We study suppression of superconductivity by disorder in $d$-wave superconductors, and predict the existence of (at least) two sequential low temperature transitions as a function of increasing disorder: a $d$-wave to $s$-wave, and then an $s$-wave to metal transition. This is a universal property of the system which is independent of the sign of the interaction constant in the $s$-channel.

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Generally the order parameter in superconductors is a function of two coordinates and two spin indices. Classification of possible superconducting phases in crystalline materials was done in [1, 2]. A majority of low-$T_c$ crystalline superconductors have a singlet order parameter with $s$-wave symmetry. It does not change its sign under rotation, and in the isotropic case can be approximated by a complex number $\Delta (r)$ changes sign under rotation by $\pi/2$, and consequently $\Delta (r) = 0$. This means that the Fourier transform $\hat{\Delta} (k)$ changes its sign under a $\pi/2$ rotation as well, as is shown schematically by the rosettes in Fig.1. Since the sign of $\Delta (k)$ in crystalline $d$-wave superconductors depends on the direction of the wave vector $k$, they are much more sensitive to disorder than $s$-wave superconductors: at temperature $T = 0$, $d$-wave superconductivity gets destroyed when the electron mean free path $l$ is of the order of the zero temperature coherence length in a pure superconductor, $l \sim l_0 = 1.78 \xi_0 \gg 1/k_F$. Here $k_F$ is the Fermi wavelength. This is in contrast with the case of $s$-wave superconductors, where according to the Anderson theorem the superconductivity is destroyed at much higher level of disorder, when $l \sim 1/k_F$. The fate of the $d$-wave superconductors at $l < \xi_0$ depends on the sign of the interaction constant $\lambda_s$ in the $s$-wave channel. If the interaction $\lambda_s$ in the $s$-wave channel is attractive, but weaker than the attraction in the $d$-wave channel $|\lambda_s| < |\lambda_D|$, then at weak disorder, $(l > \xi_0)$, the superconducting order parameter has $d$-wave symmetry, while at $l < \xi_0$ the disorder destroys the $d$-wave superconductivity and the system undergoes a phase transition into an $s$-wave superconducting state. (See, for example, [3]).

In this article we consider a more interesting case, in which the interaction in the $s$-channel is repulsive at strong enough disorder $1/k_F \ll l \ll \xi_0$ the system is in normal state. We predict at least two low-temperature phase transitions: a $d$-wave to $s$-wave, and then an $s$-wave to normal metal transition. Qualitatively the phase diagram of disordered $d$-wave superconductors is shown in Fig.1. Let us first discuss the definition of $s$- and $d$- symmetries in bulk disordered systems. Before averaging over random realizations of disorder, the system does not possess any particular spacial symmetry at all. However in bulk samples, the symmetry is restored upon configuration averaging. We can think of several different definitions of the global symmetry of the order parameter: a) An operational definition is provided by the result of a phase sensitive experiment, such as the corner SQUID experiment, for example, [3, 4]. b) The quantity $\hat{\Delta} (r, r')$ can be characterized as having $d$-wave or $s$-wave symmetry. Here the over-line stands for the averaging over the sample volume. c) A globally $s$-wave component of the order parameter can be defined in terms of the local $s$-component of the anomalous Green function $F(r = r') \equiv \langle \hat{\Delta} (r) \rangle$. If we define $P_s$ to be the volume fraction of a sample where $F^{(s)}(r)$ has a positive or negative sign, respectively, then the system has an $s$-wave component if $(P^s - P^f) \neq 0$. These definitions may be not equivalent under all circumstances. However, for the most part, we will deal with the interval of parameters in which all these definitions are approximately interchangeable.

It is important to realize that it is inevitable near criticality to have a situation in which the local pairing in disordered superconductors is "$d$-wavelike" and yet the global superconductivity has $s$-wave symmetry. The $d$-wave to $s$-wave transition can be understood at the mean field level. The electron mean free path is an average characteristic of disorder. Let us introduce a "local" value of the mean free path $l(r)$ averaged over a size of order $\xi_0$. In the region of parameters where $d$-wave superconductivity is sufficiently suppressed by disorder, the spatial dependence of the order parameter can be visualized as a system of superconducting puddles with anomalously large values of the order parameter, which are connected by Josshepsion links through non-superconducting metal. The superconductivity inside the puddles may
be enhanced because either the electron interaction constant, or the mean free path in the puddles (or both) may be larger than their average values.

