Decomposing data sets into skewness modes

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Abstract

We derive the nonlinear equations satisfied by the coefficients of linear combinations that maximize their skewness when their variance is constrained to take a specific value. In order to numerically solve these nonlinear equations we develop a gradient-type flow that preserves the constraint. In combination with the Karhunen-Loève decomposition this leads to a set of orthogonal modes with maximal skewness. For illustration purposes we apply these techniques to atmospheric data; in this case the maximal-skewness modes correspond to strongly localized atmospheric flows. We show how these ideas can be extended, for example to maximal-flatness modes.

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1. Introduction

When dealing with large data sets it is convenient and costumary to make use of the Karhunen-Loève decomposition [Karhunen 1947] in order to express the data in terms of the so-called empirical orthogonal functions (EOFs), these are linear combinations of the original variables whose second-order cross-correlations vanish, i.e. they are linearly uncorrelated, as in (5). A typical
application consists in reducing then the number of degrees of freedom to a relatively small number of EOFs with a variance larger than a certain threshold, for more applications see for example Preisendorfer [1988]. In many systems of interest the EOFs are nonlinearly correlated, a fact that can have important consequences. Similarly, the probability densities of the time-dependent EOFs’ amplitudes are often approximately Gaussian but the deviations from Gaussianity may be of great relevance. One indicator of nonlinear correlations and of the non-Gaussian character of fluctuations is the skewness, the third-order moments of the variables’ distribution. In this article we show how to construct orthogonal linear combinations of the variables that maximize the skewness and we present a numerical method in order to solve the ensuing nonlinear equations. For the purpose of illustration we apply it to meteorological data; the maximal-skewness modes found in this case correspond to spatially localized and meteorologically meaningful atmospheric flows.

2. Maximal skewness modes

Given a set of \( n \) dynamical variables,

\[ v_i(t), \quad 1 \leq i \leq n, \]

with vanishing mean we want to construct a linear combination \( \psi(t) \)

\[ \psi(t) := \sum_{i=1}^{n} a_i v_i(t) \]

such that its skewness \( \hat{s} \),

\[ \hat{s}(a_1, \ldots, a_n) := \langle \psi^3 \rangle, \]

is maximal. The angular brackets indicate an average over the data set. In order to get a sensible solution some restriction must be imposed on it. An appropriate restriction is to fix its variance, say

\[ \langle \psi^2 \rangle = 1. \]
This choice implies that the dimension of \(a_i\) is \([v_i]^{-1}\). In terms of the coefficients \(\{a_i\}\) the constraint reads,

\[
\langle \psi^2 \rangle = \sum_{i,j=1}^{n} a_i C_{ij} a_j = 1, \tag{2}
\]

where \(C_{ij}\) are the elements of the covariance matrix \(C_{il} := \langle v_l v_i \rangle\). The maximal skewness \(\hat{s}_m\) is obtained by setting equal to zero the variation of \((\hat{s} - \lambda \langle \psi^2 \rangle)\) where \(\lambda\) is the Lagrange multiplier associated with the variance constraint (1).

In this way one obtains

\[
3 \sum_{k,l=1}^{n} a_k \hat{S}_{kl} a_l - 2\lambda \sum_{l=1}^{n} C_{il} a_l = 0, \quad 1 \leq i \leq n, \tag{3}
\]

with

\[
\hat{S}_{kl} := \langle v_k v_l v_i \rangle,
\]

the elements of the skewness tensor \(\hat{S}\). We assume all the elements of the tensors \(C\) and \(\hat{S}\) to exist. The values of the \(n\) coefficients \(a_i\) and of the Lagrange multiplier \(\lambda\) satisfying the \((n + 1)\) equations (1) and (3) will be denoted as \(a_i^\alpha\) and as \(\lambda_\alpha\) respectively. The number of real solutions maybe larger than \(n\), see for example Fig. 1. If \(\{\lambda_\alpha, a_i^\alpha\}\) is a solution then also \(\{-\lambda_\alpha, -a_i^\alpha\}\) is a solution. The value \(\lambda_\alpha\) associated with a solution \(a_i^\alpha\) is proportional to the skewness \(\hat{s}(a_1^\alpha, \ldots, a_n^\alpha)\). This can be seen by multiplying the \(i\)-th equation (3) by \(a_i\) summing over all the components and making use of the constraint (2), one obtains then that

\[
2\lambda_\alpha = 3\hat{s}(a_1^\alpha, \ldots, a_n^\alpha). \tag{4}
\]

