SINGULARITY ANALYSIS TOWARDS NONINTEGRABILITY OF NONHOMOGENEOUS NONLINEAR LATTICES

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Abstract. We show non-integrability of the nonlinear lattice of Fermi-Pasta-Ulam type via singularity analysis of normal variational equations of Lamé type.

1. From a Nonlinear Lattice to Lamé Equations

We consider the following one-dimensional lattice:

\[ H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i=1}^{n+1} v(q_{i-1} - q_i), \]  

where

\[ v(X) = \frac{\mu_2}{2} X^2 + \frac{\mu_4}{4} X^4 + \cdots + \frac{\mu_{2m}}{2m} X^{2m}. \]  

(1)

Fermi-Pasta-Ulam (FPU) lattice [2] is a special type of the systems with the potential function (2) as follows:

\[ H_{FPU} = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{\mu_2}{2} \sum_{i=1}^{n+1} (q_{i-1} - q_i)^2 + \frac{\mu_4}{4} \sum_{i=1}^{n+1} (q_{i-1} - q_i)^4. \]  

(3)

If we impose the fixed boundary condition as

\[ q_0 = q_{n+1} = 0, \quad n = \text{odd}, \]  

(4)

it is easy to check that

\[ \Gamma : q_1 = C\phi(t), q_2 = 0, q_3 = -C\phi(t), \cdots, q_{n-1} = 0, q_n = (-1)^{n-1} C\phi(t) \]  

(5)

is a special solution. Thus, the equation of \( \phi(t) \) is equivalent to the following Hamiltonian system with one degree of freedom:

\[ \ddot{\phi} + 2\mu_2 \dot{\phi} + 2\mu_4 C^2 \phi^3 + \cdots + 2\mu_{2m} C^{2m-2} \phi^{2m-1} = 0, \]  

(6)
where Hamiltonian is
\[ H(\phi, \dot{\phi}) = \frac{1}{2}(\dot{\phi})^2 + \mu_2 \phi^2 + \frac{\mu_4 C^2}{2} \phi^4 + \cdots + \frac{\mu_{2m} C^{2m-2}}{m} \phi^{2m} = \text{Const.} \] (7)

Then the total energy \( \epsilon \) is given by
\[ \epsilon = H = H(\phi, \dot{\phi}) \frac{n + 1}{2} C^2 = \frac{n + 1}{2} C^2 (\mu_2 + \frac{1}{2} \mu_4 C^2 + \cdots \frac{1}{m} \mu_{2m} C^{2m-2}) \] (8)

for the initial condition (5). In the case of the FPU lattice, we can determine \( C \) as follows:
\[ C = \sqrt[\mu_4]{\frac{\mu_2^2 + 4\epsilon n + 1}{\mu_2 - \mu_2}}. \] (9)

By combining (7) with (8), the underlying equation of \( \phi(t) \) can be rewritten by the differential equation of \( \phi(t) \) as
\[ \frac{1}{2}(\dot{\phi})^2 = \gamma_2 (1 - \phi^2) + \frac{\gamma_4}{2} (1 - \phi^4) + \cdots + \frac{\gamma_{2m}}{m} (1 - \phi^{2m}), \] (10)

where
\[ \gamma_{2m}(\epsilon, \{\mu_{2j} | j = 1, \cdots, m\}) \equiv \mu_{2m} C^{2m-2}. \] (11)

In the case of the FPU lattices (3), the solution of this differential equation (10) with the condition
\[ \gamma_{2m=4} \neq 0 \] (12)

is given explicitly by the elliptic function
\[ \phi(t) = cn(k; \alpha t), \] (13)

where
\[ \alpha = \sqrt{2\gamma_2 + 2\gamma_4}, \quad k = \sqrt[\gamma_4]{\frac{\gamma_4}{2\gamma_2 + 2\gamma_4}}, \] (14)

\( cn(k; \alpha t) \) is the Jacobi \( cn \) elliptic function, and \( k \) is the modulus of the elliptic integral.

We remark that because
\[ \gamma_2 + \gamma_4 = \mu_2 + C^2 \mu_4 = \sqrt{\mu_2^2 + \frac{4\epsilon n + 1}{n + 1} \mu_4} > 0, \] (15)

holds for \( \mu_4 > 0, \mu_2 \geq 0 \), the modulus of the elliptic function \( k \) satisfies the following relation:
\[ 0 \leq k \leq \frac{1}{\sqrt{2}}. \] (16)

