Relativistic dissipative hydrodynamics with spontaneous symmetry breaking.

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In this paper we consider dissipative hydrodynamic equations for systems with continuous broken symmetries. We first present the case of superfluidity, in which the symmetry $U(1)$ is broken and then generalize to the chiral symmetry $SU(2)_L \times SU(2)_R$. The corresponding new transport coefficients are introduced.

INTRODUCTION

Relativistic hydrodynamics has often been used as a starting point in the field of heavy ion collisions in order to reproduce single-particle spectra or multi-particle correlations (see [1] for example). However, the hot and dense hadronic phase created in such experiments is mainly constituted of pions and, until recently [2], no care has been taken on the fact that the breaking of the chiral symmetry can affect the theory of hydrodynamics itself [3, 4]. Among all the applications of such a modification of the theory, two main directions emerge: first, it is highly wished to have a quantitative estimation of this effect when reproducing the experimental spectra [5] and, second, the aim of this paper, it is important to complete the whole theory, that is, to include dissipation.

In hydrodynamic regime, relevant variables are those whose variations in space and time are slow and relaxation time becomes infinite in the long wavelength, small frequency limit. Such variables are of two kinds: densities of conserved quantities and Goldstone modes associated to continuous broken symmetry [6]. The prototype of such a theory which includes both types of hydrodynamic variables is the well-established theory of superfluidity associated to the breaking of $U(1)$ and developed many years ago by Landau [7] in the non-relativistic case (see also [8, 9] in the relativistic domain). The example of the superfluidity will be used in the first part as a guideline in order to show explicitly how to include dissipation in hydrodynamic equations with a broken continuous symmetry. Then, in a second part, we will generalize to the case of $SU(2)_L \times SU(2)_R$: starting from the equations of ideal hydrodynamics, we will introduce explicitly new transport coefficients associated to the chiral charge densities and Goldstone modes.

SUPERFLUIDITY

Let us focus on our first example of hydrodynamics with a continuous broken symmetry: the superfluidity. As mentioned in the introduction, the spontaneous breaking of a continuous symmetry implies that the Goldstone modes are hydrodynamic variables, i.e. relax very slowly to equilibrium in the long wavelength, small frequency limit. Concerning the superfluid, one therefore has to introduce $\phi$, which is the phase of the condensate which breaks the $U(1)$ symmetry associated to the particle number, in our theory. More precisely, the equations for ideal (that is without dissipation) relativistic superfluid can be written as [10]:

$$\partial_\mu (n_0 u^\mu - V^2 \partial^\mu \phi) = 0 \quad (1)$$
$$\partial_\mu T^{\mu\nu} = 0 \quad (2)$$
$$u^\mu \partial_\mu \phi = -\mu_0 \quad (3)$$

where $T^{\mu\nu}$ is the energy-momentum tensor, $V^2$ is the superfluid density in the non-relativistic limit and $\mu_0 = \gamma \mu$ is the chemical potential ($\gamma$ is the Lorentz factor). We have to notice at this stage that the third equation is equivalent to the one contained in the original two-fluid model of Landau [7] when making the correspondence:

$$\mu = \mu_{\text{Landau}} + \frac{v_s^2}{2} - \vec{v}_n \cdot \vec{v}_s$$

where $\vec{v}_n$ is the spatial part of the four-velocity $u^\mu = \gamma (1, \vec{v}_n)$ and $\vec{v}_s = \frac{1}{m} \vec{\nabla} \phi$ is the superfluid velocity.

We can see that the presence of the new hydrodynamic variable $\phi$ manifests itself not only in the third equation but also in the first one (conservation of particle number) and in the second one since $T^{\mu\nu}$ is equal to:

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - pg^{\mu\nu} + V^2 \partial^\mu \phi \partial^\nu \phi$$  \quad (4)
Since we are dealing with an ideal fluid the entropy conservation must be satisfied. Actually it is already contained in the second above equation when projected along the direction of \( u^\nu \): \( u_\nu \partial_\mu T^{\mu \nu} = 0 \Rightarrow \partial_\mu (s_0 u^\mu) = 0 \).

