Robust Phase Retrieval via ADMM with Outliers

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Abstract

An outlier-resistance phase retrieval algorithm based on alternating direction method of multipliers (ADMM) is devised in this letter. Instead of the widely used least squares criterion that is only optimal for Gaussian noise environment, we adopt the least absolute deviation criterion to enhance the robustness against outliers. Considering both intensity- and amplitude-based observation models, the framework of ADMM is developed to solve the resulting non-differentiable optimization problems. It is demonstrated that the core subproblem of ADMM is the proximity operator of the $\ell_1$-norm, which can be computed efficiently by soft-thresholding in each iteration. Simulation results are provided to validate the accuracy and efficiency of the proposed approach compared to the existing schemes.

Index Terms

Phase retrieval, alternating direction method of multipliers (ADMM), outlier, least absolute deviation.

I. INTRODUCTION

In many applications, the intensity or amplitude of the signal-of-interest (SOI) can be measured, but its phase is unavailable. Signal reconstruction from phaseless measurements is referred to as phase...
retrieval [1], [2], which attracts a great attention in various fields of science and engineering, such as optical imaging [2], X-ray crystallography [3], astronomy [4], and radar [5].

The methods for phase retrieval in the early days consider the Fourier transform model, i.e., the observations are the modulus or the squared modulus of their Fourier transform. In this case, the most well-known methods are the error reduction algorithms, including Gerchberg and Saxton (GS) [6], Fienup [7] which is a modified version of GS, and other variants. The basic idea of error reduction approach is alternating projection. It starts from a random initial estimate and iterates between the time and frequency domains to correct the current estimate according to the time-domain a priori knowledge and scale Fourier coefficients to match the measured data in frequency domain. Although the GS algorithm and its modifications are useful in practice, a theoretical issue is that the guarantee of the convergence behavior is still unclear because the alternating projection onto nonconvex sets is involved in [1].

Recently, an efficient method for phase retrieval has been developed via Wirtinger flow (WF) [1], which is actually a gradient descent scheme. More importantly, Candès et al. have proved that the WF with an initialization using the spectral method exhibits geometric convergence to the solution with a high probability provided that the size of observations is on the order of $N \log N$ with $N$ being the length of the SOI. In order to accelerate the convergence rate of WF, an WF with optimal stepsize (WFOS) has been devised by selecting a more appropriate stepsize at each iteration [8].

Instead of solving the nonconvex problem directly, an alternative is to relax the original problem into a convex programming [9]. Note that phase retrieval requires solving a system of quadratic equations, which can be converted into linear ones by lifting up the $N$-dimensional vector to an $N \times N$ rank-one matrix. Therefore, it is also known as PhaseLift. Then the problem of rank minimization can be relaxed into a convex trace norm minimization and solved by semidefinite programming (SDP). Although in general the SDP-based approaches [9]–[11] can provide satisfying results, they become computationally demanding as the signal dimension increases because of matrix lifting.

Most existing methods are built upon the least squares (LS) criterion which is only optimal for Gaus-
sian noise. While they work well for the noise-free or Gaussian noise environments, their performances significantly degrade in the presence of non-Gaussian outliers. In fact, the phenomenon of outliers has been reported in different fields [12]. In order to enhance the robustness against outliers, we adopt the least absolute deviation (LAD) criterion instead of the widely used LS methodology. Considering both intensity- and amplitude-based observation models, the framework of alternating direction method of multipliers (ADMM) is proposed to solve the resulting non-differentiable optimization problems. The core subproblem of the proposed LAD-ADMM for phase retrieval can be converted to the proximity operator of the $\ell_1$-norm, which can be computed efficiently by soft-thresholding in each iteration.

The remainder of this paper is organized as follows. The problem formulation is presented in Section II. Section III describes the framework of LAD-ADMM for robust phase retrieval. In Section IV, simulation results are provided to demonstrate the high accuracy of the LAD-ADMM. Finally, conclusions are drawn in Section V.

We use bold capital upper-case and lower-case letters to represent matrices and vectors, respectively. The $\mathbf{0}$ and $\mathbf{I}$ are the zero vector and identity matrix, respectively. The superscripts $(\cdot)^T$ and $(\cdot)^*$ denote the transpose and complex conjugate, respectively. The $j = \sqrt{-1}$ is the imaginary unit. The $|\cdot|$ denotes the absolute value of a real number or the modulus of a complex number and $\|\cdot\|_2$ is the Euclidean norm of a vector. The $\odot$ is the elementwise multiplication. Finally, $\mathbb{R}$ and $\mathbb{C}$ represent the sets of real and complex numbers, respectively.

