Regularisation : many recipes, but a unique principle: Ward identities and Normalisation conditions. The case of CPT violation in QED.

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Abstract
We analyse the recent controversy on a possible Chern-Simons like term generated through radiative corrections in QED with a CPT violating term: we prove that, if the theory is correctly defined through Ward identities and normalisation conditions, no Chern-Simons term appears, without any ambiguity. This is related to the fact that such a term is a kind of minor modification of the gauge fixing term, and then no renormalised. The past year literature on that subject is discussed, and we insist on the fact that any absence of an \textit{a priori} divergence should be explained by some symmetry or some non-renormalisation theorem.

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1 Introduction

In the past years, the interesting issue of a possible spontaneous breaking of Lorentz invariance at low energy has been considered: this issue also led to CPT breaking \[1, 2, 3\]. In particular, the general Lorentz-violating extension of the minimal \( SU(3) \times SU(2) \times U(1) \) standard model has been discussed: as many breaking terms are allowed, people look for possible constraints coming from experimental results as well as from renormalisability requirements and anomaly cancellation.

In that respect, there arose a controversy on a possible Chern-Simons like term generated through radiative corrections \[4, 2, 3, 5, 6, 7\]. This phenomenon was studied in QED, an abelian gauge theory, as a part of the standard model. We think that the controversy comes from some misunderstandings on regularisation and (re)normalisation of a given theory (see a review in \[8\]), and this paper intends to clarify the situation.

In the next Section, the basis facts are recalled, with emphasis on Ward identities, and the distinct three concepts of counterterms (finite or not), quantum corrections or spurious anomalies and physical radiative corrections are defined. In Section 3, breaking terms are added to the usual QED Lagrangian density, and one loop contributions are discussed. We check that, as soon as the theory is precisely defined through its symmetries (Ward identities) and its physical parameters (Normalisation conditions), there remains no ambiguity in the results. Then we prove a kind of non-renormalisation theorem that allows for a “minimally CPT broken” theory (Subsect.3.2). Finally, the literature is discussed and some comments are also offered on the invoked difference between an invariance of the action that leaves the Lagrangian density non-invariant, and one that leaves the Lagrangian density invariant.

2 Quantum electrodynamics

The Lagrangian density is:

\[
L_0 = \overline{\psi} (i \not{\partial} - m - e \not{A}) \psi - \frac{1}{4} F_{\mu \nu}^2 - \frac{1}{2\alpha} (\partial A)^2 + \frac{1}{2} \lambda^2 A_\mu^2 ,
\]

where \( \alpha \) is the gauge parameter and \( \lambda \) an infra-red regulator photon mass. The classical field equations write:

\[
(i \not{\partial} - m)\psi(x) = e A(x) \psi(x) , \quad \bar{\psi}(x)(-i \not{\partial} - m) = e \bar{\psi}(x) A(x) ,
\]

\[
[\Box + \lambda^2] A_\mu(x) - (1 - \frac{1}{\alpha}) \partial_\mu (\partial_\nu A^\nu(x)) = e \bar{\psi}(x) \gamma_\mu \psi(x) ,
\]

and the Feynman rules are:

\[
< A_\mu(-p) A_\nu(p) >_{class.} = D_{\mu \nu}(p, -p)_{class.} = \frac{-i}{p^2 - \lambda^2} (g_{\mu \nu} - \frac{p_\mu p_\nu}{p^2}) + \frac{-i\alpha}{p^2 - \alpha \lambda^2} \frac{p_\mu p_\nu}{p^2}
\]

\[
< \bar{\psi}(-p) \psi(p) >_{class.} = S_0(p, -p)_{class.} = \frac{i}{p - m}
\]

\[
< \bar{\psi}(p) \psi(q) A_\rho(-(p + q)) >_{prop.}^{class.} = \Gamma_\rho(p, q, -(p + q))_{class.} = -ie\gamma_\rho .
\]

The Ward identity resulting from gauge invariance ensures that the non-transverse part of the 2-photon proper Green function \( \Gamma_{\mu \nu}(p, -p)_{class.} = -i(g_{\mu \nu} p^2 - p_\mu p_\nu) + i\lambda^2 g_{\mu \nu} - \frac{1}{\alpha} p_\mu p_\nu \) is not renormalised, i.e. the parameters \( \alpha \) and \( \lambda^2 \) are (unphysical) tree level parameters.
By power counting, the primitively divergent graphs are here \( \Gamma_{\mu\nu}(p, -p)_{\text{transverse}} \) and \( \Gamma^{\rho}(p, q, -(p + q)) \), respectively the transverse photon and electron 2-points proper Green functions and the photon-electron proper vertex function. All corresponding parameters (positions and residues of the poles in propagators, couplings at zero momenta,..) - but for the unphysical, non renormalised ones \( \alpha \) and \( \lambda^2 \) - require normalisation conditions, a point which has often been missed since the successes of minimal dimensionnal regularisation scheme [8]. In perturbative quantum expansion, this requires addition of definite counterterms into the Lagrangian density: the question of their being finite or not being merely a question of personal taste. For example, in the original BPHZ [9] substraction scheme, a definite Taylor expansion with respect to external momenta is subtracted from the Feynman integrand so that the integration over loops momenta becomes possible: some finite counterterms are then needed to implement the wanted normalisation conditions (of course, they will depend on some renormalisation scale \( \mu \) used to fix the normalisation conditions). In other schemes, infinite counterterms (plus finite ones of course !) are defined after some regularisation of the divergent integrals in order to achieve the same aim. In the BRS approach, the number of primitive divergences and corresponding counterterms is given by the dimension of the Fadeev-Popov 0-charge sector of the cohomology space of the Slavnov operator corresponding to the isometries that define the theory [10].

