On the notion of flat 2-functors

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Abstract

In this paper we develop the 2-dimensional theory of flat functors. We define a 2-functor \( A \to \text{Cat} \) to be flat when its left bi-Kan extension \( \text{Hom}_p(A^{op}, \text{Cat}) \to \text{Cat} \) along the Yoneda 2-functor \( A \to \text{Hom}_p(A^{op}, \text{Cat}) \) is left exact. \( \text{Hom}_p(A^{op}, \text{Cat}) \) denotes the 2-category of 2-functors, pseudonatural transformations and modifications. By left bi-Kan extension we understand the bi-universal pseudonatural transformation \( P \Rightarrow P^*h \), and by left exact we understand preservation of finite weighted bilimits. Let \((A, \Sigma)\) be a pair where \( A \) is a 2-category and \( \Sigma \) is a distinguished 1-subcategory. A \( \sigma \)-cone for a 2-functor \( A \to B \) is a lax cone such that the 2-cells corresponding to the distinguished arrows are invertible. The \( \sigma \)-limit of \( F \) is a universal \( \sigma \)-cone (characterized up to isomorphism). We introduce a notion of \( \sigma \)-filteredness of \( A \) with respect to \( \Sigma \), which we call \( \sigma \)-filtered. Our main result states the following:

A 2-functor \( A \to \text{Cat} \) is flat if and only if there is a \( \sigma \)-filtered pair \((I^{op}, \Sigma)\), a 2-diagram \( I \to A \), and \( P \) is pseudo-equivalent to the \( \sigma \)-colimit of the composition \( I^{op} \to A^{op} \to \text{Hom}_p(A, \text{Cat}) \).

Our result establishes that the 2-category of points of a “2-topos” of 2-presheaves on a (small) 2-category \( A \) is equivalent to the 2-category of \( \sigma \)-cofiltered 2-diagrams in \( A \). This fact should be essential for the development of the theory of 2-topoi, as the corresponding result for \( \text{Set} \)-valued functors is for topos theory. As Joyal pointed to us, a 2-topos could be defined as a left exact 2-localization of a 2-category of 2-presheaves.

Introduction

Size issues are in general not relevant to us in this paper, but we indicate the smallness assumption when it applies. The paper is concerned with the notion of flat functor in the context of 2-categories. Given a 2-functor \( A \to \text{Cat} \) from a 2-category \( A \) with values in the 2-category \( \text{Cat} \) of small 1-categories, we want to define when it should be considered flat, and prove a theorem that characterizes flatness using appropriate notions of filteredness and pro-representability. Recall that a \( \text{Set} \)-valued functor is flat when its left Kan extension along the Yoneda embedding is left exact (this being equivalent to its discrete cofibration being a cofiltered category). This notion is considered in [13, §6] for \( \mathcal{V} \)-enriched categories in general, and in particular for \( \mathcal{V} = \text{Cat} \).

We emphasize that 2-dimensional category theory is radically different that the theory of \( \text{Cat} \)-enriched categories, which, as well as the theory of \( \mathcal{V} \)-enriched categories for any \( \mathcal{V} \), is a part of 1-dimensional category theory.

As it is usually the case, the \( \text{Cat} \)-enriched version of flatness is too strict, and a relaxed notion is the important one. This is easily settled, but more difficult and unsolved so far is the fundamental equivalence between flatness and appropriate notions of filteredness for 2-categories and pro-representability of 2-functors, which is the problem we solve in this paper.

Though our original objective was to have results for 2-functors, the notion of pseudofunctor imposed upon us as the correct generality in which to define flatness in the 2-dimensional
context. However, for the sake of simplicity in many calculations we work primarily with 2-functors. We note that no generality is lost since, while we prove our main theorem (Theorem 4.2.7) for 2-functors, the corresponding theorem for pseudofunctors (Theorem A.6) follows as a corollary.

We define a pseudofunctor to be left exact if it preserves finite weighted bilimits. For a pseudofunctor $A \xrightarrow{P} \text{Cat}$, we define the left bi-Kan extension (as already considered in [22]) pseudofunctor $\text{Hom}_{P}(A^{op}, \text{Cat}) \xrightarrow{P^*} \text{Cat}$ along the Yoneda 2-functor $A \xrightarrow{h} \text{Hom}_{P}(A^{op}, \text{Cat})$ (namely, the bi-universal pseudonatural transformation $P \Rightarrow P^*h$, where $\text{Hom}_{P}(A^{op}, \text{Cat})$ is the 2-category of 2-functors, pseudo-natural transformations and modifications). Note that since weighted bilimits exist in $\text{Cat}$ this is actually a pointwise bi-Kan extension. Furthermore, bilimits in $\text{Cat}$ can be chosen to be pseudolimits, and then it follows that when $P$ is a 2-functor, $P^*$ can be chosen to be a 2-functor. Also note that from these definitions it follows that the flatness of a 2-functor $P$, which we define stipulating that $P^*$ is left exact, is preserved by pseudonatural equivalences, that is, equivalences in $\text{Hom}_{P}(A^{op}, \text{Cat})$.

Let $A \xrightarrow{P} \text{Cat}$ be a 2-functor, consider the 2-Grothendieck construction $\mathcal{E}l_{P} \xrightarrow{\mathcal{E}} A$ and the family $\mathcal{E}_{P}$ given by the (co)cartesian morphisms, note that we abuse the notation and consider this family both in $\mathcal{E}l_{P}$ and $\mathcal{E}l_{P}^{op}$. A crucial result that opened the door for the intended research of this paper was the realization of the following:

Given a 2-functor $A^{op} \xrightarrow{\mathcal{E}} \text{Cat}$, the category of pseudo-dicones for the 2-functor $A^{op} \times A \xrightarrow{F \times P} \text{Cat}$ is isomorphic to the category of lax cones for the 2-functor $\mathcal{E}l_{P}^{op} \xrightarrow{\mathcal{E}} A^{op} \xrightarrow{F} \text{Cat}$ such that the structural 2-cells corresponding to $\mathcal{E}_{P}$ are invertible.

This fact led us to consider a general notion that we call $\sigma$-natural transformation, already defined in [14]. Let $(A, \Sigma)$ be a pair where $A$ is a 2-category and $\Sigma$ a distinguished 1-subcategory. A $\sigma$-natural transformation is a lax natural transformation such that the structural 2-cells corresponding to the arrows of $\Sigma$ are invertible. This notion led in turn to the notion of weighted $\sigma$-limit, which became an essential tool for our work in this paper.

It would be appropriate to say that the most transcendental basic result in this paper is our theorem 2.4.10 which establishes the following:

General weighted $\sigma$-limits (or $\sigma$-colimits) can be expressed as conical ones.

This fact rescues for 2-dimensional category theory the classical fact of category theory which states that conical limits suffice to construct all weighted limits.

We establish a 2-categorical version of the canonical expression of $\text{Set}$-valued functors as colimits of representable functors: Any 2-functor $A \xrightarrow{P} \text{Cat}$ is equivalent in $\text{Hom}_{P}(A, \text{Cat})$ to a conical $\sigma$-colimit of representable 2-functors, over the indexing pair $(\mathcal{E}l_{P}^{op}, \mathcal{E}_{P})$.

We introduce a notion of 2-filteredness for pairs $(A, \Sigma)$ that we denote by $\sigma$-filteredness. When $A$ has finite weighted bilimits and $P$ is left exact, the pair $(\mathcal{E}l_{P}^{op}, \mathcal{E}_{P})$ is $\sigma$-filtered, in other words the $\sigma$-colimit in the canonical expression of $P$ is a $\sigma$-filtered $\sigma$-colimit.

We prove a key result that establishes that a $\sigma$-filtered $\sigma$-colimit of flat 2-functors is flat. This follows from the commutativity (up to equivalence) of $\sigma$-filtered $\sigma$-colimits with finite weighted bilimits in $\text{Cat}$, established in [9].

Let $A \xrightarrow{P} \text{Cat}$ be a 2-functor. Our main result, Theorem 4.2.7 states that the following are equivalent:

(i) $\mathcal{E}l_{P}$ is $\sigma$-cofiltered with respect to the family $\mathcal{E}_{P}$ of cocartesian arrows.
(ii) $P$ is equivalent to a $\sigma$-filtered $\sigma$-colimit of representable 2-functors in $\mathcal{H}om_p(\mathcal{A}, \mathbf{Cat})$.
(iii) $P$ is flat.
If $\mathcal{A}$ has finite weighted bilimits, then it is also equivalent:
(iv) $P$ is left exact.

Recall that a Grothendieck topos is a left exact localization of a category $\mathcal{H}om(\mathcal{A}^{op}, \mathbf{Set})$ of presheaves, where $\mathcal{A}$ is a small category, and that the category of models (opposite of the category of points) of the presheaf topos is equivalent to the category of ind-objects of $\mathcal{A}^{op}$.

For a small 2-category $\mathcal{A}$, a point or model of the “2-topos” $\mathcal{H}om_p(\mathcal{A}, \mathbf{Cat})$ of 2-presheaves could be appropriately defined as a $\mathbf{Cat}$-valued left exact 2-functor $\mathcal{H}om_p(\mathcal{A}, \mathbf{Cat}) \rightarrow \mathbf{Cat}$. The 2-category of models is by definition the dual 2-category of the 2-category of points. Our main theorem establishes, in particular, that these 2-categories are locally small, and equivalent, respectively, to the 2-category $\sigma$-$\mathbf{Pro}(\mathcal{A})$ of $\sigma$-pro-objects of $\mathcal{A}$ and the 2-category $\sigma$-$\mathbf{Ind}(\mathcal{A}^{op})$ of $\sigma$-ind-objects of $\mathcal{A}^{op}$. This fact should be essential for the development of the theory of 2-topoi, as the corresponding result for $\mathbf{Set}$-valued functors is for topos theory. As Joyal pointed to us, a 2-topos could be defined as a left exact 2-localization of a 2-category of 2-presheaves.

**Organization of the paper**

In Section 1 we fix notation and terminology. Through Sections 2 and 3 we fix an arbitrary pair $(\mathcal{A}, \Sigma)$ with $\Sigma$ a 1-subcategory (containing all the objects) of a 2-category $\mathcal{A}$.

In Section 2 we develop the theory of $\sigma$-limits. In §2.1 we define $\sigma$-natural transformations between 2-functors $\mathcal{A} \rightarrow \mathcal{B}$ following [14] §1.2 p.13,14. These are lax natural transformations where the 2-cells associated to the arrows in $\Sigma$ are invertible. We denote the so determined 2-category $\mathcal{H}om^\Sigma_p(\mathcal{A}, \mathcal{B})$, and whenever possible we will omit $\Sigma$ from the notation. In this way we have a chain of inclusions of categories with the same objects:

$$\mathcal{H}om_p(\mathcal{A}, \mathcal{B}) \hookrightarrow \mathcal{H}om_p(\mathcal{A}, \mathcal{B}) \xrightarrow{(1)} \mathcal{H}om_\sigma(\mathcal{A}, \mathcal{B}) \xrightarrow{(2)} \mathcal{H}om_\ell(\mathcal{A}, \mathcal{B})$$

where the sub indexes $s, p, \sigma, \ell$ indicate strict natural (i.e. 2-natural), pseudonatural, $\sigma$-natural and lax natural respectively. When $\Sigma$ is the whole underlying category of $\mathcal{A}$, (1) above is an equality, and when $\Sigma$ consists only of the identities (2) is so. This allows for a unified treatment of many results known for pseudo and lax natural transformations.

Each choice of a subindex $s, p, \sigma, \ell$ gives rise to a notion of weighted limit that we study in §2.2. Note that the three cases $s, p, \ell$ are considered in [17], but the general concept of $\sigma$-limit for an arbitrary 1-subcategory $\Sigma$ is an essential tool to work with the notion of flat 2-functor, and we use in this paper $\sigma$-limits that are neither lax nor pseudolimits.

**Notation:** In order to avoid repeating statements and, more important, to develop unified proofs whenever possible, we will use a letter $\varepsilon$, that can stand for both "$s$" and "$\sigma$", thus also for "$p$" and "$\ell$".

**Warning:** we use limit to refer to a general weighted limit, and conical limit to a classical limit (i.e., when the weight is the constant 2-functor with value 1).

In §2.3 we consider for arbitrary $\varepsilon$ the corresponding notion of $\varepsilon$-end and $\varepsilon$-coend, and establish the $\varepsilon$-end formula for the category of $\varepsilon$-natural transformations between 2-functors. We consider also tensors and cotensors, and prove the constructions of weighted $\varepsilon$-limits in terms of $\varepsilon$-ends and cotensors.
In §2.4 we study explicitly conical \( \sigma \)-limits and \( \sigma \)-colimits, and show, modifying an argument of Street [29], the fundamental property of \( \sigma \)-limits that we mention in the introduction. We choose to establish it for colimits: arbitrary \( \sigma \)-colimits can be expressed as conical \( \sigma \)-colimits. We establish then the canonical expression of a \( \text{Cat} \)-valued 2-functor as a conical \( \sigma \)-bicolimit of representable 2-functors. We finish this subsection adapting Gray’s construction of \( \sigma \)-colimits in \( \text{Cat} \) to fit our context, which is a result that we will need later.

In §2.6 we analyze the computation of weighted \( \varepsilon \)-limits in \( \varepsilon' \)-2-functor categories, and establish a general theorem about pointwise computation. This is an essential theorem in the theory of limits, which is used everywhere. In particular, we use it in §2.7 in order to prove properties of interchange of \( \varepsilon \)-limits and \( \varepsilon \)-colimits.

In Section 3 we introduce and develop the notion of 2-filteredness for pairs \( (A, \Sigma) \), which we refer to by saying that \( A \) is \( \sigma \)-filtered (with respect to \( \Sigma \)). In §3.1 we state the basic definition, which is a generalization of Kennison’s three axioms in his definition of bifiltered 2-category [20], thus it also generalizes the equivalent Dubuc-Street notion of \( \sigma \)-filtered 2-category [11]. Their notion corresponds to \( \sigma \)-filteredness when \( \Sigma \) consists of all the arrows of \( A \). We consider particular finite diagrams such that their \( \sigma \)-cones suffice for \( \sigma \)-filteredness, and show that these \( \sigma \)-cones correspond (up to equivalence) to the cones of some particular finite weighted bilimits. In §3.2 we consider the pair \( (\mathcal{E}l_P, \varepsilon_P) \) as mentioned in the introduction and we prove that the 2-functor \( \mathcal{E}l_P \mapsto A \) creates any conical \( \sigma \)-bilimit which exists in \( A \) and is preserved by \( P \) (this is a 2-dimensional version of a known 1-dimensional result, see [16 Proposition 4.87]). From this result, together with the equivalence between cones mentioned above, it follows that if \( A \) has finite weighted bilimits and \( P \) is left exact, then the pair \( (\mathcal{E}l_P^{op}, \varepsilon_P) \) is \( \sigma \)-filtered. Interestingly enough, finite conical bilimits in \( A \) do not suffice for this result. In §3.3 we consider \( \sigma \)-cofinal 2-functors and establish some of the usual properties of cofinality that we will use in the proof of our main theorem in section 4. These properties allow us to show that the canonical 2-functor \( \mathcal{E}l_P^{op} \to \mathcal{E}l_L^{op} \), where \( L \) is a left bi-Kan extension of \( P \), is \( \sigma \)-cofinal in the case considered in the theorem.

In Section 4 we consider flat pseudofunctors and we prove our main theorem. In §4.1 we define the bi-Kan extension of a pseudofunctor following [22]. It is defined by the usual representation that defines Kan extensions suitably relaxed. We focus on the pointwise case which holds when the target 2-category has all weighted bilimits, and prove some basic results on flat pseudofunctors, analogous (but independent since the two notions of flatness are different) to the ones that can be found for a general base category \( V \) in [18] §6. In §4.2 we state and prove the results mentioned in the introduction, in particular our main theorem (Theorem 4.2.7), and in Appendix A we generalize them to the case of pseudofunctors.

1 Preliminaries

1.1 Basic terminology

Terminology regarding 2-dimensional category theory vary in the literature, we list here some definitions and basic results as we will use them in this paper.

1. We refer the reader to [19] for basic notions on 2-categories. Size issues are not relevant to us here, when it is not clear from the context we indicate the smallness condition if it applies.
2. In any 2-category, we use $\circ$ to denote vertical composition and juxtaposition to denote horizontal composition. We consider juxtaposition more binding than "$\circ$", thus $\alpha \beta \circ \gamma$ means $(\alpha \beta) \circ \gamma$. We will abuse notation by writing $f$ instead of $id_f$ for arrows $f$ when there is no risk of confusion.

3. Given any arrow or 2-cell "$x$", we use "$x^*$", "$x^\ast$" to denote precomposition, postcomposition with "$x$" respectively.

4. By $\mathbf{Cat}$ we denote the 2-category of (small) categories, with functors as morphisms and natural transformations as 2-cells.

5. For a 2-category $\mathcal{A}$ and objects $A, B \in \mathcal{A}$, we use the notation $A(A, B)$ to denote the category whose objects are the morphisms between $A$ and $B$ and whose arrows are the 2-cells between those morphisms.

6. We use $\cong$ to denote isomorphisms and $\approx$ to denote equivalences in a 2-category.

7. A 2-functor $F : \mathcal{A} \to \mathcal{B}$ is said to be pseudo-fully-faithful if for each $A, B \in \mathcal{A}$, $A(A, B) \xrightarrow{F_{A,B}} B(FA, FB)$ is an equivalence of categories, 2-fully-faithful if each $F_{A,B}$ is an isomorphism and locally-fully-faithful if each $F_{A,B}$ is full and faithful.

8. For a 2-category $\mathcal{A}$, $\mathcal{A}^{op}$ denotes the 2-category with the same objects as $\mathcal{A}$ but with $\mathcal{A}^{op}(A, B) = \mathcal{A}(B, A)$, i.e. we reverse the 1-cells but not the 2-cells. We use the notation $\mathcal{B} \xrightarrow{f} \mathcal{A}$ for the arrow in $\mathcal{A}^{op}$ that corresponds to the arrow $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathcal{A}$. 2-cells keep their names.

9. For a 2-category $\mathcal{A}$, $\mathcal{A}^{co}$ denotes the 2-category with the same objects and arrows as $\mathcal{A}$, but with $\mathcal{A}^{co}(A, B) = \mathcal{A}(A, B)^{op}$, i.e. we reverse the 2-cells but not the 1-cells.

10. The 2-category $\mathbf{Cat}$ has a duality 2-functor $\mathbf{Cat}^{co} \xrightarrow{D} \mathbf{Cat}$ that maps each category $C$ to its dual $C^{op}$. Clearly $D$ is an isomorphism of 2-categories and it is its own inverse.

