Secure Classical Bit Commitment using Fixed Capacity Communication Channels

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Abstract

If mutually mistrustful parties A and B control two or more appropriately located sites, special relativity can be used to guarantee that a pair of messages exchanged by A and B are independent. In earlier work, we used this fact to define a relativistic bit commitment protocol, RBC1, in which security is maintained by exchanging a sequence of messages whose transmission rate increases exponentially in time. We define here a new relativistic protocol, RBC2, which requires only a constant transmission rate and could be practically implemented. We prove that RBC2 allows a bit commitment to be indefinitely maintained with unconditional security against all classical attacks. We examine its security against quantum attacks, and show that it is immune from the class of attacks shown by Mayers and Lo-Chau to render non-relativistic quantum bit commitment protocols insecure.

Key words: bit commitment, relativistic cryptography, quantum cryptography.

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I. INTRODUCTION

The field of quantum cryptography was opened up by Wiesner’s early investigation [21] of the application of quantum information to cryptography, and the ensuing discoveries by Bennett and Brassard of secure quantum key distribution [1] and by Ekert of entanglement-based quantum key distribution [5]. Quantum cryptography is now a flourishing area of theoretical research. Its successes raise a broader theoretical question: it would be very interesting to know precisely which cryptographic tasks, other than key distribution, can be implemented in such a way that their security is guaranteed by physical principles, without any additional computational assumptions. An important first step would be to establish precisely which physical principles can guarantee cryptographic security in some serious application.

This paper focusses on one task, bit commitment, and one physical principle, the impossibility of superluminal signalling: that is, of sending signals faster than light speed. We begin by briefly explaining both.

Roughly speaking — precise definitions are given in the next section — bit commitment is the cryptographic version of a securely sealed envelope. In the commitment phase of a bit commitment protocol, Alice supplies Bob with data that commit her to the value of a bit, without allowing Bob to infer that value. In the unveiling phase, which takes place after commitment, if and when Alice wishes, she supplies Bob with further data that reveal the value of the bit to which she was committed. We are particularly interested in protocols which are unconditionally secure, in the sense that the laws of physics imply that neither party can cheat, regardless of the technology or computing power available to them.

The impossibility of faster than light signalling is guaranteed if the standard understanding of causality within Einstein’s special theory of relativity is correct. To characterise its cryptographic relevance requires a little more discussion: readers without a background in physics may wish to skip the technical details in the remainder of this paragraph, which are not needed to understand the main part of the paper. We assume that physics takes place in flat Minkowski spacetime, with the Minkowski causal structure. This is not exactly correct, of course: according to general relativity and to experiment, spacetime is curved. But it is true to a good enough approximation for any protocol implemented on or near Earth. In principle, the timing constraints of our protocol should take into account the error in the
approximation. Other than this, the known corrections arising from general relativity do not affect our security discussion. Discussion of the hypothetical cryptographic relevance of other more exotic and speculative general relativistic phenomena can be found in Ref. [6].

Nothing in science is beyond doubt, but it seems fair to say that the impossibility of superluminal signalling provides as solid a foundation for a security argument as a cryptologist could wish for. It would take a major scientific revolution for our protocols to be revealed as insecure because special or general relativity turned out to be incorrect — just as it would for quantum key distribution or other quantum cryptography protocols to be revealed as insecure because quantum theory proved to be incorrect. (That said, it should be added that at the time of writing the parallel between the two cases is not perfect: some quantum key distribution protocols have been proven secure against eavesdroppers equipped with arbitrarily powerful quantum computers, while we have not so far been able to prove that our bit commitment protocols are secure against general quantum attack.)

The recent history of work on secure physical implementations of bit commitment is complex and interesting. Initially, attention was focussed entirely on non-relativistic quantum protocols. Bennett and Brassard [1] described a simple quantum bit commitment protocol which is secure against both parties given current technology; they also pointed out that it would be insecure against a committer who is able to make and store entangled singlet states. Some time later, Brassard, Crépeau, Jozsa and Langlois (BCJL) [3] produced a quantum bit commitment protocol which, at the time, they claimed to have proved to be unconditionally secure against attacks by either party.

The subject was then transformed by the remarkable and celebrated insights of Mayers and Lo-Chau. A detailed history would need to consider unpublished work and also earlier papers that were circulated or temporarily archived (e.g. [12]). Those who actively participated in these developments are best placed to supply such a history, and Mayers’ and Lo-Chau’s accounts can be found in the references cited below. Not having been an active participant, I here discuss only Mayers’ and Lo-Chau’s published or permanently archived papers.

Lo and Chau [10, 11] pointed out the existence of an inherently quantum cheating strategy which implies that quantum bit commitment protocols that are perfectly secure against Bob are completely insecure against Alice. Essentially the same point was made by Mayers [13, 14, 15] at around the same time. Mayers showed further that the essential intuition
underlying this strategy can be extended to imply that, for a large class of quantum schemes, not only is perfect security against both Alice and Bob impossible, but security in the usual cryptographic sense — in which non-zero cheating probabilities are tolerated provided that they can be made arbitrarily small by adjusting a security parameter — is also impossible. In particular, as Lo and Chau argued [11] and Mayers rigorously proved [13], the BCJL protocol is insecure.

Mayers [15] briefly discussed the possibility of using relativistic signalling constraints for bit commitment, and suggested that his version of the no-go theorem should also apply to relativistic protocols. Brassard, Crépeau, Mayers and Salvail (BCMS) [4] subsequently examined one possible strategy for bit commitment based on temporary relativistic signalling constraints, based on an earlier protocol of Ben-Or, Goldwasser, Kilian and Widgerson (BGKW) [2], and showed that it was indeed insecure against quantum attacks.

In fact, though, relativistic protocols can evade the Mayers and Lo-Chau no-go theorems [6]. Ref. [6] describes a relativistic bit commitment protocol which iteratively combines BGKW’s bit commitment technique with relativistic signalling constraints. The protocol does not belong to the classes considered by Mayers and Lo-Chau, and their no-go results do not apply: in particular, the model is demonstrably not vulnerable to the Mayers-Lo-Chau cheating strategy. It allows bit commitments to be maintained indefinitely, and is conjectured to be unconditionally secure against both classical and quantum attacks. However, it has a serious practical weakness, as it requires the communication rates between the parties to increase exponentially in time in order to sustain the commitment [6].

The main point of this paper is to describe a new relativistic protocol which allows a committer and recipient, who each control more than one suitably located separated site, to sustain a bit commitment indefinitely. The protocol requires only a constant rate of communication between the parties. The communications need to be continued indefinitely to maintain the commitment. It is shown that the protocol is unconditionally secure against all classical attacks by either party. I conjecture that it is also unconditionally secure against all quantum attacks.

This represents theoretical progress, and also opens up new possibilities for practical cryptography. The relativistic protocol described in Ref. [6] is impractical in a rather strong sense: the laws of physics appear to preclude using it to maintain a commitment indefinitely, since current physical theories imply an upper bound on the attainable rate of communica-
tions emanating from any finite spatial region. That protocol aside, all previous methods of bit commitment have been able to guarantee security only modulo the assumed computational difficulty of some task (such as factoring a large number) or modulo the assumed physical difficulty of some process (such as breaking into a locked safe, or storing entangled quantum states [1]). That is, all previous bit commitment protocols have in principle been insecure against a sufficiently technologically advanced cheater.

The new protocol described here not only has the theoretical virtue of needing only a constant communication rate, but also, as we explain below, can be implemented with current technology. If, as we conjecture, the protocol is secure against all quantum attacks as well as classical attacks, it gives a practical solution to the bit commitment problem which is unconditionally secure in the sense that it relies on no computational complexity assumptions, classical or quantum. It relies on no physical assumptions either, other than the well-established principle that signals cannot travel faster than light.

For completeness, we note that Popescu [19] subsequently pointed out that Mayers’ and Lo-Chau’s models of cryptographic protocols neglect the existence of quantum superselection rules. One might perhaps think this gives another way of using a physical principle to ensure the security of a quantum bit commitment scheme, but this possibility was later excluded by Mayers, Kitaev and Preskill [17].

II. DEFINITIONS AND CONVENTIONS

For the main part of this paper, we assume that Alice’s and Bob’s actions are limited to those allowed by classical physics, under which heading we include special relativity but not general relativity nor, of course, quantum theory. This means that in particular we allow the parties in the protocol to send, share and manipulate classical information but not quantum information. We are thus content for the moment to use a definition of bit commitment that is adequate in the context of classical information theory, and postpone till section 7 discussion of the subtleties that arise when considering quantum information.

