An Exactly Solvable Model of Fermions with Disorder.

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Abstract

Non-perturbative results are obtained for multi-point correlation functions of the model of (2 + 1)-dimensional relativistic fermions in a random non-Abelian gauge potential. The results indicate that the replica symmetry for this model is unbroken. We calculate the diffusion propagator and show that DC-conductivity for this model is finite.

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In this letter we continue to study the model of (2 + 1)-dimensional relativistic fermions interacting with random non-Abelian gauge potential. This model was introduced in Ref. 1 in relation with effects of disorder in two dimensional d-wave superconductors. It was shown that after averaging over the static disorder one gets the following Euclidean action:

\[ S = S_0 + \int d^2 x \left[ i \bar{\psi}_{\alpha,p} \gamma_\mu \partial_\mu \psi_{\alpha,p} + \frac{i \omega + 0}{2} \bar{\psi}_{\alpha,p} \Lambda^3_{pq} \psi_{\alpha,q} \right] \]  (1)

\[ S_0 = \int d^2 x \left[ \bar{\psi}_{\alpha,p} \gamma_\mu \partial_\mu \psi_{\alpha,p} + c J_\mu^a J_\mu^a \right] \]  (2)

Spinor fields \( \psi_{\alpha,p} \) with \( p = 1, \ldots, R \) and \( p = R + 1, \ldots, 2R \) describe Fourier components of the original fermions with frequencies \( \epsilon + \omega/2 \) and \( \epsilon - \omega/2 \) respectively, where \( R \to 0 \) is the number of replicas. Greek indices run from 1 to \( N \). The flavour currents are given by

\[ J_\mu^a = \sum_{p=1}^{2R} \bar{\psi}_{\alpha,p} \tau_\alpha^a \gamma_\mu \psi_{\beta,p} \]

with \( \tau^a \)'s being generators of the SU(N) group. In the context of disordered superconductor \( N \) denotes the number of nodes of the order parameter on the Fermi surface. The coupling constant \( c \) is related to the random potential. The matrix \( \Lambda^3 \) is the diagonal matrix with \( R \) elements +1 and \( R \) elements −1 on the main diagonal: \( \Lambda^3 = diag(1, \ldots, 1; -1, \ldots, -1) \). In what follows we shall also use the matrix \( \Lambda^1 \) whose only nonzero elements are situated on the main antidiagonal: \( \Lambda^1 = antidiag(1,1,\ldots,1) \).

The Abelian version of this model has been studied by Ludwig et al.\(^2\). The density of states (DOS) is expressed as

\[ \rho(\epsilon) = \frac{1}{\pi} Re \lim_{R \to 0} R^{-1} Tr < \bar{\psi}(x) \psi(x) > |_{\omega=0} \]  (3)

The main difference between the present problem and the conventional localization is that in the present case DOS is strongly affected by the disorder. Being linear in energy in the absence of disorder (\( \rho(\epsilon) \sim |\epsilon| \)), DOS changes its behavior at low energies \( \rho(\epsilon) \sim |\epsilon|^\nu \) where \( \nu = 1/(2N^2 - 1) \). It will be shown later that with DOS vanishing at zero energy the low energy states remain delocalized. Another difference is that the conductivity of (2 + 1)-dimensional massless Dirac fermions with an arbitrary damping is constant\(^2,3\).
\[ \sigma_{xx} = N e^2 / 2\pi^2 h \]

The insensitivity of the conductivity to dumping makes one suspect that it may remain finite in the presence of disorder. We shall prove that this is indeed the case.

The model (1,2), as it stands, contains fast and slow degrees of freedom. The latter are SU\((N)\)-singlets described by the well known \(Q\)-matrices \((2R \times 2R)-\text{matrices}\). The symmetry of the model dictates that \(Q\)-matrices belong to the \(U(2R) = U(1) \times SU(2R)\) group. The integration over the fast degrees of freedom performed in Ref. 1 gives the effective action for the matrices \(Q = g \exp[i\gamma \Phi]\) \((\gamma = \sqrt{2\pi/Nc})\) in the form of the Wess-Zumino model on the group \(U(1) \times SU(2R)\):

\[
S = S_0 + M \epsilon Tr(Q + Q^+) + iM \frac{\omega}{2} Tr[\Lambda(Q + Q^+)] \\
S_0 = \frac{1}{2} \int d^2 x (\partial_\mu \Phi)^2 + NW[SU(2R); g] \\
W[SU(2R); g] = \frac{1}{16\pi} \int d^2 x Tr(\partial_\mu g^+ \partial_\mu g) + \frac{1}{24\pi} \int d^3 x \epsilon_{abc} Tr(g^+ \partial_\alpha gg^+ \partial_\beta gg^+ \partial_\gamma g) \tag{5}
\]

The quantity \(M\) is the energy scale introduced by the disorder \(M \sim \exp[-2\pi/Nc]\), which marks the crossover from the bare DOS \(\rho(\epsilon) \sim |\epsilon|\) to the renormalized DOS \(\rho(\epsilon) \sim |\epsilon|^\nu\). In this Wess-Zumino model \(M\) serves as the ultraviolet cutoff. It is worth remarking that the Wess-Zumino action is not equivalent to the action with the topological term which describes the conventional localization problem in a strong magnetic field.

