Hochschild Cohomology and Twisted Complexes on Complex Manifolds

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Abstract

We use the theory of twisted resolutions and twisted complexes to give a proof of Kontsevich’s assertion that Yoneda product and cup product are preserved in a canonical isomorphism

\[ \text{Ext} (X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \oplus H^\cdot (X, \wedge T_X) \]

where \( X \) is a complex manifold and \( \Delta \) is the diagonal in \( X \times X \).

Introduction

The categorical mirror symmetry [K1] has been an impetus for the considerable interest recently in the Hochschild cohomology of algebraic varieties [C, K2, S, W1, Y]. One of the themes is to unify the different possible definitions of Hochschild cohomology, whether one takes it to be the \( \text{Ext} \) groups, \( \text{Ext} (X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \), where \( \Delta \) is diagonal in \( X \times X \), or hypercohomology of the standard Hochschild cochain complex, or variations thereof. Then using the degeneration of spectral sequence for \( \text{Ext} \), one gets via the Hochschild-Kostant-Rosenberg isomorphism a decomposition:

\[ \text{Ext} (X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \oplus H^\cdot (X, \Lambda T_X) \]

(0.1)

An interesting problem, inspired by Kontsevich’s assertion at the end of [K2], is the existence of a natural isomorphism between the two sides of

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(0.1) which preserves the Yoneda product in $Ext$ and the cup product in $H(H(X, \Lambda TX))$. In a recent paper [C], it is conjectured that such an isomorphism must involve a correction of the HKR isomorphism by the square root of the Todd class in the right hand side of (0.1).

The purpose of this paper is to show that the theory of twisted resolution and twisted complexes [TT1, 2] is ideally suited to unify the different versions of Hochschild cohomology on any complex manifold, and furthermore we show that HKR isomorphism already preserves products in (0.1), so no correction is needed. In comparison to the statement of Kontsevich [K2], whose approach presumably involves his deformation quantization techniques, no such techniques are used here.

The sheaf $\mathcal{O}_\Delta$ in $X \times X$ has concrete local Koszul resolutions in any coordinate neighborhood of $\Delta$ in $X \times X$. Of course such local resolutions do not patch up to a global resolution of $\mathcal{O}_\Delta$ in $X \times X$. In [TT1, 2], a technique is introduced which builds a twisted resolution of $\mathcal{O}_\Delta$ from such local resolutions. Using this data one can define twisted complexes to represent various cohomology functors on $X \times X$ with coefficients in $\mathcal{O}_\Delta$, in particular the Hochschild cohomology groups. The reliance on local resolutions overcomes a key difficulty which arises when the bar resolution is used to represent $Ext(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)$. The bar resolution is only defined on affine open sets near the diagonal in $X \times X$. For example the differentials are generally not defined at $(z, \zeta) \in X \times X$ if $z$ and $\zeta$ are not in the same affine open set.

The restriction of local Koszul complex and its dual to $\Delta$ readily yield the sheaves $\Omega_X$ and $\Lambda TX$, so the twisted complexes provide a particularly direct link between the two sides of (0.1). In fact the theory developed in [TT2] already suffices to give an isomorphism in (0.1) which is compatible with the products on both sides. However some work is required to show that the isomorphism coincides with the HKR isomorphism. For this purpose we construct in §3 a chain map between local bar and Koszul resolutions. The standard HKR map emerges naturally when the chain map is restricted to $\Delta$. In §1 & 2 we recall the theory of twisted resolutions and twisted complexes and make some adjustments for present applications. We have tried to make the exposition reasonably self contained. In §4 we represent Hochschild homology by the cohomology of a twisted complex which naturally leads to $Tor(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)$. The details here are mostly paralleled to that of Hochschild cohomology developed in §1-3.
Twisted resolutions also provide a convenient framework to consider deformations of coherent sheaves. We hope to discuss related matters elsewhere.

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§1. Twisted Resolutions and Twisted Complexes

We recall the construction of twisted resolutions and twisted complexes for the sheaf $\mathcal{O}_\Delta[T T 1, 2]$. Let $X$ be a complex manifold of dimension $n$, and $\mathcal{U} = \{ U_\alpha \}$ be a locally finite Stein open cover of $X$ for which there are complex analytic coordinates $z_\alpha^i$ on each $U_\alpha$. Let $p_1, p_2$ be projections $X \times X \to X$ onto the first and second factors. On $U_\alpha \times U_\alpha$ write the local coordinates $p_1^* z_\alpha^i, p_2^* z_\alpha^i$ as $z_\alpha^i$ and $\zeta_\alpha^i$ respectively. Take $\mathcal{W}$ as the open cover of $X \times X$ consisting of the cover $U \times U = \{ U_\alpha \times U_\alpha; U_\alpha \in \mathcal{U} \}$ together with a Stein open cover $\mathcal{V} = \{ V_\mu \}$ of the complement of the diagonal.

Given an open set $W_\mu \in \mathcal{W}$ let $F_\mu$ be the free $\mathcal{O}_{W_\mu}$ module of rank $n$, $F_\mu = \sum_{i=1}^n \mathcal{O}_{W_\mu} e_i$ and $K_\mu^{-q} = \Lambda^q (F_\mu), 0 \leq q \leq n$. Also let $\tilde{F}_\mu = \sum_{i=1}^n \mathcal{O}_{W_\mu} \tilde{e}_i$ be the dual module of $F_\mu$, and $\tilde{K}_\mu^q = \Lambda^q (\tilde{F}_\mu), 0 \leq q \leq n$.

If $W_\alpha = U_\alpha \times U_\alpha \in \mathcal{U} \times \mathcal{U}$ we have a differential

$$(d_K)_{\alpha} : K_\alpha^{-q} \to K_\alpha^{-q+1}$$

given by contraction by $(z_\alpha^1 - \zeta_\alpha^1)e_1 + \ldots + (z_\alpha^n - \zeta_\alpha^n)e_n$, while a differential $$(d_{\tilde{K}})_{\alpha} : \tilde{K}_\alpha^q \to \tilde{K}_\alpha^{q+1}$$

is wedge product with the same element. If $W_\mu \in \mathcal{V}$, $(d_K)_{\mu}$ is contraction by $\tilde{e}_1$ while $(d_{\tilde{K}})_{\mu}$ is wedge product by $\tilde{e}_1$.

