On Roots of Eigenpolynomials for Degenerate Exactly-Solvable Differential Operators

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Abstract

In this paper we partially settle our conjecture from [1] on the roots of eigenpolynomials for degenerate exactly-solvable operators. Namely, for any such operator we establish a lower bound (which supports our conjecture) for the largest modulus of all roots of its unique and monic eigenpolynomial $p_n$ as the degree $n$ tends to infinity. The main theorem below thus extends earlier results obtained in [1] for a restrictive class of operators.

1 Introduction

We are interested in roots of eigenpolynomials satisfying certain linear differential equations. Namely, consider an operator

$$T = \sum_{j=1}^{k} Q_j D^j$$

where $D = d/dz$ and the $Q_j$ are complex polynomials in one variable satisfying the condition $\text{deg } Q_j \leq j$, with equality for at least one $j$, and in particular $\text{deg } Q_k < k$ for the leading term. Such operators are referred to as degenerate exactly-solvable operators, see [1]. We are interested in eigenpolynomials of $T$, that is polynomials satisfying

$$T(p_n) = \lambda_n p_n$$

for some value of the spectral parameter $\lambda_n$, where $n$ is a positive integer and $\text{deg } p_n = n$. The importance of studying eigenpolynomials for these operators is among other things motivated by numerous examples coming from classical orthogonal polynomials, such as the Laguerre and Hermite polynomials, which

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1 Correspondingly, operators for which $\text{deg } Q_k = k$ are called non-degenerate exactly-solvable operators. We have treated roots of eigenpolynomials for these operators in [2].
appear as solutions to (1) for certain choices on the polynomials $Q_j$ when $k = 2$. Note however that for the operators considered here the sequence of eigenpolynomials $\{p_n\}$ is in general not an orthogonal system.

Let us briefly recall our previous results:

A. In [2] we considered eigenpolynomials of non-degenerate exactly-solvable operators, that is operators of the above type but with the condition $\deg Q_k = k$ for the leading term. We proved that when the degree $n$ of the unique and monic eigenpolynomial $p_n$ tends to infinity, the roots of $p_n$ stay in a compact set in $\mathbb{C}$ and are distributed according to a certain probability measure which is supported by a tree and which depends only on the leading polynomial $Q_k$.

B. In [1] we studied eigenpolynomials of degenerate exactly-solvable operators ($\deg Q_k < k$). We proved that there exists a unique and monic eigenpolynomial $p_n$ for all sufficiently large values on the degree $n$, and that the largest modulus of the roots of $p_n$ tends to infinity when $n \to \infty$. We also presented an explicit conjecture and partial results on the growth of the largest root. Namely,\footnote{It remains to prove the existence of $\mu_T$ and to describe its support explicitly.}

Conjecture (from [1]). Let $T = \sum_{j=1}^k Q_j D^j$ be a degenerate exactly-solvable operator of order $k$ and denote by $j_0$ the largest $j$ for which $\deg Q_j = j$. Let $r_n = \max\{|\alpha| : p_n(\alpha) = 0\}$, where $p_n$ is the unique and monic $n$th degree eigenpolynomial of $T$. Then

$$\lim_{n \to \infty} \frac{r_n}{n^d} = c_0,$$

where $c_0 > 0$ is a positive constant and

$$d := \max_{j \in [j_0+1,k]} \left( \frac{j - j_0}{j - \deg Q_j} \right).$$

Extensive computer experiments listed in [1] confirm the existence of such a constant $c_0$. Now consider the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$. We construct the probability measure $\mu_n$ by placing a point mass of size $1/n$ at each zero of $q_n$. Numerical evidence indicates that for each degenerate exactly-solvable operator $T$, the sequence $\{\mu_n\}$ converges weakly to a probability measure $\mu_T$ which is (compactly) supported by a tree. In [1] we deduced the algebraic equation satisfied by the Cauchy transform of $\mu_T$: Namely, let

$$T = \sum_{j=1}^k Q_j D^j = \sum_{j=1}^k \left( \sum_{i=0}^{\deg Q_j} q_{j,i} z^i \right) D^j$$

and denote by $j_0$ the largest $j$ for which $\deg Q_j = j$. Assuming wlog that $Q_{j_0}$ is monic, i.e. $q_{j_0,j_0} = 1$, we have

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} q_{1,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where $C(z) = \int \frac{d\mu_T(z)}{z-\xi}$ is the Cauchy transform of $\mu_T$ and $A = \{ j : (j-j_0)/(j - \deg Q_j) = d \}$, where $d$ is defined in the conjecture. Below we present some
typical pictures of the roots of the scaled eigenpolynomial \( q_n(z) = p_n(n^d z) \).