The Wess-Zumino model is well studied at finite \(R\), where it is a critical theory with power law decay of correlation functions. Our primary objective is to show that this model remains well defined in the replica limit, that is its correlation functions have finite values at \(R \to 0\). The closest implication of this fact is that the replica symmetry remains unbroken.

In determining the replica limit we follow the general principle: calculating any \(N\)-point correlation function \(F_{2R}(1, 2, \ldots N)\) one treats \(R\) as an arbitrary integer number on all intermediate steps of calculations until the finite expression is obtained. The replica limit is then defined as follows:

\[
F(1, 2, \ldots N) = \lim_{R \to 0} \frac{1}{2R} F_{2R}(1, 2, \ldots N) \tag{6}
\]
Let us study the correlation functions of the $Q$-fields. The problem of indices is simplified by the fact that $Q_{pq}$ matrices are slow parts of the operators

$$Q_{pq} \sim \sum_{\alpha=1}^{N} \psi^{+}_{R,\alpha,p} \psi_{L,\alpha,q}$$

(7)

This fact provides us with a simple recipe: any $N$-point correlation function has the same index structure as the $N$-point function of the $\sum_{\alpha} \psi^{+}_{R,\alpha,p} \psi_{L,\alpha,q}$-fields in the theory of massless free fermions. The simplest example is the 2-point function:

$$<Q_{p_1r_1}(z, \bar{z})Q^+_{r_2p_2}(0, 0)> = \delta_{r_1r_2} \delta_{p_1p_2} \frac{1}{(M|z|)^{4\Delta_1}}$$

(8)

where $\Delta_1$ is the conformal dimension of the composite operator $Q$ given by the sum of the dimensions of the bosonic exponent $\exp[i \sqrt{2\pi/RN} \Phi]$ and the operator field $g_{pr}$ from the fundamental representation of the SU(2R) group. Using the results of Ref. 4 we get:

$$\Delta_1 = \lim_{R \to 0} \frac{1}{4RN} + (2R - 1/2R)/(N + 2R) = \frac{1}{2N^2}$$

(9)

In the replica limit we get from (8)

$$D(z, \bar{z}) \equiv \lim_{R \to 0} R^{-1} <Tr[\Lambda^1 Q(z, \bar{z})]Tr[\Lambda^1 Q^+(0, 0)]> = (M|z|)^{-2/N^2}$$

(10)

We identify this correlation function with the propagator of diffusion in the zero frequency limit. Its Fourier transformation gives the propagator in momentum space:

$$D(\omega = 0, q) \sim \frac{D_0(q/M)^{2/N^2}}{q^2}$$

(11)

where $D_0$ is the bare diffusion constant.

We can identify other operators and their conformal dimensions. Thus, the operator expansion of two $Q$ matrices contain the symmetric and antisymmetric fields $O_S$ and $O_A$:

$$Q(1)Q(2) = \exp[i \sqrt{2\pi/RN} \Phi(1)]g(1) \exp[i \sqrt{2\pi/RN} \Phi(2)]g(2) \sim \exp[i \sqrt{8\pi/RN} \Phi(1)]O_S(1) + \exp[i \sqrt{8\pi/RN} \Phi(1)]O_A(1) + ...$$

(12)

where dots stand for nonsingular terms. Taking the corresponding conformal dimensions from [4] we get the conformal dimensions of the composite fields $\hat{O}_{S,A} = \exp[i \sqrt{8\pi/RN} \Phi]O_{S,A}$ in the replica limit:
\[ \Delta_A = \lim_{R \to 0} \left[ \frac{1}{RN} + \frac{(R - 1)(2R + 1)}{R(N + 2R)} \right] = \frac{2 - N}{N^2} \]

\[ \Delta_S = \lim_{R \to 0} \left[ \frac{1}{RN} + \frac{(R + 1)(2R - 1)}{R(N + 2R)} \right] = \frac{2 + N}{N^2} \] (13)

We see that the scaling dimension of the antisymmetric field is negative for \( N > 2 \), which indicates that the replica limit of the model (1,2) is a nonunitary theory. The next field is the adjoint field \( O_{ad} \) appearing in the operator expansion of \( QQ^* \):

\[ Q(1)Q^+(2) \sim g(1)g^+(2) \sim Tr[\tau^a \tau^b g] \equiv O_{ad}^{ab} \] (14)

According to Ref. 6 the corresponding conformal dimension is \( \Delta_{ad} = 2R/(2R + N) \). It vanishes in the replica limit and, as we shall see later, this field just does not exist in the replica limit.

