On F-modeling based Empirical Bayes Estimation of Variances

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Abstract

We consider the problem of empirical Bayes estimation of multiple variances when provided with sample variances. Assuming an arbitrary prior on the variances, we derive different versions of the Bayes estimators using different loss functions. For one particular loss function, the resulting Bayes estimator relies on the marginal cumulative distribution function of the sample variances only. When replacing it with the empirical distribution function, we obtain an empirical Bayes version called F-modeling based empirical Bayes estimator of variances. We provide theoretical properties of this estimator and further demonstrate its advantages through extensive simulations and real data analysis.

Keywords: uniform convergence, empirical distribution function, selective inference.

1 Introduction

The empirical Bayes approach was introduced as a compound decision procedure in (Robbins, 1951) and has been widely studied thereafter (Robbins, 1956; Dvoretzky et al., 1956; Efron & Morris, 1972, 1973, 1975; Laird & Louis, 1987; Jiang & Zhang, 2009; Koenker & Gu, 2017). This approach plays an important role in the kinds of data analysis conducted during gene expression experiments, which often involve a large number of parallel inference problems.

The core idea of the empirical Bayes approach is to estimate the prior distribution either directly or indirectly using the available data, wherein the final inference is based on the posterior distribution when using this estimated prior. Efron (2014) classified empirical Bayes approaches as pursuing one of two strategies: (i) f-modeling, which is modeling on the data scale; and (ii) g-modeling, which is modeling on the parameter scale. Under f-modeling, the resulting empirical Bayes rule usually depends on the prior indirectly via the marginal probability density function; under g-modeling, the prior distribution is estimated and then plugged into the posterior calculation. It is further commented in that paper that the g-modeling approach has been widely used in theoretical investigations (Laird & Louis, 1987; Morris, 1983; Jiang & Zhang, 2009), whereas the f-modeling approaches are more prevalent in applications (Robbins, 1956; Brown & Greenshtein, 2009; Efron, 2011).

The simultaneous estimation of variances and the covariance matrix has a long history, dating back to James & Stein (1961). Haff (1980) provided a parametric empirical Bayes estimator of the covariance matrix by assuming an inv-Wishart prior distribution on the covariance matrix. Efron et al. (1976) proposed an estimator to
dominate the sample covariance. Wild (1980) considered simultaneous estimation of the variances under different loss functions. Robbins (1982) discussed a parametric empirical Bayes methods for scale mixture of Gaussians. Champion (2003) considered the shrinkage estimator of variances based on the Kullback-Leibler distance.

Heteroskedasticity is prevalent in many applications, such as microarray experiments, rendering the simultaneous estimation of variances even more important. There have been many attempts to estimate these parameters with different approaches (Tusher et al., 2001; Lönnstedt & Speed, 2002; Storey & Tibshirani, 2003; Lin et al., 2003; Tong & Wang, 2007; Koenker & Gu, 2017). Among these, there are a few widely used parametric empirical Bayes estimators which are widely used. When assuming an inverse gamma prior, Smyth (2004) developed a parametric empirical Bayes estimator of the variances. Cui et al. (2005) approximated both the chi-square distribution and the inverse gamma prior by log-normal random variables and derived the exponential Lindley-James-Stein estimator. Lu & Stephens (2016) assumed that the prior of the variances follows a mixture of inverse gamma distributions to derive a flexible empirical Bayes estimator. These parametric empirical Bayes methods have the advantage of providing the full posterior distribution of the variances for further inference such as constructing credible intervals and performing hypothesis testing. Koenker & Gu (2017) took the g-modeling approach by estimating the probability density function of the prior distribution using non-parametric maximum likelihood estimator (Koenker & Mizera, 2014; Kiefer & Wolfowitz, 1956).

In this work, we assume an arbitrary prior distribution \( g(\sigma^2) \) for the variances to produce a nonparametric empirical Bayes estimator. When assuming some commonly used loss functions, we derive empirical Bayes estimators for the variances by modeling on the data scale. For a particular loss function, the resulting Bayes estimator depends only on the marginal cumulative distribution function of the sample variances, \( F(s^2) \). To the best of the authors’ knowledge, this is the first estimator for the variances which relies on the marginal cumulative distribution function rather than the marginal probability density function. To differentiate our method from the terminology used in Efron (2014), we call this estimator an F-modeling based estimator. The advantage of the F-modeling based estimator is that one can simply replace the marginal cumulative distribution function with the empirical distribution function to obtain the proposed empirical Bayes version, which we call F-modeling based empirical Bayes estimator for the variances. The computation of the proposed method is instantaneous without any tuning parameters.

It is known that the empirical distribution function converges to the true distribution function uniformly (Dvoretzky et al., 1956). As shown in Section 3, the proposed empirical Bayes estimator converges to the Bayes version uniformly over a set \( D_\delta = (0, D_\delta) \) where \( D_\delta \) is a large value and tends to infinity when \( \delta \) goes to zero. We impose this condition for technical reasons so as to prevent the denominator of the Bayes estimator to be arbitrarily small. It causes little practical concern because most often one would be interested in parameters corresponding to the small and moderate sample variances which fall in \( D_\delta \). We have also derived the estimator of the variances for the post selection inference and finite Bayes inference (Efron, 2019).

2 Empirical Bayes Estimator for Variances

Let \( \sigma^2_{[1:N]} = (\sigma^2_1, \sigma^2_2, \ldots, \sigma^2_N) \) be the parameters of interest and \( s^2_{[1:N]} = (s^2_1, s^2_2, \ldots, s^2_N) \) be the corresponding sample variances. In this paper, we consider the following model,

\[
\begin{align*}
    & s^2_i | \sigma^2_i \sim p (s^2_i | \sigma^2_i) \sim \sigma^2_i \chi^2_k, \\
    & \sigma^2_i \sim g (\sigma^2_i).
\end{align*}
\]

Here, \( \chi^2_k \) denotes the random variable which follows a chi-square distribution with \( k \) degrees of freedom. We assume an arbitrary prior \( g(\sigma^2_i) \) on the variances. When integrating the variance out, the marginal probability
density function of the sample variances is \( f(s_i^2) = \int_0^\infty p(s_i^2|\sigma_i^2)g(\sigma_i^2)d\sigma_i^2 \). Let
\[
F(s_i^2) = \int_0^{s_i^2} f(s_i^2)ds_i^2
\]
be the corresponding marginal cumulative distribution function of \( s_i^2 \)’s.

