Dynamics of two Rydberg atoms successively interacting with a detuned thermal field

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Abstract. The entanglement between two identical two-level atoms successively passing a vacuum or a thermal cavity has been investigated taking into account the detuning. The cases when atoms are initially prepared in the separable two-atom or Bell type entangled atomic states have been considered. It has been shown that thermal field can produce the entanglement between Rydberg atoms successively passing a cavity for all considered initial atomic states. For separable initial atomic states and vacuum state of the cavity the presence of detuning leads to decreasing of the degree of atomic entanglement. For entangled initial atomic states and vacuum state of the cavity the presence of detuning leads to decreasing of the entanglement amplitude oscillations. We have also derived that for thermal field the increasing of the mean photon number leads to decreasing of the entanglement, but the entanglement increases as the detuning increases. For thermal field and small detuning we have established that the effect of sudden death and birth of entanglement takes place and that for large detuning such effect vanishes. Results of the computer simulation are presented.

1. Introduction

Quantum entanglement has now been considered as a significant resource of quantum information processing and many quantum protocols [1]. Cavity quantum electrodynamics (QED) offers a good system to manipulate entanglement and to realize quantum information schemes [2]. In recent years it became feasible to investigate the complex of light with two-level systems: neutral atoms, spin in solids, superconducting qubits, quantum dots etc. [3]. In the context of quantum information processing, interaction between cavity field and qubits may be useful to generate entangled states. Cavity QED with Rydberg atoms has been a favorite tool to checking the foundations of quantum mechanics including entanglement. Rydberg atoms was proposed a two decade ago to implement quantum gates between neutral atoms. The entanglement between two initially independent atoms successively passing the vacuum cavity have been demonstrated by S. Haroche et all. [4]. The entanglement procedure involves the resonant coupling, one by one, of the atoms to a high $Q$ microwave superconducting cavity. The atoms, prepared in circular Rydberg states, exchange a single photon in the cavity and become entangled by this indirect interaction. Later a number of theoretical works have been devoted to investigations of entanglement properties of two atoms passing through a cavity one after another, thereafter interacting with a various state radiation field inside the cavity. Ghosh et al. [5] have investigated the entanglement properties of two Rydberg atoms with Stark-shifted two-photon atomic transitions passing through a single-mode cavity. They have shown that the Stark shift can be used to enhance the magnitude of atomic entanglement over that obtained.
in the resonant condition for certain parameter values. Liao et al. [6, 7] have investigated the effects of the atomic coherence and mean photon number on the time evolution of entanglement of two atoms passing through a cavity one after another when the field is initially in a Fock state [6] or in a thermal state [7] and found that the phenomenon of sudden birth of entanglement occurs in some certain conditions. In [8] the analogous behavior has been investigated for model with two-photon interaction. Yan [9] also has investigated the entanglement properties of two atoms successively passing a cavity with Fock or thermal field but especially focused on the case when two atoms are initially in an entangled state. In [10]-[12] we have reexamined the dynamics of entanglement of two atoms successively passing a thermal cavity for another type of initial entangled atomic state. In this paper we have investigated the influence of detuning on entanglement of two Rydberg atoms prepared in the separable or entangled initial state and consequently passing a thermal lossless cavity.

2. Model and initial system states

The physical system under consideration consists of two separate identical two-level Rydberg atoms passing through a cavity one after another and interacting with the cavity field. With the exception of the initial atomic state and cavity temperature, the parameters of the considered model are assumed to be the same as in experiment [4]. In [4] the authors used two circular Rydberg states of Rb with principal quantum number 51 or 50 before crossing the cavity. The relevant cavity mode has is slightly off resonant with the transition at $\omega_0 = 51.1$ GHz between exited and ground states (detuning $\delta = 170$ kHz). The cavity was cooled to 0.6 K, i.e. the mean thermal photon number was negligible. The cavity photon damping time was 112 ms, much shorter than the interval between two experimental sequences. In our theoretical consideration we took into account the detuning and thermal photons. We also considered the entangled initial atomic states.

The Hamiltonian of the joint "one atom+field" system with the dipole and rotating wave approximation can be written as

$$H = (1/2)\hbar \omega_0 \sigma_z^+ + \hbar \omega a^+ a + \hbar g (a^+ \sigma^- + a \sigma^+), \quad (1)$$

where $(1/2)\sigma_z^+$ is the inversion operator, $\sigma^+ = |e\rangle\langle g|$, and $\sigma^- = |g\rangle\langle e|$ are the transition operators between the excited $|e\rangle$ and the ground $|g\rangle$ states, $a^+$ and $a$ are the creation and the annihilation operators of photons of the cavity mode, $g$ is the constant of atom-field interaction, $\omega_0$ is the frequency of the atomic transition and $\omega$ is the frequency of the cavity mode. We introduce the detuning as $\delta = \omega_0 - \omega$.

