A TRANSVERSALITY THEOREM FROM THE VIEWPOINT OF
HAUSDORFF MEASURES AND ITS APPLICATIONS

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Abstract. In this paper, we establish a transversality theorem from the viewpoint of Hausdorff measures. The transversality theorem is an improvement of transversality theorems from the viewpoint of Lebesgue measures such as the basic transversality result and its strengthening which was given by Mather. Moreover, we give applications of the transversality theorem from the viewpoint of singularity theory. Furthermore, we also give an application of singularity theory to multiobjective optimization.

1. Introduction

In this paper, unless otherwise stated, all manifolds are without boundary and assumed to have countable bases.

First, we give the definition of transversality.

Definition 1. Let $X$ and $Y$ be $C^r$ manifolds, and $Z$ be a $C^r$ submanifold of $Y$ ($r \geq 1$). Let $f : X \to Y$ be a $C^1$ mapping.

(1) We say that $f : X \to Y$ is transverse to $Z$ at $x$ if $f(x) \notin Z$ or in the case of $f(x) \in Z$, the following holds:

$$df_x(T_x X) + T_{f(x)} Z = T_{f(x)} Y.$$ 

(2) We say that $f : X \to Y$ is transverse to $Z$ if for any $x \in X$, the mapping $f$ is transverse to $Z$ at $x$.

Let $X$, $A$, and $Y$ be $C^r$ manifolds ($r \geq 1$) and $U$ be an open set of $X \times A$. In what follows, by $\pi_1 : U \to X$ and $\pi_2 : U \to A$, we denote the natural projections defined by

$$\pi_1(x, a) = x,$$
$$\pi_2(x, a) = a.$$ 

Let $F : U \to Y$ be a $C^1$ mapping. For any element $a \in \pi_2(U)$, let

$$F_a : \pi_1(U \cap (X \times \{ a \})) \to Y$$

be the mapping defined by $F_a(x) = F(x, a)$. Here, note that $\pi_1(U \cap (X \times \{ a \}))$ is open in $X$. Set

$$\Sigma(F, Z) = \{ a \in \pi_2(U) \mid F_a \text{ is not transverse to } Z \}.$$ 

The main purposes of this paper are to establish a transversality theorem from the viewpoint of Hausdorff measures (Theorem 3) and to give its applications.

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Some significant transversality results from the viewpoint of Lebesgue measures have been given so far (Proposition 1 and Theorems 1 and 2). The following basic transversality result lies at the heart of most application of transversality.

**Proposition 1** ([1]). Let $X$, $A$ and $Y$ be $C^\infty$ manifolds, $Z$ be a $C^\infty$ submanifold of $Y$ and $F : X \times A \to Y$ be a $C^\infty$ mapping. If $F$ is transverse to $Z$, then $\Sigma(F, Z)$ has Lebesgue measure zero in $A$.

In [12], an improvement of Proposition 1 is given by Mather (for the result, see Theorem 1). In order to state the result, we give the following definition.

**Definition 2.** Let $X$ and $Y$ be $C^r$ manifolds, and $Z$ be a $C^r$ submanifold of $Y$ ($r \geq 1$). Let $f : X \to Y$ be a $C^1$ mapping. For any $x \in X$, set

$$
\delta(f, x, Z) = \begin{cases} 
0 & \text{if } f(x) \notin Z, \\
\dim Y - \dim (df_x(T_xX) + T_{f(x)}Z) & \text{if } f(x) \in Z,
\end{cases}
$$

$$
\delta(f, Z) = \sup \{ \delta(f, x, Z) \mid x \in X \}.
$$

In the case that all manifolds and mappings are of class $C^\infty$, Definition 2 is the definition of [12, p. 230]. As in [1], $\delta(f, x, Z)$ measures the extent to which $f$ fails to be transverse to $Z$ at $x$. Since $\delta(f, Z) = 0$ if and only if $f$ is transverse to $Z$, the following result by Mather is a natural strengthening of Proposition 1.

**Theorem 1** ([12]). Let $X$, $A$ and $Y$ be $C^\infty$ manifolds, $Z$ be a $C^\infty$ submanifold of $Y$ and $F : X \times A \to Y$ be a $C^\infty$ mapping. If for any $(x, a) \in X \times A$, $\delta(F_a, x, Z) = 0$ or $\delta(F, (x, a), Z) < \delta(F_a, x, Z)$, then $\Sigma(F, Z)$ has Lebesgue measure zero in $A$.

Theorem 1 is an important transversality result for investigating global properties of differentiable mappings. For example, this result is an essential tool for the proofs of [12, Theorem 1] and [1, Theorem 2.2]. However, it is difficult to apply to mappings with elements $(x, a) \in X \times A$ satisfying $\delta(F, (x, a), Z) = \delta(F_a, x, Z) > 0$. On the other hand, in [9], an improvement of Theorem 1 is given, which can be applied in this case (see Theorem 2).

**Definition 3.** Let $X$, $A$ and $Y$ be $C^r$ manifolds, and $Z$ be a $C^r$ submanifold of $Y$ ($r \geq 1$). Let $F : U \to Y$ be a $C^1$ mapping, where $U$ is an open set of $X \times A$. Then, we define

$$
W(F, Z) = \{ (x, a) \in U \mid \delta(F_a, x, Z) = \delta(F, (x, a), Z) > 0 \},
$$

$$
\delta^*(F, Z) = \dim X + \dim Z - \dim Y + \delta(F, Z) = \dim X - \text{codim } Z + \delta(F, Z),
$$

where $\text{codim } Z = \dim Y - \dim Z$.

**Theorem 2** ([1]). Let $X$, $A$ and $Y$ be $C^r$ manifolds, $Z$ be a $C^r$ submanifold of $Y$ and $F : U \to Y$ be a $C^r$ mapping, where $U$ is an open set of $X \times A$. If $r > \max \{ \delta^*(F, Z), 0 \}$, then the following (α) and (β) are equivalent.

(α) The set $\pi_2(W(F, Z))$ has Lebesgue measure zero in $\pi_2(U)$.

(β) The set $\Sigma(F, Z)$ has Lebesgue measure zero in $\pi_2(U)$.

If a given $C^\infty$ mapping $F : X \times A \to Y$ satisfies the assumption of Theorem 1, then $W(F, Z) = \emptyset$. Thus, Theorem 2 implies Mather’s Theorem 1.

Although Proposition 1 and Theorems 1 and 2 are useful for investigating properties of generic mappings, it is difficult to estimate the Hausdorff dimension of the
bad set \( \Sigma(F, Z) \) by these results from the viewpoint of Lebesgue measures. Hence, as an improvement of Theorem 2 (and hence Proposition 1 and Theorem 1), we establish a transversality theorem from the viewpoint of Hausdorff measures (see Theorem 3). By the theorem, we can also estimate the Hausdorff dimension of \( \Sigma(F, Z) \). In what follows, we denote the image of a given mapping \( f \) by \( \text{Im} f \).

**Theorem 3.** Let \( X, A \) and \( Y \) be \( C^r \) manifolds, \( Z \) be a \( C^r \) submanifold of \( Y \) and \( F : U \to Y \) be a \( C^r \) mapping, where \( U \) is an open set of \( X \times A \) and \( r \) is a positive integer. Then, the following hold:

1. Suppose \( \delta^*(F, Z) \geq 0 \). Then, for any real number \( s \) satisfying
   \[
   s \geq \dim A - 1 + \frac{\delta^*(F, Z) + 1}{r},
   \]
   the following (a) and (\( \beta \)) are equivalent.
   
   (a) The set \( \pi_2(W(F, Z)) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(U) \).
   (\( \beta \)) The set \( \Sigma(F, Z) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(U) \).

2. Suppose \( \delta^*(F, Z) < 0 \). Then, the following hold:
   
   (2a) We have \( W(F, Z) = \emptyset \).
   (2b) For any non-negative real number \( s \) satisfying \( s > \dim A + \delta^*(F, Z) \), the set \( \Sigma(F, Z) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(U) \).
   (2c) For any \( a \in \pi_2(U) - \Sigma(F, Z) \), we have \( \text{Im} F_a \cap Z = \emptyset \).

Finally, we give some important remarks on Theorem 3.

**Remark 1.**

1. If \( r < \delta^*(F, Z) + 1 \), both (a) and (\( \beta \)) of Theorem 3 trivially hold since
   \[
   s \geq \dim A - 1 + \frac{\delta^*(F, Z) + 1}{r} > \dim A.
   \]
   Namely, in Theorem 3, note that the case \( r \geq \delta^*(F, Z) + 1 \) is essential.

2. We will show that Theorem 3 is an improvement of Theorem 2 (and hence Proposition 1 and Theorem 1).

   Let \( F : U \to Y \) be a \( C^r \) mapping satisfying the assumption \( r > \max \{ \delta^*(F, Z), 0 \} \) of Theorem 2.

   First, we consider the case \( \delta^*(F, Z) \geq 0 \). Since \( r \geq \delta^*(F, Z) + 1 \), we can set \( s = \dim A \) in (2.1). Since a subset of \( \pi_2(U) \) has \( (\dim A) \)-dimensional Hausdorff measure zero in \( \pi_2(U) \) if and only if the subset has Lebesgue measure zero in \( \pi_2(U) \), the assertion (a) (resp., (\( \beta \))) of Theorem 3 is the same as (a) (resp., (\( \beta \))) of Theorem 2. Thus, in the case \( \delta^*(F, Z) \geq 0 \), Theorem 3 implies Theorem 2.

   Next, we consider the case \( \delta^*(F, Z) < 0 \). Since \( W(F, Z) = \emptyset \) by Theorem 3, we obtain (a) of Theorem 2. By Theorem 3, \( \Sigma(F, Z) \) has \( (\dim A) \)-dimensional Hausdorff measure zero in \( \pi_2(U) \). Thus, \( \Sigma(F, Z) \) also has Lebesgue measure zero in \( \pi_2(U) \). Namely, we obtain (\( \beta \)) of Theorem 2.

3. There exists a \( C^r \) mapping such that (a) \( \Rightarrow \) (\( \beta \)) of Theorem 3 does not hold if a given non-negative real number \( s \) does not satisfy (1.1). Namely, in general, we cannot improve (1.1) (see Example 1 in Section 2).

4. In Theorem 3, if all manifolds and mappings are of class \( C^\infty \), then for any real number \( s \) such that \( s > \dim A - 1 \), there exists a positive integer \( r \) satisfying (1.1). Thus, in the \( C^\infty \) case, we can replace (1.1) by
   \[
   s > \dim A - 1.
   \]
Moreover, in the $C^\infty$ case, there exists an example such that $(\alpha) \Rightarrow (\beta)$ of Theorem 3 (1) does not hold if we replace $s > \dim A - 1$ by $s \geq \dim A - 1$ (see Example 2 in Section 2).

(5) As opposed to $(\alpha) \Rightarrow (\beta)$ of Theorem 3 (1), $(\beta) \Rightarrow (\alpha)$ always holds for any positive integer $r$ and any non-negative real number $s$ which do not necessarily satisfy (1.1) (see Lemma 6 in Section 4).

(6) There exists an example such that Theorem 3 (2) does not hold if we replace $s > \dim A + \delta^*(F, Z)$ by $s \geq \dim A + \delta^*(F, Z)$ (see Example 3 in Section 2).

(7) In the case $F(U) \cap Z = \emptyset$, there exists an example such that $\dim A + \delta^*(F, Z) < 0$ (see Example 4 in Section 2). Hence, in Theorem 3 (2), it is necessary to assume that $s$ is non-negative.

On the other hand, note that in the case $F(U) \cap Z \neq \emptyset$, we always have $\dim A + \delta^*(F, Z) \geq 0$ (see Lemma 3 in Section 3).

The remainder of this paper is organized as follows. In Section 2, we give some examples on Theorem 3 and Remark 1. In Section 3, some standard definitions are reviewed and some lemmas for the proof of Theorem 3 are prepared. Section 4 is devoted to the proof of Theorem 3. In Sections 5 and 6, Theorems 5 and 6 are shown, respectively. Furthermore, in Section 8, we give some applications of Theorems 5 and 6. Finally, Section 9 provides an application of singularity theory to multiobjective optimization (see Theorem 9), and we prove Theorem 9 in Section 10.

2. Some examples on Theorem 3 and Remark 1

First, as mentioned in Remark 1 (3), (4) and (6), we give Examples 1 to 3, respectively.

Example 1 (An example of Theorem 3 (1)). As in [2, Example 4.2], there exists a $C^r$ mapping $\eta = (\eta_1, \ldots, \eta_n) : \mathbb{R}^n \to \mathbb{R}^n$ such that the Hausdorff dimension of the set consisting of all critical values of $\eta$ is $n - 1 + \frac{1}{r}$, where $r$ is a positive integer (for the definition of critical values, see Section 3).

Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the mapping defined by

$$F(x, a) = (a_1 - \eta_1(x), \ldots, a_n - \eta_n(x)),$$

with $Z = \{ (0, \ldots, 0) \} \subset \mathbb{R}^n$, where $a = (a_1, \ldots, a_n)$. Since $\delta(F, Z) = 0$, we have $\delta^*(F, Z) = 0$. Thus, by Theorem 3 (1), for any real number $s$ satisfying

$$s \geq n - 1 + \frac{1}{r},$$

the following $(\alpha)$ and $(\beta)$ are equivalent.

$(\alpha)$ The set $\pi_2(W(F, Z))$ has $s$-dimensional Hausdorff measure zero in $\mathbb{R}^n$.