To derive the Bayes rule \( \hat{\sigma}_2^2_{[1:N]} = (\hat{\sigma}_1^2, \hat{\sigma}_2^2, \cdots, \hat{\sigma}_N^2) \), a loss function must be specified. Sinha & Ghosh (1985) summarized many commonly used loss functions as follows:
\[
L_0 \left( \sigma_{[1:N]}^2, \hat{\sigma}_{[1:N]}^2 \right) = \sum_{i=1}^N \left( \sigma_i^2 - \hat{\sigma}_i^2 \right)^2,
L_1 \left( \sigma_{[1:N]}^2, \hat{\sigma}_{[1:N]}^2 \right) = \sum_{i=1}^N \left( \frac{\sigma_i^2}{\hat{\sigma}_i^2} - 1 \right)^2,
L_1' \left( \sigma_{[1:N]}^2, \hat{\sigma}_{[1:N]}^2 \right) = \sum_{i=1}^N \left( \frac{\sigma_i^2}{\hat{\sigma}_i^2} - 1 \right)^2,
L_2 \left( \sigma_{[1:N]}^2, \hat{\sigma}_{[1:N]}^2 \right) = \sum_{i=1}^N \left( \frac{\sigma_i^2}{\hat{\sigma}_i^2} - \ln \frac{\hat{\sigma}_i^2}{\sigma_i^2} - 1 \right).
\]

The squared error loss function, \( L_0(\cdot) \), is not scale-invariant. The other three loss functions are scale-invariant. The loss function \( L_1'(\cdot) \) is equivalent to using \( L_1(\cdot) \) when estimating the precision parameters (Ghosh & Sinha 1987). The loss function \( L_1'(\cdot) \) by nature favors under-estimation because “underestimation has only a finite penalty, while overestimation has an infinite penalty” (Casella & Berger 2001). This could lead to an estimator which works extremely poor when focusing on the parameter with the smallest sample variance. On the contrary, both the loss function \( L_1(\cdot) \) and Stein’s Loss function \( L_2(\cdot) \) have an infinite penalty for the underestimation. In addition, the loss function \( L_2(\cdot) \) is tied to the Kullback-Leibler divergence and the entropy loss (Ghosh & Sinha 1987, Wild 1980, Haft 1977, 1980). A potential drawback of the loss function \( L_1(\cdot) \) is that it imposes a finite penalty on the overestimation.

In this article, we derive empirical Bayes estimators with respect to the scale-invariant loss functions \( L_1'(\cdot), L_1(\cdot), \) and \( L_2(\cdot) \) by modeling on the data scale. We start with the loss function \( L_1'(\cdot) \) where \( \hat{\sigma}_{B,[1:N]}^2 = (\hat{\sigma}_{1,B}^2, \hat{\sigma}_{2,B}^2, \cdots, \hat{\sigma}_{N,B}^2) \) is the corresponding Bayes rule.

**Theorem 2.1.** Assume Model (1) and the loss function \( L_1'(\cdot) \), then
\[
\hat{\sigma}_{i,B}^2 = \frac{k(k-2)s_i^2 f(s_i^2) - 2ks_i^4 f'(s_i^2)}{4s_i^4 f''(s_i^2) - 4(k-2)s_i^2 f'(s_i^2) + k(k-2)f(s_i^2)}.
\]

Formula (3) could be viewed as generalizing Tweedie’s formula (Efron 2011) to the simultaneous estimation of variances. It is seen that the estimator \( \hat{\sigma}_{i,B}^2 \) depends on the marginal probability density function \( f(s_i^2) \), its first and second derivatives. We can get an empirical Bayes version by replacing \( f(s_i^2) \) and its derivatives with the corresponding estimators using the kernel density estimator (Brown & Greenshtein 2009), or Lindsey’s method (Efron 2010, 2019). We call this method the f-modeling based empirical Bayes estimator for variances:
\[
\hat{\sigma}_{i,EBV}^2 = \frac{k(k-2)s_i^2 f(s_i^2) - 2ks_i^4 f'(s_i^2)}{4s_i^4 f''(s_i^2) - 4(k-2)s_i^2 f'(s_i^2) + k(k-2)f(s_i^2)}.
\]

Next, consider the Stein’s loss \( L_2(\cdot) \) and let \( \hat{\sigma}_{Stein,[1:N]}^2 = (\hat{\sigma}_{1,Stein}^2, \hat{\sigma}_{2,Stein}^2, \cdots, \hat{\sigma}_{N,Stein}^2) \) be the corresponding Bayes rule. Then we have the following theorem.
Assume Model (1) and Stein’s loss function $L_2(\cdot)$, then

$$\hat{\sigma}^2_{i,\text{Stein}} = \left(\frac{k - 2}{ks_i^2} - \frac{2}{k} \cdot \frac{f'(s_i^2)}{f(s_i^2)}\right)^{-1}.$$  \hspace{1cm} (5)

When replacing $f(s^2)$ and $f'(s^2)$ with the corresponding estimators, we have the following f-modeling based empirical Bayes estimator of the variances when assuming Stein’s loss:

$$\hat{\sigma}^2_{i,f-\text{EBVS}} = \left(\frac{k - 2}{ks_i^2} - \frac{2}{k} \cdot \frac{f'(s_i^2)}{f(s_i^2)}\right)^{-1}.$$  \hspace{1cm} (6)

When assuming Stein’s loss, the empirical Bayes estimator does not require the estimation of the second derivative of the marginal probability density function. However, it still relies on the marginal density function and its first order derivative. The non-parametric estimation of the density function and its derivatives is a challenging problem, not to mention that the estimation accuracy on the tail becomes even worse. Additionally, the commonly used approaches such as the kernel density estimation relies on the choice of tuning parameters, which are difficult to choose in practice.

Next, we consider the loss function $L_1(\cdot)$ and the corresponding Bayes decision rule $\hat{\sigma}^2_{B,[1:N]} = (\hat{\sigma}^2_{1,B}, \hat{\sigma}^2_{2,B}, \ldots, \hat{\sigma}^2_{N,B})$. We have the following theorem.

**Theorem 2.3.** Assume Model (1) and the loss function $L_1(\cdot)$. If

$$\int_0^\infty (s^2)^{-(\frac{k}{2} - 2)} \, dF(s^2) < \infty \quad \text{and} \quad \int_0^\infty (s^2)^{-(\frac{k}{2} - 1)} \, dF(s^2) < \infty,$$

then

$$\hat{\sigma}^2_{i,B} = \frac{k}{2} \left\{ \frac{\int_0^\infty (s^2)^{-(\frac{k}{2} - 2)} \, dF(s^2)}{\int_0^\infty (s^2)^{-(\frac{k}{2} - 1)} \, dF(s^2)} - s_i^2 \right\}.$$  \hspace{1cm} (7)

According to Model (1), we know that

$$\int_0^\infty (s^2)^{-(k/2-j)} \, dF(s^2) = \int_0^\infty \int_0^\infty C_k (s^2)^{j-1} \exp \left(-\frac{ks^2}{2\sigma^2}\right) g(\sigma^2) \sigma^2 \, ds^2, \quad j = 1, 2,$$

where $C_k = \frac{k^{k/2}}{\Gamma(k/2)^{2/k}}$. When assuming an inverse gamma prior (Smyth, 2004) and a mixture of inverse gamma prior (Lu & Stephens, 2016), basic arithmetic calculations show that the conditions in the theorem hold.

Our F-modeling approach constructs a Bayes estimator of the variances which relies on $F(s^2)$, the cumulative distribution function of the sample variances. The advantage of using an F-modeling based estimator is that one can avoid the daunting task of estimating the marginal probability density function and its derivatives, which usually requires some kind of assumptions. Instead, to obtain an empirical Bayes version of the Bayes rule, we simply replace $F(s^2)$ with the empirical distribution function $F_N(s^2) = \frac{1}{N} \sum_{i=1}^N I(s_i^2 \leq s^2)$. After the substitution, we have the following proposed empirical Bayes estimator, which we refer to as the F-modeling based empirical Bayes estimator of the variances:

$$\hat{\sigma}^2_{i,F-\text{EBV}} = \left\{ \begin{array}{ll}
\frac{s_i^2}{k} = \max_{1 \leq j \leq N} s_j^2, & \text{if } s_i^2 = \max_{1 \leq j \leq N} s_j^2; \\
\frac{\sum_{j \geq i} (s_j^2)^{-(k/2)} - s_i^2}{\sum_{j \geq i} (s_j^2)^{-(k/2-1)} - s_i^2}, & \text{otherwise}.
\end{array} \right.$$  \hspace{1cm} (8)
The proposed estimator is calculated instantaneously and does not involve any tuning parameters.

