On the tightness of Tietäväinen’s bound for distributions with limited independence

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Abstract

In 1990, Tietäväinen showed that if the only information we know about a linear code is its dual distance $d$, then its covering radius $R$ is at most $\frac{n}{2} - (\frac{1}{2} - o(1))\sqrt{dn}$. While Tietäväinen’s bound was later improved for large values of $d$, it is still the best known upper bound for small values including the $d = o(n)$ regime. Tietäväinen’s bound holds also for $(d-1)$-wise independent probability distributions on $\{0,1\}^n$, of which linear codes with dual distance $d$ are special cases. We show that Tietäväinen’s bound on $R - \frac{n}{2}$ is asymptotically tight up to a factor of 2 for $k$-wise independent distributions if $k \leq n^{1/3} \log^2 n$. Namely, we show that there exists a $k$-wise independent probability distribution $\mu$ on $\{0,1\}^n$ whose covering radius is at least $\frac{n}{2} - \sqrt{kn}$. Our key technical contribution is the following lemma on low degree polynomials, which implies the existence of $\mu$ by linear programming duality. We show that, for sufficiently large $k \leq n^{1/3} \log^2 n$ and for each polynomial $f(v) \in \mathbb{R}[v]$ of degree at most $k$, the expected value of $f$ with respect to the binomial distribution cannot be positive if $f(w) \leq 0$ for each integer $w$ such that $|w - n/2| \leq \sqrt{kn}$. The proof uses tools from approximation theory.

1 Introduction

The covering radius of a subset $C$ of the Hamming cube $\{0,1\}^n$ is the minimum $R$ such that any vector in $\{0,1\}^n$ is within Hamming distance at most $R$ from $C$. Studying the relation between the covering radius of a binary linear code and its dual code goes back to Delsarte [1] (see also Helleseth, Kløve, and Mykkeltveit [2] and Sole [3]). For a general reference on covering codes, see Cohen, Honkala, Litsyn, and Lobstein’s book [4].

Based on Delsarte linear programming relaxation [5], Tietäväinen showed in 1990 that if the only information we know about a linear code $C$ is its dual distance $d$, then its covering radius $R$ cannot be too large:

Theorem 1.1 (Tietäväinen [6, 7]) (Upper bound on the covering radius of codes in terms of dual distance) Let $C \subset \mathbb{F}_2^n$ an $\mathbb{F}_2$-linear code whose dual has minimum distance $d \geq 2$. Then the covering radius $R$ of $C$ is at most

\[
\begin{align*}
&\text{if } d = 2s \text{ is even} \quad \frac{n}{2} - \sqrt{s(n-s) + s^{1/6}\sqrt{n-s}} & \text{if } d = 2s + 1 \text{ is odd},
&\text{if } d = 2s - s^{1/6}\sqrt{n-s} + s^{1/6}\sqrt{n-1-s} - \frac{1}{2}
\end{align*}
\]

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Tietäväinen’s bound was later improved in the $d = \Theta(n)$ regime in a sequence of works [8] - [17] by Sole, Stokes, Honkala, Litsyn, Tietäväinen, Struik, Honkala, Laihonen, Ashikhmin, and Barg. See also Fazekas and Levenshtein [18] for extensions to polynomial metric spaces and Chapters 8 and 12 in [4].

For sufficiently small values of $d = \Theta(n)$, Tietäväinen’s bound is still the best known upper bound on the covering radius as a function of dual distance. Actually, Tietäväinen argued in [7] that improving his bound in the $d = o(n)$ regime is difficult since this regime includes dual BCH codes and accordingly improvements would give new interesting results on character sums.

The focus of this paper is on $d = o(n)$, i.e., on rate-zero linear codes of subexponential size. A natural question is how tight Tietäväinen’s bound is in this regime. That is, if $d$ is sub-linear in $n$, what can we say about the covering radius of a code given only its dual distance $d$?

As noted by Tietäväinen [6], we know from dual BCH codes that if $n = 2^m - 1$, where $m \geq 2$ and $s \geq 1$ are integers such that $s < \frac{1}{2} \sqrt{n+1} + 1$, then there are codes with dual distance $2s + 1$ and covering radius $R$ satisfying the lower bound [1]

$$R \geq \frac{n}{2} - (s - 1)\sqrt{n+1} - \frac{1}{2}.$$  

(1)

Asymptotically, the lower bound on $R - \frac{n}{2}$ in (1) is away from Tietäväinen’s bound by a $\sqrt{\frac{n}{2}}$ factor, which is considerable for $d = w(1)$.

While we do not resolve in this paper the question of tightness of Tietäväinen’s bound for linear codes in the $d = o(n)$ regime, we show that it is essentially tight for the bigger class of $k$-wise independent distributions in the $k \leq n^{1/3}$ regime.

A probability distribution $\mu$ on $\{0, 1\}^n$ is called $k$-wise independent if sampling $x \sim \mu$ gives a random vector $x = (x_1, \ldots, x_n)$, where each $x_i$ is equally likely to be 0 or 1 and any $k$ of the $x_i$’s are statistically independent. Linear codes with dual distance $d$ are special cases of $k$-wise independent probability distributions on $\{0, 1\}^n$, where $k = d - 1$; if $\mu$ is a probability distribution on $\{0, 1\}^n$ uniformly distributed on an $F_2$-linear code $C \subset F_2^n$, then $\mu$ being $k$-wise independent is equivalent to $C$ having dual minimum distance at least $k + 1$. If $\mu$ is a probability distribution on $\{0, 1\}^n$, define the covering radius of $\mu$ to be the covering radius of its support.

Tietäväinen’s bound is based on the following lemma which asserts the existence of certain low degree polynomials. Let $B_n$ be the binomial distribution on $[0 : n] := \{0, \ldots, n\}$, i.e., $B_n(w) := \binom{n}{w}$.

**Lemma 1.2 (Tietäväinen [6, 7]) (Low degree polynomials lower bound)** Let $1 \leq k \leq n - 1$ be integers. There exists a polynomial $p(v) \in \mathbb{R}[v]$ of degree at most $k$ such that $\mathbb{E}_{B_n} p > 0$ and $p(w) \leq 0$, for each $w \in [0 : n]$ such that

$$w \geq \left\{ \begin{array}{ll} \frac{n}{2} - \sqrt{s(n-s) + s^{1/6} \sqrt{n-s}} + 1 & \text{if } k = 2s - 1 \text{ is odd} \\ \frac{n}{2} - \sqrt{s(n-1-s) + s^{1/6} \sqrt{n-1-s}} + 1 & \text{if } k = 2s \text{ is even.} \end{array} \right.$$  

Tietäväinen established his bound using Krawtchouk polynomials. It is not hard to see that Lemma [12] actually shows more than Theorem [14]: it gives the following upper bound on the covering radius of $k$-wise independent distributions:

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1 The lower bound in (1) follows immediately from Weil-Carlitz-Uchiyama’s bound (it is also slightly better than the lower bound $R \geq \frac{n}{2} - \sqrt{n+1}$ stated on p. 1473 in [6]). Let $n = 2^m - 1$, where $m \geq 2$ an integer, and let $s \geq 1$ be an integer such that $2s - 2 < 2^{m/2}$, i.e., $s < \frac{1}{2} \sqrt{n+1} + 1$. Weil-Carlitz-Uchiyama’s bound (see [19]) asserts that for each non-zero codeword $x \in BCH(s, m)$, we have $|x| - 2^{m-1} \leq (s-1)2^{m/2}$. Thus, (1) holds because the all-ones vector $1 \notin BCH(s, m)$ because $n$ is odd and $1 \notin BCH(s, m)$. 

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2
Corollary 1.3 (Upper bound on the covering radius of $k$-wise independent distributions) Let $1 \leq k \leq n-1$ be integers and let $\mu$ be a $k$-wise independent probability distribution on $\{0,1\}^n$. Then the covering radius of $\mu$ is at most

$$
\begin{align*}
\left\{ \begin{array}{ll}
\frac{n}{2} - \sqrt{s(n-s)} + \frac{s^{1/6}}{\sqrt{n-s}} & \text{if } k = 2s - 1 \text{ is odd} \\
\frac{n}{2} - \sqrt{s(n-1-s)} + \frac{s^{1/6}}{\sqrt{n-1-s - \frac{1}{2}}} & \text{if } k = 2s \text{ is even.}
\end{array} \right.
\end{align*}
$$

Actually, Corollary 1.3 is equivalent to Lemma 1.2. First, we note that this follows from the linear programming duality between low degree polynomials and $k$-wise independent distributions:

Lemma 1.4 (Duality between low degree polynomials and $k$-wise independence distributions) Let $1 \leq k \leq n$ be integers and $R > 0$ a real number. Then the following are equivalent:

I) Each $k$-wise independent probability distribution on $\{0,1\}^n$ has covering radius less than $R$.

II) There exists a polynomial $p(v) \in \mathbb{R}[v]$ of degree at most $k$ such that $\mathbb{E}_{B_n}p > 0$ and $p(w) \leq 0$, for each $w \in [0:n]$ such that $w \geq R$.

The implication from (II) to (I) was implicitly used by Tietäväinen in his proof of Theorem 1.1 in the context of linear codes.

We show that, for $k$-wise independent distributions, Tietäväinen’s bound on $R - \frac{n}{2}$ is asymptotically tight up to a factor of 2 if $k \leq \frac{n^{1/3}}{\log^2 n}$:

Theorem 1.5 (Lower bound on the covering radius of $k$-wise independent distributions) There exist absolute constants $k_0, n_0 > 0$ such that for each integer $n \geq n_0$ and each integer $k$ satisfying $k_0 \leq k \leq \frac{n^{1/3}}{\log^2 n}$, there exists a $k$-wise independent probability distribution on $\{0,1\}^n$ whose covering radius is at least $\frac{n}{2} - \sqrt{kn}$.