The second problem is related to the symmetries of the classical action: the physical meaningfulness of the quantum extension requires that the symmetries still hold at the quantum level, which is possible as soon as the coresponding Ward identity has no anomaly. As is well known, this may involve the addition of finite counterterms to the Lagrangian density (the so-called quantum corrections or spurious anomalies). In particular, each time one regularises a theory without respecting its symmetries, such non-symmetric quantum corrections are needed, and moreover the classical currents have to be redefined (renormalised) in order that their conservation should lead to the correct Ward identity (the correct "contact terms"...). A pedagogical example may be found in the second reference in [11]. In the BRS approach, the absence of anomaly corresponds to an empty Fadeev-Popov charge-1 sector of the cohomology space of the upper mentioned Slavnov operator.

Finally, the success of a perturbative theory lies in its ability to compute with precision some quantities such as S matrix elements whose classical values acquire radiative corrections, for example the Bremsstrahlung in annihilation processes: \( e^+ + e^- \rightarrow \text{photon} + \text{final state} \ X \), or the magnetic moment of the electron which is found to be 2 in the classical theory with Lagrangian (1) and gets some definite corrections in higher orders in the electric charge, or the Lamb-shift that results from higher loop contributions to the photon self-energy \( \Gamma^{\mu\nu}_{\text{transverse}} \), e.t.c...

One should not confuse these three concepts: Lagrangian counterterms (finite or not) required to get a definite perturbative expansion with definite values for the physical parameters of the theory; Lagrangian finite spurious anomalies or quantum corrections required to compensate for the use of a non symmetry preserving regularisation scheme; calculable radiative corrections to physical processes.

For instance, in QED the electromagnetic current is conserved, thanks to the gauge invariance of the theory; so one obtains the Ward identity:

\[
p^{\mu}\Gamma_{\mu\nu\cdots}^L (p_i ; q_j ; q_k ; q'_j) = -\frac{i}{\alpha} (p^2 - \alpha \lambda^2) p_{i\nu} \delta_{L}^i \delta_{N}^j + \sum_{k=1}^{N} \left[ \Gamma_{\mu\nu\cdots}^{L} (p_i ; q_j ; q_k ; p ; q'_j) - \Gamma_{\mu\nu\cdots}^{L} (p_i ; q_j ; q'_j ; q_k + p) \right]
\]
where $L$ is the number of external photons (of momenta $p_i : i = 1, 2, \ldots L$) and $N$ the number of incoming and outgoing electrons (of momenta $q'_j : j = 1, 2, \ldots N$), all momenta being ingoing ones. In particular, for $L = 1$, $N = 0$ this gives the announced non-renormalisation of the longitudinal part of the photon propagator $-\frac{i}{\alpha}(p^2 - \alpha \lambda^2)$. This Ward identity is best rewritten on $\Gamma$, the generating functional for proper Green functions:

$$\partial^\mu \frac{\delta \Gamma}{\delta A^\mu(x)} - ie[\bar{\psi}_\alpha(x) \frac{\delta \Gamma}{\delta \bar{\psi}_\alpha(x)} - \psi_\beta(x) \frac{\delta \Gamma}{\delta \psi_\beta(x)}] = \frac{1}{\alpha} [\Box + \alpha \lambda^2](\partial_\mu A^\mu).$$

Then, if one uses the BPHZ scheme that breaks gauge invariance, the addition of finite counterterms into the Lagrangian and a redefinition of the electromagnetic current is required in order for the Ward Identity (6) to be satisfied to all orders of perturbation theory. On the contrary, Pauli-Villars or dimensional regularisation only need symmetric counterterms (the usual Z factors).

In the same spirit, a softly broken axial symmetry exists at the classical level; but it does not survive quantisation because of the axial anomaly. This one is readily seen, for example within dimensional regularisation \cite{12,11}, to result from the impossibility of defining in an algebraically consistent way (in complex D dimensions) a matrix $\gamma^5$ that anticommutes with all Dirac matrices $\gamma^\mu : \gamma^\mu \gamma_\mu = D$ \cite{13}. So, when computing

$$\partial_\mu [\bar{\psi} \gamma^\mu \gamma^5 \psi](x)$$

with the help of the electron field equations, an “evanescent” contribution \cite{12,11} appears:

$$[\bar{\psi} \gamma^\mu \hat{D}_\mu \psi](x)$$

where $2 \hat{\gamma}^\mu = \{\gamma^\mu, \gamma^5\}$ and $D_\mu$ denotes the covariant derivative $\partial_\mu + ieA_\mu$. Moreover the axial current has also to be redefined, a point often missed (see for example controversies on Adler Bardeen theorem in super-Yang-Mills between 1982 and 1985 \cite{8} and a recent paper by Jacquot \cite{14}).

### 3 QED with odd-CPT Lorentz violating terms

As explained in the introduction Section, let us now consider QED \text{(equ.1)} with possible presence of CPT-odd, Lorentz violating terms:

$$\mathcal{L}_1(x) = -b^\mu \bar{\psi}(x) \gamma_\mu \gamma^5 \psi(x), \quad \text{where } b^\mu \text{ is a fixed vector},$$

$$\mathcal{L}_2(x) = \frac{1}{2} c^\mu \epsilon_{\mu\nu\rho\sigma} F^{\nu\rho}(x) A^\sigma(x), \quad \text{where } c^\mu \text{ is a fixed vector}. $$

Other breakings could be considered (see a discussion in the first paper of \cite{2}), but we simplify and require charge conjugation invariance, which selects $\mathcal{L}_1(x)$ and $\mathcal{L}_2(x)$. Note for further reference that experiments on the absence of birefringence of light in vacuum put very restrictive limits on the value of $c^\mu$, typically for a timelike $c^\mu$, $c^0/m \leq 10^{-38}$ \cite{2}.

