Relation between scattering amplitude and Bethe-Salpeter wave function in quantum field theory

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Abstract

We reexamine the relations between the Bethe-Salpeter (BS) wave function of two particles, the on-shell scattering amplitude, and the effective potential in quantum field theory. It is emphasized that there is an exact relation between the BS wave function inside the interaction range and the scattering amplitude, and the reduced BS wave function, which is defined in this article, plays an essential role in this relation. Based on the exact relation, we show that the solution of Schrödinger equation with the effective potential gives us a correct on-shell scattering amplitude only at the momentum where the effective potential is calculated, while wrong results are obtained from the Schrödinger equation at general momenta. We also discuss about a momentum expansion of the reduced BS wave function and an uncertainty of the scattering amplitude stemming from the choice of the interpolating operator in the BS wave function. The theoretical conclusion obtained in this article could give hints to understand the inconsistency observed in lattice QCD calculation of the two-nucleon channels with different approaches.
I. INTRODUCTION

The finite volume (FV) method derived by Lüscher [1, 2] is the most reliable method to evaluate the scattering phase shift $\delta(k)$ in two-hadron system from lattice QCD calculations on finite volume. In this method $\delta(k)$ can be determined from the relative momentum of two particles $k^2$ on finite volume. A formula connecting $\delta(k)$ to $k^2$ on finite volume was derived from the discussion of two-particle wave function in the outside region of the interaction range in quantum mechanics [2]. It was also confirmed that the same formula is derived from the Bethe-Salpeter (BS) wave function in quantum field theory [3, 4]. Thus, the relation among $k^2$, $\delta(k)$, and the BS wave function in the outside region of the interaction range are well understood in the FV method.

On the other hand, the HALQCD method [5], which was recently proposed, is another method to evaluate $\delta(k)$ from finite volume calculation to utilize information of the BS wave function inside the interaction range. In this method an effective potential defined by the BS wave function is regarded as a potential in quantum mechanics. The scattering phase shift $\delta(k)$ is obtained by solving the Schrödinger equation in the infinite volume using the effective potential as an input.

In lattice QCD calculation of the two-nucleon channels, two methods, the direct calculation of bound state energy based on the FV method and the HALQCD method, give qualitatively different results: Bound states are confirmed by several independent groups with the former, while no bound state is observed with the latter. See more details in recent reviews of the lattice conference [6–8]. The reason of the inconsistency has not been understood at present, though several possible reasons are suggested.

In order to understand the inconsistency, we reexamine the relations between the following three quantities in the framework of quantum field theory: the BS wave function inside the interaction range, the on-shell scattering amplitude, and the effective potential. To do this, we first present the exact relation between the BS wave function inside the interaction range and the on-shell scattering amplitude in quantum field theory, which will be called the fundamental relation, although it was already implicitly used in Ref. [4]. It is emphasized that the reduced BS wave function defined in Sec. II plays an essential role in a direct relation to the on-shell scattering amplitude. This fact means that it is not necessary to introduce the effective potential in quantum field theory for calculation of the scattering phase shift,
even when we use the BS wave function inside the interaction range. Based on this relation, we show that the correct scattering phase shift is obtained from the HALQCD method only at the momentum where the effective potential is defined, while they are incorrect at other momenta. Furthermore, we discuss about a momentum expansion of the reduced BS wave function and an uncertainty of the scattering amplitude arising from the choice of operators for the BS wave function. These discussions would be also helpful to understand what is obtained from the HALQCD method.

This article is organized as follows. Section II is devoted to give definitions for various quantities in the framework of quantum field theory: the BS wave function, the on-shell scattering amplitude, the reduced BS wave function, and the fundamental relation between them. We clarify the relation between the two scattering phase shifts obtained from the fundamental relation and the HALQCD method in Sec. III. In Sec. IV we consider the expansion of the effective potential and discuss the validity of the velocity expansion used in the HALQCD method. In Sec. V an uncertainty of the scattering amplitude coming from the choice of the operators in the BS wave function is discussed employing a smeared operator as an example. We show that the smeared BS wave function gives a different scattering amplitude from the one of the unsmeared BS wave function. Section VI summarizes conclusions.

