Finite rank point perturbations of the \( p \)-adic fractional differentiation operator \( D^\alpha \) are studied. The main attention is paid to the description of operator realizations (in \( L^2(\mathbb{Q}_p) \)) of the heuristic expression \( D^\alpha + \sum_{i,j=1}^{n} b_{ij} < \delta_{x_j}, \cdot > \delta_{x_i} \), in a form that is maximally adapted for the preservation of physically meaningful relations to the parameters \( b_{ij} \) of the singular potential.

1. Introduction

The conventional description of the physical space-time uses the field \( \mathbb{R} \) of real numbers. In most cases, mathematical models based on \( \mathbb{R} \) provide quite satisfactory descriptions of the physical reality. However, the result of a physical measurement is always a rational number, so the use of the completion \( \mathbb{R} \) of the field of rational numbers \( \mathbb{Q} \) is not more than a mathematical idealization. On the other hand, by Ostrovidovski’s theorem, the only reasonable alternative to \( \mathbb{R} \) among completions of \( \mathbb{Q} \) is the fields \( \mathbb{Q}_p \) of \( p \)-adic numbers (definition of \( \mathbb{Q}_p \) see below in Section 2). For this reason, it is natural to use \( p \)-adic analysis in physical situations, where the conventional space-time geometry is known to fail, for examples in the attempts to understand the matter at sub-Planck distances or time intervals. In order to do this, at first, it is necessary to develop \( p \)-adic counterparts of the standard quantum mechanics and quantum field theory.

There are many works devoted to such an activity (see the surveys in \cite{13}, \cite{17}). However, in spite of considerable success obtained in recent years, many interesting problems of \( p \)-adic quantum mechanics are still unsolved and wait for a comprehensive study.

In the present paper, we are going to continue the investigation of the \( p \)-adic fractional differentiation operator with point interactions started by A. Kochubei \cite{12}, \cite{13}.

In ‘usual’ mathematical physics, point interactions Hamiltonians are the operator realizations in \( L^2(\mathbb{R}) \) of differential expressions \( -\Delta + V_Y \) or, more generally, \( (-\Delta)^k + V_Y \), where a zero-range potential \( V_Y = \sum_{i,j=1}^{n} b_{ij} < \delta_{x_j}, \cdot > \delta_{x_i} \) (\( b_{ij} \in \mathbb{C} \)) contains the Dirac delta functions \( \delta_{x} \) concentrated on points \( x_i \) of the subset \( Y = \{x_1, \ldots, x_n\} \subset \mathbb{R}^r \) \cite{1}.

Since there exists a \( p \)-adic analysis based on the mappings from \( \mathbb{Q}_p \) into \( \mathbb{Q}_p \), and an analysis connected with the mapping \( \mathbb{Q}_p \) into the field of complex numbers \( \mathbb{C} \), there exists two types of \( p \)-adic physical models. The present paper deals with the mapping \( \mathbb{Q}_p \rightarrow \mathbb{C} \), i.e., complex-valued functions defined on \( \mathbb{Q}_p \) will be considered. In this case, the operation of differentiation is not defined and the operator of fractional differentiation \( D^\alpha \) of order \( \alpha (\alpha > 0) \) plays a corresponding role \cite{13}, \cite{17}. In particular, \( p \)-adic Schrödinger-type operators with potentials \( V(x) : \mathbb{Q}_p \rightarrow \mathbb{C} \) are defined as \( D^\alpha + V(x) \) \cite{13}.

The definition of \( D^\alpha \) is given in the framework of the \( p \)-adic distribution theory with the help of Schwartz-type distributions \( D'(\mathbb{Q}_p) \). One of remarkable features of this theory

\footnotesize
\[ \text{2000 Mathematics Subject Classification. Primary 47A10, 47A55; Secondary 81Q10.} \]
\[ \text{Key words and phrases. \( p \)-adic analysis, fractional differentiation operator, point interactions.} \]
\[ \text{The authors thank DFG (project 436 UKR 113/88/0-1) and DFFD (project 10.01/004) for the support.} \]

\[ 1 \]
is that any distribution \( f \in \mathcal{D}'(\mathbb{Q}_p) \) with point support \( \text{supp} f = \{ x \} \) \( (x \in \mathbb{Q}_p) \) coincides with the Dirac delta function at the point \( x \) multiplied by a constant \( c \in \mathbb{C} \), i.e., \( f = c \delta_x \). For this reason, it is natural to consider the expression \( D^\alpha + V_Y \) as a \( p \)-adic analogue of Hamiltonians with finite rank point interactions.

In the present paper, the main attention is paid to the description of operator realizations of \( D^\alpha + V_Y \) in \( L_2(\mathbb{Q}_p) \) in a form that is maximally adapted for the preservation of physically meaningful relations to the parameters \( b_{ij} \) of the singular potential \( V_Y = \sum_{i,j=1}^n b_{ij} < \delta_{x_j}, > > \delta_{x_i} \).

In Section 2, we recall some elements of \( p \)-adic analysis \cite{17,13} needed for reading the paper and establish the connection between \( \alpha \) and the property of functions from \( \mathcal{D}(D^\alpha) \) to be continuous. The same problem is also analyzed for the solutions of \( D^\alpha + I = \delta \).

Section 3 contains the description of the Friedrichs extension of the symmetric operator associated with \( D^\alpha + V_Y \) (this description depends on \( \alpha \)) and the description of operator realizations of \( D^\alpha + V_Y \) in \( L_2(\mathbb{Q}_p) \). Taking into account an intensive development of consistent physical theories of quantum mechanics on the base of pseudo-Hermitian Hamiltonians that are not Hermitian in the standard sense but satisfy a less restrictive and more physical condition of symmetry in last few years \cite{6,16}, we do not restrict ourselves to the case of self-adjoint operators and consider the more general case of \( \eta \)-self-adjoint operator realizations of \( D^\alpha + V_Y \) (Theorem 3.1).

We use the following notations: \( \mathcal{D}(A) \) and \( \ker A \) denote the domain and the null-space of a linear operator \( A \), respectively. \( A |_X \) means the restriction of \( A \) onto a set \( X \).

### 2. Fractional Differential Operator \( D^\alpha \)

#### 2.1. Elements of \( p \)-adic analysis

Basically, we shall use notations from \cite{17}. Let us fix a prime number \( p \). The field \( \mathbb{Q}_p \) of \( p \)-adic numbers is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to \( p \)-adic norm \( | \cdot |_p \), which is defined as follows: \( |0|_p = 0 \); \( |x|_p = p^{-\gamma} \) if an arbitrary rational number \( x \neq 0 \) is represented as \( x = p^\gamma \frac{m}{n} \), where \( \gamma = \gamma(x) \in \mathbb{Z} \) and integers \( m \) and \( n \) are not divisible by \( p \). The \( p \)-adic norm \( | \cdot |_p \) satisfies the strong triangle inequality \( |x + y|_p \leq \max(|x|_p, |y|_p) \). Moreover, \( |x + y|_p = \max(|x|_p, |y|_p) \) if \( |x|_p \neq |y|_p \).

Any \( p \)-adic number \( x \neq 0 \) can be uniquely presented as series

\begin{equation}
\sum_{i=0}^{+\infty} x_i p^i, \quad x_i = 0, 1, \ldots, p - 1, \quad x_0 > 0, \quad \gamma(x) \in \mathbb{Z}
\end{equation}

convergent in \( p \)-adic norm (the canonical presentation of \( x \)). In this case, \( |x|_p = p^{-\gamma(x)} \).

