Energy current correlation in solvable long-range interacting systems

Shuji Tamaki$^1$ and Keiji Saito$^1$

$^1$Department of Physics, Keio University, Yokohama 223-8522, Japan

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We consider heat transfer in one-dimensional systems with long-range interactions. It is known that short-range interacting systems generally shows anomalous behavior in heat transport when total momentum is conserved, whereas momentum-nonconserving systems do not exhibit anomaly. In this study, we focus on the effect of long-range interaction. We propose an exactly solvable model that reduces to the so-called momentum-exchange model in the short-range interaction limit. We exactly calculate the asymptotic time-decay in the energy current correlation function, which is related to the thermal conductivity via the Green–Kubo formula. From the time-decay of the current correlation, we show two qualitatively crucial results. First, the anomalous exponent in the time-decay continuously changes as a function of the index of the long-range interaction. Second, even momentum-nonconserving systems can show the anomalous exponent indicating anomalous heat transport.

I. INTRODUCTION

In the past few decades, the study of dynamic and thermodynamic properties of long-range interacting systems has attracted considerable attention. These systems are characterized by interaction potentials $V(r)$ that decay with the power law

$$V(r) \propto r^{-\delta},$$

where $r$ is the distance between two interacting particles. The parameter $\delta$ controls the range of interaction; a smaller $\delta$ means longer range of interactions. When the index $\delta$ is lower than the spatial dimension, the system is called the long-range interacting system [1, 2]. In this regime, the additivity does not hold and many unusual properties appear such as negative specific heat [3–6], long-lived quasistationary state [7, 8], anomalous diffusion [5, 9, 10], and suppression of chaos [8, 9, 11–13]. In this study, we use the terminology long-range interaction in a wider sense, to refer to the interaction in the power-law form regardless of the exponent $\delta$.

However, transport properties in such systems have not yet been understood. In this study, we address the heat transfer in long-range interacting systems focusing on energy current fluctuations. We focus on one-dimensional systems, because there is a large amount of information on short-range interacting case. In short-range interacting one-dimensional systems with total momentum conservation, the energy transport is in general anomalous, in the sense that the thermal conductivity $\kappa$ diverges as $\kappa \propto N^\alpha (0 < \alpha \leq 1)$ with an increase in the system size $N$ [14–16]. The thermal conductivity is given by the Green–Kubo formula, which is the time integral of the energy current correlation. Hence, the anomalous behavior of the diverging conductivity is directly related to the slow decay in the equilibrium current correlation in a closed system:

$$C(t) := N^{-1}\langle J_{\text{tot}}(t)J_{\text{tot}} \rangle \sim t^{-\beta}, \quad 0 \leq \beta < 1, \quad (2)$$

where $J_{\text{tot}}$ is the total energy current, and $\langle \ldots \rangle$ is the equilibrium average or microcanonical average. In addition, from the microscopic viewpoint, this slow relaxation is also related to the super-diffusive behavior in the energy diffusion[17–19]. If the system has onsite pinning potentials where the law of total momentum conservation does not hold, the above anomalous behaviors are resolved, and the normal diffusion as well as normal heat transport are recovered.

From the above backgrounds on short-range interacting systems, we consider the effects of long-range potentials on energy fluctuations. Thus far, several numerical studies have proposed two paradigmatic models. In Refs.[20–22], the coupled rotor model was studied. This model shows a transition from the diffusive transport to the thermal insulator, as $\delta$ decreases from infinity. The critical point $\delta_c$ lies between $\delta = 1$ and 2; however, the explicit value depends on the temperature regime. In Refs.[22, 23], the Fermi–Pasta–Ulam (FPU) model was investigated, and it was found that the transport behavior is generally anomalous, except at $\delta = 2$, where it exhibits ballistic behavior [23]. In stud-

| $\delta$ | $\kappa$ | $\beta$ |
|--------|----------|--------|
| $\leq 2$ | — | $\beta = (\delta - 2)/2$ |
| $\leq 2$ | — | $\beta = 1/2$ |
| $3/2 < \delta < 5/2$ | $\beta = (2\delta - 3)/2$ | $\beta = (2\delta - 3)/2$ |
| $5/2 < \delta < 3$ | $\beta = (2\delta - 3)/2$ | $\beta = 3/2$ |
| $\delta > 3$ | $\beta = 3/2$ | $\beta = 3/2$ |

TABLE I: Decay rate $\beta$ for long-range interacting systems. The upper table summarizes the systems with no onsite potential. The lower table summarizes the results for systems with onsite potential. Note that the exponent gradually changes depending on $\delta$. Remarkably, even systems with onsite potentials can exhibit anomalous decay for the range $3/2 < \delta < 5/2$. At $\delta = 3$, the logarithmic correction appears in the time-decay for both cases of $k_0 = 0$ and $k_0 \neq 0$ (See Eqs.(40) and Eqs.(43)). In the regime $\delta \leq 2$ for $k_0 = 0$ and the regime $\delta \leq 3/2$ for $k_0 \neq 0$, the amplitude of current correlation diverges; hence, the exponent cannot be defined.
ies on the FPU model, long-range interactions can be added in several ways, e.g., long-range quadratic potential terms [24], long-range quartic potential terms [13, 23], and a combination of both [8, 22]. From these studies, many intriguing transport properties have been numerically shown. However, note that, in general, it is very difficult to obtain clear results through numerical calculations, because finite-size effect is very significant, especially in long-range systems [22]. Owing to this difficulty, we require clear-cut results with a solvable model for an in-depth understanding of the long-range effect.