11. A lax natural transformation between 2-functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a family of morphisms and 2-cells of $\mathcal{B}$, $\{FA \xrightarrow{\theta_A} GA\}_{A \in \mathcal{A}}$, $\{Gf \theta_A \xRightarrow{\theta_f} \theta_B Gf\}_{A \xrightarrow{f} B \in \mathcal{A}}$ satisfying the following equations:

**LN0.** For all $A \in \mathcal{A}$, $\theta_{id_A} = \theta_A$.

**LN1.** For all $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{A}$, $\theta_B Gf \circ \theta_f = \theta_g Ff \circ Gg \theta_f$.

**LN2.** For all $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{A}$, $\theta_B Fg \circ \theta_f = \theta_g Ff \circ Gg \theta_A$.

An op-lax natural transformation is defined analogously but the structural 2-cells $\theta_f$ are reversed, i.e. $\theta_B Ff \xRightarrow{\theta_f} Gf \theta_A$.

A modification $\theta \xrightarrow{\rho} \theta'$ between lax natural transformations is a family of 2-cells of $\mathcal{B}$ $\{\theta_A \xRightarrow{\rho_A} \theta'_A\}_{A \in \mathcal{A}}$ such that:

**LNM.** For all $A \xrightarrow{f} B \in \mathcal{A}$, $\theta'_f \circ Gf \rho_A = \rho_B Ff \circ \theta_f$. 

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In this way we have a 2-category $\text{Hom}_\ell(A, B)$, with arrows the lax natural transformations, and similarly $\text{Hom}_{opt}(A, B)$.

A pseudonatural transformation is a lax natural transformation where all the 2-cells $\theta_f$ are invertible, they are the arrows of a 2-category $\text{Hom}_p(A, B)$. A strict, or 2-natural transformation is a lax natural transformation where all the 2-cells $\theta_f$ are identities, they are the arrows of a 2-category $\text{Hom}_s(A, B)$. We have locally-fully-faithful inclusions

$$\text{Hom}_s(A, B) \hookrightarrow \text{Hom}_p(A, B) \hookrightarrow \text{Hom}_\ell(A, B)$$ (1.1.1)

and similarly for $\text{Hom}_{opt}(A, B)$. A pseudonatural equivalence, or pseudo-equivalence for short, is a pseudonatural transformation such that every $\theta_A$ is an equivalence in $\mathcal{B}$. This amounts to $\theta$ being an equivalence in $\text{Hom}_p(A, B)$.

12. There is a bijective correspondence between 2-functors, where $\gamma$ is either $s, p$ or $\ell$:

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{F} & \text{Hom}_\gamma(A, C) \\
\downarrow & & \downarrow \\
A & \xrightarrow{G} & \text{Hom}_{opt}(B, C)
\end{array}$$

This correspondence is given by the formulas, for $A \xrightarrow{f} A', B \xrightarrow{g} B'$:

$$FB(A) = GA(B), \quad (Fg)_A = GA(g), \quad FB(f) = (Gf)_B, \quad (Fg)_f = (Gf)_g, \quad (F\eta)_A = GA(\eta),$$

$$FB(\theta) = (G\theta)_B.$$ All the verifications are straightforward.

The expression $H(A, B) = FB(A) = GA(B)$ does not determine a 2-functor of two variables, its structure has been studied in [13, I, 4.1.] under the name of quasifunctor.

13. A lax dinatural transformation $\theta$ between 2-functors $A^{op} \times A \xrightarrow{F, G} B$ is a family of morphisms and 2-cells of $\mathcal{B}$, $\{F(A, A) \xrightarrow{\theta_A} G(A, A)\}_{A \in \mathcal{A}}, \{G(id, f)\theta_A F(f, id) \xrightarrow{\theta_f} G(f, id)\theta_B F(id, f)\}_{A \xrightarrow{\gamma} B \in \mathcal{A}}$ satisfying the following equations:

LD0. For all $A \in \mathcal{A}$, $\theta_{id_A} = \theta_A$.

LD1. For all $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{A}$, $\theta_{gf} = G(f, id)\theta_B F(id, f) \circ G(id, g)\theta_f F(g, id)$.

LD2. For all $A \xrightarrow{f} B \in \mathcal{A}$, $G(\gamma, id)\theta_B F(id, \gamma) \circ \theta_f = \theta_g \circ G(\gamma, id)\theta_A F(\gamma, id)$.

A morphism $\rho$ between two lax dinatural transformations $\theta, \theta'$ is a family of 2-cells of $\mathcal{B}$, $\{\theta_A \xrightarrow{\rho_A} \theta'_A\}_{A \in \mathcal{A}}$ such that:

LDM. For all $A \xrightarrow{f} B \in \mathcal{A}$, $\theta'_B \circ G(id, f) \rho_A F(f, id) = G(f, id)\rho_B F(id, f) \circ \theta_f$.

Note that if a pair of 2-functors $A \xrightarrow{F, G} B$ are considered as 2-functors $A^{op} \times A \xrightarrow{\tilde{F}, \tilde{G}} B$ constant in the first variable, lax dinatural transformations from $\tilde{F}$ to $\tilde{G}$ correspond to lax natural transformations from $F$ to $G$, and similarly for their morphisms.
14. The construction of item 9 defines an isomorphism $\text{Hom}_\ell(A, B) \xrightarrow{(-)_{co}} \text{Hom}_{op}(A^{co}, B^{co})$.

15. Combining the previous item with item 10, we have an isomorphism of 2-categories $\text{Hom}_\ell(A, \text{Cat}) \xrightarrow{(-)_{co}} \text{Hom}_{op}(A^{co}, \text{Cat})$ that maps a 2-functor $A \xrightarrow{P} \text{Cat}$ to a 2-functor that we will denote by $P^d$, $P^d = DP^{co}$.

16. For a 2-functor $A \xrightarrow{F} B$, and an object $E \in B$ we have isomorphisms of categories $\text{Hom}_\ell(A, \text{Cat})(\triangle 1, B(E, F)) \cong \text{Hom}_\ell(A, B)(\triangle E, F)$ $\text{Hom}_\ell(A^{op}, \text{Cat})(\triangle 1, B(F, E)) \cong \text{Hom}_{op}(A, B)(F, \triangle E)$ where $\triangle 1$ and $\triangle E$ denote the 2-functors constant at $1 = \{\ast\}$, and $E$ respectively.

For $\theta$ in the left side and $\eta$ in the right side, both isomorphisms are given by the formulas $\eta_A = \theta_{A(*)}$ for $A \in A$, $\eta_f = (\theta_f)_*$ for $A \xrightarrow{f} B \in A$.

### 1.2 The 2-category of elements

We will make extensive use of the 2-category of elements $\mathcal{E}_P$ of a $\text{Cat}$-valued 2-functor $P$. $\mathcal{E}_P$ can be defined as a particular instance of a lax comma 2-category (§1.4], [14 §1.2.5], $\mathcal{E}_P = [\triangle 1, P]$, and therefore has the universal property of Proposition 1.2.3 below.

**Definition 1.2.1.** Let $A \xrightarrow{P} \text{Cat}$ be a 2-functor. $\mathcal{E}_P$ can be described as follows:

1. **Objects:** Pairs $(x, A)$ with $A \in A$ and $x \in PA$

2. **Morphisms:** A morphism between $(x, A)$ and $(y, B)$ is a pair $(f, \varphi)$ with $A \xrightarrow{f} B \in A$ and $Pf(x) \xrightarrow{\varphi} y$

3. **2-cells:** A 2-cell between $(f, \varphi)$ and $(g, \psi)$ (from $(x, A)$ to $(y, B)$) is given by a 2-cell $A \xrightarrow{f} B \in A$ such that the following diagram commutes in $PB$:

   $\begin{array}{ccc}
   Pf(x) & \xrightarrow{\varphi} & y \\
   \downarrow{(P\theta)_x} & & \downarrow{\psi} \\
   Pg(x) & & \\
   \end{array}$

4. **Compositions in this 2-category are defined as follows:** for composable arrows $(f, \varphi)$ and $(g, \psi)$ we have $(g, \psi)(f, \varphi) = (gf, \psi Pg(\varphi))$, and both horizontal and vertical composition of 2-cells are computed in $A$.

   We consider the 1-subcategory $\mathcal{C}_P$ of $\mathcal{E}_P$ whose arrows are $(f, \varphi)$ with $\varphi$ an isomorphism.

**Remark 1.2.2.** We note that the canonical projection $\mathcal{E}_P \xrightarrow{\text{pr}^P} A$ is the opfibration (in the sense of §1.2.5 p.30]) associated to $P$, and the arrows of $\mathcal{C}_P$ are the cocartesian morphisms of $\mathcal{E}_P$. 

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**Proposition 1.2.3** ([14 §1.2.5 p.29], [2 Proposition 1.11]). The following diagram expresses the fact that (together with $\triangle$P and the lax natural transformation $\alpha$ defined by $\alpha(x,A) = x$, $\alpha(f,\varphi) = \varphi$), $\mathcal{E}l_P$ is the lax pull-back of $P$ along the 2-functor $\Delta 1 \rightarrow \text{Cat}$.

For each 2-functor $Z \rightarrow A$, and each lax natural transformation $\Delta 1 \theta \rightarrow PF$, such that $\triangleright P T = F, \alpha T = \theta$.

The formulas behind this correspondence are, for $Z \rightarrow W$ in $Z, T(Z) = (\theta Z, F(Z))$, $T(r) = (F(r), \theta r), T(\beta) = F(\beta)$. There is also a 2-categorical part of this universal property that we omit since we will not use it, the reader may consult [2, Proposition 1.11].

**Remark 1.2.4.** It is well-known (see [29 p.180], or see [2, Proposition 1.14] for a proof) that the projection $\mathcal{E}l_P \rightarrow A$ is lax dense, in the sense that for each $A \rightarrow Cat$ the pasting composition with $\alpha$ yields an isomorphism of categories

$$\text{Hom}_\ell(A, Cat)(P,Q) \cong \text{Hom}_\ell(\mathcal{E}l_P, Cat)(\Delta 1, Q\triangleright P).$$

We make explicit the formulas defining this correspondence on objects:

$$P \rightarrow Q \text{ lax natural} \quad \Delta 1 \rightarrow Q\triangleright P \text{ lax natural}$$

$$P \rightarrow Q \text{ lax natural} \quad \Delta 1 \rightarrow Q\triangleright P \text{ lax natural}$$

$$\eta_A \rightarrow QA \quad \theta \rightarrow QA$$

$$P(f) \rightarrow Q(f) \quad \eta_{(x,A)} \rightarrow \theta(\varphi)$$

$$PB \rightarrow QB \quad (\eta_f)x = \theta_{(f,\id)}$$

$$\eta_A(x) = \theta_{(x,A)} \quad \theta_{(f,\varphi)} = \eta_B(\varphi)(\eta_f)x$$

1.2.5. Combining Proposition [1.2.3] and Remark [1.2.4] we have, for each lax natural transformation $P \rightarrow Q$ between Cat-valued 2-functors, an induced 2-functor $\mathcal{E}l_P \rightarrow \mathcal{E}l_Q$ given by the formulas

$$T_\eta(x,A) = (\eta_A(x), A), \quad T_\eta(f,\varphi) = (f, \eta_B(\varphi)(\eta_f)x), \quad T_\eta(\theta) = \theta.$$ We note (see [2 Theorem 1.15]), although we will not need this result, that this assignment actually defines a 2-fully-faithful 2-functor $\text{Hom}_\ell(A, Cat) \rightarrow (2\text{-Cat}/A)$.

1.2.6. Consider now 2-functors $A \rightarrow H, B \rightarrow P \rightarrow Cat$. By the pasting lemma for lax pull-backs, we may construct the lax pull-back defining $\mathcal{E}l_{PH}$ by pasting a (strict) 2-pull-back to the lax pull-back defining $\mathcal{E}l_P$ as in the diagram below:
Then we have an induced 2-functor $E\!l\!P\!H \xrightarrow{T_H} E\!l\!P$ that is given by the formulas

$T_H(x, A) = (x, HA), \quad T_H(f, \varphi) = (Hf, \varphi), \quad T_H(\theta) = H\theta.$

## 2 σ-limits

We fix throughout this section a 1-subcategory $\Sigma$ of a 2-category $\mathcal{A}$. We introduce the use of a symbol $\sigma$ accompanying a concept, it is convenient to think that $\sigma$ means that the concept is to be taken “relative to the arrows of $\Sigma$”. Whenever possible, we will omit $\Sigma$ from the notation.

### 2.1 σ-natural transformations

**Definition 2.1.1.** Given 2-functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$, a $\sigma$-natural transformation $F \xrightarrow{\theta} G$ (with respect to $\Sigma$) is a lax natural transformation such that, if $A \xrightarrow{f} A'$ is in $\Sigma$, the structural 2-cell $\theta_f$ (see §1.1, item 11) is invertible. There is a 2-category $\text{Hom}_\Sigma(\mathcal{A}, \mathcal{B})$ whose objects are the 2-functors from $\mathcal{A}$ to $\mathcal{B}$, whose arrows are the $\sigma$-natural transformations and whose 2-cells are the modifications between them. We have locally-fully-faithful inclusions (see (1.1.1))

$$\text{Hom}_\Sigma(\mathcal{A}, \mathcal{B}) \xhookrightarrow{(1)} \text{Hom}(\mathcal{A}, \mathcal{B}) \xhookrightarrow{(2)} \text{Hom}_\ell(\mathcal{A}, \mathcal{B}).$$

(2.1.2)

Note that if $\Sigma'$ is another 1-subcategory of $\mathcal{A}$ and $\Sigma \subseteq \Sigma'$ then $\text{Hom}_{\Sigma'}(\mathcal{A}, \mathcal{B}) \xhookrightarrow{(1)} \text{Hom}_{\Sigma}(\mathcal{A}, \mathcal{B}).$

We recall that $\sigma$-natural transformations were already considered by J. W. Gray in [14, §1,2 p.13,14]. What we denote by $\text{Hom}_\Sigma(\mathcal{A}, \mathcal{B})$ is, in Gray’s notation, $\text{Fun}(\mathcal{A}, \Sigma; \mathcal{B}, \text{iso}\mathcal{B})$.

**Remark 2.1.3.** Consider a 2-category $\mathcal{A}$, its underlying category $\mathcal{A}_0$ and the 1-subcategory $\mathcal{A}_{id}$ consisting only of the identities. Then, in (2.1.2), (1) is an equality if $\Sigma = \mathcal{A}_0$, and (2) is an equality if $\Sigma = \mathcal{A}_{id}$. □

Observe that the items [14,15] and [10] in §1.1 hold with the same proof for general $\sigma$ and $\text{op}\sigma$-natural transformations, the latter being defined in an evident way. □

**Notation 2.1.4.** Even though in this paper we will work mainly with $\sigma$-limits, in order to avoid repeating statements that hold for the $\sigma$-case and the strict $s$-case, we will use a letter $\varepsilon$, that can stand for both $s$ and $\sigma$ (then also by Remark 2.1.3 for $p$ and $\ell$). This allows for a unified treatment of many results which are known for strict, pseudo and lax natural transformations.
2.2 \(\varepsilon\)-limits

**Definition 2.2.1.** Given 2-functors \(A \xrightarrow{W} \text{Cat}, A \xrightarrow{F} B\), and \(E\) an object of \(B\), we denote \(\text{Cones}_{\varepsilon}^{W}(E, F) = \mathcal{H}om_{\varepsilon}(A, \text{Cat})(W, B(E, F))\). This is the category of \(w\)-\(\varepsilon\)-cones (with respect to the weight \(W\)) for \(F\) with vertex \(E\). For a \(w\)-\(\varepsilon\)-cone \(\xi\) with vertex \(E\), \(W \xrightarrow{\xi} B(E, F-)\), we have a functor \(\theta_{B} = \xi^{*}\) given by precomposition with \(\xi\):

\[
B(E, E) \xrightarrow{\theta_{B}} \mathcal{H}om_{\varepsilon}(A, \text{Cat})(W, B(B, F-))
\]  
\[
\begin{array}{c}
B \xrightarrow{f} E \mapsto W \xrightarrow{\xi} B(E, F-) \xrightarrow{\xi^{*}} B(B, F-)
\end{array}
\]  

The \(\varepsilon\)-limit of \(F\) weighted by \(W\), denoted \(\{W, F\}_{\varepsilon}\), or more precisely \(\{W, F\}_{\varepsilon}, \xi\), is a \(w\)-\(\varepsilon\)-cone \(\xi\) with vertex \(E = \{W, F\}_{\varepsilon}\), universal in the sense that \(\theta_{B} = \xi^{*}\) in (2.2.2) is an isomorphism.

As usual, an equivalent formulation of the universal property is that there is a representation \(\theta_{B}\) natural in the variable \(B\) (as in (2.2.2)), and \(\xi\) is recovered setting \(B = E\), \(\xi = \theta_{E}(\text{id}_{E})\).

It is convenient to give an explicit definition of the dual concept, in the notation of Definition 2.2.1.

**Definition 2.2.2.** \(\varepsilon\)-colimits \(W \otimes_{\varepsilon} F\) in \(B\) are the corresponding limits in \(B^{op}\); for \(A^{op} \xrightarrow{W} \text{Cat}, A \xrightarrow{F} B\) we denote \(\text{Cones}_{\varepsilon}^{W}(F, E) = \mathcal{H}om_{\varepsilon}(A^{op}, \text{Cat})(W, B(F-, E))\) and refer to the objects of this category also as \(w\)-\(\varepsilon\)-cones, as it is clear from the context which \(w\)-\(\varepsilon\)-cones we are referring to. The \(\varepsilon\)-colimit of \(F\) weighted by \(W\), denoted \(W \otimes_{\varepsilon} F\) or more precisely \(W \otimes_{\varepsilon} F, \nu\), is a \(w\)-\(\varepsilon\)-cone \(\nu\) with vertex \(E = W \otimes_{\varepsilon} F\), \(W \xrightarrow{\nu} B(F-, B)\) universal in the sense that the functor \(\theta_{B} = \nu^{*}\) given by precomposing with \(\nu\),

\[
B(E, B) \xrightarrow{\theta_{B}} \mathcal{H}om_{\varepsilon}(A^{op}, \text{Cat})(W, B(B, F-), B))
\]  

is an isomorphism.

**Remark 2.2.5.** Considering \(\varepsilon = s\), we recover the notion of strict weighted limit ([17 §2]). Considering \(\varepsilon = \sigma, \Sigma = A_{0}\) and \(\Sigma = A_{id}\), we recover the notions of weighted lax and pseudolimits ([17 §5]). In spite of this notation, the reader should be aware that \(s\)-limits are not \(\sigma\)-limits, as it is the case for weighted lax and pseudolimits.

