Comments on fractional instantons in $\mathcal{N} = 2$ gauge theories

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Abstract

$\mathcal{N} = 1^*$ gauge theories are believed to have fractional instanton contributions in the confining vacua. D3 brane probe computations in gravitation dual of large-\(N\) $\mathcal{N} = 2^*$ gauge theories point to the absence of such contributions in the low energy gauge dynamics. We study fractional instantons in $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory from the field theoretical perspective. We present new solutions to the Seiberg-Witten $SU(2)$ monodromy problem with the same perturbative asymptotic, a massless monopole and a dyon singularity on the moduli space, and fractional instanton corrections to $\mathcal{N} = 2$ prepotential in the semi-classical region of the moduli space. We show that fractional instantons lead to infinite monopole (dyon) condensate in mass deformed $\mathcal{N} = 2$ gauge theories.
1 Introduction

Study of non-perturbative effects in 4D gauge theories is an important research direction. Unlike ordinary gauge theories where often the only way of understanding strongly coupled dynamics is to do numerical simulation, a large class of phenomena in supersymmetric gauge theories can be understood analytically and exactly. The latter is due to strong restrictions implied by the supersymmetry on the possible structure of perturbative and non-perturbative effects. Typically, the larger the supersymmetry in the theory, the more constraint is its dynamics. Perturbatively, in gauge theories with $\mathcal{N} = 1$ SUSY the superpotential is not renormalized \[1\], the beta function in $\mathcal{N} = 2$ theories is not modified beyond one-loop order \[2\], and the $\mathcal{N} = 4$ theory is finite \[3\]. There are no non-perturbative corrections to $\mathcal{N} = 4$ beta function, while perturbative beta function of gauge theories with 8 (or less) supercharges is corrected by instantons \[4\].

An interesting question to ask is whether non-perturbative corrections in supersymmetric gauge theories are due only to instantons. There is strong evidence, both from the field theory perspective \[5\], and within the framework of the D-brane engineering of gauge theories \[6\], that fractional instantons, carrying $1/N$ units of instanton charge, are responsible for gaugino condensation and the mass gap in low energy $\mathcal{N} = 1$ $SU(N)$ supersymmetric Yang-Mills theory. Furthermore, the superpotential of the mass deformed $\mathcal{N} = 4$ $SU(N)$ YM-theory (also known as the $\mathcal{N} = 1^*$ theory) was shown in the confining vacua to have fractional instanton expansion \[7\]. The latter result was confirmed in the supergravity dual of $\mathcal{N} = 1^*$ gauge theory constructed by Polchinski and Strassler \[8\]. On the contrary, D-brane construction of $\mathcal{N} = 2$ gauge theories \[9\] \[6\] and the analysis of a D3-probe dynamics in gravitational dual of $\mathcal{N} = 2^*$ gauge theories \[10\] \[11\] suggests that fractional instantons do not play role in the low energy dynamics of these theories.

The purpose of this paper is to study the consequences of presence of fractional instantons in 4D $\mathcal{N} = 2$ gauge theories from the field theoretical perspective. As it is well known, in the standard solution for the low-energy effective action of $\mathcal{N} = 2$ $SU(2)$ SYM theory \[12\], only integer instantons contribute to the prepotential. This is actually an input to the solution, rather than its prediction. In fact, we construct infinitely many new solutions to the Seiberg-Witten $SU(2)$ monodromy problem with the same (perturbative) weak coupling asymptotic, and a pair of additional singularities
on the moduli space where a monopole and a dyon, with the same charges as in [12], respectively becomes massless. These new solutions differ from the original one in that the gauge coupling of the low-energy effective action receives non-perturbative corrections of $1/2$ unit the instanton charge. Though mathematically acceptable, all the new solutions we find fail physically. Specifically, in [12], partial supersymmetry breaking by a soft mass term to the chiral superfield in the $\mathcal{N} = 2$ vector multiplet lifted all vacua of the moduli space, except for those with massless monopole and dyon. One is left then with two $\mathcal{N} = 1$ vacua as predicted in [13]. In our solutions, all the Coulomb branch is lifted under the soft SUSY breaking: the monopole (dyon) condensate is infinite at the moduli space singularities.

