Spectral functions of non-essentially self-adjoint operators

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Abstract

One of the many problems to which Dowker devoted his attention is the effect of a conical singularity in the base manifold on the behavior of the quantum fields. In particular, he studied the small-\(t\) asymptotic expansion of the heat-kernel trace on a cone and its effects on physical quantities as the Casimir energy. In this paper, we review some peculiar results found in the last decade, regarding the appearance of non-standard powers of \(t\), and even negative integer powers of \(\log t\), in this asymptotic expansion for the self-adjoint extensions of some symmetric operators with singular coefficients. Similarly, we show that the \(\zeta\)-function associated with these self-adjoint extensions presents an unusual analytic structure.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In quantum field theory, the transition amplitude for particles interacting with a background field or subject to boundary conditions can be described in terms of the effective action of the model under study. In the one-loop approximation, this effective action can be expressed in terms of the functional determinant of the differential operator appearing in the quadratic in the quantum field term of the action. Since the spectra of these operators are unbounded, these determinants must be defined in the framework of an appropriate regularization.

In 1976, Dowker and Critchley \[1\] presented a powerful and elegant regularization scheme for the definition of these functional determinants based on the \(\zeta\)-function associated with the differential operator, which is built from the spectrum of the quantum fluctuations.

Since then, this formalism has been successfully applied to the determination of one-loop effective actions, vacuum energies, anomalies and other physical quantities of interest.
in quantum field theory. It is presently a fundamental tool in the study of quantum effects in systems under the influence of external conditions [2–7].

To be more specific, the first quantum corrections to the effective action are, in general, given by (a power which depends on the nature of the fields of) the functional determinant of an elliptic differential operator on an appropriate Hilbert space—the quantum fluctuations operator—that is obtained as the second functional derivative of the classical action at vanishing quantum fields. Dowker and Critchley gave a definition of this functional determinant in terms of the derivative of the associated \( \zeta \)-function, thus relating the one-loop effective action to the spectrum of the quantum fluctuations operator and, consequently, to the external conditions applied on them.

This formalism is also related to the so-called proper time regularization through the relation between the \( \zeta \)-function and the trace of the corresponding heat-kernel [8] (see appendix A). Let \( H \) represent an elliptic boundary problem on an \( m \)-dimensional compact manifold \( M \) with smooth boundary \( \partial M \), with a discrete spectrum \( \{ \lambda_n \}_{n \in \mathbb{N}} \). Then, under appropriate conditions [9], the \( \zeta \)-function is defined as the trace of the complex power \( H^{-s} \),

\[
\zeta_H(s) := \text{Tr} \, H^{-s} := \sum_{n \in \mathbb{N}} \lambda_n^{-s}, \quad (1.1)
\]

series that converges to an analytic function in an open half-plane with \( \Re(s) \) being large enough and admits a meromorphic extension to the whole \( s \)-plane. On the other hand, for positive definite \( H \), the trace of the operator \( e^{-tH} \) is defined as [10]

\[
\text{Tr} \, e^{-tH} := \sum_{n \in \mathbb{N}} e^{-t\lambda_n} \quad (1.2)
\]

for \( t > 0 \). It has been established [9] that for a differential operator \( H \) of order \( d \) with smooth coefficients, the trace of the heat-kernel, \( \text{Tr} \, e^{-tH} \), admits a small-\( t \) asymptotic expansion given by

\[
\text{Tr} \, e^{-tH} \sim \sum_{n=0}^{\infty} a_n \cdot t^{(n-m)/d}, \quad (1.3)
\]

where the Seeley–De Witt (SDW) coefficients \( a_n \) are integrals on the manifold \( M \) and its boundary \( \partial M \) of local invariants which depend on the coefficients in \( H \), the metric on \( M \) and the boundary conditions imposed at \( \partial M \) [10]. Let us remark that for a differential operator with smooth coefficients on a base manifold \( M \) without a boundary, the SDW coefficients \( a_n \) vanish for odd \( n \).

Since the function \( \zeta_H(s) \) in equation (1.1) is related to the heat-trace \( \text{Tr} \, e^{-tH} \) in equation (1.2) by a Mellin transformation [8], the expansion in equation (1.3) implies that \( \zeta_H(s) \) has isolated simple poles at

\[
s_n = \frac{m-n}{d}, \quad \text{with } n = 0, 1, 2, \ldots, \quad (1.4)
\]

with residues being related to the SDW coefficients by (see appendix A)

\[
\text{Res} \left[ \Gamma(s) \zeta_H(s) \right]_{s = s_n} = a_n. \quad (1.5)
\]

In particular, \( \text{Res} \left[ \zeta_H(s) \right]_{s = s_n} = 0 \) when \( s_n = 0, -1, -2, \ldots \) and \( \zeta_H(s) \) is analytic in a neighborhood of the origin.

The functional determinant of \( H \) introduced in [1] can be defined by

\[
\log \text{Det} \, H := - \frac{d}{ds} \zeta_H(s) \bigg|_{s=0}, \quad (1.6)
\]

where \( |s| \to 0 \) stands for the limit of the analytic continuation to a neighborhood of \( s = 0 \).
When employed for the description of the one-loop contribution to the effective action, this scheme of regularization leads to local counter-terms expressed in terms of the SDW coefficients (or, equivalently, in terms of the $\zeta$-function). This justifies the intense research devoted to the $\zeta$-function and heat-kernel methods during the last decades in relation to its applications to quantum field theory.

In this context, one of the problems to which Dowker devoted his attention is the influence that a conical singularity in the manifold has on the quantum field behavior [11–18]. Particles in a conic singularity, a situation which can be related to the so-called Calogero models [19], appear in the study of quantum fields in a black-hole background [22], in the presence of cosmic strings [23] and in condensed matter, for example. The heat-kernel trace on a conic manifold has been studied in detail [20, 21] and shown to admit, as in the regular case, the asymptotic expansion described by equation (1.3) plus a possible logarithm. However, contrary to the case of a regular background, the SDW coefficients $a_n$ do not vanish in general for odd $n$, even in the presence of no (other) boundary.

In any case, the heat-trace on a cone satisfies the asymptotic expansion in equation (1.3) only if one assumes that the fields are regular at the singular point (see other studies and applications of this topic in [24–29]). In fact, if one considers other (square integrable) behavior of the fields at the singularity, one finds that these models present different spectral properties for which the expansion given in equation (1.3) no longer holds.

In this paper, we will review some results regarding the peculiar properties of the spectral functions associated with some differential operators with regular singularities in their coefficients. In particular, we will focus on some progress made in the last few years in the understanding of the heat-kernel trace and $\zeta$-function properties of some (symmetric but not essentially self-adjoint) differential operators which are locally homogeneous near the singularity, i.e. with the singular coefficient in the potential with the same scaling dimension as the highest derivative in the kinetic term. As we will see, an essential aspect of these models is the possibility of imposing (for a certain range of the parameters in the potential) boundary conditions at the singularity that breaks this scale homogeneity.

The Laplacian on a manifold with a conical singularity is an example of these kinds of differential operators. The asymptotic expansion of the heat-kernel trace corresponding to this operator was considered, probably for the first time, by Sommerfeld [30] and Carlslaw [31]. It was only in 1980 that it was pointed out by Callias and Taubes [32] that, for these kinds of differential operators, the heat-kernel trace small-$t$ asymptotic expansion in terms of powers of the form $t^{(n-m)/d}$ could be ill defined and conjectured that more general powers of $t$, as well as log $t$ terms, could appear. Indeed, for some second-order elliptic (essentially) self-adjoint differential operators $H$, the presence of $t^{(n-m)/2}$ log $t$ terms in the small-$t$ asymptotic expansion of the heat-kernel at the diagonal\(^1\), $e^{-tH}(x,x)$, with some distributional coefficients with support concentrated at the singularity was proved in the following years [33].

More recently, Mooers [34] studied the self-adjoint extensions (SAE) of the Laplacian acting on differential forms on a manifold with a conical singularity and found that the asymptotic expansion of the heat-trace contains powers of $t$ whose exponents depend on the deficiency angle of the singularity.

In similar settings, it was shown in a series of articles [35–39] that, for some non-essentially self-adjoint locally homogeneous differential operators, the small-$t$ asymptotic expansion of the heat-kernel trace presents non-standard powers of $t$, i.e. powers with exponents which are not determined by the order of the differential operator and the dimension of the base manifold only, as for the regular case (see equation (1.3)), but also depend on the coefficient

\(^1\) By ‘heat-kernel’ we mean the kernel $e^{-tH}(x,y)$ of the integral operator $e^{-tH}$.
of the singular term in the potential. Consequently, in these models, the presence of a regular singularity in the potential term of the differential operator leads to non-standard poles in the associated $\zeta$-function, which lie at positions that depend on parameters other than the order of the operator and the dimension of the manifold.

Later, Kirsten et al. [40–44] considered a limit case of a symmetric second-order locally homogeneous differential operator, finding that integer powers of log $t$ terms appear in the small-$t$ asymptotic expansion of the heat-kernel trace and a logarithmic cut in the corresponding $\zeta$-function, so correcting an error in appendix A in [37].

It is our aim to review these results concerning the non-standard behavior of the small-$t$ asymptotic expansion of the heat-kernel trace and the singularity structure of the $\zeta$-function corresponding to this kind of locally homogeneous differential operator. We will consider the case of some symmetric operators with a regular singularity in the potential term with the same scaling dimension as the derivative term, which makes them admit a continuous family of SAE [45]. As we will see, the (local) scaling homogeneity is in general broken in the domains of definition of these SAE, and this fact has consequences on the behavior of the associated spectral functions which will be studied in the following.

In section 2, we will consider a Dirac operator $D$ (see equation (2.1)) defined on a space of two-component functions $\Phi(x)$ with $x$ taking values in the compact segment $[0, 1]$. We will introduce in the operator $D$ a singular term proportional to $1/x$, which has the same scaling dimension as the kinetic term $\partial x$. As we will see, this simple model shows the above-mentioned characteristic.

This first-order differential operator is not positive definite and one cannot define the associated heat-kernel. Instead, we will consider the $\zeta$-function as defined in (1.1) and show that it presents a non-standard pole structure due to the presence of the singular term $\sim 1/x$ in $D$. In doing so, we will first determine the large-$|\lambda|$ asymptotic expansion of the resolvent $(D - \lambda)^{-1}$, since the powers of $\lambda$ in this expansion determine the positions of the poles of the $\zeta$-function (see [8], for example).

We will proceed as follows. Firstly, we will construct the resolvents for two particular SAE for which the boundary conditions at the singular point $x = 0$ are invariant under the scaling $x \rightarrow cx$. The resolvent expansion for these particular SAE displays the standard powers of $\lambda$, leading to the standard poles for the $\zeta$-function. Secondly, we will show that the resolvent for a general SAE is a convex linear combination of these special resolvents with coefficients which depend on $\lambda$. This additional dependence on $\lambda$ leads to the non-standard powers in the resolvent asymptotic expansion of a general SAE and, hence, to non-standard poles for the associated $\zeta$-function. The SAE of $D$ are not, in general, locally scale invariant at the singularity in the sense that the conditions that the functions in its domain satisfy near the origin are not invariant under the scaling $x \rightarrow cx$. As $c \rightarrow 0$, they tend to the conditions satisfied by the functions in the domain of one of the locally scale-invariant SAE, and as $c \rightarrow \infty$, they tend to the other. The dependence of the residues at the anomalous poles on the SAE will also be explained by a scaling argument.

This model describes the central idea of our work. We consider differential operators $H$ which contain singular terms of the same scaling dimension as the highest derivative term; these operators—which we call locally homogeneous near the singularity—admit SAE whose domains are characterized by boundary conditions at the singularity that break, in general, this local homogeneity. This introduces in the definition of the SAE dimensionful parameters which are not present in its expression as differential operators. This determines that the large-$|\lambda|$ asymptotic expansion of the resolvent-trace of a general SAE, $\text{Tr}(H - \lambda)^{-1}$, presents non-standard powers of $\lambda$, with dependence on the coupling in the singular term. This, in turn, is the origin of non-standard poles of the $\zeta$-function, $\text{Tr}H^{-s}$, in the complex $s$-plane and
(for $H$ positive definite) non-standard powers of $t$ in the small-$t$ asymptotic expansion of the heat-kernel trace, $\text{Tr} e^{-tH}$. Here, by 'non-standard' we mean 'not determined by the dimension of the base manifold and the order of the differential operator only', as happens for the regular case.

Indeed, in section 3, we consider a second-order differential operator on the segment $[0, 1]$ with a singular potential term of the form $g(g - 1)/x^2$. We show that, for $|g| < 1/2$, one obtains similar results for the associated $\zeta$-function as those described for the first-order operator case we will consider in section 2. One finds that there are two locally scale-invariant SAE for which the $\zeta$-function presents the usual poles. The resolvent of the general SAE is obtained as a convex linear combination of the resolvents of these particular SAE, with a coefficient dependent on the $\lambda$-parameter. This implies the presence of $g$-dependent poles in the $\zeta$-function with residues which also depend on the SAE. On the other hand, the small-$t$ asymptotic expansion of the heat-kernel trace of the general SAE presents non-standard $g$-dependent powers of $t$ with SDW coefficients being dependent on the SAE.

This second-order differential operator taken for $g = 1/2$ also admits a continuous family of the SAE. This is a limit case in the sense that beyond this range, the operator becomes unbounded. It has the peculiarity that only one SAE is locally scale invariant near the singularity at $x = 0$. This is reflected in even more pathological properties of its associated spectral functions. In section 4, we briefly review the singularities of the $\zeta$-function, which also presents a branch cut, and the behavior of the heat-kernel trace, which in the general case admits a small-$t$ asymptotic expansion in terms of negative integer powers of $\log t$.

In section 5, we consider a locally homogeneous second-order differential operator on the half-line $\mathbb{R}^+$. In order to get discrete spectra for the SAE of this operator, we also include a quadratic term in the potential $V(x) = x^2 + \left(\nu^2 - \frac{1}{4}\right)/x^2$. We obtain similar results as in the compact case discussed in section 3: only two of the continuous family of SAE admitted by this Schrödinger operator for a certain range of the coupling $\nu$ have domains which remain invariant under scaling transformations. The associated spectral functions show the same properties as for the regular potential case. In contrast, the conditions satisfied near the singularity by the functions belonging to the domains of definition of the remaining SAE introduce a dimensionful parameter which explicitly break this local scale homogeneity, which implies unusual behavior of the $\zeta$-function poles and powers of $t$ in the asymptotic expansion of the heat-kernel trace already described. These results are obtained by employing the von Neumann theory of the SAE of symmetric operators to characterize the spectrum of the general SAE. They are also confirmed by an argument based on the asymptotic growth of the eigenvalues.

Let us mention that this model corresponds to a classically integrable system whose quantum spectrum—subject to Dirichlet boundary conditions at the singularity—has been determined in [19]. This operator also appears as an effective radial Hamiltonian for an isotropic harmonic oscillator in multi-dimensional Euclidean space with given angular momentum. For this model several results are known: the heat-kernel and the resolvent have been determined (also imposing Dirichlet boundary conditions) in [47–49]; the resolvent for a different boundary condition can be obtained from the Dirichlet case by means of the so-called Krein’s formula [50, 51], which will be examined in detail in section 6.

In section 6, we derive a generalization of Krein’s formula which can be applied to the kind of locally homogeneous operator with singular coefficients considered in this paper. Krein’s formula establishes a relation between the resolvents corresponding to different SAE. We generalize this relation to the case of a Schrödinger operator with a potential that contains...
terms with the same scaling dimension as the derivative term. Our generalized Krein’s formula will make manifest the presence of non-standard powers of $\lambda$ in the large-$|\lambda|$ expansion of these resolvents. The operators considered in the present paper can be understood as the radial problem resulting from a separation of variables for particular models, and some results described in the following sections rely on the knowledge of the behavior of the special functions appearing in their resolutions. It should be mentioned that a more general (and abstract) approach to these kind of problems has been developed in [56] (See also [57–62] and references therein). In [56] the asymptotic behavior of the trace of the resolvent of elliptic cone operators on compact manifolds has been considered under general conditions, showing that it admits an expansion which may present non-integer powers of $\lambda$ with coefficients which are in general rational functions of powers of $\lambda$ and $\log \lambda$, not mere polynomials. As pointed out in this reference, this result implies that the associated $\zeta$-functions might have poles at unusual locations, or that they might even not extend meromorphically to $\mathbb{C}$ at all, which is consistent with what is described in the following.

2. The first-order operator and its self-adjoint extensions

Following [36], in this section we consider the first-order symmetric differential operator

$$D = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A} & 0 \end{pmatrix},$$

where

$$\tilde{A} = -\partial_x + \frac{g}{x}, \quad \tilde{A} = \partial_x + \frac{g}{x}$$

(2.2)

and $|g| < 1/2$, defined on the domain $\mathcal{D}(D) := \mathbb{C}^2 \times \mathcal{C}^0_{\text{loc}}(0, 1)$, i.e. the set of smooth two component functions with compact support in the open segment $(0, 1)$. For the given range of the coupling, $g$, $D$ is not essentially self-adjoint, admitting a continuous family of SAE.

We will show that, for a general SAE of $D$, the large-$|\lambda|$ asymptotic expansion of the resolvent-trace $\text{Tr}((D - \lambda)^{-1})$ presents non-standard powers of $\lambda$ whose exponents depend on the parameter $g$. Besides, the corresponding $\zeta$- and $\eta$-functions present simple poles lying at $g$-dependent points in the complex plane, with residues which depend on the considered SAE.

We proceed as follows: firstly, we construct the resolvents for two particular SAE for which the behavior of the functions in the definition domain of the SAE near the singular point $x = 0$ is invariant under the scaling $x \rightarrow cx$. The asymptotic expansion of the resolvent for these special extensions displays the usual powers of $\lambda$, then leading to the usual poles for the corresponding $\zeta$- and $\eta$-functions. Secondly, we show that the resolvents for the remaining SAE are convex linear combinations of these special extensions with $\lambda$-dependent coefficients. This additional dependence on $\lambda$ leads to non-standard powers in the asymptotic expansion of the resolvent, and hence to non-standard poles for the $\zeta$- and $\eta$-functions.

These SAE are not locally invariant under the scaling $x \rightarrow cx$ in the sense that the conditions defining the behavior of the functions belonging to their domain near $x = 0$ are not scale invariant. Rather, as $c \rightarrow 0$, they tend to the conditions for the domain of one of the locally scale-invariant SAE, while as $c \rightarrow \infty$ they tend to the other. As a consequence, the residues at the anomalous poles of the $\zeta$-function tend to zero as $c \rightarrow 0$ and diverge as $c \rightarrow \infty$. This behavior will be explained by means of a scaling argument.

Integration by parts shows that the operator $D$ is symmetric on $\mathcal{D}(D)$. The adjoint operator $D^\dagger$, which is the maximal extension of $D$, is defined on the domain $\mathcal{D}(D^\dagger)$ of functions $\Phi(x) = (\phi_1(x), \phi_2(x)) \in \mathbb{C}^2 \times \mathcal{L}_2(0, 1)$ having a locally summable first derivative and such that

$$D\Phi(x) = \begin{pmatrix} \tilde{A}\phi_2(x) \\ \tilde{A}\phi_1(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in \mathbb{C}^2 \times \mathcal{L}_2(0, 1).$$

(2.3)
Since $D$ is symmetric, $D(D) \subset D(D^1)$. The following lemma characterizes the behavior at the singular point $x = 0$ of the two components of the functions in $D(D^1)$.

**Lemma 2.1.** If $\Phi(x) \in D(D^1)$ and $-\frac{1}{2} < g < \frac{1}{2}$, then

$$|\phi_1(x) - C_1[\Phi]x^g| + |\phi_2(x) - C_2[\Phi]x^{-g}| \leq K_g\|D\Phi(x)\| x^{1/2},$$

for some constants $K_g$, $C_1[\Phi]$ and $C_2[\Phi]$, where $\| \cdot \|$ is the $L_2$-norm.

