Irregular conformal block and its matrix model

Chaiho Rim

Department of Physics and Center for Quantum Spacetime (CQUeST)
Sogang University, Seoul 121-742, Korea

Abstract

Irregular conformal block is a new tool to study Argyres-Douglas theory, whose irregular vector is represented as a simultaneous eigenstate of a set of positive Virasoro generators. One way to find the irregular conformal block is to use the partition function of the $\beta$-ensemble of hermitian matrix model. So far the method is limited to the case of irregular singularity of even degree. In this letter, we present a new matrix model for the case of odd degree and calculate its partition function. The model is different from the previous one in that its potential has additional factor of square root of matrix.

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*email: rimpine@sogang.ac.kr
1 Introduction

Irregular conformal block (ICB) is obtained from the conformal block by taking the colliding limit which puts some of primary fields at the same point maintaining its limit finite [1]. ICB generalizes the AGT relation [2] and is considered to reproduce [1, 3] the Argyres-Douglas (AD) theory [4].

ICB is first obtained as the eigenstate of Virasoro generator $L_1$ or $(L_1, L_2)$ in [5]. The eigenstate is not annihilated by $L_k$ with $k > 0$ but is an eigenstate. This coherent state is called irregular vector. This idea is generalized so that the irregular vector is the simultaneous eigenstate of $L_k$’s ($0 < n \leq k \leq 2n$) and is constructed in terms of the free field representation in the presence of background charge in [1]. In fact, this ICB is described by the Seiberg-Witten curve $x^2 = \phi_2$ where $\phi_2$ is a quadratic differential with a pole of degree $2n + 2$, which is associated to a punctured Riemann sphere and is classified as $D_{2n}$ of AD theory [7].

On the other hand, it is noted in [6] that ICB is conveniently obtained from the hermitian matrix model whose potential consists of inverse powers of matrix and logarithmic one. Its partition function is identified with the inner product of an irregular vector at the origin and a regular one at infinity. Because this matrix model can be obtained as the appropriate limit of the Penner-type matrix model [8, 9, 10], this matrix model approach shares the same idea as that of the colliding limit of the regular conformal block.

The matrix model is, however, so far limited to the case of even degree. The presence of the pole of degree $2n + 1$ requires the irregular state annihilated by $L_{2n}$ but simultaneous eigenstates of $L_k$ with $0 < n \leq k \leq 2n - 1$. This ICB may be constructed in terms of the twisted free field modes which introduce a new square root branch-cut at the origin as shown in [1].

In this letter, we present a new matrix model with square root potential and calculate the corresponding partition function. In section 2 we define the matrix model and obtain the loop equation. The square root potential not only introduces the branch-cut at the origin but also changes the details of the loop equation. In section 3 we provide the solution to the simplest non-trivial case namely, the case $n = 2$ (or $D_3$) explicitly. It is straightforward to generalize to the case with $n > 2$. Section 4 is for discussion and a detailed derivation of the loop equation is provided in appendix A.
2 Partition function and loop equation

We consider the partition function of a hermitian matrix

\[ Z_M = \int \prod_{I=1}^{N} dz_I \Delta(\lambda_I)^{2\beta} e^{\frac{g}{\beta} \sum \lambda_I V(\lambda_I)} \]  \hspace{1cm} (2.1) 

where \( \Delta(\lambda_I) \) is the Vandermonde determinant. \( \lambda_I \) is the eigenvalue of the matrix and \( N \) is the size of the matrix. The potential has the form

\[ V(z) = \alpha \log z - \sum_{s \in S} c_s z^s \]  \hspace{1cm} (2.2) 

where \( S = \{1/2, 3/2, \ldots, n-1/2\} \) is a finite set of \( n \) half-odd integers. It is convenient to denote the potential as \( V(z) = V_e(z) + V_o(z) \) where \( V_e(z) = \alpha \log z \) and \( V_o(z) = -\sum_{s \in S} c_s z^s \) so that \( V_o(z) \) has a cut \( \Gamma \) along the negative real axis. We assume that there is an appropriate domain of the parameter space \( \{\alpha, c_S\} \) so that the partition function is well defined when the integration range lies along the positive real axis.

