Symmetric texture-zero mass matrices with eigenvalues quark mass

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Working within the context of texture-zeros mechanism for fermionic mass matrices, we provide necessary and sufficient conditions on the characteristic polynomial coefficients such that it has real, simple and positive roots. We translate these conditions in terms of invariants from congruent matrices. Then, all symmetric texture-zero matrices are counted and classified. Next we apply the result from the first part to analyze the three, two and one zero texture matrices in a systematic way. Finally we solve analytically the $V_{ckm}$ mixing matrix for the four zero sets; we also analyze the $V_{ckm}$ for a particular case of four zero, four zero-perturbed and three zero sets.

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I. INTRODUCTION

In the Standard Model (SM) with $SU(2) \times U(1)$ as the gauge group of electroweak interactions, the masses of quarks and charged leptons are contained in the Yukawa Sector.

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After Spontaneous Symmetry Breaking (SSB), the mass matrix is defined as:

\[ M_f = \frac{v}{\sqrt{2}} Y_f, \quad (f = u, d, l), \]

where \( v \) is the vacuum expectation value of the Higgs field and \( Y_f \) are the 3 × 3 Yukawa matrices. The physical masses of the particles are defined as the eigenvalues of the mass matrix \( M_f \). Within the SM context the mass matrix is unknown, the only trail of the quarks mass matrices is the \( V_{ckm} \) matrix, which is built by the product of left matrices that diagonalize the \( u \) and \( d \)-quark mass matrix.

In 1977 Harald Fritzsch proposed a phenomenological study \(^1\), the so called texture-zeros mechanism \(^1\), that consist of looking for the simplest pattern of mass matrices, which can result in a self-consistent way and it reproduce the \( V_{ckm} \) parameters obtained experimentally. From all possible texture-zero matrices (symmetric, non-symmetric and triangular \(^3\) matrices) we restrict our study to symmetric textures. Mathematically speaking, a symmetric mass matrix always guarantees that the physical masses are real, however, the positivity condition for the eigenvalues is not fulfilled by any symmetric matrix, moreover a positive definite matrix has real and positive eigenvalues, but not necessarily they are different. In the texture-zeros formalism it is possible to have negative eigenvalues, in this case, these negative signs can be removed with a rotation, however in our proposal, in this paper we will not consider this extra rotation, we take as starting point strictly that all eigenvalues must be the quark masses, in other words, each eigenvalue must be real, different and positive. Going in this direction, we discuss what kind of symmetric texture-zeros are self consistent considering by definition that, the eigenvalues of the mass matrix are the masses of quarks or charged leptons, and they must be positive (and different) real numbers.

The organization of this paper goes as follows. In Sec. II, we show analytically how the mass matrices appears in the SM context. In Sec. III, we find necessary and sufficient conditions on the characteristic polynomial coefficients such that its roots are real, simple and positive quantities. These conditions are rewritten in terms of the invariants of the congruent matrices, \( i.e. \) trace, determinant and trace of the power matrix. In Sec. IV, we develop a simple notation that counts and classifies the texture-zero matrices, and we show that all symmetric matrices of 3 × 3 can be grouped into 1-zero, 2-zero and 3-zero texture, \(^1\) For excellent reviews see \( ^2 \), and references there in.
in order to complete the counting the matrix without zeros is included. In Sec. V, we apply systematically the results of Sec. III to all matrices of the Sec. IV, and we show what kind of texture matrices have real, different and positive eigenvalues. Finally in Sec. VI, we derive analytical expressions for all the $V_{ckm}$ elements arising from the 4-zero sets, then by choosing a particular case of a four zero set, we compute the $V_{ckm}$ matrix, next we perturb this case in order to improve the expressions for the $V_{ckm}$ elements, finally we took this case to the three zero sets.

II. PRELIMINARIES

In the Yukawa sector of the SM, the mass terms for quarks and charged leptons can be expressed as

$$
\bar{u}_L M_u u_R + \bar{d}_L M_d d_R + \bar{l}_L M_l l_R,
$$

(1)

where $u_{L(R)}$, $d_{L(R)}$ and $l_{L(R)}$ are the left(right)-handed quark and charged leptons fields for the u-sector ($u, c, t$), d-sector ($d, s, b$) and charged leptons ($e, \mu, \tau$) respectively. $M_u$, $M_d$ and $M_l$ are the mass matrices. Expressing the above equation in terms of the physical fields, one diagonalize the mass matrices by bi-unitary transformations

$$
\bar{M}_u = U_{UL}^\dagger M_u U_{UR} = \text{Diag}[m_u, m_c, m_t],
$$

$$
\bar{M}_d = U_{DL}^\dagger M_d U_{DR} = \text{Diag}[m_d, m_s, m_b],
$$

$$
\bar{M}_l = U_{IL}^\dagger M_l U_{IR} = \text{Diag}[m_e, m_\mu, m_\tau],
$$

(2)

where $U_{fL}$ and $U_{fR}$ ($f = u, d, l$) are in general complex unitary matrices. The quantities $m_u, m_d, \ldots$ etc. denote the eigenvalues of the mass matrices, i.e. the physical quark masses and they must have real and nonnegative quantities.

Re-expressing Eq. (1) in terms of physical fermion fields ($f'_{L(R)}$) as

$$
\bar{u'}_L \bar{M}_u u'_R + \bar{d'}_L \bar{M}_d d'_R + \bar{l'}_L \bar{M}_l l'_R,
$$

(3)

where $f'_{L} = \bar{f}_L U_{fL}$ and $f'_{R} = U_{fR}^\dagger f_R$, ($f' = u', d', l'$).

Eq. (2) implies that $M_f$ and $M_f$, ($f = u, d, l$) are congruent matrices, the relation of congruence is an equivalence relation, which implies a space partition into cosets. Any two elements that belong at the same coset have the following invariants: determinant, trace, trace of the power matrix, characteristic polynomial and their eigenvalues, on the other hand, if $M_f$ and
If $M_f$ are congruent matrices then: $\det \bar{M}_f = \det M_f$, $\text{tr} \bar{M}_f = \text{tr} M_f$, $\text{tr} \bar{M}_f^n = \text{tr} M_f^n$, where $n$ is a positive integer, $\det(\bar{M}_f - \lambda I) = \det(M_f - \lambda I)^2$.

Considering $M_f$ as a $3 \times 3$ symmetric matrix with real coefficients then $\bar{M}_f$ is built as a diagonal matrix where its elements are the eigenvalues of $M_f$, these eigenvalues are found as the roots its characteristic polynomial. In the following section we give conditions on the coefficients of the characteristic polynomial from $M_f$, i.e on the $M_f$ elements, such that this polynomial has three real, positive and simple roots.

