EXISTENCE OF AN INVARIANT FORM UNDER A LINEAR MAP

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ABSTRACT. Let $\mathbb{F}$ be a field of characteristic different from 2 and $V$ be a vector space over $\mathbb{F}$. Let $J : \alpha \to \alpha^J$ be a fixed involutory automorphism on $\mathbb{F}$. In this paper we answer the following question: given an invertible linear map $T : V \to V$, when does the vector space $V$ admit a $T$-invariant non-degenerate $J$-hermitian, resp. $J$-skew-hermitian, form?

1. Introduction

Let $\mathbb{F}$ be a field of characteristic different from 2. Let $J : \alpha \to \alpha^J$ be a fixed involutory automorphism on $\mathbb{F}$, i.e. $(\alpha + \beta)^J = \alpha^J + \beta^J$, $(\alpha\beta)^J = \alpha^J\beta^J$, $(\alpha^J)^J = \alpha$. If there exists a non-zero $\alpha$ such that $J(\alpha) = \alpha^J = -\alpha$, we say: “$x^J = -x$ has a solution in $\mathbb{F}$”. This is always the case if $J$ is non-trivial. Let $V$ be a finite dimensional vector space over $\mathbb{F}$. In this paper we ask the following question.

**Question 1.** Given an invertible linear map $T : V \to V$, when does the vector space $V$ over $\mathbb{F}$ admit a $T$-invariant non-degenerate $J$-hermitian, resp. $J$-skew-hermitian, form?

We have answered the question in this note. This generalizes earlier work by Gongopadhyay and Kulkarni [GK11] where the authors obtained conditions for an invertible linear map to admit an invariant non-degenerate quadratic and symplectic form assuming that the underlying field is of large characteristic. de Seguins Pazzis [dSP12] extended the work of [GK11] over arbitrary characteristic. The technicalities are slightly different in these works due to the underlying field characteristic. In this work we follow ideas from [GK11].

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}$. Let $f(x) = \sum_{i=0}^{d} a_i x^i$, $a_d = 1$, be a monic polynomial of degree $d$ over $\mathbb{F}$ such that $-1$, $0$, $1$ are not its roots. The dual of $f(x)$ is defined to be the polynomial $f^*(x) = (f(0))^d x^d f'(x^{-1})$, where $f'(x) = \sum_{i=0}^{d} a_i^i x^i$. Thus, $f^*(x) = \frac{1}{a_d^d} \sum_{i=0}^{d} a_{d-i}^i x^i$. In other words, if $\alpha$ in $\mathbb{F}$ is a root of $f(x)$ with multiplicity $k$, then $(\alpha^J)^{-1}$ is a root of $f^*(x)$ with the same multiplicity. The polynomial $f(x)$ is said to be self-dual if $f(x) = f^*(x)$.

Let $T : V \to V$ be a linear transformation. A $T$-invariant subspace is said to be indecomposable with respect to $T$, or simply $T$-indecomposable if it can not be expressed as a direct sum of two proper $T$-invariant subspaces. $V$ can be written as a direct sum $V = \oplus_{i=1}^{m} V_i$, where each $V_i$ is $T$-indecomposable for $i = 1, 2, \ldots, m$. In general, this decomposition is not canonical. But for each $i$, $(V_i, T|_{V_i})$ is “dynamically equivalent” to $(\mathbb{F}[x]/(p(x)^k), \mu_x)$, where $p(x)$ is an irreducible monic factor of the minimal polynomial of $T$, and $\mu_x$ is the operator $[u(x)] \mapsto [xu(x)]$, for eg. see [Kul08]. Such $p(x)^k$ is an elementary divisor of $T$. If $p(x)^k$ occurs $d$ times in the decomposition, we call $d$ the multiplicity of the elementary divisor $p(x)^k$.

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\end{itemize}
Let $\chi_T(x)$ denote the characteristic polynomial of an invertible linear map $T$. Let

$$\chi_T(x) = (x - 1)^e(x + 1)^f \chi_{oT}(x),$$

where $e, f \geq 0$, and $\chi_{oT}(x)$ has no root $1$, or $-1$. The vector space $\mathbb{V}$ has a $T$-invariant decomposition $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_o$, where for $\lambda = 1, -1$, $\mathbb{V}_\lambda$ is the generalized eigenspace to $\lambda$, i.e.

$$\mathbb{V}_\lambda = \{ v \in \mathbb{V} \mid (T - \lambda I)^n v = 0 \},$$

and $\mathbb{V}_o = \ker \chi_{oT}(T)$. Let $T_o$ denote the restriction of $T$ to $\mathbb{V}_o$. Clearly $T_o$ has the characteristic polynomial $\chi_{oT}(x)$ and does not have any eigenvalue $1$ or $-1$.