In this paper, we propose an analytically solvable model, which mimics the FPU dynamics with the long-range quadratic potential. The dynamics of the model consists of the Hamiltonian dynamics of harmonic interactions and stochastic perturbation exchanging momentums of nearest neighbors. The explicit expression of the Hamiltonian is given Eqs. (3) and (4). For \( \delta = \infty \), without onsite pinning potential, this model is equivalent to the so-called momentum exchange (ME) model [25–27]. The ME model rigorously explains the anomalous transport properties showing the slow decay in the current correlation \( \beta = 1/2 \) [25, 26], super-diffusion in the energy diffusion [17–19], and nonequilibrium steady current under finite thermal gradient [27, 28]. Herein, we extend the technique developed in [25, 26] to the long-range interacting case with and without onsite pinning potential. In particular, we focus on the current correlation. We present a brief summary in Table I. In brief, the exponent \( \beta \) in the current correlation depends on the index \( \delta \) in the long-range potential. Moreover, even the systems with the onsite pinning potential can show anomalous behavior, i.e., \( \beta < 1 \). These findings indicate that the exponent in the anomalous transport can also have various realistic systems with long range potentials, even if equilibrium thermodynamic property is normal.

This paper is organized as follows. In Sec. II, we introduce our model and explain some notations and definitions. In Sec. III, our main result about analytical solution of the energy current correlation is presented, and we derive the results listed in the TableI. Finally, we summarize and discuss our results in Sec. IV.

II. MODEL

A. Long-range interacting Harmonic chain

We consider a classical one-dimensional system composed of \( N \) particles. The position and momentum of the \( x \)th particle are denoted by \( \tilde{q}_x \) and \( p_x \), respectively. For the convenience of analysis, we consider the structure that is schematically shown in figure 1. That is, the infinite particles are arranged along the infinite line, and we focus on the dynamics of \( N \) particles by imposing the boundary conditions appropriately. For momentum variables, we always impose the condition \( p_x = p_{x+N} \). For the position variables, we impose different boundary conditions depending on whether the system has momentum conservation, as discussed below.

We employ the hybrid dynamics containing the deterministic dynamics from the Hamiltonian and stochastic exchange of momentum variables between the nearest neighbor sites, which is introduced in the subsequent subsection. The deterministic dynamics is induced by the following Hamiltonian that describes the long-range interacting harmonic chain:

\[
H = \sum_x \frac{p_x^2}{2} + \frac{k_0}{2} (\tilde{q}_x - (x - 1)\ell)^2 + \sum_{x=1}^{N/2} V_{x,x'}, \quad (3)
\]

\[
V_{x,x'} = \frac{1}{N r^\delta} \frac{(\tilde{q}_{x+x'} - \tilde{q}_{x-x'})^2}{2}, \quad (N = \sum_{r=1}^{N/2} 1/r^\delta), \quad (4)
\]

where the index \( \delta \) controls the range of the harmonic interactions. When \( \delta = \infty \), the interaction is reduced to the nearest neighbor interaction, while \( \delta \to 0 \) implies global coupling. The factor \( \tilde{N} \) is introduced to ensure extensivity on the equilibrium thermodynamic variables such as the free energy and entropy for \( \delta < 1 \), whereas when \( \delta > 1 \), the extensivity is satisfied even without this term. The quantity \( \ell \) is the natural length of the springs.

1. Momentum-conserving case

First, we explain the structure of the system without the onsite potential, i.e., \( k_0 = 0 \), called the momentum-conserving case. In this case, the spatially translational invariance is satisfied; hence, the total momentum is conserved. We define the stretch variable as

\[
s_x := \tilde{q}_{x+1} - \tilde{q}_x - \ell. \quad (5)
\]