Suppose that before the first atom enters the cavity the Rydberg atoms have been prepared in one of separable initial atomic states such as

$$|e, e\rangle, \quad |e, g\rangle, \quad |g, e\rangle, \quad |g, g\rangle$$

or in the Bell-type entangled states of the form

$$|\Psi(0)\rangle_{A1A2} = \cos \theta |e, g\rangle + \sin \theta |g, e\rangle, \quad (2)$$

where $\theta$ is the parameter which depends the degree of atomic entanglement. and one-mode cavity field is in a thermal state

$$\rho_F(0) = \sum_n p_n |n\rangle \langle n|,$$

where the probabilities

$$p_n = \frac{n^n}{(1 + \bar{n})^{n+1}}.$$

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The matrix elements of (3) for separable initial states are cumbersome calculations we derived the reduced atomic density operator for separable initial state leaves the cavity. Taking a partial trace over the field variables we obtained from the density matrix of the whole system the reduced atomic density operator \( \rho_{A_1A_2}(t, \tau) \). Omitting the cumbersome calculations we derived the reduced atomic density operator for separable initial states \(|e, e\rangle, |e, g\rangle, |g, e\rangle, |g, g\rangle\) and entangled initial state (2)

\[
\rho_{A_1A_2}(t, \tau) = \begin{pmatrix} \rho_{11}(t, \tau) & 0 & 0 & 0 \\ 0 & \rho_{22}(t, \tau) & \rho_{23}(t, \tau) & 0 \\ 0 & \rho_{23}(t, \tau)^* & \rho_{33}(t, \tau) & 0 \\ 0 & 0 & 0 & \rho_{44}(t, \tau) \end{pmatrix}.
\]

The matrix elements of (3) for separable initial state \(|e, e\rangle\) are

\[
\rho_{11}(t, \tau) = \sum_{n=1}^{\infty} p_n |Y_n(t)|^2 |Y_n(\tau)|^2 + p_0 |Y_0(t)|^2 |Y_0(\tau)|^2,
\]
\[
\rho_{22}(t, \tau) = \sum_{n=1}^{\infty} p_n |Z_n(t)|^2 |Y_n(\tau)|^2 + p_0 |Z_0(t)|^2 |Y_0(\tau)|^2,
\]
\[
\rho_{33}(t, \tau) = \sum_{n=1}^{\infty} p_n |Y_{n+1}(t)|^2 |Z_n(\tau)|^2 + p_0 |Z_1(t)|^2 |Y_1(\tau)|^2,
\]
\[
\rho_{44}(t, \tau) = \sum_{n=1}^{\infty} p_n |Z_{n+1}(t)|^2 |Z_n(\tau)|^2 + p_0 |Z_0(t)|^2 |Z_0(\tau)|^2,
\]
\[
\rho_{23}(t, \tau) = \sum_{n=1}^{\infty} p_n Y_n(\tau) Z_n(t) Z_n^*(\tau) Y_{n+1}^*(t) + p_0 Y_0(\tau) Z_0(t) Z_0^*(\tau),
\]

For separable initial state \(|e, g\rangle\) these take the form

\[
\rho_{11}(t, \tau) = \sum_{n=1}^{\infty} p_n |Z_{n-1}(t)|^2 |Y_n(\tau)|^2,
\]
\[
\rho_{22}(t, \tau) = \sum_{n=1}^{\infty} p_n [X_{n-1}(t)|^2 |Y_n(\tau)|^2 + p_0 |Y_0(\tau)|^2,
\]
\[
\rho_{33}(t, \tau) = \sum_{n=1}^{\infty} p_n |Z_n(t)|^2 |Z_n(\tau)|^2 + p_0 |Z_0(t)|^2 |Z_0(\tau)|^2,
\]
\[
\rho_{44}(t, \tau) = \sum_{n=1}^{\infty} p_n |X_n(t)|^2 |Z_n(\tau)|^2 + p_0 |X_0(t)|^2 |Z_0(\tau)|^2,
\]

Here \( \bar{n} \) is the mean photon number in the cavity mode

\[
\bar{n} = (\exp[h\omega/k_BT] - 1)^{-1}
\]

