THE REGIONALLY-IMPlicit DISCONTINUOUS GALERKIN
METHOD: IMPROVING THE STABILITY OF DG-FEM∗

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Abstract. Discontinuous Galerkin (DG) methods for hyperbolic partial differential equations (PDEs) with explicit time-stepping schemes such as strong stability-preserving Runge-Kutta (SSP-RK) suffer from time-step restrictions that are significantly worse than what a simple Courant-Friedrichs-Lewy (CFL) argument requires. In particular, the maximum stable time-step scales inversely with the highest degree in the DG polynomial approximation space and becomes progressively smaller with each added spatial dimension. In this work we introduce a novel approach that we have dubbed the regionally implicit discontinuous Galerkin (RIDG) method to overcome these small time-step restrictions. The RIDG method is based on an extension of the Lax-Wendroff DG (LxW-DG) method, which previously had been shown to be equivalent to a predictor-corrector approach, where the predictor is a locally implicit spacetime method (i.e., the predictor is something like a block-Jacobi update for a fully implicit spacetime DG method). The corrector is an explicit method that uses the spacetime reconstructed solution from the predictor step. In this work we modify the predictor to include not just local information, but also neighboring information. With this modification we show that the stability is greatly enhanced; in particular, we show that we are able to remove the polynomial degree dependence of the maximum time-step and show how extend this to multiple spatial dimensions. A semi-analytic von Neumann analysis is presented and the resulting methods are implemented and tested in a MATLAB code that can be freely downloaded.

Key words. discontinuous Galerkin, hyperbolic conservation laws, Courant-Friedrichs-Lewy condition, time-stepping, numerical stability

AMS subject classifications. 65M12, 65M60, 35L03

1. Introduction. Hyperbolic conservation laws model phenomena characterized by waves propagating at finite speeds; examples include the shallow water (gravity waves), compressible Euler (sound waves), Maxwell (light waves), magnetohydrodynamic (magnetostatic and Alfvén waves), and Einstein (gravitational waves) equations. In recent years, the discontinuous Galerkin (DG) finite element method (FEM) has become a standard approach for solving hyperbolic conservation laws alongside other methods such as weighted essentially non-oscillatory (WENO) schemes (e.g., see Shu [17]) and various finite volume methods (e.g., see LeVeque [12]). The DG method was first introduced by Reed and Hill [16] for neutron transport, and then fully developed for time-dependent hyperbolic conservation laws in a series of papers by Cockburn, Shu, and collaborators (see [2] and references therein for details). An important feature of DG methods is that they can, at least in principle, be made arbitrarily high-order in space by increasing the polynomial order in each element; and therefore, the DG method is an example of a spectral element method (e.g., see Chapter 7.5 of Karniadakis and Sherwin [9]).

If DG is only used to discretize the spatial part of the underlying PDE, it remains to also introduce a temporal discretization. Many time-stepping methods are possible, including various explicit and implicit schemes. In general one time-step of an explicit scheme is significantly cheaper than an implicit one; the trade-off is that implicit schemes usually allow for larger time-steps. In many applications involving hyperbolic

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conservation laws, however, it is necessary to resolve the fastest time scales, in which case explicit methods are more efficient and easier to implement than implicit ones.

An upper bound on the largest allowable time-step for explicit schemes is provided by the Courant-Friedrichs-Lewy (CFL) condition, which requires that the domain of dependence of the numerical discretization subsumes the domain of dependence of the continuous PDE [3]. For example, a 1D hyperbolic PDE for which information propagates at a maximum wave speed of \( \lambda_{\text{max}} \), on a uniform mesh of elements of size \( h = \Delta x \), and with a time-stepping method that updates the solution on the element \( T_i^h \) only using existing solution values from \( T_{i-1}^h \), \( T_i^h \), and \( T_{i+1}^h \), has the following constraint on \( \Delta t \):

\[
\nu := \frac{\lambda_{\text{max}} \Delta t}{\Delta x} \leq 1.
\]

This has a clear physical interpretation: a wave that emanates from the boundaries of element \( i \) that is traveling at the maximum speed, \( \lambda_{\text{max}} \), is not allowed to propagate further than one element width. If we wanted to allow the wave to travel more than one element width, we would need to widen the numerical stencil.

The CFL condition as described above is a necessary condition for stability (and therefore convergence), but it is not sufficient. For high-order DG methods with explicit time-stepping, a fact that is well-known in the literature is that the actual maximum linearly stable value of the CFL number, \( \nu = \lambda_{\text{max}} \Delta t / \Delta x \), is significantly smaller than what the CFL condition predicts (see for example Liu et al. [14] and Sections 4.7 and 4.8 of Hesthaven and Warburton [8]).

Two popular explicit time-stepping schemes for DG are strong-stability-preserving Runge-Kutta DG (SSP-RK) [5, 6] and Lax-Wendroff [4, 15]. SSP-RK time-steps are one-step multistage Runge-Kutta methods that can be written as convex combinations of forward Euler steps. Lax-Wendroff utilizes the Cauchy-Kovalevskaya [19] procedure to convert temporal derivatives into spatial derivatives; the name Lax-Wendroff is due to the paper [11]. In Table 1 we illustrate for both the SSP-RK and Lax-Wendroff DG methods the gap between the CFL condition, a necessary but not sufficient condition for stability, and the semi-analytically computed maximum CFL number needed for linear stability. Shown are the methods with space and time order \( k = 1, 2, 3, 4 \). The SSP-RK DG numbers are reported from Liu et al. [14], while the Lax-Wendroff DG numbers are from von Neumann analysis done in this paper. Note that the maximum CFL number from the CFL condition for SSP-RK DG grows with \( k \) due to the fact that the number of Runge-Kutta stages grows with \( k \); and therefore, the numerical domain of dependence is increased. For both sets of methods, the clear trend is that the maximum CFL numbers are much smaller than what a simple CFL domain of dependence argument would dictate. In particular, the relationship between the maximum CFL number and the order of the method is roughly: \( \nu_{\text{max}} \propto 1/k \).

The goal of this paper is to develop an alternative time discretization for DG that allows for a linearly stable time-step that is closer to what is predicted by the CFL condition. The starting point of this work is the interpretation of the Lax-Wendroff DG method developed by Gassner et al. [4], where it was shown that Lax-Wendroff DG can be formulated as a predictor-corrector method. The predictor is a local version of a spacetime DG method [10, 18] (i.e., the predictor is something like a block-Jacobi update for a fully implicit spacetime DG method), and the corrector is an explicit method that uses the spacetime reconstructed solution from the predictor step. In this work we modify the predictor to include not just local information, but also neighboring information. The name that we are giving to this new approach is
the regionally-implicit discontinuous Galerkin (RIDG) scheme, which contrasts with the locally-implicit (LIDG) formulation of the Lax-Wendroff DG scheme developed by Gassner et al. [4]. In this new formulation, we are able to achieve all of the following:

- Develop RIDG schemes for 1D, 2D, and 3D advection;
- Show that RIDG has larger maximum CFL numbers than explicit SSP-RK and Lax-Wendroff DG;
- Show that the maximum linearly stable CFL number is bounded below by a constant that is independent of the polynomial order;
- Demonstrate experimentally the correct convergence rates on 1D, 2D, and 3D advection examples.

All of the methods described in this work are written in a MATLAB code that can be freely downloaded [7].

The organization of this paper is as follows. In section 2 we briefly review how space is discretized in the discontinuous Galerkin (DG) method. In section 3 we review the Lax-Wendroff DG scheme, then develop the one-dimensional version of the proposed regionally implicit DG (RIDG) scheme, and carry out von Neumann analysis for both methods. The generalization to multiple dimensions is done in section 4. Finally in section 5 we carry out numerical convergence tests to validate the new approach and to quantify the computational efficiency of RIDG relative to the Lax-Wendroff method.

