Conservation Laws and 2D Black Holes in Dilaton Gravity

R.B. Mann
Department of Physics
University of Waterloo
Waterloo, Ontario
N2L 3G1

June 9, 1992
WATPHYS-TH92/03
Abstract

A very general class of Lagrangians which couple scalar fields to gravitation and matter in two spacetime dimensions is investigated. It is shown that a vector field exists along whose flow lines the stress-energy tensor is conserved, regardless of whether or not the equations of motion are satisfied or if any Killing vectors exist. Conditions necessary for the existence of Killing vectors are derived. A new set of 2D black hole solutions is obtained for one particular member within this class of Lagrangians. One such solution bears an interesting resemblance to the 2D string-theoretic black hole, yet contains markedly different thermodynamic properties.
1 Introduction

The study of gravitation in two spacetime dimensions has been a subject of much study in recent years, motivated both by string theory and by a desire to study quantum gravitational effects in a mathematically tractable setting. Significant progress has been made in recent years as a consequence of the realization that two-dimensional spacetimes with non-trivial event horizons (i.e. black holes) exist [1, 2, 3, 4].

The fundamental problem in constructing a relativistic theory of gravitation in (1 + 1) dimensions (2D) is the triviality of the Einstein tensor: $G_{\mu \nu}[g] \equiv 0$ for all metrics $g_{\mu \nu}$. The most popular method of addressing this difficulty has been to use the Polyakov action [5], a non-local action which becomes the local Liouville action for a certain choice of coordinates.

However in the last few years it has been shown that other approaches exist which are of significant interest in their own right. These approaches typically couple a scalar field (the dilaton) to gravity in such a way that the Ricci scalar is a non-vanishing function of the co-ordinates over some region of spacetime, and so the spacetime can develop interesting features, i.e. black-hole horizons and singularities. These approaches include setting the Ricci scalar equal to a constant [6], setting the Ricci scalar equal to the trace of the 2D stress-energy tensor [2, 3], or setting the one-loop beta functions of the bosonic non-linear sigma model to zero in a two-dimensional target space [7]. This latter approach yields a set of equations which are equivalent to those derived from an effective action coupling the target space metric and dilaton fields and is motivated from string theory [4]. This action has recently been extended to include more general dilaton couplings to gravity and matter [8, 9].

In the present paper a quite general class of actions which couple a dilaton field to gravity and matter in two-dimensional spacetime is analyzed. It will be shown that there always exists a flow line along which the stress-energy tensor is conserved, regardless of the other equations of motion in the theory or of the existence of a Killing vector. Conditions necessary for the tangent vector to this flow line to be a Killing vector are constructed. These results are then applied to the theories discussed in refs. [2, 3, 8, 9]. Finally, for one particular member in the class of Lagrangians considered here, a new set of black hole solutions is derived and their thermodynamic properties explored.
2 Two-Dimensional Gravity and Dilaton Fields

The action considered here is taken to be

\[ S[g, \Psi] = \int d^2 x \sqrt{-g} \left( H(\Psi) g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi + D(\Psi) R + V(\Psi; \Phi^M) \right) \]  

where \( R \) is the Ricci scalar and \( D \) and \( H \) are arbitrary functions of the scalar field \( \Psi \) (referred to as the dilaton in string theory). The potential \( V \) is the matter Lagrangian, and depends on both \( \Psi \) and the matter fields \( \Phi^M \). This action is a generalization of one considered recently by Banks and O'Loughlin \[10\], in which no matter fields \( \Phi^M \) were present and \( H = 1/2 \).