\( k_B \) is the Boltzmann constant and \( T \) is the equilibrium temperature of the cavity mirrors. Solving the evolution equation we derived the density matrix of considered system for time moment \( \tau \) when the first atom leaves the cavity. The considered density matrix on the other hand is the initial state of the system prior to entering into the cavity of the second atom. We deduced the density matrix of the whole system at the moment \( t \) when the second atom leaves the cavity. Taking a partial trace over the field variables we obtained from the density matrix of the whole system the reduced atomic density operator \( \rho_{A_1A_2}(t, \tau) \).
\[ \rho_{23}(t, \tau) = \sum_{n=1}^{\infty} p_n X_{n-1}(t)Y_n(\tau)Z_n(\tau)^* + p_0 Y_0(\tau)Z_0(t)^*Z_0(\tau). \]

For initial state \( |g, g\rangle \) these are

\[ \rho_{11}(t, \tau) = \sum_{n=2}^{\infty} p_n |Z_{n-1}(\tau)|^2 |Z_{n-2}(t)|^2, \]
\[ \rho_{22}(t, \tau) = \sum_{n=1}^{\infty} p_n |Z_{n-1}(\tau)|^2 |X_{n-2}(t)|^2, \]
\[ \rho_{33}(t, \tau) = \sum_{n=1}^{\infty} p_n |X_{n-1}(\tau)|^2 |Z_{n-1}(t)|^2, \]
\[ \rho_{44}(t, \tau) = \sum_{n=1}^{\infty} p_n |X_{n-1}(\tau)|^2 |X_{n-1}(t)|^2 + p|0\rangle, \]
\[ \rho_{23}(t, \tau) = \sum_{n=1}^{\infty} p_n Z_{n-1}(\tau)X_{n-2}(t)X_{n-1}(\tau)^*Z_{n-1}(t). \]

At last, for entangled initial state (2) the matrix elements are

\[ \rho_{11}(t, \tau) = \sum_{n=1}^{\infty} p_n \left( \cos^2 \theta |Z_{n-1}(t)|^2 |Y_{n-1}(\tau)|^2 \right) + \sin^2 \theta |Y_{n-1}(t)|^2 |Z_{n-1}(\tau)|^2, \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n (Z_{n-1}(t)Y_{n}(\tau)Y_{n-1}(t)^*Z_{n-1}(\tau) + Y_{n-1}(t)Z_{n-1}(\tau)Z_{n-1}(t)^*Y_{n}(\tau)^*) \right), \]
\[ \rho_{22}(t, \tau) = \cos^2 \theta \left( \sum_{n=1}^{\infty} p_n |X_{n-1}(t)|^2 |Y_{n}(\tau)|^2 + p_0 |Y_0(\tau)|^2 \right) + \]
\[ + \sin^2 \theta \sum_{n=1}^{\infty} p_n |Z_{n-1}(t)|^2 |Z_{n-1}(\tau)|^2, \]
\[ + \cos \theta \sin \theta \sum_{n=1}^{\infty} p_n (X_{n-1}(t)Y_{n}(\tau)Z_{n-1}(\tau)^*Z_{n-1}(t) + Z_{n-1}(t)Z_{n-1}(\tau)X_{n-1}(t)^*Y_{n}(\tau)^*), \]
\[ \rho_{33}(t, \tau) = \cos^2 \theta \left( \sum_{n=1}^{\infty} p_n |Z_{n}(t)|^2 |Z_{n}(\tau)|^2 + p_0 |Z_0(t)|^2 |Z_0(\tau)|^2 \right) + \]
\[ + \sin^2 \theta \left( \sum_{n=1}^{\infty} p_n |X_{n-1}(t)|^2 |Y_{n}(t)|^2 + p_0 |Y_0(t)|^2 \right) + \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n Z_{n}(t)Z_{n}(\tau)Y_{n}(t)^*X_{n-1}(\tau)^* + p_0 Z_0(t)Z_0(\tau)Y_0(t)^* \right) + \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n X_{n-1}(\tau)Y_{n}(t)Z_{n}(\tau)^* + p_0 Y_0(t)Z_0(t)^*Z_0(\tau)^* \right), \]
\[ \rho_{44}(t, \tau) = (\cos \theta)^2 \left( \sum_{n=1}^{\infty} p_n |X_{n}(t)|^2 |Z_{n}(\tau)|^2 + p_0 |X_0(t)|^2 |Z_0(\tau)|^2 \right) + \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n X_{n}(t)Z_{n}(\tau)Z_{n}(t)^*X_{n-1}(\tau)^* + p_0 X_0(t)Z_0(\tau)Z_0(t) \right) + \]
curves were obtained under the assumption that atomic state (2) and different values of the model parameters are shown in Figs. 1-5. The transpose of the reduced atomic density matrix
\[ \rho_{23}(t, \tau) = (\cos \theta)^2 \left( \sum_{n=1}^{\infty} p_n X_n(t) Z_n(t) Z_n(\tau) + p_0 Z_0(t) Z_0(\tau) \right) + \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n X_n(t) X_n(\tau) + p_0 Z_0(t) Z_0(\tau) \right) \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n X_n(t) X_n(\tau) Z_n(\tau) + p_0 Z_0(t) Z_0(\tau) \right) \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} p_n Z_n(t) Z_n(\tau) X_n(\tau) + p_0 Z_0(t) Z_0(\tau) \right) \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} \sin \Omega_n \sin(\Delta_n t/2) \right) \]
\[ + \cos \theta \sin \theta \left( \sum_{n=1}^{\infty} \sin \Delta_n t/2 \right) \]
\[ \text{Here} \]
\[ X_n = \cos(\Delta_n t/2) + \frac{i \delta}{\Delta_n} \sin(\Delta_n t/2), \]
\[ Y_n = \cos(\Delta_n t/2) - \frac{i \delta}{\Delta_n} \sin(\Delta_n t/2), \]
\[ Z_n = \frac{i \Omega_n}{\Delta_n} \sin(\Delta_n t/2), \]
\[ \Delta_n = g \sqrt{\alpha^2 + \Omega_n^2}, \quad \Omega_n = 2\sqrt{n+1}, \quad \alpha = \delta/g. \]

