Maximal Syntactic Complexity of Regular Languages Implies Maximal Quotient Complexities of Atoms

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Abstract. We relate two measures of complexity of regular languages. The first is syntactic complexity, that is, the cardinality of the syntactic semigroup of the language. That semigroup is isomorphic to the semigroup of transformations of states induced by non-empty words in the minimal deterministic finite automaton accepting the language. If the language has $n$ left quotients (its minimal automaton has $n$ states), then its syntactic complexity is at most $n^n$ and this bound is tight. The second measure consists of the quotient (state) complexities of the atoms of the language, where atoms are non-empty intersections of complemented and uncomplemented quotients. A regular language has at most $2^n$ atoms and this bound is tight. The maximal quotient complexity of any atom with $r$ complemented quotients is $2^n - 1$, if $r = 0$ or $r = n$, and $1 + \sum_{k=1}^{r} \sum_{h=k+1}^{n} \binom{n}{h} \binom{h}{k}$, otherwise. We prove that if a language has maximal syntactic complexity, then it has $2^n$ atoms and each atom has maximal quotient complexity, but the converse is false.

Keywords: atom, finite automaton, quotient complexity, regular language, reversal, semigroup, state complexity, syntactic complexity

1 Introduction

In recent years much of the theory of the so-called descriptional complexity of regular languages has been concerned with state complexity. The state complexity of a regular language $[13]$ is the number of states in the minimal complete deterministic finite automaton (DFA) recognizing the language. An equivalent notion is quotient complexity $[1]$, which is the number of left quotients of the language, where the left quotient (or simply quotient) of a language $L$ over an alphabet $\Sigma$ by a word $w \in \Sigma^*$ is $w^{-1}L = \{ x \mid wx \in L \}$. The (state/quotient) complexity of an operation on regular languages is the maximal complexity of the language resulting from the operation as a function of the complexities of the arguments. The operations considered may be basic, for example, union, star

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or product (concatenation), or combined, for example, star of union or reversal of product. Basic operations were first studied by Maslov [7] in 1970, and later by Yu, Zhuang and K. Salomaa [12] in 1994. Combined operations were first considered by A. Salomaa, K. Salomaa and Yu [10] in 2007. See also the 2012 paper on this topic by Brzozowski [2] and the references in that paper.

It has been suggested in [5] by Brzozowski and Ye that syntactic complexity can be a useful measure of complexity. It has its roots in the Myhill congruence [8] defined by a language \( L \subseteq \Sigma^* \) as follows: For \( x, y \in \Sigma^* \),

\[
x \approx_L y \text{ if and only if } uxv \in L \iff uyv \in L \text{ for all } u, v \in \Sigma^*.
\]

The syntactic semigroup [9] of \( L \) is the quotient semigroup \( \Sigma^+/\approx_L \). It is isomorphic to the semigroup of transformations of states by non-empty words in the minimal DFA of \( L \). This semigroup is called the transition semigroup and is often used to represent the syntactic semigroup. Syntactic complexity is the cardinality of the syntactic semigroup. Syntactic complexity may be able to distinguish between two regular languages with the same quotient complexity. For example, a language with three quotients may have syntactic complexity as low as 2 or as high as 27.

Atoms of regular languages were introduced in 2011 [3], and their quotient complexities were studied in 2012 [4]. An atom \([\mathcal{K}]_i\) of a regular language \( L \) with quotients \( K_0, \ldots, K_{n-1} \) is a non-empty intersection of the form \( \widetilde{K}_0 \cap \cdots \cap \widetilde{K}_{n-1} \), where \( \widetilde{K}_i \) is either \( K_i \) or \( \overline{K}_i \), and \( \overline{K}_i = \Sigma^* \setminus K_i \). Thus the number of atoms is bounded from above by \( 2^n \), and it was proved in [4] that this bound is tight.

Since every quotient of \( L \) (including \( L \) itself) and every quotient of every atom of \( L \) is a union of atoms, the atoms of \( L \) are its basic building blocks. It was proved in [4] that the quotient complexity of the atoms with 0 or \( n \) complemented quotients is bounded from above by \( 2^n - 1 \), and that of any atom with \( r \) complemented quotients, where \( 1 \leq r \leq n - 1 \), by

\[
f(n, r) = 1 + \sum_{k=1}^{r} \sum_{h=k+1}^{n} \binom{n}{h} \binom{k}{h}.
\]

These bounds are tight [4]. When we say that a language has maximal quotient complexity of atoms we mean that (a) it has all \( 2^n \) atoms, and (b) they all reach their maximal bounds, as stated above.

It was argued in [2] that it is useful to consider several measures of complexity of regular languages, including syntactic complexity and atom complexity, along with the more traditional measures such as the state complexity of operations. If one does consider several measures, the question arises whether these measures are related. There are only two such results. The first is the following proposition which restates for our purposes the 2004 result of A. Salomaa, Wood, and Yu [11].

\[1\] The definition of [3], has been slightly modified in [4]. The newer model, which admits up to \( 2^n \) atoms, is used here.
**Proposition 1 (Syntactic Semigroup and Reversal).** Maximal syntactic complexity of a regular language implies maximal quotient complexity of its reverse.

