BCS-BEC crossover in spatially modulated fermionic condensates

Armen Sedrakian

Institute for Theoretical Physics, J. W. Goethe-University,
Max-von-Laue-Str. 1, 60438 Frankfurt am Main, Germany
E-mail: sedrakian@th.physik.uni-frankfurt.de

Abstract. Several novel multi-component fermionic condensates show universal behavior under imbalance in the number of fermionic species. Here I discuss their phase structure, thermodynamics, and the transition from the weak (BCS) to strong (BEC) coupling regime. The inhomogeneous superconducting phases are illustrated on the example of the Fulde-Ferrell phase which appears in the weak coupling regime, at low temperatures and large asymmetries. The inhomogeneous phases persist through the crossover up to (and possibly beyond) the transition to the strong coupling regime.

1. Introduction

The last decade has seen impressive advances in the research, both theoretical and experimental, on pairing in novel fermionic systems which share a number of common features. One example is encountered in experiments with ultracold atomic vapors in magnetic traps. The atomic gases allow for a remarkable control over the parameter space, including variations in the strength of the pairing force via Feshbach resonance. The second example of interest is the nuclear/quark matter which can be either spin polarized (by strong magnetic fields) or isospin polarized, as required by the beta equilibrium in the interiors of compact stars.

The possibility of transition from Bardeen-Cooper-Schrieffer (BCS) pairing to Bose-Einstein condensation (BEC) in ensembles of attractive degenerate fermions was conjectured long ago [1]. This transition takes place when the dimensionless coupling $\lambda$, which is a product of the density of states at the Fermi surface and the dimensional coupling of the theory increases from weak coupling ($\lambda \ll 1$) to strong coupling ($\lambda \gg 1$). Experimentally, it is achieved by variations of the coupling constant via the mechanism of Feshbach resonance in atomic gases. For particles interacting via the strong force the interactions cannot be varied at will, but the range over which they are effective and the density of states at the Fermi surface will change with density. Therefore, the combined effect of density variation on the density of states and effective range of the coupling can lead to a crossover in strongly interacting systems with density gradients.

The generalization of the Nozières-Schmitt-Rink (NSR) theory to ensembles with population imbalance was carried out in the context of the strongly interacting matter in ref. [2]. Experimental realizations in the atomic vapors were achieved somewhat later [3]. The novel feature of the Hamiltonian describing these imbalanced gases or (super)fluids is the non-invariance under exchange of paired particles, i.e., there is a new scale in the problem $\delta \mu = (\mu_\alpha - \mu_\beta)/2$, where $\mu_\alpha$ and $\mu_\beta$ are the chemical potentials of the species. Furthermore,
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BCS: \( k = -k, \delta \mu = 0 \)

asymmetric BCS: \( k = -k, \delta \mu \neq 0 \)

Figure 1. Schematic illustration of the Fermi spheres of two components (solid - majority component, dashed - minority component) and the momenta of paired fermions for four phases: BCS, asymmetric BCS, LOFF and DFSP.

for certain values of asymmetry \( \delta \mu \) the system may support a current carrying state, where the counter-propagating super-current and normal current cancel each other. This phase is known as the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) phase [4, 5]. The LOFF condensate is spatially modulated, therefore, minimally, it breaks the \( O(3) \) rotational symmetry of the system down to \( O(2) \). However, in general, the condensate may assume a complicated structure which breaks also translational symmetries. The emergence of the LOFF phase becomes energetically favorable for the following reason: setting up the condensate and the normal excitations in motion costs kinetic energy, which is always positive and thus unfavorable; however, the current changes the way the phase-space is sampled in the vicinity of the Fermi surface, which in turn enhances the condensation energy. The interplay of these two effects tells us whether the LOFF phase is favored or not. Indeed, in the low-temperature and large asymmetry sector of the phase diagram the LOFF phase is favored over the BCS state. The way the LOFF phase evolves into the BEC regime is not fully understood. I shall provide some answers to this problem below. An alternative to the LOFF phase, is the phase which allows deformations of the Fermi surfaces [6]. The emergence of deformed Fermi surface pairing (DFSP) can be explained in full analogy to the LOFF phase and I will not repeat the arguments above.

2. Homogeneous phases

Let us start our discussion with the homogeneous phases, i.e., the ordinary BCS phase and the asymmetrical BCS phase. Once the pairing interaction is specified, the mean-field solutions are obtained by solving the gap equation together with the equations for the densities for each fermionic component (see, e. g., refs. [6, 7]). The solutions of the coupled integral equations then provide the value of the gap parameter for fixed net density and asymmetry in the populations of fermions measured either in terms of the density asymmetry (hereafter \( \alpha \)) or the shift in the chemical potentials \( \delta \mu \). Figures 2 and 3 show the temperature dependence of various quantities for fixed density asymmetries [8]. The pairing correlations are clearly suppressed for large asymmetries; in the high-temperature domain \( T \rightarrow T_c \) the temperature dependence of the functions is analogous to the BCS theory for balanced systems. The low temperature domain
$T \to 0$, on the contrary, is anomalous, i.e., all the quantities show temperature dependence that is different from the BCS theory. The reduction of the gap, for example, is the result of the loss of coherence between the paired fermions (or equivalently, overlap between the thermal bands around the Fermi surfaces). For large asymmetries this leads to a complete disappearance of the gap in the limit of $T \to 0$, which manifests itself in the anomalous jump of the specific heat (Fig. 3). While such a jump could be a signature of the asymmetrical BCS state, we will see below that the temperature anomalies are removed if one allows for the LOFF (or DFSP) phase. This, of course, does not exclude the possibility of observing the second jump in the specific heat if the gas is prepared at high temperatures, where the BCS phase is stable, and is cooled down into the low temperature unstable phase [8].

