ON S. MAZUR’S PROBLEMS 8 AND 88
FROM THE SCOTTISH BOOK

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Abstract. The paper discusses Problems 8 and 88 posed by Stanislaw Mazur in the Scottish Book [SB]. It turns out that negative solutions to both problems are immediate consequences of the results of §5 of [P1]. We discuss here some quantitative aspects of Problems 8 and 88 and give answers to open problems discussed in a recent paper [PS] in connection with Problem 88.

1. Introduction

We are going to discuss in this paper Problems 8 and 88 posed by Stanislaw Mazur in the Scottish Book [SB]. Problem 88 asks whether a Hankel matrix in the injective tensor product $\ell^1 \hat{\otimes} \ell^1$ of two spaces $\ell^1$ must have finite sum of the moduli of its matrix entries. Problem 8 asks whether for an arbitrary sequence $\{z_n\}_{n \geq 0}$ in the space $c$ of converging sequences there exist sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ in the space $c$ such that

$$z_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k y_{n-k}, \quad n \geq 0.$$  

We give precise statements of the problems and all necessary definitions later.

It turned out that both problems have negative solutions. Independently, solutions were obtained by Kwapień and Pełczyński [KP] and Eggermont and Leung [EL]. In a recent paper by Pełczyński and Sukochev [PS] in Section 6 certain quantitative results related to negative solutions of Problems 8 and 88 are obtained and certain open problems are raised.

It turns out, however, that the results of Section 5 of my earlier paper [P1] immediately imply negative solutions to Problems 8 and 88. Moreover, Section 5 of [P1] contains much stronger results. In particular, a complete description of the Hankel matrices\(^1\) in the injective tensor product of two spaces $\ell^1$ is obtained in [P1] in terms of the Besov space $B_{\infty, 1}$\(^1\). Unfortunately, I was not aware about the Problems 8 and 88 when I wrote the paper [P1].

In Sections 3 and 4 of this paper we explain why the results of [P1] immediately imply negative solutions to Problems 8 and 88 and we give a solution to the problems raised in [PS].

\(^1\)Note that Hankel matrices and Hankel operators play an important role in many areas of mathematics and applications, see [P2].

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In §2 we collect necessary information on tensor products and Besov spaces.

2. Preliminaries

1. Projective and injective tensor products. We define the projective tensor product $\ell_\infty \hat{\otimes} \ell_\infty$ as the space of matrices $\{q_{jk}\}_{j,k \geq 0}$ of the form

$$q_{jk} = \sum_{n \geq 0} a_j^{(n)} b_k^{(n)},$$

(2.1)

where the $a^{(n)} = \{a_j^{(n)}\}_{j \geq 0}$ and $b^{(n)} = \{b_j^{(n)}\}_{j \geq 0}$ are sequences in $\ell_\infty$ such that

$$\sum_{n \geq 0} \|a^{(n)}\|_{\ell_\infty} \|b^{(n)}\|_{\ell_\infty} < \infty.$$  

(2.2)

The norm of the matrix $\{q_{jk}\}_{j,k \geq 0}$ in $\ell_\infty \hat{\otimes} \ell_\infty$ is defined as the infimum of the left-hand side of (2.2) over all sequences $a^{(n)} = \{a_j^{(n)}\}_{j \geq 0}$ and $b^{(n)} = \{b_j^{(n)}\}_{j \geq 0}$ satisfying (2.1).

Similarly, one can define the projective tensor products $c \hat{\otimes} c$ and $c_0 \hat{\otimes} c_0$, where $c$ is the subspace of $\ell_\infty$ that consists of the converging sequences and $c_0$ is the subspace of $c$ that consists of the sequences with zero limit.

We consider the space $V^2$ that is a kind of a “weak completion” of $\ell_\infty \hat{\otimes} \ell_\infty$. $V^2$ consists of the matrices $Q = \{q_{jk}\}_{j,k \geq 0}$ for which

$$\sup_{n > 0} \|P_n Q\|_{\ell_\infty \hat{\otimes} \ell_\infty} < \infty,$$

where the projections $P_n$ are defined by

$$(P_n Q)_{jk} = \begin{cases} q_{jk}, & j \leq n, \ k \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $c \hat{\otimes} c \subset \ell_\infty \hat{\otimes} \ell_\infty \subset V^2$.

