A connection with parallel torsion on almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics

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Abstract

Almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics are considered. A linear connection $D$ is introduced such that the structure of these manifolds is parallel with respect to $D$. Of special interest is the class of the locally conformally equivalent manifolds of the manifolds with covariantly constant almost complex structures and the case when the torsion of $D$ is $D$-parallel. Curvature properties of these manifolds are studied. An example of 4-dimensional manifolds in the considered basic class is constructed and characterized.

Keywords: almost hypercomplex manifold, pseudo-Riemannian metric, anti-Hermitian metric, Norden metric, indefinite metric, neutral metric, natural connection, parallel structure.

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Introduction

In this work we continue the investigations on a manifold $M$ with an almost hypercomplex structure $H$. We provide this almost hypercomplex manifold $(M, H)$ with a metric structure $G$, generated by a pseudo-Riemannian metric $g$ of neutral signature $([1], [2])$.

It is known that, if $g$ is a Hermitian metric on $(M, H)$, the derived metric structure $G$ is the known hyper-Hermitian structure. It consists of the given Hermitian metric $g$ with respect to the three almost complex structures of $H$ and the three Kähler forms associated with $g$ by $H$.

Dedicated to Professor Kostadin Gribachev, the author’s teacher in the theory of manifolds, on the occasion of his 70th birthday

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Here, the considered metric structure $G$ has a different type of compatibility with $H$. The structure $G$ is generated by a neutral metric $g$ such that the first (resp., the other two) of the almost complex structures of $H$ acts as an isometry (resp., act as anti-isometries) with respect to $g$ in each tangent fibre. Let the almost complex structures of $H$ act as isometries or anti-isometries with respect to the metric, then the existence of an anti-isometry generates exactly the existence of one more anti-isometry and an isometry. Thus, $G$ contains the metric $g$ and three $(0,2)$-tensors associated by $H$ – a Kähler form and two metrics of the same type. The existence of such bilinear forms on an almost hypercomplex manifold is proved in [1]. The neutral metric $g$ is Hermitian with respect to the first almost complex structure of $H$ and $g$ is an anti-Hermitian (i.e. Norden) metric regarding the other two almost complex structures of $H$. For this reason we call the derived manifold $(M, H, G)$ an almost hypercomplex manifold with Hermitian and anti-Hermitian metrics or shortly an almost $(H, G)$-manifold.

Recently, manifolds with neutral metrics and various tensor structures have been object of interest in theoretical physics.

The geometry of an arbitrary almost $(H, G)$-manifold is the geometry of the hypercomplex structure $H = \{J_1, J_2, J_3\}$ and the neutral metric $g$ or equivalently – the geometry of the metric structure $G = \{g, g_1, g_2, g_3\}$. In this geometry, there are important so-called natural connections for the $(H, G)$-structure (briefly, the $(H, G)$-connections), i.e. those linear connections, with respect to which the parallel transport determines an isomorphism between the tangent spaces with $(H, G)$-structure. This holds if and only if $H$ and $g$ are parallel with respect to such a connection.

If the three almost complex structures of $H$ are parallel with respect to the Levi-Civita connection $\nabla$ of $g$, then we call such $(H, G)$-manifolds of Kähler type pseudo-hyper-Kähler manifolds and we denote their class by $\mathcal{K}$. Therefore, outside of the class $\mathcal{K}$, the Levi-Civita connection $\nabla$ is no longer an $(H, G)$-connection. There exist countless natural connections on an almost $(H, G)$-manifold in the general case.

The class $\mathcal{W}$ of the locally conformally equivalent manifolds of the pseudo-hyper-Kähler manifolds is an object of special interest in this paper. The covariant derivatives of the elements of $H$ are explicitly expressed by the structure tensors of $H$ and $G$ on the manifolds in $\mathcal{W}$.

In this work we construct and characterize an $(H, G)$-connection on $\mathcal{W}$-manifolds. In the case of almost hyper-Hermitian manifolds such a connection is known as the Lichnerowicz connection [3].

In the first section we show that as the pseudo-hyper-Kähler manifolds have a zero curvature tensor for $\nabla$ ([2]), thus the tensors with the same properties are zero on any almost $(H, G)$-manifold.

In the second section we introduce a linear connection $D$ with respect to which the structure tensors of the almost $(H, G)$-manifolds are parallel. Then we characterize the torsion tensor and the curvature tensor of $D$.

In the third section we consider the special case when the torsion of $D$ is $D$-parallel. There we characterize geometrically the manifolds with that spe-
cialization.

In the fourth section we construct a class of 4-dimensional Lie groups as \( W \)-manifolds where the torsion of \( D \) is not \( D \)-parallel.

The basic problem of this work is the existence and the geometric characteristics of the considered manifolds with \( D \)-parallel torsion of a natural connection \( D \). The main result of this paper is that every locally conformal pseudo-hyper-Kähler manifold with \( D \)-parallel torsion of \( D \) is a \( D \)-flat Lie group.

1. Almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics

1.1. The almost \((H,G)\)-manifolds

Let \((M,H)\) be an almost hypercomplex manifold, i.e. \( M \) is a \( 4n \)-dimensional differentiable manifold and \( H = (J_1, J_2, J_3) \) is a triple of almost complex structures with the properties:

\[
J_\alpha = J_\beta \circ J_\gamma = - J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\) and \( I \) denotes the identity.

The standard structure of \( H \) on a \( 4n \)-dimensional vector space with a basis \( \{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}\}_{k=0,1,...,n-1} \) has the form [4]:

\[
\begin{align*}
J_1 X_{4k+1} &= X_{4k+2}, & J_2 X_{4k+1} &= X_{4k+3}, & J_3 X_{4k+1} &= -X_{4k+4}, \\
J_1 X_{4k+2} &= -X_{4k+1}, & J_2 X_{4k+2} &= X_{4k+4}, & J_3 X_{4k+2} &= X_{4k+3}, \\
J_1 X_{4k+3} &= -X_{4k+4}, & J_2 X_{4k+3} &= -X_{4k+1}, & J_3 X_{4k+3} &= -X_{4k+2}, \\
J_1 X_{4k+4} &= X_{4k+3}, & J_2 X_{4k+4} &= -X_{4k+2}, & J_3 X_{4k+4} &= X_{4k+1}.
\end{align*}
\]  

(2)

Let \( g \) be a pseudo-Riemannian metric on \((M,H)\) with the properties

\[
g(x, y) = \varepsilon_\alpha g(J_\alpha x, J_\alpha y),
\]

where

\[
\varepsilon_\alpha = \begin{cases}
1, & \alpha = 1; \\
-1, & \alpha = 2; 3.
\end{cases}
\]

(4)

In other words, for \( \alpha = 1 \), the metric \( g \) is Hermitian with respect to \( J_1 \), and in the case \( \alpha = 2 \) or \( \alpha = 3 \), the metric \( g \) is an anti-Hermitian (i.e., Norden) metric with respect to \( J_\alpha \) (\( \alpha = 2 \) or \( \alpha = 3 \), respectively) [3]. Moreover, the associated bilinear forms \( g_1, g_2, g_3 \) are determined by

\[
g_\alpha(x, y) = g(J_\alpha x, y) = -\varepsilon_\alpha g(x, J_\alpha y), \quad \alpha = 1, 2, 3.
\]

(5)

Because of [3] and [5], the metric \( g \) and the associated bilinear forms \( g_2 \) and \( g_3 \) are necessarily pseudo-Riemannian metrics of neutral signature \((2n, 2n)\). The associated bilinear form \( g_1 \) is the associated (Kähler) 2-form.
A structure \((H, G) = (J_1, J_2, J_3, g, g_1, g_2, g_3)\) is introduced in [2] and [1]. The cases when the original metric \(g\) is Hermitian or anti-Hermitian with respect to the almost complex structures of \(H\) are considered. In [1] it is proved that the unique possibility that an anti-Hermitian metric be considered on an almost hypercomplex manifold is the case when the given metric is Hermitian with respect to the first and moreover it is an anti-Hermitian metric with respect to other two structures of \(H\). Therefore, we call \((H, G)\) an almost hypercomplex structure with Hermitian and anti-Hermitian metrics on \(M\) (or, in short, an almost \((H, G)\)-structure on \(M\)). Then, we call briefly a manifold with such a structure an almost \((H, G)\)-manifold.