Let us consider the classical Lagrangian density $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$. In order to avoid the difficulties resulting from the new poles in the propagators, we take into account the smallness of the breakings and include them into the interaction Lagrangian density as super-renormalisable
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couplings. Moreover, we define the photon and electron masses by the same normalisation
conditions as in ordinary Q.E.D., e.g.

\[ <\bar{\psi}(p)\psi(-p)>^{prop.} \big|_{\gamma=m, \ b=c=0} = 0, \ \cdots \]

According to standard results in renormalisation theory, these breakings add new terms in
the primitively divergent functions and require 2 new normalisation conditions to fix their
parameters \( b^\mu \) and \( c^\nu \):

\[
\begin{align*}
  b^\mu &= -\frac{i}{4} Tr[\gamma^\mu \gamma^5 <\bar{\psi}(p)\psi(-p)>^{prop.}] \big|_{p=0}, \\
  c^\mu &= \frac{1}{12} \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial p^\sigma} <A_\nu(p)A_\rho(-p)>^{prop.} \big|_{p=0}.
\end{align*}
\]  

Note that, contrary to \( L_1(x) \), the \( L_2(x) \) term also breaks the local gauge invariance of the
Lagrangian density, even if the action remains gauge invariant. If fields and the gauge parameter
function \( \Lambda(x) \) vanish sufficiently rapidly at infinity, there will be no difference; however, the
literature on this subject [4] emphasizes the difference, and we want to clarify this point. We
shall prove that the variation of \( L_2(x) \) in a local gauge transformation being linear in the
quantum field, no essential difference occurs.

So it is tempting to separate the discussion into two cases:

- QED with the two CPT odd, Lorentz breaking, C conserving terms : \( L_1(x) \) and \( L_2(x) \).
- QED with the sole breaking term \( L_1(x) \) : is it a consistent quantum theory?

### 3.1 Lagrangian of QED with CPT-odd, Lorentz and gauge breaking
terms.

Starting from the Lagrangian density \( L_0 + L_1 + L_2 \), let us analyse the Ward identity corre-
spanding to the gauge invariance of the action. The classical field equations are now:

\[
\begin{align*}
  (i \gamma^\mu \gamma^5) \psi(x) &= e A(x) \psi(x), \\
  \bar{\psi}(x)(-i \gamma^\mu \gamma^5) &= e \bar{\psi}(x) A(x), \\
  [\square + \lambda^2] A_\mu(x) - (1 - \frac{1}{\alpha}) \partial_\mu(\partial_\nu A^\nu(x)) - \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}(x) &= e \bar{\psi}(x) \gamma_\mu \psi(x); \\
\end{align*}
\]

then equations (9) ensure that the vector current \( [\bar{\psi} \gamma_\mu \psi](x) \) is conserved, and equation (10)
the fact that the “scalar” field \( \partial_\nu A^\nu(x) \) is a free field of squared mass \( \alpha \lambda^2 \).

After addition of adequate counterterms, these equations of motion may always be extended
to the quantum level [4, 13], and, as a consequence, the same is true for the vector current
conservation and the free-character of the longitudinal photon \( \partial_\nu A^\nu(x) \).

Let \( \Gamma \) be the classical action

\[
\Gamma = \int d^4x [L_0 + L_1 + L_2],
\]

the Ward identity writes:

\[
\int d^4x \left\{ \frac{1}{e} \partial_\mu \Lambda(x) \frac{\delta \Gamma}{\delta A_\mu(x)} + i \Lambda(x) \gamma^\mu \frac{\delta}{\delta \psi(x)} + \frac{\Gamma}{\delta \psi(x)} \frac{\delta}{\delta \bar{\psi}(x)} \right\} = \]
its quantum non-conservation due to the axial anomaly is not dangerous. However, should
Of course, as for ordinary QED, the axial current is not coupled to the fields of the model, and
The last equation is exactly the same as the one for ordinary QED (6): so, to select the
desired action, one needs an extra symmetry such as Lorentz invariance if one wants ordinary
QED (1), or some non-renormalisation theorem if one wants to consistently suppress the
\( L_2 \) term through a normalisation condition (Subsec. 3.2).

As soon as we use a regularisation that respects the symmetries (gauge, Lorentz covariance
and charge conjugation invariance), the perturbative proof of renormalisability reduces to the
check that the \( \mathcal{O}(\hbar) \) quantum corrections to the classical action \( \Gamma : \Gamma_1 = \Gamma_{\text{class.}} + \hbar \Delta \),
constrained by the Ward identity (11):

\[
W_x \Delta \equiv \frac{\delta \Delta}{\delta A_\mu(x)} - ie[\bar{\psi}(x) \frac{\delta}{\delta \psi}(x) - \frac{\Delta}{\delta \psi(x)} \psi(x)] = 0 ,
\]

may be reabsorbed into the classical action through suitable renormalisations of the fields
and parameters of the theory. Thanks to the quantum action principle \([17, 10]\), \( \Delta \) is a
charge conjugation invariant integrated local polynomial in the fields, their derivatives and
the parameters of the theory, of dimension 4 (recall that the photon field and the parameters
\( m, b^\mu, c^\nu \) have dimension 1, the electron field dimension 3/2): then the general solution of
(12) is readily shown to be of the same form as the classical action \( \Gamma \) (without the gauge fixing
and photon mass terms) \( Q.E.D. \).

