Exactly solvable two-dimensional quantum spin models

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A method is proposed for constructing an exact ground-state wave function of a two-dimensional model with spin 1/2. The basis of the method is to represent the wave function by a product of fourth-rank spinors associated with the sites of a lattice and the metric spinors corresponding to bonds between nearest neighbor sites. The function so constructed is an exact wave function of a 14-parameter model. The special case of this model depending on one parameter is analyzed in detail. The ground state is always a nondegenerate singlet, and the spin correlation functions decay exponentially with distance. The method can be generalized for models with spin 1/2 to other types of lattices.

1. INTRODUCTION

There has been growing interest lately in quantum spin systems with frustrated interactions. Of special importance are models in this category for which it is possible to construct an exact ground state. The first example of such a representation is the well-known Majumdar-Ghosh model. It comprises a chain of spins 1/2 with antiferromagnetic interactions $J_1$ and $J_2$ of nearest neighbor and next-nearest neighbor spins, where $J_2 = J_1/2$. The ground state of this model is two-fold degenerate and consists of dimerized singlets; moreover, there is a gap in the spectrum of excitations. Another example of an exactly solvable model is the one-dimensional model with bilinear and biquadratic interactions and spin 1, investigated by Affleck, Kennedy, Lieb, and Tasaki (AKLT model). Its ground state has a structure of the type where each neighboring pair of spins has valence bonds. It is not degenerate, the spin correlation functions in the ground state decrease exponentially with distance, and there is a gap in the spectrum of excitations. This model therefore has properties predicted by Haldane for the one-dimensional Heisenberg antiferromagnetic model with spin 1. The valence-bond ground state is also exact for systems with many dimensions, but with spin $d/2$ ($d$ is the coordination number of the lattice). The one-dimensional AKLT model has subsequently been generalized and investigated in a number of papers, where it has also been shown that the wave function of the ground state can be represented by the trace of the product of matrices describing the spin states of sites of a chain (the “matrix” form). These two examples are characterized by the fact that the total Hamiltonian of the model is written as a sum of cell Hamiltonians (which are not mutually commuting), and the exact ground-state wave function of the total system is the eigenfunction having the lowest energy of each cell Hamiltonian.

We have previously investigated an exactly solvable, one-dimensional, frustrated model with spin 1/2, whose properties by and large are similar to those of the AKLT model. The ground-state wave function has a special recursion formula, and we have shown that it can be reduced to matrix form. It must be noted, however, that both the recursive form and the matrix form are essentially one-dimensional constructions and cannot be extended directly to higher-dimensional systems. We cite Ref. in this regard, where a method has been proposed for constructing an exact wave function of the ground state for models with spin 3/2 on a hexagonal lattice. The same method is applicable to other systems with spin $d/2$.

In this paper we consider a class of models with spin 1/2 for which the exact wave function of the ground state can be represented in an alternative form. In the one-dimensional case this wave function reduces to a wave function that we have found previously, but it admits generalization to higher-dimensional systems. The present study is devoted primarily to an analysis of the two-dimensional model.

The article is organized as follows. In Sec. we discuss the method of construction of the exact wave function for a one-dimensional model with $s = 1/2$. In Sec. we formulate an exactly solvable two-dimensional model. In Sec. we investigate the properties of this model with the aid of numerical calculations. In Sec. we discuss the possibility of generalizing our treatment to other types of lattices. The Appendix gives a proof of the nondegeneracy of the ground state of the two-dimensional model in the presence of cyclic boundary conditions.

2. ONE-DIMENSIONAL MODEL

We have previously investigated a one-dimensional, one-parameter model containing two spins 1/2 in the unit cell and admitting exchange interactions between nearest neighbor spins and spins separated by two sites of the lattice. The exact ground-state wave function of the cyclic chain can be written in the form
\[ \Psi_0 = \text{Tr} \left[ A(1, 2) A(3, 4) \ldots A(N - 1, N) \right], \]  

where \( A(2i - 1, 2i) \) is a 2 \times 2 matrix associated with the \( i \)th unit cell.

below we write the wave function \( \Psi_0 \) in a form more suitable for subsequent generalization to other types of lattices and give the general form of the Hamiltonian for which \( \Psi_0 \) is an exact wave function of the ground state.

We consider a chain of \( N = 2M \) spins 1/2. The wave function of this system is described by the \( N \)th-rank spinor

\[ \Psi = \Psi^{\lambda \mu \nu \ldots \tau}, \]  

where the indices \( \lambda, \mu, \nu, \ldots, \tau = 1, 2 \) correspond to different projections of the spin 1/2.

We partition the system into pairs of nearest neighbor spins. The wave function can then be written as the product of \( M \) second-rank spinors

\[ \Psi = \Psi^{\lambda}(1) \Psi^{\mu}(2) \ldots \Psi^{\tau}(M). \]  

We now form a scalar from Eq. (3), simplifying the latter with respect to index pairs:

\[ \Psi_s = \Psi^{\lambda}(1) \Psi^{\nu}(2) \ldots \Psi^{\sigma}(M). \]  

Here subscripts correspond to the covariant components of the spinor, which are related to the contravariant components (superscripts) through the metric spinor

\[ g_{\lambda \mu} = g^{\lambda \mu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]  

\[ \Psi_\lambda = g_{\lambda \mu} \Psi^\mu, \quad \Psi^\lambda = g^{\mu \lambda} \Psi_\mu. \]

The scalar function \( \Psi_0 \) can thus be written in the form

\[ \Psi_s = \Psi^{\lambda}(1) g_{\mu \nu} \Psi^{\nu}(2) g_{\rho \kappa} \ldots \Psi^{\sigma}(M) g_{\tau \lambda}. \]

The scalar function \( \Psi_s \) does not depend on the angle of rotation of the coordinate system and, hence, corresponds to the singlet state.

The second-rank spinor describing the pair of spins 1/2 can be written in the form

\[ \Psi^{\lambda \mu} = c_t \Psi_t^{\lambda \mu} + c_s \Psi_s^{\lambda \mu}, \]  

where \( \Psi_t^{\lambda \mu} \) and \( \Psi_s^{\lambda \mu} \) are symmetric and antisymmetric second-rank spinors, respectively, and \( c_t \) and \( c_s \) are arbitrary constants. We know that the symmetric second-rank spinor describes a system with spin 1, so that the pair of spins 1/2 in this case forms a triplet. If \( \Psi^{\lambda \mu} \) is an antisymmetric second-rank spinor reducible to a scalar multiplied by \( g_{\lambda \mu} \), the spin pair exists in the singlet state. Consequently, the ratio of the constants \( c_t \) and \( c_s \) determines the relative weights of the triplet and singlet components on the pair of spins \( s = 1/2 \) and is a parameter of the model.