The structural tensors of an almost \((H, G)\)-manifold are the three \((0, 3)\)-tensors determined by

\[
F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z), \quad \alpha = 1, 2, 3, \tag{6}
\]

where \(\nabla\) is the Levi-Civita connection generated by \(g\). The corresponding Lee forms \(\theta_\alpha\) are defined by

\[
\theta_\alpha(\cdot) = g^{ij} F_\alpha(e_i, e_j, \cdot), \quad \alpha = 1, 2, 3, \tag{7}
\]

for an arbitrary basis \(\{e_i\}_{i=1}^{4n}\).

The tensors \(F_\alpha\) have the following fundamental identities:

\[
F_\alpha(x, y, z) = -\varepsilon_\alpha F_\alpha(x, z, y) = -\varepsilon_\alpha F_\alpha(x, J_\alpha y, J_\alpha z),
\]

\[
F_\alpha(x, J_\alpha y, z) = \varepsilon_\alpha F_\alpha(x, y, J_\alpha z); \tag{8}
\]

\[
F_\alpha(x, y, z) = F_\beta(x, J_\gamma y, z) - \varepsilon_{\beta\gamma} F_\gamma(x, y, J_\beta z) - F_\gamma(x, J_\beta y, z) + \varepsilon_{\gamma\beta} F_\beta(x, y, J_\gamma z); \tag{9}
\]

\[
F_\beta(x, J_\gamma y, z) - \varepsilon_\gamma F_\beta(x, y, J_\gamma z) + F_\gamma(x, J_\beta y, z) - \varepsilon_\beta F_\gamma(x, y, J_\beta z) = 0; \tag{10}
\]

\[
F_\alpha(x, J_\beta y, J_\gamma z) = \varepsilon_\alpha F_\alpha(x, J_\beta y, J_\gamma z), \tag{11}
\]

\[
F_\alpha(x, J_\beta y, J_\gamma z) = -\varepsilon_\alpha F_\alpha(x, J_\gamma y, J_\beta z)
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\).

In [2] we study a special class \(\mathcal{K}\) of the \((H, G)\)-manifolds – the so-called there pseudo-hyper-Kähler manifolds. The manifolds from the class \(\mathcal{K}\) are the \((H, G)\)-manifolds for which the complex structures \(J_\alpha\) are parallel with respect to the Levi-Civita connection \(\nabla\), generated by \(g\), for all \(\alpha = 1, 2, 3\).

As \(g\) is an indefinite metric, there exist isotropic vectors \(x\) on \(M\), i.e. \(g(x, x) = 0\) for a nonzero vector \(x\). In [1] we define the invariant square norm

\[
\|\nabla J_\alpha\|^2 = g^{ij} g^{kl} g((\nabla_i J_\alpha) e_k, (\nabla_j J_\alpha) e_l), \tag{12}
\]

where \(\{e_i\}_{i=1}^{4n}\) is an arbitrary basis of the tangent space \(T_p M\) at an arbitrary point \(p \in M\). We say that an almost \((H, G)\)-manifold is an isotropic pseudo-hyper-Kähler manifold if \(\|\nabla J_\alpha\|^2 = 0\) for all \(J_\alpha\) of \(H\). Clearly, if \((M, H, G)\)
is a pseudo-hyper-Kähler manifold, then it is an isotropic pseudo-hyper-Kähler manifold. The inverse statement does not hold. For instance, in [1] we have constructed an almost \((H, G)\)-manifold on a Lie group, which is an isotropic pseudo-hyper-Kähler manifold but it is not a pseudo-hyper-Kähler manifold.

1.2. Properties of the Kähler-like tensors

A tensor \(L\) of type \((0,4)\) with the properties:

\[
L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),
\]
\[
L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0
\]

is called a curvature-like tensor. The last equality of (13) is known as the first Bianchi identity of a curvature-like tensor \(L\).

We say that a curvature-like tensor \(L\) is a Kähler-like tensor on an almost \((H, G)\)-manifold when \(L\) satisfies the properties:

\[
L(x, y, z, w) = \varepsilon_\alpha L(x, y, J_\alpha z, J_\alpha w) = \varepsilon_\alpha L(J_\alpha x, J_\alpha y, z, w),
\]

where \(\varepsilon_\alpha\) is determined by (4).

Let the curvature tensor \(R\) of the Levi-Civita connection \(\nabla\), generated by \(g\), be defined, as usual, by \(R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]}z\). The corresponding \((0,4)\)-tensor, denoted by the same letter, is determined by \(R(x, y, z, w) = g(R(x, y)z, w)\). Obviously, \(R\) is a Kähler-like tensor on an arbitrary pseudo-hyper-Kähler manifold.

A Kähler-like tensor \(L\) on an arbitrary almost \((H, G)\)-manifold has the same properties (13) and (14) of \(R\) on a pseudo-hyper-Kähler manifold. Then, according to [2], we have the following similar properties:

\[
L(x, y, z, w) = \varepsilon_\alpha L(x, y, J_\alpha z, J_\alpha w),
\]
\[
L(x, y, z, w) = -L(x, J_\alpha y, z, J_\alpha w) = -L(J_\alpha x, y, J_\alpha z, w).
\]

Thus, we obtain the following geometric characteristic of the Kähler-like tensors on an almost \((H, G)\)-manifold, similarly to Theorem 2.3 in [2], where it is proved that the hyper-Kähler \((H, G)\)-manifolds are flat, i.e. \(R = 0\) in \(\mathcal{K}\).