III. MAIN THEOREM

The physical quark masses are defined as the eigenvalues of the mass matrix, from mathematical point of view, to obtain the quark masses it is necessary compute the characteristic equation and its roots are the quark masses. In this section we present the conditions over the characteristic polynomial coefficients such that the polynomial characteristics roots are real, positive and different. We translate these conditions in terms of invariants of congruent matrices as Trace and Determinant of the mass matrix.

**Theorem 1** The polynomial of degree 3, $p(\lambda) = \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0$ has three different, real and positive roots if and only if the following conditions over its coefficients $a_0$, $a_1$ $a_2$ hold.

1. $a_0, a_2 < 0 < a_1$.
2. $3a_1 < a_2^2$.
3. If $\lambda_4 = \frac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3}$ and $\lambda_5 = \frac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3}$, then $p(\lambda_4) < 0$ and $p(\lambda_5) > 0$.

**Proof.** See Appendix.

We observe that in the condition $\lambda_4$ and $\lambda_5$ are the roots of the first derivative of $p(\lambda)$, and therefore the condition $\lambda_4$ and $\lambda_5$ are real numbers, in others words, $p(\lambda)$ has two critical points, this fact join to the condition 1 implies that $0 < \lambda_5 < \lambda_4$.

---

2 In this work, we will denote the product $(\text{tr} A)(\text{tr} A)$ as $\text{tr}^2 A$. In the general case $(\text{tr} A)^n = \text{tr}^n A$ for $n$ positive integer.
The condition \( p(\lambda_4) < 0 \) and \( p(\lambda_5) > 0 \), means that the maximum value is positive and the minimum value is negative, and therefore \( p(\lambda) \) has three real and different roots. This condition can be replaced by

\[
-2(a_2^2 - 3a_1)^{3/2} < 2a_2^3 - 9a_1a_2 + 27a_0 < 2(a_2^2 - 3a_1)^{3/2},
\]

the first inequality is obtaining by solving \( p(\lambda_5) > 0 \) and the second one is obtained by solving \( p(\lambda_4) < 0 \). The condition (4) can be rewriting as

\[
|2a_2^3 - 9a_1a_2 + 27a_0| < 2(a_2^2 - 3a_1)^{3/2}.
\]

It is convenient to rewrite the theorem in terms of the invariants of congruent matrices. This create directly a link between the matrix elements and its eigenvalues which facilitates subsequent computations and applications. To implement this fact, first we write the coefficients of its characteristic polynomial \( p(\lambda) \) in terms of its trace (\( \text{tr} M \)), trace of the square matrix (\( \text{tr} M^2 \)) and its determinant (\( \text{det} M \)) in the following form:

\[
p(\lambda) = \lambda^3 - \text{tr} M \lambda^2 + \frac{1}{2} \left[ \text{tr}^2 M - \text{tr} M^2 \right] \lambda - \text{det} M.
\]

Now we are ready to present the main theorem of this section

**Theorem 2** A real, symmetric matrix \( M \) has real, positive and different eigenvalues if and only if the following three conditions hold.

1. (a) \( \text{det} M > 0 \),
   
   (b) \( \text{tr} M > 0 \),
   
   (c) \( \text{tr} M^2 < \text{tr}^2 M \).
2. \( \text{tr}^2 M < 3 \text{tr} M^2 \).
3. \( |\text{tr} M(5\text{tr}^2 M - 9 \text{tr} M^2) - 54 \text{det} M| < \sqrt{2}(3\text{tr} M^2 - \text{tr}^2 M)^{3/2} \).

The theorem will be applied to texture-zero matrices.
IV. TEXTURE-ZERO FORMALISM

A texture-zero matrix is a $3 \times 3$ matrix with zeros in some entries, the way to count them is the following: a zero in the main diagonal add as 1, while zero off main diagonal add as $1/2$. We need to sum all zeros for both mass matrices $u$-quarks and $d$-quarks. For example, given $M_u$ and $M_d$ as

$$M_u = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix}, \quad M_d = \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}.$$ 

For $M_u$ we have one zero in the main diagonal, we add (+1) and 2 zeros off main diagonal that add 1($= 1/2 + 1/2$), then $M_u$ has a 2-zero texture structure. Considering now $M_d$ we have a 3-zero texture structure ($1 + 2$). Then, this set of matrices is said to have a 5-zero texture structure.

We say: a parallel structure for $M_u$ and $M_d$ mass matrices means that if $M_u$ has zeros in some places then $M_d$ has zeros in the same position than $M_u$. Non-parallel structure is when $M_u$ and $M_d$ not have the same parallel structure.

A. Notation

We start writing a symmetric matrix $M$ in the form:

$$M = \begin{pmatrix} E & D & F \\ D & C & B \\ F & B & A \end{pmatrix}.$$ 

This matrix is well determined by specifying six capital letters ($A, B, C, D, E, F$) and their corresponding positions, then we introduce the following notation:

- $M(x)$ is a matrix with a zero in the capital letter $x$, ($x = A, B, C, D, E, F$).
- $M(x, y)$ is a matrix with zeros in the capital letters $x$ and $y$, ($x, y = A, B, C, D, E, F; x \neq y$).
- $M(x, y, z)$ is a matrix with zeros in the capital letters $x$, $y$ and $z$, ($x, y, z = A, B, C, D, E, F; x \neq y \neq z$).
For example, a matrix with a zero in the position $F$ is:

$$M(F) = \begin{pmatrix} E & D & 0 \\ D & C & B \\ 0 & B & A \end{pmatrix},$$

a matrix with zeros in the positions $C$ and $D$ is,

$$M(C, D) = \begin{pmatrix} E & 0 & F \\ 0 & 0 & B \\ F & B & A \end{pmatrix},$$

finally a matrix with zeros in the positions $C$, $D$ and $F$ is,

$$M(C, D, F) = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & B \\ 0 & B & A \end{pmatrix}.$$

Using this notation, we are able to list all possible textures.

**1-zero texture structure.**

We have 6 different matrices, which are:

$M(A)$, $M(C)$, $M(E)$, $M(B)$, $M(D)$, $M(F)$.

**2-zero texture structure.**

In this case, we have 15 possibilities, which are:

$M(A, E)$, $M(A, C)$, $M(C, E)$,

$M(A, B)$, $M(A, D)$, $M(A, F)$,

$M(B, C)$, $M(C, D)$, $M(C, F)$,

$M(B, E)$, $M(D, E)$, $M(E, F)$,

$M(B, F)$, $M(B, D)$, $M(D, F)$.

