Study of quasi-static behavior of a system with partially distributed parameters under combined loading

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Abstract The problem of the existence of a quasi-static process described by the solution of a system of differential equations is considered. The possibility of quasi-static deformation of a cantilever bar, at the end of which there is a concentrated mass exceeding the mass of the bar has been studied. The bar is loaded with tracing and horizontal forces. A condition has been obtained under which the bending of a bar can be studied on the basis of the solution of the static problem used in the theory of small vibrations.

1. Introduction
Various physical and geometric nonlinearities must be taken into account when solving dynamic problems. Taking energy dissipation into account makes it much more difficult to find a correct solution. In many practical problems, it is possible to consider the deformation process in the quasi-static formulation [1-5]. When considering quasi-static processes, one neglects the rate of change of the external influence parameter and, as a consequence, the change in time of the characteristic describing the behavior of the object in question. Thus, it is assumed that the rate of change of the characteristic describing the behavior of the object continuously depends on the rate of change of the characteristic of the external influence on it. In [4] a study of the correctness of the rigid plastic model in the problems of quasi-static and dynamic bending of circular plates at small and large deflections is conducted.

In [5] the continuity property with respect to the smallness of initial perturbations and to the smallness of the quasi-static loading rate ("dynamic stability of quasi-static trajectories") is considered. The similarities, differences, and relationships between this concept and the Lyapunov stability are shown using mechanical examples.
There are known works on the formulated problem based on various sustainability criteria, for example [6-10]. In [6] the adequacy of the concepts of tangential and reduced modules in the theory of quasi-static longitudinal bending of elastic-plastic bars (in the geometrically linear formulation) is analyzed. In such works, as a rule, at a certain stage of research the loading trajectory was set and the load parameters ceased to be independent.

In [8] classical problems of longitudinal bending of an elastic-plastic bar are considered. The correctness of the quasi-static statement is given and a sufficient condition is derived. According to this condition, the quasi-static solution makes physical sense provided that the bending stiffness of the cross-section in which the maximum bending moment acts does not become less than the applied longitudinal force (in dimensionless variables).

The problem of studying the continuous dependence of the rate of change of the characteristic describing the behavior of the object on the rate of change of the characteristic of the external influence on it, that is, at what values of the parameter of external influence the process will remain quasi-static, is still relevant.

2. Study of quasi-static deformation of a system with distributed parameters
2.1 Problem description
To investigate the problem of the existence of a quasi-static process described by a solution of a differential equation, consider the following system of differential equations (in matrix form)

\[ \dot{w} = F(w, p(t)) \]  
\[ w(0) = w^0, \]

under initial conditions

where \( w \) – describes the behavior of the object, \( p(t) \) - function characterizing the external influence, \( t \) – time.

The solutions of the problem (1), (2) will be sought in the form

\[ w = \varrho + \eta^{(j)} (1 \leq j \leq s), \]

where the static component \( \eta^{(j)} \) is one of the \( s \) solutions of the system of equations

\[ F(u, p(t)) = 0, \]

and the dynamic component \( \varrho \) is a solution of the system of equations obtained by substituting (3) into (1) and (2), that is

\[ \dot{\varrho} = F(\varrho + \eta^{(j)}, p) - \frac{d\eta^{(j)}}{dp} \dot{p} \]

\[ \varrho(0) = w^0 - \eta^{(j)}(p(0)) \]

Let for (5) be satisfied:

\[ \left| \frac{d\eta^{(j)}}{dp} \right| \leq M; \quad p^0 \leq p(t) \leq p^1 \]

Let us assume that the trivial solution of the equation

\[ \dot{\varrho} = F(\varrho + \eta^{(j)}, c) \]
asymptotically stable for any \( c \in [p^0, p^1] \).

We will say that at \( p \in D \) there exists a quasi-static process described by the system of equations (1) if for any \( \varepsilon > 0 \), any \( p^0 \in D \), \( p^1 \in D \) and any \( w^0 \) there exists a corresponding solution of equation (4), such a function \( r(t) \) and a quantity \( t_0 \) that from the conditions

\[
\begin{align*}
  r(t) &= p^0 & \text{if } 0 \leq t \leq t_0, \\
  p(t) &= r(t) & \text{if } t \geq t_0
\end{align*}
\]

\[
\lim_{t \to \infty} r(t) = p^1
\]

it follows that

\[
|g(t)| \leq \varepsilon \quad \text{if } t \geq t_0
\]

Here that solution of equation (16) is called the "corresponding" one, in the region of attraction of which point \((p^0, w^0)\) falls.

