Electrostatic Bender Optics

R. Baartman

TRIUMF

Abstract: The relativistically-correct Hamiltonian and transfer matrix of electrostatic benders is derived. This is the general case where the bender electrodes have curvature in the non-bend direction.
1 Introduction

A common approach, used especially by those accustomed to only magnetic elements, is to let the third momentum be $\Delta p/p$. This is not the simplest approach, since when electric fields are included, it is not conserved. This means that when a particle enters the electrostatic element off-axis, it must receive a “kick” to get into the potential field. This kicks $p$, but to first order leaves $E$ unchanged. For an electrostatic bend of radius $A$, this kick is $\Delta p/p = \pm x/A$; the upper sign is for entry, the lower for exit.

A better approach is therefore to let the third momentum be $E$. Since the fields are static, $E$ is conserved; no kicks are required.

For independent variable $s$, on a reference trajectory curving in the $xs$-plane with curvature $h = 1/A$, the Hamiltonian $H = -p_s$ is well-known, and I will not derive it here:

$$H = -(1 + hx)\sqrt{\left(\frac{E - q\Phi}{c}\right)^2 - m^2c^2 - p_x^2 - p_y^2}$$  \hspace{1cm} (1)

We write $E = E_0 + \Delta E$, and note that the large quantity under the square root sign is $p_0^2 = E_0^2/c^2 - m^2c^2 = (\gamma^2 - 1)m^2c^2 = (\beta \gamma mc)^2$, the square of the reference momentum.

$$H = -(1 + hx)p_0\sqrt{1 + 2 \frac{2E_0(\Delta E - q\Phi)}{p_0^2c^2} + \left(\frac{\Delta E - q\Phi}{p_0c}\right)^2 - \frac{p_x^2}{p_0^2} - \frac{p_y^2}{p_0^2}}$$ \hspace{1cm} (3)

Let us transform so that the third coordinate is not time, but a relative distance deviation w.r.t. the reference particle: i.e. from $(t, -\Delta E)$ to $(\tau, p_\tau)$ where $\tau \equiv s - \beta ct, p_\tau = \Delta E/(\beta c)$. The generating function is

$$F(t, p_\tau) = (s - \beta ct)p_\tau$$  \hspace{1cm} (4)

then the new Hamiltonian is $K = H + \partial F/\partial s = H + p_\tau$

$$K = p_\tau - (1 + hx)p_0\sqrt{1 + 2 \left(\frac{p_\tau}{p_0} - \frac{q\Phi}{\beta c p_0}\right) + \beta^2 \left(\frac{p_\tau}{p_0} - \frac{q\Phi}{\beta c p_0}\right)^2 - \frac{p_x^2}{p_0^2} - \frac{p_y^2}{p_0^2}}$$ \hspace{1cm} (5)

This cleans up considerably if we scale all momenta and the Hamiltonian by $p_0$: $P_x = p_x/p_0, P_y = p_y/p_0, P_\tau = p_\tau/p_0, \tilde{K} = K/p_0$, and introduce the scaled potential $V = \frac{q\Phi}{\beta c p_0}$:

$$\tilde{K} = P_\tau - (1 + hx)\sqrt{1 + 2 (P_\tau - V) + \beta^2 (P_\tau - V)^2 - P_x^2 - P_y^2}$$ \hspace{1cm} (6)

This also makes the momenta accord with the more usual definitions, since, as we will see, $P_x = x', P_y = y'$, and outside the electric field, $P_\tau = \Delta p/p$.

2 Potential

To second order, the potential $V$ is given by

$$V = hx - h(h + k)\frac{x^2}{2} + hky^2$$ \hspace{1cm} (7)
where $h = 1/A$ and $k = 1/A_y$ is the curvature in the non-bend direction. This can be derived by solving Laplace’s equation in a curvilinear coordinate system that has different curvatures in the $xs$- and $xy$-planes, then transforming back to the chosen dynamical system, which is curved only in the $xs$-plane. We can, however, show the above potential to be correct for the two simplest cases – cylindrical and spherical.

