Control of quantum thermodynamic behaviour of a charged magneto oscillator with momentum dissipation

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In this work, we expose the role of environment, confinement and external magnetic field ($B$) in determining the low temperature thermodynamic behaviour in the context of cyclotron motion of a charged oscillator with anomalous dissipative coupling involving the momentum instead of the much studied coordinate coupling. Explicit expressions for different quantum thermodynamic functions (QTF) are obtained at low temperatures for different quantum heat bath characterized by spectral density function, $\mu(\omega)$. The power law fall of different QTF are in conformity with third law of thermodynamics. But, the sensitiveness of decay i.e. the power of the power law decay explicitly depends on $\mu(\omega)$. We also separately discuss the influence of confinement and magnetic field on the low temperature behavior of different QTF. In this process we demonstrate how to control low temperature behaviour of anomalous dissipative quantum systems by varying confining length $a$, $B$ and the temperature $T$. Momentum dissipation reduces effective mass of the system and we also discuss its effect on different QTF at low temperatures.

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I. INTRODUCTION

A most common and effective approach to deal quantum dissipation is the system-plus-reservoir (SPR) model \cite{1,2,3,4} where dissipation arises from the linear coupling of some system’s variable with a reservoir (or environment or bath) which is modelled as a collection of independent quantum harmonic oscillators which are individually weakly perturbed by the system so that the reservoir can be assumed to be at equilibrium. Usually the system’s variable is taken to be a function of system’s coordinate. The model can indeed be described by Langevin equation \cite{1} after eliminating bath variables and a connection between bath spectrum and the dissipative memory function can be established. This connection between phenomenological description and microscopic details of the bath is very much valuable as it gives us the connectivity between quantum thermodynamic behaviour of the system and phenomenology.

The standard SPR model for quantum dissipation can meaningfully be generalized to the complementary possibility of coupling of a quantum system to a quantum mechanical heat bath through the momentum variables. It is now appropriate to discuss about previous studies on quantum dissipation where coupling occurs through momentum variables. In an early paper, Leggett \cite{6} discussed about normal dissipation and anomalous dissipation by considering two possible coupling terms $q\sum_{j} d_j p_j + p\sum_{j} c_j q_j$ where $q$, $p$ are system coordinate and momentum variables respectively and $q_j$, $p_j$, $d_j$ and $c_j$ are coordinate, momentum and coupling constants of the $j^{th}$ bath oscillator respectively. But, our Hamiltonian as given in Eq.$(1)$ has no resemblance with any of the two coupling terms considered by Leggett. Our model Hamiltonian $(1)$ has the coupling term $p\sum_{j} g_j p_j$ which definitely leads to different dynamics than that of Leggett. Recently, Cuccoli et al \cite{7} and Ankerhold et al \cite{8} studied momentum coupling model which has some resemblance to our model. Infact, our model Hamiltonian can exactly be cast into the model Hamiltonian considered by Ankerhold et al \cite{8} by an Unitary transformation. As mentioned earlier the dissipation arises from this type of model is completely different from that obtained from the model studied by Leggett. To differentiate between the two models, we follow the terminology of Ankerhold et al \cite{8} and use the term momentum dissipation rather than anomalous dissipation used by Leggett. Besides some fundamental theoretical issues, the discussion about this momentum dissipation with the model Hamiltonian \cite{1} has some physical relevance. Makhnovskii and Pollok \cite{9} showed that momentum dissipation leads to stochastic acceleration \cite{10,11} without violating second law of thermodynamics \cite{12}. In general, momentum dissipation reduces effective mass of the system and leads to anti-intuitive results of amplifying quantum effect rather than destroy it. We have discussed the effect of reduction of effective mass on different QTF in the context of charged dissipative magneto-oscillator. Although, there are several instances for which mass renormalization can be negative for the position-position coupling \cite{13,14,15}. F. Sols et al \cite{16,17} have discussed about momentum dissipation and local gauge invariance in the context of spin 1/2 impurity in an antiferromagnetic environment. The gauge invariance of the Hamiltonian is taken care in our case also.

Recently, we derive the quantum Langevin equation (QLE) satisfied by the particle coordinate operator for a charged quantum oscillator moving in a harmonic potential ($\omega_0 j$) with an external uniform magnetic field ($B$) and in the presence of momentum dissipation (the Hamilton-
nian (1)) and the same system is considered in this work \[\text{[18]}.\] Later, we calculate explicitly the equilibrium free energy for the same Hamiltonian (1) in Ref. \[\text{[19]}\]. In this work, we extend our previous studies \[\text{[18] \text{[14]}\] by incorporating the discussion of low temperature behaviour of different QTF for different realistic situations and the control of low temperature behaviour of different QTF by varying external parameters. Different control parameters like magnetic field \(B\) and confining length \(a\) are identified. Low temperature behaviour of different QTF are analyzed for different realistic situations which are captured through different realistic bath spectrum \(\mu (\omega )\). The effect of reduction of effective mass of the system on different QTF is also analyzed.

