An inequality related to uncertainty principle in von Neumann algebras

Paolo Gibilisco* and Tommaso Isola†

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Abstract

Recently Kosaki proved in [8] an inequality for matrices that can be seen as a kind of new uncertainty principle. Independently, the same result was proved by Yanagi et al. in [13]. The new bound is given in terms of Wigner-Yanase-Dyson informations. Kosaki himself asked if this inequality can be proved in the setting of von Neumann algebras. In this paper we provide a positive answer to that question and moreover we show how the inequality can be generalized to an arbitrary operator monotone function.

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1 Introduction

If $A, B$ are selfadjoint matrices and $\rho$ is a density matrix, define

\[
\text{Cov}_\rho(A, B) := \text{Re}\{\text{Tr}(\rho AB) - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B)\}
\]
\[
\text{Var}_\rho(A) := \text{Cov}_\rho(A, A).
\]

The uncertainty principle reads as

\[
\text{Var}_\rho(A)\text{Var}_\rho(B) \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2.
\]

This inequality can be refined as

\[
\text{Var}_\rho(A)\text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq \frac{1}{4}|\text{Tr}(\rho[A, B])|^2,
\]

(see [5, 12]). Recently a different uncertainty principle has been found [11, 9, 10, 8, 13]. For $\beta \in (0, 1)$ define $\beta$-correlation and $\beta$-information as

\[
\text{Corr}_{\rho, \beta}(A, B) := \text{Re}\{\text{Tr}(\rho AB) - \text{Tr}(\rho^\beta A \rho^{1-\beta} B)\}
\]
\[
\text{I}_{\rho, \beta}(A) := \text{Corr}_{\rho, \beta}(A, A) = \text{Tr}(\rho A^2) - \text{Tr}(\rho^\beta A \rho^{1-\beta} A),
\]

where the latter coincides with the Wigner-Yanase-Dyson information. It has been proved that

\[
\text{Var}_\rho(A)\text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq \text{I}_{\rho, \beta}(A)\text{I}_{\rho, \beta}(B) - \text{Corr}_{\rho, \beta}(A, B)^2.
\] (1.1)
The quantities involved in the previous inequality make a perfect sense in a von Neumann algebra setting (see for example [7]). In ref. [8] Kosaki asked if the inequality (1.1) is true in this more general setting.

In this paper we provide a positive answer to Kosaki question and moreover we show that, once the inequality is formulated in the context of operator monotone functions, the result can be greatly generalized.

2 Preliminaries

Denote by $M_{n,sa}$ the space of complex self-adjoint $n \times n$ matrices, and recall that a function $f : (0, \infty) \rightarrow \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, any $A, B \in M_{n,sa}$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. Then, $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone iff for any $A, B \in B(\mathcal{H})$ such that $0 \leq A \leq B$, it holds $f(A) \leq f(B)$. An operator monotone function is said symmetric if $f(x) := x f(x - 1)$ and normalized if $f(1) = 1$. We denote by $\mathcal{F}$ the class of positive, symmetric, normalized, operator monotone functions.

Examples of operator monotone functions are the so-called Wigner-Yanase-Dyson functions $f_\beta(x) := \beta(1 - \beta)(x - 1)^2/(x^3 - 1)(x^{1-\beta} - 1)$, $\beta \in (0, 1)$.

Returning to a general $f \in \mathcal{F}$, we associate to it a function $\tilde{f} \in \mathcal{F}$ defined by

$$\tilde{f}(x) := \frac{1}{2}((x + 1) - (x - 1)^2 f(0)/f(x)),$$

$x > 0$.

For example

$$\tilde{f}_\beta(x) = \frac{1}{2}(x^\beta + x^{1-\beta}).$$

Definition 2.1. For $A, B \in M_{n,sa}$, $f \in \mathcal{F}$, and $\rho$ a faithful density matrix, define $f$-correlation and $f$-information as

$$\text{Corr}_f^\rho(A, B) := \text{Re}\{\text{Tr}(\rho AB) - \text{Tr}(\rho \tilde{f}(L_\rho R_\rho^{-1}) (A) \cdot B)\},$$

$$I_f^\rho(A) := \text{Corr}_f^\rho(A, A).$$

Recall that $f$-information is also known as metric adjusted skew information (see [4]). The following generalization of inequality (1.1) is proved in [2].

