BIRATIONAL MODELS OF THE MODULI SPACES OF STABLE VECTOR BUNDLES OVER CURVES

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Abstract. We give a method to construct stable vector bundles whose rank divides the degree over curves of genus bigger than one. The method complements the one given by Newstead. Finally, we make some systematic remarks and observations in connection with rationality of moduli spaces of stable vector bundles.

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1. Introduction

1.1. Let $X$ be a complete non-singular algebraic curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic 0 (for simplicity we assume that $k$ is the field of complex numbers $\mathbb{C}$). The moduli space $\mathcal{M}(n, L)$ of semi-stable vector bundles of rank $n$ and determinant $L$ with degree $\deg L = d$ over $X$ is an irreducible projective variety of dimension $(n^2 - 1)(g - 1)$. The question whether the moduli space $\mathcal{M}(n, L)$ is rational is very subtle. The affirmative answer is known for many cases through the works of Narasimhan-Ramanan ([NR]), Newstead ([N1, N2]), and Tjurin ([T3]). In proving rationality in [N1, N2], Newstead conjectured a systematic way to construct (generic) stable vector bundles $F$ from (generic) stable vector bundles $F'$ of lower rank via extensions of the following type:

$$
0 \to O_X^{\oplus r} \to F \to F' \to 0
$$

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where rank$(F) = n$ and deg$F = d = n(g - 1) + r$ for some $0 < r < n$. He proved the conjecture in [N1, N2] for some cases and Grzegorczyk completed the proof for all cases in her paper [G]. In [N2], this extension was used as an induction step to show that the following moduli spaces of stable vector bundles are rational.

**Theorem 1.2.** ([N2]) The moduli space $\mathcal{M}(n, L)$ is rational in the following cases:

1. $d = 1 \mod n$ or $d = -1 \mod n$;
2. gcd$(n, d) = 1$ and $g$ is a prime power;
3. gcd$(n, d) = 1$ and the sum of the two smallest distinct prime factors of $g$ is greater than $n$.

**1.3.** Clearly, from that $d = n(g - 1) + r$ and $0 < r < n$, one sees that the above method of constructing stable vector bundles leaves out the case when rank divides degree. It is one of the objectives of the current paper to find a method of constructing stable vector bundles in complement to that of [N1, N2, G].

**Theorem 1.4.** Let $n, d$ be positive integers such that $d = ng$. Let $L = \mathcal{O}_X(P_0)$ where $P_0$ is a special effective divisor defined in 2.3. Then there exists a non-empty Zariski open subset of the moduli space $\mathcal{M}(n, L)$ consisting of vector bundles $V$ such that

1. $h^1(X, V) = 0$.
2. $\mathcal{O}_X^{\oplus n}$ is a sub-sheaf of $V$.
3. there exists an exact sequence

$$0 \longrightarrow V^* \longrightarrow \mathcal{O}_X^{\oplus n} \overset{\varphi}{\longrightarrow} \mathcal{O}_P \longrightarrow 0$$

where $P$ is a divisor $P = p_1 + \ldots + p_d$ such that the map $\varphi$ in $\text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P)$ is stable with respect to the action of Aut$(\mathcal{O}_X^{\oplus n}) \times$ Aut$(\mathcal{O}_P)$.

**Remark 1.5.** We point out that Tjurin ([T1, T2]) studied these moduli spaces using the different method of matrix divisors.

For the purpose of comparison, we mention a result of Grzegorczyk [G] who proved the following conjecture of Newstead:

**Theorem 1.6.** ([N1, G]) Let, $n, d, r$, be natural numbers such that $d = n(g - 1) + r$, $0 < r < n$. Then there exists a non-empty Zariski open subset of the moduli space $\mathcal{M}(n, L)$ consisting of vector bundles $F$ such that

1. $h^1(F) = 0$. 
2. $\mathcal{O}_X^{\oplus r}$ is a sub-bundle of $F$.
3. the quotient bundle $F/\mathcal{O}_X^{\oplus r}$ is stable.

This theorem covers all moduli spaces except those where $n \mid d$. Our theorem above deals exactly the complement.

1.7. The tool that we used to construct stable vector bundles is that of elementary transformations which were probably first studied by Maruyama ([M]). We choose a special effective divisor $P_0$ (see [2.3]) whose degree equals $ng$. Take $L$ to be $\mathcal{O}_X(P_0)$. Let $U$ be a non-empty Zariski open subset of the linear system $|P_0|$ satisfying certain generic condition (see [2.3]). Let $P$ be an effective divisor in $U$. Now given a surjective map $\varphi \in \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P)$ (which is called an elementary transformation), we get an exact sequence

$$0 \rightarrow W \rightarrow \mathcal{O}_X^{\oplus n} \varphi \rightarrow \mathcal{O}_P \rightarrow 0.$$ 

Let $V = W^*$. Then we see that rank$V = n$, det$V = \mathcal{O}_X(P) = \mathcal{O}_X(P_0) = L$. Then Theorem [1.4] (see also §§3–5) basically asserts that if the elementary transformation satisfies some generic conditions, then $V$ is stable and $h^1(V) = 0$. Furthermore, the stable vector bundles $V$ obtained this way form a non-empty Zariski open subset of the moduli space and hence it leads to a birational model for the moduli space.

1.8. In the end of this paper, we shall systematically explore that the inductive method in ([N2]) can actually be extended to prove rationality of a rather larger class of moduli spaces. This class consists of much more cases than those listed in Theorem [1.2].

For example, when $g = 6$, $n = 15$, $d = 77$, the moduli space $\mathcal{M}(n, L)$ is rational. Also, the moduli spaces $\mathcal{M}(n, L)$ with

$$n = 11 + 7m, \quad \text{deg} L = 62 + 35m$$

are all rational over any genus 6 curve for $m \geq 7$. But the set \{g = 6, n = 15, d = 77\} and \{g = 6, n = 11 + 7m, d = 62 + 35m\} satisfy none of the three conditions as listed in Theorem [1.2].

In general, the moduli space $\mathcal{M}(n, L)$ is rational if $n(g - 1) < d = \text{deg} L < ng$ and the pair $(n; d)$ can be “linked” to $(1; g)$ by some successive arithmetic reductions of the following types:

1. $(n'; d') = (ng - d; d - k(ng - d))$ for some non-negative integer $k$ such that $n'(g - 1) < d' \leq n'g$; or
2. $(n''; d'') = (d - n(g - 1); n'(2g - 1) - d - k(d - n(g - 1)))$ for some non-negative integer $k$ such that $n''(g - 1) < d'' \leq n''g$.

This fact (Theorems [6.14] and [6.21]), together with some other results, is systematically explored in §6.
Also, we shall point out how our Theorem 1.4 is related to other moduli spaces by similar arithmetic reductions as above.

1.9. The paper is structured as follows. In §2, we recall the Abel-Jacobi theory and other technical results that will be used in the latter sections. §3 deals with elementary transformations and how to use them to construct stable vector bundles as stated in Theorem 1.4. §4 studies the group action of $\text{Aut}(\mathcal{O}_P)$ and $\text{Aut}(\mathcal{O}_{\mathbb{P}^n})$ on $\text{Hom}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_P)$. §5 concerns birational models for the moduli spaces and completes the proof of our main result Theorem 1.4. Finally, in §6, we make some systematic remarks and observations on rationality of moduli spaces.

1.10. We fix the following notations: $V$ is a locally free sheaf, by abuse of notation, we also call it a vector bundle. $V^*$ is the dual of $V$, i.e. $V^* = \text{Hom}(\mathcal{O}_X, V)$. $H^i(V)$ is the cohomology $H^i(X, V)$. $h^i(V)$ is the dimension of $H^i(V)$. $n$ is the rank of the vector bundle $V$. $d$ is the degree of the vector bundle $V$. $\text{det}V = \wedge^n V$ is the determinant line bundle. $L$ is a line bundle with $\text{deg}L = d$. $(M)^d = M \times \ldots \times M$ (d-times) (2.1), (2.2), ... , are the indices labeling formulas. 2.1, 2.2, ... , are the indices labeling theorems, remarks, paragraphs, and so on.

This note stems from our study of the rationality problem of moduli spaces when both of the authors were visiting Max-Planck-Institut für Mathematik in the summer of 1993. Due to the fact that there are few papers in the literature addressing the moduli spaces when the rank divides the degree, we decided to write up this note and make it available to interested readers.

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2. Preliminaries
Definition 2.1. Let $X$ be a smooth projective curve over the field of complex numbers and $V$ a rank-$n$ algebraic vector bundle over $X$. $V$ is stable (semi-stable) if for any proper sub-vector bundle $F$ of $V$,

$$\frac{\deg F}{\text{rank} F} < (\leq) \frac{\deg V}{\text{rank} V}. \tag{2.1}$$

Since any torsion free coherent sheaf over a curve is locally free, we can get a slightly modified definition of stability (or semi-stability): $V$ is stable (or semi-stable) if for any proper sub-vector bundle $F$ of $V$ whose cokernel $V/F$ is a vector bundle, (2.1) holds.

2.2. Let $L$ be a line bundle over $X$. Throughout this paper, $\mathcal{M}(n, L)$ represents the moduli space of rank-$n$ semi-stable vector bundles over $X$ with $\det V \cong L$.

2.3. We begin with a special divisor whose corresponding line bundle will be chosen as the fixed determinant of our semi-stable bundles.