Now, we can go further and add dissipative terms in the hydrodynamic equations. The method, described for example in [2], consists in introducing fluxes such as :

\[
\partial_\mu (s_0 u^\mu - V^2 \partial^\mu \phi + \nu^\mu) = 0
\]

\[
\partial_\mu \left( (\epsilon + p) u^\mu u^\nu - pg^{\mu \nu} + V^2 \partial^\mu \partial^\nu \phi + \tau^{\mu \nu} \right) = 0
\]

Moreover, our choice for the hydrodynamic velocity \( u^\mu \) in the second above equation when projected along the direction of \( u^\mu \) since we are dealing with an ideal fluid the entropy conservation must be satisfied. Actually it is already contained (here, it can be checked by comparison with [7] that \( \sigma \equiv \frac{\kappa}{T_0} \frac{\partial_\mu u^\mu}{\partial_\sigma} \frac{\partial_\sigma}{\partial_\tau} \) is proportional to the thermal conductivity and \( \kappa \) is a bilinear form between fluxes and thermodynamic forces. In the hydrodynamic regime, we are by definition near the global equilibrium so that we can express linearly the relation between these fluxes and these forces. The coefficients of proportionality are called the transport coefficients. Their physical meaning is thus to characterize the magnitude of the response of the system (flows) to a certain disturbance (thermodynamic forces).

Moreover, because of thermodynamics, \( \sigma \) must be positive. This constraint when combined with the Onsager principle leads to :

\[
\nu^\mu = \kappa (g^{\mu \nu} - u^\mu u^\nu) \frac{\partial_\nu (\frac{\mu_0}{T_0})}{\partial_\nu} \]

\[
\tau^{\mu \nu} = (g^{\mu \nu} - u^\mu u^\nu) \left[ \zeta_1 \partial_\mu (V^2 \partial^{\lambda} \phi) + (\zeta_2 - \frac{2}{3} \eta) \partial_\mu u^\lambda \right] + \eta \left[ (g^{\mu \lambda} - u^\mu u^\lambda) \partial_\lambda u^\nu + (g^{\lambda \nu} - u^\lambda u^\nu) \partial_\lambda u^\mu \right]
\]

\[
\phi'_0 = \zeta_1 \partial_\mu u^\mu + \zeta_3 \partial_\mu (V^2 \partial^{\mu} \phi)
\]

where \( \kappa \) is proportional to the thermal conductivity and \( \eta, \zeta_1, \zeta_2, \zeta_3 \) are the shear and bulk viscosities (notations are those defined in [3]). The positivity of \( \sigma \) implies that \( \eta, \zeta_2, \zeta_3 \) are positive and \( \zeta_1^2 \leq \zeta_2 \zeta_3 \). The sign of \( \zeta_1 \) has to be determined by physical considerations only (here, it can be checked by comparison with [4] that \( \zeta_1 \) is positive as well).

CHIRAL DYNAMICS

Ideal fluid

In this section, we are going to recall the main results of hydrodynamics with chiral \( SU(2) \) symmetry spontaneously broken [2]. The hydrodynamic degrees of freedom in a chiral fluid are the densities of the conserved quantities, namely entropy density \( s \), momentum density \( T^{0i} \), baryonic number density \( n \), left and right-handed charge densities written as \( SU(2) \)-matrices \( \rho_L \equiv \rho^i \tau_i / 2 \) and \( \rho_R \equiv \rho^i \tau_i / 2 \) and finally the variables associated to the Goldstone modes. For chiral symmetry, these modes are actually the pions. Then, following [2], we can write the new hydrodynamic variables as a \( SU(2) \)-matrix \( \Sigma \equiv e^{i \pi / 2} / f_\pi \) and, by analogy with the superfluid, denote \( \Sigma \) as ”phases”. The energy density \( T^{00} \) is a function of all these variables and of the first partial derivatives of \( \Sigma \) (assuming \( \Sigma \) varies slowly). Following again [2] we thus write the energy density \( T^{00} \) as :

\[
T^{00} = \epsilon_0 (s, n, T^{0i}) + \epsilon_1
\]