In particular, for \( c_s = 0 \) the wave function \( \Psi_s \) contains only a triplet component, and for \( c_t = 0 \) it contains only a singlet component.

In general, we can make the ratio of the constants \( c_s/c_t \) different in different pairs, but to preserve translational symmetry, we confine the discussion to the case in which this ratio is the same in every pair.

We note that the wave function \( \Psi_s \) has the matrix form \( \Psi_0 \), the matrices \( A(2i - 1, 2i) \) representing a mixed second-rank tensor:

\[ A_{\lambda \nu}(1, 2) = \Psi^{\lambda \nu}(1) = c_t \left( \begin{array}{cc} \frac{1}{2}(\alpha_1 \beta_2 + \beta_1 \alpha_2) & \beta_1 \beta_2 \\ -\alpha_1 \alpha_2 & -\frac{1}{2}(\alpha_1 \beta_2 + \beta_1 \alpha_2) \end{array} \right) - \frac{1}{2}(\alpha_1 \beta_2 - \beta_1 \alpha_2) I, \]

where \( \alpha_i \) and \( \beta_i \) denote the up and down projections of the spin \( s_i \), respectively, and \( I \) is the unit matrix.

We consider a Hamiltonian \( H \) for which the wave function \( \Psi_s \) is an exact ground-state wave function. To do so, we consider the part of the system (cell) consisting of two nearest neighbor spin pairs. In the wave function \( \Psi_s \) the factor corresponding to the two spin pairs is a second-rank spinor:

\[ \Psi^{\lambda \mu}(i) g_{\mu \nu} \Psi^{\nu \rho}(i + 1). \]
In the general case, therefore, only two of the six multiplets forming two pairs of spin 1/2 — one singlet and one triplet — are present in the wave function \( \Psi \). Inasmuch as four spins 1/2 form two singlets and three triplets, the specific form of the singlet and triplet components present in the wave function \( \Psi \) depends on the ratio \( c_s/c_t \). The cell Hamiltonian acting in the spin space of nearest neighbor spin pairs can be written as the sum of the projectors onto the four missing multiplets with arbitrary positive coefficients \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \):

\[
H_{i,i+1} = \sum_{k=1}^4 \lambda_k P_{k}^{i,i+1},
\]

where \( P_{k}^{i,i+1} \) is the projector onto the missing multiplets in the corresponding cell Hamiltonian.

The wave function \( \Psi \) is now an exact wave function of the ground state of the cell Hamiltonian \( H_{i,i+1} \) with zero energy, because

\[
H_{i,i+1}|\Psi_s\rangle = 0,
\]

and \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are the excitation energies of the corresponding multiplets.

The total Hamiltonian of the entire system can be written as the sum of mutually noncommuting cell Hamiltonians:

\[
H = \sum_{i=1}^N H_{i,i+1},
\]

and since each term \( H_{i,i+1} \) in (13) yields zero in its action on \( \Psi_s \), we have

\[
H|\Psi_s\rangle = 0.
\]

The nondegeneracy of the ground state of this Hamiltonian has been rigorously proved.

Since the specific form of the existing and missing multiplets in the wave function \( \Psi \) on every two nearest neighbor spin pairs depends on the model parameter \( c_s/c_t \), the projectors in (11) also depend on \( c_s/c_t \). Each projector can be written in the form

\[
P_{k}^{1,2} = J_{12}^{(k)} \mathbf{s}_1 \cdot \mathbf{s}_2 + J_{13}^{(k)} \mathbf{s}_1 \cdot \mathbf{s}_3 + J_{14}^{(k)} \mathbf{s}_1 \cdot \mathbf{s}_4 + \lambda_1 \mathbf{s}_1 \cdot \mathbf{s}_4 + J_{23}^{(k)} \mathbf{s}_2 \cdot \mathbf{s}_3 + J_{14}^{(k)} \mathbf{s}_1 \cdot \mathbf{s}_2 \cdot \mathbf{s}_4 + J_{3}^{(k)} \mathbf{s}_3 + C^{(k)},
\]

and this representation is unique for a fixed value of the parameter \( c_s/c_t \).

Substituting the above expressions for the projectors into Eq. (13), we obtain the general form of the cell Hamiltonians \( H_{i,i+1} \). Inasmuch as the Hamiltonians \( H_{i,i+1} \) have exactly the same form for any \( i \), it suffices here to give the expression for \( H_{1,2} \):

\[
H_{1,2} = J_{12} \mathbf{s}_1 \cdot \mathbf{s}_2 + J_{13} \mathbf{s}_1 \cdot \mathbf{s}_3 + J_{14} \mathbf{s}_1 \cdot \mathbf{s}_4 + J_{23} \mathbf{s}_2 \cdot \mathbf{s}_3 + J_{3} \mathbf{s}_3 + C,
\]

where all volume integrals depend on the model parameter and the spectrum of excited states \( J_i = J_i(c_s/c_t, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \).

In particular, for \( c_s = 0 \), choosing \( \lambda_2 = \lambda_3 = \lambda_4 = 3 \) and \( \lambda_1/\lambda_2 = 3 \), we obtain an expression for \( H_{1,2} \) in the form

\[
H_{1,2} = \mathbf{L}_1 \cdot \mathbf{L}_2 + \frac{1}{3} (\mathbf{L}_1 \cdot \mathbf{L}_2)^2 + \frac{2}{3},
\]

where \( \mathbf{L}_1 = \mathbf{s}_1 + \mathbf{s}_2 \) and \( \mathbf{L}_2 = \mathbf{s}_3 + \mathbf{s}_4 \).

The Hamiltonian (17) has the form of the AKLT Hamiltonian, a result that is not too surprising, because for \( c_s = 0 \) two spins 1/2 in a pair effectively form spin 1. Note, however, that for \( c_s = 0 \) a set of different forms of the Hamiltonian \( H_{1,2} \) exists, corresponding to a different choice of coefficients \( \lambda_k \).

In general, the Hamiltonian (11) contains both bilinear and four-spin interactions. The latter can be excluded by setting \( J_1 = J_2 = J_3 = 0 \) and solving these equations for \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \). However, since the condition \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0 \), generally speaking, is not satisfied over the entire range of the parameter \( c_s/c_t \), the simplified Hamiltonian will also have a ground state described by the wave function (11) only in the region where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are positive. The nonzero exchange integrals \( J_{12}, J_{13}, J_{14}, J_{23} \) and the constant \( C \) depend only on the parameter \( c_s/c_t \). The explicit form of this dependence is given in Ref. [4] in which we have also calculated the ground-state spin correlation function \( \langle s_i s_j \rangle \), which decays exponentially with correlation length \( \sim 1 \).

