COMPLETENESS OF THE BISPECTRUM ON COMPACT GROUPS

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Abstract. This paper derives completeness properties of the bispectrum for compact groups and their homogeneous spaces. The bispectrum is the Fourier transform of the triple correlation, just as the magnitude-squared spectrum is the Fourier transform of the autocorrelation. The bispectrum has been applied in time series analysis to measure non-Gaussianity and non-linearity. It has also been applied to provide orientation and position independent character recognition, as well as to analyze statistical properties of the cosmic microwave background radiation; in both cases, the data may be defined on a sphere. On the real line, it is known that the bispectrum is not only invariant under translation of the underlying function, but in many cases of interest, it is also complete, in that the function may be recovered uniquely up to a translation from its bispectrum. This paper extends the completeness theory of the bispectrum to compact groups and their homogeneous spaces, including the sphere. The main result, which depends on Tannaka-Krein duality theory, shows that every function whose Fourier coefficient matrices are always nonsingular is completely determined by its bispectrum, up to a single group action. Furthermore, algorithms are described for reconstructing functions defined on SU(2) and SO(3) from their bispectra.

Key words. bispectrum, triple correlation, pattern recognition, invariants, Tannaka-Krein duality.

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1. Introduction. The triple correlation of a complex-valued function on the real line is the integral of that function multiplied by two independently-shifted copies of itself:

\[ a_{3,f}(s_1, s_2) = \int_{-\infty}^{\infty} f^*(x) f(x + s_1) f(x + s_2) \, dx. \]  

It is easily seen that the triple correlation does not change if the function is translated. The Fourier transform of triple correlation is the bispectrum. If the Fourier transform of \( f \) is denoted \( F = F\{f\} \), then the bispectrum is

\[ A_{3,f}(u, v) = F\{a_{3,f}(s_1, s_2)\} = F(u) F(v) F^*(u + v). \]  

The triple correlation extends the concept of autocorrelation (denoted here \( a_{2,f} \)) which correlates a function with a single shifted copy of itself, thereby enhancing the function’s latent periodicities:

\[ a_{2,f}(s) = \int_{-\infty}^{\infty} f^*(x) f(x + s) \, dx. \]  

The Fourier transform of the autocorrelation is \( A_{2,f} = F\{a_{2,f}\} = |F|^2 \), which obviously lacks phase information and therefore provides a limited analysis of a function’s structure. In contrast, eq. (1.2) shows that the bispectrum contains both magnitude and phase information, while still being invariant to translation. These properties suggest applications in invariant matching for pattern recognition. More importantly for matching, the bispectrum in many cases of interest is not only invariant but also provides a complete description of the function: the function may be reconstructed from it, up to a single unknown translation [11].

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The bispectrum was perhaps first investigated by statisticians examining the cumulant structure of non-Gaussian random processes [3]. It is well known that the third order cumulant, of which the triple correlation is a sample, has zero expected value for a Gaussian process. Hence, the bispectrum is a tool for measuring non-Gaussianity. The bispectrum was also independently studied by physicists as a tool for spectroscopy. H. Gamo [6] in 1963 described an apparatus for measuring the triple correlation of a laser beam, and also showed how phase information can be completely recovered from the real part of the bispectrum—up to sign reversal and linear offset. Gamo’s is perhaps the first completeness result along the lines of what is explored in detail in this paper. However, Gamo’s method implicitly requires the Fourier transform to never be zero at any frequency. This requirement was relaxed, and the class of functions which are known to be completely identified by their bispectra was considerably expanded, by the study of Yeellott and Iverson [11]: for example, every integrable function with compact support is completely determined, up to a translation, by its bispectrum.

The statistical and physical applications described above are for data defined on Euclidean domains \( \mathbb{R}^n \). The bispectrum also finds applications on non-Euclidean domains such as the sphere \( S^2 \). For example, in astrophysics, the Cosmic Background Radiation (CBR) may be modelled as a function defined on a sphere. X. Luo [16] calculates the bispectrum of spherical CBR functions and examines the properties the bispectral coefficients have under cosmological scenarios such as inflation and late-time phase transitions.

The bispectrum’s invariance and completeness have motivated researchers in pattern recognition to apply it for template matching and shape recognition. For example, R. Kondor [14, Ch. 8] shows how position and orientation independent optical character recognition can be accomplished by projecting the character from the plane on to the sphere, and subsequently using the bispectrum on the sphere for invariant matching. Kondor’s results are discussed further in Section 5.

Despite the interest in applying the bispectrum for non-Euclidean domains, little has been published about important properties such as completeness. The contribution of this paper is to derive the completeness theory of the bispectrum for (non-commutative) compact groups and their homogeneous spaces. In order to construct the bispectrum on groups, we require concepts from harmonic analysis using group representation theory. Those concepts are presented in the next section. Then, a matrix form of the bispectrum which proves convenient for analysis is demonstrated. The matrix formulation allows a relatively simple criterion for completeness: it is shown that functions defined on compact groups that have nonsingular Fourier transform coefficients are completely determined by their bispectra. This result depends on the well-known Tannaka-Krein duality theory of compact group representations. The completeness result is extended to homogeneous spaces using the Iwahori-Sugiura duality theorem [9]. Reconstruction algorithms for functions defined on \( SU(2) \) and \( SO(3) \) in particular are described, expanding on material in a previous paper [13].

2. Preliminaries. Let us review the basic concepts of representation theory for compact groups. For more details, the reader may consult Chevalley ([4], Chapter VI). Let \( G \) be a compact group. A \( n \)-dimensional unitary representation of \( G \) is a continuous homomorphism \( D \) of \( G \) into the group \( U(n, \mathbb{C}) \) of \( n \times n \) unitary matrices. The following operations are all defined on representations: complex-conjugation, direct sum, and tensor product. The complex conjugate of any representation is simply that representation \( D^* \) whose matrices are complex-conjugates of those of \( D \). Let \( D_1 \),
$D_2$ be respectively a $n$-dimensional and a $m$-dimensional representation; their direct-sum $D_1 \oplus D_2$ is the $(n+m)$-dimensional representation which maps each element $g$ to the block-diagonal matrix $D_1(g) \oplus D_2(g)$ with $D_1(g)$ in the upper left corner and $D_2(g)$ in the lower right corner. Similarly, the tensor product $D_1 \otimes D_2$ of $D_1$ and $D_2$ is $nm$-dimensional representation whose matrices are $n \times n$ blocks, each block of size $m \times m$, where for each $g$ the $(i,j)$-th block is the matrix $D_2(g)$ multiplied by the $(i,j)$-th coefficient of $D_1(g)$. We reserve the symbol $\mathbf{1}$ for the trivial representation that maps all of $G$ into the number $1$.

We now define equivalence and reducibility of representations. Two representations $D_1$ and $D_2$ are equivalent if there exists a unitary matrix $C$ such that $D_1(g) = CD_2(g)C^\dagger$ for all $g \in G$, with $\dagger$ denoting the matrix conjugate transpose. A representation $D$ is reducible if there exists a matrix $C$ and two representations $D_1$, $D_2$ such that $D(g) = C[D_1(g) \oplus D_2(g)]C^\dagger$ for all $g \in G$; otherwise $D$ is irreducible. Every representation $D$ is the direct sum of irreducible unitary representations.

The set $\mathcal{G}$ of all equivalence classes of irreducible unitary representations of $G$ is called the dual object of $G$. Intuitively, $\mathcal{G}$ is the “frequency” domain of $G$. In general, $\mathcal{G}$ is not a group, unless $G$ is abelian. In what follows, we denote elements of $\mathcal{G}$, which are equivalence classes, by Greek letters such as $\alpha$, $\beta$, etc. For each $\alpha \in \mathcal{G}$, let $\dim(\alpha)$ denote the common dimension of the representations in $\alpha$. Let $\{D_\alpha\}_{\alpha \in \mathcal{G}}$ denote any set of representations that contain exactly one member in each equivalence class in $\mathcal{G}$. We call any such set a selection. A given selection has two properties, both consequences of the classical result known as the Peter-Weyl theorem: (1) there are at most countably many representations in any selection, and in particular there are finitely many if, and only if, $G$ is finite, i.e., $\mathcal{G}$ is finite if and only if $G$ is finite; (2) the set of all matrix coefficients $d^\alpha_{ij}(\cdot)$, where $\alpha \in \mathcal{G}$ and $1 \leq p, q \leq \dim(\alpha)$, from any selection forms an orthogonal basis for the Hilbert space $L_2(G)$.

2.1. Duality theory. Our main tool for proving completeness results is the duality theorem due to T. Tannaka and M. Krein. There are several formulations of that result, of which Chevalley’s [4] pp 188-203 is the most convenient for our purposes. The formulation is as follows. Let $\Theta(G)$ be the representative algebra of $G$, which is the algebra of complex-valued functions on $G$ that is generated by the set of matrix coefficients of any selection $\{D_\alpha\}_{\alpha \in \mathcal{G}}$. In fact, $\Theta(G)$ is independent of the selection that is used. It is well-known that every function $f \in \Theta(G)$ may be expressed in exactly one way as a finite linear combination of the set of matrix coefficients from any given selection ([4] pg 189]). The structure of $\Theta(G)$ may be understood by considering algebra homomorphisms, which are maps $\omega : \Theta(G) \rightarrow \mathbb{C}$ that are both linear and multiplicative, i.e., $\omega(c_1 f_1 + c_2 f_2) = c_1 \omega(f_1) + c_2 \omega(f_2)$ and $\omega(f_1 f_2) = \omega(f_1) \omega(f_2)$ for all scalars $c_1$, $c_2$, and functions $f_1$, $f_2$ in $\Theta(G)$. The set of all algebra homomorphisms is denoted $\Omega(G)$. Clearly, for every $g \in G$, the map $\omega_g(f) = f(g)$ is an algebra homomorphism. Note that $\omega_g(f^*) = \omega_g(f)^*$, i.e., $\omega_g$ preserves complex-conjugation. In fact, the converse is also true, an identification that is essential to the duality between groups and their representations; see [4] pg 211 for details and proofs.