In other words, if $L$ has syntactic complexity $n^n$, then the quotient complexity of $L^R$, the reverse of $L$, is necessarily $2^n$.

The converse of Proposition 1 is false. It was shown by Jirásková and Šebej that the DFA of Fig. 1 with $n \geq 2$ meets the upper bound for reversal [6]. However, it is well known that at least three inputs are required to generate all $n^n$ transformations when $n \geq 3$. Thus the cardinality of the syntactic semigroup of the language of the DFA of Fig. 1 is strictly smaller than $n^n$.

The second result is the 2011 proposition of Brzozowski and Tamm [3, 4]

**Proposition 2 (Number of Atoms and Reversal).** The number of atoms of a regular language is equal to the quotient complexity of its reverse.

The main result of this paper is the following theorem:

**Theorem 1. (Syntactic Semigroup and Atoms)** Maximal syntactic complexity of a regular language with $n$ quotients implies that the language has $2^n$ atoms and each atom has maximal quotient complexity.

The fact that the number of atoms of $L$ (quotient complexity of $L^R$) is $2^n$ does not imply that each atom has maximal quotient complexity. For example, the language of Fig. 1 for $n = 4$ (respectively, $n = 5, 6, 7$) has no atoms of quotient complexity larger than 25 (respectively, 99, 298,1053), but the maximal quotient complexity is 43 (respectively, 141, 501, 1548).

The converse of Theorem 1 is not true. The language $L$ of the minimal DFA of Fig. 2 meets all the quotient complexity bounds for the 8 atoms, but its syntactic complexity is 24, while the maximum is 27. There are also many ternary examples with higher numbers of states.

The remainder of the paper is devoted to the proof of Theorem 1.

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2 In the figure, if $n = 2$, then $a$ transposes states 0 and 1, and $b$ is as shown. For $n = 3$, state 2 goes to itself under $b$. For $n = 4$, state 3 goes to itself under $a$. 

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Fig. 1. The DFA of a language meeting the bound $2^n$ for reversal.
2 Definitions

2.1 Automata and Átomata

A nondeterministic finite automaton (NFA) is a quintuple \( \mathcal{N} = (Q, \Sigma, \eta, I, F) \), where \( Q \) is a finite, non-empty set of states, \( \Sigma \) is a finite non-empty alphabet, \( \eta : Q \times \Sigma \rightarrow 2^Q \) is the transition function, \( I \subseteq Q \) is the set of initial states, and \( F \subseteq Q \) is the set of final states. For \( a \in \Sigma \), let \( \eta_a : Q \rightarrow 2^Q \) be defined by \( \eta_a(q) = \eta(q, a) \) for \( q \in Q \). For \( a \in \Sigma \), \( x \in \Sigma^* \), and \( w = xa \), define \( \eta_w : Q \rightarrow 2^Q \) inductively by \( \eta_w(q) = \eta_a(\eta_x(q)) \).

For any function \( f : X \rightarrow Y \), we extend \( f \) to subsets of the domain in the natural way by letting \( f(S) = \bigcup_{s \in S} f(s) \) for \( S \subseteq X \). Note \( f(\emptyset) = \emptyset \) for all \( f \).

The language accepted by an NFA \( \mathcal{N} \) is \( L(\mathcal{N}) = \{ w \in \Sigma^* \mid \eta(I, w) \cap F \neq \emptyset \} \). Two NFAs are equivalent if they accept the same language. The left language of a state \( q \) is \( L_{I,q} = \{ w \in \Sigma^* \mid q \in \eta(I, w) \} \). The right language of a state \( q \) is \( L_{q,F}(\mathcal{N}) = \{ w \in \Sigma^* \mid \eta(q, w) \cap F \neq \emptyset \} \). The right language of a set \( S \) of states of \( \mathcal{N} \) is \( L_{S,F}(\mathcal{N}) = \bigcup_{q \in S} L_{q,F}(\mathcal{N}) \); so \( L(\mathcal{N}) = L_{I,F}(\mathcal{N}) \). A state is unreachable if its left language is empty and reachable otherwise. A set \( S \) of states is strongly connected if for all \( p, q \in S \), there exists \( w \in \Sigma^* \) such that \( \eta(p, w) = q \). An NFA is minimal if it has the minimal number of states among all the equivalent NFAs.

A deterministic finite automaton (DFA) is a quintuple \( \mathcal{D} = (Q, \Sigma, \delta, q_0, F) \), where \( Q \), \( \Sigma \), and \( F \) are as in an NFA, \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function, and \( q_0 \) is the initial state. It is clear that a DFA is a special type of NFA, so the definitions stated above for NFAs also apply to DFAs. It is well-known that for every regular language \( L \), there exists a unique (up to isomorphism) minimal DFA. Furthermore, there is a one-to-one correspondence between the states of the minimal DFA and the quotients of \( L \).