**Proof.** Equation (2.3) implies

$$\phi_1(x) = C_1[\Phi]x^g - x^g \int_0^x y^{-g} f_2(y) dy,$$

$$\phi_2(x) = C_2[\Phi]x^{-g} + x^{-g} \int_0^x y^g f_1(y) dy,$$

and taking into account that

$$\left| \int_0^x y^g f_1(y) dy \right| \leq \frac{x^{g+1/2}}{\sqrt{1+2g}} \|f_1\|,$$

$$\left| \int_0^x y^{-g} f_2(y) dy \right| \leq \frac{x^{-g+1/2}}{\sqrt{1-2g}} \|f_2\|,$$

we immediately obtain equation (2.4) with $K_g = (1 - 2g)^{-1/2} + (1 + 2g)^{-1/2}$. □

The following lemma will be useful to describe the SAE of $D$.

**Lemma 2.2.** Let $\Phi(x) = (\Phi_1(x), \Phi_2(x))$, $\Psi(x) = (\Psi_1(x), \Psi_2(x)) \in D(D^1)$. Then

$$(D^1\Psi, \Phi) - (\Psi, D^1\Phi) = [C_1[\Psi]^*C_2[\Phi] - C_2[\Psi]^*C_1[\Phi]] + \{\psi_2(1)^*\phi_1(1) - \psi_1(1)^*\phi_2(1)\}.$$

(2.9)

**Proof.** From equations (2.2), one easily obtains

$$(D^1\Psi, \Phi) - (\Psi, D^1\Phi) = \lim_{\epsilon \to 0} \int_{\epsilon}^1 \partial_x [x^g \psi_2(x)]^* x^{-g} \phi_1(x) - x^{-g} \psi_1(x)^* x^g \phi_2(x)] dx,$$

from which, taking into account the results in lemma 2.1, equation (2.9) follows directly. □

Next, if $\Psi(x)$ in equation (2.9) belongs to the domain of the closure of $D$, $\overline{D} = (D^1)^\dagger$, i.e.

$$\Psi(x) \in D(\overline{D}) \subset D(D^1),$$

then the right-hand side of equation (2.9) must vanish for any $\Phi(x) \in D(D^1)$. Therefore,

$$C_1[\Psi] = C_2[\Psi] = \Psi(1) = 0.$$  

(2.12)

On the other hand, if $\Psi(x)$ and $\Phi(x)$ belong to the domain of a symmetric extension of $D$—which must be contained in $D(D^1)$—then the right-hand side of equation (2.9) must also vanish. Thus, each closed extension of $D$ corresponds to a subspace of $\mathbb{C}^4$ under the map $\Phi \to (C_1[\Phi], C_2[\Phi], \phi_1(1), \phi_2(1))$. If we define the orthogonal complement in terms of the symplectic form on the right-hand side of equation (2.9), then the SAE correspond to those subspaces $S \subset \mathbb{C}^4$ such that $S = S^\perp$.

Since we are interested in the consequences of the singularity at the origin, for simplicity in the following we will consider SAE satisfying the local boundary condition

$$\phi_1(1) = 0.$$  

(2.13)

Each extension is then determined by a condition of the form

$$\alpha C_1[\Phi] + \beta C_2[\Phi] = 0,$$

(2.14)

with $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$. We denote this SAE by $D^{(\alpha, \beta)}$.  

7
2.1. The spectrum

In order to determine the spectrum of an arbitrary SAE $D^{(\alpha, \beta)}$, we need the formal solutions of

$$(D - \lambda) \Phi(x) = 0 \Rightarrow \begin{cases} \hat{A} \phi_1(x) = \lambda \phi_1(x), \\ \hat{A} \phi_2(x) = \lambda \phi_2(x), \end{cases}$$

(2.15)

satisfying the boundary conditions given by equations (2.13) and (2.14).

The zero mode, corresponding to $\lambda = 0$, is given by

$$\Phi(x) = \begin{pmatrix} C_1 x^g \\ C_2 x^{-g} \end{pmatrix}.$$  

(2.16)

The boundary condition at $x = 1$ chosen by equation (2.13) implies $C_1 = 0$. The boundary condition at $x = 0$ given by equation (2.14) tells that $C_2 = 0$. Consequently, only the SAE $D^{(1,0)}$ has a zero mode.

For $\lambda \neq 0$, we apply $\hat{A}$ to the second line in equation (2.15) so that, using the first line in equation (2.15), one obtains

$$\left\{ - \frac{\partial^2}{\partial x^2} - \frac{g(g-1)}{x^2} + \lambda^2 \right\} \phi_1(x) = 0.$$  

(2.17)

The solutions of this equation take the form

$$\phi_1(x) = K_1 \sqrt{x} J_{\frac{1}{2} - g}(x) + K_2 \sqrt{x} J_{\frac{1}{2} + g}(x),$$

(2.18)

where we have made the rescaling $X := \lambda x$ and defined $\tilde{\lambda} = +\sqrt{\lambda^2}; K_1$ and $K_2$ are complex constants. The lower component results

$$\phi_2(x) = \sigma \left\{ - K_1 \sqrt{x} J_{-\frac{1}{2} - g}(x) + K_2 \sqrt{x} J_{\frac{1}{2} + g}(x) \right\},$$

(2.19)

where $\sigma = \tilde{\lambda}/\lambda$.

From the behavior of $\phi_1(x)$ and $\phi_2(x)$ at the origin, we can obtain $C_1[\Phi]$ and $C_2[\Phi]$, respectively. The boundary condition now reads

$$\alpha C_1[\Phi] + \beta C_2[\Phi] = - \frac{\alpha K_2 \tilde{\lambda}^{\frac{1}{2}}}{2^{g-\frac{1}{2}} \Gamma \left( \frac{1}{2} + g \right)} - \frac{\beta K_1 \tilde{\lambda}^{-g}}{2^{-g-\frac{1}{2}} \Gamma \left( \frac{1}{2} - g \right)} = 0.$$  

(2.20)

Let us first consider a particular SAE, namely $\alpha = 0$ and $\beta = 1$. For $\alpha = 0$, equation (2.20) implies $K_1 = 0$. Therefore, from the condition $\phi_1(1) = 0$ we obtain the spectrum equation $J_{-\frac{1}{2} - g}(\tilde{\lambda}) = 0$. Thus, the spectrum of $D^{(0,1)}$ is given by

$$\tilde{\lambda}_{\pm,n} = \pm j_0^{(n)} \frac{1}{\sqrt{\frac{1}{2} - g}}, \quad n = 1, 2, \ldots,$$

(2.21)

where $j_0^{(n)}$ is the $n$th positive zero of the Bessel function $J_0(z)$. The spectrum is non-degenerate and symmetric with respect to the origin.

For $\alpha \neq 0$, equation (2.20) can be written as

$$\frac{K_2}{K_1} = \sigma \tilde{\lambda}^{-2g} \left[ \frac{4\Gamma \left( \frac{1}{2} + g \right)}{\Gamma \left( \frac{1}{2} - g \right)} \right] \left( \frac{\beta}{\alpha} \right).$$  

(2.22)

In this case, the boundary condition at $x = 1$ determines the eigenvalues as the solutions of the transcendental equation

$$\lambda^{2g} J_{\frac{1}{2} - g}(\tilde{\lambda}) J_{\frac{1}{2} + g}(\tilde{\lambda}) = \sigma \rho(\alpha, \beta),$$

(2.23)
where we have defined
\[ \rho(\alpha, \beta) := -\frac{4\Gamma\left(\frac{1}{2} + g\right)}{\Gamma\left(\frac{1}{2} - g\right)} \left(\frac{\beta}{\alpha}\right). \]  
(2.24)

For the positive eigenvalues, \( \tilde{\lambda} = \lambda, \sigma = 1 \) and equation (2.23) reduces to
\[ F(\lambda) := \lambda^{2g} J_{\frac{1}{2}}^{1-g}(\lambda) \]  
(2.25)

This expression has been plotted in figure 1 for a particular value of \( \rho(\alpha, \beta) \) and \( g \).

On the other hand, for negative eigenvalues \( \lambda = e^{i\pi} \tilde{\lambda}, \sigma = e^{-i\pi} \) and equation (2.23) reads
\[ \tilde{F}(\tilde{\lambda}) = e^{-i\pi} \rho(\alpha, \beta) = \rho(\alpha, -\beta). \]  
(2.26)

Therefore, the negative eigenvalues of \( D^{(\alpha,\beta)} \) have the same absolute value as the positive eigenvalues of \( D^{(\alpha,-\beta)} \).

Note that for any SAE, the spectrum is non-degenerate and that there is a positive eigenvalue between each pair of consecutive zeros of \( J_{\frac{1}{2}}^{1-g}(\lambda) \). Moreover, the spectrum is symmetric with respect to the origin only for the SAE corresponding to \( \alpha = 0 \) (from now on, the ‘\( D \)-extension’, see equation (2.21)) and for the extension corresponding to \( \beta = 0 \) (which we will call the ‘\( N \)-extension’). Indeed, from equations (2.23) and (2.24), one can see that the eigenvalues of \( D^{(0,\beta)} \) are given by
\[ \lambda_0 = 0, \quad \lambda_{\pm,n} = \pm j_{\frac{1}{2}}^{(n)}, \quad n = 1, 2, \ldots. \]  
(2.27)

2.2. The resolvent

In this section, we will construct the resolvent of \( D \)
\[ G(\lambda) = (D - \lambda)^{-1}, \]  
(2.28)

for its different SAE.

We will first consider the two limiting cases in equation (2.14), namely the ‘\( D \)-extension’ and the ‘\( N \)-extension’. The resolvent for a general SAE will be later evaluated as a linear combination of those obtained for these two limiting cases.

The kernel of the resolvent
\[ G(x, y; \lambda) = \begin{pmatrix} G_{11}(x, y; \lambda) & G_{12}(x, y; \lambda) \\ G_{21}(x, y; \lambda) & G_{22}(x, y; \lambda) \end{pmatrix} \]  
(2.29)
must satisfy the boundary conditions

\[ \Phi \mid_{x=10} = 0, \]

from which we straightforwardly obtain for the diagonal elements

\[ \begin{align*}
\frac{d^2}{d x^2} - \frac{g(g - 1)}{x^2} + \lambda^2 \end{align*} \begin{align*}
G_{11}(x, y; \lambda) &= -\lambda \delta(x, y), \\
G_{22}(x, y; \lambda) &= -\lambda \delta(x, y),
\end{align*} \tag{2.31}

while for the non-diagonal ones, we have

\[ \begin{align*}
G_{21}(x, y; \lambda) &= \frac{1}{\lambda} \left\{ \partial_x + \frac{g}{x} \right\} G_{11}(x, y; \lambda), \\
G_{12}(x, y; \lambda) &= \frac{1}{\lambda} \left\{ \partial_x + \frac{g}{x} \right\} G_{22}(x, y; \lambda),
\end{align*} \tag{2.32}

assuming \( \lambda \neq 0 \).

Since the resolvent is analytic in \( \lambda \), it is sufficient to evaluate it on the open right half-plane. In order to do that, we use the upper and lower components of some particular solutions of the homogeneous equation (2.15).

Let us define

\[ \begin{align*}
L^D_0(X) &= \sqrt{X} J_{g - \frac{1}{g}}(X), \\
L^D_1(X) &= \sqrt{X} J_{g + \frac{1}{g}}(X), \\
L^N_1(X) &= \sqrt{X} J_{\frac{1}{g} - g(x)}, \\
L^N_2(X) &= \sqrt{X} J_{\frac{1}{g} + g(x)}, \\
R_1(X; \lambda) &= \sqrt{X} [J_{g - \frac{1}{g}}(\lambda) J_{\frac{1}{g} - g}(X) - J_{\frac{1}{g} - g}(\lambda) J_{g - \frac{1}{g}}(X)], \\
R_2(X; \lambda) &= \sqrt{X} [J_{\frac{1}{g} - g}(\lambda) J_{\frac{1}{g} + g}(X) + J_{\frac{1}{g} + g}(\lambda) J_{\frac{1}{g} - g}(X)].
\end{align*} \tag{2.33}

Note that \( R_1(\lambda; \lambda) = \tilde{A} R_2(\lambda; \lambda) \mid_{x=1} = 0 \). We will also need the Wronskians

\[ \begin{align*}
W \left[ L^D_0(X), R_1(X; \lambda) \right] &= -\frac{2}{\pi} \cos(\pi g \nu) J_{\frac{1}{g} - \frac{1}{g}}(\lambda) = \frac{1}{\gamma_D(\nu)}, \\
W \left[ L^D_1(X), R_2(X; \lambda) \right] &= \frac{2}{\pi} \cos(\pi g \nu) J_{\frac{1}{g} + \frac{1}{g}}(\lambda) = \frac{1}{\gamma_D(\nu)}, \\
W \left[ L^N_1(X), R_1(X; \lambda) \right] &= -\frac{2}{\pi} \cos(\pi g \nu) J_{\frac{1}{g} - \frac{1}{g}}(\lambda) = \frac{1}{\gamma_N(\nu)}, \\
W \left[ L^N_2(X), R_2(X; \lambda) \right] &= \frac{2}{\pi} \cos(\pi g \nu) J_{\frac{1}{g} + \frac{1}{g}}(\lambda) = \frac{1}{\gamma_N(\nu)},
\end{align*} \tag{2.34}

which vanish only at the zeros of \( J_{\nu}(\lambda) \), for \( \nu = \pm \left( \frac{1}{g} \right) \).

### 2.3. The resolvent for the D-extension

In this case, the function

\[ \Phi(x) = \int_0^1 G_D(x, y; \lambda) \begin{bmatrix} f_1(y) \\ f_2(y) \end{bmatrix} \, dy \tag{2.35} \]

must satisfy the boundary conditions \( \phi_1(1) = 0 \) and \( C_2[\Phi] = 0 \), for any functions \( f_1(x), f_2(x) \in L_2(0, 1) \).
This requires that

\[ G_{11}^D(x, y; \lambda) = \gamma_D(\lambda) \times \begin{cases} L_1^D(X) R_1(Y; \lambda), & \text{for } x \leq y, \\ R_1(X; \lambda) L_1^D(Y), & \text{for } x \geq y, \end{cases} \]

(2.36)

and

\[ G_{22}^D(x, y; \lambda) = -\gamma_D(\lambda) \times \begin{cases} L_2^D(X) R_2(Y; \lambda), & \text{for } x \leq y, \\ R_2(X; \lambda) L_2^D(Y), & \text{for } x \geq y. \end{cases} \]

(2.37)

whereas the components \( G_{12}^D(x, y; \lambda) \) and \( G_{21}^D(x, y; \lambda) \) are given by equation (2.32).

The fact that the boundary conditions are satisfied, as well as \((D - \lambda)\Phi(x) = (f_1(x), f_2(x))\), can be straightforwardly verified from equations (2.33) and (2.34). Indeed, from equations (2.32)–(2.37), one obtains

\[ \phi_1(x) = C_1^D[\Phi] x^\varnothing + O(\sqrt{x}), \quad \phi_2(x) = O(\sqrt{x}), \]

(2.38)

with

\[ C_1^D[\Phi] = \frac{-\pi \lambda^{1+\varnothing}}{2^{1+\varnothing} \cos(\varnothing \pi) \Gamma(\frac{1}{2} + \varnothing)} \int_0^1 [R_1(\lambda y; \lambda) f_1(y) - R_2(\lambda y; \lambda) f_2(y)] \, dy, \]

(2.39)

for \( \lambda \) not a zero of \( J_{1-\varnothing}(\lambda) \).

Note that \( C_1^D[\Phi] \neq 0 \) if the integral on the right-hand side of equation (2.39) is non-vanishing.

### 2.4. The resolvent for the N-extension

In this case, the function

\[ \Phi(x) = \int_0^x G_N(x, y; \lambda) \left( \frac{f_1(y)}{f_2(y)} \right) \, dy \]

(2.40)

satisfies the boundary conditions \( \phi_1(1) = 0 \) and \( C_1[\Phi] = 0 \), for any functions \( f_1(x), f_2(x) \in L_2(0, 1) \).

This requires that

\[ G_{11}^N(x, y; \lambda) = \gamma_N(\lambda) \times \begin{cases} L_1^N(X) R_1(Y; \lambda), & \text{for } x \leq y, \\ R_1(X; \lambda) L_1^N(Y), & \text{for } x \geq y, \end{cases} \]

(2.41)

and

\[ G_{22}^N(x, y; \lambda) = \gamma_N(\lambda) \times \begin{cases} L_2^N(X) R_2(Y; \lambda), & \text{for } x \leq y, \\ R_2(X; \lambda) L_2^N(Y), & \text{for } x \geq y, \end{cases} \]

(2.42)

whereas the components \( G_{12}^N(x, y; \lambda) \) and \( G_{21}^N(x, y; \lambda) \) are given by equation (2.32).

These boundary conditions, as well as the fact that \((D - \lambda)\Phi(x) = (f_1(x), f_2(x))\), can be straightforwardly verified from equations (2.33) and (2.34). In this case, from equations (2.32)–(2.34) and (2.40)–(2.42) one obtains

\[ \phi_1(x) = O(\sqrt{x}), \quad \phi_2(x) = C_2^N[\Phi] x^\varnothing + O(\sqrt{x}), \]

(2.43)

with

\[ C_2^N[\Phi] = \frac{\pi \lambda^{1-\varnothing}}{2^{1-\varnothing} \cos(\varnothing \pi) \Gamma(\frac{1}{2} - \varnothing)} \int_0^1 [R_1(\lambda y; \lambda) f_1(y) - R_2(\lambda y; \lambda) f_2(y)] \, dy, \]

(2.44)

for \( \lambda \) not a zero of \( J_{1-\varnothing}(\lambda) \).

Note that \( C_2^N[\Phi] \neq 0 \) if the integral on the right-hand side of equation (2.44) (the same integral as the one appearing in the D-extension, equation (2.39)) is non-vanishing.
2.5. The resolvent for a general SAE of $D^{(\alpha, \beta)}$

For the general case, we can implement the boundary conditions

$$\phi_1(1) = 0, \quad \alpha C_1[\Phi] + \beta C_2[\Phi] = 0, \quad \alpha, \beta \neq 0$$

(2.45)

on

$$\Phi(x) = \int_0^1 G(x, y; \lambda) \left( f_1(y) f_2(y) \right) dy$$

(2.46)

for any $f_1(x), f_2(x) \in L_2(0, 1)$ by taking a linear combination of the $D$- and $N$-resolvents,

$$G(x, y; \lambda) = [1 - \tau(\lambda)] G_D(x, y; \lambda) + \tau(\lambda) G_N(x, y; \lambda).$$

(2.47)

Since the boundary condition at $x = 1$ is already satisfied, we must now impose

$$\alpha [1 - \tau(\lambda)] C_1^N[\Phi] + \beta \tau(\lambda) C_2^N[\Phi] = 0.$$  

(2.48)

Note that, by virtue of equations (2.39), (2.44) and (2.25),

$$\alpha C_1^N[\Phi] - \beta C_2^N[\Phi] = 0$$

(2.49)

whenever $\lambda$ is an eigenvalue of $D^{(\alpha, \beta)}$. Therefore, from equation (2.48) we obtain the resolvent of $D^{(\alpha, \beta)}$ by setting

$$\tau(\lambda) \equiv \alpha C_1^D[\Phi]/(\alpha C_1^N[\Phi] - \beta C_2^N[\Phi]) \equiv \frac{1}{1 - \rho(\alpha, \beta)/F(\lambda)}$$

$$= 1 - \frac{1}{\lambda^2 g} \frac{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)}{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)},$$

(2.50)

for $\lambda$ not a zero of $J_{\frac{g}{2} - \frac{g}{2}}(\lambda)$.

2.6. The trace of the resolvent

It follows from equation (2.47) that the resolvent of a general SAE of $D$ can be expressed in terms of the resolvents of the two limiting cases, $G_D(\lambda)$ and $G_N(\lambda)$. Moreover, since the eigenvalues of any extension grow linearly with $n$ (see section 2.1), these resolvents are Hilbert–Schmidt operators and their $\lambda$-derivatives are trace class.

