ASYMPTOTIC ESTIMATE FOR THE POLYNOMIAL COEFFICIENTS

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Abstract. The polynomial coefficient \((^n_{c,n})\) is defined to be the coefficient of \(x^k\) in the expansion of \((1 + x + x^2 + \cdots + x^{q-1})^n\). In this note we give an asymptotic estimate for \((^n_{c,n})\) as \(n\) tends to infinity, where \(c\) is a positive integer. Based on experimental results, it was conjectured that for any \(n\), \((^n_{c,n}) - (^{n-1}_{c,n})\) is unimodal and its maximum value occurs \(q = \lfloor \log_{1+ \frac{1}{c}} n \rfloor \) or \(q = \lfloor \log_{1+ \frac{1}{c}} n \rfloor + 1\). In particular, when \(c = 1\), its maximum value occurs for \(q = \lfloor \log_2 n \rfloor \) or \(q = \lfloor \log_2 n \rfloor + 1\).

1. Introduction

The polynomial coefficient, \((^n_{q,k})\), defined to be the coefficient of \(x^k\) in the expansion of \((1 + x + x^2 + \cdots + x^{q-1})^n\). That is,

\[
\sum_{k=0}^{\infty} \left( ^n_{q,k} \right) x^k = (1 + x + x^2 + \cdots + x^{q-1})^n.
\]

In the case \(q = 2\) it is the binomial coefficient \((^n_k)\) and in the case \(q = 3\) it is the trinomial coefficient. Clearly \((^n_{q,k})\) can be regarded as a generalization of binomials and is one of the fundamental combinatorial coefficients. It was studied extensively by many mathematicians since the time of Euler’s. For details we refer to [4, 6, 8, 19]. Some applications in coding theory and communication theory can be found in [9, 13, 17, 18].

The polynomial coefficient has close relation with a composition. Let \(k\) be a positive integer. A composition (also called ordered partitions) of \(k\) is a finite sequence of positive integers \(x_1, x_2, \ldots, x_r\) such that \(x_1 + x_2 + \cdots + x_r = k\). The \(x_i\) are called parts of the composition. Let \(b(k, n, q)\) be the number of compositions of \(k\) with \(n\) parts such that each part is bounded by \(q\). Thus \(b(k, n, q)\) is exactly the coefficient of \(x^k\) in the expansion of \((x + x^2 + \cdots + x^{q-1})^n = x^n(1 + x + \cdots + x^{q-1})^n\) and \(b(k, n, q) = \left( ^{n,q}_{k,n} \right)\). It also equals the number of distinct ways in which \(k\) identical balls can be distributed in \(n\) labeled boxes with each box containing at most \(q - 1\) balls and being nonempty. This classical bounded model of compositions also has many other combinatorial meanings such as restricted multi-combinations and forbidden 0,1-sequences.

It follows by above discussion that the combinatorial properties of both \(b(k, n, q)\) and \((^n_{q,k})\) are essentially the same. In this paper we will focus on the study of \((^n_{q,k})\). In [24], it was proved that when \(q > 2\), \((^n_{q,k})\) has no closed form, that is, it cannot be expressed to be the sum of a fixed number of hypergeometric terms. Thus it is a natural question to ask some asymptotical estimate. There seems to have many questions on the distributions of \((^n_{q,k})\).
Denote $f(x) \sim g(x)$ if
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$  
In the case $q = 2$, it is well known that the binomial coefficient \( \binom{n,2}{cn} = \binom{n}{cn} \) has asymptotical estimate
$$\binom{n,2}{cn} \sim \frac{1}{\sqrt{2\pi(cn-c^2)}}(e^{-c}(c-1)^{-c+1})^n,$$
where $0 < c < 1$ is a constant.

In the case $q = 3$, it is well known that the trinomial coefficient has asymptotical estimate
$$\binom{n,3}{n} \sim \frac{3n+1/2}{2\sqrt{\pi n}}.$$  

The first result of this paper is an asymptotical estimate for \( \binom{n,q}{cn} \) for general $q > 3$ by applying Hayman’s method in the case $k = cn$ and $c$ is fixed positive integer.

