On the relation between $U_q(\widehat{sl}(2))$ vertex operators and $q$-zonal functions

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Abstract

We show how the states constructed from the action of the modes of bosonized vertex operators, that intertwine $U_q(\widehat{sl}(2))$ modules, are related to $q$-zonal functions.

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I Introduction

The space of states of an exactly-solvable two-dimensional model typically consists of a finite number of infinite-dimensional highest weight modules of an affine algebra or of its $q$-deformation. These modules can be constructed using either the currents, or the intertwining vertex operators. They can also be embedded in the Fock spaces constructed from the action of the Heisenberg subalgebras and certain group algebras on the weight lattice of the Lie subalgebra.

It has been shown \cite{1} that the currents generating $U_q(\widehat{sl}(2))$ are related to certain symmetric functions introduced by Kerov in \cite{2}. Since the correlation functions of exactly-solvable models are expectation values of the intertwiners, it is natural to relate the latter to symmetric functions also.

Let $\Lambda_i$, $i = \{0, 1\}$ be the $sl(2)$ fundamental weights, $V(\Lambda_i)$ the level-one $U_q(\widehat{sl}(2))$ highest weight modules, $V$ the $U_q(sl(2))$ two-dimensional module with basis vectors $v_\pm$, $z$ a complex variable, and $V(z) = V \otimes \mathbb{C}[z, z^{-1}]$ the level-zero $U_q(\widehat{sl}(2))$-evaluation module. In §2, we show that the states constructed from the modes of the vertex operators $\phi^{i,\pm}(z)$, referred to as type-I in \cite{3} and which intertwine the $U_q(\widehat{sl}(2))$-modules as

$$\phi^{i,+}(z) \otimes v_+ + \phi^{i,-}(z) \otimes v_- : V(\Lambda_i) \to V(\Lambda_{i-1}) \otimes V(z),$$

(1)

are related to the Macdonald functions $P_\lambda(x; q^4, q^2)$ \cite{3}. In the limit $q \to 1$, these functions reduce to the zonal functions $Z_\lambda(x)$ and we shall hence refer to them as the $q$-zonal functions. (The functions $Z_\lambda(x)$ are related to the Jack polynomials \cite{5} in the following way $Z_\lambda(x) = J_\lambda(x; 2)$.) In §3, we compare our results with those of Jing \cite{3} who considered some general vertex operators, related to Macdonald functions, but not to $U_q(\widehat{sl}(2))$. In particular, Jing’s operators are not related to intertwiners of $U_q(\widehat{sl}(2))$ modules. We conclude by discussing a few open problems.
II Relation between $q$-zonal functions and $U_q(sl(2))$ intertwining vertex operators

In this section, we review the bosonization of the vertex operators (VO) (1.1), following [6], and consider the states of the Fock space constructed by acting with the modes of these VO on the bosonic vacuum state. We refer to these states as the vertex operator states (VOS). The VO $\phi_{i,\pm}(z)$ are bosonized as follows [6]:

$$\phi_{i,-}(z) = e^{\sum_{n=1}^{\infty} \frac{zn/2}{n!} a_n z^n} e^{-\sum_{n=1}^{\infty} \frac{zn/2}{n!} a_n z^{-n}} e^{\alpha/2 (-q^3 z) (\partial + i)/2} = \sum_{n \in \mathbb{Z}} \phi_{i,-n} z^{-n},$$

$$\phi_{i,+}(z) = -\frac{q-q^{-1}}{2\pi i} \oint_{|z|<|w|<|zq^2|} dw \frac{E^{-}(w)}{(w-zq^{-2})(w-zq^2)} = \sum_{n \in \mathbb{Z}} \phi_{i,n} z^n,$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$. The normal ordering symbol $\text{\text{::}}$ means that the creation modes $a_n, e^{\alpha}; n < 0$ are placed to the left of the annihilation modes $a_n, \partial; n > 0$, and $E^{-}(z)$ is one of the quantum currents that generate $U_q(sl(2))$. The explicit realization of this current is given by

