UNCENTERED MAXIMAL FUNCTION FOR ELLIPTIC
PARTIAL DIFFERENTIAL OPERATOR

CHOKRI ABDELKEFI* AND SAF A CHABCHOUB**

Abstract. In the present paper, we study in the harmonic analysis associated to the Weinstein operator, the boundedness on $L^p$ of the uncentered maximal function. First, we establish estimates for the Weinstein translation of characteristic function of a closed ball with radius $\varepsilon$ centered at 0 on the upper half space $\mathbb{R}^{d-1} \times [0, +\infty[$. Second, we prove weak-type $L^1$-estimates for the uncentered maximal function associated with the Weinstein operator and we obtain that it is bounded on $L^p$ for $1 < p \leq +\infty$.

1. Introduction

For a real parameter $\alpha > -\frac{1}{2}$ and $d \geq 2$, the Weinstein operator (also called Laplace-Bessel operator) is the elliptic partial differential operator $\Delta_{d,\alpha}$ defined on the upper half space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, +\infty)$ by

$$\Delta_{d,\alpha} = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha + 1}{x_d} \frac{\partial}{\partial x_d}.$$

The operator $\Delta_{d,\alpha}$ can be written as

$$\Delta_{d,\alpha} = \Delta_{d-1} + L_\alpha,$$

where $\Delta_{d-1}$ is the Laplacian operator on $\mathbb{R}^{d-1}$ and $L_\alpha$ is the Bessel operator on $(0, +\infty)$ with respect to the variable $x_d$ given by

$$L_\alpha = \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha + 1}{x_d} \frac{\partial}{\partial x_d}.$$

For $d > 2$, the operator $\Delta_{d,\alpha}$ arises as the Laplace-Beltrami operator on the Riemannian space $\mathbb{R}^d_+$ equipped with the metric $ds^2 = x_d^{4\alpha+2} \sum_{i=1}^d dx_i^2$. The Weinstein

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operator \( \Delta_{d,\alpha} \) has important applications in both pure and applied mathematics, especially in the fluid mechanics (see [19]). Many authors were interested in the study of the Weinstein equation \( \Delta_{d,\alpha} u = 0 \), one can cite for instance M. Brelot [5] and H. Leutwiler [12]. The harmonic analysis associated with the Weinstein operator was studied in [2, 9]. In particular, the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator also called the Weinstein transform.

The Hardy-Littlewood maximal function was first introduced by Hardy and Littlewood in 1930 for functions defined on the circle (see [10]). Later it was extended to various Lie groups, symmetric spaces, some weighted measure spaces (see [6, 9, 13, 15, 16, 17]) and different hypergroups (see [7, 8, 14]). In this paper, we denote by \( B_{+}^{d}(0, \varepsilon) = \{ y \in \mathbb{R}^{d}, \| y \| < \varepsilon \} \), the closed ball on \( \mathbb{R}^{d} \) with radius \( \varepsilon \) centered at \( 0 \). For \( x \in \mathbb{R}^{d} \), we establish estimates of the Weinstein translation (see next section) of the characteristic function of \( B_{+}^{d}(0, \varepsilon) \), based on the inversion formula, where we put \( y = (y', y_d) \) in \( \mathbb{R}^{d} \), with \( y' = (y_1, ..., y_{d-1}) \). Using these estimates, we prove the weak-type \((1, 1)\) of the uncentered maximal function \( M_{\alpha} f \), defined for each integrable function \( f \) on \( (\mathbb{R}_{+}^{d}, \nu_{\alpha}) \) and \( x \in \mathbb{R}^{d}_{+} \) by

\[
M_{\alpha} f(x) = \sup_{\varepsilon > 0, z \in B_{+}^{d}(x, \varepsilon)} \frac{1}{\nu_{\alpha}(B_{+}^{d}(0, \varepsilon))} \int_{\mathbb{R}_{+}^{d}} f(y) \tau_z(\chi_{B_{+}^{d}(0, \varepsilon)})(-y', y_d) \, d\nu_{\alpha}(y),
\]

where \( \nu_{\alpha} \) is a weighted Lebesgue measure associated with the Weinstein operator (see next section) and \( B_{+}^{d}(x, \varepsilon) = B(x, \varepsilon) \cap \mathbb{R}^{d}_{+} \) the closed ball on \( \mathbb{R}^{d}_{+} \) with radius \( \varepsilon \) centered at \( x \). Finally, we obtain the \( L^{p} \)-boundedness of \( M_{\alpha} f \) when \( 1 < p \leq +\infty \).

