WELL-POSEDNESS AND SCATTERING OF INHOMOGENEOUS CUBIC-QUINTIC NLS

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ABSTRACT. In this paper we consider inhomogeneous cubic-quintic NLS in space dimension $d = 3$:

$$iu_t = -\Delta u + K_1(x)|u|^2 u + K_2(x)|u|^4 u.$$  

We study local well-posedness, finite time blowup, and small data scattering and non-scattering for the ICQNLS when $K_1, K_2 \in C^4(\mathbb{R}^3 \setminus \{0\})$ satisfy growth condition $|\partial^j K_1(x)| \leq |x|^{b_j - j} (j = 0, 1, 2, 3, 4)$ for some $b_j \geq 0$ and for $x \neq 0$. To this end we use the Sobolev inequality for the functions $f \in H^n (n = 1, 2)$ such that $\|L^\ell f\|_{H^n} < \infty (\ell = 1, 2)$, where $L$ is the angular momentum operator defined by $L = x \times (-i \nabla)$.

1. INTRODUCTION

In this paper we consider the following Cauchy problem for inhomogeneous cubic-quintic nonlinear Schrödinger equations of the form:

$$\begin{cases}
i\partial_t u = -\Delta u + K_1(x)Q_1(u) + K_2(x)Q_2(u) \\
u(x, 0) = \varphi(x),
\end{cases}$$  

(1.1)

where $Q_1(u) = |u|^2 u, Q_2(u) = |u|^4 u$, and $K_1 \in C^4(\mathbb{R}^3 \setminus \{0\}; \mathbb{C})$. The model of ICQNLS (1.1) can be a dilute BEC when both the two- and three-body interactions of the condensate are considered. For this see [2, 22] and the references therein. Also it has been considered to study the laser guiding in an axially nonuniform plasma channel. For this see [15, 21, 22].

In this paper we consider ICQNLS with $K_1$ satisfying the growth condition: for some constants $b_1, b_2 \geq 0$

$$|\partial^j K_1| \lesssim |x|^{b_j - j}, \quad j = 0, 1, 2, 3, 4, \quad l = 1, 2,$$

(1.2)

where $\partial$ is one of the partial derivatives $\partial_j, j = 1, 2, 3$. Some basic notations are listed at the end of this section.

By Duhamel’s formula, (1.1) is written as an integral equation

$$u = e^{it\Delta} \varphi - i \int_0^t e^{i(t-t')\Delta} [K_1(x)Q_1(u(t')) + K_2(x)Q_2(u(t'))] dt'.$$

(1.3)

Here we define the linear propagator $e^{it\Delta}$ given by the linear problem $i\partial_t v = -\Delta v$ with initial datum $v(0) = f$. It is formally given by

$$e^{it\Delta} f = \mathcal{F}^{-1}(e^{-it|\xi|^2} \mathcal{F}(f)) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x - \xi - t|\xi|^2)} \hat{f}(\xi) d\xi,$$

(1.4)

where $\hat{f} = \mathcal{F}(f)$ denotes the Fourier transform of $f$ and $\mathcal{F}^{-1}$ the inverse Fourier transform such that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot \xi} f(x) dx, \quad \mathcal{F}^{-1}(g)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot \xi} g(\xi) d\xi.$$
If $K_l$ are real-valued, then we can define mass and energy of the solution $u$ of (1.1) as follows:

\[
m(u(t)) := \| u(t) \|_{L_x^2}^2, \]
\[
E(u(t)) := \frac{1}{2} \| \nabla u \|_{L_x^2}^2 + \frac{1}{4} \int K_1(x)|u(t,x)|^4 \, dx + \frac{1}{6} \int K_2(x)|u(t,x)|^6 \, dx.
\]

We say that the mass and the energy of solutions are conserved if they are constant with respect to time.

The aim of this paper is to establish a well-posedness theory, a finite time blowup, and a scattering theory for suitable growth rate $b_1, b_2 \geq 0$. In case that $K_l$ are radially symmetric, the authors [11, 12, 18] considered well-posedness, finite time blowup and stability of radial solutions. The main obstacle of that problems is the growth of $K_l$ at infinity. To avoid this Sobolev inequalities of radial $H^1$ functions were utilized. However, nothing in general cases has been known about the global behavior like scattering as far as we know. For other work treating bounded or decaying coefficients like $|x|^{-b}$ see [12, 17, 19, 20] or [11, 8, 11], respectively.

To circumvent the lack of symmetry of $K_l$ and growth at space infinity, we suggest alternatives of radial symmetry, the angular momentum conditions, for which we introduce the angular momentum operator $L$:

\[
L = (L_1, L_2, L_3) = x \times (-i \nabla).
\]

It is well-known that $|L|^2 := L \cdot L = \sum_{j=1,2,3} L_j^2 = -\Delta S^2$, where $\Delta S^2$ is the Laplace-Beltrami operator on the unit sphere. Now we define Sobolev spaces $H^{n,\ell}_{L,p}(n, \ell = 0, 1, 2, 1 \leq p \leq \infty)$ associated with $L$ as follows:

\[
H^{0,\ell}_{L,p} = L_x^p, \quad H^{1,0}_{L,p} = H^1_x,
\]
\[
H^{n,1}_{L,p} = \{ f \in H^{n-1,2}_{L,p} \cap H^n : \| f \|_{H^{n,1}_{L,p}} := \| f \|_{H^{n-1,2}_{L,p}} + \| f \|_{H^1_x} + \| Lf \|_{H^1_x} < \infty \},
\]
\[
H^{n,2}_{L,p} = \{ f \in H^{n,1}_{L,p} : \| f \|_{H^{n,1}_{L,p}} := \| f \|_{H^{n-1,2}_{L,p}} + \| L^2 f \|_{H^1_x} < \infty \}.
\]

Here $H^s_p$ denotes the standard $L^p$ Sobolev space. If $p = 2$, then we drop $p$ and denote $H^{n,\ell}_{L,2}$ by $H^{n,\ell}_{L}$. These spaces give us Sobolev type inequalities associated angular momentum such as $\| |x|^b f \|_{L_x^p} \lesssim \| f \|_{H^{1,1}}$ for $0 < b < 1$ and $\| |x|^b f \|_{L_x^p} \lesssim \| f \|_{H^{1,2}}$ (see Lemma 2.3 below).

Our first result is on the local well-posedness, whose definition is the following.

**Definition 1.1.** The equation (1.1) is said to be locally well-posed $H^{n,\ell}_{L}$ if there exist maximal existence time interval $I_* = (-T_*, T_*^*)$ and a unique solution $u \in C(I_*, H^{n,\ell}_{L})$ with continuous dependency on the initial data and blowup alternative ($T^* < \infty \Rightarrow \lim_{t \to T^*} \| u(t) \|_{H^{n,\ell}_{L}} = \infty$).