This model actually only depends upon the one function \( V(\Psi; \Phi) \), since reparametrizations of \( \Psi \) accompanied by \( \Psi \)-dependent Weyl rescalings of the metric allow one to relate models with different \( H \)'s and \( D \)'s. Explicitly, under the transformation

\[ g_{\mu\nu} = e^{\sigma(\Psi)} \tilde{g}_{\mu\nu} \quad \Psi = J(\Psi) \]  

the action (1) with \( V = 0 \) is form invariant provided \( \tilde{D}(\Psi) = D(J(\Psi)) \) and

\[ (J')^2 (H(J) + D'(J) \sigma'(J)) = H(\Psi) \]  

where \( \prime \) denotes the derivative with respect to the functional argument. Only the critical points and overall sign of \( H \) and \( D \) contain reparametrization invariant information \[10\]. In order for there to be non-trivial evolution for the spacetime metric, it is necessary that \( D' \neq 0 \).

Of course a Weyl rescaling is information external to that derived from the action \[1\] via a variational principle. The matter potential \( V \) breaks Weyl invariance, and so the field equations which follow from \[1\] determine the evolution of the spacetime metric and matter fields. General coordinate invariance implies that locally the evolution of the metric is determined by the evolution of its conformal factor.

The action \[1\] is the most general action linear in the curvature and quadratic in the derivatives of \( \Psi \) and the matter fields. It has been employed a number of times in constructing two-dimensional theories of gravity. For \( H = 0 \) and \( V = -D \Lambda \) this action reduces to that considered in ref.\[3\]. For \( H = 1/2 \), \( D = \psi \) and \( V = -8\pi G L_M(\Phi) \), the action is that considered in \[4, 5, 11\], in which the field equations set the Ricci scalar equal to the trace
of the stress energy. For $H = \gamma e^{-2\phi}$, $D = e^{-2\phi}$ and $V = -\frac{1}{4}e^{(e-2)\phi}L_M(\Phi)$, this yields the class of theories considered in [8, 9]. For $\gamma = 4$ and $\epsilon = 0$, the action reduces to that of the effective target space action for non-critical string theory [7, 8] the matter action being that of the tachyon field.

3 Conservation Laws and Killing Vectors

Consider a potential $V$ such that the stress-energy tensor associated with the action (1) is

$$T_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = H \left( H^{-1} \nabla_\mu \nabla_\nu \Psi - \frac{1}{2}(\nabla \Psi)^2 \right) - \nabla_\mu \nabla_\nu D + g_{\mu\nu} \nabla^2 D - \frac{1}{2} g_{\mu\nu} V ,$$

(4)

essentially restricting the potential to have no metric dependence. While this does omit kinetic energy terms for scalars and spinors, it does permit couplings to gauge field strengths since these are always dual to scalars. The divergence of the stress energy is

$$\nabla^\nu T_{\mu\nu} = \frac{1}{2} \nabla_\nu \Psi \left[ H' (\nabla \Psi)^2 - D'R - V' + 2H \nabla^2 \Psi \right] - \frac{1}{2} V' \nabla_\nu S_I ,$$

(5)

where $I$ denotes the derivative with respect to $\Psi$ and $S_I$ are scalar quantities formed from the matter fields $\Phi_M$ and their derivatives, with $V' \equiv \frac{\delta V}{\delta S_I}$. The right hand side of this equation will vanish when the equations of motion of $\Psi$ and the matter fields $\Phi_M$ are satisfied.

However even if these equations are not satisfied, the stress-energy tensor obeys a conservation law. Consider the divergence of the quantity $J_\mu \equiv T_{\mu\nu} \epsilon^{\nu\alpha} \nabla_\alpha F(\Psi)$:

$$\nabla^\mu J_\mu = \frac{1}{2} \epsilon^{\nu\alpha} \left[ \nabla_\nu \Psi \nabla_\alpha ((\nabla \Psi)^2) \right] (D'F'' - F'D'' + HF') - V' \nabla_\nu S_I \epsilon^{\nu\alpha} \nabla_\alpha F .$$

(6)

The result (6) holds regardless of whether $T_{\mu\nu}$ is conserved or whether $\xi'' \equiv \epsilon^{\nu\alpha} \nabla_\alpha F(\Psi)$ is a Killing vector. It is clear that $J_\mu$ is conserved provided

$$\nabla_\nu S_I \epsilon^{\nu\alpha} \nabla_\alpha F = 0 \quad (7)$$

and

$$F = F_0 \int^\Psi d\Sigma D' e^{-\int^\Psi d\Sigma \frac{H(\Sigma)}{D'(\Sigma)} } ,$$

(8)
where $F_0$ is a constant. This constant may be chosen so that $\frac{dF}{dx} \to 1$ for large $|x|$.

