QUANTUM, GRAVITY, AND GEOMETRY

Ali Shojai$^{1,2,*}$

$^1$Department of Physics, Tarbiat Modares University,

P.O.Box 14155–4838, Tehran, IRAN,

$^2$Institute for Studies in Theoretical Physics and Mathematics,

P.O.Box 19395–5531, Tehran, IRAN.

*Email: SHOJAI@THEORY.IPM.AC.IR
Abstract

Recently [1, 2, 3], it is shown that, the quantum effects of matter are well described by the conformal degree of freedom of the space–time metric. On the other hand, it is a well-known fact that according to Einstein’s gravity theory, gravity and geometry are interconnected. In the new quantum gravity theory [1, 2, 3], matter quantum effects completely determine the conformal degree of freedom of the space–time metric, while the causal structure of the space–time is determined by the gravitational effects of the matter, as well as the quantum effects through back reaction effects. This idea, previously, is realized in the framework of scalar–tensor theories. In this work, it is shown that quantum gravity theory can also be realized as a purely metric theory. Such a theory is developed, its consequences and its properties are investigated. The theory is applied, then, to black holes and the radiation-dominated universe. It is shown that the initial singularity can be avoided.

1 INTRODUCTION AND SURVEY

This paper concerns with the generalization of some specific approach to quantum gravity presented in [1, 2]. We shall construct a purely tensor theory instead of the scalar–tensor theory presented in [2]. Our approach would be applicable to purely quantum gravity effects as well as to quantum effects of matter in a curved space–time. First of all we shall briefly review the motivation and basic points of the quantum gravity theory presented
In this century, physicists have been departed from 19th century physics, in two ways. The first was the generalization and bringing the old idea of frame independence or general covariance, in a manifest form. The result of this effort was the pioneer general relativity theory, in which the gravitational effects of matter are identified with the geometry of the space–time. The enigmatic character of this theory is just the above–mentioned property, i.e., the interconnection of gravity and general covariance. When one tries to make a general covariant theory, one is forced to include gravity. The main root of this interconnection is the equivalence principle. According to the equivalence principle, it is possible to go to a frame in which gravity is locally absent, and thus special theory of relativity is applicable locally. Now using the general covariance and writting down anything in a general frame, we will get the general relativity theory.

The second was the investigation of the quantal behaviour of matter, that leads to the quantum theory. According to which a great revolution is appeared in physics. In order to explain the atomic world, quantum theory threw out two essensial classical concepts, the principle of causality and the dogma of formulation of physics in terms of motion in the space–time (motion dogma). The first one is violated during a measurement process, while the second does not exist at any time.

After appearance of quantum mechanics, it was proved that not only the ordinary particles show quantal behaviour but also mediators of the fundamental forces do so. In this way quantum electrodynamics, quantum chromodynamics and quantum flavourdy-
namics were born. But the construction of quantum gravitodynamics or quantum gravity, leads to several structural and conceptual difficulties[5]. For example, the meaning of the wavefunction of the universe is a conceptual difficulty while the time independence of the wavefunction is a structural one.

In contrast to general theory of relativity which is the best theory for gravity, standard quantum mechanics is not the only satisfactory way of understanding the quantal behaviour of matter. One of the best theories explaining the quantal behaviour of matter but remaining faithful to the principle of causality and the motion dogma, is the de-Broglie–Bohm quantum theory.[6] According to this theory, all the enigmatic quantal behaviour of the matter is resulted from a self–interaction of the particle. In fact, any particle exerts a quantum force on itself which can be expressed in terms of a quantum potential and which is derived from the particle wavefunction.

In the non–relativistic de-Broglie–Bohm theory, the quantum potential is given by[6]:

\[ Q = -\frac{\hbar^2}{2m} \nabla^2 |\Psi| \]  

(1)

in which \( \Psi \) is the particle’s wavefunction satisfying an appropriate wave equation. (in this case the nonrelativistic Schrödinger equation) The particle’s trajectory then can be derived from Newton’s law of motion in which in addition to the classical force \(-\nabla V\), the quantum force \(-\nabla Q\) is also present. In this way the de-Broglie–Bohm quantum theory presents a formulation of physics in terms of motion in the space–time.

The celebrated property of the de-Broglie–Bohm quantum theory is the following prop-
At anytime, even when a measurement is done, the particle is on the trajectory given by Newton’s law of motion, including the quantum force. During a measurement, the system is in fact a many-body system (including the particle itself, the probe particle, and the registrating system particles). When one writes down the appropriate equation of motion of all the particles and when one considers the very fact that we know nothing about the initial conditions of the registering system particles, one sees how the projection postulate of quantum mechanics came out. According to which the result of any measurement is one of the eigenvalues of the operator related to the measured quantity with some calculable probability distribution.