**Proposition 1.** Every Kähler-like tensor on an almost \((H, G)\)-manifold is zero. \(\square\)

For the sake of brevity we introduce the following notation for an arbitrary \((0,4)\)-tensor

\[
(L \circ J_\alpha)(x, y, z, w) = L(x, y, J_\alpha z, J_\alpha w).
\]

Let us recall the Ricci identity for an almost complex structure \(J\) and a curvature tensor \(R\) for the Levi-Civita connection \(\nabla\) of \(g\):

\[
(\nabla_x \nabla_y J) z - (\nabla_y \nabla_x J) z = R(x, y)J z - JR(x, y)z.
\]

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Then, according to (13), (14), (15), and $\nabla g = 0$, we obtain the following corollaries of the Ricci identity for the almost complex structures $J_\alpha (\alpha = 1, 2, 3)$ of $H$:

$$
(\nabla_x F_\alpha) (y, z, w) - (\nabla_y F_\alpha) (x, z, w) = R(x, y, J_\alpha z, w) + \varepsilon_\alpha R(x, y, z, J_\alpha w),
$$

$$
R(x, y, z, w) - \varepsilon_\alpha R(x, y, J_\alpha z, J_\alpha w)
$$

$$
= -\varepsilon_\alpha \{(\nabla_x F_\alpha) (y, z, J_\alpha w) - (\nabla_y F_\alpha) (x, z, J_\alpha w)\}
$$

$$
= - (\nabla_x F_\alpha) (y, J_\alpha z, w) + (\nabla_y F_\alpha) (x, J_\alpha z, w).
$$

1.3. The class $W$ of the locally conformal pseudo-hyper-Kähler manifolds

According to (3) for $\alpha = 1$, the manifold $(M, J_1, g)$ is almost Hermitian. The following class $W(J_1)$ is the class where the tensor $F_1$ is expressed explicitly by the structure tensors. It is denoted by $W_4$ in [3]. For dimension $4n$ this class is determined by

$$
W(J_1) : F_1(x, y, z) = \frac{1}{2(2n - 1)} \{g(x, y)\theta_1(z) - g(x, z)\theta_1(y)
$$

$$
- g(x, J_1 y)\theta_1(J_1 z) + g(x, J_1 z)\theta_1(J_1 y)\}.
$$

On the other hand, having in mind (5) for $\alpha = 2$ or 3, the manifold $(M, J_\alpha, g)$ is an almost complex manifold with Norden metric (i.e. anti-Hermitian or B-metric). The class $W_1$ in [3] is the class where the tensor $F_\alpha$ is expressed explicitly by the structure tensors. We denote this class by $W(J_\alpha)$. For dimension $4n$ and $\alpha = 2, 3$, it is determined by

$$
W(J_\alpha) : F_\alpha(x, y, z) = \frac{1}{4n} \{g(x, y)\theta_\alpha(z) + g(x, z)\theta_\alpha(y)
$$

$$
+ g(x, J_\alpha y)\theta_\alpha(J_\alpha z) + g(x, J_\alpha z)\theta_\alpha(J_\alpha y)\}.
$$

The three special classes $W_0(J_\alpha): F_\alpha = 0$ for $\alpha = 1, 2, 3$ of the Kähler-type manifolds belong to any other class within the corresponding classification.

Let us denote the class $W = \bigcap_{\alpha=1}^{3} W(J_\alpha)$ of the considered manifolds. It is known from [2], that if a manifold $(M, H, G)$ belongs to the class $W(J_\alpha) \bigcap W(J_\beta)$, then $(M, H, G)$ is of the class $W(J_\gamma)$ for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$.

It is well known that the almost hypercomplex structure $H = (J_\alpha)$ is a hypercomplex structure if $N_\alpha$ vanishes for all $\alpha = 1, 2, 3$, where $N_\alpha(\cdot, \cdot) = [J_\alpha, \cdot] - J_\alpha [\cdot, \cdot] - J_\alpha [\cdot, \cdot] - [\cdot, \cdot]$ are the Nijenhuis tensors for $J_\alpha$. Moreover, it is known that an almost hypercomplex structure $H$ is hypercomplex if and only if two of the three structures $J_\alpha$ are integrable. This means that two of the three tensors $N_\alpha$ vanish [3]. According to [2], the class $W$ is a subclass of the
class of the (integrable) \((H,G)\)-manifolds. An \((H,G)\)-manifold belonging to \(W\) will be called in brief a \(W\)-manifold.

Let us recall from [2] that if \((M, H, G)\) belongs to the class \(W\) then
\[
\frac{2n}{1-2n}\theta_1 \circ J_1 = \theta_2 \circ J_2 = \theta_3 \circ J_3,
\]
where \(\theta_\alpha (\alpha = 1, 2, 3)\) are defined in (7). Hence, having in mind the last equalities, we introduce an 1-form \(\theta\) as follows
\[
\theta = \frac{4n}{1-\varepsilon(4n-1)}\theta_\alpha \circ J_\alpha,
\]  
where \(\alpha = 1, 2, 3\) and \(\varepsilon, \theta_\alpha\) are determined by (4), (7), respectively.

Let \(c : g \mapsto \bar{g} = e^{2u}g\), where \(u\) is a differential function on \(M\), be the usual conformal transformation of the given metric. Then the manifolds \((M, H, G)\) and \((M, H, \bar{G})\), generated by the metrics \(g\) and \(\bar{g}\), respectively, are called \textit{locally conformally equivalent manifolds}. Let \((M, H, G)\) be an arbitrary pseudo-hyper-Kähler manifold. Then we consider the class of the locally conformally equivalent manifolds \((M, H, \bar{G})\) to the pseudo-hyper-Kähler manifolds (i.e. in short, locally conformal pseudo-hyper-Kähler manifolds).

\textbf{Theorem 2.} The class \(W\) is the class of the locally conformally equivalent manifolds to the pseudo-hyper-Kähler manifolds.

\textbf{Proof.} Let \(c : \bar{g} = e^{2u}g\) be the usual conformal transformation, where \(u\) is a differential function on \(M\). Then, according to Theorem 3.2 in [2], the subclass \(W^0\) of \(W\) with the condition \(d\theta = 0\) is the class of the locally conformally equivalent manifolds of the pseudo-hyper-Kähler manifolds.

According to [7], the 1-form \(\theta_\alpha \circ J_\alpha\) is closed on the complex manifold with anti-Hermitian metric \((M, J_\alpha, g)\) in \(W(J_\alpha)\) \((\alpha = 2, 3)\) for any even dimension of \(M\). Therefore, having in mind (19), the 1-forms \(\theta_1 \circ J_1\) and \(\theta\) are closed for any \(4n\)-dimensional \(W\)-manifold \((n \in \mathbb{N})\). Hence, \(W^0\) coincides with \(W\).

Let us remark that for a Hermitian manifold \((M, J_1, g)\) in \(W(J_1)\) the 1-form \(\theta_1 \circ J_1\) is closed for a dimension greater than 4.

From the last theorem and the fact that every pseudo-hyper-Kähler manifold is flat (2), we have

\textbf{Corollary 3.} Every locally conformal pseudo-hyper-Kähler manifold is conformally flat.

By virtue of (17), (18) and (19), the tensors \(F_\alpha\) have the following form on a \(W\)-manifold:
\[
F_\alpha(x, y, z) = \frac{1}{4n} \left\{ \varepsilon_\alpha g(x, y)\theta(J_\alpha z) - g(x, z)\theta(J_\alpha y) - \varepsilon_\alpha g(J_\alpha x, y)\theta(z) + g(J_\alpha x, z)\theta(y) \right\}.
\]  
(20)

On a \(W\)-manifold the relations in (19) between the 1-forms \(\theta\) and \(\theta_\alpha (\alpha = 1, 2, 3)\) imply immediately corresponding equalities for the squares of the corresponding vectors with respect to \(g\) given in the following
Corollary 4. On a locally conformal pseudo-hyper-Kähler manifold the following equalities are valid

\[ \theta(\Omega) = \frac{\varepsilon_\alpha 16n^2}{[1 - \varepsilon_\alpha (4n - 1)]^2} \theta_\alpha(\Omega_\alpha), \]

where \( \Omega \) and \( \Omega_\alpha \) are the Lee vectors associated to \( \theta \) and \( \theta_\alpha \), respectively, i.e. \( \theta = g(\cdot, \Omega) \) and \( \theta_\alpha = g(\cdot, \Omega_\alpha) \). Obviously, if one of the Lee vectors is isotropic then all Lee vectors and their \( J_\alpha \)-images are isotropic. \( \square \)

According to (12), (6), (20) and Corollary 4 we calculate immediately the square norms of \( \nabla J_\alpha \) on a W-manifold.

