LQG Differential Stackelberg Game under Nested Observation Information Pattern

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Abstract—We investigate the linear quadratic Gaussian Stackelberg game under a class of nested observation information pattern. Two decision makers implement control strategies relying on different information sets: The follower uses its observation data to design its strategy, whereas the leader implements its strategy using global observation data. We show that the solution requires solving a new type of forward-backward stochastic differential equations whose drift terms contain two types of conditional expectation terms associated to the adjoint variables. We then propose a method to find the functional relations between each adjoint pair; i.e., each pair formed by an adjoint variable and the conditional expectation of its associated state. The proposed method follows a layered pattern. More precisely, in the inner layer, we seek the functional relation for the adjoint pair under the $\sigma$-sub-algebra generated by follower’s observation information; and in the outer layer, we look for the functional relation for the adjoint pair under the $\sigma$-sub-algebra generated by leader’s observation information. Our result shows that the optimal open-loop solution admits an explicit feedback type representation. More precisely, the feedback coefficient matrices satisfy tuples of coupled forward-backward differential Riccati equations, and feedback variables are computed by Kalman-Bucy filtering.

Index Terms—Differential Stackelberg game, separation principle, linear quadratic Gaussian optimal control, backward stochastic differential equations (BSDEs), Kalman filter.

I. INTRODUCTION

Stackelberg game was introduced by H. Von Stackelberg as a static game in the context of static economic equilibrium in 1934 [1], when he defined a concept of a hierarchical solution for markets where some firms have power of domination over others. This solution is now known as the Stackelberg equilibrium which, in terms of two-person nonzerosum games, involves two players with asymmetric roles, one leader and one follower. On the other hand, the study of differential games was initiated by Isaacs in 1954 [2]. Different from the static optimization problem, the related equality constraint becomes a differential dynamic system. With the rise of differential games, the differential Stackelberg game entered the control literature through the early works [3]–[7]. Since then, stochastic differential Stackelberg games have been investigated by many authors and have been used in many applications. For example, Castanon et al. [8] gave explicit solutions to the linear quadratic Gaussian (LQG) Stackelberg game under asymmetric information where the follower knows the leader’s control; Øksendal et al. [9] derived a maximum principle for the controlled jump diffusion process and applied it to the news vendor problem; Bensoussan et al. [10] derived a global maximum principle for both open-loop and closed-loop Stackelberg differential games. Xiu et al. [11] researched the Stackelberg strategy for the time-delay Stackelberg game in discrete-time dynamic setting and presented a necessary and sufficient condition. Mukaidani et al. [12] investigated the infinite horizon linear quadratic discrete-time Stackelberg game problem with multiple decision makers and derived the necessary conditions for the existence of the optimal strategy set. Moon et al. [13] considered a class of stochastic differential games with the Stackelberg mode of play, with one leader and $N$ uniform followers and designed an approximate Stackelberg equilibrium for the games with sufficiently many finite agents. Lin et al. [14] concerned an open-loop linear quadratic Stackelberg game of the mean-field stochastic systems, and shown that the open-loop Stackelberg equilibrium admits a feedback representation involving a new state and its mean.

Recently, Shi et al. [15] investigated the stochastic differential Stackelberg game for nonlinear systems under asymmetric information, i.e., the leader and the follower implement control strategies using different information patterns. By applying the Girsanov’s theorem, the maximum principle is derived for this problem and solutions for control strategies are given. In this paper, we aim to develop more explicit solutions for LQG Stackelberg game under nested observation information pattern. This pattern implies that the information available to the leader contains the information available to the follower. In our model, the leader is able to access two linear noisy observation data and the follower is only able to access one of them.

The difference between our work and the one in Shi et al. [15] is mainly on the following aspect. The control strategy processes of their work are adapted to the filtration generated by two Wiener processes, i.e., they consider the case of the non-anticipative information pattern. In contrast, we deal with the noisy information pattern [16]–[18], i.e., two decision makers implement strategies relying on two observation processes, respectively. This leads to our estimators being
different from ones in [15]. In their situation, two decision makers’ estimators are obtained directly by taking conditional expectation with respect to the $\sigma$-sub-algebras generated by two Wiener processes stated above respectively.

In Castanon et al. [8], the follower uses its observation and the leader’s strategy to design its own strategy, whereas the leader implements its strategy using global observation data, thus the information set of the follower to implement strategy contains the observation itself and a subset of both decision makers’ observation data.

Comparing with [8], the follower in our model uses only the observation data to implement its strategy. In this setting, the separation principle is no longer valid [19]. Different from the method in [15], we take the dynamic system equation and the observation equation as equality constraints to overcome this difficulty.

A comparison between this work and the previous one [20] is as follows: In both works the follower implements its strategy relying on its observation data, and this data is also available to the leader for implementing its strategy. In this work we generalize this setup by making available to the leader also its local observation data. That is, the leader is no longer able to access the system state information. In this sense, the setup studied in this work is a generalization of the one studied in [20].

In addition to the above point, the technical results derived in the current paper are more complete than those in [20]. In this work we provide conditions to guarantee existence of the solution, and give sufficient and necessary condition for the optimal open-loop solution. In contrast with the current paper, we only discuss the necessary condition for the optimal open-loop solution. Neither the existence condition and the sufficient condition for the optimal open-loop solution nor the existence and uniqueness analysis for the solution of the related FB-SDE are given in the previous paper. Finally, technically speaking, the different setup studied in this work leads to a FB-SDE which is different from the one in [20]. More precisely, it contains two kind of conditional expectation terms in its drift, rather than only one kind. As a result, the layered analysis is technically more complex, whereas two kind of (non-independent) innovation processes appear.

With respect to this class of FB-SDEs and their solving methods, as we know, there are few reports. We give a layered calculation method to solve it. Noticing that there are two kind of conditional mean terms of of adjoint states, we then look for the functional relation between these two terms and their corresponding state variables respectively. This procedure is done by a layered way. We first look for the functional relation for the adjoint pair under the $\sigma$-sub-algebra generated by follower’s observation information in the inner layer part; then we look for the functional relation for the adjoint pair under the $\sigma$-sub-algebra generated by leader’s observation information in the outer layer part.

The main innovations are the following.

1. Propose a new layered calculation method to find the feedback type strategies for both decision makers.

The rest of the paper is organized as follows. In Section II, the LQG Stackelberg game under nested information patterns is formulated. In Section III, the sufficient and necessary conditions for the follower and the leader to be optimal are given respectively. In addition, the existence and uniqueness property for the solution of the FB-SDE appeared in leader’s problem is derived. In section IV, we deal with the computation aspect and obtain the feedback type strategies. In section V, we give some examples to show its application of our main result.

Notation: We use $M'$ to denote the transpose of matrix or vector $M$ and use $I_d$ to denote the identity matrix. $\text{diag}(A_{11},...,A_{nn})$ denotes the block diagonal matrix whose diagonal entries are $A_{11},...,A_{nn}$, and $\text{col}(A_{11};...;A_{n1}) := (A_{11},...,A_{n1})'$. We also use $S > 0$ and $S \succeq 0$ to denote the symmetric matrix $S$ to be positive definite and positive semi-definite respectively. $\langle \cdot,\cdot \rangle$ and $|\cdot|$ denote the inner product and the norm in the Euclidean space, respectively. For a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{E}[\cdot]$ and $\text{cov}(\cdot,\cdot)$ denote the expectation and covariance operators defined by the probability measure $\mathbb{P}$ respectively.

