Continued Fractions, Quadratic Fields, and Factoring: Some Computational Aspects

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Abstract

Legendre discovered that the continued fraction expansion of \( \sqrt{N} \) having odd period leads directly to an explicit representation of \( N \) as the sum of two squares. In this vein, it was recently observed that the continued fraction expansion of \( \sqrt{N} \) having even period directly produces a factor of composite \( N \). It is proved here that these apparently fortuitous occurrences allow us to propose and apply a variation of Shanks’ infrastructural method which significantly reduces the asymptotic computational burden with respect to currently used factoring techniques.

1 Introduction

In a letter to Pierre de Carcavi, August 14th 1659, Pierre de Fermat reported several propositions; in particular, he stated the following theorem: Every prime \( p \) of the form \( 4k + 1 \) is uniquely expressible as the sum of two squares, i.e. \( p = X^2 + Y^2 \iff p \equiv 1 \mod 4 \), whose first known proof was given by Euler using Fermat’s infinite descent method. Many other proofs have been given, some constructive, others non-constructive; in particular, among the latter, Zagier’s one-sentence proof deserves to be mentioned for its conciseness [18]. Among the numerous constructive proofs, two different proofs by Gauss stand out. The first is direct, and gives \( x = \left(\frac{(2k)!}{2(2k)!}\right)^2 \mod p \) and \( y = \left(\frac{(2k)!^2}{2(2k)!}\right)^2 \mod p \); the partially incomplete proof was completed, a century later, by Jacobsthal. The second proof is based on quadratic forms of discriminant \(-4\), and considers two equivalent principal quadratic forms with discriminant \(-4\): \( pX^2 + 2b_1XY + b_1^2 + 1Y^2 \) and \( x^2 + y^2 \), where \( b_1 \) is a root of \( z^2 + 1 \) modulo \( p \). The first form represents \( p \) trivially with \( X = 1 \) and \( Y = 0 \), thus Gauss’ reduction produces the unique reduced form in the class \([11]\), and meanwhile yields \( x \) and \( y \).

Jacobsthal’s constructive solution (1906) is based on counting the number of points on the elliptic curve \( y^2 = n(n^2 - a) \) in \( \mathbb{Z}_p \). He considers the sum of Legendre symbols

\[
S(a) = \sum_{n=1}^{p-1} \left(\frac{n(n^2 - a)}{p}\right) \Rightarrow x = \frac{1}{2} S(q_R), \quad y = \frac{1}{2} S(q_N)
\]

where \( q_R, q_N \in \mathbb{Z}_p \) are any quadratic residue and non-residue, respectively, [8]. Legendre’s proof is reported on pages 59-60 of [10]. It is constructive, since it yields \( X \) and \( Y \) from the complete remainder of the continued fraction expansion of \( \sqrt{7} \). It is well explained in his own words.

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... Donc tous les fois que l’équation $x^2 - Ay^2 = -1$ est résoluble (ce qui a lieu entre autres cas lorsque $A$ est un nombre premier $4n + 1$) le nombre $A$ peut toujours être décomposé en deux quarrés; et cette décomposition est donnée immédiatement par le quotient-complet $\frac{\sqrt{A} + 1}{2}$ qui répond au second des quotients moyens compris dans la première période du développement de $\sqrt{A}$; le nombres $I$ et $D$ étant ainsi connus, on aura $A = D^2 + 1^2$.

Cette conclusion ranferme un des plus beaux théorèmes de la science des nombres, savoir, que tout nombre premier $4n + 1$ est la somme de deux quarrés; elle donne en même temps le moyen de faire cette décomposition d’une manière directe et sans aucun tâtonnement.

Thus, Legendre’s proof gives the representation of any composite $N$ such that the period of the continued fraction for $\sqrt{N}$ is odd, or equivalently, $x^2 - Ny^2 = -1$ is solvable in integers [10, 15, 5].

As a counterpart to Legendre’s finding, when the period of the continued fraction expansion of $\sqrt{N}$ is even, we directly obtain, under mild conditions, a factor of a composite $N$. In particular, this is certainly the case when both prime factors of $N = pq$ are congruent 3 modulo 4 [5]. Legendre’s solution of Fermat’s theorem tacitly introduces a connection between continued fractions and the ramified primes of quadratic number fields, obviously without using this notion more than a century before Dedekind’s invention. To explain these singular connections, the paper is organized as follows. Section 2 summarizes the properties of the continued fraction expansion of $\sqrt{N}$. Section 3 discusses the factorization of composite numbers $N$. Lastly, Section 4 draws conclusions.

