priors, we put \(\mu \sim \text{No}(0, 100), \nu \sim \text{U}(1, 4), \) and \(\tau \sim \text{C+}(0, 1)\), the standard Cauchy distribution. The prior on \(\mu\) is a default, rather vague, prior on a location parameter, and the prior on \(\tau\) is recommended as a prior on variance components in hierarchical models by Gelman [4] and Polson and Scott [12]. The support of the uniform prior on \(\nu\) is rather informative and was chosen based on preliminary analysis using Quantile-Quantile (Q-Q) plots.

We run MCMC in Stan for 5,000 iterations, discarding 2,500 iterations as burn-in. This resulted in estimates of \(\hat{\nu} = 2.31, \hat{\mu} = -0.02, \) and \(\hat{\tau} = 0.18; \) these estimates are the posterior mean estimate of these parameters. A Q-Q plot of the empirical quantiles of \(x_i\) versus the fitted quantiles of \(x_i\) in the model in (9) with \((\nu, \mu, \tau) = (\hat{\nu}, \hat{\mu}, \hat{\tau})\) is shown in Figure 2.

In all of the analysis that follows, we use the posterior mean estimates \((\hat{\nu}, \hat{\mu}, \hat{\tau})\) to make a plug-in estimate of \(\pi\). This is somewhat nonstandard in that we use a fully Bayesian procedure to estimate the parameters of \(\pi\), but then follow an empirical Bayes approach to the rest of the analysis by fixing these parameters at the estimated posterior means. In other words, we use the Bayes machinery and MCMC simply to obtain lightly regularized point estimates of the parameters of \(\pi\), in lieu of a more traditional non-regularized type II maximum likelihood approach. Experience with fitting t distributions with unknown degrees of freedom to data suggests that some regularization in our setting is wise.

We now approximate the Bayes-optimal threshold \(\beta\) by numerically estimat-
Figure 2: Q-Q plot of empirical quantiles of $x_i$ against fitted quantiles from the model in (9) with $\Delta_i \sim t_{2.35}(-0.02, 0.18)$.

The Q-Q plot is used to assess the goodness-of-fit of the fitted model to the empirical data. The points on the plot should ideally lie close to the straight line that represents the expected quantiles under the fitted model. Deviations from this line indicate discrepancies between the empirical and fitted distributions.

By examining the Q-Q plot, we can determine whether the model adequately captures the distribution of the data. If the plot shows a systematic pattern away from the line, it suggests that the model may not be appropriate for the data. The specific parameters of the fitted model, such as the degrees of freedom and the centering of the t-distribution, are critical in understanding the nature of these deviations.

The t-distribution, in this case, is chosen due to its flexibility in modeling data with heavier tails compared to the normal distribution. The use of a t-distribution with 2.35 degrees of freedom indicates that the model may account for more variability in the data than a standard normal distribution.

The empirical quantiles are calculated from the observed data, while the fitted quantiles are derived from the theoretical distribution of the fitted model. By comparing these two sets of quantiles, we can evaluate the model's performance.

The Q-Q plot is a valuable tool in statistical analysis, allowing for a visual and intuitive understanding of how well a theoretical model approximates the empirical data.
Figure 3: Bayes risk vs $\beta$ for the model in (9) using $\sigma = 0.3$ and the estimated values of $\nu, \mu, \tau$. The vertical red line shows the minimum value of $\beta$, the horizontal green line indicates zero risk.

Figure 4: Top: $\beta_{opt}(\sigma)$. Bottom: Bayes risk vs $\sigma$ for $\beta_{opt}(\sigma)$ for the model in (9) at the estimated values of $\nu, \mu, \tau$. The blue lines show a local linear smooth.
4 Theoretical Results

Considering still the case of the loss function in (3) with \( F \) is a location-scale family CDF and frequentist risk given by (7), we now derive some simple results under the assumption that \( \Delta \sim \pi(\Delta; \eta) \) with \( \pi \) also a location-scale family, with both having densities symmetric about the location. This covers the examples in the previous section, and is arguably the most common type of model that would arise in applied settings. We have the following general result.

