

Stability of a mixed type quadratic, cubic and quartic functional equation

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Abstract. In this paper, we obtain the general solution and the generalized Hyers-Ulam Rassias stability of the functional equation

\[3(f(x + 2y) + f(x - 2y)) = 12(f(x + y) + f(x - y)) + 4f(3y) - 18f(2y) + 36f(y) - 18f(x).\]

1. Introduction

The stability problem of functional equations originated from a question of Ulam [17] in 1940, concerning the stability of group homomorphisms. Let \((G_1,\cdot)\) be a group and let \((G_2,\ast)\) be a metric group with the metric \(d(\cdot,\cdot)\). Given \(\epsilon > 0\), dose there exist a \(\delta > 0\), such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality \(d(h(x \cdot y), h(x) \ast h(y)) < \delta\) for all \(x, y \in G_1\), then there exists a homomorphism \(H : G_1 \to G_2\) with \(d(h(x), H(x)) < \epsilon\) for all \(x \in G_1\)? In the other words, Under what condition dose there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [8] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \(f : E \to E'\) be a mapping between Banach spaces such that

\[\|f(x + y) - f(x) - f(y)\| \leq \delta\]

for all \(x, y \in E\), and for some \(\delta > 0\). Then there exists a unique additive mapping \(T : E \to E'\) such that

\[\|f(x) - T(x)\| \leq \delta\]

for all \(x \in E\). Moreover if \(f(tx)\) is continuous in \(t\) for each fixed \(x \in E\), then \(T\) is linear. In 1978, Th. M. Rassias [15] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda [4] answered the question for the

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The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.1} \]
is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function \( f \) between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive function \( B \) such that \( f(x) = B(x,x) \) for all \( x \) (see [1,11]). The bi-additive function \( B \) is given by
\[ B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.2} \]

Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions \( f : A \rightarrow B \), where \( A \) is normed space and \( B \) Banach space (see [16]). Cholewa [2] noticed that the Theorem of Skof is still true if relevant domain \( A \) is replaced an abelian group. In the paper [3], Czerwik proved the Hyers-Ulam-Rassias stability of the equation (1.1). Grabiec [6] has generalized these result mentioned above.

Jun and Kim [10] introduced the following functional equation
\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.3} \]
and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3). The \( f(x) = x^3 \) satisfies the functional equation (1.3), which is called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function \( f \) between real vector spaces \( X \) and \( Y \) is a solution of (1.3) if and only if there exits a unique symmetric multi-additive function \( C : X \times X \times X \rightarrow Y \) such that \( f(x) = C(x,x,x) \) for all \( x \in X \), and \( C \) is symmetric for each fixed one variable and is additive for fixed two variables.

In [12], Won-Gil Prak and Jea Hyeong Bae, considered the following quartic functional equation:
\[ f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y). \tag{1.4} \]
In fact they proved that a function \( f \) between real vector spaces \( X \) and \( Y \) is a solution of (1.3) if and only if there exits a unique symmetric multi-additive function \( B : X \times X \times X \rightarrow Y \) such that \( f(x) = B(x,x,x) \) for all \( x \). It is easy to show that the function \( f(x) = x^4 \) satisfies the functional equation (1.4), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

We deal with the next functional equation deriving from quadratic, cubic and quartic functions:
\[ 3(f(x+2y)+f(x-2y)) = 12(f(x+y)+f(x-y))+4f(3y) - 18f(2y) + 36f(y) - 18f(x). \tag{1.5} \]

It is easy to see that the function \( f(x) = ax^2 + bx^3 + cx^4 \) is a solution of the functional equation (1.5). In the present paper we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.5).

2. General solution

In this section we establish the general solution of functional equation (1.5).
Theorem 2.1. Let $X, Y$ be vector spaces, and let $f : X \rightarrow Y$ be a function. Then $f$ satisfies (1.5) if and only if there exist a unique symmetric function $Q_1 : X \times X \rightarrow Y$, a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multi-additive function $Q_2 : X \times X \times X \rightarrow Y$ such that $f(x) = Q_1(x, x) + C(x, x, x) + Q_2(x, x, x, x)$ for all $x \in X$, and that $Q_1$ is additive for each fixed one variable, $C$ is symmetric for each fixed one variable and is additive for fixed two variables.

Proof. Suppose there exist a symmetric function $Q_1 : X \times X \rightarrow Y$, a function $C : X \times X \times X \rightarrow Y$ and a symmetric multi-additive function $Q_2 : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q_1(x, x) + C(x, x, x) + Q_2(x, x, x, x)$ for all $x \in X$, and that $Q_1$ is additive for each fixed one variable, $C$ is symmetric for each fixed one variable and is additive for fixed two variables. Then it is easy to see that $f$ satisfies (1.5). For the converse let $f$ satisfies (1.5).

We decompose $f$ into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)),$$

for all $x \in X$. By (1.5), we have

\[
3(f_e(x + 2y) + f_e(x - 2y)) = \frac{1}{2}[3f(x + 2y) + 3f(-x - 2y) + 3f(x - 2y) + 3f(-x + 2y)]
\]

\[
= \frac{1}{2}[3f(x + 2y) + 3f(x - 2y)] + \frac{1}{2}[3f((-x) + (-2y)) + 3f((-x) - (-2y))]
\]

\[
= \frac{1}{2}[12f(x + y) + 12f(x - y) + 4f(3y) - 18f(2y) + 36f(y) - 18f(x)]
\]

\[
+ \frac{1}{2}[12f(-x + y) + 12f(-x - y) + 4f(-3y) - 18f(-2y) + 36f(-y) - 18f(-x)]
\]

\[
= 12[\frac{1}{2}(f(x + y) + f((-x + y))) + \frac{1}{2}(f(x - y) + f((-x - y)))]
\]

\[
+ 4[\frac{1}{2}(f(3y) + f(-3y))] - 18[\frac{1}{2}(f(2y) + f(-2y))]
\]

\[
+ 36[\frac{1}{2}(f(y) + f(-y))] - 18[\frac{1}{2}(f(x) + f(-x))]
\]

\[
= 12(f_e(x + y) + f_e(x - y)) + 4f_e(3y) - 18f_e(2y) + 36f_e(y) - 18f_e(x)
\]

for all $x, y \in X$. This means that $f_e$ satisfies (1.5), or

\[
3(f_e(x + 2y) + f_e(x - 2y)) = 12(f_e(x + y) + f_e(x - y))
\]

\[
+ 4f_e(3y) - 18f_e(2y) + 36f_e(y) - 18f_e(x).
\] (1.5(e))