Theorem 2.1 (Tannaka-Krein). To every algebra homomorphism $\omega \in \Omega(G)$ that preserves complex-conjugation, i.e., $\omega(f^*) = \omega(f)^*$, there corresponds a unique element $g \in G$ such that $\omega(f) = f(g)$ for all $f \in \Theta(G)$.

We recast this duality theorem in a slightly different form (see also [20], pp 303-306). Let $\{D_\alpha\}_{\alpha \in \mathcal{G}}$ be any selection of irreducible representations, and let $\{U(\alpha)\}_{\alpha \in \mathcal{G}}$ be a corresponding sequence of unitary matrices, such that for each $\alpha$ the matrix $U(\alpha)$
has the same dimension as the representation $D_\alpha$. We determine the necessary and sufficient conditions under which the latter sequence arises from the former by an element of $\Theta(G)$, i.e., when $U(\alpha) = \omega(D_\alpha)$ for some fixed homomorphism $\omega \in \Omega(G)$. Consider the tensor product $D_\sigma \otimes D_\delta$ of any two representations in our selection; that representation is, in general, reducible, and we write its decomposition into irreducibles (taken from our selection) as follows:

$$D_\sigma \otimes D_\delta = C_{\sigma\delta} \left[ D_{\alpha_1} \oplus \cdots \oplus D_{\alpha_k} \right] C_{\sigma\delta}^\dagger.$$  

The indices $\alpha_1, \ldots, \alpha_k$ appearing on the right are unique up to permutation (pg 175). Suppose now that there exists $\omega$ such that $U(\alpha) = \omega(D_\alpha)$ for all $\alpha$. By applying $\omega$ to both sides of eq. (2.1), and using the fact that $\omega$ is both linear and multiplicative, we obtain that (writing $U(\alpha)$ for $\omega(D_\alpha)$)

$$U(\sigma) \otimes U(\delta) = C_{\sigma\delta} \left[ U(\alpha_1) \oplus \cdots \oplus U(\alpha_k) \right] C_{\sigma\delta}^\dagger.$$  

Equation (2.2) is not only necessary, but also sufficient, as the following result shows.

**Theorem 2.2.** Let $\{D_\alpha\}_{\alpha \in G}$ and $\{U(\alpha)\}_{\alpha \in G}$ be as above. If, whenever eq. (2.1) is true, we have that eq. (2.2) is also true for same $\sigma, \delta$, and matrix $C_{\sigma\delta}$, then there exists a fixed $g \in G$ such that $U(\alpha) = D_\alpha(g)$ for all $\alpha$.

**Proof.** To any sequence $\{U(\alpha)\}_{\alpha \in G}$, there exists a unique linear map $\omega : \Theta(G) \to \mathbb{C}$ such that $U(\alpha) = \omega(D_\alpha)$. To see this, note that the set of matrix coefficients in any selection is linearly independent, and furthermore, any function $f \in \Theta(G)$ may be written uniquely as a finite linear combination of the matrix coefficients. Thus it is always possible to construct a linear map $\omega : \Theta(G) \to \mathbb{C}$ that gives any desired set of values to the corresponding coefficient functions; in particular, there exists an $\omega$ such that $\omega(D_\alpha) = U(\alpha)$. We now show that $\omega$ is multiplicative and conjugate preserving. Applying the linear map $\omega$ to both sides of eq. (2.1) results in the following identity:

$$\omega(D_\sigma \otimes D_\delta) = C_{\sigma\delta} \left[ \omega(D_{\alpha_1}) \oplus \cdots \oplus \omega(D_{\alpha_k}) \right] C_{\sigma\delta}^\dagger.$$  

Substituting the matrices $U$ on the right side and using (2.2) reveals that

$$\omega(D_\sigma \otimes D_\delta) = \omega(D_\sigma) \otimes \omega(D_\delta).$$  

Consequently, $\omega$ is multiplicative. To prove that $\omega$ preserves conjugation, note that the equation $D_\sigma D_\delta^\dagger = I$ implies that $\omega(D_\sigma) \omega(D_\delta^\dagger) = I$ for all $\alpha$. Since the matrices $U(\alpha) = \omega(D_\alpha)$ are unitary, we also have $\omega(D_\alpha) \omega(D_\alpha)^\dagger = I$. Matrix inverses are unique, and thus $\omega(D_\delta^\dagger) = \omega(D_\delta)$, showing that $\omega$ preserves conjugation. By the Tannaka-Krein theorem, there exists a unique $g \in G$ such that $U(\alpha) = \omega(D_\alpha) = D_\alpha(g)$ for all $\alpha$. \(\Box\)

**3. Bispectrum.** We use this result to establish sufficient conditions for a function to be described uniquely by its bispectrum. It is convenient to establish the Fourier transform domain for compact groups. Let $\{D_\alpha\}_{\alpha \in G}$ be any selection of irreducible representations. The Fourier transform of any $f \in L_1(G)$ is the matrix-valued function $F$, such that for each $\alpha \in G$, we have

$$F(\alpha) = \int_G f(g) D_\alpha(g) \, dg.$$  

Here the integral uses the Haar measure $dg$ on $G$; because $G$ is compact, $dg$ is both left and right invariant.
We use two important properties of the Fourier transform in what follows ([5 pp 73-78]): (i) the Fourier transform of any \( f \in L_1(G) \) determines \( f \) uniquely up to a set of Haar measure zero; (ii) \( s(g) = r(xg) \) for all \( g \), and only if, \( S(\alpha) = R(\alpha)D_\alpha(x) \) for all \( \alpha \).

For \( f \in L_1(G) \), the triple correlation \( a_{3,f} \) is defined as follows:

\[
(3.2) \quad a_{3,f}(g_1, g_2) = \int_G f(g) f(g_1) f(g_2) dg.
\]

(Compare with eq. (1.1)). Note that the triple-correlation is invariant under left-translation, i.e., if there exists \( x \) such that \( r(g) = s(xg) \) for all \( g \), then \( a_{3,f} = a_{3,g} \). This follows directly from the left-invariance of the Haar measure \( dg \). Similarly, we may define a right-translation invariant version of eq. (3.2) by integrating \( f(g)^* f(g_1) f(g_2) \), but we will not pursue this minor variation in what follows.

Because \( f \in L_1(G) \), we have that \( a_{3,f} \) is a function in \( L_1(G \times G) \). It is known that any irreducible representation of \( G \times G \) is equivalent to a tensor product \( D_\sigma \otimes D_\delta \), where \( D_\sigma \otimes D_\delta \) are irreducible representations of \( G \) ([17 pg 45]). Thus the Fourier transform of \( a_{3,f} \) with respect to the selection \( \{D_\alpha\}_{\alpha \in \mathcal{G}} \) is the function on \( G \times G \) that is defined as follows:

\[
(3.3) \quad A_{3,f}(\sigma, \delta) = \int_G \int_G a_{3,f}(g_1, g_2) [D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger] \, dg_1 \, dg_2.
\]

There exists a convenient formula for computing \( A_{3,f} \).

**Lemma 3.1.** For any pair \( \sigma, \delta \), let \( C_{\sigma\delta} \) be the matrix and let \( \alpha_1, \ldots, \alpha_k \) be the indices appearing in eq. (3.4). Then

\[
(3.4) \quad A_{3,f}(\sigma, \delta) = [F(\sigma) \otimes F(\delta)] C_{\sigma\delta} [F(\alpha_1)^\dagger \otimes \cdots \otimes F(\alpha_k)^\dagger] C_{\sigma\delta}^\dagger.
\]

**Proof.** Since \( a_{3,f} \) is integrable, we use the Fubini theorem to interchange the order of integration in the following derivation:

\[
A_{3,f}(\sigma, \delta) = \int_G \int_G a_{3,f}(g_1, g_2) [D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger] \, dg_1 \, dg_2,
\]

\[
= \int_G \int_G f(g)^* f(g_1) f(g_2) [D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger] \, dg_1 \, dg_2,
\]

\[
= \int_G f(g)^* \int_G f(g_1) f(g_2) [D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger] \, dg_1 \, dg_2 \, dg.
\]

By making a change of variables, we find that the double integral inside simplifies as follows:

\[
\int_G \int_G f(g_1) f(g_2) [D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger] \, dg_1 \, dg_2 = [F(\sigma) \otimes F(\delta)] [D_\sigma(g) \otimes D_\delta(g)].
\]

Upon substituting into the expression for \( A_{3,f} \), we find that

\[
(3.5) \quad A_{3,f} = [F(\sigma) \otimes F(\delta)] \int_G f(g)^* [D_\sigma(g) \otimes D_\delta(g)] \, dg.
\]

Upon substituting the tensor product decomposition (2.1) into the above, we obtain

\[
(3.6) \quad A_{3,f}(\sigma, \delta) = [F(\sigma) \otimes F(\delta)] C_{\sigma\delta} \left[ \int_G f(g)^* (D_{\alpha_1}(g) \otimes \cdots \otimes D_{\alpha_k}(g)) \, dg \right] C_{\sigma\delta}^\dagger.
\]
After evaluating the integral, the result (3.4) follows.

