DOUGLAS FACTORIZATION THEOREM REVISITED

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Dedicated to the memory of R. G. Douglas (1938-2018)

Abstract. Inspired by the Douglas factorization theorem, we investigate the solvability of the operator equation $AX = C$ in the framework of Hilbert $C^*$-modules. Utilizing partial isometries, we present its general solution when $A$ is a semi-regular operator. For such an operator $A$, we show that the equation $AX = C$ has a positive solution if and only if the range inclusion $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ holds and $CC^* \leq tCA^*$ for some $t > 0$. In addition, we deal with the solvability of the operator equation $(P + Q)^{1/2}X = P$, where $P$ and $Q$ are projections. We provide a tricky counterexample to show that there exist a $C^*$-algebra $\mathfrak{A}$, a Hilbert $\mathfrak{A}$-module $\mathcal{H}$ and projections $P$ and $Q$ on $\mathcal{H}$ such that the operator equation $(P + Q)^{1/2}X = P$ has no solution. Moreover, we give a perturbation result related to the latter equation.

1. Introduction

The significant equation $AX = C$ and systems of equations including it have been intensely studied for matrices [9, 18], bounded linear operators on Hilbert spaces [2, 3, 11], and operators on Hilbert $C^*$-modules [14, 20]. For any operator $A$ between linear spaces, the range and the null space of $A$ are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. In 1966, R. G. Douglas proved an equivalence of factorization, range inclusion, and majorization, known as the Douglas factorization theorem (Douglas lemma) in the literature. It reads as follows.

Theorem 1.1. [4, Theorem 1] If $\mathcal{H}$ is a Hilbert space and $A, B \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent:

(i) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$;
(ii) The equation $AX = C$ has a solution $X \in \mathcal{B}(\mathcal{H})$;
(iii) $CC^* \leq k^2AA^*$ for some $k \geq 0$.

Moreover, if (i), (ii), and (iii) are valid, then there exists a unique operator $C$ (known as the Douglas Solution in the literature) so that

(a) $\|X\|^2 = \inf \{\mu | CC^* \leq \mu AA^*\}$;
(b) $\mathcal{N}(C) = \mathcal{N}(X)$;
(c) $\mathcal{R}(X) \subseteq \mathcal{R}(A^*)$.

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There are several applications of the Douglas factorization theorem in investigation of operator equations. For instance, Nakamoto [16] studied the solvability of $XAX = B$ by employing the Douglas factorization theorem [11]. In 2008, Arias, Corach, and Gonzalez [2] introduced the notion of reduced solution which is a generalization of the concept of Douglas solution. More precisely, let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ be operators between Hilbert spaces such that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and let $M$ be a closed subspace of $\mathcal{H}$ such that $\mathcal{N}(A) \oplus M = \mathcal{H}$. Then there exists a unique solution $X_M$ of the equation $AX_M = C$ such that $\mathcal{R}(X_M) \subseteq M$. The operator $X_M$ is called the reduced solution of the equation $AX = C$ for the subspace $M$ in the framework of Hilbert spaces. They parametrized these solutions by employing generalized inverses.

Inner product $C^*$-modules are generalizations of inner product spaces by allowing inner products to take values in some $C^*$-algebras instead of the field of complex numbers. More precisely, an inner-product module over a $C^*$-algebra $\mathfrak{A}$ is a right $\mathfrak{A}$-module equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathfrak{A}$. If $\mathcal{H}$ is complete with respect to the induced norm defined by $\|x\| = \|\langle x, x \rangle\|^\frac{1}{2}$ $(x \in \mathcal{H})$, then $\mathcal{H}$ is called a Hilbert $\mathfrak{A}$-module.

Throughout the rest of this paper, $\mathfrak{A}$ denotes a $C^*$-algebra and $\mathcal{E}, \mathcal{H}, \mathcal{K},$ and $\mathcal{L}$ denote Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of operators $A : \mathcal{H} \to \mathcal{K}$ for which there is an operator $A^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$. It is known that any element $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ must be bounded and $\mathfrak{A}$-linear. In general, a bounded operator between Hilbert $C^*$-modules may be not adjointable. We call $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the set of all Hermitian (adjointable) operators from $\mathcal{H}$ to $\mathcal{K}$. In the case when $\mathcal{H} = \mathcal{K}$, $\mathcal{L}(\mathcal{H}, \mathcal{H})$, abbreviated to $\mathcal{L}(\mathcal{H})$, is a $C^*$-algebra. An operator $A \in \mathcal{L}(\mathcal{H})$ is positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ (see [10, Lemma 4.1]), and we then write $A \geq 0$. For Hermitian operators $A, B \in \mathcal{L}(\mathcal{H})$, we say $B \geq A$ if $B - A \geq 0$. Let $\mathcal{L}(\mathcal{H})_{sa}$ and $\mathcal{L}(\mathcal{H})_+$ denote the set of Hermitian elements and positive elements in $\mathcal{L}(\mathcal{H})$, respectively.

A closed submodule $M$ of $\mathcal{H}$ is said to be orthogonal complemented if $\mathcal{H} = M \oplus M^\perp$, where $M^\perp = \{x \in \mathcal{H} : \langle x, y \rangle = 0$ for any $y \in M\}$. In this case, the projection from $\mathcal{H}$ onto $M$ is denoted by $P_M$. If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ does not have closed range, then neither $\mathcal{N}(A)$ nor $\overline{\mathcal{R}(A)}$ needs to be orthogonal complemented. In addition, if $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $\overline{\mathcal{R}(A^*)}$ is not orthogonally complemented, then it may happen that $\mathcal{N}(A)^\perp \neq \overline{\mathcal{R}(A^*)}$; see [10, 13]. The above facts show that the theory Hilbert $C^*$-module are much different and more complicated than that of Hilbert spaces.