**Theorem 1.2.** (1) If $b_1 = b_2 = 0$, then (1.1) is locally well-posed in $H^1$.
(2) If $n - 1 < b_1 < n$ and $0 \leq b_2 < n + 2$ for $n = 1, 2$, then (1.1) is locally well-posed in $H^{n,1}_{L}$.
(3) If $b_1 = n$ and $0 \leq b_2 \leq n + 2$ for $n = 1, 2$, then (1.1) is locally well-posed in $H^{n,2}_{L}$.
(4) If $K_l$ are real-valued, then in any cases the mass and the energy are conserved.

We prove this theorem via standard contraction mapping theorem. If $b_1 \leq 2, b_2 \leq 4$, we can control the growing coefficients by using Sobolev inequality associated with angular momentum. For example we need to estimate $\| |x|^n \partial u^{(2)}(x) \|_{L_x^2}$, which can be done by the bound $\| |x|^n \partial_x u \|_{L_x^2} \lesssim \| u \|_{H^{1,1}}^3$. In case that $b_1 > 2$ we cannot control it only with Sobolev inequality. To this end one can try to show the local well-posedness for the initial data with higher regularity and additional weight condition ($|x|^k \varphi \in L_x^2, k = 1, 2, \cdots$). We will not pursue this issue here. The local well-posedness results are far away from the sharpness of regularity on the space and angle. One may improve them via fractional Sobolev space and fractional Leibniz rule 13.
The next result is on the finite time blowup when the initial energy is negative.

**Theorem 1.3.** Let $K_1$ be real-valued function such that $K_1 - x \cdot \nabla K_1 \leq \alpha K_1$ and $4K_2 - x \cdot \nabla K_2 \leq \alpha K_2$ for some $\alpha \geq 0$. Let $u$ be the local solution of (1.1) as in Theorem 1.2 with $|x|\varphi \in L^2_t$. Suppose that $E(\varphi) < 0$. Then the solution blows up in finite time.

If $K_1 = -|x|^{-b_1}$ and $K_2 = -|x|^{-b_2}$, then the condition on $K_1$ implies that $b_1 \leq \alpha + 1$ and $b_2 \leq \alpha + 4$. We use the standard virial argument for which the weight condition $|x|\varphi \in L^2_t$ and the sign condition of the coefficients $K_1$ are necessary. Once a regular solution exists even for $b_1 > 2$, the finite time blowup can be shown by the same argument.

Now we consider a small data scattering.

**Definition 1.4.** We say that a solution $u$ to (1.1) scatters (to $u_{\pm}$) in a Hilbert space $\mathcal{H}$ if there exist $\varphi_{\pm} \in \mathcal{H}$ (with $u_{\pm}(t) = e^{it\Delta} \varphi_{\pm}$) such that $\lim_{t \to \pm \infty} \|u(t) - u_{\pm}\|_\mathcal{H} = 0$.

Our small data scattering is the following.

**Theorem 1.5.** Let $0 \leq b_1 < \frac{2}{3}$ and $0 \leq b_2 < \frac{8}{3}$. If $\|\varphi\|_{L^1_t}$ is sufficiently small, then there exists a unique $u \in (C \cap L^\infty)(\mathbb{R}; H^1_t)$ to (1.1) and $u^\pm \in H^1_t$ to which $u$ scatters in $H^1_t$.

For the proof we carry out nonlinear estimates with constants not depending on the local time. This is possible due to the endpoint Strichartz estimates and Sobolev inequality associated with angular momentum.

**Theorem 1.6.** Assume that $K_1(x) = |x|^{b_1}$ and $K_2(x) = |x|^{b_2}$ for $b_1, b_2 > 0$, and $\lambda \in \mathbb{R}$. Let $u$ be a smooth global solution of (1.1) with $b_1 \geq 2$ and $0 < b_2 < 3 + b_1$, which scatters to $u_{\pm} = e^{it\Delta} \varphi_{\pm}$ in $L^2_t$ for some smooth function $\varphi_{\pm}$. Then $u, u_{\pm} \equiv 0$.

For the proof we use pseudo-conformal identity to get the potential energy bound $\frac{1}{4} \int |x|^{b_1} |u|^4 \, dx + \frac{1}{6} \int |x|^{b_2} |u|^6 \, dx \leq \|\varphi\|_{L^1_t}$, which is crucial to the estimate of quintic term. Theorem 1.6 implies that the scattering in the sense of Definition 1.3 does not occur in the long-range case $b_1 \geq 2$. We think the case $b_1 = 2$ will be borderline of the scattering and non-scattering. In this critical case it is highly expected that a modified scattering will occur. This will be another interesting issue to be pursued. The scattering problem still remains open in short-range cases $\frac{2}{3} \leq b_1 < 2$. This short range together with critical case may be taken into account by utilizing the generator of Galilean transformation $J$ (see (5.1) below).

This paper is organized as follows: In Section 2 we introduce angular Sobolev inequality and some properties of angular momentum operators. We give a proof for Theorem 1.2 in Section 3 by standard contraction argument and for Theorem 1.3 in Section 4 via virial argument. In Sections 5, 6 we prove small data scattering, Theorem 1.5 and non-scattering, Theorem 1.6.

**Basic notations.**

- Fractional derivatives: $D^s = (-\Delta)^{\frac{s}{2}} F^{-1} |\xi|^s \hat{F}$, $\Lambda^s = (1 - \Delta)^{\frac{s}{2}} F^{-1} (1 + |\xi|^2)^{\frac{s}{2}} \hat{F}$ for $s > 0$. 


• Function spaces: $\dot{H}^s_t = D^{-s} L^r_t$, $\dot{H}^s = \dot{H}^s_t$, $H^{-s}_t = \Lambda^{-s} L^r_t$, $H^s = H^s_2$, $L^r_t = L^r_x(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 \leq r \leq \infty$.

• Mixed-normed spaces: For a Banach space $X$, $u \in L^1_t I$ if $u(t) \in X$ for a.e. $t \in I$ and $\|u\|_{L^1_t I} := \|\|u(t)\|_X\|_{L^1_t} < \infty$. Especially, we denote $L^q_t L^r_x = L^q_t(I; L^r_x(\mathbb{R}^d))$, $L^q_t L^r_x = L^q_t L^r_2$ and $L^q_t L^r_x = L^q_t L^r L^r$.

• As usual different positive constants depending are denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

2. USEFUL LEMMATA

If a pair $(q, r)$ satisfies that $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{2}{r} = \frac{2}{s}$, then it is said to be admissible.

Lemma 2.1 (16). Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any admissible pair. Then we have

$$\|e^{it\Delta} \varphi\|_{L^q_t L^r_x} \lesssim \|\varphi\|_{L^2},$$

$$\|\int_0^t e^{i(t-t')\Delta} F(t') dt'\|_{L^q_t L^r_x} \lesssim \|F\|_{L^q_t L^r_x}.$$

Lemma 2.2. For any $f \in \dot{H}^s_2(\mathbb{R}^d)(1 < p < \infty, 0 < s < \frac{3}{p})$ we have

$$\|x|^{-s} f\|_{H^s_2} \lesssim \|f\|_{H^s_2}.$$

Proof. This can be done by interpolation between Theorem 2 of [15] and critical Sobolev inequality $\|f\|_{BMO} \lesssim \|f\|_{H^s_2}$ (For instance see [23]).

