Lyapunov exponents at anomalies of $\text{SL}(2, \mathbb{R})$-actions

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Abstract

Anomalies are known to appear in the perturbation theory for the one-dimensional Anderson model. A systematic approach to anomalies at critical points of products of random matrices is developed, classifying and analysing their possible types. The associated invariant measure is calculated formally. For an anomaly of so-called second degree, it is given by the groundstate of a certain Fokker-Planck equation on the unit circle. The Lyapunov exponent is calculated to lowest order in perturbation theory with rigorous control of the error terms.

1 Introduction

Anomalies in the perturbative calculation of the Lyapunov exponent and the density of states were first found and analysed by Kappus and Wegner [KW] when they studied the center of the band in a one-dimensional Anderson model. Further anomalies, albeit in higher order perturbation theory, were then treated by Derrida and Gardner [DG] as well as Bovier and Klein [BK]. More recently, anomalies also appeared in the study of random polymer models [JSS]. Quite some effort has been made to understand anomalies in the particular case of the Anderson model also from a more mathematical point of view [CK, SVW]. However, Campanino and Klein [CK] need to suppose decay estimates on the characteristic function of the random potential, and Shubin, Vakilian and Wolff [SVW] appeal to rather complicated techniques from harmonic analysis (allowing only to give the correct scaling of the Lyapunov exponent, but not a precise perturbative formula for it).

It is the purpose of this work to present a more conceptual approach to anomalies of products of random matrices. In fact, various types may appear and only those of second degree (in the sense of the definition below) seem to have been studied previously. Indeed, this is the most difficult and interesting case to analyse, and the main insight of the present work is to exhibit an associated Fokker-Planck operator, the spectral gap of which is ultimately responsible for the positivity of the Lyapunov exponent. In the special case of the Anderson model, a related operator already appeared in [BK]. Here it is, however, possible to circumvent the spectral analysis of the Fokker-Planck operator and prove the asymptotics of the Lyapunov exponent...
more directly (cf. Section 5.4). The other cases of various first degree anomalies are more elementary to analyse. Examples for different types of anomalies are given in Section 6.

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2 Definition of anomalies

Let us consider families \((T_{\lambda,\sigma})_{\lambda \in \mathbb{R}, \sigma \in \Sigma}\) of matrices in \(\text{SL}(2, \mathbb{R}) = \{ T \in \text{Mat}_{2 \times 2}(\mathbb{R}) \mid \det(T) = 1 \}\) depending on a random variable \(\sigma\) in some probability space \((\Sigma, p)\) as well as a real coupling parameter \(\lambda\). In order to avoid technicalities, we suppose that \(p\) has compact support. The dependence on \(\lambda\) is supposed to be smooth. The expectation value w.r.t. \(p\) will be denoted by \(E\).

Definition 1 The value \(\lambda = 0\) is anomaly of first order of the family \((T_{\lambda,\sigma})_{\lambda \in \mathbb{R}, \sigma \in \Sigma}\) if for all \(\sigma \in \Sigma\):

\[
T_{0,\sigma} = \pm 1 ,
\]

with a sign that may depend on \(\sigma \in \Sigma\). In order to further classify the anomalies and for later use, let us introduce \(P_\sigma, Q_\sigma \in \text{sl}(2, \mathbb{R})\) by

\[
MT_{\lambda,\sigma}M^{-1} = \pm \exp (\lambda P_\sigma + \lambda^2 Q_\sigma + \mathcal{O}(\lambda^3)) ,
\]

where \(M \in \text{SL}(2, \mathbb{R})\) is a \(\lambda\)- and \(\sigma\)-independent basis change to be chosen later. An anomaly is said to be of first degree if \(E(P_\sigma)\) is non-vanishing, and then it is called elliptic if \(\det(E(P_\sigma)) > 0\), hyperbolic if \(\det(E(P_\sigma)) < 0\) and parabolic if \(\det(E(P_\sigma)) = 0\). Note that all these notions are independent of the choice of \(M\).

If \(E(P_\sigma) = 0\), but the variance of \(P_\sigma\) is non-vanishing, then an anomaly is said to be of second degree.

Furthermore, for \(k \in \mathbb{N}\), set \(\hat{\sigma} = (\sigma(k), \ldots, \sigma(1)) \in \hat{\Sigma} = \Sigma^k\), as well as \(\hat{p} = p^k\) and \(T_{\lambda,\hat{\sigma}} = T_{\lambda,\sigma(k)} \cdots T_{\lambda,\sigma(1)}\). Then \(\lambda = 0\) is anomaly of \(k\)th order of the family \((T_{\lambda,\sigma})_{\lambda \in \mathbb{R}, \sigma \in \Sigma}\) if the family \((T_{\lambda,\hat{\sigma}})_{\lambda \in \mathbb{R}, \hat{\sigma} \in \hat{\Sigma}}\) has an anomaly of first order at \(\lambda = 0\) in the above sense. The definitions of degree and nature transpose to \(k\)th order anomalies.

As is suggested in the definition and will be further explained below, we may (and will) restrict ourselves to the analysis of anomalies of first order. In the examples, however, anomalies of higher order do appear and can then be studied by the present techniques (cf. Section 6). Furthermore, by a change of variables in \(\lambda\), anomalies of degree higher than 2 can be analysed like an anomaly of second degree.
Anomalies are particular cases of so-called critical points studied in [JSS, SSS], namely $\lambda = 0$ is by definition a critical point of the family $(T_{\lambda, \sigma})_{\lambda \in \mathbb{R}, \sigma \in \Sigma}$ if for all $\sigma, \sigma' \in \Sigma$:

$$[T_{0, \sigma}, T_{0, \sigma'}] = 0 , \quad \text{and} \quad |\text{Tr}(T_{0, \sigma})| < 2 \quad \text{or} \quad T_{0, \sigma} = \pm 1 .$$

(3)

Critical points appear in many applications like the Anderson model and the random polymer model. In these situations anomalies appear for special values of the parameters, such as the energy or the coupling constant, cf. Section 6.