Proposition 5. On a locally conformal pseudo-hyper-Kähler manifold the square norms of \( \nabla J_\alpha \) for all \( \alpha = 1, 2, 3 \) are proportional to the square of the Lee vector \( \Omega \), corresponding to the Lee form \( \theta \) from (19), with respect to \( g \). More precisely,

\[ \| \nabla J_\alpha \|^2 = \frac{(4n - 1)\varepsilon_\alpha - 1}{4n^2} \theta(\Omega), \quad \alpha = 1, 2, 3. \]

\( \square \)

Therefore we have

Corollary 6. A locally conformal pseudo-hyper-Kähler manifold is an isotropic pseudo-hyper-Kähler manifold if and only if the Lee vector \( \Omega \) is isotropic. \( \square \)

2. A natural \((H, G)\)-connection

2.1. Definition of the connection \( D \)

Let \((M, H, G)\) be an almost \((H, G)\)-manifold generated by \( H = (J_1, J_2, J_3) \) and the metric \( g \). We consider a linear connection on \((M, H, G)\) determined by

\[ D_yz = \nabla_y z + Q(y, z), \quad Q(y, z) = \frac{1}{4} \sum_{\alpha=1}^{3} (\nabla_y J_\alpha) J_\alpha z, \quad (21) \]

where \( \nabla \) is the Levi-Civita connection generated by \( g \). Obviously, the connections \( D \) and \( \nabla \) coincide if and only if the manifold belongs to the class \( \mathcal{K} \) of the pseudo-hyper-Kähler manifolds.

By direct computations we establish that \( D \) is a natural connection for the structure \((H, G)\) or the following is valid

Proposition 7. The connection \( D \) preserves the almost \((H, G)\)-structure, i.e.

\[ DJ_\alpha = Dg = Dg_\alpha = 0, \quad \alpha = 1, 2, 3. \quad (22) \]

\( \square \)
Having in mind Proposition\[7\] we call the natural connection $D$, determined by (21), an \((H, G)\)-connection.

Further, we use the corresponding tensor $Q$ of type \((0, 3)\) determined by

$$Q(y, z, w) = g(Q(y, z), w) = \frac{1}{4} \sum_{\alpha=1}^{3} F_\alpha(y, J_\alpha z, w).$$

(23)

Hence, according to (8), we obtain

$$Q(x, y, z) = -Q(x, z, y).$$

(24)

Since $DJ_\alpha = 0$, we have $(\nabla_y J_\alpha)z = -Q(y, J_\alpha z) + J_\alpha Q(y, z)$ and consequently

$$F_\alpha(x, y, z) = -Q(x, J_\alpha y, z) - \varepsilon_\alpha Q(x, y, J_\alpha z), \quad \alpha = 1, 2, 3.$$ 

(25)

Therefore, because of (23), the following property of $Q$ is valid:

$$Q(x, y, z) + \sum_{\alpha=1}^{3} \varepsilon_\alpha Q(x, J_\alpha y, J_\alpha z) = 0.$$ 

(26)

2.2. The torsion tensor of $D$

For the torsion $T$ of an arbitrary linear connection $D = \nabla + Q$, where $\nabla$ is the Levi-Civita connection, we have

$$T(x, y, z) = Q(x, y, z) - Q(y, x, z).$$

(27)

By virtue of (24) and (27) we obtain

$$Q(x, y, z) = \frac{1}{2} \left\{ T(x, y, z) - T(y, z, x) + T(z, x, y) \right\}.$$ 

(28)

If the considered \((H, G)\)-manifold belongs to the class $W$, using (27), (23) and (20), we obtain the following explicit form of the torsion of $D$ on a $W$-manifold:

$$T(x, y, z) = \frac{1}{16n} \left\{ -2g_1(x, y)\theta(J_1 z) + 3g(x, z)\theta(y) - 3g(y, z)\theta(x) \\
+ g_1(x, z)\theta(J_1 y) + g_2(x, z)\theta(J_2 y) + g_3(x, z)\theta(J_3 y) \\
- g_1(y, z)\theta(J_1 x) - g_2(y, z)\theta(J_2 x) - g_3(y, z)\theta(J_3 x) \right\}.$$ 

(29)

Hence we establish directly that the torsion of $D$ on a $W$-manifold has the property

$$T(\cdot, \cdot, \Omega) = 0,$$

(30)

where $\Omega$ is the Lee vector, corresponding to the Lee form $\theta$ from (19), with respect to $g$. 

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2.3. The curvature tensor of $D$

Let us consider the curvature tensor $K$ of a connection $D$, i.e. $K(x,y)z = [D_x, D_y]z - D_{[x,y]}z$ and $K(x,y,z,w) = g(K(x,y)z, w)$. The relation between the connections $D$ and $\nabla$ generates the corresponding relation between their curvature tensors $K$ and $R$.

**Proposition 8.** On an almost $(H, G)$-manifold the curvature tensors $K$ of the $(H, G)$-connection $D$, determined by (21), and $R$ of the Levi-Civita connection $\nabla$ are related as follows

$$K = \frac{1}{4} \{ R + R \circ J_1 - R \circ J_2 - R \circ J_3 \} - \frac{1}{16} P, \quad (31)$$

where

$$P = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} P_{\alpha\beta} - 4 \sum_{\alpha=1}^{3} P_{\alpha\alpha}, \quad (32)$$

and

$$P_{\alpha\beta}(x, y, z, w) = g((\nabla_x J_\alpha) J_\alpha z, (\nabla_y J_\beta) J_\beta w) - g((\nabla_y J_\alpha) J_\alpha z, (\nabla_x J_\beta) J_\beta w). \quad (34)$$

**Proof.** For an arbitrary linear connection $D$ given by $D = \nabla + Q$ the following equality is known [8]

$$K(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) + Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \quad (33)$$

Applying (23) for the considered connection $D$, we obtain consecutively

$$Q(x, Q(y, z), w) - Q(y, Q(x, z), w) =$$

$$= g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)) \quad (34)$$

$$= \frac{1}{16} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} P_{\alpha\beta}(x, y, z, w),$$

$$\quad (\nabla_x Q)(y, z, w) = \frac{1}{4} \sum_{\alpha=1}^{3} \{ (\nabla_x F_\alpha)(y, J_\alpha z, w)$$

$$\quad - \varepsilon_\alpha g((\nabla_x J_\alpha) z, (\nabla_y J_\alpha) w) \} \quad (35)$$

and therefore by (16) we have

$$\quad (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) =$$

$$= \frac{1}{4} \sum_{\alpha=1}^{3} \{ [-R + \varepsilon_\alpha (R \circ J_\alpha) - P_{\alpha\alpha}](x, y, z, w) \}. \quad (36)$$

We replace (36) and (34) in (33) and then we obtain (31).

□
When we consider $W$-manifolds instead of arbitrary almost $(H, G)$-manifolds, we have to specify the tensor $P$ introduced in (32).

