ON GENERATING SERIES OF FINITELY PRESENTED OPERADS

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Abstract. Given an operad \( P \) with a finite Gröbner basis of relations, we study the generating functions for the dimensions of its graded components \( P(n) \). Under moderate assumptions on the relations we prove that the exponential generating function for the sequence \( \{ \dim P(n) \} \) is differential algebraic, and in fact algebraic for \( P \) is a symmetrization of a non-symmetric operad. If, in addition, the growth of the dimensions \( P(n) \) is bounded by an exponent of \( n \) (or a polynomial of \( n \), in the non-symmetric case) then, moreover, the ordinary generating function for the above sequence \( \{ \dim P(n) \} \) is rational. We give a number of examples of calculations and discuss conjectures about the above generating functions for more general classes of operads.

0. Introduction

We study the generating series for the dimensions of the components of finitely generated operads. We conjecture (motivated by an analogy with graded algebras, see Subsection 0.2) that, for a wide class of generic operads, these series are differential algebraic for symmetric operads and algebraic for non-symmetric ones (for symmetric operads, we consider exponential generating functions). We prove that this is indeed the case if the operad has a finite Gröbner basis and satisfies an additional (mild) condition. Moreover, if the dimensions of components of the operad are bounded by an exponential function (or by a polynomial function for nonsymmetric case), then the corresponding generating series is rational. We also describe several algorithms for calculating these series in various situations, and provide a number of examples of calculations. In particular, there are several natural examples of operads for which the generating series was not previously known.

0.1. Main results. The concept of so-called Gröbner basis for operads was recently introduced in [DK]. A number of important operads admit a finite Gröbner basis of the ideal of relations. Here we discuss the dimensions of components of operads with a finite Gröbner basis of relations in the form of their generating functions.

Let \( P \) be a finitely generated operad over a field \( k \) of characteristic zero. Recall that an exponential generating series of \( P \) is defined as

\[
E_P(z) := \sum_{n \geq 1} \frac{\dim P(n)}{n!} z^n.
\]

We also consider an ordinary generating series

\[
G_P(z) := \sum_{n \geq 1} \dim P(n) z^n.
\]

In particular, if \( P \) is a symmetrization of a non-symmetric operad \( \mathcal{P} \), then \( E_P(z) = G_{\mathcal{P}}(z) \).

Our first result deals with non-symmetric operads.

Theorem 0.1.3 (Theorem 2.3.1). The ordinary generating series of a non-symmetric operad with a finite Gröbner basis is an algebraic function.

In the more general symmetric case, the analogous result is true under some additional assumptions explained below.

Recall that the leading terms of the elements of a reduced Gröbner basis of a symmetric operad are shuffle monomials, that is, rooted trees with labeled vertices and leaves. Each shuffle monomial has a unique planar realization (see details in Section 1.3), so we can identify a shuffle monomial with a planar...
tree whose leaves are enumerated by an initial segment of positive integers permuted by a so-called shuffle substitution. We call a set $M$ of shuffle monomials shuffle regular (Definition 3.2.1) if for each shuffle monomial $m$ in $M$ and for each shuffle substitution $\sigma$ of its leaves, the monomial $m'$ obtained from $m$ by acting with $\sigma$ on its leaves also belongs to $M$. An operad with a given Gröbner basis of relations is called shuffle regular if the set of the leading monomials of the elements of the Gröbner basis is shuffle regular.

Recall that a function or a formal power series is called differential algebraic if it satisfies a non-trivial algebraic differential equation with polynomial coefficients.

**Theorem 0.1.4** (Corollary 3.2.7). Let $P$ be a shuffle regular symmetric operad with a finite Gröbner basis of relations. Then its generating series $E_P$ is differential algebraic.

We also consider two special classes of operads which give rise to generating series of more special form. The first class consists of monomial shuffle regular operads with an additional symmetry of relations. Namely, if the set of relations of a monomial operad $P$ forms a set of all planar realizations of a given set of non-planar trees, then the generating series $E_P$ is algebraic (Theorem 3.3.2). The second class is defined by the following bounds for the dimension growth of the components of the operad.

**Theorem 0.1.5** (Corollaries 2.4.1 and 3.4.1). Let $P$ be an operad with a finite Gröbner basis of relations. Suppose that either

(i) $P$ is non-symmetric and the numbers $\dim P_n$ are bounded by some polynomial in $n$

or

(ii) $P$ is shuffle regular and the dimensions $\dim P_n$ are bounded by an exponential function $a^n$ for some $a > 1$.

Then the generating series $G_P$ is rational.

In fact, the growth conditions we need in Theorem 0.1.5 are even a little weaker, see Corollaries 2.4.1 and 3.4.1.

All our proofs are constructive and provide effective methods of obtaining the corresponding algebraic or differential algebraic relation for the generating series. We give a number of methods; depending on the situation, one of these may be the most effective. These methods are based on relations between right sided ideals in monomial operads, on homological computations, and on using symmetries of the relations of operads. For some classes of operads, we derive a general formula for the generating series or prove that the generating series is algebraic. We also give a number of examples of calculations.

**Remark 0.1.6.** Sometimes there exists an additional integer grading on the operad $P$ such that all vector spaces $P_n$ are graded, and such that the grading is additive with respect to the compositions of homogeneous elements. Then one can also consider the two-variable generating functions $G_P(z,t)$ and $E_P(z,t)$, see Sections 2.1 and 3.1. All the our above results remain valid after replacing the coefficient field $\mathbb{Q}$ in the differential and algebraic equations by the ring $\mathbb{Q}[t]$.

**0.2. Motivation.** In this paper we study generating series for operads. A lot of examples of such series are recently calculated, see [Z]. Moreover, for binary symmetric operads with one or two generators (essentially, for quotients of the operad of associative algebras), such generating series has been extensively studied for decades under the name of codimension series of varieties of algebras, see [GZ, BD]. Here we present a new approach to such series based on the theory of Gröbner bases for operads developed in [DK].

Many results in the theory of operads have analogy in the theory of graded associative algebras, which we now partly review.

For a graded finitely generated associative algebra $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$ over a field $k$, its Hilbert series is defined as the generating function for the dimensions of the graded homogeneous components:

$$H_A(z) = \sum_{n \geq 0} (\dim_k A_n) z^n.$$

Whereas in general the series $H_A(z)$ can be very complicated (e.g., is a transcendental function), for a number of important classes of algebras it is a rational function

$$\frac{p(z)}{q(z)}, \text{ where } p(z), q(z) \in \mathbb{Z}[z].$$
Indeed, $H_A(z)$ is rational if $A$ is commutative, or Noetherian with a polynomial identity, or relatively free, or Koszul (according to a conjecture of [PP]), etc. Probably the most general class of algebras with rational Hilbert series is the class of algebras with a finite Gröbner basis of relations. Finite presented monomial algebras, PBW algebras, and many other natural types of algebras are particular examples of this class. For a survey of these and other results on Hilbert series of associative algebras, we refer the reader to [U] and references therein.

We conjecture that the generating series of the most important general examples of operads belong to some analogous reasonable class of formal power series that we now describe. Let us try to bound this hypothetical class.

Certainly, the generating series of the most useful operads such as operads of commutative, Lie, and associative algebras should belong to the hypothetical class of simplest generating series. Since we have

$$E_{\text{Com}} = e^z - 1, E_{\text{Assoc}} = \frac{z}{1 - z}, E_{\text{Lie}} = -\ln(1 - z),$$

we conclude that, at least, the generating series of a generic symmetric operad may not be rational and may be exponential or logarithmic. In addition, finite-dimensional operads seem to be simple enough to have “general” generating series. Hence the described class should also include the polynomials with rational coefficients. Free finitely generated operads are also general; if the generating series of the vector space (more precisely, the symmetric module) $V$ of generators of such an operad $\mathcal{F}$ is a polynomial $p(z) = E_V$, then the generating series of $\mathcal{F}$ is

$$E_{\mathcal{F}} = (f^{-1}_V)(z),$$

where $f_V(z) = z - p(z)$ and $^{-1}$ stands for the inverse function. This means that our class of formal power series should also include algebraic (over $\mathbb{Q}$) functions.

Note that the sets of quadratic, finitely presented, and other main types of operads are closed under a number of operations (such as direct sum with common identity component, coproduct, and composition). Thus, it seems reasonable to assume that the set of common generating series should be closed under corresponding formal power series operations, that is, the operations that send the pair of formal power series $f(z) = E_P(z)$ and $g(z) = E_Q(z)$ to

$$f(z) + g(z) - z,$$

$$f(g(z)),$$

and $^1$

$$(f^{-1}(z) + g^{-1}(z) - z)^{-1}.$$  

Finally, since the Koszul duality plays an important role in the theory of operads and its applications, one would want to have that if a generating series of a Koszul operad $\mathcal{P}$ belongs to this class, then the series of its Koszul dual $\mathcal{P}^!$ should also belong to this class. To ensure this, we assume that the described class of formal power series should be closed under the operation which sends $f(z)$ to $-(f^{-1})(-z)$.

Note that while the generating series of $\text{Com}$ and $\text{Lie}$ are not rational nor algebraic, both these series satisfy simple first-order differential equations. In particular, these functions are differential algebraic. In view of the consideration above, we state the following claim.

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$^1$To prove the equality $h(z) = (f^{-1}(z) + g^{-1}(z) - z)^{-1}$, where $h(z) = E_{\mathcal{P}\bigoplus\mathcal{Q}}$ is the generating series of the coproduct (=free product with common identity component) of operads $\mathcal{P}$ and $\mathcal{Q}$, consider the (minimal) differential graded models $M_{\mathcal{P}}, M_{\mathcal{Q}}$, and $M_{\mathcal{P}\bigoplus\mathcal{Q}}$ of these three operads. They are DG operads which are free as operads. Let $H_{\mathcal{P}}, H_{\mathcal{Q}}, H_{\mathcal{P}\bigoplus\mathcal{Q}}$ be the symmetric modules which minimally generates these three DG operads; then $H_{\mathcal{P}\bigoplus\mathcal{Q}} \simeq H_{\mathcal{P}} \oplus H_{\mathcal{Q}}$. One can consider these symmetric modules as complexes (with trivial differential but non-trivial homological grading); let $\chi(\mathcal{P}), \chi(\mathcal{Q})$, and $\chi(\mathcal{P}\bigoplus\mathcal{Q}) = \chi(\mathcal{P}) + \chi(\mathcal{Q})$ be the exponential generating functions of their Euler characteristics. For each operad $\mathcal{O}$, the analogous exponential Euler characteristic of its minimal model is

$$\chi(M_{\mathcal{O}})^{-1}(z) = z - \chi(H_{\mathcal{O}})(z), \quad \text{where} \quad \chi(M_{\mathcal{O}})(z) = E_{\mathcal{O}}(z).$$

Thus,

$$E_{\mathcal{P}\bigoplus\mathcal{Q}}^{-1}(z) = (\chi(M_{\mathcal{P}}) + \chi(M_{\mathcal{Q}}))^{-1}(z) = z - \chi(\mathcal{P}\bigoplus\mathcal{Q}) = z - \chi(\mathcal{P}) + \chi(\mathcal{Q}) = f^{-1}(z) + g^{-1}(z) - z.
Expectation 1. The exponential generating series of a generic finitely presented symmetric operad is differential algebraic.

As for the ordinary generating series of non-symmetric operads (i.e., the exponential generating series of symmetric operads which are symmetrizations of the non-symmetric ones), the class of such general functions should include, at least, the above generating function of Assoc, polynomials over nonnegative integers (for finite dimensional operads) and formal power series which are roots of some polynomials with integer coefficients (for free operads). In particular, such generating series are algebraic, that is, they satisfy some non-trivial algebraic equations over \( \mathbb{Z}[z] \).

Expectation 2. The generating series of a generic finitely presented non-symmetric operads are algebraic over \( \mathbb{Z}[z] \).

Suppose that the sequence of dimensions \( \dim P_n \) of components of an operad has slow growth (e.g., \( \dim P_n \) is bounded by some polynomial of \( n \)). Then in many examples the ordinary generating series \( G_P \) of the operad \( P \) is quite simple. For example, the operad \( \text{Com} \) has a rational generating series:

\[
G_{\text{Com}}(z) = z + z^2 + z^3 + \cdots = \frac{z}{1-z}.
\]

Such examples suggest the following claim.

Expectation 3. The ordinary generating series \( G_P \) of a generic symmetric or non-symmetric operad \( P \) is rational if the sequence \( \{ \dim P_n \} \) is bounded by a polynomial in \( n \).

In particular, if the sequence \( \{ \dim P_n \} \) is bounded by a constant, then it is eventually periodic.

To support our claims, we consider operads with a finite Gröbner bases of relations. The analogy with the theory of associative algebras leads us to expect that the generating series of operads with finite Gröbner bases are in our class. This expectation is suggested by the results of this article.

0.3. Outline of the paper. In Section 1 we give a brief introduction to algebraic operads and Gröbner bases for them. In particular, Subsections 1.2 and 1.3 contain detailed description of monomials in free non-symmetric and symmetric operads respectively.

In Section 2, we illustrate our ideas in the simpler case of non-symmetric operads. In Subsection 2.2.1 we begin to prove Theorem 0.1.3 by finding an algebraic relation for the generating series of an arbitrary non-symmetric operad with a finite Gröbner basis. Our proof is based on considering of the principal ideals with bounded degree (or level) of generators in the finitely presented monomial operad. We provide an algorithm to construct polynomial relations of the form \( y_i = F(y_1, \ldots, y_N) \) for the generating functions \( y_1, \ldots, y_N \) of these ideals. In Subsections 2.2.2 and 2.2.3 we give two other versions of this algorithm optimized for the case of few relations and for the case of simple structure of relations, respectively. Since the generating series of the operad itself is a linear combination of some \( y_i \), we deduce in Subsection 2.3 that the latter generating series does satisfy an algebraic equation. Then we discuss a bound for the degree of the equation. In Subsection 2.4 we investigate the case of operads with subexponential growth of the dimensions of its components and prove the first part of Theorem 0.1.5.

In Section 3, we deal with the case of symmetric operads. In Subsection 3.1 we prove a formula for the generating series of the shuffle composition of sequences of linear subspaces. This is the first place where integral and differential equations appear. In Subsection 3.2 we prove Theorem 0.1.4. In Subsection 3.3 we consider a class of shuffle regular operads with additional symmetries among the relations. For these operads (called symmetric regular), the sets of leading monomials of a Gröbner basis of relations are, by definition, closed under arbitrary re-numbering of the leaves. We show that exponential generating series of symmetric regular operads are algebraic functions. In Subsection 3.4 we consider shuffle regular operads with slow growth of the dimensions of the components and prove the shuffle part of Theorem 0.1.5. All provided examples of symmetric operads are collected in Subsection 3.5.

Some remarks and conjectures are collected in the final Section 4. First, we give some evidence for the conjecture that each binary operad with quadratic Gröbner basis has the same generating series as some shuffle regular operad (hence, this series is differential algebraic). Second, we discuss some evidence showing that operads of associative algebras with polynomial identities are “generic” in the sense of our Expectation 1 (and in the sense of Expectation 3 in the case of polynomial growth). Third, we remark that the operation of shuffle composition induces a structure of Zinbiel algebra on the ring of formal power series. Therefore, it should be interesting to describe the minimal Zinbiel subalgebras.
which contain generating series of some classes of operads, and to describe the class of “Zinbiel algebraic”
functions which contains these generating series.

0.4. Considered examples. We illustrate our methods by a number of examples. For the reader’s
convenience, we give below a brief list of calculations (more or less explicit) of generating series for some
classes of operads. In most of these examples, the operads are generated by a single binary operation
(multiplication). As usual, we denote \([a, b] := ab - ba\) and \((a, b, c) := (ab)c - a(bc)\). These examples are:

- a non-symmetric operad with the identities \((ab)c = 0\) and \((a(b(cd)e)) = 0\) (Example 2.2.11);
- a Lie-admissible operad defined by the set of identities \([x_1, x_2, x_3] + \{[x_2, x_3], x_1\} + \{[x_3, x_1], x_2\} = 0\), where \(\{a, b\} := ab + ba\), \([x_1, x_2]x_3 + [x_2, x_3]x_1 + [x_3, x_1]x_2 = 0\), and
  \(x_1[x_2, x_3] + x_2[x_3, x_1] + x_3[x_1, x_2] = 0\), respectively (Example 3.5.1).

- an operad \(\mathcal{N}_{\text{L}}^n\) of upper triangular matrices of order \(n\) over a nonassociative commutative ring
  (Example 3.5.3) for \(n = 2\), Example 3.5.6 for \(n = 3\) and Lemma 3.5.4 for general case.

- a Lie-admissible operad defined by the set of identities \(\{g([\ldots, \ldots, [g, \ldots, g]]|g \in \Phi\)\), where the
  generator \([\ldots, \ldots, [g, \ldots, g]]\) satisfies the Jacobi identity and \(g\) is a composition of other generators
  \(\Phi\) is a given set of operations on some generators and the additional generator (Example 3.5.7);

- an operad with the identities \(\{x, [y, z]\} + [y, [z, x]] + [z, [x, y]] = 0\) and \([x, y]|z, t| + [z, t]|x, y] = 0\)
  (Example 3.5.11) as a special case of the previous example).

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1. Operads and trees

In the brief and rough explanation below, we refer the reader to the books [LV, MSS] for the details
on the basic facts on operads and to [DK] and [LV, Ch. 8] for the details on shuffle operads and Gröbner
bases in operads.

