Inhomogeneous Universe Models with Varying Cosmological Term

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Abstract

The evolution of a class of inhomogeneous spherically symmetric universe models possessing a varying cosmological term and a material fluid, with an adiabatic index either constant or not, is studied.

KEY WORDS: Decay of vacuum energy.
One of the main puzzles concerning our current understanding of the physical world is the present small value of the effective cosmological constant $|\Lambda| < 10^{-120} M_{Pl}^2$, or, which amounts to the same thing, the small value of the vacuum energy density $\rho_v = \Lambda/(8\pi G)$, as witnessed by cosmic observation. The status of the problem was reviewed by Weinberg [1]. A current of thought holds the view that the cosmological term is not really constant but its value decreases as the universe expands. The rationale behind this is that the energy of the vacuum should spontaneously decay into massive and massless particles, hence reducing $\Lambda$ to a value compatible with astronomical constraints, see e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. All these papers adhere to one or another “law” for the decay of $\rho_v$. However, as shown by Pavón [14] these laws should be restricted by statistical physics considerations, and so several of them may be ruled out.

All these studies were carried out on the assumption that the universe is homogeneous and isotropic, which is to some extent very reasonable, for it is believed the universe has been so at least since shortly after the beginning of the expansion. To our knowledge the only work departing from a Robertson-Walker background is the one by Beesham who simultaneously considers $\Lambda$ and $G$ evolving with time in a Bianchi type-I universe [15].

In this short article we consider a varying cosmological term in a class of inhomogeneous (but isotropic) universe. The reason to undertake such a study is that there is no reason a priori to believe that the universe was homogeneous and isotropic right back to the beginning of the expansion, and on the other hand there is the fact that the homogeneous and isotropic models cannot account for the high degree of homogeneity and isotropy so frequently ascribed to the present state of our universe. Further motivations to study inhomogeneous cosmologies can be found in [16] as well as in [17].

Let us consider the isotropic but inhomogeneous spherically-symmetric spacetime described by the plane Lemaître-Tolman-Bondi metric [18]

$$ds^2 = -dt^2 + Y^2 dr^2 + Y^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (Y = Y(r,t))$$

(1)

(the “prime” denotes partial derivative with respect to the radial coordinate $r$), the source of the metric being a material perfect fluid of energy density $\rho(r,t)$ and pressure $P(r,t)$ plus the quantum vacuum. The corresponding stress-energy tensor reads

$$T_{ab} = (\rho + P)u_au_b + \left(P - \frac{\Lambda}{8\pi G}\right) g_{ab},$$

(2)
where \( u_a = \delta_a^t \) is the fluid fourvelocity. The nontrivial Einstein’s equations take the form

\[
\rho + \Lambda = \frac{1}{Y^2 Y'} (\dot{Y}^2 Y')', \quad (3)
\]

\[
P - \Lambda = -\frac{1}{Y^2 Y'} (\dot{Y}^2 Y'), \quad (4)
\]

\[
\frac{\ddot{Y}}{Y} + \left( \frac{\dot{Y}}{Y} \right)^2 - \frac{\ddot{Y}'}{Y'} - \frac{\dot{Y} \dot{Y}'}{Y Y'} = 0, \quad (5)
\]

where the upper dot means partial derivative with respect to time and we have set \( 8 \pi G = 1 \). Introducing the change of variables

\[ Y = f^{2/3} \]

the last three equations become into

\[
\rho + \Lambda = \frac{4}{3} \frac{\dot{f} \ddot{f}}{f f'}, \quad (6)
\]

\[
P - \Lambda = -\frac{4}{3} \frac{\ddot{f}}{f}, \quad (7)
\]

and

\[
\dddot{f} - f \dddot{f} - f' \dddot{f} = 0, \quad (8)
\]

respectively. We incorporate to this system the widely-used equation of state for the material fluid, namely

\[
P = (\gamma - 1) \rho \quad (9)
\]

where very often the adiabatic index \( \gamma \) is considered constant, though on physical grounds it may also depend on time. The latter possibility is very natural since if the quantum vacuum decays into a mixture of massive and massless particles, \( \gamma \) must vary with time because both species of particles redshift at different rates. Equation (8) readily implies

\[
\ddot{f} - F(t) f = 0, \quad (10)
\]

