Changing the preferred direction of the refined topological vertex

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Abstract

We consider the issue of the slice invariance of refined topological string amplitudes, which means that they are independent of the choice of the preferred direction of the refined topological vertex. We work out two examples. The first example is a geometric engineering of five-dimensional $U(1)$ gauge theory with a matter. The slice invariance follows from a highly non-trivial combinatorial identity which equates two known ways of computing the $\chi_y$ genus of the Hilbert scheme of points on $\mathbb{C}^2$. The second example is concerned with the proposal that the superpolynomials of the colored Hopf link are obtained from a refinement of topological open string amplitudes. We provide a closed formula for the superpolynomial, which confirms the slice invariance when the Hopf link is colored with totally anti-symmetric representations. However, we observe a breakdown of the slice invariance for other representations.
1 Introduction

All genus topological string amplitudes on local toric Calabi-Yau 3-fold can be computed
by a diagrammatic rule, in terms of the topological vertex \[ \frac{1}{1}, 2, 3\];
\[
C_{\mu \lambda \nu}(q) = q^{\kappa(\nu)} s_\lambda(q^\rho) \sum_\eta s_{\mu/\eta}(q^{\lambda^\vee + \rho}) s_{\nu/\eta}(q^{\lambda^\vee + \rho}) ,
\] (1.1)
with three \( U(\infty) \) representations, or partitions \( \mu, \nu \) and \( \lambda \), which will be identified as
Young diagrams throughout the present paper. In (1.1) \( s_{\mu/\eta}(x) \) is the (skew) Schur
function and \( q^{\lambda^\vee + \rho} \) means the substitution \( x_i := q^{\lambda_i^\vee + \rho} \). The dual partition \( \lambda^\vee \)
is defined by the transpose of the corresponding Young diagram. The index \( \kappa(\nu) \) is related to
the (relative) framing of the topological vertex. The relation of Nekrasov’s partition
function \[ \frac{4}{4}, 5, 6\] to the topological string amplitudes moti vates us to seek a refinement
of the topological vertex \[ \frac{7}{7} \]. We first proposed such a refinement in \[ \frac{8}{8} \] by employing the
(skew) Macdonald function \( P_{\lambda/\mu}(x; q, t) \). Later it has been improved by incorporating
the framing factor \[ \frac{9}{9} \] as follows:
\[
C_{\mu \lambda \nu}(q, t) = f_\nu(q, t)^{-1} P_\lambda(t^\rho; q, t) \sum_\eta \left( \frac{q}{t} \right)^{|\eta| - |\mu|} \nu P_{\mu^\vee/\eta^\vee}(-t^{\lambda^\vee} q^\rho; t, q) P_{\nu/\eta}(q^{\lambda^\vee + \rho}; q, t) ,
\] (1.2)
where \( q^{\lambda^\vee + \rho} \) etc. means the specialization \( x_i := q^{\lambda_i^\vee + \rho + i} \) and \( f_\nu(q, t) \) is the framing factor.
See \[ \frac{10}{10} \] for more details on notations. A slightly different version of the refined topological
vertex was introduced in \[ \frac{11}{11} \];
\[
C_{\mu \nu \lambda}^{(IKV)}(q_1, q_2) = \left( \frac{q_1}{q_2} \right)^{|\nu| + |\mu| - |\lambda|} \nu \sqrt{q_1 q_2}^{|\eta| - |\mu|} \nu P_{\lambda^\vee}(q_2^{-\rho}; q_1, q_2)
\times \sum_\eta \left( \frac{q_1}{q_2} \right)^{|\eta| - |\mu| - |\lambda|} \nu s_{\mu^\vee/\eta^\vee}(q_1^{-\lambda} q_2^{-\rho}) s_{\nu/\eta}(q_2^{-\lambda} q_1^{-\rho}) ,
\] (1.3)
which allows an interesting interpretation in terms of “unisotropic” plane partitions. This
is also related to the statistical model of melting crystal \[ \frac{3}{3}, 11\]. It has been proposed
that a certain topological open string amplitude computed from \( C_{\mu \nu \lambda}^{(IKV)}(q_1, q_2) \) gives
homological invariants of the Hopf link \[ \frac{12}{12} \]. The relation of the refined topological vertex
and homological invariants is one of the main subjects of this paper.

Though it is not manifest in the expression \[ \frac{11}{11} \], the topological vertex is symmetric
under the cyclic permutation of three partitions. However, it seems impossible to keep
the cyclic symmetry for the refinements. Consequently, both $C_{\mu\lambda}^\nu(q, t)$ and $C_{\mu\nu}^{(I\overline{K}V)}(q_1, q_2)$ have a preferred direction, which gives us an issue of the choice of the preferred direction in the rule of diagrammatic computation by the refined topological vertex. In [10] we have proved that our refined topological vertex gives a building block of Nekrasov’s partition function. To obtain Nekrasov’s partition function we consider toric diagrams of the geometric engineering of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [13, 14, 15, 16, 7]. For such toric diagrams we required that the preferred direction at each vertex should be parallel each other. This requirement largely restricts a possible choice of the preferred direction. However, for a certain diagram there are more than one choice. In this paper we consider the possibility of changing the preferred direction. It is often the case that a summation over partitions remains for one choice of the preferred direction, while we can compute the summation completely for the other choice. Thus the independence of the preferred direction gives a kind of summation formula for partitions. In this paper we consider two examples of such summation formulae (see (1.4) and (1.7) below). The slice invariance is important not only for the consistency of the formalism but also for practical computations.

The first example is a topological closed string amplitude that appears in the geometric engineering of five-dimensional $U(1)$ theory with a matter. When we compute the closed string amplitude based on the refined topological vertex, there are alternative choices of the preferred direction in the toric diagram of the five-dimensional $U(1)$ theory. The agreement of the amplitude requires a highly non-trivial combinatorial identity;

$$\sum_\lambda \Lambda^{(\lambda)} \prod_{s \in \lambda} \frac{1 - Qq^{a(s)}t^{\ell(s)+1}}{1 - q^{a(s)}t^{\ell(s)+1}} \frac{1 - Qq^{-a(s)-1}t^{-\ell(s)}}{1 - q^{-a(s)-1}t^{-\ell(s)}}$$

where the light hand side is a summation over partitions $\lambda$. $a(s)$ and $\ell(s)$ in the product are the arm length and the leg length that are defined by the corresponding Young diagram. The identity (1.4) was discussed before in [17, 18, 10]. It is known that this amplitude gives the generating function of the equivariant $\chi_y$ genus of the Hilbert scheme of points on $\mathbb{C}^2$. The validity of (1.4) has a deep geometrical meaning. Namely the left hand side of (1.4) comes from a computation of the generating function by the localization for a toric action on the Hilbert scheme which is induced by $(z_1, z_2) \rightarrow (e^{i\epsilon_1}z_1, e^{i\epsilon_2}z_2)$ on $\mathbb{C}^2$, where the fixed points are labeled by Young diagrams. On the other hand a formula that computes the $\chi_y$ genus of the Hilbert scheme of a surface $S$ from the $\chi_y$ genus of
the underlying surface $S$ has been established in [19]. There is also a proposal of more general formula of the elliptic genus based on string theory or the orbifold conformal field theory [20]. The right hand side of (1.4) can be derived, if we apply these formulae to the equivariant $\chi_y$ genus of $\mathbb{C}^2$. Thus the identity (1.4) equates these two ways of computing the equivariant $\chi_y$ genus of the Hilbert scheme, one by the localization principle, the other from the corresponding $\chi_y$ genus of the underlying manifold.

The second example comes from a refinement of topological open string amplitudes, which is concerned with the relation of the refined topological vertex to homological refinements of the link polynomial. According to the proposal in [12], the homological invariant of the Hopf link colored with representations $\lambda$ and $\mu$ is proportional to

$$Z_{\lambda,\mu}(q_1, q_2, Q) = \sum_{\eta} (-Q)^{|\eta|} q_2^{|\eta^\vee||\eta|/2} q_1^{|\eta||\eta^\vee|/2} \tilde{Z}_\eta(q_2, q_1) Z_\eta(q_1, q_2) s_{\lambda}(q_1^{-n} q_2^{-\rho}) s_{\mu}(q_1^{-n} q_2^{-\rho}) ,$$

(1.5)

where

$$\tilde{Z}_\eta(q_2, q_1) = \prod_{(i,j) \in \eta} \left(1 - q_1^{\eta_i^{-i+1}} q_2^{\eta_j^{-j}}\right)^{-1} = \prod_{s \in \eta} \left(1 - q_1^{\ell(s)+1} q_2^{\rho(s)}\right)^{-1} .$$

(1.6)

Note that the summation over the partition $\eta$ remains in the definition of $Z_{\lambda,\mu}$. Therefore the fact that the homological invariants are polynomials in $Q$ is not clear at all in the above proposal. In this paper we work out the summation in (1.5), when both $\lambda$ and $\mu$ are totally antisymmetric representations, $(\lambda, \mu) = (1^r, 1^s)$ and prove

$$\frac{Z_{1^r, 1^s}(Q; q, t)}{Z_{1^r, 1^s}(Q; q, t)} = (-1)^{s} t^{-\frac{1}{2}s(s-1)} e_r(t^\rho) e_s(\sqrt{q} Q q^{-1} t^\rho, t^{-\rho}) \prod_{i=1}^{r} \left(1 - Q q^{\frac{1}{2}} t^{-i+1}\right) ,$$

(1.7)

where $(q, t) = (q_1^{-1}, q_2^{-1})$ and $\bullet$ means the trivial representation. $e_r(x)$ and $e_s(x, y)$ are the elementary symmetric functions. The fundamental summation formula on the space of symmetric functions is the Cauchy formula [21]. However, we cannot apply the Cauchy formula directly to (1.5). We will heavily rely on many properties of the Macdonald functions and the Macdonald operator to be introduced in section 4, so that we can use the Cauchy formula to perform the summation over partitions. Though our proof of the formula (1.7) can be made without referring to the refined topological vertex, it is closely related to a change of the preferred direction as has been argued by Taki [22], (see section 3 for the explicit form of his conjecture). The above formula agrees with his conjecture (up to normalization), when both $\lambda$ and $\mu$ are totally anti-symmetric representations. This means that we can show the invariance under the change of the preferred direction in this case. However, for general representations Taki’s conjecture
should be appropriately modified as discussed in section 6. Recently the homological invariants for the pair \((1^k,1)\) is constructed by Yonezawa \[23\] by the method of matrix factorization. If we apply his result to the Hopf link, we find a complete agreement to our formula (1.7) up to an overall normalization.

The paper is organized as follows; in section 2 we consider the toric diagram of the geometric engineering of five-dimensional \(U(1)\) theory and show that the slice independence of the refined closed topological string amplitude gives the combinatorial identity (1.4). It turns out that the validity of (1.4) is equivalent to the agreement of two known ways of computing the \(\chi_y\) genus of the Hilbert scheme of points on \(\mathbb{C}^2\). In section 3 we review the relation of the homological invariants of the colored Hopf link and the refined open topological string amplitudes following [12]. We also introduce the conjecture presented in [22] and explain how it is related to the problem of changing the preferred direction. Section 4 is the main part of the paper. We prove the formula (1.7) by making use of the Macdonald operator on the space of the symmetric functions. In section 5 by using the formula we provide a general formula for the superpolynomial of the Hopf link colored with any pair \((\lambda,\mu) = (1^r,1^s)\) of totally anti-symmetric representations. We also show that homological invariants obtained from (1.5) are actually two variable Laurent polynomials in \(q\) and \(t\) with positive integer coefficients. Some examples of the superpolynomial for representations other than the totally anti-symmetric ones are worked out in section 6. By the base change between the Schur functions and the Macdonald functions, we can replace one of the anti-symmetric representations \((1^r,1^s)\) with any representation \(\lambda\). We compute the superpotential explicitly when \(\lambda\) is the symmetric representation or the hook representation. We also argue some general structure of the superpolynomial of the Hopf link colored with \((\lambda,1^s)\). In appendix combinatorial identities coming from the slice invariance of the toric diagram of \(U(1)^N\) theory are presented.