Return to Model (1) with \( g(\sigma^2) \) being arbitrary. Assume that one additional sample variance \( s_0^2 \) which is independent of \( s_{[1:N]}^2 \) has been observed. Let \( \sigma_0^2 \) be the corresponding variance which is assumed to be generated from \( g(\sigma^2) \) and \( s_0^2 \sim \sigma_0^2 \frac{\chi_2^2}{k} \). The goal is to estimate \( \sigma_0^2 \) based on the posterior distribution \( \sigma_0^2 | s_0^2 \). When \( N \) goes to infinity, the prior distribution \( g(\sigma^2) \) could be fully recovered and this reduces to the standard Bayes approach. For a finite \( N \), this problem is called the finite Bayes inference (Efron, 2019). Assume the loss function

\[
L^{FB}_1(\sigma_0^2, \sigma_0^2) = \left( \frac{\sigma_0^2}{\sigma_0^2} - 1 \right)^2.
\]  

(9)

Based on the proof of Theorem 2.3 we know that the Bayes rule is

\[
\hat{\sigma}_{0,B}^2 = \frac{k}{2} \left\{ \int_{s_0^2}^{\infty} (s^2)^{-(4-2)/2} dF(s^2) - s_0^2 \right\}.
\]

Consequently, we propose to estimate \( \sigma_0^2 \) by

\[
\hat{\sigma}_{0,F-EBV}^2 = \begin{cases} 
  s_0^2, & \text{if } s_0^2 \geq \max_{1 \leq j \leq N} s_j^2, \\
  \frac{\sum_{s_j^2 \geq s_0^2} (s_j^2)^{-(4-2)/2}}{\sum_{s_j^2 \geq s_0^2} (s_j^2)^{-(4-1)/2}} - s_0^2, & \text{otherwise.}
\end{cases}
\]

(10)

Similarly, we estimate \( \sigma_0^2 \) based on f-modeling methods by

\[
\hat{\sigma}_{0,f-EBV}^2 = \frac{k(k-2)s_0^2 \overline{f(s_0^2)} - 2ks_0^2 \overline{f'(s_0^2)}}{4s_0^2 \overline{f''(s_0^2)} - 4(k-2)s_0^2 \overline{f'(s_0^2)} + k(k-2) \overline{f(s_0^2)}}.
\]

(11)

and

\[
\hat{\sigma}_{0,f-EBVS}^2 = \left( \frac{k - 2}{ks_0^2} - 2 \right) \cdot \left( \frac{\overline{f'(s_0^2)}}{\overline{f(s_0^2)}} \right)^{-1}.
\]

(12)

We can similarly construct estimators for variances relating to a set of indices, even if the indices have been chosen using the data. Given the data \( s_{[1:N]}^2 = (s_1^2, s_2^2, \ldots, s_N^2) \), let \( C \) be the set of indices selected using a certain procedure. Our target is to estimate \( \sigma_i^2, \forall i \in C \) under the loss function

\[
L^{PS}_1(\sigma^2, \sigma^2) = \sum_{i \in C} \left( \frac{\sigma_i^2}{\sigma_i^2} - 1 \right)^2.
\]

(13)

As an example, we might be interested in the variances corresponding to the \( K \) smallest sample variances. In other words, order the sample variances \( s_i^2 \)’s increasingly as \( s_{(1)}^2 \leq s_{(2)}^2 \leq \cdots \leq s_{(N)}^2 \). Let \( \sigma_i^2 \) be the parameter corresponding to \( s_{(i)}^2 \). Set \( C = \{ i : s_i^2 \leq s_{(K)}^2 \} \).

For any \( i \in C \),

\[
\pi(\sigma_i^2 | s_{[1:N]}, i \in C) = \pi(\sigma_i^2 | s_{[1:N]}).
\]

This implies that the posterior distribution of \( \sigma_i^2 \) when conditioning on both the data and the selection set is the same as the posterior distribution of \( \sigma^2 \) conditioning on the data. Consequently, the Bayes rule based on the
The reason is that the denominator converges to those of the Bayes rule uniformly. However, it does not guarantee that the ratio converges uniformly. Namely, for a number $\delta > 0$, let $D \equiv \{ u \in (0, \infty) \mid \int_u^\infty F(s^2) \, ds^2 < \delta \}$. We therefore propose to estimate $\sigma_i^2, i \in C$ according to (8) without adjustment. We would like to point out that this argument is true because the full data set is available for the post-selection inference. Otherwise, the Bayes rule might be affected by the selection. For instance, if only the data post the selection is available for further inference, then the Bayes rule needs to be corrected for such a selection rule. See Yekutieli (2012) for a full discussion on this issue.

3 Theoretical Properties

In this section, we study the theoretical properties of the proposed method. To ease our notation, we define two functions $l_1(s^2, u) = (s^2)^{(k/2-2)}I(s^2 \geq u)$ and $l_2(s^2, u) = (s^2)^{-(k/2-1)}I(s^2 \geq u)$ where $I(\cdot)$ is an indicator function. Then the Bayes decision rule and the proposed method can be respectively written as

$$\hat{\sigma}_{i,B}^2 = \frac{k}{2} \left\{ \int_0^\infty l_1(s^2, s_i^2) \, dF(s^2) - s_i^2 \right\}, \quad \text{and} \quad \hat{\sigma}_{i,EBV}^2 = \frac{k}{2} \left\{ \int_0^\infty l_1(s^2, s_i^2) \, dF_N(s^2) - s_i^2 \right\}.$$

First, we study the numerator and denominator separately.

**Theorem 3.1.** Assume the same conditions in Theorem 2.3 and $F(s^2)$ is continuous with the support of $(0, \infty)$, then

$$\sup_u \left| \int_0^\infty l_1(s^2, u) \, dF_N(s^2) - \int_0^\infty l_1(s^2, u) \, dF(s^2) \right| \overset{a.s.}{\rightarrow} 0,$$

and

$$\sup_u \left| \int_0^\infty l_2(s^2, u) \, dF_N(s^2) - \int_0^\infty l_2(s^2, u) \, dF(s^2) \right| \overset{a.s.}{\rightarrow} 0.$$

This theorem implies that both the numerator and the denominator of the proposed empirical Bayes estimator converge to those of the Bayes rule uniformly. However, it does not guarantee that the ratio converges uniformly.