With the notations as given above, we prove the following theorem that answers the above question. When $J$ is the trivial automorphism, the following theorem descends to Theorem 1.1 of [GK11] in view of Lemma 2.1 in section 2.

**Theorem 1.1.** Let $\mathbb{F}$ be a field with characteristic different from two. Let $J : \alpha \to \alpha^J$ be a fixed non-trivial involutory automorphism on $\mathbb{F}$. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ of dimension at least $2$. Let $T : \mathbb{V} \to \mathbb{V}$ be an invertible linear map. Then $\mathbb{V}$ admits a $T$-invariant non-degenerate $J$-hermitian, resp. $J$-skew-hermitian form if and only if an elementary divisor of $T_o$ is either self-dual, or its dual is also an elementary divisor with the same multiplicity.

We prove this theorem in the next section.

Suppose $T : \mathbb{V} \to \mathbb{V}$ is a linear map that admits an invariant hermitian or skew-hermitian form $H$. Then canonical forms for $T$ in known in the literature, for example, see [Wal63]. This provides the necessary condition in the above theorem. The focus of this article is the converse part, i.e. the sufficient conditions for a linear map $T$ to admit a non-degenerate hermitian form.

After finishing this work, we found the papers by Sergeichuk [Ser87, Theorem 5], [Ser08, Theorem 2.2], where the author has obtained canonical forms for the pairs $(A, B)$, where $B$ is a non-degenerate form and $A$ is an isometry of $B$ over a field of characteristic not $2$. The work of Sergeichuk not only gives the necessary condition stated above, but the sufficient condition is also implicit there. However, it has not been stated in a precise form as in the above theorem. Sergeichuk’s approach involves quivers to derive the results. Our approach is simpler here.

2. Proof of Theorem 1.1

For a matrix $A = (a_{ij})$, let $A^J$ be the matrix $A^J = (a^J_{ij})$. Let $H$ be a $J$-sesquilinear form on $\mathbb{V}$. The form $H$ is $J$-hermitian or simply hermitian, resp. skew-hermitian if for all $u, v$ in $\mathbb{V}$, $H(u, v) = H(v, u)^J$, resp. $H(u, v) = -H(v, u)^J$. For a linear map $T : \mathbb{V} \to \mathbb{V}$, we say $H$ is $T$-invariant or $T$ is an isometry of $H$ if for all $u, v \in \mathbb{V}$, $H(Tu, Tv) = H(u, v)$.

**Lemma 2.1.** Let $T : \mathbb{V} \to \mathbb{V}$ be a unipotent linear map with minimal polynomial $(x - 1)^n$, $n \geq 2$. Let $\mathbb{V}$ be $T$-indecomposable.

(i) If $n$ is even, resp. odd, then $\mathbb{V}$ admits a $T$-invariant skew-hermitian, resp. hermitian form.

(ii) If $n$ is even, resp. odd, and $x^J = -x$ has a non-zero solution in $\mathbb{F}$, then $\mathbb{V}$ admits a $T$-invariant hermitian, resp. skew-hermitian form.
Proof. Let $T$ be an unipotent linear map. Suppose the minimal polynomial of $T$ is $m_T(x) = (x - 1)^n$. Without loss of generality we can assume that $T$ is of the form

$$T = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 1
\end{pmatrix}$$

(2.0.1)

Suppose $T$ preserves a $J$-sesquilinear form $H$. In matrix form, let $H = (a_{ij})$. Then, $(T^J)^t HT = H$. This gives the following relations: For $1 \leq i \leq n - 1$,

$$a_{i+1,n} = 0 = a_{n,i+1},$$

(2.0.2)

$$a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} = a_{i,j},$$

(2.0.3)

i.e. $a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} = 0$.