Further we use the notation $g \odot h$ for the Kulkarni-Nomizu product of two $(0,2)$-tensors, i.e.

$$(g \odot h)(x, y, z, w) = g(x, z)h(y, w) - g(y, z)h(x, w) + g(y, w)h(x, z) - g(x, w)h(y, z).$$

Obviously, the tensor $g \odot h$ is a curvature-like tensor when the $(0,2)$-tensors $g$ and $h$ are symmetric.

**Proposition 9.** On a locally conformal pseudo-hyper-Kähler manifold the curvature tensors $K$ and $R$ of the $(H, G)$-connection $D$ and the Levi-Civita connection $\nabla$, respectively, are related by (31) and the auxiliary tensor $P$ has the form

$$P = \frac{1}{16n^2}\{V + V \circ J_1 - V \circ J_2 - V \circ J_3\}, \quad (37)$$

where

$$V = g \odot B + \frac{1}{2}U,$$

$$B = 3\theta \otimes \theta + \theta \circ J_1 \otimes \theta \circ J_1 + \theta \circ J_2 \otimes \theta \circ J_2 + \theta \circ J_3 \otimes \theta \circ J_3 - \frac{3}{4}\theta(\Omega)g - \frac{1}{4}\theta(J_2\Omega)g_2 - \frac{1}{4}\theta(J_3\Omega)g_3, \quad (38)$$

$$U(x, y, z, w) = g_1(x, y)E(z, w),$$

$$E = \theta \otimes \theta \circ J_1 - \theta \circ J_1 \otimes \theta - \theta \circ J_2 \otimes \theta \circ J_3 + \theta \circ J_3 \otimes \theta \circ J_2.$$

**Proof.** The introduced above tensors have the following properties:

(i) The $(0,2)$-tensor $B$ is symmetric and $B(J_\alpha \cdot, J_\alpha \cdot) \neq \pm B(\cdot, \cdot)$;

(ii) The $(0,2)$-tensor $E$ is antisymmetric and $E(J_\alpha \cdot, J_\alpha \cdot) = \varepsilon_\alpha E(\cdot, \cdot)$;

(iii) The $(0,4)$-tensor $U$ has the following properties:

$$U(x, y, z, w) = -U(y, x, z, w) = -U(x, y, w, z),$$

$$U(x, y, J_\alpha z, J_\alpha w) = \varepsilon_\alpha U(x, y, z, w); \quad (39)$$

(iv) The $(0,4)$-tensor $V$ shares the properties (39).

Equations (32) and (20) imply directly

$$P = \frac{1}{16n^2}\left\{(g \odot B) + \sum_{\alpha=1}^3 \varepsilon_\alpha (g \odot B) \circ J_\alpha\right\} + 2U. \quad (40)$$

By virtue of the second line in (39), we have

$$U = \frac{1}{4}\{U + U \circ J_1 - U \circ J_2 - U \circ J_3\}. \quad (41)$$

Then, from (40) and (41) we obtain (37). \qed
Now, let us consider the tensor

\[ S = R - \frac{1}{64n^2}(g \circ B), \]  

(42)

where \( B \) is determined by (38). Since \( B \) is symmetric, then \( S \) is a curvature-like tensor, where \( S \circ J_\alpha \neq \pm S \). Having in mind Proposition 8 and Proposition 9, we have

**Proposition 10.** On a locally conformal pseudo-hyper-Kähler manifold the curvature tensor \( K \) of the \((H, G)\)-connection \( D \) has the form

\[ K = \frac{1}{4}\{\hat{S} + \hat{S} \circ J_1 - \hat{S} \circ J_2 - \hat{S} \circ J_3\}, \]  

(43)

where

\[ \hat{S} = S - \frac{1}{128n^2}U, \]

and \( S, U \) are determined by (42), (38), respectively.

**Proof.** From (31) and (37), according to (32) and (38), we have

\[ K = \frac{1}{4}\{S + S \circ J_1 - S \circ J_2 - S \circ J_3 - \frac{1}{32n^2}U\}. \]  

(44)

Then, taking into account (39), we obtain (43).

□

3. Almost \((H, G)\)-manifolds with \( D \)-parallel torsion of \( D \)

3.1. The condition that the torsion of \( D \) be \( D \)-parallel

In (31) we consider the special case when the torsion \( T \) of \( D \) and the structural tensors \( F_\alpha \) are covariantly constant with respect to the \((H, G)\)-connection \( D \) or briefly \( T \) and \( F_\alpha \) are \( D \)-parallel, i. e. \( DT = 0 \) and \( DF_\alpha = 0 \). In the first subsection we give some interpretations of these conditions.

**Proposition 11.** On an almost \((H, G)\)-manifold the torsion \( T \) of the \((H, G)\)-connection \( D \) is \( D \)-parallel if and only if the structural tensors \( F_\alpha \) are \( D \)-parallel for all \( \alpha = 1, 2, 3 \), i. e. \( DT = 0 \Leftrightarrow DF_\alpha = 0, \alpha = 1, 2, 3 \).

**Proof.** Let \( T \) be \( D \)-parallel. Then, from (28) we obtain \( DQ = 0 \). After that, taking into account (25) and \( DJ_\alpha = 0 \), we obtain \( DF_\alpha = 0, \alpha = 1, 2, 3 \).

Vice versa, since we have \( DF_\alpha = 0 \) and \( DJ_\alpha = 0 \) for all \( \alpha = 1, 2, 3 \), from (26) we obtain \( DQ = 0 \). Hence \( DT = 0 \) holds, because of (27).

□

**Proposition 12.** On a locally conformal pseudo-hyper-Kähler manifold the torsion \( T \) of \( D \) is \( D \)-parallel if and only if the 1-form \( \theta \) is \( D \)-parallel, i. e. \( DT = 0 \Leftrightarrow D\theta = 0 \).
Proof. Let we have \( DT = 0 \). Then the conditions \( DF_\alpha = 0, Dg = 0 \) and \( DJ_\alpha = 0 \) for all \( \alpha = 1, 2, 3 \) imply \( D\theta = 0 \).

Vice versa, supposing conditions (20) and \( D\theta = 0 \) hold, we obtain \( DF_\alpha = 0, \alpha = 1, 2, 3, \) because of (22). Then we have \( DT = 0 \), according to Proposition 11.

If \( \theta \) is \( D \)-parallel then we obtain directly the following

**Corollary 13.** On an almost \((H,G)\)-manifold with \( D \)-parallel torsion of \( D \) the quantities \( \theta(J_\alpha \Omega) \) are constant and, in particular, \( \theta(J_1 \Omega) \) is zero.

3.2. Curvature properties of a \( \mathcal{W} \)-manifold with \( D \)-parallel torsion of \( D \)

Here we characterize the locally conformal pseudo-hyper-Kähler manifolds with \( D \)-parallel torsion of the \((H,G)\)-connection \( D \). The main result is the following

**Theorem 14.** Every locally conformal pseudo-hyper-Kähler manifold \( M \) with \( D \)-parallel torsion of the complete natural connection \( D \) is \( D \)-flat and:

(i) \( M \) has a structure of a Lie group when is simply connected;

(ii) \( M \) is an isotropic pseudo-hyper-Kähler manifold;

(iii) \( M \) is scalar flat.

In the proof of the theorem we need to prove some lemmas.

