FRACTIONAL OPERATORS ON WEIGHTED MORREY SPACES

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Abstract. A necessary condition and a sufficient condition for one weight norm inequalities on Morrey spaces to hold are given for the fractional maximal operator and the fractional integral operator. We clarify the difference between the behavior of the fractional maximal operator and the one of the fractional integral operator which is originated from the structure of Morrey spaces. Both the necessary condition and the sufficient condition are also verified for the power weights.

1. Introduction

The purpose of this paper is to develop a theory of weights for fractional maximal and integral operators on Morrey spaces. There are several results concerning the weight theory on Morrey spaces by assuming the $A_p$ conditions (for example, [9, 24]). However, it was pointed in [27, 32] that the $A_p$ condition is not suitable for the Morrey setting. In fact, it is too strong. So, the problem we address in is to establish the weight theory on Morrey spaces without the $A_p$ condition. After C. Morrey introduced Morrey spaces, many people realized that Morrey spaces are used for various purpose. One of the reasons is that Morrey spaces describe local regularity more precisely than Lebesgue spaces. As a result we can use Morrey spaces widely not only in harmonic analysis but also in partial differential equations (cf. [7]).

To define Morrey spaces, we shall consider all cubes in $\mathbb{R}^n$ which have their sides parallel to the coordinate axes. We denote by $Q$ the family of all such cubes. For a cube $Q \in Q$ we use $\ell(Q)$ to denote the sides length of $Q$, $c(Q)$ to denote the center of $Q$, $|Q|$ to denote the volume of $Q$ and $cQ$ to denote the cube with the same center as $Q$ but with side-length $\ell(Q)$. Let $0 < p \leq p_0 < \infty$ be two real parameters. For $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ define

\begin{equation}
\|f\|_{\mathcal{M}^{p_0}_{p}} = \sup_{Q \in Q} |Q|^{1/p_0} \left( \int_Q |f|^p \, dx \right)^{1/p},
\end{equation}

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where we have used a barred integral to denote the integral average
\[ \int_Q f \, dx = \frac{1}{|Q|} \int_Q f \, dx. \]

The Morrey space \( \mathcal{M}^p_0(\mathbb{R}^n) \) is defined to be the subset of all \( L^p \) locally integrable functions \( f \) on \( \mathbb{R}^n \) for which \( \| f \|_{\mathcal{M}^p_0} \) is finite. It is easy to see that \( \| \cdot \|_{\mathcal{M}^p_0} \) is a norm if \( p \geq 1 \) and is a quasi-norm if \( p \in (0, 1) \). The completeness of Morrey spaces follows easily by that of Lebesgue spaces.

Applying Hölder’s inequality, we see also that
\[ \| f \|_{\mathcal{M}^p_0} \geq \| f \|_{\mathcal{M}^q_0} \]
for all \( p_0 \geq p_1 \geq p_2 > 0 \).

This tells us that
\[ L^p_{p_1}(\mathbb{R}^n) = \mathcal{M}^p_{p_0}(\mathbb{R}^n) \subset \mathcal{M}^p_{p_1}(\mathbb{R}^n) \subset \mathcal{M}^p_{p_2}(\mathbb{R}^n) \quad \text{for all} \quad p_0 \geq p_1 \geq p_2 > 0. \]

Sometimes it is convenient to define Morrey spaces in an equivalent form. Let \( 1 < p < p_0 < \infty \) and define \( \lambda \) by \( \lambda/n = 1 - p/p_0 \). We will use the notation
\[ \| f \|_{L^p, \lambda} = \sup_{Q \in \mathcal{Q}} \left( \frac{1}{|Q|^{\lambda/n}} \int_Q |f|^p \, dx \right)^{1/p} \]
and \( L^{p, \lambda}(\mathbb{R}^n) \) to denote \( \| f \|_{\mathcal{M}^p_{p_0}} \) and \( \mathcal{M}^p_{p_0}(\mathbb{R}^n) \), respectively.

As we mentioned above, Morrey spaces reflect local properties of the functions. Due to this property, we can describe the boundedness property of the linear (or sublinear) operators more precisely than Lebesgue spaces.

We envisage the following operators in this paper.

- Given \( 0 < \alpha < n \) and a measurable function \( f \), we define the fractional integral operator \( I_\alpha \) by
  \[ I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy. \]

- Given \( 0 \leq \alpha < n \) and a measurable function \( f \), we define the fractional maximal operator \( M_\alpha \) by
  \[ M_\alpha f(x) = \sup_{Q \in \mathcal{Q}} 1_Q(x)|Q|^{\alpha/n} \int_Q |f| \, dy, \]

  where \( 1_Q \) denotes the characteristic function of the cube \( Q \). If \( \alpha = 0 \) we drop the subscript \( \alpha \). Thus, \( M = M_0 \) is the Hardy-Littlewood maximal operator.

Based on the definition above, let us see two remarkable results asserting for what parameters \( p, p_0, q, q_0 \) the fractional integral operator \( I_\alpha \) is bounded from \( \mathcal{M}^{p_0}_p \) to \( \mathcal{M}^{q_0}_q \), where \( 1 < p \leq p_0 < \infty \) and \( 1 < q \leq q_0 < \infty \). The first one is due to Spanne (unpublished): the inequality
\[ \| I_\alpha f \|_{\mathcal{M}^q_{q_0}} \leq C \| f \|_{\mathcal{M}^{p_0}_p} \]
holds if
\[ \frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \]
The second one is due to Adams [1] (see also [4]): the inequality (1.3) holds if
\[
\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.
\]
A simple arithmetic shows that
\[
\frac{1}{p} - \frac{\alpha}{n} = \frac{p_0}{p} \left( \frac{1}{p_0} - \frac{p}{p_0} \cdot \frac{\alpha}{n} \right) \geq \frac{p_0}{p} \left( \frac{1}{p_0} - \frac{\alpha}{n} \right) = \frac{p_0}{pq_0}.
\]
This inequality together with (1.2) says that the Spanne target space is larger than the Adams target space. Thus, we can say that Adams improved the result of Spanne. Furthermore, Olsen [25] showed by an example that the result of Adams is optimal.

By weights we will always mean non-negative, locally integrable functions which are positive on a set of positive measure. Given a measurable set $E$ and a weight $w$, $w(E) = \int_E w$. Given $1 < p < \infty$, $p' = p/(p - 1)$ will denote the conjugate exponent number of $p$.

Given $p > 1$, one says that a weight $w$ on $\mathbb{R}^n$ belongs to the Muckenhoupt class $A_p$ if
\[
[w]_{A_p} = \sup_{Q \in \mathcal{Q}} \left( \frac{w(Q) \sigma(Q)^{p-1}}{|Q|^p} \right) < \infty, \quad \sigma = w^{1-p'}.
\]

For $p = 1$, one says that a weight $w$ on $\mathbb{R}^n$ belongs to the Muckenhoupt class $A_1$ if
\[
[w]_{A_1} = \sup_{Q \in \mathcal{Q}} \left( \frac{w(Q)/|Q|}{\text{ess inf}_{x \in Q} w(x)} \right) < \infty.
\]

In [22], Muckenhoupt showed that, for $p > 1$, the weights satisfying the $A_p$ condition are exactly the weights for which the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(w)$.

Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $q$ be defined by $1/q = 1/p - \alpha/n$. In [23], Muckenhoupt and Wheeden characterized the weighted strong type inequality for fractional maximal and integral operators in terms of the so-called $A_{p,q}$ condition. They showed that the inequality
\[
(1.4) \quad \|T_{\alpha} f\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)},
\]
where $T_{\alpha}$ is the operator $I_{\alpha}$ or $M_{\alpha}$, holds if and only if $w \in A_{p,q}$. That is,
\[
[w]_{A_{p,q}} = \sup_{Q \in \mathcal{Q}} \left( \frac{\int_Q w^{q} }{\int_Q w^{p} } \right)^{1/q} \left( \frac{\int_Q w^{-p'}}{\int_Q w^{-q'}} \right)^{1/p'} < \infty.
\]

If $p > 1$, we have that $w \in A_{p,q}$ if and only if $w^q \in A_{1+q/p'}$; this follows at once from the definition.

The following is the sharp weighted bound for the fractional integral operator $I_{\alpha}$.

**Theorem 1.1** ([18, Theorem 2.6]). Let $w \in A_{p,q}$. Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $q$ be defined by $1/q = 1/p - \alpha/n$. Then
\[
\|I_{\alpha}\|_{L^p(w^p) \to L^q(w^q)} \leq \left( \frac{1}{w_{A_{p,q}}} \right)^{\max\{q,p\}}.
\]
Furthermore, the power $(1 - \alpha/n) \max\{q,p\}$ is sharp.
For $E \subset \mathbb{R}^n$ and $0 < \alpha \leq n$, the $\alpha$-dimensional Hausdorff content of $E$ is defined by

$$H^\alpha(E) = \inf \left\{ \sum_j l(Q_j)^\alpha \right\},$$

where the infimum is taken over all coverings of $E$ by countable families of cubes $\{Q_j\} \subset \mathcal{Q}$. The Choquet integral of $\phi \geq 0$ with respect to the Hausdorff content $H^\alpha$ is defined by

$$\hat{\int}_{\mathbb{R}^n} \phi \, dH^\alpha = \int_0^\infty H^\alpha(\{y \in \mathbb{R}^n : \phi(y) > t\}) \, dt.$$  

**Definition 1.2.** Let $0 < \lambda < n$. Define the basis $\mathcal{B}_\lambda$ to be the set of all weights $b$ such that $b \in A_1$ and $\int_{\mathbb{R}^n} b \, dH^\lambda \leq 1$.

Let $1 < p < p_0 < \infty$ and set $\lambda/n = 1 - p/p_0$. Then one has (see [3] and also [32])

$$\tag{1.5} \|f\|_{\mathcal{M}_{p_0}^\alpha} = \|f\|_{L^{p,\lambda}} \approx \sup_{b \in \mathcal{B}_\lambda} \left( \int_{\mathbb{R}^n} |f|^p b \, dx \right)^{1/p}.$$  

**Definition 1.3.** Let $1 < p < \infty$ and $0 < \lambda < n$. The space $H^{p,\lambda}(\mathbb{R}^n)$ is defined by the set of all measurable functions $f$ on $\mathbb{R}^n$ with the quasi norm

$$\|f\|_{H^{p,\lambda}} = \inf_{b \in \mathcal{B}_\lambda} \left( \int_{\mathbb{R}^n} |f|^p b^{1-p} \, dx \right)^{1/p} < \infty.$$  

Let $1 < p < p_0 < \infty$ and set $\lambda/n = 1 - p/p_0$. For any $b \in \mathcal{B}_\lambda$ and for all non-negative functions $f \in L^{p,\lambda}(\mathbb{R}^n)$ and $g \in H^{p',\lambda}(\mathbb{R}^n)$, by Hölder’s inequality that

$$\int_{\mathbb{R}^n} fg \, dx = \int_{\mathbb{R}^n} f b^{1/p} g b^{-1/p} \, dx \leq \left( \int_{\mathbb{R}^n} f^p b \, dx \right)^{1/p} \left( \int_{\mathbb{R}^n} g^{p'} b^{1-p'} \, dx \right)^{1/p'},$$

which implies by (1.5) Hölder’s inequality for Morrey spaces

$$\tag{1.6} \int_{\mathbb{R}^n} fg \, dx \leq C \|f\|_{L^{p,\lambda}} \|g\|_{H^{p',\lambda}}.$$  

In this paper we shall establish the following theorems:

**Theorem 1.4.** Let $0 \leq \alpha < n$, $1 < p < p_0 < \infty$, $1 < q < q_0 < \infty$ and $w$ be a weight. Suppose that

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0},$$

Set $\lambda/n = 1 - p/p_0 = 1 - q/q_0$. Consider the following three statements:

(a) There exists a constant $C_1 > 0$ such that

$$\|(M_\alpha f)w\|_{L^{q,\lambda}} \leq C_1 \|fw\|_{L^{p,\lambda}}$$

holds for every measurable function $f$ with $fw \in L^{p,\lambda}(\mathbb{R}^n)$;

(b) There exists a constant $C_2 > 0$ such that

$$\sup_{Q \in \mathcal{Q}} |Q|^{\alpha/n-1} \|w^1_Q\|_{L^{q,\lambda}} \|w^{-1}Q\|_{H^{p',\lambda}} \leq C_2;$$

(1.7)
(c) For any \( Q_0 \in \mathcal{Q} \), there exists \( b_{Q_0} \in \mathcal{B}_\lambda \) satisfying the following:

\[
\sup_{Q \in \mathcal{Q}} \left( \frac{1}{Q} \int_Q w^q \, dx \right)^{1/q} \left( \frac{1}{Q} \int_Q [wb_{Q_0}^{1/p} - p'] \, dx \right)^{1/p'} \leq C_3 \ell(Q_0)^{\lambda/p}
\]

and

\[
[w_{Q_0}^{1/p}]_{A_s} \leq C_3 \text{ for some } s \geq 1,
\]

where the constant \( C_3 \) is independent of the choices \( Q_0 \).

Then,

(I) That (a) implies (b) with \( C_2 \leq CC_1 \);

(II) Those (b) and (c) imply (a) with \( C_1 \leq CC_2^{(q-p)/q}C_3^{(p+1)/q} + C_2 \).

Unfortunately, because of the additional condition (c), Theorem 1.4 does not completely characterize the boundedness of \( M_\alpha \) on weighted Morrey spaces. However, by employing Theorem 1.4, we can still settle down the problem at least for power weights; see Proposition 4.1.

For the fractional integral operator \( I_\alpha \), we have the following.