The key technical contribution of the this paper is the following result about low degree polynomials:

Theorem 1.6 (Low degree polynomials upper bound) There exist absolute constants $k_0, n_0 > 0$ such that for each integer $n \geq n_0$ and each integer $k$ satisfying $k_0 \leq k \leq \frac{n^{1/3}}{\log^2 n}$, the following holds. For each polynomial $f(v) \in \mathbb{R}[v]$ satisfying

i) $\deg(f) \leq k$

ii) $f(w) \leq 0$, for each integer $w \in [0:n]$ such that $|w - n/2| \leq \sqrt{kn}$, we must have $\mathbb{E}_{B_n} f \leq 0$.

By further constraining (ii) in Theorem 1.6, we get the following:

Corollary 1.7 (Low degree polynomials upper bound) There exist absolute constants $k_0, n_0 > 0$ such that for each integer $n \geq n_0$ and each integer $k$ satisfying $k_0 \leq k \leq \frac{n^{1/3}}{\log^2 n}$, the following holds. For each polynomial $f(v) \in \mathbb{R}[v]$ satisfying

i) $\deg(f) \leq k$

ii) $f(w) \leq 0$, for each integer $w \in [0:n]$ such that $w \geq n/2 - \sqrt{kn}$, we must have $\mathbb{E}_{B_n} f \leq 0$.

Thus, Theorem 1.5 follows from Corollary 1.7 via the duality in Lemma 1.4.

The proof of Theorem 1.6 uses tools from approximation theory. At high level, we will bound $\mathbb{E}_{B_n} f$ by examining the values of $f$ on integer sequences of length $k+1$ contained in the interval of points $w \in [0:n]$ such that $|w - n/2| \leq \sqrt{kn}$. The sequence are disjoints and they
have small Lebesgue constant. We will show that each sequence contains a point on which the negative value of $f$ is large in absolute value (assuming that $f$ is not identically zero). Those points will be used to show that $E_{B_n} f \leq 0$. The sequences will be constructed from translates of a quantized Chebyshev sequence whose Lebesgue constant will be estimated using Markov’s theorem.

The use of approximation theory tools in the proof was inspired by the works of Paturi \[22\] and Linial and Nissan \[23\]. Paturi implicitly used the Lebesgue constant of equally-spaced sequences and he used Markov’s theorem to estimate the approximate degree of symmetric boolean functions. Linial and Nissan used properties of quantized zeros of Chebyshev polynomials to approximate the inclusion-exclusion formula. At a high level, the new ingredient in our argument is the use of multiple sequences and in particular the translated sequences technique.

Another related work which builds on \[22\] is the author’s joint work with Nahas on read-once CNF formulas and small-bias spaces \[26\]. See also the aforementioned papers \[9\], \[10\], \[12\] - \[14\] which use Chebyshev polynomials to improve on Tietäväinen’s bound in the $d = \Theta(n)$ regime.

2 Paper outline

After summarizing the notations and terminology used throughout the paper in Section 3, we prove Lemma \[14\] in Section 4. The proof of of Theorem \[16\] uses tools from approximation theory, which we explain in Sections 5 and 6. After explaining the proof technique and outline in Section 7, we establish Theorem \[16\] in Sections 8 and 9. We conclude in Section 10 with open questions.

3 Preliminaries

The following section summarizes the terminology used in the paper. Section 3.2 contains Fourier analysis notions on the hypercube used in the proof of Lemma \[14\].

3.1 Terminology

Throughout the paper, $n \geq 1$ is an integer. We will use the following notations. If $x \in \{0, 1\}^n$, the Hamming weight of $x$, which we denote by $|x|$, is the number of nonzero coordinates of $x$. The set $\{0, \ldots, n\}$ is denoted by $[0 : n]$. The binomial distribution on $[0 : n]$ is denoted by $B_n$, i.e., $B_n(w) = \binom{n}{w}$. The uniform distribution on $\{0, 1\}^n$ is denoted by $U_n$, i.e., $U_n(x) = \frac{1}{2^n}$, for all $x \in \{0, 1\}^n$. The finite field structure on $\{0, 1\}$ is denoted by $\mathbb{F}_2$. The minimum distance of a non-empty $\mathbb{F}_2$-linear code is the minimum weight of a nonzero codeword. Throughout this paper, log means $\log_2$.

If $\mu$ is a probability distribution, $E_{\mu}$ denotes the expectation with respect to $\mu$ and “$x \sim \mu$” denotes the process of sampling a random point $x$ according to $\mu$. A probability distribution $\mu$ on $\{0, 1\}^n$ is called $k$-wise independent if sampling $x \sim \mu$ gives a random vector $x = (x_1, \ldots, x_n)$, where each $x_i$ is equally likely to be 0 or 1 and any $k$ of the $x_i$’s are statistically independent \[20\] \[21\]. See also Section 3.2 for an equivalent definition.

If $r \geq 0$ is a real number and $x \in \{0, 1\}^n$, $H_n(x; r)$ denotes the radius-$r$ Hamming ball in $\{0, 1\}^n$ centered at $x$, i.e., $H_n(x; d) = \{x \in \{0, 1\}^n : |x + y| \leq r\}$. If $C$ is a subset of $\{0, 1\}^n$, $H_n(C; r)$ denotes the $r$-neighborhood of $C$ with the respect to the Hamming distance, i.e., $H_n(C; r) = \cup_{x \in C} H_n(x; r)$. The covering radius of $C$ is the minimum $r$ such that $H_n(C; r) = \{0, 1\}^n$. Equivalently, the covering radius of $C$ is the minimum $r$ such that $H_n(x; r) \cap C \neq \emptyset$ for each $x \in \{0, 1\}^n$. If $\mu$ is a probability distribution on $\{0, 1\}^n$, the covering radius of $\mu$ is the covering radius of its support. Equivalently, the covering radius of $\mu$ is the minimum $r$ such that $\mu(H_n(x; r)) \neq 0$ for each $x \in \{0, 1\}^n$. 
3.2 Fourier transform preliminaries

The use of Harmonic analysis methods in coding theory dates back to MacWilliams [25]. We give below some preliminary notions used in the proof of Lemma 1.4; see also 26.

Identify the hypercube \( \{0,1\}^n \) with the abelian group \( \mathbb{Z}_2^n = (\mathbb{Z}/2\mathbb{Z})^n \) and consider the characters \( \{\chi_z\}_{z \in \mathbb{Z}_2^n} \) of \( \mathbb{Z}_2^n \), where \( \chi_z : \{0,1\}^n \to \{-1,1\} \) is given by \( \chi_z(x) = (-1)^{x \cdot z} \) and \( (x,z) = \sum_{i=1}^n x_iz_i \). Consider the \( \mathbb{C} \)-vector space \( L(\mathbb{Z}_2^n) \) of complex valued functions defined on \( \mathbb{Z}_2^n \) and consider the inner product on \( L(\mathbb{Z}_2^n) \):

\[
(f,g) = \mathbb{E}_u f g = \frac{1}{2^n} \sum_x f(x)\overline{g(x)}.
\]

The characters \( \{\chi_z\}_{z} \) form an orthonormal basis of \( L(\mathbb{Z}_2^n) \), i.e., \( \langle \chi_z, \chi_{z'} \rangle = \delta_{z,z'} \), for each \( z, z' \in \{0,1\}^n \), where \( \delta \) is the Kronecker delta function.

If \( f \in L(\mathbb{Z}_2^n) \), its Fourier transform \( \hat{f} \in L(\mathbb{Z}_2^n) \) is given by the coefficients of the unique expansion of \( f \) in terms of the characters:

\[
f(x) = \sum_z \hat{f}(z)\chi_z(x) \quad \text{and} \quad \hat{f}(z) = \langle f, \chi_z \rangle = \mathbb{E}_u f \chi_z.
\]

The degree of \( f \in L(\mathbb{Z}_2^n) \) is the smallest degree of a polynomial \( p \in \mathbb{C}[x_1, \ldots, x_n] \) such that \( p(x) = f(x) \) for all \( x \in \{0,1\}^n \). Equivalently, in terms of the Fourier transform \( \hat{f} \), the degree of \( f \) is equal to the maximal weight of \( z \in \mathbb{Z}_2^n \) such that \( \hat{f}(z) \neq 0 \).

In terms of the characters \( \{\chi_z\}_{z} \), we have the following equivalent definition of \( k \)-wise independence. A probability distribution \( \mu \) on \( \{0,1\}^n \) is \( k \)-wise independent iff \( \mathbb{E}_z \chi_z = 0 \) for each nonzero \( z \in \{0,1\}^n \) such that \( |z| \leq k \). Equivalently, \( \mu \) is \( k \)-wise independent iff \( \mathbb{E}_z p = \mathbb{E}_x p \) for each polynomial \( p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) of degree at most \( k \). This follows from the fact that the evaluation \( f \) of \( p \) on \( \{0,1\}^n \) has degree at most \( k \), hence its Fourier transform \( \hat{f} \) is zero on all frequencies of weight larger than \( k \), i.e., \( p(x) = \sum_{z \in \{0,1\}^n: |z| \geq k} \hat{f}(z)\chi_z(x) \), for all \( x \in \{0,1\}^n \).

4 Proof of Lemma 1.4

The lemma is restated below for convenience.

Lemma 1.4 Let \( 1 \leq k \leq n \) be integers and \( R > 0 \) a real number. Then the following are equivalent:

- I) Each \( k \)-wise independent probability distribution on \( \{0,1\}^n \) has covering radius less than \( R \).
- II) There exits a polynomial \( p(v) \in \mathbb{R}[v] \) of degree at most \( k \) such that \( \mathbb{E}_{B_n} p > 0 \) and \( p(w) \leq 0 \), for each \( w \in \{0 : n\} \) such that \( w \geq R \).