The following notations are used through this article. \(q\) and \(t\) are formal parameters otherwise stated. Let \(\lambda\) be a Young diagram, i.e., a partition \(\lambda = (\lambda_1, \lambda_2, \cdots)\), which is a sequence of non-negative integers such that \(\lambda_i \geq \lambda_{i+1}\) and \(|\lambda| = \sum \lambda_i < \infty\). \(\lambda^\vee\) denotes its conjugate (dual) diagram. \(\ell(\lambda) = \lambda_1^\vee\) is the length and \(|\lambda| = \sum \lambda_i\) is the weight. For each square \(s = (i,j)\) in \(\lambda\), \(a(s) := \lambda_i - j\) and \(\ell(s) := \lambda_j^\vee - i\) are the arm length and the leg length, respectively.\(^1\) The notations for the symmetric functions in \(x = (x_1, x_2, \cdots)\) are the Macdonald function \(P_\lambda(x; q, t)\), the Schur function \(s_\lambda(x)\) and the elementary symmetric function \(e_\lambda(x)\), respectively. The algebra of symmetric functions \(\Lambda\) can be identified as a polynomial ring in the power sums \(\{p_n(x)\}_{n=1}^\infty\) with

\(^1\)In [11] the definitions of the arm length and the leg length are exchanged.
$p_n(x) = \sum_{i=1}^{\infty} x^n$. All symmetric functions in this article are treated as polynomials in the power sum symmetric functions $(p_1, p_2, \cdots)$. For $|t^{-1}| < 1$, $x = q^\lambda t^\rho$ means $x_i = q^{\lambda_i} t^{\frac{i}{2} - \frac{1}{2}}$. By definition a specialization of the symmetric functions is an algebra homomorphism $\rho : \Lambda \to R$. For example one may take $R = \mathbb{C}$. Since we treat the symmetric functions as polynomials in $p_n$, we can define a specialization by giving the images $\rho(p_n)$ of the power sums under $\rho$. For example the specialization of the symmetric functions with $x = q^\lambda t^\rho$ is defined by

$$p_n(q^\lambda t^\rho) := \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^{n(\frac{1}{2} - \frac{i}{2})} = \sum_{i=1}^{N} q^{n\lambda_i} t^{n\left(\frac{1}{2} - \frac{i}{2}\right)} + \frac{t^{-N}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}, \quad (1.8)$$

which is independent of $N$ as long as $N \geq \ell(\lambda)$. In this specialization the algebra $R = \mathbb{Q}(q, t)$ is the field of rational functions of $q$ and $t$. We also note that for $p_n(x, y) := p_n(x) + p_n(y)$,

$$p_n(cq^\lambda t^\rho, Lt^{-\rho}) = c^n \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^{n\left(\frac{1}{2} - \frac{i}{2}\right)} + \frac{c^n - L^n}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}, \quad c, L \in \mathbb{C}. \quad (1.9)$$

An involution $\iota$ acting on the power sum function $p_n(x)$ is defined as $p_n(\iota x) := -p_n(x)$.

The definition of the $q$-integer in this article is $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{1-n} \frac{1 - q^{2n}}{1 - q^2}$. In terms of $[n]_q = [n]_q [n-1]_q \cdots [1]_q$, the $q$-binomial coefficient is

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[n-k]_q! [k]_q!} = q^{k(n-k)} \prod_{i=1}^{k} \frac{1 - q^{2(n-k+i)}}{1 - q^{2i}}. \quad (1.10)$$

For $n, k \in \mathbb{N}$ both the $q$-integer and the $q$-binomial coefficient are polynomials in $q$ with positive integer coefficients. Finally, we often use $\widetilde{Q} := vQ$ with $v := (q/t)^{\frac{1}{2}}$.

## 2 Five-dimensional $U(1)$ theory with extra matter

Changing the preferred direction in the diagrammatic computation of the refined topological vertex often gives a highly non-trivial combinatorial equality that involves a summation over partitions. In this section we provide an example which shows a curious connection of the change of the preferred direction and a combinatorial identity in enumerative geometry.

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2 We follow the standard convention in knot theory. The definition $[n]_q := q^{n/2} - q^{-n/2}$ is also used in the literatures.
By using the Macdonald function $P_{\lambda}(x; q, t)$, the refined topological vertices are defined as [10]

$$C_{\mu\lambda}^{\nu}(q, t) := P_{\lambda}(t^{\rho}; q, t) \sum_{\sigma} P_{\mu/\sigma}^{\nu/\sigma}(t^{\rho}; q, t) P_{\nu/\sigma}(q^{\rho}; q, t) v^{(|\sigma| - |\nu|)} f_{\nu}(q, t)^{-1},$$

$$C_{\mu
u}^{\lambda}(q, t) := C_{\mu\lambda}^{\nu}(q, t)(-1)^{|\lambda|+|\mu|+|\nu|}
= P_{\lambda}(q^{\rho}; t, q) \sum_{\sigma} P_{\nu/\sigma}^{\nu/\sigma}(t^{\rho}; q, t) P_{\mu/\sigma}(iq^{\rho}; q, t) v^{(|\sigma|+|\nu|)} f_{\nu}(q, t),$$

$$C_{\mu
u}^{\lambda}(q, t) := C_{\mu\lambda}^{\nu}(q, t)v^{(|\mu|+|\nu|)} f_{\mu}(q, t) f_{\nu}(q, t),$$

$$C_{\mu
u}^{\lambda}(q, t) := C_{\mu\lambda}^{\nu}(q, t)v^{-|\mu|-|\nu|} f_{\mu}(q, t)^{-1} f_{\nu}(q, t)^{-1},$$

with the framing factor

$$f_{\lambda}(q, t) := \prod_{(i,j) \in \lambda} (-1)q^{\lambda_{i}-j+\frac{1}{2}t^{\lambda_{j}-i+\frac{1}{2}}}.$$  

(2.3)

The lower and the upper indices correspond to the incoming and the outgoing representations, respectively, and the edges of the topological vertex are ordered clockwise. The middle index $\lambda$ is the representation for the preferred direction.

As has been discussed in [17, 10], the refined topological string amplitude for the diagrams in Fig. 1 gives the generating function of the equivariant $\chi_y$ genus of the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of $n$ points on $\mathbb{C}^2$. These diagrams are one-loop diagrams where we identify two external vertical edges and arise from the geometric engineering of five-dimensional $U(1)$ gauge theory with adjoint matter [7]. In the left diagram the preferred direction is along the internal line, while it is along the external lines in the right diagram.

The gluing rule of the refined topological vertex gives the amplitude for the left diagram

$$Z_L := \sum_{\lambda, \nu} Q^{[\mu]}\Lambda^{[\lambda]} C_{\lambda}^{\nu}(q, t) C_{\nu}(q, t)$$
\[= \sum_{\lambda, \nu} Q^{[\nu]} L^{|\lambda|} P_\lambda(t^\rho; q, t) P_{\lambda^\nu}(-q^\rho; t, q) P_{\nu}(q^\lambda t^\rho; q, t) P_{\nu^\nu}(-t^\nu q^\rho; t, q). \]  

(2.4)

Here \( \bullet \) stands for the trivial representation, i.e., the zero Young diagram \((0, 0, \ldots)\). On the other hand the right diagram gives us the following partition function;

\[ Z_R := \sum_{\mu, \nu} Q^{[\nu]} L^{|\lambda|} C_{\bullet}(q, t) C_{\nu}(q, t) \]

\[ = \sum_{\mu, \nu, \sigma_1, \sigma_2} Q^{[\nu]} L^{|\lambda|} P_{\nu^\nu/\sigma_1^\nu}(\mu^\nu; t, q) P_{\lambda/\sigma_1}(t^\rho; t, q) P_{\lambda/\sigma_2^\nu}(-q^\rho; t, q) P_{\nu/\sigma_2}(\nu^\rho; t, q)^{\nu^\sigma_1^\nu-\nu^\sigma_2^\nu}. \]  

(2.5)

The computation of \( Z_L \) is made by the Cauchy formula for the Macdonald function

\[ \sum_{\lambda} P_{\lambda/\mu}(x; q, t) P_{\lambda^\nu/\nu^\nu}(y; t, q) = \Pi_0(x, y) \sum_\eta P_{\mu^\nu/\eta^\nu}(y; t, q) P_{\nu/\eta}(x; q, t), \]  

(2.6)

where

\[ \Pi_0(-x, y) := \exp \left\{ -\sum_{n>0} \frac{1}{n} p_n(x) p_n(y) \right\} = \prod_{i,j} (1 - x_i y_j). \]  

(2.7)

Note that for \( c \in \mathbb{C}, \Pi_0(cx, cy) = \Pi_0(x, cy) \) and for the involution \( \iota(p_n) = -p_n \), \( \Pi_0(\iota x, y) = \Pi_0(x, \iota y) = \Pi_0(x, y)^{-1} \). We also use the following adding formula

\[ \sum_{\mu} P_{\lambda/\mu}(x; q, t) P_{\mu/\nu}(y; q, t) = P_{\lambda/\nu}(x, y; q, t). \]  

(2.8)

From the formula of the principal specialization

\[ P_\lambda(t^\rho; q, t) = \prod_{s \in \lambda} \frac{(-1)^{1/2} q^{a(s)}q^{b(s)}}{1 - q^{a(s)} t^{\ell(s)+1}}, \quad P_{\lambda^\nu}(-q^\rho; t, q) = \prod_{s \in \lambda} \frac{(-1) q^{-1/2} q^{-a(s)}}{1 - q^{-a(s)} - t^{-\ell(s)}}, \]  

(2.9)

and the Cauchy formula (2.6) for \( \mu = \nu = \bullet \), we have

\[ Z_L = \sum_{\lambda} \Pi_0(-Qq^\Lambda t^\rho, t^\nu q^\rho) \prod_{s \in \lambda} \frac{v^{-1} \Lambda}{(1 - q^{a(s)} t^{\ell(s)+1})(1 - q^{-a(s)} - t^{-\ell(s)})}. \]  

(2.10)

If we define the perturbative part by \( Z_L^{pert} := \sum_\nu Q^{[\nu]} C_{\bullet \nu}(q, t) C_{\nu}(q, t) = \Pi_0(-Q t^\rho, q^\rho) \), which is independent of \( \Lambda \), then the instanton part \( Z_L^{inst} := Z_L/Z_L^{pert} \) is

\[ Z_L^{inst} = \sum_{\lambda} \prod_{s \in \lambda} \frac{v^{-1} \Lambda}{1 - q^{a(s)} t^{\ell(s)+1}} \frac{1 - \tilde{Q} q^{-a(s)} - t^{-\ell(s)}}{1 - q^{-a(s)} - t^{-\ell(s)}}. \]  

(2.11)
where $\tilde{Q} = vQ$. Here we use \cite[(2.10) in \cite{10}]{10}

\[
N_{\lambda,\mu}(Q; q, t) := \prod_{(i,j) \in \lambda} \left(1 - Q q^{\lambda_{i,j}^+ t^{\lambda_{j,i}^+} - i^+ 1}\right) \prod_{(i,j) \in \mu} \left(1 - Q q^{-\mu_{i,j}^+ - j^+ 1 t^{-\lambda_{j,i}^+}}\right)
= \Pi_0(-v^{-1}Q q^\rho, t^\rho, q^\rho) / \Pi_0(-v^{-1}Q t^\rho, q^\rho).
\] (2.12)

The computation of $Z_R$ is more involved and we have to employ the trace formula (see \cite[B.26]{8}), which is obtained by successively using \cite[(2.6) in \cite{10}]{10},

\[
\sum_{\{\lambda,\eta,\mu,\sigma\}} P_{\mu/\eta} (x; t, q) P_{\lambda/\sigma} (y; q, t) P_{\lambda/\eta} (z; t, q) P_{\mu/\eta} (w; q, t) \alpha^{[\lambda]} \beta^{[\eta]} \gamma^{[\mu]} \delta^{[\sigma]}
= \prod_{k \geq 0} \frac{1}{1 - c^{k+1}} \Pi_0(y, \alpha c^k z) \Pi_0(y, \delta^{-1} c^{k+1} x) \Pi_0(w, \gamma c^k x) \Pi_0(w, \beta^{-1} c^{k+1} z)
= \exp \left\{- \sum_{n > 0} \frac{(-1)^n}{n} \frac{1}{1 - c^n} \left( \alpha^n p_n (y) p_n (z) + \frac{c^n}{\beta^n} p_n (y) p_n (x)
+ \gamma^n p_n (w) p_n (x) + \frac{c^n}{\beta^n} p_n (w) p_n (z) - c^n \right) \right\},
\] (2.13)

with $c := \alpha \gamma / \beta \delta$. Using this trace formula, we can compute the partition function $Z_R$ as follows; if we put $(\alpha, \beta, \gamma, \delta) = (\Lambda, v^{-1}, Q, v)$ and $(x, y, z, w) = (-\imath q^\rho, t^\rho, -q^\rho, \imath t^\rho)$, then $c = Q \Lambda$. Hence from \cite[(2.13)]{10}

\[
Z_R = \prod_{k \geq 0} \frac{\Pi_0(t^\rho, -\Lambda c^k q^\rho) \Pi_0(t^\rho, -Q c^k q^\rho)}{\Pi_0(t^\rho, -v c^{k+1} q^\rho) \Pi_0(t^\rho, -v^{-1} c^{k+1} q^\rho)} \frac{1}{1 - c^{k+1}}
= \exp \left\{- \sum_{n > 0} \frac{1}{n} \frac{1}{1 - c^n} \left( \frac{\Lambda^n + Q^n - (v^n + v^{-n}) c^n}{(t^\frac{\pi}{2} - t^{-\frac{\pi}{2}})(q^{\frac{\pi}{2}} - q^{-\frac{\pi}{2}}) - c^n} \right) \right\}.
\] (2.14)