From the relation

$$G(\lambda)^2 = \partial_\lambda G_D(\lambda) - \partial_\lambda G_D(\lambda) - \tau'(\lambda)[G_D(\lambda) - G_N(\lambda)] - \tau(\lambda)[\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)],$$

(2.51)

it follows that the difference $G_D(\lambda) - G_N(\lambda)$ is a strongly analytic function of $\lambda$ (except at the zeros of $\tau'(\lambda)$) taking values in the trace class operators ideal.

From the explicit expressions of $G_D(\lambda)$ and $G_N(\lambda)$ (see equations (2.36), (2.37), (2.41) and (2.42)), we straightforwardly obtain

$$\text{Tr}[\partial_\lambda G_D(\lambda)] = \int_0^1 \text{Tr}[\partial_\lambda G_D(x, x; \lambda)] dx$$

$$= \partial_\lambda \left\{ \frac{J_{\frac{g}{2} + \frac{g}{2}}(\lambda)}{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)} \right\} = 1 - \frac{g^2}{\lambda^2} + \left( \frac{1}{2 \lambda} + \frac{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)}{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)} \right)^2.$$  

(2.52)

Similarly,

$$\text{Tr}[G_D(\lambda) - G_N(\lambda)] = \frac{2g}{\lambda} \frac{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)}{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)} + \frac{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)}{J_{\frac{g}{2} - \frac{g}{2}}(\lambda)}.$$  

(2.53)
Moreover, since
\[ \partial_\lambda \text{Tr}[G_D(\lambda) - G_N(\lambda)] = \text{Tr}[\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)], \]
we obtain
\[ \text{Tr}[\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)] = \frac{2g}{\lambda^2} + \left( \frac{1}{2\lambda} + \frac{J'_{\frac{1}{2}-g}(\lambda)}{J_{\frac{1}{2}-g}(\lambda)} \right)^2 \frac{1}{2\lambda} + \left( \frac{1}{2\lambda} + \frac{J'_{\frac{1}{2}+g}(\lambda)}{J_{\frac{1}{2}+g}(\lambda)} \right)^2. \]  

Finally, we can also write
\[ \text{Tr}[G(\lambda)^2] = \text{Tr}[\partial_\lambda G_D(\lambda)] - \partial_\lambda \tau(\lambda) \text{Tr}[G_D(\lambda) - G_N(\lambda)]. \]

2.7. Asymptotic expansion for the trace of the resolvent

Using the Hankel asymptotic expansion for the Bessel functions [63] (see appendix B), we get for the first term on the right-hand side of equation (2.56),
\[ \text{Tr}[\partial_\lambda G_D(\lambda)] \sim \sum_{k=2}^{\infty} \frac{A_k(g, \sigma)}{\lambda^k} = \frac{g}{\lambda} + i\sigma \frac{g (g - 1)}{\lambda^2} - \frac{3g (g - 1)}{2\lambda^3} + \frac{g (g - 3)(g - 1) (g + 2)}{2\lambda^5} + O\left(\frac{1}{\lambda}\right)^{6}, \]
where \(\sigma = 1\) for \(\Im(\lambda) > 0\), and \(\sigma = -1\) for \(\Im(\lambda) < 0\). The coefficients in this series can be straightforwardly evaluated from equations (2.52) and (B.14). Note that \(A_k(g, -1) = A_k(g, 1)^\ast\), since \(A_{2k}(g, 1)\) is real and \(A_{2k+1}(g, 1)\) is purely imaginary.

Similarly, from equations (2.53), (2.55) and (B.15), we obtain
\[ \text{Tr}[G_D(\lambda) - G_N(\lambda)] \sim \frac{2g}{\lambda^2}, \]
\[ \text{Tr}[\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)] \sim \frac{2g}{\lambda^2}. \]

On the other hand, taking into account equation (B.7), we have
\[ \tau(\lambda) \sim \begin{cases} -\sum_{k=1}^{\infty} \frac{e^{\sigma i\pi (\frac{1}{2}-g)\lambda^2\rho}}{\rho(\alpha, \beta)} \lambda^k, & \text{for } -\frac{1}{2} < g < 0, \\ \sum_{k=0}^{\infty} (\rho(\alpha, \beta) e^{-\sigma i\pi (\frac{1}{2}-g)\lambda^2\rho})^k, & \text{for } 0 < g < \frac{1}{2}, \end{cases} \]
where \(\sigma = 1\) \((\sigma = -1)\) corresponds to \(\Im(\lambda) > 0\) \((\Im(\lambda) < 0)\). Note the appearance of non-integer, \(g\)-dependent powers of \(\lambda\) in this asymptotic expansion.

Similarly
\[ \tau'(\lambda) \sim \begin{cases} -\frac{2g}{\lambda} \sum_{k=1}^{\infty} k \left( \frac{e^{\sigma i\pi (\frac{1}{2}-g)\lambda^2\rho}}{\rho(\alpha, \beta)} \right)^k, & \text{for } -\frac{1}{2} < g < 0, \\ -\frac{2g}{\lambda} \sum_{k=1}^{\infty} k (\rho(\alpha, \beta) e^{-\sigma i\pi (\frac{1}{2}-g)\lambda^2\rho})^k, & \text{for } 0 < g < \frac{1}{2}, \end{cases} \]
which are the term by term derivatives of the corresponding asymptotic series in equation (2.60).
Collecting these results, we have
\[
\sigma[\tau(\lambda) \text{ Tr}(G_D(\lambda) - G_N(\lambda))] \sim \begin{cases}
2 g \sum_{k=1}^{\infty} \left( \frac{e^{i \alpha \pi (\xi^2 + \eta^2)}}{\rho(\alpha, \beta)} \right)^k (2 g k - 1) \lambda^{2 k - 2}, & \text{for } -\frac{1}{2} < g < 0, \\
2 g \sum_{k=0}^{\infty} \left( \rho(\alpha, \beta) e^{-i \pi (\xi^2 + \eta^2)} \right)^k (2 g k + 1) \lambda^{-2 k - 2}, & \text{for } 0 < g < \frac{1}{2}.
\end{cases}
\tag{2.62}
\]
Note the \( g \)-dependent powers of \( \lambda \) appearing in these asymptotic expansions.

### 2.8. The \( \zeta \)- and \( \eta \)-functions

The \( \zeta \)-function for a general self-adjoint extension of \( D \) can be defined, for \( \Re(s) > 1 \), as
\[
\zeta(s) = -\frac{1}{2 \pi i} \oint_{C} \frac{\lambda^{1-s}}{s-1} \text{Tr}(G(\lambda)^2) \, d\lambda,
\tag{2.63}
\]
where the curve \( C \) encircles counterclockwise the spectrum of \( D \). According to equation (2.56), we have
\[
\zeta(s) = \zeta^D(s) + \frac{1}{2 \pi i} \oint_{C} \frac{\lambda^{1-s}}{s-1} \partial_{\lambda} \sigma[\tau(\lambda) \text{ Tr}(G_D(\lambda) - G_N(\lambda))] \, d\lambda,
\tag{2.64}
\]
where \( \zeta^D(s) \) is the \( \zeta \)-function for the \( D \)-extension.

Since the negative eigenvalues of the self-adjoint extension \( D_{+}^{(\alpha, \beta)} \) are minus the positive eigenvalues corresponding to extension \( D_{-}^{(\alpha, \beta)} \) (as discussed in section 2.1), we define a partial \( \zeta \)-function by means of a path of integration that encircles the positive eigenvalues only,
\[
\zeta_{+}^{(\alpha, \beta)}(s) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\lambda^{1-s}}{s-1} \text{Tr}(G(\lambda)^2) \, d\lambda = \zeta_{+}^D(s) - \frac{1}{2 \pi i} \oint_{C} \frac{\lambda^{1-s}}{s-1} \partial_{\lambda} \sigma[\tau(\lambda) \text{ Tr}(G_D(\lambda) - G_N(\lambda))] \, d\lambda,
\tag{2.65}
\]
where \( \zeta_{+}^D(s) \) is the partial \( \zeta \)-function for the \( D \)-extension. Note that we can write
\[
\zeta_{+}^{(\alpha, \beta)}(s) = \frac{1}{2 \pi i} \int_{1}^{\infty} e^{i \frac{\pi}{2} (1-s) \mu^{1-s}} \frac{1}{s-1} \text{Tr}\left[ \left( g e^{i \frac{\pi}{2} \mu} \right)^2 \right] \, d\mu + \frac{1}{2 \pi} \int_{1}^{\infty} e^{-i \frac{\pi}{2} (1-s) \mu^{1-s}} \frac{1}{s-1} \text{Tr}\left[ \left( g e^{-i \frac{\pi}{2} \mu} \right)^2 \right] \, d\mu + h(s) \frac{1}{s-1},
\tag{2.66}
\]
where \( h(s) \) is an entire function. In order to determine the poles of \( \zeta_{+}^{(\alpha, \beta)}(s) \), we add and subtract a partial sum of the asymptotic expansion of \( \text{Tr}[G(\lambda)^2] \) (obtained in section 2.7) to the integrands on the right-hand side of equation (2.66).

In this way, we get for the \( D \)-extension and for real \( s > 1 \),
\[
\zeta_{+}^D(s) = \frac{1}{2 \pi} \left( s - 1 \right) \int_{1}^{\infty} e^{i \frac{\pi}{2} (1-s) \mu^{1-s}} \mu^{1-s} \left\{ \sum_{k=2}^{N} e^{-i \frac{\pi}{2} k} A_k(g, 1) \mu^{-k} \right\} \, d\mu + \frac{1}{2 \pi} \left( s - 1 \right) \int_{1}^{\infty} e^{-i \frac{\pi}{2} (1-s) \mu^{1-s}} \mu^{1-s} \left\{ \sum_{k=2}^{N} e^{i \frac{\pi}{2} k} A_k(g, 1)^* \mu^{-k} \right\} \, d\mu + \frac{h_N(s)}{s-1}
\]
\[
= \frac{1}{\pi} \left( s - 1 \right) \sum_{k=0}^{N} \frac{1}{s + k} \Re\left\{ e^{-i \frac{\pi}{2} (s+k+1)} A_{k+2}(g, 1) \right\} + \frac{h_N(s)}{s-1},
\tag{2.67}
\]
where
\[
\Re\left\{ e^{-i \frac{\pi}{2} (s+k+1)} A_{k+2}(g, 1) \right\} = \begin{cases}
\frac{1}{2}, & \Re\left\{ e^{-i \frac{\pi}{2} (s+k+1)} A_{k+2}(g, 1) \right\} > 0, \\
-\frac{1}{2}, & \Re\left\{ e^{-i \frac{\pi}{2} (s+k+1)} A_{k+2}(g, 1) \right\} < 0.
\end{cases}
\]
where \( H_N(s) \) is an analytic function in the open half-plane \( \Re(s) > 1 - N \). In consequence, the meromorphic extension of \( \zeta_+^D(s) \) presents a simple pole at \( s = 1 \) (see equation (2.67)) with a residue given by (see equation (2.52))

\[
\text{Res} \left. \zeta_+^D(s) \right|_{s=1} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \lambda^0 \partial_\lambda \left\{ \frac{J_{-\lambda}^+, (\lambda)}{J_{-\lambda}^+, (\lambda)} \right\} \, d\lambda = \frac{1}{\pi}. \tag{2.68}
\]

It also presents simple poles at \( s = -k \), for \( k = 0, 1, 2, \ldots \), with residues given by

\[
\text{Res} \left. \zeta_+^D(s) \right|_{s=-k} = \frac{\text{Res} \left( A_k + (g, 1) \right)}{\pi (k + 1)}, \tag{2.69}
\]

with the coefficients \( A_k (g, 1) \) given by equation (2.57). In particular, note that these residues vanish for even \( k \).

For a general self-adjoint extension \( D^{(\alpha, \beta)} \), we must also consider the singularities coming from the asymptotic expansion of \( \partial_\lambda \left[ \tau (\lambda) \right] \text{Tr} \{ G_D (\lambda) - G_N (\lambda) \} \} \) (see equation (2.62)). For definiteness, let us consider in the following the case \(- \frac{1}{2} < g < 0\) (the case \( 0 < g < \frac{1}{2} \) leads to similar results).

From equation (2.65), and taking into account equation (2.66), for real \( s > 1 \), we write

\[
\zeta_+^{(\alpha, \beta)}(s) - \zeta_+^D(s) = \frac{H_N(s)}{s-1}
\]

\[
- \frac{g}{\pi (s-1)} \int_1^{\infty} e^{\frac{1}{2} (s-1) - \frac{k}{2}} \mu^{1-s} \left\{ \sum_{k=1}^{N} \left( \frac{e^{\frac{1}{2} (\alpha, \beta)}}{\mu (\alpha, \beta)} \right)^k (2gk-1) \mu^{2gk-2} \right\} \, d\mu.
\]

\[
- \frac{g}{\pi (s-1)} \int_1^{\infty} e^{i \frac{1}{2} (s-1) - \frac{k}{2}} \mu^{1-s} \left\{ \sum_{k=1}^{N} \left( \frac{e^{-i \frac{1}{2} (\alpha, \beta)}}{\mu (\alpha, \beta)} \right)^k (2gk-1) \mu^{2gk-2} \right\} \, d\mu.
\]

\[
- \frac{2g}{\pi (s-1)} \sum_{k=1}^{N} \left( 2gk-1 \right) \left( \frac{2gk-1}{2gk} \right) \left[ \frac{e^{\frac{1}{2} (\alpha, \beta)}}{\mu (\alpha, \beta)} \right]^k + \frac{H_N(s)}{s-1}.
\]  \tag{2.70}

where \( H_N(s) \) is holomorphic for \( \Re(s) > 2g (N + 1) \). We conclude that \( \zeta_+^{(\alpha, \beta)}(s) - \zeta_+^D(s) \) has a meromorphic extension which presents a simple pole at \( s = 1 \), with a vanishing residue,

\[
\text{Res} \left. \left( \zeta_+^{(\alpha, \beta)}(s) - \zeta_+^D(s) \right) \right|_{s=1} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \lambda^0 \partial_\lambda \left[ \tau (\lambda) \text{Tr} \{ G_D (\lambda) - G_N (\lambda) \} \} \, d\lambda = 0,
\]  \tag{2.71}

as follows from equations (2.58) and (2.60).

Note also the presence of simple poles located at negative non-integer \( g \)-dependent positions \( s = 2gk \) for \( k = 1, 2, \ldots \), with residues which also depend on the SAE, given by

\[
\text{Res} \left. \left( \zeta_+^{(\alpha, \beta)}(s) - \zeta_+^D(s) \right) \right|_{s=2gk} = \frac{2g}{\pi \rho (\alpha, \beta)} \sin \left( \frac{1}{2} - g \right) k \pi.
\]  \tag{2.72}

Following the discussion after equation (2.26), we obtain for the complete \( \zeta \)-function

\[
\zeta_+(\alpha, \beta)(s) = \zeta_+^{(\alpha, \beta)}(s) + e^{-i\pi s} \zeta_+^{(\alpha, -\beta)}(s).
\]  \tag{2.73}

In particular, for the \( D \)-extension—whose spectrum is symmetric with respect to the origin (see equation (2.21))—we obtain

\[
\zeta_+^D(s) = (1 + e^{-i\pi s}) \zeta_+^D(s).
\]  \tag{2.74}

Note that \( \zeta_+^D(s) \) is an entire function. Indeed, from equation (2.69), the residue at \( s = -k \) vanishes for \( k \) even, whereas for \( k = 2l + 1 \), with \( l = 0, 1, 2, \ldots \), we obtain

\[
\text{Res} \left. \left( \zeta_+^D(s) \right) \right|_{s=-2l-1} = (1 + e^{i\pi (2l+1)}) \text{Res} \left. \left( \zeta_+^D(s) \right) \right|_{s=-2l} = 0.
\]  \tag{2.75}
On the other hand, for a general SAE, the singularities of $\zeta^{(\alpha,\beta)}(s)$ are simple poles located at $s = g k - 1/2$, for $k = 1, 2, \ldots$, with residues

$$\text{Res} \left\{ \zeta^{(\alpha,\beta)}(s) - \zeta_+^D(s) \right\}_{s g k} = \left. \pi \rho(\alpha,\beta) \right|_{s g k} = \left. e^{-\frac{1}{2} g k \pi} \zeta^{(\alpha,\beta)}(s) \right|_{s g k},$$  \hspace{1cm} (2.76)

where we have used $\rho(\alpha, -\beta) = -\rho(\alpha, \beta)$ (see equation (2.24)).

Similarly, for the spectral asymmetry [64], we have

$$\eta^{(\alpha,\beta)}(s) = \zeta^{(\alpha,\beta)}(s) - \zeta^{(\alpha,\beta)}(s).$$  \hspace{1cm} (2.77)

In particular, $\eta^{(0,1)}(s) = \eta^{(1,0)}(s) \equiv 0$, since the corresponding spectra are symmetric (see equations (2.21) and (2.27)). For a general self-adjoint extension and $-1/2 < g < 0$, the function $\eta^{(\alpha,\beta)}(s)$ presents simple poles at $s = 2 k g$, for odd $k = 1, 3, 5, \ldots$, with residues given by

$$\text{Res} \left\{ \eta^{(\alpha,\beta)}(s) \right\}_{s g k} = \left. -\frac{4 g \sin \left( \frac{1}{2} g k \pi \right)}{\rho(\alpha,\beta) k} \right|_{s g k}.$$  \hspace{1cm} (2.78)

For the case $0 < g < 1/2$, an entirely similar calculation shows that $\zeta^{(\alpha,\beta)}(s) - \zeta_+^D(s)$ has a meromorphic extension which presents simple poles at negative non-integer $g$-dependent positions, $s = -2 g k$, for $k = 1, 2, \ldots$, with residues depending on the SAE, given by

$$\text{Res} \left\{ \zeta^{(\alpha,\beta)}(s) - \zeta_+^D(s) \right\}_{s g k} = \left. -\frac{2 g}{\rho(\alpha,\beta)} \sin \left( \frac{1}{2} g k \pi \right) \right|_{s g k}.$$  \hspace{1cm} (2.79)

From this result, it is immediate to obtain the residues for the $\zeta$- and $\eta$-functions. One obtains the same expressions as on the right-hand sides of equations (2.76) and (2.78), with $\rho(\alpha,\beta)$ and $e^{\frac{1}{2} g k \pi}$ replaced by their inverses.

2.9. Scale invariance

Let us remark that when neither $\alpha$ nor $\beta$ is 0, the residue of $\zeta^{(\alpha,\beta)}(s)$ at $s = -2 g k$ is a constant times $(\beta/\alpha)^{\text{sign}(\alpha)}$. This is consistent with the behavior of $D$ under the scaling isometry $T_L u(x) := L^{-1/2} u(s/L)$ mapping $L_2(0, 1)$ onto $L_2(0, L)$. The extension $D^{(\alpha,\beta)}$ is unitarily equivalent to the operator $LD^{(\alpha,\beta)}$ defined on $L_2(0, L)$, with $\alpha := L/\alpha$ and $\beta := L^{-1/2}\beta$:

$$T_L D^{(\alpha,\beta)} := LD^{(\alpha,\beta)} T_L.$$

Note that only for the extensions with $\alpha = 0$ or $\beta = 0$ is the boundary condition at the singular point $x = 0$—given by equation (2.14)—scale invariant.

