Jet Riemann-Hamilton geometrization for the 
conformal deformed Berwald-Moór quartic metric 
depending on momenta

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Abstract

In this paper we expose on the dual 1-jet space $J^1(\mathbb{R}, M^4)$ the distinguished (d-) Riemannian geometry (in the sense of d-connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) for the $(t, x)$-conformal deformed Berwald-Moór Hamiltonian metric of order four.

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1 Introduction

The geometric-physical Lagrangian or Hamiltonian Berwald-Moór structure ([4], [7]) was intensively investigated by P.K. Rashevski ([13]) and further fundamental and developed by D.G. Pavlov, G.I. Garas’ko and S.S. Kokarev ([11], [5], [12]). In their works, the preceding Russian scientists emphasize the importance in the theory of space-time structure, gravitation and electromagnetism of the geometry produced by the classical Berwald-Moór Lagrangian metric

$$F : TM \to \mathbb{R}, \quad F(y) = \sqrt{y_1y_2\ldots y_n}, \quad n \geq 2,$$

or by the corresponding Berwald-Moór Hamiltonian metric

$$H : T^*M \to \mathbb{R}, \quad H(p) = \sqrt{p_1p_2\ldots p_n}.$$

In such a perspective, according to the recent geometric-physical ideas proposed by Garas’ko ([14]), we consider that a distinguished Riemannian geometry (in the sense of d-connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) for the conformal deformations of the jet Berwald-Moór Hamiltonian metric of order four is required. Note that a similar geometric-physical study for the $(t, x)$-conformal deformations of the jet Berwald-Moór Lagrangian metric of order four is now completely developed.
in the paper [8]. Also, few elements of distinguished Hamiltonian geometry produced by the cotangent quartic Berwald-Moór metric depending of momenta are already presented in the paper [1].

In such a geometrical and physical context, this paper investigates on the dual 1-jet space $J^1(\mathbb{R}, M^4)$ the Riemann-Hamilton distinguished geometry (together with a theoretical-geometric field-like theory) for the $(t,x)$-conformal deformed Berwald-Moór Hamiltonian metric of order four

$$H(t,x,p) = 2e^{-\sigma(x)}\sqrt{h_{11}(t)[p_1^1p_2^1p_3^1p_4^1]}^{1/4},$$  (1.1)

where $\sigma(x)$ is a smooth non-constant function on $M^4$, $h_{11}(t)$ is a Riemannian metric on $\mathbb{R}$, and $(t,x,p) = (t,x^1,x^2,x^3,x^4,p_1^1,p_2^1,p_3^1,p_4^1)$ are the coordinates of the momentum phase space $J^1(\mathbb{R}, M^4)$; these transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\bar{t} = \bar{t}(t), \quad \bar{x}^i = \bar{x}^i(x^i), \quad \bar{p}^1_i = \frac{\partial x^j}{\partial x^i} \frac{d\bar{t}}{dt} \bar{p}^1_j,$$  (1.2)

where $i,j = 1,4$, rank $\langle \partial x^j/\partial x^i \rangle = 4$ and $d\bar{t}/dt \neq 0$. It is important to note that, based on the geometrical ideas promoted by Miron, Hrimiuc, Shimada and Sabău in the classical Hamiltonian geometry of cotangent bundles ($\mathbb{R}$, together with those used by Atanasiu, Neagu and Oaţa in the geometry of dual 1-jet spaces, the differential geometry (in the sense of d-connections, d-torsions, d-curvatures, abstract gravitational-like and electromagnetic-like geometrical theories) produced by a jet Hamiltonian function $H : J^1(\mathbb{R}, M^n) \to \mathbb{R}$ is now completely done in the papers [2], [3], [9] and [10]. In what follows, we apply the general geometrical results from [9] and [10] to the square of Hamiltonian metric (1.1), which is given by $(n = 4)$

$$\bar{H}(t,x,p) = H^2(t,x,p) = 4e^{-2\sigma(x)}h_{11}(t)[p_1^1p_2^1p_3^1p_4^1]^{1/2}.  \tag{1.3}$$