II. DEFINITIONS

A. BS wave function and scattering amplitude

In this article, we consider only the simplest two-particle scattering, which is an $S$-wave scattering of two distinguishable scalar particles, whose mass is $m$, below the threshold $2E_p \leq 4m$, where $E_p = \sqrt{m^2 + p^2}$ with the relative momentum $p$. We follow the definitions in Refs. [3, 4]. The BS wave function in the infinite volume is defined by\footnote{Since we mainly discuss $\phi(\vec{x}; \vec{k})$ in the infinite volume, the subscript $\infty$ of the BS wave function employed in Ref. [4] is omitted. On the other hand, we express the BS wave function on finite volume as $\phi_L(\vec{x}; k)$.}

$$\phi(\vec{x}; \vec{k}) = \langle 0| \pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\hat{\pi}_1(\vec{k})\hat{\pi}_2(-\vec{k}); \text{in} \rangle, \quad (1)$$

$$\phi(\vec{x}; \vec{k}) = \langle 0| \pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\hat{\pi}_1(\vec{k})\hat{\pi}_2(-\vec{k}); \text{in} \rangle, \quad (1)$$
where \( \pi_i \) is an interpolating operator of the \( i \)th scalar particle \((i = 1, 2)\), and \( |\tilde{\pi}_1(\vec{k})\tilde{\pi}_2(-\vec{k}); \text{in} \rangle \) is an asymptotic two-particle state with the relative momenta \( \vec{k} \) and \(-\vec{k}\).

As discussed in Refs. [3, 4], in quantum field theory, the relation between \( \phi(\vec{x}; \vec{k}) \) and the two-particle scattering amplitude \( M(p; k) \) with \( k = |\vec{k}| \) is derived using the LSZ reduction formula in the case that inelastic scattering effects are negligible. Here \( M(p; k) \) is the off-shell amplitude defined by the 4-point Green function:

\[
e^{-i\mathbf{q} \cdot \mathbf{x}} \frac{-i\sqrt{Z}M(p; k)}{-\mathbf{q}^2 + m^2 - i\varepsilon} = \\
\int d^4z d^4y_1 d^4y_2 K(p, z)K(-\mathbf{k}_1, y_1)K(-\mathbf{k}_2, y_2)\langle 0|T[\pi_1(z)\pi_2(x)\pi_1(y_1)\pi_2(y_2)]|0\rangle, \tag{2}
\]

where the bold faced momenta and coordinates are four-dimensional vectors, and

\[
K(p, z) = \frac{i}{\sqrt{Z}} e^{i\mathbf{p} \cdot \mathbf{z}}(-\mathbf{p}^2 + m^2), \tag{3}
\]

with \( Z \) the renormalization factor of the operator \( \pi_i \). As given in Ref. [4], \( p, k_1, \) and \( k_2 \) are on-shell momenta,

\[
p = (E_p, \vec{p}), \quad k_1 = (E_k, \vec{k}), \quad k_2 = (E_k, -\vec{k}), \tag{4}
\]

while \( \mathbf{q} \) is generally off-shell momentum and determined by the total energy momentum conservation,

\[
\mathbf{q} = (2E_k - E_p, -\vec{p}) \tag{5}
\]

with \( \mathbf{q} = (E_p, -\vec{p}) \) at on-shell. \( M(p; k) \) is related to the scattering phase shift \( \delta(k) \) at on-shell,

\[
M(k; k) = \frac{16\pi E_k}{k} e^{i\delta(k)} \sin(\delta(k)). \tag{6}
\]

In Refs. [3, 4] the relation between \( \phi(\vec{x}; \vec{k}) \) and \( M(p; k) \) is given by

\[
\phi(\vec{x}; \vec{k}) = e^{i\vec{k} \cdot \vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p; k)}{p^2 - k^2 - i\varepsilon} e^{i\vec{p} \cdot \vec{x}}, \tag{7}
\]

where \( H(p; k) \) is related to \( M(p; k) \) as

\[
H(p; k) = \frac{E_p + E_k}{8E_p E_k} M(p; k). \tag{8}
\]

In Eq. (7), irrelevant overall constants are neglected. In this article we consider only the \( S \)-wave scattering so that \( \phi(\vec{x}; \vec{k}) = \phi(x; k) \) where the first term \( e^{i\vec{k} \cdot \vec{x}} \) in the right-hand side of Eq. (7) is replaced by the spherical Bessel function \( j_0(kx) \).