First, we note that (I) is equivalent to:

\( \Gamma \) For each \( k \)-wise independent probability distribution on \( \{0,1\}^n \), we have \( \mu(H_n(0;r)) \neq 0 \), where \( r = \lceil R \rceil - 1 \).

The reason is that \( \mu(H_n(x;r)) = (\sigma_x \mu)(H_n(0;r)) \), where \( \sigma_x \mu \) is the translation of \( \mu \) by \( x \) (i.e., \( (\sigma_x \mu)(y) = \mu(x+y) \)). The equivalence between (I) and (\( \Gamma \)) then follows from the fact that if \( \mu \) is \( k \)-wise independent, then so is \( \sigma_x \mu \) because \( \mathbb{E}_{\sigma_x \mu} \chi_z = \chi_z(x)\mathbb{E}_\mu \chi_z \). That is, we may assume without loss of generality that \( x = 0 \).

The equivalence between (\( \Gamma \)) and (II) follows from Linear Programming duality. The use of LP duality in such problems goes back to Delsarte [5]. Before going to the LP formulation, it
is instructive to directly establish the implication from (II) to (I') by appropriately translating Tietävänäinen’s argument to the distributions framework. Assume that (II) holds and let $p$ be such a polynomial. Consider any $k$-wise independent distribution $\mu$ on $\{0,1\}^n$. Let $I$ be the set of $w \in [0:n]$ such that $w \leq R$, i.e., $w \leq r$, and let $I^c$ be the complement of $I$ in $[0:n]$. Let $M$ be the maximum value of $p$ in $I$. Since $E_{B\mu} f > 0$ and $f$ is non-positive on $I^c$, $M$ must be positive. Let $p' = \frac{f}{M}$. Thus $p' \leq 1$ on $I$ and $p \leq 0$ on $I^c$, i.e., $p'(w) \leq \delta_I(w)$ for each $w \in [0:n]$, where $\delta_I$ is the indicator function of $I$ (for each $w \in [0:n]$, $\delta_I(w) = 1$ if $w \leq r$ and, otherwise, $\delta_I(w) = 0$). Therefore,

$$\mu(H_n(0;r)) = E_{x \sim \mu} \delta_I(|x|) \geq E_{x \sim \mu} p'(|x|) = E_{x \sim U_n} p'(|x|) = E_{B\mu} p' > 0,$$

where the second equality follows from the fact the $\mu$ is $k$-wise independent and $p'(x_1+\ldots+x_n) \in \mathbb{R}[x_1,\ldots,x_n]$ is a polynomial on the variables $x_1,\ldots,x_n$ of degree at most $k$.

Now, we establish the lemma using linear programming duality. Note that the above argument is not enough for our purposes since Theorem 4.3 follows from Theorem 4.6 via the other implication from (I') to (II). Consider the linear program

$$A = \min_{\mu} \mu(H_n(0;r)),$$

where the minimum is over all $k$-wise independent probability distributions on $\{0,1\}^n$. Note that objective function is $\mu(H_n(0;r)) = E_n f$, where $f$ is the indicator function of $H_n(0;r)$, i.e., $f(x) = 1$ if $|x| < R$ and $f(x) = 0$ if $|x| \geq R$. The linear constraints are $\mu \geq 0$, $\sum x \mu(x) = 1$, and $E_{\mu} \chi_z = 0$ for each nonzero $z \in \{0,1\}^n$ such that $|z| \leq k$.

Taking the dual, we get

$$B = \max_q E_{U_n} q,$$

where the maximum is over all functions $q : \{0,1\}^n \to \mathbb{R}$ such that the degree of $q$ is at most $k$, i.e., $\hat{q}(z) = 0$, for each $z \in \{0,1\}^n$ such that $|z| > k$, and $q \leq f$ pointwise, i.e., $q(x) \leq f(x)$ for each $x \in \{0,1\}^n$. See Lemma 5.2.10 in [24] for the underlying duality calculations.

Since the primal is feasible ($U_n$ is a feasible solution) and bounded (at least 0), we get that $A = B$. That is, (I') is equivalent to:

**II'** There exists $q : \{0,1\}^n \to \mathbb{R}$ such that the degree of $q$ is at most $k$, $E_{U_n} q > 0$, and $q(x) \leq 0$, for each $x \in \{0,1\}^n$ such that $|x| \geq R$.

Note that we dropped the condition $q(x) \leq 1$, for $|x| < R$, since it follows from appropriately scaling $q$. Thus (II') is the special case of (II') corresponding to the case when $q(x)$ is symmetric, i.e., $q(x)$ depends on the weight $|x|$ of $x$. The fact that (II) and (II') are equivalent follows from a classical symmetrization argument. Let $p$ be the symmetric polynomial associated with $q$, i.e., $p(w) = E_{x:|x|=w} q(x)$, for all $w \in [0:n]$. Thus $E_{B\mu} p = E_{U_n} q > 0$ and $p(w) \leq 0$ for each $w \geq R$. To see why $p$ has degree at most $k$ in $w$, consider the Fourier expansion of $q$:

$$q(x) = \sum_{|z| \leq k} \hat{q}(z) \chi_z(x).$$

Thus

$$p(w) = \sum_{|z| \leq k} \hat{q}(z) E_{|x|=w} \chi_z(x) = \sum_{|z| \leq k} \hat{q}(z) \frac{1}{\binom{n}{|z|}} K^{(n)}_w(|z|) = \sum_{|z| \leq k} \hat{q}(z) \frac{1}{\binom{n}{|z|}} K^{(n)}_w(|z|),$$

where $K^{(n)}_w(t) = \sum_{|z|=t} \chi_z(x) = \sum_{t=-\infty}^\infty (-1)^{(t)} \binom{n-w}{t-i}$ is the degree-$t$ Krawtchouk polynomial and $x$ is any element of $\{0,1\}^n$ of weight $w$. Note that (2) uses the Krawtchouk polynomials identity $\binom{n}{t} K^{(n)}_w(t) = \binom{n}{w} K^{(n)}_t(w)$ (e.g., see (2.3.15) in [24]).
5 Approximation theory machinery

Consider the space $C[-1, 1]$ of continuous function on the interval $[-1, 1]$ endowed with the max norm:

$$\|f\|_{[-1,1]} = \max_{-1 \leq x \leq 1} |f(x)|.$$ 

**Lebesgue Constant.** Let $X = \{x_i\}_{i=1}^{k+1}$ be an increasing sequence of real points in the interval $[-1, 1]$. In what follows, we assume that $k \geq 1$. The *Lebesgue constant* of $X$ is given by

$$\Lambda_k(X) = \max_p \|p\|_{[-1,1]},$$

where the maximum is over the choice of a polynomial $p \in \mathbb{R}[x]$ of degree at most $k$ such that $|p(x_i)| \leq 1$ for $i = 1, \ldots, k + 1$.

In interpolation theory, $\Lambda_k(X)$ captures how good are interpolations on $X$ of functions in $C[-1, 1]$ by degree-$k$ polynomials in comparison to optimal degree-$k$ polynomial approximations with respect the max norm on $[-1, 1]$. For our purposes, the above simple equivalent definition is enough. We also need the following estimates of the Lebesgue constant of specific sequences; e.g., see Section 1.4 in [28].

**Equally-spaced sequences.** Let $E(k)$ be the sequence of $k + 1$ equally-spaced points starting with $-1$ and ending with 1. Then, as $k$ tends to infinity, $\Lambda_k(E(k)) \sim \frac{2^k}{ek \log e}$. We also have the bound $\Lambda_k(E(k)) < \frac{2^{k+3}}{k}$, which holds for all $k \geq 1$.

**Extended Chebyshev sequences.** The extended Chebyshev sequence $C(k) = \{c_i\}_{i=1}^{k+1}$ is the increasing sequence given by

$$c_i = -\frac{\cos(2i - 1)\phi_k}{\cos \phi_k}.$$ 

Thus $c_1 = -1$ and $c_{k+1} = 1$. Extended Chebyshev sequences have much better Lebesgue constants than equally-spaced ones. As $k$ tends to infinity, we have the estimate $\Lambda_k(C(k)) \sim \frac{2}{\pi} \log k$. We also have the bound:

$$\Lambda_k(C(k)) < \frac{2}{\pi} \log (k + 1) + 0.7213 \quad \text{for all } k \geq 1.$$ 

(3)

For our purposes, the fact that $\Lambda_k(C(k)) = O(\log k)$ is sufficient.

**Bounds outside $[-1, 1]$.** We need the following basic tool from approximation theory which bounds the absolute value of a polynomial on points outside the interval $[-1, 1]$ in terms of its max norm on $[-1, 1]$ and its degree.

