Corrigendum: On Schrödinger systems with cubic dissipative nonlinearities of derivative type (2016 Nonlinearity 29 1537–63)

Chunhua Li\textsuperscript{1} and Hideaki Sunagawa\textsuperscript{2,3}

\textsuperscript{1} Department of Mathematics, College of Science, Yanbian University, 977 Gongyuan Road, Yanji, Jilin Province, 133002, People’s Republic of China
\textsuperscript{2} Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan

E-mail: sxlch@ybu.edu.cn and sunagawa@math.sci.osaka-u.ac.jp

Received 15 August 2016
Accepted for publication 30 September 2016
Published 14 October 2016

Recommended by Dr Jean-Claude Saut
Mathematics Subject Classification numbers: 35Q55, 35B40

In this note, we correct errors which have occurred in the proof of lemma 5.2 in [1]. There is no change in the statement of this lemma, so this correction does not affect the main results or any other parts in [1].

(A) On p 1553 in [1], we mentioned the following estimate for \(r_0\):
\[
\|r_0\|_{L^\infty} \lesssim C t^{-1/4} \|u_1\|_{H^3}\|u_2\|_{H^3}\|u_3\|_{H^3}.
\]

This estimate is incorrect and should be replaced with
\[
\|r_0\|_{L^\infty} \lesssim C t^{-1/4} \sum_{k=1}^3 (\|u_k\|_{H^3} + \|J_{m_k} u_k\|_{H^3}). \tag{0.1}
\]

For the proof of (0.1), we first remember that \(r_0\) can be written as
\[
r_0 = (W_{m_1}^{-1} - 1)(W_{m_2} \alpha_1^{(1)} W_{m_2} \overline{\alpha_2^{(1)}} W_{m_3} \alpha_3^{(1)}) + (W_{m_1}^{-1} \alpha_1^{(1)} W_{m_2} \overline{\alpha_1^{(1)}} W_{m_3} \alpha_3^{(1)}) + \alpha_1^{(1)} (W_{m_2}^{-1} \alpha_1^{(1)} W_{m_3} \alpha_3^{(1)}) + \alpha_1^{(1)} \alpha_2^{(1)} W_{m_1}^{-1} \alpha_3^{(1)}.
\]

We will estimate each term of the expression above. By using the inequalities

\textsuperscript{3} Author to whom any correspondence should be addressed.
\[ \| \mathcal{W}_m^{1} \mathcal{F}_m \phi \|_{H^s} \leq C \| \mathcal{U}_m^{1} \phi \|_{H^{s+1}} \leq C (\| \phi \|_{L^2} + \| J_m \phi \|_{L^2}) \quad (0.2) \]

and
\[ \| (\mathcal{W}_m^{1} - 1) \phi \|_{L^\infty} \leq C r^{-1/4} \| \phi \|_{H^s} \quad (0.3) \]

(see (3.4) in [1]), we have
\[
\begin{aligned}
\| (\mathcal{W}_m^{1} - 1) \mathcal{W}_m \alpha_1 \mathcal{W}_m \alpha_2 \mathcal{W}_m \alpha_3 \|_{L^\infty} & 
\leq C r^{-1/4} \| \mathcal{W}_m \alpha_1 \|_{H^s} \| \mathcal{W}_m \alpha_2 \|_{H^s} \| \mathcal{W}_m \alpha_3 \|_{H^s} \\
\leq C r^{-1/4} \| \alpha_1 \|_{H^s} \| \alpha_2 \|_{H^s} \| \alpha_3 \|_{H^s} \\
\leq C r^{-1/4} \prod_{k=1}^{3} (\| u_k \|_{H^s} + \| J_m u_k \|_{H^s}).
\end{aligned}
\]

As for the second term, we use (0.2), (0.3) and the Sobolev imbedding \( H^1(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \) to obtain
\[
\begin{aligned}
\| (\mathcal{W}_m^{1} - 1) \mathcal{W}_m \alpha_1 \mathcal{W}_m \alpha_2 \mathcal{W}_m \alpha_3 \|_{L^\infty} & 
\leq C \| (\mathcal{W}_m^{1} - 1) \alpha_1 \|_{L^\infty} \| \mathcal{W}_m \alpha_2 \|_{H^s} \| \mathcal{W}_m \alpha_3 \|_{H^s} \\
\leq C r^{-1/4} \| \alpha_1 \|_{H^s} \| \alpha_2 \|_{H^s} \| \alpha_3 \|_{H^s} \\
\leq C r^{-1/4} \prod_{k=1}^{3} (\| u_k \|_{H^s} + \| J_m u_k \|_{H^s}).
\end{aligned}
\]

The third and fourth term can be estimated in the same way. Piecing them together, we arrive at (0.1).

(B) By the same reason as above, the estimates for \( r_1 \) on p 1554 and for \( r_2 \) on p 1555 should be corrected as
\[
\| r_1 \|_{L^\infty} \leq C r^{-1/4} \prod_{k=1}^{3} (\| u_k \|_{H^s} + \| J_m u_k \|_{H^s})
\]

and
\[
\| r_2 \|_{L^\infty} \leq C r^{-1/4} \prod_{k=1}^{3} (\| u_k \|_{H^s} + \| J_m u_k \|_{H^s}),
\]

respectively.

**Acknowledgments**

The authors thank Daisuke Sakoda and Yuji Sagawa for pointing out these errors.

**Reference**

[1] Li C and Sunagawa H 2016 On Schrödinger systems with cubic dissipative nonlinearities of derivative type Nonlinearity 29 1537–63
On Schrödinger systems with cubic dissipative nonlinearities of derivative type

Chunhua Li\textsuperscript{1} and Hideaki Sunagawa\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, College of Science, Yanbian University, 977 Gongyuan Road, Yanji, Jilin Province 133002, People’s Republic of China
\textsuperscript{2} Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan

E-mail: sxlch@ybu.edu.cn and sunagawa@math.sci.osaka-u.ac.jp

Received 17 June 2015, revised 9 January 2016
Accepted for publication 3 March 2016
Published 29 March 2016

Recommended by Dr Jean-Claude Saut

Abstract
Consider the initial value problem for systems of cubic derivative nonlinear Schrödinger equations in one space dimension with the masses satisfying a suitable resonance relation. We give structural conditions on the nonlinearity under which the small data solution gains an additional logarithmic decay as $t \to +\infty$ compared with the corresponding free evolution.

Keywords: derivative nonlinear Schrödinger systems, nonlinear dissipation, logarithmic time-decay
Mathematics Subject Classification: 35Q55, 35B40

1. Introduction
Consider the initial value problem for the system of nonlinear Schrödinger equations of the following type:

\begin{align}
\mathcal{L}_m u_j &= F_j(u, \partial_x u), \quad t > 0, \ x \in \mathbb{R}, \ j = 1, \ldots, N, \\
u_j(0, x) &= \varphi_j(x), \quad x \in \mathbb{R}, \ j = 1, \ldots, N,
\end{align}

where $\mathcal{L}_m = i\partial_t + \frac{1}{2m_j} \partial_x^2$, $i = \sqrt{-1}$, $m_j \in \mathbb{R} \setminus \{0\}$, and $u = (u_j(t, x))_{1 \leq j \leq N}$ is a $\mathbb{C}^N$-valued unknown function. The nonlinear term $F = (F_j)_{1 \leq j \leq N}$ is always assumed to be a cubic homogeneous polynomial in $(u, \partial_x u, u_x, \partial_x^2 u)$. Our main interest is how the combinations of $(m_j)_{1 \leq j \leq N}$ and the structures of $(F_j)_{1 \leq j \leq N}$ affect large-time behavior of the solution $u$ to (1.1). Before going into details, let us first recall some known results briefly and clarify our motivation.
We begin with the single case \((N = 1)\). One of the most typical nonlinear Schrödinger equations appearing in various physical settings is

\[
 i\partial_t u + \frac{1}{2} \partial_x^2 u = \lambda |u|^2 u, \quad t > 0, \ x \in \mathbb{R}
\]  

(1.2)

with \(\lambda \in \mathbb{R}\). What is interesting in (1.2) is that the large-time behavior of the solution is actually affected by the nonlinearity even if the initial data is sufficiently small, smooth and decaying fast as \(|x| \to \infty\). To be more precise, it is shown in [7] that the solution to (1.2) with small initial data behaves like

\[
 u(t, x) = \frac{1}{\sqrt{it}} \alpha(x/t) e^{i \left( \frac{\lambda}{2} x^2 - \lambda |\alpha(x/t)|^2 \log t \right)} + o(t^{-1/2}) \quad \text{as} \quad t \to \infty
\]

with a suitable \(C\)-valued function \(\alpha\). An important consequence of this asymptotic expression is that the solution decays like \(O(t^{-1/2})\) in \(L^\infty(\mathbb{R})\), while it does not behave like the free solution unless \(\lambda = 0\). In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension.

This result is extended in [12] to the case where the nonlinearity depends also on \(\partial_x u\). If \(\lambda \in \mathbb{C}\) in (1.2), another kind of long-range effect can be observed. Indeed, it is verified in [32] that the small data solution to (1.2) decays like \(O(t^{-1/2} \log t)^{-1/2})\) in \(L^\infty(\mathbb{R})\) as \(t \to \infty\) if \(\text{Im} \lambda < 0\). This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect (see also [36], [26, 27, 29], etc, for related works). This result is generalized in [11] to the derivative nonlinear case:

\[
 i\partial_t u + \frac{1}{2} \partial_x^2 u = G(u, \partial_x u), \quad t > 0, \ x \in \mathbb{R},
\]  

(1.3)

where \(G\) is a cubic homogeneous polynomial in \((u, \partial_x u, \bar{u}, \bar{\partial_x u})\) with complex coefficients, and satisfies the gauge invariance

\[
 G(e^{i\theta}v, e^{i\theta}w) = e^{i\theta}G(v, w), \quad \theta \in \mathbb{R}, \ (v, w) \in \mathbb{C} \times \mathbb{C}.
\]  

(1.4)

It is shown in [11] that the solution to (1.3) decays like \(O(t^{-1/2} \log t)^{-1/2})\) in \(L^\infty(\mathbb{R})\) as \(t \to \infty\) if

\[
 \sup_{\xi \in \mathbb{R}} \text{Im} \ G(1, i\xi) < 0.
\]  

(1.5)

Remark that (1.4) and (1.5) are satisfied by \(G = \lambda |u|^2 u\) if \(\text{Im} \lambda < 0\). Next let us turn our attention to the system case \((N \geq 2)\). Recently, a lot of efforts have been made for the study on systems of nonlinear Schrödinger equations (see e.g. [2, 5, 6, 28, 13], [30, 18, 17, 14, 24], etc). An interesting feature in the system case is that the behavior of solutions are affected by the combinations of the masses as well as the structure of the nonlinearity. Note that similar phenomena can be observed in nonlinear Klein–Gordon systems (see e.g. [33, 3, 34, 23, 20]). For nonlinear Schrödinger systems, an additional logarithmic decay result is first obtained by [18]. Strictly saying, two-dimensional quadratic nonlinear Schrödinger systems are treated in [18], but we can adopt the method of [18] directly to one-dimensional cubic nonlinear Schrödinger systems, as pointed in [24]. When we restrict ourselves to a two-component model

\[
 \begin{cases}
 L_{m_1} u_1 = \lambda_1 |u_1|^2 u_1 + \nu_1 \bar{u}_1^2 u_2, \\
 L_{m_2} u_2 = \lambda_2 |u_2|^2 u_2 + \nu_2 \bar{u}_2^2 u_1,
 \end{cases}
\]  

\[
 t > 0, \ x \in \mathbb{R}
\]  

(1.6)

with \(\lambda_1, \lambda_2, \nu_1, \nu_2 \in \mathbb{C}\) and \(m_1, m_2 \in \mathbb{R} \setminus \{0\}\), then the result of [18] can be read as follows: the solution to (1.6) decays like \(O(t^{-1/2} \log t)^{-1/2})\) in \(L^\infty(\mathbb{R})\) as \(t \to \infty\) if
\[ m_2 = 3m_1, \quad (1.7) \]
\[ \text{Im } \lambda_j < 0, \quad j = 1, 2, \quad (1.8) \]
and
\[ \kappa_1 \mu_1 = \kappa_2 \mu_2 \quad \text{with some } \kappa_1, \kappa_2 > 0 \quad (1.9) \]
(see example 2.1 in [24] for the detail). However, it should be pointed out that the approach of [18] is not directly applicable to the derivative nonlinear case, because we need suitable pointwise \textit{a priori} estimates not only for the solution itself but also for its derivatives without breaking good structure in order to apply the method of [18].