Let $\omega_1, \ldots, \omega_g$ be a basis of $H^0(K_X)$. By the Abel-Jacobi theory (see [GH]), for general points $x_1, \ldots, x_g \in X$, the determinant

$$\begin{vmatrix}
\omega_1(x_1) & \omega_1(x_2) & \ldots & \omega_1(x_g) \\
\vdots & \vdots & \ddots & \vdots \\
\omega_g(x_1) & \omega_g(x_2) & \ldots & \omega_g(x_g)
\end{vmatrix}$$

is not zero. We now choose, once and for all, $d = ng$ many distinct points $q_1, \ldots, q_{ng}$ such that the determinants

$$\begin{vmatrix}
\omega_1(q_{i+1}) & \omega_1(q_{i+2}) & \ldots & \omega_1(q_{i+g}) \\
\vdots & \vdots & \ddots & \vdots \\
\omega_g(q_{i+1}) & \omega_g(q_{i+2}) & \ldots & \omega_g(q_{i+g})
\end{vmatrix} \neq 0 \tag{2.2}$$

for $i = 0, g, 2g, \ldots, (n - 1)g$. Here $n$ is a positive integer bigger than one which will be taken later on as the rank of vector bundles. Let $P_0$ be the effective divisor $P_0 = q_1 + \ldots + q_{ng}$. The linear system $|P_0|$ is a projective space of dimension $(n - 1)g$. This can be easily seen by Riemann-Roch.
2.4. For a technical reason as we shall see in §3, we need to consider the determinant

\[
\begin{vmatrix}
  b_{11}\omega_1(x_1) & b_{21}\omega_1(x_2) & \cdots & b_{d1}\omega_1(x_d) \\
  b_{11}\omega_2(x_1) & b_{21}\omega_2(x_2) & \cdots & b_{d1}\omega_2(x_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{11}\omega_g(x_1) & b_{21}\omega_g(x_2) & \cdots & b_{d1}\omega_g(x_d) \\
  b_{12}\omega_1(x_1) & b_{22}\omega_1(x_2) & \cdots & b_{d2}\omega_1(x_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{1n}\omega_1(x_1) & b_{2n}\omega_1(x_2) & \cdots & b_{dn}\omega_1(x_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{1n}\omega_g(x_1) & b_{2n}\omega_g(x_2) & \cdots & b_{dn}\omega_g(x_d)
\end{vmatrix}.
\tag{2.3}
\]

Take $b_{ij}$'s as unknowns, the determinant at the point $(q_1, \cdots, q_d) \in (X)^d$ (chosen as in (2.3)) is a non-vanishing polynomial in variables $b_{ij}$. The way to see this is as follows. We expand the determinant of (2.3) and check the coefficient of

\[
b_{11}b_{21} \cdots b_{g1}b_{g+1,2} \cdots b_{(2g),2} \cdots b_{(d-g)n} \cdots b_{dn}.
\]

By the technique of minor expansions in the determinant theory, the coefficient is

\[
\prod_{i=0}^{(n-1)g} \begin{vmatrix}
  \omega_1(q_{i+1}) & \omega_1(q_{i+2}) & \cdots & \omega_1(q_{i+g}) \\
  \vdots & \vdots & \ddots & \vdots \\
  \omega_g(q_{i+1}) & \omega_g(q_{i+2}) & \cdots & \omega_g(q_{i+g})
\end{vmatrix}
\]

which is not zero by the choices of $q_i$'s. Hence the determinant (2.3) as a polynomial in variables $b_{ij}$ is not identically zero.

2.5. Choose a collection of $b_{ij}$'s such that the determinant at $(q_1, \cdots, q_d)$ is not zero. With these $b_{ij}$'s fixed, the zero locus of the determinant (2.3) defines a divisor in $|P_0|$. So we can choose a non-empty Zariski open subset $U$ of $|P_0|$ such that every divisor $P = p_1 + \cdots + p_d$ in $U$ satisfies the property that $p_i$'s are distinct and that the determinant (2.3) with the chosen $b_{ij}$'s is non-zero at $P \in U$.

2.6. It is known that $\mathcal{M}(n, L) \cong \mathcal{M}(n, L')$ if $\deg L = \deg L'$. This can be seen as follows. There exists a line bundle $\bar{L} \in \text{Pic}^0(X)$ such that $\bar{L}^\otimes n = L^* \otimes L' \in \text{Pic}^0(X)$. We have that $\det(V \otimes \bar{L}) = \det V \otimes \bar{L}^\otimes n = L \otimes L^* \otimes L' = L' = \det V'$. Hence there is a bijection

\[
\mathcal{M}(n, L) \xrightarrow{\otimes \bar{L}} \mathcal{M}(n, L').
\tag{2.4}
\]
Since these two moduli spaces are coarse, the map (2.4) must be an isomorphism. Therefore, we have some freedom in choosing a line bundle $L$ in a way we like without affecting the isomorphic type of the moduli space $\mathcal{M}(n, L)$. In this paper, we choose $L = \mathcal{O}_X(P_0)$.

3. Stable vector bundles whose ranks divide their degrees

This section is entirely devoted to the case when the rank of the bundles divides the degree. We shall use elementary transformations to construct generic rank-$n$ stable vector bundles $V$ with degree $ng$ whose $H^1(V)$ is trivial. In the section that follows, we shall discuss the necessary group actions on the space of elementary transformations, which will lead to a birational model for the moduli space.

Definition 3.1. Let $W_1, W_2$ be two rank-$n$ vector bundles over an algebraic variety. Suppose that there exists an injective morphism from $W_1$ to $W_2$, then we can write a short exact sequence

$$0 \to W_1 \to W_2 \xrightarrow{\varphi} Q \to 0$$

where $Q$ is a torsion sheaf. In this case, we say that $W_1$ is an elementary transformation of $W_2$ with respect to the surjective map $W_2 \xrightarrow{\varphi} Q \to 0$; or we may simply say that the map $\varphi$ is an elementary transformation.

3.2. Let $P = p_1 + \ldots + p_d$ be an effective divisor over the curve $X$ such that $P \in U$ (see 2.5). Consider an elementary transformation:

$$0 \to W \to \mathcal{O}_X^\oplus n \xrightarrow{\varphi} \mathcal{O}_P \to 0.$$  

$W$ has to be a vector bundle. Let $V = W^*$ or $W = V^*$. Then the above exact sequence can be rewritten as

$$0 \to V^* \to \mathcal{O}_X^\oplus n \xrightarrow{\varphi} \mathcal{O}_P \to 0. \quad (3.1)$$

A simple computation leads to

$$\text{rank} V = n, \quad \text{det} V = \mathcal{O}_X(P) = \mathcal{O}_X(P_0) = L, \quad \text{and} \deg V = d.$$  

Such elementary transformations $\varphi$ are classified by surjective maps in

$$\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P) = \bigoplus_{i=1}^d \text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_{P_i}) = (\mathbb{C}^n)^d. \quad (3.2)$$
Notation 3.3. A map $\varphi$ in (3.1) can be expressed, under the isomorphisms in (3.2), as a matrix
\[
\begin{pmatrix}
\varphi_1 \\
\vdots \\
\varphi_d
\end{pmatrix} =
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{d1} & \cdots & a_{dn}
\end{pmatrix}
\] (3.3)
where $\varphi_i$ represents the vector $(a_{i1}, \ldots, a_{in})$ in $\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P) \cong \mathbb{C}^n$.

Remark 3.4. $\varphi$ is a surjection if and only if no rows $\varphi_i$ in (3.3) are zero vectors.

3.5. Tensor the exact sequence (3.1) by $K_X$ and then take the long cohomological exact sequence, we get
\[
0 \to H^0(V^* \otimes K_X) \to H^0(\mathcal{O}_X^\oplus n \otimes K_X) \xrightarrow{\varphi^0} H^0(\mathcal{O}_P) \\
\to H^1(V^* \otimes K_X) \to H^1(\mathcal{O}_X^\oplus n \otimes K_X) \to 0. \tag{3.4}
\]
We need to introduce two generic conditions on elementary transformations (3.1). We shall then show that if an elementary transformation satisfies these two conditions, then the vector bundle $V$ is stable and $h^1(V) = 0$.

Condition 3.6. We define the following:
1. (A) An elementary transformation $\varphi$ in $\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P)$ is said to satisfy Condition A if the induced map $\varphi^0$ of $\varphi$ in (3.4) is an isomorphism.
\[
\varphi^0 : H^0(K_X^\oplus n) \to H^0(\mathcal{O}_P)
\]
2. (B) An elementary transformation $\varphi$ in $\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P)$ is said to satisfy Condition B if any $n$ many $\varphi_i$’s in (3.3) are linearly independent.

Remark 3.7. 1. From the exact sequence (3.4), we see that if $\varphi$ satisfies Condition A, then $h^0(V^* \otimes K_X) = 0$, or $h^1(V) = 0$ by Serre duality. Converse is also true. That is, if $h^1(V) = 0$, then $\varphi^0$ is injective. Because $h^0(\mathcal{O}_X^\oplus n \otimes K_X) = h^0(\mathcal{O}_P) = d$, it has to be an isomorphism. Hence Condition A is equivalent to $h^1(V) = 0$. By Riemann-Roch, it is also equivalent to $h^0(V) = n$.
2. Generic $d \times n$ matrix satisfies Condition B. Also this property is clearly invariant under the natural action of $\text{Aut}(\mathcal{O}_X^\oplus n)$ and $\text{Aut}(\mathcal{O}_P)$ on $\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P)$.

Lemma 3.8. Fix any divisor $P$ in $U$. Generic elementary transformation $\varphi$ in (3.1) satisfies Condition A.
Proof. First notice that \( h^0(K_X^{\otimes n}) = ng = d = h^0(\mathcal{O}_P) \). Let \( \omega_1, \ldots, \omega_g \) be a basis of \( H^0(K_X) \) as chosen in 2.3. Take
\[
(\omega_1, 0, \ldots, 0), \ldots, (\omega_g, 0, \ldots, 0), \ldots, (0, \ldots, 0, \omega_1), \ldots, (0, \ldots, 0, \omega_g)
\]as a basis of \( H^0(K_X^{\otimes n}) \). One checks that
\[
\varphi^0(\omega_1, 0, \ldots, 0) = (a_{11}\omega_1(p_1), a_{21}\omega_1(p_2), \ldots, a_{d1}\omega_1(p_d)).
\]
By the similar calculation for other elements of the basis, we get a natural matrix representation of the map \( \varphi^0 \):
\[
\begin{pmatrix}
  a_{11}\omega_1(p_1) & a_{21}\omega_1(p_2) & \cdots & a_{d1}\omega_1(p_d) \\
  a_{11}\omega_2(p_1) & a_{21}\omega_2(p_2) & \cdots & a_{d1}\omega_2(p_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{11}\omega_g(p_1) & a_{21}\omega_g(p_2) & \cdots & a_{d1}\omega_g(p_d) \\
  a_{12}\omega_1(p_1) & a_{22}\omega_1(p_2) & \cdots & a_{d2}\omega_1(p_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{12}\omega_g(p_1) & a_{22}\omega_g(p_2) & \cdots & a_{d2}\omega_g(p_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1n}\omega_1(p_1) & a_{2n}\omega_1(p_2) & \cdots & a_{dn}\omega_1(p_d) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{1n}\omega_g(p_1) & a_{2n}\omega_g(p_2) & \cdots & a_{dn}\omega_g(p_d)
\end{pmatrix}
\]
(3.5)
Hence \( \varphi \) satisfies Condition A iff the determinant of (3.5) is not zero.