### 2 Preliminaries

A regular continued fraction is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where $a_0, a_1, a_2, \ldots, a_i, \ldots$ is a sequence, possibly infinite, of positive integers. A convergent of a continued fraction is the sequence of fractions $\frac{A_m}{B_m}$, each of which is obtained by truncating the continued fraction at the $(m+1)$-th term. The fraction $\frac{A_m}{B_m}$ is called the $m$-th convergent [4, 7]. A continued fraction is said to be definitively periodic, with period $\tau$, if, starting from a finite position $n_0$, a fixed pattern $a'_1, a'_2, \ldots, a'_\tau$ repeats indefinitely. Lagrange showed that any definitively periodic continued fraction, of period length $\tau$, represents a positive number of the form $a + b\sqrt{N}$, $a, b \in \mathbb{Q}$, i.e. an element of $\mathbb{Q}(\sqrt{N})$, and conversely any such positive number is represented by a definitively periodic continued fraction [4, 15]. The period of the continued fraction expansion of $\sqrt{N}$ begins immediately after the first term $a_0$, and is written as $\sqrt{N} = [a_0, a_1, a_2, \ldots, a_1, a_2, \ldots, a_1]$, where the over-lined part is the period, which includes a palindromic part formed by the $\tau - 1$ terms $a_1, a_2, \ldots, a_1$. In Carr’s book [2] p.70-71] we find a good collection of properties of the continued fraction expansion of $\sqrt{N}$, which are summarized in the following, along with some properties taken from [4] [15].

1. Let $c_n$ and $r_n$ be the elements of two sequences of positive integers defined by the relation

$$\frac{\sqrt{N} + c_n}{r_n} = a_{n+1} + \frac{r_{n+1}}{\sqrt{N} + c_{n+1}}$$
with \( c_0 = \lfloor \sqrt{N} \rfloor \), and \( r_0 = N - a_0^2 \); the elements of the sequence \( a_1, a_2, \ldots, a_n \ldots \) are thus obtained as the integer parts of the left-side fraction, which is known as the complete quotient.

2. Let \( a_0 = \lfloor \sqrt{N} \rfloor \) be initially computed, and set \( c_0 = a_0, r_0 = N - a_0^2 \), then sequences \( \{c_n\}_{n \geq 0} \) and \( \{r_n\}_{n \geq 0} \) are produced by the recursions

\[
  a_{m+1} = \frac{a_m + c_m}{r_m}, \quad c_{m+1} = a_{m+1}r_m - c_m, \quad r_{m+1} = \frac{N - c_{m+1}^2}{r_m}.
\]  

(2)

These recursions allow us to compute the sequence \( \{a_m\}_{m \geq 1} \) using only rational arithmetical operations, and the iterations may be stopped when \( a_m = 2a_0 \), having completed a period.

3. If the period length \( \tau \) is odd, set \( \ell = \frac{\tau - 1}{2} \): Legendre discovered and proved that the complete quotient \( \frac{N + c_0}{r_\ell} \) gives a representation of \( N = c_\ell^2 + r_\ell^2 \) as the sum of two squares.

4. Numerator \( A_n \) and denominator \( B_n \) of the \( n \)-th convergent to \( \sqrt{N} \) can be recursively computed as

\[
  A_n = a_nA_{n-1} + A_{n-2} \quad \text{and} \quad B_n = a_nB_{n-1} + B_{n-2}, \quad n \geq 1,
\]

respectively, with initial conditions \( A_1 = 1, \quad B_1 = 0, \quad A_0 = a_0, \quad \text{and} \quad B_0 = 1 \). The numerator \( A_m \) and the denominator \( B_m \) of any convergent are shown to be relatively prime by the relation \( A_mB_{m-1} - A_{m-1}B_m = (-1)^{m-1} \). [11, p.85].