**Remark 1.1** The momentum Hamiltonian metric (1.3) is exactly the natural Hamiltonian attached to the square of the jet Berwald-Moór Lagrangian metric, which has the expression

$$\bar{L}(t,x,y) = e^{2\sigma(x)}h_{11}(t)[y_1^1y_2^1y_3^1y_4^1]^{1/2}.  \tag{1.4}$$

In other words, we have

$$\bar{H}(t,x,p) = p_1^1y_1^1 - \bar{L}(t,x,y),$$

where $p_1^1 = \partial \bar{L}/\partial y_1^1$. Note that the jet Lagrangian metric (1.4) is even the square of the conformal deformed jet quartic Berwald-Moër Finslerian metric

$$F(t,x,y) = e^{\sigma(x)}\sqrt{h_{11}(t)[y_1^1y_2^1y_3^1y_4^1]}^{1/4}.$$

In other words, we have $\bar{L} = F^2$.\footnote{We assume that we have $p_1^1p_2^1p_3^1p_4^1 > 0$. This is one domain where we can $p$-differentiate the Hamiltonian function $H(t,x,p)$.}
2 The canonical nonlinear connection

Using the notation $\mathcal{P}^{1111} := p_1^1 p_2^1 p_3^1 p_4^1$ and taking into account that we have

$$\frac{\partial \mathcal{P}^{1111}}{\partial p_i} = \frac{\mathcal{P}^{1111}}{p_i},$$

then the fundamental metrical d-tensor produced by the metric (1.3) is given by the formula (no sum by $i$ or $j$)

$$\ast g_{ij} (t, x, \mathcal{P}^{1111}) = h_{11} (t) \frac{\partial^2 H}{\partial p_i \partial p_j} = \frac{e^{-2 \sigma (x)} (1 - 2 \delta_{ij}) \left[ \mathcal{P}^{1111} \right]^{1/2}}{2 p_1^1 p_j^1}.$$ (2.1)

Moreover, the matrix $\ast g = (\ast g_{ij})$ admits the inverse $\ast g^{-1} = (\ast g_{jk})$, whose entries are given by

$$\ast g_{jk} = \frac{e^{2 \sigma (x)} (1 - 2 \delta_{jk}) \left[ \mathcal{P}^{1111} \right]^{-1/2}}{2 p_j^1 p_k^1} (\text{no sum by } j \text{ or } k).$$ (2.2)

Let us consider the Christoffel symbol of the Riemannian metric $h_{11} (t)$ on $\mathbb{R}$, which is given by

$$\kappa_{11}^1 = \frac{h_{11} dh_{11}}{2 dt},$$

where $h_{11} = 1/h_{11} > 0$. Then, using the notation $\sigma_i := \partial \sigma/\partial x^i$, we find the following geometrical result:

**Proposition 2.1** For the $(t, x)$-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.3), the canonical nonlinear connection on the dual 1-jet space $J^1 \ast (\mathbb{R}, M^4)$ has the components

$$N = \left( N_{1 (i) 1}^{(1)} = \kappa_{11}^1 p_i^1, N_{2 (i) j}^{(1)} = -4 \sigma_i p_j^1 \delta_{ij} \right).$$ (2.3)

**Proof.** The canonical nonlinear connection produced by $\dot{H}$ on the dual 1-jet space $J^1 \ast (\mathbb{R}, M^4)$ has the following components (see [2]): $N_{1 (i) 1}^{(1)} \overset{df}{=} \kappa_{11}^1 p_i^1$ and

$$N_{2 (i) j}^{(1)} = \frac{h_{11}}{4} \left[ \frac{\partial g_{ij}}{\partial x^k} \frac{\partial H}{\partial p_k} - \frac{\partial g_{ij}}{\partial p_k} \frac{\partial H}{\partial x^k} + g_{ik} \frac{\partial^2 H}{\partial x^j \partial p_k} + g_{jk} \frac{\partial^2 H}{\partial x^i \partial p_k} \right].$$