Theorem 2.2.

$$\text{Var}_\rho(A) \text{Var}_\rho(B) - \text{Cov}_\rho(A, B)^2 \geq I_f^\rho(A) I_f^\rho(B) - \text{Corr}_f^\rho(A, B)^2.$$

In the next Section we prove that the above inequality holds true in a general von Neumann algebra, thus answering, in particular, the question raised by Kosaki in [8], and recalled above. A different generalization of Theorem 2.2 has been proved in [3].

3 The main result

Let $\mathcal{M}$ be a von Neumann algebra, and $\omega$ a normal faithful state on $\mathcal{M}$, and denote by $\mathcal{H}_\omega$ and $\xi_\omega$ the GNS Hilbert space and vector, and by $S_\omega$, $J_\omega$ and $\Delta_\omega$ the modular operators associated to $\omega$.

The proof of the main result is divided in a series of Lemmas. In order to deal with unbounded operators, we introduce some sesquilinear forms on $\mathcal{H}_\omega$, and take [6] as our standard reference.
Definition 3.1. Let \( f \in \mathfrak{g} \), and define the following sesquilinear forms
\[
\mathcal{E}(\xi, \eta) := \langle \Delta_{\omega}^{1/2} \xi, \Delta_{\omega}^{1/2} \eta \rangle, \\
\mathcal{E}_1(\xi, \eta) := \mathcal{E}(\xi, \eta) + \langle \xi, \eta \rangle, \\
\mathcal{F}^f(\xi, \eta) := \langle \hat{f}(\Delta_{\omega})^{1/2} \xi, \hat{f}(\Delta_{\omega})^{1/2} \eta \rangle, \\
\mathcal{G}^f(\xi, \eta) := \frac{1}{2} \mathcal{E}_1(\xi, \eta) - \mathcal{F}^f(\xi, \eta).
\]

It follows from [6], Example VI.1.13, that \( \mathcal{E}, \mathcal{E}_1, \mathcal{F}^f \) are closed, positive and symmetric sesquilinear forms.

Lemma 3.2. Let \( \xi, \eta \in \mathcal{D}(\Delta_{\omega}^{1/2}) \), and \( \{\xi_n\}, \{\eta_n\} \subset \mathcal{D}(\Delta_{\omega}) \) be such that \( \xi_n \to \xi, \mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0, n \to \infty \), and analogously for \( \eta_n \) and \( \eta \). Then
\[
\mathcal{E}(\xi, \eta) = \lim_{n \to \infty} \mathcal{E}(\xi_n, \eta_n), \\
\mathcal{F}^f(\xi, \eta) = \lim_{n \to \infty} \mathcal{F}^f(\xi_n, \eta_n).
\]

Proof. It follows from [6] Theorem VI.1.12 that \( \mathcal{D}(\Delta_{\omega}) \) is a core for \( \mathcal{D}(\mathcal{E}) \equiv \mathcal{D}(\Delta_{\omega}^{1/2}) \), so that, from [6] Theorem VI.1.21, for any \( \xi \in \mathcal{D}(\Delta_{\omega}^{1/2}) \) there is \( \{\xi_n\} \subset \mathcal{D}(\Delta_{\omega}) \) such that \( \xi_n \to \xi \), and \( \mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0, n \to \infty \). Then \( \mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0, m, n \to \infty \). Now observe that \( 0 \leq f(x) \leq \frac{1}{x+1} \), for \( x > 0 \) [2], so that
\[
\mathcal{F}^f(\xi_n - \xi_m, \xi_n - \xi_m) = \langle \hat{f}(\Delta_{\omega})^{1/2}(\xi_n - \xi_m), \hat{f}(\Delta_{\omega})^{1/2}(\xi_n - \xi_m) \rangle \\
= \langle \xi_n - \xi_m, \hat{f}(\Delta_{\omega})(\xi_n - \xi_m) \rangle \\
\leq \frac{1}{2} \langle \xi_n - \xi_m, \xi_n - \xi_m \rangle + \frac{1}{2} \langle \xi_n - \xi_m, \Delta_{\omega}(\xi_n - \xi_m) \rangle \\
= \frac{1}{2} \|\xi_n - \xi_m\|^2 + \frac{1}{2} \mathcal{E}(\xi_n - \xi_m, \xi_n - \xi_m) \to 0, m, n \to \infty.
\]
This implies \( \xi \in \mathcal{D}(\mathcal{F}^f) \) and \( \mathcal{F}^f(\xi_n - \xi, \xi_n - \xi) \to 0, n \to \infty \).

