DEFORMATIONS AND OBSTRUCTIONS OF PAIRS \((X, D)\)

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Abstract. We study infinitesimal deformations of pairs \((X, D)\) with \(X\) smooth projective variety and \(D\) a smooth or a normal crossing divisor, defined over an algebraically closed field of characteristic 0. Using the differential graded Lie algebras theory and the Cartan homotopy construction, we are able to prove in a completely algebraic way the unobstructedness of the deformations of the pair \((X, D)\) in many cases, e.g., whenever \((X, D)\) is a log Calabi-Yau pair, in the case of a smooth divisor \(D\) in a Calabi-Yau variety \(X\) and when \(D\) is a smooth divisor in \(|-mK_X|\), for some positive integer \(m\).

Introduction

Let \(X\) be a smooth projective variety over an algebraically closed field \(\mathbb{K}\) of characteristic 0. If \(X\) has trivial canonical bundle (torsion is enough), then the deformations of \(X\) are unobstructed: this is the well known Bogomolov-Tian-Todorov theorem. The first proofs of this theorem, due to Bogomolov in [Bo78], Tian [Ti87] and Todorov [To89], are transcendental and rely on the underlying differentiable structure of the variety \(X\). More algebraic proof, based on the \(T^1\)-lifting theorem and the degeneration of the Hodge-to-de Rham spectral sequence, are due to Ran [Ra92], Kawamata [Kaw92] and Fantechi-Manetti [FM99]. The Bogomolov-Tian-Todorov theorem is also a consequence of the stronger fact that the differential graded Lie algebra associated with the infinitesimal deformations of \(X\) is homotopy abelian, i.e., quasi-isomorphic to an abelian differential graded Lie algebra. For \(\mathbb{K} = \mathbb{C}\), this was first proved in [GM90], see also [Ma04]. For any algebraically closed field \(\mathbb{K}\) of characteristic 0, this was proved in a completely algebraic way in [IM10], using the degeneration of the Hodge-to-de Rham spectral sequence and the notion of Cartan homotopy.

The aim of this paper is then to extend these techniques to analyse the infinitesimal deformations of pairs; indeed, we prove that the DGLA associated with these deformations is homotopy abelian in many cases, and hence the deformations are unobstructed. This extension can be viewed as an application of the Iitaka’s philosophy: “whenever we have a theorem about non singular complete varieties whose statement is dictated by the behaviour of the regular differentials forms (the canonical bundles), there should exist a corresponding theorem about logarithmic paris (pairs consisting of nonsingular complete varieties and boundary divisors with only normal crossings) whose statement is dictated by the behaviour of the logarithmic forms (the logarithmic canonical bundles) and vice versa” [Mat07, Principle 2-1-4].

More precisely, let \(X\) be a smooth projective variety, \(D\) a smooth divisor and consider the deformations of the pair \((X, D)\), i.e., the deformations of the closed embedding \(j : D \hookrightarrow X\). As first step, we give an explicit description of a differential graded Lie algebra controlling the deformations of \(j\). Namely, let \(\Theta_X(-\log D)\) be the sheaf of germs of the tangent vectors to \(X\) which are tangent to \(D\). Once we fix an open affine cover \(U\) of \(X\), the Thom-Whitney construction applied to \(\Theta_X(-\log D)\) provides a differential graded Lie algebra \(TW(\Theta_X(-\log D)(U))\) controlling the deformations of \(j\) (Theorem...
If the ground field is \( \mathbb{C} \), then we can simply take the DGLA associated with the Dolbeault resolution of \( \Theta_X(-\log D) \), i.e., \( A^{0,*}_X(\Theta_X(-\log D)) = \bigoplus_i \Gamma(X, A^{0,i}_X(\Theta_X(-\log D))) \) (Example 3.5).

Then, we provide a condition that ensures that the DGLA \( TW(\Theta_X(-\log D)(U)) \) is homotopy abelian.

**Theorem.** Let \( X \) be a smooth projective variety of dimension \( n \), defined over an algebraically closed field of characteristic 0 and \( D \subset X \) a smooth divisor \( D \). If the contraction map

\[
H^*(X, \Theta_X(-\log D)) \xrightarrow{i} \text{Hom}^*(H^*(X, \Omega^*_X(\log D)), H^*(X, \Omega^{n-1}_X(\log D)))
\]

is injective, then the DGLA \( TW(\Theta_X(-\log D)(U)) \) is homotopy abelian, for every affine open cover \( U \) of \( X \).

As in [IM10], we recover this result using the power of the Cartan homotopy construction and the degeneration of the Hodge-to-de Rham spectral sequence associated in this case with the complex of logarithmic differentials \( \Omega^*_X(\log D) \).

As corollary, we obtain an alternative (algebraic) proof, that, in the case of a log Calabi-Yau pair (Definition 1.5), the DGLA controlling the infinitesimal deformations of the pair \( (X, D) \) is homotopy abelian (Corollary 4.4). In particular, we are able to prove the following result about smoothness of deformations (Corollary 4.5).

**Theorem.** Let \( X \) be a smooth projective \( n \)-dimensional variety defined over an algebraically closed field of characteristic 0 and \( D \subset X \) a smooth divisor. If \( (X, D) \) is a log Calabi-Yau pair, i.e., the logarithmic canonical bundle \( \Omega^n_X(\log D) \cong O(K_X + D) \) is trivial, then the pair \( (X, D) \) has unobstructed deformations.

The unobstructedness of the deformations of a log Calabi-Yau pair \( (X, D) \) is also interesting from the point of view of mirror symmetry. The deformations of the log Calabi Yau pair \( (X, D) \) should be mirror to the deformations of the (complexified) symplectic form on the mirror Landau-Ginzburg model. Therefore, these deformations are also smooth [Au07, Au09, KKP08].

Then, we focus our attention on the deformations of pairs \( (X, D) \), with \( D \) is a smooth divisor in a smooth projective Calabi Yau variety \( X \). Also in this case, we provide an alternative (algebraic) proof that the DGLA controlling these infinitesimal deformations is homotopy abelian (Theorem 4.7). We also show the following statement about smoothness of deformations (Corollary 4.8).

**Theorem.** Let \( X \) be a smooth projective Calabi Yau variety defined over an algebraically closed field of characteristic 0 and \( D \subset X \) a smooth divisor. Then, the pair \( (X, D) \) has unobstructed deformations.

The previous results are also sketched in [KKP08], see also [Ra92, K09], where the authors work over the field of complex number and make a deep use of transcendental methods. More precisely, using Dolbeault type complexes, one can construct a differential Batalin-Vilkovisky algebra such that the associated DGLA controls the deformation problem (Definition 6.1). If the differential Batalin-Vilkovisky algebras has a degeneration property then the associated DGLA is homotopy abelian [Te08, KKP08, BL13]. Using our approach and the powerful notion of the Cartan homotopy, we are able to give an alternative proof of this result (Theorem 6.6).

In a very recent preprint [Sa13], the \( T^1 \)-lifting theorem is applied in order to prove the unobstructedness of the deformations \( (X, D) \), for \( X \) smooth projective variety and
A smooth divisor in $| - mK_X|$, for some positive integers $m$, under the assumption $H^1(X, O_X) = 0$ [Sa13 Theorem 2.1]. Inspired by this paper, we also study the infinitesimal deformations of these pairs $(X, D)$. Using the cyclic covers of $X$ ramified over $D$, we relate the deformations of the pair $(X, D)$ with the deformations of the pair (ramification divisor, cover) and we show that the DGLA associated with the deformations of the pair $(X, D)$ is homotopy abelian. In particular, we can prove the following result about smoothness of deformations (Proposition 5.4).

**Theorem.** Let $X$ be a smooth projective variety and $D$ a smooth divisor such that $D \in | - mK_X|$, for some positive integer $m$. Then, the pair $(X, D)$ has unobstructed deformations.

We refer the reader to [Sa13] for examples in the Fano setting and the relation with the unobstructedness of weak Fano manifold.

Once the unobstructedness of a pair $(X, D)$ is proved, then studying the forgetting morphism of functors $\phi : \text{Def}_{(X,D)} \to \text{Def}_X$, one can prove the unobstructedness of $\text{Def}_X$, for instance when $D$ is stable in $X$, i.e., $\phi$ is smooth [Se06, Definition 3.4.22].

The paper goes as follows. With the aim of providing a full introduction to the subject, we include Section 1 on the notion of the logarithmic differentials and Section 2 on the DGLAs, Cartan homotopy and cosimplicial constructions, such as the Thom-Whitney DGLA. In Section 3 we review the definition of the infinitesimal deformations of the pair $(X, Z)$, for any closed subscheme $Z \subset X$ of a smooth variety $X$, describing the DGLA controlling these deformations. Section 4 is devoted to the study of obstruction and it contains the proof of the first three theorems. In Section 5 we study cyclic covers of a smooth projective variety $X$ ramified on a smooth divisor $D$ and we prove the last theorem stated above. In the last section, we apply the notion of Cartan homotopy construction to the the differential graded Batalin-Vilkovisky algebra setting, providing a new proof of the fact that if the differential Batalin-Vilkovisky algebras has a degeneration property then the associated DGLA is homotopy abelian (Theorem 6.6).

**Notation.** Unless otherwise specified, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0. Throughout the paper, we also assume that $X$ is always a smooth proper variety over $\mathbb{K}$. Actually, the main ingredient of the proofs is the degeneration at the $E_1$-level of some Hodge-to-de Rham spectral sequences and it holds whenever $X$ is smooth proper over a field of characteristic 0 [DeIl87].

By abuse of notation, we denote by $K_X$ both the canonical divisor and the canonical bundle of $X$. $\textbf{Set}$ denotes the category of sets (in a fixed universe) and $\textbf{Art}$ the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Unless otherwise specified, for every objects $A \in \textbf{Art}$, we denote by $m_A$ its maximal ideal.

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1. Review of logarithmic differentials

Let $X$ be a smooth projective variety of dimension $n$ and $j : Z \hookrightarrow X$ a closed embedding of a closed subscheme $Z$. We denote by $\Theta_X(- \log Z)$ the sheaf of germs
of the tangent vectors to $X$ which are tangent to $Z$ [Se06, Section 3.4.4]. Note that, denoting by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of $Z$ in $X$, then $\Theta_X(-\log Z)$ is the subsheaf of the derivations of the sheaf $\mathcal{O}_X$ preserving the ideal sheaf $\mathcal{I}$ of $Z$, i.e.,

$$
\Theta_X(-\log Z) = \{ f \in \text{Der}(\mathcal{O}_X, \mathcal{O}_X) \mid f(\mathcal{I}) \subset \mathcal{I} \}.
$$

**Remark 1.1.** If $Z$ is smooth in $X$, then we have the short exact sequence

$$
0 \to \Theta_X(-\log Z) \to \Theta_X \to N_{Z/X} \to 0.
$$

Note also that if the codimension of $Z$ is at least 2, then the sheaf $\Theta_X(-\log Z)$ is not locally free, see also Remark 1.4.