Another important property of the de-Broglie–Bohm quantum theory is that using the Schrödinger equation and the Newton’s law of motion, one observes that the phase of the wavefunction is proportional to the Hamilton–Jacobi function of the system. In fact if one set $\Psi = \sqrt{\rho} \exp[iS/\hbar]$ then one arrives at:

$$\frac{\partial S}{\partial t} + \frac{|
abla S|^2}{2m} + V + Q = 0 \quad (2)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{\nabla S}{m} \right) = 0 \quad (3)$$

The first of these equations (the Hamilton–Jacobi equation) is identical to the Newton’s

\begin{equation}
Q = -\frac{\hbar^2}{2} \sum_{i=1}^{N} \frac{1}{m_i} \frac{\nabla_i^2 |\Psi|}{|\Psi|} \nonumber
\end{equation}

where $m_i$ is the mass of $i$th particle and $\nabla_i$ represents differentiating with respect to the position of the $i$th particle, and $N$ is the number of particles.
second law. It is in fact the energy condition:

\[
E = \frac{|\vec{p}|^2}{2m} + V + Q \tag{4}
\]

Remember that in the Hamilton–Jacobi theory\[7\] \(-\partial S/\partial t = E\) and \(\vec{\nabla}S = \vec{p}\). The second equation is then the continuity equation of an hypothetical ensemble of the particle.

The relativistic extension of the Bohmian quantum theory is straightforward\[6\]. All we must do is to generalize the relativistic energy equation:

\[
\eta_{\mu\nu} P^\mu P^\nu = m^2 c^2 \tag{5}
\]

to:

\[
\eta_{\mu\nu} P^\mu P^\nu = m^2 c^2 + Q = M^2 c^2 \tag{6}
\]

where:

\[Q = \hbar^2 \Box |\Psi| \frac{1}{|\Psi|} \tag{7}\]

or

\[M^2 = m^2 \left(1 + \alpha \frac{\Box |\Psi|}{|\Psi|}\right) \tag{8}\]

In fact one can derive this by setting \(\Psi = \sqrt{m} e^{iS/\hbar}\) in the Klein–Gordon equation, and separating the real and imaginary parts. The result is the relativistic Hamilton–Jacobi equation:

\[
\eta_{\mu\nu} \partial^\mu S \partial^\nu S = M^2 c^2 \tag{9}
\]

which is identical to \(2\) (Note that \(P^\mu = -\partial^\mu S\) ) and the continuity equation

\[\partial_\mu (\rho \partial^\mu S) = 0 \tag{10}\]

6
An important problem about the relativistic de-Broglie–Bohm theory is that the mass square defined by equation (8), is not positive–definite. Later, we shall show that this problem would be solved in our present theory.

The generalization of (9) to an arbitrary curved space–time with metric $g_{\mu\nu}$ is:

$$g_{\mu\nu}P^\mu P^\nu = M^2 c^2$$  \hspace{1cm} (11)

The particle trajectory can be derived from the guidance relation:

$$P^\mu = M \frac{dx^\mu}{d\tau}$$  \hspace{1cm} (12)

or by differentiating (11) leading to the Newton’s equation of motion:

$$M \frac{d^2 x^\mu}{d\tau^2} + M \Gamma^\mu_{\nu\kappa} u^\nu u^\kappa = (g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu M$$  \hspace{1cm} (13)

The essential observation in references [1, 2, 3] is that equation (11) can be transformed to the classical (i.e. non quantum) one:

$$g_{\mu\nu}P^\mu P^\nu = m^2 c^2$$  \hspace{1cm} (14)

via a conformal transformation:

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \left( \frac{M^2}{m^2} \right)^{-1} g_{\mu\nu}$$  \hspace{1cm} (15)

In an equivalent manner, equation (13) reduces to the standard geodesic equation via the above conformal transformation.