First, we define the general form of a classical bit commitment protocol. A protocol must require Alice and Bob to exchange classical information according to prescribed rules, with probability distributions prescribed whenever they are required to make random choices, and perhaps with some prescribed constraints on the places from and times at which the
information is sent. In Alice’s case, the rules should depend on the value of the bit $b$ to which she wishes to commit herself. The protocol must include a commitment phase with a definite end, after which Alice is, if she has followed the protocol correctly, committed to a bit $b$ of her choice. It may also include a sustaining phase after the commitment, during which both parties continue communications and/or other operations in order to maintain the commitment. And it must include an unveiling phase, which Alice can initiate if she chooses, in which she unveils the committed bit to Bob by supplying further information which is supposed to convince him that she was committed to the bit $b$.

An attempted unveiling of the bit $b$ is any attempt by Alice to use the unveiling procedure, at any point after the commitment, by sending information to Bob in the hope that it constitutes satisfactory evidence (as defined by the protocol) that Alice was committed to the bit $b$. An attempted unveiling of the bit $b$ might fail either by producing satisfactory evidence that Alice was actually committed to the bit $\bar{b}$, or else by failing to produce satisfactory evidence that she was committed to either bit value. A successful unveiling of the bit $b$ is an attempt by Alice to use the unveiling procedure which succeeds in producing satisfactory evidence (as defined by the protocol) that she was committed to the bit $b$. Note that in principle this could happen even though Alice has not actually followed the protocol for committing and unveiling the bit $b$.

For the protocol to be perfectly secure against Bob it must guarantee that, however he proceeds, he cannot obtain any information about the committed bit unless and until Alice chooses to unveil it. For it to be perfectly secure against Alice, it must guarantee that, however she proceeds, after the point at which she is supposed to be committed, either her probability of successfully unveiling 0 is zero or her probability of successfully unveiling 1 is zero. The protocol is secure against Bob if it includes some security parameter $n$ and has the property that the expectation value of the Shannon information Bob can obtain about the committed bit is bounded by some function $\epsilon(n)$ which tends to zero as $n$ tends to infinity. It is secure against Alice if, after the point at which she is supposed to be committed, it guarantees that either her probability of successfully unveiling 0 or her probability of successfully unveiling 1 is less than $\epsilon'(n)$, which again should tend to zero as $n$ tends to infinity. For the protocol to be perfectly reliable, it must guarantee that, if Alice honestly follows the protocol to commit and later unveil the bit $b$, the probability of her unveiling being successful is one. For it to be reliable, it must guarantee that this probability is greater
than $1 - e''(n)$, where $e''(n)$ again should tend to zero as $n$ tends to infinity.

There may be a case for treating reliability and security on an equal footing. However, the usual definition of bit commitment requires security and perfect reliability, but not necessarily perfect security, and we follow this convention here, since the distinction makes no essential difference to our discussion. That is, we require that a bit commitment protocol should be secure against both parties and perfectly reliable. If a protocol can guarantee this, regardless of the technology or computing power available to the parties, within the model defined by some physical theory, then it is *unconditionally secure* within that theory. The relevant physical theory here — until we consider the role of quantum information, in section [VII] onward — is relativistic classical physics in Minkowski space.

We use units in which the speed of light $c = 1$ and choose inertial coordinates, so that the minimum possible time for a light signal to go from one point in space to another is equal to their spatial separation. We consider a cryptographic scenario in which coordinates are agreed by Alice and Bob, who also agree on two points $x_1, x_2$. Alice and Bob are required to erect laboratories, including sending and receiving stations, within an agreed distance $\delta$ of the points, where $\Delta x = |x_1 - x_2| \gg \delta$. These laboratories need not be restricted in size or shape, except that they must not overlap.

We refer to the laboratories in the vicinity of $x_i$ as $A_i$ and $B_i$, for $i = 1$ or 2. To avoid unnecessarily proliferating notation, we use the same labels for the agents (sentient or otherwise) assumed to be occupying these laboratories. The agents $A_1$ and $A_2$ may be separate individuals or devices, but we assume that they are collaborating with complete mutual trust and with completely prearranged agreements on how to proceed, to the extent that for cryptanalytic purposes we can identify them together simply as a single entity, Alice ($A$); similarly $B_1$ and $B_2$ are identified as Bob ($B$).

As usual in defining a cryptographic scenario for a protocol between mistrustful parties, we suppose Alice and Bob each trust absolutely the security and integrity of their own laboratories, in the sense that they are confident that all their sending, receiving and analysing devices function properly and also that nothing within their laboratories can be observed by outsiders. They also have confidence in the locations of their own laboratories in the agreed coordinate system, and in clocks set up within their laboratories. However, neither of them trusts any third party or channel or device outside their own laboratory. We also assume that $A_1$ and $A_2$ either have, or can securely generate as needed during the protocol,
an indefinite string of shared secret random bits.

Strictly speaking, if $A_1$ and $A_2$ are disconnected laboratories, this arrangement requires $A$ to trust something outside her laboratories — for instance the coordinates of distant stars — in order to establish their relative locations, and similarly for $B$. Purists in the matter of cryptographic paranoia might thus instead require that $A_1$ and $A_2$ are separated regions within one large laboratory controlled by $A$ (which must be long in one dimension, of size order $\Delta x$, but could be small in the other two dimensions), and similarly for Bob’s agents, with $A$’s and $B$’s laboratories still of course kept disjoint. In this case, agents in $A_1$ and $A_2$ can establish their relative separation by measurements within $A$’s laboratory; similarly agents in $B_1$ and $B_2$ need only carry out measurements within $B$’s laboratory. Though the participants still need to believe that Minkowski space is a good approximate description of the world outside their laboratories in order to have confidence in the protocol, they then need trust nothing outside their laboratories in order to trust their implementation of it. This arrangement also allows $A_1$ and $A_2$ to use a secure channel within their shared laboratory to share secret random bits as needed during the protocol, eliminating the need for them either to share an unbounded string of secret random bits or to establish a secure external channel between two disconnected laboratories.

To ensure in advance that their clocks are synchronised and that their communication channels transmit at sufficiently near light speed, the parties may check that test signals sent out from each of Bob’s laboratories receive a response within time $4\delta$ from Alice’s neighbouring laboratory, and vice versa. However, the parties need not disclose the precise locations of their laboratories in order to implement the protocol. Nor need Alice or Bob take it on trust that the other has set up laboratories in the stipulated region. (A protocol which required such trust would, of course, be fatally flawed.) The reason is that each can verify that the other is not significantly deviating from the protocol by checking the times at which signals from the other party arrive. For each party can verify from these arrival times, together with the times of their own transmissions, that particular specified pairs of signals, going from Alice to Bob and from Bob to Alice, were generated independently — and this guarantee is all that is required for security.

Given a laboratory configuration as above, one can set out precise timing constraints for all communications in a protocol in order to ensure the independence of all pairs of signals which are required to be generated independently. We may use the time coordinate in the
agreed frame to order the signals in the protocol. (Without such a convention there would be some ambiguity, since the time ordering is frame dependent).

In fact, the protocols we consider are unconditionally secure against Bob regardless of his actions: Alice needs no guarantees about the location of his laboratories or the independence of the messages he sends during the protocol. However, Bob needs some guarantees about the independence of Alice’s messages, which can be ensured by the following arrangement.

We say that two spacetime regions $P$ and $Q$ are \textit{sufficiently spacelike separated} if it is the case that if Alice receives a message sent from within the region $P$ and sends a reply which is received by Bob within that region, and receives a message from within $Q$ and sends a reply which is received by Bob within $Q$, then Bob will be assured that the first reply was generated independently of the second received message and the second reply was generated independently of the first received message. Now we choose two sequences of spacetime regions $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ with the properties that:

(i) $P_1$ is sufficiently spacelike separated from $Q_1$, $Q_1$ from $P_2$, $P_2$ from $Q_2$, $Q_2$ from $P_3$ and so on;

(ii) the spacetime regions $P_1, P_2, \ldots$ are defined by the same region $P$ in space, namely the ball of radius $\delta$ around the point $x_1$, and successive disjoint time intervals $[s_1, s'_1], [s_2, s'_2], \ldots$; similarly the $Q_j$ are defined by the same spatial region $Q$, namely the ball of radius $\delta$ around the point $x_2$, and successive disjoint time intervals $[t_1, t'_1], [t_2, t'_2], \ldots$;

(iii) the regions are strictly time ordered in the agreed time coordinate, with ordering $P_1, Q_1, P_2, Q_2, \ldots$, so that we have $s_1 < s'_1 < t_1 < t'_1 < s_2 < s'_2 < t_2 < t'_2 < \ldots$;

(iv) the time intervals are all of the same length $\Delta t$, where $\Delta t > 4\delta$ and $\Delta t \ll \Delta x$.