Without loss of generality, we can take the \(\{v_i(t)\}\) to be EOFs, i.e. their covariance matrix \(C_{il} := \langle v_l v_i \rangle\) is

\[
C_{il} = w_i^2 \delta_{il}. \tag{5}
\]

Then the equations to solve become,

\[
3 \sum_{k,l} a_k \hat{S}_{kl} a_l - 2\lambda w_i^2 a_i = 0, \quad 1 \leq i \leq n.
\]
It is convenient to introduce the dimensionless quantities
\[ \beta_i := w_ia_i, \]
and
\[ S_{kli} := \frac{\langle v_kv_lv_i \rangle}{w_kw_tw_i}. \]

In terms of these equation (3) reads,
\[ 3 \sum_{k,l} \beta_k S_{kli} \beta_l - 2 \lambda \beta_i = 0, \quad 1 \leq i \leq n, \quad (6) \]
and the unity covariance constraint (1) is,
\[ \sum_i \beta_i^2 = 1. \quad (7) \]

A more compact way of writing equation (6) is
\[ -\vec{\sigma}(\vec{\beta}) = 2\lambda \vec{\beta}, \]
where \( \vec{\sigma} \) is the gradient of the skewness function \( s(\beta_1, \ldots, \beta_n) := \hat{s}(a_1, \ldots, a_n) \), i.e.
\[ \sigma_i(\vec{\beta}) := \frac{\partial s}{\partial \beta_i} = 3 \sum_{j,k} S_{ijk} \beta_j \beta_k. \quad (8) \]

The solutions to equations (6) and (7) may correspond to saddle points of \( s(\beta_1, \ldots, \beta_n) \), this is further analyzed at the end of Section 3. With \( n = 2 \) the solutions can be expressed analytically, see the Appendix. With \( n > 2 \) one has to find them numerically, in the next Section we present a way of doing this.

If all the \( w_i \)'s have the same dimensionality then we can associate a variance to each solution \( \beta^\alpha \), namely
\[ W_{\alpha}^2 := \sum_i (\beta_i^\alpha)^2 w_i^2. \quad (9) \]

It may happen that a solution has a relatively large skewness \( s(\beta_1^\alpha, \ldots, \beta_n^\alpha) \) while its variance \( W_{\alpha}^2 \) is relatively small. Usually one is interested in solutions with both quantities relatively large. Accordingly one can consider a dimensional skewness parameter, call it \( \Sigma_{\alpha} \)
\[ \Sigma_{\alpha} := s(\beta_1^\alpha, \ldots, \beta_n^\alpha)W_{\alpha}^3. \quad (10) \]
In closing, let us mention that we take the space of the coefficients \( \{\beta_1, \ldots, \beta_n\} \) to be Euclidean, i.e. the inner product of two \( \beta \)-vectors is

\[
\vec{\beta}^a \cdot \vec{\beta}^b = \sum_{i=1}^{n} \beta^a_i \beta^b_i,
\]

and the constraint \( \sum \beta_i^2 = 1 \) means that the vectors \( \vec{\beta} \) are of length 1.