2. DG-FEM spatial discretization. Consider hyperbolic conservation laws of the form

\[ \frac{\partial q}{\partial t} + \nabla \cdot F(q) = 0, \]

where \( q(t, x) : \mathbb{R}^+ \times \mathbb{R}^{M_{\text{dim}}} \rightarrow \mathbb{R}^{M_{\text{eqn}}} \) is the vector of conserved variables, \( F(q) : \mathbb{R}^{M_{\text{eqn}}} \rightarrow \mathbb{R}^{M_{\text{eqn}} \times M_{\text{dim}}} \) is the flux function, \( M_{\text{dim}} \) is the number of spatial dimensions, and \( M_{\text{eqn}} \) is the number of conserved variables. We assume that the system is hyperbolic, which means that the flux Jacobian,

\[ A(q; n) = \frac{\partial (n \cdot F)}{\partial q}, \]

for all \( q \in \mathcal{S} \subset \mathbb{R}^{M_{\text{eqn}}} \), where \( \mathcal{S} \) is some physically meaningful convex subset of \( \mathbb{R}^{M_{\text{eqn}}} \), and for all directions, \( n \in \mathbb{R}^{M_{\text{dim}}} \) such that \( \|n\| = 1 \), must be diagonalizable with only real eigenvalues (e.g., see Chapter 18 of LeVeque [12]).
Next consider discretizing system (2) in space via the discontinuous Galerkin (DG) method, which was first introduced by Reed and Hill [16] for neutron transport, and then fully developed for time-dependent hyperbolic conservation laws in a series of papers by Bernardo Cockburn, Chi-Wang Shu, and collaborators (see [2] and references therein for details). We define $\Omega \subset \mathbb{R}^{\dim}$ to be a polygonal domain with boundary $\partial \Omega$ and discretize $\Omega$ using a finite set of non-overlapping elements, $T_i$, such that $\bigcup_{i=1}^{M_{\text{elem}}} T_i = \Omega$, where $M_{\text{elem}}$ is the total number of elements. Let $P(M_{\text{deg}}, M_{\text{dim}})$ denote the set of polynomials from $\mathbb{R}^{M_{\text{dim}}}$ to $\mathbb{R}$ with maximal polynomial degree $M_{\text{deg}}$\(^1\). On the mesh of $M_{\text{elem}}$ elements we define the broken finite element space:

$$W^h := \left\{ w^h \in [L^\infty(\Omega)]^{M_{\text{eqn}}} : w^h|_{T_i} \in [P(M_{\text{deg}}, M_{\text{dim}})]^{M_{\text{eqn}}} \quad \forall T_i \right\},$$

where $h$ is the grid spacing, $M_{\text{dim}}$ is the number of spatial dimensions, $M_{\text{eqn}}$ is the number of conserved variables, and $M_{\text{deg}}$ is the maximal polynomial degree in the finite element representation. The above expression means that $w \in W^h$ has $M_{\text{eqn}}$ components, each of which when restricted to some element $T_i$ is a polynomial in $P(M_{\text{deg}}, M_{\text{dim}})$, and no continuity is assumed across element faces.

Let $\varphi_k(\mathbf{z})$ for $k = 1, \ldots, M_{\text{basis}}$ be an appropriate basis that spans $P(M_{\text{deg}}, M_{\text{dim}})$ over $T_i$ (e.g., Legendre or Lagrange polynomials). In order to get the DG semi-discretization, we multiply (2) by $\varphi_k \in P(M_{\text{deg}}, M_{\text{dim}})$, integrate over the element $T_i$, use integration-by-parts in space, and replace the true solution, $q$, by the following ansatz:

$$q^h(t, \mathbf{z}) |_{T_i} = \sum_{\ell=1}^{M_{\text{basis}}} Q^\ell(t) \varphi_\ell(\mathbf{z}).$$

All of these steps results in the following semi-discrete system:

$$\sum_{\ell=1}^{M_{\text{basis}}} \left[ \int_{T_i} \varphi_k \varphi_\ell \, dx \right] \frac{dQ^\ell}{dt} = \int_{T_i} F(q^h) \cdot \nabla \varphi_k \, dx - \int_{\partial T_i} \varphi_k F(q^h) \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is an outward-pointing normal vector to $\partial T_i$, $q^h$ and $q^h_+$ are the states on either side of the boundary $\partial T_i$, and $F$ is the numerical flux, which must satisfy the following two conditions:

- **Consistency:** $F(q, q; \mathbf{n}) = F(q) \cdot \mathbf{n}$;
- **Conservation:** $F(q^h, q^h_+; \mathbf{n}) = -F(q^h_-, q^h_+; -\mathbf{n})$.

Equation (6) represents a large system of coupled ordinary differential equations in time.

3. RIDG in one space dimension. We present in this section the proposed regionally-implicit discontinuous Galerkin (RIDG) method as applied to a one-dimensional advection equation. Each RIDG time-step is comprised of two key steps: a predictor and a corrector. The predictor is a truncated version of an implicit space-time DG approximation, which is not consistent, at least by itself, with the PDE

\(^1\)In 1D (i.e., $M_{\text{dim}} = 1$) this definition is unambiguous. In higher dimensions $P(M_{\text{deg}}, M_{\text{dim}})$ could refer the set of polynomials that have a total degree $\leq M_{\text{deg}}$ (we refer to this as the $P(M_{\text{deg}}, M_{\text{dim}})$ basis), it could refer to the set of polynomials that have degree $\leq M_{\text{deg}}$ in each independent variable (we refer to this as the $Q(M_{\text{deg}}, M_{\text{dim}})$ basis), or it could be something in between.
that it endeavors to approximate. The corrector is a modified forward Euler step that makes use of the predicted solution; this step restores consistency, and indeed, high-order accuracy, with the underlying PDE.

In the subsections below we begin with a brief description of the advection equation in subsection 3.1. We then review the Lax-Wendroff (aka locally-implicit) prediction step in subsection 3.2, which provides the motivation for RIDG. The RIDG prediction step is developed in subsection 3.3. The correction step for both predictors is detailed in subsection 3.4. Finally, we carry out semi-analytic von Neumann analysis for both schemes in subsection 3.5 and demonstrate the improved stability of RIDG over Lax-Wendroff DG.

### 3.1. 1D advection equation. We consider here the 1D advection equation for 
\((t, x) \in [0, T] \times \Omega\), along with some appropriate set of boundary conditions:
\[
q_t + u q_x = 0. 
\]
Next, we introduce a uniform Cartesian spacetime mesh with spacetime elements:
\[
S_i^{n+1/2} = [t^n, t^{n+\Delta t}] \times [x_i - \Delta x/2, x_i + \Delta x/2],
\]
which can be written in local coordinates, \([\tau, \xi] \in [-1, 1]^2\), where
\[
t = t^{n+1/2} + \tau (\Delta t/2) \quad \text{and} \quad x = x_i + \xi (\Delta x/2).
\]
In these local coordinates the advection equation \((7)\) becomes
\[
q_\tau + \nu q_\xi = 0, \quad \text{where} \quad \nu = \frac{u \Delta t}{\Delta x},
\]
where \(|\nu|\) is the CFL number.

### 3.2. Lax-Wendroff DG (aka LIDG) prediction step. We review here the prediction step for the Lax-Wendroff DG scheme as formulated by Gassner et al. \[4\]. In order to contrast with the proposed RIDG method, we will refer to this method as the locally-implicit DG (LIDG) method.