**Lemma 15.** On a locally conformal pseudo-hyper-Kähler manifold with \( D \)-parallel torsion of \( D \) the curvature tensor \( K \) of \( D \) has the form

\[
K = L - \frac{1}{256n^2} \{ W + W \circ J_1 - W \circ J_2 - W \circ J_3 \},
\]

where

\[
L = R + \frac{1}{64n^2} (g \circ A),
\]

\[
A = -\theta \circ \theta + \theta \circ J_1 \circ \theta \circ J_1 + \theta \circ J_2 \circ \theta \circ J_2 + \theta \circ J_3 \circ \theta \circ J_3
\]

\[
- \theta(\Omega)g - \theta(J_2 \Omega)g_2 - \theta(J_3 \Omega)g_3,
\]

\[
W = g \circ C + \frac{1}{2} U,
\]

\[
C = A + B = 2\theta \circ \theta + 2 \sum_{\alpha=1}^{3} \{ \theta \circ J_\alpha \circ \theta \circ J_\alpha \}
\]

\[
- \frac{5}{2} \theta(\Omega)g - \frac{3}{2} \theta(J_2 \Omega)g_2 - \frac{3}{2} \theta(J_3 \Omega)g_3.
\]
Proof. Let the torsion $T$ of the $(H,G)$-connection $D$ be $D$-parallel or equivalently $DF_\alpha = 0$ ($\alpha = 1, 2, 3$) hold. Then, for the covariant derivative of $F_\alpha$ ($\alpha = 1, 2, 3$) with respect to $\nabla$, we obtain
\[
(\nabla_x F_\alpha)(y, z, w) = F_\alpha(Q(x, y), z, w) + F_\alpha(y, Q(x, z), w) + F_\alpha(y, z, Q(x, w)).
\] (50)

Hence, having in mind (16), (23) and (20), we obtain the following properties of the curvature tensor $R$ for the Levi-Civita connection $\nabla$ of $g$:
\[
R - \varepsilon_\alpha R \circ J_\alpha = -\frac{1}{64n^2} \{(g \circ A) - \varepsilon_\alpha (g \circ A) \circ J_\alpha\},
\] (51)
where $A$ is determined by (47). It is obvious that $A$ is symmetric, then $g \circ A$ is a curvature-like tensor. Clearly, $R$ is a Kähler-like tensor if and only if $A = 0$ or equivalently $\theta = 0$. In other words, $R$ is a Kähler-like tensor if and only if $(M, H, G)$ is a pseudo-hyper-Kähler manifold, which is known.

Now we transform (51) in the following form
\[
R + \frac{1}{64n^2} (g \circ A) = \varepsilon_\alpha \left\{ R \circ J_\alpha + \frac{1}{64n^2} (g \circ A) \circ J_\alpha \right\},
\] (52)
which implies the property $L = \varepsilon_\alpha L \circ J_\alpha$ for the tensor $L$ from (46). Therefore, $L$ is a Kähler-like tensor with respect to each $J_\alpha$.

After that, applying (51) to (31), we obtain
\[
K = L - \frac{1}{256n^2} \left\{(g \circ C) + \sum_{\alpha=1}^3 \varepsilon_\alpha (g \circ C) \circ J_\alpha\right\} + 2U.
\] (53)

Obviously from (49) the tensor $C$ is symmetric. Then $g \circ C$ is a curvature-like tensor.

Let us remark that $W$, introduced by (48), has the Kähler-like property $W = \varepsilon_\alpha W \circ J_\alpha$.

Finally, from (53) and (48), we obtain the form of $K$ in terms of $L$ and $W$ given in (45). □

Lemma 16. On a locally conformal pseudo-hyper-Kähler manifold with $D$-parallel torsion of $D$ the following tensors are zero:

(i) the curvature tensor $K$ of the connection $D$ defined by (21);

(ii) the tensor $W$ defined by (48);

(iii) the tensor $L$ defined by (46).

Proof. It is known that $K$ is a Kähler-like tensor, i.e. $K = \varepsilon_\alpha K \circ J_\alpha$, because of $DJ_\alpha = 0$ and (43). Moreover, we have established that $L$, defined by (46), is also a Kähler-like tensor. Then the tensor $(W + W \circ J_1 - W \circ J_2 - W \circ J_3)$ is

14
also a Kähler-like tensor, according to (45). Therefore, using Proposition 1, we establish that the tensors \( K, L \) and \((W + W \circ J_1 - W \circ J_2 - W \circ J_3)\) are zero. It is easy to obtain that the last tensor in brackets is zero if and only if \( W = 0 \). □

PROOF OF THEOREM 14. Here we give some geometric interpretations of the results of Lemma 16.

(i) The annulment of \( K \) means that the manifold is \( D \)-flat. Moreover, since \( DT = 0 \) and \( K = 0 \), according to the first Bianchi identity of \( K \) with torsion \( T \), we obtain

\[
T(T(x, y), z) + T(T(y, z), x) + T(T(z, x), y) = 0. \quad (54)
\]

Indeed, if we define the commutator by \([\cdot, \cdot] = \mathcal{L}(\cdot, \cdot)\) for instance, then (54) becomes the Jacobi identity. Therefore, the manifold has a structure of a Lie group.

(ii) If we denote the trace \( \tau(L) = \sum_{i=1}^{4n} \epsilon_i \) of an arbitrary tensor \( L \) with respect to an arbitrary basis \( \{e_i\}_{i=1}^{4n} \) then we have for \( W \) from (48)

\[
\tau(W) = 20n(4n-1)\theta(\Omega). \tag{55}
\]

Since \( W = 0 \) according to Lemma 16 then \( \theta(\Omega) = 0 \) and because of Corollary 6 we obtain that the considered manifold is an isotropic pseudo-hyper-Kähler manifold.

(iii) Since \( L = 0 \) by Lemma 16 then according to (49) the curvature tensor \( R \), the Ricci tensor \( \rho \) and the scalar curvature \( \tau \) have the form

\[
R = -\frac{1}{64n^2} (g \otimes A), \quad \rho = \frac{1}{32n^2} \{ (2n-1)A - (2n+1)\theta(\Omega)g \}, \quad \tau = \frac{(2n+1)(1-4n)}{16n^2} \theta(\Omega). \tag{56}
\]

Hence from \( \theta(\Omega) = 0 \) it follows that

\[
\tau = 0, \quad \rho = \frac{2n-1}{32n^2} A. \]

So, the considered manifold is scalar flat.

In that way, we complete the proof of Theorem 14. □

The following identity for the curvature tensor \( R \) of the Levi-Civita connection \( \nabla \) on a complex manifold with Norden metric \((M, J, g)\) is known from [9]:

\[
\mathcal{S}_{x,y,z} \{ R(Jx, Jy, z, w) + R(x, y, Jz, Jw) + g((\nabla_x J) y - (\nabla_y J) x, (\nabla_z J) w - (\nabla_w J) z) \} = 0,
\]

where \( \mathcal{S} \) denotes the cyclic sum by three arguments. Using the last identity and the form (55) of \( R \), we obtain
Corollary 17. On a locally conformal pseudo-hyper-Kähler manifold with D-parallel torsion of $D$ it is valid the following identity for $\alpha = 2, 3$

$$\mathcal{S}_{x, y, z} \left( g_\alpha \circ A_\alpha \right)(x, y, z, w) = 0,$$

where $A_\alpha = \theta \circ (\theta \circ J_\alpha) + (\theta \circ J_\alpha) \circ \theta$.

