Approximate quantum error correction can lead to better codes

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(November 6, 2018)

We present relaxed criteria for quantum error correction which are useful when the specific dominant quantum noise process is known. These criteria have no classical analogue. As an example, we provide a four-bit code which corrects for a single amplitude damping error. This code violates the usual Hamming bound calculated for a Pauli description of the error process, and does not fit into the GF(4) classification.

I. INTRODUCTION

Quantum error correction is the reversal of part of the adverse changes due to an undesired and unavoidable quantum process. Code criteria for perfect error correction have been developed, and all presently known codes satisfy these criteria exactly. Furthermore, most of these codes belong to a general GF(4) classification. In these schemes, quantum bit (qubit) errors are described using a Pauli (I, X, Y, Z) error basis, and coding is performed to allow correction of arbitrary unknown errors.

However, in the usual case in the laboratory, one works with a specific apparatus with a particular dominant quantum noise process. The GF(4) codes do not take advantage of this specific knowledge, and may thus be sub-optimal, in terms of transmission rate and code complexity. Unfortunately, there is no general method to construct quantum codes in a non-Pauli basis, and few such codes are known. The code criteria are generally very difficult to satisfy, and without the Pauli basis, no way is known to apply classical coding techniques for quantum error correction.

In this paper, we develop a novel approach to quantum error correction based on approximate satisfaction of the existing quantum error correction criteria. These approximate criteria are simpler and less restrictive, and in certain cases, such as for amplitude damping, codes can be found relatively more easily. The approximate criteria also admit more codes, therefore, codes requiring shorter block lengths may also be possible.

For example, using this approach we have discovered a four bit code which corrects for single qubit amplitude damping errors. Such a short non-degenerate code is impossible using the Pauli basis. The reason is that the effects [I, X] to be corrected in a non-degenerate code have to map the codeword space to orthogonal spaces if the syndrome is to be detected unambiguously. Hence, the minimum allowable size for the encoding space is the product of the dimension of the codeword space and the number of effects to be corrected. The single qubit amplitude damping effect operators expressed in the Pauli basis are:

\[ A_0 = \frac{1}{2} \left( (1 + \sqrt{1-\gamma})I + (1 - \sqrt{1-\gamma})\sigma_z \right) \]
\[ A_1 = \frac{\sqrt{\gamma}}{2} \left[ \sigma_x + i\sigma_y \right]. \]

These describe the changes due to the loss of zero or one excitation to the environment. To first order in the scattering probability \( \gamma \), \( n + 1 \) possible effects may happen to an \( n \)-qubit code using the \( A_0 \), \( A_1 \) basis, so it follows that \( n \geq 3 \) is required. In contrast, in the Pauli basis, any \( \sigma_x \) or \( \sigma_y \) error must be corrected by the code, so that \( 2n + 1 \) possible effects must be dealt with. It follows that at least \( n \geq 5 \) is required for a non-degenerate Pauli basis code, in contrast to the four bit code which we demonstrate in this paper.

The lessons are that (1) better codes may be found for specific error processes, and (2) approximate error correction simplifies code construction and admits more codes. Approximate error correction is a property with no classical analogue, because it makes use of slight non-orthogonalities possible only between quantum states. We describe our approach to this problem by first exhibiting our four-bit example code in detail. We then generalize our results to provide new, relaxed error correction criteria and specific procedures for decoding and recovery. We conclude by discussing possible extensions to our work.
II. FOUR Bit AMPLITUDE DAMPING CODE

Consider the single qubit quantum noise process defined by

$$\mathcal{E}(\rho) = \sum_{k=0,1} A_k \rho A_k^\dagger$$ \hspace{3cm} (3)

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix},$$ \hspace{3cm} (4)

known as amplitude damping. The probability of losing a photon, $\gamma$, is assumed to be small. To correct errors induced by this process, we encode one qubit using four, with the logical states

$$|0_L\rangle = \frac{1}{\sqrt{2}} \left[ |0000\rangle + |1111\rangle \right]$$ \hspace{3cm} (5)

$$|1_L\rangle = \frac{1}{\sqrt{2}} \left[ |0011\rangle + |1100\rangle \right].$$ \hspace{3cm} (6)

A circuit for encoding the logical state is shown in Fig. 1.