The aim of this paper is to introduce structural conditions on \((F_j)_{j \in \mathbb{N}}\) and \((m_j)_{j \in \mathbb{N}}\) under which the small data solution to the derivative nonlinear Schrödinger system (1.1) gains an additional logarithmic decay as \(t \to +\infty\) compared with the corresponding free evolution. We intend to give a unified approach of the previous results of [11] for scalar derivative nonlinear case and [18] for the case of systems without derivatives in the nonlinearity. Another novelty of the present work is that our results cover not only for \(L^\infty\)-decay but also for \(L^2\)-decay of solutions. Moreover, a condition for asymptotically free behavior is also presented which can be comparable to the Klein–Gordon case.

2. Main results

In the subsequent sections, we will use the following notations: We set \(I_N = \{1, \ldots, N\}\) and \(I'_N = \{1, \ldots, N, N+1, \ldots, 2N\}\). For \(z = (z_j)_{j \in \mathbb{N}} \in \mathbb{C}^N\), we write
\[ z^0 = (z^0_k)_{k \in I'_N} := (z_1, \ldots, z_N, \overline{z}_1, \ldots, \overline{z}_N) \in \mathbb{C}^{2N}. \]
Then general cubic nonlinear term \(F = (F_j)_{j \in I'_N}\) can be written as
\[ F_j(u, \partial_x u) = \sum_{k, l, k', l' = 0}^1 \sum_{k, k', k'' \in I'_N} C_{j, k, k', k''}^{l, l', l''} (\partial_x^{l_1} u_k) (\partial_x^{l''} u_{k''}) \]
with suitable \(C_{j, k, k', k''}^{l, l', l'', \nu} \in \mathbb{C}\). With this expression of \(F\), we define \(p = (p_j(\xi; Y))_{j \in I'_N} : \mathbb{R} \times \mathbb{C}^N \to \mathbb{C}^N\) by
\[ p_j(\xi; Y) := \sum_{k, l, k', l' = 0}^1 \sum_{k, k', k'' \in I'_N} C_{j, k, k', k''}^{l, l', l''} (i\tilde{m}_k \xi)^{l_1} (i\tilde{m}_{k'} \xi)^{l''} Y_{k''}^{l''} Y_{k'}^{l'} Y_k^{l_1}, \]
for \(\xi \in \mathbb{R}\) and \(Y = (Y_j)_{j \in I'_N} \in \mathbb{C}^N\), where
\[ \tilde{m}_k = \begin{cases} m_k & (k = 1, \ldots, N), \\ -m_{2N-k} & (k = N+1, \ldots, 2N). \end{cases} \]
In what follows, we denote by \(\langle \cdot, \cdot \rangle_{\mathbb{C}^N}\) the standard scalar product in \(\mathbb{C}^N\), i.e.
\[ \langle z, w \rangle_{\mathbb{C}^N} = \sum_{j=1}^N z_j w_j \]
for \(z = (z_j)_{j \in I'_N}\) and \(w = (w_j)_{j \in I'_N} \in \mathbb{C}^N\).
Now let us introduce the following conditions:

(a) For all \( j \in I \) and \( k_1, k_2, k_3 \in \mathbb{N} \),
\[
m_j = m_{k_1} + m_{k_2} + m_{k_3},
\]
implies \( C_{j, k_1, k_2, k_3} = 0 \), \( l_1, l_2, l_3 \in \{0, 1\} \).

(b_0) There exists an \( N \times N \) positive Hermitian matrix \( A \) such that
\[
\text{Im}(p(\xi; Y), AY)_{C^0} \leq 0
\]
for all \( \xi, Y \in \mathbb{R} \times \mathbb{C}^N \).

(b_1) There exist an \( N \times N \) positive Hermitian matrix \( A \) and a positive constant \( C \), such that
\[
\text{Im}(p(\xi; Y), AY)_{C^0} \leq -C|Y|^4
\]
for all \( \xi, Y \in \mathbb{R} \times \mathbb{C}^N \).

(b_2) There exist an \( N \times N \) positive Hermitian matrix \( A \) and a positive constant \( C^* \), such that
\[
\text{Im}(p(\xi; Y), AY)_{C^0} \leq -C^*|\xi|^2|Y|^4
\]
for all \( \xi, Y \in \mathbb{R} \times \mathbb{C}^N \), where \( (\xi) = \sqrt{1 + \xi^2} \).

To state the main results, we introduce some function spaces. For \( s, \sigma \in \mathbb{Z}_{\geq 0} \), we denote by \( H^s \) the \( L^2 \)-based Sobolev space of order \( s \), and the weighted Sobolev space \( \sigma H^s \), is defined by
\[
\{ \phi \in L^2 | \langle \xi \rangle^\sigma \phi \in H^s \}
\]
equipped with the norm \( \|\phi\|_{H^s} = \|\langle \xi \rangle^\sigma \phi\|_{L^2} \). The main results are as follows:

**Theorem 2.1.** Assume the conditions (a) and (b_0) are satisfied. Let \( \varphi = (\varphi_j)_{j \in I} \in H^3 \cap H^2, \) and assume \( \varepsilon := \|\varphi\|_{H^3} + \|\varphi\|_{H^2} \) is sufficiently small. Then (1.1) admits a unique global solution \( u = (u_j)_{j \in I} \in C([0, \infty); H^3 \cap H^2). \) Moreover we have
\[
\|u(t)\|_{L^\infty} \leq \frac{C\varepsilon}{\sqrt{1 + t}}, \quad \|u(t)\|_{L^2} \leq C\varepsilon
\]
for \( t \geq 0 \), where \( C \) is a positive constant not depending on \( \varepsilon \).

**Theorem 2.2.** Assume the conditions (a) and (b_1) are satisfied. Let \( u \) be the global solution to (1.1), whose existence is guaranteed by theorem 2.1. Then we have
\[
\|u(t)\|_{L^\infty} \leq \frac{C\varepsilon}{\sqrt{(1 + t)(1 + \varepsilon^2 \log(2 + t))}}
\]
for \( t \geq 0 \), where \( C \) is a positive constant not depending on \( \varepsilon \). We also have
\[
\lim_{t \to +\infty} \|u(t)\|_{L^2} = 0.
\]

**Theorem 2.3.** Assume the conditions (a) and (b_2) are satisfied. Let \( u \) be as above. Then we have
\[
\|u(t)\|_{L^2} \leq \frac{C\varepsilon}{\sqrt{1 + \varepsilon^2 \log(2 + t)}}
\]
for \( t \geq 0 \), where \( C \) is a positive constant not depending on \( \varepsilon \).
Theorem 2.4. Assume the conditions (a) and (b) are satisfied. Let $u$ be as above. For each $j \in I_N$, there exists $\varphi_j^+ \in L^2(\mathbb{R})$ with $\hat{\varphi}_j^+ \in L^\infty(\mathbb{R})$ such that

$$u_j(t) = e^{\frac{t}{2m} \hat{\varphi}_j^+} + O(t^{-1/4 + \delta}) \quad \text{in } L^2(\mathbb{R})$$

and

$$u_j(t, x) = \sqrt{\frac{m_j}{|t|}} \varphi_j^+(\frac{m_j x^2}{t}) e^{\frac{m_j x^2}{2t}} + O(t^{-3/4 + \delta}) \quad \text{in } L^\infty(\mathbb{R}),$$

as $t \to +\infty$, where $\delta > 0$ can be taken arbitrarily small, and $\hat{\varphi}$ denotes the Fourier transform of $\varphi$, i.e.

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int \xi e^{-iy\xi} \phi(y) dy.$$

Remark 2.1. In view of the proof of theorem 2.4 below, we can see that $\varphi_+ = (\varphi_j^+) \in \mathcal{E}$ does not identically vanish if the initial data $\varphi$ is suitably small and does not identically vanish (see remark 6.1 for the detail). Therefore the solution does not gain an additional logarithmic decay under the conditions (a) and (b).

For the convenience of the readers, let us explain the origin of the function $p_j(\xi;Y)$ and give a heuristic explanation for the roles played by our conditions (a) and (b0)–(b3). First of all, let us recall the fact that for the solution $u_j^0$ to the free Schrödinger equation

$$\partial_t u_j + i u_j = -\sum_{s \neq j} H_{js}(u_j u_s) + \sum_{s \neq j} H_{js}(u_j u_s)$$

we have

$$\partial_t^0 u_j^0(t, x) \sim \left(\frac{im_j x}{t}\right)^i \sqrt{\frac{m_j}{|t|}} \hat{\varphi}_j\left(\frac{m_j x^2}{t}\right) e^{\frac{m_j x^2}{2t}} + \cdots$$

as $t \to +\infty$. Viewing it as a rough approximation of the solution $u_j$ for (1.1), we may expect that $\partial_t^0 u_j(t, x)$ could be better approximated by

$$\left(\frac{im_j x}{t}\right)^i Y_j\left(\log t, \frac{x}{t}\right) e^{\frac{m_j x^2}{2t}}$$

with a suitable function $Y_j = (Y_j(\tau, \xi))_{\xi \in \mathbb{R}^n}$, where $\tau = \log t$, $\xi = x/t$ and $t \gg 1$. Note that $Y_j(0, \xi) = \sqrt{|m_j|} \hat{\varphi}_j(m_j \xi)$ and that the extra variable $\tau = \log t$ is responsible for possible long-range nonlinear effect. Substituting the above expression into (1.1) and keeping only the leading terms, we can see (at least formally) that $Y_j$ should satisfy the ordinary differential equation

$$i\partial_\tau Y_j(\tau, \xi) = p_j(\xi;Y_j(\tau, \xi))$$

under the condition (a). This is where the function $p_j(\xi;Y)$ comes from. Note also that (a) plays the role of a resonance condition among the masses in the derivation of (2.2) (see [17] for similar observation in quadratic nonlinear case). Next let $A$ be a positive Hermitian matrix. Then (2.2) yields
\[ \partial_t \langle Y(\tau, \xi), AY(\tau, \xi) \rangle_{C^0} = 2 \text{Im}(p(\xi; Y(\tau, \xi)), AY(\tau, \xi))_{C^0}, \]

and the condition (b0) is just what makes this quantity non-positive. Since \( |Y|^2 \) and \( (Y, AY)_{C^0} \) are equivalent, the inequality \( \partial_t \langle Y(\tau, \xi), AY(\tau, \xi) \rangle_{C^0} \leq 0 \) implies that \( Y(\tau, \xi) \) remains bounded when \( \tau \) becomes large. Going back to the original variables, we see that the solution \( u(t, x) \) for (1.1) decays like \( O(t^{-1/2}) \) in \( L^\infty_x \) as \( t \to +\infty \) under (b0). Under the stronger condition (b1), we have

\[ \partial_t \langle Y(\tau, \xi), AY(\tau, \xi) \rangle_{C^0} \leq -2C_4 |Y(\tau, \xi)|^4 \lesssim -\langle Y(\tau, \xi), AY(\tau, \xi) \rangle_{C^0}, \]

which makes the solution decay strictly faster than the free evolution. It is also natural to regard (b2) as a stronger dissipative condition than (b1). On the other hand, (b3) makes the solution decay strictly faster than the free evolution. It is also natural to regard (b2) as a stronger dissipative condition than (b1).

Our strategy of the proof of theorems 2.1–2.4 is to justify the above heuristic argument. Let us give a more detailed summary of our approach. The key is to introduce

\[ \alpha(t, \xi) = \mathcal{F}_m[U_m(t)^{-1}u_0(t, \cdot)](\xi), \]

where \( \mathcal{F}_m \) and \( U_m(t) \) are given in section 3 below. Roughly speaking, this \( \alpha(t, \xi) \) is expected to play the role of \( Y(t, \xi) \). We will see in section 5.2 that \( \alpha = (\alpha(t, \xi))_{t \in \mathbb{R}} \) satisfies

\[ i\partial_t \alpha(t, \xi) = \frac{1}{t} \beta_j(\cdot; \alpha(t, \xi)) + O(t^{-5/4 + \gamma}) \]

and

\[ \|u_0(t)\|_{L^\infty} \lesssim t^{-1/2} \|\alpha(t)\|_{L^\infty} + O(t^{-3/4 + \gamma/2}) \]

with some \( 0 < \gamma \ll 1 \). To control the remainder terms, we need several \( L^2 \)-estimates involving the operator \( J_m = x + i\frac{\gamma}{m} \partial_x \) as well as smoothing properties of the Schrödinger equations. The global existence part of theorem 2.1 will be proved by means of an \textit{a priori} estimate combined with the standard local existence theorem. To derive the expected decay of \( \alpha \), we will apply an ODE lemma developed in [18, 25, 19] (see lemma 6.2 below).