5. Using the sequences \( \{A_m\}_{m \geq 0} \) and \( \{B_m\}_{m \geq 0} \), two sequences \( \Delta = \{\Delta_m = A_m^2 - NB_m^2\}_{m \geq 0} \), and \( \Omega = \{\Omega_m = A_mA_{m-1} - NB_mB_{m-1}\}_{m \geq 1} \) are introduced. It can easily be checked that \( \Omega_m - \Delta_m\Delta_{m-1} = N, \quad \forall m \geq 1 \). The elements of \( \Delta \) and \( \Omega \) satisfy a system of linear recurrences

\[
  \begin{align*}
  \Delta_{m+1} &= a_m^2\Delta_m + 2a_m\Omega_m + \Delta_{m-1} \\
  \Omega_{m+1} &= \Omega_m + a_{m+1}\Delta_m
  \end{align*}
\]

(3)

with initial conditions \( \Delta_0 = a_0^2 - N, \Delta_1 = (1 + a_0a_1)^2 - N a_1^2 \) and \( \Omega_1 = (1 + a_0a_1)a_0 - N a_1 \). By [5], it is immediate to see that \( c_{m+1} = |\Omega_m| \) and \( r_{m+1} = |\Delta_m| \).

6. The period of \( \Delta \) and \( \Omega \) is \( \tau \) or \( 2\tau \), depending on whether \( \tau \) is even or odd.

7. The sequence of ratios \( \frac{A_m}{B_m} \) assumes the limit value \( \sqrt{N} \) as \( n \) goes to infinity, due to the inequality

\[
  \left| \frac{A_n}{B_n} - \sqrt{N} \right| \leq \frac{1}{B_n|B_{n+1}|}, \quad \text{since} \quad A_n \quad \text{and} \quad B_n \quad \text{go to infinity along with} \quad n.
\]

Since \( \frac{A_n}{B_n} < \sqrt{N} \), if \( n \) is even, and \( \frac{A_n}{B_n} > \sqrt{N} \), if \( n \) is odd [7], any convergent of even index is smaller than any convergent of odd index. This property implies that the terms of the sequence \( \Delta \) have alternating signs, with \( \Delta_1 > 0 \).

8. The value \( c_0 = a_0 \) is the greatest value that \( c_n \) may assume. No \( a_n \) or \( r_n \) can be greater than \( 2a_0 \). If \( r_n = 1 \) then \( a_{n+1} = a_0 \). For all \( n \) greater than 0, we have \( a_0 - c_n < r_n \leq 2a_0 \). The first complete quotient that is repeated is \( \sqrt{N + c_0} \), and \( a_1, r_0, \text{ and } c_0 \) commence each cycle of repeated terms.

9. Through the first period, we have the equalities \( a_{\tau - j} = a_j, \quad r_{\tau - j - 2} = r_j, \) and \( c_{\tau - j - 1} = c_j \).

10. The period \( \tau \) has the tight upper bound \( 0.72\sqrt{N \ln N} \), \( \quad N > 7 \), as was shown by Kraitchik [15, p.95]. However, the period length has irregular behavior as a function of \( N \), because it may assume any value from 1, when \( N = M^2 + 1 \), to values close to the order \( O(\sqrt{N \ln N}) \) [15].
11. Define the sequence of quadratic forms \( f_m(x, y) = \Delta_m x^2 + 2\Omega_m xy + \Delta_{m-1} y^2 \), \( m \geq 1 \), which has the same period as \( \Delta \). Every \( f_m(x, y) \) is a reduced form of discriminant \( 4N \). Within the first block, all quadratic forms \( f_m(x, y), 1 \leq m \leq \tau \) are distinct, and constitute the principal class \( \Gamma(f) \) of reduced forms, with the ordering of the elements inherited from \( \Delta \). The definition of reduced form used here is slightly different from the classic one: set \( \kappa = \min\{|\Delta_m|, |\Delta_{m-1}|\} \); it is easily checked that \( \Omega_m \) is the sole integer such that \( \sqrt{N} - |\Omega_m| < \kappa < \sqrt{N} + |\Omega_m| \), with the sign of \( \Omega_m \) chosen opposite to the sign of \( \Delta_m \). Since the sign of \( \Delta_m \) is the same as that of \( \Omega_m \), which is opposite to that of \( \Delta_m \) in \( \Gamma(f) \) the two triples of signs (signatures) \((-,-,+)\) and \((+,-,-)\) alternate.