FIG. 1. Circuit for encoding a qubit. The third mode contains the input qubit. The rotation gate in the first qubit performs $\exp(-i \pi \sigma_y/4)$.

The possible outcomes after amplitude damping may be written as

$$|\psi_{out}\rangle = \bigoplus_k \phi_k \equiv \bigoplus_k A^\dagger_k |\psi_{in}\rangle,$$ \hspace{3cm} (7)

where $k$ are strings of 0’s and 1’s serving as indices for each error, and we use the notation $A_{010\ldots} = A_0 A_1 A_0 \cdots$. $|\cdot\rangle$ is convenient shorthand [11] for a mixed state (tensor sum of un-normalized pure states). In the following, the squares of the norm of $|\cdot\rangle$ states will give their probabilities for occurring in a mixture. For the input qubit state

$$|\psi_{in}\rangle = a|0\rangle + b|1\rangle,$$ \hspace{3cm} (8)

all possible final states occurring with probabilities $\mathcal{O}(\gamma)$ or above are

$$|\phi_{0000}\rangle = a \frac{|0000\rangle + (1-\gamma)^2 |1111\rangle}{\sqrt{2}}$$ \hspace{3cm} (9)

$$+ b \frac{(1-\gamma)(|0011\rangle + |1100\rangle)}{\sqrt{2}}$$

$$|\phi_{1000}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}} \left[ a(1-\gamma) |0111\rangle + b|0100\rangle \right]$$

$$|\phi_{0010}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}} \left[ a(1-\gamma) |1101\rangle + b|0001\rangle \right]$$

$$|\phi_{0001}\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}} \left[ a(1-\gamma) |1110\rangle + b|0010\rangle \right].$$

The usual criteria used to study quantum codes (reviewed in section [11]) require that $\langle 0_L | A_k^\dagger A_k | 0_L \rangle = \langle 1_L | A_k^\dagger A_k | 1_L \rangle$, but

$$\langle 0_L | A_{0000}^\dagger A_{0000} | 0_L \rangle = 1 - 2\gamma + 3\gamma^2 + \mathcal{O}(\gamma^3)$$ \hspace{3cm} (10)

$$\langle 1_L | A_{0000}^\dagger A_{0000} | 1_L \rangle = 1 - 2\gamma + \gamma^2.$$ \hspace{3cm} (11)

So the code we have constructed does not satisfy the usual criteria. We will demonstrate that the code satisfies new approximate error correction conditions later on, and revisit the recovery procedure afterwards. First, we exhibit how it works.

A. Decoding and Recovery Circuit

Let us denote each of the four qubits by $|n_1\rangle, \ldots, |n_4\rangle$. Error correction is performed by distinguishing the five possible outcomes of Eq. (9), and then applying the appropriate correction procedure. The first step is syndrome calculation, which may be done using the circuit shown in Fig. 2A. There are three possible measurement results from the two meters: $(M_2, M_4) = (0,0), (1,0)$ and $(0,1)$. Conditioned on $(M_2, M_4)$, recovery processes $W_k$ implemented by the other three circuits of Fig. 2 can be applied to the output $|n_1 n_3\rangle$ from the syndrome calculation circuit.

