We propose and analyze a scanning microscope to monitor ‘live’ the quantum dynamics of cold atoms in a Cavity QED setup. The microscope measures the atomic density with subwavelength resolution via dispersive couplings to a cavity and homodyne detection within the framework of continuous measurement theory. We analyze two modes of operation. First, for a fixed focal point the microscope records the wave packet dynamics of atoms with time resolution set by the cavity lifetime. Second, a spatial scan of the microscope acts to map out the spatial density of stationary quantum states. Remarkably, in the latter case, for a good cavity limit, the microscope becomes an effective quantum non-demolition (QND) device, such that the spatial distribution of motional eigenstates can be measured back-action free in single scans, as an emergent QND measurement.

Spatially resolved observation of individual atoms is a key ingredient in exploring quantum many-body dynamics with ultracold atoms. This is highlighted by the recent development of the quantum gas microscope [1] where fluorescence measurements provide us with single shot images of atoms in optical lattices. Fluorescence imaging is, however, an inherently destructive quantum measurement, as it is based on multiple resonant light scattering resulting in recoil heating (see, however, [2]). In contrast, quantum motion of cold atoms can also be observed in non-destructive, weak measurements, realizing the paradigm of continuous measurement of a quantum system [3–5]. Below we describe and analyze a quantum optical setup for a scanning atomic microscope employing dispersive interactions in a Cavity QED (CQED) setup [6], where the goal is to achieve continuous observation of the density of cold atoms [7] with subwavelength resolution [8]. We will be interested in operating modes, where we either map out spatial densities of energy eigenstates in single scans as an emergent QND measurement [9, 10], or we monitor at a fixed position, the time resolved response to ‘see’ quantum motion of atoms.

The operating principle of the microscope is illustrated as a CQED setup in Fig. 1: We assume that an atom traversing the focal region of the microscope signals its presence with an internal spin flip, i.e. the position, and thus motion of the atom, is correlated with its internal spin degree of freedom. While subwavelength spatial resolution can in principle be achieved by driving transition between spin states in the presence of external fields generating energy shifts with strong spatial gradients [11, 12], this spatial resolution is typically accompanied with strong forces acting on the atom. Instead we will describe below a setup with diminished disturbance, based on position dependent ‘dark state’ in a Λ-system [13], involving a pair of longlived atomic ground state levels, representing the spin. We can detect this spin flip nondestructively with a dispersive interaction, e.g. as shift of a cavity mode of an optical resonator. Thus the atom traversing the focal region of the microscope, as defined by lasers generating the atomic dark state, becomes visible as a phase shift of the laser light reflected from the cavity. This phase shift is revealed in homodyne detection. Such CQED schemes are timely in view of both the recent progress with cold atoms in cavity and nano-photonic setups [14–22], and the growing interests in conditional dynamics of cold atoms under measurement [23–28].

Below we will develop a quantum optical model of con-
taneous measurement [4, 5, 29] of atomic density, via measurement of the homodyne current for the setup described in Fig. 1. We adopt the language of the Stochastic Master Equation (SME) for the conditional density matrix $\rho_c(t)$ of the joint atom-cavity system, which describes time evolution conditional to observation of a given homodyne current trajectory, as ‘seen’ in a single run of an experiment, and including the backaction on the atom. This will allow us to address to what extent the observed homodyne current in a spatial scan provides a faithful measurement of atomic density, and the expected signal-to-noise ratio (SNR).

Quantum Optical Model – We consider a model system of an atom moving in 1D along the $z$-axis, placed in a driven optical cavity. To detect the atom at $z_0$ with resolution $\sigma$ we introduce a spatially localized dispersive coupling of the atom to a single cavity mode of the form

$$\hat{H}_{\text{coup}} = \phi_{z_0}(\hat{z}) \hat{c}^\dagger \hat{c}. \quad (1)$$

Here $\phi_{z_0}(z)$ defines a sharply peaked focusing function of support $\sigma$ around $z_0$, and $\hat{c}^\dagger \hat{c}$ is the photon number operator for the cavity mode with destruction (creation) operators $\hat{c} (\hat{c}^\dagger)$. An implementation of $\phi_{z_0}(\hat{z})$ achieving optical subwavelength resolution $\sigma \ll \lambda$ based on atomic dark states in a Λ-system will be described below. We find it convenient to write $\phi_{z_0}(z) \equiv Af_{z_0}(z)$ with $f_{z_0}(z)$ normalized, and $A$ a constant with the dimensions of energy.

According to Eq. (1), the presence of an atom inside the focal region results in a shift of the cavity resonance. This can be detected with homodyne measurement, where the output field of the cavity is superimposed with a local oscillator with phase $\phi$. The homodyne current can, for a single measurement trajectory, be written as $I(t) = \sqrt{\kappa} \langle \hat{X}_\phi \rangle_c + \xi(t)$, i.e. follows the expectation value of the quadrature operator of the intracavity field, $\hat{X}_\phi \equiv e^{i\phi} \hat{c}^\dagger + e^{-i\phi} \hat{c}$, up to the (white) shot noise $\xi(t)$. Here $\kappa$ represents the cavity damping rate, and $\langle \ldots \rangle_c \equiv \text{Tr}\{\ldots \rho_c(t)\}$ refers to an expectation value with respect to the conditional density matrix of the joint atom-cavity system.

On a more formal level, we write for the evolution under homodyne detection the Itô stochastic differential equations for the homodyne current

$$dX_{\phi}(t) \equiv I(t) dt = \sqrt{\kappa} \langle \hat{X}_\phi \rangle_c dt + dW(t), \quad (2)$$

with $dW(t)$ Wiener noise increments, and the SME for the conditional density matrix

$$d\rho_c = -i[H_c, \rho_c] dt + \kappa D[\hat{c}] \rho_c dt + \sqrt{\kappa} \mathcal{H}[\hat{c} e^{-i\phi}] \rho_c dW(t). \quad (3)$$

Eq. (2) identifies the homodyne current as measurement of the quadrature component $dX_{\phi}(t)$ of the output field in a time step $[t, t + dt]$. The SME (3) contains the total Hamiltonian $\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_c + \hat{H}_{\text{coup}}$ with $\hat{H}_{\text{sys}} = \hat{p}_z^2 / 2m + V(\hat{z})$ the Hamiltonian of the atomic system in an external potential $V$, $\hat{H}_c = i\hbar \sqrt{\kappa} \mathcal{E} (\hat{c} - \hat{c}^\dagger)$ the Hamiltonian for the driven cavity in the rotating frame (we assume resonant driving for simplicity), and $\mathcal{E}$ the coherent amplitude of the cavity mode driving field. The last two terms in Eq. (3) account for the back-action of homodyne measurement. The Lindblad operator $D[\hat{c}] \equiv \epsilon \hat{c}^\dagger \hat{c} - \frac{1}{2} \hat{c}^\dagger \hat{c}^2 - \frac{1}{2} \hat{c} \hat{c}^\dagger \epsilon$ describes the system decoherence (the cavity field damping) due to the coupling to the outside electromagnetic modes, and the nonlinear operator $\mathcal{H}(\hat{c}) \rho_c = \epsilon \hat{c}^\dagger \hat{c} \rho_c - \epsilon \rho_c \hat{c}^\dagger \hat{c} + \text{H.c.}$ updates the density matrix conditioned on the observation of the homodyne photocurrent $I(t)$.

The relation between the homodyne current and the local atomic density is most transparent in the limit where the cavity response time $\tau_c = 1/\kappa$ is much faster than other time scales including atomic motion $\hat{H}_{\text{sys}}$, and the dispersive coupling $f_{z_0}$, i.e. the bad cavity limit. Adiabatic elimination of the cavity gives

$$dX_{\phi}(t) \equiv I(t) dt = 2\sqrt{\kappa} \langle f_{z_0}(\hat{z}) \rangle_c dt + dW(t), \quad (4)$$

with the atomic conditional density matrix $\tilde{\rho}_c(t)$ obeying the SME

$$d\tilde{\rho}_c = -i[\hat{H}_{\text{sys}}, \tilde{\rho}_c] dt + \gamma D[f_{z_0}(\hat{z})] \tilde{\rho}_c dt + \sqrt{\gamma \mathcal{H}[f_{z_0}(\hat{z})]} \tilde{\rho}_c dW(t). \quad (5)$$

Here $\gamma = [4A\mathcal{E}/(\hbar\kappa)]^2$ is an effective measurement rate, and we have chosen $\phi = -\pi/2$ (see Appendix D). According to Eq. (4) the homodyne current $I(t)$ is a direct probe of the local atomic density at $z_0$ with spatial resolution $\sigma$ [30]. Eqs. (4) and (5), or (2) and (3) in the general case, provide us with the tools to study dynamics of the ‘microscope’ in various modes of operation (see below).