Note that these conditions allow a single agent (person or device) $A_1$ to be responsible for all Alice’s communications from the regions $P_i$ and a second agent $A_2$ to be responsible for all her communications from the regions $Q_i$, and similarly for Bob, as suggested by the earlier discussion.

It is then easy to define timing constraints, in terms of $\delta$, $\Delta x$ and $\Delta t$, which, if respected by $A$ and $B$, and if $A$’s and $B$’s laboratories are sited as prescribed, ensure that $A_1$ can receive a message from $B_1$ sent from within $P_1$ and send a reply which will be received within $P_1$, then that $A_2$ can receive a message from $B_2$ sent from within $Q_1$ and send a reply...
which will be received within $Q_1$, then that $A_1$ can receive a message from $B_1$ sent from within $P_2$ and send a reply which will be received within $P_2$, and so on. Each message-reply exchange constitutes one \textit{round} of the commitment protocol. To keep the notation simple in the following discussion, we take these constraints as implicitly specified, and identify rounds of the protocol by the spacetime region $P_i$ or $Q_i$ in which they take place.

III. THE RELATIVISTIC BIT COMMITMENT PROTOCOL \textit{RBC1}

We now recall the bit commitment protocol — call it \textit{RBC1} — described in Ref. [6]. Alice and Bob first agree a large number $N$, a security parameter for \textit{RBC1}. In what follows we take $N = 2^m$ to be a power of 2, and where convenient also refer to $m$ as the security parameter. Taking $N$ to be a power of 2 simplifies the description of the protocol, since it allows the various random numbers generated and transmitted by the parties to be efficiently coded in binary. Note, however, that both \textit{RBC1} and \textit{RBC2} (defined below) can be defined for any $N$. Optimally efficient implementations — those which attain a given security level with minimal communication requirements — may generally require $N$ to be other than a power of 2.

All arithmetic in the protocol is carried out modulo $N$. Before the protocol begins, $A_1$ and $A_2$ agree on a list $\{m_1, m_2, \ldots\}$ of independently chosen random integers in the range $0 \leq m_i < N$. $B_1$ and $B_2$ also need to generate lists of random pairs of integers $(n_{j,0}, n_{j,1})$ in the range $0 \leq n_{j,0}, n_{j,1} < N$; these numbers, which need not be agreed between the $B_i$ in advance, are drawn from independent uniform distributions, with the constraint that $n_{j,0} \neq n_{j,1}$ for each $j$.

In the first round of the protocol, which takes place within $P_1$, $B_1$ sends $A_1$ the labelled pair $(n_{1,0}, n_{1,1})$. On receiving these numbers, $A_1$ returns $n_{1,b} + m_1$ in order to commit the bit $b$. This completes the commitment phase.

The second and later rounds of the protocol constitute the sustaining phase. In the second round, which takes place within $Q_1$, $B_2$ asks $A_2$ to commit to him the binary form $a_{m-1}^1 \ldots a_0^1$ of $m_1$. This is achieved by sending $A_2$ the labelled list of $m$ pairs $(n_{2,0}, n_{2,1}), \ldots, (n_{m+1,0}, n_{m+1,1})$. $A_2$ returns $n_{2,a_0^1} + m_2, \ldots, n_{m+1,a_{m-1}^1} + m_{m+1}$.

In the third round, which takes place within $P_2$, $B_1$ asks $A_1$ to commit in similar fashion the binary forms of the random numbers $m_2, \ldots, m_{m+1}$ used by $A_2$; in the fourth round,
which takes place within $Q_2$, $B_2$ asks $A_2$ to commit the binary forms of the random numbers $m_{m+2}, \ldots, m_{m^2+m+1}$ used by $A_1$ in the third round commitment. And so on; communications are continued indefinitely in order to sustain the commitment, unless and until Alice chooses to unveil the committed bit or either party chooses to abandon the protocol.

In fact, the protocol’s security is ensured provided that the differences $n_{j,1} - n_{j,0}$ are random elements of \{0, \ldots, N - 1\}, even if the integers $n_{j,0}$ or $n_{j,1}$ are not. A natural alternative convention would thus be to choose $n_{j,0} = 0$ for all $j$ and take the $n_{j,1}$ to be independently randomly chosen in the range $0 < n_{j,1} < N$. If $A$ and $B$ adopt this convention, there is obviously no need for $B$ to transmit the values of $n_{j,0}$. We have kept the option of general $n_{j,0}$ partly for notational convenience, but also because we have no proof that every possible level of security is most efficiently attained by setting $n_{j,0} = 0$.

We now define the unveiling protocol. Define the index $\bar{i}$ to be the alternate value to $i$; e.g. $A_1 = A_2$. Either or both of the $A_i$ may choose to unveil the originally committed bit. For $A_i$ to unveil, she reveals to $B_i$ the set of random numbers used by $A_{\bar{i}}$ in $A_{\bar{i}}$’s last set of commitments, sending the signal sufficiently early that when $B_i$ receives it he will be guaranteed that it was generated independently of the message sent by $B_{\bar{i}}$ to $A_{\bar{i}}$ to initiate this last set of commitments. To check the unveiling, $B_i$ sends the unveiling data to $B_{\bar{i}}$, who checks through the data to ensure that all the commitments in the protocol are consistent and correspond to a valid commitment of a bit $b$ in the first round. If so, he accepts that Alice was genuinely committed to the bit $b$ from the point at which the first round was completed.

To allow $A_1$ to unveil soon after the first round — which the protocol $RBC2$, discussed below, requires — we need to vary this procedure, since there is no previous round of commitments about which she can supply data. $A_1$ can unveil after the first round by revealing to $B_1$ the set of random numbers which will be used by $A_2$ in the second round, sending the signal sufficiently early that when $B_1$ receives it he will be guaranteed that it was generated independently of the messages sent by $B_2$ to $A_2$ to initiate the second round of commitments.

Since it takes a finite time for the relevant signals to reach Bob’s agents, the protocol is not immediately completed by Alice’s unveiling message. It is the need to wait for receipt of information that is unknown to the unveiler $A_i$ (since it depends on the last set of pairs sent by $B_i$) and to the unveilee $B_i$ (since it includes the last set of commitments sent by $A_i$)
which ensures that the protocol is not vulnerable to a Mayers-Lo-Chau quantum attack.

*RBC1* requires the \( A_i \) and \( B_i \) to exchange sequences of strings of exponentially increasing length, each exchange taking place within a time interval of length less than \( \Delta t \). This requires exponentially increasing communication rates. It also requires either that the \( A_i \) previously generated a list of shared secret random numbers whose length depends exponentially on the duration of the protocol, or else that they generate and securely share such a list during the protocol. The second option requires an exponentially increasing secure communication rate, and also requires the generation of random numbers at an exponentially increasing rate. The \( B_i \) also need to generate a sequence of random numbers whose length depends exponentially on the duration of the protocol, though they do not need to share them. In the following sections we define a refinement of *RBC1* that avoids all these impractical features.

It is perhaps worth noting here that the arrangement of \( A_1 \) and \( A_2 \) used above for *RBC1*, and in a later section for the improved protocol *RBC2*, represents just one possible configuration of agents. It might sometimes be useful in practice to have more agents participating in the protocol, or to allow the agents to be more mobile, or both. For instance, one might imagine applications in which it is important for \( A \) to be able to allow any of her agents, anywhere, to initiate an unveiling as soon as they learn some critical fact. One way of doing this is to use the above arrangement for the commitment and sustaining phases of the protocol, but to arrange in addition that \( A \) and \( B \) maintain large numbers of additional agents densely distributed in the same spatial region, with all Alice’s random bits shared among all her agents.

**IV. RUDICH’S SCHEME FOR LINKING COMMITTED BITS**

In the following section we improve *RBC1* using a technique developed (for a different purpose) by Rudich [20]. Since Rudich’s idea applies to any type of classical bit commitment, it is most naturally described abstractly — so, in this section, we need not consider separated laboratories, relativistic signalling constraints, timings or locations.