3. An algorithm in order to solve the system of equations (6) and (7)

Consider a gradient flow, i.e. let \( \vec{\beta}(\tau) \) move downhill a potential \( V(\vec{\beta}) \),

\[
\frac{d\beta_i}{d\tau} = -\frac{\partial V}{\partial \beta_i},
\]

so that \( V(\vec{\beta}) \) is a Lyapunov function,

\[
\frac{dV}{d\tau} = \sum \frac{\partial V}{\partial \beta_i} \frac{d\beta_i}{d\tau} = -\sum \left( \frac{\partial V}{\partial \beta_i} \right)^2 \leq 0,
\]

and the evolution of \( \vec{\beta}(\tau) \) stops when an extremum of \( V(\vec{\beta}) \) is reached. In general, such an evolution will not conserve \( \vec{\beta} \) itself. In this context notice that if \( \{\lambda, \beta^\alpha\} \) is a solution of (6) then also \( \{\mu \lambda, \mu \beta^\alpha\} \) is a solution of (6). Therefore, we can work with \( \beta \)-vectors of arbitrary length if we replace \( \lambda \) by \( \|\vec{\beta}\| \lambda \), i.e. instead of (6) we can solve

\[
\vec{\sigma}(\vec{\beta}) = 2\lambda \vec{\beta} \cdot \|\vec{\beta}\| \vec{\beta}.
\]

(11)

Since both the lhs and the rhs are proportional to \( \|\vec{\beta}\|^2 \), in this formulation the length of \( \vec{\beta} \) does not play any role. Once a vector satisfying this equation is found, then we normalize its length in order to obtain a solution of the equations (6) and (7).

The previous considerations lead us to introduce the following deviation vector \( \vec{\Delta}(\vec{\beta}) \),

\[
\vec{\Delta}(\vec{\beta}) := \frac{\vec{\sigma}(\vec{\beta}) - 2\lambda \vec{\beta} \cdot \|\vec{\beta}\| \vec{\beta}}{\|\vec{\beta}\|^2}.
\]
and a potential $V(\vec{\beta})$ given by

$$V(\vec{\beta}) := |\Delta|^2 = |\vec{\beta}|^{-4} |\vec{\sigma}(\vec{\beta})|^2 - 12\lambda |\vec{\beta}|^{-3} s(\vec{\beta}) + 4\lambda^2 \geq 0.$$  

(12)

The deviation vector $\Delta(\vec{\beta})$ does not depend upon the length of $\vec{\beta}$, it vanishes when $\vec{\beta}$ solves (11) and $\lambda$ equals the corresponding Lagrange multiplier $\lambda_\alpha$. Therefore, the value of $V(\vec{\beta})$ measures the departure of $\{\lambda, \vec{\beta}\}$ from a solution to (11) and, after normalization, from a solution to equations (6,7). This specific form of the potential has been chosen because, as it will be seen, it conserves the length $|\vec{\beta}(\tau)|$. From it one finds that,

$$\frac{d\beta_i}{d\tau} = -\frac{\partial V}{\partial \beta_i} = \left[\frac{4 |\vec{\sigma}|^2}{|\vec{\beta}|^6} - \frac{36\lambda s}{|\vec{\beta}|^5}\right] \beta_i + \frac{12\lambda}{|\vec{\beta}|^3} \sigma_i - \frac{12}{|\vec{\beta}|^4} \sum_{k,j} S_{ijk} \beta_j \sigma_k. \quad (13)$$

One can check that indeed this implies $d |\vec{\beta}|^2 /d\tau = 0$. Therefore, we can fix $|\vec{\beta}(\tau)|^2 = 1$ and get

$$\frac{d\beta_i}{d\tau} \bigg|_{|\beta|=1} = \left[4\sigma^2 - 36\lambda s\right] \beta_i + 12\lambda \sigma_i - \frac{12}{|\vec{\beta}|^4} \sum_{k,j} S_{ijk} \beta_j \sigma_k \quad (14)$$

with $\vec{\beta} \cdot \frac{d\vec{\beta}}{d\tau} \bigg|_{|\beta|=1} = 0$. \quad (15)