Bloom and Xu in [4] have obtained analogous results for the Chébli-Trimèche hypergroups. Later, similar results have been established in [1] for the harmonic analysis involving the Dunkl operator on the real line.

The contents of this paper are as follows.
In section 2, we collect some basic definitions and results about harmonic analysis associated with Weinstein operator.
In section 3, we establish in a first step, estimates of \( \tau_x(\chi_{B_{+}^{d}(0, \varepsilon)}), x \in \mathbb{R}^{d}_{+} \). In the second step, we prove that the uncentered maximal function \( M_{\alpha} f \) is of weak-type \((1, 1)\) and we obtain that is strong type \((p, p)\) for \( 1 < p \leq +\infty \).
Along this paper, we denote \((.,.)\) the usual Euclidean inner product in \( \mathbb{R}^{d} \) as well as its extension to \( \mathbb{C}^{d} \times \mathbb{C}^{d} \), we write for \( x \in \mathbb{R}^{d}, \| x \| = \sqrt{(x, x)} \). In the sequel \( c \) represents a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- \( D_{c}(\mathbb{R}^{d}) \) the space of \( C^{\infty} \)-functions which are of compact support, even with respect to the last variable.
2. Preliminaries

In this section, we recall some notations and results about harmonic analysis associated with the Weinstein operator.

For every $1 \leq p \leq +\infty$, we denote by $L^p(\mathbb{R}^d_+, \nu_\alpha)$ the space of measurable functions $f$ on $\mathbb{R}^d_+$ such that

$$
\|f\|_{p,\alpha} = \left( \int_{\mathbb{R}^d_+} |f(x)|^p d\nu_\alpha(x) \right)^{1/p} < +\infty, \quad \text{if } p < +\infty
$$

and

$$
\|f\|_{\infty} = \text{ess sup}_{x \in \mathbb{R}^d_+} |f(x)| < +\infty,
$$

where $\nu_\alpha$ is a measure defined by

$$
d\nu_\alpha(x) = \frac{x_{d}^{2\alpha+1}}{(2\pi)^{\frac{d+1}{2}} 2^{\alpha} \Gamma(\alpha + 1)} \, dx_1 \ldots dx_d.
$$

For a radial function $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$, the function $F$ defined on $\mathbb{R}^d_+$ such that

$$f(x) = F(\|x\|), \quad \|x\| \in \mathbb{R}^d_+$$

is integrable with respect to the measure $r^{2\alpha+d} dr$.

More precisely, we have

$$
\int_{\mathbb{R}^d_+} f(x) d\nu_\alpha(x) = \frac{1}{2^{\alpha+d} \Gamma(\alpha + \frac{d+1}{2})} \int_{0}^{+\infty} F(r) r^{2\alpha+d} dr. \quad (2.1)
$$

For all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$, the system

$$
\begin{cases}
\frac{\partial^2}{\partial x_j^2} u(x) = -\lambda_j^2 u(x), \quad j = 1, \ldots, d-1, \\
L_\alpha u(x) = -\lambda_d^2 u(x) \\
u(0) = 1, \quad \frac{\partial u}{\partial x_j}(0) = 0, \quad j = 1, \ldots, d-1,
\end{cases}
$$

has a unique solution on $\mathbb{R}^d$, denoted by $\Psi_\lambda$ called the Weinstein kernel and given by

$$
\Psi_\lambda(x) = e^{-i(x',\lambda')} j_\alpha(x_d \lambda_d). \quad (2.2)
$$

Here $x = (x', x_d) \in \mathbb{R}^d_+$, $x' = (x_1, \ldots, x_{d-1})$, $\lambda' = (\lambda_1, \ldots, \lambda_{d-1})$ and $j_\alpha$ is the normalized Bessel function of the first kind and order $\alpha$, defined by

$$
j_\alpha(\lambda x) = \begin{cases} 
2^{\alpha} \Gamma(\alpha + 1) \frac{J_\alpha(\lambda x)}{\lambda^\alpha} & \text{if } \lambda x \neq 0 \\
1 & \text{if } \lambda x = 0,
\end{cases} \quad (2.3)
$$

where $J_\alpha$ is the Bessel function of first kind and order $\alpha$ (see [18]).

We have for all $x \in \mathbb{R}$, the function $\lambda \to j_\alpha(\lambda x)$ is even on $\mathbb{R}$. 