Suppose that \( A \) and \( B \) are equipped with a black box oracle that generates secure commitments to \( B \) of bits specified by \( A \), and unveils the bits to \( B \) if (and only if) \( A \) requests it to. Suppose now that \( A \) wishes to commit to two bits, \( b_1 \) and \( b_2 \), in a way that will allow her subsequently to demonstrate to \( B \) that the committed bits are equal, without giving \( B \) any
information about their shared bit value $b = b_1 = b_2$, and while still retaining a commitment of this shared bit $b$, which can subsequently be unveiled if she chooses. Rudich some time ago [20] pointed out a simple and elegant way to achieve this. A version of Rudich’s scheme, slightly modified to adapt it for relativistic bit commitments, follows.

First, $A$ and $B$ agree on the value of a large integer $M$, which serves as a security parameter. Now, $A$ makes a redundant commitment of the bit $b_1$ using $2M$ elementary bit commitments of bits $b_{1j}$, where the index $i$ runs from 1 to 2 and $j$ runs from 1 to $M$, with the property that the $b_{1j}$ are chosen randomly and independently and the $b_{2j}$ are defined by the constraint that $b_{1j} \oplus b_{2j} = b_1$ for each $j$. Alice also makes a redundant commitment of the bit $b_2$ using $4M$ elementary commitments of bits, $b_{2j}$, defined similarly but with $j$ running from 1 to $2M$, with the $b_{1j}$ chosen randomly and independently of the $b_{2j}$ as well as each other.

To test the equality of the committed bits $b_1$ and $b_2$, $B$ proceeds as follows. $B$ chooses a random one-to-one map $f$ from $\{1, \ldots, M\}$ to $\{1, \ldots, 2M\}$. For each $j$ from 1 to $M$, $B$ then asks $A$ whether the pairs $(b_{1j}, b_{2j})$ and $(b_{1f(j)}, b_{2f(j)})$ are equal (i.e. $b_{1j} = b_{1f(j)}$ and $b_{2j} = b_{2f(j)}$) or opposite (i.e. $b_{1j} = 1 - b_{1f(j)}$ and $b_{2j} = 1 - b_{2f(j)}$). If $A$ has followed the protocol correctly, and the bits $b_1$ and $b_2$ are indeed equal, then one of these two cases must apply for any given $j$, and $A$ states which, for each $j$. $B$ then tests $A$’s answers by choosing further independent random numbers $m(j) \in \{1, 2\}$ for each $j$ from 1 to $M$ and asking $A$ to unveil the two bits $b_{1m(j)j}$ and $b_{2m(j)f(j)}$. $A$ does so, and $B$ checks that these bits are indeed equal, or opposite, as claimed.

If $A$ passes all these tests, $B$ accepts that indeed $A$ was committed to two bits $b_1$ and $b_2$ with $b_1 = b_2$. $B$ accepts also that the $M$ remaining unopened pairs of bits $(b_{1k}, b_{2k})$, corresponding to values $k$ not in the image of $f$, together constitute a redundant commitment to the common bit value $b = b_1 = b_2$ of the same form as the original commitment to $b_1$.

The following security features are sufficient for our purposes.

**Security against $B$:** Clearly, if the protocol is correctly followed by $A$, $B$ obtains no information about $b$.

**Security against $A$:** The following definitions assume a fixed parameter $\gamma$. Any value of $\gamma$ in the range $0 < \gamma < 1/2$ suffices to prove the security of Rudich’s linking scheme. For the moment we will not specify $\gamma$ further; we will make a specific choice later. We say $A$ is effectively $R$-committed (R here stands for Rudich) to $b_1$ if at least $(1 - \gamma)M$ of the $M$
pairs that are supposed to define the redundant commitment to $b_1$ are of the correct form: i.e. if $b_1^j \oplus b_2^j = b_1$ for at least $(1 - \gamma)M$ different values of $j$. We say $A$ is effectively $R$-committed to $b_2$ if at least $(2 - \gamma)M$ of the $2M$ pairs that are supposed to define the redundant commitment to $b_2$ are of the correct form.

Note that, according to these definitions, if $A$ is effectively $R$-committed to $b_1$ by $M$ pairs of bit commitments and effectively $R$-committed to $b_2$ by $2M$ pairs, then, after the tests (and regardless of their result), the remaining $M$ unopened pairs will again effectively $R$-commit her to $b_2$ in the first sense. That is, at least $(1 - \gamma)M$ of the remaining $M$ pairs define a redundant commitment to $b_2$.

**Lemma 1** Suppose $A$’s Rudich linking does not effectively $R$-commit her to both $b_1 = b$ and $b_2 = \bar{b}$, for some bit value $b$. (That is, either she is not effectively $R$-committed to at least one of the bits, or she is effectively $R$-committed to two different bit values.) Let $\epsilon(M)$ be the maximum probability of $A$’s passing $B$’s tests, optimised over all bit configurations satisfying these constraints. Then $\epsilon(M) \approx \exp(-CM)$ as $M$ tends to infinity, for some positive constant $C$.

**Proof**

Suppose $(1 - \gamma_1)M$ of the $M$ pairs defining $A$’s first Rudich commitment define the bit $b_1 = b$, and the remaining $\gamma_1M$ define the bit $b_1 = \bar{b}$. Suppose $(2 - 2\gamma_2)M$ of the $M$ pairs defining $A$’s second Rudich commitment define the bit $b_2 = b$, and the remaining $2\gamma_2M$ define the bit $b_2 = \bar{b}$. As $A$ is not effectively $R$-committed to either $b$ or $\bar{b}$ with both commitments, we have that $\max(\gamma_1, 2\gamma_2) > \gamma$ and $\max(1 - \gamma_1, 2 - 2\gamma_2) > \gamma$.

Each of the $\gamma_1M$ bits from the first commitment that are committed to $\bar{b}$ has a probability $(1 - \gamma_2)$ of being tested against a commitment to $b$ from the second commitment. Each of the $(1 - \gamma_1)M$ bits from the first commitment that are committed to $b$ has a probability $\gamma_2$ of being tested against a commitment to $\bar{b}$ from the second commitment. The probability of passing all these tests is of order

$$\left(\frac{1}{2}\right)^{M(\gamma_1(1-\gamma_2)+\gamma_2(1-\gamma_1))} \leq \left(\frac{1}{2}\right)^{\frac{M\gamma}{2}}.$$  

(The inequality follows since $x(1-y)+y(1-x) \geq \frac{x+y}{2}$ for any $x, y$ in the range $(0, 1)$ such that $\max(x, y) > \frac{x+y}{2}$ and $\max(1-x, 1-y) > \frac{x+y}{2}$.)

**Remark** To keep a sequence of linked bit commitments secure against $A$, $B$ wants to ensure that he will almost certainly detect cheating unless $A$ is effectively $R$-committed
to the same bit for the first and last commitments. From the note preceding the above
lemma, we see that, if \( A \)’s first and last commitments are not effectively \( R \)-committed to the
same bit value, some intermediate adjacent pair of commitments must fail to be effectively
\( R \)-committed. Any value of \( \gamma < \frac{1}{2} \) such that \( \gamma M \) is an integer can be used in the definition
of effective \( R \)-commitment here. We can now apply the lemma and minimise our bound on
the probability of \( A \) escaping detection by taking \( \gamma M = \lfloor \frac{M-1}{2} \rfloor \), where \( |x| \) is the largest
integer less than or equal to \( x \). We thus see \( A \)’s probability of successful cheating is bounded
by a term of order
\[
\left( \frac{1}{2} \right)^{\frac{1}{2}(\lfloor \frac{M-1}{2} \rfloor)} \approx \left( \frac{1}{2} \right)^{\frac{M}{4}}.
\]

V. USE OF RUDICH’S LINKING IN FINITE CHANNEL RELATIVISTIC BIT
COMMITMENT

Our key idea for improving \( RBC1 \) is the following. If \( b_1 \) represents a relativistic bit
commitment, and \( b_2 \) another relativistic bit commitment begun at a later round than \( b_1 \),
then \( A \) can use Rudich’s technique to link these commitments by showing that \( b_1 = b_2 = b \).
After doing so, she maintains a commitment to their common bit value, drawn from a subset
of the elementary commitments used for \( b_2 \). She can thus abandon the commitment to \( b_1 \) at
this point, while remaining committed to \( b \) via a subset of the elementary commitments for
\( b_2 \), and start a new commitment for a bit \( b_3 \). Letting \( b_i \) take the role of \( b_{i-1} \) for \( i = 2, 3 \), she
then repeats the procedure above iteratively. \( A \)’s commitment to the bit \( b \) is thus always
defined by elementary commitments made in a recent round, so avoiding the exponential
blowup that makes \( RBC1 \) impractical.

