ITERATED CLUB SHOOTING AND THE STATIONARY-LOGIC
CONSTRUCTIBLE MODEL

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Abstract. We investigate iterating the construction of $C(aa)$, the $L$-like inner model constructed using the so-called "stationary-logic". We show that it is possible to force over generic extensions of $L$ to obtain a model of $V = C(aa)$, and to obtain models in which the sequence of iterated $C(aa)$s is decreasing of arbitrarily large order types. For this we prove distributivity and stationary-set preservation properties for countable iterations of club-shooting forcings using mutually stationary sets, and introduce the notion of mutually fat sets which yields better distributivity results even for uncountable iterations.

1. Introduction and preliminaries

1.1. Introduction. The model $C(aa)$, introduced by Kennedy, Magidor and Väänänen in [9], is the model of sets constructible using the so-called stationary-logic $L(aa)$ – first order logic augmented with the quantifiers $aa$ and $\text{stat}$, meaning roughly "for club/stationarily many countable subsets" (see definition 6 below). This is a model of ZF, and one can phrase the formula $\forall x \exists \alpha (x \in L'_\alpha)$ where $L'_\alpha$ is the $\alpha$-th level in the construction of $C(aa)$ (see definition 7). However, it is not always true that $C(aa) \models "V = C(aa)"$, which is equivalent to the question whether $(C(aa))^{C(aa)} = C(aa)$. This is clearly the case if $V = L$, so the interesting question is whether this can hold with $C(aa) \neq L$. In section 3 we show that this is consistent relative to the consistency of ZFC. Next we investigate the possibilities of $C(aa) \neq V = C(aa)$. In such a case, it makes sense to define recursively the iterated $C(aa)$s:

$$C(aa)^{0} = V$$

$$C(aa)^{(\alpha+1)} = C(aa)^{C(aa)^{\alpha}} \text{ for any } \alpha$$

$$C(aa)^{\alpha} = \bigcap_{\beta < \alpha} C(aa)^{\beta} \text{ for limit } \alpha.$$
This type of construction was first investigated by McAloon [11] regarding HOD, where he showed that it is equiconsistent with ZFC that there is a strictly decreasing sequence of iterated HOD of length $\omega$, and the intersection of the sequence can be either a model of ZFC or of ZF + $\neg$AC. Harrington also showed (in unpublished notes, cf. [17]) that the intersection might not even be a model of ZF. Jech [5] used forcing with Suslin trees to show that it is possible to have a strictly decreasing sequence of iterated HOD of any arbitrary ordinal length, and later Zadroźni [16] improved this to an $\text{Ord}$ length sequence. In [17] Zadroźni generalized McAloon’s method and gave a more flexible framework for coding sets by forcing, which he used to give another proof of this result. As HOD can also be described as the model constructed using second order logic (as shown by Myhill and Scott [12]), it is natural to ask which of the results for HOD apply to other such models.

Our goal in this paper is to use Zadroźni’s framework in the context of $C(aa)$, and our main challenges would be in finding the appropriate coding tools for this case. As we are dealing with stationary-logic, the natural candidates are club shooting forcings. We will show how to use such forcings to code sets into the $C(aa)$ construction by choosing exactly which stationary sets we are destroying (out of a predetermined list). The first challenge would be to find a way to iterate such forcings without destroying what we’ve already coded. The second challenge would be to investigate the limit stages of the iterated $C(aa)$ sequence. After some preliminaries regarding stationary sets and intersections of generic extensions, in section 2.2 we use the notion of mutually stationary sets to form countable iterations of club shooting forcings, which will allow us, in sections 3.1 and 3.2, to obtain models of $V = C(aa) \neq L$, and models with descending sequences of iterated $C(aa)$s of countable length.

In order to obtain longer descending sequences of iterated $C(aa)$s, in section 2.3 we introduce the notion of mutually fat sets, and show that these kinds of sets allows us to form arbitrarily large iterations of club shooting forcings. We then provide two ways of obtaining mutually fat sets – using specific $\Box$-sequences (section 2.3.1) and by forcing non-reflecting stationary sets (section 2.3.2). We eventually use the first option, in theorem 49, to obtain $C(aa)$-sequences of any predetermined order-type.

These results should be contrasted with the case of $C^*$ – the model constructed from the cofinality $\omega$ logic. $C^*$ is contained in $C(aa)$, but in [15] we show that having a model with a descending sequence of iterated $C^*$ is equiconsistent with the existence of an inner model with a measurable cardinal, while here we obtain such models and even more over $L$. This further demonstrate the difference in the expressive power of stationary-logic with respect to the $\omega$-cofinality quantifier.

1.2. Stationary sets. In this paper we will have two notions of club and stationary sets – one regarding sets of ordinals, and one regarding countable subsets of some given set.
In most cases it will be clear from the context which notion is used, and otherwise we will state it explicitly. We first recall the definitions. For an ordinal \( \alpha \), a subset \( C \subseteq \alpha \) is called closed if it contains all its limit points and it is club if it is closed and unbounded in \( \alpha \). \( C \) is \( \sigma \)-closed if it contains all limit of its countable subsets. \( S \subseteq \alpha \) is stationary if it intersects every club in \( \alpha \). For regular \( \lambda < \kappa \) we denote by \( E^\kappa_\lambda \) the set \( \{ \alpha < \kappa \mid \text{cf} (\alpha) = \lambda \} \) and similarly for \( \lambda \leq \kappa \) \( E^\kappa_{\leq \lambda} = \{ \alpha < \kappa \mid \text{cf} (\alpha) < \lambda \} \). It is well known that these are stationary sets.

For an arbitrary set \( X \), \( C \subseteq P_\omega_1 (X) \) is called club in \( X \) if there is some algebra \( \mathfrak{A} = \langle X, f_n \rangle_{n < \omega} \) (where \( f_n : X^k \to X \) for some \( k \)) such that \( C = C_\mathfrak{A} := \{ z \in P_\omega_1 (X) \mid \forall n (f_n^0 z^k \subseteq z) \} \) i.e the collection of all subsets of \( X \) closed under all functions of \( \mathfrak{A} \). \( S \subseteq P_\omega_1 (X) \) is called stationary in \( P_\omega_1 (X) \) if for every algebra \( \mathfrak{A} \) on \( X \), \( S \cap C_\mathfrak{A} \neq \emptyset \). Note that club subsets form a filter on \( P_\omega_1 (X) \), and the stationary sets are the positives sets with respect to this filter. Using Skolem functions, the club filter is also generated by club sets which consist of all elementary substructures of some structure on \( X \). And similarly – a stationary set is one that contains an elementary substructure of any structure on \( X \). The notions of club and stationary sets can also be defined using single functions \( F : [X]^{< \omega} \to X \), where a club is the set of all sets closed under such \( F \) and a stationary is such that for each such \( F \) there is a member closed under \( F \); and a club in \( P_\omega_1 (X) \) can also be characterized as a set \( C \) closed under countable unions and unbounded in the sense that for every \( a \in P_\omega_1 (X) \) there is \( c \in C \) such that \( a \subseteq c \). If we say that some set of countable sets, \( S \), is stationary without mentioning any ambient set, we mean that it stationary in \( P_\omega_1 (\cup S) \).

The following lemma connects the two notions. For \( A \subseteq \kappa \) and \( B \subseteq P_\omega_1 (\kappa) \) denote
\[
\tilde{A} = \{ X \in P_\omega_1 (\kappa) \mid \sup X \cap \kappa \in A \}
\]
\[
\tilde{B} = \{ \sup X \cap \kappa \mid X \in B \} .
\]
Notice that \( \tilde{A} = A \), and \( \tilde{B} \supseteq B \), but in the second case there might not be equality, as there may be more subsets having the same suprema as \( B \).

**Lemma 1.** Let \( \kappa > \omega \) be regular, \( A \subseteq E^\kappa_\omega \), \( B \subseteq P_\omega_1 (\kappa) \).

1. \( B \) is club in \( P_\omega_1 (\kappa) \) \( \implies \) \( \tilde{B} \) is \( \sigma \)-closed and unbounded in \( \kappa \).
2. \( A \) is \( \sigma \)-closed and unbounded in \( \kappa \) \( \implies \) \( \tilde{A} \) contains a club in \( P_\omega_1 (\kappa) \).
3. \( \tilde{A} \) is club in \( P_\omega_1 (\kappa) \) \( \implies \) \( A \) is \( \sigma \)-closed and unbounded in \( \kappa \).
4. \( \tilde{B} \) is stationary in \( P_\omega_1 (\kappa) \) \( \implies \) \( \tilde{B} \) is stationary in \( \kappa \).
5. \( A \) is stationary in \( \kappa \) \( \iff \) \( \tilde{A} \) is stationary in \( P_\omega_1 (\kappa) \).

**Proof.** If \( B \) is club in \( P_\omega_1 (\kappa) \), then, using the last characterization we gave, closure under unions gives \( \sigma \)-closure of \( \tilde{B} \) and unboundedness gives unboundedness of \( \tilde{B} \).
Let \( A \subseteq E_\kappa^\omega \). If \( A \) is \( \sigma \)-closed and unbounded, then \( \hat{A} \) contains the set of all \( X \in \mathcal{P}_{\omega_1}(\kappa) \) closed under the function \( \alpha \mapsto \min A \setminus \alpha \).

If \( \hat{A} \) is club in \( \mathcal{P}_{\omega_1}(\kappa) \) then \( A = \hat{A} \) and we’ve already shown that this is \( \sigma \)-closed and unbounded in \( \kappa \).

Now assume \( B \) is stationary in \( \mathcal{P}_{\omega_1}(\kappa) \). If \( C \subseteq \kappa \) is club, then we saw that \( \hat{C} \) contains a club in \( \mathcal{P}_{\omega_1}(\kappa) \) so there is some \( X \in \hat{C} \cap B \), and \( \sup X \cap \kappa \in C \cap \hat{B} \). So \( \hat{B} \) is stationary in \( \kappa \).

At last, assume \( A \) is stationary in \( \kappa \), and let \( C \subseteq \mathcal{P}_{\omega_1}(\kappa) \) be club, then \( \hat{C} \) is \( \sigma \)-closed and unbounded in \( \kappa \). Let \( \hat{C} \) be the closure of \( \check{C} \), which is club, so there is some \( \alpha \in \check{C} \cap A \).

But recall that \( A \) contains only points of cofinality \( \omega \) so \( \alpha \) is a limit point of \( \check{C} \) of countable cofinality, so by \( \sigma \)-closure it is in \( \check{C} \). Hence by definition of \( \check{C} \) there is some \( X \in C \) such that \( \sup X \cap \kappa = \alpha \in A \cap \check{C} \), so \( X \in \hat{A} \cap C \). Hence \( \hat{A} \) is stationary in \( \mathcal{P}_{\omega_1}(\kappa) \). On the other hand if \( \hat{A} \) is stationary in \( \mathcal{P}_{\omega_1}(\kappa) \) then we’ve already shown that \( A = \hat{A} \) is stationary in \( \kappa \).

An important property of stationary sets in \( \mathcal{P}_{\omega_1}(\kappa) \) is that they project upwards and downwards, i.e. they form a tower:

**Lemma 2.** Let \( Y \supseteq X \neq \emptyset \).

1. If \( S \subseteq \mathcal{P}_{\omega_1}(Y) \) is stationary then \( S \downarrow X := \{ Z \cap X \mid Z \in S \} \) is stationary.
2. If \( S \subseteq \mathcal{P}_{\omega_1}(X) \) is stationary then \( S \uparrow Y := \{ Z \in \mathcal{P}_{\omega_1}(Y) \mid Z \cap X \in S \} \) is stationary.

**Proof.**

1. Let \( H : [X]^{<\omega} \to X \), define \( \check{H} : [Y]^{<\omega} \to Y \) by \( \check{H}(\bar{a}) = H(\bar{a} \cap X) \), and let \( \bar{Z} \in S \) closed under \( \check{H} \). Then \( \check{Z} \cap X \in S \downarrow X \) is closed under \( H \).

2. Let \( G : [Y]^{<\omega} \to Y \). Given the variables \( \langle x_i \mid i < \omega \rangle \), enumerate all terms in \( G \) as \( \langle t_i \mid i < \omega \rangle \) such that \( t_i \) is a term in the variables \( x_0, \ldots, x_{i-1} \) (usually there will be dummy variables), and wrap them all in one function \( G' : [Y]^{<\omega} \to Y \) where \( G(\bar{a}) \) is \( t_0(a_0, \ldots, a_{i-1}) \) for \( i = \|\bar{a}\| \).

Let \( x \in X \) and set \( G''(\bar{a}) = G'(\bar{a}) \) if it is in \( X \) and \( x \) otherwise. Let \( Z \in S \) closed under \( G'' \), and \( Z' \) the closure of \( Z \) under \( G \). \( Z \) is countable so it’s closure is also countable. \( Z' \cap X = Z \) since for \( a \in Z' \cap X \), \( a \) is some term in variables from \( Z \), that is it is \( G'(\bar{b}) \) for some \( \bar{b} \in [Z]^{<\omega} \), and if it is in \( X \), by closure under \( G'' \), it would be in \( Z \). So \( Z' \in S \uparrow Y \) and it is closed under \( G \).
Lemma 4. Let $\mathbb{P}$ be a forcing notion and $T \subseteq \mathcal{P}_{\omega_1}(\kappa)$ stationary. Let $\theta \geq \kappa$ be large so that $\mathbb{P} \in H(\theta)$. If for every $a \prec H(\theta)$ with $\mathbb{P} \in a$ such that $a \cap \kappa \in T$ and every $p \in a$ there is an $a$-generic below $p$, then $\mathbb{P} \Vdash T$ is stationary.

Proof. Let $p \in \mathbb{P}$ and $\dot{F}$ a name for a function from $\kappa^{<\omega}$ to $\kappa$. Let $a = \langle H(\theta), \mathbb{P}, p, \dot{F} \rangle$ for large enough $\theta$, and by stationarity there is $a \prec \mathfrak{A}$ such that $a \cap \kappa \in T$. By the assumption there is $p' \leq p$ a-generic. For every $x \in a \cap \kappa^{<\omega}$ there is in $a$ a dense subsets of $\mathbb{P}$ determining the value of $\dot{F}(x)$. $p'$ meets this set, so there is some $y \in a \cap \kappa$ such that $p' \Vdash \dot{F}(x) = y$. So $p'$ forces that $a \cap \kappa \in T$ is closed under $\dot{F}$. For every $p \in \mathbb{P}$ there is $p' \leq p$ forcing that there is an element of $T$ closed under $\dot{F}$. $\dot{F}$ was arbitrary, so indeed $\mathbb{P}$ forces that $T$ is stationary. □

Lemma 5. If $\mathbb{P}$ is a $\sigma$-closed forcing and $T \subseteq \mathcal{P}_{\omega_1}(\kappa)$ is stationary then $\mathbb{P} \Vdash T$ is stationary.