The paper is organized as follows. In the next section we discuss the would-be signature of fractional instantons in supergravity dual of large-$N$ gauge theories with 8 supercharges. Field-theoretical analysis of $\mathcal{N} = 2$ $SU(2)$ YM theory with fractional instanton contributions to the prepotential is presented in section 3. We conclude in section 4.

2 Fractional instantons in $\mathcal{N} = 2$ gauge theories in the framework of Maldacena duality

AdS/CFT duality of Maldacena [14] relates strongly coupled superconformal gauge theories to supergravity backgrounds. In [14] it was shown that four dimensional $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory, at large values of ’t Hooft coupling and $N$, has a weakly coupled description as type IIB string theory compactified on $AdS_5 \times S^5$ with $N$ units of the five-form flux through the $S^5$. $AdS_5 \times S^5$ gravitational background represents a near horizon geometry of coincident $N$ D3 branes, and is dual to the origin of the Coulomb moduli space of the gauge theory. The classical $3(N-1)$ complex dimensional moduli space of the gauge theory is not corrected quantum mechanically, and has a very simple interpretation in supergravity as a moduli space of multi-centered solutions with D3-brane charge [14, 15]. The fact that there are no quantum corrections to the moduli space relates to the possibility of moving without obstruction a D3-brane probe in $AdS_5 \times S^5$ background.

The situation is different in the case of gravitation dual of gauge theories with reduced supersymmetry. Gauge theories with 8 supercharges has quantum moduli space which is however different from the classical one.
A typical example is the classical vacuum of the four dimensional $\mathcal{N} = 2$ $SU(2)$ YM theory with unbroken gauge symmetry which does not survive the quantization \cite{12}. This has profound implications for the probe dynamics in dual gravitational backgrounds. To be more specific we concentrate now on the probe computation in the gravitational background dual to the mass deformed $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ $SU(N)$ Yang-Mills theory, also known as $\mathcal{N} = 2^*$ gauge theory. The corresponding supergravity background, which we refer to as PW, was constructed in \cite{10} and the D-brane probe dynamics was discussed in \cite{10, 11}. In the language of four-dimensional $\mathcal{N} = 1$ supersymmetry, the mass deformed $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory consists of a vector multiplet $V$, an adjoint chiral superfield $\Phi$ related by $\mathcal{N} = 2$ supersymmetry to the gauge field, and two additional adjoint chiral multiplets $Q$ and $\tilde{Q}$ which form the $\mathcal{N} = 2$ hypermultiplet. In addition to the usual gauge-invariant kinetic terms for these fields, the theory has additional interactions and hypermultiplet mass term summarized in the superpotential\footnote{The classical Kähler potential is normalized $(2/g_{YM}^2)\text{tr}[(\Phi \Phi + Q \tilde{Q} + \tilde{Q} \tilde{Q})].$}

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \text{tr}(Q \tilde{Q}) + \frac{m}{g_{YM}^2} (\text{tr}Q^2 + \text{tr}\tilde{Q}^2). \quad (2.1)$$

The theory has a classical moduli space of Coulomb vacua parameterized by expectation values of the adjoint scalar

$$\Phi = \text{diag}(a_1, a_2, \cdots, a_N), \quad \sum_i a_i = 0, \quad (2.2)$$

in the Cartan subalgebra of the gauge group. For generic values of the moduli $a_i$ the gauge symmetry is broken to that of the Cartan subalgebra $U(1)^{N-1}$, up to the permutation of individual $U(1)$ factors. Classically, when two or more moduli $a_i$ coincide, the gauge symmetry is appropriately enhanced. The moduli space of a D3-brane probing the PW supergravity background is dual to the projection of the Coulomb branch vacua of $SU(N+1) \rightarrow U(1) \times U(1)^{N-1}$ to that of the probe $U(1)$. If $u$ is the modulus of the $U(1)$ representing the probe, the classical parametrization of the full moduli space (2.2) is given by