The canonical presentation (2.1) enables one to determine the fractional part \( \{ x \}_p \) of \( x \in \mathbb{Q}_p \) by the rule: \( \{ x \}_p = 0 \) if \( x = 0 \) or \( \gamma(x) \geq 0 \); \( \{ x \}_p = p^{\gamma(x)} \sum_{i=0}^{-\gamma(x)-1} x_i p^i \) if \( \gamma(x) < 0 \).

Denote by

\[ B_{\gamma}(a) = \{ x \in \mathbb{Q}_p \mid |x - a|_p \leq p^\gamma \} \quad \text{and} \quad S_{\gamma}(a) = \{ x \in \mathbb{Q}_p \mid |x - a|_p = p^\gamma \}, \]

respectively, the ball and the sphere of radius \( p^\gamma \) with the center at a point \( a \in \mathbb{Q}_p \) and set \( B_1(0) = B_1, S_1(0) = S_1, \gamma \in \mathbb{Z} \).

The ring \( \mathbb{Z}_p \) of \( p \)-adic integers coincides with the disc \( B_0 (\mathbb{Z}_p = B_0) \), which is the completion of integers with respect to the \( p \)-adic norm \( | \cdot |_p \).

As usual, in order to define some classes of distributions on \( \mathbb{Q}_p \), one has first to introduce an appropriate class of test functions.

A complex-valued function \( f \) defined on \( \mathbb{Q}_p \) is called \textit{locally-constant} if for any \( x \in \mathbb{Q}_p \) there exists an integer \( l(x) \) such that \( f(x + x') = f(x), \; \forall x' \in B_l(x) \).

Denote by \( \mathcal{D}(\mathbb{Q}_p) \) the linear space of locally constant functions on \( \mathbb{Q}_p \) with compact supports. For any test function \( \phi \in \mathcal{D}(\mathbb{Q}_p) \) there exists \( l \in \mathbb{Z} \) such that \( \phi(x + x') = \phi(x) \),
$x' \in B_l$, $x \in \mathbb{Q}_p$. The largest of such numbers $l = l(\phi)$ is called the parameter of constancy of $\phi$. Typical examples of test functions are indicator functions of spheres and balls:

$$
\delta(|x|_p - p^\gamma) := \begin{cases}
1, & x \in S_\gamma, \\
0, & x \notin S_\gamma,
\end{cases} \quad \Omega(|x|_p) := \begin{cases}
1, & |x|_p \leq 1, \\
0, & |x|_p > 1.
\end{cases}
$$

In order to furnish $D(\mathbb{Q}_p)$ with a topology, let us consider a subspace $D^l_\gamma \subset D(\mathbb{Q}_p)$ consisting of functions with supports in the ball $B_\gamma$ and the parameter of constancy $\geq l$. The convergence $\phi_n \to 0$ in $D(\mathbb{Q}_p)$ has the following meaning: $\phi_k \in D^l_\gamma$, where the indices $l$ and $\gamma$ do not depend on $k$ and $\phi_k$ tends uniformly to zero. This convergence determines the Schwartz topology in $D(\mathbb{Q}_p)$.

Denote by $D'(\mathbb{Q}_p)$ the set of all linear functionals (Schwartz-type distributions) on $D(\mathbb{Q}_p)$. In contrast to distributions on $\mathbb{R}^n$, any linear functional $D(\mathbb{Q}_p) \to \mathbb{C}$ is automatically continuous. The action of a functional $f$ upon a test function $\phi$ will be denoted $\langle f, \phi \rangle$.

It follows from the definition of $D(\mathbb{Q}_p)$ that any test function $\phi \in D(\mathbb{Q}_p)$ is continuous on $\mathbb{Q}_p$. This means the Dirac delta function $\langle \delta_x, \phi \rangle = \phi(x)$ is well posed for any point $x \in \mathbb{Q}_p$.

On $\mathbb{Q}_p$ there exists the Haar measure, i.e., a positive measure $d_p x$ invariant under shifts $d_p(x + a) = d_p x$ and normalized by the equality $\int_{|x|_p \leq 1} d_p x = 1$.

Denote by $L_2(\mathbb{Q}_p)$ the set of measurable functions $f$ on $\mathbb{Q}_p$ satisfying the condition $\int_{\mathbb{Q}_p} |f(x)|^2 d_p x < \infty$. The set $L_2(\mathbb{Q}_p)$ is a Hilbert space with the scalar product

$$
\langle f, g \rangle_{L_2(\mathbb{Q}_p)} = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} d_p x.
$$

The Fourier transform of $\phi \in D(\mathbb{Q}_p)$ is defined by the formula

$$
F[\phi](\xi) = \overline{\phi}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x) \phi(x) d_p x, \quad \xi \in \mathbb{Q}_p,
$$

where $\chi_p(\xi x) = e^{2\pi i \langle \xi x \rangle_p}$ is an additive character of the field $\mathbb{Q}_p$ for any fixed $\xi \in \mathbb{Q}_p$. The Fourier transform $F[\cdot]$ maps $D(\mathbb{Q}_p)$ onto $D(\mathbb{Q}_p)$. Its extension by continuity onto $L_2(\mathbb{Q}_p)$ determines an unitary operator in $L_2(\mathbb{Q}_p)$.

The Fourier transform $F[f]$ of a distribution $f \in D'(\mathbb{Q}_p)$ is defined by the standard relation $\langle f, \phi \rangle = \langle F[f], \phi \rangle$, $\forall \phi \in D(\mathbb{Q}_p)$. The Fourier transform is a linear isomorphism of $D'(\mathbb{Q}_p)$ onto $D'(\mathbb{Q}_p)$.

2.2. The operator $D^\alpha$. The operator of differentiation is not defined in $L_2(\mathbb{Q}_p)$. Its role is played by the operator of fractional differentiation $D^\alpha$ (the Vladimirov pseudo-differential operator) which is defined as

$$
D^\alpha f = \int_{\mathbb{Q}_p} |\xi|^\alpha F[f](\xi) \chi_p(-\xi x) d_p \xi, \quad \alpha > 0.
$$

It is easy to see [13] that $D^\alpha f$ is well defined for $f \in D(\mathbb{Q}_p)$. Note that $D^\alpha f$ need not belong necessarily to $D(\mathbb{Q}_p)$ (since the function $|\xi|^\alpha_p$ is not locally constant), however $D^\alpha f \in L_2(\mathbb{Q}_p)$.

Since $D(\mathbb{Q}_p)$ is not invariant with respect to $D^\alpha$ we cannot define $D^\alpha$ on the whole space $D'(\mathbb{Q}_p)$. For a distribution $f \in D'(\mathbb{Q}_p)$, the operation $D^\alpha$ is well defined only if the right-hand side of (2.3) exists.