The lemma helps to quickly establish the basic completeness result for the bi-

spectrum on compact groups.

**Theorem 3.2.** Let $G$ be any compact group, and let $r$ in $L_1(G)$ be such that its
Fourier coefficients $R(\alpha)$ are nonsingular for all $\alpha \in \mathcal{G}$. Then $a_{3,s} = a_{3,r}$ for some
$s \in L_1(G)$ if and only if there exists $x \in G$ such that $s(g) = r(xg)$ for all $g$.

**Proof.** If $s(g) = r(xg)$, then the translation-invariance of the triple correlation
implies that $a_{3,s} = a_{3,r}$. We now prove the converse. Let $s$ be such that $a_{3,s} = a_{3,r}$;
then $A_{3,r} = A_{3,s}$, and by Lemma 3.1 we obtain that for $\sigma$, $\delta$ that

\[ [R(\sigma) \otimes R(\delta)] C_{\sigma \delta} \left[ R(\alpha_1)^\dagger \oplus \cdots \oplus R(\alpha_k)^\dagger \right] = \\
(3.7) [S(\sigma) \otimes S(\delta)] C_{\sigma \delta} \left[ S(\alpha_1)^\dagger \oplus \cdots \oplus S(\alpha_k)^\dagger \right] \]

Set $\sigma = \delta = 1$, where 1 is the trivial representation $g \mapsto 1$ of $G$. Both $R(1)$ and $S(1)$
are complex numbers, and the equality above becomes

\[ R(1)R(1)^* = S(1)S(1)^*. \]

Thus $R(1) = S(1)$. Now set $\delta = 1$; for any $\sigma$, we have $D_\sigma \otimes D_1 = D_\sigma$, and thus

\[ \text{eq. (3.7) becomes} \]

\[ (3.8) [R(\sigma) \otimes R(1)] R(\sigma)^\dagger = [S(\sigma) \otimes S(1)] S(\sigma)^\dagger. \]

By assumption $R(1) = S(1)$ is a non-zero scalar, and we cancel it from both sides
to obtain that $R(\sigma)R(\sigma)^\dagger = S(\sigma)S(\sigma)^\dagger$ for all $\sigma$. Such an equality between matrices
holds if and only if there exists a unitary matrix $U(\sigma)$ such that $S(\sigma) = R(\sigma)U(\sigma)$.
Substituting for $S$ in (3.7) yields, upon rearranging terms,

\[ [R(\sigma) \otimes R(\delta)] C_{\sigma \delta} \left[ R(\alpha_1)^\dagger \oplus \cdots \oplus R(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger = \\
[\sigma R(\sigma) \otimes U(\delta)] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] \left[ R(\alpha_1)^\dagger \oplus \cdots \oplus R(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger. \]

We cancel the nonsingular matrices $R(\sigma)$ from both sides and rearrange the remaining
terms to obtain the identity

\[ (3.10) U(\sigma) \otimes U(\delta) = C_{\sigma \delta} \left[ U(\alpha_1) \oplus \cdots \oplus U(\alpha_k) \right] C_{\sigma \delta}^\dagger. \]

Since the identity above holds for all $\sigma$, $\delta$, Theorem 2.2 implies that there exists
$x \in G$ such that $U(\sigma) = D_\sigma(x)$ for all $\sigma$. Thus $S(\sigma) = R(\sigma)D_\sigma(x)$ for all $\sigma$, and the
translation property of the Fourier transform now implies that $s(g) = r(xg)$ for all $g$.

The hypothesis that all coefficients $R(\sigma)$ are nonsingular is satisfied generically, in
the sense that almost every $n \times n$ matrix is nonsingular with respect to the Lebesgue
measure on the set of $n \times n$ matrices. Nevertheless, it is desirable to weaken the
hypothesis, to include for example functions on $G$ that are invariant under the trans-
formations of a normal subgroup $N$ of $G$. We prove a result for this case.

We review some facts concerning group representations and normal subgroups
([8 pg 64]). Let $N$ be a closed normal subgroup of $G$. Any irreducible representation
$\tilde{D}$ of the quotient group $G/N$ extends to an irreducible representation $D$ of $G$ by
composition: $D = \tilde{D} \circ \pi$, where $\pi$ is the canonical coset map $\pi : G \to G/N$. The
converse is also true: any representation $D$ of $G$ such that $D(n) = I$ for all $n \in N$ is
of the form \( D = \overline{D} \circ \pi \) for some representation \( \overline{D} \) of \( G/N \). Moreover, letting \( (G/N) \) represent the dual object of the group \( G/N \), the set

\[
(3.11) \quad \mathcal{G}[N] = \{ D = \overline{D} \circ \pi, \overline{D} \in (G/N) \},
\]

is closed under both conjugation and tensor-product decomposition, i.e., the tensor product of any two representations from the set decomposes into irreducible representations that are also contained in the set. Conversely, to each subset \( A \) of \( \mathcal{G} \) that contains \( 1 \) and that is closed under both conjugation and tensor-product decomposition, there corresponds a unique closed and normal subgroup \( N \) of \( G \) such that \( A = \mathcal{G}[N] \).

Now let \( f \) be a function in \( L_2(G) \) that is invariant under \( N \), i.e., \( f(ng) = f(gn) = f(g) \) for all \( n \in N \). If \( \alpha \notin \mathcal{G}[N] \), then the Fourier coefficient matrix \( F(\alpha) \) is a zero matrix. To prove this, note that the Peter-Weyl theorem implies that the matrix coefficients \( \overline{d}^{\alpha}(\cdot) \) from any selection \( \{ D_\alpha \}_{\alpha \in (G/N)} \) form an orthogonal basis for \( L_2(G/N) \); thus the corresponding functions \( \overline{d}^{\alpha} = \overline{d}^{\alpha} \circ \pi \) on \( G \) form an orthogonal basis for the closed subspace in \( L_2(G) \) of functions invariant under \( N \). Consequently, any \( N \)-invariant function in \( L_2(G) \) has zero inner product with the coefficients of \( D_\alpha \) when \( \alpha \notin \mathcal{G}[N] \). We use those facts to produce a stronger version of Theorem 3.2.

**Theorem 3.3.** Let \( r \in L_2(G) \) be such that its Fourier coefficients \( R \) satisfy the following conditions:

1. Each \( R(\alpha) \) is either zero or nonsingular;
2. The set of \( \alpha \) such that \( R(\alpha) \) is non-singular includes \( 1 \), and is closed under conjugation and tensor product decomposition.

Then there exists a normal subgroup \( N \) of \( G \) such that \( r \) is \( N \)-invariant, and furthermore \( r \) is uniquely determined up to left translation by its bispectrum \( A_{3,r} \).

**Proof.** As discussed above, the set of \( \alpha \) such that \( R(\alpha) \) is nonsingular corresponds to \( \mathcal{G}[N] \) for some normal subgroup \( N \) of \( G \). Furthermore, \( r = \tilde{r} \circ \pi \) for a unique function \( \tilde{r} \) on \( G/N \). We obtain \( A_{3,r} \) from \( A_{3,\tilde{r}} \) by restricting the latter to the arguments \((\sigma, \delta)\) for which \( R(\sigma) \) and \( R(\delta) \) are nonsingular. Theorem 3.2 now shows that \( \tilde{r} \) is uniquely determined up to a left translation by \( A_{3,\tilde{r}} \), and thus \( r = \tilde{r} \circ \pi \) is uniquely determined up to a left translation by \( A_{3,r} \).

**Remark.** The hypotheses of Thms 3.2 and 3.3 have an interesting interpretation in the context of the Tauberian theorems for compact groups. The latter theorems determine what functions lie in the span of translates of a single function \( f \) in \( L_1(G) \). Edwards ([5, pp 121-125]) describes one such result: If \( f_1, f_2 \in L_1(G) \) have Fourier transforms \( F_1, F_2 \), such that \( F_2(\alpha) = F_1(\alpha)M(\alpha) \) for each \( \alpha \), where \( M(\alpha) \) is an arbitrary matrix whose dimensions match that of \( F_1(\alpha) \), then \( f_2 \) lies in the span of left translates of \( f_1 \), i.e., \( f_2 \) may be approximated arbitrarily closely in \( L_1 \) by linear combinations of left translates of \( f_1 \). Suppose now that \( f_1 \) satisfies the hypothesis of Theorem 3.2 i.e., \( F_1(\alpha) \) is nonsingular for all \( \alpha \). Then the aforementioned Tauberian theorem implies that any function \( f_2 \in L_1(G) \) lies in the span of translates of \( f_1 \), i.e., the translates of \( f_1 \) span \( L_1(G) \). Similarly, if \( f_1 \) satisfies the hypothesis of Theorem 3.3 then the translates of \( f_1 \) span the closed subspace of \( L_1(G) \) that consists of functions invariant under some fixed normal subgroup \( N \). As our theorems show, the bispectrum of \( f_1 \) identifies exactly which functions are its translates.