The Koszul complexes $K^{-}$ give local resolutions of the sheaf of the diagonal $\mathcal{O}_\Delta$ in $X \times X$. In fact chain homotopies on $U_\alpha \times U_\alpha : P_\alpha : K_\alpha^{-q} \to K_\alpha^{-q-1}$ are given by

$$P_\alpha (f e^I) = \sum_{j=1}^n \left( \int_0^1 t^{\mid I \mid} \frac{\partial f}{\partial z_\alpha^j} (\zeta_\alpha + t(z_\alpha - \zeta_\alpha), \zeta_\alpha) dt \right) e^j \wedge e^I$$

where $I = (i_1, \ldots, i_q)$ is an increasing multi-index, and $\mid I \mid$ its length. We have $[TT1, (9, 19)]$.

$$(d_K)e P_\alpha + P_\alpha (d_K) = 1 - res$$
\[
P_\alpha^2 = 0
\]

where \( \text{res} : K^{-q}_\alpha \to K^{-q}_\alpha \) is zero except when \( q = 0 \), and in that case for \( f \in K^0_\alpha \),

\[
\text{res}(f)(z, \zeta) = f(z, z)
\]

If \( W_\mu \in \mathcal{V} \), \( P_\mu : K^{-q}_\mu \to K^{-q-1}_\mu \) is wedge product by \( e^1 \), then

\[
(dK)_\mu P_\mu + P_\mu (dK)_\mu = 1
\]

(1.4)

The local resolutions \( K^{\alpha}_\alpha \to \mathcal{O}_{(U_\alpha \times U_\alpha) \cap \Delta} \) cannot be expected to patch up to a global resolution of \( \mathcal{O}_\Delta \) in a neighborhood of \( \Delta \) in \( X \times X \). Nevertheless it is possible to build a global twisted resolution from which one can represent the various cohomological functors on \( X \times X \) with coefficients in \( \mathcal{O}_\Delta \).

In \( W_{\alpha_0} \cap \ldots \cap W_{\alpha_p} = W_{\alpha_0 \ldots \alpha_p} \) we set

\[
\text{Hom}^q(K^{\alpha_p}_\alpha, K^{\alpha_0}_\alpha) = \oplus_i \text{Hom}(K^{-i+q}_{\alpha_p}, K^{-i+q}_{\alpha_0}) \cong \oplus_i K^{-i+q}_{\alpha_0} \otimes \tilde{K}^i_{\alpha_p}
\]

(1.5)

The differential acting on \( f \in K^{-i+q}_{\alpha_0} \otimes \tilde{K}^i_{\alpha_p} \) is

\[
(d_{\text{Hom}})_{\alpha_0 \ldots \alpha_p}(f) = (dK)_{\alpha_0} f + (-1)^{q+1} f (dK)_{\alpha_p} = (dK)_{\alpha_0} f + (-1)^{i+q} (d\tilde{K})_{\alpha_p} f
\]

(1.6)

so that

\[
d_{\text{Hom}}(fg) = (d_{\text{Hom}} f) g + (-1)^{|f|} f (d_{\text{Hom}} g)
\]

(1.7)

where \(|f|\) is degree of \( f \) in \( \text{Hom}^{\alpha} \). A homotopy operator \( P_H \) is defined on \( \text{Hom}^{\alpha} \) by using the tensor product representation in (1.5):

\[
(P_H)_{\alpha_0 \ldots \alpha_p}(f) = \sum_{i \geq 0} (-1)^i P_{\alpha_0} \left( \sigma (dK)_{\alpha_p} P_{\alpha_0} \right)^i (f)
\]

(1.8)

where \( \sigma \) is the sign appearing in (1.6). Then by \([TT1, (8,17), TT2, (1,10)]\)

\[
d_{\text{Hom}} P_H + P_H d_{\text{Hom}} = 1 - r
\]

(1.9)

where

\[
r = \sum_{i \geq 0} (-1)^i (P_H \sigma dK)^i \text{res}
\]
Let \( C^p(W, K^q) \) denote the group of Čech \( p \) cochains whose values \( c_{\alpha_0..\alpha_p} \) lie in \( K^q_{\alpha_0}|_{W_{\alpha_0..\alpha_p}} \). Similarly \( C^p(W, \text{Hom}^r(K^\cdot, K^\cdot)) \) denote Čech \( p \) cochains taking value in \( \text{Hom}^r(K^\cdot_{\alpha_p}, K^\cdot_{\alpha_0})|_{W_{\alpha_0..\alpha_p}} \). An associative cup product on \( C^p(W, \text{Hom}^r(K^\cdot, K^\cdot)) \) and an action of \( C^p(W, \text{Hom}^r(K^\cdot, K^\cdot)) \) on \( C^p(W, K^\cdot) \) are defined by

\[
(f^{pq} \cdot g^{rs})_{\alpha_0..\alpha_{p+r}} = (-1)^{qr} f^{pq}_{\alpha_0..\alpha_p} g^{rs}_{\alpha_p..\alpha_{p+r}} \tag{1.10}
\]

The Čech coboundary operators \( \delta \) are given by the formulas

\[
(\delta f)_{\alpha_0..\alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i f_{\alpha_0..\hat{\alpha}_i..\alpha_{p+1}} \bigg|_{W_{\alpha_0..\alpha_{p+1}}} \tag{1.11}
\]

for \( f \in C^p(W, \text{Hom}^r(K^\cdot, K^\cdot)) \) and

\[
(\delta c)_{\alpha_0..\alpha_{p+1}} = \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0..\hat{\alpha}_i..\alpha_{p+1}} \bigg|_{W_{\alpha_0..\alpha_{p+1}}}
\]

for \( c \in C^p(W, K^\cdot) \). It follows readily that

\[
\delta(f \cdot g) = (\delta f) \cdot g + (-1)^{\text{deg} f} f \cdot (\delta g) \tag{1.12}
\]

where \( \cdot \) is the product (1.10) and \( \text{deg} f \) refers to the total degree (Čech + \( \text{Hom}^r \)). These operators satisfy \( \delta^2 = 0 \), but they are different from standard Čech coboundary operators where the sum in (1.11) is over \( 0 \leq i \leq p + 1 \).

