Structures and propagation in globally coupled systems
with time delays

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(March 21, 2022)

Abstract

We consider an ensemble of globally coupled phase oscillators whose interaction is transmitted at finite speed. This introduces time delays, which make the spatial coordinates relevant in spite of the infinite range of the interaction. We show that one-dimensional arrays synchronize in an asymptotic state where all the oscillators have the same frequency, whereas their phases are distributed in spatial structures that –in the case of periodic boundaries– can propagate, much as in coupled systems with local interactions.

PACS: 05.45.Xt, 05.65.+b
I. INTRODUCTION

Standard models for studying collective complex behavior in natural systems consist typically of ensembles of interacting dynamical elements \cite{1}. Such kind of models has proven to be extremely versatile in the mathematical description, both analytical and numerical, of a wide variety of phenomena within the scopes of physics, biology, and other branches of science \cite{2}. According to the range of the involved interactions, these models can be divided into two well distinct classes. Local interactions –which are paradigmatically represented in reaction-diffusion systems \cite{1}– give rise to macroscopic evolution in which space variables play a relevant role, such as spatial structures and propagation phenomena. On the other hand, with global interactions –where the coupling range is of the order of, or larger than, the system size– space becomes irrelevant and collective behavior is observed to develop in time, typically, in the form of synchronization \cite{3}.

An essential model of globally coupled elements is given by a set of \(N\) identical oscillators described, in the so-called phase approximation, by phase variables \(\phi_i(t) \ (i = 1, \ldots, N)\). Their evolution is governed by the equations

\[
\dot{\phi}_i = \omega + \epsilon \sum_{j=1}^{N} \sin(\phi_j - \phi_i). \tag{1}
\]

It is known that, for any value of the coupling intensity \(\epsilon\), all the elements converge to a single orbit whose frequency \(\omega\) coincides with that of an individual oscillator \cite{3}. In this case, \(\epsilon^{-1}\) measures the time required to reach such synchronized state.

In this note we present results on a generalization of the above model, where time delays are introduced. The effect of time delays in synchronization phenomena has already been considered for two-oscillator systems, both periodic \cite{4} and chaotic \cite{5}. Ensembles with local interactions \cite{6} and globally interacting inhomogeneous systems have also been studied \cite{7}. None of these contributions make however explicit reference to the relevant case where interactions are global but their propagation occurs at a finite velocity \(v\). This situation, which naturally introduces time delays, provides a realistic description of highly connected systems.
systems where the time scales associated with individual evolution and with mutual signal transmission are comparable. Instances of such systems are neural and informatic networks and biological populations with relatively slow communication media—such as sound. Our main result is that, since a finite signal velocity makes spatial variables relevant even when interactions are global, globally coupled ensembles with time delays exhibit typical features of systems driven by local interactions, in particular, structure formation and propagation.

II. THE MODEL AND ITS SOLUTION FOR SHORT DELAYS

We consider an ensemble of $N$ identical oscillators in the phase approximation, governed by the equations

$$\dot{\phi}_i(t) = \omega + \frac{\epsilon}{N} \sum_{j=1}^{N} \sin[\phi_j(t - \tau_{ij}) - \phi_i(t)],$$

(2)

where $\tau_{ij} = d_{ij}/v$ is the time required for the signal to travel from element $j$ to element $i$ at velocity $v$, and $d_{ij}$ is the distance between $i$ and $j$. Note that coupling is still global, since its intensity $\epsilon$ does not depend on the distance between elements. However, the relative position of the oscillators becomes now relevant through time delays.

