INFINITELY MANY NON-RADIAL SOLUTIONS TO A CRITICAL EQUATION ON ANNULUS

YUXIA GUO, BENNIAO LI, ANGELA PISTOIA AND SHUSEN YAN

Abstract. In this paper, we build infinitely many non-radial sign-changing solutions to the critical problem:
\[
\begin{aligned}
-\Delta u &= |u|^{\frac{4}{N-2}}u, & &\text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, & &\text{on } \partial\Omega.
\end{aligned}
\]
\[(P)\]
on the annulus \(\Omega := \{ x \in \mathbb{R}^N : a < |x| < b \}, N \geq 3.\) In particular, for any integer \(k\) large enough, we build a non-radial solution which look like the unique positive solution \(u_0\) to \((P)\) crowned by \(k\) negative bubbles arranged on a regular polygon with radius \(r_0\) such that
\[
\max_{a \leq r \leq k} r^{\frac{N-2}{2}} u_0(r) =: \max_{a \leq r \leq b} r^{\frac{N-2}{2}} u_0(r).
\]

1. Introduction

This paper deals with the existence of solutions to the critical elliptic problem:
\[
\begin{aligned}
-\Delta u &= |u|^{\frac{4}{N-2}}u, & &\text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, & &\text{on } \partial\Omega,
\end{aligned}
\]
\[(1.1)\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) and \(N \geq 3.\)

It is well known that the geometry of the domain \(\Omega\) plays a crucial role in the solvability of the problem \((1.1).\) Indeed, if \(\Omega\) is a star-shaped domain, the classical Pohozaev identity [30] implies that \((1.1)\) does not have any solutions. While if \(\Omega = \{ x \in \mathbb{R}^N : a < |x| < b \}\) is an annulus, Kazdan and Warner [21] found a positive solution and infinitely many radial sign-changing solutions. Without any symmetry assumptions, the existence of solutions is a delicate issue. The first existence result is due to Coron in [10] who proved that problem \((1.1)\) has a positive solution in domain \(\Omega\) with a small hole. Later, Bahri and Coron in [2] proved that actually a positive solution always exists as long as the domain has non-trivial homology with \(\mathbb{Z}_2\)-coefficients. However, this last condition is not necessary since solutions to problem \((1.1)\) in contractible domains have been found by Dancer [11], Ding [17], Passaseo [28, 29] and Clapp and Weth [6]. The existence of sign-changing solutions is an even more delicate issue and it is known only for domains which have some symmetries or a small hole. The first existence result is due to Marchi and Pacella [24] for symmetric domains with thin channels. Successively, Clapp and Weth [6] found sign-changing solutions in a symmetric domain with a small hole. A first attempt to remove the symmetry assumption is due to Clapp and Weth [7], who found a second solution to \((1.1)\) in a domain with a small hole but they were not able to say if it changes sign or not. Sign-changing solutions in a domain with a small hole have been found by Clapp, Musso and Pistoia in [8]. Recently, Musso and Pistoia [25] and Ge, Musso...
and Pistoia [18] (see also [19]) proved that in a domain (not necessarily symmetric) with a small hole the number of sign-changing solutions to problem (1.1) becomes arbitrary large as the size of the hole decreases. The existence of a large number of sign-changing solutions in a domain with a hole of arbitrary size is due to Clapp and Pacella in [5], provided the domain has enough symmetry.

It is largely open for the problem of the existence of infinitely many sign-changing solutions in a general domain with non-trivial homology in the spirit of the famous Bahri and Coron’s result.

Here, we will focus on the existence of infinitely many sign-changing solutions to problem (1.1) when \( \Omega := \{ x \in \mathbb{R}^N : a < |x| < b \} \) is an annulus. The existence of infinitely many radial solutions was established by Kazdan and Warner in [21]. On the other hand, an annulus is invariant under many group actions and then it is natural to expect non-radial solutions which are invariant under these group actions. Indeed, Y.Y. Li in [22] improved a previous result by Coffman [9] and he found for any integer \( k \geq 1 \) in a sufficiently thin annulus some non-radial solutions which are invariant under the action of the group \( \mathfrak{G}_k \times \mathfrak{O}(N-2) \), when \( N \geq 4 \). Here \( \mathfrak{O}(N-2) \) denotes the group of orthogonal \( (N-2) \times (N-2) \) matrices and \( \mathfrak{G}_k \) is the subgroup of matrices which rotates \( \mathbb{R}^2 \) with angles equal to integer multiple of \( \frac{2\pi}{k} \). Recently, Clapp in [4] found infinitely many non-radial solutions which are invariant under the action of a suitable group whose orbits are infinite, provided \( N = 4 \) or \( N \geq 6 \).

In this paper we prove the existence of infinitely many new non-radial solutions which are invariant under the action of a group whose orbits are finite and they are not invariant under the action of the group \( \mathfrak{G}_k \times \mathfrak{O}(N-2) \). Moreover, as far as we know, this is the first example of non-radial solutions in the 3-dimensional annulus.

Let us state our main result. Let \( \Omega := \{ x \in \mathbb{R}^N : a < |x| < b \} \) be an annulus. Assume that

\[ \text{the unique positive radial solution } u_0 \text{ to (1.1) is non-degenerate.} \] (1.2)

The uniqueness has been proved by Ni and Nussbaum [26]. The non-degeneracy will be studied in Appendix A and it is true for most radii \( a \) and \( b \). Let us introduce the functions:

\[
U_{\xi, \lambda}(y) = C_N \lambda^{\frac{N+2}{2}} \left( \frac{1}{1 + \lambda^2 |y - \xi|^2} \right)^{\frac{N-2}{2}}, \quad \xi, y \in \mathbb{R}^N, \quad \lambda > 0
\]

which are all the positive solutions of the following critical problem on the whole space:

\[
-\Delta U = U^{\frac{N+2}{2}} \text{ in } \mathbb{R}^N, \quad (1.3)
\]

where \( C_N \) is a constant dependent on \( N \) (see [1, 27, 31]). We call \( U_{\xi, \lambda}(y) \) the bubble centered at the point \( \xi \) with scaling parameter \( \lambda \). Let us introduce its projection \( PU_{\xi, \lambda} \) onto \( H_0^1(\Omega) \), namely the solution of the Dirichlet problem:

\[
\begin{aligned}
-\Delta PU_{\xi, \lambda} &= U^{\frac{N+2}{2}}_{\xi, \lambda}, & \text{in } \Omega, \\
PU_{\xi, \lambda} &= 0, & \text{on } \partial \Omega.
\end{aligned}
\] (1.4)

Let \( k \geq 1 \) be an integer. Let us choose the centers of the bubbles as the \( k \) vertices of a regular \( k \)-polygon with radius \( r \) inside \( \Omega \) as:

\[
\xi_j = r\xi_j^*, \quad \xi_j^* : = (e^{j\frac{2\pi}{k}}, 0), \quad 0 \in \mathbb{R}^{N-2}, \quad j = 1, 2, \ldots, k, \quad r \in (a, b)
\] (1.5)
and the concentration parameter as:
\[ \lambda = \ell k^2, \quad \ell \in [\eta, \eta^{-1}] \] for some \( \eta > 0 \) small enough. (1.6)

Finally, we introduce the space
\[ H_s := \left\{ u \in H_0^1(\Omega) : u(x_1, x_2, \ldots, x_i, \ldots, x_N) = u(x_1, x_2, \ldots, -x_i, \ldots, x_N), \quad i = 2, \ldots, N, \\
\quad u(re^{i\theta}, x_3, \ldots, x_N) = u\left(re^{i(\theta + \frac{2\pi}{k}j)}, x_3, \ldots, x_N\right), \quad j = 1, \ldots, k \right\} \]

Now, we can state our main result.