**3-zero texture structure.**

For this case, there are 20 different matrices, which are:

$M(A, B, C)$, $M(A, C, F)$, $M(A, C, D)$,

$M(A, B, E)$, $M(A, E, F)$, $M(A, D, E)$,

$M(B, C, E)$, $M(C, E, F)$, $M(C, D, E)$,

$M(A, B, F)$, $M(A, B, D)$, $M(A, D, F)$,

$M(B, C, F)$, $M(B, C, D)$, $M(C, D, F)$,
\[ M(B, E, F), M(B, D, E), M(D, E, F), \\
M(A, C, E), M(B, D, F). \]

Now we are ready to analyze which kind of textures have three different and positive eigenvalues, applying in each case one of the theorems presented in previous sections.

\section*{V. COMBINED ANALYSIS}

The aim of this section is give to quark mass matrices the structure of zero textures and find which of these structures have real, positive and different eigenvalues. In order to start the analysis in a systematic way, we need to implement another sub-classification, which depends on whether the matrix has or not zeros in the main diagonal, doing this, first we analyze the 3-zeros textures, after this, we study the 2-zero textures and finally the 1-zeros textures.

\subsection*{A. 3-zero analysis}

According the sub-classification given above, the 3-zero textures present the following cases:

1. Without zeros in the main diagonal there is one case \( M(B, D, F) \).

2. With one zero in the main diagonal exist 9 cases: \( M(A, B, F) \), \( M(A, B, D) \), \( M(A, D, F) \), \( M(B, C, F) \), \( M(B, C, D) \), \( M(C, D, F) \), \( M(B, E, F) \), \( M(B, D, E) \), \( M(D, E, F) \).

3. With two zeros in the main diagonal there are 9 cases: \( M(A, B, C) \), \( M(A, C, F) \), \( M(A, C, D) \), \( M(A, B, E) \), \( M(A, E, F) \), \( M(A, D, E) \), \( M(B, C, E) \), \( M(C, E, F) \), \( M(C, D, E) \).

4. With three zeros in the main diagonal we have only 1 case (\( M(A, C, E) \)).

We obtain a total of 20 different possibilities. We only present the analysis of the following three cases.

- Applying the Theorem \( \mathbb{E} \) \( \mathbb{D} \) the trivial \( M(A, C, E) \) case is ruled out\(^3\)

\(^3\) In this work, we are looking for textures with positive and different eigenvalues, therefore, textures with two equal eigenvalues or one of them negative, we say that, they are ruled out.
Now, we analyze the Fritzsch 6-zero texture given by $M(C, E, F)$ \[1\]. Applying again the Theorem 2 (1b) we must have $\text{tr} \, M(C, E, F) = A > 0$, from the condition (1a) $\text{det} \, M(C, E, F) = -D^2 A < 0$ that is a contradiction, because of that this 6-zero texture is ruled out.

Next, we analyze the following texture $M(A, D, F)$. The condition (1b) of the Theorem 2 we have that $\text{tr} \, M(A, D, F) = C + E > 0$ and from (1a) $\text{det} \, M(A, D, F) = -EB^2 > 0$, $\iff E < 0 \Rightarrow C > 0 \Rightarrow EC < 0$. The condition (1c) of the Theorem 2 implies that $0 < E^2 + C^2 + 2B^2 < E^2 + C^2 + 2EC \Rightarrow 0 < EC$ and we have a contradiction and this texture is also ruled out.

We have analyzed the others 17 cases and we found that the only case that is not excluded is $M(B, D, F)$, obviously being $A$, $B$ and $C$ the eigenvalues ($A \neq C \neq E > 0$).

B. 2-zero analysis

These kind of textures have the following cases:

1. Without zeros in the main diagonal there are 3 cases: $M(B, F)$, $M(B, D)$, $M(D, F)$.

2. With one zero in the main diagonal exist 9 cases: $M(A, B)$, $M(A, D)$, $M(A, F)$, $M(B, C)$, $M(C, D)$, $M(C, F)$, $M(B, E)$, $M(D, E)$, $M(E, F)$

3. With two zeros in the main diagonal there are 3 cases: $M(A, E)$, $M(A, C)$, $M(C, E)$.

We present the analysis of some cases more representative:

- We start with the matrix $M(C, E)$. If we compute $\text{tr}^2 \, M(C, E)$, $\text{tr} \, M(C, E)^2$ and we apply the condition (1c) of the Theorem 2 we obtain:

$$2(D^2 + F^2 + B^2) + A^2 < A^2,$$

that is a contradiction. We have found that $M(A, E)$ and $M(A, C)$ are ruled out too.

- The second example is the Fritzsch 4-zero texture given by $M(E, F)$ \[6\]. From the Theorem 2 follows that the condition (1a) $\text{det} \, M(E, F) = -AD^2 > 0$ implies $A < 0$, and of the condition (1b) $\text{tr} \, M(E, F) = C + A > 0$ we have that $C > 0$ and then
AC < 0. Now we compute tr² M(C, E), tr M(C, E)² and using the condition (1c) of the Theorem 2 we obtain:

\[ 2(D^2 + B^2) + C^2 + A^2 < C^2 + A^2 + 2AC, \]

then AC > 0, that is a contradiction.

We have analyzed the eight cases M(A, B), M(A, D), M(A, F), M(B, C), M(C, D), M(C, F), M(B, E), M(D, E) and we found that are ruled out.

The cases that are in agreement with the condition (1) of the Theorem 2 are M(B, F), M(B, D) and M(D, F), this means that, it exist a range of values of (B, F), (B, D) and (D, F) where these textures have real, positive and different eigenvalues.