**Lemma 5.1** If $p \in \mathbb{R}[x]$ of degree at most $k$, then for each real $x$ such that $|x| > 1$,

$$|p(x)| \leq \|p\|_{[-1,1]} |(2|x|)^k.$$ 

Lemma 5.1 follows from properties of Chebyshev polynomial. If $k \geq 0$ is an integer, the $k$’th Chebyshev polynomial of the first kind is a degree-$k$ polynomial $T_k(x) \in \mathbb{R}[x]$ given by

$$T_k(x) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^k + (x - \sqrt{x^2 - 1})^k \right).$$

See [29] and [30] for a general reference on Chebyshev polynomials. Lemma 5.1 is a consequence of the following basic basic facts about Chebyshev polynomials:

- If $p \in \mathbb{R}[x]$ is of degree at most $k$, then for each real $x$ such that $|x| > 1$,

$$|p(x)| \leq \|p\|_{[-1,1]} |T_k(x)|.$$ 

- If $|x| \geq 1$, then $|T_k(x)| \leq (2|x|)^k$. This follows immediately from the definition of $T_k$.
6 Scaling, translation, and distortion

We are interested in integer sequences in the interval $[0 : n]$. Since the Lebesgue constant is invariant under scaling and translations, the above machinery directly translates from the interval $[-1, 1]$ to any interval in $\mathbb{R}$. We introduce in this section the needed notations. Then we note that a direct consequence of Markov’s theorem is that the Lebesgue constant of a sequence does not significantly increase after small distortions of its points. Distortions will result in this paper from quantizing real sequences in the real interval interval $[0, n]$ to integer values in the discrete interval $[0 : n]$. Hence interval $[0, 1]$ is invariant under scaling and translations, the above machinery directly translates from the

We are interested in integer sequences in the interval $[0 : n]$. Define the Lebesgue constant $\Lambda(X)$ of $X$ as

$$\Lambda(X) = \Lambda_k(\bar{X}),$$

where $\bar{X} = \{\bar{x}_i\}_{i=1}^{k+1}$ is the sequence obtained by translating and scaling $X$ so that $\bar{x}_1 = -1$ and $\bar{x}_{k+1} = 1$.

Define the interval, center, and radius of $X$ by $I(X) = [x_1, x_{k+1}]$, $C(X) = \frac{x_1 + x_{k+1}}{2}$, and $R(X) = \frac{x_{k+1} - x_1}{2}$, respectively.

If $I$ is a closed real interval and $f$ is continuous, let

$$\|f\|_I = \max_{x \in I} |f(x)|.$$

Also define

$$\|f\|_X = \max_{i=1}^{k+1} |f(x_i)|.$$

Thus

$$\Lambda(X) = \max\{|p|_{I(X)} : p \in \mathbb{R}[x] \text{ of degree at most } k \text{ such that } \|p\|_X \leq 1\}.$$

Therefore, using Lemma 5.1 with Lemma 6.1 we get the following bound.

**Corollary 6.1 (Key tool)** Let $p \in \mathbb{R}[x]$ be a polynomial of degree at most $k$ and let $X$ be a sequence of $k+1$ increasing points in $\mathbb{R}$. Then, for each real $x$ outside $I(X)$,

$$|p(x)| \leq \|p\|_X \Lambda(X) \left(\frac{2(|x| - C(X))}{R(X)}\right)^k.$$

Corollary 6.1 is the key tool in the proof of Theorem 1.6. To handle distortions, we need Markov’s theorem.

**Lemma 6.2 (Markov’s theorem; see [31])** If $p \in \mathbb{R}[x]$ is a degree $k$ polynomial, consider the derivative $p'$ of $p$. Then $\|p'\|_{[-1, 1]} \leq k^2 \|p\|_{[-1, 1]}$.

**Corollary 6.3 (Distortion)** Let $X = \{x_i\}_{i=1}^{k+1}$ and $X' = \{x'_i\}_{i=1}^{k+1}$ be two increasing sequences of points in $\mathbb{R}$. Let $\gamma > 0$ be such that $\gamma k^2 \Lambda(X) < 1$. Assume that $|x'_i - x_i| \leq \gamma R(X)$, for $i = 1, \ldots, k+1$, and that $I(X') \subset I(X)$ (i.e., $x_1 \leq x'_1 \leq x_{k+1}$ and $x_1 \leq x'_{k+1} \leq x_{k+1}$). Then

$$\Lambda(X') \leq \frac{\Lambda(X)}{1 - \gamma k^2 \Lambda(X)}.$$

**Proof:** Since the Lebesgue constant is invariant under scaling and translation, assume without loss of generality that $I(X) = [-1, 1]$, i.e., $C(X) = 0$ and $R(X) = 1$. Let $p \in \mathbb{R}[x]$ be of degree at most $k$ such that $\|p\|_{X'} \leq 1$. For any $1 \leq i \leq k+1$, we have $|p(x_i) - p(x'_i)| \leq \gamma \|p'\|_{[-1, 1]}$ since $x'_i \in I(X) = [-1, 1]$. Applying Markov’s theorem, we get $|p(x_i) - p(x'_i)| \leq \gamma k^2 \|p\|_{[-1, 1]}$. Hence $|p(x_i)| \leq |p(x'_i)| + \gamma k^2 \|p\|_{[-1, 1]}$. It follows that

$$\|p\|_X \leq \|p\|_{X'} + \gamma k^2 \|p\|_{[-1, 1]} \leq 1 + \gamma k^2 \Lambda(X) \|p\|_X.$$
Therefore 
\[ \|p\|_X \leq \frac{1}{1 - \gamma^2 \Lambda(X)}. \]
Hence 
\[ \|p\|_{I(X')} \leq \|p\|_{[-1,1]} \leq \frac{\Lambda(X)}{1 - \gamma^2 \Lambda(X)}, \]
where the first inequality holds because \( I(X') \subset I(X) = [-1,1] \).

7 Proof Technique

Consider the following linear program.

**Definition 7.1** If \( n, k \geq 1 \) are integers and \( \Delta > 0 \) is a real number such that \( \Delta \leq n/2 \), let
\[ E_n(k, \Delta) = \max \mathbb{E}_{B_n} f, \]
where the maximum is over all polynomials \( f \in \mathbb{R}[x] \) such that the degree of \( f \) is at most \( k \) and \( f(w) \leq 0 \), for each integer \( w \in [0 : n] \) such that \( |w - n/2| \leq \Delta \).

Note that, by setting \( f \) to the identically zero polynomial, we get \( E_n(k, \Delta) \geq 0 \).

**Definition 7.2** If \( n, k \geq 1 \) are integers, let \( \Delta^*_n(k) \) be the minimum value of \( \Delta > 0 \) such that \( E_n(k, \Delta) = 0 \).

In the above terms terms, Theorem 1.6 can be restated as follows.

**Theorem 1.6** There exist absolute constants \( k_0, n_0 > 0 \) such that for each integer \( n \geq n_0 \) and each integer \( k \) satisfying \( k_0 \leq k \leq n^{1/3} \log^* n \), we have have \( E_n(k, \sqrt{k}n) = 0 \), or equivalently, \( \Delta^*_n(k) \leq \sqrt{k}n \).

We will actually prove a slightly stronger statement; we will show that \( \Delta^*_n(k) \leq \sqrt{\alpha k}n \), where \( \alpha = 0.93 \). This bound is asymptotically tight up to a factor less than 2; it follows from Tietävänäinen’s bound (Lemma 1.3) that if \( 1 \leq k \leq n - 1 \) are integers, then
\[
\begin{cases} 
\Delta^*_n(k) > \sqrt{s(n-s)} - s^{1/6} \sqrt{n-s} & \text{if } k = 2s - 1 \text{ is odd} \\
\Delta^*_n(k) > \sqrt{s(n-1-s)} - s^{1/6} \sqrt{n-1-s} + \frac{1}{2} & \text{if } k = 2s \text{ is even}.
\end{cases}
\]

To illustrate the technique, we assume below that \( \Delta > 0 \) is any number such that \( \Delta \leq n/2 \). Given \( k \), we would like to make \( \Delta \) as small as possible while guaranteeing that \( E_n(k, \Delta) = 0 \). We illustrate in this section how to reduce the task of showing that \( E_n(k, \Delta) = 0 \) to that of constructing a sequence with appropriate parameters (Lemma 4.4). The outline of the rest of the proof is Section 7.4.

Let \( L \) be the set integers \( w \in [0 : n] \) such that \( |w - n/2| \leq \Delta \) and let \( L^c \) be the complement of \( L \) in \( [0 : n] \). Let \( f(x) \in \mathbb{R}[x] \) be a polynomial of degree at most \( k \), where \( k \geq 1 \) is an integer. Assume that \( f(w) \leq 0 \), for each \( w \in L \). We want to show that \( \mathbb{E}_{B_n} f \leq 0 \) if \( k \) is small enough compared to \( \Delta \).

Let \( W = \{ w_i \}_{i=1}^{k+1} \) be a length-\((k+1)\) integer sequence of increasing points contained in the interval \( L \) and centered at \( n/2 \). By Corollary 6.5, for each \( w \in L^c \),
\[ |f(w)| \leq \|f\|_W \Lambda(W) \left( \frac{2(w-n/2)}{R(W)} \right)^k. \]
Let \( w^* \) be the point in \( W \) which maximizes \( \|f\|_W \), i.e., \( w^* = w_{i^*} \), where \( i^* \) is such that \( |f(w_{i^*})| = \|f\|_W \). Since \( f \leq 0 \) on \( L \), \( f(w^*) = -\|f\|_W \). Note that \( \|f\|_W \neq 0 \) unless \( f \) is identically zero.
since the degree of $f$ is at most $k$ and $W$ has $k+1$ points. They key is to try to use the point $w^*$ to bound $E_{B_n} f$ as follows. We have

$$E_{B_n} f = \sum_{w=0}^n B_n(w) f(w) \leq \sum_{w \in L^c} B_n(w) |f(w)| - \|f\|_W B_n(w^*).$$

As we don’t have information about the position of $w^*$ in $W$, we use the following bound

$$B_n(w^*) \geq B_n\left(\frac{n}{2} + R(W)\right),$$

which follows from the fact that the binomial distribution $B_n$ is bell shaped around $n/2$. Note that, even if $n$ is odd, $\frac{n}{2} + R(W)$ is an integer since $W$ is an integer sequence centered at $n/2$ with radius $R(W)$. It follows that

$$E_{B_n} f \leq \|f\|_W \left(\Lambda(W) \sum_{w \in L^c} B_n(w) \left|\frac{2(w - n/2)}{R(W)}\right|^k - B_n\left(\frac{n}{2} + R(W)\right)\right) = \|f\|_W \left(2\Lambda(W) \sum_{w > \frac{n}{2} + \Delta} B_n(w) \left(\frac{2(w - n/2)}{R(W)}\right)^k - B_n\left(\frac{n}{2} + R(W)\right)\right).$$

In summary, we get the following:

**Lemma 7.3 (One-sequence approach)** Let $n, k \geq 1$ be integers and let $\Delta > 0$ be a real number such that $\Delta \leq n/2$. Let $L$ be the set integers $w \in [0 : n]$ such that $|w - \frac{n}{2}| \leq \Delta$. Let $W$ be a length-$(k+1)$ integer sequence of increasing points contained in the interval $L$ and centered at $n/2$. Let

$$\nu = 2\Lambda(W) \sum_{w > \frac{n}{2} + \Delta} B_n(w) \left(\frac{2(w - n/2)}{R(W)}\right)^k.$$ 

If $\nu \leq B_n\left(\frac{n}{2} + R(W)\right)$, then $\mathcal{E}_n(k, \Delta) = 0$.