The breakings (7) introduce two operators which may be defined through a modification of
the classical action: to \( \Gamma \) we add two source terms for the \( C = +1 \), dimension 3 insertions

\[
J_5^\mu = [\bar{\psi} \gamma_\mu \gamma_5 \psi](x) \quad \text{and} \quad K_5^\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\nu\rho} A^\sigma(x) : \quad \bar{\Gamma} = \Gamma + \int d^4x \left[ \alpha^\mu(x) J_5^\mu(x) + \beta^\mu(x) K_5^\mu(x) \right].
\]

Then, as soon as the renormalisation is properly done, the operators being bilinear in the
quantum fields, the Ward identity (11) holds true, to all orders of perturbation theory (all
orders in \( \hbar, e, b^\mu, c^\nu, m \), for the Green functions with one insertion of either of these
operators:

- action on (11) of \( \frac{\delta}{\delta \alpha^\lambda(y)} \) for the gauge invariant axial current \( J_5^\mu(y) \), the right-hand side
  vanishing: \( W_x \frac{\delta \Gamma}{\delta \alpha^\lambda(y)} \big|_{\alpha=\beta=0} = 0 \),

- action on (11) of \( \frac{\delta}{\delta \beta^\lambda(y)} \) for the non gauge invariant operator \( K_5^\lambda(y) \), the right-hand side
  reducing itself to a tree contribution as the variation is linear in the photon field:
  \( W_x \frac{\delta \Gamma}{\delta \beta^\lambda(y)} \big|_{\alpha=\beta=0} = -\frac{1}{2} \varepsilon_{\lambda\nu\rho\sigma} F^{\nu\rho}(y) \partial_y^\sigma \delta(y - x) \).

All Green functions are of course computed with the complete Lagrangian density \( \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \).
Of course, as for ordinary QED, the axial current is not coupled to the fields of the model, and
its quantum non-conservation due to the axial anomaly is not dangerous. However, should
one consider the CPT breaking extensions of the standard $SU(3) \times SU(2) \times U(1)$ model, the generalisation of the Adler-Bardeen non renormalisation theorem would be necessary in order for the one loop cancellation of the axial anomaly to stay to all orders.

As particular consequences of the previous discussion, the complete proper 2-photon Green function $\Gamma_{\mu\nu}(p, -p)$ satisfies

$$p^\mu \Gamma_{\mu\nu}(p, -p) = p^\nu \Gamma_{\mu\nu}(p, -p) = 0,$$  

up to the classical longitudinal contribution, (13)

the one with one axial-current insertion verifies:

$$p^\mu < A_\mu(p) A_\nu(q) \bar{\psi} \gamma^\rho \gamma^5 \psi >^{prop.} =$$

$$q^\nu < A_\mu(p) A_\nu(q) \bar{\psi} \gamma^\rho \gamma^5 \psi >^{prop.} = 0,$$  

(14)

and the one with a $K^5_{\mu}$ insertion

$$p'^\mu < A_{\mu'}(p) A_\mu(q) \frac{1}{2} \epsilon_{\mu \nu \sigma \lambda} F^{\nu \sigma} A^\lambda(-(p + q)) >^{prop.} =$$

$$q'^\nu < A_\mu(p) A_{\mu'}(q) \frac{1}{2} \epsilon_{\mu \nu \sigma \lambda} F^{\nu \sigma} A^\lambda(-(p + q)) >^{prop.} = -i \epsilon_{\mu \rho \sigma} p^\sigma q^\lambda.$$  

(15)

Note that the validity of these equations is not a question of personal taste or choice: if the renormalisation is correctly done, they do hold true as a consequence of the gauge invariance of the action (moreover, as we shall show, in the absence of (14, 15), the finiteness of the CPT-odd part of the photon self energy will remain accidental).

We now illustrate these results by a one-loop calculation. We first need a regularisation to give a meaning to the loops integrals involved in those Green functions. Any consistent one is as good as any other, but it is simpler to consider regularisations that respect the invariances of the classical theory, here gauge invariance. A question arises about Lorentz non-invariance: a priori there is no longer symmetric integration and averaging formulas for $l^\mu l^\nu$ (where $l$ is a loop momentum variable): any computation will become fairly hard; in particular, linear divergences would stay\footnote{For instance as:}

$$\int_{-\Lambda}^{\Lambda'} \frac{x \, dx}{\sqrt{(x + p)^2 + m^2}} = \Lambda' - \Lambda + 2p - p \log \frac{4\Lambda \Lambda'}{m^2} + O(1/\Lambda, 1/\Lambda').$$

\footnote{Let us emphasize that if one uses correct (anti-)commutation relation of Dirac matrices $\gamma^\mu$ and $\gamma^5$ (Breitenlhoner and Maison\footnote{\cite{13}}), there will be no ambiguity in the results and a minimal subtraction can be safely done (it corresponds to some implicit but definite normalisation conditions).} and require extra subtractions and counterterms. However, as discussed in\footnote{\cite{13}}, the spirit of the spontaneously broken Lorentz invariance is that, except for the vacuum expectation values $b^\mu$ and $c^\nu$ of some fields, Lorentz invariance holds: Colladay and Kostelecky speak of true “observer Lorentz invariance”\footnote{\cite{12}}. So we shall use the Lorentz preserving consistent dimensional regularisation of t’Hooft and Veltmann\footnote{\cite{12, 13}} and, in particular, check at the one-loop order that the correction to the self-energy of the photon of first order in $b^\mu$ unambiguously vanishes at $p^2 = 0$, in agreement with the theorem in the appendix of\footnote{\cite{3}}.