There are a variety of ways of implementing this basic strategy. One of the simplest
is the following protocol, which we call \( RBC2 \). This uses iterations of the relativistic bit
commitment protocol \( RBC1 \) as sub-protocols. \( RBC2 \) has two security parameters: \( N \), the
security parameter for the \( RBC1 \) sub-protocols, and \( M \), the security parameter used in the
Rudich linking protocol.

The first iteration of the linking mechanism is as follows. In region \( P_1 \), \( A_1 \) redundantly
commits a bit \( b_1 = b \) using a Rudich coding with \( M \) pairs of individual relativistic bit
commitments made using \( RBC1 \). This relativistic commitment is sustained by \( A_2 \) in the
region \( Q_1 \). A second redundant bit commitment, \( b_2 = b \), using \( 2M \) pairs of individual
relativistic bit commitments, is begun by $A_1$ in the region $P_2$. Also in the region $P_2$, $A_1$ and $B_1$ go through the Rudich linking protocol for $b_1$ and $b_2$. In the case of $b_1$, $A_1$ can unveil elementary relativistic bit commitments, when required by the linking protocol, by revealing the random numbers used by $A_2$ in $Q_1$ to sustain these commitments. In the case of $b_2$, $A_1$’s required unveilings are made by revealing the random numbers that will be used by $A_2$ in $Q_2$ to sustain the relevant commitments.

In both cases, $B$ needs to collect together the information supplied by $A_1$ (in $P_2$) and $A_2$ (in $Q_1$ or $Q_2$) in order to verify that $A$ has passed the tests in the linking protocol. Once this is done, and $B$ has established that $b_1 = b_2$, the remaining unopened elementary commitments defining $b_1$ may be discontinued. All the $2M$ pairs of elementary commitments defining $b_2$ are sustained by $A_2$ in $Q_2$, since she does not know which of the elementary commitments was unveiled by $A_1$ in $P_2$. Of these $2M$ pairs, $M$ remain unopened and define a redundant commitment for $b$, and both $A_1$ and $B_1$ know the identity of these $M$ after communications in $P_2$ have ended. They can thus use the redundant bit commitment defined by these $M$ pairs for another iteration of the linking mechanism in $P_3$.

In this second iteration, in the region $P_3$, $A_1$ makes a third redundant bit commitment, $b_3 = b$, to $B_1$, using $2M$ pairs of individual relativistic bit commitments, and goes through a second Rudich linking, proving (eventually) to $B_1$ that $b_3 = b_1$.

And so on: $A_1$ initiates a new redundant commitment to, and goes through a Rudich linking with, $B_1$, for each round they participate in after the first. $A_2$ and $B_2$ simply sustain each of the commitments initiated by $A_1$ and $B_1$ for one round.

VI. PROOF OF SECURITY OF RBC2 AGAINST CLASSICAL ATTACKS

Security against $B$: If the $A_i$ honestly follow the protocol using secret shared independently generated random numbers, then whatever strategy the $B_i$ use, the information they receive is uncorrelated with the committed bit. The protocol is thus perfectly secure against Bob.

Security against $A$: A key constraint on Alice is that $B_1$ may request $A_1$ to unveil any of the individual $RBC1$ relativistic bit commitments made during the protocol. $B_1$ will request $M$ of the first batch of $2M$ commitments to be unveiled during the linking protocol in $P_2$. Of every successive batch of $4M$ commitments, $B_1$ will request that $A_1$ unveil $M$ of
them as soon as they are committed, and another $M$ two rounds later. Thus, if $M$ is large, unless $A_1$ is able to provide a valid unveiling of a bit for almost every commitment in every batch, she will almost certainly be discovered to be cheating.

Now, the relativistically enforced security of $RBC_1$ means that $A_1$ has no strategy that will allow her a high probability of successfully producing valid unveilings for both 0 and 1 for any of the elementary $RBC_1$ commitments. To unveil a commitment that she has just initiated in $P_i$, she must supply $m$ $m$-bit numbers that correspond to a valid commitment by $A_2$ in $Q_i$ of an $m$-bit number that corresponds to a valid commitment of a bit $b$ by $A_1$ in $P_i$. Similarly, to unveil in $P_{i+1}$ a commitment she initiated in $P_i$, she needs to supply $m$ $m$-bit numbers that correspond to a valid commitment by $A_2$ in $Q_i$ of an $m$-bit number that corresponds to a valid commitment of a bit $b$ by $A_1$ in $P_i$. Either way, if she is able, with significant probability, to produce valid unveilings of both values of $b$, she is also able to infer, with significant probability, the differences between the $m$ pairs of random numbers, $(n_{i,0} - n_{i,1})$ sent by $B_2$ to $A_2$ in the commitment round in $Q_i$. (With our convention that $n_{i,0} = 0$, this is equivalent to inferring the $n_{i,1}$.) But, if $B$ follows the protocol honestly, all $(N - 1)^m$ possible sets of difference values are equiprobable, and $A_1$ can have no information available to her, either in $P_i$ or $P_{i+1}$, about the actual difference values, assuming that our current understanding of physics correctly assures us that superluminal signalling is impossible.

Suppose that $N \geq 4$, so that $m \geq 2$ and $(N - 1)^m \geq 9$. Then in particular, for each elementary commitment, there can be at most one set of $m$ numbers that $A_1$ can use for unveiling with the knowledge that the probability of unveiling a valid bit commitment is $> 1/3$. Thus, for each elementary commitment, $A_1$ can successfully unveil at most one of the bit values 0 and 1 with probability $> 1/3$. We say $A_1$ is probabilistically committed to the bit value $b$ if she can successfully unveil $b$ with probability $> 1/3$.

By analogy with our earlier definition, we say Alice is probabilistically effectively RR-committed (RR here stands for relativistic Rudich) to the bit $b$ by the redundant commitment initiated in round 1 if she is probabilistically committed by each elementary bit commitment in a subset of $(1 - \gamma)M$ of the $M$ pairs of elementary commitments and these bit values constitute a Rudich coding for the bit $b$ among these pairs. We say she is probabilistically effectively RR-committed to the bit $b$ by the redundant commitment initiated in any later round if she is probabilistically committed by each elementary bit commitment in a subset
of $(2 - \gamma)M$ of the $2M$ pairs of elementary commitments and these bit values constitute a Rudich coding for the bit $b$ among these pairs.

We take $\gamma < \frac{1}{2}$, so that Alice can be probabilistically effectively RR-committed to at most one bit value in any given round.

**Lemma 2** Suppose $A$ is not probabilistically effectively RR-committed to both $b_1 = b$ and $b_2 = b$, for some bit value $b$, on any two successive rounds of RBC2. (That is, either she is not probabilistically effectively RR-committed to at least one of the bits, or she is probabilistically effectively RR-committed to two different bit values.) Let $\epsilon(M)$ be the maximum probability of $A$’s passing $B$’s tests, optimised over all bit configurations satisfying these constraints. Then $\epsilon(M) \approx \exp(-CM)$ as $M$ tends to infinity, for some positive constant $C$.

**Proof**

Suppose $y_1M$ of the $M$ pairs defining $A$’s first Rudich commitment define the bit $b_1 = b$, and that $\bar{y}_1M$ define the bit $b_1 = \bar{b}$. Suppose $y_2M$ of the $M$ pairs defining $A$’s second Rudich commitment define the bit $b_2 = b$, and that $\bar{y}_2M$ define the bit $b_2 = \bar{b}$. Each of the $y_1M$ bits from the first commitment that are committed to $b$ has probability $(1 - y_2)$ of being tested against a pair from the second commitment that does not commit to $b$. Similar calculations for the other possibilities lead to an expected total of $(1 - y_1y_2 - \bar{y}_1\bar{y}_2)M$ tests that have probability $\geq \frac{1}{3}$ of failure.

As $A$ is not probabilistically effectively RR-committed to either $b$ or $\bar{b}$ with both commitments, we have that max$(1 - y_1, 2 - 2y_2) > \gamma$ and max$(1 - \bar{y}_1, 2 - 2\bar{y}_2) > \gamma$. By an argument similar to that used in the proof of Lemma 1, the probability of passing all the tests is bounded by a term of order

$$\left(\frac{2}{3}\right)^M(\frac{1}{2}\gamma).$$

QED.