Instead of single trajectories, we can also consider ensemble averages corresponding to repeated preparation and measurement cycles. We define a density operator for the atom-cavity system $\rho(t) = \langle \rho_c(t) \rangle_{st}$ as statistical average over the conditional density matrices, and an averaged homodyne current $\langle I(t) \rangle_{st} = \sqrt{\kappa} \langle \hat{X}_\phi \rangle_{st}$.

This density operator obeys a master equation (ME), obtained from the SME (3) by averaging over trajectories. Thus $\rho_c(t) \to \rho(t)$ in Eq. (3) with the stochastic term dropped according to the Itô property $\langle \ldots dW(t) \rangle_{st} = 0$. An analogous ME for the atom $\tilde{\rho}(t) = \langle \tilde{\rho}_c(t) \rangle_{st}$ can be derived from the adiabatically eliminated SME (5), see Appendix D.

Implementation of the focusing function $\phi_{z_0}(\hat{z})$ – The atom-cavity coupling Eq. (1) with subwavelength resolution can be achieved using the position-dependent dark state of a Λ-system [13, 31, 32]. We consider the level scheme of Fig. 2a, where two atomic ground (spin) states $|g\rangle$ and $|r\rangle$ are coupled to the excited state $|e\rangle$ with Rabi frequencies $\Omega_0$ and $\Omega_1(z)$, respectively. This configuration supports a dark state $|D(z)\rangle = \sin \theta(z) |g\rangle - \cos \theta(z) |r\rangle$ with $\tan \theta(z) = \Omega_1(z)/\Omega_0$, which via destructive interference is decoupled from the dissipative excited state $|e\rangle$. We note that in spatial regions $\Omega_1(z) \gg \Omega_0$ the
atom will be (dominantly) in state $|g\rangle$, while in regions $\Omega_1(z) \ll \Omega_0$ the atom will be in $|r\rangle$. This allows us to define via the spatial dependence of $\Omega_1(z)$ regions with subwavelength resolution $|z - z_0| \lesssim \sigma < \pi/k = \lambda/2$, characterized by atoms in $|r\rangle$. Atoms in $|r\rangle$ can be dispersively coupled to the cavity mode, resulting in a shift $g^2(z)/\Delta c \tilde{c}$, with $g(z)$ the cavity coupling much smaller than the detuning $\Delta_0$ (c.f. Fig. 2a). Thus atoms prepared in the dark state experience a shift $1$ with

$$\phi_{z_0}(z) = \frac{\Delta g^2(z)}{\Delta_0} |\langle r| D(z) \rangle|^2 = \frac{\Delta g^2(z)}{\Delta_0} \cos^2 \theta(z).$$

We illustrate this focusing function with subwavelength resolution in Fig. 2b for a specific laser configuration.

**Microscope Operation** – The parameters characterizing the microscope are the spatial resolution $\sigma \ll \lambda$, the temporal resolution $\tau_c$ (given essentially by the cavity linewidth $1/k$) and the dispersive atom-cavity coupling controlling the strength of the measurement. To be specific we illustrate below the operation of the microscope as continuous observation of an atom moving in a harmonic oscillator (HO) potential with an oscillation frequency $\omega$ and vibrational eigenstates $|n\rangle$ ($n = 0, 1, \ldots$). The generic physical realization includes a neutral atom in an optical trap (lattice), or an ion in a Paul trap [5], where we require a spatial resolution better than the length scale set by the HO ground state $\sigma \lesssim t_0 = \sqrt{\hbar/m\omega}$ with $m$ the atomic mass.

We consider below two modes of operation. In the first, the microscope is placed at a given $z_0$, and we wish to ‘record a movie’ of the time dynamics of an atomic wave packet (e.g. a coherent state) passing (repeatedly) through the observation zone. This requires a time resolution better than the oscillation period, and corresponds to the ***bad cavity limit $\kappa \gg \omega$***, where according to Eq. (5) the homodyne current as a function of time mirrors directly the wave packet motion at $z_0$ (c.f. Fig. 3, and discussion below). As the second case we consider the ***good cavity limit $\kappa \ll \omega$***. Here the observed homodyne signal traces the atomic dynamics at $z_0$ cavity-averaged over many oscillation periods [33]. However, as we show below, in this regime a **slow scan** of the focal point $z_0 \equiv z_0(t)$ across the spatial region of interest will turn the microscope into an effective QND device, which maps out the spatial density associated with a particular energy eigenfunction of the trapped particle with resolution $\sigma$. This will be discussed below in the context of Fig. 1c-e, where a particle is prepared initially in a state $|\alpha\rangle$ and in the spirit of QND measurements a **single scan** with the microscope first collapses the atomic state into a particular motional eigenstate, and subsequently ‘takes a picture’ of its spatial density. This ability of a **single scan** to reveal the density of energy eigenfunctions is in contrast to the first case above, where the measurement is inherently destructive and a good SNR is only obtained with repeated runs of the experiment.

**Bad cavity limit and time-resolved dynamics** – In Fig. 3a we plot the ensemble averaged homodyne current $\langle I(t) \rangle_{st}$ for a microscope positioned at $z_0 = 0$, which monitors the periodic motion of an atomic wave packet in the HO. The atom is initially prepared in a coherent state $|\alpha\rangle$ displaced from trap center with $|\alpha| \gg 1$, and the microscope detects the transit of the wave packet with velocity $v = 2t_{sec} |\alpha| \omega$ through the trap center at times $t = 1/4, 3/4T_{sec}$ etc., with $T_{sec} = \pi\tau_c/\omega$ the oscillator period. The time dependence of the homodyne current reveals the shape of the wave packet for the given resolution $\sigma = 0.3t_0$. Fig. 3a plots $(I(t))_{\text{st}} = 2\sqrt{\gamma} \text{Tr} \{f_{z_0}(z)\hat{\rho}(t)\}$ for increasing measurement strengths $\gamma$, with $\hat{\rho}(t) = \langle \hat{\rho}_c(t) \rangle_{\text{st}}$ obeying Eq. (5). For the given parameters, Fig. 3a displays the ability of the homodyne current to faithfully represent the temporal shape of the wave packet, and re-
induces strong additional noises.

The SNR associated with these measurements is shown in Figs. 3c-d. We define the SNR as \( \frac{\langle I(t) \rangle}{\sigma} \), with \( I(t) \) the homodyne current (2) after a lowpass filter with bandwidth \( \tau^{-1} \), and the variance \( \langle \delta I^2(t) \rangle = \langle I^2(t) \rangle - \langle I(t) \rangle^2 \). We choose an integration time \( \tau \) sufficiently long to suppress the shot noise, but short enough to resolve the temporal shape of the wave packet. An optimal \( \tau \) is related to the microscope spatial resolution, \( \tau \sim \sigma/v = (\sigma/\ell_0) \tau_r \), with \( \tau_r \) the transit time of the wave packet through the focal region. In Fig. 3c we show the homodyne current \( I_f(t) \) averaged over an increasing number of measurements, and the convergence to the results of Fig. 3a. In Fig. 3d the SNR in a single scan is plotted vs. the measurement strength \( \gamma \). It shows the general behavior of non-QND measurements [34]: For small \( \gamma \), the SNR grows with increasing \( \gamma \) due to suppression of the shot noise. For large \( \gamma \), SNR eventually drops down as the measurement backaction induces strong additional noises.