We fix the largest polynomial degree to \(M_{\text{deg}}\) in order to eventually achieve an approximation that has an order of accuracy \(O(\Delta x^{M_{\text{deg}}+1} + \Delta t^{M_{\text{deg}}+1})\). At the old time, \(t = t^n\), we are given the following approximate solution on each space element, \(T_i = [x_i - \Delta x/2, x_i + \Delta x/2]\):
\[
q(t^n, x) \big|_{T_i} \approx q_i^n := \Phi^T Q_i^n,
\]
where \(Q_i^n \in \mathbb{R}^{M_C}\), \(\Phi \in \mathbb{R}^{M_C}\), \(M_C := M_{\text{deg}} + 1\), and
\[
\Phi = \left(1, \sqrt{3} \xi, \frac{\sqrt{5}}{2} (3 \xi^2 - 1), \cdots\right), \quad \text{s.t.} \quad \frac{1}{2} \int_{-1}^{1} \Phi \Phi^T d\xi = I \in \mathbb{R}^{M_C \times M_C},
\]
are the orthonormal space Legendre polynomials.

In order to compute a predicted solution on each spacetime element \((8)\) we make the following ansatz:
\[
q(t, x) \big|_{S_i^{n+1/2}} \approx w_i^{n+1/2} := \Psi_i^T W_i^{n+1/2},
\]
where \( W_i^{n+1/2} \in \mathbb{R}^{M_p} \), \( \Psi \in \mathbb{R}^{M_p} \), \( M_p := (M_{\text{deg}} + 1)(M_{\text{deg}} + 2)/2 \), and

\[
\Psi = \left(1, \sqrt{3}t, \sqrt{3}\xi, \cdots\right), \quad \text{s.t.} \quad \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \Psi \Psi^T \, dt \, d\xi = I \in \mathbb{R}^{M_p \times M_p},
\]

are the spacetime Legendre basis functions. Next, we pre-multiply (10) by \( \Psi \) and integrate over \( S_i^{n+1/2} \) to obtain:

\[
\frac{1}{4} \int_{-1}^1 \int_{-1}^1 \Psi \left[q_0 + \nu Q_1\right] \, dt \, d\xi = 0.
\]

We then replace the exact solution by (13). We integrate-by-parts in time, first forwards, then backwards, which introduces a jump term at the old time \( t = t^n \). No integration-by-parts is done in space – this is what gives the local nature of the predictor step. All of this results in the following equation:

\[
\int \int \Psi \left[\Psi_\tau + \nu \Psi_{\xi}\right]^T W_i^{n+1/2} \, dt \, d\xi + \int \Psi \bigg|_{t=-1}^1 \left[\Psi_\tau^{T} W_i^{n+1/2} - \Phi^T Q^n\right] \, d\xi = 0,
\]

where all 1D integrals are over \([-1, 1]\), which can be written as

\[
\frac{L^0}{4} W_i^{n+1/2} = T Q^n_i,
\]

\[
\frac{L}{4} = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \Psi \left[\Psi_\tau + \nu \Psi_{\xi}\right]^T \, dt \, d\xi + \frac{1}{4} \int_{-1}^1 \Psi \bigg|_{t=-1}^1 \Psi_\tau \, d\xi \in \mathbb{R}^{M_p \times M_p},
\]

\[
T = \frac{1}{4} \int_{-1}^1 \Psi \bigg|_{t=-1}^1 \Phi^T \, d\xi \in \mathbb{R}^{M_p \times M_c}.
\]

As is evident from the formulas above, the predicted spacetime solution as encoded in the coefficients \( W_i^{n+1/2} \) is entirely local – the values only depend on the old values from the same element: \( Q^n_i \). Therefore, we refer to this prediction step as locally-implicit.

Remark 1. Gassner et al. [4] argued that the locally-implicit prediction step as presented above produces the key step in the Lax-Wendroff DG scheme [15]. We briefly illustrate this point here.

Lax-Wendroff [11] (aka the Cauchy-Kovalevskaya [19] procedure) begins with a Taylor series in time. All time derivatives are then replaced by spatial derivatives using the underlying PDE – in this case (7). For example, if we kept all the time derivatives up to the third derivative we would get:

\[
q^{n+1} \approx q^n + 2q_\tau^n + 2q_{\tau,\tau}^n + \frac{4}{3} q_{\tau,\tau,\tau}^n - \frac{2}{3} \nu q_\xi^3 + \frac{2}{3} \nu q_{\xi,\xi,\xi}^3 - \frac{4}{3} \nu^3 q_{\xi,\xi,\xi}^3 = q^n - 2\nu Q_L W.
\]

Using the third-order DG approximation,

\[
q^n := \varphi_1 Q_1^n + \varphi_2 Q_2^n + \varphi_3 Q_3^n = Q_1^n + \sqrt{3} \xi Q_2^n + \frac{\sqrt{5}}{2} (3\xi^2 - 1) Q_3^n,
\]

we obtain

\[
G_L W = \varphi_1 \left( Q_1^n - \sqrt{3} \nu Q_2^n + 2 \sqrt{5} \nu^2 Q_3^n \right) + \varphi_2 \left( Q_2^n - \sqrt{3} \nu Q_3^n \right) + \varphi_3 Q_3^n.
\]
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\[ \dot{\Psi}^T W^{n+1/2}_i \]

\[ \dot{\Psi}^T W^{n+1/2}_i \]

\[ \dot{\Psi}^T W^{n+1/2}_{i-1} \]

\[ \dot{\Psi}^T W^{n+1/2}_{i} \]

\[ \dot{\Psi}^T W^{n+1/2}_{i+1} \]

\[ \nabla \text{upwind-in-time} \]

\[ \text{interior flux} \]

\[ \text{proper upwind flux} \]

Fig. 1. Shown are the domains of dependence for the LIDG (left) and RIDG (right) prediction steps for spacetime element \( S^{n+1/2}_i \) in one spatial dimension. Note that on the \( t = t^n \) and \( t = t^{n+1} = t^n + \Delta t \) faces, the “proper upwind flux” values are always on the “past” side of the face – we refer to these as the “upwind-in-time” values. The LIDG prediction step is purely local in space – there is no spatial communication with neighboring cells. The RIDG prediction step computes the proper upwind flux on the \( x = x_i \pm \Delta x/2 \) faces, but nowhere else. In the RIDG prediction step the states to the immediate left and right of element \( i \) are only temporary variables and will be discarded once the predicted solution on element \( i \) has been computed – to make note of this we place hats over the temporary variables.

Alternatively, the time-averaged flux, \( G^{LxW} \), can be directly obtained from the locally-implicit predictor described above. We first calculate the predicted spacetime solution, \( w^{n+1/2} \), via (17), (18), and (19). From this we compute the time averaged flux:

\[ \mathbf{G}^{LxW} = \frac{1}{2} \int_{-1}^{1} w^{n+1/2} (\tau, \xi) d\tau = \frac{1}{2} \int_{-1}^{1} \Psi (\tau, \xi) d\tau \left( \frac{L^0}{\mathbf{Q}^n} \right)^{-1} \mathbf{T}^Q. \]

A straightforward calculation shows that (23) with (21) and the corresponding \( P(2,2) \) spacetime basis\(^2\) is exactly the same as (22).

3.3. RIDG prediction step. As we have argued in section 1 and with Table 1, the locally-implicit predictor described above in subsection 3.2 will result in a scheme that has a maximum linearly stable CFL number that is both small and becomes progressively smaller with increasing order of accuracy. We introduce a modified prediction step here to remedy these shortcomings.

The starting point is the same as in subsection 3.2: ansatz (11) and (13), but now with the full spacetime \( Q(M_{\text{deg}}, M_{\text{dim}} + 1) \) basis in the prediction step:

\[ M_C = M_{\text{deg}} + 1 \quad \text{and} \quad M_P = (M_{\text{deg}} + 1)^2. \]

We integrate the advection equation in spacetime to get (15), but this time we integrate-by-parts in both space and time, which yields:

\[ \int \Psi (\dot{\Psi} + \nu \dot{\Psi}^T \xi) W_{i}^{n+1/2} d\tau d\xi + \int \Psi_{\xi = -1} \left[ \Psi^T_{\xi = -1} W_{i}^{n+1/2} - \Psi^T Q^i \right] d\xi - \right. \]

\[ \left. \int \{ \Psi_{\xi = 1} \left[ \nu \Psi^T_{\xi = 1} W_{i}^{n+1/2} - F^*_{i+1/2} \right] - \Psi_{\xi = -1} \left[ \nu \Psi^T_{\xi = -1} W_{i}^{n+1/2} - F^*_{i-1/2} \right] \} d\tau = 0. \]

\[^2\text{Actually, for LIDG the result is the same whether we use } P(M_{\text{deg}}, M_{\text{dim}} + 1), Q(M_{\text{deg}}, M_{\text{dim}} + 1), \text{or something in between.}\]
where all 1D integrals are over \([-1, 1]\). \(\Phi\) is the Legendre basis (12), \(\Psi\) is the spacetime Legendre basis (14), and \(\mathcal{F}^*\) are some appropriately defined numerical fluxes.