**Theorem 1.1.** Let \( \Omega := \{ x \in \mathbb{R}^N : a < |x| < b \} \) be an annulus. Assume (1.2). Then there exists an integer \( k_0 > 0 \), such that for any integer \( k \geq k_0 \), problem (1.1) has a solution
\[ u_k(x) = u_0(x) - \sum_{j=1}^{k} PU_{r_k \xi_j, \lambda_k}(x) + \phi_k(x). \]

Where as \( k \to \infty \)

(i) \( r_k \to r_0 \in (a, b) \) and \( r_0 := \max_{a \leq r \leq b} u_0(r) \)

(ii) \( \lambda_k/k^2 \to \ell_0 > 0 \)

(iii) \( \phi_k \in H_s \) and \( \|\phi_k\|_{H_s^1(\Omega)} \to 0 \)

The paper is inspired by recent results obtained by Del Pino, Musso, Pacard and Pistoia [15, 16], where the authors constructed for any \( N \geq 3 \) infinitely many sign-changing solutions to (1.3) which look like the solution \( U_{0,1} \) crowned with \( k \) negative bubbles arranged on a regular polygon with radius near 1.

For the proof of our theorem, it relies on a Ljapunov-Schmidt procedure which allows us to reduce the problem of finding a solution to (1.1) whose profile at main order is \( u_0 - \sum_{j=1}^{k} PU_{r_k \xi_j, \lambda} \) to a 2–dimensional problem, namely finding the concentration parameter \( \lambda > 0 \) in (1.6) and the radius \( r \in (a, b) \) of the \( k \)–regular polygon whose vertices are the concentration points as in (1.5). The basic outline is similar to that in [15], but we carry out the reduction argument in a different way. Indeed, the invariance by Kelvin’s transform which is one of the main ingredient in the proof of [15], does not hold for problem (1.1). In particular, all our estimates are more straightforward than those used in [15].

This paper is organized as follows. In Section 2 we study the linearized equation around the approximate solution and we reduce the problem to a finite dimensional one. In Section 3 we study the reduced problem and we complete the proof of Theorem 1.1. Appendix A is devoted to the study of the non-degeneracy of the positive radial solution \( u_0 \).

2. Finite-dimensional reduction

Let us introduce the norms:
\[ \|u\|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{1 + \lambda|y - \xi_j|^N + r} \right)^{-1} \lambda^{-\frac{N-2}{2}} |u(y)| \] (2.1)
and
\[ \|f\|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^k \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{\alpha+2}{2} + \tau}} \right)^{-1} \lambda^{-\frac{N+2}{2}} |f(y)|, \]  
where \( \tau = \frac{1}{2} \). Since we assume that \( \lambda \sim k^2 \), it holds
\[ \sum_{j=1}^k \frac{1}{(1 + \lambda|\xi_j - \xi_1|)^{\tau}} \leq \frac{Ck}{\lambda^2} \leq C. \]

Set \( U_j = U_{\xi_j, \lambda}(y), P_j = PU_{\xi_j, \lambda}(y) \) and \( U_* = U_0 - \sum_j P_j \).

Denote
\[ Z_{j,1} = \frac{\partial P_j}{\partial \lambda}, \quad Z_{j,2} = \frac{\partial P_j}{\partial r}, \quad j = 1, 2, \ldots, k. \]

We consider the following linearized problem:
\[
\begin{aligned}
L_k \varphi &:= -\Delta \varphi - (2^* - 1)|U_*|^{2^*-2} \varphi - h + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l}, \quad \text{in } \Omega, \\
\varphi &\in H_\delta, \quad \sum_{j=1}^k \int_\Omega U_j^{2^*-2} Z_{j,l} \varphi = 0, \quad l = 1, 2,
\end{aligned}
\]  
for some real numbers \( c_l \).

**Lemma 2.1.** Suppose that \( \varphi_k \) solves (2.3) for \( h = h_k \). If \( \|h_k\|_* \) goes to zero as \( k \to +\infty \), so does \( \|\varphi_k\|_* \).

**Proof.** We argue by contradiction. Suppose that there exist \( k \to +\infty, r_k \to r_0, \lambda_k \in [L_0 k^2, L_1 k^2] \) and \( \varphi_k \) solving (2.3) for \( h = h_k, \lambda = \lambda_k, r = r_k \) with \( \|h_k\|_* \to 0 \) and \( \|\varphi_k\|_* \geq c > 0 \). Without loss of generality, we may assume that \( \|\varphi_k\|_* = 1 \). In the following, for simplicity reason, we drop the subscript \( k \).

Since we assume \( U_0 \) is non-degenerate, the following linear operator:
\[ \tilde{L}_0 \varphi := -\Delta \varphi - (2^* - 1)u_0^{2^*-2} \varphi, \quad \varphi \in H^1_0(\Omega), \]
is invertible. Let \( G(y, x) \) be the corresponding Green’s function. It is easy to prove that there exists a constant \( C > 0 \), such that
\[ |G(y, x)| \leq \frac{C}{|y - x|^{N-2}}. \]  

We rewrite (2.3) as:
\[
\begin{aligned}
L_0 \varphi &= (2^* - 1)(|U_*|^{2^*-2} - u_0^{2^*-2}) \varphi + h + \sum_{l=1}^2 c_l \sum_{j=1}^k U_j^{2^*-2} Z_{j,l}, \quad \text{in } \Omega, \\
\varphi &\in H_\delta, \quad \sum_{j=1}^k \int_\Omega U_j^{2^*-2} Z_{j,l} \varphi = 0, \quad l = 1, 2.
\end{aligned}
\]  
Then
\[ \varphi(y) = \int_{\Omega} G(z, y) \left[ (2^*-1)(|U_0|^{2^*-2} - u_0^{2^*-2}) \varphi + \sum_{i=1}^{2} \sum_{j=1}^{k} c_i \sum_{j=1}^{k} U_j^{2^*-2} Z_{j,l} \right]. \]

Using (2.4), we obtain
\[ |\varphi(y)| \leq C \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left[ (2^*-1)(|U_0|^{2^*-2} - u_0^{2^*-2}) \varphi + h + \sum_{i=1}^{2} \sum_{j=1}^{k} c_i \sum_{j=1}^{k} U_j^{2^*-2} Z_{j,l} \right]. \]

As in [32], we have
\[ \int_{\Omega} \frac{1}{|z-y|^{N-2}} |U_0|^{2^*-2} - u_0^{2^*-2} |\varphi| \leq C \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^{k} U_j \right)^{2^*-2} |\varphi| \]
\[ \leq C \|\varphi\|^* \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^{k} U_j \right)^{2^*-2} \left( \sum_{j=1}^{k} \frac{\lambda^{N-2}}{(1 + \lambda|z-\xi_j|)^{N-2+\tau}} \right) \]
\[ \leq C \|\varphi\|^* \lambda^{N-2} \sum_{j=1}^{m} \frac{1}{(1 + \lambda|y-\xi_j|)^{N-2+\tau+\theta}}, \]
\[ \int_{\Omega} \frac{1}{|z-y|^{N-2}} |h(z)| dz \leq C \|h\|^* \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^{k} U_j \right)^{2^*-2} \left( \sum_{j=1}^{k} \frac{\lambda^{N-2}}{(1 + \lambda|z-\xi_j|)^{N-2+\tau}} \right) \]
\[ \leq C \|h\|_{\alpha,*} \lambda^{N-2} \sum_{j=1}^{k} \frac{1}{(1 + \lambda|y-\xi_j|)^{N-2+\tau}}, \]

and
\[ \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left| \sum_{j=1}^{k} U_j^{2^*-2} Z_{j,l} \right| dz \]
\[ \leq C \lambda^{N-2+n_i} \int_{\Omega} \frac{1}{|z-y|^{N-2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + \lambda|z-\xi_j|)^{N+2}} \right) \]
\[ \leq C \lambda^{N-2+n_i} \sum_{j=1}^{k} \frac{1}{(1 + \lambda|y-\xi_j|)^{N-2}} \]
\[ \leq C \lambda^{N-2+n_i} \sum_{j=1}^{k} \frac{1}{(1 + \lambda|y-\xi_j|)^{N-2+\tau}}, \]
where $n_2 = 1$, $n_1 = -1$.