Let us assume that the distance between the puddles is larger than both their size and the mean free path. In this case the system is already in a state with the "global s-wave" symmetry. Its origin is illustrated qualitatively in Fig. 2, where a system of superconducting puddles of arbitrary shape embedded into a metal is shown. The order parameter inside the puddles has d-wave symmetry, and the orientation of the gap nodes is assumed to be pinned by the crystalline anisotropy. In a d-wave superconductor, in addition to an overall phase of the order parameter on individual puddles, one can calculate the Josephson coupling between a pair of far separated puddles. Since the time that it takes for electrons to travel between puddles is shorter than the characteristic time of fluctuations of the order parameter on individual puddles, one can calculate $J_{ij}$ using the mean-field Usadel equation for the configuration-averaged anomalous Green function 

$$
\Delta(r) = \frac{1}{2} \sum_{i \neq j} J_{ij} \cos(\phi_i - \phi_j),
$$

(3)

where $J_{ij} > 0$ characterizes the strength of the exchange interaction between the d-wave components of the order parameter. Typically, at small $|r_i - r_j|$, $J_{ij} > J_{ij}^{(s)}$, but at large $|r_i - r_j|$ the coupling strength $J_{ij}^{(s)}$ decays more slowly than $J_{ij}^{(d)}$. Here $r_i$, $r_j$ are coordinates of the puddles. Thus it is likely that in this intermediate region the system may exhibit spin glass features and/or coexistence of d-wave and s-wave ordering. In this article, however, we will not further explore this fascinating but complex aspect of this problem.

To quantify the picture presented above one has to compute the Josephson coupling between a pair of far separated puddles. Since the time that it takes for electrons to travel between puddles is shorter than the characteristic time of fluctuations of the order parameter on individual puddles, one can calculate $J_{ij}^{(s)}$ using the mean-field Usadel equation for the configuration-averaged anomalous Green function 

$$
\langle F_x(r) \rangle = -\sin(\theta(\epsilon, r)) \text{ in the metal,}
$$

(4)

Here $D_{tr}$ is the transport diffusion coefficient of electrons in the metal, $F_x(r)$ is the Fourier transform of $F(\epsilon, r, (t - t'))$, $\Delta^{(s)} = \lambda^{(s)} F(r)$, and the brackets $\langle \rangle$ indicate averaging over random scattering potential between the puddles at a given shape of the puddles. The only, but crucial difference with the conventional case of s-n junctions (See, for example, [13, 14]), is the boundary conditions for

For the case when the size of the puddle is larger than the coherence length and the Andreev reflection on the puddles is effective the boundary conditions for
Eq.[4] on the d-n boundary have been derived in Ref. [12]. Since the relevant energy for computing the Josephson coupling, $\epsilon \approx D_{12}/|r_i - r_j|^2$, is much smaller than the value of the order parameter in the puddles, the boundary condition for $\theta(\mathbf{r}, \epsilon)$ is independent of $\epsilon$ and depends only on the angle between the unit vector parallel to the direction of a gap node $\hat{n}_2$ and a unit vector, $\hat{n}(\mathbf{r})$, normal to the boundary at point $\mathbf{r}$ at the surface, $: \theta_s(\mathbf{r}, \epsilon) = f(\alpha(\mathbf{r}))$, $\sin[\alpha(\mathbf{r})] \equiv \hat{n}(\mathbf{r}) \cdot \hat{n}_2$. Here $f(\alpha)$ is a smooth, approximately odd and periodic function, $f(\alpha) \approx f(-\alpha)$, $f(\alpha) \approx f(\alpha + \pi)$, which grows from $f(\alpha) \approx 0$ at $\alpha = 0$, to $f(\alpha) \approx \pm \zeta$ for $\alpha = \pi/4$, where $\zeta \sim 1$. Solving Eq.[4] with these boundary conditions, and using the standard procedure of calculation of the Josephson energy we get

\[ J_{ij}^s \sim C \left| \frac{V}{|r_i - r_j|^2} \right| \exp(-\frac{|r_i - r_j|}{L_T}) \]

\[ \eta_i = \text{sign} \left\{ \int ds f(\alpha) \right\} \]

and $C \sim G_{ej}/R_{22}$, $V$ is the puddle volume, the integral is taken over the surface of the $i$th puddle, and $G_{ej}$ is the conductance of a metal of a size of order of the size of the superconducting puddle. In this case the magnitude of the $s$-component of the order parameter generated at the superconductor-normal metal boundary is of order of the magnitude of the $d$-wave component. Thus it is not surprising that the value of $J_{ij}^s$ in Eq.[4] turns out to be of the same order as in the case of SNS junction.