The physical conclusion from this observation is the following: Suppose in the metric $g_{\mu\nu}$ there is no quantal effect at all (i.e. the trajectory of the particle is a geodesic), the quantal effect can be brought in via a specific conformal transformation as above.
In ref.\cite{1} the authors as a first step towards the formulation of the above conclusion, introduced the quantum conformal degree of freedom via the method of Lagrange multipliers. In this way they have a set of equations of motion describing the background metric, the conformal degree of freedom and the particle trajectory. As a brief introduction to their work we present here their action principle and equations of motion. The action is:\cite{1}:

\[ A[g_{\mu\nu}, \Omega, S, \rho, \lambda] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( \mathcal{R}\Omega^2 - 6\nabla_\mu \Omega \nabla^\mu \Omega \right) + \int d^4x \sqrt{-g} \left( \frac{\rho}{m} \Omega^2 \nabla_\mu S \nabla^\mu S - m\rho \Omega^4 \right) + \int d^4x \sqrt{-g} \lambda \left[ \Omega^2 - \left( 1 + \frac{\hbar^2}{m^2} \frac{\Box}{\sqrt{\rho}} \right) \right] \] 

(16)
in which any overlined quantity is calculated in the background metric, \( \lambda \) is the Lagrange multiplier, and \( \Omega^2 \) is the conformal factor. The equations of motion are:\cite{1}:

1. The equation of motion for \( \Omega \):

\[ \mathcal{R}\Omega + 6 \Box \Omega + \frac{2\kappa}{m} \rho \Omega \left( \nabla_\mu S \nabla^\mu S - 2m^2\Omega^2 \right) + 2\kappa \lambda \Omega = 0 \] 

(17)

2. The continuity equation for particles:

\[ \nabla_\mu \left( \rho \Omega^2 \nabla^\mu S \right) = 0 \] 

(18)

3. The equation of motion for particles:

\[ \left( \nabla_\mu S \nabla^\mu S - m^2\Omega^2 \right) \Omega^2 \sqrt{\rho} + \frac{\hbar^2}{2m} \left[ \Box \left( \frac{\lambda}{\sqrt{\rho}} \right) - \lambda \frac{\Box}{\rho} \right] = 0 \] 

(19)
4. The modified Einstein equations for $\mathcal{R}_{\mu\nu}$:

$$
\Omega^2 \left[ \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \right] - \left[ g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right] \Omega^2 - 6 \nabla_\mu \Omega \nabla_\nu \Omega + 3 g_{\mu\nu} \nabla_\alpha \Omega \nabla^\alpha \Omega \\
+ \frac{2\kappa}{m} \rho \Omega^2 \nabla_\mu S \nabla_\nu S - \frac{\kappa}{m} \rho \Omega^2 \mathcal{R}_{\mu\nu} \nabla_\alpha S \nabla^\alpha + \kappa m \rho \Omega^4 g_{\mu\nu}\nabla_\mu \nabla_\nu \Omega^2 \\
+ \frac{\kappa \hbar^2}{m^2} \left[ \nabla_\mu \sqrt{\rho} \nabla_\nu \left( \frac{\lambda}{\sqrt{\rho}} \right) \nabla_\nu \sqrt{\rho} \nabla_\mu \left( \frac{\lambda}{\sqrt{\rho}} \right) \right] - \frac{\kappa \hbar^2}{m^2} g_{\mu\nu} \nabla_\alpha \left[ \frac{\nabla^\alpha}{\sqrt{\rho}} \right] = 0
$$

(20)

5. The constraint equation:

$$
\Omega^2 = 1 + \frac{\hbar^2}{m^2} \frac{\sqrt{\rho}}{\sqrt{\rho}}
$$

(21)

If one assumes that anything can be expanded in terms of powers of Planck’s constant, one gets zero for $\lambda$ and the equations of motion are:

$$
\nabla_\mu \left( \rho \Omega^2 \nabla^\mu S \right) = 0
$$

(22)

$$
\nabla_\mu S \nabla^\mu S = m^2 \Omega^2
$$

(23)

$$
\Omega^2 \left[ \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \right] - \left[ g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right] \Omega^2 - 6 \nabla_\mu \Omega \nabla_\nu \Omega + 3 g_{\mu\nu} \nabla_\alpha \Omega \nabla^\alpha \Omega \\
+ \frac{2\kappa}{m} \rho \Omega^2 \nabla_\mu S \nabla_\nu S - \frac{\kappa}{m} \rho \Omega^2 g_{\mu\nu} \nabla_\alpha S \nabla^\alpha + \kappa m \rho \Omega^4 g_{\mu\nu} = 0
$$

(24)

$$
\Omega^2 = 1 + \alpha \frac{\sqrt{\rho}}{\sqrt{\rho}}
$$

(25)