The full specification of our system requires to fix the topology and the metric properties of the ensemble, by fixing the values $d_{ij}$ for all $i, j = 1, \ldots, N$. Moreover, initial conditions for $\phi_i$ must be provided. In the case of delay equations like (2), it is necessary to specify the evolution of $\phi_i$ at times prior to $t = 0$ up to a time $T_i = -\max\{\tau_{ij}\}$. In the following we shall assume that for $t < 0$ the oscillators evolve independently from each other at their proper frequency $\omega$ and with random relative phases. Namely, for $t < 0$ we have $\phi_i(t) = \omega t + \phi_i(0)$, where $\phi_i(0)$ is chosen at random from a uniform distribution in $[-\pi, \pi)$. At $t = 0$ coupling is switched on, so that we formally have a time-dependent coupling intensity $\epsilon(t) = \epsilon \theta(t)$, where $\theta$ is the Heaviside step function.

Through extensive numerical calculations for a variety of topologies, ranging from one-dimensional arrays to tree (ultrametric) structures, we have found that the system evolves
to a state where all the oscillators have the same frequency. On the other hand, in contrast with the case without time delays [3], their phases can be different. This asymptotic state corresponds thus to a situation of frequency synchronization. The long-time evolution of each oscillator can then be written as \( \phi_i(t) = \Omega t + \psi_i \), where in general \( \psi_i \neq \psi_j \) for \( i \neq j \). The fact that these phases are different could have been expected for topologies where not all the elements are equivalent—for instance, when boundaries are present. As we show later, however, such states are also found in homogeneous topologies. In this case, they are associated with propagating structures.

In general, the synchronization frequency is different from the proper frequency of each oscillator, \( \Omega \neq \omega \). According to (2), the synchronization frequency satisfies
\[
\Omega = \omega - \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\Omega \tau_{ij} - \psi_j + \psi_i).
\]
Note that the sums \( S_i = \sum_j \sin(\Omega \tau_{ij} - \psi_j + \psi_i) \) are in general different for each \( i \). However, their numerical value must coincide if the synchronization frequency is to be well defined. For a given value of \( \Omega \), this constraint provides \( N-1 \) independent equations for the phases \( \psi_i \):
\[
S_1 = S_2 = \ldots = S_N.
\]
Since phases are defined up to an additive constant we can choose for instance \( \psi_1 = 0 \), and solve these equations for \( \psi_2, \ldots, \psi_N \). Then, \( \Omega \) can be found self-consistently from (3). For large values of \( N \), and due to the involved nonlinearities, this results to be a quite hard numerical problem.

An approximate solution can however been found in the case of short delays, i.e. close to the situation where the system is also synchronized in phase, \( \psi_i = \psi_j \) for all \( i, j \). Assuming that \( |\Omega \tau_{ij} - \psi_j + \psi_i| \ll 1 \), we can write
\[
S_i \approx \sum_j (\Omega \tau_{ij} - \psi_j + \psi_i) = N \Omega \langle \tau_i \rangle - \sum_j \psi_j + N \psi_i,
\]
where \( \langle \tau_i \rangle = N^{-1} \sum_j \tau_{ij} \) is the average of the time delays associated with element \( i \). Taking into account Eq. (4), we get
\[ \psi_i \approx \Psi - \Omega \langle \tau_i \rangle. \]  

(6)

where \( \Psi \) is a constant, independent of \( i \), that can be chosen arbitrarily. This result indicates that, in this short-delay limit, oscillators with small average delays are relatively ahead in the evolution, as their phases are larger. This is plausibly due to the fact that, in average, they receive the information on the system state before other elements with larger values of \( \langle \tau_i \rangle \), which are thus relatively retarded. Note moreover that in a homogeneous topology all the elements are equivalent, so that \( \langle \tau_i \rangle \) is the same for all oscillators. In this case, the system is also synchronized in phase.

Within the approximation of short delays, the synchronization frequency is given by

\[ \Omega \approx \frac{\omega}{1 + \epsilon \langle \langle \tau \rangle \rangle}, \]  

(7)

where \( \langle \langle \tau \rangle \rangle = N^{-1} \sum_i \langle \tau_i \rangle \) is the overall average delay. It therefore results that \( \Omega \) is smaller than the proper frequency of each oscillator.