1.1. Operads in few words. Roughly speaking operad is a way to define a type, or a variety, of
(linear) algebraic systems. An operad \(\mathcal{P}\) consists of a sequence (or a disjoint union) of sets (vector
spaces) \(\mathcal{P}(n)\) of all multilinear operations (numbered by the amount of inputs) and composition rules
\(\mathcal{P}(n) \circ \mathcal{P}(m) \mapsto \mathcal{P}(n + m - 1)\) which prescribe how one can substitute an input of an operation by the
result of another one.

One typically separates two versions of this notion (the so-called symmetric and non-symmetric op-
erads) depending on whether we allow and do not allow to permute the inputs of operations. The
permutations of inputs define the action of symmetric groups on operations and compositions, hence,
the direct computations becomes much harder in the symmetric case. On the other hand, the symmetric
operads are much more important for applications. The general theory of non-symmetric operads is
similar but more transparent, since one needs not take care of the symmetric group action.

Given a discrete set \(\Upsilon = \cup_{n \geq 1} \Upsilon_n\) of \((\text{abstract})\) multilinear operations, where \(\Upsilon_n\) consists of the \(n\)-ary
operations, one can define a free operad \(\mathcal{F} = \mathcal{F}(\Upsilon) = \bigoplus_{n \geq 1} \mathcal{F}_n\) generated by \(\Upsilon\). Here \(\mathcal{F}_n\) is the vector space
spanned by all \(n\)-ary operations which are compositions of the elements of \(\Upsilon\) with each other (and
the first component \(\mathcal{F}_1\) is assumed to be spanned by the identity operation which, formally, does not belong
to \(\Upsilon\)). Note that in the symmetric case a ‘composition’ may permute the inputs, so that the set of \(n\)-ary
generators \(\Upsilon_n\) should span a representation of the symmetric group. To separate the non-symmetric and
symmetric cases, we will denote the corresponding free operads by \(\mathcal{F}(\Upsilon)\) and \(\mathcal{F}(\Upsilon)\), respectively.

Each operad can be defined in terms of generators and relations. Therefore, each operad may be
considered as a quotient of a free operad. That is why we first describe linear bases in the free operads
(referred to as monomial bases) which are compatible with compositions of operations. The difference be-
tween the symmetric and the non-symmetric cases is large already on this level. Namely, the definition of
monomial basis for non-symmetric operads is natural whereas to define monomials for symmetric operads
we need an additional structure of shuffle operads introduced in [DK]. We hope that the definitions and combinatorics of non-symmetric and shuffle monomials will be clear from the forthcoming subsections.

1.2. A basis for a free non-symmetric operad: tree monomials. Let us recall the combinatorics of trees involved in the description of monomials in operads.

Basis elements of the free operad are represented by (decorated) trees. By a (rooted) tree we mean a non-empty connected oriented graph $T$ of genus zero for which each vertex has at least one incoming edge and exactly one outgoing edge. Some edges of a tree have vertex at one end only. Such edges are called external. All other edges (having vertices at both sides) are called internal. Each tree has one outgoing external edge (= the output or the root) and several ingoing external edges, called leaves or inputs. The number of leaves of a tree $v$ is called an arity of a tree and is denoted by $ar(v)$.

Consider a set of generators $\Upsilon = \cup_{n \geq 1} \Upsilon_n$ of a free operad. We mark a tree with a single vertex and $n$ incoming edges by an arbitrary element of $\Upsilon_n$ and we call such tree a corolla. We say that a rooted tree $T$ is internally labeled by the set $\Upsilon$ if each vertex is labeled by an element from $\Upsilon$ such that the number of inputs in this vertex coincide with the arity of corresponding operation.

A rooted tree is called planar if for each vertex there is an ordering of all incoming edges. (Therefore, there is a canonical way to project a tree on a plane.)

**Proposition–Definition 1.2.1.** A canonical basis of a free non-symmetric operad $\mathcal{T}(\Upsilon)$ is enumerated by the set $B(\Upsilon)$ of all planar rooted trees internally labeled by the elements of the set $\Upsilon$. We shall refer to the elements of this basis as planar tree monomials.

There are two standard ways to think of elements of an operad in terms of its generators. The first way in terms of tree monomials represented by planar trees and the second one is in terms of compositions of operations presented by formulas with brackets. Our approach is somewhat in the middle: in most cases, we prefer (and strongly encourage the reader) to think of tree monomials, but to write formulas required for definitions and proofs in the language of operations since it makes things more compact.

A particular example of a planar tree monomial is presented in Figure 1. The corresponding operation may be written in a following way $g(f(f(\cdot, \cdot), g(\cdot, \cdot, \cdot)), \cdot, f(\cdot, \cdot))$. It is also convenient to use variables/letters for the inputs of operation. In a non-symmetric operad, we enumerate the inputs from the left to the right, so that the input variables are (from the left to the right) $x_1, x_2, x_3, \ldots$. For example, the operation from Figure 1 will be written in the following form:

$$g(f(f(x_1, x_2), g(x_3, x_4, x_5)), x_6, f(x_7, x_8)).$$

The composition rules for tree-monomials are given by the concatenation of trees (or composition of operations). The notion of divisibility is defined in the following way: A monomial $v$ is divisible by $w$ if there exists a planar subtree of $v$ isomorphic to $w$ as a planar labeled tree. (Notice that the root of $w$ not need to coincide with the root of $v$.) For example, the tree-monomial that represents the operation $f(\cdot, g(\cdot, \cdot, \cdot))$ is a divisor of a tree-monomial drawn in Figure 1 which corresponds to a unique minimal subtree which contains the internal edge going from the ternary operation $g$ in the upper level to the binary operation $f$ in the intermediate level.

Let us specialize two cases of divisibility. A monomial $w$ of some arity $m$ is a left divisor of $v$ if $v = w(v_1, \ldots, v_m)$ for some monomials $v_1, \ldots, v_m$. By the other words, $w$ is a left divisor of $v$ if there is a planar subtree $w'$ in $v$ which is isomorphic to $w$ and has the same root as $v$. Similarly, a monomial $u$ of some arity $l$ is a right divisor of $v$ if $v = w(x_1, \ldots, u(x_i, \ldots, x_{i+l-1}), x_{i+l}, \ldots, x_{l+m-1})$ (where $u$ is in $i$th place) for some monomial $w$ of some arity $m$. By the other words, $u$ is a right divisor of $v$ if all leaves of the subtree $u'$ in $v$ isomorphic to $u$ are also leaves of $v$. For example, the empty monomial and the monomial $v$ itself are both left and right divisors of $v$.

1.3. A basis in free symmetric operad: shuffle monomials. The free symmetric operad may not have a monomial basis compatible with all possible compositions. The way to avoid this problem is to construct a basis compatible with some compositions. A class of compositions we would like to preserve is called shuffle compositions. A collection of all multilinear operations with the prescribed rules for...
shuffle compositions form a **shuffle operad** \([\text{DK}]\). Any symmetric operad \(P\) may be considered as a shuffle operad denoted by \(P^f\). Reversely, one can recover from \(P^f\) all information concerning \(P\) but the action of the symmetric groups. In particular, the generating series of the operad \(P\) and \(P^f\) are the same. All definitions and conclusions below are given for shuffle operads and their monomial bases. However, symmetric operads are close to the nature, hence all examples of shuffle operads discussed below are examples of symmetric operads considered as shuffle ones.

Let us define a basis in the free symmetric operad \(F(\Upsilon)\) compatible with the shuffle compositions. The elements of the basis will be called **shuffle monomials**. The main difference from a non-symmetric world is that one should have an external labeling of an operation (i.e., inputs should be ordered). We say that a rooted tree with \(n\) incoming edges (=leaves) has an **external** labeling if the set of leaves is numbered by distinct natural numbers from 1 to \(n\).

**Proposition–Definition 1.3.1.** The basis of a free shuffle operad \(F(\Upsilon)\) generated by a set \(\Upsilon\) is numbered by the set \(B(\Upsilon)\) of all rooted (nonplanar) trees internally labeled by the set \(\Upsilon\) and having external labeling. We shall refer to the elements of this basis as **shuffle tree monomials** or simply **shuffle monomials**.

Note that whereas the shuffle tree monomials are originally considered as abstract graphs, we would need their particular planar representatives that we now describe.

In general, an embedding of a (rooted) tree in the plane is determined by an ordering of inputs for each vertex. To compare two inputs of a vertex \(v\), we find for each of the inputs the minimal leaf that one can reach from \(v\) via it. The input for which the minimal leaf is smaller is considered to be less than the other one. The corresponding embedding of a shuffle monomial to the plane is called its **canonical realization** (in \([\text{DK}, 3.1]\), it is referred to as **canonical planar representative**). Thus we identify a shuffle monomial with its canonical realization. Note that a canonical realization is a planar tree whose leaves are enumerated by an initial segment of positive integers. The leaves may be permuted by substitutions of a particular kind; we call them **shuffle substitutions**. The only assumption is that it should preserve the ordering of minima of the leaves in each vertex.

Similarly to the non-symmetric case there are two languages to think about the tree-monomials and operations in the free operad. We still recommend the reader to think on shuffle tree-monomials as labeled planar trees but most of the formulas in examples are written in the more compact language of operations. See Figure 2 below for the comparison of this two languages, with the monomials \(g(f(f(x_1, x_3), g(x_2, f(x_4, x_9), g(x_5, x_6, x_{11}))), x_7, f(x_8, x_{10}))\) and \(f(x_1, g(x_2, x_3, x_4))\) represented as trees.

Figure 2 provides an example of divisibility of shuffle monomials.

In the shuffle monomial represented by the left tree we encircle a subtree with one internal edge. This subtree is a divisor of the left monomial. The tree in the middle is the circled divisor where the leaves are numbered by the minima of the corresponding subtrees.

The right tree represents a shuffle monomial where we put the subsequent numbers on leaves according
to their local ordering. Thus we see that the shuffle monomial
\[ g(f(f(x_1,x_3),g(x_2,f(x_4,x_9),g(x_5,x_6,x_11))), x_7, f(x_8,x_{10})) \]
is divisible by
\[ f(x_1,g(x_2,x_3,x_4)) \].

Let us define two particular cases of divisibility. We say that a shuffle monomial \( v \) is \textit{right divisible} by a shuffle monomial \( w \) if \( v \) is divisible by \( w \) and, in the notation above, all external vertices of the subtree \( u \) are also external as vertices of \( v \). This means that there exists another shuffle monomial \( t \) such that the monomial \( v \) is obtained from \( t \) by replacing one of its input (say, the \( m \)-th) by a subtree isomorphic to \( u \). In this situation, we call the monomial \( v \) a \textit{shuffle composition} of \( t \) and \( w \) and write \( v = t \circ_{Sh}^{Sh} w \). Note that this notation is insufficient to define the composition uniquely because the former should contain also an information on “shuffling” substitution of the inputs of \( t \) and \( w \) [DK, Prop. 2].

Analogously, a shuffle monomial \( u \) (say, of arity \( m \)) is a \textit{left divisor} of \( v \) if there is a labeled subtree \( u \) inside \( v \) with common root with \( v \) which is isomorphic to \( w \) as a labeled tree and with the local ordering of the inputs with the same property as above. In this case one can represent \( v \) as a multiple shuffle composition
\[ (1.3.2) \quad v = (u \circ_{m}^{Sh} v_m) \circ_{m-1}^{Sh} (v_m-1) \cdots \circ_{1}^{Sh} v_1 \]
for some shuffle monomials \( v_1, \ldots, v_m \). We denote this multiple composition simply by \( u(v_1, \ldots, v_m)_{Sh} \).

Again, this notation is insufficient since it does not contain a part of information on the permutation of the inputs of \( v \).

1.4. Monomial operads and Gröbner bases. Let \( \Upsilon \) be a set of \textit{generators} of a non-symmetric or shuffle operad \( \mathcal{P} \), that is, each element of \( \mathcal{P} \) is a linear combination of compositions of elements of \( \Upsilon \) with each other and the identity operator. Each component \( \mathcal{P}_n \) has a natural structure of the quotient of the component \( \mathcal{F}(\Upsilon)_n \) of the (non-symmetric or symmetric) free operad \( \mathcal{F}(\Upsilon) \) by some vector space \( C_n \). Then the suboperad \( C = \bigcup_{n \geq 1} C_n \) in \( \mathcal{F}(\Upsilon) \), being a kernel of the natural surjection \( \mathcal{F}(\Upsilon) \to \mathcal{P} \), forms an ideal in \( \mathcal{F}(\Upsilon) \) called the \textit{ideal of relations} of \( \mathcal{P} \). The operad \( \mathcal{P} \) is called \textit{monomial} if each vector space \( C_n \) is spanned by monomials (non-symmetric or shuffle, according to the type of the operad \( \mathcal{P} \)). In this case, the ideal \( C \) is called monomial as well. For a monomial operad \( \mathcal{P} \), all monomials which do \textit{not} belong to \( C_n \) form a linear basis of \( \mathcal{P}_n \) for each \( n \).

Given an order on monomials compatible with compositions (for the discussion on admissible orderings, see [DK §3.2]), one can define a \textit{leading monomial} for each element of the free operad. For the ideal \( C \) of the relations of an operad \( \mathcal{P} \) as above, let \( \hat{C} \) be the span of the leading monomials of all its elements. Then \( \hat{C} \) is a monomial ideal such that the monomial basis of the quotient operad \( \hat{\mathcal{P}} = \mathcal{F}(\Upsilon)/\hat{C} \) is also a basis of \( \mathcal{P} \) (if one identifies the monomials in \( \mathcal{F}(\Upsilon) \) with their images in \( \mathcal{P} \)). For instance, the dimensions of the corresponding components of \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) are the same, so that these two operads have the same generating series. A \textit{Gröbner basis} of the ideal \( C \) is a set of elements of \( C \) such that their leading monomials generate the monomial ideal \( \hat{C} \) [DK §3.5]. Gröbner bases gives a way to construct the monomial operad \( \hat{\mathcal{P}} \) starting from \( \mathcal{P} \). This leads to a linear basis of \( \mathcal{P} \) and to the dimensions of its components.

Note that if an operad \( \mathcal{P} \) admits a finite Gröbner basis then the generating series of \( \mathcal{P} \) is the same as the generating series of the operad \( \hat{\mathcal{P}} \) defined by a finite set of monomial relations (i.e., the leading terms of the elements of the Gröbner basis). In this paper, we study the generating series of such operads \( \mathcal{P} \). The operads which admits finite Gröbner basis includes PBW operads ([H], [DK Cor.3]), operads coming from commutative algebras [DK §4.2] and many others. Some new nonquadratic examples are presented below.

We split the main part of this article into two sections. Section 2 contains the results for generating series of the non-symmetric operads. Section 3 contains a (somewhat more complicated) version for symmetric operads. In the additional Section 4 we discuss some conjectures and connections with other types of operads.

2. Nonsymmetric operads

2.1. Generating series and compositions. Suppose that the subset \( \mathcal{M} \subset \mathcal{B}(\Upsilon) \) defines a monomial basis of a non-symmetric operad \( \mathcal{P} := \mathcal{F}(\Upsilon)/(\Phi) \) (where \( \Phi \) denotes the ideal in the free non-symmetric
Operad $\mathcal{F}(\Upsilon)$ generated by the set $\Phi \subset \mathcal{F}(\Upsilon)$ called the set of relations of $\mathcal{P}$). The (ordinary) generating series of $\mathcal{P}$ is defined as the generating function of the dimensions of its components as follows

$$G_{\mathcal{P}}(z) := \sum_{n \geq 1} \dim \mathcal{P}(n) z^n = \sum_{v \in \mathcal{M}} z^{ar(v)}.$$

Moreover, if there exists an additional grading of the set of generators $\Upsilon$ such that all relations from $\Phi$ are homogeneous with respect to this internal grading, one can consider in addition a generating series with two parameters

$$G_{\mathcal{P}}(z, t) := \sum_{n \geq 1} \dim_t \mathcal{P}(n) z^n = \sum_{v \in \mathcal{M}} t^{[v]} z^{ar(v)}.$$

On the other hand, we will omit the parameter $t$ if the internal grading is defined by the arity of operations. For example, if all generating operations has degree 1 with respect to internal grading and has same arity $k > 1$, then all operations of arity $n(k-1)+1$ should have the same grading $n$.

Let us consider in details the case of one composition. Namely, let $\mu$ be an $m$-ary generator in the free operad $\mathcal{F}(\Upsilon)$ (i.e., $\mu \in \Upsilon_m$), and let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be some graded vector subspaces in $\mathcal{F}(\Upsilon)$). A non-symmetric composition $\mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)$ of vector spaces $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is the subspace of $\mathcal{F}(\Upsilon)$ whose $n$-th component is spanned by all possible compositions $\mu(p_1, \ldots, p_m)$, where $p_i \in \mathcal{P}_i(k_i)$ and $\sum k_i = n$.

**Lemma 2.1.1.** The generating series of a non-symmetric composition of vector spaces $\mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)$ is the product of generating series of inputs:

$$G_{\mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)}(z, t) = t^{[\mu]} (G_{\mathcal{P}_1}(z, t) \cdots G_{\mathcal{P}_m}(z, t)).$$

**Proof.** It is obvious from the description of the canonical basis in a free non-symmetric operad that $\mu(\mathcal{P}_1, \ldots, \mathcal{P}_m) \simeq \mathcal{P}_1 \otimes \ldots \otimes \mathcal{P}_m$ as a graded vector space. $\square$

### 2.2. System of equations for generating series

We give here a result which is a key point of Theorem 0.1.3. It gives a system of algebraic equations which will lead (via elimination of variables) to the algebraic equation for $G_{\mathcal{P}}$, see Subsection 2.3. The proof of this result given here contains some core algorithms enabling computations in particular examples.