**Example: Cylindrical bend**

Here $k = 0$, the potential is $V = -\log(A/r) = \log(1 + hx)$, since the distance $r$ to the bend centre is $x + A$. Expanding, we find

$$V = hx - \frac{h^2 x^2}{2} + \frac{h^3 x^3}{3} - \cdots$$

This agrees with eqn. (7) for $k = 0$.

**Example: Spherical bend**

Here $k = h$, the potential is $V = 1 - A/r = 1 - 1/\sqrt{1 + 2hx + h^2x^2 + h^2y^2}$, since the distance $r$ to the bend centre is $\sqrt{(x + A)^2 + y^2}$. Expanding, we find

$$V = hx - \frac{h^2 x^2}{2} + \frac{h^3 y^2}{2} + \frac{h^3 x^3}{2} - \frac{3h^3 xy^2}{2} + \cdots$$

This agrees with eqn. (7) for $k = h$.

In any case, the first term, needed to ensure that the reference trajectory is $x = 0$, yields the required electric field on the reference orbit:

$$\mathcal{E} = -\frac{\partial \Phi}{\partial x} = -\frac{\beta cp_0}{q} \left. \frac{\partial V}{\partial x} \right|_{x=0} = \frac{\beta^2 E_0}{A}$$

In the non-relativistic limit, the electric field is twice the beam kinetic energy divided by charge and bend radius: $q\mathcal{E} = mv^2/A$.

### 3 Hamiltonian and Transfer Matrix

The first order terms in the resulting Hamiltonian all cancel, so when expanded to second order it is

$$\tilde{K} = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{P_r^2}{2\gamma^2} - \frac{2 - \beta^2}{A} xP_r + \frac{\xi^2}{2A^2} x^2 + \frac{\eta^2}{2A^2} y^2$$

The parameters $\xi$ and $\eta$ are introduced as they parameterize the $x$ and $y$ focusing strengths: $\xi^2 + \eta^2 = 2 - \beta^2$, $\eta^2 = k/h = A/A_y$, $A_y$ being the curvature radius in the non-bend direction. In the non-relativistic limit, for a cylindrical bend, $\xi = \eta = 0$; for a spherical bend, $\xi = \eta = 1$. 
The transfer matrix is easily derived from this Hamiltonian $\tilde{K}$:

\[
\begin{pmatrix}
\cos \xi \theta & \frac{4}{\xi} \sin \xi \theta & 0 & 0 & 0 & \frac{2-\beta^2}{\xi} A (1 - \cos \xi \theta) \\
-\frac{2-\beta^2}{\xi} \sin \xi \theta & \cos \xi \theta & 0 & 0 & 0 & \frac{2-\beta^2}{\xi} \sin \xi \theta \\
0 & 0 & \cos \eta \theta & \frac{4}{\eta} \sin \eta \theta & 0 & 0 \\
0 & 0 & -\frac{2-\beta^2}{\eta} \sin \eta \theta & \cos \eta \theta & 0 & 0 \\
-\frac{2-\beta^2}{\xi} \sin \xi \theta & -\frac{2-\beta^2}{\xi} A (1 - \cos \xi \theta) & 0 & 0 & 1 & A \theta \left[ \frac{1}{\gamma^2} - \left( \frac{2-\beta^2}{\xi} \right)^2 \left( 1 - \frac{\sin \xi \theta}{\xi \theta} \right) \right] \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(12)

Interestingly, in the extreme relativistic limit ($\beta = 1$), this is identical to the matrix for a magnetic bend with field index.

References

[1] R. Baartman: End Effects of Beam Transport Elements, Talk at Snowmass 2001, [http://lin12.triumf.ca/text/Talks/2001Snowmass/FF.ps](http://lin12.triumf.ca/text/Talks/2001Snowmass/FF.ps)