The injective tensor product $\ell^1 \hat{\otimes} \ell^1$ of two spaces $\ell^1$ is, by definition, the space of matrices $Q = \{q_{jk}\}_{j,k \geq 0}$ such that

$$\|Q\|_{\ell^1 \hat{\otimes} \ell^1} = \sup_{j,k=0}^N \sum_{j,k=0}^N q_{jk} x_j y_k < \infty,$$

where the supremum is taken over all sequences $\{x_j\}_{j \geq 0}$ and $\{y_k\}_{k \geq 0}$ in the unit ball of $\ell^\infty$ and over all positive integers $N$. The space $\ell^1 \hat{\otimes} \ell^1$ can naturally be identified with the space of bounded linear operators from $c_0$ to $\ell^1$ (note that every bounded operator from $c_0$ to $\ell^1$ is compact).

2. Besov spaces. In this paper we consider only Besov spaces of functions analytic in the unit disk $\mathbb{D}$. Besov spaces $B_{p,q}^s$ admit many equivalent descriptions. We give a definition in terms of dyadic Fourier expansions. We define the polynomials $W_n$, $n \geq 0$, as follows. If $n \geq 1$, then $\hat{W}_n(2^n) = 1$, $\hat{W}_n(k) = 0$ for $k \not\in (2^{n-1}, 2^{n+1})$, and $\hat{W}_n$ is a
linear function on $[2^{n-1}, 2^n]$ and on $[2^n, 2^{n+1}]$. We put $W_0(z) = 1 + z$. It is easy to see that
\[ \|W_n\|_{L^1} \leq \frac{3}{2}, \quad n \geq 0, \]
and
\[ f = \sum_{n \geq 0} f \ast W_n \]
for an arbitrary analytic function $f$ in $\mathbb{D}$.

For $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $s \in \mathbb{R}$, we define the Besov space $B^{s}_{p,q}$ as the space of analytic functions in $\mathbb{D}$ satisfying
\[ f \in B^{s}_{p,q} \iff \{2^{ns} \|f \ast W_n\|_{L^p}\}_{n \geq 0} \in c_0. \quad (2.3) \]
If $q = \infty$, the space $B^{s}_{p,q}$ is nonseparable. We denote by $b^{s}_{p,\infty}$ the closure of the set of polynomials in $B^{s}_{p,\infty}$. It is easy to verify that
\[ f \in b^{s}_{p,\infty} \iff \{2^{ns} \|f \ast W_n\|_{L^p}\}_{n \geq 0} \in c_0. \]

Besov spaces admit many other descriptions (see [Pe] and [P2]).

3. Problem 8

To state Mazur’s Problem 8 of the Scottish Book [SB], consider the bilinear form $B$ on $c \times c$ defined by
\[ B(\{x_n\}_{n \geq 0}, \{y_n\}_{n \geq 0}) = \{z_n\}_{n \geq 0}, \]
where
\[ z_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k y_{n-k}, \quad n \geq 0, \]
and $c$ is the space of sequences that have a limit at infinity.

It is easy to see that $B$ maps $c \times c$ into $c$. S. Mazur asked in Problem 8 whether $B$ maps $c \times c$ onto $c$.

As mentioned in the Introduction, a negative solution to problem 8 follows immediately from the results of §5 of [P1]. To state Theorem 5.1 of [P1], we define the operator $A$ on the space of matrices. Let $Q = \{q_{jk}\}_{j,k \geq 0}$. Then $AQ$ is the sequence defined by
\[ AQ = \{z_n\}_{n \geq 0}, \quad \text{where} \quad z_n = \frac{1}{n+1} \sum_{j+k=n} q_{jk}. \]

**Theorem 5.1 of [P1].** $A$ maps the space $V^2$ onto the space of Fourier coefficients of the Besov space $B^{0}_{1,\infty}$.

