Singular surface points of steady-state traveling solutions of mathematical models of a falling fluid film

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Abstract. The solutions to a divergent system of equations were studied numerically for a freely flowing fluid film. Various families of steady-state traveling solutions in a wide range of wave numbers were found. The shapes of the fluid film surface corresponding to these solutions are given. Specific ranges of parameter values, where re-closing of different families occurs, were determined. It is shown that they are the singular points of the saddle type.

1. Introduction

Falling film evaporators and crystallizers are especially popular in the chemical and food industry. Heat exchange units based on the phenomenon of laminar film condensation are widely used in refrigeration technology [1-3]. Theoretical foundations of the problem of heat transfer between a plane inclined wall covered by a thin layer of viscous fluid and surrounding gas were laid by Nusselt [4] in 1916. The Nusselt theory takes into account the condensation and evaporation processes and is known to significantly underestimate the experimental data on heat transfer rates due to the presence of waves on the film surface. A series of pioneering works of Kapitza and Kapitza [5] describes numerous wave regimes of a laminar film flow, earlier considered to be trivial. The authors also attempt to explain the additional wave-induced heat transfer rate. Although their simple model provides only a 20% increase in the corresponding coefficient in contrast to its almost 100% increase in experiments. The first realistic theory of wave influence on the heat transfer of evaporating and condensing film is proposed in [3]. Two different wave regimes are investigated: a quasi-sinusoidal high frequency mode and the “intermediate waves.” The latter is characterized by large humps, separated by rather long relatively plane thin liquid layers. In this case, authors discovered a significant influence of waves on thermal processes. More recent developments usually consider the heat problem to be inseparable from an established hydrodynamic wave regime [6].

Observations of waves in natural conditions mostly confirm their irregular behavior, determined by a large number of steady modes and complex transient regimes. It was shown [7] that even in the case of small Reynolds number (flow rate) periodic solutions of a model equation (Kuramoto-Sivashinsky) demonstrate a chaotic dynamics. To classify and describe the data obtained in both mathematical modeling and experiments, the topological structure of steady-state traveling solutions of the model equation was intensively investigated in [8, 9]. Tsvelodub and Trifonov [10] extended these results to the region of moderate Reynolds numbers and provided new data on stability of the found regimes. In a series of experimental works by Liu and Gollub [10,11] the basic mechanisms of onset and
development of spatio-temporal chaos in wavy film flows were experimentally found. Among them the most significant was the subharmonic instability leading to wavelength doubling. This mechanism is very sensitive to the noise and, therefore, has an irregular character. However, the successive period doubling is not typical for this scenario, and the subharmonic spectral peaks usually do not become dominant [12].

2. Problem statement and governing equations

Governing equations of fluid dynamics were written in new coordinates in the following dimensionless form [13]:

\[
\begin{align*}
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{h} \right) + \frac{\partial}{\partial \eta} \left( \frac{QV}{h} \right) = & \frac{1}{\varepsilon \text{Re}^2} \frac{\partial^2 Q}{\partial \eta^2} + \frac{3h}{\varepsilon \text{Re}} + \frac{18}{5} h \frac{\partial^3 h}{\partial x^3} \\
\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} + \frac{\partial V}{\partial \eta} = & 0
\end{align*}
\]

With boundary conditions:

\[
Q = V = 0 \text{ at } \eta = -1 \\
\frac{\partial Q}{\partial \eta} = 0, V = 0, \text{ at } \eta = 0.
\]

Here, \( h(x,t) \) is the dimensionless film thickness, \( \eta = y/h(x,t)-1 \) is the transformed transverse coordinate, \( Q = hu \), \( V = hv \), where \( u \) and \( v \) are the dimensionless longitudinal and transverse contravariant components of velocity vector, \( \text{Re} = g h_0^2 / 3 v^2 \) is the Reynolds number, and \( \varepsilon = h_0 / l_0 \) is the wavelength parameter. In derived equations, perturbations are assumed to have long wavelengths \( \varepsilon \ll 1 \), and liquid flow rates are supposed to be moderate \( \varepsilon \text{Re} \sim 1 \). The selected characteristic scales are: the average thickness \( h_0 \) and the superficial velocity of a waveless Nusselt flow \( u_0 = g h_0^3 / 3 v \). The scales of the length \( l_0 \) and time \( l_0 / u_0 \) are determined from the relationship:

\[
\frac{h_0}{l_0} = \frac{\varepsilon}{\sqrt{\frac{18 \text{Re}^{5/3}}{5 \sqrt[3]{3 \sqrt[3]{3} \varepsilon} \text{Fi}^{1/3}}}}
\]

where \( \text{Fi} = \sigma^3 / \rho^3 g v^4 \) is the film number, characterizing liquid properties. As it is shown in [14], this choice provides the value of neutral wave number \( \alpha_n \approx 1 \).

The current study considers periodic steady-state traveling perturbations of the free surface, therefore, solutions to the system are presented in the following form:

\[
[Q,h,V] = [Q,h,V](x',\eta), \quad x' = \alpha (x - ct).
\]

Here, \( \alpha \) is the wave number of solution, and \( c \) is the phase velocity.

The equations (1) are solved by the pseudospectral (collocation) method, described in detail in [14]. Solutions to problem (1)–(2) are expanded in a series on Chebyshev polynomials \( T_n \):

\[
\begin{align*}
Q(x,\eta) = & \sum Q_{2j}(x) (T_{2j}(\eta) - 1) \\
V(x,\eta) = & \sum V_{2j+1}(x) (T_{2j+1}(\eta) - \eta)
\end{align*}
\]

Representation (3) satisfies automatically the boundary conditions on the free surface and on the solid wall. At that, the coefficients dependent on the longitudinal coordinate were expanded into the Fourier series.