C. 1-zero analysis

Here we only have two cases,

1. Without zeros in the main diagonal belong three different possibilities M(B), M(D) and M(F).

2. With one zero in the main diagonal also belong three different possibilities M(A), M(C) and M(E).

We only present the analysis of M(A). The condition (1b) produces \( E + C > 0 \), the condition (1a) implies that \( 2BDF - B^2E - F^2C > 0 \) and the condition (1c) gives \( 2(B^2 + D^2 + F^2) + E^2 + C^2 < E^2 + C^2 + 2EC \), the last three inequalities are equivalents with

\[ E + C > 0, \quad (7) \]
\[ 2BDF > B^2E + F^2C, \quad (8) \]
\[ 0 < B^2 + D^2 + F^2 < EC, \quad (9) \]

from (7) and (9) we have that \( E > 0 \) and \( C > 0 \), therefore

\[ -2BF\sqrt{EC} < B^2E + F^2C, \quad (10) \]
\[ 2BF\sqrt{EC} < B^2E + F^2C, \quad (11) \]
now if $BF > 0$, the inequalities (9, 11, 8) produce the following chain of inequalities

$$2BF\sqrt{B^2 + D^2 + F^2} < 2BF\sqrt{EC} < B^2E + F^2C < 2BDF,$$

and then

$$\sqrt{B^2 + D^2 + F^2} < D,$$

that is a contradiction. If $BF < 0$ use (10). We have analyzed the other 2 cases $M(C)$, $M(E)$ and we found that are ruled out.

The cases that are in agreement with the condition (1) of the Theorem 2 are $M(B)$, $M(D)$ and $M(F)$.

Summing up this section, the zero texture mass matrices that they have real, positive and different eigenvalues are:

$$M(B, F), M(B, D), M(D, F), M(B), M(D) and M(F).$$

Our results are in agreement with [5], where the authors using Weak Basic Transformations they have shown that any symmetric texture with (1,1) zero entry has at least one negative eigenvalue.

VI. $V_{ckm}$ PROPERTIES

Another important quantity that any quark mass matrices need to satisfied it is reproduce the experimental values of the $V_{ckm}$ for this reason, in this section we analyze the $V_{ckm}$ phenomenology, in the first part and considering a set of four zeros for mass matrices, we note the presence of zeros in the $V_{ckm}$ that depend if we have a parallel and non parallel structures in the quark mass matrices, in the second part we choose a particular non parallel case and compute the $V_{ckm}$ matrix. In order to fit this $V_{ckm}$ matrix with the experimental $V_{ckm}$ matrix we introduce a perturbation analysis. Finally we present a set of three zeros where the $V_{ckm}$ fits numerically.

A. $V_{ckm}$ from 4-zero texture set

In the previous sections it was shown that $M(B, F)$, $M(B, D)$ and $M(D, F)$ are matrices with simple, real and different eigenvalues. When the mass matrix of u-type quarks and the
mass matrix of and d-type quarks both have a parallel structure (e.g. \( M_u = M_u(B_u, F_u) \) and \( M_d = M_d(B_d, F_d) \)), one direct implication is that the \( V_{CKM} \) has the same texture structure as the mass matrices \( V_{CKM} = V_{CKM}(B_{CKM}, F_{CKM}) \) and we cannot reproduce the experimental values of the \( V_{CKM} \) elements because of that, all these three cases are ruled out.

Now, if the mass matrix of u-type quarks and the mass matrix of and d-type quarks have not a parallel structure, all nine cases were analyzed and always we find one zero element (off main diagonal) in the \( V_{CKM} \) matrix. We present the case where the best fit of the \( V_{CKM} \) is found, this is because we can obtain analytic expressions as well as a lot of information about the mass matrices. For this, we choose the mass matrix \( M(D, F) \) texture for u-type quarks, and the matrix \( M(B, F) \) texture for d-type quarks. Then we have that

\[
M_u = \begin{pmatrix}
m_u & 0 & 0 \\
0 & C_u & B \\
0 & B & A_u
\end{pmatrix}, \quad M_d = \begin{pmatrix}
E_d & D & 0 \\
D & C_d & 0 \\
0 & 0 & m_b
\end{pmatrix}.
\]

From the appendix \[B3\] and \[B4\], the above matrices take the form:

\[
M_u = \begin{pmatrix}
m_u & 0 & 0 \\
0 & \mu_{ct} + \sqrt{\delta_{tc}^2 - B^2} & B \\
0 & B & \mu_{ct} - \sqrt{\delta_{tc}^2 - B^2}
\end{pmatrix},
\]

\[
M_d = \begin{pmatrix}
\mu_{ds} + \sqrt{\delta_{sd}^2 - D^2} & D & 0 \\
D & \mu_{ds} - \sqrt{\delta_{sd}^2 - D^2} & 0 \\
0 & 0 & m_b
\end{pmatrix},
\]

where \( \mu_{qi,qj} = \frac{m_{qi} + m_{qj}}{2} \) and \( \delta_{qi,qj} = \frac{m_{qi} - m_{qj}}{2} \) (with \( m_{qi} > m_{qj} \), \( i, j = 1, 2, 3 \) and \( q = u, d \)).

The quantities \( \mu_{qi,qj} \) and \( \delta_{qi,qj} \) have a interesting physical meaning, the first one is the average mass, and for the second one we can rewriting as \( 2\delta_{qi,qj} + m_{qj} = m_{qi} \), then \( 2\delta_{qi,qj} \) is the quantity that distinguishes the masses, i.e. the particles \( m_{qi} \) and \( m_{qj} \) are different because their mass are different and the factor of difference is \( 2\delta_{qi,qj} \). The matrices that diagonalize the mass matrices are

\[
U_u = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \beta & \sin \beta \\
0 & -\sin \beta & \cos \beta
\end{pmatrix}, \quad U_d = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (12)
\]
where \( \sin \alpha = \frac{D}{\sqrt{D^2 + (y_d - m_d)^2}} \) and \( \sin \beta = \frac{B}{\sqrt{B^2 + (y_u - m_c)^2}} \).

Now we computing the \( V_{ckm} = U^T U_d \) matrix

\[
V_{ckm} = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
- \cos \beta \sin \alpha & \cos \beta \cos \alpha & - \sin \beta \\
- \sin \beta \sin \alpha & \sin \beta \cos \alpha & \cos \beta
\end{pmatrix} .
\] (13)

Setting:

\[
\sin \alpha = V_{us} = \lambda, \quad \sin \beta = -V_{cb} = -A \lambda^2 ,
\] (14)

where \( \lambda \) is the Wolfenstein parameter and \( A \) is a real number of order one.