From the above two equations we have, for $1 \leq l \leq n - 3$ and $l + 2 \leq i \leq n - 1$,

$$a_{i,n-l} = 0 = a_{n-l,i}.$$

(2.0.5)

This implies that $H$ is a matrix of the form

$$H = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \ldots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \ldots & a_{2,n-2} & a_{2,n-1} & 0 \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \ldots & a_{3,n-2} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{n-1,1} & a_{n-1,2} & 0 & 0 & \ldots & 0 & 0 & 0 \\
a_{n,1} & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix},$$

(2.0.6)

where,

$$a_{i+1,j} + a_{i,j+1} + a_{i+1,j+1} = 0.$$

Choose a basis $e_1, e_2, e_3, \ldots, e_n$ of $V$ such that $T$ and $H$ has the above forms with respect to the basis. From (2.0.4) we have

$$a_{i,n-l+1} = (-1)^{n+1-2l} a_{n-l+1,i}.$$  

(2.0.7)

Note that it follows from (2.0.4) that $a_{i,n-l+1} = (-1)^{l-1} a_{1,n}$ for $1 \leq l \leq n$. Hence $H$ is non-singular if and only if $a_{1,n} \neq 0$. Continuing the procedure, all entries of $H$ except $a_{1,1}$ can be expressed as a scalar multiple of $a_{1,n}$.

For $H$ to be hermitian, from (2.0.7) we must have $a_{i,n-l+1}^t = a_{n-l+1,i} = (-1)^{n+1} a_{l,n-l+1}$. This implies, $a_{1,n}^t = (-1)^{n+1} a_{1,n}$. So, a non-zero choice of $a_{1,n}$ is possible only if either $n$ is odd or, in case $n$ is even, then $x^f = -x$ must have a solution in $\mathbb{F}$, which is always the case. The other case is similar. □

Corollary 2.2. Let $\mathbb{F}$ be a field of characteristic different from 2 such that $x^f = -x$ has a solution in $\mathbb{F}$. Let $V$ be an $n$-dimensional vector space of dimension $\geq 2$ over $\mathbb{F}$. Let $T : V \to V$ be a unipotent linear map such that $V$ is $T$-indecomposable. Then $V$ admits a $T$-invariant non-degenerate hermitian, as well as skew-hermitian form.
Lemma 2.3. Let $T : \mathbb{V} \to \mathbb{V}$ be an invertible linear map. If $T$ has no eigenvalue 1 or $-1$, then there exists a $T$-invariant non-degenerate hermitian form on $\mathbb{V}$ if and only if there exists a $T$-invariant non-degenerate skew-hermitian form on $\mathbb{V}$.

Proof. Assume, $H$ is a $T$-invariant hermitian form. Define a form $H_T$ on $\mathbb{V}$ as follows:

$$H_T(u,v) = H((T - T^{-1})u,v).$$

Note that

$$H_T(u,v) = H(Tu,v) - H(T^{-1}u,v) = H(u,Tv) - H(u,Tv),$$

since $T$ is an isometry.

$$= H(u,T^{-1}v - Tv) = -H(u,(T - T^{-1})v) = -H_T^I((T - T^{-1})v,u),$$

since $H$ is hermitian.

$$= -H_T^I(v,u).$$

Thus $H_T$ is a $T$-invariant non-degenerate skew-hermitian form on $\mathbb{V}$. Also it follows by the same construction that corresponding to each $T$-invariant skew-hermitian form, there is a canonical $T$-invariant hermitian form.\[\square\]

Lemma 2.4. Let $T : \mathbb{V} \to \mathbb{V}$ be an invertible linear map with characteristic polynomial $\chi_T(x) = p(x)^d$, where $p(x) \neq x \pm 1$, is irreducible over $\mathbb{F}$, and is self-dual. Let $\mathbb{V}$ be $T$-indecomposable. Then there exists a $T$-invariant non-degenerate hermitian, resp. skew-hermitian form on $\mathbb{V}$.

Proof. Since $\mathbb{V}$ is $T$-indecomposable, $(\mathbb{V}, T)$ is dynamically equivalent to the pair $(\mathbb{F}[x]/(p(x)^d), \mu_x)$, where $\mu_x$ is the operator $\mu_x[u(x)] \mapsto [xu(x)]$, cf. [Kul08]. Hence without loss of generality we assume $\mathbb{V} = \mathbb{F}[x]/(p(x)^d)$, $T = \mu_x$ and let

$$\mathcal{B} = \{e_1 = 1, e_2 = x, \ldots, e_k = x^{k-1}\},$$

be the corresponding basis. Let $\chi_T(x) = p(x)^d = \sum_{i=0}^{k} c_i x^i$. Since $\chi_T(x)$ is self-dual, we must have $c_i = \frac{c_{k+1-i}}{c_0}$, $1 \leq i \leq k - 1$, $c_0 c_k^J = 1$. If possible, suppose $H = (h_{ij})$ be a $T$-invariant sesquilinear form on $\mathbb{V}$. Then

$$h_{ij} = H(e_i, e_j) = H(\mu_x e_i, \mu_x e_j) = h_{i+1, j+1}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq k.$$}