Next, assume to be in the divisor setting, i.e., let $D \subset X$ be a globally normal crossing divisor in $X$. With the divisor assumption, we can define the sheaves of logarithmic differentials, see for instance [De70, p. 72], [Kaw85], [EV92, Chapter 2] or [Vo02, Chapter 8]. For any $k \leq n$, we denote by $\Omega^k_X(\log D)$ the locally free sheaf of differential $k$-forms with logarithmic poles along $D$. More explicitly, let $\tau : V = X - D \to X$ and $\Omega^k_X(\log D) = \lim_{\tau} \Omega^k_{X,\nu}(\nu \cdot D) = \tau_* \Omega^k_{X,\nu}(\nu \cdot D)$. Then, $(\Omega^k_X(\log D), d)$ is a complex and $(\Omega^k_X(\log D), d)$ is the subcomplex such that, for every open $U$ in $X$, we have

$$
\Gamma(U, \Omega^k_X(\log D)) = \{ \alpha \in \Gamma(U, \Omega^k_X(\log D)) \mid \alpha \text{ and } d\alpha \text{ have simple poles along } D \}.
$$

**Remark 1.2.** For every $p$, the following short exact sequence of sheaves

$$
0 \to \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D) \to \Omega^p_X \to \Omega^p_D \to 0
$$

is exact [EV92, 2.3] or [La04, Lemma 4.2.4].

**Example 1.3.** [Vo02, Chapter 8] In the holomorphic setting, $\Omega^k_X(\log D)$ is the sheaf of meromorphic differential forms $\omega$ that admit a pole of order at most 1 along each component of $D$, and the same holds for $d\omega$. Let $z_1, z_2, \ldots, z_n$ be holomorphic coordinates on an open set $U$ of $X$, in which $D \cap U$ is defined by the equation $z_1 z_2 \cdots z_r = 0$. Then, $\Omega^k_X(\log D)|_{U}$ is a sheaf of free $\mathcal{O}_U$-modules, for which

$$
\frac{dz_i}{z_i} \wedge \cdots \wedge \frac{dz_l}{z_l} \wedge dz_1 \wedge \cdots \wedge dz_m
$$

with $i_s \leq r$, $j_s > r$ and $l + m = k$ form a basis.

**Remark 1.4.** The sheaves of logarithmic $k$-forms $\Omega^k_X(\log D) = \wedge^k \Omega^1_X(\log D)$ are locally free and the sheaf $\Theta_X(-\log D)$ is dual to the sheaf $\Omega^k_X(\log D)$, so it is in particular locally free for $D$ global normal crossing divisor. The sheaf of logarithmic $n$-forms $\Omega^n_X(\log D) \cong \mathcal{O}_X(K_X + D)$ is a line bundle called the logarithmic canonical bundle for the pair $(X, D)$.

**Definition 1.5.** A log Calabi-Yau pair $(X, D)$ is a pair where $D$ is a smooth divisor in a smooth projective variety $X$ of dimension $n$, and the logarithmic canonical bundle $\Omega^n_X(\log D)$ is trivial.

**Example 1.6.** Let $X$ be a smooth projective variety and $D$ an effective smooth divisor such that $D \in |-K_X|$. Then, the sheaf $\Omega^n_X(\log D) \cong \mathcal{O}_X(K_X + D)$ is trivial, i.e., the pair $(X, D)$ is a log Calabi Yau pair.

The complex $(\Omega^k_X(\log D), d)$ is equipped with the Hodge filtration, which induces a filtration on the hypercohomology $H^*(X, \Omega^k_X(\log D))$. As for the algebraic de Rham complex, the spectral sequence associated with the Hodge filtration on $\Omega^k_X(\log D)$ has its first term equal to $E_1^{p,q} = H^q(X, \Omega^p_X(\log D))$. The following degeneration properties hold.
Theorem 1.7. (Deligne) Let $X$ be a smooth proper variety and $D \subset X$ be a globally normal crossing divisor. Then, the spectral sequence associated with the Hodge filtration

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D)) \implies H^{p+q}(X, \Omega^*_X(\log D))$$

degenerates at the $E_1$-level.

Proof. This is the analogous of the degeneration of the Hodge-to-de Rham spectral sequence. As in this case, there is a complete algebraic way to prove it, avoiding analytic technique, see [De71, Section 3], [DeIl87], [EV92, Corollary 10.23] or [Vo02, Theorem 8.35]).

Theorem 1.8. Let $X$ be a smooth proper variety and $D \subset X$ be a globally normal crossing divisor. Then, the spectral sequence associated with the Hodge filtration

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D) \otimes O_X(-D)) \implies H^{p+q}(X, \Omega^*_X(\log D) \otimes O_X(-D))$$

degenerates at the $E_1$-level.

Proof. See [Fuj09, Section 2.29] or [Fuj11, Section 5.2].

2. Background on DGLAs and Cartan Homotopies

2.1. DGLA. A differential graded Lie algebra is the data of a differential graded vector space $(L,d)$ together with a bilinear map $[-,-] : L \times L \to L$ (called bracket) of degree 0, such that the following conditions are satisfied:

1. (graded skewsymmetry) $[a,b] = -(−1)^{\overline{a}\overline{b}}[b,a]$.
2. (graded Jacobi identity) $[a, [b,c]] = [[a,b],c] + (−1)^{\overline{a}\overline{b}}[b,[a,c]]$.
3. (graded Leibniz rule) $d[a,b] = [da,b] + (−1)^{\overline{a}}[a,db]$.

In particular, the Leibniz rule implies that the bracket of a DGLA induces a structure of graded Lie algebra on its cohomology. Moreover, a DGLA is abelian if its bracket is trivial.

A morphism of differential graded Lie algebras $\chi : L \to M$ is a linear map that commutes with brackets and differentials and preserves degrees.

A quasi-isomorphism of DGLAs is a morphism that induces an isomorphism in cohomology. Two DGLAs $L$ and $M$ are said to be quasi-isomorphic or homotopy equivalent, if they are equivalent under the equivalence relation generated by: $L \sim M$ if there exists a quasi-isomorphism $\chi : L \to M$. A DGLA is homotopy abelian if it is quasi-isomorphic to an abelian DGLA.

Remark 2.1. The category DGLA of DGLAs is too strict for our purpose and we require to enhance this category allowing $L_\infty$ morphisms of DGLAs. Therefore, we work in the category whose objects are DGLAs and whose morphisms are $L_\infty$ morphisms of DGLAs. This category is equivalent to the homotopy category of DGLA, obtained inverting all quasi-isomorphisms. Using this fact, we do not give the explicit definition of an $L_\infty$ morphism of DGLAs: by an $L_\infty$ morphism we mean a morphism in this homotopy category (a zig-zag morphism) and we denote it with a dash-arrow. We only emphasize that an $L_\infty$ morphism of DGLAs has a linear part that is a morphism of complexes and therefore it induces a morphism in cohomology. For the detailed descriptions of such structures we refer to [LS93, LM95, Ma02, Fu03, Kou03, Get04, Ma04, FiMa07, Ia08].

Lemma 2.2. Let $f_\infty : M_1 \to M_2$ be a $L_\infty$ morphism of DGLAs with $M_2$ homotopy abelian. If $f_\infty$ induces an injective morphism in cohomology, then $M_1$ is also homotopy abelian.
Proof. See [KKP08, Proposition 4.11] or [IM10, Lemma 1.10]. □

The homotopy fibre of a morphism of DGLA \( \chi : L \to M \) is the DGLA
\[
TW(\chi) := \{(l, m(t, dt)) \in L \times M[t, dt] \mid m(0, 0) = 0, m(1, 0) = \chi(l)\}.
\]

Remark 2.3. If \( \chi : L \to M \) is an injective morphism of DGLAs, then its cokernel \( M/\chi(L) \)
is a differential graded vector space and the map
\[
TW(\chi) \to (M/\chi(L))[-1], \quad (l, p(t)m_0 + q(t)dtm_1) \mapsto \left( \int_0^1 q(t)dt \right) m_1 \mod \chi(L),
\]
is a surjective quasi-isomorphism.

Lemma 2.4. Let \( \chi : L \to M \) be an injective morphism of differential graded Lie algebras such that: \( \chi^* : H^*(L) \to H^*(M) \) is injective. Then, the homotopy fibre \( TW(\chi) \) is homotopy abelian.

Proof. [IM10, Proposition 3.4] or [IM13, Lemma 2.1]. □

Example 2.5. [IM10, Example 3.5] Let \( W \) be a differential graded vector space and let \( U \subset W \) be a differential graded subspace. If the induced morphism in cohomology \( H^*(U) \to H^*(W) \) is injective, then the inclusion of DGLAs
\[
\chi : \{ f \in \text{Hom}_\mathbb{K}(W, W) \mid f(U) \subset U \} \to \text{Hom}_\mathbb{K}(W, W)
\]
satisfies the hypothesis of Lemma 2.4 and so the DGLA \( TW(\chi) \) is homotopy abelian.

2.2. Cartan homotopies. Let \( L \) and \( M \) be two differential graded Lie algebras. A Cartan homotopy is a linear map of degree \(-1\)
\[
i : L \to M
\]
such that, for every \( a, b \in L \), we have:
\[
i_{[a,b]} = [i_a, d_M i_b] \quad \text{and} \quad [i_a, i_b] = 0.
\]

For every Cartan homotopy \( i \), it is defined the Lie derivative map
\[
l : L \to M, \quad l_a = d_M i_a + i_{d_L a}.
\]
It follows from the definition of a Cartan homotopy \( i \) that \( l \) is a morphism of DGLAs. Therefore, the conditions of Cartan homotopy become
\[
i_{[a,b]} = [i_a, l_b] \quad \text{and} \quad [i_a, i_b] = 0.
\]
Note that, as a morphism of complexes, \( l \) is homotopic to 0 (with homotopy \( i \)).