As before we define the perturbative part by

\[
Z_R^{\text{pert}} := Z_R(\Lambda = 0) = \exp \left\{- \sum_{n > 0} \frac{Q^n}{n(t^\frac{\pi}{2} - t^{-\frac{\pi}{2}})(q^{\frac{\pi}{2}} - q^{-\frac{\pi}{2}})} \right\}.
\] (2.15)

Then the instanton part $Z_R^{\text{inst}} := Z_R / Z_R^{\text{pert}}$ is

\[
Z_R^{\text{inst}} = \exp \left\{- \sum_{n > 0} \frac{\Lambda^n}{n} \left( \frac{Q^n - u^n}{(t^\frac{\pi}{2} - t^{-\frac{\pi}{2}})(q^{\frac{\pi}{2}} - q^{-\frac{\pi}{2}})} \right) \right\}.
\] (2.16)

\footnote{This is obtained by applying the automorphism $\omega_{q,t}(p_n) := (-1)^{n+1} - q^{\frac{n}{2}} p_n$ on $y$ and $w$ in \cite[B.26]{8}.}
with \( u := (qt)^{\frac{1}{2}} \). If the topological partition function is independent of the choice of the preferred direction in our one-loop diagram, \( Z^\text{inst}_L = Z^\text{inst}_R \) and we should have

\[
\sum_{\lambda} \frac{1}{\lambda} \prod_{s \in \lambda} \frac{1 - yq^{a(s)} t^{\ell(s) + 1}}{1 - q^{a(s)} t^{\ell(s) + 1}} \frac{1 - yq^{-a(s)-1} t^{-\ell(s)}}{1 - q^{-a(s)-1} t^{-\ell(s)}} = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1 - t^n(1 - q^{-n})} \left( 1 - t^n y^n \right) \right\} .
\]

Thus the slice invariance requires the highly non-trivial identity (2.17). One can check the validity of (2.17) for special cases \( y = 0 \) and \( y = 1 \). When \( y = 0 \) it reduces to

\[
\sum_{\lambda} \Lambda^{\lambda} = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1 - t^n(1 - q^{-n})} \right\} ,
\]

which was proved by Nakajima and Yoshioka [6]. The proof in [6] is geometric. Namely we can see that the right hand side is the generating function of the Hilbert series of \( \text{Hilb}^n(C^2) \) as follows;

\[
\sum_{n=0}^{\infty} \Lambda^n \chi^y(H^0(\text{Hilb}^n(C^2), \mathcal{O})) = \prod_{k, \ell \geq 0} \frac{1}{(1 - t^k q^{-\ell} \Lambda)} = \exp \left\{ \sum_{n>0} \frac{\Lambda^n}{n} \frac{1}{(1 - t^n)(1 - q^{-n})} \right\} .
\]

On the other hand the left hand side arises from a computation of the generating function by the localization theorem for toric action, where the fixed points of the toric action are labeled by partitions. We also have a combinatorial proof based on the Cauchy formula (2.6) for \( \mu = \nu = \bullet \). The formula of the principal specialization (2.9) implies the desired identity. For \( y = 1 \) the conjecture is simply

\[
\sum_{\lambda} \Lambda^{\lambda} = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1 - \Lambda^n} \right\} = \prod_{n=1}^{\infty} (1 - \Lambda^n)^{-1} ,
\]

which is nothing but the generating function of the number of partitions. Physically \( y = e^{-m} \) in terms of the mass \( m \) of the adjoint matter. Thus the equality (2.17) interpolates the massless theory (\( \mathcal{N} = 4 \) theory) and the infinitely massive theory (\( \mathcal{N} = 2 \) theory).

As has been pointed out in [24], the right hand side of (2.17) is identified as the generating function of the orbifold equivariant \( \chi^y \) genera of the symmetric product \( \text{Sym}^n(C^2) := (C^2)^n / S_n \). The Hirzebruch \( \chi^y \) genus of a manifold \( M \) is defined by

\[
\chi^y(M) := \sum_{p,q} (-1)^q y^p \dim H^q(M, \wedge^p T^* M) ,
\]
and we will consider the equivariant version of (2.21) where we use the equivariant cohomology $H^q_G(M, \wedge^p T^*M)$. Since

\[
H^0_{T^2}(\mathbb{C}^2, \Lambda^p T^*\mathbb{C}^2) = \begin{cases} \mathbb{C}[z_1, z_2], & (p = 0) \\ \mathbb{C}[z_1, z_2]dz_1 \oplus \mathbb{C}[z_1, z_2]dz_2, & (p = 1) \\ \mathbb{C}[z_1, z_2]dz_1 \wedge dz_2, & (p = 2) \end{cases}
\]

(2.22)

the generating function of the equivariant $\chi_y$ genus of $\mathbb{C}^2$ is given by

\[
\chi_{-y}(\mathbb{C}^2) = \sum_{k, \ell=0}^{\infty} t^{k} t^{\ell} (1 - y(t_1 + t_2) + y^2 t_1 t_2) = \frac{(1 - yt_1)(1 - yt_2)}{(1 - t_1)(1 - t_2)}.
\]

(2.24)

A formula of the elliptic genera of the symmetric product $\text{Sym}^n M$ was proposed in [20], which gives the elliptic genera of $\text{Sym}^n M$ in terms of those of the underlying Kähler manifold $M$, see also [25]. The formula tells us that

\[
\sum_{n=0}^{\infty} z^n \text{Ell}_{\text{orb}}(\text{Sym}^n M; y, q) = \prod_{i=1}^{\infty} \prod_{\ell, m} \frac{1}{(1 - z^i y^\ell q^m)^{c(m, \ell)}},
\]

(2.25)

where $c(m, \ell)$ are the coefficients of the elliptic genus of $M$;

\[
\text{Ell}(M; y, q) = \sum_{m, \ell} c(m, \ell) y^\ell q^m.
\]

(2.26)

Since the $\chi_y$ genus is obtained from the elliptic genus by $\text{Ell}(M; y, q = 0) = y^{\dim M/2} \chi_{y}(M)$, from DMVV formula (2.25) we find

\[
\sum_{n \geq 0} \Lambda^n \chi_{-y}(\text{Sym}^n(\mathbb{C}^2)) = \prod_{m \geq 1} \prod_{k, \ell \geq 0} \frac{(1 - \Lambda^n y^m t_1^k t_2^\ell)(1 - \Lambda^n y^{m+1} t_1^k t_2^{\ell+1})}{(1 - \Lambda^n y^{m-1} t_1^k t_2^\ell)(1 - \Lambda^n y^{m+1} t_1^{k+1} t_2^{\ell+1})},
\]

(2.27)

which means

\[
\log \left\{ \sum_{n \geq 0} \Lambda^n \chi_{-y}(\text{Sym}^n(\mathbb{C}^2)) \right\} = \sum_{n \geq 1} \sum_{m \geq 1} \sum_{k, \ell \geq 0} \frac{1}{n} \Lambda^{nm} y^{(m-1) t_1^k t_2^\ell} (1 - y^{n t_1^k})(1 - y^{n t_2^{\ell}})
\]

\[
= \sum_{n \geq 1} \frac{1}{n} \Lambda^n \left( \frac{(1 - y^{n t_1^n})(1 - y^{n t_2^n})}{(1 - t_1^n)(1 - t_2^n)} \right).
\]

(2.28)

On the other hand the localization theorem tells us that the left hand side is the equivariant $\chi_y$ genera of the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of points on $\mathbb{C}^2$. Thus the slice invariance
(2.17) geometrically means that the equivariant $\chi_y$ genera of $\text{Hilb}^n(C^2)$ which gives a resolution of the symmetric product agrees with the orbifold equivariant $\chi_y$ genera of $\text{Sym}^n(C^2)$. One may wonder how they agree in mathematically rigorous sense. For equivariant Euler character the agreement was proved in [6]. Furthermore, the following formula of the generating function of the $\chi_y$ genera of the Hilbert scheme of a smooth projective surface $S$ was established in [19];

$$\sum_{n=0}^{\infty} z^n \chi_y(\text{Hilb}^n(S)) = \exp \left\{ \sum_{m=1}^{\infty} \frac{\chi_{y^m}(S)}{1 - y^m z^m} \frac{z^m}{m} \right\}. \quad (2.29)$$

This formula for the equivariant case would give us a proof of (2.17).

3 Refined topological vertex and homological link invariants

In [26] it was argued that homological link invariants are related to a refinement of the BPS state counting in topological open string theory. Based on this proposal, in [12] a conjecture on homological link invariants of the Hopf link from the refined topological vertex has been provided. Let us review their proposal briefly. Let $L$ be an oriented link in $S^3$ with $\ell$ components. We consider homological invariants of $L$ whose components are colored by representations $R_1, \cdots, R_\ell$ of the Lie algebra $\mathfrak{sl}(N)$. This means that we have a doubly graded homology theory $\mathcal{H}^{\mathfrak{sl}(N);R_1,\cdots,R_\ell}(L)$ whose graded Poincaré polynomial is

$$\sum_{i,j} q^i t^j \dim \mathcal{H}^{\mathfrak{sl}(N);R_1,\cdots,R_\ell}_{i,j}(L). \quad (3.1)$$

Substitution of $t = -1$ gives the unnormalized link polynomial $\overline{P}_{\mathfrak{sl}(N);R_1,\cdots,R_\ell}(q) = \overline{P}_{\mathfrak{sl}(N);R_1,\cdots,R_\ell}(q,-1)$. The normalized invariants $P_{\mathfrak{sl}(N);R_1,\cdots,R_\ell}(q)$ are obtained by dividing by the invariants of the unknot. However, in the following we only consider the unnormalized one.

The physical interpretation of homological invariants as the BPS state counting leads to a prediction on the dependence of the link homologies on the rank $N - 1$. It has been conjectured [27, 28] that there exists a “superpolynomial” $\overline{P}_{R_1,\cdots,R_\ell}(a,q,t)$ which is a rational function in three variables such that

$$\overline{P}_{\mathfrak{sl}(N);R_1,\cdots,R_\ell}(q,t) = \overline{P}_{R_1,\cdots,R_\ell}(a = q^N, q,t). \quad (3.2)$$
We can reproduce the polynomial invariants of the colored Hopf link from topological open string amplitudes on the conifold with appropriate brane configuration \[29, 30, 31, 32, 33\]. Since they are computed by the method of topological vertex, it is natural to expect that a refined version of the topological vertex gives a homological version of the Hopf link invariants. Based on the refined topological vertex introduced in \[11\], the superpolynomial for the colored Hopf link was proposed in \[12\];

\[
\overline{P}_{\lambda,\mu}(a, q, t) = (-1)^{|\lambda|+|\mu|} \left( Q^{-1} \sqrt{\frac{q_1}{q_2}} \right)^{\frac{1}{2}|\lambda|+|\mu|} \left( \frac{q_1}{q_2} \right)^{|\lambda|+|\mu|} \frac{Z_{\lambda,\mu}(q_1, q_2, Q)}{Z_{\bullet,\bullet}(q_1, q_2, Q)},
\]

(3.3)

where

\[
Z_{\lambda,\mu}(q_1, q_2, Q) = \sum_{\eta} (-Q)^{|\eta|} q_2^{\frac{1}{2}||\eta||^2} q_1^{\frac{1}{2}||\eta||^2} \tilde{Z}_{\mu}^{-1}(q_1, q_2) \tilde{Z}_{\eta}(q_2, q_1) s_{\lambda}(q_1^{-\eta} q_2^{-\rho}) s_{\mu}(q_1^{-\eta} q_2^{-\rho}).
\]

(3.4)