**Theorem 2.2.1.** For a given non-symmetric operad $\mathcal{P}$ with a finite set of generators and a finite Gröbner basis there exist an integer $N$ and a system of algebraic equations on $N+1$ functions $y_0 = y_0(z, t), \ldots, y_N = y_N(z, t)$

$$y_i = t^{a_i} \sum_{s \in [0, N]^d_i} q^i_s \cdot y_{s_1} \cdots y_{s_{d_i}} \quad \text{for } i = 1, \ldots, N, \quad (2.2.2)$$

such that $G_{\mathcal{P}}(z, t) = \sum_{i=0}^N y_i(z, t)$, $y_0 = z$ and $y_i(0, t) = y_i(z, 0) = 0$, $i = 1, \ldots, N$. The numbers $q^i_s \in \{0, 1\}$ and the nonnegative integer numbers $d_i, a_i$ and $N$ are bounded from above by some functions of the degrees and the numbers of generators and relations of the operad $\mathcal{P}$.

Note that, under the initial conditions the solution of system (2.2.2) is unique. It follows that one can consider (2.2.2) as a system of recursive equations on the coefficients of series $y_i$. We will refer to (2.2.2) as the system of recursive equations.

Below we present two different proofs of this theorem (in Subsections 2.2.1 and 2.2.2 below). It is better to say that we present two different algorithms to construct the above system of equations. In addition, we give also an idea of a third algorithm.

The first one is more efficient in the case of many relations of relatively low arity. Additionally, the first proof gives the positivity of coefficients $q^i_s$ which we use in Corollary 2.3.1. The second algorithm seems much more useful in case of few relations. Namely, the number $N$ of additional variables is typically much lower than in the first algorithm. On the other hand, the second algorithm do not allow to have only positive coefficients $q^i_s$, that is, it gives a slightly weaker version of Theorem 2.2.1. Moreover, in Subsection 2.2.3 we present an example of computations based on a third idea, that is, we use combinatorics of homology of the operad. We hope that a reader can get the general idea of this method from the given example in order to compute the functional inverse to the generating series of a given operad.
Note that the systems of equations similar to the system \((2.2.2)\) above are sometimes appear in combinatorics and computer science in symbolic methods for combinatorial structures (see e.g. [FS]) as well as for the context-free languages (see e.g. [ChSch]).

2.2.1. First proof of Theorem 2.2.1

Proof. Suppose that an operad \(\mathcal{P}\) has a finite set of generators \(\Upsilon\) and a finite set of monomial relations \(\Phi\). (It suffices to consider monomial relations since we are dealing with generating series. Therefore, there is no difference between the relations that form a Gröbner basis and corresponding monomial relations presented by the leading terms of the Gröbner basis.) Let \(d\) be the maximum level of leaves of elements of \(\Phi\) (by the level of a vertex/leaf in a tree we mean the number of the edges in the path from the root to this vertex/leaf). As mentioned in Section 1.2.1 every monomial \(v\) in the free operad \(\mathcal{F}(\Upsilon)\) can be identified with a rooted planar tree whose vertices are marked by elements of \(\Upsilon\). Given such a monomial \(v\), by its stump \(b(v)\) we mean its maximal monomial left divisor such that the leaves of \(b(v)\) have levels strictly less than \(d\). In other words, \(b(v)\) is the submonomial (rooted subtree) of \(v\) which consists of the root and all vertices and leaves of \(v\) of level less or equal to \((d - 1)\) and all edges connected them.

Let \(\text{Stump}\) be the set of all stumps of all nonzero monomials in \(\mathcal{P}\). Let \(N\) be the cardinality of this set. The elements \(b_1, \ldots, b_N\) of \(\text{Stump}\) are partially ordered by the following relation:

\[ b_i < b_j \iff i \neq j \text{ and } b_i \text{ is a left divisor of } b_j. \]

Let \(M_{b_i}\) be the set of all monomials in \(=\) the monomial basis of \(\mathcal{P}\) generated by \(b_i\), and let

\[ \overline{M}_{b_i} = M_{b_i} \setminus \bigcup_{j: b_i < b_j} M_{b_j}. \]

Then the pairwise intersections of the sets \(\overline{M}_{b_i}\) are empty. Moreover, the disjoint union \(\bigcup_{i=1}^{N} \overline{M}_{b_i}\) is the monomial basis of the operad \(\mathcal{P}\). We have

\[ G_\mathcal{P}(z) = \sum_{i=1}^{N} y_i(z), \]

where \(y_i(z) = G_{\text{span}(\overline{M}_{b_i})}(z)\) is the generating series of the span of the set \(\overline{M}_{b_i}\). For every element (=operation) \(\mu \in \Upsilon\) of some arity \(n\), let us define the numbers \(j_\mu(i_1, \ldots, i_n)\) for all \(1 \leq i_1, \ldots, i_n \leq N\) as follows:

\[ j_\mu(i_1, \ldots, i_n) = \begin{cases} 0, & \text{if } \mu(\overline{M}_{b_{i_1}}, \ldots, \overline{M}_{b_{i_n}}) = 0 \text{ in } \mathcal{P} \\ j, & \text{if the stump } b(\mu(\overline{M}_{b_{i_1}}, \ldots, \overline{M}_{b_{i_n}})) = b_j. \end{cases} \]

Note that the nonzero sets of the type \(\mu(\overline{M}_{b_{i_1}}, \ldots, \overline{M}_{b_{i_n}})\) have empty pairwise intersections. Let \(v\) be a nonzero monomial in \(\mathcal{P}\) with the root vertex labeled by \(\mu\). Then \(v \in \mu(\overline{M}_{b_{i_1}}, \ldots, \overline{M}_{b_{i_n}})\) for some \(\mu, i_1, \ldots, i_n\), that is, \(v = \mu(v_{i_1}, \ldots, v_{i_n})\) where the monomial subtrees \(v_{i_j} \in \overline{M}_{b_{i_j}}\) are uniquely determined by \(v\). Hence, \(v \in \overline{M}_{b_j}\) where \(j = j_\mu(i_1, \ldots, i_n)\) from \((2.2.3)\). As soon as the degrees of the relations are less than or equal to \(d\) we come up with the following disjoint union decomposition for all \(j = 1 \ldots N\)

\[ \overline{M}_{b_j} = \bigcup_{j_\mu(i_1, \ldots, i_n) = j} \mu(\overline{M}_{b_{i_1}}, \ldots, \overline{M}_{b_{i_n}}), \]

where \(\mu\) is the root vertex of any \(v \in \overline{M}_{b_j}\).

This equality implies equations \((2.2.2)\) for the generating functions \(y_i(z) = G_{\text{span}(\overline{M}_{b_i})}(z)\). \(\square\)

Notice that if a stump \(b_j\) do not have leaves of level \((d - 1)\) then the corresponding set \(\overline{M}_{b_j}\) consists of one element \(b_j\). For example, \(M_1\) always consists of one element representing the identity operation and \(G_{M_1} = z\). In fact, the substitution \(y_0 = G_{M_1} = z\) is already made in the system \((2.2.2)\). Therefore, one can reduce the number of algebraic equations in the system \((2.2.2)\) to the number of the stumps of level \((d - 1)\) which contain at least one leave of level \((d - 1)\). In particular, if the operad \(\mathcal{P}\) is a PBW operad then the number of recursive equations is one greater than the number of generators.
Example 2.2.4. Let \( \text{Assoc} \) be the operad of associative algebras considered as a non-symmetric operad. Namely, \( \text{Assoc} \) is generated by one binary operation \( \mu(, ) \) subject to one quadratic relation
\[
\mu(\mu(a,b), c) = \mu(a, \mu(b,c)).
\]
It is well known that this relation forms a Gröbner basis according to the standard lexicographical ordering of monomials \([DK, H]\). The set \( \text{Stump} \) of stumps of level less or equal to 1 consists of the identity operator 1 and the unary operation \( \mu \). So, there are two sets of the type \( \mathcal{M} \), that is, \( \mathcal{M}_1 = \{1\} \) and \( \mathcal{M}_{\mu} \). Thus, we get the following system of equations:

\[
\begin{cases}
G_{\text{Assoc}} = z + y_{\mu} \\
y_{\mu} = z^2 + z y_{\mu}
\end{cases}
\implies G_{\text{Assoc}} = z + z G_{\text{Assoc}} \implies G_{\text{Assoc}}(z) = \frac{z}{1 - z}.
\]

2.2.2. Decreasing the number of equations. Now we present one more algorithm that allows to derive the system of equations \([2.2.2]\). The advantage here is that in many examples the number of equations is much less than in the algorithm given in Subsection 2.2.1 above. On the other hand, the coefficients \( q_k \) might be negative (i.e., \( q_k \in \{-1, 0, 1\} \)). One may consider this algorithm as a generalization of the algorithm for binary trees with one relation presented in \([R]\).

The key point of the algorithm is the following. In the notation of Subsection 2.2.1 this new algorithm gives a system in the generating series of the sets \( \mathcal{M}_b \) instead of \( \mathcal{M}_b \). Then \( G_P \) is equal to the sum of \( z \) and those \( G_{\mathcal{M}_{b_i}} \) where \( b_i \) runs the generators of the operad \( \mathcal{P} \). Therefore, to find \( G_P \) it is sometimes sufficient to solve only a part of the system which allows to express these new variables. Thus, the number of equations (=the number of variables) of the reduced system can be less than the number \( N \) of equations (and variables) of the system constructed in Subsection 2.2.1.

Second proof of Theorem 2.2.1. Let \( \mathcal{P} \) be as above, i.e. \( \mathcal{P} \) is a finitely presented (non-symmetric) monomial operad with a finite set of generators \( \Upsilon \) and a finite set of relations \( \Phi \). We suppose that the set of relations is reduced, namely, \( g \) is not divisible by \( g' \) for every distinct pair \( g, g' \in \Phi \). Let \( \mathcal{M} \) be the set of monomials in the free operad \( \mathcal{F}(\Upsilon) \) which are not divisible by the relations from \( \Phi \). In other words, \( \mathcal{M} \) is a monomial basis of \( \mathcal{P} \). We also suppose that the unary identity operation 1 belongs to the set of monomials \( \mathcal{M} \). We call it the trivial monomial.

Consider a free operad \( \mathcal{F}(\Upsilon) \) and the set of all tree monomials \( \mathcal{B}(\Upsilon) \) generated by the same set \( \Upsilon \). With each monomial \( v \in \mathcal{B}(\Upsilon) \) one can associate the subset \( \mathcal{F}_v \subset \mathcal{B}(\Upsilon) \) consisting of those basis elements that are left divisible by \( v \). In other words, the set \( \mathcal{F}_v \) is a monomial basis of a right ideal \( v \circ \mathcal{F} \) generated by \( v \).

Obviously, for the identity operation 1, we have \( \mathcal{F}_1 = \mathcal{B}(\Upsilon) \). Each collection \( \{v_1 \cup \ldots \cup v_l\} \) of monomials defines the left common multiple denoted by \( [v_1 \cup \ldots \cup v_l] \). The monomial \( [v_1 \cup \ldots \cup v_l] \) is defined as the smallest element in \( \mathcal{F}_{v_1} \cap \ldots \cap \mathcal{F}_{v_l} \) if \( \mathcal{F}_{v_1} \cap \ldots \cap \mathcal{F}_{v_l} \) is non-zero. The left common multiple (if exists) should be the unique tree given as an union of its subtrees \( v_i \). As the definition of the left common multiple one may use the following identity:

\[
\mathcal{F}_{v_1} \cap \ldots \cap \mathcal{F}_{v_l} = \mathcal{F}_{[v_1 \cup \ldots \cup v_l]} = \mathcal{F}_{[v_1 \cup \ldots \cup v_l]}
\]

For the case of empty intersection \( \mathcal{F}_{v_1} \cap \ldots \cap \mathcal{F}_{v_l} \) we set \( [v_1 \cup \ldots \cup v_l] \) to be zero.

With each monomial \( v \in \mathcal{M} \) we associate the monomial basis \( \mathcal{M}_v \) of the corresponding right ideal \( v \circ \mathcal{M} \). Obviously, we have \( \mathcal{M}_v = \mathcal{M} \cap \mathcal{F}_v \).

Consider a given non-trivial monomial \( v \in \mathcal{M} \). Suppose that the root generator of \( v \) is a \( k \)-ary operation \( \mu \), that is \( v = \mu \circ (v^1, \ldots, v^k) \) where \( v^i \) denotes the subtree which grows from the \( i \)-th incoming arrow of the root vertex of \( v \). Let \( \Phi_v \) be the subset of the set of generating relations \( \Phi \) such that the corresponding relations have a non-trivial left common multiple with \( v \) in the free operad \( \mathcal{F}(\Upsilon) \), i.e.

\[
g \in \Phi_v \subset \Phi \iff [v \cup g] \neq 0 \text{ & } g \in \Phi \iff \mathcal{F}_v \cap \mathcal{F}_g \neq 0 \text{ & } g \in \Phi
\]

If the left common multiple \( [v \cup g] \) is non-zero then the \( \mu \) should be a left divisor of \( v \), hence, we have a decomposition \( g = \mu \circ (g^1, \ldots, g^k) \) for some \( g^1, \ldots, g^k \). We get the following recursive relation for the set \( \mathcal{M}_v \):

\[
\text{(2.2.5)} \quad \mathcal{M}_v = \mu \circ (\mathcal{M}_{v^1}, \ldots, \mathcal{M}_{v^k}) \setminus \bigcup_{g \in \Phi_v} \mu \circ \left( \mathcal{M}_{[v^1 \cup g^1]}, \ldots, \mathcal{M}_{[v^k \cup g^k]} \right).
\]

Identity \( \text{(2.2.5)} \) should be clear from the observation that each monomial from \( \mu \circ (\mathcal{M}_{v^1}, \ldots, \mathcal{M}_{v^k}) \) either belongs to \( \mathcal{M}_v \) or is divisible from the left by the relation \( g \in \Phi_v \).
Denote by $y_v(z)$ (or $y_v(z, t)$ if the operad is $\mathbb{Z}^2$-graded) the generating series of the span of the set $\mathcal{M}_v$. Combining the formula (2.2.5) with the inclusion-exclusion principle, we count the cardinalities and the generating series of the sets $\mathcal{M}_v$ and their intersections. We have

\begin{equation}
(2.2.6) \quad y_v = t^{|\mu|} \left| \Phi_v \right| \sum_{s=0}^{\left| \Phi_v \right|} (-1)^s \left\{ \sum_{\{g_1, \ldots, g_s\} \subseteq \Phi_v: |v \cup g_1 \cup \ldots \cup g_s| \neq 0} \left[ y_{[v \cup g_1 \cup \ldots \cup g_s]} \right] \right\},
\end{equation}

where (as above) $\mu$ is the generator placed in the root vertex of both $v$ and $g_j$, and the subtrees growing from the $i$-th incoming arrow of $\mu$ are denoted by $v^i$ and $g_j^i$ respectively. The internal summation is taken over the subsets $\{g_1, \ldots, g_s\} \subseteq \Phi_v$ of cardinality $s$ such that the corresponding left common multiple $[v \cup g_1 \cup \ldots \cup g_s]$ is different from zero and is not divisible by any element of $\Phi$. In order to finish the proof of Lemma 2.2.1 we have to explain the finiteness of the number of the recursive equations (2.2.6). This is made in the next Lemma.

**Lemma 2.2.7.** There exists a minimal finite set of monomials $T(\mathcal{P}) \subset \mathcal{M}$ satisfying the following conditions.

- The identity operator $\mathcal{I}$ and all generators $\mathcal{Y}$ belongs to $T(\mathcal{P})$.
- Suppose that $v = \mu(v_1^1, \ldots, v_k^k) \in T(\mathcal{P})$ and a collection $\{g_1, \ldots, g_s\} \subset \Phi_v$ has a nontrivial left common multiple $w = \left[ v \cup g_1 \cup \ldots \cup g_s \right] \neq 0$ with $v$ (in particular, each $g_j$ has the form $g_j = \mu(g_j^1, \ldots, g_j^k)$). Then for each $i = 1, \ldots, k$ the monomial $[v^i \cup g_1^i \cup \ldots \cup g_s^i]$ either belongs to $T(\mathcal{P})$ or is divisible by a relation from $\Phi$.

**Proof.** Suppose that $d > 1$ is the maximum of the levels of the relations in $\Phi$. Since the level of the left common multiple $[v \cup w]$ is bounded from above by the maximum of the levels of $v$ and $w$, we conclude that the level of any monomial from $T(\mathcal{P})$ should be strictly less than $d$. The set of monomials with the bounded level is finite thus $T(\mathcal{P})$ is finite.

In fact, this bound is quite large. We will see in examples below that the cardinality of $T$ is much less than the number of monomials of height not greater than $(d - 1)$.

It is clear that

\begin{equation}
(2.2.8) \quad y_1 = z + \sum_{\mu \in T} y_\mu.
\end{equation}

Therefore, for $v \in T(\mathcal{P})$ we get a system of algebraic equations (2.2.5) and (2.2.6) using a finite set of unknown functions $\{y_v | v \in T(\mathcal{P})\}$.

