QUASI-ALTERNATING LINKS AND ODD HOMOLOGY:
COMPUTATIONS AND CONJECTURES

SLAVIK JABLAN AND RADMILA SAZDANOVIĆ

Abstract. We present computational results about quasi-alternating knots and links and odd homology obtained by looking at link families in the Conway notation. More precisely, we list quasi-alternating links up to 12 crossings and the first examples of quasi-alternating knots and links with at least two different minimal diagrams, where one is quasi-alternating and the other is not. We provide examples of knots and links with \( n \leq 12 \) crossings which are homologically thin and have no minimal quasi-alternating diagrams. These links are candidates for homologically thin links that are not quasi-alternating. For one of our candidates [JaSa1], knot 11n$_{50}$, J. Greene proved that it is not quasi-alternating, so this is the first example of homologically thin knot which is not quasi-alternating [Gr]. Computations were performed by A. Shumakovitch’s program \textit{KhoHo}, the program \textit{Knotscape}, and our program \textit{LinKnot}. 
1. Introduction

In this paper we present computational results addressing the classification of quasi-alternating (short QA) links based on the thickness of Khovanov homology $\overline{Kh}$ [Kh1] and odd Khovanov homology $\overline{Kh}$ [OzRaSz] and analyzing their minimal diagrams.

Our motivation stems from the results obtained from the classical Khovanov homology $Kh$. D. Bar-Natan was the first to notice [BN] that the vast majority of prime knots up to 10 crossings (238 among 250) are $Kh$-thin. In particular, all alternating links are $Kh$-thin [Le] and all adequate non-alternating knots are $\overline{Kh}$-thick [Kh2]. The complete list of $\overline{Kh}$-thick knots up to 13 crossings is computed by A. Shumakovitch [Sh1].

Moreover, C. Manolescu and P. Ozsváth show that both Khovanov homology $Kh$ and Heegaard-Floer homology $\hat{HF}_K$ can be used to detect links which are not QA. Quasi-alternating links are Khovanov homologically $\sigma$-thin (over $\mathbb{Z}$) and Floer homologically $\sigma$-thin (over $\mathbb{Z}/2\mathbb{Z}$) [MaOz]. The same property extends to odd Khovanov homology $Kh'$ [OzRaSz] and we will use this property in the rest of the paper. A knot or link is called homologically thin (without qualification), if it is simultaneously thin with respect to $Kh$, $\hat{HF}_K$, and $Kh'$ [Gr, Def. 1.2].

Definition 1.1. The set $Q$ of quasi-alternating links is the smallest set of links such that

- the unknot is in $Q$;
- if the link $L$ has a diagram $D$ with a crossing $c$ such that
  1. both smoothings of $c$, $L_0$ and $L_\infty$, are in $Q$;
  2. $\det(L) = \det(L_0) + \det(L_\infty)$

then $L$ is in $Q$. We say that a crossing $c$ satisfying the properties above is a quasi-alternating crossing of the diagram $D$ or that $D$ is quasi-alternating at the crossing $c$ [OzSz ChKo1].

The recursive definition makes it difficult to determine if a knot is quasi-alternating. It is a challenge to find candidates for homologically thin knots that are not QA. For a long time, knots $9_{46} = 3, 3, -3$ and $10_{140} = 4, 3, -3$ have been the main candidates. However, according to A. Shumakovitch’s computations [Sh2] they are not QA, since are $\overline{Kh}$-thick, although they are both $\hat{HF}_K$ and $\overline{Kh}$-thin.

According to Theorem 1 [ChKo1], quasi-alternating links with a higher number of crossings can be obtained as extension of links which are already recognized as quasi-alternating [ChKo1 Wi].

Consider the crossing $c$ in Definition 1 as 2-tangle with marked endpoints. Using Conway’s notation for rational tangles, let $\varepsilon(c) = \pm 1$, according to whether the overstrand has positive or negative slope. We will say that a rational 2-tangle $\tau = C(a_1, \ldots, a_m)$ extends $c$ if $\tau$ contains $c$ and $\varepsilon(c) \cdot a_i \geq 1$ for $i = 1, \ldots, m$. In particular, $\tau$ is an alternating rational tangle.