Now, by a direct calculation, we obtain (2.3). ■

3 The Cartan canonical $N$-linear connection. Its d-torsions and d-curvatures

The nonlinear connection (2.3) produces the dual adapted bases of d-vector fields (no sum by $i$)

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} - \kappa_{11}^1 p_i^1 \frac{\partial}{\partial p_i^1} ; \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + 4 \sigma_i p_j^1 \frac{\partial}{\partial p_j^1} ; \frac{\partial}{\partial p_i^1} \right\} \subset \mathcal{X}(E^*)$$ (3.1)
Proposition 3.1 The Cartan canonical N-linear connection produced by the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four \( \text{(1.3)} \) has the following adapted local components (no sum by \( i, j \) or \( k \)):

\[
CT(N) = \left( k_{11}, A_{1j}^i = 0, H_{jk}^i = 4\delta_j^i \delta_k^i \sigma_i, C_{i(1)}^{j(k)} = C_i^{jk} \cdot \frac{p_1^j}{p_1^j p_k^1} \right),
\]

(3.3)

where

\[
C_i^{jk} = \frac{1 - 2\delta_j^k - 2\delta_j^i - 8\delta_k^i \delta_j^i}{8} = \begin{cases} 
\frac{1}{8}, & i \neq j \neq k \neq i \\
\frac{1}{8}, & i = j \neq k \text{ or } i = k \neq j \text{ or } j = k \neq i \\
\frac{3}{8}, & i = j = k.
\end{cases}
\]

Proof. The adapted components of the Cartan canonical connection are given by the formulas (see [9]),

\[
A_{1j}^i \overset{\text{def}}{=} \frac{\partial g^{ij}}{\partial t} \delta g_{ij}, \quad H_{jk}^i \overset{\text{def}}{=} \frac{\partial g^{ij}}{\partial x^k} + \frac{\partial g^{jk}}{\partial x^i},
\]

\[
C_{i(1)}^{j(k)} \overset{\text{def}}{=} \frac{\partial g^{jr}}{\partial p_k} + \frac{\partial g^{kjr}}{\partial p^r} - \frac{\partial g^{jr}}{\partial p_j} = \frac{\partial g^{jr}}{\partial p_k}.
\]

Using the derivative operators \( \text{(3.1)} \), the direct calculations lead us to the required results. ■

Remark 3.2 It is important to note that the vertical d-tensor \( C_{i(1)}^{j(k)} \) also has the properties (sum by \( m \)):

\[
C_{i(1)}^{j(k)} = C_{i(1)}^{k(j)}, \quad C_{i(1)}^{j(m)} p_m^1 = 0, \quad C_{m(1)}^{i(m)} = 0, \quad C_{i(1)m}^{j(m)} = 0,
\]

(3.4)

where

\[
C_{i(1)j}^{(k)} \overset{\text{def}}{=} \frac{\partial C_{i(1)}^{j(k)}}{\partial x^j} + C_{i(1)}^{r(k)} H_{rj}^i - C_{r(1)}^{i(k)} H_{ij}^r + C_{i(1)}^{(r)} H_{rj}^k.
\]
Proposition 3.3 The Cartan canonical connection of the \((t, x)\)-conformal deformed Berwald-Mo\'er Hamiltonian metric of order four \((1.3)\) has two effective local torsion d-tensors:

\[ R_{ij}^{(1)}(r) = -4\sigma_{ij} \left(p^1_i \delta_{ir} - p^1_j \delta_{jr}\right), \quad P_{ij}^{(1)}(r) = \frac{p^1_i}{p^1_j} p^1_j, \]

where \(\sigma_{ij} := \frac{\partial^2 \sigma}{\partial x^i \partial x^j} \).