Because we impose the boundary condition on the momentum \( p_{x+N} \equiv p_x \), we immediately find that the summation of the stretch variables \( \sum_{x' = x}^{x' = x+1-N} s_{x'} = \tilde{q}_{N+x} - \tilde{q}_x - N\ell \) is conserved. Namely, for a given initial state, the length of the \( N \) particles is conserved. For the momentum conserving systems, a specific value of \( \tilde{q}_x \) does not matter because the global shift gives arbitrary values without changing the dynamics; hence, we impose the boundary condition for stretch variables as \( s_{x+N} \equiv s_x \), and not for position variables. This boundary condition

\[
\begin{array}{c}
\cdot \cdot \cdot \bullet \cdot \cdot \cdot \bullet \cdot \cdot \cdot \bullet \cdot \cdot \cdot \bullet \cdot \cdot \cdot \cdot \cdot \end{array}
\]

\[
\begin{array}{c}
\tilde{q}_0; \quad \tilde{q}_1; \quad \tilde{q}_2; \quad \tilde{q}_{N-1}; \quad \tilde{q}_N; \quad \tilde{q}_{N+1}; \quad \tilde{q}_{N+2}
\end{array}
\]

\[
\begin{array}{c}
s_0 + \ell; \quad s_1 + \ell; \quad s_{N-1} + \ell; \quad s_{N+1} + \ell; \quad s_{N+2} + \ell
\end{array}
\]

FIG. 1: Schematic of the structure of spring system
can be achieved once we set an initial configuration cor-
rectly. For each initial state, we define an average stretch
as \( s = \sum_{x=1}^{N} s_x/N \). In addition, we introduce a displace-
ment variable

\[
q_x := \tilde{q}_x - (x - 1)(\bar{s} + \ell).
\]  

(6)

Then, we find that \( q_{N+x} = q_x \) automatically holds from the conser-
vation of the length of \( N \) particles. In this paper, for the momen-
tum conserving case, we consider the initial states satisfying \( \bar{s} = 0 \). In such initial states, the potential \( V_{x,r} \) is rewritten as \( V_{x,r} = (1/(\bar{N} r^\delta)) (q_{x+r} - q_x)^2/2 \).

In the momentum conserving case, there are three con-
served quantities, total stretch, momentum, and energy. For the calculation of current correlation, we take a micro-
canonical average over the phase space with zero total stretch, zero total momentum, and a finite fixed total en-
ergy.

2. Momentum-nonconserving case

We next explain the structure of the system with the
onsite potential, \( k_0 \neq 0 \), which we call the momentum-
nonconserving case. In this case, there is a mechanical equilib-
rium position for each particle, i.e., \( x - 1 \ell \) for the \( x \)th particle. In this system, a specific value in the
position does matter; hence, we impose the boundary condi-
tion \( \tilde{q}_x + N \ell \) in addition to \( p_{x+N} = p_x \). It is con-
venient to introduce a displacement variable

\[
q_x := \tilde{q}_x - (x - 1)\ell.
\]  

(7)

Then we find that \( q_{x+N} = q_x \) is also satisfied. The po-
tential \( V_{x,r} \) is also rewritten as \( V_{x,r} = (1/(\bar{N} r^\delta)) (q_{x+r} - q_x)^2/2 \).

In momentum-nonconserving systems, only the total en-
ergy is an important conserved quantity relevant to the
detailed calculation of the current correlation. To cal-
culate the current correlation, we take a microcanonical
average over the phase space with a finite fixed energy.

3. Dispersion relation

We consider the dispersion relation by which the sound
velocity is defined. The dispersion relation and sound
velocity are fundamental properties to characterize the
macroscopic dynamics. A recent work [31] pointed out
that the sound velocity can be important, especially in
an open system attached to two reservoirs of different
temperatures. Although in this study we focus on
the current fluctuation in the closed setup, we list the clas-
sification of the sound velocities for different classes of
system.

Note that for momentum conserving and nonconserv-
ing cases, the potential term is reduced to the same ex-
pression with appropriately defined displacement vari-
ables \( q_x \). We define the Fourier transform as follows

\[
q_x = \frac{1}{\sqrt{N}} \sum_k q_k e^{-ikx},
\]  

(8)

\[
p_x = \frac{1}{\sqrt{N}} \sum_k p_k e^{-ikx},
\]  

(9)

where the wave number is \( k = 2\pi/N, 4\pi/N, \cdots, 2\pi \).

In this study, the same Fourier transform is applied for dif-
ferent variables. Through straightforward calculation for the
potential term, the dispersion relation can be obtained as

\[
\omega_k = \left[ k_0 + \frac{1}{N} \sum_{r=1}^{N/2} \frac{4 \sin^2(k r / 2)}{r^\delta} \right]^{1/2}.
\]  

(10)

In the proximity of \( k = 0 \), this has the following asymptotic
behavior

\[
\omega_k^2 \sim \begin{cases} 
\text{const.}, & (0 < \delta < 1) \\
 k_0 + a_1 (\ln k^{-1})^{-1}, & (\delta = 1) \\
 k_0 + a_2' k^\delta, & (1 < \delta < 3) \\
 k_0 + a_2'' k^2 \ln k^{-1}, & (\delta = 3) \\
 k_0 + a_3''' k^2, & (\delta > 3) 
\end{cases}
\]  

(11)

where \( a_1, a_2', a_2'', \) and \( a_3''' \) are constants dependent on \( \delta \).

The sound velocity is given by the slope of \( \omega_k \) at \( k = 0 \).