Therefore, if \( \xi, \eta \in \mathcal{D}(\Delta_{\omega}^{1/2}) \), and \( \{\xi_n\}, \{\eta_n\} \subset \mathcal{D}(\Delta_{\omega}) \) approximate \( \xi, \eta \) in the above sense, we obtain, from [6] Theorem VI.1.12, that \( \mathcal{F}^f(\xi, \eta) = \lim_{n \to \infty} \mathcal{F}^f(\xi_n, \eta_n) \), and analogously for \( \mathcal{E} \). □

Lemma 3.3. 
(i) \( \mathcal{D}(\mathcal{F}^f) \supset \mathcal{D}(\Delta_{\omega}^{1/2}) \),
(ii) \( \mathcal{G}^f \) is a symmetric sesquilinear form on \( \mathcal{D}(\mathcal{G}^f) \supset \mathcal{D}(\Delta_{\omega}^{1/2}) \), which is positive on \( \mathcal{D}(\Delta_{\omega}^{1/2}) \).

Proof. (i) It follows from the proof of the previous Lemma.
(ii) We only need to prove positivity. To begin with, let \( \xi \in \mathcal{D}(\Delta_{\omega}) \). Then, setting \( g(x) := \frac{1}{2} (x+1) - \hat{f}(x) \geq 0 \), for all \( x > 0 \), we have
\[
\mathcal{G}^f(\xi, \xi) = \frac{1}{2} \mathcal{E}_1(\xi, \xi) - \mathcal{F}^f(\xi, \xi) = \frac{1}{2} \langle \xi, \xi \rangle + \frac{1}{2} \langle \xi, \Delta_{\omega} \xi \rangle - \langle \xi, \hat{f}(\Delta_{\omega}) \xi \rangle = \langle \xi, g(\Delta_{\omega}) \xi \rangle \geq 0.
\]
Moreover, if \( \xi \in \mathcal{D}(\Delta_{\omega}^{1/2}) \), and \( \xi_n \in \mathcal{D}(\Delta_{\omega}) \) is such that \( \xi_n \to \xi \), and \( \mathcal{E}(\xi_n - \xi, \xi_n - \xi) \to 0 \), then, from Lemma 3.2 it follows \( \mathcal{G}^f(\xi, \xi) = \lim_{n \to \infty} \mathcal{G}^f(\xi_n, \xi_n) \geq 0 \). □

We can now introduce the main objects of study. In the sequel, we denote by \( T \in \mathfrak{M} \) the fact that \( T \) is a closed, densely defined, linear operator on \( \mathcal{H}_{\omega} \), and is affiliated with \( \mathfrak{M} \).