**Remark**  Note that if $A_1$ is probabilistically effectively committed to a Rudich coding for a bit $b$ by a commitment initiated in $P_i$, she cannot be probabilistically effectively committed to a coding for the opposite bit $\bar{b}$ in $P_{i+1}$, since she learns no information in the mean time about the round that takes place in $Q_i$. Thus, unless she is probabilistically effectively committed to the same bit in both $P_i$ and $P_{i+1}$, she must fail to be probabilistically effectively committed to any bit value in at least one of the two regions.

As before, we take $\gamma M = \lfloor \frac{M - 1}{2} \rfloor$. We thus see any cheating by $A$ will be detected with
probability of order
\[ (1 - \left( \frac{2}{3} \right)^{\frac{1}{2}\left\lfloor \frac{M-1}{2} \right\rfloor}) \approx (1 - \left( \frac{2}{3} \right)^{\frac{M}{2}}). \]

**Remark** Alice can only usefully cheat either by failing to be probabilistically effectively Rudich committed to any bit by some set of \( M \) pairs at some point, or by being effectively Rudich committed to some bit \( b \) by a set of \( M \) pairs and failing to be effectively Rudich committed to \( b \) by the corresponding set of \( 2M \) pairs at some point. It is not hard to show that, among the commitment strategies covered by these options, the one that minimises her probability of being detected cheating in the relevant Rudich test is for all \( M \) pairs from the first set to probabilistically commit to the same bit \( b \), while \( \left\lfloor \frac{2M-1}{4} \right\rfloor + 1 \) (i.e. \( \frac{M}{2} \) if \( M \) is even, and \( \frac{M+1}{2} \) if \( M \) is odd) of the pairs from the second set commit to \( \bar{b} \) and the remainder to \( b \). We can thus find an exact lower bound on the probability \( p(M) \) of being detected: for example, if \( M \) is even we have
\[
p(M) \leq \sum_{r=0}^{M/2} \left( \frac{2}{3} \right)^r \left( \frac{M}{2} \right)^r \left( \frac{3M}{2} \right)^r \left( \frac{2M}{M-r} \right)^{-1}
\]

For suitably large \( M \), deviating from the protocol risks detection while giving her essentially no possibility of successfully cheating. If Alice is rational and values her reputation for integrity, she will thus honestly follow the protocol throughout, committing (not just probabilistically effectively committing) herself to a Rudich coding of the same bit value \( b \) in each round.

**VII. DEFINITIONS OF SECURITY FOR QUANTUM BIT COMMITMENT**

We want to consider the security of \( RBC2 \) when the possibility of the parties sharing, storing and manipulating quantum information is taken into account. We still assume that physics takes place in Minkowski spacetime, with the Minkowski causal structure, but now suppose that the correct description of physics is some relativistic version of quantum theory. Without specifying the details of this theory, we simply assume that the parties can devise their own consistent labelling of physically realisable orthogonal basis states. To allow as much scope for cheaters as possible, we also assume that any localised party is able to apply arbitrary local quantum operations to quantum states in their possession.

Before considering the specifics of \( RBC2 \), we need to consider the general definition of security for a bit commitment protocol that may involve quantum information. The
definitions of reliability and perfect reliability and of security and perfect security against Bob are as for classical bit commitment. However, we need a more general definition of security against Alice. Suppose that we have reached the first point in the protocol at which Alice is supposed to be committed. We define $p_0^{\text{sup}}$ to be supremum of the set of her probabilities of successfully unveiling 0 over all strategies she could pursue from this point onwards; similarly $p_1^{\text{sup}}$. We say that the protocol is perfectly secure against Alice if, for any possible initial commitment strategy, it guarantees that $p_0^{\text{sup}} + p_1^{\text{sup}} \leq 1$. It is secure against Alice if, for any possible initial commitment strategy, it guarantees that $p_0^{\text{sup}} + p_1^{\text{sup}} \leq 1 + \epsilon'(n)$, where $\epsilon'(n)$ tends to 0 as $n$ tends to infinity.

To see the point of the definitions, consider the standard model of classical bit commitment in which Alice writes her committed bit on a piece of paper and places it in a safe, which she then locks, before handing the safe over to Bob. In this model, she unveils the bit by giving Bob the combination, allowing him to open the safe and read the paper. Modulo physical assumptions about the impossibility of remotely manipulating the contents of the safe, this is perfectly secure against Alice.

Now, in an idealised quantum version of this protocol, Alice could create a superposition $\alpha|0\rangle_A|0\rangle_S + \beta|1\rangle_A|1\rangle_S$, where $|0\rangle_S$ and $|1\rangle_S$ are the commitment states for 0 and 1 respectively, and then place the second system, described by the $|\rangle_S$ states, in the safe. The probabilities of Bob reading the commitments for 0 and 1 when he opens the safe are respectively $|\alpha|^2$ and $|\beta|^2$, which sum to 1.

Modulo the same physical assumptions as before, this is a perfectly secure protocol that commits Alice, in the sense that she cannot alter the contents of the safe or affect the probabilities of 0 and 1 being unveiled. However, it does not satisfy the definition of perfect security we gave for classical bit commitment, which requires that either the probability of successfully unveiling 0 or that of successfully unveiling 1 should vanish. This distinction between classical and quantum definitions of security was noted and discussed in Ref. [4].

This suggests that the natural definition for perfect security for a quantum bit commitment protocol is that given above, while the classical definition implies something stronger, which we call bit commitment with a certificate of classicality [7]. The quantum definition of imperfect security is then a natural extension: as in the classical case, it allows that Alice may have scope for cheating but requires that any cheating advantage can be made arbitrarily small.
Another historical reason for favouring the definitions of quantum security just given, and against applying the classical definitions to quantum bit commitment, is that the former, unlike the latter, are in line with Mayers’ and Lo-Chau’s analyses of cheating strategies for quantum bit commitment protocols. Mayers’ and Lo-Chau’s essential results were that, in non-relativistic quantum bit commitment protocols conforming to their cryptographic models, perfect security against Bob implies that $p_0^{\text{sup}} + p_1^{\text{sup}} = 2$, and that in a protocol with security parameter $n$, security against Bob implies that $p_0^{\text{sup}} + p_1^{\text{sup}} = 2 - \delta(n)$, where $\delta(n)$ tends to 0 as $n$ tends to infinity. In other words, all such quantum bit commitment protocols are insecure (in fact, maximally or near-maximally insecure) by our quantum definition. On the other hand, if we regarded a certificate of classicality as part of the definition of a quantum bit commitment, showing that unconditionally secure quantum bit commitment is impossible would only require showing that no quantum protocol can guarantee to prevent the commitment of superposed bits — a much simpler result that can be proven in one line, without appealing to Mayers’ and Lo-Chau’s analyses [8].

VIII. SECURITY OF RBC2 AGAINST QUANTUM ATTACKS

Security against $B$: Allowing for the possibility of Bob storing or manipulating quantum information does not affect the security argument given earlier. If the $A_i$ honestly follow the protocol using secret shared independently generated random numbers, then whatever strategy the $B_i$ use, the information they receive is uncorrelated with the committed bit. The protocol is thus perfectly secure against Bob.

Security against $A$: Alice certainly has the option, allowed by our definition of security for quantum bit commitment, of committing a qubit belonging to an entangled superposition. To see this, consider first a quantum strategy applicable to $RBC1$. Suppose that before the protocol $A_1$ and $A_2$ share a commitment state $\alpha|0\rangle_1|0\rangle_2 + \beta|1\rangle_1|1\rangle_2$ and a string of superposed number states of the form $\sum_{r=0}^{N-1} |r\rangle_1|r\rangle_2$. If $A_1$ receives the pair $(n_{1,0}, n_{1,1})$ from $B_1$ and runs the first round of the protocol on a quantum computer, using her part of the commitment state and the first superposed number state as input, and returns $k$ as output, the joint state becomes

$$\alpha|0\rangle_1|k - n_{1,0}\rangle_1|0\rangle_2|k - n_{1,0}\rangle_2 + \beta|1\rangle_1|k - n_{1,1}\rangle_1|1\rangle_2|k - n_{1,1}\rangle_2$$
tensored with the remaining superposed number states. \( A_2 \) and \( A_1 \) can proceed similarly at each successive round, maintaining a superposed commitment. Either of them can consistently unveil the bit when required, by measuring their commitment state and their random number states, producing a consistent unveiling of 0 (with probability \( |\alpha|^2 \)) or of 1 (with probability \( |\beta|^2 \)). But now it is clear that the same basic strategy extends to \( RBC_2 \), provided the \( A_i \) share appropriate superposed entangled states: in particular, every unveiling required in \( RBC_2 \) can be consistently made without destroying the quantum superposition of the committed bit.