**Theorem 1.5.** Let \( 0 < \alpha < n \), \( 1 < p < p_0 < \infty \), \( 1 < q < q_0 < \infty \) and \( w \) be a weight. Suppose that

\[
\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.
\]

Set \( \lambda/n = 1 - p/p_0 = 1 - q/q_0 \). Consider the following three statements:

(a) There exists a constant \( C_1 > 0 \) such that

\[
\| (I_\alpha f) w \|_{L^{q,\lambda}} \leq C_1 \| f w \|_{L^{p,\lambda}}
\]

holds for every function \( f \) with \( fw \in L^{p,\lambda}(\mathbb{R}^n) \);

(b) There exists a constant \( C_2 > 0 \) such that

\[
\| (M_\alpha f) w \|_{L^{q,\lambda}} \leq C_2 \| f w \|_{L^{p,\lambda}}
\]

holds for every function \( f \) with \( fw \in L^{p,\lambda}(\mathbb{R}^n) \);

(c) There exists \( \kappa > 1 \) such that

\[
2 \| w_{1Q} \|_{L^{q,\lambda}} \leq \| w_{1\kappa Q} \|_{L^{q,\lambda}}
\]

holds for every \( Q \in \mathcal{Q} \).

Then,

(I) That (a) implies (b) and (c) with \( C_2 \leq CC_1 \);

(II) Those (b) and (c) imply (a) with \( C_1 \leq CC_2^{n+1} \).

Theorem 1.5 implies that the boundedness of \( I_\alpha \) is equivalent to the one of \( M_\alpha \) and the additional condition (1.10). Note that the additional condition (1.10) was introduced in [24] as the weighted integral condition to ensure the boundedness of the singular integral operator on weighted Morrey spaces.

It is well known that Muckenhoupt introduced the class of weight \( A_p \) in his paper [22, 23]. In fact Muckenhoupt was successful in characterizing the condition for \( M \) to be bounded on \( L^p(w) \). In establishing the theory of
weights for Lebesgue spaces, it is difficult to obtain the strong $A_p$ estimates. Muckenhoupt established the strong weight theory in [22, Section 4] for $n=1$ and Coifman and Fefferman considered the higher dimensional case [3]. We can say that the key tool is the Calderón-Zygmund decomposition. The Calderón-Zygmund decomposition is skillfully used to solve the $A_2$ conjecture [15] and develop a modern weighted theory [17, 16]. However, it seems that the Calderón-Zygmund theory is not enough when we prove the boundedness of the operators on Morrey spaces. A standard technique to prove the boundedness of the operators on Morrey spaces is to fix a cube $Q$, as is seen from the definition (1.1). Accordingly, when we are given a function $f$, we decompose it according to $3Q$. Let $f_1 = f \chi_{3Q}$ and $f_2 = f - f_1$. Then we can benefit a lot from the Calderón-Zygmund theory for the function $f_1$. However, it seems that some different approaches are necessary for $f_2$. In this paper, we applied this strategy in the proof of Lemma 5.1. See (5.3) and (5.5) for the estimates for $f_1$ and $f_2$, respectively, where a special tool (5.4) is necessary for $f_2$ in order to do without the Calderón-Zygmund decomposition.

One of the striking achievements in the theory of weighted Lebesgue spaces is that the classes $\{A_p\}_{p>1}$ enjoy the openness property. Originally, Muckenhoupt used to show that the strong boundedness on $L^p(w)$ is equivalent to $w \in A_p$ [22]. From the definition of $A_p$, we can show that $M$ is weak bounded on $L^p(w)$ using the covering lemma. It is not so hard to show that the strong boundedness on $L^p(w)$ implies $w \in A_p$. We follow the same line in our proof of (1.7) based on the strong boundedness of Morrey spaces in Theorem 1.4. However, even in the case of Lebesgue spaces, it was hard to show that $w \in A_p$ implies the strong boundedness on $L^p(w)$.

In fact, the proof hinged upon the openness property asserting that $w \in A_q$ for some $1 < q < p$. Since $M$ is weak bounded on $L^q(w)$ and bound on $L^\infty$ trivially, we see that $M$ is bounded on $L^p(w)$. When we want to run this program, we are faced with the problem of showing the weak boundedness on weighted Morrey spaces although we still have some openness property, see [20]; once again, the Calderón-Zygmund decomposition is not enough. We remark that the results in [20] are available in weighted Morrey spaces by reexamining the proof.

Although the openness property seems to have been essential in early 80’s, it turned out that we can prove the $L^p(w)$ boundedness of $M$ without using the openness property [8, 12, 19]. Among others, Lerner used a universal estimate (5.1) for the weighted dyadic Hardy-Littlewood maximal operator. His main idea is to convert the Hardy-Littlewood maximal operator adapted to the weighted Lebesgue space $L^p(w)$ [19, p. 2831]. Although we still have a counterpart to weighted Morrey spaces of the universal estimate Lerner used, the gap exists between the condition (1.7) and the universal estimate we obtain, see Lemma 5.1.

Another barrier for us to study Morrey spaces is that Morrey spaces are not rearrangement invariant as is seen from the example in [29, Proposition 4.1]. In fact, another example shows that the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ with
1 < \lambda < n is not embedded into \( L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n) \), see \cite{11} Section 6. This fact prevents us from using the theory developed in \cite{21} Theorem 2.4. Since Morrey spaces are not rearrangement invariant, it is convenient for us to use the decreasing rearrangement.

We can locate the function space \( H^{p,\lambda}(\mathbb{R}^n) \) as a new tool to overcome these problems.

Here and below, the letter \( C \) will be used for constants that may change from one occurrence to another. Constants with subscripts, such as \( C_1, C_2 \), do not change in different occurrences. By \( A \approx B \) we mean that \( c^{-1}B \leq A \leq cB \) with some positive constant \( c \) independent of appropriate quantities.

2. Proof of Theorem 1.4

In what follows we shall prove Theorem 1.4. We need three lemmas (cf. \cite{32} for the first lemma).

**Lemma 2.1.** Let \( 1 < p < p_0 < \infty \) and set \( \lambda/n = 1 - p/p_0 \). Then, for any measurable function \( g \) on \( \mathbb{R}^n \), we have the estimate
\[
\|g\|_{H^{p',\lambda}} \approx \sup_f \int_{\mathbb{R}^n} |fg| \, dx,
\]
where the supremum is taken over all functions \( f \in L^{p,\lambda}(\mathbb{R}^n) \) with unit norm.