The partial $\zeta$-function of the scaled operator can be written as

$$\zeta^{(\alpha,\beta)}(s) = L^s \zeta^{(\alpha,\beta)}(s),$$

whereas its residues satisfy

$$\text{Res} \left\{ \zeta^{(\alpha,\beta)}(s) \right\}_{s g k} = L^{-2 g k} \text{Res} \left\{ \zeta^{(\alpha,\beta)}(s) \right\}_{s g k}.$$  \hspace{1cm} (2.82)

The factor $L^{-2 g k}$ exactly cancels the effect the change under the boundary condition at the singularity has on $\rho(\alpha,\beta)$,

$$\rho(\alpha,\beta)^{g \text{sign}(g)} = L^{2 g k} \rho(\alpha,\beta)^{g \text{sign}(g)}.$$  \hspace{1cm} (2.83)

Thus the length of the intervals $(0, 1)$ and $(0, L)$ has no effect on the structure of these residues, which presumably are determined locally in a neighborhood of $x = 0$.

Finally, let us point out that these anomalous poles are not present in the $g = 0$ case. Indeed, in this case $\tau(\lambda)$ in equation (2.50) has a constant asymptotic expansion, whereas $\text{Tr}[G_D(\lambda) - G_N(\lambda)] \sim 0$ (see equation (2.58)). Moreover, the residues of the poles coming
from $\zeta_D^0(s)$ vanish (see equations (2.69) and (2.57)), except for the one at $s = 1$, whose residue is $1/\pi$ (see equation (2.68)).

Consequently, the presence of poles in the spectral functions located at non-integer positions is a consequence of the singular behavior of the zeroth order term in $D$ near the origin, together with a boundary condition which breaks the scale invariance.

### 3. The second-order case

In this section, we will show that one obtains similar results for the SAE of the second-order differential operator

$$H = -\partial_x^2 + \frac{g(g - 1)}{x^2},$$

(3.1)

defined on a set of functions $\phi(x) \in \mathcal{C}_0^\infty(0, 1)$. Since we are interested in the effects of the singularity at $x = 0$, definiteness we impose again $\phi(1) = 0$. Proceeding as in the previous section [37], for $|g| < \frac{1}{2}$ (to be congruent with the first-order case analyzed in section 2), one can show that $H$ admits a continuous family of self-adjoint extensions in $L_2(0, 1)$, $H^{(\alpha, \beta)}$, characterized by two real parameters $\alpha$ and $\beta$ satisfying $\alpha^2 + \beta^2 = 1$ and defined on functions which behave near the origin as

$$\phi(x) = C_1 x^g + C_2 x^{1-g} + O(x^{3/2}),$$

(3.2)

where the coefficients $C_1$ and $C_2$ satisfy

$$\alpha C_1 + \beta C_2 = 0.$$  

(3.3)

The spectrum of $H^{(\alpha, \beta)}$ is determined by the relation (analogous to equation (2.25))

$$F(\mu) := \frac{1}{\mu} F(\mu) = \varrho(\alpha, \beta),$$

(3.4)

where now we define

$$\varrho(\alpha, \beta) := \left(\frac{\beta}{\alpha}\right) 2^{2-\frac{1}{2}} \Gamma\left(\frac{1}{2} + g\right) \Gamma\left(\frac{1}{2} - g\right).$$

(3.5)

Also in this case, $\alpha = 0$ and $\beta = 0$ correspond to two scale-invariant boundary conditions at the singularity. For these two limiting extensions, it is easily seen from equations (2.31), (2.38) and (2.43) that the resolvent of $H^{(\alpha, \beta)}$ satisfies

$$\mathcal{G}^{D,N}(x, y; \mu^2) = \frac{1}{\mu} \mathcal{G}^{D,N}_{11}(x, y; \mu).$$

(3.6)

The resolvent for a general SAE $H^{(\alpha, \beta)}$ is constructed as a convex linear combination of $\mathcal{G}^D(\mu^2)$ and $\mathcal{G}^N(\mu^2)$ as in (2.47), with a coefficient

$$\tau(\mu) = \frac{1}{1 - \frac{\varrho(\alpha, \beta)}{F(\mu)}}.$$  

(3.7)

Following the methods employed for the first-order case [37], one can show that the $\zeta$-function associated with $H^{(\alpha, \beta)}$ has a meromorphic extension which presents simple poles located at negative $g$-dependent positions,

$$s_k = -\left(\frac{1}{2} - g\right)k, \quad \text{for } k = 1, 2, \ldots,$$

(3.8)
with residues which depend on the SAE given by

$$\text{Res} \left\{ \zeta^{(\alpha, \beta)}(s) - \zeta^{(\alpha)}(s) \right\}_{s=\infty} = -\left( \frac{2g-1}{2\pi} \right) \varrho(\alpha, \beta) k \sin \left[ \frac{\pi}{2} (2g-1) k \right].$$  \hspace{1cm} (3.9)

Note that these poles are irrational for irrational values of \(g\). Moreover, these residues vanish for the ‘N-extension’ \(\varrho(\alpha, 0) = 0\), and have a singular limit for \(\alpha \to 0\). As in the first-order case, this behavior can also be explained through scaling arguments \[37\].

Finally, let us remark that the relation between the \(\zeta\)-function and the trace of the heat-kernel of \(D^{(\alpha, \beta)}\) (see appendix A) straightforwardly leads to the following small-\(t\) asymptotic expansion:

$$\text{Tr} \left\{ e^{-tD^{(\alpha, \beta)}} - e^{-tD^{(\alpha)}} \right\} \sim \left( g - \frac{1}{2} \right) - \sum_{k=1}^{\infty} \left\{ \Gamma \left( \left[ \frac{1}{2} - g \right] k \right) \frac{2g-1}{2\pi} \varrho(\alpha, \beta) k \sin \left[ \frac{\pi}{2} (2g-1) k \right] \right\} t^{\left( \frac{1}{2} - g \right) k}. \hspace{1cm} (3.10)$$

The first term on the right-hand side coincides with the result reported in \[34\]. Note also the \(g\)-dependent powers of \(t\) appearing in the asymptotic series on the right-hand side of equation (3.10) for the general SAE and the dependence of the SDW coefficients on the \((\alpha, \beta)\)-parameters. In particular, the first term in this series reduces to

$$-\frac{\beta}{\alpha} \frac{2^{2g-1}}{\Gamma \left( \frac{1}{2} - g \right)} t^{\frac{1}{2} - g}.$$  \hspace{1cm} (3.11)

This power of \(t\) also coincides with the result quoted in \[34\], but we find a different coefficient.

4. The \(g = \frac{1}{2}\) case

The second-order differential operator given by equation (3.1) with \(g = \frac{1}{2}\),

$$H_0 = -\frac{d^2}{dx^2} - \frac{1}{4x^2},$$  \hspace{1cm} (4.1)

defined on the domain \(\phi(x) \in C_0^\infty(0, 1)\), also admits a continuous family of nontrivial SAE in \(L_2(0, 1)\), which reflects in even more unusual properties of its spectral functions. This problem has been considered by Kirsten et al in \[40–43\], articles where an error in appendix A of \[37\] has been corrected.

Also here, for definiteness, we impose \(\phi(1) = 0\). As before, the SAE \(H_0^{(\alpha, \beta)}\) are characterized by two real parameters satisfying \(\alpha^2 + \beta^2 = 1\) and defined on a domain of functions which behave near the origin as

$$\phi(x) = C_1 \sqrt{x} + C_2 \sqrt{x} \log x + O(x^{3/2}),$$  \hspace{1cm} (4.2)

where the coefficients \(C_1\) and \(C_2\) satisfy equation (3.3).

The eigenfunction of \(H_0^{(\alpha)} := H_0^{(0,1)}\) corresponding to an eigenvalue \(\lambda = \mu^2\) is given by

$$\phi(x) = \sqrt{x} J_0(\mu x).$$  \hspace{1cm} (4.3)

The condition \(\phi(1) = 0\) tells that \(\mu\) is a (positive) zero of the Bessel function \(J_0(z)\). On the other hand, for an arbitrary SAE \(H_0^{(\alpha, \beta)}\) with \(\alpha \neq 0\), the eigenfunction corresponding to an eigenvalue \(\lambda = \mu^2\) is given by

$$\phi(x) = \left[ C_1 - C_2 (\log \mu/2 + \gamma) \right] \sqrt{x} J_0(\mu x) + \frac{\pi}{2} C_2 \sqrt{x} N_0(\mu x),$$  \hspace{1cm} (4.4)
where $C_1$ and $C_2$ are constrained by equation (3.3). The condition $\phi(1) = 0$ leads to the equation

$$2(\theta - \log \mu) J_0(\mu) + \pi N_0(\mu) = 0,$$

(4.5)

where $\theta := -\beta/\alpha + \log 2 - \gamma$, which determines the spectrum of $H_0^{(\alpha, \beta)}$. Note that there are no negative eigenvalues.

The trace of the resolvent $G^D_0(\mu^2) := (H_0^D - \mu^2)^{-1}$ can be readily computed to obtain

$$\text{Tr}\{G^D_0(\mu^2)\} = \frac{1}{2\mu} J_1(\mu)$$

(4.6)

This trace admits the following asymptotic expansion in integer powers of $\mu$

$$\text{Tr}\{G^D_0(\mu^2)\} \sim \frac{e^{i\sigma}}{2\mu} \left( \frac{P(1, \mu) - i\sigma Q(1, \mu)}{P(0, \mu) - i\sigma Q(0, \mu)} \right)$$

$$\sim \frac{i\sigma}{2\mu} + \frac{1}{4\mu^2} + \frac{i\sigma}{16\mu^3} - \frac{1}{16\mu^4} + O(\mu^{-5}),$$

(4.7)

where $\sigma = +1$ ($-1$) for $\Im(\mu) > 0$ ($\Im(\mu) < 0$). From this asymptotic expansion, one concludes that the poles of the corresponding $\zeta$-function $\text{Tr}\{H_0^D\}^{-s}$ are located at $s = 1/2 - k$, for $k = 0, 1, 2, \ldots$.

On the other hand, the trace of the resolvent $G^{(\alpha, \beta)}_0(\mu^2) := (H_0^{(\alpha, \beta)} - \mu^2)^{-1}$, corresponding to a general self-adjoint extension, gives

$$\text{Tr}\{G^{(\alpha, \beta)}_0(\mu^2)\} = \frac{1}{2\mu} \left( 2(\theta - \log \mu) J_0(\mu) + \pi N_0(\mu) + \frac{\mu^2 \sigma J_0(\mu)}{\mu^2[\pi N_0(\mu) + 2(\theta - \log \mu)] J_0(\mu)} \right).$$

(4.8)

The asymptotic expansion of the first term on the RHS of equation (4.8) is also given by expression (4.7). Therefore, it leads to the standard poles of the corresponding $\zeta$-function, $\text{Tr}\{H_0^{(\alpha, \beta)}\}^{-s}$, located at $s = 1/2 - k$, for $k = 0, 1, 2, \ldots$. However, the asymptotic expansion of the second term on the RHS of equation (4.8) is given by

$$\frac{J_0(\mu)}{\mu^2[\pi N_0(\mu) + 2(\theta - \log \mu)] J_0(\mu)} \sim \frac{1}{\mu^2[\pi \sigma + 2(\theta - \log \mu)]},$$

(4.9)

with $\sigma = 1$ ($\sigma = -1$) for $\mu$ in the upper (lower) half-plane. This expression gives an additional contribution to the $\zeta$-function given by

$$\text{Tr}\{H_0^{(\alpha, \beta)}\}^{-s} - \text{Tr}\{H_0^D\}^{-s} = -\frac{e^{-2i\theta}}{2\pi i} \left[ e^{i\pi s} \Gamma\left(0, \left(\frac{i\pi}{2} - 2\theta\right)s\right) - e^{-i\pi s} \Gamma\left(0, \left(-\frac{i\pi}{2} - 2\theta\right)s\right) \right]$$

$$= -\frac{1}{\pi} e^{2\gamma s - 2\gamma} \sin(\pi s) \log s + H(s),$$

(4.10)

where $H(s)$ is an entire function of $s$. Therefore, the $\zeta$-function of the general SAE of $H_0$ develops a cut on the negative real axis, which is present even for $\beta = 0$ and has a singular behavior for $\alpha \to 0$.

As a consequence, the derivative of the $\zeta$-function in the neighborhood of the origin behaves as [40, 41]

$$\frac{d}{ds} \text{Tr}\{H_0^{(\alpha, \beta)}\}^{-s} = -\log s + O(1).$$

(4.11)

This logarithmic branch cut from the origin for the general SAE makes the functional determinant in equation (1.6) ill defined. (See the discussions in [40, 41].) Note that this logarithmic cut is absent only for the unique SAE of $H_0$ which is locally scale invariant, namely $H_0^D = H_0^{(0, 1)}$. 

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On the other hand, the term (4.9) gives the following contribution to the heat-kernel trace of a general SAE of $H_0$

\[ \text{Tr}\{e^{-tH_0^{\alpha,\beta}}\} - \text{Tr}\{e^{-tH_0^{\alpha,\beta}}\} = -\frac{\mu}{\pi^2} \int_0^\infty \frac{J_0(\mu)}{\pi N_0(\mu) + 2(\theta - \log \mu)J_0(\mu)} \]

\[ = \frac{1}{\pi} \Im \int_1^\infty \frac{e^{-tx}}{x(\log x - 2\theta + i\pi)} \, dx + R(t), \tag{4.12} \]

where $R(t)$ is a smooth function at $t = 0$. It was shown in [40] that the integral on the right-hand side of equation (4.12) has an asymptotic expansion for small $t$ in terms of negative integer powers of $\log t$.

5. Non-compact case

In this section, we will consider a locally homogeneous (near a singularity) symmetric second-order differential operator on the noncompact one-dimensional manifold $M = \mathbb{R}^+$. We will see that also in this case the associated $\zeta$-function presents a non-standard singularity structure, related to the breaking of this scale homogeneity by the definition domains of the SAE of this operator. We will employ a different approach to this problem, based on the von Neumann theory of SAE of symmetric operators. This allows us to express the spectrum of the SAE in terms of a transcendental equation from which we are able to derive an asymptotic expansion of the eigenvalues. This eventually leads to the pole structure of the $\zeta$-function we are interested in.

Then, following [35], let us consider the operator

\[ H = -\frac{d^2}{dx^2} + V(x), \tag{5.1} \]

with

\[ V(x) = \nu^2 - \frac{1}{x^2} + x^2, \tag{5.2} \]

densely defined on the domain $\mathcal{D}(H) = C_0^\infty(\mathbb{R}^+)$, the linear space of smooth functions $\phi(x)$ with $x \in \mathbb{R}^+$ and compact support out of the origin. We have added to the singular at the origin term an $x^2$ term in order to get discrete spectra.

According to the von Neumann theory of deficiency indices [45], in order to get the SAE of $H$ in $L_2(\mathbb{R}^+)$ we need to compute the adjoint $H^\dagger$ and determine the deficiency subspaces $\mathcal{K}_\pm := \text{Kernel}(H^\dagger \mp i)$. The domain of $H^\dagger$ is the subspace of square integrable functions having an absolutely continuous first derivative such that

\[ H^\dagger \phi = -\psi''(x) + V(x)\psi(x) \in L_2(\mathbb{R}^+). \tag{5.3} \]

Note that no boundary condition is imposed on $x = 0$. To compute the deficiency indices $n_\pm := \dim \mathcal{K}_\pm$ of $H$, we must solve the eigenvalue problem

\[ H^\dagger \phi_\lambda = -\psi''(x) + V(x)\psi(x) = \lambda \phi_\lambda, \tag{5.4} \]

for $\phi_\lambda \in \mathcal{D}(H^\dagger)$ and $\lambda \in \mathbb{C}$ with the imaginary part $\Im(\lambda) \neq 0$. Let us define the parameter

\[ \alpha := \frac{i}{2} + \nu. \tag{5.5} \]

For the case $0 \leq \nu < 1$, i.e. $1/2 \leq \alpha < 3/2$, for any $\lambda \in \mathbb{C}$, equation (5.4) has a unique nontrivial square-integrable solution given by

\[ \phi_\lambda(x) = x^\alpha e^{-\frac{x^2}{4}} U\left(\frac{2\alpha + 1 - \lambda}{4}; \alpha + \frac{1}{2}; x^2\right), \tag{5.6} \]
where $U$ is the confluent hypergeometric function as defined in [63]. Thus, the deficiency subspaces $K_{\pm}$ are one dimensional, $n_{\pm} = 1$, and $H$ admits a one-parameter family of SAE\(^3\), which are in one-to-one correspondence with the isometries from $K_{+}$ onto $K_{-}$. The deficiency subspaces $K_{+}$ and $K_{-}$ are spanned by $\phi_{+} := \phi_{\alpha=i}$ and $\phi_{-} := \phi_{\alpha=-i} = \phi_{+}^*$, respectively. Each isometry $U_{\gamma} : K_{+} \rightarrow K_{-}$ can be characterized by a parameter $\gamma \in [0, \pi)$ defined by

$$U_{\gamma} \phi_{+} = e^{-2i\gamma} \phi_{-}. \quad (5.7)$$

Each isometry is identified with a SAE $H_{\gamma}$, a closed restriction of $H^\dagger$ to a linear subspace

$$D(H_{\gamma}) \subset D(H^\dagger) = D(\overline{H}) \oplus K_{+} \oplus K_{-}, \quad (5.8)$$

where $\overline{H}$ is the closure of $H$. The domain $D(H_{\gamma})$ is defined as the set of functions that can be written as

$$\phi = \phi_{0} + A(\phi_{+} + e^{-2i\gamma} \phi_{-}), \quad (5.9)$$

with $\phi_{0} \in D(\overline{H})$ and $A$ being a complex constant, and the action of the operator $H_{\gamma}$ is given by

$$H_{\gamma} \phi = H^\dagger \phi = H^\dagger \phi_{0} + tA(\phi_{+} - e^{-2i\gamma} \phi_{-}). \quad (5.10)$$

Let us consider, for simplicity, the repulsive case ($v \geq \frac{1}{2}$), that is, $1 \leq \alpha < 3/2$. In appendix C, we show that $\phi_{0}(x) = o(x^\alpha)$ and $\phi_{0}'(x) = o(x^{\alpha-1})$. Therefore, from equations (5.9) and (5.6) with $\lambda = \pm i\epsilon$ near the origin\(^4\),

$$\partial \log \phi(x) = \frac{1 - \alpha}{x} - 2 \frac{\Gamma \left( \frac{1}{2} - \alpha \right) \cos (\gamma - \gamma_1)}{\Gamma \left( \alpha - \frac{1}{2} \right) \cos (\gamma - \gamma_2)} x^{2\alpha - 2} + o(x^{2\alpha - 2}), \quad (5.12)$$

where we have defined $\gamma_1 = \arg \left[ \Gamma \left( -2\alpha + 3 - i \right)/4 \right]$ and $\gamma_2 = \arg \left[ \Gamma \left( 2\alpha + 1 - i \right)/4 \right]$.

The boundary condition specified in equation (5.12) characterizes the domain of the SAE $H_{\gamma}$. In order to determine its spectrum, we select from the set of eigenfunctions of $H^\dagger$ given in equation (5.6) those which satisfy equation (5.12). This leads to the following transcendental equation for the eigenvalues $\lambda$ \([35]\):

$$\frac{\Gamma \left( \kappa - \frac{1}{2} \right)}{\Gamma \left( 1 - \kappa - \frac{1}{2} \right)} = \beta(\gamma, \kappa), \quad (5.13)$$

where we have defined the constants

$$\kappa := \frac{2\alpha + 1}{4} \in \left[ \frac{3}{4}, 1 \right), \quad (5.14)$$

$$\beta(\gamma, \kappa) := \cos (\gamma - \gamma_1)/\cos (\gamma - \gamma_2).$$

\(^3\) This is in accordance with Weyl’s criterion \([45]\) according to which, for continuous $V(x)$, $H$ is essentially self-adjoint if and only if it is in the limit point case, both at infinity and at the origin. In addition, if $V(x) \geq M > 0$, for $x$ large enough, then $H$ is in the limit point case at infinity. In consequence, in the present case $H$ is essentially self-adjoint if and only if it is in the limit point case at zero. In particular, for positive $V(x)$ (i.e. $v^2 \geq \frac{1}{4}$), if $V(x) \geq \frac{1}{4} x^{-2}$ for $x$ sufficiently close to zero then $H$ is in the limit point case at the origin. On the contrary, if $V(x) \leq (\frac{1}{4} - \varepsilon)x^{-2}$, for some $\varepsilon > 0$, then $H$ is in the limit circle case at zero.