Example 2.6. Let \( X \) be a smooth algebraic variety. Denote by \( \Theta_X \) the tangent sheaf and by \((\Omega_X^*, d)\) the algebraic de Rham complex. Then, for every open subset \( U \subset X \), the contraction of a vector space with a differential form
\[
\Theta_X(U) \otimes \Omega_X^k(U) \longrightarrow \Omega_X^{k-1}(U)
\]
induces a linear map of degree \(-1\)
\[
i : \Theta_X(U) \to \text{Hom}^*(\Omega_X^*(U), \Omega_X^*(U)), \quad i_\xi(\omega) = \xi \omega
\]
that is a Cartan homotopy. Indeed, the above conditions coincide with the classical Cartan’s homotopy formulas.

We are interested in the logarithmic generalization of the previous example.
Example 2.7. Let \( X \) be a smooth algebraic variety and \( D \) a normal crossing divisor. Let \( (\Omega^*_X(\log D), d) \) be the logarithmic differential complex and \( \Theta_X(-\log D) \) the subsheaf of the tangent sheaf \( \Theta_X \) of the derivations that preserve the ideal sheaf of \( D \) as in the previous section. It is easy to prove explicitly that for every open subset \( U \subset X \), we have

\[
(\Theta_X(-\log D)(U) \cup \Omega^*_X(\log D)(U)) \subset \Omega^{k-1}_X(\log D)(U).
\]

Then, as above, the induced linear map of degree \(-1\)

\[
i: \Theta_X(-\log D)(U) \to \text{Hom}^*(\Omega^*_X(\log D)(U), \Omega^*_X(\log D)(U)), \quad i_\xi(\omega) = \xi \omega
\]

is a Cartan homotopy.

Lemma 2.8. Let \( L, M \) be DGLAs and \( i: L \to M \) a Cartan homotopy. Let \( N \subset M \) be a differential graded Lie subalgebra such that \( l(L) \subset N \) and

\[
TW(\chi) = \{ (x, y(t)) \in N \times M[t, dt] \mid y(0) = 0, \ y(1) = x \}
\]

the homotopy fibre of the inclusion \( N \hookrightarrow M \). Then, it is well defined an \( L_\infty \) morphism \( L \to TW(\chi) \).

Proof. See \cite{IM13, Corollary 7.5} for an explicit description of this morphism. We only note that the linear part, i.e., the induced morphism of complexes, is given by \((l, i)(a) := (l_a, t_a + dt_i)\), for any \( a \in L \). \( \square \)

2.3. Simplicial objects and Cartan homotopies. Let \( \Delta_{\text{mon}} \) be the category whose objects are finite ordinal sets and whose morphisms are order-preserving injective maps between them. A semicosimplicial differential graded Lie algebra is a covariant functor \( \Delta_{\text{mon}} \to \text{DGLA} \). Equivalently, a semicosimplicial DGLA \( g^\Delta \) is a diagram

\[
g_0 \to g_1 \to g_2 \to \cdots,
\]

where each \( g_i \) is a DGLA, and for each \( i > 0 \), there are \( i + 1 \) morphisms of DGLAs

\[
\partial_{k,i}: g_{i-1} \to g_i, \quad k = 0, \ldots, i,
\]

such that \( \partial_{k+1,i+1} \partial_{l,i} = \partial_{l,i+1} \partial_{k,i} \), for any \( k \geq l \).

In a semicosimplicial DGLA \( g^\Delta \), the maps

\[
\partial_i = \partial_{0,i} - \partial_{1,i} + \cdots + (-1)^i \partial_{i,i}
\]

endow the vector space \( \prod_i g_i \) with the structure of a differential complex. Moreover, being a DGLA, each \( g_i \) is in particular a differential complex; since the maps \( \partial_{k,i} \) are morphisms of DGLAs, the space \( g^\bullet \) has a natural bicomplex structure. We emphasise that the associated total complex

\[
(\text{Tot}(g^\Delta), d_{\text{Tot}}) \quad \text{where} \quad \text{Tot}(g^\Delta) = \prod_i g_i[-i], \quad d_{\text{Tot}} = \sum_i \partial_i + (-1)^i d_j
\]

has no natural DGLA structure. However, there is another bicomplex naturally associated with a semicosimplicial DGLA, whose total complex is naturally a DGLA.

For every \( n \geq 0 \), let \( (A_{PL})_n \) be the differential graded commutative algebra of polynomial differential forms on the standard \( n \)-simplex \( \{(t_0, \ldots, t_n) \in \mathbb{K}^{n+1} \mid \sum t_i = 1\} \)

\cite{FHT01}:

\[
(A_{PL})_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(1 - \sum t_i, \sum dt_i)}.
\]
Denote by $\delta^{k,n}: (A_{PL})_n \rightarrow (A_{PL})_{n-1}, k = 0, \ldots, n$, the face maps; then, there are well-defined morphisms of differential graded vector spaces

$$\delta^k \otimes \text{Id}: (A_{PL})_n \otimes g_n \rightarrow (A_{PL})_{n-1} \otimes g_n,$$

$$\text{Id} \otimes \partial_k: (A_{PL})_{n-1} \otimes g_{n-1} \rightarrow (A_{PL})_{n-1} \otimes g_n,$$

for every $0 \leq k \leq n$. The Thom-Whitney bicomplex is then defined as

$$C^{i,j}_{TW}(g^\Delta) = \{ (x_n)_{n \in \mathbb{N}} \in \prod_n (A_{PL})_n^i \otimes g_n^j \mid (\delta^k \otimes \text{Id})x_n = (\text{Id} \otimes \partial_k)x_{n-1}, \forall 0 \leq k \leq n\},$$

where $(A_{PL})_n^i$ denotes the degree $i$ component of $(A_{PL})_n$. Its total complex is denoted by $(TW(g^\Delta), d_{TW})$ and it is a DGLA, called the Thom-Whitney DGLA. Note that the integration maps

$$\int_{\Delta^n} \otimes \text{Id}: (A_{PL})_n \otimes g_n \rightarrow \mathbb{K}[n] \otimes g_n = g_n[n]$$

give a quasi-isomorphism of differential graded vector spaces

$$I: (TW(g^\Delta), d_{TW}) \rightarrow (\text{Tot}(g^\Delta), d_{\text{Tot}}).$$

For more details, we refer the reader to [Whi57, NaA87, Get04, FiMa07, CG08].

**Remark 2.9.** For any semicosimplicial DGLA $g^\Delta$, we have just defined the Thom-Whitney DGLA. Therefore, using the Maurer-Cartan equation, we can associate with any $g^\Delta$ a deformation functor, namely

$$\text{Def}_{TW(g^\Delta)}: \text{Art} \rightarrow \text{Set},$$

$$\text{Def}_{TW(g^\Delta)}(A) = \frac{\text{MC}_{TW(g^\Delta)}(A)}{\text{gauge}} = \frac{\{ x \in TW(g^\Delta)^1 \otimes m_A \mid dx + \frac{1}{2}[x,x] = 0 \}}{\exp(TW(g^\Delta)^0 \otimes m_A)}.$$

In particular, the tangent space to $\text{Def}_{TW(g^\Delta)}$ is

$$T\text{Def}_{TW(g^\Delta)} := \text{Def}_{TW(g^\Delta)}(\mathbb{K}[e]/e^2) \cong H^1(TW(g^\Delta)) \cong H^1(\text{Tot}(g^\Delta))$$

and obstructions are contained in

$$H^2(TW(g^\Delta)) \cong H^2(\text{Tot}(g^\Delta)).$$

**Example 2.10.** Let $\mathcal{L}$ be a sheaf of differential graded vector spaces over an algebraic variety $X$ and $\mathcal{U} = \{ U_i \}$ an open cover of $X$; assume that the set of indices $i$ is totally ordered. We can then define the semicosimplicial DG vector space of Čech cochains of $\mathcal{L}$ with respect to the cover $\mathcal{U}$:

$$\mathcal{L}(\mathcal{U}) := \prod_i \mathcal{L}(U_i) \Longrightarrow \prod_{i<j} \mathcal{L}(U_{ij}) \Longrightarrow \prod_{i<j<k} \mathcal{L}(U_{ijk}) \Longrightarrow \cdots,$$

where the coface maps $\partial_h : \prod_{i_0 < \cdots < i_{k-1}} \mathcal{L}(U_{i_0 \cdots i_{k-1}}) \rightarrow \prod_{i_0 < \cdots < i_k} \mathcal{L}(U_{i_0 \cdots i_k})$ are given by

$$\partial_h(x)_{i_0 \cdots i_k} = x_{i_0 \cdots i_k} |_{U_{i_0 \cdots i_k}}, \text{ for } h = 0, \ldots, k.$$

The total complex $\text{Tot}(\mathcal{L}(\mathcal{U}))$ is the associated Čech complex $C^*(\mathcal{U}, \mathcal{L})$ and we denote by $TW(\mathcal{L}(\mathcal{U}))$ the associated Thom-Whitney complex. The integration map $TW(\mathcal{L}(\mathcal{U})) \rightarrow C^*(\mathcal{U}, \mathcal{L})$ is a surjective quasi-isomorphism. If $\mathcal{L}$ is a quasicoherent DG-sheaf and every $U_i$ is affine, then the cohomology of $TW(\mathcal{L}(\mathcal{U}))$ is the same of the cohomology of $\mathcal{L}$. 
Example 2.11. [FIM09] [FMM12] If each $g_i$ is concentrated in degree zero, i.e., $g^\Delta$ is a semicosimplicial Lie algebra, then the functor $\text{Def}_{TW(g^\Delta)}$ has another explicit description; namely, it is isomorphic to the following functor:

\[
H^1_{sc}(\exp g^\Delta) : \text{Art} \to \text{Set}
\]

\[
H^1_{sc}(\exp g^\Delta)(A) = \{x \in g_1 \otimes m_A \mid e^{\partial_0 x}e^{-\partial_1 x}e^{\partial_2 x} = 1\},
\]

where $x \sim y$ if and only if there exists $a \in g_0 \otimes m_A$, such that $e^{-\partial_1 a}e^{x}e^{\partial_0 a} = e^y$.

In particular, let $Z \subset X$ be a closed subscheme of a smooth variety $X$, $U = \{U_i\}$ an open affine cover of $X$ and consider $g^\Delta = TW(\Theta_X(-\log Z)(U))$. Then, for every $A \in \text{Art}$, we have

\[
\text{Def}_{TW(g^\Delta)}(A) \cong \{(x_{ij}) \in \prod_{i<j} \Theta_X(-\log Z)(U_{ij}) \otimes m_A \mid e^{x_{jk} - x_{ik} e^{x_{ij}}} = 1\}
\]

where $x \sim y$ if and only if there exists $(a_i)_i \in \prod_i \Theta_X(-\log Z)(U_i) \otimes m_A$, such that $e^{-a_i}e^{x_{ij}}e^{a_j} = e^{y_{ij}}$ [FMM12] Theorem 4.1].

