The integral representations of the $q$-Bessel-Macdonald functions

V.-B.K.Rogov
101475, Moscow, MIIT
e-mail vrogov@cemi.rssi.ru

Abstract

The $q$-Bessel-Macdonald functions of kinds 1, 2 and 3 are considered. Their representations by classical integral are constructed.

1 Introduction

The definitions of the $q$-Bessel-Macdonald functions ($q$-BMF) and their properties were given in [2]. Their representations by the Jackson $q$-integral have been constructed in [3, 4].

The double integral appears necessarily if we consider the problems of the harmonic analysis on the quantum Lobachevsky space. As the double Jackson $q$-integral is connected hardly with the $q$-lattice we have not any possibility to pass to the polar coordinates. So we are forced to introduce the usual double integral for $q$-functions.

In the first sections we remind the known formulas for $q$-Bessel functions and $q$-binomial.

2 The modified $q$-Bessel functions and the $q$-BMF

In [1] the $q$-Bessel functions were defined as follows:

\[ J_{\nu}^{(1)}(z, q) = \left( \frac{q^{\nu+1}, q}{q, q} \right)_{\infty} \left( \frac{z}{2} \right)^{\nu} {}_2\Phi_1(0, 0; q^{\nu+1}; q, -\frac{z^2}{4}), \]  
(2.1)

\[ J_{\nu}^{(2)}(z, q) = \left( \frac{q^{\nu+1}, q}{q, q} \right)_{\infty} \left( \frac{z}{2} \right)^{\nu} {}_0\Phi_1(-; q^{\nu+1}; q, -\frac{z^2 q^{\nu+1}}{4}), \]  
(2.2)

\[ J_{\nu}^{(3)}(z, q) = \left( \frac{q^{\nu+1}, q}{q, q} \right)_{\infty} \left( \frac{z}{2} \right)^{\nu} {}_1\Phi_1(0; q^{\nu+1}; q, -\frac{z^2 q^{\nu+1}}{4}). \]  
(2.3)

where $\Phi_q$ is basic hypergeometric function [5].

\[ \Phi_q(a_1, \cdots, a_r; b_1, \cdots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_r, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_s, q)_n} (-1)^n q^{n(n-1)/2} 1+s-r z^n, \]

The work was supported by the Russian Foundation for Fundamental Research (grant no. 0001-00143) and the NIOKR MPS RF.
\[(a, q)_n = \begin{cases} 1 & \text{for } n = 0 \\ (1 - a)(1 - aq) \ldots (1 - aq^{n-1}) & \text{for } n \geq 1, \end{cases} \]

\[(a, q)_\infty = \lim_{n \to \infty} (a, q)_n, \quad (a_1, \ldots, a_k, q)_\infty = (a_1, q) \ldots (a_k, q). \]

It allows to introduce the modified \(q\)-Bessel functions (\(q\)-MBFs) using (2.1), (2.2) and (2.3) similarly to the classical case \([6]\).

**Definition 2.1** The modified \(q\)-Bessel functions are the functions

\[ I^{(j)}_\nu(z, q) = \frac{(q^{\nu+1}; q)_\infty}{(q, q)_\infty} (z/2)^\nu \Phi_1 \left( \begin{array}{c} 0, \ldots, 0; q^{\nu+1}; q, \\
\delta \end{array} \frac{z^2 q^{\frac{\nu+1}{2}(2-\delta)}}{4} \right). \]

Here

\[ \delta = \begin{cases} 2 & \text{for } j = 1 \\ 0 & \text{for } j = 2 \\ 1 & \text{for } j = 3. \end{cases} \] (2.4)

Obviously,

\[ I^{(j)}_\nu(z, q) = e^{-i\pi j/2} I^{(j)}_\nu(e^{i\pi/2} z, q), \quad j = 1, 2, 3. \]