Lemma 2.3 (9). For any smooth function $f$ there holds

$$\|L_f\|_{L^2} \sim \|L f\|_{L^2} = \|(-\Delta)^{\frac{1}{2}} f\|_{L^2},$$

$$\sum_{1 \leq j, k \leq 3} \|L_j L_k f\|_{L^2} \sim \|L f\|_{L^2}.$$ 

Lemma 2.4 (7, 10). Let $0 < b < 1$. Then for any $f \in H^1_{L^2}$ there holds

$$\|x|^{-b} f\|_{L^\infty} \lesssim \|f\|_{H^1_{L^2}}.$$

And also for any $f \in H^1_{L^2}$

$$\|x| f\|_{L^\infty} \lesssim \|f\|_{H^1_{L^2}}.$$

Lemma 2.5. Let $0 < b < 1$, $0 < \epsilon < 1 - b$, and $2 \leq p < \infty$. Then for any $f \in H^1_{L^2} \cap L^p_\epsilon$ we have

$$\|x|^{-b} f\|_{L^p_\epsilon} \lesssim \|f\|_{H^1_{L^2}}, \|f\|_{L^p_\epsilon}.$$

Proof. Since $\epsilon < 1 - b$, $\frac{b}{1 - \epsilon} < 1$ and thus we get from Lemma 2.4 that

$$\|x|^{-b} f\|_{L^p_\epsilon} \lesssim \|x|^{-b} f\|_{L^p_\epsilon} \lesssim \|f\|_{H^1_{L^2}} \|f\|_{L^p_\epsilon} \lesssim \|f\|_{H^1_{L^2}} \|f\|_{L^p_\epsilon}.$$

By direct calculation we have the following.

Lemma 2.6. (1) Let $s \geq 0$. Then $L D^s f = D^s L f$ and $A^s f = \Lambda^s L f$ for any smooth function $f$.

(2) Let $\psi$ be smooth and radially symmetric. Then

$$L(\psi \ast f) = \psi \ast (L f).$$
3. Local well-posedness: Proof of Theorem 1.2

Let $I_T = [-T, T]$. Let us define a complete metric spaces $X_{L_1}^{n, \ell}(T, \rho), n, \ell = 0, 1, 2$ with metric $d^{n, \ell}$ by

$$X_{L_1}^{1, 0}(T, \rho) := \left\{ u \in S_{L_1}^{1, 0} = L^{10}_{I_T} H^{\frac{10}{3}} \cap (C \cap L^\infty)(I_T; H^1) : \|u\|_{S_{L_1}^{1, 0}} \leq \rho \right\}, \quad d^{1, 0}(u, v) = \|u - v\|_{S_{L_1}^{1, 0}}.$$

$$X_{L_1}^{n, \ell}(T, \rho) := \left\{ u \in S_{L_1}^{n, \ell} = L^{10}_{I_T} H^{\frac{10}{3}} \cap (C \cap L^\infty)(I_T; L^{10}_{H^{\ell}}) : \|u\|_{S_{L_1}^{n, \ell}} \leq \rho \right\}, \quad d^{n, \ell}(u, v) = \|u - v\|_{S_{L_1}^{n, \ell}}.$$

From the assumption (1.2) it follows that for each $j = 0, 1, 2$

$$(3.1) \quad |\partial^j K_1(x)| + |\partial^j L K_1(x)| + |\partial^j |L|^2 K_1(x)| \lesssim |x|^{h-j}.$$

We will show that the nonlinear functional $\Psi(u) = e^{it\Delta} \varphi + \mathcal{N}(u)$ is a contraction on $X_{L_1}^{n, \ell}(T, \rho)$ for each case. Here

$$\mathcal{N}(u) = -i \int_0^t e^{i(t-t')\Delta} [K_1 Q_1(u) + K_2 Q_2(u)] dt'.$$

By $N_i^{n, \ell}$ we denote the derivatives of Duhamel part as follows:

$$N_i^{n, 0} = -i \partial^n \int_0^t e^{i(t-t')\Delta} [K_1 Q_1(u)] dt',$$

$$N_i^{n, 1} = -i \partial^n L \int_0^t e^{i(t-t')\Delta} [K_1 Q_1(u)] dt',$$

$$N_i^{n, 2} = -i \partial^n |L|^2 \int_0^t e^{i(t-t')\Delta} [K_1 Q_1(u)] dt'.$$

We have by Leibniz rule and Lemma [276] that for $n = 1, 2$

$$N_i^{n, 0} = -i \sum_{k=0}^n \binom{n}{k} \int_0^t e^{i(t-t')\Delta} [\partial^{n-k} K_1] \partial^k Q_1(u),$$

$$N_i^{n, 1} = -i \sum_{k=0}^n \binom{n}{k} \int_0^t e^{i(t-t')\Delta} [\partial^{n-k} L K_1] \partial^k Q_1(u) + (\partial^{n-k} K_1) \partial^k L Q_1(u)] dt',$$

$$N_i^{n, 2} = -i \sum_{k=0}^n \binom{n}{k} \int_0^t e^{i(t-t')\Delta} [(\partial^{n-k} |L|^2 K_1) \partial^k Q_1(u) + 2(\partial^{n-k} K_1) \partial^k |L|^2 Q_1(u) + (\partial^{n-k} K_1) \partial^k |L|^2 Q_1(u)] dt'.$$

3.1. Case: $b_1 = b_2 = 0$. Given $\rho$, it follows from Lemmas [241] and [242] that for any $u \in X_{L_1}^{1, 0}(T, \rho)$

$$\|N_i^{1, 0}\|_{L_1^{10} L_2^{\frac{10}{3}} \cap L_1^{10} L_2^{\frac{10}{3}}} \lesssim \| |x|^{-1} |u|^3 \|_{L_1^{10} L_2^{\frac{10}{3}}} + \| |u|^2 |\partial u| \|_{L_1^{10} L_2^{\frac{10}{3}}} \lesssim T^{\frac{1}{4}} \left( \|u\|_{L_1^{10} L_2^{\frac{10}{3}}} \| |x|^{-1} \|_{L_1^{10} L_2^{\frac{10}{3}}} + \| |u|^2 \|_{L_1^{10} L_2^{\frac{10}{3}}} \| \partial u \|_{L_1^{10} L_2^{\frac{10}{3}}} \right) \lesssim T^{\frac{1}{4}} \|u\|_{H^1}^{\frac{3}{2}} \lesssim T^{\frac{1}{4}} \rho^3.$$