We emphasize that the spin correlation functions \( \langle s_i s_j \rangle \) do not depend on the choice of \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) for a fixed parameter \( c_s/c_t \), because the ground-state wave function of the four-parameter set of Hamiltonians is the same.
3. TWO-DIMENSIONAL MODEL

We consider an $M \times M$-site square lattice with cyclic boundary conditions. We replace each site of the lattice by a square (Fig. 1) with spins $s = 1/2$ at its corners, making the total number of spins equal to $4M^2$. To avoid misunderstanding, however, from now on we continue to refer to these squares as sites. The wave function of the system is described by the product of fourth-rank spinors

$$\Psi = \prod_n \Psi_{\lambda\mu\nu\rho}^n(n).$$

(18)

By analogy with (13), from Eq. (18) we form the scalar

$$\Psi = \prod_n \psi_{\lambda\mu\nu\rho}^n(n)g_{\lambda\mu\nu\rho}g_{\lambda\lambda\nu\rho}g_{\lambda\mu\nu\rho}g_{\lambda\mu\nu\rho}.$$

(19)

where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors in the $x$ and $y$ directions.

The singlet wave function (19) is conveniently identified graphically with a square lattice, each site corresponding to a fourth-rank spinor $\Psi_{\lambda\mu\nu\rho}$ (whose form is identical for all sites), and each segment linking sites corresponds to a metric spinor $g_{\lambda\mu}$ (Fig. 2).

To completely define the wave function (19), it is necessary to know the form of the site spinor $\Psi_{\lambda\mu\nu\rho}$. For this purpose we classify an arbitrary fourth-rank spinor, simplifying and symmetrizing it with respect to different pairs of indices. We have the following types of spinors as a result:

1) a fourth-rank spinor $Q_{\lambda\mu\nu\rho}$ symmetric with respect to all indices;
2) three linearly independent products of a symmetric and an antisymmetric second-rank spinor: \( \varphi_{1}^{\lambda\mu} g_{\nu\rho}, \varphi_{2}^{\lambda\nu} g_{\mu\rho}, \) and \( \varphi_{3}^{\lambda\mu} g_{\nu\rho}; \)

3) two linearly independent products of two metric spinors and a scalar function: \( g_{\lambda\mu} g_{\nu\rho} \chi_1 \) and \( g_{\lambda\nu} g_{\mu\rho} \chi_2. \)

According to this classification, any fourth-rank spinor can be written in the form

\[
\Psi_{\lambda\mu\nu\rho} = c_1 Q_{\lambda\mu\nu\rho} + c_2 \varphi_{1}^{\lambda\mu} g_{\nu\rho} + c_3 \varphi_{2}^{\lambda\nu} g_{\mu\rho} + c_4 \varphi_{3}^{\lambda\mu} g_{\nu\rho} + c_5 g_{\lambda\nu} g_{\mu\rho} \chi_1 + c_6 g_{\lambda\mu} g_{\nu\rho} \chi_2. \tag{20}
\]

We note, however, that because the system of four spins \( 1/2 \) contains one quintet, three triplets, and two singlets, Eq. (20) still does not completely determine the form of \( \Psi_{\lambda\mu\nu\rho} \), and it is necessary to determine the specific form of the spinors \( \varphi_{1}^{\lambda\mu}, \varphi_{2}^{\lambda\nu}, \) and \( \varphi_{3}^{\lambda\mu} \) and the scalar functions \( \chi_1 \) and \( \chi_2. \)

Each symmetric second-rank spinor \( \varphi_{1}^{\lambda\mu} \) describes a triplet state of the system, representing a linear combination of the three basis triplet functions \( \varphi_{11}^{\lambda\mu}, \varphi_{12}^{\lambda\mu}, \) and \( \varphi_{13}^{\lambda\mu}. \) We can now specify nine linearly independent spinors describing triplet states of four spins \( s = 1/2: \)

\[
\varphi_{11}^{\lambda\mu} g_{\nu\rho}, \varphi_{12}^{\lambda\mu} g_{\nu\rho}, \varphi_{13}^{\lambda\mu} g_{\nu\rho},
\varphi_{11}^{\lambda\nu} g_{\mu\rho}, \varphi_{12}^{\lambda\nu} g_{\mu\rho}, \varphi_{13}^{\lambda\nu} g_{\mu\rho},
\varphi_{11}^{\lambda\mu} g_{\nu\rho}, \varphi_{12}^{\lambda\mu} g_{\nu\rho}, \varphi_{13}^{\lambda\mu} g_{\nu\rho}. \tag{21}
\]

The products of two metric spinors and a scalar function \( g_{\lambda\mu} g_{\nu\rho} \chi_1 \) and \( g_{\lambda\nu} g_{\mu\rho} \chi_2 \) describe singlets states of four spins \( s = 1/2. \) Since there are two independent singlet functions \( \chi_{s1} \) and \( \chi_{s2}, \) we have four linearly independent scalars describing singlet states of four spins \( s = 1/2: \)

\[
g_{\lambda\mu} g_{\nu\rho} \chi_{s1}, g_{\lambda\mu} g_{\nu\rho} \chi_{s2},
g_{\lambda\nu} g_{\mu\rho} \chi_{s1}, g_{\lambda\nu} g_{\mu\rho} \chi_{s2}. \tag{22}
\]

As a result, the specific form of the fourth-rank spinor \( \Psi_{\lambda\mu\nu\rho} \) [and, hence, the wave function \( \Psi \)] describing the system of four spins \( s = 1/2 \) is governed by \( 1 + 9 + 4 = 14 \) quantities, which are parameters of the model.

We now choose a Hamiltonian \( H \) for which the wave function \( \Psi \) is an exact ground-state wave function. As in the one-dimensional case, we seek the required Hamiltonian in the form of a sum of cell Hamiltonians acting in the space of two nearest neighbor spin quartets:

\[
H = \sum_n H_{n,n+a} + \sum_n H_{n,n+b}. \tag{23}
\]

The first term in Eq. (23) is the sum of the cell Hamiltonians in the horizontal direction, and the second term is the same for the vertical. The cell Hamiltonians along each direction have the same form, but the “horizontal” and “vertical” Hamiltonians differ in general. In the ensuing discussion, therefore, we consider only the Hamiltonians \( H_{1,2} \) and \( H_{1,3} \) (Fig. 3), which describe interactions of “sites” in the \( x \) and \( y \) directions, respectively.