**Example 2.2.9.** Let $\mathcal{Q}$ be a non-symmetric operad generated by one binary operation $(.)$ satisfying the following two monomial relation of arities 4 and 5:

\begin{equation}
(2.2.9) \quad ((a, b), c), d) = 0 \quad \text{and} \quad (a, (b, ((c, d), e))) = 0.
\end{equation}

The set $T(\mathcal{Q})$ from Lemma 2.2.7 consists of the following 5 elements:

$$T(\mathcal{Q}) := \{1; (ab); ((ab)c); (a((bc)d)); ((ab)((cd)e))\}.$$  

The corresponding system of recursive equations is

\begin{equation}
(2.2.10) \quad \left\{ \begin{array}{l}
y_1 = z + y_{(ab)}, \\
y_{(ab)} = y_1^1 - y_{(ab)}y_1 - y_1y_{((ab)c)} + y_{((ab)c)}y_{(a(bc)d))}, \\
y_{((ab)c)} = y_1y_{(ab)} - y_{((ab)c)}y_1 - y_{(ab)}y_{(a(bc)d))} + y_{((ab)c)}y_{(a(bc)d))}, \\
y_{(a((bc)d))} = y_1y_{((ab)c)} - y_{((ab)c)}y_1 - y_{((ab)c)}y_{((cd)e))} + y_{((ab)c)}y_{((cd)e))}, \\
y_{((ab)((cd)e))} = y_1y_{((ab)c)} - y_{((ab)c)}y_1 - y_{((ab)c)}y_{((cd)e))} + y_{((ab)c)}y_{((cd)e)).}
\end{array} \right.
\end{equation}

To exclude additional variables we make the following linear change of variables: $y := y_1$, $v_3 := y_1 - y_{((ab)c)}$, $v_4 := y_1 - y_{(a((bc)d))}$, $v_5 := y_{((ab)c)} - y_{((ab)((cd)e))}$. The system (2.2.11) is then equivalent to the following

\begin{equation}
(2.2.12) \quad \left\{ \begin{array}{l}
y - z = v_3v_4, \\
y - v_3 = (v_3 - z)v_4 \Rightarrow v_3 - z = zv_4, \\
y - v_4 = v_3v_5, \\
y - v_3 - v_5 = (v_3 - z)v_5 \Rightarrow v_3 - v_4 + v_5 = zv_5.
\end{array} \right. \Rightarrow G_\mathcal{Q}(z) = y = z + \frac{z^2(1 - z^2)}{(1 - z^2)^2}.
\end{equation}
2.2.3. Computations via homology. In this section we show how one can simplify computations in some cases using the monomial resolutions of operads with finite Gröbner bases introduced in [DK1]. First, we remind a description of a monomial basis in this resolutions. Second, we present a particular example where such a description allows to compute the generating series. The corresponding computation using the first two methods (given in two previous examples) became extremely hard compare to what the homological method can give. Theoretically, the monomial description of a resolution from [DK1] allows to produce an algorithm similar to the one given in Section 2.2.1 starting from a given finite Gröbner basis. But the involved combinatorics became extremely hard as soon as the complexity of intersections of leading terms of monomials grows up. Therefore we decided not to give all the details of this algorithm. However, a couple of examples given for the non-symmetric (Example 2.2.14) and the symmetric (Example 2.2.15) cases should convince the readers that in real computations they should not close their eyes on this method.

Suppose that $R$ is a free resolution (=DG model) of an operad $P$ generated by a differential graded vector space $Q$. (In other words, $R \simeq \mathcal{F}(Q)$ as an operad and there exists a differential $d$ on $R$ such that the homology operad of $R$ is isomorphic to $P$). Then the generating series of $P$ and $Q$ are related by the equality

\begin{equation}
G_P(z) - G_Q(G_P(z)) = \sum_{k \geq 0} \chi(Q(n))z^n.
\end{equation}

Let $P$ be a finitely presented operad with a set of generators $\Upsilon$ and a set of monomial relations $\Phi$. Let us recall a basis in a free monomial resolution of the operad $P$.

**Proposition** (see [DK1]). There exists a free resolution $(R, d) \rightarrow P$ such that the set of free generators of $R$ consists of the union of the set $\Upsilon$ and the set $\mathcal{K}$ elements of which are numbered by the following pairs: a monomial $v \in B(\Upsilon)$ and a set $\{w_1, \ldots, w_n\}$ of labeled subtrees (=submonomials) of $v$ satisfying the following two conditions.

(h1) Each $w_i$ is isomorphic to one of the elements of $\Phi$ as a planar labeled tree.
(h2) Each internal edge of the monomial $v$ should be covered by at least one of subtrees $w_i$. In other words, there is no decomposition $v = v' \circ v''$ such that each $w_i$ is a subtree of $v'$ or a subtree of $v''$.

The homological degree of each generator $v \in \Upsilon$ is set to be zero and the homological degree of the generator $(v, \{w_1, \ldots, w_n\}) \in \mathcal{K}$ is set to be $k$.

Note that the monomial $v$ in a pair $(v, \{w_1, \ldots, w_n\}) \in \mathcal{K}$ is uniquely defined by the set of submonomials $\{w_1, \ldots, w_n\}$. For such a pair we will use a notation $\overline{w}$.

Let us present an example where we use this description of a basis in a monomial resolution to get a functional equation for the generating series.

**Example 2.2.14.** Consider a non-symmetric operad $Q_k$ generated by one binary operation $(,)$ which satisfies the following relation of degree $k > 2$ (i.e., of arity $k + 1$):

\[ r_k := (x_1, \ldots, (x_{k-2}, (x_{k-1}, x_k, x_{k+1})) \ldots) = 0, \]

where $(a, b, c)$ denotes the associator $(ab)c - a(bc)$.

The operad $Q_2$ defined in the same way is $\text{Assoc}$. So, one might consider the identity $r_k = 0$ of $Q_k$ for $k > 2$ as a weak version of associativity.

**Proposition 2.2.15.** The unique generator $r_k$ of the ideal of relations forms a Gröbner basis of relations in $Q_k$. 

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that both pairs \((H, \sigma)\) where \(\sigma \neq 1\) and

de present the decomposition of the set of submonomials

\[
S_{k+l} := (x_1(x_2 \ldots (x_{k-1}x_k)(x_{k+1} \ldots (x_{k+l-1}(x_{k+l}x_{k+l+1})x_{k+l+2})) \ldots) \quad \text{for } l = 1, \ldots, k - 2
\]

and

\[
S_{2k} := (x_1(x_2 \ldots (x_{k-1}(x_k \ldots (x_{2k-4}(x_{2k-3}x_{2k-2})x_{2k-1}) \ldots)x_{2k}) \ldots).
\]

The lower index in the left hand side corresponds to the number of leaves/inputs in a monomial. The corresponding \(s\)-polynomials looks as follows:

\[
S_{k+l} \rightsquigarrow (x_1((x_{k+l}x_{k+l+1})x_{k+l+2}) \ldots) - (x_1((x_{k-1}x_k)x_{k+1}(x_{k+l}x_{k+l+2}) \ldots))
\]

\[
S_{2k} \rightsquigarrow (x_1(x_{k-1}((x_k((x_{2k-4}(x_{2k-3}x_{2k-2})x_{2k-1}) \ldots)x_{2k})) \ldots)
\]

It is easy to see that each monomial of the form

\[
(x_1((x_{k-2}(f(x_k, \ldots, x_j)) \ldots)),
\]

where \(j \geq k + 1\) and \(f\) is an arbitrary iterated composition of the operation \((,\)\), is reducible via \(r_k\) to the monomial

\[
m_j = (x_1((x_{j-2}(x_{j-1}, x_j)) \ldots)).
\]

Thus, both monomials in each \(s\)-polynomial above are reduced to the same monomial \(m_j\) for suitable choice of \(j\). This means that each \(s\)-polynomial is reduced to \(m_j - m_j = 0\). \(\square\)

As soon as the \(\text{Gröbner}\) basis is chosen it remains to compute the generating series of homology for the corresponding monomial replacement.

The operad \(Q_k\) is generated by one binary operation. Therefore, the tree-monomials under consideration are rooted planar binary trees where all internal vertices are labeled by the same operation \((,\)\). Therefore, we will omit this labeling with no loss. In order to specify the tree-type of monomials we say that the monomial operation \((x_1x_2)x_3\) corresponds to a planar rooted binary tree with two internal vertices and one internal edge that goes to the left and the monomial operation \((x_1x_2x_3)\) corresponds to a binary planar rooted tree with one internal edge that goes to the right. In particular, the leading term of the unique element \((r_k)\) of a \(\text{Gröbner}\) bases corresponds to a planar rooted binary tree of level \(k\) with \(k\) internal vertices and the unique path that contains all \((k - 1)\) internal edges of a tree. This path starts at the root vertex and first have \((k - 2)\) edges that goes to the right and the last edge goes to the left.

For any given element \((v, \{w_1, \ldots, w_n\}) \in \mathcal{H}\) all \(w_i\) should be isomorphic to the leading term of the relation \(r_k\) as a planar binary tree. Therefore, there exists exactly one submonomial which contains the root vertex of \(v\). Without loss of generality we assume that this submonomial is \(w_1\). The pair \((v, \{w_2, \ldots, w_n\})\) will no longer satisfy the property \((h2)\) of the elements in \(\mathcal{H}\) but can be presented as the composition of a generator \((,\)\) taken several times and pairs from \(\mathcal{H}\). In other words, one has to present the decomposition of the set of submonomials \(\{w_2, \ldots, w_n\}\) into a disjoint union of subsets such that \(w_i\) and \(w_j\) belong to the same subset if and only if there exists a submonomial \(v'\) in \(v\) and a subset \(\{w_i, w_j, \ldots\} \subset \{w_2, \ldots, w_n\}\) such that a pair \((v', \{w_i, w_j, \ldots\})\) is isomorphic to a pair from \(\mathcal{H}\). Let us show that in the case of operad \(Q_k\) this decomposition contains at most two subsets. Indeed the unique left internal edge of a submonomial \(w_1\) may not belong to any other submonomial \(w_i\) for \(i > 1\). Consider a decomposition of a monomial \(v = v' \circ v''\) according to this left internal edge. Then each \(w_i\) should be a submonomial of \(v'\) or \(v''\) and we have a decomposition

\[
\{w_2, \ldots, w_n\} = \{w_{\sigma(2)}, \ldots, w_{\sigma(l)}\} \sqcup \{w_{\sigma(l+1)}, \ldots, w_{\sigma(n)}\}
\]

for an appropriate permutation \(\sigma\). There also might be the cases when the first or the second subset is empty. There exists a pair of uniquely defined (probably empty) submonomials \(v_1 \subset v'\) and \(v_2 \subset v''\) such that both pairs \((v_1, \{w_{\sigma(2)}, \ldots, w_{\sigma(l)}\})\) and \((v_2, \{w_{\sigma(l+1)}, \ldots, w_{\sigma(n)}\})\) are isomorphic to the elements of \(\mathcal{H}\). Moreover, the monomial tree \(v\) can be uniquely presented as a composition of several generators and
monomials $v_1$ and $v_2$. We get a recursive formula for the generating series $y$ of the Euler characteristics of elements in $\mathcal{H}$:

$$y = -z^{k+1} - \sum_{l=2}^{k-1} z^{l+1} y - \sum_{l=1}^{k-2} z^l y^2 - z^{k-1} y.$$ 

Here the first summand corresponds to the empty set $\{w_2, \ldots, w_n\}$; the second summand corresponds to the empty set $\{w_{\sigma(t+1)}, \ldots, w_{\sigma(n)}\}$; the index $l$ corresponds to the number of internal vertices in a subtree $v'$ which do not belong to none of the vertices of submonomials in $\{w_{\sigma(2)}, \ldots, w_{\sigma(l)}\}$; the third summand deals with both nonempty sets in decomposition (2.2.16); and the fourth summand corresponds to the empty set $\{w_{\sigma(2)}, \ldots, w_{\sigma(l)}\}$. The powers of $z$ are equal to the number of leaves coming from the internal vertices that do not belong to submonomials $v_1$ and $v_2$. The minus sign comes from the homological degree since we remove exactly one element $w_1$ from the set of submonomials.

Finally we have the following quadratic equation for the functional inverse series $G_{Q_k}(z)^{-1} = z - z^2 - y$:

$$(G_{Q_k}(z))^{-1} (z k^{k-1} z) + G_{Q_k}(z)^{-1} (z k^{k+1} - 3 z^k + z k^{k-3} + 2 z^2 - 2 z + 1) - (z k^{k+2} - 2 z^k + z k^2 - 2 z^3 - 2 z^2 + z) = 0,$$

which is equivalent to the algebraic equation of degree $(k + 2)$ on the generating series $G_{Q_k}$.

2.3. Single algebraic equation for generating series. The classical elimination theory implies existence of algebraic equation on a function $G_{\mathcal{P}}$ from the system (2.2.2) (see explanation below). See also [CHSh] and the appendix B.1 in [FS] and references therein where the same theorem is proven for context-free specifications and languages.

**Theorem 2.3.1.** The generating series $G_{\mathcal{P}}$ of a non-symmetric operad $\mathcal{P}$ with a finite Gröbner basis is an algebraic function.

Starting from the system (2.2.2) it remains to use the following

**Lemma 2.3.2.** Suppose that the formal power series $f_1, \ldots, f_n$ without constant terms in variables $t$ and $z$ over $\mathbb{Q}$ satisfy a system of algebraic equations of the form

$$f_i t^{\mu_i} = g_i (f_1, \ldots, f_n)$$

for each $i = 1, \ldots, n$, where $g_i$ is a homogeneous polynomial in $n$ variables of degree $d_i \geq 2$. Then the power series $f_1$ satisfy a polynomial equation

$$Q(f) = 0,$$

where $Q$ is a non-constant polynomial with coefficients in $\mathbb{Q}[t, z]$ such that $\deg Q \leq (\prod_{i=1}^{n} d_i)^2$.

**Proof of Lemma 2.3.2** The above system has the form

$$F = G(P),$$

where $F = (f_1, \ldots, f_n)$ and $G = (t^{-\mu_1} g_1, \ldots, t^{-\mu_n} g_n)$, or

$$H(F) = 0$$

with $H = \text{Id} - G$. Note that the Jacobi matrix $J = \partial H/\partial F$ is non-degenerate, because $\det J = 1 + O(F) \neq 0$. Let $K$ be the field of rational fractions $\mathbb{Q}(t, z)$ and let $L$ be some its algebraically closed extension which contains the ring of formal power series in $t$ and $z$. Obviously, the variety $V \subset L^n$ of the solutions of the above system is 0-dimensional.

Therefore, there exists a non-trivial polynomial $T(x)$ over $L$ such that $T(f_1) = 0$. By Bezout’s theorem, one can take $T$ such that $\deg T \leq \prod_{i=1}^{n} d_i$. By effective Hilbert Nullstellensatz (see [Ko] Corollary 1.7), for some $j \leq \prod_{i=1}^{n} d_i$ the polynomial $T(x)^j$ lies in the polynomial ideal $I$ generated over $K$ by $G_i$’s. It follows that some divisor $Q(x)$ of $T(x)^j$ belongs to the reduced Gröbner basis of $I$ (w. r. t. the “lex” order). Since this element $Q(x)$ can be constructed via Buchberger’s algorithm, its coefficients belong to $K$. In addition, we have $Q(f) = 0$ and $\deg Q \leq j \deg T \leq (\prod_{i=1}^{n} d_i)^2$. □

**Remark 2.3.3.** Note that the existence of such a polynomial $Q$ follows also from Artin’s Approximation Theorem [1] Theorem 1.7].

**Corollary 2.3.4.** Let $\mathcal{P}$ be a PBW operad with $k$ binary generating operations. Then the generating series of this operad is a solution of an algebraic equation of degree not greater than $4^k$. 


Proof. The algorithm described above implies the existence of a system of \( k \) quadratic and one linear equation on \( k + 1 \) functions. Then we apply Lemma 2.3.2.

\[ \square \]

2.4. Operads of subexponential growth. Below we present an application of the above theory for non-symmetric operads with a small growth. Recall that a sequence \( \{a_n\}_{n \geq 0} \) of nonnegative real numbers is said to have a subexponential growth if its growth is strictly less than exponential, that is, for each \( d > 1 \) there exists \( C > 0 \) such that \( a_n < C d^n \) for all \( n > 0 \).

**Corollary 2.4.1.** Let \( \mathcal{P} \) be a non-symmetric operad with a finite Gröbner basis of relations. Suppose that the growth of dimensions \( \mathcal{P}(n) \) is subexponential. Then the generating series \( G_\mathcal{P} = \sum_{n \geq 1} \dim \mathcal{P}(n) z^n \) is rational. In particular, the sequence of dimensions \( \dim \mathcal{P}(n) \) should have polynomial growth \( [n^d] \) for some integer \( d \).

The proof of Corollary 2.4.1 is based on the general facts about the Taylor coefficients of algebraic functions and on the positivity of coefficients in the system (2.2.2).

Consider the system of equations (2.2.2) obtained while using the first algorithm. Let us remind that the series \( y_i = \sum_{n \geq 0} y_{i,n} z^n \) is a generating series of the set of monomials in \( \tilde{M}_0 \) and the right hand side of any equation from this system has strictly positive coefficients. Therefore, for all \( i \) the series \( y_i \) is an algebraic function with nonnegative integer coefficients bounded by the dimensions of \( \mathcal{P}(n) \). We first explain the technical result on algebraic functions with subexponential growth of coefficients and then present a proof of Corollary 2.4.1.

**Lemma 2.4.2.** Suppose that \( f(z) := \sum_{n \geq 1} f_n z^n \) is an algebraic function, such that the sequence of coefficients form a sequence of nonnegative real numbers with subexponential growth. Then there exists a rational number \( s \), integer number \( m \) and a pair of constants \( C_-, C_+ \) such that for \( n_0 \) sufficiently large

(a) the upper bound \( f_n < C_+ n^s \) is true for all \( n > n_0 \),

(b) the lower bound \( f_n > C_- n^s \) is true for at least one index \( n \) in each consecutive collection \( \{N, N + 1, \ldots, N + m - 1\} \) of \( m \) integer numbers, where \( N > n_0 \).