To include the missing terms we need to generalize the notion of transition functions of a complex of vector bundles. We consider the following chain maps between the local Koszul complexes. Since \( K^0_\alpha = \mathcal{O}_{W_\alpha} \) define \( A^0_{\alpha\beta} : K^0_\beta \to K^0_\alpha \) to be just the identity map. The \( A^0_{\alpha\beta} \) extends to a chain map of the complexes by the formula

\[
A_{\alpha\beta} = A^0_{\alpha\beta} - (P_H)_{\alpha\beta}(d_{K^0_\beta} A^0_{\alpha\beta}) \tag{1.13}
\]

Clearly \( A^q_{\alpha\beta} = A^0_{\alpha\beta} \) when \( q = 0 \), and \( A_{\alpha\beta} \) gives a chain map, i.e. \( A_{\alpha\beta} \) is a \( d_{\text{Hom}} \) cocycle from the equation, using (1.8):

\[
d_{\text{Hom}} A_{\alpha\beta} = d_{K^0_\beta} A^0_{\alpha\beta} - (1 - P_H d_{\text{Hom}} - r)d_{K^0_\beta} A^0_{\alpha\beta} = 0,
\]
since $res(d_{K^*} A_{\alpha\beta}^0) = 0$.

Let $a^{1,0} \in C^1(W, Hom^0(K^*, K^*))$ be given by $a_{\alpha\beta}^{1,0} = A_{\alpha\beta}$. Then since these are not true transition functions $a_{\alpha\beta}^{1,0} a_{\beta\gamma}^{1,0} - a_{\alpha\gamma}^{1,0}$ does not vanish. However this difference is chain homotopic to zero, and thus gives rise to

$$a^{i,-i+1} \in C^i(W, Hom^{-i+1}(K^*, K^*))$$

$0 \leq i \leq n$, where $a^{0,1} = d_{Hom}$, $a^{1,0}$ as above, and inductively,

$$a^{i+1,-i} = (-1)^i P_H(\delta a^{i,-i+1} + \sum_{r+s=i-1} a^{r+1,-r} a^{s+1,-s}) \quad (1.14)$$

Let $a = \sum_{i \geq 0} a^{i,-i+1}$, then $a$ has total degree 1 and it satisfies the twisting cochain equation: $[TT2, \S2][OTT, \S1]$.

$$\delta a + a \cdot a = 0 \quad (1.15)$$

The data $(W, K^*, a)$ is called a twisted resolution of $O_\Delta$ on $X \times X$. This generalizes the 1 cocycle condition for transition functions of a global complex of vector bundles. The equations (1.12) and (1.15) enable one to define differential $D_a$ on $C^\cdot(W, K^*)$ by

$$D_a c = \delta c + a \cdot c \quad (1.16)$$

and similarly differential $D_{a,a}$ on $C^\cdot(W, Hom^\cdot(K^*, K^*))$ by

$$D_{a,a} f = \delta f + a \cdot f + (-1)^{deg} f^{+1} f \cdot a \quad (1.17)$$

We denote the singly graded complexes by $C^\cdot_a(W, K^*)$ and $C^\cdot_{a,a}(W, Hom^\cdot(K^*, K^*))$ respectively. Thus

$$C^p_a(W, K^*) = \bigoplus_{i \geq 0} C^{p+i}(W, K^{-i})$$

with the differential $D_a$ and similarly for $C^p_{a,a}(W, Hom^\cdot(K^*, K^*))$. These differentials are compatible with products and actions (1.10).

\section{Cohomology of Twisted Complexes}

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Consider the restriction of the twisting cochain \( a \) to the diagonal. From (1.1) it follows that \( a^{0,1}|_{\Delta} = 0 \). From [TT2, p.53(9), (6.1)] it follows that for \( i \geq 2 \)
\[
a^{i-1,1}|_{\Delta} = 0 \tag{2.1}
\]
For the remaining component
\[
a^{1,0}_{\alpha\beta}|_{\Delta} = \bigwedge \left( \frac{\partial z^{i}_{i}}{\partial z^{j}_{\alpha}} \right) \tag{2.2}
\]
This follows from the statement [TT2, (3.10)], we fill in some details here as an exercise in the formulas in §1. From (1.8), (1.13) it follows that
\[
A^{-1}_{\alpha\beta} = -P_{\alpha}(dK_{\beta} A^{0}_{\alpha\beta}) = P_{\alpha}(A^{0}_{\alpha\beta} dK_{\beta}) \tag{2.3}
\]
thus \( A^{-1}_{\alpha\beta}(e^{i}) = P_{\alpha}(z^{i}_{\beta} - \zeta^{i}_{\beta}) \), and from (1.2)
\[
A^{-1}_{\alpha\beta}(e^{i})|_{\Delta} = \sum_{j} \frac{\partial z^{j}_{i}}{\partial z^{j}_{\alpha}} e^{j}
\]
Now assume \( A^{-k}_{\alpha\beta}|_{\Delta} \) acts as \( \bigwedge^{k} \left( \frac{\partial z^{i}_{i}}{\partial z^{j}_{\alpha}} \right) \) for \( k < q \) and consider a multi index \( I = (i_{1}, ..., i_{q}) \). We denote by \( I_{r} \) the multi index \( (i_{1}, ..., \hat{i}_{r}, ..., i_{q}) \).
\[
A^{-q}_{\alpha\beta}(e^{I}) = P_{\alpha}(dK_{\beta} A^{q-1}_{\alpha\beta})(e^{I})
\]
\[
= P_{\alpha} A^{-q-1}_{\alpha\beta} \left( \sum_{r=1}^{q} (-1)^{r-1}(z^{i}_{\beta} - \zeta^{i}_{\beta})e^{I_{r}} \right) \tag{2.4}
\]
It is clear that in \( A^{-q}_{\alpha\beta}(e^{I})|_{\Delta} \), the nonzero terms come from \( P_{\alpha} \) differentiating the \( (z^{i}_{\beta} - \zeta^{i}_{\beta}) \). Thus when restricted to \( \Delta \) a summand in (2.4) is given by
\[
\frac{(-1)^{r-1}}{|I|} \sum_{j} \frac{\partial z^{j}_{i}}{\partial z^{j}_{\alpha}} e^{k} \bigwedge^{q-1} \left( \frac{\partial z^{i}_{i}}{\partial z^{j}_{\alpha}} \right)(e^{I_{r}}) = \frac{1}{q} \bigwedge^{q} \left( \frac{\partial z^{i}_{\beta}}{\partial z^{j}_{\alpha}} \right)(e^{I})
\]
where we have used induction hypothesis and formula (1.2). This finishes the proof of (2.2).