The \( V_{ckm} \) matrix takes the form:

\[
V_{ckm} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & 0 \\
-\lambda & 1 - \frac{\lambda^2}{2} & A \lambda^2 \\
A \lambda^3 & -A \lambda^2 & 1
\end{pmatrix} + O(\lambda^4).
\]

With this election of texture structure of the mass matrices of quarks, we can reproduce (in Wolfenstein parametrization) eight \( V_{ckm} \) parameters and the \( (V_{ckm})_{13} \) element is zero. Now, with this information we can know explicitly each element of the mass matrices, from (14) we have that

\[
\sin \alpha = V_{us} = \lambda, \quad \sin \beta = -V_{cb} = -A \lambda^2 ,
\] (14)

the solutions for \( D \) and \( B \) are:

\[
D_0 = \pm 2 \delta_{sd} V_{us} \sqrt{1 - V_{us}^2} \quad \approx \pm 2 \delta_{sd} V_{us} ,
\] (17)

\[
B_0 = \pm 2 \delta_{tc} V_{cb} \sqrt{1 - V_{cb}^2} \quad \approx \pm 2 \delta_{tc} V_{cb} ,
\] (18)

and the mass matrices are:

\[
M_u = \begin{pmatrix}
m_u & 0 & 0 \\
0 & m_c + 2 \delta_{tc} V_{cb}^2 & \pm 2 \delta_{tc} V_{cb} \\
0 & \pm 2 \delta_{tc} V_{cb} & m_t - 2 \delta_{tc} V_{cb}^2
\end{pmatrix}, \quad M_d = \begin{pmatrix}
m_d + 2 \delta_{sd} V_{us}^2 & \pm 2 \delta_{sd} V_{us} & 0 \\
\pm 2 \delta_{sd} V_{us} & m_s - 2 \delta_{sd} V_{us}^2 & 0 \\
0 & 0 & m_b
\end{pmatrix} .
\] (19)
Finally the mass matrices can be written as:

\[ M_u = \bar{M}_u + 2\delta_{tc} V_{cb}^2 \Delta M_u \pm 2\delta_{tc} V_{cb} \delta M_u, \quad (20) \]
\[ M_d = \bar{M}_d + 2\delta_{sd} V_{us}^2 \Delta M_d \pm 2\delta_{sd} V_{us} \delta M_d, \quad (21) \]

where the matrices \( \Delta M_u, \Delta M_d, \delta M_u \) and \( \delta M_d \) are given by:

\[
\begin{align*}
\Delta M_u & = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \Delta M_d & = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \delta M_u & = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \delta M_d & = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

We observe that the mass matrices have three contributions; the first one (\( \bar{M} \)) comes from a diagonal matrix, where its elements correspond to mass quarks, the second contribution (\( \Delta M \)) is a correction of diagonal entries and it is characterized by the square of \((V_{ckm})_{12}\) and \((V_{ckm})_{13}\) elements respectively. The last contribution (\( \delta M \)) is off-diagonal correction characterized by the \((V_{ckm})_{12}\) and \((V_{ckm})_{13}\) elements. Note that: off diagonal contribution is bigger than the diagonal ones.

From [13] we can see that, we get one zero in \((V_{ckm})_{13}\) element, the experimental value for this element is around 0.00351, this invite us to apply perturbation theory to 4-zero texture (especially in the example presented above) in order to remove this zero and get a better approximation for this \(V_{ckm}\) element.

1. Perturbative analysis of 4-zero texture set

As we saw in previous section, when we consider a 4-zero texture set as structure of mass matrices of quarks, the presence of zeros in the \(V_{ckm}\) matrix is unavoidable, the aim of this part of the paper is use perturbation theory to remove these zeros and get small quantities.

We consider that quark mass matrices can be divide in two parts:

\[ M_q = M_q(2T) + \epsilon N_q, \quad (22) \]

where \( M_q(2T) \) is a 2-zero texture, \( N_q \) is known mass matrix and \( \epsilon \) a small parameter (See appendix for more details). The new contributions to \(V_{ckm}\) matrix comes from a antisymmetric matrix \( X_q \).
In the example presented before, where $M_u = M_u(D_u, F_u)$ and $M_d = M_d(B_d, F_d)$ are the structures for the quark mass matrices, one can reproduce eight experimental values of $V_{ckm}$ elements and one get that the $(V_{ckm})_{13}$ element is zero. To remove this zero first we consider a perturbation on $M_u = M_u(D_u, F_u)$ and keeping $M_d = M_d(B_d, F_d)$ unchanged, after that, we will interchange the roles.

We consider that, $M'_u$ mass matrix differs a small quantity $^4 \epsilon$ from $M_u$ in the positions $(1,2)$, $(2,1)$, $(1,3)$ and $(3,1)$.

$$
M'_u = \begin{pmatrix}
M_u & \epsilon a_u & \epsilon b_u \\
\epsilon a_u & C_u & B \\
\epsilon b_u & B & A_u
\end{pmatrix},
$$

where $\epsilon$ is a real parameter in the interval $0 \leq \epsilon \leq 1$ and $a_u$, $b_u$ are parameters with mass units. $M'_u$ matrix can be written in the form

$$
M'_u = M_u(D_u, F_u) + \epsilon N_u,
$$

where $M_u(D_u, F_u)$ matrix is given in (19) and $N_u$ matrix is given by

$$
N_u = \begin{pmatrix}
0 & a_u & b_u \\
a_u & 0 & 0 \\
b_u & 0 & 0
\end{pmatrix}.
$$

Following the analysis given in the appendix and applying right perturbation at first order in $\epsilon$, we find

$$
O_u = U_u(1 + \epsilon X_u),
$$

where the $U_u$ matrix is given in (12) and $X_u$ matrix is:

$$
X_u = \begin{pmatrix}
0 & x_{1u} & x_{2u} \\
x_{1u} & 0 & x_{3u} \\
x_{2u} & x_{3u} & 0
\end{pmatrix},
$$

and its elements are: $x_{1u} = \frac{a_u \cos \beta}{m_c - m_u} - \frac{b_u \sin \beta}{m_c - m_u}$, $x_{2u} = \frac{a_u \sin \beta}{m_t - m_u} + \frac{b_u \cos \beta}{m_t - m_u}$, and $x_{3u} = 0$.