**Lemma 2.2.** Let \( w \in A_p, p \geq 1, \) and \( Q \in \mathcal{Q} \). Then, for any measurable set \( S \subset Q \),
\[
\left( \frac{|S|}{|Q|} \right)^p w(Q) \leq C[w]_{A_p} w(S).
\]

**Proof.** Using the well-known fact that
\[
\sup_{t>0} t^p w(\{x : Mf(x) \geq t\}) \leq C[w]_{A_p} \|f\|_{L^p(w)},
\]
we have that
\[
w(Q) \leq w(\{x : M1_S(x) \geq |S|/|Q|\}) \leq C[w]_{A_p} (|S|/|Q|)^{-p} w(S),
\]
which proves the lemma. \( \square \)

To describe the third lemma, we need terminology. We say that a family \( \mathcal{S} \) of cubes from \( \mathbb{R}^n \) is \( \eta \) sparse, \( 0 < \eta < 1 \), if for every \( Q \in \mathcal{S} \), there exists a measurable set \( E_Q \subset Q \) such that \( |E_Q| \geq \eta |Q| \), and the sets \( \{E_Q\}_{Q \in \mathcal{S}} \) are pairwise disjoint. Given a cube \( Q_0 \in \mathcal{Q} \), let \( \mathcal{D}(Q_0) \) denote the set of all dyadic cubes with respect to \( Q_0 \), that is, the cubes obtained by repeated subdivision of \( Q_0 \) and each of its descendants into \( 2^n \) congruent subcubes. By convention \( Q_0 \) itself belongs to \( \mathcal{D}(Q_0) \).

**Lemma 2.3.** Let \( 0 \leq \alpha < n \). Suppose that the non-negative and bounded function \( f \) has compact support. Then, for any cube \( Q_0 \in \mathcal{Q} \), there exists a 1/2 sparse family \( \mathcal{S} \subset \mathcal{D}(Q_0) \) such that, for all \( x \in Q_0 \),
\[
M_\alpha f(x) \leq CL_\alpha^S f(x) + c_\infty,
\]
where
\[ L^S_\alpha f(x) = \sum_{Q \in S} 1_{E_Q}(x)|Q|^{\alpha/n} \int_{3Q} f \, dy \]
and
\[ c_\infty = \sup_{Q \in \mathbb{Q}} |Q|^{\alpha/n} \int_Q f \, dx. \]

Proof. Fix \( Q_0 \in \mathbb{Q} \). We write
\[ \widetilde{M}_\alpha f(x) = \sup_{Q \in \mathcal{D}(Q_0)} 1_Q(x)|Q|^{\alpha/n} \int_{3Q} f \, dy. \]
It is easy to see that, for all \( x \in Q_0 \),
\[ M_\alpha f(x) \leq C \widetilde{M}_\alpha f(x) + c_\infty. \]

Let \( a_0 = |Q_0|^{\alpha/n} \int_{3Q_0} f \, dx \) and \( a = 9^n 2^{n+1-\alpha} \). For each \( k = 0, 1, 2, \ldots \), define
\[ D_k = \bigcup \left\{ Q \in \mathcal{D}(Q_0) : |Q|^{\alpha/n} \int_{3Q} f \, dx \geq a_0 a^k \right\} (\subset Q_0). \]
Considering the maximal cubes with respect to inclusion, we can write
\[ D_k = \bigcup_j Q_j^k, \]
where the cubes \( \{Q_j^k\} \) are pairwise disjoint. By the maximality of \( Q_j^k \), we see that
\[ a_0 a^k \leq |Q_j^k|^{\alpha/n} \int_{3Q_j^k} f \, dx \leq 2^{n-\alpha} a_0 a^k. \]

We shall verify that the family \( S = \{Q_j^k\} \) is 1/2 sparse. To this end, we let
\[ E_{Q_j^k} = Q_j^k \setminus D_{k+1}, \]
then we see that the sets \( \{E_{Q_j^k}\} \) are pairwise disjoint and decompose \( Q_0 \).
So, we need only verify that
\[ |E_{Q_j^k}| \geq \frac{1}{2} |Q_j^k|. \]
Notice that, if \( Q_i^{k+1} \subset Q_j^k \), then by (2.1)
\[ a_0 a^{k+1} \leq |Q_i^{k+1}|^{\alpha/n} \int_{3Q_i^{k+1}} f \, dx < |Q_j^k|^{\alpha/n} \int_{3Q_j^k} f \, dx \leq |Q_j^k|^{\alpha/n} M[f1_{3Q_j^k}](x) \text{ for all } x \in Q_i^{k+1}. \]
This entails
\[ Q_j^k \cap D_{k+1} \subset \left\{ x \in \mathbb{R}^n : M[f1_{3Q_j^k}](x) \geq \frac{a_0 a^{k+1}}{|Q_j^k|^{\alpha/n}} \right\}. \]
The weak-\( (1,1) \) boundedness of \( M \) together with (2.1) yields
\[
\left| Q_j^k \cap D_{k+1} \right| \leq 3^n \cdot \frac{|Q_j^k|^{\alpha/n}}{a_0 a^{k+1}} \cdot \int_{3Q_j^k} f \leq 3^n \cdot 2^{n-\alpha} a_0 a^{k+1} \cdot \frac{|Q_j^k|^{\alpha/n}}{|Q_j^k|^{\alpha/n}} \cdot 3^n |Q_j^k| = 9^n 2^n - \alpha a_0 a^{k+1} \cdot \frac{|Q_j^k|}{|Q_j^k|} = 1,
\]
which implies (2.2).

Finally, for each \( Q = Q_j^k \in S \) and any \( x \in E_Q \), we have by (2.1) that
\[
\tilde{M}_\alpha f(x) \leq a_0 a^{k+1} \leq a |Q|^{\alpha/n} \int_{3Q} f \, dy.
\]
Since the sets \( \{ E_Q \}_{Q \in S} \) are pairwise disjoint and decompose \( Q_0 \), we conclude that
\[
\tilde{M}_\alpha f(x) \leq C L^S f(x) \text{ for all } x \in Q_0.
\]
This completes the proof. \( \square \)

2.1. Proof of Theorem 1.4 (I). Assume the statement (a). Then the inequality
\[
\| (M_\alpha f) w \|_{L^{q,\lambda}} \leq C_1 \| f w \|_{L^{p,\lambda}}
\]
holds for every function \( f \) with \( f w \in L^{p,\lambda}(\mathbb{R}^n) \). For any cube \( Q \in Q \) and any function \( f \) with \( f w \in L^{p,\lambda}(\mathbb{R}^n) \),
\[
|Q|^{\alpha/n} \int_Q |f| \, dx \times \| w_1 Q \|_{L^{q,\lambda}} \leq \| M_\alpha [f 1_Q] w \|_{L^{q,\lambda}} \leq C_1 \| f w 1_Q \|_{L^{p,\lambda}}.
\]
Taking the supremum over all functions \( f \) with \( \| f w 1_Q \|_{L^{p,\lambda}} \leq 1 \), we have by Lemma 2.1
\[
|Q|^{\alpha/n-1} \| w_1 Q \|_{L^{q,\lambda}} \| w^{-1} 1_Q \|_{H^{p,\lambda}} \leq C C_1,
\]
which is the statement (b).