Good cavity limit as emergent QND measurement – A QND measurement requires that the associated observable commutes with the system Hamiltonian. While \( f_z(\hat{z}) \) does not commute with \( H_{sys} \), an effective QND measurement emerges in the good cavity limit \( \kappa \ll \omega \). We can see this by transforming the SME (3) to an interaction picture with respect to \( \hat{H}_{sys} \). This transformation results in the replacement \( f_z(\hat{z}) \to \sum f_z(\hat{z}) e^{-i\omega t} \), where \( f_z(\hat{z}) = \sum_f f_{n_f,n+\ell}(\hat{z}) \langle n + \ell | \hat{z} | n \rangle \). In a homodyne measurement, where the current \( I(t) \) is monitored with time resolution \( 1/\kappa \), as filtered by the cavity, the terms rapidly oscillating with frequencies \( \ell \omega \) (motional sidebands) will not be resolved. Thus homodyne detection provides a continuous measurement of \( f_z(\hat{z}) = \sum f_{n_f,n}(\hat{z})\langle n | \hat{z} | n \rangle \) representing the emergent QND observable [35].

A formal derivation of these results is provided in Appendix D starting from the SME (3). There we derive for the homodyne current \( dX_\phi(t) = I(t)dt = 2\sqrt{\gamma} \langle f_z(\hat{z}) \rangle + dW(t) \) with \( \langle \ldots \rangle = Tr\{\ldots \hat{p}_c(t)\} \), where the conditional density operator \( \hat{p}_c(t) \) obeys the SME

\[
d\hat{p}_c = -\frac{i}{\hbar} [\hat{H}_{sys}, \hat{p}_c] dt + \sum_{\ell \neq 0} \frac{\gamma}{1 + (2\omega \ell/\kappa)^2} D[f_z(\hat{z})] \hat{p}_c dt + \gamma D[f_z(\hat{z})] \hat{p}_c dt + \sqrt{\gamma} \hat{H}[f_z(\hat{z})] \hat{p}_c dW(t),
\]

with \( \gamma \) the measurement strength defined above (assuming resonant driving). To provide a physical interpretation, we take matrix elements of Eq. (6) in the energy eigenbases and obtain a (nonlinear) stochastic rate equation (SRE) for the trap-state populations \( p_n = \langle n | \hat{p}_c | n \rangle \):

\[
dp_n = \frac{\gamma}{1 + (2\omega \ell/\kappa)^2} \left[ A_n^{(+)} p_{n+1} + A_n^{(-)} p_{n-1} - B_n p_n \right] dt + 2\sqrt{\gamma} p_n \left[ \sum_m f_{m,n} p_m \right] dW(t).
\]

Here \( A_n^{(+)} = |f_{n+1,n}|^2, B_n = A_n^{(+)} + A_n^{(-)}, \) and for simplicity we have kept only the dominant terms \( \ell = 0, \pm 1 \) for \( \kappa/\omega \ll 1 \). We emphasize that Eq. (7) involves two time scales. The stochastic term in the second line describes the collapse of the density operator to a particular trap eigenstate \( \hat{p}_c(t) \to |n\rangle \langle n | \) within a collapse time \( T_{coll} \sim 1/\gamma \). In contrast, the first line is a redistribution of population between the trap levels, for a much longer dwell time \( T_{dwell} \sim (2\omega/\kappa)^2 \gamma^{-1} \gg T_{coll} \). As a result, the time evolution consists of a rapid collapse to an energy eigenstate \( |n\rangle \), followed by a sequence of rare quantum jumps \( n \to n \pm 1 \) on the time scale \( T_{dwell} \). The QND mode of the microscope exploits these two time scales by scanning the focal point across the system, \(-L/2 < z_0(t) < L/2\), in a time \( T_{coll} \ll T \lesssim T_{dwell} \). Starting the measurement scan, the motional state will first collapse to a particular \( |n\rangle \), with the subsequent scan revealing the spatial density profile \( \langle n | f_z(\hat{z}) | n \rangle = \int dz f_{z_0}(z) \langle z | n \rangle^2 \).

Fig. 1c shows a simulation representing a single run in the QND regime \( \kappa/\omega = 0.1 \) based on integrating the SME (6). The atom at \( t = 0 \) is prepared in a thermal motional state of the HO, \( \hat{p}(0) = \sum_n p_n | n \rangle \langle n | \) with \( \langle n \rangle = 0.6 \). We perform three consecutive spatial scans covering \(-L/2 < z_0(t) < L/2 \) (\( L = 10\ell_0 \)), each in a time interval \( T (\gamma T = 5000) \). For the run shown in Figs. 1c-e, the QND measurement in scan 1 first projects the atomic trap population into \( |0\rangle \), followed by a transition to \( |1\rangle \) at time \( t_1 \), and \( |1\rangle \to |0\rangle \) at \( t_2 \) in scan 2, and no transition in scan 3. The homodyne current \( I_f(t) \) associated with these single scans is a faithful representation of the spatial density distributions of eigenfunctions \( \langle z | n \rangle^2 \). In Fig. 4a the SNR of single scans of a pure state is shown against the (dimensionless) measurement strength \( \gamma T \). By decreasing \( \kappa/\omega \) we greatly suppress the measurement back-action, rendering them into rarer quantum jumps, thus improving the SNR.
The concept of a scanning microscope to observe in vivo cold atom dynamics is readily adapted to a quantum many-body system, and we show in Fig. 4b a single spatial scan of the Frenkel oscillation of an non-interacting Fermi sea in the presence of a single impurity (see Appendix E for details). While we have focused on homodyne measurement in CQED for continuous readout (with experimental feasibility discussed in Appendix C), atomic physics setups provide interesting alternative routes to achieve weak continuous measurement, e.g. coupling to atomic ensembles via Rydberg interactions [36–38].