Note that gauge invariance\footnote{\cite{13}} ensures, as in standard Q.E.D., that the 4-photon Green function is not primitively divergent. Then $\partial \Delta L_0$, the standard counterterms of Q.E.D., should be added to $L_0$. Let us now compute the $b^\mu$ and $c^\nu$ dependent counterterms, to one-loop order.
3.1.1 The one-loop electron self-energy

To first order in the small breaking parameters $b^\mu$ and $c^\mu$, a one-loop calculation gives:

\[-i < \psi(p)\bar{\psi}(-p) >^{prop} = \left[ \gamma^\mu \left[ B - m - B^5 \right] \right] (1 + \alpha I_\infty) - 3mI_\infty - 3 \alpha \gamma^5 I_\infty + \]

\[
+ \text{ regular } (\mu^2 \text{ independent}) \text{ terms, where } I_\infty = \frac{\hbar e^2}{16\pi^2} \frac{2}{4 - D} - (C + \log \frac{m^2}{4\pi\mu^2}) ;
\]

$C$ is the Euler constant and $\mu^2$ the scale needed in the dimensional scheme,

\[
\frac{d^4k}{(2\pi)^4} \Rightarrow \mu^{(4-D)} \frac{d^Dk}{(2\pi)^D}.
\]

In such a calculation as there is no need for traces of Dirac matrices, there is no unconsistency in using a fully anticommuting $\gamma^5$ matrix. So we have the standard Q.E.D. renormalisations of the fermionic field (wave function and mass) and a $c^\mu$ dependent renormalisation of the $b^\mu$ parameter in $\mathcal{L}_1$.

Higher orders in $b^\mu$ and $c^\mu$ are given by convergent integrals, then contribute as regular $(\mu^2$ independent) functions at $D = 4$, and do not change the result (16). Of course, should one implement the normalisation condition (8), extra finite counterterms renormalizing $\mathcal{L}_1$ should be required. Indeed, Lorentz covariance and (16) ensures that

\[
-\frac{i}{4} Tr \left[ \gamma^\mu \gamma^5 < \psi(p)\bar{\psi}(-p) >^{prop} \right] \bigg|_{p=0} \simeq (\alpha b^\mu + 3c^\mu) I_\infty +
\]

\[
+ b^\mu a_1 \left[ \frac{b^2}{m^2}, \frac{c^2}{m^2}, \frac{b.c}{m^2} \right] + c^\mu a_2 \left[ \frac{b^2}{m^2}, \frac{c^2}{m^2}, \frac{b.c}{m^2} \right].
\]

Note also that, as we choose to fix the wave function and the mass of the electron by the usual normalisation conditions at vanishing $b^\mu$ and $c^\mu$, no finite new counterterms, such as $b^\mu b^\nu [\bar{\psi} \gamma_\alpha \partial_\beta \psi] / m^2$, will be required. At this point, additive renormalisation requires a $\hbar \Delta \mathcal{L}_1(c, b)$ counter-lagrangian.

3.1.2 The one-loop photon self-energy

In the same way, the self energy of the photon may be expanded in powers of $b^\mu$ (up to one-loop order, $\mathcal{L}_2$ does not modify the photon self energy, except at the tree level). The $b^\mu$ independent contribution comes from ordinary Q.E.D, the $b^\mu$ linear one being given by the two “triangle-like” graphs with a zero-momentum “axial vertex”:

\[
I_{\mu\nu}(p) = -\hbar e^2 b^\mu \int \frac{d^4q}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{q} + m)\gamma_\nu(\not{q} + \not{p} + m)\gamma_\alpha\gamma^5(\not{q} + \not{p} + m)]}{(q^2 - m^2)[(q + p)^2 - m^2]^2} ;
\]

and

\[
-\hbar e^2 b^\mu \int \frac{d^4q}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{q} + m)\gamma_\alpha\gamma^5(\not{q} + m)\gamma_\nu(\not{q} + \not{p} + m)]}{[q^2 - m^2]^2[(q + p)^2 - m^2]} =
\]

\[
= -\hbar e^2 b^\mu \int \frac{d^4q}{(2\pi)^4} \frac{Tr[\gamma_\nu(\not{q} + \not{p} + m)\gamma_\mu(\not{q} + m)\gamma_\alpha\gamma^5(\not{q} + m)]}{[q^2 - m^2]^2[(q + p)^2 - m^2]} \bigg|_{q \rightarrow -p} I_{\nu\mu}(p) .
\]

\(^3\)We used $\gamma_\mu\gamma^\mu = D$ and $k_\mu k_\nu = k^2 q_{\mu\nu}/D$ : this allows shifting of internal momenta and ensures gauge invariance. Other prescription would lead to different finite counterterms (spurious anomalies). This point is sometimes missed, in particular in reference (8) (after eq. 7), leading to differences proportional to $4 - D$ which give extra finite contributions, when involved in divergent integrals.
as soon as shift of loop momenta is allowed and thanks to the cyclicity of the trace of a product of matrices (this last property is independant of the dimension of the matrices and should hold in any consistent dimensional regularisation). The only algebraic properties we need for the calculation in D-dimension are the algebraically consistent ones [15, 13, 8] :

- as previously emphasized, $\gamma_\mu \gamma^\mu = D$ and $k_\mu k_\nu = k^2 g_{\mu\nu} / D$;
- the trace of $\gamma^5$ times an odd number of Dirac matrices vanishes,
- the trace of $\gamma^5$ times an even number of Dirac matrices can be reduced with the Clifford algebra to the quantity $Tr[\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma^5]$; consistency enforces $Tr[\gamma_\alpha \gamma_\beta \gamma^5] = Tr[\gamma^5] = 0$.