To estimate $c_l$, $l = 1, 2$, multiplying the both sides of (2.3) by the function $Z_{1,l}$, ($l = 1, 2$) and integrating on $\Omega$, we see that $c_l$ satisfies:

$$
\sum_{l=1}^{2} c_l \sum_{j=1}^{k} \int_{\Omega} U_j^{2^* - 2} Z_{j,h} Z_{1,l} = \int_{\Omega} \left( -\Delta \varphi - (2^* - 1)|U_*|^{2^* - 2} \varphi \right) Z_{1,l} - \int_{\Omega} h Z_{1,l}.
$$

(2.9)

We have

$$
\left| \int_{\Omega} h Z_{1,l} \right| \leq C \|h\|_{\ast \ast} \int_{R^N} \frac{\lambda^{N-2} \sum_{j=1}^{k} \lambda^{N-2}}{(1 + \lambda|z - \xi_j|)^{N-2}} \leq C \lambda^n \|h\|_{\ast \ast} \sum_{j=2}^{k} \frac{1}{(\lambda|\xi_j - \xi_1|)^{\tau}} \leq C \lambda^n \|h\|_{\ast \ast}.
$$

(2.10)

On the other hand, direct calculation gives

$$
\left| \int_{\Omega} \left( -\Delta \varphi - (2^* - 1)|U_*|^{2^* - 2} \varphi \right) Z_{1,l} \right| \\
= \left| \int_{\Omega} \left( -\Delta Z_{1,l} - (2^* - 1)|U_*|^{2^* - 2} Z_{1,l} \right) \varphi \right| \\
= (2^* - 1) \left| \int_{\Omega} \left( U_1^{2^* - 2} - |U_*|^{2^* - 2} \right) Z_{1,l} \varphi \right| \\
\leq C \lambda^n \|\varphi\|_{\ast \ast} \int_{\Omega} \left( u_0^{2^* - 2} + \sum_{j=2}^{k} U_j^{2^* - 2} \right) U_1 \sum_{j=1}^{k} \frac{\lambda^{N-2}}{(1 + \lambda|z - \xi_j|)^{N-2 + \tau}} \\
\leq \mathcal{O} \left( \lambda^n \|\varphi\|_{\ast \ast} \left( \frac{1}{\lambda^{N-2}} + \frac{1}{\lambda^{N-2 + \tau}} \right) \right).
$$

(2.11)

And it is easy to check that

$$
\sum_{j=1}^{k} \int_{\Omega} U_j^{2^* - 2} Z_{j,h} Z_{1,l} = (\tilde{c} + o(1)) \delta h \lambda^{2n},
$$

(2.12)

for some constant $\tilde{c} > 0$.

Now inserting (2.12) into (2.9), we find

$$
c_l = \frac{1}{\lambda^n} \left( o(\|\varphi\|_{\alpha,\ast}) + O(\|h\|_{\alpha,\ast \ast}) \right).
$$

(2.13)

So,

$$
\|\varphi\|_{\ast \ast} \leq \left( o(1) + \|h\|_{\ast \ast} + \frac{\sum_{j=1}^{k} (1 + \lambda|y - \xi_j|)^{-1}}{\sum_{j=1}^{k} (1 + \lambda|y - \xi_j|)^{-1 + \tau}} \right).
$$

(2.14)
Since $\|\varphi\|_* = 1$, we obtain from (2.14) that there is $R > 0$ such that
\[ \|\lambda^{\frac{N-2}{2}}\varphi\|_{L^\infty(B_R(\lambda(x_j)))} \geq a > 0, \tag{2.15} \]
for some $j$. But $\bar{\varphi}(y) = \lambda^{\frac{N-2}{2}} \varphi(\lambda(y-x_j))$ converges uniformly in any compact set to a solution $u$ of
\[ -\Delta u - (2^* - 1)U_{0,\Lambda}^{2^*-2}u = 0, \quad \text{in } \mathbb{R}^N, \tag{2.16} \]
for some $\Lambda \in [\Lambda_1, \Lambda_2]$, where $\Lambda_1, \Lambda_2$ are two constants, and $u$ is perpendicular to the kernel of (2.16). So $u = 0$. This is a contradiction to (2.15).

From Lemma 2.1, applying the same argument as in the proof of Proposition 4.1 in [13], we can prove the following result:

**Lemma 2.2.** There exist $k_0 > 0$ and a constant $C > 0$ independent of $k$, such that for $k \geq k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (2.3) has a unique solution $\varphi_k = L_k(h)$. Moreover,
\[ \|\varphi_k\|_* \leq C\|h\|_{**}, \quad |c_l| \leq \frac{C}{\lambda^{n_l}} \|h\|_{**}. \tag{2.17} \]

Now we consider the following non-linear problem:
\[
\left\{ \begin{array}{l}
-\Delta (U_* + \varphi) = |U_* + \varphi|^{2^*-2}(U_* + \varphi) + \sum_{i=1}^2 c_i \sum_{j=1}^k U_j^{2^*-2}Z_{j,i}, & \text{in } \Omega, \\
\varphi \in H_* \quad \int_{\Omega} \sum_{j=1}^k U_j^{2^*-2}Z_{j,i}\varphi = 0, \quad l = 1, 2.
\end{array} \tag{2.18} \right.
\]

The main result of this section is:

**Proposition 2.3.** There exists a positive integer $k_0$ such that for each $k \geq k_0$, $\lambda \in [\eta k^2, \eta^{-1}k^2], r \in [a + \tau, b - \tau]$, where $\tau$ and $\eta$ are positive and small, (2.18) has a unique solution $\varphi = \varphi_{r,\lambda} \in H_*$ satisfying
\[ \|\varphi\|_* \leq C\lambda^{-\frac{N-2}{4}}\sigma, \quad |c_l| \leq C\lambda^{-\frac{N-2}{4}}\sigma - n_l, \tag{2.19} \]
where $\sigma > 0$ is a small constant.

Rewrite (2.18) as:
\[
\left\{ \begin{array}{l}
-\Delta \varphi - (2^* - 1)|U_*|^{2^*-2}\varphi = N(\varphi) + l_k + \sum_{i=1}^2 c_i \sum_{j=1}^k U_j^{2^*-2}Z_{j,i}, & \text{in } \Omega, \\
\varphi \in H_* \quad \int_{\Omega} \sum_{j=1}^k U_j^{2^*-2}Z_{j,i}\varphi = 0, \quad l = 1, 2,
\end{array} \right. \tag{2.20} \]

where
\[ N(\varphi) = |U_* + \varphi|^{2^*-2}(U_* + \varphi) - |U_*|^{2^*-2}U_* - (2^* - 1)|U_*|^{2^*-2}\varphi, \]
and
\[ l_k = |U_*|^{2^*-2}U_* - u_{0,2^*-1} + \sum_{j=1}^k U_j^{2^*-1}. \]

In order to apply the contraction mapping principle to prove that (2.20) is uniquely solvable, we have to estimate $N(\varphi)$ and $l_k$ respectively.
Lemma 2.4. We have

\[ |N(\varphi)||_{**} \leq C\|\varphi\|_{*}^{\min(2^* - 1, 2)}. \]

