Quantum error-correcting codes are many-body entangled states that are prepared and measured using complex sequences of entangling operations. Each element of such an entangling sequence introduces noise to delicate quantum information during the encoding or reading out of the code. It is important therefore to find efficient entangling protocols to avoid the loss of information. Here we propose an experiment that uses only global entangling operations to encode an arbitrary logical qubit to either the five-qubit repetition code or the five-qubit code, with a six-ion Coulomb crystal architecture in a Penning trap. We show that the use of global operations enables us to prepare and read out these codes using only six and ten global entangling pulses, respectively. The proposed experiment also allows the acquisition of syndrome information during readout. We provide a noise analysis for the presented protocols, estimating that we can achieve a six-fold improvement in coherence time with noise as high as $\sim 1\%$ on each entangling operation.

In this Letter we propose an alternative to the local circuit approach for realising many-body entangled states. We describe an experimental protocol to prepare two QECCs using global entangling operations alone, avoiding the need to decouple subsets of the qubit register. We consider a six-ion Coulomb crystal in a Penning trap (Fig. 1), and provide a protocol to encode an arbitrary quantum state to either the five-qubit code (5QC), or the five-qubit repetition code (5RC). We extract syndrome information while teleporting information from the code, allowing us to correct for errors that may have occurred while the logical state was stored. We provide a noise analysis showing we that double coherence times with an entangling pulse noise of 1.7% and 0.2% for 5RC and 5QC respectively. Such performance is attainable using current technology.

In general, it is difficult to find global unitary operations that realise a chosen target state from a given input state. However, many target states share a common symmetry with the crystal architecture. This simplifies protocols to realise desired states, as all the terms in the state that are equivalent up to the symmetry evolve identically. The global pulse sequences we discover enable us to execute entire error correcting protocols, including syndrome extraction, using a very low number of operations compared with other trapped-ion protocols [1–3]. We can perform 5RC(5QC) using only 14(18) discrete operations.

The difficulties in the realisation of QECCs in ion traps are technical, the first among them, the experimental finesse necessary to perform two-ion entangling gates on an array of many ions, as any spectator ions must be decoupled from the entangling interaction. This can be achieved physically, via shuttling [1], or spectroscopically, either by ‘hiding’ ions in ancillary states [2], or by using dynamic decoupling techniques [3]. Both spectroscopic methods require the repeated addressing of individual ions.

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The number of operations required to perform these protocols with two-qubit entangling gates is typically an order of magnitude greater. Conventional non-demolition stabilizer measurements are even more taxing: the 5QC stabilizers require several hundred operations.

![Quantum error correcting codes are prepared using six ions trapped in a rotationally symmetric planar crystal.](image)
per cycle. In this respect our protocol is favourable, as it substantially reduces the total gate noise introduced to the system \[12\] [13].

We consider a qubit register in a two-dimensional ion Coulomb crystal confined using a Penning trap. Physical qubits are written to the two Zeeman sublevels of the \( S_{1/2} \) ground state of each ion. These levels are separated by \( \sim 100 \text{ GHz} \) at typical magnetic field strengths. The state of the qubit register of \( N \) ions is written in the spin basis \( |s\rangle_R = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_N\rangle \) for \( s_j = \uparrow, \downarrow \), where arbitrary states are written \( |\chi\rangle_R = \sum_s r_s |s\rangle_R \) such that \( \sum_s |r_s|^2 = 1 \). We do not consider a particular species of ion, but a suitable example are beryllium ions, shown in Fig. 2.

Projective readout of the physical qubits in the computational basis is given by fluorescence from a laser resonant with a suitable dipole transition, for instance \( S_{1/2} \rightarrow P_{3/2} \). This transition is also used for Doppler cooling the crystal. A microwave field at the qubit frequency \( \omega_0 \) permits global Pauli X and Y rotations, while Pauli Z rotations are effected by applying the laser normally used for readout, detuned far from resonance, producing an AC Stark shift of the qubit levels.