\(^4\) Note that

$$\phi_{\pm}(x) = x^{1-\alpha} \left( \frac{\Gamma \left( \frac{1}{2} - \alpha \right)}{\Gamma \left( \frac{1}{2} (2\alpha + (1 - i)) \right)} + O(x^2) \right) + x^\alpha \left( \frac{\Gamma \left( \frac{1}{2} - \alpha \right)}{\Gamma \left( \frac{1}{2} (\frac{1}{2} - \frac{1}{2}) - \frac{1}{2} \right)} + O(x^2) \right). \quad (5.11)$$

Then, it is easy to see that there are two locally scale-invariant SAE for which the functions in their domains behave near the origin as $x^\alpha$ and $x^{1-\alpha}$. For any other SAE, the simultaneous presence of both powers of $x$ in the boundary condition near the singularity breaks this local scale homogeneity.
Equation (5.13) determines the discrete spectrum of the SAE characterized by the parameter \( \gamma \). From now on, we will refer to the SAE as \( H(\beta) \), identifying it by the value of \( \beta \in \mathbb{R} \cup \{-\infty\} \) defined above.

As expected, the spectrum of \( H(\beta) \) is bounded from below; however, it presents a negative eigenvalue for those SAE characterized by \( \beta > \Gamma(\kappa)/\Gamma(1-\kappa) \) (even though the potential \( V(x) \geq 0 \)). Moreover, there is no common lower bound; instead, any negative real number is in the spectrum of some SAE.

For any value of \( \nu \in [1/2, 1) \), there are two particular SAE for which the spectrum can be easily worked out (see equation (5.13)).

- For \( \beta = 0 \), the spectrum is given by
  \[
  \lambda_n = 4(n + 1 - \kappa), \quad n = 0, 1, 2, \ldots
  \]
  \( (5.15) \)

- For \( \beta = -\infty \), the spectrum is given by
  \[
  \lambda_n = 4(n + \kappa), \quad n = 0, 1, 2, \ldots
  \]
  \( (5.16) \)

For any other value of \( \beta \), the eigenvalues also grow linearly with \( n \),

\[
4(n - 1 + \kappa) < \lambda_n < 4(n + \kappa).
\]

\( (5.17) \)

### 5.1. Pole structure of the \( \zeta \)-function

We will now study the pole structure of \( \zeta(\beta)(s) \) corresponding to an arbitrary SAE \( H(\beta) \). Note that, since the eigenvalues grow linearly with \( n \) (see equation (5.17)), \( \zeta(\beta)(s) \) is analytic in the open half-plane \( \Re(s) > 1 \).

Let us begin by considering the \( \zeta \)-functions for the SAE characterized by \( \beta = 0 \) and \( \beta = -\infty \), which can be explicitly evaluated from the expression of its spectra in equations (5.15) and (5.16). We obtain

\[
\zeta_{\beta=0}(s) = 4^{-s} \sum_{n=0}^{\infty} (n + 1 - \kappa)^{-s} = 4^{-s} \zeta(s, 1 - \kappa),
\]

\( (5.18) \)

\[
\zeta_{\beta=-\infty}(s) = 4^{-s} \sum_{n=0}^{\infty} (n + \kappa)^{-s} = 4^{-s} \zeta(s, \kappa),
\]

\( (5.19) \)

where \( \zeta(s, q) \) is the Hurwitz \( \zeta \)-function, whose analytic extension presents a unique simple pole at \( s = 1 \) with residue \( \text{Res} \zeta(s, q)|_{s=1} = 1 \). Therefore, for \( \beta = 0 \) and \( \beta = -\infty \), the \( \zeta \)-function presents a unique simple pole at \( s = 1 \), with residue 1/4.

For finite \( \beta \), let us define the holomorphic function,

\[
f(\lambda) = \frac{1}{\Gamma(1 - \kappa - \frac{1}{4})} - \frac{\beta}{\Gamma(\kappa - \frac{1}{4})};
\]

\( (5.20) \)

recall that we are considering the repulsive case \( \frac{1}{4} \leq \kappa < 1 \). The eigenvalues of the self-adjoint operator \( H(\beta) \) correspond to the zeros of \( f(\lambda) \) which, consequently, are all real. They are also positive, with the only possible exception of the lowest one, according to the discussion in the previous section. Moreover, the zeros of \( f(\lambda) \) are simple and isolated; thus, the \( \zeta \)-function can be represented as the integral on the complex plane

\[
\zeta_{\beta}(s) = \frac{1}{2\pi i} \oint_{C} \lambda^{-s} f(\lambda) + \Theta(-\lambda s) \lambda_{0, \beta}^{-s},
\]

\( (5.21) \)
where $C$ is a curve which encloses counterclockwise the positive zeros of $f(\lambda)$, $\lambda_{0,\beta}$ is the lowest eigenvalue and $\Theta(\cdot)$ is the Heaviside function.

For $\Re(s) > 1$, the path of integration in (5.21) can be deformed to a vertical line to obtain

$$
\zeta(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f'(\lambda)}{f(\lambda)} d\lambda + h_1(s)
$$

where $h_1(s)$ and $h_2(s)$ are the holomorphic functions. Equation (5.22) gives an integral representation of $\zeta(s)$, analytic in the half-plane $\Re(s) > 1$. To compute its meromorphic extension to the whole complex $s$-plane and its pole structure, we need the asymptotic expansion of $f'(\lambda)/f(\lambda)$, which is given by [35]

$$
\frac{f'(\lambda)}{f(\lambda)} \sim \frac{1}{4} \log (-\lambda) + \frac{i}{4} \sum_{k=0}^{\infty} \epsilon_k(\kappa)(-\lambda)^{-k} + \sum_{N=1}^{\infty} \sum_{n=0}^{\infty} C_{N,n}(\kappa, \beta)(-\lambda)^{-N(2\kappa-1)-2n-1}.
$$

(5.23)

The coefficients $\epsilon_k(\kappa)$ are polynomials in $\kappa$. As we will see, these terms do not contribute to the pole structure of $\zeta(s)$. On the other hand, the coefficients $C_{N,n}(\kappa, \beta)$ are defined through the following relations:

$$
C_{N,n}(\kappa, \beta) := -\left(4^{2\kappa-1}\beta\right)^N \left(2\kappa - 1 + \frac{2n}{N}\right) b_n(\kappa, N),
$$

(5.24)

$$
\sum_{n=0}^{\infty} b_n(\kappa, N) z^{-2n} := \exp \left\{ N \sum_{m=1}^{\infty} a_m(\kappa) z^{-2m} \right\}.
$$

(5.25)

$$
a_m(\kappa) := \frac{2^{4m-1}}{2m+1} \left\{ \left[ (1 - \kappa)^{2m} - \kappa^{2m} \right] + \left( \frac{1}{2} \right)^m \left[ (1 - \kappa)^{2m} + \kappa^{2m} \right] + (2m+1) \right\}
$$

$$
\times \sum_{p=1}^{m} \frac{B_{2p}}{p(2p-1)} \left( \frac{2m-1}{2p-2} \right) (2m-2p+1) \left( (1 - \kappa)^{2m-2p+1} - (1 - \kappa)^{2m-p+1} \right).
$$

(5.26)

Note that $C_{N,n}(\kappa, \beta) = 0$ for $\beta = 0$.

Replacing into equation (5.22) the dominant logarithmic term in equation (5.23), we get a simple pole at $s = 1$ with residue $1/4$. The remaining terms in the asymptotic expansion of $f'(\lambda)/f(\lambda)$ are of the form $A_j(-\lambda)^{-j}$, for some $j \geq 0$ (see equation (5.23)). Replacing these terms into equation (5.22) gives simple poles at $s = 1 - j$, with residues given by $-A_j(\pi) \sin(\pi j)$.

Note that these residues vanish for integer values of $j$.

In conclusion, there is a simple pole at $s = 1$ with residue $1/4$ as for the SAE with $\beta = 0$, $-\infty$. But for a general SAE $H_{\beta}$ with $\beta \neq 0$, $-\infty$ and for $\kappa < 1$ there are also simple poles at non-integer values of $s$. Indeed, for each pair of integers $(N, n)$ with $N = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$, the function $\zeta(s)$ has a simple pole at the negative value

$$
s_{N,n} = -N(2\kappa - 1) - 2n,
$$

(5.27)

with a $\beta$-dependent residue given by

$$
\text{Res} [\zeta(s)]_{s=s_{N,n}} = \frac{(-1)^N}{\pi} C_{N,n}(\kappa, \beta) \sin(2\pi N\kappa).
$$

(5.28)

Let us remark that when $\kappa$ is a rational number, there can be several (but a finite number of) pairs $(N, n)$ contributing to the same pole. In contrast, when $\kappa$ is irrational the poles coming from different pairs $(N, n)$—which are also irrational—do not coincide.
Finally, note that a pole of $\zeta_\nu(s)$ at a non-integer $s_{\nu,\nu} = -N(2\kappa - 1) - 2n$ implies the presence of a term proportional to $t^{N(2\kappa - 1) + 2n}$ in the small-$t$ asymptotic expansion of $\text{Tr}[e^{-tH_\nu}]$ (see appendix A).

Therefore, for this second-order differential operator on the half-line, we obtain similar results as in section 3 for a compact segment. There are two SAE whose definition domains are locally scale invariant near the singularity (see footnote 4) and show the usual properties in their spectral functions. Moreover, there exists a continuous family of other SAE which present anomalous poles in their $\zeta$-functions, dependent on an external parameter (the coupling $\nu$), with residues dependent on the SAE.

We finish this section by presenting an alternative method based on the relation between the pole structure we have already found.

By solving equation (5.13) order by order in $n$, we obtain [35]

$$
\lambda_n = 4n + 4(1 - \kappa) + \frac{4\beta}{\pi} \sin(2\pi \kappa) n^{1-2\kappa} + \frac{4\beta}{\pi} (1 - 3\kappa + 2\kappa) \sin(2\pi \kappa) n^{-2\kappa} - \frac{2\beta^2}{\pi} \sin(4\pi \kappa) n^{2-4\kappa} + \ldots, 
$$

(5.29)

where we have retained only powers of $n$ greater than $-2$. This leads to

$$
\zeta_\nu(s) = 4^{-x} \xi(s) + s 4^{-x} (\kappa - 1) \xi(s + 1) + s (s + 1) 4^{-x} (\kappa - 1)^2 \frac{\pi}{2} \xi(s + 2) - s 4^{-x} \frac{\beta}{\pi} \sin(2\pi \kappa) \xi(s + 2\kappa)
$$

$$
- s (s + 2\kappa) 4^{-x} \frac{\beta}{\pi} (\kappa - 1) \sin(2\pi \kappa) \xi(1 + s + 2\kappa)
$$

$$
+ s 4^{-x} \frac{\beta^2}{2\pi} \sin(4\pi \kappa) \xi(s - 1 + 4\kappa) + \ldots,
$$

(5.30)

where $\xi(z)$ is the Riemann $\zeta$-function. Taking into account that $\zeta(z)$ presents a unique simple pole at $z = 1$ with residue 1, equation (5.30) confirms, order by order in this development, the pole structure we have already found.

6. Krein’s formula

In this section, we study the behavior of the resolvent of a locally homogeneous second-order differential operator in relation to its SAE, and the consequences this has on the properties of other spectral functions. This will be done in the framework of the Krein formula [55], which relates the resolvent of two SAE of the given operator.

We consider the differential operator

$$
A = -\partial_x^2 + \frac{\nu^2 - 1/4}{x^2} + V(x),
$$

(6.1)

where $\nu \in (0, 1) \subset \mathbb{R}$ and $V(x)$ is an analytic function of $x \in \mathbb{R}^+$. The operator (6.1) defined on $D(A) := C_0^\infty(\mathbb{R}^+)$ admits a continuous family of SAE $A^\theta$ characterized by a real parameter which we call $\theta$. Since the operator $e^{-tA^\theta}$ corresponding to a general SAE $A^\theta$ is not trace class (note that the base manifold $\mathbb{R}^+$ is non-compact), we will consider the trace of the difference $e^{-tA^\theta} - e^{-tA^{\infty}}$, where $A^{\infty}$ corresponds to the Friedrichs
extension \[45\]. We will show in theorem 6.15 that this trace admits an asymptotic expansion given by

\[
\text{Tr}\left[e^{-tA^0} - e^{-tA^\infty}\right] \sim \sum_{n=0}^\infty a_n(v, V) t^{\frac{n}{2}} + \sum_{N,n=1}^\infty b_{N,n}(v, V) t^{N+\frac{n}{2}-\frac{1}{2}}. \tag{6.2}
\]

As we will see, the SDW coefficients \(a_n(v, V)\) and \(b_{N,n}(v, V)\) can be recursively computed for each given potential \(V(x)\). Let us remark that the singular term in (6.1) not only contributes to the coefficients \(a_n(v, V)\) of the standard powers of \(t\) but also leads to the presence of non-standard powers of \(t\) whose exponents are not half-integers but depend on the ‘external’ parameter \(v\). We will also show that these terms are absent only for the SAE with \(\theta = 0\) and \(\theta = \infty\), which correspond to SAE characterized by scale-invariant domains. In section 6.5, we will consider the case \(A^\infty\) and has already been established for the case of operators with regular coefficients. We will therefore extend this result to \(A^\infty\) and then use this generalization to prove the asymptotic behavior (6.2).

6.1. The regular coefficients case

In this section, we state without proof two theorems valid for the case of differential operators with regular coefficients. Theorem 6.1 describes the self-adjoint extensions of a symmetric operator in terms of the (regular) boundary values of the functions belonging to its domains \[65\]. The statement of theorem 6.3 is the Krein formula (see \[50\]), which relates the resolvents corresponding to an arbitrary SAE and to \(A^\infty\). This relation is called Krein’s formula \[55\] and has already been established for the case of operators with regular coefficients. We will therefore extend this result to \(A\) in (6.1) and then use this generalization to prove the asymptotic behavior (6.2).

**Theorem 6.1.** Let \(A\) be a symmetric operator densely defined on a subspace \(D(A)\) of a Hilbert space \(\mathcal{H}\), for which the deficiency indices are equal: \(n_+ = n_- = : n < \infty\). Then,

- there exist two surjective maps \(\Gamma_1, \Gamma_2 : D(A^*) \to \mathbb{C}^n\) such that \(\forall \phi, \psi \in D(A^*)\)
  \[
  (\phi, A^*\psi)_{\mathcal{H}} - (A^*\phi, \psi)_{\mathcal{H}} = (\Gamma_1\phi, \Gamma_2\psi)_{\mathbb{C}^n} - (\Gamma_2\phi, \Gamma_1\psi)_{\mathbb{C}^n},
  \tag{6.3}
  \]
  where \((\cdot, \cdot)_{\mathcal{H}}\) is the inner product in \(\mathcal{H}\) and \((\cdot, \cdot)_{\mathbb{C}^n}\) is the usual inner product in \(\mathbb{C}^n\).

- The self-adjoint extensions \(A^{(M,N)}\) of \(A\) are characterized by two matrices \(M, N \in \mathbb{C}^{n \times n}\), such that \(M \cdot N^*\) is Hermitian and \((M|N) \in \mathbb{C}^{n \times 2n}\) has rank \(n\). The domain of definition of \(A^{(M,N)}\) is defined as
  \[
  D\left(A^{(M,N)}\right) := \{\phi \in D(A^*) : M\Gamma_1\phi = N\Gamma_2\phi\}.
  \tag{6.4}
  \]

Since the restrictions of \(\Gamma_1, \Gamma_2\) to \(\ker(A^* - \lambda)\) are invertible, we can establish the following definitions:

**Definition 6.2.**

\[
\Gamma_1^{-1}(\lambda) := \left(\Gamma_1 |_{\ker(A^* - \lambda)}\right)^{-1},
\tag{6.5}
\]

\[
K(\lambda) := -\Gamma_2 \cdot \Gamma_1^{-1}(\lambda).
\tag{6.6}
\]
Now we can write down the Krein’s formula, which expresses the resolvent \((A^{(M,N)} - \lambda)^{-1}\) corresponding to an arbitrary self-adjoint extension in terms of the resolvent \((A^{(1,0)} - \lambda)^{-1}\) corresponding to the self-adjoint extension characterized by the matrices \(M = 1\) and \(N = 0\) [66].

**Theorem 6.3 (Krein’s formula).**
\[
(A^{(M,N)} - \lambda)^{-1} = (A^{(1,0)} - \lambda)^{-1} + \Gamma^{-1}(\lambda) \cdot \frac{N}{(M + NK(\lambda))} \cdot (\Gamma^{-1}(\lambda^*))^T. 
\] (6.7)

**Example.** Let us write down the Krein formula for the case of the one-dimensional second-order differential operator
\[
A = -\frac{d^2}{dx^2} + U(x), 
\] (6.8)
defined on \(C_0^\infty(\mathbb{R}^+) \subset L_2(\mathbb{R}^+)\). We assume that the potential \(U(x) \in C(\mathbb{R}^+)\) is in the limit point case [45] at infinity and in the limit circle case [45] at \(x = 0\). This is the case if there exist \(\delta, \epsilon > 0\) and a positive differentiable function \(f(x) \geq -U(x)\) such that
\[
0 \leq U(x) \leq \frac{3/4 - \epsilon}{x^2} \quad \forall x \in (0, \delta), 
\] (6.9)
\[
\int_1^\infty \frac{dx}{\sqrt{f(x)}} \quad \text{diverges, and} \quad \frac{f'(x)}{f(x)^{3/2}} \quad \text{is bounded near } \infty. 
\] (6.10)

Then, the deficiency indices of \(A\) are \(n_\pm = 1\) [45]. In relation to theorem 6.1, we define the boundary operators \(\Gamma_1\) and \(\Gamma_2\) as
\[
\Gamma_1\phi(x) := \phi(0), 
\] (6.11)
\[
\Gamma_2\phi(x) := \phi'(0). 
\] (6.12)

According to the second statement in theorem 6.1, the SAE \(A^0\) of \(A\) are characterized by a real parameter \(\theta\) corresponding to \(M^{-1} N \in \mathbb{R}\) and their domains of definition \(D(A^0)\) are given by (see equation (6.4))
\[
D(A^0) = \{ \phi \in D(A^1) : \phi'(0) = -\theta \phi(0) = 0 \}. 
\] (6.13)

As expected, we obtain the classical boundary conditions of the Robin type. The extension characterized by \(N = 0\) (Dirichlet boundary condition) corresponds to \(\theta = \infty\), while \(M = 0\) (Neumann boundary conditions) corresponds to \(\theta = 0\). Note that there are only two boundary conditions, namely \(\theta = 0\) and \(\theta = \infty\), which are scale invariant.

Let us now determine the operators \(\Gamma_1^{-1}(\lambda)\) and \(K(\lambda)\) defined in (6.2). Since the deficiency indices of the operator \(A\) are \(n_\pm = 1\), the deficiency subspace \(\text{Ker}(A^1 - \lambda)\) is generated by a normalized function which we denote by \(\phi_\pm(x)\). Consequently,
\[
\Gamma_1^{-1}(\lambda) : \mathbb{C} \rightarrow \text{Ker}(A^1 - \lambda), 
\]
\[
\Gamma_1^{-1}(\lambda) \cdot c = \phi_\pm(x)/\phi_\pm(0) \cdot c. 
\] (6.14)

The operator \(K(\lambda)\) is therefore given by
\[
K(\lambda) = \frac{-\phi_\pm'(0)}{\phi_\pm(0)}. 
\] (6.15)

The Krein formula (equation (6.7)) can then be written as
\[
(A^0 - \lambda)^{-1} - (A^\infty - \lambda)^{-1} = \frac{(A^0 - \lambda)^{-1} - (A^\infty - \lambda)^{-1}}{1 + \theta K(\lambda)}. 
\] (6.16)
Equation (6.16) gives the resolvent corresponding to an arbitrary self-adjoint extension \( A^\theta \) in terms of the resolvents corresponding to the boundary conditions which are scale invariant, namely \( \theta = 0 \) (Neumann) and \( \theta = \infty \) (Dirichlet).