$(\beta)$ The set $\Sigma(F, Z)$ has $s$-dimensional Hausdorff measure zero in $\mathbb{R}^n$.

Since $\pi_2(W(F, Z)) = \emptyset$, we have $(\alpha)$ for any $s \geq 0$. Hence, we obtain $(\beta)$ for any $s$ satisfying (2.1).

On the other hand, $(\beta)$ does not hold in the case $0 \leq s < n - 1 + \frac{1}{r}$ as follows. Since $a \in \Sigma(F, Z)$ if and only if there exists $x \in \mathbb{R}^n$ such that $F_a(x) = (0, \ldots, 0)$ (i.e., $a = \eta(x)$) and rank $d\eta_x < n$, it follows that $\Sigma(F, Z)$ is equal to the set consisting of all critical values of $\eta$. Therefore, the Hausdorff dimension of $\Sigma(F, Z)$ is $n - 1 + \frac{1}{r}$. 
Thus, in the case $0 \leq s < n - 1 + \frac{1}{r}$, (β) does not hold. Namely, we cannot improve the assumption (2.1).

**Example 2** (An example of Theorem 3 (1)). Let $F : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by

$$F(x, a) = (0, a_1^2 - a_2^2),$$

where $a = (a_1, a_2)$. Set $Z = \{(0, 0)\} (\subset \mathbb{R}^2)$.

Since

$$JF(x,a) = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 2a_1 & -2a_2 \end{pmatrix},$$

we have $\delta(F, Z) = 2$ and $\delta^*(F, Z) = n \geq 0$. Since $F$ is of class $C^\infty$, by Theorem 3 (1) (and Remark 4), for any real number $s$ satisfying $s > \dim \mathbb{R}^2 - 1 = 1$,

the following (α) and (β) are equivalent.

(α) The set $\pi_2(W(F, Z))$ has $s$-dimensional Hausdorff measure zero in $\mathbb{R}^2$.

(β) The set $\Sigma(F, Z)$ has $s$-dimensional Hausdorff measure zero in $\mathbb{R}^2$.

Since

$$\delta(F, (x, a), Z) = \begin{cases} 0 & \text{if } a_1^2 - a_2^2 \neq 0, \\ 1 & \text{if } a_1^2 - a_2^2 = 0 \text{ and } a_1 \neq 0, \\ 2 & \text{if } a_1 = a_2 = 0, \end{cases}$$

and

$$\delta(F, a, x, Z) = \begin{cases} 0 & \text{if } a_1^2 - a_2^2 \neq 0, \\ 2 & \text{if } a_1^2 - a_2^2 = 0, \end{cases}$$

we obtain $W(F, Z) = \mathbb{R}^n \times \{(0, 0)\}$. Since $\pi_2(W(F, Z)) = \{(0, 0)\} (\subset \mathbb{R}^2)$, we have (α) for any $s > 1$. Hence, we also have (β) for any $s > 1$.

On the other hand, since

$$\Sigma(F, Z) = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 - a_2^2 = 0\},$$

(β) does not hold in the case $s = 1$ although (α) holds even in the case $s \geq 1$. Hence, (α) $\Rightarrow$ (β) does not hold if we replace $s > 1$ by $s \geq 1$.

**Example 3** (An example of Theorem 3 (2)). Let $F : \mathbb{R} \times \mathbb{R}^\ell \to \mathbb{R}^\ell \ (\ell \geq 2)$ be the mapping defined by

$$F(x, a) = (x + a_1, \ldots, x + a_\ell),$$

where $a = (a_1, \ldots, a_\ell) \in \mathbb{R}^\ell$. Set $Z = \{(0, \ldots, 0)\} (\subset \mathbb{R}^\ell)$.

Since $\delta(F, Z) = 0$, we have $\delta^*(F, Z) = 1 - \ell \ (< 0)$. Hence, by Theorem 3 (2), the following hold:

(2a) We have $W(F, Z) = \emptyset$.

(2b) For any non-negative real number $s$ satisfying

$$s > \dim \mathbb{R}^\ell + \delta^*(F, Z) = 1,$$

the set $\Sigma(F, Z)$ has $s$-dimensional Hausdorff measure zero in $\mathbb{R}^\ell$.

(2c) For any $a \in \mathbb{R}^\ell - \Sigma(F, Z)$, we have $\text{Im} F_a \cap Z = \emptyset$. 

On the other hand, since
\[ \Sigma(F, Z) = \{ (a_1, \ldots, a_t) \in \mathbb{R}^t \mid a_1 = \cdots = a_t \}, \]
the set \( \Sigma(F, Z) \) does not have 1-dimensional Hausdorff measure zero in \( \mathbb{R}^t \). Thus, we cannot replace \( s > 1 \) in (2b) by \( s \geq 1 \).

Finally, as mentioned in Remark (14), we give the following example.

Example 4. Let \( F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n (n \geq 3) \) be the mapping defined by \( F(x, a) = (0, \ldots, 0) \), with \( Z = \{ (1, 0, \ldots, 0) \} (\subset \mathbb{R}^n) \). Since \( \delta^*(F, Z) = 1 - n \), we have \( \dim \mathbb{R} + \delta^*(F, Z) = 2 - n \) (< 0). Namely, there exists an example satisfying \( \dim A + \delta^*(F, Z) < 0 \).

3. Preliminaries for the proof of Theorem 3

Let \( s \) be an arbitrary non-negative real number. Then, the \( s \)-dimensional Hausdorff outer measure on \( \mathbb{R}^n \) is defined as follows. Let \( B \) be a subset of \( \mathbb{R}^n \). The \( 0 \)-dimensional Hausdorff outer measure of \( B \) is the number of points in \( B \). For \( s > 0 \), the \( s \)-dimensional Hausdorff outer measure of \( B \) is defined by
\[ \lim_{\delta \to 0} \mathcal{H}_s^*(B), \]
where for each \( 0 < \delta \leq \infty \),
\[ \mathcal{H}_s^*(B) = \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } C_j)^s \mid B \subset \bigcup_{j=1}^{\infty} C_j, \text{ diam } C_j \leq \delta \right\}. \]

Here, for a subset \( C \) of \( \mathbb{R}^n \), we write
\[ \text{diam } C = \sup \{ ||x - y|| \mid x, y \in C \}, \]
where \( ||z|| \) denotes the Euclidean norm of \( z \in \mathbb{R}^n \). Note that the infimum in \( \mathcal{H}_s^*(B) \) is over all coverings of \( B \) by subsets \( C_1, C_2, \ldots \) of \( \mathbb{R}^n \) satisfying \( \text{diam } C_j \leq \delta \) for all positive integers \( j \).

Let \( s \) be an arbitrary non-negative real number. Let \( N \) be a \( C^r \) manifold \( (r \geq 1) \) of dimension \( n \), and \( \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda} \) be a coordinate neighborhood system of \( N \). Then, a subset \( \Sigma \) of \( N \) has \( s \)-dimensional Hausdorff measure zero in \( N \) if for any \( \lambda \in \Lambda \), the set \( \varphi_\lambda(\Sigma \cap U_\lambda) \) has \( s \)-dimensional Hausdorff (outer) measure zero in \( \mathbb{R}^n \).

Note that this definition does not depend on the choice of a coordinate neighborhood system of \( N \). Moreover, for a subset \( \Sigma \) of \( N \), set
\[ \text{HD}_N(\Sigma) = \inf \{ s \in [0, \infty) \mid \Sigma \text{ has } s \text{-dimensional Hausdorff measure zero in } N \}, \]
which is called the Hausdorff dimension of \( \Sigma \).

Let \( X \) and \( Y \) be \( C^r \) manifolds \( (r \geq 1) \), and let \( f : X \to Y \) be a \( C^1 \) mapping. A point \( x \in X \) is called a critical point of \( f \) if rank \( df_x < \dim Y \). A point \( x \in X \) is called a regular point of \( f \) if it is not a critical point. We say that a point \( y \in Y \) is a critical value if it is the image of a critical point. A point \( y \in Y \) is called a regular value if it is not a critical value.

The assertions (11) and (2) of the following result follow from [13 3.4.3 (p. 316)] and [14 Theorem 1 (p. 253)], respectively.

Theorem 4 ([13 14]). Let \( X \) and \( Y \) be \( C^r \) manifolds, and \( f : X \to Y \) be a \( C^r \) mapping, where \( r \) is a positive integer.
Proof. Since there exists (Lemma 3. Let $\tilde{\pi}$ it follows that

Next, let $a$ and $F$ Lemma 1, we have $(x,a)$ It is clearly seen that $\pi$.)

Proof of Lemma 2. Lemma 2 \cite{9}. For the sake of readers' convenience, in this paper, we explicitly give the part as a lemma with the proof.

Lemma 2 \cite{9}. Let $X, A$ and $Y$ be $C^r$ manifolds, $Z$ be a $C^r$ submanifold of $Y$ and $F : U \rightarrow Y$ be a $C^1$ mapping, where $U$ is an open set of $X \times A$ ($r \geq 1$). Then, it follows that

$$\pi_2(W(F,Z)) \cup \pi_2(\tilde{W}(F,Z)) = \Sigma(F,Z),$$

where $\tilde{W}(F,Z) = \{ (x,a) \in U \mid \delta(F_a, x, Z) > \delta(F, (x,a), Z) \}$. \hfill \blacksquare

Proof of Lemma 2. It is clearly seen that $\pi_2(W(F,Z)) \cup \pi_2(\tilde{W}(F,Z)) \subset \Sigma(F,Z)$. Next, let $a \in \Sigma(F,Z)$. Then, there exists $x \in \pi_1(U)$ satisfying $\delta(F_a, x, Z) > 0$. By Lemma 1, we have $(x,a) \in W(F,Z) \cup \tilde{W}(F,Z)$. Thus, we obtain $a \in \pi_2(W(F,Z)) \cup \pi_2(\tilde{W}(F,Z))$. \hfill \blacksquare

Lemma 3. Let $X, A$ and $Y$ be $C^r$ manifolds, $Z$ be a $C^r$ submanifold of $Y$ and $F : U \rightarrow Y$ be a $C^1$ mapping, where $U$ is an open set of $X \times A$ ($r \geq 1$). If $F(U) \cap Z \neq \emptyset$, then we have $\dim A + \delta^*(F,Z) \geq 0$.

Proof. Since there exists $(x, a) \in U$ satisfying $F(x, a) \in Z$, we have

$$\delta(F,Z) \geq \dim Y - \dim(\text{Im} \ dF(x,a) + T_{F(x,a)}Z)$$

$$\geq \dim Y - (\dim \text{Im} \ dF(x,a) + \dim T_{F(x,a)}Z)$$

$$\geq \dim Y - (\dim U + \dim Z)$$

$$= \dim Y - (\dim X + \dim A + \dim Z).$$

Thus, it follows that

$$\dim A + \delta^*(F,Z) = \dim A + \dim X + \dim Z - \dim Y + \delta(F,Z) \geq 0.$$ \hfill \blacksquare

In the following, for two sets $V_1, V_2$, a mapping $f : V_1 \rightarrow V_2$, and a subset $V_3$ of $V_1$, the restriction of the mapping $f$ to $V_3$ is denoted by $f|_{V_3} : V_3 \rightarrow V_2$.

Lemma 4 \cite{9}. Let $X, A$ and $Y$ be $C^r$ manifolds, $Z$ be a $C^r$ submanifold of $Y$ and $F : U \rightarrow Y$ be a $C^1$ mapping, where $U$ is an open set of $X \times A$ ($r \geq 1$). For any integer $\rho$ satisfying $0 \leq \rho \leq \delta(F,Z)$, set

$$\tilde{W}_\rho = \{ (x,a) \in U \mid \delta(F_a, x, Z) \mid \delta(F, (x,a), Z) = \rho \}.$$
Then, for any \((x_0, a_0) \in \tilde{W}_\rho\), there exist an open neighborhood \(\tilde{U}\) of \((x_0, a_0)\) and a \(C^r\) submanifold \(\tilde{Z}\) of \(Y\) satisfying the following:

(a) \(\dim \tilde{Z} = \dim Z + \rho\).
(b) \(F(\tilde{U}) \cap Z \subset \tilde{Z}\).
(c) The mapping \(F|_{\tilde{U}} : \tilde{U} \to Y\) is transverse to \(\tilde{Z}\).
(d) For any \((x, a) \in \tilde{U}\), it follows that \(\delta(F_a, x, Z) - \delta(F_a, x, \tilde{Z}) \leq \rho\).

4. Proof of Theorem 3

In Section 4.1, we give an essential result for the proof of Theorem 3 (see Proposition 2). In Section 4.2 (resp., Section 4.3), we show Theorem 3 (1) (resp., Theorem 3 (2)).

4.1. An essential result for the proof of Theorem 3.

Proposition 2. Let \(X\), \(A\) and \(Y\) be \(C^r\) manifolds, \(Z\) be a \(C^r\) submanifold of \(Y\) and \(F : U \to Y\) be a \(C^r\) mapping, where \(U\) is an open set of \(X \times A\) and \(r\) is a positive integer. Set

\[
\tilde{W}(F, Z) = \{ (x, a) \in U \mid \delta(F_a, x, Z) > \delta(F, (x, a), Z) \}.
\]

Then, the following hold:

(1) Suppose \(\delta^*(F, Z) \geq 0\). Then, for any real number \(s\) satisfying

\[
s \geq \dim A - 1 + \frac{\delta^*(F, Z) + 1}{r},
\]

the set \(\pi_2(\tilde{W}(F, Z))\) has \(s\)-dimensional Hausdorff measure zero in \(\pi_2(U)\).