Of course, this strategy of using shared entanglement gives Alice no control over which bit will be unveiled, so its existence is consistent with the quantum security of \( RBC_1 \) and \( RBC_2 \). To establish that security, we need arguments parallel to those in the classical case, establishing that the \( A_i \) have essentially no alternative but to implement this strategy honestly, since any cheating will be detected with near certainty. We conjecture that this is indeed the case, and that both protocols are secure against quantum attacks.

One can easily show that \( RBC_1 \) and \( RBC_2 \) are temporarily secure against quantum attack, round by round, in the following sense.

**Lemma 3** Let \( \sup p_{0,j} \) be the supremum of the set of probabilities of the relevant \( A_i \) successfully unveiling 0 in round \( j \), for all possible unveiling strategies she could implement in that round; similarly \( \sup p_{1,j} \). Then \( \sup p_{0,j} + \sup p_{1,j} \leq 1 + \epsilon \), where \( \epsilon \equiv \epsilon(N) \) (in the case of \( RBC_1 \)) or \( \epsilon \equiv \epsilon(M,N) \) (in the case of \( RBC_2 \)), and in either case tends to 0 as the security parameter(s) tend(s) to infinity.

**Proof** First note that at any given point in the protocol \( A_1 \) and \( A_2 \) share a quantum state. We may suppose without loss of generality that, whenever the \( B_i \) receive a state from the \( A_i \) during the protocol, they measure it in the computational basis. The \( A_i \) thus share no entanglement with the \( B_i \). Hence, if we include in the definition of their shared state all ancillae that the \( A_i \) may have ready for use later in the protocol (either for continuing the commitment or for unveiling), then without loss of generality we can take their shared state to be pure: call it \( |\psi\rangle \).

Let \( A_i \) be the party attempting an unveiling on round \( j \). To simplify the discussion a little we assume that the suprema defined above are attainable: that is, optimal strategies exist. (The argument below can easily be extended to cover the possibility that the suprema are not attained, by considering near-optimal strategies.)
Her optimal strategy for attempting to unveil a 0 — the optimal strategy that she can construct given the knowledge available to her, that is — must be defined by a projective decomposition of the identity \( \{ A_i \}_{i=0}^{m} \), representing a von Neumann measurement she will carry out on \( |\psi\rangle \), together with an assignment of distinct lists of numbers \( r_i \) to each of the \( A_i \). (We can represent general measurements in this way, as we allowed ancillae to be included in the definition of \( |\psi\rangle \).) Her unveiling will then consist of carrying out the measurement, and announcing the \( r_i \) corresponding to the result \( A_i \). At most one of the \( r_i \) can correspond to a valid unveiling of 0. Without loss of generality we can assume that precisely one of them does — otherwise there is nothing to prove — and let it be \( r_0 \).

Similarly, her optimal strategy for attempting to unveil a 1 must be defined by a projective decomposition of the identity \( \{ B_i \}_{i=0}^{n} \), together with an assignment of distinct lists of numbers \( s_i \) to each of the \( B_i \). At most one of the \( s_i \) can correspond to a valid unveiling of 1. Without loss of generality we can assume that precisely one of them does — otherwise there is nothing to prove — and let it be \( s_0 \).

Note that we make no assumption here about the relation between the \( A_i \) and \( B_i \); in particular, we do not assume that they represent the same decomposition or commuting decompositions.

We have that
\[
\begin{align*}
p_0^{i,\text{sup}} &= |A_0|\psi\rangle|^2, \\
p_1^{i,\text{sup}} &= |B_0|\psi\rangle|^2.
\end{align*}
\]

But now
\[
|A_0B_0|\psi\rangle|^2 \geq (|A_0|\psi\rangle| - |(1-B_0)|\psi\rangle|)^2 = ((p_0^{i,\text{sup}})^{1/2} - (1 - p_1^{i,\text{sup}})^{1/2})^2.
\]

Hence if \( (p_0^{i,\text{sup}} + p_1^{i,\text{sup}} - 1) \) is significantly positive, there is a strategy available to \( A_i \) — applying the \( B \) projections followed by the \( A \) projections — that has a significant probability of yielding a valid unveiling for both 0 and 1. This means that, with significant probability, \( A_i \) learns information that reveals the differences \( n_{k,1} - n_{k,0} \) in the random numbers sent to \( A_i \) in the previous round. But it is impossible for her to obtain any information about these differences at this point, since a light signal cannot yet have reached her. QED

Lemma 3 shows in particular that RBC1 and RBC2 are not vulnerable to the type of attack shown by Mayers and Lo-Chau to imply the insecurity of non-relativistic quantum bit commitment schemes. In a Mayers-Lo-Chau attack on a protocol perfectly secure against Bob, Alice can successfully unveil either 0 or 1, each with probability one. This is impossible
in RBC1 and RBC2, essentially because the Mayers-Lo-Chau attack would require Alice to implement one of two unitary operations when unveiling, and, while both operations do indeed exist, she would need to know the random data supplied by Bob on the previous round in order to be able to construct both of them simultaneously.

A full quantum security analysis would need to consider all possible quantum operations that Alice might perform. In general, the $A_i$ may initially share arbitrarily many entangled states of their choice, they may generate and share further entangled states during the protocol, and they may also communicate classically during the protocol. Whenever one of them is required to respond to Bob’s queries, she may carry out arbitrary quantum operations and measurements on the states in her possession before doing so. These quantum operations may depend on all communications previously received from Bob or from her partner. We conjecture that RBC1 and RBC2 are secure against general quantum attacks, but have no proof at present.

**IX. COMMENTS ON PRACTICALITY**

Security definitions need only consider the asymptotic behaviour of a protocol, but real world implementations require finite values of the security parameters. Here we give some estimates of the degree of security attainable against classical attacks, for realistic security parameter choices. From these we can deduce what is needed in practice to implement RBC2 with near-perfect security against classical attacks.

RBC2 requires $A_2$ to maintain $4M$ individual relativistic bit commitments on each round. Sustaining each of these commitments requires her to commit $m$ bits, and each of these last commitments requires $m$ bits to be transmitted. Her total bit transmission rate is thus $4Mm^2$ per round. To allow these commitments, $B_2$ needs to send her $4Mm^2$ bits per round (we assume here that they use the convention $n_{i,0} = 0$, saving $B_2$ a factor of 2). These communications thus take a total of $8Mm^2$ bits per round.

On the third and later rounds, $A_1$ needs to send $4Mm$ bits to initiate $4M$ commitments, $M$ bits to respond to $B_1$’s queries about pair relations during the linking tests, and $2Mm^2$ bits for the unveiling required in the Rudich linking subprotocol. $B_1$ needs to send $4Mm$ bits to initiate the commitments (again assuming the convention $n_{i,0} = 0$), send $\approx M \log_2 M$ bits to make a random choice among the $\binom{2M}{M}$ possible linkings of pairs, and a further $2M$
bits to query the pair relations. These communications thus take a total of \( \approx M(2m^2 + 8m + \log_2 M + 3) \) bits per round.

For \( m = 2 \), the maximum communication cost per round is thus approximately \( \max(32M, M(27 + \log_2 M)) \). For small \( m \) and \( M \) we can easily calculate the communication cost precisely, as we do in the following illustrative calculations.

We assume a 10 GHz transmission rate and require each round to take place within a tenth of the time it would take light to travel between \( A_1 \) and \( A_2 \). We consider two possibilities

**Case I:** \( m = 2 \) and \( M = 40 \);
**Case II:** \( m = 2 \) and \( M = 200 \).

For case I, the maximum communication cost per round is 1280 bits, requiring \( 1.3 \times 10^{-7} \) sec. Conservatively allowing a factor of 3 to include the \( A_i \)'s data processing time, we obtain a total elapsed time of \( 4 \times 10^{-7} \) sec per round. This allows a separation of \( 4 \times 10^{-6} \) light seconds, about 1.3 km.

For Case II, the maximum communication cost is 7041 bits per round. Calculating similarly, we obtain separation of about 9 km.

To get an indication of the security levels attained, we use the bound derived in section VI on the probability \( p(M) \) of Alice’s cheating being detected. This gives bounds on Alice’s successful cheating probability of \( 2.4 \times 10^{-2} \) in Case I and \( 7.3 \times 10^{-9} \) in Case II.