In the sequel we consider the functions

\[ I^{(1)}_\nu((1 - q^2)z; q^2) = \sum_{k=0}^{\infty} \frac{(1 - q^2)^k (z/2)^{\nu+2k}}{(q^2, q^2)_k \Gamma q^2 (\nu + k + 1)}, \quad |z| < \frac{2}{1 - q^2}, \] (2.5)

\[ I^{(2)}_\nu((1 - q^2)z; q^2) = \sum_{k=0}^{\infty} \frac{q^{2k(\nu+1)}(1 - q^2)^k (z/2)^{\nu+2k}}{(q^2, q^2)_k \Gamma q^2 (\nu + k + 1)} \] (2.6)

\[ I^{(3)}_\nu((1 - q^2)z; q^2) = \sum_{k=0}^{\infty} \frac{q^{k(\nu+1)}(1 - q^2)^k (z/2)^{\nu+2k}}{(q^2, q^2)_k \Gamma q^2 (\nu + k + 1)}, \] (2.7)

where

\[ \Gamma_{q^2}(\nu) = \frac{(q^2, q^2)_\infty}{(q^{2\nu}, q^2)_\infty} (1 - q^2)^{1-\nu} \]

is the \(q^2\)-gamma function. If \(|q| < 1\), the series (2.6) and (2.7) are absolutely convergent for all \(z \neq 0\). Consequently, \(I^{(2)}_\nu((1 - q^2)z; q^2)\) and \(I^{(3)}_\nu((1 - q^2)z; q^2)\) are holomorphic functions outside a neighborhood of zero.

**Remark 2.1**

\[ \lim_{q \to 1-0} I^{(j)}_\nu((1 - q^2)z; q^2) = I_\nu(z), \quad j = 1, 2, 3. \]

**Proposition 2.1** The function \(I^{(j)}_\nu((1 - q^2)z; q^2)\) is a solution of the difference equation

\[ f(q^{-1} z) - (q^{-\nu} + q^\nu) f(z) + f(q z) = q^{-\delta}(1 - q^2)^2 z^2 f(q^{1-\delta} z), \] (2.8)

where \(j = 1, 2, 3\) are connected with \(\delta = 2, 0, 1\) by relations (2.4).
Corollary 2.1 The function $I_{-\nu}^{(j)}((1 - q^2)z; q^2)$ satisfies equation (2.8).

Definition 2.2 We define the q-Bessel-Macdonald function (q-BMF) for $j = 1, 2, 3$ as follows [2, 4]:

$$K_{-\nu}^{(j)}((1 - q^2)z; q^2) = \frac{1}{2}q^{-\nu + \nu'}\Gamma_{q^2}(\nu)\Gamma_{q^2}(1 - \nu) \left[ A_{\nu}^{[1 - \delta]} I_{-\nu}^{(j)}((1 - q^2)z; q^2) - A_{-\nu}^{[1 - \delta]} I_{\nu}^{(j)}((1 - q^2)z; q^2) \right], \quad (2.9)$$

where

$$A_{\nu} = \sqrt{\frac{I_{\nu}^{(2)}(2; q^2)}{I_{-\nu}^{(2)}(2; q^2)}}. \quad (2.10)$$

As in the classical case, this definition must be extended to integral values of $\nu = n$ by passing to the limit in (2.9).

Definition 2.3 The $q$-Wronskian of two solutions $f_{\nu}^1(z)$ and $f_{\nu}^2(z)$ of a second-order difference equation is defined as follows:

$$W(f_{\nu}^1, f_{\nu}^2)(z) = f_{\nu}^1(z)f_{\nu}^2(qz) - f_{\nu}^1(qz)f_{\nu}^2(z).$$

If the $q$-Wronskian does not vanish, then any solution of the second-order difference equation can be written in form

$$f_{\nu}(z) = C_1 f_{\nu}^1(z) + C_2 f_{\nu}^2(z).$$

In this case the functions $f_{\nu}^1(z)$ and $f_{\nu}^2(z)$ form a fundamental system of the solutions of the given equation.

Proposition 2.2 The functions $I_{-\nu}^{(j)}((1 - q^2)z; q^2)$ and $K_{\nu}^{(j)}((1 - q^2)z; q^2)$ form a fundamental system of the solutions of equation (2.8) ($z \neq \pm \frac{2q - r}{1 - q}, r = 0, 1, \ldots$ if $j = 1$).