Now let us give several examples which satisfy the conditions (a) and (b0)–(b3):

**Example 2.1.** The first example is taken from physics. The following system is derived in [15] and [16] to investigate nonlinear short pulse propagations in birefringent optical fiber:

\[
\begin{align*}
\begin{cases}
    i\partial_t U + \frac{1}{2} \partial_x^2 U + \lambda \left[ |U|^2 + \frac{2}{3} |V|^2 \right] U + \frac{1}{3} V^2 \bar{U} + i\gamma \partial_x \left[ \left( |U|^2 + \frac{2}{3} |V|^2 \right) U + \frac{1}{3} V^2 \bar{U} \right] &= 0, \\
    i\partial_t V + \frac{1}{2} \partial_x^2 V + \lambda \left[ |V|^2 + \frac{2}{3} |U|^2 \right] U + \frac{1}{3} U^2 \bar{V} + i\gamma \partial_x \left[ \left( |V|^2 + \frac{2}{3} |U|^2 \right) U + \frac{1}{3} U^2 \bar{V} \right] &= 0,
\end{cases}
\end{align*}
\]

where \( \lambda \) and \( \gamma \) are real constants. This system can be regarded as a special case of (1.1) by putting \( u_1 = U, u_2 = V, m_1 = m_2 = 1, N = 2 \) and

\[
\begin{align*}
F_1 &= -\lambda \left[ (|u_1|^2 + \frac{2}{3} |u_2|^2) u_1 + \frac{1}{3} |u_2|^2 \bar{u}_1 \right] - i\gamma \partial_x \left[ (|u_1|^2 + \frac{2}{3} |u_2|^2) u_1 + \frac{1}{3} |u_2|^2 \bar{u}_1 \right], \\
F_2 &= -\lambda \left[ (|u_2|^2 + \frac{2}{3} |u_1|^2) u_2 + \frac{1}{3} |u_1|^2 \bar{u}_2 \right] - i\gamma \partial_x \left[ (|u_2|^2 + \frac{2}{3} |u_1|^2) u_2 + \frac{1}{3} |u_1|^2 \bar{u}_2 \right].
\end{align*}
\]
For this system, we can easily check that (a) is satisfied and that
\[
\begin{align*}
p_1(\xi; Y) &= (-\lambda + \gamma \xi) \left[ (|Y_1|^2 + \frac{2}{3} |Y_2|^2) Y_1 + \frac{1}{3} Y_2^* Y_2 \right], \\
p_2(\xi; Y) &= (-\lambda + \gamma \xi) \left[ (|Y_2|^2 + \frac{2}{3} |Y_1|^2) Y_2 + \frac{1}{3} Y_1^* Y_1 \right],
\end{align*}
\]
whence (b0) is also satisfied with \( A = I \) (identity matrix).

**Example 2.2.** In the single case (i.e. \( N = 1 \)), we may assume \( m_1 = 1 \) without loss of generality. Then we can check that the condition (a) is equivalent to the gauge invariance (1.4), and that the condition (1.5) is equivalent to the condition (b1). Therefore our results above can be viewed as an extension of [11] except the explicit asymptotic profile of the solution. We can also see that our results cover the system (1.6) under the assumptions (1.7)–(1.9). Indeed, (1.7) plays the role of (a), and (1.8), (1.9) correspond to (b1) with \( \kappa = A_0 \).

**Example 2.3.** Next let us consider the following two-component system
\[
\begin{align*}
L_{m_1} u_1 &= \lambda_1 |u_1|^2 u_1 + \lambda_2 m_1 (\partial_s u_1)^2 + i u_2 \partial_s (m_2^2), \\
L_{m_2} u_2 &= \lambda_1 |u_2|^2 \partial_s u_2 - i (|u_2|^2 + |\partial_s u_2|^2) u_2 - i u_1 \partial_s u_1
\end{align*}
\]
with \( \lambda_1, \lambda_2, \kappa \in \mathbb{C} \) and \( m_1, m_2 \in \mathbb{R} \setminus \{0\} \), which is a bit more complicated than (1.6). It is easy to check that the condition (a) is satisfied by this system when \( m_2 = 3m_1 \). Also it follows from simple calculations that
\[
\begin{align*}
p_1(\xi; Y) &= (\lambda_1 - \lambda_2 m_2^2 \xi^2) |Y_1|^2 Y_1 + 2m\xi Y_1 Y_2, \\
p_2(\xi; Y) &= i(3\lambda_2 m_2 \xi - 1 - 9m_2^2 \xi^2) |Y_2|^2 Y_2 + 3m\xi Y_1^2
\end{align*}
\]
when \( (m_1, m_2) = (m, 3m) \). With \( A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \), we have
\[
\langle p(\xi; Y), AY \rangle = 3(\lambda_1 - \lambda_2 m_2^2 \xi^2) |Y_1|^4 - 2i(1 - 3\lambda_2 m_2 \xi + 9m_2^2 \xi^2) |Y_2|^4 + 12m\xi \text{Re}(Y_1^3 Y_2),
\]
whence
\[
\text{Im}(p(\xi; Y), AY) = 3(\text{Im} \lambda_1 - \text{Im} \lambda_2 m_2^2 \xi^2) |Y_1|^4 - \left\{ 2 - \frac{(\text{Re} \lambda_1)^2}{2} + 2\left(3m\xi - \frac{\text{Re} \lambda_3}{2}\right)^2 \right\} |Y_2|^4.
\]
Therefore we see that
- (b0) is satisfied if \( \text{Im} \lambda_1 \leq 0, \text{Im} \lambda_2 \geq 0 \) and \( |\text{Re} \lambda_3| \leq 2 \).
- (b1) is satisfied if \( \text{Im} \lambda_1 < 0, \text{Im} \lambda_2 \geq 0 \) and \( |\text{Re} \lambda_3| < 2 \).
- (b2) is satisfied if \( \text{Im} \lambda_1 < 0, \text{Im} \lambda_2 > 0 \) and \( |\text{Re} \lambda_3| < 2 \).

**Example 2.4.** Let us focus on the three-component system
\[
\begin{align*}
L_{m_1} u_1 &= u_2 \partial_s (\overline{m}_1 u_2), \\
L_{m_1} u_2 &= \overline{m}_1^2 \partial_s u_3 + 3 m_1 u_2 \partial_s (m_2^2), \\
L_{m_1} u_3 &= 2 m_1^2 \partial_s u_2 - u_2 \partial_s (u_1^2)
\end{align*}
\]
with $m_1 : m_2 : m_3 = 1 : 1 : 3$. We can immediately check that this system satisfies (a) and (b3). Note that this example should be compared with [17], where the null structure in quadratic derivative nonlinear Schrödinger systems in $\mathbb{R}^2$ is considered in detail (see also [21, 22, 35]). Analogous results in the Klein–Gordon case can be found in [3, 23] and [20].

**Example 2.5.** Finally, to indicate what happens when these conditions fail, let us focus on the very simple example

$$
\begin{align*}
\mathcal{L}_m u_1 &= 0, \\
\mathcal{L}_m u_2 &= cu_1^2 \partial_t u_1
\end{align*}
$$

(2.3)

with $c \in \mathbb{C} \setminus \{0\}$. Note that (a) is satisfied by this system only if $m_2 = 3m_1$, and that (b0) is never satisfied (unless $c = 0$). When $m_2 = 3m_1$, we can show by direct calculations that

$$
\|u_2(t)\|_{L^\infty} \gtrsim \varepsilon^2 \log t, \quad \|u_2(t)\|_{L^2} \gtrsim \varepsilon^3 \log t
$$

for $t \gg 1$ with a suitable choice of the data $\varphi$ satisfying $\|\varphi\|_{H^{1/4},H^{1/2}} = \varepsilon$ (see [30] for the detail). This tells us that (2.1) fails to hold without (b0) in general. On the other hand, when $m_2 \neq 3m_1$, it is possible to show that $u_2$ is asymptotically free by taking into account the oscillating factor caused by the non-resonance relation of the masses (see [33, 34] for the closely related works in the Klein–Gordon case). It should be noted that the presence of $\partial_\xi$ in the nonlinearity of (2.3) seems to be essential, whence the situation is slightly different from the Klein–Gordon case. This example suggests that another structure comes into play when (a) is violated. For the single Schrödinger equation ($N = 1$), some works concern non-gauge-invariant cubic derivative nonlinearities (see e.g. [8, 9, 31]). However, when $N \geq 2$, we have not been successful to treat this case in a unified way, and it is our next problem to be considered. A related issue will be discussed in a forthcoming paper.

The rest part of this paper is organized as follows: The next section is devoted to preliminaries on basic properties of the operator $J_m$. In section 4, we recall the smoothing properties of the linear Schrödinger equations. In section 5, we will get an $a$ priori estimate. After that, the main theorems will be proved in section 6 along the strategy mentioned above. The appendix is devoted to the proof of technical lemmas. In what follows, we will denote several positive constants by the same letter $C$, which is possibly different from line to line.

### 3. The operator $J_m$

We set $J_m = x + \frac{i}{m} \partial_\xi$ for non-zero real constant $m$. In this section, we collect several identities and inequalities related to $J_m$. This operator has a good compatibility with the Schrödinger equation including the mass $m$, and it is quite useful in our analysis. Our point of departure is the commutation relations $[\mathcal{L}_m, J_m] = 0$ and $[\partial_\xi, J_m] = 1$, where $[\cdot, \cdot]$ denotes the commutator of two linear operators. We also note that

$$
J_m \phi = \frac{i}{m} e^{im\xi^2 \xi} \partial_\xi \left( e^{-im\xi^2 \xi} \phi \right),
$$

(3.1)

which yields the following useful identities.
Lemma 3.1. Let \( m, \mu_1, \mu_2, \mu_3 \) be non-zero real constants satisfying \( m = \mu_1 + \mu_2 + \mu_3 \). We have
\[
J_m(f_1, f_2, f_3) = \frac{\mu_1}{m} (J_{\mu_1} f_1 f_2 f_3) + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2 f_3) + \frac{\mu_3}{m} f_1 f_2 (J_{\mu_3} f_3),
\]
\[
J_m(f_1 f_2 f_3) = \frac{\mu_1}{m} (J_{\mu_1} f_1) f_2 f_3 + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2) f_3 + \frac{\mu_3}{m} f_1 f_2 (J_{\mu_3} f_3),
\]
\[
J_m(f_1 f_2 f_3) = \frac{\mu_1}{m} (J_{\mu_1} f_1) f_2 f_3 + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2) f_3 + \frac{\mu_3}{m} f_1 f_2 (J_{\mu_3} f_3),
\]
\[
J_m(f_1 f_2 f_3) = \frac{\mu_1}{m} (J_{\mu_1} f_1) f_2 f_3 + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2) f_3 + \frac{\mu_3}{m} f_1 f_2 (J_{\mu_3} f_3)
\]
for smooth \( C \)-valued functions \( f_1, f_2 \) and \( f_3 \).

Proof. We set \( \theta = x^2/(2t) \). It follows from (3.1) that
\[
m J_m(f_1, f_2, f_3) = it e^{i\mu_1 \theta} J_m(f_1, f_2, f_3) + i t e^{i\mu_2 \theta} J_m(f_1, f_2, f_3) + i t e^{i\mu_3 \theta} J_m(f_1, f_2, f_3)
\]
which gives the second identity. The other three identities can be shown in the same way. \( \square \)

Remark 3.1. The above identities can be viewed as the Leibniz rule for the operator \( J_m \) acting on the cubic terms satisfying the condition (a). This should be compared with the fact that, if \( m \neq \mu_1 + \mu_2 + \mu_3 \), we have
\[
J_m(f_1, f_2, f_3) = \frac{\mu_1}{m} (J_{\mu_1} f_1) f_2 f_3 + \frac{\mu_2}{m} f_1 (J_{\mu_2} f_2) f_3 + \frac{\mu_3}{m} f_1 f_2 (J_{\mu_3} f_3)
\]
and so on. The last term implies a loss of time-decay when we use \( J_m \) without (a).

Lemma 3.2. Let \( m, \mu_1, \mu_2 \) be non-zero real constants. We have
\[
\partial_x (f_1 f_2 f_3) = \frac{m}{\mu_1} (\partial_x f_1) f_2 f_3 + \frac{R_1}{t},
\]
and
\[
\partial_x^2 (f_1 f_2 f_3) = \frac{m^2}{\mu_1 \mu_2} (\partial_x f_1) (\partial_x f_2) f_3 + \frac{R_2}{t},
\]
where \( R_1 = -i m J_m f_2 f_3 \) and \( R_2 = \frac{i m^2}{\mu_1} (\partial_x f_1) f_2 f_3 + \frac{i m^2}{\mu_1} (\partial_x f_1) (\partial_x f_2) f_3 + \partial_x R_1 \).
Remark 3.2. We do not assume any relations among $\mu_1$, $\mu_2$ and $m$ in lemma 3.2.