**Fig.1:**

![Roots of roots of roots of \( q_{100}(z) = p_{100}(100z) \)]

**Fig.2:**

![Roots of \( q_{100}(z) = p_{100}(100z) \)]

**Fig.3:**

![Roots of \( q_{100}(z) = p_{100}(100z) \)]

In this paper we extend the results from [1] by establishing a lower bound for \( r_n \) for all degenerate exactly-solvable operators and which supports the above conjecture. This is our main result:

**Main Theorem.** Let \( T = \sum_{j=1}^{k} Q_j D^j \) be a degenerate exactly-solvable operator and denote by \( j_0 \) the largest \( j \) for which \( \deg Q_j = j \). Let \( p_n \) be the unique and monic \( n \)th degree eigenpolynomial of \( T \) and \( r_n = \max\{|\alpha| : p_n(\alpha) = 0\} \). Then there exists a positive constant \( c > 0 \) such that

\[
\lim_{n \to \infty} \frac{r_n}{n^d} \geq c,
\]

where

\[
d := \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{j - \deg Q_j} \right).
\]

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## 2 Proofs

**Lemma 1.** For any monic polynomial \( p(z) \) of degree \( n \geq 2 \) for which all the zeros are contained in a disc of radius \( A \geq 1 \), there exists an integer \( n(j) \) and an absolute constant \( C_j \) depending only on \( j \), such that for every \( j \geq 1 \) and

\footnote{It is still an open problem to prove the upper bound.}
For every $n \geq n(j)$ we have

$$\frac{1}{C_j} \cdot \frac{n^j}{A^j} \leq \left\| \frac{p^{(j)}(z)}{p(z)} \right\|_{2A} \leq C_j \cdot \frac{n^j}{A^j}$$

where $p^{(j)}(z)$ denotes the $j$th derivative of $p(z)$, and where we have used the maximum norm $\|p(z)\|_{2A} = \max_{|z|=2A} |p(z)|$.

**Remark.** The right-hand side of the above inequality actually holds for all $n \geq 2$, whereas the left-hand side holds for all $n \geq n(j)$.

**Proof.** To obtain the inequality on the right-hand side we use the notation 
$p(z) = \prod_{i=1}^{n}(z - \alpha_i)$ where by assumption $|\alpha_i| \leq A$ for every complex root of $p(z)$. Then $p^{(j)}(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms, each being the product of $(n-j)$ factors $(z - \alpha_i)$. Thus $p^{(j)}(z)/p(z)$ is the sum of $n(n-1)\cdots(n-j+1)$ terms, each equal to 1 divided by a product consisting of $n - (n-j) = j$ factors $(z - \alpha_i)$. If $|z| = 2A$ we get $|z - \alpha_i| \geq A \Rightarrow \frac{1}{|z - \alpha_i|} \leq \frac{1}{A}$, and thus

$$\left\| \frac{p^{(j)}}{p} \right\|_{2A} \leq \frac{n(n-1)\cdots(n-j+1)}{A^j} \leq C_j \cdot \frac{n^j}{A^j}.$$ 

Here we can choose $C_j = 1$ for all $j$, but we refrain from doing this since we will need $C_j$ large enough to obtain the constant $1/C_j$ in the left-hand side inequality. To prove the left-hand side inequality we will need inequalities (i)-(iv) below, where we need (i) to prove (ii), and we need (ii) and (iii) to prove (iv), from which the left-hand side inequality of this lemma follows.

For every $j \geq 1$ we have

(i) $\left\| \frac{d}{dz} \left( \frac{p^{(j)}(z)}{p(z)} \right) \right\|_{2A} \leq j \cdot \frac{n^j}{A^{j+1}}$.