where \( \epsilon_0 \) is the ”normal fluid” part and \( \epsilon_1 \) contains all the non trivial terms, at lowest order, compatible with chiral symmetry :

\[
\epsilon_1 = \frac{f_\pi^2}{4} \left( \delta_{ij} - \frac{1 - v_\pi^2}{1 - v_\pi^2} v_i v_j \right) \text{tr}(\partial_i \Sigma \partial_j \Sigma^\dagger) + \frac{1}{\gamma^2 f_\pi^2 (1 - v_\pi^2 v^2)} \text{tr}(\rho_L - \Sigma \rho \Sigma^\dagger)^2
\]

\[
+ \frac{1}{\gamma^2 f_\pi} \text{tr}(\rho_L + \Sigma \rho R \Sigma^\dagger)^2 - \frac{1}{\gamma^2 (1 - v_\pi^2 v^2)} \eta^2 \text{tr}(\rho_{L} \partial_\xi \Sigma^\dagger + \Sigma^\dagger \partial_\xi \rho \Sigma R)
\]
where $v_\tau \equiv f_s/f_t$ is the pion velocity and $f_s$, $f_t$, and $f_\nu$ are functions to be determined by thermodynamics of the underlying fundamental theory, namely QCD.

We have fourteen hydrodynamics variables. Therefore we must have fourteen hydrodynamics equations. With the hamiltonian density above, it is possible to show that the equations can be written in a covariant way only if we make some combinations of the initial variables. Finally the result is:

$$\partial_\mu (n_0 u^\mu) = 0 \quad (12)$$

$$\partial_\mu T^{\mu\nu} = 0 \quad (13)$$

$$\partial_\mu (\alpha u^\mu) + \frac{1}{2} [A, \alpha] = 0 \quad (14)$$

$$i \partial_\mu ((f_1^2 - f_s^2) u^\mu A + f_s^2 \Sigma \partial^\mu \Sigma^\dagger) + [A, \alpha] = 0 \quad (15)$$

with $T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu} + \frac{f_1^2}{4} \operatorname{tr}(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger + \partial^\nu \Sigma \partial^\mu \Sigma^\dagger) + A = u^\mu \Sigma \partial_\mu \Sigma^\dagger$.

The nine equations which are specific to chiral dynamics ($\partial_\mu j_{L/R}^\mu = 0$ and the first order equation for $\Sigma$) are actually contained in the first order equation (13) for $\alpha \equiv (\rho_L + \Sigma \rho_R \Sigma^\dagger)/\gamma$ and in the second order equation (15) for $\Sigma$ constructed from the combination ($\rho_L - \Sigma \rho_R \Sigma^\dagger$).

Once again, starting from equation (13) and using all the other (hydrodynamic and thermodynamic) equations, it is possible to deduce the entropy conservation : $\partial_\mu (s_0 u^\mu) = 0$ as it should be for a perfect fluid. At this stage, we can already make some comparisons with the superfluid. First of all if $\Sigma$ was associated with the breaking of $U(1)$ type symmetry, namely $\Sigma = e^{i\phi}$, then $i \Sigma \partial_\mu \Sigma^\dagger$ would be equal to $\partial_\mu \phi$. This means that $i \nabla^\mu \Sigma^\dagger$ plays the role of superfluid velocity. Then, by direct comparison between the energy-momentum tensors (for $U(1)$ and $SU(2)$), we see that $\frac{f_1^2}{2}$ can be interpreted as "superfluid density".

\section*{Dissipation}

We are now in position to treat the dissipation case. As for the superfluid, we add some dissipative flux densities to conservation equations : $\nu^\mu$ for baryonic number, $j_{L,R}^\mu \equiv j_{L,R}^\mu \tau_i/2$ for chiral charges and $\tau^{\mu\nu}$ for energy-momentum tensor; and we add also a dissipative term for the other non-conserved variables : $\frac{1}{2} \Sigma^0 \equiv \frac{1}{2} \Sigma_0^{\mu} \tau_i/2$.