So when restricted to $\Delta$, Koszul complexes may be identified with the sheaf of holomorphic forms.

$$K_{\alpha}^{-q} \mid _{\Delta} \cong \Omega_{U_{\alpha}}^{q}$$ (2.5)

Similarly the dual Koszul complex restricts to sheaf of multi tangent fields.

$$\tilde{K}_{\alpha}^{q} \mid _{\Delta} \cong \bigwedge \limits^{q} T_{U_{\alpha}}$$ (2.6)

The twisted complexes $C_{a}$ and $C_{a,a}$ have natural filtrations given by Čech degrees which are preserved by differentials $D_{a}$ and $D_{a,a}$ respectively.

**Proposition (2.7)**

$$H^{\bullet}(\text{Hom}(K^{\cdot}, K^{\cdot})) \cong \text{Ext}_{X \times X}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta})$$

$$H^{\cdot}(C_{a,a}(\mathcal{W}, \text{Hom}(K^{\cdot}, K^{\cdot}))) \cong \text{Ext}(X \times X; \mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}).$$

**Proof:** Let

$$R : K_{\alpha}^{0} \to \mathcal{O}_{\Delta \cap U_{\alpha}}$$ (2.8)

be the quasi isomorphism $R : K_{\alpha}^{0} \to \mathcal{O}_{\Delta \cap U_{\alpha}}$ the projection, and $R = 0$ acting on $K_{\alpha}^{q}$, $q \neq 0$, $\text{Hom}(K^{\cdot}, K^{\cdot})$ is the total complex of the bicomplex $K^{\cdot} \otimes \tilde{K}^{\cdot}$ whose first differential has cohomology concentrated in top degree, and so by [TT2, (2.9)]

$$H^{\cdot}(\text{Hom}(K^{\cdot}, K^{\cdot})) \xrightarrow{R} H^{\cdot}(K^{\cdot}, \mathcal{O}_{\Delta}) \cong \text{Ext}_{X \times X}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta})$$

Next $R$ also induces a chain map of twisted complexes

$$C_{a,a}(\mathcal{W}, \text{Hom}(K^{\cdot}, K^{\cdot})) \xrightarrow{R} C_{a,1}(\mathcal{W}, \text{Hom}(K^{\cdot}, \mathcal{O}_{\Delta}))$$ (2.9)

$R$ preserves the Čech filtrations and the corresponding spectral sequences. At $E_{2}$ level $R$ is just the identity map on $H^{\cdot}(X \times X, \text{Ext}_{X \times X}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}))$ by preceding discussions. Hence we have again by [TT2, (2.9)]

$$H^{\cdot}(C_{a,a}(\mathcal{W}, \text{Hom}(K^{\cdot}, K^{\cdot}))) \xrightarrow{R} H^{\cdot}(C_{a,1}(\mathcal{W}, \text{Hom}(K^{\cdot}, \mathcal{O}_{\Delta}))) \cong \text{Ext}(X \times X; \mathcal{O}_{\Delta}, \mathcal{O}_{\Delta})$$
The map $R$ in (2.8) can be given more detailed interpretations. By (2.6)

$$
R : \text{Hom}^q(K^{\cdot}, K^{\cdot}) \to \text{Hom}(K^{-q}, \mathcal{O}_\Delta) \cong q\mathbin{\wedge} T_X
$$

and by (2.1), $a^{i,-i+1} |_{\Delta} = 0$ for $i \neq 1$. This shows that in fact we have a chain map of complexes

$$
R : C_{a,a}(\mathcal{W}, \text{Hom}^{\cdot}(K^{\cdot}, K^{\cdot})) \to \bigoplus C^\cdot(\mathcal{U}, \mathbin{\wedge} T_X) \quad (2.10)
$$

where the right hand side is the single complex associated to the Čech bi-complex with the differential in the coefficient sheaves $\mathbin{\wedge} T_X$ being zero. Note further that $R$ is obviously compatible with the cup products in the chain complexes. The cup product in $C_{a,a}(\mathcal{W}, \text{Hom}^{\cdot}(K^{\cdot}, K^{\cdot}))$ is given by (1.10), the usual Čech cup product, together with compositions in coefficients. In cohomology this corresponds to the Yoneda product in $\text{Ext}^\cdot(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)(\text{cf}[G, M])$. The restriction of this cup product to $\bigoplus C^\cdot(\mathcal{U}, \mathbin{\wedge} T_X)$ via $R$ clearly gives the Čech cup product together with wedge product in the $\mathbin{\wedge} T_X$.

**Theorem (2.11)** The map (2.8) induces a chain map preserving cup products

$$
R : C_{a,a}(\mathcal{W}, \text{Hom}^{\cdot}(K^{\cdot}, K^{\cdot})) \to \bigoplus C^\cdot(\mathcal{U}, \mathbin{\wedge} T_X).
$$

In cohomology $R$ gives an isomorphism

$$
R : \text{Ext}^\cdot(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \to \bigoplus H^\cdot(X, \mathbin{\wedge} T_X)
$$

preserving the Yoneda product in $\text{Ext}^\cdot$ and the cup product in $H^\cdot(X, \mathbin{\wedge} T_X)$.