For every $j \geq 1$ there exists a positive constant $C'_j$ depending only on $j$, such that

(ii) $\left\| \frac{p^{(j)}}{p} - \frac{(p')^j}{p^j} \right\|_{2A} \leq C'_j \cdot \frac{n^{j-1}}{A^j}$.

(iii) $\left\| \frac{p'}{p} \right\|_{2A} \geq \frac{n}{3A}$.

For every $j \geq 1$ there exists a positive constant $C''_j$ and some integer $n(j)$ such that for all $n \geq n(j)$ we have

(iv) $\left\| \frac{p^{(j)}}{p} \right\|_{2A} \geq C''_j \cdot \frac{n^j}{A^j}$.

To prove (i), let $p(z) = \prod_{i=1}^{n}(z - \alpha_i)$, where $|\alpha_i| \leq A$ for each complex root $\alpha_i$.
\(\alpha_i\) of \(p(z)\). Then again \(p^{(j)}(z)/p(z)\) is the sum of \(n(n-1)\cdots(n-j+1)\) terms and each term equals 1 divided by a product consisting of \(j\) factors \((z-\alpha_i)\).

Differentiating each such term we obtain a sum of \(j\) terms each being on the form \((-1)\) divided by a product consisting of \((j+1)\) factors \((z-\alpha_i)\). Thus \(\frac{d}{dz}(\frac{p^{(j)}(z)}{p(z)})\) is a sum consisting of \(j \cdot n(n-1)\cdots(n-j+1)\) terms, each on the form \((-1)\) divided by \((j+1)\) factors \((z-\alpha_i)\). Using \(\frac{1}{|z-\alpha_i|} \leq \frac{1}{A}\) for \(|z| = 2A\) since \(|\alpha_i| \leq A\) for all \(i \in [1,n]\), we thus get

\[
\left|\frac{d}{dz}\left(\frac{p^{(j)}(z)}{p(z)}\right)\right|_{2A} \leq \frac{j \cdot n(n-1)\cdots(n-j+1)}{A^{j+1}} \leq j \cdot \frac{n^j}{A^{j+1}}.
\]

To prove (ii) we use (i) and induction over \(j\). The case \(j = 1\) is trivial since \(\frac{p'}{p} - \frac{(p')^2}{p^2} = 0\). If we put \(j = 1\) in (i) we get \(\left|\frac{d}{dz}\left(\frac{p^{(1)}}{p}\right)\right|_{2A} \leq \frac{n}{A}\). But 
\[
\left|\frac{d}{dz}\left(\frac{p^{(j)}}{p}\right)\right|_{2A} \leq C_p \cdot \frac{n^{j-1}}{A^p}.
\]

Also note that with \(j = p\) in (i) we have
\[
\left|\frac{d}{dz}\left(\frac{p^{(p)}}{p}\right)\right|_{2A} = \left|\frac{d}{dz}\left(\frac{p^{(p)}}{p}\right)\right|_{2A} \leq p \cdot \frac{n^p}{A^{p+1}},
\]
and also \(\left|\frac{p'}{p}\right|_{2A} \leq \frac{n}{A}\) (from the right-hand side inequality of this lemma). Thus we have

\[
\left|\frac{p^{(p+1)}}{p} - \frac{(p')^{p+1}}{p^{p+1}}\right|_{2A} = \left|\frac{p^{(p+1)}}{p} - \frac{(p')^{p+1}}{p^{p+1}}\right|_{2A} \leq \left|\frac{p^{(p+1)}}{p} - \frac{(p')^{p+1}}{p^{p+1}}\right|_{2A} \leq p \cdot \frac{n^p}{A^{p+1}} + C_p \cdot \frac{n^{p-1}}{A^p} = (p + C_p) \cdot \frac{n^p}{A^{p+1}}.
\]

