Hamiltonian formalism and spacetime symmetries in generic DSR models

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Abstract

We study the structure of the phase space of generic models of deformed special relativity which gives rise to a definition of velocity consistent with the deformed Lorentz symmetry. In this way we can also determine the laws of transformation of spacetime coordinates.

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1. Introduction

Recently, the idea that the symmetry group of spacetime at energy close to the Planck scale could be a deformation of the Poincaré group has been widely investigated [1-4]. This hypothesis is motivated by the observation that the Planck energy $\kappa$, whose role is essential in the formulation of the theories of quantum gravity, might be a fundamental constant of physics on the same ground as the speed of light, and should therefore be left invariant by the group of transformation of spacetime. This may be achieved by deforming the Poincaré group in such a way that its action on momentum space leaves the energy $\kappa$ invariant [3-4].

Unfortunately, this assumption is not sufficient to single out a unique deformation of the Poincaré group, even if one introduces further physical requirements, as for example the request that in the low energy limit the deformation tends to zero. One is lead therefore to define a large class of different models, usually called deformed (or doubly) special relativity (DSR) theories. The first example of these, obtained from purely algebraic investigations, was the $\kappa$-Poincaré group [1-2]. Later, different models were derived from physical arguments [3-4].

All these models are characterized by the property that the deformations are realized as a nonlinear action of the Lorentz group on the momentum space [4-5]. This definition however leaves the action of the Lorentz group on the coordinate space undetermined. It is evident that such action cannot be the same as in special relativity, and in particular cannot be independent of the momentum of the particle on which it is applied. A further complication is the possibility, suggested by the $\kappa$-Poincaré approach, that the geometry of spacetime be noncommutative [1-2].

From these considerations, it is also clear that the kinematics and the dynamics of point particles must be modified if one wants to obtain a picture consistent with the deformed spacetime symmetries. In particular, the definition of the velocity of a particle is problematic in DSR models [6-11]. In absence of a definite description of spacetime, the velocity of a particle must in fact be defined in terms of its momentum, but since the dispersion relations are deformed in DSR, several inequivalent prescriptions are possible. For example, if one adopts the naive definition $v_i = p_i/p_0$, the velocity of a particle depends on its mass [9], and the speed of light is energy dependent. The same problems arise if one defines the velocity as $v_i = \partial p_0/\partial p_i$, as proposed by some authors [3,12]. These drawbacks can be overcome if one requires that the velocity be a property of the reference frame rather than of a specific object and hence defines it in terms of boosts [10].

The expression for the velocity obtained in this way can be derived from a Hamiltonian description of the motion of free particles only by postulating noncanonical Poisson brackets [2,7-8]: in particular the Poisson brackets between space and time coordinates cannot vanish. This property can be interpreted as the classical counterpart of a noncommutative geometry. Although several specific examples are given in the literature [7-8,13-14], no general prescription is known for defining the Hamiltonian structure for generic DSR models.

Fixing the Hamiltonian structure is also useful for the determination of the transformation laws of coordinates. In [13], in fact, the transformation laws were derived from the requirement that the action functional be a scalar under deformed Lorentz transformations (DLT). Although the methods of [13] worked well for the Maguejo-Smolin model [4], they
led to inconsistencies in the case of the κ-Poincaré model of ref. [1]. In fact, the same definition of velocity can be obtained from several inequivalent Hamiltonian structures, but in general these do not lead to the correct transformation rules under DLT. One must therefore check that the velocity transforms in the correct way. This was not the case for the Hamiltonian structure of the κ-Poincaré model discussed in ref. [13].

In this paper, we try to extend the results of [13] to generic DSR models, giving the conditions that must be satisfied by the Poisson brackets in order to obtain the definition of velocity of ref. [10] obeying the correct transformation laws. Although in the general case these conditions are too difficult to be solved, we give some explicit examples where this can be done.

2. Hamilton equations

According to the approach of ref. [5], since the symmetry group of DSR theories is a nonlinear realization of the Lorentz group, there must exist a function \( \pi = \Phi(p) \), with inverse \( p = \Phi^{-1}(\pi) \), that maps the physical momentum \( p \) into an unphysical momentum \( \pi \) which transforms linearly under Lorentz transformations.

The action of a deformed Lorentz transformations on \( p \) will then be given by the composition

\[
p' = \Phi \circ \Lambda \circ \Phi(p),
\]

where \( \pi' = \Lambda(\pi) \) is the linear action of a Lorentz transformations on \( \pi \). The kinematical quantities of the physical theory transforming in the correct way under deformed Lorentz transformations should then be defined through this mapping. For example, this method has been used to obtain a consistent definition of the addition law for momenta [5].