For two-qubit system described by the density operator $\rho_{A_1A_2}(t, \tau)$, a measure of entanglement or negativity can be defined in terms of the negative eigenvalues $\mu_i^-$ of partial transpose of the reduced atomic density matrix $\rho_{A_1A_2}^T$ [13, 14]. The negativity is
\[ \varepsilon = -2 \sum_i \mu_i^- \quad (4) \]
When $\varepsilon = 0$ two qubits are separable and $\varepsilon > 0$ means the atom-atom entanglement. The case $\varepsilon = 1$ indicates maximum entanglement. The partial transpose of the reduced atomic density matrix (3) is
\[ \rho_{A_1}^T(t, \tau) = \left( \begin{array}{cccc}
\rho_{11}(t, \tau) & 0 & 0 & \rho_{23}(t, \tau)^* \\
0 & \rho_{22}(t, \tau) & 0 & 0 \\
0 & 0 & \rho_{33}(t, \tau) & 0 \\
\rho_{23}(t, \tau) & 0 & 0 & \rho_{44}(t, \tau)
\end{array} \right) \quad (5) \]
This matrix (5) has only one eigenvalues which can be negative. Therefore, the negativity (4) can be written as
\[ \varepsilon(t, \tau) = \sqrt{(\rho_{11}(t, \tau) - \rho_{44}(t, \tau))^2 + 4|\rho_{23}(t, \tau)|^2 - \rho_{11}(t, \tau) - \rho_{44}(t, \tau)} \quad (6) \]
The results of numerical calculations of negativity (6) for separable and entangled initial atomic state (2) and different values of the model parameters are shown in Figs. 1-5. The curves were obtained under the assumption that $\tau = t/2$ as in experiment [2].
3. Results and discussion
In numerical calculations we have turned our attention to the effects of the detuning as for the vacuum and the thermal field. In Figs. 1-3 we plot the entanglement parameter as a function of a scaled time $gt$ for initial atomic state $|e, e\rangle$, $|e, g\rangle$ and $|g, g\rangle$ and different values of mean photon numbers and detunings. One can see from mentioned pictures that thermal field can induced the entanglement between Rydberg atoms successively passing the cavity of one-atom maser. But for all types of separable initial atomic states the detuning sharply reduces the maximum degree of entanglement both for the vacuum and the thermal field. When the atom-field detuning is large enough, there is no energy exchange between the cavity and atom. The field, which acts as a medium, is virtually excited during the atom-atom coupling process. In the considered case the interaction between atoms through virtual media doesn’t lead to entanglement. Note that for atoms simultaneously interact with common field the virtually excited cavities may produce the maximally entangled two-atom states [15].