$$E^{-}(z) = e^{-\sum_{n=1}^{\infty} \frac{zn/2}{n!} a_n z^n} e^{\sum_{n=1}^{\infty} \frac{zn/2}{n!} a_n z^{-n}} e^{-\alpha z^{-\partial}},$$

where $\alpha$ is the simple positive root of $sl(2)$. Moreover, $e^\alpha$ is a translation operator acting on the $sl(2)$ weight lattice, while $\partial$ is the zero mode (momentum) operator which is conjugate to $\alpha$ in the following sense:

$$[\partial, \alpha] = 2,$$

$$z^\partial e^\alpha = z^2 : e^\alpha z^\partial : .$$

In actual computations invoking to the zero mode $\partial$ and $\alpha$, only the last relation is used. The non-zero modes satisfy the following commutation relations:

$$[a_n, a_m] = \frac{[n]}{n} \delta_{n+m,0}. $$

For the purposes of §3, where we compare our results with Jing’s, it is useful to redefine the non-zero modes as follows:

$$a_n \equiv \frac{[n]}{n} q^{-|n|} b_n, \quad \forall n \in \mathbb{Z}\setminus\{0\},$$

$$[b_n, b_m] = n(1 + q^{2|n|}) \delta_{n+m,0}.$$
In terms of the modes $b_n$, the vertex operators are then expressed as follows
\[
\phi^i_{-}(z) = e^\sum_{n=1}^{\infty} \frac{q^n}{n(1+q^2n)} b_{-n} z^n e^{-\sum_{n=1}^{\infty} \frac{a^{-2n}}{n(1+q^2n)} b_{-n} z^{-n}} e^{\alpha/2} (-q^3 z)^{(\partial+i)/2} \\
= \sum_{n\in\mathbb{Z}} \phi^i_{-} n \cdot z^{-n},
\]
with
\[
\phi^{i+}(z) = -\frac{q^{-q^{-1}}}{2\pi i} \int_{|zq^{|} < |w| < |zq^2|} dw \phi^{i-}(z) E^{-}(w);
\]
\[
\phi^{i-}(z) E^{-}(w) := e^\sum_{n=1}^{\infty} \frac{q^n}{n(1+q^2n)} b_{-n} z^n e^{-\sum_{n=1}^{\infty} \frac{b_{-n} w^n}{n}} e^{\alpha/2} w^{-\partial} (-q^3 z)^{(\partial+i)/2}. \tag{7}
\]

Let us now construct the VOS from the above VO. It is possible to consider the most general VOS, which is obtained from an arbitrary number of successive applications of the modes of $\phi^{i-}(z)$ on the bosonic vacuum state. This, however, is very complicated. We will instead focus on the case where the VOS are obtained from the action of at most two $\phi^{i-}(z)$ on the bosonic vacuum state, and where the dual VOS are obtained from the action of at most two $\phi^{i+}(z)$ on the dual bosonic vacuum state. We believe that these cases embody many of the qualitative features of the general case. Before constructing the VOS, here are some operator product expansions that we shall be using:
\[
\phi^{1-i-}(z) \phi^{i-}(w) = (-q^3 z)^{1/2} \frac{q^2 w z^{-1} q^4}{(q^4 w z^{-1} q^4)_{\infty}} : \phi^{1-i-}(z) \phi^{i-}(w) :,
\]
\[
E^{-}(z) \phi^{i-}(w) = \frac{1}{z(1-q^w z^{-1})} : E^{-}(z) \phi^{i-}(w) :,
\]
\[
\phi^{i-}(z) E^{-}(w) = -q^{-1} \frac{1}{1-q^{-2w z^{-1}}} : \phi^{i-}(z) E^{-}(w) :,
\]
\[
E^{-}(z) E^{-}(w) = z^2 (1-w z^{-1}) (1-q^2 w z^{-1}) : E^{-}(z) E^{-}(w) :, \tag{9}
\]
where in $(1-z)^{-1}, |z| < 1$ is meant, and where
\[
(x,q)_\infty = \prod_{n=0}^{\infty} (1-xq^n). \tag{10}
\]