For an NFA \( \mathcal{N} \) (or DFA \( \mathcal{D} \)), let \( \mathcal{N}^R \) (or \( \mathcal{D}^R \)) denote the result of performing the reversal operation which interchanges the final and initial states, and reverses all the transitions. Let \( \eta^R \) (or \( \delta^R \)) denote the transition function of \( \mathcal{N}^R \) (or \( \mathcal{D}^R \)).

Let \( \mathcal{N}^D \) denote the result of performing the determinization operation, which is the well-known subset construction. Unreachable subsets are not included in the determinization, but the empty state, if present, is included. Let \( \eta^D \) denote the transition function of \( \mathcal{N}^D \).
For $S \subseteq Q$, let $A_S$ denote the following intersection of uncomplemented and complemented quotients:

$$A_S = \left( \bigcap_{i \in S} K_i \right) \cap \left( \bigcap_{j \in Q \setminus S} \overline{K_j} \right).$$

(2)

An atom $[3,4]$ of $L$ is such an intersection $A_S$, provided it is not empty. If the intersection with all quotients complemented is non-empty, then it constitutes the negative atom; all the other atoms are positive. Let $A = \{A_0, \ldots, A_{m-1}\}$ be the set of atoms of $L$, and let the number of positive atoms be $p$. The only atom containing $\varepsilon$ is the one in which all the quotients containing $\varepsilon$ are uncomplemented and all the remaining quotients are complemented. This atom is called final, and is $A_{p-1}$ by convention. The negative atom can never be final if $L$ is non-empty, since there must be at least one final quotient in its intersection.

Atoms containing $L$, rather than $L$ in their intersection are called initial.

We use the one-to-one correspondence between atoms $A_i$ and atom symbols $A_i$. Let $A = \{A_0, \ldots, A_{m-1}\}$ be the set of atom symbols.

Definition 1. The atomaton of $L$ is the NFA $A = (A, \Sigma, \eta, A_I, \{A_{p-1}\})$, where $A$ is the set of atom symbols, $A_I$ corresponds to the set of initial atoms, $A_{p-1}$ corresponds to the final atom, and $A_j \in \eta(A_i, a)$ if and only if $aA_j \subseteq A_i$, for all $A_i, A_j \in A$ and $a \in \Sigma$.

In the atomaton, the right language of any state $A_i$ is the atom $A_i[3]$. Also, all the positive atoms are reachable, but the negative atom is not.

It was shown in [3,4] that $A^R$ is a minimal DFA that accepts $L^R$, and that $A^R$ is isomorphic to $D^{RD}$. The following makes this isomorphism precise [4]:

**Proposition 3 (Atomaton Isomorphism).** Let $L$ be a regular language and let $K$ be its set of quotients. Let $\varphi : A \rightarrow 2^K$ be the mapping assigning to state $A_j$, corresponding to atom $A_j = \left( \bigcap_{i \in S} K_i \right) \cap \left( \bigcap_{j \in Q \setminus S} \overline{K_j} \right)$ of $A^R$, the set $S$. Then $\varphi$ is a DFA isomorphism between $A^R$ and $D^{RD}$.

**Corollary 1.** The mapping $\varphi$ is an NFA isomorphism between $A$ and $D^{RDR}$.

2.2 Transformations

A transformation of a set $Q$ is a mapping of $Q$ into itself. We consider only transformations $t$ of a finite set $Q$. For a transformation $t$ of $Q$ and a subset $S$ of $Q$, let $t^{-1}(S) = \{ q \in Q \mid \text{there exists } i \in S \text{ such that } t(q) = i \}$. We say $t^{-1}(S)$ is the preimage of $S$ under $t$: the maximal set of elements of $Q$ that is mapped onto $S$ by $t$. When discussing preimages of singletons such as $t^{-1}(\{i\})$, we drop the braces and write $t^{-1}(i)$. If $P \subseteq Q$ is in the set preim $t = \{P \mid \text{there exists } S \subseteq Q \text{ such that } P = t^{-1}(S)\}$, then we say $P$ is a preimage of $t$ (as opposed to calling it the preimage of some $S$). The set preim $t$ is the set of all preimages of $t$. 

5
The image of \( t \) is \( \text{im} \ t = \{ q \in Q \mid \text{there exists } p \in Q \text{ such that } t(p) = q \} \); this is the subset of \( Q \) that \( t \) maps onto. The coimage of \( t \) is \( \text{coim} \ t = Q \setminus \text{im} \ t \); this is the set of elements of \( Q \) that are not mapped onto \( \text{im} \ t \). For \( P \subseteq Q \), the set \( t(P) \) obtained by applying \( t \) to each element of \( P \) is called the image of \( P \) under \( t \).