The general concept of \(\sigma\)-limit for an arbitrary 1-subcategory \(\Sigma\) is an essential tool to work with the notion of flat 2-functor, and we will consider in this paper \(\sigma\)-limits that are neither lax nor pseudolimits.

**Remark 2.2.6.** We also consider (but omit to write explicitly) analogous statements for \(op\)-\(\sigma\)-natural transformations, thus defining \(op\)-\(\sigma\)-limits. Recall §11 item 14. Every \(\sigma\)-limit in \(B\) is an \(op\)-\(\sigma\)-limit in \(B^{op}\), and vice versa. If \(A \xrightarrow{W} \text{Cat}, A \xrightarrow{F} B\), then \(\{W, F\}_{\sigma} = \{W^{d}, F^{co}\}_{op\sigma}\). See [2] Proposition 1.5 for a proof for lax natural transformations, that can be easily adapted to a general \(\sigma\).

Therefore one can think there is only one main or “primitive” notion between the four possible choices in \((\text{op})\)-\(\sigma\)-(co)limits, and the other three can be obtained from that one. Then, as it is usual in the literature, we can state and prove general results for \(\sigma\)-limits, and use them for any of the four choices mentioned above.
Remark 2.2.7. If the universal property in Definition 2.2.1 is in the weak sense (equivalence instead of isomorphism) we have the notion of \( \sigma \)-bilimit \( \mathcal{B}_\sigma \{W,F\}_\sigma \) (and \( \sigma \)-bicollimit \( W_\sigma \otimes_\sigma F \)). Clearly, any \( \sigma \)-limit is in particular a \( \sigma \)-bilimit. Note however that the defining universal properties characterize \( \sigma \)-bilimits up to equivalence and \( \sigma \)-limits up to isomorphism. We abuse nevertheless the language by referring to “the” \( \sigma \)-bilimit, or “the” \( \sigma \)-limit, and use equalities to express that a certain object satisfies the corresponding universal property, but it is important to be aware that for a given data, both a \( \sigma \)-limit and a \( \sigma \)-bilimit may be constructed independently, and they will be equivalent objects, but not necessarily isomorphic.

As in \([17] (2.5), (5.5)\) (see Remark 2.3.13 below for a proof) we have the basic result:

Proposition 2.2.8. The 2-category \( \text{Cat} \) has all (small) weighted \( \varepsilon \)-limits. In fact, given \( \mathcal{A} \xrightarrow{W} \text{Cat}, \mathcal{A} \xrightarrow{P} \text{Cat}, \{W,P\}_\varepsilon = \mathcal{H}om_\varepsilon(\mathcal{A},\text{Cat})(W,P). \)

As an immediate corollary, it follows that representable 2-functors preserve weighted \( \varepsilon \)-limits. That is, they “come out of the second variable”. More precisely:

Corollary 2.2.9. Let \( \mathcal{A} \xrightarrow{W} \text{Cat}, \mathcal{A} \xrightarrow{P} \mathcal{B} \) be 2-functors, then we have the following isomorphism (equivalence), 2-natural in the variable \( B \):

\[ \mathcal{B}(B,\{W,F\}_\varepsilon) \xrightarrow{\cong} \{W,\mathcal{B}(B,F-)\}_\varepsilon, \quad \mathcal{B}(B,W_\varepsilon\{W,F\}_\varepsilon) \xrightarrow{\cong} b_\varepsilon\{W,\mathcal{B}(B,F-)\}_\varepsilon \]

Proof. Consider \( P = \mathcal{B}(B,F-) \) in Proposition 2.2.8 and the Definition 2.2.1 of \( \varepsilon \)-limit. The case of \( \varepsilon \)-bilimits is analogous. \( \square \)

It is well known (\([16] (3.11)\)) that weighted strict limits behave functorially both in the weight and the argument. Here we establish the fact that \( \varepsilon \)-limits (recall that \( \varepsilon \) stands for \( \sigma \) or \( s \)) behave functorially respect to any natural transformation stronger than \( \varepsilon \)-natural, more precisely:

Notation 2.2.10. Let \( \mathcal{A} \) be any 2-category. Consider the set \( \mathcal{L}_\mathcal{A} \) consisting of the label \( s \) and one label \( \sigma \) for each 1-subcategory \( \Sigma \) of \( \mathcal{A} \). Note that in particular we have labels that we denote \( p = \sigma^{A_\ell}, \ell = \sigma^{A_0} \) (see Remark 2.1.3). Consider the order in \( \mathcal{L}_\mathcal{A} \) induced by the inclusions in \((2.1.2)\), that is \( s \leq \sigma^\Sigma \) for every \( \Sigma \), and \( \sigma^{\Sigma''} \leq \sigma^\Sigma \) if \( \Sigma \subseteq \Sigma'' \).

Note that if we are considering only one 1-subcategory \( \Sigma \), and omit it from the notation, we have \( s \leq p \leq \sigma \leq \ell \) (cf. \((2.1.2)\)).

Remark 2.2.11. Let \( \alpha, \beta \in \mathcal{L}_\mathcal{A} \), if \( \alpha \leq \beta \) then weighted \( \beta \)-limits behave functorially respect to \( \alpha \)-natural transformations. That is:

Let \( \mathcal{A} \xrightarrow{V} \text{Cat}, \mathcal{A} \xrightarrow{W} \mathcal{C} \) be \( \alpha \)-natural transformations, by \((2.2.2)\), we have

\[
\begin{align*}
V \xrightarrow{\xi} \mathcal{C}(\{V,F\}_\beta,F-) & \quad \{V,F\}_\beta & \quad W \xrightarrow{\xi} \mathcal{C}(\{W,G\}_\beta,G-) & \quad \{W,G\}_\beta \\
\theta \downarrow & \quad \downarrow \varepsilon_f & \quad \downarrow \varepsilon_f & \quad \downarrow \varepsilon_f & \quad \downarrow \varepsilon_f \\
W \xrightarrow{\xi} \mathcal{C}(\{W,F\}_\beta,F-) & \quad \{W,F\}_\beta & \quad \mathcal{C}(\{W,F\}_\beta,G-) \xrightarrow{\eta} \mathcal{C}(\{W,F\}_\beta,G-) & \quad \{W,F\}_\beta
\end{align*}
\]

A standard line of reasoning by the uniqueness in the universal properties yields that these constructions define 2-functors

\[
(\mathcal{H}om_\alpha(\mathcal{A},\text{Cat}^\op))_+ \xrightarrow{\{\cdot,F\}_\beta} \mathcal{C}, \quad (\mathcal{H}om_\alpha(\mathcal{A},\mathcal{C}))_+ \xrightarrow{\{\cdot,-\}_\beta} \mathcal{C},
\]
\((\text{Hom}_\alpha(A, \mathcal{C})^\op \times \text{Hom}_\alpha(A, \mathcal{C}))_+ \xrightarrow{(-,-)_\alpha} \mathcal{C}\),

where the subscript “+” indicates the full-subcategories with objects such that the corresponding \(\beta\)-limits exist.

### 2.3 \(\varepsilon\)-ends and cotensors

**\(\varepsilon\)-ends and \(\varepsilon\)-coends.**

The relation of strict ends (coends) of of 2-functors \(A^\op \times A \xrightarrow{T} \mathcal{B}\) with weighted limits (colimits) is well understood, they are given by the weight \(A^\op \times A \xrightarrow{\mathcal{A}(\cdot,\cdot)} \text{Cat}\)

\((A \times A^\op) \xrightarrow{A^\op(-,\cdot)} \text{Cat}\), see [16, 3.10], [30, 5.2.2]. However for general \(\sigma\) the situation is not at all the same. Some particular cases have been considered, for example, in [22, 9.6] the pseudoend of a \(\text{Cat}\) valued 2-functor, which requires the explicit construction of weighted pseudolimits in \(\text{Cat}\), in [30, 5.3] the lax coend of a \(\text{Cat}\) valued 2-functor of the form \(A^\op \times S \times T \xrightarrow{\pi} \text{Cat}\), which requires a non-trivial change in the weight.

We will now define the \(\varepsilon\)-end of a 2-functor \(A^\op \times A \xrightarrow{T} \mathcal{B}\) (recall Notation 2.1.1). The notion of \(\varepsilon\)-dinatural transformation is obtained, for \(\varepsilon = \sigma\), by the requirement of invertibility on the \(\theta_f\) for \(f \in \Sigma\) in §1.1, item 13. The case \(\varepsilon = s\) yields the notion of strict dinaturality, that corresponds to \(\mathcal{V}\)-naturality when \(\mathcal{V} = \text{Cat}\), [10, I.3.1, I.3.5], [16, §2.1, §3.10].

**Definition 2.3.1.** Let \(T : A^\op \times A \rightarrow \mathcal{B}\) be a 2-functor and \(E \in \mathcal{B}\). A \(\varepsilon\)-dicone \(\theta\) (with respect to \(\Sigma\)) for \(T\) with vertex \(E\) is a \(\varepsilon\)-dinatural transformation from the 2-functor which is constant at \(E\) to \(T\). This amounts to a lax dicone given by a family of morphisms \(\{E \xrightarrow{\pi_A} T(A,A)\}_{A \in A}\) and a family of 2-cells \(\{T(A,f)\pi_A \xrightarrow{\theta_f} T(f,B)\pi_B\}_{A \xrightarrow{f} B \in A}\) such that:

1. If \(\varepsilon = \sigma\), \(\theta_f\) is invertible for every \(f \in \Sigma\).
2. If \(\varepsilon = s\), \(\theta_f = \text{id}\) for every \(f\).

For each \(E\), \(\varepsilon\)-dicones with vertex \(E\) form a category \(\text{Dicones}_\varepsilon(E,T)\), whose arrows are the morphisms as lax dicones.

The \(\varepsilon\)-end in \(\mathcal{B}\) (with respect to \(\Sigma\)) of the 2-functor \(T\) is the universal \(\varepsilon\)-dicone, denoted

\(\{\varepsilon \int_A T(A,A) \xrightarrow{\pi_A} T(A,A)\}_{A \in A}\), \(\{T(A,f)\pi_A \xrightarrow{\pi_f} T(f,B)\pi_B\}_{A \xrightarrow{f} B \in A}\).

It is universal in the sense that for each \(E \in \mathcal{B}\) postcomposition with \(\pi\) yields an isomorphism of categories

\(\mathcal{B}(E, \varepsilon \int_A T(A,A)) \xrightarrow{\pi} \text{Dicones}_\varepsilon(E,T)\) \hspace{1cm} (2.3.2)

**Proposition 2.3.3.** For the 2-functor \(A^\op \times A \xrightarrow{\mathcal{B}(E,T(\cdot,\cdot))} \text{Cat}\), there is an obvious \(\varepsilon\)-dicone with vertex \(\text{Dicones}_\varepsilon(E,T)\). It can be checked that it is universal, therefore there is an isomorphism of categories

\(\text{Dicones}_\varepsilon(E,T) \xrightarrow{\cong} \varepsilon \int_A \mathcal{B}(E,T(A,A))\) \hspace{1cm} (2.3.4)
As usual, then, the universal property \( (2.3.2) \) defining \( \varepsilon \int_A T(A, A) \) is equivalent to stating that there is an isomorphism of categories

\[
B(E, \varepsilon \int_A T(A, A)) \cong \varepsilon \int_A B(E, T(A, A)) \quad (2.3.5)
\]

commuting with the \( \varepsilon \)-dicones. □

It is convenient to have at hand the explicit definition of the dual concept \( \varepsilon \)-coend.

**Definition 2.3.6.** \( \varepsilon \)-coends are defined as \( \varepsilon \)-ends in \( B^{\text{op}} \), for \( T : A^{\text{op}} \times A \to B \) we define

\[
\varepsilon \int^A T(A, A) = \varepsilon \int_A T^{\text{op}}(A, A),
\]

and we denote the universal \( \varepsilon \)-dicone by

\[
\{ T(A, A) \} \quad \lambda_A \varepsilon \int^A T(A, A) \quad \{ \lambda_B T(B, f) \} \quad \lambda_f \varepsilon \int^A T(f, A)
\]

A dual reasoning to the one used for \( (2.3.5) \) proves that the universal property defining \( \varepsilon \int^A T(A, A) \) can be stated as

\[
B(\varepsilon \int^A T(A, A), E) \cong \varepsilon \int_A B(T(A, A), E) \quad (2.3.7)
\]

We denote the weak concept of \( \sigma \)-biend of \( T \), where \( \pi^* \) in Definition 2.3.1 is required to be only an equivalence, by \( \sigma \int^A T(A, A) \).

The following key formula remains valid for the \( \text{Hom}_\varepsilon \) categories:

**Proposition 2.3.8.** For 2-functors \( P, Q : A \to B \), we have the formula

\[
\text{Hom}_\varepsilon(A, B)(P, Q) = \varepsilon \int_A B(PA, QA)
\]

**Proof.** It can be readily checked (by a straightforward but necessary arguing) that the following data defines a universal \( \varepsilon \)-dicone with vertex \( \text{Hom}_\varepsilon(A, B)(P, Q) \). Projections are given by \( \pi_A(\theta) = \theta_A \) for \( \varepsilon \)-natural transformations \( \theta \), \( \pi_A(\rho) = \rho_A \) for modifications \( \rho \). And the structural 2-cells of the \( \varepsilon \)-cone are given by \( \pi_f^\theta = \theta_f : Qf\theta_A \Rightarrow \theta_B Pf \) for \( A \xrightarrow{f} B \in A \), \( \theta \in \text{Hom}_\varepsilon(A, B)(P, Q) \). □

**Tensors and cotensors.**

**Definition 2.3.9.** \( \varepsilon \)-cotensor (resp. \( \varepsilon \)-tensor) are \( \varepsilon \)-limits (resp. \( \varepsilon \)-colimits) with \( A = 1 \). In this case the choice of \( \varepsilon \) is irrelevant, since \( \text{Hom}_\varepsilon(1, \text{Cat}) = \text{Hom}_s(1, \text{Cat}) \) for any choice of \( \varepsilon \). We identify \( 1 \to \text{Cat} \) with \( C \in \text{Cat} \), \( 1 \to B \) with \( B \in B \) (note that \( 1^{\text{op}} = 1 \)), and denote cotensor products by \( \{ C, B \} \) and tensor products by \( C \otimes B \). (see also [17, (3.1)]).

Recall that in the base 2-category \( \text{Cat} \), cotensors and tensors are given by the internal hom and the cartesian product, \( \{ C, B \} = \text{Cat}(C, B) \), and \( C \otimes B = C \times B \).

As in the case of enriched category theory, from Proposition 2.3.8 and Remark 2.2.11 it easily follows that for any \( \varepsilon \) the 2-functor categories have cotensors and that they are computed pointwise. The proof is very similar to [10, §3.3] so we omit it.
Proposition 2.3.10. Cotensor products are computed pointwise in $\text{Hom}_\varepsilon(A, B)$. This means precisely that for $C \in \text{Cat}$, $A \xrightarrow{G} B$, if $\{C, GA\}$ exist for each $A \in A$ then the formula $\{C, G\}A = \{C, GA\}$ defines a 2-functor that is the cotensor product of $C$ and $G$ in $\text{Hom}_\varepsilon(A, B)$. 

We finish this subsection establishing in the $\varepsilon$ case some well known formulas of enriched category theory. We omit to explicitly state the corresponding formulas for bilimits, which hold with the same proofs.

Proposition 2.3.11. For $A \xrightarrow{W} \text{Cat}$, $A \xrightarrow{P} B$, if $B$ is cotensored we have

$$\{W, P\}_\varepsilon = \varepsilon \int_A \{WA, PA\}$$

Proof. We have the following chain of natural isomorphisms

$$\text{Hom}_\varepsilon(A, \text{Cat})(W, B(B, P-)) \cong \varepsilon \int_A \text{Cat}(WA, B(B, PA))$$

$$\cong \varepsilon \int_A B(B, \{WA, PA\})$$

$$\cong B(B, \varepsilon \int_A \{WA, PA\})$$

given in turn by Proposition 2.3.8, definition of cotensor and (2.3.5). Then the statement follows by Definition 2.2.1.

With a dual proof we have

Corollary 2.3.12. For $A^{op} \xrightarrow{W} \text{Cat}$, $A \xrightarrow{P} B$, if $B$ is tensored we have

$$W \otimes_\varepsilon P = \varepsilon \int_A WA \otimes PA$$

Remark 2.3.13. For the case $B = \text{Cat}$, we have

1. For $A \xrightarrow{W} \text{Cat}$, $A \xrightarrow{P} \text{Cat}$, $\{W, P\}_\varepsilon \cong \varepsilon \int_A \text{Cat}(WA, PA)$

2. For $A^{op} \xrightarrow{W} \text{Cat}$, $A \xrightarrow{P} \text{Cat}$, $W \otimes_\varepsilon P \cong \varepsilon \int_A WA \times PA$

2.4 Conical $\sigma$-colimits

Remark 2.4.1. Let $F : A \rightarrow B$ be a 2-functor, and $E$ an object of $B$. It is immediate to check that the isomorphism in §1.1 item 16 restricts to an isomorphism

$$\text{Hom}_\sigma(A^{op}, \text{Cat})(\Delta 1, B(F-, E)) \cong \text{Hom}_{op^\sigma}(A, B)(F, \triangle E)$$

(2.4.2)

By considering the weight $\Delta 1$ in (2.2.4), it follows

$$\text{B}(\Delta 1 \otimes_\sigma F, E) \xrightarrow{\varepsilon^*} \text{Hom}_\sigma(A^{op}, \text{Cat})(\Delta 1, B(F-, E)) \cong \text{Hom}_{op^\sigma}(A, B)(F, \triangle E)$$

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Definition 2.4.3. Let $F : A \to B$ be a 2-functor, and $E$ an object of $B$. We define the category of $\sigma$-cones, $\text{Cones}_\Sigma^\sigma(F,E) = \text{Cones}_\Sigma^\sigma(A,F,E)$, see Definition 2.2.7. By Remark 2.4.1 a $\sigma$-cone for $F$ (with respect to $\Sigma$) with vertex $E$ corresponds to an $\sigma$-$\sigma$-natural transformation $F \xrightarrow{\theta} \triangle E$, this amounts to a lax cone $\{FA \xrightarrow{\theta_A} E\}_{A \in A}$, $\{\theta_B F \xrightarrow{\theta_A} \lambda_A\}_{A \xrightarrow{\lambda} B \in A}$ such that $\theta_f$ is invertible for every $f \in \Sigma$. The morphisms between two $\sigma$-cones correspond to their lax cones.