In the definition of the potential $V(\vec{\beta})$, equation (12), and in all the equations we have derived from it, $\lambda$ appears as a free parameter. In order to find the solutions to the equations (6) and (7) $\lambda$ must take the value $\lambda_\alpha$ associated with the vector solution $\vec{\beta}_\alpha$. Therefore one should let also the Lagrange multiplier $\lambda$ evolve until it reaches the searched value $\lambda_\alpha$. This is achieved by taking $\lambda(\tau) = \lambda_M(\vec{\beta})$ where $\lambda_M(\vec{\beta})$ is the $\lambda$ value that minimizes $V(\vec{\beta})$,

$$\frac{\partial V}{\partial \lambda} \bigg|_{\lambda=\lambda_M} = 0 \iff 8\lambda_M |\vec{\beta}|^3 - 12s = 0,$$

i.e. $\lambda_M(\vec{\beta}) = \frac{3}{2} |\vec{\beta}|^{-3} s(\vec{\beta})$. 


Compare this with \(4\). At this value of \(\lambda\) the potential \(V(\vec{\beta})\) equals,

\[
V_M(\vec{\beta}) := V(\vec{\beta})\bigg|_{\lambda=\lambda_M} = |\vec{\beta}|^{-4} \big|\vec{\sigma}(\vec{\beta})\big|^2 - 9 |\vec{\beta}|^{-6} s^2(\vec{\beta}) \geq 0.
\]

The evolution of the \(\vec{\beta}(\tau)\) induced by this potential is,

\[
\frac{d\beta_i}{d\tau} = -\frac{\partial V_M}{\partial \beta_i} = \left[\frac{4\sigma^2}{|\vec{\beta}|^6} - \frac{54s^2}{|\vec{\beta}|^8}\right] \beta_i + \frac{18s\sigma_i}{|\vec{\beta}|^6} - \frac{12}{|\vec{\beta}|^4} \sum_{kj} S_{ijk} \beta_j \sigma_k.
\]

In agreement with the results in the previous paragraph, also these equations lead to \(d |\vec{\beta}|^2 / d\tau = 0\). Therefore, one could simplify them by taking \(|\vec{\beta}(\tau)|^2 = 1\), however, in order to control numerical errors, it is preferable to use these equations as they stand. As before, one has that

\[
\frac{dV_M}{d\tau} = \sum \frac{\partial V_M}{\partial \beta_i} \frac{d\beta_i}{d\tau} = -\sum \left(\frac{\partial V_M}{\partial \beta_i}\right)^2 \leq 0,
\]

i.e. \(V_M(\vec{\beta})\) is the corresponding Lyapunov function.

In Fig. 1 we see an example with \(n = 3\) constructed from the meteorological data used in Section (6). Fig. 1 shows the level plots of \(s(\beta_2, \beta_7, \beta_9)\) and of the corresponding potential \(V(\beta_2, \beta_7, \beta_9)\) on one hemisphere of the sphere \(\beta_2^2 + \beta_7^2 + \beta_9^2 = 1\). The angle \(\alpha_1 = \arccos(\beta_2)\) and the angle \(\alpha_2 = \arctan(\beta_7/\beta_9)\). The blacks dots indicate the positions where the potential vanishes, these co-incide with the positions of the two maxima, two minima and three saddles of \(s(\beta_2, \beta_7, \beta_9)\). In addition, Fig. 1 shows how, starting from initial values of \(\beta_2, \beta_7\) and \(\beta_9\), on a regular lattice the gradient-flow equations \(17\) lead to these critical values. Whether a solution \(\vec{\beta}^\alpha\) is an extremum or a saddle depends upon the character of \([h^\alpha_{ij}]\) the Hessian matrix at the solution projected on the sphere \(|\vec{\beta}|^2 = 1\), i.e. by the character of the \((n - 1) \times (n - 1)\) matrix with elements

\[
h^\alpha_{ij} := H^\alpha_{ij} - \frac{\beta^\alpha_i H^\alpha_{in} + \beta^\alpha_n H^\alpha_{nj}}{\beta^\alpha_n} + \frac{H^\alpha_{nn} \sigma^\alpha_i \sigma^\alpha_j - \sigma^\alpha_n \left[\beta^\alpha_n \delta_{ij} - \beta^\alpha_i \beta^\alpha_j \right]}{(\beta^\alpha_n)^2},
\]

with \(1 \leq i, j \leq n - 1\).
where
\[ H_{li}^i := 6 \sum_k^n \beta_k^i S_{kli} \quad 1 \leq l, i \leq n, \]
is the $n \times n$ Hessian at the solution $\vec{\beta}^i$ and $\sigma_j^i := \sigma_j(\vec{\beta}^i)$. In these expressions it is assumed that $\beta_n^i \neq 0$.