In ref. some features of the above formulation is presented. For example the solution for a Robertson–Walker metric is derived, and the formulation is extended to contain the radiation quantum effects. Also nonlocal effects in the framework of such theory is investigated. We shall come back to this point later.
As a next step, currently\textsuperscript{3, 4} it is shown that there is a close connection with scalar–tensor theories. In fact it is possible to set up a scalar–tensor theory in which there is no need to use the method of lagrange multipliers and in which the correct conformal degree of freedom, i.e. one given by quantum effects as in equation (15), is derived using the equations of motion. The action functional for this theory is\textsuperscript{4}:

\begin{equation}
\mathcal{A} = \int d^4x \left\{ \phi \mathcal{R} - \frac{\omega}{\phi} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda \phi + \mathcal{L}_m \right\}
\end{equation}

\begin{equation}
\mathcal{L}_m = \frac{\rho}{m} \nabla_\mu S \nabla^\mu S - \rho m \phi - \Lambda (1 + Q)^2
\end{equation}

in which $\Lambda$ is the cosmological constant. The equations of motion are\textsuperscript{4}:

\begin{equation}
\phi = 1 + Q - \frac{\alpha}{2} \Box Q
\end{equation}

\begin{equation}
\nabla^\mu S \nabla_\mu S = m^2 \phi - \frac{2\Lambda m}{\rho} (1 + Q)(Q - \bar{Q}) + \frac{\alpha \Lambda m}{\rho} \left( \Box Q - 2 \nabla_\mu \frac{\nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} \right)
\end{equation}

\begin{equation}
\nabla^\mu (\rho \nabla_\mu S) = 0
\end{equation}

\begin{equation}
G^{\mu\nu} - \Lambda g^{\mu\nu} = -\frac{1}{\phi} T^{\mu\nu} - \frac{1}{\phi} [\nabla^\mu \nabla^\nu - g^{\mu\nu} \Box] \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla_\nu \phi - \frac{1}{2} \frac{\omega}{\phi^2} g^{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi
\end{equation}

where

\begin{equation}
\bar{Q} = \alpha \frac{\nabla_\mu \sqrt{\rho} \nabla^\mu \sqrt{\rho}}{\rho}
\end{equation}

Both the above mentioned theories\textsuperscript{3, 4}, have a problem. In these theories, it is assumed that one deals with an ensemble of similar particles with density $\rho$. In Bohm’s theory, the quantum potential exists for a single particle as well as for an ensemble. In the case of a single particle, the interpretation of the quantum potential is in terms of an
hypothetical ensemble. Note that in the above theories, the ensemble is a real one, not an hypothetical one, because, the energy–momentum tensor of the ensemble is appeared and has physical effects. As we shall show later, in this paper we have solved this problem and the theory would work both for a single particle and for an ensemble.

In addition to the above problem there is still another one. The theory considers only the quantal effects of matter, the pure quantum gravity effects are not included.

In this paper we shall show that it is possible to make a pure tensor theory for quantum gravity. As a result we shall show that the correct quantum conformal degree of freedom would be achieved, and that the theory works for a particle as well as for a real ensemble of the particle under consideration and that it includes the pure quantum gravity effects. We shall do all of these by trying to write the quantum potential terms in term of geometrical parameters not in terms of ensemble properties.

2 A TOY QUANTUM–GRAVITY THEORY

In this section we first examine how we can translate the quantum potential in a complete geometrical manner, i.e. we write it in a form that there is no explicite reference to matter parameters. Only after using the field equations one deduce the original form of the quantum potential. This has the advantage that lets our theory work both for a single particle and an ensemble. Next we write a special field equation as a toy theory and extract some of its consequences.
2.1 Geometry of the Quantum Conformal Factor

In order to construct a purely metric theory for quantum gravity, let us first examine the geometrical properties of the conformal factor given by:

\[ g_{\mu\nu} = e^{4\Lambda} \eta_{\mu\nu}; \quad e^{4\Lambda} = \frac{\mathcal{M}^2}{m^2} = 1 + \alpha \frac{\Box_{\eta} \sqrt{\rho}}{\sqrt{\rho}} \]  

(33)

where \( \eta_{\mu\nu} \) is the flat space–time metric, \( \Box_{\eta} \) is the d'Alambertian operator evaluated using the \( \eta_{\mu\nu} \) metric.