The first of these conditions states that the gradients of the scalars formed from matter fields and their derivatives (e.g. for gauge fields the gradient of the dual of the gauge field strength) must be orthogonal to surfaces of constant $F$. The second condition guarantees that $T_{\mu\nu}$ is always conserved along the flow lines of $\xi^\nu$, even if $\nabla^\mu T_{\mu\nu} \neq 0$. Since in two dimensions a divergenceless current is always dual to the gradient of a scalar, $J_\mu = \epsilon_{\mu}^\alpha \nabla_\alpha M$; from (4,8)

$$M = \frac{1}{2} \left[ (\nabla D)^2 e^{-\int^t dt' \frac{H(t')}{D(t')}} - F_0 \int dDV \int^t dt' \frac{H(t')}{D(t')} \right].$$

The quantity $M$ is a generalization of the mass-function $m(x)$ considered in ref. [8]. Note that $M$ is constant whenever the equation of motion for the metric (i.e. $T_{\mu\nu} = 0$ in (4)) is satisfied.

Consider next the conditions under which $\xi^\mu$ is a Killing vector. For $H \neq D''$, (5) and (8) give

$$\nabla_\mu (\mu \xi^\nu) = \frac{D''}{H} \nabla_\mu (\mu \xi^\nu) + \frac{F''}{H} T_{\alpha \mu} \epsilon_{\nu}^\alpha + F_0 \int dDV \int^t dt' \frac{H(t')}{D(t')} \left[ \left( \nabla_{\alpha} F \right) \epsilon_{\mu}^\alpha \right].$$

and so,

$$\nabla_\mu (\mu \xi^\nu) = -e^{-\int^t dt' \frac{H(t')}{D(t')}} T_{\alpha \mu} \epsilon_{\nu}^\alpha.$$ 

A similar computation when $H = D''$ (which is the case for string theory [7, 9]) yields

$$\nabla_\mu (\mu \xi^\nu) = -\frac{F_0}{D'} T_{\alpha \mu} \epsilon_{\nu}^\alpha.$$ 

In either case it is clear that $\xi$ is a Killing vector only if $T_{\mu\nu} = K(\Psi, \Phi_M)g_{\mu\nu}$. This is the condition used in ref. [10] in deriving the general form $\xi^\nu = \epsilon^\nu_\alpha \nabla_\alpha F$ (with $F$ given by (8)) of the Killing vector associated with the action (1) with $\Phi_M = 0$.

To close this section, the above results will be applied to a few simple examples. For string theory $H = 4e^{-2\phi} = 4D$ and $V = \lambda e^{-2\phi}$, and so (8) becomes

$$M = -2F_0e^{-2\phi} \left( (\nabla \phi)^2 - \frac{\lambda}{4} \right).$$
When the field equations are satisfied

\[ ds^2 = -(1 - ae^{-Qx})dt^2 + \frac{dx^2}{1 - ae^{-Qx}} \] (14)
\[ \phi = -\frac{Q}{2}x \] (15)

where \( a \) is a constant of integration and \( \lambda = Q^2 \). Normalizing \( F_0 \) as stated above yields

\[ M = \frac{1}{2}aQ \] (16)

which is the expression for the ADM mass for the black hole obtained in refs. [4, 8, 12]. For the \( R = T \) theory of ref.[2, 3], \( H = \frac{1}{2}, D = \psi \) and \( V = 0 \) in the absence of matter, and (9) becomes

\[ M = \frac{F_0}{2} (\nabla \psi)^2 e^{-\psi/2} \] (17)

Outside a symmetrically placed distribution of matter the exact solution to the field equations is [13]

\[ ds^2 = -(2M|x| - 1)dt^2 - \frac{dx^2}{(2M|x| - 1)} \] (18)
\[ \psi = -2 \ln((2M|x| - 1)) \], and so \( M \) becomes

\[ M = M \] (19)

where \( F_0 \) has been normalized as above.