**Theorem 1.** Suppose \( F \) is the distribution function of a location-scale family with a density \( f \) that is symmetric about the location, and \( \pi \) is the density of a location-scale family also symmetric about the location. Then if \( \mathbb{E}_\pi(\Delta) = 0 \), \( \beta = 0 \) is a critical point of the Bayes risk.

**Proof.** We have

\[
\mathbb{E}_\pi[R(\Delta, \delta)] = \int -\Delta F \left( \frac{\Delta - \beta}{\sigma} \right) \frac{1}{\tau} \pi_0 \left( \frac{\Delta}{\tau} \right) d\Delta
\]

where \( \pi_0 \) is the density of the standard member of the location-scale family. We have

\[
\frac{\partial}{\partial \beta} \mathbb{E}_\pi[R(\Delta, \delta)] = \int \frac{\partial}{\partial \beta} -\Delta F \left( \frac{\Delta - \beta}{\sigma} \right) \frac{1}{\tau} \pi_0 \left( \frac{\Delta}{\tau} \right) d\Delta
\]

\[
= \int \Delta \tau f \left( \frac{\Delta - \beta}{\sigma} \right) \pi_0 \left( \frac{\Delta}{\tau} \right) d\Delta.
\]

Observe that

\[
\left. \frac{\partial}{\partial \beta} \mathbb{E}_\pi[R(\Delta, \delta)] \right|_{\beta = 0} = \int \frac{\Delta}{\tau} f \left( \frac{\Delta}{\sigma} \right) \pi_0 \left( \frac{\Delta}{\tau} \right) d\Delta = 0,
\]

since the integrand is symmetric about zero, so \( \beta = 0 \) is always a critical point of the Bayes risk.

If this critical point is unique, it follows that if there exists a unique minimizer of the Bayes risk, it must be \( \beta = 0 \). Put another way, in a “generic” setup of this problem, when experiments have on average zero lift, then the optimal cutoff to use is \( \beta = 0 \), corresponding to a \( p \)-value cutoff of 0.5. We now show that for the case where both \( F \) and \( \pi \) are Gaussian, the optimal \( \beta \) can be obtained in closed form for any values of the parameters of \( F \) and \( \pi \).

**Theorem 2.** Suppose \( F \) is Gaussian, and \( \pi \) is the density of a \( \text{No}(\mu, \tau^2) \) random variable. Then

\[
\beta = -\frac{\mu \sigma^2}{\tau^2}
\]

minimizes the Bayes risk.
Proof. The Bayes risk is

$$
E_{\pi}[R(\Delta, \delta)] = \int -\Delta \Phi\left(\frac{\Delta - \beta}{\sigma}\right) \phi\left(\frac{\Delta - \mu}{\tau}\right) d\Delta
$$

with \(\phi\) the standard Gaussian density, so

$$
\frac{\partial}{\partial \beta} E_{\pi}[R(\Delta, \delta)] = \int \frac{\Delta}{\sigma} \phi\left(\frac{\Delta - \beta}{\sigma}\right) \phi\left(\frac{\Delta - \mu}{\tau}\right) d\Delta
$$

$$
= \frac{(\mu \sigma^2 + \beta \tau^2) \tau}{\sqrt{2\pi}(\sigma^2 + \tau^2)^{3/2}} e^{-\frac{(\beta - \mu)^2}{2(\sigma^2 + \tau^2)}},
$$