Theorem 1.2. If $L$ is a quasi-alternating link, let $L'$ be obtained by replacing any quasi-alternating crossing $c$ with an alternating rational tangle that extends $c$. Then $L'$ is quasi-alternating [ChKo1].

In this paper we give new computational results for QA links up to 12 crossings and the examples of QA links with at least two different minimal diagrams, where one is QA and the other is not. We provide examples of knots and links (short $KL$s) with $n \leq 12$ crossings which are homologically thin and have no minimal quasi-alternating diagrams. In the first version of this paper we proposed these $KL$s as the candidates for prime homologically thin links that are not QA, and J. Greene proved that knot $11n_{50}$ is not QA [Gr]. Using the method described in his paper it can be shown that the link $L11n_{990}$ is not QA, although it is homologically thin. The remaining candidates may require additional ideas to prove that
they are non-QA, if indeed this is the case.

Knots and links are given in Conway notation [Con, JaSa], which is implemented in Mathematica package LinKnot used for deriving families of KLs and their distinct diagrams. Odd Khovanov homology $\mathcal{Kh}$ is computed using KhoHo by A. Shumakovitch [Sh1]. All flype-equivalent minimal diagrams of non-alternating KLs up to $n \leq 12$ crossings are derived from alternating link diagrams, using software LinKnot. KnotFind, the part of the program Knotscape [HosThi], is used for recognition of knots, and Jones and Kauffman polynomials are used for distinguishing links. In addition, we used the criterion that homologically thick knots are not QA and that for a quasi-alternating crossing both smoothings must be homologically thin knots or links.

2. QUASI-ALTERNATING KNOTS UP TO 12 CROSSINGS

Since an homologically thick knot cannot be QA, we first selected knots which are homologically thin. Table 1 gives an overview of the numbers of non-alternating knots with $8 \leq n \leq 12$ crossings and how many among them are homologically thin:

| No. of crossings | 8  | 9  | 10 | 11 | 12 |
|------------------|----|----|----|----|----|
| No. of non-alternating knots | 3  | 8  | 42 | 185 | 888 |
| No. of homologically thin non-alternating knots | 2  | 6  | 31 | 142 | 663 |

Table 1.

Among 11-crossing knots, the following six knots are $\mathcal{Kh}$-homologically thin [Sh1] and have a minimal diagram which is not QA:

| Knot | Conway  |
|------|---------|
| $9_{46}$ | 3,3,−3 |
| $11n_{139}$ | 5,3,−3 |
| $10_{140}$ | 4,3,−3 |
| $11n_{107}$ | −212,3,3 |
| $11n_{65}$ | (3,−21)(21,2) |
| $11n_{65}$ | (3,−21)(21,2) |

Table 2.

Even columns in Table 2 contain the Conway symbols of these knots. Four of them, $9_{46}$, $10_{140}$, $11n_{139}$, and $11n_{107}$ are $\mathcal{Kh}$-thick [Sh2], so they are not QA.

The knot $11n_{65}$ has two minimal diagrams: $(3,−21)(21,2)$ (Fig. 1a) and $6^*2.21.−20.−1.−2$ (Fig. 1b). The first diagram is not QA, and the second is QA. By smoothing at the crossing $c_1$, the second diagram resolves into unknot and QA link $(2,2+)−(21,2)$, which resolves into QA knots $3,21,−2$ and $211,21,−2$ by smoothing the crossing $c_1$ (Fig. 1b). Moreover, all minimal KL diagrams of the family derived from knot $11n_{65}$, $(3,−21)(p,1,2)$ and $6^*2,p1.−20.−1.−2$ ($p \geq 2$) which represent the same KL have this property: the first is not QA, and the other is QA.