The saddle point equation holds on the positive real axis

\[ 2w(\lambda_I) + V'(\lambda_I) = 0, \quad w(\lambda_I) = g\sqrt{\beta} \sum_{J \neq I} \left( \frac{1}{\lambda_I - \lambda_J} \right). \]  \hspace{1cm} (2.3) 

A nice way to find the functional relation is to introduce the resolvent \[ \mathbb{1} \], \( W(z) = g\sqrt{\beta} \langle \sum_{I} \frac{1}{z-\lambda_I} \rangle \). Here \( \langle \cdots \rangle \) stands for the expectation value with the given potential \( V(z) \). The saddle point equation (2.3) shows that the resolvent is discontinuous along a certain integration range \( \Lambda \) of the partition function

\[ W(\xi + i0) + W(\xi - i0) + V'(\xi) = 0 \quad \text{when} \quad \xi \in \Lambda. \]  \hspace{1cm} (2.4) 

This discontinuity is encoded in terms of \( G(z) \equiv 2W(z) + V'(z) \) so that \( G(\xi - i\epsilon) + G(\xi + i\epsilon) = 0 \) is automatically satisfied on the square-root branch cut. Furthermore, \( G(z) \) is not continuous on the branch cut \( \Gamma \) and consists of the monodromy even and odd parts.

The resolvent satisfies the loop equation of the form

\[ f(z) = 4W(z)^2 + 4V'(z)W(z) + 2\hbar QW'(z) - \hbar^2 W(z, z). \]  \hspace{1cm} (2.5) 

where we switch the notation from \( g \) and \( \beta \) to \( \sqrt{\beta} = -ib \), \( \hbar = -2ig \) and \( Q = b + 1/b \). \( W(z_1, z_2) = \beta \langle \sum_{I_1, I_2} \frac{1}{z_1 - \lambda_{I_1}} \frac{1}{z_2 - \lambda_{I_2}} \rangle_{\text{conn}} \) is a two-point (connected) resolvent and \( f(z) \) is the quantum correction defined as \( f(z) = 2\hbar b \sum_{I=1}^{N} \langle \frac{V'(z) - V'(\lambda_I)}{z - \lambda_I} \rangle \). As shown in appendix \[ \mathbb{A} \] its explicit form is

\[ f(z) = \sum_{k=0}^{n-1} \frac{d_k}{z^{2+k}} + \frac{4V'(z)}{\sqrt{z}} \tau(z) \]  \hspace{1cm} (2.6)
where \( d_k = v_k(-\hbar^2 \log Z_M) \) and \( v_k = \sum_{s \in S} S c_{s+k} \frac{\partial}{\partial c_s} \). Here we use the definition \( c_s = 0 \) if \( s \) does not belong to the set \( S \). \( \tau(z) = \frac{\hbar b}{2} \sum_I \left( \frac{1}{\sqrt{z+\Delta_I}} \right) \) is a new analytic function, continuous on \( \Lambda \) but discontinuous on \( \Gamma \).

## 3 Solution to the lowest order in \( \hbar \)

The coefficient \( d_k \) in \((2.6)\) is the key element to find the partition function. Once \( d_k \)'s are known, one can find the partition function using the differential equation with respect to the parameters of the potential. Our strategy is to find \( d_k \)'s from the loop equation order by order in \( \hbar \). To make the expansion feasible, we assume that \( V'(z) \) is the order of \( \hbar \) so that the expansion is equivalent to the large \( N \) expansion (\( \hbar \propto 1/N \)). This is achieved if \( \alpha \) and \( c_S \) are proportional to \( \hbar \).