Let us recall that the vacuum states 1 and $e^{\alpha/2}$ are identified with the highest weight vectors of the modules $V(\Lambda_0)$ and $V(\Lambda_1)$ respectively. Consequently, only $\phi^{0,\pm}(z)$ act on 1, and only $\phi^{1,\pm}(z)$ act on $e^{\alpha/2}$. Since 1 is annihilated by $\partial$ and $b_n$, $n > 0$, and since $e^{\alpha/2} \cdot 1 = e^{\alpha/2}$, we find that $\phi^{0-}(z)$ acts on 1 according to:
\[
\phi^{0-}(z) \cdot 1 = e^{\sum_{n=1}^{\infty} \frac{q^n}{n(1+q^2n)} b_{-n} z^n} e^{\alpha/2} = \prod_{n=1}^{\infty} \sum_{m_n=0}^{\infty} \frac{b_{-n} q^{4mn} z^{m_n} q^{mn}}{m_n (1+q^{2m_n}) m_n} e^{\alpha/2} \tag{11}
\]
\[
= \sum_{\lambda \in \mathcal{P}} \frac{b_{-\lambda} q^{4 \lambda z} q^{|\lambda|}}{z_{\lambda}(q)} e^{\alpha/2} = \sum_{n \geq 0} Z_n (q^4 z)^n e^{\alpha/2},
\]
4
The sum in (11) is over the set of all partitions \( \mathcal{P} \), and to the partition \( \lambda = (\lambda_1, \ldots, \lambda_s) = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k}) \) we have associated the symbols

\[
|\lambda| = \sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{k} m_i
\]

is the weight of the partition \( \lambda \).

Let \( p_\lambda \) denote the power sum symmetric functions: \( p_\lambda = p_{\lambda_1}p_{\lambda_2}\ldots \), with \( p_i = \sum_k x_i^k \).

Upon using the scalar product

\[
\langle p_\lambda, p_\mu \rangle_q = z_\lambda(q)\delta_{\lambda,\mu},
\]

the following identification can be made

\[
b_{-\lambda} = p_\lambda
\]

As a result, the \( Z_n \) become homogeneous symmetric functions and are recognized to be particular one-row Macdonald functions. In general, the symmetric functions \( P_\lambda(x; s, t) \) of Macdonald with two parameters \( s \) and \( t \), are orthogonal with respect to the scalar product

\[
\langle p_\lambda, p_\mu \rangle_{s, t} = z_\lambda(s, t)\delta_{\lambda,\mu},
\]

where

\[
z_\lambda(s, t) = \prod_{i=1}^{k} \frac{(1 - s i^{m_i})(1 - t i^{m_i})}{m_i!}.
\]

The one-row functions \( P_{(n)}(x; s, t) \) are proportional [4] to \( \sum_{|\lambda|=n} z_\lambda(s, t)^{-1}p_\lambda(x) \). If we set \( s = t^2 = q^4 \), we see in fact that \( \langle \ , \rangle_{q^4,q^2}\rangle_{q^4,q^2} = \langle \ , \rangle_{q^4,q^2} \) and that \( z_\lambda(q^4,q^2) = z_\lambda(q) \). We thus observe that the \( Z_n \) are one-row \( q \)-zonal functions.