4. **Homogeneous spaces.** The definition of a homogeneous space is as follows. Let \( G \) be any topological group and \( X \) any topological space. We say that \( G \) acts (on the right) on \( X \) if for each \( g \in G \) there exists a homeomorphism \( \tau_g : X \to X \),
such that \( \tau_e(x) = x \) for the identity \( e \) in \( G \), and furthermore, for \( g_1, g_2 \) in \( G \), we have \( \tau_{g_1 g_2}(x) = \tau_{g_1} (\tau_{g_2}(x)) \). The group \( G \) acts transitively on \( X \) if for each \( x_1, x_2 \) in \( X \), there exists \( g \in G \) such that \( \tau_g(x_1) = x_2 \). The space \( X \) is a homogeneous space for \( G \) if \( G \) acts on \( X \) transitively and continuously. An important example of a homogeneous space is the quotient space of right cosets \( G \setminus H = \{ Hg : g \in G \} \) of a closed subgroup \( H \) in \( G \). In fact, it is a theorem that any locally compact homogeneous space \( X \) of a separable and locally compact group \( G \) can be represented as a quotient space \( G \setminus H \) for some closed subgroup \( H \) of \( G \) ([8 pg 124]).

Our goal in this section is to investigate the bispectrum’s completeness for functions on arbitrary homogeneous spaces of compact groups. By the result cited above, we lose no generality by focusing on spaces of the form \( G \setminus H \), where \( G \) is some compact group and \( H \) some closed subgroup of \( G \). To any function \( f \) on \( G \setminus H \) there corresponds a unique function \( \tilde{f} \) on \( G \) such that \( f = \tilde{f} \circ \pi \), where \( \pi : G \to G \setminus H \) is the canonical coset map; conversely, to any function \( f \) on \( G \) that is invariant under left \( H \)-translations, i.e., \( f(hg) = f(g) \) for all \( g \in G \) and \( h \in H \), there corresponds a unique function \( \tilde{f} \) on \( G \setminus H \) such that \( f = \tilde{f} \circ \pi \). Thus we lose no generality by further restricting our study of functions on homogeneous spaces to functions on \( G \) that are left \( H \)-invariant for some closed subgroup \( H \).

Our main tool for proving completeness results is the Iwahori-Sugiura duality theorem for homogeneous spaces of compact groups [9]. Let \( G \) be any compact group, \( \{ D_\alpha \}_{\alpha \in \mathbb{G}} \) be any selection of irreducible representations, and \( \Theta(G) \) the representative algebra of \( G \). For any closed subgroup \( H \) of \( G \), let \( \Theta_H(G) \) denote the subalgebra of \( \Theta(G) \) consisting of functions that are invariant under left \( H \)-translations. For each \( f \in \Theta_H(G) \), let \( f(Hg) \) denote the common value given to elements of the coset \( Hg \) by \( f \). The algebraic structure of \( \Theta_H(G) \) is revealed to a large extent by the multiplicative linear functionals \( \omega : \Theta_H(G) \to \mathbb{C} \), i.e., algebra homomorphisms of \( \Theta_H(G) \). The Iwahori-Sugiura theorem characterizes those algebra homomorphisms that preserve conjugation.

**Theorem 4.1 (Iwahori-Sugiura).** To each algebra homomorphism \( \omega : \Theta_H(G) \to \mathbb{C} \) that preserves conjugation, there corresponds a unique coset \( Hg \) in the quotient space \( G \setminus H \) such that for all \( f \in \Theta_H(G) \),

\[
\omega(f) = f(Hg).
\]

We describe an equivalent formulation of the Iwahori-Sugiura theorem that is analogous to Theorem 2.2. Several preliminary results are required for the new formulation, with some of the longer proofs being relegated to the appendices.

**Lemma 4.2.** Any function \( f \in \Theta_H(G) \) can be expressed as a unique finite linear combination of the left \( H \)-invariant matrix coefficients of a given selection.

The proof is given in Appendix [A]

Let \( G, H \), and \( \{ D_\alpha \}_{\alpha \in \mathbb{G}} \) be as before. Let us define a corresponding sequence of matrices \( \{ P_\alpha \}_{\alpha \in \mathbb{G}} \) as follows:

\[
P_\alpha = \int_H D_\alpha(h)dh,
\]

where \( dh \) denotes the normalized Haar measure on \( H \). It is easy to show that each \( P_\alpha \) is a projection, i.e., a self-adjoint matrix such that \( P_\alpha P_\alpha = P_\alpha \) ([8 pg 190]). Moreover, the projection matrices as defined above inherit some of the tensor product properties of the corresponding representations ([8 pg 190]).
Let \( \{ P_\alpha \}_{\alpha \in G} \) be as above. For each \( \sigma, \delta \), let \( C_{\sigma\delta} \) be the Clebsch-Gordan matrix and \( \alpha_1, \ldots, \alpha_k \) be the indices in the tensor product decomposition in eq. (2.1). Then

\[
P_\sigma \otimes P_\delta = C_{\sigma\delta} \left[ P_{\alpha_1} \oplus \cdots \oplus P_{\alpha_k} \right] C^\dagger_{\sigma\delta} \left[ P_\sigma \otimes P_\delta \right],
\]

\[
= \left[ P_\sigma \otimes P_\delta \right] C_{\sigma\delta} \left[ P_{\alpha_1} \oplus \cdots \oplus P_{\alpha_k} \right] C^\dagger_{\sigma\delta}.
\]

It proves convenient to apply the following similarity transformations to the \( P \) matrices. For each \( \omega \), let \( \omega = (2.1) \). Then

\[
\text{In the decomposition above, } 1 \text{ is the trivial representation of } H, \text{ and the last term } D_H^H \text{ is some unitary representation of } H \text{ that does not contain } 1.
\]

Rather than starting with an arbitrary selection of \( \{ D_\alpha \}_{\alpha \in G} \), suppose now that we choose one in which each matrix \( D_\alpha(h) \) is exactly equal to a direct sum where the first \( \text{rank}(\alpha) \) representations that appear in the sum are \( 1 \), i.e.,

\[
D_\alpha(h) = \left[ \bigoplus_{q=1}^{\text{rank}(\alpha)} 1(h) \right] \oplus D_H^H(h), \quad h \in H.
\]

We always obtain such a convenient selection (that is what we shall call it henceforth) from a given one by applying similarity transformations as described above. For a convenient selection, the projection matrices in eq. (4.1) are simply \( P_\alpha = I(\text{rank}(\alpha)) \) for all \( \alpha \).

Lemma 4.4. Let \( \{ D_\alpha \}_{\alpha \in G} \) be a convenient selection and \( \{ P_\alpha \}_{\alpha \in G} \) be its projections. The nonzero coefficients in the matrices \( \{ P_\alpha D_\alpha \} \) are precisely those coefficients of the selection that are left \( H \)-invariant.

Proof. Each matrix \( P_\alpha D_\alpha = I(\text{rank}(\alpha)) D_\alpha \) has its first \( \text{rank}(\alpha) \) rows equal to those of \( D_\alpha \), while the remaining rows are identically zero. Moreover, \( P_\alpha D_\alpha(hg) = P_\alpha D_\alpha(g) \) for all \( h \) and \( g \), so (simply substitute \( \int_H D_\alpha dh \) for \( P_\alpha \) and use the translation invariance of the Haar measure \( dh \)), and thus the nonzero coefficient functions in each \( P_\alpha D_\alpha \) are left \( H \)-invariant. We now show the converse: any left \( H \)-invariant coefficient \( \bar{d}_{\alpha}^{pq} \) of \( D_\alpha \) is one of the nonzero coefficients in \( P_\alpha D_\alpha \). Left \( H \)-invariance requires that

\[
\bar{d}_{\alpha}^{pq} = \sum_{\ell=1}^{\text{dim}(\alpha)} \bar{d}_{\alpha}^{\ell q}(h) d_{\alpha}^{\ell p}(g).
\]

The linear independence of the coefficients implies that \( d_{\alpha}^{\ell p}(h) = 1 \) for all \( h \) if \( \ell = p \), and \( d_{\alpha}^{\ell p}(h) = 0 \) for all \( h \) if \( \ell \neq p \). But the assumption on \( D_\alpha \) requires that \( d_{\alpha}^{\ell p}(h) = 1 \) on \( H \) only if \( p \leq \text{rank}(\alpha) \), and thus any left \( H \)-invariant coefficient \( \bar{d}_{\alpha}^{pq} \) must appear in one of the first \( \text{rank}(\alpha) \) rows of \( P_\alpha D_\alpha \). \( \square \)

Since the left \( H \)-invariant coefficients are a basis for \( \Theta_H(G) \), any linear map \( \omega : \Theta_H(G) \to \mathbb{C} \) is uniquely determined by the values that it gives to those coefficients.
For each matrix $P_\alpha D_\alpha$, the map $\omega$ produces a corresponding matrix $\omega(P_\alpha D_\alpha)$. We now determine terms in the characteristics of the matrices $\omega(PD)$ under which $\omega$ is not only linear but also multiplicative and conjugate-preserving. In the following, we use the standard inner product $<\zeta_1, \zeta_2> = \bar{\zeta}_1 \zeta_2$ for complex-valued row vectors $\zeta_1$, $\zeta_2$, and the standard norm $\|\zeta\| = (\zeta, \zeta)^{\frac{1}{2}}$.