(2) Suppose \(\delta^*(F, Z) < 0\). Then, for any non-negative real number \(s\) satisfying

\[
s > \dim A + \delta^*(F, Z),
\]

the set \(\pi_2(\tilde{W}(F, Z))\) has \(s\)-dimensional Hausdorff measure zero in \(\pi_2(U)\).

Proof of Proposition 2 The proof will be separated into STEP 1, STEP 2.1 and STEP 2.2. In STEP 1, we give the former common part of the proofs of (1) and (2). In STEP 2.1 and STEP 2.2, we give the latter parts of the proofs of (1) and (2), respectively.

STEP 1. In this step, we give the former common part of the proofs of Proposition 2 (1) and (2).

If \(F(U) \cap Z = \emptyset\), then \(\pi_2(\tilde{W}(F, Z)) = \emptyset\) has 0-dimensional Hausdorff measure zero in \(\pi_2(U)\). Thus, in this case, both (1) and (2) hold.

Now, we will consider the case \(F(U) \cap Z \neq \emptyset\). By Lemma 3, we have \(\dim A + \delta^*(F, Z) \geq 0\). Hence, in the proof of not only (1) but also (2), we can assume \(s > 0\).

For any integer \(\rho\) satisfying \(0 \leq \rho \leq \delta(F, Z)\), set

\[
\tilde{W}_\rho = \{ (x, a) \in U \mid \delta(F_a, x, Z) > \delta(F, (x, a), Z) = \rho \}.
\]

Then, we have

\[
\tilde{W}(F, Z) = \bigcup_{0 \leq \rho \leq \delta(F, Z)} \tilde{W}_\rho.
\]

For any integer \(\rho\) satisfying \(0 \leq \rho \leq \delta(F, Z)\), by Lemma 4, there exist countably many open neighborhoods \(\tilde{U}_{\rho, 1}, \tilde{U}_{\rho, 2}, \ldots\) such that \(\tilde{W}_\rho \subset \bigcup_{i=1}^{\infty} \tilde{U}_{\rho, i}\) (\(\tilde{W}_\rho \cap \tilde{U}_{\rho, i} \neq \emptyset\))
for any \( i \) and countably many \( C^r \) submanifolds \( \tilde{Z}_{\rho,1}, \tilde{Z}_{\rho,2}, \ldots \) satisfying for any positive integer \( i \),

(a) \( \dim \tilde{Z}_{\rho,i} = \dim Z + \rho \).
(b) \( F(\tilde{U}_{\rho,i}) \cap Z \subset \tilde{Z}_{\rho,i} \).

The set \( B_{\rho,i} = (F|_{\tilde{U}_{\rho,i}})^{-1}(\tilde{Z}_{\rho,i}) \).

Now, we prepare the following lemma.

**Lemma 5.** In the above situation, for any integer \( \rho \) satisfying \( 0 \leq \rho \leq \delta(F,Z) \) and any positive integer \( i \), we have the following:

1. For any \( (x,a) \in \tilde{W}_\rho \cap \tilde{U}_{\rho,i} \), it follows that \( \delta(F_a, x, \tilde{Z}_{\rho,i}) > 0 \).
2. We have \( \tilde{W}_\rho \cap \tilde{U}_{\rho,i} \subset B_{\rho,i} \), and hence \( B_{\rho,i} \neq \emptyset \).
3. The set \( B_{\rho,i} \) is a \( C^r \) submanifold of \( \tilde{U}_{\rho,i} \) satisfying

\[ \dim B_{\rho,i} = \dim X + \dim A + \dim \tilde{Z}_{\rho,i} - \dim Y. \]

4. We have \( \dim B_{\rho,i} \leq \dim B_{\delta(F,Z),i} \).

5. It follows that

\[ \delta^*(F, Z) = \dim X + \dim Z + \delta(F, Z) - \dim Y \]
\[ = \dim X + \dim \tilde{Z}_{\delta(F,Z),i} - \dim Y \]
\[ = \dim B_{\delta(F,Z),i} - \dim A \]
\[ \geq \dim B_{\rho,i} - \dim A. \]

**Proof of Lemma 5** Let \( (x,a) \in \tilde{W}_\rho \cap \tilde{U}_{\rho,i} \) be any element. Since \( \delta(F_a, x, \tilde{Z}_{\rho,i}) > 0 \), we have \( \delta(F_a, x, \tilde{Z}_{\rho,i}) > 0 \) by (d). Thus, (1) holds.

Let \( (x,a) \in \tilde{W}_\rho \cap \tilde{U}_{\rho,i} \) be an arbitrary element. Since \( F_a(x) \in \tilde{Z}_{\rho,i} \), we obtain \( (x,a) \in B_{\rho,i} \). Since \( \tilde{W}_\rho \cap \tilde{U}_{\rho,i} \neq \emptyset \), we also have \( B_{\rho,i} \neq \emptyset \). Hence, (2) holds.

The assertion (3) (resp., (4)) can be obtained by (c) (resp., (a)) and (3). In (5), the second equality, the third equality and the last inequality can be obtained by (a), (3) and (4), respectively.

By Lemma 5, note that for any positive integer \( i \), we have

\[ \delta^*(F, Z) \geq 0 \iff \dim B_{\delta(F,Z),i} \geq \dim A. \]

Since \( \tilde{W}_\rho = \bigcup_{i=1}^{\infty} (\tilde{W}_\rho \cap \tilde{U}_{\rho,i}) \), we have

\[ \tilde{W}(F, Z) = \bigcup_{0 \leq \rho \leq \delta(F,Z)} \left( \bigcup_{i=1}^{\infty} (\tilde{W}_\rho \cap \tilde{U}_{\rho,i}) \right). \]

Hence, in order to show that \( \pi_2(\tilde{W}(F, Z)) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(U) \), it is sufficient to show that \( \pi_2(\tilde{W}_\rho \cap \tilde{U}_{\rho,i}) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{\rho,i}) \) for any \( \rho \) and \( i \).

Now, for the proof of Proposition 2, let \( \rho \) and \( i \) be arbitrary integers satisfying \( 0 \leq \rho \leq \delta(F,Z) \) and \( i \geq 1 \), respectively.
We consider the case \( \dim B_{p,i} = 0 \). Since \( B_{p,i} \) is countable, we see that \( \tilde{W}_{p} \cap \tilde{U}_{p,i} \) (and hence \( \pi_2(\tilde{W}_{p} \cap \tilde{U}_{p,i}) \)) is also countable by Lemma 3 (2). Since \( s > 0 \), the set \( \pi_2(\tilde{W}_{p} \cap \tilde{U}_{p,i}) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \).

Therefore, in what follows, it is sufficient to consider the case \( \dim B_{p,i} > 0 \). Let \( \Sigma_{p,i} \) be the set of the critical values of \( \pi_2|_{B_{p,i}} : B_{p,i} \to \pi_2(\tilde{U}_{p,i}) \). Let \( a \in \pi_2(\tilde{W}_{p} \cap \tilde{U}_{p,i}) \) be an arbitrary element. Then, there exists \( x \in \pi_1(\tilde{U}_{p,i}) \) such that \( (x, a) \in \tilde{W}_{p} \cap \tilde{U}_{p,i} \). Since \( (F|_{\tilde{U}_{p,i}})_a \) is not transverse to \( \tilde{Z}_{p,i} \) by Lemma 5 (1), it is easy to see that \( a \in \Sigma_{p,i} \). Namely, we obtain

\[
\pi_2(\tilde{W}_{p} \cap \tilde{U}_{p,i}) \subset \Sigma_{p,i}.
\]

Thus, in order to show that \( \pi_2(\tilde{W}_{p} \cap \tilde{U}_{p,i}) \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \), it is sufficient to show that \( \Sigma_{p,i} \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \).

**STEP 2.1.** In this step, we give the latter part of the proof of Proposition 2 (1).

For the proof of Proposition 2 (1), it is sufficient to show that \( \Sigma_{p,i} \) has \( s \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \) in the case \( \delta^*(F, Z) \geq 0 \), where \( s_1 \) is a real number satisfying

\[
s_1 \geq \dim A - 1 + \frac{\delta^*(F, Z) + 1}{r}.
\]

Here, by Lemma 5 (4) and (4.1), note that in the case \( \delta^*(F, Z) \geq 0 \), it is necessary to consider the two cases \( \dim B_{p,i} \geq \dim A \) and \( \dim B_{p,i} < \dim A \).

We consider the first case \( \dim B_{p,i} \geq \dim A \). Since \( \delta^*(F, Z) \geq \dim B_{p,i} - \dim A \) by Lemma 5 (5), we have

\[
s_1 \geq \dim A - 1 + \frac{\dim B_{p,i} - \dim A + 1}{r}.
\]

Hence, by applying Theorem 4 (1) to \( \pi_2|_{B_{p,i}} \), it follows that \( \Sigma_{p,i} \) has \( s_1 \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \).

In the second case \( \dim B_{p,i} < \dim A \), since

\[
s_1 > \dim A - 1 \geq \dim B_{p,i},
\]

by applying Theorem 3 (2) to \( \pi_2|_{B_{p,i}} \), the set \( \Sigma_{p,i} \) has \( s_1 \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \).

**STEP 2.2.** In this step, we give the latter part of the proof of Proposition 2 (2).

For the proof of Proposition 2 (2), it is sufficient to show that \( \Sigma_{p,i} \) has \( s_2 \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \) in the case \( \delta^*(F, Z) < 0 \), where \( s_2 \) is a real number satisfying

\[
s_2 > \dim A + \delta^*(F, Z) \geq 0.
\]

In the case \( \delta^*(F, Z) < 0 \), by the last inequality of Lemma 5 (5), it is sufficient to consider only the case \( \dim B_{p,i} < \dim A \). By the same inequality, we have

\[
\dim B_{p,i} \leq \dim A + \delta^*(F, Z) < s_2.
\]

Thus, by applying Theorem 4 (2) to the mapping \( \pi_2|_{B_{p,i}} \), it follows that \( \Sigma_{p,i} \) has \( s_2 \)-dimensional Hausdorff measure zero in \( \pi_2(\tilde{U}_{p,i}) \). \( \square \)
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4.2. Proof of Theorem 3 (1). By combining Lemma 2 and Proposition 2 (1), we obtain $(\alpha) \Rightarrow (\beta)$.

In order to prove $(\beta) \Rightarrow (\alpha)$, it is sufficient to prepare the following.

Lemma 6. Let $X$, $A$ and $Y$ be $C^r$ manifolds, $Z$ be a $C^r$ submanifold of $Y$ and $F : U \to Y$ be a $C^1$ mapping, where $U$ is an open set of $X \times A$ ($r \geq 1$). Let $s$ be any non-negative real number. If $\Sigma(F, Z)$ has $s$-dimensional Hausdorff measure zero in $\pi_2(U)$, then $\pi_2(W(F, Z))$ has $s$-dimensional Hausdorff measure zero in $\pi_2(U)$.

Proof of Lemma 7. Since $\pi_2(W(F, Z)) \subset \Sigma(F, Z)$ by Lemma 2, this lemma clearly holds.

4.3. Proof of Theorem 3 (2). First, we show $W(F, Z) = \emptyset$. Suppose $W(F, Z) \neq \emptyset$. Then, there exists $(x, a) \in U$ such that

$$\delta(F_a, x, Z) = \delta(F, (x, a), Z) > 0.$$ 

Thus, we have

$$\delta^*(F, Z) = \dim X + \dim Z - \dim Y + \delta(F, Z) \geq \dim X + \dim Z - \dim Y + \delta(F_a, x, Z) = \dim X + \dim Z - \dim(\text{Im}(dF_a)_x + T_{F_a(x)}Z) \geq \dim X + \dim Z - (\dim \text{Im}(dF_a)_x + \dim T_{F_a(x)}Z) \geq 0$$

This contradicts $\delta^*(F, Z) < 0$. Hence, we obtain $W(F, Z) = \emptyset$.

Thus, we have $\Sigma(F, Z) = \pi_2(W(F, Z))$ by Lemma 2 where $W(F, Z)$ is defined in Lemma 2. By Proposition 2 (2), we obtain (2b).

Now, suppose that there exists $a \in \pi_2(U) - \Sigma(F, Z)$ such that $\text{Im} F_a \cap Z \neq \emptyset$. Then, there exists $x \in \pi_1(U)$ satisfying

$$\text{Im}(dF_a)_x + T_{F_a(x)}Z = T_{F_a(x)}Y.$$ 

Since

$$\dim \text{Im}(dF_a)_x + \dim T_{F_a(x)}Z - \dim T_{F_a(x)}Y \geq 0,$$

we have

$$\dim X + \dim Z - \dim Y \geq 0.$$ 

This contradicts $\delta^*(F, Z) < 0$. Hence, we have (2c). □

5. APPLICATIONS OF THEOREM 3

Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. In what follows, we will regard $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ as the Euclidean space $(\mathbb{R}^m)^\ell$ in the obvious way.