We exploit the coupling between transverse mechanical modes of the Coulomb crystal and spin-dependent optical dipole forces (ODFs) to realise complex unitary operations on the qubit register. We show that we can generate global unitary operations of the form

\[
U(\phi) = \sum_s e^{i\phi_s} |s\rangle \langle s|_R ,
\]

where \( \phi \) is a vector of phases \( \phi_s \) determined by the spin-mode couplings.

We generate a spin-dependent ODF by application of a pair of crossed Raman lasers with detuning \( \Delta_R \sim 40 \text{GHz} \) from a suitable dipole transition, shown in Fig. 2 and the supplementary material of Britton et al [13]. The polarisations of the two beams can be adjusted such that the AC Stark shift of the qubit transition frequency \( \omega_0 \) due to each individual beam is zero, while their interference produces a one-dimensional optical lattice with wavevector \( k_L = (0, 0, k_L) \), the polarisation gradient of which provides the force. Additionally, we introduce a frequency difference \( \mu_L \) between the lasers causing the optical lattice to scan across the crystal and leading to a periodic driving force. If this frequency difference is tuned close to one of the normal mechanical modes of the crystal, the ODF can be made to excite collective vibrations of the ions.

In general, a quantum harmonic oscillator driven off-resonance will traverse a closed loop in phase space with radius proportional to the driving force [15]. The system returns to its initial motional state at times \( \tau = 2\pi/\delta \) where \( \delta \) is the detuning from resonance, acquiring a geometric phase proportional to the area of the loop. If the driving force is spin-dependent then this can be used to entangle the spin degree of freedom, as in the two-ion phase gate [16, 17]. We demonstrate the extension of this technique to arbitrary numbers of ions driven in modes that do not couple identically to all ions. Crucially, unlike previous applications of such ODF beams, we do not require the beams to produce forces of equal magnitude on opposing spins, allowing their ratio \( R = F_\uparrow/F_\downarrow \) to be varied between approximately -0.5 and -2 through adjustment of \( \Delta_R \). This modification greatly increases the range of unitaries that may be produced.

Following the treatment in [18] we obtain the transverse modes of the Coulomb crystal by solving the eigenvector equation for the \( N \times N \) axial stiffness matrix \( K_{zz} \). The mode eigenvectors \( a_m \) are the eigenvectors of \( K_{zz} \) and the corresponding mode eigenfrequencies \( \omega_m \) are related to its eigenvalues \( \lambda \) by the mass of a single ion, \( M \), as \( \omega_m = \sqrt{\lambda_m/M} \).

Writing the set of mode eigenvectors as a matrix \( A = \sum_i |i\rangle \langle a_m| \) with elements \( A_{mi} \), any forces \( f_i \) on the constituent ions can be decomposed into generalised forces \( Q_m \) acting on each mode, where

\[
Q_m = \sum_i A_{mi} f_i .
\]  

Applying a uniform ODF beam across the crystal, we can express the forces on the whole register as a matrix \( F \) with elements \( F_{is} \) representing the forces on ion \( i \) for each \( |s\rangle_R \) in \( |\chi\rangle_R \). The product of the modal matrix \( A \) and this new dipole force matrix gives the spin-mode coupling matrix \( M \) with elements

\[
M_{ms} = \sum_i A_{mi} F_{is} .
\]
M describes the generalised force acting on each normal mode for each state $|s\rangle_R$ of the register. In the case of a symmetric crystal, many of the modes will have a degenerate partner of the same frequency. A pair of degenerate modes will be driven simultaneously, so it is proper to consider the Pythagorean sum of the generalised forces on each pair and the corresponding rows in M are combined in quadrature.

We apply a set of pulses near-resonant with each mode such that $\mu_\ell = \omega_m - \delta$, with areas proportional to the elements $v_m$, where we use the freedom to adjust $\delta$ to ensure the loops close. The geometric phases acquired by each basis state in $|\chi\rangle_R$ is proportional to the square of the generalised forces, and given by

$$\phi_s = \sum_m (M_{ms})^2 v_m,$$

determining the unitary of Eqn. [1].