$$\Phi = \text{diag}(u, a_1 - u/N, a_2 - u/N, \cdots, a_N - u/N), \quad \sum_i a_i = 0. \quad (2.3)$$
Up to coordinate change, $u$ identifies the position of the probe brane in the supergravity background. Classically, all values of $u$ are allowed. As we already mentioned, this is true quantumly for the corresponding modulus of $\mathcal{N} = 4$ YM theory, resulting in the fact that a D3 probe can be moved freely in $AdS_5 \times S^5$ background. In $\mathcal{N} = 2^*$ gauge theory, the classical $U(1)$ probe modulus $u$ receives quantum corrections \cite{17}. Here, there are no perturbative corrections, but there are instanton corrections which become increasingly important as $u(1 + 1/N) \to a_i$ in (2.3)\footnote{More precisely, in the large $N$ limit, instanton corrections become important as $|u - a_i| \ll |u|/N$ \cite{18,11}.}. As in the case of $SU(2)$ YM theory, the vacua with classical nonabelian gauge symmetry, $u(1 + 1/N) = a_i$, are not preserved by quantum corrections. As the result, one should expect boundaries in the D3-probe moduli space of the $\mathcal{N} = 2^*$ gravitational dual. The boundary of the D3-probe (or more generally Dp-probe) moduli space in gravitational dual of gauge theories with 8 supercharges is nothing but the enhanco of \cite{13}. We would like to emphasize that this boundary is infinitely sharp in the $N \to \infty$ limit only if there are no fractional instanton corrections in theories with 8 supercharges. This is precisely what was found in \cite{13,16,10,11}. The metric on the D3-probe moduli space is related by $\mathcal{N} = 2$ supersymmetry to the imaginary part of the complexified probe $U(1)$ gauge coupling $\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2_{YM}}$. This coupling receives both perturbative and non-perturbative corrections. The perturbative corrections are one-loop exact \cite{4}. From the field theory perspective, it was argued in \cite{11} that nonperturbative corrections due to instantons do not survive 't Hooft limit, thus the metric on the D3 probe moduli space is one-loop exact even non-perturbatively. This result has been confirmed explicitly in \cite{14} by comparing the one-loop $\tau$ computation in the $\mathcal{N} = 2^*$ gauge theory with induced metric on the D3 probe moduli space in the PW geometry. Nonperturbative corrections carrying $p/N$ unit the instanton charge are suppressed as $\exp(-8\pi^2 p/(N^2 g^2_{YM}))$, and thus would contribute in the 't Hooft limit for any finite $p$, invalidating one-loop gauge theory/supergravity agreement of \cite{14}.
3 Fractional instantons in $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory

In this section we study fractional instantons in Seiberg-Witten theory \cite{12}. We will follow the steps of \cite{12} while relaxing the requirement of only integer instanton contributions to the low energy effective action.

Consider $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory in four dimensions. The theory is asymptotically free in the UV and is strongly coupled below dynamically generated scale $\Lambda$. We would like to study the low energy physics of this theory. Classically, the theory has a Coulomb moduli space parameterized by the expectation value of the adjoint scalar

$$\Phi = \text{diag}(a, -a),$$

in the Cartan subalgebra of the gauge group. At the generic point on the moduli space the gauge symmetry is broken to $U(1)$. The entire low-energy effective action $\mathcal{L}$ of an Abelian $\mathcal{N} = 2$ vector multiplet is completely determined in terms of the single prepotential $F \equiv F(\Lambda, a)$ which depends holomorphically on the strong coupling scale of the theory $\Lambda$, and the Coulomb modulus $a$

$$8\pi \mathcal{L} = -\text{Im}[\tau] \left( \partial_\mu a \partial^\mu \bar{a} + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi \right) + \text{Re} \left\{ \tau \left( \frac{i}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} F_{\mu\nu} \tilde{F}^{\mu\nu} - 2 \bar{\lambda} \bar{\sigma}^\mu \partial_\mu \lambda \right) \right\},$$

with

$$\tau = \frac{d^2 F}{d^2 a}. \quad (3.6)$$

In Eq. (3.5) $\psi$ and $\lambda$ are fermionic superpartners of the scalar and the gauge boson respectively. Classically, the prepotential is given by