To prove (iii) observe that \(\frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{(z-\alpha_i)} = \sum_{i=1}^{n} \frac{1}{z - \alpha_i} = \frac{1}{\frac{1}{z} - \frac{1}{\alpha_i}} = \frac{|\alpha_i|}{|z|} \leq \frac{1}{\frac{1}{2} |w_i|}\) for all \(i \in [1,n]\). Writing \(w_i = \frac{1}{z - \alpha_i}\) we obtain

\[
|w_i - 1| = \left|\frac{1}{1 - \alpha_i} - \frac{1 - \alpha_i}{1 - \alpha_i}\right| = \left|\frac{1 - \alpha_i}{1 - \alpha_i}\right| \leq \frac{1}{2} |w_i|,
\]

5 With \(D = d/dz\) consider for example \(D \frac{1}{\prod_{i=1}^{n} (z - \alpha_i)} = -1 \cdot \frac{1}{\prod_{i=1}^{n} (z - \alpha_i)}\), which is a sum of \(j\) terms, each being on the form \((-1)\) divided by a product consisting of \(2j - (j - 1) = (j + 1)\) factors \((z-\alpha_i)\).
which implies
\[
Re\left(\frac{1}{1 - \frac{1}{z}}\right) = Re(w_i) \geq \frac{2}{3} \quad \forall i \in [1, n] \Rightarrow Re\left(\sum_{i=1}^{n} \frac{1}{1 - \frac{1}{z}}\right) \geq \frac{2n}{3}.
\]

Thus
\[
\left\| \frac{p'(z)}{p(z)} \right\|_{2A} = \max_{|z|=2A} \left| \frac{p'(z)}{p(z)} \right| = \max_{|z|=2A} \frac{1}{|z|} \cdot \left| \sum_{i=1}^{n} \frac{1}{1 - \frac{1}{z}}\right|
\geq \frac{1}{2A} \cdot \left| \sum_{i=1}^{n} \frac{1}{1 - \frac{1}{z}}\right|_{2A} \geq \frac{1}{2A} \cdot Re\left(\sum_{i=1}^{n} \frac{1}{1 - \frac{1}{z}}\right)
\geq \frac{n}{3A}.
\]

To prove (iv) we note that from (iii) we obtain \(\left\| \left(\frac{p'_j}{p}\right)^j \right\|_{2A} \geq \frac{n^j}{3A} \), and this together with (ii) yields
\[
\left\| \frac{p'(z)}{p} \right\|_{2A} = \left\| \left(\frac{p'_j}{p}\right)^j + \frac{p'(z)}{p} \right\|_{2A} \geq \left\| \left(\frac{p'_j}{p}\right)^j \right\|_{2A} - \left\| \frac{p'(z)}{p} \right\|_{2A} \geq C'_j \cdot n^j \geq C''_j \cdot n^j,
\]
where \(C''_j\) is a positive constant such that \(C''_j \leq \left(\frac{1}{3^j} - \frac{C'_j}{n}\right)\) for all \(n \geq n(j)\).

The left-hand side inequality in this lemma now follows from (iv) if we choose the constant \(C_j\) on right-hand side inequality so large that \(\frac{1}{C_j} \leq C''_j\).

To prove Main Theorem we will need the following lemma, which follows from Lemma 1:

**Lemma 2.** Let \(0 < s < 1 \) and \(d > 0\) be real numbers. Let \(p(z)\) be any monic polynomial of degree \(n \geq 2\) such that all its zeros are contained in a disc of radius \(A = s \cdot n^d\), and let \(Q_j(z)\) be arbitrary polynomials. Then there exists some positive integer \(n_0\) and positive constants \(K_j\) such that
\[
\frac{1}{K_j} \cdot n^{d(\deg Q_j - j)} \cdot \frac{s^{\deg Q_j}}{s^j} \leq \left\| Q_j(z) \cdot \frac{p'(z)}{p} \right\|_{2s^n} \leq K_j \cdot n^{d(\deg Q_j - j)} \cdot \frac{s^{\deg Q_j}}{s^j}
\]
for every \(j \geq 1\) and all \(n \geq \max(n_0, n(j))\), where \(n(j)\) is as in Lemma 1.