We now describe the universal property defining the (conical) $\sigma$-colimit of $F$. The $\sigma$-colimit in $B$ (with respect to $\Sigma$) of the 2-functor $F : A \to B$ is the universal $\sigma$-cone, denoted $\{FA \xrightarrow{\lambda_A} \text{colim}_A\Sigma FA\}_{A \in A}$, $\{\lambda_B F \xrightarrow{\lambda_A} \lambda_A\}_{A \xrightarrow{\lambda} B \in A}$ in the sense that for each $E \in B$, pre-composition with $\lambda$ is an isomorphism of categories

$$\mathcal{B}(\text{colim}_A\Sigma FA, E) \xrightarrow{\lambda^*} \text{Cones}_\sigma(F,E)$$

We denote the weak notion of (conical) $\sigma$-bicolimit as in Remark 2.2.7, where $\lambda^*$ is an equivalence, $\text{bicolim}_A\Sigma FA$. By definition we have, for $F : A \to B$,

$$\text{colim}_A\Sigma FA = \Delta 1 \otimes \sigma F, \quad \text{bicolim}_A\Sigma FA = \Delta 1 \otimes_{\sigma} F.$$ (2.4.5)

Conical $\sigma$-bicolimits are special cases of Cartesian quasi limits considered by J. W. Gray in [14, I, 7.9.1 iii)]. What we would denote by $\sigma$ in the lax naturality involved (because there is no “op” in the first isomorphism of §1.1 item 16), and they correspond in Gray’s notation to $\text{Cart q-\lim}_A\Sigma FA$. We also refer to $\sigma$-natural transformations $\Delta E \xrightarrow{\theta} F$ as $\sigma$-cones (as in Definition 2.4.3 above), since the direction of $\theta$ is clear from the context.

The following diagram illustrates the correspondence between $\sigma$-cones in $\mathcal{B}^{op}$ and $\sigma$-cones in $\mathcal{B}$ (recall that 2-cells in $\mathcal{B}^{op}$ keep their direction and that we denote objects in $\mathcal{B}^{op}$ with an overline, see §1.1 item §).

$$\text{For } A \xrightarrow{f} B \text{ in } A, \quad \overline{E} \xrightarrow{\theta_B} \overline{FB} \text{ in } \mathcal{B}^{op} \quad \text{corresponds to } \quad FA \xrightarrow{\theta_A} \overline{FB} \xrightarrow{\theta_B} E \text{ in } \mathcal{B}.$$ (2.4.7)

Definition 2.4.8. Recall Definition 1.2.7. We denote by $\mathcal{C}_\Sigma$ the 1-subcategory of $\mathcal{E}1_\mathcal{P}$ with arrows $(f, \varphi)$ that satisfy $f \in \Sigma$, $\varphi$ invertible (i.e. the intersection of $\mathcal{C}_\mathcal{P}$ and $\mathcal{P}^{-1}(\Sigma)$).

In [29] Theorem 15 Street shows that for each weight $A \xrightarrow{W} \text{Cat}$, $s$-limits weighted by $W$ are equivalent to a special type of Gray’s cartesian quasi-limit (see Remark 2.4.1) over $\mathcal{E}1_\mathcal{W}$. A slight modification of this procedure shows that weighted $\sigma$-limits can be expressed as conical $\sigma$-limits. Since we will use this result for colimits, we prefer to prove the colimit version.
Proposition 2.4.9. Let $\mathcal{A}^{op} \rightarrow \mathbf{Cat}$, $\mathcal{A} \rightarrow \mathcal{B}$, then we have

$$\text{Hom}^\Sigma_{\sigma}(\mathcal{A}^{op}, \mathbf{Cat})(W, \mathcal{B}(P-, B)) \cong \text{Hom}^\Sigma_{\sigma}(\mathcal{E}l_W, \mathbf{Cat})(\Delta 1, \mathcal{B}(P\sigma^{op}_W, B))$$

Proof. Consider the isomorphism

$$\text{Hom}_\ell(\mathcal{A}^{op}, \mathbf{Cat})(W, H) \cong \text{Hom}_\ell(\mathcal{E}l_W, \mathbf{Cat})(\Delta 1, H\sigma_W)$$

of Remark 1.2.4 and the explicit formulas therein. Then it can be seen at once that the isomorphism restricts to

$$\text{Hom}^\Sigma_{\sigma}(\mathcal{A}^{op}, \mathbf{Cat})(W, H) \cong \text{Hom}^\Sigma_{\sigma}(\mathcal{E}l_W, \mathbf{Cat})(\Delta 1, H\sigma_W)$$

In particular, for $H = \mathcal{B}(P-, B)$ we have the desired isomorphism.

This Proposition has as a corollary the following fundamental result:

Theorem 2.4.10. Let $\mathcal{A}^{op} \rightarrow \mathbf{Cat}$, $\mathcal{A} \rightarrow \mathcal{B}$, then

$$W \otimes_{\sigma} P = \Delta 1 \otimes_{\sigma} P\sigma^{op}_W = \sigma\text{Lim}_{(x,A) \in \mathcal{E}l_W^{op}}^{\Sigma_{\sigma}} PA, \quad W_{bi} \otimes_{\sigma} P = \Delta 1_{bi} \otimes_{\sigma} P\sigma^{op}_W = \sigma\text{biLim}_{(x,A) \in \mathcal{E}l_W^{op}}^{\Sigma_{\sigma}} PA,$$

which means that the universal properties defining each object are equivalent. In particular, by considering $\Sigma = A_0$, we have

$$W \otimes_p P = \sigma\text{Lim}_{(x,A) \in \mathcal{E}l_W^{op}}^{\Sigma_{\sigma}} PA, \quad W_{bi} \otimes_p P = \sigma\text{biLim}_{(x,A) \in \mathcal{E}l_W^{op}}^{\Sigma_{\sigma}} PA.$$

Proof. The first equality (in both expressions) is given by Proposition 2.4.9, the second one is (2.4.5).

In particular for $\epsilon = p$ using Theorem 2.4.10 we have the following expressions of a pseudocoen of tensors as a conical $\sigma$-colimit (for the first equality consider $(\mathcal{A}^{op})^{op} \rightarrow \mathbf{Cat}$, \mathcal{A}^{op} \rightarrow \mathbf{Cat}, and the first coend as indexed by $\mathcal{A}^{op}$).

Proposition 2.4.11.

$$\sigma\text{Lim}_{(x,A) \in \mathcal{E}l_W^{op}}^{\Sigma_{\sigma}} WA = p \int^{\mathcal{A}} PA \times WA = p \int^{\mathcal{A}} WA \times PA = \sigma\text{Lim}_{(y,A) \in \mathcal{E}l_W^{op}}^{\Sigma_{\sigma}} PA$$

The expression of a $\mathbf{Cat}$-valued 2-functor as a conical $\sigma$-bicolimit of representable 2-functors.

It is a classical result that any $\mathbf{Set}$-valued functor has a canonical expression as a colimit of representable functors. We now establish a 2-categorical version of this result. Consider a 2-functor $\mathcal{A} \rightarrow \mathbf{Cat}$, and the Yoneda embedding $\mathcal{A}^{op} \rightarrow \mathbf{Hom}_p(\mathcal{A}, \mathbf{Cat})$, $hA = \mathcal{A}(A, -)$. Recall the Pseudo-Yoneda Lemma, see [1.1.18] for a proof.
2.4.12 (Pseudo-Yoneda Lemma).

a) For any 2-functor \( A \rightarrow \text{Cat} \), evaluation at the identity for each \( A \in A \) provides the components:

\[
\text{Hom}_p(A, \text{Cat})(A(A, -), Q) \rightarrow QA
\]
of a pseudo-equivalence, that is an equivalence in \( \text{Hom}_p(A, \text{Cat}) \), between \( Q \) and the 2-functor on the left side. Furthermore, this equivalence is pseudonatural in the variable \( Q \).

From this, as usual, it follows:

b) The Yoneda embedding is pseudo-fully-faithful. That is, there is an equivalence of categories:

\[
\text{Hom}_p(A, \text{Cat})(A(A, -), A(B, -)) \rightarrow A(B, A).
\]

Proposition 2.4.13. For any 2-functor \( A \rightarrow \text{Cat} \), we have a pseudo-equivalence, that is an equivalence in \( \text{Hom}_p(A, \text{Cat}) \):

\[
P \approx p \int^A PA \otimes A(A, -).
\]

Proof. Consider \( P \) as a weight for the colimit of the Yoneda embedding \( h \), then Corollary 2.3.12 shows \( = \). Then, we have the following chain of equivalences, pseudonatural in the variable \( Q \):

\[
\text{Hom}_p(A, \text{Cat}(p \int^A PA \otimes A(A, -), Q) \approx p \int_A \text{Hom}_p(A, \text{Cat})(PA \otimes A(A, -), Q)
\]

\[
\approx p \int_A \text{Cat}(PA, \text{Hom}_p(A, \text{Cat})(A(A, -), Q))
\]

\[
\approx p \int_A \text{Cat}(PA, QA)
\]

\[
\approx \text{Hom}_p(A, \text{Cat})(P, Q).
\]

justified, in turn, by (2.3.7), Proposition 2.3.10, Pseudo Yoneda a) and Proposition 2.3.8.

By the pseudonaturality in \( Q \), a use of Pseudo Yoneda b), applied this time to the category \( \text{Hom}_p(A, \text{Cat}) \), finishes the proof.

Remark 2.4.15. In particular, since \( \sigma \)-bilimits are defined up to equivalence, it follows that any \( \text{Cat} \)-valued 2-functor \( P \) is a conical \( \sigma \)-bilimit in \( \text{Hom}_p(A, \text{Cat}) \) of a 2-diagram in \( \text{Hom}_{s}(A, \text{Cat}) \) of representable 2-functors, indexed by the pair \((\mathcal{E}^{op}_{P}, \mathcal{E}_{P})\)
2.5 A construction of conical $\sigma$-colimits of categories.

In [14, I,7.11.4 i] Gray proves that conical op$\sigma$-colimits in $\mathsf{Cat}$ exist and gives an explicit construction of them. In Proposition 2.5.1 below, we interpret this result according to our notation. We will use the left adjoint $\pi_0$ of the inclusion $\mathsf{Cat} \xrightarrow{d} 2\mathsf{Cat}$ (where $2\mathsf{Cat}$ is the 2-category of small 2-categories, 2-functors and 2-natural transformations) and the existence of the usual category of fractions [13]. For a subcategory $\Sigma$ of a category $C$, we will denote this category by $C[\Sigma^{-1}]$.

We observe that Gray only makes invertible those morphisms of the form $(f, id) \in \mathcal{E}_\Sigma$ while we invert every morphism in $C[\Sigma^{-1}]$. Since every morphism $(f, \varphi) \in \mathcal{E}_\Sigma$ can be factorized as $(id, \varphi)(f, id)$ and $(id, \varphi)$ is already invertible because $\varphi$ is an isomorphism, both constructions are isomorphic.

**Proposition 2.5.1.** Let $A \xrightarrow{Q} \mathsf{Cat}$ be a 2-functor. Then

$$\text{op}$\sigma$Lim_{A \in A} QA = (\pi_0 \mathcal{E}_Q)[\mathcal{E}_\Sigma^{-1}]$$

**Remark 2.5.2.** Let $A \xrightarrow{Q} \mathsf{Cat}$ be a 2-functor. There is a construction dual to the 2-category of elements $\mathcal{E}_Q$, that we will denote by $\Gamma_Q$, where the direction of $\varphi$ in Definition 1.2.1 is reversed. Note that Definition 2.4.8 can be easily adapted to define the 1-subcategory $C[\Sigma^{-1}]$ of $\Gamma_Q$. Then a proof dual to the one of Proposition 2.5.1 shows that, for a 2-functor $A \xrightarrow{Q} \mathsf{Cat}$,

$$\sigma$Lim_{A \in A} QA = (\pi_0 \Gamma_Q)[\mathcal{E}_\Sigma^{-1}]$$

The interested reader can also see [2, Proposition 1.17] for the $\ell$-case, and [6, Theorem 5.2] for the $p$-case, where it is shown the formula

$$\sigma$Lim_{A \in A} QA = \pi_0(\Gamma_Q[\mathcal{E}_\Sigma^{-1}]),$$

where now $\Gamma_Q[\mathcal{E}_\Sigma^{-1}]$ is the 2-category of fractions in the sense of [26]. Note that from the adjunction $\pi_0 \dashv d$ for any 1-subcategory $\Sigma$ of a 2-category $A$ it follows $(\pi_0 A)[\Sigma^{-1}] = (\pi_0 A)[\Sigma^{-1}]$, thus the two constructions are the same.

**Remark 2.5.3.** Note that, since computing $\pi_0$ and the category of fractions doesn’t change the objects, the objects of $\sigma$Lim_{A \in A} QA can be taken to be the objects of $\Gamma_Q$, which are pairs $(x, A)$ with $A \in \mathcal{A}$, $x \in QA$. By looking at the proof of [14, I,7.11.1] (which has [14, I,7.11.4 i]), i.e. Proposition 2.5.1 as corollary), we have a formula for the $\sigma$-colimit $\sigma$-cone $Q \xrightarrow{\lambda} \triangle(\sigma$Lim_{A \in A} QA), in particular on objects $\lambda_A(x) = (x, A)$. Note that for each object $c \in \sigma$Lim_{A \in A} QA, there are $A \in \mathcal{A}$, $x \in QA$ such that $\lambda_A(x) = c$.

**Lemma 2.5.4.** Let $A^{op} \xrightarrow{W} \mathsf{Cat}$, $\mathcal{A} \xrightarrow{P} \mathsf{Cat}$, and consider the universal $w$-$\sigma$-cone $W \xrightarrow{\nu} \mathcal{C}at(P, -)$, where $C = W \otimes_\sigma P$ (see (2.2.1)), note that $\nu$ is a $\sigma$-natural transformation. Then for each object $c \in C$, there exist $A \in \mathcal{A}$, $x \in WA$, $a \in PA$ such that $\nu_A(x)(a) = c$. 

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Proof. By Theorem 2.4.10 we have $C = \sigma \lim_{\ell} P A$. In other words, $W \otimes_\sigma P$ is the $\sigma$-colimit of the 2-functor $\mathcal{E}_W^{op} \to A \to \text{Cat}$. We may compute this colimit using Remark 2.5.2, then we have the colimit $\sigma$-cone $P \otimes_\sigma W \to \Delta C$. By Remark 2.5.3, there are $(x, A) \in \mathcal{E}_W$, $a \in PA$ such that $\lambda_{(x, A)}(a) = c$.

Now, as in the proof of Theorem 2.4.10, the correspondence between the $\sigma$-colimit $\sigma$-cone $P \otimes_\sigma W \to \Delta C$ and the $\sigma$-colimit $w$-$\sigma$-cone $W \otimes W \to C$ at $\lambda$ is given by Proposition 2.4.9 and (2.4.2), and therefore by the formulas in Remark 1.2.4 and §1, item 16. Then we have $\nu_A(x)(a) = \lambda_{(x, A)}(a) = c$.

2.6 Pointwise limits

We analyze now the computation of weighted $\varepsilon$-limits in functor categories. The pointwise computation of arbitrary weighted $\varepsilon$-limits is a much more delicate matter than that of cotensors (Proposition 2.3.10), we give below a general result regarding the pointwise computation of $\alpha$-limits in $\mathcal{O}p\beta$-functor categories (with pseudo or strict diagrams). Note in particular the appearance of the “op” prefix, this is reminiscent of the lifting of $\mathcal{O}p$-lax limits to the 2-category of strict algebras and lax morphisms for a 2-monad ([21]), which has as a particular case the case $\gamma = \ell$ of our Proposition 2.6.2.

Remark 2.6.1. Let $\mathcal{B}, \mathcal{C}$ be 2-categories, and $\gamma \in \{s, p, \ell\}$. Then we have a 2-functor $\mathcal{B} \to \mathcal{H}om_{\gamma}(\mathcal{H}om_{\mathcal{O}p\gamma}(\mathcal{B}, \mathcal{C}), \mathcal{C})$ given by the formulas, for $B \xrightarrow{f} B'$ in $\mathcal{B}$, $F \xrightarrow{\rho} G$ in $\mathcal{H}om_{\mathcal{O}p\gamma}(\mathcal{B}, \mathcal{C})$,

1. $ev_B(F) = FB, \quad ev_B(\theta) = \theta_B, \quad ev_B(\rho) = \rho_B$
2. $(ev_f)_{\theta} = \theta_f$
3. $(ev_{\mu})_F = F\mu$

For each $B \in \mathcal{B}$ and any $\varepsilon$ the definition in [1] determines 2-functors

$\mathcal{H}om_\varepsilon(\mathcal{B}, \mathcal{C}) \xrightarrow{ev_B} \mathcal{C}, \quad \mathcal{H}om_{\mathcal{O}pe}(\mathcal{B}, \mathcal{C}) \xrightarrow{ev_B} \mathcal{C}$,

that is, functors

$\mathcal{H}om_\varepsilon(\mathcal{B}, \mathcal{C})(F, G) \xrightarrow{ev_B} \mathcal{C}(FB, GB), \quad \mathcal{H}om_{\mathcal{O}pe}(\mathcal{B}, \mathcal{C})(F, G) \xrightarrow{ev_B} \mathcal{C}(FB, GB)$.

All the verifications are straightforward. □

Recall Notation 2.2.10. Given 2 categories $\mathcal{A}$, $\mathcal{B}$, in the next proposition we let $\alpha \in \mathcal{L}_A$, $\beta \in \mathcal{L}_B$ be the label “$s$”, or labels corresponding to arbitrary 1-subcategories of $\mathcal{A}$, $\mathcal{B}$ respectively. Among all the possible labels, the three labels $\ell, p, s$ always make sense for any $\mathcal{A}$ and $\mathcal{B}$. We will use the letter $\gamma$ to refer to these labels. With this in mind, we have:
Proposition 2.6.2. Let $\gamma \in \{\ell, p, s\}, \alpha \in \mathcal{L}_A$, $\beta \in \mathcal{L}_B$, such that $\alpha \geq \gamma$, $\beta \geq \gamma$. Then weighted $\alpha$-limits of $\op\gamma$-diagrams are computed pointwise in the 2-functor 2-categories $\mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})$ (in particular in $\mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})$), and are preserved by the inclusion 2-functor $\mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C}) \xrightarrow{\imath} \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})$. This means precisely that given 2-functors $A \xrightarrow{W} \mathbf{Cat}$, $A \xrightarrow{F} \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})$, if the $\alpha$-limits $\{W, ev B F\}_\alpha$ exist in $\mathcal{C}$ for each $B \in \mathcal{B}$, the definition $LB = \{W, ev B F\}_\alpha$ determines a 2-functor $B \xrightarrow{L} \mathcal{C}$ which is the $\alpha$-limit $L = \{W, i F\}_\alpha$ of $i F$ weighted by $W$ in $\mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})$. Denoting the composition $ev B F$ by $(F\cdotp)B : A \rightarrow \mathcal{C}$, we can write $\{W, F\}_\alpha(B) = \{W, (F\cdotp)(B)\}_\alpha$. Note that when $\gamma = \ell$, this forces that also $\alpha = \ell$ and $\beta = \ell$.