As for $N_2^{1, 0}$ we have

$$\|N_2^{1, 0}\|_{L_1^{10} L_2^{\frac{10}{3}} \cap L_1^{10} L_2^{\frac{10}{3}}} \lesssim \| |x|^{-1} |u|^5 \|_{L_1^{10} L_2^{\frac{10}{3}}} + \| |u|^4 |\partial u| \|_{L_1^{10} L_2^{\frac{10}{3}}} \lesssim \left( \|u\|_{L_1^{10} L_2^{\frac{10}{3}}} \| |x|^{-1} \|_{L_1^{10} L_2^{\frac{10}{3}}} + \| |u|^4 \|_{L_1^{10} L_2^{\frac{10}{3}}} \| \partial u \|_{L_1^{10} L_2^{\frac{10}{3}}} \right) \lesssim \|u\|_{H^1}^{\frac{5}{2}} \lesssim \rho^5.$$

Hence we obtain

$$\|\Psi(u)\|_{S_{L_1}^{1, 0}} \lesssim \|e^{it\Delta} \varphi\|_{S_{L_1}^{1, 0}} + C(1 + T^{\frac{1}{4}})(\rho^3 + \rho^5).$$
The choice of $T = T(\varphi)$ and $\rho$ such that $\|e^{i\alpha \varphi}\|_{S^{1,0}_p} \leq \rho/2$ and $C(1 + T^{\frac{1}{2}})(\rho^3 + \rho^5) \leq \rho/2$ shows the self-mapping of $\Psi$ from $X^{1,0}_L(T, \rho)$ to $X^{1,0}_L(T, \rho)$. We can also readily show that for a little smaller $T$

\[ d^{1,0}(\Psi(u), \Psi(v)) \leq \frac{1}{2} d^{1,0}(u, v), \]

because we have only to replace a $u$ with $u - v$ in the proof of self-mapping. Then the local well-posedness in $H^1$ is clear from the contraction.

3.2. **Case:** $0 < b_1 < 1$. Given $\rho$, from Lemmas 2.3, 2.2, 2.1 and 2.1 we obtain that for any $u \in X^{1,1}_L(T, \rho)$

\[
\|N^{1,1}_1\|_{L^{10}_{L^2} \cap L^\infty_{L^2}} + \|x|b_1-1|u|^3\|_{L^1_{L^2} \cap L^\infty_{L^2}} + \|x|b_1|u|^2 \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} + \|x|b_1-1|u|^2 \|L^1_{L^2} \cap L^\infty_{L^2} \|
\]

\[ \lesssim T\left(\|x|b_1-1|u|^3\|_{L^1_{L^2} \cap L^\infty_{L^2}} + \|x|b_1|u|^2 \|L^1_{L^2} \cap L^\infty_{L^2} \|ight) + T^{\frac{1}{2}}\left(\|x|b_1-1|u|^3\|_{L^1_{L^2} \cap L^\infty_{L^2}} + \|x|b_1|u|^2 \|L^1_{L^2} \cap L^\infty_{L^2} \|ight) \]

\[ \lesssim (T + T^{\frac{1}{2}}) \|u\|_{L^{1,1}_L} \lesssim (T + T^{\frac{1}{2}})\rho. \]

On the other hand, $N^{1,1}_2$ consists of $u, \partial u, L^1_{L^2} \partial u, L^1_{L^2} \partial u$, and additional $|u|^2$. For simplicity we only consider $\|x|b_2|u|^3 \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}}$. If $b_2 > 0$, then

\[ \|x|b_2|u|^3 \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} \lesssim T^{\frac{1}{2}} \|x|b_2|u|^3 \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} \|L^1_{L^2} \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} \lesssim T^{\frac{1}{2}} \|u\|_{L^{1,1}_L} \lesssim T^{\frac{1}{2}} \rho. \]

If $b_2 = 0$, then

\[ \|x|b_2|u|^3 \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} \leq \|x|b_2|u|^3 \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} \|L^1_{L^2} \partial u\|_{L^1_{L^2} \cap L^\infty_{L^2}} \lesssim \|u\|_{L^{1,1}_L} \lesssim \rho. \]

Hence we obtain that for $b_2 > 0$

\[ \|\Psi(u)\|_{S^{1,1}_L} \lesssim \|\varphi\|_{H^{1,1}_L} + (T + T^{\frac{1}{2}})(\rho^3 + \rho^5) \]

and for $b_2 = 0$

\[ \|\Psi(u)\|_{S^{1,1}_L} \lesssim \|e^{i\alpha \varphi}\|_{S^{1,1}_L} + (1 + T^{\frac{1}{2}})(\rho^3 + \rho^5). \]

Now we can choose $T$ and $\rho$ so that $\Psi$ becomes self-mapping from $X^{1,1}_L(T, \rho)$ to $X^{1,1}_L(T, \rho)$, and also choose a little smaller $T$ so that

\[ d^{1,1}(\Psi(u), \Psi(v)) \leq \frac{1}{2} d^{1,1}(u, v). \]

This completes the proof of part (2) of Theorem 1.2.
3.3. Case: $b_1 = 1$. In view of the proof in Section 3.2 we have only to estimate $N_1^{1,2}$ for the contraction on $X^{1,2}_L(T, \rho)$. From (3.1) we get

$$
\|N_1^{1,2}\|_{L^{10}_L L^{\infty}_T \cap L^{\infty}_T L^2_x} \lesssim \|u\|^3_{L^{5}_T L^2_x} + \|x\| \|u\|^2 \|\partial u\|_{L^{5}_T L^2_x} + \|x\| \|u\|^2 \|L u\|_{L^{5}_T L^2_x} + \|u\|^3 \|L u\|_{L^{5}_T L^2_x} + \|u\|^3 \|L u\|_{L^{5}_T L^2_x} + \|u\|^3 \|L u\|_{L^{5}_T L^2_x} + \|x\| \|u\|^2 \|\partial u\|_{L^{5}_T L^2_x} + \|x\| \|u\|^2 \|\partial L u\|_{L^{5}_T L^2_x} + \|x\| \|u\|^2 \|\partial u\|_{L^{5}_T L^2_x}
$$

Similarly obtain

$$
\|N_2^{1,2}\|_{L^{10}_L L^{\infty}_T \cap L^{\infty}_T L^2_x} \lesssim \begin{cases} (T + T^\frac{1}{2}) \rho^5 & \text{if } b_2 > 0, \\ \rho^5 & \text{if } b_2 = 0. \end{cases}
$$