Now we show that the mapping $g : X \rightarrow Y$ defined by $g(x) := f_e(2x) - 16f_e(x)$ is quadratic and the mapping $h : X \rightarrow Y$ defined by $h(x) := f_e(2x) - 4f_e(x)$ is quartic. Putting $x = y = 0$ in (1.5(e)), we get $f_e(0) = 0$. Setting $x = 0$ in (1.5(e)), by evenness of $f_e$ we obtain

$$f_e(3y) = 6f_e(2y) - 15f_e(y).$$ (2.1)

Hence, according to (2.1), (1.5(e)) can be written as

$$f_e(x + 2y) + f_e(x - 2y) = 4f_e(x + y) + 4f_e(x - y) - 8f_e(y) + 2f_e(2y) - 6f_e(x).$$ (2.2)

Interchanging $x$ with $y$ in (2.2) gives the equation
Replacing \( f_e(x+y) + f_e(2x-y) = 4f_e(x+y) + 4f_e(x-y) - 8f_e(x) + 2f_e(2x) - 6f_e(y). \) \hspace{1cm} (2.3)

With the substitution \( y := x + y \) in (2.3), we have

\[
f_e(3x+y) + f_e(x-y) = 4f_e(2x+y) - 6f_e(x+y) + 4f_e(y) + 2f_e(2x) - 8f_e(x).
\] \hspace{1cm} (2.4)

Replacing \( y \) by \(-y\) in (2.4), gives

\[
f_e(3x-y) + f_e(x+y) = 4f_e(2x-y) - 6f_e(x-y) + 4f_e(y) + 2f_e(2x) - 8f_e(x).
\] \hspace{1cm} (2.5)

If we add (2.4) to (2.5), we have

\[
f_e(3x + y) + f_e(3x - y) = 4f_e(2x + y) + 4f_e(2x - y) - 7f_e(x + y) - 7f_e(x - y) + 8f_e(y) + 4f_e(2x) - 16f_e(x).
\] \hspace{1cm} (2.6)

Setting \( x + y \) instead of \( x \) in (2.3), we get

\[
f_e(2x + 3y) + f_e(2x + y) = 4f_e(x + 2y) - 8f_e(x + y) + 2f_e(2(x + y)) - 6f_e(y) + 4f_e(x).
\] \hspace{1cm} (2.7)

Which on substitution of \(-y\) for \( y \) in (2.7) gives

\[
f_e(2x - 3y) + f_e(2x - y) = 4f_e(x - 2y) - 8f_e(x - y) + 2f_e(2(x - y)) - 6f_e(y) + 4f_e(x).
\] \hspace{1cm} (2.8)

By adding (2.7) and (2.8), we lead to

\[
f_e(2x + 3y) + f_e(2x - 3y) = 4f_e(x + 2y) + 4f_e(x - 2y) - f_e(2x + y) - f_e(2x - y) + 2f_e(2(x + y)) + 2f_e(2(x - y)) - 8f_e(x + y) - 8f_e(x - y) - 12f_e(y) + 8f_e(x).
\] \hspace{1cm} (2.9)

Putting \( y := 2y \) in (2.6) to obtain

\[
f_e(3x + 2y) + f_e(3x - 2y) = 4f_e(2(x + y)) + 4f_e(2(x - y)) - 7f_e(x + 2y) - 7f_e(x - 2y) + 8f_e(2y) + 4f_e(2x) - 16f_e(x).
\] \hspace{1cm} (2.10)

Interchanging \( x \) and \( y \) in (2.9) to get

\[
f_e(3x + 2y) + f_e(3x - 2y) = 4f_e(2x + y) + 4f_e(2x - y) - f_e(x + 2y) - f_e(x - 2y) + 2f_e(2(x + y)) + 2f_e(2(x - y)) - 8f_e(x + y) - 8f_e(x - y) - 12f_e(x) + 8f_e(y).
\] \hspace{1cm} (2.11)

If we compare (2.10) and (2.11) and utilizing (2.2) and (2.3), we conclude that
Then there exist a unique symmetric function $Q$ such that

$$[f_e(2(x + y)) - 16f_e(x + y)] + [f_e(2(x - y)) - 16f_e(x - y)] = 2[f_e(2x) - 16f_e(x)] + 2[f_e(2y) - 16f_e(y)]$$

for all $x, y \in X$. The last equality means that

$$g(x + y) + g(x - y) = 2g(x) + 2g(y)$$

for all $x, y \in X$. Therefore the mapping $g : X \rightarrow Y$ is quadratic.

With the substitutions $x := 2x$ and $y := 2y$ in (2.3), we have

$$f_e(2(2x + y)) + f_e(2(2x - y)) = 4f_e(2(x + y)) + 4f_e(2(x - y)) - 6f_e(2x) + 2f_e(4x) - 8f_e(2x).$$

Let $g : X \rightarrow Y$ be the quadratic mapping defined above. Since $g(2x) = 4g(x)$ for all $x \in X$, then

$$f_e(4x) = 20f_e(2x) - 64f_e(x)$$

(2.13)

for all $x, y \in X$.

Hence, according to (2.13), (2.12) can be written as

$$f_e(2(2x + y)) + f_e(2(2x - y)) = 4f_e(2(x + y)) + 4f_e(2(x - y)) - 6f_e(2y) + 32f_e(2x) - 128f_e(x).$$

(2.14)

Interchanging $x$ with $y$ in (2.14) gives the equation

$$f_e(2(x + 2y)) + f_e(2(x - 2y)) = 4f_e(2(x + y)) + 4f_e(2(x - y)) - 6f_e(2x) + 32f_e(2y) - 128f_e(y).$$

(2.15)

By multiplying by 4 in (2.2) and subtract the last equation from (2.15), we arrive at

$$h(x + 2y) + h(x - 2y) = [f_e(2(x + 2y)) - 4f_e(x + 2y)] + [f_e(2(x - 2y)) - 4f_e(x - 2y)] = 4[f_e(2(x + y)) - 4f_e(x + y)] + 4[f_e(2(x - y)) - 4f_e(x - y)] + 24[f_e(2y) - 4f_e(y)] - 6[f_e(2x) - 4f_e(x)] = 4h(x + y) + 4h(x - y) + 24h(y) - 6h(x)$$

for all $x, y \in X$. Therefore the mapping $h : X \rightarrow Y$ is quartic. On the other hand we have

$$f_e(x) = \frac{1}{12}h(x) - \frac{x^2}{12}g(x)$$

for all $x \in X$. This means that $f_e$ is quartic-quadratic function.