In Fig. 4 we plot the entanglement parameter as a function of a scaled time $gt$ for entangled state (2) and a vacuum field. Fig.4 shows that detunings lead to decreasing of the entanglement parameter oscillations and stabilization of the entanglement. For large detunings the virtually excited medium doesn’t destroy the initial atomic quantum correlations or entanglement. Fig. 5 shows the same for thermal field. Fig 5(a) shows that for slight detunings the entanglement behavior is similar to that for a vacuum field, but the time periodicity destroys because system has a set of Rabi frequencies. One can see from Fig. 5(a) that for a thermal cavity field and small detunings the sudden death and birth of entanglement takes place. Figs. 5(b) demonstrates that for large detunings the amplitude of entanglement oscillations are scarcely affected by detuning. This result is more or less to be expected from the fact that the quantum correlations of atoms are affected by the medium involving not only the virtual but the real photons of thermal field. The frequencies of oscillations increase with increasing the Rabi frequencies $\Omega_n \approx \delta$ ($\delta \gg g$ and $\bar{n} \sim 1$).

![Figure 1](image1.png)

**Figure 1.** Time dependence of entanglement for separable atomic state $|e, e\rangle$ with $\alpha = 0$ (solid) and $\alpha = 1$ (dashed) and $\alpha = 3$ (dotted). The mean photon number $\bar{n} = 0$ (a) and $\bar{n} = 0.5$ (b).

4. Conclusions
We used the negativity to study the entanglement of the system of two Rydberg atoms successively passing a vacuum or a thermal cavity for different initial separable and entangled states taking into account the detuning. We derived the exact expressions for the reduced atomic density matrixes and calculated the negativity formulae for separable and Bell’s initial atomic state and thermal cavity field. We investigated the entanglement turning our attention to the
Figure 2. Time dependence of entanglement for separable atomic state $|\epsilon, g\rangle$ with $\alpha = 0$ (solid), $\alpha = 1$ (dashed) and $\alpha = 3$ (dotted). The mean photon number $\bar{n} = 0$ (a) and $\bar{n} = 0.5$ (b).

Figure 3. Time dependence of entanglement for separable atomic state $|g, g\rangle$ with $\alpha = 0$ (solid), $\alpha = 1$ (dashed) and $\alpha = 3$ (dotted). The mean photon number $\bar{n} = 0.2$ (a) and $\bar{n} = 0.5$ (b).

Figure 4. Time dependence of entanglement for entangled atomic state (2). The detuning $\alpha = 0$ (solid) and $\alpha = 1$ (dashed), $\alpha = 3$ (dotted) (a) and $\alpha = 10$ (solid) and $\alpha = 30$ (dashed), $\alpha = 75$ (dotted) (b). The mean photon number $\bar{n} = 0$.

role of detuning in entanglement behavior. Our numerical results reveal that for entangled
initial atomic state and small detunings the presence of detuning leads to decreasing of the entanglement amplitude oscillations and stabilization of the degree of entanglement both for the vacuum and for the thermal field. For vacuum field and large detunings the initial entanglement ceases to vary in time. For thermal field and large detunings the entanglement amplitude oscillations not decrease with detuning increasing. For thermal field and small detunings the effect of sudden death and birth of entanglement takes place. For large detunings such effect vanishes. For separable initial atomic states the detuning reduces the maximum degree of entanglement both for the vacuum and the thermal field. These results show that the atom-atom entanglement can be controlled by changing the system parameters, such as the mean photon numbers and detuning.

5. References
[1] Georgescu I, Ashhab S and Nori F 2014 Rev. Mod. Phys. 86 153
[2] Xiang Z L, Ashhab S, You J Q and Nori F 2013 Rev. Mod. Phys. 85 623
[3] Buluta I, Ashhab S and Nori F 2011 Rep. Prog. Phys. 74 104401
[4] Hagley E et al. 1997 Phys. Rev. Lett. 79 1
[5] Ghosh B, Majumdar A S and Nayak N 2008 J. Phys. B 41 065503
[6] Liao Q, Fang G, Wang Y, Ahmad M A and Liu S 2010 Chin. Phys. Lett. 8 1191
[7] Liao Q, Fang G, Ahmad M A and Liu S 2011 Optics Commun. 384 201
[8] Bashkirov E K and Nikiforova Y A 2012 Computer Optics 36 468
[9] Yan X Q 2009 Chaos Solit. Fract. 41 1645
[10] Bashkirov E K and Mastuygina M S 2016 J. Phys.: Conf. Ser. 735 012026
[11] Bashkirov E K 2017 Procedia Engineering 201 593
[12] Bashkirov E K and Mastuygina T S 2017 J. Phys.: Conf. Ser. 929(1) 012087
[13] Peres A 1996 Phys. Rev. Lett. 77 1413
[14] Horodecki R, Horodecki M and Horodecki P 1996 Phys. Lett. A 223 333
[15] Zheng S B and Guo G C 2000 Phys. Rev. Lett. 85 2392