II. Problem Formulation

A. Model Description

We study the linear quadratic Stackelberg game described by

$$
\mathcal{J}^i(U^L, U^F) = \mathbb{E}\int_0^T \frac{1}{2} \left( \langle Q_iX_t, X_t \rangle + \langle R_iU_t^L, U_t^L \rangle + \langle R_iU_t^F, U_t^F \rangle \right) dt + \mathbb{E}\left( \frac{1}{2} (G_iX_T, X_T) \right),
$$

(1)

subject to the signal model

$$
\begin{align*}
    dX_t &= (AX_t + BF_tU_t^F + BL_tU_t^L)dt + DdW_t, \\
    dZ_t^1 &= H_1X_t dt + dW_t^1, \\
    dZ_t^2 &= H_2X_t dt + dW_t^2,
\end{align*}
$$

(2)

where $i = F, L$. $X_t \in \mathbb{R}^n$ is the state of the linear time-invariant system, $Z_t^1 \in \mathbb{R}^d$ and $Z_t^2 \in \mathbb{R}^m$ are the linear noisy observations at time $t$, initialized by $Z_0^1 = 0$ and $Z_0^2 = 0$ respectively. Between them, the follower is able to access to both observation process $\{Z_t^1\}$, whereas the leader is able to access to both $\{Z_t^1\}$ and $\{Z_t^2\}$. $U_t^F \in U^F \subset \mathbb{R}^k$ and $U_t^L \in U^L \subset \mathbb{R}^d$ are the strategies of two decision makers, respectively, where $U^F$ and $U^L$ are convex and compact. $(W_t^1, V_t^1, V_t^2) \in \mathbb{R}^m$ is the standard $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted Wiener process on the filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$. The initial value $X_0 = \xi$ is a Gaussian random variable which is independent of the standard Wiener process above. $A, B, H_1, H_2, \xi$ are constant matrices with adequate dimensions respectively; Weights matrices $Q_i, R_{ij}, G_i, Q_i, R_{ii}$ are symmetric matrices, with $i, j = F, L$. The detail conditions on them are listed in Assumption I.

The two information patterns available to the decision makers $L$ and $F$ are denoted by two filtrations $\{I_t^L\}_{t \in [0,T]}$ and $\{I_t^F\}_{t \in [0,T]}$. We use $I_t$ to denote the $\sigma$-algebra generated by $\{I_t^F\}_{t \in [0,T]} \cup \{I_t^L\}_{t \in [0,T]}$. For each $i = F, L$, we also use $\mathcal{F}_t^i = \sigma(I_t^i \cup \mathcal{F}_t)$ to denote the $\sigma$-algebra generated by $\{I_t^i\}_{t \in [0,T]} \cup \{\mathcal{F}_t\}_{t \in [0,T]}$.
The admissible strategy sets of the follower and the leader are defined, respectively, by

\[ U^F := \{ U^F | U^F_t \text{ is } \mathbb{F} \text{-valued and } \{ I^F_t \} \text{ - adapted process with } \mathbb{E} \int_0^T |U^F_t|^2 \, dt < \infty \}, \]

\[ U^L := \{ U^L | U^L_t \text{ is } \mathbb{L} \text{-valued and } \{ I^L_t \} \text{- adapted process with } \mathbb{E} \int_0^T |U^L_t|^2 \, dt < \infty \}. \]

The open-loop LQG Stackelberg game under asymmetric information pattern is stated as follows: Determine the control functions \( U^{F*} \in U^F \) and \( U^{L*} \in U^L \) such that \( U^{F*} = \eta(U^{L*}) \), with \( \eta \) being the reaction function in the sense that the follower responds to the leader’s strategy. \( U^{L*} \) and the mapping \( \eta: U^L \to U^F \) satisfy the following conditions:

\[ (\text{Follower’s Optimality Condition}) \]

\[ \mathcal{J}^F(U^L, \eta(U^L)) \leq \mathcal{J}^F(U^L, U^F), \quad \forall U^L \in U^L, \]

\[ U^F \in U^F, \quad (3) \]

\[ (\text{Leader’s Optimality Condition}) \]

\[ \mathcal{J}^L(U^{L*}, \eta(U^{L*})) \leq \mathcal{J}^L(U^L, \eta(U^L)), \quad \forall U^L \in U^L. \]

**Assumption 1.** Suppose the weight matrices satisfy the following conditions:

\[ A_{1a}. \quad Q_F \succeq 0, \quad G_F \succeq 0, \quad R_{FF} \succ 0, \]

\[ A_{1b}. \quad Q_L \succeq 0, \quad G_L \succeq 0, \quad R_{LL} \succ 0, \quad R_{LF} \succ 0. \]

With regard to (2), we consider an additive linear system rather than a one with multiplicative noise. As is well known, the filter of the latter case is nonlinear.

**B. Optimization Problem**

As stated above, the follower is able to access to the observation process \( \{ Z^F_t \} \), whereas the leader is able to access to both \( \{ Z^L_t \} \) and \( \{ Z^F_t \} \). We suppose the follower and the leader depend respectively on their respective information available to implement their strategies at time \( t, t \in [0, T] \).

To describe it, let \( I^L_t = Z^L_t, I^F_t = Z^F_t \). They are generated by their respective information sets, for detail

\[ Z^L_t := \sigma(Z^L_s, Z^L_s | 0 \leq s \leq t), \]

\[ Z^F_t := \sigma(Z^F_s | 0 \leq s \leq t), \quad t \in [0, T]. \]

Notice that, for each \( t \in [0, T] \), \( Z^F_t \subseteq Z^L_t \), implying that the \( \sigma \)-algebras are nested [3]. For other works in linear quadratic control under nested information pattern we refer the reader to [21].

**The Follower’s Problem** Based on the *Follower’s Optimality Condition* [3], given the leader’s strategy \( U^L \in U^L \), the follower is faced with the following optimal control problem:

\[ \inf_{U^F \in U^F} \mathcal{J}^F(U^L, U^F), \quad (4) \]

subject to

\[ \begin{cases} 
  dX_t = (AX_t + B_F U^F_t + B_L U^L_t) \, dt + DdW_t, \\
  dZ^F_t = H_2 X_t \, dt + dV^2_t, \\
  X_0 = \xi, \quad Z^2_0 = 0. 
\end{cases} \]

**The Leader’s Problem** Based on the *Leader’s Optimality Condition* of [3], after substituting \( U^{F*} \), obtained from solving the follower’s problem into [5], we get the optimization problem below faced by the leader:

\[ \inf_{U^L \in U^L} \mathcal{J}^L(U^L, U^{F*}), \quad (6) \]

subject to

\[ \begin{cases} 
  dX_t = (AX_t + B_F U^{F*} + B_L U^L_t) \, dt + DdW_t, \\
  dZ^L_t = H_1 X_t \, dt + dV^1_t, \\
  dZ^2_t = H_2 X_t \, dt + dV^2_t, \\
  X_0 = \xi, \quad Z^1_0 = 0, \quad Z^2_0 = 0. 
\end{cases} \]

**III. Optimality Conditions**

For the follower’s problem, for every \( U^L \in U^L \), it is a classic LQG regulation problem under assumption A1a. Thus, the optimization problem of the follower’s is well-posed and has a unique solution [22, 23].

**Lemma 1.** Let assumption A1a be satisfied. Then, a necessary and sufficient condition for \( U^F \) to be an open-loop optimal strategy of the follower’s problem described by [21-23] is

\[ U^{F*}_t = -R^{-1}_{FF} B_F \hat{p}^F_t, \quad t \in [0, T], \]

with \( \hat{p}^F_t := \mathbb{E} [p^F_t | Z^F_t]. \) Here, \((p^F_t, (q^F_t, r^F_t))\) (called the adjoint process) is the unique solution of the BSDE (conventionally called the adjoint equation):

\[ \begin{cases} 
  dp^F_t = -\left[ A^T p^F_t + Q_F X_t \right] \, dt + q^F_t \, dW_t + r^F_t \, dV^2_t, \\
  p^F_T = G_F X_T. \nonumber \end{cases} \]

Notice that there appears a adjoint process when solve the follower’s problem, a similar situation may appear later in leader’s problem, see [14]. Its proof is similar to the work in [16]. For easy of reading, we give its proof in Appendix.