Proof. If \( N \geq 6 \), then \( 2^* - 2 \leq 1 \). So we have

\[ |N(\varphi)| \leq C|\varphi|^{2^* - 1}, \]

which gives

\[
|N(\varphi)| \leq C\|\varphi\|^{2^* - 1}\left( \sum_{j=1}^{k} \frac{\lambda^{\frac{N-2}{2}}}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^* - 1} \\
\leq C\|\varphi\|^{2^* - 1}\lambda^{\frac{N-2}{2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^* - 1} \\
\leq C\|\varphi\|^{2^* - 1}\lambda^{\frac{N-2}{2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right).
\]

Therefore,

\[ |N(\varphi)||_{**} \leq C\|\varphi\|^{2^* - 1}. \]

Similarly, if \( 3 \leq N \leq 5 \), then \( 2^* - 3 > 0 \). In view of \( U_j \geq c_0 > 0 \) in \( \Omega \), we find

\[
|N(\varphi)| \leq C(\|u_0\|^{2^* - 3} + \sum_{j=1}^{k} (U_j)^{2^* - 3})|\varphi|^2 + C|\varphi|^{2^* - 1} \\
\leq C(\|\varphi\|^2 + \|\varphi\|^{2^* - 1})\lambda^{\frac{N-2}{2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^* - 1} \\
\leq C\|\varphi\|^2 \lambda^{\frac{N-2}{2}} \sum_{j=1}^{k} \left( \frac{1}{(1 + \lambda|y - \xi_j|)^{\frac{N-2}{2} + \tau}} \right).
\]

Next, we estimate \( l_k \).

Lemma 2.5. There is a constant \( \sigma > 0 \), such that

\[ \|l_k\||_{**} \leq C\lambda^{-\frac{N-2}{4} + \sigma}. \]

Proof. Write

\[
l_k = \left[ |U_0|^{2^* - 2}u_0^{2^* - 1} + \sum_{j=1}^{k} P_j^{2^* - 1} \right] + \sum_{j=1}^{k} (U_j^{2^* - 1} - P_j^{2^* - 1}) \\
=: J_1 + J_2.
\]

First, we estimate \( \|J_2\||_{**}. \) We have

\[ 0 \leq U_j^{2^* - 1} - P_j^{2^* - 1} \leq \frac{CU_j^{2^* - 2}}{\lambda^{\frac{N-2}{4}}}. \]

Let us determine the number \( \alpha > 0 \), such that

\[ \|l_k\||_{**} \leq C\lambda^{-\frac{N-2}{4} + \sigma}. \]
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\[
\frac{CU_j^{2^* - 2}}{\lambda^{\frac{2^*}{2^* - 2}}} \leq \frac{C\lambda^{-\alpha} \lambda^{\frac{N+2}{N-2}}}{(1 + \lambda|y - x_j|)^{\frac{N+2}{2} + \tau}}.
\]

The above inequality is equivalent to

\[
(1 + \lambda|y - x_j|)^{\frac{N+2}{2} + \tau - 4} \leq C\lambda^{-\alpha} \lambda^{N-2}.
\]

Note that \( \tau = \frac{1}{2} \). We find that \( \frac{N+2}{2} + \tau - 4 \geq 0 \) if \( N \geq 5 \). In view of \( 1 + \lambda|y - x_j| \leq C\lambda \) in \( \Omega \). We have

\[
(1 + \lambda|y - x_j|)^{\frac{N+2}{2} + \tau - 4} \leq C\lambda^{N-2}.
\]

As an result, \( \alpha = \frac{N-1}{2} \). Thus, we get

\[
\|J_2\|_{**} \leq C\lambda^{-\frac{N-1}{2}}, \quad \text{if } N \geq 5.
\]

(2.21)

If \( N \leq 5 \), it holds \( \frac{N+2}{2} + \tau - 4 < 0 \). Thus

\[
(1 + \lambda|y - x_j|)^{\frac{N+2}{2} + \tau - 4} \leq C = C\lambda^{2-N} \lambda^{N-2}.
\]

So \( \alpha = N - 2 \). Hence, we obtain

\[
\|J_2\|_{**} \leq C\lambda^{2-N}, \quad \text{if } N \leq 5.
\]

(2.22)

In order to estimate \( \|J_1\|_{**} \). We define

\[
\Omega_j = \{ y : y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \left( \frac{y'}{|y'|}, \frac{\xi_j}{|\xi_j|} \right) \geq \cos \frac{\pi}{k} \}.
\]

Using the assumed symmetry, we just need to estimate \( J_1 \) in \( \Omega_1 \). Let \( S = \Omega_1 \cap B_{1/\sqrt{\lambda}}(\xi_1) \).

Note that, it holds \( P_1 \geq c_0 > 0 \) in \( S \), and

\[
|U_*|^{2^* - 2} U_* = |u_0 - \sum_{j=2}^{k} P_j - P_1|^{2^* - 2}(u_0 - \sum_{j=2}^{k} P_j - P_1),
\]

we have

\[
|J_1| \leq P_1^{2^* - 2} (u_0 + \sum_{j=2}^{k} P_j) + J_3,
\]

where \( |J_3| \leq C \) in \( S \).

Since

\[
\frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - x_1|)^{\frac{N+2}{2} + \tau}} \geq \frac{\lambda^{\frac{N+2}{2}}}{(1 + \sqrt{\lambda})^{\frac{N+2}{2} + \tau}} \geq \alpha_0 \lambda^{\frac{N+2}{2} - \frac{k}{2}}, \quad y \in S,
\]

it holds

\[
|J_3| \leq C\lambda^{-\frac{N+2}{2}} \frac{\lambda^{\frac{N+2}{2}}}{(1 + \lambda|y - x_1|)^{\frac{N+2}{2} + \tau}}, \quad y \in S.
\]

On the other hand

\[
|P_1^{2^* - 2} (u_0 + \sum_{j=2}^{k} P_j)| \leq CU_1^{2^* - 2},
\]
and if $N \geq 5$, 

$$(1 + \lambda|y - \xi_1|)^{N + 2 - \frac{\lambda}{2} + \tau - 4} \leq C \lambda^{\frac{N + 2}{2} - \frac{\lambda}{2}} ((1 + \lambda|y - \xi_1|)^{N + 2 - \frac{\lambda}{2}})^{\frac{\lambda}{2} + \tau - 4},$$

which gives 

$$|P_2^2 - u_0 + \sum_{j=2}^k P_j) \leq C U_2^2 \leq \lambda^{- \frac{N + 2}{2} + \tau - 4} \left(1 + \lambda|y - \xi_1|\right)^{N + 2 - \frac{\lambda}{2} + \tau - 4}, y \in S.$$ 

If $N = 3, 4$, 

$$(1 + \lambda|y - \xi_1|)^{N + 2 - \frac{\lambda}{2} + \tau - 4} \leq C,$$

which gives 

$$|P_2^2 - u_0 + \sum_{j=2}^k P_j) \leq C U_2^2 \leq \lambda^{- \frac{N + 2}{2} + \tau - 4} \left(1 + \lambda|y - \xi_1|\right)^{N + 2 - \frac{\lambda}{2} + \tau - 4}, y \in S.$$ 

Therefore, we have proved 

$$|J_1| \leq C \lambda^{\frac{N + 2 - \sigma}{2}} \frac{\lambda^{N + 2}}{(1 + \lambda|y - \xi_1|)^{N + 2 - \frac{\lambda}{2} + \tau}}, y \in S. \quad (2.23)$$

On the other hand, we note that, in $\Omega_1 \setminus S$, it holds $P_1 \leq C$. Thus 

$$|J_1| \leq C \sum_{j=1}^k U_j \leq \frac{C}{\lambda^{\frac{N + 2}{2} - \frac{\lambda}{2}}|y - \xi_1|^{N - 2 - \frac{\lambda}{2}}} + \frac{C}{\lambda^{\frac{N + 2}{2} - \frac{\lambda}{2}}|y - \xi_1|^{N - 2 - \frac{\lambda}{2}}} \sum_{j=2}^k \frac{1}{|\xi_j - \xi_1|^{\tau}} \leq \frac{C}{\lambda^{\frac{N + 2}{2} - \frac{\lambda}{2} - \tau}|y - \xi_1|^{N - 2 - \frac{\lambda}{2}}. \quad (2.24)$$