Regarding the determinant of (3.5) as a polynomial in variables \( a_{ij} \) with \( \omega_i(p_j) \)'s fixed, either the polynomial is identically zero or the zero locus of this polynomial is a divisor in the space \( \text{Hom}(\mathcal{O}_X^{\otimes n}, \mathcal{O}_P) \). However we know that if \( P \) is in \( U \), the determinant of the matrix (3.5) is not zero at \( a_{ij} = b_{ij} \) (see 2.3). Hence the set of the elementary transformations \( \varphi \) satisfying Condition A is a non-empty open dense subset of the space \( \text{Hom}(\mathcal{O}_X^{\otimes n}, \mathcal{O}_P) \).

\[ \square \]

Corollary 3.9. For generic elementary transformations \( \varphi \in \text{Hom}(\mathcal{O}_X^{\otimes n}, \mathcal{O}_P) \),
\[
0 \longrightarrow V^* \longrightarrow \mathcal{O}_X^{\otimes n} \xrightarrow{\varphi} \mathcal{O}_P \longrightarrow 0 \quad \text{where } P \in U,
\]
h^0(V) = n or equivalently h^1(V) = 0.

Theorem 3.10. Fix a divisor \( P \in U \). If an elementary transformation \( \varphi \) satisfies Condition A and B, then \( V \) is stable and satisfies:
1. \( \deg V = ng \) and \( \det V = \mathcal{O}_X(P) = L \);
2. \( h^1(V) = 0 \).
Proof. Suppose $V$ is not stable. Then there exist rank-$f$ vector bundle $F$, rank-$f'$ vector bundle $F'$, and an exact sequence:

$$0 \to F \to V \to F' \to 0 \quad (3.6)$$

such that $\deg F \geq \frac{f \cdot \deg V}{n} = fg$.

Take the dual of (3.1), we get an exact sequence

$$0 \to \mathcal{O}_X^{\oplus n} \to V \to \mathcal{O}_P \to 0. \quad (3.7)$$

Combining (3.6) and (3.7), we obtain a commutative diagram of exact sequences:

$$\begin{array}{c}
0 \\
\uparrow \\
F' \\
\uparrow \\
\mathcal{O}_X^{\oplus n} \\
\uparrow \\
V \\
\uparrow \\
\mathcal{O}_P \\
\downarrow \\
\downarrow \\
0
\end{array}$$

where $E$ is a rank-$f$ vector bundle and $Q$ is a subset of $P$.

Diagram (3.8) can be extended into another commutative diagram of exact sequences:

$$\begin{array}{c}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
E' & F' & \mathcal{O}_R \\
\uparrow & \uparrow & \uparrow \\
\mathcal{O}_X^{\oplus n} & \to V & \mathcal{O}_P \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & E & F & \mathcal{O}_Q \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}$$

where $R$ is a subset of $P$.

From Riemann-Roch, we get $h^0(F) = \deg F + f(1-g) + h^1(F) \geq f$ and equality holds iff $h^1(F) = 0$ and $\deg F = fg$. 
Take long cohomological exact sequence of the diagram (3.9), we get a commutative diagram of exact sequences:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(E') & \overset{\alpha_1}{\rightarrow} & H^0(F') & \overset{\alpha_2}{\rightarrow} & H^0(O_R) & \\
\uparrow & & \uparrow & & \uparrow \phi_1 & & \uparrow \phi_1 & \\
0 & \rightarrow & H^0(O_X^{\oplus n}) & \overset{\beta_1}{\rightarrow} & H^0(V) & \overset{\beta_2}{\rightarrow} & H^0(O_P) & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \rightarrow & H^0(E) & \overset{\gamma_1}{\rightarrow} & H^0(F) & \overset{\gamma_2}{\rightarrow} & H^0(O_Q) & \\
\end{array}
\]

(3.10)

Since the map \( \beta_2 \) is a zero map by Condition A, we get \( \gamma_2 \) is also a zero map, hence \( h^0(E) = h^0(F) \).

Assume that \( h^0(F) > f \), then \( h^0(E) = h^0(F) > f \). Consider the sub-sheaf \( E_1 \) of \( E \) generated by global sections of \( E \). Since \( E \) is a sub-sheaf of \( O^{\oplus n}_X \), \( H^0(E) \) is a subspace of \( H^0(O^{\oplus n}_X) \). Hence the sub-sheaf \( E_1 \) is a trivial sheaf with rank equal to \( h^0(E) > f \). This contradicts to the fact that \( \text{rank} E = f \).

Thus \( h^0(F) \) must equal \( f \), therefore \( h^1(F) = 0 \) and \( \deg F = fg \). In this case, we get

\[ O^{\oplus f} \hookrightarrow E \hookrightarrow O^{\oplus n} \]

The first map forces \( \deg E \geq 0 \) and the second map forces \( \deg E \leq 0 \). Hence \( \deg E = 0 \) and the map \( O^{\oplus f}_X \rightarrow E \) is an isomorphism. From the exact sequence \( 0 \rightarrow O^{\oplus f}_X \rightarrow E \rightarrow O_Q \rightarrow 0 \), we can see that the number of points in \( Q \) is \( \ell(Q) = \deg F = fg \). Hence \( \ell(R) = ng - fg = f'g \).

Now we consider the dual diagram of (3.9):

\[
\begin{array}{ccccccccc}
0 & \rightarrow & O_R & \overset{x^*}{\leftarrow} & E^* & \overset{y^*}{\leftarrow} & F^* & \leftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & O_P & \overset{z^*}{\leftarrow} & O^{\oplus n}_X & \overset{w^*}{\leftarrow} & V^* & \leftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & O_Q & \overset{a^*}{\leftarrow} & O^{\oplus f}_X & \overset{b^*}{\leftarrow} & F^* & \leftarrow & 0 \\
\end{array}
\]

(3.11)
Since \( h^1(V) = 0 \) (due to Condition A), \( h^0(V^* \otimes K_X) = 0 \) by the Serre duality. Also \( h^0(K_X) \geq 1 \) because we have assumed that \( g \geq 2 \). Hence \( h^0(V^*) = 0 \) and \( h^0(F'^*) = 0 \) as well. Since \( h^1(F) = 0 \), by the similar argument, \( h^0(F^*) = 0 \). Now we take the cohomology of the diagram (3.11), we get

\[
\begin{array}{c}
0 \\
\downarrow \\
H^0(\mathcal{O}_R) \\
\downarrow \pi_2 \\
H^0(\mathcal{O}_P) \\
\downarrow \pi_1 \\
H^0(\mathcal{O}_Q) \\
\downarrow \\
0
\end{array} \quad \begin{array}{c}
\phi_2 \\
\sigma_2 \\
\phi_1 \\
\sigma_1 \\
0 \\
0
\end{array}
\begin{array}{c}
\leftarrow 0 \\
\leftarrow 0 \\
\leftarrow 0 \\
\end{array}
\]

where \( \phi_0 \) is the induced map of \( \varphi \) on the cohomology. (Notice that the map \( \varphi_0 \) here is different from the map \( \varphi^0 \) we defined in (3.4).)

The surjectivity of \( \sigma_1 \) is due to the nature of the map \( \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X^{\oplus f} \).

Hence \( h^0(E'^*) = n - f = f' \). (In fact, we can show that \( E'^* = \mathcal{O}_X^{\oplus f'} \), but we don’t need such a strong statement.)

\( \pi_1 \) is a natural coordinate projection. By Condition B, \( \pi_1|\text{Im}\varphi_0 \) is either a surjection if \( \ell(Q) \leq n \) or an injection if \( \ell(Q) \geq n \).

If \( \ell(Q) \leq n \), \( \pi_1|\text{Im}\varphi_0 \) is a surjection. Hence \( \rho_1 \) has to be a surjection. But \( h^0(\mathcal{O}_X^{\oplus f}) = f < fg = h^0(\mathcal{O}_Q) \), a contradiction.

If \( \ell(Q) \geq n \), \( \pi_1|\text{Im}\varphi_0 \) is an injection. \( \text{Im}(\pi_1 \circ \varphi_0) \) has rank \( n \). But \( \text{Im}(\pi_1 \circ \varphi_0) = \text{Im}(\rho_1 \circ \sigma_1) \) and \( \text{Im}(\rho_1 \circ \sigma_1) \) has rank \( f < n \), a contradiction.

Hence \( V \) has to be stable.

\[ \square \]

4. Group actions on \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \)

In this section, we study the actions of \( \text{Aut}(\mathcal{O}_X^{\oplus n}) \) and \( \text{Aut}(\mathcal{O}_P) \) on \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \).

4.1. Recall that every surjective map \( \varphi \) in \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \) gives an exact sequence

\[ 0 \rightarrow V^* \xrightarrow{\phi} \mathcal{O}_X^{\oplus n} \xrightarrow{\varphi} \mathcal{O}_P \rightarrow 0. \]

Hence for an elementary transformation \( \varphi \), we get a vector bundle \( V^* \), or equivalent \( V \). The group \( \text{Aut}(\mathcal{O}_P) \) and \( \text{Aut}(\mathcal{O}_X^{\oplus n}) \) act on \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \) as follows. Let \( \rho \in \text{Aut}(\mathcal{O}_X^{\oplus n}) \) and \( \sigma \in \text{Aut}(\mathcal{O}_P) \). The action of \( \rho \) and
σ on ϕ gives a new elementary transformation ϕ' = σ ◦ ϕ ◦ ρ⁻¹. It fits into the following exact sequence:

\[ 0 \rightarrow V^* \xrightarrow{\rho \circ \sigma} \mathcal{O}_X^\oplus \xrightarrow{\phi'} \mathcal{O}_P \rightarrow 0. \]

This means that ϕ' gives arise to the same bundle V*, or equivalently, V. So we have:

**Lemma 4.2.** Suppose two elementary transformations ϕ, ϕ' ∈ Hom(\(\mathcal{O}_X^\oplus\), \(\mathcal{O}_P\)) are in the same orbit under the actions of Aut(\(\mathcal{O}_X^\oplus\)) and Aut(\(\mathcal{O}_P\)). Then the corresponding vector bundles (which are kernels of them) are isomorphic.