\[\text{We omit } t \text{ dependence of } \phi(\vec{x}; \vec{k}), \text{ because it is expressed by an overall factor } e^{i2E_x t}. \text{ All the overall factors will be neglected in the later discussion.}\]
B. Reduced BS wave function

We define \( h(x; k) \) using \( \phi(x; k) \) as in Ref. [4],
\[
h(x; k) = (\Delta + k^2)\phi(x; k).
\] (9)

Using Eq. (7) we also obtain
\[
h(x; k) = -\int \frac{d^3p}{(2\pi)^3} H(p; k)e^{ip\cdot \vec{x}}.
\] (10)

We assume that \( h(x; k) = 0 \) except for exponential tail in the outside region of the interaction range \( R \) as in Ref. [4]. We shall call \( h(x; k) \) as the reduced BS wave function in the following, because \( h(x; k) \) is more directly related to the scattering amplitude defined through the reduction formula of Eq. (2) than the BS wave function itself. The reduced BS wave function plays an essential role to calculate \( \delta(k) \) in quantum field theory, when we use the information of \( \phi(x; k) \) in \( x \leq R \), which will be shown below.

C. Fundamental relation

Using the definition of the reduced BS wave function Eq. (10), \( H(k; k) \), which is related to the on-shell amplitude Eq. (6), is obtained by the Fourier transformation of \( h(x; k) \),
\[
H(k; k) = -\int d^3x \ h(x; k)e^{-ik\cdot \vec{x}} = \frac{4\pi}{k}e^{i\delta(k)} \sin \delta(k).
\] (11)

This is the relation between the reduced BS wave function \( h(x; k) \), i.e., \( \phi(x; k) \) inside the interaction range, and the on-shell scattering amplitude \( \delta(k) \) in quantum field theory. Although this relation is not explicitly written in Ref. [4], it was used to show the relation between \( \delta(k) \) and \( \phi(\vec{x}; \vec{k}) \) in \( x \to \infty \) in the reference. We shall call the relation as the fundamental relation through this article. It should be noted that it is not necessary to introduce the effective potential, which is defined in the next section, for the calculation of \( \delta(k) \) in quantum field theory, if we use this relation using the reduced BS wave function \( h(x; k) \).

In later sections, based on the fundamental relation, we will discuss a relation of \( \delta(k) \) in quantum field theory and the one obtained from Schrödinger equation in quantum mechanics, and an uncertainty arising from the choice of the interpolating operator in \( \phi(x; k) \).

3 \( h(x; k) \) is defined in the opposite sign to Eq. (A10) in Ref. [4] to coincide the sign of lattice effective potential Eq. (29) in the reference.
III. SCATTERING PHASE SHIFT FROM SCHRÖDINGER EQUATION USING EFFECTIVE POTENTIAL

A. Effective potential

The effective potential $V(x; k)$ was first introduced in Ref. [4]. We follow its definition with a factor $1/m$ assuming $V(x; k) = 0$ in $x > R$,

$$V(x; k) = \begin{cases} \frac{1}{m} \frac{h(x; k)}{\phi(x; k)} & (x \leq R) \\ 0 & (x > R) \end{cases}. \quad (12)$$

Note that $V(x; k)$ diverges, if $\phi(x; k)$ has a node in $x \leq R$. Although we call it as the effective potential here, the relation between this quantity and the potential in quantum mechanics is not trivial; The former is the reduced BS wave function normalized by the BS wave function being manifestly momentum dependent, while the latter is defined to be momentum independent in principle. This is an essential difference between relativistic quantum field theory and nonrelativistic quantum mechanics [1, 2].

In Ref. [4] the BS wave function $\phi_L(\vec{x}; k)$ in the $I = 2$ two-pion channel on finite volume was calculated in lattice QCD, and the effective potential was evaluated from $\phi_L(\vec{x}; k)$. In that study, the effective potential was used only for the purpose to determine the interaction range $R$. Using $\phi_L(\vec{x}; k)$ outside the interaction range, $k^2$ was obtained from two types of analyses: $-\Delta \phi_L(\vec{x}; k)/\phi_L(\vec{x}; k)$ and a fit of $\phi_L(\vec{x}; k)$ with the form of the Green function on finite volume. Both gave consistent results and also agreed with the one from the two-pion energy. Once $k^2$ on finite volume is determined, $\delta(k)$ in the infinite volume is directly obtained through the FV method [1, 2]. The relations between $\delta(k)$, $k^2$ on finite volume, and $\phi_L(\vec{x}; k)$ in $x > R$ are understood in the FV method without any ambiguity.