Now, having in mind (55), we obtain

$$\nabla_u R = -\frac{1}{64n^2} (g \circ \nabla_u A)$$

(58)

and according to (56) we have

$$\nabla_u R = \frac{1}{2(1 - 2n)} (g \circ \nabla_u \rho).$$

(59)

Since $\nabla_u \rho$ is a symmetric tensor, the last property implies the second Bianchi identity without any additional condition. Then a locally conformal pseudo-hyper-Kähler manifold $M$ with D-parallel torsion is Ricci-symmetric (i.e. $\nabla \rho = 0$) if and only if $M$ is locally symmetric (i.e. $\nabla R = 0$). The same proposition also follows from Corollary 3.

4. An example of a 4-dimensional $W$-manifold

In [1] it is given an example of a 4-dimensional Lie group as an almost $(H, G)$-manifold, which is an isotropic pseudo-hyper-Kähler manifold. It belongs to the class $W(J_1)$, but is not a $W$-manifold. The manifold is quasi-Kählerian with respect to $J_2$ and $J_3$ (i.e. $F_\alpha(x, y, z) + F_\alpha(y, z, x) + F_\alpha(z, x, y) = 0$, $\alpha = 2, 3$).

In this section we construct an example of a 4-dimensional Lie group as a $W$-manifold.

Let $L$ be a 4-dimensional real connected Lie group, and $l$ be its Lie algebra with a basis $\{X_1, X_2, X_3, X_4\}$.

Now we introduce an almost hypercomplex structure $H = (J_1, J_2, J_3)$ by a standard way as in (2) for $k = 0$:

$$J_1X_1 = X_2, \quad J_1X_2 = -X_1, \quad J_1X_3 = -X_4, \quad J_1X_4 = X_3,$$
$$J_2X_1 = X_3, \quad J_2X_2 = X_4, \quad J_2X_3 = -X_1, \quad J_2X_4 = -X_2,$$
$$J_3X_1 = -X_4, \quad J_3X_2 = X_3, \quad J_3X_3 = -X_2, \quad J_3X_4 = X_1.$$

(60)

Let $g$ be a pseudo-Riemannian metric such that

$$g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,$$
$$g(X_i, X_j) = 0, \ i \neq j.$$  

(61)

Let us consider $(L, H, G)$ with the Lie algebra $l$ determined by the following nonzero commutators:

$$[X_1, X_4] = [X_2, X_3] = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4,$$
$$[X_1, X_3] = -[X_2, X_4] = \lambda_2 X_1 - \lambda_1 X_2 + \lambda_4 X_3 - \lambda_3 X_4.$$  

(62)
where \( \lambda_i \in \mathbb{R} \) (\( i = 1, 2, 3, 4 \)).

We check that \( J_\alpha \) (\( \alpha = 1, 2, 3 \)) are Abelian structures for the Lie algebra \( \mathfrak{l} \), i.e. \([J_\alpha X_i, J_\alpha X_j] = [X_i, X_j]\).

In [10], five types of 4-dimensional Lie algebras which admit integrable invariant hypercomplex structures are classified. Obviously, the introduced Lie algebra \( \mathfrak{l} \) in (62) with the hypercomplex structure from (60) is of the third type in [10] when \( \lambda_2 = 1, \lambda_1 = \lambda_3 = \lambda_4 = 0 \). Namely, the Lie algebra \( \mathfrak{l} \) is isomorphic to \( \text{aff}(\mathbb{C}) \).

Then we prove the following

**Theorem 18.** Let \((L, H, G)\) be the almost \((H, G)\)-manifold, determined by (60), (61) and (62). Then:

(i) it belongs to the class of the locally conformal pseudo-hyper-Kähler manifolds;

(ii) it is locally conformally flat and the curvature tensor \( R \) has the form

\[
R = -\frac{1}{2}(g \circ \rho) + \frac{1}{72}(g \circ g);
\]

(iii) it is \( D \)-flat with non-\( D \)-parallel torsion of \( D \).

**Proof.** According to (61), (62) and the well-known property of a Levi-Civita connection \( \nabla \), we obtain the nonzero components of \( \nabla \) generated by \( g \), as follows:

\[
\begin{align*}
\nabla_{X_1} X_1 &= \nabla_{X_2} X_2 = \lambda_2 X_3 + \lambda_1 X_4, \\
\nabla_{X_1} X_3 &= \nabla_{X_2} X_3 = \lambda_2 X_1 - \lambda_3 X_4, \\
\nabla_{X_1} X_4 &= -\nabla_{X_2} X_2 = \lambda_1 X_3 + \lambda_3 X_3, \\
\nabla_{X_2} X_3 &= -\nabla_{X_1} X_1 = \lambda_2 X_3 + \lambda_4 X_4, \\
\nabla_{X_2} X_4 &= \nabla_{X_2} X_2 = \lambda_1 X_3 - \lambda_4 X_3, \\
\nabla_{X_3} X_3 &= \nabla_{X_4} X_4 = -\lambda_2 X_1 - \lambda_3 X_2.
\end{align*}
\]

The components of \( \nabla J_\alpha \) follow from the last equalities and (60). Then, having in mind (61), we obtain the following nonzero components \((F_\alpha)_{ijk} = F_\alpha (X_i, X_j, X_k) = g((\nabla X_\alpha) X_j, X_k)\) of the tensors \( F_\alpha \), \( \alpha = 1, 2, 3 \):

\[
\begin{align*}
\lambda_1 &= (F_1)_{113} = (F_1)_{124} = -(F_1)_{131} = -(F_1)_{142} \\
&= -(F_1)_{214} = (F_1)_{223} = -(F_1)_{232} = -\frac{1}{2}(F_1)_{241} \\
&= (F_2)_{112} = (F_2)_{121} = (F_2)_{134} = (F_2)_{143} = \frac{1}{2}(F_2)_{222} \\
&= -\frac{1}{2}(F_2)_{244} = (F_2)_{314} = -(F_2)_{323} = -(F_2)_{332} = (F_2)_{341} \\
&= -\frac{1}{2}(F_3)_{111} = -\frac{1}{2}(F_3)_{144} = -(F_3)_{212} = -(F_3)_{221} = (F_3)_{234} \\
&= (F_3)_{243} = (F_3)_{313} = (F_3)_{324} = (F_3)_{331} = (F_3)_{342},
\end{align*}
\]

\[\text{(65a)}\]
the curvature tensor \( R \) formally flat. Thus, the corresponding Weyl tensor vanishes and consequently of the 1-form \( \theta \) manifold \((L, H, G)\) (iii) Using (64), (60) and the components of (i) We verify, using (62) and (66), that \((\theta)\) and (66)