After substituting \( U^{F*} \) in (8) into (5), the optimization problem faced by the leader changes to:

\[ \inf_{U^L \in U^L} \mathcal{J}^L(U^L, U^{F*}), \quad (10) \]

subject to

\[ \begin{cases} 
  dX_t = \left( AX_t - B_F R^{-1}_{FF} B_F \hat{p}^F_t + B_L U^L_t \right) \, dt + DdW_t, \\
  dZ^L_t = H_1 X_t \, dt + dV^1_t, \\
  dZ^2_t = H_2 X_t \, dt + dV^2_t, \\
  dp^F_t = -\left[ A^T p^F_t + Q_F X_t \right] \, dt + q^F_t \, dW_t + r^F_t \, dV^2_t, \\
  X_0 = \xi, \quad Z^1_0 = 0, \quad Z^2_0 = 0; \quad p^F_T = G_F X_T. \nonumber \end{cases} \]

Notice that the existence and the uniqueness of the follower’s strategy implies that, for every \( U^L \in U^L \), there
exists a solution for (11). Suppose \((X^1, (p^{F1}, q^{F1}, r^{F1}))\) and \((X^2, (p^{F2}, q^{F2}, r^{F2}))\) are two different solutions of (11). Let
\[
\phi_t(X) := X_t^1 - X_t^2, \quad \phi_t(p^F) := p_t^{F1} - p_t^{F2}, \\
\phi_t(q^F) := q_t^{F1} - q_t^{F2}, \quad \phi_t(r^F) := r_t^{F1} - r_t^{F2}. \tag{12}
\]
Suppose assumption A1a holds. From an argument similar to the one in [24], it follows that
\[
E \sup_{t \in [0, T]} \left| \phi_t(X) \right|^2 + E \sup_{t \in [0, T]} \left| \phi_t(p^F) \right|^2 + E \int_0^T \left[ \left| \phi_t(q^F) \right|^2 + \left| \phi_t(r^F) \right|^2 \right] dt \leq K \left( E \left| \phi_0(X) \right|^2 + E \int_0^T \left| \phi_t(U^L) \right|^2 dt \right),
\]
where \(K\) is a positive constant and \(\phi_t(U^L) := U^L_t - U^L_t\) denotes the error of two different strategies of the leader.

Notice that (13) implies all left terms can be dominated by initial data error and the energy of the strategy process error. Based on this, uniqueness of solution of (11) is derived. From now on, we have got, for every \(U^L \in U^L\), that the FB-SDE in (11) has a unique solution. To proceed, we need a lemma.

**Lemma 2.** Let assumptions A1a, A1b be satisfied. Then the leader’s problem has an optimal solution.

**Proof.** Let \((X_n, p_{nF}, q_{nF}, r_{nF}, U_{nL})\) be a minimizing sequence, i.e., \(J^L(U^L_{nL}, U^F) \rightarrow \inf_{U^L \in U^L} J^L(U^L, U^F)\) as \(n \rightarrow +\infty\), where \((X_n, p_{nF}, q_{nF}, r_{nF})\) denotes the state processes of (11) corresponding with the strategy process \(U^L_{nL}\) for every \(n \in \mathbb{N}\). From the boundedness of \(U^L\), we have
\[
E \int_0^T \left| U^L_{nL} \right|^2 dt \leq K, \quad \forall \ n \in \mathbb{N},
\]
with \(K\) being a positive constant. Thus, there is a subsequence \((U^L_{kL})\) such that \(U^L_{kL}\) converges weakly to \(U^L\) as \(k \rightarrow +\infty\) in the square integrable space containing all \((Z^L_t)\)-adapted \(\mathbb{R}^d\)-valued random processes.

By Mazur theorem [25], there exists a sequence \(V_{nL}\) made up of convex combinations of the \(U^L_{kL}\)’s that converges strongly to \(U^L\):
\[
U^L_{nL} = \sum_{k \geq 1} a_{nk} U^L_{kL}, \quad a_{nk} \geq 0, \quad \sum_{k \geq 1} a_{nk} = 1.
\]
Since the subset \(U^L \subset \mathbb{R}^d\) is convex and closed, it follows that \(U^L \in U^L\). Let \((\tilde{X}_{t,n}, \tilde{p}_{t,n}^F, \tilde{q}_{t,n}^F, \tilde{r}_{t,n}^F)\) and \((\hat{X}_{t}, \hat{p}_{t}^F, \hat{q}_{t}^F, \hat{r}_{t}^F)\) be the states under the strategies \(U^L_{nL}\) and \(U^L\) respectively, then it follows, from (13), that
\[
E \sup_{t \in [0, T]} \left| \tilde{X}_{t,n} - \hat{X}_{t} \right|^2 = 0, \\
E \sup_{t \in [0, T]} \left| \tilde{p}_{t,n}^F - \hat{p}_{t}^F \right|^2 = 0, \quad n \rightarrow +\infty.
\]
Finally, from the convexity of the quadratic performance index of (10) with their respective variables, we have
\[
J^L(U^L_t, -R^L_{1F}B^F F^L_t) = \lim_{n \rightarrow +\infty} J^L(U^L_{nL}, -R^L_{1F}B^F F^L_{nL}) \leq \lim_{n \rightarrow +\infty} \sum_{k \geq 1} a_{nk} J^L(U^L_{kL}, -R^L_{1F}B^F F^L_{kL}) = \inf_{U^L \in U^L} J^L(U^L_t, U^F_t).
\]
Hence, \(U^L\) is optimal.

**Remark 1.** This lemma develops the result of stochastic LQ problems in [26], pp.68. With regard to the uniqueness of the optimal control, we show later the optimal control is computed by a FB-SDE. Under our assumption, we show the FB-SDE admits a unique solution. Therefore, the optimal control is unique.

Next, we introduce a new FB-SDE (14). This FB-SDE is needed for the proof of Lemma 3. We write it here in advance and discuss the existence and uniqueness property of its solution to avoid the proof of Lemma 3 being too long. Comparing with (11), there are two additional equations appear. They are the equations satisfied by two adjoint state processes, \((p^F, r^F, k^F)\) and \(Y\), of the original state processes \(X\) and \((p^F, q^F, r^F)\) respectively.

\[
\begin{aligned}
\frac{dp_t^F}{dt} &= -\left[ A p_t^F + Q_F Y_t + Q_L X_t \right] dt + q_t^F dW_t + k_t^F dV_t^1 + r_t^F dV_t^2, \\
\frac{dp_t^F}{dt} &= -\left[ A p_t^F + Q_F Y_t \right] dt + q_t^F dW_t + r_t^F dV_t^2, \\
dX_t &= \left[ AX_t + B_t^L U^L_t - R^L_{1F}B^F F^L_t \right] dt + D dW_t, \\
U^L_t &= -R^L_{1F}B^F F^L_t, \quad t \in [0, T], \\
dY_t &= \left( A Y_t - B_F R^L_{1F}B^F F^L_t \right) dt + B_F R^L_{1F}R^1_{LE}R^L_{F}B^F F^L_t dt, \\
dZ_t &= H_t X_t dt + dV_t^1, \\
p_t^F &= G_L X_T + G_F Y_T, \quad p^F_t = G_F X_t, \\
Y_0 = \xi, \quad Z_0 = 0, \quad Z^2 = 0, \quad \left| \hat{p}_t^F \right| = E \left| p_t^F \right|^2, \quad \left| \hat{p}_t^F \right| = E \left| p_t^F \right|^2 \tag{14}
\end{aligned}
\]

where \(\hat{p}_t^F := E \left[ p_t^F | Z_t^1 \right], \hat{p}_t^F := E \left[ p_t^F | Z_t^1 \right] \) and \(\hat{p}_t^F := E \left[ p_t^F | Z_t^1 \right] \).
\( \phi_t (q^L), \phi_t (k^L), \phi_t (p^L), \phi_t (p^F), \phi_t (r^F) \) are defined similarly as (12), all of them denote the deviation of two tuple of different solutions of (14). We first show the uniqueness of \( X \), which relies on the following two priori estimates: 

\[
E(\langle G_F \phi_T (X), \phi_T (X) \rangle) + \mathbb{E} \int_0^T \left( \langle B_L R_{LL}^{-1} B_L \phi_t (p^L) \rangle, \phi_t (p^L) \rangle \right) dt + \mathbb{E} \int_0^T \left( \langle B_F R_{FF}^{-1} B_F \phi_t (p^F) \rangle, \phi_t (p^F) \rangle \right) dt = \mathbb{E}(\phi_0 (X), \phi_0 (p^L)) \leq \mathbb{E}(\mathbb{E}(\phi_0 (X) \mid | \phi_0 (p^L)) |),
\]

and

\[
\mathbb{E} \int_0^T e^{-Kt} | \phi_t (X) |^2 dt \leq \mathbb{E} | \phi_0 (X) |^2
\]

where they are obtained by applying Itô’s formula to \( \langle \phi_t (X), \phi_t (p^L) \rangle - \langle \phi_t (Y), \phi_t (p^L) \rangle \) and \( e^{-Kt} | \phi_t (X) |^2 \) and then taking expectation respectively. We provide the latter one as follow:

\[
\mathbb{E} e^{-Kt} | \phi_t (X) |^2 + \mathbb{E} \int_0^T K e^{-Kt} | \phi_t (X) |^2 dt \]

\[
= \mathbb{E} | \phi_0 (X) |^2 + \mathbb{E} \int_0^T \left( 2e^{-Kt} | \phi_t (X) |^2 + A \phi_t (X) - B_L R_{LL}^{-1} B_L \phi_t (p^L) - B_F R_{FF}^{-1} B_F \phi_t (p^F) \right) dt
\]

\[
\leq \mathbb{E} | \phi_0 (X) |^2 + C_1 \mathbb{E} \int_0^T e^{-Kt} | \phi_t (X) |^2 dt + \mathbb{E} \int_0^T \left( e^{-Kt} \phi_t (p^L), B_L R_{LL}^{-1} B_L \phi_t (p^L) \right) dt + \mathbb{E} \int_0^T \left( e^{-Kt} \phi_t (p^F), B_F R_{FF}^{-1} B_F \phi_t (p^F) \right) dt,
\]

where \( C_1 = \text{tr}(AA^T) + \text{tr}(B_L R_{LL}^{-1} B_L) + \text{tr}(B_F R_{FF}^{-1} B_F) + 1. \)