The natural variables of symmetric functions \((q_1, q_2)\) are related to the variables of the superpolynomial by

\[
\sqrt{q_2} = q, \quad \sqrt{q_1} = -tq, \quad Q = -ta^{-2}.
\]

(3.5)

The factor \(\tilde{Z}_{\eta}\) in (3.4) is given by

\[
\tilde{Z}_{\eta}(q_2, q_1) = \prod_{(i,j) \in \eta} \left( 1 - q_2^{\eta_i+1} q_1^{-\eta_j} \right)^{-1} = \prod_{s \in \eta} \left( 1 - q_2^{\ell(s)+1} q_1^{-a(s)} \right)^{-1},
\]

(3.6)

and, therefore, related to the following specialization of the Macdonald function;

\[
P_{\lambda}(t^\rho; q, t) = t^{-\frac{1}{2}||\lambda||^2} \prod_{s \in \lambda} \left( 1 - q_{-a(s)} t^{-\ell(s)+1} \right)^{-1}.
\]

(3.7)

Namely if we identify \((q_1, q_2) \equiv (q^{-1}, t^{-1})\), then we find

\[
Z_{\lambda,\mu}(q_1, q_2, Q) = \sum_{\eta} (-Q)^{|\eta|} P_{\eta,\nu}(q^{a(s)}; t, q) P_{\eta}(t^\rho; q, t) s_{\lambda}(q^{a(s)} t^\rho) s_{\mu}(q^{a(s)} t^\rho).
\]

(3.8)

Note that this expression is manifestly symmetric in \(\lambda\) and \(\mu\). But due to the summation over \(\eta\), it is not clear at all that \(Z_{\lambda,\mu}\) is a polynomial in \(Q\). Furthermore, since the superpolynomial \(\overline{P}_{\lambda,\mu}\) is defined by (3.1) and (3.2), we would like to check that, when \(a = q^N\), the superpolynomials derived from (3.8) are in fact Laurent polynomials in \(q\) and \(t\) with positive integer coefficients. Thus it is desirable to have a closed formula of \(Z_{\lambda,\mu}\) without a summation over partitions.

\[\text{The definitions of the arm length } a(s) \text{ and the leg length } \ell(s) \text{ in [12] are exchanged, compared with the standard ones, for example in [21].}\]
The configuration that leads to the proposal (3.3) is the toric diagram of the resolved conifold with two Lagrangian brane insertions (see Fig. 2). Two branes are inserted on the same side of external edges. But they are on the different vertices. In [12] the refined topological string amplitude for this diagram was computed by choosing the internal line as the preferred direction. This was the reason why the summation over the partition $\eta$, which was attached to the internal line, remained in the proposal (3.3) for the superpolynomial. As has been pointed out by Taki [22], if we assume the slice invariance that means the partition function computed by the refined topological vertex is independent of the choice of the preferred direction, we may have a formula of $\overline{P}_{\lambda,\mu}$ in a closed form without a summation over partitions. Namely we first change the preferred direction from the internal line to the external lines as shown in Fig. 2. Then we make a flop operation to move two Lagrangian branes to the same vertex. When we compute the topological string amplitude based on the final diagram, we can perform the summation over the partitions of the internal line and obtain

$$\overline{P}_{\lambda,\mu}(a, q, t) = \left(Q^{-1} \sqrt{\frac{q_1}{q_2}}\right)^{-\frac{1}{2}(|\lambda|+|\mu|)} \left(\frac{q_1}{q_2}\right)^{-n(\mu') + |\lambda||\mu|} q_2^{-\frac{1}{2}(\kappa(\lambda)+\kappa(\mu))}$$

$$s_\mu(q_2^{-\rho})s_\lambda(q_1^{-\mu}q_2^{-\rho}, Q^{-1} \sqrt{\frac{q_1}{q_2}}) \prod_{(i,j) \in \mu} \left(1 - Q^{-1} q_2^{-i+\frac{1}{2}} q_1^{-j+\frac{1}{2}}\right)^2.$$  \hspace{1cm} (3.9)

Since the flop operation in the computation of the refined topological vertex was worked out completely in [10, 22], we only have to assume the slice independence to derive (3.9). In other words the validity of (3.9) is equivalent to the slice invariance. In the next section we prove (3.9) agrees to the original proposal (3.3) up to an overall normalization, if both $\lambda$ and $\mu$ are totally antisymmetric representations. The proof relies on various properties of the Macdonald functions and is independent of the computation of the refined topological vertex. This means the slice invariance holds in this case. However,
in section 6 we will see that we need a few correction terms to the formula \((3.9)\) for general representations.

## 4 Macdonald operator and summation over partitions

In this section we will work out the summation over the partitions \(\eta\) in \((3.8)\), when both \(\lambda\) and \(\mu\) are totally anti-symmetric representations. The most fundamental summation formula on the space of symmetric functions is the Cauchy formula

\[
\sum_{\lambda} \frac{1}{\langle P_\lambda | P_\lambda \rangle_{q,t}} P_\lambda(x; q, t) P_\lambda(y; q, t) = \Pi(x, y; q, t) := \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - t^n}{1 - q^n p_n(x)p_n(y)} \right\}
\]

(4.1)

where the scalar product is given by

\[
\langle P_\lambda | P_\lambda \rangle_{q,t} := \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{\ell(s)}}{1 - q^{a(s)} t^{\ell(s)+1}}.
\]

(4.2)

If we try to apply it to \((3.8)\), the problem is the existence of two Schur functions whose specialization depends on the partition \(\eta\). Thus our strategy is to “remove” these Schur functions from the summand. As we will see below this is possible for totally anti-symmetric representations. The starting point is the observation that for totally anti-symmetric representation \(\lambda = (1^r)\) both the Schur function \(s_\lambda(x)\) and the Macdonald function \(P_\lambda(x; q, t)\) coincide with the elementary symmetric function \(e_r(x)\). Thus we can use a trick of replacing one of the Schur functions with the Macdonald function to make use of the following remarkable symmetry of the specialization of the Macdonald functions [21] (Ch. VI.6):

\[
P_\lambda(t^\rho; q, t) P_\mu(q^\lambda t^\rho; q, t) = P_\mu(t^\rho; q, t) P_\lambda(q^\mu t^\rho; q, t).
\]

(4.3)

We have

\[
Z_{1^r,1^r}(Q; q, t) = \sum_{\eta} (-Q)^{|\eta|} P_{\eta^\vee}(q^{\rho}; t, q) P_{\eta}(t^\rho; q, t) P_{(1^r)}(q^n t^\rho; q, t) e_\lambda(q^n t^\rho)
\]

\[
= e_r(t^\rho) \sum_{\eta} (-Q)^{|\eta|} P_{\eta^\vee}(q^{\rho}; t, q) P_{\eta}(q^{(1^r)} t^\rho; q, t) e_\lambda(q^n t^\rho).
\]

(4.4)

Thus we have succeeded in getting rid of one of the Schur functions from the summand. To eliminate the remaining elementary symmetric function in \((4.4)\), we introduce the
Macdonald operator on the space of symmetric functions \[21\]. The Macdonald functions are characterized by the property that they are simultaneous eigen-functions of the Macdonald operator. The Macdonald operator in \(N\) variables \(x = (x_1, \cdots, x_N)\) is given by

\[
D_N^1 := \sum_{i=1}^N \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q, x_i},
\]

where \(T_{q, x}\) is the \(q\)-shift operator defined by \(T_{q, x} f(x) = f(qx)\). It is known that the Macdonald polynomials are the eigen-functions of \(D_N^1\);

\[
D_N^1 P_\lambda(x; q, t) = \varepsilon_{N, \lambda}^1 P_\lambda(x; q, t), \quad \varepsilon_{N, \lambda}^1 := t^{N-\frac{1}{2}} \sum_{i=1}^N q^{\lambda_i} t^{\frac{1}{2} - i}. \quad (4.6)
\]

More generally for non-negative integer \(r\), we define the higher Macdonald operators \(D_N^r\) by \(D_N^0 := 1\) and

\[
D_N^r := t^{(r-1)/2} \sum_{|I|=r} \prod_{j \in I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i}, \quad (1 \leq r \leq N)
\]

where the sum is over all \(r\)-element subsets \(I\) of \(\{1, 2, \cdots, N\}\). We set \(D_N^r := 0\) for \(r > N\). Let \(D_N := \sum_{r=0}^N D_N^r \bar{z}^r\), then the Macdonald polynomial is the eigen-function of \(D_N\)

\[
D_N P_\lambda(x; q, t) = P_\lambda(x; q, t) \varepsilon_N, \quad \varepsilon_N := \prod_{i=1}^N (1 + \bar{z} q^{\lambda_i} t^{N-i}) =: \sum_{r=0}^N \bar{z}^r e_r(q^{\lambda_i} t^{N-i}). \quad (4.8)
\]

Therefore, \(D_N^r\) are simultaneously diagonalized by the Macdonald polynomials

\[
D_N^r P_\lambda(x; q, t) = P_\lambda(x; q, t) e_r(q^{\lambda_i} t^{N-i}), \quad (4.9)
\]

and \(D_N^r\) commute with each other; \([D_N^r, D_N^s] = 0\), on the space of the symmetric functions in \(N\) variables.

Since \(D_N\) is not compatible with the restriction of the variables defined by \(x_N = 0\), we need to modify it so that we can take the limit \(N \to \infty\) \[34\] \[35\]. Let us define \(H\) and \(H^r\) which act on \(x = (x_1, x_2, \cdots)\) by the limits \(H := \lim_{N \to \infty} H_N\) and \(H^r := \lim_{N \to \infty} H_N^r\), where

\[
H_N := \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{(-\bar{z})^n}{1 - t^n} \right\} \quad D_N := \sum_{r \geq 0} \bar{z}^r H_N^r, \quad z := \bar{z} t^{N-\frac{1}{2}},
\]

15
\[ H_N^r = t^{(i-r-N)} \sum_{s=0}^{\min(r,N)} \prod_{i=1}^{r-s} \frac{1}{t^i - 1}; \quad D_N^s, \quad r = 0, 1, 2, \ldots \]  

Here we use \( \sum_{n \geq 0} z^n \prod_{i=1}^n (t^i - 1)^{-1} = \exp \left\{ \sum_{n \geq 0} (-z)^n (1 - t^n)^{-1}/n \right\} \) and define a new parameter \( z \) by scaling \( \tilde{z} \). Then

\[
E_{N,\lambda} := \exp \left\{ \sum_{n > 0} \frac{1}{n} \frac{(-\tilde{z})^n}{1 - t^n} \right\} \varepsilon_{N,\lambda} = \exp \left\{ - \sum_{n > 0} \frac{(-\tilde{z})^n}{n} \left( \sum_{i=1}^N \frac{N^{n_i}}{i^{n_i}(N-n)} + \frac{1}{t^n - 1} \right) \right\} = \exp \left\{ - \sum_{n > 0} \frac{(-\tilde{z})^n}{n} p_n(q^{\lambda} t^\rho) \right\} = \sum_{r \geq 0} z^r e_r(q^{\lambda} t^\rho). \tag{4.11}
\]

Since this is independent of \( N \) as long as it is larger than the length \( \ell(\lambda) \) of the partition, we have

\[
H_N P_\lambda(x; q, t) = P_\lambda(x; q, t) E_{N,\lambda}, \quad H_N^r P_\lambda(x; q, t) = P_\lambda(x; q, t) e_r(q^{\lambda} t^\rho). \tag{4.12}
\]

Now by (4.12) we can eliminate the elementary symmetric function in (4.4) and apply the Cauchy formula (4.11) as follows;

\[
Z_{1^r,1^s}(Q; q, t) = e_r(t^\rho) \sum_\eta \frac{(\tilde{Q}|\eta|}{\langle \eta | P_\eta \rangle_{q,t}} P_\eta(t^{-\rho}; q, t) P_\eta(q^{(1^r)} t^\rho; q, t) e_s(q^{(1^s)} t^\rho)
\]

\[
= e_r(t^\rho) \sum_\eta \frac{P_\eta(\tilde{Q}^{(1^r)} t^\rho; q, t)}{\langle \eta | P_\eta \rangle_{q,t}} (H^s(x) P_\eta(x; q, t)) \big|_{x=t^{-\rho}}
\]

\[
= e_r(t^\rho) \left( H^s(x) \Pi(x, \tilde{Q}^{(1^r)} t^\rho; q, t) \right) \big|_{x=t^{-\rho}}, \tag{4.13}
\]

where we have also used the following formula proved in [8]

\[ P_\rho(q^{\lambda} t^\rho; q, t) = \frac{(q/t)^{[n]} \frac{x^n}{P_\mu}}{(P_\rho P_\mu)_{q,t}} P_\mu(q^{-\lambda} t^{-\rho}; q, t). \tag{4.14}
\]