Using the above, the following theorem is formulated and proved in [11]:

If \( p^0 \in D \), \( p^1 \in D \) and domain \( D \) do not contain any singular points of the system of equations (4), that is,

\[
\left| \frac{dn^{(j)}}{dp} \right| \leq M \text{ for all } (1 \leq j \leq s),
\]

and a trivial solution of the system of equations (8) for the corresponding \( \eta^{(j)} \) is asymptotically stable for all \( c \in D \), then a quasi-static process exists (in the sense (9)-(12)).

Thus, the condition for the existence of a quasi-static process when the external action changes from \( p^0 \) to \( p^1 \) is reduced to the study of falling within this interval of singular points of equation (4), provided that the trivial solution of the system of equations (8) is asymptotically stable.

Obviously, for systems with distributive parameters, the requirement of existence and boundedness of \( \frac{dn^{(j)}}{dp} \) leads to checking the fulfillment of the conditions of the theorem on implicit functions for the differential equations of statics [11, 12]. But some restrictions should be made on the choice of state space.

We will use only those pairs of state spaces in which the derivative of the Frechette mapping is isomorphism [13, 14]. Such spaces with appropriate norms are usually chosen when solving elastic-plastic deformation problems, such as \( C^2([a, b], \mathbb{R}^{m}) \), \( C^4([a, b], \mathbb{R}^{n}) \), Gelder spaces, Hilbert space, and others.

Following [4, 5], let us consider the simplest version, which consists in the assumption that at the end of the cantilever bar there is a concentrated mass, exceeding in magnitude the mass of the bar.

\[2.2\] Formulation of the problem of bending a bar

Consider the quasi-static deformation of a bar with a concentrated mass, loaded with forces \( p_1 \) and \( p_2 \), as shown in figure 1.
In the linear formulation, the equation [9, 10] describing the behavior of the bar will be:
\[
EI(\dddot{u} - \dddot{f}) - (p_1 + p_2)(\dddot{u}(\ell, t) - \dddot{u}(\bar{x}, t)) + (\ell - \bar{x})(p_2 \ddot{u}(\ell, t) + m\dddot{u}(\ell, t) + b\dddot{u}(\ell, t)) + \\
+ mp^2\dddot{u}(\ell, t) + b\dddot{u}(\ell, t) = 0
\]
with boundary conditions
\[
\dddot{u}(0, t) = \dddot{u}(0, t) = 0
\]
where \( \dddot{f}(\bar{x}) \) - is the function characterizing the bar axis in the unloaded state (initial imperfection), with \( \dddot{f}(0) = \dddot{f}(0) = 0 \). The dashed line denotes differentiation by \( \bar{x} \), and the point denotes differentiation by \( t \).

Let us convert (13), (14) to dimensionless variables using the following relations
\[
\bar{x} = x\ell, \quad \dddot{u} = u\ell, \quad \dddot{f} = f\ell, \quad p_1 = \frac{\alpha, EI}{\ell^2}, \quad m = \gamma, EI, \quad b = \frac{\beta, EI}{\ell^3}, \quad mp^2 = \frac{\gamma_2, EI}{\ell^2}
\]

Then the problem (13), (14) will be written as follows (at \( \alpha_1 + \alpha_2 > 0 \))
\[
u^* + \alpha^2u = f^* + \alpha^2u(1, t) - (1 - x)(\alpha_2\dddot{u}(1, t) + \gamma_1\dddot{u}(1, t) + \beta\dddot{u}(1, t)) - \gamma_2\dddot{u}(1, t) - \beta\dddot{u}(1, t) \\
u(0, t) = \dddot{u}(0, t) = 0
\]
where \( \alpha^2 = \alpha_1 + \alpha_2 \).