A transformation \( t \) is a cycle of length \( k \), where \( k \geq 2 \), if there exist pairwise different elements \( i_1, \ldots, i_k \) such that \( t(i_1) = i_2, t(i_2) = i_3, \ldots, t(i_{k-1}) = i_k, \) and \( t(i_k) = i_1 \), and the remaining elements are mapped to themselves. A cycle is denoted by \((i_1, i_2, \ldots, i_k)\). For \( i < j \), a transposition is the cycle \((i, j)\). A singular transformation, denoted by \((i \rightarrow j)\), has \( t(i) = j \) and \( t(h) = h \) for all \( h \neq i \). A constant transformation, denoted by \((Q \rightarrow j)\), has \( t(i) = j \) for all \( i \). If \( s \) and \( t \) are transformations, the composition \( s \circ t \) is defined by \( s \circ t(i) = s(t(i)) \).

## 3 Proof of the Main Result

To establish Theorem [1] we need several intermediate results. In the sequel we represent the states of the átomaton \( A \) of a regular language \( L \) by sets of quotients of \( L \), that is, by sets of states of the minimal DFA \( D \) recognizing \( L \), as allowed by Proposition [3]. Since the states of \( A \) are sets of states and \( A \) is an NFA, the outputs of \( A \)'s transition function are sets of sets of states. To reduce confusion, we refer to these as collections of sets of states.

In some case, the collections of sets that arise as outputs of \( A \)'s transition function can be described as “intervals”. If \( U \) and \( V \) are sets, the interval \([V, U]\) between \( V \) and \( U \) is the collection of all subsets of \( U \) that contain \( V \). Note that if \( V \) is not a subset of \( U \), this interval is empty.

### 3.1 Transition Function of the Átomaton

**Lemma 1.** Let \( L \subseteq \Sigma^* \) be a regular language with quotient complexity \( n \) and syntactic complexity \( n^n \). Let \( D \) be the minimal DFA for \( L \) with state set \( Q \) and transition function \( \delta \). Let \( A \) be the átomaton of \( L \) with transition function \( \eta \).

1. Let \( S \subseteq Q \) and \( a \in \Sigma \). Then the transition function \( \eta \) of \( A \) satisfies:
   \[
   \eta_a(S) = \begin{cases} 
   [\delta_a(S), \delta_a(S) \cup \text{coim} \delta_a], & \text{if } S \in \text{preim} \delta_a; \\
   \emptyset, & \text{otherwise.}
   \end{cases}
   \]

2. Let \( U, V \subseteq Q \) and let \( a \in \Sigma \). If every set in the interval \([V, U]\) is a preimage of \( \delta_a \), then the transition function \( \eta \) of \( A \) satisfies:
   \[
   \eta_a([V, U]) = [\delta_a(V), \delta_a(U) \cup \text{coim} \delta_a].
   \]

3. Let \( U, V \subseteq Q \) and let \( w \in \Sigma^* \). If \( \delta_w \) is a permutation, then the transition function \( \eta \) of \( A \) satisfies:
   \[
   \eta_w([V, U]) = [\delta_w(V), \delta_w(U)].
   \]
Proof. \(\square\): In \(D^R\), the letter \(a\) induces the function \(\delta_a^R : Q \rightarrow 2^Q\) which maps each state \(i\) to its preimage \(\delta_a^{-1}(i)\). Furthermore, by Proposition \(\square\) every subset of \(Q\) is reachable in \(D^{RD}\) since \(L\) has maximal syntactic complexity. So the set of states of \(D^{RD}\) is \(2^Q\), and the empty set of states of \(D\) is a state of \(D^{RD}\). Thus in \(D^{RD}\), the letter \(a\) induces the function \(\delta_a^{RD} : 2^Q \rightarrow 2^Q\), defined as follows:

\[
\delta_a^{RD}(S) = \bigcup_{i \in S} \delta_a^{-1}(i) = \delta_a^{-1}(S).
\] (3)

In \(D^{RDR}\), a induces the function \(\delta_a^{RDR} : 2^Q \rightarrow 2^Q\). This function maps a subset \(S\) of \(Q\) to its preimage under \(\delta_a^{RD}\), that is, to the collection of sets each of which maps to \(S\) under \(\delta_a^{RD}\). Since \(D^{RDR}\) is isomorphic to \(A\), \(\delta^{RDR}\) is equivalent to \(\eta\).

We now show this function satisfies the statement from the lemma.

Notice that if \(S \notin \text{preim}\ \delta_a\), then \(S\) cannot be an output of \(\delta_a^{RD}\). It follows that \(\delta_a^{RDR}(S) = \emptyset\), since the collection of sets that map to \(S\) under \(\delta_a^{RD}\) is empty.