Notice that the exponent \( s \) in the above Lemma 2.4.2 is one less then the so-called Gelfand–Kirillov dimension

\[ \dim[f] := \lim_{n \to \infty} \frac{\ln \left( \sum_{i=0}^{n} f_i \right)}{\ln n} \]

(see [KL] for more details about GK-dimensions of algebras). Lemma 2.4.3 below implies that the integer number \( m \) is bounded from above by the number of singular points on the unit circle of the function \( f \).

**Proof.** Let us remind the following well known theorem about asymptotic of coefficients of algebraic function based on the Puiseux expansion near critical points.

**Theorem** (see [EF, Theorem D]). If \( f(z) \) is an algebraic function over \( \mathbb{Q} \) that is analytic at the origin, then its \( n \)-th Taylor coefficient \( f_n \) is asymptotically equivalent to a term of the form

\[ f_n = \beta^n n^s \sum_{i=0}^{m} C_i \omega_i^n + O(\beta^n n^t), \]

where \( s \) is a rational number, \( t < s \), \( \beta \) is a positive algebraic number and \( \omega_i \) are algebraic with \( |\omega_i| = 1 \).

It follows from the proof in [EF, Theorem D] that \( \beta \) is an inverse of the radius of convergence. Therefore in our case (where the sequence of nonnegative integer coefficients \( f_n \) has subexponential growth) \( \beta \) is equal to 1. The numbers \( \omega_i \) are equal to the singular points on the unit circle of function \( f(z) \). The upper bound (a) of Lemma 2.4.2 follows from the upper bound:

\[ \sum_{i=1}^{m} C_i |\omega_i^n| \leq \sum_{i=1}^{m} |C_i| \Rightarrow f_n \leq (\sum_{i=1}^{m} |C_i|) n^s + O(n^t) < (1 + \sum_{i=1}^{m} |C_i|) n^s \text{ for } n \gg 0. \]

The following simple sub-lemma implies the lower bound (b) in Lemma 2.4.2 and finishes its proof. \[ \square \]

**Lemma 2.4.3.** For any given collection of different complex numbers \( \omega_1, \ldots, \omega_m \) with \( |\omega_i| = 1 \) and a given collection of constants \( C_1, \ldots, C_m \) there exists a constant \( C \) such that for all \( n \) there exists an index \( k = k(n) \in \{n, n + 1, \ldots, n + m - 1\} \) such that \( \sum_{i=1}^{m} C_i |\omega_i^k| > C \).
Proof. The proof is by induction on \( m \) (the number of summands). The induction base is trivial since the absolute value \(|C_1 \omega_1^n| = |C_1| |\omega_1|^n| = |C_1|\) does not depend on \( n \).

**Induction step.** Consider an integer number \( n \). The induction hypothesis implies the existence of a constant \( C \) which does not depend on \( n \) and an integer \( k = k(n) \in \{n, n+1, \ldots, n+m-1\} \) such that

\[
\left| \sum_{i=1}^{m} \left( C_i \left( \frac{\omega_i}{\omega_{m+1}} - 1 \right) \right) \left( \frac{\omega_i}{\omega_{m+1}} \right)^k \right| > 2C.
\]

Therefore the absolute value of either the number \( \left( C_{m+1} + \sum_{i=1}^{m} C_i \left( \frac{\omega_i}{\omega_{m+1}} \right)^{k+1} \right) \) or the next element of the sequence \( \left( C_{m+1} + \sum_{i=1}^{m} C_i \left( \frac{\omega_i}{\omega_{m+1}} \right)^{k+2} \right) \) should be greater than \( C \) since their difference coincides with the left hand side of the inequality \((2.4.4)\). The obvious equality of the absolute values

\[
\left| \sum_{i=1}^{m+1} C_i \omega_i^n \right| = \left| C_{m+1} + \sum_{i=1}^{m} C_i \left( \frac{\omega_i}{\omega_{m+1}} \right)^n \right|
\]

finishes the proof of the induction step. \(\square\)

**Lemma 2.4.5.** Let \( f(z) \) and \( g(z) \) be a pair of algebraic functions such that their Taylor expansion at the origin has nonnegative coefficients with subexponential growth. Then the Gelfand–Kirillov dimension of the product \( f(z)g(z) \) is a sum of the Gelfand–Kirillov dimension of multiples:

\[
\text{Dim}[f(z)g(z)] = \text{Dim}[f(z)] + \text{Dim}[g(z)]
\]

**Proof.** Let us denote the \( n \)-th Taylor coefficient of the product \( f(z)g(z) \) by \( (fg)_n \). We have \( (fg)_n = \sum_{i+j=n} f_ig_j \). The upper bound on Gelfand–Kirillov dimension is obvious:

\[
\sum_{i=1}^{n} (fg)_i < \sum_{i+j \leq n} C_+(f)^{\text{Dim}[f]-1} C_+(g)^{\text{Dim}[g]-1} \leq \frac{n(n+1)}{2} C_+(f) C_+(g) n^{\text{Dim}[f]+\text{Dim}[g]-2} < Cn^{\text{Dim}[f]+\text{Dim}[g]},
\]

where \( C_+(f), C_+(g) \) are the constants from the upper bound in Lemma 2.4.2 for functions \( f \) and \( g \) respectively. We use a similar sequence of inequalities valid for all sufficiently large \( n \) in order to prove the lower bound:

\[
\sum_{i=1}^{n} (fg)_i = \sum_{i+j \geq n} f_ig_j \geq \sum_{i+j \leq n} f_ig_j \geq \left[ \frac{n}{4} \right]^2 C_-(f) \left[ \frac{n}{4m_f} \right]^{\text{Dim}[f]-1} C_-(g) \left[ \frac{n}{4m_g} \right]^{\text{Dim}[g]-1} \geq Cn^{\text{Dim}[f]+\text{Dim}[g]},
\]

where \( (C_-(f), m_f) \) and \( (C_-(g), m_g) \) are the constants and integer numbers from the lower bound in Lemma 2.4.2 chosen for algebraic functions \( f \) and \( g \) respectively. \(\square\)

Finally, we can prove Corollary 2.4.1

**Proof.** Let \( P \) be a non-symmetric operad with a finite Gröbner basis such that the sequence of dimensions \( \text{dim} P(n) \) has subexponential growth.

Consider a system of equations \((2.2.2)\) obtained while using the first algorithm. We know that the functions \( y_i(z) \) are algebraic with subexponential growth of coefficients. Lemma 2.4.2 implies the existence of finite nonzero GK-dimension of any infinite series \( y_i \). Let us reorder the set of unknowns according to the value of their GK-dimension. I.e. we suppose that \( y_1, \ldots, y_0 \) are polynomials that have zero GK-dimension; \( y_{l_0+1}, \ldots, y_{l_1} \) has GK-dimension \( \alpha_1 \); \( \ldots \); \( y_{l_r+1}, \ldots, y_{l_{r+1}} \) has GK-dimension \( \alpha_r \), where \( 0 < \alpha_1 < \ldots < \alpha_r \) and \( l_r = N \). For any given \( s \) and for any given \( i \in \{l_{s-1}+1, \ldots, l_s\} \) the \( i \)-th equation in System \((2.2.2)\) should be of the following form:

\[
y_i = \sum_{j=l_{s-1}+1}^{l_s} p_j(z) y_j + f_i(z, y_1, \ldots, y_{l_{s-1}}),
\]

(2.4.6)
where \( p_j(z) \) are polynomials and \( f_i \) is a polynomial in \((l_{s-1} + 1)\) variables with non-negative coefficients. Namely, \( f_i \) depends only on the first \( l_{s-1} \) unknown variables and do not depend on \( y_i \)’s with \( i > l_{s-1} \). Indeed, Lemma 2.4.5 implies that the GK-dimension of the product of functions \( y_{i_1} \ldots y_{i_k} \) should be greater than \( \alpha_s \) if there exists at least one \( i_j > l_s \) or if there exists at least two different multiples \( y_{i_j} \) and \( y_{i_j’} \) with \( i_j > l_{s-1} \) and \( i_j’ > l_0 \).

It remains to show that the solutions of the system (2.4.6) is rational. Actually, it is well known in the theory of generating functions that the solutions of a linear system of equations with polynomial coefficients are rational functions (see, e.g., [S1]). However, our system is not generally linear, so, we need an induction argument.

By the induction on \( s \), for the vector \( y_s = (y_{l_{s-1}+1}, \ldots, y_l)^T \) we get the system of the form

\[
y_s = A_s y_s + B_s,
\]

where \( A_s \in \text{Mat}_{(l_s-l_{s-1})\times(l_s-l_{s-1})}(z\mathbb{Z}[z]) \) and \( B_s \) is a vector of rational functions which are equal to zero at the origin. The vector \( B_s \) is obtained by substitution of solutions \( y_i \)’s with \( i < l_{s-1} \) which should be rational by induction hypothesis. Then

\[
y_s = (\text{Id} - A_s)^{-1} B_s,
\]

so that all infinite series \( y_i \) are rational functions. Then the function \( G_P = \sum y_i \) is also rational. \( \square \)

3. Symmetric and shuffle operads

3.1. Generating series for a shuffle composition. The first change we should do in the case of symmetric operads (compared to what we have explained for non-symmetric operads) is to change the type of generating series. Suppose that a subset \( \Phi \subset B(\Upsilon) \) defines a monomial basis of a shuffle operad \( \mathcal{P} := \mathcal{F}(\Upsilon)/\Phi \) (meaning that \( \mathcal{P} \) is the quotient of the free operad \( \mathcal{F}(\Upsilon) \) by the ideal generated by a subset \( \Phi \subset \mathcal{F}(\Upsilon) \)). The exponential generating series of the dimensions of \( \mathcal{P} \) is defined as follows:

\[
E_\mathcal{P}(z) := \sum_{n \geq 1} \dim \mathcal{P}(n) \frac{z^n}{n!} = \sum_{v \in \mathcal{M}} \frac{z^{ar(v)}}{ar(v)!}, \text{ where } ar(v) \text{ means the arity of } v.
\]

If there is an additional grading of the set of generators \( \Upsilon \) such that all relations from \( \Phi \) are homogeneous, one can also consider an exponential generating series in two variables. Let \( \mathcal{M}_n = \mathcal{M} \cap \mathcal{P}(n) \) be a homogeneous basis of \( \mathcal{P}(n) \) and let \( \mathcal{M}_{n,k} \) be the subset of \( \mathcal{M}_n \) consisting of the elements of degree \( k \). Then we define

\[
E_\mathcal{P}(z,t) := \sum_{n \geq 1} \frac{z^n}{n!} \sum_{k \in \mathbb{Z}} \#(\mathcal{M}_{n,k}) t^k = \sum_{n \geq 1} \frac{z^n}{n!} \sum_{m \in \mathcal{M}_n} t^{\deg m}.
\]

One can equivalently define

\[
E_\mathcal{P}(z,t) = \sum_{n \geq 1} \frac{H_{\mathcal{P}(n)}(t)}{n!} z^n,
\]

where \( H_{\mathcal{P}(n)}(t) \) is the Hilbert series of the graded vector space \( \mathcal{P}(n) \). As well as in the non-symmetric case (Section 2.1), we do not need the additional parameter \( t \) in the most of our examples. We provide our proofs mostly for the one-variable series meaning that they remain valid also for the two-variable ones with minimal additions.

Similar to the case of non-symmetric operads, one can define a shuffle composition of vector spaces \( \mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)_\text{Sh} \) (where \( \mu \) is an element of a free shuffle operad \( \mathcal{F} \) and \( \mathcal{P}_1, \ldots, \mathcal{P}_m \) are the graded vector subspaces of \( \mathcal{F} \)) as the vector space generated by all possible shuffle compositions

\[
(3.1.1) \quad \mu(p_1, \ldots, p_m)_\text{Sh},
\]

where \( p_i \) belongs to the graded component \( \mathcal{P}_i(k_i) \) for all \( i = 1, \ldots, m \) with \( k_1 + \cdots + k_m = n \). Similarly to the non-symmetric case, the set of underlying internally labeled trees in \( \mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)_\text{Sh} \) should have \( \mu \) as a root vertex and the \( i \)-th subtree should belong to the basis of \( \mathcal{P}_i \). The main difference with the non-symmetric operads concerns the external labeling. As it was mentioned in the definition of divisibility in Section 1.3, a possible external labelings of a tree from \( \mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)_\text{Sh} \) should preserve the local order of minima of leaves in subtrees (see the proof of Lemma 3.1.4 below).
Let $R = \mathbb{Q}[z]$ be the ring of formal power series. Define a multilinear map $C : R^n \to R$ as follows:

\[(3.1.2)\quad C(f, g)(z) := \int_0^z f'(w)g(w) \, dw \quad \text{for } n = 2\]
\[(3.1.3)\quad \text{and } C(f_1, \ldots, f_n) := C(f_1, C(f_2, \ldots, f_n)) \quad \text{for } n > 2.\]

The next Lemma establishes a connection of this operation with shuffle composition.

**Lemma 3.1.4.** Let $\mu, \mathcal{P}_1, \ldots, \mathcal{P}_m$ be as above and let $S = \mu(\mathcal{P}_1, \ldots, \mathcal{P}_m)_{Sh}$. Then

$$E_S(z) = C(E\mathcal{P}_1, \ldots, E\mathcal{P}_m).$$

**Proof.** By linearity, it is sufficient to check the above relation in the case of one-dimensional vector spaces $\mathcal{P}_1, \ldots, \mathcal{P}_m$. Assume that $\mathcal{P}_i$ is spanned by the basis element $p_i$ of arity $n_i$. Let $n = n_1 + \cdots + n_m$. Then

$$E_S(z) = z^n c(n_1, \ldots, n_m),$$

where $c = c(n_1, \ldots, n_m)$ is equal to $\dim S(n)$. For each $k = 1, \ldots, m$, denote by $N_k$ the $n_k$-element set $\{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}$. It follows from the definition (cf. [DK, Def. 2]) that the number $c(n_1, \ldots, n_m)$ is equal to the number of permutations $\sigma \in \Sigma_n$ such that $\min \sigma(N_1) < \min \sigma(N_2) < \cdots < \min \sigma(N_m)$ and the restriction of $\sigma$ to every $N_k$ is an isomorphism of ordered sets. Therefore, $c(n_1, \ldots, n_m)$ is equal to the number of decompositions $[1..n] = Q_1 \cup \cdots \cup Q_m$ with $|Q_k| = n_k$ and $\min Q_1 < \cdots < \min Q_m$ (here $Q_k = \sigma(N_k)$ for some $\sigma$ as above). The first inequality is equivalent to the condition $1 \in Q_1$, hence for every $Q_1 \supseteq 1$ (there are $\binom{n-1}{m-1}$ ways to choose it) there is exactly $c(n_2, \ldots, n_m)$ ways to get decompositions $Q_2 \cup \cdots \cup Q_m$ of the same kind for the set $[1..n] \setminus Q_1$. Thus, we have the relations

$$c(n_1, n_2) = \binom{n_1 + n_2 - 1}{n_1 - 1} \quad \text{and} \quad c(n_1, \ldots, n_m) = \binom{n - 1}{n_1 - 1} c(n_2, \ldots, n_m).$$

For the generating functions, we obtain the equalities

$$n \frac{z^{n_1 + n_2}}{n!} c(n_1, n_2) = \left( \frac{n}{n_1} \frac{z^{n_1}}{n_1!} \right) \left( \frac{z^n}{n_2!} \right),$$

$$n \frac{z^n}{n!} c(n_1, \ldots, n_m) = \left( \frac{n}{n_1} \frac{z^{n_1}}{n_1!} \right) \left( \frac{z^{n_2 + \cdots + n_m}}{(n_2 + \cdots + n_m)!} c(n_2, \ldots, n_m) \right),$$

which are equivalent to the desired integration equalities. \hfill \square

**Remark 3.1.5.** The equation in Lemma 3.1.4 is equivalent to the following system of ordinary differential equations for the functions $h_k(z) = E_{\mu(\mathcal{P}_k, \ldots, \mathcal{P}_m)_{Sh}}(z)$:

$$\begin{cases} 
  h'_1(z) = E_{\mathcal{P}_1}(z)h_2(z), \\
  h'_2(z) = E_{\mathcal{P}_2}(z)h_3(z), \\
  \vdots \\
  h'_{m-1}(z) = E_{\mathcal{P}_{m-1}}(z)E_{\mathcal{P}_m}(z)
\end{cases}$$

with the initial conditions $h_k(0) = 0$ for $0 \leq j \leq m - k$. This system uniquely determines the functions $h_1, \ldots, h_{m-1}$.

The following easy verified property of the operation $C$ will be used later in Theorem 3.3.2.

**Proposition 3.1.6.** One has $C(f, g) + C(g, f) = fg$ and, generally,

$$\sum_{\sigma \in S_n} C(f_{\sigma(1)}, \ldots, f_{\sigma(n)}) = f_1f_2\cdots f_n.$$ 

In particular, $C(f, \ldots, f) = \frac{f^n}{n!}$.

In the view of Lemma 3.1.4, this means that the sum of shuffle compositions of some vector spaces with respect to all orderings is equal to their non-symmetric composition of the same arity.
3.2. System of differential equations. So far we were not able to formulate any statement about generating series of an arbitrary shuffle operad with a finite Gröbner basis. To establish some properties of these series, we require additional assumptions the main of which is given in Definition 3.2.1 below. This assumption hold in a number of examples some of which are discussed below. We have checked also that for all known symmetric PBW operads there exists a monomial shuffle operad with the same generating series as the initial PBW operad but with the following property being satisfied (see Conjecture 4.1.2 below).

Definition 3.2.1. The planar skeleton of a shuffle monomial $m$ in a free shuffle operad $F$ is the corresponding planar internally labeled tree, that is, it is obtained from $m$ by erasing the labels (numbers) of all leaves.