FIG. 2. (A) Circuit for error syndrome detection. The measurement result is used to select $W_k$ out of three actions. If the result $(M_2, M_4)$ is 00, 10, or 01, circuits (B), (C), or (D) are applied, respectively, to recover the state. The angles $\theta, \theta'$ are given by $\tan \theta = (1-\gamma)^2$ and $\cos \theta' = 1 - \gamma$. The rotation gate and controlled-rotation gate specified by the angle $\theta$ perform the functions $\exp(i \theta \sigma_y)$ and $\Lambda_1(\exp(i \theta \sigma_y))$ respectively in the notation of [14].
\[
\text{If } (M_2, M_4) = (0, 0), \text{ then } |n_1n_3\rangle \text{ is:}
\]
\[
a \left[ \frac{|00\rangle + (1 - \gamma)^2|11\rangle}{\sqrt{2}} \right] + b \left[ \frac{(1 - \gamma)(|01\rangle + |10\rangle)}{\sqrt{2}} \right]. \tag{12}
\]

To regenerate the original qubit, the circuit of Fig. 2B is used: a controlled-not is applied using \(n_4\) as the control, giving
\[
a|0\rangle \left[ \frac{|0\rangle + (1 - \gamma)^2|1\rangle}{\sqrt{2}} \right] + b|1\rangle \left[ \frac{(1 - \gamma)(|1\rangle + |0\rangle)}{\sqrt{2}} \right]. \tag{13}
\]

\(|n_1\rangle\) can now be used as a control to rotate \(|n_3\rangle\) to be parallel to \(|0\rangle\). We obtain as the final output:
\[
|n_1n_3\rangle = a \sqrt{(1 - \gamma)^4 + \frac{1}{4}} |0\rangle + b(1 - \gamma)|1\rangle \left| 0 \right\rangle \tag{14}
\]
\[
= \left[ (1 - \gamma)(a|0\rangle + b|1\rangle) + \mathcal{O}(\gamma^2)|0\rangle \right] \left| 0 \right\rangle, \tag{15}
\]

with the corrected and decoded qubit left in \(|n_1\rangle\) as desired.

If \((M_2, M_4) = (1, 0)\), the inferred state before syndrome measurement is \(\phi_{1000}\) or \(\phi_{0100}\). In either case, \(|n_1n_3\rangle\) is in a product state and the third qubit has the distorted state:
\[
|n_3\rangle = \sqrt{(1 - \gamma)^2} \left[ a(1 - \gamma)|1\rangle + b|0\rangle \right]. \tag{16}
\]

To undo the distortion, we apply the non-unitary transformation in Fig. 2C. The combined operation on \(|n_3\rangle\) due to the NOT gate, the controlled-rotation gate and the measurement of the ancilla bit can be expressed in the operator sum representation: \(\mathcal{N}(\rho) = N_0\rho N_0^\dagger + N_1\rho N_1^\dagger\), where
\[
N_0 = |0\rangle\langle 1| + (1 - \gamma)|1\rangle\langle 0|, \quad N_1 = \sqrt{\gamma(2 - \gamma)}|1\rangle\langle 1|.
\]
\[
\tag{17}
\]

The \(N_0\) and \(N_1\) operators correspond to measuring the ancilla to be in the \(|0\rangle\) and \(|1\rangle\) states respectively. If the ancilla state is \(|0\rangle\), we obtain the state:
\[
|\psi_{\text{out}}\rangle = \sqrt{(1 - \gamma)^4 + \frac{1}{4}} \left[ a|0\rangle + b|1\rangle \right], \tag{18}
\]
in the third mode because \(N_0\) preferentially damps out the \(|b\rangle\) component in Eq. (14). We get an error message if the ancilla is in the \(|1\rangle\) state. Finally, if \((M_2, M_4) = (0, 1)\) the same procedure can be applied as in the \((M_2, M_4) = (1, 0)\) case, with \(n_1\) and \(n_3\) swapped.

The fidelity, defined as the worst (over all input states) possible overlap between the original qubit and the recovered qubit is
\[
\mathcal{F} = (1 - \gamma)^2 + 4 \frac{(1 - \gamma)^3}{2} \gamma \tag{19}
\]
\[
= 1 - 5\gamma^2 + \mathcal{O}(\gamma^3), \tag{20}
\]

Note that the final state Eq. (17) is slightly distorted. This occurs because the recovery operation is not exact, due to the failure to satisfy the code criteria exactly. Furthermore, the circuits in Fig. 2C and 2D have a finite probability for failure. However, these are second order problems, and do not detract from the desired fidelity order.