The reason is that the denominator $\int_0^\infty l_2(s^2, u) \, dF(s^2)$ converges to zero when $u$ goes to $\infty$. To prove that the proposed method converges to the Bayes estimator uniformly, we consider the set such that the denominator of the Bayes rule is greater than some positive number. Namely, for a number $\delta > 0$, let $D^\delta$ be a set defined as

$$D^\delta \equiv \{ u \mid \int_u^\infty (s^2)^{-(k/2-1)} \, dF(s^2) > \delta \}.$$

Since $\int_0^\infty (s^2)^{-(k/2-1)} \, dF(s^2) < \infty$, then $D^\delta = (0, D_\delta)$ for some positive number $D_\delta$. We then have the following theorem:

**Theorem 3.2.** Assume the same conditions in Theorem 3.1, then

$$\sup_{s_i^2 \in D^\delta} \left| \hat{\sigma}_{i,EBV}^2 - \hat{\sigma}_{i,B}^2 \right| \overset{a.s.}{\rightarrow} 0.$$

The constant $D_\delta$ is a quantity depending on the marginal distribution function of the sample variances only and $D_\delta$ tends to infinity when $\delta$ tends to 0. For any $0 < \tau < 1$, let $s^2_{[1:]N}$ be a random sample consisting of $N$ sample variances. Let $s^2_\tau$ be the $\tau$-th sample quantile. We can always choose $\delta$ sufficiently small, such that
\{s^2_i, s^2_j \leq s^2_r\} \in D^d \text{ with large probability. For a sample variance which doesn’t fall in } D^d, \text{ one could estimate the corresponding parameter by this sample variances. Namely, we could modify the proposed estimator as }

\[ \hat{\sigma}^2_{i,mF-EBV} = \begin{cases} 
  s^2_i, & \text{if } s^2_i \geq s^2_{(N^2)} \\
  \frac{k}{\tau} \frac{\sum_{i,j \geq \tau} (s^2_j)^{-(k-2)}}{\sum_{i,j \geq \tau} (s^2_j)^{-(k-1)}} - s^2_i, & \text{otherwise.}
\end{cases} \]  

(15)

In practice, especially when focusing on parameters with small sample variances, this modification does not make much difference.

We can extend the result to the post-selection inference and finite Bayes inference.

**Corollary 3.1.** Assume the same conditions in Theorem 3.1 then

\[
\sup_{s^2_i \in D^d, i \in C} \left| \hat{\sigma}^2_{i,F-EBV} - \hat{\sigma}^2_{i,B} \right| \xrightarrow{a.s.} 0.
\]

As commented in Section 2 the Bayes estimator is immune to the selection rule \(\mathcal{C}\), and the empirical Bayes estimator could be a good approximation of the Bayes estimator. However, the discrepancy between these two widens when focusing on the selected case (Pan et al. 2017), and some correction is needed (Hwang & Zhao 2013). On the other hand, Corollary 3.1 indicates that the proposed F-modeling based empirical Bayes estimator converges to the corresponding Bayes version if \(s^2_i \in D^d, i \in \mathcal{C}\). In other words, we don’t need to make further correction for the selection.

Similarly, when considering the finite Bayes inference, the uniform convergence of the proposed estimator guarantees a good estimation as long as \(s^2_0 \in D^d\).

**Corollary 3.2.** Assume the conditions in Theorem 3.1 then

\[
\sup_{s^2_0 \in D^d} \left| \hat{\sigma}^2_{0,F-EBV} - \hat{\sigma}^2_{0,B} \right| \xrightarrow{a.s.} 0.
\]

4 Numerical studies

In this section, we compare the numerical performances of the proposed methods with existing methods, including the sample variance \((s^2)\), exponential Lindley-James-Stein estimator (ELJS, Cui et al. 2005), Tong and Wang’s method (TW, Tong & Wang 2007), Smyth method (Smyth 2004), variance adaptive shrinkage method (Vash, Lu & Stephens 2016), and REBayes method (Koenker & Gu 2017). As suggested by a referee, we consider two more estimators based on the Smyth method and variance adaptive shrinkage method by considering the loss function \(L_1(\cdot)\). Assume that the prior distribution \(g(\sigma^2_i)\) in Model 1 is inverse gamma \((a_0, b_0)\), then the posterior distribution of \(\sigma^2_i\) is inverse gamma \((a_1, b_1)\) where \(a_1 = a_0 + k/2, b_1 = b_0 + k s^2_i/2\). The hyper parameters \(a_0\) and \(b_0\) are estimated by using the method of moments (Smyth 2004). The Smyth method, which minimizes \(EL'_1(\cdot)\), is given as \(\frac{b_1}{a_1}\). The modified Smyth method, which minimizes \(EL_1(\cdot)\), is given as

\[
\frac{\sigma^2_{i,mSmyth}}{\sigma^2_{1,EBV}} = \frac{E(\sigma^2_i | s^2_i)}{E(\hat{\sigma}^2_{i,mF-EBV} | s^2_i)} = \frac{b_1}{a_1 - 2}.
\]

Similarly, we include two versions of variance adaptive shrinkage estimators, the original version (Vash) and modified version (mVash) in our simulation studies.

Let \((\sigma^2_1, s^2_2, \ldots, N)\) be the parameters and the sample variances be generated according to Model 1 where the degrees of freedom \(k\) is chosen as 5 and the prior \(g(\sigma^2_i)\) is chosen from
Setting I: $\sigma^2_i \sim$ inverse gamma distribution: IG($a, 1$) where $a = 10$ and 6;

Setting II: $\sigma^2_i \sim$ Mixture of inverse gamma distributions: $0.2IG(a, 1) + 0.4IG(8, 6) + 0.4IG(9, 19)$, where $a = 10$ and 6;

Setting III: $\sigma^2_i = a$ with 0.4 probability and $1/a$ with 0.6 probability, where $a = 3$ and 4;

Setting IV: $\sigma^2_i \sim$ Mixture of inverse Gaussian distributions: $0.4InvGauss(1/a, 1) + 0.6InvGauss(a, a^4)$, where $a = 2$ and 3.

For all simulations, we set $N = 1,000$ and the number of replications as 500. For each replication, we generate the data $(\sigma^2_i, s^2_i)$ and order them according to the sample variances increasingly. We consider three different selection rules: (i) the parameters corresponding to the 1% smallest sample variances; (ii) the parameters corresponding to the 5% smallest sample variances; and (iii) all the parameters. We calculate the estimated values based on the aforementioned methods. The risks associated with the loss function (13) are calculated and reported in Table 1 and the table in Appendix B. In our numerical studies, it is shown that two f-modelling estimators defined in (4) and (6) perform poorly, and the results are not reported in the tables. The proposed F-modeling based empirical Bayes estimator performs the best among all the estimators considered. The modified Smyth method and modified variance adaptive shrinkage method perform similarly under these settings. Under Setting I when the prior of the variance is an inverse gamma distribution, the proposed method, the modified Smyth method and modified variance adaptive shrinkage method are essentially the same. However, for Settings II to IV when the prior distribution is not an inverse gamma distribution, the proposed method outperforms all other competing methods, including the modified Smyth method and the modified variance adaptive shrinkage method.