To the lowest order in \( \hbar \) (planar limit), \( G(z) \) satisfies the loop equation \((2.5)\)

\[
G(z)^2 - 2V'_o(z)H(z) = \varphi_2(z)
\]  
(3.1)

where \( H(z) = 2\tau(z)/\sqrt{z} + V'_e(z) \) and \( \varphi_2(z) = \sum_{k=0}^{n-1} \frac{d_k}{z+k} + V'_e(z)^2 + V'_o(z)^2 \) is the expectation value of the energy-momentum tensor which has the pole of odd degree \( 2n+1 \). At large \( z \), the dominant term of \((3.1)\) is the order of \( 1/z^2 \) and its coefficient provides an identity

\[
(hbN+\alpha)^2 = d_0 + \alpha^2.
\]  
(3.2)

For the simplest case, \( n = 1 \), \( d_0 = \frac{1}{2}c_{1/2}^2 \frac{\partial}{\partial c_{1/2}} (-\hbar^2 \log Z_M) \) determines the partition function, \( Z_M = \zeta \left( c_{1/2} \right)^{-2(hbN)^2 + 2bN_\alpha/\hbar} \). \( \zeta \) is a \( c_{1/2} \)-independent constant and can be absorbed to the definition of the partition function. Furthermore, one has \( G(z) = \frac{hbN+\alpha}{z} + \frac{c_{1/2}^2}{z^{1/2}} \) and \( H(z) = \frac{hbN+\alpha}{z} \) which shows no no branch cut \( \Lambda \). In fact, \( \Lambda \) reduces to the origin \( z = 0 \) as seen from the \( n = 2 \) case below (if \( c_{3/2} \) is turned off).

When \( n \geq 2 \), the solution is more involved. It is natural to assume there are \( n-1 \) cuts \( \Lambda_k \)'s each of which lies between \( a_k \) and \( b_k \) (\( 0 < a_k < b_k \)) on the positive real axis. In addition, according to the monodromy at \( z = 0 \), one may put \( G(z) = G_e(z) + G_o(z) \);

\[
G_e(z) = \frac{g_e(z)}{z^n} \prod_{k=0}^{n-1} \sqrt{(z-a_k)(z-b_k)}, \quad G_o(z) = \frac{g_o(z)}{z^{n+1/2}} \prod_{k=0}^{n-1} \sqrt{(z-a_k)(z-b_k)}.
\]  
(3.3)

\( g_e(z) \) and \( g_o(z) \) are holomorphic functions whose large \( z \) behavior is \( g_e(z) = O(z^0) \) and \( g_o(z) = O(z^0) \). The loop equation \((3.1)\) is split into two parts

\[
G_e(z)^2 + G_o(z)^2 = \varphi_2(z), \quad G_e(z)G_o(z) = V'_o(z)H(z).
\]  
(3.4)

The way of splitting is consistent with the fact that \( H(z) \) has no cut along \( \Lambda \).
For $n = 2$, the non-trivial simplest case, the partition function satisfies the differential equations $d_0 = v_0(-\hbar^2 \log Z_M)$ and $d_1 = v_1(-\hbar^2 \log Z_M)$. It is convenient to re-parametrize as $t = c_3/2/c_{1/2}^3$ and $x = c_{3/2}$ and regard the partition function as the function of $t$ and $x$. The merit of parametrization is that any function of $t$ is the homogeneous solution of $v_0$ since $v_0(t) = 0$. Employing the fact that $v_0(x) = 3x/2$, $v_1(x) = 0$ and $v_1(t) = -(3/2)t^2c_{1/2}^2$, one has the differential equations

$$d_0 = -\frac{3\hbar^2}{2} x \frac{\partial}{\partial x} (\log Z_M), \quad \delta_1 \equiv d_1/c_{1/2}^2 = \frac{3\hbar^2}{2} t^2 \frac{\partial}{\partial t} (\log Z_M). \quad (3.5)$$

With $d_0$ in (3.2), we put log $Z_M$ as

$$\log Z_M = \frac{2}{3\hbar^2} \left(-d_0 \log x + Y(t)\right) \quad (3.6)$$

and $Y(t)$ satisfies the differential equation, $t^2 dY(t)/dt = \delta_1$. This implies that $\delta_1$ is the function of $t$ only and vanishes as $t \to 0$.