It is shown in Ref. \[\text{[10]}\] that the equilibrium free energy for the Hamiltonian (1) can be expressed as \(F_0(T, B) = F_0(T, 0) + \Delta_1 F_0(T, B) + \Delta_2 F_0(T, B)\), where \(T\) and \(B\) are temperature and external magnetic field of the system respectively. In this paper, we explicitly show how to control the low temperature behaviour of different QTF obtainable from \(F_0(T, B)\) by varying three characteristic frequencies of the system: thermal frequency \(\omega_{th} = \frac{k_B T}{\hbar}\), cyclotron frequency \(\omega_c = \frac{eaB}{mc}\), and confining potential frequency \(\omega_0 = \frac{1}{2m a^2}\). Here \(k_B\), \(\hbar\), \(a\), \(c\) are, respectively, the Boltzmann constant, Planck constant and the velocity of light in free space, while \(q\), \(m\) and \(a\) are charge, mass, and confining length of the particle respectively. It is now useful to state our principal results at the outset: Considering a generalized quantum heat bath, we find three different regimes depending on the ratio \(\frac{\omega_0}{\omega_{th}}\) of three characteristic frequencies \(\omega_c\), \(\omega_0\) and \(\omega_{th}\) of the system: (i) Region I where \(F_0(T, 0)\) dominates the low temperature thermodynamic properties; (ii) Region II, a fuzzy regime, where all the three terms of free energy are important in determining low temperature thermodynamic properties; (iii) Region III where the power of the power law fall of different QTF are determined by \(\Delta_1 F_0(T, B) = \Delta_2 F_0(T, B)\); (iv) In the absence of confining potential (\(\omega_0 = 0\)) the low temperature behaviour is fully determined by \(F_0(T, 0)\); (v) Unlike the normal dissipation, the low temperature behaviour for \(\omega_0 = 0\) is independent of \(\omega_c\) with momentum dissipation for the arbitrary heat bath; (vi) Finally, momentum dissipation decreases mass of the system (anti-intuitive quantum effects) and as a result the quantum contribution to different QTF increases with the increase of dissipation. Although, there are certainly a number of studies which demonstrate that the quantum corrections arise for a free particle only in the presence of position-position coupling \[\text{[20] \text{[22]}\]. On the other hand, all the above mentioned results hold for the radiation heat bath, but the low temperatures phase diagram is fully controlled by the ratio \(\frac{\omega_0}{\omega_c}\). Like the normal dissipation, the low temperature behaviour of different QTF for the radiation bath depends on \(\omega_c\) and friction constant \(\gamma\) for the case \(\omega_0 = 0\) with momentum dissipation.

With this we summarize the construction of our paper. In the following section, we describe our model and summarize the basic expressions which are required for our further calculations. In Sec. III, we consider a generalized quantum heat bath spectrum and derive several QTF at low temperatures for our model system. As a matter of fact we identify different external control parameters like \(B\) and \(a\) to control the low temperature behaviour of our model system. Separately we discuss the case for \(\omega_0 = 0\). Further, we consider low temperature behaviour for the charged magneto-oscillator coupled with radiation bath through momentum variables in this Sec. (III). We also discuss the effect of the negative renormalization of the mass of the system due to momentum coupling and its effect on different QTF at low temperatures in this section. We conclude this section with a possible demonstration of our control mechanism of different QTF for a physically realizable system. The paper is concluded in Sec. (IV).

II. MODEL SYSTEM & BASIC EXPRESSIONS

We consider the dissipative dynamics of a charged quantum oscillator in the presence of an external magnetic field. Such a situation is often arises in many problems of theoretical and experimental relevance, e.g., Landau dissipative diamagnetism \[\text{[23]}\], quantum Hall effect \[\text{[24]}\], and high temperature superconductivity \[\text{[25]}\]. Recently, we analyze the dissipative dynamics of such system by considering bilinear coupling between the system and the environment through momentum variables by invoking a gauge-invariant SPR model \[\text{[18] \text{[19]}\]. The Hamiltonian of the system is

\[
H_0 = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c} \mathbf{A}\right)^2 + \frac{1}{2}\omega_0^2 \mathbf{q}^2 + \sum_{j=1}^{N} \left[ \frac{1}{2m_j}\left(\mathbf{p}_j - g_j \mathbf{p} + \frac{g_j q}{c} \mathbf{A}\right)^2 + \frac{1}{2}m_j \omega_j^2 \mathbf{q}_j^2 \right], \tag{1}
\]