Proof. From the relation $-\frac{1}{m} \partial_t - \frac{1}{t} J_m = i \partial_x$, we see that

$$\frac{1}{m} \partial_x(f_1 f_2 f_3) - \frac{1}{t} J_m(f_1 f_2 f_3) = i \frac{x}{t} f_1 f_2 f_3 = \left( \frac{1}{\mu_1} \partial_x f_1 - \frac{1}{t} J_{\mu_1} f_1 \right) f_2 f_3,$$

which yields (3.2). We also have (3.3) by using (3.2) twice. □

Next we set

$$(\mathcal{U}_m(t) \phi)(x) := e^{\frac{ix^2}{2m}} \phi(x) = \frac{\sqrt{m}}{2\pi} e^{-\frac{x^2}{4m}} \int e^{i(m(x-y)^2/2)} \phi(y)dy$$

for $m \in \mathbb{R} \setminus \{0\}$ and $t > 0$. We also introduce the scaled Fourier transform $\mathcal{F}_m$ by

$$(\mathcal{F}_m \phi)(\xi) := |m|^{1/2} e^{-i\frac{\xi^2}{4m}} \hat{\phi}(m\xi) = \frac{\sqrt{m}}{2\pi} e^{-i\frac{\xi^2}{4m}} \int e^{-im\xi y} \phi(y)dy,$$

as well as auxiliary operators

$$(\mathcal{M}_m(t) \phi)(x) := e^{\frac{x^2}{2m} t} \phi(x), \quad (\mathcal{D}(t) \phi)(x) := \frac{1}{\sqrt{t}} \phi \left( \frac{x}{t} \right), \quad \mathcal{W}_m(t) \phi := \mathcal{F}_m \mathcal{M}_m(t) \mathcal{F}_m^{-1} \phi,$$

so that $\mathcal{U}_m$ can be decomposed into $\mathcal{U}_m = \mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{M}_m = \mathcal{M}_m \mathcal{D} \mathcal{W}_m \mathcal{F}_m$. The following lemmas are well-known (see e.g. [7, 30]).

Lemma 3.3. Let $m$ be a non-zero real constant. We have

$$\|\phi - \mathcal{M}_m \mathcal{D} \mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} \leq C t^{-3/4} (\|\phi\|_{L^2} + \|\mathcal{J}_m \phi\|_{L^2})$$

and

$$\|\phi\|_{L^\infty} \leq t^{-1/2} \|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{L^\infty} + C t^{-3/4} (\|\phi\|_{L^2} + \|\mathcal{J}_m \phi\|_{L^2})$$

for $t \geq 1$.

Proof. For the convenience of the readers, we give the proof. By the relation $J_m = \mathcal{U}_m \mathcal{U}_m^{-1}$, we see that

$$\|\mathcal{F}_m \mathcal{U}_m^{-1} \phi\|_{H^s} \leq C \|\mathcal{U}_m^{-1} \phi\|_{H^{s+1}} \leq C (\|\phi\|_{L^2} + \|\mathcal{J}_m \phi\|_{L^2}).$$

Also it follows from the inequalities $\|\phi\|_{L^\infty} \leq \sqrt{2} \|\phi\|_{L^2}^{1/2} \|\partial_x \phi\|_{L^2}^{1/2}$ and $|e^{\theta} - 1| \leq C |\theta|^{1/2}$ that

$$\|(\mathcal{M}_m^{s+1} - 1) \phi\|_{L^\infty} \leq C \|(\mathcal{M}_m^{s+1} - 1) \mathcal{F}_m^{-1} \phi\|_{L^2}^{1/2} \|\partial_x \mathcal{M}_m^{s+1} (\mathcal{M}_m^{s+1} - 1) \phi\|_{L^2}^{1/2}$$

$$\leq C (t^{-1/2} \|\mathcal{F}_m^{-1} \phi\|_{H^{s+1}})^{1/2} \|\partial_x \phi\|_{L^2}^{1/2}$$

$$\leq C t^{-1/4} \|\phi\|_{H^s}. \quad (3.4)$$

Combining with the inequalities obtained above, we have
Using the result derived above, we also have
\[ \|\phi\|_{L^\infty} \leq \|M_m \mathcal{D} F_m U_m^{-1} \phi\|_{L^\infty} + \|\phi - M_m \mathcal{D} F_m U_m^{-1} \phi\|_{L^\infty} \leq t^{-1/2} \|F_m U_m^{-1} \phi\|_{L^1} + C t^{-3/4} (\|\phi\|_{L^2} + \|F_m \phi\|_{L^2}). \]

Lemma 3.4. Let \( m \) be a non-zero real constant. We have
\[ \|F_m U_m^{-1} (f_1 f_2 f_3)\|_{L^\infty} \leq C \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^\infty}. \]

Proof. By the relation \( F_m U_m^{-1} = W_m^{-1} \mathcal{D}^{-1} M_m^{-1} \) and the estimate \( \|W_m^{-1} \phi\|_{L^\infty} \leq C t^{1/2} \|\phi\|_{L^2} \), we have
\[ \|F_m U_m^{-1} (f_1 f_2 f_3)\|_{L^\infty} \leq C t^{1/2} \|D^{-1} M_m^{-1} (f_1 f_2 f_3)\|_{L^1} \leq C t^{1/2} \cdot t^{-1/2} \|D^{-1} f_1\|_{L^2} \|D^{-1} m^{-1} f_2\|_{L^2} \|D^{-1} f_3\|_{L^\infty} \leq C t^{1/2} \|f_1\|_{L^2} \|f_2\|_{L^2} \cdot t^{1/2} \|f_3\|_{L^\infty} \leq C \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^\infty}. \]

4. Smoothing properties

In this section, we recall smoothing properties of the linear Schrödinger equations. As is well-known, the standard energy method causes a derivative loss when the nonlinear term involves derivatives of the unknown functions. Smoothing effect is a useful tool to overcome this obstacle. Among various kinds of such techniques, we will follow the approach of [10], which is a version of the so-called positive commutator method whose original idea goes back to [4] (see also [1] and the references cited therein for the history and more information on this subject).

Let \( \mathcal{H} \) be the Hilbert transform, that is,
\[ \mathcal{H} \psi(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\psi(y)}{x - y} \, dy. \]

With a non-negative weight function \( \Phi(x) \) and a non-zero real constant \( m \), let us also define the operator \( S_{\Phi,m} \) by
\[ S_{\Phi,m} \psi(x) := \left\{ \cosh \left( \int_{-\infty}^x \Phi(y) \, dy \right) \right\} \psi(x) - i \text{sgn}(m) \left\{ \sinh \left( \int_{-\infty}^x \Phi(y) \, dy \right) \right\} \mathcal{H} \psi(x). \]

Note that \( S_{\Phi,m} \) is \( L^2 \)-automorphism and that both \( \|S_{\Phi,m}\|_{L^2 \to L^2} \), \( \|S_{\Phi,m}^{-1}\|_{L^2 \to L^2} \) are dominated by \( C \exp(\|\Phi\|_{L^1}) \). Roughly speaking, the operator \( S_{\Phi,m} \) is chosen so that
\[ [\mathcal{L}_m, S_{\Phi,m}] \simeq -\frac{i}{|m|} \Phi S_{\Phi,m} |\partial_x| + \text{‘harmless terms’}, \]

where $|\partial_x|$ is interpreted as the Fourier multiplier (see (A.1) below for the explicit form of the ‘harmless terms’). The first term in the right-hand side enables us to gain the half-derivative $|\partial_x|^{1/2}$. More precisely, we have the following:

**Lemma 4.1.** Let $m, \mu_1, \ldots, \mu_N$ be non-zero real constants. Let $v$ be a $\mathbb{C}$-valued smooth function of $(t, x)$, and let $w = (w_j)_{j \in I_n}$ be a $\mathbb{C}^N$-valued smooth function of $(t, x)$. We set $\Phi = \eta(|w|^2 + |\partial_x w|^2)$ with $\eta \geq 1$, and $S = S_{\Phi,m}$. Then we have

\[
\frac{d}{dt} \| Sv(t) \|_{L^2}^2 + \frac{1}{|m|} \int_{\mathbb{R}} \Phi(x) |S| |\partial_x|^{1/2} v(t,x)^2 \, dx 
\leq 2 \left( \| Sv(t), S \mathcal{L}_m v(t) \|_{L^2} + CB(t) \| v(t) \|_{L^2}^2 \right),
\]

where

\[
B(t) = e^{C \eta \mathbb{R}^2} \left\{ \eta \| w(t) \|_{W^{1,\infty}}^2 + \eta^2 \| w(t) \|_{W^{2,\infty}}^2 + \eta \sum_{k \in I_n} \| w_k(t) \|_{H^k} \| \mathcal{L}_m w_k(t) \|_{H^{1/2}} \right\}
\]

and the constant $C$ is independent of $\eta$. We denote by $W^{s,\infty}$ the $L^\infty$-based Sobolev space of order $s \in \mathbb{Z}_{\geq 0}$.

This lemma is essentially the same as lemma 2.1 in [10], although we need slight modifications to fit for our purpose. For the convenience of the readers, we will give the proof of this lemma in the appendix.

By using lemma 4.1 combined with the following auxiliary lemma, we can get rid of the derivative loss coming from the nonlinear terms, as we will do in section 5.1 below.

**Lemma 4.2.** Let $m_1, \ldots, m_N$ be non-zero real constants. Let $v = (v_j)_{j \in I_n}$, $w = (w_j)_{j \in I_n}$ be $\mathbb{C}^N$-valued smooth functions of $x \in \mathbb{R}$. Suppose that $q_{1,j,k}$ and $q_{2,j,k}$ are quadratic homogeneous polynomials in $(w, \partial_x w, \overline{w}, \overline{\partial_x w})$. We set $\Phi = \eta(|w|^2 + |\partial_x w|^2)$ with $\eta \geq 1$, and $S_j = S_{\Phi,m_j}$ for $j \in I_n$. Then we have

\[
\sum_{j \in I_n} \left( \| S v_j, S_j (q_{1,j,k} \partial_x v_k) \|_{L^2} + \| S v_j, S_j (q_{2,j,k} \overline{\partial_x v_k}) \|_{L^2} \right) 
\leq \frac{C e^{Co \mathbb{R}^2}}{\eta} \sum_{k \in I_n} \int_{\mathbb{R}} \Phi(x) |S_k| |\partial_x|^{1/2} v_k(x)^2 \, dx 
\leq C e^{Co \mathbb{R}^2} \left( 1 + \eta^2 \| w \|_{H^1}^2 + \eta^2 \| w \|_{H^{2,\infty}}^2 \right) \| w \|_{W^{2,\infty}}^2 \| v \|_{L^2}^2,
\]

where the constant $C$ is independent of $\eta$.

We skip the proof of lemma 4.2 because this is nothing more than a paraphrase of lemma 2.3 in [10].

**5. A priori estimate**

This section is devoted to getting an *a priori* estimate for the solution to (1.1), which will play an important role in the proof of the global existence part of theorem 2.1 (see section 6.1
below). Let $T \in (0, +\infty)$, and let $u = (u_j)_{j \in \mathbb{N}} \in C([0, T); H^2 \cap H^{2,1})$ be a solution to (1.1) for $t \in [0, T)$. We set $\alpha_j(t, \xi) = \mathcal{F}_n^{-1} u_j(t, \cdot)(\xi)$, $\alpha(t, \xi) = (\alpha_j(t, \xi))_{j \in \mathbb{N}}$, and define

$$E(T) = \sup_{0 \leq t < T} \sum_{j \in \mathbb{N}} \left(1 + t \right)^2 \left( \|u_j(t)\|_{H^1} + \|J_{m_j}u_j(t)\|_{H^2} + \sup_{\xi \in \mathbb{R}} (\xi^2 |\alpha_j(t, \xi)|) \right)$$

with $\gamma > 0$. We are going to show the following:

**Lemma 5.1.** Assume the conditions (a) and (b) are satisfied. Let $\gamma \in (0, 1/4)$. There exist positive constants $\varepsilon_1$ and $K$, not depending on $T$, such that

$$E(T) \leq \varepsilon^{2/3}$$  \hspace{1cm} (5.1)

implies

$$E(T) \leq K\varepsilon,$$

provided that $\varepsilon = \|\varphi\|_{H^1 \cap H^{2,1}} \leq \varepsilon_\ell$.

The proof of this lemma will be divided into two parts.

### 5.1. $L^2$-estimates

In the first part, we consider the bounds for $\|u_j(t)\|_{H^1}$ and $\|J_{m_j}u_j(t)\|_{H^2}$. It is enough to show

$$\sum_{j \in \mathbb{N}} \sum_{l=0}^{1} \|J_{m_j}^l u_j(t)\|_{L^2}^2 \leq C \varepsilon + C \varepsilon^2 (1 + t)^{2/3}$$  \hspace{1cm} (5.2)

and

$$\sum_{j \in \mathbb{N}} \sum_{l=0}^{1} \|\partial_x^3 J_{m_j}^l u_j(t)\|_{L^2}^2 \leq C \varepsilon^2 (1 + t)^{2/3}$$  \hspace{1cm} (5.3)

for $t \in [0, T)$ under the assumption (5.1). First we remark that (5.1) implies a rough $H^1$-bound

$$\|u_j(t)\|_{H^1} \leq C \|\alpha_j(t)\|_{H^{1,1}} \leq C \left( \int_{\mathbb{R}} \frac{d\xi}{(\xi^2 \|\alpha_j(t, \xi)\|)^{1/2}} \right)^{1/2} \sup_{\xi \in \mathbb{R}} (\xi^2 |\alpha_j(t, \xi)|) \leq C \varepsilon^{2/3}$$  \hspace{1cm} (5.4)

for $t \in [0, T)$. We also deduce from (5.1) that

$$\|u_j(t)\|_{W^{2,\infty}} \leq \frac{C \varepsilon^{2/3}}{(1 + t)^{1/2}}$$

for $t \in [0, T)$. Indeed, it follows from lemma 3.3 and the relation $[\partial_x, J_{m_j}] = 1$ that

$$\|u_j(t)\|_{W^{2,\infty}} \leq C \sup_{\xi \in \mathbb{R}} (\xi^2 |\alpha_j(t, \xi)|) + \frac{C}{t^{1/2}} (\|u_j(t)\|_{H^1} + \|J_{m_j}u_j(t)\|_{H^2}) \leq \frac{C \varepsilon^{2/3}}{t^{1/2}}$$

for $t \geq 1$, and $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yields $\|u_j(t)\|_{W^{2,\infty}} \leq C \|u_j(t)\|_{H^1} \leq C \varepsilon^{2/3}$ for $t \leq 1$. 