Let us now construct the one-row VOS. They are obtained from the modes of \( \phi^{-\alpha}(z) \) as follows:

\[
\phi_{-n}^{0, -1} = \frac{1}{2\pi i} \int dzz^{-n-1}\phi_{0, -}(z).1
\]

\[
= \frac{1}{2\pi i} \int dzz^{-n-1}\sum_{m \geq 0} Z_m(q^4 z)^m e^{\alpha/2} = q^{4n} Z_n e^{\alpha/2},
\]
with \( n > 0 \). These states belong to \( V(\Lambda_1) \). Next, we construct the most general two-row VOS from the modes of \( \phi^0(z) \). This time, these VOS lie in \( V(\Lambda_0) \). We need

\[
\phi^{1,-}(z)\phi^{0,-}(w) : 1 = (-q^3z)^{1/2}C \sum_{n=1}^{\infty} \frac{4n}{n(n+q^2)}b_{-n}z^n C \sum_{n=1}^{\infty} \frac{4n}{n(n+q^2)}b_{-n}w^n e^{\alpha} = (-q^3z)^{1/2} \sum_{m_1,m_2 \geq 0} q^{4(m_1+m_2)} Z_{m_1} Z_{m_2} z^{m_1} w^{m_2} e^{\alpha}.
\]

(19)

Using this relation and (15), we find

\[
\phi^1_{-r} \phi^0_s : 1 = \left(\frac{2\pi i}{2}\right) \oint dz dw w^{-1} z^{-s} \phi^{1,-}(z) \phi^{0,-}(w),
\]

\[
= \frac{1}{(2\pi i)^2} \oint dz dw (q^3 w^{-1} q^4) \sum_{m_1,m_2 \geq 0} q^{4(m_1+m_2)} Z_{m_1} Z_{m_2} z^{m_1-r} w^{m_2-s} e^{\alpha}
\]

\[
= \frac{-q^3}{(2\pi i)^2} \oint dz \sum_{m_1,n \geq 0} q^{4(m_1+s-n)} C_n Z_{m_1} z_n z^{m_1-r} e^{\alpha}
\]

\[
= -q^3 \sum_{n \geq 0} q^{4(r+s-1)} C_n Z_{r+s} e^{\alpha}
\]

\[
= -q^{4(r+s-1)} \sum_{n \geq 0} C_n (R_{12})^n Z_{r+s} e^{\alpha} = -q^{4(r+s-1)}(q^{2R_{12};q^4}) Z_{r+s} e^{\alpha},
\]

where \( R_{12} \) is the raising operator acting on \( Z_r Z_s \) according to

\[
R_{12}(Z_r Z_s) = Z_{r+1} Z_{s-1},
\]

(21)

and the coefficients \( C_n \), \( n \geq 0 \) are such that

\[
\sum_{n \geq 0} C_n x^n \frac{(q^2 x; q^4)^{\infty}}{(q^4 x; q^4)^{\infty}} = e \sum_{k \geq 1} \frac{q^{2k} q^k}{k(1+q^2)} = \sum_{\lambda \in \mathbb{P}} (-1)^\ell(\lambda) (q^2 x)^{\lambda} z_\lambda(q),
\]

(22)

that is,

\[
C_n = \sum_{|\lambda|=n} (-1)^\ell(\lambda) q^{2|\lambda|} z_\lambda(q).
\]

(23)

Let us now consider the construction of a single-row dual VOS from the action of the conjugate VO \( \phi^{1,+}(z^{s-1}) \) on the dual bosonic vacuum 1. For this purpose let us remark that, unlike the situation in the classical case (i.e., \( q = 1 \)), the dual VOS cannot simply be obtained from the VOS through the usual conjugation \( b^*_n = b_{-n} \) because this leads to a dual Fock space that cannot be identified with any of the dual modules of \( V(\Lambda_i) \). For this reason, we construct the dual VOS by operating with \( \phi^{1,+}(z^{s-1}) \). The action of the modes \( b_n \), \( \partial \), and \( e^\alpha \) on the dual bosonic vacuum are given by

\[
1.b_{-n} = 1.\partial = 0, \quad n > 0,
\]

\[
1.e^{-\alpha} = e^{-\alpha},
\]