| Setting | $a$ | % | $s^2$ | ELJS | TW | Smyth | mSmyth | Vash | mVash | REBayes | Proposed |
|---------|-----|---|-------|------|-----|-------|--------|------|-------|---------|----------|
| I       | 10  | 1%| 2.60  | -0.48| -0.72| -0.90 | -1.06  | -0.87| -1.06 | -0.65   | -1.06    |
|         |     | 5%| 2.00  | -0.70| -0.87| -0.89 | -1.05  | -0.88| -1.05 | -0.92   | -1.05    |
|         |     | all| 0.77  | -0.94| -0.98| -0.91 | -1.05  | -0.92| -1.05 | -0.97   | -1.03    |
| II      | 10  | 1%| 2.34  | 1.05 | 0.45 | -0.14 | -0.21  | 0.87 | -0.10 | -0.05   | -0.22    |
|         |     | 5%| 1.79  | 0.62 | 0.17 | -0.10 | -0.20  | 0.74 | -0.11 | -0.06   | -0.22    |
|         |     | all| 0.75  | 0.01 | 0.00 | 0.14  | -0.43  | 0.26 | -0.48 | -0.38   | -0.52    |
| III     | 4   | 1%| 2.22  | 1.15 | 0.88 | -0.28 | -0.48  | -0.26| -0.49 | -0.50   | -0.60    |
|         |     | 5%| 1.72  | 0.74 | 0.53 | -0.06 | -0.36  | -0.05| -0.37 | -0.22   | -0.39    |
|         |     | all| 0.69  | 0.10 | 0.16 | 0.26  | -0.35  | 0.27 | -0.35 | -0.32   | -0.58    |
| IV      | 4   | 1%| 2.28  | 1.26 | 0.97 | -0.08 | -0.28  | -0.06| -0.28 | -0.13   | -0.28    |
|         |     | 5%| 1.73  | 0.77 | 0.53 | -0.13 | -0.28  | -0.11| -0.29 | -0.22   | -0.32    |
|         |     | all| 0.72  | 0.14 | 0.20 | 0.29  | -0.34  | 0.30 | -0.34 | -0.30   | -0.56    |

Table 1: The log_{10}(risk) associated with the loss function (13) of the different estimators for the variances under different simulation settings. For each setting, we consider three selection rule: (i) the parameters corresponding to the 1% smallest sample variances; (ii) the parameters corresponding to the 5% smallest sample variances; and (iii) all the parameters.

Next, we consider the finite Bayes inference problem. Namely, for each generated data set $s^2_{1:N}$ and a new observation $s^2_0$, we calculate the estimated values based on different approaches and calculate the risk according to the loss function (9). The risks are reported in Table 2 and the table in Appendix B. Overall, the proposed F-modeling based empirical Bayes estimator performs the best among all the estimators considered. The modified Smyth method and modified variance adaptive shrinkage method are essentially the same. Under Setting I when
the prior of the variance is an inverse gamma distribution, the proposed method, the modified Smyth method and modified variance adaptive shrinkage method perform similarly with negligible differences. However, for Settings II to IV when the prior distribution is not an inverse gamma distribution, the proposed method outperforms all other competing methods.

| Setting | (a, b) | $s^2$ | ELJS | TW | Smyth | mSmyth | Vash | mVash | REBayes | Proposed |
|---------|--------|-------|------|----|-------|--------|------|-------|---------|----------|
| I       | 10     | 0.38  | 0.16 | -1.05 | -0.96 | -1.06 | -0.96 | -1.07 | -1.02 | -1.03    |
| II      | 10     | 0.36  | 0.14 | -0.11 | 0.01  | -0.48 | -0.02 | -0.5  | -0.51  | -0.55    |
| III     | 4      | 0.92  | 0.72 | 0.23  | 0.23  | -0.36 | 0.25  | -0.36 | -0.31  | -0.47    |
| IV      | 4      | 0.7   | 0.49 | 0.25  | 0.37  | -0.3  | 0.38  | -0.29 | -0.1   | -0.51    |

Table 2: The log_{10}(\text{risk}) associated with the loss function (9) of the different estimators for the finite Bayes inference problem.

## 5 Real data Analysis

In this section, we apply different variance estimators to two microarray dataset: colon cancer (Alon et al., 1999) and Leukemia data (Golub et al., 1999). The colon cancer data contains gene expressions of genes ($N=2,000$) for 22 patients and 40 normal people. The leukemia data includes the expressions of genes ($N = 7,128$) extracted from 72 patients with two types of leukemia: Acute Lymphoblastic Leukemia (47 patients) and Acute Myeloid Leukemia (25 patients). For the Leukemia data set, we first randomly split the subjects into two subgroups such that both subgroups contain similar numbers of subjects from the Acute Lymphoblastic Leukemia patients and Acute Myeloid Leukemia patients. For each sub-group, we then constructed $1 - \gamma$ ($\gamma = 0.05$) confidence intervals for $\theta_i$, the mean parameter of the $i$-th gene, following the work of Hwang et al. (2009) by considering

$$CI_i = \hat{\theta}_i \pm \sqrt{\hat{M}_i \hat{\sigma}_i^2 \cdot \sqrt{\hat{\tau}_i^2 / 2 \log \hat{M}_i}}$$

$$\hat{\theta}_i = \hat{M}_i X_i + (1 - \hat{M}_i) \bar{X}, \hat{M}_i = \hat{\tau}_i^2 / (\hat{\sigma}_i^2 + \hat{\tau}_i^2),$$

and

$$\hat{\tau}_i^2 = \max \left\{ \frac{1}{N} \sum_{i=1}^{N} (X_i - \hat{\mu})^2 - \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2, \tau_0^2 \right\}.$$

We declare the $i$-th gene, where $i = 1, 2, \cdots, N$, to be significant if the corresponding interval does not enclose zero. We do the same for the other sub-group. We call the decision of the $i$-th gene discordant if the interval based on the first subgroup does (does not) enclose zero while the interval based on the second subgroup does not (does) enclose zero. If a decision is discordant, this implies that a significant conclusion based on one subgroup cannot be replicated by the other. We repeat these steps 500 times to calculate the average proportions of discordant decisions. We perform the same calculation for the colon cancer data by splitting the patients group and normal people group.

In Figure [1] we plot the box-plots of the rate of discordant decisions. The average percentage of discordant decisions are reported in Table 3. It is seen that the proposed method, the modified Smyth and modified variance adaptive shrinkage estimator produce a similar number of discordance decisions. This number is substantially smaller than all the other competing methods.

To further investigate why these three methods perform similarly, we test the hypothesis that the distribution of the sample variances is the convolution of a scaled chi-square distribution and an inverse gamma distribution. The Kolmogorov-Smirnov test statistics for the Colon data set and Leukemia data set are 0.014 and 0.017, respectively.
Table 3: The average percentages of discordant decisions of different intervals when applied to the Colon Cancer data and Leukemia Data based on 500 replications.

| Data      | $s^2$ | ELJS | TW  | Smyth | mSmyth | Vash | mVash | REBayes | F-EBV |
|-----------|------|------|-----|-------|--------|------|-------|---------|-------|
| Colon     | 0.27 | 0.20 | 0.20| 0.20  | 0.17   | 0.20 | 0.17  | 0.19    | 0.17  |
| Leukemia  | 0.23 | 0.15 | 0.15| 0.15  | 0.13   | 0.16 | 0.13  | 0.14    | 0.13  |

Figure 1: The boxplots of the percentage of discordant decisions for colon cancer and leukemia data based on 500 replications. Left panel: Colon cancer data. Right panel: Leukemia data.

The resulting p-values are 0.80 and 0.031, respectively. In other words, there is no evidence to reject the null hypothesis which states that the prior is an inverse gamma distribution for the colon data and there is only moderate evidence to reject the null hypothesis for the Leukemia data. It is expected to see similar performances for these three methods.

The code for simulations and real data analysis are available on github [https://github.com/zhaozhg81/FEBV](https://github.com/zhaozhg81/FEBV).