**Theorem 1.1.** Suppose $q > 3$ and $k = cn$, where $c < q$ is an absolute positive integer. Then we have
$$\binom{n,q}{cn} \sim \frac{\phi(r)}{\sqrt{2\pi n}} \left(\frac{1}{r-r^2}\right)^n,$$
as $n \to \infty$, where
$$\phi(r) = \left(\frac{r}{(1-r)^2} - \frac{q^2 r^q}{(1-r^q)^2}\right)^{-1/2},$$
$$r = \frac{1}{d} + \frac{q}{c^2 n^{q+2}} + \theta \frac{q^3}{d^2 q},$$
$$|\theta| \leq 1 \text{ and } d = 1 + \frac{1}{c}. \text{ In particular, when } c = 1 \text{ we have}$$
$$r = \frac{1}{2} + \frac{q}{2q^2 + \theta \frac{q^3}{2q}}.$$  

Star \cite{29} gave an asymptotic formula only for the case $k = \frac{1}{2} (n-s)(q+1)$, where $s = Kn^\theta$, $0 \leq \theta \leq 1/2$ and $K$ is a positive constant.

We are interested in the unimodality of the polynomial coefficients. A sequence \( \{a_0, a_1, \cdots, a_n\} \) is \textbf{unimodal} if there exits index $k$ with $0 \leq k \leq n$ such that
$$a_0 \leq a_1 \leq \cdots a_{k-1} \leq a_k \geq a_{k+1} \cdots \geq a_n.$$  

A sequence \( \{a_0, a_1, \cdots, a_n\} \) is called reciprocal if $a_i = a_{n-i}$ for $0 \leq i < n$.

**Proposition 1.2.** For given $n, k$, \( \binom{n,q}{k} \) is unimodal and reaches its maximum at $k = qn/2$ provided $qn$ is even and $k = (qn-1)/2$ provided $qn$ is odd.

This well known proposition has many proofs. Perhaps the most simple one is that the product of two unimodal reciprocal polynomials is also unimodal reciprocal, for details please refer to \cite{4}.

Let $a(k,n,q)$ be the number of compositions of $k$ with $n$ parts such that the largest part is $q$. Clearly $a(k,n,q) = b(k,n,q) - b(k,n,q-1)$ and thus $a(2n,n,q) = \binom{n,q}{n} - \binom{n,q-1}{n}$.

Let $b(k,q) = \sum_{n=1}^{k} b(k,n,q)$ be the number of all compositions of a positive integer $k$ with parts bounded by $q$ and $a(k,q) = \sum_{n=1}^{k} a(k,n,q)$ be the number of
all compositions of a positive integer $k$ such that the largest part is $q$. It is well known that
\[
\sum_{k=0}^{\infty} b(k, q)x^k = \frac{1 - x}{1 - 2x + x^{q+1}}.
\]

Based on this formula and analytical tools, Odlyzko and Richmond [19] proved the following theorem.

**Theorem 1.3** (Odlyzko and Richmond). Let $a(k, q)$ be the number of all compositions of a positive integer $k$ with parts bounded by $q$. Then $a(k, q)$ is unimodal for any $k$ and the maximum value occurs for $q = \lfloor \log_2 k \rfloor$ infinitely often and $q = \lfloor \log_2 k \rfloor + 1$ infinitely often and always at one of these two values and no other.

Our conjecture is a more subtle one:

**Conjecture 1.4.** Let $a(k, n, q)$ be the number of compositions of $k$ with $n$ parts such that the largest part is $q$. Let $c$ be a positive integer. Then for any $n$, $a((c+1)n, n, q) = \binom{n-1}{c} \binom{n-q-1}{c} = 1$ is a unimodular function on $q$ and the maximum value occurs for $q = \lfloor \log_2 (n+1) \rfloor + 1$ or $q = \lfloor \log_2 (n+1) \rfloor + 1$.

In particular, $a(2n, n, q) = \binom{n-1}{c} \binom{n-q-1}{c} = 1$ is a unimodular function on $q$ and the maximum value occurs for $q = \lfloor \log_2 (n+1) \rfloor$ or $q = \lfloor \log_2 (n+1) \rfloor + 1$.