\[
\lambda_2 = -(F_1)_{114} = (F_1)_{123} = -(F_1)_{132} = (F_1)_{141} \\
= -(F_1)_{213} = -(F_1)_{224} = (F_1)_{231} = (F_1)_{242} \\
= \frac{1}{2}(F_2)_{111} = \frac{1}{2}(F_2)_{133} = (F_2)_{212} = (F_2)_{221} = (F_2)_{234} \\
= (F_2)_{243} = -(F_2)_{414} = (F_2)_{423} = (F_2)_{432} = -(F_2)_{441} \\
= -(F_3)_{112} = (F_3)_{121} = -(F_3)_{134} = -(F_3)_{143} = \frac{1}{2}(F_3)_{222} \\
= \frac{1}{2}(F_3)_{233} = -(F_3)_{413} = -(F_3)_{424} = -(F_3)_{431} = -(F_3)_{442}. \\
\lambda_3 = (F_1)_{313} = (F_1)_{324} = -(F_1)_{331} = -(F_1)_{342} \\
= (F_1)_{414} = -(F_1)_{423} = (F_1)_{432} = -(F_1)_{441} \\
= (F_2)_{114} = -(F_2)_{123} = -(F_2)_{132} = (F_2)_{141} = -(F_2)_{312} \\
= -(F_2)_{321} = -(F_2)_{334} = -(F_2)_{343} = -\frac{1}{2}(F_2)_{422} = -\frac{1}{2}(F_2)_{444} \\
= (F_3)_{113} = (F_3)_{124} = (F_3)_{131} = (F_3)_{142} = -\frac{1}{2}(F_3)_{322} \\
= -(F_3)_{313} = (F_3)_{324} = (F_3)_{331} = -(F_3)_{342} = -(F_3)_{344}, \\
\lambda_4 = (F_1)_{314} = -(F_1)_{323} = (F_1)_{332} = -(F_1)_{341} \\
= -(F_1)_{413} = -(F_1)_{424} = (F_1)_{431} = (F_1)_{442} \\
= -(F_2)_{214} = (F_2)_{223} = (F_2)_{232} = -(F_2)_{241} = -\frac{1}{2}(F_2)_{311} \\
= -\frac{1}{2}(F_2)_{323} = -(F_2)_{412} = -(F_2)_{421} = -(F_2)_{434} = -(F_2)_{443} \\
= -(F_3)_{213} = -(F_3)_{224} = -(F_3)_{231} = -(F_3)_{242} = -(F_3)_{312} \\
= -(F_3)_{321} = (F_3)_{334} = -(F_3)_{343} = \frac{1}{2}(F_3)_{411} = \frac{1}{2}(F_3)_{444}. \\
\]

Using (65), we establish that (17) and (18) are satisfied. Therefore the manifold \((L, H, G)\) belongs to the class \(W\).

After that, by (7), (19) and (60), we obtain the components \((\theta)_k = \theta(X_k)\) of the 1-form \(\theta\):

\[
(\theta)_1 = 4\lambda_4, \quad (\theta)_2 = 4\lambda_3, \quad (\theta)_3 = -4\lambda_2, \quad (\theta)_4 = -4\lambda_1. \\
\]

(i) We verify, using (62) and (66), that \(\theta\) is closed (i.e. \(d\theta = 0\)) and therefore \((L, H, G)\) is a locally conformal pseudo-H\-K\-ähler manifold.

(ii) It is clear, because of (i) and Corollary 8 that \((L, H, G)\) is locally conformally flat. Thus, the corresponding Weyl tensor vanishes and consequently the curvature tensor \(R\) has the form in (63).

(iii) Using (64), (60) and the components of \(\nabla J_\alpha\), we obtain the components
of the \((H,G)\)-connection \(D\), determined by (21):
\[
D_{X_1}X_1 = -\frac{1}{2}\lambda_3 X_2, \quad D_{X_1}X_2 = \frac{1}{2}\lambda_3 X_1, \\
D_{X_1}X_3 = -\frac{1}{2}\lambda_2 X_1, \quad D_{X_1}X_4 = \frac{1}{2}\lambda_3 X_3, \\
D_{X_2}X_1 = \frac{1}{2}\lambda_4 X_2, \quad D_{X_2}X_2 = -\frac{1}{2}\lambda_4 X_1, \\
D_{X_2}X_3 = \frac{1}{2}\lambda_4 X_4, \quad D_{X_2}X_4 = -\frac{1}{2}\lambda_4 X_3, \\
D_{X_3}X_1 = \frac{1}{2}\lambda_1 X_2, \quad D_{X_3}X_2 = -\frac{1}{2}\lambda_1 X_1, \\
D_{X_3}X_3 = \frac{1}{2}\lambda_1 X_4, \quad D_{X_3}X_4 = -\frac{1}{2}\lambda_1 X_3, \\
D_{X_4}X_1 = -\frac{1}{2}\lambda_2 X_2, \quad D_{X_4}X_2 = \frac{1}{2}\lambda_2 X_1, \\
D_{X_4}X_3 = -\frac{1}{2}\lambda_2 X_4, \quad D_{X_4}X_4 = \frac{1}{2}\lambda_2 X_3.
\]

(67)

Hence we establish that the corresponding curvature tensor \(K\) vanishes, therefore \((L,H,G)\) is D-flat.

We verify immediately that \((L,G,H)\) has no \(D\)-parallel torsion of \(D\). If we put the condition \(DT = 0\), then the Lie algebra \(\mathfrak{l}\) from (62) becomes Abelian.

\[
\square
\]

According to (61) and (64), the curvature tensor \(R\) has the following nonzero components \(R_{ijkl} = R(X_i, X_j, X_k, X_l)\):
\[
R_{1221} = \lambda_1^2 + \lambda_2^2, \quad R_{1331} = \lambda_1^2 - \lambda_2^2, \quad R_{1441} = \lambda_3^2 - \lambda_4^2, \\
R_{2332} = \lambda_3^2 - \lambda_2^2, \quad R_{2442} = \lambda_3^2 - \lambda_2^2, \quad R_{3443} = -\lambda_3^2 - \lambda_4^2, \\
R_{1341} = R_{2342} = -\lambda_1 \lambda_2, \quad R_{2132} = -R_{4134} = -\lambda_1 \lambda_3, \quad (68) \\
R_{1231} = -R_{2434} = \lambda_1 \lambda_4, \quad R_{2142} = -R_{3143} = \lambda_2 \lambda_3, \\
R_{1241} = -R_{3243} = -\lambda_2 \lambda_4, \quad R_{3123} = R_{4124} = \lambda_3 \lambda_4.
\]

and the rest are determined by (63) and properties (13).

Hence, (68) implies the components of the Ricci tensor \(\rho\) and the value of the scalar curvature \(\tau\):
\[
\rho_{11} = 2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2), \quad \rho_{12} = \rho_{21} = -2\lambda_3 \lambda_4, \quad \rho_{23} = \rho_{32} = 2\lambda_1 \lambda_4, \\
\rho_{22} = 2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2), \quad \rho_{13} = \rho_{31} = -2\lambda_1 \lambda_3, \quad \rho_{24} = \rho_{42} = -2\lambda_2 \lambda_4, \\
\rho_{33} = 2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2), \quad \rho_{14} = \rho_{41} = 2\lambda_2 \lambda_3, \quad \rho_{34} = \rho_{43} = -2\lambda_1 \lambda_2, \\
\rho_{44} = 2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2), \quad \tau = 6(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2).
\]

**Corollary 19.** The manifold \((L,H,G)\) is scalar flat and isotropic hyper-Kählerian if and only if the structural constants from (62) satisfy the following condition
\[
\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0.
\]

**Proof.** It follows immediately from the square norms of \(\nabla J_\alpha\):
\[
-2 \|\nabla J_1\|^2 = \|\nabla J_2\|^2 = \|\nabla J_3\|^2 = 16 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2)
\]
and the last equation of (69).  \(\square\)
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