**Theorem 4.5.** Let $\{D_\alpha\}_{\alpha \in \mathcal{G}}$ be a convenient selection and $\{P_\alpha\}_{\alpha \in \mathcal{G}}$ be its projections. Any linear map $\omega : \Theta_H(G) \rightarrow \mathbb{C}$ is both multiplicative and conjugate-preserving if and only if the following two conditions hold for all $\sigma, \delta, \alpha$ in $\mathcal{G}$:

\[
(4.7) \quad \omega(P_\alpha D_\alpha) \otimes \omega(P_\sigma D_\sigma) = [P_\sigma \otimes P_\delta] C_{\sigma \delta}\omega([\omega(P_{\alpha_1} D_{\alpha_1}) \oplus \cdots \oplus \omega(P_{\alpha_k} D_{\alpha_k})]) C_{\sigma \delta}^\dagger;
\]
\[
(4.8) \quad \omega(P_\alpha D_\alpha)\omega(P_\alpha D_\alpha)^\dagger = P_\alpha.
\]

These conditions in eq. (4.7), the matrix $C_{\sigma \delta}$ and the indices $\alpha_1, \ldots, \alpha_k$ are as in eq. (2.1).

The proof is given in Appendix B.

Let $f$ be a function in $L_1(G)$ such that $f(hg) = f(g)$ for all $h$ in a given closed subgroup $H$ of $G$. The translation property of the Fourier transform ensures that each Fourier coefficient $F(\alpha)$ satisfies the identity $F(\alpha) = F(\alpha) D_\alpha(h)$ for all $h$ in $H$. Integrating over $h$, we find that $F(\alpha) = F(\alpha)P_\alpha$ for all $\alpha$. We say that each Fourier coefficient $F(\alpha)$ is of maximal $H$-rank if the rank of $F(\alpha)$ equals the rank of $P_\alpha$. We now show that if $f$ is any left $H$-invariant function whose Fourier coefficients $F$ all have maximal rank, then $f$ is uniquely determined by its bispectrum $A_{3,f}$ up to a left translation. The proof of our assertion uses the standard notation from linear algebra [15]. For each matrix $A$, let image($A$) and ker($A$) denote respectively the image and kernel of $A$. For each $\alpha \in G$, let $H_\alpha$ denote the Hilbert space on which the corresponding representations $D_\alpha$ act.

**Theorem 4.6.** Let $G$ be any compact group, and let $H$ be any closed subgroup of $G$. Let $r \in L_1(G)$ be invariant under left $H$-translations. If the Fourier coefficients $\{R(\alpha)\}_{\alpha \in \mathcal{G}}$ all have maximal $H$-rank, then $a_{3,r} = a_{3,s}$ for some $s \in L_1(G)$ if and only if there exists $x \in G$ such that $s(g) = r(xg)$ for all $g$.

The proof is given in Appendix C.

In the theorem above, we did not require that the function $s$ also be left $H$-invariant. (Equality of bispectra may hold regardless of whether both functions are $H$-invariant.) Suppose now that two left $H$-invariant functions $r$, $s$ are such that both have maximal $H$-rank coefficients and both have exactly the same bispectrum. The theorem just proved demonstrates that under those conditions, there exists $x \in G$ such that $s(g) = r(xg)$ for all $g$. Yet the element $x$ cannot be arbitrary, for $s$ is left $H$-invariant, and thus $s(hg) = s(g)$, implying that $r(xhg) = r(xg)$ for all $h \in H$ and $g \in G$. But since $r$ is also left $H$-invariant, we must have $r(xg) = r(hxg)$, and thus $r(xhg) = r(hxg)$ for all $g$ and $h$. The last identity is always satisfied if $x$ lies in the normalizer of $H$ in $G$, which is the subgroup $N_H$ of $G$ defined as follows:

\[
(4.9) \quad N_H = \{x \in G : xH = Hx\}.
\]

(The normalizer of $H$ is the largest subgroup $N_H$ of $G$ such that $G$ itself is a normal subgroup of $N_H$.) In fact, we show that $x$ must lie in $N_H$ in the following theorem.

**Theorem 4.7.** Let $r$, $s$ in $L_1(G)$ be two left $H$-invariant functions whose Fourier coefficients $R(\alpha)$ and $S(\alpha)$ both have maximal $H$-rank for all $\alpha$. Then $a_{3,r} = a_{3,s}$ if and only if $s(g) = r(xg)$ for some $x \in N_H$.

Proof. The “if” assertion is shown above, so we prove the “only if” part. Suppose that $a_{3,r} = a_{3,s}$, and that $r$, $s$, both have maximal $H$-rank coefficients. Under those
conditions, Theorem 4.4 shows that there exits \( x \in G \) such that \( r(g) = s(xg) \) for all \( g \). Then \( R(\alpha) = S(\alpha)D_{\alpha}(x) \) for all \( \alpha \in G \). Furthermore, the left invariance of \( r \) implies that \( R(\alpha) = R(\alpha)P_{\alpha} \) for each \( \alpha \), Thus \( S(\alpha)D_{\alpha}(x) = S(\alpha)D_{\alpha}(x)P_{\alpha} \) for each \( \alpha \), and combining that with the identity \( S(\alpha) = S(\alpha)P_{\alpha} \) yields \( S(\alpha)P_{\alpha}D_{\alpha}(x) = S(\alpha)P_{\alpha}D_{\alpha}(x)P_{\alpha} \), and thus \( S(\alpha)[P_{\alpha}D_{\alpha}(x) - P_{\alpha}D_{\alpha}(x)P_{\alpha}] = 0. \) By the maximal \( H \)-rank hypothesis, we obtain that

\[
P_{\alpha}D_{\alpha}(x) = P_{\alpha}D_{\alpha}(x)P_{\alpha}.
\]

Since \( P_{\alpha} = I(\text{rank}(\alpha)) \) for a convenient selection, the element \( x \) satisfies the above equality if and only if the unitary matrix \( D_{\alpha}(x) \) is the direct sum of two smaller unitary matrices, the first with dimensions \( \text{rank}(\alpha) \times \text{rank}(\alpha) \) and the second with dimensions \( (n - \text{rank}(\alpha)) \times (n - \text{rank}(\alpha)) \). For such an \( x \), it follows for any \( h \in H \) that

\[
P_{\alpha}D_{\alpha}(x)^{\dagger}D_{\alpha}(h)D_{\alpha}(x) = P_{\alpha}D_{\alpha}(x^{-1}hx) = P_{\alpha}.
\]

But we now see by Lemma 4.4 that \( P_{\alpha}D_{\alpha}(x^{-1}hx) = P_{\alpha} \) if and only if \( x^{-1}hx \in H \).

The last inclusion holds for all \( h \in H \), and thus \( x^{-1}hx = H \), or equivalently, \( x \in N_H \).

In the interesting special case when \( G \) is the group \( SO(3) \) and \( H \) is the subgroup of rotations that fix the \( z \)-axis, we have that \( N_H = H \). In that case, if \( r \) is any left \( H \)-invariant function with maximal \( H \)-rank coefficients, then there are no other left \( H \)-invariant functions with the same bispectrum besides \( r \) itself. However, that does not mean that the bispectrum uniquely determines \( r \); any function \( s \) such that \( s(g) = r(xg) \) on \( G \) has the same bispectrum, although \( s \) is not necessarily \( H \)-invariant.

If \( G = SO(3) \) and \( H \) as above, then the maximal \( H \)-rank condition is easy to satisfy. Here it is well-known that \( \text{rank}(P_{\alpha}) = 1 \) for all \( \alpha \in \hat{G} \) (19). Thus an arbitrary left \( H \)-invariant function \( r \) has maximal \( H \)-rank coefficients if for all \( \alpha \), the matrix \( R(\alpha) \) contains at least one nonzero coefficient. That is evidently true if any noise is present in measuring \( r \).

5. Reconstruction algorithms. The completeness theory for arbitrary compact groups in the preceding sections can be refined further for the special case when the group is \( SU(2) \), which is the group of all \( 2 \times 2 \) unitary matrices with determinant +1. The group \( SU(2) \) arises frequently in applications because it is a double-covering of the rotation group \( SO(3) \), and in many problems it is more convenient to model three-dimensional rotations by elements of \( SU(2) \) rather than the corresponding elements of \( SO(3) \)—one reason is that the addition of rotations is much simpler for \( SU(2) \) (Cayley-Klein parameters) than for \( SO(3) \) (Euler parameterization). The representation theory of \( SU(2) \) is known in extensive detail, and we take advantage of the special properties of \( SU(2) \)’s irreducible representations to analyze the bispectrum of bandlimited functions. The latter are functions whose Fourier coefficients are identically zero except for a finite set of indices. One reason why bandlimited functions are important is that any \( L_2 \)-function can be approximated as closely as desired in the \( L_2 \)-metric by a bandlimited function.

The irreducible representations of \( SU(2) \) have several properties that simplify our analysis of the bandlimited case ([18 Chapter 2]). First, there exists one and only one irreducible representation (up to equivalence) in each dimension. It is thus possible to index the set of all irreducible representations (modulo equivalence) by the nonnegative integers, in such a way that for each \( \ell \geq 0 \), the representation \( D_\ell \) has
The unitary matrix $H_{SU}$ is the self-representation following facts from matrix theory. First, any positive definite matrix with nonsingular coefficients from its bispectrum. Our algorithm makes use of the determinant of $A$ has a unique polar decomposition $H_{SU}$ has a unique polar decomposition if $A = H_{SU}U$ and $U$ is a unitary matrix. The polar decomposition is unique in the sense that if $A = H_{SU}U = H_{SU}'U'$ for positive definite matrices $H_{SU}$, $H_{SU}'$, and unitary matrices $U$, $U'$, then $H_{SU} = H_{SU}'$ and $U = U'$. It is easy to see that $H_{SU}$ as chosen above is such that $\det(H_{SU}) = |\det(A)|$. If the determinant of $A$ is real, then we may choose a square root $H$ of $(AA^\dagger)$ such that $A = HU$, where $U$ is unitary and $\det(H) = \det(A)$. The last observation becomes important in our analysis of $SO(3)$ below.