4 New Black Hole Solutions

In this section the action

\[ S = \int d^2x \sqrt{-g} \left( \frac{1}{2} (\nabla \psi)^2 + \psi R + 2b(\nabla \phi)^2 - 8\pi G \left[ -f(\phi)\Lambda + \frac{1}{4} h(\phi) F_{\mu\nu} F^{\mu\nu} \right] \right) \] (20)

will be investigated, where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( f \) and \( h \) are (at this point) unspecified functions of the scalar field \( \phi \).
This action is clearly a special case of (1), and corresponds to the R=T theory mentioned earlier [2, 3]. Such a theory is of interest because it yields a two dimensional theory which closely resembles (3 + 1) dimensional general relativity in that the evolution of the gravitational field is driven by the stress-energy and no other Brans-Dicke type fields [13]. Its classical and semi-classical properties and solutions are also markedly similar [2, 11, 15, 16, 17]. Indeed, the field equations which follow from (20) may be obtained from a reduction of the Einstein equations from D to 2 spacetime dimensions [13].

These field equations are

\[ \nabla^2 \psi - R = 0 \] (21)

\[ \frac{1}{2} \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} (\nabla \psi)^2 \right) + g_{\mu\nu} \nabla^2 \psi - \nabla_\mu \nabla_\nu \psi = 8\pi GT_{\mu\nu} \] (22)

\[ -4b \nabla^2 \phi + 8\pi G \left( \Lambda \frac{df}{d\phi} - \frac{1}{4} \frac{dh}{d\phi} F_{\mu\nu} F^{\mu\nu} \right) = 0 \] (23)

\[ \nabla_\mu \left( h(\phi) F^{\mu\nu} \right) = 0 \] (24)

where

\[ T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \Lambda f(\phi) + \frac{1}{2} h(\phi) \left( F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\rho\tau} F^{\rho\tau} \right) \] (25)

is the stress-energy tensor due to the matter fields \( \phi \) and \( A_\mu \).

Taking the metric in the static case to be of the form

\[ ds^2 = -\alpha(x) dt^2 + \alpha^{-1}(x) dx^2 \] (26)

the gravity/matter system reduces to

\[ \alpha'' = -8\pi G \Lambda f(\phi) + 4\pi G \frac{Q^2}{h(\phi)} \] (27)

and

\[ (\alpha' \phi')' = \frac{2\pi G}{b} \Lambda f'(\phi) + \frac{\pi G}{b} \left( \frac{Q^2}{h(\phi)} \right)' \] (28)

where

\[ F_{\mu\nu} = \epsilon_{\mu\nu} \frac{Q}{h(\phi)} \] (29)
is the solution to (24). It is possible to use (27) to integrate (28) when one of $Q$ or $\Lambda$ vanishes; the result is

$$\left(\phi'\right)^2 = \pm \frac{1}{4b} \frac{d}{d\alpha} \left[ \frac{\left(\alpha'\right)^2 - X_0}{\alpha} \right]$$

(30)

where the plus (minus) sign corresponds to $\Lambda = 0$ ($Q = 0$). Equation (24) always has the solution

$$\psi = -\frac{\alpha' + K}{\alpha}$$

(31)

for the auxiliary field $\psi$. Here $K$ and $X_0$ are constants of integration. Comparison with the 00 component of (22) indicates $2X_0 = K^2$.