The following theorems are taken, without proof, from [5].

**Theorem 1.** Starting with \( m = 1 \), the sequences \( \Delta = \{\Delta_m\}_{m \geq 0} \) and \( \Omega = \{\Omega_m\}_{m \geq 0} \) are periodic with the same period \( \tau \) or \( 2\tau \) depending on whether \( \tau \) is even or odd. The elements of the blocks \( \{\Delta_m\}_{m = 0}^{\tau} \) and \( \{\Omega_m\}_{m = 1}^{\tau} \) satisfy the symmetry relations \( \Delta_m = (-1)^j \Delta_{\tau-m-2}, \forall m \leq \tau - 3 \) and \( \Omega_{\tau-m-1} = (-1)^j \Omega_m, \forall m \leq \tau - 2 \), respectively.

If \( \tau \) is odd, the ordered set \( \{\Delta_m\}_{m=1}^{\tau} \) has a central term of index \( \ell = \left\lfloor \frac{\tau - 2}{2} \right\rfloor \), \( \Delta_{\ell} = -\Delta_{\ell-1} \) since \( \tau - \ell - 2 = \ell - 1 \), and the equation \( \Omega_{\ell}^2 - \Delta_{\ell} \Delta_{\ell-1} = N \) gives a solution of the Diophantine equation \( x^2 + y^2 = N \) with \( x = \Delta_{\ell} \) and \( y = \Omega_{\ell} \), the situation first recognized by Legendre.

If \( \tau \) is even, the ordered set \( \{\Delta_m\}_{m=1}^{\tau} \) has no central term; in this case, with \( \ell = \left\lfloor \frac{\tau - 2}{2} \right\rfloor \) we have \( \Omega_{\ell+1} = -\Omega_{\ell} \) and \( \Delta_{\ell+1} = \Delta_\ell \), hence \( f_{\ell+1}(x, y) = f_\ell(y, -x) \).

**Theorem 2.** Let the period \( \tau \) of the continued fraction expansion of \( \sqrt{N} \) be even; we have \( \Omega_{\tau-1} = -a_0, \Delta_{\tau} = \Delta_{\tau-2} \), and \( \Omega_\tau = -\Omega_{\tau-1} \). Defining the integer \( \gamma \in \mathbb{Q}(\sqrt{N}) \) by the product

\[
\gamma = \prod_{m=1}^{\tau} \left( \sqrt{N} + (-1)^m \Omega_m \right),
\]

let \( \sigma \) denote the Galois automorphism of \( \mathbb{Q}(\sqrt{N}) \) (i.e. \( \sigma(\sqrt{N}) = -\sqrt{N} \)), then \( \frac{\gamma}{\sigma(\gamma)} = A_{\tau-1} + B_{\tau-1} \sqrt{N} \) is a positive fundamental unit (or the cube of the fundamental unit) of \( \mathbb{Q}(\sqrt{N}) \).

Based on this theorem, we say that the unit \( c_{\tau-1} = A_{\tau-1} + B_{\tau-1} \sqrt{N} \) in \( \mathbb{Q}(\sqrt{N}) \) splits \( N \), if \( N_1 = \gcd\{A_{\tau-1} - 1, N\} \) is neither 1 nor \( N \). Then we have the proper factorization \( N = N_1 N_2 \). Further, using the following involutory matrix, [5], whose square is \((-1)^\tau I_2\)

\[
M_{\tau-1} = \begin{bmatrix}
-A_{\tau-1} & NB_{\tau-1} \\
-B_{\tau-1} & A_{\tau-1}
\end{bmatrix},
\]

it is shown that

\[
A_{\tau-m-2} = (-1)^{m-1}(A_{\tau-1}A_m - N B_{\tau-1} B_m) \quad 1 \leq m \leq \tau - 2 .
\]

As an immediate consequence of this equation, if the unit \( c_{\tau-1} \) splits \( N \), then any pair \( (A_m, A_{\tau-m-2}) \) splits \( N \), since taking \( A_{\tau-m-2} \) modulo \( N \) we have \( A_{\tau-m-2} = (-1)^{m-1} A_m A_{\tau-1} \) mod \( N \), thus \( A_{\tau-m-2} \) is certainly different from \( A_m \), because \( A_{\tau-1} \neq \pm 1 \) mod \( N \).