Hence, a possible $T$-invariant sesquilinear form should be represented necessarily by a matrix of the following form:

$$(2.0.8) \quad X = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\beta_2 & \alpha_1 & \alpha_2 & \ldots & \alpha_{k-2} & \alpha_{k-1} \\
\beta_3 & \beta_2 & \alpha_1 & \ldots & \alpha_{k-3} & \alpha_{k-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{k-1} & \beta_{k-2} & \beta_{k-3} & \ldots & \alpha_1 & \alpha_2 \\
\beta_k & \beta_{k-1} & \beta_{k-2} & \ldots & \beta_2 & \alpha_1
\end{pmatrix}.$$}

Let

$$S = \{A = (a_{ij}) \in M_k(\mathbb{F}) \mid (T^J)^t AT = A\}.$$
Thus $T$ has an invariant form if and only if $S \neq \phi$.

Let $C$ be the companion matrix of $\mu_x$ given by:
\[
C = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & -c_0 \\
1 & 0 & 0 & \ldots & 0 & -c_1 \\
0 & 1 & 0 & \ldots & 0 & -c_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -c_{k-1}
\end{pmatrix}.
\]

Let
\[
H_1 = \begin{pmatrix}
\beta_1 & 0 & 0 & \ldots & 0 & a_1 \\
\beta_2 & \beta_1 & 0 & \ldots & 0 & a_2 \\
\beta_3 & \beta_2 & \beta_1 & \ldots & 0 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{k-1} & \beta_{k-2} & \beta_{k-3} & \ldots & \beta_1 & a_{k-1} \\
\beta_k & \beta_{k-1} & \beta_{k-2} & \ldots & \beta_2 & \beta_1
\end{pmatrix}.
\]

We consider the equation $(C^J)^t H_1 C = H_1$. Simplifying the left hand side, we get
\[
(C^J)^t H_1 C = \sum_{i=0}^{k-1} c_i \begin{pmatrix}
\beta_i & 0 & 0 & \ldots & 0 & a_1 \\
\beta_i & \beta_i & 0 & \ldots & 0 & a_2 \\
\beta_i & \beta_i & \beta_i & \ldots & 0 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_i & \beta_i & \beta_i & \ldots & \beta_i & a_{k-1} \\
\beta_i & \beta_i & \beta_i & \ldots & \beta_i & \beta_i
\end{pmatrix},
\]

where
\[
a_i = -c_0 \beta_{i+1} - c_1 \beta_i - c_2 \beta_{i-1} - \ldots - c_{i-1} \beta_1,
\]
\[
b_i = -c_{k-1} \beta_i - c_{k-2} \beta_{i-1} - \ldots - c_{k-i} \beta_1.
\]

Comparing both sides of $(C^J)^t H_1 C = H_1$ gives
\[
(2.0.9) \quad \beta_{i+1} = \frac{1}{c_0} (-c_1 \beta_i - c_2 \beta_{i-1} - \ldots - c_{i-1} \beta_1), \quad 1 \leq i \leq k - 2.
\]

Now, by back substitution it is easy to see that all $\beta_i$ can be expressed as a multiple of $\beta_1$ by an expression in $\frac{c_1}{c_0}, \ldots, \frac{c_{k-1}}{c_0}$. Next, consider
\[
H_2 = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{k-2} & \alpha_{k-1} \\
0 & 0 & \alpha_1 & \ldots & \alpha_{k-3} & \alpha_{k-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \alpha_1 & \alpha_2 \\
0 & 0 & 0 & \ldots & 0 & \alpha_1
\end{pmatrix}.
\]

Comparing both sides of the equation $(C^J)^t H_2 C = H_2$ we get
\[
(2.0.10) \quad c_0' \alpha_{i+1} = -c_1' \alpha_i - c_2' \alpha_{i-1} - \ldots - c_{i-1}' \alpha_1, \quad 1 \leq i \leq k - 1.
\]