There are several extensions of the Douglas factorization theorem in various settings; see [7, 17] as well as the recent survey [15]. A generalization of the Douglas factorization theorem to the Hilbert $C^*$-module case was given as follows in which we do not need to assume that $\overline{\mathcal{R}(A^*)}$ is orthogonally complemented.

**Theorem 1.2.** [5, Corollary 2.5] Let $\mathfrak{A}$ be a $C^*$-algebra, $\mathcal{E}, \mathcal{H}$ and $\mathcal{K}$ be Hilbert $\mathfrak{A}$-modules. Let $A \in \mathcal{L}(\mathcal{E}, \mathcal{H})$ and $A' \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:

(i) $A'(A')^* \leq \lambda AA^*$ for some $\lambda > 0$;
There exists $\mu > 0$ such that $\| (A')^* z \| \leq \mu \| A^* z \|$, for any $z \in \mathcal{K}$.

In general, $A'(A')^* \leq \lambda AA^*$ for some $\lambda > 0$ does not imply $\mathcal{R}(A') \subseteq \mathcal{R}(A)$. As an example, let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, let $\mathfrak{A} = \mathcal{H} = L^2(\mathcal{H})$ and let $\mathcal{E}$ be the algebra $\mathbb{K}(\mathcal{H})$ of all compact operators. Suppose that $S = \text{diag}(1, 1/2, 1/3, \ldots)$ is the diagonal operator with respect to some orthonormal basis and define $A : \mathcal{E} \to \mathcal{H}$ by $A(T) := ST$ for $T \in \mathfrak{A}$, and set $A' := (AA^*)^{1/2}$.

We, however, have the following interesting result.

**Lemma 1.3.** ([5, Theorem 3.2] and [6, Theorem 1.1]) Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:

(i) $\mathcal{R}(A^*)$ is orthogonally complemented in $\mathcal{H}$;

(ii) Let $C \in \mathcal{L}(\mathcal{L}, \mathcal{K})$ be any such that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. Then the equation

$$AX = C, X \in \mathcal{L}(\mathcal{L}, \mathcal{H})$$

(1.1)

has a reduced solution $D$, that is,

$$AD = C, D \in \mathcal{L}(\mathcal{L}, \mathcal{H}) \text{ and } \mathcal{R}(D) \subseteq \overline{\mathcal{R}(A^*)}. \quad (1.2)$$

It is remarkable that such a reduced solution (if it exists) is unique, and for Hilbert space operators as well as adjointable operators on Hilbert $C^*$-modules, most literatures on the solvability of equation (1.1) are only focused on the regular case [3, 20], that is, the ranges of $A$ and the other associated operators are assumed to be closed. Very little has been done in the case when the associated operators are non-regular, which is the concern of this paper.

In view of the equivalence of Lemma 1.3 (i) and (ii), the term of the semi-regularity for adjointable operators is introduced in this paper (see Definition 2.1). Such a semi-regularity condition is somehow natural in dealing with the solvability of equation (1.1), since it is always true for Hilbert space operators and if it fails to be satisfied, then equation (1.1) may be unsolvable. Furthermore, it is noted that for an adjointable operator $A$, $A$ is semi-regular if and only if $A$ has the polar decomposition $A = U|A|$ [21, Proposition 15.3.7]. So instead of the Moore-Penrose inverse in the regular case, one might use the partial isometry in the semi-regular case. By utilizing partial isometries, we present the general solution of equation (1.1) when $A$ is a semi-regular operator. For such an operator $A$, the Hermitian solutions and the positive solutions of equation (1.1) have been completely characterized in Section 2 of this paper; see Theorems 2.8 and 2.14. As a result, certain mistakes in [6, Section 1] are corrected for adjointable operators on Hilbert $C^*$-modules, and some generalizations of [11, Section 3] are obtained from the Hilbert space case to the Hilbert $C^*$-module case.

The shorted operators initiated in [1] for Hermitian positive semi-definite matrices and generalized in [8] for Hilbert space operators, are closely related to the operator equation $(A + B)^{1/2}X = A^{1/2}$, where $A$ and $B$ are two positive operators. Such an operator equation is always solvable when the underlying spaces are Hilbert spaces. To show that the same is not true for adjointable operators on Hilbert $C^*$-modules, we focus on the special case that both $A$ and $B$ are projections. In the last section of this paper, we provide a tricky counterexample to
show that there exist a C*-algebra $\mathfrak{A}$, a Hilbert $\mathfrak{A}$-module $\mathcal{H}$ and two projections $P$ and $Q$ on $\mathcal{H}$ such that the operator equation $(P + Q)^{1/2}X = P, X \in \mathcal{L}(\mathcal{H})$ has no solution. Moreover, given projections $P, Q \in \mathcal{L}(\mathcal{H})$, we show that for any $\varepsilon \in (0, 1)$, there exists a projection $Q' \in \mathcal{L}(\mathcal{H})$ such that $\|Q - Q'\| < \varepsilon$ and the equation $(P + Q')^{1/2}X = P, X \in \mathcal{L}(\mathcal{H})$ has a solution.

2. Solutions of the operator equation $AX = C$

We begin with the definition of the semi-regularity as follows:

**Definition 2.1.** An operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be semi-regular if $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are orthogonally complemented in $\mathcal{H}$ and $\mathcal{K}$, respectively.