3 Phase shift dynamics

The bijective action $S_T$ of a matrix $T \in \text{SL}(2, \mathbb{R})$ on $S^1 = [0, 2\pi)$ is given by

$$e_{S_T(\theta)} = \frac{Te_\theta}{\|Te_\theta\|} , \quad e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} , \quad \theta \in [0, 2\pi) .$$

(4)

This defines a group action, namely $S_{TT'} = S_T S_{T'}$. In particular the map $S_T$ is invertible and $S_T^{-1} = S_{T^{-1}}$. Note that this is actually an action on $\mathbb{R}P(1)$, and $S^1$ appears as a double cover here. In order to shorten notations, we write

$$S_{\lambda, \sigma} = S_{MT_{\lambda, \sigma} M^{-1}} .$$

Next we need to iterate this dynamics. Associated to a given semi-infinite code $\omega = (\sigma_n)_{n \geq 1}$ with $\sigma_n \in \Sigma$ is a sequence of matrices $(T_{\lambda, \sigma_n})_{n \geq 1}$. Codes are random and chosen independently according to the product law $p \otimes N$. Averaging w.r.t. $p \otimes N$ is also denoted by $E$. Then one defines iteratively for $N \in \mathbb{N}$

$$S_{\lambda, \omega}^N(\theta) = S_{\lambda, \sigma_N}(S_{\lambda, \omega}^{N-1}(\theta)) , \quad S_{\lambda, \omega}^0(\theta) = \theta .$$

(5)

This is a discrete time random dynamical system on $S^1$. Let us note that at an anomaly of first order, one has $S_{\lambda, \sigma}(\theta) = \theta + \mathcal{O}(\lambda)$ or $S_{\lambda, \sigma}(\theta) = \theta + \pi + \mathcal{O}(\lambda)$ depending on the sign in $[1]$. As all the functions appearing below will be $\pi$-periodic we can neglect the summand $\pi$, meaning that we may suppose that there is a sign $+$ in $[1]$ for all $\sigma$ (this reflects that the action is actually on projective space).

In order to do perturbation theory in $\lambda$, we need some notations. Introducing the unit vector $v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$, we define the first order polynomials in $e^{2i\theta}$

$$p_\sigma(\theta) = \Im m \left( \frac{\langle v | P_\sigma | e_\theta \rangle}{\langle v | e_\theta \rangle} \right) , \quad q_\sigma(\theta) = \Im m \left( \frac{\langle v | Q_\sigma | e_\theta \rangle}{\langle v | e_\theta \rangle} \right) ,$$

as well as

$$\alpha_\sigma = \langle v | P_\sigma | v \rangle , \quad \beta_\sigma = \langle \overline{v} | P_\sigma | v \rangle .$$

Hence $p_\sigma(\theta) = \Im m (\alpha_\sigma - \beta_\sigma e^{2i\theta})$. Now starting from the identity

$$3$$
$$e^{2\mathcal{S}_{\lambda,\sigma}(\theta)} = \frac{\langle v| MT_{\sigma} M^{-1} | e_\theta \rangle}{\langle v| MT_{\sigma} M^{-1} | e_\theta \rangle},$$

the definition (2) and the identity $\langle v| e_\theta \rangle = \frac{1}{\sqrt{2}} e^{i\theta}$, one can verify that

$$\mathcal{S}_{\lambda,\sigma}(\theta) = \theta + 3m\left( \frac{\lambda \langle v| P_\sigma | e_\theta \rangle}{\langle v| e_\theta \rangle} + \frac{\lambda^2 \langle v| Q_\sigma + P^2_\sigma | e_\theta \rangle}{\langle v| e_\theta \rangle} - \frac{\lambda^2}{2} \frac{\langle v| P_\sigma | e_\theta \rangle^2}{\langle v| e_\theta \rangle^2} \right) + O(\lambda^3).$$

As one readily verifies that

$$P^2_\sigma = -\det(P_\sigma) 1,$$

$$3m\left( \frac{\langle v| P_\sigma | e_\theta \rangle^2}{\langle v| e_\theta \rangle^2} \right) = -p_\sigma(\theta) \partial_\theta p_\sigma(\theta),$$

it follows that

$$\mathcal{S}_{\lambda,\sigma}(\theta) = \theta + \lambda p_\sigma(\theta) + \lambda^2 q_\sigma(\theta) + \frac{1}{2} \lambda^2 p_\sigma(\theta) \partial_\theta p_\sigma(\theta) + O(\lambda^3). \quad (6)$$

Finally let us note that

$$\mathcal{S}_{\lambda,\sigma}^{-1}(\theta) = \theta - \lambda p_\sigma(\theta) - \lambda^2 q_\sigma(\theta) + \frac{1}{2} \lambda^2 p_\sigma(\theta) \partial_\theta p_\sigma(\theta) + O(\lambda^3), \quad (7)$$

as one verifies immediately because $\mathcal{S}_{\lambda,\sigma}(\mathcal{S}_{\lambda,\sigma}^{-1}(\theta)) = \theta + O(\lambda^3)$, or can deduce directly just as above from the identity $\exp(\lambda P_\sigma + \lambda^2 Q_\sigma + O(\lambda^3))^{-1} = \exp(-\lambda P_\sigma - \lambda^2 Q_\sigma + O(\lambda^3))$.