where \(q, m, \mathbf{p}, \mathbf{r}\) are, respectively, the charge, the mass, the momentum operator and the coordinate operator of the particle, while \(\omega_0\) is the frequency characterizing its motion in the harmonic potential. The \(j\)th heat-bath oscillator has mass \(m_j\), frequency \(\omega_j\), coordinate operator \(\mathbf{q}_j\), and momentum operator \(\mathbf{p}_j\). The dimensionless parameter \(g_j\) describes the coupling between the particle and the \(j\)th oscillator. The speed of light in vacuum is denoted by \(c\). The vector potential \(\mathbf{A} = \mathbf{A}(\mathbf{r})\) is related to the uniform external magnetic field \(\mathbf{B} = (B_x, B_y, B_z)\) through \(\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})\). The field has the magnitude \(B = \sqrt{B_x^2 + B_y^2 + B_z^2}\). The commutation relations for the different coordinate and momentum operators are

\[
[r_\alpha, p_\beta] = i\hbar \delta_{\alpha\beta}, [q_\alpha, p_\beta] = i\hbar \delta_{\alpha\beta}, \tag{2}
\]

while all other commutators vanish. In the above equation, \(\delta_{jk}\) denotes the Kronecker Delta function. Here,
Greek indices \((\alpha, \beta, \ldots)\) refer to the three spatial directions, while Roman indices \((i, j, k, \ldots)\) represent the heat-bath oscillators. Let us remark that momentum-momentum coupling has been considered earlier in the literature \(^R\), and, in particular, to model the physical situation of a single Josephson junction interacting with the blackbody electromagnetic field in the dipole approximation \(^S\). Our model Hamiltonian is similar to that considered in Refs. \(^R\), \(^S\); the additional interesting feature that we consider here is the inclusion of the effects of an external magnetic field. In Ref. \(^{19}\), we derived the equilibrium free energy of Hamiltonian (1) in the following form:

\[
F_0(T, B) = F_0(T, 0) + \Delta_1 F_0(T, B) + \Delta_2 F_0(T, B),
\]

where

\[
F_0(T, 0) = \frac{3}{\pi} \int_0^\infty d\omega \ f(\omega, T) \text{Im} \left[ \frac{d}{d\omega} \ln \alpha^{(0)}(\omega) \right]
\]

is the free energy of the charged particle in the absence of the magnetic field. The contribution from the latter is contained in the two terms \(\Delta_1 F_0(T, B)\) and \(\Delta_2 F_0(T, B)\), given by

\[
\Delta_1 F_0(T, B) = -\frac{1}{\pi} \int_0^\infty d\omega \ f(\omega, T) \text{Im} \left[ \frac{d}{d\omega} \ln \left( 1 - \frac{(G(\omega))^2}{\omega B q c} \right)^2 \alpha^{(0)}(\omega)^2 \right],
\]

and

\[
\Delta_2 F_0(T, B) = \frac{1}{\pi} \int_0^\infty d\omega \ f(\omega, T) \text{Im} \left[ \frac{(G(\omega))^2}{\omega B q c} \alpha^{(0)}(\omega)^2 \right] \left[ 1 - \frac{(G(\omega))^2}{\omega B q c} \alpha^{(0)}(\omega)^2 \right]^{-1}.
\]

Here \(f(\omega, T)\) is the free energy of a free oscillator of frequency \(\omega\):

\[
f(\omega, T) = k_B T \ln \left[ 2 \sinh \left( \frac{\hbar \omega}{2k_B T} \right) \right].
\]

\(\alpha^{(0)}(\omega) = [-m_r \omega^2 + m \omega_0^2 G(\omega)]^{-1}\) is the susceptibility in the absence of the magnetic field, \(G(\omega) = 1 - \sum_{j=1}^{N} \frac{(g_j)^2 m \omega_0^2}{m_r \omega_j^2 - \omega^2}\) and renormalized mass \(m_r = \frac{m}{1 + \sum_{j=1}^{N} \frac{(g_j)^2 m \omega_0^2}{m_r \omega_j^2}}\). Let us now comment on the form of the free energy (3) with respect to that for coordinate-coordinate coupling between the particle and the heat-bath oscillators, obtained in Ref. \(^{26}\). In the latter case, the free energy is given by

\[
F_0(T, B) = F_0(T, 0) + \Delta_1 F_0(T, B),
\]

where \(F_0(T, 0)\) has the same form as in Eq. (4), while \(\Delta_1 F_0(T, B)\) is of the same form as given in Eq. (5) except the \(G(\omega)\) term. This is to be remembered that the existence of \(G(\omega)\) and negative renormalized mass \(m_r\) in the free energy for momentum dissipation change thermodynamic properties dramatically. One can also compare our results Eqs. (3)-(6) with that of Wang et al \(^{27}\) who have discussed the momentum dissipation in the context of a dissipative oscillator. One can easily show that Eqs. (5) and (6) identically vanish for vanishing magnetic field. Thus, our results exactly matches with that of Eq. (4) of Wang et al \(^{27}\) except a prefactor of 3 which is coming due to 3-dimensions considered in our problem.