So (16) holds by taking \( K = C_1 + 1. \)

Suppose A1b holds. From (15), it always has a proper positive constant \( C_2 \) such that

\[
\mathbb{E} \int_0^T e^{-Kt} | \phi_t (X) |^2 dt \leq \mathbb{E} | \phi_0 (X) |^2 + C_2 \mathbb{E} \left( | \phi_0 (X) | \mid \phi_0 (p^L) \right),
\]

thus, \( \phi_t (X) = 0, t \in [0, T]. \) The uniqueness of \( p^F \) follows from the classic estimate for BSDEs below (27):

\[
\mathbb{E} \sup_{t \in [0, T]} | \phi_t (p^F) |^2 + \mathbb{E} \int_0^T \left( | \phi_t (q^F) |^2 + | \phi_t (r^F) |^2 \right) dt \leq C_3 \left( \mathbb{E} |G_F \phi_T (X)| + \mathbb{E} \int_0^T |Q_F \phi_t (X) | dt \right),
\]

where \( C_3 \) is a positive constant. Therefore, we have from the uniqueness of state process \( X \) that \( \phi_t (p^F) = 0, \phi_t (q^F) = 0, \) and \( \phi_t (r^F), t \in [0, T]. \) Finally, we prove the uniqueness of the adjoint process \( p^L \) and the state process \( Y. \) To achieve it, consider the first and the fifth equations of (14) and compute the respective deviation equations:

\[
d\phi_t (p^L) = - \left[ A' \phi_t (p^L) + Q_F \phi_t (Y) \right] dt + \phi_t (q^L) dW_t + \phi_t (k^L) dV_t + \phi_t (r^L) dV_t^2,
\]

\[
d\phi_t (Y) = \left( A \phi_t (Y) - B_F R_{FF}^{-1} B_F \phi_t (p^L) \right) dt,
\]

\[
\phi_T (p^L) = G_F \phi_T (Y), \quad \phi_0 (Y) = 0,
\]

where there are no influence to be caused by \( \{ \phi_t (X), t \in [0, T] \} \) and \( \{ \phi_t (p^F), t \in [0, T] \} \) since the processes \( X \) and \( p^F \) have been shown to be unique. In another word, we take \( X_1 = X^2 \) and \( p^F_1 = p^F_2. \)

Applying Itô’s formula to \( \langle \phi_t (p^L), \phi_t (Y) \rangle \) and then taking expectation, it yields

\[
\mathbb{E}(G_F \phi_T (Y), \phi_T (Y)) + \mathbb{E} \int_0^T \left( \langle Q_F \phi_t (Y), \phi_t (Y) \rangle + \langle B_F R_{FF}^{-1} B_F \phi_t (p^L), \phi_t (p^L) \rangle \right) dt
\]

\[
= \mathbb{E}(\phi_0 (p^L), \phi_0 (Y)) \leq \mathbb{E}(\mathbb{E}(\phi_0 (p^L) \mid | \phi_0 (Y)) \right).
\]

A similar analysis with the above proofs leads to \( \phi_t (Y) = 0, \) and then \( \phi_t (p^F) = 0, \phi_t (q^F) = 0, \phi_t (k^F) = 0, \phi_t (r^F) = 0, t \in [0, T]. \)

\[ \Box \]

Lemma 4. Let assumptions A1a, A1b be satisfied. Then, a necessary and sufficient condition for \( U^L \) to be an open-loop optimal strategy of the leader’s problem described by (12)-(14) is

\[
U_t^{L*} = - R_{LL}^{-1} B_L \tilde{p}_t^L, \quad t \in [0, T],
\]

with \( \tilde{p}_t^L = \mathbb{E} [p_t^L | Z_t]. \) Here, \( (p_t^L, (q_t^L, k_t^L), r_t^L) \) is computed from the unique solution of the FB-SDE (27).

Proof. Based on Lemma 3 now we are able to show arguments of this lemma. The necessary part is left to Appendix. We show the sufficient part. Let \( U_t^{L*} \) be the optimal strategy for the leader and \( U_t^L \) be any other. We use \( X_t^* := X(U_t^{L*}) \) and \( X_t^* \) to distinguish the state processes under the two strategies above respectively, the similar denotations will be used for the processes \( p_t^L, q_t^L, r_t^L \) and so on.

Applying Itô’s formula to \( \langle \psi_t (L), \psi_t (p^F) \rangle - \langle Y_t, \psi_t (p^F) \rangle, \) it gives

\[
\langle \psi_t (X), G_t X_t^* \rangle = \int_0^T \left( \langle B_L \tilde{p}_t^L, \psi_t (U_t^L) \rangle - \langle Q_L X_t^*, \psi_t (X) \rangle \right)
\]

\[
- \langle B_F R_{FF}^{-1} B_F \psi_t (p^L), \psi_t (p^L) \rangle \right) dt
\]

where \( \psi_t (U_t^L) := U_t^L - U_t^{L*}, \psi_t (X) := X_t^* - X_t, \) and \( \psi_t (p_t^L) := p_t^L - p_t^{L*} \).
The convexity of the integrand of $J^L$ with respect to their respective variables leads to
\[
J^L(U^L, U^{F*}) - J^L(U^{L*}, U^{F*}) 
\geq \mathbb{E}(\psi_T(X), G_L X_T^p) 
+ \int_0^T \left( \langle Q_L X_t^p \psi_t(X) \rangle + \langle R_{LL} U_t^{L*} \psi_t(U_L) \rangle \right) dt 
+ \langle B_F^t R_{FF}^t R_{LF}^t R_{FF}^t B_F^t F_t X_t, \psi_t(p(t)) \rangle dt,
\]
Inserting the relationship (23) into (24), we have
\[
J^L(U^L, U^{F*}) - J^L(U^{L*}, U^{F*}) 
\geq \mathbb{E} \int_0^T \left( R_{LL} U_t^{L*} + B_L^t p_t \psi_t(U_t) \right) dt.
\]
Hence, sufficiency of (22) is shown.

Remark 2. With respect to the existence of the FB-SDE (14), we show it from the optimal control theory. It could be proven from the stochastic dynamic system point of view. The works in this aspect may refer to Antonelli [23], Hu et al. [24], Yong [25], Peng et al. [26], etc. We point out that their models do not contain ours; it still needs new method to solve it. In the procedure of finding the solution, we are not able to compute the relationship between the adjoint state and the state directly by the traditional methods, e.g. the four-step method [22] or by the nonlinear Feynman-Kac formula [23]. From the optimal control point of view, the reason is that the Riccati equations appearing in the control and filtering parts are coupled. The randomness of one of them will affect another; certainly, one is deterministic, so is the other. To verify if it is deterministic or not is what we are interested in.