The notion of Cartan homotopy is related to the notion of calculus and it can be extended to the semicosimplicial setting.

Definition 2.12. [TT05] [IM13] Let $L$ be a differential graded Lie algebra and $V$ a differential graded vector space. A bilinear map

\[
L \times V \longrightarrow V
\]

of degree $-1$ is called a calculus if the induced map

\[
i : L \to \text{Hom}^*_K(V, V), \quad i_l(v) = l \cdot v,
\]

is a Cartan homotopy.

Definition 2.13. Let $g^\Delta$ be a semicosimplicial DGLA and $V^\Delta$ a semicosimplicial differential graded vector space. A semicosimplicial Lie-calculus

\[
g^\Delta \times V^\Delta \longrightarrow V^\Delta,
\]

is a sequence of calculi $g_n \times V_n \longrightarrow V_n$, $n \geq 0$, commuting with coface maps, i.e., \(\partial_k(l \cdot v) = \partial_k(l) \cdot \partial_k(v)\), for every $k$.

Lemma 2.14. Every semicosimplicial calculus $g^\Delta \times V^\Delta \longrightarrow V^\Delta$ extends naturally to a calculus

\[
TW(g^\Delta) \times TW(V^\Delta) \longrightarrow TW(V^\Delta),
\]

Therefore, the induced map

\[
i : TW(g^\Delta) \to \text{Hom}^*_K(TW(V^\Delta), TW(V^\Delta))
\]

is a Cartan homotopy.

Proof. [IM10] Proposition 4.9. □
Example 2.15. Let $X$ be a smooth algebraic variety and $D$ a normal crossing divisor. Denote by $(\Omega^*_X(\log D), d)$ the logarithmic differential complex and $\Theta_X(-\log D)$ the usual subsheaf of $\Theta_X$ preserving the ideal of $D$.

According to Example 2.7, for every open subset $U \subset X$, we have a contraction

$$\Theta_X(-\log D)(U) \times \Omega^*_X(\log D)(U) \rightarrow \Omega^*_X(\log D)(U).$$

Since it commutes with restrictions to open subsets, for every affine open cover $\mathcal{U}$ of $X$, we have a semicosimplicial contraction

$$\Theta_X(-\log D)(U) \times \Omega^*_X(\log D)(U) \rightarrow \Omega^*_X(\log D)(U).$$

According to Lemma 2.14, it is well defined the Cartan homotopy

$$i: TW(\Theta_X(-\log D)(\mathcal{U})) \rightarrow Hom^*(TW(\Omega^*_X(\log D)(\mathcal{U})), TW(\Omega^*_X(\log D)(\mathcal{U}))).$$

3. Deformations of pairs

Let $Z \subset X$ be a closed subscheme of a smooth variety $X$ and denote by $j: Z \hookrightarrow X$ the closed embedding. Note that at this point we are not assuming neither $Z$ divisor nor $Z$ smooth. We recall the definition of infinitesimal deformations of the closed embedding $j: Z \hookrightarrow X$, i.e., infinitesimal deformations of the pair $(X, Z)$; full details can be found for instance in [Se06, Section 3.4.4] or [Kaw78].

**Definition 3.1.** Let $A \in \textbf{Art}$. An infinitesimal deformation of $j: Z \hookrightarrow X$ over $\text{Spec}(A)$ is a diagram

$$
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(A) & & \\
\end{array}
$$

where $p$ and $\pi$ are flat maps, such that the diagram is isomorphic to $j: Z \hookrightarrow X$ via the pullback $\text{Spec}(\mathbb{K}) \rightarrow \text{Spec}(A)$. Note that $J$ is also a closed embedding [Se06, pag 185].

Given another infinitesimal deformation of $j$:

$$
\begin{array}{ccc}
Z' & \rightarrow & X' \\
\downarrow & & \downarrow \\
\text{Spec}(A) & & \\
\end{array}
$$

an isomorphism between these two deformations is a pair of isomorphisms of deformations:

$$\alpha: Z \rightarrow Z', \quad \beta: X \rightarrow X'$$

such that the following diagram

$$
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow^\alpha & & \downarrow^\beta \\
Z' & \rightarrow & X' \\
\end{array}
$$

is commutative. The associated infinitesimal deformation functor is

$$\text{Def}_{(X, Z)}: \textbf{Art} \rightarrow \textbf{Set},$$

$$\text{Def}_{(X, Z)}(A) = \{\text{isomorphism classes of infinitesimal deformations of } j \text{ over } \text{Spec}(A)\}.$$
Furthermore, we define the sub-functor
\[
\text{Def}^t_{(X,Z)} : \text{Art} \to \text{Set},
\]
\[
\text{Def}^t_{(X,Z)} = \left\{ \text{isomorphism classes of locally trivial infinitesimal deformations } j \text{ over Spec}(A) \right\}.
\]

**Remark 3.2.** Since every affine non-singular algebraic variety is rigid [Se06, Theorem 1.2.4], whenever \( Z \subset X \) is smooth, every deformation of \( j \) is locally trivial and so \( \text{Def}_{(X,Z)} \cong \text{Def}^t_{(X,Z)} \).

Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be an affine open cover of \( X \) and \( TW(\Theta_X(-\log Z)(\mathcal{U})) \) the DGLA associated with the sheaf of Lie algebras \( \Theta_X(-\log Z) \) as in Example 2.10.

**Theorem 3.3.** In the assumption above, the DGLA \( TW(\Theta_X(-\log Z)(\mathcal{U})) \) controls the locally trivial deformation of the closed embedding \( j : Z \hookrightarrow X \), i.e., there exists an isomorphism of deformation functors
\[
\text{Def}_{TW(\Theta_X(-\log Z)(\mathcal{U}))} \cong \text{Def}^t_{(X,Z)}.
\]

In particular, if \( Z \subset X \) is smooth, then \( \text{Def}_{TW(\Theta_X(-\log Z)(\mathcal{U}))} \cong \text{Def}_{(X,Z)} \).

**Proof.** See also [Se06, Proposition 3.4.17], [Ia10, Theorem 4.2].

Denote by \( \mathcal{V} = \{V_i = U_i \cap Z\}_{i \in I} \) the induced affine open cover of \( Z \). Every locally trivial deformation of \( j \) is obtained by the gluing of the trivial deformations
\[
\begin{array}{ccc}
V_i & \to & V_i \times \text{Spec}(A) \\
U_i & \to & U_i \times \text{Spec}(A),
\end{array}
\]
in a compatible way along the double intersection \( V_{ij} \times \text{Spec}(A) \) and \( U_{ij} \times \text{Spec}(A) \).

Therefore, it is determined by automorphisms of the trivial deformations, that glues over triple intersections, i.e., by pairs of automorphisms \((\varphi_{ij}, \phi_{ij})\), where \( \varphi_{ij} : V_{ij} \times \text{Spec}(A) \to V_{ij} \times \text{Spec}(A) \) and \( \phi_{ij} : U_{ij} \times \text{Spec}(A) \to U_{ij} \times \text{Spec}(A) \) are automorphisms of deformations, satisfying the cocycle condition on triple intersection and such that the following diagram
\[
\begin{array}{ccc}
V_{ij} \times \text{Spec}(A) & \to & V_{ij} \times \text{Spec}(A) \\
U_{ij} \times \text{Spec}(A) & \to & U_{ij} \times \text{Spec}(A)
\end{array}
\]
commutes. Equivalently, we have \( \phi_{ij}|_{V_{ij}} = \varphi_{ij} \). Since we are in characteristic zero, we can take the logarithms so that \( \varphi_{ij} = e^{d_{ij}} \), for some \( d_{ij} \in \Theta_Z(V_{ij}) \otimes m_A \), and \( \phi_{ij} = e^{D_{ij}} \), for some \( D_{ij} \in \Theta_X(U_{ij}) \otimes m_A \). The compatibility condition is equivalent to the condition \( D_{ij} \in \Gamma(U_{ij}, \Theta_X(-\log Z)) \otimes m_A \). Summing up, a deformation of \( j \) over \( \text{Spec}(A) \) corresponds to the datum of a sequence \( \{D_{ij}\}_{ij} \in \prod_{ij} \Theta_X(-\log Z)(U_{ij}) \otimes m_A \) satisfying the cocycle condition
\[
e^{D_{jk}} e^{-D_{ik}} e^{D_{ij}} = \text{Id}, \quad \forall \ i < j < k \in I.
\]

Next, let \( J' \) be another deformation of \( j \) over \( \text{Spec}(A) \). To give an isomorphism of deformations between \( J \) and \( J' \) is equivalent to give, for every \( i \), an automorphism \( \alpha_i \).
of $V_i \times \text{Spec}(A)$ and an automorphism $\beta_i$ of $U_i \times \text{Spec}(A)$, that are isomorphisms of deformations of $X$ and $Z$, respectively, i.e., for every $i < j$, $\varphi_{ij} = \alpha_i^{-1} \varphi_{ij}'^{-1} \alpha_j$ and $\phi_{ij} = \beta_i^{-1} \phi_{ij}'^{-1} \beta_j$. Moreover, they have to be compatible, i.e., the following diagram
\[
\begin{array}{ccc}
V_i \times \text{Spec}(A) & \longrightarrow & U_i \times \text{Spec}(A) \\
\alpha_i \downarrow & & \beta_i \downarrow \\
V_i \times \text{Spec}(A) & \longrightarrow & U_i \times \text{Spec}(A)
\end{array}
\]
has to commute, for every $i$.

Taking again logarithms, an isomorphism between the deformations $J$ and $J'$ is equivalent to the existence of a sequence $\{\alpha_i\}_i \in \prod_i \Theta_X(-\log Z)(U_i) \otimes \mathfrak{m}_A$, such that $e^{-\alpha_i} e^{D_{ij}} e^{\alpha_j} = e^{D_{ij}}$. Then, the conclusion follows from the explicit description of the functor $\text{Def}_{TW}(\Theta_X(-\log Z)(U))$ given in Example 2.11.

**Remark 3.4.** If $Z = 0$, then we are analysing nothing more than the infinitesimal deformations of the smooth variety $X$ and they are controlled by the tangent sheaf, i.e., $\text{Def}_{TW}(\Theta_X(U)) \cong \text{Def}_X$, for any open affine cover $U$ of $X$ [[MI06], Theorem 5.3].