Another interesting observation is the emergence of the empty shell around the Fermi surface of the minority component for large asymmetries seen in Fig. 4. This leads to the notion of “breached pairing” superconductivity [9, 10]. Clearly the topology of the minority Fermi surface changes as the asymmetry is increased. However, the topological phase transition disappears in the high temperature regime (Fig. 4 lower panel).
Figure 5. BCS (right column) to BEC (left column) crossover in an imbalanced system. Upper panel: the occupation number of minority component; middle panel: the pairing field; lower panel: the condensate wave-function. Figure taken from ref. [2].

Next we turn to the question on how the Nozières–Schmitt-Rink theory is modified in asymmetrical systems. This problem was first studied in the context of pairing in low-density nuclear matter in the attractive $^3S_1 - ^3D_1$ coupled channel, where neutron-proton Cooper pairs in the BCS limit transform to a BEC of deuterons [2]. Following the NSR conjecture one can directly extrapolate the BCS equations in the strong coupling regime to obtain the properties of the deuteron condensate. The pair wave-function, which is defined as the correlation function of two fermionic creation operators $\psi(k) = \langle a_k^\dagger b_{-k}^\dagger \rangle$, where $a_k^\dagger$ and $b_k^\dagger$ are the creation operators of neutrons and protons, obeys a Schrödinger type equation with an eigenvalue equal $2\mu$, with $\mu$ being the chemical potential in the symmetrical case. The state that emerges in the strong coupling limit corresponds to a mixture of deuterons (strongly bound two-body states) plus a fluid of unbound neutrons at arbitrary large asymmetries. At low densities the Pauli-blocking plays a minor role and its modifications to the deuteron wave-function are marginal. As a consequence large asymmetries are compatible with the existence of a deuteron condensate. Figure 5 shows the evolution of occupation numbers of the minority component (upper panel), pairing gap (intermediate panel) and condensate wave-function (lower panel) across the BCS-BEC transition (from left to right). In this particular case, the transition is enforced via the reduction of the net density. A remarkable feature of the transition is the change in the occupation numbers of the minority component: the gap (breach) is widened across the transition and in the BEC regime an empty (unoccupied) sphere emerges inside the Fermi sphere. Clearly, the topology of the Fermi sphere changes from having a strip of empty states to a empty interior region; thus, the BCS-BEC transition is associated with a topological phase transition in the shape of the minority Fermi sphere at non-zero asymmetries. The BCS vs BEC nature of the condensate is most clearly seen in the form of the condensate wave-function. In the momentum space it is strongly peaked in the BCS limit and is flat in the BEC limit (Fig. 5 lower panel). In the real space this picture translates into loosely bound state whose wave-function oscillates over many periods in the BCS limit. In the opposite BEC limit one finds tightly bound state
with a strongly localized wave-function.

3. Inhomogeneous phases

In this section I will explore the modifications that arise when inhomogeneous phases are allowed to exist along with the symmetrical and asymmetrical BCS phases. I will illustrate the main points on the example of color superconductivity in cold quark matter with pairing among the lightest quarks of different flavors ($u$ and $d$) in the color anti-triplet state (see e.g. refs. [11, 12]). Our Ansatz for the inhomogeneous condensate corresponds to the Fulde-Ferrell (FF) phase which postulates a single plane-wave dependence of the gap on the total momentum $\Delta(Q) \propto \Delta_0 \exp(iQr)$ [13]. Let us first concentrate on the BCS limit. One remarkable feature of the FF state is that, once it is allowed for, the anomalies seen in Figs. 2 and 3 are removed, i.e., the functional dependence of the physical quantities on the temperature are the same as in the ordinary BCS theory. The only effect of asymmetry is the reduction of the magnitude the gap [13, 14]. Thus, the $T$-dependence of quantities for $\alpha \neq 0$ are self-similar to the symmetrical BCS phase. The low-temperature topology of the Fermi spheres in the FF state is the same as in the asymmetrical BCS case, but only for selected directions, such as along in the direction orthogonal to the axis of symmetry breaking. This direction is chosen by the condensate spontaneously. At other angles the appearance of the empty strip (breach) mentioned above is suppressed. The FF phase occupies the low-temperature and large asymmetry portion of the $T-\alpha$ phase diagram and there exists a tri-critical Lifshitz point where the normal phase, the BCS phase, and the FF phase all meet (for more details see ref. [13]). Now let us comment on the transition from the BCS-FF state to the BEC state. Generally, as one moves towards the BEC limit, the portion of the $T-\alpha$ phase diagram occupied by the FF phase becomes smaller. A key question is whether the FF state exists only in the weakly coupled regime of the phase diagram or does it persist into the strongly coupled regime. The answer is best provided by the form of the condensate wave-function. Fig. 6 displays the evolution of the condensate wave-function from the weak (left column) to the strong (right column) coupling regime. In the weak coupling limit the wave-function and the quantity $r^2|\psi(r)|^2$ show oscillations with a period given by
$l \simeq 2\pi k_F^{-1}$, where $k_F$ is the Fermi wave-number. These sustained oscillations in space evidence the long-range correlations characteristic to BCS state; in this regime Cooper-pair size is much larger than the inter-particle spacing. In the strong coupling limit and for small asymmetries (solid line) a pronounced peak in these quantities corresponds to a well-localized bound state. However, for large asymmetries the picture is more complicated; there are several peaks which are not necessarily localized at the origin. These features indicate that a perfect BEC condensate has not been formed yet. Since the condensate is in the FF state sector of the $\alpha$-$T$ phase diagram (see ref. [13]) one may conjecture that the FF state has a strong-coupling counterpart, where tightly bound states carry a super-current.

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