III. APPROXIMATE SUFFICIENT CONDITIONS

We now explain why our code works despite its violation of the usual error correction criteria. The reason is simple: small deviations from the criteria are allowed as long as they do not detract from the desired fidelity order. This section presents a simple generalization of the usual error correction criteria which makes this idea mathematically concrete. These approximate error correction criteria are sufficient to do approximate error correction. We expect that a more complete theory giving necessary and sufficient conditions to do approximate error correction is possible, but have not obtained such a theory. Nevertheless we hope that the simple sufficient conditions presented here will inspire other researchers to develop a general theory of approximate error correction.

Quantum error correction is performed by encoding logical basis states in a subspace \(\mathcal{C}\) of the total Hilbert space \(\mathcal{H}\) so that some effects of \(\mathcal{E}\) can be reversed on \(\mathcal{C}\). Let the noise process be described in some operator sum representation \(\mathcal{E}(\rho) = \sum_{n,m} \tilde{A}_m \rho \tilde{A}_m^\dagger\), where \(\mathcal{K}\) is the index set of \(\mathcal{A}\), the set of all effects \(A_n\) appearing in the sum. We denote by \(\mathcal{A}_{re} \subseteq \mathcal{A}\) the reversible subset on \(\mathcal{C}\), and let \(\mathcal{K}_{re} = \{ n | A_n \in \mathcal{A}_{re} \}\) be the index set of \(A_{re}\). In other words, the process \(\mathcal{E}'(\rho) = \sum_{n \in \mathcal{K}_{re}} A_n \rho A_n^\dagger\) is reversible on \(\mathcal{C}\). \(\mathcal{A}_{re}\) includes all the effects satisfying the condition
\[
P_C \tilde{A}_m \tilde{A}_n P_C = g_{mn} P_C \quad \forall m, n \in \mathcal{K}_{re}, \tag{21}
\]
where \(P_C\) is the projector onto \(\mathcal{C}\), and \(g_{mn}\) are entries of a positive matrix. When Eq. (21) is true for some subset of effects \(A_n\) which form an operator-sum representation for the operation \(\mathcal{E}\), there exists some operator-sum representation of \(\mathcal{E}\) such that \(\mathcal{E}'(\rho) = \sum_{n \in \mathcal{K}_{re}} A_n \rho A_n^\dagger\) and:
\[
P_C \tilde{A}_m^\dagger \tilde{A}_n P_C = p_{mn} \delta_{mn} P_C \quad \forall m, n \in \mathcal{K}_{re}, \tag{22}
\]
where \(p_n\) are non-negative c-numbers. Eq. (22) (or equivalently Eq. (21)) must be satisfied for some subset of effects \(A_n\) \((A_n)\) which form an operator-sum representation for the operation \(\mathcal{E}\). We emphasize that the \(A_n\) operators in Eq. (21) and the \(A_n\) operators in Eq. (22) are not
necessarily the same, and similar distinction holds for the reversible subsets. For simplicity of notation, we will drop the tilde even when we refer to the operators and reversible subset in Eq. (22).

Eq. (22) is equivalent to the usual (exact) orthogonality and non-deformation conditions for a non-degenerate code with logical states $|c_i⟩$, and these conditions can be stated concisely as: $\forall i, j$

$$\langle c_i | A_m^\dagger A_n | c_j⟩ = \delta_{ij} \delta_{mn} p_n \quad \forall m, n \in K_{rc}$$  \hspace{1cm} (23)

where $p_n$ are the non-negative c-numbers in Eq. (21), and they represent the error detection probability of the effects $A_n$.