We now show we can use the described experimental setup to prepare and read-out the five-qubit code (5QC), and the five-qubit repetition code (5RC), in a six-ion register. The 5QC and 5RC encode a single logical qubit in the subspace of the Hilbert space of five physical qubits. Additionally, the measurement of syndrome information enables the identification of errors to the encoded state that can then be corrected with an appropriate single qubit operation. The 5QC protects the encoded information against arbitrary Bloch sphere rotations suffered by a single physical qubit of the register. Alternatively, the 5RC protects against dephasing channel errors acting on up to two physical qubits, but cannot correct for spin-flip noise. In many ion qubits, the T$_1$ lifetime is several hundred seconds. In these cases, the 5RC provides sufficient protection against a realistic noise model.

We consider the explicit example of a qubit register of six ions in a crystal in the configuration shown in Fig. 3(a). The qubit of the central ‘hub’ ion is initialised in the logical state $|\psi\rangle_H = \alpha |\uparrow\rangle_H + \beta |\downarrow\rangle_H$. The five ‘code’ ions in the ring around the hub will encode the quantum information. They are initially prepared in the logical state $|\chi\rangle_R = |\psi\rangle_H \otimes |S\rangle_C$, and to measure qubits of the register in the Pauli-X basis, we require only two global unitary operations, $U_{\text{ring}}$ and $U_{\text{spokes}}$, to encode and read out the 5QC, and only $U_{\text{spokes}}$ to encode and read out 5RC. The unitaries $U_{\text{spokes}}$ will globally apply a controlled phase gate between each of the qubits in the register that share a grey edge in Fig. 3(b)(c)). We summarise the error correcting protocol in the caption of Fig. 3.

The protocol requires the ability to prepare and measure the hub qubit independently of the code qubits and vice versa. This can be achieved with a focussed beam, or globally, using two isotopes of the same ion species, where the isotopic shift would be sufficient to allow resonant frequency addressing during state preparation and readout.

Without affecting the off-resonant entangling pulses.

Given suitable control to prepare the initial state and the ability to perform the necessary measurements, we also require suitable $\phi_s$ to perform $U_{\text{ring}}$ and $U_{\text{spokes}}$. A six-ion planar crystal has six transverse modes including two degenerate pairs, shown in the matrix $A^T$ in order of decreasing frequency

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -5 \\
1 & (G - 2) & 0 & -(G + 1) & (1 - G) & 1 \\
1 & 2(1 - G) & G & 2G & 0 & 1 \\
1 & (G - 2) & -G & -(G + 1) & (G - 1) & 1 \\
1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

where $G = (1 + \sqrt{5})/2$ and the mode vectors are shown un-normalised for clarity.

We have five free parameters with which to engineer the desired unitary, the pulse areas, $P_i$, applied to each of the four modes, and the force ratio, $R$. In general, it is difficult to find suitable pulse sequences, but by exploiting our freedom to include a global phase and to find elements of $\phi_s$ modulo $2\pi$ we are able to find a very large number of sparsely distributed solutions for these two codes and we are free to select those that behave well under predicted experimental noise.
FIG. 4: Average fidelity as a function of time with imperfectly prepared 5RC states, where we have pulse noise of $\sigma = 0, 0.005, 0.01$ and $0.015$ in red, yellow and green and blue respectively. Also plotted is the fidelity of a single qubit under the dephasing map in blue. Dashed line marks 0.99 fidelity. Inset shows $\tau/\tau_{1Q}$ plotted as a function of $\sigma$. We find fitting $\tau \sim \tau_0(2 - \exp(1858\sigma^2))$.

One solution, $(P_1, P_2, P_3, P_4, R)$, for $U_{\text{spokes}}$ is

$$(3.1250, 2.6043, 2.6043, 0, 1.4000),$$

where the pulse areas are normalised such that a centre-of-mass pulse $P_1 = 1$ would produce a $\pi$ radian phase shift on the $|\uparrow\rangle_H \otimes |\uparrow\downarrow\rangle_C$ state. We give a detailed analysis of the pulse sequence leading to this solution in Tab. 8 in Supplementary material. Additionally we find pulses (10.9862, 7.6768, 19.6502, 10.9861, 0.6737) will produce $U_{\text{ring}}$. We remark that we can vary the ratio of forces for each pulse, but for the present purposes it is sufficient to keep this constant throughout the sequence.