**Example 3.5.** In the case $\mathbb{K} = \mathbb{C}$, we can consider the DGLA $(\Omega^*_X(\Theta_X(-\log Z))) = \bigoplus_i \Gamma(X, A^0_X(\Theta_X(-\log Z)), \partial_i [\cdot])$ as an explicit model for $TW(\Theta_X(-\log Z)(U))$ [[Ma07], Section 5.1 [Ma06], Corollary V.4.1].

4. Obstructions of pairs

In this section, we analyse obstructions for the infinitesimal deformations of pairs, whenever the sub variety is a divisor, so that we can make use of the logarithmic differential complex $(\Omega_X^*(\log D), d)$.

**Theorem 4.1.** Let $X$ be a smooth projective variety of dimension $n$, defined over an algebraically closed field of characteristic 0 and $D \subset X$ a smooth divisor. If the contraction map
\begin{equation}
H^*(X, \Theta_X(-\log D)) \longrightarrow \text{Hom}^*(H^*(X, \Omega_X^*(\log D)), H^*(X, \Omega_X^{n-1}(\log D)))
\end{equation}
is injective, then the DGLA $TW(\Theta_X(-\log D)(U))$ is homotopy abelian, for every affine open cover $U$ of $X$.

**Proof.** According to Lemma 2.2, it is sufficient to prove the existence of a homotopy abelian DGLA $H$ and an $L_\infty$-morphism $TW(\Theta_X(-\log D)(U)) \rightarrow H$, such that the induced map of complexes is injective in cohomology. We use the Cartan homotopy to construct the morphism, as in Lemma 2.8 and the homotopy fibre construction to provide an homotopy abelian DGLA $H$, as in Lemma 2.4.

Let $U$ be an affine open cover of $X$. For every $i \leq n$, denote by $\check{C}(U, \Omega_X^i(\log D))$ the Čech complex of the coherent sheaf $\Omega_X^i(\log D)$, and $\check{C}(U, \Omega_X^*(\log D))$ the total complex of the logarithmic de Rham complex $\Omega_X^*(\log D)$ with respect to the cover $U$, as in Example 2.11. We note that
\[
\check{C}(U, \Omega_X^i(\log D)) = \bigoplus_{a+b=i} \check{C}(U, \Omega_X^a(\log D))^b.
\]
and that $\check{C}(U, \Omega_X^a(\log D))$ is a subcomplex of $\check{C}(U, \Omega_X^*(\log D))$.

We also have a commutative diagram of complexes with horizontal quasi-isomorphisms:
Following commutative diagram of complexes induces injective morphisms in cohomology.

\[
\begin{array}{c}
\mathcal{C}(\mathcal{U}, \Omega^n_X(\log D)) \\
\downarrow \\
\mathcal{C}(\mathcal{U}, \Omega^n_X(\log D)) \\
\end{array} \longrightarrow \left. \begin{array}{c}
TW(\Omega^n_X(\log D)(\mathcal{U})) \\
\end{array} \right| \begin{array}{c}
TW(\Omega_X^*(\log D)(\mathcal{U})). \\
\end{array}
\]

According to Theorem 1.7, the spectral sequence associated with the Hodge filtration degenerates at the \( E_1 \)-level, where \( E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \); this implies that we have the following injections:

\[
H^*(X, \Omega^n_X(\log D)) = H^*(\mathcal{C}(\mathcal{U}, \Omega^n_X(\log D))) \hookrightarrow H^*(\mathcal{C}(\mathcal{U}, \Omega^n_X(\log D))).
\]

\[
H^*(X, \Omega^{n-1}_X(\log D)) = H^*(\mathcal{C}(\mathcal{U}, \Omega^{n-1}_X(\log D))) \hookrightarrow H^*(\mathcal{C}(\mathcal{U}, \Omega^n_X(\log D)))/C(\mathcal{U}, \Omega^n_X(\log D)).
\]

Thus, the natural inclusions of complexes

\[
TW(\Omega_X^k(\log D)(\mathcal{U})) \rightarrow TW(\Omega_X^*(\log D)(\mathcal{U})),
\]

\[
TW(\Omega_X^{n-1}(\log D)(\mathcal{U})) \rightarrow \frac{TW(\Omega_X^n(\log D)(\mathcal{U}))}{TW(\Omega_X^*(\log D)(\mathcal{U}))},
\]

induces injective morphisms in cohomology.

Consider the differential graded Lie algebras

\[
M = \text{Hom}^*(TW(\Omega_X^*(\log D)(\mathcal{U})), TW(\Omega_X^*(\log D)(\mathcal{U}))),
\]

\[
L = \{f \in M \mid f(TW(\Omega_X^*(\log D)(\mathcal{U}))) \subset TW(\Omega^n_X(\log D)(\mathcal{U}))\},
\]

and denote by \( \chi : L \rightarrow M \) the inclusion. Lemma 2.3 implies that the homotopy fibre \( TW(\chi^\Delta) \) is homotopy abelian. Next, we provide the existence of a morphism to this homotopy abelian DGLA.

According to Example 2.15 it is well defined the Cartan homotopy

\[
i : TW(\Theta_X(\log D)(\mathcal{U})) \longrightarrow \text{Hom}^*(TW(\Omega_X^*(\log D)(\mathcal{U})), TW(\Omega_X^*(\log D)(\mathcal{U}))).
\]

In particular, for every \( \xi \in TW(\Theta_X(\log D)(\mathcal{U})) \) and every \( k \), we note that

\[
i_\xi(TW(\Omega_X^*(\log D)(\mathcal{U}))) \subset TW(\Omega_X^{k-1}(\log D)(\mathcal{U})),
\]

\[
i_\xi(TW(\Omega_X^*(\log D)(\mathcal{U}))) \subset TW(\Omega_X^k(\log D)(\mathcal{U})), \quad i_\xi = d\xi + i_{d\xi}.
\]

Therefore, \( i(TW(\Theta_X(\log D)(\mathcal{U}))) \subset L \) and so, by Lemma 2.8 there exists an \( L_\infty \)-morphism

\[
TW(\Theta_X(\log D)(\mathcal{U})) \xrightarrow{(i,i)} TW(\chi^\Delta).
\]

Finally, since the map \( \chi \) in injective, according to Remark 2.3, the homotopy fibre \( TW(\chi^\Delta) \) is quasi-isomorphic to the suspension of its cokernel

\[
\text{Coker} \chi[-1] = \text{Hom}^* \left( \frac{TW(\Omega_X^*(\log D)(\mathcal{U})), TW(\Omega^n_X(\log D)(\mathcal{U}))}{TW(\Omega_X^*(\log D)(\mathcal{U}))} \right) [-1].
\]

Summing up, since the \( L_\infty \)-morphism induces a morphism of complexes, we have the following commutative diagram of complexes

\[
\begin{array}{c}
TW(\Theta_X(\log D)(\mathcal{U})) \\
\downarrow i \\
\end{array} \longrightarrow \left. \begin{array}{c}
TW(\chi^\Delta) \\
\end{array} \right| \begin{array}{c}
\text{Hom}^*(TW(\Omega^n_X(\log D)(\mathcal{U})), TW(\Omega^{n-1}_X(\log D)(\mathcal{U}))) \\
\end{array} \longrightarrow \text{Coker} \chi[-1].
\]

\[\text{q-iso}\]
By the assumption of the theorem, together with \cite[3.1]{NaA87}, the left-hand map is injective in cohomology. Since $\alpha$ is also injective in cohomology, we conclude that the $L_\infty$-morphism $(l, i)$ is injective in cohomology.

**Theorem 4.2.** Let $X$ be a smooth projective variety defined over an algebraically closed field of characteristic 0 and $D \subset X$ a smooth divisor. Then, the obstructions to the deformations of the pair $(X, D)$ are contained in the kernel of the contraction map

$$H^2(\Theta_X((- \log D))) \to \prod_p \text{Hom}(H^p(\Omega^n_X(\log D)), H^{p+2}(\Omega^{n-1}_X(\log D))).$$

**Proof.** Following the proof of Theorem 4.1 for every affine open cover $U$ of $X$, there exists an $L_\infty$-morphism $TW(\Theta_X((- \log D)(U)) \to TW(\chi^\Delta)$ such that $TW(\chi^\Delta)$ is homotopy abelian. Therefore, the deformation functor associated with $TW(\chi^\Delta)$ is unobstructed and the obstructions of $\text{Def}_{(X, D)}$ and $\text{Def}_{TW(\Theta_X((- \log D)(U))}$ are contained in the kernel of the obstruction map $H^2(TW(\Theta_X((- \log D)(U))) \to H^2(TW(\chi^\Delta)).$

**Remark 4.3.** In the previous theorem, we prove that all obstructions are annihilated by the contraction map; in general, the $T^1$-lifting theorem is definitely insufficient for proving this kind of theorem, see also \cite{Ia11, Ma09}.

**Corollary 4.4.** Let $U = \{U_i\}$ be an affine open cover of a smooth projective variety $X$ defined over an algebraically closed field of characteristic 0 and $D \subset X$ a smooth divisor. If $(X, D)$ is a log Calabi-Yau pair, then the DGLA $TW(\Theta_X((- \log D))(U)$ is homotopy abelian.

**Proof.** Let $n$ be the dimension of $X$, then by definition the sheaf $\Omega^n_X(\log D)$ is trivial (Definition 1.5). Therefore, the cup product with a nontrivial section of it gives the isomorphisms $H^i(X, \Theta_X((- \log D)) \simeq H^i(X, \Omega^{n-1}_X(\log D)).$ Then, the conclusion follows from Theorem 4.1.

**Corollary 4.5.** Let $X$ be a smooth projective $n$-dimensional variety defined over an algebraically closed field of characteristic 0 and $D \subset X$ a smooth divisor. If $(X, D)$ is a log Calabi-Yau pair, i.e., the logarithmic canonical bundle $\Omega^n_X(\log D) \cong \mathcal{O}(K_X + D)$ is trivial, then the pair $(X, D)$ has no obstructed deformations.

**Proof.** According to Theorem 4.3 for every affine open cover $U$ of $X$, there exists an isomorphism of functor $\text{Def}_{(X, D)} \cong \text{Def}_{TW(\Theta_X((- \log D))(U))}$. Then, Corollary 4.4 implies that they are both smooth.

**Remark 4.6.** For the degeneration of the spectral sequence associated with the logarithmic complex, it is enough to have a normal crossing divisor $D$ in a smooth proper variety $X$ (Theorem 1.7). Therefore, we can still perform the same computations of Theorem 4.1 and prove that the obstructions to the locally trivial deformations of a pair $(X, D)$, with $X$ smooth proper variety and $D$ normal crossing divisor, are contained in the kernel of the contraction map (2). Analogously, if the sheaf $\Omega^n_X(\log D)$ is trivial, the above computations prove the unobstructedness for the locally trivial deformations of the pair $(X, D)$.