Using this prescription, Kosinski and Maslanka [10] have shown that a definition of the velocity vector compatible with the group structure of the DLT is given by

\[
v = \frac{\pi_1}{\pi_0} = \frac{\Phi(p_1)}{\Phi(p_0)},
\]

i.e. the velocity of DSR must coincide with that defined in the standard way from the unphysical momentum \( \pi \). It follows that under DLT the velocity transforms as in special relativity. In particular, under a boost,

\[
\delta v = 1 - v^2.
\]

In [9] it was also argued that the definition (2) is the only one that satisfies the natural requirement that the velocity of a particle be independent of its mass. Moreover, it implies that the speed of light is energy independent and always equal to 1.

In order to write down a Hamiltonian formalism in which (2) arises as the natural definition of the velocity, \( v = \dot{q}_1/\dot{q}_0 \), we examine in more detail the theory. According to

\footnote{For simplicity, we work in two dimensions, but the results can be easily generalized. We denote 2-vectors indices by \( a, b... = 0, 1 \), and 1-vectors by bold letters. The signature is \((+, -)\), and the coordinates of a particle are denoted by \( q_a \).}
the previous considerations, we assume that the components of the unphysical momentum \( \pi_a (a = 0, i) \), satisfying the mass-shell constraint \( \pi_0^2 - \pi_i^2 = m^2 \), can be written in terms of the physical momentum \( p_a \) as

\[
\pi_0 = F(p_0, p_1^2), \quad \pi_1 = G(p_0, p_1^2)p_1, \tag{4}
\]

We write the inverse relations as

\[
p_0 = \bar{F}(\pi_0, \pi_1^2), \quad p_1 = \bar{G}(\pi_0, \pi_1^2)\pi_1. \tag{5}
\]

In this notation, the definition (2) of velocity is

\[
v = \frac{G(p_0, p_1)}{F(p_0, p_1)}p_1. \tag{6}
\]

In general, this expression of the velocity cannot be obtained from the Hamiltonian formalism with canonical Poisson brackets. Nevertheless, as we shall see, it can be recovered if one admits a more general symplectic structure [7-8].

It must also be remarked that the transformation laws for the momenta can be derived from (1). In fact,

\[
p'_a = W_a(p), \tag{7}
\]

where

\[
W_0 = \bar{F}(\pi'_0, \pi'_1) , \quad W_1 = \bar{G}(\pi'_0, \pi'_1). \tag{8}
\]

As usual, the Hamiltonian \( H \) for a free particle can be defined as the Casimir operator of the deformed algebra,

\[
H = \frac{m}{2} = \frac{1}{2m}(\pi_0^2 - \pi_1^2) = \frac{1}{2m}(F^2 - G^2 p_1^2). \tag{9}
\]

Then the velocity of a particle is by definition

\[
v = \frac{\dot{q}_1}{\dot{q}_0} = \frac{\omega_{10}\partial H/\partial p_0 + \omega_{11}\partial H/\partial p_1}{\omega_{00}\partial H/\partial p_0 + \omega_{01}\partial H/\partial p_1}, \tag{10}
\]

where \( \omega_{ab} = \{q_a, p_b\} \), and \( \dot{q}_a \equiv dq_a/d\tau \) is the derivative of the position coordinate \( q_a \) with respect to the variable \( \tau \) that parametrizes the trajectory. The second equality follows from the Hamilton equations. In the following, we shall assume that the \( \omega_{ab} \) are functions of the momenta, but not of the coordinates. We also postulate \( \{p_a, p_b\} = 0 \).

Equating the expressions (6) and (10) for \( v \), one can obtain an algebraic relation between the \( \omega_{ab} \). This relation is not sufficient to fix them uniquely. However, further constraints arise from the transformation law of the velocity. Consider first the infinitesimal transformation laws of the momenta arising from (7),

\[
\delta p_a \equiv \{J, p_a\} = w_a(p), \tag{11}
\]
where $J$ is the generator of the deformed boosts. Inserting (11) into the Jacobi identities

$$\{\{J, q_a\}, p_b\} + \{\{q_a, p_b\}, J\} + \{\{p_b, J\}, q_a\} = 0,$$

(12)

one can derive the infinitesimal transformation laws of the coordinates

$$\delta q_a \equiv \{J, q_a\} = u_{ab}(p) q_b,$$

(13)

where the functions $u_{ab}$ depend on $w_a, \omega_{ab}$ and their derivatives with respect to $p_a$.