### III. DIFFERENT QTF WITH DIFFERENT ILLUSTRATIVE \(\mu(\omega)\)

In this section, we utilize the main results obtained in Ref. \(^{19}\) i.e., the equilibrium free energy for Hamiltonian (1) to obtain and analyze different QTF at low temperatures for the dissipative cyclotron motion with momentum dissipation. To proceed, we need to express \(G(\omega)\) and \(\alpha(\omega)\) in terms of the Fourier transform of the diagonal part of the memory function, given by \(^{18}\)

\[
\mu_d(\omega) = i \sum_{j=1}^{N} \frac{g_j^2 m \omega_0^2}{m_r (\omega_j^2 - \omega^2)}
\]

Thus we can have the free oscillator susceptibility

\[
\alpha^{(0)}(\omega) = \frac{1}{m_r (\omega_0^2 - \omega^2) - i\omega \mu_d(\omega)},
\]

and

\[
G(\omega) = \frac{m_r}{m} - \frac{i\omega \mu_d(\omega)}{m \omega_0^2}
\]

At this point, we consider different \(\mu_d(\omega)\) for different heat bath spectrum and derive explicit expression of different QTF at low temperatures.

#### A. Arbitrary heat bath

We consider here a very general class of heat bath for which in the small \(\omega\) regime we have \(\mu(\omega) \approx mb^{1-\nu} \omega^{\nu}\), with \(b\) is a positive constant with the dimension of frequency \(^{28}\). The Ohmic, sub-Ohmic and super-Ohmic heat bath spectra are characterized by considering \(\nu = 1, 0 < \nu < 1, \) and \(\nu > 1\), respectively. These three cases are also relevant for a real physical system. To describe quantum tunneling in a metallic environment one can use the Ohmic spectrum \(^2\). The super-Ohmic spectrum corresponds to the phonon bath in \(d > 1\) spatial dimensions and it refers to \(\nu = d\) or \(\nu = d + 2\) cases depending on the underlying symmetry of the strain field \(^2\). The sub-Ohmic spectrum is useful in describing the type of noise in some solid state devices or \(1/f\) noise in Josephson junction \(^{29}\). This is to mention here that there is no harm in choosing Ohmic, sub-Ohmic or super-Ohmic bath in the context of momentum dissipation until and unless these choices make any conflict with the conditions (49) and
We can rearrange the free energy expressions as follows:

\[ F_0(T, B) = F_0(T, 0) + \Delta_1 F_0(T, B) + \Delta_2 F_0(T, B) \]

\[ = \frac{1}{\pi} \int_0^\infty d\omega f(\omega, T)[3I_0 - I_1 + I_2], \quad (11) \]

with

\[ I_0 = \Im \left[ \frac{d}{d\omega} \ln\alpha^{(0)}(\omega) \right], \]

\[ I_1 = \Im \left[ \frac{d}{d\omega} \ln \left\{ 1 - (G(\omega))^2 \left( \frac{qB}{c} \right)^2 \left[ \alpha^{(0)}(\omega) \right]^2 \right\} \right], \]

\[ I_2 = \Im \left[ \left\{ 1 - (G(\omega))^2 \left( \frac{qB}{c} \right)^2 \left[ \alpha^{(0)}(\omega) \right]^2 \right\} \right]. \quad (12) \]

Now, \( f(\omega, T) \) vanishes exponentially for \( \omega >> \frac{kbT}{\omega} \). Therefore, in order to evaluate the free energy of the dissipative charged oscillator at low temperatures, we need to consider only low-\( \omega \) contributions of integrands in evaluating the integral in Eq. (11). With this we can show that at low frequencies the magnetic field independent integrand \( I_0 \) becomes:

\[ \lim_{\omega \to 0} I_0(\omega) \simeq \frac{C(1 + \nu)}{\omega_0^{\nu}}, \quad (13) \]

with \( C = \frac{\pi^2}{m_e b^{1-\nu}} \cos \left( \frac{\nu \pi}{2} \right) \). On the other hand, we obtain for the magnetic field dependent integrands \( I_1 \) and \( I_2 \) as follows:

\[ \lim_{\omega \to 0} I_1 = \frac{2C(\nu + 1)\omega^2}{\omega_0^3}\omega^{\nu+2} = \lim_{\omega \to 0} I_2 \quad (14) \]

Now, we use the result

\[ \int_0^\infty dy y^\nu \log(1 - e^{-y}) = -\Gamma(\nu + 1)\zeta(\nu + 2), \quad (15) \]

where \( \Gamma(z) \) is the gamma function, while \( \zeta(z) \) is the Riemann Zeta function, to obtain the free energy at low temperatures:

\[ F_0(T, 0) \simeq -\frac{3\Gamma(\nu + 2)\zeta(\nu + 2)\cos\left( \frac{\nu \pi}{2} \right)mhb}{m_\pi} \left( \frac{b}{\omega_0} \right)^2 \left( \frac{\omega_{th}}{b} \right)^{\nu+2} \]

\[ \Delta_1 F_0(T, B) \simeq -\frac{2(\nu + 1)\Gamma(\nu + 3)\zeta(\nu + 4)\cos\left( \frac{\nu \pi}{2} \right)mhb}{m_\pi} \left( \frac{\omega_{th}}{\omega_0} \right)^2 \left( \frac{b}{\omega_0} \right)^4 \left( \frac{\omega_{th}}{b} \right)^{\nu+4} = \Delta_2 F_0(T, B). \quad (16) \]