Proof. Let $\theta \geq \kappa$ be large so that $\mathbb{P} \in H(\theta)$. Let $a \prec H(\theta)$ with $\mathbb{P} \in a$ such that $a \cap \kappa \in T$ and $p \in a$. We show there is a generic over $a$ below $p$, which will be enough by the previous lemma. Since $a$ is countable, we can enumerate all its dense open sets $\langle D_n | n < \omega \rangle$, and inductively define a decreasing sequence of conditions in $\mathbb{P} \cap a$ where $p_0 = p$ and $p_{n+1}$ is chosen from $D_n$ below $p_n$ (possible by denseness). By $\sigma$-closure there is $p'$ such that for every $n$, $p' \leq p_n$ and by openness, $p' \in D_n$ for every $n$. So $p'$ is generic over $a$ below $p$. □

1.3. Stationary-logic and its constructible model.

Definition 6. Stationary-logic, denoted $\mathcal{L}(aa)$, is the extension of first-order logic by the following quantifiers:

- $\mathcal{M} \models \operatorname{aas} \varphi(s, t, a) \iff \{ A \in \mathcal{P}_{\omega_1}(M) | \mathcal{M} \models \varphi(A, t, a) \}$ contains a club in $\mathcal{P}_{\omega_1}(M)$

- $\mathcal{M} \models \operatorname{stats} \varphi(s, t, a) \iff \{ A \in \mathcal{P}_{\omega_1}(M) | \mathcal{M} \models \varphi(A, t, a) \} \iff \mathcal{M} \not\models \operatorname{aas} \neg \varphi(s)$ is stationary in $\mathcal{P}_{\omega_1}(M)$

where $a$ is a finite sequence of elements of $M$ and $t$ a finite sequence of countable subsets of $M$.

Following their general framework for models constructed from extended logics of [10], Kennedy Magidor and Väänänen introduce the model constructed from stationary-logic in [9]:
**Definition 7.** \( C(aa) \) is defined by induction:

\[
\begin{align*}
L'_0 &= \emptyset \\
L'_{\alpha+1} &= \text{Def}_{L(aa)}(L'_\alpha) \\
L'_{\beta} &= \bigcup_{\alpha<\beta} L'_{\alpha} \quad \text{for limit } \beta \\
C(C^*) &= \bigcup_{\alpha \in Ord} L'_{\alpha}
\end{align*}
\]

where

\[
\text{Def}_{L(aa)}(M) = \left\{ a \in M \mid (M, \in) \models_{L(aa)} \varphi(a, \vec{b}) \right\} \mid \varphi \in L(aa); \ \vec{b} \in M^{<\omega}
\]

\( C(aa) \) is a model of ZF, but recent results of Magidor and Väänänen show it might not always be a model of AC. However in this paper it will always turn out to be a forcing extension of \( L \), so it will satisfy AC.

**Definition 8.** The sequence of iterated \( C(aa) \)'s is defined recursively by:

\[
\begin{align*}
C(aa)^0 &= V \\
C(aa)^{\alpha+1} &= C(aa)^{C(aa)^\alpha} \text{ for any } \alpha \\
C(aa)^\alpha &= \bigcap_{\beta<\alpha} C(aa)^\beta \quad \text{for limit } \alpha.
\end{align*}
\]

This will be our main object of study in this paper.

### 1.4. Intersections of forcing extensions.

In order to investigate the limit stages of the iterated \( C(aa) \) construction, we need to understand intersections of generic extensions. The basic facts are the following:

**Fact 9.** Let \( B \) be a complete Boolean algebra,

\[
B_0 \supseteq B_1 \supseteq \cdots \supseteq B_\alpha \supseteq \cdots \quad (\alpha < \kappa)
\]

a descending sequence of complete subalgebras of \( B \), \( B_\kappa = \bigcap_{\alpha<\kappa} B_\alpha \), \( G \) a \( V \)-generic filter on \( B \) and for every \( \alpha \leq \kappa, G_\alpha = G \cap B_\alpha \). Then

1. (Grigorieff [4]) \( \bigcap_{\alpha<\kappa} V[G_\alpha] \) satisfies ZF.
2. (Jech [6, lemma 26.6]) If \( B \) is \( \kappa \)-distributive then \( \bigcap_{\alpha<\kappa} V[G_\alpha] = V[G_\kappa] \), and in particular satisfies ZFC.

So one of our challenges would be to obtain the distributivity of the forcing notions we wish to use. To get a more precise result, we will use a characterization by Sakarovitch [14, 13] giving an exact form to \( V[G_\kappa] \) in the \( \kappa \)-distributive case.

**Definition 10.** Let \( P, Q \) be forcing notions. A function \( f : P \to Q \) is called normal iff it is order preserving, \( f''P \) is dense in \( Q \) and

\[
\forall p \in P \forall q \in Q \left( q \leq f(p) \rightarrow \exists p' \in P \left( p' \leq p \land f(p') \leq q \right) \right).
\]
A collection \( \langle \mathbb{P}_\alpha, f_{\alpha\beta} \mid \alpha, \beta < \kappa \rangle \) is called a \( \kappa \)-normal system if for every \( \alpha < \beta < \kappa \), \( f_{\alpha\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta \) is normal, and \( \alpha < \beta < \gamma \rightarrow f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta} \).

**Definition 11.** Let \( \langle \mathbb{P}_\alpha, f_{\alpha\beta} \mid \alpha, \beta < \kappa \rangle \) be a \( \kappa \)-normal system, We define an equivalence relation \( \sim \) on \( \mathbb{P}_0 \) by \( p \sim q \) iff \( \exists \alpha < \kappa (f_{0,\alpha}(p) = f_{0,\alpha}(q)) \), let \( \mathbb{P}_0/\sim \) be the set of equivalence classes \( \{[p] \mid p \in \mathbb{P}_0\} \), and denote \( f_{0,\kappa}(p) = [p] \). We give \( \mathbb{P}_0/\sim \) the natural order:

\[
[p] \leq [q] \iff \exists \alpha < \kappa \forall \beta < \kappa (\beta \geq \alpha \rightarrow f_{0,\beta}(p) \leq f_{0,\beta}(q)).
\]

Note that this doesn’t depend on the choice of representatives so this could be stated for some/all \( p' \in [p] \) and \( q' \in [q] \). Then we have the following fact ([14], see also [13] and [17, section 8.1]):

**Fact 12.** Let \( \langle \mathbb{P}_\alpha, f_{\alpha\beta} \mid \alpha, \beta < \kappa \rangle \) be a \( \kappa \)-normal system, for every \( \alpha \) let \( B_\alpha = \text{ro}(\mathbb{P}_\alpha) \) and \( B_\kappa = \bigcap_{\alpha < \kappa} B_\alpha \). Then the separative quotient of \( \mathbb{P}_0/\sim \) is isomorphic to a dense subset of \( B_\kappa \).

If \( G \) is \( \mathbb{P}_0 \) generic over \( V \), then the set \( f''_{0,\kappa}G \), denoted by \( G/\sim \), is \( \mathbb{P}_0/\sim \)-generic over \( V \), and hence \( V[G_\kappa] = V[G/\sim] \).

**Remark 13.** One should be warned that even if \( \bigcap_{\alpha < \kappa} \mathbb{P}_\alpha \) is trivial, this does not mean that \( \bigcap_{\alpha < \kappa} B_\alpha \) is trivial as well, and in general we cannot claim that \( \bigcap_{\alpha < \kappa} B_\alpha = \text{ro} \left( \bigcap_{\alpha < \kappa} \mathbb{P}_\alpha \right) \).

We sketch a proof in the following case, from which the general case can be derived:

**Proposition 14.** Let \( B \) be a complete Boolean algebra,

\[
B_0 \supseteq B_1 \supseteq \cdots \supseteq B_\alpha \supseteq \cdots (\alpha < \kappa)
\]

a descending sequence of complete subalgebras of \( B \), \( B_\kappa = \bigcap_{\alpha < \kappa} B_\alpha \). For \( b, c \in B \) let \( b \sim c \) iff \( \exists \alpha < \kappa \) such that

\[
\inf \{d \in B_\alpha \mid d \geq b\} = \inf \{d \in B_\alpha \mid d \geq c\}.
\]

Then the separative quotient of \( B_0/\sim \) is isomorphic to \( B_\kappa \).

**Proof.** Recall that the separative quotient of a poset \( P \) is the unique (up to isomorphism) separative \( Q \) such that there is \( h : P \to Q \) order preserving such that \( x \) is compatible with \( y \) iff \( h(x) \) is compatible with \( h(y) \) (see [7, p. 205]). Since \( B_\kappa \) is separative as a Boolean algebra, we want to provide such \( h : B_0/\sim \to B_\kappa^+ \).

Let \( b \in B_0^+ \). For every \( \alpha < \kappa \) let \( b_\alpha = \inf \{d \in B_\alpha \mid d \geq b\} \), and let \( \bar{b} = \sup \{b_\alpha \mid \alpha < \kappa\} \). Note that \( \{b_\alpha \mid \alpha < \kappa\} \) is an ascending sequence, so in fact for every \( \beta \), \( \bar{b} = \sup \{b_\alpha \mid \beta \leq \alpha < \kappa\} \), and this is an element of \( B_\beta \), so all-in-all \( \bar{b} \in B_\kappa \). Now if \( b' \sim b \) then for all large enough \( \alpha \), \( b_\alpha = b'_\alpha \) so \( \bar{b} = \bar{b}' \). So the function \( h([b]) = \bar{b} \) is well defined, and since in particular \( \bar{b} \geq b > 0 \), it is into \( B_\kappa^+ \).

Let \( [\bar{b}], \bar{c} \in B_0/\sim \). We show they are compatible iff \( \bar{b} \) and \( \bar{c} \) are compatible:

- If \( [\bar{b}], [\bar{c}] \) are compatible, \( [d] \leq [\bar{b}], [\bar{c}] \), then for all large enough \( \alpha d_\alpha \leq b_\alpha, c_\alpha \) so \( \bar{d} \leq \bar{b}, \bar{c} \).
Proof. Let $\langle p_i \mid i < \kappa \rangle$ be sequence such that $i < j$ implies $[p_i] > [p_j]$. Let $a_0 = 0$ and for every $i < \kappa$, if $a_i$ is defined then fix $a_{i+1} > a_i$ such that $\forall \beta \geq a_{i+1}, f_{0\beta}(p_{i+1}) \geq f_{0\beta}(p_i)$. For limit $i$ set $a_i = \sup_{j<i} a_j$. Note that $i \leq a_i$ so $\sup_{i < \kappa} a_i = \kappa$. By induction, using the fact that the sequence $\langle a_i \mid i < \kappa \rangle$ is strictly increasing, we get that for every $\beta \geq a_{i+1}$ and $m < n \leq i$ we have $f_{0\beta}(p_m) \geq f_{0\beta}(p_n)$.

We now define $q \in \mathbb{P}_0$ by $q(\xi) = p_i(\xi)$ for the unique $i < \kappa$ such that $\xi \in [a_i, a_{i+1})$. We claim that for every $i$ $[p_i] \geq [q]$, in particular that $\forall \beta \geq a_{i+1}, f_{0\beta}(p_{i+1}) \geq f_{0\beta}(q)$. So fix $\beta \geq a_{i+1}$ and $\xi \geq \beta$. Let $j$ be such that $\xi \in [a_j, a_{j+1})$, so $q(\xi) = p_j(\xi)$. Note that $a_{j+1} > \xi \geq \beta \geq a_{i+1}$ so $j > i$, so using the above claim we have $f_{0\beta}(p_{i+1}) \geq f_{0\beta}(p_j)$ so:

$$f_{0\beta}(p_i)(\xi) \geq f_{0\beta}(p_j)(\xi) = g_{\xi}^{0\beta}(p_j(\xi)) = g_{\xi}^{0\beta}(q(\xi)) = f_{0\beta}(q(\xi))$$

as required. \qed
2. Iterated club shooting

2.1. Shooting one club. The basic tool for changing the notion of stationarity over some model is “club shooting” – adding a club to the complement of a stationary set. For this to be possible, the complement must be “fat”:

**Definition 18.** Let $\kappa$ be a regular cardinal. A set $S \subseteq \kappa$ is called fat iff for every club $C \subseteq \kappa$, $S \cap C$ contains closed sets of ordinals of arbitrarily large order types below $\kappa$.

We say that $T \subseteq \kappa$ is co-fat if $\kappa \setminus T$ is fat.

**Lemma 19** (Abraham and Shelah [1, Lemma 1.2]). Assume $\mu < \kappa$, $\kappa$ regular, and $S \subseteq \kappa$ has the property that for club $C \subseteq \kappa$, $S \cap C$ contains a closed set of ordinals of order-type $\mu + 1$. Then for any $\tau < \mu + 1$ and every club $C \subseteq \kappa$, $S \cap C$ contains a closed set of order-type $\tau + 1$.

**Definition 20.** Let $S$ be a co-fat subset of some regular $\kappa$. We denote by $\operatorname{Des}(S)$ the poset that adds a closed unbounded subset of $\kappa \setminus S$ using bounded conditions. That is, conditions in $\operatorname{Des}(S)$ are closed bounded subsets of $\kappa \setminus S$, ordered by end-extension – $p$ is stronger than $q$ ($p \leq q$) iff $p \cap (\sup q + 1) = q$.

**Fact 21** (Abraham and Shelah [1, Theorem 1]). Let $\kappa$ be either a strongly inaccessible cardinal or the successor of a regular cardinal $\mu$ such that $\mu = \mu^\mu$. Let $S \subseteq \kappa$ be co-fat.

1. Forcing with $\operatorname{Des}(S)$ adds a club $C \subseteq \kappa \setminus S$.
2. $\operatorname{Des}(S)$ is $< \kappa$-distributive i.e. forcing with $\operatorname{Des}(S)$ does not add new sets of size $< \kappa$ (hence cardinals and cofinalities $\leq \kappa$ remain unchanged in an extension by $\operatorname{Des}(S)$).
3. Cardinality of $\operatorname{Des}(S)$ is $2^{< \kappa}$, so if $2^{< \kappa} = \kappa$ cardinals above $\kappa$ are not collapsed.

One way of obtaining co-fat stationary sets is by using $\square$-sequences. Recall that for a regular cardinal $\kappa$, the principle $\square_\kappa$ (see Jensen’s [8]) asserts the existence of a sequence $C = \langle C_\alpha \mid \alpha \in \text{Lim}(\kappa^+) \rangle$ such that

1. $C_\alpha$ is club in $\alpha$;
2. If $\beta \in \text{Lim}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$;
3. If $\text{cf}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$.

**Lemma 22.** Let $\kappa$ be regular such that $\square_\kappa$ holds, and let $S \subseteq \mathcal{E}^\kappa_{< \kappa}$ be stationary. Then there is some stationary $\bar{S} \subseteq S$ which is co-fat.