Further progress requires specifying the functions $f$ and $h$. For ‘clumped’ matter, coordinates may be chosen so that the edges of the matter are symmetric about the origin and located a finite proper distance away. In this case one expects the metric to have a spatial dependence such that $\alpha(x)$ is positive for large $|x|$. A black hole is a region of spacetime for which $\alpha$ is negative; in the static case this will occur for some range of $x$ [14]. This range may either be finite, as in the case of the collapse of localized ‘clumped’ matter [13, 18], or infinite, as in the case of dilatonic black holes [7, 4, 8]. It is also worthwhile to note that the criterion for asymptotic flatness is slightly more general than in higher dimensions [13]. One need only require that $\alpha(x) \to K|x| + C$ for large $|x|$, (or perhaps just for large $x$) since in this case the metric (24) will become asymptotically like a Rindler spacetime; a Rindler transformation may then be applied locally to rewrite the metric in Minkowskian form.

A particularly interesting class of static solutions to the system (21)–(25) results when $\phi$ is chosen to be a Liouville field i.e. $f = e^{-2a\phi}$ and $h = f$. For both $Q$ and $\Lambda$ non-vanishing there exists a solution with $\beta = 1$:

$$\alpha(x) = A - \frac{8\pi G\Lambda}{C^2} e^{-2aE} e^{-Cx} + \frac{4\pi GQ^2}{C^2} e^{2aE} e^{Cx}$$

(32)

$$\phi = \frac{C}{2a} x + E$$

(33)

where $A$, $C$ and $E$ are constants of integration. This solution is not asymptotically flat. However if either of $\Lambda$ or $Q$ is non-vanishing, a more physically interesting set of solutions results.
Consider first the case $Q = 0$. Solving (27) for $\phi$ and substituting this into (30) yields

$$\beta \left( \frac{\alpha'''}{\alpha''} \right)' - \alpha'' = 0$$

(34)

where $\beta \equiv -b/a^2$. There is a discretely infinite set of solutions which are asymptotically Rindlerian at large $x$ whenever $\beta = \frac{a}{p+2}$

$$\alpha(x) = C(x - x_0) - \frac{A_p}{(x - x_0)^p}$$

(35)

where $A_p$ and $C$ are constants of integration, and

$$\phi = -\frac{1}{2a} \ln \left( \frac{p(p + 1)A_p}{8\pi GA(x - x_0)^{p+2}} \right) + \phi_0$$

(36)

For large $p$ (i.e. $\beta = 1$) the solution is

$$\alpha(x) = A - \frac{8\pi GA}{C^2} e^{-2aE} e^{-Cx}$$

(37)

with $\phi$ the same as in (33), and $A$, $C$, $E$ are constants of integration. When $\Lambda = 0$ and $Q \neq 0$, the solutions to (21)–(25) are given by (35)–(33) with $a \to -a$ and $\Lambda$ replaced by $-Q^2/2$. For each of the solutions in (32,37), the constant $K^2 = A^2C^2$ in (31).

The temperature of the black hole solutions is straightforwardly obtained by naive Wick-rotation arguments and is easily seen to be $T = \left| \frac{d\alpha}{dx} / 4\pi \right|_{x_H}$ where $x_H$ is the location of the horizon ($\alpha(x_H) = 0$). For (35) this is

$$T = \frac{C(p + 1)}{4\pi}$$

(38)

whereas for (37) this is

$$T = \frac{C}{4\pi}$$

(39)

whenever $C$ and $A_p$ are of appropriate relative sign so that an event horizon exists.
5 Discussion

The solutions (35,37) represent an interesting new class of 2D metrics. For large $x$ they are asymptotic to either Rindler space (as in (35)) or flat space (as in (37)) and for appropriate choices of the signs of $C, A_p$, they have black hole event horizons. If $x$ is replaced by $|x|$ in (35,37) then they remain solutions of the system (21)-(25) provided an appropriate point-source stress-energy [2] is inserted at the origin. Such metrics would represent the endpoint of gravitational collapse of localized matter. Due to the presence of the $\phi$ field, a full treatment of such a problem would involve consideration of matching conditions at the boundary [18, 19]. For (37) the curvature tensor would have a delta-function singularity at the origin, and a power-law singularity at $x = x_0$.