We require one last fact: the coefficient matrix $F(1)$ of any real-valued function $f$ on $SU(2)$ has nonnegative determinant. To see that, recall from above that $D_1$ is the self-representation of $SU(2)$, and thus

$$F(1) = \int_G f(g)D_1g^\dagger dg = \int_G f(g) \begin{bmatrix} d_{11}^1(g)^* & -d_{11}^2(g)^* \\ d_{11}^2(g) & d_{11}^1(g) \end{bmatrix} dg.$$
By evaluating the matrix coefficients above, and using the assumption of real $f$, we find that $\det [F(1)] \geq 0$.

Putting all the facts above together, we obtain the following result.

**Proposition 5.1.** Let $L > 0$, and let $f$ be any real-valued function on $SU(2)$ whose Fourier coefficients are such that $F(\ell)$ is a nonsingular matrix for each $\ell \leq L$, and furthermore, $F(\ell) = 0$ if $\ell > L$. Then $f$ can be uniquely recovered up to a left translation from its bispectrum $A_{3,f}$.

**Proof.** Since $f$ is real-valued, it follows that $F(0)$ is a real number. Equation (5.6) shows that $A_{3,f}(0,0) = F(0)^3$, and thus we obtain $F(0)$ by taking cube roots. By assumption, $F(0)$ is nonzero, and thus we obtain from (5.6) that

$$
\frac{A_{3,f}(1,0)}{F(0)} = F(1)F(1)^\dagger.
$$

The matrix on the right hand side above is positive definite. Let $\hat{F}(1) = \left( \frac{A_{3,f}(1,0)}{F(0)} \right)^\dagger$ be the positive square-root as constructed above. In polar form $F(1) = \hat{F}(1)U$, and thus $\hat{F}(1) = F(1)U^\dagger$. The determinant of $\hat{F}(1)$ is positive, as is the determinant of $F(1)$, and thus $\det [U^\dagger] = +1$. Consequently, $U^\dagger \in SU(2)$, and we may write $\hat{F}(1) = F(1)D_1(x)$ for $x = U^\dagger$. If $L = 1$, then we are done. Otherwise, the following algorithm produces matrices $\hat{F}(2), \ldots, \hat{F}(L)$, such that $\hat{F}(\ell) = F(\ell)D_\ell(x)$ for the same $x$ and for all $2 \leq \ell \leq L$. Since we know $\hat{F}(1)$ and $A_{3,f}(1,1)$, we obtain $\hat{F}(2)$ from the upper-left $3 \times 3$ submatrix of the following $4 \times 4$ matrix:

$$
C_{11}^\dagger \left[ \hat{F}(1)^{-1} \otimes \hat{F}(1)^{-1} \right] A_{3,f}(1,1)C_{11}.
$$

The reason we use the matrix above is as follows. All terms above are known, and if we substitute for $\hat{F}(1)$ and $A_{3,f}(1,1)$, then we find that

$$
C_{11}^\dagger \left[ \hat{F}(1)^{-1} \otimes \hat{F}(1)^{-1} \right] A_{3,f}C_{11} = C_{11}^\dagger \left[ D_1(x)^\dagger \otimes D_1(x)^\dagger \right] \left[ F(1)^{-1} \otimes F(1)^{-1} \right] \left[ F(1) \otimes F(1) \right] C_{11} \left[ F(2)^\dagger \otimes F(0) \right] C_{11}.
$$

Cancelling terms, and using the reduction formula (5.1), shows that

$$
C_{11}^\dagger \left[ \hat{F}(1)^{-1} \otimes \hat{F}(1)^{-1} \right] A_{3,f}C_{11} = [D_2(x)^\dagger F(2)^\dagger] \otimes F(0).
$$

The upper left $3 \times 3$ submatrix of the right hand side is exactly the matrix $[F(2)D_2(x)]^\dagger$, and we set its adjoint equal to $\hat{F}(2)$. Having obtained $\hat{F}(2)$ in that way, we obtain $\hat{F}(\ell)$ for any $\ell > 2$ from the upper $(\ell + 1) \times (\ell + 1)$ submatrix of the following matrix

$$
C_{(\ell-1)1}^\dagger \left[ \hat{F}(\ell - 1)^{-1} \otimes \hat{F}(1)^{-1} \right] A_{3,f}(\ell - 1,1)C_{(\ell-1)1}
$$

The same argument as above shows that

$$
C_{(\ell-1)1}^\dagger \left[ \hat{F}(\ell - 1)^{-1} \otimes \hat{F}(1)^{-1} \right] A_{3,f}(\ell - 1,1)C_{(\ell-1)1} = [D_\ell(x)^\dagger F(\ell)^\dagger] \otimes \hat{F}(\ell - 2)^\dagger.
$$

We set $\hat{F}(\ell)$ equal to the adjoint of the upper $(\ell + 1) \times (\ell + 1)$ submatrix of the right hand side, and thus obtain that $\hat{F}(\ell) = F(\ell)D_\ell(x)$. On doing so for all $\ell \leq L$, the
function \( \hat{f} \) on \( SU(2) \) obtained by Fourier series expansion with the coefficients \( F(0), \hat{F}(1), \ldots, \hat{F}(L) \) is such that \( \hat{f}(g) = f(xg) \) for all \( g \).

It is easy to prove the same completeness result for bandlimited functions on \( SO(3) \). We describe the few differences that exist, drawing on standard facts about representations of \( SO(3) \) (ES Chap II). First, the irreducible representations of \( SO(3) \) occur only in odd dimensions, and there is exactly one representation (modulo equivalence) in each odd dimension. Thus we may list any selection of irreducible representations as \( \{ D_\ell \}_{\ell=0}^\infty \), where for each \( \ell \), the representation \( D_\ell \) has dimension \( (2\ell + 1) \).

In that indexing, \( D_0 \) is the trivial representation, and \( D_1 \) is equivalent to the self-representation \( g \mapsto g \) of \( SO(3) \), i.e., \( D_1(g) = UgU^\dagger \) for some unitary matrix \( U \). (Recall that \( SO(3) \) is the set of all real-valued \( 3 \times 3 \) orthogonal matrices with determinant \( +1 \).) For each \( n, m \), the tensor-product \( D_n \otimes D_m \) reduces explicitly as follows:

\[
(5.10) \quad D_n \otimes D_m = C_{nm} [D_{n+m} \oplus D_{n+m-1} \oplus \cdots \oplus D_{|n-m|}] C_{nm}^\dagger.
\]

With the formula above, it is easy to see that the recursive algorithm given in the proof of Proposition 5.1 generalizes to recover all real-valued bandlimited functions on \( SO(3) \). To initialize the algorithm, we require an estimate \( \hat{F}(1) \) of the first coefficient \( F(1) \) from the data \( F(1)F(1)^\dagger \), such that \( F(1) = F(1)D_1(g) \) for some element \( g \) of \( SO(3) \). Assuming that \( F(1) \) is nonsingular, we obtain the estimate as follows. The representation \( D_1 \) is such that \( D_1(g) = UgU^\dagger \), where \( U \) is fixed as \( g \) varies in \( SO(3) \). Thus

\[
F(1) = \int_G f(g)D_1(g)^\dagger dg, = U \left[ \int_G f(g)g^\dagger \right] U^\dagger.
\]

Let \( F_s(1) \) denote the matrix that results by evaluating the integral in brackets. Since \( f \) is real-valued, and every matrix \( g \) has real coefficients, the matrix \( F_s(1) \) has only real coefficients. Thus the determinant of \( F(1) = UF_s(1)U^\dagger \) is a real number. Assume for the moment that \( \det [F(1)] = \det [F_s(1)] > 0 \). Let \( \hat{F}(1) \) and \( \hat{F}_s(1) \) denote respectively the (unique) positive square roots of \( F(1)F(1)^\dagger \) and \( F_s(1)F_s(1)^\dagger \). Since \( F(1)F(1)^\dagger = UF_s(1)F_s(1)^\dagger U^\dagger \), it is easily seen that \( \hat{F}(1) = UF_s(1)U^\dagger \). Now consider the polar decomposition \( F_s(1) = HV \), where \( H \) is positive definite and \( V \) is unitary. Recall from the earlier discussion for \( SU(2) \) that \( H = (F_s(1)F_s(1)^\dagger)^{\frac{3}{2}} \), and thus \( H = \hat{F}_s(1) \). Since \( F_s(1) \) is real-valued, \( V \) must be real-valued orthogonal matrix. Matching determinants on both sides of the equation \( F_s(1) = \hat{F}_s(1)V \) reveals that \( \det[V] = +1 \), and thus \( V = g \), for some \( g \in SO(3) \). Substitution reveals that

\[
(5.11) \quad \hat{F}(1) = U\hat{F}_s(1)U^\dagger = UF_s(1)gU^\dagger = UF_s(1)U^\dagger UgU^\dagger = F(1)D_1(g),
\]

The assumption that \( \det[F(1)] > 0 \) is not critical. We use it only to obtain that \( \det[V] = +1 \), where \( V = F_s(1)^{-1}F_s(1) \). Instead of selecting \( \hat{F}(1) \) to be the positive definite square root of \( F(1)F(1)^\dagger \), we may choose \( \hat{F}(1) \) to be any square root such that \( \det[\hat{F}(1)] = \det[F(1)] \), e.g., by multiplying the top row of the positive definite square root matrix by \(-1\) if necessary. We do not know \( \det[F(1)] \text{ a priori} \), but if we store it as “side information” along with the bispectrum, then we obtain a complete rotation-invariant description for any real-valued bandlimited function on \( SO(3) \). Note that \( \det[F(1)] \) remains invariant under translation on \( SO(3) \), i.e., if
\( f(g) = s(hg) \), then \( F(1) = S(1)D_1(h) \), but since \( \det[D_1(h)] = +1 \), we obtain that \( \det[F(1)] = \det[S(1)] \). To sum up, any real-valued bandlimited function \( f \) on \( SO(3) \), whose coefficient matrices are all nonsingular up to the bandlimit, can be recovered completely—up to a single translation on \( SO(3) \)—if both its bispectrum and the value of \( \det[F(1)] \) is known, and the algorithm described above is used.