Now we determine $\beta > 0$, such that 

$$\frac{1}{\lambda^{\frac{N + 2}{2} - \frac{\lambda}{2} - \tau}|y - \xi_1|^{N - 2 - \frac{\lambda}{2}}} \leq C \lambda^{- \beta} \frac{\lambda^{\frac{N + 2}{2}}}{(1 + \lambda|y - \xi_1|)^{N + 2 - \frac{\lambda}{2} + \tau}}, y \in \Omega_1 \setminus S. \quad (2.24)$$

It holds 

$$\frac{\lambda^{\frac{N + 2}{2}}}{(1 + \lambda|y - \xi_1|)^{N + 2 - \frac{\lambda}{2} + \tau}} \geq \frac{c'}{\lambda^{\tau}|y - \xi_1|^{N + 2 - \frac{\lambda}{2} + \tau}}, y \in \Omega_1 \setminus S.$$ 

So (2.24) holds if 

$$\frac{1}{\lambda^{\frac{N + 2}{2} - \tau}|y - \xi_1|^{N - 2 - \frac{\lambda}{2}}} \leq \frac{C \lambda^{- \beta}}{\lambda^{\tau}|y - \xi_1|^{N + 2 - \frac{\lambda}{2} + \tau}}, y \in \Omega_1 \setminus S.$$ 

which is equivalent to 

$$C|y - \xi_1|^{N - 2 - \frac{\lambda}{2} - \tau} \geq \lambda^{\beta + 2 \tau - \frac{\lambda}{2}}, y \in \Omega_1 \setminus S. \quad (2.25)$$
Since $|y - \xi_1| \geq \frac{1}{\sqrt{N}}$, we can take

$$\beta = \frac{N - 2}{2} - 2\tau - \frac{1}{2}(N - 2 - 2\tau - \frac{N + 2}{2}) = \frac{N + 2}{4} - \tau,$$

if $N - 2 - 2\tau - \frac{N + 2}{2} \geq 0$. That is $N \geq 8$. If $N \leq 8$, we can take $\beta = \frac{N - 2}{2} - 2\tau$.

So, we have proved

$$|J_1| \leq C\lambda^{-\frac{N - 2}{4} - \sigma} \frac{\lambda^{\frac{N + 2}{4}}}{(1 + \lambda|y - \xi_1|)^{\frac{N + 2}{4} + \tau}}, \quad y \in \Omega \setminus S.$$  \hspace{1cm} (2.26)

Combining (2.23) and (2.26), we find that there exists $\sigma > 0$, such that

$$|J_1| \leq C\lambda^{-\frac{N - 2}{4} - \sigma} \frac{\lambda^{\frac{N + 2}{4}}}{(1 + \lambda|y - \xi_1|)^{\frac{N + 2}{4} + \tau}}, \quad y \in \Omega.$$  \hspace{1cm} (2.27)

This gives

$$\|J_1\|_{**} \leq C\lambda^{-\frac{N - 2}{4} - \sigma}.$$

\[\square\]

Now we are ready to the proof of Proposition 2.3.

**Proof of Proposition 2.3.** First we recall that $\lambda \in [\eta k^2, \eta^{-1} k^2]$ for some $\eta > 0$. Set

$$\mathcal{N} = \left\{ w : w \in C(\mathbb{R}^N) \cap H_s, \|w\|_s \leq \frac{1}{\lambda^{\frac{N - 2}{4}}, \int_{\Omega} \sum_{j=1}^{k} U_j^{2\tau - 2} Z_{j,l}w = 0} \right\},$$

where $l = 1, 2$. Then (2.20) is equivalent to

$$\varphi = A(\varphi) =: L_k(N(\varphi)) + L_k(I_1),$$

here $L_k$ is defined in Lemma 2.2. We will prove that $A$ is a contraction map from $\mathcal{N}$ to $\mathcal{N}$.

First, we have

$$\|A(\varphi)\|_s \leq C\|N(\varphi)\|_{**} + C\|I_k\|_{**} \leq C\|\varphi\|_{**} + C\lambda^{\frac{1}{2\tau - 2}},$$

Hence, $A$ maps $\mathcal{N}$ to $\mathcal{N}$.

On the other hand, we see

$$\|A(\varphi_1) - A(\varphi_2)\|_s = \|L_k(N(\varphi_1)) - L_k(N(\varphi_2))\|_s \leq C\|\varphi_1 - \varphi_2\|_{**}.$$  \hspace{1cm} (2.28)

It is easy to check that if $N \geq 6$, then

$$|N(\varphi_1) - N(\varphi_2)| = |N'(\varphi_1 + \theta\varphi_2)||\varphi_1 - \varphi_2| \leq C(|\varphi_1|^{2\tau - 2} + |\varphi_2|^{2\tau - 2})|\varphi_1 - \varphi_2| \leq C(|\varphi_1|^{2\tau - 2} + |\varphi_2|^{2\tau - 2})\|\varphi_1 - \varphi_2\|_s \left(\sum_{j=1}^{k} \frac{\lambda^{\frac{N - 2}{4}}}{(1 + \lambda|y - \xi_j|)^{\frac{N - 2}{4} + \tau}}\right)^{2\tau - 1}. $$
As before, we have
\[
\left( \sum_{j=1}^{k} \frac{1}{(1 + \lambda |y - \xi_j|)^{\frac{N+2}{2} + \tau}} \right)^2 \leq C \sum_{j=1}^{k} \frac{1}{(1 + \lambda |y - \xi_j|)^{\frac{N+2}{2} + \tau}}.
\]
Hence,
\[
\|A(\varphi_1) - A(\varphi_2)\|_* \leq C \left( \|\varphi_1\|_*^{2^* - 2} + \|\varphi_2\|_*^{2^* - 2} \right) \|\varphi_1 - \varphi_2\|_* \leq \frac{1}{2} \|\varphi_1 - \varphi_2\|_*.
\]
Therefore, \( A \) is a contraction map.

The case \( N \leq 5 \) can be proved in a similar way.

By using the contraction mapping theorem, there exists a unique \( \varphi = \varphi_{r,\lambda} \in N \) such that (2.28) holds. Moreover, by Lemmas 2.2, 2.4 and 2.5, we deduce
\[
\|\varphi\|_* \leq \|L_k(N(\varphi))\|_* + \|L_k(I_k)\|_* \leq C\|N(\varphi)\|_{**} + C\|I_k\|_{**} \leq C\left( \frac{1}{r} \right)\frac{N+2}{2} + \sigma.
\]
Moreover, we get the estimate of \( c_l \) from (2.17).

3. The Proof of the Main theorem

We look for a solution to (1.1) as \( u = U_* + \varphi \), where \( \varphi = \varphi_k \) is the function obtained in Proposition 2.3. Let us introduce the energy functional whose critical points are solutions to (1.1)
\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{2^{*}} \int_{\Omega} |u|^{2^*}.
\]
and the reduced energy
\[
I_k(\ell, r) := I(U_* + \varphi).
\]

Where
\[
\lambda = \ell k^2, \quad \ell \in [\eta, \eta^{-1}] \text{ for some } \eta > 0 \text{ small enough.}
\]

We have the following result

**Proposition 3.1.**
(i) \( U_* + \varphi \) is a critical point of \( I \) if and only if \((\ell, r)\) is a critical point of the reduced energy \( I_k \).
(ii) We have
\[
I_k(\ell, r) = I(u_0) + kA + \frac{1}{kN-2} F(\ell, r) + o\left( \frac{1}{kN-2} \right)
\]
uniformly in compact sets of \((0, +\infty) \times (a, b)\), where
\[
F(\ell, r) := B\frac{u_0(r)}{\ell^{N-2}} - C\frac{1}{r^{N-2} \ell^{N-2}}
\]
for some positive constants \( A, B \) and \( C \).