This Proposition is following from

$$W(z) = \begin{cases} q^{-\nu}(1 - q^2)A_{\nu}e_{q^2}(\frac{(1-q^2)^2}{4}z^2) & \text{for } \delta = 2 \\ q^{-\nu}(1 - q^2) & \text{for } \delta = 1 \\ q^{-\nu}(1 - q^2)A_{\nu}e_{q^2}(\frac{(1-q^2)^2}{4}q^2z^2) & \text{for } \delta = 0. \end{cases}$$

(See [4].) Obviously, this function is defined for $z \neq \pm \frac{2q - r}{1 - q}, r = 0, 1, \ldots$ if $j = 1$ ($\delta = 2$) and does not vanish.
3 Some ancillary formulas

There is a $q$-analog of the classical binomial formula 

$$(1 - z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}, \quad |z| < 1,$$

$$(q^\alpha z, q)^\infty = \sum_{k=0}^{\infty} \frac{(q^\alpha, q)_k}{(q, q)_k} z^k, \quad |z| < 1.$$

We need in two generalizations of the $q$-binomial

$$r(a, b, z, q) = \frac{(az, q)^\infty}{(bz, q)^\infty}$$

$$R(a, b, \gamma, z, q^2) = \frac{(az^2, q^2)^\infty}{(bz^2, q^2)^\infty} z^\gamma$$

**Proposition 3.1** The function $R(a, b, \gamma, z, q^2)$ satisfies the difference equation

$$z^2[bq^\gamma R(a, b, \gamma, z, q^2) - aR(a, b, \gamma, qz, q^2)] = q^\gamma R(a, b, \gamma, z, q^2) - R(a, b, \gamma, qz, q^2).$$

The **Proof** see in [3]

It was shown in [3] that if $\alpha > \beta$ then

$$\frac{(-q^{2\alpha} z^2, q^2)^\infty}{(-q^{2\beta} z^2, q^2)^\infty} = \frac{(q^{2(\alpha-\beta)}, q^2)^\infty}{(q^2, q^2)^\infty} \sum_{k=0}^{\infty} \frac{(q^{2(\beta-\alpha+1)}, q^2)_k q^2(\alpha-\beta-1)k}{(q^2, q^2)_k (1 + z^2 q^{2\beta + 2k})}.$$  

**Remark 3.1** Let $a = \epsilon q^{2\alpha}, b = \epsilon q^{2\beta}, \epsilon = \pm 1$, in (3.3). Then if $q \to 1 - 0$ the difference equation (3.3) takes the form of the differential equation

$$z(1 - \epsilon z^2) R'(z) + [\gamma + \epsilon(2\alpha - 2\beta - \gamma) z^2] R(z) = 0$$

with solution

$$R(z) = C z^{\gamma} (1 - \epsilon z^2)^{\beta - \alpha}.$$ 

Designate the $q$-derivative of $g(x)$ by

$$\partial_x g(x) = \frac{g(x) - g(qx)}{(1 - q)x}.$$ 

It follows from

$$J_{0}^{(3)}((1 - q^2) z; q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{(2-\delta)k^2} (1 - q^2)^{2k} (sz/2)^{2k}}{(q^2, q^2)_k^2}$$
that
\[ 2\partial_s \frac{J^j_0}{1 + q} ((1 - q^2)q^{-\frac{1}{2}}s; q^2) = -q^{1 - \frac{j}{2}} s J^j_1((1 - q^2)q^{-\frac{j}{2}}s; q^2), \]  
(3.7)
\[ 2\partial_s \frac{J^j_0}{1 + q} ((1 - q^2)zs; q^2) = -q^{1 - \frac{j}{2}} s J^j_1((1 - q^2)q^{-\frac{j}{2}}zs; q^2), \]  
(3.8)

and
\[ 2\partial_s \frac{J^j_1}{1 + q} ((1 - q^2)q^{-\frac{j}{2}}zs; q^2) = q^{-\frac{j}{2}} zs J^j_0((1 - q^2)q^{-\frac{j}{2}}zs; q^2) \]  
(3.9)