It is noted that $\varphi_2(z)$ in (3.1) is cast into the form $\varphi_2(z) = P_3(z)/z^5$ where

$$P_3(z) = (d_0 + \alpha^2)z^3 + (d_1 + c_{1/2}^2)z^2 + 2c_{3/2}c_{1/2}z + c_{3/2}^2. \quad (3.7)$$

Rescaling $z = uc_{1/2}^2$ and $P_3(z) = c_{1/2}^2Q_3(u)$, one puts $Q_3(u)$ in terms of $t$ and $x$,

$$Q_3(u) = (d_0 + \alpha^2)u^3 + (\delta_1 + 1)u^2 + 2tu + t^2$$

$$\equiv (d_0 + \alpha^2)(u - A)(u - B)(u + \gamma^2). \quad (3.8)$$

$Q(u) = 0$ has two positive roots $A$ and $B$ and one negative root $-\gamma^2$. Two positive roots are related with branch points $a = Ac_{1/2}^2$ and $b = Bc_{1/2}^2$. The negative root is related with holomorphic function $g_e(z)$ and $g_o(z)$ through the loop equation (3.4)

$$g_e(z)^2u + g_o(z)^2/c_{1/2}^2 = (d_0 + \alpha^2)(u + \gamma^2). \quad (3.9)$$

Noting that $g_e(z)^2$ and $g_o(z)^2$ are even degree of $u$, we conclude that they are constant and $g_e(z) = \sqrt{d_0 + \alpha^2}$ and $g_o(z) = c_{1/2}\gamma\sqrt{d_0 + \alpha^2}$.

The relation between three roots are obtained from (3.8)

$$t^2 = (d_0 + \alpha^2)\gamma^2 AB, \quad 2t = (d_0 + \alpha^2)(AB - (A + B)\gamma^2)$$

$$\delta_1 + 1 = (d_0 + \alpha^2)(\gamma^2 - (A + B)). \quad (3.10)$$

In addition, $H(z)$ in (3.1) is the order of $1/z$ whose coefficient provides an additional identity; $\hbar bN + \alpha = \gamma(d_0 + \alpha^2)$ or $\gamma(\hbar bN + \alpha) = 1$. This together with (3.10) solves the algebraic equations

$$AB = t^2, \quad A + B = -2t + t^2(d_0 + \alpha^2)$$

$$\delta_1 = 2(d_0 + \alpha^2)t - (d_0 + \alpha^2)^2t^2. \quad (3.11)$$
Therefore, \( Y(t) \) in (3.6) is determined to the lowest order in \( \hbar \) to give

\[
\log\left( \frac{Z_M}{\zeta} \right) = \frac{2}{3\hbar^2} \left( -d_0 \log x + 2(d_0 + \alpha^2) \log t - (d_0 + \alpha^2)^2 t \right)
\]

\[
Z_M/\zeta = \left( \frac{c_{3/2}}{2} \right)^{2((bN+\hat{\alpha})^2+\hat{\alpha}^2)/3} \left( \frac{c_{1/2}}{2} \right)^{4(bN+\hat{\alpha})^2} \exp\left( -\frac{2c_{3/2}(bN+\hat{\alpha})^4}{3c_{1/2}^3} \right)
\]

where \( \hat{\alpha} = \alpha/\hbar \).

4 Discussion and comments

We presented how to find the partition function of matrix model which corresponds to the irregular conformal block in the presence of the irregular singularity of odd degree. Explicit solution is obtained for \( n = 1 \) and \( n = 2 \) cases.

It is obvious to generalize to \( n > 2 \). One may have \( \varphi_2(z) = P_{2n-1}(z)/z^{2n+1} \) in (3.4) where \( P_{2n-1}(z) \) is a polynomial of degree \( (2n-1) \). The \( 2(n-1) \) positive zeros of \( P_{2n-1}(z) \) will provide the \( 2(n-1) \) branch points. The remaining factor linear in \( z \) will fix the holomorphic functions \( g_e(z) \) and \( g_o(z) \). \( d_0 \) is fixed as in (3.2) and other \( d_k \)'s are determined from the filling fractions, \( \oint_{\Lambda_k} \frac{dz}{\pi i} W(z) = \hbar b N_k \) with \( \sum N_k = N \). It is noted that the contour surrounding all \( \Lambda_k \)'s is trivial. Instead, one additional constraint comes from the \( 1/z \) behavior of \( H(z) \) in (3.4).