4. Numerical implementation of the gradient algorithm

In the general case one has that the number of extrema grows explosively with $n$. We deal with this problem by taking first the ten largest EOFs and using (17) with 1000 random initial values of $\{\beta_1, \ldots, \beta_{10}\}$ in order to find the combination with the largest skewness, then we increase the number of EOFs by taking the fifteen largest EOFs and as initial $\beta$-values the solution found in the previous step supplemented by $\beta_{11} = \cdots = \beta_{15} = 0$ and let the $\beta$’s evolve according to (17) with noise added to them. Once a new solution is found the last step is repeated but now with twenty EOFs, etc. The noise is added in order to explore larger sections of the phase space and not to remain trapped in a local extremum. A 4th-order Runge-Kutta algorithm is used in order to integrate the system of ODEs (17). A solution is found when the value of the potential $V$ is close enough to zero and the value of the skewness has virtually ceased to change.

In Fig. 2 we show the results obtained when this procedure is applied to the meteorological data used in Section 6. One can see an increasing maximal-skewness value as $n$ increases from 10 to 50.

5. Orthogonal set of maximal-skewness modes

In order to generate a set of linearly uncorrelated orthogonal modes ordered according to their skewness one should proceed as follows. Firstly, the method presented in the previous sections is used in order to create from the data $m(x, t), 1 \leq x \leq n,$ the linear combination with the largest skewness, which we write as follows,
\[ \psi^1(t) = \sum_{i=1}^n \beta_i e_i(t) = \sum_{i=1}^n a_i v_i(t), \]
i.e. the \( e_i(t) \) are the normalized EOFs,

\[
e_i(t) = w_i^{-1} v_i(t), \quad \text{with} \quad \langle v_i v_l \rangle = w_i^2 \delta_{il} \quad 1 \leq i, l \leq n.
\]

Thanks to the Karhunen-Loève theorem (Karhunen, 1947) each \( v_i \) is associated with a unique \( n \)-dimensional eigenvector \( \pi_i(x) \) satisfying

\[
\sum_{x=1}^{n} C_{yx}^{(m)} \pi_i(x) = \sum_{x=1}^{n} \pi_i(y), \quad 1 \leq i \leq n,
\]

\[
C_{yx}^{(m)} = \langle m(y,t) m(x,t) \rangle, \quad 1 \leq x, y \leq n.
\]

Therefore, if \( w_i^2 \neq w_j^2 \), the corresponding eigenvectors are orthogonal,

\[
\sum_{x=1}^{n} \pi_i(x) \pi_j(x) = \delta_{ij},
\]

and can be taken to be normalized. If the discrete indices \( x \) and \( y \) are associated with positions in physical space then each of the eigenvectors \( \pi_i(x) \) describes a spatial pattern. In such a case, to \( \psi^1(t) = \sum_{i=1}^{n} a_i v_i(t) \) there corresponds a unique spatial pattern \( \psi^1(x) := \sum_{i=1}^{n} a_i \pi_i(x) \).

Using the covariance metric (5) this \( \psi^1(t) \) mode is projected out from the \( n \)-dimensional set \( \{e_1(t), e_2(t), \ldots, e_n(t)\} \). The new set so obtained is

\[
\tilde{m}_i(t) = (1 - \beta_i^2) e_i(t) - \beta_i \sum_{k \neq i}^{n} \beta_k e_k(t),
\]

\[
\langle \psi^1(t) \tilde{m}_i(t) \rangle = 0, \quad 1 \leq i \leq n.
\]