Lemma 3.1 If \( F(x) \) is a differentiable (in classical sense) function in some neighborhood of zero then
\[ \lim_{\epsilon \to 0} \int_{q\epsilon}^{\epsilon} \frac{F(x)}{x} dx = -F(0) \ln q. \]  
(3.10)

Proof. Integrating by parts we have
\[
\lim_{\epsilon \to 0} \int_{q\epsilon}^{\epsilon} \frac{F(x)}{x} dx = \lim_{\epsilon \to 0} \left[ F(x) \ln x|_{q\epsilon}^\epsilon - \int_{q\epsilon}^{\epsilon} F'(x) \ln x dx \right] = \\
= \lim_{\epsilon \to 0} \left[ (F(\epsilon) - F(q\epsilon)) \ln \epsilon - F(q\epsilon) \ln q - \int_{q\epsilon}^{\epsilon} F'(x) \ln x dx \right].
\]
Using Lagrange's theorem and the theorem about mean value we obtain in the left side
\[
\lim_{\epsilon \to 0} [F'(\theta_1\epsilon)(1 - q)\epsilon \ln \epsilon - F(q\epsilon) \ln q - (1 - q)\epsilon F'(\theta_2\epsilon) \ln(\theta_2\epsilon)],
\]
where \( \theta_1 \in (0, 1), \ \theta_2 \in (0, 1) \). As \( \lim_{\epsilon \to 0} \epsilon \ln \epsilon = 0 \) we have (3.11).

Lemma 3.2 If \( f(x) \) and \( g(x) \) are integrable on \((0, \infty)\) and differentiable ones in zero (in classical sense) then the \( q \)-analog of formula of integration by parts for the \( q \)-derivative takes place
\[
\int_0^\infty \partial_x f(x) g(x) dx = f(0)g(0) \frac{\ln q}{1 - q} - \int_0^\infty f(qx) \partial_x g(x) dx.
\]  
(3.11)

Proof.
\[
\int_0^\infty \partial_x f(x) g(x) dx = \int_0^\infty \frac{f(x) - f(qx)}{(1 - q)x} g(x) dx = \\
= \int_0^\infty \frac{f(x)g(x) - f(qx)g(qx)}{(1 - q)x} dx - \int_0^\infty \frac{f(qx)g(x) - g(qx)}{(1 - q)x} dx = \\
= \int_0^\infty \partial_x (f(x)g(x)) dx - \int_0^\infty f(qx) \partial_x g(x) dx.
\]
Using Lemma 3.1 calculate the first integral in the right side.
\[
\int_0^\infty \partial_x (f(x)g(x)) dx = \frac{1}{1 - q} \lim_{\epsilon \to 0} \left[ \int_\epsilon^\infty \frac{f(x)g(x)}{x} dx - \int_\epsilon^\infty \frac{f(qx)g(qx)}{x} dx \right] = \\
= -\frac{1}{1 - q} \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{f(x)g(x)}{x} dx = \frac{\ln q}{1 - q} f(0)g(0).
\]
The statement of the Lemma follows from here. ■
Corollary 3.1 If \( f(s)g(s)s^{-1} \) is integrable function on \((0, \infty)\) then
\[
\int_0^\infty f(s)\partial_s g(s)ds = -\int_0^\infty \partial_s f(s)g(qs)ds.
\]
(3.12)
The proof follows from \( f(0)g(0) = 0 \) in this case. ■

4 The integral representations of the \( q \)-BMFs

We will assume that \( z \) and \( s \) are the commuting variables.