Our results show that we can execute both QECC protocols using a low number of pulses. Short pulse sequences are advantageous as errors in the protocol are minimised. We numerically simulate both protocols using imperfect unitary operations to estimate the engineering requirements necessary to realise improved quantum coherence compared with the coherence of a single qubit.

We examine the decoherence of the encoded state over some time $t$ where we perform the protocol using imperfect unitary operations. We numerically simulate the 5RC protocol using the imperfect $U_{\text{spokes}}^\sigma$ unitary, where we disrupt the optimal pulse sequence for $U_{\text{spokes}}$ by replacing pulse areas $P_j$ and force ratio $R$ with $\epsilon_j(\sigma)P_j$ and $\epsilon_R(\sigma)R$ respectively, where $\epsilon_j(\sigma)$ and $\epsilon_R(\sigma)$ are a random variables chosen from a normal distribution with standard deviation $\sigma$ and unity mean. The spin of each individual ion $\rho$ dephases via the map $\gamma_t(\rho) = (1 + e^{-t})\rho/2 + (1 - e^{-t})Z\rho Z/2$ where $Z$ is the Pauli-Z matrix. Additionally, we assume that each application of $U_{\text{spokes}}^\sigma$ is performed in time $t_U = 5 \cdot 10^{-4}$, equivalent to a 50\&mu s gate for qubits with 100\&mu s coherence time. We model this by applying the dephasing map for an additional $t_U$ before the instantaneous application of $U_{\text{spokes}}^\sigma$.

We measure decoherence using the average fidelity, $F$, calculated with $\sim 500$ random samples of $|\psi\rangle_H$ and $U_{\text{spokes}}^\sigma$ for each given $t$. We plot $F(\rho_H^t)$, the absolute value of $\text{Tr}[\rho_H^t|\psi\rangle\langle\psi|\rho_H^t]$, where $\rho_H^t$ is the state that is read out from the code. The results for 5RC are shown in Fig. 4.

To quantify the improvement in coherence time attained using the code, we define the high-fidelity time, $\tau$, to be the time the prepared code maintains fidelity above 0.99, and compare this to $\tau_{1Q}$, the high-fidelity time of a single qubit. The inset of Fig. 4 shows $\tau/\tau_{1Q}$ for 5RC as a function of $\sigma$. We observe scaling of the form

$$\tau/\tau_{1Q} = \tau_0(2 - e^{\sigma^2/\alpha})$$

where $\tau_0 = \tau/\tau_{1Q}$ is the code-dependent optimal coherence time improvement we can achieve for the noiseless $\sigma = 0$ case. Our results show that we converge exponentially towards $\tau_0$ as $\sigma^2$ decreases. For 5RC, we find $\alpha \sim 5.4 \cdot 10^{-4}$, and $\tau_0 \sim 6.92$.

We perform a similar simulation for 5QC, using disrupted $U_{\text{ring}}$ and $U_{\text{spoke}}$ unitaries, and where ions decohere via the depolarising map $\xi_t(\rho) = (1+3e^{-t})\rho/4 + (1-e^{-t})(X\rho X + Y\rho Y + Z\rho Z)/4$. Under these conditions we find $\alpha \sim 3.0 \cdot 10^{-5}$, and $\tau_0 \sim 2.45$ for the 5QC protocol.

We extrapolate the present data to the value $\tau/\tau_{1Q} = 1$ to obtain a threshold pulse noise, $\sigma_{\text{th}}$, of $\sigma_{\text{th}} \sim 0.0038$ for 5QC, and $\sigma_{\text{th}} \sim 0.018$ for 5RC. Producing gates that meet these threshold values is feasible with existing technology. We expect that further improvements can be achieved by repeating the protocol periodically, thus suppressing errors before significant decoherence has occurred.