Conversely, suppose \(S \in \text{preim}\ \delta_a\). Then clearly \(S\) is the preimage of \(\delta_a(S)\) under \(\delta_a\). It follows by Equation (3) that \(\delta_a^{RD}(\delta_a(S)) = S\), and thus \(\delta_a(S)\) is in the collection of sets produced by \(\delta_a^{RDR}(S) = \eta_a(S)\).

Consider which other sets map to \(S\) under \(\delta_a^{RD}\). Notice no strict subset of \(\delta_a(S)\) maps to \(S\); if \(S\) is the preimage of \(\delta_a(T) \subset \delta_a(S)\) under \(\delta_a\) this means \(\delta_a^{-1}(\delta_a(T)) = S\), and applying \(\delta_a\) to both sides gives \(\delta_a(T) = \delta_a(S)\). A strict superset of \(\delta_a(S)\), say \(\delta_a(S) \cup T\), maps to \(S\) only if \(\delta_a^{-1}(T) \subseteq S\), since we have

\[
\delta_a^{RD}(\delta_a(S) \cup T) = \delta_a^{-1}(\delta_a(S) \cup T) = \delta_a^{-1}(\delta_a(S)) \cup \delta_a^{-1}(T) = S \cup \delta_a^{-1}(T).
\]

Suppose \(\delta_a^{-1}(T)\) is non-empty. Since \(\delta_a^{-1}(T) \subseteq S\), we have \(T \subseteq \delta_a(S)\). Thus if \(\delta_a(S) \cup T\) is a strict superset of \(\delta_a(S)\), then \(\delta_a^{-1}(T)\) must be empty. Therefore \(T\) must be a subset of \(\text{coim}\ \delta_a\), since \(\text{coim} \ \delta_a\) contains all the elements of \(Q\) with empty preimages under \(\delta_a\).

In fact, for all \(T \subseteq \text{coim} \ \delta_a\) we have \(\delta_a^{RD}(\delta_a(S) \cup T) = S\). This means that the collection of sets produced by \(\delta_a^{RDR}(S)\) (and thus \(\eta_a(S)\)) is the set of all supersets of \(\delta_a(S)\) which are subsets of \(\delta_a(S) \cup \text{coim} \ \delta_a\). Thus, as required, we have:

\[
\eta_a(S) = [\delta_a(S), \delta_a(S) \cup \text{coim} \ \delta_a].
\]

\(\square\): We proceed by induction on the number of sets in the interval. If there are no sets, that is, if \([V, U] = \emptyset\), then \(\eta_a([V, U]) = \emptyset\) as required. If there is only one set, say \([V, U] = [S, S] = \{S\}\), then the proof of the previous part shows the statement is true.

Suppose that the statement holds if \(|[V, U]| < k\). We must show it also holds if \(|[V, U]| = k\). If \(V \supset U\) then \([V, U] = \emptyset\), and if \(V = U\) then \(|[V, U]| = 1\). These are the base cases, so we can assume that \(V \subset U\).

If \(V \subset U\), then we have some \(u \in U\) such that \(u \notin V\). Notice that we can write \([V, U]\) as \([V \cup \{u\}, U] \cup [V, U \setminus \{u\}]\). Also, we have \(\eta_a([V \cup \{u\}, U] \cup [V, U \setminus \{u\}]) = \eta_a([V \cup \{u\}, U]) \cup \eta_a([V, U \setminus \{u\}]).\) It follows that:

\[
\eta_a([V, U]) = \eta_a([V \cup \{u\}, U]) \cup \eta_a([V, U \setminus \{u\}]).
\]
These two intervals have strictly fewer sets than \([V, U]\); so by the induction hypothesis we have:

\[
\eta_a([V \cup \{u\}, U]) = [\delta_a(V \cup \{u\}), \delta_a(U) \cup \coim \delta_a],
\]

and

\[
\eta_a([V, U \setminus \{u\}]) = [\delta_a(V), \delta_a(U \setminus \{u\}) \cup \coim \delta_a].
\]

Notice that \(U\) and \(U \setminus \{u\}\) are both in \([V, U]\), and thus are preimages of \(\delta_a\). Since preimages are maximal, distinct preimages map to distinct sets under \(\delta_a\). Thus \(\delta_a(U \setminus \{u\}) \neq \delta_a(U)\). It follows that \(\delta_a(u) \notin \delta_a(U \setminus \{u\})\), since otherwise the two sets would be equal. Furthermore, \(\delta_a(u)\) is the only element which is present in \(\delta_a(U)\) but not present in \(\delta_a(U \setminus \{u\})\). Thus \(\delta_a(U \setminus \{u\}) = \delta_a(U \setminus \{\delta_a(u)\})\). It follows that:

\[
\eta_a([V, U \setminus \{u\}]) = [\delta_a(V), \delta_a(U \setminus \{\delta_a(u)\}) \cup \coim \delta_a].
\]