Now we consider the easier estimate (5.2). It follows from the standard energy method that
\[ \frac{d}{dt} \|u(t)\|_{L^2}^2 \leq \|F_j(u(t), \partial_t u(t))\|_{L^2}^2 \]
\[ \leq C \|u(t)\|_{W^{2,\infty}}^2 \|u(t)\|_{H^1} \]
\[ \leq C \left( \frac{\varepsilon^{2/3}}{(1+t)^{1/2}} \right)^2 \cdot C \varepsilon^{2/3} \]
\[ \leq C \varepsilon^2 + \frac{1}{1+t}. \]

Also we see from lemma 3.1 that
\[ L_m J_m u_j = \sum_{k \in I_k} (q_1_{jk} J_m \partial_{t} u_k + q_2_{jk} J_m \partial_{x} u_k + q_3_{jk} J_m u_k + q_4_{jk} J_m u_k), \]
where \( q_{1,jk}, \ldots, q_{4,jk} \) are quadratic homogeneous polynomials in \( (u, \partial_t u, \vec{u}, \overline{\partial_t u}) \). Then the standard energy method again implies
\[ \frac{d}{dt} \|J_m u_j(t)\|_{L^2}^2 \leq C \|u\|_{W^{2,\infty}}^2 \sum_{k \in I_k} (\|u_k\|_{H^1} + \|J_m u_k\|_{H^1}) \leq C \varepsilon^2 \frac{1}{1+t}. \]

These lead to (5.2).

Next we consider (5.3). We set \( v_{ij} = \partial_x^{-1} J_m^l u_j \) for \( l \in \{0, 1 \} \) and \( j \in I_N \). We apply lemma 4.1 with \( m = m_j, \mu_k = m_k, v = v_{ij}, w = u, \eta = \varepsilon^{-2/3}, \) then we obtain
\[ \frac{d}{dt} \|S v_{ij}(t)\|_{L^2}^2 + \frac{1}{|m_j|} \int R \Phi(t, x) |S_j |\partial_{x} |^2 v_{ij}(t)|\) \]
\[ \leq 2 \left( \sum_{l} (S v_{ij}, S_j \partial_{x}^{-1} J_m^l F_j(u, \partial_t u))_{L^2} \right) + CB(t) \|v_{ij}(t)\|_{L^2}^2. \]

where
\[ B(t) = \varepsilon^{2} \|u\|_{L^2}^2 \left( \int e^{2/3} \|u\|_{L^{2, \infty}}^2 \right) \]
\[ \leq \frac{C \varepsilon^{2/3}}{1+t}. \]

To estimate the first term of the right-hand side of (5.5), we use lemma 3.1 and the usual Leibniz rule to split \( \partial_x^{-1} J_m^l F_j(u, \partial_t u) \) into the following form:
\[ \sum_{k \in I_k} (g_{1,jl} \partial_{x} u_k + g_{2,jl} \overline{\partial_{x} u_k}) + h_{jl}, \]
where \( g_{1,jl} \) and \( g_{2,jl} \) are quadratic homogeneous polynomials in \( (u, \partial_t u, \vec{u}, \overline{\partial_t u}) \), and \( h_{jl} \) is a cubic term satisfying
\[ \|h_{jl}\|_{L^2} \leq C \|u(t)\|_{W^{2,\infty}}^2 \sum_{k \in I_k} (\|u_k(t)\|_{H^1} + \|J_m u_k(t)\|_{H^1}) \leq \frac{C \varepsilon^{2}}{(1+t)^{1-\gamma/3}}. \]

Then lemma 4.2 and the \( L^2 \)-automorphism of \( S_j \) lead to
\[ \sum_{j \in I} \left| \int S_y \partial_j \partial_x^{-1} J_{m}^{I} F_j (u, \partial_x u) \right| L^2 \]
\[ \leq \sum_{j, k \in I} \left( \left| \int S_y \partial_j \partial_x^{-1} J_{m}^{I} F_j (u, \partial_x u) \right| L^2 + \left| \int S_y \partial_j \partial_x^{-1} J_{m}^{I} F_j (u, \partial_x u) \right| L^2 \right) + \sum_{j \in I} \| S_y \partial_j \| L^2 \]
\[ \leq C e^{2/3} C e^{2/3} \| u \|_{H^1} \sum_{k \in I} \int_{R} \Phi (t, x) \left| S_{k} \partial_x^{-1/2} v_{k} (t, x) \right| dx \]
\[ + C e^{2/3} \| u \|_{H^1} (1 + e^{-4/3}) \| u \|_{H^1} + e^{-4/3} \| u \|_{H^1} ) \| u \|_{H^2} \sum_{k \in I} \| v_{k} \| L^2 \]
\[ \leq C e^{2/3} \sum_{k \in I} \int_{R} \Phi (t, x) \left| S_{k} \partial_x^{-1/2} v_{k} (t, x) \right| dx + \frac{C e^{2/3}}{(1 + t)^{1 - 2 \gamma / 3}} \]
with some positive constant \( C_0 \) not depending on \( \varepsilon \). Summing up, we obtain
\[ \frac{d}{dt} \sum_{j \in I} \| S_y \partial_j (t) \| L^2 \leq \sum_{k \in I} \left( 2 C e^{2/3} - \frac{1}{|m_k|} \right) \int_{R} \Phi (t, x) \left| S_{k} \partial_x^{-1/2} v_{k} (t, x) \right| dx \]
\[ + \frac{C e^{2/3}}{(1 + t)^{1 - 2 \gamma / 3}} + \frac{C e^{2/3}}{(1 + t)^{1 - 2 \gamma / 3}} \]
\[ \leq \frac{C e^{2/3}}{(1 + t)^{1 - 2 \gamma / 3}} , \]
provided that
\[ 2 C e^{2/3} \leq \frac{1}{\min_{l \in \mathcal{N}} |m_l|} . \]
Integrating with respect to \( t \), we have
\[ \sum_{j \in I} \| S_y \partial_j (t) \| L^2 \leq C e^{2} + C e^{2} (1 + t)^{2 \gamma / 3} \leq C e^{2} (1 + t)^{2 \gamma / 3} , \]
whence
\[ \sum_{j \in I} \| \partial_j^{-1} J_{m}^{I} F_j (t) \| L^2 \leq C e^{2} (1 + t)^{2 \gamma / 3} , \]
as required. \( \square \)

### 5.2. Estimates for \( \alpha \)

In the second part, we are going to show \( \langle \xi \rangle^2 |\alpha(t, \xi) | \leq C e \) for \( (t, \xi) \in [0, T] \times \mathbb{R} \) under the assumption (5.1). If \( t \in [0, 1] \), the Sobolev imbedding yields this estimate immediately. Hence we have only to consider the case of \( t \in [1, T] \). We set
\[ \rho (t, \xi) = \mathcal{F} J_{m}^{I} \mathcal{L}^{-1} \left[ F_j (u, \partial_x u) - \frac{1}{t} \right] (\xi; \alpha(t, \xi)) \]
and \( \rho = (\rho_j)_{j \in I_N} \), so that

\[
i\partial_t \alpha(t, \xi) = \mathcal{F}_{m N}^{-1} [\mathcal{L}_m u_j] = \mathcal{F}_{m N}^{-1} [F_j(u, \partial_x u)] = \frac{1}{t} \rho_j(\xi; \alpha(t, \xi)) + \rho_j(t, \xi).
\]

(5.6)

We also put \( \nu(t, \xi) = \sqrt{\langle \alpha(t, \xi), A\alpha(t, \xi) \rangle_{\mathbb{C}^N}} \), where \( A \) is the positive Hermitian matrix appearing in the condition (b_0). Remark that

\[
\sqrt{\kappa} |\alpha(t, \xi)| \leq \nu(t, \xi) \leq \sqrt{\kappa'} |\alpha(t, \xi)|,
\]

(5.7)

where \( \kappa \) and \( \kappa' \) are the smallest and largest eigenvalues of \( A \), respectively. It follows from (b_0) that

\[
\partial_t \nu(t, \xi)^2 = 2 \text{Im}(i \partial_t \alpha(t, \xi) A \alpha(t, \xi))_{\mathbb{C}^N}
\]

\[
= \frac{2}{t} \text{Im}(\rho(\xi; \alpha(t, \xi)), A\alpha(t, \xi))_{\mathbb{C}^N} + 2 \text{Im}(\rho(t, \xi), A\alpha(t, \xi))_{\mathbb{C}^N}
\]

\[
\leq 0 + C |\rho(t, \xi)| \nu(t, \xi).
\]

This leads to

\[
\nu(t, \xi) \leq \nu(1, \xi) + C \int_1^t |\rho(\tau, \xi)| d\tau \leq \frac{C \varepsilon}{\langle \xi \rangle^2} + \frac{C \varepsilon^2}{\langle \xi \rangle^2} \int_1^\infty \frac{d\tau}{t^{3/4 - \gamma}} \leq \frac{C \varepsilon}{\langle \xi \rangle^2},
\]

(5.8)

provided that we have

\[
|\rho_j(t, \xi)| \leq \frac{C \varepsilon^2}{t^{3/4 - \gamma}}.
\]

(5.9)

As we shall see soon later, (5.9) holds true under the condition (a) and the assumption (5.1). By (5.7) and (5.8), we arrive at

\[
\langle \xi \rangle^2 |\alpha_j(t, \xi)| \leq C |\xi|^2 \nu(t, \xi) \leq C \varepsilon,
\]

as required.

It remains to prove (5.9). To this end, we introduce the following lemma:

**Lemma 5.2.** Suppose that the condition (a) is satisfied. For a \( \mathbb{C}^N \)-valued function \( u = (u_j(t, x))_{j \in I_N} \), we set \( \alpha(t, \xi) = \mathcal{F}_{m N} (U_m(t^{-1})^{-1} u(t, \cdot))(\xi) \) and \( \alpha_j(t, \xi) \). Then we have

\[
\left\| \mathcal{F}_{m N}^{-1} [\partial_x F_j(u, \partial_x u)] - \frac{(im\xi)^j}{t} \rho_j(\xi; \alpha) \right\|_{L^\infty} \leq \frac{C}{t^{3/4}} \sum_{k=1}^N (\|u_k(t)\|_{L^p} + \|Abs u_k(t)\|_{L^p})^3
\]

for \( j \in I_N \), \( l \in \{0, 1, 2\} \) and \( t \geq 1 \).

Once this lemma is verified, the desired estimate (5.9) follows immediately. Indeed,
\[
|\rho(t, \xi)| \leq \frac{C}{\langle \xi \rangle^2} \sum_{i=0}^{2} |(\text{im} \xi) \rho(t, \xi)| = \frac{C}{\langle \xi \rangle^2} \sum_{i=0}^{2} F_{m}^i \mathcal{U}^{-1}_{m} [\partial_{x} F_j(u, \partial_{x} u)] - \frac{(\text{im} \xi)^{y}}{t} |\xi|^2 \alpha(t, \xi) | \leq \frac{C}{\langle \xi \rangle^2} \cdot \frac{C}{t^{3/4}} \left( E(T) \gamma \right)^{3} \leq \frac{C \varepsilon}{\langle \xi \rangle^2} \cdot \frac{C}{t^{3/4-\gamma}}.
\]

The rest part of this section is devoted to the proof of lemma 5.2. For simplicity of exposition, we treat only the case where \( F_j = (\partial_{x} u_1)(\partial_{x} u_2)(\partial_{x} u_3) \) with \( m_j = m_1 - m_2 + m_3 \). The general case can be shown in the same way.