At \( p_i = \text{const} \) and \( f(x) = f_i(x) \) problem (15) admits some solution that does not depend on \( t \), i.e. \( u = u_0(x, p_1, p_2) \), and that is a solution of the statics problem
\[
u^* + \alpha^2u = f_0^* + \alpha^2u(1) - (1 - x)\alpha_2\dddot{u}(1) \\
u(0) = \dddot{u}(0) = 0
\]

Let us find out under what conditions the quasi-static deformation of the bar corresponding to \( u = u_0(x, p_1, p_2) \) is possible.

2.3 Continuous dependence of the solution describing bending of the bar on the initial data
To do this, as follows from [11-15], it is necessary to compose the following auxiliary problem with respect to the function \( \zeta(x) \):
\[
u_0^* + \zeta^* + \alpha^2(u_0 + \zeta) = f_0^* + \alpha^2(u_0(1) + \zeta(1)) - (1 - x)\alpha_2(u_0(1) + \zeta'(1)) \\
u_0(0) + \zeta(0) = u_0(0) + \zeta(0) = 0
\]
Since \( u_0 \) is the solution to problem (16), from (17) we obtain the problem for \( \zeta(x) \)
\[
\zeta'' + \alpha^2 \zeta = \alpha^2 \zeta(1) - (1 - x) \alpha \zeta'(1)
\]
\[
\zeta(0) = \zeta'(0) = 0
\]

(18)

The general solution of the equation from (18) we find in the form
\[
\zeta = A \sin \alpha x + B \cos \alpha x + \zeta(1) - \frac{\alpha_2}{\alpha}(1 - x) \alpha \zeta'(1)
\]

(19)

Substituting (19) into the conditions from (18), we obtained a system for finding the unknown constants
\[
A \sin \alpha + B \cos \alpha = 0
\]
\[
A \left( \cos \alpha - \frac{\alpha_2}{\alpha} \right) - B \sin \alpha = 0
\]

(20)

The condition for the existence of a nontrivial solution of problem (18) takes the form
\[
\left( 1 - \frac{\alpha_2}{\alpha} \right) \cos \alpha - 1 = 0
\]

(21)

From (21) we have the relation describing the statically special curve
\[
k = \frac{\alpha_2}{\alpha_1} = -\cos \alpha \quad \text{or} \quad \alpha_1 \cos \sqrt{\alpha_1 + \alpha_2 + \alpha_2} = 0
\]

(22)

In figure 2 we can see a curve corresponding to (22) in the space of parameters of external influences

![Figure 2. Lines bounding the area of continuous dependence.](image)

Line (22) is the upper boundary of the quasi-static state corresponding to \( u = u_0(x, p_1, p_2) \). If there is no tracking force \( \alpha_2 = 0 \), then we have compression of the cantilever bar by the longitudi-
nal force $p_2$. For the problem under consideration, at $\alpha_2 = 0$ there will be $\alpha_1 = 2.465$, which corresponds to the Eulerian force $p_\nu = \frac{\pi^2 EI}{4 \ell^2}$.

2.4 Research on asymptotic stability

Since there is a non-potential force $p_2$, a sufficient condition for the existence of a quasi-static process is the existence of regions of attraction for each equilibrium state corresponding to $u = u_0(x, p_1, p_2)$.

We will search for the solution of problem (15) in the form

$$ u(x, t) = u_0(x) + w(x, t) \quad (23) $$

Substituting (23) into (15), taking into account that $u_0$ is a solution of problem (16), we obtain the following problem concerning the function $w$

$$ w'' + \alpha^2 w = \alpha^2 T - (1 - x)(\alpha_2 \varphi + \gamma_1 \ddot{T} + \beta \ddot{T}) - \gamma_2 \ddot{\varphi} - \beta \ddot{\varphi} \quad (24) $$

Here

$$ T = w(1, t), \quad \varphi = w'(1, t) \quad (25) $$

By integrating equation (24) over $x$, we found the function $w$

$$ w = A \sin \alpha x + B \cos \alpha x + T - \frac{1}{\alpha^2} (1 - x)(\alpha_2 \varphi + \gamma_1 \ddot{T} + \beta \ddot{T}) - \frac{1}{\alpha^2} (\gamma_2 \ddot{\varphi} + \beta \ddot{\varphi}) \quad (26) $$