**Proof.** A Cartan canonical connection on the dual 1-jet space \(J^1(\mathbb{R}, M^4)\) generally has six effective local d-tensors of torsion (for more details, see [9]). For the particular Cartan canonical connection \((3.3)\) these reduce only to two (the other four are zero):

\[ R_{ij}^{(1)}(r) \overset{\text{def}}{=} \frac{\delta N_{(1)}^{(1)}(r) i}{\delta x^j} - \frac{\delta N_{(1)}^{(1)}(r) j}{\delta x^i}, \quad P_{ij}^{(1)}(r) \overset{\text{def}}{=} C_{(1)}^{(1)}(i) \cdot \frac{p^1_i}{p^1_j} p^1_j. \]

Proposition 3.4 The Cartan canonical connection of the \((t, x)\)-conformal deformed Berwald-Mo\'er Hamiltonian metric of order four \((1.3)\) has three effective local curvature d-tensors:

\[ R_{ijk}^l = \frac{\partial H_{ij}^l}{\partial x^k} - \frac{\partial H_{ik}^l}{\partial x^j} + H_{ij}^r H_{rk}^l - H_{ik}^r H_{rj}^l + C_{(1)}^{(r)}(i) R_{ijkl}^{(1)}(r), \quad P_{ij}^{(1)}(k) = -C_{(1)}^{(1)}(i). \]

**Proof.** A Cartan canonical connection on the dual 1-jet space \(J^1(\mathbb{R}, M^4)\) generally has five effective local d-tensors of curvature (for all details, see [9]). For the particular Cartan canonical connection \((3.3)\) these reduce only to three (the other two are zero). These are \(S_{ij}^{(j)(k)}(r)\) and

\[ R_{ijk}^l \overset{\text{def}}{=} \frac{\delta H_{ij}^l}{\delta x^k} - \frac{\delta H_{ik}^l}{\delta x^j} + H_{ij}^r H_{rk}^l - H_{ik}^r H_{rj}^l + C_{(1)}^{(r)}(i) R_{ijkl}^{(1)}(r), \quad P_{ij}^{(1)}(k) = -C_{(1)}^{(1)}(i). \]

where

\[ P_{ij}^{(1)}(k) = \frac{\partial N_{(1)}^{(1)}(r) i}{\partial p^1_k} + H_{rj}^k = 0. \]
4 From \((t, x)\)-conformal deformations of the quartic Berwald-Moór Hamiltonian metric to field-like geometrical models

4.1 Momentum gravitational-like geometrical model

The \((t, x)\)-conformal deformed Berwald-Moór Hamiltonian metric of order four produces on the momentum phase space \(J^1(\mathbb{R}, M^4)\) the adapted metrical d-tensor (sum by \(i\) and \(j\))

\[
G = h_{11} dt \otimes dt + \ast g_{ij} dx^i \otimes dx^j + h_{11} \ast g_{ij} \delta p_1^i \otimes \delta p_1^j,
\]

(4.1)

where \(\ast g_{jk}\) and \(\ast g_{ij}\) are given by (2.2) and (2.1), and we have (no sum by \(i\)) \(\delta p_1^i = dp_1^i + k_1 p_1^i dt - 4 \sigma_i p_1^i dx^i\). We believe that, from a physical point of view, the metrical d-tensor (4.1) may be regarded as a “gravitational potential depending on momenta”. In our abstract geometric-physical approach, one postulates that the momentum gravitational potential \(G\) is governed by the geometrical Einstein equations

\[
\text{Ric} (CN) - \frac{\text{Sc} (CN)}{2} G = K T,
\]

(4.2)

where

- \text{Ric} (CN) is the Ricci d-tensor associated to the Cartan canonical linear connection (3.3); the Cartan canonical linear connection plays in our geometric-physical theory the same role as the Levi-Civita connection in the classical Riemannian theory of gravity;
- \text{Sc} (CN) is the scalar curvature;
- \(K\) is the Einstein constant and \(T\) is an intrinsic momentum stress-energy d-tensor of matter.