The remaining Montesinos knot

$$11n_{50} = −22,22,3 = M(0;(5,−2),(5,2),(3,1)) = M(1;(5,3),(5,2),(3,1))$$

(Fig 2) is homologically thin, and with no minimal quasi-alternating diagrams. In the first version of this paper we proposed this knot as the smallest candidate for an homologically
thin knot which is not quasi-alternating. J. Greene [Gr] proved that \(11n_{50}\) is not QA, so it is the first example of homologically thin knot which is not QA.

12-crossing knots \(12n_{196} = (−31, 3)(21, 2)\), \(12n_{393} = 8*2.20 : −210\) and \(12n_{397} = 211 : −210 : 20\) have another minimal diagram which is QA, and the knots given in Table 3 are candidates for homologically thin non-QA knots.

Table 3. Candidates for 12-crossing homologically thin non-QA knots.

| \(12n\) | Minimal Diagram | \(12n\) | Minimal Diagram |
|--------|----------------|--------|----------------|
| 12n_{139} | \(2.(-21, 2), 2\) | 12n_{145} | \(-22, 22, 4\) |
| 12n_{296} | \(-22, 21, 2, 3\) | 12n_{331} | \(3, 2 +) − (21, 3)\) |
| 12n_{708} | \(2 : −310 : 30\) | 12n_{838} | \(-2, −2, −20.2.2.20\) |

Among them, the only Montesinos knot is \(12n_{145} = −22, 22, 4 = M(0; (5, −2), (5, 2), (4, 1)) = M(1; (5, 3), (5, 2), (4, 1))\).

Kanenobu knots \(K(n, 3 − n)\) are knots of the form \(2.2.−p0.−2.2−q0\), where \(|p−q| = 3\), \(n ≥ 0\). J. Greene proved that except knot \(11n_{132} = K(1, 2) = K(2, 1)\) all Kanenobu knots \(K(n, 3 − n)\) are homologically thin non-QA knots. The first member of this family is \(11n_{50} = K(0, 3)\), and the next members are knots \(12n_{414} = −210.3.2.20\), \(15n_{54616}\), etc.

All other 141 \(Kh\)-thin knots with \(n = 11\) crossings and 656 \(Kh\)-thin knots with \(n = 12\) crossings crossings are QA.

As the additional candidates for non-QA homologically thin knots, we propose the following 13-crossing Montesinos knots:
\(13n_{1408} = −32, 22, 22 = M(0; (7, −3), (5, 2), (5, 2)) = M(1; (7, 4), (5, 2), (5, 2))\), \(13n_{2006} = −32, 32, 3 = M(0; (7, −3), (7, 3), (3, 1)) = M(1; (7, 4), (7, 3), (3, 1))\), and \(13n_{3142} = −22, 312, 3 = M(0; (5, −2), (11, 4), (3, 1)) = M(1; (5, 3), (11, 4), (3, 1))\). For 13-crossing knots we did not check their other minimal diagrams.
3. Quasi-alternating links up to 12 crossings

We provide examples of QA links with two distinct minimal diagrams, where the first is not QA, and the other is: \((3, -21) (2, 2) = .2 - 30.2, (-21, 4) (2, 2) = 6^* - 3 - 2.20 : 2 - 1, \)

e tc.

Among 11-crossing links there are five candidates for homologically thin non QA links:
\[L_{11n77} = 6^* 2. (2, -2) : 2. 0, L_{11n226} = 6^* - 2. 2. - 2 : 2, (2, 2) = 6^* - 2. - 2. 2 : 2, 6^* 2, (2, 2) = 6^* - 2 - 2. \]

J. Greene [Gr] proved that link \(L_{11n90} = (2, 2+) - (21, 3)\) is homologically-thin non QA link.

Moreover, the following families of homologically thin links are potentially infinite families of non QA homologically thin \(KLs: (p, 2+) - (21, 3) (p \geq 2), 6^* p. (2, -2) : 20 (p \geq 2).\)

Members of the first family (beginning with the 11-crossing link \((2, 2+) - (21, 3)\) and knot \(12n331 = (3, 2+) - (21, 3)\) are links for even \(p\) and knots for odd \(p\), and the members of the other family are 2-component links.