The crux of the idea of the regionally-implicit DG scheme in one spatial dimension can be summarized as follows:

- We define a region to be the current spacetime element, \(S_i^{n+1/2}\), and its immediate neighbors: \(S_{i-1}^{n+1/2}\) and \(S_{i+1}^{n+1/2}\). This is illustrated in Figure 1.
- For \(S_i^{n+1/2}\), we use the correct upwind fluxes to define the numerical fluxes, \(\mathcal{F}^*\), on its faces.
- For the immediate neighbors, \(S_{i-1}^{n+1/2}\) and \(S_{i+1}^{n+1/2}\), we again use the correct upwind fluxes on the faces that are shared with \(S_i^{n+1/2}\), but on the outer faces we use one-sided interior fluxes. See Figure 1.
- We use the \(Q(M_{deg}, M_{dim}+1)\) spacetime basis in the prediction step (i.e., the full tensor product spacetime basis). Numerical experimentation showed us that using the \(Q(M_{deg}, M_{dim}+1)\) basis for the prediction step, rather than the \(P(M_{deg}, M_{dim}+1)\) basis, produces significantly more accurate results; in the case of linear equations, this creates little additional computational expense since all the relevant matrices can be precomputed.

The result of this is a collection of three elements with solutions that are coupled to each other, but that are completely decoupled from all remaining elements. This RIDG setup is depicted in Figure 1, where we also show the LIDG setup as a point of comparison.

The precise form of the fluxes for the RIDG prediction step on spacetime element \(S_i^{n+1/2}\) can be written as follows:

\[
(26) \quad \mathcal{F}_{i-3/2}^* = \nu^\Psi T_{\xi=1} W_{i-1}^{n+1/2}, \quad \mathcal{F}_{i-1/2}^* = \nu^\Psi T_{\xi=1} W_{i-1}^{n+1/2} + \nu^\Psi T_{\xi=-1} W_{i+1}^{n+1/2},
\]

\[
(27) \quad \mathcal{F}_{i+1/2}^* = \nu^\Psi T_{\xi=1} W_{i+1}^{n+1/2} + \nu^- \Psi T_{\xi=-1} W_{i-1}^{n+1/2}, \quad \mathcal{F}_{i+3/2}^* = \nu^\Psi T_{\xi=1} W_{i+1}^{n+1/2}.
\]

Combining (25) with numerical fluxes (26) and (27), yields the following block 3 \(\times\) 3 system:

\[
(28) \quad \begin{bmatrix}
L_0^+ + L_0^- & X^- & V_{i-1}^{n+1/2} \\
X^+ & L_0^+ + L_0^- + L_0^+ & X^- \\
L_0^+ & L_0^+ + L_0^+ & X^-
\end{bmatrix} \begin{bmatrix}
V_{i-1}^n \\
V_{i+1}^{n+1/2} \\
V_{i+1}^n
\end{bmatrix} = \begin{bmatrix}
T Q_i^n \\
T Q_{i+1}^n \\
T Q_{i+1}^n
\end{bmatrix},
\]

where \(L_0^+\) is given by (18), \(T\) is given by (19), and

\[
(29) \quad L_0^+ = \frac{\nu^+}{4} \int_{-1}^{1} \Psi_{\xi=-1} \Psi_{\xi=1} T_{\xi=1} \, d\tau, \quad L_0^- = -\frac{\nu^-}{4} \int_{-1}^{1} \Psi_{\xi=1} \Psi_{\xi=1} T_{\xi=1} \, d\tau,
\]

\[
(30) \quad X^+ = -\frac{\nu^+}{4} \int_{-1}^{1} \Psi_{\xi=-1} \Psi_{\xi=1} T_{\xi=1} \, d\tau, \quad X^- = \frac{\nu^-}{4} \int_{-1}^{1} \Psi_{\xi=1} \Psi_{\xi=1} T_{\xi=1} \, d\tau,
\]

where \(L_0^+, L_0^-, X^+, X^- \in \mathbb{R}^{M_p \times M_p}\), \(\nu^+ = \max(\nu, 0)\), and \(\nu^- = \min(\nu, 0)\). Note that the states to the immediate left and right of the current spacetime element \(S_i^{n+1/2}\) are only temporary variables and will be discarded once the predicted solution in element \(i\) has been computed – to make note of this we place hats over the temporary variables. This also means that we have to solve a block 3 \(\times\) 3 system of the form (28) on every single element \(S_i^{n+1/2}\).
3.4. Correction step for both LIDG and RIDG. In order to go from predictor to corrector step, we multiply (10) by $\Phi \in \mathbb{R}^{Mc}$ and integrate in spacetime:

$$Q^{n+1}_i = Q^n_i + \frac{\nu}{2} \int_{-1}^{1} \int_{-1}^{1} \Phi q d\tau d\xi - \frac{1}{2} \int_{-1}^{1} [\Phi_{\xi=1} f_{i+1/2} - \Phi_{\xi=-1} f_{i-1/2}] d\tau,$$

where $f$ is the numerical flux. Next we replace $q$ by the predicted solution from either subsection 3.2 or subsection 3.3, and use the upwind flux:

$$f_{i-1/2} = \nu^+ \Psi^T_{\xi=1} W^{n+1/2}_{i-1} + \nu^- \Psi^T_{\xi=-1} W^{n+1/2}_{i},$$

which results in

$$Q^{n+1}_i = Q^n_i + C^- W^{n+1/2}_{i-1} + C^0 W^{n+1/2}_{i} + C^+ W^{n+1/2}_{i+1},$$

$$C^- = \frac{\nu^+}{2} \int_{-1}^{1} \Phi_{\xi=1} \Psi^T_{\xi=1} d\tau,$$

$$C^+ = -\frac{\nu^-}{2} \int_{-1}^{1} \Phi_{\xi=1} \Psi^T_{\xi=-1} d\tau,$$

where $C^0, C^-, C^+ \in \mathbb{R}^{M_C \times MP}$.

3.5. Von Neumann stability analysis for both LIDG and RIDG. The Lax-Wendroff DG scheme (aka LIDG) with prediction step detailed in subsection 3.2 and correction step given by (33), (34), and (35) uses a stencil involving three elements: $T_{i-1}, T_i,$ and $T_{i+1};$ the RIDG scheme detailed in subsection 3.3 with the same correction step as LIDG uses a stencil involving five elements: $T_{i-2}, T_{i-1}, T_i, T_{i+1},$ and $T_{i+2}. This resulting CFL conditions for the LIDG and RIDG schemes are

$$|\nu| = \frac{|u| \Delta t}{\Delta x} \leq 1 \quad \text{and} \quad |\nu| = \frac{|u| \Delta t}{\Delta x} \leq 2,$$

respectively. In reality, the CFL number, $|\nu|$, for which linear stability is achieved is smaller than what this CFL argument provides; we investigate this in more detail here.
In order to study linear stability, we employ the technique of von Neumann stability analysis (e.g., see Chapter 10.5 of LeVeque [13]). In particular, we assume the following Fourier ansatz:

\[ Q_{n+1}^j = \tilde{Q}_{n+1}^j e^{i\omega j} \quad \text{and} \quad Q_i^n = \tilde{Q}_i^n e^{i\omega j}, \]