Proposition 4.1 \( q \)-BMF \( K^{(j)}(1-q^2z; q^2) \) for \( Re \nu > 0 \) can be represented as the integral
\[
K^{(j)}(1-q^2z; q^2) = \frac{-q^{-\nu^2+\nu(1-\delta)}(1-q^2)^{2\nu}(\nu+1)A_{\nu}^{1-\delta}}{2\ln q} \times \]
\[
\times (z/2)^{-\nu} \int_0^\infty \frac{(-q^{2\nu+2-\delta}r_s^2q^2)^\infty}{(-q^{-\delta}s^2,q^2)^\infty} sJ_0^{(j)}((1-q^2)zs; q^2)ds,
\]
where the constant \( A_{\nu} \) is defined by (2.10) and \( j = 1, 2, 3 \) are connected with \( \delta = 2, 0, 1 \) by relations (2.4).

Proof. Consider the absolutely convergent integral
\[
S^{(j)}(z) = \int_0^\infty f^{(j)}(s)sJ_0^{(j)}((1-q^2)zs; q^2)ds,
\]
and require that \( S^{(j)}(z)(z/2)^{-\nu} \) satisfies the difference equation (2.8). Then \( S^{(j)}(z) \) satisfies the equation
\[
S^{(j)}(q^{-1}z) - S^{(j)}(z) = q^{-2\nu}(S^{(j)}(z) - S^{(j)}(qz)) = q^{\nu(\delta-1)-\delta}(1-q^2)^{2\nu}zS^{(j)}(q^{1-\delta}z).
\]
(4.3)
Substituting (3.2) in (4.3) and multiplying it on \( \frac{2z^{-1}}{1-q^2} \) we obtain
\[
\int_0^\infty f^{(j)}(s)s\frac{2z^{-1}}{1-q^2}[J_0^{(j)}((1-q^2)q^{-1}zs; q^2) - J_0^{(j)}((1-q^2)zs; q^2)]ds -
\]
\[
-q^{2\nu}\int_0^\infty f^{(j)}(s)s\frac{2z^{-1}}{1-q^2}[J_0^{(j)}((1-q^2)zs; q^2) - J_0^{(j)}((1-q^2)qzs; q^2)]ds =
\]
\[
= q^{\nu(\delta-2)-\delta}\frac{1-q^2}{2z} \int_0^\infty f^{(j)}(s)sJ_0^{(j)}((1-q^2)q^{1-\delta}zs; q^2)ds.
\]
Due to (3.7) - (3.9) we can write
\[
\int_0^\infty f^{(j)}(s)sJ_1^{(j)}((1-q^2)q^{-\delta}zs; q^2)ds - q^{-2\nu+1}\int_0^\infty f^{(j)}(s)s^2J_1^{(j)}((1-q^2)q^{1-\delta}zs; q^2)ds =
\]
\[ q^{\nu(\delta-2)-\frac{\delta}{2} \frac{1}{2}-\frac{q^2}{2} s} \int_0^\infty f^{(j)}(s) s f_0^{(j)}((1-q^2)q^{1-\delta}zs; q^2)^2 ds \]

or

\[ \int_0^\infty f^{(j)}(s)s^{2\nu+2} \frac{2\delta s}{1+q} [s^{-2\nu+1} f^{(j)}_1((1-q^2)q^{-\frac{\delta}{2}}zs; q^2)] ds = \]

\[ = -q^{\nu(\delta-2)} \int_0^\infty f^{(j)}(s) \frac{2\delta s}{1+q} f^{(j)}_1((1-q^2)q^{-\frac{\delta}{2}}zs; q^2)^2 ds. \]

Using (3.12) we obtain

\[ \int_0^\infty \partial_s(f^{(j)}_s(s)s^{2\nu+2})q^{-2\nu+1}s^{-2\nu+1}f^{(j)}_1((1-q^2)q^{-\frac{\delta}{2}}zs; q^2)^2 ds = \]

\[ = -q^{\nu(\delta-2)} \int_0^\infty \partial_s f^{(j)}(s)qs J_2^{(j)}((1-q^2)q^{-\frac{\delta}{2}}zs; q^2)^2 ds. \]

Thus we come to the difference equation for \( f^{(j)}(s) \)

\[ q^{-\delta s} \frac{2}{2}[f^{(j)}(s) + q^{2
u+2}f^{(j)}(qs)] = f^{(j)}(s) - f^{(j)}(qs). \]  

(4.4)

It follows from Proposition 3.1 that the function

\[ f^{(j)}(s) = \frac{(-q^{2\nu+2-\delta s}q^2)_{\infty}}{(-q^{-\delta s}q^2)_{\infty}} \]

satisfies to (4.4), and it follows from (3.4) integral (4.2) is absolutely convergent.