Setting equal to zero and solving gives the unique solution

$$
\beta = \frac{-\mu \sigma^2}{\tau^2},
$$

and noting that

$$
\frac{\partial}{\partial \beta} \left(\frac{\mu \sigma^2 + \beta \tau^2}{\sqrt{2\pi}(\sigma^2 + \tau^2)^{3/2}} e^{-\frac{(\beta - \mu)^2}{2(\sigma^2 + \tau^2)}}\right) \times \frac{\tau}{\sqrt{2\pi}(\sigma^2 + \tau^2)^{3/2}} \left(\tau^2 - \frac{(\beta - \mu)(\mu \sigma^2 + \beta \tau^2)}{\sigma^2 + \tau^2}\right)
$$

which evaluated at \(-\mu \sigma^2/\tau^2\) is

$$
\frac{\tau^3}{\sqrt{2\pi}(\sigma^2 + \tau^2)^{3/2}} e^{-\frac{\mu^2(\sigma^2 + \tau^2)}{2(\sigma^2 + \tau^2)}} > 0,
$$

we conclude that \(\beta = \frac{-\mu \sigma^2}{\tau^2}\) is the unique minimizer of the Bayes risk.

This result is intuitive. The optimal cutoff is decreasing in the prior mean \(\mu\). In other words, if most experiments tend to have large positive lifts, we become less conservative and accept proposed changes to the platform with weaker evidence that they are beneficial. The optimal threshold is also a linear function of the ratio \(\sigma^2/\tau^2\) of the observation noise to the prior variance. Thus, when the observation noise is small relative to the variation in the true lifts, the optimal threshold is shrunk toward zero, meaning we accept an experiment with a small positive lift more readily than when the observation noise is large relative to \(\tau^2\). This makes sense since in the former case we typically have smaller uncertainty about whether the true lift is positive than in the latter case.

Although we do not have a theoretical result for all \(\mu, \tau\) for the model in (9), we can similarly evaluate the optimal \(\beta\) empirically by fixing \(\nu = 2.31\), \(\sigma_i = 0.30\), and \(\tau = 0.18\) and varying \(\mu\). The resulting optimal \(\beta\) value is shown in Figure 5 along with the line \(-\mu \sigma^2/\tau^2\) for comparison to the Gaussian case. Interestingly, in the region between -0.5 and 0.5, \(\beta_{\text{opt}}\) is a decreasing function
Figure 5: Bayes optimal $\beta$ vs $\mu$ with $\nu = 2.31$, $\sigma = 0.3$, and $\tau = 0.18$ for model in (9).

of $\mu$, just as for the Gaussian, but the slope is smaller than the $\sigma^2/\tau^2$ slope for the Gaussian prior. However, for larger or smaller values of $\mu$, $\beta_{opt}$ moves back toward zero.

An intuitive explanation of this phenomenon is that it is caused indirectly by the heavier tails of the prior relative to the (Gaussian) sampling model. When $|\mu| \ll \sigma$, $x$ and $\Delta$ will often have different signs, and thus the optimal threshold is an approximately linear function of the prior expectation of $\Delta$. We obtain relatively little information from $x$ and use more prior information in making the decision. When $|\mu| \gg \sigma$, an observed value of $x$ that is very far from $\mu$ most likely reflects a value of $\Delta$ that is very far from $\mu$, since outliers are much more common in the prior than in the sampling model. Thus, a threshold closer to zero makes sense, since variation in the prior swamps the observation noise. This is why the value of $\beta_{opt}$ flattens around the value $|\mu| = 0.3 = \sigma$ and then moves back toward zero in Figure 5.

5 Hierarchical Structure of Experiments

We have until now ignored the fact that some experiments may be more related than others, opting for a simple hierarchical model. Often, if a feature is developed and is not selected after the first A/B test, the team that developed the feature will modify the algorithm and then re-test. This gives rise to sequences of closely related tests. If we treat all the observations $x_i$ as having means $\Delta_i$ that are iid from the random effect distribution, we are ignoring this structure in the data.