We end this section, by proving that the DGLA associated with the infinitesimal deformations of the pair $(X, D)$, with $D$ a smooth divisor in a smooth projective Calabi Yau variety $X$ is homotopy abelian; hence, we show that the deformations of these pairs $(X, D)$ are unobstructed.
Theorem 4.7. Let $\mathcal{U} = \{U_i\}$ be an affine open cover of a smooth projective variety $X$ of dimension $n$ defined over an algebraically closed field of characteristic 0 and $D \subset X$ a smooth divisor. If $\Omega^n_X$ is trivial, i.e., $X$ is Calabi Yau, then the DGLA $TW(\Theta_X(-\log D))(\mathcal{U})$ is homotopy abelian.

Proof. The proof is similar to the one of Theorem 4.11. According to Theorem 1.8, the Hodge-to-de Rham spectral sequences associated with the complex $\Omega^*_X(\log D) \otimes \mathcal{O}_X(-D)$ degenerates at the $E_1$ level. Therefore, we have injective maps

$$H^*(\check{C}(\mathcal{U}, \Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))),$$
$$H^*(\check{C}(\mathcal{U}, \Omega^{n-1}_X(\log D) \otimes \mathcal{O}_X(-D))) \hookrightarrow H^*\left(\frac{\check{C}(\mathcal{U}, \Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))}{\check{C}(\mathcal{U}, \Omega^{n-1}_X(\log D) \otimes \mathcal{O}_X(-D))}\right),$$

and so the natural inclusions of complexes

(3) $TW(\Omega^n_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U})) \rightarrow TW(\Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U}))$,

(4) $TW(\Omega^{n-1}_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U})) \rightarrow TW(\Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U}))$,

are injective in cohomology. Observe that $\Omega^n_X(\log D) \otimes \mathcal{O}_X(-D) \cong \Omega^n_X$, according to Remark 1.2. Consider the inclusion of DGLAs $\chi : L \rightarrow M$, where

$$M = \text{Hom}^*(TW(\Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U})),$$
$$TW(\Omega^{n-1}_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U}))$$

and

$$L = \{ f \in M \mid f(TW(\Omega^n_X(\mathcal{U}))) \subset TW(\Omega^n_X(\mathcal{U})) \}.$$

is the sub-DGLA preserving $TW(\Omega^*_X(\mathcal{U})) = TW(\Omega^n_X(\mathcal{U}))$.

Note that

$$\text{Coker} \chi[-1] = \text{Hom}^*\left(\frac{TW(\Omega^n_X(\mathcal{U})), TW(\Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U}))}{TW(\Omega^n_X(\mathcal{U}))}\right)[-1].$$

Lemma 2.4 together with Equation (3) implies that $TW(\chi^\Delta)$ is homotopy abelian.

As in the proof of Theorem 4.11 it is well defined the Cartan homotopy

$$i : TW(\Theta_X(-\log D))(\mathcal{U})) \longrightarrow M$$

and, in particular, $U(TW(\Theta_X(-\log D))(\mathcal{U})) \subset L$. Therefore, Lemma 2.8 implies the existence of an $L_\infty$-morphism

$$TW(\Theta_X(-\log D))(\mathcal{U})) \xrightarrow{(l,i)} TW(\chi^\Delta).$$

According to Lemma 2.2 to conclude the proof it is enough to show that this map is injective in cohomology. As morphism of complexes, the previous map fits in the following commutative diagram of complexes

$$\begin{array}{ccc}
TW(\Theta_X(-\log D))(\mathcal{U})) & \xrightarrow{(l,i)} & TW(\chi^\Delta) \\
\downarrow i & & \downarrow \text{q-iso} \\
\text{Hom}^* \left(\frac{TW(\Omega^n_X(\mathcal{U})), TW(\Omega^{n-1}_X(\log D) \otimes \mathcal{O}_X(-D))(\mathcal{U}))}{\chi^\Delta}\right) & \xrightarrow{\alpha} & \text{Coker} \chi[-1],
\end{array}$$

where $\alpha$ is injective in cohomology by Equation (4).

At this point, we use the fact that $X$ is a smooth projective Calabi Yau variety. Since $\Omega^n_X$ is trivial, the cup product with a nontrivial section gives the isomorphisms $H^i(X, \Theta_X(-\log D)) \simeq H^i(X, \Omega^{n-1}_X(\log D) \otimes \mathcal{O}_X(-D))$, for every $i$. Therefore, the left map in the diagram is injective in cohomology and so the same holds for $(l,i)$.
Corollary 4.8. Let $X$ be a smooth projective Calabi Yau variety defined over an algebraically closed field of characteristic 0 and $D \subset X$ a smooth divisor. Then, the pair $(X, D)$ has unobstructed deformations.

5. Application to cyclic covers

Let $X$ be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic 0. If $X$ has trivial canonical bundle, then the deformations of $X$ are unobstructed. It is actually enough that the canonical bundle $K_X$ is a torsion line bundle, i.e., there exists $m > 0$ such that $K_X^m = \mathcal{O}_X$, see for instance [Ra92, Corollary 2], [Ma04a, Corollary B], [IM10, Corally 6.5]. Indeed, consider the unramified $m$-cyclic cover defined by the line bundle $L = K_X$, i.e., $\pi : Y = \text{Spec}(\bigoplus_{i=0}^{m-1} L^{-i}) \to X$. Then, $\pi$ is a finite flat map of degree $m$ and $Y$ is a smooth projective variety with trivial canonical bundle ($K_Y \cong \pi^* K_X \cong \mathcal{O}_Y$) and so it has unobstructed deformations. Let $\mathcal{U} = \{U_i\}$ be an affine cover of $X$ and fix $\mathcal{V} = \{\pi^{-1}(U_i)\}$, the induced cover of $Y$. Then, the pull back map induces a morphism of DGLAs $TW(\Theta_X(\mathcal{U})) \to TW(\Theta_Y(\mathcal{V}))$ that is injective in cohomology; since the DGLA $TW(\Theta_Y(\mathcal{V}))$ is homotopy abelian, Lemma 2.2 implies that $TW(\Theta_X(\mathcal{U}))$ is also homotopy abelian and so $\text{Def}_X$ is unobstructed [IM10, Theorem 6.2].

As observed in Remark 3.4, the infinitesimal deformations of $X$ can be considered as deformations of the pair $(X, D)$ with $D = 0$. Then, according to the Iitaka’s philosophy and inspired by [Sa13], the idea is to extend the previous computations to the logarithmic case, i.e., a pair $(X, D)$ with $D$ a smooth divisor, by considering cyclic covers of $X$ branched on the divisor $D$ (indeed, if $D = 0$ we obtain the unramified covers).

We firstly recall some properties of these covers; for full details see for instance [Pa91, EV92, Section 3] or [KM98, Section 2.4]. Suppose we have a line bundle $L$ on $X$, a positive integer $m \geq 1$ and a non zero section $s \in \Gamma(X, L^m)$ which defines the smooth divisor $D \subset X$ (as usual $L^m$ stands for $L^\otimes m$). The cyclic cover $\pi : Y \to X$ of degree $m$ and branched over $D$ is, in the language of [Pa91], the simple abelian cover determined by its building data $L$ and $D$, such that $mL \equiv D$, associated with the cyclic group $G$ of order $m$. More explicitly, the variety $Y = \text{Spec}(\bigoplus_{i=0}^{m-1} L^{-i})$ is smooth and there exists a section $s' \in \Gamma(Y, \pi^* L)$, with $(s')^m = s^* s$. The divisor $\Delta = (s')$ is also smooth and maps isomorphically to $D$ so that $\pi^* D = m \Delta$ and $\pi^* L = \mathcal{O}_Y(\Delta)$. Moreover,

$$\pi_* \mathcal{O}_Y \left( \bigoplus_{i=0}^{m-1} L^{-i} \right), \quad K_Y = \pi^* K_X \otimes \mathcal{O}_Y((m-1) \Delta) = \pi^* (K_X \otimes L^{m-1})$$

and

$$\pi^* \Omega_X^i (\log D) \cong \Omega_Y^i (\log \Delta) \quad \text{for all } i;$$

in particular, $K_Y \otimes \mathcal{O}_Y(\Delta) = \pi^*(K_X \otimes \mathcal{O}_Y(m \Delta)) = \pi^*(K_X \otimes \mathcal{O}_X(D))$ [EV92, Lemma 3.16] or [La04, Proposition 4.2.4].

Since $\pi : Y \to X$ is a finite map, for any sheaf $\mathcal{F}$ on $Y$, the higher direct images sheaves vanish and so the Leray spectral sequence $E_2^{pq} = H^p(X, R^q \pi_* \mathcal{F}) \Rightarrow H^{p+q}(Y, \mathcal{F})$ degenerates at level $E_2$; therefore, it induces isomorphisms: $H^p(X, \pi_* \mathcal{F}) \cong H^p(Y, \mathcal{F})$, $\forall p$. In particular, for any locally free sheaf $\mathcal{E}$ on $X$ we have:

$$H^p(X, \pi_* \pi^* \mathcal{E}) \cong H^p(Y, \pi^* \mathcal{E}), \quad \forall p.$$
By the projection formula
\[ \pi_*\pi^*\mathcal{E} \cong \pi_*(\pi^*\mathcal{E} \otimes \mathcal{O}_Y) \cong \mathcal{E} \otimes \pi_*\mathcal{O}_Y \cong \mathcal{E} \otimes \bigoplus_{i=0}^{m-1} L^{-i}; \]
then, for any locally free sheaf \( \mathcal{E} \) on \( X \)
\[ H^p(X, \mathcal{E} \otimes L^{-i}) \subseteq H^p(X, \pi_*\pi^*\mathcal{E}) \cong H^p(Y, \pi^*\mathcal{E}), \quad \forall \ p, \ i. \]
and in particular
\[ H^p(X, \mathcal{E}) \subseteq H^p(X, \pi_*\pi^*\mathcal{E}) \cong H^p(Y, \pi^*\mathcal{E}), \quad \forall \ p. \]

Remark 5.1. Note that, the m-cyclic group \( G \) acts on \( \pi_*\pi^*\mathcal{E} \): the invariant summand of \( \pi_*\pi^*\mathcal{E} \) is \( (\pi_*\pi^*\mathcal{E})^{inv} = \mathcal{E} \), while \( \mathcal{E} \otimes L^{-i} \) is the direct summand of \( \pi_*\pi^*\mathcal{E} \) on which \( G \) acts via multiplication by \( \zeta^i \) (\( \zeta^m = 1 \)).