It is natural to ask whether the converse is also true. In general, it might well be that two elementary transformations in different group orbits give arise to two isomorphic vector bundles. However we have the following lemma:

**Lemma 4.3.** Let V and V' be obtained from two elementary transformations ϕ and ϕ' in Hom(\(\mathcal{O}_X^\oplus\), \(\mathcal{O}_P\)) and Hom(\(\mathcal{O}_X^\oplus\), \(\mathcal{O}_Q\)) respectively both of which satisfy Condition A (P and Q are not assumed to be the same). If V and V' are isomorphic, then P = Q and ϕ and ϕ' are in the same group orbit under the actions of Aut(\(\mathcal{O}_X^\oplus\)) and Aut(\(\mathcal{O}_P\)).

**Proof.** Let f : V → V' be the isomorphism. Consider the following exact sequences

\[ 0 \rightarrow V^* \rightarrow \mathcal{O}_X^\oplus \xrightarrow{\varphi} \mathcal{O}_P \rightarrow 0, \]

\[ 0 \rightarrow V'^* \rightarrow \mathcal{O}_X^\oplus \xrightarrow{\varphi'} \mathcal{O}_Q \rightarrow 0. \]

Take the duals of these two exact sequences, we have

\[ 0 \rightarrow \mathcal{O}_X^\oplus \xrightarrow{\alpha} V \rightarrow \mathcal{O}_P \rightarrow 0, \quad (4.1) \]

\[ 0 \rightarrow \mathcal{O}_X^\oplus \xrightarrow{\alpha'} V' \rightarrow \mathcal{O}_Q \rightarrow 0. \quad (4.2) \]

Since ϕ and ϕ' satisfy Condition A, from Remark 3.7, we have that \(h^0(V) = h^0(V') = n\). Take the cohomology of the two exact sequences (4.1) and (4.2), we get two isomorphisms

\[ H^0(\mathcal{O}_X^\oplus) \xrightarrow{\alpha^0} H^0(V), \quad H^0(\mathcal{O}_X^\oplus) \xrightarrow{\alpha^0'} H^0(V'). \]

where \(\alpha^0\) (or \(\alpha^0'\)) is the induced map of \(\alpha\) (or \(\alpha'\)) on cohomologies. Hence Imα (or Imα') is the sub-sheaf of V (or V') generated by \(H^0(V)\) (or \(H^0(V')\)). Since \(H^0(V)\) is mapped isomorphically to \(H^0(V')\) by the map \(f^0\) which is the induced map of the given isomorphism \(f : V \rightarrow V'\), Imα is mapped isomorphically to Imα' by the map f from V to V'.
Therefore, there is an induced morphism $\rho \in Aut(O_X^{\oplus n})$ making the following diagram commute:

$$
\begin{array}{ccc}
0 & \rightarrow & O_X^{\oplus n} \\
\downarrow & & \downarrow f \\
0 & \rightarrow & O_X^{\oplus n}
\end{array}
\quad (\rho)
\begin{array}{ccc}
\rightarrow & V & \rightarrow O_P \\
\downarrow & & \downarrow \sigma \\
\rightarrow & V' & \rightarrow O_Q
\end{array}
$$

which in turn induces a morphism $\sigma : O_P \rightarrow O_Q$. That is to say, there exists a commutative diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & O_X^{\oplus n} \\
\downarrow & & \downarrow f \\
0 & \rightarrow & O_X^{\oplus n}
\end{array}
\quad (\rho)
\begin{array}{ccc}
\rightarrow & V & \rightarrow O_P \\
\downarrow & & \downarrow \sigma \\
\rightarrow & V' & \rightarrow O_Q
\end{array}
$$

Notice that points in $P$ (or $Q$) are the points where the map $\alpha$ (or $\alpha'$) fails to be an isomorphism. Hence $Q$ must equal $P$ and $\sigma$ must be an automorphism of $O_P$.

Take the dual of (4.3), we get

$$
\begin{array}{ccc}
0 & \rightarrow & V^* \\
\downarrow & & \downarrow f^* \\
0 & \rightarrow & V'^*
\end{array}
\quad (\rho^*)
\begin{array}{ccc}
\rightarrow & \phi^* & \rightarrow O_P \\
\downarrow & & \downarrow \sigma^* \\
\rightarrow & \phi'^* & \rightarrow O_P
\end{array}
$$

where $\rho^* \in Aut(O_X^{\oplus n})$ and $\sigma^* \in Aut(O_P)$. So $\phi = \sigma^* \circ \phi' \circ (\rho^*)^{-1}$.

Hence $\phi$ and $\phi'$ are in the same group orbit. 

4.4. Lemma 4.2 and Lemma 4.3 tell us that for generic elementary transformation $\phi \in Hom(O_X^{\oplus n}, O_P)$, its group orbit $Aut(O_P) \cdot \phi \cdot Aut(O_X^{\oplus n})$ classifies stable vector bundles up to bundle isomorphisms. This leads us to the quotient space

$$
Aut(O_P) \backslash Hom(O_X^{\oplus n}, O_P) / Aut(O_X^{\oplus n}).
$$

4.5. We have the following identifications:

$$
Hom(O_X^{\oplus n}, O_P) \cong H^0(O_P \otimes O_X^{\oplus n}) \\
\cong H^0(O_{P_{i_1}}^{\oplus n}) \oplus \ldots \oplus H^0(O_{P_{i_d}}^{\oplus n}) \cong (\mathbb{C}^n)^d. 
$$

Under the isomorphism in (4.4), the group actions become the natural ones:

1. $Aut(O_X^{\oplus n}) = GL(n)$ acts on the space $\bigoplus_{i=1}^d H^0(O_{P_{i}}^{\oplus n}) = (\mathbb{C}^n)^d$ diagonally (on each component $H^0(O_{P_{i}}^{\oplus n}) = \mathbb{C}^n$ the action is the standard one);
2. $Aut(O_P^{\oplus n}) = (\mathbb{C}^*)^d$ acts on $Hom(O_X^{\oplus n}, O_P) = (\mathbb{C}^n)^d$ by component-wise scalar multiplications.
In a more down-to-earth way, we may write an element $\varphi \in (\mathbb{C}^n)^d$ as $(\varphi_1, \ldots, \varphi_d)$ where each $\varphi_i \in \mathbb{C}^n$. Then for $g \in GL(n)$,
$$g \cdot \varphi = (g\varphi_1, \ldots, g\varphi_d),$$
and for $c = (c_1, \ldots, c_d) \in (\mathbb{C}^*)^d$,
$$c \cdot \varphi = (c_1\varphi_1, \ldots, c_d\varphi_d).$$

**Lemma 4.6.** Condition A (or B) is invariant under the actions of $\text{Aut}(\mathcal{O}_P)$ and $\text{Aut}(\mathcal{O}_X^\oplus n)$. That is, if $\varphi$ is an elementary transformation in $\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P)$ satisfying Condition A (or B), then every element in the orbit $\text{Aut}(\mathcal{O}_P) \cdot \varphi \cdot \text{Aut}(\mathcal{O}_X^\oplus n)$ also satisfies Condition A (or B).

**Proof.** From Remark 3.7, we know that Condition A is equivalent to $h^1(V) = 0$. From Lemma 4.2, we know that $\varphi$ and $\sigma \circ \varphi \circ \rho^{-1}$ induce the same vector bundles $\tilde{V}$ for any $\sigma \in \text{Aut}(\mathcal{O}_P)$ and $\rho \in \text{Aut}(\mathcal{O}_X^\oplus n)$. Thus if $\varphi$ satisfies Condition A, so does $\sigma \circ \varphi \circ \rho^{-1}$.

The invariance of Condition B under the actions of $\text{Aut}(\mathcal{O}_P)$ and $\text{Aut}(\mathcal{O}_X^\oplus n)$ is straightforward from the definition of Condition B. ☐

**Notation 4.7.**
1. Define $N$ to be the space of elementary transformations in $\text{Hom}(\mathcal{O}_X^\oplus n, \mathcal{O}_P)$, i.e.
$$N \cong (\mathbb{C}^n - \{0\})^d$$
where 0 is the zero vector in $\mathbb{C}^n$.
2. Define $\overline{N}$ to be the quotient space of $N$ by the action of $\text{Aut}(\mathcal{O}_P)$, i.e.
$$\overline{N} = (\mathbb{C}^*)^d \backslash (\mathbb{C}^n - \{0\})^d \cong (\mathbb{P}^n_{n-1})^d.$$  
3. Define $\widehat{N}$ to be the (non-separated) quotient space of $N$ by the action of groups $\text{Aut}(\mathcal{O}_P)$ and $\text{Aut}(\mathcal{O}_X^\oplus n)$, i.e.
$$\widehat{N} \cong (\mathbb{C}^*)^d \backslash (\mathbb{C}^n - \{0\})^d / GL(n) \cong (\mathbb{P}^n_{n-1})^d / PGL(n)$$
where $PGL(n)$ acts on $(\mathbb{P}^n_{n-1})^d$ diagonally.
4. Define $N_A$ (or $N_B$) to be the subset of elementary transformations in $N$ satisfying Condition A (or B); define $\overline{N}_A$ (or $\overline{N}_B$) to be the quotient space of $N_A$ (or $N_B$) by the action of the group $\text{Aut}(\mathcal{O}_P)$; and define $\widehat{N}_A$ (or $\widehat{N}_B$) to be the quotient space of $N_A$ (or $N_B$) by the action of the groups $\text{Aut}(\mathcal{O}_P)$ and $\text{Aut}(\mathcal{O}_X^\oplus n)$. Clearly $\widehat{N}_A$ and $\widehat{N}_B$ are non-empty Zariski open subsets of $\widehat{N}$ and we shall see that $\widehat{N}_B$ is separated, quasi-projective, and rational (see 4.8 below).
4.8. Now take an element \( \varphi = (\varphi_1, \ldots, \varphi_d) \) in \( N \). Consider its image \( \overline{\varphi} = (\overline{\varphi}_1, \ldots, \overline{\varphi}_d) \) in \( \overline{N} \). \( PGL(n) \) acts on \( \overline{N} = (\mathbb{P}^{n-1})^d \). The geometric invariant theory of this standard \( PGL(n) \) action on \( \overline{N} = (\mathbb{P}^{n-1})^d \) can be found in \([MF]\). If \( \varphi \) satisfies Condition B, then any \( n \)-many \( \overline{\varphi}_{i1}, \ldots, \overline{\varphi}_{in} \) will be linearly independent. In the context of geometric invariant theory, such \( \overline{\varphi} \)'s are necessarily stable with respect to all linearizations. It follows from \([MF]\) that \( \hat{N}_B \) is separated, quasi-projective, and rational.