Then there exist a unique symmetric function $Q_1 : X \times X \rightarrow Y$ and a unique symmetric multi-additive function $Q_2 : X \times X \times X \rightarrow Y$ such that $f_e(x) = Q_1(x, x) + Q_2(x, x, x)$ for all $x \in X$, and $Q_1$ is additive for each fixed one variable.

On the other hand we can show that $f_o$ satisfies (1.5), or

$$3(f_o(x + 2y) + f_o(x - 2y)) = 12(f_o(x + y) + f_o(x - y)) + 4f_o(3y) - 18f_o(2y) + 36f_o(y) - 18f_o(x).$$

(1.5(o))

Setting $x = y = 0$ in (1.5(o)) to obtain $f_o(0) = 0$. Putting $x = 0$ in (1.5(o)), then by oddness of $f_o$, we have...
\[ 2f_o(3y) = 9f_o(2y) - 18f_o(y). \]

Hence (1.5(o)) can be written as
\[ f_o(x + 2y) + f_o(x - 2y) = 4f_o(x + y) + 4f_o(x - y) - 6f_o(x). \]

Replacing \( y \) by \( x \) in (1.5(o)) to get
\[ f_o(3y) = 6f_o(2y) - 21f_o(y). \]

By comparing (2.16) with (2.18), we arrive at
\[ f_o(2y) = 8f_o(y). \]

From the substitution \( x := 2x \) in (2.17) and (2.19), it follows that
\[ f_o(2x + y) + f_o(2x - y) = 2f_o(x + y) + 2f_o(x - y) + 12f_o(x). \]

This shows that \( f_o \) is cubic. Thus there exists a unique function \( C : X \times X \times X \rightarrow Y \) such that \( f_o(x) = C(x, x, x) \) for all \( x \in X \), and \( C \) is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all \( x \in X \), we have
\[ f(x) = f_e(x) + f_o(x) = Q_1(x, x) + Q_2(x, x, x) + C(x, x, x). \]

This completes the proof of Theorem. \( \square \)

The following Corollary is an alternative result of above Theorem.

**Corollary 2.2.** Let \( X, Y \) be vector spaces, and let \( f : X \rightarrow Y \) be a function satisfies (1.5). Then the following assertions hold.

a) If \( f \) is even function, then \( f \) is quartic-quadratic.

b) If \( f \) is odd function, then \( f \) is cubic.

3. **Stability**

We now investigate the generalized Hyers-Ulam-Rassias stability problem for functional equation (1.5). From now on, let \( X \) be a real vector space and let \( Y \) be a Banach space. Now before taking up the main subject, given \( f : X \rightarrow Y \), we define the difference operator \( D_f : X \times X \rightarrow Y \) by
\[ D_f(x, y) = 3[f(x+2y) + f(x-2y)] - 12[f(x+y) + f(x-y)] - 4f(3y) + 18f(2y) - 36f(y) + 18f(x) \]
for all \( x, y \in X \). We consider the following functional inequality:
\[ ||D_f(x, y)|| \leq \phi(x, y) \]
for an upper bound \( \phi : X \times X \rightarrow [0, \infty) \).

**Theorem 3.1.** Let \( s \in \{1, -1\} \) be fixed. Suppose that an odd mapping \( f : X \rightarrow Y \) satisfies
\[ ||D_f(x, y)|| \leq \phi(x, y) \]
for all \( x, y \in X \). If the upper bound \( \phi : X \times X \rightarrow [0, \infty) \) is a mapping such that
\[ \sum_{i=1}^{\infty} 8^{-i} [\phi(2^{-i} x, 2^{-i} y) + 4\phi(0, 2^{-i} x)] < \infty, \]
and that \( \lim_{n} 8^{sn}\phi(2^{-sn}x, 2^{-sn}y) = 0 \) for all \( x, y \in X \). Then the limit \( C(x) := \lim_{n} 8^{sn}f(2^{-sn}x) \) exists for all \( x \in X \), and \( C : X \to Y \) is a unique cubic function satisfies (1.5), and

\[
\|f(x) - C(x)\| \leq \frac{1}{6} \sum_{i=\frac{|x|+1}{2}}^{\infty} 8^{i-1}\phi(0, 2^{-ix}x) + \frac{4}{6} \sum_{i=\frac{|x|+1}{2}}^{\infty} 8^{i-1}\phi(2^{-ix}x, 2^{-ix}x), \tag{3.2}
\]

for all \( x \in X \).

**Proof.** Putting \( x = 0 \) in (3.1) to get

\[
\|4f(3y) - 18f(2y) + 36f(y)\| \leq \phi(0, y). \tag{3.3}
\]

Now replacing \( y \) by \( x \) in (3.1) to obtain

\[
\|f(3y) - 6f(2y) + 21f(y)\| \leq \phi(y, y). \tag{3.4}
\]

combining (3.3) with (3.4) yields

\[
\|\frac{f(2y)}{8} - f(y)\| \leq \frac{1}{6} \times 8\phi(0, y) + \frac{4}{6} \times 8\phi(y, y). \tag{3.5}
\]

From the inequality (3.5) we use iterative methods and induction on \( n \) to prove our next relation.

\[
\|\frac{f(2^n)x}{8^n} - f(x)\| \leq \frac{1}{6} \sum_{i=0}^{n-1} \frac{\phi(0, 2^ix)}{8^{i+1}} + \frac{4}{6} \sum_{i=0}^{n-1} \frac{\phi(2^ix, 2^{i+1})}{8^{i+1}}. \tag{3.6}
\]

Dividing (3.6) by \( 8^m \), and then replacing \( x \) by \( 2^mx \), it follows that

\[
\|\frac{f(2^{m+n}x)}{8^{m+n}} - \frac{f(2^mx)}{8^m}\| \leq \frac{1}{6} \sum_{i=0}^{m+n-1} \frac{\phi(0, 2^{m+i}x)}{8^{m+i+1}} + \frac{4}{6} \sum_{i=0}^{m+n-1} \frac{\phi(2^{m+i}x, 2^{m+i+1})}{8^{m+i+1}}
\]

\[
= \frac{1}{6} \sum_{i=m}^{m+n-1} \frac{\phi(0, 2^ix)}{8^{i+1}} + \frac{4}{6} \sum_{i=m}^{m+n-1} \frac{\phi(2^ix, 2^i2x)}{8^{i+1}}. \tag{3.7}
\]