As a result of substitution of the conditions from (24) we have

$$ A = -\frac{1}{\alpha^2} (1 - x)(\alpha_2 \varphi + \gamma_1 \ddot{T}) \quad (27) $$

$$ B = \frac{1}{\alpha^2} (\gamma_1 \ddot{T} + \beta \ddot{T} + \alpha_2 \varphi + \gamma_2 \ddot{\varphi} + \beta \ddot{\varphi}) - T $$

From (25)-(27) we obtain the following problem concerning the unknown function $T$

$$ a_{11} \ddot{T} + a_{12} \ddot{T} + a_{13} T + b_{11} \ddot{\varphi} + b_{12} \ddot{\varphi} + b_{13} \varphi = 0 \quad (28) $$

where

$$ a_{11} = \gamma_1 (\alpha \cos \alpha - \sin \alpha), \quad a_{12} = \beta (\alpha \cos \alpha - \sin \alpha) $$

$$ a_{13} = -\alpha^3 \cos \alpha, \quad b_{11} = \gamma_2 (\alpha \cos \alpha - \sin \alpha), \quad b_{12} = \alpha \beta (\cos \alpha - 1) $$

$$ b_{13} = \alpha_2 (\alpha \cos \alpha - \sin \alpha), \quad a_{21} = \gamma_1 (1 - \cos \alpha - \alpha \sin \alpha) $$

$$ a_{22} = \beta (1 - \cos \alpha - \alpha \sin \alpha), \quad a_{23} = \alpha^3 \sin \alpha, \quad b_{21} = -\gamma_2 \alpha \sin \alpha $$

$$ b_{22} = \alpha \beta \sin \alpha, \quad b_{23} = \alpha_2 (1 - \cos \alpha - \alpha \sin \alpha) - \alpha^2 $$

The characteristic equation corresponding to (28) is

$$ a_0 \mu^4 + a_1 \mu^3 + a_2 \mu^2 + a_3 \mu + a_4 = 0 \quad (29) $$
where

\[ a_0 = \alpha \gamma_2 \gamma_1 (2 - 2 \cos \alpha - \alpha \sin \alpha), \quad a_1 = \alpha \beta (\gamma_2 + \gamma_1)(2 - 2 \cos \alpha - \alpha \sin \alpha) \]

\[ a_2 = \alpha^2 \gamma_1 (\sin \alpha - \alpha \cos \alpha) + \alpha^4 \gamma_2 \sin \alpha + \alpha \beta^2 (2 - 2 \cos \alpha - \alpha \sin \alpha) \]

\[ a_3 = \alpha^2 \beta ((1 + \alpha^2) \sin \alpha - \alpha \cos \alpha) \]

\[ a_4 = \frac{\alpha^5}{1 + k} (k + \cos \alpha) \]

The negativity conditions for the real parts of the roots of equation (29) in the Leenard-Shypar form [9] in this case are

\[ a_i > 0 \quad (i = 0, \ldots, 4) \]

\[ (a_2 a_3 - a_1 a_4) a_i > a_0 a_3^2 \]  \hspace{1cm} (30)

Since \( \alpha > 0, ~ \gamma_1 > 0, ~ \gamma_2 > 0, ~ \beta > 0 \), the conditions for \( a_0 \) and \( a_1 \) coincide, and the requirement that the coefficients of \( a_i > 0 \) are positive will be

\[ (1 + \alpha^2) \sin \alpha - \alpha \cos \alpha > 0 \]

\[ 2 - 2 \cos \alpha - \alpha \sin \alpha > 0 \]

\[ \alpha \gamma_1 (\sin \alpha - \alpha \cos \alpha) + \alpha^3 \gamma_2 \sin \alpha + \beta^2 (2 - 2 \cos \alpha - \alpha \sin \alpha) > 0 \]

\[ k + \cos \alpha > 0 \]

The signs \( a_0, ~ a_3, ~ a_4 \) do not depend on \( \gamma_1, ~ \gamma_2, ~ \beta \). Moreover, for any \( \alpha \)

\[ 2 - 2 \cos \alpha - \alpha \sin \alpha > (1 + \alpha^2) \sin \alpha - \alpha \cos \alpha \]