**Remark 2.2.** Recall that $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be regular if $\mathcal{R}(A)$ is closed. In this case, the Moore-Penrose inverse $A^\dagger$ of $A$ exists. This is an operator $A^\dagger$ such that $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, and $AA^\dagger = P_{\mathcal{R}(A)}$ and $A^\dagger A = P_{\mathcal{R}(A^*)}$ are projections ([22, Theorem 2.2]). Hence $A$ is semi-regular.

**Lemma 2.3.** [21, Proposition 15.3.7] Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be semi-regular. Then there exists a unique partial isometry $U_A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that

$$A = U_A(A^*A)^{\frac{1}{2}} \text{ and } U_A^*U_A = P_{\mathcal{R}(A^*)}. \quad (2.1)$$

**Theorem 2.4.** Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be semi-regular. Then for any $C \in \mathcal{L}(\mathcal{L}, \mathcal{K})$, operator equation (1.1) has a solution if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. In such case, the general solution of (1.1) has the form

$$X = D + (I - U_A^*U_A)Y, \quad (2.2)$$

where $D \in \mathcal{L}(\mathcal{L}, \mathcal{K})$ is the reduced solution of (1.1), $U_A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is the partial isometry satisfying (2.1), and $Y \in \mathcal{L}(\mathcal{L}, \mathcal{K})$ is arbitrary.

**Proof.** Suppose that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. By Lemma 1.3 equation (1.1) is solvable and its reduced solution $D$ satisfies (1.2). Since $U_A^*U_A = P_{\mathcal{R}(A^*)}$ and $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$, we have

$$\mathcal{N}(U_A) = \mathcal{N}(A) \text{ and } A(I - U_A^*U_A) = 0. \quad (2.3)$$

Therefore, any $X$ of the form (2.2) is a solution of equation (1.1).

On the other hand, given any solution $X$ of equation (1.1), we have

$$X - D \in \mathcal{N}(A) = \mathcal{N}(U_A) = \mathcal{N}(U_A^*U_A),$$

which leads to

$$X - D = (I - U_A^*U_A)(X - D),$$

hence $X$ has the form of (2.2) with $Y = X - D$ therein. $\square$

**Remark 2.5.** Suppose that $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is regular and $C \in \mathcal{L}(\mathcal{L}, \mathcal{K})$ is given such that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. Then $A^\dagger C$ is the reduced solution of (1.1), so the general solution of (1.1) has the form (2.2) with $D$ and $U_A^*U_A$ being replaced by $A^\dagger C$ and $A^\dagger A$, respectively.

To study the Hermitian solutions of (1.1), we need the following lemmas.
Lemma 2.6. [12, Proposition 2.7] Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $B, C \in \mathcal{L}(\mathcal{E}, \mathcal{H})$ be such that $\mathcal{R}(B) = \mathcal{R}(C)$. Then $\mathcal{R}(AB) = \mathcal{R}(AC)$.

Lemma 2.7. Let $A, C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be such that $A$ is semi-regular and $\mathcal{R}(C) \subseteq \mathcal{R}(A)$. Let $D \in \mathcal{L}(\mathcal{H})$ be the reduced solution of (1.1) with $\mathcal{L} = \mathcal{H}$ therein and $P = U_A^*U_A$, where $U_A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is the partial isometry satisfying (2.1). Then the following statements are valid:

(i) $DP$ is Hermitian if and only if $CA^*$ is Hermitian;
(ii) $DP$ is positive if and only if $CA^*$ is positive;
(iii) If $CA^*$ is Hermitian and regular, then $DP$ is also regular.

Proof. By (2.1) and (1.2), we have $PD = D$ and thus $D^*(I - P) = 0$, (2.4)

which leads to

$$\langle DPx, y \rangle = \langle DPx, Py \rangle$$

and

$$\langle PD^*x, y \rangle = \langle D^*Px, Py \rangle,$$

for any $x, y \in \mathcal{H}$. (2.5)

(i) $\implies$: If $DP$ is Hermitian (positive), then $CA^* = (AD)(PA^*) = A(DP)A^*$ is also Hermitian (positive).

“$\impliedby$”: For any $u, v \in \mathcal{H}$,

$$\langle DA^*u, A^*v \rangle = \langle A^*u, D^*A^*v \rangle = \langle A^*u, C^*v \rangle = \langle CA^*u, v \rangle$$

$$= \langle AC^*u, v \rangle = \langle C^*u, A^*v \rangle = \langle D^*A^*u, A^*v \rangle,$$

which implies that

$$\langle DPx, y \rangle = \langle D^*Px, Py \rangle,$$

for all $x, y \in \mathcal{H}$.

The equation above together with (2.5) yields $DP = (DP)^*$.

(ii) $\impliedby$: For any $u \in \mathcal{H}$,

$$\langle DA^*u, A^*u \rangle = \langle A^*u, C^*u \rangle = \langle CA^*u, u \rangle \geq 0,$$

which gives, by (2.5), that

$$\langle DPx, x \rangle = \langle DPx, Px \rangle \geq 0,$$

for any $x \in \mathcal{H}$.

(iii) By Lemma 2.6, we have

$$\mathcal{R}(CA^*) = \mathcal{R}(CA^*) = \mathcal{R}(CP) \supseteq \mathcal{R}(CP) \supseteq \mathcal{R}(CA^*),$$

hence

$$\mathcal{R}(CP) = \mathcal{R}(CP) = \mathcal{R}(CA^*).$$

(2.6)

Given any $x \in \mathcal{R}(DP)$, there exists a sequence $\{x_n\}$ in $\mathcal{H}$ such that $DPx_n \to x = Px$ (since $D = PD$). Then

$$CPx_n = ADPx_n \to Ax = CA^*u = AC^*u$$

for some $u \in \mathcal{H}$ (see (2.6)).