This shows that \( \{\frac{f(2^nx)}{8^n}\} \) is a Cauchy sequence in \( Y \), by taking the limit \( m \to \infty \) in (3.7). Since \( Y \) is a Banach space, it follows that the sequence \( \{\frac{f(2^nx)}{8^n}\} \) converges. Now we define \( C : X \to Y \) by \( C(x) := \lim_{n} \frac{f(2^nx)}{8^n} \) for all \( x \in X \). Obviously (3.2) holds for \( s = -1 \). It is easy to see that \( C(-x) = -C(x) \) for all \( x \in X \). By using (3.1) we have

\[
\|D_C(x,y)\| \leq \lim_{n \to \infty} \frac{1}{8^n} \|D_f(2^nx, 2^ny)\| \leq \lim_{n \to \infty} \frac{1}{8^n} \phi(2^n x, 2^n y) = 0
\]

for all \( x, y \in X \). Hence by Corollary 2.2, \( C \) is cubic. It remains to show that \( C \) is unique. Suppose that there exists a cubic function \( C' : X \to Y \) which satisfies (1.5) and (3.2). Since \( C(2^nx) = 8^nC(x) \), and \( C'(2^nx) = 8^nC'(x) \), for all \( x \in X \), we have

\[
\|C(x) - C'(x)\| = \frac{1}{8^n} \|C(2^n x) - C'(2^n x)\|
\]

\[
\leq \frac{1}{8^n} \|C(2^n x) - f(2^n x)\| + \frac{1}{8^n} \|f(2^n x) - f(2^n x)\|
\]

\[
\leq \frac{1}{8^n} \sum_{i=0}^{\infty} \frac{1}{8^{n+i}} \phi(0, 2^{n+i}x) + \frac{4}{6} \sum_{i=0}^{\infty} \frac{1}{8^{n+i}} \phi(2^{n+i}x, 2^{n+i}x)
\]
for all $x \in X$. By taking $n \to \infty$ in this inequality, it follows that $C(x) = C'(x)$ for all $x \in X$. Which gives the conclusion for $s = -1$. On the other hand by replacing $2y$ by $x$ in (3.5) and multiplying the result by 8, we get

$$
\|f(x) - 8f(x)\| \leq \frac{1}{6} \phi(0, \frac{x}{2}) + \frac{4}{6} \phi(\frac{x}{2}, \frac{x}{2}).
$$

From (3.8) we use iterative methods and induction on $n$ to obtain

$$
\|f(x) - 8^n f(x)\| \leq \frac{1}{6} \sum_{i=0}^{n-1} 8^i \phi(0, \frac{x}{2^{i+1}}) + \frac{4}{6} \sum_{i=0}^{n-1} 8^i \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})
$$

(3.9)

for all $x \in X$.

Now multiplying both sides of (3.9) with $8^n$ and replacing $x$ by $\frac{x}{2^n}$ in (3.9) to get

$$
\|f(\frac{x}{2^n}) - 8^n f(\frac{x}{2^{n+m}})\| \leq \frac{1}{6} \sum_{i=0}^{m+n-1} 8^{m+i} \phi(0, \frac{x}{2^{n+i+1}}) + \frac{4}{6} \sum_{i=0}^{m+n-1} 8^{m+i} \phi(\frac{x}{2^{n+i+1}}, \frac{x}{2^{n+i+1}})
$$

$$
= \frac{1}{6} \sum_{i=m}^{m+n-1} 8^i \phi(0, \frac{x}{2^{i+1}}) + \frac{4}{6} \sum_{i=m}^{m+n-1} 8^i \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}).
$$

(3.10)

By taking $m \to \infty$ in (3.10), it follows that $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $Y$. Then $C(x) := \lim_n 8^n f(\frac{x}{2^n})$ exists for all $x \in X$. Obviously (3.2) holds for $s = 1$. The rest of proof is similar to the proof of the case $s = -1$.

\[\square\]

**Theorem 3.2.** Suppose an even function $f : X \to Y$ satisfies

$$
\|D_f(x, y)\| \leq \phi(x, y)
$$

(3.11)

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that

$$
\sum_{i=1}^{\infty} 4^i [\phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \phi(\frac{x}{2^i}, \frac{x}{2^i})] < \infty
$$

(3.12)

for all $x \in X$, and that $\lim_n 4^n \phi(\frac{x}{2^n}, \frac{x}{2^n}) = 0$ for all $x, y \in X$. Then the limit

$$
Q_1(x) := \lim_n 4^n [f(\frac{x}{2^n}) - 16f(\frac{x}{2^n})]
$$

exists for all $x \in X$, and $Q_1 : X \to Y$ is a unique quadratic function satisfies (1.5), and

$$
\|f(2x) - 16f(x) - Q_1(x)\| \leq \sum_{i=0}^{\infty} 4^i \frac{1}{3} \phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}).
$$

(3.13)

for all $x \in X$.

**Proof.** Replacing $x$ by $2y$ in (3.11) to obtain

$$
\|3f(4y) - 16f(3y) + 36f(2y) - 48f(y)\| \leq \phi(2y, y).
$$

(3.14)

Replacing $x$ by $y$ in (3.11) to get

$$
\|f(3y) - 6f(2y) + 15f(y)\| \leq \phi(y, y).
$$

(3.15)

By combining (3.14) and (3.15) we lead to
there exists a quadratic function $Q$ replacing $x$ for all $\{x\}$. Then by (3.16) we have

$$
\|f(4x) - 20f(2x) + 64f(x)\| = \|\frac{1}{3}[3f(4y) - 16f(3y) + 36f(2y) - 48f(y)] + \frac{16}{3}[f(3y) - 6f(2y) + 15f(y)]\| \\
\leq \frac{1}{3}\phi(2x, x) + \frac{16}{3}\phi(x, x)
$$

(3.16)

for all $x \in X$. Put $g(x) = f(2x) - 16f(x)$ for all $x \in X$. Then by (3.16) we have

$$
\|g(2x) - 4g(x)\| \leq \frac{1}{3}\phi(2x, x) + \frac{16}{3}\phi(x, x).
$$

(3.17)