If one evaluates the Einstein’s tensor for the above metric, one gets:

\[ G_{\mu\nu} = 4g_{\mu\nu}e^{\Lambda} \Box_{\eta} e^{-\Lambda} + 2e^{-2\Lambda} \partial_{\mu} \partial_{\nu} e^{2\Lambda} \]  

(34)

Now replacing \( |\mathcal{T}| \) for \( \rho \) in equation (33) (as it is true for a dust):

\[ \frac{\Lambda}{\alpha} = e^{-4\Lambda} \frac{\Box \sqrt{|\mathcal{T}|} - 4(\nabla \Lambda)^2}{\sqrt{|\mathcal{T}|}} \]  

(35)

and expanding anything up to first order in \( \alpha \) one gets:

\[ \Lambda = \alpha \frac{\Box \sqrt{|\mathcal{T}|}}{\sqrt{|\mathcal{T}|}} \]  

(36)

As an ansatz, we suppose that in the presence of gravitational effects, the field equations are:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu} + 4g_{\mu\nu}e^{\Lambda} \Box e^{-\Lambda} + 2e^{-2\Lambda} \nabla_{\mu} \nabla_{\nu} e^{2\Lambda} \]  

(37)

This equation is written in such a way that in the limit \( \mathcal{T}_{\mu\nu} \to 0 \) the solution (33) achieved.

\(^2\)The absolute value sign is introduced to make the square root always meaningful.
Making the trace of the above equation one gets:

\[- R = \kappa T - 12\Box \Lambda + 24(\nabla \Lambda)^2\]  

(38)

which has the iterative solution:

\[\kappa T = - R + 12\alpha \Box \left( \frac{\Box \sqrt{R}}{\sqrt{R}} \right) + \cdots\]  

(39)

leading to:

\[\Lambda = \alpha \frac{\Box \sqrt{|T|}}{\sqrt{|T|}} \simeq \alpha \frac{\Box \sqrt{|R|}}{\sqrt{|R|}}\]  

(40)

### 2.2 Field Equations of a Toy Quantum Gravity

In the previous subsection we have learned that $T$ can be replaced with $R$ in the expression for the quantum potential or for the conformal factor of the space–time metric. This replacement is in fact an important improvement, because the explicit reference to ensemble density is removed. This lets the theory to work for both a single particle and an ensemble.

So with a glance at equation (37) for our toy quantum–gravity theory, we assume the following field equations:

\[G_{\mu\nu} - \kappa T_{\mu\nu} - Z_{\mu\nu\alpha\beta} \exp \left[ \frac{\alpha}{2} \Phi \right] \nabla^\alpha \nabla^\beta \exp \left[ - \frac{\alpha}{2} \Phi \right] = 0\]  

(41)

where

\[Z_{\mu\nu\alpha\beta} = 2 \left[ g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} \right]\]  

(42)
\[
\Phi = \frac{\Box \sqrt{|\mathcal{R}|}}{\sqrt{|\mathcal{R}|}} \tag{43}
\]

Note that the number 2 and the minus sign of the second term of the last equation are chosen so that the energy equation derived later be correct. It would be very useful to take the trace of equation (41):

\[
\mathcal{R} + \kappa \mathcal{T} + 6 \exp[\alpha \Phi/2] \Box \exp[-\alpha \Phi/2] = 0 \tag{44}
\]

In fact this equation represents the connection of Ricci scalar curvature of the space–time and the trace of matter energy–momentum tensor. This is the analogous of the Einstein’s relation \( \mathcal{R} = -\kappa \mathcal{T} \). In the cases when a perturbative solution is admitted, i.e. when we can expand anything in terms of powers of \( \alpha \), one can find the relation between \( \mathcal{R} \) and \( \mathcal{T} \) perturbatively. In the zeroth approximation one has the classical relation:

\[
\mathcal{R}^{(0)} = -\kappa \mathcal{T} \tag{45}
\]

As a better approximation up to first order in \( \alpha \), one gets:

\[
\mathcal{R}^{(1)} = -\kappa \mathcal{T} - 6 \exp[\alpha \Phi^{(0)}/2] \Box \exp[-\alpha \Phi^{(0)}/2] \tag{46}
\]

where:

\[
\Phi^{(0)} = \frac{\Box \sqrt{|\mathcal{T}|}}{\sqrt{|\mathcal{T}|}} \tag{47}
\]