The solution (37) is of particular interest. Formally, it is identical to the string-theoretic solution (14). However its interpretation is quite different. The coefficient $C$ inside the exponential in (37) is a constant of integration, in contrast to the $Q$ parameter in (14) which is a coupling parameter of the theory. The ADM mass computed from (14) is proportional to $aQ$, leading one to interpret $a$ as a mass parameter (although an alternate interpretation has been proposed [18]), whose dimensionality is given by the fundamental coupling $Q$.

However for the theory described by (20), the constant of integration in (37) is $C$. This suggests that it ought to be interpreted as a mass. In computing the ADM mass associated with (37), the formula (3) cannot be used, since the $V$ term in (20) explicitly depends on the metric. However one can proceed directly, using the fact that $\xi^\mu = (1,0)$ is a Killing vector for the spacetime (37); defining the mass-function $M$ analogously to (3) so that $\nabla_\mu M = -\epsilon_{\mu\nu}T^\nu\xi^\rho$ one obtains from (31,33)

$$M = \alpha' - \frac{b}{a}\alpha\dot{\phi}' - \int \left[ \frac{1}{4} \left( \frac{(a')^2 - K^2}{\alpha} \right) - b\alpha(\phi')^2 \right]$$

and using (34) this becomes

$$M = \frac{AC}{2}$$

which upon normalizing (37) so that $A = 1$, yields $C = 2M$.

This interpretation of the mass parameter is in striking contrast to that considered in string theoretic contexts [4, 7, 8] and leads to significantly dif-
ferent thermodynamic properties for the black hole described by (41). In contrast to the metric (14) in which the temperature is independent of the black hole mass, and the entropy is proportional to it, in this case the temperature (39) varies linearly with the mass and the entropy logarithmically. Such an interpretation is in keeping with previous results in the R=T theory [2, 17]. Consequences of a black hole entropy $S$ for which $S \sim \ln(M)$ have been investigated previously [17]. The implications of this for black hole evaporation remain to be explored.

Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] J.D. Brown, M. Henneaux and C. Teitelboim, Phys. Rev. D33 (1986) 319; J.D. Brown, Lower Dimensional Gravity, (World Scientific, 1988).

[2] R.B. Mann, A. Shiekh, and L. Tarasov, Nucl. Phys. B341 (1990) 134.

[3] R.B. Mann, Found. Phys. Lett. 4 (1991) 425.

[4] E. Witten, Phys. Rev. D44 (1991) 314.

[5] A.M. Polyakov, Phys. Lett B103 (1981) 207.

[6] R. Jackiw, Nucl. Phys. B252 (1985) 343; C. Teitelboim, Phys. Lett. B126 (1983) 41, 46.

[7] G. Mandal, A.M. Sengupta, and S.R. Wadia, Mod. Phys. Lett. 6 (1991) 1685.

[8] V. Frolov, ‘2D Black Hole Physics’ Copenhagen Preprint (1992).

[9] O. Lechtenfeld and C. Nappi, Princeton preprint IASSNS-HEP-92/22.
[10] T. Banks and M. O’Loughlin, Nucl. Phys. B362 (1991) 649.

[11] R.B. Mann, S.M. Morsink, A.E. Sikkema and T.G. Steele, Phys. Rev. D43 (1991) 3948.

[12] M. MacGuigan, C. Nappi and S. Yost, Princeton preprint IASSNS-HEP-91/57.

[13] R.B. Mann, Gen. Rel. Grav. 24 (1992) 433.

[14] J.D. Christensen and R.B. Mann, Class. Quantum Grav. (to be published).

[15] A.E. Sikkema and R.B. Mann, Class. Quantum Grav. 8 (1991) 219.

[16] S.M. Morsink and R.B. Mann, Class. Quant. Grav. 8 (1991) 2257.

[17] R.B. Mann and T.G. Steele, Class. Quant. Grav. 9 (1992) 475.

[18] R.B. Mann, M.S. Morris and S.F. Ross, University of Waterloo preprint, WATPHYS-TH91/04.

[19] R.B. Mann and S.F. Ross, University of Waterloo preprint, WATPHYS-TH92/02.