When Eq. (23) is satisfied, the effects $A_n$ have polar decompositions

$$A_n P_C = \sqrt{p_n} U_n P_C \quad \forall n \in K_{rc}, \hspace{1cm} (24)$$

where the $U_n$’s are unitary. Note that $P_C U_n^\dagger U_m P_C = \delta_{mn} P_C$ is required for syndrome detection to be unambiguous. This is simple to see by writing the recovery operation $\mathcal{R}$ as

$$\mathcal{R}(\rho) = \sum_{k \in K_{rc}} R_k \rho R_k^\dagger + P_E \rho P_E, \hspace{1cm} (25)$$

where $R_k = P_C U_k^\dagger$ is the appropriate reversal process for each effect $A_k$ ($k \in K_{rc}$), and $P_E = I - \sum_{k \in K_{rc}} U_k P_C U_k^\dagger$.

When we apply $\mathcal{R}$ on $\mathcal{E}(|ψ_{in}⟩⟨ψ_{in}|)$, where $|ψ_{in}⟩$ is a pure input state, the first term in Eq. (23) is proportional to $|ψ_{in}⟩⟨ψ_{in}|$, and represents the recovered state. $\mathcal{R}(\rho)$ is trace preserving as a result of orthogonality of different $U_k P_C$.

It is convenient to define $P_{det} = \sum_{n \in K_{rc}} p_n$ to be the total detection probability for the reversible subset. It is immediate that to achieve a desired fidelity $\mathcal{F}$, we have to include in $A_{rc}$ a sufficient number of highly probable effects so that $P_{det} \geq \mathcal{F}$. Thus, $A_{rc}$ is also a high probability subset.

Now we generalize Eq. (22)-(24) based on the following assumption: the error is parameterized by a certain number of small quantities with physical origins such as the strength and duration of the coupling between the system and the environment. For simplicity, we consider only one-parameter processes, and let $\epsilon$ be the small parameter. For example, $\epsilon$ can be the single qubit error probability. Suppose the aim is to find a code for a known $\mathcal{E}$ with fidelity:

$$\mathcal{F} \geq 1 - \mathcal{O}(\epsilon^{t+1}). \hspace{1cm} (26)$$

In the new criteria, it is still necessary that $P_{det} \geq \mathcal{F}$, that is, $A_{rc}$ has to include all effects $A_n$ with maximum detection probability $\max_{|ψ_{in}⟩ \in C} \text{tr}(⟨ψ_{in}| A_m^\dagger A_n |ψ_{in}⟩) \approx \mathcal{O}(\epsilon^t)$, $s \leq t$. However, it is not necessary to recover the exact input state; only a good overlap between the input and output states is needed. In terms of the condition on the codeword space, it suffices for the effects to be approximately unitary and mutually orthogonal. These observations can be expressed as relaxed sufficient conditions for error correction. Suppose

$$A_n P_C = U_n \sqrt{P_C A_n^\dagger A_n} P_C, \hspace{1cm} (27)$$

is a polar decomposition for $A_n$. We define c-numbers $p_n$ and $λ_n$ so that $p_n$ and $p_n λ_n$ are the largest and the smallest eigenvalues of $P_C A_n^\dagger A_n P_C$, considered as an operator on $\mathcal{C}$. The relaxed conditions for error correction are that:

$$p_n (1 - λ_n) \leq \mathcal{O}(\epsilon^{t+1}) \quad \forall n \in K_{rc} \hspace{1cm} (28)$$

$$P_C U_n^\dagger U_n P_C = \delta_{mn} P_C \hspace{1cm} (29)$$

Note that when $λ_n = 1$, Eq. (27) - (29) reduce to the exact criteria. In the approximate case, $P_{det} = \sum_{n \in K_{rc}} \text{tr}(⟨ψ_{in}| A_n^\dagger A_n |ψ_{in}⟩)$ is not a constant, but depends on the input state $|ψ_{in}⟩$. Since $A_{rc}$ includes enough efficient operators so that $P_{det} \geq 1 - \mathcal{O}(\epsilon^{t+1})$, when Eq. (28) is satisfied, we also have $\sum_{n \in K_{rc}} p_n \geq 1 - \mathcal{O}(\epsilon^{t+1})$ and

$$\sum_{n \in K_{rc}} p_n λ_n \geq 1 - \mathcal{O}(\epsilon^{t+1}). \hspace{1cm} (30)$$