**Proof.** Let \(Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i}z^i\). Then for \(|z| = 2A >> 1\) we have
\[
|Q(z)|_{2A} = |q_{j,\deg Q_j}| 2^{\deg Q_j} A^{\deg Q_j} \left(1 + O\left(\frac{1}{A}\right)\right).
\]
Proof of Main Theorem. Let \( n = n_0 \) such that \( n \geq n_0 \Rightarrow A \geq A_0 >> 1 \), and thus by Lemma 1 there exists a positive constant \( K_j \) such that the following inequality holds for all \( n \geq \max(n(j), n_0) \) and all \( j \geq 1 \):

\[
\frac{1}{K_j} \cdot \frac{n^j}{A^j} \cdot A^{\deg Q_j} \leq \left\| Q_j(z) \cdot \frac{p(j)}{p} \right\|_{2s} \leq K_j \cdot \frac{n^j}{A^j} \cdot A^{\deg Q_j}.
\]

Inserting \( A = s \cdot n^d \) in this inequality we obtain

\[
\frac{1}{K_j} \cdot \frac{n^j}{s^j n^d} \cdot s^{\deg Q_j} \cdot n^d \cdot \deg Q_j \leq \left\| Q_j(z) \cdot \frac{p(j)}{p} \right\|_{2s} \leq K_j \cdot \frac{n^j}{s^j n^d} \cdot s^{\deg Q_j} \cdot n^d \cdot \deg Q_j
\]

\[
\frac{1}{K_j} \cdot n^{d(\deg Q_j - j)} \cdot s^{\deg Q_j} \leq \left\| Q_j(z) \cdot \frac{p(j)}{p} \right\|_{2s} \leq K_j \cdot n^{d(\deg Q_j - j)} \cdot s^{\deg Q_j}
\]

for every \( j \geq 1 \) and all \( n \geq \max(n_0, n(j)) \).

\( \square \)

Proof of Main Theorem. Let \( d = \max_{j \in [j_0 + 1, k]} \left( \frac{j - j_0}{\deg Q_j} \right) \) where \( j_0 \) is the largest \( j \) for which \( \deg Q_j = j \) in the degenerate exactly-solvable operator \( T = \sum_{j=1}^{k} Q_j D^j \), where \( Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i} z^i \). Let \( p_n(z) \) be the \( n \)th degree monic eigenpolynomial of \( T \) and denote by \( \lambda_n \) the corresponding eigenvalue. Then the eigenvalue equation can be written

\[
\sum_{j=1}^{k} Q_j(z) \cdot \frac{p_n^{(j)}(z)}{p_n(z)} = \lambda_n
\]

where \( \lambda_n = \sum_{j=1}^{j_0} q_{j,j} \cdot \frac{n!}{(n-j)!} \). We will now use the result in Lemma 2 to estimate each term in (3).

* Denote by \( j_m \) the largest \( j \) for which \( d \) is attained. Then \( d = (j_m - j_0) / (j_m - \deg Q_{j_m}) \Rightarrow d(\deg Q_{j_m} - j_m) + j_m = j_0 \), and \( j_m - \deg Q_{j_m} = (j_m - j_0) / d \). By Lemma 2 we have:

\[
\frac{1}{K_{j_m}} \cdot n^{j_0} \cdot \frac{1}{s^{j_m - j_0 - \deg Q_{j_m}}} \leq \left\| Q_{j_m}(z) \cdot \frac{p(j_m)}{p} \right\|_{2s} \leq K_{j_m} \cdot n^{j_0} \cdot \frac{1}{s^{j_m - j_0 - \deg Q_{j_m}}} \]

Note that the exponent of \( s \) is positive since \( j_m > j_0 \) and \( d > 0 \). In what follows we will only need the left-hand side of the above inequality.

* Consider the remaining (if there are any) \( j_0 < j < j_m \) for which \( d \) is attained. For such \( j \) we have (using the right-hand side inequality of Lemma 2):

\[
\left\| Q_j(z) \cdot \frac{p(j)}{p} \right\|_{2s} \leq K_j n^{j_0} \cdot \frac{1}{s^{j_m - j}} = K_j n^{j_0} \cdot \frac{1}{s^{j_m - j_0}} \cdot s^{j_m - j_0}
\]

\[
\leq K_j n^{j_0} \cdot \frac{1}{s^{j_m - j_0}} \cdot s^{j/m - 1}
\]

\( \square \)
where we have used that \((j_{m} - j) \geq 1\) and \(s < 1 \Rightarrow s^{(j_{m} - j)/d} \leq s^{1/d}\).