For the convenience to the reader, we shall give a brief account of these standard results. Consider the diagonal action of \( PGL(n+1) \) on \( (\mathbb{P}^n)^{m+1} \). Let \( p_i \) be the projection of \( (\mathbb{P}^n)^{m+1} \) to the \( i \)-th factor. Let \( L_i = p_i^* \mathcal{O}_{\mathbb{P}^n}(1) \). Then \( L = (L_1 \otimes \ldots \otimes L_{m+1})^{n+1} \) admits a \( PGL(n+1) \)-linearization. Assume that \( m \geq n+1 \).

The following is the Definition 3.7 / Proposition 3.4 in \([MF]\).

**Proposition 4.9.** The set of stable points in \( (\mathbb{P}^n)^{m+1} \) with respect to \( L \) is the open subset of \( (\mathbb{P}^n)^{m+1} \) whose geometric points \( x = (x^{(0)}, \ldots, x^{(m)}) \) are those points such that for every proper linear subspaces \( L \subset \mathbb{P}^n \),

\[
\text{number of points } x^{(i)} \text{ in } L \frac{m+1}{m+1} < \frac{\dim L + 1}{n+1}.
\]

(4.5)

It is an easy exercise to check the following:

**Corollary 4.10.** Let \( x = (x^{(0)}, \ldots, x^{(m)}) \) be a point in \( (\mathbb{P}^n)^{m+1} \) such that any \( (n+1) \)-many \( x^{(i)} \) are linearly independent in \( \mathbb{P}^n \), then \( x \) is a stable point with respect to \( L \).

**Definition 4.11.** We say an elementary transformation \( \varphi \) in \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \) is stable if \( \overline{\varphi} \in \overline{N} \) is stable with respect to the linearization \( L \).

Apply the geometric invariant theory to our situation, we get:

**Corollary 4.12.** An elementary transformation \( \varphi \in \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \) is stable if \( \varphi \) satisfies Condition B.

Now we may rewrite Theorem 3.10 as follows:

**Proposition 4.13.** Fix a divisor \( P \) in \( U \). Let \( \varphi \) be an elementary transformation in \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \). If \( \varphi \) satisfies Condition A and B, then the vector bundle \( V \) obtained from \( \varphi \) is a stable vector bundle with \( \deg V = d = ng \) and \( \text{rank} V = n \) such that

1. \( h^1(V) = 0 \).
2. \( \varphi \) is stable (in the sense of Definition 4.11).
3. The set of such stable bundles is isomorphic to the set \( \hat{N}_A \cap \hat{N}_B \). Furthermore, \( \hat{N}_A \cap \hat{N}_B \) is quasi-projective, rational, and of dimension \( d(n-1) - n^2 + 1 \).
4.14. Now we see that we can construct a stable vector bundle such that it satisfies (1), (2) and (3) of Theorem 1.4. Next, we need to prove that such vector bundles are generic in the moduli space $\mathcal{M}(n, L)$.

For this, we have to let $P$ move in $U$. Then we get a set $B$ of stable vector bundles. $B$ has dimension

$$\dim \hat{N}_A \cap \hat{N}_B + \dim U = (n - 1)ng - n^2 + 1 + (n - 1)g = (n^2 - 1)(g - 1)$$

which is the same as the dimension of the moduli space $\mathcal{M}(n, L)$. It is natural to ask if this space $B$ is actually a non-empty Zariski open subset of the moduli space. Indeed, we shall show that this is the case in the next section.

5. A birational model for $\mathcal{M}(n, L)$

In this section, using relative extension sheaf, we will construct a family of stable bundles which provides an injection to the moduli space $\mathcal{M}(n, L)$ and gives a non-empty Zariski open subset of $\mathcal{M}(n, L)$.

In the previous sections, the method we used to construct stable vector bundles is the elementary transformations. Here we shall use extensions instead. The two approaches are related: one is the dual of the other. Below we will elaborate on this.

5.1. Given an elementary transformation $\varphi \in \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P)$, i.e.,

$$0 \longrightarrow V^* \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{\varphi} \mathcal{O}_P \longrightarrow 0 \quad (5.1)$$

Take the dual of the exact sequence (5.1), we get

$$0 \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow V \longrightarrow \mathcal{O}_P \longrightarrow 0. \quad (5.2)$$

Hence $V$ is an extension of $\mathcal{O}_P$ by $\mathcal{O}_X^{\oplus n}$. Such extensions are classified by the extension group $\text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n})$. There is an isomorphism

$$\text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) \cong H^0(\mathcal{E}xt^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n})) \cong H^0(\mathcal{O}_P \otimes \mathcal{O}_X^{\oplus n})$$

$$\cong H^0(\mathcal{O}_X^{\oplus n}|_{p_1}) \oplus \ldots \oplus H^0(\mathcal{O}_X^{\oplus n}|_{p_d}) \cong (\mathbb{C}^n)^d.$$

The extension (5.2) in general does not give arise to a vector bundle in the middle. However, writing an extension class $e$ as $(e_1, \ldots, e_d) \in (\mathbb{C}^n)^d$, we have the following:

**Lemma 5.2.** The extension $e$ gives a vector bundle $V$ in (5.2) if and only if $e_i \neq 0$ for all $i$.

**Proof.** See Lemma 16 of [B1].
Remark 5.3. The above lemma is equivalent to Remark 3.4. Actually there exists a dictionary between elementary transformations and their corresponding extensions. For example, under this dictionary, the groups \( \text{Aut}(\mathcal{O}_P) \) and \( \text{Aut}(\mathcal{O}_X^{\oplus n}) \) act on \( \text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) \) in the same way as they act on \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \) after we identify \( \text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) \) with \( \text{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_P) \). We leave the existence of the dictionary to the reader.

Notation 5.4. 1. Define \( Z' \subset X \times U \) to be the universal divisor
\[
Z' = \{ (x, D) \in X \times U | x \in D \}.
\]
2. Define \( \pi_i \) to be the natural projection from \( X \times U \) to the \( i \)-th factor. 3. Define \( \mathcal{E} \) to be the relative extension sheaf
\[
\mathcal{E} = \text{Ext}^1_{\pi_2}((\mathcal{O}_{Z'}, \pi_1^*\mathcal{O}_X^{\oplus n})).
\]
The fiber of \( \mathcal{E} \) over a point \( P \in U \) is isomorphic to \( \text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) \).
4. The general linear group \( \text{GL}(n) \) acts on \( \mathcal{E} \) via acting on \( \mathcal{O}_X^{\oplus n} \). Meanwhile, the group scheme \( \mathcal{T} = (\pi_2')_*\mathcal{O}_{Z'}^{\times} \) also acts on \( \mathcal{E} \) where \( \pi_2' \) is the restriction of \( \pi_2 \) to \( Z' \). This group scheme is a twisted torus. The fiber of \( \mathcal{T} \) over a point \( P \in U \) is \( \text{Aut}(\mathcal{O}_P^{\oplus n}) \cong (\mathbb{C}^*)^d \). The fiber-wise action of \( \mathcal{T} \) on \( \mathcal{E} \) is just the action of \( \text{Aut}(\mathcal{O}_P) \) on \( \text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) \). The actions of the group scheme \( \mathcal{T} \) and the (global) group \( \text{GL}(n) \) commute. Hence we get an action of \( \mathcal{G} = \mathcal{T} \times \text{GL}(n) \) where the diagonal multiplicative group \( \mathbb{C}^* \) appears in both \( \mathcal{T} \) and \( \text{GL}(n) \).

Lemma 5.5. The natural morphism:
\[
\mathcal{E}_{[P]} \rightarrow \text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) \tag{5.3}
\]
is an isomorphism and \( \mathcal{E} \) is a locally free sheaf over \( U \).

Proof. Note that \( \dim \text{Ext}^1(\mathcal{O}_P, \mathcal{O}_X^{\oplus n}) = nd \) is a constant. By Satz 3 of [BPS], \( \mathcal{E} \) is a locally free sheaf and the natural morphism (5.3) is an isomorphism. \( \square \)

Proposition 5.6. Let \( \mathcal{V} = \mathcal{V}(\mathcal{E}^*) \) be the vector bundle associated to the locally free sheaf \( \mathcal{E} \) following Grothendieck’s notation. Then over \( \mathcal{V} \), there is a universal extension
\[
0 \rightarrow q_1^*\mathcal{O}_X^{\oplus n} \otimes q_2^*\mathcal{A} \rightarrow \mathcal{V} \rightarrow \gamma^*\mathcal{O}_{Z'} \rightarrow 0 \tag{5.4}
\]
where \( q_i \) is the projection from \( X \times \mathcal{V} \) to its \( i \)-th factor and \( \gamma \) is the projection from \( X \times \mathcal{V} \) to \( X \times U \) and \( \mathcal{A} \) is some line bundle on \( \mathcal{V} \). Moreover, \( \mathcal{V} \) is flat over \( \mathcal{V} \).
Proof. Since \( \text{Ext}^0(\mathcal{O}_P, \mathcal{O}_{\mathbb{P}^n}^\oplus) = 0 \) for all \( P \in U \), \( \text{Ext}^0_{\pi_2}(\mathcal{O}_Z^\prime, \pi_1^* \mathcal{O}_X^\oplus) = 0 \). By Lemma 5.5, \( \text{Ext}^1_{\pi_2}(\mathcal{O}_Z^\prime, \pi_1^* \mathcal{O}_X^\oplus) \) commutes with base change (according to Lange’s terminology \([L]\)). Hence by Corollary 3.4 of \([L]\), we get the universal extension (5.4).