Using Feynman parameters to combine the denominators in (18), the first triangle contribution gives

$$-\frac{\hbar c^2}{8\pi^2} b^\alpha p^\beta Tr[\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma^5] \times (A + B)$$

where the divergent part comes from :

$$A = \int_0^1 dx [x(4 - 6x) - (4 - D)x(2 - x)] \int \frac{(4\pi \mu^2)^{2-D/2} dD q'}{\pi^{D/2}} \frac{q^2 / D}{[q^2 + x(1 - x)p^2 - m^2]^3} ;$$

and $B$, a convergent one, may be evaluated at $D = 4$ :

$$B = -\int_0^1 dx [x^2(1 - x)^2 p^2 + x(2 - x)m^2] \int \frac{1}{\pi^2} \frac{dD q'}{[q^2 + x(1 - x)p^2 - m^2]^3} .$$

As $\int_0^1 dx x(4 - 6x) = 0$, the divergent part vanishes and, after Wick rotation and integration on $q'$, one is left with the finite quantity :

$$A + B = \frac{i}{2} \int_0^1 dx \left[ x(2 - 3x) \log \frac{x(1 - x)p_E^2 + m^2}{4\pi \mu^2} - x(2 - x) - \frac{x^2(1 - x)p_E^2 - x(2 - x)m^2}{x(1 - x)p_E^2 + m^2} \right] .$$

An integration by parts on $x$ for the log term finally gives :

$$A + B = -\frac{i}{2} b^2 K(p^2) = -i p_E^2 \int_0^1 dx \frac{x^2(1 - x)}{x(1 - x)p_E^2 + m^2} \ni \rho - \frac{1}{2} \frac{p^2}{12m^2} ,$$

a finite result, which moreover vanishes for $p^2 = 0$. The complete one-loop 2-photon proper Green function is :

$$< A_\mu (-p) A_\nu (p) >_{\text{prop.}}^{\text{transverse}} =$$

$$-i (g_{\mu\rho} p^\rho - p^\mu p^\rho) [1 - 4 \rho I_\infty + \text{regular } (\mu^2 \text{ independant terms})]$$

$$-\varepsilon_{\mu\nu\rho\sigma} b^\sigma [2c^\rho + b^\rho \frac{\hbar c^2}{2\pi^2} b^\beta K(p^2)] , \quad p^2 K(p^2) \text{ is analytic for } p^2 < 4 m^2 ,$$

$$p^2 K(p^2) = 1 - \frac{\log \rho \sqrt{1 + \rho^2}}{\rho \sqrt{1 + \rho^2}} \sim \frac{p_E^2}{6m^2} \text{ when } p_E^2 \to 0 \text{ (with } \rho = \sqrt{\frac{p_E^2}{4m^2}}) .$$

Note that, here also, contributions with higher order dependence on $b^\mu$ are given by convergent integrals (then contributing as $\mu^2$ independent terms), thanks to gauge invariance (the unique logarithmically divergent polynomial term $b^\alpha b^\beta \Pi_{\mu\nu,\alpha\beta}$ cannot be transverse on $p^\mu$ and $p^\nu$).

A few remarks are in order to understand the results :
• The finiteness does not come from a cancellation between the two “triangle-like” graphs: on the contrary, each of the triangle being superficially linearly divergent, its divergent part is a dimension-1 polynomial in masses and external momentum $p$. The only possible structure (in any regularisation that respects Lorentz covariance) is $\epsilon_{\mu\nu\alpha\beta}p^\beta$, then the divergence is at most a logarithmic one; moreover, thanks to Bose symmetry ($\mu \leftrightarrow \nu, p \leftrightarrow -p$), it doubles itself when the two triangles are added.

• Then, to understand the convergence, it is necessary to use gauge invariance and the Ward identity (14) for the operator $J_\alpha^5$. First, note that the $\mathcal{O}(b^\alpha)$ term is given by the axial vertex $< A_\mu(q)A_\nu(p)J_\alpha^5(-(p+q)) >^{prop.}$ for $q + p = 0$. Second, the divergent part of this vertex is a dimension-1 polynomial in masses and external momenta $p$ and $q : A \epsilon_{\mu\nu\alpha\beta}(p-q)^\beta$, thanks to Bose symmetry, constrained by gauge invariance (14), then it vanishes. Third, the value of this vertex, thus given by convergent integrals, is uniquely fixed, independently of the regularisation used, as soon as the vector current conservation is ensured (see the nice ”old” discussion in Adler’s 1970 lectures [14], and of course the same occurs for the photon self energy correction $b^\alpha[I_{\mu\nu\alpha}(p) + I_{\nu\mu\alpha}(p)]$.

• As argued by Coleman and Glashow [3], due to analyticity, “from the once well known theory of Feynman-diagram singularities”, this Chern-Simons like contribution will vanish. Indeed, Lorentz covariance and a generalisation of Coleman and Glashow’s argument ensure that

\[
\frac{1}{12} \epsilon_{\mu\nu\rho\sigma} \frac{\partial}{\partial p^\rho} < A_\nu(p)A_\rho(-p) >^{prop.} \bigg|_{p=0} = 0
\]

(consider the proper Green function of two photons $A^\alpha(p)$ and $A^\beta(q)$ with one insertion of $b^\mu J_\mu^5(-(p+q))$, computed with the complete Lagrangian density $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$, i.e. to all orders in $b^\nu$, and the corresponding Ward identity (14)). Note also that, as we choose to fix the wave function of the photon by the usual normalisation conditions at vanishing $b^\mu$ and $c^\nu$, no finite new counterterm, such as $b^2[F_{\mu\nu}F^{\mu\nu}]/m^2$ e.t.c., will be required.

At this point, additive renormalisation requires no $\hbar \Delta \mathcal{L}_2(c, b)$ counter-lagrangian.

3.1.3 The one-loop photon-electron vertex

Finally, simple power counting shows that the vertex function $\langle \psi(p)\bar{\psi}(q) A_\mu(-(p+q)) \rangle^{prop.}$ has no $b^\mu$ or $c^\beta$ dependent divergence: so, as we choose to fix the electron-photon vertex by the usual normalisation condition at vanishing $b^\mu$ and $c^\nu$, the Q.E.D. counter-lagrangian $\hbar \Delta \mathcal{L}_0$ ensures finiteness and correct normalisation conditions.