**Proof:** It only remains to point out that the same map $R$ is shown in (2.7) to be an isomorphism to its image.

§3. Local Comparisons of Bar and Koszul Resolutions and Global consequences.
Let $U \subset X$ be a Stein open set and $A = \Gamma(U, \mathcal{O})$ the ring of holomorphic functions on $U$. For any integer $q \geq 0$, let

$$\mathcal{B}^{-q}(A) = A \otimes \mathbb{C} \cdots \otimes \mathbb{C} A = A^{\otimes(q+2)}$$

(3.1)

$\mathcal{B}^{-q}(A)$ is an $A^e = A \otimes A$ module by multiplication in the first and last factors. The bar resolution of $A$ is

$$\rightarrow \mathcal{B}^{-q}(A) \xrightarrow{\partial} \cdots \rightarrow \mathcal{B}^{-1}(A) \xrightarrow{\partial} \mathcal{B}^0(A) \rightarrow A \rightarrow 0$$

where $\partial$ is the $A^e$-linear map

$$\partial(f_0 \otimes \cdots \otimes f_{q+1}) = \sum_{i=0}^{q} (-1)^i f_0 \otimes \cdots \otimes f_i f_{i+1} \otimes \cdots \otimes f_{q+1}$$

On $U \times U$ we have the Koszul complex $K_U$ defined in (1.1). We proceed to construct a quasi isomorphism

$$\Phi^e : \mathcal{B}^e \rightarrow K^e$$

Let $(z, \zeta) = (z^1, \ldots, z^n, \zeta^1, \ldots, \zeta^n)$ be coordinates on $U \times U$. We will identify $A^e$ with $p_1^* \mathcal{O}_U \otimes p_2^* \mathcal{O}_U$. $\Phi^0 : \mathcal{B}^0 \rightarrow K^0$ is the $A^e$-linear map.

$$\Phi^0(f_0 \otimes f_1) = f_0(z) f_1(\zeta) \in K^0$$

Next for $\eta = f_0 \otimes f_1 \otimes f_2 \in \mathcal{B}^{-1}$, we set

$$\Phi^{-1}(\eta) = f_0(z) \{ P \Phi^0(\partial(1 \otimes f_1 \otimes 1)) \} f_2(\zeta)$$

where $P : K^0 \rightarrow K^{-1}$ is the $\mathbb{C}$ linear homotopy operator (1.2) on $U \times U$, $\Phi^{-1}$ is $A^e$ linear by definition.

Assume inductively that $\Phi^{-k}$ is defined for $k \leq q - 1$, then set

$$\Phi^{-q}(f_0 \otimes \cdots \otimes f_{q+1}) = f_0(z) \{ P \Phi^{-(q-1)}(\partial(1 \otimes f_1 \otimes \cdots \otimes f_q \otimes 1)) \} f_{q+1}(\zeta)$$

(3.2)

If $q > n$, set $\Phi^{-q} = 0$. 

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Lemma (3.3) \( \Phi : B \rightarrow K^* \) is a quasi isomorphism.

Proof: Both \( B \) and \( K^* \) give resolutions of \( \mathcal{O}_{\Delta \setminus U \times U} \) and clearly \( \Phi^0 \) induces identity map on \( \mathcal{O}_{\Delta \setminus U \times U} \). To see that \( \Phi \) commutes with differentials, let 
\[ \eta = f_0 \otimes f_1 \otimes f_2 \in B^{-1} \]
then
\[ \Phi^0(\partial \eta) = f_0(z)(f_1(z) - f_1(\zeta))f_2(\zeta) \]

On the other hand
\[
d_K \Phi^{-1}(\eta) = d_K f_0(z)[P \Phi^0(\partial(1 \otimes f_1 \otimes 1))]f_2(\zeta) \\
= f_0(z)f_2(\zeta)(d_K P)(f_1(z) - f_1(\zeta)) \\
= f_0(z)f_2(\zeta)(f_1(z) - f_1(\zeta)) = \Phi^0(\partial \eta).
\]

where we have used the identity (1.3) on \( K^0 \):
\[ d_K P = 1 - \text{res} \]
and the fact \( \text{res}(f_1(z) - f_1(\zeta)) = 0 \). Now assume \( d_K \Phi^{-k} = \Phi^{-(k-1)} \partial \) for \( 1 \leq k \leq q - 1 \). Consider

\[
d_K \Phi^{-q}(f_0 \otimes ... \otimes f_{q+1}) \\
= f_0(z)f_{q+1}(\zeta)[d_K P \Phi^{-(q-1)}(\partial(1 \otimes f_1... \otimes f_9 \otimes 1))] \quad (3.4) \\
= f_0(z)f_{q+1}(\zeta)[(1 - Pd_K) \Phi^{-(q-1)}(\partial(1 \otimes f_1... \otimes f_q \otimes 1))]
\]

By inductive hypothesis \( d_K \Phi^{-(q-1)} = \Phi^{-(q-1)} \partial \). So (3.4) reduces to
\[ f_0(z)f_{q+1}(\zeta)\Phi^{-(q-1)}(\partial(1 \otimes f_1... \otimes f_q \otimes 1)) \\
= \Phi^{-(q-1)}(\partial(f_0 \otimes ... \otimes f_{q+1})). \]

This finishes the proof of lemma.

To compare with the standard formula of the Hochschild-Kostant-Rosenberg isomorphism, we consider the map
\[
\Phi : B \xrightarrow{\Phi} K^* \rightarrow K^* \mid_{\Delta} \quad (3.5)
\]
By (2.5), \( K \mid_{\Delta} \) is naturally identified with \( \Omega_U \).