Since \( \mathcal{O}_Z^\prime \) is flat over \( U \), both \( \gamma^* \mathcal{O}_Z^\prime \) and \( q_1^* \mathcal{O}_X^\oplus \otimes q_2^* \mathfrak{A} \) are flat over \( V \). Hence \( V \) is flat over \( \mathbb{V} \).

\[ \begin{align*} \text{Remark 5.7.} & \quad \text{If we use } e \text{ to represent a point in } \mathbb{V} \text{ corresponding to an extension (5.2), then the restriction of (5.4) to } X \times e \text{ is just the extension (5.2) and } \mathbb{V}|_{X \times e} \cong V. \\ \text{Theorem 5.8.} & \quad \text{The moduli space } \mathcal{M}(n, L) \text{ is birational to the quotient space of a non-empty Zariski open subset of } \mathbb{V} \text{ by the action of the group } \mathcal{G} = T \times GL(n). \end{align*} \]

Proof. Let \( \mathbb{V}^0 \) be the subset of \( \mathbb{V} \) consisting of extensions (5.2) whose corresponding elementary transformations (via the dictionary) satisfy Conditions A and B. By Lemma 4.8, \( \mathbb{V}^0 \) is invariant under the action of \( \mathcal{G} \). By Proposition 4.13 and semi-continuity using the flatness of \( \mathcal{V} \), \( \mathbb{V}^0 \) is a non-empty Zariski open subset of \( \mathbb{V} \). Restrict \( \mathcal{V} \) to \( X \times \mathbb{V}^0 \), we get a family of stable bundles over \( X \times \mathbb{V}^0 \). Since the moduli space \( \mathcal{M}(n, L) \) is coarse, the family induces a morphism:

\[ \Phi: \mathbb{V}^0 \longrightarrow \mathcal{M}(n, L). \] (5.5)

Now by Lemma 4.2 and Lemma 4.3, we see that points of \( \mathbb{V}^0 \) are in the same \( \mathcal{G} \) orbit if and only if they correspond to isomorphic stable bundles. Thus by passing to the quotient, we get a natural induced map

\[ \overline{\Phi}: \mathbb{V}^0/(T \times GL(n)) \longrightarrow \mathcal{M}(n, L) \]

which is an injective morphism. By calculating dimensions, we get

\[ \dim(\mathbb{V}^0/\mathcal{G}) = \dim\mathbb{V}^0 - \dim(T \times GL(n)) + 1 = \dim\mathcal{M}(n, L). \]

(Notice that a “+1” modification appears in the first equality because the multiplicative group \( \mathbb{C}^* \) appears in both \( \mathcal{T} \) and \( GL(n) \).) Hence the morphism \( \overline{\Phi} \) is birational.

\[ \begin{align*} \text{Remark 5.9.} & \quad \text{It can be seen that the non-empty Zariski open subset } \mathbb{V}_0/\mathcal{G} \text{ of } \mathcal{M}(n, L) \text{ is a fibration over the rational variety } U \text{ whose typical fiber is also rational since it is birational to the geometric quotients of } (\mathbb{P}^n_{n-1})^d \text{ by } PGL(n) \text{ (which are known to be rational) (see also } [T1, T2]). \text{ Unfortunately, this fibration seems not to be locally trivial in Zariski topology. The twisted torus } \mathcal{T} \text{ is responsible for the problem.} \end{align*} \]
5.10. proof of Theorem 1.4. Theorem 1.4 clearly follows from the combination of Proposition 4.13 and Theorem 5.8.

6. Some remarks on rationality

We shall provide a thorough and systematic account of the inductive method in [N1] and prove several theorems that are stronger than Theorem 1.2. The essential ideas are, however, contained in [N1].

Throughout, \((n; d)\) stands for a pair of integers; while \(\gcd(n, d)\) stands for the greatest common factor of the two integers \(n\) and \(d\).

6.1. Let \(V\) be a stable vector bundle in \(\mathcal{M}(n, L)\) and \((n; d)\) satisfies
\[
n(g - 1) < d < ng.
\]
By Riemann-Roch, \(\chi(V) = \deg V + n(1 - g) = d - n(g - 1)\). Set \(r = \chi(V)\).

Then \(0 < r < n\) by (6.1).

In ([N1]) and ([G]), it was shown that for generic bundles \(V\) in \(\mathcal{M}(n, L)\), \(V\) satisfies \(h^1(V) = 0\) and there exists an exact sequence
\[
0 \to \mathcal{O}_X^r \to V \to V' \to 0
\]
(6.2) such that \(V'\) is stable. That is \(V' \in \mathcal{M}(n', L)\) where \(n' = n - r\). One can always find a non-negative integer \(k\) such that
\[
n'(g - 1) < d' = d - kn' \leq n'g.
\]
Then \(\mathcal{M}(n', L) \cong \mathcal{M}(n', L')\) where \(L' = L \otimes (M^*)^{\otimes n'}\) for some line bundle \(M\) of degree \(k\) and \(\deg L' = d'\).

This suggests that there exist a rational map from \(\mathcal{M}(n, L)\) to \(\mathcal{M}(n', L')\).

Definition 6.2. Let \((n; d)\) be a pair of positive integers satisfying (6.1). Let \(n' = ng - d\), \(d' = d - kn'\) for some non-negative integer \(k\) so that \(n'(g - 1) < d' \leq n'g\). We call \((n'; d')\) the reduction of \((n; d)\), we denote this process of reduction by
\[
(n; d) \to (n'; d').
\]

6.3. Sometimes, it is necessary to apply the same procedure to the dual bundle \(V^*\). In other words, instead of constructing \(V\) by an extension (6.2), we may do it for \(V^*\). Precisely, let again \((n; d)\) satisfy (6.1). Choose a line bundle \(M\) with \(\deg M = 2g - 1\). Then
\[
n(g - 1) < \deg(V^* \otimes M) = -d + n(2g - 1) < ng.
\]
Hence \(V^* \otimes M\) is a stable vector bundle in the moduli space \(\mathcal{M}(n, L^* \otimes M^{\otimes n})\). The pair \((n; \deg(L^* \otimes M^{\otimes n}))\) satisfies (6.1). We then apply the reduction to \((n; n(2g - 1) - d)\), and obtain \((n'; d')\) where \(n' = d - n(g - 1)\), \(d' = n(2g - 1) - d - kn'\), and \(n'(g - 1) < d' \leq n'g\).
for some non-negative integer \( k \).

**Definition 6.4.** Let \((n; d)\) be a pair of integers satisfying the hypothesis (6.1). Let \( n' = d - n(g - 1) \) and \( d' = n(2g - 1) - d - kn' \) for some non-negative integer \( k \) such that \( n'(g - 1) < d' \leq n'g \). We call \((n'; d')\) the dual reduction of \((n; d)\). We denote this process also by

\[
(n; d) \longrightarrow (n'; d').
\]

**Proposition 6.5.** Let \((n; d)\) be a pair satisfying (6.1). Let \((n'; d')\) be the reduction (or dual reduction) of \((n; d)\). Let \( \mathcal{M}(n, L) \) and \( \mathcal{M}(n', L') \) be the moduli spaces such that \( \deg L = d \), \( \deg L' = d' \) and \( L = L' \otimes M \otimes n' \) for some line bundle \( M \) of non-negative degree \( k \) with \( h^0(M) \neq 0 \). Then there exists a non-empty Zariski open subset \( \mathcal{M}^0(n, L) \subset \mathcal{M}(n, L) \) and a morphism \( \Phi : \mathcal{M}^0(n, L) \longrightarrow \mathcal{M}(n', L') \) such that the image of \( \Phi \) is a non-empty Zariski open subset of \( \mathcal{M}(n', L') \).

**Proof.** We only prove the proposition when \((n; d) \to (n'; d')\) is a reduction. The same arguments work for a dual reduction equally well.

Take \( \mathcal{M}^0(n, L) \) to be the non-empty Zariski open subset of \( \mathcal{M}(n, L) \) as in Theorem 1.8. Recall that a stable vector bundle \( V \) in \( \mathcal{M}^0(n, L) \) has \( h^1(V) = 0 \) and sits in the exact sequence (6.2) where \( r = n - n' \) and \( V' \in \mathcal{M}(n', L) \).

Let \( F' = V' \otimes (M^*) \otimes n' \). Then \( F' \) is a stable bundle in \( \mathcal{M}(n', L') \).

Define the map

\[
\Phi : \mathcal{M}^0(n, L) \longrightarrow \mathcal{M}(n', L')
\]

by setting

\[
\Phi(V) = F'.
\]

First of all, the map is well-defined. This amounts to saying that if \( V_1 \) and \( V_2 \) are two isomorphic stable bundles in \( \mathcal{M}^0(n, L) \), i.e. there exist two exact sequences

\[
0 \longrightarrow O_X^{\oplus r} \xrightarrow{\alpha_1} V_1 \xrightarrow{\beta_1} V_1' \longrightarrow 0,
\]

\[
0 \longrightarrow O_X^{\oplus r} \xrightarrow{\alpha_2} V_2 \xrightarrow{\beta_2} V_2' \longrightarrow 0
\]

where \( h^0(V_1) = h^0(V_2) = r \) and \( V_1' \) and \( V_2' \) are stable, then \( V_1' \) and \( V_2' \) are isomorphic. In fact, since \( h^0(V_1) = r \), \( O_X^{\oplus r} \) is the sub-sheaf of \( V_1 \) generated by global sections of \( H^0(V_1) \). Same is true for \( V_2 \). Hence if \( f \) is an isomorphism from \( V_1 \) to \( V_2 \), \( f \), restricted to \( O_X^{\oplus r} \), maps \( O_X^{\oplus r} \subset V_1 \) to \( O_X^{\oplus r} \subset V_2 \) isomorphically, hence \( V_1' \) is isomorphic to \( V_2' \).

Next we need to show that \( \Phi \) is a morphism. Consider the construction of \( \mathcal{M}(n, L) \) via geometric invariant theory, we let \( Q \) be the Quot scheme, \( U \) be the universal quotient sheaf over \( X \times Q \) and \( G \) be the group such that \( \mathcal{M}(n, L) \) is the GIT quotient of \( Q \) by \( G \). Let \( \Pi \) be the
quotient map from \( \mathcal{Q} \) to \( \mathcal{M}(n, L) \). Let \( \mathcal{Q}_0 \subset \mathcal{Q} \) be the inverse image of \( \mathcal{M}(n, L) \) under the map \( \Pi \). Since bundles in \( \mathcal{M}(n, L) \) are stable bundles, the pre-image of \( V \in \mathcal{M}(n, L) \) under the map \( \Pi \) consists of bundles isomorphic to \( V \).

Clearly we can define a map

\[
\Phi : \mathcal{Q}_0 \to \mathcal{M}(n', L')
\]

in the same way as we defined \( \Phi \).

Because \( \mathcal{M}(n', L') \) is a coarse moduli space and since \( \mathcal{U} \) is a universal quotient sheaf over \( X \times \mathcal{Q}_0 \), we conclude that \( \Phi \) is a morphism.