### II. Problem Formulation

In general, there are two types of observation models for phase retrieval. One is the intensity-based model given by

$$y = |Ax|^2 + \nu,$$  \hspace{1cm} (1)

where $x = [x_1, \cdots, x_N]^T \in \mathbb{C}^N$ is the SOI to be estimated, $y = [y_1, \cdots, y_M]^T \in \mathbb{R}^M$ is the recorded intensity, $A = [a_1^*, \cdots, a_M^*]^T \in \mathbb{C}^{M \times N}$ is the measurement matrix and $\nu$ is the noise component. A
common approach to determining $x$ from (1) is to employ the LS criterion:

$$\min_x \|y - |Ax|\|^2_2.$$  \hfill (2)

The recently reported WF algorithm \cite{1}, which applies a gradient descent scheme, has been demonstrated as an efficient method for solving (2). The WFOS method can further accelerate the convergence rate of WF \cite{8}.

Another type of observation model is based on the amplitude. That is,

$$b = |Ax| + v,$$  \hfill (3)

where $b \in \mathbb{R}^M$. The corresponding LS criterion is given by

$$\min_x \|b - |Ax|\|^2_2,$$  \hfill (4)

which can be reformulated as

$$\min_{x, \phi} \|Ax - b \odot e^{j\phi}\|^2_2,$$  \hfill (5)

where $\phi = [\phi_1, \cdots, \phi_M]^T$ is phase vector. The problem of (5) can be solved based on alternating projections, say, GS algorithm \cite{6}. That is, given $\phi$, $x$ can be obtained by solving the following classic LS problem: $\min_x \|Ax - b \odot e^{j\phi}\|^2_2$ while $\phi = \angle(Ax)$, where $\angle(Ax) \in [0, 2\pi)$ is the phase angle of $Ax$, with $x$ being fixed. In the case of Fourier transform, $A$ is the Fourier matrix. The algorithm essentially adjusts the modulus of the Fourier transform of the current estimate so that it is consistent with the amplitude data. Although the LS criterion of (2) and (4) works well with Gaussian noise, its performance significantly degrades with outliers. In this letter, we aim to devise an outlier-robust phase retrieval via ADMM.

\section{Outlier-Robust Phase Retrieval}

\subsection{Framework of ADMM}

To enhance the robustness against outliers, we adopt the LAD criterion instead of (2), which is given by

$$\min_x \|y - |Ax|\|_1.$$  \hfill (6)

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where \( \| \cdot \|_1 \) is the \( \ell_1 \)-norm of a vector defined as \( \| h \|_1 = \sum_{i=1}^{N} |h_i| \) for \( h \in \mathbb{C}^N \). Since the objective function of (6) is non-differentiable, the gradient-based approaches are not applicable anymore. We rewrite (6) as

\[
\min_{x,z} \| z \|_1 \\
\text{s.t.} \quad z = |Ax|^2 - y
\]

by introducing a real-valued auxiliary vector \( z \in \mathbb{R}^M \). The augmented Lagrangian function of (7) is [13]

\[
L_\rho(x, z, \lambda) = \| z \|_1 + \lambda^T (|Ax|^2 - y - z) + \frac{\rho}{2} \| |Ax|^2 - y - z \|_2^2
\]

(8)

where the vector \( \lambda \in \mathbb{R}^M \) contains the \( M \) Lagrange multipliers (dual variables) and \( \rho > 0 \) is the penalty parameter. The augmented Lagrangian reduces to the unaugmented one with \( \rho = 0 \). ADMM is proved to converge for all positive values of \( \rho \) under quite mild conditions [13], which makes the selection of \( \rho \) rather flexible [14]. Hence, we can simply use a fixed positive constant for \( \rho \). Of course, using possibly different penalty parameter for each iteration may improve the convergence rate in practice [13], [14]. The Lagrange multiplier method solves (7) by finding a saddle point of the augmented Lagrangian. That is,

\[
\max_{\lambda} \min_{x,z} L_\rho(x, z, \lambda),
\]

(9)

with the following steps at each iteration:

\[
x^{k+1} = \arg \min_x L_\rho(x, z^k, \lambda^k)
\]

(10)

\[
z^{k+1} = \arg \min_z L_\rho(x^{k+1}, z, \lambda^k)
\]

(11)

\[
\lambda^{k+1} = \lambda^k + \rho \left( |Ax^{k+1}|^2 - y - z^{k+1} \right)
\]