**Theorem 3.** If the period \( \tau \) of the continued fraction expansion of \( \sqrt{N} \) is even, the element \( c_{\tau-1} \) in \( \mathbb{Q}(\sqrt{N}) \) splits \( 4N \), and a factor of \( 4N \) is located at positions \( \frac{x-2}{2} + j\tau, j = 0, 1, \ldots \), in the sequence \( \Delta = \{\epsilon_m \sigma(\epsilon_m)\}_{m \geq 1} \).
3 Factorization

Gauss recognized that the factoring problem was to be important, although very difficult,

\[ \ldots \text{Problema, numeros primos a compositis dignoscendi, hosque in factores suos primos resolvendi, ad gravissima ac utilissima totius arithmeticae pertinere, et geometrarum tum veterum tum recentiorum industriam ac sagacitatem occupavisse, tam notum est, ut de hac re copiose loqui superfluum foret.} \ldots \]

C. F. GAUSS [Disquisitiones Arithmeticae Art. 329]

In spite of much effort, various different approaches, and the increased importance stemming from the large number of cryptographic applications, no satisfactorily factoring method has yet been found. However, approaches to factoring based on continued fractions have led to some of the most efficient factoring algorithms. In the following, a new variant of Shanks’ infrastructural method \cite{14} is described which exploits the property of the block \( \Delta_1 = \{ \Delta_m \}_{m=1}^\tau \), which is made more precise in the following theorem taken without proof from \cite{5}.

**Theorem 4.** Let \( N \) be a positive square-free integer. If the norm of the positive fundamental unit \( u \in \mathbb{Q}(\sqrt{N}) \) is 1, and some factor of \( N \) is a square of a principal integral ideal in \( \mathbb{Q}(\sqrt{N}) \), then \( u \) is split for \( N \). A proper factor of \( N \) is found in position \( \frac{\tau}{2} - 2 \) of \( \Delta_1 \).

It should be noted that \( \Delta_1 \) offers several different ways for factoring a composite number \( N \):

1. If \( \tau \) is even and 2 is not a quadratic residue modulo \( N \), then in position \( \frac{\tau}{2} - 2 \) of the sequence \( \Delta_1 \) we find a factor of \( N \).

2. If \( \tau \) is odd, then by Legendre’s results we find a representation \( N = X^2 + Y^2 \), which implies that \( s_1 = \frac{X}{Y} \mod N \) is a square root of \(-1\). If we are able to find another square root \( s_2 \) different from \(-\frac{X}{Y} \mod N \) (we have four different square roots of a quadratic residue modulo \( N = pq \)), then the difference \( s_1 - s_2 \) contains a proper factor of \( N \).

3. If some square \( d_2 \) is found in the sequence \( \Delta_1 \), it implies the equation \( A_m^2 - B_n^2 = d_2 \), thus there is a chance that some proper factor of \( N \) divides \( (A_m - d_2) \) or \( (A_m + d_2) \).

The number of squares in \( \Delta_1 \) is \( O(\sqrt{\tau}) \), and about \((\frac{1}{2})\) of these squares factor \( N \). This method was introduced by Shanks.

4. If equal terms \( \Delta_m = \Delta_n \), \( m \neq n \) occur in \( \Delta_1 \), with \( m, n < \frac{\tau}{2} \), then \( A_m^2 - A_n^2 = 0 \mod N \) allows us to find two factors of \( N \) by computing \( \gcd\{A_m - A_n, N\} \) and \( \gcd\{A_m + A_n, N\} \). This is an implementation of an old idea of Fermat’s.