Proof. The definition of $L$ is given by the composition (see Remarks 2.2.11 and 2.6.1):

$$L : B \xrightarrow{ev(-)} \mathcal{H}om_{\gamma}(\mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C}), \mathcal{C}) \xrightarrow{F^*} \mathcal{H}om_{\gamma}(\mathcal{A}, \mathcal{C}) + \xrightarrow{(W, L)^*} \mathcal{C}$$

Note that by hypothesis the limits $\{W, ev B F\}_\alpha$ exist for each $B$. It follows then that the composite $F^* ev(-)$ actually lands in $\mathcal{H}om_{\gamma}(\mathcal{A}, \mathcal{C})$, so $L$ is defined.

Clearly $LB = \{W, ev B F\}_\alpha$. For each $B \in \mathcal{B}$, let $W \xrightarrow{\xi_B} \mathcal{C}(LB, (F\cdotp)B)$ be a $\alpha$-limit $w$-$\alpha$-cone in $\mathcal{C}$, $LB = \{W, (F\cdotp)(B)\}_\alpha$, let us denote the components of $\xi_B$ by $WA \xrightarrow{\xi_B} \mathcal{C}(LB, (FA)B)$. Then:

a) For each $A \in \mathcal{A}$, $WA \xrightarrow{\xi_B} \mathcal{C}(LB, (FA)B)$ are the components of a $s$-dinatural cone in the variable $B$.

\textit{proof:} Let $B \xrightarrow{h} B'$ in $\mathcal{B}$, and consider the following diagram as in the proof of Remark 2.2.11:

$$\begin{array}{ccc}
W & \xrightarrow{\xi_B} & \mathcal{C}(LB, (F\cdotp)B') \\
\downarrow{\xi_B} & & \downarrow{(Lh)^*} \\
\mathcal{C}(LB, (F\cdotp)B) & \xrightarrow{((F\cdotp)h)^*} & \mathcal{C}(LB, (F\cdotp)B')
\end{array} \quad LB'
$$

Then, evaluating at $A$ finishes the proof. \quad end proof of a) $\square$

b) The arrows in a) determine a $w$-$\alpha$-cone:

$WA \xrightarrow{\xi_B} \mathcal{H}om_{\gamma}(\mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C}), \mathcal{C}) \xrightarrow{F^*} \mathcal{H}om_{\gamma}(\mathcal{A}, \mathcal{C}) + \xrightarrow{(W, L)^*} \mathcal{C}$

\textit{proof:} Let $A \xrightarrow{g} A'$ in $\mathcal{A}$, and consider the following diagram:

$$\begin{array}{ccc}
WA & \xrightarrow{\xi_A} & \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})(L, FA) \\
W_g & \downarrow{\psi_g} & \downarrow{(Fg)_*} \\
WA' & \xrightarrow{\xi_{A'}} & \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})(L, FA')
\end{array} \quad \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})(L, FA) \xrightarrow{ev_B} \mathcal{C}(LB, (FA)B) \quad \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})(L, FA') \xrightarrow{ev_B} \mathcal{C}(LB, (FA')B)$$

By Proposition 2.3.3 it follows there are arrows $\xi_A$, $\xi_{A'}$ such that $ev_B \xi_A = \xi_B A, ev_B \xi_{A'} = \xi_B A'$. By hypothesis, for each $B$, $\xi_B$ is $\alpha$-natural (in the variable $A$). Thus there is a 2-cell $((Fg)_* \xi_B A) \xrightarrow{(\xi_B)_g} (Fg)_* \xi_B A \xrightarrow{Wg}$. It is straightforward to check that $((Fg)_* \xi_B A$ and $\xi_B A' Wg$ are $op\gamma$-dicones for the 2-functor $\mathcal{C}(L\cdotp, (F\cdotp)'\cdotp)$, and the $((Fg)_* \xi_B A$ determine a morphism of dicones. Then, the existence of $\xi_g$ as indicated in the diagram, $((Fg)_* \xi_B A$, follows from the isomorphism of categories:

$$\mathbf{Cat}(WA, \mathcal{H}om_{\op\gamma}(\mathcal{B}, \mathcal{C})(L\cdotp, (FA')\cdotp)) \xrightarrow{\cong} \mathcal{D}icones_{\op\gamma}(WA, \mathcal{C}(L, FA')).$$
This shows we have a $w$-$\alpha$-cone: $W \xrightarrow{\xi} \mathcal{H}om_{op\gamma}(B, C)(L, F-)$, thus also one into $\mathcal{H}om_{op\alpha}(B, C)(L, F-)$. The axioms of $\alpha$-naturality for $\xi_B$ can be checked using the corresponding axioms for $(\xi_B)_g$ and the isomorphism of categories above. 

\[\text{end proof of b)} \square\]

c) The $w$-$\alpha$-cone in b) is a $\alpha$-limit cone in $\mathcal{H}om_{op\beta}(B, C)$, $L = \{W, F\}_\alpha$.

proof: It only remains to show the universal property. Let $B \xrightarrow{H} C$ be a 2-functor and $W \xrightarrow{\rho} \mathcal{H}om_{op\beta}(B, C)(H, F-)$ be a $w$-\(\alpha\)-cone. We have:

\[
\begin{align*}
W & \xrightarrow{\xi_B} C(LB, (F-)B) \\
& \xrightarrow{\rho_B} C(HB, (F-)B) \\
& \xrightarrow{\exists \eta_B} LB \\
& \xrightarrow{\exists \eta_B} HB
\end{align*}
\]

We now prove that $\eta_B$ is $op\beta$-natural in the variable $B$. Let $B \xrightarrow{h} B'$ in $B$. Consider the isomorphism in the definition of $\xi_{B'}$,

\[C(HB, LB') \xrightarrow{(\xi_{B'})^*} \mathcal{H}om_\alpha(A, \text{Cat})(W, C(HB, (F-)B'))\]

We have the $\alpha$-natural structural 2-cell $\eta_h$ defined as follows:

\[(Hh)^*(\eta_{B'})^*\xi_{B'} = (Hh)^*\rho_{B'} \xRightarrow{\rho_{B'}} ((F-)h)_*\rho_B = ((F-)h)_*(\eta_B)^*\xi_B = \eta_B^*(((F-)h)_*\xi_B = (\eta_B)^*(Lh)^*\xi_{B'}).

We suggest the reader to use the diagram below to check the equations in this definition.
Thus, we have a $2$-cell $(\eta_B\cdot Hh)^*\xi_{B'} \Rightarrow (Lh\eta)_*\xi_{B'}$. By the isomorphism (1) above, it follows that there exist a unique $$ \begin{array}{lcl} \eta h & \eta h' & L h \\ H B & \eta h & L h \\ B' & \eta h' & \eta h' \\ \end{array} $$ such that $\rho_h = (\eta_h)^*\xi_{B'}$. The $\beta$-naturality axioms for $\eta$ follow from the $\beta$-dicone axioms for $\rho$ and the $op\beta$-naturality of $\rho_A(x)$, $A \in \mathcal{A}$, $x \in WA$. We leave to the reader the verification of the $2$-dimensional aspect of the universal property.

This finishes the proof of the proposition. Note that if $\mathcal{C}$ has tensor products with $\mathbf{2} = \{0 \rightarrow 1\}$, by Proposition 2.3.10 so does $\text{Hom}_{op\beta}(\mathcal{B}, \mathcal{C})$ and thus (as in [17, p.306]) the $2$-dimensional aspect of the universal property follows from the $1$-dimensional one. However we think it is pertinent not to assume that $\mathcal{C}$ has tensors, for example, in practice, $\mathcal{C}$ may only have all finite conical $p$-limits.

**Remark 2.6.3.** Note that to compute pointwise $\alpha$-limits in the $\text{Hom}_{op\beta}(\mathcal{B}, \mathcal{C})$ categories we use $\alpha$-limits in $\mathcal{C}$. Since $\text{Hom}_{op\beta}(\mathcal{B}, \mathcal{C}) \cong \text{Hom}_{\beta}(\mathcal{B}^{op}, \mathcal{C}^{co})$ (§1.1, item 14), to compute $\alpha$-limits in $\text{Hom}_{\beta}(\mathcal{B}, \mathcal{C})$ we use $\alpha$-limits in $\mathcal{C}^{co}$, that is, $op\alpha$-limits in $\mathcal{C}$ (Remark 2.2.6).

Since for $\beta = p$ or $s$ we have isomorphisms $\text{Hom}_{op\beta}(\mathcal{B}, \mathcal{C}) \cong \text{Hom}_{\beta}(\mathcal{B}, \mathcal{C})$, it follows:

**Corollary 2.6.4.** Weighted $\sigma$-limits are computed pointwise in the $2$-functor $2$-categories $\text{Hom}_s(\mathcal{B}, \mathcal{C})$ and $\text{Hom}_p(\mathcal{B}, \mathcal{C})$. The inclusion $\text{Hom}_s(\mathcal{B}, \mathcal{C}) \xrightarrow{\sigma} \text{Hom}_p(\mathcal{B}, \mathcal{C})$ preserves these limits, we have $i\{W, F\}_\sigma = \{W, iF\}_\sigma$.

**Remark 2.6.5.** In general, $s$-limits are not computed pointwise in $\text{Hom}_p(\mathcal{A}, \mathcal{B})$, see [3, Example 6.2] for a counterexample. The obstruction in the proof of Proposition 2.6.2 if one tries to prove this statement is that the definition $LB$ in the beginning of the proof would not be functorial in the variable $B$, as we do not have a $2$-functor $\text{Hom}_p(\mathcal{A}, \mathcal{C}) \xrightarrow{\{W, -\}} \mathcal{C}$.

### 2.7 Interchange formulas

As usual, from the pointwise computation it follows the commutativity of limits with limits. Recall the notation considered before Proposition 2.6.2.

**Proposition 2.7.1.** Let $\gamma \in \{\ell, p, s\}$, $\alpha \in \mathcal{L}_A$, $\beta \in \mathcal{L}_B$, such that $\alpha \geq \gamma$, $\beta \geq \gamma$. Let $\mathcal{A} \xrightarrow{F_1} \text{Hom}_{op\gamma}(\mathcal{B}, \mathcal{C})$, $\mathcal{B} \xrightarrow{F_2} \text{Hom}_{\gamma}(\mathcal{A}, \mathcal{C})$ be $2$ functors in correspondence as in §1.1, item 12. Consider weights $\mathcal{A} \xrightarrow{W_1} \text{Cat}$, $\mathcal{B} \xrightarrow{W_2} \text{Cat}$. Then, the following holds:

$$ \{W_1, \{W_r, F_r\}_\beta\}_\alpha \cong \{W_r, \{W_i, F_i\}_\alpha\}_\beta $$

**Proof.** By the usual reasoning it suffices to show it for the case $\mathcal{C} = \text{Cat}$. We have the following isomorphisms given by Proposition 2.2.8 and Corollary 2.2.9

$$ \{W_1, \{W_r, F_r\}_\beta\}_\alpha \cong \text{Hom}_\alpha(\mathcal{A}, \text{Cat})(W_1, \{W_r, F_r\}_\beta) \cong \{W_r, \text{Hom}_\alpha(\mathcal{A}, \text{Cat})(W_1, F_r\cdot -)\}_\beta $$

We conclude the proof by showing that $\{W_1, F_1\}_\alpha = \text{Hom}_\alpha(\mathcal{A}, \text{Cat})(W_1, F_r\cdot -)$. By Proposition 2.6.2 we can compute pointwise:

$$ \{W_1, F_1\}_\alpha(B) = \{W_1, (F_1\cdot -)(B)\}_\alpha = \text{Hom}_\alpha(\mathcal{A}, \text{Cat})(W_1, (F_1\cdot -)(B)) = \text{Hom}_\alpha(\mathcal{A}, \text{Cat})(W_1, F_r\cdot (-)) = \text{Hom}_\alpha(\mathcal{A}, \text{Cat})(W_1, F_r\cdot -)(B). $$

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The second equality is justified by Proposition 2.2.8 the third one follows from the formulas in §11 item 12 and the last is clear.

It is convenient to state with a slightly different notation a particular case which we will need in this paper;

**Proposition 2.7.2.** Let \( \sigma \in \mathcal{L}_I \). Consider a 2-functor \( I \to \text{Hom}_p(A,B) \) and a weight \( A \to \text{Cat} \). Then the following holds:

\[
\{ W, \sigma \text{Lim}_{i \in I} F_i \}_p = \sigma \text{Lim}_{i \in I} \{ W, F_i \}_p.
\]

The commutativity of weighted pseudolimits with conical \( \sigma \)-colimits, which is (as it is usual) a much deeper subject, is treated in [9] for \( \text{Cat} \)-valued 2-functors. We recall now this result noting that, while in [9, Theorem 3.2] it is stated for a 2-functor \( I \to \text{Hom}_s(A,\text{Cat}) \), since pseudolimits are also computed pointwise in \( \text{Hom}_p(A,\text{Cat}) \) a careful inspection of the proof yields:

**Theorem 2.7.3.** Let \( I \) be a \( \sigma \)-filtered 2-category, \( A \) a 2-category. Consider a 2-functor \( I \to \text{Hom}_p(A,\text{Cat}) \) and a finite weight (see Definition 3.2.2) \( A \to \text{Cat} \). Then the canonical comparison functor

\[
\diamond : \sigma \text{Lim}_{i \in I} \{ b_{W, F_i} \}_p \overset{\cong}{\longrightarrow} b_{W, \sigma \text{Lim}_{i \in I} F_i}_p
\]

is an equivalence of categories.

## 3 \( \sigma \)-filtered 2-categories

We fix throughout this section a 2-category \( \mathcal{C} \) and a 1-subcategory \( \Sigma \) of \( \mathcal{C} \) such that \( \Sigma \) contains all the objects of \( \mathcal{C} \). Note that this amounts to a family, that we will also denote by \( \Sigma \), of arrows of \( \mathcal{C} \) such that all the identities belong to \( \Sigma \) and \( \Sigma \) is closed by composition. We don’t ask \( \Sigma \) to contain neither the isomorphisms nor the equivalences of \( \mathcal{C} \).

### 3.1 The notion of \( \sigma \)-filtered

Recall that a non empty 1-category is filtered if and only if every finite diagram has a cone (see [23, § VII.6]). This happens if and only if two particular diagrams, corresponding to binary products and equalizers, have a cone (the two usual axioms of filtered category).

In the 2-dimensional case the notion of filteredness has been considered under the name of bifiltered in [20], and 2-filtered in [11], and it holds if and only if every finite diagram has a pseudocone (see [6]).

We introduce now the concept of 2-filteredness with respect to a family \( \Sigma \). It generalizes the definition of bifiltered in [20], which corresponds to the case where \( \Sigma \) consists of all the arrows of \( A \).

**Notation 3.1.1.** We add a circle to an arrow \( \cdot \longrightarrow \cdot \) to indicate that it belongs to \( \Sigma \).
**Definition 3.1.2.** We say that a pair \((C, \Sigma)\) is \(\sigma\)-filtered, or for brevity, that \(C\) is \(\sigma\)-filtered (with respect to \(\Sigma\)), if it is non empty and the following hold:

\(\sigma \text{F0.} \) Given \(A, B \in C\), there exist \(E \in C\) and morphisms \(\xymatrix{A \ar[r]^f & E} \)

\(\sigma \text{F1.} \) Given \(\xymatrix{A \ar[r]^f & B \in C}\), there exist a morphism \(\xymatrix{B \ar[r]^h & E}\) and a 2-cell \(hf \xrightarrow{\alpha} hg\).

If \(f \in \Sigma\), we may choose \(\alpha\) invertible.

\(\sigma \text{F2.} \) Given \(\xymatrix{A \ar@{..>}[r]^f & B \in C}\), there exists a morphism \(\xymatrix{B \ar[r]^h & E}\) such that \(h \alpha = h \beta\).

We say that \(C\) is \(\sigma\)-cofiltered if \(C^{op}\) is \(\sigma\)-filtered. We keep the same labels for the axioms.

**Proposition 3.1.3.** Consider the following finite diagrams:

1. \(\{0, 1\} \xrightarrow{F_1} C, \quad \{C, D\}\)
2. \(0 \xrightarrow{u \atop \nu \atop \nu} 1 \xrightarrow{F_2} C, \quad \{C \ar@{..>}[r]^f & D\}\)
3. \(0 \xrightarrow{u \atop \nu \atop \nu} 1 \xrightarrow{F_3} C, \quad \{C \ar@{..>}[r]^f & D\}\)

Let \(\Sigma_1, \Sigma_2, \Sigma_3\) (respectively) be the family of arrows that are mapped to arrows of \(\Sigma\), i.e. \(\Sigma_i = F^{-1}_i(\Sigma)\). Then, for each \(i\), the category \(\text{Cones}_{\Sigma}^{\Sigma_i}(F_i, E)\) (recall Definition 2.4.3) is equivalent (naturally in \(E\)) to the category \(\mathcal{A}_i\), whose objects and arrows are:

\(\mathcal{A}_1 \) Objects: Pairs of morphisms \(\xymatrix{C \ar[r]^h & E}\)

Arrows: Pairs of 2-cells \(f \Rightarrow f', g \Rightarrow g'\).

\(\mathcal{A}_2 \) Objects: An object consists of a morphism \(\xymatrix{D \ar[r]^h & E}\) together with a 2-cell \(hf \xrightarrow{\gamma} hg\), invertible if \(f \in \Sigma\).

Arrows: 2-cells \(h \xrightarrow{\eta} h'\) such that \(\gamma'(\eta f) = (\eta g)\gamma\).

\(\mathcal{A}_3 \) Objects: Morphisms \(\xymatrix{D \ar[r]^h & E}\) such that \(h \alpha = h \beta\).

Arrows: 2-cells \(h \xrightarrow{\eta} h'\).

**Proof.** Certainly item 1 requires no proof.

For item 2, we define the equivalence \(\text{Cones}_{\sigma}^{\Sigma_2}(F_2, E) \xrightarrow{\phi} \mathcal{A}_2\), and leave the verification of the details to the reader. Given a \(\sigma\)-cone \(\theta\), define \(h = \theta_1\), \(\gamma = \theta_0^{-1} \theta_u\). Given a morphism of \(\sigma\)-cones \(\theta \xrightarrow{\varphi} \theta'\), define \(\eta = \varphi_1\). Then it is easy to check that \(\phi\) is actually surjective on objects, and given \(\phi(\theta) \xrightarrow{\eta} \phi(\theta')\), the unique \(\theta \xrightarrow{\varphi} \theta'\) such that \(\phi(\varphi) = \eta\) is defined by \(\varphi_1 = \eta, \varphi_0 = \theta_0(\eta g)\theta_v^{-1}\).