In what follows we will consider the operator $D^\alpha$, $\alpha > 0$, as an unbounded operator in $L_2(\mathbb{Q}_p)$. In this case, its domain of definition $D(D^\alpha)$ consists of those $u \in L_2(\mathbb{Q}_p)$ for which $|\xi|^\alpha_p F[u](\xi) \in L_2(\mathbb{Q}_p)$. Since $D^\alpha$ is unitarily equivalent to the operator of multiplication by $|\xi|^\alpha_p$, it is a positive self-adjoint operator in $L_2(\mathbb{Q}_p)$, its spectrum consists of the eigenvalues $\lambda_\gamma = p^{\alpha \gamma}$ ($\gamma \in \mathbb{Z}$) of infinite multiplicity, and their accumulation point $\lambda = 0$. 
It is easy to see from (2.3) that an arbitrary (normalized) eigenfunction $\psi$ of $D^\alpha$ corresponding to the eigenvalue $\lambda_\gamma = p^{\alpha \gamma}$ admits the description

$$
\tilde{\psi}(\xi) = \delta(\xi|_p - p^{\gamma}) \rho(\xi), \quad \int_{S_\gamma} |\rho(\xi)|^2 d_p \xi = 1,
$$

where the function $\rho(\xi)$ defined on the sphere $S_\gamma$ serves as a parameter of the description. Choosing $\rho(\xi)$ in different ways one can obtain various orthonormal bases in $L_2(\mathbb{Q}_p)$ formed by eigenfunctions of $D^\alpha$ [13, 14, 17]. In particular, the choice of $\rho(\xi)$ as a system of locally constant functions on $S_\gamma$ leads to the well-known Vladimirov functions [13, 17]. The selection of $\rho(\xi)$ as indicators of a special class of subsets of $S_\gamma$ gives the $p$-adic wavelet basis $\{\psi_{N,j}\}$ recently constructed in [14]. Precisely, it was shown [14] that the set of eigenfunctions $D^\alpha$

$$
\psi_{N,j}(x) = p^{-\gamma} \chi(p^{N-1}j) \Omega(|p^N x - \epsilon|_p), \quad N \in \mathbb{Z}, \quad \epsilon \in \mathbb{Q}_p/\mathbb{Z}_p, \quad j = 1, \ldots, p - 1
$$

forms an orthonormal basis in $L_2(\mathbb{Q}_p)$ such that

$$
D^\alpha \psi_{N,j} = p^{\alpha(1-N)} \psi_{N,j}.
$$

Here the indexes $N, j, \epsilon$ serve as parameters of the basis. In particular, elements $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ can be described as $\epsilon = \sum_{i=1}^{m} \epsilon_i p^{-i}$ $(m \in \mathbb{N}, \epsilon_i = 0, \ldots, p - 1)$.

**Theorem 2.1.** An arbitrary function $u \in \mathcal{D}(D^\alpha)$ is continuous on $\mathbb{Q}_p$ if and only if $\alpha > 1/2$.

**Proof.** Let $u \in \mathcal{D}(D^\alpha)$ and let

$$
u(x) = \sum_{N=1}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} (u, \psi_{N,j}) \psi_{N,j}(x) + \sum_{N=-\infty}^{0} \sum_{j=1}^{p-1} \sum_{\epsilon} (u, \psi_{N,j}) \psi_{N,j}(x)
$$

be its expansion into the $p$-adic wavelet basis (2.4).

It is easy to see that $\psi_{N,j}(x) \in \mathcal{D}(\mathbb{Q}_p)$ and hence, the functions $\psi_{N,j}(x)$ are continuous on $\mathbb{Q}_p$. Thus, to prove the continuity of $u(x)$, it suffices to verify that the series in (2.6) converges uniformly.

First of all we remark that for fixed $N$ and $x$ there is at most one $\epsilon$ such that $\psi_{N,j}(x) \neq 0$. Indeed, if there exist $\epsilon_1$ and $\epsilon_2$ such that $\psi_{N,j}(x) \neq 0$, then $\Omega(|p^N x - \epsilon_1|_p) = 1$. But then $|p^N x - \epsilon_1|_p \leq 1$ and $|p^N x - \epsilon_2|_p \leq 1$. By the strong triangle inequality $|\epsilon_1 - \epsilon_2|_p \leq 1$. The latter relation and the condition $\epsilon_1 \in \mathbb{Q}_p/\mathbb{Z}_p$ imply the equality $\epsilon_1 = \epsilon_2$.

Thus, for fixed $N$ and $x$, the sum corresponding to the parameter $\epsilon$ consists of at most one non-zero term.

Further, it follows from (2.4) and (2.6) that

$$
|\psi_{N,j}(x)| \leq p^{-N/2} \quad \text{and} \quad |(u, \psi_{N,j})| \leq \|u\|_{L_2(\mathbb{Q}_p)}.
$$

For a fixed $N > 0$, relations (2.7) ensure the following estimate

$$
\sum_{j=1}^{p-1} \sum_{\epsilon} (u, \psi_{N,j}) \psi_{N,j}(x) \leq p^{-N/2} \|u\|_{L_2(\mathbb{Q}_p)}(p - 1), \quad \forall x \in \mathbb{Q}_p,
$$

which gives the uniform convergency of the first series in (2.6).
The condition \( u \in \mathcal{D}(D^\alpha) \) and (2.3) imply \((u, \psi_{Nj\epsilon}) = p^{\alpha(N-1)}(D^\alpha u, \psi_{Nj\epsilon})\). Using this equality and (2.7), we obtain

\[
\left| \sum_{j=1}^{p-1} \sum_{\epsilon \in \mathbb{F}_p} (u, \psi_{Nj\epsilon}) \psi_{Nj\epsilon}(x) \right| = \left| \sum_{j=1}^{p-1} \sum_{\epsilon \in \mathbb{F}_p} p^{\alpha(N-1)}(D^\alpha u, \psi_{Nj\epsilon}) \psi_{Nj\epsilon}(x) \right| \leq \left( \sum_{j=1}^{p-1} \sum_{\epsilon \in \mathbb{F}_p} |(D^\alpha u, \psi_{Nj\epsilon})|^2 \right)^{1/2} \left( \sum_{j=1}^{p-1} \sum_{\epsilon \in \mathbb{F}_p} p^{2\alpha(N-1)} |\psi_{Nj\epsilon}(x)|^2 \right)^{1/2} \leq \|D^\alpha u\|_{L^2(\mathbb{Q}_p)} \left\{ \sum_{j=1}^{p-1} \|p^{-\alpha(N-1)} \psi_{Nj\epsilon}\| \right\}^{1/2}
\]

The obtained estimate implies that the second series in (2.6) is uniformly convergent for \( \alpha > 1/2 \). Therefore, any function \( u \in \mathcal{D}(D^\alpha) \) is continuous on \( \mathbb{Q}_p \) for \( \alpha > 1/2 \).

In the case \( \alpha \leq 1/2 \), we show that the function

\[
f(x) = \sum_{N=-\infty}^{N=\infty} \frac{1}{|N|^\alpha} p^{(N-1)/2} \psi_{Nj\epsilon}(x).
\]

(determined in \( p \)-adic wavelet basis) belongs to \( \mathcal{D}(D^\alpha) \) but \( f(x) \) is not continuous on \( \mathbb{Q}_p \).

Obviously, \( f \in L^2(\mathbb{Q}_p) \) and its Fourier transform is

\[
\hat{f}(\xi) = \sum_{N=-\infty}^{N=\infty} \frac{1}{|N|^\alpha} p^{(N-1)/2} \hat{\psi}_{Nj\epsilon}(\xi).
\]

By (2.3) and (2.5), \( \xi^\alpha \hat{\psi}_{Nj\epsilon}(\xi) = p^{\alpha(N-1)} \hat{\psi}_{Nj\epsilon}(\xi) \). Hence,

\[
|\xi^\alpha \hat{f}(\xi) = \sum_{N=-\infty}^{N=\infty} \frac{1}{|N|^\alpha} p^{(N-1)/2} \cdot p^{\alpha(N-1)} \hat{\psi}_{Nj\epsilon}(\xi)
\]

and (since \( \{\psi_{Nj\epsilon}(\xi)\}_{N \leq -1} \) is orthonormal) \( |\xi^\alpha \hat{f}(\xi) \in L^2(\mathbb{Q}_p) \) for \( \alpha \leq 1/2 \) and \( |\xi^\alpha \hat{f}(\xi) \notin L^2(\mathbb{Q}_p) \) for \( \alpha > 1/2 \). Hence, \( f(x) \in \mathcal{D}(D^\alpha) \) for \( \alpha \leq 1/2 \) only.