By back substitution it is easy to see that each $\alpha_i$ can be expressed as a multiple of $\alpha_1$ by an expression in $\frac{c_1'}{c_0'}, \ldots, \frac{c_{k-1}'}{c_0'}$. 
Thus $H_1$ and $H_2$ are elements from the set $S$ and for $\beta_1 \neq 0$, resp. $\alpha_1 \neq 0$, they give non-degenerate sesquilinear forms. We also see that $H = H_1 + H_2$ is an element in the set $S$ and is of the form (2.0.8). If we choose $\beta_1 = \alpha_1^j$, it follows from (2.0.9) and (2.0.10) that $\alpha_{i+1} = \beta_{i+1}^j$, $1 \leq i \leq k - 1$, and hence the form $H$ is hermitian. If we choose $\beta_1 = -\alpha_1^j$, it follows $H$ is skew-hermitian. It can be seen that $H$ may be chosen to be non-degenerate. □

**Remark 2.5.** We would like to clarify a small inaccuracy in the statement of [GK11, Lemma 3.2(i)]. It has been stated there that for $T$ unipotent and $(V, B)$ a $T$-indecomposable bilinear space, the bilinear form $B$ degenerate implies $B = 0$. This statement needs slight modification. For example, if $V$ is of dimension 2, $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $V$ is $T$-indecomposable, preserves $B$ which is degenerate, but $B \neq 0$. Following similar computations as in the proof of Lemma 2.4, we can modify the statement as follows: with the hypothesis as in Lemma 3.2 of [GK11], $B$ degenerate implies that the radical of $B$ is a co-dimension one subspace of $V$. What had been used in the relevant parts of [GK11], especially in the proof of Theorem 1.3., is actually the following lemma.

**Lemma.** Let $V$ be a vector space equipped with a non-degenerate symmetric or skew-symmetric bilinear form $B$ over a field $F$ of large characteristic. Suppose $T : V \rightarrow V$ is a unipotent isometry. Let $W$ be a $T$-indecomposable subspace of $V$. Then either $B|_W = 0$ or, $B|_W$ is non-degenerate.

The proof of the lemma is a slight modification of the proof of Lemma 2.2(i) in [Gon10].

### 2.1. Proof of Theorem 1.1

**Proof.** Suppose that the linear map $T$ admits an invariant non-degenerate hermitian, resp skew-hermitian, form $H$. Then the necessary condition follows from existing literatures, for example see [Wal63, Ser87, Ser08].

Conversely, let $V$ be a vector space of dim $n \geq 2$ over the field $F$ and $T : V \rightarrow V$ an invertible map such that the an elementary divisor of $T$, is either self-dual or, its dual is also an elementary divisor. For an elementary divisor $h(x)$, let $V_h$ denote the $T$-indecomposable subspace isomorphic to $F[x]/(h(x))$. From the structure theory of linear maps, for eg. see [Jac75, Kul08], it follows that $V$ has a primary decomposition of the form

$$V = \bigoplus_{i=1}^{m_1} V_{f_i} \oplus \bigoplus_{j=1}^{m_2} (V_{g_i} \oplus V_{g_i}^*),$$

(2.1.1)

where for each $i = 1, 2, ..., m_1$, $f_i(x)$ is either self-dual, or one of $(x+1)^k$ and $(x-1)^k$, for each $j = 1, 2, ..., m_2$, $g_i(x)$, $g_i^*(x)$ are dual to each other and $g_i(x) \neq g_i^*(x)$. To prove the theorem it is sufficient to induce a $T$-invariant hermitian (resp. skew-hermitian) form on each of the summands.

Let $W$ be an $T$-indecomposable summand in the above decomposition and let $p(x)k$ be the corresponding elementary divisor. Suppose $p(x)k$ is self-dual. It follows from Lemma 2.4 that there exists a $T$-invariant non-degenerate hermitian, as well as skew-hermitian form on $W$.
Suppose $p(x)^k$ is not self-dual. Then there is a dual elementary divisor $p^*(x)^k$. $W_p = \ker p(T)^k$, $W_{p^*} = \ker p^*(T)^k$. Then $W_{p^*}$ can be considered as dual to $W_p$ and the dual pairing gives a $T$-invariant non-degenerate form $h$ on $W_p \oplus W_{p^*}$, where $h|_{W_p} = 0 = h|_{W_{p^*}}$.

Suppose, $p(x)^k = (x - 1)^k$. Then the respective forms are obtained from Lemma 2.1. Suppose $p(x)^k = (x + 1)^k$. Let $T_w$ denote the restriction of $T$ to $W$. Then the minimal polynomial of $T_w$ is $(x + 1)^k$. Thus the minimal polynomial of $-T_w$ is $(x - 1)^k$. Further $T_w$ preserves a hermitian (resp. skew-hermitian) form if and only if $-T_w$ also preserves it. Thus this case reduces to the previous case and the existence of the required forms are clear.

This completes the proof. □

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