2.2. Proof of Theorem 1.4 (II). To prove sufficiency we may assume that the function \( f \) is non-negative and bounded and that \( f \) has compact support. Fix \( Q_0 \in Q \). We have to evaluate the quantity
\[
\left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} [(M_\alpha f) w]^q \, dx \right)^{1/q}.
\]
By Lemma 2.3 we can select a 1/2 sparse family \( S \subset D(Q_0) \) such that, for all \( x \in Q_0 \),
\[
M_\alpha f(x) \leq C L^S_\alpha f(x) + c_\infty,
\]
where
\[
L^S_\alpha f(x) = \sum_{Q \in S} 1_{E_Q}(x) |Q|^{\alpha/n} \int_Q f \, dy
\]
and
\[
c_\infty = \sup_{Q \in Q \supset Q_0} |Q|^{\alpha/n} \int_Q f \, dx.
\]
We first estimate
\[ (i) = c_\infty \left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} w^q \, dx \right)^{1/q}. \]
For any cube $Q \supset Q_0$,
\[
|Q|^{\alpha/n} \int_{Q} f \, dx \left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} w^q \, dx \right)^{1/q} \\
\leq |Q|^{\alpha/n-1} \|w1_Q\|_{L^q,\lambda} \int_{Q} w^{-1} f \, w \, dx \\
\leq C |Q|^{\alpha/n-1} \|w1_Q\|_{L^q,\lambda} \|w^{-1}1_Q\|_{H^r,\lambda} \|f w\|_{L^{p,\lambda}} \\
\leq CC_2 \|fw\|_{L^{p,\lambda}},
\]
where we have used Hölder’s inequality (1.6) and (1.7). This implies, since the cube $Q \supset Q_0$ is arbitrary,
\[ (i) \leq CC_2 \|fw\|_{L^{p,\lambda}}. \]

We next estimate
\[ (ii) = \int_{Q_0} [(L_S f)^w]^q \, dx. \]
Take $b = b_{9Q_0} \in B_\lambda$ satisfying (1.8) and (1.9) (replacing $Q_0$ with $9Q_0$). Set $u = w^q$ and $\sigma = [wb^{1/p}]^{-p'}$. Since the sets $E_Q$, $Q \in \mathcal{S}$, are pairwise disjoint, we have that
\[ (ii) = 3^{-nq} \sum_{Q \in \mathcal{S}} \left( |Q|^{\alpha/n-1} \int_{3Q} f \, dx \right)^q u(E_Q). \]

We recall the following: Since
\[ \frac{\int_{3Q} f \, dx}{\sigma(9Q)} = \frac{\int_{3Q} f \sigma^{-1} \, d\sigma}{\sigma(9Q)} \leq \inf_{y \in Q} M^c_\sigma[f \sigma^{-1}](y), \]
where $M^c_\sigma$ is the centered weighted Hardy-Littlewood maximal operator with respect to $\sigma$, we obtain
\[ \sum_{Q \in \mathcal{S}} \left( \frac{\int_{3Q} f}{\sigma(9Q)} \right)^p \sigma(E_Q) \leq \sum_{Q \in \mathcal{S}} \int_{E_Q} M^c_\sigma[f \sigma^{-1}]^p \, d\sigma \leq \|M^c_\sigma[f \sigma^{-1}]\|_{L^p(\sigma)}^p \leq \left( \frac{p}{p-1} \right)^{p} \|f \sigma^{-1}\|_{L^p(\sigma)}^p, \]
where we have used (1.5) and the well-known fact that $M^c_\sigma$ is bounded on $L^p(\sigma)$.

With this in mind, we shall estimate the quantity
\[ (iii) = \left( |Q|^{\alpha/n-1} \int_{3Q} f \, dx \right)^{q} \frac{u(E_Q)}{\sigma(E_Q)}. \]
To this end, we first define

$$X = (iii) \cdot \sigma(E_Q)$$

and

$$Y = (iii) \cdot \sigma(E_Q)^{1-q/p}. $$

Then an arithmetic shows that

$$(iii) = X^{1-p/q} Y^{p/q}. $$

It follows that

$$X = u(Q) \left( |Q|^{\alpha/n-1} \int_{3Q} f \, dx \right)^q$$

and

$$Y = \sigma(E_Q)^{-q/p} \left( |Q|^{\alpha/n-1} u(Q)^{1/q} \int_{3Q} f \, dx \right)^q$$

are

$$\leq C |Q|^{\lambda/n} \left( |Q|^{\alpha/n-1} \|w1_{1Q}\|_{L^{\alpha,n}} \|w^{-1}1_{3Q}\|_{H^{p',\lambda}} \cdot \|fw1_{3Q}\|_{L^{p,\lambda}} \right)^q$$

and

$$\leq C |Q|^{\lambda/n} (C_2 \|fw\|_{L^{p,\lambda}})^q,$$

where we have used Hölder’s inequality (1.6) and our assumption (1.7).

It follows also that

$$Y = \sigma(E_Q)^{-q/p} \left( |Q|^{\alpha/n-1} u(Q)^{1/q} \sigma(9Q) \int_{3Q} f \, dx \right)^q$$

are

$$\leq \left\{ \left( \frac{\sigma(9Q)}{\sigma(E_Q)} \right)^{1/p} \cdot |Q|^{\alpha/n-1} u(9Q)^{1/q} \sigma(9Q)^{1/p'} \cdot \int_{3Q} f \, dx \right\}^q.$$

By (1.9) together with $|9Q| = 9^n |Q| \leq 2 \cdot 9^n |E_Q|$, Lemma 2.2 gives

$$\left( \frac{\sigma(9Q)}{\sigma(E_Q)} \right)^{1/p} \leq CC_3^{1/p}.$$

Meanwhile, an arithmetic shows that

$$|Q|^{\lambda/n} \left( |Q|^{\alpha/n-1} u(9Q)^{1/q} \sigma(9Q)^{1/p'} \right)^p$$

are

$$= \left( |Q|^{\lambda/n} \left( |Q|^{\alpha/n-1} u(9Q)^{1/q} \sigma(9Q)^{1/p'} \right)^p \right)^{1/p}$$

by using $\frac{\lambda}{n} = 1 - \frac{p}{p_0} = 1 - \frac{q}{q_0}$

$$= \left( |Q|^{(1-p/p_0) + (1/q_0) - (1/q)} \left( |Q|^{\alpha/n-1} u(9Q)^{1/q} \sigma(9Q)^{1/p'} \right)^p \right)^{1/p}$$

by using $\frac{1}{q_0} - \frac{1}{p_0} + \frac{\alpha}{n} = 0$

$$= \left( |Q|^{-q/p_0} u(9Q)^{1/q} \sigma(9Q)^{1/p'} \right)^p$$

are

$$= C \left\{ \left( \frac{u(9Q)}{|9Q|} \right)^{1/q} \left( \frac{\sigma(9Q)}{|9Q|} \right)^{1/p'} \right\}^p \leq CC_3^{p \ell(Q_0)\lambda},$$

where we have used (1.8) for the last inequality.
Altogether,
\[
\ell(Q_0)^{-\lambda} \cdot (ii) \leq C C_2^{q-p} C_3^{p+1} \|f w\|_{L^{p,\lambda}}^{q-p} \sum_{Q \in S} \left( \int_{3Q} f \sigma(9Q) \right)^p \sigma(E_Q)
\]
\[
\leq C C_2^{q-p} C_3^{p+1} \|f w\|_{L^{p,\lambda}}^q.
\]
This proves sufficiency.