Following [39], in the following section we will prove a generalization of the Krein formula that relates in a similar way the resolvents corresponding to different self-adjoint extensions of the operator in (6.1). The method employed follows the lines given by Mooers in [34]. We will obtain an expression similar to (6.16) in which the \( K(\lambda) \) factor, although not given by (6.15) as in the regular case, is also related to the behavior near the origin of the functions in \( \text{Ker}(A^1 - \lambda) \).

6.2. Locally homogeneous second-order differential operators

So, we consider the symmetric differential operator \( A \),

\[
A = -\partial_x^2 + \frac{\nu^2 - 1/4}{x^2} + V(x),
\]

defined on \( C_0^\infty(\mathbb{R}^+) \subset L^2(\mathbb{R}^+) \). We assume that \( V(x) \) is an analytic function of \( x \in \mathbb{R}^+ \) bounded from below and the parameter \( \nu \in (0, 1) \subset \mathbb{R} \).

The following theorem describes the behavior of the functions in \( D(A^1) \) near the singular point \( x = 0 \).

**Theorem 6.4.** If \( \psi \in D(A^1) \), then

\[
\psi(x) = C[\psi](x^{-\nu+1/2} + \theta_\psi x^{\nu+1/2}) + O(x^{3/2}),
\]

for \( x \to 0^+ \) and some constants \( C[\psi], \theta_\psi \in \mathbb{C} \).

**Proof.** By virtue of the Riesz representation lemma

\[
\psi \in D(A^1) \quad \exists \tilde{\psi} \in L^2(\mathbb{R}^+) : (\psi, A\phi) = (\tilde{\psi}, \phi) \quad \forall \phi \in D(A).
\]

Consequently,

\[
A^1 \psi := \tilde{\psi}.
\]

If we define \( \chi := x^{-\nu-1/2} \psi \), we obtain

\[
\partial_x (x^{2\nu+1} \partial_x \chi) = -x^{\nu+1/2} (\tilde{\psi} - V(x)) \psi \in L^1(\mathbb{R}^+).
\]

Therefore, there exists a constant \( C_1 \in \mathbb{R} \) such that

\[
\partial_x \chi = C_1 x^{-1-2\nu} - x^{-1-2\nu} \int_0^x y^{\nu+1/2} \left( -\partial_y^2 + \frac{\nu^2 - 1/4}{y^2} \right) \psi \, dy.
\]

The Cauchy–Schwartz inequality implies

\[
\left| x^{-1-2\nu} \int_0^x y^{\nu+1/2} \left( -\partial_y^2 + \frac{\nu^2 - 1/4}{y^2} \right) \psi \, dy \right| \leq C_2 \left\| \left( -\partial_y^2 + \frac{\nu^2 - 1/4}{y^2} \right) \psi \right\|_{L^2(0,x)} x^{-\nu},
\]

for some \( C_2 \in \mathbb{R} \). In consequence,

\[
\left| \int_0^x z^{-1-2\nu} \int_0^z y^{\nu+1/2} \left( -\partial_y^2 + \frac{\nu^2 - 1/4}{y^2} \right) \psi \, dy \, dz \right| \leq C_3 + C_4 x^{1-\nu},
\]

where \( C_3, C_4 \in \mathbb{R} \). Thus, there exist \( C_5, C_6 \in \mathbb{R} \), such that

\[
\psi = C_5 x^{-\nu+1/2} + C_6 x^{\nu+1/2} + O(x^{3/2}),
\]

for \( x \to 0^+ \).
Moreover,

Let us point out that in the regular case limit, when $SAE$ of $\theta_2$ characterizes the real parameter $\theta$, this parameter vanishes, this parameter $\theta$ thus determines the boundary condition at the singularity.

Remark 2. By writing expression (6.26) for $\psi = \phi$, we conclude that for all $\psi \in \mathcal{D}(A^1)$ the parameter $\theta_2$ defined by theorem 6.2 is real.

Proof. Expression (6.26) follows from an integration by parts on its LHS using expression (6.18).

As a consequence of corollary 6.5, the differential operator $A$ admits a family of $SAE A^\theta$, characterized by the real parameter $\theta$, whose domains are given by

$$\mathcal{D}(A^\theta) := \{ \phi \in \mathcal{D}(A^1) : \theta_0 = \theta \},$$

where $\theta_0$ is defined according to theorem 6.2. The parameter $\theta$, with dimensions $[\text{length}]^{-2\nu}$, thus determines the boundary condition at the singularity.

There exists another $SAE$, which we denote by $A^\infty$, whose domain is given by

$$\mathcal{D}(A^\infty) := \{ \phi \in \mathcal{D}(A^1) : \phi(x) = C[\phi] x^{\nu+1/2} + O(x^{3/2}), \quad \text{with } C[\phi] \in \mathbb{C} \}. \quad (6.28)$$

Let us point out that in the regular case limit, when $\nu \to 1/2$, where the singular coefficient in the operator vanishes, this parameter $\theta$ coincides with the one characterizing Robin boundary conditions for the regular case (see equation (6.13)).

6.3. Generalization of Krein’s formula

Our purpose now is to establish a relation between the resolvents corresponding to different $SAE$ of $A$. This relation will prove to be useful to show that the trace of the heat-kernel $\text{Tr} e^{-tA^\nu}$ corresponding to a general $SAE$ admits, for $\theta \neq 0, \infty$, a small-$t$ asymptotic expansion with $t$-dependent powers of $t$.

We begin by stating the following theorem.

Theorem 6.6. For any $f(x) \in L_2(\mathbb{R}^+)$ and $\lambda \notin \sigma(A^\theta)$, there exists a unique function $\phi^\theta(x, \lambda) \in \mathcal{D}(A^\theta)$ such that

$$(A^\theta - \lambda)\phi^\theta(x, \lambda) = f(x). \quad (6.29)$$

Moreover,

$$\phi^\theta(x, \lambda) = \int_0^\infty G_\theta(x, x', \lambda) f(x') \, dx', \quad (6.30)$$

with $G_\theta(x, x', \lambda)$ being the kernel of the resolvent $(A^\theta - \lambda)^{-1}$.

The kernel $G_\theta(x, x', \lambda)$ can be written as

$$G_\theta(x, x', \lambda) = -\frac{\Theta(x' - x)L_\theta(x, \lambda)R(x', \lambda) + \Theta(x - x')L_\theta(x', \lambda)R(x, \lambda)}{W[L_\theta, R](\lambda)}, \quad (6.31)$$

where $\Theta(\cdot)$ is the Heaviside function. The functions $L_\theta(x, \lambda)$ and $R(x, \lambda)$ satisfy equation (6.29) for $f(x) \equiv 0$. The latter is square integrable at $x \to \infty$ and the former satisfies the boundary condition

$$L_\theta(x, \lambda) = x^{-\nu+1/2} + \theta x^{\nu+1/2} + O(x^{3/2}), \quad (6.32)$$

at $x \to 0^+$. $W[L_\theta, R](\lambda)$ is the Wronskian of $L_\theta(x, \lambda)$ and $R(x, \lambda)$, and is independent of $x$.
To obtain the generalization of the Krein formula we begin by relating the resolvents corresponding to $\theta = \infty$ and $\theta = 0$. In particular, for these SAE the boundary condition (6.32) reads
\[ L_\infty(x, \lambda) = x^{\nu+1/2} + O(x^{3/2}), \]  
(6.33)
and
\[ L_0(x, \lambda) = x^{-\nu+1/2} + O(x^{3/2}). \]  
(6.34)
Since these functions determine the behavior at the origin of the kernels $G_\infty(x, x', \lambda)$ and $G_0(x, x', \lambda)$, the following definitions are in order:

Definition 6.7.
\[ G_\infty(x', \lambda) := \lim_{x \to 0} x^{-\nu-1/2} G_\infty(x, x', \lambda), \]  
(6.35)
\[ G_0(x', \lambda) := \lim_{x \to 0} x^{-1/2} G_0(x, x', \lambda). \]  
(6.36)

The new functions $G_\infty(x, \lambda)$ and $G_0(x, \lambda)$ determine the behavior at the singularity of the solutions $\phi^\infty(x, \lambda)$ and $\phi^0(x, \lambda)$ of (6.29) corresponding to $\theta = \infty$ and $\theta = 0$, respectively. Indeed,
\[ \phi^\infty(x, \lambda) = \int_0^\infty G_\infty(x, x', \lambda) f(x') \, dx' = \phi^\infty(\lambda) x^{\nu+1/2} + O(x^{3/2}), \]  
(6.37)
\[ \phi^0(x, \lambda) = \int_0^\infty G_0(x, x', \lambda) f(x') \, dx' = \phi^0(\lambda) x^{-\nu+1/2} + O(x^{3/2}), \]  
(6.38)
being
\[ \phi^\infty(\lambda) := \int_0^\infty G_\infty(x', \lambda) f(x') \, dx', \]  
(6.39)
\[ \phi^0(\lambda) := \int_0^\infty G_0(x', \lambda) f(x') \, dx'. \]  
(6.40)
To obtain a relationship between the kernels $G_\infty(x, x', \lambda)$ and $G_0(x, x', \lambda)$ we will relate the solutions $\phi^\infty(x, \lambda)$ and $\phi^0(x, \lambda)$ of (6.29) corresponding to the same inhomogeneity $f(x)$. To do that we need the following two lemmas.

Lemma 6.8. Let $\varphi_0(x) \in D(A^0)$ such that $\varphi_0(x) = x^{-\nu+1/2} + O(x^{3/2})$ for $x \to 0^+$. Then, the solutions $\phi^\infty(x, \lambda)$ and $\phi^0(x, \lambda)$ of (6.29) are related by
\[ \phi^\infty(x, \lambda) = \phi^0(x, \lambda) - \phi^0(\lambda) \left[ \varphi_0(x) - \int_0^\infty G_\infty(x, x', \lambda)(A^0 - \lambda)\varphi_0(x') \, dx' \right]. \]  
(6.41)

Proof. On the one hand,
\[ (A^\dagger - \lambda) \phi^\infty(x, \lambda) = f(x). \]  
(6.42)
Moreover,
\[ (A^\dagger - \lambda) \left[ \phi^0(x, \lambda) - \phi^0(\lambda) \left[ \varphi_0(x) - \int_0^\infty G_\infty(x, x', \lambda)(A^0 - \lambda)\varphi_0(x') \, dx' \right] \right] = f(x) - \phi^0(\lambda) \left[ (A^0 - \lambda)\varphi_0(x) - (A^0 - \lambda)\varphi_0(x) \right] = f(x). \]  
(6.43)
On the other hand, for $x \to 0^+$,
\[ \phi^\infty(x, \lambda) \equiv \phi^\infty(\lambda) x^{\nu+1/2} + O(x^{3/2}), \]  
(6.44)
Lemma 6.10. Let \( \phi^0(x, \lambda) = \phi^\infty(x, \lambda) + 2\nu G_\infty(x, \lambda) \phi^0(\lambda). \) 

We are interested in rewriting expression (6.49) so that \( \phi^0(x, \lambda) \) is given in terms of quantities corresponding to the extension characterized by \( \theta = \infty \). In doing so, we take the \( x \to 0^+ \) limit in equation (6.49) obtaining
\[
G_\infty(x, \lambda) = \frac{1}{2\nu} (x^{\nu+1/2} - K(\lambda)^{-1}x^{\nu+1/2}) + O(x^{3/2}),
\] (6.50)
where
\[
K(\lambda) := \frac{\phi^0(\lambda)}{\phi^\infty(\lambda)}.
\] (6.51)
The term \( K(\lambda) \) defined in (6.51) relates the behavior at the singularity of the solutions to equation (6.29) corresponding to the SAE \( \theta = \infty \) and \( \theta = 0 \). Note that equation (6.50) allows us to compute \( K(\lambda) \) by studying the behavior at the singularity of the kernel of the resolvent corresponding to the extension \( \theta = \infty \). Therefore, the kernel \( G_\infty(x, x', \lambda) \) determines \( K(\lambda) \).
and, consequently, also \( \phi^0(\lambda) \). Note that \( K(\lambda) \) reduces to expression (6.15) for the regular limit \( v \to 1/2 \).

We can finally express the solution \( \phi^0(x, \lambda) \) to (6.29) corresponding to \( \theta = 0 \) by means of the data obtained by imposing the boundary conditions corresponding to \( \theta = \infty \) (see lemma 6.10),

\[
\phi^0(x, \lambda) = \phi^\infty(x, \lambda) + 2vK(\lambda)G_\infty(x, \lambda)\phi^\infty(\lambda). \tag{6.52}
\]

Since this equation is valid for any inhomogeneity \( f(x) \), by virtue of equations (6.37), (6.38) and (6.39), we obtain the following theorem.

**Theorem 6.11.**

\[
G_\theta(x, x', \lambda) = G_\infty(x, x', \lambda) + 2vK(\lambda)G_\infty(x, \lambda)G_\infty(x', \lambda). \tag{6.53}
\]

Next, we will establish an expression similar to (6.53) giving the resolvent for an arbitrary SAE in terms of data related to the boundary conditions corresponding to \( \theta = \infty \). The first step is to state the following lemma.

**Lemma 6.12.** The solution \( \phi^\theta(x, \lambda) \) to (6.29) is given by

\[
\phi^\theta(x, \lambda) = \phi^\infty(x, \lambda) + 2v(K(\lambda)^{-1} + \theta)^{-1}G_\infty(x, \lambda)\phi^\infty(\lambda). \tag{6.54}
\]

**Proof.** By means of equation (6.49) it is immediate to show that the difference between both sides of expression (6.54) belongs to \( \text{Ker}(A^\theta - \lambda) \).

On the other hand, both sides of (6.54) belong to \( \mathcal{D}(A^\theta) \) since the behavior of its RHS at the singularity is given by (see equations (6.37), (6.50) and (6.51))

\[
\frac{\phi^\theta(\lambda)\phi^\infty(\lambda)}{\phi^\infty(\lambda) + \theta \phi^\theta(\lambda)}(x^{-v+1/2} + \theta x^{v+1/2}) + O(x^{3/2}). \tag{6.55}
\]

Once more, uniqueness established in theorem 6.6 leads us to equation (6.54). \( \Box \)

From lemma 6.12, together with equations (6.30), (6.37) and (6.39), we straightforwardly get the following theorem.

**Theorem 6.13 (Generalization of Krein’s formulas).**

\[
G_\theta(x, x', \lambda) = G_\infty(x, x', \lambda) + 2v(K(\lambda)^{-1} + \theta)^{-1}G_\infty(x, \lambda)G_\infty(x', \lambda). \tag{6.56}
\]

Expressions (6.53) and (6.56) readily lead to

\[
G_\theta(x, x', \lambda) - G_\infty(x, x', \lambda) = \frac{G_\theta(x, x', \lambda) - G_\infty(x, x', \lambda)}{1 + \theta K(\lambda)} \tag{6.57}
\]

Therefore, we obtain the following relation between the resolvents of the different SAE:

\[
(A^\theta - \lambda)^{-1} - (A^\infty - \lambda)^{-1} = \frac{(A^\theta - \lambda)^{-1} - (A^\infty - \lambda)^{-1}}{1 + \theta K(\lambda)}. \tag{6.58}
\]

This expression formally coincides with the Krein formula (6.16), which is valid for regular operators. However, while the factor \( K(\lambda) \) in (6.16) is given by (6.15), in the singular case under study \( K(\lambda) \) in (6.58) corresponds to equation (6.51). As already mentioned, (6.16) and (6.58) coincide in the \( v \to 1/2 \) limit.

We summarize our results in the following theorem, which will allow us to prove the non-standard behavior of the spectral functions of the SAE of the operator \( A \) in (6.17).

**Theorem 6.14.**

\[
\text{Tr}[(A^\theta - \lambda)^{-1} - (A^\infty - \lambda)^{-1}] = \frac{\text{Tr}[(A^\theta - \lambda)^{-1} - (A^\infty - \lambda)^{-1}]}{1 + \theta K(\lambda)}. \tag{6.59}
\]

In the following section, we will show that the asymptotic expansion of \( K(\lambda) \) for large \(|\lambda|\) presents powers of \( \lambda \) whose exponents depend on the parameter \( v \). This will finally lead to the asymptotic series (6.2).
6.4. Asymptotic expansion of the resolvent. In this section, we will make use of theorem 6.14 to obtain the large-|\lambda| asymptotic expansion for the resolvent \((A - \lambda)^{-1}\) of an arbitrary SAE of the operator \(A\) defined in equation (6.17). According to this theorem it suffices to study the solutions to

\[(A + z)\psi = 0,\]  

(6.60)
satisfying the boundary conditions corresponding to \(\theta = \infty\) and \(\theta = 0\). If we consider \(\lambda\) in the negative real semi-axis, we can take \(z \in \mathbb{R}^+\). In particular, we will focus on the behavior of the solutions for large \(z\).

Taking into account the scaling properties of the first two terms in (6.17) it will be convenient to define a new variable \(y := \sqrt{\lambda} x \in \mathbb{R}^+\). The solution to equation (6.60) can then be written as \(\psi = \psi(\sqrt{\lambda} x, z)\), with \(\psi(\sqrt{\lambda} x, z)\) being a solution to

\[\left(-\frac{\partial^2}{\partial y^2} + \frac{\nu^2 - 1/4}{y^2} + 1 + \frac{1}{z} V(y/\sqrt{\lambda})\right)\psi(y, z) = 0.\]  

(6.61)

We propose the following ansatz:

\[\psi(y, z) = \phi(y) + \sum_{n=0}^{\infty} \psi_n(y) z^{-1-n/2},\]  

(6.62)
to be consistent with the analytic series for the potential,

\[V(x) = \sum_{n=0}^{\infty} V_n x^n,\]  

(6.63)
where \(V_n := V^{(i)}(0)/n!\). Equation (6.61) can now be solved order by order in \(z\). The solution to (6.61), which is square integrable at \(y \to \infty\), can be written as

\[R(y, z) = \sqrt{\lambda} K_\nu(y) + \sum_{n=0}^{\infty} \psi_n(y) z^{-1-n/2},\]  

(6.64)
where

\[\psi_n(y) = -y^{1/2} K_\nu(y) \int_0^y \left[V_n y^{(n+1)/2} K_\nu(y') + \sum_{l+m=n-2} V_l y^l \psi_m(y')\right] \sqrt{y'} I_\nu(y') \, dy'\]  

\[-y^{1/2} I_\nu(y) \int_y^{\infty} \left[V_n y^{(n+1)/2} K_\nu(y') + \sum_{l+m=n-2} V_l y^l \psi_m(y')\right] \sqrt{y'} K_\nu(y') \, dy'.\]  

(6.65)

Therefore, the behavior of \(R(y, z)\) at \(y \to 0^+\) is given by

\[R(y, z) \sim \frac{\Gamma(v)}{2^{1-v}} y^{-v+1/2} + \frac{\Gamma(-v)}{2^{1+v}} H(z) \cdot y^{v+1/2} + \cdots,\]  

(6.66)
where we have defined

\[H(z) := 1 + \frac{2 \sin(\pi v)}{\pi} \sum_{n=0}^{\infty} z^{-1-n/2} \int_0^{\infty} \left[V_n y^{n+1/2} K_\nu(y) + \sum_{l+m=n-2} V_l y^l \psi_m(y)\right] \sqrt{y'} K_\nu(y') \, dy'.\]  

(6.67)

It is important to note that \(H(z)\) admits a large-\(z\) asymptotic expansion in half-integer powers of \(z\).
Next, we find a relation between \( K(z) \) in equation (6.59) (defined in equation (6.51)) and \( H(z) \). To obtain an expression for \( K(z) \), we study the kernel of the resolvent \( G_\infty(x, x', z) \), which, for \( x < x' \), is given by (see equation (6.31)),

\[
G_\infty(x, x', z) = z^{-1/2} \frac{L(y, z)R(y', z)}{W[L, R](z)} \bigg|_{y=\sqrt{z}, y'=\sqrt{z'}}. \tag{6.68}
\]

The function \( L(y, z) \) is a solution to (6.61) whose leading term at the origin is proportional to \( y^{v+1/2} \). \( W[L, R](z) \) is the Wronskian of \( L(y, z) \) and \( R(y, z) \), which is independent of \( y \).