4. Families of odd-homology thick links

According to M. Khovanov [Kh2], there is no doubt that for large \(n\), most \(n\)-crossing links are \(\overline{Kh}\)-thick. In the case of odd Khovanov homology \(\overline{Kh}\), preliminary computational results show that some families of thin links can become thick. Typical example is the family of \(\overline{Kh}\)-thin knots \(-22, 22, p (2 \leq p \leq 5)\), giving \(\overline{Kh}\)-thick knots with torsion of order 5 for \(p \geq 6\).

According to the computational results, most of the \(\overline{Kh}\)-thick links are members of the families of \(\overline{Kh}'\)-thick links obtained for some lower value of \(n\). Table 4 contains the number of \(\overline{Kh}'\)-thick links for \(6 \leq n \leq 10\), and the number of links which generate new families of \(\overline{Kh}'\)-thick links:

| No. of crossings | 6 | 7 | 8 | 9 | 10 |
|------------------|---|---|---|---|----|
| No. of \(\overline{Kh}'\)-thick KLs | 1 | 1 | 6 | 15 | 61 |
| No. of links which generate new families of \(\overline{Kh}'\)-thick links | 1 | 0 | 4 | 6 | 28 |

Table 4.

Recall that families of \(KLs\) given in Conway notation \([\text{Con}, \text{Cau}, \text{JaSa}]\) are determined by their generating links whose Conway symbol contains integer parameters \(p, q, r, \ldots\) greater or equal to 2. There is only one family \(p, q, r--\) of odd homology thick links beginning for \(n = 6\) crossings. Tables 5, 6, 7 contain families of \(\overline{Kh}\)-thick knots beginning from \(n = 8, 9, 10\) crossings, together with additional conditions on parameters insuring that the family contains only \(\overline{Kh}'\)-thick links.

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1 Odd homology was computed for these families up to 16 crossings.
2 Conditions are based on the computational results for \(KLs\) with at most 16 crossing in each of the families.
We introduce the idea of Jp-special KLs, inspired by the work of M. Khovanov, as analogy with Ap-special knots [Kh2].

The Jones polynomial $J(L) = \sum c_i q^{2i}$ of a link $L$ is alternating, if $c_i c_j > 0$ means that $j = i \mod 2$ and if $c_i c_j < 0$, then $j \neq i \mod 2$. The Jones polynomial has no gaps if $c_i \neq 0$, $c_{i+k} \neq 0$ implies $c_{i+m} \neq 0$ for all $m$ between 1 and $k - 1$. A link is called Jp-special if its Jones polynomial is either non-alternating or has gaps.

Adequacy of links is determined using program LinKnot. Among all above-listed families of $\overline{Kh}'$-thick links, families that consist exclusively of Jp-special links are the families of adequate links given in Table 8, and three families of semi-adequate Jp-special links:

- $p, q, r$–;
- $p : - q 0 : - r 0$ with $min(p, q, r) \geq 3$; and
- $- p 1 0 : q 0 : r 0$.

| $p, q, r, s$ | $p, q, r$– |
|------------|------------|
| $p 1, q, r$–; $min(q, r) > p$ | $p 1 q, r$– |
| $(p, q +) (r, s)$; $max(p, q) \leq min(r, s)$ | $(p 1, q) - (r, s)$ |
| $(p, q +) - (r, s)$ | $(p 1, q) - (r, s)$ |

Table 6. Families of $\overline{Kh}'$-thick links beginning from $n = 9$ crossings.