3. Proof of Theorem 1.5

In what follows we shall prove Theorem 1.5. We need a lemma which is similar to Lemma 2.2.

Lemma 3.1. Let \(0 < \alpha < n\) and \(\kappa > 1\). Suppose that the function \(f\) is non-negative and bounded and that \(f\) has compact support. Then, for any cube \(Q_0 \in Q\), there exists a \(1/2\) sparse family \(S \subset D(Q_0)\) such that, for all \(x \in Q_0\),
\[
I_\alpha f(x) \leq C \left( I_\alpha^S f(x) + C_\infty \right),
\]
where
\[
I_\alpha^S f(x) = \sum_{Q \in S} 1_Q(x) |Q|^{\alpha/n} \int_{3Q} f dy
\]
and
\[
C_\infty = \sum_{k=0}^{\infty} |\kappa^k Q_0|^{\alpha/n} \int_{\kappa^k Q_0} f dx.
\]

Proof. For all \(x \in Q_0\) it follows that
\[
I_\alpha f(x) \leq C \left( I_\alpha^{D(Q_0)} f(x) + C_\infty \right),
\]
where
\[
I_\alpha^{D(Q_0)} f(x) = \sum_{Q \in D(Q_0)} 1_Q(x) |Q|^{\alpha/n} \int_{3Q} f dy.
\]
Indeed, for \(x, y \in Q_0\) with \(x \neq y\), we notice that
\[
\sum_{Q \in D(Q_0)} |Q|^{\alpha/n-1} \approx \frac{1}{\|x-y\|^{n-\alpha}}.
\]
This implies together with Fubini’s theorem
\[
\int_{3Q_0} \frac{f(y)}{|x-y|^{n-\alpha}} dy \approx I_\alpha^{D(Q_0)} f(x).
\]
We have also that
\[
\int_{\mathbb{R}^n \setminus 3Q_0} \frac{f(y)}{|x-y|^{n-\alpha}} dy \leq CC_\infty.
\]

We now construct the sparse set \(S\). Let \(a_0 = \int_{3Q_0} f dx\) and fix \(a = 9n^2 2^{n+1}\). For each \(k = 0, 1, 2, \ldots\), define
\[
D_k = \bigcup \left\{ Q \in D(Q_0) : \int_{3Q} f dx \geq a_0 a^k \right\}.
\]
Considering the maximal cubes with respect to inclusion, we can write
\[ D_k = \bigcup_j Q_j^k, \]
where the cubes \( \{Q_j^k\} \) are pairwise disjoint. By the maximality of \( Q_j^k \),
\[ a_0 a^k \leq \int_{3Q_j^k} f \, dx \leq 2^n a_0 a^k. \]
By the same way as the proof of Lemma 2.3 letting 
\[ E_{Q_j^k} = Q_j^k \setminus D_{k+1}, \]
we can verify that the family \( S = \{Q_j^k\} \) is 1/2 sparse.

Finally, if we let 
\[ D_{j}^k = \left\{ Q \in D(Q_0) : Q \subset Q_j^k, \ a_0 a^k \leq \int_{3Q} f \, dx < a_0 a^{k+1} \right\}, \]
then we see that 
\[ D(Q_0) = \bigcup_{k,j} D_j^k. \]
For all \( x \in Q_j^k \)
\[ \sum_{Q \in D_j^k} 1_Q(x)|Q|^{\alpha/n} \int_{3Q} f \, dy \leq a_0 a^{k+1} \sum_{Q \in D_j^k} 1_Q(x)|Q|^{\alpha/n} \leq C a_0 a^{k+1} |Q_j^k|^{\alpha/n} \leq a |Q_j^k|^{\alpha/n} \int_{3Q_j^k} f \, dy, \]
where we have used (3.1). Thus, for all \( x \in Q_0 \),
\[ I_{\alpha} f_m(x) \leq a \sum_{Q \in S} 1_Q(x)|Q|^{\alpha/n} \int_{3Q} f \, dy. \]
This complete the proof. \( \square \)

3.1. **Proof of Theorem 1.5 (I).** Assume that (a) holds. The assertion (b) follows from the pointwise inequality \( M_{\alpha} f(x) \leq C |I_{\alpha} f(x)| \). To prove (c), we shall obtain the contradiction. So, we assume that (1.10) fails. Then, for any \( m \in \mathbb{N} \), there exists \( Q_m \in Q \) such that
\[ 2 \| w1_{Q_m} \|_{L^{q,\lambda}} > \| w1_{mQ_m} \|_{L^{q,\lambda}}. \]
Now we define for \( m > 2 \)
\[ f_m(y) = \frac{1_{mQ_m \setminus 2Q_m}(y)}{|y - c(Q_m)|^{\alpha}}. \]
Then we notice that, for any \( x \in Q_m \),
\[ I_{\alpha} f_m(x) \geq C \int_{mQ_m \setminus 2Q_m} \frac{dy}{|y - c(Q_m)|^{\alpha}} \approx \int_{m\ell(Q_m)}^{\ell(Q_m)} \frac{dt}{t} \approx \log m. \]
This implies by (a)

\[ \log m \| w \|_{L^{q,\lambda}} \leq C \| (I_\alpha f_m)w \|_{L^{q,\lambda}} \leq C \| f_mw \|_{L^{p,\lambda}}. \]

We recall that

\[ \frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}. \]

If we define \( r \) by \( 1/p = 1/q + 1/r \), then

\[ \frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \frac{p_0}{p} \left( \frac{1}{p_0} - \frac{1}{q_0} \right) > \left( \frac{1}{p_0} - \frac{1}{q_0} \right) = \frac{\alpha}{n}, \]

which means \( 1 < r < n/\alpha \). By Hölder’s inequality for the Morrey norms with exponents \( 1/p = 1/q + 1/r \) and \( 1/p_0 = 1/q_0 + \alpha/n \),

\[ \| f_mw \|_{L^{p,\lambda}} = \| f_m \|_{M^{p_0}_{p_0}} \leq \| w \|_{M^{\alpha/n}_{\lambda}} \leq \| f_m \|_{M^{\alpha/n}_{\lambda}}. \]

Since,

\[ \| f_m \|_{M^{\alpha/n}_{\lambda}} \leq \| f_m \|_{L^{\alpha/n}} \approx (\log m)^{\alpha/n}, \]

the inequalities (3.2)–(3.4) yield the contradiction \((\log m)^{1-\alpha/n} \leq C\). Thus, the statement (c) holds.