Their covariance matrix is

\[
\langle \tilde{m}_i^2(t) \rangle = (1 - \beta_i^2),
\]

\[
\langle \tilde{m}_j(t) \tilde{m}_i(t) \rangle = -\beta_i \beta_j, \quad i \neq j.
\]

It has one vanishing eigenvalue with eigenvector \( \psi^1(t) = \sum_{i=1}^{n} \beta_i e_i(t) \) and \((n-1)\)-times degenerate eigenvalue 1 with normalized eigenvectors

\[
\tilde{e}_k(t) := (\beta_k^2 + \beta_1^2)^{-1} \left[ -\beta_k \tilde{m}_1(t) + \beta_1 \tilde{m}_k(t) \right]
\]

\[
= (\beta_k^2 + \beta_1^2)^{-1} \left[ -\beta_k e_1(t) + \beta_1 e_k(t) \right], \quad 2 \leq k \leq n.
\]
By construction the \( \tilde{e}_k(t) \)'s and \( \tilde{\psi}_1(t) \) are linearly uncorrelated \( \langle \tilde{\psi}_1(t) \tilde{e}_k(t) \rangle = 0 \). To each \( \tilde{e}_k(t) \) there corresponds a unique pattern \( \tilde{e}_k(x) \propto -a_k \pi_1(x) + a_1 \pi_k(x) \).

It follows that \( \tilde{\psi}_1(x) \) and the \((n-1)\) patterns \( \tilde{e}_k(x) \) are orthogonal since

\[
\tilde{\psi}_1(x) \tilde{e}_k(x) \propto -a_k \tilde{\psi}_1(x) \pi_1(x) + a_1 \tilde{\psi}_1(x) \pi_k(x)
\]

\[
\to \sum_{x=1}^{n} \tilde{\psi}_1(x) \tilde{e}_k(x) \propto -a_k a_1 + a_1 a_k = 0.
\]

As indicated in (10) instead of \( \tilde{\psi}_1(t) \) one may consider the associated dimensional mode \( W_1 \tilde{\psi}_1(t) \).

6. An application to meteorological data

Using the data available at the ECWMF ERA40 website we computed the daily values of the streamfunction anomalies on the Northern-hemisphere 500 hPa level during the months December, January and February from 1958 to 2001. The skewness in this field was already noticed by White (1980); Nakamura and Wallace (1991). The dimensionless skewness field

\[
s(x) := \frac{\langle m^3(x,t) \rangle}{\langle m^2(x,t) \rangle^{3/2}}
\]

is shown in Fig. 3. Two black dots indicate the two positions with the largest positive and negative skewness \( s(x) \), to wit: (168 East, 47 North) with skewness 0.76 and (158 East, 19 North) with skewness -0.39.

Fig. 4 shows the spatial patterns corresponding to \( \tilde{\psi}_1 \) the largest and \( \tilde{\psi}_2 \) the second-largest skewness mode. The skewness of these modes are \( s_1 = 1.13 \) and \( s_2 = 0.96 \). That these values are larger than the maximal values of the local skewness \( s(x) \) indicated by the black dots in Fig. 3 is possible due to the non-vanishing third-order cross-correlations \( \langle m(y,t)m(x,t)m(z,t) \rangle \neq 0 \) for \( x, y \) and \( z \) not simultaneously identical.

For these data one has that \( \sum_{i=1}^{50} w_i^2 \approx 0.9 \sum_{i=1}^{50} w_i^2 \), i.e. the first 50 EOFs describe 90% of the total variance of the daily streamfunction fields. In particular, \( w_1^2 \) and \( w_2^2 \) describe 7.7% and 6.4% of the total variance respectively.

One also finds that the variances \( W_1^2 \) and \( W_2^2 \) corresponding to the maximal-skewness modes \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \), confer (9), account for 3.3% and 3.9% of the total variance. Their contributions to the total variance is comparable to those of
the EOFs $v_9$ and $v_7$ respectively. Therefore, these maximal-skewness modes are physically relevant.