For item 3, define \(h, \gamma\) and \(\eta\) as in item 2, but now since \(\theta\) is a \(\sigma\)-cone we have \(h \alpha = \gamma = h \beta\).

Then in this case the condition \(\gamma'(\eta f) = (\eta g)\gamma\) for a 2-cell \(h \xrightarrow{\eta} h'\) is \((h' \alpha)(\eta f) = (\eta g)(h \alpha)\), which holds by the interchange law. 

The following proposition expresses a basic property of $\sigma$-filteredness and it is a generalization of [24 \S VII.6, Lemma 1] to the 2-dimensional case. See also [6] where the case of $\Sigma$ consisting of all the arrows of $C$ is analyzed.

**Notation 3.1.4.** For a 2-functor $\Delta \overset{F}{\to} C$, we say that a $\sigma$-cone $\theta$ with vertex $E$ has *arrows in $\Sigma$* if the structural arrows $F(i) \overset{\theta_i}{\to} E$ are in $\Sigma$ for all $i \in \Delta$.

**Proposition 3.1.5.** The following are equivalent

i) $\mathcal{C}$ is $\sigma$-filtered.

ii) Each of the diagrams $F_1, F_2, F_3$ in Proposition 3.1.3 has a $\sigma$-cone (with respect to $F^{-1}(\Sigma)$) with arrows in $\Sigma$.

iii) Every finite 2-diagram $\Delta \overset{F}{\to} C$ (i.e. every 2-functor $\Delta \overset{F}{\to} C$ with $\Delta$ a finite 2-category) has a $\sigma$-cone (with respect to $F^{-1}(\Sigma)$) with arrows in $\Sigma$.

**Proof.** iii) $\Rightarrow$ ii) is trivial. ii) $\Rightarrow$ i) follows from the description of the $\sigma$-cones in Proposition 3.1.3. To show i) $\Rightarrow$ iii), suppose that $\mathcal{C}$ is $\sigma$-filtered and let $\Delta \overset{F}{\to} C$ be a finite 2-diagram.

Since $\Delta$ is finite, by axiom $\sigma\text{F0}$, we have morphisms $\{F_i \overset{\theta_i}{\to} E\}_{i \in \Delta}$.

We will modify $E$ and the arrows $\theta_i$ by going further, in order to have a $\sigma$-cone with arrows in $\Sigma$. We will do this one arrow $u$ of $\Delta$ at a time. Using axiom $\sigma\text{F1}$, there is a morphism $E \overset{h}{\to} E'$ and a 2-cell $\varnothing_j F(u) \overset{\theta_j}{\Rightarrow} \theta_i$, invertible if $F(u) \in \Sigma$. We denote $E'$ by $E$ again, the compositions $h \theta_i$ by $\theta_i$, and $h \theta_u$ by $\theta_u$ for all the pre-existing $\theta_u$. We repeat the procedure to have $\{\theta_j F(u) \overset{\theta_j}{\Rightarrow} \theta_i\}_{1 \leq j \leq \Delta}$, with $\theta_u$ invertible for all $u$ such that $F(u) \in \Sigma$.

Now we consider the equations $\text{LN0, LN1, LN2}$ of §1.1 item 11 expressing the lax naturality of $F \overset{\theta}{\Rightarrow} \Delta E$ (see Definition 2.4.3). A similar procedure, considering one equation at a time and using axiom $\sigma\text{F2}$ instead of $\sigma\text{F1}$, allows one to go further and make $\theta$ a $\sigma$-cone with arrows in $\Sigma$.

**Remark 3.1.6.** The reason why we consider $\sigma$-cones with arrows in $\Sigma$ will be clear in Proposition 3.2.6 below. We note nevertheless that a *weaker* notion of $\sigma$-filteredness where we ask that every finite 2-diagram has a $\sigma$-cone could also be worth considering in another context.

### 3.2 Exact 2-functors

**Definition 3.2.1.** Consider 2-functors $\mathcal{A} \overset{W}{\to} \text{Cat}$, $\mathcal{A} \overset{F}{\to} \mathcal{B} \overset{H}{\to} \mathcal{C}$. We say that $H$ preserves a $\sigma$-bilinear $b[W,F]_{\sigma}$ if $b[W,HF]_{\sigma}$ exists, and the canonical map $H b[W,F]_{\sigma} \to b[W,HF]_{\sigma}$ is an equivalence.

We will define the notion of finite weight and finite bilimit below. We note that there is a more general notion of finite (or finitary) weight ([18 \S 4], [29]) which we don’t consider here as it is not necessary for our purposes.

**Definition 3.2.2.**

1. We say that a 2-functor $\mathcal{A} \overset{W}{\to} \text{Cat}$ is a finite weight if $\mathcal{A}$ is a finite 2-category and for each $A \in \mathcal{A}$, $WA$ is a finite category. A finite bilimit is a weighted bilimit with finite weight.

2. Assume that $\mathcal{B}$ is a 2-category with finite weighted bilimits. We say that a 2-functor $\mathcal{B} \overset{H}{\to} \mathcal{C}$ is left exact if it preserves all finite bilimits.
Note that all finite weighted bilimits are required to exist in the domain category of exact 2-functors, but not necessarily in the codomain category.

**Remark 3.2.3.** There are some particular finite bilimits which will be relevant to us, as they are related to the \( \sigma \)-bilimits of the finite diagrams considered in Proposition 3.1.3. These are biproducts, biequalizers, biinserters and biequifiers.

We note, though we won’t use this fact, that finite biproducts, biequalizers and bicotensor products with \( 2 = \{ 0 \to 1 \} \) suffice to construct all finite weighted bilimits (The general proof in [27] can be restricted to the finite case, see [5, §6.2] for details). Since the bicotensor product with 2 can be constructed from the biinserter and the biequifier, the four bilimits above are also sufficient to construct all finite weighted bilimits. In particular we have that a 2-functor \( H \) is left exact if and only if these four bilimits exist and are preserved by \( H \).

1. **biinserter** \((f, g)\): for a pair of arrows \( C \xrightarrow{f} D \), this is the bilimit of the diagram

   \[
   \begin{array}{ccc}
   a & \xrightarrow{u} & b \\
   \downarrow & F_1 & \downarrow \\
   C & \xrightarrow{f} & D
   \end{array}
   \]

   weighted by \( a \xrightarrow{v} b \xleftarrow{u} \) \( \xrightarrow{0} \xrightarrow{1} \xrightarrow{2} \), see [17, (4.1)] for details.

   Note that if we consider the category \( I \) (consisting of two objects and an isomorphism) instead of 2, the weighted bilimit is the biequalizer \((f, g)\) (note that the biequalizer \((f, g)\) is also the biisoinserter \((f, g)\) see [17, p.308] and [5, Observation 5.23]).

2. **biequifier** \((\alpha, \beta)\): for a pair of 2-cells \( C \xrightarrow{\alpha} \xrightarrow{\beta} D \), this is the bilimit of the diagram

   \[
   \begin{array}{ccc}
   a & \xrightarrow{u} & b \\
   \downarrow & F_2 & \downarrow \\
   C & \xrightarrow{f} & D
   \end{array}
   \]

   weighted by \( a \xrightarrow{v} b \xleftarrow{u} \) \( \xrightarrow{0} \xrightarrow{1} \xrightarrow{2} \), see [17, (4.5)] for details.

**Proposition 3.2.4.** With the definitions above, for \( i = 1, 2 \), the category \( \text{Cones}_{W_i}^{\Sigma_i} (F_i, E) \) (recall Definition [2.2.7]) is equivalent (naturally in \( E \)) to the category \( \mathcal{B}_i \) whose objects and arrows are:

- **\( \mathcal{B}_1 \)**
  - Objects: An object consists of a morphism \( E \xrightarrow{h} C \) together with a 2-cell \( \gamma : f h \Rightarrow gh \).
  - Arrows: Natural transformations \( h \Rightarrow h' \) such that \( \gamma'(f \eta) = (g \eta) \gamma \).

  For the case where we replace 2 by \( I \), we have the same description of \( \mathcal{B}_1 \), except that the 2-cell \( \gamma \) is required to be invertible.

- **\( \mathcal{B}_2 \)**
  - Objects: Morphisms \( E \xrightarrow{h} C \) such that \( \alpha h = \beta h \).
  - Arrows: Natural transformations \( h \Rightarrow h' \).

**Proof.** We denote 2 = \( \{ 0 \xrightarrow{\ell} 1 \} \). Note that a cone \( W_1 = \theta \xrightarrow{\theta_0(0)} C(E, F_1(-)) \) amounts to \( E \xrightarrow{\theta_0(\ell)} C \), \( E \xrightarrow{\theta_0(\ell) \psi} D \), and invertible 2-cells \( \theta_a f \Rightarrow \theta_b(0), \theta_a g \Rightarrow \theta_b(1) \). The definition of the equivalence \( \text{Cones}_{\Sigma_i}^{\Sigma_i} (F_i, E) \xrightarrow{\phi} \mathcal{B}_1 \) on objects is by the formulas \( h = \theta_a, \gamma = \theta_v^{-1} \theta_b(\ell) \theta_u \), this is easily seen to be surjective.
A morphism of cones $W_1 \xrightarrow{\theta} C(E, F_1(-))$ is a modification given by natural transformations $\theta_u \xrightarrow{\varphi_u} \theta'_u$, $\theta_b \xrightarrow{\varphi_b} \theta'_{b, u}$, therefore by 2-cells $\varphi$, $(\varphi_b)_0$, $(\varphi_b)_1$ such that $\theta'_b(0)(\varphi_b)_0 = (\varphi_b)_1$ and $\theta'_{b, u}(0) = (\varphi_b)_1$. The definition of $\varphi$ is by the formula $\eta = \varphi_u$, then from the equations above we note that $(\varphi_b)_0$ and $(\varphi_b)_1$ are determined by $\varphi_u$, and the condition $\gamma'(f \eta) = (g \eta) \gamma$ is equivalent to the equation $\theta'_b(0)(\varphi_b)_0 = (\varphi_b)_1$. Then $\varphi$ is full and faithful.

In the case where we replace 2 by 1, the formulas are the same, simply note that $\gamma$ is invertible. If $i = 2$, we consider $h, \gamma, \eta$ as in the case $i = 1$, then from the 2-naturality of $\theta$ it follows $\alpha h = \gamma = \beta h$ and the proof finishes like the proof of Proposition 3.1.3.

**Remark 3.2.5.** Let $\mathcal{B}$ be a 2-category. A comparison of the dual of the descriptions of the $\sigma$-cones in Proposition 3.1.3 (see 2.4.17) and the descriptions in Proposition 3.2.4 immediately yields:

1. For $\{0, 1\} \xrightarrow{G_1} \{C, D\}$, $\varphi \text{biLim} G_1 = \text{biproduct}(C, D)$.

2. (i) For $\{0 \xrightarrow{u} 1\} \xrightarrow{G_2} \{C \xrightarrow{f} D\}$, $\varphi \text{biLim} G_2 = \text{biequalizer}(f, g)$.

   (ii) For $\{0 \xrightarrow{u} 1\} \xrightarrow{G_2} \{C \xrightarrow{f} D\}$, $\varphi \text{biLim} G_2 = \text{binserter}(f, g)$.

3. For $\{0 \xrightarrow{u} 1\} \xrightarrow{G_3} \{C \xrightarrow{f} D\}$, and for $\{0 \xrightarrow{u} 1\} \xrightarrow{G_3} \{C \xrightarrow{f} D\}$, $\varphi \text{biLim} G_3 = \text{biequifier}(\alpha, \beta)$.

It follows that if a 2-functor $P$ is left exact then $P$ preserves the $\varphi \text{biLim} G_i$, for $i = 1, 2, 3$ with the 1-subcategories $\Sigma$ considered above.

The following result is a 2-dimensional version of a result that is known for $\text{Set}$-valued functors, see for example [10, Proposition 4.87].

**Proposition 3.2.6.** Let $\mathcal{A} \xrightarrow{P} \text{Cat}$ be a 2-functor. Let $\Delta \xrightarrow{F} \mathcal{E} l_P$ be a 2-functor, and set $\Sigma = F^{-1}(\mathcal{E} l_P)$. Assume that $\varnothing_P F$ has a $\sigma$-bilimit $L$ in $\mathcal{A}$ that is preserved by $P$. Then there exist $c \in PL$ and a $\sigma$-cone for $F$ with arrows in $\mathcal{E} l_P$ with vertex $(c, L)$, which is the $\sigma$-bilimit of $F$.

**Proof.** We are going to denote the action of $F$ by $Fi = (a_i, F_i)$ for each $i \in \Delta$, $Fu = (F_u, \sigma_u)$ for each $i \xrightarrow{u} j \in \Delta$, $F\theta = F\theta$ for each $i \xrightarrow{u} \theta_j v \xrightarrow{\theta_j} j \in \Delta$.

Consider the $\sigma$-bilimit $L$ of the 2-functor $\Delta \xrightarrow{\varnothing_P F} \mathcal{A}$, then $L$ is furnished with a $\sigma$-cone $\{F_i \xrightarrow{h_i} F_j\}_{i \in \Delta}$, $\{F_u h_i \xrightarrow{h_j} F_j\}_{i \xrightarrow{u} j \in \Delta}$. This $\sigma$-bilimit is preserved by $P$, this means that if we
Proposition 3.2.7. Let \( A \) be a 2-category with finite weighted bilimits and let \( A \to \mathbf{Cat} \) be a 2-functor. If \( P \) is left exact then \( \mathcal{E}l_{P} \) is \( \sigma \)-cofiltered with respect to \( \mathcal{C}_{P} \).
Proof. By Proposition 3.1.5 it suffices to show that each of the diagrams $F_1, F_2, F_3$ considered in Proposition 3.1.3 has a $\sigma$-cone with arrows in $\mathcal{C}_P$. Let $i = 1, 2, 3$, by Remark 3.2.5 the bi_limits $\circ \mathcal{C}_P F_i$ exists in $\mathcal{A}$ and is preserved by $P$ (note that all the possible $\Sigma = F_i^{-1}(\mathcal{C}_P)$ were considered in the remark). Then by Proposition 3.2.6 $F_i$ has a $\sigma$-cone with arrows in $\mathcal{C}_P$ and we are done.

\section*{3.3 $\sigma$-cofinal 2-functors}

In this section we define $\sigma$-cofinal 2-functors and establish some properties that will be used in the proof of Theorem 4.2.7. Our definition is a 2-dimensional $\sigma$-version of the definition in SGA4 for the case when $\mathcal{C}$ is filtered (see [1, 8.1.1]). If $\Sigma = \mathcal{C}_0$, we recover [7, Definition 1.3.1]. We do not deal with a more general concept of $\sigma$-cofinality since this particular case is relevant enough and it is the only one that we need in this paper. We leave for future work the development of the full theory of $\sigma$-cofinal 2-functors.

\begin{definition}
Let $\mathcal{C}, \mathcal{C}'$ be 2-categories and $\Sigma, \Sigma'$ 1-subcategories of $\mathcal{C}, \mathcal{C}'$ respectively. Suppose that $\mathcal{C}$ is $\sigma$-filtered. We say that a 2-functor $\mathcal{C} \xrightarrow{T} \mathcal{C}'$ is $\sigma$-cofinal (with respect to $\Sigma$ and $\Sigma'$) if it satisfies:

\begin{enumerate}
  \item[$\sigma \text{C0.}$] Given $\mathcal{C}' \in \mathcal{C}'$, there exist $\mathcal{C} \in \mathcal{C}$ and a morphism $\mathcal{C}' \xrightarrow{\alpha} TC$ in $\mathcal{C}'$.
  \item[$\sigma \text{C1.}$] Given $\mathcal{C}' \xrightarrow{f} \mathcal{C}' \in \mathcal{C}'$, there exist a morphism $\mathcal{C} \xrightarrow{\beta} \mathcal{D}$ and a 2-cell $T(u)f \xrightarrow{\alpha} T(u)g$. If $f \in \Sigma'$, we may choose $\alpha$ invertible.
  \item[$\sigma \text{C2.}$] Given $\mathcal{C} \in \mathcal{C}, \mathcal{C}' \in \mathcal{C}'$ and 2-cells $\mathcal{C}' \xrightarrow{\alpha \beta} \mathcal{C}' \xrightarrow{f} \mathcal{C}' \xrightarrow{\gamma} \mathcal{D} \in \mathcal{C}$ such that $T(u)\alpha = T(u)\beta$.
\end{enumerate}

If $\mathcal{C}$ is $\sigma$-cofiltered, we say that $\mathcal{C} \xrightarrow{T} \mathcal{C}'$ is $\sigma$-initial if $\mathcal{C}^{\op} \xrightarrow{T^{\op}} \mathcal{C}'^{\op}$ is $\sigma$-cofinal. We keep the same labels for the axioms.

The following proposition is the only result concerning $\sigma$-cofinal functors that we need in this paper, and it is the analogous to item c) of [1, 8.1.3].

\begin{proposition}
Let $\mathcal{C}, \mathcal{C}'$ be 2-categories, $\mathcal{C} \xrightarrow{T} \mathcal{C}'$ a 2-functor, $\Sigma'$ a subcategory of $\mathcal{C}'$, and $\Sigma = T^{-1}(\Sigma')$. If the following hold:

1. $\mathcal{C}'$ is $\sigma$-filtered,
2. $T$ is pseudo-fully-faithful,
3. Condition $\sigma \text{C0}$ from Definition 3.3.1

Then $\mathcal{C}$ is $\sigma$-filtered and $T$ is $\sigma$-cofinal.

\end{proposition}

\begin{proof}
We observe that since $T$ is pseudo-fully-faithful and $\Sigma = T^{-1}(\Sigma')$ we have:

(1) For every arrow $\mathcal{C} \xrightarrow{h} \mathcal{D}$ in $\mathcal{C}'$, there exists $\mathcal{C} \xrightarrow{u} \mathcal{D}$ such that $T(u) \cong h$.

We are going to check first that axioms $\sigma \text{C1}$ and $\sigma \text{C2}$ from Definition 3.3.1 are satisfied:
σC1. Given \( C \in \mathcal{C}, \ C' \in \mathcal{C}', \ C' \xrightarrow{f/g} TC \in \mathcal{C} \), since \( C' \) is \( \sigma \)-filtered, there exist a morphism \( TC \xrightarrow{h} D' \) and a 2-cell \( hf \Rightarrow hg \), that we may take invertible if \( f \in \Sigma' \). Then, by the fact that condition \( \sigma \text{CO} \) is satisfied, there exist an object \( D \in \mathcal{C} \) and a morphism \( D' \xrightarrow{l} TD \in \mathcal{C}' \). Now, by (1) above, there exists a morphism \( C \xrightarrow{w} D \) such that \( T(u) \cong lh \) and so we have a 2-cell \( T(u)f \cong lhf = \alpha \), \( lhg \cong T(u)g \), which is invertible if \( f \in \Sigma' \).