Therefore, using the adapted basis of vector fields (3.1), we can locally describe the global geometrical Einstein equations (4.2). Consequently, some direct computations lead to:

**Lemma 4.1** The Ricci tensor of the Cartan canonical connection of the \((t, x)\)-conformal deformed Berwald-Moór Hamiltonian metric of order four has the following two effective local Ricci d-tensors (no sum by \(i, j, k\) or \(l\)):

\[
R_{ij} = \begin{cases} 
-2 \sigma_{ij} - \frac{p_i^l}{p_k^l} \sigma_{jk} - \frac{p_j^l}{p_i^l} \sigma_{ji}, & i \neq j, \quad \{i, j, k, l\} = \{1, 2, 3, 4\} \\
0, & i = j,
\end{cases}
\]

(4.3)

\[
R_{(i)(j)}^{(1)(1)} = -S_{(i)(j)}^{(1)(1)} = \frac{4 \delta_{ij} - 1}{8} \frac{1}{p_i^l p_j^l}.
\]
Proof. Generally, the Ricci tensor of a Cartan canonical connection $CT(N)$ (produced by an arbitrary momentum Hamiltonian function) is determined by six effective local Ricci $d$-tensors (for more details, see [10]). For the particular Cartan canonical connection (3.3) these reduce only to two (the other four are zero), where (sum by $r$ and $m$):

$$R_{ij} \overset{def}{=} R_{ij}^m = \frac{\partial H^m_{ij}}{\partial x^m} - \frac{\partial H^m_{im}}{\partial x^j} + H^j_{ij} H^m_{rm} - H^j_{i} H^r_{mj} + C^{m(r)}_{i(1)} R^{(1)}_{(r)jm},$$

$$S^{(ij)(j)}_{m(1)(1)} \overset{def}{=} S^{(ij)(m)}_{m(1)(1)} = \frac{\partial C^{(ij)(m)}_{m(1)}}{\partial p^l_m} - \frac{\partial C^{(i(m)}_{m(1)}}{\partial p^l_j} + C^{(r(m)}_{m(1)} C_{r(1)}^{(j)} = \frac{\partial C^{(ij)(m)}_{m(1)}}{\partial p^l_m} + C_{m(1)}^{(r)} C_{r(1)}^{(j)}.$$

Lemma 4.2 The scalar curvature of the Cartan canonical connection of the $(t,x)$-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.3) has the value

$$Sc \ (CT(N)) = -4 e^{-2\sigma} \left[ p^{1111} \right]^{1/2} \Sigma_{11} - \frac{3}{2} h^{11} e^{2\sigma} \left[ p^{1111} \right]^{-1/2},$$

where

$$\Sigma_{11} = \frac{\sigma_{12}}{p^1 p^2} + \frac{\sigma_{13}}{p^1 p^3} + \frac{\sigma_{14}}{p^1 p^4} + \frac{\sigma_{23}}{p^2 p^3} + \frac{\sigma_{24}}{p^2 p^4} + \frac{\sigma_{34}}{p^3 p^4}.$$

Proof. The scalar curvature of the Cartan canonical connection (3.3) is given by the general formula

$$Sc \ (CT(N)) = \ast_{ij} R_{ij} - h^{11} g_{ij} S^{(ij)(1)}.$$

The local description in the adapted basis of vector fields (3.1) of the global geometrical Einstein equations (4.2) leads us to