3.1 Computational issues

By Theorem 4 we know that a factor of \( N \) is \( \Delta_{\frac{\tau}{2} - 2} \), which can be directly computed from the continued fraction of \( \sqrt{N} \) in \( \frac{\tau}{2} \) steps. Unfortunately, this number is usually prohibitively large. However, if \( \tau \) is known, using the baby-step/giant-step artifice, the number of steps can be reduced to the order \( O(\log_2 \tau) \). To this end, we can move through the principal class \( \Gamma(f) \), of ordered quadratic forms \( f_m(x, y) \), by introducing a notion of distance between pairs of quadratic forms compliant with Gauss’ quadratic
form composition. The distance between two adjacent quadratic forms \( f_{m+1}(x, y), f_m(x, y) \in \Gamma(f) \) is defined as
\[
d(f_{m+1}, f_m) = \frac{1}{2} \ln \left( \frac{\sqrt{N} + (-1)^m \Omega_m}{\sqrt{N} - (-1)^m \Omega_m} \right),
\]
and the distance between two quadratic forms \( f_m(x, y) \) and \( f_n(x, y) \), with \( m > n \), is defined as the sum \( d(f_m, f_n) = \sum_{j=n}^{m-1} d(f_{j+1}, f_j) \). The distance of \( f_m(x, y) \) from the beginning of \( \Gamma(f) \) is defined referring to a properly-chosen quadratic form \( f_0 = \Delta_0 x^2 - 2\sqrt{N - \Delta_0} xy + y^2 \) hypothetically located before \( f_1 \). Thus we have \( d(f_m, f_0) = \sum_{j=0}^{m-1} d(f_{j+1}, f_j) \) if \( m \leq \tau \). The notion is also extended to index \( k\tau \leq m < (k+1)\tau \) by setting \( d(f_m, f_0) = d(f_m \pmod{\tau}, f_0) + kR_\varepsilon \). The distance \( d(f, f_0) \) is exactly equal to \( R^* = \ln \varepsilon_{\tau-1} \), which is the regulator \( R_\varepsilon \), or three times \( R_\varepsilon \), and the distance \( d(f_\tau, f_0) \) is exactly equal to \( \frac{R^*}{2} \), see \cite{5} for a straightforward proof. Now, a celebrated formula of Dirichlet’s gives the product
\[
h_\varepsilon R_\varepsilon = \frac{\sqrt{D}}{2} L(1, \chi) = - \sum_{n=1}^{D+1} \left( \frac{D}{n} \right) \ln \left( \sin \frac{n\pi}{D} \right)
\]
where \( h_\varepsilon \) is the class field number, \( L(1, \chi) \) is a Dedekind \( L \)-function, \( D = N \) if \( N \equiv 1 \mod 4 \) or \( D = 4N \) otherwise, and character \( \chi \) is the Jacobi symbol in this case. If we know \( h_\varepsilon \) exactly, we know \( R^* \) exactly and we can proceed to factorization, with complexity \( O((\log_2 N)^4) \) \cite{9}, conditioned on the computation of \( L(1, \chi_N) \). The Dirichlet \( L(1, \chi_N) \) function can be efficiently evaluated using the following expression for the product \( h_\varepsilon R_\varepsilon \) as a function of \( N \)
\[
h_\varepsilon R_\varepsilon = \frac{1}{2} \sum_{x \geq 1} \left( \frac{D}{x} \right) \left( \sqrt{\frac{D}{x}} \operatorname{erfc} \left( x \sqrt{\frac{\pi}{D}} \right) + E_1 \left( \frac{\pi x^2}{D} \right) \right).
\]
where the complementary error function \( \operatorname{erfc}(x) \), and the exponential integral function \( E_1(x) \), can be quickly evaluated. Once we know \( R^* \), with the Shanks' infrastructural method \cite{14} or some of its improvements \cite{17, 3, 13}, we can find \( f_{\tau\tau}(x, y) \), thus a factor of \( N \). The goal is to obtain \( f_{m\tau}(x, y) \) with as few steps as possible. To this end we can perform 1) giant-steps within \( \Gamma(f) \) which are realized by the Gauss composition law of quadratic forms, followed by a reduction of this form to \( \Gamma(f) \), and 2) babysteps moving from one quadratic form to the next in \( \Gamma(f) \). Two operators \( \rho^+ \) and \( \rho^- \) are further defined \cite[p.259]{3} to allow small (baby) steps, precisely

\begin{itemize}
  \item \( \rho^+ \) transforms \( f_m(x, y) \) into \( f_{m+1}(x, y) \) in \( \Gamma(f) \), and is defined as \( \rho^+(a, 2b, c) = \left[ \frac{b^2 - N}{a}, 2b, a \right] \), where \( b_1 \) is \( 2b_1 = \mbox{int} \left( \frac{2b}{a} \right) + 2ka \) with \( k \) chosen in such a way that \( -|a| < b_1 < |a| \).
  \item \( \rho^- \) transforms \( f_m(x, y) \) into \( f_{m-1}(x, y) \) in \( \Gamma(f) \) and is defined as \( \rho^-(a, 2b, c) = \left[ c, 2b_1, \frac{b^2 - N}{c} \right] \), where \( b_1 \) is \( 2b_1 = \mbox{int} \left( \frac{-2b}{c} \right) + 2kc \) with \( k \) chosen in such a way that \( -|c| < b_1 < |c| \).
\end{itemize}

The composed form \( f_m \circ f_n \) has the distance \( d(f_m \circ f_n, f_0) \approx d(f_m, f_0) + d(f_n, f_0) \).