According to definition (6.55)

\[
G_\infty(x', z) = z^{-1/2} y^{-v-1/2} \frac{L(y, z)R(y', z)}{W[L, R](z)} \bigg|_{y=0, y'=\sqrt{z'}}. \tag{6.69}
\]

Replacing (6.66) into equation (6.69), we obtain the behavior of \( G_\infty(x, z) \) for \( x \to 0^+ \),

\[
G_\infty(x \leq 0, z) \approx - \frac{z^{-1/2} y^{-v-1/2} L(y, z)}{W[L, R](z)} \bigg|_{y=0} \times \left[ \frac{\Gamma(v)}{2^{-v} (\sqrt{z})^{v+1/2}} + \frac{\Gamma(-v)}{2^{1-v}} H(z) \cdot (\sqrt{z})^{v+1/2} \right] + \cdots. \tag{6.70}
\]

Comparing equations (6.50) and (6.70), we obtain a relation between \( K(z) \) and \( H(z) \),

\[
K(z) = 4^{v} \Gamma(1 + v) \Gamma(1 - v) z^{-v} H(z)^{-1}. \tag{6.71}
\]

Since \( H(z) \) admits an asymptotic expansion in half-integer powers of \( z \), the large-\( z \) asymptotic expansion of \( K(z) \) contains powers of \( z \) whose exponents depend on the parameter \( v \). Theorem 6.14 shows that these powers are also present in the large-\( z \) asymptotic expansion of the resolvent trace, which after equation (6.59) can be written as

\[
\text{Tr} \left( (A^0 + z)^{-1} - (A^\infty + z)^{-1} \right) = \frac{\text{Tr} \left( (A^0 + z)^{-1} - (A^\infty + z)^{-1} \right)}{1 + 4^{v} \frac{\Gamma(1 + v)}{\Gamma(1 - v)} z^{-v} H(z)^{-1}}. \tag{6.72}
\]

The trace on the RHS of equation (6.72) can be readily obtained from equation (6.53):

\[
\text{Tr} \left( (A^0 + z)^{-1} - (A^\infty + z)^{-1} \right) = 2v K(z) \int_0^\infty G_\infty^2(x, z) \, dx. \tag{6.73}
\]

To evaluate this expression, we compare equations (6.50) and (6.70) and obtain

\[
- \frac{z^{-1/2} y^{-v-1/2} L(y, z)}{W[L, R](z)} \bigg|_{y=0} = \frac{1}{2^{v} \Gamma(v)} \sqrt{z}^{v-1/2}. \tag{6.74}
\]

Therefore, equation (6.69) reads

\[
G_\infty(x', z) = \frac{1}{2^{v} \Gamma(v)} \sqrt{z}^{v-1/2} R(\sqrt{z}, z). \tag{6.75}
\]

Replacing this equation, together with (6.71), into expression (6.73), we obtain

\[
\text{Tr} \left( (A^0 + z)^{-1} - (A^\infty + z)^{-1} \right) = \frac{2H(z)^{-1} z^{-1/2}}{\Gamma(v) \Gamma(1 - v)} \int_0^\infty R(\sqrt{z}, z)^2 \, dx. \tag{6.76}
\]

This shows that \( \text{Tr} \left( (A^0 + z)^{-1} - (A^\infty + z)^{-1} \right) \) admits an asymptotic expansion in half-integer powers of \( z \). The following theorem summarizes equations (6.72) and (6.76) regarding the resolvent trace. We also state the corresponding result as regards the heat-kernel trace, which is the inverse Laplace transform of the resolvent trace.
The trace of the difference between the resolvents \((A^0 - \lambda)^{-1}\) and \((A^\infty - \lambda)^{-1}\) admits an asymptotic expansion for large \(|\lambda|\) given by

\[
\text{Tr}(A^0 - \lambda)^{-1} - (A^\infty - \lambda)^{-1} \sim \sum_{n=2}^{\infty} \alpha_n(v, V) \lambda^{-\frac{n}{2}} + \sum_{N,n=1}^{\infty} \beta_{N,n}(v, V) \theta^N \lambda^{-(2N+n/2)}. 
\]  

(6.77)

The coefficients \(\alpha_n(v, V)\) and \(\beta_n(v, V)\) depend on the parameter \(v\) characterizing the singularity and are also determined by the coefficients \(V_n\) characterizing the analytic potential \(V(x)\) by means of equations (6.72), (6.76), (6.64), (6.65) and (6.67) with \(z = e^{i\pi} \lambda\).

The trace of the difference \(e^{-tA^0} - e^{-tA^\infty}\) admits a small-\(t\) asymptotic expansion given by

\[
\text{Tr}(e^{-tA^0} - e^{-tA^\infty}) \sim \sum_{n=2}^{\infty} \frac{\alpha_n(v, V)}{\Gamma(n/2)^2} t^{\frac{n}{2}-1} + \sum_{N,n=1}^{\infty} \frac{\beta_{N,n}(v, V)}{\Gamma(2N+n/2)} \theta^N t^{N+n/2}. 
\]  

(6.78)

Let us give a dimensional analysis argument to explain the non-standard powers of \(t\) in the expansion (6.78). First of all, note that the parameter \(\theta\) introduced by the boundary conditions has dimensions [length] \(^{-2n}\) and, after the analyticity of \(V(x)\), the dimension of every other parameter in the problem is an integer power of the length. Since \(t\) has dimensions [length]\(^2\), if the coefficients of the asymptotic expansion of the heat-trace were to depend analytically on \(\theta\), then it would be necessary that this expansion contains integer powers of \(t^n\). The only SAE for which these powers are to be absent are those with \(\theta = 0\) and \(\theta = \infty\).

**Example.** Let us consider \(V(x) = x^2\). We will use expansion (6.78) to describe the pole structure of the difference between the \(\zeta\)-functions \(\zeta^0_\kappa(s) - \zeta^\infty_\kappa(s)\) corresponding to operator (6.17), which will confirm the results obtained in section 5 with two other techniques.

First of all, note that for \(V(x) = x^2\) only one of the coefficients \(V_n\) defined in (6.63) is non-vanishing, namely

\[
V_n = \delta_{n,2}. 
\]  

(6.79)

As a consequence, the only functions \(\psi_n(y)\) which are non-trivial correspond to \(n = 2 + 4k\) with \(k = 0, 1, \ldots\) (see equation (6.65)). According to equations (6.67) and (6.64),

\[
H(z)^{-1} \sim 1 + \sum_{k=1}^{\infty} C_k(v) z^{-2k}, 
\]  

(6.80)

\[
R(v, y) \sim \sqrt{2} K_v(y) + \sum_{k=1}^{\infty} C'_k(v, y) z^{-2k}, 
\]  

(6.81)

for some \(C_k(v)\) and \(C'_k(v, y)\). Substituting these equations into (6.76), we obtain

\[
\text{Tr}(A^0 + z)^{-1} - (A^\infty + z)^{-1} \sim vz^{-1} + \sum_{k=0}^{\infty} C'_k(v) z^{-3-2k}, 
\]  

(6.82)

where \(C'_k(v)\) can be written in terms of \(C_k(v)\) and \(C'_k(v, y)\).

Replacing equations (6.82) and (6.80) into (6.72), we obtain the asymptotic expansion of the trace of the difference between the resolvents \((A^0 + z)^{-1}\) and \((A^\infty + z)^{-1}\),

\[
\text{Tr}(A^0 + z)^{-1} - (A^\infty + z)^{-1} \sim \left[ vz^{-1} + \sum_{k=0}^{\infty} C'_k(v) z^{-3-2k} \right] 
\times \sum_{N=0}^{\infty} (-1)^N 4^N \left[ \frac{\Gamma(1+v)}{\Gamma(1-v)} \right]^N \theta^N z^{-N-1} \left[ 1 + \sum_{k=1}^{\infty} C_k(v) z^{-2k} \right]^N. 
\]  

(6.83)
It is straightforward to see that the first series in (6.83) gives no contribution to the pole structure of the difference $\zeta_\theta^A(s) - \zeta_\infty^A(s)$. On the other hand, from the second series in the asymptotic expansion (6.83), one shows that $\zeta_\theta^A(s) - \zeta_\infty^A(s)$ has simple poles which are located at

$$s_{N,n} = -N\nu - 2n \quad \text{with } N = 1, 2, \ldots \text{ and } n = 0, 1, \ldots$$

(6.84)

This result coincides with equation (5.27). In particular, the leading term in (6.83) leads to a simple pole at

$$s_{1,0} = -\nu,$$

(6.85)

whose residue is given by

$$\frac{4\nu}{\Gamma^2(-\nu)}\theta,$$

(6.84)

in agreement with the result quoted in equation (5.28).

7. Conclusions

We have studied some symmetric non-essentially self-adjoint first- and second-order differential operators with a singular potential term with the same scaling dimension as the highest derivative term, a characteristic which we have mentioned as local scale invariance (at the singular point in the potential). For a certain range of the ‘external parameters’ that weight the singular term (i.e. the coupling constants in the potential – $g$ or $\nu$ throughout the paper) these operators admit a continuous family of self-adjoint extensions. Each self-adjoint extension describes a different physical system, with its own spectrum determined by the behavior of the functions belonging to its domain near the singularity. This is an essential point in our discussion for it implies the existence of infinitely many physically admissible boundary conditions at the singularity, identified in our examples by an additional dimensionful parameter (which is not present in the expression of the differential operator).

This situation takes place when the adjoint of the differential operators considered above admits two different types of square integrable behavior near the singularity for the functions in its domain of definition. Then, a symmetric extension will contain, in general, both possible types of behavior, and then it must incorporate an additional dimensionful parameter necessary to specify its domain. The only exceptions are those combinations which make the domain (locally) scale invariant.

Therefore, the spectral functions associated with a self-adjoint extension of these operators will, in general, depend on this additional dimensionful parameter (with a scaling dimension which depends on the external parameters in the singular potential term), opening the possibility of having, for example, non-standard powers of $t$ in the asymptotic expansion of the heat-kernel trace and, consequently, non-standard poles in the associated $\zeta$-function. By non-standard we mean that these powers and poles are not determined by the dimension of the base manifold and the order of the differential operator only (as in the smooth coefficients case) but also depend on the external parameters that characterize the singularity.

Our results show that those self-adjoint extensions which break in this way the local scale invariance of the dominant scaling dimension terms in the differential operator present these non-standard poles in the associated $\zeta$-function as well as non-standard powers of $\lambda$ in the large-$|\lambda|$ asymptotic expansion of the resolvent. As a consequence, for second-order differential operators one finds non-standard powers of $t$ in the small-$t$ asymptotic expansion of the heat-kernel trace. Moreover, the Seeley–De Witt coefficients and the residues of the $\zeta$- and $\eta$-functions depend on the self-adjoint extension.

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In fact, also the large-$n$ asymptotic behavior of the eigenvalues \( \lambda_n \) of the self-adjoint extensions contains powers of \( n \) which depend on these external parameters. One also expects non-standard singularities in the corresponding Green functions at coincident points. This issue is relevant for the definition of physical states and the regularization of the stress tensor, for example.

This non-standard behavior of the spectral functions has been explicitly shown by solving some examples of first- and second-order non-essentially self-adjoint differential operator on both compact and non-compact one-dimensional base manifold. We have also proved that this phenomenon is not affected by the introduction of arbitrary smooth potentials.

Let us finally mention that, in establishing these results, we have derived an extension of Krein’s formula which applies to this kind of differential operator defined on functions which, generically, do not have regular behavior at the singularity. This formula (equations (6.56) and (6.58)) relates the resolvents of different self-adjoint extensions and directly leads to the non-standard small-$t$ asymptotic expansion of the heat-kernel trace we have found.

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Appendix A. Spectral functions and their relations

Given an elliptic differential operator \( A \) in a manifold \( M \) with a complete orthogonal set of eigenfunctions corresponding to the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}} \), the associated \( \xi \)-function is defined as [9, 10]

\[
\xi_A(s) = \text{Tr} A^{-s} = \sum_{n \in \mathbb{N}} \lambda_n^{-s},
\]

which is a convergent series for \( \Re(s) \) large enough.

If \( |\lambda_n| \to \infty \) fast enough when \( n \to \infty \), then there also exists the trace of the resolvent, given by

\[
\text{Tr}(A - \lambda)^{-1} = \sum_{n \in \mathbb{N}} \frac{1}{\lambda_n - \lambda}.
\]

Moreover, if the set \( \{\Re(\lambda_n)\} \) is bounded below and \( \Re(\lambda_n) \to \infty \) for \( n \to \infty \), then the trace of the associated heat-kernel is given by

\[
\text{Tr} e^{-tA} = \sum_{n \in \mathbb{N}} e^{-\lambda_n t}.
\]

For positive definite operators, we have for the Laplace transform

\[
\text{Tr}(A - \lambda)^{-1} = \int_0^\infty e^{\lambda t} \text{Tr} e^{-tA} \, dt,
\]

for \( \Re(\lambda) < \lambda_n, \forall n \), and for the Mellin transform

\[
\text{Tr} A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} e^{-tA} \, dt,
\]

for \( \Re(s) > m/d \), where \( m \) is the dimension of the manifold and \( d \) is the order of the differential operator.
The $\zeta$-function singularities are related to the asymptotic expansion of $\text{Tr} \left( A - \lambda \right)^{-1}$ for large $|\lambda|$ and with the asymptotic expansion of $\text{Tr} e^{-tA}$ for small values of $t$. Indeed, if $\zeta_A(s)$ has simple poles at $s = s_n \leq s_0$, for $n \in \mathbb{N}$, then $\text{Tr} \left( A - \lambda \right)^{-1}$ has an asymptotic expansion in powers of the form $\lambda^{s_n - 1}$, while $\text{Tr} e^{-tA}$ admits an asymptotic expansion in powers of the form $t^{-s_n}$. In particular, under the hypothesis considered in [9], these powers depend only on $d$ and $m$. Moreover, the coefficients of both expansions are determined by the residues of $\zeta_A(s)$ at the corresponding poles.

For example, if

$$\text{Tr}[e^{-tA}] \sim \sum_{n=0}^{\infty} c_n(A) t^{-s_n},$$

with $s_n \leq s_0$, then for $\Re(s) > s_0$, we have

$$\Gamma(s)\zeta_A(s) = \sum_{n=0}^{n_0} c_n(A) \int_0^1 t^{s-1-s_n} \, dt + \int_0^1 t^{s-1} \left( \text{Tr}[e^{-tA}] - \sum_{n=n_0}^{n_0} c_n(A) t^{-s_n} \right) \, dt$$

$$+ \int_1^{\infty} t^{s-1} \text{Tr}[e^{-tA}] \, dt = \sum_{n=n_0}^{n_0} c_n(A) \left( \frac{1}{s-s_n} \right) + h(s),$$

where $h(s)$ is analytic on the open half-plane $\Re(s) > s_0$. Therefore, the residue of $\Gamma(s)\zeta_A(s)$ at $s_n$ is related to the coefficient $c_n(A)$ through the equality

$$\text{Res} \left[ \Gamma(s)\zeta_A(s) \right]_{s=s_n} = c_n(A).$$

Since $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$, the residues $\text{Res} \left[ \zeta_A(s) \right]_{s=s_n}$ vanish when $s_n = 0, -1, -2, \ldots$. In particular, $\zeta_A(s)$ is analytic in a neighborhood of the origin.

Even for nonpositive elliptic differential operators $A$, the complex $s$-power of $A$ is defined in terms of the resolvent as [9]

$$A^{-s} = -\frac{1}{2\pi i} \oint_C \lambda^{-s}(A - \lambda)^{-1} \, d\lambda,$$

where $C$ is a curve enclosing anti-clockwise the eigenvalues of $A$. From this, one obtains

$$\text{Tr} A^{-s} = -\frac{1}{2\pi i} \oint_C \lambda^{-s} \text{Tr} (A - \lambda)^{-1} \, d\lambda.$$  

(A.10)

The resolvent $(A - \lambda)^{-1}$, the complex power $A^{-s}$ and the heat-kernel $e^{-tA}$ can be considered as integral operators characterized by its kernels, $G(x,x',\lambda)$, $\zeta_A(x,x',s)$ and $K(x,x',t)$, defined for $\lambda \in \mathbb{C} \setminus \{\lambda_n\}_{n \in \mathbb{N}}$, $R(s)$ sufficiently large and $t > 0$, respectively. In this case, their traces are expressed as

$$\text{Tr} (A - \lambda)^{-1} = \int_M G(x,x,\lambda) \, dx,$$

(A.11)

$$\zeta_A(s) := \text{Tr} A^{-s} = \int_M \zeta_A(x,x,s) \, dx$$

(A.12)

and

$$\text{Tr} e^{-tA} = \int_M K(x,x,t) \, dx.$$  

(A.13)
Appendix B. The Hankel expansion

In this appendix, we write down some of the asymptotic expansions of the Bessel functions that are used in section 2.7 [63]. For \(|z| \to \infty\), with \(v\) fixed and \(|\arg z| < \pi\), we have

\[
J_v(z) \sim \left( \frac{2}{\pi z} \right)^{1/4} \left[ P(v, z) \cos \chi(v, z) - Q(v, z) \sin \chi(v, z) \right],
\]

(B.1)

where

\[
\chi(v, z) = z - \left( \frac{v}{2} + \frac{1}{4} \right) \pi,
\]

(B.2)

\[
P(v, z) \sim \sum_{k=0}^{\infty} (-1)^k (v, 2k) \frac{1}{(2z)^{2k}},
\]

(B.3)

\[
Q(v, z) \sim \sum_{k=0}^{\infty} (-1)^k (v, 2k + 1) \frac{1}{(2z)^{2k+1}},
\]

(B.4)

and the coefficients

\[
(v, k) = \frac{\Gamma \left( \frac{1}{4} + v + k \right)}{k! \Gamma \left( \frac{1}{4} + v - k \right)} = (-v, k)
\]

(B.5)

are the Hankel symbols. Therefore, \(P(-v, z) = P(v, z), Q(-v, z) = Q(v, z)\) and

\[
J_v(z) \sim \frac{e^{i\pi z} e^{i\pi (\frac{1}{4} + i)} \sum_{k=0}^{\infty} (v, k) \left( \frac{\pi}{2z} \right)^k}{\sqrt{2\pi z}},
\]

(B.6)

where the upper (lower) signs correspond to \(z\) in the upper (lower) open half-plane. In particular, the quotient

\[
\frac{J_{\frac{1}{2} - v}(\lambda)}{J_{\frac{1}{2} - (-v)}(\lambda)} \sim e^{i\pi (\frac{1}{4} - s)},
\]

(B.7)

for \(\Re(\lambda) > 0\) and \(\Re(\lambda) < 0\), respectively.