### 3.2. Proof of Theorem 1.5 (II)

Assume the statements (b) and (c). To prove sufficiency we may assume that the function \( f \) is non-negative and bounded and that \( f \) has compact support. Fix \( Q_0 \in \mathcal{Q} \). We shall evaluate the quantity

\[ \left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} [(I_{\alpha} f)w]^q \, dx \right)^{1/q}. \]

By Lemma 3.1 we can select a \( 1/2 \) sparse family \( S \subset D(Q_0) \) such that, for all \( x \in Q_0 \),

\[ I_{\alpha} f(x) \leq C \left( I_{\alpha}^S f(x) + C_{\infty} \right), \]

where

\[ I_{\alpha}^S f(x) = \sum_{Q \in S} \chi_Q(x) |Q|^{\alpha/n} \int_{3Q} f \, dy \]

and

\[ C_{\infty} = \sum_{k=0}^\infty |\kappa^k Q_0|^{\alpha/n} \int_{\kappa^k Q_0} f \, dx. \]

It follows from (1.10) that

\[ C_{\infty} \left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} w^q \, dx \right)^{1/q} \leq C_{\infty} \| w \|_{L^{\alpha/n}} \leq \sum_{k=0}^\infty 2^{-k} |\kappa^k Q_0|^{\alpha/n-1} \| w \|_{L^{\alpha/n}} \int_{\kappa^k Q_0} f \, dx, \]

by Hölder’s inequality (1.5), that

\[ \int_{\kappa^k Q_0} f \, dx \leq C \| w^{-1} \|_{L^{\alpha/n}} \| fw \|_{L^{p,\lambda}}, \]
by the use of Theorem 1.4 (I), and that
\[ C_\infty \left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} w^q \right)^{1/q} \leq CC_2 \| fw \|_{L^{p,\lambda}} \sum_{k=0}^{\infty} 2^{-k} = CC_2 \| fw \|_{L^{p,\lambda}}. \]

Let \( u = w^q \). We wish to estimate \( \| I^S f \|_{L^q(u)} \) by way of a duality argument. To this end, we take a function \( g \), which is non-negative, supported in \( Q_0 \) and satisfies \( \| g \|_{L^{q'}(u)} = 1 \), and evaluate the quantity
\[
(i) = \sum_{Q \in \mathcal{S}} |Q|^{\alpha/n} \int_{3Q} f \, dy \int_{Q} g \, du.
\]

By the statement (b),
\[
|Q|^{\alpha/n} \left( \frac{w^{-1}(Q)}{|Q|} \right) \cdot |Q|^{1/q_0} \left( \frac{u(Q)}{|Q|} \right)^{1/q} \leq \| (M_\alpha[w^{-1}1_Q])w \|_{L^{q,\lambda}} \leq C_2 \| 1_Q \|_{L^{p,\lambda}} = C_2 |Q|^{1/p_0}.
\]

Since \( 1/q_0 = 1/p_0 - \alpha/n \),
\[
\left( \frac{u(Q)}{|Q|} \right) \left( \frac{w(Q)}{|Q|} \right)^{q} \leq C_2^q,
\]
which means that \( u \) belongs to \( A_{q+1} \) with the estimate \( [u]_{A_{q+1}} \leq C_2^q \). This and Lemma 2.2 give us that
\[
(3.5) \quad u(Q) \leq CC_2^q u(E_Q) \quad \text{for all } Q \in \mathcal{S}.
\]

It follows from (3.5) that
\[
(i) = \sum_{Q \in \mathcal{S}} |Q|^{\alpha/n} \int_{3Q} f \, dy \int_{Q} g \, du \leq CC_2^q \sum_{Q \in \mathcal{S}} |Q|^{\alpha/n} \int_{3Q} f \, dy \int_{Q} g \, du \frac{u(E_Q)^{1/q+1/q'}}{u(Q)}
\]
\[
\leq CC_2^q \left\{ \sum_{Q \in \mathcal{S}} \left( \int_{3Q} f \, dy \right)^q u(E_Q) \right\}^{1/q'}
\]
\[
\times \left\{ \sum_{Q \in \mathcal{S}} \left( \frac{\int_{Q} g \, du}{u(Q)} \right)^{q'} u(E_Q) \right\}^{1/q'}
\]
\[
\leq CC_2^q \left( \int_{Q_0} [(M_\alpha f)w]^{q} \, dx \right)^{1/q},
\]
where in the last inequality we have used the \( L^{q'}(u) \) boundedness of the dyadic weighted Hardy-Littlewood maximal operator with respect to \( u \) with the norm less than or equal to \( q \); see (5.1).
This and the statement (b) yield
\[
\left( \frac{1}{|Q_0|^{\lambda/n}} \int_{Q_0} [(I_0^S f)w]^q \, dx \right)^{1/q} \leq CC^{q+1} \|fw\|_{L^{p,\lambda}},
\]
which completes the proof.

4. The Power Weight Cases and Some Equivalences

In this section we investigate the case of the power weight cases and introduce some equivalence conditions for our theorems for the purpose.

We first give the certain range of the power for which the boundedness of $M_\alpha$ on power weighted Morrey spaces as follows.

Proposition 4.1. Suppose that the parameters satisfy the same conditions as in Theorem 1.4 and let $w_\rho(x) = |x|^\rho$ with $\rho > -n$. Then the following are equivalent.

(a) There exists a constant $C_1 > 0$ such that
\[
\|(M_\alpha f)w_\rho\|_{L^{q,\lambda}} \leq C_1 \|fw_\rho\|_{L^{p,\lambda}}
\]
holds for every function $f$ with $fw_\rho \in L^{p,\lambda} (\mathbb{R}^n)$

(b) There exist constants $C_2 > 0$ such that
\[
\sup_{Q \in \mathcal{Q}} |Q|^{\alpha/n-1} \|w_\rho 1_Q\|_{L^{q,\lambda}} \|w_\rho^{-1} 1_Q\|_{H^{p',\lambda}} \leq C_2.
\]

(c) The parameter $\rho$ satisfies
\[-n + \lambda \leq q\rho, \quad pp < n(p-1) + \lambda.
\]

Since the proof of Proposition 4.1 is almost the same as the one of [32, Proposition 4.2], we omit the proof. Meanwhile, as we observed in [24], the condition (1.10) with the power weight $w = w_\rho$ is equivalent to $q\rho > -n + \lambda$. Hence, we obtain the power weight result for $I_\alpha$ as follows.

Proposition 4.2. Suppose the parameters satisfy the same conditions as in Theorem 1.5 and let $w_\rho(x) = |x|^\rho$ with $\rho > -n$. Then the following are equivalent.

(a) There exists a constant $C_1 > 0$ such that
\[
\|(I_\alpha f)w_\rho\|_{L^{q,\lambda}} \leq C_1 \|fw_\rho\|_{L^{p,\lambda}}
\]
holds for every function $f$ with $fw_\rho \in L^{p,\lambda} (\mathbb{R}^n)$

(b) There exist constants $C_2 > 0$ such that
\[
|Q|^{\alpha/n-1} \|w_\rho 1_Q\|_{L^{q,\lambda}} \|w_\rho^{-1} 1_Q\|_{H^{p',\lambda}} \leq C_2 \text{ and } 2\|w_\rho 1_Q\|_{L^{q,\lambda}} \leq \|w_\rho^\kappa 1_Q\|_{L^{q,\lambda}}
\]
hold for some $\kappa > 1$ and all $Q \in \mathcal{Q}$.