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[1] For a review, see S. Kuhr, National Science Review 3, 170 (2016) and references therein.
[2] Y. Ashida and M. Ueda, Phys. Rev. Lett. 115, 095301 (2015).
[3] V. B. Braginsky and F. Y. Khalili, Quantum Measurement (CUP, Cambridge, 1992).
[4] H. M. Wiseman and G. J. Milburn, Quantum measurement and control (CUP, Cambridge, 2009).
[5] C. Gardiner and P. Zoller, The Quantum World of Ultra-Cold Atoms and Light Book II (ICP, London, 2015).
[6] For recent reviews, see I. B. Mekhov and H. Ritsch, Journal of Physics B: Atomic, Molecular and Optical Physics 45, 102001 (2012); H. Ritsch, P. Domokos, F. Brennecke, and T. Esslinger, Rev. Mod. Phys. 85, 533 (2013); T. E. Northrup and R. Blatt, Nat. Photon. 8, 356 (2014); A. Reiserer and G. Rempe, Rev. Mod. Phys. 87, 1379 (2015).
[7] Continuous observation of classical atomic motion in Cavity QED is demonstrated in C. J. Hood, T. W. Lynn, A. C. Doherty, A. S. Parkins, and H. J. Kimble, Science 317 (2007); T. Puppe, I. Schuster, A. Grothe, A. Kuhnaneck, K. Murr, P. W. H. Pinkse, and G. Rempe, Phys. Rev. Lett. 99, 013002 (2007); M. L. Terraciano, R. Olson Knell, D. G. Norris, J. Jing, A. Fernandez, and L. A. Orozco, Nat. Phys. 5, 480 (2009).
[8] Strong measurement of quantum particles with sub-wavelength resolution is demonstrated, e.g. in P. C. Maurer, J. R. Maze, P. L. Stanwix, L. Jiang, A. V. Gorshkov, A. A. Zibrov, B. Harke, J. S. Hodges, A. S. Zibrov, A. Yacoby, D. Twitchen, S. W. Hell, R. L. Walsworth, and M. D. Lukin, Nat. Phys. 6, 912 (2010); F. Zähringer, G. Kirchmair, R. Gerritsma, E. Solano, R. Blatt, and C. F. Roos, Phys. Rev. Lett. 104, 100503 (2010).
[9] Note that the local density operator does not commute with the Hamiltonian. Nevertheless, in our setup, we can measure local densities for energy eigenstates with no back-action — an emergent QND measurement, see text and [35].
[10] For QND measurements in quantum optics, see, e.g., S. Gleyzes, S. Kuhr, C. Guerlin, J. Bernu, S. Deleglise, U. Busk Hoff, M. Brune, J.-M. Raimond, and S. Haroche, Nature 446, 297 (2007); B. R. Johnson, M. D. Reed, A. A. Houck, D. I. Schuster, L. S. Bishop, E. Ginosar, J. M. Gambetta, L. DiCarlo, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf, Nat. Phys. 6, 663 (2010); J. Volz, R. Gehr, G. Dubois, J. Estève, and J. Reichel, Nature 475, 210 EP (2011); G. Barontini, L. Hohmann, F. Haas, J. Estève, and J. Reichel, Science 349, 1317 (2015); C. B. Møller, R. A. Thomas, G. Vasilakis, E. Zeuthen, Y. Tsaturyan, M. Balabas, K. Jensen, A. Schliesser, K. Hammerer, and E. S. Polzik, Nature 547, 191 (2017).
[11] K. D. Stokes, C. Schnurr, J. R. Gardner, M. Marable, G. R. Welch, and J. E. Thomas, Phys. Rev. Lett. 67, 1997 (1991).
[12] C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schauss, T. Fukuhara, I. Bloch, and S. Kuhr, Nature 471, 319 (2011).
[13] A. V. Gorshkov, L. Jiang, M. Greiner, P. Zoller, and M. D. Lukin, Phys. Rev. Lett. 100, 093005 (2008).
[14] A. Stute, B. Casabone, P. Schindler, T. Monz, P. O. Schmidt, B. Brandstatter, T. E. Northup, and R. Blatt, Nature 485, 452 (2012).
[15] M. Wolke, J. Klimmer, H. Kelller, and A. Hemmerich, Science 337, 75 (2012).
[16] B. Hacker, S. Welte, G. Rempe, and S. Ritter, Nature 536, 193 (2016).
[17] J. Léonard, A. Morales, P. Zupancic, T. Esslinger, and T. Donner, Nature 543, 87 (2017).
[18] A. J. Kollár, A. T. Papageorge, V. D. Vaidya, Y. Guo, J. Keeling, and B. L. Lev, Nat. Commun. 8, 14386 (2017).
[19] J. D. Thompson, T. G. Tiecke, N. P. de Leon, J. Feist, A. V. Akimov, M. Mallans, A. S. Zibrov, V. Vuletić, and M. D. Lukin, Science 340, 1202 (2013).
[20] C. Junge, D. O’Shea, J. Volz, and A. Rauschenbeutel, Phys. Rev. Lett. 110, 213604 (2013).
[21] A. A. Zibrov, Y. Liu, G. Barontini, S. Kuhr, P. Schindler, T. E. Northup, and K. Hammerer, Nature Comm. 8, 13084 (2017).
[22] H. L. Sørensen, J.-B. Béguin, K. W. Kluge, I. Iakoupov, A. S. Sørensen, J. H. Müller, E. S. Polzik, and J. Appel, Phys. Rev. Lett. 117, 133604 (2016).
[23] D. A. Steck, K. Jacobs, H. Mabuchi, T. Bhattacharya, and S. Habib, Phys. Rev. Lett. 92, 223004 (2004).
[24] M. D. Lee and J. Ruostekoski, Phys. Rev. A 90, 023628 (2014).
[25] A. C. J. Wade, J. F. Sherson, and K. Mølmer, Phys. Rev. Lett. 115, 060401 (2015).
[26] G. Mazzucchi, S. F. Caballero-Benitez, D. A. Ivanov, and I. B. Mekhov, Optica 3, 1213 (2016).
Appendix A: Quantum non-Demolition vs. Emergent Quantum non-Demolition Measurements

In this section we summarize the concept of emergent quantum non-demolition (QND) measurements in a more formal way.

A familiar definition of a QND measurement [3, 34] considers an observable $\hat{A}$ to be QND, if it commutes with the system Hamiltonian $[\hat{H}, \hat{A}] = 0$. Such a QND observable can be continuously measured with an arbitrary high Signal-to-Noise Ratio (SNR) [3, 34].

In general, for a system observable $\hat{O}$, which does not commute with the Hamiltonian, $[\hat{H}, \hat{O}] \neq 0$, we define as emergent QND observable

$$\hat{O}_{\text{eQND}} \equiv \sum_n |n\rangle \langle n| \hat{O} |n\rangle \langle n|,$$

with $|n\rangle$ the energy eigenstates. Measurement of $\hat{O}_{\text{eQND}}$ provides the same information as of $\hat{O}$ for energy eigenstates, but in a non-destructive way. This enables studying properties of energy eigenstates of various quantum systems with very high precision and from different perspectives provided by the corresponding observables $\hat{O}$. The emergent QND measurement in context of the quantum scanning microscope, as discussed in the main text, considers the eQND observable defined from the $\delta$-like probe $f(\hat{z})$, where $\hat{z}$ is the position operator. This allows in particular to map out atomic densities of energy eigenstates for harmonic oscillator and Freidel oscillations for many-body systems in a single scan with high SNR, as illustrated in Figs. 1 and 4 of the main text.

Appendix B: Engineering of The Sub-wavelength Focusing Function $\phi_{2n}(z)$

Here we discuss in detail the realization of the focusing function $\phi_{2n}(z)$ [c.f. Eq. (1) of the main text], showing that subwavelength resolution can be achieved along with negligible additional forces on the atom.

1. Sub-wavelength spin structure with negligible non-adiabatic potential

The atomic internal levels for implementing the focusing function, shown in Fig. 2a of the main text, consists of a $\Lambda$-system formed by $|g\rangle, |r\rangle, |e\rangle$, described by the Hamiltonian

$$\hat{H}_a = -\hbar \left( \Delta_e + i \frac{\Gamma_e}{2} \right) \sigma_{ee} + \frac{\hbar}{2} \left[ \Omega_0(z) \sigma_{eg} + \Omega_1(z) \sigma_{ee} + \text{H.c.} \right],$$

where $\Gamma_e$ is the decay rate of the excited state, and we assume Raman resonance $\Delta_r = 0$. Diagonalizing $\hat{H}_a$ gives
the eigenstates

\[ |D(z)\rangle = \sin \theta(z) |g\rangle - \cos \theta(z) |r\rangle , \]

\[ |+ (z)\rangle = \cos \chi(z) |e\rangle + \sin \chi(z) [\cos \theta(z) |g\rangle + \sin \theta(z) |r\rangle] , \]

\[ |- (z)\rangle = \sin \chi(z) |e\rangle - \cos \chi(z) [\cos \theta(z) |g\rangle + \sin \theta(z) |r\rangle] , \]

\[ (B2) \]

with the corresponding eigenenergies \( E_D = 0 \) and \( E_\pm(z) = -\frac{\hbar^2}{2m}[\Delta_c \mp \Omega_c^2(z) + \Delta_c^2]^{1/2} \), where \( \Delta_c = \Delta_c + i\Gamma_c/2 \), and the mixing angles defined by \( \theta(z) = \arctan[\Omega(z)/\Omega_0(z)] \) and \( \chi(z) = -(1/2) \arctan[\Omega_0(z) + \Omega_1(z)/\Delta_c] \). We note that the dark state \( |D(z)\rangle \) is decoupled from the dissipative excited state \( |e\rangle \), and its spin structure is varying in space controlled by the Rabi frequency configuration.