For the wave function \( \Psi \) to be an exact eigenfunction of the Hamiltonian \( H, \) it is sufficient that the sixth-rank spinors

\[
\begin{align*}
\Psi_{\lambda\mu\nu\rho} &= c_1 Q_{\lambda\mu\nu\rho} + c_2 \varphi_{1}^{\lambda\mu} g_{\nu\rho} + c_3 \varphi_{2}^{\lambda\nu} g_{\mu\rho} + c_4 \varphi_{3}^{\lambda\mu} g_{\nu\rho} + c_5 g_{\lambda\nu} g_{\mu\rho} \chi_1 + c_6 g_{\lambda\mu} g_{\nu\rho} \chi_2. \\
&= \sum_n H_{n,n+a} + \sum_n H_{n,n+b}. \\
\end{align*}
\]
four multiplets with \( (19) \). The sixth-rank spinors \( (24) \) and \( (25) \) are symmetric with respect to two triplets of indices and, hence, contain

If we set

\[
\psi^{\lambda_1 \mu_1 \nu_1 \rho_1 (1)} \psi^{\lambda_2 \mu_2 \nu_2 \rho_2 (2)} g_{\nu_1 \lambda_2},
\]

\[
\psi^{\lambda_1 \mu_1 \nu_1 \rho_1 (1)} \psi^{\lambda_3 \mu_3 \nu_3 \rho_3 (3)} g_{\rho_1 \mu_3},
\]

be eigenfunctions of the corresponding cell Hamiltonians \( H_{1,2} \) and \( H_{1,3} \).

In general, when the site spinor \( \Psi^{\lambda \mu \nu \rho} \) is not symmetric with respect to any indices, the possible states of two quartets of spins \( s = 1/2 \) consist of 70 multiplets. A wave function represented by a sixth-rank spinor contains only 20 of them. Accordingly, the cell Hamiltonians \( H_{1,2} \) and \( H_{1,3} \) can be represented by the sum of projectors onto the 50 missing multiplets:

\[
H_{1,2} = \sum_{k=1}^{50} \lambda_k P_k^{1,2},
\]

\[
H_{1,3} = \sum_{k=1}^{50} \mu_k P_k^{1,3},
\]

where the positive constants \( \lambda_k \) and \( \mu_k \) are the excitation energies of \( H_{1,2} \) and \( H_{1,3} \), and the specific form of the projectors depends on 14 model parameters.

Inasmuch as

\[
H_{n,n+a} |\Psi_s\rangle = 0, \quad H_{n,n+b} |\Psi_s\rangle = 0,
\]

for the total Hamiltonian \( (23) \) we have the expression

\[
H |\Psi_s\rangle = 0.
\]

Consequently, \( \Psi_s \) is the ground-state wave function of the total Hamiltonian \( H \), because it is a sum of nonnegative definite cell Hamiltonians. Also, it can be rigorously proved (see the Appendix) that the ground state of \( H \) is nondegenerate.

As mentioned above, the specific form of the projectors depends on 14 model parameters, and in general the cell Hamiltonians \( (23) \), expressed in terms of scalar products of the type \( s_i \cdot s_j \), \( (s_i \cdot s_j)(s_k \cdot s_l) \), etc., have an extremely cumbersome form. We therefore consider a few special cases.

When the site spinor \( \Psi^{\lambda \mu \nu \rho} \) is a symmetric fourth-rank spinor \( Q^{\lambda \mu \nu \rho} \) (corresponding to the two-dimensional AKLT model \( (24) \)), only the quintet component out of the six multiplets on each spin quartet is present in the wave function \( (25) \). The sixth-rank spinors \( (24) \) and \( (25) \) are symmetric with respect to two triplets of indices and, hence, contain four multiplets with \( S = 0, 1, 2, 3 \) formed from two quintets. Consequently, the cell Hamiltonian \( (H_{1,2} \) and \( H_{1,3} \) coincide in this case) has the form

\[
H_{1,2} = \sum_{k=1}^{66} \lambda_k P_k^{1,2}.
\]

If we set \( \lambda_k = 1 \) \((k = 1, 66)\), we can write Eq. \( (29) \) in the form

\[
H_{1,2} = P_4(S_1 + S_2) + [1 - P_2(S_1) P_2(S_2)],
\]

where \( S_i \) is the total spin of the quartet of spins \( s = 1/2 \) on the \( i \)th site, \( S_i = s_i(1) + s_i(2) + s_i(3) + s_i(4) \), and \( P_i(S) \) is the projector onto the state with spin \( S = l \).

If the four spins \( s = 1/2 \) at each site are replaced by a single spin \( S = 2 \) and if the wave function \( (25) \) is treated as a wave function describing a system of \( M^4 = 2 \) spins \( S = 2 \), the second term in the Hamiltonian \( (30) \) vanishes, and we arrive at the Hamiltonian of the two-dimensional AKLT model:

\[
H_{1,2} = P_4(S_1 + S_2) = \frac{1}{28} S_1 \cdot S_2 + \frac{1}{40} (S_1 \cdot S_2)^2 + \frac{1}{180} (S_1 \cdot S_2)^3 + \frac{1}{2520} (S_1 \cdot S_2)^4.
\]

Another interesting special case is encountered when the system decomposes into independent one-dimensional chains. This happens if the site spinor \( \Psi^{\lambda \mu \nu \rho} \) reduces to a product of two second-rank spinors, each describing two spins 1/2. For example,

\[
\Psi^{\lambda \mu \nu \rho}(s_1, s_2, s_3, s_4) = \varphi^{\lambda \mu}(s_1, s_3) \varphi^{\nu \rho}(s_2, s_4).
\]
In this case the Hamiltonians $H_{1,2}$ and $H_{1,3}$ contain interactions of four rather than eight spins $1/2$ and have the form (14).

The simplest case is when the site spinor $\Psi^{\lambda\mu\nu\rho}$ is a product of four first-rank spinors:

$$
\Psi^{\lambda\mu\nu\rho}(s_1, s_2, s_3, s_4) = \varphi^\lambda(s_1)\varphi^\mu(s_2)\varphi^\nu(s_3)\varphi^\rho(s_4).
$$

(33)

Now the system decomposes into independent singlet pairs (Fig. 4), and the total Hamiltonian of the system has the form

$$
H = \sum_{i,j} \left( s_i \cdot s_j + \frac{3}{4} \right),
$$

(34)

where $s_i$ and $s_j$ are the spins forming the singlet pairs.