In [7], for a $C^\infty$ immersion (resp., a $C^\infty$ injection) $f : X \to V$ and an arbitrary $C^\infty$ mapping $g : V \to \mathbb{R}^\ell$, a transversality theorem on the 1-jet extension (resp., on crossings) of $(g + \pi) \circ f : X \to \mathbb{R}^\ell$ ($\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$) is given, where $X$ is a $C^\infty$ manifold, $V$ is an open subset of $\mathbb{R}^m$ and $\Sigma$ is a subset of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero (for details, see Theorem 1 (resp., Theorem 2) in [7]). In [8], the two transversality theorems on generic linear perturbations described above are improved so that these work even in the case where manifolds and mappings...
are not necessarily of class $C^\infty$. However, these are transversality theorems from the viewpoint of Lebesgue measures (for details, see Theorems 1 and 2 in [8]).

On the other hand, in this section, as applications of Theorem [8] we give two transversality theorems on generic linear perturbations from the viewpoint of Hausdorff measures (see Theorems [9] and [10], which are further improvements of the transversality theorems from the viewpoint of Lebesgue measures described above.

For the statement of Theorem [9] we prepare some definitions. Let $X$ be a $C^r$ manifold ($r \geq 2$) of dimension $n$ and $J^1(X, \mathbb{R}^\ell)$ be the space of 1-jets of mappings of $X$ into $\mathbb{R}^\ell$. Then, note that $J^1(X, \mathbb{R}^\ell)$ is a $C^{r-1}$ manifold. For a given $C^r$ mapping $f : X \to \mathbb{R}^\ell$, the 1-jet extension $j^1f : X \to J^1(X, \mathbb{R}^\ell)$ is defined by $q \mapsto j^1f(q)$. Then, notice that $j^1f$ is of class $C^{r-1}$. For details on $J^1(X, \mathbb{R}^\ell)$ and $j^1f$, see for example [1].

Now, let $\{(V_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ be a coordinate neighborhood system of $X$. Let $\Pi : J^1(X, \mathbb{R}^\ell) \to X \times \mathbb{R}^\ell$ be the natural projection defined by $\Pi(j^1f(q)) = (q, f(q))$. Let $\Phi_\lambda : \Pi^{-1}(V_\lambda \times \mathbb{R}^\ell) \to \varphi_\lambda(V_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ be the homeomorphism defined by

$$
\Phi_\lambda(j^1f(q)) = (\varphi_\lambda(q), f(q), j^1(\psi_\lambda \circ f \circ \varphi_\lambda^{-1} \circ \varphi_\lambda)(0)),
$$

where $J^1(n, \ell) = \{ j^1f(0) \mid f : (\mathbb{R}^n, 0) \to (\mathbb{R}^\ell, 0) \}$ and $\varphi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ (resp., $\psi_\lambda : \mathbb{R}^\ell \to \mathbb{R}^\ell$) is the translation given by $\varphi_\lambda(0) = \varphi_\lambda(q)$ (resp., $\psi_\lambda(f(q)) = 0$). Then, $\{(\Pi^{-1}(V_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda}$ is a coordinate neighborhood system of $J^1(X, \mathbb{R}^\ell)$.

Set

$$
S^k = \{ j^1f(0) \in J^1(n, \ell) \mid \text{corank } Jf(0) = k \},
$$

where corank $Jf(0) = \min \{ n, \ell \} - \text{rank } Jf(0)$ and $k = 1, 2, \ldots, \min \{ n, \ell \}$. Set

$$
S^k(X, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}(\varphi_\lambda(V_\lambda) \times \mathbb{R}^\ell \times S^k).
$$

Then, the set $S^k(X, \mathbb{R}^\ell)$ is a submanifold of $J^1(X, \mathbb{R}^\ell)$ satisfying

$$
\text{codim } S^k(X, \mathbb{R}^\ell) = \dim J^1(X, \mathbb{R}^\ell) - \dim S^k(X, \mathbb{R}^\ell) = (n - v + k)\ell - v + k,
$$

where $v = \min \{ n, \ell \}$. (For details on $S^k$ and $S^k(X, \mathbb{R}^\ell)$, see [1] pp. 60–61).

The following is the first transversality theorem on generic linear perturbations from the viewpoint of Hausdorff measures.

**Theorem 5.** Let $f : X \to V$ be a $C^r$ immersion and $g : V \to \mathbb{R}^\ell$ be a $C^r$ mapping, where $r$ is an integer satisfying $r \geq 2$, $X$ is a $C^r$ manifold and $V$ is an open subset of $\mathbb{R}^m$. Let $k$ be an integer satisfying $1 \leq k \leq \min \{ \dim X, \ell \}$. Set

$$
\Sigma_k = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid j^1((g + \pi) \circ f) \text{ is not transverse to } S^k(X, \mathbb{R}^\ell) \}.
$$

Then, the following hold:

1. Suppose $\dim X - \text{codim } S^k(X, \mathbb{R}^\ell) \geq 0$. Then, for any real number $s$ satisfying

   $$
   (5.1) \quad s \geq m\ell - 1 + \frac{\dim X - \text{codim } S^k(X, \mathbb{R}^\ell) + 1}{r - 1},
   $$

   the set $\Sigma_k$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

2. Suppose $\dim X - \text{codim } S^k(X, \mathbb{R}^\ell) < 0$. Then, we have the following:
(2a) For any real number \( s \) satisfying
\[
 s > m\ell + \dim X - \text{codim} S^k(X, \mathbb{R}^\ell),
\]
the set \( \Sigma_k \) has \( s \)-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \).

(2b) For any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_k \), we have
\[
 j^1((g + \pi) \circ f)(X) \cap S^k(X, \mathbb{R}^\ell) = \emptyset.
\]

Remark 2.

(1) In Theorem 5, if all manifolds and mappings are of class \( C^\infty \), then we can replace (5.1) by \( s > m\ell - 1 \) by the same argument as in Remark 4.

(2) In Theorem 5 since \( f \) is an immersion, we have \( n \leq m \), where \( n = \dim X \).

Thus, in Theorem 5 (2), since
\[
m\ell + \dim X - \text{codim} S^k(X, \mathbb{R}^\ell) \geq m\ell + n - n\ell = (m - n)\ell + n \geq n,
\]
it is not necessary to assume that \( s \) is non-negative.

For the statement of Theorem 6 we prepare some definitions. Let \( X \) be a \( C^r \) manifold \((r \geq 1)\). Set
\[
 X^{(d)} = \{ (q_1, \ldots, q_d) \in X^d \mid q_i \neq q_j \ (i \neq j) \}.
\]
Note that \( X^{(d)} \) is an open submanifold of \( X^d \). For any mapping \( f : X \to \mathbb{R}^\ell \), let \( f^{(d)} : X^{(d)} \to (\mathbb{R}^\ell)^d \) be the mapping given by
\[
f^{(d)}(q_1, \ldots, q_d) = (f(q_1), \ldots, f(q_d)).
\]
Set \( \Delta_d = \{ (y, \ldots, y) \in (\mathbb{R}^\ell)^d \mid y \in \mathbb{R}^\ell \} \). Then, \( \Delta_d \) is a submanifold of \((\mathbb{R}^\ell)^d\) satisfying
\[
 \text{codim} \Delta_d = \dim ((\mathbb{R}^\ell)^d - \Delta_d) = \ell(d - 1).
\]

As in [7], for any injection \( f : X \to \mathbb{R}^m \), set
\[
d_f = \max \left\{ d \mid \forall (q_1, \ldots, q_d) \in X^{(d)}, \dim \sum_{i=2}^d \mathbb{R}f(q_1)f(q_i) = d - 1 \right\}.
\]
Since the mapping \( f \) is an injection, we have \( 2 \leq d_f \). Since \( f(q_1), \ldots, f(q_d) \) are points of \( \mathbb{R}^m \), it follows that \( d_f \leq m + 1 \). Hence, we get
\[
 2 \leq d_f \leq m + 1.
\]

The following is the second transversality theorem on generic linear perturbations from the viewpoint of Hausdorff measures.

**Theorem 6.** Let \( f : X \to V \) be a \( C^r \) injection and \( g : V \to \mathbb{R}^\ell \) be a \( C^r \) mapping, where \( r \) is a positive integer, \( X \) is a \( C^r \) manifold and \( V \) is an open subset of \( \mathbb{R}^m \).

Set
\[
 \Sigma_d = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid ((g + \pi) \circ f)^{(d)} \text{ is not transverse to } \Delta_d \},
\]
where \( d \) is an integer satisfying \( 2 \leq d \leq d_f \). Then, the following hold:

(1) Suppose \( \dim X^{(d)} - \text{codim} \Delta_d \geq 0 \). Then, for any real number \( s \) satisfying
\[
 (5.2) \quad s \geq m\ell - 1 + \frac{\dim X^{(d)} - \text{codim} \Delta_d + 1}{r},
\]
the set \( \Sigma_d \) has \( s \)-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \).

(2) Suppose \( \dim X^{(d)} - \text{codim} \Delta_d < 0 \). Then, the following hold:
(2a) For any real number $s$ satisfying
\[ s > m\ell + \dim X(d) - \text{codim} \Delta_d, \]
the set $\Sigma_d$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

(2b) For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_d$, we have $((g + \pi) \circ f)^{(d)}(X(d)) \cap \Delta_d = \emptyset$.

**Remark 3.**

(1) In Theorem 5, if all manifolds and mappings are of class $C^\infty$, then we can replace (5.2) by $s > m\ell - 1$ by the same argument as in Remark 4.1.

(2) Set $n = \dim X$. In Theorem 6 (2), since
\[ m\ell + \dim X(d) - \text{codim} \Delta_d = m\ell + n\ell - (d - 1) \geq n\ell, \]
it is not necessary to assume that $s$ is non-negative.

(3) As in [7], there is a case of $d_f = 3$ as follows. If $n + 1 \leq m$, $X = S^n$ and $f : S^n \to \mathbb{R}^m$ is the inclusion $f(x) = (x, 0, \ldots, 0)$, then $d_f = 3$. Indeed, suppose that there exists a point $(q_1, q_2, q_3) \in (S^n)^{(3)}$ satisfying $\dim \sum_{i=1}^3 \mathbb{R}f(q_1)f(q_i) = 1$. Since the number of the intersections of $f(S^n)$ and a straight line of $\mathbb{R}^m$ is at most two, this contradicts the assumption. Thus, we have $d_f \geq 3$. On the other hand, since $S^1 \times \{0\} \subset f(S^n)$, we get $d_f < 4$, where $0 = (0, 0, \ldots, 0)$. Therefore, it follows that $d_f = 3$.

Remark 4. In Theorems 5 and 6 there is an advantage that the domain of $g : V \to \mathbb{R}^\ell$ is not $\mathbb{R}^m$ but an arbitrary open subset $V$ of $\mathbb{R}^m$. Suppose $V = \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = |x|$. Since $g$ is not differentiable at $x = 0$, we cannot apply Theorems 5 and 6 to $g : \mathbb{R} \to \mathbb{R}$. On the other hand, if $V = \mathbb{R} - \{0\}$, then we can apply these theorems to $g|_V$.

**6. Proof of Theorem 5**

Set $n = \dim X$. For a positive integer $\tilde{n}$, we denote the $\tilde{n} \times \tilde{n}$ unit matrix by $E_{\tilde{n}}$. Let $\Gamma : X \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to J^1(X, \mathbb{R}^\ell)$ be the $C^{r-1}$ mapping defined by
\[ \Gamma(q, \pi) = j^1((g + \pi) \circ f)(q). \]

First, we will show that $\delta(\Gamma, S^k(X, \mathbb{R}^\ell)) = 0$. Let $(\tilde{q}, \tilde{\pi}) \in X \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be an arbitrary element satisfying $\Gamma(\tilde{q}, \tilde{\pi}) \in S^k(X, \mathbb{R}^\ell)$. Then, in order to show that $\delta(\Gamma, S^k(X, \mathbb{R}^\ell)) = 0$, it is sufficient to show that
\begin{equation}
\text{dim} \left( \text{Im} \, d\Gamma_{(\tilde{q}, \tilde{\pi})} + T_{\Gamma(\tilde{q}, \tilde{\pi})}S^k(X, \mathbb{R}^\ell) \right) = n + \ell + n\ell. \tag{6.1} \end{equation}

As in Section 5 let $\{ (V_\lambda, \varphi_\lambda) \}_{\lambda \in \Lambda}$ (resp., $\{ (\Pi^{-1}(V_\lambda \times \mathbb{R}^\ell), \Phi_\lambda) \}_{\lambda \in \Lambda}$) be a coordinate neighborhood system of $X$ (resp., $J^1(X, \mathbb{R}^\ell)$). There exists a coordinate neighborhood $(V_\lambda, \varphi_\lambda) \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ containing $(\tilde{q}, \tilde{\pi})$ of $X \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, where $\text{id}$ is the identity mapping of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ into $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, and $\varphi_\lambda \times \text{id} : V_\lambda \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to \varphi_\lambda(V_\lambda) \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ is given by $(\varphi_\lambda \times \text{id})(q, \pi) = (\varphi_\lambda(q), \text{id}(\pi))$. Then, $(\Pi^{-1}(V_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)$ is a coordinate neighborhood containing the point $\Gamma(\tilde{q}, \tilde{\pi})$ of $J^1(X, \mathbb{R}^\ell)$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a local coordinate on $\varphi_\lambda(V_\lambda)$. 