3.4. Case: $1 < b_1 < 2$. In this case we use a modified complete metric space $X^{1,2,1}_L(T, \rho) = X^{1,2}_L(T, \rho) \cap X^{2,1}_L(T, \rho)$ with metric $d^{1,2,1} = d^{1,2} + d^{2,1}$. To show the contraction on $X^{1,2,1}_L(T, \rho)$ we consider $N_1^{2,1}$ and $N_1^{1,2}$. Using (3.1), and Lemmas 2.4 and 2.2 we have

$$
\|N_1^{1,2}\|_{L^{10}_L L^{\infty}_T \cap L^{\infty}_T L^2_x} \lesssim \begin{cases} (T + T^\frac{1}{2}) \rho^5 & \text{if } b_2 > 0, \\ \rho^5 & \text{if } b_2 = 0. \end{cases}
$$

$$
\|N_2^{1,2}\|_{L^{10}_L L^{\infty}_T \cap L^{\infty}_T L^2_x} \lesssim \begin{cases} (T + T^\frac{1}{2}) \rho^5 & \text{if } b_2 > 0, \\ \rho^5 & \text{if } b_2 = 0. \end{cases}
$$
and
\[
\| N_{1,2}^{2,1} \|_{L^{10,q}_{1,2}, \mathbb{T}^{20} \cap L^{20}_{1,2}} \lesssim \| x^{b_1-2} u^2 \|_{L^{10}_{1,2}} + \| x^{b_1-1} |u| \|_{L^{10}_{1,2}} + \| x^{b_1} |u| \partial u \|_{L^{10}_{1,2}} + \| x^{b_1} |u| \partial^2 u \|_{L^{10}_{1,2}} \\
+ \| x^{b_1-2} |u|^2 \|_{L^{10}_{1,2}} + \| x^{b_1-1} |u| \|_{L^{10}_{1,2}} + \| x^{b_1} |u| \partial u \|_{L^{10}_{1,2}} + \| x^{b_1} |u| \partial^2 u \|_{L^{10}_{1,2}} \\
+ \| x^{b_1} |u| \partial^2 u \|_{L^{10}_{1,2}} + \| x^{b_1} |u| \partial u \|_{L^{10}_{1,2}} + \| x^{b_1} |u| \partial u \|_{L^{10}_{1,2}} \\
+ \| x^{b_1} |u| \partial^2 u \|_{L^{10}_{1,2}} \lesssim T^{\rho^5}
\]
for \( b_2 > 0 \) and \( \lesssim \rho^5 \) for \( b_2 = 0 \).

3.5. Case: \( b_2 = 2 \). As above we consider \( N_{1,2}^{2,1} \). Together with Lemmas 2.1, 2.2 and 2.4, the bound 3.1 of \( K \) gives us
\[
\| N_{1,2}^{2,2} \|_{L^{10,q}_{1,2}, \mathbb{T}^{20} \cap L^{20}_{1,2}} \lesssim \rho^5
\]
for \( b_2 > 0 \) and \( \rho^5 \) for \( b_2 = 0 \).
3.6. Mass and energy conservation. According to the nonlinear estimates above, one can readily show that if \( \varphi \in H_{L}^{2,2} \) (or \( \in H^{2} \)) then the solution \( u \in C(I_{x}; H_{L}^{2,2}) \) for \( b_{1} > 0 \) (or \( \in C(I_{x}; H^{2}) \) for \( b_{1} = 0 \), respectively). So we first assume that \( \varphi \in H_{L}^{2,2} \) (or \( \in H^{2} \)). Then the map \( g(u) = K_{1}Q_{1}(u) + K_{2}Q_{2}(u) \in C(H_{L}^{2,2}, L_{x}^{2}) \). Hence for any \( I_{T} \subset I_{x} \) if \( u \in C(I_{T}; H_{L}^{2,2}) \) (or \( \in C(I_{T}; H^{2}) \)), then \( u_{t} \in C(I_{T}; L_{x}^{2}) \). The mass or energy conservation follows from \( H_{L}^{2,2} \) (or \( H^{2} \)) regularity. By continuous dependency and standard limiting argument for the sequence \( \varphi_{k} \in H_{L}^{2,2} \) (or \( H^{2} \)) with \( \varphi_{k} \to \varphi \) in \( H_{L}^{1,1} \) or \( H_{L}^{1,2} \) (or \( H^{1} \), respectively), we get

4. Proof of Blowup

We show the finite time blowup via standard virial argument. To avoid duplication of proof we consider the case \( b_{1} > 0 \). For the case of constant \( K_{i} \) see [3].

Lemma 4.1. Let \( \varphi \in H_{L}^{1,1} \) and \( x\varphi \in L_{x}^{2} \), and let \( u \) be the solution of (1.1) in \( C([-T,T]; H_{L}^{1,1}) \). Then \( xu \in C([-T,T]; L_{x}^{2}) \) and it satisfies that

\[
\begin{align*}
\frac{d}{dt} \| xu(t) \|_{L_{x}^{2}}^2 &= \| x\varphi \|_{L_{x}^{2}}^2 + 4 \int_{0}^{t} A(s) \, ds,
\end{align*}
\]

where \( A(t) = \text{Im} \int \varphi \cdot \nabla u \, dx \).

\[
A(t) = 4t E(\varphi) + \frac{1}{2} \int_{0}^{t} \int (K_{1} - x \cdot \nabla K_{1}) |u|^{4} \, dx \, ds + \frac{1}{3} \int_{0}^{t} \int (4K_{2} - x \cdot \nabla K_{2}) |u|^{6} \, dx \, ds.
\]

Proof. Let \( \theta_{\varepsilon}(x) = e^{-\varepsilon |x|^{2}} \) and \( f_{\varepsilon}(t) = \| \theta_{\varepsilon}(x) |u(t)| \|_{L_{x}^{2}}^2 \, dx \). Then since \( g(u) = \sum_{i=1,2} K_{i}Q_{i}(u) \in C([-T,T], L_{x}^{2}) \), by direct differentiation one can easily obtain that

\[
\theta_{\varepsilon}(x) = \varepsilon(1 - 2\varepsilon|x|^{2}) \theta_{\varepsilon}(x) \text{Im} x \cdot \nabla u, \, dx \, dt,
\]

and thus

\[
\sqrt{f_{\varepsilon}(t)} \leq \| x\varphi \|_{L_{x}^{2}} + C \int_{0}^{t} \| \nabla u(t) \|_{L_{x}^{2}} \, ds.
\]

Using Fatou’s lemma, we obtain that \( xu(t) \in (L^{\infty} \cap C)([-T,T]; L_{x}^{2}) \) and (4.1).

Here one can also show that if a sequence \( \{ \varphi_{k} \} \subset H_{L}^{2,2} \) satisfies that \( \varphi_{k} \to \varphi \) in \( H_{L}^{n,\ell} \) and \( x\varphi_{k} \to x\varphi \) in \( L_{x}^{2} \), then the solution sequence \( \{ u_{k} \} \) satisfies that

\[
xu_{k} \to xu \quad \text{in} \quad (L^{\infty} \cap C)([-T,T]; L_{x}^{2}).
\]

Due to the continuous dependency on the initial data and (4.3) we may assume that \( \varphi \in H^{2} \) and \( u \in C([-T,T]; H^{2} \cap H_{L}^{n,\ell} \cap C^{1}([-T,T]; L_{x}^{2}) \) and \( xu \in C([-T,T]; L_{x}^{2}) \). Let us consider a modified quantity \( \text{Im} \int \varphi \cdot \nabla u \, dx \). Then the identity (4.2) follow from direct differentiation of this quantity and standard limiting argument \( \varepsilon \to 0 \).