where \( I = \sqrt{-1} \) and \( 0 \leq \omega \leq 2\pi \) is the wave number. After using this ansatz, the next step is to write the resulting update in the form:

\[ \tilde{Q}_{n+1}^j = M(\nu, \omega) \tilde{Q}_n^j, \]

for some matrix \( M \in \mathbb{R}^{MC \times MC} \). If we apply ansatz (37) to LIDG and RIDG, assuming w.l.o.g. that \( \nu \geq 0 \), we obtain the following:

\[ M_{\text{LIDG}}(\nu, \omega) = \left( I + C^0_0 \left( L_0^0 \right)^{-1} T \right) + e^{-i\omega} C^- \left( L_0^0 \right)^{-1} T, \]

\[ M_{\text{RIDG}}(\nu, \omega) = \left( I + C^0_0 \left( L_0^0 + L_+^0 \right)^{-1} T \right) \]
\[ + e^{-i\omega} \left( C^- \left( L_0^0 + L_+^0 \right)^{-1} T - C^0_0 \left( L_0^0 + L_+^0 \right)^{-1} X^+ \left( L_0^0 \right)^{-1} T \right) \]
\[ - e^{-2i\omega} C^- \left( L_0^0 + L_+^0 \right)^{-1} X^+ \left( L_0^0 \right)^{-1} T, \]

where \( I \in \mathbb{R}^{MC \times MC} \) is the identity matrix.

The final step in the stability analysis is to study the spectral properties \( M \) as a function of the CFL number \( \nu \). In particular, to find the largest \( \nu \) for which LIDG or RIDG are linearly stable, we define the following function:

\[ f(\nu) = \max_{0 \leq \omega \leq 2\pi} \rho \left( M(\nu, \omega) \right) - 1, \]

where \( \rho \) is the spectral radius of \( M \). For both LIDG and RIDG, the function \( f(\nu) \) satisfies \( f(0) = 0 \), there exists a finite range of \( \nu \) for which \( f(\nu) \approx 0 \), and there exists a value of \( \nu \) for which \( f(\nu) \) transitions from being approximately zero to rapidly increasing with increasing \( \nu \). We illustrate this point in Figure 2 for both LIDG and RIDG for the case \( M_{\text{deg}} = 5 \) (MC = 6, LIDG: \( M_P = 21 \), RIDG: \( M_P = 36 \)); for each scheme we note approximately where the linear stability transition occurs. In order to numerical estimate the location of the linear stability transition we look for the value of \( \nu \) that satisfies \( f(\nu) = \varepsilon \). We do this via a simple bisection method where we set \( \varepsilon = 0.0005 \) and we replace the true maximization in (41) over the maximization of 2001 uniformly spaced wave numbers over \( 0 \leq \omega \leq 2\pi \).

The result of this bisection procedure for both LIDG and RIDG for \( M_{\text{deg}} = 0, 1, 2, 3, 4, 5 \), is summarized in Table 2. In all cases we have also run the full numerical method at various grid resolution to verify that the simulations are indeed stable at the various CFL numbers shown in Table 2. There are three key take-aways from Table 2:

- Both methods give stability regions smaller than their CFL conditions (36);
- The LIDG CFL number degrades roughly as the inverse of the method order;
- The RIDG CFL number is roughly one, independent of the method order.

### 4. Generalization to higher dimensions

We present in this section the generalization of the proposed regionally-implicit discontinuous Galerkin (RIDG) method to the case of the two and three-dimensional versions of the advection equation. The
key innovation beyond what was developed in section 3 for the one-dimensional case is the inclusion of transverse cells in the prediction step. With these inclusions, the prediction gives enhanced stability for waves propagating at all angles to the element faces.

4.1. RIDG method in 2D. We consider here the two-dimensional advection equation for \((t, x, y) \in [0, T] \times \Omega\) with appropriate boundary conditions:

\[ q_t + u_x q_x + u_y q_y = 0. \]

We define a uniform Cartesian mesh with grid spacings \(\Delta x\) and \(\Delta y\) in each coordinate direction. On each spacetime element:

\[ S_{ij}^{n+1/2} = [t^n, t^{n+\Delta t}] \times [x_i - \Delta x/2, x_i + \Delta x/2] \times [y_j - \Delta y/2, y_j + \Delta y/2], \]

we define the local coordinates, \([\tau, \xi, \eta] \in [-1, 1]^3\), such that

\[ t = t^{n+1/2} + \tau (\Delta t/2), \quad x = x_i + \xi (\Delta x/2), \quad \text{and} \quad y = y_j + \eta (\Delta y/2). \]

In these local coordinates, the advection equation is given by

\[ q_\tau + \nu_x q_\xi + \nu_y q_\eta = 0, \quad \nu_x = \frac{u_x \Delta t}{\Delta x}, \quad \nu_y = \frac{u_y \Delta t}{\Delta y}, \]

where \(|\nu_x|\) and \(|\nu_y|\) are the CFL numbers in each coordinate direction and the multidimensional CFL number is

\[ |\nu| := \max \{|\nu_x|, |\nu_y|\}. \]

At the old time, \(t = t^n\), we are given the following approximate solution on each space element, \(T_{ij} = [x_i - \Delta x/2, x_i + \Delta x/2] \times [y_j - \Delta y/2, y_j + \Delta y/2]::

\[ q(t^n, x, y) \big|_{T_{ij}} \approx q^n_{ij} := \Phi^T Q^n_{ij}, \]

where \(Q^n_{ij} \in \mathbb{R}^{M_C}\), \(\Phi \in \mathbb{R}^{Mc}\), \(M_C := (M_{deg} + 1)(M_{deg} + 2)/2\), and

\[ \Phi = \left(1, \sqrt{3} \xi, \sqrt{3} \eta, \cdots \right), \quad \text{s.t.} \quad \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Phi \Phi^T d\xi d\eta = \mathbb{I} \in \mathbb{R}^{Mc \times Mc}, \]

are the orthonormal space Legendre polynomials. In order to compute a predicted solution on each spacetime element we make the following ansatz:

\[ q(t, x) \big|_{S_{ij}^{n+1/2}} \approx w_{ij}^{n+1/2} := \Psi^T W_{ij}^{n+1/2}, \]
LIDG

RIDG

\begin{equation}
\Psi^T \hat{W}^{n+1/2}_{i-1,j+1} + \Psi^T \hat{W}^{n+1/2}_{j+1} - \Psi^T \hat{W}^{n+1/2}_{i+1,j+1} - \Psi^T \hat{W}^{n+1/2}_{i-1,j-1} + \Psi^T \hat{W}^{n+1/2}_{j-1} - \Psi^T \hat{W}^{n+1/2}_{i+1,j-1} - \Psi^T \hat{W}^{n+1/2}_{i-1,j} + \Psi^T \hat{W}^{n+1/2}_{j} - \Psi^T \hat{W}^{n+1/2}_{i+1,j}
\end{equation}

\( \Psi^T W^+_{i,j} \) \hspace{1cm} proper upwind flux

\( \Psi^T \) interior flux

Fig. 3. Shown are the stencils for the LIDG (left) and RIDG (right) prediction steps in two spatial dimension. The LIDG prediction step is purely local – there is no communication with neighboring cells. The RIDG prediction step computes the proper upwind flux on the spacetime faces shared with immediate neighbors, as well as the four corner elements. In the RIDG prediction step, all of the states, excepting only the one belonging to the middle element, are only temporary variables and will be discarded once the predicted solution in element \( ij \) has been computed – to make note of this we place hats over the temporary variables.

where \( W^+_{i,j} \in \mathbb{R}^{M_p} \), \( \Psi \in \mathbb{R}^{M_p}, M_p := (M_{\text{deg}} + 1)^3 \), and

\begin{equation}
\Psi = (1, \sqrt{3} \tau, \sqrt{3} \xi, \sqrt{3} \eta, \cdots), \quad \text{s.t.} \quad \frac{1}{8} \iiint \Psi \Psi^T d\tau d\xi d\eta = I \in \mathbb{R}^{M_p \times M_p},
\end{equation}

are the spacetime Legendre basis functions, where all 1D integrals are over \([-1, 1]\).