The practice of modifying and re-testing may offer a partial explanation for the use of 0.05 as a $p$-value threshold, which our analysis suggests is much too conservative when performing single, exchangeable experiments. Recall that we computed a Bayes-optimal threshold $\beta = 0.04$, corresponding to a one-tailed $p$ value of 0.45. If instead of a single experiment yielding a single noisy measurement $x_i$ of the true lift $\Delta_i$, we performed $n_i$ experiments yielding $n_i$ noisy measurements of $\Delta_i$, a simple Bonferroni correction would indicate performing
each test by thresholding the \( p \) value at \( 0.45/n_i \), which is 0.05 for \( n_i = 9 \). Figure 6 shows the empirical distribution function (ECDF) for the number of replicate tests of each experiment conducted by eBay in 2016. The 95th percentile is 6, and the 99th percentile is 9. Thus, if we translate the optimal threshold for single tests into a Bonferroni-corrected \( p \)-value threshold for multiple tests, a threshold of 0.05 would be appropriate to uniformly control multiplicity for 99 percent of the features tested at eBay.

This simple analysis is unsatisfactory because it lapses back into the testing framework that we have sought to avoid. To extend the decision-theoretic approach to the case of repeated observations, we now analyze the Bayes risk in this setting. For simplicity, we consider the case where modifications to a feature after the first experiment have no affect on the true lift.

If modifications have no effect, then instead of observing one noisy realization of the lift \( x \sim F(x \mid \Delta) \), we observe \( n_i \) noisy data points \( x_{i1}, \ldots, x_{in_i} \) for each feature. Our decision rule \( \delta(x) \) is now a function of \( n_i \) many observations, so \( \delta : \mathbb{R}^{n_i} \to \{0, 1\} \). If, as before, \( x_{ij} \sim F(x; \Delta_i, \sigma_{ij}^2) \), then

\[
Y_i := \sum_{j=1}^{n_i} \frac{X_{ij}}{\sigma_{ij}^2} \sim \text{No} \left( \frac{\Delta_i}{S_i}, \frac{1}{S_i} \right),
\]

where \( X \sim \mathcal{L} \) indicates that the random variable \( X \) approximately follows the law \( \mathcal{L} \), and

\[
S_i \equiv \left( \sum_j \frac{1}{\sigma_{ij}^2} \right)^{-1},
\]

\( n_i \) times the harmonic mean of the observation variances. The motivation to weight by the inverse variances will soon become apparent. Thus, if \( \delta \) corre-
responds to thresholding the inverse variance-weighted sum, so that

$$\delta(x) = 1_{\{y_i > \beta\}},$$

then the risk is

$$R(\Delta, \delta) = -\Delta \int 1_{\{y_i > \beta\}} F(y | \Delta, \sigma_1, \ldots, \sigma_n) dy$$

$$= -\Delta \Pr[Y > \beta]$$

$$= -\Delta \Pr\left[\frac{Y - \Delta/S}{1/\sqrt{S}} > \frac{\beta - \Delta/S}{1/\sqrt{S}}\right]$$

$$= -\Delta \Phi\left(\frac{\Delta/S - \beta}{1/\sqrt{S}}\right),$$

where we have dropped subscripts above to simplify notation. Now we derive the Bayes risk in the case where $\Delta \sim \text{No}(\mu, \tau^2)$.

**Theorem 3.** Suppose $\delta(x) = 1_{\{y_i > \beta\}}$, and $\pi$ is the density of a $\text{No}(\mu, \tau^2)$ random variable. Then

$$\beta = -\frac{\mu}{\tau^2}$$

minimizes the Bayes risk.

**Proof.** The Bayes risk is

$$\mathbb{E}_\pi[R(\Delta, \delta)] = \int -\Delta \Phi\left(\frac{\Delta/S - \beta}{1/\sqrt{S}}\right) \phi\left(\frac{\Delta - \mu}{\tau}\right) d\Delta$$

with $\phi$ the standard Gaussian density, so

$$\frac{\partial}{\partial \beta} \mathbb{E}_\pi[R(\Delta, \delta)] = \int \sqrt{S} \Delta \phi\left(\frac{\Delta/S - \beta}{1/\sqrt{S}}\right) \phi\left(\frac{\Delta - \mu}{\tau}\right) d\Delta$$

$$= \frac{S^2 \tau (\mu + \beta \tau^2)}{\sqrt{2\pi(S + \tau^2)^{3/2}}} e^{-\frac{(S\beta - \mu)^2}{2(S + \tau^2)}}.$$

Setting equal to zero and solving gives the unique solution

$$\beta = -\frac{\mu}{\tau^2}.$$

The remaining details are similar to the proof of Theorem 2 and are omitted. 