The essential idea of the HALQCD method [5] is to regard $V(x; k)$ as a potential in quantum mechanics. To obtain the scattering phase shift $\delta(p)$ at any $p$ below the threshold, which means that $p \neq k$ is available, Schrödinger equation in the infinite volume is solved using $V(x; k)$ as a potential,

$$(-\Delta + p^2)\phi(x; p) = 2\mu V(x; k)\phi(x; p), \quad (13)$$

4 The relation between $V(x; k)$ in this article and the effective potential $U(\vec{x}; k)$ in Eq. (29) of Ref. [4] is $V(x; k) = k^2 U(\vec{x}; k)/m$. 

where \( \phi(x; p) \) is the solution of the equation, and \( \mu \) is the reduced mass \( \mu = m/2 \). \( \delta(p) \) is obtained from the \( x \) dependence of \( \phi(x; p) \) in \( x \to \infty \). Up to now, however, we have not known any explicit relation between the scattering phase shift \( \delta(p) \) obtained from the Schrödinger equation with \( V(x; k) \) and the one in the fundamental relation of Eq. (11). We will clarify this point in the following subsection.

B. Scattering phase shift from Schrödinger equation

As explained in a textbook of quantum mechanics\(^5\), if \( V(x; k) \) in the Schrödinger equation of Eq. (13) is a central potential going to zero faster than \( 1/x \) as \( x \to \infty \), the scattering amplitude \( f(p) \) of the two-particle scattering with the interaction \( V(x; k) \) is given by

\[
f(p) = -\frac{2\mu}{4\pi} \int d^3x V(x; k) \phi(x; p) e^{-ip \cdot x} = -\frac{1}{4\pi} \int d^3x \frac{h(x; k)}{\phi(x; k)} \phi(x; p) e^{-ip \cdot x}.
\]

(14)

On the other hand, \( f(p) \) can be also expressed by the scattering phase shift \( \delta(p) \),

\[
f(p) = \frac{e^{i\delta(p)} \sin \delta(p)}{p}.
\]

(15)

In the case of \( p = k \), at which the effective potential \( V(x; k) \) is defined, the Schrödinger equation of Eq. (13) should reduce to Eq. (9) so that \( \phi(x; k) = \phi(x; k) \) holds in the Schrödinger equation as well as in Eq. (14). Thus, \( f(k) \) is written by \( H(k; k) \) through the fundamental relation of Eq. (11):

\[
f(k) = -\frac{1}{4\pi} \int d^3x h(x; k) e^{-ik \cdot x} = \frac{1}{4\pi} H(k; k) = \frac{e^{i\delta(k)} \sin \delta(k)}{k}.
\]

(16)

Comparing this equation with Eq. (15), one obtains \( \delta(k) = \delta(k) \). Therefore, the same scattering phase shift is obtained from the Schrödinger equation and the fundamental relation of Eq. (11) at the momentum \( k \) where \( V(x; k) \) is defined.

In the case of \( p \neq k \), however, the two scattering phase shifts obtained from the Schrödinger equation with \( V(x; k) \) and the fundamental relation do not coincide anymore\(^6\).

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\(^5\) See for example Eq. (7.1.34) in p. 384 of Ref. 9. Note that the normalization of the wave function in the equation differs by \( (2\pi)^{3/2} \) from our case.

\(^6\) While in a special system, where \( V(x; k) \) is assumed to be independent of \( k \) as defined in quantum mechanics, a correct \( \delta(p) \) is obtained through the Schrödinger equation, in a general relativistic system one needs \( V(x; p) \) at \( p \neq k \) to obtain \( \delta(p) \).
IV. EXPANSION OF REDUCED BS WAVE FUNCTION

In quantum field theory, when we focus on $\phi(x; k)$ in $x \leq R$, most important quantity to calculate the on-shell scattering amplitude is the reduced BS wave function $h(x; k)$, because one can evaluate $\delta(k)$ from $h(x; k)$ through the fundamental relation of Eq. (11). A simple momentum expansion of $h(x; k)$ is allowed by the Taylor expansion in the vicinity of a given momentum.