**Limitations of the one-sequence approach.** Consider the setup when $\Delta = \sqrt{\alpha kn}$, where $\alpha > 0$ is any constant. To motivate the translated sequences approach explained below, we note below that the one-sequence approach is not useful if $k$ is small. It can be used to establish Theorem 1.5 for $k = w(\log n)$, but it fails for smaller values of $k$. Namely, for all constants $\alpha > 0$, it fails to show that $\mathcal{E}_n(k, \sqrt{\alpha kn}) = 0$ if $k = o(\log n)$.

Assume that $k = o(\log n)$. Since $W$ is contained in $L$, $R(W) \leq \Delta$. Using the loose lower bounds $\frac{2(w - n/2)}{R(W)} > \frac{2\Delta}{R(W)} \geq 2 > 1$, for $w > \frac{n}{2} + \Delta$, and $\Lambda(W) \geq 1 > \frac{1}{2}$, we get

$$\nu > \sum_{w > \frac{n}{2} + \sqrt{\alpha kn}} B_n(w) \geq \sqrt{\alpha kn} B_n\left(\left[\frac{n}{2} + 2\sqrt{\alpha kn}\right]\right) = \Omega\left(\frac{\sqrt{kn} \sqrt{e^{-8\alpha k}}}{}\right) = \Omega\left(\sqrt{e^{-8\alpha k}}\right),$$

via de Moivre-Laplace normal approximation of the binomial (see Theorem 9.2). On the other hand, we have

$$B_n\left(\frac{n}{2} + R(W)\right) \leq B_n\left(\left[\frac{n}{2}\right]\right) = \Theta\left(\frac{1}{\sqrt{n}}\right).$$

Thus, to conclude that $\nu \leq B_n\left(\frac{n}{2} + R(W)\right)$, we need $k$ to be at least $\Omega(\log n)$ to compensate for the $\frac{1}{\sqrt{n}}$ term.

The one-sequence approach exhibits one point $w^*$ in the sequence $W$ on which $f$ is negative (assuming that $f$ is not identically zero). To resolve the $\frac{1}{\sqrt{n}}$ issue, we will use multiple disjoint
sequences and exhibit one point in each sequence on which \( f \) is negative. The sequences will be translates of \( W \). To guarantee that they are disjoint, their number is limited by the minimum distance \( t \) between consecutive points in \( W \). Eventually, we will overcome the \( \frac{1}{\sqrt{n}} \) term by using a sequence with \( t = \Omega(\frac{\sqrt{n}}{\log n}) \).

**Lemma 7.4 (Translated sequences approach)** Let \( n, k \geq 1 \) integers and let \( \Delta > 0 \) be a real number such that \( \Delta \leq n/2 \). Let \( W = \{ w_i \}_{i=1}^{k+1} \) be a length-\((k+1)\) integer sequence of increasing points centered at \( n/2 \). Let \( t \) be the minimum distance between two consecutive points in \( W \), i.e., \( t = \min_{i=1}^{k} w_{i+1} - w_i \), and let \( p = \left\lfloor \frac{t-1}{2} \right\rfloor \). Assume that \( R(W) + p \leq \Delta \). Let

\[
\nu = 2 \Lambda(W) \sum_{w > \frac{n}{2} + \Delta} B_n(w) \left( \frac{2(w - n/2) + 2p}{R(W)} \right)^k.
\]

If \( \nu \leq tB_n \left( \frac{2}{t} + R(W) + p \right) \), then \( E_n(k, \Delta) = 0 \).

**Proof:** Consider the \( t \) translated sequences \( W_0, \ldots, W_{t-1} \), where for \( s = 0, \ldots, t-1, W_s = \{ w_i \}_{i=1}^{k+1} \) and \( w_{i,s} = w_{i+s-p} \). Thus \( W = W_p \). By the definition of \( t \), the sequences \( W_0, \ldots, W_{t-1} \) are disjoint, i.e., \( w_{s,i} \neq w'_{s',i'} \) if \( (s, i) \neq (s', i') \). Moreover, for each \( s \), \( R(W_s) = R(W) \), \( \Lambda(W_s) = \Lambda(W) \), and, since \( W \) is centered at \( n/2 \), \( C(W_s) = n/2 + s - p \). Thus

\[
|C(W_s) - n/2| \leq p.
\]

As above, let \( L \) be the set integers \( w \in [0 : n] \) such that that \( |w - \frac{n}{2}| \leq \Delta \) and let \( L^c \) be the complement of \( L \) in \([0 : n]\). Since \( R(W) + p \leq \Delta \), each \( W_s \) is contained in \( L \).

By Corollary 6.1 for each \( w \in L^c \) and for each \( 0 \leq s \leq t-1 \),

\[
|f(w)| \leq \|f\| W_s, \Lambda(W) \left| \frac{2(w - C(W_s))}{R(W)} \right|^k.
\]

Averaging over \( s \), we get

\[
|f(w)| \leq \frac{\Lambda(W)}{t} \sum_{s=0}^{t-1} \|f\| W_s, \frac{2(w - C(W_s))}{R(W)} \right|^k.
\]

Now we argue as above on each \( W_s \). For each \( s \), let \( w^*_s \) be the point in \( W_s \) which maximizes \( \|f\| W_s \), i.e., \( w^*_s = w_{s,i^*} \), where \( i^* \) is such that \( |f(w_{s,i^*})| = \|f\| W_s \). Since \( f \leq 0 \) on \( L \), \( f(w^*_s) = -\|f\| W_s \). Here again, \( \|f\| W_s \neq 0 \) if \( f \) is not identically zero. The key is to use the integer points \( \{w^*_s\}_{s=0}^{t-1} \) to bound \( E_{B_n} \). Note that \( w^*_0, \ldots, w^*_t \) are distinct points contained in \( L \) since the sequences \( W_0, \ldots, W_{t-1} \) are disjoint and contained in \( L \). Therefore,

\[
E_{B_n} = \sum_{w=0}^{\infty} B_n(w) f(w) \leq \sum_{w \in L^c} B_n(w) f(w) \leq \sum_{s=0}^{t-1} \|f\| W_s B_n(w^*_s).
\]

We have

\[
|w^*_s - \frac{n}{2}| \leq |w_s - C(W_s)| + \left| C(W_s) - \frac{n}{2} \right| \leq R(W_s) + \left| C(W_s) - \frac{n}{2} \right|.
\]

Using [5] and the fact that \( R(W_s) = R(W) \), we obtain \( |w^*_s - \frac{n}{2}| \leq R(W) + p \), and hence

\[
B_n(w^*_s) \geq B_n \left( \frac{n}{2} + R(W) + p \right).
\]

As before, note that, even of \( n \) is odd, \( \frac{n}{2} + R(W) \) is an integer since \( W \) is an integer sequence centered at \( n/2 \). If follows also from [5] that

\[
|w - C(W_s)| \leq \left| w - \frac{n}{2} \right| + \left| C(W_s) - \frac{n}{2} \right| \leq \left| w - \frac{n}{2} \right| + p.
\]
Therefore, by replacing (6) and (8) in (7), using (9), and then interchanging the summations, we get

\[ E_{B_n} f \leq \frac{1}{t} \sum_{s=0}^{t-1} \|f\|_{W_{s}} \left( \Lambda (W) \sum_{w \in L^c} B_n (w) \left| \frac{2 \left( \left| \frac{w-n}{2} \right| + p \right)^k}{R (W)} \right| - t B_n \left( \frac{n}{2} + R (W) + p \right) \right) = \left( \frac{1}{t} \sum_{s=0}^{t-1} \|f\|_{W_{s}} \right) \left( 2 \Lambda (W) \sum_{w > \frac{n}{2} + \Delta} B_n (w) \left( \frac{2 (w - n/2 + p)}{R (W)} \right)^k - t B_n \left( \frac{n}{2} + R (W) + p \right) \right). \]

### 7.1 Discussion and proof outline

Lemma 7.4 reduces the problem of showing that \( E_n (k, \Delta) = 0 \) to that of constructing the sequence \( W \). The relevant parameters of \( W \) are its radius \( R (W) \), its Lebesgue constant \( \Lambda (W) \), and its minimum distance \( t \). We need \( t \) to be large and \( \Lambda (W) \) small. We also need to optimize on \( R (W) \) since increasing \( R (W) \) decreases both \( \nu \) and \( t B_n \left( \frac{n}{2} + R (W) + p \right) \).