### III. ONE-DIMENSIONAL ARRAYS

In this note, we specifically focus the attention on the case of one-dimensional arrays. Two different topologies are considered, namely, with periodic boundary conditions –where all the elements are equivalent– and with free boundaries –where the neighborhood of each element depends on its distance to the center of the array. For periodic boundary conditions, which we consider first, the distance between two elements is not uniquely defined, since it can be measured around the ring in both directions. We fix \( d_{ij} \) by taking the minimum of these values, namely, \( d_{ij} = \min \{|i - j|, N - |i - j|\} \). The delay time is thus \( \tau_{ij} = \tau_0 \min \{|i - j|, N - |i - j|\} \), where \( \tau_0 \) is the time required for the signal to travel between nearest neighbors.

In equations (2), the proper frequency \( \omega \) can be used to define time units so that, without loss of generality, we fix \( \omega = 1 \). Moreover, our numerical simulations are restricted to the case \( \epsilon = 1 \). As a matter of fact, we have found that other coupling intensities do not produce
qualitatively different results. Note that this would not be the case if the oscillators had chaotic individual dynamics. In such situation, the value of $\epsilon$ is expected to control the existence of synchronized states.

We have solved numerically equations (2) for ensembles of $N = 10^2$ to $10^4$ oscillators with a standard finite-difference scheme. For small values of $\tau_0$ we find that the above described random-phase initial conditions evolve to a state of synchronized frequency where the phases of all oscillators coincide, $\psi_i = \psi_j$ for all $i, j$. This fully synchronized state is completely analogous to that of globally coupled identical oscillators without time delays, and corresponds to the approximate solution (3) for the present homogeneous topology. In this case, (3) becomes an autonomous equation for $\Omega$. The sum in the right-hand side can in fact be explicitly evaluated –though its expression depends on $N$ being even or odd– and the synchronization frequency can be found numerically by standard methods. In general, this equation admits several solutions. For the values of $\tau_0$ where the state of phase synchronization is encountered, however, there is only one possible value of $\Omega$.

At $\tau_0 \approx 5N^{-1}$ a qualitative change occurs. Above this critical value, the asymptotic synchronized state is not characterized by a homogeneous phase anymore. Instead, the phase varies linearly along the system, in such a way that a phase difference $\Delta \psi = \pm 2\pi$ accumulates in a whole turn around. The sign of $\Delta \psi$ is defined by the initial condition. Symmetry considerations, in fact, indicate that both signs will be found with equal probability over the set of initial conditions that lead to this kind of asymptotic state. The individual phases are given by $\psi_i = \psi_0 + i\delta \psi$, with $\delta \psi = \pm 2\pi/N$ and $\psi_0$ an arbitrary constant. Due to the time evolution of $\phi_i(t) = \Omega t + i\delta \psi + \psi_0$ a structure propagates around the system at velocity $V_1 = -\Omega/\delta \psi$.

Similar qualitative changes are found at larger values of $\tau_0$. For $\tau_0 \approx 11N^{-1}, 16N^{-1}, \ldots$, the asymptotic states modify their phase structure in such a way that the phase difference around the whole system, $m\Delta \psi = \pm 2\pi m$ with $m = 2, 3, \ldots$, increases progressively. The corresponding individual evolution is $\phi_i(t) = \Omega t + im\delta \psi + \psi_0$, which defines a propagation velocity $V_m = -\Omega/m\delta \psi$. The synchronization frequency is given by
\[ \Omega = \omega - \frac{\epsilon}{N} \sum_j \sin[\Omega \tau_0 \min\{|i-j|, N-|i-j|\}] + (i-j)m\delta\psi. \quad (8) \]

For \( m = 0 \) this reduces to the case of full synchronization found for small \( \tau_0 \). Figure [1] shows the solutions of equation \((8)\) for various values of \( m \), and \( N = 100 \). Bolder curves indicate the intervals where each mode has been observed in the numerical calculations with random-phase initial conditions. Note the zones where more than one solution exist for \( m = 0 \) and \( m = 1 \).