We set \( \alpha_{k}^{(i)} = (\text{im} \xi)^{y} \alpha_{k} \) for \( s \in \mathbb{Z}_{20} \), so that

\[
\partial_{x} u_k = \mathcal{U}_{m} \alpha_{m}^{-1} \alpha_{k}^{(i)} = \mathcal{M}_{m} \mathcal{D} \mathcal{W}_{m} \alpha_{m}^{(i)}, \quad \partial_{x} u_k = \mathcal{U}_{m} \alpha_{m}^{-1} \alpha_{k}^{[i]}.
\]

Remark that

\[
\alpha_{k}^{(i)} = (\text{im} \xi)(-\text{im} \xi)(\text{im} \xi) \alpha_{1} \alpha_{2} \alpha_{3} = \alpha_{1}^{(i)} \alpha_{2}^{(i)} \alpha_{3}^{(i)}.
\]

Now we consider the simplest case \( l = 0 \). By the factorization of \( \mathcal{U}_{m} \) and the condition \( m_j = m_1 - m_2 + m_3 \), we have

\[
F_{m}^i \mathcal{U}^{-1}_{m} F_j = \mathcal{W}^{-1}_{m} \mathcal{D} \mathcal{W}^{-1}_{m} \left( (\mathcal{M}_{m} \mathcal{D} \mathcal{W}_{m} \alpha_{m}^{(i)})(\mathcal{M}_{-m} \mathcal{D} \mathcal{W}_{m} \alpha_{m}^{(i)}) \right) = \frac{1}{t} \mathcal{W}^{-1}_{m} \left[ (\mathcal{W}_{m} \alpha_{m}^{(i)})(\mathcal{W}_{-m} \alpha_{m}^{(i)}) \right] = \frac{1}{t} \rho(t, \xi) + \frac{1}{t} r_{0},
\]

where

\[
r_{0} = \mathcal{W}^{-1}_{m} \left[ (\mathcal{W}_{m} \alpha_{m}^{(i)})(\mathcal{W}_{-m} \alpha_{m}^{(i)}) \right] = \alpha_{1}^{(i)} \alpha_{2}^{(i)} \alpha_{3}^{(i)}.
\]

Since we can rewrite it as

\[
r_{0} = (\mathcal{W}^{-1}_{m} - 1) \left[ (\mathcal{W}_{m} \alpha_{m}^{(i)})(\mathcal{W}_{-m} \alpha_{m}^{(i)}) \right] + (\mathcal{W}_{m} - 1) \alpha_{1}^{(i)} \alpha_{2}^{(i)} \alpha_{3}^{(i)}
\]

we can apply (3.4) and the Sobolev imbedding \( H^{1}(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R}) \) to obtain

\[
\|r_{0}\|_{L^{\infty}} \leq C t^{-1/4} \|u\|_{H^{1}} \|\alpha\|_{H^{2}} \|\alpha\|_{H^{2}}.
\]

Next we consider the case of \( l = 1 \). By (3.2) with \( m = m_j \), \( \mu = m_1 \), \( f_1 = \partial_{x} u_1 \), \( f_2 = \partial_{x} u_2 \), \( f_3 = \partial_{x} u_3 \), we have

\[
\partial_{x} F_j = \frac{m_j}{m_1} (\partial_{x}^{2} u_1)(\partial_{x} u_2)(\partial_{x} u_3) + \frac{R_{i}}{t}, \quad (5.10)
\]

where
\[ R_1 = -im_j J_m \left[ (\partial_t u_1)(\overline{\partial_x u_2})(\overline{\partial_t u_3}) \right] + im_j (J_m \partial_t u_1)(\overline{\partial_x u_2})(\partial_t u_3). \]

By applying lemma 3.1 to the first term and using lemma 3.4, we see that
\[
\| \mathcal{F}_m \mathcal{U}_m^{-1/2} R_1 \|_{L^{\infty}} \leq C \| J_m \partial_t u_1 \|_{L^2} \| \partial_x u_2 \|_{L^2} \| \partial_t u_3 \|_{L^2} + C \| \partial_t u_1 \|_{L^2} \| J_m \partial_x u_2 \|_{L^2} \| \partial_t u_3 \|_{L^2} + C \| \partial_x u_2 \|_{L^2} \| J_m \partial_x u_2 \|_{L^2} \| \partial_t u_3 \|_{L^2} \\
\leq \frac{C}{t^{1/2}} \sum_{k=1}^3 \| u_k \|_{H^k}^3 + \| J_m u_k \|_{H^k}^3, \tag{5.11}
\]

where we have used the inequality \( \| \phi \|_{L^\infty} \leq C t^{-1/2} \| \phi \|_{L^2} \| J_m \phi \|_{L^2}^3 \) and the commutation relation \([\partial_x, J_m] = 1\) in the last line. As for the first term of (5.10), similar computations as in the previous case lead to
\[
\mathcal{F}_m \mathcal{U}_m^{-1} \left[ (\partial_x u_2)(\overline{\partial_t u_3})(\partial_t u_3) \right] = \frac{1}{t} \mathcal{W}_m^{-1} (\mathcal{W}_m \alpha_1(2) \overline{\mathcal{W}_m \alpha_2}) (\mathcal{W}_m \alpha_3(1)) \]
\[
= \frac{1}{t} \alpha_1(2) \overline{\alpha_2(1)} \alpha_3(1) + \frac{r_1}{t},
\]

where
\[
r_1 = \mathcal{W}_m^{-1} (\mathcal{W}_m \alpha_1(2) \overline{\mathcal{W}_m \alpha_2(1)} (\mathcal{W}_m \alpha_3(1))) = \alpha_1(2) \overline{\alpha_2(1)} \alpha_3(1).
\]

This can be estimated as follows:
\[
\| r_1 \|_{L^\infty} \leq C t^{-1/4} \| \partial_t u_1 \|_{H^k} \| u_2 \|_{H^k} \| u_3 \|_{H^k}.
\]

Moreover, we observe that
\[
\frac{im_j \xi}{t} p_j(\xi; \alpha) = \frac{m_j}{m_1} \frac{im_j \xi}{t} (\partial_{\xi j}^3) = \frac{m_j}{m_1} \frac{1}{t} \alpha_1(2) \overline{\alpha_2(1)} \alpha_3(1).
\]

Piecing them together, we arrive at
\[
\left\| \mathcal{F}_m \mathcal{U}_m^{-1} \partial_j F_j - \frac{im_j \xi}{t} p_j(\xi; \alpha) \right\|_{L^{\infty}} = \frac{1}{t} \left\| \frac{m_j}{m_1} r_1 + \mathcal{F}_m \mathcal{U}_m^{-1} R_1 \right\|_{L^{\infty}} \leq \frac{C}{t} (\| r_1 \|_{L^\infty} + \mathcal{F}_m \mathcal{U}_m^{-1} R_1) \leq \frac{C}{t^{5/4}} \sum_{k=1}^3 \| u_k(t, \cdot) \|_{H^k}^3 + \| J_m u_k(t, \cdot) \|_{H^k}^3 \tag{5.12},
\]
as desired. Finally we consider the case of \( l = 2 \). By (3.3) with \( m = m_j, \mu_1 = m_1 \) and \( \mu_2 = -m_2 \), we have
\[
\partial_j^2 F_j = \frac{m_j^2}{-m_1 m_2} (\partial_x^2 u_1)(\partial_x u_2)(\partial_t u_3) + \frac{R_2}{t},
\]

where
\[
R_2 = \frac{im_j^2}{m_1} J_m \left[ (\partial_x^2 u_1)(\overline{\partial_t u_2})(\partial_t u_3) \right] + \frac{im_j^2}{m_1} (\partial_x^2 u_1)(\overline{J_m \partial_x u_2})(\partial_t u_3) \]
\[
- im_j \partial_x J_m \left[ (\partial_t u_1)(\overline{\partial_x u_2})(\partial_t u_3) \right] + im_j \partial_x \left[ (J_m \partial_t u_1)(\overline{\partial_x u_2})(\partial_t u_3) \right].
\]
As in the derivation of (5.11), we see that
\[
\|F_{m}\mathcal{U}_{m}^{-1}R_{f}\|_{L^{\infty}} \leq C_{t}^{-1/2} \sum_{k=1}^{3} (\|u_{k}\|_{H^{3}} + \|J_{m}u_{k}\|_{H^{3}})^{3}.
\]

Similarly to the previous cases, we can also show that
\[
\mathcal{F}_{m}\mathcal{U}_{m}^{-1}[(\partial_{\xi}^{2}u_{1})(\partial_{\xi} u_{2})(\partial_{\xi} u_{3})] = \mathcal{W}_{m}^{-1}[\mathcal{W}_{m}\alpha_{1}^{(2)}(\mathcal{W}_{m}\alpha_{2}^{(2)})(\mathcal{W}_{m}\alpha_{3}^{(1)})] - \alpha_{1}^{(2)}\alpha_{2}^{(2)}\alpha_{3}^{(1)}.
\]

where
\[
r_{2} = \mathcal{W}_{m}^{-1}[\mathcal{W}_{m}\alpha_{1}^{(2)}(\mathcal{W}_{m}\alpha_{2}^{(2)})(\mathcal{W}_{m}\alpha_{3}^{(1)}) - \alpha_{1}^{(2)}\alpha_{2}^{(2)}\alpha_{3}^{(1)}].
\]

Note that
\[
\|r_{2}\|_{L^{\infty}} \leq C_{t}^{-1/4} \|\partial_{\xi} u_{1}\|_{H^{3}} \|\partial_{\xi} u_{2}\|_{H^{3}} \|\partial_{\xi} u_{3}\|_{H^{3}}.
\]

Therefore we have
\[
\mathcal{F}_{m}\mathcal{U}_{m}^{-1}[(\partial_{\xi}^{2}u_{1})(\partial_{\xi} u_{2})(\partial_{\xi} u_{3})] - \frac{(im\xi)^{2}}{m_{j}} \mathcal{P}_{j}(\xi; \alpha) + \frac{r_{2}}{t}.
\]

which completes the proof of lemma 5.2.

\[\Box\]

6. Proof of the main theorems

Now we are in a position to prove theorems 2.1–2.4.

6.1. Proof of theorem 2.1

First let us recall the local existence theorem. For fixed $t_{0} \geq 0$, let us consider the initial value problem
\[
\begin{align*}
\mathcal{L}_{m} u_{j} &= F_{j}(u, \partial_{\xi} u), & t > t_{0}, & x \in \mathbb{R}, & j \in I_{N},
\end{align*}
\]
\[
\begin{align*}
u_{j}(t_{0}, x) &= \psi_{j}(x), & x \in \mathbb{R}, & j \in I_{N},
\end{align*}
\]

(6.1)

**Lemma 6.1.** Let $\psi = (\psi_{j})_{j \in I_{N}} \in H^{3} \cap H^{2.1}$. There exists a positive constant $\varepsilon_{0}$ which is independent of $t_{0}$ such that the following holds: for any $\varepsilon \in (0, \varepsilon_{0})$ and $M \in (0, \infty)$, one can choose a positive constant $\tau^{*} = \tau^{*}(\varepsilon, M)$, which is independent of $t_{0}$, such that (6.1) admits a unique solution $u = (u_{j})_{j \in I_{N}} \in C([t_{0}, t_{0} + \tau^{*}); H^{3} \cap H^{2.1})$, provided that
\[
\|\psi\|_{H^{3}} \leq \varepsilon \quad \text{and} \quad \sum_{l=0}^{1} \sum_{j \in I_{N}} \left\| x + \frac{t_{0}}{m_{j}} \partial_{\xi} \right\|_{H^{3-l}} \psi_{j} \leq M.
\]
We omit the proof of this lemma because it is standard (see e.g. appendix of [17] for the proof of similar lemma in the quadratic nonlinear case).

Now we are going to prove the global existence by the so-called bootstrap argument. Let $T^*$ be the supremum of all $T \in (0, \infty)$ such that the problem (1.1) admits a unique solution $u \in C([0, T); H^3 \cap H^3_1)$. By lemma 6.1 with $t_0 = 0$, we have $T^* > 0$ if $\| \varphi \|_{H^{3/2}} \leq \varepsilon < \varepsilon_0$. We also set

$$T_* = \sup \{ \tau \in [0, T^*) \mid E(\tau) \leq \varepsilon^{2/3} \}.$$ 

Note that $T_* > 0$ because of the continuity of $[0, T^*) \ni \tau \mapsto E(\tau)$ and $\| \varphi \|_{H^{3/2}} = \varepsilon \leq \frac{1}{2} \varepsilon^{2/3}$ if $\varepsilon \leq 1/8$.

We claim that $T_* = T^*$ if $\varepsilon$ is small enough. Indeed, if $T_* < T^*$, lemma 5.1 with $t_0 = 0$ yields

$$E(T_*) \leq K\varepsilon \leq \frac{1}{2} \varepsilon^{2/3}$$

for $\varepsilon \leq \varepsilon_2 := \min \{ \varepsilon_1, 1/(2K)^3 \}$, where $K$ and $\varepsilon_1$ are mentioned in lemma 5.1. By the continuity of $[0, T^*) \ni \tau \mapsto E(\tau)$, we can take $T^* \in (T_*, T^*)$ such that $E(T^*) \leq \varepsilon^{2/3}$, which contradicts the definition of $T_*$. Therefore we must have $T_* = T^*$. By using lemma 5.1 with $T = T_*$ again, we see that

$$\sum_{l=0}^{1} \sum_{j \in I_j} \| J^l_m u_j(t, \cdot) \|_{H^{3/2}} \leq K\varepsilon(1 + t)^{\frac{3}{2}}$$

for $t \in [0, T^*)$. In particular we have

$$\sup_{t \in [0, T^*)} \| u(t) \|_{H^3} \leq C \sup_{(t, \xi) \in [0, T^*) \times \mathbb{R}} \langle |\xi|^3, \alpha(t, \xi) \rangle \leq C' \varepsilon$$

with some $C' > 0$.