**Lemma (3.6)** \( \tilde{\Phi}(f_0 \otimes \ldots \otimes f_{q+1}) = \frac{1}{q!} f_0(z) f_{q+1}(z) df_1 \wedge \ldots \wedge df_q \)

**Proof:** We prove by induction on \( q \).

When \( q = 1 \)

\[
\Phi^{-1}(f_0 \otimes f_1) = f_0(z) f_2(\zeta) P(f_1(z) - f_1(\zeta)).
\]

Hence the restriction to \( \Delta \) gives

\[
\tilde{\Phi}^{-1}(f_0 \otimes f_1) = f_0(z) f_2(z) df_1.
\]

Assume the lemma is valid for \( k < q \). Then by (3.2):

\[
\Phi^{-q}(f_0 \otimes \ldots \otimes f_{q+1})
\]

\[
= f_0(z) f_{q+1}(\zeta) P \{ \Phi^{-(q-1)}[f_1 \otimes f_2 \otimes \ldots \otimes f_q \otimes 1 - 1 \otimes f_1 f_2 \otimes \ldots \otimes 1 \\
\ldots \pm 1 \otimes f_1 \ldots \otimes f_q] \}
\]

(3.7)

In (3.7) we expand again using (3.2) for \( \Phi^{-(q-1)} \),

\[
f_0(z) f_{q+1}(\zeta) P \{ f_1(z) \Phi^{-(q-2)} \partial(1 \otimes f_2 \otimes \ldots \otimes f_q \otimes 1) - P \Phi^{-(q-2)} \partial(1 \otimes f_1 f_2 \otimes \ldots \otimes 1) \\
\ldots \pm f_q(\zeta) \Phi^{-(q-2)} \partial(1 \otimes f_1 \ldots \otimes 1) \}
\]

(3.8)

Using the fact that \( P^2 = 0((1,3)) \), and \( P \) differentiates in \( z \) variables and is therefore linear in functions in \( \zeta \), it is clear that all the terms in the sum in (3.8) drop out except the first one. Upon restriction to \( \Delta \), by induction hypothesis

\[
P \Phi^{-(q-2)} \partial(1 \otimes f_2 \otimes \ldots \otimes f_q \otimes 1) \mid_{\Delta} = \frac{1}{(q-1)!} df_2 \wedge \ldots \wedge df_q.
\]

Substitute this into (3.8) we get by (1.2)

\[
\tilde{\Phi}^{-q}(f_0 \otimes \ldots \otimes f_{q+1}) = f_0(z) f_{q+1}(z) \frac{1}{q!} df_1 \wedge df_2 \wedge \ldots \wedge df_q.
\]

This proves the lemma.
Consider the bar complex $\mathcal{B} \otimes_{A^e} A$ where $f_0 \otimes f_1 \otimes \ldots \otimes f_q \otimes f_{q+1}$ is identified with $f_0 f_{q+1} \otimes f_1 \otimes \ldots \otimes f_q \otimes 1$ in (3.1). We will denote the bar complex by $\mathcal{B} \mid_\Delta$ or $\mathcal{B} \otimes \mathcal{O}_\Delta$. We also denote by $\tilde{\Phi}$ the restriction of the $A^e$ linear map (3.6) to $\mathcal{B} \mid_\Delta$.

$$\tilde{\Phi} : \mathcal{B} \mid_\Delta \to K^* \mid_\Delta$$

(3.9)

The definition of $\mathcal{B}$ does not extend globally to make it a complex of sheaves on $X \times X$. However $\mathcal{B} \otimes \mathcal{O}_\Delta$ is well defined to give a global complex of sheaves on $X$, which is the standard Hochschild chain complex on $X$. By (3.6) the map (3.9) is just the map of Hochschild-Kostant-Rosenberg.

$\mathcal{B} \otimes \mathcal{O}_\Delta$ has a natural shuffle product $[W2, 9.4]$. For elements $f_0 \otimes f_1 \otimes \ldots \otimes f_p$ and $f_0' \otimes f_{p+1} \otimes \ldots \otimes f_{p+q}$ their shuffle product $f_0 \otimes f_1 \otimes \ldots \otimes f_p \# f_0' \otimes f_{p+1} \otimes \ldots \otimes f_{p+q}$ is given by

$$\sum_{\sigma \in \Sigma_{p+q}^{(p,q)} \text{shuffle}} (-1)^\sigma f_0 f_0' \otimes f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(p)} \otimes f_{\sigma(p+1)} \otimes \ldots \otimes f_{\sigma(p+q)}$$

(3.10)

It follows from $[W2, Cor9.4.4]$ that (3.9) is compatible with the shuffle product in $\mathcal{B} \mid_\Delta$ and exterior product of forms in $K^* \mid_\Delta$.

Combining the chain map $\Phi : \mathcal{B} \to K^*$ with the projection $R : K^* \to \mathcal{O}_\Delta$, (2.8), in second factor of $\text{Hom}^*(K^*, K^*)$, we have a chain map

$$\Psi : \text{Hom}^*(K^*, K^*) \to \text{Hom}^*(\mathcal{B}, \mathcal{O}_\Delta)$$

(3.11)

which in cohomology gives identity on the sheaves $\check{\text{Ext}}^\bullet_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. $\text{Hom}^*(\mathcal{B}, \mathcal{O}_\Delta)$ is the standard Hochschild cochain complex of $X$. Let $HH^*(X)$ be the hyperhomology of $\text{Hom}^*(\mathcal{B}, \mathcal{O}_\Delta)[S, W1, WG]$. Using a Stein covering of $X$ the Hochschild cohomology of $X$ is given by the total cohomology of the Čech–Hom' bicomplex:

$$HH^*(X) = H^*(C^*(U, \text{Hom}'(\mathcal{B}, \mathcal{O}_\Delta)))$$

(3.12)
In the setting of the cochain complex $\text{Hom}(\mathcal{B}, \mathcal{O}_\Delta)$ the Hochschild-Kostant-Rosenberg map (abbreviated HKR) is given by

$$\bigwedge T_X \xrightarrow{\text{HKR}} \text{Hom}(\mathcal{B}, \mathcal{O}_\Delta). \quad (3.13)$$

where $v_1 \wedge ... \wedge v_q \in \bigwedge^q T_X$ acts on $f_0 \otimes f_1 \otimes ... \otimes f_q$ by

$$\frac{1}{q!} \sum_{\sigma \in \Sigma_q} (-1)^{\sigma} f_0 v_{\sigma(1)}(f_1) ... v_{\sigma(q)}(f_q).$$

This is the dual of (3.9), therefore we get a commutative diagram:

$$\text{Hom}^*(K^*, K^*) \xrightarrow{\Psi} \text{Hom}^*(\mathcal{B}, \mathcal{O}_\Delta) \xrightarrow{\bigwedge T_X} \text{Hom}^*(K^*, K^*) \xrightarrow{\text{HKR}} \text{Hom}^*(\mathcal{B}, \mathcal{O}_\Delta). \quad (3.14)$$

By the remark following (3.10), HKR preserves products, and by §2 $R$ preserves products, therefore $\Psi$ also preserves products.