IV. OPTIMAL STRATEGIES
By grouping variables $X_t$ with $Y_t$ and $p_t^L$ with $p_t^F$, we turn the expression (14) to a more compact form
\[
\begin{aligned}
&dX_t = (A X_t + \dot{S} p_t + \dot{S} p_t) dt + D dW_t, \\
dp_t = -A p_t + Q X_t dt + q_t dW_t \\
&+ k_t dV_t + r_t dV_t^2, \\
dZ_t = H X_t dt + dV_t, \\
dZ_t^2 = H_F X_t dt + dV_t^2, \\
X_0 = \gamma, \\
p_T = G X_T, \\
Z_0 = 0, \\
Z_0^2 = 0,
\end{aligned}
\]
where
\[
\begin{aligned}
X_t &= \text{col}(X_t; Y_t), \\
p_t &= \text{col}(p_t^L; p_t^F), \\
qu_t &= \text{col}(q_t^L; q_t^F), \\
r_t &= \text{col}(r_t^L; r_t^F), \\
\gamma &= \text{col}(\gamma^L; 0), \\
A &= \text{diag}(A, A), \\
\dot{S} &= \text{diag}(-B_L R_{LL}^{-1} B_L^t, 0), \\
D &= \text{col}(D; 0), \\
H_L &= (H_1, 0), \\
\dot{S} &= \begin{pmatrix} 0 & -B_F R_{FF}^{-1} B_F^t \\ -B_F R_{FF}^{-1} B_F^t & B_F R_{FF}^{-1} R_{LF} R_{FF}^{-1} B_F^t \end{pmatrix}, \\
Q &= \begin{pmatrix} Q_L & Q_F \\ Q_F & 0 \end{pmatrix}, \\
G &= \begin{pmatrix} G_L & G_F \\ G_F & 0 \end{pmatrix}, \\
H &= \begin{pmatrix} H_1 & 0 \\ H_2 & 0 \end{pmatrix}, \\
H_F &= (H_2, 0), \\
Z_t &= \text{col}(Z_t^L; Z_t^F), \\
V_t &= \text{col}(V_t^L; V_t^F).
\end{aligned}
\]
Moreover, $\dot{p}_t := \mathbb{E} [p_t | Z_t^F]$ and $\dot{p}_t := \mathbb{E} [p_t | \tilde{Z}_t^F]$.

Inspired by the ordinary differential equation theory, the main idea is to change a two-point boundary value problem to a Cauchy initial value problem [23]. To do it, some functional relationships between the adjoint state variables and the state variables need to be found, relating $p$ with $X$, $\dot{p}$ with $X$, $\ddot{p}$ with $X$, three layers, where $X := \{X_t, t \in [0, T]\}$ and $X := \{X_t, t \in [0, T]\}$ with $X_t := \mathbb{E}[X_t | \tilde{Z}_t^F]$ and $X_t := \mathbb{E}[X_t | Z_t^F]$. Among them, the last two parts are enough to design feedback type strategy, we state it in the following Theorem.

Theorem 1. Let assumptions A1a, A1b be satisfied. With notations in (27), the feedback type strategies of the leader and the follower given by (28) and (29) are given, respectively, by
\[
\begin{aligned}
U_t^{F*} &= -R_{LL}^{-1} B_L^t \left( P_{t,11}^1 + P_{t,12}^1 \right) X_t, \\
U_t^{L*} &= -R_{LL}^{-1} B_L^t \left( P_{t,11} - P_{t,11}^1 - P_{t,12} - P_{t,12}^1 \right) X_t,
\end{aligned}
\]
where $P_{t,ij}, P_{t,ij}^1, i, j = 1, 2$ are, respectively, the $ij$-element of the $2 \times 2$-blocked matrices $P_t$ and $P_t^1$, computed by (30) below
\[
\begin{aligned}
P_t^1 &= A^t + P_t^1 + P_t^1 A + P_t^1 S P_t + Q, \\
P_t^1 \Xi_{t,11} H_F H_F = 0, \\
\dot{P}_t + A^t P_t + P_t A + (\dot{S} + \dot{S}) P_t + Q = 0, \\
\dot{\Xi}_t = A \Xi_t + \Sigma_t A^t - \Sigma_t H \Sigma_t + \dot{D} D^t, \\
\dot{\Xi}_t &= F_t \Xi_t + \Xi_t F_t + \dot{D} D^t + G_t^t G_t^t + G_t^t G_t^t \\
&- (\Xi_t H_F + G_t^t) (H_F \Xi_t + G_t^t), \\
\Xi_t &= F_t \Xi_t + \Xi_t F_t + \dot{D} D^t + G_t^t G_t^t + G_t^t G_t^t \\
&- (\Xi_t H_F + G_t^t) (H_F \Xi_t + G_t^t), \\
F_t &= \begin{pmatrix} A + \dot{S} P_t^1 & -\dot{S} P_t^1 \\ 0 & A - \Sigma_t H H \end{pmatrix}, \\
\dot{D} &= \text{col}(D_t; D_t), \\
G_t &= \text{col}(0; -\Sigma_t H_t), \\
G_t &= \text{col}(0; -\Sigma_t H_t), \\
H_F &= \text{col}([H_2, 0], [0, 0]).
\end{aligned}
\]
In addition, the augmented system state estimates $\tilde{X}_t$ and $\tilde{X}_t$ appeared in (28) and (29) are given, respectively, by the two Kalman-Bucy filters
\[
\begin{aligned}
\dot{dX}_t &= \begin{pmatrix} A + (\dot{S} + \dot{S}) P_t | \tilde{X}_t \end{pmatrix} dt + \Xi_{t,11} H_F dW_t, \\
\dot{dW}_t &= dZ_t^2 - H_F \tilde{X}_t dt, \\
X_0 &= \mathbb{E}[\gamma],
\end{aligned}
\]
and
\[
\begin{aligned}
\dot{d\tilde{X}}_t &= \begin{pmatrix} A + \dot{S} P_t^1 | \tilde{X}_t + (\dot{S} + \dot{S}) P_t - \dot{S} P_t^1 | \tilde{X}_t \end{pmatrix} dt \\
&+ \Sigma_t H_t dW_t, \\
\dot{dL}_t &= dZ_t - H_X \tilde{X}_t dt, \\
\tilde{X}_0 &= \mathbb{E}[\gamma].
\end{aligned}
\]
Proof. (Inner Layer Calculation) We seek the relationship between $\mathbf{p}_t$ and $\hat{\mathbf{X}}_t$ in this step. We first calculate $\hat{\mathbf{p}}_t$ from the second equation of (26). By a simple calculation, it reads

$$
\hat{\mathbf{p}}_t = \Phi'(T,t)G\mathbf{X}_T + \int_t^T \Phi'(s,t)Q\mathbf{X}_s ds - \int_t^T \Phi'(s,t) [\mathbf{q}_d dW_s + \mathbf{k}_s dV_s^1 + \mathbf{r}_s dV_s^2],
$$

where the state transition matrix $\Phi'(s,t)$ satisfies

$$
\left\{
\begin{array}{l}
\frac{d\Phi'(s,t)}{ds} = -\mathbf{A}' \Phi'(s,t), \\
\Phi'(s,s) = I_{id},
\end{array}
\right.
$$

(35)

Next, we calculate the estimate $\hat{\mathbf{p}}_t$. For simplicity we use

$$
\check{\mathbf{X}}_{s,t} := \mathbb{E}[\mathbf{X}_s|\mathbf{Z}^F_t], \quad \check{\mathbf{p}}_{s,t} := \mathbb{E}[\mathbf{p}_s|\mathbf{Z}^F_t], \quad s \geq t,
$$

to denote the predictions of $\mathbf{X}_s$ and $\mathbf{p}_s$ with respect to $\mathbf{Z}^F_t$.

Taking conditional expectation, with respect to $\mathbf{Z}^F_t$, on both sides of (34), it yields

$$
\hat{\mathbf{p}}_t = \Phi'(T,t)G\mathbf{X}_T + \int_t^T \Phi'(s,t)Q\mathbf{X}_{s,t} ds.
$$

(36)

Noticing the left-hand side of (36) involved the prediction terms $\check{\mathbf{X}}_{T,t}$ and $\check{\mathbf{p}}_{s,t}$, we are therefore going to calculate the estimate $\hat{\mathbf{X}}_t$. To do it, consider the signal model constructed by the first and the forth equations of (26). Based on Theorem 6.6 [34], this estimate can be obtained by the following nonlinear filter equation

$$
\left\{
\begin{array}{l}
d\check{\mathbf{X}}_t = \left( A\check{\mathbf{X}}_t + [\hat{\mathbf{S}} + \hat{\mathbf{S}}\check{\mathbf{p}}_t] \right) dt + \gamma_t H_p dt, \\
d\mathbf{Z}^F_t := d\mathbf{Z}^F_t - H\check{\mathbf{X}}_t dt, \\
\gamma_t := \mathbb{E}[\mathbf{X}_t|\mathbf{Z}^F_t] - \check{\mathbf{X}}_t,
\end{array}
\right.
$$

(37)

Based on the filter (37), we now are able to derive the prediction terms stated above. Utilizing $\mathbb{E}[\mathbb{E}[\mathbf{X}_s|\mathbf{Z}^F_t] | \mathbf{Z}^F_t] = \mathbb{E}[\mathbf{X}_s|\mathbf{Z}^F_t]$, $s \geq t$, the predictor, referring to Section 3.7 [22], is given by

$$
d\check{\mathbf{X}}_{s,t} = \left( A\check{\mathbf{X}}_{s,t} + [\hat{\mathbf{S}} + \hat{\mathbf{S}}\check{\mathbf{p}}_{s,t}] \right) ds,
$$

(38)

where

$$
\check{\mathbf{X}}_{s,t} = \check{\mathbf{X}}_t, \quad s \geq t.
$$

Observing (36) and (38), it hints us to let

$$
\varphi_t = \hat{\mathbf{p}}_t - P_t \check{\mathbf{X}}_t, \quad t \in [0,T],
$$

(39)

where $P, \varphi$ are two processes to be determined.