3.1.4 Higher-loop orders

To summarise, to one-loop order, given the classical Lagrangian $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$, renormalisation only requires the counterterms $\hbar(\Delta \mathcal{L}_0 + \Delta \mathcal{L}_1)$ : if $b^\mu$ is renormalised to $b^\mu_c(b, c)$, no $\hbar \mathcal{L}_2$ correction has appeared. The Chern-Simons like term is not renormalised - even if the $c^\mu$ parameter
is rescaled by a $Z_3^{-1}$ factor, to compensate for the usual photon field renormalisation $\sqrt{Z_3}$. Does this stay in higher orders?

Let us suppose that, up to $N$-loop order, renormalisation has been done with counterterms $\Delta L_0 + \Delta L_1$ and that the only dependence on $b^r$ and $c^r$ of the Lagrangian plus counterterms is through the classical $L_2$ term and $b_N^r(b, c)[\bar{\psi}\gamma_\mu\gamma^5\psi](x)$, $b_N^r(b, c)$ being an order-$N$ polynomial in $h$.

- One of the primitively divergent Green functions is the electron-photon vertex. As previously argued, its divergence and normalisation condition are the same for ordinary Q.E.D., and so taken into account by the counterterm $\Delta L_0$.

- To $(N+1)$-order, the $b^\mu$, $c^\mu$ dependent part of the divergence of the self-energy of the electron (subdivergences being properly subtracted) is proportional to $b^\mu\gamma_\mu\gamma^5$ and $c^\mu\gamma_\mu\gamma^5$. Moreover, as

$$-\frac{i}{4} Tr[\gamma^\mu\gamma^5] <\psi(p)\bar{\psi}(-p)>^{prop.}|_{p=0} = b^\mu a_{N+1}[\frac{\mu^2}{m^2}] + c^\mu b_{N+1}[\frac{\mu^2}{m^2}] + b^\mu c_{N+1}[\frac{b^2}{m^2}, \frac{c^2}{m^2}, \frac{b}{m}, \frac{c}{m}, \frac{b}{m^2}, \frac{c}{m^2}] + c^\mu \beta_{N+1}[\frac{b^2}{m^2}, \frac{c^2}{m^2}, \frac{b}{m}, \frac{c}{m}, \frac{b}{m^2}, \frac{c}{m^2}],$$

the normalisation conditions (5) require only a $L_1$ like counterterm.

- On the other hand, the derivative with respect to $b^\alpha$ of the regularised photon self-energy at $(N+1)$-loop order is given by \[7\] by:

$$\frac{\partial}{\partial b^\alpha} \Gamma^{(N+1)}(p, -p) = -i \left[ \frac{\partial b_N^\beta}{\partial b^\alpha} < A_\mu(p) A_\nu(-p) | \int d^4x \{ \bar{\psi}\gamma_\beta\gamma^5\psi \}(x) >^{prop.} \right]^{(N+1)} =$$

$$= -i \sum_{l=0}^{N} \left[ \frac{\partial b_N^\beta}{\partial b^\alpha} \right]^{(l)} \left[ < A_\mu(p) A_\nu(-p) | \int d^4x \{ \bar{\psi}\gamma_\beta\gamma^5\psi \}(x) >^{prop.} \right]^{(N+1-l)}.$$

First, this quantity is finite: the proof is the same as the one given in Subsection (3.1.2), based on the vanishing of a dimension-one, Lorentz covariant 3-tensor $a_{\mu\nu\beta}$, polynomial in masses, parameter $b^\alpha$, external momenta $p$ and $q$, transverse with respect to $p^\mu$ and $q^\nu$, and associated to the divergence of any n-loop order ($n \leq N + 1$) axial (unintegrated) vertex $< A_\mu(p) A_\nu(q) | \bar{\psi}\gamma_\beta\gamma^5\psi | (-p + q) >^{prop.}$, with subdivergences properly subtracted.

- Second, we look for possible finite counterterms required by the normalisation conditions (5); following Coleman and Glashow’s argument analyticity and gauge invariance of the n-loop order axial (unintegrated) vertex, i.e. vector current conservation, enforce its proportionality to some $p^\alpha q^\beta G_{\mu\nu\rho\sigma}^{(n)}(p, q, b, m)$, $\forall n$, giving after integration $-p^\alpha q^\beta G_{\mu\nu\rho\sigma}^{(n)}(p, -p, b, m)$.

On the other hand, the derivative with respect to $c^\alpha$ of the regularised photon self-energy at $(N+1)$-loop order is given by \[7\] by:

$$\frac{\partial}{\partial c^\alpha} \Gamma^{(N+1)}(p, -p) = -i \left[ < A_\mu(p) A_\nu(-p) | \int d^4x \{ \frac{1}{2} \epsilon_{\alpha\beta\lambda} F_{\rho\sigma} A^\lambda \}(x) >^{prop.} \right]^{N+1}$$

$^4$The existence of an anomaly for the axial current conservation law does not enter the game as we used only vector current conservation.
In that equation, the vertex insertions are at least at one loop order, so the same argument as before holds, thanks to the Ward identity (15).

As a consequence, to (N+1)-loop order, the quantity

\[
\frac{1}{12} \varepsilon_{\mu\nu\rho\sigma} \frac{\partial}{\partial p^\sigma} < A_\nu(p) A_\rho(-p) >^{\text{prop.}} |_{p=0}
\]

is given by its value for vanishing \(b^a\) and \(c^3\). Then the normalisation conditions (8) may be ensured without any \(\Delta L_2\) counterterm.

- The \(b^a\) and \(c^\nu\) independent part of the electron and photon self-energy being correctly renormalised by the counterterm \(\Delta L_0\), \(\Delta L_0 + \Delta L_1\) ensures the correct renormalisation up to order \((N+1)\).

Q.E.D.

So, to all orders of perturbation theory, the theory described by \(L_0 + L_1 + L_2\) is a quantum consistent theory, with an (infinite) renormalisation of the photon field, electron mass, electric charge and \(b^a\) parameter, and no \(L_2\) renormalisation.