(24)
and 1.\(b_n\), \(n > 0\) are non-zero states. Due to the integral form of this VO, the following new technical problem arises: the contour in the definition of \(\phi^{1,+}(z^{s-1})\) winds around the pole \(w = 0\) and it is not easy to compute the residue at this pole in a closed form. Therefore, we make a change of variable to remove this pole when \(\phi^{1,+}(z^{s-1})\) acts on the dual bosonic vacuum \(1\). The appropriate change of variable is simply \(w = \xi^{-1}\). Let us also set \(z^{s-1} = \eta\), then we obtain

\[
1. : \phi^{1,-}(\eta)E^-(\xi^{-1}) : = 1.e^{\sum_{n=1}^{\infty} \frac{n^{-2n}}{n(1+q^{2n})} b_n \eta^{-n}} e^{\sum_{n=1}^{\infty} \frac{b_n \xi^n}{n} e^{-\alpha/2} \xi^\partial (-q^3 \eta)^{(\partial+1)/2}} = e^{-\alpha/2} e^{\sum_{n=1}^{\infty} \frac{n^{-2n}}{n(1+q^{2n})} b_n \eta^{-n}} e^{\sum_{n=1}^{\infty} \frac{b_n \xi^n}{n} \xi (-q^3 \eta)} \tag{25}
\]

From these relations we arrive at

\[
1. : \phi^{1,-}(\eta)E^-(\eta^q) : = -q e^{-\alpha/2} e^{\sum_{n=1}^{\infty} \frac{1}{n(1+q^{2n})} b_n \eta^{-n}}. \tag{26}
\]

where

\[
b_{\lambda} = b_{\lambda_1} b_{\lambda_2} \ldots, \tag{27}
\]

\[
Z_n^* = \sum_{|\lambda|=n} b_{\lambda} z_{\lambda}(q). \tag{28}
\]

Keeping with (2.15), we make the identification

\[
b_n = n(1 + q^{2n}) \frac{\partial}{\partial p_n} = D(p_n), \quad n > 0. \tag{29}
\]

Here the first equality is consistent with the Heisenberg algebra \(\mathfrak{h}\) and the second one defines the adjoint operator \(D\) in the space of symmetric functions. We thus have

\[
Z_n^* = D(Z_\lambda). \tag{30}
\]

From this, we obtain the following scalar product

\[
< Z_n, Z_m >_q = \delta_{n,m} \sum_{|\lambda|=n} \frac{1}{z_{\lambda}(q)}. \tag{31}
\]

The one-row dual VOS are hence found to be

\[
1.\phi_n^{1,+} = \frac{1}{2\pi i} \oint d\eta \eta^{n-1} 1.\phi^{1,+}(\eta) = \frac{e^{-\alpha/2}}{2\pi i} \oint d\eta \eta^{n-1} \sum_{m \geq 0} Z_m^* \eta^{-m} = e^{-\alpha/2} Z_n^*. \tag{32}
\]
As a check, we now show that the scalar products of 1-row VOS are consistent with the q-KZ equation \[8\]. Indeed, from (18), (30) and (31) we get the following matrix element:

\[1.\Phi^{1+}_m \Phi^{0-}_n.1 = q^{4n} \Phi^{1+}_{m} \Phi^{0-}_{-n.1} = q^{4n} \sum_{|\lambda|=n} \frac{1}{z_{\lambda}(q)},\]  

(32)

which in turn leads to

\[1.\Phi^{1+}(z)\Phi^{0-}(w).1 = \sum_{n,m\in\mathbb{Z}} z^{-m} w^n \Phi^{1+}_{m} \Phi^{0-}_{-n.1} = \sum_{n\in\mathbb{Z}} \sum_{|\lambda|=n} z^{-n} w^n q^{4n} \frac{1}{z_{\lambda}(q)} = (q^{\delta_{wz^{-1},q^4}})_{\infty},\]  

(33)

but this is precisely the matrix element of the VO obtained through solving the q-KZ in \[3\].