$$F_{\text{class}} = \frac{1}{2} \tau_0 a^2. \quad (3.7)$$

where $\tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{9a}$ is the bare coupling constant. This prepotential receives quantum corrections. The tree level and one-loop contributions add up to

$$F_{\text{pert}} = \frac{i a^2}{\pi} \ln \left[ \frac{a^2}{\Lambda^2} \right]. \quad (3.8)$$
Higher order perturbative corrections are absent, although there are nonperturbative corrections due to instantons. $N = 2$ lagrangian (3.5) has $U(1)_R$ global symmetry, which is broken by anomaly to a $Z_8$. Thus, a full pre-potential $F$ should at most respect $Z_8$ subgroup of $U(1)_R$. One instanton action violates $U(1)_R$ symmetry by eight units, so assuming that there are no nonperturbative effects that further break this R-symmetry, Seiberg arrived at the following form of the full prepotential at weak coupling [4]

$$F = \frac{ia^2}{\pi} \ln \left[ \frac{a^2}{\Lambda^2} \right] + \frac{1}{2\pi i} a^2 \sum_{\ell=1}^{\infty} c_\ell \left( \frac{\Lambda}{a} \right)^{4\ell}, \quad (3.9)$$

where the $\ell$'th term arises as a contribution of $\ell$ instantons. It is this assumption of the exact $Z_8$ symmetry of the low energy effective $SU(2)$ prepotential that we want to relax. Specifically, we assume that in addition to instanton corrections, there are nonperturbative corrections which carry 1/2 unit the instanton charge. So we demand only $Z_4$ R-symmetry of the quantum prepotential

$$F_{1/2} = \frac{ia^2}{\pi} \ln \left[ \frac{a^2}{\Lambda^2} \right] + \frac{1}{2\pi i} a^2 \sum_{\ell=1}^{\infty} c_\ell \left( \frac{\Lambda}{a} \right)^{2\ell}. \quad (3.10)$$

To proceed with the full solution of the model subject to (3.10), we review physical assumptions of the original Seiberg-Witten solution [12]. We would like to emphasize that in our solution we adopt all constraints listed below. First, the unitarity constrains $\text{Im}\{\tau\} > 0$ throughout the moduli space. As a result, $F$, $a^D \equiv \frac{\partial F}{\partial a}$, $\tau$ are defined only locally on the moduli space. Low-energy electric-magnetic duality [12] implies that $a$ is a multi-valued section on the moduli space as well, and thus can not be a nice global coordinate. Seiberg and Witten thus introduce global coordinate $u$, such that the period section

$$(\frac{a^D}{a}) \sim \left( \frac{i}{\pi} \sqrt{2u} \ln \left[ \frac{u}{\Lambda^2} \right] \right), \quad |u| \gg |\Lambda^2| \quad (3.11)$$

The monodromy of the period section due to the semi-classical singularity is determined by the asymptotic (3.11)

$$(\frac{a^D}{a}) \rightarrow M_\infty \left( \frac{a^D}{a} \right) \equiv \left( \begin{array}{cc} -1 & 4 \\ 0 & -1 \end{array} \right) \left( \frac{a^D}{a} \right). \quad (3.12)$$

We assign $U(1)_R$ charge two to $a$. 

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Second, in addition to the semi-classical singularity on the moduli space at \(u = \infty\), there are precisely two other singularities at \(u = \pm \Lambda^2\). The \(u = -\Lambda^2\) singularity is generated by integrating out massless dyon of charge \((1, -2)\) (the BPS formula determines its exact mass to be \(m^{(1,-2)} = |a^D - 2a|\)), and the \(u = \Lambda^2\) singularity is due to the massless monopole of charge \((1, 0)\) \((m^{(1,0)} = |a^D|)\). These two singularities generate the following monodromies of the period section

\[
\begin{pmatrix} a^D \\ a \end{pmatrix} \rightarrow M^{(1,-2)} \begin{pmatrix} a^D \\ a \end{pmatrix} \equiv \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a^D \\ a \end{pmatrix}, \quad |u + \Lambda^2| \ll |\Lambda^2|, (3.13)
\]

\[
\begin{pmatrix} a^D \\ a \end{pmatrix} \rightarrow M^{(1,0)} \begin{pmatrix} a^D \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a^D \\ a \end{pmatrix}, \quad |u - \Lambda^2| \ll |\Lambda^2|. (3.14)
\]