Here, let \((a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}\) be a representing matrix of a linear mapping \(\pi \in L(\mathbb{R}^m, \mathbb{R}^\ell)\). Then, \((g + \pi) \circ f : X \to \mathbb{R}^\ell\) is given as follows:

\[
(g + \pi) \circ f = \left( g_1 \circ f + \sum_{j=1}^{m} a_{1j} f_j, \ldots, g_\ell \circ f + \sum_{j=1}^{m} a_{\ell j} f_j \right),
\]

where \(f = (f_1, \ldots, f_m)\), \(g = (g_1, \ldots, g_\ell)\) and \((a_{11}, \ldots, a_{1m}, \ldots, a_{\ell 1}, \ldots, a_{\ell m}) \in (\mathbb{R}^m)^\ell\).

Hence, the mapping \(\Gamma\) is locally given by the following:

\[
\Phi_\chi \circ \gamma \circ (\varphi_\lambda \times \text{id})^{-1}(x, \pi) = (x, (g + \pi) \circ \varphi_\lambda^{-1}(x), (g + \pi)_1 \circ \varphi_\lambda^{-1}(x), \ldots, (g + \pi)_\ell \circ \varphi_\lambda^{-1}(x)) = (x, (g + \pi), (g + \pi)_1 \circ \varphi_\lambda^{-1}(x), \ldots, (g + \pi)_\ell \circ \varphi_\lambda^{-1}(x)) \in (\mathbb{R}^m)^\ell.
\]

\[
\frac{\partial (g + \pi)_1 \circ \varphi_\lambda^{-1}(x)}{\partial x_1}, \ldots, \frac{\partial (g + \pi)_\ell \circ \varphi_\lambda^{-1}(x)}{\partial x_n} = (\tilde{f}_1, \ldots, \tilde{f}_m) = (f_1 \circ \varphi_\lambda^{-1}, \ldots, f_m \circ \varphi_\lambda^{-1}) = f \circ \varphi_\lambda^{-1}. \quad \text{The Jacobian matrix of} \ \Gamma \ \text{at} \ \bar{q}, \bar{\pi} \ \text{is the following:}
\]

\[
J \Gamma_{\bar{q}, \bar{\pi}} = \begin{pmatrix}
E_n & 0 & \cdots & \cdots & 0 \\
\ast & \cdots & \cdots & \ast \\
\ast & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
0 & 0 & \ast & \ast & \ast
\end{pmatrix}
\]

where \(J f_{\bar{q}}\) is the Jacobian matrix of \(f\) at \(\bar{q}\). Notice that \((J f_{\bar{q}})^T\) is the transpose of \(J f_{\bar{q}}\) and that there are \(\ell\) copies of \((J f_{\bar{q}})^T\) in the above description of \(J \Gamma_{\bar{q}, \bar{\pi}}\). Since \(S^k(X, \mathbb{R}^\ell)\) is a subfiber-bundle of \(J^1(X, \mathbb{R}^\ell)\) with the fiber \(S^k\), in order to show \([\ref{6.1}]\), it is sufficient to prove that the matrix \(M_1\) given below has rank \(n + \ell + n\ell\):

\[
M_1 = \begin{pmatrix}
E_{n+l} & * & \cdots & \cdots & * \\
0 & \ast & \cdots & \cdots & \ast \\
0 & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast & \ast
\end{pmatrix}
\]
Notice that there are $\ell \leq 16$ SHUNSUKE ICHIKI
\[ \pi \delta \]
Since $S$, by Theorem 3 (1), the set $\Sigma(\Gamma_{\pi}(X,\ell))$ have $\text{Im } \Gamma_{\pi}$ arbitrary element satisfying $\Gamma_{\pi}(\tilde{q},\pi)$ is equal to $\pi$. Therefore, we get \[ (6.1) \]
Next, we will show Theorem 5 (1). Since $\text{dim } L_{\pi}(S) = \dim S^k(X,\ell)$, we have $\text{Im } \Gamma_{\pi}$ of class $C^{r-1}$ ($r \geq 2$) and we have
\[ s > m\ell + \dim X - \text{codim } S^k(X,\ell) = m\ell + \delta^*(\Gamma, S^k(X,\ell)), \]
the set $\Sigma(\Gamma, S^k(X,\ell))$ has s-dimensional Hausdorff measure zero in $L(\mathbb{R}^m, \ell)$ by Theorem 3 (2). By Theorem 3 (2), for any $\pi \in L(\mathbb{R}^m, \ell) - \Sigma(\Gamma, S^k(X,\ell))$, we have $\text{Im } \Gamma_{\pi} \cap S^k(X,\ell) = \emptyset$. Since $\Sigma_k = \Sigma(\Gamma, S^k(X,\ell))$, we obtain Theorem 5 (2).

7. PROOF OF THEOREM 6

Set $n = \dim X$. For a positive integer $\bar{n}$, we denote the $\bar{n} \times \bar{n}$ unit matrix by $E_{\bar{n}}$. Let $\Gamma : X(d) \times L(\mathbb{R}^m, \ell) \to (\ell)^d$ be the $C^r$ mapping given by
\[ (\Gamma(q, \pi) = (((g + \pi) \circ f)(q_1), \ldots, ((g + \pi) \circ f)(q_d)), \]
where $q = (q_1, \ldots, q_d)$. \[ (7.1) \]
First, we will show that $\delta(\Gamma, \Delta_d) = 0$. Let $(\tilde{q}, \tilde{\pi}) \in X(d) \times L(\mathbb{R}^m, \ell)$ be an arbitrary element satisfying $\Gamma(\tilde{q}, \tilde{\pi}) \in \Delta_d$. Then, in order to show that $\delta(\Gamma, \Delta_d) = 0$, it is sufficient to show that
\[ \dim (\text{Im } d\Gamma_{(\tilde{q}, \tilde{\pi})} + T_{\Gamma(\tilde{q}, \tilde{\pi})}\Delta_d) = d\ell. \]
Let $\{(V_{\lambda}, \varphi_{\lambda})\}_{\lambda \in \Lambda}$ be a coordinate neighborhood system of $X$. There exists a coordinate neighborhood $(V_{\lambda_1} \times \cdots \times V_{\lambda_d} \times L(\mathbb{R}^m, \ell), \varphi_{\lambda_1} \times \cdots \times \varphi_{\lambda_d} \times \text{id})$ containing $(\tilde{q}, \tilde{\pi})$ of $X(d) \times L(\mathbb{R}^m, \ell)$, where $\text{id} : L(\mathbb{R}^m, \ell) \to L(\mathbb{R}^m, \ell)$ is the identity mapping, and $\varphi_{\lambda_1} \times \cdots \times \varphi_{\lambda_d} \times \text{id} : V_{\lambda_1} \times \cdots \times V_{\lambda_d} \times L(\mathbb{R}^m, \ell) \to \varphi_{\lambda_1}(V_{\lambda_1}) \times \cdots \times \varphi_{\lambda_d}(V_{\lambda_d}) \times L(\mathbb{R}^m, \ell)$ is defined by $(\varphi_{\lambda_1}(q_1), \ldots, \varphi_{\lambda_d}(q_d), \text{id}(\pi))$. Let $x = (x_1, \ldots, x_d) \in (\mathbb{R}^n)^d$ be a local coordinate on $\varphi_{\lambda_1}(V_{\lambda_1}) \times \cdots \times \varphi_{\lambda_d}(V_{\lambda_d})$. 

Let \((a_{ij})_{1 \leq i \leq 1, 1 \leq j \leq m}\) be a representing matrix of a linear mapping \(\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)\). As in the proof of Theorem 5, the mapping \((g + \pi) \circ f : \mathbb{R} \to \mathbb{R}^d\) is given by the same expression as (6.2). Hence, \(\Gamma\) is locally given by the following:

\[
\Gamma \circ \left(\varphi_{\lambda_1} \times \cdots \times \varphi_{\lambda_d} \times \text{id}\right)^{-1}(x_1, \ldots, x_d, \pi)
= \left((g + \pi) \circ f \circ \varphi_{\lambda_1}^{-1}(x_1), \ldots, (g + \pi) \circ f \circ \varphi_{\lambda_d}^{-1}(x_d)\right)
= \left(g_1 \circ \tilde{f}(x_1) + \sum_{j=1}^{m} a_{1j} \tilde{f}_j(x_1), \ldots, g_\ell \circ \tilde{f}(x_1) + \sum_{j=1}^{m} a_{\ell j} \tilde{f}_j(x_1),
\right.
\left.
g_1 \circ \tilde{f}(x_2) + \sum_{j=1}^{m} a_{1j} \tilde{f}_j(x_2), \ldots, g_\ell \circ \tilde{f}(x_2) + \sum_{j=1}^{m} a_{\ell j} \tilde{f}_j(x_2),
\right.
\left.
\ldots \ldots \ldots,
\right.
\left.
g_1 \circ \tilde{f}(x_d) + \sum_{j=1}^{m} a_{1j} \tilde{f}_j(x_d), \ldots, g_\ell \circ \tilde{f}(x_d) + \sum_{j=1}^{m} a_{\ell j} \tilde{f}_j(x_d)\right),
\]

where \(\tilde{f}(x_i) = (\tilde{f}_1(x_i), \ldots, \tilde{f}_m(x_i)) = (f_1 \circ \varphi_{\lambda_1}^{-1}(x_i), \ldots, f_m \circ \varphi_{\lambda_d}^{-1}(x_i))\) \((1 \leq i \leq d)\).

For simplicity, set \(\bar{x} = (\varphi_{\lambda_1} \times \cdots \times \varphi_{\lambda_d})(\bar{q})\).

The Jacobian matrix of \(\Gamma\) at \((\bar{q}, \bar{x})\) is the following:

\[
J\Gamma_{(\bar{q}, \bar{x})} = \begin{pmatrix}
\star & B(x_1) \\
\star & B(x_2) \\
\vdots & \vdots \\
\star & B(x_d)
\end{pmatrix}_{(x, \pi) = (\bar{x}, \bar{x})},
\]

where

\[
B(x_i) = \begin{pmatrix}
b(x_i) & 0 \\
0 & \ddots \\
& b(x_i)
\end{pmatrix}, \quad \text{\ell rows}
\]

and \(b(x_i) = (\tilde{f}_1(x_i), \ldots, \tilde{f}_m(x_i))\). By the construction of \(T_{\Gamma(\bar{q}, \bar{x})}\Delta_d\), in order to prove (7.1), it is sufficient to prove that the rank of the following matrix \(M_2\) is equal to \(d\ell\):

\[
M_2 = \begin{pmatrix}
E_\ell & B(x_1) \\
E_\ell & B(x_2) \\
\vdots & \vdots \\
E_\ell & B(x_d)
\end{pmatrix}_{x = \bar{x}}.
\]

There exists a \(d\ell \times d\ell\) regular matrix \(Q_1\) satisfying

\[
Q_1M_2 = \begin{pmatrix}
E_\ell & B(x_1) \\
0 & B(x_2) - B(x_1) \\
\vdots & \vdots \\
0 & B(x_d) - B(x_1)
\end{pmatrix}_{x = \bar{x}}.
\]
There exists an \((\ell + m\ell) \times (\ell + m\ell)\) regular matrix \(Q_2\) satisfying

\[
Q_1 M_2 Q_2 = \begin{pmatrix}
E_{\ell} & 0 \\
0 & B(x_2) - B(x_1) \\
\vdots & \vdots \\
0 & B(x_d) - B(x_1)
\end{pmatrix}_{x=\bar{x}}
\]

where \(\bar{f}(x_1)\bar{f}(x_2) = (\bar{f}_1(x_1) - \bar{f}_1(x_1), \ldots, \bar{f}_m(x_1) - \bar{f}_m(x_1)) (2 \leq i \leq d)\) and \(x = \bar{x}\).

From \(d - 1 \leq d_f - 1\) and the definition of \(d_f\), we have

\[
\dim \sum_{i=2}^{d} \mathbb{R} \bar{f}(x_1)\bar{f}(x_i) = d - 1,
\]

where \(x = \bar{x}\). Hence, by the construction of \(Q_1 M_2 Q_2\) and \(d - 1 \leq m\), the rank of \(Q_1 M_2 Q_2\) is equal to \(d\ell\). Therefore, the rank of \(M_2\) must be equal to \(d\ell\). Hence, we get \((\ref{eq:1})\).