### 4 Formal perturbative formula for the invariant measure

For each $\lambda$, the family $(MT_{\lambda,\sigma}M^{-1})_{\sigma \in \Sigma}$ and the probability $p$ define an invariant probability measure $\nu_\lambda$ on $S^1$ by the equation

$$\int d\nu_\lambda(\theta) f(\theta) = E \int d\nu_\lambda(\theta) f(\mathcal{S}_{\lambda,\sigma}(\theta)) , \quad f \in C(S^1). \quad (8)$$

Furstenberg proved that this invariant measure is unique whenever the Lyapunov exponent of the associated product of random matrices (discussed below) is positive (e.g. [BL]) and in this situation $\nu_\lambda$ is also known to be Hölder continuous, so, in particular, it does not contain a point component. For the study of the invariant measure at an anomaly of order $k$, it is convenient to iterate (8):

$$\int d\nu_\lambda(\theta) f(\theta) = E \int d\nu_\lambda(\theta) f(\mathcal{S}_{\lambda,\sigma}^k(\theta)) , \quad f \in C(S^1).$$

Replacing $(\Sigma, p)$ by $(\tilde{\Sigma}, \hat{p})$ therefore shows that the families $(T_{\lambda,\sigma})_{\lambda \in \mathbb{R}, \sigma \in \Sigma}$ and $(T_{\lambda,\hat{\sigma}})_{\lambda \in \mathbb{R}, \hat{\sigma} \in \Sigma}$ have the same invariant measure. Hence it is sufficient to study anomalies of first order.
The aim of this section is to present a formal perturbative expansion of the invariant measure under the hypothesis that it is absolutely continuous, that is $d\nu_\lambda(\theta) = \rho_\lambda(\theta)\frac{d\theta}{2\pi}$ with $\rho_\lambda = \rho_0 + \lambda \rho_1 + O(\lambda^2)$. Then (8) leads to

$$E\left(\partial_\theta S^{-1}_\lambda(\theta) \rho_\lambda(S^{-1}_\lambda(\theta))\right) = \rho_\lambda(\theta), \quad (9)$$

which with equation (7) gives

$$\rho_\lambda - \lambda\partial_\theta \left(E(p_\sigma)\rho_\lambda\right) + \lambda^2 \frac{1}{2} \partial_\theta \left(E(p_\sigma^2) \partial_\theta \rho_\lambda + E(p_\sigma \partial_\theta p_\sigma)\rho_\lambda - 2E(q_\sigma)\rho_\lambda\right) + O(\lambda^3) = \rho_\lambda.$$

We first consider an anomaly of first degree. As $E(P_\sigma) \neq 0$, it follows that $E(p_\sigma)$ is not vanishing identically. Therefore the above perturbative equation is non-trivial to first order in $\lambda$, hence $E(p_\sigma)$ should be constant. If now the first degree anomaly is elliptic, then $\det E(P_\sigma) > 0$ which can easily be seen to be equivalent to $|E(\alpha_\sigma)| > |E(\beta_\sigma)|$, which in turn is equivalent to the fact that $E(p_\sigma(\theta))$ does not vanish for any $\theta \in S^1$. For an elliptic anomaly of first degree, the lowest order of the invariant measure is therefore

$$\rho_0 = \frac{c}{E(p_\sigma)},$$

with an adequate normalization constant $c \in \mathbb{R}$. If on the other hand, the anomaly is hyperbolic (resp. parabolic), then $E(p_\sigma)$ has four (resp. two) zeros on $S^1$. In this situation, the only possible (formal) solution is that $\rho_0$ is given by Dirac peaks on these zeros (which is, of course, only formal because the invariant measure is known to be Hölder continuous). In Section 5.2 we shall see that for the calculation of certain expectation values w.r.t. the invariant measure, it looks as if it were given by a sum of two Dirac peaks, concentrated on the stable fixed points of the averaged phase shift dynamics. These fixed points are two of the zeros of $E(p_\sigma)$.

Next we consider an anomaly of second degree. As then $E(p_\sigma) = 0$, it follows that the equation for the lowest order of the invariant measure is

$$\frac{1}{2} \partial_\theta \left(E(p_\sigma^2) \partial_\theta \rho_0 + E(p_\sigma \partial_\theta p_\sigma)\rho_0 - 2E(q_\sigma)\rho_0\right) = 0.$$

Now $E(p_\sigma^2) > 0$ unless $p$-almost all $p_\sigma$ vanish simultaneously for some $\theta$, a (rare) situation which is excluded throughout the present work. Then this is an analytic Fokker-Planck equation on the unit circle and it can be written as $\mathcal{L} \rho_0 = 0$ where $\mathcal{L}$ is by definition the Fokker-Planck operator. Its spectrum contains the simple eigenvalue 0 with eigenvector given by the (lowest order of the) invariant measure $\rho_0$ calculated next. Indeed,

$$\frac{1}{2} E(p_\sigma^2) \partial_\theta \rho_0 + \frac{1}{2} E(p_\sigma \partial_\theta p_\sigma)\rho_0 - E(q_\sigma)\rho_0 = C,$$

where the real constant $C$ has to be chosen such that the equation admits a positive, $2\pi$-periodic and normalized solution $\rho_0$. It is a routine calculation to determine the solution using the method of variation of the constants. Setting
\[ \kappa(\theta) = \int_0^\theta d\theta' \frac{2 E(q_{\sigma}(\theta'))}{E(p_{\sigma}^2(\theta'))}, \quad K(\theta) = \int_0^\theta d\theta' 2 E(p_{\sigma}^2(\theta'))^{-\frac{1}{2}} e^{-\kappa(\theta')}, \quad C = \frac{e^{-\kappa(2\pi)} - 1}{K(2\pi)}, \]

it is given by

\[ \rho_0(\theta) = \frac{c e^{\kappa(\theta)}}{E(p_{\sigma}^2(\theta))^{\frac{1}{2}} (C K(\theta) + 1)}, \]

where \( c \) is a normalization constant. It is important to note at this point that \( \rho_0(\theta) \) is an analytic function of \( \theta \).

The rest of the spectrum of \( \mathcal{L} \) is discrete (\( \mathcal{L} \) has a compact resolvent), at most twice degenerate and has a strictly negative real part, all facts that can be proven as indicated in [Ris]. As already stated in the introduction, we do not need to use this spectral information directly.