**Proof.** Let $C = \langle C_\alpha \mid \alpha \in \text{Lim}(\kappa^+) \rangle$ be a $\square_\kappa$ sequence. Note first that if $\alpha < \kappa^+$ is of cofinality $\kappa$, then the order type of $C_\alpha$ is $\kappa$ – otherwise, consider the $\kappa + \omega$th element of $C_\alpha$, say $\beta$. By condition (2), $C_\beta = C_\alpha \cap \beta$ so it is of size $\kappa$, which contradicts condition (3) since $\text{cf}(\beta) = \omega$. Now for every $\gamma \in S$ let $f(\gamma) = \text{otp}(C_\gamma)$. Since $\gamma \in \mathcal{E}^\kappa_{< \kappa}$, $f(\gamma) < \kappa$,
so for every $\gamma \in S \setminus \kappa$ $f(\gamma) < \kappa \leq \gamma$, hence by Fodor’s lemma there is some stationary $\bar{S} \subseteq T$ such that $f$ is constant on $\bar{S}$, say with value $\nu$. Denote $T = \kappa^+ \setminus \bar{S}$ and we want to show that $T$ is fat. By lemma 19 it is enough to show that for any club $C \subseteq \kappa^+$ there is a closed subset of $T \cap C$ of order-type $\kappa + 1$. $C$ is club in $\kappa^+$ and $T$ contains all ordinals $< \kappa^+$ of cofinality $\kappa$ (since $\bar{S} \subseteq E^\kappa_\omega$) so let $\lambda \in T \cap \text{Lim} C$ of cofinality $\kappa$. So $C \cap C_\lambda$ is club in $\lambda$. Let $\gamma$ be a limit point of $C_\lambda \cap C$ such that $\text{otp}(C_\gamma) > \nu$. Since $C_\gamma = C_\lambda \cap \gamma$ has order type $> \nu$, $\gamma \notin \bar{S}$. Since $\lambda$ is of cofinality $\kappa$, $C_\lambda \cap C$ is of order-type $\kappa$ and so also the set of limit points of $C_\lambda \cap C$ with order-type $> \nu$ is of order-type $\kappa$, it is contained in $T$ and converges to $\lambda \in T \cap C$, we get a closed subset of $T \cap C$ of order-type $\kappa + 1$. □

In order to code sets into $C(aa)$, we would like to destroy the stationarity of specific sets. This in itself can be done by a single club shooting, but to get models with $V = C(aa)$ we’d need to iterate this construction and to code this coding into $C(aa)$, and then repeat until we catch our tail. For this we need to be able to iterate club shooting forcing, in a way that later stages don’t destroy our previous codings. This is where the notion of mutual stationarity, introduced by Foreman and Magidor in [3], steps in.

2.2. Shooting countably many clubs.

**Definition 23** ([3, Definition 6]). Let $K$ be a collection of regular cardinals with supremum $\delta$ and suppose that we have $S_\kappa \subseteq \kappa$ for each $\kappa \in K$. Then the collection $\{S_\kappa \mid \kappa \in K\}$ is mutually stationary if and only if for all algebras $\mathfrak{A}$ on $\delta$ (or more generally on $H(\theta)$ for some large enough $\theta$) there is an $N \prec \mathfrak{A}$ such that for all $\kappa \in N \cap K$, $\sup(N \cap \kappa) \in S_\kappa$. We call such $N$ a witness with respect to $\mathfrak{A}$ to the mutual stationarity of the collection.

Every set of stationary sets of cofinality $\omega$ is mutually stationary, witnessed by a countable structure:

**Theorem 24** ([3, Theorem 7]). Let $\langle \kappa_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of regular cardinals with supremum $\delta$, and $\langle S_\alpha \mid \alpha < \theta \rangle$ a sequence of stationary sets such that $S_\alpha \subseteq \kappa_\alpha$ consists of points of cofinality $\omega$. Then for every algebra $\mathfrak{A}$ on $\delta$ there is a countable $N \prec \mathfrak{A}$ such that for all $\alpha \in N \cap \theta$, $\sup(N \cap \kappa_\alpha) \in S_\alpha$.

For our purposes we’ll need to strengthen the above theorem a bit, to incorporate also a stationary subset of $P_{\omega_1}(\lambda)$ for $\lambda < \kappa_0$:

**Theorem 25.** Let $\langle \kappa_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of regular cardinals with supremum $\delta$. Let $\lambda$ be a regular cardinal such that $\lambda^\omega < \kappa_0$. Let $T \subseteq P_{\omega_1}(\lambda)$ be stationary in $P_{\omega_1}(\lambda)$ and for $\alpha < \theta$ let $S_\alpha \subseteq \kappa_\alpha$ stationary of points of cofinality $\omega$. Then for every $\rho \geq \delta$ and an algebra $\mathfrak{A}$ on $\rho$, there is a countable $N \prec \mathfrak{A}$ such that $N \cap \lambda \in T$ and for $\alpha < \theta$ such that $\kappa_\alpha \in N$ we have $\sup(N \cap \kappa_\alpha) \in S_\alpha$.

**Proof.** Fix an algebra $\mathfrak{A}$ on $\delta$. We wish to find a tree $\mathcal{T} \subseteq \delta^{<\omega}$ such that there is a labeling $l : \mathcal{T} \rightarrow \{\kappa_\alpha \mid \alpha < \theta\} \cup \{\lambda\}$, and the following hold (sk$^\mathfrak{A}$ denotes the Skolem hull in $\mathfrak{A}$):
(1) If $\sigma \in \mathcal{T}$ and $l(\sigma) = \kappa_\alpha$ then \( \{ \gamma \mid \sigma \leftarrow \gamma \in \mathcal{T} \} \subseteq \kappa_\alpha \) and has cardinality $\kappa_\alpha$.

(2) If $\sigma \in \mathcal{T}$ and $\kappa_\alpha \in sk^\mathcal{A}(\sigma)$ then there are infinitely many $n < \omega$ such that if $\tau \supset \sigma, \tau \in \mathcal{T}$ has length $n$, then $l(\tau) = \kappa_\alpha$.

(3) If $\sigma \in \mathcal{T}$ and $l(\sigma) = \lambda$ then there is there is a unique $\gamma$ such that $\sigma \leftarrow \gamma \in \mathcal{T}$, and this $\gamma$ is $< \lambda$.

(4) For every branch $b$ of $\mathcal{T}$, $sk^\mathcal{A}(b) \cap \lambda \in T$.

This will suffice as the proof of theorem 24 actually shows the following:

Lemma 26. If $\mathcal{T} \subseteq \delta^{<\omega}$ and $l : \mathcal{T} \rightarrow \{ \kappa_\alpha \mid \alpha < \theta \}$ satisfy (1)+(2) above, then there is a decreasing sequence of subtrees $\mathcal{T}_n$, the length of the stem of $\mathcal{T}_n$ is at least $n$, and for the branch $b = \bigcap_{n < \omega} \mathcal{T}_n, N = sk^\mathcal{A}(b)$ satisfies that for every $\kappa_\alpha \in N, N \cap \kappa_\alpha \in S_\alpha$.

So if $\mathcal{T}$ also satisfies (4) then we get that $N \cap \lambda = sk^\mathcal{A}(b) \cap \lambda \in T$, and incorporating (3) into the proof of the lemma is straightforward.

Claim 27. There are $\mathcal{T}, l$ satisfying (2) as above and also

(1)* If $\sigma \in \mathcal{T}$ then $\{ \gamma \mid \sigma \leftarrow \gamma \in \mathcal{T} \} = l(\sigma)$.

Proof. We build $\mathcal{T}, l$ by induction on the length of $\sigma$. Let $\langle p_n \mid n < \omega \rangle$ be an increasing enumeration of all prime numbers. For each $\sigma \in \lambda^{<\omega}$, the set $\{ \alpha < \theta \mid \kappa_\alpha \in sk^\mathcal{A}(\sigma) \} \}$ is at most countable. Enumerate it as $\langle \alpha_n^\sigma \mid n < \omega \rangle$.

Start with $l(\langle \rangle) = \lambda$. Assume that $\sigma \in \mathcal{T}$ and $l(\sigma)$ are defined. Then we let the successors of $\sigma$ in $\mathcal{T}$ be $\{ \sigma \leftarrow \gamma \mid \gamma < l(\sigma) \}$ to get condition (1)*. If the length of $\sigma$ is of the form $p_k \cdot p_{k+n+1}^{m+1} - 1$ for some $k, n, m < \omega$, then we let $l(\sigma \leftarrow \gamma) = \kappa_{\alpha_n^\sigma \leftarrow p_k}$ for every $\gamma$. Otherwise $l(\sigma \leftarrow \gamma)$ is arbitrary. In this way, if $\sigma \in \mathcal{T}$ and $\kappa_\alpha \in sk^\mathcal{A}(\sigma)$, let $k = \text{length}(\sigma)$ and $n$ such that $\kappa_\alpha = \kappa_{\alpha_n^\sigma}$, then for every $\tau \supset \sigma$ of length of the form $p_k \cdot p_{k+n+1}^{m+1}$ we have that $l(\tau) = \kappa_{\alpha_n^\sigma \leftarrow p_k} = \kappa_{\alpha_n^\sigma} = \kappa_\alpha$, so condition (2) holds.

Assume $\mathcal{T}, l$ are as in the claim, satisfying (1)*+(2). We want to define a tree $\mathcal{T}'$, satisfying (1)-(4). We begin by defining games $\mathcal{G}_n$ for $a \in \mathcal{P}_{\omega_1}(\lambda)$ as follows. We have two players, B and G, such that the choices of G define a branch through $\mathcal{T}$. Fix in advance an enumeration $\langle \gamma_n \mid n < \omega \rangle$ of $a$. At each play, if $\sigma \in \mathcal{T}$ is defined by the choices of G, then:

- If $l(\sigma) \neq \lambda$, B chooses some $D \subseteq l(\sigma)$ of cardinality $< l(\sigma)$, and G chooses some $\gamma \in l(\sigma) \setminus D$ such that $\sigma \leftarrow \gamma \in \mathcal{T}$.
- If $l(\sigma) = \lambda$ then G chooses the first $\gamma_n \in a$ not chosen yet.

G wins if the game defines a branch $b$ through $\mathcal{T}$, such that $sk^\mathcal{A}(b) \cap \lambda = a$. Otherwise B wins. Note that this is an open game for B: Since by the construction there are infinitely many $n$s such that any node of length $n$ is labeled by $\lambda$, we must have $a \subseteq sk^\mathcal{A}(b)$. So B wins if at some stage the game constructs a $\sigma \in \mathcal{T}$ such that $(sk^\mathcal{A}(\sigma) \cap \lambda) \setminus a \neq \emptyset$, hence B’s payoff is open. Hence the game is determined.
Claim 28. There is a club $C \subseteq P_{\omega_1}(\lambda)$ such that for every $a \in C$, $G$ has a winning strategy in $\mathcal{G}_a$.

Proof. Otherwise, there is a stationary $S \subseteq P_{\omega_1}(\lambda)$ such that for every $a \in S$ $B$ has a winning strategy $s_a$ in $\mathcal{G}_a$. So, for $\theta$ large enough, let

$$N \prec \langle H(\theta), \langle s_a \mid a \in S \rangle, \mathfrak{A}, \mathcal{I}, l, \lambda, \ldots \rangle$$

be countable such that

$$T \upharpoonright H(\theta) = \{ X \in P_{\omega_1}(H(\theta)) \mid X \cap \lambda \in T \}.$$ 

Denote $N_0 = N \cap \lambda$. The assumption is that in every play of $\mathcal{G}_{N_0}$ where $B$ plays according to $s_{N_0}$, if the play gives the branch $b$ then $sk^\mathfrak{A}(b) \cap \lambda \neq N_0$. We wish to arrive at a contradiction by describing a play where this fails. We construct this play inductively such that every choice $G$ makes is of an ordinal from $N$. So we assume $\sigma \in \mathcal{I} \cap N$ was defined, and consider two cases:

1. $l(\sigma) \neq \lambda$. In this case $B$ plays according to the strategy $s_{N_0}$ a subset $s_{N_0}(\sigma) \subseteq l(\sigma)$ of cardinality $< l(\sigma)$. For every $a \in T$, $s_a(\sigma)$ is a subset of $l(\sigma)$ of cardinality $< l(\sigma)$. $|T| = \lambda^\omega$ which is less than the regular $l(\sigma)$, so the cardinality of $U = \{ s_a(\sigma) \mid a \in T \}$ is less than $l(\sigma)$. Since $\sigma, (s_a \mid a \in T) \in N$, also $l(\sigma), U \in N$, and so $N \models |U| < l(\sigma)$. So there is some $\gamma \in N \cap l(\sigma) \setminus U$ (in particular $\gamma \notin s_{N_0}(\sigma)$) such that $\sigma \upharpoonright \gamma \in \mathcal{I}$. Then $G$ chooses such $\gamma$.

2. $l(\sigma) = \lambda$. In this case by the rules of the game $G$ chooses the first element of $N_0$ which wasn’t chosen yet (according to some enumeration of order-type $\omega$).

Let $b$ be the branch constructed in the game. We wish to show that $sk^\mathfrak{A}(b) \cap \lambda = N_0$. As we noted earlier, by the cases $l(\sigma) = \lambda$ we get $sk^\mathfrak{A}(b) \cap \lambda \supseteq N_0$. On the other hand, since $N$ is in particular closed under the functions of $\mathfrak{A}$, and all elements of $b$ were chosen from $N$, we get $sk^\mathfrak{A}(b) \cap \lambda \subseteq N \cap \lambda = N_0$. So indeed $sk^\mathfrak{A}(b) \cap \lambda = N_0$, so $G$ wins, contradicting the fact that this is a game where $B$ plays according to $s_{N_0}$.

Now fix $C$ as in the lemma, let $a \in T \cap C$, and let $s$ be a winning strategy for $G$ in $\mathcal{G}_a$. We wish to define a tree $\mathcal{I}' \subseteq \mathcal{I}$ such that (1)+(2) hold, and every branch in $\mathcal{I}'$ is obtained as a play in $\mathcal{G}_a$ where $G$ plays according to $s$. This will give us (3). Note that (2) will remain true in any subtree of $\mathcal{I}$ such that any node is contained in an infinite branch, so the main task is to choose nodes in a way preserving (1). We work by induction – assume that $\sigma \in \mathcal{I}'$ was constructed as a partial play in $\mathcal{G}_a$ where $G$ plays according to $s$, and we determine which of its successors to put in next. If $l(\sigma) = \lambda$ then there is exactly one option – $\sigma \upharpoonright s(\sigma)$ (note (1) doesn’t talk about the case $l(\sigma) = \lambda$). Otherwise, we inductively define a sequence $\langle \gamma_\nu \mid \nu < l(\sigma) \rangle \subseteq l(\sigma)$ as follows. If $\langle \gamma_\nu \mid \nu < \mu \rangle$ is already defined, let $\gamma_\mu$ be the response of $G$ to $B$ playing $\{ \gamma_\nu \mid \nu < \mu \}$. To conclude, we let the
successors of $\sigma \in T'$ be $\{\sigma \smallsetminus \gamma_\nu \mid \nu < l(\sigma)\}$. So condition (1) is indeed satisfied, and by the construction any branch through $T'$ is given as a play according to $s$, as required. □

The following is a useful remark made at the beginning of [3, section 7]:

Remark 29.  
(1) Any subset of a sequence of mutually stationary sets is mutually stationary.
(2) If $N < \langle H(\theta), \in, \Delta, \ldots \rangle = A$ (where $\theta > \delta$ is regular) and $\nu \in N$, then for all regular $\mu \in N \smallsetminus (\nu + 1)$,
   \[
   \sup(N \cap \mu) = \sup(\sup(A(N \cup \nu) \cap \mu)).
   \]
   In particular, if $N$ witnesses the mutual stationarity of a sequence $\langle S_\kappa \mid \kappa \in K \rangle$ with respect to $A$, $K \subseteq N$ and $\nu \in N \cap \min K$ then also $\sup(A(N \cup \nu))$ is such a witness.