The final step is the evaluation of \( H^s(x) \Pi(x, \tilde{Q}^{(1^r)} t^\rho; q, t) \big|_{x=t^{-\rho}}. \) For this purpose we have to compute the action of the shift operator \( T_{q,x_i} \) on the Cauchy kernel \( \Pi(x, y; q, t) \). Since \( T_{q,x_i} p_n = ((q^n - 1)x_i^n + p_n) \) for the power sum \( p_n = \sum_i x_i^n \),

\[
\frac{T_{q,x_i} \Pi(x, y; q, t)}{\Pi(x, y; q, t)} = \exp \left\{ \sum_{n > 0} \frac{t^n}{n} x_i^n p_n(y) \right\}. \tag{4.15}
\]
Hence,
\[
\prod_{i \in I} \frac{(1 - t^{1/2}x^i)T_{q,x_i} \cdot \Pi(x, y; q, t)}{\Pi(x, y; q, t)} = \exp \left\{ \sum_{n>0} \frac{t^n - 1}{n} \sum_{i \in I} x_i^n \left( p_n(y) - \frac{t^{n-1}}{t^n - 1} \right) \right\}
= \exp \left\{ \sum_{n>0} \frac{t^n - 1}{n} \sum_{i \in I} x_i^n p_n(y, t^{-\rho}) \right\}. \quad (4.16)
\]

When \( x = t^{-\rho} \), since \( tx_i - x_j = 0 \) for \( j = i + 1 \), only \( I = \{N, N - 1, \ldots, N - r + 1\} \) contributes to the summation in (4.17). Therefore,
\[
D_N^r \bigg|_{x=t^{-\rho}} = \prod_{i=1}^r \frac{t^{i-1}}{1 - t^i} \frac{1 - t^{N-i+1}}{1 - t^i} T_{q,xN-i+1},
\]
\[
H_N^r \bigg|_{x=t^{-\rho}} = (t^{(1/2-N)r}) \sum_{s=0}^{r-s} \prod_{i=1}^r \frac{1}{t^i - 1} \prod_{j=1}^s \frac{1 - t^{N-j+1}}{1 - t^j} T_{q,xN-j+1}. \quad (4.17)
\]

Then we have

**Proposition.**

\[
\frac{D_N^r \Pi(x, y; q, t)}{\Pi(t^{-\rho}, y; q, t)} \bigg|_{x=t^{-\rho}} = \exp \left\{ \sum_{n>0} \frac{1 - t^{-rn}}{n} t^{n(N+\frac{1}{2})} p_n(y, t^{-\rho}) \right\} \prod_{j=1}^r \frac{t^{j-1}}{1 - t^j},
\]
\[
\frac{H_N^r \Pi(x, y; q, t)}{\Pi(t^{-\rho}, y; q, t)} \bigg|_{x=t^{-\rho}} = (-1)^r t^{-r(r-1)/2} e_r(y, t^{-\rho}) + O(t^N), \quad (4.18)
\]

with \( \sum_{r \geq 0} (-z)^r e_r(x, y) := \exp \left\{ - \sum_{n>0} p_n(x, y) z^n / n \right\} \).

**Proof.** By (4.16),
\[
\prod_{i=N-r+1}^N (1 - t^{i}) T_{q,x_i} \cdot \Pi(x, y; q, t) \bigg|_{x=t^{-\rho}} = \exp \left\{ \sum_{n>0} \frac{t^n - 1}{n} \sum_{i=1}^r t^{n(i-\frac{1}{2})} p_n(y, t^{-\rho}) \right\}
= \exp \left\{ \sum_{n>0} \frac{1 - t^{-rn}}{n} t^{n(N+\frac{1}{2})} p_n(y, t^{-\rho}) \right\}. \quad (4.19)
\]

we get the first equation. We also have
\[
\frac{H_N^r \Pi(x, y; q, t)}{\Pi(t^{-\rho}, y; q, t)} \bigg|_{x=t^{-\rho}} = (t^{(1/2-N)r}) \sum_{s=0}^r \exp \left\{ \sum_{n>0} \frac{1 - t^{-rn}}{n} (t^{N+\frac{1}{2}}) p_n(y, t^{-\rho}) \right\} \prod_{i=1}^r \frac{t^{i-1}}{1 - t^i} \prod_{j=1}^s \frac{t^{j-1}}{1 - t^j}. \quad (4.20)
\]

Then by the following lemma, we obtain the second equation. □
Lemma.  
\[
\sum_{s=0}^{r} \exp \left\{ \sum_{n=1}^{\infty} \frac{(1 - t^{-sn})}{n} t^n p_n x^n \right\} \left( -1 \right)^s t^{\frac{1}{2} s(s-1)} \left[ r \right]_s = t^{-\frac{1}{2} r(r-1)} \prod_{k=1}^{r} (1 - t^k) x^r e_r + O(x^{r+1}),
\]
(4.21)

where \( \sum_{r \geq 0} (-x)^r e_r = \exp \left\{ - \sum_{n=1}^{\infty} p_n x^n / n \right\} \) and \( \left[ r \right]_s := \prod_{i=1}^{s} \frac{1 - t^{r-s+i}}{1 - t^i} \).

Proof. Let us fix \( k \) and consider the coefficient of \( x^k \) on the left hand side. By expanding the exponential we see that it consists of the terms with \( t^{-\ell s} \) \((0 \leq \ell \leq k)\). Hence we have terms proportional to

\[
\sum_{s=0}^{r} (-t^{-\ell})^s t^{\frac{1}{2} s(s-1)} \left[ r \right]_s , \quad (0 \leq \ell \leq k).
\]
(4.22)

However by the formula ([21], Chap. I-2, Example 3);

\[
\prod_{i=0}^{r-1} (1 + t^i z) = \sum_{s=0}^{r} z^s t^{\frac{1}{2} s(s-1)} \left[ r \right]_s ,
\]
(4.23)

we see that it vanishes as long as \( 0 \leq \ell \leq k < r \). Finally among the coefficients of \( x^r \) only the term with \( t^{-rs} \) survives by the same reasoning as above. Hence we may compute it with

\[
\sum_{s=0}^{r} \exp \left\{ - \sum_{n=1}^{\infty} \frac{t^{-sn}}{n} (tx)^n p_n \right\} \left( -1 \right)^s t^{\frac{1}{2} s(s-1)} \left[ r \right]_s
\]
(4.24)

by omitting “1” in the left hand side of (4.21). Comparing the coefficient of \( x^r \) in the relation of \( p_n \) and \( e_r \), we see the coefficient of \( x^r \) is

\[
t^r (-1)^r e_r \sum_{s=0}^{r} (-t^{-r})^s t^{\frac{1}{2} s(s-1)} \left[ r \right]_s = (-1)^r e_r t^r \prod_{i=0}^{r-1} (1 - t^{i-r}),
\]
(4.25)

which completes the proof. \( \square \)

The following is the main result of this section.

Proposition. For \( |t| < 1 \),

\[
\frac{Z_{1^{r},1^{s}}(Q; q, t)}{Z_{*,*}(Q; q, t)} = (-1)^r t^{-\frac{1}{2} s(s-1)} e_r(t^\rho) e_s(\tilde{Q} q^{(1)} t^\rho, t^{-\rho}) \prod_{i=1}^{r} \left( 1 - Q q^{\frac{t}{2}} t^{-i+\frac{1}{2}} \right) 
\]
(4.26)

\[
= (-1)^{r+s} t^{-\frac{1}{2} r(r-1)-\frac{1}{2} s(s-1)} e_r(\tilde{Q} t^{\rho}, t^{-\rho}) e_s(\tilde{Q} q^{(1)} t^\rho, t^{-\rho}),
\]

where \( e_r(x) \) and \( e_s(x, y) \) are the elementary symmetric functions in variables \( x = (x_1, x_2, x_3, \cdots) \) and two sets of variables \( (x_1, x_2, \cdots, y_1, y_2, \cdots) \), respectively.
Proof. By taking the large $N$ limit of the proposition proved above, we see

$$Z_{1r,1s}(Q; q, t) = e_r(t^\rho) \left( H^s(x) \Pi(x, \tilde{Q} q^{(1r)} t^\rho; q, t) \right) |_{x = t^{-\rho}} = (-1)^s t^{-\frac{2(s-1)}{2}} e_r(t^\rho) \Pi(t^{-\rho}, \tilde{Q} q^{(1r)} t^\rho; q, t) e_s(\tilde{Q} q^{(1r)}, t^{-\rho}).$$ (4.27)

Finally the proof is completed by noting

$$\frac{\Pi(t^{-\rho}, \tilde{Q} q^{(1r)} t^\rho; q, t)}{\Pi(t^{-\rho}, \tilde{Q} t^\rho; q, t)} = N_{1r,1s}(Q; q, t) = \prod_{i=1}^{r} \left( 1 - Q q^\frac{1}{2} t^{-i+\frac{1}{2}} \right),$$ (4.28)

where $N_{\lambda,\mu}$ is the denominator factor of Nekrasov’s partition function (2.12), which satisfies (2.12) in [10]

$$N_{\lambda,\mu}(Q; q, t) = \frac{\Pi(Q q^{\lambda t^\rho}, q^{-\mu} t^{-\rho}; q, t)}{\Pi(Q t^\rho, t^{-\rho}; q, t)}.$$ (4.29)

The condition $|t| < 1$ in the above proposition can be eliminated [36]. By the proposition it is clear that for totally antisymmetric representations the superpolynomial $\mathcal{P}_{\lambda,\mu}(a, q, t)$ is actually a polynomial in $Q$ and consequently in $a$, which is not manifest in the proposal (3.3). We can also prove that it is a polynomial in $q$ and $t$ with positive integer coefficients when $\tilde{Q} = t^N$ with $N \in \mathbb{N}$.

5 Homological invariants for totally anti-symmetric representations

To accommodate the standard convention of link polynomials, we make the following change of variables\footnote{Note that compared with [12], we have changed the translation rule by $q \rightarrow q^{-1}$ and $t \rightarrow t^{-1}$. Accordingly the identification or the $t$-grading of $Q$ is also changed.}:

$$\sqrt{t} = q^{-1}, \quad \sqrt{q} = -t^{-1} q^{-1}, \quad Q = -ta^{-2},$$ (5.1)

so that $\tilde{Q} := (q/t)^{\frac{1}{2}} Q = a^{-2}$. Then

$$\mathcal{P}_{\lambda,\mu}(a, q, t) = (-a)^{|\lambda| + |\mu|} (-t)^{|\lambda| |\mu|} Z_{\lambda,\mu}(Q; q, t) Z_{\star,\star}(Q; q, t).$$ (5.2)
Substituting the formula (4.26), we obtain

\[
\overline{P}_{(1^r),(1^s)}(a, q, t) = (-a)^{r+s}(-t)^{-r-s}(1)^s q^{s(s-1)} e_r(t^\rho) e_s(\tilde{Q}q^{(1^r)}t^\rho, t^{-\rho}) \prod_{i=1}^{r+s} \left( 1 - \tilde{Q}t^{-i+1} \right),
\]

\[
= a^{-r-s}(-t)^{-r-s} q^{s(s-1)} e_r(t^\rho) e_s(q^{(1^r)}t^\rho, a^2t^{-\rho}) \prod_{i=1}^{r+s} \left( q^{2(i-1)} - a^2 \right). \tag{5.3}
\]

This is a rather simple closed formula of the superpolynomial for the Hopf link colored with totally anti-symmetric representations \((1^r, 1^s)\). Note that apart from the overall factor \((-t)^{-r-s}\), \(t\)-dependence only appears in the specialization \(x = q^{(1^r)}t^\rho\) of \(e_s(x, y)\). By putting \(t = -1\) (or \(q = t\)) and taking the limit \(a \to 0\) we find

\[
\lim_{a \to 0} \overline{P}_{(1^r),(1^s)}(a, q, -1) \sim a^{-r-s} q^{r(r-1)+s(s-1)} e_r(q^\rho) e_s(q^{(1^r)}+\rho). \tag{5.4}
\]

We note that there is an additional factor \(q^{r(r-1)+s(s-1)}\) in the normalization of (5.2), compared with the large \(N\) behavior of the polynomial invariants of the colored Hopf link. When \(s = 0\) the superpolynomial becomes independent of \(t\). The formula (5.3) gives

\[
\overline{P}_{(1^r),(1^s)}(a, q, t) = a^{-r} q^{2r(r-1)} \prod_{i=1}^{r} \frac{q^{2(i-1)} - a^2}{q^i - q^{-i}} = a^{-r} q^{r^2} \prod_{i=1}^{r} \frac{q^{2(i-1)} - a^2}{1 - q^{2i}}. \tag{5.5}
\]

This agrees with the superpolynomial of the unknot with a totally anti-symmetric representation in \([12, 22]\). For \(a = q^N\) it reproduces the homological invariants of the unknot, which has been proposed to give the Hilbert series of the Grassmannian \(Gr(N, r)\) \([28]\).