Now from (4.1) and (4.2), and from the condition of \( K_{i} \) it follows that

\[
\| xu(t) \|_{L_{x}^{2}}^2 \leq \| x\varphi \|_{L_{x}^{2}}^2 + 4t \text{Im} \int \varphi \cdot \nabla \varphi \, dx + (8 + 4\alpha)t^{2} E(\varphi).
\]

Since \( E(\varphi) < 0 \), (4.4) gives us the finite time blowup. \( \square \)
5. Scattering: Proof of Theorem \ref{thm:scattering}

5.1. Nonlinear estimates.

**Lemma 5.1.** Let $0 \leq b_1 < \frac{2}{3}$. Then we have for any $f \in H^{1,1}_L$, $g \in H^1$, and $h \in L^6_x$ we have
\[
\| |x|^{b_1-1} fgh\|_{L^\infty_x} + \| |x|^{b_1} fgh\|_{L^6_x} \lesssim \| f \|_{H^{1,1}_L} \| g \|_{H^1} \| h \|_{L^6_x}.
\]

**Proof.** For the first term we have from Lemma \ref{lem:sobo} that
\[
\| |x|^{b_1-1} fgh\|_{L^\infty_x} \lesssim \| |x|^{b_1} f\|_{L^6_x} \| g\|_{L^6_x} \| h\|_{L^6_x} \lesssim \| |x|^{b_1} f\|_{L^6} \| g\|_{H^1} \| h\|_{L^6_x}.
\]

If $b_1 = 0$, then we are done by Sobolev embedding. If $b_1 > 0$, then let us choose $\varepsilon$ such that $\frac{1}{3} < \varepsilon < 1 - b_1$ and set $p = 6\varepsilon$. Then by Lemma \ref{lem:sobo} with $b = b_1$ and $p = 6\varepsilon$ we have
\[
\| |x|^{b_1} f\|_{L^6_x} \lesssim \| f \|_{H^{1,1}_L} \| f \|_{L^6_x}.
\]

Since $2 < p < 6$, Sobolev embedding gives the desired bound.

By the same way we can treat the second term as follows. If $b_1 > 0$
\[
\| |x|^{b_1} fgh\|_{L^6_x} \lesssim \| |x|^{b_1} f\|_{L^6_x} \| g\|_{L^6_x} \| h\|_{L^6_x} \lesssim \| f \|_{H^{1,1}_L} \| g\|_{H^1} \| h\|_{L^6_x}.
\]

If $b_1 = 0$, then for small positive $\varepsilon$ we have
\[
\| fgh\|_{L^6_x} \lesssim \| |x|^{\varepsilon} f\|_{L^6_x} \| g\|_{L^6_x} \| h\|_{L^6_x} \lesssim \| f \|_{H^{1,1}_L} \| g\|_{H^1} \| h\|_{L^6_x}.
\]
\[\Box\]

**Lemma 5.2.** (1) Let $0 \leq b_2 < \frac{2}{3}$. Then we have for any $f_1, f_2, f_3 \in H^{1,1}_L$, $g \in H^1$, and $h \in L^6_x$ we have
\[
\| |x|^{b_2-1} f_1 f_2 f_3 gh\|_{L^\infty_x} + \| |x|^{b_2} f_1 f_2 f_3 gh\|_{L^6_x} \lesssim \prod_{j=1}^3 \| f_j \|_{H^{1,1}_L} \| g \|_{H^1} \| h \|_{L^6_x}.
\]

(2) If $b_2 = 0$, then we have for any $f_1, f_2, f_3 \in H^{1,1}_{L^6}$, $g \in H^{1,1}_{L^6}$, and $h \in L^{20}_x$ we have
\[
\| |x|^{-1} f_1 f_2 f_3 gh\|_{L^\infty_x} + \| f_1 f_2 f_3 gh\|_{L^{20}_x} \lesssim \prod_{j=1}^3 \| f_j \|_{H^{1,1}_{L^6}} \| g \|_{H^{1,1}_{L^6}} \| h \|_{L^{20}_x}.
\]

**Proof.** We first consider the case $b_2 > 0$. Choose $\frac{1}{3} \leq \varepsilon < \min(\frac{1}{3}, 1 - \frac{b_2}{3})$. Then from Lemma \ref{lem:sobo} with $b = \frac{b_2}{3}, p = 18\varepsilon$ and Lemma \ref{lem:sobo} it follows that
\[
\| |x|^{b_2-1} f_1 f_2 f_3 gh\|_{L^\infty_x} \lesssim \prod_{j=1}^3 \| |x|^{\frac{b_2}{3}} f_j\|_{L^{18}_x} \| |x|^{-1}\|_{L^6_x} \| g\|_{L^6_x} \| h\|_{L^6_x} \lesssim \prod_{j=1}^3 \| f_j \|_{H^{1,1}_L} \| g\|_{H^1} \| h\|_{L^6_x}.
\]

Since $2 < p < 6$, Sobolev embedding ($H^1 \to L^p$) leads us to the desired estimate.

On the other hand, one can easily see that
\[
\| |x|^{b_2} f_1 f_2 f_3 gh\|_{L^{20}_x} \lesssim \prod_{j=1}^3 \| |x|^{\frac{b_2}{3}} f_j\|_{L^{18}_x} \| g\|_{L^6_x} \| h\|_{L^6_x} \lesssim \prod_{j=1}^3 \| f_j \|_{H^{1,1}_L} \| g\|_{H^1} \| h\|_{L^6_x}.
\]
If \( b_2 = 0 \), then we have
\[
\| |x|^{-1} f_1 f_2 f_3 g h \|_{L_x^\delta} + \| f_1 f_2 f_3 g h \|_{L_x^\delta} \lesssim \prod_{j=1}^3 \| f_j \|_{L^2} \| |x|^{-1} \|_{L_x^\delta} + \| g \|_{L_x^\delta} \| h \|_{L_x^\delta}
\]
\[
\lesssim \prod_{j=1}^3 \| f_j \|_{H_x^{1,1}}^{\frac{1}{3}} \| g \|_{H_x^{1,1}} \| h \|_{L_x^\delta}.
\]

\[ \square \]

Remark 1. We can apply the above estimate to non-algebraic cases \(|x|^b |f|^{\alpha} g h \) with \( \frac{1}{3} < \alpha < 3 \), \( 0 < b < \alpha - \frac{1}{3} \).