| $p, q, r, s, t$ | $p, q, r, s$ |
|-----------|------------|
| $p, q, r, s, t$– | $p 1 1, q, r$– |
| $p 1 1, q, r$–; $min(q, r) \geq 3$ | $p 1 1, q, r$– |
| $p, q, r, s, t$– | $p q, r, s, t$– |
| $p, q, r, s, t$– | $p q, r, s, t$– |
| $(p 1 1, q) - (s, t)$ | $(p 1, q) - (r 1, s)$ |
| $(p, q) - (r, s)$ | $(p, q) - (r, s)$ |
| $(p, q) - (r, s)$ | $(p, q) - (r, s)$ |
| $(p, q) - (r, s)$ | $(p, q) - (r, s)$ |
| $(p, q) - (r, s)$ | $(p, q) - (r, s)$ |
| $p : - q 0 : - r 0$; $min(p, q, r) \geq 3$ | $p : - q 0 : - r 0$; $min(p, q, r) \geq 3$ |
| $- p 1 0 : q 0 : r 0$ | $- (p, q) r$ |
| $- (p, q) r 0$ | $- (p, q) : r 0$ |
| $- (p, q) : r$ | $(p, q) - 1$ |

Table 7. Families of $\overline{Kh}'$-thick links beginning from $n = 10$ crossings.
Conjecture 4.1. All alternating knots or links, except those belonging to the family given by Conway symbol \( n, \) (i.e. \( 2^1_1, 3^1_1, 4^2_1, \ldots \) consisting of torus links \((2, n), n \geq 2\)), are not \( Jp \)-special.

Conjecture 4.2. Every adequate non-alternating link is \( Jp \)-special.

Conjecture 4.3. All minimal positive braids are \( Jp \)-special.

Among links with 11 crossings at least 126 generate families of \( Kh^\prime \)-thick links. Sixty-one of these families consist only of adequate links, see Table 9.

The remaining 65 families containing \( Kh^\prime \)-thick links are given in the following table, together with estimated conditions for \( Kh^\prime \)-thick links, see Table 10.

The majority of the remaining families derived from \( Kh^\prime \)-thick links with \( n = 11 \) crossings contain both \( Jp \)-special and not \( Jp \)-special links, but two families, \( 6^p(p, q) - \) \( (p \geq 3, q \geq 3) \) and \( 8^p.p. - q \), contain only links which are not \( Jp \)-special.

5. Recognition of odd-homology thickness based on computational results for different classes of links

After computing odd homology of different classes of links by using the program \( KhoHo \) [Sh1], we propose several conjectures about odd homology thick links.

Conjecture 5.1. Every link given by a positive minimal \( k \)-braid is \( Kh^\prime \)-thick \((k \geq 3)\).

According to computational results, this conjecture holds for all positive minimal 3-, 4- and 5-braids with at most \( n = 20 \) crossings.

If Conjecture 5.1 is true in general, it implies that all non-alternating Lorenz links [Lo, BiWi, Gh, GhLe] are \( Kh^\prime \)-thick. M. Stošić proved that non-alternating torus knots, a subset of Lorenz links, are \( Kh \)-thick [St].

A rational tangle is called positive if it has only positive numbers in its Conway symbol. Recall that up to taking mirror image and permuting rational tangles, every non-alternating Montesinos link can be denoted by Conway symbol of the form \( p_1, ..., p_m, -q_1, ..., -q_n \) where \( p_i \) and \( q_j \) denote positive rational tangles which do not start by one, \( i = \in \{1, 2, ..., m\}, j = \in \{1, 2, ..., n\}, m \geq n \geq 1, m \geq 2.\)

Every Montesinos link with \( n > 1 \) is \( Kh^\prime \)-thick, since it is an adequate non-alternating link [Ja]. Hence, we restrict our consideration to the Montesinos links of the form \( p_1, ..., p_m, -q \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\((p, q) - (r, s)\) & \((p, q, r, s) -\) \\
\((p, 1, q) - (r, s)\) & \((p, q, r, s, t) -\) \\
\(p, q, r, s -\) & \(p, q, r, s, t -\) \\
\(p, q, r, s, t -\) & \((p, 1, q) - (r, 1, s)\) \\
\((p, 1, q) - (r, s)\) & \((p, q, r, s) - (s, t)\) \\
\((p, q, r) - (s, t)\) & \((p, q, r) - (s, t)\) \\
\((p, q, r, s) - (s, t)\) & \((p, q, r) - (s, t)\) \\
\((p, q, r, s) - (s, t)\) & \((p, q, r) - (s, t)\) \\
\((p, q, r) - (s, t)\) & \((p, q, r) - (s, t)\) \\
\(\cdots - (p, q) : r 0\) & \(\cdots - (p, q) : r 0\) \\
\hline
\end{tabular}
\caption{Families of adequate \( Jp \)-special links}
\end{table}

\(^3\)See [Kr2], Problem 6.2.