Next we assume $T^* < \infty$. Then, by setting $\varepsilon_3 = \min \{ \varepsilon_2, \varepsilon_0/2C' \}$ and $M = K\varepsilon_3(1 + T^*)^{3/2}$, we have

$$\sup_{t \in [0, T^*)} \| u(t) \|_{H^3} \leq \varepsilon_0/2 < \varepsilon_0$$

as well as

$$\sup_{t \in [0, T^*)} \| u(t) \|_{H^3} \leq \varepsilon_0/2 < \varepsilon_0$$

for $\varepsilon \leq \varepsilon_3$. By lemma 6.1, there exists $\tau^* > 0$ such that (1.1) admits the solution $u \in C([0, T^* + \tau^*); H^3 \cap H^3_1)$. This contradicts the definition of $T^*$, which means $T^* = +\infty$ for $\varepsilon \in (0, \varepsilon_3)$. Moreover, we have

$$\| u(t) \|_{L^2} \leq C \sup_{\xi \in \mathbb{R}} \langle |\xi|^3, \alpha(t, \xi) \rangle \leq C\varepsilon \varepsilon_3 \varepsilon$$

By using lemma 3.3 and the inequality obtained above, we also have

$$|u_j(t, x)| \leq \frac{C}{t^{3/2}} |\alpha_j(t, \xi)| + \frac{C}{t^{1/2}} (\| u_j(t) \|_{L^2} + \| J^l_m u_j(t) \|_{L^2}) \leq \frac{C\varepsilon}{t^{3/2}}$$

for $t \geq 1$ and $j \in I_0$. This completes the proof of theorem 2.1. \hfill \Box
6.2. Proof of theorems 2.2 and 2.3

The proof of theorems 2.2 and 2.3 heavily relies on the following lemma due to [19]. Note that special cases of this lemma have been used previously in [18] and [25] less explicitly.

**Lemma 6.2 ([19]).** Let $C_0 > 0$, $C_1 > 0$, $p > 1$ and $q > 1$. Suppose that $\Psi(t)$ satisfies

$$\frac{d\Psi}{dt}(t) \leq -\frac{C_0}{t} |\Psi(t)|^p + \frac{C_1}{t^q}$$

for $t \geq 2$. Then we have

$$\Psi(t) \leq \frac{C_2}{(\log t)^{\rho^* - 1}}$$

for $t \geq 2$, where $\rho^*$ is the Hölder conjugate of $p$ (i.e. $1/p + 1/\rho^* = 1$), and

$$C_2 = \left( \frac{p^*}{C_0 p} \right)^{\rho^* - 1} + (\log 2)^{\rho^* - 1}p(2) + \frac{C_1}{\log 2} \int_2^{\infty} (\log t)^{\rho^*} \frac{dt}{t^q}.$$

With $\xi \in \mathbb{R}$ fixed, we set $\Psi(t) = (\alpha(t, \xi), \alpha(t, \xi))_{C_0}$, where $A$ is the positive Hermitian matrix appearing in the condition (b). Then we deduce from (5.6) and (5.9) that $\Psi(t)$ satisfies

$$\frac{d\Psi}{dt}(t) \leq -\frac{2C_1}{t} |\alpha(t)|^4 + C |\rho(t, \xi)||\alpha(t, \xi)| \leq \frac{-2C_1}{t} |\Psi(t)|^2 + \frac{C_1}{\epsilon(t)^{1/54 - 1/4}}$$

for $t \geq 2$, where $C_+$ is the positive constant appearing in the condition (b1) and $\kappa_*$ is the smallest eigenvalue of $A$. We also have $\Psi(t) \leq C|\alpha(2, \xi)|^2 \leq C\epsilon^2(\xi)^{-2}$. So we can apply lemma 6.2 with $p = 2$, $q = 5/4 - \gamma$ to obtain

$$|\alpha(t, \xi)|^2 \leq C\Psi(t) \leq \frac{1}{(\log t)^{-1}} \left( \frac{\kappa_*^2}{2C_1} + \frac{C_1^2}{\epsilon(t)} \right) \leq \frac{C}{\log t}.$$

From lemma 3.3 it follows that

$$|u(t, x)| \leq \frac{C}{t \log t} \sup_{\xi \in \mathbb{R}} |\alpha(t, \xi)| + \frac{C}{t \log t} (\|u(t, x)\|_{L^2} + \|J_\alpha u(t, x)\|_{L^2})$$

$$\leq \frac{C}{(t \log t)^{1/2}} + \frac{C_1}{t^{3/4 - 1/3}}$$

$$\leq \frac{C}{(t \log t)^{1/2}},$$

for $t \geq 2$, $x \in \mathbb{R}$ and $j \in I_\nu$. On the other hand, we already know that $|u(t, x)| \leq C\epsilon(1 + t)^{-1/2}$ for $t \geq 0$. Hence we arrive at

$$(1 + t)(1 + s^2 \log(t + 2))|u(t, x)|^2 \leq C\epsilon^2$$

for $t \geq 0$, which implies the desired pointwise decay estimate. By the Fatou lemma we also have

$$\limsup_{t \to +\infty} \|u(t, x)\|^2_{L^2} \leq \int_{\mathbb{R}} \limsup_{t \to +\infty} |\alpha(t, \xi)|^2 d\xi = 0,$$

which leads to decay of $\|u(t)\|_{L^2}$ as $t \to +\infty$, as stated in theorem 2.2.
Under the stronger condition (b2), we have
\[
\frac{d\Psi}{dt}(t) \leq -\frac{2C_\ast\langle\xi\rangle^2/\xi^2}{t}\vert\Psi(t)\vert^2 + \frac{C\varepsilon^3}{\langle\xi\rangle^2 t^{3/4-\gamma}}.
\]
for \( t \geq 2 \). Therefore lemma 6.2 again yields
\[
|\alpha(t, \xi)|^2 \leq \frac{1}{\log t} \left( \frac{\kappa^2}{2C_\ast\langle\xi\rangle^2} + \frac{C^2}{\langle\xi\rangle^4} \right) \leq \frac{C}{\langle\xi\rangle^2 \log t},
\]
whence
\[
\|u(t)\|_{L^2} = \|\alpha(t)\|_{L^2} \leq C \sup_{\xi \in \mathbb{R}}|\alpha(t, \xi)| \leq \frac{C}{\sqrt{\log t}}
\]
for \( t \geq 2 \). This yields theorem 2.3. □

6.3. Proof of theorem 2.4

For given \( \delta > 0 \), we set \( \gamma = \min \{\delta, 1/5\} \in (0, 1/4) \). Remember that we have already shown that
\[
|\alpha_j(t, \xi)| \leq \frac{C\varepsilon}{\langle\xi\rangle^2}, \quad |\rho_j(t, \xi)| \leq \frac{C\varepsilon^2}{\langle\xi\rangle^2 t^{3/4-\gamma}}
\]
for \( t \geq 1, \xi \in \mathbb{R} \) and \( j \in I_N \). These estimates allow us to define \( \alpha^+ = (\alpha^+_j)_{j \in I_N} \in L^2 \cap L^\infty \) by
\[
\alpha^+_j(\xi) := \alpha_j(1, \xi) - i \int_1^\infty \rho_j(t', \xi) dt'.
\]
On the other hand, the condition (b3) and (5.6) lead to
\[
\alpha_j(t, \xi) = \alpha_j(1, \xi) - i \int_1^t \rho_j(t', \xi) dt',
\]
whence
\[
\|\alpha(t) - \alpha^+_j\|_{L^2 \cap L^\infty} \leq \int_j^\infty \|\rho(t', \cdot)\|_{L^2 \cap L^\infty} dt' \leq C\varepsilon^2 t^{-1/4+\gamma}.
\]
Now we set \( \varphi^+_j := \mathcal{F}_m^{-1}[\alpha^+_j] \). Then we have
\[
\|u(t) - U_m \varphi^+_j\|_{L^2} = \|\mathcal{F}_m U_m^{-1} u(t) - \mathcal{F}_m \varphi^+_j\|_{L^2}
\]
\[
= \|\alpha_j(t) - \alpha^+_j\|_{L^2} \leq C\varepsilon^2 t^{-1/4+\gamma}.
\]
By lemma 3.3 and the inequality obtained above, we also have
\[ \|u(t) - M_{m}DF_{m}\phi^{+}\|_{L^\infty} \]
\[ \leq \|u(t) - M_{m}DF_{m}U_{m}^{-1}\|_{L^\infty} + \|M_{m}D(\alpha(t) - \alpha^{+})\|_{L^\infty} \]
\[ \leq CT^{-3/4}(\|u(t)\|_{L^2} + \|u_{m}u(t)\|_{L^2}) + C\sqrt{2t}^{-1/2-1/4+\gamma} \]
\[ \leq C(T^{-3/4+\gamma/3} + C\sqrt{2t}^{-1/2-1/4+\gamma}) \]
for \( t \geq 1 \).

**Remark 6.1.** We put \( \varphi = \epsilon'\psi_j \) with \( \psi_j \not\equiv 0 \) and \( \epsilon' \in (0, \epsilon^*) \), where \( \epsilon^* > 0 \) is chosen suitably small so that theorem 2.4 is valid. Then we can check that the corresponding \( \varphi_j^+ \) satisfies
\[ \|\varphi_j^+\|_{L^2} = \|\alpha_j^+\|_{L^2} \geq \epsilon'\|\psi_j\|_{L^2} - C^*(\epsilon')^3 \]
with some \( C^* > 0 \). Therefore \( \varphi_j^+ \) does not identically vanish if \( \epsilon' < \min\{\epsilon^*, \sqrt{|\psi_j\|_{L^2}/C^*}\} \).

**Acknowledgments**

One of the authors (HS) would like to express his gratitude for warm hospitality of Department of Mathematics, Yanbian University. Main parts of this work were done during his visit there. The authors thank Profs S Katayama and M Ohta for their useful conversations on this subject. The work of CL is supported by NNSFC under Grant No. 11461074. The work of HS is supported by Grant-in-Aid for Scientific Research (C) (No. 25400161), JSPS.

**Appendix. Proof of lemma 4.1**

In this appendix, we shall give the proof of lemma 4.1 in the similar way as section 2 of [10] with slight modifications. We first state the following useful lemma without proof, which is a special case of lemma 2.1 of [10].

**Lemma A.1.** We have
\[ |\partial| + |\partial| \|g\|_{L^2} \]
\[ \leq C\|g\|_{w} \|\varphi\|_{L^2}. \]

**Proof of lemma 4.1.** As in the standard energy method, we compute
\[ \frac{1}{2}\frac{d}{dt}\|S\|_{L^2}^2 = \text{Im}\langle L_{m}Sv, Sv \rangle_{L^2} = \text{Im}\langle S L_{m}v, Sv \rangle_{L^2} + \text{Im}\langle [L_{m}, S]v, Sv \rangle_{L^2}. \]

We also note that
\[ [L_{m}, S]v = -\frac{i}{|m|}\phi S|\partial_{x}|v + Q, \]
where
\[ Q = \frac{1}{2m}\phi^2 Sv - \frac{i}{2|m|}(\partial_{x}\phi)\mathcal{H}v + \text{sgn}(m)\left( \int_{-\infty}^{x} \partial_{y}\phi(t, y)dy \right)\mathcal{H}v. \]
Remark that \( |\partial_{x}| = \sqrt{\alpha}, \partial_{x}^{2} = -1 \), and that \( \mathcal{H} \) is \( L^2 \)-bounded. Now we set \( v_{k}^{(l)} = \partial_{x}w_{k}^{(l)} \)
for \( l \in \mathbb{Z}_{2l} \). Then, since
\[ \partial_t \Phi = 2\eta \sum_{l=0}^{1} \sum_{k \in \mathbb{N}} \text{Im} \left( (i\partial_l w_k^{(l)}) w_k^{(l)} \right) \]
\[ = 2\eta \sum_{l=0}^{1} \sum_{k \in \mathbb{N}} \text{Im} \left\{ \left( -\frac{1}{2\mu_k} \partial^2 w_k^{(l)} + \partial_x \mathcal{L}_{\mu_k} w_k \right) w_k^{(l)} \right\} \]
\[ = 2\eta \sum_{l=0}^{1} \sum_{k \in \mathbb{N}} \text{Im} \left\{ \partial_l \left( -\frac{1}{2\mu_k} (\partial_l w_k^{(l)}) w_k^{(l)} \right) + \frac{1}{2\mu_k} \left( \partial_w w_k^{(l)} \right)^2 + (\partial_x \mathcal{L}_{\mu_k} w_k) w_k^{(l)} \right\} \]
\[ = 2\eta \sum_{l=0}^{1} \sum_{k \in \mathbb{N}} \left\{ \partial_l \left( -\frac{1}{2\mu_k} (\partial_l w_k^{(l)}) w_k^{(l)} \right) + (\partial_x \mathcal{L}_{\mu_k} w_k) w_k^{(l)} \right\} , \]

we see that
\[
\left| \int_{-\infty}^{\infty} \partial_t \Phi(t, y) dy \right| = 2\eta \sum_{l=0}^{1} \sum_{k \in \mathbb{N}} \text{Im} \left\{ -\frac{1}{2\mu_k} (\partial_l w_k^{(l)}) w_k^{(l)} + \int_{-\infty}^{\infty} (\partial_x \mathcal{L}_{\mu_k} w_k) w_k^{(l)} dy \right\} \]
\[ \leq C\eta \left( \|w\|^2_{H^\infty} + \sum_{k \in \mathbb{N}} \|\mathcal{L}_{\mu_k} w_k\|_{H^\infty} \|w_k\|_{H^r} \right) . \]