By taking \( \frac{1}{3a} \leq \varepsilon < 1 - \frac{b}{\alpha} \), one can get
\[
\| |x|^{b-1} |f|^\alpha g h \|_{L_x^\delta} \leq \| |x|^b |f|^\alpha \|_{L_x^\delta} \| |x|^{-1} \|_{L_x^\delta} \| h \|_{L_x^\delta}
\]
\[
\lesssim \| |x|^b f \|_{L_x^\delta} \| g \|_{H^1} \| h \|_{L_x^\delta}
\]
\[
\lesssim \| f \|_{L_x^\delta} \| g \|_{H^1} \| h \|_{L_x^\delta}
\]
\[
\lesssim \| f \|_{H_x^{1,1}} \| g \|_{H^1} \| h \|_{L_x^\delta}.
\]

5.2. Proof of scattering. Let us define a complete metric space \( X_\rho \) by
\[
X_\rho := \{ u \in L^{1,0}_t H^{1,1}_{\rho, \frac{1}{30}} \cap (C \cap L^\infty_t)(\mathbb{R}; H^{1,1}_L) : \| u \|_{L^{\infty}_t H^{1,1}_{L,0}} + \| u \|_{L^{1,0}_t H^{1,1}_{L,0}} + \| u \|_{L^{1,0}_t H^{1,1}_{\rho, \frac{1}{30}}} \leq \rho \}
\]
equipped with the metric \( d \) such that
\[
d(u, v) = \| u - v \|_{X_\rho} := \| u - v \|_{L^{\infty}_t H^1_L} + \| u - v \|_{L^{2}_t H^0_{L,0}} + \| u - v \|_{L^{1,0}_t H^{1,1}_{\rho, \frac{1}{30}}}.
\]

Let us show that the nonlinear functional \( \Psi(u) = e^{it\Delta} \varphi + \mathcal{N}(u) \) is a contraction on \( X_\rho \). For this we have only to show
\[
\| \mathcal{N}(u) \|_{X_\rho} \leq \| u \|_{X_\rho}^3 + \| u \|_{X_\rho}^2,
\]
\[
\mathcal{N}(u) - \mathcal{N}(u) \|_{X_\rho} \leq \| u \|_{X_\rho} + \| v \|_{X_\rho} + \| u \|_{X_\rho} + \| v \|_{X_\rho}.
\]

Clearly, \( \| e^{it\Delta} \varphi \|_{X_\rho} \leq \| \varphi \|_{H^{1,1}_L} \) by Strichartz estimates, and thus we can find \( \rho \) small enough for \( \Psi \) to be a contraction mapping on \( X_\rho \), and for the equation (1.1) to be globally well-posed in \( H^{1,1}_L \).

Once (1.1) is globally well-posed, the scattering is straightforward. In fact, let us define a scattering state \( u_\pm \) with
\[
\varphi_\pm := \varphi + \lim_{t \to \pm \infty} e^{-it\Delta} \mathcal{N}(u).
\]
Then we get the desired result by the duality argument:
\[
\| u(t) - u_\pm(t) \|_{H^{1,1}_L} = \|(1 - \Delta)^{\frac{1}{2}} L(u(t) - u_\pm(t))\|_{L^2}
\]
\[
= \sup_{\| \psi \|_{L^2} \leq 1} \left| \int_{\pm \infty} \langle (1 - \Delta)^{\frac{1}{2}} L[K_1 Q_1(u) + K_2 Q_2(u)], e^{-it\Delta} \psi \rangle dt \right|
\]
\[
\leq \|(1 - \Delta)^{\frac{1}{2}} L[K_1 Q_1(u) + K_2 Q_2(u)]\|_{L^2(t, \pm \infty; L^2)} \| e^{-it\Delta} \psi \|_{L^2_x}
\]
\[
\lesssim \rho^2 (\| u \|_{L^2(t, \pm \infty; H^{1,1}_{L,0})} + \| u \|_{L^{1,0}(t, \pm \infty; H^{1,1}_{\rho, \frac{1}{30}})}) \to 0 \text{ as } t \to \pm \infty.
\]

Here \( (t, \pm \infty) \) means that \( (t, + \infty) \) if \( t > 0 \) and \( (-\infty, t) \) if \( t < 0 \).
Now it remains to show (5.1) and (5.2). Given $\rho$, for $u \in X_L(\rho)$ we consider $N_t^{1,1}$ as in Section 3. From the bound (3.1) and the endpoint Strichartz estimate it follows that

$$\left\| N_t^{1,1} \right\|_{L_t^2 L_{x}^{\infty} \cap L_t^6 L_x^{\frac{30}{7}}} \lesssim \left\| (\partial L K_1) Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\| \partial K_1 Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\| L K_1 \partial Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\| K_1 L Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}}$$

$$\lesssim \left\| x \right\|^{b_1-1} \left\| Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\| x \right\|^{b_1-1} \left\| L Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\| x \right\|^{b_1-1} \left\| \partial Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}} + \left\| \partial L Q_t(u) \right\|_{L_t^2 L_x^{\frac{6}{5}}}$$

Since $|L Q_1(u)| \lesssim |u|^2 |Q_1|$, $|L Q_1(u)| \lesssim |u|^2 |\partial u|$, and $|\partial L Q_1(u)| \lesssim |u| |\partial u| |L u| + |u|^2 |\partial L u|$, applying Lemma 5.1 with $f = u$, $g = u$ or $|L u|$, and $h = u$, $\partial u$, or $|\partial L u|$, we get

$$\left\| N_t^{1,1} \right\|_{L_t^2 L_{x}^{\infty} \cap L_t^6 L_x^{\frac{30}{7}}} \lesssim \left\| u \right\|_{X_L}^3.$$

As for $Q_2(u)$, there hold $|L Q_2(u)| \lesssim |u|^4 |L u|$, $|L Q_2(u)| \lesssim |u|^4 |\partial u|$, and $|\partial L Q_2(u)| \lesssim |u|^3 |\partial u| |L u| + |u|^4 |\partial L u|$. Thus by taking $f_j = u$, $g = u$ or $|L u|$, and $h = u$, $\partial u$, or $|\partial L u|$ we obtain from Lemma 5.2

$$\left\| N_t^{1,1} \right\|_{L_t^2 L_{x}^{\infty} \cap L_t^6 L_x^{\frac{30}{7}}} \lesssim \left\| u \right\|_{X_L}^5.$$

These show the estimate (5.1).

To treat (5.2) let us set $w = u - v$. Then $Q_1(u) - Q_1(v)$ can be decomposed by new cubic and quintic terms of $u$, $v$ and only one $w$. Applying the same argument as above to these terms, one readily get the second part (5.2). This completes the proof of Theorem 1.3.

6. PROOF OF NON-SCATTERING

We follow the argument as in [10]. By contradiction we assume that $\|\varphi_+\|_{L_t^2} \neq 0$. Since $K_1$ are real-valued, $m(u(t)) = m(\varphi)$. We consider $H(t) = -\text{Im} \int u(t)\overline{u_+}(t) \, dt$ for $t \gg 1$. Differentiating $H$, we get

$$\frac{d}{dt} H(t) = \text{Re} \int (K_1 Q_1 + K_2 Q_2) \overline{u_+} \, dx,$$

where $\lambda = 0, 1$. We decompose this as follows:

$$\frac{d}{dt} H(t) = \sum_{j=1}^{3} J_j^1 + J_2,$$

where

$$J_j^1 = \int |x|^{b_1} |u_+|^4 \, dx,$$

$$J_2 = \text{Re} \int |x|^{b_1} |u|^2 |u_+|^2 \, dx,$$

$$J_3 = \text{Re} \int |x|^{b_1} |u|^2 (u - u_+) \overline{u_+} \, dx,$$

and

$$J_2 = \text{Re} \int |x|^{b_2} |u|^4 uu_+ \, dx.$$

We estimate $J_1^1$ as follows: for $0 < \delta \ll 1 \ll k$

$$\int_{|t| \leq |x| \leq kt} |u_+|^2 \, dx \lesssim \left\| |x|^{-\delta} \right\|_{L_t^2 (\delta \leq |x| \leq kt)} (J_1^1)^{\frac{1}{2}} \lesssim t^{\frac{3}{2}} (J_1^1)^{\frac{1}{2}}.$$