Are the transitions observed at the above quoted values of \( \tau_0 \) actual bifurcations, associated with changes in the stability of the asymptotic states? In view of the difficulty of dealing with the linear stability problem for a many-dimensional system with time delays such as \((2)\) \[10\], we choose to answer this question by numerical means. For a given value of \( \tau_0 \) we calculate the frequency \( \Omega \) of a given mode \( m \) from equation \((8)\) and generate an initial condition which corresponds to that mode added with a certain –typically random– small perturbation. Then, we run the evolution and study the asymptotic behavior. This has been carried out for \( m = 0, \ldots, 3 \) at several values of \( \tau_0 \) in \((0, 0.2)\), for a 100-oscillator ensemble. In almost all cases, it has been found that for sufficiently small perturbations the considered states are stable for any value of \( \tau_0 \). The only exceptions seem to be the states whose frequencies are multiple solutions of equation \((8)\), since in this case the only stable state correspond to the smallest frequency.

The observed transitions are therefore not related to stability changes in the propagation modes. Rather, several modes coexist and the system is multistable. The specific asymptotic state is thus selected by the initial condition. The fact that from the random-phase initial conditions considered previously the system evolves to a well defined synchronous mode, whose order \( m \) grows with \( \tau_0 \), suggests that the attraction basins of the various solutions could considerably vary in size as \( \tau_0 \) changes. Indeed, from a probabilistic viewpoint, most initial conditions are of the random-phase type. Initial conditions that, for a given value of \( \tau_0 \), do not evolve to the mode marked with a bold line in Fig. [1] should be considered probabilistically rare.
We consider now the case of a one-dimensional array with free boundaries. Here, the distance between elements can be defined in the standard form, \( d_{ij} = |i - j| \), so that the time delays are \( \tau_{ij} = \tau_0|i - j| \). In this topology sites are not equivalent. Delays for elements near the center of the array are in average lower than for elements towards the boundaries. As a consequence, no homogeneous stable states are expected for the coupled ensemble. Numerical results show that, in fact, in the asymptotic evolution all the oscillators have the same frequency, given by

\[
\Omega = \omega - \frac{\epsilon}{N} \sum_j \sin(\Omega \tau_0|i - j| - \psi_j + \psi_i),
\]

(9)

but \( \psi_i \neq \psi_j \) if \( i \neq j \) for any nearest-neighbor time delay \( \tau_0 \). Unexpectedly, however, the associated spatial structures not always preserve the topological symmetry of the system, as shown in the following.

Our numerical calculations for the case of free boundaries correspond to a 100-oscillator ensemble with the random-phase initial conditions described above. For small values of \( \tau_0 \) we find a symmetric asymptotic pattern, \( \psi_i = \psi_{N/2-i} \), where the central elements have larger phases than near the boundaries (Fig. 2 for \( \tau = 0.02 \)). This structure corresponds to the approximate solution (3) which, for this topology, predicts a parabolic phase profile with a maximum at the center of the array. Beyond a critical value \( \tau_0 \approx 0.025 \) random-phase initial conditions are instead attracted towards an asymmetric structure, where the phase varies in \( |\psi_N - \psi_1| \approx \pi \) from one end to the other, and attains a maximum in between. Figure 2 shows such structure for \( \tau_0 = 0.05 \). In average, of course, half of the realizations produce the symmetric counterpart of this asymptotic state. The situation changes again at \( \tau_0 \approx 0.06 \). Beyond this point, stationary structures are again symmetric, as shown in Fig. 2 for \( \tau_0 = 0.1 \). They result however to be more complicated than for small \( \tau_0 \), with inflection points at \( i \approx N/4 \) and a much flatter maximum. A new critical point occurs at \( \tau_0 \approx 0.11 \), where phase structures become asymmetric once more (see Fig. 2 for \( \tau_0 = 0.12 \)). The phase variation between the ends is similar to that observed for smaller \( \tau_0 \) but the intermediate geometry is considerably more complex.
An analytical or semi-analytical study of these structures—including their existence and stability properties—requires considering the consistency problem discussed in connection with Eq. (3). Fixing \( \psi_1 = 0 \), the \( N - 1 \) equations for \( \psi_i \) (\( i = 2, \ldots, N \)) read here

\[
\sum_j \sin(\Omega \tau_0 |i - j| - \psi_j + \psi_i) = \sum_j \sin[\Omega \tau_0 (j - 1) - \psi_j].
\] (10)