Proposition 5.2. In the above notation, let \( \pi : Y \to X \) be the m-cyclic cover branched over \( D \) with \( \pi^*D = m\Delta \). Let \( U = \{ U_i \}_i \) be an affine open cover of \( X \) and \( V = \{ \pi^{-1}(U_i) \}_i \) the induced affine open cover of \( Y \); then, the pull back define a morphism of DGLAs
\[ TW(\Theta_X(-\log D)(U)) \to TW(\Theta_Y(-\log \Delta)(V)) \]
that is injective in cohomology.

Proof. Let \( U \subset X \) be an affine open subset and \( V = \pi^{-1}(U) \). Then, the pull back map induce a morphism \( \Theta_X(-\log D)(U) \to \pi^*\Theta_X(-\log D)(V) \), that behaves well under the restriction to open sets. Therefore, fixing an affine cover \( U = \{ U_i \}_i \) of \( X \) and denoting by \( V = \{ \pi^{-1}(U_i) \}_i \); the induced affine cover of \( Y \), the pull back map induces a morphism of DGLAs
\[ TW(\Theta_X(-\log D)(U)) \to TW(\pi^*\Theta_X(-\log D)(V)). \]
Since \( \pi^*\Omega^1_X(-\log D) \cong \Omega^1_Y(-\log \Delta) \), we have \( \pi^*\Theta_X(-\log D) \cong \Theta_Y(-\log \Delta) \). Moreover, the pull back morphism
\[ \Theta_X(-\log D) \xrightarrow{\pi^*} \pi^*\Theta_X(-\log D) \cong \Theta_Y(-\log \Delta) \]
induces injective morphisms on the cohomology groups. Indeed, \( H^i(X, \Theta_X(-\log D)) \) is a direct summand of \( H^i(X, \pi_*\pi^*\Theta_X(-\log D)) \cong H^i(Y, \pi^*\Theta_X(-\log D)) \cong H^i(Y, \Theta_Y(-\log \Delta)) \). It follows that the induced DGLAs morphism
\[ TW(\Theta_X(-\log D)(U)) \to TW(\Theta_Y(-\log \Delta)(V)) \]
is injective in cohomology.

\[ \square \]

Remark 5.3. The DGLAs morphism \( TW(\Theta_X(-\log D)(U)) \to TW(\Theta_Y(-\log \Delta)(V)) \), induces a morphism of the associated deformation functor
\[ \text{Def}_{(X,D)} \to \text{Def}_{(Y,\Delta)}. \]
According to Lemma 2.2, if \( TW(\Theta_Y(-\log \Delta)(V)) \) is homotopy abelian, so that the deformations of the pair \( (Y,\Delta) \) are unobstructed, then \( TW(\Theta_X(-\log D)(U)) \) is also homotopy abelian and so the deformations of the pair \( (X,D) \) are also unobstructed. In particular, this happen if the pair \( (Y,\Delta) \) is a log Calabi-Yau. Note that this is a sufficient but not necessary condition for the unobstructedness of \( (X,D) \), as we can observe in the following example.
Proposition 5.4. Let $X$ be a smooth projective variety and $D$ a smooth divisor such that $D \in |-mK_X|$, for some positive integer $m$. Then, the DGLA $TW(\Theta_X(-\log D)(U))$ is homotopy abelian and so the deformations of the pair $(X, D)$ are unobstructed.

Proof. Let $n$ be the dimension of $X$ and consider the m-cyclic cover $\pi : Y \to X$ branched over $D$ defined by the line bundle $L = K_X^{-1}$ together with a section $s \in H^0(X, L^m)$ defining $D$. Note that $\Omega_Y^1(\log D) \otimes L^{-m+1} \cong L^{-m} \otimes \mathcal{O}_X(D) \cong \mathcal{O}_X(mK_X + D) \cong \mathcal{O}_X$. Defining $\Delta$ as before, i.e., $\pi^*D = m\Delta$, we also have

$$K_Y = \pi^*K_X \otimes \mathcal{O}_Y((m-1)\Delta) = \pi^*(K_X \otimes L^{m-1}) = \pi^*(L^{m-2})$$

and in particular,

$$K_Y \otimes \mathcal{O}_Y(\Delta) = \pi^*(K_X \otimes \mathcal{O}_X(D)) = \pi^*(L^{m-1}).$$

According to Equations (5) and (6), we have the following inclusions

$$H^p(Y, \Theta_Y(-\log \Delta)) \supset H^p(X, \pi_*(\Theta_Y(-\log \Delta))) \cong H^p(X, \Theta_X(-\log D)) \forall p,$$

$$H^p(Y, \Omega_Y^1(\log \Delta)) \supset H^p(X, \Omega_X^1(\log D) \otimes L^{-1}) \forall p, a, i;$$

in particular for $a = n, p = 0$ and $i = m - 1$, we have

$$H^0(Y, \Omega_Y^1(\log \Delta)) \supset H^0(X, \Omega_X^1(\log D) \otimes L^{-m+1}) \cong H^0(X, \mathcal{O}_X).$$

Then the constant section of $\mathcal{O}_X$ gives a section $\omega$ of the logarithmic canonical bundle $\Omega_Y^1(\log \Delta)$, vanishing only on $\Delta$ (of order $m - 1$). In particular, the cup product with $\omega \in H^0(X, \Omega_X^1(\log D) \otimes L^{-m+1})$, gives isomorphisms $H^p(X, \Theta_X(-\log D)) \cong H^p(X, \Omega_X^n(\log \Delta) \otimes L^{-m+1})$, for all $p$.

Therefore, the following composition

$$H^p(Y, \Theta_Y(-\log \Delta)) \xrightarrow{i} \prod_j \text{Hom}(H^j(Y, \Omega_Y^1(\log \Delta)), H^{j+p}(X, \Omega_X^{n-1}(\log \Delta))) \xrightarrow{j} H^p(X, \Theta_X(-\log D)) \xrightarrow{j\omega} H^p(X, \Omega_X^n(\log \Delta) \otimes L^{-m+1})$$

is injective and in particular the composition $i \circ j$ is injective, for all $p$.

According to Proposition 5.2, fixing an affine cover $U = \{U_i\}_i$ of $X$ and denoting by $V = \{\pi^{-1}(U_i)\}_i$ the induced affine cover of $Y$, the pull back map induces a morphism of DGLAs

$$TW(\Theta_X(-\log D)(U)) \to TW(\Theta_Y(-\log \Delta)(V))$$

that is injective in cohomology. Finally, as in the proof of Theorem 4.1 denote by $TW(\chi^{\Delta})$ the homotopy abelian differential graded Lie algebra associated with the inclusion $\chi : L \to M$, with

$$M = \text{Hom}^*(TW(\Omega_Y^1(\log \Delta)(V)), TW\Omega_Y^1(\log \Delta)(V)),$$

$$L = \{f \in M \mid f(TW(\Omega_Y^1(\log \Delta)(V))) \subset TW(\Omega_Y^1(\log \Delta)(V))\}.$$}

Then, the composition morphism

$$TW(\Theta_X(-\log D)(U)) \to TW(\Theta_Y(-\log \Delta)(V)) \to TW(\chi^{\Delta}).$$

is injective in cohomology and so by Lemma 2.2 the DGLA $TW(\Theta_X(-\log D)(U))$ is homotopy abelian.

$\square$

Remark 5.5. In the case $m = 2$, the result is a consequence of Theorem 4.7 and Remark 5.3. Indeed, in this case the canonical line bundle $K_Y$ of $Y$ is trivial, i.e., $Y$ is a smooth Calabi Yau variety and so the DGLA associated with the pair $(Y, \Delta)$ is homotopy abelian.
Remark 5.6. The previous proposition is a generalisation of [Sa13, Theorem 2.1], avoiding the assumption \( H^1(X, \mathcal{O}_X) = 0 \). Moreover, if \( H^1(D, N_{D/X}) = 0 \), then \( D \) is stable in \( X \), i.e., the forgetting morphism \( \phi : \text{Def}(X,D) \to \text{Def}_X \) is smooth; this implies that the deformations of \( X \) are unobstructed, e.g., deformations of weak Fano manifolds are unobstructed [Sa13, Theorem 1.1].

6. Application to differential graded Batalin-Vilkovisky algebras

If the ground field is \( \mathbb{C} \), we already noticed in Example 3.5.1 that the differential graded Lie algebra \( (A^\bullet_X, (\Theta_X(- \log D)), \overline{\partial}, [\cdot, \cdot]) \) controls the deformations of the pair \((X, D)\), for \( D \) a smooth divisor in a projective smooth manifold \( X \). In [Te08, KKP08, BL13], the authors use the differential Batalin-Vilkovisky algebras and a degeneration property for these algebras to prove that the associated DGLA is homotopy abelian. Using the power of the notion of Cartan homotopy, we can give an alternative proof of these results and so we provide alternative proofs, over \( \mathbb{C} \), of Corollary 4.4 and Corollary 4.8.

First of all we recall the fundamental definitions in this setting, for more details we refer the reader to [BV81, Ge94, KKP08].

Definition 6.1. Let \( k \) be a fixed odd integer. A differential Batalin-Vilkovisky algebra (dBV for short) of degree \( k \) over \( \mathbb{K} \) is the data \((A,d,\Delta)\), where \((A,d)\) is a differential \( \mathbb{Z} \)-graded commutative algebra with unit \( 1 \), and \( \Delta \) is an operator of degree \(-k\), such that \( \Delta^2 = 0, \Delta(1) = 0 \) and

\[
\Delta(ab) + \Delta(a)bc + (-1)^{\overline{b}\overline{c}} \Delta(b)ac + (-1)^{\overline{a}\overline{c}} \Delta(c)ab = \\
= \Delta(ab)c + (-1)^{\overline{b}\overline{c}} \Delta(bc)a + (-1)^{\overline{a}\overline{c}} \Delta(ac)b.
\]

The previous equality is often called the seven-term relation. It is well known [Ko85] or [KKP08, Section 4.2.2], that given a graded dBV algebra \((A,d,\Delta)\) of degree \( k \), it is canonically defined a differential graded Lie algebra \((g,d,[-,-])\), where: \( g = A[k] \), \( d_g = -d_A \) and,

\[
[a,b] = (-1)^p(\Delta(ab) - \Delta(a)b - a\Delta(b)), \quad a \in A^p.
\]

Next, let \((A,d,\Delta)\) be a dBV algebra and \( t \) a formal central variable of (even) degree \( 1 + k \). Denote by \( A[[t]] \) the graded vector space of formal power series with coefficients in \( A \) and by \( A((t)) = \bigcup_{p \in \mathbb{Z}} t^p A[[t]] \) the graded vector space of formal Laurent power series. We have \( d(t) = \Delta(t) = 0 \) and \( d - t\Delta \) is a well-defined differential on \( A((t)) \).