We are interested in the regime where \( \text{Re}[E_\pm(z)] \) is much larger than the other energy scales in the model. In the spirit of the Born-Oppenheimer (BO) approximation, we study the slow dynamics by assuming the atomic internal state remains in \( |D(z)\rangle \) adiabatically. This allows us to design the desired sub-wavelength spin structure \(|\langle r | D(z)\rangle|^2| = \cos^2 \theta\). Such a spatially varying internal spin is, however, necessarily accompanied by non-adiabatic corrections to the atomic external motion \([31, 32]\)

\[ V_{na}(z) = \langle D(z)\rangle \left[ \frac{\hbar^2}{2m} \left| D(z)\right|^2 \right] = \frac{\hbar^2}{2m} \left| \partial_z \theta(z) \right|^2 . \] \[ (B3) \]

We now show that \(|\langle r | D(z)\rangle|^2|\) can be made nano-scale with negligible \( V_{na}(z) \). We consider the Rabi frequencies

\[ \Omega_0(z) = \epsilon \Omega_c , \]

\[ \Omega_1(z) = \Omega_c [1 + \beta - \cos k_1(z - z_0)] , \] \[ (B4) \]

where \( \Omega_c \) is a large reference frequency (assumed real positive) and \( 0 < \epsilon \sim \beta \ll 1 \). Physically, \( \Omega_1(z) \) can be realized, e.g., by super-imposing three phase-coherent laser beams where the first two lasers form the standing wave \( \Omega_c \cos k_1(z - z_0) \), and the third propagates perpendicularly, to provide the offset \( \Omega_1(z) [1 + \beta] \).

For Rabi frequencies in Eq. \((B4)\), the resolution \( \sigma \), quantified by the full width at half maximum (FWHM) of \(|\langle r | D(z)\rangle|^2|\), is given in the limit \( \epsilon \ll 1 \) by

\[ \sigma = \frac{\sqrt{2} \lambda_1}{\pi} \left( \sqrt{\epsilon^2 + 2 \beta^2} - \beta \right)^{1/2} , \] \[ (B5) \]

with \( \lambda_1 = 2\pi/k_1 \). The non-adiabatic potential is

\[ V_{na}(z) = \frac{\hbar^2 k_1^2}{2 m \epsilon^2} \left( \frac{\sin k_1(z - z_0)}{1 + [1 + \beta - \cos k_1(z - z_0)]^2/\epsilon^2} \right)^2 . \] \[ (B6) \]

Importantly, \( V_{na}(z) \) decreases rapidly by increasing the ratio \( \beta/\epsilon \), as shown in Fig. 5. Physically, increasing \( \beta/\epsilon \) reduces the maximal population transfer onto the state \( |r\rangle \) during the adiabatic motion, \(|\langle r | D(z)\rangle|^2|_{\text{max}} = (1 + \beta^2/\epsilon^2)^{-1}\), thus suppressing the corresponding non-adiabatic potential. This sub-wavelength spin structure, with negligible \( V_{na}(z) \), is exploited to realize the focusing function \( \phi_{z_0}(z) \), as we show below.

\[ \text{FIG. 5. The resolution } \sigma \text{ and the maximum of the non-adiabatic potential } V_{na}(z) \text{ [c.f., Eq. (B3)] in unit of the recoil energy } E_r = \hbar^2 k_1^2/2m, \text{ vs. } \beta/\epsilon, \text{ for the laser configuration Eq. (B4). Also shown is the maximum overlap between } |D(z)\rangle \text{ and } |r\rangle. \text{ Parameters: } \epsilon = 0.1. \text{ We note that } V_{na}(z) \text{ is strongly suppressed for } \beta/\epsilon \gg 1. \]

2. Sub-wavelength focusing function \( \phi_{z_0}(z) \)

Being a part of the A-system, the level \(|r\rangle \) is also coupled to another level \(|t\rangle \) through a cavity mode \( \hat{c} \), resulting in an effective dispersive coupling \( \hat{H}_{\text{ac}} = \hbar g(z) \hat{c}^{\dagger} \hat{c} / \Delta_c \) between \(|r\rangle \) and the cavity mode (the effects of the spontaneous decay of the state \(|t\rangle \) will be discussed below). After projecting onto the dark state \(|D(z)\rangle \) in the BO approach, one obtains the desired sub-wavelength atom-cavity coupling

\[ \hat{H}_{\text{coup}} = \frac{\hbar g(z)^2}{\Delta_t} |\langle r | D(z)\rangle|^2 \hat{c}^{\dagger} \hat{c} \equiv \phi_{z_0}(z) \hat{c}^{\dagger} \hat{c} , \] \[ (B7) \]

with the spatial resolution given by Eq. \((B5)\) [notice that \( g(z) \) varies slowly on the scale \( \sigma \)].

As mentioned in the main text, it is convenient to write the focusing function in the form \( \phi_{z_0}(z) \equiv A f_{z_0}(z) \), where \( A \) has the dimension of energy and \( f_{z_0}(z) \) is dimensionless and normalized. We choose the normalization \( \int dz f_{z_0}(z) = \ell_0 \) with \( \ell_0 \) being the characteristic length scale of the system under measurement, such that the matrix elements of \( f_{z_0}(z) \) are of order 1.

Note that through this coupling, the stationary coherent field inside the driven cavity exerts a force on the atom, \( V_{\text{OL}}(z) = \hbar g(z) |r| |D(z)|^2 / \Delta_t \), where \( \alpha = \sqrt{\pi \mathcal{E}(i\delta - \kappa)/2} \) is the amplitude of the stationary field, \( \delta \text{ and } \mathcal{E} \text{ are the detuning and the strength of the cavity driving laser, respectively. This force can be compensated by simply detuning the Raman resonance with the offset } \Delta_r = g^2(z_0) |\alpha|^2 / \Delta_t \text{ (c.f. Fig. 2a in the main text), which results in nearly perfect compensation of } V_{\text{OL}}(z) \text{ for the dark state } |D(z)\rangle \).

We also note that the focusing function \( \phi_{z_0}(z) \) in by Eq. \((B4)\) has a periodic set of peaks separated by \( \lambda_1 = 2\pi/k_1 \). To design a single-peak \( \phi_{z_0}(z) \) one can simply choose a spatially dependent \( \Omega_0(z) \) which is tightly \( (\sim \lambda_1) \) focused at position \( z_0 \).
3. Spontaneous emission

Here we discuss the spontaneous emission of the dark state originated from the spontaneous decay of the states \(|e\rangle\) and \(|t\rangle\) entering the construction of the focusing function (see Fig. 2a of the main text). A more formal derivation of the same results using the stochastic master equation including the atomic internal states and the associated spontaneous decay terms will be presented in a follow-up paper [39].

The spontaneous decay rate of the state \(|e\rangle\) enters through the residual coupling between the dark state \(|D(z)\rangle\) and the bright states \(|\pm(z)\rangle\) in the \(\Lambda\)-configuration, due to the atomic kinetic term. As shown in [31], the corresponding decay rate of \(|D(z)\rangle\) scales with the Rabi frequencies as \(\propto [\Omega_2^2(z) + \Omega_0^2(z)]^{-1}\), and can be strongly suppressed by choosing large Rabi frequencies.

The spontaneous decay of \(|D(z)\rangle\) due to virtual population of the state \(|t\rangle\) (resulting from \(\hat{H}_{\text{coup}}\)) is the dominant decay channel, and the corresponding decay rate can be calculated as

\[
\gamma_D(z) = \frac{g^2(z)}{\Delta_i^2} \left( \frac{4\xi^2}{\kappa} \right) \Gamma_t |\langle r|D(z)\rangle|^2 = \gamma_{sp} f_{\text{sp}}(z),
\]

where \(4\xi^2/\kappa\) is the mean photon number in the driven cavity, \(\Gamma_t\) is the spontaneous emission rate of the state \(|t\rangle\), and we introduce the average spontaneous decay rate

\[
\gamma_{sp} = \frac{1}{\ell_0} \int dz \gamma_D(z) = \frac{4A_E^2 \Gamma_t}{\hbar \kappa \Delta_i}.
\]

This has to be compared with the measurement strength \(\gamma = [4A_E^2/(\hbar \kappa)]^2\), with the result

\[
\frac{\gamma}{\gamma_{sp}} = \frac{4A \Delta_i}{\hbar \kappa \Gamma_t} = \frac{4}{\ell_0} \frac{\int dz g^2(z) |\langle r|D(z)\rangle|^2}{\int dz g^2(z)} \simeq 4\xi^2 \sigma_{\text{sp}}^2 (z_0) |\langle r|D(z)\rangle|^2_{\text{max}},
\]

where \(C = g^2(z_0)/(\kappa \Delta_i)\) is the cavity cooperativity, and we use the approximation \(\int dz g^2(z) |\langle r|D(z)\rangle|^2 \simeq \sigma g^2(z_0) |\langle r|D(z)\rangle|^2_{\text{max}}\).