Let us show that \( f(x) \) is not continuous on \( \mathbb{Q}_p \). First of all, using (2.4), we rewrite the definition (2.9) of \( f \) as:

\[
f(x) = \sum_{N=-\infty}^{N=\infty} \frac{1}{|N|^\alpha} p^{-\alpha-N} x \Omega(\chi(\alpha-N) x) p^N x^N.
\]

It is easy to see that the restriction of the left-hand side of (2.10) onto any ball \( B_\epsilon(u) \subset \mathbb{Q}_p \) contains a finite number of non-zero terms. Therefore, \( f(x) \) is continuous on \( \mathbb{Q}_p \) and it is represented by point-wise convergent series (2.9).

Let us consider the sequence \( x_n = p^n, (n \in \mathbb{N}) \). Obviously, \( x_n \to 0, \ (n \to \infty) \) in the \( p \)-adic norm \( |\cdot|_p \). Furthermore, \( \Omega(|p^N x^N|_p) = \Omega(|p^N x|^N) = 0 \) when \( N + n \leq -1 \). On the other hand, if \( N + n \geq 1 \), then \( p^{N-1} x_n \) is an integer \( p \)-adic number and, hence \( \chi(p^{N-1} x_n) = 1 \). Taking these relations into account, we deduce from (2.10) that

\[
f(x_n) = f(p^n) = p^{\alpha-N} x \Omega(\chi(\alpha-N) x) \to \infty \quad \text{as} \quad n \to \infty.
\]

Thus, \( f(x) \) cannot be continuous at \( x = 0 \). Theorem (2.1) is proved.
2.3. Properties of solutions of $\mathcal{D}^{\alpha} + I = \delta$. Let us consider an equation
\begin{equation}
\label{2.11}
\mathcal{D}^{\alpha} h + h = \delta_{x_k}, \quad h \in L_2(\mathbb{Q}_p), \quad x_k \in \mathbb{Q}_p, \quad \alpha > 0,
\end{equation}
where $\mathcal{D}^{\alpha} : L_2(\mathbb{Q}_p) \to \mathcal{D}(\mathbb{Q}_p)$ is understood in the distribution sense.

It is known \cite{19} that Eq. \eqref{2.11} has a unique solution $h = h_k \in L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$ and has no solutions belonging to $L_2(\mathbb{Q}_p)$ for $\alpha \leq 1/2$. The next statement continues the investigation of $h_k$.

**Lemma 2.1.** The solution $h_k$ of \eqref{2.11} is a function continuous on $\mathbb{Q}_p$ when $\alpha > 1$ and continuous on $\mathbb{Q}_p \setminus \{x_k\}$ when $1/2 < \alpha \leq 1$.

**Proof.** Reasoning as in the proof of (\cite[Lemma 3.7]{19}), where the basis of Vladimirov eigenfunctions was used, we establish the expansion of $\delta_{x_k}$ in terms of the $p$-adic wavelet basis.

Let $u \in \mathcal{D}(\mathcal{D}^{\alpha})$. By analogy with the proof of Theorem 2.1 we expand $u$ in an uniformly convergent series with respect to the complex-conjugated $p$-adic wavelet basis $\{\psi_{Njr}\}$. Since $\{\psi_{Njr}\}$ are continuous functions on $\mathbb{Q}_p$, we can write: $u(x_k) = \sum_{N=\infty}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} (u, \psi_{Njr})\psi_{Njr}(x_k) x_k$.

Consider $\psi_{Njr}(x_k) = p^{-N/2} \chi(p^{-1}jx_k)\Omega(|p^N x_k - \epsilon|_p) = p^{-N/2} \chi(-p^{-1}jx_k)\Omega(|p^N x_k - \epsilon|_p)$.

Obviously, $\psi_{Njr}(x_k) \neq 0 \iff |p^N x_k - \epsilon|_p \leq 1$. Here $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ and hence, $|\epsilon|_p > 1$ for $\epsilon \neq 0$. It follows from the strong triangle inequality and the condition $\epsilon \in \mathbb{Q}_p/\mathbb{Z}_p$ that $|p^N x_k - \epsilon|_p \leq 1 \iff \epsilon = \{p^N x_k\}_p$ (if $\epsilon \neq 0$). Moreover, if $\epsilon = 0$, then condition $|p^N x_k|_p \leq 1$ implies $\{p^N x_k\}_p = 0$. Combining these two cases we arrive at the conclusion that
\[
\psi_{Njr}(x_k) = \begin{cases} 0, & \epsilon \neq \{p^N x_k\}_p \\ p^{-N/2} \chi(-p^{-1}jx_k), & \epsilon = \{p^N x_k\}_p. \end{cases}
\]

But then
\begin{equation}
\label{2.12}
\langle \delta_{x_k}, u \rangle = u(x_k) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{-1}jx_k)(u, \psi_{Njr(p^N x_k)}) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{-1}jx_k) \langle \psi_{Njr(p^N x_k)}, u \rangle.
\end{equation}

Since $\mathcal{D}(\mathbb{Q}_p) \subseteq \mathcal{D}(\mathcal{D}^{\alpha})$, the equality \eqref{2.12} means that
\begin{equation}
\label{2.13}
\delta_{x_k} = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{-1}jx_k)\psi_{Njr(p^N x_k)}(x_k),
\end{equation}
where the series converges in $\mathcal{D}'(\mathbb{Q}_p)$.

Suppose that a function $h_k \in L_2(\mathbb{Q}_p)$ is represented as a convergent series in $L_2(\mathbb{Q}_p)$:
\[h_k(x) = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} c_{Njr}\psi_{Njr}(x).\]

Applying the operator $\mathcal{D}^{\alpha} + I$ termwise, we get a series
\begin{equation}
\label{2.14}
\mathcal{D}^{\alpha} h_k + h_k = \sum_{N=-\infty}^{\infty} \sum_{j=1}^{p-1} \sum_{\epsilon} c_{Njr}(1 + p^{\alpha(1-N)})\psi_{Njr},
\end{equation}
converging in $\mathcal{D}'$ (since $D^\alpha \mathcal{D}(\mathbb{Q}_p) \subset L_2(\mathbb{Q}_p)$). Comparing the terms of (2.13) and (2.14) gives
\[
c_{Nj^c} = \begin{cases} 
0, & \epsilon \neq \{p^N x_k\}_p \\
p^{-N/2} \chi(-p^{N-1} j x_k) [p^{\alpha(1-N)} + 1]^{-1}, & \epsilon = \{p^N x_k\}_p
\end{cases}
\]
Thus,
\[
h_k(x) = \sum_{N = -\infty}^{\infty} \sum_{j = 1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) [p^{\alpha(1-N)} + 1]^{-1} \psi_{N j (p^N x_k)}(x).
\]
Let us show that the series (2.15) is uniformly convergent on $\mathbb{Q}_p$ for $\alpha > 1$ and is uniformly convergent on any ball not containing $x_k$ for $1/2 < \alpha \leq 1$.
Indeed, by virtue of (2.7) the general term of (2.15) does not exceed
\[
p^{-N} [p^{\alpha(1-N)} + 1]^{-1} \leq p^{-N}.
\]
Hence, the subseries of (2.15) formed by terms with $N \geq 0$ converges uniformly. For $N < 0$ the general term of (2.15) does not exceed
\[
p^{-N} [p^{\alpha(1-N)} + 1]^{-1} \leq \frac{1}{p^N} p^{-N(1-\alpha)}.
\]
The obtained estimate implies that for $\alpha > 1$ the subseries of (2.15) formed by terms with $N < 0$ also converges uniformly. So, the series (2.15) converges uniformly for $\alpha > 1$. This proves the assertion of Lemma 2.1 for $\alpha > 1$ (since $\psi_{Nj^c}$ are continuous on $\mathbb{Q}_p$).
Let $B_\gamma(a)$ be a ball such that $x_k \notin B_\gamma(a)$. To prove Lemma 2.1 for $1/2 < \alpha \leq 1$ it suffices to verify that the restriction of (2.15) onto $B_\gamma(a)$ contains finite number of terms with negative parameter $N < 0$.
Indeed, it follows from the strong triangle inequality and the definitions of $\{\cdot\}_p$ and $\Omega(\cdot)$ (see (2.2)) that
\[
\Omega(|p^N x - \{p^N x_k\}_p|_p) = \Omega(|p^N x - p^N x_k|_p).
\]
Hence, the restriction of $\psi_{N j (p^N x_k)}(x)$ onto $B_\gamma(a)$ is equal to 0 if $|x - x_k|_p > p^N$ for all $x \in B_\gamma(a)$. Since $|x - x_k|_p > p^N$, the relation $\psi_{N j (p^N x_k)}(x) \equiv 0$ ($\forall x \in B_\gamma(a)$) holds for all $N \geq \gamma$. Using the estimation (2.10) we arrive at the conclusion that the series (2.15) converges uniformly for any ball $B_\gamma(a) \subset \mathbb{Q}_p \setminus \{x_k\}$. Lemma 2.1 is proved.