The paper is organized as follows. A brief (but not complete) historical review is given in Section 2 and the main results are proved in Section 3. For simplicity of the computations, we only gives the proof for the special case $k = n$ i.e., $c = 1$. In the last Section, we give some analysis to support Conjecture 1.4.

2. **Historical results on $\binom{n}{k}$**

Many equalities on the polynomial coefficients were found since the time of Euler’s [8]. For instance, the following equations are well-known and proofs can be found in [10]:

**Proposition 2.1.** $\binom{n}{k}$ satisfies:

\[
\binom{n}{k} = \sum_{i=0}^{q-1} \binom{n-1}{k-i};
\]
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n}{k-1} - \binom{n-1}{k-q};
\]
\[
\binom{n}{k} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+k-iq-1}{n-1};
\]
\[
\binom{n}{k} = 2 \pi \int_0^\frac{\pi}{2} \frac{\sin q\theta}{\sin \theta} \theta \cos((q-1)n-2k))d\theta.
\]

André proved that [10]

\[
\sup_k \binom{n}{k} \sim q^n \sqrt{\frac{6}{(q^2-1)\pi n}}, \quad n \to \infty.
\]

Star [29] gave an asymptotic formula only for the case $k = \frac{1}{2}(n - s)(q + 1)$, where $s = Kn^\theta, 0 \leq \theta \leq 1/2$ and $K$ is a positive constant. Note that this formula generalized the result of André.
Theorem 2.2 (Star). Let \( k = \frac{1}{2}(n - s)(q + 1) \), where \( s = Kn^\theta, 0 \leq \theta < 1/2 \) and \( K \) is a positive constant. As \( n \) tends to infinity,

\[
\binom{n, q + 1}{k - n} = \frac{1}{\sqrt{\pi}} \left( \frac{6}{q^2 - 1} \right)^{1/2} \frac{q^n}{n^{3/2}} \cdot \left( 1 + \frac{h_{1,0}(q) + h_{1,1}(q)s^2}{n} + \sum_{j=0}^{m-1} \frac{h_{m-1,j}(q)s^{2j}}{n^{m-1}} + O\left( \frac{1 + s^{2m}}{n^m} \right) \right),
\]

where \( h_{i,j} \) are some rational functions of \( q \).

3. Main result

Definition 3.1. Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is a complex analytic function for \(|z| < R\), where \( 0 < R \leq \infty \). Define

\[
M(r) = \max_{|z|=r} |f(z)|.
\]

If for large enough \( r \), we have \( M(r) = f(r) \), then \( f(z) \) is called an admissible function. Please refer to [14, 30] for detailed theory on admissible functions.

Note that this one is different from the current definition of admissible functions, which actually defines a function satisfying (3.1).

In the well known paper [14] Hayman proved that such good functions have very good asymptotic estimate on their coefficients. The following lemma due to Hayman gives a subtle estimate on controlling its coefficients for an admissible analytical function.

Lemma 3.2 (Hayman). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an admissible function, which is analytic in the disk \(|z| < R\). Denote

\[
a(r) = r \frac{f'(r)}{f(r)}, \quad b(r) = ra'(r),
\]

and suppose \( 0 < r_n < R \) is a positive real root satisfying

\[
a(r_n) = n, \quad \forall n \in \mathbb{N}.
\]

Then

\[
a_n \sim \frac{f(r_n)}{r_n^2 \sqrt{2\pi b(r_n)}}, \quad n \to \infty.
\]

Example 3.3. \( f(z) = (1 + z + z^2 + \cdots + z^{q-1})^n \) is an admissible function analytical in the disk \(|z| < 1.\)

Lemma 3.4. Assume \( q \geq 3 \). Then the equation

\[
(q - 2)x^{q+1} - (q - 1)x^q + 2x - 1 = 0, \quad q \in \mathbb{N}
\]

has only two positive real roots including 1 as a trivial one. The nontrivial one \( r \) satisfies

\[
|r - \frac{1}{2} - \frac{q}{2q+2}| \leq \frac{q^2}{2q}.
\]
Proof. Since the cases $q = 3, 4$ can be verified directly, we may assume $q > 4$. Suppose $f(x) = (q - 2)x^{q+1} - (q - 1)x^q + 2x - 1$. Then $f''(x) = qx^q - 2x - xq - q^2 + 2q - 1 = 0$ gives two inflection points $\frac{(q-1)^2}{(q+1)(q-2)}$, 0. This proves that there are only two positive real roots including 1.