Recall that the space $V^2$ is defined in the introduction. In particular, it follows from Theorem 5.1 of [P1] that
\[ A(c \hat{\otimes} c) \subset A(\ell^\infty \hat{\otimes} \ell^\infty) \subset A(V^2) \subset \left\{\{\hat{f}(n)\}_{n \geq 0} : f \in B^{0}_{1,\infty}\right\}, \]
and so
\[ B(c \times c) \subset \left\{ \{\hat{f}(n)\}_{n \geq 0} : f \in B_{1,\infty}^0 \right\}. \]

It is easy to see that
\[ c \not\subset \left\{ \{\hat{f}(n)\}_{n \geq 0} : f \in B_{1,\infty}^0 \right\}. \]

Indeed, if \( f \in B_{1,\infty}^0 \), then it follows immediately from (2.3) and from [R], §8.6 that
\[
\sup_{n \geq 0} \sum_{k=0}^{n} |\hat{f}(2^n + 2^k)|^2 < \infty.
\]

This gives a negative solution to Problem 8.

In fact, Theorem 5.1 of [P1] allows one to describe \( A(c \hat{\otimes} c) \). First, let us observe that Theorem 5.1 of [P1] easily implies the following description of \( A(c_0 \hat{\otimes} c_0) \).

**Theorem 3.1.**
\[
A(c_0 \hat{\otimes} c_0) = \left\{ \{\hat{f}(n)\}_{n \geq 0} : f \in b_{1,\infty}^0 \right\}.
\]

Recall that \( b_{1,\infty}^0 \) is the closure of the polynomials in \( b_{1,\infty}^0 \) (see §2). Theorem 3.1, in turn, easily implies the following description of \( A(c \hat{\otimes} c) \).

**Theorem 3.2.**
\[
A(c \hat{\otimes} c) = \left\{ \{\hat{f}(n) + d\}_{n \geq 0} : f \in b_{1,\infty}^0, \ d \in \mathbb{C} \right\}.
\]

4. Problem 88

Recall that in Problem 88 of [SB] S. Mazur asked whether a Hankel matrix \( \{\gamma_{j+k}\}_{j,k \geq 0} \) in the injective tensor product \( \ell^1 \hat{\otimes} \ell^1 \) must satisfy the condition:
\[
\sum_{k=0}^{\infty} (1+k)|\gamma_k| < \infty,
\]
i.e., whether the sum of the moduli of the matrix entries must be finite.

As mentioned in the Introduction, a negative solution to Problem 88 follows immediately from the results of §5 of [P1]. A complete description of Hankel matrices in \( \ell^1 \hat{\otimes} \ell^1 \) is given by Theorem 5.2 of [P1]:

**Theorem 5.2 of [P1].** A Hankel matrix \( \{\gamma_{j+k}\}_{j,k \geq 0} \) belongs to \( \ell^1 \hat{\otimes} \ell^1 \) if and only if the function \( f \) defined by
\[
f(z) = \sum_{n \geq 0} \gamma_n z^n
\]
belongs to the Besov class \( B_{\infty,1}^1 \).

Let us obtain the best possible estimate on the moduli of the matrix entries of Hankel matrices in \( \ell^1 \hat{\otimes} \ell^1 \).
Since \( \|f * W_n\|_{L^2} \leq \|f\|_{L^2} \|W_n\|_{L^1} \leq 3/2\|f\|_{L^2} \), it follows easily from (2.3) that if \( f \in B^{1}_{\infty, 1} \), then
\[
\sum_{n=0}^{\infty} 2^n \left( \sum_{k=2^n}^{2^{n+1}-1} |\hat{f}(k)|^2 \right)^{1/2} < \infty.
\]
(4.1)

Let us show that this is the best possible estimate for the moduli of the Fourier coefficients of functions in \( B^{1}_{\infty, 1} \). To show this, we are going to use a version of the de Leeuw–Katznelson–Kahane theorem. It was proved in [dLKK] that if \( \{\beta_n\}_{n \in \mathbb{Z}} \) is a sequence of nonnegative numbers in \( \ell^2(\mathbb{Z}) \), then there exists a continuous function \( f \) on \( T \) such that
\[
|\hat{f}(n)| \geq \beta_n, \quad n \in \mathbb{Z}.
\]
We refer the reader to [K1], [K2], and [N] for refinements of the de Leeuw–Katznelson–Kahane theorem and different proofs. We need the following version of the de Leeuw–Katznelson–Kahane theorem:

**Lemma 4.1.** There is a positive number \( K \) such that for arbitrary nonnegative numbers \( \beta_0, \beta_1, \cdots, \beta_m \), there exists a polynomial \( f \) of degree \( m \) such that
\[
|\hat{f}(j)| \geq \beta_j, \quad 0 \leq j \leq n, \quad \text{and} \quad \|f\|_{L^\infty} \leq K \left( \sum_{j=0}^{n} \beta_j^2 \right)^{1/2}.
\]
Lemma 4.1 follows easily from the results of [K2].