Let us now construct the most general two-row dual VOS from the successive actions of two modes, one from \(\Phi^{1+}(z^{-1})\) and the other from \(\Phi^{0+}(w^{-1})\), on the dual bosonic vacuum. Let \(\eta = z^{-1}\) and \(\theta = w^{-1}\), we find that

\[1.\Phi^{1+}(\eta)\Phi^{0+}(\theta) = q^{-4} \frac{(q^2 \theta^{-1}; q^4)_{\infty}}{(q^{4\theta^{-1}}; q^4)_{\infty}} e^{-\alpha} \sum_{n,m\geq 0} Z^*_n Z^n_{m} \eta^{-n} \theta^{-m-1}.\]  

(34)

Thus

\[1.\Phi^{1+}_r \Phi^{0+}_s = \frac{1}{(2\pi i)^2} \oint \eta d\theta \eta^{-1} \Phi^{1+}_r(\eta) \Phi^{0+}_s(\theta) = q^{-4e^{-\alpha}} \frac{(q^2 \theta^{-1}; q^4)_{\infty}}{(q^{4\theta^{-1}}; q^4)_{\infty}} \sum_{n,m,k\geq 0} C_k Z^*_n Z^n_{m} \eta^{-n-k} \theta^{-m-k}\]  

(35)

where the coefficients \(C_n\) are the same as those given in \(\[23\]\) and \(\tilde{R}_{12}\) is the lowering operator defined by

\[\tilde{R}_{12} Z_r Z_s = Z_{r-1} Z_{s+1}.\]  

(36)

We know through the relations \(\[18\]\) and \(\[26\]\) that the VOS obtained from the action of a single mode of the vertex operators on the vacuum are proportional to the one-row \(q\)-zonal functions. This is no longer true in the case of VOS obtained from the action of two modes, that is, they are not proportional to two-row \(q\)-zonal functions. The two are however related and we now display this relation. For this purpose, we use the following relation obtained
in [3], which expresses the two-row \(q\)-zonal functions \(Z_{n-1,m}\) as the product of two one-row \(q\)-zonal functions:

\[
Z_{n-1,m} = 2\phi_1(q^2, q^{4(n-m)}; q^{2+4(n-m)}; q^4, q^2 R)(1 - R)Z_{n-1}Z_m.
\]

We have set \(R = R_{12}\) in (2.37) and as usual \(2\phi_1(a, b; c, z)\) denotes the \(q\)-hypergeometric function

\[
2\phi_1(a, b; c, z) = \sum_{n \geq 0} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n.
\]

Combining the relations (37) and (20), we get

\[
Z_{n-1,m} = -q^{-4(n+m)+1} 2\phi_1(q^2, q^{4(n-m)}; q^{2+4(n-m)}; q^4, q^2 \tilde{R})(1 - \tilde{R})(Z_{n-1})^*(Z_m)^*,
\]

which expresses the two-row \(q\)-zonal functions in terms of the VOS. To also relate the dual VOS (35) to the two-row \(q\)-zonal functions, we use the relation

\[
(Z_{n-1,m})^* = 2\phi_1(q^2, q^{4(n-m)}; q^{2+4(n-m)}; q^4, q^2 \tilde{R})(1 - \tilde{R})(Z_m)^*(Z_{n-1})^*,
\]

which is deduced from (37). Then, (35) leads to

\[
(Z_{n-1,m})^* = -q^{-4} 2\phi_1(q^2, q^{4(n-m)}; q^{2+4(n-m)}; q^4, q^2 \tilde{R})(1 - \tilde{R})\frac{(\tilde{R}; q^4)_\infty}{(q^2 \tilde{R}; q^4)_\infty} 1_{\phi_{n+1}^1, \phi_{n-1}^0}.1,
\]

where \(\tilde{R} = \tilde{R}_{12}\).