$\bullet$ $S_\star(\mathbb{R}^d)$ the space of $C^\infty$-functions which are rapidly decreasing together with their derivatives, even with respect to the last variable.
The Weinstein kernel $\Psi_\lambda(x)$ has a unique extension to $\mathbb{C}^d \times \mathbb{C}^d$. It has the following properties:

i) $\forall \lambda, z \in \mathbb{C}^d$, $\Psi_\lambda(z) = \Psi_{-\lambda}(z)$.

ii) $\forall \lambda, z \in \mathbb{C}^d$, $\Psi_\lambda(-z) = \Psi_z(\lambda)$.

iii) $\forall \lambda, x \in \mathbb{R}^d$, $|\Psi_\lambda(x)| \leq 1$.  

There exists an analogue of the classical Fourier transform with respect to the Weinstein kernel called the Weinstein transform and denoted by $\mathcal{F}_W$. The Weinstein transform enjoys properties similar to those of the classical Fourier transform and is defined for $f \in L^1(\mathbb{R}_+^d, \nu_\alpha)$ by

$$\mathcal{F}_W(f)(\lambda) = \int_{\mathbb{R}_+^d} f(y) \Psi_\lambda(y) \, d\nu_\alpha(y), \quad \lambda \in \mathbb{R}_+^d.$$  

We list some known properties of this transform:

i) For all $f \in L^1(\mathbb{R}_+^d, \nu_\alpha)$, we have

$$\|\mathcal{F}_W(f)\|_\infty \leq \|f\|_{1,\alpha}. \quad (2.6)$$

ii) Let $f \in L^1(\mathbb{R}_+^d, \nu_\alpha)$. If $\mathcal{F}_W(f) \in L^1(\mathbb{R}_+^d, \nu_\alpha)$, then we have the inversion formula

$$f(x) = \int_{\mathbb{R}_+^d} \Psi_\lambda(-x', x_d) \mathcal{F}_W(f)(\lambda) \, d\nu_\alpha(\lambda), \quad x \in \mathbb{R}_+^d. \quad (2.7)$$

iii) The Weinstein transform $\mathcal{F}_W$ on $S_*(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}_+^d, \nu_\alpha)$.

iv) Plancherel formula: For all $f \in L^2(\mathbb{R}_+^d, \nu_\alpha)$, we have

$$\int_{\mathbb{R}_+^d} |f(x)|^2 \, d\nu_\alpha(x) = \int_{\mathbb{R}_+^d} |\mathcal{F}_W(f)(x)|^2 \, d\nu_\alpha(x)$$

v) Let $f \in L^1(\mathbb{R}_+^d, \nu_\alpha)$ be a radial function, then the function $F$ such that $f(x) = F(\|x\|)$ is integrable on $(0, +\infty)$ with respect to the measure $r^{2\alpha+d} \, dr$ and its Weinstein transform is given for $y \in \mathbb{R}_+^d$, by

$$\mathcal{F}_W(f)(y) = \mathcal{F}_B^{\gamma + \frac{1}{2}}(F)(\|y\|), \quad (2.8)$$

where $\mathcal{F}_B^\gamma$ is the Fourier-Bessel transform of order $\gamma$, $\gamma > -\frac{1}{2}$, given by

$$\mathcal{F}_B^\gamma(F)(\lambda) = \frac{1}{2\pi \Gamma(\gamma + 1)} \int_0^{+\infty} F(r) j_\gamma(\lambda r) r^{2\gamma+1} \, dr, \quad \lambda \in (0, +\infty).$$

For $x, y \in \mathbb{R}_+^d$ and $f$ a continuous function on $\mathbb{R}^d$ which is even with respect to the last variable, the Weinstein translation operator $\tau_x$ is given by

$$\tau_x(f)(y) = \int_0^{+\infty} f(x' + y', \rho) W_\alpha(x_d, y_d, \rho) \rho^{2\alpha+1} \, d\rho, \quad (2.9)$$
where the kernel $W_\alpha$ is given by

$$W_\alpha(x_d, y_d, \rho) = \frac{\Gamma(\alpha + 1)((x_d + y_d)^2 - \rho^2)^{-\frac{\alpha}{2}}(\rho^2 - (x_d - y_d)^2)^{\frac{\alpha}{2}}}{2^{2\alpha - 1}\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})(x_d y_d \rho)^{2\alpha}} \chi|\rho| |x_d - y_d| |x_d + y_d|(\rho).$$

For all $x_d, y_d > 0$, we have

$$\int_0^{+\infty} W_\alpha(x_d, y_d, \rho) \rho^{2\alpha + 1} d\rho = 1.$$

The Weinstein translation operator satisfies the following properties.