(12)

where \( \{x^k, z^k, \lambda^k\} \) denotes the result at the \( k \)th iteration.
B. Solving the Subproblems

Now we investigate the three subproblems of (10)–(12). Note that the gradient of \( L_\rho(x^{k+1}, z^{k+1}, \lambda) \) with respect to (w.r.t.) \( \lambda \) is calculated from (8) as
\[
\frac{\partial L_\rho(x^{k+1}, z^{k+1}, \lambda)}{\partial \lambda} = |Ax^{k+1}|^2 - y - z^{k+1}.
\] (13)

Therefore, it is clear that (12) adopts a gradient ascent with a step size \( \rho \) updating the dual variable \( \lambda \). The ADMM updates \( x \) and \( z \) in an alternating way to avoid jointly minimizing w.r.t. two primal variables. By ignoring the constant term independent of \( x \), the subproblem of (10) can be written as
\[
\min_x ||Ax||_2^2 - \left( z^k + y - \lambda^k/\rho \right) ||_2^2,
\] (14)

which is equivalent to the LS problem in (2) and hence can be solved by WF or WFOS efficiently.

The subproblem of (11) can also be expressed as
\[
\min_z \frac{1}{2} \|z - c^k\|_2^2 + \frac{1}{\rho} \|z\|_1,
\] (15)

where \( c^k \triangleq |Ax^{k+1}|^2 - y + \lambda^k/\rho \). Note that the minimizer of (15) defines the proximity operator of the \( \ell_1 \)-norm, which can be computed by soft-thresholding \[15\]:
\[
z^{k+1}_i = \text{soft} \left( c^k_i, \frac{1}{\rho} \right), \quad i = 1, \cdots, M
\] (16)

where \( z_i \) and \( c_i \) represent the \( i \)th component of \( z \) and \( c \), respectively, and
\[
\text{soft} \left( u, a \right) \triangleq \text{sgn}(u) \cdot \max(|u| - a, 0)
\] (17)
is the well-known soft-thresholding function.

We summarize the steps of ADMM with intensity-based model for robust phase retrieval in Algorithm \[1\].

C. Extension to Amplitude-based Model

Similarly, we adopt the LAD criterion for the amplitude-based model:
\[
\min_{x,z} \|z\|_1 \\
\text{s.t. } z = |Ax| - b
\] (18)
Algorithm 1 ADMM with Intensity-based Model

Input: $A \in \mathbb{C}^{M \times N}$, $y \in \mathbb{R}^M$, and $\rho > 0$.

Initialize: $z^0 = 0$, $\lambda^0 = 0$.

for $k = 0, 1, \cdots$, do until converge

1) $x^{k+1} = \arg \min_x \|Ax\|^2 - \left( z^k + y - \frac{\lambda^k}{\rho} \right)$, which is solved by WF or WFOS.

2) Calculate $c^k = |Ax^{k+1}|^2 - y + \lambda^k/\rho$.

3) $z_i^{k+1} = \text{soft} \left( c_i^k, \frac{1}{\rho} \right)$, $i = 1, \cdots, M$.

4) $\lambda^{k+1} = \lambda^k + \rho \left( |Ax^{k+1}| - y - z^{k+1} \right)$.

end for

Output: $x^{k+1}$.

and the corresponding augmented Lagrangian function is

$$\bar{L}_\rho(x, z, \lambda) = \|z\|_1 + \lambda^T (|Ax| - b - z) + \frac{\rho}{2} \| |Ax| - b - z \|^2_2.$$ \hspace{1cm} (19)

To find the saddle point of

$$\max_{\lambda} \min_x \bar{L}_\rho(x, z, \lambda),$$ \hspace{1cm} (20)

the following steps are performed iteratively:

$$x^{k+1} = \arg \min_x \bar{L}_\rho(x, z^k, \lambda^k)$$ \hspace{1cm} (21)

$$z^{k+1} = \arg \min_z \bar{L}_\rho(x^{k+1}, z, \lambda^k)$$ \hspace{1cm} (22)

$$\lambda^{k+1} = \lambda^k + \rho \left( |Ax^{k+1}| - b - z^{k+1} \right).$$ \hspace{1cm} (23)

It is not difficult to verify that the subproblem of (21) is equivalent to

$$\min_x \|Ax\| - \left( z^k + y - \frac{\lambda^k}{\rho} \right),$$ \hspace{1cm} (24)

which can be solved in the same way as (4) by alternating minimization, while (22) is computed via soft-thresholding, which is given by

$$z_i^{k+1} = \text{soft} \left( |Ax^{k+1}| - b + \lambda^k/\rho, 1/\rho \right), \ i = 1, \cdots, M.$$ \hspace{1cm} (25)
Now we summarize the steps of ADMM with amplitude-based model in Algorithm 2.