Remark 2. Just as in the one-dimensional case outlined in section 3, for the RIDG scheme we make use of the \( P(M_{\text{deg}}, M_{\text{dim}}) \) spatial basis for the correction step (i.e., \( M_C = (M_{\text{deg}} + 1)(M_{\text{deg}} + 2)/2 \)), and the \( Q(M_{\text{deg}}, M_{\text{dim}} + 1) \) spacetime basis for the prediction step (i.e., \( M_P = (M_{\text{deg}} + 1)^3 \)).

We integrate the advection equation over a spacetime element and apply integrate-by-parts in all three independent variables: \( \tau, \xi, \eta \), which yields:

\begin{equation}
\begin{aligned}
\iiint &\Psi \mathcal{R} (\Psi)^T W^+_{i,j} \frac{1}{2} dS + \iiint \Psi_{i-1} \left[ \nu_x \Psi_{i+\frac{1}{2},j}^T W^+_{\frac{i+\frac{1}{2},j}^*} - \Psi_{i+\frac{1}{2},j}^T Q_{ij}^n \right] dS_x - \\
&\iiint \left\{ \Psi_{i+\frac{1}{2}} \left[ \nu_x \Psi_{i+\frac{1}{2},j}^T W^+_{\frac{i+\frac{1}{2},j}^*} - \mathcal{F}_{i+\frac{1}{2}} \right] - \Psi_{i+\frac{1}{2}} \left[ \nu_x \Psi_{i-\frac{1}{2},j}^T W^+_{\frac{i-\frac{1}{2},j}^*} - \mathcal{F}_{i-\frac{1}{2}} \right] \right\} dS_x - \\
&\iiint \left\{ \Psi_{j+\frac{1}{2}} \left[ \nu_y \Psi_{i+\frac{1}{2},j}^T W^+_{\frac{i+\frac{1}{2},j}^*} - \mathcal{G}_{\frac{j+\frac{1}{2}}^*} \right] - \Psi_{j+\frac{1}{2}} \left[ \nu_y \Psi_{i-\frac{1}{2},j}^T W^+_{\frac{i-\frac{1}{2},j}^*} - \mathcal{G}_{\frac{j-\frac{1}{2}}^*} \right] \right\} dS_y = 0.
\end{aligned}
\end{equation}
where \( R(\Psi) = \Psi_{x} + \nu_{x}\Psi_{\xi} + \nu_{y}\Psi_{\eta} \),
\( dS = d\tau d\xi d\eta \),
\( dS_{\xi} = d\tau d\xi \),
and all 1D integrals are over \([-1, 1]\).

The crux of the idea of the regionally-implicit DG scheme in two spatial dimensions can be summarized as follows:

- We define a region to be the current spacetime element, \( S_{ij}^{n+1/2} \), and the eight neighbors that have a face that shares at least one point in common with one of the faces of \( S_{ij}^{n+1/2} \). This is illustrated in Figure 3.

- For the current spacetime element, \( S_{ij}^{n+1/2} \), we use the correct upwind fluxes to define \( \mathcal{F}^{*} \) and \( \mathcal{G}^{*} \).

- For the remaining eight elements we use the correct upwind fluxes on all faces that are interior to the region and one-sided fluxes on all faces that are on the boundary of the region. See Figure 3.

- We use the \( Q(M_{\text{deg}}, M_{\text{dim}} + 1) \) spacetime basis in the prediction step (i.e., the full tensor product spacetime basis). Numerical experimentation showed us that using the \( Q(M_{\text{deg}}, M_{\text{dim}} + 1) \) basis for the prediction step, rather than the \( P(M_{\text{deg}}, M_{\text{dim}} + 1) \) basis, produces significantly more accurate results; in the case of linear equations, this creates little additional computational expense since all the relevant matrices can be precomputed.

The result of this is a collection of nine elements with solutions that are coupled to each other, but that are completely decoupled from all remaining elements. This RIDG setup is depicted in Figure 3, where we also show the LIDG setup as a point of comparison.

Remark 3. One of the key innovations in going from the 1D RIDG scheme to its 2D counterpart is the inclusion of the transverse elements in the prediction step: \( S_{i-1j-1}^{n+1/2}, S_{i+1j-1}^{n+1/2}, S_{i-1j+1}^{n+1/2}, \) and \( S_{i+1j+1}^{n+1/2} \). Without these transverse cells, the maximum allowable two-dimensional CFL number (46) remains small for any waves traveling transverse to the mesh. In the recent literature, there exist several variants of genuinely multidimensional Riemann solvers (e.g., Balsara [1]); by including the transverse elements, the current work can be viewed as an example of a novel type of multidimensional Riemann solver.

Applying all of the above principles to (51) for all of the nine elements that are in the current region yields a block \( 9 \times 9 \) linear system. The left-hand side of this system can be written as

\[
\begin{bmatrix}
L^{1010} & X^{-} & Y^{+} & & & & & & & \\
X^{+} & L^{1100} & X^{-} & Y^{+} & & & & & & & \\
X^{+} & X^{+} & L^{0110} & Y^{-} & & & & & & & \\
Y^{+} & X^{+} & L^{0111} & X^{-} & Y^{-} & & & & & & \\
Y^{+} & Y^{+} & L^{1111} & X^{-} & Y^{-} & & & & & & \\
Y^{+} & Y^{+} & X^{+} & L^{1110} & Y^{-} & & & & & & \\
Y^{+} & Y^{+} & X^{+} & X^{+} & L^{0101} & X^{-} & & & & & \\
Y^{+} & Y^{+} & X^{+} & X^{+} & X^{+} & L^{0100} & X^{-} & & & & \\
Y^{+} & Y^{+} & X^{+} & X^{+} & X^{+} & X^{+} & L^{0101} & \hat{W}_{n+1/2}^{i-1j-1} & & & \\
& & & & & & & & & & \hat{W}_{n+1/2}^{i+j-1} \\
& & & & & & & & & & \hat{W}_{n+1/2}^{i-1j+1} \\
& & & & & & & & & & \hat{W}_{n+1/2}^{i+j+1} \\
\end{bmatrix}
\]
and the right-hand side can be written as

\[
\begin{bmatrix}
TQ^n_{i,j,k-1},
TQ^n_{i,j-1},
\cdots,
TQ^n_{i,j},
\cdots,
TQ^n_{i,j,k+1},
TQ^n_{i,j+1,1}
\end{bmatrix}^T,
\]

where

\[
X^\pm = \mp \frac{\nu^\pm}{8} \int \Psi_{\xi=\pm 1} \Psi_T^T d\xi, \quad Y^\pm = \mp \frac{\nu^\pm}{8} \int \Psi_{\eta=\pm 1} \Psi_T^T d\eta,
\]

\[
L_\alpha^{\beta\gamma\delta} = L_0^0 + \alpha L_{\xi} + \beta L_{\eta} + \gamma L_{\tau} + \delta L_{\eta}, \quad \alpha, \beta, \gamma, \delta \in \{0, 1\},
\]

\[
L_\tau^0 = \frac{1}{8} \int \Psi R(\Psi)^T d\Psi + \frac{1}{8} \int \Psi_{\tau=1} \Psi_T^T d\Psi, \quad T = \frac{1}{8} \int \Psi_{\tau=1} \Phi_T d\Psi,
\]

where \(T \in \mathbb{R}^{Mr \times MC}\) and \(X^\pm, Y^\pm, L_\alpha^{\beta\gamma\delta}, L_\tau^0 \in \mathbb{R}^{Mr \times Mr}\).