The central charge of our theory is the sum of central charges of the free bosonic field \( (C = 1) \) and the Wess-Zumino-Witten model on the \( SU(2R) \) group:

\[ C = 1 + \frac{N(4R^2 - 1)}{N + 2R} = \frac{2R}{N} + O(R^2) \] (15)

Thus the resulting central charge vanishes; however, according to the definition of the replica limit (8) the physical correlation function of the stress-energy tensors does not vanish:

\[ < T(z)T(0) > = \lim_{R \to 0} \frac{C_R}{2R} \frac{1}{2z^4} = \frac{1}{2Nz^4} \] (16)

Let us now study the four point correlation function of the \( Q \) fields and show that it has the correct replica limit. For this purpose we use the result obtained in Ref. 4:

\[ < Q_{p_1r_1}(z_1, \bar{z}_1)Q_{r_2p_2}^+(z_2, \bar{z}_2)Q_{p_3r_3}(z_3, \bar{z}_3)Q_{r_4p_4}^+(z_4, \bar{z}_4) >= \]

\[ \left[ \frac{|z_{14}z_{23}|}{z_{12}z_{14}z_{13}z_{24}} \right]^{2/N^2} (F + \tilde{F}), \] (17)

\[ \tilde{F} = [\delta_{p_1p_2}\delta_{p_3p_4}\delta_{q_1q_2}\delta_{q_3q_4}G_{11}(x, \bar{x}) + \delta_{p_1p_4}\delta_{p_3p_2}\delta_{q_1q_2}\delta_{q_3q_4}G_{22}(x, \bar{x})], \] (18)

\[ F = [\delta_{p_1p_2}\delta_{p_3p_4}\delta_{q_1q_2}\delta_{q_3q_4}G_{12}(x, \bar{x}) + \delta_{p_1p_4}\delta_{p_3p_2}\delta_{q_1q_2}\delta_{q_3q_4}G_{21}(x, \bar{x})] \] (19)

where
\[ x = \frac{z_{12}z_{44}}{z_{14}z_{32}}, \bar{x} = \frac{\bar{z}_{12}\bar{z}_{44}}{\bar{z}_{14}\bar{z}_{32}} \]

and the functions \( G_{AB}, A, B = 1, 2 \) are known functions. Now note that in our theory we shall deal only with correlation functions of \( TrQ, Tr\Lambda^3Q \) and \( Tr\Lambda^1Q \). Since all correlation functions must be proportional to \( R \), only the term with \( F \) (that is the term with all indices equal) survives in the replica limit, the term with \( \bar{F} \) being proportional \( R^2 \). Therefore we have

\[
R^{-1} < Tr\Lambda^1Q(1)TrQ^+(2)TrQ(3)Tr\Lambda^1Q^+(4) > = \left[ \frac{|z_{14}z_{23}|}{|z_{12}z_{14}z_{13}z_{24}|} \right]^{1/N^2} \times \\
[\bar{x}F(1/N,-1/N,1,x)F(1+1/N,1-1/N,2,\bar{x}) + (x \to \bar{x})] \quad (20)
\]

In order to get the operator algebra of the model we consider various limits of this formula. In the limit \( z_{12} = z_{34} = \delta \to 0 \) we get

\[
D(1, 1 + 0; 3, 3 + 0) = (\delta)^{-4/N^2} + \frac{1}{z_{13}^2} + \frac{1}{\bar{z}_{13}^2} \quad (21)
\]

From this we derive the following operator expansion:

\[
Q(1)Q^+(2) \sim |z_{12}|^{-2/N^2}(I + z_{12}J(2) + \bar{z}_{12}\bar{J}(2) + \ldots) \quad (22)
\]

where \( J, \bar{J} \) are the left and right current operators. Notice that the primary operator in the adjoint representation present for \( R \neq 0 \) does not appear in this expansion.

In the limit \( z_{13} = z_{24} = \delta \to 0 \) we have

\[
D(1, 2; 1 + 0, 2 + 0) = |z|^{4(N-2)/N^2}C_1 + |z|^{-4(N+2)/N^2}C_2 \quad (23)
\]

\( (C_1, C_2 \) are constants) which shows the presence of the symmetric and antisymmetric fields with conformal dimensions (13) in the operator expansion of \( QQ \) and \( Q^+Q^+ \). These operators describe mesoscopic fluctuations of the local density of states.