5 The Lyapunov exponent

The asymptotic behavior of the products of the random sequence of matrices \( (T_{\lambda,\sigma_n})_{n \geq 1} \) is characterized by the Lyapunov exponent [BL, A.III.3.4]

\[ \gamma(\lambda) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \left( \left\| \prod_{n=1}^N T_{\lambda,\sigma_n} e^\theta \right\| \right), \]

where \( \theta \) is an arbitrary initial condition. One may also average over \( \theta \) w.r.t. an arbitrary continuous measure before taking the limit [JSS, Lemma 3]. A result of Furstenberg states a criterion for having a positive Lyapunov exponent [BL]. A quantitative control of the Lyapunov exponent in the vicinity of a critical point is given in [SSS, Proposition 1], however, only in the case where the critical point is not an anomaly of first or second order. The latter two cases are dealt with in the present work.

Let us first suppose that the anomaly is of first order. Because the boundary terms vanish in the limit, it is possible to use the matrices \( MT_{\lambda,\sigma_n} M^{-1} \) instead of \( T_{\lambda,\sigma_n} \) in \((\text{II})\). Furthermore, the random dynamical system (5) allows to expand \((\text{II})\) into a telescopic sum:

\[ \gamma(\lambda) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \log \left( \left\| M T_{\lambda,\sigma_{n+1}} M^{-1} e^{S_{\lambda,\omega}(\theta)} \right\| \right). \]

Up to terms of order \( O(\lambda^3) \), we can expand each contribution of this Birkhoff sum:

\[
\log \left( \left\| M T_{\lambda,\sigma} M^{-1} e^\theta \right\| \right) = \lambda \langle e^\theta | P_\sigma | e^\theta \rangle + \frac{1}{2} \lambda^2 \left( \langle e^\theta | (|P_\sigma|^2 + Q_\sigma + P_\sigma^2) | e^\theta \rangle - 2 \langle e^\theta | P_\sigma | e^\theta \rangle^2 \right) \\
= \frac{1}{2} \text{Re} \left[ 2 \lambda \beta_\sigma e^{2i\theta} + \lambda^2 \left( |\beta_\sigma|^2 + \langle \overline{v} | (|P_\sigma|^2 + 2 Q_\sigma) | v \rangle e^{2i\theta} - \beta_\sigma^2 e^{4i\theta} \right) \right],
\]
where we used the identity

\[ \langle e_\theta | T | e_\theta \rangle = \frac{1}{2} \text{Tr}(T) + \Re \left( \langle \tau | T | \nu \rangle e^{2i\theta} \right) , \]

holding for any real matrix \( T \), as well as \( \text{Tr}(P_\sigma) = \text{Tr}(Q_\sigma) = 0 \) and \( \text{Tr}(|P_\sigma|^2 + P_\sigma^2) = 4|\beta_\sigma|^2 \).

Let us set

\[ I_j(N) = \frac{1}{N} \sum_{n=0}^{N-1} E(e^{2ijS_{\lambda,\omega}(\theta)}) , \quad j = 1, 2 , \]

and introduce \( I_j \) by \( I_j(N) = I_j + \mathcal{O}(\lambda) \) for \( N \) sufficiently large. We therefore obtain for an anomaly of first order

\[ \gamma(\lambda) = \frac{1}{2} \text{E} \Re \left( 2\lambda \beta_\sigma I_1 + \lambda^2 \left( |\beta_\sigma|^2 + \langle \tau | (|P_\sigma|^2 + 2Q_\sigma)|\nu \rangle I_1 - \beta_\sigma^2 I_2 \right) \right) + \mathcal{O}(\lambda^3) . \]

For an anomaly of second order, one regroups the contributions pairwise as in Definition 1, namely works with the family \( (T_\lambda, \hat{\sigma})_{\lambda \in \mathbb{R}, \hat{\sigma} \in \Sigma^2} \) where \( T_\lambda, \hat{\sigma} = T_{\lambda,\sigma(2)}T_{\lambda,\sigma(1)} \) for \( \hat{\sigma} = (\sigma(2), \sigma(1)) \), furnished with the probability measure \( \tilde{p} = p \times 2 \). This family has an anomaly of first order, and its Lyapunov exponent is exactly twice that of the initial family \( (T_{\lambda,\sigma})_{\lambda \in \mathbb{R}, \sigma \in \Sigma} \). It is hence sufficient to study anomalies of first order.

5.1 Elliptic first degree anomaly

Let us consider the matrix \( E(\partial_{\lambda}T_{\lambda,\sigma}|_{\lambda=0}) \in \text{sl}(2, \mathbb{R}) \). For an elliptic anomaly, the determinant of this matrix is positive. Its eigenvalues are therefore a complex conjugate pair \( \pm i \frac{\eta}{2} \), so that there exists a basis change \( M \in \text{SL}(2, \mathbb{R}) \) such that

\[ E(M\partial_{\lambda}T_{\lambda,\sigma}|_{\lambda=0}M^{-1}) = E(P_\sigma) = \frac{1}{2} \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix} . \]

It follows that \( E(\alpha_\sigma) = i \frac{\eta}{2} \) and \( E(\beta_\sigma) = 0 \), implying \( \gamma(\lambda) = \mathcal{O}(\lambda^2) \). Furthermore, from (3),

\[ e^{2ijS_{\lambda,\sigma}(\theta)} = \left( 1 + 2ij \lambda \Im(\alpha_\sigma - \beta_\sigma e^{2i\theta}) \right) e^{2i\theta} + \mathcal{O}(\lambda^2) . \]

Hence

\[ I_j(N) = \frac{1}{N} \sum_{n=0}^{N-1} (1 + ij\lambda\eta) E(e^{2ijS_{\lambda,\omega}^{n-1}(\theta)}) + \mathcal{O}(\lambda^2) = (1 + ij\lambda\eta) I_j(N) + \mathcal{O}(N^{-1}, \lambda^2) , \]

so that \( I_j(N) = \mathcal{O}(\lambda, (N\lambda)^{-1}) \) and \( I_j = 0 \). Replacing into (14), this leads to:

**Proposition 1** If \( \lambda = 0 \) is an elliptic anomaly of first order and first degree, then

\[ \gamma(\lambda) = \frac{1}{2} \lambda^2 E(|\beta_\sigma|^2) + \mathcal{O}(\lambda^3) . \]
In order to calculate $\beta_\sigma$ in an application, one first has to determine the basis change (15), then $P_\sigma$ before deducing $\beta_\sigma$ and $E(|\beta_\sigma|^2)$, see Section 6 for two examples. Let us note that after the basis change, (15) implies $E(P_\sigma(\theta)) = \eta$. Hence the invariant measure is to lowest order given by the Lebesgue measure after the basis change.

The term 'elliptic' indicates that the mean dynamics at the anomaly is to lowest order a rotation. For a hyperbolic anomaly it is an expansion in a given direction and a contraction into another one. These directions (i.e. angles) can be chosen to our convenience through the basis change $M$, as will be done next.

5.2 Hyperbolic first degree anomaly

For a hyperbolic anomaly of first degree, the eigenvalues of $E(\partial_\lambda T_{\lambda,\sigma}|_{\lambda=0})$ are $\pm \frac{\mu}{2}$ and there exists $M \in \text{SL}(2,\mathbb{R})$ such that

$$E(M\partial_\lambda T_{\lambda,\sigma}|_{\lambda=0}M^{-1}) = E(P_\sigma) = \frac{1}{2} \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}.$$  
(16)

It follows that $E(\alpha_\sigma) = 0$ and $E(\beta_\sigma) = \frac{\mu}{2}$, so that (14) leads to

$$\gamma(\lambda) = \frac{1}{2} \lambda \mu \Re(I_1) + O(\lambda^2).$$

We now need to evaluate $I_1$. Introducing the reference dynamics $\tilde{S}_\lambda(\theta) = \theta - \frac{1}{2} \lambda \mu \sin(2\theta)$ as well as the centered perturbation $r_\sigma(\theta) = 3m(\alpha_\sigma - (\beta_\sigma - \frac{\mu}{2}) e^{2i\theta})$, it follows from (6) that the phase shift dynamics is

$$S_{\lambda,\sigma}(\theta) = \tilde{S}_\lambda(\theta) + \lambda r_\sigma(\theta) + O(\lambda^2).$$  
(17)

The non-random dynamics $\tilde{S}_\lambda$ has four fixed points, $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. If $\lambda \mu > 0$, $\theta = 0$ and $\pi$ are stable, while $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ are unstable. For $\lambda \mu < 0$, the roles are exchanged, and this case will not be considered here. Unless the initial condition is the unstable fixed point, one has $\tilde{S}_\lambda^n(\theta) \to 0, \pi$ as $n \to \infty$. Furthermore, if $\theta$ is not within a $O(\lambda^2)$-neighborhood of an unstable fixed point, it takes $n = O(\lambda^{-\frac{1}{2}})$ iterations of $\tilde{S}_\lambda$ in order to attain a $O(\lambda^{\frac{1}{2}})$-neighborhood of 0. We also need to expand iterations of $S_\lambda$:

$$\tilde{S}_\lambda^k(\theta + \lambda r_\sigma(\theta') + O(\lambda^2)) = \tilde{S}_\lambda^k(\theta) + \lambda \partial_\sigma \tilde{S}_\lambda^k(\theta) r_\sigma(\theta') + O(\lambda^2),$$

where the corrective term $O(\lambda^2)$ on the r.h.s. is bounded uniformly in $k$ as one readily realizes when thinking of the dynamics induced by $\tilde{S}_\lambda$. Furthermore, $\partial_\theta \tilde{S}_\lambda^k(\theta) = O(1)$ uniformly in $k$.

Iteration thus shows:

$$S_{\lambda,\omega}^n(\theta) = \tilde{S}_{\lambda,\omega}(\theta) + \lambda \sum_{k=1}^n \partial_\sigma \tilde{S}_{\lambda,\omega}^{n-k}(\tilde{S}_{\lambda,\omega}(\tilde{S}_{\lambda,\omega}^{k-1}(\theta))) r_\sigma(\tilde{S}_{\lambda,\omega}^{k-1}(\theta)) + O(n\lambda^2).$$
Let us denote the coefficient in the sum over $k$ by $s_k$. Then $s_k$ is a random variable that depends only on $\sigma_l$ for $l \leq k$. Moreover, $s_k$ is centered when a conditional expectation over $\sigma_k$ is taken. Taking successively conditional expectations thus shows

$$E(e^{2iS_{n,\omega}(\theta)}) = e^{2i\tilde{S}_n(\theta)} \left( E(e^{2i\lambda \sum_{k=1}^{n-1} s_k} + O(n\lambda^2)) + O(\lambda^2) \right) = e^{2i\tilde{S}_n(\theta)} + O(n\lambda^2). \quad (18)$$

Choosing $n = \lambda^{-\frac{3}{2}}$ gives according to the above

$$E(e^{2iS_{n,\omega}(\theta)}) = 1 + O(\lambda^{\frac{1}{2}}), \quad (19)$$

unless $\theta$ is within a $O(\lambda^{\frac{1}{2}})$-neighborhood of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. In the latter cases, an elementary argument based on the central limit theorem shows that it takes of order $E(r_\sigma(\frac{\pi}{2})^2)\lambda^{-\frac{3}{2}}$ iterations to diffuse out of these regions left out before. Supposing that one does not have $r_\sigma(\frac{\pi}{2}) = 0$ for $p$-almost all $\sigma$, one can conclude that (19) holds for all initial conditions $\theta$. Consequently $I_1 = 1 + O(\lambda^{\frac{1}{2}})$ so that:

**Proposition 2** If $\lambda = 0$ is an hyperbolic anomaly of first order and first degree, and $r_\sigma(\frac{\pi}{2})$ does not vanish for $p$-almost all $\sigma$, one has

$$\gamma(\lambda) = \frac{1}{2} |\lambda \mu| + O(\lambda^{\frac{3}{2}}).$$

The argument above shows that the random phase dynamics is such that the angles $S_{n,\omega}(\theta)$ are for most $n$ and $\omega$ in a neighborhood of size $\lambda^{\frac{1}{2}}$ of the stable fixed points $\theta = 0, \pi$. This does not mean that for some $n$ and $\omega$, the angles are elsewhere; in particular, the rotation number of the dynamics does not vanish. However, this leads to corrections which do not enter into the lowest order term for the Lyapunov exponent.