The correction step can be written as

\[
Q^{n+1}_{i,j} = Q^n_{i,j} + C_{\alpha} W^{n+\frac{1}{2}}_{i,j-1} + C_{\beta} W^{n+\frac{1}{2}}_{i,j+1} + C_{\gamma} W^{n-\frac{1}{2}}_{i+1,j} + C_{\delta} W^{n+\frac{1}{2}}_{i,j+1} + C_{\eta} W^{n+\frac{1}{2}}_{i,j},
\]

\[
C_\alpha^0 = \frac{1}{4} \int \mathcal{U}(\Phi) \Psi_T d\Psi - \frac{1}{4} \int \left[ \nu_x^+ \Phi_{\xi=1} \Psi_T^T - \nu_x^- \Phi_{\xi=-1} \Psi_T^T \right] d\xi
\]

\[
- \frac{1}{4} \int \left[ \nu_y^+ \Phi_{\eta=1} \Psi_T^T - \nu_y^- \Phi_{\eta=-1} \Psi_T^T \right] d\eta,
\]

\[
C_\eta^+ = \pm \frac{\nu^\pm}{4} \int \Phi_{\tau=1} \Psi_T^T d\Psi, \quad C_\tau^+ = \pm \frac{\nu^\pm}{4} \int \Phi_{\eta=1} \Psi_T^T d\Psi,
\]

where \(\mathcal{U}(\Phi) = \nu_x \Phi_\xi + \nu_y \Phi_\eta, \quad C_\alpha^0, C_\beta^\pm, C_\gamma^\pm, C_\delta^\pm \in \mathbb{R}^{MC \times MP}\).

### 4.2. RIDG method in 3D

We consider here the three-dimensional advection equation for \((t, x, y, z) \in [0, T] \times \Omega\) with appropriate boundary conditions:

\[
q_t + u_x q_x + u_y q_y + u_z q_z = 0.
\]

We define a uniform Cartesian mesh with grid spacings \(\Delta x, \Delta y,\) and \(\Delta z\) in each coordinate direction. On each spacetime element:

\[
S^{n+1/2}_{i,j,k} = S^{n+1/2}_{i,j} \times [z_k - \Delta z/2, z_k + \Delta z/2],
\]

where \(S^{n+1/2}_{i,j}\) is defined by (43), we define the local coordinates, \([\tau, \xi, \eta, \zeta] \in [-1, 1]^4\), such that

\[
t = t^{n+1/2} + \tau (\Delta t/2), \quad x = x_i + \xi (\Delta x/2), \quad y = y_j + \eta (\Delta y/2), \quad z = z_k + \zeta (\Delta z/2).
\]

In these local coordinates, the advection equation is given by

\[
q_\tau + \nu_x q_\xi + \nu_y q_\eta + \nu_z q_\zeta = 0, \quad \nu_x = \frac{u_x \Delta t}{\Delta x}, \quad \nu_y = \frac{u_y \Delta t}{\Delta y}, \quad \nu_z = \frac{u_z \Delta t}{\Delta z},
\]

where \(|\nu_x|, |\nu_y|,\) and \(|\nu_z|\) are the CFL numbers in each coordinate direction and the multidimensional CFL number is

\[
|\nu| := \max \{|\nu_x|, |\nu_y|, |\nu_z|\}.
\]
The development of the RIDG scheme in 3D is completely analogous to the 2D RIDG scheme from subsection 4.1. In 1D the prediction step requires a stencil of 3 elements, in 2D we need \(3^2 = 9\) elements, and in 3D we need \(3^3 = 27\) elements. For the sake of brevity we omit the details.

4.3. Von Neumann stability analysis for both LIDG and RIDG. Linear stability analysis proceeds in 2D and 3D in the same manner as in 1D. We take the numerical update and make the Fourier ansatz:

\[
\bar{Q}^{n+1}_{ijk} = \bar{Q}^n e^{I(\omega_x i + \omega_y j + \omega_z k)}
\]

for some matrix \(\bar{M} \in \mathbb{R}^{MC \times MC}\). Finally, we define the function

\[
f(\nu_x, \nu_y, \nu_z) = \max_{0 \leq \omega_x, \omega_y, \omega_z \leq 2\pi} \rho(\bar{M}(\nu_x, \nu_y, \nu_z, \omega_x, \omega_y, \omega_z)) - 1,
\]

where \(\rho(\bar{M})\) is the spectral radius of \(\bar{M}\).

Just as in 1D, we estimate the maximum CFL numbers of LIDG and RIDG by studying the values of (68). Our numerically obtained estimates for the maximum value of \(|\nu|\) as defined by (46) and (65) are summarized in Table 3. Again we see the following:

- LIDG: the maximum stable CFL number tends to zero as the polynomial degree is increased; and
- RIDG: the maximum stable CFL number has a finite lower bound with increasing polynomial degree (approximately 0.75 in 2D and 0.60 in 3D).

To get a more detailed view of the stability function (68) in 2D, we show false color plots of \(f(\nu_x, \nu_y) + 1\) in Figure 4 for both LIDG and RIDG for various method orders. The transverse elements that were included in the prediction step for RIDG (see subsection 4.1) are critically important in achieving a stability region that does not significantly degrade in going from 1D to 2D.

5. Numerical convergence studies. In this section we present convergence studies in 1D, 2D, and 3D for both LIDG and RIDG, and compare the errors and runtimes for the two methods. In all cases, we compute an approximate order of accuracy using the following approximation:

\[
\text{error}(h) = ch^M + \mathcal{O}(h^{M+1}) \implies M \approx \frac{\log(\text{error}(h_1)/\text{error}(h_2))}{\log(h_1/h_2)}.
\]
1D LIDG: \((M_{\text{deg}} = 3, \nu = 0.104)\)

| mesh | \(T_r/0.031793s\) | \(L^1\) error | \(L^2\) error | \(L^\infty\) error |
|------|----------------|---------------|---------------|----------------|
| 40   | 6.93926        | 1.83e–01     | –             | 1.92e–01     |
| 80   | 24.6472        | 1.08e–02     | 4.08          | 4.13e–02     |
| 160  | 96.6439        | 6.52e–04     | 4.05          | 6.66e–04     |
| 320  | 385.189        | 4.01e–05     | 4.02          | 4.10e–05     |
| 640  | 1538.96        | 2.49e–06     | 4.01          | 2.79e–06     |

1D RIDG: \((M_{\text{deg}} = 3, \nu = 0.9)\)

| mesh | \(T_r/0.031793s\) | \(L^1\) error | \(L^2\) error | \(L^\infty\) error |
|------|----------------|---------------|---------------|----------------|
| 40   | 1.00000        | 8.46e–02     | –             | 1.02e–01     |
| 80   | 3.79518        | 3.67e–03     | 4.53          | 4.68e–03     |
| 160  | 14.8923        | 1.51e–04     | 4.61          | 1.76e–04     |
| 320  | 59.2929        | 7.96e–06     | 4.24          | 8.95e–06     |
| 640  | 239.622        | 4.75e–07     | 4.07          | 5.57e–07     |

1D LIDG: \((M_{\text{deg}} = 5, \nu = 0.04)\)

| mesh | \(T_r/0.031793s\) | \(L^1\) error | \(L^2\) error | \(L^\infty\) error |
|------|----------------|---------------|---------------|----------------|
| 40   | 16.2847        | 1.11e–03     | –             | 1.25e–03     |
| 80   | 64.1997        | 1.74e–05     | 6.00          | 1.88e–05     |
| 160  | 253.524        | 2.73e–07     | 5.99          | 2.86e–07     |
| 320  | 1012.68        | 4.24e–09     | 6.01          | 4.36e–09     |
| 640  | 4062.91        | 6.61e–11     | 6.00          | 6.78e–11     |

1D RIDG: \((M_{\text{deg}} = 5, \nu = 0.9)\)

| mesh | \(T_r/0.031793s\) | \(L^1\) error | \(L^2\) error | \(L^\infty\) error |
|------|----------------|---------------|---------------|----------------|
| 40   | 1.02457        | 1.50e–04     | –             | 4.64e–04     |
| 80   | 3.88293        | 2.68e–06     | 5.81          | 5.19e–06     |
| 160  | 15.1584        | 3.91e–08     | 6.10          | 4.89e–08     |
| 320  | 60.8939        | 5.85e–10     | 6.06          | 8.37e–10     |
| 640  | 243.242        | 8.94e–12     | 6.03          | 1.36e–11     |

Table 4: Convergence and runtime study for the 1D LIDG and RIDG methods with \(M_{\text{deg}} = 3\) and \(M_{\text{deg}} = 5\) on 1D advection equation (7) with initial condition (70). Shown for various mesh sizes are the runtimes \((T_r)\) measured in seconds normalized by the runtime of the fastest simulation \((0.031793s)\), the relative errors in \(L^1\), \(L^2\), and \(L^\infty\), as well as the estimated convergence rates according to formula (69). We see that for any fixed number of elements, the RIDG method has a shorter runtime (i.e., computational cost) and smaller error.