We now show that using mutual stationarity allows us to iterate countably many club shooting forcings, in a way which is $\omega$-distributive and preserves the stationarity of sets we do not wish to destroy.

Theorem 30. Let $K = (\kappa_\alpha \mid \alpha < \theta)$ be an increasing sequence of uncountable cardinals such that for every regular $\lambda$ and every $\kappa \in K$, $\lambda < \kappa \Rightarrow \lambda^\omega < \kappa^1$. Let $\delta = \sup K$. For each $\alpha$ fix some stationary $T_\alpha \subseteq E_\omega^{\omega^\omega}$. Assume that $\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \theta \rangle$ is a forcing iteration with countable support, and for each $\alpha$ there is a $P_\alpha$ name $\dot{S}_\alpha$ such that
\[
P_\alpha \Vdash \dot{S}_\alpha \text{ is a co-fat subset of } \kappa_\alpha, T_\alpha \cap \dot{S}_\alpha = \emptyset, \text{ and } \dot{Q}_\alpha = \text{Des}(\dot{S}_\alpha).
\]
(1) $P_0$ is $\omega$-distributive.

Let $\lambda$ be regular
(2) Let $T \subseteq E_\omega^{\omega^\omega}$ stationary (in $V$) such that if $\lambda = \kappa_\alpha$ for some $\alpha$, then $P_0 \Vdash \dot{T} \cap \dot{S}_\alpha = \emptyset$. Then $P_0 \Vdash \dot{T}$ is stationary.
(3) Let $T \subseteq P_{\omega_1}(\lambda)$ be stationary (in $V$) such that for every $a \in T$ and every $\kappa_\alpha \in a$ (and if $\lambda = \kappa_\alpha$, then also for $\lambda$), $P_0 \Vdash \sup(a \cap \kappa_\alpha) \notin \dot{S}_\alpha$. Then $P_0 \Vdash T$ is stationary.
(4) Assume that for every $\alpha$ $\dot{Q}_\alpha = \text{Des}(\dot{S}_\alpha)$ for $S_\alpha \in V$.
   (a) Let $T \subseteq P_{\omega_1}(\lambda), T \in V$. Then $P_0 \Vdash T$ is stationary iff $V \Vdash \dot{T}$ is stationary where
   \[
   \dot{T} = \{a \in T \mid \forall \kappa_\alpha \in a, \sup(a \cap \kappa_\alpha) \notin \dot{S}_\alpha\}.
   \]
   (b) $C(aa)^{V[P_0]} \subseteq V$.

Proof. For (1), let $\{D_n \mid n < \omega\} \subseteq P_0$ be open dense, $p \in P_0$ and we wish to find $p' \leq p$ in $\bigcap_{n < \omega} D_n$. Consider the set $\{T_\alpha \mid \alpha < \theta\}$ of stationary sets, each consisting of points of cofinality $\omega$. By theorem 24, there is some countable
\[
N \smallsetminus \langle \delta, K, P_0, p, \{T_\alpha \mid \alpha < \theta\}, D_k, \alpha \rangle_{k < \omega, \alpha \in \text{supp}(p)}
\]
\[\overset{\text{This will hold for example if there is } \mu < \kappa_0 \text{ such that GCH holds starting at } \mu.}{\text{(1)}}\]
(note that supp(p) is countable so this is possible) such that for every \(\kappa_\alpha \in N\), \(\sup N \cap \kappa_\alpha \in T_\alpha\). Enumerate all the dense open sets of \(P_\theta\) in \(N\) by \(\langle d_n \mid n < \omega\rangle\), and inductively define \(p_n \in N \cap P_\theta\), by setting \(p_0 = p\) and choosing \(p_{n+1} \in N\) to be some extension of \(p_n\) in \(d_n\). Let \(s = \bigcup \text{supp}(p_n)\) (note that \(s \subseteq N\) and is countable), and define \(p'\) to be the function on \(\theta\) such that for \(\alpha \in s\), \(p'(\alpha) = \bigcup_{n<\omega} p_n(\alpha) \cup \{\sup \bigcup_{n<\omega} p_n(\alpha)\}\) and for \(\alpha \in \theta \setminus s\), \(p'(\alpha) = \emptyset\). By a density argument, for every \(\alpha \in s\), \(\bigcup_{n<\omega} p_n(\alpha)\) is cofinal in \(\sup N \cap \kappa_\alpha\), and by assumption \(\sup N \cap \kappa_\alpha \in T_\alpha\) which is forced to be disjoint from \(\hat{S}_\alpha\), so \(p'\) is indeed a condition in \(P_\theta\) with support \(s\) extending each \(p_n\). Since each \(d_n\) is open, \(p' \in d_n\), and since every \(D_k\) is also open dense in \(N\), it equals some \(d_n\), so \(p' \in \bigcap_{k<\omega} D_k\) as required.

(2) follows from (3) by using lemma 1.5, so we only prove (3). Let \(p \in P_\theta\) and \(\hat{F}\) be a \(P_\theta\) name forced by \(p\) to be a function from \(\lambda^{<\omega}\) to \(\lambda\). We wish to find \(p' \leq p\) forcing that there is an element of \(T\) closed under \(\hat{F}\). Let \(\mathcal{A} = \langle H(\mu), f_n \mid n < \omega\rangle\) be an algebra (for \(\mu\) large enough) extending
\[
\langle H(\mu), K, P_\theta, p, \lambda, \hat{F}, \{T_\alpha \mid \alpha < \theta\}, T, \alpha \rangle_{\alpha \in \text{supp}(p)}
\]
(note that supp(p) is countable so this is possible) such that the functions of \(\mathcal{A}\) are closed under composition.

By theorem 25 applied to \(T\) and \(\{T_\alpha \mid \alpha < \theta \land \kappa_\alpha > \lambda\}\), there is \(N < \mathcal{A}\) countable such that \(N \cap \lambda \in T\) and for every \(\kappa_\alpha \in N \setminus \lambda + 1\), \(\sup N \cap \kappa_\alpha \in T_\alpha\) (if \(\lambda \geq \sup K\) then simply use the stationarity of \(T\)). Now as in the previous clause, we can define a decreasing sequence of conditions \(\langle p_n \mid n < \omega\rangle \subseteq N\), starting with \(p_0 = p\), that meets every dense open set of \(N\). As before let \(s = \bigcup \text{supp}(p_n) \subseteq N\), and define \(p'\) to be the function on \(\theta\) such that for \(\alpha \in s\), \(p'(\alpha) = \bigcup_{n<\omega} p_n(\alpha) \cup \{\sup \bigcup_{n<\omega} p_n(\alpha)\}\) and for \(\alpha \in \theta \setminus s\), \(p'(\alpha) = \emptyset\). \(p'\) will be a condition if we show that for every \(\alpha \in s\), \(\sup \bigcup_{n<\omega} p_n(\alpha) \notin \hat{S}_\alpha\). Again by a density argument, for every \(\alpha \in s\) (in particular \(\alpha \in N\)), \(\sup \bigcup_{n<\omega} p_n(\alpha) = \sup N \cap \kappa_\alpha\). If \(\kappa_\alpha > \lambda\) then this is in \(T_\alpha\), which is forced to be disjoint from \(\hat{S}_\alpha\). If \(\kappa_\alpha \leq \lambda\), then since \(N \cap \lambda \in T\), the assumption is exactly that \(P_\theta \models \sup(N \cap \kappa_\alpha) \notin \hat{S}_\alpha\). Hence \(p' \in P_\theta\).

Now, \(p_0\) forces that \(\hat{F}\) is a function on \(\lambda^{<\omega}\), so for each \(x \in N \cap \lambda^{<\omega}\) there is in \(N\) a dense set of conditions determining the value of \(F(x)\), which is an element of \(N \cap \lambda\), and so \(p'\) forces some value for \(F(x)\). So eventually \(p'\) determines some function \(\hat{F}\) with domain \(\lambda^{<\omega} \cap N\) and range in \(N\) such that \(p' \models \hat{F}^N \cap \lambda = \hat{F}\). So \(p'\) forces that \(N \cap \lambda \in T\) is closed under \(\hat{F}\), as required.

For (4)(a), first note that by (3) if \(V \models \hat{T}\) is stationary then \(P_\theta \models \hat{T}^\theta\) is stationary but \(\hat{T} \subseteq T\) so also \(P_\theta \models \hat{T}^\theta\) is stationary. Second, assume \(P_\theta \models \hat{T}^\theta\) is stationary but \(V \models \hat{T}\) is not stationary. Let \(G \subseteq P_\theta\) be generic. In \(V[G]\), \(T\) is stationary while \(\hat{T}\) is not, hence
\[
T \setminus \hat{T} = \{a \in T \mid \exists \kappa_\alpha \in a(\sup(a \cap \kappa_\alpha) \in S_\alpha)\}
\]
is stationary in \( \mathcal{P}_{\omega_1}((\lambda)^{V[G]}) \). From \( G \) we can derive a function \( F(\cdot, \cdot) \) such that for \( \alpha < \theta \), \( F(\kappa_\alpha, \cdot) \) is the function corresponding to the club \( G \cap \kappa_\alpha \) i.e. \( \alpha \) is closed under \( F(\kappa_\alpha, \cdot) \)

\[ \sup(a \cap \kappa_\alpha) \in G \cap \kappa_\alpha. \]

So let \( \mathfrak{A} = \langle H(\mu), F \rangle \) for \( \mu \) large enough. The set \( R = \{ b \in \mathcal{P}_{\omega_1}(\mu) \mid b \cap \lambda \in T \setminus \tilde{T} \} \) is stationary, so there is some \( b \in R \) closed under \( F \). Then on one hand, there is \( \kappa_\alpha \in b \cap \lambda \) such that \( \sup(b \cap \kappa_\alpha) \in S_\alpha \). But on the other hand, \( b \) is closed under \( F(\kappa_\alpha, \cdot) \) so \( \sup(b \cap \kappa_\alpha) \in G \cap \kappa_\alpha \), but \( G \cap \kappa_\alpha \) is disjoint from \( S_\alpha \), a contradiction.

For (4)(b), let \( W \) be a generic extension of \( V \) by \( P_b \) and denote the \( \alpha \)th level in the construction of \( C(aa)^W \) by \( L'_\alpha \). We prove by induction that for every \( \alpha \) the construction up to \( \alpha \) can be done in \( V \) — so the sequence \( \langle L'_\alpha \mid \alpha \in \text{Ord} \rangle \) is definable in \( V \) and in particular every level is both contained in \( V \) and an element of \( V \). \( L'_\alpha \) is clear and the limit case is immediate from the induction hypothesis. So assume the assumption holds up to (and including) \( \alpha \) and \( X \in \text{Def}_{L(aa)}(L'_\alpha) \), i.e. for some \( \varphi \in L(aa), b \in L'_\alpha <^\omega \) and \( t \in \mathcal{P}_{\omega_1}(L'_\alpha)^{<\omega} \)

\[ X = \{ a \in L'_\alpha \mid W \models [L'_\alpha, \in] \models L(aa) \varphi(a, b, t) \}. \]

We prove by induction on the complexity of \( \varphi \) that the relation

\[ W \models [L'_\alpha, \in] \models L(aa) \varphi(a, b, t) \]

can in fact be determined in \( V \). The only interesting case is when \( \varphi = \text{stats}\psi \) for some \( \psi \in L(aa) \), so set \( \mathcal{M} = (L'_\alpha, \in) \) and we need to show that

\[ W \models (\mathcal{M} \models \text{stats}\psi(s, a, b, t)) \]

can by determined in \( V \). Note that by the induction assumption on \( L'_\alpha \) and \( \omega \)-distributivity, \( a, b, t \in V \), and the countable sets \( s \) considered in evaluating \( \psi \) are the same in \( V \) and \( W \). So we assume inductively that for any such \( s, a, b, t \), the relation \( W \models (\mathcal{M} \models \psi(s, a, b, t)) \) can be determined in \( V \). Now for fixed \( a, b, t \), the relation \( W \models (\mathcal{M} \models \text{stats}\psi(s, a, b, t)) \) is equivalent to "\( W \models T \) is stationary" where

\[ T = \{ s \in \mathcal{P}_{\omega_1}(L'_\alpha) \mid W \models (\mathcal{M} \models \psi(s, a, b, t)) \}. \]

\( T \in V \) since \( W \models (\mathcal{M} \models \psi(s, a, b, t)) \) can be determined in \( V \), so by clause (4)(a), "\( W \models T \) is stationary" is equivalent to "\( V \models \tilde{T} \) is stationary" where

\[ \tilde{T} = \{ a \in T \mid \forall \kappa_\alpha \in a (\sup(a \cap \kappa_\alpha) \notin S_\alpha) \} \in V, \]

so \( W \models (\mathcal{M} \models \text{stats}\psi(s, a, b, t)) \) can also be determined in \( V \).

Wrapping up, we can correctly determine in \( V \) for every \( a \in L'_\alpha \) if it will satisfy \( \varphi = \text{stats}\psi \) after the extension, so in fact we can calculate \( X \) in \( V \). So every element of \( L'_{\alpha+1} \) is in \( V \), and the entire \( L'_{\alpha+1} \) can be computed in \( V \).

This will allow us to force over \( L \), and even over generic extensions of \( L \), with a countable iteration of club shooting forcing. This will be enough to get a proper extension of
the ground model satisfying $V = C(aa)$ (see theorem 41), and also extensions with a decreasing $C(aa)$ sequence of length $\omega$.

However, to get longer sequences we will need longer iterations while still preserving enough distributivity. For this it will not be enough to have a mutually stationary sequence of sets each of which is co-fat on its own – we will need a notion of “mutual fatness”.

2.3. **Shooting uncountably many clubs.**

**Definition 31.** Let $K$ be a collection of regular cardinals with supremum $\delta$ and suppose that we have $S_\kappa \subseteq \kappa$ for each $\kappa \in K$. Then the collection $\{S_\kappa \mid \kappa \in K\}$ is **mutually $\nu$-fat** (for $\nu < \min K$) if and only if for all algebras $\mathfrak{A}$ on $\delta$ (or more generally on $H(\theta)$ for some large enough $\theta$) there is an increasing and continuous sequence of models $N_\alpha \prec \mathfrak{A}$, $\alpha \leq \nu$ such that for all $\alpha \leq \nu$ and $\kappa \in N_\alpha \cap K$, $\sup(N_\alpha \cap \kappa) \in S_\kappa \cap N_{\alpha + 1}$. We say the collection is **mutually $\mu$-fat** if it is mutually $\nu$-fat for every regular $\nu < \mu$ and that it is simply **mutually fat** if it is mutually $< \min K$-fat. We say that the collection is **strongly mutually ($\nu$-)fat** if for every algebra, mutual fatness is witnessed by models containing $K$ as a subset (or equivalently by models containing $|K|$ as a subset).

**Remark 32.** Note that if $\{S_\kappa \mid \kappa \in K\}$ are mutually $\nu$-fat witnessed with respect to some $\mathfrak{A}$ by a sequence $\langle N_\alpha \mid \alpha \leq \nu \rangle$ then for every regular $\mu \leq \nu$, $N_\mu$ witnesses with respect to $\mathfrak{A}$ that the sequence $\{S_\kappa \cap E_\mu^\kappa \mid \kappa \in K\}$ is mutually stationary.