On the other hand, $h(x; k)$ is expressed by the nonlocal effective potential $U(x, x')$ as

$$h(x; k) = \int d^3x' U(x, x')\phi(x'; k),$$

where $U(x, x')$ does not explicitly depend on $k$. As in Ref. [5], one can construct $U(x, x')$ by $k$ integration of Eq. (17) up to the threshold momentum $k_{\text{max}}$ with an appropriate function $g(x; k)$,

$$U(x, x') = \int_0^{k_{\text{max}}} \frac{d^3k}{(2\pi)^3} h(x; k)g(x'; k),$$

where $g(x; k)$ is assumed to satisfy

$$\int_0^{k_{\text{max}}} \frac{d^3k}{(2\pi)^3} \phi(x; k)g(x'; k) = \delta(\vec{x} - \vec{x}').$$

In the expression of Eq. (18), the degree of freedom of $k$ in $h(x; k)$ is integrated out, and then it is converted to the degree of freedom of $x'$ in $U(x, x')$.

In the HALQCD method, $U(x, x')$ is approximated by the so-called velocity expansion defined by

$$U(x, x') = \delta^3(\vec{x} - \vec{x}')(V_0(x) + V_1(x)\Delta + \cdots),$$

where the dots express higher order terms of $O(\Delta^n)$ ($n \geq 2$), and $V_i(x)$ is assumed to be independent of $k$. This assumption provides a theoretical base for the time-dependent HALQCD method [10], where the $k$ independent $V_i(x)$ is extracted from $\phi(x; k)$ at any $k$ below the threshold.

We point out that the expansion parameter in this approximation is not clear. Formally, it is not an expansion in terms of the velocity, but of the Laplacian. Substituting Eq. (20) into Eq. (17) one obtains

$$h(x; k) = V_0(x)\phi(x; k) + V_1(x)(h(x; k) - k^2\phi(x; k)) + \cdots.$$
Since \( h(x; k) \) appears both sides in the equation, this is not a systematic expansion of \( h(x; k) \) in terms of \( k^2 \) in contrast to the Taylor expansion. From the above equation we obtain the following expressions for the reduced BS wave function,

\[
h(x; k) = \left\{ \frac{V_0(x)}{1 - V_1(x)} - k^2 \frac{V_1(x)}{1 - V_1(x)} + O(k^0, k^2, k^4, \ldots) \right\} \phi(x; k) \tag{22}
\]

\[
= \left\{ V_0(x) + V_1(x)\Delta + O(k^0, k^2, k^4, \ldots) \right\} \phi(x; k) \tag{23}
\]

\[
= \left[ U(x, x') \right]_{\text{local}} \phi(x; k). \tag{24}
\]

The coefficients of \( k^{2n} (n \geq 0) \) should vary according to the inclusion of the higher order terms of \( O(\Delta^m) (m > n) \) in Eq. (20). We point out that \( [U(x, x')]_{\text{local}} \), which is regarded as a local approximation of \( U(x, x') \) in Eq. (17), has \( k \) dependence. Since \( h(x; k)/\phi(x; k) \) has the two independent degrees of freedom of \( x \) and \( k \), \( [U(x, x')]_{\text{local}} \) needs to keep the same number of the degrees of freedom after the degree of freedom of \( x' \) is integrated out.

The problem in this expansion becomes manifest in the practical determination of \( V_i(x) \). The simplest example is the determination of the leading term \( V_0(x) \), which is approximated by \( h(x; k)/\phi(x; k) \). We find that it should contain the contributions of \( O(k^{2n}) \) (\( n \geq 0 \)) from the higher order terms of the velocity expansion in order to properly describe the \( k \) dependence of \( h(x; k) \). The inclusion of the term of \( V_1(x) \) does not change the argument. Both \( V_0(x) \) and \( V_1(x) \) are determined in \( k \) dependent way with the use of \( \phi(x; k) \) and \( \Delta\phi(x; k) \). Otherwise the constructed \( [U(x, x')]_{\text{local}} \) has the uncertainties of \( O(k^{2n}) \) (\( n \geq 0 \)). This is a consequence of the fact that \( [U(x, x')]_{\text{local}} \) has the two independent degrees of freedom of \( x \) and \( k \), even if \( U(x, x') \) is superficially expanded using \( \Delta \) and \( V_i(x) \) without any explicit \( k \) dependence.