If the distribution function of the mean-free paths is unbounded, and with certain probabilities one can find arbitrary large values of $l(\mathbf{r})$, the mean field superconducting solution always exists. However, if the puddle concentration is small enough, the transition from the state with global $s$-wave symmetry to the normal metal is triggered by a competition between the inter-puddle Josephson coupling energy and the thermal (or quantum) fluctuations. Thermal fluctuations destroy the coherence between two puddles when $J_{ij} \sim kT$, which gives us an expression for the critical temperature $T_c$ of the $s$-wave superconductor-metal transition

\[ T_c \sim CV \frac{R}{R^D} \]

where $R$ is the inter-puddle distance.

We would like to stress that the existence of the $s$-wave superconducting phase is a generic property of the system because the long-range nature of the decay of Eq.[5] ensures that near the superconductor-normal metal transition and at small enough temperatures the superconducting puddles are separated by a distance larger than their size.

In principle, the situation described above can be realized when grains of $d$-wave superconductors are embedded into a normal metal artificially. In random systems the critical point can be identified by finding the set of “optimal puddles” which lie on the critical links of “the percolating cluster”. In this case the properties of the $s$-wave phase and the dependence of the critical temperature $T_c$ on the parameters of the system depends on details of the distribution function of the disordered potential. To illustrate the situation we consider here a simple model where the mean free part $l(\mathbf{r})$ is a random function of coordinates with a Gaussian distribution characterized by an average $\bar{l}$, a variance $\sigma l_0$, and a correlation length which is of order $\xi_0$. To be concrete, we consider the 2D case. Then the distance between the puddles becomes of order of their size, the amplitude of fluctuations of the order parameter becomes of order of the average, and the system has a transition to the $s$-wave state when $l \sim l_{c1}$ and $T < T_{c1}$

\[ \bar{l}_{c1} - l_0 \sim \sigma^2 l, \quad T_{c1} \sim \sigma T_{c0}. \]

Here $T_{c0}$ is the critical temperature of a pure $d$-wave superconductor. If $l_0 - \bar{l} > \sigma^2 l$ the distance between ”the optimal puddles” is much bigger than their size. We can characterize such puddles by a value of the mean free path $l_{opt} > l_0$ averaged over the volume of the puddle. In this case $\Delta_{opt} \sim \Delta_0 l_0/(l_{opt} - l_0)^{1/2} \ll \Delta_0$, the size of the puddle is of order of the zero temperature coherence length $\xi_{opt} \sim \xi_0 l_0/(l_{opt} - l_0)^{1/2} \gg \xi_0$, and the characteristic distance between the puddles is of order of $\xi_{opt}$.

\[ \xi_{opt} \sim \xi_0 l_0/(l_{opt} - l_0). \]

Here $\Delta_0$ is the magnitude of the order parameter in a pure $d$-wave superconductor at $T = 0$. This expression has a minimum at $(l_{opt} - l_0) \sim (l_0 - \bar{l})$, and therefore $R_{opt} \sim \exp[(l_0 - l)/(l_0 l_0^2)]$. Using Eq.[6] we get

\[ T_{c1} \sim T_{c0} \sigma \exp\left(-\frac{(l_0 - \bar{l})}{l_0 l_0^2}\right). \]

At very small values of $T_{c1}$ the phase transition between the $s$-wave superconducting phase and the normal metal is triggered by quantum fluctuations of the order parameter. In this case to determine the point of the transition one has to compare $J_{ij}$ with the ”quantum temperature” characterized by the zero temperature superconducting susceptibility $\chi_1$ of individual puddles. [13, 14, 15]. Their
strongly as a function of the doped hole concentration, findings apply to the cuprates as applicable. We assume, some of the more robust of our 

thus, the present considerations are not appear to undergo a superconductor to insulator transition, with increasing underdoping, these materials frequently 

On the underdoped side of the superconducting dome, 

the “underdoped” side, and an upper critical concentration, \( x_2 \), on the “overdoped” side of the phase diagram. On the underdoped side of the superconducting dome, with increasing underdoping, these materials frequently appear to undergo a superconductor to insulator transition [6, 7, 8]. Thus, the present considerations are not applicable. We assume, some of the more robust of our findings apply to the cuprates as \( T_c \to 0 \) with overdoping. There are a number of interesting predictions we can make. 1) There should be a transition from a globally \( d\)-wave to a globally \( s\)-wave superconducting state at a doping concentration \( x = x_2 \). (Some evidence of such a transition may already be present in the experiments of Ref. [3].) 2) In the metallic state with \( x > x_2 \), the conductivity at low temperature should diverge as \( x \to x_2 \), the Hall resistance should vanish, and the Weideman-Franz law should be increasingly strongly violated.

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