- A subset $M$ of monomials in the free shuffle operad $F(Υ)$ is called shuffle regular if for each monomial $m \in M$ all monomials with the same shuffle skeleton belongs to $M$.

For example, the set $\alpha(\beta(x_1, x_2), \gamma(x_3, x_4)), \alpha(\beta(x_1, x_3), \gamma(x_2, x_4)), \alpha(\beta(x_1, x_4), \gamma(x_2, x_3))$ forms a shuffle regular subset with a shuffle skeleton $\alpha(\beta(.), \gamma(.))$.

- A monomial operad $P$ is shuffle regular iff the corresponding monomial basis is a shuffle regular subset.

It is obvious that a monomial operad is shuffle regular iff the set of generating monomial relations is shuffle regular.

- Given a set of generators $Υ$ of a symmetric or shuffle operad $P$ and an admissible ordering of monomials, the operad $P$ is called shuffle regular if the set of leading terms of the corresponding monomial ideal of relations is shuffle regular. In other words, there exists a reduced Gröbner basis of the ideal of relations of $P$ in $F(Υ)$ with shuffle regular set of leading terms.

Example 3.2.2. According to the Gröbner bases calculated in [DK], one can see that the operads Com, AntiCom and Assoc are shuffle regular [DK Examples 8,10] whereas the operads Lie and PreLie are not (with respect to given orders on shuffle monomials) [DK Examples 9,11]. On the other hand, if we change the ordering of monomials by the dual path-lexicographical ordering, the both operads Lie and PreLie become shuffle regular.

For instance, consider the shuffle operad of associative algebras Assoc. The leading terms of a Gröbner basis of the ideal of its relations are listed in [DK Example 10]. They are the shuffle monomials with the shuffle skeletons

$\alpha(\alpha(a_1, -) -), \alpha(\beta(a_1, -) -), \beta(\beta(a_1, -) -),$\

where $\alpha : \alpha(x, y) = x \cdot y$ and $\beta : \beta(x, y) = y \cdot x$ are the generator operations for Assoc. The cases of other operads listed above are analogous.

Theorem 3.2.3. Let $P$ be a shuffle regular symmetric operad such that the corresponding set of generators and a Gröbner basis of relations are finite. Then there exists an integer $N$ and a system of integral equation on $N + 1$ functions $y_0 = y_0(z, t), \ldots, y_N = y_N(z, t)$

\[(3.2.4)\]

\[y_i = t^{a_i} \sum_{s \in \{0, \ldots, N\}^{d_i}} q_s^i C(y_{s_1}, \ldots, y_{s_{d_i}}) \quad \text{for } i = 1, \ldots, N,\]

such that $E_P(z, t) = \sum_{i=0}^{N} y_i(z, t)$, $y_0 = z$ and $y_i(0, t) = y_i(z, 0) = 0$ for all $i > 0$. The numbers $q_s^i \in \{0, 1\}$ and the nonnegative integer numbers $d_i, a_i$ and $N$ are bounded from above by some functions of the degrees and the numbers of generators and relations of the operad $P$.

Our proof of Theorem 3.2.3 (as well as the proof of Theorem 3.3.2 below) is close to that of Theorem 2.2.1. The main difference is in the counting the number of external labels of a planar tree. This reduces to a simple change in the right-hand side of the formula (2.2.2):

\[y_{s_1} \cdots y_{s_{d_i}} \sim C(y_{s_1}, \ldots, y_{s_{d_i}}).\]

Namely, one should replace the product of the series by the sign of the operator $C(\ldots)$ applied to them. In order to make our exposition in symmetric case more self-contained we repeat one of the proofs-algorithms in all details.
Proof of Theorem 3.2.3. Suppose that an operad $\mathcal{P}$ has a finite set of generators $\Upsilon$ and a finite set of monomial relations $\Phi$. (It is enough to consider the monomial relations since we are dealing with generating series, therefore there is no difference between the relations that form a Gröbner basis and the monomial relations presented by the leading terms of the first ones.) Let $d$ be the maximum level of leaves of elements in $\Phi$ (by the level of a vertex/leaf in a tree we mean the number of vertices in a path from the root to this vertex/leaf). As it was mentioned in Proposition 1.3.1 every monomial $v$ in a free operad $\mathcal{F}(\Upsilon)$ generated by $\Upsilon$ may be identified with a rooted planar tree whose vertices are marked by elements of $\Upsilon$ and leaves are numbered by natural numbers $1, \ldots, ar(v)$ in such a manner that this numbering preserves the ordering of minimums in each internal vertex. Given such a monomial $v$, by its stump $b(v)$ we mean the shuffle skeleton of its maximal monomial left divisor such that the leaves and the internal vertices of $b(v)$ have levels strictly less than $d$.

Let $\text{Stump}$ be the set of all stumps of all nonzero monomials in $\mathcal{P}$. Let $N$ be the cardinality of this set. The elements $b_1, \ldots, b_N$ of $\text{Stump}$ are partially ordered by the following relation: $b_i < b_j$ iff $i \neq j$ and $b_i$ is a left divisor of $b_j$ as a rooted planar tree. Let $\mathcal{M}_{b_i}$ be the set of all monomials in (＝the monomial basis of) the right-sided ideal generated by all possible versions of the internal labeling of a stump $b_i$, and set

$$\mathcal{M}_{\tilde{b}_i} = \mathcal{M}_{b_i} \setminus \bigcup_{j : b_i < b_j} \mathcal{M}_{b_j}.$$ 

The sets $\mathcal{M}_{\tilde{b}_i}$ have empty pairwise intersections. Moreover, the set $\bigcup_{i=1}^N \mathcal{M}_{\tilde{b}_i}$ form a monomial basis of the operad $\mathcal{P}$. We have

$$E_\mathcal{P}(z) = \sum_{i=1}^N y_i(z),$$

where $y_i(z) = E_{\text{span}(\mathcal{M}_{\tilde{b}_i})}(z)$ is the exponential generating series of the span of the set $\mathcal{M}_{\tilde{b}_i}$. For every element (=operation) $\mu \in \Upsilon$ of some arity $n$, define the numbers $j_\mu(i_1, \ldots, i_n)$ for all $1 \leq i_1, \ldots, i_n \leq N$ as follows:

$$(3.2.5) \quad j_\mu(i_1, \ldots, i_n) = \begin{cases} 0, & \text{if } \mu(\mathcal{M}_{b_{i_1}}, \ldots, \mathcal{M}_{b_{i_n}})_{Sh} = 0 \text{ in } \mathcal{P} \\ j, & \text{if the stump } b(\mu(\mathcal{M}_{b_{i_1}}, \ldots, \mathcal{M}_{b_{i_n}})) = b_j. \end{cases}$$

(where the shuffle compositions for monomial sets are defined as the union of all compositions of type (3.1.1)). Note that the sets of the type $\mu(\mathcal{M}_{b_{i_1}}, \ldots, \mathcal{M}_{b_{i_n}})_{Sh}$ have vanishing pairwise intersections.

Let $v$ be a nonzero monomial in $\mathcal{P}$ with the root vertex labeled by $\mu$. Then $v \in \mu(\mathcal{M}_{b_{i_1}}, \ldots, \mathcal{M}_{b_{i_n}})_{Sh}$ for some $\mu, i_1, \ldots, i_n$, that is, $v = \mu(v_{i_1}, \ldots, v_{i_n})_{\sigma}$ where the monomials $v_{i_j} \in \mathcal{M}_{b_{i_j}}$ and a shuffle composition $\sigma$ are uniquely determined by $v$. Suppose that the index $j = j_\mu(i_1, \ldots, i_n)$ from (3.2.5) is different from zero. Then the shuffle regularity condition of a Gröbner basis and the bound on the level of relations and stumps implies that $v$ should belong to $\mathcal{M}_{b_j}$. We comes up with the following disjoint union decomposition for all $j = 1 \ldots N$

$$(3.2.6) \quad \mathcal{M}_{\tilde{b}_j} = \bigcup_{j_\mu(i_1, \ldots, i_n) = j} \mu(\mathcal{M}_{b_{i_1}}, \ldots, \mathcal{M}_{b_{i_n}})_{Sh}, \text{ where } \mu \text{ is a root vertex of each } v \in \mathcal{M}_{\tilde{b}_j}. \quad \Box$$

Then the equation (3.2.4) corresponds to the generating functions $y_i(z) = E_{\text{span}(\mathcal{M}_{\tilde{b}_j})}(z)$ (where $a_j$ is the value of the corresponding grading on the operation $\mu$). Similarly to the non-symmetric case, it follows that $\mathcal{M}_0$ consists of the identity operation and all $\mathcal{M}_{b_j}$ should contain elements of positive degrees in generators. This imply the initial conditions on series $y_i$.

Corollary 3.2.7. Let $\mathcal{P}$ be a finitely presented symmetric operad with a finite shuffle regular Gröbner basis of relations. Then there exists a system of ordinary differential equations

$$(3.2.8) \quad y_i'(z) + \sum_{j,l=1}^n q_{j,l}^i y_j y_l' = g_i(z) \text{ for } i = 1, \ldots, n$$
where \( q_{j,i}^t \in \mathbb{Q} \) and \( g_i(z) \in \mathbb{Q}[z] \) with the initial conditions \( y_1(0) = \cdots = y_n(0) = 0 \), whose unique formal power series solution \((y_1(z), \ldots, y_n(z))\) satisfies the equality
\[
E_P(z) = y_1(z) + \cdots + y_N(z)
\]
for some \( N \leq n \).

**Proof.** Let us introduce the functions \( h_i \) as in Remark 3.1.5 for all combinations of the power series \( y_k \) which appear in the equations in the statement of Theorem 3.2.3. Then the \( i \)-th equation given in the statement of Theorem 3.2.3 is equivalent to an equation of the form
\[
y'_i(z) = \sum_{p=1}^n \sum_{s \in [1..N]^p} q_{s}^i y^{s}_i(z) h_{j(s)} + f'_i(z)
\]
(which is obtained by the differentiation) with the initial condition \( y_i(0) = 0 \). After the re-naming \( y_{N+j} = h_j \) and adding the equations of the form
\[
y'_{N+j} = y_k y_i
\]
with the same initial conditions \( y_{N+j}(0) = 0 \) (cf. Remark 3.1.5), we obtain a system of equations of the desired form which is equivalent (up to the ghost variables \( y_{N+j} = h_j \)) to the initial system. \( \square \)

**Remark 3.2.9.** The number of the equations in the system of differential equations in some cases can be reduced. To do these, one can apply to the shuffle regular monomial operads the same methods as we have discussed in Subsections 2.2.2 and 2.2.3 for the non-symmetric operads. The first of these methods is illustrated in Example 3.5.6 below. We leave the detailed description of the algorithms to an interested reader.

**Corollary 3.2.10.** The exponential generating series \( E_P(z) \) of a finitely presented operad \( P \) with a shuffle regular Gröbner basis is differential algebraic over \( \mathbb{Q} \text{.}^2 \) That is, there exist a number \( n \geq 0 \) and a non-constant polynomial \( \theta \) in \( n+2 \) variables such that
\[
\theta(z, E_P(z), E'_P(z), \ldots, E^{(n)}_P(z)) = 0.
\]

**Proof.** By Artin’s Approximation Theorem for differential equations [DL Theorem 2.1], for each positive integer \( a \) there exists another power series solution \((\overline{y}_1(z), \ldots, \overline{y}_n(z))\) of the system (3.2.8) such that all functions \( \overline{y}_i \) are differentially algebraic and
\[
\overline{y}_i(z) = y_i(z) \mod z^a
\]
for all \( i = 1, \ldots, n \). Taking \( a = 1 \) and using the fact that the solution with a zero constant term is unique, we conclude that \( \overline{y}_i(z) = y_i(z) \) for all \( i \). Since the sum of differential algebraic functions is again differential algebraic, we conclude that \( E_P(z) = y_1(z) + \cdots + y_N(z) \) does satisfy a differential algebraic equation. \( \square \)

### 3.3. Relation sets with additional symmetries

In some cases there exists even more symmetries of a given Gröbner basis of a symmetric (or shuffle) operad. We present below an additional definition of **symmetric regular** operads. This condition implies even more restrictions on the set of generating series (Theorem 3.3.2 and Corollary 3.3.4 below).

**Definition 3.3.1.**
- The **tree skeleton** of a shuffle monomial \( m \) in the free shuffle operad \( F \) is the corresponding internally labeled rooted (non-planar) tree, that is, we erase the labels (numbers) of all leaves and forget the planar realization of the labeled tree \( m \).
- A subset \( M \) of monomials in the free shuffle operad \( F(V) \) is called **symmetric regular** if for each monomial \( m \in M \) all monomials with the same tree skeleton belongs to \( M \).

For example, the set
\[
\{ \alpha(\beta(x_1, x_2), \gamma(x_3, x_4), \alpha(\beta(x_1, x_3), \gamma(x_2, x_4))) \}
\]
forms a symmetric regular subset with a tree skeleton \( \alpha(\beta(\cdot, \cdot), \gamma(\cdot, \cdot)) \).

---

\(^2\)If we consider power series on two variables \( t \) and \( z \), then the coefficient ring \( \mathbb{Q} \) is replaced by the ring \( \mathbb{Q}[t] \) with the trivial differentiation \( \frac{\partial}{\partial t} = 0 \).
The definitions of symmetric regular monomial operad and arbitrary symmetric regular operad is analogous to the ones given in Definition 3.2.1 for shuffle regular case.

Obviously, the standard monomial basis of a symmetric regular operad is again symmetric regular.

**Theorem 3.3.2.** If the set of leading terms of a finite Gröbner basis of a shuffle regular operad $P$ form a symmetric regular set then the corresponding system of recursive differential algebraic equations (3.2.4) reduces to the system of algebraic equations

$$y_i = t^{a_i} \frac{1}{d_i!} \sum_{s \in [0..m]^d} q_s \cdot y_{s_1} \cdot \ldots \cdot y_{s_d}, \quad \text{for each } i = 1, \ldots, N$$

for some formal power series $y_1, \ldots, y_N$ with non-negative coefficients such that $E_P(z) = m_1y_1 + \cdots + m_ny_N$ for some integers $m_1, \ldots, m_n$.

**Proof.** Consider the algorithm given in the proof of Theorem 3.2.3 applied to a symmetric regular operad $P$. Note that the generating series of $\mathcal{M}_b$ and $\mathcal{M}_{b'}$ coincide if the stumps $b$ and $b'$ has the same tree skeleton. Moreover, for each collection of monomials $p_1, \ldots, p_n$ and each $n$-ary operation $\mu$ and a permutation $\sigma \in S_n$, there is a bijection between the tree skeletons of the elements of the set $\mu(p_1, \ldots, p_n), s_b$ and the ones of the elements of the set $\mu(p_{\sigma(1)}, \ldots, p_{\sigma(n)})$. Consider a relation from the system (3.2.4) which corresponds to a given stump $b$:

$$y_i = t^{a_i} \sum_{s \in [0..N]^d} q_s C(y_{s_1}, \ldots, y_{s_d}).$$

While changing the subtrees of the root operation in a shuffle monomial one may change the planar skeleton, whereas the tree skeleton remains the same. Therefore,

$$y_i = t^{a_i} \sum_{s \in [0..N]^d} q_s \frac{1}{d_l!} \left( \sum_{\sigma \in S_d} C(y_{s_\sigma(1)}, \ldots, y_{s_\sigma(d)}) \right) = t^{a_i} \sum_{s \in [0..N]^d} q_s y_{s_1} \cdot \ldots \cdot y_{s_d}$$

The last equality follows from Proposition 3.1.6. Thus, the system (3.2.4) of integration relations can be replaced by the system of algebraic equations. Moreover the equations are numbered by the appropriate subset of tree-skeletons. Note that these algebraic equations are numbered by the tree skeletons of the monomials whose levels are less than the maximal level of the relations.

The above statement is illustrated in Example 3.5.6.

Analogously to the case of non-symmetric operads, the classical elimination theory implies the following

**Corollary 3.3.4.** The exponential generating series $E_P$ of a symmetric regular finitely presented operad $P$ is an algebraic function.

### 3.4. Operads of restricted growth

We present here an application of the above theory to symmetric and shuffle operads of a restricted growth. We say that a sequence $(a_n)_{n \geq 0}$ of nonnegative real numbers has a subfactorial growth if for each positive constants $A, B > 0$ there exists a constant $C > 0$ such that $a_n < C A^n B^n$ for all $n$ sufficiently large. In other words, this means that the growth $[a_n]$ of this sequence is less than the growth $[n!]$. In particular, if the sequence is bounded by an exponent $C^n$ then its growth is subfactorial.

**Corollary 3.4.1.** Let $P$ be a symmetric or shuffle operad with a shuffle regular finite Gröbner basis. Suppose that the growth of the sequence of the dimensions $\dim P(n)$ is subfactorial. Then the exponential generating series $E_P$ should satisfy a linear differential equation with constant coefficient. Equivalently, the usual generating series $G_P = \sum_{n \geq 1} \dim P(n) z^n$ should be rational. In particular, the sequence of dimensions $\dim P(n)$ should have an exponential or polynomial growth with integer exponent.

The proof is similar to the one of Corollary 2.4.1. The key point is to reduce the system of equations (3.2.4) for generating series to a system of linear recursive equations on coefficients.

First, let us discuss some general conclusions of the subfactorial growth. The series $y_b = \sum_{n \geq 0} y_{b,n} z^n$ is an exponential generating series of the set of monomials in $\mathcal{M}_b$, hence the right hand side of any equation from the system (3.2.4) has nonnegative coefficients. Therefore, for each stump $b$ the coefficient $y_{b,n}$ of the exponential series $y_b$ should be a nonnegative integer bounded by the dimension $P(n)$. Each equation
is numbered by an appropriate stump. However, the proof presented below uses rather the type of
the combinatorics of stumps and the statement of Corollary 3.4.1 is still true in a much more
general case arising in some areas of combinatorics.