6. Applications. As mentioned in the introduction, the invariance and completeness properties of the bispectrum lend themselves to applications in pattern matching problems. One particular application is described here. R. Kondor [14] demonstrates how translation- and rotation-invariant matching of hand-written characters is accomplished with bispectral invariants. To do so, Kondor notes that, for practical purposes, the characters themselves may defined as intensity-valued functions on a compact patch on \( \mathbb{R}^2 \) of radius 1. A transformation may be constructed that maps the planar patch to the upper hemisphere of the sphere \( S^2 \) as follows:

\[
\omega : (r, \phi_{\mathbb{R}^2}) \mapsto (\theta, \phi_{S^2}) \quad \text{with} \quad r = \sin(\theta); \quad \phi_{\mathbb{R}^2} = \phi_{S^2}.
\]

The subscripts denote the domain of the angle involved, whether it be the plane \( \mathbb{R}^2 \) or the sphere \( S^2 \). Kondor shows that rigid body motions on the patch, each of which consists of a rotation by an angle \( \alpha \) and a translation by a vector \( T = (t_x, t_y) \) with \( \|T\| \leq 1 \), may be mapped to 3-D rotations through the use of Euler angles \( (\theta, \phi, \psi) \) as follows

\[
\alpha = \psi; \quad t_x = \sin \theta \cos \phi; \quad t_y = \sin \theta \sin \phi.
\]

This mapping produces a local isomorphism between planar rigid motions and spherical rotations.

By using the transformation (6.1), every intensity function defined on the planar patch may be converted to a function on \( S^2 \). The problem of finding rigid-motion invariants on \( \mathbb{R}^2 \) now becomes one of finding rotation invariants on the sphere \( S^2 \). Since the sphere is a homogeneous space for \( SO(3) \), every function \( \tilde{f} \) on \( S^2 \) may in turn be lifted to a function \( f \) on \( SO(3) \) using the “north-pole” mapping: if \( z = [0, 0, 1] \), then \( f(R) = \tilde{f}(Rz) \) for every \( R \in SO(3) \). We may now construct the bispectrum of \( f \) from eq. (3.4) using the Fourier transform on \( SO(3) \), which may be calculated using spherical harmonic basis functions. Kondor calculates bispectral invariants in this way, and shows, in an experiment using 1000 hand-written characters from a standard dataset, that the invariants perform well in matching over arbitrary orientations and starting positions of the characters.

A second application of the bispectrum occurs in astrophysical models of primordial fluctuations, as mentioned in the introduction. Cosmic inflation [7] predicts a Gaussian pattern of temperature anisotropies in the cosmic microwave background radiation (CBR). The CBR anisotropy is a function defined on \( S^2 \), and therefore we may calculate its bispectrum using eq. (3.4). If the anisotropy is Gaussian, then the expected value of the angular bispectrum is zero. However, as X. Luo [16] shows, the stochastic nature of anisotropies means that cosmic variance makes it difficult to extract non-Gaussian structure from CBR data. In that paper, as in much of the physics literature, expressions of the bispectrum follow the “summation notation”, which implicitly focuses attention at the level of individual elements of bispectral matrices. It is hoped that the approach of this paper, including the matrix form derived in (3.4), proves useful in allowing insight into higher level properties, such as matrix rank, decomposition, and completeness.
Both of the applications mentioned apply the bispectrum to functions on the sphere. Healy et al. [10] describe a fast “divide-and-conquer” discrete Legendre transform, which leads to an “FFT” on $S^2$. They show how this transform leads to efficient computation of bispectrum on the sphere.

7. Summary and future directions. This paper derives completeness properties of the bispectrum for functions defined on compact groups and their homogeneous spaces. A matrix form of the bispectrum is derived, and it is shown that every function with nonsingular coefficients is completely determined, up to a group translation, by its bispectrum. A reconstruction algorithm for functions defined on the groups $SU(2)$ and $SO(3)$ is described.

Results similar to those in this paper may be established for non-compact, non-commutative groups [12]. Those results rely on the duality theorem of Tatsuuma. The Tannaka-Krein duality theorem, which is central to this paper, has been extended to compact groupoids [4]. It would be interesting to see if a corresponding bispectral theory may be constructed there.

Appendix A. Proof of Lemma 4.2. Since $\Theta_H \subset \Theta(G)$, each $f \in \Theta_H(G)$ is a unique finite linear combination of matrix coefficients $d^{pq}_\alpha$ (not necessarily $H$-invariant):

\[ f(g) = \sum_{\alpha, p, q} c^{pq}_\alpha d^{pq}_\alpha(g). \]  

We set $f(hg) = f(g)$ to obtain

\[ \sum_{\alpha, p, q} c^{pq}_\alpha d^{pq}_\alpha(g) = \sum_{\alpha, p, q} c^{pq}_\alpha d^{pq}_\alpha(hg). \]  

The multiplication rule $D_\alpha(hg) = D_\alpha(h)D_\alpha(g)$ for representation matrices reveals that

\[ d^{pq}_\alpha(hg) = \sum_{\ell=1}^{\dim(\alpha)} d^{p\ell}_\alpha(h)d^{\ell q}_\alpha(g), \]  

and thus

\[ \sum_{\alpha, p, q} c^{pq}_\alpha d^{pq}_\alpha(g) = \sum_{\alpha, p, q} c^{pq}_\alpha d^{pq}_\alpha(h)d^{pq}_\alpha(g) + \sum_{\alpha, p, q} c^{pq}_\alpha \sum_{\ell \neq p} d^{p\ell}_\alpha(h)d^{\ell q}_\alpha(g). \]  

The linear independence of the matrix coefficients implies that in the equation above, $d^{pq}_\alpha(h) = 1$ and $d^{pq}_\alpha(h) = 0$ if $\ell \neq p$. Thus $d^{pq}_\alpha(hg) = d^{pq}_\alpha(g)$ for every coefficient function in eq. (A.1).

Appendix B. Proof of Theorem 4.5. If $\omega$ preserves multiplication and complex-conjugation, then the Iwahori-Sugiura theorem shows that there exists a unique coset $Hg$ such that $\omega(P_\alpha D_\alpha) = P_\alpha D_\alpha(Hg)$ for all $\alpha$. From this, equations (4.7) and (4.8) follow immediately. Suppose now that $\omega$ is some linear map that also satisfies eq. (4.7). Applying Lemma 4.3 to both sides of the tensor product decomposition in eq. (2.1) yields

\[ (P_\sigma D_\sigma) \otimes (P_\delta D_\delta) = [P_\sigma \otimes P_\delta] C_{\sigma\delta} \left[ (P_{\alpha_1} D_{\alpha_1}) \oplus \cdots \oplus (P_{\alpha_k} D_{\alpha_k}) \right] C_{\sigma\delta}^\dagger. \]
Now apply $\omega$ to both sides to obtain
\[
\omega \left( (P_\sigma D_\sigma) \otimes (P_\delta D_\delta) \right) = [P_\sigma \otimes P_\delta] C_{\sigma \delta} [\omega(P_\sigma D_\alpha_1) \oplus \cdots \oplus \omega(P_\sigma D_\alpha_k)] C_{\sigma \delta}^\dagger.
\]
Because eq. (4.7) holds, we obtain that for all $\alpha$,
\[
(B.2) \quad \omega \left( (P_\sigma D_\sigma) \otimes (P_\delta D_\delta) \right) = \omega(P_\sigma D_\sigma) \otimes \omega(P_\delta D_\delta).
\]
Thus $\omega$ is multiplicative. Suppose now that the linear and multiplicative map $\omega$ also satisfies eq. (4.8). Applying $\omega$ to both sides of the identity $(P_\sigma D_\sigma)(P_\sigma D_\sigma)^\dagger = P_\alpha$ yields
\[
(B.3) \quad \omega(P_\alpha D_\alpha) \omega \left( (P_\alpha D_\alpha)^\dagger \right) = P_\alpha.
\]
We show that $\omega \left( (P_\alpha D_\alpha)^\dagger \right) = \omega(P_\alpha D_\alpha)^\dagger$, proving that $\omega$ preserves conjugation. Let $\zeta_\alpha$ be any nonzero row of the matrix $P_\alpha D_\alpha$. We establish the following three equalities:
\[
(B.4) \quad < \omega(\zeta_\alpha), \omega(\zeta_\alpha)^* > = 1,
\]
\[
(B.5) \quad < \omega(\zeta_\alpha), \omega(\zeta_\alpha)^* > = 1,
\]
\[
(B.6) \quad < \omega(\zeta_\alpha^*), \omega(\zeta_\alpha^*) > = 1.
\]
The first equality (B.4) follows from (4.8) (recall that we are working with a convenient selection, for which $P_\alpha = I(\text{rank}(\alpha))$). The second is derived from eq. (B.3). The final equality requires more work, but is a straightforward consequence of (4.8) and the linearity of $\omega$. We give its proof later, but for now assume that it is true. The three inequalities above imply that
\[
(B.7) \quad < \omega(\zeta_\alpha), \omega(\zeta_\alpha^*) > = ||\omega(\zeta_\alpha)|| ||\omega(\zeta_\alpha)^*||.
\]
The Cauchy-Schwartz inequality shows that the identity above holds if and only if $\omega(\zeta_\alpha) = c \omega(\zeta_\alpha)^*$, and from eq. (B.5) we see that $c = 1$. Thus $\omega(\zeta_\alpha) = \omega(\zeta_\alpha^*)^*$. Since the preceding argument applies to any nonzero-row $\zeta_\alpha$ of any matrix $P_\alpha D_\alpha$, it follows that $\omega$ preserves conjugation.