In the next section, we will construct \( W \) by starting with translated and a scaled Chebyshev sequence \( X \) and quantizing its points to integer values. We will see that, for a suitable choice of parameters, the effect of quantizing is negligible as it increases its Lebesgue constant by at most a factor of 2. This follows from Markov’s theorem via Corollary 6.3. For \( \Delta = \Theta \left( \frac{n}{\log n} \right) \), we will get \( t = \Theta \left( \frac{n}{\sqrt{k/2}} \right) \). For such values, \( B_n \left( \frac{n}{2} + R (W) + p \right) = \Theta \left( \frac{1}{n} \right) \). Hence multiplying \( B_n \left( \frac{n}{2} + R (W) + p \right) \) by \( t \) cancels out the \( \frac{1}{n} \) term and replaces it with a \( O \left( \frac{1}{\sqrt{k}} \right) \) term, which barely affects the exponent. We conclude the proof of Theorem 1.6 in Section 9 by optimizing on \( R (W) \) and estimating \( \nu \) using Moivre-Laplace normal approximation of the binomial and Hoeffding’s inequality.

### 7.2 Note on equally-spaced sequences

Note that the largest possible values of \( t \) is around \( \frac{2 R (W)}{k} = \Theta \left( \frac{n}{\sqrt{k}} \right) \) and it is achieved by a sequence of equally-spaced points. Compared to Chebyshev sequences, the gain is negligible since the effect of \( \frac{1}{k} \) vanishes asymptotically. The issue with equally-spaced sequences is that \( \Lambda (W) \) is exponential in \( k \) and namely around \( \frac{2^k}{k^{3/2} \log k} \) (see Section 5). Ignoring non-exponential terms, the effect of using an equally-spaced sequence boils down to turning the \( 2 \left( \frac{w - n}{2} + p \right)^k \) term in the expression of \( \nu \) into \( 4 (w - n/2) \). This increases the upper bound on \( \Delta^*_n (k) \) by a constant factor. It can be shown that equally-spaced sequences lead to a weaker version of Theorem 1.6 and namely that \( \Delta^*_n (k) \leq \sqrt{1.43} \nu \), if \( k \leq \frac{n^{1/3}}{\log n} \) and \( n \) are sufficiently large.

### 8 Quantized Chebyshev sequences

Lemma 8.1 below summarizes the parameters of the sequence \( W \) on which Lemma 7.4 will be applied. The sequence is a quantized version of a scaled and translated Chebyshev sequence.

Let \( \Delta = \sqrt{\alpha n} \), where \( k \leq \frac{n^{1/2}}{\log n} \) and \( \alpha > 0 \) is a constant (which, as previously mentioned, will be eventually set to \( \alpha = 0.93 \)). The sequence consists of integer points. It is centered at \( n/2 \) and its radius is \( R (W) = \lfloor \beta \Delta + 1 \rfloor \), where \( 0 < \beta < 1 \) is a constant which we will optimize on in Section 9. Eventually, we will set \( \beta = 0.5204 \). The smallest distance \( t \) between consecutive points in \( W \) is in the order of \( t = \Omega \left( \frac{n}{R (W)} \right) \). We also have \( R (W) + p \leq (\beta + \epsilon) \Delta \) and \( 2p \leq \epsilon \Delta \), where \( \epsilon > 0 \) will be set to a sufficiently small value in Section 9. We will use \( \epsilon \) in Section 9 to handle the term \( \frac{2p}{R (W)} = \Theta \left( \frac{1}{n} \right) \) and the offset \( p \) added to the radius of \( W \) in Lemma 7.4.
Eventually, we will set $\epsilon = 0.004$. The Lebesgue constant of the quantized Chebyshev sequence is at most twice that of the original Chebyshev sequence.

**Lemma 8.1** Let $\alpha, \beta, \epsilon > 0$ be positive constants such that $\beta + \epsilon < 1$. Then there exist $k_1 > 0$ and $n_1 > 0$, depending on $\alpha, \beta$, and $\epsilon$, such that for each integer $n \geq n_1$ and each integer $k$ satisfying $k_1 \leq k \leq \frac{n^{1/3}}{\log n}$, the following holds.

Let $\Delta = \sqrt{\alpha kn}$. Then there exists a length-$(k + 1)$ integer sequence $W$ of increasing points centered at $n/2$ such that:

a) The minimum distance between consecutive points in $W$ is at least $b(k)\sqrt{n}$, where $b(k) = \frac{3\pi}{2} \left( \frac{\pi}{2(k + 1)} \right)^2 \sqrt{\alpha k}$.

b) $R(W) \geq \beta \Delta$

c) $R(W) + p \leq (\beta + \epsilon)\Delta$, where $p = \lceil \frac{n-k}{4} \rceil$

d) $2p \leq \epsilon \Delta$

e) $\Lambda(W) \leq \frac{\pi}{\beta} \log (k + 1) + 2$

f) $R(W) \geq \Delta \leq \frac{n}{2}$.

**Proof** Let $n, k \geq 1$ and assume that:

\[
\sqrt{\alpha kn} \leq \frac{n}{2} \tag{10}
\]

\[
\frac{4}{3\beta \sqrt{\alpha k}} \left( \frac{2(k + 1)}{\pi} \right)^2 \leq \sqrt{n} \tag{11}
\]

\[
\frac{2k^{3/2}}{\beta \sqrt{\alpha k}} \left( \frac{2}{\pi} \log (k + 1) + 1 \right) \leq \sqrt{n} \tag{12}
\]

\[
4 \left( \frac{\pi}{2(k + 1)} \right)^2 \left( \beta + \frac{1}{\sqrt{\alpha kn}} \right) + \frac{3}{\sqrt{\alpha kn}} \leq \epsilon. \tag{13}
\]

We will verify the lemma under the above assumption. Then we show that they hold for $k$ and $n$ large enough if $k \leq \frac{n^{1/3}}{\log n}$.

Note first that since $\beta + \epsilon < 1$, (f) follows trivially from (c) and (10).

Recall from Section 5 the extended Chebyshev sequence $C = \{c_i\}_{i=1}^{k+1}$, where

\[c_i = -\frac{\cos (2i - 1)\phi_k}{\cos \phi_k} \quad \text{for} \quad i = 1, \ldots, k + 1\]

and $\phi_k = \frac{\pi}{2(k + 1)}$. Thus $c_1 = -1$ and $c_{k+1} = 1$. By scaling and translation, map $C$ into a real sequence $X = \{x_i\}_{i=1}^{k+1}$ centered at $n/2$ with radius $R(X) = \beta \sqrt{\alpha kn} + 1$. That is, $x_i = (\beta \sqrt{\alpha kn} + 1)x_i + n/2$. Quantize $X$ to construct an integer sequence $W = \{w_i\}_{i=1}^{k+1}$ centered at $n/2$ as follows. To make sure that $W$ is centered at $n/2$ and that $I(W) \subset I(X)$, let $w_1 = [x_1]$ and $w_{k+1} = [x_{k+1}]$. For $i = 2, \ldots, k$, set $w_i$ arbitrarily to $[x_i]$ or $[x_i]$. Note that the condition $I(W) \subset I(X)$ is needed by Corollary 6.3. Thus the radius of $W$ is

\[R(W) = \left| \beta \sqrt{\alpha kn} + 1 \right| \geq \beta \sqrt{\alpha kn},\]

which proves (b). Moreover, $|w_i - x_i| \leq \gamma R(X)$, where $\gamma = \frac{1}{R(X)} \leq \frac{1}{\beta \sqrt{\alpha kn}}$.

First, we need to verify that the distortion did not collide points in $W$, and hence the length of $W$ is equal to the length $k + 1$ of $C$. The minimum distance between points in $C$ is $t_C = c_2 - c_1 = -\frac{\cos \phi_k - \cos^2 \phi_k}{\cos \phi_k}$. We have the bounds $3\phi_k^2 \leq t_C \leq 4\phi_k^2$, which hold for any
Thus the minimum distance $t$ between consecutive points in $W$ is satisfies $t \leq \tilde{t}$, where
\[
\tilde{t} = 3\beta \left( \frac{\pi}{2(k+1)} \right)^2 \sqrt{\alpha kn} - 2 \quad \text{and} \quad \bar{t} = 4 \left( \frac{\pi}{2(k+1)} \right)^2 (\beta \sqrt{\alpha kn} + 1) + 2.
\]
Condition (11) is equivalent to
\[
3\beta \left( \frac{\pi}{2(k+1)} \right)^2 \sqrt{\alpha kn} \geq 4,
\]
hence $\tilde{t} \geq 2$. Therefore, $t > 0$, and hence the points in $W$ are distinct.

**Proof of (a).** We have
\[
t \geq \tilde{t} \geq \bar{t} - \left( \frac{3\beta}{2} \left( \frac{\pi}{2(k+1)} \right)^2 \sqrt{\alpha kn} - 2 \right) = \frac{3\beta}{2} \left( \frac{\pi}{2(k+1)} \right)^2 \sqrt{\alpha kn} = b(k)\sqrt{n},
\]
where the second inequality follows from (14).

**Proof of (c).** We have $R(W) \leq \beta \sqrt{\alpha kn} + 1$ and $p = \lceil \frac{t-1}{2} \rceil \leq \bar{t}$, hence $R(W) + p \leq \beta \sqrt{\alpha kn} + 1 + \bar{t}$. Therefore,
\[
R(W) + p \leq \beta \sqrt{\alpha kn} + 4 \left( \frac{\pi}{2(k+1)} \right)^2 (\beta \sqrt{\alpha kn} + 1) + 3 \leq (\beta + \epsilon)\sqrt{\alpha kn},
\]
where the last inequality is equivalent to condition (13).

**Proof of (d).** We have
\[
\frac{2p}{\sqrt{\alpha kn}} \leq \tilde{t} + 1 \sqrt{\alpha kn} = 4 \left( \frac{\pi}{2(k+1)} \right)^2 \left( \beta + \frac{1}{\sqrt{\alpha kn}} \right) + \frac{3}{\sqrt{\alpha kn}} \leq \epsilon,
\]
where the last inequality is condition (13).