Replacing $x$ by $\frac{x}{2}$ in (3.17) to get

$$
\|g(x) - 4g(\frac{x}{2})\| \leq \frac{1}{3}\phi(x, \frac{x}{2}) + \frac{16}{3}\phi(\frac{x}{2}, \frac{x}{2}).
$$

(3.18)

An induction argument now implies that

$$
\|g(x) - 4^n g(\frac{x}{2^n})\| \leq \sum_{i=0}^{n-1} 4^i [\frac{1}{3}\phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3}\phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}})]
$$

(3.19)

for all $x \in X$. Multiplying both sides of above inequality by $4^m$ and replacing $x$ by $\frac{x}{2^m}$ to get

$$
\|4^m g(\frac{x}{2^m}) - 4^{m+n} g(\frac{x}{2^{m+n}})\| \leq \sum_{i=0}^{n-1} 4^{i+m} [\frac{1}{3}\phi(\frac{x}{2^{m+i}}, \frac{x}{2^{m+i+1}}) + \frac{16}{3}\phi(\frac{x}{2^{m+i+1}}, \frac{x}{2^{m+i+2}})]
$$

$$
\leq \sum_{i=m}^{m+n-1} 4^i [\frac{1}{3}\phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3}\phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}})].
$$

Since the right hand side of the above inequality tends to 0 as $m \to \infty$, the sequence $\{4^n g(\frac{x}{2^n})\}$ is Cauchy. Then the limit $Q_1(x) := \lim_n 4^n g(\frac{x}{2^n}) = \lim_n 4^n (f(\frac{x}{2^n}) - 16f(\frac{x}{2^n}))$ exists for all $x \in X$. On the other hand we have

$$
\|Q_1(2x) - 4Q_1(x)\| = \lim_n [4^n g(\frac{x}{2^{n-1}}) - 4^{n+1} g(\frac{x}{2^n})]
$$

$$
= 4 \lim_n [4^n g(\frac{x}{2^{n-1}}) - 4^{n} g(\frac{x}{2^n})] = 0
$$

(3.20)

for all $x \in X$. Let $D_y(x, y) := D_f(2x, 2y) - 16D_f(x, y)$ for all $x \in X$. Then we have

$$
D_{Q_1}(x, y) = \lim_n \|4^n D_y(\frac{x}{2^n}, \frac{y}{2^n})\| = \lim_n \|4^n D_f(\frac{x}{2^n}, \frac{y}{2^n}) - 16D_f(\frac{x}{2^n}, \frac{y}{2^n})\|
$$

$$
\leq \lim_n 4\|4^{n-1} D_f(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}})\| + 16\|4^n D_f(\frac{x}{2^n}, \frac{y}{2^n})\|
$$

$$
\leq 4\lim_n 4^{n-1} \phi(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}) + 16\lim_n 4^n \phi(\frac{x}{2^n}, \frac{y}{2^n}) = 0
$$

This means that $Q_1$ satisfies (1.5). Thus by (3.20), it follows that $Q_1$ is quadratic. It remains to show that $Q_1$ is unique quadratic function which satisfies (3.13). Suppose that there exists a quadratic function $Q'_1 : X \to Y$ satisfies (3.13). Since $Q_1(2^n x) = 4^n Q_1(x)$, and $Q'_1(2^n x) = 4^n Q'_1(x)$ for all $x \in X$, it follows that
Theorem 3.3. Suppose that an even function $f$ exists for all $x \in X$. Replacing $x$ by $x$ in (3.26) and then multiplying the result by $16^n$ for all $x \in X$, we have

$$\|Q_1(x) - Q_1'(x)\| = 4^n \|Q_1(\frac{2x}{2^n}) - Q_1'(\frac{2x}{2^n})\| \leq 4^n \|Q_1(\frac{x}{2^n}) - f(\frac{2x}{2^n}) - 16f(\frac{x}{2^n})\| + \|Q_1(\frac{2x}{2^n}) - f(\frac{2x}{2^n}) - 16f(\frac{x}{2^n})\|

\leq \sum_{i=n}^\infty 4^i \left[ \frac{1}{3} \phi(\frac{x}{2^n}, \frac{x}{2^{n+i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{n+i+1}}, \frac{x}{2^{n+i+1}}) \right]$$

for all $x \in X$. By taking $n \to \infty$ the right hand side of above inequality tends to 0. Thus we have $Q_1(x) = Q_1'(x)$ for all $x \in X$, and the proof of Theorem is complete.

\[\square\]

**Theorem 3.3.** Suppose that an even function $f : X \to Y$ satisfies

$$\|D_f(x, y)\| \leq \phi(x, y) \tag{3.21}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that

$$\sum_{i=1}^\infty 16^i [\phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \phi(\frac{x}{2^i}, \frac{x}{2^i})] < \infty \tag{3.22}$$

for all $x \in X$ and that $\lim_{n} 16^n \phi(\frac{x}{2^n}, \frac{x}{2^n}) = 0$ for all $x, y \in X$, then the limit

$$Q_2(x) := \lim_{n} 16^n [f(\frac{x}{2^n-1}) - 4f(\frac{x}{2^n})]$$

exists for all $x \in X$, and $Q_2 : X \to Y$ is a unique quartic function satisfies (1.5) and

$$\|f(2x) - 4f(x) - Q_2(x)\| \leq \sum_{i=0}^\infty 16^i \left[ \frac{1}{3} \phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \right], \tag{3.23}$$

for all $x \in X$.