i) For all continuous function $f$ on $\mathbb{R}^d$ which is even with respect to the last variable and $x, y \in \mathbb{R}^d_+$, we have

$$\tau_x f(y) = \tau_y f(x), \quad \tau_0 f = f.$$

ii) For all $f \in S_*(\mathbb{R}^d)$ and $y \in \mathbb{R}^d_+$, the function $x \to \tau_x f(y)$ belongs to $S_*(\mathbb{R}^d)$.

iii) For all $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, $1 \leq p \leq +\infty$ and $x \in \mathbb{R}^d_+$, we have

$$\|	au_x f\|_{p, \alpha} \leq \|f\|_{p, \alpha}.$$

For a function $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, $p = 1$ or $2$ and $x \in \mathbb{R}^d_+$, the Weinstein translation $\tau_x$ is also defined by the following relation:

$$\mathcal{F}_W(\tau_x f)(\lambda) = \Psi_{y}(x_d) \mathcal{F}_W(f)(\lambda), \quad \lambda \in \mathbb{R}^d_+.$$

By using the Weinstein translation, we define the convolution product $f *_{\alpha} g$ of functions $f, g \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ as follows:

$$(f *_{\alpha} g)(x) = \int_{\mathbb{R}^d_+} \tau_x f(-y', y_d) g(y) d\nu_{\alpha}(y), \quad x \in \mathbb{R}^d_+.$$

This convolution is commutative and associative and satisfies the following results.

i) Let $1 \leq p, q, r \leq +\infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ (the Young condition). If $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$ and $g \in L^q(\mathbb{R}^d_+, \nu_\alpha)$, then $f *_{\alpha} g \in L^r(\mathbb{R}^d_+, \nu_\alpha)$ and we have

$$\|f *_{\alpha} g\|_{r, \alpha} \leq \|f\|_{p, \alpha} \|g\|_{q, \alpha}.$$

ii) For all $f \in L^1(\mathbb{R}^d_+, \nu_\alpha)$ and $g \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, $p = 1$ or $2$, we have

$$\mathcal{F}_W(f *_{\alpha} g) = \mathcal{F}_W(f) \mathcal{F}_W(g).$$
3. Weak-type (1,1) of the uncentered maximal function

In this section, we establish estimates of $\tau_x(\chi_{B^+(0,\epsilon)})(-y', y_d)$, $x, y \in \mathbb{R}^d_+$, and we prove the weak-type (1,1) of the uncentered maximal function $M_\alpha f$ and we obtain that is bounded on $L^p$ for $1 < p \leq +\infty$. 

The following remark plays a key role.

**Remark 3.1.** For any $x, y \in \mathbb{R}^d_+$ and $\epsilon > 0$, we have

$$\tau_x(\chi_{B^+(0,\epsilon)})(-y', y_d) = \int_0^{+\infty} \chi_{B^+(0,\epsilon)}(x' - y', \rho)W_\alpha(x_d, y_d, \rho)^{2\alpha+1}d\rho. \quad (3.1)$$

Put $u = (x' - y', \rho)$, we have $\|u\|^2 = \sum_{i=1}^{d-1} (x_i - y_i)^2 + \rho^2$. Then by (2.9) and (2.10), we have

$$\sqrt{\sum_{i=1}^{d-1} (x_i - y_i)^2 + (x_d - y_d)^2} < \|u\| < \sqrt{\sum_{i=1}^{d-1} (x_i - y_i)^2 + (x_d + y_d)^2}. \quad (3.2)$$

From (3.1), we have $u \in B^+(0, \epsilon)$, which gives according to (3.2),

$$\tau_x(\chi_{B^+(0,\epsilon)})(-y', y_d) = 0 \text{ when } \|x - y\| = \sqrt{\sum_{i=1}^{d-1} (x_i - y_i)^2 + (x_d - y_d)^2} \geq \epsilon.$$

Then we can assume that $y \in \mathbb{R}^d_+$ satisfies $\|x - y\| < \epsilon$. Note that $\|x - y\| < \epsilon$ implies $|x_d - y_d| < \epsilon$.

**Lemma 3.1.** Let $\lambda \in \mathbb{R}^d_+$ and $\epsilon \in [0, +\infty[$, then we have

i) $|\mathcal{F}_W(\chi_{B^+(0,\epsilon)})(\lambda)| \leq c \epsilon^{2\alpha+d+1}, \quad (3.3)$

ii) $|\mathcal{F}_W(\chi_{B^+(0,\epsilon)})(\lambda)| \leq c \epsilon^{\alpha+\frac{d}{2}}\|\lambda\|^{-(\alpha+\frac{d}{2}+1)}. \quad (3.4)$

Here $c$ is a constant which depends only on $\alpha$ and $d$.