Suppose now $r = \frac{1}{2} + \frac{c}{2^{q+2}}$ is a positive real root of $f(x)$, where $c$ is regarded as a variable depending on $q$ and will be specified.

By the definition we have

$$f(r) = (q - 2)(\frac{1}{2} + \frac{c}{2^{q+2}})^{q+1} - (q - 1)(\frac{1}{2} + \frac{c}{2^{q+2}})^q + 2(\frac{1}{2} + \frac{c}{2^{q+2}}) - 1$$

$$= (\frac{-q}{2^{q+1}} + \frac{(q-2)c}{2^{2q+2}})(1 + \frac{c}{2^{q+1}})^q + \frac{c}{2^{q+1}}$$

$$= 0.$$

We assume without of generality that $0 \leq c \leq q^{3/2}$. By the Taylor’s expansion, for any $q$ we have,

$$\left| (1 + \frac{c}{2^{q+1}})^q - 1 - \frac{cq}{2^{q+1}} \right| \leq \frac{c^2q^2}{2^q} \leq \frac{q^5}{2^q}.$$

And hence

$$\left| f(r) + \frac{q-c}{2^{q+1}} + \frac{c(q^2-q+2)}{2^{2q+2}} \right| \leq \frac{q^6}{2^{2q+1}}.$$

Since $f(r) = 0$, we then get

$$q - c - \frac{c(q^2-q+2)}{2^{q+1}} + \frac{\theta q^6}{2^q} = 0,$$

where $0 \leq |\theta| \leq 1$. Set $c = q + c'$ and substitute this into the above equality one has

$$c'(1 - \frac{q^2-q+2}{2^{q+1}}) = \frac{q^3-q^2+2q}{2^{q+1}} + \frac{\theta q^6}{2^q}.$$

Finally we get that

$$\left| c - q - \frac{q^3-q^2+2q}{2^{q+1}-q^2+q-2} \right| \leq \frac{2q^6}{2^q}.$$

Substitute this into $r = \frac{1}{2} + \frac{c}{2^{q+2}}$ and when $q > 16$ we then have

$$\frac{q^3-q^2}{2^{2q+2}} \leq r - \frac{1}{2} - \frac{q}{2^{q+2}} \leq \frac{q^3}{2^{2q+2}}.$$

The cases for $3 < q < 16$ can be easily checked by computers and thus the proof is complete. \(\square\)

Theorem 3.5. Assume $q > 3$. Then we have the asymptotic estimate

$$\left( \frac{n}{q} \right) \sim \frac{\phi(r)}{\sqrt{2\pi n}} \left( \frac{1 - r^q}{r - r^q} \right)^n, \ n \to \infty$$

where

$$\phi(r) = \left( \frac{r}{(1-r)^2} - \frac{q^2r^q}{(1-r^q)^2} \right)^{-1/2},$$

$$\left| r - \frac{1}{2} - \frac{q}{2^{q+2}} \right| \leq \frac{q^3}{2^{2q}}.$$
Proof. Let \( f(z) = (1 + z + z^2 + \cdots + z^{q-1})^n = (\frac{1 - z^q}{1 - z})^n \). One checks that \( f(z) \) is indeed an admissible analytical function on \( \mathbb{C} - \{\infty\} \). Applying Hayman’s theorem Lemma 3.2 we have

\[
a(x) = x f'(x) = \frac{-n x (q x^{q-1} - q x^q - 1 + x^q)}{(1 - x^q)(1 - x)},
\]

\[
b(x) = x a'(x) = \frac{n x (1 - x^{q-1} - 2 x^q + x^{2q} + 2 x^2 q^q - x^{q+1} q^q)}{(-1 + x^q)^2 (x - 1)^2},
\]

and thus

\[
\frac{-n x_n (q x_n^{q-1} - q x_n^q - 1 + x_n^q)}{(1 - x_n^q)(1 - x_n)} = n,
\]

and the formula follows from Theorem 3.5. \( \square \)

**Corollary 3.6.** When \( q > 3 \), for large \( n \) we have the estimate

\[
\left( \frac{n}{q} \right) \sim \frac{(1 + \frac{q^2 - 6q}{2q} + \theta_1 \frac{q^2}{2q}) 2^n (1 - \frac{1}{2q - 2} + \theta_2 \frac{q^2}{2q^2})^n}{\sqrt{\pi n} (\frac{1}{x_n^q} - \frac{3q}{2q - 1})}, \quad n \to \infty,
\]

where \( |\theta_i| \leq 1 \) for \( i = 1, 2 \).