**Proof.** The proof of (i) is quite standard. We only prove (ii). First of all we prove that
\[
I(U_* + \varphi) = I(U_*) + kO\left( \lambda^{-\frac{N^2}{2}} \frac{1}{r^{N-2} \ell^{N-2}} \right), \text{ for some } \sigma < 0.
\]
First of all, we have
Thus, we obtain

\[ I(U_+ + \varphi) = I(U_+) + \frac{1}{2} \int |\nabla \varphi|^2 + \int_{\Omega} (\mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1}) \varphi \]

\[ - \frac{1}{2^*} \int_{\Omega} \left( |U_+ + \varphi|^{2^*} - |U_+|^{2^*} \right). \tag{3.4} \]

It follows from (2.18) that

\[ \int_{\Omega} |\nabla \varphi|^2 = \int_{\Omega} |U_+ + \varphi|^{2^* - 2}(U_+ + \varphi) - \int_{\Omega} \left( \mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1} \right) \varphi. \tag{3.5} \]

Thus, we obtain

\[
\begin{align*}
I(U_+ + \varphi) &= I(U_+) + \frac{1}{2} \int |U_+ + \varphi|^{2^* - 2}(U_+ + \varphi) \varphi + \frac{1}{2} \int_{\Omega} \left( \mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1} \right) \varphi \\
&\quad - \frac{1}{2^*} \int_{\Omega} \left( |U_+ + \varphi|^{2^*} - |U_+|^{2^*} \right) \\
&= I(U_+) + \frac{1}{2} \int \left( \mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1} \right) \varphi \\
&\quad + \frac{1}{2} \int \left( |U_+ + \varphi|^{2^* - 2}(U_+ + \varphi) - |U_+|^{2^* - 2} U_+ \right) \varphi \\
&\quad - \frac{1}{2^*} \int_{\Omega} \left( |U_+ + \varphi|^{2^*} - |U_+|^{2^*} - 2^*|U_+|^{2^*} \varphi \right). \tag{3.6} \end{align*}
\]

Write

\[ l_k = \left[ |U_+|^{2^* - 2} U_+ - \mathcal{U}_0^{2^* - 1} + \sum_{j=1}^{k} P_j^{2^* - 1} \right] + \sum_{j=1}^{k} (U_j^{2^* - 1} - P_j^{2^* - 1}). \]

It follows from Lemma 2.5, there is a constant \( \sigma > 0 \), such that

\[ ||l_k||_{\ast\ast} \leq C \lambda^{-\frac{\sigma}{2^* - 2^*}}. \]

By Proposition 2.3, we can obtain from (3.6) that if \( N \geq 6 \),

\[
\begin{align*}
I(U_+ + \varphi) &= I(U_+) + O \left( \frac{\|\varphi\|_{\ast\ast} \|\varphi\|_{\ast\ast}}{k} \right) \sum_{j=1}^{k} \int \left( \sum_{j=1}^{k} \frac{\lambda^{N/2} (y - x_j)^{2^* - 2^*}}{(1 + \mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1})^{2^* - 1 + \tau}} \right) \left( \sum_{j=1}^{k} \frac{\lambda^{N/2 - 2} (y - x_j)^{2^* - 2^*}}{(1 + \mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1})^{2^* - 1 + \tau}} \right) \\
&\quad + O \left( \|\varphi\|_{\ast\ast}^{2^*} \right) \int_{\Omega} \left( \sum_{j=1}^{k} \frac{\lambda^{N/2 - 2} (y - x_j)^{2^* - 2^*}}{(1 + \mathcal{U}_0^{2^* - 1} - \sum_{j=1}^{k} U_j^{2^* - 1})^{2^* - 1 + \tau}} \right)^{2^*} \tag{3.7} \\
&= I(U_+) + kO \left( \lambda^{-\frac{\sigma}{2^* - 2^*}} \right). \end{align*}
\]
While if $N \leq 5$, then

$$I(U_\ast + \varphi)$$

$$= I(U_\ast) + O\left(\|l_k\|_* \|\varphi\|_*\right) \sum_{j=1}^k \int_\Omega \left(\sum_{j=1}^k \frac{\lambda_{N+2}^{\frac{\gamma}{2}}}{{(1 + \lambda|y - x_j|)^{\frac{N-2}{2}}}^{1 + \tau}}\right) \left(\sum_{j=1}^k \frac{\lambda_{N-2}^{\frac{\gamma}{2}}}{{(1 + \lambda|y - x_j|)^{\frac{N-2}{2}}}^{1 + \tau}}\right)$$

$$+ O\left(\|\varphi\|_*^2\right) \int_\Omega \left(\sum_{j=1}^k \frac{\lambda_{N+2}^{\frac{\gamma}{2}}}{{(1 + \lambda|y - x_j|)^{\frac{N-2}{2}}}^{1 + \tau}}\right) 2^\gamma$$

$$+ O\left(\|\varphi\|_*^2\right) \int_\Omega |U_j|^{2^\gamma - 3} \left(\sum_{j=1}^k \frac{\lambda_{N+2}^{\frac{\gamma}{2}}}{{(1 + \lambda|y - x_j|)^{\frac{N-2}{2}}}^{1 + \tau}}\right)^2$$

$$= I(U_\ast) + kO\left(\lambda^{-N+2}\right).$$

That concludes the proof of (3.3).

Next, we prove that

$$I(U_\ast) = I(u_0) + k \left[ A - \frac{B_1 \lambda_{N-2}^{\frac{\gamma}{2}}}{{r^{N-2}\lambda^{N-2}}} + \frac{B_2 u_0(r)}{\lambda^{N-2}} + O\left(\frac{1}{\lambda^{\frac{N-2}{2}(1+\delta)}}\right)\right]$$

(3.9)

where $A, B_1, B_2$ and are positive constants, $\delta > 0$ is small.

Recall that $P_j$ satisfies (1.4) and set $V = \sum_{j=1}^k P_j$. We have

$$\int_\Omega |\nabla U_\ast|^2$$

$$= \int_\Omega |\nabla u_0|^2 + \int_\Omega |\nabla U|^2 - 2 \int_\Omega \nabla U \nabla u_0$$

$$= \int_\Omega |\nabla u_0|^2 + \int_\Omega |\nabla U|^2 - 2 \int_\Omega u_0^{2^\gamma - 1} V.$$  

(3.10)

Let $\Omega_j = \{(r \cos \theta, r \sin \theta, x')|\frac{2\pi(j-1)}{k} - \frac{\pi}{k} \leq \theta \leq \frac{2\pi(j)}{k} + \frac{\pi}{k}, x' \in \mathbb{R}^{N-2}\} \cap \Omega, j = 1, ..., k$. Then by the symmetry, we have

$$\int_\Omega u_0^{2^\gamma - 1} V = k \int_{\Omega_1} u_0^{2^\gamma - 1} V,$$

and

$$\int_\Omega |U_j|^{2^\gamma} = k \int_{\Omega_1} |U_\ast|^{2^\gamma}.$$

Let $S =: \Omega_1 \cap B_{\lambda^{-\frac{1}{4}}}(\xi_1)$, by suing the following inequality:

$$|1 - t|^p = 1 - pt + O(t^2) = 1 - pt + O(t^\alpha), 1 < \alpha \leq 2, \forall 0 \leq t \leq c,$$

where $c$ is some constant, we obtain
\[ \int_S |U_*|^2 \]
\[ = \int_S V^{2^*} - 2^* \int_S V^{2^* - 1} u_0 + O(\int_S \sigma^{2^* - 1 - \delta} u_0^\delta) \]
\[ = \int_S V^{2^*} - 2^* \int_S V^{2^* - 1} u_0 + O(\lambda^{-\frac{(1+\delta)(N-2)}{2}}), \tag{3.11} \]

where \( \delta > 0 \) is small.