**Proof.** By (2.8), we can write for $\lambda \in \mathbb{R}^d_+$ and $\epsilon \in (0, +\infty)$,

$$\mathcal{F}_W(\chi_{B^+(0,\epsilon)})(\lambda) = \frac{1}{2^{\alpha+d+1}\Gamma(\alpha + \frac{d+1}{2})} \int_0^{+\infty} \chi_{B^+(0,\epsilon)}(r)j_{\alpha + \frac{d}{2} - \frac{1}{2}}(\|\lambda\|r)r^{2\alpha+d}dr = \frac{\epsilon^{2\alpha+d+1}}{2^{\alpha+d+1}\Gamma(\alpha + \frac{d+3}{2})} j_{\alpha + \frac{d}{2} + \frac{1}{2}}(\|\lambda\|\epsilon). \quad (3.5)$$

Since $|j_{\alpha + \frac{d}{2} + \frac{1}{2}}(\|\lambda\|\epsilon)| \leq 1$, we get

$$|\mathcal{F}_W(\chi_{B^+(0,\epsilon)})(\lambda)| \leq c \epsilon^{2\alpha+d+1}.$$
Now, from (2.3), (3.5) and the fact that the function \( z \mapsto \sqrt{z} J_\alpha(z) \) is bounded on \((0, +\infty)\), we can see that
\[
|\mathcal{F}W(\chi_{B^+(0,c)})(\lambda)| = \frac{\varepsilon^{2\alpha+d+1}}{2^{\alpha+d+1} \Gamma(\alpha + \frac{d+3}{2})} |j_\alpha + \frac{\varepsilon}{\lambda} + \frac{1}{2}(\|\lambda\| \varepsilon)|
\]
\[
= \frac{1}{2^{\alpha+d+1} \Gamma(\alpha + \frac{d+3}{2})} \|\lambda\| \varepsilon |J_\alpha + \frac{\varepsilon}{\lambda} + \frac{1}{2}(\|\lambda\| \varepsilon)|
\]
\[
\leq c \varepsilon^{\alpha+\frac{d+1}{2}} \|\lambda\|^{-(\alpha+\frac{d+1}{2})}.
\]
Hence the lemma is proved. \(\square\)

**Lemma 3.2.** For \( \alpha > \frac{d}{2} - 1 \), there exists \( c > 0 \) such that for any \( x \in \mathbb{R}^d_+ \) with \( x_d > 0 \) and \( \varepsilon > 0 \), we have
\[
0 \leq \tau_x(\chi_{B^+(0,c)})(-y', y_d) \leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha+1}, \text{ a.e. } y \in \mathbb{R}^d_+.
\] (3.6)

Here \( c \) is a constant which depends only on \( \alpha \) and \( d \).

**Proof.** Let \( x \in \mathbb{R}^d_+ \) and \( \varepsilon > 0 \). Using (2.9) and (2.11) we have
\[
0 \leq \tau_x(\chi_{B^+(0,c)})(-y', y_d) \leq 1, \text{ a.e. } y \in \mathbb{R}^d_+.
\] (3.7)

If \( 0 < x_d \leq 2\varepsilon \), we obtain that
\[
1 \leq \left( \frac{2\varepsilon}{x_d} \right)^{2\alpha+1},
\]
hence, by (3.7) we deduce (3.6).

Therefore we can assume in the following argument that \( x_d > 2\varepsilon \), and in view of Remark 3.1 that \( y \in \mathbb{R}^d_+ \) satisfies \( |x_d - y_d| < \varepsilon \).

Take \( \psi \in D_*(\mathbb{R}^d) \) satisfying \( 0 \leq \psi(x) \leq 1 \), \( \text{supp } \psi \subset B^+(0,1) \) and \( \|\psi\|_{1,\alpha} = 1 \). Put
\[
\psi_t(x) = \frac{1}{t^{2\alpha+d+1}} \psi\left( \frac{x}{t} \right), \text{ } t > 0, \text{ } x \in \mathbb{R}^d_+,
\]
the dilation of \( \psi \). We have \( \psi_t \in D_*(\mathbb{R}^d) \) which gives \( \mathcal{F}W(\psi_t) \in S_*(\mathbb{R}^d) \), then we can assert that both of \( \tau_x(\chi_{B^+(0,c)} \ast \alpha \psi_t) \) and \( \mathcal{F}W(\tau_x(\chi_{B^+(0,c)} \ast \alpha \psi_t)) \) are in \( L^1(\mathbb{R}^d_+, \nu_\alpha) \). Using (2.7), (2.13) and (2.15), we obtain for \( y \in \mathbb{R}^d_+ \)
\[
\tau_x(\chi_{B^+(0,c)} \ast \alpha \psi_t)(y)
\]
\[
= \int_{\mathbb{R}^d_+} \Psi_x(-x', x_d) \Psi_x(-y', y_d) \mathcal{F}W(\chi_{B^+(0,c)})(\lambda) \mathcal{F}W(\psi_t)(\lambda) d\nu_\alpha(\lambda)
\]
\[
= \int_{\mathbb{R}^d_+} e^{i\langle x', \lambda \rangle + \langle y', \lambda \rangle} j_\alpha(\lambda d x_d) j_\alpha(\lambda d y_d) \mathcal{F}W(\chi_{B^+(0,c)})(\lambda) \mathcal{F}W(\psi_t)(\lambda) d\nu_\alpha(\lambda).
\] (3.8)
Clearly we have $\| \psi_t \|_{1, \alpha} = 1$. According to (2.6), we have