Definition 3.4. For any \( A, B \in \mathfrak{M}_{\omega} \), such that \( \xi_{\omega} \in \mathcal{D}(A) \cap \mathcal{D}(B) \), and any \( f \in \mathfrak{g} \), we set \( A_0 := A - \langle \xi_{\omega}, A \xi_{\omega} \rangle \), \( B_0 := B - \langle \xi_{\omega}, B \xi_{\omega} \rangle \), and define the bilinear forms
\[
\text{Cov}_{\omega}(A, B) := \text{Re}(A_0 \xi_{\omega}, B_0 \xi_{\omega}), \\
\text{Var}_{\omega}(A) := \text{Cov}_{\omega}(A, A), \\
\text{Corr}^f_{\omega}(A, B) := \text{Re}(A_0 \xi_{\omega}, B_0 \xi_{\omega}) - \text{Re}(\hat{f}(\Delta_{\omega})^{1/2} A_0 \xi_{\omega}, \hat{f}(\Delta_{\omega})^{1/2} B_0 \xi_{\omega}), \\
I^f_{\omega}(A) := \text{Corr}^f(A, A).
\]
Remark 3.5. Observe that in the matrix case $\omega = \text{Tr}(\rho)$, for some density matrix $\rho$, and $\Delta_\omega = L_\rho R_\rho^{-1}$, so that the previous Definition is a true generalization of covariance and $f$-correlation in the matrix case.

For the reader’s convenience, we prove the following folklore result.

Lemma 3.6. $\mathcal{D}(\Delta^{1/2}_\omega) = \{T\xi_\omega : T\in \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

Proof. (1) Let us first prove that $\mathcal{D}(\Delta^{1/2}_\omega) \subseteq \{T\xi_\omega : T\in \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$. Indeed, let $\eta \in \mathcal{D}(\Delta^{1/2}_\omega)$, and define the linear operator $T_0 : x'\xi_\omega \in M_\omega \rightarrow x'\eta \in \mathcal{H}_\omega$, which is densely defined, and affiliated with $M$. Let us show that $T_0$ is precsed: indeed, if $x'\xi_\omega \rightarrow 0$, and $x'\eta \rightarrow \zeta$, then, for any $y' \in M'$, we get

$$\langle \zeta, y' \xi_\omega \rangle = \lim_{n \to \infty} \langle x'\eta, y' \xi_\omega \rangle = \lim_{n \to \infty} \langle \eta, x'_n y' \xi_\omega \rangle = \lim_{n \to \infty} \langle \eta, S_{\omega}(y' x'_n \xi_\omega) \rangle = \lim_{n \to \infty} \langle y' x'_n \xi_\omega, S_{\omega} \eta \rangle = \lim_{n \to \infty} \langle x'_n \xi_\omega, y' S_{\omega} \eta \rangle = 0,$$

which shows that $T_0$ is precsed. Let $T_\eta := T_0$. Then, $T_\eta \in \mathcal{M}$, and $T_\eta \xi_\omega = \eta$. It remains to be proved that $\xi_\omega \in \mathcal{D}(T_\eta^*)$. Since $S_{\omega} \eta \in \mathcal{D}(\Delta^{1/2}_\omega)$, we can also consider $T_{S_{\omega} \eta}$. Let us show that $T_{S_{\omega} \eta} \subseteq T_\eta^*$. Indeed, for any $x' \in M'$, we have

$$\langle T_{S_{\omega} \eta} x' \xi_\omega, y' \xi_\omega \rangle = \langle x' S_{\omega} \eta, y' \xi_\omega \rangle = \langle S_{\omega} \eta, x' y' \xi_\omega \rangle = \langle y' x' \xi_\omega, \eta \rangle = \langle x' \xi_\omega, y' S_{\omega} \eta \rangle = \langle x' \xi_\omega, T_\eta y' \xi_\omega \rangle.$$  

Then, $\xi_\omega \in \mathcal{D}(T_{S_{\omega} \eta}) \subseteq \mathcal{D}(T_\eta^*)$, which shows that $\mathcal{D}(\Delta^{1/2}_\omega) \supseteq \{T\xi_\omega : T\in \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