4. SPIN CORRELATION FUNCTIONS IN THE GROUND STATE

We now look at the problem of calculating the norm and the correlation function of the model described by the wave function (19). The expression for the norm of the wave function $G = \langle \Psi_s | \Psi_s \rangle$ has the form

$$
G = \prod_n \left( \Psi^\lambda_{\alpha\beta\gamma\delta}(n) \right) \Psi^\lambda_{\alpha\beta\gamma\delta}(n) g_{\alpha\beta\gamma\delta},
$$

(35)

where $R_{\alpha\beta\gamma\delta}$ is a $4 \times 4 \times 4 \times 4$ matrix.

According to the selection rules for the projection of the total spin $S_z$, only 70 of the 256 elements in the expression $\langle \Psi^\lambda_{\alpha\beta\gamma\delta}(n) | \Psi^\lambda_{\alpha\beta\gamma\delta}(n) \rangle$ are nonvanishing. Consequently, the matrix $R$ also contains at most 70 elements. If we regard the elements of $R$ as Boltzmann vertex weights, the problem of calculating the norm reduces to the classical 70-vertex model.

Since the exact solution for the 70-vertex model is unknown, numerical methods must be used to calculate the norm and the expected values.

To calculate the above-indicated expected values, we carry out Monte Carlo calculations on $20 \times 20$-site lattices. As mentioned, the ground-state wave function of the model depends on 14 parameters and, of course, cannot possibly be analyzed completely. We confine the numerical calculations to the case in which the spinor $\Psi^{\lambda\mu\nu\rho}$ depends on one parameter $\alpha$:

$$
\Psi^{\lambda\mu\nu\rho} = \cos \alpha \cdot Q^{\lambda\mu\nu\rho} + \sin \alpha \cdot \left( A^{\lambda\mu\nu\rho} - Q^{\lambda\mu\nu\rho} \right),
$$

(36)
where \( \alpha \in [-\pi/2; \pi/2]\), the spinor \( Q^{\lambda\mu\nu\rho} \) is symmetric with respect to all indices, and

\[
A^{\lambda\mu\nu\rho} = \varphi^\lambda(s_1)\varphi^\mu(s_2)\varphi^\nu(s_3)\varphi^\rho(s_4).
\]  

In this case we have a one-parameter model with two well-known limiting cases. One corresponds to \( \alpha = \pi/4 \), for which \( \Psi^{\lambda\mu\nu\rho} = A^{\lambda\mu\nu\rho} \), and the system decomposes into independent singlet pairs (Fig. 4); the other limiting case corresponds to \( \alpha = 0 \) (our model reduces to the two-dimensional AKLT model in this case, the spins at each site forming a quintet).

In the given model there are four spins \( s = 1/2 \) at each site, and the enumeration of each spin is determined by the order number of the lattice site to which it belongs and by its own number at this site. The spin correlation function therefore has the form

\[
f_{ij}(r) = \langle s_i(n) \cdot s_j(n+r) \rangle.
\]  

In determining the spin structure of the ground state, however, it is more practical to consider the more straightforward quantity \( F(r) \):

\[
F(r) = \sum_{i,j=1}^{4} \langle s_i(n) \cdot s_j(n+r) \rangle = \langle S(n) \cdot S(n+r) \rangle.
\]  

The function \( F(r) \) is left unchanged by a change of sign of \( \alpha \). This invariance is attributable to the fact that the spinor \( A^{\lambda\mu\nu\rho} - Q^{\lambda\mu\nu\rho} \) does not contain a quintet component, so that all the functions of this spinor are orthogonal to all functions of the symmetric spinor

\[
\langle Q^{\lambda'\mu'\nu'\rho'} \big| \big( A^{\lambda\mu\nu\rho} - Q^{\lambda\mu\nu\rho} \big) \rangle = 0.
\]  

for all \( \lambda, \mu, \nu, \rho \) and \( \lambda', \mu', \nu', \rho' \).

In addition, since the total spin operator \( S \) at a site commutes with \( S^2 = \sum_{i,j=1}^{4} s_i \cdot s_j \), we then have

\[
\langle Q^{\lambda'\mu'\nu'\rho'} \big| \sum_{i=1}^{4} s_i \big| \big( A^{\lambda\mu\nu\rho} - Q^{\lambda\mu\nu\rho} \big) \rangle = 0
\]  

It follows from Eqs. (35), (40), and (41) that \( \sin \alpha \) and \( \cos \alpha \) enter into the norm and into the expected value \( \langle \Psi|S(n) \cdot S(n+r)\rangle\Psi \) only in even powers, so that \( F(r) \) is invariant against a change of sign of \( \alpha \). We note, however, that only the total correlation function, and not \( f_{ij}(r) \), possesses symmetry under a change of sign of \( \alpha \). This assertion is evident, for example, in Fig. which shows the dependence of \( f_{31}(n) \) on \( \alpha \) as an illustration.

Figure 6 shows plots of \( F(r) \) for certain values of the parameter \( \alpha \). In every case it is found that the correlation function decays exponentially as \( r \) increases, differing from the one-dimensional model in that the preexponential factor...
also depends on $r$. Figure 7 shows the dependence of the correlation length $r_c$ on the parameter $\alpha$. The correlation length is a maximum at the point $\alpha = 0$ (two-dimensional AKLT model), decreases as $\alpha$ increases, and at $\alpha = \pi/4$, when the system decomposes into independent singlet pairs (Fig. 4), it is equal to zero. With a further increase in $\alpha$ the correlation length increases and attains a second maximum at $\alpha = \pi/2$. Like the correlation function $F(r)$, the function $r_c(\alpha)$ is symmetric with respect to $\alpha$. It is evident from Fig. 7 that the parameter $\alpha$ has two ranges corresponding to states with different symmetries. In the range $|\alpha| < \pi/4$ the correlation function $F(r)$ exhibits antiferromagnetic behavior:

$$F(r) \propto (-1)^{r_x + r_y} e^{-|r|/r_c}, \quad (42)$$

whereas the spins at one site are coupled ferromagnetically, $\langle s_i(n) \cdot s_j(n) \rangle > 0$. On the other hand, in the range $\pi/4 < |\alpha| < \pi/2$ the correlation function $F(r)$ is always negative:

$$F(r) \propto -e^{-|r|/r_c} \quad (43)$$

and all the correlation functions at one site are also negative (Fig. 8).