We have shown, by utilising symmetries of the mechanical modes of an ion Coulomb crystal, that we can find global entangling operations to produce interesting quantum states. We have demonstrated this by describing two quantum error-correcting protocols using the presented scheme. Further, we have shown that the simplicity of the proposed QECC protocols leads to improvements in quantum coherence with relatively modest experimental demands. We expect the present experimental proposal will produce results that are competitive with current state of the art trapped-ion error-correcting protocols [1-3]. We hope the methods we have used will motivate further study into other entangled states that can be produced entirely or in part by simple global operations.

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SUPPLEMENTARY MATERIAL

Table I shows how each basis state evolves under the application of the series of pulses for $U_{\text{spoke}}$, from the initial product state in the leftmost column, towards the target cluster state in the rightmost column. The columns list the absolute phases and phase angles after each pulse (we group all cyclic permutations of the ring qubits as these acquire the same phase). The bottom row lists the fidelity of each intermediate state relative to the target.
| Basis $|s_i\rangle$ | Initial Phase/π | Phase after $\omega_1$ Pulse/π | Phase after $\omega_2$ Pulse/π | Phase after $\omega_3$ Pulse/π | Target Phase/π |
|--------------|-----------------|------------------|------------------|------------------|-----------------|
| $|0\rangle$ 0 | 0  | 3.125  | 1.125  | 3.125  | 1.125  | 0  |
| + Perms      | $\phi_i$        | $\phi_i$         | $\phi_i$ (mod 2π) | $\phi_i$         | $\phi_i$ (mod 2π) | $\phi_i$         |
| 00000        | 1.125  | 1.125  | 2.125  | 0.125  | 3.125  | 1.125  | 0  |
| 00001        | 0.125  | 0.125  | 2.743  | 2.743  | 3.125  | 2.743  | 0  |
| 00010        | 0.125  | 0.125  | 0.507  | 0.507  | 3.125  | 0.507  | 0  |
| 00011        | 0.125  | 0.125  | 2.743  | 2.743  | 3.125  | 2.743  | 0  |
| 00101        | 0.125  | 0.125  | 0.507  | 0.507  | 3.125  | 0.507  | 0  |
| 00111        | 0.125  | 0.125  | 2.743  | 2.743  | 3.125  | 2.743  | 0  |
| 01001        | 1.125  | 1.125  | 2.125  | 0.125  | 3.125  | 0.125  | 1  |
| 01011        | 1.125  | 1.125  | 2.125  | 0.125  | 3.125  | 0.125  | 1  |
| 01101        | 0.125  | 0.125  | 1.125  | 1.125  | 2.125  | 1.125  | 0  |
| 01111        | 0.125  | 0.125  | 1.125  | 1.125  | 2.125  | 1.125  | 0  |
| 10000        | 1.125  | 1.125  | 1.125  | 1.125  | 2.125  | 1.125  | 0  |
| 10001        | 0.125  | 0.125  | 2.743  | 2.743  | 3.125  | 2.743  | 0  |
| 10011        | 0.125  | 0.125  | 0.507  | 0.507  | 3.125  | 0.507  | 0  |
| 10101        | 0.125  | 0.125  | 2.743  | 2.743  | 3.125  | 2.743  | 0  |
| 10111        | 1.125  | 1.125  | 1.507  | 1.507  | 4.125  | 1.507  | 1  |
| 11001        | 3.125  | 1.125  | 1.507  | 1.507  | 4.125  | 1.507  | 1  |
| 11011        | 6.125  | 0.125  | 6.125  | 0.125  | 6.125  | 0.125  | 1  |
| 11111        | 6.125  | 0.125  | 6.125  | 0.125  | 6.125  | 0.125  | 1  |

Fidelity 0.5 0.25 0.634 1 -

TABLE I: Phases $\phi_i$ acquired by basis states $|s_i\rangle$ after each stage of the composite entangling pulse. Note that the final state is equivalent to the target up to a global phase of $\epsilon = 1.125\pi$. The fidelity with the target state at each stage is shown on the final row.