Furthermore, noting that \(\delta_a(V \cup \{u\}) = \delta_a(V) \cup \{\delta_a(u)\}\), we have:

\[
\eta_a([V \cup \{u\}, U]) = [\delta_a(V) \cup \{\delta_a(u)\}, \delta_a(U) \cup \coim \delta_a].
\]

Thus, as required, the union of these two intervals is:

\[
\eta_a([V, U \setminus \{u\}]) \cup \eta_a([V \cup \{u\}, U]) = [\delta_a(V), \delta_a(U) \cup \coim \delta_a].
\]

\(\Box\). We proceed by induction on the length of \(w\). Every subset of \(Q\) is a preimage of \(\delta_w\), since \(\delta_w\) is a permutation. Also, \(\coim \delta_w = \emptyset\). Thus the base case (where \(w\) is a single letter) is covered by the proof of the previous part.

Now suppose \(w = a_1a_2 \cdots a_k\) and the lemma holds for words of length less than \(k\). Let \(w' = a_1a_2 \cdots a_{k-1}\). By the inductive hypothesis, we have

\[
\eta_{w'}([V, U]) = [\delta_{w'}(V), \delta_{w'}(U)].
\]

Notice that \(\delta_w = \delta_{a_k} \circ \delta_{w'}\), and similarly \(\eta_w = \eta_{a_k} \circ \eta_{w'}\). Furthermore, \(\delta_{a_k}\) must be a permutation (or else \(\delta_w\) would not be a permutation). Thus by Part 2 of this lemma, we have:

\[
\eta_w([V, U]) = \eta_{a_k}(\eta_{w'}([V, U])) = \eta_{a_k}([\delta_{w'}(V), \delta_{w'}(U)]) = [\delta_{a_k}(\delta_{w'}(V)), \delta_{a_k}(\delta_{w'}(U))] = [\delta_w(V), \delta_w(U)].
\]

This proves that the statement holds for \(k\) and thus for all natural numbers. \(\Box\)

\textbf{Example 1.} Consider the DFA \(D\) with \(Q = \{0, 1, 2\}\), \(\Sigma = \{a, b, c, d\}\), \(q_0 = 0\), \(F = \{2\}\), and transition function \(\delta\) defined by \(\delta_a = (0, 1), \delta_b = (1, 2), \delta_c = (2 \rightarrow 0)\), and \(\delta_d = (Q \rightarrow 1)\). The language \(L = L(D)\) has syntactic complexity \(n^0 = 3^3 = 27\).

The transition functions of \(D\), \(D^R\), \(D^{RD}\) and \(A = D^{RRD}\) are shown in Tables 1 to 4. For conciseness, we represent sets like \(\{0, 1, 2\}\) and \(\{0, 2\}\) by 012.
and 02, respectively, and collections of sets like \( \{0\}, \{0,1\}, \{0, 2\}, \{0, 1, 2\} \), by 0, 01, 02, 012. We use \( \Phi \) to denote the “empty-set state” that arises when performing determinization of an NFA \( N \) (that is, the state in \( N^D \) which corresponds to the empty subset of states of \( N \)) and \( \emptyset \) to denote the actual empty set. The arrows in the leftmost column of each table denote initial states (→) and final states (←).

One can check that the definition of the transition function \( \eta = \delta^{RDR} \) of the automaton matches that of Part 1 of the lemma. For example, we have \( \eta_{d}(\{0,1,2\}) = \{\{1\}, \{0,1\}, \{1,2\}, \{0,1,2\}\} = \{\{1\}, \{0,1,2\}\} \). The lower bound of this interval is \( \{1\} = \delta_0(\{0,1,2\}) \). Since \( \text{coim} \delta_0 = \{0,2\} \), the upper bound of this interval is \( \delta_0(\{0,1,2\}) \cup \text{coim} \delta_0 = \{1\} \cup \{0,2\} = \{0,1,2\} \).

Notice that \( \{0,1,2\} \) is a preimage of \( \delta_0 \) (in particular, \( \delta_0^{-1}(\{0,1,2\}) \)) so \( \eta_{d}(\{0,1,2\}) \) is not the empty set. The only other preimage of \( \delta_0 \) is \( \Phi \), and we have \( \eta_{d}(\Phi) = [\Phi, \{0,2\}] \) as required. For all other subsets \( S \) of \( \{0,1,2\} \), we see that \( S \) is not a preimage of \( \delta_0 \) and \( \eta_{d}(S) = \emptyset \) as required.

3.2 Strong Connectedness and Reachability

To show that each atom has maximal quotient complexity if the associated language has maximal syntactic complexity, we follow the approach of [4]. Let \( L \subseteq \Sigma^* \) be a regular language, let \( D \) be the quotient DFA for \( L \) with state set \( Q \), and let \( \mathcal{A} \) be the automaton of \( L \). For \( S \subseteq Q \), we derive \( A_S^R \) (the minimal DFA of the atom \( A_S \)) by making \( S \) the starting state of \( \mathcal{A} \), and then determinizing. The initial state of \( A_S^R \) is \( \{S\} \), or equivalently the interval \( [S,S] \). To prove the quotient complexity of \( A_S \) is maximal, we use our results on the transition function of \( \mathcal{A} \) (Lemma 1) to count the number of intervals that are reachable from \([S,S]\) in
\( A_S \). If the number of reachable intervals meets the quotient complexity bound for the atom \( A_S \), it follows \( A_S \) has maximal quotient complexity.