**Theorem 4.2.** Let \( \{\alpha_k\}_{k \geq 0} \) be a sequence of nonnegative numbers such that
\[
\sum_{n=0}^{\infty} 2^n \left( \sum_{k=2^n}^{2^{n+1}-1} \alpha_k^2 \right)^{1/2} < \infty.
\]
(4.2)
Then there exists a function \( \varphi \) in the space \( B^{1}_{\infty, 1} \) such that \( |\hat{\varphi}(k)| \geq \alpha_k \) for \( k \geq 0 \).

**Proof.** By Lemma 4.1, there exists \( K > 0 \) and a sequence of polynomials \( f_n, n \geq 0 \), such that
\[
f_0(z) = \hat{f}_0(0) + \hat{f}_0(1)z, \quad f_n(z) = \sum_{k=2^n}^{2^{n+1}-1} \hat{f}_n(k)z^k, \quad \text{for} \quad n \geq 1,
\]
\[
|\hat{f}_0(k)| \geq \alpha_k, \quad \text{for} \quad k = 0, 1, \quad |\hat{f}_n(k)| \geq \alpha_k, \quad \text{for} \quad n \geq 1, \quad 2^n \leq k \leq 2^{n+1} - 1,
\]
and
\[
\|f_0\|_{L^\infty} \leq K(\alpha_0^2 + \alpha_1^2)^{1/2}, \quad \|f_n\|_{L^\infty} \leq K \left( \sum_{k=2^n}^{2^{n+1}-1} \alpha_k^2 \right)^{1/2}, \quad \text{for} \quad n \geq 1.
\]
We can define now the function \( \varphi \) by
\[
\varphi = \sum_{n \geq 0}^{\infty} f_n.
\]
Obviously, $|\hat{\varphi}(k)| \geq \alpha_k$ for $k \geq 0$. Let us show that $\varphi \in B^1_{\infty}$. We have
\[
\sum_{n \geq 1} 2^n \| \varphi \ast W_n \|_{L^\infty} = \sum_{n \geq 1} 2^n \| (f_{n-1} + f_n + f_{n+1}) \ast W_n \|_{L^\infty} 
\leq \sum_{n \geq 1} 2^n \| (f_{n-1} + f_n + f_{n+1}) \|_{L^\infty} \| W_n \|_{L^1}
\leq 3 \sum_{n \geq 1} 2^n \| f_n \|_{L^\infty} \| W_n \|_{L^1} \leq 3 \cdot \frac{3}{2} \sum_{n \geq 1} 2^n \| f_n \|_{L^\infty}
\leq 9 \frac{1}{2} K \sum_{n=0}^{\infty} 2^n \left( \sum_{k=2^n}^{2^{n+1}-1} \alpha_k^2 \right)^{1/2} < \infty. \quad \blacksquare
\]

In [PS] the following problem was considered. Let $\Psi$ be the function on $(0, \infty)$ defined by
\[
\Psi(t) = \begin{cases} 
\frac{3}{2} t - 1, & 0 < t \leq 2, \\
t, & t > 2.
\end{cases}
\]

Let $\{\gamma_{j+k}\}_{j,k \geq 0}$ be a Hankel matrix. The following result was proved in [PS] (Theorem 6.7):

(i) if $\beta < \Psi(t)$, then
\[
\sum_{k \geq 0} |\gamma_k|^t (1 + k)^\beta < \infty \quad \text{whenever} \quad \{\gamma_{j+k}\}_{j,k \geq 0} \in \ell^1 \otimes \ell^1;
\]

(ii) if $\beta > \Psi(t)$, then
\[
\sum_{k \geq 0} |\gamma_k|^t (1 + k)^\beta = \infty \quad \text{for some} \quad \{\gamma_{j+k}\}_{j,k \geq 0} \in \ell^1 \otimes \ell^1;
\]

(iii) if $\beta = \Psi(t)$ and $t < \infty$, then
\[
\sum_{k \geq 0} |\gamma_k|^t (1 + k)^\beta < \infty \quad \text{whenever} \quad \{\gamma_{j+k}\}_{j,k \geq 0} \in \ell^1 \otimes \ell^1.
\]

In [PS] the problem is raised to find out whether
\[
\sum_{k \geq 0} |\gamma_k|^t (1 + k)^{\Psi(t)}
\]
has to be finite for $t \in \left(0, \frac{4}{3}\right)$ whenever $\{\gamma_{j+k}\}_{j,k \geq 0} \in \ell^1 \otimes \ell^1$.