Since $\dot{p}_a = 0$, deriving (13) with respect to $\tau$, one obtains that the $\dot{q}_a$ transform as the $q_a$. For consistency, the velocity defined by (10) must also transform as (3). Hence,

$$\delta \dot{\mathbf{v}} = \delta \left( \frac{\dot{q}_1}{\dot{q}_0} \right) = \frac{\dot{q}_0 u_{1a} \dot{q}_a - \dot{q}_1 u_{0a} \dot{q}_a}{\dot{q}_0^2} = 1 - \frac{\dot{q}_1^2}{\dot{q}_0^2},$$

(14)

and therefore the $u_{ab}$ must satisfy

$$u_{01} = u_{10} = 1 \quad u_{00} = u_{11} = f(p),$$

(15)

for some function $f(p)$. Substituting (15) into (12) and equating (6) and (10) one obtains a system of one algebraic and four partial differential equations for the five functions $\omega_{ab}$ and $f$.

After solving them, from the Jacobi identities

$$\{\{q_a, q_b\}, p_c\} + \text{perms.} = 0,$$

(16)

one can obtain the Poisson brackets between the coordinates. In general $\{q_0, q_1\} \neq 0$, indicating the necessity of a noncommutative geometry.

It is interesting to note that, if the conditions (15) hold, the line element $d\sigma^2 = dq_0^2 - dq_1^2$ transforms in a simple way, as

$$\delta(d\sigma^2) = 2f(p) d\sigma^2,$$

(17)

and is therefore possible to construct an invariant "metric" by multiplying $d\sigma^2$ by a suitable function of the momentum $p_a$.

Moreover, following [8], we notice that (6) and (10) imply that

$$\dot{q}_0 = \frac{A(p)}{m} F(p), \quad \dot{q}_1 = \frac{A(p)}{m} G(p) p_1,$$

(18)

for some function $A(p)$. In term of differentials,

$$dq_0 = \frac{A(p)}{m} F(p) d\tau, \quad dq_1 = \frac{A(p)}{m} G(p) p_1 d\tau,$$

(19)

and hence

$$dq_0^2 - dq_1^2 = A^2 \frac{(F^2 - G^2 p_1^2)}{m^2} d\tau^2 = A^2(p) d\tau^2.$$  

(20)
Consequently, the proper time $d\tau$ is given by the line element $d\sigma$ times a function of the momentum. It is easy to see that the proper time so defined must be invariant under DLT and can then be identified with the above defined "metric". It must also be remarked that since in general the variable $\sigma$ is different from the proper time $\tau$ that parametrizes the trajectories, the modulus of the 2-velocity is not unitary. A more detailed discussion of the interpretation of these results will be given elsewhere.

3. Examples

The conditions introduced in the previous section give rise to a system of partial differential equations for the $\omega_{ab}$, that in general is extremely difficult to solve. However, if $F$ and $G$ depend only on the energy $p_0$, it reduces to a system of ordinary differential equations, and there is some chance to obtain a solution. For example, this is possible in the case of the Maguejo-Smolin model, as discussed in [13]. We present here two other models where an explicit solution can be obtained, namely the Poincaré subalgebra of the deformed conformal algebra introduced by Herrantz in ref. [15], and the Heuson model of ref. [16].

a) The Herrantz model

This model is defined by the functions [15]

$$F = \kappa \left( e^{p_0/\kappa} - 1 \right), \quad G = 1,$$

that give for the velocity

$$v = \frac{p_1}{\kappa (e^{p_0/\kappa} - 1)}.$$  (22)

The transformation laws of the momentum under a boost of rapidity $\xi$ are

$$p'_0 = \kappa \log \Delta, \quad p'_1 = p_1 \cosh \xi + \kappa (e^{p_0/\kappa} - 1) \sinh \xi,$$

with

$$\Delta = 1 + (e^{p_0/\kappa} - 1) \cosh \xi + \frac{p_1}{\kappa} \sinh \xi.$$  (23)

In infinitesimal form,

$$\delta p_0 = p_1 e^{-p_0/\kappa}, \quad \delta p_1 = \kappa (e^{p_0/\kappa} - 1).$$  (24)

The Hamiltonian reads

$$H = \frac{1}{2m} \left[ \kappa^2 (e^{p_0/\kappa} - 1)^2 - p_1^2 \right],$$  (25)

and the conditions of the previous section are satisfied by the Poisson structure

$$\omega_{01} = \omega_{10} = 0, \quad \omega_{00} = 1, \quad \omega_{11} = -e^{p_0/\kappa}.$$  (26)

In view of (16), $\{ q_0, q_1 \} = q_1/\kappa$. From (25) and (26) follow the Hamilton equations

$$\dot{q}_0 = \frac{\kappa}{m} e^{p_0/\kappa} (e^{p_0/\kappa} - 1), \quad \dot{q}_1 = e^{p_0/\kappa} \frac{p_1}{m}.$$  (27)
from which one recovers (22).