A more better result can be obtained in the second order as:

\[
\mathcal{R}^{(2)} = -\kappa \mathcal{T} - 6 \exp[\alpha \Phi^{(0)}/2] \Box \exp[-\alpha \Phi^{(0)}/2] - 6 \exp[\alpha \Phi^{(1)}/2] \Box \exp[-\alpha \Phi^{(1)}/2] - 6 \exp[\alpha \Phi^{(1)}/2] \Box \exp[-\alpha \Phi^{(1)}/2] \tag{48}
\]
with:
\[
\phi^{(1)} = \frac{\Box \sqrt{|-\kappa T - 6 \exp[\alpha \Phi(0)/2] \Box \exp[-\alpha \Phi(0)/2]|\} - \kappa T \exp[\alpha \Phi(0)/2] \Box \exp[-\alpha \Phi(0)/2]|}}{\sqrt{|-\kappa T \exp[\alpha \Phi(0)/2] \Box \exp[-\alpha \Phi(0)/2]|}}
\] (49)

The energy relation can be obtained via taking the four divergence of the field equations. Using the fact that the divergence of the Einstein’s tensor is zero, one gets:
\[
\kappa \nabla^\nu T_{\mu\nu} = \alpha R_{\mu\nu} \nabla^\nu \Phi - \frac{\alpha^2}{4} \nabla_\mu (\nabla \Phi)^2 + \frac{\alpha^2}{2} \nabla_\mu \Phi \Box \Phi
\] (50)

For a dust with:
\[
T_{\mu\nu} = \rho u_\mu u_\nu
\] (51)

and assuming the conservation law for mass:
\[
\nabla^\nu (\rho M u_\nu) = 0
\] (52)

up to first order in \(\alpha\) one arrives at:
\[
\frac{\nabla_\mu M}{M} = -\frac{\alpha}{2} \nabla_\mu \Phi
\] (53)

or:
\[
M^2 = m^2 \exp(-\alpha \Phi)
\] (54)

where \(m\) is some integration constant. Note that the first two terms of the above equation is in accordance with the relation (8). Higher order terms which are smaller than the first two terms, are interesting, because \(M^2\) given by (54) is positive-definite, while \(M^2\) given by (8) is not. So in this way an important problem of the de-Broglie–Bohm theory is solved.
3 SOME GENERAL SOLUTIONS

In this section we shall study some general solutions of the field equation (41).

3.1 Conformally Flat Solution

Suppose we search for a solution which is conformally flat, and that the conformal factor is near unity. Such a solution is of the form:

\[ g_{\mu\nu} = e^{2\Lambda} \eta_{\mu\nu}; \quad \Lambda \ll 1 \quad (55) \]

As a result one can derive the following relations:

\[ g = -e^{8\Lambda} = -1 - 8\Lambda \quad (56) \]
\[ R_{\mu\nu} = \eta_{\mu\nu} \Box \Lambda + 2\partial_\mu \partial_\nu \Lambda \quad (57) \]
\[ R = 6\Box \Lambda \quad (58) \]
\[ G_{\mu\nu} = 2\partial_\mu \partial_\nu \Lambda - 2\eta_{\mu\nu} \Box \Lambda \quad (59) \]

In order to solve for \( \Lambda \) one can use the relation (44), and solve it iteratively as it is discussed in the previous section. The result is:

\[ R^{(0)} = -\kappa T = 6\Box \Lambda^{(0)} \implies \Lambda^{(0)} = -\frac{\kappa}{6} \Box^{-1} T \quad (60) \]
\[ R^{(1)} = -\kappa T + 3\alpha \frac{\Box \sqrt{|T|}}{\sqrt{|T|}} = 6\Box \Lambda^{(1)} \implies \]
\[ \Lambda^{(1)} = -\frac{\kappa}{6} \Box^{-1} T + \frac{\alpha}{2} \frac{\Box \sqrt{|T|}}{\sqrt{|T|}} \quad (61) \]
and so on. Thus:

\[ \Lambda = -\frac{\kappa}{6} \square^{-1} T + \frac{\alpha}{2} \frac{\Box \sqrt{|T|}}{\sqrt{|T|}} \] + higher terms including gravity – quantum interactions. \hspace{1cm} (62)

where \( \square^{-1} \) represents the inverse of the d'Alembertian operator. Note that the solution is in complete agreement with de-Broglie–Bohm theory.