where \( F(t) \) does not depend on the radial coordinate. Note in passing that, in the particular case, \( \gamma = 1 \) equations (7) and (10) lead to \( \Lambda = \Lambda(t) \). For \( \gamma \neq 1 \) we will have in general that \( \Lambda = \Lambda(r, t) \), though \( \Lambda = \Lambda(t) \) is also possible.
Introducing the factorization

$$f(r, t) = R(r) T(t)$$

in (10) we get

$$\ddot{T} - F T = 0. \quad (11)$$

The latter has two independent solutions, and therefore the general solution to (10) is

$$f(r, t) = R_1 T_1(t) + R_2 T_2(t).$$

However we will restrict ourselves to the simpler case

$$f(r, t) = R(r) T(t), \quad (12)$$

which is nonetheless fairly general. In this way (8) becomes an identity and equations (6) and (7) reduce to

$$\rho = \frac{4}{3} \left( \frac{\dot{T}}{T} \right)^2 - \Lambda \quad (13)$$

and

$$P = -\frac{4}{3} \ddot{T} + \Lambda, \quad (14)$$

respectively. Note that because of the above factorization any regular function $R(r)$ will satisfy (10). To get an equation for $T(t)$ we substitute the right hand sides of (13) and (14) in (9); it follows

$$T \dddot{T} + (\gamma - 1) \ddot{T}^2 - \frac{3}{4} \Lambda T^2 = 0. \quad (15)$$

To solve it we resort to the change of variables

$$T = Z^n \quad (n = \text{constant}), \quad (16)$$

which leads to $n = 1/\gamma$ and

$$\dddot{Z} - \frac{3}{4} \gamma \Lambda Z = 0. \quad (17)$$

Depending on the expression of $\Lambda$, which in what follows will be assumed position independent (the latter automatically implies that both $\rho$ and $P$ will depend on $t$ only), different cases arise.
1. For $\gamma$ and $\Lambda$ constants one obtains

\[ Z_1 = C_1 \cosh \left( \frac{\sqrt{3\gamma\Lambda}}{2} t + \varphi_1 \right), \quad (18) \]

\[ Z_2 = C_2 \sinh \left( \frac{\sqrt{3\gamma\Lambda}}{2} t + \varphi_2 \right), \quad (19) \]

and therefore

\[ Y_1 = R^{2/3}(r) C_1^{2/3\gamma} \cosh^{2/3\gamma} \left( \frac{\sqrt{3\gamma\Lambda}}{2} t + \varphi_1 \right), \quad (20) \]

\[ Y_2 = R^{2/3}(r) C_2^{2/3\gamma} \sinh^{2/3\gamma} \left( \frac{\sqrt{3\gamma\Lambda}}{2} t + \varphi_2 \right). \quad (21) \]

Note that solution (20) does not present initial singularity, but solution (21) has a singularity at $t = -\frac{2\varphi_2}{\sqrt{3\gamma\Lambda}}$. However both sets of solutions have a final inflationary stage.

2. For $\gamma = \text{constant}$ and

\[ \Lambda(t) = \frac{\lambda^2}{t^2} \quad (\lambda^2 = \text{constant}), \quad (22) \]

the corresponding differential equation can be integrated by using the ansatz $Z \propto t^m$ with $m = \text{constant}$. The general solution reads

\[ Z(t) = C_1 t^{m_+} + C_2 t^{m_-} \quad (23) \]

where $m_\pm = \frac{1}{2} \pm \sqrt{1 + 3\gamma\lambda^2 t^2}/2$. Inflationary solutions may occur for large enough $\lambda^2_0$.

3. For $\gamma = \text{constant}$ and

\[ \Lambda = \lambda^2 t^{n-2} \quad (n \neq 0, 2), \quad (24) \]

equation (17) becomes

\[ \ddot{Z} - \frac{3}{4} \gamma \lambda^2 t^{n-2} Z = 0 \quad (25) \]

and the general solution can be expressed as a combination of Bessel functions (see Ref. 18).
\[ Z = C_1 t^{1/2} J_{1/n} \left( \frac{\lambda_0}{n} \sqrt{-3\gamma} t^{n/2} \right) + C_2 t^{1/2} J_{-1/n} \left( \frac{\lambda_0}{n} \sqrt{-3\gamma} t^{n/2} \right). \]  