In the momentum-conserving case \( k_0 = 0 \), we find that
the sound wave does not exist for \( 0 < \delta < 1 \), whereas the
sound velocity is infinite for \( 1 < \delta \leq 3 \), and it is finite for
\( \delta > 3 \). In the momentum-nonconserving case, the sound
wave does not exist for \( 0 < \delta < 1 \), whereas the sound
velocity is infinite for \( 1 < \delta < 2 \), and it is zero for \( \delta > 2 \).

B. Momentum-exchange dynamics with long-range interaction

The dynamics is hybrid dynamics consisting of the de-
terministic part from the Hamiltonian and the stochastic
part described by the random exchange of momentums
between the nearest neighbor sites. For both momentum-
conserving and nonconserving cases, the microscopic dy-
namics for variables \( q_x \) and \( p_x \) are the same. The in-
finities in the variables from time \( t \) to \( t + dt \) are described as follows

\[
dq_x = p_x \, dt,
\]  

(12)

\[
dp_x = \left[ - k_0 q_x + N^{-1} \sum_{r=1}^{N/2} r^{-\delta} (q_{x+r} + q_{x-r} - 2q_x) \right] \, dt
+ dn_x (p_{x+1} - p_x) + dn_{x-1} (p_{x-1} - p_x),
\]  

(13)

where \( \{dn_x\}_{x=1}^N \) are independent stochastic variables,
which take the value 0 or 1 with the Poisson process
satisfying the noise average \( \langle dn_x \rangle_n = \gamma dt \). The noises stochastically exchange momentums between the nearest neighbor sites. This hybrid dynamics conserves total energy. In addition, for the momentum conserving case, the dynamics still satisfies the conservation of total momentum. When \( \delta \) is infinite, the interaction between the particles contains only nearest neighbor harmonic interaction; hence, the dynamics reduces to the original ME model discussed in Refs. [25–27].

The corresponding dynamics for the distribution function can be obtained easily. Here, we only show for the momentum nonconserving case, where the distribution function for the phase space \( (q,p) := (q_1, \cdots, q_N, p_1, \cdots, p_N) \) is defined [29]. Let \( P(q,p,t) \) be the probability distribution. Because the stochastic noise is generated according to the Poisson process, the time evolution is calculated according to

\[
\frac{\partial}{\partial t} P(q,p,t) = (-\mathcal{A} + \gamma \mathcal{S}) P(q,p,t),
\]

(14)

where the operator \(-\mathcal{A}\) denotes the deterministic dynamics given by Liouville’s operator

\[
\mathcal{A} := \sum_{x=1}^{N} \left( \frac{\partial H}{\partial p_x} \frac{\partial}{\partial q_x} - \frac{\partial H}{\partial q_x} \frac{\partial}{\partial p_x} \right).
\]

(15)

The operator \( \gamma \mathcal{S} \) is the part of the stochastic dynamics that acts as

\[
\mathcal{S} P(q,p,t) := \sum_{x=1}^{N} \left[ P(q,p,x+1,t) - P(q,p,t) \right]
\]

(16)

where \( p^{x+1}_x \) is obtained by substituting \( p_x, p_{x+1} \) with \( p_{x+1}, p_x \) in \( p \).

C. Energy current

Energy current is defined by the continuity equation of the local energy. Therefore, we define the local energy as

\[
\epsilon_x = \frac{p_x^2}{2} + \frac{1}{2N} \sum_{r=1}^{N/2} \left[ \frac{(q_{x+r} - q_x)^2}{2r^\delta} + \frac{(q_x - q_{x-r})^2}{2r^\delta} \right].
\]

(17)

The evolution of the local energy is calculated according to (12) and (13). We need to be careful as the time evolution involves stochastic terms. In addition, we also note that the dynamics contains nonlocal interaction, which inevitably leads to nonlocal expression on the energy current. First, we consider the infinitesimal change in the local energy:

\[
d\epsilon_x = \frac{1}{N} \sum_{r=1}^{N/2} \frac{1}{2r^\delta} \left[ -(q_x - q_{x+r})(p_{x+r} + p_x)dt + (q_{x-r} - q_x)(p_{x-r} + p_x)dt \right] + dn_x \left( \frac{p_{x+1}^2}{2} - \frac{p_x^2}{2} \right) + dn_{x-1} \left( \frac{p_x^2}{2} - \frac{p_{x-1}^2}{2} \right).
\]

(18)

where the expression containing the noise terms denote the exchange of kinetic energies caused by the exchange of momentums between the nearest neighbor sites.