**Theorem 33.** Let $K = \langle \kappa_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of uncountable cardinals, $\theta < \kappa_0$. Let $\delta = \sup_{\alpha < \theta} \kappa_\alpha$. Assume that $\mathcal{T} = \langle T_\alpha \mid \alpha < \theta \rangle$ is a strongly mutually fat sequence $(T_\alpha \subseteq \kappa_\alpha)$. Assume that $\langle P_\alpha, Q_\alpha \mid \alpha < \theta \rangle$ is a forcing iteration with full support, and for each $\alpha$ there is a $P_\alpha$-name $\dot{S}_\alpha$ such that

$$P_\alpha \not\Vdash \dot{S}_\alpha \subseteq E_{\kappa_\alpha} \setminus T_\alpha \text{ and } Q_\alpha = \text{Des}(\dot{S}_\alpha).$$

(1) $P_0$ is $< \kappa_0$-distributive.

Assume additionally that $\text{GCH}$ holds starting below $\kappa_0$. Let $\lambda$ be regular.

(2) Let $T \subseteq E^\lambda_{\kappa_\alpha}$ stationary (in $V$) such that if $\lambda = \kappa_\alpha$ for some $\alpha$, then $P_0 \not\Vdash \dot{T} \cap \dot{S}_\alpha = \emptyset$. Then $P_0 \not\Vdash \dot{T}$ is stationary.

(3) Let $T \subseteq P_{\omega_1}(\lambda)$ be stationary (in $V$) such that for every $\alpha \in T$ and every $\kappa_\alpha \in a$ (and if $\lambda = \kappa_\alpha$ then also for $\lambda$), $P_0 \not\Vdash \sup(a \cap \kappa_\alpha) \notin S_\alpha$. Then $P_0 \not\Vdash \dot{T}$ is stationary.

(4) Assume that for every $\alpha$ $\dot{Q}_\alpha = \text{Des}(\dot{S}_\alpha)$ for $S_\alpha \in V$.

(a) Let $T \subseteq P_{\omega_1}(\lambda), T \in V$. Then $P_0 \not\Vdash T$ is stationary iff $V \not\Vdash \dot{T}$ is stationary where

$$\dot{T} = \{a \in T \mid \forall \kappa_\alpha \in a \sup(a \cap \kappa_\alpha) \notin S_\alpha\}.$$

(b) $C(aa)^{V_{P_0}} \subseteq V$.

**Proof.** For (1) let $\bar{D} = \langle D_\alpha \mid \alpha < \nu \rangle$, $\nu < \kappa_0$, be a sequence of dense open sets in $P_0$ and let $p \in P_0$. We can assume without loss of generality that $\nu$ is a regular cardinal (for
singular \( \nu \) we can work by induction and set \( \bigcap_{\alpha<\nu} D_\alpha = \bigcap_{\beta<\cf(\nu)} (\bigcap_{\alpha<\gamma_\beta} D_\alpha) \) for some \( \langle \gamma_\beta \mid \beta < \cf(\nu) \rangle \) cofinal in \( \nu \). For convenience we will assume the dense open sets are indexed only by successor ordinals. From the assumption of strong mutual fatness applied to \( (H(\lambda), \in, K, P_0, p, D, \ldots) \) there is an increasing continuous sequence of elementary submodels, \( \langle N_\alpha \mid \alpha \leq \nu \rangle \) such that \( P_\theta, p, D \in N_0, \max(\theta, \nu) \subseteq N_0, \) such that for all \( \alpha \leq \nu \) and \( \beta < \theta \), \( \sup(N_\alpha \cap \kappa_\beta) \in T_\beta \). We define a decreasing sequence of conditions in \( P_\theta - \langle p_\alpha \mid \alpha \leq \nu \rangle \) such that for every \( \alpha < \nu \) \( p_\alpha \in N_{\alpha+1} \), for successor \( \alpha \) \( p_\alpha \in D_\alpha \) and for every \( \beta < \theta \) \( \max p_\alpha(\beta) = \sup(N_\alpha \cap \kappa_\beta) \). Begin with \( p_0 = p \). If \( p_\alpha \) is defined, choose some \( p'_{\alpha+1} \in D_{\alpha+1} \cap N_{\alpha+1} \) extending \( p_\alpha \) and define \( p_{\alpha+1} \) such that for every \( \beta < \theta \) \( p_{\alpha+1}(\beta) = p'_{\alpha+1}(\beta) \cup \{ \sup(N_\alpha \cap \kappa_\beta) \} \). By our choice \( \sup p'_{\alpha+1}(\beta) < \sup(N_\alpha \cap \kappa_\beta) \in T_\beta \cap N_{\alpha+2} \) so this is indeed a condition in \( N_{\alpha+2} \). If \( \alpha \leq \nu \) is limit and \( \langle p_\gamma \mid \gamma < \alpha \rangle \) are defined, we define for every \( \beta < \theta \) \( p_\alpha(\beta) = \bigcup_{\gamma<\alpha} p_\gamma(\beta) \cup \{ \sup \bigcup_{\gamma<\alpha} p_\gamma(\beta) \} \). Since the sequence \( \langle N_\gamma \mid \gamma < \alpha \rangle \) is increasing and continuous, by the induction hypothesis

\[
\sup_{\gamma<\alpha} \bigcup p_\gamma(\beta) = \sup \{ \sup(N_\gamma \cap \kappa_\beta) \mid \gamma < \alpha \} = \sup(N_\alpha \cap \kappa_\beta) \in T_\beta
\]

and if \( \alpha < \nu \) then this is also an element of \( N_{\alpha+1} \) by assumption. So this is indeed a condition in \( P_\theta \). Now, since the sets \( D_\alpha \) are open, we get inductively that \( p_\alpha \in \bigcap_{\gamma<\alpha} D_\gamma \) and in particular \( p_\nu \in \bigcap_{\gamma<\nu} D_\gamma \) as required.

For (2) and (3), first assume that \( \lambda \notin K \). Then

\[
P_\theta = P_{<\lambda} * P_{\lambda<} = \{ p \mid \lambda \in K, p \in P_\theta \}.
\]

\( P_{<\lambda} \) is \( \omega \)-distributive and, by the GCH assumption, of size \( \lambda \), and \( P_{\lambda<} \) is \( \lambda \)-distributive, so neither one destroys stationary sets at \( \lambda \) or at \( P_{\omega_1}(\lambda) \). Now assume \( \lambda \in K \). Then \( P_\theta = P_{<\lambda} * P_{\lambda} * P_{\lambda<} \) where as before \( P_{<\lambda} \) and \( P_{\lambda<} \) don’t destroy stationarity at \( \lambda \), and for \( P_\lambda \) we can apply theorem 30 to get that stationarity of \( T \) is preserved.

The proof of (4) is the same as in (4) of theorem 30. \( \square \)

We will now present two methods of obtaining strongly mutually fat sets – by using \( \Box \)-sequences, and by forcing a sequence of non-reflecting stationary sets.

2.3.1. \( \Box \)-sequence on singulars. Recall the so-called “Global \( \Box \) principle” (see Jensen’s [8]) that asserts the existence of a sequence \( C = \langle C_\alpha \mid \alpha \) a singular ordinal \rangle \) such that

1. \( C_\alpha \) is club in \( \alpha \);
2. \( \otp(C_\alpha) < \alpha \);
3. If \( \beta \in \text{Lim}(C_\alpha) \) then \( C_\beta = C_\alpha \cap \beta \).

For an ordinal \( \theta \) we say that \( \Box(\theta) \) holds if there is a sequence

\[
C = \langle C_\alpha \mid \alpha < \theta \text{ a singular limit ordinal} \rangle
\]

2Strictly speaking we should say that for every \( \beta < \theta \), \( p_\alpha \mid \beta \) forces that \( \max p_\alpha(\beta) = \sup(N_\alpha \cap \kappa_\beta) \), but for clarity we would not mention this explicitly.
Theorem 34. Let $K = \{\kappa_\xi \mid \xi < \theta\}$ be a set of regular uncountable cardinals such that $\kappa^{+} < \kappa_0$, $\kappa_0$ not the successor of a singular, and that the GCH holds starting from some cardinal $\psi$ such that $\psi^{+} < \kappa_0$. Let $\kappa^* = \sup K$ and assume $C = \langle C_\alpha \mid \alpha \in \text{Sing}_{\kappa^*+1} \rangle$ is a $\Box_{(\kappa^*+1)}$-sequence and $\langle S_\kappa \mid \kappa \in K \rangle$ is a sequence of sets such that $S_\kappa \subseteq E^\kappa_\kappa$ is stationary in $\kappa$. Then there is a sequence $\langle S_\kappa \mid \kappa \in K \rangle$ such that for every $\kappa \in K$ $S_\kappa \subseteq S_\kappa$ is stationary in $\kappa$ and the sequence $\langle T_\kappa = \kappa \cap S_\kappa \mid \kappa \in K \rangle$ is strongly mutually fat.

Proof. We begin as in lemma 22. Considering the function $\gamma \mapsto \text{otp}(C_\gamma)$, we apply Fodor’s lemma at every $S_\kappa$ for $\kappa \in K$ to obtain $\langle S_\kappa \mid \kappa \in K \rangle$ a sequence of stationary sets, $\bar{S}_\kappa \subseteq S_\kappa$ and $\bar{\nu} = \langle \nu_\kappa \mid \kappa \in K \rangle \in \prod_{\kappa \in K} \kappa$ such that for every $\gamma \in \bar{S}_\kappa$ $\text{otp}(C_\gamma) = \nu_\kappa$. Set $T_\kappa = \kappa \setminus \bar{S}_\kappa$. Note that if $\alpha$ is a limit point of some $C_\gamma$, then $C_\alpha = \alpha \cap C_\gamma$ implies that $\text{otp}(C_\alpha)$ equals $\alpha$’s place in $C_\gamma$, so if $\alpha$’s place in $C_\gamma$ is not $\nu_\kappa$, then it is in $T_\kappa$.

Let $\mathfrak{A}$ be an algebra on some large enough $H(\lambda)$. We can assume it has predicates for $\kappa^*$, $K$, $C$ and $\bar{\nu}$, and fix a cardinal $\mu < \kappa_0$ which we can assume is regular (as $\kappa_0$ is not the successor of a singular). We need to find an increasing and continuous sequence of models $N_\alpha \prec \mathfrak{A}$, $\alpha \leq \mu$ such that $K \subseteq N_0$ and for all $\alpha \leq \mu$ and $\kappa \in K$, $\sup(N_\alpha \cap \kappa) \in S_\kappa \cap N_{\alpha+1}$.

Let $\bar{\mu} = \max\{\theta^+, \psi^{++}, \mu\}$ (note $\bar{\mu} < \kappa_0$) and let $M \prec \mathfrak{A}$ of size $\bar{\mu}$ such that $\bar{\mu} + 1, K \subseteq M$ and $\bar{\mu} M \subseteq M$ (the last is possible by the GCH assumption). For every $\kappa \in K$ let $\gamma_\kappa = \sup M \cap \kappa$. Since $|M| = \bar{\mu} < \kappa$, $\bar{\mu} \in M$ and $\bar{\mu} M \subseteq M$, we must have $|M \cap \kappa| = \bar{\mu}$ (otherwise $M$ would have a function from $\bar{\mu}$ onto $M \cap \kappa$ so $M$ would think $\kappa$ is not a cardinal, contradicting elementarity). So $\gamma_\kappa$ is singular of cofinality $\bar{\mu}$. Note that $M \cap \gamma_\kappa = M \cap \kappa$ is club in $\gamma_\kappa$, so $C^M_{\gamma_\kappa} := C_{\gamma_\kappa} \cap M$ is also a club in $\gamma_\kappa$ (note the slight abuse of notation - we think of $C^M_{\gamma_\kappa}$ as “the $C_{\gamma_\kappa}$ of $M$” even though $\gamma_\kappa$ and $C_{\gamma_\kappa}$ are not in $M$).

We now inductively define a sequence $\langle M_{\alpha, \kappa} \mid \langle \alpha, \kappa \rangle \in \mu \times \{(0) \cup K\} \rangle \in M$ of elementary submodels of $M$. The order of the induction is the left-lexicographic order on $\mu \times \{(0) \cup K\}$ - for $\alpha \in \mu$ we first define $\langle M_{\alpha, \kappa} \mid \kappa \in \{(0) \cup K\} \rangle$ before moving to $\alpha + 1$. $M_{0,0}$ is the closure of $K$ in $M$ (so of size $|K| < \theta^+ \leq \bar{\mu}$). Assume that for some $\langle \alpha, \kappa \rangle \in \mu \times K$, $\langle M_{\alpha, \kappa'} \mid \kappa' \in \{(0) \cup (K \cap \kappa)\} \rangle$ is defined, is an element of $M$, and is such that each $M_{\alpha, \kappa'}$ is of size $< \bar{\mu}$ (in $M$). So $\bigcup_{\kappa'} M_{\alpha, \kappa'} < \bar{\mu}$, and in particular $\bigcup_{\kappa'} M_{\alpha, \kappa'} \cap \kappa = \bigcup_{\kappa'} M_{\alpha, \kappa'} \cap \gamma_\kappa$ is bounded in $\gamma_\kappa$ (whose cofinality is $\bar{\mu}$), say by $\delta$. $C^M_{\gamma_\kappa}$ is club in $\gamma_\kappa$, so we can pick some $\xi_{\alpha, \kappa} \in C^M_{\gamma_\kappa} \setminus \delta$. If $\text{otp}(C_{\gamma_\kappa}) \leq \nu_\kappa$ then any such $\xi_{\alpha, \kappa}$ will do. Otherwise, since $C^M_{\gamma_\kappa}$ must also be of cofinality $\bar{\mu} > \omega = \text{cf} (\nu_\kappa)$ (\nu_\kappa$ is the
order type of a club in some element of \( S_{\kappa} \subseteq E_{\kappa}^\kappa \), there must be cofinally many elements of \( C_{\gamma_{\kappa}}^\kappa \) above the \( \nu_{\kappa} \) place of \( C_{\gamma_{\kappa}} \), so we can choose one of those. Then we set \( M_{\alpha,\kappa} \) as the closure of \( \bigcup_{\gamma_{\kappa}} M_{\alpha,\kappa'} \cup \{ \sup (\bigcup_{\gamma_{\kappa}} M_{\alpha,\kappa'}) \cap \kappa, \xi_{\alpha,\kappa} \} \) in \( M \).

If \( M_{\alpha,\kappa} \) is defined for all \( \kappa \in \{ 0 \} \cup K \) then \( M_{\alpha+1,0} = \bigcup_{\kappa \in K} M_{\alpha,\kappa} \), and if \( M_{\beta,0} \) is defined for all \( \beta < \alpha \) for limit \( \alpha \), then \( M_{\alpha,0} = \bigcup_{\beta < \alpha} M_{\beta,0} \). We also define \( M_{\mu,0} = \bigcup_{\beta < \mu} M_{\beta,0} \).