Then, performing the same combinations as in the previous section, we can finally write after some calculations :

$$\partial_\mu (n_0 u^\mu + \nu^\mu) = 0 \quad (16)$$

$$\partial_\mu \left( (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu} + \frac{f_1^2}{4} \operatorname{tr}(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger + \partial^\nu \Sigma \partial^\mu \Sigma^\dagger) + \tau^{\mu\nu} \right) = 0 \quad (17)$$

$$\partial_\mu (\alpha u^\mu) + \frac{1}{2} [A^{(0)}, \alpha] = -\partial_\mu j_L^\mu - \Sigma \partial_\mu j_R^\mu \Sigma^\dagger \quad (18)$$

$$i \partial_\mu \left( (f_1^2 - f_s^2) u^\mu A^{(0)} + f_s^2 \Sigma \partial^\mu \Sigma^\dagger \right) + [A^{(0)}, \alpha] = 2 \partial_\mu j_L^\mu - 2 \Sigma \partial_\mu j_R^\mu \Sigma^\dagger \quad (19)$$

with $A = A^{(0)} + \frac{1}{2} \Sigma^0$ which is the dissipative first order equation for $\Sigma$. $A^{(0)}$ represents the non-dissipative part of $A$. With these new equations, it is a simple task to express the equation for the entropy (note that, since the expressions for $\epsilon$ and $p$ must be given by the same formula as in the non-dissipative case, $\epsilon$ and $p$ depend actually only on $A^{(0)}$ and not on $A$) :

$$\partial_\mu \left( s_0 u^\mu - \frac{\mu_0}{T_0} u^\mu - \operatorname{tr} \left( \frac{\mu_L}{T_0} j_L^\mu + \frac{\mu_R}{T_0} j_R^\mu \right) \right) = -\nu^\mu \partial_\mu \frac{\mu_0}{T_0} + \frac{\tau^{\mu\nu}}{T_0} \partial_\mu u^\nu - \operatorname{tr} \left( j_L^\mu \partial_\mu \left( \frac{\mu_L}{T_0} \right) + j_R^\mu \partial_\mu \left( \frac{\mu_R}{T_0} \right) + \frac{\Sigma^0}{2 T_0} \partial_\mu \left( \frac{f_1^2}{2} \Sigma \partial^\mu \Sigma^\dagger \right) \right) \quad (20)$$
where $\mu_{L0} = \frac{2\alpha}{f^2} - iA^{(0)}$ and $\mu_{R0} = \Sigma(\frac{2\alpha}{f^2} + iA^{(0)})\Sigma$.

The introduction of the above shorthand notations $\mu_{R0}$ and $\mu_{L0}$ is motivated by the fact that they can be identified as chemical potentials but for the left and right chiral densities (they are thermodynamic conjugate variables of $\rho_L$ and $\rho_R$). Nevertheless, due to the fact that we are considering the $SU(2)$ case and no more $U(1)$, new features appear, that is we have now some couplings with the derivatives of these chemical potentials $\mu_{R0}$ and $\mu_{L0}$.

On the right-hand side of the equation, the entropy production is, as usual, a bilinear form between “thermodynamic forces” and dissipative fluxes. Again, we can make linear combinations with those thermodynamic forces and introduce transport coefficients. Our choice for the hydrodynamic velocity imposes also that $u_{R} j_{L}^{\mu} = 0$ and $u_{R} j_{R}^{\mu} = 0$ and we have to make the entropy production positive. These prescriptions allow us to eliminate some coefficients and to obtain some constraints on the remaining others. We get:

$$\Sigma_0' = \left(\zeta_{4,i} \partial_\mu u^\mu + [\zeta_3]_{i,j} \partial_\mu \left(\frac{f^2}{2} i(\Sigma \partial^\mu \Sigma^\dagger)\right)\right) \tau_i$$

(21)

$$\nu^\mu = (g^{\mu\nu} - u^\mu u^\nu) \left(\kappa \partial_\nu \left(\frac{\mu_0}{T_0}\right) + \kappa_{L,i} \partial_\nu \left(\frac{\mu_{L0,i}}{T_0}\right) + \kappa_{R,i} \partial_\nu \left(\frac{\mu_{R0,i}}{T_0}\right)\right)$$