$$|\mathcal{F}_W(\psi_t)(\lambda)| \leq \| \mathcal{F}_W(\psi_t) \|_{\infty} \leq \| \psi_t \|_{1, \alpha} = 1, \quad \text{a.e } \lambda \in \mathbb{R}_+^d.$$  \hfill (3.9)

Let us decompose (3.8) as a sum of three terms:

$$\tau_x(\chi_{B^{+}(0, \varepsilon)} * \alpha \psi_t)(y) = \int_{\| \lambda \| \leq x_d^{-1}} + \int_{x_d^{-1} \leq \| \lambda \| \leq \varepsilon^{-1}} + \int_{\varepsilon^{-1} \leq \| \lambda \|}$$

$$= I_1 + I_2 + I_3. \hfill (3.10)$$

From (2.1), (2.4), (3.3) and (3.9), we obtain

$$|I_1| \leq c \varepsilon^{2\alpha+d+1} \int_{\| \lambda \| \leq x_d^{-1}} d\nu_\alpha(\lambda)$$

$$\leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha+d+1}$$

$$\leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha+1}, \quad \text{for } x_d > 2\varepsilon. \hfill (3.11)$$

To estimate $I_2$, we observe that for $x_d > 2\varepsilon$ and $|x_d - y_d| < \varepsilon$, we have

$$\frac{1}{2} x_d < x_d - \varepsilon < y_d < x_d + \varepsilon < \frac{3}{2} x_d,$$

so we deduce

$$0 < y_d^{-(\alpha + \frac{1}{2})} < c x_d^{-(\alpha + \frac{1}{2})}. \hfill (3.12)$$

By (2.3) and the fact that the function $z \mapsto \sqrt{z} J_\alpha(z)$ is bounded on $(0, +\infty)$, we can write

$$|j_\alpha(z)| \leq c z^{-(\alpha + \frac{1}{2})}, \hfill (3.13)$$

then from (2.1), (3.3), (3.9), (3.12) and (3.13), we get

$$|I_2| \leq c \varepsilon^{2\alpha+d+1} x_d^{-(\alpha + \frac{1}{2})} y_d^{-(\alpha + \frac{1}{2})} \int_{x_d^{-1} \leq \| \lambda \| \leq \varepsilon^{-1}} \lambda_d^{-(2\alpha+1)} d\nu_\alpha(\lambda)$$

$$\leq c \varepsilon^{2\alpha+d+1} x_d^{-2\alpha-1} \left( \frac{1}{x_d} - \frac{1}{x_d^2} \right)$$

$$\leq c \varepsilon^{2\alpha+1} x_d^{-2\alpha-1}$$

$$\leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha+1}, \quad \text{for } x_d > 2\varepsilon. \hfill (3.14)$$

For $I_3$, we use (2.1), (3.4), (3.9), (3.12) and (3.13) and we find that

$$|I_3| \leq c \varepsilon^{\alpha+\frac{d}{2}} x_d^{-(\alpha + \frac{1}{2})} y_d^{-(\alpha + \frac{1}{2})} \int_{\varepsilon^{-1} \leq \| \lambda \| \leq (\alpha + \frac{1}{2})} \lambda_d^{-(2\alpha+1)} d\nu_\alpha(\lambda)$$

$$\leq c \varepsilon^{\alpha+\frac{d}{2}} x_d^{-2\alpha-1} \int_{\varepsilon^{-1}}^{+\infty} r^{-(\alpha + \frac{d}{2} - 2)} dr.$$
Since $\alpha > \frac{d}{2} - 1$, we obtain

$$|I_3| \leq c \varepsilon^{\alpha + \frac{d}{2} - (2\alpha + 1)} \varepsilon^{\frac{d}{2} - 1} \leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha + 1}, \quad \text{for } x_d > 2\varepsilon.$$  \hspace{1cm} (3.15)