Remark. The solution $h_k(x)$ of (2.11) constructed in Lemma 2.1 is a real-valued function. This fact can be obtained directly from the expansion (2.15). Another way to establish it is based on the invariance of the space $\mathcal{D}(\mathbb{Q}_p)$ and the operator $D^\alpha$ with respect to the complex conjugation. Combining these properties with the uniqueness of the solution of $D^\alpha + I = \delta_{x_k}$ in $L_2(\mathbb{Q}_p)$, we get $\overline{h_k}(x) = h_k(x)$.

**Corollary 2.1.** Let the index $\alpha > 1/2$ and points $x_1, \ldots, x_n \in \mathbb{Q}_p$ be fixed and let $\text{Sp}\{h_k\}_n^1$ be the linear span of solutions $h_k$ ($1 \leq k \leq n$) of (2.11). Then $\text{Sp}\{h_k\}_n^1 \cap \mathcal{D}(D^{\alpha/2}) = \{0\}$ for $1/2 < \alpha \leq 1$ and $\text{Sp}\{h_k\}_n^1 \subset \mathcal{D}(D^{\alpha/2})$ for $\alpha > 1$.

Proof. The solution $h_k$ of (2.11) is determined by (2.15). Taking the expansion (2.15) and the “semigroup property”
\[
D^\alpha_1 D^\alpha_2 = D^{\alpha_1 + \alpha_2}, \quad \alpha_1, \alpha_2 > 0
\]
of $D^\alpha$ into account, it is easy to see that $h_k \in \mathcal{D}(D^{\alpha/2})$ if and only if the following series converge in $L_2(\mathbb{Q}_p)$:
\[
\sum_{N = 1}^{\infty} \sum_{j = 1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) [p^{\alpha(1-N)} + 1]^{-1} p^{\frac{\alpha}{2}(1-N)} \psi_{N j (p^N x_k)} + \\
\sum_{N = -\infty}^{0} \sum_{j = 1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) [p^{\alpha(1-N)} + 1]^{-1} p^{\frac{\alpha}{2}(1-N)} \psi_{N j (p^N x_k)}
\]
(if the limit exists then it coincides with $D^{\alpha/2}h_k$). For the general term of the first series we have
\[
\left|p^{-N/2}p^{\frac{1}{2}(1-N)}\chi(-p^{N-1}jx)\left[p^{\alpha(1-N)}+1\right]^{-1}\right|^2 \leq p^{-N(\alpha+1)+\alpha}, \quad N \geq 1
\]
that implies its convergence in $L_2(\mathbb{Q}_p)$ for any $\alpha > 1/2$.

Similarly, the general term of the second series can be estimated as follows:
\[
\left|p^{-N/2}p^{\frac{1}{2}(1-N)}\chi(-p^{N-1}jx)\left[p^{\alpha(1-N)}+1\right]^{-1}\right|^2 \leq Cp^{\alpha(1-N)}, \quad N \leq 0.
\]
Obviously this series converges for $\alpha > 1$. Thus $\text{Sp}\{h_k\}^n_0 \subset \mathcal{D}(D^{\alpha/2})$ for $\alpha > 1$.

Since $p^{\alpha(1-N)}+1 \leq 2p^{\alpha(1-N)}$ for $N \leq 0$, we can estimate from below the general term of the second series
\[
(2.19) \quad \frac{1}{4p^n}p^{\alpha(1-N)} \leq \left|p^{-N/2}p^{\frac{1}{2}(1-N)}\chi(-p^{N-1}jx)\left[p^{\alpha(1-N)}+1\right]^{-1}\right|^2 \quad (N \leq 0)
\]
that implies its divergence in $L_2(\mathbb{Q}_p)$ for $\alpha \leq 1$.

Thus $h_k \not\in \mathcal{D}(D^{\alpha/2})$. From this, taking into account that the estimate (2.19) does not depend on the choice of $h_k$ and the functions $\{\psi_{Nj}(p^\alpha x_k)\} \ (N < 0)$ of the basis $\{\psi_{Nj}(x)\}$ corresponding to $h_k \ (1 \leq k \leq n)$ in (2.16) are different for sufficiently small negative indexes $N$, we conclude that $\text{Sp}\{h_k\}^n_1 \cap \mathcal{D}(D^{\alpha/2}) = \{0\}$ for $1/2 < \alpha \leq 1$. Corollary 2.1 is proved.

### 3. Operator $D^\alpha$ with Point Interactions

#### 3.1. The Friedrichs extension.
Let $\mathcal{H}_2 \subset \mathcal{H}_1 \subset L_2(\mathbb{Q}_p) \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2}$ be the standard scale of Hilbert spaces ($A$-scale) associated with the positive self-adjoint operator $A = D^\alpha$. Here $\mathcal{H}_s = \mathcal{D}(A^{s/2})$, $s = 1, 2$, with the norm $\|u\|_s = \|\langle A^\alpha + I \rangle^{s/2}u\|$ and $\mathcal{H}_{-s}$ are the completion of $L_2(\mathbb{Q}_p)$ with respect to the norm $\|u\|_{-s}$ (see [3], [7] for details).

Recalling that $h_k(x)$ is a real-valued function and employing (2.12), (2.14) (with $u$ and $\psi_{Nj}, \alpha$ instead of $h_k$ and $\psi_{Nj}$, respectively), and (2.16), we get
\[
(3.1) \quad \langle \delta_{x_k}, u \rangle = u(x_k) = \langle (D^\alpha + I)u, h_k \rangle_{L_2(\mathbb{Q}_p)}, \quad \forall u \in \mathcal{D}(D^\alpha), \ x_k \in \mathbb{Q}_p.
\]

Thus, the Dirac delta function $\delta_{x_k}$ is well posed on $\mathcal{H}_2 = \mathcal{D}(D^\alpha)$ and $\delta_{x_k} \in \mathcal{H}_{-2}$ for $\alpha > 1/2$.