(26)

The behavior at the asymptotic limits depends on \( n \). For \( 0 < n < 2 \) one has the following. (i) When \( t \to 0 \) one obtains

\[ Z \sim C_1 t + C_2. \]  

(27)

One can choose \( C_2 = 0 \) to have the initial singularity at \( t = 0 \). (ii) When \( t \to \infty \) there follows

\[ Z \sim t^{\frac{1}{2} - \frac{n}{4}} \cos t^{n/2}. \]  

(28)

For \( n < 0 \) one has the following: (i) when \( t \to 0 \) one obtains \( Z \sim t^{\frac{1}{2} - \frac{n}{4}} \cos (t^{n/2} + \varphi) \). (ii) When \( t \to \infty \) one obtains \( Z \sim t \).

4. For \( \gamma = \text{constant} \) and

\[ \Lambda(t) = \lambda_0^2 + ce^{-\alpha t} \]  

(29)

where \( \lambda_0^2, c \) and \( \alpha \) are constants (with \( c < 0 \) for mathematical convenience), equation (17) becomes

\[ \ddot{Z} - \frac{3}{4} \gamma \left[ \lambda_0^2 + ce^{-\alpha t} \right] Z = 0 \]  

(30)

and the general solution can be expressed as a combination of Bessel functions (see Ref. 18)

\[ Z = C_1 J_{-\lambda_0/\sqrt{3\gamma}} \left( \frac{\sqrt{-3\gamma c}}{\alpha} e^{-\alpha t} \right) + C_2 J_{-\lambda_0/\sqrt{3\gamma}} \left( \frac{\sqrt{-3\gamma c}}{\alpha} e^{-\alpha t} \right), \]  

(31)

with
\[
C_2 = -\frac{J_\lambda_0 \alpha}{\lambda_0 \sqrt{-3\gamma}} \left( \frac{\sqrt{-3\gamma c}}{\alpha} \right) C_1
\]  

(32)

in order to fix the initial singularity at \( t = 0 \).

The asymptotic behavior near the initial singularity, when \( t \to 0 \), is given by

\[
Z \sim t.
\]  

(33)

At the final stage, when \( t \to \infty \) and \( \Lambda \to \lambda_0^2 \), one obtains the following asymptotic behavior

\[
Y \approx R^{2/3} (r) e^{\frac{\lambda_0}{\sqrt{-3\gamma}}} t.
\]  

(34)

Besides, from (21) we recover the same result in the far future.

For the particular case \( \lambda_0^2 = 0 \) the general solution of (30) is given by

\[
Z = C_1 J_0 \left( \frac{\sqrt{-3\gamma c}}{\alpha} e^{-\frac{\alpha}{2} t} \right) + C_2 Y_0 \left( \frac{\sqrt{-3\gamma c}}{\alpha} e^{-\frac{\alpha}{2} t} \right),
\]  

(35)

where \( Y_0 \) is the Weber function of the second kind and zero order. In the limit \( t \to \infty \) and \( \Lambda \to 0 \) the final behaviour of the solutions, obtained from (33) are

\[
Y \approx R^{2/3} (r) t^{2/3\gamma}.
\]  

(36)

The same result can be easily obtained from (7) by setting \( \Lambda = 0 \).

5. For \( \gamma = \gamma(t) \) and \( \Lambda = \Lambda(t) \) we shall find the expressions of both quantities and analyse the behavior of the solutions at late time. To do this we introduce the new variable \( s(t) \) in the following way

\[
T = T_0 e^{\int \frac{1}{\gamma} dt},
\]  

(37)

inserting (37) into (15) we get

\[
\ddot{s} - \frac{\gamma}{\gamma} \dot{s} + \frac{3}{4} \Lambda \gamma s = 0.
\]  