### 5.3 Parabolic first degree anomaly

This may seem like a pathological and exceptional case. It turns out to be the mathematically most interesting anomaly of first degree, though, and its analysis is similar to that of the Lyapunov exponent near band edges (this will be discussed elsewhere). First of all, at a parabolic anomaly of first degree, there exists $M \in SL(2, \mathbb{R})$ allowing to attain the Jordan normal form:

$$E(M \partial_\lambda T_{\lambda,\sigma}|_{\lambda=0} M^{-1}) = E(P_\sigma) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (20)$$

Thus $E(\alpha_\sigma) = E(\beta_\sigma) = -i \frac{1}{\lambda}$ so that $\gamma(\lambda) = \frac{1}{2} \lambda \Im(I_1) + O(\lambda^2)$. Introducing the reference dynamics $\tilde{S}_n(\theta) = \theta + \frac{1}{2} (\cos(2\theta) - 1)$ as well as the centered perturbation $r_\sigma(\theta) = \Im(\alpha_\sigma + \frac{1}{2} - (\beta_\sigma + \frac{1}{2}) e^{2\theta})$, the dynamics can then be decomposed as in (17). Moreover, the argument leading to (18) directly transposes to the present case. However, this does not allow to calculate the leading order contribution, but shows that $\gamma(\lambda) = O(\lambda^{\frac{3}{2}})$. 

9
5.4 Second degree anomaly

At a second degree anomaly one has $E(\beta \sigma) = 0$, so in order to calculate the lowest order of the Lyapunov exponent one needs to, according to (14), evaluate $I_1$ and $I_2$. For this purpose let us introduce an analytic change of variables $Z : S^1 \to S^1$ using the density $\rho_0$ given in (10):

$$\hat{\theta} = Z(\theta) = \int_0^\theta d\theta' \rho_0(\theta').$$

According to Section 4, one expects that the distribution of $\hat{\theta}$ is the Lebesgue measure. We will only need to prove that this holds perturbatively in a weak sense when integrating analytic functions.

We need to study the transformed dynamics $\hat{S}_{\lambda, \sigma} = Z \circ S_{\lambda, \sigma} \circ Z^{-1}$ and write it again in the form:

$$\hat{S}_{\lambda, \sigma}(\hat{\theta}) = \hat{\theta} + \lambda \hat{p}_\sigma(\hat{\theta}) + \lambda^2 \hat{q}_\sigma(\hat{\theta}) + \frac{1}{2} \lambda^2 \hat{p}_\sigma(\hat{\theta}) \frac{\partial}{\partial \hat{\theta}} \hat{p}_\sigma(\hat{\theta}) + O(\lambda^3). \quad (21)$$

As, up to order $O(\lambda^3)$,

$$Z \circ S_{\lambda, \sigma}(\theta) = Z(\theta) + \left( \lambda p_\sigma(\theta) + \frac{1}{2} \lambda^2 p_\sigma(\theta) \frac{\partial}{\partial \theta} p_\sigma(\theta) + \lambda^2 q_\sigma(\theta) \right) \frac{\partial}{\partial \theta} Z(\theta) + \frac{1}{2} \lambda^2 p_\sigma^2 \frac{\partial^2}{\partial \theta^2} Z(\theta),$$

one deduces from

$$\frac{\partial}{\partial \theta} Z = \rho_0, \quad \frac{\partial^2}{\partial \theta^2} Z = \frac{2}{E(p_\sigma^2)} \left( C - \frac{1}{2} E(p_\sigma \frac{\partial}{\partial \theta} p_\sigma) \rho_0 + E(q_\sigma) \rho_0 \right),$$

that, with $\theta = Z^{-1}(\hat{\theta})$,

$$\hat{p}_\sigma(\hat{\theta}) = p_\sigma(\theta) \rho_0(\theta), \quad \hat{q}_\sigma(\hat{\theta}) = q_\sigma(\theta) \rho_0(\theta).$$

A short calculation shows that the expectation values satisfy

$$E(\hat{p}_\sigma(\hat{\theta})) = 0, \quad E(\hat{q}_\sigma(\hat{\theta})) = \frac{1}{2} E \left( \hat{p}_\sigma(\hat{\theta}) \frac{\partial}{\partial \hat{\theta}} \hat{p}_\sigma(\hat{\theta}) \right) - C, \quad (22)$$

where $C$ is as in (14).

Given any analytic function $\hat{f}$ on $S^1$, let us introduce its Fourier coefficients

$$\hat{f}(\hat{\theta}) = \sum_{m \in \mathbb{Z}} \hat{f}_m e^{im\hat{\theta}}, \quad \hat{f}_m = \int_0^{2\pi} \frac{d\hat{\theta}}{2\pi} \hat{f}(\hat{\theta}) e^{-im\hat{\theta}}.$$

There exist $a, \xi > 0$ such that $\hat{f}_m \leq a e^{-\xi|m|}$. We are interested in

$$\hat{I}_f(N) = \frac{1}{N} E \sum_{n=0}^{N-1} \hat{f}(\hat{\theta}_n),$$

where $\hat{\theta}_n = \theta_{n/N}$.
where, for sake of notational simplicity, we introduced \( \hat{\theta}_n = \hat{S}_{\lambda, \omega}^{\hat{\theta}} \) for iterations defined just as in (5).