Therefore we obtain
\[
\frac{d}{dt} \|Sv\|_{L^2}^2 + \frac{2}{|m|} \text{Re} \langle \Phi_S | \partial_x | v, Sv \rangle_{L^2} \leq 2 \|S L_{\mu_k} v, Sv \|_{L^2}^2 + C(t) \|Sv\|_{L^2}^2 , \quad (A.2) \]
where
\[
B(t) = e^{C\|\Phi\|_{L^\infty}} \left( \|\Phi\|_{L^\infty} + \|\partial_x \Phi\|_{L^\infty} + \eta \|w\|_{H^\infty}^2 + \eta \sum_{k \in \mathbb{N}} \|\mathcal{L}_{\mu_k} w_k\|_{H^r} \|w_k\|_{H^r} \right) . \]

Next we observe that
\[
w_k^{(l)} | \partial_x | v = w_k^{(l)} S \partial_x H v
\[ = \partial_t (w_k^{(l)} S H v) + [w_k^{(l)} S, \partial_t] H v
\[ = -|\partial_t|^{1/2} |\partial_x|^{1/2} H w_k^{(l)} S H v + [w_k^{(l)} S, \partial_t] H v
\[ = |\partial_t|^{1/2} (w_k^{(l)} S |\partial_x|^{1/2} v) + [w_k^{(l)} S, \partial_t] H v - |\partial_t|^{1/2} [|\partial_x|^{1/2} H, w_k^{(l)} S] H v , \]

which leads to
\[
\left\{ w_k^{(l)} S |\partial_x| v, w_k^{(l)} S v \right\}_{L^2} = \left\{ w_k^{(l)} S |\partial_x|^{1/2} v, |\partial_x|^{1/2} (w_k^{(l)} S v) \right\}_{L^2} + \left\{ [w_k^{(l)} S, \partial_t] H v, w_k^{(l)} S v \right\}_{L^2}
\[ \quad - \left\{ [|\partial_t|^{1/2} H, w_k^{(l)} S] H v, |\partial_x|^{1/2} (w_k^{(l)} S v) \right\}_{L^2}
\[ = \|w_k^{(l)} S |\partial_x|^{1/2} v\|_{L^2}^2 + X_{kl} , \]

where
By using Lemma A.1, we can see that all the commutators appearing in $X_{kl}$ are $L^2$-bounded and their operator norms are dominated by

$$B_2(t) = C e^{t|\Phi|_2} (\|w\|_{W^{1,\infty}} + \|w\|_{W^{1,\infty}} \|\Phi\|_{L^\infty}).$$

Hence we obtain

$$\|\sqrt{\Phi} S|\partial_i|^{1/2} v\|_{L^2}^2 - \text{Re} \langle \Phi S|\partial_i| v, Sv \rangle_{L^2}$$

$$= \sum_{l=0}^1 \sum_{k \in I_k} \eta \text{Re} \left( \|w_k^{(l)} S|\partial_i| v\|_{L^2}^2 - \langle w_k^{(l)} S|\partial_i| v, w_k^{(l)} Sv \rangle_{L^2} \right)$$

$$\leq \sum_{l=0}^1 \sum_{k \in I_k} \eta \|X_{kl}\|_{L^2}$$

$$\leq C \eta B_2(t) \sum_{l=0}^1 \sum_{k \in I_k} \|w_k^{(l)} S|\partial_i|^{1/2} v\|_{L^2} \|v\|_{L^2} + C \eta B_2(t)^2 \|v\|_{L^2}^2$$

$$\leq \frac{1}{2} \|\sqrt{\Phi} S|\partial_i|^{1/2} v\|_{L^2}^2 + C \eta B_2(t)^2 \|v\|_{L^2}^2,$$

where we have used the Young inequality in the last line. Therefore,

$$\frac{2}{|m|} \text{Re} \langle \Phi S|\partial_i| v, Sv \rangle_{L^2} \geq \frac{1}{|m|} \|\sqrt{\Phi} S|\partial_i|^{1/2} v\|_{L^2}^2 - C \eta B_2(t)^2 \|v\|_{L^2}^2. \quad (A.3)$$

From (A.2) and (A.3) it follows that

$$\frac{d}{dt} \|Sv\|_{L^2}^2 + \frac{1}{|m|} \|\sqrt{\Phi} S|\partial_i|^{1/2} v\|_{L^2}^2 \leq 2 \left( |S L_m v, Sv|_{L^2} \right) + C(B_1(t) + \eta B_2(t)^2) \|Sv\|_{L^2}^2.$$

Finally, by using $\|\Phi\|_{L^2} \leq C \eta \|w\|_{W^{1,\infty}} + \|\Phi\|_{L^2} \leq C \eta \|w\|_{W^{1,\infty}}$ and $\|\partial_i \Phi\|_{L^2} \leq C \eta \|w\|_{W^{1,\infty}}$, we have

$$B_1(t) + \eta B_2(t)^2 \leq C e^{C \eta \|w\|_{H^2}} \left( \eta^2 \|w\|_{W^{1,\infty}}^2 + \eta \|w\|_{W^{1,\infty}} + \eta \sum_{k \in B_k} \|L_m w_k\|_{H^1} \|w_k\|_{H^1} \right)$$

$$+ C \eta e^{C \eta \|w\|_{H^2}} (\|w\|_{W^{1,\infty}}^2 + C \eta^2 \|w\|_{W^{1,\infty}})$$

$$\leq C e^{C \eta \|w\|_{H^2}} \left( \eta \|w\|_{W^{1,\infty}}^2 + \eta^2 \|w\|_{W^{1,\infty}} + \eta \sum_{k \in B_k} \|L_m w_k\|_{H^1} \|w_k\|_{H^1} \right),$$

which yields the desired conclusion. \qed

References

[1] Chihara H 1999 Gain of regularity for semilinear Schrödinger equations Math. Ann. 315 529–67
[2] Colin M and Colin T 2004 On a quasilinear Zakharov system describing laser-plasma interactions Differ. Integral Equ. 17 297–330
[3] Delort J-M, Fang D and Xue R 2004 Global existence of small solutions for quadratic quasilinear Klein–Gordon systems in two space dimensions J. Funct. Anal. 211 288–323
[4] Doi S 1994 On the Cauchy problem for Schrödinger type equations and the regularity of solutions J. Math. Kyoto Univ. 34 319–28
[5] Hayashi N, Li C and Naumkin P I 2011 On a system of nonlinear Schrödinger equations in 2d Differ. Integral Equ. 24 417–34
[6] Hayashi N, Li C and Ozawa T 2011 Small data scattering for a system of nonlinear Schrödinger equations Differ. Equ. Appl. 3 415–26
[7] Hayashi N and Naumkin P I 1998 Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations Am. J. Math. 120 369–89
[8] Hayashi N and Naumkin P I 1999 Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations without a self-conjugate property Funkcial. Ekvac. 42 311–24
[9] Hayashi N and Naumkin P I 2002 Asymptotics of small solutions to nonlinear Schrödinger equations with cubic nonlinearities Int. J. Pure Appl. Math. 3 255–73
[10] Hayashi N, Naumkin P I and Pipolo P N 1999 Smoothing effects for some derivative nonlinear Schrödinger equations Discrete Contin. Dyn. Syst. 5 685–95
[11] Hayashi N, Naumkin P I and Sunagawa H 2008 On the Schrödinger equation with dissipative nonlinearities of derivative type SIAM J. Math. Anal. 40 278–91
[12] Hayashi N, Naumkin P I and Uchida H 1999 Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations Publ. Res. Inst. Math. Sci. 35 501–13
[13] Hayashi N, Ozawa T and Tanaka K 2013 On a system of nonlinear Schrödinger equations with quadratic interaction Ann. Inst. Henri Poincaré Anal. Non Linéaire 30 661–90
[14] Hirayama H 2014 Well-posedness and scattering for a system of quadratic derivative nonlinear Schrödinger equations with low regularity initial data Commun. Pure Appl. Anal. 13 1563–91
[15] Hisakado M, Izuka T and Wadati M 1994 Coupled hybrid nonlinear Schrödinger equation and optical solitons J. Phys. Soc. Japan 63 2887–94
[16] Hisakado M and Wadati M 1995 Integrable multi-component hybrid nonlinear Schrödinger equations J. Phys. Soc. Japan 64 408–13
[17] Ikeda M, Katayama S and Sunagawa H 2015 Null structure in a system of quadratic derivative nonlinear Schrödinger equations Ann. Henri Poincaré 16 535–67
[18] Katayama S, Li C and Sunagawa H 2014 A remark on decay rates of solutions for a system of quadratic nonlinear Schrödinger equations in 2D Differ. Integral Equ. 27 301–12
[19] Katayama S, Matsumura A and Sunagawa H 2015 Energy decay for systems of semilinear wave equations with dissipative structure in two space dimensions NonDEA Nonlinear Differ. Equ. Appl. 22 601–28
[20] Katayama S, Ozawa T and Sunagawa H 2012 A note on the null condition for quadratic nonlinear Klein–Gordon systems in two space dimensions Commun. Pure Appl. Math. 65 1285–302
[21] Katayama S and Tsutsumi Y 1994 Global existence of solutions for nonlinear Schrödinger equations in one space dimension Commun. PDE 19 1971–97
[22] Kawahara Y and Sunagawa H 2006 Remarks on global behavior of solutions to nonlinear Schrödinger equations Proc. Japan Acad. A 82 117–22
[23] Kawahara Y and Sunagawa H 2011 Global small amplitude solutions for two-dimensional nonlinear Klein–Gordon systems in the presence of mass resonance J. Differ. Equ. 251 2549–67
[24] Kim D 2014 A note on decay rates of solutions to a system of cubic nonlinear Schrödinger equations in one space dimension Asymptotic Anal. to appear
[25] Kim D and Sunagawa H 2014 Remarks on decay of small solutions to systems of Klein–Gordon equations with dissipative nonlinearities Nonlinear Anal. 97 94–105
[26] Kita N and Shimomura A 2007 Asymptotic behavior of solutions to Schrödinger equations with a subcritical dissipative nonlinearity J. Differ. Equ. 242 192–210
[27] Kita N and Shimomura A 2009 Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data J. Math. Soc. Japan 61 39–64
[28] Li C 2012 Decay of solutions for a system of nonlinear Schrödinger equations in 2D Discrete Contin. Dyn. Syst. 32 4265–85
[29] Li C and Hayashi N 2014 Critical nonlinear Schrödinger equations and scale invariant spaces J. Math. Anal. Appl. 419 1214–34
[30] Ozawa T and Sunagawa H 2013 Small data blow-up for a system of nonlinear Schrödinger equations J. Math. Anal. Appl. 399 147–55
[31] Sagawa Y and Sunagawa H 2015 The lifespan of small solutions to cubic derivative nonlinear Schrödinger equations in one space dimension preprint (arXiv:1511.03126 [math.AP])
[32] Shimomura A 2006 Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities *Commun. PDE* **31** 1407–23

[33] Sunagawa H 2003 On global small amplitude solutions to systems of cubic nonlinear Klein–Gordon equations with different mass terms in one space dimension *J. Differ. Equ.* **192** 308–25

[34] Sunagawa H 2005 Large time asymptotics of solutions to nonlinear Klein–Gordon systems *Osaka J. Math.* **42** 65–83

[35] Sunagawa H 2006 Lower bounds of the lifespan of small data solutions to the nonlinear Schrödinger equations *Osaka J. Math.* **43** 771–89

[36] Sunagawa H 2006 Large time behavior of solutions to the Klein–Gordon equation with nonlinear dissipative terms *J. Math. Soc. Japan* **58** 379–400