There is a standard combinatorial data attached to the system (3.2.4). (See, e.g., [FS, p. 33]). We say
that a stump \( b \) depends on a stump \( b' \) if the right hand side of the corresponding recursive equation
for generating series \( y_b \) contains a nonzero summand of the form \( C(y_{s_1}, \ldots, y_{s_m}) \), where some \( s_k \) is equal to
\( b' \). In other words, the right hand side of the equation (3.2.4) corresponding to \( y_b \) depends on \( y_{b'} \) in a
nontrivial way. We say that the dependence is nonlinear if the right hand side of the recursive equation
for \( y_b \) contains a multiple \( C(y_{s_1}, \ldots, y_{s_k}, \ldots, y_{s_m}) \), where \( y_{s_k} = y_b \) and at least one of the series \( y_{s_j} \) for
\( j \neq k \) is infinite.

Let us define a graph of dependencies for a system of recursive equations (3.2.4). It is a directed graph
with vertices numbered by all possible stumps. A pair of stumps \( b \) and \( b' \) is connected by an arrow if the
stump \( b \) depends on \( b' \). This graph is called the dependence graph and will be denoted by \( \Gamma(P) \).
Whereas that dependence graph do not contain all information about the system, it sometimes gives
growth conditions which we illustrate in the lemma below.

**Lemma 3.4.2.** Given an edge \( b \to b' \) in the dependence graph \( \Gamma(P) \), there exists an integer \( d \) and a
polynomial \( a(n) \) with positive integer values for all sufficiently large integer \( n \) such that the following
inequality is satisfied for the coefficients of the corresponding generating series:

\[
y_{b,n} \geq a(n)y_{b',n-d}.
\]

Moreover, if the dependence \( b \to b' \) is nonlinear one may choose a polynomial \( a(n) \) different from constant.

**Proof.** By definition (see (3.2.6) for details), there is an arrow \( b \to b' \) in the dependence graph if and only
if there exists a collection of stumps \( (b_1, \ldots, b_r) \) such that \( b' = b_j \) for some \( j \) and there is an embedding
of sets

\[
\mu(\mathcal{M}_{b_1}, \ldots, \mathcal{M}_{b_r})_{Sh} \subset \mathcal{M}_b.
\]

For each \( i \neq j \) let us choose an element \( v_i \in \mathcal{M}_{b_i} \). Let \( d_i \) be the arity of the corresponding monomial
\( v_i \in P \). Consider the subset

\[
\mathcal{M}_{b \to b'} := \left\{ \mu(w_1, \ldots, w_{j-1}, \mathcal{M}_{b'}, w_{j+1}, \ldots, w_r)_{Sh}
\text{ s.t. } w_i \text{ has the same shuffle skeleton as } v_i \text{ for all } i \neq j \right\} \subset \mu(\mathcal{M}_{b_1}, \ldots, \mathcal{M}_{b_r})_{Sh}.
\]

The number of elements of arity \( n \) in the set \( \mathcal{M}_{b \to b'} \) should not be greater than the number of elements
of the same arity in \( \mathcal{M}_b \). The detailed counting of the elements in \( \mathcal{M}_{b \to b'} \) using Lemma 3.1.4 provides
the inequality

\[
\left( \frac{n-1}{d_1-1} \right) \times \ldots \times \left( \frac{n-\sum_{i\leq j-2} d_i - 1}{d_{j-1} - 1} \right) \times \left( \frac{n-\sum_{i\leq j-1} d_i - 1}{\sum_{i>j} d_i} \right) \times \ldots \times \left( \frac{d_r - 1}{d_{r-1} - 1} \right) \times \left( \frac{d_{r-1} - 1}{y_{b,n-d}} \right) \leq y_{b,n},
\]

which shows the existence of \( d \) and \( a(n) \) as prescribed in the lemma. Here each binomial coefficient in the
left hand side is equal to number of shuffle monomials in the corresponding set \( \mathcal{M}_{b_i} \) with shuffle skeleton
\( v_i \). Thus, if the dependence \( b \to b' \) is nonlinear, then there exists a collection of monomials \( v_i \) such that the
corresponding product of binomial coefficients is different from constant. \( \square \)

**Lemma 3.4.2** has a simple corollary for operads with small growth of dimensions:

**Corollary 3.4.4.** If the growth of dimensions \( P(n) \) is subfactorial then any loop \( b_1 \to b_2 \to \ldots \to b_l \to b_1 \n\)
in the dependence graph \( \Gamma(P) \) does not contain nonlinear dependencies.

**Proof.** Consider a collection of pairs \( [(d_1, a_1(n)), \ldots, (d_l, a_l(n))] \) such that the sequence of following
inequalities holds:

\[
y_{b_1,n} \geq a_1(n)y_{b_2,n-d_1} \geq a_1(n)a_2(n-d_1)y_{b_3,n-d_2} \geq \ldots \geq \left( \prod_{j=1}^l a_j(n-\sum_{i=1}^{j-1} d_i) \right) y_{b_1,n-(d_1+\ldots+d_n)}.
\]
Suppose that there is a non-linear dependence in the given loop. Then the degree of the polynomial $a(n) := \left( \prod_{j=1}^{l} a_j (n - \sum_{i=1}^{j-1} d_i) \right)$ is positive, so that $a(n) > Cn^k$ for some $k \geq 1, C > 0$ and for all $n$ sufficiently large. Therefore, $y_{b_{1}, n} \geq Cn^{k}y_{b_{1}, n-d} \geq C^{m}n^{k}(n-d) \cdots (n-(m+1)d)^{k}y_{b_{1}, n-md}$ for each $m \leq n/d$. In particular, if the series $y_{b_{1}}$ is different from zero there should exists an infinite arithmetical progression of indices $\{n_0, n_0 + d, n_0 + 2d, \ldots \}$ such that the corresponding sequence of coefficients $\{y_{b_{1}, n_0}, y_{b_{1}, n_0 + d}, \ldots \}$ does not contain zeros. Put $m = \lfloor \frac{n}{d} \rfloor$. Then

$$y_{b_{1}, n} \geq \left( \frac{\lfloor \frac{n}{d} \rfloor}{2} \right)^{C} C^{m} \cdot \frac{n}{A}, \quad \text{if} \quad y_{b_{1}, n-\lfloor \frac{n}{d} \rfloor d} \neq 0$$

where $A = 2/C$ and $B = 2d+1$. Therefore, the only chance for $y_{b_{1}}$ to have a subfactorial growth is to have all polynomials $a_i(n)$ equal to positive constants. In particular, all dependencies should be linear. \qed

**Proof of Corollary 3.4.4.** The proof is by induction on the number of possible stumps or by the number of vertices in the dependence graph $\Gamma(P)$. The induction base easily follows from Corollary 3.4.4 since any arrow in a graph with one vertex should be a loop. Therefore, the recursive equation on a coefficients of generating series should be linear.

**Induction step.** Let $V$ be a maximal proper subset of vertices in a graph $\Gamma(P)$ such that there is no outgoing arrows to the remaining set of vertices $\bar{V}$. Let $G$ be a maximal subgraph spanned by $V$ respectively, let $\bar{G}$ be a subgraph spanned by $\bar{V}$ and all arrows between them. (We omit arrows coming from $\bar{V}$ to $V$.) Notice that the subgraph $G$ may be empty and, on the contrary, $\bar{G}$ should contain at least one vertex. Moreover, the subgraph $G$ should be a dependence graph for the subset of stumps numbered by vertices in $V$. From the induction hypothesis it follows that for each $b \in V$ the corresponding usual generating series $G_{b} = \sum_{n \geq 0} y_{b_{n}} z^{n}$ is rational. Lemma 3.4.6 given below shows that any summand of the form $C(y_{b_{1}}, \ldots, y_{b_{k}})$ where all $b_{i}$ belong to $V$ should be an exponential generating series of a sequence such that the corresponding ordinary generating function is rational. On the other hand, the maximality property of $V$ implies that each two vertices from $\bar{V}$ should be connected by a directed path. Therefore, any arrow $b \rightarrow b' \subset \bar{G}$ should belong to a loop where the remaining part of the loop is a directed path from $b'$ to $b$. Corollary 3.4.4 implies that all dependencies in this wheel should be linear. Hence, the system for the usual generating series $G_{b}$ with $b \in \bar{V}$ reduces to a system of linear equations with rational coefficients. As we have seen in the end of Section 2.4 this implies that all ordinary generating series are rational. \qed

The following Lemma is well known. We include its simple proof for completeness.

**Lemma 3.4.5.** Let $G(z) = \sum_{i \geq 1} a_{i} z^{i}$ and $E(z) = \sum_{i \geq 1} \frac{a_{i}}{i!} z^{i}$ be the ordinary and exponential generating functions of the same sequence of complex numbers $\{a_{i}\}_{i \geq 1}$. Then the function $G(z)$ is rational if and only if the function $E(z)$ satisfies a non-trivial linear differential equation with scalar coefficients.

**Proof.** The condition “$G(z)$ is rational” means that a recurrent equation

$$a_{n+k} = \sum_{j=0}^{k-1} c_{j}a_{n+j}$$

holds for all $n \geq 1$. It is equivalent to the recurrent relation

$$\frac{(n+k)!}{n!} b_{n+k} = \sum_{j=0}^{k-1} c_{j} \frac{(n+j)!}{n!} b_{n+j}$$

for the numbers $b_{n} = a_{n}/n!$. This is equivalent to the differential relation

$$E(z)^{(k)} = \sum_{j=0}^{k-1} c_{j} E(z)^{(j)}$$

for the generating function $E(z) = \sum_{n \geq 1} b_{n} z^{n}$. \qed

Let $E$ be the set of all exponential generating series such that the corresponding ordinary generating series are rational functions. In other words, $E$ is the set of all exponential generating functions which are the solutions of non-trivial linear differential equations with scalar coefficients. Now, the next Lemma is obvious.
Lemma 3.4.6. The set $E$ is closed under multiplication, differentiation and integration. In particular if $E_f, E_g \in E$ then $C(E_f, E_g) = \int_0^\infty E'_f(w)E_g(w)dw$ also belongs to $E$, that is, the corresponding ordinary generating series is also rational.

3.5. Examples for symmetric operads.

3.5.1. Examples of shuffle regular PBW operads.

Example 3.5.1. Consider the class of so-called alia algebras introduced by Dzhumadil’daev [Dzh], that is, the algebras with one binary operation (multiplication) and satisfying the identity

$\{[x_1, x_2], x_3\} + \{[x_2, x_3], x_1\} + \{[x_3, x_1], x_2\} = 0,$

where $[x_1, x_2] = x_1x_2 - x_2x_1$ and $\{x_1, x_2\} = x_1x_2 + x_2x_1$ (these algebras are also referred in [Dzh] as 1alia algebras).

Let us choose the generators $\alpha : (x_1, x_2) \mapsto [x_1, x_2]$ and $\beta : (x_1, x_2) \mapsto \{x_1, x_2\}$ for the corresponding symmetric/shuffle operad $\mathcal{A}l\mathcal{i}a$. Then the leading term of the relation corresponding to the above identity (with respect to the path-lex order with $\beta > \alpha$) is $\beta(x_1, \alpha(x_2, x_3))$, that is, the only shuffle monomial corresponding to the shuffle skeleton $\beta(-, \alpha(-, -))$.

Obviously, there is no overlapping of the leading term $\beta(x_1, \alpha(x_2, x_3))$ of the relation with itself, hence the relation is the unique element of the Gröbner basis of the relations of $\mathcal{A}l\mathcal{i}a$. Then we have the following three elements of the set of stumps (in terms of the proof of Theorem 3.2.3):

$B_0 = \text{Id}, B_1 = \alpha, B_2 = \beta.$

We get the relations

$$
\begin{cases}
  y_0 = z,
  y_1 = C(E_{\mathcal{A}l\mathcal{i}a}, E_{\mathcal{A}l\mathcal{i}a}),
  y_2 = C(E_{\mathcal{A}l\mathcal{i}a}, y_0 + y_2),
  E_{\mathcal{A}l\mathcal{i}a} = y_0 + y_1 + y_2,
\end{cases}
$$

(here we use the linearity of the operation $C$ to shorten the summation in the right hand sides). Using the linearity of $C$ and Proposition 3.4.6 we get the system

$$
\begin{cases}
  y_0 = z,
  y_1 = E_{\mathcal{A}l\mathcal{i}a}^2/2,
  y_2 = E_{\mathcal{A}l\mathcal{i}a}^2/2 - C(E_{\mathcal{A}l\mathcal{i}a}, y_1),
  E_{\mathcal{A}l\mathcal{i}a} = y_0 + y_1 + y_2,
\end{cases}
$$

which leads to the equation

$C(y, y^2/2) = z - y + y^2$

for $y = E_{\mathcal{A}l\mathcal{i}a}(z)$. After differentiation, we get the equation

$y'y^2/2 = 1 - y' + 2yy'$.

Using the initial condition $y(0) = 0$, we get the algebraic equation

$y^3/6 - y^2 + y = z$

for $y$. In particular, the function $y = E_{\mathcal{A}l\mathcal{i}a}(z)$ is algebraic.

Since the operad $\mathcal{A}l\mathcal{i}a$ has a quadratic Gröbner basis of relations, it follows from [H] (see also [DK, Cor. 3]) that the operad $\mathcal{A}l\mathcal{i}a$ is Koszul. By the Ginzburg–Kapranov relation, its exponential generating series $y$ satisfy the relation

$f(-y) = -z,$

where $f(z)$ is the exponential generating series of the quadratic dual operad $\mathcal{A}l\mathcal{i}a^!$. It follows from the equation above that

$E_{\mathcal{A}l\mathcal{i}a}!(z) = z + z^2 + z^3/6.$

This polynomial coincides with the result of a direct calculation given in the concluding remark of [Dzh].

Recall now two another classes of algebras from [Dzh]. A nonassociative algebra is called left (respectively, right) alia if it satisfies the identity

$l(x_1, x_2, x_3) = [x_1, x_2]x_3 + [x_2, x_3]x_1 + [x_3, x_1]x_2 = 0,$
or, respectively, the identity
\[ r(x_1, x_2, x_3) = x_1[x_2, x_3] + x_2[x_3, x_1] + x_3[x_1, x_2] = 0. \]
Consider the left hand side \( r(x_1, x_2, x_3) \) of the last identity. Using the above generators \( \alpha \) and \( \beta \) with the substitution \( 2x_1x_2 = \alpha(x_1, x_2) + \beta(x_1, x_2) \), we see that the leading monomial of \( r(x_1, x_2, x_3) \) with respect to the same path-lex order with \( \beta > \alpha \) is the same monomial \( \beta(x_1, \alpha(x_2, x_3)) \) as for alia algebras. By the same reasons as above, we see that the operad of right alia algebras is PBW and Koszul with the same exponential generating series as the operad \textit{Alia}. By the right-left symmetry, the same is true for left alia algebras. Thus, we get

**Proposition 3.5.2.** The three operads for alia algebras, left alia algebras and right alia algebras are Koszul with the same exponential generating series \( y = E_P(z) \) satisfying the equation
\[ y^3/6 - y^2 + y = z, \]
that is,
\[ y(z) = z + z^2 + \frac{11}{6} z^3 + \frac{25}{6} z^4 + \frac{127}{12} z^5 + \frac{259}{9} z^6 + \frac{1475}{18} z^7 + \frac{17369}{72} z^8 + \frac{943855}{1296} z^9 + \frac{2906189}{1296} z^{10} + O(z^{11}). \]
Each of their three Koszul dual operads is finite-dimensional and has exponential generating series
\[ E_P(z) = z + z^2 + z^3/6. \]

### 3.5.2. Examples of symmetric regular operads.

In the next two examples, we consider the operad of upper triangular matrices over non-associative commutative rings.

**Example 3.5.3.** Let \( R \) be a commutative non-associative ring (or a \( k \)-algebra). Then it is easy to see that the algebra \( UT_2(R) \) of upper triangular \( 2 \times 2 \) matrices over \( R \) satisfies the identity
\[ [x_1, x_2][x_3, x_4] = 0, \]
where \([a, b] = ab - ba\).