Hence, from (2.3), we have $x - C^*u \in \mathcal{N}(A) = \mathcal{N}(U_A)$. Therefore, $Px = PC^*u$. It follows that

$$x = Px = PC^*u = PD^*A^*u = DPA^*u \in \mathcal{R}(DP),$$

since $PD^* = DP$ by item (i) of this lemma (as $CA^*$ is Hermitian). This completes the proof that $\mathcal{R}(DP) = \mathcal{R}(DP)$. □
Now, we consider the Hermitian solutions of equation (1.1).

**Theorem 2.8.** Let \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) be semi-regular. Then for any \( C \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \), the system

\[
AX = C, \quad X \in \mathcal{L}(\mathcal{H})_{sa}
\]

has a solution if and only if

\[
\mathcal{R}(C) \subseteq \mathcal{R}(A) \quad \text{and} \quad CA^* \quad \text{is Hermitian.}
\]

In such case, the general solution of (2.7) has the form

\[
X = D + (I - U_A^*U_A)Y_0 \quad \text{for some} \quad Y_0 \in \mathcal{L}(\mathcal{H}),
\]

which leads by \( X^*_0 = X_0 \) to

\[
D + (I - U_A^*U_A)Y_0 = D^* + Y_0^*(I - U_A^*U_A).
\]

Moreover, from (2.4) we have

\[
(I - U_A^*U_A)(D - D^*)(I - U_A^*U_A) = 0.
\]

This together with (2.11) yields \( Z_0 = Z_0^* \), where

\[
Z_0 = (I - U_A^*U_A)Y_0^*(I - U_A^*U_A).
\]

In view of (2.4), (2.11), and (2.12), we have

\[
(I - U_A^*U_A)Y_0 = (I - U_A^*U_A)[D + (I - U_A^*U_A)Y_0] = (I - U_A^*U_A)[D^* + Y_0^*(I - U_A^*U_A)] = (I - U_A^*U_A)D^* + Z_0.
\]

Substituting the above into (2.10) yields

\[
X_0 = D + (I - U_A^*U_A)D^* + Z_0,
\]

therefore \( U_A^*U_A D^* = (D + D^*) + Z_0 - X_0 \), whence \( U_A^*U_A D^* = DU_A^*U_A \) since both \( X_0 \) and \( Z_0 \) are Hermitian. Furthermore, it is clear from (2.12) that

\[
Z_0 = (I - U_A^*U_A)Z_0(I - U_A^*U_A),
\]

which indicates by (2.13) that \( X_0 \) has the form (2.9).

Conversely, assume that (2.8) is fulfilled. Then by Lemma 2.7 (i) \( DU_A^*U_A \) is Hermitian, and it is easy to verify that any \( X \) of the form (2.9) is a solution of system (2.7).
Remark 2.9. Let $A,C \in \mathcal{L}(\mathcal{H},\mathcal{H})$ be such that $A$ is regular and (2.8) is satisfied. Unlike the assertion given in [6, Theorem 1.2], the reduced solution $A^\dagger C$ of (1.1) may fail to be Hermitian. An interpretation can be given by using block matrices as follows:

Evidently, the operators $A,A^\dagger$ and $C$ can be partitioned in the following way:

\[ A = \begin{pmatrix} \mathcal{R}(A) & \mathcal{N}(A^*) \\ \mathcal{N}(A) & \mathcal{R}(A^*) \end{pmatrix} \], \quad \text{where } A_{11} \text{ is invertible,} \\
\[ A^\dagger = \begin{pmatrix} \mathcal{R}(A^*) & \mathcal{N}(A) \\ \mathcal{N}(A) & \mathcal{R}(A) \end{pmatrix} \]
\[ C = \begin{pmatrix} \mathcal{R}(A) & \mathcal{N}(A) \\ \mathcal{N}(A) & \mathcal{R}(A) \end{pmatrix} \].

Conditions of $AA^\dagger C = C$ and $CA^* = (CA^*)^*$ can then be rephrased as

\[ C_{21} = 0, C_{22} = 0 \text{ and } C_{11}A_{11}^* = A_{11}C_{11}, \]

which gives the partitioned form of $A^\dagger C$ as

\[ A^\dagger C = \begin{pmatrix} \mathcal{R}(A^*) & \mathcal{N}(A^*) \\ \mathcal{N}(A) & \mathcal{R}(A) \end{pmatrix} \begin{pmatrix} A_{11}^{-1}C_{11} & A_{11}^{-1}C_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{R}(A) & \mathcal{N}(A) \\ \mathcal{N}(A) & \mathcal{R}(A) \end{pmatrix}. \]

Clearly, $A^\dagger C$ is Hermitian if and only if $C_{12} = 0$.

In view of the observation above, a concrete counterexample to [6, Theorem 1.2] can be constructed as follows:

Example 2.10. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AX = C$ and $X = Y^*Y \geq 0$, whereas $A^\dagger C \neq (A^\dagger C)^*$. 

To study the positive solutions of equation (1.1), we need the following lemma.

Lemma 2.11. [22, Corollary 3.5] Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be Hermitian, where $A_{11} \in \mathcal{L}(\mathcal{H})$ is regular. Then $A \geq 0$ if and only if

(i) $A_{11} \geq 0$;
(ii) $A_{12} = A_{11}A_{11}^*A_{12}$;
(iii) $A_{22} - A_{12}^*A_{11}^*A_{12} \geq 0$.