### 3.2 Conformally Quantic Solution

As a generalization of the solution found in the previous subsection, suppose we set:

\[ g_{\mu\nu} = e^{2\Lambda} \overline{g}_{\mu\nu} = (1 + 2\Lambda) \overline{g}_{\mu\nu}; \quad \Lambda \ll 1 \] \hspace{1cm} (63)

One can evaluate the following relations:

\[ \mathcal{R}_{\nu\rho} = \overline{\mathcal{R}}_{\nu\rho} + \overline{g}_{\mu\nu} \Box \Lambda + 2 \left( \nabla_{\nu} \nabla_{\rho} \Lambda + \overline{g}_{\nu\rho} \nabla_{\alpha} \Lambda \nabla^{\alpha} \Lambda - \nabla_{\nu} \Lambda \nabla_{\rho} \Lambda \right) \] \hspace{1cm} (64)

\[ \mathcal{R} = e^{-2\Lambda} \left( \overline{\mathcal{R}} + 6 \Box \Lambda + 6 \nabla_{\alpha} \Lambda \nabla^{\alpha} \Lambda \right) \] \hspace{1cm} (65)

\[ \mathcal{G}_{\nu\rho} = \overline{\mathcal{G}}_{\nu\rho} - 2 \overline{g}_{\nu\rho} \Box \Lambda + 2 \nabla_{\nu} \nabla_{\rho} \Lambda \] \hspace{1cm} (66)

On using these relations in the field equations (41), one gets the following solution:

\[ \overline{g}_{\mu\nu} = \kappa \overline{T}_{\mu\nu}; \quad \Lambda = \frac{\alpha}{2} \Phi \] \hspace{1cm} (67)

provided the energy–momentum tensor be conformally invariant. So under this condition we have:

\[ g_{\mu\nu}^{\text{quantum+gravity}} = (1 + \alpha \Phi) g_{\mu\nu}^{\text{gravity}} \] \hspace{1cm} (68)
3.3 Conformally Highly Quantic Solution

Now we can generalize the result of the previous subsection. Suppose in the overlined metric there is no quantum effect, so that:

\[ \mathcal{G}_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu} \quad (69) \]

and assuming that the quantum effects could bring in via a conformal transformation like:

\[ g_{\mu\nu} = e^{2\Lambda} \mathcal{G}_{\mu\nu} \quad (70) \]

Using the field equations (41) and the transformation properties of the Einstein’s equation one gets:

\[ 2 \Box \Lambda + 2 \nabla_\alpha \Lambda \nabla^\alpha \Lambda = \alpha \Box \Phi + 2 \alpha \nabla_\alpha \Phi \nabla^\alpha \Lambda - \frac{\alpha^2}{2} \nabla_\alpha \Phi \nabla^\alpha \Phi \quad (71) \]

which has the solution:

\[ \Lambda = \frac{\alpha}{2} \Phi \quad (72) \]

In the above solution it is assumed that the energy–momentum tensor is either zero or conformally invariant. So, under this condition, no matter how large is the quantum effects, the general solution is:

\[ g^\text{quantum+gravity}_{\mu\nu} = e^{\alpha \Phi} g^\text{gravity}_{\mu\nu} \quad (73) \]
4 QUANTUM EFFECTS NEAR THOSE REGIONS OF THE SPACE–TIME WHERE GRAVITY IS LARGE

In the previous section we have derived the exact solution of our toy field equations in terms of the classical solution, i.e. the solution of Einstein’s equation. Now, in this section we shall use that solution to examine the quantum effects near those regions of the space–time where the gravitational effects of matter are large. Black holes and the initial singularity are two examples we consider.

4.1 Quantum Black Hole

For a spherically symmetric black hole we have

\[ g_{\mu\nu}^{\text{gravity}} = \begin{pmatrix}
1 - \frac{r_s}{r} & 0 & 0 & 0 \\
0 & -\frac{1}{1 - \frac{r_s}{r}} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2 \sin^2 \theta
\end{pmatrix} \]  

Using the fact that the Ricci scalar is zero for the above metric and the transformation properties of the Ricci scalar under conformal transformations we have:

\[ \Phi = \frac{\Box \sqrt{|\mathcal{R}|}}{\sqrt{|\mathcal{R}|}} \]  

\[ \mathcal{R} = 3\alpha e^{-\alpha \Phi} \left[ \Box \Phi + \frac{\alpha}{2} \nabla \mu \Phi \nabla^\mu \Phi \right] \]
The above equations are in fact a differential equation for the conformal factor and can be solved for different regimes, giving the following solution:

\[
g_{\mu\nu}^{\text{quantum+gravity}} = g_{\mu\nu}^{\text{gravity}} \times \begin{cases} 
\exp(-\alpha r_s/r^3) & r \to 0 \\
\text{Constant} & r \to r_s \\
\exp(r^2/3\alpha) & r \to \infty 
\end{cases}
\]  
(77)

The conformal factor is plotted in figure (1). It can be seen that the above conformal factor does not remove the metric singularity at \( r = 0 \).
4.2 Initial Singularity