Consider the composition on global Čech complexes:

$$C_{a,a}(\mathcal{W}, \text{Hom}^*(K^*, K^*)) \xrightarrow{R} C_{a,1}(\mathcal{W}, \text{Hom}^*(K^*, \mathcal{O}_\Delta)) \xrightarrow{\Psi} C(\mathcal{U}, \text{Hom}^*(\mathcal{B}, \mathcal{O}_\Delta)) \xrightarrow{\bigwedge T_X}$$

which we still denote by $\Psi$. Recall that $a^{i,-i+1}_{\Delta} = 0$ for $i \geq 2$. This implies that the composition $\Psi$ is a chain map. $\Psi$ induces isomorphism in total cohomology since it preserves spectral sequences induced by Čech filtrations and it is isomorphism on $E_2$. We summarize the results in the following theorem.

**Theorem (3.16)** There is a commutative diagram of chain complexes preserving cup products:
The induced maps on cohomology are isomorphisms preserving cup products:

\[
\begin{align*}
\Ext^\cdot(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) & \xrightarrow{\Psi} \HH^\cdot(X) \\
& \xrightarrow{\oplus H^\cdot(X, \wedge T_X)} \HH^\cdot(X, \wedge T_X)
\end{align*}
\]

§4. Hochschild Homology and Tor Functors

One way to represent the global Tor functor Tor\((X; \mathcal{F}, \mathcal{G})\) for a pair of coherent sheaves \(\mathcal{F}\) and \(\mathcal{G}\) on \(X\) is to construct twisted resolutions of \(\mathcal{F}\) and \(\mathcal{G}\) over a covering \(\mathcal{U}\) of \(X\), and then make use of the tensor product of twisting cochains [OTT, §5]. This involves more complicated constructions. For our present purpose which is to represent Tor\((X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)\) and have an action by \(\Ext^\cdot(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)\), a simpler approach is possible.

Let \(K_\alpha\) be the Koszul complex from §1. We set

\[
(K_\alpha \otimes K_\beta)^{-q} = \sum_{i+j=q} K_\alpha^{-i} \otimes K_\beta^{-j} \approx \sum_{i+j=q} \Hom(\tilde{K}_\beta^j, K_\alpha^{-i}) = \Hom^{-q}(\tilde{K}_\beta, K_\alpha)
\]  

This tensor product has differential

\[
d_\otimes(\eta_\alpha \otimes \xi_\beta) = (d_K)_{\alpha} \eta_\alpha \otimes \xi_\beta + (-1)^{\deg \eta} \eta_\alpha \otimes (d_K)_{\beta} \xi_\beta
\]  

\[
= (d_K)_{\alpha}(\eta_\alpha \otimes \xi_\beta) + (-1)^{\deg \eta + \deg \xi + 1}(\eta_\alpha \otimes \xi_\beta)(d_K)_{\beta}
\]

Let \(R : (K \otimes K) \to (K \otimes \mathcal{O}_\Delta)\) be the map induced by (2.8) in the second
factor. As $d_K$ has homology concentrated at the top degree we have, by (2.5), over $U_\alpha \times U_\alpha \cap U_\beta \times U_\beta$

\[
\text{Tor}^1_{\mathcal{O}_X} (\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong H^i((K_\alpha \otimes K_\beta)^\vee) \cong R \Omega^i_\Delta
\]  

Next to put a twisted differential on

\[
C^\cdot(\mathcal{W}, (K^\cdot \otimes K^\cdot)^\cdot) \cong C^\cdot(\mathcal{W}, \text{Hom}(\check{K}^\cdot, K^\cdot))
\]

where $\mathcal{W}$ is a covering as in §1, we will construct a twisted resolution for $\check{K}$ and define $D_{\alpha, \check{a}}$ as in [OTT, §1]. A chain homotopy operator $\check{P}_\alpha$ on $\check{K}_\alpha$ over the open set $U_\alpha \times U_\alpha$ is given by

\[
\check{P}_\alpha(f \check{e}^I) = \sum_{j=1}^\eta \left( \int_0^1 t^{n-|I|} \frac{\partial f}{\partial z_\alpha}(z_\alpha + t(z_\alpha - \zeta_\alpha), \zeta_\alpha) dt \right) \iota_{\check{e}^j} \check{e}^I
\]

where $\iota$ denotes contraction. It satisfies ([TT1, (9.19)])

\[
(d_{\check{K}})_\alpha \check{P}_\alpha + \check{P}_\alpha (d_{\check{K}})_\alpha = 1 - r\check{e}s
\]

\[
(\check{P}_\alpha)^2 = 0
\]

where $r\check{e}s(f \check{e}^1 \wedge ... \wedge \check{e}^n)(z, \zeta) = f(z, z) \check{e}^1 \wedge ... \wedge \check{e}^n$, $r\check{e}s|K^q = 0$ if $q \neq n$. Since $\check{K}^\cdot|_\Delta \cong \wedge^n T_X$ it follows that locally we have a resolution

\[
0 \rightarrow \check{K}^\cdot \rightarrow \bigwedge^n T_X \rightarrow 0
\]

Now using $\check{P}$ we construct a twisting cochain $\check{a}$ for $K^\cdot$ by exactly the same recipe as the twisting cochain $a$ for $K^\cdot$ (cf.[TT2, §2]). It should be noted that

\[
\check{a}_{a_0...a_p}^{i-1+i+1} \in \text{Hom}(\check{K}^\cdot_{a_0} \check{K}^\cdot_{a_p}) \cong \check{K}^\cdot_{a_0} \otimes \check{K}^\cdot_{a_p}
\]

and this is different from the one obtained by dualizing $a$ in [TT2, (1.5), (1.7)],
where the cochain takes value in $K_{\alpha_0} \otimes \tilde{K}_{\alpha_r}$. Here $C^r(W, (K^c \otimes K^c)^{-s})$ are the Čech $r$-cochains
\[ c_{\alpha_0...\alpha_r} \in (K_{\alpha_0} \otimes K_{\alpha_r})^{-s} \]