**Proof.** Similar to the proof of Theorem 3.2, we can show that $f$ satisfies (3.16). Put $h(x) = f(2x) - 4f(x)$ for all $x \in X$. Then by (3.16) we have

$$\|h(2x) - 6h(x)\| \leq \frac{1}{3} \phi(2x, x) + \frac{16}{3} \phi(x, x). \tag{3.24}$$

Replacing $x$ by $x$ in (3.24) to obtain

$$\|h(x) - 6h(x)\| \leq \frac{1}{3} \phi(x, x) + \frac{16}{3} \phi(x, x). \tag{3.25}$$

By (3.25) we use iterative methods and induction on $n$ to prove our next relation.

$$\|h(x) - 6^n h(\frac{x}{2^n})\| \leq \sum_{i=0}^{n-1} 16^i \left[ \frac{1}{3} \phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \right]. \tag{3.26}$$

Replacing $x$ by $x$ in (3.26) and then multiplying the result by $16^n$ to get

$$\|16^n h(\frac{x}{2^m}) - 16^{m+n} h(\frac{x}{2^{m+n}})\| \leq \sum_{i=0}^{n-1} 16^i \left[ \frac{1}{3} \phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \right] \tag{3.27}$$

$$= \sum_{i=m}^{m+n-1} 16^i \left[ \frac{1}{3} \phi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \right].$$

By taking $m \to \infty$ in above inequality, it follows that
This means that $\{16^n h(x, \frac{x}{2^n})\}$ is a Cauchy sequence in $Y$. Thus the limit $Q_2(x) = \lim_n 16^n h(x, \frac{x}{2^n}) = \lim_n 16^n [f(\frac{x}{2^n}) - 4f(\frac{x}{2^n})]$ exists for all $x \in X$. On the other hand we have

$$\|Q_2(2x) - 16Q_2(x)\| = \lim_n \|16^n h(x, \frac{x}{2^n}) - 16^{n+1} h(x, \frac{x}{2^n})\|$$

$$= 16 \lim_n \|16^n h(x, \frac{x}{2^n}) - 16^n h(\frac{x}{2^n})\| = 0. \quad (3.27)$$

Set $D_0(x, y) = D_f(2x, 2y) - 4D_f(x, y)$ for all $x, y \in X$. Then we have

$$D_{Q_2}(x, y) = \lim_n \|16^n D_0(x, \frac{y}{2^n})\| = \lim_n 16^n \|D_f(x, \frac{y}{2^n}) - 4D_f(\frac{x}{2^n}, \frac{y}{2^n})\|$$

$$\leq \lim_n 16^{n-1} \|D_f(x, \frac{y}{2^n-1})\| + 4\|16^n D_f(\frac{x}{2^n}, \frac{y}{2^n})\|$$

$$\leq 16 \lim_n 16^{n-1} \phi(\frac{x}{2^n-1}, \frac{y}{2^n-1}) + 4 \lim_n 16^n \phi(\frac{x}{2^n}, \frac{y}{2^n}) = 0.$$

This means that $Q_2$ satisfies (1.5). By (3.27) it follows that $Q_2$ is quartic function. To prove the uniqueness property of $Q_2$, let $Q_2 : X \to Y$ be a quartic function which satisfies (1.5) and (3.23). Since $Q_2(2^n x) = 16^n Q_2(x)$, and $Q_2(2^n x) = 16^n Q_2(x)$ for all $x \in X$, then

$$\|Q_2(x) - Q_2(x)\| = 16^n \|Q_2(x) - Q_2(\frac{x}{2^n})\| \leq 16^n \|Q_2(\frac{x}{2^n}) - f(\frac{2x}{2^n}) - 4f(\frac{x}{2^n})\|$$

$$+ \|Q_2(\frac{x}{2^n}) - f(\frac{2x}{2^n}) - 4f(\frac{x}{2^n})\|$$

$$\leq 2 \sum_{i=0}^{n} 16^i \frac{1}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})$$

for all $x \in X$. Let $n \to \infty$ in above inequality. Then by (3.22), we have $Q_2(x) = Q_2(x)$ for all $x \in X$. This complete the proof of Theorem. \hfill $\Box$

**Theorem 3.4.** Suppose that an even mapping $f : X \to Y$ satisfies $\|D_f(x, y)\| \leq \phi(x, y)$ for all $x, y \in X$. If the upper bound $\phi : X x X \to [0, \infty)$ satisfies

$$\sum_{i=1}^{\infty} 16^i \phi(\frac{x}{2^i}, \frac{x}{2^i}) \leq \sum_{i=1}^{\infty} 16^i \phi(\frac{x}{2^i}, \frac{x}{2^i}) < \infty, \quad (3.28)$$

and $\lim_n 16^n \phi(\frac{x}{2^n}, \frac{y}{2^n}) = 0$ for all $x, y \in X$. Then there exist a unique quadratic function $Q_1 : X \to Y$ and a unique quartic function $Q_2 : X \to Y$ such that

$$\|f(x) - Q_1(x) - Q_2(x)\| \leq \frac{1}{12} \sum_{i=0}^{\infty} (4^i + 16^i) \left[ \frac{1}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \right]$$

for all $x \in X$. \hfill $\Box$

**Proof.** By Theorems 3.2 and 3.3, there exist a quadratic mapping $Q_{o1} : X \to Y$ and a quartic mapping $Q_{o2} : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q_{o1}(x)\| \leq \sum_{i=0}^{\infty} 4^i \frac{1}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})$$

$$\|f(2x) - 16f(x) - Q_{o2}(x)\| \leq \sum_{i=0}^{\infty} 4^i \frac{1}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) + \frac{16}{3} \phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})$$

for all $x \in X$. \hfill $\Box$
and
\[
\|f(2x) - 4f(x) - Q_{o2}(x)\| \leq \sum_{i=0}^{\infty} 16^i \left[ \frac{1}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right] \tag{3.31}
\]
for all \(x \in X\). Combining (3.30) and (3.31) to obtain
\[
\|f(x) + \frac{1}{12}Q_{o1}(x) - \frac{1}{12}Q_{o2}(x)\| \leq \frac{1}{12} \left[ \sum_{i=0}^{\infty} \left( 4^i + 16^i \right) \left( \frac{1}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right) \right].
\]
By putting \(Q_1(x) := -\frac{1}{12}Q_{o1}(x)\), and \(Q_2(x) := -\frac{1}{12}Q_{o2}(x)\) we get (3.29). To prove the uniqueness property of \(Q_1\) and \(Q_2\), let \(Q'_1, Q'_2 : X \to Y\) be another quadratic and quartic maps satisfying (3.29). Set \(Q''_1 = Q_1 - Q'_1\), \(Q''_2 = Q_2 - Q'_2\). Then by (3.28) we have
\[
\lim_{n \to \infty} 16^n \|Q''_1\left(\frac{x}{2^n}\right) - Q''_2\left(\frac{x}{2^n}\right)\| \leq \lim_{n \to \infty} 16^n \left[ \frac{1}{3} \phi\left(\frac{x}{2^n}, \frac{x}{2^{n+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^n}, \frac{x}{2^{n+1}}\right)\right] + \frac{2}{12} \sum_{i=0}^{\infty} \left( 16^i \times (2^i + 16^i) \right) \left[ \frac{1}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right]
\]
\[
\leq \frac{1}{6} \sum_{i=n}^{\infty} 2 \times 16^i \left[ \frac{1}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right] = 0 \quad \tag{3.32}
\]
for all \(x \in X\). On the other hand \(Q_2\) and \(Q'_2\) are quartic, then \(16^nQ''_2\left(\frac{x}{2^n}\right) = Q''_2(x)\). Thus by (3.32) it follows that \(Q''_2(x) = 0\) for all \(x \in X\). It is easy to see that \(Q''_1\) is quadratic. Then by putting \(Q'_2(x) = 0\) in (3.32), it follows that \(Q''_1(x) = 0\) for all \(x \in X\) and the proof is complete. \(\square\)