For an isotropic and homogeneous universe we have

\[ g_{\mu\nu}^{\text{gravity}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{a^2}{1-kr^2} & 0 & 0 \\
0 & 0 & -a^2r^2 & 0 \\
0 & 0 & 0 & -a^2r^2\sin^2\theta
\end{pmatrix} \] (78)

As in the previous subsection the equations governing the conformal factor are:

\[ \Phi = \frac{\Box \sqrt{|\mathcal{R}|}}{\sqrt{|\mathcal{R}|}} \] (79)

\[ \mathcal{R} = e^{-\alpha\Phi} \left[ \mathcal{R} + 3\alpha \left( \Box \Phi + \frac{\alpha}{2} \nabla_{\mu} \Phi \nabla^\mu \Phi \right) \right] \] (80)

As \( t \to 0 \) one can solve the above equations approximately:

\[ g_{\mu\nu}^{\text{quantum+gravity}} = g_{\mu\nu}^{\text{gravity}} \exp(\alpha/2t^2) \] (81)

so the universe scale is given by:

\[ a(t) = a^{\text{classic}}(t) \exp(\alpha/4t^2) = \sqrt{t} \exp(\alpha/4t^2) \] (82)

As it can be seen easily, the curvature singularity at \( t = 0 \) is removed because as time goes to zero, the universe scale goes to infinity. The behaviour of the universe scale is plotted in figure (2).
Figure 2: The universe scale v.s. time
5 CONCLUDING REMARKS

Bohmian quantum theory is a theory based on the idea of bringing back the ideas of causality and the dogma of formulation of physics in terms of motion of particles, to quantum theory. It is a satisfactory and successful theory. It is not only a causal theory in terms of particles trajectories, but also presents a beautiful explanation of how wavefunction collapse happens and what is the meaning of the uncertainty relations. The key stone of Bohm’s theory is the quantum potential. Any particle is acted upon by a quantum force derived from the quantum potential. Quantum potential is itself resulted from some self-field of the particle, the wavefunction. Since quantum potential is related only to the norm of the wavefunction and because of the Born’s postulate asserting that the ensemble density of the particle under consideration is given by the square of the norm of the wavefunction, quantum potential is resulted from the ensemble density. The nonunderstandable point of Bohm’s theory is just this. How a particle knows about its hypothetical ensemble? When the hypothetical ensemble is a real one, i.e. when there are a large number of similar particles just like the particle under consideration, quantum potential can be understood. It is a kind of interaction between the particles in the real ensemble. But when one deals with only one particle the quantum potential is interaction with other hypothetical particles!!!

On the other hand, quantum potential is highly related to the conformal degree of freedom of the space–time metric. In fact, the presence of the quantum force is just
like to have a curved space–time which is conformally flat and the conformal factor is expressed in terms of the quantum potential. In this way one sees that quantum effects are in fact geometric effects. Geometrization of quantum theory can be done successfully. But still there is the problem of ensemble noted above.

In this paper we have shown that if one tries to geometrize the quantum effects in a purely metric way, the ensemble problem would be overcomed. In addition it provides the framework for bringing in the purely quantum gravity effects.

We presented a toy model, and investigate its solutions. It is shown that the initial singularity is removed by quantum effects.

References

[1] F. Shojai, and M. Golshani, Int. J. Mod. Phys. A., Vol. 13, No. 4, 677, 1998.

[2] F. Shojai, A. Shojai, and M. Golshani, Mod. Phys. Lett. A., Vol. 13, No. 36, 2915, 1998.

[3] F. Shojai, A. Shojai, and M. Golshani, Mod. Phys. Lett. A., Vol. 13, No. 34, 2725, 1998.

[4] S. Weinberg, *Gravitation and Cosmology*, John Wiley and sons, NewYork, 1972.

[5] *Conceptual problem of quantum gravity*, ed. A. Ashtekar, and J. Stachel, (The center for Einstein studies, Boston University, 1991).
[6] D. Bohm, Phys. Rev., 85, 166, 1952;

D. Bohm, Phys. Rev., 85, 180, 1952;

D. Bohm, and B.J. Hiley, The undivided universe, Routledge, 1993;

P.R. Holland, The quantum theory of motion, Cambridge University Press, 1993.

[7] H. Goldstein, Classical Mechanics, Addison–Wesley, 1980.

[8] A. Shojai, F. Shojai, and M. Golshani, Mod. Phys. Lett. A., Vol. 13, No. 37, 2965, 1998.