Next, we compare the above expression with the continuity equation with respect to energy \( d\epsilon_x = -dj_x + dj_{x-1} \). Note that the Hamiltonian satisfies translational invariance; hence, the current expressions should be constructed such that the expressions of \( dj_x \) and \( dj_{x-1} \) are identical to each other once we shift the site index. From this criterion, we can derive the following expression of energy current:

\[
dj_x := (j_x^A + \gamma j_x^S) dt + di_x,
\]

(19)

\[
j_x^A := \frac{1}{N} \sum_{x'=x+1}^{x+N/2} \sum_{r'=x-x'}^{N/2} \frac{q_{x'} - q_{x'-r}}{r^\delta} \frac{p_{x'} + p_{x'-r}}{2},
\]

(20)

\[
j_x^S := \frac{-p_{x+1}^2 - p_x^2}{2},
\]

(21)

\[
di_x := \frac{-p_{x+1}^2 - p_x^2}{2} dm_x.
\]

(22)

Here, \( dm_x \) is the Martingale noise defined as \( dm_x := dn_x - \gamma dt \) [30]. The currents \( j_x^A \) and \( j_x^S \) are the instantaneous currents from the deterministic dynamics and average stochastic noise, respectively. The third current \( di \) is a current from the Martingale noise. Note that the expression of \( j_x^A \) is nonlocal, which is a direct consequence of long-range interactions. By considering all contributions of the energy transmissions across the surface between the sites \( x \) and \( x+1 \), the expression \( j_x^A \) is defined. Figure 2 shows the schematic for the interpretation.

![Figure 2: Interpretation of energy current](image)

**FIG. 2:** Interpretation of energy current \( j_x^A \). Each arrow indicates the direct transmission of energy from one site to another. The energy current is defined by counting all transmissions through the surface between the site \( x \) and \( x+1 \).
plified from the expression of local current (20) and is

\[ C_N(t) := \frac{1}{N} \langle j_{\text{tot}}(t)j_{\text{tot}} \rangle_{\text{mc:n}} \]

\[ = \langle j_{\text{tot}}(t)j_{0}^{'} \rangle_{\text{mc:n}}. \tag{23} \]

where the symbol \( \langle ... \rangle_{\text{mc:n}} \) denotes the microcanonical average (mc) as well as the noise average (n). The variable \( j_{0}^{'} \) is defined by noting that the total current \( j_{\text{tot}} \) is simplified from the expression of local current (20) and is rewritten with the new variable \( j_{x}^{'} \) as

\[ j_{\text{tot}}^{A} = -\frac{1}{N} \sum_{x}^{N/2} g_{x} - g_{x-r} p_{x} + p_{x-r} / 2 = \sum_{x} j_{x}^{'} , \tag{24} \]

\[ j_{x}^{'} := -\frac{1}{N} \sum_{r=1}^{N/2} g_{x} - g_{x-r} p_{x} + p_{x-r} / 2 . \tag{25} \]

Using the translational invariance in the system, we selected one site in Eq.(23).

The current correlation is directly related to the thermal conductivity via the Green–Kubo formula, if the time integral is finite [33]. In anomalous heat transport, the combination of power law decay in the current correlation and time integration up to the cut-off time \( N/c \), where \( c \) is the sound velocity, is thought to explain the system size dependence of the diverging thermal conductivity.

Note that the current expression (19) also has the Martingale part. However, it is known that its contribution to the thermal conductivity is constant, and the correlations between the Martingale part and \( j_{\text{tot}}^{A}, j_{x}^{'} \) vanish [25–27]. Hence, we do not involve the Martingale current in the definition of the current correlation.

**B. Laplace transform of \( C_N(t) \)**

We outline the calculation below. First, we consider the Laplace transform:

\[ \tilde{C}_N(\lambda) := \int_{0}^{\infty} dt e^{-\lambda t} C_N(t) \]

\[ = \int_{0}^{\infty} dt e^{-\lambda t} \langle [j^{A}+\gamma S] j_{\text{tot}}^{A} j_{0}^{'} \rangle_{\text{mc:n}} , \]

\[ = \langle [\lambda - \mathbb{A} - \gamma S]^{-1} j_{\text{tot}}^{A} j_{0}^{'} \rangle_{\text{mc:n}} , \tag{26} \]

where, from the second line, the expression is the form by taking the noise average. The dynamics for the variables is given by the operator \( \mathbb{A} \) and \( S \); the dynamics is conjugate to the distribution function (14).

For further calculation (26), we consider the equation

\[ (\lambda - \mathbb{A} - \gamma S) u(\lambda) = j_{\text{tot}}^{A} . \tag{27} \]

Note that using the quantity \( u(\lambda) \), the Laplace transform of the current correlation can be written as \( \tilde{C}_N(\lambda) = \langle u(\lambda) j_{0}^{'} \rangle_{\text{mc:n}} \). To obtain the explicit expression of \( u(\lambda) \), we impose the following form

\[ u(\lambda) = \sum_{x,x'} g_{x-x'} q_x p_{x'} , \tag{28} \]

where we also assume the relation \( g_{-x} = -g_{x} \) and \( g_{x+N} = g_{x} \). Substituting this expression into Eq.(27) and comparing the coefficients of the term \( q_x p_{x'} \), the relation is satisfied:

\[ (\lambda - \gamma \Delta_{x'}) g_{x-x'} = -\frac{1}{2N} \delta_{x-x',r} - \delta_{x-x',r'} , \tag{29} \]