$^4 |\epsilon a_u| \sim |\epsilon b_u| \ll |C_u|, |A_u|, |B|, m_u$
The new $V'_{ckm}$ matrix takes the following form:

$$V'_{ckm} = O_u^T U_d,$$

$$= (1 - \epsilon X_u)U_u^T U_d,$$  \hfill (24)  

$$= (1 - \epsilon X_u)V_{ckm}. \hfill (25)$$

After some algebra, using (16) and considering $m_t > m_c >> m_u$, we get that, the element $(V'_{ckm})_{13}$ has the form:

$$(V'_{ckm})_{13} = \left(\frac{V_{cb}}{m_c} \frac{1}{m_t}\right) \epsilon b_u - \left(\frac{V_{cb}}{m_c}\right) \epsilon a_u. \hfill (26)$$

We have non zero element, which its magnitude depend on $V_{cb}$ and the perturbation parameters. The numerical contribution from $b_u$ goes like $10^{-6}$, while the numerical contribution from $a_u$ goes like $10^{-5}$. The smallest numerical element of $M_u(D_u, F_u)$ matrix is $m_u$, then we consider that the maximum value of the perturbation is $m_u/10$. We scanned all allowed range of $\epsilon a_u$ and $\epsilon b_u$ parameters and we get that the best numerical absolute value is $8 \times 10^{-6}$. For left and left-right perturbations (See Appendix), the numerical values were the same order. The absolute values of new $V'_{ckm}$ elements are:

$$|V'_{ckm}| = \begin{pmatrix} 0.9753 & 0.2208 & 8 \times 10^{-6} \\ 0.2206 & 0.9745 & 0.039 \\ 0.0086 & 0.0380 & 0.9992 \end{pmatrix}.$$

The values of the mass matrix parameters of $M'_u$ were: $|\epsilon a_u| = |\epsilon b_u| = 0.2 \sim \frac{|m_u|}{10} \ll, m_u = 2.3$, $|A_u| = 172739$, $|C_u| = 1531.2$, $|D| = 6697.47$, all quantities in MeV. The numerical values that corresponding to second order in $\epsilon$ are $O(10^{-8})$ or less.

Now we consider that $M'_d$ mass matrix differs a small quantity $^5 \epsilon a_d, \epsilon b_d$ from $M_d$ in the positions $(1, 3), (3, 1), (3, 2)$ and $(2, 3)$.

$$M'_d = \begin{pmatrix} E_d & D_0 & \epsilon a_d \\ D_0 & C_d & \epsilon b_d \\ \epsilon a_d & \epsilon b_d & m_b \end{pmatrix},$$

$M'_d$ matrix can be written in the form

$$M'_d = M_d(B_d, F_d) + \epsilon N_d,$$

$^5 |\epsilon a_d| \sim |\epsilon b_d| \ll |E_d|, |C_d|, |D|, m_b$
where $M_d(B_d, F_d)$ matrix is given in (19) and $N_d$ matrix is given by

$$N_d = \begin{pmatrix} 0 & 0 & a_d \\ 0 & 0 & b_d \\ a_d & b_d & 0 \end{pmatrix}.$$  

Applying right perturbation at first order in $\epsilon$, we find

$$O_d = U_d(1 + \epsilon X_d),$$  

where the matrix $X_d$ is:

$$X_d = \begin{pmatrix} 0 & x_{1d} & x_{2d} \\ -x_{1d} & 0 & x_{3d} \\ -x_{2d} & -x_{3d} & 0 \end{pmatrix},$$

and its elements are $x_{1d} = 0$, $x_{2d} = \frac{a_d \cos \alpha}{m_b - m_d} - \frac{b_d \sin \alpha}{m_b - m_d}$ and $x_{3d} = \frac{a_d \sin \alpha}{m_b - m_s} + \frac{b_d \cos \alpha}{m_b - m_s}$.

The new $V'_{ckm}$ matrix takes the following form:

$$V'_{ckm} = V_u^T O_d,$$

$$= V_u^T V_d(1 + \epsilon X_d),$$

$$= V_{ckm}(1 + \epsilon X_d).$$  

(27)

(28)

(29)

After some algebra, using (19) and considering $m_t >> m_c >> m_u$, we get that the element $(V'_{ckm})_{13}$ has the form:

$$(V'_{ckm})_{13} = V_{us} \left( \frac{m_s}{m_b^2} \right) \epsilon b_d + \left( \frac{1}{m_b} \right) \epsilon a_d.$$  

(30)

We have non zero element, which its magnitude depend on $V_{us}$ and the perturbation parameters. The numerical contribution from $b_d$ goes like $10^{-6}$, while the numerical contribution from $a_d$ goes like $10^{-4}$. The smallest numerical element from matrix $M_d(B_d, F_d)$ is $E_d$, then we consider that the maximum value of the perturbation is $E_d/10$. We scanned all allowed range of $\epsilon a_d$ and $\epsilon b_d$ parameters and we get that, the best numerical absolute value is $2 \times 10^{-4}$. For left and left-right perturbations, the numerical values were the same order of magnitude. The absolute values of new $V'_{ckm}$ elements are:

$$|V_{ckm}| = \begin{pmatrix} 0.9742 & 0.2253 & 0.0002 \\ 0.2251 & 0.9734 & 0.0406 \\ 0.0089 & 0.0396 & 0.9991 \end{pmatrix}.$$
The values of the parameters were: $|\epsilon a_d| = |\epsilon b_d| = 0.9 \sim |E_d| \ll |E_d| = 9.37$, $|C_d| = 90.42$, $|D| = 20.32$, $m_b = 4180$, all quantities in MeV. The numerical values that corresponding to second order in $\epsilon$ are $O(10^{-8})$ or less.

Also we have numerically analyzed all possibilities to get a perturbation on both mass matrices without get better numerically values in the $V_{ckm}$ matrix.

From the analysis of this section, we conclude that 4-zero texture set in the normal and perturbative cases are ruled out, because they can not reproduce the experimental values of the $V_{ckm}$ matrix.

**B. $V_{ckm}$ from 3-zero texture set**

The next case of structure is a 3-zero texture set, which it born when one type of quarks has as mass matrix $M(B, F)$, $M(B, D)$ or $M(D, F)$ and the other type of quarks has mass matrix $M(B)$, $M(D)$ or $M(F)$. We have in total 18 possible combinations$^6$.

From the analysis presented before, we can point out two issues:

- We can introduce a $(V_{ckm})_{13}$ element different from zero, setting in appropriate way the values (1,3) and (3,1) in $M_d$ matrix. From (30), we can note a lineal dependence between $(V_{ckm})_{13}$ and the perturbation, $\epsilon a_d$ and if $|\epsilon a_d| \sim E_d$ we obtain the numerical value of $(V_{ckm})_{13}$ very close that the experimental one. Then we will consider that $F_d$ is the same order than $E_d$.

- The quark mass matrix can be split in two parts, a diagonal part plus off-diagonal contributions, which both of them are in power series of $V_{us}$ and $V_{cb}$ elements.