Lemma 6.2. In the above notation, the map

\[
i : g \to \text{Hom}_A^*(A((t)), A((t))), \quad a \mapsto i_a(b) = \frac{1}{t}ab
\]

is a Cartan homotopy.

Proof. We have to verify the two conditions of being a Cartan homotopy, given in Section 2.2. The former identity \([i_a, i_b] = 0\) is trivial. As regard the latter \([i_a, l_b] - i_{[a,b]} = 0\), we recall that \( l_b = [d - t\Delta, i_b] - i_{db} \) (note that the differential changes sign on the \( k \)-fold suspension). Moreover, we have the following explicit description

\[
l_b(c) = -\Delta(bc) + (-1)^{\overline{b}\overline{c}} b\Delta(c).
\]

Indeed,

\[
l_b(c) = [d - t\Delta, i_b](c) - \frac{(db)c}{t}
\]
\[ dBV \text{ algebra } (A,d,\Delta) \text{ is a dBV algebra } \]

Then, \[
\begin{align*}
[i_a, l_0](c) - i_{[a,b]}(c) &= i_a(-\Delta(bc) + (-1)^{\overline{a}b}\Delta(c)) - (-1)^{\overline{a}b+c}l_0\left(\frac{ac}{t}\right) - \frac{1}{t}[a,b]c \\
&= \frac{1}{t}(-a\Delta(bc) + (-1)^{\overline{a}b}ab\Delta(c) - (-1)^{\overline{a}b+c}(-\Delta(bac) + (-1)^{\overline{a}b}\Delta(ac))) \\
&\quad - (-1)^{\overline{a}b}(\Delta(ab)c - \Delta(a)bc) + a\Delta(b)c = 0.
\end{align*}
\]

Example 6.4. Let \((A,d,\Delta)\) be a dBV algebra and suppose that it is bigraded, i.e., \(A = \bigoplus_{i,j \geq 0} A^{i,j}\) and \(d : A^{i+1,j} \to A^{i,j}\) and \(\Delta : A^{i,j} \to A^{i+1,j+1}\). Then, the filtration \(F_p = \bigoplus_{j \geq p} A^{i,j}\) define a decreasing filtration of the double complex and therefore a spectral sequence. If this spectral sequence degenerates at the first page \(E_1\), then the dBV algebra \((A,d,\Delta)\) has the degeneration property [Mor78 Lemma 1.5], [DSV12 Proposition 1.5].

Example 6.5. Let \((A,d,\Delta)\) be a dBV algebra. On the complex \((A((t)),d-t\Delta)\), consider the filtration \(F^*\), defined by \(F^p = t^p A[[t]]\), for every \(p \in \mathbb{Z}\). Note that \(A((t)) = \bigcup_{p \in \mathbb{Z}} F^p\) and \(F^0 = A[[t]]\). Then, the dBV algebra \((A,d,\Delta)\) has the degeneration property if and only if the morphism of complexes \((A[[t]],d-t\Delta) \to (A,d)\), given by \(t \mapsto 0\) is surjective in cohomology, if and only if the inclusion of complexes \((tA[[t]],d-t\Delta) \to (A[[t]],d-t\Delta)\) is injective in cohomology. In particular, the degeneration property implies that the inclusion \(F^p \to A((t))\) is injective in cohomology, for every \(p\), and so \(A[[t]] \to A((t))\) is also injective in cohomology.

Theorem 6.6. Let \((A,d,\Delta)\) be a dBV algebra with the degeneration property. Then, the associated DGLA \(g = A[k]\) is homotopy abelian.

Proof. According to the previous Lemma 6.2, it is well defined a Cartan homotopy \(i : g \to Hom^*(A((t)), A((t)))\), whose associated Lie derivative has the following explicit expression

\[ l_0(c) = -\Delta(bc) + (-1)^{\overline{a}b}\Delta(c). \]

Therefore, considering the filtration \(F^p = t^p A[[t]]\) of the complex \((A((t)),d-t\Delta)\) as in Example 6.5, we note that \(i : g \to Hom^*(F^p,F^{p-1})\) and \(l : g \to Hom^*(F^p,F^p)\), \(\forall p\).

Next, consider the differential graded Lie algebra

\[ M = Hom^*(A((t)), A((t))), \]

the sub-DGLA

\[ N = \{ \varphi \in M \mid \varphi(A[[t]]) \subset A[[t]] \}, \]
and let \( \chi : N \to M \) be the inclusion. Since \( U(g) \subset N \), according to Lemma 2.8, there exists an induced \( L_\infty \)-morphism \( \psi : g \to TW(\chi) \).

As observed in the Example 6.5, the degeneration property implies that the inclusion \( A[[t]] \to A((t)) \) is injective in cohomology. Therefore, the DGLA \( TW(\chi) \) is homotopy abelian by Lemma 2.4. According to Lemma 2.2, to conclude the proof it is enough to show that \( \psi \) induces an injective morphism in cohomology.

As observed in Remark 2.3, \( TW(\chi) \) is quasi-isomorphic to \( \text{Coker}(\chi)[-1] = \text{Hom}_K^\bullet \left( A[[t]], \frac{A((t))}{A[[t]]} \right)[-1] \); therefore, it is sufficient to prove that the morphism of complexes

\[
i : A \to \text{Hom}_K^\bullet \left( A[[t]], \frac{A((t))}{A[[t]]} \right)[-k-1]
\]

is injective in cohomology. It is actually enough to prove the injectivity for the composition with the evaluation at \( 1 \in A[[t]], \) i.e., the map

\[
A \to \frac{A((t))}{A[[t]]}, \quad a \mapsto \frac{a}{t},
\]

is injective in cohomology. Note that this is equivalent to the statement that the inclusion

\[
\frac{F^{-1}}{F^0} \to \frac{A((t))}{F^0}
\]

is injective in cohomology, since the map \( a \mapsto \frac{a}{t} \) defines an isomorphism of DG-vector spaces \( A \to F^{-1}/F^0 \). The claim follows considering the short exact sequences

\[
0 \to F^0 \to F^{-1} \to \frac{F^{-1}}{F^0} \to 0
\]

and keeping in mind that the inclusion \( j \) is injective in cohomology by the degeneration property (Example 6.5).

\[ \Box \]

Remark 6.7. The original proof of this theorem (for \( k = 1 \)) can be found in [Te08, Theorem 1] or [KKP08, Theorem 4.14]. This proof was suggested to the author by Marco Manetti.

Example 6.8. [KKP08, Theorem 4.18] Let \( X \) be a compact projective Calabi Yau variety of dimension \( n \) over \( \mathbb{C} \). In this situation, the relevant dBV algebra is \((A,d,\Delta)\) with \( A = \Gamma(X,\mathcal{A}^*_X(\wedge^*\Omega_X)) \), \( d = \overline{\partial} \) and \( \Delta = \text{div}_{\omega} = i_{\omega}^{-1} \circ \partial \circ i_{\omega} \). Here \( \omega \) is a non-vanishing section of \( \Omega^n_X \) and \( i_{\omega} : \wedge^*\Omega_X \to \Omega^n_X \) is the isomorphism given by the contraction with \( \omega \). The contraction \( i_{\omega} \) gives an isomorphism of bicomplexes between the dBV algebra \((A,d,\Delta)\) and the Dolbeault bicomplex \((A^{**}(X),\overline{\partial},\partial)\). According to Example 6.4, the degeneration of the Hodge-to-de Rham spectral sequence implies that \((A,d,\Delta)\) has the degeneration property. Therefore, Theorem 6.6 implies that the associated DGLA \( L = \Gamma(X,\mathcal{A}^*_X(\wedge^*\Theta_X)) \), is homotopy abelian. The Kodaira Spencer DGLA of \( X \Gamma(X,\mathcal{A}^{**}_X(\Theta_X)) \) is embedded in \( L \) and it is also an embedding in cohomology. According to Lemma 2.2, the Kodaira Spencer DGLA is also homotopy abelian and the deformations of \( X \) are unobstructed.
Example 6.9. \[\text{KKP08}\] Section 4.3.3 (i)] Let $X$ be a smooth projective $n$-dimensional variety over $\mathbb{C}$ and $D$ a smooth divisor, such that $\Omega^n_{X}(\log D)$ is trivial. In this case, the relevant dBV algebra is $(A,d,\Delta)$ with $A = \Gamma(X,\mathcal{A}_X^{0,+}(\wedge^\ast \Theta_X(- \log D)))$ $d = \partial \bar{\partial}$ and $\Delta = \text{div}_\omega = i_\omega^{-1} \circ \partial \circ i_\omega$. Here $\omega$ is the non vanishing section of $\Gamma(X,\Omega^n_{X}(\log D))$ and $i_\omega : \wedge^\ast \Theta_X(- \log D) \to \Omega^n_{X}(- \ast \log D)$ is the isomorphism given by the contraction with $\omega$. The map $i_\omega$ identifies $(A,d,\Delta)$ with the logarithmic Dolbeault bicomplex $(A^{\ast,+}(\log D),\partial,\bar{\partial})$.

Arguing as in the previous example and using the degeneration of the spectral sequence of Theorem [17] we can conclude that the DGLA $(A^{\ast,+}_{X}(\Theta_X(- \log D)),\partial,\bar{\partial})$ is homotopy abelian and so the deformations of the pair $(X,D)$ are unobstructed [KKP08 Lemma 4.19].

Example 6.10. \[\text{KKP08}\] Section 4.3.3 (ii)] Let $X$ be a smooth projective $n$-dimensional Calabi Yau variety over $\mathbb{C}$ and $D$ a smooth divisor. In this case the relevant dBV algebra $(A,d,\Delta)$ is similar to the one introduced in the previous example, indeed $A = \Gamma(X,\mathcal{A}_X^{0,+}(\wedge^\ast \Theta_X(- \log D)))$ $d = \partial \bar{\partial}$ and $\Delta = \text{div}_\omega = i_\omega^{-1} \circ \partial \circ i_\omega$. Here $\omega$ is a non vanishing section of $\Omega^n_{X}$. The degeneration of the spectral sequence of Theorem [18] implies that $(A,d,\Delta)$ has the degeneration property and so that $(A^{\ast,+}_{X}(\Theta_X(- \log D)),\partial,\bar{\partial})$ is homotopy abelian [KKP08 Lemma 4.20].

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