V. UNCERTAINTY OF SCATTERING AMPLITUDE FROM THE CHOICE OF OPERATOR IN \( \phi(\vec{x}; \vec{k}) \)

In this section, let us consider the case of smeared operator as a choice of operator in \( \phi(\vec{x}; \vec{k}) \). We discuss the relation between the scattering amplitude and the BS wave function with the use of a smeared operator through the fundamental relation, and show that the

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7 It is inappropriate to consider only the leading term \( V_0(x) \), because \( U(x, x') \) is a nonlocal potential.

8 As discussed in Sec. II C, \( h(x; k) \) directly provides the scattering amplitude through the fundamental relation of Eq. (11) so that it does not make sense to approximate \( V_0(x) \) by \( h(x; k)/\phi(x; k) \) and solve the Schrödinger equation with \( V_0(x) \).
scattering amplitude obtained from the BS wave function depends on the details of the smeared operator.

At first suppose $\phi(x; k)$ is constructed by the local operators. For convenience in the following discussion, we change the coordinates of the two-particle operator in $\phi(x; k)$ of Eq. (1) from $\pi_1(\vec{x})/2 \pi_2(-\vec{x}/2)$ to $\pi_1(\vec{x}) \pi_2(\vec{0})$. This transformation does not lose any generality, because $\phi(x; k)$ is a function of the relative position of the two interpolating operators. Thus, the modified $\phi(x; k)$ also satisfies Eq. (9) yielding the correct scattering amplitude in the fundamental relation.

For simplicity we will discuss the smeared-local BS wave function given by

$$\tilde{\phi}(\vec{x}; \vec{k}) = \langle 0 | \hat{\pi}_1(\vec{x}) \pi_2(\vec{0}) | \hat{\pi}_1(\vec{k}) \hat{\pi}_2(-\vec{k}) ; \text{in} \rangle,$$

(25)

where only one of the interpolating operators is smeared. The smeared operator $\tilde{\pi}_1(\vec{x})$ is defined by

$$\tilde{\pi}_1(\vec{x}) = \int d^3 y \ s(|\vec{x} - \vec{y}|) \pi_1(\vec{y})$$

(26)

with $s(x)$ a smearing function as $s(x) \to 0$ for $x \to \infty$. As discussed in Sec. III we consider only the $S$-wave scattering so that $\tilde{\phi}(\vec{x}; \vec{k}) = \tilde{\phi}(x; k)$. The Laplacian of $\tilde{\pi}(\vec{x})$ gives

$$\Delta \tilde{\pi}_1(\vec{x}) = \Delta_x \int d^3 y \ s(|\vec{x} - \vec{y}|) \pi_1(\vec{y}) = \int d^3 y \ [\Delta_y s(|\vec{x} - \vec{y}|)] \pi_1(\vec{y}) = \int d^3 y \ s(|\vec{x} - \vec{y}|) [\Delta_y \pi_1(\vec{y})],$$

(27)

where $\Delta_x$ is the Laplacian with respect to $\vec{x}$, and we use $\Delta_x s(|\vec{x} - \vec{y}|) = \Delta_y s(|\vec{x} - \vec{y}|)$ and the integration by parts in the second and the third equalities, respectively. The corresponding reduced BS wave function $\tilde{h}(x; k)$ is defined as

$$\tilde{h}(x; k) = (\Delta + k^2) \tilde{\phi}(x; k) = \int d^3 y \ s(|\vec{x} - \vec{y}|) h(y; k).$$

(28)

The Fourier transformation of $\tilde{h}(x; k)$ gives a different scattering amplitude from that for $h(x; k)$ in the fundamental relation of Eq. (11)

$$\tilde{H}(k; k) = - \int d^3 x \ \tilde{h}(x; k) e^{-i\vec{k} \cdot \vec{x}} = - \int d^3 x \ \int d^3 y \ s(|\vec{x} - \vec{y}|) h(y; k) e^{-i\vec{k} \cdot \vec{x}},$$