Proposition 4.3 The geometrical Einstein-like equations produced by the $(t,x)$-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.3) are locally described by

$$\begin{align*}
2 e^{-2\sigma} h_{11} \left[ p^{1111} \right]^{1/2} \Sigma_{11} + \frac{3}{4} e^{2\sigma} \left[ p^{1111} \right]^{-1/2} &= \mathcal{K} \Sigma_{11}, \\
R_{ij} + \left\{ 2 e^{-2\sigma} \left[ p^{1111} \right]^{1/2} \Sigma_{11} + \frac{3}{4} h^{11} e^{2\sigma} \left[ p^{1111} \right]^{-1/2} \right\} \ast_{ij} &= \mathcal{K} \Sigma_{ij}, \\
- S^{(ij)(j)}_{(1)(1)} + \left\{ 2 e^{-2\sigma} h_{11} \left[ p^{1111} \right]^{1/2} \Sigma_{11} + \frac{3}{4} e^{2\sigma} \left[ p^{1111} \right]^{-1/2} \right\} \ast^{ij} &= \mathcal{K} \Sigma^{(ij)(j)}_{(1)(1)}, \\
0 &= T_{11}, \quad 0 = T_{11}^{(i)}, \quad 0 = T_{11}^{(i)}; \quad 0 = T_{ij}^{(i)}; \quad 0 = T_{ij}^{(i)}; \\
0 &= T^{(i)}_{11}, \quad 0 = T^{(j)}_{i1}, \quad 0 = T^{(i)}_{(i)1}; \\
0 &= T^{(i)}_{(i)1}, \quad 0 = T^{(j)}_{i1}, \quad 0 = T^{(i)}_{(i)1}.
\end{align*}$$

(4.4)
Corollary 4.4  The momentum stress-energy $d$-tensor of matter $T$ satisfies the following geometrical conservation-like laws (sum by $m$):

\[
\begin{align*}
T^{1}_{i/m} + T^{m}_{i(1)} &= 0 \\
T^{1}_{i/1} + T^{m}_{i}(i) &= E^{m}_{i}
\end{align*}
\]

where (sum by $r$):

\[
\begin{align*}
T^{1}_{i} &= h^{11} T^{1}_{11} = \kappa^{-1} \left\{ 2 e^{-2\sigma} [p^{1111}]^{1/2} \Sigma_{11} + \frac{3}{4} h^{11} e^{2\sigma} [p^{1111}]^{-1/2} \right\}, \\
T^{m}_{i} &= g^{mr} T^{1}_{r1} = 0, \quad T^{(1)}_{(m)1} = h^{11} g^{mr} T^{(r)}_{11} = 0, \quad T^{1}_{i} = h^{11} T^{1}_{i} = 0, \\
T^{m}_{i} &= g^{mr} T^{1}_{ri} = E^{m} := \kappa^{-1} \left\{ \frac{g^{mr} R_{ri}}{2} + \delta_{i}^{m} \left\{ 2 e^{-2\sigma} [p^{1111}]^{1/2} \Sigma_{11} + \frac{3}{4} h^{11} e^{2\sigma} [p^{1111}]^{-1/2} \right\} \right\},
\end{align*}
\]

\[
\begin{align*}
T^{(1)}_{(m)1} &= h^{11} g^{mr} T^{(r)}_{11} = 0, \quad T^{(1)}_{(i)1} = h^{11} T^{(1)}_{11} = 0, \quad T^{(m)1}_{(i)} = g^{mr} T^{(1)}_{r1} = 0,
\end{align*}
\]

\[
\begin{align*}
T^{(1)(i)}_{(m)(1)} &= h^{11} g^{mr} T^{(r)(i)}_{11} = \frac{h^{11} e^{2\sigma} [p^{1111}]^{-1/2}}{8K} \frac{p^{m}}{p_{i}} + \\
&\quad + \delta_{i}^{m} \left\{ \frac{h^{11} e^{2\sigma} [p^{1111}]^{-1/2}}{4K} \frac{2 e^{-2\sigma} [p^{1111}]^{1/2} \Sigma_{11}}{K} \right\},
\end{align*}
\]

and we also have (summation by $m$ and $r$, but no sum by $i$)