Our technical result on the positive solutions of (1.1) is as follows:

Lemma 2.12. Let $A,C \in \mathcal{L}(\mathcal{H},\mathcal{H})$ be such that $A$ is semi-regular. Then the system

\[ AX = C, \quad X \in \mathcal{L}(\mathcal{H}+) \tag{2.14} \]

has a solution if and only if

\[ \mathcal{R}(C) \subseteq \mathcal{R}(A), \quad CA^* \in \mathcal{L}(\mathcal{H}+) \text{ and } \lambda = \sup \{ \|T_n\| : n \in \mathbb{N} \} < +\infty, \tag{2.15} \]

where

\[ T_n = (I - U_A U_A^*) D^* \left[ \frac{1}{n} I_{\mathcal{H}_1} + DU_A U_A^*, \mathcal{H}_1 \right]^{-1} D(I - U_A U_A^*), \quad \text{for each } n \in \mathbb{N}, \tag{2.16} \]
in which \( U_A \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \) is the partial isometry satisfying (2.1), \( \mathcal{H}_1 = U_A^* U_A \mathcal{H} \) and \( D \in \mathcal{L}(\mathcal{H}) \) is the reduced solution of (1.1) with \( \mathcal{L} = \mathcal{H} \) therein. If (2.15) is fulfilled, then the general solution of (2.14) has the form (2.9) with \( Y \in \mathcal{L}(\mathcal{H})_+ \) therein such that
\[
(I - U_A^* U_A) Y (I - U_A^* U_A) \geq T_n \quad \text{for all } n \in \mathbb{N}.
\] (2.17)

**Proof.** For simplicity, we put
\[
P = U_A^* U_A \quad \text{and} \quad \mathcal{H}_1 = P \mathcal{H}.
\] (2.18)
Suppose that \( X \in \mathcal{L}(\mathcal{H})_+ \) is a solution of system (2.14). Then (2.8) is fulfilled and \( X \) has form (2.9). Therefore, by (2.4) and (2.9),
\[
DP = PXP \geq 0 \quad \text{and} \quad CA^* \geq 0
\]
by Lemma 2.7 (ii).
For each \( n \in \mathbb{N} \), let \( X_n = X + \frac{1}{n} P \in \mathcal{L}(\mathcal{H})_+ \). Then from (2.4), we get
\[
X_n = \mathcal{H}_1 \left( \frac{1}{n} I_{\mathcal{H}_1} + DP|_{\mathcal{H}_1} \begin{pmatrix} D(I - P)|_{\mathcal{H}_1} & (I - P)X(I - P)|_{\mathcal{H}_1} \end{pmatrix} \right) \mathcal{H}_1.
\]
A direct application of Lemma 2.11 to the operator \( X_n \) above yields
\[
0 \geq T_n \leq (I - P)X(I - P) \quad \text{for all } n \in \mathbb{N}.
\]

hence
\[
\lambda = \sup \{ \| T_n \| : n \in \mathbb{N} \} \leq \| (I - P)X(I - P) \| < +\infty.
\]

Conversely, suppose that (2.15) is fulfilled. Then by Lemma 2.7 (ii) \( DP \) is positive. For each \( n \in \mathbb{N} \), let
\[
Z_n = \frac{1}{n} P + D + (I - P)D^* + \lambda(I - P).
\]
Then \( Z_n \) is positive by Lemma 2.11 since it has the partitioned form
\[
Z_n = \mathcal{H}_1 \left( \frac{1}{n} I_{\mathcal{H}_1} + DP|_{\mathcal{H}_1} \begin{pmatrix} D(I - P)|_{\mathcal{H}_1} & (I - P)\lambda|_{\mathcal{H}_1} \end{pmatrix} \right) \mathcal{H}_1
\]
and
\[
\lambda(I - P) - T_n \geq \lambda(I - P) - \| T_n \|(I - P) = (\lambda - \| T_n \|)(I - P) \geq 0.
\]

Let
\[
X = \lim_{n \to \infty} Z_n = D + (I - P)D^* + \lambda(I - P).
\]
Then \( X \) is positive and \( AX = C \).

Finally, suppose that (2.15) is satisfied. Given any \( X \in \mathcal{L}(\mathcal{H})_{sa} \) of form (2.9), it is clear that \( X \geq 0 \) if and only if \( X + \frac{1}{n} P \geq 0 \) for any \( n \in \mathbb{N} \). Based on such an observation and the direct application of Lemma 2.11 the asserted form of the general solution of (2.14) follows. \( \square \)

**Remark 2.13.** In the preceding lemma, there is no regularity or semi-regularity assumption on \( DU_A^* U_A \). It is interesting to determine conditions under which the number \( \lambda \) defined by (2.15) is finite. With the notations and the conditions of Lemma 2.12 (except for \( \lambda < +\infty \)), for each \( n \in \mathbb{N} \) let
\[
S_n = \frac{1}{n} I_{\mathcal{H}_1} + DU_A^* U_A|_{\mathcal{H}_1} \in \mathcal{L}(\mathcal{H}_1).
\] (2.19)
In this case, for any \( n \) which leads to
\[
\left( I - U_A^* U_A \right) D^* S_n^{-1/2} \left( I - U_A^* U_A \right) D^* S_n^{-1/2} \right)^* ,
\]

hence
\[
\|T_n\| = \left\| S_n^{-1/2} D \left( I - U_A^* U_A \right) D^* S_n^{-1/2} \right\| = \| U_n + V_n \|
\]
where
\[
U_n = S_n^{-1/2} \left( D U_A^* U_A \right)^2 S_n^{-1/2} \quad \text{and} \quad V_n = S_n^{-1/2} D D^* S_n^{-1/2}
\]
are such that \( \| U_n \| \leq \| D U_A^* U_A \| \) for all \( n \in \mathbb{N} \), and \( \| V_n \| = \| D^* S_n^{-1} D \| \) for each \( n \in \mathbb{N} \). Therefore,
\[
\lambda < +\infty \iff \sup \{ \| V_n \| : n \in \mathbb{N} \} < +\infty \quad \text{(2.21)}
\]
\[
\iff \sup \{ \| D^* S_n^{-1} D \| : n \in \mathbb{N} \} < +\infty . \quad \text{(2.22)}
\]