These ranges have two end points in common, $\alpha = \pm \pi/4$, where $r_c = 0$. Whereas $\alpha = \pi/4$ corresponds to the trivial partition of the system into independent singlet pairs, the case $\alpha = -\pi/4$ is more interesting. In this case we have

$$\Psi^{\lambda\mu\nu\rho} = 2Q^{\lambda\mu\nu\rho} - A^{\lambda\mu\nu\rho}, \quad (44)$$

and the matrix $\langle \Psi^{\lambda\mu\nu\rho}, \Psi^{\lambda\mu\nu\rho'} \rangle$, which enters into the equation for the norm (35) and the expected values, is transformed to
FIG. 8. Dependence of the spin correlation function at one site on the parameter $\alpha$. 

\[
\langle s_i(1) | s_j(1) \rangle = \begin{cases} 
4 \langle Q_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | Q_{\lambda \mu \nu \rho} \rangle & - 2 \langle A_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | Q_{\lambda \mu \nu \rho} \rangle \\
- 2 \langle Q_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | A_{\lambda \mu \nu \rho} \rangle & + \langle A_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | A_{\lambda \mu \nu \rho} \rangle.
\end{cases}
\] (45)

The symmetry of the spinor $Q_{\lambda \mu \nu \rho}$ with respect to all the indices leads to the relation

\[
\langle Q_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | Q_{\lambda \mu \nu \rho} \rangle = \langle Q_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | A_{\lambda \mu \nu \rho} \rangle = \langle Q_{\lambda \mu \nu \rho} | A_{\lambda \mu \nu \rho} \rangle.
\] (46)

Equation (45) therefore acquires the form

\[
\langle \Psi_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | \Psi_{\lambda \mu \nu \rho} \rangle = \langle A_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | A_{\lambda \mu \nu \rho} \rangle = \delta_{\lambda \lambda^\prime} \delta_{\mu \mu^\prime} \delta_{\nu \nu^\prime} \delta_{\rho \rho^\prime}.
\] (47)

From the equation for the norm (13) we have then

\[
G = \prod_n \langle \Psi_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | \Psi_{\lambda \mu \nu \rho} \rangle g_{\rho \mu \lambda^\prime n} g_{\rho^\prime \nu \lambda n} = 2^{2M^2}.
\] (48)

The latter equation has been derived with allowance for the relation $g_{\rho \lambda^\prime} g_{\rho^\prime \lambda} = \delta_{\lambda \lambda^\prime}$.

We now calculate the expected value $\langle \Psi | s_i(n) \cdot s_j(n + r) | \Psi \rangle$. If sites $n$ and $n + r$ are not nearest neighbors, $\langle \Psi | s_i(n) \cdot s_j(n + r) | \Psi \rangle$ decomposes into the product of the expected values

\[
\langle \Psi | s_i(n) \cdot s_j(n + r) | \Psi \rangle = 2^{2M^2 - 8} \langle \Psi_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | s_i(n) | \Psi_{\lambda \mu \nu \rho} \rangle \times \langle \Psi_{\lambda \nu^\prime \nu^\prime \rho^\prime} | s_j(n + r) | \Psi_{\lambda \nu^\prime \nu^\prime \rho^\prime} \rangle \times \delta_{\lambda \lambda^\prime} \delta_{\mu \mu^\prime} \delta_{\nu \nu^\prime} \delta_{\rho \rho^\prime} = 0.
\] (49)

Consequently, for $\alpha = -\pi/4$ all the correlation functions at non-nearest neighbor sites are equal to zero. But if sites $n$ and $n + r$ are nearest neighbors, the corresponding correlation function assumes the form

\[
\langle \Psi | s_i(1) \cdot s_j(2) | \Psi \rangle = 2^{2M^2 - 7} \langle \Psi_{\lambda^\prime \mu^\prime \nu^\prime \rho^\prime} | s_i(1) | \Psi_{\lambda \mu \nu \rho} \rangle \times \langle \Psi_{\lambda \nu^\prime \nu^\prime \rho^\prime} | s_j(2) | \Psi_{\lambda \nu^\prime \nu^\prime \rho^\prime} \rangle \times g_{\nu \lambda} g_{\nu^\prime \lambda^\prime} \delta_{\lambda \lambda^\prime} \delta_{\mu \mu^\prime} \delta_{\rho \rho^\prime} \delta_{\nu^\prime \nu^\prime} \delta_{\rho^\prime \rho^\prime}.
\] (50)
The exact calculation of the latter expression yields the following results (Fig. 3):

\[ \langle s_i(1) \cdot s_j(2) \rangle = -\frac{25}{768}, \quad i = 1, 2, 4, \quad j = 2, 3, 4, \]

\[ \langle s_i(1) \cdot s_1(2) \rangle = \langle s_3(1) \cdot s_j(2) \rangle = -\frac{15}{256}, \]

\[ \langle s_3(1) \cdot s_1(2) \rangle = -\frac{27}{256}. \]

It follows from Eqs. (51) that

\[ \sum_{i,j=1}^{4} \langle s_i(1) \cdot s_j(2) \rangle = -\frac{3}{4}, \]

as in the case of independent singlets \((\alpha = \pi/4)\). It can also be shown that all the correlations functions at one site are equal to zero.

To write the cell Hamiltonian \(H_{1,2}\) in explicit form for \(\alpha = -\pi/4\), we introduce the notation

\[
\begin{align*}
\{ l_1 &= s_1(1) + s_2(1) + s_4(1), \\
\{ l_2 &= s_2(2) + s_3(2) + s_4(2), \\
\{ h_1 &= l_1 \cdot s_1 + l_2 \cdot s_2, \\
\{ h_2 &= l_1 \cdot s_2 + l_2 \cdot s_1. \\
\end{align*}
\]

(52)

Accordingly, choosing \(\lambda_k = 1\) \((k = 1, 50)\), we can write the cell Hamiltonian \(H_{1,2}\) in Eq. (53) in the form

\[
H_{1,2} = P_{1/2}(l_1)P_{1/2}(l_2)P_{3/2}(s_1 + s_2) + P_{3/2}(l_1)P_{3/2}(l_2)h_3 \\
+ P_{3/2}(l_1)P_{1/2}(l_2)h_4 + P_{1/2}(l_1)P_{3/2}(l_2)h_5,
\]

(53)

where

\[
\begin{align*}
h_3 &= \frac{207}{256} + \frac{49}{64}s_1 \cdot s_2 + \frac{3}{64}l_1 \cdot l_2 + \frac{1}{16}(s_1 \cdot s_2)(l_1 \cdot l_2) - \frac{15}{64}h_2 - \frac{1}{32}h_2^2 \\
+ \frac{3}{64}h_1(l_1 \cdot l_2) + 4h_2^2(l_1 \cdot l_2)^2 + \text{H.c.}, \\
h_4 &= \frac{3}{4} - \frac{3}{8}s_1 \cdot s_2 + \frac{1}{4}l_1 \cdot l_2 + \frac{1}{4}(l_1 \cdot s_1)(s_2 \cdot l_2) + \text{H.c.}, \\
h_5 &= \frac{3}{4} - \frac{3}{8}s_1 \cdot s_2 + \frac{1}{4}l_2 \cdot s_1 + \frac{1}{4}(l_2 \cdot s_2)(l_1 \cdot s_1) + \text{H.c.}.
\end{align*}
\]