where \( \delta_{x,a} \) is the Kronecker’s delta function, i.e., \( \delta_{x,a} = 1 \) for \( x = a \), and \( \delta_{x,a} = 0 \) otherwise. The symbol \( \Delta_{x'} \) is the discrete Laplacian that acts as \( \Delta_{x'} f_{x'} := f_{x'+1} + f_{x'-1} - 2f_{x'} \). Through the Fourier transform for both sides in the equation, the explicit form of the function \( g_{x-x'} \) is easily obtained as

\[ g_{x-x'} = \frac{1}{\sqrt{N}} \sum_{k} g_k e^{-ik(x-x')} , \tag{30} \]

\[ g_k = -\frac{i}{\sqrt{N}} \Phi_k \Phi_k , \tag{31} \]

where

\[ \Phi_k := \frac{1}{N} \sum_{r=1}^{N/2} \sin (kr) / r^{3/2} . \tag{32} \]

The function \( \Phi_k \) is related to the Fourier representation of the total instantaneous current as \( j_{\text{tot}} = -i \sum_k \Phi_k q_{-k} p_k \).

Finally, we consider taking the average over the microcanonical average, where we proceed computation based on the ensemble equivalence between microcanonical and canonical distribution. Note the expression

\[ \tilde{C}_N(\lambda) = \frac{1}{2N} \sum_{r} \frac{1}{r^{3/2}} \sum_{x,x'} g_{x-x'} \langle q_x p_{x'} (q_0 - q_{-r}) (p_0 + p_{-r}) \rangle_{\text{mc}} \]

\[ = \frac{1}{2N} \sum_{r} \frac{1}{r^{3/2}} \sum_{x,x'} g_{x-x'} \times \langle q_x (q_0 - q_{-r}) \rangle_{\text{mc}} (p_{x'} (p_0 + p_{-r}) \rangle_{\text{mc}} \]

\[ = -\frac{(g_{0}^{2})_{\text{mc}}}{2N} \sum_{r} \frac{1}{r^{3/2}} \sum_{x} g_{x} h_{r}(x) , \tag{33} \]

\[ h_{r}(x) := \langle q_x (q_0 - q_{-r}) \rangle_{\text{mc}} , \tag{34} \]

where to obtain the function \( h_{r}(x) \), we used the translational invariance to shift the site index in the correlation. For the expression of the function \( h_{r}(x) \), we further use the ensemble equivalence between the microcanonical and canonical ensemble (the detailed calculation is provided in the Appendix), and we have the following expression for the Fourier transform

\[ h_{r}(k) = \frac{4ikB T}{\sqrt{N}} \sin (kr) / \omega_{k}^{2} . \tag{35} \]
The Fourier transform of the resultant expression of \( \langle u(\lambda) \rangle_{0_{me}} \) is as follows:

\[
\hat{C}_N(\lambda) = \frac{2(k_B T)^2}{N} \sum_k \frac{1}{\lambda + \gamma [2 \sin(k/2)]^2} \frac{\Phi_k^2}{\omega_k^2} \\
\sim \frac{2(k_B T)^2}{\pi} \int_{N^{-1}}^{\pi} \frac{1}{\lambda + \gamma [2 \sin(k/2)]^2} \frac{\Phi_k^2}{\omega_k^2}, \quad (36)
\]

where we take the continuous expression in terms of the wave number in the last line. We reduced the interval of integration to \([N^{-1}, \pi] \) using the symmetry with respect to \(k = \pi\). The constant \(k_B\) is the Boltzmann constant.

C. Asymptotic behavior of current correlation

Now, we can analyze the asymptotic behavior of the current correlation. The correlation function in the time domain is obtained with the inverse Laplace transform

\[
C_N(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda \hat{C}_N(\lambda)e^{\lambda t} \\
= \frac{2(k_B T)^2}{\pi} \int_{N^{-1}}^{\pi} \frac{\Phi_k^2}{\omega_k^2} e^{-\gamma [2 \sin(k/2)]^2 t} \cdot (37)
\]

Here, we have selected the pole \(-\gamma [2 \sin(k/2)]^2\) in the \(\lambda\)-plane. From this, one can recognize that the asymptotic behavior in time is obtained from the small wave number regime. The asymptotic behavior in the time domain considering the behavior of small wave numbers is discussed below.

1. Momentum-conserving case

In the short-range interacting case, i.e., in the ME model, it is exactly shown that the exponent of the time-decay in the current correlation function is \(1/2\) (< 1), which implies anomalous transport. Now we consider the long-range interacting case satisfying total momentum conservation [25].