3. Results

Parameter \( \varepsilon \) was chosen in such a way that the wave number of neutral perturbations for any Reynolds number was \( \alpha_n = 1 \). At this point, the first family of nonlinear steady-state traveling periodic regimes arises. This family can be continuously extended to the area of small wave numbers. In this area, a
region, where the shape of solution $h(x)$ varies rapidly, is identified at the wavelength together with the region, where this function is almost constant. This situation takes place for other families of the steady-state traveling periodic regimes. We will conditionally call such regimes “quasi-solitons”. Depending on what is larger in magnitude, the value of the maximum or minimum, these quasi-solitons are often called positive or negative, respectively. According to this classification, in the region of small wave numbers, the shape of the film surface for the waves of the first family is a negative quasi-soliton (quasisoliton-dent).

**Figure 1** – Dependence of amplitude of the first harmonic on the wave number. $Re = 5$.
Red curve – the first family; black curve – the second family; blue curve – the third family; green curve – the fourth family.

Investigation of stability of solutions of this family allows us to find the bifurcation points, where new families of steady-state traveling solutions arise. As an example, the dependence of the amplitude modulus of the first harmonic on wave number $\alpha$ is shown in figure 1 for four families. Here, the Reynolds number is $Re = 5$. The red curve represents the first family branching-off of the unperturbed flow with a flat free boundary at wave number $\alpha = 1$. Three other families of solutions branch off from the first family at points: $\alpha = 0.4947$ (the second family, the black curve); $\alpha = 0.3307$ (the third family, the blue curve); and $\alpha = 0.3897$ (the fourth family, the green curve). Examples of the film surface profiles for these families at one wavelength ($\lambda = 2\pi/\alpha$) are presented in figure 2; the color of the profiles coincides with the color corresponding to the family in figure 1. Here, the value of the wave number is $\alpha = 0.15$. In this example, the solution of the second family (see figure 1) is a positive “quasi-soliton” (quasi-soliton-elevation or hump). For the remaining three families at this value of $Re$, the solutions are the negative “quasi-solitons” of various shapes.
With small changes in Reynolds numbers, the structure of these families does not change qualitatively, in particular, for small wave numbers $\alpha$, they have the form of “quasi-soliton” solutions of the same type as in figure 2 (hereinafter, the prefix “quasi” will be omitted, as it is often done in other works). With an increase in Reynolds number, the pattern of the wave regimes in the vicinity of certain critical points varies radically. For instance, such a singular point is found at the Reynolds number $Re=9.005$. In its vicinity, significant changes take place in the structure of these families. Thus, in figure 3 for $Re = 9.0$, the second family branches from the solution of the first family at point $\alpha = 0.477$, and for small wave numbers $\alpha$ of solutions, it still has the form of a positive soliton. Solutions of the third family, arising from the first one at point $\alpha = 0.32$, as in the case of $Re = 5$, have the form of a two-humped negative soliton (soliton with two dents). With a slight change in the Reynolds number to $Re = 9.01$, significant changes occur in these families (see figure 4). Now, the two-humped negative soliton belongs to the second family, and for the third family the positive single-hump soliton becomes the limiting one. As it can be seen from these two figures, in the given range of Reynolds numbers, there is re-closing of these families.

Figure 2 – Profiles of the film surface for the soliton solutions at Reynolds number $Re=5$. Wave number $\alpha = 0.15$
Figure 3 – Dependence of the amplitude of the first harmonic on the wave number at Reynolds number $Re = 9.0$. Black curve – the second family; blue curve – the third family.

Figure 4 – Dependence of the amplitude of the first harmonic on the wave number at Reynolds number $Re = 9.01$. Black curve – the second family; blue curve – the third family.

The picture of this re-closing in plane $(\alpha, c)$ is shown in figure 5. Here, a vicinity of the critical point is presented. Two curves representing the second and the third families correspond to each value of the Reynolds number. It is seen that they form a saddle with the critical point coordinates $(\alpha = 0.352, \ c = 2.963)$.

Figure 5 – Dependence of phase velocity on wave number in a vicinity of Reynolds number $Re=9.005$

With a further increase in the Reynolds number, the topological structure of families of steady-state traveling solutions continues becoming more complicated. For example, the family of solutions, which have the form of a single-hump positive soliton for small wave numbers $\alpha$, at $Re = 14$, branches from the first family with wavelength $\lambda = 4 \lambda_1$, at $Re = 17$, they branch off with $\lambda = 5 \lambda_1$, and at $Re = 20$, they branch off with $\lambda = 6 \lambda_1$. Here, $\lambda_1$ is the wavelength of solution of the first family.
Conclusion

The steady-state traveling solutions to a divergent system of equations have been studied numerically for a freely flowing fluid film. Solutions have been presented in the form of a Chebyshev polynomials series along the transverse coordinate and in the form of a Fourier series along the longitudinal coordinate. At point $\alpha_n = 1$, corresponding to the wave number of neutral perturbations, the first family of nonlinear steady-state traveling periodic regimes arises. In the range of small wave numbers, the shape of the film surface for the first family is a negative “quasi-soliton”. Investigation of stability of solutions of this family, has resulted in obtaining the bifurcation points, where new families of steady-state traveling solutions arise. Examples of film surface profiles at one wavelength for solutions of these families are presented. At small changes in the Reynolds number, the structure of these families does not change qualitatively, in particular, for small wave numbers $\alpha$ they have the form of “quasi-soliton” solutions of the same type. With an increase in the Reynolds number, the pattern of wave regimes in the vicinity of the critical points varies radically, and the re-closing of these families occurs. Such singular points are found at Reynolds numbers $Re = 9$, $Re = 14$, $Re = 17$, and $Re = 20$. It is shown that these singular points belong to the saddle type.

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