To provide more examples of our formula, let us look at the case where one of the representations is the fundamental representation. Then the formula (5.3) is evaluated to be

\[
\overline{P}_{(1^k), (1)}(a, q, t) = a^{-k-1}(-t)^k e_k(t^\rho) e_1(q^{(1^k)}t^\rho, a^2t^{-\rho}) \prod_{i=1}^{k} \left( q^{2i-2} - a^2 \right)
\]

\[
= a^{-k-1} t^k q^{k(k-1)} \prod_{i=1}^{k} \frac{q^{2i-2} - a^2}{q^i - q^{-i}} \left( t^{-2} \frac{q^{2k-2} - q^{-2}}{q - q^{-1}} + \frac{a^2 - q^{2k}}{q - q^{-1}} \right). \tag{5.6}
\]

When \(k = 1\), we find

\[
\overline{P}_{(1), (1)}(a, q, t) = t \left( a^{-2} - \frac{1}{q - q^{-1}} \left( t^{-2} \frac{1 - q^{-2}}{q - q^{-1}} + \frac{a^2 - q^2}{q - q^{-1}} \right) \right)
\]

\[
= \frac{(-t)^{-1}}{a^2(1 - q^2)^2} \left( a^4 t^2 q^2 - a^2 (t^2 q^4 + t^2 q^2 - q^2 + 1) + t^2 q^4 - q^2 + 1 \right). \tag{5.7}
\]
Up to the factor \((-t)^{-1}\) or a shift of \(t\)-grading by \(-1\), this result agrees with [12], where it was shown that for \(a = q^N\) this superpolynomial gives rise to the Khovanov-Rozansky invariants of the Hopf link.

It is remarkable that for any \(k\) the substitution of \(a = q^N\) reduces the superpolynomial \(\overline{P}_{(1^k)}(a, q, t)\) to a polynomial in \(q^{\pm 1}\) and \(t^{\pm 1}\):

\[
\overline{P}_{(1^k)}(a, q, t) = t^k q^{Nk-N+k(k-1)} \prod_{i=1}^{k} \frac{q^{2i-2-2N} - 1}{q^i - q^{-i}} \left( t^{-2} q^{k-2} \frac{q^k - q^{-k}}{q - q^{-1}} + \frac{q^{2N} - q^{2k}}{q - q^{-1}} \right) = q^{-2N} q^{k(k-1)} (-t)^k \left[ \binom{N}{k} \right]_q \left( [k]_q q^{N+k-2} + [N-k]_q q^{2N+k} \right),
\]

(5.8)

which gives homological invariants of the Hopf link with representations \((1^k, \mathfrak{q})\) of \(\mathfrak{sl}(N)\). Note that this expression vanishes for \(k > N\) as it should be for the totally antisymmetric representation \((1^k)\) of \(\mathfrak{sl}(N)\). We have factorized the standard normalization factor \(q^{-2N}\) for the Hopf link. Since both the \(q\)-integer and the \(q\)-binomial coefficient are polynomials with positive integer coefficients, \((5.8)\) shows that \(\overline{P}_{(1^k)}(a, q, t)\) is a Laurent polynomial in \(q\) and \(t\) with positive integer coefficients. Up to an overall factor \(q^{k(k-1)} (-t)^k\), \((5.8)\) agrees to the recent result by Yonezawa by matrix factorization [23].

More generally we can show that the superpolynomial \(\overline{P}_{(1^r), (1^s)}(a, q, t)\) always gives a polynomial in \(q^{\pm 1}, t^{\pm 1}\), when we substitute \(a = q^N\). In fact after the substitution we have

\[
\overline{P}_{(1^r), (1^s)}(q, t) = (-1)^{r(s-1)} q^{N(r-s)+s(s-1)+\frac{1}{2}r(r-1)} t^{r-s} \prod_{i=1}^{r} \frac{q^{2i-2-2N} - 1}{q^i - q^{-i}} e_s(q^{1^r} t^\rho, q^{2N} t^{-\rho}) = (-t)^{r-s} q^{-sN+s(s-1)+r(r-1)} \left[ \binom{N}{r} \right]_q e_s(q^{1^r} t^\rho, q^{2N} t^{-\rho}).
\]

(5.9)

Hence what we have to show is that \(e_s(q^{1^r} t^\rho, q^{2N} t^{-\rho})\) is a polynomial. Since the elementary symmetric function \(e_s\) is a polynomial in the power sums \(p_1, \ldots, p_s\), let us look at \(p_k(q^{1^r} t^\rho, q^{2N} t^{-\rho})\). We find

\[
p_k(q^{1^r} t^\rho, q^{2N} t^{-\rho}) = (tq)^{-2k} \frac{q^{2kr} - 1}{q^k - q^{-k}} + \frac{q^{2kN} - q^{2kr}}{q^k - q^{-k}} = t^{-2k} q^{k(r-2)} \frac{[kr]_q}{[k]_q} + \frac{k(kN+r)[k(N-r)]_q}{[k]_q}.
\]

(5.10)

which shows that they are polynomials. Thus we see that \(e_s(q^{1^r} t^\rho, q^{2N} t^{-\rho})\) is a polynomial in \(q^{\pm 1}, t^{\pm 1}\).
6 Case of other representations

Using the fact that the Macdonald functions and the Schur functions are related by a base change of the space of symmetric functions, we can compute the homological invariants for other representations than totally anti-symmetric representations. The base change is rather complicated in general. But let us illustrate the method of computation in the simplest examples.

6.1 \((\lambda, \mu) = ((2), (1^s))\) case

When \(|\lambda| = 2\), the base change is

\[
\begin{bmatrix}
    s(2)(x) \\
    s(1^2)(x)
\end{bmatrix} = \begin{bmatrix}
    1 & t - q \\
    0 & 1 - qt
\end{bmatrix} \begin{bmatrix}
    P(2)(x; q, t) \\
    P(1^2)(x; q, t)
\end{bmatrix}.
\]

By transforming the Schur functions to the Macdonald functions, we can apply a similar way of computation to the case of totally anti-symmetric representations. We obtain

\[
\frac{Z_{(2), (1^s)}(Q; q, t)}{Z_{(1^2), (1^s)}(Q; q, t)} = (-1)^s q^{-\frac{s}{2}(s-1)} \begin{bmatrix}
    1 & t - q \\
    0 & 1 - qt
\end{bmatrix} \begin{bmatrix}
    P(2)(t^\rho; q, t)N_{(2), \bullet}(\tilde{Q}; q, t)e_s(\tilde{Q}q^{(2)}t^\rho, t^{-\rho}) \\
    P(1^2)(t^\rho; q, t)N_{(1^2), \bullet}(\tilde{Q}; q, t)e_s(\tilde{Q}q^{(1^2)}t^\rho, t^{-\rho})
\end{bmatrix},
\]

and hence

\[
\overline{P}_{(2), (1^s)}(a, q, t) = a^{s+2}t^{4s}q^{s(s-1)} \left( (s(2)(t^\rho) - \frac{t - q}{1 - qt} s(1^2)(t^\rho))N_{(2), \bullet}(\tilde{Q}; q, t)e_s(\tilde{Q}q^{(2)}t^\rho, t^{-\rho}) + \frac{t - q}{1 - qt} s(1^2)(t^\rho)N_{(1^2), \bullet}(\tilde{Q}; q, t)e_s(\tilde{Q}q^{(1^2)}t^\rho, t^{-\rho}) \right).
\]

The terms proportional to \(\frac{t - q}{1 - qt}\) give a discrepancy to the conjecture (3.9). It holds only for totally anti-symmetric representations. As we argued in section 3, this means that the slice invariance of open topological string amplitudes is broken, if the representation attached to the topological brane is not totally anti-symmetric. The above formula for \(s = 0\) provides the following superpolynomial for the unknot colored by the symmetric representation;

\[
\overline{P}_{(2), \bullet}(a, q, t) = a^{-2}t^{-4}q^{-7} \frac{(a^2 - 1)}{(q - q^{-1})(q^2 - q^{-2})}(t^2q^3(a^2 - q^2) + q^4 - 1).
\]
When $s = 1$, we find the following superpolynomial for the Hopf link colored by the symmetric representation and the fundamental representation:

$$\overline{P}_{(2),(1)}(a, q, t) = a t^4 \frac{q}{(q - q^{-1})(q^2 - q^{-2})} \left( (q^2 - \frac{q^{-2} - t^{-2}q^{-2}}{1 - t^{-2}q^{-4}})(1 - a^{-2})(1 - a^{-2}t^{-2}q^{-2})(t^{-4}q^{-3} + \frac{a^2 - q^2}{q - q^{-1}}) + \frac{q^{-2} - t^{-2}q^{-2}}{1 - t^{-2}q^{-4}}(1 - a^{-2})(1 - a^{-2}q^{-2})(t^{-2}q^{-2}(q + q^3) + \frac{a^2 - q^4}{q - q^{-1}}) \right) = a^{-3}t^4q^{-1} \frac{(a^2 - 1)}{(q - q^{-1})^2(q^2 - q^{-2})} \left( (a^2 - q^4)(a^2 - q^2) + t^{-2}q^{-2}(q^2 - 1)(q^2 + 1)^2(a^2 - q^2) - t^{-4}q^{-4}(q^2 - 1)^2(q^2 + 1)^2(a^2 - q^2) + t^{-6}q^{-6}(q^2 - 1)^2(q^2 + 1) \right). \tag{6.5}$$

Note that there is a cancellation of the factor $(1 - qt) = (1 - t^{-2}q^{-4})$ in the denominator. Finally by substituting $a = q^N$ we can eliminate the remaining factors in the denominator to get a polynomial in $q^{\pm 1}$ and $t^{\pm 1}$. We believe this is a highly non-trivial consistency check of our formula $[4, 26]$.

### 6.2 $(\lambda, \mu) = (21), (1^s)$ case

The second example is the hook representation $\lambda = (21)$ with $|\lambda| = 3$. In this case we have

$$\begin{bmatrix} s_{(3)}(x) \\ s_{(21)}(x) \\ s_{(13)}(x) \end{bmatrix} = \begin{bmatrix} 1 & (t - q)(1 + q) & (t - q)(t^2 - q) \\ 1 - q^2t & (1 - qt)(1 - q^2t) & (t - q)(t + 1) \\ 0 & 1 & 1 - qt^2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{(3)}(x; q, t) \\ P_{(21)}(x; q, t) \\ P_{(13)}(x; q, t) \end{bmatrix}. \tag{6.6}$$

By the same manner as above we obtain the superpolynomial

$$\begin{align*}
\overline{P}_{(21),(1^s)}(a, q, t) &= -a^{s+3}t^{6s}q^{s(s-1)} \left( s_{(21)}(t^\rho) - \frac{(t - q)(t + 1)}{1 - qt^2} s_{(13)}(t^\rho) \right) N_{(21)}, \star (\widetilde{Q}; q, t) e_s(\widetilde{Q}q^{(21)}t^\rho, t^{-\rho}) \\
&\quad + \frac{(t - q)(t + 1)}{1 - qt^2} s_{(13)}(t^\rho) N_{(13)}, \star (\widetilde{Q}; q, t) e_s(\widetilde{Q}q^{(13)}t^\rho, t^{-\rho}) \right). \tag{6.7}
\end{align*}$$

$^6$Examples of the superpolynomials provided in appendix of [12] are only for totally antisymmetric representations and this is a new example of the superpolynomial for the colored Hopf link.
When \( s = 0 \) the superpolynomial for the unknot is
\[
\mathcal{P}_{(21),*}(a, q, t) = a^{-3} t^{-4} q^{-9} \frac{(a^2 - 1)(a^2 - q^2)}{(q - q^{-1})(q^3 - q^{-3})^2}(t^2 q^2(a^2 - q^4) + q^6 - 1).
\]

(6.8)

When \( s = 1 \) the superpolynomial is
\[
\mathcal{P}_{(21),*}(a, q, t) = a^2 t^6 \frac{(1 - a^{-2})(1 - a^{-2}q^2)}{(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})} \left[ (q + q^{-1} - \frac{q(1 - t^{-2})(1 + q^{-2})}{1 - t^{-2}q^{-6}})(1 - a^{-2}t^{-2}q^{-2})(t^{-4}q^{-3} + t^{-2}q + \frac{a^2 - q^4}{q - q^{-1}}) + \frac{q(1 - t^{-2})(1 + q^{-2})}{1 - t^{-2}q^{-6}}(1 - a^{-2}q^4)(t^{-2}(q^{-1} + q + q^3) + \frac{a^2 - q^6}{q - q^{-1}}) \right] = a^{-4} \frac{(a^2 - 1)(a^2 - q^2)}{(q - q^{-1})^3(q^3 - q^{-3})}(t^6 P_6(a, q) + t^4 q^{-2} P_4(a, q) + t^2 q^{-4} P_2(a, q) + q^{-6} P_0(a, q)),
\]

(6.9)

where
\[
\begin{align*}
P_6(a, q) &= (a^2 - q^4)(a^2 - q^6), \\
P_4(a, q) &= (a^2 - q^4)(q^2 - 1)(q^2 + 1)(q^2 + q + 1)(q^2 - q + 1), \\
P_2(a, q) &= -(a^2 - q^2 - q^4)(q^2 - 1)(q^2 + 1)(q^2 - q + 1), \\
P_0(a, q) &= (q^2 - 2)(q^2 + q + 1)(q^2 - q + 1).
\end{align*}
\]

We can again confirm a cancellation of the factor \((1 - q^2t^2) = (1 - t^{-2}q^{-6})\) in the denominator and the fact that the superpolynomial reduces to a polynomial in \(q^{\pm 1}\) and \(t^{\pm 1}\) when \(a = q^N\).