It is important that (3.13), (3.14) are determined using low-energy electric-magnetic dualities once the charges of the massless states on the moduli space are specified\(^4\). Using above assumptions, Seiberg and Witten identified \(\tau(u)\) with the complex structure of the one-parameter family of tori

\[
y^2 = (x^2 - u)^2 - \Lambda^4, \quad (3.15)
\]

and the section \((a^D, a)\) with the integral of one-form \(\lambda \equiv \frac{1}{\sqrt{2\pi}} \frac{dx}{y} \ln f\) over their homology basis. Note that the strong coupling scale enters as \(\Lambda^4\) in (3.13). As a result, in the semi-classical region \(|a| \gg |\Lambda|\) the prepotential is guaranteed to have only integer instanton expansion (3.9).

We now discuss solution of the SU(2) YM theory with low energy effective prepotential (3.10) in the weakly coupled region of the moduli space. We put \(\Lambda = 1\) and assume the existence of a global coordinate \(f\) on the moduli space. We assume the moduli space singularities to be at \(f = \{0, 1, \infty\}\) with the monodromies of the period section \((a^D(f), a(f))\) given by \(\{M^{(1,-2)}, M^{(1,0)}, M_\infty\}\) respectively. The weak coupling asymptotic is assumed to be

\[
\begin{pmatrix} a^D \\ a \end{pmatrix} \sim \left( \frac{2i}{\pi} \sqrt{f} \ln f \right), \quad f \gg 1. \quad (3.16)
\]

\(^4\)Actually, one has to specify the nature of only one of the two non-perturbative singularities on the moduli space. The monodromy (and charges of a state that generates it) due to the other one is determined from the monodromy algebra: \(M_\infty = M^{(1,0)} \cdot M^{(1,-2)}\).
Comparing (3.16) with (3.11) we thus have
\[ f \sim \frac{u}{2A^2}, \quad |u| \gg |A^2|. \]  
(3.17)

The construction of \( SL(2, \mathbb{Z}) \) sections \((a^D(f), a(f))\) with required weak coupling asymptotic (3.16) and monodromies (3.12), (3.14) and (3.13) is rather simple. We start with the following ansatz for \( a^D \):
\[ a^D = A \left( 1 - \frac{1}{f} \right)^{d_1+1} f^{\delta_2 + 1/2} {}_2F_1 \left( \alpha, \beta, \alpha + \beta + m, 1 - \frac{1}{f} \right), \]  
(3.18)

where \( A \) is a normalization constant and \( m \) is an integer. Above ansatz insures that the only singularities of \( a^D \) occur at \( f = \{0, 1, \infty\} \). The third parameter in the hypergeometric function, \((\alpha + \beta + m)\), is chosen to get a logarithmic singularity in \( a^D \) as \( f \to \infty \). Furthermore, we assume that \( \Gamma(\alpha + \beta + m) \) is finite to have (3.18) well-defined. A set of useful identities among hypergeometric functions can be found in [20]. Comparing the asymptotics of (3.18) as \( f \to \infty \) with (3.16) we find
\[ m = 0, \quad \delta_2 = 0, \quad A = \frac{2i \Gamma(\alpha) \Gamma(\beta)}{\pi \Gamma(\alpha + \beta)}. \]  
(3.19)

The monodromy of (3.18) around \( f = \infty \) determines \( a \). With (3.19), we find
\[ a = f^{1/2} \left( 1 - \frac{1}{f} \right)^{\delta_1+1} {}_2F_1 \left( \alpha, \beta, 1, \frac{1}{f} \right). \]  
(3.20)