Using (12) one can then obtain the explicit infinitesimal transformation laws for the coordinates
\[
\delta q_0 = q_1 + \frac{p_1}{\kappa} e^{-p_0/\kappa} q_0, \quad \delta q_1 = q_0 + \frac{p_1}{\kappa} e^{-p_0/\kappa} q_1.
\] (28)
The line element \(d\sigma^2 = dq_0^2 - dq_1^2\) transforms therefore as
\[
\delta (d\sigma^2) = 2\frac{p_1}{\kappa} e^{-p_0/\kappa} d\sigma^2,
\] (29)
and hence the form
\[
d\tau^2 = e^{-2p_0/\kappa} (dq_0^2 - dq_1^2)
\] (30)
is invariant under infinitesimal DLT. This is the proper time, as defined in (20).

Following ref. [13], it is also possible to find the finite form of the deformed transformation laws for the coordinates consistent with the Hamiltonian structure. It is known that the Hamilton equations for systems with nonstandard symplectic structure can be derived from an action principle [18]. Given a phase space with symplectic structure \(\{Q_A, Q_B\} = \Omega_{AB}\), where \(Q_A\) denotes either the coordinates or the momenta, one defines the functions \(R^A(Q_A)\) such that
\[
\frac{\partial R^A}{\partial Q_B} - \frac{\partial R^B}{\partial Q_A} = \Omega^{AB},
\] (31)
where \(\Omega^{AB}\) is the inverse of \(\Omega_{AB}\). The Hamilton equations can then be obtained varying with respect to \(Q_A\) the action
\[
I = \int (R^A \dot{Q}_A - H) d\tau.
\] (32)
Note that in general the action so defined contains derivatives of the momenta.

In our case, we define \(Q_1 = q_0, Q_2 = q_1, Q_3 = p_0, Q_4 = p_1\). Inverting \(\Omega_{AB}\), one finds for \(\Omega^{AB}\) the nonvanishing components \(\Omega^{13} = -\Omega^{31} = -1, \Omega^{24} = -\Omega^{42} = e^{-p_0/\kappa}\), and \(\Omega^{34} = -\Omega^{43} = q_1 e^{-p_0/\kappa}\). Solving (31), one has then
\[
R^1 = p_0, \quad R^2 = -p_1 e^{-p_0/\kappa}, \quad R^3 = -\frac{p_1 q_1}{\kappa} e^{-p_0/\kappa}, \quad R^4 = 0.
\]
Substituting in (32) and integrating by parts one obtains
\[
I = -\int \left[q_0 \dot{q}_0 - q_1 e^{-p_0/\kappa} \dot{q}_1 + H\right] d\tau,
\] (33)
and can identify the variables conjugated to the momenta \(p_a\) as
\[
r_0 = q_0, \quad r_1 = -q_1 e^{-p_0/\kappa}.
\] (34)
In order for the action to be invariant under DLT, the \( r_a \) must transform contravariantly, i.e. as

\[
r'_a = \Lambda_{ab}(p) \, r_b,
\]

where \( \Lambda_{ab} = (\partial W_b/\partial p_a)^{-1} \). From (34) and (35), it follows after some calculations that the coordinates transform as

\[
q'_0 = e^{-p_0/\kappa} \Delta(\cosh \xi \, q_0 + \sinh \xi \, q_1), \quad q'_1 = e^{-p_0/\kappa} \Delta(\sinh \xi \, q_0 + \cosh \xi \, q_1),
\]

i.e. as a Lorentz transformations times a momentum-dependent factor.