### 6.3 \((\lambda, \mu) = (\lambda, 1^s)\) case

Generalizing the above computations, we can show the “finiteness” of the superpolynomial for \((\lambda, \mu) = (\lambda, 1^s)\). As a Laurent polynomial in \(a\) the degree of \(\mathcal{P}_{\lambda,(1^s)}(a, q, t)\) ranges from \(-|\lambda| - s\) to \(|\lambda| + s\). When we substitute \(a = q^N\) with a fixed \(N\), \(\mathcal{P}_{\lambda,(1^s)}(q^N, q, t)\) vanishes if \(N < \max(|\lambda|, s)\). By the relation (5.2) these properties follow from the propositions below. Since the set of the Macdonald functions \(\{P_\lambda(x; q, t)\}_{|\lambda| = d}\) is a basis of the symmetric functions of homogeneous degree \(d\), we can write the Schur function \(s_\lambda(x)\) by the Macdonald functions as
\[
s_\lambda(x) = \sum_{\mu \vdash |\lambda|} U_{\lambda, \mu}(q, t) P_\mu(x; q, t),
\]

(6.11)
where \( U_{\lambda,\mu}(q,t) \) is a rational function. Then we have

**Proposition.** For \(|t| < 1\),

\[
\frac{Z_{\lambda,1^*}(Q;q,t)}{Z_{*,1^*}(Q;q,t)} = g_1 \sum_{|\mu|=|\lambda|} U_{\lambda,\mu}(q,t)g_\mu P_\mu(vQt^\rho,t^{-\rho};q,t)e_\mu(vQq^\mu t^\rho,t^{-\rho}),
\]

\[
g_\lambda := \prod_{s \in \lambda} (-1)^{q^s} t^{N-\ell(s)},
\]

which is a polynomial of degree \(|\lambda| + s\) in \(Q\).

**Proof.** From (6.11)

\[
Z_{\lambda,1^*}(Q;q,t) = \sum_\eta (-Q)^{|\eta|} P_\eta(t^\rho; q, t)P_\eta(t^\rho; q, t)s_\lambda(q^\eta t^\rho)\sum_\mu s_{1^*}(q^\eta t^\rho) = \sum_{|\mu|=|\lambda|} U_{\lambda,\mu}\tilde{Z}_{\mu,1^*}(Q;q,t),
\]

where

\[
\tilde{Z}_{\mu,1^*}(Q;q,t) := \sum_\eta (-Q)^{|\eta|} P_\eta(t^\rho; q, t)P_\eta(t^\rho; q, t)P_\eta(q^\eta t^\rho; q, t)s_{1^*}(q^\eta t^\rho).
\]

Then by (4.3) and (4.14),

\[
\tilde{Z}_{\mu,1^*}(Q;q,t) = \sum_\eta \frac{(vQ)^{|\eta|}}{\langle P_\eta \rangle_q |P_\eta_q|} P_\eta(t^\rho; q, t)P_\eta(t^\rho; q, t)P_\eta(q^\eta t^\rho; q, t)s_{1^*}(q^\eta t^\rho).
\]

Since \( s_{1^*}(q^\eta t^\rho) = e_s(q^\eta t^\rho) \), by (4.12) and (4.18),

\[
\tilde{Z}_{\mu,1^*}(Q;q,t) = P_\mu(t^\rho; q, t) \sum_\eta \frac{(vQ)^{|\eta|}}{\langle P_\eta \rangle_q |P_\eta_q|} P_\eta(q^\eta t^\rho; q, t)H^*P_\eta(x; q, t)\big|_{x=t^\rho}.
\]

\[
= P_\mu(t^\rho; q, t)H^*\Pi(x, vQq^\mu t^\rho; q, t)|_{x=t^\rho}
\]

\[
= P_\mu(t^\rho; q, t)(-1)^s t^{-\frac{s(s-1)}{2}} e_s(vQq^\mu t^\rho, t^{-\rho})\Pi(t^{-\rho}, vQq^\mu t^\rho; q, t).
\]

But by (4.29) and ((5.20) in [10])

\[
P_\mu(t^\rho; q, t)N_{\mu,\bullet}(vQ; q, t) = P_\mu(vQt^\rho, t^{-\rho}; q, t)v^{-|\mu|} f_\mu(q, t),
\]

with the framing factor \( f_{\mu}(q,t) \) defined by (2.3), we have

\[
\tilde{Z}_{\mu,1^*}(Q;q,t) = (-1)^s t^{-\frac{s(s-1)}{2}} v^{-|\mu|} P_\mu(vQt^\rho, t^{-\rho}; q, t)e_s(vQq^\mu t^\rho, t^{-\rho}) f_\mu(q, t).
\]
Note that, for \( N \in \mathbb{Z} \) and \( N \geq \ell(\lambda) \),

\[
p_n(q^\lambda t^\rho, t^{-N-\rho}) = \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^n(\frac{1}{2} - i) + \frac{1 - t^{-N}}{1 - q} = \sum_{i=1}^{N} q^{n\lambda_i} t^n(\frac{1}{2} - i).
\]  

(6.19)

Hence, \( P_\mu(q^\lambda t^\rho, t^{-N-\rho}) \) is the Macdonald polynomial in \( N \) variables \( \{q^\lambda t^{\frac{1}{2} - i}\}_{1 \leq i \leq N} \) and vanishes for \( \ell(\lambda) \leq N < \ell(\mu) \). Therefore when \( vQ = t^N \), we have

**Proposition.** If \( N \in \mathbb{Z}_{\geq 0} \) and \( |t| < 1 \),

\[
\frac{Z_{\lambda,1^*}(v^{-1}t^N; q, t)}{Z_{\ast, \ast}(v^{-1}t^N; q, t)} = g_1 \sum_{|\mu| = |\lambda|} U_{\lambda, \mu} g_\mu P_\mu(t^{N+\rho}, t^{-\rho}; q, t) P_1(q^\mu t^{N+\rho}, t^{-\rho}; q, t)
\]

\[
= g_1 P_1(t^{N+\rho}, t^{-\rho}; q, t) \sum_{|\mu| = |\lambda|} U_{\lambda, \mu} g_\mu P_\mu(q^{(1^*)} t^{N+\rho}, t^{-\rho}; q, t),
\]

(6.20)

which vanishes for \( 0 \leq N < \max(|\lambda|, s) \).

**Proof.** If \( 0 \leq N < |\mu| \), \( P_\mu(t^\rho, t^{-N-\rho}; q, t) = 0 \). If \( |\mu| \leq N < s \), \( P_1(q^\mu t^\rho, t^{-N-\rho}; q, t) = 0 \). Similarly, if \( 0 \leq N < s \), \( P_1(t^\rho, t^{-N-\rho}; q, t) = 0 \). If \( s \leq N < |\mu| \), \( P_\mu(q^{(1^*)} t^\rho, t^{-N-\rho}; q, t) = 0 \). Hence the middle and the right hand side of (6.20) vanish for \( 0 \leq N < \max(|\mu|, s) \).

On the other hand, if \( N \geq \max(|\mu|, s) \), by the finite \( N \) version of (4.3),

\[
P_1(t^\rho, t^{-N-\rho}; q, t) P_\mu(q^{1^*} t^\rho, t^{-N-\rho}; q, t) = P_\mu(t^\rho, t^{-N-\rho}; q, t) P_1(q^\mu t^\rho, t^{-N-\rho}; q, t),
\]

(6.21)

we have the proposition. \( \Box \)

The condition \( |t| < 1 \) in the last two propositions can be eliminated [36].

Finally we should make a remark on the fact that the transition function \( U_{\lambda, \mu}(q, t) \) in (6.11) is a rational function in \( q \) and \( t \). It is not obvious that the formulae in the above propositions are in fact polynomials in \( q \) and \( t \). However, we have checked the following conjecture up to \( d = 7 \) by direct calculation;

**Conjecture.** For \( |\lambda| = d \), \( \sum_{\mu, |\mu| = d} U_{\lambda, \mu} g_\mu (U^{-1})_{\mu, \nu} \) is a polynomial of degree \( d(d-1)/2 \) in \( q \) and of degree \( d(d-1)/2 \) in \( t^{-1} \).

Under the above conjecture, if \( N \in \mathbb{Z}_{\geq 0} \),

\[
\frac{Z_{\lambda,1^*}(v^{-1}t^N; q, t)}{Z_{\ast, \ast}(v^{-1}t^N; q, t)} = g_1 e_n(t^{N+\rho}, t^{-\rho}) \sum_{|\mu| = |\nu| = |\lambda|} U_{\lambda, \mu} g_\mu (U^{-1})_{\mu, \nu} s_\nu(q^{(1^*)} t^{N+\rho}, t^{-\rho}),
\]

(6.22)

is a polynomial in \( q \) and \( t \).
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Appendix: Five-dimensional $U(1)^N$ theory

Here we generalize the argument in section 2 to the five-dimensional $U(1)^N$ theory. We compare two partition functions associated with the diagrams in Fig. 3. Since the difference is just the choice of the preferred directions, we expect that they coincide.