Based on the observation above, an application of Lemma 2.12 is as follows:

**Theorem 2.14.** (cf. [11, Theorem 3.1 (iii)]) Let \( A, C \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) be such that \( A \) is semi-regular. Then system (2.14) has a solution if and only if
\[
\mathcal{R}(C) \subseteq \mathcal{R}(A) \quad \text{and} \quad CC^* \leq t CA^* \quad \text{for some} \ t > 0 . \quad \text{(2.23)}
\]

**Proof.** Suppose that \( X \in \mathcal{L}(\mathcal{H})_+ \) is such that \( AX = C \). Then \( \mathcal{R}(C) \subseteq \mathcal{R}(A) \) and
\[
CC^* = AX^2 A^* \leq \| X \| AXA^* = \| X \| CA^* .
\]

Therefore, (2.23) is satisfied for any \( t \geq \| X \| . \)

Conversely, suppose that (2.23) is satisfied. Let \( P, \mathcal{H}_1, S_n \) and \( V_n \) be defined by (2.18), (2.19), and (2.20), respectively. Then \( DP \) is positive by Lemma 2.7(ii), and from the latter condition in (2.23) we have
\[
\langle DD^* A^* x, A^* x \rangle = \langle ADD^* A^* x, x \rangle = \langle CC^* x, x \rangle \leq \t \langle CA^* x, x \rangle = \t \langle DA^* x, A^* x \rangle
\]
for any \( x \in \mathcal{H} \). Thus
\[
\langle DD^* Pu, u \rangle = \langle DD^* Pu, Pu \rangle \leq \t \langle DPu, Pu \rangle \leq \t \langle DPu, u \rangle,
\]
whence \( DD^* P \leq \t DP \). Accordingly,
\[
\| V_n \| = \left\| S_n^{-1/2} DD^* P S_n^{-1/2} \right\| \leq \t \left\| S_n^{-1/2} \cdot DP \cdot S_n^{-1/2} \right\| \leq \t , \quad \text{for any} \ n \in \mathbb{N} .
\]

The conclusion then follows from (2.21) and Lemma 2.12.

**Remark 2.15.** Let \( S_n \) be defined by (2.19), where \( P = U_A^* U_A \) and \( DP \) is positive. Obviously, a sufficient condition for \( \lambda < +\infty \) can be derived from (2.22) as
\[
M = \sup \{ \| S_n^{-1} \| : n \in \mathbb{N} \} < +\infty . \quad \text{(2.24)}
\]

In this case, for any \( n \in \mathbb{N} \) and \( x \in \mathcal{H}_1 \) we have
\[
\| x \| \leq \| S_n^{-1} \| \| S_n(x) \| \leq M \| S_n(x) \| ,
\]
which leads to
\[
\| DP x \| = \lim_{n \to \infty} \| S_n(x) \| \geq \frac{1}{M + 1} \| x \| , \quad \text{for any} \ x \in \mathcal{H}_1 .
\]
Therefore, $DP|_{\mathcal{H}_i}$ and furthermore $DP$ is regular, since $DP = PDP$.

Our next result on the positive solutions of (1.1) is as follows:

**Theorem 2.16.** Let $A, C ∈ \mathcal{L}(\mathcal{H}, \mathcal{K})$ be such that $A$ is semi-regular and $CA^*$ is regular. Then system (2.14) has a solution if and only if

$$\mathcal{R}(C) ⊆ \mathcal{R}(A), CA^* ∈ \mathcal{L}(\mathcal{H})_+ \text{ and } \mathcal{R}(D) = \mathcal{R}(DP),$$

(2.25)

where $U_A ∈ \mathcal{L}(\mathcal{H}, \mathcal{K})$ is the partial isometry satisfying (2.1), $P = U_A^*U_A$ and $D ∈ \mathcal{L}(\mathcal{H})$ is the reduced solution of (1.1) with $\mathcal{L} = \mathcal{K}$ therein. In such case, the general solution of (2.14) has the form

$$X = X_0 + PZP,$$

(2.26)

in which $Z ∈ \mathcal{L}(\mathcal{H})_+$ is arbitrary, and

$$X_0 = D + (I - P)D^* + (I - P)D^*(DP)^\dagger D(I - P).$$

(2.27)

**Proof.** Let $\mathcal{H}_i$ be defined by (2.13). Suppose that $X ∈ \mathcal{L}(\mathcal{H})_+$ is a solution of system (2.14). Then the first two conditions in (2.25) is satisfied by Lemma 2.12; $DP$ is positive and regular by Lemma 2.7 (ii) and (iii); and by Theorem 2.8 there exists $Y ∈ \mathcal{L}(\mathcal{H})_+$ such that $X$ has form (2.3), which leads to

$$X = \mathcal{H}_i \begin{pmatrix} DP|_{\mathcal{H}_i} & D(I - P)|_{\mathcal{H}_i}^\dagger \\ (I - P)D^*|_{\mathcal{H}_i} & (I - P)Y(I - P)|_{\mathcal{H}_i} \end{pmatrix} \mathcal{H}_i,$$