As \( S^{(j)}(z)(z/2)^{-\nu} \) is a solution to (2.8) it can be represented as

\[ S^{(j)}(z)(z/2)^{-\nu} = A I^{(j)}_\nu((1-q^2)z; q^2) + B K^{(j)}_\nu((1-q^2)z; q^2). \]  

(4.6)

Let \( j = 1 \). As it follows from (2.9) \( f^{(1)}(s)((1-q^2)z; q^2) \) is a meromorphic function with the ordinary poles \( z = \pm \frac{2q^r}{1-q^2}, \) \( r = 0, 1, \ldots \), and \( K_\nu^{(1)}((1-q^2)z; q^2) \) and the left side of (4.6) are the holomorphic functions in region \( \text{Re} z > 0 \).

Let \( j = 2, 3 \). It is easily to show (see (1)) that \( \lim_{z \to \infty} f^{(j)}_\nu((1-q^2)z; q^2) = \infty, \) \( \lim_{z \to \infty} K^{(j)}_\nu((1-q^2)z; q^2) = 0 \) and the left side of (4.6) tends to zero if \( z \to \infty \).

So for any \( j = 1, 2, 3 \) \( A = 0 \). Multiplying the both sides of (4.6) on \( (z/2)^{\nu} \) and putting \( z = 0 \) we obtain from (2.9) and (2.3) - (2.7)

\[ \int_0^\infty \frac{(-q^{2\nu+2-\delta s}q^2)_{\infty}}{(-q^{-\delta s}q^2)_{\infty}} sds = \frac{B}{2} q^{\nu+2} \Gamma(q^2)(1-\delta) A^{(1-\delta)}_\nu \]  

(4.7)

Now calculate the integral in the left side of (4.7).

\[ \int_0^\infty \frac{(-q^{2\nu+2-\delta s}q^2)_{\infty}}{(-q^{-\delta s}q^2)_{\infty}} sds = \frac{q^{\delta s}}{2} \int_0^\infty \frac{(-q^{1+1}x, q_1)_{\infty}}{(-x, q_1)_{\infty}} dx = \frac{q^{\delta s}}{2} \int_0^\infty e_{q_1}(-x) E_{q_1}(q^{1+1}x) dx, \]

where \( x = q^{-\delta s}q^2 \) and \( q_1 = q^2 \).
Note, that
\[ \partial_x e_{q_1}(-x) = -\frac{1}{1-q_1} e_{q_1}(-x), \quad \partial_x E_{q_1}(q_1^{\nu+1}x) = \frac{q_1^{\nu+1}}{1-q_1} E_{q_1}(q_1^{\nu+2}x). \]

Using Lemma 3.2 we obtain
\[
\int_0^\infty e_{q_1}(-x)E_{q_1}(q_1^{\nu+1}x)dx = -\ln q_1 + q_1^\nu \int_0^\infty e_{q_1}(-x)E_{q_1}(q_1^{\nu+1}x)dx.
\]
So
\[
\int_0^\infty e_{q_1}(-x)E_{q_1}(q_1^{\nu+1}x)dx = -\frac{\ln q_1}{1-q_1},
\]
and we have
\[
\int_0^\infty -q^{2\nu+2-\delta^2}s^2 q_2\infty sds = q^{\delta^2} \frac{\ln q}{1-q^{2\nu}}.
\]
(4.8)

It follows from (4.8) that
\[
B = -\frac{2q^{2-\nu(1-\delta)} \ln q}{(1-q^2)\Gamma(q^2(\nu+1)A_{\nu}[1-\delta])},
\]
and we have (4.1).