Let

$$Z_L := \sum_{\{\lambda_\alpha\}} \prod_{\alpha=1}^{N} C_{\lambda_2 \lambda_2^{-1}}^{\lambda_2 \lambda_2^{-1}}(q, t) C_{\lambda_2 \lambda_2^{-1}}^{\lambda_2 \lambda_2^{-1}}(q, t) \prod_{\alpha=1}^{2N} Q_{\lambda_\alpha}^{\lambda_\alpha}$$

$$= \sum_{\{\lambda_\alpha\}} \prod_{\alpha=1}^{N} P_{\lambda_2 \lambda_2^{-1}}^{\lambda_2 \lambda_2^{-1}}(t^\rho; q, t) P_{\lambda_2 \lambda_2^{-1}}^{\lambda_2 \lambda_2^{-1}}(q^{\lambda_2 \lambda_2^{-1}} t^\rho; q, t)$$

$$\times P_{\lambda_2 \lambda_2^{-1}}^{\lambda_2 \lambda_2^{-1}}(-q^\rho; t, q) P_{\lambda_2 \lambda_2^{-1}}^{\lambda_2 \lambda_2^{-1}}(-q^{\lambda_2 \lambda_2^{-1}} q^\rho; t, q) \prod_{\alpha=1}^{2N} Q_{\lambda_\alpha}^{\lambda_\alpha},$$

(A.1)

with $\lambda_0 = \lambda_2$ and $\sigma_0 = \sigma_2$. The summations over the partitions with odd suffices $\lambda_1$, $\lambda_3$, $\lambda_5$, $\cdots$, are performed by the Cauchy formula (2.6) for $\mu = \nu = \bullet$. From the specialization formula:

$$P_\lambda(t^\rho; q, t) P_\lambda^\vee(-q^\rho; t, q) = \frac{v^{-|\lambda|}}{N_{\lambda\lambda}(1; q, t)},$$

(A.2)

with (2.12) we have

$$Z_L = \sum_{\{\lambda_2\}} \prod_{\alpha=1}^{N} \left(\frac{v^{-1} Q_{2\alpha}^{\lambda_{2\alpha}}}{N_{\lambda_{2\alpha}, \lambda_{2\alpha}}(1; q, t)}\right) \prod_{0}(q^{\lambda_{2\alpha-2} t^\rho}, -Q_{2\alpha-1} t^{\lambda_{2\alpha}^\vee} q^\rho).$$

(A.3)
If we separate out the perturbative part $Z_L^{\text{pert}} := Z_L(Q_{2\alpha} = 0) = \prod_{\alpha=1}^{N} \Pi_{0}(t^\rho, -Q_{2\alpha-1}q^\rho)$, then from (2.12), $Z_L^{\text{inst}} := Z_L/Z_L^{\text{pert}}$ is

$$Z_L^{\text{inst}} = \sum_{\{\lambda_2\}} \prod_{\alpha=1}^{N} \overrightarrow{Q}_{2\alpha} |^{\lambda_2\alpha} N_{\lambda_{2\alpha} \lambda_{2\alpha}}(1; q, t) N_{\lambda_{2\alpha-2} \lambda_{2\alpha}}(Q_{2\alpha-1}; q, t).$$ (A.4)

On the other hand, let

$$Z_R := \sum_{\{\lambda_2\}} \prod_{\alpha=1}^{N} C_{\lambda_{2\alpha-1}}^{\lambda_{2\alpha-2}}(q, t) C_{\lambda_{2\alpha-1}}^{\lambda_{2\alpha}}(q, t) \prod_{\alpha=1}^{2N} Q_{\alpha|\lambda_\alpha}$$

$$= \sum_{\{\lambda_2\}} \prod_{\alpha=1}^{N} \sum_{\sigma_{2\alpha-1}} t^{P_{\lambda_{2\alpha-1} \cdot \sigma_{2\alpha-1}}(-q^\rho; t, q)} P_{\lambda_{2\alpha-2} \cdot \sigma_{2\alpha-1}^{-1}}(t^\rho; q, t)$$

$$\times \sum_{\sigma_{2\alpha}} P_{\lambda_{2\alpha} \cdot \sigma_{2\alpha}}(-q^\rho; t, q) t^{P_{\lambda_{2\alpha-1} \cdot \sigma_{2\alpha}}(t^\rho; q, t)} \prod_{\alpha=1}^{N} t^{\sigma_{2\alpha-1} \cdot |\sigma_{2\alpha}|} \prod_{\alpha=1}^{2N} Q_{\alpha|\lambda_\alpha}. \quad \text{(A.5)}$$
with \( \lambda_0 = \lambda_{2N} \) and \( \sigma_0 = \sigma_{2N} \). The following trace formula is useful for calculating \( Z_R \).

**Proposition.** For \( N \in \mathbb{N} \), let \( x^\alpha = x^{\alpha+2N} \)’s be sets of variables, \( \lambda_\alpha \)'s be the Young diagrams, \( \lambda_0 = \lambda_{2N} \), \( c_{\alpha,\alpha+1} = c_{\alpha+2N,\alpha+1+2N} \in \mathbb{C} \), \( c_{\alpha,\beta} := \prod_{\frac{\alpha}{n=\alpha}}^{\frac{\beta-1}{n=\alpha}} c_{\alpha,\alpha+1} \) and \( c := c_{1,1+2N} = \prod_{\alpha=1}^{2N} c_{\alpha,\alpha+1} \). If \( |c| < 1 \), then

\[
\sum_{\{\lambda_1, \lambda_2, \ldots, \lambda_{2N}\}} \frac{1}{N!} P_{\lambda_2-2/\lambda_2-1}(x^{2\alpha-1}; q, t) P_{\lambda_2+1/\lambda_2+1}(x^{2\alpha}; t, q) \cdot \prod_{\alpha=1}^{2N} \left| c_{\lambda_\alpha} \right| \prod_{\alpha=1}^{2N} \prod_{\beta=0}^{N-1} \Pi_0\left( x^{2\alpha}, c_{2\alpha,2\alpha+2\beta+1} x^{2\alpha+2\beta+1} \right) \]
\[
= \exp \left\{ - \sum_{n>0} \frac{1}{n} \frac{1}{1-c^n} \left\{ \sum_{\alpha=1}^{N-1} \sum_{\beta=0}^{N-1} c_{2\alpha,2\alpha+2\beta+1} P_{\alpha,\beta} (x^{2\alpha}) p_{\alpha,\beta} (-x^{2\alpha+2\beta+1}) + c^n \right\} \right\}. \tag{A.6}
\]

Let \( \left( c_{4\alpha-3,4\alpha-2}, c_{4\alpha-1,4\alpha}, c_{4\alpha+1,4\alpha} \right) := (v, Q_{2\alpha-1}, v^{-1}, Q_{2\alpha}) \) with \( v := (q/t)^{\frac{1}{2}} \) and \( (x^{4\alpha-3}, x^{4\alpha-2}, x^{4\alpha-1}, x^{4\alpha}) := (t^\rho, -tq^\rho, t\tau, -q^\rho) \), then we have \( c = \prod_{\alpha=1}^{2N} Q_\alpha \),

\[
c_{2\alpha,2\alpha+2\beta+1} = v^{-1} \frac{1}{\gamma=0} \frac{(-1)^\gamma}{\prod_{\gamma=0}^{\beta}} Q_{\alpha+\gamma}, \tag{A.7}
\]

with \( Q_{\alpha+2N} := Q_\alpha \) and

\[
p_{\alpha,\beta} (x^{2\alpha}) p_{\alpha,\beta} (-x^{2\alpha+2\beta+1}) = \frac{(-1)^\beta}{\left( q^{\frac{\rho}{\tau}} - q^{-\frac{\rho}{\tau}} \right) \left( t^{\frac{\rho}{\tau}} - t^{-\frac{\rho}{\tau}} \right)} \tag{A.8}
\]

Therefore, from \((A.6)\), it follows that

\[
Z_R = \exp \left\{ - \sum_{n>0} \frac{1}{n} \frac{1}{1-c^n} \left\{ \sum_{\alpha=1}^{N-1} \sum_{\beta=0}^{N-1} \frac{g(\{Q_\alpha\}; q^n, t^n)}{\left( t^{\frac{\rho}{\tau}} - t^{-\frac{\rho}{\tau}} \right) \left( q^{\frac{\rho}{\tau}} - q^{-\frac{\rho}{\tau}} \right)} + c^n \right\} \right\}. \tag{A.9}
\]

with

\[
g(\{Q_\alpha\}; q, t) := \sum_{\alpha=1}^{2N} \sum_{\beta=0}^{2N-1} \left( -1 \right)^\beta v^{(-1)^{\alpha-1} \frac{1}{\gamma=0} \frac{(-1)^\gamma}{\prod_{\gamma=0}^{\beta}} Q_{\alpha+\gamma} \]
\[
= \sum_{\alpha=1}^{2N} v^{(-1)^{\alpha} \prod_{\gamma=0}^{\beta} \tilde{Q}_{\alpha+\gamma} \]
\[
= \sum_{\alpha=1}^{2N} v^{(-1)^{\alpha} \tilde{Q}_{\alpha} (1 - \tilde{Q}_{\alpha+1} (1 - \tilde{Q}_{\alpha+2} (\cdots (1 - \tilde{Q}_{\alpha+2N-1} \cdots))))}. \tag{A.10}
\]

Here \( \tilde{Q}_\alpha := v^{(-1)^{\alpha+1}} Q_\alpha \).
If we separate out the perturbative part

\[ Z_R^{\text{pert}} := Z_R(Q_{2\alpha} = 0) = \exp \left\{ - \sum_{n>0} \frac{1}{n} \left( \sum_{\alpha=1}^{N} Q_{2\alpha-1}^n \right) \right\}, \quad (A.11) \]

then \( Z_R^{\text{inst}} := Z_R/Z_R^{\text{pert}} \) is

\[ Z_R^{\text{inst}} = \exp \left\{ - \sum_{n>0} \frac{1}{n} \left( \frac{g^{\text{inst}}(\{Q_{\alpha}\}; q^n, t^n)}{t_k^{\frac{1}{r}} - t^{-\frac{1}{r}}} (q^\lambda - q^{-\lambda}) - c^n \right) \right\}, \quad (A.12) \]

with

\[ g^{\text{inst}}(\{Q_{\alpha}\}; q, t) := \sum_{\alpha=1}^{N} \left( v \sum_{\beta=0}^{2N-1} (-1)^{\beta} \prod_{\gamma=0}^{\beta} \tilde{Q}_{2\alpha+\gamma} + v^{-1} \sum_{\beta=1}^{2N} (-1)^{\beta} \prod_{\gamma=0}^{\beta} \tilde{Q}_{2\alpha-1+\gamma} \right) \]

\[ = \sum_{\alpha=1}^{N} (v - v^{-1} \tilde{Q}_{2\alpha-1}) \tilde{Q}_{2\alpha} (1 - \tilde{Q}_{2\alpha+1}(1 - \tilde{Q}_{2\alpha+2}(1 - \tilde{Q}_{2\alpha+3} \cdots (1 - \tilde{Q}_{2\alpha+2N-1} \cdots)))). \quad (A.13) \]

Since \( Z_L^{\text{pert}} = Z_R^{\text{pert}} \), the slice invariance \( Z_L^{\text{inst}} = Z_R^{\text{inst}} \) is equivalent to the following conjecture:

**Conjecture.**

\[ \sum_{\{\lambda_{2\alpha}\}}^{N} \prod_{\alpha=1}^{2N-1} Q_{2\alpha}^{\lambda_{2\alpha}} \frac{N_{\lambda_{2\alpha-1+2}}(Q_{2\alpha+1}; q, t)}{N_{\lambda_{2\alpha-2+2}}(1; q, t)} = \exp \left\{ \sum_{n>0} \frac{1}{n} \left( f(\{Q_{\alpha}\}; q^n, t^n) (1 - t^n)(1 - q^{-n}) + c^n \right) \right\}, \quad (A.14) \]

with \( c := \prod_{\alpha=1}^{2N} Q_{\alpha}, \quad Q_{\alpha+2N} := Q_{\alpha} \) and

\[ f(\{Q_{\alpha}\}; q, t) := \sum_{\alpha=1}^{N} \left( \sum_{\beta=0}^{2N-1} (-1)^{\beta} \prod_{\gamma=0}^{\beta} Q_{2\alpha+\gamma} + \frac{t}{q} \sum_{\beta=1}^{2N} (-1)^{\beta} \prod_{\gamma=0}^{\beta} Q_{2\alpha-1+\gamma} \right) \]

\[ = \sum_{\alpha=1}^{N} \left( 1 - \frac{t}{q} Q_{2\alpha-1} \right) Q_{2\alpha} (1 - Q_{2\alpha+1}(1 - Q_{2\alpha+2}(1 - Q_{2\alpha+3} \cdots (1 - Q_{2\alpha+2N-1} \cdots)))). \quad (A.15) \]

Computer calculations support this conjecture. We have checked this by Maple for \( N \leq 3 \) and \(|\lambda_{2\alpha}| \leq 5\).

For \( N = 1 \), \( Q_1 = Q \) and \( Q_2 = \Lambda \), then \( Z_L \) and \( Z_R \) are equal to those in section 2 and the above conjecture reduces to (2.17).

When \( N = 2 \) and \( Q_3 = Q_4 = 0 \), the conjecture reduces to

\[ \exp \left\{ \sum_{n>0} \frac{1}{n} t^n Q_1^n Q_2^n - q^n Q_2^n \right\} = \sum_{\lambda} Q_{2|^\lambda|} \frac{N_{\lambda}(Q_1; q, t)}{N_{\lambda, \lambda}(1; q, t)} \]
\[
\sum_\lambda Q_2^{\lambda} \frac{\prod_{(i,j)\in\lambda} (1 - Q_1 q^{-\lambda_i+j-1} t^i)}{\prod_{s\in\lambda} (1 - q^{a(s)} t^{\ell(s)+1})(1 - q^{-a(s)} t^{\ell(s)})}. \tag{A.16}
\]

From the specialization formula (2.9) or (A.2), we find that this coincides with the identity of the slice invariance discussed in subsection 5.9 of [11]. It is interesting that the slice independence implies the same condition in spite of the difference of our refinement of the topological vertex and the refined topological vertex proposed in [11]. Thus the above conjecture generalizes that in [11].

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32
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