(2) Let us now prove that $\mathcal{D}(\Delta^{1/2}_\omega) \supseteq \{T\xi_\omega : T\in \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$. Indeed, if $T\in \mathcal{M}$ is such that $\xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)$, we can consider its polar decomposition $T = vt|T|$, and let $e_n := \chi[0,n](|T|)$, $T_n := v|T| e_n$, for any $n \in \mathbb{N}$. Since $\xi_\omega \in \mathcal{D}(T)$, we have $T_n \xi_\omega = v|T| \xi_\omega \rightarrow T \xi_\omega$. Moreover, since $\xi_\omega \in \mathcal{D}(T^*)$, we have $T_n^* \xi_\omega = |T| v^* \xi_\omega = e_n T^* \xi_\omega \rightarrow T^* \xi_\omega$. Since $S_{\omega}$ is a closed operator, it follows that $T \xi_\omega \in \mathcal{D}(S_{\omega}) = \mathcal{D}(\Delta^{1/2}_\omega)$ and $S_{\omega} T \xi_\omega = T^* \xi_\omega$, which is what we wanted to prove.

Lemma 3.7. For any $A,B \in \mathbb{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathcal{F}$, we have

(i) $\text{Cov}_\omega(A,B) = \frac{1}{2} \text{Re} E_1(A_0 \xi_\omega, B_0 \xi_\omega)$ is a positive bilinear form,

(ii) $\text{Corr}_\omega^f(A,B) = \text{Re} \mathcal{G}^f(A_0 \xi_\omega, B_0 \xi_\omega)$ is a positive bilinear form.

Proof. (i) Observe that

$$\langle B_0 \xi_\omega, A_0 \xi_\omega \rangle = \langle B_0^* \xi_\omega, A_0^* \xi_\omega \rangle = \langle J_\omega \Delta^{1/2}_\omega B_0 \xi_\omega, J_\omega \Delta^{1/2}_\omega A_0 \xi_\omega \rangle = \langle \Delta^{1/2}_\omega A_0 \xi_\omega, \Delta^{1/2}_\omega B_0 \xi_\omega \rangle = \mathcal{E}(A_0 \xi_\omega, B_0 \xi_\omega).$$

The thesis follows from this and the fact that $\mathcal{D}(\Delta^{1/2}_\omega) = \{T\xi_\omega : T\in \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

(ii) It follows from (i) and Lemma 3.6 (ii).

Lemma 3.8. Let $\xi, \eta \in \mathcal{H}_\omega$, $\Delta_\omega = \int_0^\infty t \text{d}e(t)$, and define, for $\Omega$ a Borel subset of $[0, \infty)$, $\mu_{\xi \eta}(\Omega) := \text{Re} \langle \xi, e(\Omega) \eta \rangle$, and

$$\mu := \mu_{\xi \xi} \otimes \mu_{\eta \eta} + \mu_{\xi \eta} \otimes \mu_{\eta \xi} - 2 \mu_{\xi \xi} \otimes \mu_{\eta \eta}.$$ 

Then, $\mu$ is a bounded positive Borel measure on $[0, \infty)^2$.

Proof. Let $\Omega_1, \Omega_2$ be Borel subsets of $[0, \infty)$, and set $e_j := e(\Omega_j)$, $j = 1, 2$. Observe that $\text{Re} \langle \xi, e_1 \eta \rangle \cdot \text{Re} \langle \xi, e_2 \eta \rangle \leq \|e_1 \xi\| \cdot \|e_2 \xi\| \cdot \|e_1 \eta\| \cdot \|e_2 \eta\|$, so that

$$\mu(\Omega_1 \times \Omega_2) \geq \|e_1 \xi\|^2 \cdot \|e_2 \eta\|^2 + \|e_2 \xi\|^2 \cdot \|e_1 \eta\|^2 - 2 \|e_1 \xi\| \cdot \|e_1 \eta\| \cdot \|e_2 \xi\| \cdot \|e_2 \eta\| \geq 0.$$ 

The thesis follows by standard measure theoretic arguments.