An implementation of the proposed microscope would require \(\gamma \ll \gamma_{sp}\), so that large SNR can be achieved during the time when spontaneous emission is still negligible. This condition can be met with today’s high-Q optical cavities, as shown below.

Appendix C: Experimental feasibility

In this section we show that the proposed microscope can be implemented in the state-of-the-art experiments involving cold atoms/trapped ions and optical cavities, and discuss typical experimental parameters.

First, as discussed in Appendix B 3, a prerequisite for the operation of the microscope is \(\gamma \gg \gamma_{sp}\). The cooperativity \(C\) of high-Q optical cavity can exceed 100 in state-of-the-art experiments [40]. To make an estimation we choose \(C = 150\), \(\sigma = 0.3\ell_0\) and \(|\langle r|D(z)\rangle|^2_{\text{max}} = 0.4\) (which suffices to render \(V_{\text{max}}\) being negligible, c.f. Fig. 5), yielding \(\gamma/\gamma_{sp} \simeq 75\). Such a high ratio guarantees that spontaneous emission is indeed negligible for the observation of the key predictions in the main text: for \(\gamma T \approx 75\) with \(T\) being the total measurement time, one gets SNR \(\gg 1\) (c.f. Fig. 4a of the main text) for a single scan of atomic motional eigenstates in the QND mode of the microscope.

Second, the two operation modes of the microscope require either \(\omega \ll \kappa\) or \(\omega \gg \kappa\). While the first region \(\omega \ll \kappa\) us naturally obtained using cavities with a sufficiently large linewidth, the second condition is also realistic. For example, Ref. [15] reports a coupled BEC-cavity setup with \(\kappa \approx 2\pi \times 4.5\text{kHz}\) which is far smaller than the recoil energy of light-mass alkali’s (e.g., \(E_r \approx 2\pi \times 60\text{kHz}\) for \(^7\text{Li}\) at the D2 line). Trapped ions provides another platform for reaching the good cavity limit, due to their large oscillation frequency (~MHz).

Appendix D: Perturbative Elimination of the Cavity Field

In this section we consider the relation between the homodyne current \(I(t)\) and the localised microscope probe \(f_{\text{sp}}(z)\). To obtain the connection we eliminate the cavity field from the stochastic dynamics described by the Eq. (3) in the main text. The aim is to derive effective stochastic master equations (5) and (6) in the main text, with corresponding photocurrents in ‘bad’ and ‘good’ cavity limits respectively.

To make the discussion more general, first, we consider an arbitrary system which is coupled to the cavity field \(\hat{c}\) via interaction \(\hat{H}_{\text{int}} = \hbar \epsilon \hat{c} + \hat{c}^\dagger\) where \(\hat{f}\) is a system operator and the coupling is assumed to be weak compared to the cavity decay rate \(\varepsilon \ll \kappa\). This model is related to the atomic system from the main text coupled to a driven cavity via the interaction Eq. (1) linearised around the steady state intracavity field.

Transforming to an interaction picture with respect to the system Hamiltonian \(\hat{H}_{\text{sys}}\) we obtain the following SME describing the dynamics of the full setup under continuous homodyne monitoring of the cavity output field:

\[
d\rho_c = -i[\epsilon \hat{f}(t)\hat{c} + \hat{c}^\dagger, \rho_c]dt + \kappa D[e^{i\phi}]\rho_c dt + \sqrt{\kappa} \hat{H}[e^{-i\phi}]\rho_c dW(t),
\]

where \(\delta\) is the cavity detuning, \(\phi\) is the homodyne angle, and \(\hat{f}(t) = e^{iH_{\text{sys}}t} \hat{f} e^{-iH_{\text{sys}}t}\). The corresponding homodyne current reads:

\[
dX_\phi(t) \equiv I(t) dt = \frac{\sqrt{\kappa}}{N} \langle e^{-i\phi} + e^{i\phi}\rangle_c dt + dW(t)
\]

where \(\langle\ldots\rangle_c \equiv \text{Tr}\{\ldots \rho_c(t)\}\) refers to an expectation value with respect to the conditional density matrix. We
eliminate the cavity field along the lines of [41]. First, we simply trace out the cavity dynamics from the SME (D1) to obtain a stochastic equation for the system density matrix only \((\tilde{\rho}_c = \text{Tr}_T \rho_c)\):

\[
d\tilde{\rho}_c = -i\varepsilon \left[ \hat{f}(t), \tilde{\eta} + \hat{\eta} \right] dt + \sqrt{\kappa} \left( \mu e^{-i\phi} + \mu^* e^{i\phi} \right) dW(t)
\]  

(D3)

where we define operators \(\tilde{\eta} = \text{Tr}_T \{ \hat{c} \rho_c \} \) and \(\hat{\mu} = -i(\hat{c}) \tilde{\rho}_c \) such that \(\{ \hat{c} \} = \text{Tr}_S \tilde{\eta} \) and operations \(\text{Tr}_T \) and \(\text{Tr}_S \) stand for the partial traces over states of the cavity \((T \) for transducer \) and the system respectively. We derive an effective equation for \(\tilde{\rho}_c \) up to the second order in the perturbation \(\varepsilon \) for the deterministic term and up to a linear stochastic term: \(d\tilde{\rho}_c = O(\varepsilon^2) dt + O(\varepsilon) dW(t)\).

This restricts solutions for \(\eta \) operator reads

\[
d\eta = \text{Tr}_T \{ \hat{c} d\rho_c \} = -i\varepsilon \text{Tr}_T \left\{ \hat{c} \left[ \hat{f}(t), \hat{c} + \hat{c}^\dagger \right] \right\} dt - \left( \frac{\kappa}{2} - i\delta \right) \eta dt + \sqrt{\kappa} \text{Tr}_T \left\{ \hat{c} (\hat{c} - 1) \rho_c e^{-i\phi} + \hat{c} (\hat{c}^\dagger - 1) \rho_c e^{i\phi} \right\} dW(t)
\]

\(\sim -i\varepsilon \hat{f}(t) \tilde{\rho}_c dt - \left( \frac{\kappa}{2} - i\delta \right) \eta dt, \) (D4)

where in the first deterministic term and in the stochastic term we used the fact that \(\rho_c = \tilde{\rho}_c \otimes \rho_T \) to zeroth order in \(\varepsilon \) and that the unperturbed cavity is in a vacuum steady state such that \((\hat{c}^\dagger)^\dagger = 1, (\hat{c}^\dagger)^2 = 0\). Next, for the cavity mean field we have:

\[
d(\hat{c}) = \text{Tr}_S d\eta
\]

\(\sim -i\varepsilon \hat{f}(t) dt - \left( \frac{\kappa}{2} - i\delta \right) \langle \hat{c} \rangle dt \)

(D5)

This equation is first order in \(\varepsilon \) which means, to define operator \(\hat{\mu} \), we need to know \(\tilde{\rho}_c \) to the zeroth order in \(\varepsilon \). It is constant in this approximation \((d\tilde{\rho}_c = 0)\) and we obtain an equation for the \(\tilde{\mu} \) operator using Itô rule:

\[
d\tilde{\mu} = d\eta - \{(d(\hat{c})) \hat{\rho}_c + (d(\hat{c}^\dagger)) \rho_c d\tilde{\rho}_c + (d(\hat{c}^\dagger) \hat{\rho}_c \}
\]

\(\sim -i\varepsilon \left( \hat{f}(t) - \langle \hat{f}(t) \rangle \right) \tilde{\rho}_c dt - \left( \frac{\kappa}{2} - i\delta \right) \tilde{\mu} dt \) (D6)

Plugging the solutions of the Eqs. (D4) and (D6) into the equation of motion for the system density operator (D3) we recover an effective equation with the necessary precision in \(\varepsilon \). There are two cases to consider.