(2.28)

In view of (2.28) and the regularity together with the positivity of $DP$, we conclude from Lemma 2.11 that

$$\mathcal{R}(D(I - P)|_{\mathcal{H}_i^\perp}) ⊆ \mathcal{R}(DP|_{\mathcal{H}_i})$$

(2.29)

and

$$Z \overset{def}{=} (I - P)Y(I - P) - (I - P)D^*(DP)^\dagger D(I - P) ⩾ 0.$$

(2.30)

Formula (2.28) for $X$ then follows from (2.9), (2.27), and (2.30), since it is obvious that $(I - P)Z(I - P) = Z$. Furthermore, it is clear that (2.29) is satisfied if and only if $\mathcal{R}(D(I - P)) ⊆ \mathcal{R}(DP)$, which can obviously be rephrased as $\mathcal{R}(D) = \mathcal{R}(DP)$.

The discussion above indicates that when (2.25) is satisfied, any $X ∈ \mathcal{L}(\mathcal{H})_+$ is a solution of system (2.14) if and only if it has the form (2.26). □

**Remark 2.17.** Let $A, C ∈ \mathcal{L}(\mathcal{H}, \mathcal{K})$ be such that $A$ is semi-regular, $\mathcal{R}(C) ⊆ \mathcal{R}(A)$ and $DU_A^*U_A$ is regular. Then from the proof of Theorem 2.16 we can conclude that system (2.14) has a solution if and only if $DU_A^*U_A$ is positive and $\mathcal{R}(D) = \mathcal{R}(DU_A^*U_A)$. In such case, the general solution of (2.14) also has form (2.26).

It is noticeable that $CA^*$ may be non-regular even if $DU_A^*U_A$ is positive and regular. For example, let $A$ be semi-regular and meanwhile be non-regular, and put $C = A$. Then clearly, $U_A^*U_A$ is the reduced solution of $AX = A$. It is known that $A$ is regular if and only if $AA^*$ is regular (see [10] Theorem 3.2 and [22] Remark 1.1]), so in this case $CA^*$ fails to be regular.
Remark 2.18. The conclusion stated in Theorem 2.16 may be false if the last condition in (2.25) is not fulfilled. For example, let \( \mathcal{A} = \mathbb{C}, \mathcal{H} = \mathcal{H} = \mathbb{C}^3 \) and put
\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \in \mathcal{L}(\mathcal{H}).
\]
Then \( \mathcal{R}(C) = \mathcal{R}(A), D = A^*C = C, P = U_A^*U_A = A \) and \( CA^* = DP = \text{diag}(1, 0, 0) \in \mathcal{L}(\mathcal{H})_+ \). Therefore, \( \mathcal{R}(D) \neq \mathcal{R}(DP) \). Let \( X = (x_{ij})_{1 \leq i, j \leq 3} \) be Hermitian such that \( AX = C \). Then direct computation yields
\[
X = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & x_{33}
\end{pmatrix},
\]
which can never be positive for any \( x_{33} \in \mathbb{C} \).

Remark 2.19. Example 2.10 also indicates the wrong assertion in [6, Theorem 1.3].

Remark 2.20. Based on quite different methods from ours, the solvability, Hermitian solvability and positive solvability of operator equation (1.1) were considered recently in [11, Section 3] for Hilbert space operators. With the restriction of the regularities of the Hilbert space operators \( A \) and \( CA^* \), the positive solvability of equation (1.1) was considered in [3, Theorem 5.2]. The real positive solvability of equation (1.1) was studied in [11, Section 4] for Hilbert space operators. The latter topic can also be dealt with by following the line in the proof of Lemma 2.12.

3. Solvability of \( AX = C \) associated with projections

In this section, we study the solvability of the following operator equation
\[
(P + Q)^{1/2}X = P, \ X \in \mathcal{L}(\mathcal{H}), \tag{3.1}
\]
where \( P, Q \in \mathcal{L}(\mathcal{H}) \) are projections.

Theorem 3.1. There exist a \( C^*- \) algebra \( \mathcal{A} \), a Hilbert \( C^* \) module \( \mathcal{H} \) over \( \mathcal{A} \) and two projections \( P \) and \( Q \) in \( \mathcal{L}(\mathcal{H}) \) such that operator equation (3.1) has no solution.

Proof. Let \( M_2(\mathbb{C}) \) be the set of \( 2 \times 2 \) complex matrices, and \( \mathcal{B} = C([0, 1]; M_2(\mathbb{C})) \) be the set of continuous matrix-valued functions from \([0, 1]\) to \( M_2(\mathbb{C}) \). Put
\[
\mathcal{A} = \{ f \in \mathcal{B} : f(0) \) and \( f(1) \) are both diagonal\}, \tag{3.2}
\]
and \( \mathcal{H} = \mathcal{A} \). With the inner product given by
\[
(x, y) = x^*y \text{ for any } x, y \in \mathcal{H},
\]
\( \mathcal{H} \) becomes a Hilbert \( \mathcal{A} \)-module such that \( \mathcal{L}(\mathcal{H}) = \mathcal{A} \).