Thus, the special solutions of the FPU lattices for \( \mu_4 > 0, \mu_2 \geq 0 \) have the two fundamental periods in the complex time plane as follows:
\[ T_1(\epsilon, \mu) = \frac{2K(k)}{\alpha}, \quad T_2(\epsilon, \mu) = \frac{2K(k) + 2iK'(k)}{\alpha}, \] (17)
where $K(k)$ and $K'(k)$ are the complete elliptic integrals of the first kind:

$$ K(k) = \int_0^1 \frac{dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} , \quad K'(k) = \int_0^1 \frac{dv}{\sqrt{(1 - v^2)(1 - (1 - k^2)v^2)}} . \quad (18) $$

Poles are located at $t = \tau$, where $\tau = \frac{2K(k)}{a} + i \frac{K'(k)}{a} \pmod{T_1,T_2}$ in the parallelogram of each period cell. Let us consider the variational equations along these special solutions. The variational equations are obtained by

$$ \eta_j = \xi_j = -\sum_{k=1}^n \frac{\partial^2 V}{\partial \eta_k \partial \eta_j} \xi_k $$

$$ = -(\gamma_2 + 3 \gamma_4 \phi^2 + 5 \gamma_6 \phi^4 + \cdots + (2m - 1) \gamma_{2m} \phi^{2m-2}) (2\xi_j - \xi_{j-1} - \xi_{j+1}) \quad \text{for} \quad 1 \leq j \leq n , \quad (19) $$

where $\xi_0 = \xi_{n+1} = \eta_0 = \eta_{n+1} = 0$ and $\xi_j = \delta q_j, \eta_j = \delta p_j \quad (1 \leq j \leq n)$. Moreover, these linear variational equations in the form of the vector

$$ d^2 \xi \over dt^2 = -(\gamma_2 + \cdots + (2m - 1) \gamma_{2m} \phi^{2m-2}) [ \begin{array}{cccccc} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & 2 \end{array} ] \xi \quad (20) $$

can be decoupled as follows. After we note that the eigenvalues of the $n \times n$ symmetric matrix

$$ G = [ \begin{array}{cccccc} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & 2 \end{array} ] \quad (21) $$

are obtained as $\{ 4 \sin^2 (\frac{j \pi}{2(n+1)}) | 1 \leq j \leq n \}$ by a normal orthogonal transformation $G \to OGO^{-1}$, the variational equations (19) are rewritten in the decoupled form:

$$ \ddot{\xi}_j(t) = -4 \sin^2 (\frac{j \pi}{2(n+1)})(\gamma_2 + 3 \gamma_4 \phi^2 + \cdots + (2m - 1) \gamma_{2m} \phi^{2m-2}) \ddot{\xi}_j(t) \quad (1 \leq j \leq n) , \quad (22) $$

where $\xi' = O \xi$. Clearly, these equations are written in the form of vector Hill’s equation[3]

$$ \frac{d^2 \xi'}{dt^2} + A(t) \xi' = 0 , \quad A(t + T) = A(t) , \quad (23) $$

where $T = T_1, T_2$ in the case of $m = 2$. For $j = \frac{n+1}{2}$, we have the relation

$$ \xi_{n+1}' = \sqrt{\frac{2}{n+1}} (\xi_1 - \xi_3 + \xi_5 + \cdots + (-1)^{\frac{n-1}{2}} \xi_n) . \quad (24) $$

Thus, the corresponding variational equation

$$ \ddot{\xi}_{n+1}' = -2(\gamma_2 + 3 \gamma_4 \phi^2 + \cdots + (2m - 1) \gamma_{2m} \phi^{2m-2}) \xi_{n+1}'(t) \quad (25) $$
has a time-dependent integral \( I(\xi, \dot{\xi}; t) \equiv I(\xi, \eta; t) \) because

\[
I(\xi, \eta; t) = DH \equiv (\eta \cdot \frac{\partial}{\partial p} + \xi \cdot \frac{\partial}{\partial q})H = \eta \cdot p + \xi \cdot V_q
\]

\[
= C\dot{\phi}(\eta_1 - \eta_3 + \eta_5 + \cdots + (-1)^{\frac{n-1}{2}}\eta_n)
+ 2(C\gamma_2\phi + C\gamma_4\phi^3 + \cdots + C\gamma_{2m}\phi^{2m-1})(\xi_1 - \xi_3 + \xi_5 + \cdots + (-1)^{\frac{n-1}{2}}\xi_n),
\]

where

\[
\frac{1}{C} \frac{df}{dt} = \phi(\ddot{\xi}_1 - \ddot{\xi}_3 + \cdots + (-1)^{\frac{n-1}{2}}\ddot{\xi}_n)
+ 2\phi(\gamma_2 + 3\gamma_4\phi^2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})(\xi_1 - \xi_3 + \cdots + (-1)^{\frac{n-1}{2}}\xi_n) = 0.
\]