There is an action
\[ C^p(W, \text{Hom}^q(K^c, K^c)) \times C^r(W, (K^c \otimes K^c)^{-s}) \to C^{p+r}(W, (K^c \otimes K^c)^{-s+q}) \]
\[ (f^{pq}, c^{r,-s})_{\alpha_0...\alpha_p+\alpha_r} = (-1)^{qr} f^{pq} c^{r,-s}_{\alpha_0...\alpha_p...\alpha_p+\alpha_r} \]

The differential $D_{a,\tilde{a}}$ on $C^c(W, \text{Hom}^c(\tilde{K}^c, K^c))$ is given by
\[ D_{a,\tilde{a}} c = \delta c + a \cdot c + (-1)^{\text{deg} c + 1} c \cdot \tilde{a} \]

where $\text{deg} c$ is total degree as in (1.17).

For $f \in C^c(W, \text{Hom}^c(K^c, K^c))$ we have
\[ D_{a,\tilde{a}} (f \cdot c) = D_{a,a}(f) \cdot c + (-1)^{\text{deg} f} f \cdot D_{a,\tilde{a}}(c) \]

$C^c_{a,\tilde{a}}(W, \text{Hom}^c(\tilde{K}^c, K^c))$ denotes the singly graded complex with the differential $D_{a,\tilde{a}}$.

The global functor $\text{Tor}_m(X \times X; O_\Delta, O_\Delta)$ is represented by
\[ \text{Tor}_m(X \times X; O_\Delta, O_\Delta) \cong H^{-m}(C^c_{a,\tilde{a}}(W, \text{Hom}^c(\tilde{K}^c, K^c))) \]

and arguments paralleled to §2 show that
\[ R : C^c_{a,\tilde{a}}(W, \text{Hom}^c(\tilde{K}^c, K^c)) \to C^c_{1,\tilde{a}}(W, \text{Hom}^c(\tilde{K}^c, O_\Delta)) \cong \oplus C^c(\mathcal{U}, \Omega^1_X) \]
\[ (4.5) \]

induces an isomorphism.
\[ H^{-m}(C^c_{a,\tilde{a}}(W, \text{Hom}^c(\tilde{K}^c, K^c))) \xrightarrow{R} \bigoplus_i H^{-m}(C^c(\mathcal{U}, \Omega^1_X)) \]
\[ (4.6) \]
which gives degeneration of spectral sequence for $Tor$:

$$Tor_m(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \bigoplus_i H^{i-m}(X \times X; \mathcal{T}or^{\mathcal{O}_{\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$$

The action $\text{Hom}^q(K^\bullet, K^\bullet) \times \text{Hom}^{-s}(\tilde{K}^\bullet, K^\bullet) \to \text{Hom}^{-s+q}(\tilde{K}^\bullet, K^\bullet)$ gives in cohomology:

$$\text{Ext}_{\mathcal{O}_{\times X}}^q(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \times \text{Tor}^{\mathcal{O}_{\times X}}_s(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \to \text{Tor}^{\mathcal{O}_{\times X}}_{s-q}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

or the contractions:

$$\bigwedge^q T_X \times \Omega_X^s \to \Omega_X^{s-q}$$

Globally the action (4.4) induces in cohomology

$$\text{Ext}^k(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \times Tor_m(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

$$\to Tor_{m-k}(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

Finally the Hochschild homology of $HH_m(X)$ is given by the total cohomology $[W1, WG]$:

$$HH_m(X) = H^{-m}(C^\bullet(\mathcal{U}, \mathcal{B} \otimes \mathcal{O}_\Delta))$$

and via the chain map $\Phi : \mathcal{B} \to \mathcal{K}$ (§2), we have the isomorphism

$$H^{-m}(C^\bullet(\mathcal{U}, \mathcal{B} \otimes \mathcal{O}_\Delta)) \xrightarrow{\sim} H^{-m}(C_{1,\delta}^\bullet(\mathcal{W}, \text{Hom}(\tilde{K}^\bullet, \mathcal{O}_\Delta)))$$

$$\xrightarrow{\sim} \bigoplus_i H^{i-m}(C^\bullet(\mathcal{U}, \Omega_X^i)).$$

By (3.6) this is just the standard HKR isomorphism on Hochschild homology. This also coincides with the map (4.6).
We summarize in the following theorem.

**Theorem (4.8)** The maps $\Phi$, (3.2), and $R$, (2.8), give a commutative diagram of chain maps.

\[
\begin{align*}
C^\cdot(\mathcal{U}, \mathcal{B} \otimes \mathcal{O}_\Delta) & \xrightarrow{\tilde{\Psi}} C_{1,\Delta}^\cdot(\mathcal{W}, \text{Hom}(\tilde{\mathcal{K}}, \mathcal{O}_\Delta)) \\
\downarrow \text{HKR} & \quad \downarrow \text{R} \\
C^\cdot(\mathcal{U}, \Omega_X) & \quad C^\cdot(\mathcal{U}, \Omega_X)
\end{align*}
\]

which induce isomorphisms in cohomology:

\[
\begin{align*}
HH_m^\cdot(X) & \xrightarrow{\tilde{\Psi}} \text{Tor}_m^\cdot(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \\
\downarrow \text{HKR} & \quad \downarrow \text{R} \\
\bigoplus_i H^{i-m}(X, \Omega_X^i) & \quad \bigoplus_i H^{i-m}(X, \Omega_X^i)
\end{align*}
\]

Furthermore the action (4.4) induces in cohomology

\[
\text{Ext}^l(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \times \text{Tor}_m(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta) \rightarrow \text{Tor}_{m-l}(X \times X; \mathcal{O}_\Delta, \mathcal{O}_\Delta)
\]

which corresponds, via $R$, to the contractions.

\[
H^j(X, \wedge^k T_X) \times H^p(X, \Omega^q_X) \rightarrow H^{j+p}(X, \Omega^{q-k}_X).
\]

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