On the other hand, we have
\[ \int_{\Omega \setminus S} |U_*|^2 \]
\[ = \int_{\Omega_0 \setminus S} u_0^{2^*} - 2^* \int_{\Omega_0 \setminus S} u_0^{2^* - 1} V + O(\int_{\Omega_0 \setminus S} u_0^{2^* - 2} V^2) \]
\[ = \int_{\Omega_0} u_0^{2^*} - 2^* \int_{\Omega_0} u_0^{2^* - 1} V + O(\int_{\Omega_0} u_0^{2^* - 2} V^2) + O(\lambda^{\frac{-N}{2}}), \tag{3.12} \]

since
\[ \int_{\Omega_1 \setminus S} u_0^{2^*} = \int_{\Omega_1} u_0^{2^*} + O(\lambda^{\frac{-N}{2}}). \tag{3.13} \]

Note that, for any \( y \in \Omega_0 \), we have
\[ \sum_{j=2}^{k} \frac{1}{|y - \xi_j|^{N-2}} \leq \frac{C}{|y - \xi_1|^{N-2-\tau}} \sum_{j=2}^{k} \frac{1}{|\xi_j - \xi_1|^{\tau}} \leq \frac{Ck}{|y - \xi_1|^{N-2-\tau}} \]
for \( \tau \in (0, 1) \). So we obtain
\[ \int_{\Omega_0 \setminus S} u_0^{2^* - 2} V^2 \leq C \int_{\Omega_0 \setminus S} V^2 \]
\[ \leq C \int_{\Omega_0 \setminus S} \left( \sum_{j=1}^{k} \frac{1}{\lambda^{\frac{N-2}{2}} |y - \xi_j|^{N-2}} \right)^2 \]
\[ \leq C \int_{\Omega_0 \setminus S} \left( \frac{1}{|y - \xi_1|^{N-2-\tau}} \right)^2 \]
\[ \leq C \int_{\Omega_0 \setminus S} \left( \frac{1}{|y - \xi_1|^{2(N-2)}} + \frac{k}{|y - \xi_1|^{2(N-2-\tau)}} \right) \]
\[ \leq C \frac{\lambda^{\frac{N}{2}} + k^2 \lambda^{\frac{N}{2}} (N-4-2\tau)}{\lambda^{\frac{N}{2}} (N-4) + k^2 \lambda^{\frac{N}{2}} (N-4-2\tau)} \]
\[ \leq \frac{C}{\lambda^{\frac{N}{2} - 1 + 2\tau}}, \tag{3.14} \]

since \( k^2 \sim \lambda \). As a consequence,
\[ \int_{\Omega \setminus S} |U_*|^2 \]
\[ = \int_{\Omega_1} u_0^{2^*} - 2^* \int_{\Omega_1 \setminus S} u_0^{2^* - 1} V + O(\frac{1}{\mu^{\frac{N}{2} - 1 + \delta}}), \tag{3.15} \]
Combining the above obtained results, we get
\[
I(U_*) = I(u_0) + \frac{1}{2} \int_{\Omega} |DV|^2 - \frac{k}{2} \int_{S} V^{2^*} + k \int_{S} V^{2^*-1} u_0 - \int_{\Omega} u_0^{2^*-1} V + k \int_{\Omega \setminus S} u_0^{2^*-1} V + O\left(\frac{k}{\lambda^{\frac{N-2}{2}}(1+\delta)}\right).
\]
(3.16)

Now we compute those integrals in (3.16) one by one:
\[
\begin{align*}
-\int_{\Omega} u_0^{2^*-1} V + k \int_{\Omega \setminus S} u_0^{2^*-1} V \\
= -k \int_{\Omega} u_0^{2^*-1} V + k \int_{\Omega \setminus S} u_0^{2^*-1} V \\
= -k \int_{\Omega} u_0^{2^*-1} V \\
= O\left(k \int_{\Omega} \frac{1}{\lambda^{\frac{N-2}{2}}} \sum_{j=1}^{k} \frac{1}{|y - \xi_j|^{N-2}}\right) \\
= O\left(k \int_{\Omega} \frac{1}{\lambda^{\frac{N-2}{2}}} \left(\frac{1}{k} \sum_{j=1}^{k} \frac{1}{|y - \xi_j|^{N-2}} + \frac{k}{|y - \xi_1|^{N-2}}\right)\right) \\
= O\left(\frac{k}{\lambda^{\frac{N-2}{2}}} \left(\frac{1}{\lambda} + \frac{1}{\lambda^{1+2}}\right)\right) = O\left(\frac{k}{\lambda^{\frac{N-2}{2}}} \left(\frac{1}{\lambda} + \frac{1}{\lambda^{1+2}}\right)\right).
\end{align*}
\]
(3.17)

We have that for any \(y \in S\)
\[
\sum_{j=2}^{k} \frac{P_j(\xi_1 + \lambda^{-1} y)}{\lambda^{\frac{N-2}{2}}} \leq C \sum_{j=2}^{k} \frac{1}{(1 + |y - \lambda(\xi_j - \xi_1)|)^{N-2}} \\
\leq C \sum_{j=2}^{k} \frac{1}{|\lambda(\xi_j - \xi_1)|^{N-2}} \\
\leq C \frac{\ln k^{1-\sigma_N} k^{N-2}}{\lambda^{N-2}} \leq C \frac{\ln k^{\sigma_N}}{\lambda^{\frac{N-2}{2}}},
\]
where \(\sigma_N = 0\) if \(N \geq 4\) and \(\sigma_N = 1\), if \(N = 3\). So
\[
\left|\left(\frac{V(\xi_1 + \lambda^{-1} y)}{\lambda^{\frac{N-2}{2}}}\right)^{2^*-1} - \left(\frac{P_1(\xi_1 + \lambda^{-1} y)}{\lambda^{\frac{N-2}{2}}}\right)^{2^*-1}\right| \\
\leq C \left(\frac{1}{(1 + |y|)^4} \frac{\ln k^{\sigma_N}}{\lambda^{\frac{N-2}{2}}} + \frac{\ln k^{(2^*-1)\sigma_N}}{\lambda^{\frac{N+2}{2}}}\right).
\]

Thus, we have
\[ \int_S V^{2^* - 1} u_0 \]

\[ = \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_r(\xi_1)} \left( \frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^* - 1} u_0(\xi_1 + \lambda^{-1}y) \]

\[ = \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_r(\xi_1)} \frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} u_0(\xi_1 + \lambda^{-1}y) \]

\[ + \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_r(\xi_1)} \left[ \frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} - \frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right] u_0(\xi_1 + \lambda^{-1}y) \]

\[ = \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_r(\xi_1)} \frac{P_1(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} u_0(\xi_1 + \lambda^{-1}y) \]

\[ + \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_r(\xi_1)} \left( \frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^* - 1} u_0(\xi_1 + \lambda^{-1}y) \]

\[ = \frac{1}{\lambda^{\frac{N-2}{2}}} \int_{B_r(\xi_1)} \left( \frac{V(\xi_1 + \lambda^{-1}y)}{\lambda^{\frac{N-2}{2}}} \right)^{2^* - 1} \left( u_0(\xi_1) + O\left( \frac{1}{\lambda^2} \right) \right) \]

\[ = \frac{u_0(\xi_1)}{\lambda^{\frac{N-2}{2}}} \left( \int_{\mathbb{R}^N} U^{2^* - 1} + O\left( \frac{1}{\lambda^2} \right) \right). \]

Finally, it is standard to prove

\[ \frac{1}{2} \int_{\Omega} |\nabla V|^2 - \frac{k}{2} \int_{\Omega} V^{2^*} = k \left( \frac{1}{2} \int_{\Omega} |\nabla V|^2 - \frac{k}{2} \int_{S} V^{2^*} \right) \]

\[ = k \left[ \int_{\mathbb{R}^N} |\nabla U_{0,1}|^2 - \frac{1}{2} \int_{\mathbb{R}^N} U_{0,1}^{2^*} + \sum_{j=2}^k \frac{B_0}{\lambda^{N-2}|x_j - x_1|} + O\left( \frac{1}{\lambda^{\frac{N-2}{2}(1+\delta)}(1+\delta)} \right) \right] \]

Combining the above obtained results, we get (3.9).