Differentiating (36), and recalling that \( \dot{p}_a = 0 \) by the field equations, one easily sees that the \( \dot{q}_a \) transform as the \( q_a \) and that the velocity \( \mathbf{v} \) transforms in the required fashion. Moreover, the ”metric” (30) is invariant also under finite boosts.

b) The Heuson model

This model was introduced in ref. [16] (see also [17]) and is analogous to that of ref. [4]. It is defined by

\[
F = \frac{p_0}{\sqrt{1 - p_0^2/\kappa^2}}, \quad G = \frac{1}{\sqrt{1 - p_0^2/\kappa^2}}.
\]

From (6) one gets the velocity

\[
\mathbf{v} = \frac{p_1}{p_0}.
\]

The transformation laws of the momentum under a boost of rapidity \( \xi \) are

\[
p'_0 = \frac{p_0 \cosh \xi + p_1 \sinh \xi}{\Gamma}, \quad p'_1 = \frac{p_0 \sinh \xi + p_1 \cosh \xi}{\Gamma},
\]

with \( \Gamma = \sqrt{1 - p_0^2/\kappa^2 + 1/\kappa^2 (p_0 \cosh \xi + p_1 \sinh \xi)^2} \).

In infinitesimal form,

\[
\delta p_0 = p_1 \left( 1 - \frac{p_0^2}{\kappa^2} \right), \quad \delta p_1 = p_0 \left( 1 - \frac{p_1^2}{\kappa^2} \right).
\]

The Hamiltonian reads

\[
H = \frac{1}{2m} \, \frac{p_0^2 - p_1^2}{1 - p_0^2/\kappa^2},
\]

and the conditions of consistency are solved by the functions

\[
\omega_{01} = -\frac{p_0 p_1}{\kappa^2}, \quad \omega_{10} = 0, \quad \omega_{00} = 1 - \frac{p_0^2}{\kappa^2}, \quad \omega_{11} = -1.
\]

From (41) and (42) follow the Hamilton equations

\[
\dot{q}_0 = \frac{p_0/m}{1 - p_0^2/\kappa^2}, \quad \dot{q}_1 = \frac{p_1/m}{1 - p_0^2/\kappa^2}.
\]
which yield the velocity (38). The Jacobi identities (16) imply a nontrivial Poisson bracket between space and time coordinates, \( \{q_0, q_1\} = p_0 p_1 / \kappa^2 \).

Using (12) one can deduce the infinitesimal transformation laws for the coordinates

\[
\delta q_0 = q_1 + \frac{p_0 p_1}{\kappa^2} q_0, \quad \delta q_1 = q_0 + \frac{p_0 p_1}{\kappa^2} q_1,
\]

and for the line element \( d\sigma^2 \),

\[
\delta (d\sigma^2) = 2 \frac{p_0 p_1}{\kappa^2} d\sigma^2.
\]

It follows that the form

\[
d\tau^2 = \left( 1 - \frac{p_0^2}{\kappa^2} \right) (dq_0^2 - dq_1^2)
\]

is invariant under infinitesimal DLT.

Proceeding as in the previous example, one can obtain also the finite transformations. The equations of motion can be derived by the action

\[
I = - \int \left[ \frac{q_0 + p_0 p_1 q_1 / \kappa^2}{1 - p_0 / \kappa^2} \dot{q}_0 - q_1 \dot{p}_1 + H \right] d\tau,
\]

and therefore the variables conjugated to the momenta \( p_a \) are

\[
q_0 = \frac{q_0 + p_0 p_1 q_1 / \kappa^2}{1 - p_0 / \kappa^2}, \quad q_1 = - q_1.
\]

From the transformation rules of the \( r_a \), one can then obtain the transformation laws for the coordinates

\[
q'_0 = \Gamma(\cosh \xi \ q_0 + \sinh \xi \ q_1), \quad q'_1 = \Gamma(\sinh \xi \ q_0 + \cosh \xi \ q_1).
\]

As in the previous example, they look like the Lorentz transformations except for a momentum-dependent factor. One can easily check that under (49) the velocity \( \mathbf{v} \) transforms in the correct way and the ”metric” (46) is invariant.

4. Conclusions

We have established the conditions that the Poisson structure of DSR models must satisfy so that the Hamilton equations yield an expression for the velocity of a particle which is consistent with the DLT, and have given two explicit examples of their application.

We have obtained one algebraic and four differential equations for five unknown functions, and this should be sufficient to fix uniquely the Poisson structure. However, a general algorithm to solve these conditions is not available. In particular, we have not been able to find a solution in the case of the \( \kappa \)-Poincaré model of ref. [1-2].

From the requirement of invariance of the action it is also possible to deduce the laws of transformation of the coordinates of a particle. Their main peculiarity is the dependence on the momentum of the particle, and this suggests that a consistent description of spacetime in DSR theories should involve the full phase space. In particular, the proper time, invariant under DLT, is given by the product of the line element \( d\sigma \) with a suitable function of the momentum.
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