6 Conclusion

The proposed method is developed under Model (1) assuming a scaled chi-square distribution with equal degrees of freedom. The Bayes estimator in Theorem 3.1 still applies when the degrees of freedom are different. However, the estimation of the cumulative distribution function requires that the sample variances are identically distributed. Therefore, the proposed method could not be directly applied to cases with unequal degrees of freedom. In practice, we take a slightly conservative approach by considering the smallest degrees of freedom as the common one. We would like to point out that many parametric empirical Bayesian approaches based on the g-modeling estimate the prior distribution explicitly and can handle unequal degrees of freedom.

In the real data analysis, we use the estimator of the variances as a plug-in estimator for inferring the mean parameters. One natural follow-up challenge to address is how to obtain a non-parametric empirical Bayes estimator of the means assuming arbitrary priors for both the means and the variances. Given the observed advantages of the F-modeling based approach, we would like to further extend this framework to broader settings in future research. We will further study the properties of the F-modeling based approach under the decision theoretical framework.
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### A Technical proofs.

**Proof of Theorem 2.1** According to the loss function $L_1'$,

$$
E[L_1'(\sigma^2_{[1:N]}, \hat{\sigma}^2_{[1:N]})|\sigma^2_{[1:N]}] = \sum_i \hat{\sigma}_i^4 E[(\sigma_i^2)^{-2}|\sigma^2_{[1:N]}] - 2\hat{\sigma}_i^2 E[(\sigma_i^2)^{-1}|\sigma^2_{[1:N]}] + 1.
$$
Consequently,

\[
\sigma_{i:B}^2 = \frac{E \left[ (\sigma_i^2)^{-1} | s_{i:1:N}^2 \right]}{E \left[ (\sigma_i^2)^{-2} | s_{i:1:N}^2 \right]}.
\]

For ease of notation, we drop the subscript “i” in the proof. Recall that \( p(s^2 | \sigma^2) \) is the density function of \( s^2 | \sigma^2 \) and \( g(\sigma^2) \) is the prior distribution of \( \sigma^2 \). Note that

\[
p(s^2 | \sigma^2) = \frac{(s^2)^{-1/2} e^{-\frac{ks^2}{2\sigma^2}}}{\Gamma \left( \frac{k}{2} \right) 2^{k/2}} \cdot \left( \frac{k}{\sigma^2} \right)^{k/2}, \quad s^2 > 0.
\]

Then

\[
f(s^2) = \int p(s^2 | \sigma^2) g(\sigma^2) d\sigma^2 = \int C_k \omega(s^2, \sigma^2) d\sigma^2,
\]

where

\[
C_k = \frac{k^{k/2}}{\Gamma(k/2) 2^{k/2}}, \quad \text{and} \quad \omega(s^2, \sigma^2) = \frac{(s^2)^{k/2-1}}{(\sigma^2)^{k/2}} e^{-\frac{ks^2}{2\sigma^2}} g(\sigma^2).
\]

Take the derivative of \( f(s^2) \) with respect to \( s^2 \), we know that

\[
f'(s^2) = \int C_k \frac{k-2}{2s^2} \omega(s^2, \sigma^2) d\sigma^2 - \frac{k}{2} \int C_k \frac{1}{\sigma^2} \omega(s^2, \sigma^2) d\sigma^2
\]

\[
= \frac{k-2}{2s^2} f(s^2) - \frac{k}{2} E \left( \frac{1}{\sigma^2} | s^2 \right) \cdot f(s^2).
\]

This leads to

\[
\frac{k}{2} E \left( \frac{1}{\sigma^2} | s^2 \right) \cdot f(s^2) = \frac{k-2}{2s^2} f(s^2) - f'(s^2).
\]

(16)

Take the second order derivative of \( f(s^2) \) with respect to \( s^2 \), we have

\[
f''(s^2) = -\frac{k-2}{2s^4} f(s^2) + \frac{k-2}{2s^2} f'(s^2) - \frac{k}{2} \int C_k \frac{1}{\sigma^2} \left( \frac{k-2}{2s^2} - \frac{k}{2\sigma^2} \right) \omega(s^2, \sigma^2) d\sigma^2
\]

\[
= -\frac{k-2}{2s^4} f(s^2) + \frac{k-2}{2s^2} f'(s^2) - \frac{k(k-2)}{4s^4} E \left( \frac{1}{\sigma^2} | s^2 \right) \cdot f(s^2) + \frac{k^2}{4} E \left( \frac{1}{\sigma^4} | s^2 \right) \cdot f(s^2).
\]

Consequently,

\[
\frac{k^2}{4} E \left( \frac{1}{\sigma^4} | s^2 \right) \cdot f(s^2) = f''(s^2) - \frac{k-2}{s^2} f'(s^2) + \frac{k(k-2)}{4s^4} f(s^2).
\]

(17)

Combining (16) and (17), we know that

\[
\sigma_{B}^2 = \frac{E \left[ (\sigma^2)^{-1} | s_i^2 \right]}{E \left[ (\sigma^2)^{-2} | s_i^2 \right]} = \frac{k(k-2)s^2 f(s^2) - 2k s^4 f'(s^2)}{4s^4 f''(s^2) - 4k(k-2)s^2 f'(s^2) + k(k-2) f(s^2)}.
\]

Proof of Theorem 2.2: For ease of notation, we drop the subscript “i” in the proof. Recall that Stein loss function is defined as

\[
L_2(\sigma^2, \hat{\sigma}^2) = \frac{\hat{\sigma}^2}{\sigma^2} - \ln \left( \frac{\hat{\sigma}^2}{\sigma^2} \right) - 1.
\]

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Consequently,

\[ EL_2(\sigma^2, \hat{\sigma}^2|s^2) = \hat{\sigma}^2 E \left[ (\sigma^2)^{-1}|s^2 \right] - \ln \hat{\sigma}^2 + E(\ln \sigma^2|s^2) - 1. \]

Therefore, the estimator \( \hat{\sigma}^2_{\text{Stein}} \) which minimizes the above expression is

\[ \hat{\sigma}^2_{\text{Stein}} = \frac{1}{E[(\sigma^2)^{-1}|s^2]} \]

According to the proof of Theorem 2.1,

\[ \frac{k}{2} E \left[ (\sigma^2)^{-1}|s^2 \right] \cdot f(s^2) = \frac{k - 2}{2s^2} f(s^2) - f'(s^2). \]

Therefore,

\[ \hat{\sigma}^2_{\text{Stein}} = \frac{1}{E[(\sigma^2)^{-1}|s^2]} = \left( \frac{k - 2}{ks^2} - \frac{2f'(s^2)}{kf(s^2)} \right)^{-1}. \]

**Proof of Theorem 2.3.** For ease of notation, we drop the subscript “i” in the proof. Recall that \( p(s^2|\sigma^2) \) is the density function of \( s^2|\sigma^2 \) and \( g(\sigma^2) \) is the prior distribution of \( \sigma^2 \). Note that \( p(s^2|\sigma^2) \) is given as

\[ p(s^2|\sigma^2) = \frac{(s^2)^{k/2 - 1} e^{-k s^2/2\sigma^2}}{\Gamma(k/2) 2^{k/2} \sigma^{k}}, \quad s^2 > 0. \tag{18} \]

Define \( f(s^2), n(s^2) \) and \( h(s^2) \) as

\[ f(s^2) = \int_0^\infty p(s^2|\sigma^2)g(\sigma^2) \, d\sigma^2, \tag{19} \]

\[ n(s^2) = \int_0^\infty \sigma^2 p(s^2|\sigma^2)g(\sigma^2) \, d\sigma^2, \tag{20} \]

and

\[ h(s^2) = \int_0^\infty (\sigma^2)^2 p(s^2|\sigma^2)g(\sigma^2) \, d\sigma^2. \tag{21} \]