Let us denote by \( NU_2 \) the operad generated by the operation of non-symmetric multiplication \( \mu : (x_1, x_2) \mapsto x_1x_2 \) (i. e., the arity two component \( NU_2 \) is spanned by \( \mu \) and \( \mu' : (x_1, x_2) \mapsto x_2x_1 \)) subject to this identity. Consider the corresponding shuffle operad \( NU_2 \) generated by two binary generators, namely, the operations \( \mu \) and \( \alpha : (x_1, x_2) \mapsto [x_1, x_2] \). Then the above identity is equivalent to the pair of shuffle regular monomial identities
\[ f_1 = \mu(\alpha(-,-), \alpha(-,-)) = 0 \quad \text{and} \quad f_2 = \alpha(\alpha(-,-), \alpha(-,-)) = 0. \]
Therefore, the ideal of relations of the shuffle operad \( NU_2 \) is generated by the following six shuffle monomials obtained from \( f_1 \) and \( f_2 \) by substituting all shuffle compositions of four variables (which we denote for simplicity by 1,2,3,4):
\[ m_1 = \mu(\alpha(1,2), \alpha(3,4)), \quad m_2 = \mu(\alpha(1,3), \alpha(2,4)), \quad m_3 = \mu(\alpha(1,4), \alpha(2,3)), \quad m_4 = \alpha(\alpha(1,2), \alpha(3,4)), \quad m_5 = \alpha(\alpha(1,3), \alpha(2,4)), \quad m_6 = \alpha(\alpha(1,4), \alpha(2,3)). \]
Let us describe the set \( B \) of all stumps of all nonzero monomials in \( NU_2 \). Since the relations have their leaves at level 2, \( B \) includes all monomials of level at most one, that is, the monomials
\[ B_0 = Id, B_1 = \mu(-,-), B_2 = \alpha(-,-). \]
For the corresponding generating series \( y_i = y_i(z) \) with \( i = 0, 1, 2 \) we have
\[
\begin{cases}
  y_0 = z, \\
  y_1 = C(y_0, y_0) + C(y_1, z) + C(z, y_1) + C(y_2, z) + C(z, y_2) + C(y_1, y_1) + C(y_1, y_2) + C(y_2, y_1), \\
  y_2 = C(y_0, y_0) + C(y_1, z) + C(z, y_1) + C(y_2, z) + C(z, y_2) + C(y_1, y_1) + C(y_1, y_2) + C(y_2, y_1).
\end{cases}
\]
We see that \( y_1(z) = y_2(z) \) and \( E_{NU_2}(z) = y(z) = y_0(z) + y_1(z) + y_2(z) = z + 2y_1(z). \) The second equation of the above system gives, after differentiation, the ordinary differential equation (ODE)
\[ y'_1 = z + 2zy_1 + 2zy'_1 + 3y_1y'_1 \]
with the initial condition \( y_1(0) = 0 \), which is equivalent to the ODE
\[ (y'(z) - 1)(2 - 3y(z)) = 4y(z) \]
on \( y(z) \), again with the initial condition \( y(0) = 0 \). It follows that
\[
E_{\mathcal{NU}_4}(z) = y(z) = \frac{1}{3} \left( 2 - z - 2\sqrt{1 - 4z + z^2} \right)
= z + z^2 + 2z^3 + \frac{19}{4}z^4 + \frac{25}{6}z^5 + \frac{281}{8}z^6 + \frac{413}{4}z^7 + \frac{20071}{64}z^8 + \frac{31249}{32}z^9 + \frac{396887}{128}z^{10} + o(z^{10})
\]

Let us generalize Example 4.5.3 to the case of matrices of order \( n \). The following description of identities easily follows from the fact that the diagonal elements of the commutator of two upper triangular matrices are zero.

**Lemma 3.5.4.** Let \( U_n(R) \) be the algebra of upper triangular matrices of order \( n \) over a (non-associative) commutative ring \( R \). Then for each \( n \)-ary multiple composition function \( f \) of multiplications of matrices, the identity of \( 2n \) arguments

\[
f([x_1, x_2], \ldots, [x_{n-1}, x_n]) = 0,
\]

holds in \( U_n(R) \), where, as usually, \([a, b]\) stands for \( ab - ba\).

For example, for \( n = 2 \) we get the only identity \([x_1, x_2] [x_3, x_4]\) as above. For \( n = 3 \) we have the identities
\[
f_i([x_1, x_2], [x_3, x_4], [x_5, x_6]) \text{ with } i = 1, 2,
\]
where \( f_1(a_1, a_2, a_3) = (a_1 a_2) a_3 \) and \( f_2(a_1, a_2, a_3) = a_1 (a_2 a_3) \) (all other identities are obtained from these two by permutations of variables).

Consider the operad \( \mathcal{NU}_n \) of upper triangular matrices generated by the non-symmetric operation of multiplication \( \mu : (x_1, x_2) \mapsto x_1 x_2 \) subject to all these identities. Consider a natural generators of the corresponding shuffle operad \( \alpha : (x_1, x_2) \mapsto [x_1, x_2] \) and \( \beta : (x_1, x_2) \mapsto x_1 x_2 + x_2 x_1 \). We immediately see that the set of shuffle relations of this operad is spanned by monomials and is symmetric regular.

**Corollary 3.5.5.** For all \( n \geq 2 \) the exponential generating series \( E_{\mathcal{NU}_n} \) is algebraic.

**Example 3.5.6.** Let us find a relation for the exponential generating series for the operad \( \mathcal{P} = \mathcal{NU}_3 \). To do this, we use the appropriate version of the method used in Subsection 2.2.2.

Subject to the binary generators \( \alpha, \beta \) the minimal sets of the monomial relations of this operad consists of the monomials with one of the following 4 tree skeletons:

\[
\xi(\zeta(\alpha(-,-), \alpha(-,-), \alpha(-,-)), \text{ where } \xi, \zeta \in \{\alpha, \beta\}.
\]

Then the set \( T(\mathcal{P}) \) from Lemma 2.2.7 consists of the following five tree skeletons:

\[
I (\text{the identical operation}), a := \alpha(-,-), b := \beta(-,-), A := \alpha(\alpha(-,-), \alpha(-,-)), B := \beta(\alpha(-,-), \alpha(-,-)).
\]

We get the following equations for the corresponding generating functions \( a, b, y_A, y_B, \) and \( y_I = E_{\mathcal{P}} \):

\[
\begin{align*}
&\begin{cases}
y_I = z + y_a + y_b, \\
y_a = y_b = \frac{1}{2} y_I - \frac{1}{2} (y_I y_a + y_b y_a + y_A y_A + y_A y_B) + \frac{1}{2} y_A^2, \\
y_A = y_B = \frac{1}{2} y_a - \frac{1}{2} (y_A y_a + y_A y_B) + \frac{1}{2} y_A^2,
\end{cases}
&\Leftrightarrow \begin{cases}
y_I = z + 2y_a, \\
y_a = \frac{1}{2} y_I - y_A y_a + \frac{1}{2} y_A^2, \\
y_A = \frac{1}{2} y_a - y_A y_a + \frac{1}{2} y_A^2.
\end{cases}
\end{align*}
\]

After elimination of the variables \( y_A \) and \( y_a \), we obtain the following algebraic equation for \( y_I = E_{\mathcal{NU}_3} \):

\[
y_I^4 + (12z - 24)y_I^3 + (30z^2 + 8z + 80)y_I^2 + (-36z^3 + 24z^2 - 32z - 64)y_I + 9z^4 - 8z^3 + 16z^2 + 64z = 0.
\]

It follows that

\[
E_{\mathcal{NU}_3} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \frac{167}{4}z^6 + 130z^7 + \frac{26745}{64}z^8 + \frac{44045}{32}z^9 + \frac{36969}{8}z^{10} + O(z^{11}).
\]

3.5.3. **Examples of computations via homology.**

**Example 3.5.7.** Let us define a class of operads where all relations are relations on the commutators.

Namely, any given finite set of operations \( \Upsilon \) and finite set of linear independent elements \( \Phi \subset \mathcal{F}(\Upsilon) \) defines an operad

\[
\mathcal{Q}_\Phi := \mathcal{F}(\Upsilon \cup \{-, -\}) / \left( [[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [x_3, x_1], x_2] = 0 \right)
\]

In other words, the operad \( \mathcal{Q}_\Phi \) is generated by the union of the set \( \Upsilon \) and a Lie bracket \([\cdot, \cdot]\). The first identity is the Jacobi identity for the Lie bracket. The set of additional identities in the lower line consists of substitutions of the Lie bracket into the arguments of the operations in \( \Phi \). (I. e., we put the bracket \([\cdot, \cdot]\) in each leaf of \( g \)).

**Proposition 3.5.8.** The above set of relations of the operad \( \mathcal{Q}_\Phi \) form a shuffle regular Gröbner basis with respect to the dual path-lexicographical ordering. (The ordering of generators does not matter.)
Proposition 3.5.9. There exists a resolution of $Q_Φ$ generated by the union of the following sets:

- the set of generators $Υ$ with homological degree zero,
- the set of right-normalized commutators $L := \{[x_1,[x_2,\ldots,[x_{k-1},x_k]\ldots]] | k \geq 2\}$, (The homological degree of the commutator on $k$-letters is equal to $(k-2)$.)
- the union of sets $\{g(L,\ldots,L)_{Sh}\}$ for all $g \in Φ$ (The homological degree of $g(l_1,\ldots,l_k)_{Sh}$ is equal to $\sum \deg(l_i) + 1$.)

The generating series of the right normalized commutators $L$ is $(e^{-z} + z - 1)$. Multiplying the generating series of each of the above sets by $(-1)^h$, where $h$ is the homological degree, we get the equation

\[ E^{-1}_{Q_Φ}(z) = 1 - e^{-z} - E_Υ(z) - E_Φ(e^{-z} + z - 1) \]

for the functional inverse $E^{-1}_{Q_Φ}$ of the exponential generating series of the operad $Q_Φ$.

Example 3.5.11. Consider a particular example of the above construction consisting of the class of Lie-admissible non-associative algebras. Let $P$ be an operad generated by one binary non-symmetric operation (multiplication) subject to the following identities:

\[ [a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0, \]
\[ [a,b][c,d] + [c,d][a,b] = 0 \]

where, as usually, $[a,b] = ab - ba$. The operad $P$ satisfies the conditions of Proposition 3.5.8 with $Υ = Φ = \{β : (x_1,x_2) \mapsto x_1x_2 + x_2x_1\}$. It follows that $E_Υ(z) = E_Φ(z) = z^2/2$. The identity (3.5.10) gives the following equation on the generating series $E_P(z)$ after appropriate substitutions:

\[ E_P^{-1}(z) = 1 - e^{-z} - \frac{z^2}{2} - \frac{(e^{-z} + z - 1)^2}{2} \Leftrightarrow 3 - 2E_P + 2E_P e^{-EP} + e^{-2EP} - 4e^{-EP} = 2z. \]

It follows that

\[ E_P(z) = z + z^2 + \frac{11}{6}z^3 + \frac{49}{12}z^4 + \frac{1219}{120}z^5 + \frac{811}{30}z^6 + \frac{75919}{1008}z^7 + \frac{97175}{448}z^8 + \frac{25827439}{40320}z^9 + \frac{116679221}{60480}z^{10} + O(z^{11}). \]

4. Remarks and conjectures

4.1. Operads with quadratic Gröbner bases. Let us formulate a couple of conjectures about PBW operads (i.e. the operads which admits a quadratic Gröbner basis).

Conjecture 4.1.1. The exponential generating series of a symmetric PBW operad with binary generators is differential algebraic.

This conjecture is motivated by the following stronger conjecture that has been checked in all known examples.
Conjecture 4.1.2. Let \( P \) be a symmetric operad generated by binary operations and with quadratic relations that form a Gröbner basis according to one of the admissible orderings of shuffle monomials. Then there exists a monomial shuffle regular operad \( Q \) with the same dimensions of operations \( (\dim P(n) = \dim Q(n)) \) for all \( n \).

Why this conjecture might be true? The most important assumption is that the operad \( P \) is symmetric. Therefore, one should use the representation theory of the permutation groups \( \Sigma_n \). First, using the upper-triangular change of basis one can choose the set of generators \( \Upsilon \) of an operad \( \mathcal{P} \) such that the transposition element \((12) \in \Sigma_2\) preserves this set. This means that \( \forall \alpha \in \Upsilon \) \((12) \cdot \alpha = (-1)^{\ell} \alpha'\) where \( \alpha' \in \Upsilon \). Suppose, for simplicity, that \((12) \cdot \alpha = \pm \alpha\) for all \( \alpha \in \Upsilon \). For any given pair of generators \( \alpha, \alpha' \) the set of all operations of the form \( \alpha \oplus \alpha' \) form a 3-dimensional representation of \( \Sigma_3 \) which is the induction from a one-dimensional \( \Sigma_2 \)-representation. Therefore, it decomposes into the sum of one-dimensional and 2-dimensional irreducible representations. Relations for the 2-dimensional representation looks as follows

\[
\alpha(\alpha'(x_1, x_2), x_3) + \alpha(\alpha'(x_2, x_3), x_1) + \alpha(\alpha'(x_3, x_1), x_2)
\]

Relations for the one-dimensional representation looks as follows:

\[
\pm \alpha(\alpha'(x_1, x_2), x_3) = \pm \alpha(\alpha'(x_2, x_3), x_1) = \pm \alpha(\alpha'(x_3, x_1), x_2)
\]

On the other hand, there are two shuffle regular sets of quadratic monomials:

\[
\{\alpha(\alpha'(x_1, x_2), x_3), \alpha(\alpha'(x_1, x_3), x_2)\} \text{ and } \{\alpha(x_1, \alpha'(x_2, x_3))\}
\]

The hope is that the monomial operad \( Q \) can be constructed by taking the leading term \( \{\alpha(x_1, \alpha'(x_2, x_3))\} \)

of the relations that nontrivially projects on corresponding one-dimensional \( \Sigma_3 \)-irreducible subspace and the set \( \{\alpha(\alpha'(x_1, x_2), x_3), \alpha(\alpha'(x_1, x_3), x_2)\} \) for those relations which projects on the two-dimensional \( \Sigma_3 \)-irreducible subspace.

4.2. Varieties of associative algebras. The symmetric quotient operads of the operad \( \mathcal{A}ssoc \) (i.e., the operads corresponding to the varieties of associative algebras with polynomial identities; called here associative operads) form, probably, the widest extensively classified and studied class of operads. Note that the traditional terminology for these operads differs from the one we use. Particularly, the dimensions \( \dim P_n \) of an associative operad \( \mathcal{P} \) are usually referred to as codimensions of the corresponding varieties of algebras with polynomial identities; the exponential and the usual generating series of the operad are referred as the codimension series and the exponential codimension series of the corresponding variety. (See, e.g., [GZ].)

We do not know whether each symmetric quotient operad of \( \mathcal{A}ssoc \) has a finite Gröbner basis. On the other hand, one can construct examples of shuffle monomial quotients of \( \mathcal{A}ssoc \) which are infinitely presented, that is, which have no finite Gröbner basis, at least with the standard choice of generators. However, the following properties of generating series of associative operads are close to the properties of our operads with finite shuffle regular Gröbner bases.

Firstly, in many cases the dimensions \( \dim P_n \) for such an operad \( \mathcal{P} \) can be calculated explicitly (see [U, GZ, Dr, BD]). In such cases, these dimensions are presented as linear combinations of integer exponents of \( n \) with polynomial coefficients. In particular, the corresponding exponential generating functions are differential algebraic, that is, they satisfy the conclusion of our Corollary 3.2.10.

Secondly, the only non-symmetric associative operads are the finite-dimensional ones defined by the identity \( x_1x_2\ldots x_n = 0 \) (operads of nilpotent associative algebras); obviously, the generating series of such operads do satisfy the conclusions of our Theorem 2.3.1 and Corollary 2.4.1. Thirdly, the sequence of dimensions of an associative operad has either polynomial or exponential growth (by the results of Regev and Kemer, see [GZ, Theorems 4.2.4 and 7.2.2]). This fact is similar to our Corollary 3.4.1.

After these observations, we think that our Corollary 3.2.10 (which suggests Expectation 1) holds in a more general situation, that is, for a wider class of symmetric operads containing both operads with a finite shuffle regular Gröbner basis and associative operads.

Conjecture 4.2.1. The exponential generating series of any associative operad is differential algebraic.

Note that the two languages of universal algebra (based on varieties vs the operads) are equivalent provided that the basic field \( k \) has zero characteristic. In this case, our Conjecture 4.2.1 is equivalent to the following: the exponential codimension series of each variety of PI algebras is differential algebraic.
A stronger version of Conjecture 4.2.1 holds for associative operads of polynomial growth, that is, their ordinary generating series are rational \([BD]\) Ex. 13. However, for some natural associative operads (e.g., for the operad defined by the identities of \(3 \times 3\) matrices) the ordinary generating series are not rational \([BR]\). This means that the conclusion of our Corollary 3.3.1 does not hold for general associative operad. In particular, this means that some associative operads have no finite shuffle regular Gröbner basis (whereas they are always finitely presented, by the Kemer’s famous solution of the Specht problem \([Kd]\)). This means also that some associative operads are not generic in the sense of our Expectation \([3]\).

### 4.3. Detailed description of generating series

It would be interesting to find a closer class of formal power series than the class of differential algebraic ones which include all (exponential) generating series of generic symmetric operads. For the operads with a finite shuffle regular Gröbner basis, we propose the following approach.

Let us define a binary operation on \(Q[[z]]\) by \(f \ast g := C(f, g)\), where \(C(f, g)(z) = \int_0^z f'(w)g(w)\, dw\) is defined in (3.1.2). Recall that the multiple operation \(C\) from equation (3.1.3) is also defined via this binary operation: \(C(f_1, \ldots, f_n) = (f_1 \ast (f_2 \ast \ldots (f_{n-1} \ast f_n) \ldots))\). The algebra \(Q[[z]]\) with respect to the operation \(f \ast g\) satisfy the identity

\[(a \ast b) \ast c = a \ast (b \ast c + c \ast b),\]

that is, this is a Zinbiel algebra (also known as dual Leibniz algebra, or chronological algebra)\(^3\). Then one can consider the main system (3.2.4) as a set of algebraic equations over the Zinbiel subalgebra \(Q[z] \subset Q[[z]]\).

In this connection, the following questions arise.

1. Does it follows that the exponential series function \(E_P(z)\) satisfy a single algebraic equation over \(Q[z]\) with respect to the operation \(\ast'\), that is, does it follows that \(E_P(z)\) is Zinbiel algebraic over \((Q[z], \ast)\)?
2. What are the conditions for the (non-negative rational) coefficients for a formal power series to be Zinbiel algebraic (or for a collection \(y_0(z), \ldots, y_N(z)\) of such formal power series to satisfy a system of Zinbiel algebraic equations)?

The answer to the second question could give a detailed description of generating series of, at least, operads with shuffle regular Gröbner basis.

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