Then it is always \( a_0 > a_3 \). Therefore for any \( \gamma_1, ~ \gamma_2, ~ \beta \), the corresponding lines shown in figure 3

![Figure 3. Lines \( a_3 = a_4 = 0 \).](image)

The particular curve corresponding to the conditions (30) consists of the graphs
\[ \alpha \gamma_1 (\sin \alpha - \alpha \cos \alpha) + \alpha^3 \gamma_2 \sin \alpha + \beta^2 (2 - 2 \cos \alpha - \alpha \sin \alpha) = 0 \]

\[ (1 + \alpha^2) \sin \alpha - \alpha \cos \alpha = 0 \]

\[ k + \cos \alpha = 0 \]

\[
\left( \alpha \gamma_1 (\sin \alpha - \alpha \cos \alpha) + \alpha^3 \gamma_2 \sin \alpha + \beta^2 (2 - 2 \cos \alpha - \alpha \sin \alpha) \right) (1 + \alpha^2) \sin \alpha - \alpha \cos \alpha (1 + k) - \\
- (\gamma_1 + \gamma_2) \alpha^3 (2 - 2 \cos \alpha - \alpha \sin \alpha) (\cos \alpha + k) \cos \alpha + k - \\
- \gamma_1 \gamma_2 \beta (2 - 2 \cos \alpha - \alpha \sin \alpha) (1 + k) ((1 + \alpha^2) \sin \alpha - \alpha \cos \alpha)^2 = 0
\] (32)

In figure 4 we can see the graphs of lines (32) limiting the region of existence of the quasi-static state corresponding to \( u = u_0(x, p_1, p_2) \) for different values of the parameters \( \gamma_1, \gamma_2, \beta \) on the plane of the parameters of external forces.

Figure 4. Lines \( a_2 = 0 \) (---), \( a_3 = 0 \) (---), \( a_4 = 0 \) (---), \((a_2 a_3 - a_4 a_4)a_1 = a_9 a_3^2\) (---)
The case corresponding to $\gamma_1 < \gamma_2$, $\beta < \gamma_2$ is shown in figure 4a. For $\gamma_1 < \gamma_2 < \beta$ the line corresponding to (32) is shown in figure 4b. The case corresponding to $\gamma_2 < \gamma_1$, $\beta < \gamma_1$ is shown in figure 4c. In figure 4d the values of the parameters of the same order. It can be seen that the main lines limiting the region of existence of quasi-static bending are (22) and the last of (32).

For sufficiently small $\beta$ and $\gamma_2$ the curve corresponding to the last of (32) has the form

$$k = \frac{\alpha^2 g_2 \cos\alpha - \alpha^2 g_1 \sin\alpha - g_1^2}{\alpha^2 g_1 \sin\alpha + g_1^2 - \alpha^2 g_2}$$

where $g_1 = \sin\alpha - \alpha \cos\alpha$, $g_2 = 2 - 2 \cos\alpha - \alpha \sin\alpha$.

In this case the area of existence of the quasi-static process will be as follows (figure 5).

![Figure 5](image)

**Figure 5.** Lines limiting the area of existence of the quasi-static process for small $\beta$ and $\gamma_2$.

3. Conclusion

As a result of this study, the necessary condition for the existence of a quasi-static process corresponding to $u = u_0(x, p_1, p_2)$ was obtained on the basis of the criterion of continuous dependence. It gives the upper boundary of the area of existence of the quasi-static process. A sufficient condition for the existence of a quasi-static process is the existence of regions of attraction for each equilibrium state corresponding to $u = u_0(x, p_1, p_2)$. The lower bound of this region is found on the basis of the study of the Lyapunov stability of the trivial solution of the problem (13).

In the space of parameters of external influences, the curves were obtained, limiting the area of existence of the quasi-static process corresponding to $u = u_0(x, p_1, p_2)$. When crossing the boundary of the region, the type of quasi-static process changes. Outside of it, the study of the bending of the considered bar should not be carried out on the basis of the solution of the static problem in the formulation (16).

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