**Proof of (e).** Recall from Section 5 that
\[
\Lambda(X) = \Lambda(C) < \frac{2}{\pi} \log(k+1) + 0.7213 < \frac{2}{\pi} \log(k+1) + 1.
\]
Invoking Corollary 6.3 we get
\[
\Lambda(W) \leq \frac{\Lambda(X)}{1 - \gamma k^2 \Lambda(X)} \leq 2\Lambda(X)
\]
if $\gamma k^2 \Lambda(X) \leq \frac{1}{2}$. Since $\gamma \leq \frac{1}{\beta \sqrt{\alpha kn}}$, this condition follows from
\[
2k^2 \left( \frac{2}{\pi} \log(k+1) + 1 \right) \leq \beta \sqrt{\alpha kn},
\]
which is equivalent to condition (12).

**Asymptotics.** It remains to show that for each constants $\alpha, \beta, \epsilon > 0$, there exist $k_1 > 0$ and $n_1 > 0$ such that conditions (10), (11), (12), and (13) hold for each $n \geq n_1$ and each $k$ satisfying $k_1 \leq k \leq \frac{n_1^{1/3}}{\log n}$.

The claim is straight forward for condition (13) and it does not require a relation between $k$ and $n$. Condition (10) is equivalent to $k \leq \frac{n^{1/3}}{\log n}$, which holds, for $n$ large enough, since $k \leq \frac{n^{1/3}}{\log n}$. 

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To verify (11), let $k_2$ be large enough so that $\frac{(k+1)^2}{\sqrt{k}}$ is increasing in $k$ for $k \geq k_2$. Thus, for $k \geq k_2$,

$$\frac{4}{3\beta\sqrt{\alpha k}} \left( \frac{2(k+1)}{\pi} \right)^2 \leq \frac{16}{3\beta\sqrt{\alpha k^2}} \left( \frac{k^{3/4}}{4} + 1 \right)^2 \leq \frac{16}{3\beta\sqrt{\alpha k^2}} \left( \frac{n^{1/4}}{\log^{3/4} n} + 1 \right)^2 \leq \sqrt{n},$$

where the last inequality holds for sufficiently large $n$. To see why the claim holds for (12), note that

$$\frac{2k^{3/2}}{\beta\sqrt{\alpha}} \left( \frac{\log (k+1) + 1}{\pi} \right) \leq \frac{2}{\beta\sqrt{\alpha}} \frac{n^{1/2}}{\log^{3/2} n} \left( \frac{\log (k+1) + 1}{\pi} \right) \leq \sqrt{n},$$

where the last inequality holds for $n$ large enough.

\[\square\]

9 Putting things together

Let $\alpha > 0$ be a constant. We will show that for $\alpha = 0.93$, there exist absolute constants $k_0, n_0 > 0$ such that for all integers $n \geq n_0$ and $k$ satisfying $k_0 \leq k \leq \frac{n^{1/3}}{\log^{2/3} n}$, we have $\mathcal{E}_n(k, \sqrt{\alpha kn}) = 0$.

Let $\beta, \epsilon > 0$ be real number such that $\beta + \epsilon < 1$. Assume that $k$ and $n$ are sufficiently large so that Lemma 8.1 is applicable. Note that the condition $k \leq \frac{n^{1/3}}{\log^{2/3} n}$ is stronger than the condition $k \leq \frac{n^{1/3}}{\log n}$ required by Lemma 8.1 (the stronger condition is needed by Lemma 9.1 below).

Let $\Delta = \sqrt{\alpha kn}$. By lemma 8.1, there exists a length-$(k+1)$ integer sequence $W$ of increasing points centered at $n/2$ such that:

a) The minimum distance $t$ between any two consecutive points in $W$ is at least $b(k)\sqrt{n}$, where $b(k) = \frac{4\beta}{\pi} \left( \frac{\pi}{2(k+1)} \right)^2 \sqrt{\alpha k}$.

b) $R(W) \geq \beta\sqrt{\alpha kn}$

c) $R(W) + p \leq (\beta + \epsilon)\sqrt{\alpha kn}$, where $p = \left\lceil \frac{t-1}{2} \right\rceil$

d) $2p \leq \epsilon\sqrt{\alpha kn}$

e) $\Lambda(W) \leq \frac{\epsilon}{2} \log (k+1) + 2$

f) $R(W) + p \leq \Delta \leq \frac{4}{\pi}$.

Applying Lemma 7.4 to $W$, we get that

$$\mathcal{E}_n(k, \sqrt{\alpha kn}) = 0$$

if

$$\nu \leq tB_n \left( \frac{n}{2} + R(W) + p \right),$$

where

$$\nu = 2\Lambda(W) \sum_{w>\frac{n}{2}+\Delta} B_n(w) \left( \frac{2(w-n/2) + 2p}{R(W)} \right)^k.$$

It follows from (a), (b), (c), (d), and (e) that

$$tB_n \left( \frac{n}{2} + R(W) + p \right) \geq b(k)\sqrt{n}B_n \left( \left\lfloor \frac{n}{2} + (\beta + \epsilon)\sqrt{\alpha kn} \right\rfloor \right)$$

and

$$\nu \leq a(k) \sum_{w>\frac{n}{2}+\sqrt{\alpha kn}} B_n(w) \left( \frac{2(w-n/2) + \epsilon\sqrt{\alpha kn}}{\beta\sqrt{\alpha kn}} \right)^k,$$

where $a(k) = \frac{8}{\pi} \log (k+1) + 4$. 

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Lemma 9.1 Let \( \alpha, \beta > 0 \) and \( \epsilon \geq 0 \) be constants such that:

\[
\frac{2 + \epsilon}{\beta} \geq \sqrt{c} \tag{17}
\]

\[
\alpha > \frac{1}{2} \log \left( \frac{2 + \epsilon}{\beta} \right) \tag{18}
\]

There are constants \( n_1, k_1 > 0 \), depending only on \( \alpha \) and \( \beta \) and \( \epsilon \), such that for all \( n \geq n_1 \) and all \( k \) satisfying \( k \leq k_1 \leq \frac{n^{1/3}}{\log^2 n} \), we have

\[
a) \quad B_n \left( \left\lfloor n/2 + (\beta + \epsilon)\sqrt{\alpha kn} \right\rfloor \right) \geq \frac{1}{\sqrt{2\pi}} e^{-2(\beta + \epsilon)^2 a k} \tag{19}
\]

\[
b) \quad \sum_{w > \frac{n}{2} + \sqrt{\alpha kn}} B_n(w) \left( \frac{2(w-n/2) + \sqrt{\alpha kn}}{\beta \sqrt{\alpha kn}} \right)^k \leq (1 + \epsilon) e^{-ck}, \text{ where } c = 2 \left( \alpha - \frac{1}{2} \log \left( \frac{2a}{\beta} \right) \right). \tag{19}
\]

Note: Conditions (17) and (18) are not needed in (a).

The proof of Lemma 9.1 is in Section 9.1.

Therefore to make sure that (16) holds we need to choose \( \alpha, \beta, \epsilon > 0 \), with \( \alpha \) as small as possible, so that

\[
\frac{1}{\sqrt{2}} b(k) e^{-2(\beta + \epsilon)^2 a k} > a(k)(1 + \sqrt{\alpha k}) e^{-ck},
\]

for \( k \) large enough, or equivalently,

\[
2(\beta + \epsilon)^2 \alpha < c. \tag{19}
\]

Note that \( b(k) \) decays polynomially \( b(k) = \Omega\left(\left\lfloor \frac{1}{k^{1/2}} \right\rfloor \right) \) and \( a(k) \) increases logarithmically \( a(k) = O(\log k) \), hence the effect of \( \frac{1}{\sqrt{2}} b(k) \) and \( a(k)(1 + \sqrt{\alpha k}) \) vanishes for \( k \) large enough if \( 2(\beta + \epsilon)^2 \alpha < c \).

We are free to choose \( \alpha, \beta, \epsilon > 0 \) as long as \( \beta + \epsilon < 1 \) and conditions (17) and (18) of Lemma 9.1 are satisfied. We can ignore (17) since it is implied by the condition \( \beta + \epsilon < 1 \). We can also ignore (18) since (19) is stronger as (18) is equivalent to \( \epsilon > 0 \). Writing (19) as \( \alpha > h(\beta, \epsilon) \), where

\[
h(\beta, \epsilon) = \frac{\log(2 + \epsilon) - \log \beta}{2(1 - (\beta + \epsilon)^2)},
\]

we see that can attain any value of \( \alpha > \alpha^* \), where \( \alpha^* = \min h(\beta, \epsilon) \), over the choice of \( \beta, \epsilon > 0 \) such that \( \beta + \epsilon \leq 1 \). Numerical evaluation shows that \( \alpha^* \approx 0.9232 \). For instance, for \( \beta = 0.5204 \) and \( \epsilon = 0.004 \), we have \( h(\beta, \epsilon) \approx 0.9299 \). Setting \( \alpha = 0.93 \), we get (16), and hence (15), i.e., \( \mathcal{E}_n(k, \sqrt{\alpha kn}) = 0 \), for \( k \) and \( n \) sufficiently large.

9.1 Proof of Lemma 9.1

The proof uses the following estimates.