Considering the above statements, we take $M(D, F)$ as 2-zero structure for u-type quarks, i.e it has the form given in (20) and the matrix that diagonalize it is (12). For d-quarks we take $M(B)$ as 1-zero structure given by

$$
M_d = \begin{pmatrix}
E_d & D_d & F_d \\
D_d & C_d & 0 \\
F_d & 0 & A_d
\end{pmatrix},
$$

$^6$ We shall discuss these kind of textures in a forthcoming paper
where each element is parameterized as:

\[
\begin{align*}
A_d &= m_b + x V_{us}^3, \\
C_d &= m_s - 2\delta_{sd} V_{us}^2 + y V_{us}^3, \\
E_d &= m_d + 2\delta_{sd} V_{us}^2 + z V_{us}^3, \\
D_d &= +2\delta_{sd} V_{us}, \\
F_d &= m_d + 2\delta_{sd} V_{us}^2,
\end{align*}
\]

where \((x, y, z)\) are variables to find. Now as \(M_d\) is congruent with \(\text{Diag} [m_d, m_s, m_b]\) we can write the following equations:

\[
\begin{align*}
Tr M_d &= m_d + m_s + m_b, \\
det M_d &= m_d m_s m_b, \\
\frac{1}{2} \left[\text{tr}^2 M_d - \text{tr} M_d^2\right] &= m_d m_s + m_d m_b + m_s m_b,
\end{align*}
\]

(31)

this set of equations has six solutions for \((x, y, z)\), and we choose the solution that \(E_d < C_d < A_d\) is hold, i.e. the numerical values for \((x, y, z)\) are \((1.84199, x + z, -1.84417)\), then numerically the matrix \(M_d\) results

\[
M_d = \begin{pmatrix}
9.42827 & 19.802896 & 9.380186 \\
19.802896 & 90.1575 & 0 \\
9.380186 & 0 & 4180.21
\end{pmatrix},
\]

and the numerical absolute values of \(V_{ckm}\) elements are:

\[
|V_{ckm}| = \begin{pmatrix}
0.974118 & 0.226027 & 0.00224906 \\
0.225925 & 0.973273 & 0.0412108 \\
0.00712578 & 0.0406523 & 0.999148
\end{pmatrix},
\]

that is in agreement with the experimental value of \(V_{ckm}\) matrix.

This is a good example that shows that 3-zero texture sets are viable candidates to model the quark mass matrices.

VII. CONCLUSIONS

In this paper, in the understanding that by definition the physical mass of the quarks and charged leptons are the eigenvalues of the mass matrices. We found the necessary and sufficient conditions over the characteristic polynomial coefficients from any symmetric 3
by 3 matrix, so that it has real, simple and positive roots. We apply this formalism to analyze the symmetric texture-zero quark matrices, we found that a lot of them are ruled out (i.e. they have two equal eigenvalues or one of them negative). Only the zero texture matrices \( M(B, F) \), \( M(B, D) \), \( M(D, F) \), \( M(B) \), \( M(D) \) and \( M(F) \) are in agreement with this condition. In the texture-zero formalism, the matrices have variable coefficients, the conditions 2 and 3 impose restrictions over these coefficients, this means that, we need to find the complete domain of the coefficients in both mass matrices, u-type quarks and d-type quarks, in order to approximate the experimental values of the \( V_{ckm} \) matrix. We develop analytically the case of four zero sets, and we show the set that gives the best approximation to the \( V_{ckm} \) matrix and always a zero element in the theoretical \( V_{ckm} \) matrix is found, to remove this zero, we implement a perturbation method and we analyze the 4 zero texture set, even with these results the four-zero texture sets are ruled out. The quark mass matrix can be split in in two parts, a diagonal part plus off-diagonal contributions, which both of them are in power series of \( V_{ckm} \) elements, statement that is valid for three and four zero texture sets. With an example, we show that 3-zero texture sets are viable to model the quark mass matrices, and this structure in the minimal which satisfy that they have real, positive and different eigenvalues and also it reproduce the experimental values of the \( V_{ckm} \) matrix.

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Appendix A: Proof Theorem

In this appendix we proof the theorem 1 for this we need the following statement

Lemma 1 The polynomial of second degree \( p(\lambda) = \lambda^2 + a_1 \lambda + a_0 \) has two real, simple and positive roots if and only if the following condition hold.

\[
a_1 < 0 < a_0 < \frac{a_1^2}{4}.
\]
Proof. Follows from a simple computation. ■

Now we remember and proof the theorem [1]

Theorem 1 The polynomial of degree 3, \( p(\lambda) = \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 \) has three different, real and positive roots if and only if the following conditions over its coefficients \( a_0, a_1, a_2 \) hold.

1. \( a_0, a_2 < 0 < a_1 \).
2. \( 3a_1 < a_2^2 \).
3. If \( \lambda_4 = \frac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3} \) and \( \lambda_5 = \frac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3} \), then \( p(\lambda_4) < 0 \) and \( p(\lambda_5) > 0 \).

Proof. If exists three different \( \lambda_i \in \mathbb{R}^+ \), \( i = 1, 2, 3 \) such that \( p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 \), then by equality of polynomials we obtain

- \( a_2 = -(\lambda_1 + \lambda_2 + \lambda_3) < 0 \),
- \( a_0 = -\lambda_1\lambda_2\lambda_3 < 0 \),
- \( a_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0 \),

and the condition 1 hold.

Without loss of generality, we suppose that \( 0 < \lambda_1 < \lambda_2 < \lambda_3 \), if we have a polynomial of degree 3 with three real, simple roots then there exits two critical points, they are roots of the first derivative, i.e. the condition 2 hold. Now if the roots of polynomial are positive then the critical points are positive too and the follow chain of inequalities hold \( \lambda_1 < \lambda_5 < \lambda_2 < \lambda_4 < \lambda_3 \). From the coefficient of \( \lambda^3 \) is 1, we have that \( \lim_{\lambda \to \infty} p(\lambda) = \infty \) and \( \lim_{\lambda \to -\infty} p(\lambda) = -\infty \), then for points less than \( \lambda_1 \) the polynomial is negative, we applied the Rolle theorem to the roots \( \lambda_1 \) and \( \lambda_2 \), therefore the polynomial has a maximum value between \( \lambda_1 \) and \( \lambda_2 \), and therefore \( p(\lambda_5) > 0 \). Similarly for points greater than \( \lambda_3 \) the polynomial is positive and we applied the Rolle theorem to the roots \( \lambda_2 \) and \( \lambda_3 \), and the polynomial has a minimum, this value is negative i.e. \( p(\lambda_4) < 0 \).