Now we establish the generalized Hyers-Ulam -Rassias stability of functional equation (1.5) as follows:

**Theorem 3.5.** Suppose that a mapping \(f : X \to Y\) satisfies \(f(0) = 0\) and
\[
\|Df(x, y)\| \leq \phi(x, y)
\]
for all \(x, y \in X\). If the upper bound \(\phi : X \times X \to [0, \infty)\) is a mapping such that
\[
\sum_{i=0}^{\infty} \left[ 16^i \left( \frac{1}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right) + \frac{8}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right] < \infty
\]
and that \(\lim_{n \to \infty} 16^n\phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right) = 0\) for all \(x, y \in X\). Then there exist a unique quadratic function \(Q_1 : X \to Y\), a unique cubic function \(C : X \to Y\) and a unique quartic function \(Q_2 : X \to Y\) such that
\[
\|f(x) - Q_1(x) - C(x) - Q_2(x)\| \leq \frac{1}{12} \sum_{i=0}^{\infty} \left( 4^i + 16^i \right) \left[ \frac{1}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{16}{3} \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\right]
\]
\[
+ \frac{1}{6} \sum_{i=1}^{\infty} 2^{i-1} \phi(0, \frac{x}{2^i}) + \frac{2}{3} \sum_{i=1}^{\infty} \phi(\frac{x}{2^i}, \frac{x}{2^i}) \tag{3.33}
\]
for all $x \in X$.

Proof. Let $f_ε(x) = \frac{1}{ε}(f(x) + f(-x))$ for all $x \in X$. Then $f_ε(0) = 0$, $f_ε(-x) = f_ε(x)$, and $\|D_{f_ε}(x,y)\| \leq \frac{1}{ε}[(φ(x,y) + φ(-x,-y)]$ for all $x,y \in X$. Hence in view of Theorem 3.4, there exist a unique quadratic function $Q_1 : X \to Y$ and a unique quartic function $Q_2 : X \to Y$ satisfies (3.29). Let $f_φ(x) = \frac{1}{φ}(f(x) - f(-x))$. Then $f_φ$ is an odd function, satisfies $\|D_{f_φ}(x,y)\| \leq \frac{1}{φ}[φ(x,y) + φ(-x, -y)]$. From Theorem 3.1, it follows that there exists a unique cubic function $C : X \to Y$ satisfies (3.2). Now it is easy to see that (3.33) holds true for all $x \in X$, and the proof of Theorem is complete.

Corollary 3.6. Let $p > 4$, $θ \geq 0$. Suppose that a mapping $f : X \to Y$ satisfies $f(0) = 0$, and

$$\|D_f(x,y)\| \leq θ(\|x\|^p + \|y\|^p)$$

for all $x,y \in X$. Then there exist a unique quadratic function $Q_1 : X \to Y$, a unique cubic function $C : X \to Y$ and a unique quartic function $Q_2 : X \to Y$ satisfying

$$\|f(x) - Q_1(x) - C(x) - Q_2(x)\| \leq \left[\frac{33 + 2p}{36} + \left(\frac{1}{2^p - 4} + \frac{1}{2^p - 16}\right) + \frac{3}{2(2^p - 8)}\right]θ\|x\|^p$$

for all $x \in X$.

Theorem 3.7. Suppose that an even function $f : X \to Y$ satisfies

$$\|D_f(x,y)\| \leq φ(x,y)$$

for all $x,y \in X$. If the upper bound $φ : X \times X \to [0, \infty)$ is a mapping such that

$$\sum_{i=1}^{∞} \frac{1}{4^i}[φ(2^{i+1}x, 2^i y) + φ(2^i x, 2^i y)] < ∞,$$

(3.34)

and that $\lim_n \frac{1}{4^n}φ(2^n x, 2^n y) = 0$ for all $x,y \in X$. Then the limit

$$Q_1(x) = \lim_n \frac{1}{4^n}[f(2^{n+1}x) - 16f(2^n x)]$$

is a unique quadratic function satisfies (1.5) and

$$\|f(2x) - 16f(x) - Q_1(x)\| \leq \frac{1}{4} \sum_{i=0}^{∞} \frac{1}{4^i}[\frac{1}{3}φ(2^{i+1}x, 2^i y) + \frac{16}{3} φ(2^i x, 2^i y)]$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 3.2, we can show that $f$ satisfies (3.20). Let $g(x) = f(2x) - 16f(x)$. Then by (3.20) we have

$$\|g(2x) - g(x)\| \leq \frac{1}{4} \sum_{i=0}^{∞} \frac{1}{4^i}[\frac{1}{3}φ(2^{i+1}x, 2^i y) + \frac{16}{3} φ(2^i x, 2^i y)].$$

(3.35)

By induction on $n$ and by (3.35) we have

$$\|g(2^n x) - g(x)\| \leq \frac{1}{4} \sum_{i=0}^{n-1} \frac{1}{4^i}[\frac{1}{3}φ(2^{i+1}x, 2^i y) + \frac{16}{3} φ(2^i x, 2^i y)]$$

(3.36)

for all $x \in X$. Dividing both sides of (3.36) by $4^m$ and replacing $x$ by $2^m x$ to get the relation
\[
\left\| g(2^{m+n}) - g(2^m) \right\| \leq \frac{1}{4} \sum_{i=0}^{n-1} \frac{1}{4^{m+i}} \left[ \frac{1}{3} \phi(2^{m+i+1}, x, 2^{m+i}) + \frac{16}{3} \phi(2^m, x, 2^m) \right] \\
\leq \frac{1}{4} \sum_{i=n}^{m+n-1} \frac{1}{4^{m+i}} \left[ \frac{1}{3} \phi(2^{m+i+1}, x, 2^{m+i}) + \frac{16}{3} \phi(2^m, x, 2^m) \right].
\]