Similarly, the derivative of the Bessel function has the following asymptotic expansion for \(|\arg z| < \pi\):

\[
J_v'(z) \sim -\frac{2}{\sqrt{2\pi z}} \left[ R(v, z) \sin \chi(v, z) + S(v, z) \cos \chi(v, z) \right],
\]

(B.8)

where

\[
R(v, z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{v^2 + (2k)^2 - 1/4 (v, 2k)}{v^2 - (2k - 1/2)^2 (2z)^{2k}},
\]

(B.9)

\[
S(v, z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{v^2 + (2k + 1)^2 - 1/4 (v, 2k + 1)}{v^2 - (2k + 1 - 1/2)^2 (2z)^{2k+1}}.
\]

(B.10)

Then,

\[
J_v'(z) \sim \mp i \frac{e^{i\pi z} e^{i\pi (\frac{1}{4} + i)} \sum_{k=0}^{\infty} (v, k) \left( \frac{\pi}{2z} \right)^k}{\sqrt{2\pi z}} \left[ R(v, z) \mp i S(v, z) \right],
\]

(B.11)

where the upper sign is valid for \(\Re(z) > 0\) and the lower sign for \(\Re(z) < 0\). From the following relation

\[
R(v, z) \pm i S(v, z) = P(v, z) \pm i Q(v, z) + T_{\pm}(v, z),
\]

(B.12)
with
\[ T_k(v, z) \sim \sum_{k=1}^{\infty} (2k - 1) (v, k - 1) \left( \frac{\pm i}{2z} \right)^k, \]  
we obtain
\[ \frac{J'_a(z)}{J_a(z)} \sim \mp i \left\{ 1 \pm \frac{T_\mp(v, z)}{P(v, z) \mp i Q(v, z)} \right\} \sim \mp i \left\{ 1 \mp \frac{i}{2z} + O \left( \frac{1}{z^2} \right) \right\}. \]  
where the upper sign is valid for \( \Im(z) > 0 \) and the lower one for \( \Im(z) < 0 \). Finally, since the Hankel symbols are even in \( v \) (see equation (B.5)), we have
\[ \frac{J'_a(z)}{J_a(z)} \sim \frac{J'_o(z)}{J_o(z)}. \]  

Appendix C. Closure of \( H \)

In section 5, we omit the contributions to the boundary condition in equation (5.12) of the functions in the domain of the closure of the operator \( H \) defined in equation (5.1); in this appendix, we will justify this procedure. Indeed, we will show that if \( \phi \in D(\Gamma) \), then
\[ \phi(x) = o(x^\alpha) \quad \text{and} \quad \phi'(x) = o(x^{\alpha-1}) \]  
near the origin, for any \( \alpha := v + \frac{1}{2} < \frac{3}{2} \).

In order to determine the domain \( D(\Gamma) \) of the closure of \( H \) we must consider those Cauchy sequences \( \{\phi_n\}_{n \in \mathbb{N}} \) in \( D(H) = C_0^\infty(\mathbb{R}^+) \), such that \( \{H\phi_n\}_{n \in \mathbb{N}} \) are also Cauchy sequences. Since the coefficients of \( H \) are real, it will suffice to consider real functions. Throughout this section \( \varphi := \varphi_n - \varphi_m \) (with \( n, m \in \mathbb{N} \)) so that \( \varphi \to 0 \) and \( H\varphi \to 0 \) as \( n, m \to \infty \).

In the following, write \( g := \sqrt{v^2 - 1} \). Note first that the scalar product
\[ \langle \varphi, H\varphi \rangle = \int_0^\infty \left( \varphi^2 + \frac{g}{x^2} \varphi^2 + x^2 \varphi^2 \right) \ dx \leq ||\varphi|| \ ||H\varphi|| \to 0 \]  
as \( n, m \to \infty \); \( || \cdot || \) represents the usual norm in \( L_2(\mathbb{R}^+) \). Therefore, for \( g > 0 \), we conclude that
\[ \{\varphi_n(x)\}_{n \in \mathbb{N}}, \quad \{\varphi_n(x) x^{-\alpha}\}_{n \in \mathbb{N}} \quad \text{and} \quad \{x \varphi_n(x)\}_{n \in \mathbb{N}} \]  
are also Cauchy sequences. We will now prove the following.

**Lemma C.1.** Let \( \{\psi_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence in \( D(H) = C_0^\infty(\mathbb{R}^+) \) such that, for \( g > 0 \), \( 1 \leq a < 2 \) and \( g \neq (a^2 - 1)/4 \),
\[ \{H\psi_n\}_{n \in \mathbb{N}} : \quad \left\{ \begin{array}{l} \frac{\varphi_n(x)}{x^a} \quad \text{and} \quad \frac{\varphi'_n(x)}{x^{a-1}} \end{array} \right\}_{n \in \mathbb{N}} \]  
are also Cauchy sequences. Then,
\[ \left\{ \begin{array}{l} \frac{\varphi_n(x)}{x^{1+a/2}} \quad \text{and} \quad \frac{\varphi'_n(x)}{x^{a/2}} \end{array} \right\}_{n \in \mathbb{N}} \]  
are Cauchy sequences, too.

**Proof.** Taking into account that the sum of fundamental sequences is also a Cauchy sequence, we see that
\[ \left( A \frac{\varphi(x)}{x^a} + B \frac{\varphi'(x)}{x^{a-1}}, H\varphi(x) \right) \to 0 \]  
(C.6)
as \( n, m \to \infty \), for any pair of real numbers \( A \) and \( B \). It is easily seen that appropriately choosing the coefficients \( A, B \) (whenever \( g = (a^2 - 1)/4 \)) equation (C.6) proves the lemma.

Let us now assume that \( g \) is an irrational number. Then, applying iteratively lemma C.1 to the sequences (C.3), one can show that, for any positive integer \( k \),

\[
\left\{ \frac{\varphi_n(x)}{x^{\left[1/(2^k)\right]}} \right\}_{n \in \mathbb{N}} \quad \text{and} \quad \left\{ \frac{\varphi'_n(x)}{x^{\left[1/(2^k)\right]-1}} \right\}_{n \in \mathbb{N}} \tag{C.7}
\]

are Cauchy sequences, too. One immediately concludes that \( \{x^{-1+\varepsilon}\varphi_n(x)\}_{n \in \mathbb{N}} \) and \( \{x^{\varepsilon}\varphi'_n(x)\}_{n \in \mathbb{N}} \) are also Cauchy sequences. If \( g \) were a rational number, one could choose an irrational \( a \in (1, 3/2) \) from which lemma C.1 could also be iteratively applied to arrive at the same conclusions.

In the following, we will consider the behavior of the functions near the origin. For any \( \varepsilon > 0 \), we can write

\[
x^{-\alpha} \varphi(x) = \int_0^x (y^{-\alpha} \varphi(y))^\varepsilon \ dy = \int_0^x y^{-\alpha+1-\varepsilon} \left\{-\alpha \frac{\varphi(y)}{y^{2-\varepsilon}} + \varphi'(y) \frac{1}{y^{1-\varepsilon}} \right\} \ dy \tag{C.8}
\]

Therefore, for \( x \leq 1 \), \( \alpha < 3/2 \) and \( \varepsilon \) small enough, we have

\[
|x^{-\alpha} \varphi(x)| \leq \left( \int_0^1 y^{2-(\alpha+1-\varepsilon)} \ dy \right)^{1/2} \left\{ |\alpha| \left\| \frac{\varphi(y)}{y^{2-\varepsilon}} \right\| + \left\| \frac{\varphi'(y)}{y^{1-\varepsilon}} \right\| \right\} \to 0, \tag{C.9}
\]

as \( n, m \to \infty \). We conclude that the sequence \( \{x^{-\alpha}\varphi_n(x)\}_{n \in \mathbb{N}} \), for \( \alpha < 3/2 \), is uniformly convergent in \([0, 1]\) and its limit is a continuous function vanishing at the origin, which we write as \( x^{-\alpha} \phi(x) \):

\[
\lim_{n \to \infty} (x^{-\alpha}\varphi_n(x)) = x^{-\alpha} \phi(x), \tag{C.10}
\]

\[
\lim_{n \to \infty} (x^{-\alpha}\phi(x)) = 0. \tag{C.11}
\]

In particular, for \( \alpha = 0 \) we have the uniform limit

\[
\lim_{n \to \infty} \varphi_n(x) = \phi(x), \tag{C.12}
\]

which coincides with the limit of this sequence in \( L_2(\mathbb{R}^+) \).

Similarly, we can write

\[
x^{-\alpha+1} \varphi'(x) = -\int_0^x y^{-\alpha+1} H\varphi(y) \ dy + \int_0^x y^{-\alpha+1-\varepsilon} \left\{ (-\alpha + 1) \frac{\varphi(y)}{y^{2-\varepsilon}} + g \frac{\varphi'(y)}{y^{1-\varepsilon}} \right\} \ dy + \int_0^x y^{-\alpha+2} \varphi(y) \ dy. \tag{C.13}
\]

Therefore, for \( x \leq 1 \), \( \alpha < 3/2 \) and \( \varepsilon \) sufficiently small, we have

\[
|x^{-\alpha+1} \varphi'(x)| \leq \left( \int_0^1 y^{2-(\alpha+1)} \ dy \right)^{1/2} \left\| H\varphi(y) \right\| + \left( \int_0^1 y^{2-(\alpha+1-\varepsilon)} \ dy \right)^{1/2} \left\{ |\alpha-1| \left\| \frac{\varphi'(y)}{y^{1-\varepsilon}} \right\| + g \left\| \frac{\varphi(y)}{x^{2-\varepsilon}} \right\| \right\} + \left( \int_0^1 y^{2-(\alpha+2)} \ dy \right)^{1/2} \left\| \varphi(y) \right\| \to 0, \tag{C.14}
\]

as \( n, m \to \infty \). Consequently, the sequence \( \{x^{-\alpha+1}\varphi'_n(x)\}_{n \in \mathbb{N}} \), with \( \alpha < 3/2 \), is uniformly convergent in \([0, 1]\) and its limit is a continuous function vanishing at the origin, which we write as \( x^{-\alpha+1} \chi(x) \):

\[
\lim_{n \to \infty} (x^{-\alpha+1}\varphi'_n(x)) = x^{-\alpha+1} \chi(x), \tag{C.15}
\]
\[ \lim_{x \to 0^+} (x^{-\alpha+1} \chi(x)) = 0. \]  

(C.16)

In particular, for \( \alpha = 1 \), we have the uniform limit

\[ \lim_{n \to \infty} \psi_n(x) = \chi(x), \]

(C.17)

which coincides with the limit of this sequence in \( L^2(\mathbb{R}^+) \). Let us now show that \( \chi(x) = \phi'(x) \). Indeed, for \( x \leq 1 \), we have

\[
\begin{align*}
|\phi(x) - \int_0^x \chi(y) \, dy| & \leq |\phi(x) - \psi_n(x)| + \left| \int_0^x \left( \chi(y) - \psi_n(y) \right) \, dy \right| \\
& \leq |\phi(x) - \psi_n(x)| + \| \chi - \psi_n \| \to 0, \quad (C.18)
\end{align*}
\]

as \( n \to \infty \). Then, \( \phi(x) \) is a differentiable function whose first derivative is \( \chi(x) \). On the other hand, equations (C.11) and (C.16) imply that any \( \phi \in \mathcal{D}(\mathcal{H}) \) satisfies equation (C.1).

References

[1] Dowker J S and Critchley R 1976 Effective Lagrangian and energy momentum tensor in de Sitter space Phys. Rev. D 13 3224
[2] Elizalde E 1994 Zeta Regularization Techniques with Applications (Singapore: World Scientific)
[3] Elizalde E 1995 Ten physical applications of spectral zeta functions Lect. Notes Phys. M 351
[4] Bytsenko A A, Cognola G, Vanzo L and Zerbini S 1996 Quantum fields and extended objects in space-times with constant curvature spatial section Phys. Rep. 266 1
[5] Kirsten K 2001 Spectral Functions in Mathematics and Physics (Boca Raton, FL: CRC Press)
[6] Bordag M, Mohideen U and Mostepanenko V M 2001 New developments in the Casimir effect Phys. Rep. 353 1
[7] Fursaev D and Vassilevich D 2011 Operators, Geometry and Quanta: Methods of Spectral Geometry in Quantum Field Theory (Theoretical and Mathematical Physics) (Berlin: Springer)
[8] Vassilevich D V 2003 Heat kernel expansion: user’s manual Phys. Rep. 388 279
[9] Seeley R T 1967 Complex powers of an elliptic operator Proc. Symp. Pure Math. 10 288
[10] Seeley R T 1969 The resolvent of an elliptic boundary problem Am. J. Math. 91 889
[11] Seeley R T 1969 Analytic extension of the trace associated with elliptic boundary problems Am. J. Math. 91 963
[12] Gilkey P B 1994 Invariance Theory, the Heateqution and the Atiyah–Singer Index Theorem (Studies in Advanced Mathematics vol 16) (Boca Raton, FL: CRC Press)
[13] Dowker J S 1977 Quantum field theory on a cone J. Phys. A: Math. Gen. 10 115
[14] Dowker J S 1978 Thermal properties of Green’s functions in Rindler, de Sitter, and Schwarzschild spaces Phys. Rev. D 18 1856
[15] Dowker J S 1987 Casimir effect around a cone Phys. Rev. D 36 3095
[16] Dowker J S 1989 Heat kernel expansion on a generalized cone J. Math. Phys. 30 770
[17] Dowker J S 1990 Quantum field theory around conical defects The Formation and Evolution of Cosmic Strings (Cambridge: Cambridge University Press) p 251
[18] Dowker J S 1994 Heat kernels on curved cones Class. Quantum Grav. 11 L137
[19] Bordag M, Kirsten K and Dowker J S 1996 Heat kernels and functional determinants on the generalized cone Commun. Math. Phys. 182 371
[20] Dowker J S and Kirsten K 2001 Smearred heat kernel coefficients on the ball and generalized cone J. Math. Phys. 42 434
[21] Calogero F 1969 Solution of a three-body problem in one-dimension J. Math. Phys. 10 2191
[22] Calogero F 1969 Ground state of one-dimensional N body system J. Math. Phys. 10 2197
[23] Calogero F 1971 Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials J. Math. Phys. 12 419
[24] Cheeger J 1983 Spectral geometry of singular Riemannian spaces J. Differ. Geom. 18 575
[25] Brüning J and Seeley R 1987 The resolvent expansion for second order regular singular operators J. Funct. Anal. 73 360
[26] Larsen F and Wilczek F 1996 Renormalization of black hole entropy and of the gravitational coupling constant Nucl. Phys. B 458 249
[27] Fursaev D V 1994 The heat kernel expansion on a cone and quantum fields near cosmic strings Class. Quantum Grav. 11 1431
Callias C J 1983 The heat equation with singular coefficients: I. Operators of the form $-\partial^2/\partial x^2 + k/\partial x^2$ in dimension 1 Commun. Math. Phys. 88 357

Callias C J and Uhlmann G A 1984 Singular asymptotics approach to partial differential equations with isolated singularities in the coefficients Bull. Am. Math. Soc. 11 172

Callias C J 1988 The resolvent and the heat kernel for some singular boundary problems Commun. Partial Differ. Equas. 13 1113

Mooers E A 1999 Heat kernel asymptotics on manifolds with conic singularities J. Anal. Math. 78 1

Falorni H, Pisani P A G and Wipf A 2002 Pole structure of the Hamiltonian $\zeta$-function for a singular potential J. Phys. A.: Math. Gen. 35 5427

Falorni H, Muschietti M A, Pisani P A G and Seeley R 2003 Unusual poles of the $\zeta$-functions for some regular singular differential operators J. Phys. A: Math. Gen. 36 9991

Falorni H, Muschietti M A and Pisani P A G 2004 On the resolvent and spectral functions of a second order differential operator with a regular singularity J. Math. Phys. 45 4560

Falorni H and Pisani P A G 2005 Self-adjoint extensions and SUSY breaking in supersymmetric quantum mechanics J. Phys. A: Math. Gen. 38 4665

Falorni H and Pisani P A G 2006 Krein’s formula and heat-kernel expansion for some differential operators with a regular singularity J. Phys. A: Math. Gen. 39 6333

Kirsten K, Loya P and Park J 2006 The very unusual properties of the resolvent, heat kernel, and zeta function for the operator $-\partial^2/\partial r^2 - 1/(4r^2)$ J. Math. Phys. 47 043506

Kirsten, K, Loya P and Park J 2008 Exotic expansions and pathological properties of $\zeta$-functions on conic manifolds J. Geom. Anal. 18 835

Kirsten K, Loya P and Park J 2008 Functional determinants for general self-adjoint extensions of Laplace-type operators resulting from the generalized cone Manuscr. Math. 125 95

Kirsten K, Loya P and Park J 2008 The ubiquitous $\zeta$-function and some of its ‘usual’ and ‘unusual’ meromorphic properties J. Phys. A: Math. Theor. 41 164070

Kirsten K and Loya P 2010 Spectral functions for the Schrödinger operator on $\mathbb{R}^+$ with a singular potential J. Math. Phys. 51 053512

Reed M and Simon B 1980 Methods of Modern Mathematical Physics vol I and II (New York: Academic)

Oshanskyi M A and Perelomov A M 1981 Classical integrable finite dimensional systems related to Lie algebras Phys. Rep. 71 313

Oshanskyi M A and Perelomov A M 1983 Quantum integrable systems related to Lie algebras Phys. Rep. 94 313

Peak D and Inomata A 1969 Summation over Feynman histories in polar coordinates J. Math. Phys. 10 1422

Kleinert H 1995 Path Integrals in Quantum Mechanics, Statistics and Polymer Physics 2nd edn (Singapore: World Scientific)

Fischer W, Leschke H and Muller P 1992 Changing dimension and time: two well-founded and practical techniques for path integration in quantum physics J. Phys. A: Math. Gen. 25 3835

Akhiezer N I and Glazman I M 1993 Theory of Linear Operators in Hilbert Space (New York: Dover)

Albeverio S and Kurasov P 2000 Singular Perturbations of Differential Operators: Solvable Schrödinger Type Operators (London Mathematical Society Lecture Note vol 271) (Cambridge: Cambridge University Press)

Simon B 1973 Quadratic forms and Klauder’s phenomenon: a remark on very singular perturbations J. Funct. Anal. 14 295

Rellich F 1943/44 Die zulässigen Randbedingungen bei den singulären Eigenwertproblemen der mathematischen Physik Math. Z. 49 702 (in German)
[54] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics (Texts and Monographs in Physics)* (Berlin: Springer)

[55] Krein M G 1944 On Hermitian operators with deficiency indices one *Dokl. Akad. Nauk SSSR* 43 339 (in Russian)

Krein M G 1946 Resolvents of Hermitian operators with defect index \((m, m)\) *Dokl. Akad. Nauk SSSR* 52 657 (in Russian)

[56] Gil J, Krainer T and Mendoza G 2011 Dynamics on Grassmannians and resolvents of cone operators *Anal. PDE* 4 115

[57] Gil J, Krainer T and Mendoza G 2010 Trace expansions for elliptic cone operators with stationary domains *Trans. Amer. Math. Soc.* 362 6495

[58] Mendoza G, Gil J and Krainer T (ed) 2010 The resolvent trace of an elliptic cone operator *Analysis and Geometric Singularities (Oberwolfach Reports)* vol 28 ed J Brüning, R Mazzeo and P Piazza pp 1672–5

[59] Gil J, Krainer T and Mendoza G 2007 On rays of minimal growth for elliptic cone operators *Oper. Theory Adv. Appl.* 172 33

[60] Krainer T 2007 Resolvents of elliptic boundary problems on conic manifolds *Comm. Partial Differential Equations* 32 257

[61] Gil J, Krainer T and Mendoza G 2007 Geometry and spectra of closed extensions of elliptic cone operators *Canad. J. Math.* 59 742

[62] Gil J, Krainer T and Mendoza G 2006 Resolvents of elliptic cone operators *Journal of Functional Analysis* 241 1

[63] Abramowitz M and Stegun I (ed) 1970 *Handbook of Mathematical Functions* (New York: Dover)

[64] Atiyah M F, Patodi V K and Singer I M 1975 Spectral asymmetry and Riemannian geometry I *Math. Proc. Camb. Phil. Soc.* 77 43

[65] Albeverio S and Pankrashkin K 2005 A remark on Krein’s resolvent formula and boundary conditions *J. Phys. A: Math. Gen.* 38 4859

[66] Derkach V A and Malamud M M 1991 Generalized resolvents and the boundary value problems for Hermitian operators with gaps *J. Funct. Anal.* 95 1