We now prove that $\sum_{n \in K_{rc}} p_n λ_n$ is a lower bound on the fidelity. Defining the residue operator

$$π_n = \sqrt{P_C A_n^\dagger A_n} P_C - \sqrt{p_n λ_n} P_C, \hspace{1cm} (30)$$

we find, for the operator norm of $π_n$,

$$0 \leq |π_n| \leq \sqrt{p_n - p_n λ_n}, \hspace{1cm} (31)$$

and

$$A_n P_C = U_n (\sqrt{p_n λ_n} I + π_n) P_C. \hspace{1cm} (32)$$

The sufficiency of our conditions to obtain the desired fidelity may be proved as follows. Though Eq. (23) is not satisfied, as long as Eq. (24) is true, we can still define the approximate recovery operation

$$\mathcal{R}(\rho) = \sum_{k \in K_{rc}} R_k \rho R_k^\dagger + P_E \rho P_E \hspace{1cm} (33)$$

with $R_k = P_C U_k^\dagger$ as the approximate recovery operation for $A_k$ and $P_E$ defined as in the case of exact error correction. For a pure input state $|ψ_{in}⟩⟨ψ_{in}|$, applying $\mathcal{R}(\rho)$ on $\mathcal{E}(|ψ_{in}⟩⟨ψ_{in}|)$, and ignoring the last term which is positive definite produces an output with fidelity

$$\mathcal{F} \equiv \min_{|ψ_{in}⟩ \in C} \text{tr}(⟨ψ_{in}| \mathcal{E}(|ψ_{in}⟩⟨ψ_{in}|)) \hspace{1cm} (34)$$

Omitting all terms for which $k ≠ n$ and applying Eq. (32) gives
\[ F \geq \min_{|\psi_{in}\rangle \in C} \sum_{n \in K_{re}} |\langle \psi_{in} | \sqrt{P_n A_n} + \pi_n | \psi_{in} \rangle|^2 \]
\[ \geq \sum_{n \in K_{re}} p_n \lambda_n , \quad (35) \]

where in the last step, we have used Eq. (31). Hence, the fidelity is at least \( \sum_{n \in K_{re}} p_n \lambda_n \geq 1 - O(e^{t+1}) \) and the desired fidelity order is achieved as claimed.

An explicit procedure for performing this recovery is as follows. First, a measurement of the projectors \( P_k \equiv U_k P_C U_k^\dagger \) is performed. Conditional on the result \( k \) of the measurement, the unitary operator \( U_k \) is applied to complete the recovery.

Note that when the exact criteria hold, it is not necessary for the set \( A \) to form an operator sum representation for the error process; they may instead give a generalized description of the quantum process such as

\[ \mathcal{E}(\rho) = \sum_{m,n} A_{mn} \rho A_{mn}^\dagger , \quad (36) \]

where \( \chi_{mn} \) is a matrix of c-numbers \([5]\). In this case, approximate criteria can be obtained straightforwardly along the lines we have described for the operator sum representation (where \( \chi_{mn} \) is diagonal). The main issue is to ensure that interference terms coming off-diagonal in \( \chi_{mn} \) are sufficiently small. We will not describe this calculation here. Instead, we return to our four-bit code, and analyze it using the new criteria.

**IV. 4-BIT CODE REVISITED**

In terms of the approximate quantum error correction criteria Eqs. (28-29), we may understand why our four-bit amplitude damping code works as follows. Let the orthonormal basis for the Hilbert space be ordered as:

\[ |0000\rangle, |0011\rangle, |1100\rangle, |1111\rangle, |0111\rangle, |0100\rangle, \ldots \quad (37) \]

We include only the first few non-zero rows and columns in our matrices. The projection operator \( |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L| \) onto the two dimensional codeword space \( C \) is

\[ P_C = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad (38) \]