Finally, we use a undetermined coefficient method to pin down $P, \varphi$ above. Applying (39) to (38), the explicit solution of the predictor is

$$
\check{\mathbf{X}}_{s,t} = \int_t^s \Psi(s,\tau) \left[ (\hat{\mathbf{S}} + \hat{\mathbf{S}}) \varphi_t \right] d\tau + \Psi(s,t) \check{\mathbf{X}}_t,
$$

(40)

where the state transition matrix $\Psi(s,t)$ is given by

$$
\left\{
\begin{array}{l}
\frac{d\Psi(s,t)}{ds} = (A + [\hat{\mathbf{S}} + \hat{\mathbf{S}}] P_s) \Psi(s,t), \\
\Psi(t,t) = I_{id},
\end{array}
\right.
$$

(41)

Inserting (40) into (36) and comparing with (39), it can be verified that

$$
P_t = \int_t^T \Phi'(s,t)\mathbf{Q}\Psi(s,t) ds + \Phi'(T,t)\mathbf{G}\Psi(T,t),
$$

$$
\varphi_t = \Phi'(T,t)\mathbf{G} \int_t^T \Psi(s,t) (\hat{\mathbf{S}} + \hat{\mathbf{S}}) \varphi_t ds + \int_t^T \int_t^s \Phi'(s,t)Q\Psi(s,\tau) (\hat{\mathbf{S}} + \hat{\mathbf{S}}) \varphi_t d\tau ds.
$$

Taking partial derivative with respect to the time variable $t$, it yields the second equation of (40) and $\varphi_t \equiv 0$, $t \in [0,T]$.

(Outer Layer Calculation) From now on, the first three equations of (26) have changed to

$$
\left\{
\begin{array}{l}
d\mathbf{X}_t = \left( A\mathbf{X}_t + \hat{\mathbf{S}}\mathbf{p}_t + \hat{\mathbf{S}}P_t\check{\mathbf{X}}_t \right) dt + \mathbf{D} dW_t, \\
d\mathbf{p}_t = - \left[ A' \mathbf{p}_t + Q\mathbf{X}_t \right] dt + \mathbf{q}_d dW_t + k_d dV_t^1 + r_d dV_t^2, \\
d\mathbf{Z}_t = H\mathbf{X}_t dt + dV_t,
\end{array}
\right.
$$

(42)

where the initial data and $\Sigma$ are the same with the ones in (43) and (30) respectively.

Observing (43) and (44), it hints us to let

$$
\check{\mathbf{p}}_t = P_t^\dagger \check{\mathbf{X}}_t + P_t^\dagger \check{\mathbf{X}}_t,
$$

(45)

where $P^\dagger$ is a deterministic matrix-valued process to be determined.

Next, we write the first system equation in (42) and the filer equations (37) and (44) together. It is in a form of

$$
\left\{
\begin{array}{l}
d\mathbf{X}_t = \Theta_t \left( \begin{array}{c}
\mathbf{X}_t \\
\check{\mathbf{X}}_t
\end{array} \right) dt + \left( \begin{array}{c}
\mathbf{D} \\
0
\end{array} \right) dW_t + \left( \begin{array}{c}
\Sigma_t \mathbf{H}_p \\
0
\end{array} \right) dV_t^2,
\end{array}
\right.
$$

(46)

with

$$
\Theta_t = \left[ \begin{array}{cc}
A & \hat{\mathbf{S}}P_t^\dagger + \hat{\mathbf{S}}p_t \\
0 & A + \hat{\mathbf{S}}P_t^\dagger + \hat{\mathbf{S}}p_t + \hat{\mathbf{S}}P_t^\dagger + \hat{\mathbf{S}}p_t
\end{array} \right],
$$

(47)
where the observation equation (the forth equation in (26) ) is used to substitute the innovation term \( I_t^P \) with the observation noise \( V_t^2 \).

Define the following state transition matrix
\[
\begin{cases}
\frac{\partial \Pi(s, t)}{\partial s} = \Theta, \Pi(s, t), & s \geq t, \\
\Pi(t, t) = I_{td}.
\end{cases}
\]
(48)

Via (48), the prediction of the state variable of (46) is expressed as:
\[
\begin{align*}
& (\mathbb{E}[X_t | Z^t_t]) = \Pi(s, t) \left( \begin{array}{c}
\hat{X}_s \\
\hat{X}_t
\end{array} \right), & s \geq t,
\end{align*}
\]
(49)
where the last three stochastic integrals disappear, since the independent increment property of the \( \{ Z^t_t, t \in [0, T] \} \)-adapted Wiener processes \( \{ I_t, t \in [0, T] \} \) and the following equalities:
\[
\begin{align*}
& \mathbb{E} \left[ \int_t^s \Pi(s, \tau) [\cdot] dW_\tau \right] | Z^t_t = 0, \\
& \mathbb{E} \left[ \int_t^s \Pi(s, \tau) [\cdot] dW^2_\tau | Z^t_t = 0.
\end{align*}
\]
In the above, the tower property of the conditional expectation has been used.

Applying (49) to (43) and comparing with (45), we have
\[
\begin{align*}
P_t^{1+} + P_t^{2+} + P_t^1 = \Phi'(T, t) G [0, I_{td}, 0] \Pi(T, t) + \int_t^T \Phi'(s, t) Q [0, I_{td}, 0] \Pi(s, t) ds,
\end{align*}
\]
(50)
where \( P_t^{1+} = P_t^{1-} + P_t^{2+}, t \in [0, T]. \)

Taking partial derivation with respect to the time variable \( t \), it yields:
\[
\begin{align*}
P_t^{1+} + A' P_t^{1+} + P_t^{1-} A + P_t^{1-} T_t \Xi_t \Phi_t = 0, \\
P_t^{2+} + A' P_t^{2+} + P_t^{2-} A + P_t^{2-} T_t \Xi_t \Phi_t = 0,
\end{align*}
\]
(51)
(52)
where \( P_t^{1+} = 0, P_t^{2+} = G. \)

Thus, \( P_t^1 \) satisfies
\[
\begin{align*}
P_t^1 + A' P_t^1 + P_t^1 A + P_t^1 T_t \Xi_t \Phi_t + P_t^1 T_t H_t \Phi_t + Q = 0,
\end{align*}
\]
(53)
\[
\begin{align*}
P_t^1 = G.
\end{align*}
\]
Finally, we calculate the filtering equations for \( \hat{X} \) and \( \bar{X} \).