\(^7\) In this work, a chain of inequalities \( a < b < c < \ldots \), the first inequality is \( a < b \), the second one is \( b < c \) and so on.
Conversely, we have that \( p(\lambda) = \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 \) such that the conditions 1, 2 and 3 hold. The conditions 1, 2 join to lemma \( \text{Lemma } 1 \) implies that \( p'(\lambda) \) has two real, simple and positive roots given by:

\[
\lambda_4 = \frac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3}, \quad \lambda_5 = \frac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3}.
\]

First we observe that \( 0 < \lambda_5 < \lambda_4 \), now computing \( p''(\lambda_4) = 2\sqrt{a_2^2 - 3a_1} > 0 \), this implies in \( \lambda_4 \) we have a minimum whereas \( p''(\lambda_5) = -2\sqrt{a_2^2 - 3a_1} < 0 \) and then in \( \lambda_5 \) we have a maximum. We applied repeatedly the Intermediate Value theorem. From 1, we have that \( p(0) = a_0 < 0 \) and from 3 it follows that \( p(\lambda_5) > 0 \), then we have a positive root. The condition 3, \( p(\lambda_4) < 0 \) and \( p(\lambda_5) > 0 \), guarantee that exits a second root between \( \lambda_5 \) and \( \lambda_4 \), finally due to the coefficient to \( \lambda^3 \) is positive \( p(\lambda) \) we have that \( \lim_{\lambda \to \infty} p(\lambda) = \infty \) and this implies \( p(\lambda_4) < 0 \), then \( p(\lambda) \) intersects to horizontal axis one more time in the third root. 

Appendix B: 2-zero textures

In the above sections we show that the matrices \( M(B, F) \), \( M(B, D) \) and \( M(D, F) \) can have three positive, real and different eigenvalues, these matrices are diagonal by blocks (one block \( 1 \times 1 \) and other block \( 2 \times 2 \)). They can be diagonalize by matrices that are also diagonal by blocks.

Pay attention only in the \( 2 \times 2 \) block. The mass matrix can be rewritten as:

\[
M_{2\times2} = \begin{pmatrix}
y & K \\
K & x
\end{pmatrix}, \quad (K = B, D, F).
\]

This matrix has to be congruent with

\[
\bar{M}_{2\times2} = \begin{pmatrix}
m_i & 0 \\
0 & m_j
\end{pmatrix}, \quad (i, j) = (1, 2), (2, 3), (1, 3).
\]

This implies the following relations among their elements:

\[
x + y = m_i + m_j, \quad (B1)
\]

\[
xy - K^2 = m_i m_j, \quad (B2)
\]
the solutions for $x$ and $y$ are:

$$x(K) = \mu_{ij} \pm \sqrt{\delta_{ij}^2 - K^2},$$  
(B3)

$$y(K) = \mu_{ij} \pm \sqrt{\delta_{ij}^2 - K^2},$$  
(B4)

where $\mu_{ij} = \frac{m_i + m_j}{2}$, if $m_i > m_j$, $\delta_{ij} = \frac{m_i - m_j}{2}$ and the parameter $K$ has to satisfy $|K| \leq \delta_{ij}$.

The matrix that diagonalize the matrix $M_{2 \times 2}$ always can be set as:

$$\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix},$$

where: $\sin \theta = \frac{K}{\sqrt{K^2 + (y - m_i)^2}}$ and $\theta \in [0, \pi/4]$.

**Appendix C: Perturbation Theory**

In this appendix we applied perturbation theory to texture formalism.

We start dividing the complete mass matrix in two parts

$$M = M_0 + \epsilon N,$$
(C1)

where $M_0$ and $N$ are known mass matrices and $\epsilon$ is a small parameter. We look for a $O$ matrix that diagonalize the $M$ matrix in the following way:

$$O^T M O = \bar{M},$$
(C2)

where $\bar{M}$ is a diagonal matrix.

We have three different versions of the perturbation method according to the way that $O$ matrix is proposed, namely:

1. **Right Perturbation**, when the $O$ matrix takes the form

$$O = O_0(1 + \epsilon X).$$
(C3)

2. **Left Perturbation**, when the matrix $O$ takes the form

$$O = (1 + \epsilon X) O_0.$$
(C4)

---

8 The first work in this direction was [7] and it applies non-hermitian perturbations to the 6-zero texture
3. **Left-Right Perturbation**, when the matrix $O$ takes the form
\[
O = (1 + \epsilon X) O_0 (1 + \epsilon X). \tag{C5}
\]

Where the $O_0$ matrix diagonalize the $M_0$ matrix ($O_0^T M_0 O_0 = \bar{M}$) and the $X$ matrix is determined in this process.

From the orthogonality condition of the $O$ matrix it is find that $X$ is antisymmetric matrix $X^T = -X$ and $Y + Y^T = X^2$ for all cases.

Notation: We are considering $\tilde{A} = O_0^T A O_0$ for any matrix $A$.

1. **Right Perturbation**

Substituting the form the $O$ matrix (Eq. C3) into (Eq.C2):
\[
[O_0(1 + \epsilon X)]^T M [O_0(1 + \epsilon X)] = \bar{M}.
\]

After some algebra one gets, at first order in $\epsilon$ parameter, that the $X$ matrix has to satisfied:
\[
\bar{N} = [X, \bar{M}]. \tag{C6}
\]

At second order in $\epsilon$, the $Y$ matrix has to satisfied:
\[
\bar{N} X + X \bar{N} = [Y + Y^T, \bar{M}]. \tag{C7}
\]

2. **Left Perturbation**

Substituting the form the $O$ matrix (Eq. C4) into (Eq.C2):
\[
[(1 + \epsilon X)O_0]^T M [(1 + \epsilon X)O_0] = \bar{M}.
\]

After some algebra one gets, at first order in $\epsilon$ parameter, that the $\bar{X}$ matrix has to satisfied:
\[
\bar{N} = [\bar{X}, \bar{M}]. \tag{C8}
\]

At second order in $\epsilon$, the $\bar{Y}$ matrix has to satisfied:
\[
\bar{N} \bar{X} + \bar{X} \bar{N} = [\bar{Y} + \bar{Y}^T, \bar{M}]. \tag{C9}
\]
3. Left-Right Perturbation

Substituting the form the \( O \) matrix (Eq. C5) into (Eq.C2):

\[
[(1 + \epsilon X)O_0(1 + \epsilon X)]^T M [(1 + \epsilon X)O_0(1 + \epsilon X)] = \bar{M}.
\]

After some algebra one gets, at first order in \( \epsilon \) parameter, that the \( \bar{X} \) matrix has to satisfied:

\[
\bar{N} = [X + \bar{X}, \bar{M}] .
\]  

(C10)

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