σC2. Given \( C \in \mathcal{C}, \ C' \in \mathcal{C}' \) and 2-cells \( C' \xrightarrow{\alpha/\beta, \psi} TC \in \mathcal{C} \), since \( C' \) is \( \sigma \)-filtered, there exists a morphism \( TC \xrightarrow{h} D' \) such that \( h\alpha = h\beta \). Then, by the fact that condition \( \sigma \text{CO} \) is satisfied, there exist an object \( D \in \mathcal{C} \) and a morphism \( D' \xrightarrow{l} TD \). Now, by (1) above, there exists a morphism \( C \xrightarrow{w} D \) such that \( T(u) \cong lh \). This, together with the fact that \( h\alpha = h\beta \) can be used to prove that \( T(u)\alpha = T(u)\beta \).

It only remains to check that \( \sigma \text{FO}, \sigma \text{FI} \) and \( \sigma \text{F2} \) from Definition 3.1.2 are satisfied for the 2-category \( \mathcal{C} \):

σF0. Given \( C, D \in \mathcal{C} \), since \( C' \) is \( \sigma \)-filtered, there exist morphisms \( TC \xrightarrow{f/g} E' \). Then, since \( T \) is \( \sigma \)-cofinal, there exist an object \( E \in \mathcal{C} \) and a morphism \( E' \xrightarrow{h} TE \). Now, by (1) above, this yields morphisms \( C \xrightarrow{h} E \).

σF1. Given \( C \xrightarrow{u/v} D \in \mathcal{C} \), consider \( TC \xrightarrow{Tu/Tv} TD \in \mathcal{C}' \). Since \( T \) is \( \sigma \)-cofinal, there exist a morphism \( D \xrightarrow{x} E \) and a 2-cell \( T(wu) = TwTu \xrightarrow{\alpha} TwTv = T(wv) \), that we may take invertible if \( u \in \Sigma \). Then, since \( T \) is pseudo-fully-faithful, this gives a 2-cell \( u = v \), which is invertible if \( u \in \Sigma \).

σF2. Given \( C \xrightarrow{\theta/\psi} D \in \mathcal{C} \), consider \( TC \xrightarrow{T\theta/T\psi} TD \in \mathcal{C}' \). Since \( T \) is \( \sigma \)-cofinal, there exists a morphism \( D \xrightarrow{x} E \) such that \( T(w\theta) = TwT\theta = TwT\eta = T(\psi) \). Then, since \( T \) is pseudo-fully-faithful, \( w\theta = \psi \).

\[ \square \]

**Proposition 3.3.3.** Let \( P, Q : \mathcal{A} \rightarrow \text{Cat} \) be 2-functors, and \( P \xrightarrow{\eta} Q \) a pseudonatural transformation. If \( \eta_A \) is full and faithful for each \( A \in \mathcal{A} \), then the induced 2-functor \( \mathcal{E}_P \xrightarrow{T_\eta} \mathcal{E}_Q \) (recall [1.2.5]) is 2-fully-faithful and the 1-subcategories given by the cocartesian arrows satisfy \( \mathcal{E}_P = T_{\eta}^{-1}(\mathcal{E}_Q) \).

\[ \text{Proof.} \] Recall the formulas in [1.2.5]. Let \( (\eta_A(x), A) \xrightarrow{(f,\psi)} (\eta_B(y), B) \), consider then \( \eta_B(Pf(x)) \xrightarrow{(\psi)^{-1}x} Qf\eta_A(x) \xrightarrow{\psi} \eta_B(y) \), since \( \eta_B \) is full and faithful there is a unique \( Pf(x) \xrightarrow{\eta_B(y)} y \) such that \( T_\eta(f, \psi) = (f, \psi) \). This shows that \( T_\eta \) is 2-fully-faithful (the fact that we have an isomorphism between 2-cells is trivial).
Remark 4.1.3. Let \( \eta \) be given by the cocartesian arrows satisfy \( \sigma \). It is immediate from the formulas that the case results that will be needed later. For the sake of simplicity in the exposition, we consider only there are pseudofunctors between pseudofunctors. With the same arguing as the one in Remark 2.2.11 it follows that there are pseudofunctors.

4 Flat pseudofunctors and the main theorems

4.1 Flat pseudofunctors

In this subsection we will consider pseudofunctors between 2-categories. Though our objective when starting the research that led to this paper was to have results for 2-functors, it turned out that the correct generality in which to define flat 2-functors is to consider flat pseudofunctors. For 2-categories \( \mathcal{A}, \mathcal{B} \), we denote by \( \text{pHom}_p(\mathcal{A}, \mathcal{B}) \) the 2-category of pseudofunctors, pseudonatural transformations and modifications. We refer the reader to the Appendix A for the complete definitions of these concepts, noting that we will not need the explicit formulas of Definition A.1 in this section.

Weighted bilimits for pseudofunctors are considered for example in [28], [12], [22, § 2]. We recall this notion, with an approach more similar to the one of Definition A.1 in this section.

Proposition 3.3.4. Consider 2-functors \( \mathcal{A} \xrightarrow{H} \mathcal{B} \xrightarrow{F} \text{Cat} \), and the induced 2-functor \( \mathcal{E}l_{PH} \xrightarrow{T_H} \mathcal{E}l_{P} \) as in 1.2.6. If \( H \) is 2-fully-faithful, then so is \( T_H \) and the 1-subcategories given by the cocartesian arrows satisfy \( \mathcal{E}P_{PH} = T_H^{-1}(\mathcal{E}P) \).

Proof. It is immediate from the formulas \( T_H(f, \varphi) = (Hf, \varphi) \), \( T_H(\theta) = H\theta \) in 1.2.6. □

4 Flat pseudofunctors and the main theorems
where the subscript \( + \) indicates the full-subcategories with objects such that the corresponding bilimits exist.

As for 2-functors, we refer to equivalences in \( p\text{Hom}_p(A, B) \) as pseudo-equivalences. Since pseudofunctors send equivalences to equivalences, we have:

1. If \( \alpha \) is a pseudo-equivalence, then \[ p\{W, F\}_p \xrightarrow{b(\alpha, F)_p} p\{V, F\}_p \] is an equivalence.

2. If \( \beta \) is a pseudo-equivalence, then \[ p\{W, F\}_p \xrightarrow{b(W, \beta)_p} p\{W, G\}_p \] is an equivalence. \( \square \)

Note that the definitions of preservation of bilimits (Definition 3.2.1), and left exact (Definition 3.2.2) make perfect sense for pseudofunctors. From Remark 4.1.3 item 2, it follows:

**Corollary 4.1.4.** Let \( A \xrightarrow{\beta} B \) be a pseudo-equivalence between pseudofunctors. Then any weighted bilimit preserved by \( F \) is also preserved by \( G \). In particular, \( F \) is left exact if and only if \( G \) is so. \( \square \)

Recall that a Set-valued functor is flat when its left Kan extension along the Yoneda embedding is left exact (see for example [24, § VII.5]). This notion is considered in [18, § 6] for \( V \)-enriched categories in general, and in particular for \( V = \text{Cat} \). However, as it is usually the case (for example with limits), the \( \text{Cat} \)-enriched version is too strict, and a relaxed version is the important notion.

In Definition 4.1.11 below, we will introduce the notion of flat pseudofunctor into \( \text{Cat} \). The reader should be aware that if \( A \xrightarrow{P} \text{Cat} \) is a 2-functor (as we will consider in § 4.2), both Kelly’s notion of flat and ours make sense, but are not at all equivalent. We will always be referring to our notion.

A relaxed notion of Kan extension was already considered in [22], where it was denoted pseudo Kan extension. We review the main results while, as it is defined by a bicolimit, changing the notation into the one adopted in this paper. We will use (and therefore choose to define) the left bi-Kan extension.

Let \( C \) be a 2-category with weighted bicolimits. We will only use the case \( C = \text{Cat} \). We review the main results while, as it is defined by a bicolimit, changing the notation into the one adopted in this paper. We will use (and therefore choose to define) the left bi-Kan extension.

Let \( C \) be a 2-category with weighted bicolimits. We will only use the case \( C = \text{Cat} \) in this paper. Given two pseudofunctors \( A \xrightarrow{P} C \), \( A \xrightarrow{H} \mathcal{E} \), consider the composite \( \mathcal{E} \xrightarrow{h} p\text{Hom}_p(\mathcal{E}^{op}, \text{Cat}) \xrightarrow{H} p\text{Hom}_p(A^{op}, \text{Cat}) \) of the Yoneda embedding with the pseudofunctor determined by precomposition with \( H \), which we denote \( E(H, -) = H^* \circ h \).

We have:

**Definition 4.1.5 ([22, 9.3]).** The left (pointwise) bi-Kan extension of \( P \) along \( H \) is the pseudofunctor \( L = \text{Lan}_H P : \mathcal{E} \longrightarrow C \) given by the formula \( LE = \mathcal{E}(H, E)_{bi \otimes_p P} \) for \( E \in \mathcal{E} \), that is \( L = (H^* \circ h)_{bi \otimes_p P} \) (see Remark 4.1.7).

The pointwise bi-Kan extension has the following important (but, as it is well known in the strict case, not always characteristic) property:

**Proposition 4.1.6 ([22, 9.6]).** Given pseudofunctors \( A \xrightarrow{P} C \), \( A \xrightarrow{H} \mathcal{E} \), for each pseudofunctor \( \mathcal{E} \xrightarrow{Q} C \) we have an equivalence

\[ p\text{Hom}_p(\mathcal{E}, C)(\text{Lan}_H P, Q) \xrightarrow{\sim} p\text{Hom}_p(A, C)(P, QH) \] (4.1.7)

pseudonatural in \( Q \). \( \square \)
Remark 4.1.8. Equation (4.1.7) expresses a biadjunction between precomposition with $H$ and $\text{Lan}_H$. The unit of this biadjunction consists of a pseudonatural transformation $P \xrightarrow{\eta} \text{Lan}_H P \circ H$, which is given by $\eta = r(id_{\text{Lan}_H P})$ in (4.1.7):

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & \mathcal{E} \\
\downarrow \eta & & \downarrow \text{Lan}_H P \\
\mathcal{C} & \xleftarrow{P} & \mathcal{E}
\end{array}
\]

It can be seen (following the $id_{\text{Lan}_H P}$ in the chain of equivalences in the proof of [22, 9.6]) that $\eta_A = (\nu_{HA})_A(id_{HA})$, where for each $E \in \mathcal{E}$ we denote by $\nu_E$ the unit of the bicolimit $(\text{Lan}_H P)E = \mathcal{E}(H,E)_{bi} \otimes P$, $\mathcal{E}(H,E) \xrightarrow{\nu_E} \mathcal{C}(P-, (\text{Lan}_H P)E)$.

From Remark 4.1.3, item 2, we have:

Proposition 4.1.9. Consider pseudofunctors $A \xrightarrow{P,Q} \mathcal{C}$, $A \xrightarrow{H} \mathcal{E}$. If $P$ and $Q$ are pseudo-equivalent, then so are $\text{Lan}_H P$ and $\text{Lan}_H Q$.

If $H$ is pseudo-fully-faithful, then the bi-Kan extension is really a (pseudo) extension:

Proposition 4.1.10 ([22, 9.5]). With the notation of Definition 4.1.2, if $H$ is pseudo-fully-faithful, then the unit $\eta$ of Remark 4.1.8 is a pseudo-equivalence (recall that this amounts to each $\eta_A$ being an equivalence of categories).

Definition 4.1.11. Let $A \xrightarrow{P} \mathcal{C}$ be a pseudofunctor, consider the Yoneda embedding $\mathcal{A} \xrightarrow{h} \mathcal{C}$, $A \xrightarrow{h} \mathcal{C}$, we denote $P^* = \text{Lan}_h P$. We say that $P$ is flat if $P^*$ is left exact (note that this is well defined by Corollary 4.1.4). Note that, by Proposition 4.1.10, the following diagram commutes up to pseudo-equivalence:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{h} & \mathcal{C}, \mathcal{A} \xrightarrow{P} \mathcal{C} \\
\downarrow P & & \downarrow \text{Lan}_h P \\
\mathcal{C} & \xleftarrow{P^*} & \mathcal{C}
\end{array}
\] (4.1.12)

Proposition 4.1.13. A flat pseudofunctor $A \xrightarrow{P} \mathcal{C}$ preserves any finite (weighted) bilimit that exists in $A$.

Proof. It follows immediately from diagram (4.1.12) and Corollary 4.1.4 (note that $h$ preserves weighted bilimits by Corollary 2.2.9).

Consider $P$, $h$, $P^*$ as in Definition 4.1.11. It follows from Remark 4.1.3 item 1, that for a 2-functor $A^{op} \xrightarrow{F} \mathcal{C}$ the formula $P^*F = \text{Hom}_p(A^{op},\mathcal{C})(h,F)_{bi} \otimes P$ in Definition 4.1.5 is pseudo-equivalent by Yoneda to the usual coend formula $P^*F = F_{bi} \otimes P$ (recall Corollary 2.3.12).

For a 2-functor $A^{op} \xrightarrow{F} \mathcal{C}$, from the dual case of Proposition 2.4.13 we have $F \approx F \otimes_p h$. Since this pseudo-colimit is computed pointwise by Proposition 2.6.2 for $A \in \mathcal{A}$ we have $FA \approx F \otimes_p A(-, -)$. It follows:

Proposition 4.1.14. The bi-Kan extension of a representable 2-functor $A(A, -)$ along $h$ can be chosen to be the evaluation 2-functor $\mathcal{C} \xrightarrow{\text{ev}_A} \mathcal{C}$. Since by Proposition 2.6.2 the evaluations preserve any weighted pseudolimit, we have in particular that the representable 2-functors are flat.
4.2 The main theorem

Let $C$ be a 2-category with weighted pseudo-colimits. We will only need the case $C = \text{Cat}$ in this paper.

**Remark 4.2.1.** Consider a 2-functor $\mathcal{A} \xrightarrow{P} C$, and a 2-functor $\mathcal{A} \xrightarrow{H} \mathcal{E}$. Note that we can compute the bi-Kan extension $L = \text{Lan}_HP$ of Definition 4.1.5 as a 2-functor $\mathcal{E} \xrightarrow{L} C$. The definition of $L$ is given by the formula $LE = \mathcal{E}(H,E) \circ_P P$, but we can compute it by the equivalent pseudo-colimit $LE = \mathcal{E}(H,E) \otimes_P P$. □

**Remark 4.2.2.** Consider 2-functors $\mathcal{I} \xrightarrow{F} \mathcal{Hom}_p(\mathcal{A},\mathcal{C})$, and $\mathcal{A} \xrightarrow{H} \mathcal{E}$. From the dual of Proposition 2.7.2 we have the equation $\mathcal{E}(H,E) \circ_P \sigma\text{Lim}_{i \in \mathcal{I}} F_i = \sigma\text{Lim}_{i \in \mathcal{I}} \mathcal{E}(H,E) \circ_P F_i$, which together with the fact that $\sigma$-colimits are computed pointwise, implies immediately the equation $\text{Lan}_H(\sigma\text{Lim}_{i \in \mathcal{I}} F_i)E = (\sigma\text{Lim}_{i \in \mathcal{I}} \text{Lan}_H F_i)E$. That is, the left bi-Kan extension commutes with $\sigma$-colimits. □

With this, and using again that $\sigma$-colimits in $\mathcal{Hom}_p(\mathcal{A},\text{Cat})$ and are computed pointwise, we have the following immediate corollary of Theorem 2.7.3.

**Proposition 4.2.3.** A $\sigma$-filtered $\sigma$-colimit in $\mathcal{Hom}_p(\mathcal{A},\text{Cat})$ of left exact 2-functors is left exact. □

From Remark 4.2.2 and Corollary 4.2.3 it follows (all $\sigma$-colimits below are considered in $\mathcal{Hom}_p(\mathcal{A},\text{Cat})$):

**Corollary 4.2.4.** A $\sigma$-filtered $\sigma$-colimit of flat 2-functors is flat. In particular, by Proposition 4.1.14 a $\sigma$-filtered $\sigma$-colimit of representable 2-functors is flat. □

**Lemma 4.2.5.** Let $A \xrightarrow{P} \text{Cat}$, $\mathcal{A} \xrightarrow{H} \mathcal{E}$, $\mathcal{E} \xrightarrow{L} \text{Cat}$ as in 4.2.1. Consider the 1-subcategories $\mathscr{C}_P$ of $\mathcal{E}l_P$, and $\mathscr{C}_L$ of $\mathcal{E}l_L$ as in Definition 1.2.11 Then there exists a canonical 2-functor

$$T : \mathcal{E}l_P \longrightarrow \mathcal{E}l_L$$

satisfying (the dual of) axiom $\sigma\text{C0}$ in Definition 3.3.1. If $H$ is 2-fully-faithful, then so is $T$ and $\mathscr{C}_P = T^{-1}(\mathscr{C}_L)$.

**Proof.** $T$ is defined as the composition of the 2-functors $\mathcal{E}l_P \xrightarrow{T_\eta} \mathcal{E}l_{HL} \xrightarrow{T_H} \mathcal{E}l_L$ considered in propositions 3.3.3 and 3.3.4, where $\eta$ is the pseudonatural transformation of Remark 4.1.8. Then we have the formula $T(x,A) = (\eta_A(x), HA)$. Let $(c,E) \in \mathcal{E}l_L$, we will show that there is an arrow in $\mathscr{C}_L$ of the form $(\eta_A(x), HA) \xrightarrow{\theta \circ id} (c,E)$.

We have $c \in LE = \mathcal{E}(H-,E) \circ_P P$, then by Lemma 2.5.4 there exist $A \in \mathcal{A}$, $HA \xrightarrow{\theta} E$ and $\nu \in P$ such that $(\nu_{HA})_{\theta}(x) = c$.

We consider the following diagram, which commutes by definition of $L$ on the arrow $\theta$ (see Remark 2.2.11).

\[
\begin{array}{ccc}
\mathcal{E}(HA,E) & \xrightarrow{(\nu_{HA})_{\theta}} & \text{Cat}(PA,LE) \\
\eta \downarrow & & \downarrow (L\theta)_* \\
\mathcal{E}(HA,HA) & \xrightarrow{(\nu_{HA})_{\theta}} & \text{Cat}(PA,LHA)
\end{array}
\]

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Element chasing \(id_{HA}\) and then evaluating at \(x\), we have the equality
\[
c = (\nu_E)_A(\theta)(x) = L\theta \circ (\nu_{HA})_A(id_{HA})(x) = L\theta(\eta_A(x)),
\]
which expresses the fact that \((\theta, id)\) is an arrow of \(\mathcal{E}_L\) as desired.