Theorem 9.2 (de Moivre-Laplace normal approximation of the binomial; see [32], p. 184) Let \( \delta : \mathbb{N} \to \mathbb{N} \) be such that \( \delta(n) = o(n^{2/3}) \). Then, for each \( \epsilon > 0 \), there exists \( n_0 > 0 \) such that for each \( n \geq n_0 \) and each integer \( w \) such that \( |w - n/2| \leq \delta(n) \), we have

\[
(1 - \epsilon) \sqrt{\frac{2}{\pi n}} e^{-\frac{2(w-n/2)^2}{n}} \leq B_n(w) \leq (1 + \epsilon) \sqrt{\frac{2}{\pi n}} e^{-\frac{2(w-n/2)^2}{n}}.
\]

Theorem 9.3 (Hoeffding’s inequality [33]) Let \( X_1, \ldots, X_n \) be independent random variables such that \( a_i \leq X_i \leq b_i \), for each \( i \). Then, for each \( t \geq 0 \),

\[
Pr \left[ \sum_{i} X_i - E[X_i] \geq t \right] \leq e^{-2 \frac{t^2}{\Sigma (b_i - a_i)^2}}.
\]
Therefore, using Theorem 9.2 assume that \( n \) is sufficiently large so that the following bounds hold for all \( w \) such that \( |w - n/2| \leq \frac{n^{2/3}}{\log^{1/2} n} \):

\[
B_n(w) \geq \frac{1}{\sqrt{2n}} e^{-\frac{2(w-n/2)^2}{n}}
\]

(20)

\[
B_n(w) \leq \frac{1}{\sqrt{n}} e^{-\frac{2(w-n/2)^2}{n}}.
\]

(21)

To upper bound \( B_n(w) \) for \( w > n/2 + \frac{n^{2/3}}{\log^{1/2} n} \), we use the following weak consequence of Hoeffding’s inequality. It follows from Theorem 9.3 that for each integer \( w \in [0 : n] \),

\[
B_n(w) \leq e^{-\frac{2(w-n/2)^2}{n}}.
\]

(22)

We use the lower bound (20) to estimate \( B_n \left( \left\lfloor n/2 + (\beta + \epsilon) \sqrt{\alpha k} n \right\rfloor \right) \). We use the upper bound (21) to estimate the summation in (b) for \( n/2 + \sqrt{\alpha k} n < w \leq n/2 + \frac{n^{2/3}}{\log^{1/2} n} \). For \( w > n/2 + \frac{n^{2/3}}{\log^{1/2} n} \), we use (22).

**Proof of (a)** We have

\[
\frac{2}{n} \left( \left\lfloor n/2 + (\beta + \epsilon) \sqrt{\alpha k} n \right\rfloor - \frac{n}{2} \right)^2 \leq 2(\beta + \epsilon)^2 \alpha k.
\]

Moreover, since \( k \leq \frac{n^{1/3}}{\log^2 n} \),

\[
\left| \left\lfloor n/2 + (\beta + \epsilon) \sqrt{\alpha k} n \right\rfloor - \frac{n}{2} \right| \leq (\beta + \epsilon) \sqrt{\alpha k} n \leq (\beta + \epsilon) \sqrt{\alpha} \frac{n^{2/3}}{\log n} \leq \frac{n^{2/3}}{\log^{1/2} n},
\]

for \( n \) large enough. It follows that

\[
B_n \left( \left\lfloor n/2 + (\beta + \epsilon) \sqrt{\alpha k} n \right\rfloor \right) \geq \frac{1}{\sqrt{2n}} e^{-2(\beta + \epsilon)^2 \alpha k}.
\]

**Proof of (b)** Decompose

\[
\sum_{n/2 + \sqrt{\alpha k} n < w \leq n} \left( \frac{2(w - n/2) + \epsilon \sqrt{\alpha k} n}{\beta \sqrt{\alpha k} n} \right)^k B_n(w) = A + B,
\]

where

\[
A := \sum_{n/2 + \sqrt{\alpha k} n < w \leq n/2 + \frac{n^{2/3}}{\log^{1/2} n}} \left( \frac{2(w - n/2) + \epsilon \sqrt{\alpha k} n}{\beta \sqrt{\alpha k} n} \right)^k B_n(w)
\]

\[
B := \sum_{n/2 + \frac{n^{2/3}}{\log^{1/2} n} < w \leq n} \left( \frac{2(w - n/2) + \epsilon \sqrt{\alpha k} n}{\beta \sqrt{\ln n}} \right)^k B_n(w).
\]

We will argue that \( A \leq \sqrt{\alpha k} e^{-c k} \) and \( B \leq e^{-c k} \), for sufficiently large \( n \) and \( k \).
If follows from (21) and (22) that
\[
A \leq \frac{1}{\sqrt{n}} \sum_{w > n/2 + \sqrt{n}} \left( \frac{2(w - n/2) + \epsilon \sqrt{kn}}{\beta \sqrt{n}} \right)^k e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2}.
\]
\[
B \leq \sum_{w > n/2 + \sqrt{n}} \left( \frac{2(w - n/2) + \epsilon \sqrt{kn}}{\beta \sqrt{n}} \right)^k e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2}.
\]

Let \( x = \frac{w - n/2}{\sqrt{kn}} \) and note that \( x > 1 \) for \( w > n/2 + \sqrt{n} \). Thus
\[
\left( \frac{2(w - n/2) + \epsilon \sqrt{kn}}{\beta \sqrt{n}} \right)^k e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2} = e^{-2\alpha k \left( x^2 - \frac{2 + \epsilon}{\beta} \right)}.
\]

For all \( x \geq 1 \), we have \( \log \left( \frac{2 + \epsilon}{\beta} \right) \leq \log \left( \frac{2 + \epsilon}{\beta} \right) x^2 \), where the last inequality holds for all \( x \geq 1 \) if \( \frac{2 + \epsilon}{\beta} \geq \sqrt{e} \), which is guaranteed by condition (17). It follows that, for all \( x > 1 \),
\[
2\alpha k \left( x^2 - \frac{2}{\alpha} \log \frac{2 + \epsilon}{\beta} \right) \geq 2 \left( \alpha - \frac{1}{2} \log \frac{2 + \epsilon}{\beta} \right) kx^2 = c \left( w - n/2 \right)^2.
\]

where \( c = 2 \left( \alpha - \frac{1}{2} \log \frac{2 + \epsilon}{\beta} \right) \). Note that condition (15) says that \( c > 0 \). Therefore,
\[
A \leq \frac{1}{\sqrt{n}} \sum_{w > n/2 + \sqrt{n}} e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2}
\]
\[
B \leq \sum_{w > n/2 + \sqrt{n}} e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2}.
\]

Now, in general, for any \( y \geq 1 \) and any \( a > 0 \), we have
\[
\sum_{w > n/2 + y} e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2} \leq \int_{n/2 + y}^{\infty} e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2} du = \sqrt{\pi} \int_{n/2 + y - 1}^{\infty} e^{-z^2} dz
\]
\[
\leq (y - 1) \int_{n/2 + y - 1}^{\infty} ze^{-z^2} dz = y - 1 \int_{n/2 + y - 1}^{\infty} e^{-z} e^{-z^2} dz \leq \frac{y}{2} e^{-\gamma_2 \left(\frac{w - n/2}{n}\right)^2}.
\]

It follows that
\[
A \leq \frac{\sqrt{\pi}}{2} e^{-c \left(\frac{w - n/2}{n}\right)^2} = \frac{1}{2} e^{-c \left(\frac{w - n/2}{n}\right)^2} \sqrt{\alpha k} e^{-ck} \leq \sqrt{\alpha k} e^{-ck},
\]
for \( n \) large enough. The last inequality holds for \( n \) large enough since \( \frac{\sqrt{\pi}}{n} \leq \frac{1}{n^{1/3} \log n} \) as \( k \leq \frac{n^{1/3}}{\log n} \). Finally,
\[
B \leq \frac{n^{2/3}}{2 \log^{1/3} n} e^{-c \left(\frac{w - n/2}{n}\right)^2} \leq n^{2/3} e^{-\frac{\gamma_2 \left(\frac{w - n/2}{n}\right)^2}{2 \log n}} \leq n^{2/3} e^{-\frac{\gamma_2}{2 \log n} k \log n} \leq e^{-ck},
\]
for \( n \) and \( k \) large enough, where the inequality before the last holds because \( k \leq \frac{n^{1/3}}{\log^2 n} \).

\footnote{Let \( a = \frac{2 + \epsilon}{\beta} \). The slopes at 1 of \( \log(ax) \) and \( (\log a)x^2 \) are 1 and 2 \( \log a \), respectively. Thus, to guarantee that \( \log(ax) \leq (\log a)x^2 \) for all \( x \geq 1 \), we need \( 1 \leq 2 \log a \), i.e., \( a \geq \sqrt{e} \).}
10 Conclusion

We conclude with the following open questions:

- As mentioned in the introduction, Tietäväinen’s bound is not tight for linear codes with sufficiently large dual distance $d$ in the $d = \Theta(n)$ regime. Is this also the case for $k$-wise independent distributions? Note that in the aforementioned papers [8] - [17], the techniques which improve on Tietäväinen’s bound in the $d = \Theta(n)$ regime are specific to linear codes and do not seem applicable to $k$-wise independent distributions.

- Is the upper bound $\frac{n^{1/3}}{\log^2 n}$ on $k$ in Theorem 1.6 an artifact of the proof? i.e., does the statement hold for larger values of $k$? The answer is not clear since if we ignore the log term, the requirement of $k = o(n^{1/3})$ has two independent origins in the proof. The first is conditions (11) and (12) in the analysis of quantized Chebyshev sequences in the proof of Lemma 8.1. The second is de Moivre-Laplace normal approximation of the binomial in the proof of Lemma 9.1.

- Theorem 1.6 implies the existence a $k$-wise independent probability distribution on $\{0, 1\}^n$ whose covering radius is at least $\frac{n}{2} - \sqrt{kn}$ if $k \leq \frac{n^{1/3}}{\log n}$ and $k$ and $n$ are large enough. Can such distributions be supported by linear codes or are they intrinsically non-linear? That is, assuming that $d = w(1)$ and $d = o(n)$, is there an $\mathbb{F}_2$-linear block-length-$n$ code with dual distance $d$ and covering radius at least $\frac{n}{2} - \Theta(\sqrt{dn})$?

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