Let us fix points $x_1, \ldots, x_n \ (n < \infty)$ from $\mathbb{Q}_p$ and consider a positive symmetric operator
\[
(3.2) \quad A_{\text{sym}} = D^\alpha \mathcal{D}_D, \quad \mathcal{D} = \{u \in \mathcal{D}(D^\alpha) \mid u(x_1) = \ldots = u(x_n) = 0\}.
\]

By Theorem 2.1 the formula (3.2) is well-posed for $\alpha > 1/2$. In this case, (3.1) implies that $A_{\text{sym}}$ is a closed densely defined operator in $L_2(\mathbb{Q}_p)$ and its defect subspace $\mathcal{H} = \ker(A_{\text{sym}}^* + I)$ coincides with the linear span of $\{h_k\}^n_{k=1}$. Hence, the deficiency index of $A_{\text{sym}}$ is equal to $(n, n)$.

It is clear that the domain of the adjoint $A_{\text{sym}}^*$ has the form $\mathcal{D}(A_{\text{sym}}^*) = \mathcal{D}(D^\alpha)^{\perp} + \mathcal{H}$

\[
(3.3) \quad A_{\text{sym}}^* f = A_{\text{sym}}^*(u + h) = D^\alpha u - h, \quad \forall f = u + h \in \mathcal{D}(A_{\text{sym}}^*)
\]

$(u \in \mathcal{D}(D^\alpha), \ h \in \mathcal{H})$.

**Proposition 3.1.** Let $A_F$ be the Friedrichs extension of $A_{\text{sym}}$. Then $A_F = D^\alpha$ when $1/2 < \alpha \leq 1$ and

\[
A_F = A_{\text{sym}}^* |_{\mathcal{D}(A_F)}, \quad \mathcal{D}(A_F) = \{f(x) \in \mathcal{D}(A_{\text{sym}}^*) \mid f(x_1) = \ldots = f(x_n) = 0\}
\]
when $\alpha > 1$. 


Proof. It follows from (2.18) that $\mathcal{H}_1 = \mathcal{D}(D^\alpha/2)$. This relation and Corollary 2.1 mean that $\mathcal{H} \subset \mathcal{H}_1$ ($\alpha > 1$) and $\mathcal{H} \cap \mathcal{H}_1 = \{0\}$ ($1/2 < \alpha \leq 1$).

After such a preparation work, the proof is a direct consequence of some ‘folk-lore’ results of the extension theory. For the convenience of the reader some principal stages are repeated below.

First of all, we recall that the Friedrichs extension $A_F$ of $A_{sym}$ is defined as the restriction of the adjoint $A_{sym}^*$ onto $\mathcal{D}(A_F) = \mathcal{D} \cap \mathcal{D}(A_{sym}^*)$, where $\mathcal{D}$ is the completion of $\mathcal{D}(A_{sym})$ in the Hilbert space $\mathcal{H}$. Using the obvious equality $\mathcal{H}_1 = \mathcal{D} \oplus \mathcal{H}'$ (here $\mathcal{H}' = \mathcal{H} \cap \mathcal{H}_1$ and $\oplus$ denotes the orthogonal sum in $\mathcal{H}_1$), we describe $\mathcal{D}(A_F)$ as follows:

$$\mathcal{D}(A_F) = \{ f \in \mathcal{H}_1 \cap \mathcal{D}(A_{sym}^*) \mid ((D^\alpha + I)^{1/2}f, (D^\alpha + I)^{1/2}h')_{L^2(\mathbb{Q}_p)} = 0, \forall h' \in \mathcal{H}' \}. $$

If $\mathcal{H} \cap \mathcal{H}_1 = \{0\}$ (the case $1/2 < \alpha \leq 1$), then $\mathcal{H}' = \{0\}$ and $\mathcal{D}(A_F) = \mathcal{H}_1 \cap \mathcal{D}(A_{sym}^*) = \mathcal{D}(D^\alpha)$. Thus $A_F = D^\alpha$.

If $\mathcal{H} \subset \mathcal{H}_1$ (the case $\alpha > 1$), then $\mathcal{H}' = \mathcal{H}$, $\mathcal{D}(A_{sym}^*) \subset \mathcal{H}_1$ and

$$\mathcal{D}(A_F) = \{ f \in \mathcal{D}(A_{sym}^*) \mid ((D^\alpha + I)^{1/2}f, (D^\alpha + I)^{1/2}h_k)_{L^2(\mathbb{Q}_p)} = 0, 1 \leq k \leq n \}. $$

Repeating the same arguments as in the proof of (3.1) it is easy to see that $((D^\alpha + I)^{1/2}f, (D^\alpha + I)^{1/2}h_k)_{L^2(\mathbb{Q}_p)} = f(x_k)$. Proposition 3.1 is proved. \[ \blacksquare \]

3.2. Operator realizations of $D^\alpha + V_Y$ in $L^2(\mathbb{Q}_p)$. In the additive singular perturbations theory, the algorithm of the determination of operator realizations of finite rank point perturbations of $D^\alpha$ is determined by the general expression

$$(3.4) \hspace{1cm} A_Y = D^\alpha + V_Y, \hspace{1cm} V_Y = \sum_{i,j=1}^{n} b_{ij} < \delta_{x,j}, \gamma > \delta_{x,i}, \hspace{0.5cm} b_{ij} \in \mathbb{C},$$

$Y = \{x_1, \ldots, x_n\}$ is well known [3] and it is based on the construction of some extension (regularization) $A_{Y_{\text{reg}}} = D^\alpha + V_{Y_{\text{reg}}}$ of (3.4) onto the domain $\mathcal{D}(A_{sym}^*) = \mathcal{D}(D^\alpha) + \mathcal{H}$.

The $L^2(\mathbb{Q}_p)$-part

$$(3.5) \hspace{1cm} \tilde{A} = A_{Y_{\text{reg}}} |_{\mathcal{D}(\tilde{A})}, \hspace{1cm} \mathcal{D}(\tilde{A}) = \{ f \in \mathcal{D}(A_{sym}^*) \mid A_{Y_{\text{reg}}}f \in L^2(\mathbb{Q}_p) \}$$

of the regularization $A_{Y_{\text{reg}}}$ is called the operator realization of $D^\alpha + V_Y$ in $L^2(\mathbb{Q}_p)$.

Since the action of $D^\alpha$ on elements of $\mathcal{H}$ is defined by (2.11), the regularization $A_{Y_{\text{reg}}}$ depends on the determination of $V_{Y_{\text{reg}}}$.

If $\alpha > 1$, the singular potential $V_Y = \sum_{i,j=1}^{n} b_{ij} < \delta_{x,j}, \gamma > \delta_{x,i}$ is form bounded (since all $h_k \in \mathcal{H}_1$ and hence, all $\delta_{x_k} \in \mathcal{H}_{1-1}$). In this case, $\mathcal{D}(A_{sym}^*) \subset \mathcal{H}_1$ consists of continuous functions on $\mathbb{Q}_p$ (Lemma 2.1) and the delta functions $\delta_{x_k}$ are uniquely determined on elements $f \in \mathcal{D}(A_{sym}^*)$ by continuity (cf. (3.1))

$$(3.6) \hspace{1cm} < \delta_{x_k}, f > = ((D^\alpha + I)^{1/2}f, (D^\alpha + I)^{1/2}h_k)_{L^2(\mathbb{Q}_p)} = f(x_k).$$

Thus, for $\alpha > 1$, the regularization $A_{Y_{\text{reg}}}$ is uniquely defined and formula (3.5) provides a unique operator realization of (3.4) in $L^2(\mathbb{Q}_p)$ corresponding to a fixed singular potential $V_Y$.

The case $1/2 < \alpha \leq 1$ is more complicated, because $\delta_{x_k}$ cannot be extended onto $\mathcal{D}(A_{sym}^*)$ by continuity.