Note that \( f(s^2) \) is the marginal distribution of \( s^2 \). Then

\[ \hat{\sigma}^2_B = \frac{E \left[ (\sigma^2)^2|s^2 \right]}{E[\sigma^2|s^2]} = \frac{\int_0^\infty (\sigma^2)^2 p(\sigma^2|s^2) \, d\sigma^2}{\int_0^\infty \sigma^2 p(\sigma^2|s^2) \, d\sigma^2} = \frac{h(s^2)}{n(s^2)}. \]

By differentiating \( n(s^2)(s^2)^{-(k/2 - 1)} \) with respect to \( s^2 \), we have

\[ \left[ n(s^2)(s^2)^{-(k/2 - 1)} \right]' = -\frac{k}{2} f(s^2)(s^2)^{-(k/2 - 1)}. \tag{22} \]

Namely,

\[ n(s^2)(s^2)^{-(k/2 - 1)} = -\frac{k}{2} \int_0^{s^2} f(t) t^{-(k/2 - 1)} \, dt + C, \tag{23} \]

for some constant \( C \).
On the other hand, from (20), the left hand side of (23) can be expressed as

\[ n(s^2)(s^2)^{-(k/2-1)} = \int_0^\infty (s^2)^{-(k/2-1)} \sigma^2 p(s^2|\sigma^2)g(\sigma^2) \, d\sigma^2 \]

\[ = \int_0^\infty \frac{(k/2)^{k/2}}{\Gamma(k/2)} \left( \frac{1}{\sigma^2} \right)^{k/2-1} e^{-\frac{k^2}{2\sigma^2}} g(\sigma^2) \, d\sigma^2. \tag{24} \]

From (23) and (24), as \( s^2 \) approaches to zero,

\[ C = \lim_{s^2 \to 0} n(s^2)(s^2)^{-(k/2-1)} = \frac{(k/2)^{k/2}}{\Gamma(k/2)} E \left( \frac{1}{\sigma^2} \right)^{k/2-1} = \frac{k}{2} E \left( \frac{1}{S^2} \right)^{k/2-1}, \]

since, for \( j = 1, 2 \),

\[ E \left( \frac{1}{S^2} \right)^{k/2-j} = \frac{(k/2)^{k/2-j}}{\Gamma(k/2)} E \left( \frac{1}{\sigma^2} \right)^{k/2-j}. \]

Therefore,

\[ n(s^2)(s^2)^{-(k/2-1)} = -\frac{k}{2} \int_0^{s^2} f(t) t^{-(k/2-1)} \, dt + \frac{k}{2} E \left( \frac{1}{S^2} \right)^{k/2-1} \]

\[ = -\frac{k}{2} \int_0^{s^2} f(t) t^{-(k/2-1)} \, dt + \frac{k}{2} \int_0^\infty f(t) t^{-(k/2-1)} \, dt \]

\[ = \frac{k}{2} \int_0^\infty t^{-(k/2-1)} dF(t). \]

We can calculate \( h(s^2) \) in the similar way. Take the first and second order derivatives of \( h(s^2)(s^2)^{-(k/2-1)} \) with respect to \( s^2 \), we then have

\[ [h(s^2)(s^2)^{-(k/2-1)}]' = \frac{k}{2} n(s^2)(s^2)^{-(k/2-1)}, \tag{25} \]

\[ [h(s^2)(s^2)^{-(k/2-1)}]'' = \frac{k^2}{4} f(s^2)(s^2)^{-(k/2-1)}. \tag{26} \]

Consequently,

\[ [h(s^2)(s^2)^{-(k/2-1)}]' = \int_0^{s^2} \frac{k^2}{4} f(t) t^{-(k/2-1)} \, dt + C_1, \tag{27} \]

and

\[ h(s^2)(s^2)^{-(k/2-1)} = \int_0^{s^2} \int_0^y \frac{k^2}{4} f(t) t^{-(k/2-1)} \, dt \, dy + C_1 s^2 + C_2 \]

\[ = \frac{k^2}{4} \int_0^{s^2} f(t) t^{-(k/2-1)} (s^2 - t) \, dt + C_1 s^2 + C_2, \tag{28} \]

for some constants \( C_1 \) and \( C_2 \).
From (25) and (27), as \( s^2 \) approaches to zero, similar argument shows that

\[
C_1 = \lim_{s^2 \to 0} \left[ h(s^2)(s^2)^{-(\frac{k}{2}-1)} \right] = -\frac{k^2}{4} E \left( \frac{1}{s^2} \right)^{\frac{k}{2}-1}.
\]

Similarly, combine equations (26) and (28) and let \( s^2 \) approach to zero,

\[
C_2 = \lim_{s^2 \to 0} h(s^2)(s^2)^{-(\frac{k}{2}-1)} = \frac{k^2}{4} E \left( \frac{1}{s^2} \right)^{\frac{k}{2}-2}.
\]

Thus,

\[
h(s^2)(s^2)^{-(\frac{k}{2}-1)} = \frac{k^2}{4} \int_0^{s^2} f(t) t^{-(\frac{k}{2}-1)} (s^2 - t) \, dt - \frac{k^2}{4} E \left( \frac{1}{s^2} \right)^{\frac{k}{2}-1} s^2 + \frac{k^2}{4} E \left( \frac{1}{s^2} \right)^{\frac{k}{2}-2} = \frac{k^2}{4} \left[ - \left( s^2 \int_{s^2}^{\infty} f(t) t^{-(\frac{k}{2}-1)} \, dt \right) + \left( \int_{s^2}^{\infty} f(t) t^{-(\frac{k}{2}-2)} \, dt \right) \right] = \frac{k^2}{4} \int_{s^2}^{\infty} t^{-(\frac{k}{2}-1)} (t - s^2) \, dF(t).
\]

Therefore,

\[
\sigma_B^2 = \frac{h(s^2)}{n(s^2)} = \frac{k}{2} \left[ \int_{s^2}^{\infty} t^{-(\frac{k}{2}-2)} \, dF(t) \right] - s^2.
\]

\( \Box \)

**Proof of Theorem 3.1.** We restate one of the monumental theorems in the empirical process, on which our proof is based (Blum, 1955; DeHardt, 1971).

Let \( \mathcal{F} \) be a set of measurable functions. The bracket \([a, b]\) is the set of all the functions \( l \in \mathcal{F} \) with \( a \leq l \leq b \). An \( \epsilon \)-bracket is a bracket with \( \| b - a \| \leq \epsilon \). The bracketing number \( N_{[\epsilon]}(\mathcal{F}, L_1(P)) \) is the minimum number of \( \epsilon \)-brackets with which \( \mathcal{F} \) can be covered.

**Theorem (Blum-DeHardt)** Let \( \mathcal{F} \) be a class of measurable functions such that \( N_{[\epsilon]}(\mathcal{F}, L_1(P)) < \infty \), for every \( \epsilon > 0 \). Then \( \mathcal{F} \) is a P-Glivenko-Cantelli.