Thus we get by (3.10), (3.11), (3.14) and (3.15)

$$0 \leq \tau_x(\chi_{B^+(0, \varepsilon)} * \psi_t)(-y', y_d) \leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha + 1}, \quad \text{for } x_d > 2\varepsilon.$$  \hspace{1cm} (3.16)

Now using (2.9) and Fatou’s Lemma, we can assert that

$$\lim_{t \to 0^+} \tau_x(\chi_{B^+(0, \varepsilon)} * \psi_t)(-y', y_d) = \tau_x(\chi_{B^+(0, \varepsilon)})(-y', y_d), \quad \text{a.e } y \in \mathbb{R}^d_+.$$  \hspace{1cm} (3.17)

Hence, we deduce that

$$0 \leq \tau_x(\chi_{B^+(0, \varepsilon)})(-y', y_d) \leq c \left( \frac{\varepsilon}{x_d} \right)^{2\alpha + 1},$$

for $x_d > 2\varepsilon$, a.e $y \in \mathbb{R}^d_+$ with $|x_d - y_d| < \varepsilon$, therefore (3.6) is established.  \hspace{1cm} \Box

**Notation:** For $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$, we put

$$C^+(x, \varepsilon) = B_{d-1}(x', \varepsilon) \times [0, x_d + \varepsilon[,$$

with $x = (x', x_d)$ and $B_{d-1}(x', \varepsilon)$ is the closed ball on $\mathbb{R}^{d-1}$ with radius $\varepsilon$ centred at $x'$.

**Lemma 3.3.** For $\alpha > \frac{d}{2} - 1$, there exists $c > 0$ such that for any $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$, we have

$$0 \leq \tau_x(\chi_{B^+(0, \varepsilon)})(-y', y_d) \leq c \frac{\nu_\alpha(B^+(0, \varepsilon))}{\nu_\alpha(C^+(x, \varepsilon))}, \quad \text{a.e } y \in \mathbb{R}^d_+.$$  \hspace{1cm} (3.18)

Here $c$ is a constant which depends only on $\alpha$ and $d$.

**Proof.** Let $x \in \mathbb{R}^d_+$ and $\varepsilon > 0$. Using (2.1), we have

$$\nu_\alpha(B^+(0, \varepsilon)) = \frac{\varepsilon^{2\alpha + d + 1}}{2^{\alpha + \frac{d+1}{2}}(2\alpha + d + 1)\Gamma(\alpha + \frac{d+1}{2})}.$$  \hspace{1cm} (3.19)

On the one hand, we get for $x_d \leq \varepsilon$,

$$C^+(x, \varepsilon) = B_{d-1}(x', \varepsilon) \times [0, x_d + \varepsilon[,$$

then, we obtain

$$\nu_\alpha(C^+(x, \varepsilon)) = \frac{1}{(2\pi)^{\frac{d+1}{2}}2^{\alpha+1}\Gamma(\alpha+1)} \int_{B_{d-1}(x', \varepsilon)} dy_1...dy_{d-1} \int_0^{x_d + \varepsilon} \int_0^{x_d + \varepsilon} y^{2\alpha + 1} dy_d \leq c \varepsilon^{2(\alpha+1)} \int_{B_{d-1}(x', \varepsilon)} dy_1...dy_{d-1} \leq c \varepsilon^{2\alpha + d + 1}.$$
Using \((3.17)\), we deduce
\[
\nu_\alpha(C^+(x, \varepsilon)) \leq c \nu_\alpha(B^+(0, \varepsilon)),
\]
then by \((3.7)\), we obtain \((3.16)\) for \(x_d \leq \varepsilon\).

On the other hand, we have for \(x_d > \varepsilon\),
\[
C^+(x, \varepsilon) = B_{d-1}(x', \varepsilon) \times [x_d - \varepsilon, x_d + \varepsilon],
\]
then , we obtain
\[
\nu_\alpha(C^+(x, \varepsilon)) = \frac{1}{(2\pi)^\frac{d-1}{2} 2^{\alpha} \Gamma(\alpha + 1)} \int_{B_{d-1}(x', \varepsilon)} dy_1 \cdots dy_{d-1} \int_{x_d - \varepsilon}^{x_d + \varepsilon} y_d^{2\alpha + 1} dy_d
\]
\[
\leq c \varepsilon^{d-1} (x_d + \varepsilon)^{2\alpha + 1} \times \varepsilon
\]
\[
\leq c \varepsilon^d x_d^{2\alpha + 1}.
\]
Using \((3.17)\), we get
\[
\nu_\alpha(C^+(x, \varepsilon)) \leq c \nu_\alpha(B^+(0, \varepsilon)) \left(\frac{x_d}{\varepsilon}\right)^{2\alpha + 1},
\]
then by \((3.6)\), we obtain \((3.16)\) for \(x_d > \varepsilon\), which proves the result. \(\Box\)

According to \((13)\), Lemma 1.6, we have the following Vitali covering lemma.