Consider all \(j_0 < j \leq k\) for which \(d\) is not attained. Then \((j - \deg Q_j) > 0\)
and \((j - j_0)/(j - \deg Q_j) < d \Rightarrow d(\deg Q_j - j) + j < j_0\) and we can write
\(d(\deg Q_j - j) + j \leq j_0 - \delta\) where \(\delta > 0\). Then we have:

\[
\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|_{2sn^d} \leq K_j \cdot n^{d(\deg Q_j - j) + j} \cdot \frac{s^{\deg Q_j}}{s^j} \leq K_j \cdot n^{j_0 - \delta} \cdot \frac{s^{\deg Q_j}}{s^j} \leq K_j \cdot n^{j_0 - \delta} \cdot \frac{1}{s^k}, \tag{6}
\]

where the last inequality follows since \(\deg Q_j \geq 0 \Rightarrow s^{\deg Q_j} \leq s^0 = 1\) and
\(j \leq k \Rightarrow s^j \geq s^k\) since \(0 < s < 1\).

For \(j = j_0\) by definition \(\deg Q_{j_0} = j_0\) and thus:

\[
\left| Q_{j_0}(z) \cdot \frac{p^{(j_0)}}{p} \right|_{2sn^d} \leq K_{j_0} \cdot n^{d(\deg Q_{j_0} - j_0) + j_0} \cdot \frac{s^{\deg Q_{j_0}}}{s^{j_0}} = K_{j_0} \cdot n^{j_0}. \tag{7}
\]

Now consider all \(1 \leq j < j_0 - 1\). Since \(n \geq n_0 \Rightarrow A = sn^d >> 1\) we get
\((sn^d)^{j - \deg Q_j} \geq 1\) and thus:

\[
\left| Q_j(z) \cdot \frac{p^{(j)}}{p} \right|_{2sn^d} \leq K_j \cdot n^{d(\deg Q_j - j) + j} \cdot \frac{s^{\deg Q_j}}{s^j} = K_j \cdot n^j \cdot (sn^d)^{\deg Q_j - j} \leq K_j \cdot n^j \cdot \frac{1}{(sn^d)^{\deg Q_j}} \leq K_j \cdot n^j \leq K_j \cdot n^{j_0 - 1}. \tag{8}
\]

Finally we estimate the eigenvalue \(\lambda_n = \sum_{j=1}^{j_0} q_{j,j} \cdot \frac{n!}{(n-j)!}\), which grows as
\(n^{j_0}\) for large \(n\), since there exists an integer \(n_{j_0}\) and some positive constant \(K'_{j_0}\)
such that for all \(n \geq n_{j_0}\) we obtain:

\[
|\lambda_n| \leq \sum_{j=1}^{j_0} |q_{j,j}| \cdot \frac{n!}{(n-j)!} = |q_{j_0,j_0}| \cdot \frac{n!}{(n-j_0)!} \left[ 1 + \sum_{1 \leq j < j_0} \frac{|q_{j,j}|}{|q_{j_0,j_0}|} \cdot \frac{(n-j_0)!}{(n-j)!} \right] \leq K'_{j_0} \cdot n^{j_0}. \tag{9}
\]

Finally we rewrite the eigenvalue equation \((\ref{3})\) as follows:

\[
Q_{j_m}(z) \cdot \frac{p^{(j_m)}}{p_n(z)} = \lambda_n + \sum_{j \neq j_m} Q_j(z) \cdot \frac{p^{(j)}}{p_n(z)}. \tag{10}
\]
Applying inequalities (5)-(9) to this we obtain

\[
\left| Q_{j_m} \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)} \right|_{2sn^d} \leq |\lambda_n| + \sum_{j \neq j_m} \left| Q_j \cdot \frac{p_n^{(j)}(z)}{p_n(z)} \right|_{2sn^d} \\
\leq K_j \cdot n^{j_0} + K_j \cdot n^{j_0} + \sum_{1 \leq j < j_0} K_j \cdot n^{j_0 - 1} \\
+ \sum_{j_0 < j < k} K_j \cdot n^{j_0 - \delta} \frac{s^1}{s^{jm - j_0}} + \sum_{j_0 < j < j_0} K_j \cdot n^{j_0} \frac{s^1}{s^{jm - j_0}} \\
\leq K \cdot n^{j_0} + K \cdot \frac{n^{j_0 - \delta}}{s^k} + K \cdot n^{j_0} \frac{s^1}{s^{jm - j_0}} \quad (10)
\]

for all \( n \geq \max(n_0, n(j), n_0) \), where \( K \) is some positive constant and \( 0 < s < 1 \).