For small \(k\), the function \(\frac{\Phi_k^2}{\omega_k^2}\) behaves as

\[
\frac{\Phi_k^2}{\omega_k^2} \sim \begin{cases} 
  k^{-2}, & (\delta < 1) \\
  k^{-2}(\ln k^{-1})^{-1}, & (\delta = 1) \\
  k^{-(\delta-3)}, & (1 < \delta < 3) \\
  \ln k^{-1}, & (\delta = 3) \\
  \text{const.}, & (\delta < 3)
\end{cases} \quad (38)
\]

Hence, for \(\delta \leq 2\), the integral in Eq. (36) exhibits infrared divergence in the limit of \(N \to \infty\). The asymptotic behavior is expressed as

\[
\lim_{N \to \infty} C_N(t) \sim \begin{cases} 
  N, & (\delta < 1) \\
  N^{2-\delta}, & (1 < \delta < 2) \\
  \ln N, & (\delta = 2) \\
  C(t), & (\delta > 2)
\end{cases} \quad (39)
\]

where \(C(t)\) is the saturated continuous function in the thermodynamics limit. The current correlation is ill-defined in the thermodynamic limit for \(\delta < 2\). Note that the appropriately scaled current correlation, i.e., \(C_N/N^{2-\delta}\) for \(1 < \delta < 2\), is independent of \(t\) for large \(t\). From the structure of the inverse Laplace transform, we can find the asymptotic behavior of the function \(C(t)\) given for \(\delta > 2\):

\[
\lim_{t \to \infty} C(t) \sim \begin{cases} 
  t^{-(\delta-2)/2}, & (2 < \delta < 3) \\
  t^{-1/2} \ln t, & (\delta = 3) \\
  t^{-1/2}, & (\delta > 3)
\end{cases} \quad (40)
\]

This result leads to the classification of the exponent \(\beta\) in Eq. (2) and the results listed in table I. A crucial observation here is that the exponent of the asymptotic time-decay continuously changes as a function of the index of long-range potential \(\delta\). Another crucial observation is that there is an ill-defined regime \((\delta < 2)\) where the current correlation diverges. The exponent 1/2 for \(\delta > 3\) implies that this regime is regarded as a short-range interaction in the context of heat transfer.

2. Momentum-nonconserving case

Next, we consider the momentum-nonconserving case, i.e., \(k_0 \neq 0\). We note that in the short-range interacting case, the onsite potential induces normal thermal conduction. It is already known that the exponent in time-decay in the current correlation is 3/2 (> 1) [25]. Here, we consider the effect of long-range potential based on the exact expression. We perform an analysis similar to the one in previous subsection. For the term \((\Phi_k^2/\omega_k^2)\), the main contribution for the small wave number regime is from \(\Phi_k^2\), because the dispersion relation is constant for the regime. Then, we have

\[
\frac{\Phi_k^2}{\omega_k^2} \sim \begin{cases} 
  k^{-2}, & (\delta < 1) \\
  (\ln k^{-1})^{-2}, & (\delta = 1) \\
  k^{2(\delta-2)}, & (1 < \delta < 3) \\
  (\ln k^{-1})^2, & (\delta = 3) \\
  k^2, & (\delta > 3)
\end{cases} \quad (41)
\]

From this expression, we can discuss the thermodynamic limit by considering the system-size dependence:

\[
\lim_{N \to \infty} C_N(t) \sim \begin{cases} 
  N, & (\delta < 1) \\
  N^{-(2\delta-3)}, & (1 < \delta < 3/2) \\
  \ln N, & (\delta = 3/2) \\
  C(t), & (\delta > 3)
\end{cases} \quad (42)
\]

From this, we find that there is a regime that the current correlation shows infrared divergence \((\delta < 3/2)\). For \(\delta > 3/2\), we have the well-defined continuous function \(C(t)\). The asymptotic behavior in the time-domain of
the function is estimated as
\[
\lim_{t \to \infty} C(t) \sim \begin{cases} 
 t^{-(2\delta-3)/2}, & (3/2 < \delta < 3) \\
 t^{-3/2}(\ln t)^2, & (\delta = 3) \\
 t^{-3/2}, & (\delta > 3)
\end{cases}.
\]

This leads to the classification of the exponent \( \beta \) in the time-decay listed in table I. A critical observation here is that the exponent can be less than 1 for \( 3/2 < \delta < 5/2 \), which indicates anomalous behavior in heat conduction. This is physically important because the momentum-nonconserving systems have been thought to show normal heat conduction. This anomalous behavior originates from the long-range interaction. Hence, one can say that long-range interactions induce the anomaly. Again, we observe that the exponent continuously changes as a function of the index of the long-range interaction \( \delta \). For \( \delta > 3 \), we have the exponent 3/2, which is the same as in the short-range interacting case. Considering the exponent, the regime \( \delta > 3 \) can be regarded as a short-range interaction. On the other hand, in the context of the normal heat conduction, the index \( \delta = 5/2 \) is critical since the exponent \( \beta \) is larger than 1 for \( \delta > 5/2 \).

IV. SUMMARY AND DISCUSSION

In this study, we consider the effect of long-range interaction in the energy current correlation. The current correlation is a key component in the Kubo formula leading to thermal conductivity. To obtain clear-cut results, we introduce the exactly soluble model, which reduces to the momentum exchange model in the short-range interaction limit, and we derive the exponent \( \beta \) in Eq.(2) exactly.