Theorem 3.9. For any $A,B \in \mathbb{M}_{sa}$, such that $\xi_\omega \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and any $f \in \mathcal{F}$, we have

$$\text{Var}_\omega(A) \cdot \text{Var}_\omega(B) - \text{Cov}_\omega(A,B)^2 \geq I^f_\omega(A) I^f_\omega(B) - \text{Corr}_\omega^f(A,B)^2.$$
Proof. Set

\[
G(A, B) := \text{Var}_\omega(A) \text{Var}_\omega(B) - \text{Cov}_\omega(A, B)^2 - I^\omega_+(A)I^\omega_+(B) + \text{Corr}^\omega_+(A, B)^2
\]

\[
\begin{align*}
&= \frac{1}{2} \mathcal{E}_1(A_0 \xi_\omega, A_0 \xi_\omega) - \frac{1}{2} \mathcal{E}_1(B_0 \xi_\omega, B_0 \xi_\omega) - \left( \frac{1}{2} \text{Re} \mathcal{E}_1(A_0 \xi_\omega, B_0 \xi_\omega) \right)^2 \\
&\quad - \left( \frac{1}{2} \mathcal{E}_1(A_0 \xi_\omega, A_0 \xi_\omega) - \mathcal{F}_1(A_0 \xi_\omega, A_0 \xi_\omega) \right) \left( \frac{1}{2} \mathcal{E}_1(B_0 \xi_\omega, B_0 \xi_\omega) - \mathcal{F}_1(B_0 \xi_\omega, B_0 \xi_\omega) \right) \\
&\quad + \left( \frac{1}{2} \text{Re} \mathcal{E}_1(A_0 \xi_\omega, B_0 \xi_\omega) - \text{Re} \mathcal{F}_1(A_0 \xi_\omega, B_0 \xi_\omega) \right)^2 \\
&\quad - \frac{1}{2} \mathcal{E}_1(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \mathcal{F}_1^\omega(B_0 \xi_\omega, B_0 \xi_\omega) + \frac{1}{2} \mathcal{F}_1^\omega(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \mathcal{E}_1(B_0 \xi_\omega, B_0 \xi_\omega) \\
&\quad - \mathcal{F}_1(A_0 \xi_\omega, A_0 \xi_\omega) \cdot \mathcal{F}_1(B_0 \xi_\omega, B_0 \xi_\omega) - \text{Re} \mathcal{E}_1(A_0 \xi_\omega, B_0 \xi_\omega) \cdot \text{Re} \mathcal{F}_1(A_0 \xi_\omega, B_0 \xi_\omega) \\
&\quad + \left( \text{Re} \mathcal{F}_1(A_0 \xi_\omega, B_0 \xi_\omega) \right)^2,
\end{align*}
\]

where in (a) we have used Lemma \ref{lem:4}. Let us now introduce the function, for \( \xi, \eta \in \mathcal{D}(\Delta^{1/2}_\omega) \),

\[
H(\xi, \eta) := \frac{1}{2} \mathcal{E}_1(\xi, \xi) \cdot \mathcal{F}_1^\omega(\eta, \eta) + \frac{1}{2} \mathcal{F}_1(\xi, \xi) \cdot \mathcal{F}_1^\omega(\eta, \eta) - \mathcal{F}_1(\xi, \xi) \cdot \mathcal{F}_1^\omega(\eta, \eta) - \text{Re} \mathcal{E}_1(\xi, \eta) \cdot \text{Re} \mathcal{F}_1(\xi, \eta) + \left( \text{Re} \mathcal{F}_1(\xi, \eta) \right)^2,
\]

and recall that \( \mathcal{D}(\Delta^{1/2}_\omega) = \{ T \xi_\omega : T \in \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*) \} \), so that, if \( A, B \) are as in the statement of the Theorem, we obtain \( G(A, B) = H(A_0 \xi_\omega, B_0 \xi_\omega) \), and to prove the theorem it suffices to show that \( H(\xi, \eta) \geq 0 \), for all \( \xi, \eta \in \mathcal{D}(\Delta^{1/2}_\omega) \). Observe that, for \( \xi, \eta \in \mathcal{D}(\Delta_\omega) \), we get