For the term on the left-hand side of the rewritten eigenvalue equation above we obtain using (4) the following estimation:

\[
\frac{1}{K} \cdot n^{j_0} \cdot \frac{1}{s^{jm - j_0}} \leq \frac{1}{K_{j_m}} \cdot n^{j_0} \cdot \frac{1}{s^{jm - j_0}} \leq \left| Q_{j_m} \cdot \frac{p_n^{(j_m)}(z)}{p_n(z)} \right|_{2sn^d} \quad (11)
\]

for some constant \( K \geq K_{j_m} \) which also satisfies (10). Now combining (10) and (11) we get

\[
\frac{1}{K} \cdot n^{j_0} \cdot \frac{1}{s^{jm - j_0}} \leq K \cdot n^{j_0} + K \cdot \frac{n^{j_0 - \delta}}{s^k} + K \cdot n^{j_0} \frac{s^1}{s^{jm - j_0}}.
\]

Dividing this inequality by \( n^{j_0} \) and multiplying by \( K \) we have

\[
\frac{1}{s^{jm - j_0}} \leq K^2 + K^2 \cdot \frac{1}{n^\delta} \cdot \frac{1}{s^k} + K^2 \cdot \frac{s^1}{s^{jm - j_0}} \\
\iff \frac{1}{s^w} \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta} + K^2 \cdot \frac{s^1}{s^w} \\
\iff \frac{1}{s^w} [1 - K^2 \cdot s^{1/d}] \leq K^2 + \frac{K^2}{s^k} \cdot \frac{1}{n^\delta}, \quad (12)
\]

where \( w = (jm - j_0)/d > 0 \).

In what follows we will obtain a contradiction to this inequality for some small properly chosen \( 0 < s < 1 \) and all sufficiently large \( n \). Since \( j_m \in [j_0 + 1, k] \) we have \( w = (jm - j_0)/d \geq 1/d \), and since \( s < 1 \) we get \( s^w \leq s^{1/d} \Rightarrow 1/s^w \geq 1/s^{1/d} \). **Now choose** \( s^{1/d} = \frac{1}{4K} \), where \( K \) is the constant in (12). Then estimating the left-hand side of (12) we get

\[
\frac{1}{s^w} [1 - K^2 \cdot s^{1/d}] \geq \frac{1}{s^{1/d}} [1 - K^2 \cdot s^{1/d}] = 4K^2 - K^2 = 3K^2
\]
and thus from (12) we have
\[ 3K^2 \leq \frac{1}{sw} \left| 1 - K^2 \cdot s^{1/d} \right| \leq K^2 + \frac{K^2}{sk} \cdot \frac{1}{n^d} \]
\[ \Leftrightarrow \]
\[ 2K^2 \leq \frac{K^2}{sk} \cdot \frac{1}{n^d} \]
\[ \Leftrightarrow \]
\[ n^d \leq \frac{1}{2} \cdot \frac{1}{sk} = \frac{1}{2} (2K)^{2dk}. \]

We therefore obtain a contradiction to this inequality, and hence to inequality (12) and thus to the eigenvalue equation (3), if
\[ n^d > \frac{1}{2} (2K)^{2dk} \]
and
\[ s = \frac{1}{2K^2} \cdot \frac{1}{n^d}. \]
We refer to this as the upper-bound conjecture. Computer experiments confirm that the zeros of the scaled eigenpolynomial \( f_n(z) = p_n(z) \cdot n^d \) are contained in a compact set when \( n \rightarrow \infty \).