Now let us discuss a crossover to small momenta. As follows from Eq.( ) the operators \( Tr(1 \pm \Lambda^3)Q \) are relevant perturbations with scaling dimensions \( 2 - 2\Delta_1 = 2 - 1/N^2 \). Therefore at finite \( \omega \) one can conjecture the following form for the diffuson propagator:
\[
D_{\epsilon=0}(\omega, q) = \frac{1}{q^2-2/N^2} f \left( \frac{\omega^2}{q^4-2/N^2} \right) \tag{24}
\]

Eq. (24) shows that the spectrum of the diffusive modes \(i \omega = D(\omega)q^2 \sim q^{2-1/N^2}\) and the diffusion coefficient scales as \(D(\omega) \sim \omega^{-1/(2N^2-1)} \sim \rho^{-1}(\omega)\). That is the quantity \(\rho(\omega)D(\omega)\) stays constant under renormalization (the presence of logarithms or finite corrections is not excluded). Since from the other side the conductivity is proportional to \(\sigma_{xx} \sim \rho(0)D(0)\), the absence of renormalization of \(\rho(\omega)D(\omega)\) at finite frequencies makes it likely that the conductivity remains finite in the presence of weak disorder.

We can go further than this qualitative analysis. Namely, using the peculiar properties of the action (5) we can calculate the propagator of diffusion in the limit \(q << |\omega|^{1-\Delta_1}\). Let us introduce the following change into the action (5):

\[
Tr(1 \pm \Lambda^3)(Q + Q^+) \rightarrow Tr(1 \pm \Lambda^3)(h^+ Q + Q^+ h) \tag{25}
\]

where \(h(x)\) is an external source - a matrix from the \(U(2R)\) group. It turns out that in the limit \(q << |\omega|^{1-\Delta_1}\) we can integrate over \(Q\)-variables and get the effective action for \(h(x)\) in the form of expansion in \(|\omega|^{-1/1-\Delta_1}\). Indeed, let us make a shift: \(\hat{Q} = h^+ Q, Q = h\hat{Q}\). Using the well known property of the Wess-Zumino action, we write

\[
S = S_0(h\hat{Q}) + M\epsilon \text{Tr}(\hat{Q} + \hat{Q}^+) + iM\frac{\omega}{2} \text{Tr}[\Lambda^3(\hat{Q} + \hat{Q}^+)] = S(\hat{Q}; \epsilon, \omega) + S_0(h) + \frac{N}{8\pi} \int d^2 x Tr[h^+ \partial_\mu h\hat{Q}\partial_\mu \hat{Q}^+] \tag{26}
\]

Since fluctuations of the \(\hat{Q}\)-field are governed by the Wess-Zumino action perturbed by relevant operators, they have a finite correlation length \(\xi \sim (|\omega|)^{-1/(1-\Delta_1)}\). Therefore the last term in the action (26) gives corrections to \(S_0(h)\) which contain powers of \(\xi \nabla h\). At small momenta such perturbations can be neglected. Thus at small wave vectors \(S_0(h)\) effectively becomes a generating functional of correlation functions of \(Q\). In particular, we have

\[
K(\epsilon = 0, \omega; q) \equiv \frac{M^2}{2R} < Tr(1 + \Lambda^3)\Lambda^1 Q Tr(1 - \Lambda^3)\Lambda^1 Q^+ > = -\frac{N}{2\pi} q^2 \omega^2 (1 + O(q^2/|s|^{1-\Delta_1})) \tag{27}
\]
This means that the crossover function $f(x)$ in Eq.(24) behaves as

$$f(x) = \frac{N}{2\pi x}$$

at large $x$. This result allows us to calculate the DC-conductivity $\sigma_{xx}$. Using the definition given in Ref. 2 we get from (27)

$$\sigma_{xx} = -\frac{e^2}{\pi}\lim_{\omega \to 0} \omega^2 \left( \frac{\partial}{\partial q^2} \right)_{q=0} K(0, \omega; q) = \frac{Ne^2}{2\pi^2 \hbar}$$

Comparing this result with Eq.(4) we see that the DC-conductivity remains unrenormalized and therefore the disorder manifest itself only at large momenta. In the limit $q \to 0$ there are no singular contributions. This is not typical for conventional disordered systems where disorder changes a statistics of low-lying energy levels. In the same time one probably should not be surprized by our results. After all they show that the model with disorder remains integrable and we know that integrability and fractality of the spectrum are inconsistent.

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REFERENCES

[1] A. A. Nersesyan, A. M. Tsvelik and F. Wenger, Phys. Rev. Lett. 72, 2628 (1994); Phys. Rev. B. (in press). (1992). (1994).

[2] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar and G. Grinstein, Phys. Rev. B in press.

[3] P. Lee, Phys. Rev. Lett. 71, 1887 (1993).

[4] V.G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247, 83 (1984).

[5] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241, 333 (1984).