Complex temperatures zeroes of partition function in spin-glass models.

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Abstract

An approximate method is proposed for investigating complex-temperature properties of real-dimensional spin-glass models. The method uses the complex-temperature data of the ferromagnetic model on the same lattice. The universality line in the complex-temperature space is obtained.

1 Introduction.

Investigating of physical quantities in the complex plane can reveal new and unexpected effects. More than four decades have passed since Dyson [1] considered imaginary charges in electrodynamics. Having $\alpha = e^2/\hbar c < 0$, he got that this theory should be essentially unstable, and expansions in $\alpha$ are, at best, asymptotic. The consideration of complex magnetic field by Lee and Yang and complex temperatures by Fisher [3] opened new and effective method for investigating phase transitions. Later the method found large variety of applications in obtaining the order and type of possible phase transitions [3], critical indices [4][8], and recently has been connected with experiments on magnetization [10].

The method can be even more useful when investigating phase transitions in disordered systems, since many well-developed analytical and numerical methods are not suitable here. This point of view was first provided in [4] where so-called Graffiths’ singularities was discovered in statistical systems with random-fields. Recently, a strongly-frustrated but nonrandom system was investigated in complex temperature plane [4], and a continuous spectra of phase transitions was obtained.
The consideration of spin-glasses in complex-field and/or complex-temperature plane was started in [5][6]. Namely, Random Energy Model (REM) was investigated, which is the simplest but typical representative of spin-glasses [4]. This has been continued in [7] where the more physical dilute (finite-connectivity) REM [10][11] was investigated. In particular, we have pointed out also how the Dilute Generalized Random Energy Model (DGREM) should be considered in the complex planes [6]. This seems very important, because DGREM is nothing else, but the quite accurate approximation to the real-dimensional (non-mean-field) Edwards-Anderson (EA) spin-glass model. We believe that the results obtained with DGREM can be relevant for that model, and will help to clarify the properties of spin-glasses in finite dimensions. Recently, the lower critical dimension of the EA model has been estimated in this fashion [8]. This line of research will be continued in the present paper. We shall give the density of the partition function’s zeroes, and discuss in details its applications to the real-dimensional EA models.

2 Dilute REM

We shall start with repeating some facts about diluted REM and GREM.

The dilute p-spin glass model is described by the following hamiltonian

\[ H = - \sum_{(i_1 < i_2 < ... < i_p)=1}^{\alpha N} j_{i_1...i_p} s_{i_1}...s_{i_p} \]  

(1)

where only \( \alpha N \) couplings \( j_{i_1...i_p} \) are non-zero \( 1 \geq i \geq N \), and \( s_i = \pm 1 \) are Ising spins. At high temperatures the system is in the paramagnetic phase with free energy

\[ \ln Z = \alpha N \ln \cosh(\beta) + N \ln 2 \]  

(2)

At critical temperature \( T_c = 1/\beta_c \) a phase transition occurs into the SG phase

\[ \alpha g(\tanh \beta_c) = 1 \]  

(3)

where the function \( g(x) \) is defined as

\[ g(x) = \frac{1}{2} (1 + x) \ln(1 + x) + \frac{1}{2} (1 - x) \ln(1 - x) \]  

(4)
Below this temperature we have

\[ \ln Z/N = \alpha N \beta y \ln(2/\alpha) \] (5)

where \( y \) is inverse function to \( g(x) \)

\[ \frac{1}{2} (1 + y) \ln(1 + y) + \frac{1}{2} (1 - y) \ln(1 - y) = x \] (6)

3 Dilute GREM

Let us consider the diluted version of GREM, with infinite levels of the hierarchy. Now we have an infinite chain of DREM-s. At the interval \([v, v + dv]\) we have \( Ns'(v)/\ln 2 \) spins with \( Nzdv \) couplings. Function \( s(v) \) is monotonic, \( s(0) = 0, s(1) = \ln 2 \). Similarly to the case of dilute GREM at real \( T \) we found:

\[ -\frac{\beta F}{N} = z(1 - v_c(\beta)) \ln \cosh \beta + (\ln 2 - s(v_c) + z\beta \int_0^{v_2(\beta)} dv_0 y(s'(v_0)/z) \] (7)

where \( v_c(\beta) \) is defined from the equation

\[ \tanh(\beta) = y(s'(v_2)/z) \] (8)

For the case of Edwards-Anderson model placed on \( d \)-dimensional hypercubic lattice

\[ z = d, \quad v = -\frac{U}{Nd}, \quad s(v) = \ln 2 - \frac{S(-vdN)}{N} \] (9)

here \( U \) is energy, and \( S(U) \) is entropy as function of the energy for corresponding ferromagnetic Ising model. It is easy to derive from the definition of temperature

\[ \frac{dS}{dU} = \frac{1}{\tau} \equiv \tilde{\beta} \] (10)

So there is a connection between \( \tilde{\beta} \) and \( v \). For a given \( \tilde{\beta} \) we find energy of corresponding ferromagnetic model, and then calculate

\[ v = -U(\tilde{\beta}(v))/(Nd) \] (11)

We obtain for the free energy

\[ -\frac{\beta F}{N} = z(1 - v_c(\beta)) \ln \cosh(\beta(\ln 2 - n(v_c(\beta)))) + z\beta \int_0^{v_c(\beta)} dv_0 y(\tilde{\beta}(v_0)) \] (12)
4 Complex temperatures

Let us consider now the case of complex temperatures. Now we have 3 phases for REM.