The first effect operator (no loss of a quantum to the environment) is

\[ A_{0000} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \gamma & 0 & 0 \\ 0 & 0 & 1 - \gamma & 0 \\ 0 & 0 & 0 & (1 - \gamma)^2 \end{bmatrix} \quad (39) \]

The eigenvalues of \( P_C A_{0000}^\dagger A_{0000} P_C \) are \((1 - \gamma)^2\) and \(\frac{1}{2}(1 + (1 - \gamma)^4)\). Interested readers can check for themselves that

\[ A_{0000} P_C = U_{0000} \left( (1 - \gamma) I + (\gamma^2 + O(\gamma^4)) \tilde{\pi}_{0000} \right) P_C \quad (40) \]

(the order of \( \gamma \) in \( \pi_{0000} \) is factored out of \( \tilde{\pi}_{0000} \)) with the choice:

\[ U_{0000} = \begin{bmatrix} \cos(\theta - \frac{\pi}{2}) & 0 & 0 & -\sin(\theta - \frac{\pi}{2}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin(\theta - \frac{\pi}{2}) & 0 & 0 & \cos(\theta - \frac{\pi}{2}) \end{bmatrix} \quad (41) \]

\[ \tilde{\pi}_{0000} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (42) \]

where \( \tan \theta = (1 - \gamma)^2 \). The exact criteria are not satisfied, as \( P_C A_{0000}^\dagger A_{0000} P_C \) has different eigenvalues. However, the difference is of order \( O(\gamma^2) \) and thus the relaxed condition Eq. (28) is satisfied.

For the second effect (loss of one quantum from the \( n_1 \) mode), we have

\[ A_{1000} = (1 - \gamma) \frac{1}{2} \sqrt{\gamma} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (43) \]

The eigenvalues of \( P_C A_{1000}^\dagger A_{1000} P_C \) are \( \gamma(1 - \gamma) \) and \( \gamma(1 - \gamma)^3 \). The difference is \( (2\gamma^2 - \gamma^3)(1 - \gamma) \). We have the decomposition

\[ A_{1000} P_C = \sqrt{(1 - \gamma)^2} U_{1000} \left( (1 - \gamma) I + \gamma \tilde{\pi}_{1000} \right) P_C \quad (44) \]

\[ U_{1000} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (45) \]

\[ \tilde{\pi}_{1000} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

Other one loss cases are similar. The \( U_k P_C \) have non-zero entries in different rows, and are orthogonal to each
other. Hence, all the approximate code criteria are satisfied. Using these explicit matrices, we obtain

\[ R(\mathcal{E}(|\psi_{in}\rangle\langle\psi_{in}|)) \approx \sum_{k \in K_{\text{re}}} P_{C} U_{k} A_{k} |\psi_{in}\rangle \langle\psi_{in}| A_{k}^{\dagger} U_{k} P_{C} \]

\[ = (1 - 3\gamma^{2}) |\psi_{in}\rangle \langle\psi_{in}| + \ldots \]  

(46)

The fidelity is thus at least 1 - 3\gamma^{2}, and is of the desired order.

The recovery procedure suggested in Eq. (46) contrasts with the decoding and recovery circuits in section 4.1. It is an interesting exercise to check that the composition of the operations in Fig. 2A and 2B, followed by re-encoding the recovered qubit has the same effect on \( \mathcal{C} \) as applying \( U_{1000} \) for recovery. For the case in which an emission occurs in the first qubit, the composition of operations in Fig. 2A and 2C, followed by re-encoding the recovered qubit has the same effect on \( \mathcal{C} \) as preferentially damp out the \( |n_{3}\rangle = |0\rangle \) component followed by applying \( U_{1000} \) for recovery. Note that it costs 2\gamma^{2} in the fidelity for removing the distortion.