**Bad’ cavity** — If the free evolution of the system can be neglected on a time scale of the cavity decay \(1/\kappa \), we choose the cavity detuning \(\delta = 0\) and the homodyne angle \(\phi = -\pi/2 \) from the signal, we obtain a homodyne current (again using Eqs. (D2) and (D5)):

\[
d\tilde{\rho}_c = -i \frac{\hbar}{\kappa} \left[ \hat{H}_\text{eff}, \tilde{\rho}_c \right] dt + \gamma D [\hat{f}] \tilde{\rho}_c dt + \sqrt{\gamma} \mathcal{H} [\hat{f}] \tilde{\rho}_c dW(t)
\]

(D7)

where \(\hat{H}_\text{eff} = \hat{H}_\text{sys} + (h\delta \varepsilon \hat{f}^\dagger)/(\{\kappa/2\}^2 + \delta^2)\) and \(\gamma = (\varepsilon^2/\kappa^2)/(\{\kappa/2\}^2 - \delta^2)\). The homodyne phase is chosen to maximize the signal in the photocurrent \((\phi = -\pi/2 + \arctan(2\delta/\kappa))\) such that

\[
dX_\phi(t) \equiv I(t) dt = 2\sqrt{\gamma} \langle \hat{f}^\dagger \hat{f} \rangle c dt + dW(t)
\]

(D8)

which is obtained by substituting solution of Eq. (D5) into Eq. (D2). In the main text we consider the cavity driven by a coherent field \(\mathcal{E}(T)\) such that the coupling coefficient is given by \(\varepsilon = (AE/\hbar) \{\kappa/(\kappa^2/4 + \delta^2)\}^{1/2}\). Defining \(\hat{f} = f_{z_0}(\hat{z})\) and setting zero detuning \(\delta = 0\) the equations (D8) and (D7) become Eqs. (4) and (5) in the main text.

Ensemble averaging the stochastic dynamics (D7) over individual trajectories results in the corresponding ME for the unconditional density matrix \(\rho(t) = \langle \tilde{\rho}_c(t) \rangle_{st}\):

\[
d\hat{\rho}_c dt = -i \frac{\hbar}{\kappa} [\hat{H}_\text{eff}, \hat{\rho}_c] + \gamma D[\hat{f}] \hat{\rho}_c dt
\]

Here we used the non-anticipating property of the stochastic differential equation in Itô form \(\langle \cdots dW(t) \rangle_{st} = 0\).

**‘Good’ cavity** — In the case of harmonic oscillator \(\hat{H}_\text{sys} = \hbar \omega (\hat{a}^\dagger \hat{a} + 1/2)\) the system coupling operator (localised probe \(f_{z_0}(\hat{z})\)) in the interaction picture reads \(\hat{f}(t) = \sum_\ell f^{(\ell)}(t) e^{-i\ell \omega t}, \) where \(f^{(\ell)}(t) = \sum_n f_{n,n+\ell} |n+\ell\rangle \langle n|\) with \(f_{mn} = \langle m | f_{z_0}(\hat{z}) | n \rangle\). This allows one to integrate Eqs. (D4) and (D6) assuming slow time dependence of \(\tilde{\rho}_c\) as follows:

\[
\tilde{\eta} \simeq -i\varepsilon \sum_\ell f^{(\ell)}(t) e^{-i\ell \omega t} \tilde{\rho}_c,
\]

\[\tilde{\mu} \simeq -i\varepsilon \sum_\ell \left( f^{(\ell)}(t) - f^{(\ell)}(t) \right) e^{-i\ell \omega t} \tilde{\rho}_c.
\]

Substituting the results into the Eq. (D3), keeping only non-rotating deterministic terms due to \(\kappa \ll \omega \) in the ‘good’ cavity limit (secular approximation), and transforming back to the Schrödinger picture we obtain:

\[
d\tilde{\rho}_c = \tilde{\rho}_c \left[ -i \frac{\hbar}{\kappa} \hat{H}_\text{eff}, \tilde{\rho}_c \right] dt + \sum_\ell \frac{\varepsilon^2 \kappa}{(\kappa/2)^2 + (\delta + \omega \ell)^2} D \left[ f^{(\ell)}(t) \right] \tilde{\rho}_c dt + \varepsilon \sqrt{\kappa} \sum_\ell \mathcal{H} \left[ \frac{-i\varepsilon e^{-i\phi} f^{(\ell)}(t)}{\kappa/2 - i(\delta + \omega \ell)} \right] \tilde{\rho}_c dW(t),
\]

(D9)

where

\[\hat{H}_\text{eff} = \hat{H}_\text{sys} + \sum_\ell \frac{\hbar \varepsilon^2 (\delta + \omega \ell)}{(\kappa/2)^2 + (\delta + \omega \ell)^2} \left( f^{(\ell)}(t) f^{(\ell)}(t) - f^{(\ell)}(t) \right).\]

To enhance the signal from the QND observable \(\hat{f}^{(0)}\) we choose the cavity detuning \(\delta = 0\) and the homodyne angle \(\phi = -\pi/2\). Then, by filtering out sidebands with \(\ell \neq 0\) from the signal, we obtain a homodyne current (again using Eqs. (D2) and (D5)):

\[
dX_\phi(t) = I(t) dt = 2\sqrt{\gamma} \langle \hat{f}^{(0)} \rangle c dt + dW(t)
\]

(D10)
with $\gamma = 4\varepsilon^2/\kappa$ and $\varepsilon$ defined above. This gives expression for the photocurrent preceding Eq. (6) in the main text. Discarding the sidebands from the photocurrent leads to averaging the effective SME (D9) over corresponding unobserved measurements. This results in dropping stochastic terms with $\ell \neq 0$ from the equation and yields the SME (6) in the paper. In the ‘good’ cavity limit $\kappa \ll \omega$, the additional part in the Hamiltonian $\hat{H}_{\text{eff}}$ is much smaller than $\hat{H}_{\text{sys}}$ and can be neglected.

Appendix E: Scanning Many-body Systems and the Friedel Oscillation

Here we extend the scanning measurement to the many-body case and provide the details on scanning Friedel oscillations discussed in the main text.

To derivation the SME describing the scan of a many-body system, we decompose the focusing function, $\phi_{z_0}(z) = A f_{z_0}(z)$ [c.f., Eq. (1) of the main text], in terms of many-body eigenstates,

$$\sum_{i=1}^N f_{z_0}(\tilde{z}_i) \to \hat{f}_{z_0} = \sum_{\vec{\nu}, \vec{\nu}'} |\vec{\nu} \rangle \langle \vec{\nu}'|,$$  \hspace{1cm} (E1)

where $\vec{\nu}$ is the set of quantum numbers specifying the many-body state $|\vec{\nu} \rangle$ with eigenenergy $E_{\vec{\nu}}$, and $f_{\vec{\nu}\vec{\nu}'} = \langle \vec{\nu}'| \sum_i f_{z_0}(\tilde{z}_i) |\vec{\nu} \rangle$ are the matrix elements (Note, being a single-particle operator, $f_{z_0}$ generates only single-particle transitions). Let us now define a set $\{\Delta E_j\}$ of difference between the eigenenergies, $\Delta E_j = E_{\vec{\nu}} - E_{\vec{\nu}'}$, for all pairs of eigenstates appearing in Eq. (E1), and define the associated operators

$$\hat{f}(\Delta E_j) = f_{\vec{\nu}\vec{\nu}'} |\vec{\nu} \rangle \langle \vec{\nu}'|,$$

so that $\hat{f}_{z_0} = \sum_j \hat{f}(\Delta E_j)$. Note that here we assume all $\Delta E_j$ being different [except for $\Delta E_j = 0$ corresponding to diagonal contributions of (E1)], as in the example of fermions in a box considered below. In situations where there are (quasi-)degenerate energy differences $\Delta E_j$, like atoms in a harmonic trap, the definition of the operators $\hat{f}(\Delta E_j)$ should include the summation over the pairs of states with (quasi) degenerate $\Delta E_j$. The operators $\hat{f}(\Delta E_j)$ are generalizations of $\hat{f}(t)$ in the single-particle case in Appendix D, and provide a ‘spectral decomposition’ of $\hat{f}_{z_0}$: In the interaction picture with respect to the Hamiltonian of the system, they evolve as $\hat{f}(\Delta E_j)(t) = \hat{f}(\Delta E_j) \exp(-i\Delta E_j t/\hbar)$. Let $\Delta E$ be a typical level spacing between physically relevant states such that $\Delta E_j \geq \Delta E$. In the good cavity regime $\kappa \leq \Delta E$, these fast rotating terms with $\Delta E_j \neq 0$ will be suppressed due to the finite time resolution $\kappa^{-1}$ of cavity, similar to the single atom case.