We claim that \( \langle M_{\alpha,0} : \alpha \in \text{Lim}(\mu + 1) \rangle \) is as required. Let \( \kappa \in K \), we claim that \( \{ \xi_{\beta,\kappa} \mid \beta < \alpha \} \) is unbounded in \( M_{\alpha,0} \cap \kappa \). If \( \zeta \in M_{\alpha,0} \cap \kappa \) then there is \( \beta < \alpha \) such that \( \zeta \in M_{\beta,\kappa} \cap \kappa \). Then in fact \( \zeta \in M_{\beta,\kappa} \cap \kappa \), and by our choice we have \( \xi_{\beta+1,\kappa} > \zeta \) as required. So \( \sup M_{\alpha,0} \cap \kappa = \sup \{ \xi_{\beta,\kappa} \mid \beta < \alpha \} \) and since \( \{ \xi_{\beta,\kappa} \mid \beta < \alpha \} \subseteq C_{\gamma_{\kappa}} \) and \( \alpha < \mu = \text{cf} (C_{\gamma_{\kappa}}) \), this is also an element of \( C_{\gamma_{\kappa}} \), and by our choices its place in \( C_{\gamma_{\kappa}} \) is not \( \nu_{\kappa} \), so as we noted before, it is an element of \( T_{\kappa} \). \( \square \)

2.3.2. Forcing non-reflecting stationary sets. Recall that a stationary set \( S \subseteq \kappa \) is said to reflect at \( \lambda < \kappa \) if \( S \cap \lambda \) is stationary at \( \lambda \). We say that \( S \) is non-reflecting if it does not reflect at any \( \lambda < \kappa \).

**Definition 35.** Let \( \kappa \) be a regular uncountable cardinal. The poset \( \text{NR}(\kappa) \) adding a Non-Reflecting stationary subset of \( \kappa \) consists of conditions which are functions \( p : \alpha + 1 \rightarrow \{ 0, 1 \} \) where \( \alpha < \kappa \) such that \( p(\beta) = 0 \Rightarrow \text{cf} (\beta) = \omega \) and for every limit \( \mu < \kappa \) of uncountable cofinality, \( p^{-1}\{ 0 \} \cap \mu \) is not stationary at \( \mu \). A condition \( p \) is stronger than \( q \) if \( q \subseteq p \). We denote \( p^{-1}\{ i \} \) by \( S_{p,i} \).

**Proposition 36.** Let \( \kappa \) be a regular uncountable cardinal. Then:

1. \( \text{NR}(\kappa) \) is \( \sigma \)-closed.
2. \( \text{NR}(\kappa) \) is \( < \kappa \) strategically-closed.
3. If \( G \subseteq \text{NR}(\kappa) \) is generic then in \( V[G] \) the set
   \[
   S_G = \bigcup_{p \in G} S_{p,0} = \{ \gamma < \kappa \mid \exists p \in G \, p(\gamma) = 0 \}
   \]
   is a non-reflecting stationary subset of \( \kappa \).

**Proof.** (1) If \( \langle p_n : n < \omega \rangle \) is a decreasing sequence of conditions then
   \[
   \bigcup_{p_n} \cup \{ \langle \bigcup \text{Dom}(p_n), 1 \rangle \}
   \]
   is also a condition since the non-reflection condition regards only limits of uncountable cofinality.

(2) Let \( \nu < \kappa \) and fix some club \( C \subseteq \kappa \). We define a strategy for player II in the game of length \( \nu \) inductively, making sure that at every limit stage \( \alpha \), if the game produced \( \langle p_{\gamma} : \gamma < \alpha \rangle \) then \( C \cap \bigcup \{ S_{p_{\gamma},1} : \gamma < \alpha \} \) is club in \( \text{sup} \{ \text{Dom}(p_{\gamma}) : \gamma < \alpha \} \). At successor stages, if player I chooses \( p \), let \( \alpha = \text{min} C \setminus \text{Dom}(p) \) and player II chooses \( p' = p \cup \{ (\gamma, 1) \mid \gamma \in \alpha + 1 \setminus \text{Dom}(p) \} \). In particular \( \alpha = \text{max} \text{Dom}(p') \in C \cap S_{p',1} \). This is
indeed a condition since $S_{p,0} = S_{p,0}$ so the non-reflection requirement holds. At limit stages, if $\langle p_\gamma \mid \gamma < \alpha \rangle$ is defined by the strategy, let $\delta = \sup \{ \text{Dom}(p_\gamma) \mid \gamma < \alpha \}$, then the successor stages ensure that $C \cap \bigcup \{ S_{p,1} \mid \gamma < \alpha \}$ is cofinal in $\delta$ and the induction assumption ensures it is closed. So player II chooses $p_\alpha = \bigcup_{\gamma < \alpha} p_\gamma \cup \{ \delta, 1 \}$ which ensures that the inductive assumptions remains true. To see this is a condition, if $\mu < \delta$ then $S_{p,0} = S_{p,0}$ for some $\gamma < \alpha$ so $S_{p,0} \cap \mu$ is not stationary by assumption. If $\mu > \delta$ then $S_{p,0}$ is bounded below $\mu$ so surely not stationary. For $\delta$, the assumption was that $C \cap \bigcup \{ S_{p,1} \mid \gamma < \alpha \}$ is club in $\delta$, and it is surely disjoint from $S_{p,0}$.

(3) For every $\alpha < \kappa$ $S_G \cap \alpha = S_{p,0} \cap \alpha$ for some $p$, so is not stationary at $\alpha$. Hence we only need to show $S_G$ is stationary in $\kappa$. Let $\dot{C}$ be a name for a club in $\kappa$. We need to show that

$$D = \left\{ p \in \text{NR}(\kappa) \mid \exists \gamma \in \text{Dom}(p) (p(\gamma) = 0 \land p \Vdash \gamma \in \dot{C}) \right\}$$

is dense. Let $p \in \text{NR}(\kappa)$, $\alpha = \max \text{Dom}(p)$. We can find some $p' \leq p$ and $\alpha^*$ some ordinal of uncountable cofinality, $\alpha < \alpha^* < \kappa$ such that $p'$ forces that $\alpha^*$ is a limit point of $\dot{C}$. By strategic closure, $\dot{C} \cap \alpha^* \in V$ and is club in $\alpha^*$, so let $\delta$ be minimal in $(\dot{C} \cap \alpha^* \cap E^\kappa_{\alpha^*}) \setminus \text{Dom}(p')$, then

$$p' \cup \{ \langle \gamma, 1 \rangle \mid \gamma \in (\alpha, \delta) \} \cup \{ \langle \delta, 0 \rangle \} \in D$$

extends $p$, and forces that $\delta \in S_G \cap \dot{C}$. So $S_G$ is indeed a non-reflecting stationary set in $V[G]$.

**Definition 37.** Let $K = \langle \kappa_\zeta \mid \zeta < \theta \rangle$ be an increasing sequence of uncountable cardinals which are successors of regulars, $\theta < \kappa_0$. Let $\text{NR}(K)$ be the full support product $\Pi_{\zeta < \theta} \text{NR}(\kappa_\zeta)$.

Note that by strategic closure of each component this is the same as the full-support iteration, and is also $< \kappa_0$-strategically closed.

**Theorem 38.** Let $G \subseteq \text{NR}(K)$ be generic, for every $\zeta$ $G_\zeta$ the projection of $G$ on the $\zeta$th coordinate, $S_\zeta = S_{G_\zeta}$ the induced stationary set and $T_\zeta := \kappa_\zeta \setminus S_\zeta = \bigcup_{p \in G} S_{p(\zeta), 1}$. Then $\langle T_\zeta \mid \zeta < \theta \rangle$ are strongly mutually fat.

**Proof.** Fix some $\nu < \kappa_0$ and let $\mathfrak{A} = \langle \lambda, \in, \theta, K, \text{NR}(K)^V, \ldots \rangle$ be, in $V[G]$, an algebra for some large enough $\lambda$ (where $K, \text{NR}(K)^V$ are coded in $V$ as ordinals$^3$). We will assume without loss of generality $\theta \leq \nu$. We aim to show that the set of conditions forcing that there is a witness with respect to $\mathfrak{A}$ to the mutual fatness of $\langle T_\zeta \mid \zeta < \theta \rangle$, is dense.

Given some $p_0 \in \text{NR}(K)$, we inductively define a decreasing sequence of conditions $\langle p_\alpha \mid \alpha \leq \nu \rangle$ and an increasing continuous sequence of sets $\langle N_\alpha \mid 0 < \alpha \leq \nu \rangle \in V$ such that $p_\alpha \in N_{\alpha+1}$ and for $\alpha > 0$ $p_\alpha \Vdash N_\alpha < \mathfrak{A}$, as follows. There is some name $\dot{N}$ such that $p_0$ forces that $p_0 \in \dot{N} \prec \mathfrak{A}$, $\theta + 1 \subseteq \dot{N}$ and $|\dot{N}| < \kappa_0$. By strategic closure there

$^3$We take the algebra to be on an ordinal so there are no “new” elements of $\mathfrak{A}$. This doesn’t limit generality.
is some \( p_1 \leq p_0 \) and some \( N_1 \in V \) such that \( p_1 \models \dot{N} = \dot{N}_1 \). If \( p_\alpha, N_\alpha \) are defined, we apply the same procedure to get \( p_{\alpha+1} \) and \( N_{\alpha+1} \), with the additional requirement that \( N_\alpha \subseteq N_{\alpha+1} \). For \( \alpha \) limit, set \( N_\alpha = \bigcup_{\beta < \alpha} N_\beta \), and \( p_\alpha \) is defined for every \( \zeta < \theta \) by \( p_\alpha(\zeta) = \bigcup_{\beta < \alpha} p_\beta(\zeta) \cup \left\{ \left( \bigcup_{\beta < \alpha} \text{Dom}(p_\beta(\zeta)), 1 \right) \right\} \). Note that in the limit case, since for every \( \beta < \alpha \), \( p_\beta \in N_{\beta+1} \subseteq N_\alpha \), for every \( \zeta < \theta \), Dom\((p_\beta(\zeta)) < \sup(N_\alpha \cap \kappa_\zeta) \), so \( \bigcup_{\beta < \alpha} \text{Dom}(p_\beta(\zeta)) \leq \sup(N_\alpha \cap \kappa_\zeta) \), and we can actually make sure they are equal. So \( \sup(N_\alpha \cap \kappa_\zeta) \in S_{p_\alpha(\zeta), 1} \). Also note that \( p_\alpha \) forces that for every \( \beta < \alpha \), \( N_\beta \prec \mathbb{A} \) so it also forces that \( N_\alpha \prec \mathbb{A} \).

To conclude, \( p_\nu \) forces that the sequence \( \langle N_\alpha \mid \alpha \leq \nu, \alpha \text{ limit} \rangle \) will be an increasing continuous sequence of elementary submodels of \( \mathbb{A} \) such that for every limit \( \alpha \leq \nu \) and every \( \zeta < \theta \), \( \sup(N_\alpha \cap \kappa_\zeta) \in S_{p_\nu(\zeta), 1} \subseteq T_\zeta \). In other words, \( p_\nu \leq p_0 \) forces that \( \langle N_\alpha \mid \alpha \leq \nu, \alpha \text{ limit} \rangle \) witnesses strong mutual fatness of \( \langle T_\zeta \mid \zeta < \theta \rangle \) with respect to \( \mathbb{A} \), as required. \( \square \)

3. Coding by shooting clubs

**Definition 39.** Let \( X \) be a set of ordinals and \( \kappa > \sup X \) a successor of a regular cardinal.

If \( \tilde{S} = \langle S^\alpha \mid \alpha < \kappa \rangle \) is a partition of a co-fat set \( \tilde{S} \subseteq E^K_\alpha \) into disjoint stationary sets, then we denote the poset \( \text{Des}(|\bigcup \{ S^\alpha \mid \alpha \in X \}) \) by \( \vartheta(X, \kappa, \tilde{S}) \). \( \hat{\vartheta}(X, \kappa, \tilde{S}) \) denotes the name for its generic, and \( \vartheta(X, \kappa, \tilde{S}) \) will denote an arbitrary generic. If \( \kappa \) and \( \tilde{S} \) are fixed and clear from the context we will often write simply \( \vartheta X \).

In a generic extension by shooting a club through the complement of the stationary set \( \bigcup \{ S^\alpha \mid \alpha \in X \} \) we get that

\[
X = \{ \alpha < \sup X \mid S^\alpha \cap \vartheta X = \varnothing \}
= \{ \alpha < \sup X \mid S^\alpha \text{ is not stationary} \}
\]

hence, if \( \tilde{S} \) is in the \( C(\text{aa}) \) of the extension (e.g. if \( \tilde{S} \in L \)), then so will \( X \) be. Thus we refer to \( \vartheta(X, \kappa, \tilde{S}) \) as "coding \( X \) into \( C(\text{aa}) \)." We make this formal in the following way:

**Proposition 40.** Let \( X \subseteq \text{Ord} \), \( \kappa > \sup X \) a successor of a regular cardinal, and assume there are \( S \subseteq E^K_\alpha \) a co-fat stationary set, and \( \tilde{S} = \langle S^\alpha \mid \alpha < \kappa \rangle \) a partition of \( S \) into disjoint stationary sets such that \( \tilde{S} \in C(\text{aa})^{V[\vartheta(X, \kappa, \tilde{S}), \vartheta]} \). Then \( X \in C(\text{aa})^{V[\vartheta(X, \kappa, \tilde{S})]} \).

In particular, if \( V = L[X] \) and \( \tilde{S} \in L \) then \( C(\text{aa})^{V[\vartheta(X, \kappa, \tilde{S})]} = L[X] \).

**Proof.** As we noted \( X = \{ \alpha < \sup X \mid S^\alpha \text{ is not stationary} \} \in C(\text{aa})^{V[\vartheta(X, \kappa, \tilde{S})]} \). So \( L[X] \subseteq C(\text{aa})^{V[\vartheta(X, \kappa, \tilde{S})]} \).

If \( \tilde{S} \in L \) then by theorem 30(4)(b) we have \( C(\text{aa})^{V[\vartheta(X, \kappa, \tilde{S})]} \subseteq V \), so if \( V = L[X] \) we get our equality. \( \square \)

3.1. **Coding a set into a model of \( V = C(\text{aa}) \).** In this section we follow the method of Zadrożny’s [17] to code sets into a model satisfying "\( V = C(\text{aa}) \)" using iterated club shooting.
Theorem 41. Let \( V = L[A] \) for some set \( A \) such that there are:

1. \( \langle \kappa_n \mid n < \omega \rangle \in L \) an increasing sequence of successors of regular cardinals (of \( V \)) above \( \kappa_1 = \sup A \) such that for every \( n \) \( 2^{\kappa_{n-1}} < \kappa_n \).

2. \( \langle T_n, \tilde{S}_n \mid n < \omega \rangle \in L \) such that for every \( n \), \( T_n \subseteq E_n^{\kappa_n} \) is stationary in \( \kappa_n \) (in \( V \)), and \( \tilde{S}_n \subseteq E_\omega^{\kappa_n} \setminus T_n \) is co-fat (again in \( V \)). It will also be convenient to assume that \( \tilde{S}_n \subseteq \kappa_n \setminus \kappa_{n-1} \).