Since \( \mathcal{M}(n, L) \) is the geometric quotient of \( \mathcal{Q}_0 \) by \( \mathcal{G} \) and any orbit (e.g., the fiber \( \Pi^{-1}(V) \)) is mapped to a single point \( F' \) in \( \mathcal{M}(n', L') \) under the map \( \Phi \), by the universality of GIT quotients, the morphism \( \Phi : \mathcal{Q}_0 \to \mathcal{M}(n', L') \) induces a morphism on the quotient \( \mathcal{M}(n, L) \to \mathcal{M}(n', L') \). Evidently, the induced morphism is just the map \( \Phi \) defined earlier.

Finally, we need to show that the image of \( \Phi \) contains a non-empty Zariski open subset of \( \mathcal{M}(n', L') \).

Recall that the combination of Lemma 5 and Lemma 6 in [N1] says that if \( V' \) is stable, \( h^1(V') = 0 \), \( \deg V' = d \), \( n(g - 1) < d < ng \) and \( r = d - n(g - 1) \), then there exists a stable vector bundle \( V \) sitting in the exact sequence (6.2) with \( h^1(V) = 0 \).

From a result in [G], we know that there exists a non-empty Zariski open subset \( \mathcal{M}_0(n', L') \) of \( \mathcal{M}(n', L') \) consisting of \( F' \) with \( h^1(F') = 0 \). Hence \( h^0(F' \otimes K_X) = 0 \) by Serre duality. Since \( h^0(M) \neq 0 \), we must have \( h^0(F' \otimes (M^*) \otimes K_X) = 0 \). Recall that \( V' = F' \otimes M^{\otimes n'} \). Hence \( h^1(V') = h^1(F' \otimes M^{\otimes n'}) = 0 \) by Serre duality. Now Lemma 5 and Lemma 6 in [N1] imply that there exists a stable vector bundle \( V \) in \( \mathcal{M}(n, L) \) such that \( \Phi(V) = F' \). So \( \text{Im} \Phi \) contains \( \mathcal{M}_0(n', L') \). Therefore it contains a non-empty Zariski open subset of \( \mathcal{M}(n', L') \).

The case when \( n' = 1, d' = g \) needs some special remarks. According to our convention, \( V' = L \) in this case. Since \( n \geq 2 \), we have

\[
\deg L = d \geq n(g - 1) + 1 \geq 2g - 2 + 1 = 2g - 1.
\]

Therefore \( h^1(L) = h^0(L^* \otimes K_X) = 0 \) because \( \deg(L^* \otimes K_X) \leq -2g + 1 + 2g - 2 = -1 \). Hence Lemma 5 and Lemma 6 in [N1] also apply.

\[\square\]

6.6. Now we need to analyze the structure of the rational map

\[
\Phi : \mathcal{M}(n, L) \to \mathcal{M}(n', L').
\]

Extensions (6.2) are classified by the extension group:

\[
\text{Ext}^1(V', \mathcal{O}_X^{\oplus r}) = H^1(V'^* \otimes L^{\oplus r}).
\]
The group $Aut(\mathcal{O}^r_X) = GL(r)$ acts on the extension group. Lemma 1 of [N1] showed that the $GL(r)$-orbits correspond to the equivalent classes of stable vector bundles.

Let $F' \in \mathcal{M}_0(n', L')$. Then $\Phi^{-1}(F')$ is a non-empty Zariski open subset of $Ext^1(V', \mathcal{O}^r_X)/GL(r)$ which is known to be rational (see Lemma 2, [N1]). The rational map $\Phi$ can be regarded as a fibration (over a non-empty Zariski open subset of $\mathcal{M}(n', L')$).

6.7. Next, we need to know when the fibration

$$\Phi: \mathcal{M}(n, L) \longrightarrow \mathcal{M}(n', L')$$

is locally trivial in Zariski topology over the range of $\Phi$. This admits the affirmative answer if the moduli space $\mathcal{M}(n', L')$ is fine, i.e. if $\mathcal{M}(n', L')$ admits a universal bundle (if and only if $\gcd(n', d') = 1$). This is because the existence of the universal bundle will allow us to get the locally triviality by using the relative extension sheaf (cf. Lemma 3, [N1]). If $\gcd(n', d') \neq 1$, we still have the map $\Phi$, but it seems very likely that the fibration is no longer locally trivial in Zariski topology.

The following is implicit in [N2], which was used to obtain Theorem [N2].

**Proposition 6.8.** Let $(n; d) \to (n'; d')$ be a reduction or a dual reduction.

1. If $(n'; d')$ is coprime, then the fibration $\mathcal{M}(n, L) \longrightarrow \mathcal{M}(n', L')$ is locally trivial in Zariski topology. Furthermore, $\mathcal{M}(n, L)$ is birational to $\mathbb{C}^m \times \mathcal{M}(n', L')$ where $m = (n^2 - n'^2)(g - 1)$.
2. If $(n'; d')$ is coprime and $\mathcal{M}(n', L')$ is rational, then $\mathcal{M}(n, L)$ is rational.

6.9. An interesting question is whether the above induction method works only for the pairs stated in Theorem [N2]. A closer examination of reductions and dual reductions tells us that the method can actually be extended to work for a much larger class of moduli spaces, which we now describe.

Given a pair $(n; d)$ satisfying (6.1), if $n > 1$, we can apply a reduction or a dual reduction to $(n; d)$ to get another pair $(n'; d')$. Three possibilities can occur:

1. $n' = 1$.
2. $n' \geq 2$ and $d' = n'g$.
3. $n' \geq 2$ and $n'(g - 1) < d - kn' < n'g$ for some integer $k \geq 0$. 


In the case that (3) occurs, we can continue reductions and dual reductions. Keep doing this, the process will eventually terminate and we get a sequence of reductions and dual reductions:

\[(n; d) \rightarrow (n_1; d_1) \rightarrow \ldots \rightarrow (n_t; d_t)\] (6.4)

such that
1. either \(n_t = 1, d_t = g\),
2. or \(d_t = n_t g, n_t \geq 2\).

**Definition 6.10.** Let \((n; d)\) be a pair of positive integers satisfying (6.1). The pair is called a nice pair if either \(n = 1\) or after successive reductions and dual reductions (6.4), we get \(n_t = 1\).

**Remark 6.11.** It can happen that after a sequence of reductions and dual reductions, \((n; d)\) is reduced to \((n_t; d_t)\) with \(n_t \geq 2\) and \(d_t = n_t g\) while after another sequence of reductions and dual reductions, \((n; d)\) is reduced to \((1; g)\). According to our definition, this kind of pair is nice. Take the pair \((7; 8)\) when \(g = 2\) as an example. After two reductions we get

\[(7; 8) \rightarrow (6; 8) \rightarrow (4; 8).\]

However, after a dual reduction, we get

\[(7; 8) \rightarrow (1; 2).\]

The latter will imply that \(\mathcal{M}(7, L)\) with \(\deg L = 8\) is rational (cf. Theorem 6.15 below), while the former won’t.

Below is a simple but useful observation.

**Lemma 6.12.** Let \((n; d)\) be a pair satisfying (6.1). Let \((n'; d')\) be the reduction or dual reduction of \((n; d)\). If \(\gcd(n', d') = 1\), then \(\gcd(n, d) = 1\).

**Proof.** Assume \((n; d) \rightarrow (n'; d')\) is a reduction:

\[(n'; d') = (ng - d; d - k(ng - d))\] (6.5)

for some non-negative integer \(k\).

If an integer \(m\) divides \(n\) and \(d\), from (6.5), it is clear that \(m\) divides \(n' = ng - d\) and \(d' = d - k(ng - d)\). Hence the lemma is proved in this case.

Assume \((n; d) \rightarrow (n'; d')\) is a dual reduction.

\[(n'; d') = (ng - n(2g - 1) + d; n(2g - 1) - d - k(ng - n(2g - 1) + d))\] (6.6)

for some non-negative integer \(k\).

If an integer \(m\) divides \(n\) and \(d\), from (6.6), it is clear that \(m\) divides \(n'\) and \(d'\) as well. Hence the lemma is also proved in this case. \(\square\)
Remark 6.13. 1. On any forwarded (dual) reduction path, $\gcd$ is a non-decreasing function.
2. All nice pairs are coprime.
3. The converse of Lemma 6.12 is not true which is exactly and the only place where the proposition of §3 in [N1] fails.

Proposition 6.14. Let $(n; d) \rightarrow (n_1; d_1) \rightarrow \cdots \rightarrow (n_s; d_s)$ be a sequence of reductions and dual reductions. Suppose $\gcd(n_s, d_s) = 1$, then $\mathcal{M}(n, L)$ is birational to $\mathbb{C}^x \times \mathcal{M}(n_s, L_s)$ where $\deg L_s = d_s$ and $x = (n^2 - n_s^2)(g - 1)$.

Proof. By assumption, we have a sequence of dominant rational maps:

$$
\mathcal{M}(n, L) \dashrightarrow \mathcal{M}(n_1, L_1) \dashrightarrow \cdots \dashrightarrow \mathcal{M}(n_s, L_s).
$$

Since $(n_s, d_s) = 1$, all the above moduli spaces are fine moduli spaces by Lemma 6.12. Hence each of the above rational map is generically a locally trivial fibration in Zariski topology whose typical fiber is rational (cf. 6.7 and Proposition 6.8). The proposition then follows immediately.

When $(n_s; d_s) = (1; g)$, we obtain:

Theorem 6.15. If $(n; d)$ is a nice pair, then the moduli space $\mathcal{M}(n, L)$ with $d = \deg L$ is rational.

One checks that all pairs in Theorem 1.2 are nice pairs. The following gives an example of a nice pair not contained in Theorem 1.2.

Example 6.16. Choose $g = 6$, $n = 15$, $d = 77$. One checks that $(n; d) = (15; 77)$ satisfies the inequality (3.1) but satisfies none of the conditions (1), (2), (3) in Theorem 1.2. Apply reductions twice, we obtain

$$(15; 77) \rightarrow (13: 77) \rightarrow (1; 6).$$

Hence the moduli space $\mathcal{M}(15, L)$ with $\deg L = 77$ is rational.

6.17. In view of Theorem 6.15, it is desirable to find ways of constructing all nice pairs. The following two lemmas help to characterize arithmetically the iterated construction of all such pairs starting from $(1; g)$.

Lemma 6.18. Fix a genus $g \geq 2$.

1. $(n; d) \rightarrow (1; g)$ is a (one-step) reduction if and only if $d = ng - 1$. Consequently, $\gcd(d, g) = 1$.
2. $(n; d) \rightarrow (1; g)$ is a (one-step) dual reduction if and only if $d = ng - n + 1$. Consequently, $\gcd(d + n, g) = 1$. 