For shortness’ sake, set
\[
c_t = \cos \frac{\pi}{2} t \text{ and } s_t = \sin \frac{\pi}{2} t, \text{ for each } t \in [0, 1].
\]
The matrix-valued functions \( P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q(t) = \begin{pmatrix} c_t^2 & s_t c_t \\ s_t c_t & s_t^2 \end{pmatrix} \) determine projections \( P_\mathcal{A} \) and \( Q_\mathcal{A} \), respectively, in \( \mathcal{A} \).
Note that $P(t) + Q(t)$ is invertible for all $t \in (0, 1]$ (and not invertible for $t = 0$). Indeed, \[\begin{vmatrix} 1 + c_t^2 & s_t c_t \\ s_t c_t & s_t^2 \end{vmatrix} = (1 + c_t^2)s_t^2 - s_t^2 c_t^2 = s_t^2.\] Standard calculation shows that
\[
(P(t) + Q(t))^{1/2} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix},
\]
where
\[
\alpha(t) = \frac{1}{2}(2 - s_t)(\sqrt{1 + c_t} + \sqrt{1 - c_t}),
\]
\[
\beta(t) = \frac{1}{2}s_t(\sqrt{1 + c_t} - \sqrt{1 - c_t}),
\]
\[
\gamma(t) = \frac{1}{2}s_t(\sqrt{1 + c_t} + \sqrt{1 - c_t}),
\]
hence
\[
(P(t) + Q(t))^{-1/2} = \frac{1}{s_t} \begin{pmatrix} \gamma(t) & -\beta(t) \\ -\beta(t) & \alpha(t) \end{pmatrix}, \text{ for all } t \in (0, 1].
\]

Suppose on the contrary that $X \in \mathfrak{A}$ is a solution of $P_\mathfrak{A} = (P_\mathfrak{A} + Q_\mathfrak{A})^{1/2} X$. Write
\[
X = X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix},
\]
where $x_{ij} \in C[0, 1]$, $i, j = 1, 2$ with $x_{12}(0) = x_{21}(0) = x_{12}(1) = x_{21}(1) = 0$. Then
\[
X(t) = (P(t) + Q(t))^{-1/2} P(t) = \frac{1}{2} \begin{pmatrix} \sqrt{1 + c_t} + \sqrt{1 - c_t} & 0 \\ -\sqrt{1 + c_t} + \sqrt{1 - c_t} & 0 \end{pmatrix}, \text{ for all } t \in (0, 1].
\]
It follows that
\[
0 = X_{21}(0) = \lim_{t \to 0} x_{21}(t) = \lim_{t \to 0} \frac{1}{2} (-\sqrt{1 + c_t} + \sqrt{1 - c_t}) = -\frac{1}{\sqrt{2}},
\]
which is a contradiction. \hfill \Box

**Theorem 3.2.** Let $\mathcal{H}$ be any Hilbert $C^*$-module and $P, Q \in \mathcal{L}(\mathcal{H})$ be two projections. Then for any $\varepsilon \in (0, 1)$, there exists a projection $Q' \in \mathcal{L}(\mathcal{H})$ such that $\|Q - Q'\| < \varepsilon$ and the equation $(P + Q')^{1/2} X = P$, $X \in \mathcal{L}(\mathcal{H})$ has a solution.

**Proof.** It is known that the $C^*$-algebra $\mathfrak{A}$ defined by (3.2) is the universal unital $C^*$-algebra generated by two projections [19]. By the universality of $\mathfrak{A}$, given two projections $P$ and $Q$ in $\mathcal{L}(\mathcal{H})$, we get a $*$-homomorphism $\psi : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ such that $\psi(P_\mathfrak{A}) = P$ and $\psi(Q_\mathfrak{A}) = Q$.

Let $h : [0, 1] \to [\varepsilon, 1]$ be a linear homeomorphism, and let $\mu : \mathfrak{A} \to \mathfrak{A}$ be a $*$-homomorphism defined by
\[
\mu(f)(t) = \begin{cases} f(0) & \text{if } t \in [0, \varepsilon]; \\ f(h^{-1} t) & \text{if } t \in [\varepsilon, 1], \end{cases} \quad f \in C([0, 1], M_2(\mathbb{C})).
\]
Set $Q'_\mathfrak{A} = \mu(Q_\mathfrak{A})$. Then $Q'_\mathfrak{A}$ is a projection, and $\lim_{\varepsilon \to 0} \|Q_\mathfrak{A} - Q'_\mathfrak{A}\| = 0$. Set $Q' = \psi(Q'_\mathfrak{A})$, then $\|Q - Q'\| \leq \|Q_\mathfrak{A} - Q'_\mathfrak{A}\|$. 

\[\Box\]
For $t \in (\varepsilon, 1]$, the equation $P_{\mathfrak{A}}(t) = (P_{\mathfrak{A}}(t) + Q'_{\mathfrak{A}}(t))^{1/2}X(t)$ has a unique solution $X_{\mathfrak{A}}(t)$ as formulated by (3.3), with $\lim_{t \to \varepsilon} x_{21}(t) = -\frac{1}{\sqrt{2}}$. Note that for $t \in [0, \varepsilon]$ we have $Q'_{\mathfrak{A}}(t) = P_{\mathfrak{A}}(t) = (1 \ 0 \ 0 \ 0)$, hence for $t \in [0, \varepsilon]$ we can take $X_{\mathfrak{A}}(t) = \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\varepsilon \sqrt{2}} \\ 0 \\ 0 \end{array} \right)$ as a solution for $P_{\mathfrak{A}}(t) = (P_{\mathfrak{A}}(t) + Q'_{\mathfrak{A}}(t))^{1/2}X(t)$, and, as $x_{21}$ is continuous and $x_{21}(0) = 0$, we have $X_{\mathfrak{A}} \in \mathfrak{A}$. Then $X = \psi(X_{\mathfrak{A}}) \in \mathcal{L}(\mathcal{H})$ is a solution for $P = (P + Q')^{1/2}X$.

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