3. \( \langle \tilde{S}_n = (S_n^\alpha \mid \alpha < \kappa_n) \mid n < \omega \rangle \in L \) such that for every \( n \), \( \tilde{S}_n \) is a partition of \( \tilde{S}_n \) into disjoint stationary sets.

Then there is a forcing extension of \( V \) satisfying \( "V = C(aa)" \).

Remark. (1) Stationarity and fatness of the sets above are with respect to \( V \) even though the sets are from \( L \).

(2) The assumptions hold for any \( A \) which is \( L \)-generic. See proposition 42.

(3) \( L \) can be replaced by other canonical inner models which are provably contained in \( C(aa) \), such as the Dodd-Jensen core model (see [9]).

Proof. We define inductively an iteration of club shooting forcings as follows. Set \( \Theta^0 A = \{1\} \) and \( \partial^0 A = A \) (which is also considered as a \( \Theta^0 A \)-name). If we’ve inductively defined \( \Theta^n A \) and \( \partial^n A \) as a \( \Theta^n A \)-name for a subset of \( \kappa_n \), then we let \( \hat{S}_n \) be a \( \Theta^n A \)-name for \( \bigcup \{ S_n^\alpha \mid \alpha \in \partial^n A \} \).

\[
\partial^{n+1} A := \Theta^n A * \text{Des}(\hat{S}_n) = \Theta^n A * \hat{\partial}(\partial^n A, \kappa_n, \tilde{S}_n),
\]

and \( \hat{\partial}^{n+1} A = \hat{\partial}(\partial^n A, \kappa_n, \tilde{S}_n) \) – the name for the generic club forced by \( \text{Des}(\hat{S}_n) \). Let \( \Theta^* A \) be the full support limit of the iteration and note that it satisfies the assumptions of theorem 30, since at each stage we destroy sets disjoint from the \( T_n \)s.

So, at stage 1 we shoot a club \( \partial^1 A \) through the complement of \( \bigcup \{ S_0^\alpha \mid \alpha \in A \} \), thus destroying the stationarity of exactly these sets out of \( \tilde{S}_0 \), so we code \( A \) into \( C(aa) \). At stage 2 we shoot a club through the complement of \( \bigcup \{ S_1^\alpha \mid \alpha \in \partial^1 A \} \) thus coding \( \partial^1 A \) into \( C(aa) \), and so on. After \( \omega \) many steps we catch our tail, so that the entire generic of \( \Theta^* A \) is coded into \( C(aa) \) (theorem 30 is used to show that what we coded at a certain stage won’t be destroyed at a subsequent stage). Lets see this formally.

Let \( G \subseteq \Theta^* A \) be generic. We claim that \( G \in C(aa)^{V[G]} \) so \( V[G] = C(aa)^{V[G]} \).

For every \( n \) let \( \partial^n A = \left( \hat{\partial}^n A \right)^G \), and it is clear from the construction that \( G \) can be obtained from \( \langle \partial^n A \mid n < \omega \rangle \), so we need to show that this sequence is in \( C(aa) \). For every \( n \) and \( \alpha < \kappa_n \), if \( \alpha \in \partial^n A \) then by the properties of the club shooting \( S_n^\alpha \) is not stationary in \( V[G] \), while if \( \alpha \notin \partial^n A \), by theorem 30 \( S_n^\alpha \) is stationary \( V[G] \). So \( \partial^n A = \{ \alpha < \kappa_n \mid V[G] = S_n^\alpha \) is stationary \} which is in \( C(aa)^{V[G]} \) since \( \tilde{S}_n \in L \). Since \( \langle \kappa_n, T_n, \tilde{S}_n \mid n < \omega \rangle \in L \), also the sequence \( \langle \partial^n A \mid n < \omega \rangle \) is in \( C(aa)^{V[G]} \). \( \Box \)

Proposition 42. If \( V = L[A] \) where \( A \) is set-forcing generic over \( L \) then the assumptions of theorem 41 hold.
Proof. In this case, for large enough cardinals, $V$ agrees with $L$ on cardinalities and the notions of stationarity and fatness, and we have for large enough $\kappa > \sup A \cdot 2^\kappa = \kappa^+$ and $\square \kappa$. So we can pick $\langle \kappa_n \mid n < \omega \rangle \in L$ an increasing sequence of cardinals (of $V$) above $\kappa_{-1} = \sup A$ which are successors of regular cardinals and such that for every $n$ $2^{\kappa_{n-1}} < \kappa_n$ and $\square_{\kappa_n}$ holds (where $\kappa^-$ denotes the predecessor of a successor $\kappa$). Then for each $n$ we split $E^\kappa_{\omega_n}$ into two disjoint stationary sets, take $T_n$ as one of them and apply lemma 22 to the other to obtain $\tilde{S}_n$, and then partition it into disjoint sets. All of this is done in $L$ but they retain the desired properties in $V$. 

In theorem 41 we only used a sequence of co-fat stationary sets, but if we add the assumption that their complements form a strongly mutually fat sequence, we can get a better result:

**Theorem 43.** Let $V = L[A]$ for some set $A$ such that the assumptions of theorem 41 hold, and further assume that $\langle T_n \mid n < \omega \rangle$ is strongly mutually fat. Then there is a forcing extension $W$ of $V$ such that $C(aa)^W = W$ and $H(\kappa_0)^W = H(\kappa_0)^V$.

Proof. We apply the same proof as of theorem 41, but now by theorem 33.1 the forcing is $< \kappa_0$ distributive, so we get $H(\kappa_0)^W = H(\kappa_0)^V$. 

This means that any "local" statement that can be forced over $L$, is consistent with $V = C(aa)$, where "local" is in fact $\Sigma_2$:

**Lemma 44.** $\Phi$ is a $\Sigma_2$ sentence iff $\exists \theta (H(\theta) \models \Phi)$.

Proof. The statement $\exists \theta (H(\theta) \models \Phi)$ is short for $\exists \theta \exists X \forall Y (|\text{trc}(y)| < \theta \iff y \in X) \land X \models \Phi$ which is a $\Sigma_2$ statement. On the other hand, a $\Sigma_2$ statement is of the form $\exists X \forall Y \Psi(X,Y)$ where all quantifiers in $\Psi$ are bounded. Assume $\Phi$ holds, and let $X$ be such that $\forall Y \Psi(X,Y)$. Then if $X \in H(\theta)$, $H(\theta) \models \forall Y \Psi(X,Y)$, so $H(\theta) \models \Phi$. For the other direction, assume $H(\theta) \models \Phi$, and let $X \in H(\theta)$ such that $H(\theta) \models \forall Y \Psi(X,Y)$. We want to show that also $V \models \forall Y \Psi(X,Y)$. Otherwise, let $Y$ be a counterexample, and let $M$ be an elementary submodel of some large enough $H(\lambda)$ such that $Y \in M$, $\text{trc}(X) \subseteq M$ and $|M| < |H(\theta)|$. Let $\pi : M \rightarrow \tilde{M}$ be the Mostowsky collapse of $M$. Then $\tilde{M} \subseteq H(\theta)$ and since $\text{trc}(X) \subseteq M$, $\pi(X) = X$. $M \models \neg \Psi(X,Y)$, so $\tilde{M} \models \neg \Psi(X,\pi(Y))$, but then this is true also in $H(\theta)$, contradiction.

**Corollary 45.** If $\Phi$ is a $\Sigma_2$ statement, perhaps with ordinal parameters, which is forceable over $L$, then there is a forcing extension of $L$ satisfying $V = C(aa) + \Phi$.

Proof. If $L[A]$ is a forcing extension of $L$ satisfying $\Phi$, then we can choose a sequence $K = \{ \kappa_n \mid n < \omega \}$ large enough (above the $H(\theta)$ which satisfies $\Phi$) so that $L$ and $L[A]$ agree on all relevant notions, use theorem 34 to obtain a sequence $\langle S_\kappa \mid \kappa \in K \rangle$ such that for every $\kappa \in K$ $\tilde{S}_\kappa \subseteq S_\kappa$ is stationary and the sequence $\langle T_\kappa \mid \kappa \in K \rangle$ where $T_\kappa =$
κ \setminus \bar{S}_κ is strongly mutually fat. Then applying the previous theorem we get the desired extension.

This gives us, for example, that for every κ of uncountable cofinality, Con(ZFC) implies the consistency of $V = C(aa) + 2^{ℵ₀} = κ$. This is in stark contrast to the case of $C^*$, where $V = C^*$ implies that $2^{ℵ₀} \in \{ ℵ_1, ℵ_2 \}$ and for any κ > ℵ₀ $2^κ = κ^+$ (cf. [10, corollary to theorem 5.20]).

3.2. Iterating $C(aa)$. In this section we want to use the coding construction to produce models with decreasing iterations of $C(aa)$. First we note that the construction above gives us finite iterations:

**Proposition 46.** In the construction of theorem 41, for every $n C(aa)^L[\partial^{n+1}A] = L[\partial^n A]$

**Proof.** Proposition 40.

So at each finite step of the iteration, where we "code $A_n$ times", we get a decreasing sequence of $C(aa)$ of length $n$, but after $ω$-stages we don’t get a decreasing sequence of length $ω$, but rather "catch our tail" and get "$V = C(aa)^*$" (this is the reason we denoted the final iteration above by $\partial^* A$ and not by $\partial^ω A$, which we will use shortly). To get an infinite decreasing sequence, we need to make sure that on each step we lose something, but we still have infinitely many things to lose. This is accomplished (also following Zadrożny) by adding $ω$ many "partial codings" – for each $n$ we code a set $A_n$ "$n$ many times", so that taking $C(aa)$ will drop the last coding of each $A_n$, giving us $n - 1$ codings of $A_n$. We will eventually lose all codings for $A_n$ after $n$ stages, but we still have infinitely more $A_m$s for which there are more codings to lose. In fact, the $A_n$s will actually be the same set, which we code at different places so the codings are different.

**Theorem 47.** Let $V = L[A]$ for some set $A$ such that the assumptions of theorem 41 hold. Then there is a forcing extension of $V$ satisfying: for every $n < ω C(aa)^n \neq C(aa)^{n+1}$, $C(aa)^ω \models$ ZFC, and $C(aa)^{ω+1} = C(aa)V$.

**Proof.** Given the sequences $⟨κ_n, T_n, \tilde{S}_n, \tilde{S}_n, n < ω⟩ \in L$ we re-index them as $⟨κ_{n,i}, T_{n,i}, \tilde{S}_{n,i}, \tilde{S}_{n,i}, i ≤ n < ω⟩ \in L$ so that $(n, i) <_{\text{Lex}} (m, j)$ iff $κ_{n,i} < κ_{m,j}$. Note that this is in fact a well order of order type $ω$. Let $κ_ω = \sup \{ κ_{n,i} \mid n < ω, i ≤ n \}$. For the sake of clarity we will now describe the forcings we use in a bit less formal way, not using names for the forcings or the sets added. We think of it as inductively defining the iterations in the corresponding extensions.

For every $n$, denote $\partial^n, i^{-1} A = A$ and for each $n$ and $0 ≤ i ≤ n$ inductively define $\partial^n, i A := \partial^n, i^{-1} A \ast \partial(\partial^n, i^{-1} A, κ_{n,i}, \tilde{S}_{n,i})$. 
Denote $\vartheta^n A = \vartheta^{n,n} A$, and let $\vartheta^\omega A$ be the full support product of the $\vartheta^n A$s. Note that this can be viewed also as a countable iteration of club shooting forcings, so theorem 30 applies. Denote the generic extension $L[\langle \vartheta^{n,i} A \mid i \leq n < \omega \rangle]$ by $W$. The idea is as follows – for each $n$, $\vartheta^n A$ codes the set $A \cap n + 1$ many times – at the cardinals $\kappa_{n,0}, \ldots, \kappa_{n,n}$. An important note is that for $n \neq m$, $\vartheta^n A$ and $\vartheta^m A$ are forced independently. So, when we take the $C(aa)$ of the extension, the last coding at each “level” will drop (see figure 3.1). But since we have infinitely many levels, we can repeat this procedure infinitely many times, and only then will all the codings drop.

Denote for every $k$ $W^k := L[\langle \vartheta^{n,i} A \mid k \leq n < \omega, i \leq n - k \rangle]$.

Claim 48. For every $k$, $C(aa)^{W^k} = W^{k+1}$.

Proof. $W^{k+1} = L[\langle \vartheta^{n,i} A \mid k + 1 \leq n < \omega, i \leq n - k - 1 \rangle]$. So 

$W^k = L[\langle \vartheta^{n,i} A \mid k \leq n < \omega, i \leq n - k \rangle] = W^{k+1} \langle \vartheta^{n,n-k} A \mid k \leq n < \omega \rangle$ 

where $Y = \langle \vartheta^{n,n-k} A \mid k \leq n < \omega \rangle$ is generic over $W^{k+1}$ for an iteration of club shooting forcings of length $\omega$, destroying sets which are in $W^{k+1}$ (as they are computed from the previous stages). So we can apply theorem 30(4)(b) to get that $C(aa)^{W^k} =$
\( C(aa)^{\omega+1} \subseteq W^{k+1} \). On the other hand, as we discussed before, for every \( k \leq n < \omega, i < n - k, \partial^{n,i}A \in C(aa)^W \). So we get equality. \( \Box \text{Claim} \)

Thus we get that for every \( k, (C(aa)^k)^W = L[\{\partial^{n,i}A | k \leq n < \omega, i \leq n - k\}] \) and we get a descending sequence. Note that for every \( k, A \in (C(aa)^k)^W \).

Now we wish to analyze \((C(aa)^\omega)^W = \bigcap_{k<\omega} (C(aa)^k)^W = \bigcap_{k<\omega} L[\{\partial^{n,i}A | k \leq n < \omega, i \leq n - k\}] \).

For every \( k \), denote by

\[
Q_k = \prod_{k \leq n < \omega, i \leq n-k} \partial^{n,i}A
\]

the poset adding the generic object \( \langle \partial^{n,i}A | k \leq n < \omega, i \leq n - k \rangle \) and by \( B_k \) the complete Boolean algebra corresponding to it. We can without loss of generality assume that \( B_{k+1} \subseteq B_k \) and clearly if \( G \) is the generic for \( B = B_0 = \text{ro}(\partial^\omega A) \), and \( G_k = G \cap B_k \), then

\[
L[A][G_k] = L[\{\partial^{n,i}A | k \leq n < \omega, i \leq n - k\}].
\]

So, denoting \( B_\omega = \bigcap_{k<\omega} B_k, G_\omega = G \cap B_\omega \), and using fact 9, we get that

\[
(C(aa)^\omega)^W = L[A][G_\omega]
\]

and in particular it satisfies ZFC, since \( Q_0 = \partial^\omega A \) (and so also \( B_0 \)) is \( \omega \)-distributive.

To show that \((C(aa)^{\omega+1})^W = C(aa)^V \) we want to show that \((C(aa)^\omega)^W \) agrees with \( V \) on the notion of stationarity. For every \( n < \omega \) and \( k < l \leq n \), let

\[
g_n^{kl} : \prod_{i \leq n-l} \partial^{n,i}A \to \prod_{i \leq n-k} \partial^{n,i}A
\]

be the natural projection. Then this induces an \( \omega \)-normal system \( f_{nl} : Q_k \to Q_l \) as in lemma 16, so we get that \( Q_0/\sim \) is \( \sigma \)-closed. So by lemma 5, for any \( \lambda \) and \( T \subseteq P_{\omega_1}(\lambda), T \in V, T \) is stationary in \( V \) iff it is stationary in \( V[G_\omega] = (C(aa)^\omega)^W \).