We call Eq. (25) the tangential variational equation. On the other hand, a \((2n-2)\)-dimensional normal variational equation (NVE) is given by the equation of (22) with the tangential variational equation (25) removed as follows:

\[
\begin{align*}
\dot{\eta}_j' &= -4\sin^2(\frac{j\pi}{2(n+1)})(\gamma_2 + 3\gamma_4\phi^2 + \cdots + (2m-1)\gamma_{2m}\phi^{2m-2})\xi_j', \\
\xi_j' &= \eta_j' \quad \text{for } 1 \leq j \neq \frac{n+1}{2} \leq n.
\end{align*}
\]

In case of the FPU lattice, the normal variational equation (28) becomes the Lamé equation [7]

\[
\frac{d^2y}{dt^2} - (E_1\sin^2(\frac{j\pi}{2(n+1)}) + E_2)y = 0,
\]

where \( E_1 = 12\frac{1}{\alpha^2k^2}\sin^2(\frac{j\pi}{2(n+1)}) \) and \( E_2 \) are constants.

2. Non-integrability Theorem

Morales and Simó obtained the following theorem on the non-integrability based on the application of Picard-Vessiot theory to Ziglin’s analysis [9, 10] for Hamiltonian systems with two degrees of freedom.

**Theorem 1 (Morales and Simó [4], 1994)** When the normal reduced variational equation is of Lamé type, if \( A \equiv E_1\alpha^2k^2 \neq m(m+1), \) \( m \in \mathbb{N} \) and the Lamé equation satisfying this condition on \( A \) is not algebraically solvable (Brioschi-Halphen-Crawford and Baldassarri solutions), then the initial Hamiltonian system does not have a first integral, meromorphic in a connected neighborhood of the integral curve \( \Gamma \), which is functionally independent together with \( H \).

In case of the present analysis, \( A \) is given by the following formula:

\[
A = E_1\alpha^2k^2 = 12\sin^2(\frac{j\pi}{2(n+1)}) = 6(1 - \cos(\frac{j\pi}{n+1})).
\]

\[\text{(30)}\]

We can easily check that \( \cos(\frac{j\pi}{n+1}) \notin \mathcal{Q} \) if and only if \( j \notin \{\frac{n+1}{3}, \frac{n+1}{2}, \frac{2(n+1)}{3}\} \). When \( A \notin \mathcal{Q} \), the above condition on the algebraic solvability of the Lamé equation is not satisfied. Thus, to check the algebraic solvability of the Lamé equations

\[
\frac{d^2\xi_j}{dt^2} - (12\sin^2(\frac{j\pi}{2(n+1)})\sin^2(k; \alpha t) + E_2)\xi_j = 0 \quad (j \neq \frac{n+1}{2}),
\]

\[\text{(31)}\]
it is sufficient to examine the following two cases:

\[ A = 6\left(1 - \cos\left(\frac{\pi}{3}\right)\right) = 3, \quad A = 6\left(1 - \cos\left(\frac{2\pi}{3}\right)\right) = 9. \]  

(32)

It is known [1] that the condition on \( A \) for the Brioschi-Halphen-Crawford solutions is given by

\[ A = m(m+1), \quad m + \frac{1}{2} \in \mathbb{N}, \]  

(33)

and that the condition on \( A \) for the Baldassarri solutions is given by

\[ A = m(m+1), \quad m + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}. \]  

(34)

However, the following relations

\[ m(m+1) = 3 \rightarrow m = \frac{-1 \pm \sqrt{13}}{2} \notin \mathbb{Q}, \quad m(m+1) = 9 \rightarrow m = \frac{-1 \pm \sqrt{37}}{2} \notin \mathbb{Q} \]  

(35)

hold, which guarantee that all \( n-1 \) Lamé equations (31) do not belong to the solvable case. In case of the systems with \( n \) degrees of freedom, we have \( n-1 \) Lamé equations which corresponds to \( n-1 \) normal variational equations.

Thus, according to the steps in Ref. [4] we obtain the following theorem:

**Theorem 2** The FPU lattice for \( \mu_4 > 0, \mu_2 \geq 0 \) does not have \( n-1 \) first integrals, meromorphic in a connected neighbourhood of the integral curve \( \Gamma \), which are functionally independent together with \( H \).

We remark here that this theorem on the non-integrability does not depend on the total energy in contrast with the result about the non-integrability proof of the FPU lattice in the low energy limit [6] based on non-resonance checking [5] and the result about the non-integrability of the FPU lattice in the high energy limit based on the Kowalevski exponents of the homogeneous systems [8]. Here, it is conjectured that more general nonhomogeneous nonlinear lattice (1) would be also non-integrable in the sense of the present analysis.

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