PM: \[ \frac{\ln Z}{N} = \alpha Re \ln \cosh(\beta) + \ln 2 \] (13)

SG: \[ \frac{\ln Z}{N} = \alpha \beta_1 y(1/\alpha) \] (14)

LYF: \[ \frac{\ln Z}{N} = \frac{\alpha}{2} Re \ln \cosh(2\beta_1) + \frac{\ln 2}{2} \] (15)

There are boundaries between SG- PM, SG - LYF , PM - LYF.

For the PM-LYF we have a line

\[ \sin^2(\beta_2) = \frac{2^{1/\alpha} - 1 - \tanh^2 \beta_1}{2^{1/\alpha}(1 - \tanh^2 \beta_1)} \] (16)

This line begins at \( \beta_1 = 0 \) and is continued till intersection with SG-LYF line. For it we have

\[ \beta_1 = \beta_0, \infty > \beta_2 > \beta_{2c} \] (17)

where \( \beta_{2c} \) is defined from the intersection with another line, and \( \beta_{1c} \) from the equation

\[ \frac{\alpha}{2} \ln \cosh(2\beta_{1c}) + \ln 2 = \alpha \beta_{1c} y(\ln 2/\alpha) \] (18)

Then the third line PM-SG is defined from the equation

\[ \alpha Re \ln \cosh \beta + \ln 2 = \beta_{1c} \alpha y(\ln 2/\alpha) \] (19)

Let us vary the parameter \( \alpha \). We can construct some universal line for the critical \( \beta_{c,1}, \beta_{c,2} \). If we define function

\[ f(s, t) = \frac{\ln(1+t)^2 - \ln(1-t)^2[1 - (1-t^2)s]}{\ln(1+t)/(1-t)} \] (20)

then for the critical \( t = \tanh(\beta_1), s = \sin^2(\beta_2) \) we have an equation

\[ \frac{1}{2}(1+f(s, t)) \ln(1+f(s, t))+\frac{1}{2}(1-f(s, t)) \ln(1-f(s, t))] = \ln(1+t^2)/[1-(1-t^2)s] \] (21)
5 DGREM at complex T

Now we have

\[- \frac{\beta F}{N} = d(1 - v_2) \text{Re} \ln \cosh \beta + (\ln 2 - s(v_2)) + d(v_2(\beta) - v_1(\beta))/2 \ln \cosh 2 \beta_1 \]

\[+ [s(v_2) - s(v_1)]/(2) + d \beta_1 \int_0^{\tilde{\beta}_1} d\tilde{\beta}_0 y(\tilde{\beta}(v_0)) \]

(22)

The values of \(v_1, v_2\) are defined from the extremum condition. Integrating by parts in the last term we get

\[- \frac{\beta F}{N} = d(1 - v_2(\beta)) \text{Re} \ln \cosh \beta + (\ln 2 - s(v_2)) + d(v_2 - v_1)/2 \ln \cosh 2 \beta_1 \]

\[+ (s(v_2(\beta)) - s(v_1(\beta)))/2 - d \beta_1 \int_0^{\tilde{\beta}_1} d\tilde{\beta}_0 \frac{2v_0(\tilde{\beta}_0)}{\ln \frac{1 + y}{1 - y}} + d\tilde{\beta}_1 y(\tilde{\beta}_1) \]

where \(y\) as a function of \(\tilde{\beta}_0\) is defined from the equation

\[y = g\left(\frac{\ln 2}{\tilde{\beta}_0}\right) \]

(24)

function \(v_0(\tilde{\beta})\) is defined from the equation

\[v_0(\tilde{\beta}) = -U(\tilde{\beta})/(Nd) \]

(25)

Here \(U(\tilde{\beta}_1)\) is the energy of ferromagnetic model on the same lattice. The variables \(v_1,v_2\) we can define from the saddle point condition of free energy.

Let us collect all these results together. We consider some point of hierarchy \(v\). At inverse temperatures \(\beta_1\) it could be in SG or PM phase. If it was in SG phase, then it stays there while adding imaginary \(\beta_2\).

When at real \(\beta\) model was in PM phase there are three possible scenarios of behavior: 1) stay in PM; 2) become SG phase; 3) become LYF. We can look under another point. What happens with our system, when we vary \(\alpha\) at fixed values of \(\beta_1, \beta_2\)?

If our point at complex \(\beta\) space is under the line \([21]\), then our system could be in either SG or PM phase. Above that line all 3 phases are allowed for a different parts of hierarchy.