We compare the momentum conserving case with nonconserving case, because it is known that the momentum-conserving case with a short range interacting case shows anomalous transport with the exponent \( \beta = 1/2 \) (< 1), whereas the momentum-nonconserving case does not show an anomaly, i.e., \( \beta = 3/2 \) (> 1). In terms of the index of long-range interaction \( \delta \), the results of the exponents are summarized in table I. We have three main results. First, the exponent \( \beta \) continuously changes as a function of the index of the long-range potential \( \delta \). Second, there is a regime where the current correlation function is ill-defined. Finally, the most remarkable finding is that even momentum-nonconserving case can exhibit anomalous transport for a certain range of \( \delta \).

These observations might be suggestive for realistic experiments with physical objects that involve long-range terms in potentials.

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Appendix A: Calculation of \( h_r(x) \)

We consider the microcanonical average for the function \( h_r(x) \):
\[
 h_r(x) := \langle q_x(q_r - q_{-r}) \rangle_{\text{mc}}.
\]

We consider this function for momentum-conserving and nonconserving cases separately.

First, we consider the momentum-nonconserving case. Note that the function \( h_r(x) \) is the correlation on the local observables. Calculating the local observable in terms of the microcanonical ensemble is equivalent to calculating the expectation value of the local observable in terms of the locally reduced distribution function from the microcanonical ensemble. Now, we impose the ensemble equivalence between the microcanonical ensemble with a fixed energy and the canonical ensemble with the corresponding temperature \( T \):
\[
 \langle q_x(q_r - q_{-r}) \rangle_{\text{mc}} \sim \int d\Gamma q_x(q_r - q_{-r})\rho_{\text{can}}(q,p)
\]
\[
\rho_{\text{can}}(q,p) = \exp(-H/(k_B T))/Z_T,
\]
where \( \int d\Gamma \ldots \) is the phase space average, i.e., \( \int dq_1 dq_2 \cdots dp_1 dp_2 \cdots \). The function \( Z_T \) is the partition function. The boundary condition for this case is \( q_{x+N} = q_x \). We obtain the expression as
\[
h_r(x) = \frac{1}{\sqrt{N}} \sum_k h_r(k)e^{-ikx},
\]
\[
h_r(k) = \frac{4ik_BT\sin(kr)}{\sqrt{N} \omega_k^2}.
\]

Next, we consider the momentum-conserving case. In this case, we use the phase space \( (s,p) \) instead of \( (q,p) \). Hence, we note the following argument on the function \( h_r(x) \). Using the discrete Laplacian, we have the following expression:
\[
\Delta h_r(x) = \langle (s_x - s_{x-1})(q_r - q_{-r}) \rangle_{\text{mc}}
\]
\[
= \langle (s_x - s_{x-1})(s_{r-1} + s_{r-2} + \cdots + s_{-r}) \rangle_{\text{mc}}
\]
\[
= \langle s_x(s_{r-1} + s_{r-2} + \cdots + s_{-r}) \rangle_{\text{mc}}
\]
\[
- \langle s_x(s_r + s_{r-1} + \cdots + s_{-r+1}) \rangle_{\text{mc}}
\]
\[
= \langle s_x(-s_r + s_{-r}) \rangle_{\text{mc}}.
\]
From the second to the third line, we used the translational invariance to shift from the index \( x-1 \) to \( x \). We consider the microcanonical ensemble with zero total stretch, momentum, and a fixed finite total energy. We then impose the ensemble equivalence between the microcanonical ensemble with the ground canonical ensemble.
with the corresponding temperature $T$ and chemical potential $\mu$:

$$\langle s_x s_r \rangle_{me} = \frac{\int ds_1 \cdots ds_N s_x s_r e^{-(V(s)-\mu \sum_{i=1}^{N} s_i)/(k_B T)}}{Z_{T,\mu}},$$

(A7)

$$V(s) = \frac{1}{2N} \sum_{x=1}^{N} \sum_{r=1}^{N/2} (s_x + s_{x+1} + \cdots + s_{x+r-1})^2$$

$$= \sum_k |s_k|^2 \frac{\sum_{r=1}^{N/2} \frac{1}{N} \frac{\sin(kr/2)}{r^2} \sin(k/2)}{2 \sin^2(k/2)},$$

(A8)

where the Fourier transform is used in the last line. The boundary condition for this case is $s_{x+N} = s_x$. Now, we take $\mu = 0$, which corresponds to $\sum_{i=1}^{N} s_i = 0$. With these parameters, we have

$$h_r(k) = \frac{ik_B T}{\sqrt{N}} \frac{\sin(kr)}{\sum_{r=1}^{N/2} \frac{1}{N} \frac{\sin^2(kr/2)}{r^2}}.$$

(A9)

This is equivalent to Eq. (A5) with $k_0 = 0$.

Note that here we do not prove the ensemble equivalence rigorously, but simply impose it. We remark that for short range interacting case, i.e., the original ME model, the ensemble equivalence can be rigorously proven [26].

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