### III Discussion and conclusions

Let us compare our results with those of Jing who considers in [5] the following vertex operators:

\[
X(z) = e^{\sum_{n=1}^{\infty} \frac{1}{n^{1-s|n|}} c_{-n} z^n} e^{-\sum_{n=1}^{\infty} \frac{1}{n^{1-t|n|}} c_{n} z^{-n}} = \sum_{n \in \mathbb{Z}} X_n z^{-n},
\]

where \(s\) and \(t\) are deformation parameters and where the bosonic modes \(c_n\) satisfy the Heisenberg algebra defining relation:

\[
[c_n, c_m] = n \frac{1 - s |n|}{1 - t |n|} \delta_{n+m,0}.
\]
The modes $X_n$ are expressed in terms of $X(z)$ as follows

$$X_n = \frac{1}{2\pi i} \oint dz z^{n-1} X(z).$$  \hspace{1cm} (45)

Jing defines the conjugate vertex operator $X^\ast(z)$ by

$$X^\ast(z) = e^{-\sum_{n=1}^{\infty} \frac{1-s^n}{1-t^n} c_{-n} z^n} e^{\sum_{n=1}^{\infty} \frac{1-s^n}{1-t^n} c_n z^{-n}} = \sum_{n \in \mathbb{Z}} X^\ast_n z^n,$$  \hspace{1cm} (46)

and he also defines the adjoint of $c_n$ in the Fock space by $c^\ast_n = c_{-n}$. It can thus easily be checked that

$$(X(z))^\ast = X^\ast(z^{-1}),$$  \hspace{1cm} (47)

from where it follows that

$$(X_n)^\ast = X^\ast_n.$$  \hspace{1cm} (48)

Thus, $X^\ast_n$ is the conjugate of $X_n$ and for this reason we call $X^\ast(z)$ the conjugate VO of $X(z)$.

Let us note that the Heisenberg algebra elements $c_n$ reduce to the elements $b_n$ when $s = t^2 = q^4$. However, there are three differences between the intertwining VO and those considered by Jing. Firstly, the conjugate VO $X^\ast(z)$ has the same simple form as the direct VO $X(z)$, whereas $\phi^{1,+}(z)$ which is the conjugate VO of $\phi^{0,-}(z)$ does not have the same simple exponential form as that of $\phi^{0,-}(z)$. Secondly, Jing’s deformation is symmetric in the following sense: if

$$[c_n, c_m] = nf(s^{[n]}, t^{[m]})\delta_{n+m,0},$$  \hspace{1cm} (49)

with

$$f(s^{[n]}, t^{[m]}) = \frac{1 - s^{[n]}}{1 - t^{[m]}},$$  \hspace{1cm} (50)

then the deformation of the VO takes the form

$$X(z) = e^{\sum_{n=1}^{\infty} \frac{1-s^n}{1-t^n} c_{-n} z^n} e^{-\sum_{n=1}^{\infty} \frac{1-s^n}{1-t^n} c_n z^{-n}}.$$  \hspace{1cm} (51)

This is of course not the case for the intertwining VO. Finally, $X(z)$ and $X^\ast(z)$ do not depend on the zero mode $\partial$ and its conjugate $\alpha$ whereas the intertwining VO’s do. For these reasons Jing’s VOS and our VOS are related differently to two-row $q$-zonal functions [5].

One important open question is to understand the relation between the matrix elements of the VOS (rather than the VOS themselves) and symmetric functions. It is known in fact
that these matrix elements satisfy the q-KZ equation [8] and it would be quite useful to find a relation between this equation and the symmetric functions. Another interesting question is to understand the relation between the intertwining vertex operators and the creation operators recently introduced in [10] which allow to construct from the ground state of the Calogero-Sutherland model the Jack functions associated to any partition.

A straightforward extension of this work is to consider the other type of VO $\psi^{i,\pm}(z)$, referred to as type-II in [3] that intertwine the $U_q(\widehat{sl}(2))$ modules as follows:

$$v_+ \otimes \psi^{i,+}(z) + v_- \otimes \psi^{i,-}(z) : V(\Lambda_i) \rightarrow V(z) \otimes V(\Lambda_{i-1}).$$

Furthermore, it would be interesting to carry this work over to the VO which intertwine the higher level modules of $U_q(\widehat{sl}(2))$ [11, 12]. Finally, and most importantly, extending this work beyond the two-row $q$-zonal functions, is certainly an interesting open problem which is also pending in the framework of Jing.

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