\(^4\)Only alternating Lorenz links are obtained from minimal 2-braids of the form \( a^n \), giving the family of the links \( 2^1_1, 3^1_1, 4^1_1, 5^1_1, \ldots \).
Every Montesinos link can be given by one or several different Conway symbols with the minimal number of crossings. For example, the link $212,211,-21$ can be written also as $212,-22,3$ or $-2111,211,3$. A rational tangle $t$ is of length $l$ if its symbol consists of $l$ integers, i.e., $t = t_1 t_2 \ldots t_l$. To rational tangles of length $l > 1$ belonging to a Conway symbol of a Montesinos link we will apply the following reduction: every positive rational tangle is replaced by its last number $t_l$, every negative rational tangle by $-t_l - 1$, and tangles of the length $l = 1$ remain unchanged. In this way, from every Conway symbol of a Montesinos link $p_1,\ldots,p_m,-q$ we obtain the reduced Conway symbol $\overline{p_1,\ldots,p_m,-q}$.

**Conjecture 5.1.** Montesinos link of the form $p_1,\ldots,p_m,-q$ is $K_h'$-thick if it has a Conway symbol with a minimal number of crossings satisfying the relationship $\min(\overline{p_1},\overline{p_2},\ldots,\overline{p_m}) \geq 7$.

| $p, q, r, s, t$ | $p, q, r, s, t$ |
|-----------------|-----------------|
| $p_1, q_1, r, s, t$ | $p_1, q, r, s, t$ |
| $(p q, r) - (s, t)$ | $(p q, r) - (s, t)$ |
| $p, q - (q, r)$ | $p, q - (q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $p_1, q, r, s, t, t$ | $p_1, q, r, s, t, t$ |
| $(p q, r) - (s, t)$ | $(p q, r) - (s, t)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $p_1, q_1$ | $p_1, q_1$ |
| $(p q, r) - (s, t)$ | $(p q, r) - (s, t)$ |
| $p_1, q_1$ | $p_1, q_1$ |
| $(p q, r) - (s, t)$ | $(p q, r) - (s, t)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |
| $6^* - (p q, r)$ | $6^* - (p q, r)$ |

Table 9. Families of adequate $K_h'$-thick links beginning from $n = 11$ crossings.
For pretzel links all tangles are of the length 1, so $p_i = p_i (i = 1, \ldots, m)$, $q_i = q$, and Conjecture 5.2 corresponds to Proposition 2.2 [Gr]:

**Proposition 5.2.** For $n \geq 2$ and $p_1, \ldots, p_n \geq 2$, and $q \geq 1$, the pretzel link $P(p_1, \ldots, p_n, -q)$ is QA iff $q > \min\{p_1, \ldots, p_n\}$ [Gr].

For $n \leq 11$ crossings all non-alternating thick Montesinos links are exactly those described in Conjecture 5.1. However, for $n \geq 12$ crossings some $\overline{K^f}$-thick links do not satisfy the conditions of Conjecture 5.1. For example, for $12 \leq n \leq 15$ crossings, we have the following families of exceptional Montesinos links:

1. $-22, 22, p; p \geq 6$;
2. $-33, 23, p; p \geq 4$;
3. $-p_2, q_2, r; p > q \geq 2, r \geq 3$;