Since \(\delta(\Gamma, \Delta_d) = 0\), we have

\[
\delta^*(\Gamma, \Delta_d) = \dim X^{(d)} - \text{codim} \Delta_d.
\]

Note that

\[
\Sigma_d = \Sigma(\Gamma, \Delta_d).
\]

Next, we will show Theorem \(\ref{thm:1}\) \(\ref{thm:1}\). Since \(\dim X^{(d)} - \text{codim} \Delta_d \geq 0\), we obtain \(\delta^*(\Gamma, \Delta_d) \geq 0\). Notice that \(\Gamma\) is of class \(C^r (r \geq 1)\) and we have

\[
s \geq m\ell - 1 + \frac{\dim X^{(d)} - \text{codim} \Delta_d + 1}{r} = m\ell - 1 + \frac{\delta^*(\Gamma, \Delta_d) + 1}{r}.
\]

Since \(\delta(\Gamma, \Delta_d) = 0\), we get \(W(\Gamma, \Delta_d) = \emptyset\). Hence, the set \(\pi_2(W(\Gamma, \Delta_d))\) has \(s\)-dimensional Hausdorff measure zero in \(L(\mathbb{R}^m, \mathbb{R}^\ell)\). Thus, by Theorem \(\ref{thm:3}\) \(\ref{thm:3}\), the set \(\Sigma(\Gamma, \Delta_d)\) has \(s\)-dimensional Hausdorff measure zero in \(L(\mathbb{R}^m, \mathbb{R}^\ell)\). Since \(\Sigma_d = \Sigma(\Gamma, \Delta_d)\), we have Theorem \(\ref{thm:1}\) \(\ref{thm:1}\).
Finally, we will show Theorem 6 (2). Since \( \dim X^{(d)} - \text{codim} \Delta_d < 0 \), we obtain
\[
\delta^* (\Gamma, \Delta_d) < 0.
\]
Since \( \Gamma \) is of class \( C^r \) (\( r \geq 1 \)) and we have
\[
s > m\ell + \dim X^{(d)} - \text{codim} \Delta_d = m\ell + \delta^* (\Gamma, \Delta_d),
\]
the set \( \Sigma(\Gamma, \Delta_d) \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}^m, \mathbb{R}^n) \) by Theorem 3 (2c). By Theorem 3 (2c), for any \( \pi \in L(\mathbb{R}^m, \mathbb{R}^n) - \Sigma(\Gamma, \Delta_d) \), we have
\[
\text{Im} \Gamma_\pi \cap \Delta_d = \emptyset. \quad \square
\]

8. Applications of Theorems 5 and 6

In Section 8.1 (resp., Section 8.2), applications of Theorem 5 (resp., Theorem 6) are stated and proved. In Section 8.2, applications obtained by combining Theorems 5 and 6 are also given. These are also improvements of some results from the viewpoint of Lebesgue measures in [7, 8].

8.1. Applications of Theorem 5

A \( C^r \) function \( f : X \to \mathbb{R} \) (\( r \geq 2 \)) is called a Morse function if all of the critical points of \( f \) are nondegenerate, where \( X \) is a \( C^r \) manifold. For details on Morse functions, see for example, [4, p. 63]. In the case \( \ell = 1 \), we have the following.

**Corollary 1.** Let \( f \) be a \( C^r \) immersion of a \( C^r \) manifold \( X \) into an open subset \( V \) of \( \mathbb{R}^m \) and \( g : V \to \mathbb{R} \) be a \( C^r \) function, where \( r \) is an integer satisfying \( r \geq 2 \). Set
\[
\Sigma = \{ \pi \in L(\mathbb{R}^m, \mathbb{R}) \mid (g + \pi) \circ f : X \to \mathbb{R} \text{ is not a Morse function} \}.
\]

Then, for any real number \( s \) satisfying
\[
(8.1) \quad s \geq m - 1 + \frac{1}{r - 1},
\]
the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}^m, \mathbb{R}) \).

**Remark 5.** In Corollary 4 if all manifolds and mappings are of class \( C^\infty \), then we can replace (8.1) by \( s > m - 1 \) by the same argument as in Remark 4.

**Proof of Corollary 4.** It is clearly seen that \( \Sigma \) is the set consisting of all elements \( \pi \in L(\mathbb{R}^m, \mathbb{R}) \) satisfying that \( j^x((g + \pi) \circ f) \) is not transverse to \( S^1(X, \mathbb{R}) \). Since \( \dim X - \text{codim} S^1(X, \mathbb{R}) = 0 \), by Theorem 5 (1), for any real number \( s \) satisfying (8.1), the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}^m, \mathbb{R}) \). \( \square \)

**Example 5** (An example of Corollary 4). Set \( X = V = \mathbb{R} \) and \( f(x) = x \) in Corollary 4. Let \( g : \mathbb{R} \to \mathbb{R} \) be a \( C^r \) function, where \( r \) is a positive integer satisfying \( r \geq 2 \). As in Corollary 4, set
\[
\Sigma = \{ \pi \in L(\mathbb{R}, \mathbb{R}) \mid g + \pi : \mathbb{R} \to \mathbb{R} \text{ is not a Morse function} \}.
\]

By Corollary 4 for any real number \( s \) satisfying \( s \geq \frac{1}{r - 1} \), the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}, \mathbb{R}) \). We regard the Cantor set \( K \) of \( \mathbb{R} \) as a subset of \( L(\mathbb{R}, \mathbb{R}) \).

In the case \( r \geq 3 \), since \( \Sigma \) has \( \frac{1}{2} \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}, \mathbb{R}) \), we obtain
\[
\text{HD}_{L(\mathbb{R}, \mathbb{R})}(\Sigma) < \text{HD}_{L(\mathbb{R}, \mathbb{R})}(K) = \frac{\log 2}{\log 3} = 0.63 \cdots .
\]

On the other hand, in the case \( r = 2 \), there exists an example such that \( \Sigma = K \) as follows. Let \( \xi : \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) function such that the set consisting of all critical values of \( \xi \) is the Cantor set. Note that the existence of the mapping \( \xi \) can be easily
shown from Proposition 2 of [13] [1]. Let \( g(x) = -\int_0^x \xi(x)dx \). Note that \( g \) is of class \( C^2 \). Then, \( \pi \in \Sigma \) if and only if there exists \( x \in \mathbb{R} \) such that \( a = \xi(x) \) and \( \frac{\partial \xi}{\partial x}(x) = 0 \), where \( \pi(x) = ax \ (a \in \mathbb{R}) \). Thus, we obtain \( \Sigma = K \).

Finally, we explain an advantage of using a result from the viewpoint of Hausdorff measures with this example. Since any subset of \( L(\mathbb{R}, \mathbb{R}) \) whose Hausdorff dimension is less than 1 has Lebesgue measure zero in \( L(\mathbb{R}, \mathbb{R}) \), we cannot investigate whether the bad set \( \Sigma \) is equal to \( K \) or not by transversality results from the viewpoint of Lebesgue measures, such as Proposition 1 and Theorems 1 and 2. On the other hand, in the case \( r \geq 3 \) of this example, we can show that \( \Sigma \) is never equal to \( K \) by Corollary 1 which is a result from the viewpoint of Hausdorff measures.

Let \( f : X \to \mathbb{R}^{2n-1} \) \((n \geq 2)\) be a \( C^\infty \) mapping, where \( X \) is a \( C^\infty \) manifold of dimension \( n \). A singular point \( q \in X \) of \( f \) is called a singular point of Whitney umbrella if there exist two germs of \( C^\infty \) diffeomorphisms \( H : (\mathbb{R}^{2n-1}, f(q)) \to (\mathbb{R}^{2n-1}, 0) \) and \( h : (X, q) \to (\mathbb{R}^n, 0) \) such that \( H \circ f \circ h^{-1}(x_1, x_2, \ldots, x_n) = (x_1^2, x_1x_2, \ldots, x_1x_n, x_2, \ldots, x_n) \), where \((x_1, x_2, \ldots, x_n)\) is a local coordinate around \( h(q) = 0 \in \mathbb{R}^n \). In the case \( \ell = 2 \dim X - 1 \) \((\dim X \geq 2)\), we have the following.

**Corollary 2.** Let \( f \) be a \( C^\infty \) immersion of an \( n \)-dimensional \( C^\infty \) manifold \( X \) \((n \geq 2)\) into an open subset \( V \) of \( \mathbb{R}^m \) and \( g : V \to \mathbb{R}^{2n-1} \) be a \( C^\infty \) mapping. Let \( \Sigma \) be the set consisting of all elements \( \pi \in L(\mathbb{R}^m, \mathbb{R}^{2n-1}) \) not satisfying that any singular point of \((g + \pi) \circ f : X \to \mathbb{R}^{2n-1}\) is a singular point of Whitney umbrella. Then, for any real number \( s \) satisfying \( s > m(2n-1) - 1 \), the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}^m, \mathbb{R}^{2n-1}) \).

**Proof of Corollary 2.** Let \( k \) be an integer satisfying \( 1 \leq k \leq n \). As in Theorem 1 set

\[
\Sigma_k = \{ \pi \in L(\mathbb{R}^m, \mathbb{R}^{2n-1}) \mid j^1((g + \pi) \circ f) \text{ is not transverse to } S^k(X, \mathbb{R}^{2n-1}) \}.
\]

Note that

\[
\dim X - \text{codim } S^k(X, \mathbb{R}^{2n-1}) = n - k(n - 1 + k).
\]

First, we consider the case \( k = 1 \). Since \( \dim X - \text{codim } S^1(X, \mathbb{R}^{2n-1}) = 0 \) and \( s > m(2n-1) - 1 \), the set \( \Sigma_1 \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}^m, \mathbb{R}^{2n-1}) \) by Theorem 1, Remark 1, and Remark 2.

Next, we consider the case \( 2 \leq k \leq n \). In this case, since

\[
s > m(2n-1) - 1 \geq m(2n-1) + \dim X - \text{codim } S^k(X, \mathbb{R}^{2n-1}),
\]

by Theorem 2, we have the following:

(a) The set \( \Sigma_k \) has \( s \)-dimensional Hausdorff measure zero in \( L(\mathbb{R}^m, \mathbb{R}^{2n-1}) \).

(b) For any \( \pi \in L(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma_k \), we have \( j^1((g + \pi) \circ f)(X) \cap S^k(X, \mathbb{R}^{2n-1}) = \emptyset \).

Note that a point \( q \in X \) is a singular point of Whitney umbrella of \((g + \pi) \circ f\) if and only if \( j^1((g + \pi) \circ f)(q) \in S^1(X, \mathbb{R}^{2n-1}) \) and \( j^1((g + \pi) \circ f) \) is transverse to \( S^1(X, \mathbb{R}^{2n-1}) \) at \( q \) (for this fact, see for example [4] p. 179). By this fact and (b),

---

1 Proposition 2 of [13] is as follows: If \( K \) is a compact subset of the closed interval \([0,1]\), then \( K \) has Lebesgue measure zero if and only if the set consisting of all critical values of \( \xi \) is equal to \( K \) for some \( C^1 \) function \( \xi : \mathbb{R} \to \mathbb{R} \).

Since the Cantor set \( K \) is a compact subset of \([0,1]\) with Lebesgue measure zero, we can guarantee the existence of the above function \( \xi : \mathbb{R} \to \mathbb{R} \).
we can easily obtain $\Sigma = \bigcup_{k=1}^{n} \Sigma_k$. Since $\Sigma_1, \ldots, \Sigma_n$ have $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$, the set $\Sigma$ also has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$. 

In the case $\ell \geq 2 \dim X$, we have the following.

**Corollary 3.** Let $f$ be a $C^r$ immersion of an $n$-dimensional $C^r$ manifold $X$ into an open subset $V$ of $\mathbb{R}^m$ and $g : V \rightarrow \mathbb{R}^\ell$ be a $C^r$ mapping, where $\ell \geq 2n$ and $r \geq 2$. Set

$$\Sigma = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid (g + \pi) \circ f : X \rightarrow \mathbb{R}^\ell \text{ is not an immersion} \}.$$ Then, for any real number $s$ satisfying

$$s > m\ell + (2n - \ell - 1),$$

the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

**Proof of Corollary**. Let $k$ be an integer satisfying $1 \leq k \leq n$. As in Theorem 5 set

$$\Sigma_k = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid j^1((g + \pi) \circ f) \text{ is not transverse to } S^k(X, \mathbb{R}^\ell) \}.$$ Since $\ell \geq 2n$, we have

$$\dim X - \text{codim } S^k(X, \mathbb{R}^\ell) \leq \dim X - \text{codim } S^1(X, \mathbb{R}^\ell) = 2n - \ell - 1 < 0.$$ Hence, since

$$s > m\ell + (2n - \ell - 1) \geq m\ell + (\dim X - \text{codim } S^k(X, \mathbb{R}^\ell),$$

by Theorem 3, we have the following:

(a) The set $\Sigma_k$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

(b) For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_k$, we have $j^1((g + \pi) \circ f)(X) \cap S^k(X, \mathbb{R}^\ell) = \emptyset$.

By (b), we can easily obtain $\Sigma = \bigcup_{k=1}^{n} \Sigma_k$. By (a), the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. 

**Example 6** (An example of Corollary 5). Set $X = V = \mathbb{R}$ and $f(x) = x$ in Corollary 5. Let $g : \mathbb{R} \rightarrow \mathbb{R}^\ell$ ($\ell \geq 2$) be the $C^\infty$ mapping defined by $g(x) = (x^2, \ldots, x^2)$. As in Corollary 5 set

$$\Sigma = \{ \pi \in \mathcal{L}(\mathbb{R}, \mathbb{R}^\ell) \mid g + \pi : \mathbb{R} \rightarrow \mathbb{R}^\ell \text{ is not an immersion} \}.$$ Then, for any real number $s$ satisfying $s > 1$, the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$ by Corollary 5. Indeed, by the following direct calculation, we obtain $\Sigma = B$, where

$$B = \{ \pi = (\pi_1, \ldots, \pi_\ell) \in \mathcal{L}(\mathbb{R}, \mathbb{R}^\ell) \mid \pi_1 = \cdots = \pi_\ell \}.$$ Since $B$ does not have 1-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$, we cannot improve the assumption $s > 1$.

Now, we prove $\Sigma = B$. First, we show $\Sigma \subset B$. Let $\pi = (\pi_1, \ldots, \pi_\ell) \in \Sigma$ be an arbitrary element. Set $\pi_i(x) = a_i x$ ($a_i \in \mathbb{R}$) for $i = 1, \ldots, \ell$. Then, there exists $\bar{x} \in \mathbb{R}$ such that $2\bar{x} + a_i = 0$ for all $i = 1, \ldots, \ell$. Since $a_1 = \cdots = a_\ell$, we have $\pi \in B$.