Using identifies of [20] it is straightforward to check that the monodromy of \((a^D, a)\) about \( f = 1 \) requires
\[ \alpha = \frac{1}{2} + n, \quad \beta = \frac{1}{2} - n + k, \quad \delta_1 = \delta, \]  
(3.21)

where \( n, d \) are arbitrary integers, and \( k \) is a non-negative integer. Altogether, \( SL(2, \mathbb{Z}) \) sections \( S(k, n, d) \)
\[ S(k, n, d) \equiv \begin{pmatrix} a^D \\ a \end{pmatrix}, \]  

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\[
\begin{aligned}
\tau_{(k,n,d)}(f) &= \frac{d a^P/df}{da/df}, \\
&= \left(\frac{2i}{\pi} \frac{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n+k)}{\Gamma(k+1)} f^{1/2} \left(1 - \frac{i}{f}\right)^{d+1} \right. \\
&\left. \quad \times \left(1 - \frac{i}{f}\right)^{d+1} 2F_1 \left(\frac{1}{2} + n, \frac{1}{2} - n + k, 1 + \frac{i}{f}\right)\right). \\
\end{aligned}
\]
\[ t_2 = 12 d^2 - 12 d \left(-1 + k + k n - n^2\right) + \left[3 \left(59 + 136 n^2 + 48 n^4\right)
+ 4 k^2 \left(17 + 32 n + 12 n^2\right) - 8 k \left(16 + 17 n + 16 n^2 + 12 n^3\right)\right]/64, \]
(3.26)

where
\[ \psi(z) \equiv \frac{d \ln \Gamma(z)}{dz}. \]  
(3.27)

Though any section \( S(k, n, d) \) with \( k \geq 0 \) solves the Seiberg-Witten monodromy problem, restrictions on \( (k, n, d) \) come from the conjectured spectrum of BPS states at the singularities. Suppressing numerical constants, we have

\[ a_D \sim a \sim f^{k-n-d} \ln f + f^{n-d} \]
\[ a_D - 2a \sim f^{k-n-d}, \quad \text{if } k > 2n, \]
\[ a_D \sim a \sim f^{k/2-n-d} \ln f \]
\[ a_D - 2a \sim f^{k/2-n-d}, \quad \text{if } k = 2n, \]
\[ a_D \sim a \sim f^{n-d} \ln f + f^{k-n-d} \]
\[ a_D - 2a \sim f^{n-d}, \quad \text{if } k < 2n, \]  
(3.28)

as \( f \to 0 \), and

\[ a_D \sim (f - 1)^{d+1} \]
\[ a \sim (f - 1)^{d+1-k} + (f - 1)^{d+1} \ln(f - 1), \quad \text{if } k > 0, \]
\[ a_D \sim (f - 1)^{d+1} \]
\[ a \sim (f - 1)^{d+1} \ln(f - 1), \quad \text{if } k = 0, \]  
(3.29)

as \( f \to 1 \). Thus, from (3.29), to have a massless monopole at \( f = 1 \) and massive all the electrically charged particles we must have

\[ k \geq d + 1 \geq 1. \]  
(3.30)

Similarly, from (3.28), to have only massless dyon of charge \((1, -2)\) at \( f = 0 \) singularity

\[ \left\{ n \leq \min[d, k - d - 1]\right\} \cup \left\{ n \geq \max[d + 1, k - d]\right\}. \]  
(3.31)

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Constraints (3.30), (3.31) are mutually compatible; thus it appears we found new solutions to the \( \mathcal{N} = 2 \) \( SU(2) \) monodromy problem with the same weak coupling asymptotic and the same massless states at nonperturbative singularities on the moduli space as in [12]. Generically, new solutions have 1/2-instanton corrections in the semi-classical region of the moduli space.

In the rest of this section we show that all new solutions (note that \( S(1,0,0) \equiv S(1,1,0) \) corresponds to the Seiberg-Witten solution) are in fact unphysical: they predict that giving mass to the chiral multiplet \( \Phi \) in \( \mathcal{N} = 2 \) vector multiplet breaks the supersymmetry completely. On the contrary, when a mass of \( \Phi \) is much larger than the strong coupling scale of the \( \mathcal{N} = 2 \) theory, we should be able to reliably integrate it out, thus ending up with \( \mathcal{N} = 1 \) \( SU(2) \) Yang-Mills theory which is predicted to have two vacua [13]. The analysis below repeat those of [12].