1. By the law \( \bullet, \Gamma(f) \) resembles a cyclic group, with \( f_{\tau-1} \) playing the role of identity.
2. Since in \( \Gamma(f) \) the two triples of signs (signatures) \((-+,+)\) and \((+,-,-)\) alternate, the composed form \( f_m(x, y) \circ f_n(x, y) \) must have one of these signatures.
3. The composition of a quadratic form with itself is called doubling and denoted $2 \bullet f_n(x, y)$. The distance is nearly maintained by the composition $\bullet$ (giant-steps). The error affecting this distance estimation is of order $O(\ln N)$ as shown by Schoof in [13], and is rigorously maintained by the one-step moves $\rho^{\pm}$ (baby-steps).

An outline of the procedure is the following, assuming that $R^*$ is preliminarily computed:

1. Let $\ell$ be a small integer. Compute an initial quadratic form $f_\ell = [\Delta_\ell, 2\Omega_\ell, +\Delta_{\ell-1}]$ and its distance $d_\ell = d(f_\ell, f_0)$ from the continued fraction expansion of $\sqrt{N}$ stopped at term $\ell + 1$.

2. Compute $j_\ell = \lceil \log_2 \frac{R^*}{d_\ell} \rceil$

3. Starting with $[f_\ell, d_\ell]$, iteratively compute and store in a vector $F_j$ the sequence $[2^{j} \bullet f_\ell, 2^{j} d_\ell]$ up to $j_\ell$. The middle term (i.e. $f_{\ell_2}$) of $\Gamma(f)$ is located between the terms $2^{j-1} \bullet f_\ell$ and $2^j \bullet f_\ell$.

4. The middle term of $\Gamma(f)$ can be quickly reached using the elements of $F_{j_\ell}$, starting by computing $f_r = (2^{j-1} \bullet f_\ell) \bullet (2^{j-2} \bullet f_\ell)$ and checking whether $2^{j-1} d_\ell + 2^{j-2} d_\ell$ is greater or smaller than $\frac{R^*}{2}$; in the first case set $f_s = f_r$, otherwise set $f_s = 2^{j-1} \bullet f_\ell$. Iterate this composition by computing $f_r = f_s \bullet (2^{i} \bullet f_\ell)$ and setting $f_s = f_r$ for decreasing $i$ up to $0$, and let the final term be $[f_s, d_s]$.

5. Iterate the operation $\rho^{\pm}$ a convenient number $O(\ln N)$ of times, until a factor of $4N$ is found.

4 Conclusions

An iterative algorithm has been described which produces a factor of a composite square-free $N$ with $O((\ln(N))^4)$ iterations at most, if $hR$ is exactly known, $h$ being the class number, and $R$ the regulator of $\mathbb{Q}(\sqrt{N})$. The bound $O((\ln(N))^4)$ is computed by multiplying the number of giant-steps, which is $O(\ln(N))$, by the number of steps at each reduction, completing a giant-step, which is upper bounded by $O((\ln(N))^3)$ as shown in [9, 12]. It is remarked that, in this bound computation, the cost of the arithmetics in $\mathbb{Z}$, i.e. multiplications and additions of big integers, is not counted [9]. Furthermore, it is not difficult to modify the algorithm to use a rough approximation of $hR$; the computations become cumbersome, but asymptotically the algorithm is polynomial, because a sufficient approximation of $hR$ is easily obtained by computing the series in equation (7) truncated at a number of terms $O(\ln(N))$, since the series converges exponentially [3, Proposition 5.6.11, p.262-263]. It remains to ascertain whether this asymptotically-good factoring algorithm is also practically better than any sub-optimal probabilistic factoring algorithm.

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