Finally, the claim follows by the choice of \( \lambda \) in (1.6).

\[ \square \]

We are now ready to prove the main theorem.

**Proof of Theorem 1.1: completed.** We apply Proposition 3.1. It is easy to check that \( F \) has a maximum point at the point \((\xi_0, r_0)\) where \( r_0 \) maximizes the function \( r \mapsto r^{\frac{N-2}{2}} u_0(r) \) and \( \xi_0 := \left( \frac{2B}{C u_0(r_0)^2} \right)^{\frac{N-2}{2}}, \) which is stable under \( C^0 \)-perturbation. Therefore, the reduced energy \( I_k \) has a critical point \((\xi_k, r_k)\), which produces the solution \( U_\ast + \phi \) to the problem (1.1).

\[ \square \]
APPENDIX A. NON-DEGENERACY OF THE POSITIVE RADIAL SOLUTION

Without loss of generality we can assume that the annulus is $A_R := \{ x \in \mathbb{R}^n : R \leq |x| \leq 1 \}$ (i.e. $a = R$ and $b = 1$).

Let $u_R$ be the unique positive radial solution to the following problem:

$$
\begin{cases}
-\Delta u = u^p & \text{in } A_R, \\
u = 0 & \text{on } \partial A_R.
\end{cases}
$$

Here we set $p := \frac{N+2}{N-2}$, $N \geq 3$.

Proposition A.1. There exists a sequence of radii $(R_k)_{k \in \mathbb{N}}$, such that $u_R$ is non-degenerate for any $R \neq R_k$.

Proof. (i) Let us consider the following linear problem:

$$
\begin{cases}
-\Delta v = p u^{p-1} v & \text{in } A_R, \\
v = 0 & \text{on } \partial A_R.
\end{cases}
$$

We denote by $\lambda_k = k(k+n-2)$ for $k = 0, 1, 2, \ldots$ the eigenvalues of $-\Delta$ on the sphere $S^{n-1}$. Let $\{\Phi_k^i : 1 \leq i \leq m_k\}$ denote a basis for the $k^{th}$ eigenspace of $-\Delta$. Then for any function $v = v(r, \theta)$ on the annulus $A_R$ we may write

$$v(r, \theta) = \sum_{k \geq 0} a_k(r) \hat{\Phi}_k^i(\theta), \quad r \in (1, R), \theta \in S^{n-1},$$

where each $a_k$ is a radial solution to

$$
\begin{cases}
a_k'' + \frac{n-1}{r} a_k' + \left( p u^{p-1}(r) - \frac{\lambda_k}{r^2} \right) a_k(r) = 0 \text{ in } (R, 1), \\
a_k(R) = a_k(1) = 0,
\end{cases}
$$

and

$$\hat{\Phi}_k^i(\theta) = \sum_{i=1}^{m_k} c_i \Phi_k^i(\theta), \quad \text{for some } c_i \in \mathbb{R}.$$  

(ii) Argue as in Proposition 2.1 in [3], we have that

$$a_0(r) = 0 \text{ for any } r \in (R, 1).$$

It means that $u_0$ is non-degenerate in the space of radial functions.

(iii) For any integer $k \geq 1$, let $\mu_{k,i} = \mu_{k,i}(R)$, $i \geq 1$ be the sequence of the eigenvalues of the problem:

$$
\begin{cases}
\phi'' + \frac{n-1}{r} \phi' + \left( p u^{p-1}(r) - \frac{\lambda_k}{r^2} \right) \phi = -\mu_{k,i} \phi \text{ in } (R, 1), \\
\phi(R) = \phi(1) = 0.
\end{cases}
$$

We point out that if

$$\mu_{k,i}(R) \neq 0 \text{ for any } k \geq 1 \text{ and } i \geq 1,$$

then any solutions to (A.4) $a_k \equiv 0$.

So by (A.3) (together with (A.5)) we deduce that any solutions to (A.2) $v \equiv 0$, i.e. $u_R$ is non-degenerate.
(iv) By Corollary 2.4 in [23] we get
\[ \mu_{k1}(R) < 0 \text{ and } \mu_{k_i}(R) > 0 \text{ for } k \geq 1 \text{ and } i \geq 2, \text{ for any } R \in (0, 1). \] (A.8)
It only remains to check the behavior of the first eigenvalue \( \mu_{k1}(R) \) for any \( k \geq 2 \). We know by Lemma 3.1 in [23] that
\[ \lim_{R \to 1} \mu_{k1}(R) = -\infty, \text{ for any } k \geq 1. \]

(v) If \( \phi \) solves (A.6) then \( \psi(t) = \phi(t(1 - R) + 2R - 1) \) solves the following problem:
\[
\begin{cases}
\psi'' + \frac{(n-1)(1-R)}{t(1-R)+2R-1} \psi' + (1-R)^2 \left( pw_R^{p-1}(t) - \frac{\lambda_k}{(t(1-R)+2R-1)^{\frac{1}{p}}} \right) \psi = \lambda_k \psi, \text{ in } (1, 2), \\
\psi(1) = \psi(2) = 0.
\end{cases}
\] (A.9)
Where \( \lambda_k = \lambda_k(R) := -(1-R)^2 \mu_{k1}(R) \).
On the other hand, we see that \( w_R(t) = u_R(t(1 - R) + 2R - 1) \) solves the following problem:
\[
\begin{cases}
w''_R + \frac{(n-1)(1-R)}{t(1-R)+2R-1} w'_R + (1-R)^2 w_R^p(t) = 0 \text{ in } (1, 2), \\
w_R(1) = w_R(2) = 0.
\end{cases}
\] (A.10)

(vi) We claim that
\begin{enumerate}
\item for any \( k \geq 2 \) there exists a finite number of radii \( R_{k1}, \ldots, R_{k\ell(k)} \) such that \( \lambda_k(R_{k_i}) = 0 \) for \( i = 1, \ldots, \ell(k) \).
\end{enumerate}
The proof for the claim could follow the same arguments as in Lemma 2.2 (c) of [14]. Indeed, using a result due to Kato (see Example 2.12, page 380 in [20]), we could prove that each function \( R \to \lambda_k(R) \) is analytic so it can only vanish at a finite number of points. We can prove that the function \( W : (0, 1) \to C^2(\mathbb{I}), \mathbb{I} = [1, 2] \), defined by \( W(R)(t) = w_R(t) \) is analytic using the same arguments developed by Dancer in [12].

\[ \square \]

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Department of Mathematical Science, Tsinghua University, Beijing, P.R.China
E-mail address: yguo@math.tsinghua.edu.cn

Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia
E-mail address: bli9@une.edu.au

Dipartimento SBAI, Via Scarpa 16, Roma, Italy
E-mail address: angela.pistoia@uniroma1.it

Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia
E-mail address: syan@turing.une.edu.au