Breaking \( \mathcal{N} = 2 \) supersymmetry down to \( \mathcal{N} = 1 \) is achieved by adding a superpotential for the chiral multiplet in \( \mathcal{N} = 2 \) vector multiplet

\[
W = m\text{Tr}\Phi^2.
\] (3.32)

In the low energy effective theory the operator \( \text{Tr}\Phi^2 \) is represented by a chiral superfield \( f \). Its first component is the scalar field \( f \) whose expectation value is

\[
< f > = < \text{Tr}\Phi^2 > .
\] (3.33)

It was argued in [12] that adding (3.32) microscopically corresponds to adding

\[
W_{eff} = mf ,
\] (3.34)

to the low energy effective superpotential. At a generic point on the moduli space there are no light chiral fields, so (3.34) is the complete superpotential. Thus perturbation (3.34) lifts all such \( \mathcal{N} = 2 \) vacua. The situation is different near the singularities on the moduli space. Near the \( f = 1 \) singularity there are massless monopoles. The monopoles can be represented by ordinary (local) chiral superfields \( M \) and \( \tilde{M} \), as long as we describe the gauge field by the dual to the original photon, \( a^D \). The complete superpotential is then

\[
W_{f=1} = \sqrt{2}a^D M \tilde{M} + mf ,
\] (3.35)

where the first term represents \( \mathcal{N} = 2 \) superpotential of the \( m = 0 \) theory. F-term equations from (3.34) give

\[
\sqrt{2}M \tilde{M} + m \frac{df}{da^D} = 0 ,
\]
\[ a^D M = a^D \bar{M} = 0. \]  
(3.36)

Using (3.29), eq. (3.30) has a solution (there is \( \mathcal{N} = 1 \) vacuum) provided

\[ \frac{df}{da^D} \neq \infty \quad \text{at} \quad f = 1. \]  
(3.37)

Along with (3.30), eq. (3.37) implies that

\[ d = 0, \ k \geq 1. \]  
(3.38)

Identical analysis at the dyon singularity, \( f = 0 \), shows that \( \mathcal{N} = 1 \) vacuum there exists provided

\[ \frac{df}{d(a^D - 2a)} \neq \infty \quad \text{at} \quad f = 0. \]  
(3.39)

Eqs. (3.31) and (3.39) give

\[ \left\{ n = k - d - 1, \ 2d + 1 \geq k \right\} \cup \left\{ n = d + 1, \ 2d + 1 \geq k \right\}. \]  
(3.40)

Combining (3.38) and (3.40) we conclude that only sections \( S(1, 0, 0) \equiv S(1, 1, 0) \) predict a pair of \( \mathcal{N} = 1 \) supersymmetric vacua for the mass deformed \( \mathcal{N} = 2 \ SU(2) \) YM theory. These sections are precisely the Seiberg-Witten solution of the model, which do not have fractional instantons in the semi-classical region of the moduli space.

### 4 Conclusion

D3 brane probe computation in gravitational dual of large-N gauge theories with \( \mathcal{N} = 2 \) supersymmetry suggests that, unlike \( \mathcal{N} = 1 \) supersymmetric gauge theories, these theories do not have fractional instantons. The evidence comes primarily from the facts that the enhancon is a sharp boundary of a D-brane probe moduli space, and the agreement of the metric on the probe moduli space with the one-loop beta-function computation in the dual gauge theory.

In this paper we studied fractional instantons in the \( \mathcal{N} = 2 \) supersymmetric gauge theories from the field theoretical perspective. On the example
of $SU(2)$ Yang-Mills theory we showed that though it is possible to construct new solutions to the Seiberg-Witten monodromy problem with the same perturbative asymptotic, but fractional instanton corrections in the semi-classical region of the moduli space, these solutions are unphysical. Specifically, they predict that the soft mass term to the chiral superfield in $\mathcal{N} = 2$ vector multiplet breaks the supersymmetry completely. Our analysis points out that allowing fractional instantons in the semi-classical prepotential would drive monopole (dyon) condensate to infinity in mass deformed $\mathcal{N} = 2$ theories.

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