Proof. (1) If \( d = ng - 1 \) for \( n \geq 2 \), it is easy to see that after one reduction we get \((1; g)\).

Now suppose \((n; d) \rightarrow (1; g)\) is a reduction. From (6.5), we get \( n = \frac{g + (k + 1)}{g} \) and \( d = g + k \) for some integer \( k \geq 0 \). Hence \( k + 1 \) must be \( mg \) for some integer \( m \geq 1 \), \( n = m + 1 \geq 2 \) and \( d = g + mg - 1 = ng - 1 \).

It is clear that \( \gcd(d, g) = 1 \).

Similar arguments prove (2).

\[ \square \]

Remark 6.19. Lemma 6.18 is equivalent to Theorem 1.2 (1).

Next, we have

Lemma 6.20. Fix a genus \( g \geq 2 \). Let \((n'; d')\) be a nice pair with either \( \gcd(d', g) = 1 \) or \( \gcd(d' + n', g) = 1 \). Then,

1. there exists a pair \((n; d)\) having \((n'; d')\) as a reduction if and only if \( \gcd(n', g) = 1 \). In particular, \( \gcd(d, g) = 1 \).
2. there exists a pair \((n; d)\) having \((n'; d')\) as a dual reduction if and only if \( \gcd(n', g) = 1 \). In particular \( \gcd(d + n, g) = 1 \).

Proof. Consider the equations in (6.5):

\[ n' = ng - d, \quad d' = d - kn'. \]

Add these equations together, we get an equation

\[ ng = d' + (k + 1)n'. \]

Since \( \gcd(d', g) = 1 \) (the case when \( \gcd(d' + n', g) = 1 \) can be proved similarly), this equation has integral solutions for \( n \) and \( k \) iff \( \gcd(n', g) = 1 \). And if such solutions exist, \( d = ng - n', \) so \( \gcd(d, g) = 1 \).

The similar arguments prove (2).

\[ \square \]

Corollary 6.21. If \((n; d)\) is a nice pair, then either \( \gcd(d, g) = 1 \) or \( \gcd(d + n, g) = 1 \).

Remark 6.22. In summary, we have the following. Start with any \((n; d)\). After successive reductions and dual reductions, we get

\[ (n; d) \rightarrow (n_1; d_1) \rightarrow \ldots \rightarrow (n_t; d_t). \]

Correspondingly, we have a sequence of dominant rational maps:

\[ \mathcal{M}(n, L) \rightarrow \mathcal{M}(n_1, L_1) \rightarrow \ldots \rightarrow \mathcal{M}(n_t, L_t) \] (6.7)

where \( \deg L_i = d_i \) fro all \( i \). The terminator is either \((1; g)\) or has that \( n_t | d_t \) and \( n_t \geq 2 \).

The first case is covered in Theorem 6.13.
For the latter case, the sequence (6.7) ends at the moduli space studied in the previous sections. In the sequence (6.7), it is possible that some fibrations are locally trivial in Zariski topology but the rest are not. The local triviality is known if the base moduli space of a fibration is fine. Otherwise, we believe that it is not. It seems that the rationality problem boils down to how well we know about the moduli spaces \( \mathcal{M}(n_t, L_t) \) when \( n_t \mid d_t \).

6.23. The above procedure of proving rationality (for coprime pairs only) can be slightly generalized to include an even larger class of pairs (essentially due to an observation by Ballico in [Ba]). The observation rests on the following easy lemma:

**Lemma 6.24.** (Transferring Lemma). Suppose that we have the following projections:

\[
\begin{align*}
X & \rightarrow Y \\
& \leftarrow Z
\end{align*}
\]

such that

1. \( X = \mathbb{C}^x \times Z \);
2. \( Y = \mathbb{C}^y \times Z \);
3. \( x \geq y \);
4. \( Y \) is rational.

Then \( X \) is rational.

**Proof.** \( X = \mathbb{C}^x \times Z \cong \mathbb{C}^{x-y} \times (\mathbb{C}^y \times Z) = \mathbb{C}^{x-y} \times Y \).

6.25. Consider a diagram

\[
\begin{align*}
(n; d) & \rightarrow (n_1; d_1) & \cdots & \rightarrow (n_{t-1}; d_{t-1}) & \rightarrow (n_t; d_t) \\
& \leftarrow (\ell_1; k_1) & \cdots & \leftarrow (\ell_{t-1}; k_{t-1}) & \leftarrow (\ell_t; k_t)
\end{align*}
\]

(6.8)

where each pair in the diagram is a pair of positive integers satisfying (6.1) and each downward arrow represents successive reductions and dual reductions. For example, \((\ell_1; k_1)\) is obtained from \((n; d)\) and \((n_1; d_1)\) by some sequences of reductions and dual reductions, respectively. It is allowed that \((n_t; d_t) = (\ell_t; k_t)\).

Correspondingly, we get a diagram of dominant rational maps:

\[
\begin{align*}
\mathcal{M}(n, L) & \rightarrow \mathcal{M}(n_1, L_1) & \cdots & \rightarrow \mathcal{M}(n_t, L_t) \\
& \leftarrow \mathcal{M}(\ell_1, K_1) & \cdots & \leftarrow \mathcal{M}(\ell_t, K_t)
\end{align*}
\]
where $L_i$ is a line bundle with $\deg L_i = d_i$ and $K_i$ is a line bundle with $\deg K_i = k_i$.

**Definition 6.26.** The diagram (6.8) is called admissible if $n \geq n_i$ and $\gcd(l_i, k_i) = 1$ for all $i$.

**Theorem 6.27.** Suppose the diagram (6.8) is admissible. Then $\mathcal{M}(n, L)$ is birational to $\mathbb{C}^z \times \mathcal{M}(n_t, L_t)$ where $z_t = (n^2 - n_t^2)(g - 1) \geq 0$ is the difference of the dimension of two moduli spaces.

**Proof.** We shall use Proposition 6.14 and Lemma 6.24 repeatedly. Let $x = \dim \mathcal{M}(n, L)$, $x_i = \dim \mathcal{M}(n_i, L_i)$, and $y_i = \dim \mathcal{M}(l_i, K_i)$. For simplicity, we use the notation $X \cong_{\text{bir}} Y$ to indicate that $X$ and $Y$ are birational equivalence. Then we have

$$
\mathcal{M}(n, L) \cong_{\text{bir}} \mathbb{C}^{x-y_1} \times \mathcal{M}(l_1, K_1) = \mathbb{C}^{x-x_1} \times \mathbb{C}^{x_1-y_1} \times \mathcal{M}(l_1, K_1)
$$

$$
\cong_{\text{bir}} \mathbb{C}^{x-x_1} \times \mathcal{M}(n_1, L_1) \cong_{\text{bir}} \mathbb{C}^{x-x_2} \times \mathbb{C}^{x_2-y_2} \times \mathcal{M}(l_2, K_2)
$$

$$
= \mathbb{C}^{x-y_2} \times \mathcal{M}(l_2, K_2) = \mathbb{C}^{x-x_2} \times \mathbb{C}^{x_2-y_2} \times \mathcal{M}(l_2, K_2)
$$

$$
\cong_{\text{bir}} \mathbb{C}^{x-x_2} \times \mathcal{M}(n_2, L_2) \cong_{\text{bir}} \ldots \cong_{\text{bir}} \mathbb{C}^{x-x_1} \times \mathcal{M}(n_t, L_t).
$$

The theorem then follows. \qed

**Definition 6.28.** A pair $(n; d)$ is called a fine pair if there exists an admissible diagram (6.8) such that $(n_t; d_t)$ is a nice pair. Fine pairs are also necessarily coprime.

**Theorem 6.29.** If $(n; d)$ is a fine pair, then $\mathcal{M}(n, L)$ is rational.

**Proof.** It follows from Theorem 6.27 and Theorem 6.15. \qed

**Example 6.30.** Assume $g = 6$. Then the pair $(7; 38)$ is not a nice pair by Corollary 6.21 ($\gcd(38, 6) = 2$ and $\gcd(38 + 7, 6) = 3$). Notice that $\gcd(7, 38) = 1$.

One can check the following statements:

1. $(7; 38)$ is the (one-step) dual reduction of $(11 + 7m; 62 + 35m)$ for all integer $m \geq 0$.
2. $m = 7$ is the smallest number such that the pair $(11 + 7m; 62 + 35m)$ is a nice pair. (When $m = 7$, the pair is $(60; 307)$ and we actually have

\[
(60; 307) \rightarrow (53; 307) \rightarrow (11; 65) \rightarrow (1; 6)
\]

where all " $\rightarrow$ " are (one-step) reductions.)
3. When $m$ is even, $(11 + 7m; 62 + 35m)$ is not a nice pair. (The dual reduction of the pair gives $(7; 38)$ which is not a nice pair; a reduction gives $(4 + 7m; 62 + 35m)$ which is not a nice pair either since 2 divides gcd$(4 + 7m, 62 + 35m)$ when $m$ is even.)

However $(11 + 7m; 62 + 35m)$ are fine pairs for all $m \geq 7$:

$$(11 + 7m; 62 + 35m) \rightarrow (7; 38) \leftarrow (60; 307)$$

Hence the corresponding moduli spaces $\mathcal{M}(11 + 7m, L)$ where $\deg L = 62 + 35m$ are rational for all $m \geq 7$ by Theorem 6.29.

The above provides infinitely many fine pairs that are not nice pairs.

**Remark 6.31.** Finally, some remarks are in order.

1. There are coprime pairs that are not fine pairs.
2. In [N2], Newstead defined a good pair to be a coprime pair $(n; d)$ whose corresponding moduli space is rational. Nice pairs and fine pairs are good. The converse may not be true. To find all good pairs (that are not nice or fine) seems requiring methods different than the one explored in this paper.
3. There is an algorithm to locate nice and fine pairs on the lattice cone in the $xy$-plane defined by inequalities:

$$x(g - 1) < y \leq xg$$

for any fixed $g \geq 2$. A reduction takes the form

$$(x, y) \rightarrow (gx - y, y - k(gx - y))$$

for some non-negative integer $k$. A dual reduction takes the form

$$(x, y) \rightarrow (y - (g - 1)x, (2g - 1)x - y - k(y - (g - 1)x))$$

for some non-negative integer $k$.
4. By using the techniques of variations of moduli spaces of parabolic bundles, H. Boden and K. Yokogawa were able to prove the rationality of certain moduli spaces. (see [B, BH].

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