(22)

$$j_{L}^\mu = (g^{\mu\nu} - u^\mu u^\nu) \left(\kappa_{L,i} \partial_\nu \left(\frac{\mu_0}{T_0}\right) + [\kappa_{LL}]_{i,j} \partial_\nu \left(\frac{\mu_{L0,i}}{T_0}\right) + [\kappa_{LR}]_{i,j} \partial_\nu \left(\frac{\mu_{R0,i}}{T_0}\right)\right) \tau_i$$

(23)

$$j_{R}^\mu = (g^{\mu\nu} - u^\mu u^\nu) \left(\kappa_{R,i} \partial_\nu \left(\frac{\mu_0}{T_0}\right) + [\kappa_{LR}]_{i,j} \partial_\nu \left(\frac{\mu_{L0,i}}{T_0}\right) + [\kappa_{RR}]_{i,j} \partial_\nu \left(\frac{\mu_{R0,i}}{T_0}\right)\right) \tau_i$$

(24)

$$\tau^{\mu\nu} = (g^{\mu\nu} - u^\mu u^\nu) \left(\zeta_2 - \frac{2}{3} \eta\right) \partial_\mu u^\lambda + \zeta_{1,j} \partial_\lambda \left(\frac{f^2}{2} i(\Sigma \partial^\lambda \Sigma^\dagger)\right) + \eta \left((g^{\mu\lambda} - u^\mu u^\lambda) \partial_\lambda u^\nu + (g^{\lambda\nu} - u^\lambda u^\nu) \partial_\lambda u^\mu\right)$$

(25)

where $[Q]$ means that $Q$ is a $3 \times 3$ matrix. Due to the Onsager reciprocity principle, all the matrices except $[\kappa_{LR}]$ are symmetric, $4\zeta_{1,i} = \zeta_{4,i}$ and there are actually 39 independant coefficients. If we represent the quadratic form of the entropy production by a $12 \times 12$ matrix, we can easily show that all coefficients appearing in the diagonal should be positive and that there exists inequalities between the 39 coefficients : as for the superfluid, all the principal minors have to be positive. We see that dissipative equations for $SU(2)_L \times SU(2)_R$ symmetry implies new transport coefficients and couplings between baryonic, left-handed and right-handed currents. We also see that spontaneous breaking of the chiral symmetry implies the existence of matricial transport coefficient for, by instance, the bulk viscosities $\zeta_1$ and $\zeta_3$ : the thermodynamic force $\partial_\lambda \left(\frac{f^2}{2} i(\Sigma \partial^\lambda \Sigma^\dagger)\right)$ which is the equivalent of $\partial_\lambda (V^2 \partial^\lambda \phi)$ has now three components. These components are nevertheless all multiplied by the same factor $\frac{f^2}{4}$ which is related to the amplitude of the order parameter of the chiral symmetry. Of course, near the transition phase this amplitude is not frozen as in our case and can fluctuate. The dynamics of these fluctuations is such that the amplitude of the order parameter has to be incorporated explicitly in the approach since then it becomes an hydrodynamic variable.

Finally let us remark that, since it is known that dissipation can affect the observables (see for the modification of the temperature profile used to describe heavy ion collisions), it will be of fundamental importance to determine quantitatively the influence of these new couplings.

**CONCLUSIONS AND OUTLOOK**

We have treated in this paper the dissipation in the hydrodynamic regime for relativistic systems with spontaneously broken symmetry $U(1)$ and $SU(2)$. For the superfluid, we recovered the non-relativistic limit of $R$. For the $SU(2)$ case, we introduced new transport coefficients associated to the right and left charges and to the Goldstone modes. Since we know that transport coefficients can affect quantitatively observables, it would be interesting to express them with Kubo-type relations and then compute them explicitly from this microscopic approach. This work is under study. An other extension of this paper is to determine the new relaxation times with the effect of the symmetry breaking in order to know if this implies some noticeable modifications to the conclusions drawn in concerning the
typical equilibration time of heavy ions collisions. It is well known that the way we introduced the dissipative effects in that paper leads formally to instabilities and can not allow to reach the underlying physics of the relaxation times. In order to solve that problem one has to introduce them explicitely. One convenient way to do this is to use the 14-moments of Grad as done in [3].

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