Now we can inductively prove that the stages of construction of \( C(aa)^V \) and \((C(aa)^{\omega+1})^W \) are exactly the same, since in the successor step we always consider the stationarity of sets which are, by the induction hypothesis, sets in \( V \), and this notion is the same in \( V \) and in \((C(aa)^\omega)^W \). So indeed \((C(aa)^{\omega+1})^W = C(aa)^V \). \( \Box \)

Note that if we apply the previous theorem to \( V = L[\partial A] \) instead of \( L[A] \), then we get that \((C(aa)^{\omega+1})^V = C(aa)^{L[\partial A]} = L[A] \). This suggests that we can get longer iterations, however for that we would need the stronger distributivity properties provided by mutually fat sets.

**Theorem 49.** Let \( V = L[A] \) where \( A \) is set-generic over \( L \). Then for any ordinal \( \delta \) there is a forcing extension of \( V \) satisfying \( \forall \alpha \leq \delta, C(aa)^\alpha \models \text{ZFC}, C(aa)^\alpha \neq C(aa)^{\alpha+1} \) and \( C(aa)^{\delta+1} = L[A] \).
Proof. As $V$ is a set-generic extension of $L$, there is some cardinal $\psi$ above which $V$ agrees with $L$ on cardinals and on the notion of stationarity, so in particular GCH and existence of global square hold above $\psi$. We assume this $\psi$ is $> \sup A$. Set $\theta = \delta^+$ and fix sequences:

1. $K = \langle \kappa_\eta \mid \eta < \theta \rangle \in L$ an increasing sequence of successors of regular cardinals (of $V$) above $\kappa_{-1} = \psi^{++}$ such that for every $\eta \ 2^{\kappa_{\eta-1}} < \kappa_\eta$.

2. $\langle \bar{S}_\eta \mid \eta < \theta \rangle \in L$ a sequence of stationary sets obtained as in theorem 34, so for every $\eta$, $S_\eta \subseteq E^\kappa_\eta$ is stationary (in $V$), and the sequence $\langle T_\eta \mid \eta < \theta \rangle$ where $T_\eta = \kappa_\eta \setminus \bar{S}_\eta$ is strongly mutually fat (again in $V$). It will also be convenient to assume that $\bar{S}_\eta \subseteq \kappa_\eta \setminus \kappa_{\eta-1}$.

3. $\langle \bar{S}_\eta \mid \alpha < \kappa_\eta \gamma \mid \eta < \theta \rangle \in L$ such that for every $\eta$ $\bar{S}_\eta$ is a partition of $S_\eta$ into disjoint stationary sets.

These will be our “coding tools”. We prove by induction on $\delta$ that for every relevant $X$ (where which $X$s are “relevant” is inductively defined as those $X$s which are used in the construction of previous stages) and any $\eta < \theta$ there is a notion of forcing, denoted $\mathcal{D}(X, \eta)$, which is a full support iteration of club shooting forcings using cardinals from $\langle \kappa_\alpha \mid \alpha \in [\eta, \eta + 1 + \delta] \rangle$, such that the following holds:

**IH:** If $Y$ is set-generic over $L$ such that $L[Y]$ agrees with $L$ on cardinalities and stationarity in the segment $\langle \kappa_\alpha \mid \alpha \in [\eta, \eta + 1 + \delta] \rangle$, and $\mathcal{D}(X, \eta)$ is generic over $L[Y]$, then in $L[Y][\mathcal{D}(X, \eta)]$ the $C(aa)$-sequence has length at least $\delta + 1$ and that

$$C(aa)^{\delta+1} L[Y][\mathcal{D}(X, \eta)] = C(aa)^{L[Y]}.$$

**Remark 50.** Note that if the above holds, then for all $\delta' > \delta$, if $\delta' = \delta + 1 + \gamma$ (we don’t require anything on $\gamma$) then

$$\left(C(aa)^{\delta'} \right)^{L[Y][\mathcal{D}(X, \eta)]} = \left(C(aa)^{\delta+1+\gamma} \right)^{L[Y][\mathcal{D}(X, \eta)]}$$

$$= \left(C(aa)^{\gamma} \right)^{L[Y][\mathcal{D}(X, \eta)]}$$

$$= \left(C(aa)^{\gamma} \right)^{C(aa)^{L[Y]}}$$

$$= \left(C(aa)^{1+\gamma} \right)^{L[Y]}.$$

The definition is as follows:

i. $\mathcal{D}(X, \eta) := \mathcal{D}(X, \kappa_\eta, \bar{S}_\eta)$.

ii. If $\delta = \alpha + \beta$ for $\alpha, \beta < \delta$, and $\beta$ is smallest such that this holds, then set

$$\mathcal{D}(X, \eta) := \mathcal{D}(X, \eta + 1 + \alpha) \ast \mathcal{D}(\mathcal{D}(X, \eta + 1 + \alpha), \eta)$$

were $\beta' = \beta - 1$ if $\beta < \omega$ and otherwise $\beta' = \beta$. 

iii. Otherwise, we can find in $\delta$ an increasing sequence of limit ordinals $\langle \eta_\alpha \mid \alpha < \delta \rangle$ such that for every $\alpha < \beta$, $[\eta_\alpha, \eta_{\alpha+1})$ has order type $\alpha$. Then

$$\theta^\delta(X, \eta) := \prod_{\alpha<\delta} \theta^\alpha(X, \eta + \eta_\alpha).$$

Let’s see that this works. The initial step is clear. Assume $\delta = \alpha + \beta$. Then

$$L[Y][\theta^\delta(X, \eta)] = L[Y][\theta^\alpha(X, \eta + 1 + \alpha)][\theta^\beta(X, \eta + 1 + \alpha, \eta)]$$

where $\theta^\alpha(\theta^\beta(X, \eta + 1 + \alpha), \eta)$ is generic over $L[Y][\theta^\beta(X, \eta + 1 + \alpha)]$ and the assumptions in IH hold, so $L[Y][\theta^\delta(X, \eta)]$ satisfies that the $C(aa)$ sequence has length at least $\alpha + 1$, and $C(aa)^{\alpha+1} = C(aa)L[Y][\theta^\delta(X,\eta+1+\alpha)]$. Now again by IH this model has a $C(aa)$-sequence of length at least $\beta' + 1$, with the $\beta' + 1$ stage being $C(aa)^{L[Y]}$. Together we get a sequence of length at least $\alpha + 1 + \beta' + 1 = \alpha + \beta + 1$ (if $\beta$ is finite then $1 + \beta' = 1 + \beta = \beta$ and if it is infinite then $1 + \beta' = 1 + \beta = \beta$), and $C(aa)^{\alpha+\beta+1}L[Y][\theta^\delta(X,\eta)] = C(aa)^{L[Y]}$.

Now, if $\theta^\beta(X, \eta + \alpha)$ uses cardinals from

$$\langle \kappa_\gamma \mid \gamma \in [\eta + 1 + \alpha, \eta + 1 + \alpha + 1 + \beta'] \rangle$$

and $\theta^\alpha(\theta^\beta(X, \eta), \eta)$ from

$$\langle \kappa_\gamma \mid \gamma \in [\eta, \eta + 1 + \alpha] \rangle$$

then $\theta^\delta(X, \eta)$ uses cardinals from

$$\langle \kappa_\gamma \mid \gamma \in [\eta, \eta + 1 + \delta] \rangle.$$

Consider now the last case (note that if $\delta = \omega$, this is exactly the construction in theorem 47). For any $\alpha, \beta < \delta$, $\alpha \neq \beta$, the forcings $\theta^\alpha(X, \eta + \eta_\alpha)$ and $\theta^\beta(X, \eta + \eta_\beta)$ are independent of one another, that is we can use the product lemma, and the generic of $\theta^\delta(X, \eta)$ is a disjoint union of generics for $\prod_{\beta<\alpha} \theta^\beta(X, \eta + \eta_\beta)$, $\theta^\alpha(X, \eta + \eta_\alpha)$ and $\prod_{\alpha<\beta<\delta} \theta^\beta(X, \eta + \eta_\beta)$, which are mutually generic. So we have

$$(C(aa)^{\alpha+1})^{L[Y][\theta^\delta(X,\eta)]} = (C(aa)^{\alpha+1})^{L[Y][\cup_{\alpha<\beta<\delta} \theta^\beta(X,\eta+\eta_\beta)][\cup_{\beta<\alpha} \theta^\alpha(X,\eta+\eta_\alpha)]}$$

Also note that the forcing $\prod_{\alpha<\beta<\delta} \theta^\beta(X, \eta + \eta_\beta)$ has the required properties to ensure that cardinalities and stationarity in the segment $\langle \kappa_\gamma \mid \gamma < \eta + 1 + \eta_\alpha \rangle$ are preserved, so we can apply IH, and specifically remark 50, and inductively get that all the codings $\cup_{\beta<\alpha} \theta^\beta(X, \eta + \eta_\beta)$ simply drop, i.e. that

$$(C(aa)^{\alpha+1})^{L[Y][\cup_{\alpha<\beta<\delta} \theta^\beta(X,\eta+\eta_\beta)][\cup_{\beta<\alpha} \theta^\alpha(X,\eta+\eta_\alpha)]} = (C(aa)^{\alpha+1})^{L[Y][\cup_{\alpha<\beta<\delta} \theta^\beta(X,\eta+\eta_\beta)]} = (C(aa)^{\alpha+1})^{L[Y][\cup_{\alpha<\beta<\delta} \theta^\beta(X,\eta+\eta_\beta)][\theta^\alpha(X,\eta+\eta_\alpha)]} = C(aa)^{L[Y][\cup_{\alpha<\beta<\delta} \theta^\beta(X,\eta+\eta_\beta)]}$$
so together we have
\[(C(aa)^{α+1})^{L[\partial^δ(X,η)]]} = C(aa)^{L[\partial^δ(X,η)]}.\]

This means that for every $α < δ$, we have a descending sequence of iterated $C(aa)$ of length at least $α$, so we get a descending sequence of length $δ$, which ends in
\[(C(aa)^δ)^{L[\partial^δ(X,η)] = \bigcap_{α<δ} C(aa)^{L[\partial^β(X,η)] \cup \partial^α(X,η+η_α)}.\]

Recall that we are forcing with an club shooting iteration of length at most $θ$ where $δ < θ < κ, 0$, so by theorem 33.1 we work in a $δ$-distributive forcing, so we can use the same methods as before to get that
\[(C(aa)^δ)^{L[\partial^δ(X,η)] = L[\partial^δ(X,η)]/\sim}\]

where $\sim$ is the equivalence relation derived from the $δ$-normal system given by the projections
\[\prod_{α<β<δ} \partial^β(X,η + η_β) \rightarrow \prod_{α'<β<δ} \partial^β(X,η + η_β)\]

for $α < α'$. So as in the last part of the proof of theorem 47, we can prove that the stages of construction of $C(aa)^{L[\partial^δ(X,η)]}$ and of $C(aa)^{L[\partial^δ(X,η)]/\sim}$ are the same, so we get
\[C(aa)^{δ+1}L[\partial^δ(X,η)] = C(aa)^{L[\partial^δ(X,η)]} = C(aa)^{L[\partial^δ(X,η)]/\sim} = C(aa)^{L[\partial^δ(X,η)]}.\]

This concludes the construction of the coding forcings $\partial^δ(X,η)$. Now we get that in the model $L[\partial^δ(\partial(A,0),1)]$ the $C(aa)$ sequence is of length at least $δ + 1$ and
\[C(aa)^{δ+1}L[\partial^δ(\partial(A,0),1)] = C(aa)^{L[\partial(A,0)]} = L[A]\]
as required.

\[\square\]

4. Open questions

The first obvious question is whether the last result can be pushed to obtain an $\text{Ord}$ length iterated $C(aa)$ sequence. The first obstacle to this is distributivity – recall that in order to get distributivity we need to have iterations which are shorter than the first cardinal used for the coding. So to get longer iterations we’ll need to choose larger and larger coding cardinals. This might be possible, but it is not straightforward. Additionally we’d need an analysis of class length iterations for this type of forcing, and this is beyond the scope of this paper. So the following is still open:

**Question 51.** Is it consistent to have a model with an $\text{Ord}$ length $C(aa)$ sequence?
A second questions arises from the fact that we force over \( L \), and use \( L \)'s □-principle to get the mutually fat sets required for the iteration. In section 2.3.2 we show that such sets can also be obtained by forcing, but then it is not clear whether we can get those sets which we use as "coding tools" into \( C(\text{aa}) \) to begin with. This raises the question whether large cardinals or failure of □ might restrict the iterated \( C(\text{aa}) \) sequence. However, the results of [15] show that in the case of \( C^* \), the restriction of the length of the \( C^* \) sequence comes from lacking large cardinals, and large cardinals enable longer sequences. In any case, the following is open:

**Question 52.** Can we force models with long \( C(\text{aa}) \) sequence over any model of ZFC? Do large cardinals determine the possible length of such sequences? What is the length of the \( C(\text{aa}) \) sequence in canonical models for large cardinals such as \( L^\mu \)?

In [9] it is shown that under the assumption of a proper class of measurable Woodin cardinals (and recent results show the measurablity assumption can be dropped) the theory of \( C(\text{aa}) \) is set-forcing absolute. However, the length of the \( C(\text{aa}) \) sequence is prima facie not a first-order statement, so it isn't clear that this length will be preserved. However it is still worth investigating:

**Question 53.** Does the assumption of a proper class of (measurable) Woodin cardinals determine the length of the \( C(\text{aa}) \) sequence?

A different kind of inquiry, in light of the results we mentioned in the beginning about HOD, is the following:

**Question 54.** Is it consistent relative to ZFC that the \( C(\text{aa}) \) sequence is of length \( \omega \), and either:

\[ \begin{align*}
\text{\ding{52}} & \quad C(\text{aa})^\omega \models ZF + \neg AC \\
\text{\ding{53}} & \quad C(\text{aa})^\omega \not\models ZF
\end{align*} \]

Finally, we have introduced the new notion of mutually fat sets, which we believe is worth further investigation in itself. The first questions concern the difference between the various notions we introduced:

**Question 55.** Let \( K \) be an increasing sequence of regular uncountable cardinals, \(|K| < \min K\), \( \langle T_\kappa \mid \kappa \in K \rangle \) a sequence with \( T_\kappa \subseteq \kappa \).

1. Assume \( \langle T_\kappa \mid \kappa \in K \rangle \) are mutually stationary and each \( T_\kappa \) is fat. Does this imply that the \( \langle T_\kappa \mid \kappa \in K \rangle \) is mutually fat?
2. Assume \( \langle T_\kappa \mid \kappa \in K \rangle \) is mutually fat. Does it imply that it is strongly mutually fat?
We have used □-sequences and forcing non-reflecting stationary sets to obtain mutually fat sequences. It is worth investigating what other methods are there for obtaining such sequences.

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