Next, we show $B \subset \Sigma$. Let $\pi \in B$ be an arbitrary element. Then, we can express $\pi_i(x) = ax$ ($a \in \mathbb{R}$) for all $i = 1, \ldots, \ell$. Set $\bar{x} = -\frac{a}{2}$. Since $d(g + \pi)\bar{x} = 0$, we obtain $\pi \in \Sigma$.

Finally, we explain an advantage of using a result from the viewpoint of Hausdorff measures with this example. Since any subset of $\mathcal{L}(\mathbb{R}, \mathbb{R}^\ell)$ whose Hausdorff
Remark 6. In Corollary 4, if all manifolds and mappings are of class $C^r \mapsto C^{\ell}$, we can replace (8.2) by

\[ \text{Let } \pi \text{ be a mapping with normal crossings, } \pi \colon X \to \mathbb{R}^\ell \text{ transverse to } \Delta_d. \]

\[ \text{Proof of Corollary 4.} \]

In the following, for a set $S$, we denote the number of its elements (or its cardinality) by $|S|$. In the case $\dim X > \ell$, we have the following.

**Corollary 4.** Let $f : X \to \mathbb{R}^\ell$ be a $C^r$ mapping, where $X$ is a $C^r$ manifold $(r \geq 1)$. Then, $f$ is called a mapping with normal crossings if for any integer $d$ satisfying $d \geq 2$, the mapping $f^{(d)} : X^{(d)} \to (\mathbb{R}^\ell)^d$ is transverse to $\Delta_d$.

In the following, for a set $S$, we denote the number of its elements (or its cardinality) by $|S|$. In the case $\dim X > \ell$, we have the following.

**Corollary 4.** Let $f : X \to \mathbb{R}^\ell$ be a $C^r$ injection of an $n$-dimensional $C^r$ manifold $X$ into an open subset $V$ of $\mathbb{R}^m$ and $g : V \to \mathbb{R}^\ell$ be a $C^r$ mapping, where $r$ is a positive integer. Suppose $n > r$. Let $\Sigma$ be the set consisting of all elements $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ not satisfying that $((g + \pi) \circ f)^{(d)} : X^{(d)} \to (\mathbb{R}^\ell)^d$ is transverse to $\Delta_d$ for any integer $d$ satisfying $2 \leq d \leq d_f$. Then, for any real number $s$ satisfying

\[ s \geq m\ell - 1 + \frac{df(n - \ell) + \ell + 1}{r}, \]

the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Moreover, if a mapping $(g + \pi) \circ f : X \to \mathbb{R}^\ell$ ($\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$) satisfies that $|((g + \pi) \circ f)^{-1}(y)| \leq d_f$ for any $y \in \mathbb{R}^\ell$, then $(g + \pi) \circ f$ is a $C^r$ mapping with normal crossings.

**Remark 6.** In Corollary 4 if all manifolds and mappings are of class $C^\infty$, then we can replace (8.2) by $s > m\ell - 1$ by the same argument as in Remark [Remark 1].

**Proof of Corollary 4.** Let $d$ be an integer satisfying $2 \leq d \leq d_f$. As in Theorem 6 set

\[ \Sigma_d = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid ((g + \pi) \circ f)^{(d)} \text{ is not transverse to } \Delta_d \}. \]

Since $n > \ell$, we obtain

\[ \dim X^{(d)} - \text{codim } \Delta_d = nd - \ell(d - 1) = d(n - \ell) + \ell \geq 0, \]

\[ df(n - \ell) + \ell \geq d(n - \ell) + \ell = \dim X^{(d)} - \text{codim } \Delta_d. \]

Thus, we also have

\[ s \geq m\ell - 1 + \frac{df(n - \ell) + \ell + 1}{r} \geq m\ell - 1 + \frac{\dim X^{(d)} - \text{codim } \Delta_d + 1}{r}. \]

Hence, by Theorem 6, $\Sigma_d$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

Since $\Sigma = \bigcup_{d=2}^{d_f} \Sigma_d$, the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

Now, suppose that a mapping $(g + \pi) \circ f : X \to \mathbb{R}^\ell$ ($\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$) satisfies that $|((g + \pi) \circ f)^{-1}(y)| \leq d_f$ for any $y \in \mathbb{R}^\ell$. Then, since $((g + \pi) \circ f)^{(d)}(X^{(d)}) \cap \Delta_d = \emptyset$ for any integer $d$ satisfying $d > d_f$, the mapping $(g + \pi) \circ f$ is a $C^r$ mapping with normal crossings. $\square$

In the case $\dim X \leq \ell \leq 2\dim X$, we have the following.
Corollary 5. Let $f$ be a $C^r$ injection of an $n$-dimensional $C^r$ manifold $X$ into an open subset $V$ of $\mathbb{R}^m$ and $g : V \to \mathbb{R}^f$ be a $C^r$ mapping, where $r$ is a positive integer. Suppose $n \leq \ell \leq 2n$. Let $\Sigma$ be the set consisting of all elements $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$ not satisfying that $((g + \pi) \circ f)^{(d)} : X^{(d)} \to (\mathbb{R}^f)^d$ is transverse to $\Delta_d$ for any integer $d$ satisfying $2 \leq d \leq d_f$. Then, for any real number $s$ satisfying
\begin{equation}
s \geq m\ell - 1 + \frac{2n - \ell + 1}{r},
\end{equation}
the set $\Sigma$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$. Moreover, if a mapping $(g + \pi) \circ f : X \to \mathbb{R}^f$ ($\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^f) - \Sigma$) satisfies that $|((g + \pi) \circ f)^{-1}(y)| \leq d_f$ for any $y \in \mathbb{R}^f$, then $(g + \pi) \circ f$ is a $C^r$ mapping with normal crossings.

Remark 7. In Corollary 5, if all manifolds and mappings are of class $C^\infty$, then we can replace (8.3) by $s > m\ell - 1$ by the same argument as in Remark 3.

Proof of Corollary 5. Let $d$ be an integer satisfying $2 \leq d \leq d_f$. As in Theorem 6 set
\[ \Sigma_d = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^f) \mid ((g + \pi) \circ f)^{(d)} \text{ is not transverse to } \Delta_d \}. \]
Since $n \leq \ell \leq 2n$, we obtain
\[ \dim X^{(d)} - \text{codim } \Delta_d = nd - \ell(d - 1) = d(n - \ell) + \ell \leq 2(n - \ell) + \ell = 2n - \ell. \]
First, we consider the case $\dim X^{(d)} - \text{codim } \Delta_d \geq 0$. Since
\[ s \geq m\ell - 1 + \frac{2n - \ell + 1}{r} \geq m\ell - 1 + \frac{\dim X^{(d)} - \text{codim } \Delta_d + 1}{r}, \]
by Theorem 6, $\Sigma_d$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$. Secondly, we consider the case $\dim X^{(d)} - \text{codim } \Delta_d < 0$. Since
\[ s \geq m\ell - 1 + \frac{2n - \ell + 1}{r} > m\ell + \dim X^{(d)} - \text{codim } \Delta_d, \]
by Theorem 6, $\Sigma_d$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$. Since $\Sigma = \bigcup_{d=2}^{d_f} \Sigma_d$, the set $\Sigma$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$.

The latter assertion can be shown by the same argument as in the proof of Corollary 4.

In the case $2 \dim X < \ell$, we have the following.

Corollary 6. Let $f$ be a $C^r$ injection of an $n$-dimensional $C^r$ manifold $X$ into an open subset $V$ of $\mathbb{R}^m$ and $g : V \to \mathbb{R}^f$ be a $C^r$ mapping ($r \geq 1$). Suppose $2n < \ell$. Let $\Sigma$ be the set consisting of all elements $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$ not satisfying that $(g + \pi) \circ f : X \to \mathbb{R}^f$ is injective. Then, for any real number $s$ satisfying $s > m\ell + 2n - \ell$, the set $\Sigma$ has s-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^f)$.

Proof of Corollary 6. As in Theorem 6 set
\[ \Sigma_2 = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^f) \mid ((g + \pi) \circ f)^{(2)} \text{ is not transverse to } \Delta_2 \}. \]
Since $2n < \ell$, we obtain
\[ \dim X^{(2)} - \text{codim } \Delta_2 = 2n - \ell < 0. \]
Hence, since
\[ s > m\ell + 2n - \ell = m\ell + \dim X^{(2)} - \text{codim } \Delta_2, \]
by Theorem 2, we have the following:
(a) The set $\Sigma_2$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.
(b) For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_2$, we have $((g + \pi) \circ f)^{(2)}(X^{(2)}) \cap \Delta_2 = \emptyset$.

By (b), we obtain $\Sigma = \Sigma_2$. Since $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ by (a), this corollary holds.

By combining Corollaries 3 and 10 we have the following.

Corollary 7. Let $f$ be a $C^r$ injective immersion of an $n$-dimensional $C^r$ manifold $X$ into an open subset $V$ of $\mathbb{R}^m$ and $g : V \to \mathbb{R}^\ell$ be a $C^r$ mapping ($r \geq 2$). Suppose $2n < \ell$. Let $\Sigma$ be the set consisting of all elements $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ not satisfying that $(g + \pi) \circ f : X \to \mathbb{R}^\ell$ is an injective immersion. Then, for any real number $s$ satisfying $s > m\ell + 2n - \ell$, the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

By Corollary 4 we obtain the following.

Corollary 8. Let $X$ be a compact $C^r$ manifold ($r \geq 2$) of dimension $n$. Let $f$ be a $C^r$ embedding of $X$ into an open subset $V$ of $\mathbb{R}^m$ and $g : V \to \mathbb{R}^\ell$ be a $C^r$ mapping. Suppose $2n < \ell$. Let $\Sigma$ be the set consisting of all elements $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ not satisfying that $(g + \pi) \circ f : X \to \mathbb{R}^\ell$ is an embedding. Then, for any real number $s$ satisfying $s > m\ell + 2n - \ell$, the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.

9. An application to multiobjective optimization

The purpose of this section is to give an application of Theorem 5 (and hence Theorem 3) to multiobjective optimization (see Theorem 9). For a positive integer $\ell$, set

$$L = \{1, \ldots, \ell\}.$$  

We consider the problem of optimizing several functions simultaneously. More precisely, let $f : X \to \mathbb{R}^\ell$ be a mapping, where $X$ is a given arbitrary set. A point $x \in X$ is called a Pareto solution of $f$ if there does not exist another point $y \in X$ such that $f_i(y) \leq f_i(x)$ for all $i \in L$ and $f_j(y) < f_j(x)$ for at least one index $j \in L$. We denote the set consisting of all Pareto solutions of $f$ by $X^*(f)$, which is called the Pareto set of $f$. The set $f(X^*(f))$ is called the Pareto front of $f$. The problem of determining $X^*(f)$ is called the problem of minimizing $f$.

Let $f = (f_1, \ldots, f_\ell) : X \to \mathbb{R}^\ell$ be a mapping, where $X$ is a given arbitrary set. For a non-empty subset $I = \{i_1, \ldots, i_k\}$ of $L$ such that $i_1 < \cdots < i_k$, set

$$f_I = (f_{i_1}, \ldots, f_{i_k}).$$

The problem of determining $X^*(f_I)$ is called a subproblem of the problem of minimizing $f$. Set

$$\Delta^{\ell-1} = \left\{(w_1, \ldots, w_\ell) \in \mathbb{R}^\ell \mid \sum_{i=1}^{\ell} w_i = 1, \ w_i \geq 0 \right\}.$$  

We also denote a face of $\Delta^{\ell-1}$ for a non-empty subset $I$ of $L$ by

$$\Delta_I = \left\{(w_1, \ldots, w_\ell) \in \Delta^{\ell-1} \mid w_i = 0 \ (i \notin I) \right\}.$$
In this section, for a $C^r$ manifold $N$ (possibly with corners) and a subset $V$ of $\mathbb{R}^\ell$, a mapping $g : N \to V$ is called a $C^r$ mapping (resp., a $C^r$ diffeomorphism) if $g : N \to \mathbb{R}^\ell$ is of class $C^r$ (resp., $g : N \to \mathbb{R}^\ell$ is a $C^r$ immersion and $g : N \to V$ is a homeomorphism), where $r$ is a positive integer or $r = \infty$. Here, $C^0$ mappings and $C^0$ diffeomorphisms are continuous mappings and homeomorphisms, respectively.

**Definition 5** (\cite{5,6}). Let $f = (f_1, \ldots, f_\ell) : X \to \mathbb{R}^\ell$ be a mapping, where $X$ is a subset of $\mathbb{R}^m$. Let $r$ be an integer satisfying $r \geq 0$ or $r = \infty$. The problem of minimizing $f$ is $C^r$ simplicial if there exists a $C^r$ mapping $\Phi : \Delta^{\ell-1} \to X^*(f)$ such that both the mappings $\Phi|_{\Delta_I} : \Delta_I \to X^*(f_I)$ and $f|_{X^*(f_I)} : X^*(f_I) \to f(X^*(f_I))$ are $C^r$ diffeomorphisms for any non-empty subset $I$ of $L$. The problem of minimizing $f$ is $C^r$ weakly simplicial\footnote{In \cite{6}, the problem of minimizing $f : X \to \mathbb{R}^\ell$ is said to be $C^r$ weakly simplicial if there exists a $C^r$ mapping $\phi : \Delta^{\ell-1} \to f(X^*(f_I))$ satisfying $\phi(\Delta_I) = f(X^*(f_I))$ for any non-empty subset $I$ of $L$. On the other hand, a surjective mapping of $\Delta^{\ell-1}$ into $X^*(f)$ is important to describe $X^*(f)$. Hence, the definition of weak simpliciality in \cite{6} is updated from that in \cite{5}. Thus, in this paper, we adopt the definition in \cite{6}.} if there exists a $C^r$ mapping $\phi : \Delta^{\ell-1} \to X^*(f)$ such that $\phi(\Delta_I) = X^*(f_I)$ for any non-empty subset $I$ of $L$.