Applying (45) to (44), it yields (33). We then compute the filtering \( \hat{X} \), it is done by the Kalman-Bucy filtering theory. Define an error variable \( e_t := X_t - \hat{X}_t \) in prior. Considering the first and the forth equations of (26) and (44), it can be verified that \( X, e \) and \( Z^2 \) construct the following signal system:
\[
\begin{align*}
dX_t &= \left( AX_t + \bar{SP}_t \bar{X}_t + [(S + \bar{S}) P_t - \bar{SP}_t] \right) \times \bar{X}_t dt + DdW_t, \\
det &= (A - \Xi_t H_t) c_t + DdW_t - \Xi_t H_t dV_t, \\
dZ_t^2 &= H_F X_t dt + dV_t^2, \\
X_0 = \gamma, & e_0 = \gamma - E[\gamma], \quad Z_0^2 = 0.
\end{align*}
\]
(54)
(55)
Via (51), they can be rewritten, in a compact form, as
\[
\begin{align*}
& dx_t = \left( F_t x_t + L(\bar{X}_t) \right) dt + \bar{D}dW_t + G_t^1 dV_t^2, \\
& \bar{L}(\bar{X}_t) = \text{col} \left( \left[ \left( \bar{S} + \bar{S} \right) P_t - \bar{SP}_t \right] \bar{X}_t; 0 \right), \\
& dZ_t^2 = H_F x_t dt + dV_t^2, \\
& x_0 = \text{col} \left( \gamma; \gamma - E[\gamma] \right), \quad Z_0^2 = 0,
\end{align*}
\]
where \( x_t := \text{col}(X_t; e_t). \)

It is a signal model with common noise. Based on the Kalman-Bucy filtering theory [33], its estimator equation is given by
\[
\begin{align*}
& d\bar{x}_t = \left( F_t \bar{x}_t + L(\bar{X}_t) \right) dt + \left( G_t^1 + \Xi_t H_t \right) \times (dZ_t^2 - H_F \bar{x}_t dt), \\
& \bar{x}_0 = \text{col}(E[\gamma]; 0), \quad \bar{Z}_0^2 = E[\bar{Z}^2],
\end{align*}
\]
(56)
where \( \bar{x}_t := \mathbb{E}[x_t | Z^t_t] \) and the covariance matrix \( \Xi \) is introduced in (30).

Note also that \( \bar{e}_t = \bar{X}_t - \mathbb{E}[X_t | Z^t_t] = 0, t \in [0, T] \). In other word, the second coordinate of the state estimate of (56) equals zero. To simplify (56), it yields (52). Comparing with (57), we get \( \bar{Y}_t = \Xi_t, t \in [0, T] \). Then, (53) is exactly the first equation of (50).

V. APPLICATION EXAMPLES

In this section we give some application examples of our main result.

Example 1. (Government Debt Stabilization Problem) We consider the following differential game, on government debt stabilization, with the fiscal authority acting as Stackelberg leader
\[
\begin{align*}
dd_t &= (rd_t + f_t - m_t) dt, \\
dz_1^2 &= \rho_1 d_1 dt + dV_t^1, \\
dz_2^2 &= \rho_2 d_2 dt + dV_t^2,
\end{align*}
\]
(57)
where the government debt \( d_t \) is the state variable, and the issue of base money \( m_t \) and primary fiscal deficits \( f_t \), are adjusted by the monetary authority and the fiscal authority respectively. \( \{ z_1^1 \} \) and \( \{ z_2^2 \} \) are two observation processes, with \( \{ z_1^2 \} \) being available by the monetary authority and \( \{ z_1^1, z_2^2 \} \) being available by the fiscal authority. We assume that these two authorities rely on their respective observation information to implement policies.
The objective of the fiscal authority is to minimize a sum of the primary fiscal deficit, base money growth and government debt
\[ J^L(f, m) = \mathbb{E} \int_0^T \frac{1}{2} \left( \lambda (d_l - d)^2 + (f_l - \bar{f})^2 \right. \]
\[ + \left. \eta (m_l - \bar{m})^2 \right) dt, \]
where \( \bar{f}, \bar{m} \) and \( d \) represent exogenous policy targets for base money growth, the primary fiscal deficit and public debt, respectively.

The monetary authority sets the growth of base money so as to minimize the following loss function
\[ J^F(f, m) = \mathbb{E} \int_0^T \frac{1}{2} \left( \kappa (d_l - d)^2 + (m_l - \bar{m})^2 \right) dt, \]
where \( \frac{1}{\kappa} (> 0) \) measures how conservative the central bank is with respect to the money growth.

Letting \( x_1^L := d_1 - \bar{d}, x_2^L := r d + \bar{f} - \bar{m}, X_1 := \text{col}(x_1^L; x_2^L), U^F = m_1 - \bar{m}, U^F = f_1 - \bar{f}, Z_1^F := x_1^F - (p_1 d) t, Z_2^F := x_2^F - (p_2 d) t, \) (60), (61) and (62) can be written in the form (1) and (2), with
\[ A = \text{diag}(r, 0), \quad B_F = \text{col}(-1, 0), \quad B_L = \text{col}(1, 0), \]
\[ D = 0, \quad H_1 = [p_1, 0], \quad H_2 = [p_2, 0], \]
\[ Q_F = \text{diag}(\kappa, 0), \quad R_{FL} = 0, \quad R_{FF} = 1, \quad G_F = 0, \]
\[ Q_L = \text{diag}(\lambda, 0), \quad R_{LL} = 1, \quad R_{LF} = \eta, \quad G_L = 0. \]

Clearly, they satisfy assumptions A1a, A1b. We can then calculate the optimal strategies for the monetary authority and the fiscal authority using Theorem I

Example 2. (Linear Optimal Servo Problem) Consider a signal model formed by the state and measurement equations of two plants (or agents)

Plant 1:
\[ dx_1^F = (A_1 x_1^F + B_1 U^L_1) dt + D_1 dW^1_t, \]
Plant 2:
\[ dx_2^F = (A_2 x_2^F + B_2 U^F) dt + D_2 dW^2_t, \]
\[ dZ_1^F = h_1 \begin{bmatrix} x_1^F \\ x_2^F \end{bmatrix} dt + dV_1^F, \]
\[ dZ_2^F = h_2 \begin{bmatrix} x_1^F \\ x_2^F \end{bmatrix} dt + dV_2^F. \]

We assume that Plant 1 has access to more information, and therefore acts as the Stackelberg leader. More precisely, the linear noisy measurement \( Z_1^F \) is available to Plant 2, whereas \( Z_2^F \) is available to Plant 1.

The objective of Plant 2 is to track a linear combination of the state of Plant 1 and a command input
\[ J^F(U^L, U^F) = \mathbb{E} \int_0^T \frac{1}{2} \left( x_1^2 - (\Gamma_{21} x_1^L + \Gamma_{22} s_t) \right)^2 \]
\[ + |U^F|^2 dt, \]
where the command input \( s_t \), generated by a command generator, is described by (57)
\[ ds_t = L_s dt. \]
The above setup includes as a particular case the one in which the command input is a prescribed time-function (57), which is often called a tracking problem.

Plant 2 has a similar objective function, given by
\[ J^L(U^L, U^F) = \mathbb{E} \int_0^T \frac{1}{2} \left( x_1^2 - (\Gamma_{12} x_2^2 + \Gamma_{11} s_t) \right)^2 \]
\[ + \theta |U^F|^2 + (1 - \theta) |U^L|^2 dt, \]
where \( \theta \in (0, 1). \)

Defining \( X_t := \text{col}(x_1^L; x_2^L; s_t), W_t := \text{col}(W_1^L; W_2^L), \) then (60), (61), (62), (63) can be written in the form (1) and (2), where
\[ A = \text{diag}(A, A, L), \quad B_F = \text{col}(0; B_2; 0), \]
\[ B_L = \text{col}(B_1; 0; 0), \quad H_1 = [h_1, 0], \quad H_2 = [h_2, 0], \]
\[ R_{FL} = 0, \quad R_{FF} = I, \quad G_F = 0, \]
\[ R_{LF} = \theta I, \quad R_{LL} = (1 - \theta) I, \quad G_L = 0. \]

It is easy to check that assumptions A1a, A1b are satisfied.

Theorem I then can be used to derive the optimal control strategies.

VI. CONCLUSION AND FUTURE WORK

In this work we studied the LQG differential Stackelberg game under nested observation information pattern. We provided conditions to guarantee existence of the solution and recasted the original problem as that of solving a new FB-SDE. Using this result we gave explicit forms for the leader’s and follower’s control strategies. The possible extension includes the related problem of the infinite horizon case or the follower containing multiple plants. Further research contains dealing with more general information patterns, referring to (16), (38), etc. and nonlinear dynamic system model with finite-dimensional filters, referring to (59), (61), etc.

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VIII. APPENDIX

Proof of Lemma 2

Necessity. We split the necessity proof in steps:

Step 1. (Convex perturbation) Given \( U^L \in U^L, \) for any \( \varepsilon \in [0, 1], \) we perturb \( U^F \) to \( U^F \) in the following way
\[ U^{F, \varepsilon} := U^F + \varepsilon \delta U^F, \quad \delta U^F := U^F - U^F, \]
where \( U^F \in U^F \) and \( U^F \in U^F \) are denoted to be the optimal strategy and any other one respectively.