It was show in [10] that $\int_{|t| \leq |x| \leq kt} |u_+|^2 \, dx \sim \|\varphi_+\|^2_{L_2}$ for some fixed large $k$ and small $\delta$, and for any large $t$. From this we deduce that

$$J_1^1 \gtrsim m(\varphi_+) t^{-(3-b_4)}.$$
Let us denote the generator of Galilean transformation by \( \mathbf{J} \), that is \( \mathbf{J} = e^{-it\Delta}xe^{it\Delta} \). On the sufficiently regular function space

\[
\mathbf{J} = x + 2it\nabla, \quad (\mathbf{J} \cdot \mathbf{J})^m = (|x|^2 - 4tA - 4t^2\Delta)^m,
\]

where \( A \) is the self-adjoint dilation operator defined by \( \frac{1}{2} (\mathbf{J} \cdot \nabla + \nabla \cdot \mathbf{x}) \), which yields \( \mathcal{A} = \int \mathcal{A} u \, dx \). Since

\[
\|u_+ (t)\|_{L^\infty} \lesssim t^{-\frac{5}{6}}\|\varphi_+\|_{L^1},
\]

and

\[
\|\|x|^{2m}u_+ (t)\|_{L^\infty} = \|e^{it\Delta} (\mathbf{J} \cdot \mathbf{J})^m \varphi_+\|_{L^\infty} \lesssim \varphi_+ t^{-\left(\frac{5}{6} - 2m\right)},
\]

by interpolation we see that

\[
\|\|x|^\theta u_+ (t)\|_{L^\infty} \lesssim \varphi_+ t^{-\left(\frac{5}{6} - \theta\right)}
\]

for any \( \theta > 0 \). By this we get \( \|\|x|^{-\frac{b_1}{2}} u_+ (t)\|_{L^2} \lesssim \frac{1}{\|\varphi_+\|_{L^2}^2} (J_1^2)^\frac{1}{2} \lesssim t^{\frac{b_1}{2}} (J_1^2)^{\frac{1}{2}} \).

For \( J_2^2 \) we have

\[
J_2^2 \lesssim \|\|x|^{b_1}u_+ (t)\|^2\|_{L^\infty} (\|u\|_{L^2}^2 + \|u_+\|_{L^2}^2)\|u - u_+\|_{L^2}.
\]

Using (6.2) we get

\[
|J_2^2| = o_{m(\varphi), \varphi_+}(t^{-(3-b_1)}).
\]

To estimate \( J_1^2 \) and \( J_2 \) we need the following lemma.

**Lemma 6.1.** Let \( u \) be a global smooth solution of (1.1) with \( K_1 = |x|^{b_1}, K_2 = |x|^{b_2} \) such that \( 0 \leq b_2 < 3 + b_1 \), and \( xu \in C(\mathbb{R}; L^2) \). Then for any large \( t \) there holds

\[
V(u) := \frac{1}{4} \int |x|^{b_1} |u|^4 \, dx + \frac{1}{6} \int |x|^{b_2} |u|^6 \, dx \leq C(m(\varphi), E(\varphi), \|\varphi\|_{L^2}) \, t^{-(3-b_1)}.
\]

From Lemma 6.1 and inequality (5.2) it follows that

\[
|J_1^2| \leq \left( \int |x|^{b_1} |u|^4 \, dx \right)^\frac{1}{2} \|u - u_+\|_{L^2} \|x|^{\frac{b_1}{2}} u_+\|_{L^\infty} = o_{\varphi, \varphi_+}(t^{-(3-b_1)}),
\]

\[
|J_2| \leq \int |x|^{\frac{5b_1}{2}} |u|^5 |x|^{\frac{b_1}{2}} |u_+| \, dx = \left( \int |x|^{b_1} |u|^6 \, dx \right)^\frac{1}{2} \|x|^{\frac{b_1}{2}} |u_+|\|_{L^\infty}^\frac{1}{2} \|u_+\|_{L^2} \lesssim \varphi_+ t^{-\left(\frac{5}{2} - \frac{b_1}{2} - \frac{b_2}{2}\right)}
\]

\[
= o_{\varphi, \varphi_+}(t^{-(3-b_1)}). \quad (:\ b_2 < 3 + b_1).
\]

Therefore we conclude that for \( t \gg 1 \)

\[
\frac{d}{dt} H(t) \gtrsim \varphi_+ t^{-(3-b_1)}.
\]

Since \( H(t) \) is uniformly bounded for any \( t \geq 0 \), the range \( b_1 \geq 2 \) leads us to the contradiction to the assumption \( \|\varphi_+\|_{L^2} \neq 0 \). By time symmetry a similar argument holds for negative time. We omit that part.

**Proof of Lemma 6.2.** From (5.1) and (5.2) we deduce the pseudo-conformal identity:

\[
\frac{d}{dt} \int |\mathbf{J} u|^2 + 8t^2 V(u) \, dx = -4t \left[ \frac{1}{2} \int (K_1 - x \cdot \nabla K_1) |u|^4 \, dx + \frac{1}{3} \int (4K_2 - x \cdot \nabla K_2) |u|^6 \, dx \right]
\]

\[
= -4t \left[ \frac{1 - b_1}{2} \int |x|^{b_1} |u|^4 \, dx + \frac{4 - b_2}{3} \int |x|^{b_2} |u|^6 \, dx \right].
\]
Since $b_1 \geq 2$ and $0 \leq b_2 < 3 + b_1$, by integrating over $[0,t]$ we obtain
\begin{align*}
i^2 V(u(t)) &\leq \frac{1}{8} \|x\varphi\|_{L^2}^2 + (b_1 - 1) \int_0^t \tau V(u(\tau)) \, d\tau \\
&\leq \frac{1}{8} \|x\varphi\|_{L^2}^2 + (b_1 - 1) \int_0^1 \tau V(u(\tau)) \tau + (b_1 - 1) \int_1^t \tau V(u(\tau)) \, d\tau \\
&\leq C(m(\varphi), E(\varphi), \|x\varphi\|_{L^2}^2) + (b_1 - 1) \int_1^t \tau V(u(\tau)) \, d\tau.
\end{align*}
Gronwall’s inequality gives us
\begin{align*}
i^2 V(u(t)) &\leq C(m(\varphi), E(\varphi), \|x\varphi\|_{L^2}^2) \exp \left[ \int_1^t \frac{b_1 - 1}{\tau} \, d\tau \right] = C(m(\varphi), E(\varphi), \|x\varphi\|_{L^2}^2) t^{b_1 - 1}.
\end{align*}
This completes the proof of Lemma 6.1.

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