**Lemma 1** Suppose \( E(p_{\sigma}^2) > 0 \) and that \( \hat{f} \) is analytic. Then

\[
\hat{I}_f(N) = \hat{f}_0 + \mathcal{O}(\lambda, (\lambda^2 N)^{-1}) ,
\]

with an error that depends on \( \hat{f} \).

**Proof.** Set \( \hat{r} = \frac{1}{2} E(p_{\sigma}^2) \). This is an analytic function which is strictly positive on \( S^1 \). Furthermore let \( \hat{F} \) be an auxiliary analytic function with Fourier coefficients \( \hat{F}_m \). Then, using (21) and the identities (22),

\[
\hat{I}_{\hat{F}}(N) = \frac{1}{N} \mathbb{E} \sum_{n=0}^{N-1} \sum_{m \in \mathbb{Z}} \hat{F}_m e^{i m \hat{\theta}_n} \left( 1 - m \lambda^2 C \right) + \mathcal{O}(m^3 \lambda^3, N^{-1})
\]

where the prime denotes the derivative. Hence we deduce that for any analytic function \( \hat{F} \)

\[
\hat{I}_{-C \hat{F}'} + (\hat{r} \hat{F}')'(N) = \mathcal{O}(\lambda, (\lambda^2 N)^{-1}) .
\]

Now, extracting the constant term, it is clearly sufficient to show \( \hat{I}_f(N) = \mathcal{O}(\lambda, (\lambda^2 N)^{-1}) \) for an analytic function \( \hat{f} \) with \( \hat{f}_0 = 0 \). But for such an \( \hat{f} \) one can solve the equation

\[
\hat{f} = -C \hat{F}' + (\hat{r} \hat{F}')'
\]

for an analytic and periodic function \( \hat{F} \) and then conclude due to (23). Indeed, by the method of variation of constants one can always solve (24) for an analytic \( \hat{F}' \). This then has an antiderivative \( \hat{F} \) as long as \( \hat{F}' \) does not have a constant term, i.e. the zeroth order Fourier coefficient of the solution of (24) vanishes. Integrating (24) w.r.t. \( \hat{\theta} \), one sees that this is precisely the case when \( \hat{f}_0 = 0 \) as long as \( C \neq 0 \). If on the other hand \( C = 0 \), then (24) can be integrated once, and the antiderivative \( \int \hat{f} \) of \( \hat{f} \) chosen such that \( \hat{r}^{-1} \int \hat{f} \) does not have a constant term. Then a second antiderivative can be taken, giving the desired function \( \hat{F} \) in this case. \( \square \)

In order to use this result for the evaluation of \( I_j(N) \) defined in (13), let us note that

\[
I_j(N) = \frac{1}{N} \mathbb{E} \sum_{n=0}^{N-1} e^{2ijZ^{-1}(\hat{\theta}_n)} .
\]

Hence up to corrections the result is given by the zeroth order Fourier coefficient of the analytic function \( \hat{f}(\theta) = e^{2ijZ^{-1}(\theta)} \), so that after a change of variables one gets:

\[
I_j(N) = \int_0^{2\pi} \frac{d\theta}{2\pi} \rho_0(\theta) e^{2ij\theta} + \mathcal{O}(\lambda, (\lambda^2 N)^{-1}) .
\]
Proposition 3 If \( \lambda = 0 \) is an anomaly of first order and second degree and \( \mathbb{E}(p_\sigma^2) > 0 \), then one has, with \( \rho_0 \) given by (10),

\[
\gamma(\lambda) = \frac{\lambda^2}{2} \Re \int_0^{2\pi} \frac{d\theta}{2\pi} \rho_0(\theta) \left[ \mathbb{E}(|\beta_\sigma|^2) + \mathbb{E}(\langle |P_\sigma|^2 + 2 Q_\sigma| \rangle |v|'\rangle) e^{2i\theta} - \mathbb{E}(\beta_\sigma^2) e^{4i\theta} \right] + \mathcal{O}(\lambda^3).
\]

In the above, anomalies of first degree were classified into elliptic, hyperbolic and parabolic. Second degree anomalies should be called 'diffusive' (strictly if \( \mathbb{E}(p_\sigma^2) > 0 \)). The random dynamics of the phases is diffusive on \( S^1 \), with a varying diffusion coefficient and furthermore submitted to a mean drift, also varying with the position. It does not seem possible to transform this complex situation into a simple normal form by an adequate basis change \( M \).

Let us note that the Lyapunov exponent at an anomaly does depend on the higher order term \( Q_\sigma \) in the expansion (2), while away from an anomaly it does not depend on \( Q_\sigma \). Of course, the coefficient of \( \lambda^2 \) in Proposition 3 cannot be negative. Up to now, no general argument could be found showing this directly (a problem that was solved in [SSS, Proposition 1] away from anomalies). For this purpose, it might be of help to choose an adequate basis change \( M \).

6 Examples

6.1 Center of band of the Anderson model

The transfer matrices of the Anderson model are given by

\[
T_{\lambda,\sigma} = \begin{pmatrix}
\lambda v_\sigma - E & -1 \\
1 & 0
\end{pmatrix},
\]

where \( v_\sigma \) is a real random variable and \( E \in \mathbb{R} \) is the energy. The band center is given at \( E = 0 \).