If \(H\) is 2-fully-faithful, by Proposition 1.1.10 each \(\eta_A\) is full and faithful and then by Propositions 3.3.3 and 3.3.4 both \(T_\eta\) and \(T_H\) are 2-fully-faithful and \(\mathcal{C}_P = T_\eta^{-1}(\mathcal{C}_{LH}) = T^{-1}(\mathcal{C}_L)\).

Using Proposition 3.3.2 for the 2-functor \(T^{op} : \mathcal{E}_P^{op} \longrightarrow \mathcal{E}_L^{op}\) it follows

**Corollary 4.2.6.** Under the hypothesis of Lemma 4.2.5, if \(\mathcal{E}_L\) is \(\sigma\)-cofiltered (with respect to \(\mathcal{C}_L\)), then \(\mathcal{E}_P\) is \(\sigma\)-cofiltered (with respect to \(\mathcal{C}_P\)).

It is a classical result (see for example [24, §VII.6]) that every \(\mathcal{S}et\)-valued functor is a filtered colimit of representable functors, that, as far as we know (see [18, (6.4)]) has no known generalization to other base categories. Here we extend this result to 2-dimensional category theory. Note that from Theorem 1.2.7 it follows that if a 2-functor \(\mathcal{A} \longrightarrow \mathcal{Cat}\) is pseudo-equivalent to any \(\sigma\)-filtered \(\sigma\)-colimit of representable 2-functors, then the \(\sigma\)-colimit in its canonical expression 2.4.14 (whose diagram is actually in \(\mathcal{H}om_s(A, \mathcal{Cat})\), see remark 2.4.15) is also \(\sigma\)-filtered.

**Theorem 4.2.7.** Let \(\mathcal{A} \longrightarrow \mathcal{Cat}\) be a 2-functor. Then the following are equivalent.

1. (i) \(\mathcal{E}_P\) is \(\sigma\)-cofiltered with respect to the family \(\mathcal{C}_P\) of cocartesian arrows.
2. (ii) \(P\) is equivalent to a \(\sigma\)-filtered \(\sigma\)-colimit of representable 2-functors in \(\mathcal{H}om_p(A, \mathcal{Cat})\).
3. (iii) \(P\) is flat.

**Proof.** (i) \(\Rightarrow\) (ii) follows immediately by the canonical expression 2.4.14. (ii) \(\Rightarrow\) (iii) holds by Corollary 1.2.7 (note that flatness is preserved by pseudo-equivalence by Corollary 4.1.9). (iii) \(\Rightarrow\) (i): If \(P^*\) is left exact, by Propositions 2.6.2 and 3.2.7 \(\mathcal{E}_P^*\) is \(\sigma\)-cofiltered with respect to \(\mathcal{C}_P^*\). Then (i) follows by Corollary 4.2.6.

**Remark 4.2.8.** Note that, since \(\sigma\)-bicategories are defined up to equivalence, we can say that the flat 2-functors are exactly the \(\sigma\)-filtered \(\sigma\)-bicategories of representable 2-functors.

Combining Proposition 4.1.13, Proposition 3.2.7 and the implication (i) \(\Rightarrow\) (iii) in the theorem above, it follows:

**Proposition 4.2.9.** If \(\mathcal{A}\) is finitely complete, then a 2-functor \(\mathcal{A} \longrightarrow \mathcal{Cat}\) is flat if and only if it is left exact.

**References**

[1] Artin M., Grothendieck A., Verdier J., SGA 4, Ch IV, (1963-64), Springer Lecture Notes in Mathematics 269 (1972).

[2] Bird G.J., Limits in 2-categories of locally-presented categories, Ph.D. Thesis, 1984.
[3] Bird G.J., Kelly G.M, Power A.J., *Flexible Limits for 2-Categories*, J. Pure Appl. Alg. 61 (1989).

[4] Buckley M., *Fibred 2-categories and bicategories*, arXiv:1212.6283 (2012).

[5] Canevali N., *2-filtered bicollimits and finite weighted bilimits commute in Cat*, degree thesis, [http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2016](http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2016).

[6] Data M. I., *Una construcción de bicollimites 2-filtrantes de categorías*, degree thesis, [http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2014](http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2014).

[7] Descotte M.E., *Una teoría de 2-pro-objetos, una teoría de 2-categorías de 2-modelos y la estructura de 2-modelos para 2-Pro(C)*, Ph.D. thesis (2015).

[8] Descotte M.E., Dubuc E.J., *A theory of 2-pro-objects*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, Tome 55 number 1 (2014).

[9] Descotte M.E., Dubuc E.J., Szyld M., *A construction of certain weak colimits and an exactness property of the 2-category of categories*, arXiv:1610.02453 (2016).

[10] Dubuc E. J., *Kan extensions in Enriched Category Theory*, Lecture Notes in Mathematics, Springer Lecture Notes in Mathematics 145 (1970).

[11] Dubuc E. J., Street R., *A construction of 2-filtered bicollimits of categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, Tome 47 number 2 (2006).

[12] Fiore T. M., *Pseudo Limits, Biadjoints, and Pseudo Algebras: Categorical Foundations of Conformal Field Theory*, Mem. Amer. Math. Soc. 182, no. 860 (2006).

[13] Gabriel P., Zisman M., *Calculus of fractions and homotopy theory*, Springer (1967).

[14] Gray J. W., *Formal category theory: adjointness for 2-categories*, Springer Lecture Notes in Mathematics 391 (1974).

[15] Grothendieck A., *SGA1 (1960-61)*, Springer Lecture Notes in Mathematics 224 (1971).

[16] Kelly G. M., *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series 64, Cambridge Univ. Press, New York (1982).

[17] Kelly G. M., *Elementary observations on 2-Categorical limits*, Bull. Austral. Math. Soc. Vol. 39 (1989).

[18] Kelly G. M., *Structures defined by finite limits in the enriched context I*, Cahiers de Topologie et Géométrie Différentielle Catégoriques 23 (1982).

[19] Kelly G. M., Street R., *Review of the elements of 2-categories*, Springer Lecture Notes in Mathematics 420 (1974).

[20] Kennison J., *The fundamental localic groupoid of a topos*, J. Pure Appl. Alg. 77 (1992).

[21] Lack S., *Limits for Lax Morphisms*, Applied Categorical Structures 13, Issue 3 (2005).

[22] Lucatelli Nunes F., *On Biadjoint Triangles*, Theory and Applications of Categories, Vol. 31, No. 9 (2016).
Appendix A  The main theorem for pseudofunctors

We will now prove a generalization of Theorem 4.2.7 and Proposition 4.2.9 to $\text{Cat}$-valued pseudofunctors $\mathcal{A} \xrightarrow{P} \mathcal{C}at$ (with $\mathcal{A}$ still a 2-category). We will prove it by applying those results to the 2-functor $\tilde{P}$ associated to the pseudofunctor $P$.

We begin by giving the explicit definition of the 2-category $p\text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{B})$ considered in Section 4. We will use the explicit formulas defining pseudofunctors and pseudonatural transformations. We refer the reader to [22, §2], [12, §3], [7, §1] among other choices for a more expanded description of the equations below.

**Definition A.1.** Let $\mathcal{A}, \mathcal{B}$ be 2-categories. A lax functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is given by the following data:

- For each object $A \in \mathcal{A}$, an object $FA \in \mathcal{B}$.
- For each hom-category $\mathcal{A}(A, B)$, a functor $\mathcal{A}(A, B) \xrightarrow{F_{A,B}} \mathcal{B}(FA, FB)$. Whenever possible we will abuse the notation $F_{A,B}$ by $F$.
- For each object $A \in \mathcal{A}$, an invertible 2-cell $\alpha^F_A : id_{FA} \Rightarrow F(id_A)$.
- For each triplet of objects $A, B, C \in \mathcal{A}$, a natural transformation

$$
\begin{array}{ccc}
\mathcal{A}(B, C) \times \mathcal{A}(A, B) & \xrightarrow{F \times F} & \mathcal{B}(FB, FC) \times \mathcal{B}(FA, FB) \\
\circ \downarrow \alpha^F \downarrow & & \circ \\
\mathcal{A}(A, C) & \xrightarrow{F} & \mathcal{B}(FA, FC)
\end{array}
$$

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This natural transformation is given, for each configuration \( A \xrightarrow{f} B \xrightarrow{g} C \) by 2-cells of \( \mathcal{B} \), \( FgFf \xrightarrow{\alpha^F_{g,f}} F(gf) \). These data are subject to the axioms

1. \( LF_0 \). For each \( A \xrightarrow{f} B \),
   \[ \alpha^F_{f,id_B} \circ (\alpha^F_{B}Ff) = Ff = \alpha^F_{id_A,f} \circ (Ff\alpha^F_A). \]
2. \( LF_1 \). For each \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \),
   \[ \alpha^F_{g,h} \circ (Fh\alpha^F_{f,g}) = \alpha^F_{f,h} \circ (\alpha^F_{g,h}Ff). \]

A lax natural transformation \( \theta \) between lax functors \( A \xrightarrow{\theta} G \xrightarrow{\theta} \mathcal{B} \) is given by families \( \{ FA \xrightarrow{\theta_A} GA \} \) satisfying the equations (cf. \( \S 1.2 \), item \( \mathbf{[1]} \)):

1. \( LN_0 \). For all \( A \in \mathcal{A} \),
   \[ \theta_{id_A} \circ \alpha^{G}_{A} \theta_A = \theta_A \alpha^F_{A}. \]
2. \( LN_1 \). For all \( A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{A} \),
   \[ \theta_{gf} \circ \alpha^{G}_{f,g} \theta_C = \theta_C \alpha^{F}_{f,g} \circ \theta_{g} Ff \circ Gg \theta_f. \]
3. \( LN_2 \). For all \( A \xrightarrow{f} B \in \mathcal{A} \),
   \[ \theta_{B} Fg \circ \theta_f = \theta_g \circ \alpha^G_{f,g} \theta_A. \]

A pseudofunctor is a lax functor \( F \) such that the structural 2-cells \( \alpha^F_A \), \( \alpha^F_{f,g} \) are all invertible. A pseudonatural transformation between lax functors is a lax natural transformation such that the structural 2-cells \( \theta_f \) are all invertible. Modifications are defined as for 2-functors.

Now we extend the Definition \( [1.2.3] \) of the 2-category of elements to the case of pseudofunctors. We note that this construction is considered in \( [6] \), and with greater generality in \( [11] 3.3.3 \), where a theory of fibred 2-categories is developed, corresponding to pseudofunctors with values in 2-categories. An idea that is useful to have in mind is that \( \text{Cat} \)-valued pseudofunctors are the 2-dimensional analogous to the “discrete” 1-dimensional fibrations.

**Definition A.2.** Let \( A \xrightarrow{P} \text{Cat} \) be a pseudofunctor. \( \mathcal{E}_P \) is the 2-category with objects, morphisms and 2-cells described exactly as in Definition \( [1.2.1] \) but now the structural 2-cells of the pseudofunctor appear in the formulas for composition and identities:

For \( (x, A) \xrightarrow{(f,\varphi)} (y, B) \xrightarrow{(g,\psi)} (z, C) \), the composition is given by the formula

\[ (g, \psi)(f, \varphi) = (gf, \psi P g(\varphi)(\alpha^P_{g,f} x^{-1}). \]

Identity morphisms are given by \( (x, A) \xrightarrow{(id_A, (\alpha^P_A)^{-1})} (x, A) \). 2-cells are composed as in \( A \).

As for 2-functors, we consider the 1-subcategory \( \mathcal{E}_P \) of \( \mathcal{E}_P \) whose arrows are \( (f, \varphi) \) with \( \varphi \) an isomorphism.

**Remark A.3.** The fact that \( (x, A) \xrightarrow{(id_A, (\alpha^P_A)^{-1})} (x, A) \) are identities follows from axiom \( LF_0 \) in Definition \( A.1 \). The fact that the composition of morphisms is associative follows from axiom \( LF_1 \). In both cases the naturality of the structural 2-cells \( \alpha^P_A, \alpha^P_{f,g} \) respectively is used. The computations are somewhat lengthy but straightforward so we omit them.

We also extend the results of \( [1.2.5] \) and Proposition \( 3.3.3 \) to pseudofunctors.

**Proposition A.4.** For each lax natural transformation \( P \xrightarrow{\eta} Q \) between \( \text{Cat} \)-valued pseudofunctors, there is an induced 2-functor \( \mathcal{E}_P \xrightarrow{T_\eta} \mathcal{E}_Q \) given by the same formulas in \( \mathcal{E}_P \).
Proof. To show that \( T_\eta \) preserves composition of morphisms strictly, consider
\((x, A) \xrightarrow{(f,x)} (y, B) \xrightarrow{(g,\psi)} (z, C)\) in \( \mathcal{E}l_P \), then the equation we have to show is
\[
(gf, \eta_C(\psi) Pg(\phi)(\alpha_{f,g}^P)^{-1})(\eta_{gf})_x = (gf, \eta_C(\psi) (\eta_g)_y Q(g)(\eta_B(\phi)(\eta_f)_x)(\alpha_{f,g}^Q)^{-1})(\eta_{gf})_x.
\]
This equation follows at once from axiom LN1 in Definition [A.1] using the naturality of \( \eta_g \) with respect to the arrow \( \phi \). The fact that \( T_\eta \) preserves identities follows immediately from axiom LN0. The rest of the verifications of the 2-functoriality are straightforward and identical to the case of 2-functors so we omit them. \( \square \)

We note that the formulas in [1.2.3] are the same formulas of [4, 3.3.12], where it is stated (for a pseudonatural transformation \( \eta \), though the same proof would work for lax natural instead) that \( T_\eta \) is a morphism of bicategories. In our case computations are simpler, and we have 2-functoriality instead. Proposition [3.3.3] holds for pseudofunctors with exactly the same proof:

**Proposition A.5.** Let \( P, Q : A \to \text{Cat} \) be pseudofunctors, and \( P \xrightarrow{\eta} Q \) a pseudonatural transformation. If \( \eta_A \) is full and faithful for each \( A \in A \), then the 2-functor \( \mathcal{E}l_P \xrightarrow{T_\eta} \mathcal{E}l_Q \) of Proposition [A.4] is 2-fully-faithful and the 1-subcategories given by the cocartesian arrows satisfy \( \mathcal{E}l_P = T_\eta^{-1}(\mathcal{E}l_Q) \). \( \square \)

We now recall (see [25, 4.2], or the nLab website on pseudofunctors) the construction of the 2-functor \( A \xrightarrow{\tilde{P}} \text{Cat} \) associated to a pseudofunctor \( A \xrightarrow{P} \text{Cat} \). We state only the facts that we will need.

Given a pseudofunctor \( A \xrightarrow{P} \text{Cat} \), there is a 2-functor \( A \xrightarrow{\tilde{P}} \text{Cat} \) and an equivalence in \( p\text{Hom}_p(A, \text{Cat}) \) between \( P \) and \( \tilde{P} \), i.e. a pseudo-equivalence \( P \xrightarrow{\eta} \tilde{P} \). The description of \( \tilde{P} \) on objects is as follows, \( \tilde{P}B \) is the category with pairs \((f, x)\) as objects, where \( A \xrightarrow{f} B \in A \) and \( x \in P A \), and arrows \((f, x) \xrightarrow{x} (f', x')\) given by an arrow \( Pf(x) \xrightarrow{x} Pf'(x') \) in \( PB \).

For \( B \xrightarrow{g} B' \in A \), we have \( \tilde{P}g(f, x) = (gf, x) \). The definition of \( PA \xrightarrow{\eta_A} \tilde{P}A \) on objects is \( \eta_A(x) = (id_A, x) \), and for the pseudo-inverse \( \tilde{P} \xrightarrow{\varepsilon} P \) we have \( \varepsilon_B(f, x) = Pf(x) \).

When applied to \( P \), Theorem [1.2.7] yields:

**Theorem A.6.** Let \( A \xrightarrow{P} \text{Cat} \) be a pseudofunctor. Then the following are equivalent.

(i) \( \mathcal{E}l_P \) is \( \sigma \)-cofiltered with respect to the family \( \mathcal{E}l_P \) of cocartesian arrows.

(ii) \( P \) is a \( \sigma \)-filtered \( \sigma \)-bilimit of representable 2-functors in \( p\text{Hom}_p(A, \text{Cat}) \).

(iii) \( P \) is flat.

Proof. We will show the equivalence of each of the items above with the corresponding statement of Theorem [1.2.7] for the 2-functor \( \hat{P} \):

(i) By Proposition [A.5] we have induced 2-functors \( \mathcal{E}l_P \xrightarrow{T_\eta} \mathcal{E}l_{\hat{P}}, \mathcal{E}l_{\hat{P}} \xrightarrow{T_\eta} \mathcal{E}l_P \), both 2-fully-faithful and satisfying \( \mathcal{E}l_P = T^{-1}_\eta(\mathcal{E}l_{\hat{P}}), \mathcal{E}l_{\hat{P}} = T^{-1}_\eta(\mathcal{E}l_P) \). In order to show that \( \mathcal{E}l_P \) is \( \sigma \)-cofiltered if and only if \( \mathcal{E}l_{\hat{P}} \) is so, by Proposition [3.3.2] it suffices to show axiom \( \sigma \text{C}0 \) for these 2-functors. 

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For $T_\eta$: given $((f,x),B)$ in $\mathcal{E}_P$, where $A \xrightarrow{f} B \in \mathcal{A}$ and $x \in PA$, consider $T_\eta(x,A) = ((id_A,x),A) \xrightarrow{(f,id_{Pf(x)})} ((f,x),B)$.

For $T_\varepsilon$: given $(x,A)$ in $\mathcal{E}_P$, consider $T_\varepsilon((id_A,x),A) = (P(id_A)(x),A) \xrightarrow{(id_A,\varphi)} (x,A)$, where $\varphi$ is the isomorphism $\varphi : P(id_A)P(id_A)(x) \xrightarrow{P(id_A)(a_\alpha^P)_{-1}} P(id_A)(x) \xrightarrow{(a_\alpha^P)_{-1}} x$.

(ii) Immediate from the pseudo-equivalence $P \cong \tilde{P}$ (recall Remark 4.2.8).

(iii) Immediate from Corollary 4.1.9.

We end the paper showing that, in the presence of finite bilimits, flat pseudofunctors coincide with left exact ones.

**Proposition A.7.** If $\mathcal{A}$ is finitely complete, then a pseudofunctor $\mathcal{A} \xrightarrow{P} \mathcal{Cat}$ is flat if and only if it is left exact.

**Proof.** By Corollary 4.1.4 $P$ is left exact if and only if $\tilde{P}$ is so. By Corollary 4.1.9 $P$ is flat if and only if $\tilde{P}$ is so. Then the proposition follows from Proposition 4.2.9 applied to $\tilde{P}$.