**X. DISCUSSION**

The secure relativistic protocol \( RBC2 \) requires the two parties each to maintain two dedicated separated secure sites. We estimate that a separation of around 10km should be adequate even if near-perfect security against cheating is required. Our security analyses suggest that \( RBC2 \) can be implemented with cheating probability bounded by roughly \( 2 \times 10^{-2} \) using roughly \( 10^3 \) bits of communication per round and by roughly \( 7 \times 10^{-9} \) using roughly \( 10^4 \) bits of communication per round. The probability of successful cheating cannot be smaller than \( 2^{-n} \) for a relativistic protocol with \( n \) bits of communication per round, so that these cheating probabilities certainly cannot be improved upon by protocols using fewer than 6 and 27 bits per round respectively. The scope for reducing the number of
bits per round (and hence the minimum site separation) for these security levels is thus not huge: one cannot possibly hope to obtain better than a factor of roughly 300, and we doubt that this bound is attainable. We cannot, though, exclude the possibility that an improved security analysis or an improved protocol could allow the site separation to be reduced by a factor of $10^1$ to $10^2$ or so. Any more substantial reductions would require faster communications technology. The required separation is inversely proportional to the achievable communication rate; our estimates assumed 10 GHz communications.

One important caveat is that there is as yet no security proof against general attacks by parties equipped with arbitrarily powerful quantum computers. This is not to say that the protocol can be broken by quantum computers. In particular, we have shown that the protocol is immune to the Mayers-Lo-Chau quantum attack, which breaks all earlier attempts at unconditionally secure bit commitment. Nonetheless, a complete quantum security analysis would obviously be highly desirable.

A subtle point, familiar by now to most experts, but potentially confusing to others, is also worth reiterating here. Using classical bit commitment protocols as a model, one can define a bit commitment with a certificate of classicality to be a perfectly secure bit commitment protocol that guarantees from the outset that there is a definite classical bit value, 0 or 1. Classically, bit commitment and bit commitment with a certificate of classicality are identical, of course, but in the quantum domain the latter is a stronger primitive. It has been known for some time that, given a protocol for unconditionally secure bit commitment with a certificate of classicality, parties who can send and receive quantum information could implement unconditionally secure oblivious transfer (and hence unconditionally secure multi-party computation, among other tasks). It is also known that neither bit commitment with a certificate of classicality nor oblivious transfer can be implemented with unconditional security, even using both quantum information and relativistic signalling constraints. Our protocols do not contradict these established results: as noted above, RBC1 and RBC2 implement bit commitment, but not bit commitment with a certificate of classicality.

An interesting feature of our protocols is that they achieve something that at first sight might appear to be impossible: secure deniable bit commitment. Suppose that, after a certain point, Alice chooses neither to sustain the commitment nor to unveil the bit, and that her two representatives then exchange all the data they received during the protocol. She can
then produce two different accounts of her actions, consistent respectively with her having made and sustained commitments to zero and one respectively during the period in which she participated in the protocol. Of course, since she has stopped sustaining the commitment at this point, there is no compelling reason for anyone to believe whichever account she produces. However, the accounts cannot be disproved by other parties. Generally speaking, deniability is useful in potentially adverse environments, in which a party may be compelled to give an account of their actions and may face sanctions if that account is inconsistent with their recorded behaviour during the protocol. The possibility of deniable bit commitment seems particularly interesting given the potential uses of bit commitment as a sub-protocol in protocols for tasks such as secure elections.

The protocol has another interesting (and related) feature. As we noted above, if Alice is equipped with quantum computers, she can use RBC2 to commit an arbitrary qubit. If her agents choose not to sustain the commitment after some point, and if they are able to exchange quantum information, they can reconstruct the committed qubit (even when it was unknown to them — i.e. when they do not have its classical description, nor any other classical or quantum information about it). In terminology proposed by Jörn Müller-Quade and Dominique Unruh, the qubit commitment is retractable: Alice can, so to speak, take the committed qubit back from Bob if she chooses to. As Müller-Quade and Unruh have pointed out [18], one can usefully develop an abstract black box model of quantum bit commitments which incorporates the property of retractability: in this model, a retractable quantum bit commitment corresponds to Alice giving Bob an ideal safe, containing a stored qubit, which has the property that Alice can at any time either give Bob the power to open the safe or repossess the safe herself and open it. This key insight pinpoints more precisely why secure oblivious transfer cannot be built from RBC2 [18]. Yao’s construction [22] requires that, if Bob chooses a suitable random subset of Alice’s commitments and finds that the unveiled commitments correctly describe choices and outcomes of quantum measurements, he may infer that the quantum measurements corresponding to the unopened commitments were also implemented irreversibly. The retractability of unopened quantum bit commitments invalidates this inference.

In summary, we have described a bit commitment protocol that is unconditionally secure against parties equipped with arbitrarily powerful classical computers, something not previously believed to be possible. The protocol has the surprising and interesting feature that it
implements deniable bit commitments. The result demonstrates that exploiting elementary relativistic signalling constraints can be a surprisingly powerful tool in classical and quantum cryptography. The protocol can be easily implemented with current technology. It would be interesting to develop practical implementations and to examine further the potential for useful applications.

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XI. APPENDIX: COMMENT ON TERMINOLOGY

The cryptographic literature on classical bit commitment includes various different illustrations of bit commitments based on computational or physical assumptions, which could inspire subtly different definitions of bit commitment. This seems not to have worried classical cryptographers in the past. After all, the protocols all achieve the same essential result: after some point, Alice is committed, and she can later unveil if she chooses. Singling out one particular protocol as the unique earthly representative of the Platonic ideal of bit commitment seems hard to justify.

However, the relation between quantum and classical cryptography involves new subtleties, and the use of relativistic signalling constraints in cryptography may also do so, so previously irrelevant distinctions perhaps need to be considered afresh. We have already noted the distinction between secure quantum bit commitment and secure quantum bit com-
mitment with a certificate of classicality. Another possible distinction that could be made deserves discussion here.

Some standard classical bit commitment schemes have the property that, once Alice is committed, neither party need do anything further unless and until Alice chooses to unveil the bit. Consider, for example, a classical bit commitment based on a suitable one-way function, in which Alice commits to Bob the value $f(x)$ of the function evaluated at some $x$, which she chose randomly subject to the constraint that $x$ has the same parity as the committed bit. (Here “suitable” means that it is computationally hard to extract any information about the parity of $x$ from $f(x)$.) In this protocol, commitment is complete once Bob has received $f(x)$, and Alice can unveil at any later time by sending $x$.

On the other hand, in $RBC1$ and $RBC2$ the $A_i$ and $B_i$ need to continue exchanging transmissions indefinitely in order to sustain a secure commitment; they stop only if and when Alice chooses to unveil the bit. If one particularly wished to make a point of stressing this feature, one could propose new definitions distinguishing *definite bit commitment* and *sustained bit commitment*, with one-way function commitments as examples of the former and relativistic commitments as examples of the latter.

It seems to me, though, that there are two persuasive arguments against this nomenclature. The first is historical. Some well-known non-relativistic bit commitment schemes also require one or both of the parties to actively maintain the commitment indefinitely. For instance, in the simplest standard illustration of bit commitment based on physical assumptions, Alice writes the bit on a piece of paper and puts it in a sealed envelope on a table between her and Bob, in sight of them both. Alice is now committed, but in order to maintain the security of the commitment each party now needs to watch the envelope to ensure that the other does not interfere with it, and they must continue doing so indefinitely unless and until Alice chooses to unveil. The original version of the BGKW protocol, in which $A_2$ is imprisoned in a Faraday cage monitored by Bob throughout the lifetime of the commitment, has the same feature. To change terminology now would retroactively delegitimise these bit commitment protocols, relabelling them as protocols for the newly defined cryptographic primitive of sustained bit commitment.

The second is a question of principle: one should not conflate the cryptographic task to be implemented and the means of implementation. A cryptographic task can be defined by the inputs and outputs into a black box version of the protocol. In the case of classical
bit commitment, Alice inputs a bit to the box, and then later, if and when Alice inputs an instruction to unveil, the bit is output to Bob. The details of how security is ensured or maintained do not enter into this definition.

There are many different ways of implementing most cryptographic tasks, and setting out the details of any given implementation has to date been considered as properly forming part of the definition of a protocol, rather than part of a definition of the task. To abandon this fundamental distinction now would requiring rewriting cryptological textbooks’ discussions of many other primitives besides bit commitment. This would remove the clear and useful distinction between task and protocol in current nomenclature, without which every different protocol might be described as an instance of a different task. It seems to me far better not to trespass onto this slippery slope.

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