### Algorithm 2: ADMM with Amplitude-based Model

**Input:** $A \in \mathbb{C}^{M \times N}$, $b \in \mathbb{R}^M$, and $\rho > 0$.

**Initialize:** $z^0 = 0$, $\lambda^0 = 0$.

**for** $k = 0, 1, \cdots$, do until converge

1. $x^{k+1} = \arg \min_x \|Ax| - (z^k + b - \frac{\lambda^k}{\rho})\|^2_2$, which is solved by GS.

2. Calculate $c^k = |Ax^{k+1}| - b + \lambda^k / \rho$.

3. $z_i^{k+1} = \text{soft} \left( \frac{c_i^k}{\rho}, \frac{1}{\rho} \right)$, $i = 1, \cdots, M$.

4. $\lambda^{k+1} \leftarrow \lambda^k + \rho (|Ax^{k+1}| - b - z^{k+1})$.

**end for**

**Output:** $x^{k+1}$.

### IV. Simulation Results

In this section, we compare the proposed LAD-ADMM with WF and GS under the intensity- and amplitude-based observation models, respectively. The normalized mean square error (NMSE) is used as the performance metric, which is defined as

$$\text{NMSE} = \frac{\|x^k - x\|^2}{\|x\|^2} \tag{26}$$

where $x$ is the true SOI. The Gaussian mixture model (GMM) is taken as the impulsive noise. The probability density function of the two-term GMM is given by

$$p_v(v) = \sum_{i=1}^2 c_i \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left( -\frac{v^2}{2\sigma_i^2} \right), \tag{27}$$

where $c_i \in [0, 1]$ and $\sigma_i^2$ are the probability and variance of the $i$th term, respectively. We have $c_1 + c_2 = 1$. If $\sigma_2^2 \gg \sigma_1^2$ and $c_2 < c_1$, noise samples of larger variance $\sigma_2^2$ occurring with a smaller probability $c_2$ can be considered as outliers embedded in Gaussian background noise of variance $\sigma_1^2$. Hence the GMM is widely used to model the scenario with both Gaussian noise and outliers.
total noise variance is $\sigma_v^2 = \sum_i c_i \sigma_i^2$ and the signal-to-noise ratio (SNR) is defined as $\|x\|^2 / \sigma_v^2$. In the simulations, we set $\sigma_i^2 = 100 \sigma_0^2$ and $c_2 = 0.1$. Therefore, 10% of noise samples can be considered as outliers. The measurement vector $a_i$ consists of independent standard complex Gaussian variables, i.e., $a_i \in \mathbb{C}^N \sim \mathcal{N}(0, I/2) + j\mathcal{N}(0, I/2)$, $i = 1, \cdots, M$. The length of the signal is $N = 32$. The number of observations is eight times the signal dimension, i.e., $M = 8N$. We set $\rho = 1$. For fair comparison, the same initial value obtained from the spectral method [1] is taken for different approaches. When plotting the performance curves, 100 Monte Carlo trials are performed.

First, we investigate the convergence behavior of the LAD-ADMM. In Figs. 1 and 2, we plot the NMSE versus iteration number with intensity- and amplitude-based observation models, respectively. We can observe that the LAD-ADMM converges in several tens of iterations and the NMSE reaches a lower bound in tens of iterations at SNR = 12 dB. Figs. 3 and 4 show the NMSE versus SNR with intensity- and amplitude measurements, respectively. It can be seen that the LAD-ADMM outperforms the WF and GS, which achieves a more accurate solution for different SNR conditions. As shown in Fig. 3 the NMSE of LAD-ADMM is about the order of $10^{-4}$ at SNR = 15 dB while that of WF is $10^{-2}$.
Fig. 2. NMSE versus SNR with amplitude-based model.

Fig. 3. NMSE versus SNR with intensity-based model.

V. CONCLUSION

An outlier-robust phase retrieval algorithm based on the ADMM is devised in this letter. Instead of the widely used LS criterion that is only optimal for Gaussian noise environment, we adopt the LAD criterion to enhance the robustness against outliers. The framework of ADMM is developed to solve the resulting non-differentiable optimization problems which is applicable for both intensity- and amplitude-based observation models. We have demonstrated that the subproblems of LAD-ADMM
can be computed efficiently in each iteration. Simulation results validated the convergence behavior of the proposed algorithm and its accuracy compared to the existing techniques.

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