By taking \( m \to \infty \) in above inequality and by using (3.34), we see that the sequence \( \{ g(2^n) \} \) is Cauchy in \( Y \). Since \( Y \) is complete, then

\[
Q_1(x) = \lim_n g(2^n) = \lim_n \frac{1}{4^n} [f(2^n) - 16f(2^n)]
\]

exists for all \( x \in X \). The rest of proof is similar to the proof of Theorem 3.2.

\[ \square \]

**Theorem 3.9.** Suppose an even function \( f : X \to Y \) satisfies

\[
\| D_f (x, y) \| \leq \phi(x, y)
\]

for all \( x, y \in X \). If the upper bound \( \phi : X \times X \to [0, \infty) \) is a mapping such that

\[
\sum_{i=0}^{\infty} \frac{1}{16^i} [\phi(2^i+1, x, 2^i) + \phi(2^i, 2^i)] < \infty
\]

and that \( \lim_n \frac{1}{16^n} \phi(2^n, 2^n) = 0 \) for all \( x, y \in X \). Then the limit

\[
Q_2(x) = \lim_n \frac{1}{16^n} [f(2^n) - 4f(2^n)]
\]

exists for all \( x \in X \), and \( Q_2 : X \to Y \) is a unique quartic function satisfies (1.5) and

\[
\| f(x) - 16f(x) - Q_2(x) \| \leq \frac{1}{16} \sum_{i=0}^{\infty} \frac{1}{4^n} [\phi(2^{i+1}, x, 2^i) + \frac{16}{3} \phi(2^i, 2^i)]
\]

for all \( x \in X \).

Proof. The proof is similar to the proof of Theorem 3.3.

\[ \square \]

**Theorem 3.9.** Suppose that an even function \( f : X \to Y \) satisfies

\[
\| D_f (x, y) \| \leq \phi(x, y)
\]

for all \( x, y \in X \). If the upper bound \( \phi : X \times X \to [0, \infty) \) satisfying

\[
\sum_{i=0}^{\infty} \frac{1}{4^i} [\phi(2^{i+1}, x, 2^i) + \phi(2^i, 2^i)] < \infty
\]

and that \( \lim_n \frac{1}{16^n} \phi(2^n, 2^n) = 0 \) for all \( x \in X \), then there exist a unique quadratic function \( Q_1 : X \to Y \), and a unique quartic function \( Q_2 : X \to Y \) such that

\[
\| f(x) - Q_1(x) - Q_2(x) \| \leq \frac{1}{12} \sum_{i=0}^{\infty} \left( \frac{1}{4^n} + \frac{1}{16^n} \right) [\phi(2^{i+1}, x, 2^i) + \frac{16}{3} \phi(2^i, 2^i)]
\]

for all \( x \in X \).
Proof. The proof is similar to the proof of Theorem 3.4.

\[ \Box \]

**Theorem 3.10.** Suppose that a function \( f : X \to Y \) satisfies \( f(0) = 0 \), and

\[ \|Df(x,y)\| \leq \phi(x,y) \]

for all \( x, y \in X \). If the upper bound \( \phi : X \times X \to [0, \infty) \) satisfies

\[ \sum_{i=1}^{\infty} \frac{1}{4^i} \left[ \phi(2^{i+1}x, 2^i x) + \phi(2^i x, 2^i x) \right] < \infty, \]

and

\[ \sum_{i=1}^{\infty} \frac{1}{8^i} \left[ \phi(2^i x, 2^i x) + 4\phi(0, 2^i x) \right] < \infty \]

for all \( x \in X \), and that \( \lim_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n y) = 0 \) for all \( x, y \in X \). Then there exist a unique quadratic function \( Q_1 : X \to Y \), a unique cubic function \( C : X \to Y \) and a unique quartic function \( Q_2 : X \to Y \) such that

\[ \| f(x) - Q_1(x) - C(x) - Q_2(x) \| \leq \frac{1}{12} \sum_{i=0}^{\infty} \left( \frac{1}{4^i} + \frac{1}{16^i} \right) \left[ \frac{1}{3} \phi(2^{i+1} x, 2^i x) + \frac{16}{3} \phi(2^i x, 2^i x) \right] \]

\[ + \frac{1}{6} \sum_{i=0}^{\infty} \frac{1}{8^{i+1}} \phi(0, 2^i x) + \frac{2}{3} \sum_{i=0}^{\infty} \frac{1}{8^{i+1}} \phi(2^i x, 2^i x) \]

for all \( x \in X \).

**Proof.** The proof is similar to the proof of Theorem 3.5.

By Theorem 3.10, we solve the following Hyers-Ulam-Rassias stability problem for functional equation (1.5).

**Corollary 3.11.** Let \( p < 3 \), and let \( \theta \) be a positive real number. Suppose that a mapping \( f : X \to Y \) satisfies \( f(0) = 0 \), and

\[ \|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \]

for all \( x, y \in X \). Then there exist a unique quadratic function \( Q_1 : X \to Y \), a unique cubic function \( C : X \to Y \) and a unique quartic function \( Q_2 : X \to Y \) satisfying

\[ \| f(x) - Q_1(x) - C(x) - Q_2(x) \| \leq \left[ (\frac{33 + 2^p}{9}) \left( \frac{1}{4 - 2^p} + \frac{4}{16 - 2^p} \right) + \frac{3}{2(8 - 2^p)} \right] \theta \|x\|^p \]

for all \( x \in X \).

By Corollary 3.11, we are going to investigate the Hyers-Ulam stability problem for functional equation (1.5).

**Corollary 3.12.** Let \( \epsilon \) be a positive real number. Suppose that a mapping \( f : X \to Y \) satisfies \( f(0) = 0 \) and \( \|Df(x,y)\| \leq \epsilon \) for all \( x, y \in X \). Then there exists a unique quadratic function \( Q_1 : X \to Y \), a unique cubic function \( C : X \to Y \) and a unique quartic function \( Q_2 : X \to Y \) satisfying
\[ \| f(x) - Q_1(x) - C(x) - Q_2(x) \| \leq \frac{431}{420} \]

for all \( x \in X \).

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