Proof.

\[
|r - \frac{1}{2} - \frac{q}{2q + 2}| \leq \frac{q^3}{2q^2},
\]

and thus one computes that

\[
\frac{r}{(1 - r)^2} - \frac{3q}{2q - 1} \leq \frac{q}{2q^2},
\]

and

\[
\frac{q^2 r^q}{(1 - r)^2} - \frac{q^2}{2q} \leq \frac{q^4}{2q^2}.
\]

Thus

\[
|\phi(r) - \sqrt{2}(1 + \frac{q^2 - 6q}{2q})| \leq \frac{q^2}{2q^2}.
\]

Similarly we have

\[
|\frac{1 - r^q}{r - r^2} - 2 + \frac{1}{2q - 1}| \leq \frac{q^2}{2q^2},
\]

and the formula follows from Theorem 3.5. \( \square \)

Fix another positive integer \( c \). The same method for \( k = cn \) gives a general result. We omit the details since the proof of this generalization is essentially the same as the case \( c = 1 \). For more details, please refer to [16].

**Theorem 3.7.**

\[
\left( \frac{n}{c n} \right) \sim \frac{\phi(r)}{\sqrt{2\pi n}} \left( \frac{1 - r^q}{r - r^2} \right)^n, \quad n \to \infty,
\]
where

\[ \phi(r) = \left( \frac{r}{(1 - r)^2} - \frac{q^2 r^q}{(1 - r)^2} \right)^{-1/2}, \]

and

\[ r = \frac{1}{d} + \frac{q}{d^q + 2} + O\left(\frac{q^2}{d^q}\right), \]

and \( d = 1 + \frac{1}{q} \).

4. Unimodality

A sequence \( \{a_0, a_1, \ldots, a_n\} \) is unimodal if there exits index \( k \) with \( 0 \leq k \leq n \) such that

\[ a_0 \leq a_1 \leq \cdots a_{k-1} \leq a_k \geq a_{k+1} \cdots \geq a_n. \]

We then consider the unimodality on the difference sequence of \( \binom{n}{q} \) on \( q \).

**Conjecture 4.1.** The sequence \( \binom{n}{q} - \binom{n}{q-1} \) is a unimodal function on \( q \) and the maximum value occurs for \( q = \lfloor \log_2 n \rfloor \) or \( q = \lceil \log_2 n \rceil + 1 \).

Furthermore, for a positive integer \( c \), the sequence \( \binom{n}{cn} - \binom{n}{cn-1} \) is a unimodal function on \( q \) and the maximum value occurs for \( q = \lfloor \log_1 n \rfloor + 1 \) or \( q = \lceil \log_1 n \rceil + 1 \).

We cannot prove this conjecture. Instead, we prove that the difference sequence of the main part of the asymptotic estimate on \( \binom{n}{q} \) is unimodal on \( q \) when \( n \) is taken to be large enough and fixed. This may be regarded as an evidence why the above conjecture should hold.

For a large integer \( n \), denote

\[ f(q) = \frac{(1 + \frac{q^2 - 6q}{2q} + \theta_1 \frac{q^2}{2q^2})(1 - \frac{1}{2q-2} + \theta_2 \frac{q^2}{2q^2})^n}{\sqrt{\pi n}}, \]

where \( |\theta_i| \leq 1 \) for \( i = 1, 2 \). Let \( g(q) = f(q + 1) - f(q) \).

**Corollary 4.2.** For large enough \( n \), the sequences \( g(q) \) is unimodal with maximum at \( q = \lfloor \log_2 n \rfloor \) or \( q = \lceil \log_2 n \rceil + 1 \).