5.1. 1D convergence tests. We consider the 1D advection equation (7) with \(u = 1\), \(\Omega = [–1, 1]\), periodic BCs, and initial condition:

\(q(0, x) = \sin(16\pi x)\).

We run the code [7] to \(t = 2\) with \(M_{\text{deg}} = 3\) (LIDG: \(\nu = 0.104\), RIDG: \(\nu = 0.9\)) and \(M_{\text{deg}} = 5\) (LIDG: \(\nu = 0.04\), RIDG: \(\nu = 0.9\)) and compare runtimes and errors; the results are shown in Table 4. We see that both methods exhibit the expected convergence rates in \(L^1\), \(L^2\), and \(L^\infty\). For a fixed number of elements, usage of the RIDG method leads to smaller errors. We also notice that for a fixed number of elements the experiment runtime for the RIDG method is shorter than that of the LIDG method – this is due to the increase in the maximum linearly stable CFL number from LIDG to RIDG.
5.2. 2D convergence tests. We consider the 2D advection equation (42) with $u_x = u_y = 1$, $\Omega = [-1, 1]^2$, double periodic BCs, and initial condition:

$$q(0, x, y) = \sin(16\pi x) \sin(16\pi y).$$

We run the code [7] to $t = 2$ with $M_{\text{deg}} = 3$ (LIDG: $\nu = 0.05$, RIDG: $\nu = 0.75$) and $M_{\text{deg}} = 5$ (LIDG: $\nu = 0.03$, RIDG: $\nu = 0.75$) and compare runtimes and errors; the results are shown in Table 5. We again see that both methods exhibit the expected convergence rates in $L^1$, $L^2$, and $L^\infty$. For a fixed number of elements, usage of the RIDG method leads to slightly smaller errors. We also notice that for a fixed number of elements the experiment runtime for the RIDG method is shorter than that of the LIDG method – this is due to the increase in the maximum linearly stable CFL number from LIDG to RIDG.

5.3. 3D convergence tests. We consider the 3D advection equation (61) with $u_x = u_y = u_z = 1$, $\Omega = [-1, 1]^3$, triple periodic BCs, and initial condition:

$$q(0, x, y, z) = \sin(2\pi x) \sin(2\pi y) \sin(2\pi z).$$
The new approach we regionally-implicit name we gave to this new approach is the RIDG method. This method exhibits better error and runtime properties than the LIDG method. For example, in Table 6, we see that both methods exhibit the expected convergence rates in $L^1$, $L^2$, and $L^\infty$. As in the one and two-dimensional settings, the RIDG method exhibits better error and runtime properties than the LIDG method.

### 6. Conclusions

The purpose of this work was to develop a novel time-stepping method for high-order discontinuous Galerkin methods that has improved stability properties over traditional approaches (e.g., explicit SSP-RK and Lax-Wendroff). The name we gave to this new approach is the regionally-implicit DG (RIDG) scheme. RIDG is composed of a prediction step, which is a localized version of a spacetime DG method, and a corrector step, which is an explicit method that uses the solution from the predictor step. With this new scheme we achieved all of the following:

- Developed RIDG schemes for 1D, 2D, and 3D advection;
- Showed that RIDG has larger maximum CFL numbers than explicit SSP-RK and Lax-Wendroff DG;
- Showed that the maximum linearly stable CFL number is bounded below by a constant that is independent of the polynomial order (1D: 1.00, 2D: 0.75, 3D: 0.60);
- Demonstrated experimentally the correct convergence rates on 1D, 2D, and 3D advection equation.

We run the code [7] to $t = 2$ with $M_{\text{deg}} = 3$ (LIDG: $\nu = 0.03$, RIDG: $\nu = 0.6$) and $M_{\text{deg}} = 5$ (LIDG: $\nu = 0.025$, RIDG: $\nu = 0.6$) and compare the error properties of the solution produced by the LIDG and RIDG methods; the results are shown in Table 6. We again see that both methods exhibit the expected convergence rates in $L^1$, $L^2$, and $L^\infty$. As in the one and two-dimensional settings, the RIDG method exhibits better error and runtime properties than the LIDG method.

### Table 6

| mesh  | $T_r$/28.5338s | $L^1$ error | $L^2$ error | $L^\infty$ error |
|-------|----------------|-------------|-------------|-----------------|
| 20$^3$ | 4.34190        | 1.21e-03    | 1.20e-03    | 6.16e-03        |
| 40$^3$ | 71.0540        | 6.82e-05    | 6.95e-05    | 3.92e-04        |
| 80$^3$ | 1143.64        | 4.21e-06    | 4.31e-06    | 2.50e-05        |

Convergence and runtime study for the 3D LIDG and RIDG methods with $M_{\text{deg}} = 3$ and $M_{\text{deg}} = 5$ on 3D advection equation (61) with initial condition (72). Shown for various mesh sizes are the runtimes ($T_r$) measured in seconds normalized by the runtime of the fastest simulation (28.5338s), the relative errors in $L^1$, $L^2$, and $L^\infty$, as well as the estimated convergence rates according to formula (69). We see that for any fixed number of elements, the RIDG method has a shorter runtime (i.e., computational cost) and slightly smaller error.
3D advection examples. All of the methods described in this work were written in a MATLAB code that can be freely downloaded [7]. We demonstrated through numerical examples in 1D, 2D, and 3D that the RIDG is more efficient than the closely related Lax-Wendroff DG.

The next step in this work will be to extend the RIDG approach to nonlinear hyperbolic conservation laws. A crucial development will be to extend existing limiter technology, including non-oscillatory limiters and positivity-preserving limiters, to the case of the RIDG scheme.

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Fig. 4. Stability plots for the two-dimensional LIDG and RIDG methods for various polynomial orders. Shown is a false color plot of $f(\nu_x, \nu_y) + 1$ as defined by (68). Note the different horizontal and vertical scales on the plots of the LIDG and RIDG schemes. The red box in each RIDG plot demonstrates the plotting bounds for the LIDG stability region for the same method order. From these plots we can estimate the maximum CFL number $|\nu|$ as defined by (46): (a) $|\nu| \leq 0.23$ ($M_{\text{deg}} = 1$, LIDG), (b) $|\nu| \leq 1.00$ ($M_{\text{deg}} = 1$, RIDG), (c) $|\nu| \leq 0.08$ ($M_{\text{deg}} = 3$, LIDG), (d) $|\nu| \leq 0.80$ ($M_{\text{deg}} = 3$, RIDG), (e) $|\nu| \leq 0.04$ ($M_{\text{deg}} = 5$, LIDG), (f) $|\nu| \leq 0.75$ ($M_{\text{deg}} = 5$, RIDG), (g) $|\nu| \leq 0.025$ ($M_{\text{deg}} = 7$, LIDG), and (h) $|\nu| \leq 0.75$ ($M_{\text{deg}} = 7$, RIDG).