(54)

The cell Hamiltonian \(H_{1,3}\) has the same form (53) but with a change of notation according to Fig. 3

\[
\begin{align*}
\{ l_1 &= s_1(1) + s_2(1) + s_3(1), \\
\{ l_2 &= s_1(3) + s_3(3) + s_4(3), \\
\{ h_1 &= l_1 \cdot s_1 + l_2 \cdot s_2, \\
\{ h_2 &= l_1 \cdot s_2 + l_2 \cdot s_1. \\
\end{align*}
\]

(55)

Of special interest is the case corresponding to \(\alpha = \pm \pi/2\). Unfortunately, exact expressions for the correlation function cannot be obtained in this case, but the Hamiltonian can be written in explicit form. Since the site spinor \(\psi^{\alpha \mu \nu \tau}\) does not contain a quintet component for \(\alpha = \pm \pi/2\), the wave function of two nearest neighbor sites \(\psi^{\alpha \mu \nu \tau}\) will lack a component with \(S = 3\). A more detailed analysis shows that 19 multiplets are present in the wave function of two nearest neighbor sites. In this case, therefore, the cell Hamiltonian has the general form

\[
H_{1,2} = \sum_{k=1}^{51} \lambda_k P_k^{1/2}. 
\]

(56)

For a definite choice of \(\lambda_k\) in Eq. (56) the cell Hamiltonian assumes the form

\[
H_{1,2} = P_2(l_1 + s_1) + P_2(l_2 + s_2) + P_{1/2}(l_1)P_{1/2}(l_2)P_1(s_1 + s_2) \\
+ P_{3/2}(l_1)P_{1/2}(l_2)P_{1/2}(l_1 + s_1 + s_2) + P_{3/2}(l_2)P_{3/2}(l_1)P_{1/2}(l_2 + s_2 + s_1), 
\]

(57)
where the notations (52) and (55) are used for nearest neighbor sites along the horizontal and along the vertical, respectively.

Our results suggest that the spin correlation functions decay exponentially with a correlation length $\sim 1$ for an arbitrary parameter $\alpha$. We also assume that the decay of the correlation function is of the exponential type for the 14-parameter model as well, i.e., for any choice of site spinor $\Psi^{\lambda \mu \rho \sigma}$. This assumption is supported in special cases: 1) the partition of the system into one-dimensional chains with exactly known exponentially decaying correlation functions; 2) the two-dimensional AKLT model, for which the exponential character of the decay of the correlation function has been rigorously proved. Further evidence of the stated assumption lies in the numerical results obtained for various values of the parameter in the one-parameter model.

5. GENERALIZATION OF THE MODEL TO OTHER TYPES OF LATTICES

The wave function (7), (19) can be generalized to any type of lattice. The general principle of wave function construction for a system of spins 1/2 entails the following:

1) Each bond on a given lattice has associated with it two indices running through the values 1 and 2, one at each end of the bond.
2) Each bond has associated with it a metric spinor $g_{\lambda \mu}$ with the indices of the ends of this bond.
3) Each site of the lattice (a site being interpreted here, of course, in the same sense as in Sec. 3) with $m$ outgoing bonds has associated with it an $m$th-rank spinor with the indices of the bonds adjacent to the site.
4) The wave function is the product of all spinors at sites of the lattice and all metric spinors.

It is obvious that each index in the formulated wave function is encountered twice, so that the wave function is scalar and, hence, singlet.

The wave function so constructed describes a system in which each lattice site contains as many spins $s = 1/2$ as the number of bonds emanating from it.

To completely define the wave function, it is necessary to determine the specific form of all site spinors. The coefficients that determine their form are then parameters of the model.

The Hamiltonian of such a model is the sum of the cell Hamiltonians acting in the spin space of the subsystem formed by the spins at two mutually coupled sites:

$$H = \sum_{\langle ij \rangle} H_{ij}.$$ (58)

Each cell Hamiltonian is the sum of the projectors with arbitrary positive coefficients onto all multiplets possible in the corresponding two-site subsystem except those present in the constructed wave function:

$$H_{i,j} = \sum_k \lambda_k P_k^{i,j}.$$ (59)

Then $H_{i,j} |\Psi_s\rangle = 0$ and, accordingly, $H |\Psi_s\rangle = 0$.

Consequently, $\Psi_s$ is an exact ground-state wave function.

We note that any two lattice sites can be joined by two, three, or more bonds, because this does not contradict the principle of construction of the wave function. Moreover, the general principle of construction of the wave function is valid not only for translationally symmetric lattices, but for any graph in general. As an example, let us consider the system shown in Fig. 9. The wave function of this system has the form

$$\Psi_s = \Psi^{\lambda_1 (1)} \Psi^{\lambda_2 \mu_1 \nu_1 \rho_1 (2)} \Psi^{\mu_2 \nu_2 \tau_1 (3)} \Psi^{\mu_2 \tau_2 (4)} g_{\lambda_1 \lambda_2} g_{\mu_1 \mu_2} g_{\nu_1 \nu_2} g_{\rho_1 \rho_2} g_{\tau_1 \tau_2}$$ (60)

and describes a system containing ten spins 1/2.

If the given lattice has dangling bonds (as occurs for systems with open boundary conditions), the resulting wave function represents a spinor of rank equal to the number of loose ends. The ground state of this kind of system is therefore $2^l$-fold degenerate (where $l$ is the number of loose ends). For an open one-dimensional chain, for example, the ground state corresponds to four functions — one singlet and three triplet components. For higher-dimensional lattices this degeneracy depends on the size of the lattice and increases exponentially as its boundaries grow.
6. CONCLUSION

We have proposed a method for the construction of an exact wave function for a class of two-dimensional spin models. In general this model depends on 14 parameters, and its Hamiltonian is written as the sum of the Hamiltonians of nearest neighbor spin quartets. The exact ground-state wave function of the total system is also the exact wave function of each cell Hamiltonian. Since 20 of the 70 multiplets of two nearest neighbor quartets are present in the exact wave function, the cell Hamiltonians are the sums of the projectors with positive coefficients onto the other 50 multiplets. These coefficients are the excitation energies of the corresponding multiplets. Different values of the coefficients correspond to different Hamiltonians. In this case, however, the ground-state wave function itself and the spin correlation functions in the ground state are identical for all Hamiltonians. This means that the ground-state wave function, as defined by Shastry and Sutherland, is superstable.