One may also consider more complicated potential which may contain polynomials together with inverse powers and a logarithmic one. Its partition function can be regarded as the inner product of the irregular states. Details of this consideration and the calculation of higher orders in \( \hbar \) will appear elsewhere.

Finally, it is noted that the matrix model we presented in this letter reduces to the \( O(n) \) matrix model on the random surface with \( n = -2 \) when \( \beta = 1 \) [12]. It will be interesting to understand the results in terms of \( O(n) \) matrix model.

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A  Explicit form of $f(z)$

We provide the explicit form of $f(z)$ presented in (2.6). Using the potential (2.2) one has

$$f(z) = 4g\sqrt{\beta} \sum_{I=1}^{N} \left\langle \left( \frac{V'(z) - V'(\lambda_I)}{z - \lambda_I} \right) \right\rangle$$

(A.1)

where $S = \{1/2, 3/2, \ldots, n-1/2\}$. Let us eliminate the $\alpha$ term by using the identity

$$0 = \left\langle \sum_{I=1}^{N} V'(\lambda_I) \right\rangle = \sum_{I=1}^{N} \left( \frac{\alpha}{\lambda_I} + \sum_{s} \frac{\zeta_s}{\lambda_I^{s+1}} \right)$$

(A.2)

which stands for the invariance of the shift of the integration. One may wonder if the identity may fail since in our case, the integration is along the positive real line. However, this is not the case thanks to the Vandermonde determinant because the infinitesimal shift $\epsilon$ around the origin contributes to the order of $\epsilon N(N-1)/2$ which does not contribute as $N \gg 1$.

$$f(z) = -4g\sqrt{\beta} \sum_{I,s} \left\langle \frac{c_s(z^s - \lambda_I^s)}{z^{s+1}\lambda_I^s(z - \lambda_I)} \right\rangle$$

(A.3)

Inside the sum in even $m$, one may add and subtract $1/\sqrt{z}$ to $1/(\sqrt{z} + \sqrt{\lambda_I})$ to rewrite

$$f(z) = -4g\sqrt{\beta} \sum_{I,s} \left\langle \sum_{m=even} \frac{c_s}{\lambda_I^{s-m/2} z^{m/2}} \right\rangle$$

$$+ 4g\sqrt{\beta} \sum_{I,s} \left\langle \frac{c_s}{\lambda_I^{s}(\sqrt{z} + \sqrt{\lambda_I})} \left( \sum_{m=even} \frac{\lambda_I^{(m+1)/2}}{z^{(m+1)/2}} - \sum_{m=odd} \frac{\lambda_I^{m/2}}{z^{m/2}} \right) \right\rangle$$

$$= -4g\sqrt{\beta} \sum_{I,s} \left\langle \sum_{k=integer \geq 0} \frac{1}{\lambda_I^{s-k} z^k} \right\rangle + 4g\sqrt{\beta} \sum_{I,s} \frac{c_s}{z^{s}(\sqrt{z} + \sqrt{\lambda_I})}$$

(A.4)

Re-arranging the order of the summation over $s$ and $k$, one has

$$f(z) = 4g^2 \sum_{k=0}^{n-1} v_k (\log Z_M) \frac{1}{z^{2+k}} + 4g\sqrt{\beta} \sum_{I} \frac{1}{\sqrt{z} + \sqrt{\lambda_I}} \left\langle \frac{V'(z)}{\sqrt{z}} \right\rangle$$

(A.5)

where $v_k = \sum_{s} s c_{s+k} \frac{\partial}{\partial \zeta_s}$ with the definition $c_s = 0$ if $s$ does not belong to $S$. 7
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