We eliminate the cavity field in the same fashion as the ‘good cavity’ case in Appendix D. The dispersive cavity-atom coupling defines the small coefficient $\varepsilon = (\mathcal{AE}/\hbar)(\kappa/(\kappa^2/4 + \delta^2))^{1/2}$. Assuming $\varepsilon \ll \kappa$ allows for eliminating the cavity in an expansion of $\varepsilon/\kappa$. Accurate to $O(\varepsilon^2)$ in the deterministic term and $O(\varepsilon)$ in the stochastic term, we arrive at the SME for the conditional density matrix of the atomic system

$$d\hat{\rho}_c = -i\hbar \hat{H}_{\text{eff}} \hat{\rho}_c dt + \gamma D[\hat{f}(0)] \hat{\rho}_c dt + \sqrt{\gamma} \hat{H}[\hat{f}(0)] \hat{\rho}_c dW(t) + \sum_{\Delta E_j \neq 0} \gamma_j D[\hat{f}(\Delta E_j)] \hat{\rho}_c dt,$$  \hspace{1cm} (E2)

where we have assumed a resonant cavity driving $\delta = 0$, the homodyne angle $\phi = -\pi/2$. In Eq. (E2), $\hat{f}(0) = \hat{f}(\Delta E_j = 0) = \sum_{\vec{\nu}\vec{\nu}'} f_{\vec{\nu}\vec{\nu}'} |\vec{\nu} \rangle \langle \vec{\nu}'|$ is the QND observable which measures the local density for an arbitrary eigenstate, with a rate $\gamma = [4A\mathcal{AE}/(\hbar\kappa)]^2$. Analogous to the single-particle case [c.f. Eq. (D9)], the last term of Eq. (E2) describes the suppressed dissipation channels, with the rates $\gamma_j = \gamma [1 + 4\Delta E_j^2/\kappa^2]^{-1}$. Finally, the Hamiltonian $\hat{H}_{\text{eff}} = \hat{H}_{\text{sys}} + \hbar^2 \sum_{\Delta E_j \neq 0} \Delta E_j \hat{f}(\Delta E_j) - \hat{f}(\Delta E_j) \hat{f}(\Delta E_j)$.

The second term comes from adiabatic elimination of the cavity, and describes cavity-mediated interactions between particles. Due to the energy hierarchy $\varepsilon \ll \kappa \leq \Delta E$, this term is far smaller than $\hat{H}_{\text{sys}}$ and only weakly disturbs the eigenspectrum of the system. We will neglect this tiny correction in the following discussion.

The associated expression for the homodyne current reads

$$I(t) = 2\sqrt{\gamma} \text{Tr}[\hat{f}(0) \hat{\rho}_c(t)] + \xi(t).$$  \hspace{1cm} (E3)

We now apply the above analysis to a simple example of a non-interacting Fermi sea, where the presence of a single impurity causes the Friedel oscillation. Consider $N$ fermions in a one-dimensional box of length $L \gg \sigma$, $-L/2 \leq z \leq L/2$, with a point-like impurity at the origin described by the potential $V_{\text{imp}}(z) = U \delta(z)$. Assuming zero boundary conditions at $z = \pm L/2$ and taking the limit $U \to \infty$ to simplify analytical expressions, the single-particle wave functions read

$$\psi_n^{(o)}(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n}{L} z\right),$$  \hspace{1cm} (E4)

$$\psi_n^{(e)}(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n}{L} |z|\right),$$

where $n = 1, 2, \ldots$ for both $\psi_n^{(o)}(z)$ (odd parity) and $\psi_n^{(e)}(z)$ (even parity). The corresponding eigenenergies are $\epsilon_n^{(o/e)} = [2\pi^2 \hbar^2/(mL^2)] n^2$. The particle density for the ground state is (we assume even $N$ for simplicity)

$$n(z) = \sum_{n=1}^{N/2} [\psi_n^{(o)}(z)]^2 + [\psi_n^{(e)}(z)]^2 = n_F + \frac{1}{L} \left(1 - \frac{\sin[2\pi(N + 1)z/L]}{\sin[2\pi z/L]}\right),$$  \hspace{1cm} (E5)
where \( n_F = N/L \) is the average fermionic density. In the vicinity of the impurity, \( |z| \ll L/2\pi \), \( n(z) \) has the form of Friedel oscillations,

\[
n(z) \approx n_F - \frac{\sin(2k_F z)}{2\pi z} = n_F \left[ 1 - \frac{\sin(2k_F z)}{2k_F z} \right],
\]

with \( k_F = \pi n_F \) the Fermi wave vector, and we omit terms \( \sim L^{-1} \). Note that for \( z \sim L/2\pi \) the “finite-size” oscillations in \( n(z) \), Eq. (E5), have the amplitude \( \sim L^{-1} \) that vanishes in the thermodynamic limit with the fixed density \( n_F \), in contrast to the Friedel oscillations Eq. (E6).

For this case it is convenient to classify the many-body states in terms of occupations of single-particle states and to use the language of second quantization. We introduce the destruction (thus the associated creation) operators as \( \hat{b}_{n,L(R)} = \frac{1}{\sqrt{2}} \int_{-L/2}^{L/2} dz [\psi_n^{(c)}(z) \mp \psi_n^{(e)}(z)] \hat{\psi}(z) \) [with \( \hat{\psi}(z) \) the fermion field operator], which correspond to the left(right) single-particle eigenmodes. The focusing function \( f_{\bar{z}_n}(\bar{z}) \) has zero matrix elements between left and right eigenmodes, \( \langle m,L|f_{\bar{z}_n}(\bar{z})|n,R \rangle = 0 \). Using these bases and for simplicity defining the single-particle quantum number \( \nu \equiv \{n,L(R)\} \), Eqs. (E2) and (E3) can be expressed explicitly: the QND observable \( \hat{f}^{(0)} \) becomes \( \hat{f}^{(0)} = \sum_{\nu} f_{\nu \nu} \hat{b}^{\dagger}_{\nu} \hat{b}_{\nu} \) whereas the last term of Eq. (E2) (the suppressed dissipations channels) reads \( \sum_{\nu \neq \nu'} \gamma_{\nu \nu'} D[\hat{b}_{\nu}^{\dagger} \hat{b}_{\nu'}] \) with the corresponding rates \( \gamma_{\nu \nu'} = \gamma f_{\nu \nu'}^2 [1 + 4(\epsilon_{\nu} - \epsilon_{\nu'})^2/\kappa^2]^{-1} \), where \( f_{\nu \nu'} = \langle \nu|f_{\bar{z}_n}(\bar{z})|\nu' \rangle \) is the single-particle matrix element and \( \epsilon_{\nu} = [2\pi^2 \hbar^2/(mL^2)] n^2 \). By truncating to a suitable number of fermi orbitals, Eqs. (E2) and (E3) can then be integrated straightforwardly.

To resolve the Friedel oscillations in the scan, their period has to be larger than the focusing region \( \sigma \), \( \pi/k_F > \sigma \). This condition puts an upper bound on the density of fermions and, therefore, on their total number, \( N < L/\sigma \), which corresponds to having not more than one fermion per length \( \sigma \). The gap to the first excited state (the level spacing) in this case can be estimated as \( \Delta E \sim \hbar^2/(m\sigma^2) \), and the condition for the non-demolition scan reads \( \kappa \leq \hbar^2/(m\sigma^2) \). For the simulation shown in Fig. 4b of the main text, we consider \( N = 16 \) fermions, scanned by a microscope with resolution \( \sigma = 0.01L \) and cavity linewidth \( \kappa = 4\pi^2\hbar^2/(mL^2) \). The dimensionless measurement strength is \( \gamma T = 400 \) with \( T \) being the total scanning time. The filter integration time for post-processing is chosen as \( \tau = \sigma T/L = 0.01T \).