(38)
This equation can be identified with
\[ \ddot{s} + s^n \dot{s} + \frac{1}{(n+2)^2} s^{2n+1} = 0 \quad (n \neq -2), \quad (39) \]
which is reduced to a linear differential equation by making the substitution \[(19)\]
\[ s^n = \frac{n+2}{n} v^n c_1 + \int v^n dt, \quad (40) \]
obtaining
\[ \ddot{v} = 0, \quad v(\tau) = c_2 + c_3 \tau, \quad (41) \]
where \(c_1, c_2\) and \(c_3\) are arbitrary integration constants. Equations \((38)\) and \((39)\) are the same if we define
\[ -\dot{\gamma} = s^n, \quad (42) \]
and
\[ \frac{3}{4} \Lambda \gamma = \frac{1}{(n+2)^2} s^{2n}. \quad (43) \]
Without loss of generality we choose \(c_2 = -t_0\), where \(t_0\) is some initial time, and \(c_3 = 1\). So, the last system of equations can be easily solved to obtain
\[ \Lambda(t) = \frac{4C^2(t-t_0)^{2n}}{3\gamma_0 n^2(n+1)^2} \left[ 1 + \frac{(t-t_0)^{n+1}}{C} \right]^{\frac{2-n}{n}}, \quad (44) \]
and
\[ \gamma(t) = \gamma_0 \left[ 1 + \frac{(t-t_0)^{n+1}}{C} \right]^{-\frac{2+n}{n}}, \quad (45) \]
where \(\gamma_0\) and \(C\) are arbitrary integration constants. On the other hand, inserting \((41)\) in \((40)\), the general solution of the nonlinear equation \((39)\) is found to be
\[ s(t) = \left[ \frac{(n+1)(n+2)}{n} \frac{(t-t_0)^n}{C + (t-t_0)^{n+1}} \right]^{1/n}. \quad (46) \]
Now, taking into account that at late time, \(t \gg t_0\), we must have \(\gamma \rightarrow \gamma_0\), the restriction \(n < -1\) readily follows as can be seen from
In addition the cosmological term vanishes in the same limit. Now, using this approximation we evaluate $T(t)$ in (37), finding

$$T(t) \approx T_0 \left[ \frac{(n+1)(n+2)}{n} (t-t_0) \right]^{1/\gamma_0}, \quad (47)$$

and therefore

$$Y \approx R^{2/3}(r) T_0^{2/3\gamma_0} \left[ \frac{(n+1)(n+2)}{n} (t-t_0) \right]^{2/3\gamma_0}. \quad (48)$$

It is worthy of note that, for $t \gg t_0$ we have both $\gamma \rightarrow \gamma_0$ and $\Lambda \rightarrow 0$. So, using these limits in (21) we recover the solution given by (48).

To investigate the singular structure of the plane Lemaître-Tolman-Bondi metric (1), we calculate the curvature scalar by resorting to change of variables $Y = f^{2/3}$ used above

$$\mathcal{R} = 2 \frac{\dddot{f}}{f} + \frac{4}{3} \frac{\dot{f} \dddot{f}'}{f'} + 2 \frac{\dddot{f}'}{f'}, \quad (49)$$

and evaluate it at the points where the coefficients of the metric $Y^2$ and/or $Y'$ vanish. To do this we insert the Einstein equation (8) along with (12), (15) and (16) in (49), obtaining

$$\mathcal{R} = 4 \left[ \frac{4}{3} \Lambda + \frac{(2-\gamma) \dot{Z}^2}{\gamma^2 Z^2} \right]. \quad (50)$$

Then we replace in the scalar curvature the expansion of $\Lambda$ and the corresponding solutions near the point where they vanish. All the solutions we have found for $\gamma = \text{constant}$, except (20), have a singularity at $t = 0$, i.e. the big-bang singularity.

In summary we have found the coefficients of the Lemaître-Tolman-Bondi metric assuming that the early universe possessed a time varying cosmological term, and that the adiabatic index of the material fluid were either constant or not. (i) All the solutions we have derived contain an arbitrary function of the radial coordinate. (ii) For $\gamma = \text{constant}$ all the solutions, except (20) have a singularity at $t = 0$, i.e. the big-bang singularity. (iii) Constant as well as varying cosmological terms give rise asymptotically to exponential inflation -see (24), (25) and (14). (iv) For a varying cosmological term there exist solutions which behave as though the universe were asymptotically matter dominated at late times when $\gamma = 1$ -see (34).
None of the solutions found has a spatially-homogeneous limit for $t \to \infty$. This is so because no homogeneization mechanism, such as anisotropic pressures [24], was assumed. Such a more general study will be undertaken soon.

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