We have already discussed that one can identify three characteristic frequencies of the system: (i) thermal frequency \( \omega_{th} \), (ii) cyclotron frequency \( \omega_c \), and (iii) the confining potential frequency \( \omega_0 \). The ratio of \( \Delta_1 F_0(T, B) \) or \( \Delta_2 F_0(T, B) \) with respect to \( F_0(T, 0) \) can be expressed in terms of the above mentioned three frequencies:

\[ \frac{\Delta_1 F_0(T, B)}{F_0(T, 0)} = \frac{\Delta_2 F_0(T, B)}{F_0(T, 0)} \simeq \left( \frac{\omega_c}{\omega_0} \right)^2 \left( \frac{\omega_{th}}{\omega_0} \right)^2. \quad (17) \]

Thus, one can control the ratios of different terms of free energies and henceforth the thermodynamic behaviour by varying the external parameters like temperature (\( T \)) associated with thermal frequency, magnetic field (\( B \)) for cyclotron frequency and confining well length (\( a \)) related to confining potential frequency. One can roughly identify three different regimes depending on the ratio \( \frac{\omega_c}{\omega_0} \). Here, we must admit that it is impossible to identify a well defined boundary between two different regimes for the present case. But, one can set a boundary between two regimes if the ratio in Eq. (17) differs by an order of magnitude. We have tried to distinguish three regimes based on the dependence of three terms of the free energy (Eq 11) on the ratios \( \frac{\omega_c}{\omega_0} \) and \( \frac{\omega_{th}}{\omega_0} \). As we have already discussed that all our discussion is valid at very low temperatures, so \( \omega_{th} \sim 10^6 \text{ Hz} \) with \( T_{\text{max}} \sim 10^{-6} \text{ K} \). Similarly, as we are interested to confine our charged particle utmost inside a microstructure (length \( a_{\text{max}} \sim 10^{-6} \)) which leads to \( \omega_0 = \frac{2\pi}{2a_{\text{max}}} \sim 10^6 \text{ Hz} \). Thus, one can set upper limit of the ratio \( \frac{\omega_c}{\omega_0} \sim 0.1 \). Similarly, the lower limit of this ratio for confinement of the charged particle inside a nanostructure (dimension \( a \sim 10^{-9} \text{ m} \)) at temperatures \( T_{\text{min}} \sim 10^{-9} \text{ K} \) can be set at \( \frac{\omega_c}{\omega_0} \sim 10^{-10} \). On the other hand, the upper limit of the ratio \( \frac{\omega_{th}}{\omega_0} \sim 10^2 \) for a maximum magnetic field \( B_{\text{max}} \sim 10^7 \text{ T} \) and confinement length \( a_{\text{max}} \sim 10^{-6} \text{ m} \). The lower limit for \( \frac{\omega_{th}}{\omega_0} \) is set to \( 10^{-8} \) for \( B_{\text{min}} \sim 10^{-5} \text{ T} \) and confinement length \( a_{\text{min}} \sim 10^{-9} \text{ m} \) (nanostructure). After setting the upper and lower bound of the ratios \( \frac{\omega_c}{\omega_0} \) and \( \frac{\omega_{th}}{\omega_0} \), one can obtain three different regimes by varying \( B \) and \( a \). For this purpose our crucial equation is Eq. (17). We have already shown that at low temperatures \( \Delta_1 F_0(T, B) = \Delta_2 F_0(T, B) \) for the arbitrary heat bath spectrum. From Eq. (17), we observe that if \( \frac{\omega_c}{\omega_0} \omega_{th} \ll 1, F_0(T, 0) >> \Delta_1 F_0(T, B) \) and we obtain region (I) where \( F_0(T, 0) \) will dominate the low temperature.
thermodynamic properties. Similarly, the region (II) can be identified for \( \frac{dF_0}{d\omega} \sim 1 \), where all the three terms of free energy are important in determining the low temperature thermodynamic properties. Lastly, the region (III) can be obtained for \( \frac{dF_0}{d\omega} \gg 1 \) (we set it to 3.5), where the low temperature thermodynamic behaviour is determined by \( \Delta_1 F_0(T, B) \) or \( \Delta_2 F_0(T, B) \). Here, we should mention that we have differentiated two regimes if two terms \(( F_0(T, 0) \) and \( \Delta_1 F_0(T, B) \)) of free energies differ by an order of magnitude. These three regimes are clearly shown in the schematic phase diagram 1. We also tabulated some typical parameter values (B,T,a) for a trapped Beryllium atom in contact with engineered phase reservoir in Table I. It is to be mentioned that we require nanostructures or microstructures to confine the charged particles and identify the three regimes by tuning external magnetic field B.