From the convexity of the admissible set \( U^F, \) it yields that
\( U^{F, \varepsilon} \in U^F. \)
Step 2. (Variation calculation) Rewrite (5) in a compact form
\[ d\left[ X_t - Z_t^\delta \right] = \left( \begin{array}{cc} A & 0 \\ H_2 & 0 \end{array} \right) \left( \begin{array}{c} X_t \\ Z_t^\delta \end{array} \right) + \left( \begin{array}{c} B_F \\ B_t \end{array} \right) U_t^F + \left( \begin{array}{c} B_t U_t^L \\ 0 \end{array} \right) dt \\
+ \left( \begin{array}{cc} D & 0 \\ 0 & 0 \end{array} \right) \frac{dW_t^1}{\sqrt{\delta}} \right]. \]

Suppose that \( \text{col}(X^*, Z^{2*}) := (X(U^{F*}), Z^{2}(U^{F*})) \) and \( \text{col}(X^*, Z^{2*}) := (X(U^{F*}), Z^{2}(U^{F*})) \) denote the augmented state processes under the two strategies above respectively. From the above equations, we can clearly decompose \( X^* \) as \( X^* + \delta \Delta X \) and \( Z^{2*} = Z^{2*} + \delta \Delta Z^2 \), where
\[ d\left[ \delta X_t - \delta Z_t^\delta \right] = \left( \begin{array}{cc} A & 0 \\ H_2 & 0 \end{array} \right) \left( \begin{array}{c} \delta X_t \\ \delta Z_t^\delta \end{array} \right) + \left( \begin{array}{c} B_F \\ B_t \end{array} \right) \delta U_t^F dt. \quad (65) \]

In another hand, calculate the Gâteaux differential of \( J^F \) at \( U_t^F \) in the direction \( \delta U_t^F \)
\[ dJ^F(\cdot, U_t^F; \delta U_t^F) = \lim_{\varepsilon \to 0} \frac{J^F(\cdot, U_t^F + \varepsilon \delta U_t^F) - J^F(\cdot, U_t^F)}{\varepsilon} \]
\[ = \mathbb{E} \int_0^T \left( \langle Q_F X_t^*, \delta X_t \rangle + \langle R_F U_t^F, \delta U_t^F \rangle \right) dt + \mathbb{E}(G_F X_T^*, \delta X_T) \quad (66) \]

The main idea is to substitute the terminal term of (65) with the right hand of (66). Thus, we pin down the terminal value \( p^F_T = G_F X_T^* \) and \( p^F_T = 0 \). Applying (65) to the left hand side of (67), then inserting (67) into (66), it can be verified that
\[ dJ^F(\cdot, U_t^F; \delta U_t^F) = \mathbb{E} \int_0^T \left( \langle dp^F_t + A p^F_t dt + Q_F X_t^* dt, \delta X_t \rangle \right. \]
\[ + \left. \langle dp^F_t + H^2 p^F_t dt, \delta Z_t^\delta \rangle \right) + \langle B_F p^F_t + R_F U_t^F, \delta U_t^F \rangle dt. \quad (68) \]

From (68), but hint us, by the arbitrariness of \( \delta X \) and \( \delta Z^2 \), that
\[ dp^F_t + A p^F_t dt + Q_F X_t^* dt = 0, \quad p^F_T = G_F X_T^*, \]
\[ dp^F_t + H^2 p^F_t dt = 0, \quad p^F_T = 0. \quad (69) \]

But the state processes of the above two equations does not satisfy the adaptiveness. The BSDEs theory provided a valid method to deal with it [26]. For detail, modify them to the following BSDEs
\[ dp^F_t = -[A p^F_t + Q_F X_t^*] dt + q^F_t dW_t + r^F_t dV^2_t, \]
\[ dp^F_t = -H^2 p^F_t dt + L^F dW_t + L^F dV^2_t, \]
\[ p^F_t = G_F X_T^*, \quad p^F_T = 0. \quad (70) \]

In the above, the martingale representation theorem is used to correct the adaptiveness, Chapter 7 [26]. This is why four stochastic integrals appear.

Notice that \( p^L = 0, q^F = 0, p^L = 0 \) satisfy the second equation. From the uniqueness of the solution of the linear BSDE, referring to the same chapter as above, it is exactly the unique solution. Thus, (9) is obtained.

Step 4. (\( \sigma \)-sub-algebra projection) Now, (68) has changed to
\[ dJ^F(\cdot, U_t^F; \delta U_t^F) = \mathbb{E} \int_0^T \left( \langle B_F p^F_t + R_F U_t^F, \delta U_t^F \rangle \right) dt \]
\[ = \mathbb{E} \int_0^T \langle B_F p^F_t + R_F U_t^F, \delta U_t^F \rangle dt, \quad (71) \]

where the tower property of conditional expectation is used in the second equality because the follower implements strategy relying on the \( \sigma \)-sub-algebra \( Z_t^F \) at time \( t \). Finally, the arbitrariness of \( \delta U_t^F \) leads to (8).

Sufficiency Given \( U_t^L \in U^* \), letting \( U_t^{F*} \) and \( U_t^F \) be the same as before, \( X^* \) and \( X \) denote the corresponding state respectively. We need calculate the difference \( J^F(\cdot, U_t^F) - J^F(\cdot, U_t^{F*}) \). The convexity of \( J^F \) with respect to the state and strategy variables lead to
\[ J^F(\cdot, U_t^F) - J^F(\cdot, U_t^{F*}) \geq \mathbb{E} \int_0^T \left( \langle Q_F X_t^*, \psi_t(X) \rangle + \langle R_F U_t^{F*}, \psi_t(U_t^F) \rangle \right) dt + \mathbb{E}(G_F X_T^*, \psi_t(T)), \quad (72) \]

where \( \psi_t(X) := X_t - X_t^* \), \( \psi_t(U_t^F) := U_t^F - U_t^{F*}. \)
Next, applying Itô’s formula to \( t \mapsto \langle p^F_t, \psi_t(X) \rangle \), it gives
\[ \langle p^F_t, \psi_t(T) \rangle = \langle p^F_0, \psi_0(X) \rangle \]
\[ + \mathbb{E} \int_0^T \left( -Q_F X_t^*, \psi_t(X) \right) + \langle p^F_t, B_F \psi_t(U_t^F) \rangle dt. \quad (73) \]

Inserting (73) into (72) and noticing \( \psi_0(X) = 0 \), it yields \( J^F(\cdot, U_t^F) \geq J^F(\cdot, U_t^{F*}) \). Sufficiency of (8) is verified.

Proof of Lemma 2

Necessity. Its proof is similar to the part of necessity proof of Lemma 1. We only give a framework. Firstly, a convex perturbation, similar with (64), is taken for \( U_t^L \). The Gâteaux differential of \( J^L \) at \( U_t^{L*} \) in the direction \( \delta U_t^L \) is
\[ dJ^L(U_t^{L*} + \varepsilon \delta U_t^L, U_t^{F*}) - J^L(U_t^{L*}, U_t^{F*}) \]
\[ = \mathbb{E} \int_0^T \left( \langle Q_L X_t^*, \delta X_t \rangle + \langle R_{L*} U_t^{L*}, \delta U_t^L \rangle \right. \]
\[ + \left. \langle R_{L*} R_{F*}^{-1} R_{F*}^t B_F p^F_t, \left( R_{F*}^{-1} B_F p^F_t \right) dt \right) + \mathbb{E}(G_L X_T^*, \delta X_T), \quad (74) \]
where \( X_t := X(U_t^{L*}) \) and \( p^F_t := p^F(U_t^{L*}) \) are the state processes corresponding with \( U_t^{L*} \).

Since there are two equations of \( X_t \)’s and \( p^F_t \’s \) in (11), we need to apply Itô’s formula to \( t \mapsto \langle \delta X_t, p_t^F \rangle - \langle \delta p^F_t, Y_t \rangle \) to substitute the terminal term of (74), where \( p^L \) and \( Y \) are two
processes to be determined. After doing it, may refer to (23), it can be checked that
\[
    d\mathcal{J}^L(U^L, U^F; \delta U^L) = \mathbb{E} \int_0^T \langle B_{LL}^L + R_{LL} U_{LL}^L, \delta U^L \rangle dt. \tag{75}
\]

Finally, a similar reason with (71) leads to the desired result.

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