We only prove the part for the numerator and the denominator can be similarly done. Let \( \mathcal{F} = \{ l_1 : l_1(s^2, u) = (s^2)^{-(k/2-2)} \} \) and \( Pl_1(s^2, u) = \int_0^\infty l_1(s^2, u) \, dF(s^2) = \int_0^\infty s^{2-(k/2-2)} \, dF(s^2) \). It suffices to show that \( \mathcal{F} \) is a P-Glivenko-Cantelli class of functions. Since \( F \) is continuous and \( \int_0^\infty (s^2)^{-(k/2-2)} \, dF(s^2) < \infty \), for any \( \epsilon > 0 \), a collection of real numbers \( 0 = v_0 < v_1 < v_2 < \cdots < v_m = \infty \) can be found such that

\[
Pl_1(s^2, v_{j-1}) - Pl_1(s^2, v_j) = \int_{v_{j-1}}^{v_j} (s^2)^{-(k/2-2)} \, dF(s^2) = \int_{v_{j-1}}^{v_j} (s^2)^{-(k/2-2)} \, dF(s^2)
\]

for all \( 1 \leq j \leq m \), with

\[
Pl_1(s^2, v_m^-) = \lim_{v_m \searrow \infty} Pl_1(s^2, v_m) = \lim_{v_m \searrow \infty} \int_{v_m}^{\infty} (s^2)^{-(k/2-2)} \, dF(s^2) = 0.
\]
Consider the collection of brackets \{[a_j, b_j], 1 \leq j \leq m\}, with \(a_j(s^2) = s^{2-(k/2-2)}[s^2 > v_j]\) and \(b_j(s^2) = s^{2-(k/2-2)}[s^2 > v_{j-1}]\). Now each \(l_i \in \mathcal{F}\) is in at least one bracket and \(|a_j - b_j|_F = P_l(s^2, v_{j-1}) - P_l(s^2, v_j) \leq \epsilon\) for all \(1 \leq j \leq m\). Thus, by Blum-DeHardt theorem, \(\mathcal{F}\) is a \(P\)-Glivenco-Cantelli Class of functions.

**Proof of Theorem 3.2** Let

\[
A_N(s_t^2) = \int_0^\infty l_1(s^2, s_t^2) dF_N(s^2), \quad A(s_t^2) = \int_0^\infty l_1(s^2, s_t^2) dF(s^2),
\]

and

\[
B_N(s_t^2) = \int_0^\infty l_2(s^2, s_t^2) dF_N(s^2), \quad B(s_t^2) = \int_0^\infty l_2(s^2, s_t^2) dF(s^2).
\]

According to the proof of Theorem 3.1, \(\sup_{s^2_i \in \mathcal{R}} |A_N(s_i^2) - A(s_i^2)| \to 0\) and \(\sup_{s^2_i \in \mathcal{R}} |B_N(s_i^2) - B(s_i^2)| \to 0\) a.s.

Let \(L = \inf_{s^2_i \in D^\delta} \{B(s_i^2)\}\). Then for any \(\epsilon > 0\), when \(N\) is sufficiently large

\[
\inf_{s^2_i \in D^\delta} B_N(s_i^2) \geq L - \epsilon \quad \text{a.s.,}
\]

and \(\sup_{s^2_i \in \mathcal{R}} A_N(s_i^2) \leq C, \text{ a.s.}\) for some constant \(C\). Then

\[
\sup_{s^2_i \in D^\delta} |\hat{\sigma}_{i,F-EBV}^2 - \hat{\sigma}_{i,B}^2|
\]

\[
= \sup_{s^2_i \in D^\delta} \left| \frac{A_N(s_t^2)}{B_N(s_t^2)} - \frac{A(s_t^2)}{B(s_t^2)} \right|
\]

\[
= \sup_{s^2_i \in D^\delta} \left| \frac{A_N(s_t^2)(B(s_t^2) - B_N(s_t^2))}{B_N(s_t^2)B(s_t^2)} + \frac{A_N(s_t^2) - A(s_t^2)}{B(s_t^2)} \right|
\]

\[
\leq \frac{C}{L^2} \sup_{s^2_i \in D^\delta} |B(s_t^2) - B_N(s_t^2)| + \frac{1}{L} \sup_{s^2_i \in D^\delta} |A(s_t^2) - A_N(s_t^2)| \to 0, \text{ a.s.}
\]

\[\blacksquare\]
Table 4: The \( \log_{10}(\text{risk}) \) associated with the loss function (13) of the different estimators for the variances under different simulation settings. For each setting, we consider three selection rule: (i) the parameters corresponding to the 1% smallest sample variances; (ii) the parameters corresponding to the 5% smallest sample variances; and (iii) all the parameters.

### B Additional simulation results

In this section, we include additional simulation results which are not listed in the paper due to the page limit. The numerical results consist of four parts: (a) results of variance estimation post-selection; and (b) results of Finite Bayes inference problem.

**(a) Results of variance estimation post-selection.**

To help the readers, we restate the simulation settings here. Let \( \sigma_i^2 \)'s be the parameters, and the sample variances \( s_i^2 \)'s are generated according to Model 1 where the degrees of freedom \( k \) is chosen as 5. We consider the following different choices of the prior \( g(\sigma^2) \):

Setting I: \( \sigma_i^2 \sim \text{inverse gamma distribution: } IG(a, 1) \) where \( a = 10 \) and 6;

Setting II: \( \sigma_i^2 \sim \text{Mixture of inverse gamma distributions: } 0.2IG(a, 1) + 0.4IG(8, 6) + 0.4IG(9, 19) \), where \( a = 10 \) and 6;

Setting III: \( \sigma_i^2 = a \) with 0.4 probability and \( 1/a \) with 0.6 probability, where \( a = 3 \) and 4;

Setting IV: \( \sigma_i^2 \sim \text{Mixture of inverse Gaussian distributions: } 0.4InvGauss(1/a, 1) + 0.6InvGauss(a, a^4) \), where \( a = 2 \) and 3.

After generating the data, order the sample variances increasingly. We consider three different selection rules: (i) select the parameters corresponding to the 1% smallest sample variances; (ii) the parameters corresponding to the 5% smallest sample variances; and (iii) all the parameters. We report \( \log_{10}(\text{risk}) \) in Table 4.

**B.1 Results of finite Bayes inference problem.**

Next, we consider the finite Bayes inference problem. Namely, for each generated data set \( s^2 \) and a new observation \( s_0^2 \), we calculate the estimated values based on different approaches and calculate the loss according to
Table 5: The log_{10}(risk) associated with the loss function (9) of the different estimators for the finite Bayes inference problem.

| Setting | (a, b) | s^2 | ELJS  | TW    | Smyth | mSmyth | Vash  | mVash | REBayes | Proposed |
|---------|--------|-----|------|-------|-------|--------|-------|-------|---------|----------|
| I       | 6      | 0.3 | 0.07 | -0.86 | -0.81 | -1     | -0.8  | -1    | -0.91   | -0.98    |
| II      | 6      | 0.64| 0.43 | -0.18 | -0.04 | -0.53  | -0.02 | -0.54 | -0.52   | -0.59    |
| III     | 3      | 0.92| 0.72 | -0.02 | 0.06  | -0.46  | 0.18  | -0.48 | -0.54   | -0.61    |
| IV      | 3      | 0.43| 0.21 | -0.08 | 0.08  | -0.43  | 0.16  | -0.46 | -0.44   | -0.59    |

The log_{10}(risk) associated with the loss function (9) of the different estimators for the finite Bayes inference problem.