**Remark 4.1** It follows from [2, Remark 5.1] \( A_\nu \to 1 \) if \( q \to 1-0 \), and it follows from Remark [7.4] that if \( q \to 1-0 \) we come to the classical integral representation of Bessel-Macdonald function [6]
\[
K_\nu(z) = \Gamma(\nu+1) \left( \frac{z}{2} \right)^{-\nu} \int_0^\infty (1+s^2)^{-\nu-1}sJ_0(zs)ds.
\]

## 5 The representation of the \( q \)-BMFs by a double integral

We take \( z, s \in \mathbb{C} \) in this section.

Consider function
\[
\xi^{(\delta)}(s) = \sum_{n=0}^{\infty} q(2-\delta)\eta^2 (1-q^2)^n s^n, \quad \eta \geq 0.
\]
(5.1)

Obviously this series converges for any \( s \) if \( (2-\delta)\eta > 0 \) and for \( |s| < \frac{1}{1-q} \) if \( (2-\delta)\eta = 0 \).

Consider three cases.

1. \( \eta = 0 \). It follows from (5.1)
\[
\xi_0^{(\delta)}(s) = e_{q^2}((1-q^2)s), \quad \delta = 2, 0, 1.
\]
(5.2)

2. \( \eta = 1 \).
\[
\xi_1^{(\delta)}(s) = \begin{cases} e_{q^2}((1-q^2)s) & \text{for } \delta = 2 \\ E_{q^2}((1-q^2)qs) & \text{for } \delta = 0 \\ \Phi_1(0;-q;q, -(1-q^2)q^2s) & \text{for } \delta = 1. \end{cases}
\]
(5.3)
3. \( \eta = \frac{1}{2} \). It follows from (5.3)

\[
\xi^{(\delta)}_{\frac{1}{2}}(\eta) = \left\{ \begin{array}{ll}
  e^{q^2(1 - q^2)s} & \text{for } \delta = 2 \\
  1\Phi_1(0; -q, -q, (1 - q^2)^{\frac{1}{2}}s) & \text{for } \delta = 0. \\
  3\Phi_3(0, 0, 0; -q^\frac{1}{2}, iq^\frac{1}{2}, -iq^\frac{1}{2}; q^\frac{1}{2}, -(1 - q^2)q^\frac{1}{2}s) & \text{for } \delta = 1.
\end{array} \right. \tag{5.4}
\]

Assume \( s = \rho e^{i\psi}, z = re^{i\psi} \), and consider integral

\[
J(r\rho) = \int_{-\pi}^{\pi} \xi^{(\delta)}_{\eta}(ir\rho e^{-i(q+\psi)})\xi^{(\delta)}_{1-\eta}(ir\rho e^{i(q+\psi)})d\phi. \tag{5.5}
\]

Let \( \delta < 2, \ \eta = \frac{1}{2} \). In this case we can calculate this integral term by term.

\[
J(r\rho) = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} \frac{q^{(2-\delta)r^2}}{(q^2, q^2)} (1 - q^2)^n (ir\rho)^n e^{-i\phi} \sum_{m=0}^{\infty} \frac{q^{(2-\delta)m^2}}{(q^2, q^2)} (1 - q^2)^m (ir\rho)^m e^{i\phi} d\phi =
\]

\[
= \sum_{n=0}^{\infty} \frac{q^{(2-\delta)n^2}}{(q^2, q^2)} (1 - q^2)^n (ir\rho)^n \sum_{m=0}^{\infty} \frac{q^{(2-\delta)m^2}}{(q^2, q^2)} (1 - q^2)^m (ir\rho)^m \int_{0}^{2\pi} e^{i(m-n)\phi} d\phi =
\]

\[
= 2\pi \sum_{n=0}^{\infty} (-1)^n \frac{q^{(2-\delta)n^2}}{(q^2, q^2)} (1 - q^2)^{2n} (r\rho)^{2n}.
\]

It follows from (3.4) the last series is \( q^2 \)-Bessel function and we have

\[
J_0^{(j)}((1 - q^2)2r\rho; q^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi^{(\delta)}_{\eta}(ir\rho e^{-i(q+\psi)})\xi^{(\delta)}_{\frac{1}{2}}(ir\rho e^{i(q+\psi)})d\phi, \tag{5.6}
\]

where \( \xi^{(\delta)}_{\frac{1}{2}} \) is defined by (5.2) and \( j = 2, 3 \) are connected with \( \delta = 0, 1 \) by relations (2.3).