First we prove the following lemma:

**Lemma 2.** Let \( L \subseteq \Sigma^* \) be a regular language with quotient complexity \( n \) and syntactic complexity \( n^a \). Let \( L \) be the minimal DFA of \( L \) with transition function \( \delta \) and state set \( Q \). Then there exists \( a \in \Sigma \) and \( w \in \Sigma^* \) such that \( \delta_a = \alpha \circ \delta_w \), where \( \alpha \) is a singular transformation and \( \delta_w \) is a permutation.

**Proof.** Let \( T = \{ \delta_a : a \in \Sigma \} \). Since \( L \) has syntactic complexity \( n^a \), the set \( T \) generates all transformations of \( Q \). We claim there exists \( \delta_a \in T \) such that \( |\text{im} \delta_a| = n - 1 \).

To see this, observe that if \( s \) and \( t \) are transformations with \( |\text{im} s| = k \) and \( |\text{im} t| = \ell \), then \( |\text{im}(s \circ t)| \leq \min\{k, \ell\} \). Now suppose for a contradiction that for all \( \delta_a \in T \), we have \( |\text{im} \delta_a| = n \) or \( |\text{im} \delta_a| = n - 2 \). Since \( T \) generates all transformations of \( Q \), there exists \( w \in \Sigma^* \) such that \( |\text{im} \delta_w| = n - 1 \). Clearly \( w \) cannot contain any letter \( b \in \Sigma \) such that \( |\text{im} \delta_b| \leq n - 2 \), or else we would have \( |\text{im} \delta_w| \leq |\text{im} \delta_{b\in \Sigma}| < n - 1 \). It follows \( w \) only contains letters \( b \) such that \( |\text{im} \delta_b| = n \). Thus \( \delta_w \) is a permutation, since it is a composition of permutations. But this implies \( |\text{im} \delta_w| = n \), which is a contradiction.

Thus there exists \( a \in \Sigma \) such that \( |\text{im} \delta_a| = n - 1 \). Suppose \( \text{im} \delta_a = \{q_1, q_2, \ldots, q_{n-1}\} \) and \( \text{coin} \delta_a = \{q_n\} \). Since \( |\text{im} \delta_a| = n - 1 \), there exists a subset \( P = \{p_1, p_2, \ldots, p_{n-1}\} \) of \( Q \) such that \( \delta_a(p_i) \neq \delta_a(p_j) \) for all \( i, j \). Suppose without loss of generality that \( \delta_a(p_i) = q_i \).

In \( Q \setminus P \) there is precisely one state, say \( p_n \). Since \( p_n \notin P \), we have \( \delta_a(p_n) = \delta_a(p_i) = q_i \) for exactly one \( p_j \in P \).

Recall that for all transformations \( t \) of \( Q \), there exists \( w \in \Sigma^* \) that induces \( t \). Pick \( w \) such that \( \delta_w : Q \to Q \) satisfies \( \delta_w(p_i) = q_i \) for all \( p_i \). Notice that \( \delta_w \) is a permutation. Now let \( \alpha : Q \to Q \) be the singular transformation \( (q_n \to q_j) \).

Then \( \alpha(\delta_w(p_i)) = \alpha(q_i) = q_i \) for all \( p_i \in P \), and \( \alpha(\delta_w(p_n)) = \alpha(q_n) = q_j \). Thus \( \alpha \circ \delta_w = \delta_a \) as required. \( \Box \)

Now we can prove the main result of this section. We assign a type to all non-empty intervals as follows: the type of \([V, U]\) is the ordered pair \( ([V], [U]) \).

**Lemma 3.** Suppose that \( L \) has quotient complexity \( n \) and syntactic complexity \( n^a \). Consider \( S \subseteq Q \) and \( A^D_S \), the minimal DFA of the atom \( A_S \).

1. All states of \( A^D_S \) which are intervals of the same type are strongly connected.
2. From a state in \( A^D_S \) which is an interval of type \( (v, u) \), if \( v \geq 2 \) we can reach a state which is an interval of type \( (v - 1, u) \) and if \( u \leq n - 2 \) we can reach a state which is an interval of type \( (v, u + 1) \).