As described in \cite{5}, simpliciality is an important property, which can be seen in several practical problems ranging from facility location studied half a century ago \cite{11} to sparse modeling actively developed today \cite{5}. If a problem is simplicial, then we can efficiently compute a parametric-surface approximation of the entire Pareto set with few sample points \cite{10}. A subset $X$ of $\mathbb{R}^m$ is convex if $tx + (1-t)y \in X$ for all $x, y \in X$ and all $t \in [0,1]$. Let $X$ be a convex set in $\mathbb{R}^m$. A function $f : X \to \mathbb{R}$ is strongly convex if there exists $\alpha > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t)\|x-y\|^2$$

for all $x, y \in X$ and all $t \in [0,1]$, where $\|z\|$ is the Euclidean norm of $z \in \mathbb{R}^m$. The constant $\alpha$ is called a convexity parameter of the function $f$. A mapping $f = (f_1, \ldots, f_\ell) : X \to \mathbb{R}^\ell$ is strongly convex if $f_i$ is strongly convex for any $i \in L$. The problem of minimizing a strongly convex $C^r$ mapping is called the strongly convex $C^r$ problem.

**Theorem 7** (\cite{3,6}). Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ be a strongly convex $C^r$ mapping, where $r$ is a positive integer or $r = \infty$. Then, the problem of minimizing $f$ is $C^r$ weakly simplicial. Moreover, this problem is $C^r$-1 simplicial if the rank of the differential $df_x$ is equal to $\ell - 1$ for any $x \in X^*(f)$.

For the assertion on simpliciality (resp., weak simpliciality) of Theorem\cite{7} in the case $r \geq 2$, see \cite{3} Theorem 1.1 (resp., \cite{6} Theorem 5)). For the case $r = 1$ of Theorem\cite{7} see \cite{6} Theorem 2).

In \cite{5}, as an application of singularity theory to a strongly convex problem, we have the following result. Here, note that strong convexity is preserved under linear perturbations (see Lemma\cite{8} in Section\cite{10}).

**Theorem 8** (\cite{5}). Let $f : \mathbb{R}^m \to \mathbb{R}^\ell$ ($m \geq \ell$) be a strongly convex $C^r$ mapping, where $r$ is an integer satisfying $r \geq 2$ or $r = \infty$. Set

$$\Sigma = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid \text{The problem of minimizing } f + \pi \text{ is not } C^r-1 \text{ simplicial} \}. $$

If $m - 2\ell + 4 > 0$, then $\Sigma$ has Lebesgue measure zero in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$.
The following is an application of singularity theory to multiobjective optimization in this paper, which is an improvement of Theorem 8 from the viewpoint of Hausdorff measures.

**Theorem 9.** Let \( f : \mathbb{R}^m \to \mathbb{R}^\ell \) (\( m \geq \ell \)) be a strongly convex \( C^r \) mapping, where \( r \) is an integer satisfying \( r \geq 2 \) or \( r = \infty \). Set

\[
\Sigma = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid \text{The problem of minimizing } f + \pi \text{ is not } C^{r-1} \text{ simplicial} \}.
\]

If \( m - 2\ell + 4 > 0 \), then for any non-negative real number \( s \) satisfying

\[
s > m\ell - (m - 2\ell + 4),
\]

the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \).

**Remark 8.** There is an example of Theorem 9 such that we cannot improve the inequality \((9.1)\) (see Example 7).

In order to show that a given mapping in Example 7 is strongly convex, we prepare Lemma 7, which is a well-known result (for the proof, for example, see [6]). Let \( X \) be a convex subset of \( \mathbb{R}^m \). A function \( f : X \to \mathbb{R} \) is said to be convex if

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

for all \( x, y \in X \) and all \( t \in [0, 1] \).

**Lemma 7.** Let \( X \) be a convex subset of \( \mathbb{R}^m \). Then, a function \( f : X \to \mathbb{R} \) is strongly convex with a convexity parameter \( \alpha > 0 \) if and only if the function \( g : X \to \mathbb{R} \) defined by \( g(x) = f(x) - \frac{\alpha}{2} \|x\|^2 \) is convex.

**Example 7** (An example of Theorem 9). Let \( f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) be the mapping defined by \( f_i(x_1, x_2) = x_1^2 + x_2^2 \) for \( i = 1, 2 \). Since \( g(x) = f_i(x) - \frac{1}{2} \|x\|^2 = 0 \) is convex, \( f \) is strongly convex by Lemma 7 where \( x = (x_1, x_2) \). As in Theorem 9, set

\[
\Sigma = \{ \pi \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \mid \text{The problem of minimizing } f + \pi \text{ is not } C^\infty \text{ simplicial} \}.
\]

Then, for any real number \( s \) satisfying \( s > 2 \), the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \) by Theorem 9.

Indeed, by the following direct calculation, we obtain \( \Sigma = B \), where

\[
B = \{ \pi = (\pi_1, \pi_2) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \mid \pi_1 = \pi_2 \}.
\]

Since \( B \) does not have 2-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) \), we cannot improve the assumption \( s > 2 \).

Now, we show \( \Sigma = B \). First, in order to show that \( \Sigma \subset B \), we will show that \( \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) - B \subset \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) - \Sigma \). Let \( \pi \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) - B \) be an arbitrary element. Let \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) be the diffeomorphism defined by \( H(X_1, X_2) = (X_1 - X_2, X_2) \). As \( f_1 = f_2 \), we obtain \( H \circ (f + \pi) = (\pi_1 - \pi_2, f_2 + \pi_2) \). Since \( \pi_1 - \pi_2 \) is a linear function satisfying \( \pi_1 - \pi_2 \neq 0 \), it follows that \( \text{rank } d(H \circ (f + \pi))_x \geq 1 \) for any \( x \in \mathbb{R}^2 \). As \( H \) is a diffeomorphism, we have that \( \text{rank } d(f + \pi)_x \geq 1 \) for any \( x \in \mathbb{R}^2 \). By Theorem 9, the problem of minimizing \( f + \pi \) is \( C^\infty \) simplicial. Namely, we obtain \( \pi \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) - \Sigma \).
Next, we will show that \( B \subset \Sigma \). Let \( \pi = (\pi_1, \pi_2) \in B \) be an arbitrary element. Set \( \pi_1(x_1, x_2) = \pi_2(x_1, x_2) = a_1 x_1 + a_2 x_2 \), where \( a_1, a_2 \in \mathbb{R} \). Since
\[
(f_i + \pi_i)(x_1, x_2) = x_i^2 + x_2^2 + a_1 x_1 + a_2 x_2
\]
for \( i = 1, 2 \), we obtain \( X^*(f + \pi) = \{ (-\frac{a_1}{2}, -\frac{a_2}{2}) \} \subset \mathbb{R}^2 \). Hence, the problem of minimizing \( f + \pi \) is not \( C^0 \) simplicial (and hence, not \( C^\infty \) simplicial). Namely, we obtain \( \pi \in \Sigma \).

10. Proof of Theorem 9

Since Theorem 9 clearly holds by combining the following two results (Lemmas 8 and 9) and Theorem 7, it is sufficient to prove Lemma 9.

Lemma 8 (5). Let \( f : \mathbb{R}^m \to \mathbb{R}^\ell \) be a strongly convex mapping. Then, for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \), the mapping \( f + \pi : \mathbb{R}^m \to \mathbb{R}^\ell \) is also strongly convex.

Lemma 9. Let \( f : \mathbb{R}^m \to \mathbb{R}^\ell \) (\( m \geq \ell \)) be a \( C^r \) mapping (\( r \geq 2 \)). Set
\[
\Sigma = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid \text{There exists } x \in \mathbb{R}^m \text{ such that } \text{rank } d(f + \pi)_x \leq \ell - 2 \}.
\]
If \( m - 2\ell + 4 > 0 \), then for any non-negative real number \( s \) satisfying
\[
s > m \ell - (m - 2\ell + 4),
\]
the set \( \Sigma \) has \( s \)-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \).

On Lemma 9 we give the following remark.

Remark 9. (1) In the case \( \ell = 1 \), note that \( \Sigma = \emptyset \) and \( m \ell - (m - 2\ell + 4) = -2 \).

Thus, in the case, since the set \( \Sigma = \emptyset \) has 0-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \), Lemma 9 clearly holds.

(2) In the case \( \ell \geq 2 \), since \( m \geq \ell \), we have \( \text{codim } S^2(\mathbb{R}^m, \mathbb{R}^\ell) = 2(\ell - m) \).

Thus, the inequality (10.1) implies that
\[
s > m \ell - (m - 2\ell + 4) = m \ell + m - 2(\ell - m) = m \ell + m - \text{codim } S^2(\mathbb{R}^m, \mathbb{R}^\ell).
\]

Proof of Lemma 9 By Remark 9 (1), it is sufficient to consider the case \( \ell \geq 2 \). As in Remark 9 (2), we have
\[
\text{codim } S^2(\mathbb{R}^m, \mathbb{R}^\ell) = 2(\ell - m).
\]
Since \( m - 2\ell + 4 > 0 \), we also have \( \text{codim } S^2(\mathbb{R}^m, \mathbb{R}^\ell) > m \).

Let \( k \) be an integer satisfying \( 2 \leq k \leq \ell \). As in Theorem 9 set
\[
\Sigma_k = \{ \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \mid j^1(f + \pi) \text{ is not transverse to } S^k(\mathbb{R}^m, \mathbb{R}^\ell) \}.
\]
It follows that
\[
m - \text{codim } S^k(\mathbb{R}^m, \mathbb{R}^\ell) \leq m - \text{codim } S^2(\mathbb{R}^m, \mathbb{R}^\ell) < 0.
\]
By Remark 9 (2), note that a given real number \( s \) satisfies that
\[
s > m \ell + m - \text{codim } S^k(\mathbb{R}^m, \mathbb{R}^\ell).
\]
Since \( r \geq 2 \), by Theorem 9 (2), we have the following:

(a) The set \( \Sigma_k \) has \( s \)-dimensional Hausdorff measure zero in \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \).

(b) For any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_k \), we have \( j^1(f + \pi)(\mathbb{R}^m) \cap S^k(\mathbb{R}^m, \mathbb{R}^\ell) = \emptyset \).
By (b), it is clearly seen that $\Sigma = \bigcup_{k=2}^{\ell} \Sigma_k$. By (a), the set $\Sigma$ has $s$-dimensional Hausdorff measure zero in $L(\mathbb{R}^m, \mathbb{R}^\ell)$. □

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References

[1] James William Bruce and Neil Kirk. Generic projections of stable mappings. *Bull. Lond. Math. Soc.*, 32(6):718–728, 2000.
[2] Carlos Gustavo Tamm de Araujo Moreira. Hausdorff measures and the morse-sard theorem. *Publ. Mat.*, 45(1):149–162, 2001.
[3] Herbert Federer. *Geometric measure theory*, volume 153. Springer-Verlag New York Inc., 1969.
[4] Martin Golubitsky and Victor Guillemin. *Stable Mappings and Their Singularities*, volume 14 of Graduate Texts in Mathematics. Springer-Verlag, New York Heidelberg, 1973.
[5] Naoki Hamada, Kenta Hayano, Shunsuke Ichiki, Yutaro Kabata, and Hiroshi Teramoto. Topology of Pareto sets of strongly convex problems. *SIAM J. Optim.*, 30(3):2659–2686, 2020.
[6] Naoki Hamada and Shunsuke Ichiki. Simpliciality of strongly convex problems. to appear in *J. Math. Soc. Japan*. [https://arxiv.org/abs/1912.09328](https://arxiv.org/abs/1912.09328)
[7] Shunsuke Ichiki. Composing generic linearly perturbed mappings and immersions/injections. *J. Math. Soc. Japan*, 70(3):1165–1184, 2018.
[8] Shunsuke Ichiki. Transversality theorems on generic linearly perturbed mappings. *Methods Appl. Anal.*, 25(4):323–335, 2018.
[9] Shunsuke Ichiki. Characterization of generic transversality. *Bull. Lond. Math. Soc.*, 51:978–988, 2019.
[10] Ken Kobayashi, Naoki Hamada, Akiyoshi Sannai, Akinori Tanaka, Kenichi Bannai, and Masashi Sugiyama. Bézier simplex fitting: Describing Pareto fronts of simplical problems with small samples in multi-objective optimization. In *Proceedings of the Thirty-Third AAAI Conference on Artificial Intelligence*, volume 33 of AAAI-19, 2304–2313, 2019.
[11] Harold W Kuhn. On a pair of dual nonlinear programs. *Nonlinear Programming*, 1:37–54, 1967.
[12] John Norman Mather. Generic projections. *Ann. of Math.* (2), 98:226–245, 1973.
[13] Alec Norton. A $C^{1,\infty}$ function with an interval of critical values. *Indiana Univ. Math. J.*, 40(4):1483–1488, 1991.
[14] Arthur Sard. Images of critical sets. *Ann. of Math.*, 48:247–259, 1958.

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