It is easy to deduce Theorem 6.7 of [PS] from (4.1) and above Theorem 4.2. Moreover, using (4.1) and Theorem 4.2, we can solve the problem posed in [PS] and settle the case $t \in \left(0, \frac{4}{3}\right)$.

**Theorem 4.3.** If $1 \leq t < \frac{4}{3}$, then
\[
\sum_{k \geq 0} |\gamma_k|^t (1 + k)^{3t/2-1} < \infty \quad \text{whenever} \quad \{\gamma_{j+k}\}_{j,k \geq 0} \in \ell^1 \otimes \ell^1.
\]
If $0 < t < 1$, then
\[ \sum_{k \geq 0} |\gamma_k|^t (1 + k)^{3t/2 - 1} = \infty \quad \text{for some} \quad \{\gamma_{j+k}\}_{j,k \geq 0} \in \ell^1 \otimes \ell^1. \]

**Proof.** Suppose that $1 \leq t < 2$. By Hölder's inequality, we have
\[
\sum_{k \geq 1} |\gamma_k|^t (1 + k)^{3t/2 - 1} \leq \const \sum_{n \geq 0} 2^{n(3t/2 - 1)} \sum_{k = 2^n}^{2^{n+1}} |\gamma_k|^t \leq \sum_{n \geq 0} 2^{\frac{3n}{2}t} 2^{-n} \left( \sum_{k = 2^n}^{2^{n+1}} |\gamma_k|^2 \right)^{t/2} 2^{n(1-t/2)} = \sum_{n \geq 0} 2^{nt} \left( \sum_{k = 2^n}^{2^{n+1}} |\gamma_k|^2 \right)^{t/2}.
\]

Since $t \geq 1$, the $\ell^t$ norm of a sequence does not exceed its $\ell^1$ norm, and so
\[
\sum_{n \geq 0} 2^{nt} \left( \sum_{k = 2^n}^{2^{n+1}} |\gamma_k|^2 \right)^{t/2} \leq \left( \sum_{n \geq 0} 2^n \left( \sum_{k = 2^n}^{2^{n+1}} |\gamma_k|^2 \right)^{1/2} \right)^t.
\]

The result follows now from Theorem 5.2 of [P1] and (4.1).

Suppose now that $0 < t < 1$. It follows from Theorem 4.2 that it suffices to find a sequence $\{\alpha_k\}_{k \geq 0}$ of nonnegative numbers that satisfies (4.2) and such that
\[ \sum_{k \geq 0} \alpha_k^t (1 + k)^{3t/2 - 1} = \infty. \]

Let $\{\delta_n\}_{n \geq 0}$ be a sequence of positive numbers such that $\{2^{3n/2}\delta_n\}_{n \geq 0} \in \ell^1$, but $\{2^{3n/2}\delta_n\}_{n \geq 0} \notin \ell^t$.

Put $\alpha_0 = 0$ and $\alpha_k = \delta_n$ if $2^n \leq k \leq 2^{n+1} - 1$.

We have
\[
\sum_{n \geq 0} 2^n \left( \sum_{k = 2^n}^{2^{n+1}-1} \alpha_k^2 \right)^{1/2} = \sum_{n \geq 0} 2^{3n/2} \delta_n < \infty.
\]

However,
\[
\sum_{k \geq 0} \alpha_k^t (1 + k)^{3t/2 - 1} \geq \const \sum_{n \geq 0} 2^{n(3t/2 - 1)} \sum_{k = 2^n}^{2^{n+1}} \alpha_k^t = \const \sum_{n \geq 0} 2^{n(3t/2 - 1)} 2^n \delta_n = \const \sum_{n \geq 0} 2^{3nt/2} \delta_n = \infty. \]

\[ \blacksquare \]
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