Let \( \delta < 2 \) and \( \eta = 0 \). Then we have the same result for \( r\rho < \frac{1}{1-q^2} \), e.i.

\[
J_0^{(j)}((1 - q^2)2r\rho; q^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{q^2(i(1 - q^2)r\rho e^{-i(q+\psi)})\xi^{(\delta)}_{1}(ir\rho e^{i(q+\psi)})d\phi, \tag{5.7}
\]

where \( \xi^{(de)}_{\frac{1}{2}} \) is defined by (5.3). The left side is a holomorphic function outside a neighborhood of zero, and so we can consider \( J_0^{(j)}((1 - q^2)2r\rho; q^2), j = 2, 3 \) as the analytic continuation of (5.8).

Let \( \delta = 2 \). It follows from (5.2) - (5.4) for \( r\rho < \frac{1}{1-q^2} \)

\[
J_0^{(1)}((1 - q^2)2r\rho; q^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{q^2(i(1 - q^2)r\rho e^{-i(q+\psi)})e^{q^2(i(1 - q^2)r\rho e^{i(q+\psi)})d\phi. \tag{5.8}
\]

In this case the both sides of (5.8) are the meromorphic functions with the ordinary poles in points \( r\rho = \pm \frac{q^2}{1-q^2} \). So we will consider \( J_0^{(1)}((1 - q^2)2r\rho; q^2) \) as a analytic continuation of (5.8).

Now we can formulate
Proposition 5.1 The $q$-BMF can be represented by double integral

$$K_{\nu}^{(j)}(2(1-q^2)|z|, q^2) = -\frac{q^{-\nu^2+\nu(1-\delta)}(1-q^2)}{8\pi \ln q} \Gamma_{q^4}(\nu(1-\delta)) A_{\nu}^{1-\delta} |z|^{-\nu} \times$$

$$\times \int \int \frac{(1-q^2)}{(-q^{-\delta \nu \bar{s} \bar{s}, q^2})} e_{q^2}(i(1-q^2)\bar{z}\bar{s}) \xi_{1}(iz\bar{s})d\bar{s}d\bar{s}$$

or

$$K_{\nu}^{(j)}(2(1-q^2)|z|, q^2) = -\frac{q^{-\nu^2+\nu(1-\delta)}(1-q^2)}{8\pi \ln q} \Gamma_{q^4}(\nu(1-\delta)) A_{\nu}^{1-\delta} |z|^{-\nu} \times$$

$$\times \int \int \frac{(1-q^2)}{(-q^{-\delta \nu \bar{s} \bar{s}, q^2})} \xi_{1}(iz\bar{s})\xi_{1}(iz\bar{s})d\bar{s}d\bar{s},$$

where $\xi_{1}(\delta)$ are defined by (5.2) - (5.4), the constant $A_{\nu}$ is defined by (2.10) and $j = 1, 2, 3$ are connected with $\delta = 2, 0, 1$ by relations (2.4).

Proof. Substituting (5.6) - (5.8) in (4.1) we obtain (5.9) or (5.10) respectively.

Remark 5.1 It is easily to show that if $q \to 1 - 0$ the all functions (5.3) - (5.4) tend to the usual exponential $e^s$, and we come to the classical integral representation of Bessel-Macdonald function

$$K_{\nu}(2\sqrt{\bar{z}\bar{s}}) = \Gamma(\nu+1)(\sqrt{\bar{z}\bar{s}})^{-\nu} \int \int (1+\bar{s}\bar{s})^{-\nu-1} \exp(i(\bar{z}\bar{s} + zs))d\bar{s}d\bar{s}.$$