\[
H(\xi, \eta) = \frac{1}{2} \mathcal{E}_1(\xi, (1 + \Delta_\omega)\xi) \cdot \mathcal{F}_1^\omega(\eta, \eta) + \frac{1}{2} \mathcal{F}_1(\xi, \xi) \cdot \mathcal{F}_1^\omega(\eta, \eta) + \left( \frac{1}{2} \mathcal{E}_1(\xi, \xi) \cdot \mathcal{F}_1^\omega(\eta, \eta) \right)^2
\]

\[
\begin{align*}
&= \frac{1}{2} \int_0^{\infty} (s + 1) \mu_{\xi \xi}(s) \int_0^{\infty} \hat{f}(t) \, d\mu_{\eta \eta}(t) + \frac{1}{2} \int_0^{\infty} \hat{f}(s) \, d\mu_{\xi \xi}(s) \int_0^{\infty} (t + 1) \, d\mu_{\eta \eta}(t) \\
&\quad - \int_0^{\infty} \hat{f}(s) \, d\mu_{\xi \xi}(s) \int_0^{\infty} \hat{f}(t) \, d\mu_{\eta \eta}(t) - \frac{1}{2} \int_0^{\infty} (s + 1) \mu_{\xi \xi}(s) \int_0^{\infty} \hat{f}(t) \, d\mu_{\eta \eta}(t) \\
&\quad - \frac{1}{2} \int_0^{\infty} \hat{f}(s) \, d\mu_{\xi \xi}(s) \int_0^{\infty} (t + 1) \mu_{\eta \eta}(t) - \int_0^{\infty} \hat{f}(s) \, d\mu_{\xi \xi}(s) \int_0^{\infty} \hat{f}(t) \, d\mu_{\eta \eta}(t) \\
&\quad - \frac{1}{2} \int_0^{\infty} \hat{f}(s) \, d\mu_{\eta \eta}(s) \int_0^{\infty} (t + 1) \mu_{\xi \xi}(t) - \int_0^{\infty} \hat{f}(s) \, d\mu_{\eta \eta}(s) \int_0^{\infty} \hat{f}(t) \, d\mu_{\xi \xi}(t) \\
&\quad + \frac{1}{2} \int_0^{(s + 1) \hat{f}(t) + (t + 1) \hat{f}(s) - 2 \hat{f}(s) \hat{f}(t)} \, d\mu_{\xi \xi} \otimes \mu_{\eta \eta}(s, t) \\
&\quad + \frac{1}{2} \int_0^{(s + 1) \hat{f}(t) + (t + 1) \hat{f}(s) - 2 \hat{f}(s) \hat{f}(t)} \, d\mu_{\eta \eta} \otimes \mu_{\xi \xi}(s, t) \\
&\quad + \frac{1}{4} \int_0^{(s + 1) \hat{f}(t) + (t + 1) \hat{f}(s) - 2 \hat{f}(s) \hat{f}(t)} \, d\mu(s, t),
\end{align*}
\]

where we used in (b) notation as in Lemma \ref{lem:4.8} in (c) Fubini-Tonelli Theorem, and in (d) the symmetries of the first integrand and notation as in Lemma \ref{lem:4.8}. Since \( \mu \) is a positive measure, and

\[
(s + 1) \hat{f}(t) + (t + 1) \hat{f}(s) - 2 \hat{f}(s) \hat{f}(t) = (s + 1 - \hat{f}(s)) \hat{f}(t) + (t + 1 - \hat{f}(t)) \hat{f}(s) \geq 0,
\]

we obtain \( H(\xi, \eta) \geq 0 \), for any \( \xi, \eta \in \mathcal{D}(\Delta_\omega) \).

It follows from Lemma \ref{lem:4.2} that, for any \( \xi, \eta \in \mathcal{D}(\Delta^{1/2}_\omega) \), we have \( H(\xi, \eta) = \lim_{n \to \infty} H(\xi_n, \eta_n) \geq 0 \), which ends the proof.

\[
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