Now to prove (B.6). For any representation $D_\alpha$ in our selection, the conjugate representation $D_\alpha^*$ is also irreducible, and there are two cases: (i) $D_\alpha^* = A_\alpha D_\alpha A_\alpha^\dagger$ for some unitary matrix $A_\alpha$; (ii) $D_\alpha^* = A_\beta D_\beta A_\beta^\dagger$ where $\beta \neq \alpha$. Assume that the first case is true. It is easy to show that any matrix $\hat{A}_\alpha$ expressing the equivalence of conjugate representations is symmetric, and thus $A_\alpha^\dagger = A_\alpha^*$ ([5, pg 15]). Furthermore, only the first rank($\alpha$) rows of $D_\alpha$ are $H$-invariant, and that must also be true of the matrix $D_\alpha^* = A_\alpha D_\alpha A_\alpha^*$. Thus $A_\alpha$ transforms the first rank($\alpha$) rows among themselves, which means that $A_\alpha$ must have the block-diagonal form $A_\alpha = A_{\alpha,1} \oplus A_{\alpha,2}$, where $A_{\alpha,1}$ is a symmetric unitary matrix with dimensions rank($\alpha$) $\times$ rank($\alpha$). Thus $P_\alpha A_\alpha = A_\alpha P_\alpha$.

Putting those facts together, we obtain the following identity by virtue of $\omega$’s linearity:
\[
(B.8) \quad \omega(P_\alpha D_\alpha^*) = \omega(P_\alpha A_\alpha D_\alpha A_\alpha^*) = A_\alpha \omega(P_\alpha D_\alpha) A_\alpha^*.
\]
By using the identity above and eq. (4.8), we find that $\omega(P_\alpha D_\alpha^*) \omega(P_\alpha D_\alpha)^\dagger = P_\alpha$. Noting that $P_\alpha = I(\text{rank}(\alpha))$, the previous equality for matrices implies eq. (B.6) for their nonzero rows. Case (ii) is similar.

Appendix C. Proof of Theorem 4.6
If \( r \) and \( s \) are left translates of each other, then \( a_{3,r} = a_{3,s} \), as follows from the definition of triple correlation and the left invariance of Haar measure. Now suppose that \( a_{3,r} = a_{3,s} \). Lemma 3.1 shows that for all \( \sigma, \delta \),

\[
(C.1) \quad [R(\sigma) \otimes R(\delta)] C_{\sigma \delta} \left[ R(\alpha_1)^\dagger \oplus \cdots \oplus R(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger = \left[ S(\sigma) \otimes S(\delta) \right] C_{\sigma \delta} \left[ S(\alpha_1)^\dagger \oplus \cdots \oplus S(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger.
\]

If we set \( \sigma = \delta = 1 \) and apply the same argument used in the proof of Theorem 3.2 we obtain \( R(1) = S(1) \). The maximal H-rank assumption implies that the scalar \( R(1) = S(1) \) is nonzero. Now set \( \delta = 1 \) in (C.2) above. Cancelling \( R(1) = S(1) \) from both sides shows that

\[
(C.2) \quad R(\sigma) R(\sigma)^\dagger = S(\sigma) S(\sigma)^\dagger.
\]

Hence, we have for each \( \sigma \) that \( S(\sigma) = R(\sigma) U(\sigma) \) for some unitary matrix \( U(\sigma) \). Substituting into eq. (C.2) reveals that

\[
\begin{align*}
[R(\sigma) \otimes R(\delta)] C_{\sigma \delta} \left[ R(\alpha_1)^\dagger \oplus \cdots \oplus R(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger = \\
\left[ R(\sigma) \otimes R(\delta) \right] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] \left[ R(\alpha_1)^\dagger \oplus \cdots \oplus R(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger.
\end{align*}
\]

We cancel \( C_{\sigma \delta}^\dagger \) from both sides. The identity \( R(\alpha) = R(\alpha) P_\alpha \) implies that \( R(\alpha)^\dagger = P_\alpha R(\alpha)^\dagger \), and thus \( \text{image}(R(\alpha)^\dagger) \subseteq P_\alpha(\mathcal{H}_\alpha) \). But the rank of \( R(\alpha)^\dagger \) equals that of \( P_\alpha \), and thus \( \text{image}(R(\alpha)^\dagger) = P_\alpha(\mathcal{H}_\alpha) \). The last identity implies that

\[
\begin{align*}
[R(\sigma) \otimes R(\delta)] C_{\sigma \delta} \left[ P_{\alpha_1} \oplus \cdots \oplus P_{\alpha_k} \right] = \\
\left[ R(\sigma) \otimes R(\delta) \right] \left[ U(\sigma) \otimes U(\delta) \right] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] \left[ P_{\alpha_1} \oplus \cdots \oplus P_{\alpha_k} \right].
\end{align*}
\]

Multiplying both sides from the right by \( C_{\sigma \delta}^\dagger \left[ P_\sigma \otimes P_\delta \right] \) and using Lemma 4.3 reveals that

\[
\begin{align*}
[R(\sigma) \otimes R(\delta)] \left[ P_\sigma \otimes P_\delta \right] = \\
\left[ R(\sigma) \otimes R(\delta) \right] \left[ U(\sigma) \otimes U(\delta) \right] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger \left[ P_\sigma \otimes P_\delta \right].
\end{align*}
\]

We substitute \( R(\sigma) = R(\sigma) P_\sigma \) and \( R(\delta) = R(\delta) P_\delta \) into the leftmost tensor product term on the right hand side of the equation above and simplify, to obtain

\[
\begin{align*}
[R(\sigma) \otimes R(\delta)] \left[ P_\sigma \otimes P_\delta \right] = \\
\left[ R(\sigma) \otimes R(\delta) \right] \left[ P_\sigma \otimes P_\delta \right] \left[ U(\sigma) \otimes U(\delta) \right] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger \left[ P_\sigma \otimes P_\delta \right].
\end{align*}
\]

For each \( \alpha \), the identity \( R(\alpha) P_\alpha = R(\alpha) \) together with the assumption that \( R(\alpha) \) has maximal H-rank imply that \( R(\alpha) \) is one-to-one on \( P_\alpha(\mathcal{H}_\alpha) \). Thus the equation above implies that

\[
\begin{align*}
P_\sigma \otimes P_\delta = [P_\sigma \otimes P_\delta] \left[ U(\sigma) \otimes U(\delta) \right] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger \left[ P_\sigma \otimes P_\delta \right]
\end{align*}
\]

The matrix in between the two orthogonal projections on the right hand side is unitary; it is easily seen that for any unitary matrix \( U \) and any orthogonal projection \( P \), the matrix equation \( P = PUP \) holds only if \( UP = P \). Thus we obtain

\[
(C.3) \quad P_\sigma \otimes P_\delta = [U(\sigma) \otimes U(\delta)] C_{\sigma \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] C_{\sigma \delta}^\dagger \left[ P_\sigma \otimes P_\delta \right]
\]
Rearranging terms, we obtain the following identity:

$$\left[ U(\sigma)^\dagger \otimes U(\delta)^\dagger \right] [P_\sigma \otimes P_\delta] = C_{\alpha \delta} \left[ U(\alpha_1)^\dagger \oplus \cdots \oplus U(\alpha_k)^\dagger \right] C_{\alpha \delta}^\dagger [P_\sigma \otimes P_\delta].$$

Substituting from Lemma 4.3 in the right hand side, and subsequently taking the matrix adjoint of both sides, reveals that

$$[(P_\sigma U(\sigma)) \otimes (P_\delta U(\delta))] = [P_\sigma \otimes P_\delta] C_{\alpha \delta}\left[ (P_\alpha U(\alpha_1)) \oplus \cdots \oplus (P_\alpha U(\alpha_k)) \right] C_{\alpha \delta}^\dagger.$$

Theorem 4.5, together with the Iwahori-Sugiura Theorem, shows that the identity above holds if and only if there exists a coset $Hx$ such that $P_\alpha U(\alpha) = P_\alpha D_\alpha(Hx)$ for all $\alpha \in \mathcal{G}$. Thus for each $\alpha$ we have the string of identities

$$(C.4) \quad S(\alpha) = R(\alpha)U(\alpha) = R(\alpha)P_\alpha U(\alpha) = R(\alpha)P_\alpha D_\alpha(Hx) = R(\alpha)D_\alpha(x).$$

The translation property of the Fourier transform now shows that $s(g) = r(xg)$ for all $g$.

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