Thus, thresholding the inverse-variance weighted average yields an optimal threshold that is independent of the variances. Notice that when $n = 1$, $Y = X/\sigma^2$, and we recover the optimal threshold for $X$ in Theorem 2.
In reality, our decision problem is typically whether to ship the \( n \)th version of the feature having already collected \( n - 1 \) noisy observations of the lifts of previous versions. If \( \Delta \sim \text{No}(\mu, \tau^2) \) then we have

\[
p(\Delta \mid x_{1:n}, \sigma_{1:n}) \propto p(\Delta) \prod_{j=1}^{n} p(x_i \mid \Delta, \sigma_j^2)
\]

\[
\propto \phi \left( \frac{\Delta - \mu}{\tau} \right) \prod_{j=1}^{n} \phi \left( \frac{x_j - \Delta}{\sigma_j} \right)
\]

so

\[
\Delta \mid x_1, \ldots, x_{n-1}, \mu, \tau \sim \text{No} \left( s^{-1} m, s^{-1} \right)
\]

where

\[
s = \left( \frac{1}{\tau^2} + \sum_{j=1}^{n-1} \frac{1}{\sigma_j^2} \right) \quad \text{and} \quad m = \left( \frac{\mu}{\tau^2} + \sum_{j=1}^{n-1} \frac{x_j}{\sigma_j^2} \right),
\]

and we can immediately apply Theorem 2 to obtain

\[
\beta_{\text{opt}} = -\frac{1}{\tau^2 + \sum_{j=1}^{n-1} \frac{1}{\sigma_j^2}} \left( \frac{\mu}{\tau^2 + \sum_{j=1}^{n-1} \frac{x_j}{\sigma_j^2}} \right) \sigma_n^2
\]

\[
= -\left( \frac{\mu}{\tau^2 + \sum_{j=1}^{n-1} \frac{x_j}{\sigma_j^2}} \right) \sigma_n^2
\]

\[
= -\frac{\mu \sigma_n^2}{\tau^2} - \sigma_n^2 \sum_{j=1}^{n-1} \frac{x_j}{\sigma_j^2},
\]

which is essentially a sum of the optimal threshold for one experiment and the weighted sum of the observations for the past \( n - 1 \) experiments. This turns out to be identical for the optimal threshold if we consider all of the experiments jointly and threshold the inverse variance weighted sum, since

\[
\sum_{j=1}^{n} \frac{x_j}{\sigma_j^2} = -\frac{\mu}{\tau^2} \iff x_n = -\frac{\mu \sigma_n^2}{\tau^2} - \sigma_n^2 \sum_{j=1}^{n-1} \frac{x_j}{\sigma_j^2},
\]

so that the only difference is operational: we either apply a threshold to \( y \), or we apply a threshold that depends on the weighted sum of the previous \( n - 1 \) experiments only the latest experiment \( x_n \). Thus, as data about the lift of a particular feature accumulates, we become more certain about the true value of the lift, and a larger effect size is necessary to convince us that the feature has the opposite effect that previous tests indicated.

We cannot perform this calculation analytically for the case where the \( \Delta \) follow a \( t \) distribution, but we can compute the optimal threshold empirically as before. To do this, we fit the model

\[
x_{ij} \sim \text{No}(\Delta_i, \sigma_{ij}^2)
\]

(11)
Figure 7: Bayes risk vs $\beta$ for thresholding decision rule $\delta(x) = 1_{\{y > \beta\}}$ computed on model in (11). Horizontal green line at zero risk, vertical red line at optimal $\beta$.

$$\Delta_i \sim t_{\nu}(\mu, \tau),$$

and compute the Bayes risk for thresholding $y_i$ as defined in (10) on a grid of $\beta$ values. The results are shown in Figure 7. The optimal value of $\beta$ is 0.31, which corresponds to a threshold for $x_i$ for a single experiment with $\sigma_i^2 = 0.3$ of $\beta \sigma_i^2 = 0.09$ and $p$-value threshold of 0.46, very similar to the optimal threshold in the model where all experiments were assigned their own lifts.