3 Open Problems and Conjectures

3.1 The upper bound

Based upon numerical evidence from computer experiments (some of which is presented in [1]) we are led to assert that there exists a constant \( C_0 \), which depends on \( T \) only, such that
\[ r_n \leq C_0 \cdot n^d \]
holds for all sufficiently large integers \( n \). We refer to this as the upper-bound conjecture. Computer experiments confirm that the zeros of the scaled eigenpolynomial \( f_n(z) = p_n(z) \cdot n^d \) are contained in a compact set when \( n \rightarrow \infty \).

3.2 The measures \( \{ \mu_n \} \)

Consider the sequence of discrete probability measures
\[ \mu_n = \frac{1}{n} \sum_{\nu=1}^{n} \delta(\alpha_{\nu}/n^d) \]
where \( \alpha_1, \ldots, \alpha_n \) are the roots of the eigenpolynomial \( p_n \) and \( d \) is as in Definition 1. Assuming (13) the supports of \( \{ \mu_n \} \) stay in a compact set in \( \mathbb{C} \). Next, by a tree we mean a connected compact subset \( \Gamma \) of \( \mathbb{C} \) which consists of a finite union of real-analytic curves and where \( \hat{\mathbb{C}} \setminus \Gamma \) is simply connected (here \( \hat{\mathbb{C}} = \mathbb{C} \cup \infty \) is the extended complex plane). Computer experiments from [1] lead us to the following

Conjecture 1. For each operator \( T \) the sequence \( \{ \mu_n \} \) converges weakly to a probability measure \( \mu_T \) which is supported on a certain tree \( \Gamma_T \).
3.3 The determination of $\mu_T$

Given $T = \sum_{j=1}^{k} Q_j(z) D^j$ and $Q_j(z) = \sum_{i=0}^{\deg Q_j} q_{j,i} z^i$ we obtain an algebraic function $y_T(z)$ which satisfies the following algebraic equation (also see [1]):

$$q_{j_0,j_0} \cdot z^{j_0} \cdot y_{j_0}^T(z) + \sum_{j \in J} q_{j,\deg Q_j} \cdot z^{\deg Q_j} \cdot y_j^T(z) = q_{j_0,j_0},$$

where $J = \{ j : (j - j_0)/ (j - \deg Q_j) = d \}$, i.e. the sum is taken over all $j$ for which $d$ is attained. In addition $y_T$ is chosen to be the unique single-valued branch which has an expansion

$$y_T(z) = \frac{1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \ldots$$

at $\infty \in \hat{C}$. Let $D_T$ be the discriminant locus of $y_T$, i.e. this is a finite set in $\mathbb{C}$ such that the single-valued branch of $y_T$ in an exterior disc $|z| > R$ can be continued to an (in general multi-valued) analytic function in $\mathbb{C} \setminus D_T$. If $\Gamma_T$ is a tree which contains $D_T$, we obtain a single-valued branch of $y_T$ in the simply connected set $\Omega_{\Gamma_T} = \mathbb{C} \setminus \Gamma_T$. It is easily seen that this holomorphic function in $\Omega_{\Gamma_T}$ defines a locally integrable function in the sense of Lebesgue outside the nullset $\Gamma_T$. A tree $\Gamma_T$ which contains $D_T$ is called $T$-positive if the distribution defined by

$$\nu_{\Gamma_T} = \frac{1}{\pi} \cdot \partial y_T / \partial \bar{z}$$

is a probability measure.

3.4 Main conjecture

Now we announce the following which is experimentally confirmed in [1]:

For each operator $T$, the limiting measure $\mu_T$ in Conjecture 1 exists. Moreover, its support is a $T$-positive tree $\Gamma_T$ and one has the equality $\mu_T = \nu_{\Gamma_T}$ which means that when $z \in \hat{C} \setminus \Gamma_T$ the following holds:

$$y_T(z) = \int_{\Gamma_T} \frac{d\mu_T(\zeta)}{z - \zeta}.$$

Remark. For non-degenerate exactly-solvable operators (i.e. when $\deg Q_k = k$) it was proved in [2] that the roots of all eigenpolynomials stay in a compact set of $\mathbb{C}$, and the unscaled sequence of probability measures $\{\mu_n\}$ converge to a measure supported on a tree, i.e. the analogue of the main conjecture above holds in the non-degenerate case.
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