This line stays for any version of dilute GREM models. Thus it means some universality. It will be interesting to find similar universality classes in other mean field models.
6 LYF zeros at the border of PM SG phases

Let us first consider the case, when our system is under the line (21) and there is not LYF phase. We have an expression

\[ \ln \frac{Z}{N} = d(1 - v_c) Re \ln \cosh \beta + \ln 2 - s(v_c) + d\beta_1 \int_{0}^{v_c} dv_0 y(\tilde{\beta}(v_0)) \] (26)

where the value of \( v_c \) is defined from the saddle point condition

\[ d Re \ln \cosh \beta + s'(v_c) = d\beta_1 y(\frac{s'(v_c)}{z}) \] (27)

For the laplacian we have an expression

\[ \frac{d^2 f}{d\beta_1^2} + \frac{d^2 f}{d\beta_2^2} = \frac{\partial^2 f}{\partial^2 v_c} \left[ (\frac{dv_c}{d\beta_1})^2 + (\frac{dv_c}{d\beta_2})^2 \right] + 2 \frac{\partial^2 f}{\partial v_c \partial \beta_1} (\frac{dv_c}{d\beta_1}) + 2 \frac{\partial^2 f}{\partial v_c \partial \beta_2} (\frac{dv_c}{d\beta_2}) \] (28)

To calculate density we need in expressions

\[ s'(v) = d\tilde{\beta}, \quad y' = 1/\tilde{\beta}, \quad s''(v) = -\frac{d^2}{c(\tilde{\beta})}, \quad y(\tilde{\beta}_1) = \frac{Re \ln \cosh \beta + \tilde{\beta}_1}{\beta_1} \] (29)

here \( c \) is specific heat of ferromagnetic phase. Eventually:

\[ \pi \rho(\beta_1, \beta_2) = \] (30)

\[ -\frac{d^2}{s''(-1 + \beta_1 y')} \left\{ \left[ (y - \tanh \beta_1 - \frac{\tanh \beta_1 \sin^2 \beta_2}{\cosh \beta_1^2 (\sin^2 \beta_2 + \cos^2 \beta_2 \tanh^2 \beta_1)} \right]^2 + \left[ \frac{\sin \beta_2 \cos \beta_2}{\cosh \beta_1^2 (\sin^2 \beta_2 + \cos^2 \beta_2 \tanh^2 \beta_1)} \right]^2 \right\} \]

7 Three phases

When our point in complex \((\beta_1, \beta_2)\) space is over the universality line, we have the following expression for the laplacian:

\[ \frac{d^2 f}{d\beta_1^2} + \frac{d^2 f}{d\beta_2^2} = f''_{\beta_1 \beta_1} + 2f''_{\beta_1 v_1} v'_1, \beta_1 + f''_{v_1 v_1} (v'_1, \beta_1)^2 + 2f''_{\beta_1 v_2} v'_2, \beta_1 + f''_{v_2 v_2} (v'_2, \beta_1)^2 \] (31)
For the $v_1$ we have an equation

\[
\frac{s'(v_1)}{2} + \frac{d}{2} \ln \cosh 2\beta_1 = \beta_1 y \left( \frac{s'(v_1)}{d} \right) d
\]  

(32)

Having values $v_1, v_2$, we can define the values of corresponding $\tilde{\beta}_1, \tilde{\beta}_2$ by means of formula (25). Equation for the $v_2$:

\[
s'(v_2) + d \ln \cosh \beta_1 + d/2 \ln (\cos^2 \beta_2 + \tanh^2 \beta_1 \sin^2 \beta_2) = s'(v_2)/2 + d/2 \ln \cosh 2\beta_1
\]  

(33)

Eventually we have for zeros density:

\[
\pi \rho(\beta_1, \beta_2) = \frac{2x(v_2 - v_1)}{\cosh^2(2\beta_1)} - \frac{d^2 (-\tanh 2\beta_1 + y(\tilde{\beta}_1))^2}{(\beta_1 y' - 1/2)s''(v_1)} + \frac{2d^2}{s''(v_2)} \left\{ \left[ \frac{\sin \beta_2 \cos \beta_2}{[\cosh \beta_1(\cos \beta_2^2 + \tanh \beta_1 \sin \beta_2^2)]^2} \right]^2 \right. \\
+ [\tanh 2\beta_1 - \tanh \beta_1 - \frac{\sin^2 \beta_2 \sinh \beta_1}{\cosh \beta_1(\cos \beta_2^2 + \tanh \beta_1 \sin \beta_2^2)}] \\
\left. + \frac{2d^2}{s''(v_2)} \right\}
\]  

(34)

One can use formulas (30) and (34) to calculate density of zeros, having the data of the ferromagnetic model at the same lattice.

8 Conclusion.

This paper devoted to the approximate method, which allows to investigate the zeroes of the statistical sum for Edwards-Anderson models in real physical dimensions. The key point of the method is in the using the rich phase spectrum of Dilute Generalized Random Energy Model. We obtained a universality line in the phase diagram of the model. It hardly controls the corresponding complex-temperature behavior. The various notions that entered our discussion give good hope for the applicability of Random Energy-type models to realistic systems.

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