Let us compare our results with that of standard coordinate-coordinate coupling case. Recently, we have shown that magnetic field dependence completely disappears \((\lim_{\omega \to 0} I_2 \simeq 0 \) or \( \Delta_1 F_0(T, B) = 0 \) ) from the low-temperature thermodynamic properties, irrespective of \( \mu(\omega) \), i.e., the nature of heat bath \([30]\). So, there is no option to control thermodynamic properties at low temperatures for standard coordinate-coordinate coupling by varying the external parameters like, external magnetic field B or the confining length a. Thus, the thermodynamic properties at low temperatures are always determined by \( F_0(T, 0) \) for coordinate-coordinate coupling case.

Now, it is time to discuss about the effect of environment in determining low temperature thermodynamic properties. Suppose we are in regime (I) where the low temperature thermodynamic properties are determined by \( F_0(T, 0) \) alone and the entropy \( S = -\frac{\partial F_0(T, B)}{\partial T} \) approaches zero as \( T \to 0 \) in conformity with third law of thermodynamics with a power law \( S \sim T^\delta \) \((\delta = \nu + 1)\). Thus, the entropy falls off linearly to zero for the Ohmic bath \((\nu = 1)\). On the other hand, entropy vanishes to zero with a power \( \delta > 2 \) and \( 1 < \delta < 2 \) for the super-Ohmic and sub-Ohmic environment, respectively. If we move to regime (III) where thermodynamics is determined by \( \Delta_1 F_0(T, B) \), the entropy falls off to zero as \( T \to 0 \) with a power law \( T^{\beta} \) with \( \beta = \nu + 3 \). Again, we find \( 3 < \beta < 4 \), \( \beta = 4 \) and \( \beta > 4 \) for the super-Ohmic, Ohmic and super-Ohmic cases respectively. Thus, we can say that the power of the power law depends on the nature of heat bath. Since, we can move from regime (I) to regime (III) for momentum dissipation by tuning B and a, the power of the power law fall of entropy to zero as \( T \to 0 \) can also be tuned by varying B and a. It is also observed that the entropy has a faster decay \((\delta)\) for momentum-momentum coupling compared to standard coordinate-coordinate coupling \((\beta)\) for which we can only have regime (I).

Let us consider the case without the confining potential, i.e., \( \omega_0 = 0 \). In this scenario, we have \( \lim_{\omega \to 0} I_0 = \frac{C(1-\nu)}{A+\nu C^2} \omega^{1-\nu} \), \( \lim_{\omega \to 0} I_1 = \frac{2C(1-\nu)}{(A^2+\nu C^2)} \omega^{1-\nu} \), and \( \lim_{\omega \to 0} I_2 = 0 \). As a result we obtain the free energy as follows:

\[
F_0(T, B) = -h\hbar(1-\nu)\Gamma(1-\nu)\zeta(2-\nu)\frac{m_r}{m_0} \cos\left(\frac{\nu \pi}{2}\right) \left(\frac{\omega m \mu}{b}\right)^{3-\nu} \tag{18}
\]

Thus, we can say that the effect of magnetic field disappears from low temperature thermodynamic properties for \( \omega_0 = 0 \), as it does not appear in energy expression (18). This is just opposite to what happened for the coordinate-coordinate coupling. For coordinate-coordinate coupling, we have shown earlier in Ref. [30] that the effect of magnetic field in the low temperature thermodynamic properties appears for \( \omega_0 = 0 \) case and the effect of B disappears for \( \omega_0 \neq 0 \). On the other hand, entropy falls off to zero as \( T \to 0 \) with a power law \( T^{\delta'} \) with \( \delta' = 2 - \nu \) for \( \omega_0 = 0 \) with momentum dissipation. The power of the power law is \( \delta' = 1 \) for Ohmic bath which is same as that of coordinate-coordinate coupling. Unlike the momentum dissipation, the prefactor of entropy, \( S(T) \), depends on the cyclotron frequency and the friction constant for coordinate-coordinate coupling.

Now, it is time to tell about the implementation of our Arbitrary Heat Bath : Beryllium Ion

| Region I | Region II | Region III |
|----------|-----------|------------|
| a=1µm   | a=1µm    | a=1µm     |
| T=10nK  | T=10nK   | T=10nK    |
| B=100µT | B=100µT  | B=100µT   |

TABLE I: Tabulated values of external parameters B, T, and a to observe three different regimes for a trapped Beryllium ion in contact with an engineered arbitrary heat reservoir.