7. Possible generalizations

For example, instead of the modes with maximal skewness one could be interested in the modes with maximal flatness $\hat{f}(a_1, \ldots, a_n) := \langle \psi^4 \rangle$ given that $\langle \psi^2 \rangle = 1$. The equation analogous to (6) is

$$4 \sum_{k,l,m=1}^{n} \beta_k \beta_l \beta_m F_{klmi} - 2\lambda \beta_i = 0, \quad 1 \leq i \leq n,$$

where $F_{klmi} := \langle v_k v_l v_m v_i \rangle w_k w_l w_m w_i$.

The Lagrange multiplier $\lambda$ is proportional now to the flatness of the solutions, $\lambda = 2\hat{f}(a_1^2, \ldots, a_n^2)$. The corresponding deviation vector and potential are

$$\Delta_4(\beta) := \frac{\phi(\beta) - 2\lambda |\beta|^2 \beta}{|\beta|^3},$$

$$V_4(\beta) := |\Delta_4|^2,$$

with

$$\phi_i(\beta) := 4 \sum_{k,l,m=1}^{n} \beta_k \beta_l \beta_m F_{klmi}.$$

The minimum value of $V_4(\beta)$ is achieved when $\lambda$ equals $\lambda_M = 2 |\beta|^{-4} f(\beta_1, \ldots, \beta_n)$, the potential is then

$$V_{4M}(\beta) := V_4(\beta) \bigg|_{\lambda=\lambda_M} = |\beta|^{-6} \phi(\beta)|^2 - 16 |\beta|^{-8} f^2(\beta_1, \ldots, \beta_n).$$

As one can check $\sum \beta_i (\partial V_{4M}/\partial \beta_i) = 0$, so that $d |\beta|^2 /d\tau = 0$.

Acknowledgment

ECMWF ERA-40 data used in this study is the Basic 2.5 Degree Atmospheric data set in the ECMWF Level III-B archive obtained from their data server at data.ecmwf.int/products/data/archive/.
Appendix

With two EOFs and after eliminating $\lambda$ there is only one equation to solve, namely

$$\beta_2 (\beta_1^2 S_{111} + \beta_2^2 S_{221} + 2 \beta_1 \beta_2 S_{121}) = \beta_1 (\beta_1^2 S_{112} + \beta_2^2 S_{222} + 2 \beta_1 \beta_2 S_{122}).$$

Dividing both sides of this equality by $\beta_1^2$ one gets that $z_2 := \beta_2 / \beta_1$ is the solution of the following third-order polynomial,

$$z_2 (S_{111} + z_2^2 S_{221} + 2 z_2 S_{121}) = (S_{112} + z_2^2 S_{222} + 2 z_2 S_{122}).$$

Once $z_2$ is known, we recover $\beta_1$ and $\beta_2$ from

$$\beta_1^2 + \beta_2^2 = 1 \iff \beta_1^2 (1 + z_2^2) = 1 \iff \beta_1^2 = (1 + z_2^2)^{-1}$$

and $\beta_2^2 = z_2^2 (1 + z_2^2)^{-1}$.

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Figure 1: The black contours correspond to isolevels of the skewness $s(\beta_2, \beta_7, \beta_9)$. The shading denotes the values of the potential $V(\beta_2, \beta_7, \beta_9)$. The black dots indicate the location of the skewness maxima, minima and saddles; the non-negative potential vanishes on these locations. The white lines are the trajectories generated by equation (17) with initial conditions on a regular lattice.

Figure 2: Skewness of the maximal-skewness mode as a function of the number of EOFs included in the calculations.
Figure 3: Skewness $s(x)$ of the daily streamfunction on the 500 hPa surface in the months December through February, years 1958 through 2002. The black points indicate the positions with maximal positive skewness $s(x) = 0.79$ and maximal negative skewness $s(x) = -0.39$.

Figure 4: The spatial patterns of two maximal-skewness modes of the daily streamfunction on the 500 hPa surface in the months December through February, years 1958 through 2002. On the left, the largest-skewness mode with skewness $s_1 = 1.13$ and, on the right, the mode with next-largest skewness $s_2 = 0.96$. 