\[
\begin{align*}
T^{1}_{1/1} &= \frac{\delta T^{1}_{1}}{\delta t} + T^{1}_{i} K^{i}_{1} - T^{1}_{i} K^{i}_{1} = \frac{\delta T^{1}_{1}}{\delta t}, \quad T^{m}_{i/m} = \frac{\delta T^{m}}{\delta x^{m}} + T^{m}_{i/m} H^{m}_{ri}, \\
T^{(1)}_{(m)1} &= \frac{\partial T^{(1)}_{(m)1}}{\partial p^{m}_{i}} - T^{(1)}_{(r)1} C^{(m)}_{1} = \frac{\partial T^{(1)}_{(m)1}}{\partial p^{m}_{i}}, \\
T^{1}_{1/1} &= \frac{\delta T^{1}}{\delta t} + T^{1}_{i} K^{i}_{1} - T^{1}_{i} A^{i}_{1} = \frac{\delta T^{1}}{\delta t} + T^{1}_{i} K^{i}_{1}, \\
T^{m}_{i/m} &= \frac{\delta T^{m}}{\delta x^{m}} + T^{m}_{i/m} H^{m}_{ri} - T^{m}_{i/m} H^{m}_{ri} = E^{m}_{i/m} := \frac{\delta E^{m}}{\delta x^{m}} + 4E^{m} \sigma_{m} - 4E^{i} \sigma_{i},
\end{align*}
\]

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\[
T^{1(1)}(m)_{j(1)} \text{ def } = \frac{\partial T^{1(1)}_m}{\partial p^1_i} - T^{(1)(1)}_c m_{(1)} - T^{(1)(1)}_c C_r(1),
\]
\[
T^{1(1)}(m)_{j(1)} \text{ def } = \frac{\partial T^{1(1)}_{m(1)}}{\partial t} + T^{1(1)}_{(1)r} A_{r1}, \quad T^{m(i)}_{(1)m} \text{ def } = \frac{\partial T^{m(i)}_{(1)}}{\partial x^m} + 4 T^{m(i)}_{(1)} \delta_{m} + 4 T^{i(i)}_{(1)} \sigma_{i},
\]
\[
T^{(1)(i)}(m)_{j(1)} \text{ def } = \frac{\partial T^{(1)(i)}_{m(1)}}{\partial p^1_i} - T^{(1)(i)}_c m_{(1)} + T^{(1)(i)}_r C_r(1).
\]

**Proof.** The local Einstein equations \([14]\), together with some direct computations, lead us to what we were looking for. \(\blacksquare\)

### 4.2 Momentum electromagnetic-like geometrical model

In the paper \([10]\), a geometrical theory for an electromagnetism depending on momenta was also created, using only a given Hamiltonian function \(H\) on the momentum phase space \(J^1(\mathbb{R}, M^4)\). In the background of the jet momentum Hamiltonian geometry from this paper, we work with the electromagnetic distinguished 2-form (sum by \(i\) and \(j\))

\[
\mathcal{F} = F^{i(j)}(1) \delta p^1_i \wedge dx^j,
\]

where (sum by \(r\) and \(m\))

\[
F^{i(j)}(1) = \frac{h_{11}}{2} \left[ g^{ir} N_{c(1)(1)} - g^{ir} N_{c(1)(1)} + \left( g_{rz} H_{r1}^m - g_{rz} H_{r1}^m \right) p^1_m \right].
\]

The above electromagnetic components depending on momenta are characterized by some natural geometrical Maxwell-like equations (for more details, see Oană and Neagu \([9, 10]\)).

By a direct calculation, we see that the \((t, x)\)-conformal deformed Berwald-Moör Hamiltonian metric of order four \([13]\) produces null electromagnetic components: \(F^{i(j)} = 0\). Consequently, our dual jet \((t, x)\)-conformal deformed Berwald-Moör Hamiltonian geometrical electromagnetic theory is trivial one. Probably, this fact suggests that the dual jet \((t, x)\)-conformal deformed Berwald-Moör Hamiltonian structure \([13]\) has rather gravitational connotations than electromagnetic ones on the momentum phase space \(J^1(\mathbb{R}, M^4)\).

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