(c) The parameter $\rho$ satisfies
\[-n + \lambda < q\rho, \quad pp < n(p-1) + \lambda.
\]
Finally, we note one observation. As we mentioned in Theorem 1.4, the weight problem for the maximal operator $M$ is still open. One finds that the problem is difficult since it is difficult to calculate the quantities
\[ \|w_1^Q\|_{L_q,\lambda}, \quad \|w^{-1}_1\|^p \]
appearing in (1.7). Indeed, in the Lebesgue setting $p_0 = p$ and $q_0 = q$, it is easy to calculate these quantities. Thus, it is important to calculate these quantities when $p_0 \neq p$ and $q_0 \neq q$. At least, we have the explicit formula for the quantity $\|w_1^Q\|_{L_q,\lambda}$ as follows.

**Proposition 4.3.** Let $0 \leq \alpha < n$, $1 < p < p_0 < \infty$, $1 < q < q_0 < \infty$ and $w$ be a weight. Suppose that $1/q_0 = 1/p_0 - \alpha/n$ and $q = p/p_0$. Set $\lambda/n = 1 - p/p_0 = 1 - q/q_0$. If we assume Theorem 1.4 (b), then, for all $Q \in \mathcal{Q}$,

\[ \|w_1^Q\|_{L_q,\lambda} \approx |Q|^{1/q_0} \left( \int_Q w_q \, dx \right)^{1/q}. \]

That is, then the Morrey norm is attained on the full cube $Q$.

**Proof.** The relation
\[ \|w_1^Q\|_{L_q,\lambda} \geq \left( \int_Q w_q \, dx \right)^{1/q} \]
follows automatically. We shall prove the converse.

Let $u = w_q$, $\beta = nq/q_0$ and $\gamma = |Q|^{\beta/n} \int_Q u \, dx$. Consider
\[ \Omega = \{ x \in Q : M_{\beta}[u_1^Q](x) > 2 \cdot 3^n \gamma \}. \]

Since, for all $x \in Q$,
\[ M_{\beta}[u_1^Q](x) \leq |Q|^{\beta/n} M[u_1^Q](x), \]
we have that
\[ \Omega \subset \left\{ x \in Q : M[u_1^Q](x) > \frac{2 \cdot 3^n \gamma}{|Q|^{\beta/n}} \right\}. \]

The weak-(1, 1) boundedness of $M$ gives us that
\[ |\Omega| \leq 3^n \frac{|Q|^{\beta/n}}{2 \cdot 3^n \gamma} \int_Q u = \frac{1}{2} |Q|. \]

Hence, if we let $E = Q \setminus \Omega$, we have $|E| \geq |Q|/2$. It follows from Hölder’s inequality (1.6) that
\[ \frac{1}{2} |Q| \leq |E| \leq C \|w_1^E\|_{L_q,\lambda} \|w^{-1}_1\|^p. \]

Because we always have
\[ \|w^{-1}_1\|^p \leq \|w^{-1}_1\|^p \leq C_2 \left( |Q|^{\alpha/n-1} \|w_1^Q\|_{L_q,\lambda} \right)^{-1}, \]
we have that
\[ \|w_1^Q\|_{L_q,\lambda} \leq C |Q|^{-\alpha/n} \|w_1^E\|_{L_q,\lambda}. \]
By the definition of the Morrey norm, using \(1/p_0 - 1/q_0 = \alpha/n > 0\) and \(q > p\), we see that
\[
\|w1_E\|_{L^{p,\lambda}} \leq |Q|^{1/p_0-1/q_0}\|w1_E\|_{L^{q,\lambda}}.
\]
Thus, noticing \(1/p_0 - 1/q_0 - \alpha/n = 0\) and the fact that
\[
\|w1_E\|_{L^{q,\lambda}} \leq (2 \cdot 3^n)^{1/q},
\]
we conclude that
\[
\|w1_Q\|_{L^{q,\lambda}} \leq C|Q|^{1/q_0} \left(\int_Q u \, dx\right)^{1/q},
\]
which proves the proposition. \(\square\)

5. Appendix–Universal estimates

Let \(\mu\) be a Radon measure on \(\mathbb{R}^n\). An example we envisage here is the weighted measure \(\mu = w \, dx\). We consider the following dyadic weighted Hardy-Littlewood maximal operator:
\[
M_{\text{dyadic},w} f(x) = \sup_Q \frac{\chi_Q(x)}{w(Q)} \int |f| w \, dy,
\]
where \(Q\) moves over all dyadic cubes in \(\mathbb{R}^n\). Using a covering lemma, we can prove
\[
w\{x \in \mathbb{R}^n : M_{\text{dyadic}} f(x) > \lambda\} \leq \frac{1}{\lambda} \|f\chi_{\{x \in \mathbb{R}^n : M_{\text{dyadic}} f(x) > \lambda\}}\|_{L^1(w)},
\]
which yields
(5.1) \[
\|M_{\text{dyadic},w} f\|_{L^p(w)} \leq p' \|f\|_{L^p(w)}.
\]
We consider the following weighted dyadic Morrey norm:
\[
\|f\|_{L^{p,\lambda}_{\text{dyadic}}(w)} = \sup_Q \left(\frac{1}{w(Q)^{\lambda/n}} \int_Q |f|^p w \, dx\right)^{1/p},
\]
where \(Q\) moves over all dyadic cubes in \(\mathbb{R}^n\).

Lemma 5.1 (Universal estimate for Morrey spaces). Let \(1 < p < \infty\) and \(0 < \lambda < n\). Then
\[
\|M_{\text{dyadic},w} f\|_{L^{p,\lambda}_{\text{dyadic}}(w)} \leq (p' + 1) \|f\|_{L^{p,\lambda}_{\text{dyadic}}(w)}.
\]

Proof. Fix a dyadic cube \(Q\). Then it suffices to show
(5.2) \[
\left(\frac{1}{|Q|^\lambda/n} \int_Q (M_{\text{dyadic},w} f)^p \, dx\right)^{1/p} \leq (p' + 1) \|f\|_{L^{p,\lambda}_{\text{dyadic}}(w)}.
\]
To this end, we decompose \(f\) according to \(Q\). Let \(f_1 = f \chi_Q\) and \(f_2 = f - f_1\). Then from (5.1) we have
(5.3) \[
\left(\frac{1}{|Q|^\lambda/n} \int_Q (M_{\text{dyadic},w} f_1)^p \, dx\right)^{1/p} \leq p' \|f\|_{L^{p,\lambda}_{\text{dyadic}}(w)}
\]
and from the pointwise equality
\[
M_{\text{dyadic},w}f_2(x) = \inf_{y \in Q} M_{\text{dyadic},w}(f \chi_{\mathbb{R}^n \setminus Q})(y)
\]
we have
\[
\left( \frac{1}{|Q|^{\lambda/n}} \int_Q (M_{\text{dyadic},w} f_2)^p \, dx \right)^{1/p} \leq \|f\|_{L_{\text{dyadic}}^{p,\lambda}(w)}.
\]
Combining (5.3) and (5.5), we conclude (5.2).

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