In order to study the behavior of the Lyapunov exponent at its vicinity, we set \( E = \epsilon \lambda^2 \) for some fixed \( \epsilon \in \mathbb{R} \). Then the associated family of i.i.d. random matrices has an anomaly of second order because

\[
T_{\lambda,\hat{\sigma}} = T_{\lambda,\sigma_2} T_{\lambda,\sigma_1} = - \exp \left( \lambda \begin{pmatrix} 0 & v_{\sigma_2} \\ -v_{\sigma_1} & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} -\frac{1}{2} v_{\sigma_1} v_{\sigma_2} & \frac{1}{2} - \epsilon \\ \epsilon & \frac{1}{2} v_{\sigma_1} v_{\sigma_2} \end{pmatrix} \right) + \mathcal{O}(\lambda^3),
\]

where \( \hat{\sigma} = (\sigma_2, \sigma_1) \). It follows that \( \alpha_{\hat{\sigma}} = \frac{1}{2\pi} (v_{\sigma_2} + v_{\sigma_1}) \) and \( \beta_{\hat{\sigma}} = \frac{1}{2\pi} (v_{\sigma_2} - v_{\sigma_1}) \). If now \( v_\sigma \) is not centered, then one has an elliptic anomaly of first degree and Proposition 3 combined with the factor 1/2 due to the order of the anomaly, implies directly (no basis change needed here) that

\[
\gamma(\lambda) = \frac{1}{8} \lambda^2 \left( \mathbb{E}(v_\sigma^2) - \mathbb{E}(v_\sigma)^2 \right) + \mathcal{O}(\lambda^3).
\]

If, on the other hand, \( v_\sigma \) is centered, one has a second degree anomaly and can apply Proposition 3. One readily verifies that \( \mathbb{E}(|\beta_\sigma|^2) = \frac{1}{2} \mathbb{E}(v_\sigma^2) \) and \( \mathbb{E}(\beta_\sigma^2) = - \frac{1}{2} \mathbb{E}(v_\sigma^2) \), and furthermore that the second term in Proposition 3 always vanishes, so that
\[ \gamma(\lambda) = \frac{1}{8} \lambda^2 E(v_\sigma^2) \int_0^{2\pi} \frac{d\theta}{2\pi} \rho_0(\theta) (1 + \cos(4\theta)) + \mathcal{O}(\lambda^3), \]

which is strictly positive unless \( v_\sigma \) vanishes identically (it was already supposed to be centered). If one wants to go further, \( E(|p_\sigma(\theta)|^2) = \frac{1}{2} E(v_\sigma^2)(1 + \cos^2(2\theta)) \). In the case \( \epsilon = 0 \), one then has \( E(Q_\sigma) = 0 \) so that \( \rho_0(\theta) = c(1 + \cos^2(2\theta))^{-\frac{1}{2}} \) with a normalization constant \( c \) that can be calculated by a contour integration. This proves the formula given in [DG].

### 6.2 A particular random dimer model

In the random dimer model, the transfer matrix is given by the square of (25), with a potential that can only take two values \( \lambda v_\sigma = \sigma v \) where \( v \in \mathbb{R} \) and \( \sigma \in \{-1, 1\} \) (e.g. [JSS]). Hence for a given energy \( E \in \mathbb{R} \) it is

\[ T^E_\sigma = \begin{pmatrix} \sigma v - E & -1 \\ 1 & 0 \end{pmatrix}^2. \]

Now the energy is chosen to be \( E = v + \lambda \). Then \( T_{\lambda,\sigma} = T^v_\sigma + \lambda \) has a critical point if \( |v| < 2 \). For the particular value \( v = \frac{1}{\sqrt{2}} \) fixing hence a special type of dimer model, one has:

\[ T_{\lambda,\sigma} = \begin{pmatrix} \frac{1}{\sqrt{2}} (\sigma - 1) - \lambda & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -\sigma - \sqrt{2} (\sigma - 1) \lambda & \frac{1}{\sqrt{2}} (1 - \sigma) + \lambda \\ \frac{1}{\sqrt{2}} (\sigma - 1) - \lambda & -1 \end{pmatrix} + \mathcal{O}(\lambda^2). \]

This family has now an anomaly of second order and first degree because

\[ (T_{\lambda,\sigma})^2 = \sigma \exp \left( \lambda \begin{pmatrix} \sqrt{2} (\sigma - 1) & -3 + \sigma \\ -3 - \sigma & -\sqrt{2} (\sigma - 1) \end{pmatrix} + \mathcal{O}(\lambda^2) \right). \]

One readily verifies that the determinant of \( E(M^{-1}P_\sigma M) \) is equal to \( 7 - 2E(\sigma) - E(\sigma)^2 \) and hence positive so that the anomaly is elliptic. Therefore the general result of Section 5.1 can be applied. Let us set \( e = E(\sigma) \). The adequate basis change (without normalization of the determinant) is

\[ M = \begin{pmatrix} \sqrt{7} - 2e - e^2 & 0 \\ \sqrt{2}(1 - e) & 3 - e \end{pmatrix}. \]

A calculation then gives

\[ P_\sigma = \frac{1}{3 - e} \begin{pmatrix} 2\sqrt{2}(\sigma - e) \\ 4(1-e)(\sigma-e)-(e-3)(7-\sigma-e-e^2) \sqrt{7} - 2e - e^2 \sqrt{7} - 2e - e^2 \end{pmatrix}, \]

allowing to extract \( \beta_\sigma \) and then \( E(|\beta_\sigma|^2) \), leading to (this contains a factor \( 1/2 \) because the anomaly is of second order)

\[ \gamma(\lambda) = \lambda^2 \frac{2(1 - e^2)}{(3 - e)^2} \left( 1 + \frac{2(e - 1)^2}{7 - 2e - e^2} \right) + \mathcal{O}(\lambda^3). \]
Note that if \( e = 1 \) or \( e = -1 \) so that there is no randomness, the coefficient vanishes. This special case was left out in [JSS] (the condition \( E(e^{4\eta}) \neq 1 \) in the theorems of [JSS] is violated). Within the wide class of polymer models discussed in [JSS], models with all types of anomalies can be constructed, and then be analyzed by the techniques of the present work.

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