**Proof.** Since \( L \) has syntactic complexity \( n^a \), every permutation of \( Q \) can be induced by a word in \( \Sigma^* \). Let \([V_1, U_1]\) and \([V_2, U_2]\) be states of \( A^D_S \) of the same type. We can assume that \( V_1 \subseteq U_1 \) and \( V_2 \subseteq U_2 \). Now consider \( w \in \Sigma^* \) and
suppose $\delta_w : Q \rightarrow Q$ is a permutation that sends $V_1$ to $V_2$ and $U_1$ to $U_2$; such a permutation exists if $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$. By Part 3 of Lemma 1, we have:

$$\eta_w([V_1, U_1]) = [\delta_w(V_1), \delta_w(U_1)] = [V_2, U_2].$$

Thus any two intervals of the same type in $A_n^Q$ are connected by a word in $\Sigma^*$.

By Lemma 2, there exists a single letter $a \in \Sigma$ and a word $w \in \Sigma^*$ such that $\delta_a$ induces a transformation $\alpha \circ \delta_w$, where $\alpha$ is a singular transformation and $\delta_w$ is a permutation. Suppose $\alpha = (k \rightarrow \ell)$ for $k, \ell \in Q$.

Note that a subset $S$ of $Q$ is a preimage of $\alpha$ only if $\{k, \ell\} \subseteq S$ or $\{k, \ell\} \cap S = \emptyset$. Since $\delta_a = \alpha \circ \delta_w$, it follows that $S$ is a preimage of $\delta_a$ only if $\{\delta_w^{-1}(k), \delta_w^{-1}(\ell)\} \subseteq S$ or $\{\delta_w^{-1}(k), \delta_w^{-1}(\ell)\} \cap S = \emptyset$. Also note that since $\delta_a = \alpha \circ \delta_w$ and $\mathrm{coim} \delta_w = \emptyset$, we have $\mathrm{coim} \delta_a = \mathrm{coim} \alpha = \{k\}$.

Let $[V, U]$ be an interval of type $(v, u)$ with $v \geq 2$. By Part 1 of this lemma, from $[V, U]$ we can reach an interval $[V', U']$ of type $(v, u)$ such that $\{k, \ell\} \subseteq V'$, and thus $k$ and $\ell$ are in every set of $[V', U']$. Since $L$ has syntactic complexity $n^n$, there exists $x \in \Sigma^*$ such that $\delta_x = \delta_w^{-1}$. By Part 3 of Lemma 1, we can apply $\eta_a$ to $[V', U']$ to obtain $[\delta_w^{-1}(V'), \delta_w^{-1}(U')]$. Every set in this interval is a preimage of $\delta_a$ since every set contains both $\delta_w^{-1}(k)$ and $\delta_w^{-1}(\ell)$.

By Lemma 1, Part 2, $\eta_a([\delta_w^{-1}(V'), \delta_w^{-1}(U')])$ is $[\alpha(V'), \alpha(U') \cup \{k\}]$ (since $\delta_a = \alpha \circ \delta_w$, $\delta_w$ cancels its inverse). Since $\{k, \ell\} \subseteq V', U'$, we have $\alpha(V') = V' \setminus \{k\}$ and $\alpha(U') \cup \{k\} = U' \setminus \{k\} \cup \{k\} = U$. Thus the resulting interval is $[V' \setminus \{k\}, U']$, which has type $(v - 1, u)$ as required.

In a similar fashion, suppose we have an interval $[V, U]$ of type $(v, u)$ such that $u \leq n - 2$. We can reach $[V', U']$ such that $\{k, \ell\} \cap U' = \emptyset$. We can then apply $\eta_a$ to get $[\delta_w^{-1}(V'), \delta_w^{-1}(U')]$. As before, each set in this interval is a preimage of $\delta_a$ since for all sets $S$ in the interval we have $\{\delta_w^{-1}(k), \delta_w^{-1}(\ell)\} \cap S = \emptyset$. Thus by Part 2 of Lemma 1, we can apply $\eta_a$ to get $[\alpha(V'), \alpha(U') \cup \{k\}] = [V', U' \cup \{k\}]$. This has type $(v, u + 1)$ as required.

\[ \square \]

### 3.3 Proof of Main Theorem

Our main theorem, restated below, now follows easily:

**Theorem 1. (Syntactic Semigroup and Atoms)** Maximal syntactic complexity of a regular language with $n$ quotients implies that the language has $2^n$ atoms and each atom has maximal quotient complexity.

**Proof.** Since $L$ has syntactic complexity $n^n$, Lemma 3 holds for minimal DFAs of atoms of $L$. It was shown in [H] that if these strong-connectedness and reachability results hold, the number of reachable intervals in the minimal DFA of an atom of $L$ is equal to the maximum possible quotient complexity of the atom. Hence these results suffice to establish that each atom has maximal quotient complexity.

\[ \square \]
4 Conclusions

Maximal quotient complexity of atoms defines a new complexity class of regular languages. We have related this new measure to syntactic complexity and quotient complexity of reversal. Such relations are important, since they often make it possible to avoid proofs of complexity results implied by other known complexity results. We believe that this subject deserves further study.

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