FIG. 1: (color online) Approximate schematic sketch of different accessible regimes which can be obtained by varying the ratios of \( \frac{\omega_0}{\omega_c} \) and \( \frac{\omega}{\omega_0} \) for a trapped Beryllium ion in contact with an arbitrary heat bath.
possibility of controlling both the environment and the system-environment coupling open the doorway of controlling the low temperature thermodynamic behaviour at nanoscale. During the last few decades, the huge advancement in the field of laser cooling and trapping experimental techniques have made the way to confine a single ion in harmonic well at very low temperatures where quantum effects are predominant. For this purpose one can use a miniature version of the linear Paul trap [32, 33]. A single laser cooled ion is theoretically equivalent to a charged particle moving in a harmonic well. Thus, it is now possible to arrange quantum Brownian motion in the context of trapped ions with the help of engineered reservoir. The advancement in reservoir engineering techniques [34] pave the way to construct possible experiments aimed at simulating paradigmatic models of open quantum systems as the one considered in this paper. It is not only possible to construct "artificial" reservoir but also one can manipulate its spectral density and the coupling with the system [34]. The same method can be extended straightforwardly to realize the Ohmic and sub-Ohmic environments considered here for a trapped Beryllium ion. In Refs. [2, 35], the cases of Ohmic and sub-Ohmic environment are modelled by an infinite RLC transmission line. As discussed in Ref. [2] (page 63), the transmission line can be thought of made of discrete building blocks which consists of inductor (L) and resistor (R) are in series along one string-board of the ladder and the capacitor (C) is on the horizontal support of the ladder as shown in figure 2. Thus, the impedance of an infinitely long transmission line is given by \( Z(\omega) = \sqrt{\frac{R^2 + (\omega L)^2}{\omega C}} \). However, it is evident that the Ohmic and sub-Ohmic environment can be realized from the LC dominant and R-dominant limit of the RLC transmission line, respectively [2]. These would allow one to test in a controlled way a fundamental and ubiquitous model such as considered in this paper through Quantum Brownian motion (QBM). In this respect, we should mention that QBM model with single trapped ion connected with Ohmic or non-Ohmic reservoir is simulated by Maniscalco et al [37, 38]. Experiments with single trapped ions have demonstrated the ability to engineer artificial environments and to control the relevant system-environment parameters [34]. In our case, the trapped ion is a single \( Be^+ \) ion which is stored in a rf Paul trap [33] with an oscillation frequency of \( \frac{\omega_0}{\hbar} = 11.2 \text{ MHz} \). Then, the ion can be laser cooled using sideband cooling with stimulated Raman transitions between the \( 2S_1/2 \) (F = 2, \( m_F = -2 \)) and \( 2S_1/2 \) (F = 1, \( m_F = -1 \)) hyperfine ground states, which are denoted by "up" and "down", respectively. These states are separated by approximately 1.25 GHz. Reference [34] represents the recent advancement to show how to couple a properly engineered reservoir with a quantum charged oscillator, e.g., the quantized center of mass motion of the trapped \( Be^{+} \) ion. This trapped ion can be capacitively coupled with the impedance \( Z(\omega) \). A schematic diagram of such an experimentally realizable system is drawn in figure 2. This situation is somewhat similar to the momentum dissipation case discussed in the context of quantum electrodynamic fluctuations of the macroscopic Josephson phase by H. Kohler et al. [40].

B. Blackbody radiation bath

In this case, the associated memory function is given by

\[
\mu_d(\omega) = \frac{2q^2\Omega^2\omega}{3e^3(\omega + i\Omega)},
\]

where \( \Omega \) is a cutoff frequency. It has been shown that in the large cut-off limit the memory function and the response function in the absence of magnetic field are given by [41, 42]:

\[
\alpha^{(0)}(\omega) = \left( -\frac{M\omega^2}{1 - i\omega\tau_r} + M\omega_0^2 \right)^{-1},
\]

where, \( M = m_r \) and \( \tau_c = \frac{2q^2}{3M\Omega} \). As a result, we have at low temperatures, i.e., only considering low frequencies contribution in (12):

\[
\begin{align*}
\lim_{\omega \to 0} I_0 &= \frac{3m_r\tau_c}{M\omega_0^2}\omega^2, \\
\lim_{\omega \to 0} I_1 &= -\frac{2m_r\omega_r^2\tau_c}{M\omega_0^2}\omega^2, \\
\lim_{\omega \to 0} I_2 &= \frac{8m_r\omega_r^2\tau_c}{M\omega_0^2}\omega^2.
\end{align*}
\]

Again using the result (15), we can obtain:

\[
\begin{align*}
F_0(T, 0) &= -\frac{3\pi^2\omega_0\tau_c m_r}{5} \left( \frac{\omega_{th}}{\omega_0} \right)^4 \hbar\omega_h, \\
\Delta F_0(T, B) &= -\frac{2\pi^3 m_r}{9} \frac{\omega_c\tau_c}{M} \left( \frac{\omega_{th}}{\omega_0} \right)^4 \hbar\omega_c.
\end{align*}
\]
with $\Delta F_0(T, B) = \Delta_1 F_0(T, B) + \Delta_2 F_0(T, B)$. Now, we can find the ratios of these two terms of the free energy:

$$\frac{F_0(T, 0)}{\Delta F_0(T, B)} = \frac{27}{10}\left(\frac{\omega_0}{\omega_c}\right)^2.$$ (23)