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| $p\overline{1}\overline{1}, q, r-$; min($q, r$) $\geq 3$ | $p\overline{1}\overline{1}, q, r, s$ |
|--------------------------------------------------|----------------------------------|
| $-p\overline{1} : q \vdash r; p \leq \min(q, r)$ or $\min(q, r) \geq 3$ | $-p\overline{1} 0 : q 0 \vdash r$ |
| $p, q 0, -r, s ; t 0$; $\min(p, q) \geq r$ | $p - q 1, -r, s 0$ |
| $p, q : -r 0 \vdash -s 0$ | $8^p, -q 0, r$ |
| $p, -q, -r, s 0$ | |
| $p, q, r, s, t -$ | $p\overline{1} q 1, r, s -$ |
| $(p, q) (r 1, s 1)$ | $-(p, q) (s, t 1)$ |
| $(p, q, r -) (s, t 1)$; $r \geq t$ | $-(p, q) (r, s, t 1)$ |
| $(p, q -) (r 1, s 1)$; $p \geq s$ | $(p, q -) (r 1, s + t)$ |
| $(p, q -) - 1 - 1 (r, s)$ | $(p, q -) - 1 - 1 (r, s, t)$ |
| $(p, q) 11 - (r, s)$ | $-(p, q) 11 (r, s)$ |
| $(p, q) 11 (r, s)$ | $-(p, q) 11 (r, s)$ |
| $(p, q) (s, t 1)$; max($q, r$) $\leq \min(s, t)$ | $(p, q, r 1) (s, t 1)$ |
| $(p, q, r 1) - (s, t)$; $\min(p, q, r) \geq \max(s, t)$ | $(p, q, r 1) (s, t 1)$ |
| $-(p, q, r) (s, t 1)$ | $(p, q, r 1, (s, t 1)$ |
| $(p, q, r 1) r 1, (s, t 1)$ | $-(p, q, r 1) r 1, (s, t 1)$ |
| $(p, q) r 1, (s, t 1)$ | $(p, q) 11 1 (r, s)$ |
| $(p, q) 11 1 (r, s)$ | $(p, q) 11 1 (r, s)$ |
| $6^* - p, q 0, r 0$; $q \leq \min(r, s)$ | $6^* - p, q 1, r - r$ |
| $6^* - p, q 0, r$; $q \geq p$ and $r \geq p$ | $6^* - p, q 1, r - 1$ |
| $6^* - p, q 0, - r$; $\min(p, q, r) > 2$ | $6^* p, q 0, r 0, - s 1$ |
| $6^* p, q 0, - r 1, s 0$ | $6^* p, q 1, - r 0, - s$ |
| $6^* - p, q 0, - r 0, t 1$; $\min(p, q) \geq r$ and $\min(p, q, r) \leq \min(s, t)$ | $6^* - p, q 1, 0, r 0, s 0$ |
| $6^* - p, q 0, - r 0, s$; $\min(p, q, r) \geq 3$ | $6^* p, q 0, - r 0, s 0$ |
| $6^* p, q - (q, r) 10$ | $6^* p, q, r 1 0$ |
| $6^* p, - (q, r) 1 0$ | $6^* p, - (q, r) 1 0$ |
| $6^* p, (q, r) - 0, - s$ | $8^p, - q, r 1 0$ |
| $6^* p, (q, r) - 0, - s$ | $8^p, - q, r 1 0$ |

Table 10. The remaining 65 families containing $\overline{K^f}$-thick links.
Table 11. Families of $\mathcal{J}_p$-special links from the Table 6

(4) $-p_{1}q_{2} \ldots q_{r}r$; $p = 2, 4 \geq 2, r \geq 4$ or $p > 2, q \geq 2, r \geq 3$, etc.

Let $p_1, p_2, \ldots, p_l - k$ ($l \geq 2, k \leq l$) denote tangle $p_1, p_2, \ldots, p_l - \ldots$ with $k$ minuses, where $p_i$ ($i = 1, 2, \ldots, l$) are positive rational tangles that do not start with 1.

**Theorem 5.3.** All algebraic links of the form $(p_1, \ldots, p_m)(q_1, \ldots, q_n)$, where $m, n \geq 2$ and all $p_i, i \in \{1, \ldots, m\}$, and $q_j, j \in \{1, \ldots, n\}$, denote positive alternating rational tangles, are $\overline{K_h}$-thick for $k \geq 2$.