### 3.2 Gauge invariant QED with a CPT-odd, Lorentz breaking term.

Consider now a possible situation without \(L_2\) in the classical lagrangian, i.e. with a value zero for the parameter \(c^\mu\) defined by the normalisation condition (8). The analysis of the previous Subsection shows that \(L_2\) is not renormalised: then, its absence at the classical level is stable against perturbative expansion, to all-loop order, and the theory described by \(L_0 + L_1\) is a quantum consistent theory, with an (infinite) renormalisation of the photon field, electron mass, electric charge and \(b^a\) parameter.

### 4 Discussion and summary

We have exemplified the fact that, as soon as a theory is correctly defined (not only by a Lagrangian density such as \(L_0 + L_1 + L_2\), but by some symmetry requirement such as gauge invariance of the action and appropriate normalisation conditions (8), the quantum corrections are unambiguous.

The opposite conclusion often given in the literature \([4, 7]\), results from an unsufficient definition of the model and some unprecised arguments:

- Jackiw and Kostelecky \([4]\) never introduce any regulator. Then some of their relations are delicate ones: see for example for a divergent integral (after equ.12), the commutation of a derivation with respect to external momentum and the integration. If the integral in their equation (11), which is twice our tensor \(I_{\mu\nu\alpha}(p)\), is computed with dimensional regularisation, a result \(\theta \sin \theta - 1\) is found, and not simply \(\frac{\theta}{\sin \theta}\) (with \(p^2 = 4m^2 \sin^2 \theta/2\)).

- Moreover, in the absence of normalisation conditions or Ward identities fixing some ambiguities, the difference of two equivalent linearly divergent integrals gives an ambiguous logarithmic divergent one. Even when one uses a symmetric integration that suppresses the linearly (and eventually the logarithmically) divergent part, the finite part remains ambiguous. The “surface term” that comes from a shift in the integration momentum in
a linearly divergent integral is a regulator dependent quantity: if one redoes the calculation in the appendix A5-2 of Jauch and Rohrlich standard book \[18\] with the dimensional scheme, one easily checks that no "surface term" occurs after a shift of the integration momenta \[14\]). Recall that this possibility of shifting internal momenta is needed to preserve gauge invariance in loop calculations (see for example \[14, subsect.17.9\]).

- If the gauge invariance constraint \[14\] is not imposed, the finiteness of the "triangles" appears as purely accidental \[4, 7\] and would not stay in higher orders. So, some authors rightly conclude that the corresponding one-loop finite contribution is ambiguous \[4\]. Note that in the main text of \[20\], Jackiw correctly summarizes the discussion of his Section 4 by the sentence "An arbitrary value persists only when no symmetry is enforced", but he gave his paper a misleading title :"When radiative corrections are finite but undetermined". Such unexpected finiteness also occurs in other situations, for example in the so-called complex sine-Gordon model: as was shown in \[21\] the origin lies, not in some isometry of the theory, but in the physical property of non-production for the S-matrix.

- Although local gauge invariance is lost at the level of the Lagrangian density, the breaking is linear in the quantum fields, and then the invariance of the action ensures the validity of the usual Ward identity that corresponds to vector current conservation. So, the difference advocated by Jackiw and others between a term in the Lagrangian density locally gauge invariant and one giving a gauge invariant contribution to the action is not relevant for the present case, as, but for the tree level, such Green functions with insertion satisfy the same usual Ward identity \[11\].

- Finally, it is difficult to see the difference advocated in \[4\] between a first order (in \(b^\mu\)) perturbative calculation and what is claimed to be a “non-perturbative unambiguous value”, but is, as a matter of fact, obtained with exactly the same triangle integrals as everyone. More precisely, after the expression given in \[4, equation (5)\] for a complete (non-perturbative) contribution of the breaking to the 2-photon one-loop Green function, it is argued that, as the linear divergences cancel among the two terms of the integral, there will be no “momentum routing” ambiguity, and then a unique value will be obtained, for example by an expansion in the parameter \(b^\mu\). As we previously explain, this argument is uncorrect. Moreover, in \[4\] the computation is also done to all orders in the breaking parameter \(b^\mu\) and it is explicitly verified that higher orders do not contribute to a possible correction to \(c^\mu\).

Note also that the Lagrangian density \(\mathcal{L}_0 + \mathcal{L}_2\) would not lead to a coherent theory as an (infinite) counterterm \(\mathcal{L}_1\) appears at the one-loop order \(16\).

To summarize, we have proven that a theory with a vanishing tree level Chern-Simons like breaking term is consistent as soon as it is correctly defined: thanks to the gauge invariance of the action, the normalisation condition \(c^\mu = 0\) may be enforced to all orders of perturbation theory. A \(\mathcal{L}_2\) term, bilinear in the gauge field, appears in facts as a minor modification of the gauge fixing term as \(\partial_\nu A^\nu\) remains a free field: then, as part of the “gauge term”, it is, as usual, not renormalised.

\[5\] The vector current conservation \[13\] only imposes the transversality of the (divergent part of the) photon self energy, which does not excludes a \(\epsilon_{\mu\nu\alpha\beta} b^\alpha p^\beta\) infinite contribution.
Of course, the 2-photon Green function receives definite radiative corrections
\[
\simeq \frac{\hbar e^2}{12\pi^2} \frac{p^2}{m^2} \epsilon_{\mu \nu \rho \sigma} \rho^\rho b^\sigma + \cdots
\]
Recall the case of the electric charge: physically measurable quantities occur only through the \( p^2 \) dependence of the photon self-energy (as the Lamb-shift is a measurable consequence of a non-measurable charge renormalisation). Unfortunately, as Coleman and Glashow explained, the absence of birefringence of light in vacuum, i.e. the vanishing of the parameter \( c^\nu \), gives no constraint on the value of the other one \( b^\mu \).

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