Since any function $f \in \mathcal{D}(A_{sym}^*) = \mathcal{D}(D^\alpha) + \mathcal{H}$ admits a decomposition $f = u + \sum_{j=1}^{n} c_j h_j$ ($u \in \mathcal{D}(D^\alpha)$, $c_j \in \mathbb{C}$), the extension of $\delta_{x_k}$ on $\mathcal{D}(A_{sym}^*)$ is well determined if the entries $r_{kj} = < \delta_{x_k}, h_j >$ of the matrix $\mathcal{R} = (r_{kj})_{n \times n}$ are known. In this case, the extended delta-function $\delta_{x_k}$ acts on functions $f \in \mathcal{D}(A_{sym}^*)$ by the rule

$$< \delta_{x_k}, f > = u(x_k) + c_1 r_{k1} + \ldots + c_n r_{kn}, \hspace{1cm} 1 \leq k \leq n.$$  

(We preserve the same notation $\delta_{x_k}$ for the extension.)
Since $\mathcal{H} \cap \mathcal{D}_1 = \{0\}$, the system $\{\delta_{x_k}\}_{k=1}^r$ is $\mathcal{D}_1$-independent (i.e., its linear span $\text{Sp}\{\delta_{x_k}\}_{k=1}^r \cap \mathcal{D}_1 = \{0\}$). Therefore, the natural restrictions on the choice of $r_{kj}$ in (3.7) induced by the fact that a functional $<\phi, \cdot>$ where $\phi \in \text{Sp}\{\delta_{x_k}\}_{k=1}^r \cap \mathcal{D}_1$ admits a natural extension by continuity onto $\mathcal{D}_1 \cap D(A^*_\eta)$ do not appear in our case (see [3] for details). This means that, in general, any Hermitian matrix $R = (r_{kj})_{k,j=1}^n$ can be used for the determination of the extended functionals $<\delta_{x_k}, \cdot>$ in (3.7).

One of the possible approaches to the definition of $r_{kj}$ deals with the fact that the functions $h_j(x)$ turn out to be continuous at the point $x = x_k$ if $j \neq k$ (see Lemma 2.1). In view of this, it is natural to assume that

$$r_{kj} = <\delta_{x_k}, h_j> = h_j(x_k), \quad j \neq k.$$ (3.8)

However this formula cannot be used for the definition of $r_{kk}$ because the substitution of $x_k$ for $x$ in (2.13) leads to the formal equality

$$h_k(x_k) = (p - 1) \sum_{N=-\infty}^{\infty} \frac{p^{-N}}{p^{N(1-N)} + 1}$$

with a divergent series at the right-hand side. Note that this series do not depend on $k$. For this reason, some choice of a real number $r = r_{kk}$ ($1 \leq k \leq n$) can be interpreted as a certain regularization of $\sum_{N=-\infty}^{\infty} \frac{p^{-N}}{p^{N(1-N)} + 1}$.

It follows from (2.13), (2.14), and (3.8) that $r_{kj} = \tau_{jk}$. Hence, the matrix $R = (r_{kj})_{k,j=1}^n$ constructed in such a way is Hermitian.

It should be noted that if we will use (3.7) instead of the direct formula (3.6) for the definition of extensions $<\delta_{x_k}, \cdot>$ in the case $\alpha > 1$, then we arrive at just the same form of the matrix $R$. The difference only is in the convergency of the series in (3.9) for $\alpha > 1$ and hence, $r_{kk} = h_k(x_k)$.

3.3. Description of operator realizations. Let $\eta$ be an invertible bounded self-adjoint operator in $L_2(\mathbb{Q}_p)$.

An operator $A$ is called $\eta$-self-adjoint in $L_2(\mathbb{Q}_p)$ if $A^* = \eta A \eta^{-1}$, where $A^*$ denotes the adjoint of $A$ [5]. Obviously, self-adjoint operators are a particular case of $\eta$-self-adjoint ones for $\eta = I$. In this case, we will use notation ‘self-adjoint’ instead of ‘$I$-self-adjoint’.

We are going to describe $\eta$-self-adjoint operator realizations $A$ (see (3.5)) of $D^\alpha + V_{\mathcal{Y}}$ in $L_2(\mathbb{Q}_p)$.

To do this, we determine linear mappings $\Gamma_i : D(A^*_\eta) \to \mathbb{C}^n$ ($i = 0, 1$):

$$\Gamma_0 f = \begin{pmatrix} <\delta_{x_1}, f> \\ \vdots \\ <\delta_{x_n}, f> \end{pmatrix}, \quad \Gamma_1 f = - \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \forall f = u + \sum_{i=1}^n c_i h_i \in D(A^*_\eta).$$ (3.10)

In what follows we will assume that

$$D^\alpha \eta = \eta D^\alpha \quad \text{and} \quad \eta : \mathcal{H} \to \mathcal{H}.$$ (3.11)

By the second relation in (3.11), the action of $\eta$ on elements of $\mathcal{H}$ can be described with the help of a matrix $\mathcal{Y} = (y_{ij})_{i,j=1}^n$, i.e.,

$$\eta \sum_{i=1}^n c_i h_i = (h_1, \ldots, h_n) \mathcal{Y}^t(c_1, \ldots, c_n)^t \quad (c_i \in \mathbb{C}),$$ (3.12)

where the upper index $^t$ denotes the operation of transposition. Since, in general, the basis $\{h_i\}_{i=1}^n$ of $\mathcal{H}$ is not orthogonal, the matrix $\mathcal{Y}$ is not Hermitian ($\mathcal{Y} \neq \overline{\mathcal{Y}}$).
Lemma 3.1. If \( \alpha > 1 \), then

\[
\Gamma_0 \eta f = \nabla^\Gamma_0 f \quad \text{and} \quad \Gamma_1 \eta f = \nabla^\Gamma_1 f \quad (\forall f \in \mathcal{D}(A_{\text{sym}}^*)).
\]

These equalities also hold for \( 1/2 < \alpha \leq 1 \) if \( \mathcal{R} \mathcal{Y} = \nabla^\mathcal{R} \mathcal{Y} \), where the matrix \( \mathcal{R} \) determines the extended functionals \( \delta x_k, \cdot \cdot \cdot \) in (3.11).

Proof. Let \( f = u + \sum_{i=1}^n c_i h_i \in \mathcal{D}(A_{\text{sym}}^*) \). By (3.12)

\[
\eta f = \eta u + (h_1, \ldots, h_n) \mathcal{Y}(c_1, \ldots, c_n),
\]

where \( \eta u \in \mathcal{D}(D^\alpha) \) (see the first relation in (3.11)). In view of (3.10),

\[
\Gamma_1 \eta f = -\mathcal{Y}(c_1, \ldots, c_n) = \mathcal{Y} \Gamma_1 f.
\]

It follows from the first relation in (3.11) that \( \eta(D^\alpha + I)^{1/2} = (D^\alpha + I)^{1/2} \eta \). Taking this equality into account, we deduce from (3.6)

\[
< \delta x_k, \eta f > = (\eta(D^\alpha + I)^{1/2} f, (D^\alpha + I)^{1/2} \eta h_k)_{\mathcal{L}_2(\mathcal{Q}_p)} =
\]

\[
= (\eta x_k, f, \ldots, \eta x_n, f)^t
\]

that implies \( \Gamma_0 \eta f = \nabla^\Gamma_0 f \) for \( \alpha > 1 \).