We have carried out Monte Carlo calculations of the spin correlation functions in the ground state for the special case of a model that depends on one parameter. For all values of the parameter the spin correlation functions decay exponentially with distance despite the complicated dependence of the correlation functions of nearest neighbor spins on the model parameter. It is justifiable to expect the spin correlations to decay exponentially in the general 14-parameter model as well.

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APPENDIX

In Sec. 3 we have constructed the singlet wave function

$$\Psi_s = \prod_n \Psi^{\lambda_n\mu_n\nu_n\rho_n}(n) G^{\lambda_n\lambda_{n+4}\mu_n\mu_{n+4}}$$

(A.1)

for a system of 4M^2 spins s = 1/2 on a square lattice. The following Hamiltonian was specially chosen for the resulting wave function:

$$H = \sum_n H_{n,n+4} + \sum_n H_{n,n+4}$$

(A.2)

for which the wave function (A.1) is the zero-energy ground state:

$$H|\Psi_s\rangle = 0.$$  

(A.3)

We now show that the ground state of the system is nondegenerate, i.e., the wave function satisfying Eq. (A.3) is unique.
Inasmuch as the Hamiltonian (A.2) is a sum of nonnegative definite cell Hamiltonians, any function satisfying Eq. (A.3) must satisfy all the cell equations

\[ H_{i,j}\ket{\Psi} = 0. \]  

(A.4)

This means that Eqs. (A.3) and (A.4) are equivalent.

We prove the nondegeneracy of the ground state of the Hamiltonian (A.2) as follows. We first write the general form of the wave function for the system in question. We then determine the general form of the wave function (A.5) to obtain the general form of the wave function satisfying Eq. (A.6):

\[
\Psi = \sum_{\lambda \mu \nu} c(\lambda \mu \nu) \cdot \prod_j \Phi^{\lambda_i \mu_i \nu_i \rho_i}(j),
\]

(A.5)

where the summation is over the 4M^2 indices \( \lambda_i, \mu_i, \nu_i, \rho_i \), \( c(\lambda \mu \nu) \) denotes coefficients that depend on these indices, and \( \Phi^{\lambda_i \mu_i \nu_i \rho_i}(j) \) are arbitrary fourth-rank site spinors (in general, spinors at different sites can differ).

We require that the wave function (A.5) obey the cell equation

\[ H_{n,n+a}\ket{\Psi} = 0. \]  

(A.6)

By the construction of the singlet wave function (A.3), which is matched by the cell Hamiltonian \( H_{n,n+a} \), any wave function at sites \( n \) and \( n + a \) that satisfies condition (A.6) is a linear combination of the 64 functions contained in the expression

\[
\Psi^{\lambda_n \mu_n \nu_n \rho_n}(n)\Psi^{\lambda_{n+a} \mu_{n+a} \nu_{n+a} \rho_{n+a}}(n + a) g_{\nu_n \lambda_{n+a}},
\]

(A.7)

because the cell Hamiltonian \( H_{n,n+a} \) by definition is the sum of the projectors onto all multiplets \( \ket{\Psi^{\lambda_n \mu_n \nu_n \rho_n}(n)} \) and \( \ket{\Psi^{\lambda_{n+a} \mu_{n+a} \nu_{n+a} \rho_{n+a}}(n + a)} \) are definite site spinors occurring in the wave function (A.1). We note that these 64 functions can be linearly dependent (as is the case, for example, for the two-dimensional AKLT model).

Thus, the general form of the wave function satisfying Eq. (A.6) can be written

\[ \Psi = \sum_{\lambda \mu \nu} c(\lambda \mu \nu) \cdot \prod_{j, j \neq n,n+a} \Phi^{\lambda_i \mu_i \nu_i \rho_i}(j), \]

(A.8)

where \( c(\lambda \mu \nu) \) are coefficients that depend on the indices \( \lambda_i, \mu_i, \nu_i, \rho_i \) exclusive of the indices \( \nu_n \) and \( \lambda_{n+a} \), and \( \Phi^{\lambda_i \mu_i \nu_i \rho_i}(j) \) are arbitrary site spinors.

Comparing the functions (A.5) and (A.8), we deduce the following conditions that must be met by the function (A.5) to obtain the general form of the wave function satisfying Eq. (A.6):

1. The spinors at sites \( n \) and \( n + a \) must coincide with the site spinors of the wave function (A.1):

\[
\Phi^{\lambda_n \mu_n \nu_n \rho_n}(n) = \Psi^{\lambda_n \mu_n \nu_n \rho_n}(n),
\]

(A.9)

2. The coefficients \( c(\lambda \mu \nu) \) have the form

\[
c(\lambda \mu \nu) = c(\lambda \mu \nu)\cdot g_{\nu_n \lambda_{n+a}}.
\]

(A.10)

From the equation

\[ H_{n,n+b}\ket{\Psi} = 0 \]

(A.11)

we deduce analogous conditions on the general form of the wave function (A.5):

\[
\Phi^{\lambda_n \mu_n \nu_n \rho_n}(n) = \Psi^{\lambda_n \mu_n \nu_n \rho_n}(n),
\]

(A.12)

\[
\Phi^{\lambda_{n+b} \mu_{n+b} \nu_{n+b} \rho_{n+b}}(n + b) = \Psi^{\lambda_{n+b} \mu_{n+b} \nu_{n+b} \rho_{n+b}}(n + b),
\]

\[
c(\lambda \mu \nu) = c(\lambda \mu \nu)\cdot g_{\nu_n \lambda_{n+b}}.
\]
The simultaneous satisfaction of all the cell equations (A.4) requires consolidation of the conditions imposed by these equations on the general form of the wave function (A.5). Combining these conditions in succession, in each step we obtain the general form of a wave function satisfying the equations corresponding to these conditions. Upon satisfying all the conditions, we obtain the general form of the wave function satisfying all the cell equations (A.4) and, hence, satisfying Eq. (A.3):

\[
\Psi_s = \sum_{\lambda\mu\nu\rho} c(\lambda\mu\nu\rho|\lambda\mu\nu\rho) \prod_j \Psi^{\lambda_j\mu_j\nu_j\rho_j}(j) g_{\nu_j\lambda_j} g_{\rho_j\mu_{j-1}},
\]  

(A.13)

where \(c(\lambda\mu\nu\rho|\lambda\mu\nu\rho) = c\) is a constant.

Comparing the wave functions (A.1) and (A.13), we readily perceive that, to within an arbitrary factor, the general form of the wave function satisfying Eq. (A.3) coincides with the wave function \(\Psi_s\). Consequently, \(\Psi_s\) is the nondegenerate ground-state wave function of the Hamiltonian (A.2).