**Proof.** It is clear that the unimodality of \( g(q) \) is equivalent to consider the unimodality of \( T(q) = B(q + 1) - B(q) \), where

\[ B(q) = (1 + \frac{q^2 - 6q}{2q} + \theta_1 \frac{q^2}{2q^2})(1 - \frac{1}{2q-2} + \theta_2 \frac{q^2}{2q^2})^n. \]

One computes

\[ T(q + 1) - T(q) = B(q + 2) - B(q) - 2B(q + 1) \]

\[ = (1 + \frac{q^2 - 2q - 8}{2q} + \theta_1 \frac{q^2}{2q^2})(1 - \frac{1}{2q+2} + \theta_2 \frac{q^2}{2q^2})^n \]

\[ + (1 + \frac{q^2 - 6q}{2q-2} + \theta_1 \frac{q^2}{2q^2})(1 - \frac{1}{2q} + \theta_2 \frac{q^2}{2q^2})^n \]

\[ + 2(1 + \frac{q^2 - 4q - 5}{2q+1} + \theta_1 \frac{q^2}{2q^2})(1 - \frac{1}{2q-1} + \theta_2 \frac{q^2}{2q^2})^n. \]

It then suffices to prove that if \( q \leq \log_2 n \), then \( T(q + 1) - T(q) < 0 \) and if \( q \geq \log_2 n + 1 \), then \( T(q + 1) - T(q) > 0 \). Assume \( n \) is large, since \( |\theta_i| \leq 1 \), if \( q \) is small enough, say \( q \ll (\log_2 n)^\epsilon \) for some small positive constant \( \epsilon \), we then have
Since the function number of compositions of $k$ be a positive absolute constant $c > 0$. The interesting cases happen when $q$ is very close to $\log_2 n$. Not that when $q = \log_2 n$, by the formula

$$(1 - \frac{1}{n})^n < e^{-1} < (1 - \frac{1}{n+1})^n$$

we then have $B(\log_2 n + i) \sim e^{-1/2^i}$ and thus when $n$ is large enough,

$$T(\log_2 n + 1) - T(\log_2 n) \sim e^{-1/4} + e^{-1} - 2e^{-1/2} < 0.$$ 

Since the function

$$\frac{e^{-2i+1} + e^{-2i-1}}{2e^{-2i}} = \frac{1}{2} e^{-2i} + \frac{1}{2} e^{-2i-1} < 1$$

for all $i \leq -1$, which means $e^{-2i+1} + e^{-2i-1} - 2e^{-2i} < 0$ for all $i \leq -1$. This shows that when $q \geq \log_2 n + 1$, $T(q + 1) - T(q) < 0$.

Similarly, since the function

$$\frac{e^{-2i+1} + e^{-2i-1}}{2e^{-2i}} = \frac{1}{2} e^{-2i} + \frac{1}{2} e^{-2i-1} > 1$$

for all $i \geq 0$, which means $e^{-2i+1} + e^{-2i-1} - 2e^{-2i} < 0$ for all $i \geq 0$. For instance,

$$T(\log_2 n) - T(\log_2 n - 1) \sim e^{-1/2} + e^{-2} - 2e^{-1} > 0.$$ 

This shows that if $q \leq \log_2 n$, then $T(q + 1) - T(q) > 0$.

\[\square\]

**Corollary 4.3.** The sequences $b(cn, n, q)$ is unimodal and reaches at its maximum at $q = \lfloor \log_{1+n} n \rfloor$ or $q = \lfloor \log_{1+n} n \rfloor + 1$.

We are expecting a combinatorial proof of our result. Recall $b(k, n, q)$ is the number of compositions of $k$ with $n$ parts such that the largest part equals $q$. Let $c > 1$ be a positive absolute constant.

**Conjecture 4.4.** Is there a combinatorial proof of the unimodality of $a(cn, n, q)$ on $q$? Equivalently, is there a combinatorial proof of the unimodality of $\binom{n,q}{cn} - \binom{n,q-1}{cn}$, where $\binom{n,q}{cn}$ is the coefficient of $x^cn$ in the polynomial $(1 + x + x^2 + \cdots + x^{q-1})^n$?

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