(29)

which explicitly depends on $s(x)^9$

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9 It is pointed out in Ref. [11] that the $s(x)$ dependence can be factorized by using the Fourier transformation of $s(x)$, i.e., $\tilde{H}(k; k) = C(k) H(k; k)$, where $C(k) = \int d^3 x \ s(x) e^{-i\vec{k} \cdot \vec{x}}$. 
Assuming that \( s(x) \) is not much broad and there is a region where \( \tilde{h}(x; k) = 0 \) in \( x > \tilde{R} \), the effective potential \( \tilde{V}(x; k) \) is given by

\[
\tilde{V}(x; k) = \frac{1}{m} \frac{\tilde{h}(x; k)}{\tilde{\phi}(x; k)}
\]

in \( x \leq \tilde{R} \) and \( \tilde{V}(x; k) = 0 \) in \( x > \tilde{R} \). Following the discussion in Sec. III, the Schrödinger equation with \( \tilde{V}(x; k) \) gives the amplitude \( \tilde{H}(k; k) \) of Eq. (29) at the momentum \( k \). The amplitude \( H(k; k) \) cannot be obtained from \( \tilde{V}(x; k) \), when we use Eq. (14).

Therefore, a smearing of the interpolating operator in the BS wave function gives a different scattering amplitude from the one obtained from the fundamental relation, which depends on the smearing function \( s(x) \). One can easily expect that the smeared-smeared BS wave function, where the two interpolating operators are smeared, also gives a different scattering amplitude from \( H(k; k) \) as well as \( \tilde{H}(k; k) \), and its relation to \( h(x; k) \) becomes more complicated than Eq. (29). Furthermore, a smearing of the quark field in the interpolating operator of hadrons utilized in lattice QCD calculations make the relation between the BS wave function and the scattering amplitude far more complicated.

The above discussion leads to the conclusion that the smearing of the interpolating operators in the BS wave function causes uncertainties both in the use of Eq. (14) and the fundamental relation. On the other hand, in the FV method, the relevant quantity is essentially \( k^2 \) on finite volume, which can be obtained even with the use of the smeared operators. For example we can extract \( k^2 \) from \( -\Delta \phi(x; k)/\phi(x; k) \) in \( x > \tilde{R} \) as shown in Ref. [4].

VI. CONCLUSION

We have reexamined the relations between the BS wave function inside the interaction range, the on-shell scattering amplitude, and the effective potential in the framework of quantum field theory. The fundamental relation allows us to obtain the scattering phase shift \( \delta(k) \) from the reduced BS wave function \( h(x; k) \) which is essentially the BS wave function inside the interaction range, while we obtain \( \delta(k) \) by analyzing the BS wave function outside the interaction range in the FV method.

The fundamental relation tells us two important facts. First, it is not necessary to introduce the effective potential \( V(x; k) \) for calculation of \( \delta(k) \) in quantum field theory. In case that one uses \( \phi(x; k) \) in \( x \leq R \) to extract \( \delta(k) \), most relevant quantity is \( h(x; k) \). Second,
the solution at the momentum $p$ of the Schrödinger equation with $V(x; k)$ as a potential gives the correct scattering phase shift only at $p = k$, while it is incorrect at $p \neq k$. In other words, the correct $\delta(k)$ is obtained only from $V(x; k)$ defined at the same momentum as $\delta(k)$, and it is not allowed to use $V(x; k)$ to calculate $\delta(p)$ at $p \neq k$.

It is possible to expand $h(x; k)$ by the Taylor expansion in the vicinity of a given momentum. On the other hand, the velocity expansion is not a systematic expansion of $h(x; k)$ in terms of $k^2$. The coefficients of the velocity expansion $V_i(x)$ should depend on $k$ to correctly describe the $k$ dependence of $h(x; k)$.

We have also shown that the BS wave function using a smeared operator yields a modified scattering amplitude, which depends on the smearing function. This conclusion holds true for both methods: the direct use of the fundamental relation and the solution of the Schrödinger equation with the effective potential.

The results obtained in this article are also applicable to the bound state, and can be extended to systems of particles with nonzero spin \cite{12}. Our theoretical conclusion could be helpful to understand the inconsistency observed in the lattice QCD results of the two-nucleon channels obtained by the two methods: the direct calculation based on the FV method and the HALQCD method.

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