This problem will be discussed in detail in a separate publication \[11\]. Let us stress for the moment that, though the appearance of spatial structures was to be expected in an inhomogeneous system as the present one-dimensional array with free boundaries, these patterns are found to exhibit an unexpected richness upon variation of \( \tau_0 \)—including, in particular, symmetry breaking.

### IV. SUMMARY AND DISCUSSION

We have found that an ensemble of identical globally coupled oscillators with finite interaction velocity, which gives origin to time delays, evolves to an asymptotic state where all the oscillators have the same frequency but different phases. Generally, the synchronization frequency differs from the proper frequency of individual oscillators, so that the dynamics of each element in the collective asymptotic motion does not coincide with its individual (uncoupled) dynamics. Phases, in turn, are distributed according to spatial patterns with nontrivial topological and dynamical properties. Specifically, in a one-dimensional periodic array several asymptotic states coexist, corresponding to propagation modes with different velocities. In a bounded one-dimensional array we have observed stationary phase structures whose symmetry properties depend on the size of time delays. These features, which are reminiscent of the behavior of reaction-diffusion systems with local interactions, point out sharp differences with the collective motion of coupled oscillators without time delays.

It is natural to ask whether any structure similar to those described above is observed in other, more complex topologies. To advance an answer to this question, we have performed a preliminary analysis of a two-dimensional array of \( 20 \times 20 \) elements with periodic boundary conditions.
conditions. In this case, each element can be labeled by two indices, \( i_x \) and \( i_y \), according to its Cartesian coordinates in the lattice. For algorithmic convenience we have defined the distance between elements as
\[
d_{ij} = ||i - j||_1 = \min\{|i_x - j_x|, L - |i_x - j_x|\} + \min\{|i_y - j_y|, L - |i_y - j_y|\},
\]
with \( L = 20 \) in our case. As above, the delay time is \( \tau_{ij} = \tau_0 d_{ij} \). In complete agreement with the one-dimensional analog, we have here found that for small \( \tau_0 \) the system synchronizes both in frequency and phase. Beyond a critical value \( \tau_0 \approx 0.025 \), instead, propagating phase patterns are observed. Figure 3 shows the simplest of these patterns, corresponding to the propagation mode with \( m_x = m_y = 1 \).

Work in progress is being devoted to the detailed characterization of the phase structures described in this note, as well as those that could arise in other topologies. The next step will be to study the effects of the present kind of time delays in ensembles formed by chaotic oscillators, where coupling competes as a stabilizing mechanism against the inherently unstable dynamics of individual elements.

**ACKNOWLEDGMENT**

This work was partially carried out at the Abdus Salam International Centre for Theoretical Physics. The author wishes to thank the Centre for hospitality.
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FIG. 1. Synchronization frequency $\Omega$ of the asymptotic modes $m = 0, \ldots, 3$ in a one-dimensional ensemble of $N = 100$ globally coupled oscillators with periodic boundary conditions, as a function of the nearest-neighbor delay time $\tau_0$.

FIG. 2. Stationary phase patterns in a one-dimensional 100-oscillator array with free boundaries, for various values of $\tau_0$. Without losing generality, we have fixed $\psi_1 = 0$. 
FIG. 3. Snapshot of a propagating structure in a two-dimensional $20 \times 20$-oscillator array with periodic boundary conditions. Dark and light zones correspond to phases near zero and $\pm \pi$, respectively.