One can again say that the power of the power law behaviour of different thermodynamic quantities can be controlled by varying external parameters magnetic field (B), and confining length (a) associated with $\omega_c$, and $\omega_0$, respectively. One can easily observe from the phase diagram (Fig 3) that it contains three regimes just like the arbitrary bath. Unlike the arbitrary bath, the three regimes for the radiation bath can be explored by varying only the ratio $\frac{\omega_0}{\omega_c}$ alone. Let us consider the case of without confining potential, i.e., $\omega_0 = 0$ case. For this situation ($\omega_0 = 0$) we have:

$$\lim_{\omega \rightarrow 0} I_0 = -\tau_e,$$

$$\lim_{\omega \rightarrow 0} I_1 = 0,$$

$$\lim_{\omega \rightarrow 0} I_2 = 4\tau_e.$$ (24)

As a result, we have the free energy

$$F_0(T, B) = -\frac{\pi}{6}\hbar \omega_c \left(\frac{\omega_0}{\omega_c}\right)^2,$$ (25)

where, $\omega_c = \tau_e^{-1}$. Unlike the coordinate-coordinate coupling, the free energy at low temperatures for the radiation heat bath with momentum dissipation is free from magnetic field [30, 31]. Also, we have observed entropy $s(T) = \frac{k_B T}{\omega_c}$ and entropy vanishes as $T \rightarrow 0$ but the temperature dependence and the prefactors are different from that of coordinate-coordinate case [30, 31]. Typical values of externally controllable parameters $(B, T$ and $a)$ for the radiation heat bath are tabulated for the trapped Calcium ion in table II. Once again we can observe three different regimes in the nanostructures or microstructures. Now, the question left, how one can mimic this radiation bath reservoir in the laboratory? This is achievable through the method discussed in Refs. [42, 43] in the context of a control of a cavity field state through an atom-driven field interaction. But, recently H. G. Barros et al have reported on the realization of an efficient single-photon source using a single calcium ion trapped within a high-finesse optical cavity. This system shares some features with that of our case of single ion interacting with a radiation bath. A detailed description of the experimental setup can be found in [46].

![FIG. 3: (color online) Different regimes for the trapped Calcium ion in contact with a radiation heat bath are plotted as a function of the ratio of the $\frac{\omega_0}{\omega_c}$ alone.](image)
bution to different thermodynamic quantities reduces for
the case of without the confining potential, i.e., $\omega_0 = 0$.
For the radiation heat bath, the effect of reduction of effec-
tive mass (as we increase $g_j$) on different low temper-
ature thermodynamic quantities is cancelled out due to
appearance of the ratio $\frac{B}{a}$ in Eqs. (22). In this context
we should mention that the negative renormalized mass is
also discussed for several cases with position-position
coupling. However, a difference between normal
dissipation and momentum dissipation is indeed the ap-
pearance of renormalized mass in the potential term of
the quantum Langevin equation derived from the Hamil-
tonian (1) with momentum dissipation [18]. Although,
this effect can be thought of as a secondary effect as the
sign of the renormalized mass is usually read off from the
inertial mass.

IV. CONCLUSIONS

In this paper, we discuss the low temperature ther-
modynamic properties of a charged oscillator in the
presence of an external magnetic field and is coupled with
a quantum heat bath through momentum-momentum
variables. Although, the validity of the third law is
confirmed for different heat bath, but the power of the
power law fall of the entropy as $T \rightarrow 0$ can be
controlled by external parameters: $B$ and $a$. Depending
on the power of the power law, we can identify different
regimes for the arbitrary heat bath and radiation heat
bath. Typical values of external parameters ($B$ and $a$) to
observe different regimes for a trapped Beryllium ion and
trapped calcium ion in contact with different engineered
reservoir are tabulated. Also, the effect of reduction of
effective mass as we increase the momentum-momentum
coupling strength $g_j$ on different QTF are discussed in
details. In this context we have described a possible
experimental realization of our control mechanism for
the quantum thermodynamics for a trapped Beryllium
ion interacting with Ohmic, sub-Ohmic or super-Ohmic
engineered reservoir at nanoscale. On the other hand, we
have described the experimental realization of engineered
radiation bath in the context of trapped calcium ion.

Now, with the advent of technological advancement
reaching into the nano and quantum regime, and in
view of the fundamentally different rules of quantum
mechanics, there is utmost requirement to understand
thermodynamics at the microscopic and nano scale
where thermal fluctuations compete with quantum
fluctuations. In that perspective our research will be
helpful in controlling thermodynamic properties as well
as understanding thermodynamics at micro and nano
scale. We can conclude that our present study is rele-
vant in the process of understanding thermodynamics at
nano-scale as well as making of small scale thermal
machines in which working fluid is a single trapped ion.

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