6 Discussion

Traditional A/B testing has treated the decision on whether to ship a feature as a hypothesis test, requiring research and development teams to make an arbitrary choice of a $p$ value threshold at which to adopt a new feature. The decision theoretic approach we outline here has the potential to automate this choice by using data on previous tests to inform about an optimal threshold via Bayesian analysis. We have not considered here the case where modifications to a feature effect the lift, though we do find some evidence of a small average improvement due to modifications at eBay. Extending the decision theoretic analysis of optimal thresholds to this more complicated setting is an interesting extension of the current work.

Acknowledgements

The authors thank Kristian Lum for suggesting consideration of the $n$th experiment after observing $n-1$ experiments in section 5, and generally helpful discussions. We thank David Dunson for useful comments on a draft.
References

[1] A. W. Correia. Bayesian sequentially monitored multi-arm experiments with multiple comparison adjustments. *arXiv preprint arXiv:1608.08076*, 2016.

[2] A. Deng. Objective Bayesian two sample hypothesis testing for online controlled experiments. In *Proceedings of the 24th International Conference on World Wide Web*, pages 923–928. ACM, 2015.

[3] A. Deng, J. Lu, and S. Chen. Continuous monitoring of A/B tests without pain: Optional stopping in Bayesian testing. In *Data Science and Advanced Analytics (DSAA), 2016 IEEE International Conference on*, pages 243–252. IEEE, 2016.

[4] A. Gelman. Prior distributions for variance parameters in hierarchical models (comment on article by Browne and Draper). *Bayesian Analysis*, 1(3):515–534, 2006.

[5] A. Javanmard and A. Montanari. Online rules for control of false discovery rate and false discovery exceedance. *arXiv preprint arXiv:1603.09000*, 2016.

[6] R. Johari, L. Pekelis, and D. J. Walsh. Always valid inference: Bringing sequential analysis to A/B testing. *arXiv preprint arXiv:1512.04922*, 2015.

[7] R. Kohavi, R. M. Henne, and D. Sommerfield. Practical guide to controlled experiments on the web: listen to your customers not to the hippo. In *Proceedings of the 13th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 959–967. ACM, 2007.

[8] R. Kohavi, R. Longbotham, D. Sommerfield, and R. M. Henne. Controlled experiments on the web: survey and practical guide. *Data mining and knowledge discovery*, 18(1):140–181, 2009.

[9] R. Kohavi, A. Deng, B. Frasca, T. Walker, Y. Xu, and N. Pohlmann. Online controlled experiments at large scale. In *Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 1168–1176. ACM, 2013.

[10] L. Pekelis, D. Walsh, and R. Johari. The new stats engine. Technical report, Optimizely, 2015.

[11] V. Perchet and P. Rigollet. The multi-armed bandit problem with covariates. *The Annals of Statistics*, 41(2):693–721, 2013.

[12] N. G. Polson and J. G. Scott. On the half-Cauchy prior for a global scale parameter. *Bayesian Analysis*, 7(4):887–902, 2012.

[13] S. L. Scott. A modern Bayesian look at the multi-armed bandit. *Applied Stochastic Models in Business and Industry*, 26(6):639–658, 2010.
[14] S. L. Scott. Multi-armed bandit experiments in the online service economy. *Applied Stochastic Models in Business and Industry*, 31(1):37–45, 2015.

[15] Stan Development Team. RStan: the R interface to Stan, 2016. URL http://mc-stan.org/. R package version 2.14.1.