The Theorem 5.3 holds since all algebraic links above are non-alternating adequate links [Ja], which are homologically thick.

Hence, we can restrict our consideration to links of the form $(p_1, \ldots, p_m)(q_1, \ldots, q_n)$. First we consider the case where $p_i$ and $q_j$ are positive integer tangles different from 1 ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; m \geq 2, n \geq 2$).

**Conjecture 5.4.** A link of the form $(p_1, \ldots, p_m)(q_1, \ldots, q_n)$, where $p_i$ and $q_j$ are positive integer tangles different from 1 ($i = 1, 2, \ldots; m; j = 1, 2, \ldots, n; m \geq 2, n \geq 2$) is $\overline{K_h}$-thick if $m \geq 4$ or $m = 3$ and $\max(p_1, \ldots, p_m) \leq \min(q_1, \ldots, q_n)$.

**Conjecture 5.5.** A link of the form $(p_1, \ldots, p_m)(q_1, \ldots, q_n)$, where $p_i$ and $q_j$ are positive rational tangles that do not start with 1 ($i = 1, 2, \ldots; m; j = 1, 2, \ldots, n$) which are not all integer tangles, is $\overline{K_h}$-thick if $m \geq 3$, $q = \min(\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_n) > 1$ and

- if $\text{length}(p_i) \geq 2$ then $\overline{p}_i = 1$ ($i = 1, 2, \ldots, m$), and
- $\max(\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m) \leq q$.

Next, we consider algebraic links of the form $(p_1, \ldots, p_m + k)(q_1, \ldots, q_n - l)$, where $p_i$ ($i = 1, \ldots, m, m \geq 2$), $q_j$ ($j = 1, \ldots, n$) are positive alternating rational tangles, $+k$ denotes a sequence of $k$ pluses ($k \geq 1$), and $-l$ denotes sequence of $l$ minuses ($n > l \geq 1$). All these links are $\overline{K_h}$-thick for $l > 1$, since they are non-alternating adequate links. Hence, we need to analyze only links of the form $(p_1, \ldots, p_m + k)(q_1, \ldots, q_n)$. We propose the following conjecture:

**Conjecture 5.6.** A link of the form $(p_1, \ldots, p_m + k)(q_1, \ldots, q_n)$ is $\overline{K_h}$-thick if $m \geq 3$, $q = \min(\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_n) > 1$ and

- if $\text{length}(p_i) \geq 2$ then $\overline{p}_i = 1$ ($i = 1, 2, \ldots, m$), and
- $\max(\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m) + k \leq q$.

**Conjecture 5.7.** An algebraic link of the form $(p_1, \ldots, p_m + k) - (q_1, \ldots, q_n)$, where $p_i$ ($i = 1, \ldots, m, m \geq 2$) and $q_j$ ($j = 1, \ldots, n, n \geq 2$) are positive alternating rational tangles and $+k$ denotes a sequence of $k$ pluses ($k \geq 1$) is $\overline{K_h}$-thick if $q = \min(\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_n) > 1$ and
However, there are \( Kh \)-thick links belonging to this class, which do not satisfy the conditions of Conjecture 5.7 for example links of the family \((2, 2, 2+) - (211, p)\) for \( p \geq 3 \).

Let \( p_1, p_2, \ldots, p_m - k \) \((m \geq 2, m \geq k)\) denote tangle \( p_1, p_2, \ldots, p_m - \ldots - \) with \( k \) minuses, where \( p_i \) \((i = 1, 2, \ldots, m)\) are positive rational tangles that do not start with 1.

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THE MATHEMATICAL INSTITUTE, KNEZ MIHAIOVA 36, P.O.BOX 367, 11001 BELGRADE, SERBIA

E-mail address: sjablan@gmail.com

THE MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 17 GAUSS WAY, BERKELEY, CA 94720-5070, USA

E-mail address: radmilas@gmail.edu