Similar arguments with the employing (3.7), (3.13), and \( \mathcal{R} \mathcal{Y} = \nabla^\mathcal{R} \mathcal{Y} \) give

\[
\Gamma_0 \eta f = \Gamma_0 \eta u + \mathcal{R} \mathcal{Y}(c_1, \ldots, c_n) = \nabla^\Gamma_0 \Gamma_1 u + \nabla^\mathcal{R} \mathcal{Y}(c_1, \ldots, c_n) = \nabla^\mathcal{R} \Gamma_0 f
\]

for \( 1/2 < \alpha \leq 1 \). Lemma 3.1 is proved. \( \blacksquare \)

Theorem 3.1. Let \( \tilde{A} \) be the operator realization of \( D^\alpha + V_\gamma \) defined by (3.11). Then \( \tilde{A} \) coincides with the operator

\[
A_B = A_{\text{sym}}^* \mid \mathcal{D}(A_B), \quad \mathcal{D}(A_B) = \{ f \in \mathcal{D}(A_{\text{sym}}^*) \mid B \Gamma_0 f = \Gamma_1 f \},
\]

where \( B = (b_{ij})_{i,j=1}^n \) is the coefficient matrix of the singular potential \( V_\gamma \).

The operator \( A_B \) is self-adjoint if and only if the matrix \( B \) is Hermitian.

If \( \eta \) satisfy (3.11) and \( \alpha > 1 \), then \( A_B \) is \( \eta \)-self-adjoint if and only if the matrix \( \mathcal{Y} B \) is Hermitian. This statement is also true for the case \( 1/2 < \alpha \leq 1 \) under the additional condition that the matrix \( \mathcal{Y} B \) is Hermitian, where \( \mathcal{Y} \) determines the extended functionals \( \delta x_k, \cdot \cdot \cdot \) in (3.11).

Proof. It follows from (2.11), (3.3), and (3.10) that

\[
A_{V_{\text{reg}}} f = A_{\text{sym}}^* f + (\delta x_1, \ldots, \delta x_n)(B \Gamma_0 f - \Gamma_1 f), \quad f \in \mathcal{D}(A_{\text{sym}}^*)
\]

This equality and (3.5) mean that the operator realization \( \tilde{A} \) of \( D^\alpha + V_\gamma \) coincides with the operator \( A_B \) determined by (3.11).

It is known (see, e.g., [4], [8]) that the triple \( (C^n, \Gamma_0, \Gamma_1) \), where \( \Gamma_i \) are defined by (3.10), is a boundary value space (BVS) of \( A_{\text{sym}} \). This means that the abstract Green identity

\[
(A_{\text{sym}}^* f, g) - (f, A_{\text{sym}}^* g) = (\Gamma_1 f, \Gamma_0 g)c^n - (\Gamma_0 f, \Gamma_1 g)c^n, \quad f, g \in \mathcal{D}(A_{\text{sym}}^*)
\]

is satisfied and the map \( (\Gamma_0, \Gamma_1) : \mathcal{D}(A_{\text{sym}}^*) \to C^n \oplus C^n \) is surjective.

It follows from the general results of the BVS-theory [9], [10], [15] that the operator \( A_B \) determined by (3.11) is self-adjoint \( \iff \) the matrix \( B \) is Hermitian.

Conditions (3.11) imposed on \( \eta \) ensure the commutativity of \( \eta \) with \( A_{\text{sym}} \) and \( A_{\text{sym}}^* \), i.e.,

\[
\eta A_{\text{sym}} = A_{\text{sym}} \eta, \quad \eta A_{\text{sym}}^* = A_{\text{sym}}^* \eta.
\]

Relations (3.16) and the definition of \( \eta \)-self-adjoint operators imply that \( A_B \) is \( \eta \)-self-adjoint \( \iff \eta A_B \) is a self-adjoint extension of the symmetric operator \( F_{\text{sym}} = \eta A_{\text{sym}} \).
Thus, the description of $\eta$-self-adjoint operators is reduced to the similar problem for self-adjoint ones.

It immediately follows from Lemma 3.1 and relations (3.15), (3.16) that the triple $(\mathbb{C}^n, \Gamma_0, \mathcal{Y}_\Gamma_1)$ is a BVS for the symmetric operator $F_{\text{sym}}$. In this BVS, the operator $\eta A_B$ is described by the formula (cf. (3.14)):

$$\eta A_B = \eta A_{\text{sym}} \upharpoonright D(\eta A_B), \quad D(\eta A_B) = \{ f \in D(A_{\text{sym}}^*) \mid \mathcal{Y} B \Gamma_0 f = \mathcal{Y} \Gamma_1 f \}$$

that completes the proof of Theorem 3.1.

Bibliography

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Springer-Verlag, Berlin/New York, 1988; 2nd ed. (with an appendix by P. Exner), AMS Chelsea Publishing, Providence, RI, 2005.

2. S. Albeverio, A.Zu. Khrennikov, and V.M. Shelkovich, *Associative algebras of $p$-adic distributions*, Proc. Steklov Instit. Math. 245 (2004), 22–33.

3. S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Math. Soc. Lecture Note Ser. 271, Cambridge Univ. Press, Cambridge, 2000.

4. S. Albeverio, S. Kuzhel, and L. Nizhnik, *Singularly perturbed self-adjoint operators in scales of Hilbert spaces*, Preprint No. 253, Universitä Bonn (2005), 29pp. submitted to IEOT

5. T. Ya. Azizov and I. S. Iokhvidov, *Linear operators in spaces with indefinite metric*, Wiley, Chichester, 1989.

6. C. M. Bender, D. C. Brody, and H. F. Jones, *Must a Hamiltonian be Hermitian?*, Amer. J. Phys. 71 (2003), no. 11, 1095–1102.

7. Yu. M. Berezansky, *Expansion in Eigenfunctions of Self-Adjoint Operators*, Naukova Dumka, Kiev, 1965 [in Russian]; English translation: AMS, Providence, R.I., 1968.

8. V. Derkach, S. Hassi, and H. de Snoo, *Singular perturbations of self-adjoint operators*, Math. Phys. Anal. Geometry 6 (2003), 349–384.

9. M. L. Gorbachuk and V. I. Gorbachuk, *Boundary-value problems for operator-differential equations*, Kluwer, Dordrecht, 1991.

10. M. L. Gorbachuk, V. I. Gorbachuk, and A. N. Kochubei, *Theory of extensions of symmetric operators and boundary-value problems for differential equations*, Ukr. Mat. J. 41 (1989), No. 10, 1299–1313.

11. A. Khrennikov, *Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models*, Kluwer, Dordrecht, 1997.

12. A.N. Kochubei, *The differentiation operator on subsets of the field of $p$-adic numbers*, Russ. Acad. Sci. Izv. 41 (1993), 289-305.

13. A.N. Kochubei, *Pseudodifferential equations and stohastics over non-archimedean fields*, Marcel Dekker, New York, 2001.

14. S.V. Kozyrev, *Wavelet analysis as a $p$-adic spectral analysis*, Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), No. 2, 149–158, arxiv:math-ph/0012019.

15. A. Kuzhel and S. Kuzhel, *Regular extensions of Hermitian operators*, VSP, Utrecht, 1998.

16. *The Proceedings of the Second International Workshop ‘Pseudo-Hermitian Hamiltonians in Quantum Physics’*, Czechoslovak J. Phys. 54 (2004), no. 10.

17. V.S. Vladimirov, I.V. Volovich, and Ye.I. Zelenov, *$p$-adic analysis and mathematical physics*, World Scientific, Singapore, 1994.

E-mail address, S. Kuzhel: kuzhel@imath.kiev.ua

E-mail address, S. Torba: sergiy.torba@gmail.com

INSTITUTE OF MATHEMATICS OF THE NATIONAL ACADEMY OF SCIENCES OF UKRAINE, TERESHCHENKOVSKAYA 3, 01601 KIEV (UKRAINE)