V. APPLICATIONS TO OTHER CODES

Our approximate criteria may also be used to simplify code construction using a non-Pauli error description basis. For example, consider the bosonic quantum codes for amplitude damping \([10]\) (these are codes which utilize bosonic states \( |0\rangle, \ldots |n\rangle \) instead of qubits). For logical states \(|c_{1}\rangle, |c_{2}\rangle, \ldots \) of the form:

\[ |c_{i}\rangle = \sqrt{\mu_{1}} |n_{11}n_{12} \ldots n_{1m}\rangle \]

\[ + \sqrt{\mu_{2}} |n_{21}n_{22} \ldots n_{2m}\rangle \]

\[ + \ldots \]

\[ + \sqrt{\mu_{n}} |n_{N1}n_{N2} \ldots n_{Nm}\rangle, \]

(47)

the original non-deformation conditions for correcting up to one photon loss require the following to be constant for all logical states:

\[ \langle c_{i}| A_{0}^{1} A_{0}^{2} |c_{i}\rangle = \sum_{i=1}^{N_{\gamma}} (1 - \gamma)^{R_{S_{i}}} \mu_{i} \]

(48)

\[ \langle c_{i}| A_{0}^{1} \ldots A_{0}^{2} \ldots |c_{i}\rangle = \sum_{i=1}^{N_{\gamma}} (1 - \gamma)^{R_{S_{i}} - 1} \gamma \mu_{i} n_{ij}. \]

(49)

In the above, \( R_{S_{i}} = \sum_{j=1}^{m} n_{ij} \) is the total number of excitation in the \( i^{th} \) quasi-classical state (QCS) in \( |c_{i}\rangle \). It is difficult to find a solution for Eq. (48) when \( R_{S_{i}} \) is not constant for all \( i \). That is the reason why previous work \([13]\) suggests using only QCS with the same number of excitations.

With the new criteria, it suffices for the QCS to be constant for all logical states:

\[ \sum_{i=1}^{N_{\gamma}} \gamma \mu_{i} n_{ij} \quad \forall j. \]

(50)

That is, instead of requiring the individual number of excitations in each QCS to be balanced, it is sufficient to balance the average number of excitations over the QCS in each codeword. This provides an alternative explanation of why the known five-bit perfect quantum codes \([3,16]\)

\[ |0_{L}\rangle = |00000\rangle + |11000\rangle - |10011\rangle - |01111\rangle \]

\[ + |11010\rangle + |00110\rangle + |01101\rangle + |10101\rangle \]

\[ |1_{L}\rangle = |11111\rangle - |00011\rangle + |01100\rangle - |10000\rangle \]

\[ - |00101\rangle + |11010\rangle + |10010\rangle - |01010\rangle \]

(51)

work for amplitude damping errors (as described in Eq. (48)); although the codes do not satisfy the exact non-deformation criterion, Eq. (48) for the error representation of Eqs. (44), they do satisfy the approximate ones leading to Eq. (50).

VI. CONCLUSION

We have shown by an example that choosing an appropriate error basis can potentially reduce the number of qubits and other requirements in coding schemes. We also suggested a method to enable code construction without the Pauli basis to be done more easily. We believe that much more can be done along these two directions. The new sufficient criteria for error correction are far from being necessary. Our results show that it is worthwhile to look for better approximate conditions or conditions which are both necessary and sufficient. Approximate error correction is particularly interesting because it is a property with no classical analogue, as it makes use of slight non-orthogonality possible only between quantum states. It would be especially useful to develop a general framework for constructing codes based on approximate conditions, similar to the group-theoretic framework now used to construct codes which satisfy the exact conditions.

ACKNOWLEDGMENTS

We thank David DiVincenzo and John Preskill, for useful and enjoyable discussions about quantum error-correction. MAN would especially like to thank Howard Barnum and Carlton Caves for explaining the error-correction conditions to him. This work was supported in part by the Office of Naval Research (N00014-93-1-0116). We thank the Institute for Theoretical Physics for its hospitality and for the support of the National Science Foundation (PHY94-07194). MAN acknowledges
financial support from the Australian-American Educational Foundation (Fulbright Commission). DWL was supported in part by the Army Research Office under grant no. DAAH04-96-1-0299.

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