**Lemma 3.4.** Let \(E\) be a measurable subset of \(\mathbb{R}^d_+\) (with respect to \(\nu_\alpha\)) which is covered by the union of a family \(\{B^+_i\}\) where \(B^+_j = B^+(x_j, r_j)\). Then from this family we can select a subfamily, \(B^+_1, B^+_2, \ldots\) (which may be finite) such that \(B^+_i \cap B^+_j = \emptyset\) for \(i \neq j\) and
\[
\sum_h \nu_\alpha(B^+_h) \geq c \nu_\alpha(E).
\]

We recall that for \(x \in \mathbb{R}^d_+\),
\[
M_\alpha f(x) = \sup_{\varepsilon > 0, z \in B^+(x, \varepsilon)} \frac{1}{\nu_\alpha(B^+(0, \varepsilon)))} \int_{B^+_x} f(y) \tau_z(\chi_{B^+(0, \varepsilon))}(-y', y_d) d\nu_\alpha(y),
\]
so, we can write also
\[
M_\alpha f(x) = \sup_{\varepsilon > 0, z \in B^+(x, \varepsilon)} \frac{1}{\nu_\alpha(B^+(0, \varepsilon)))} \left| f * \chi_{B^+(0, \varepsilon)}(z) \right|.
\]

**Theorem 3.1.** The uncentered maximal function \(M_\alpha f\) is of weak type \((1,1)\).

**Proof.** Let \(\varepsilon > 0, x \in \mathbb{R}^d_+, z \in B^+(z, \varepsilon)\) and \(f \in L^1(\mathbb{R}^d_+, \nu_\alpha)\). Using Remark 3.1, we have
\[
\left| f * \chi_{B^+(0, \varepsilon)}(z) \right| \leq \int_{B^+(z, \varepsilon)} |f(y)| \tau_z(\chi_{B^+(0, \varepsilon))}(-y', y_d) d\nu_\alpha(y).
\]
By (3.16), we obtain
\[
\left| f \ast_{\alpha} \chi_{B^+(0, \varepsilon)}(z) \right| \leq c \frac{\nu_{\alpha}(B^+(0, \varepsilon))}{\nu_{\alpha}(C^+(z, \varepsilon))} \int_{B^+(z, \varepsilon)} |f(y)| d\nu_{\alpha}(y).
\]

Since, \( B^+(z, \varepsilon) \subset C^+(z, \varepsilon) \), then
\[
\left| f \ast_{\alpha} \chi_{B^+(0, \varepsilon)}(z) \right| \leq c \frac{\nu_{\alpha}(B^+(0, \varepsilon))}{\nu_{\alpha}(B^+(z, \varepsilon))} \int_{B^+(z, \varepsilon)} |f(y)| d\nu_{\alpha}(y).
\]

Hence we deduce that
\[
M_{\alpha} f(x) \leq c \tilde{M}_{\alpha} f(x), \quad (3.18)
\]
where \( \tilde{M}_{\alpha} f \) is defined by
\[
\tilde{M}_{\alpha} f(x) = \sup_{\varepsilon > 0, x \in B^+(x, \varepsilon)} \frac{1}{\nu_{\alpha}(B^+(z, \varepsilon))} \int_{B^+(z, \varepsilon)} |f(y)| d\nu_{\alpha}(y).
\]

For \( \lambda > 0 \), put
\[
\tilde{E}_\lambda = \{ x \in \mathbb{R}^d; \tilde{M}_{\alpha} f(x) > \lambda \}.
\]

Then, for each \( x \in \tilde{E}_\lambda \), there exists \( \varepsilon > 0 \) and \( z \in B^+(x, \varepsilon) \), such that
\[
\int_{B^+(z, \varepsilon)} |f(y)| d\nu_{\alpha}(y) > \lambda \nu_{\alpha}(B^+(z, \varepsilon)). \quad (3.19)
\]

Furthermore, note that \( x \in B^+(z, \varepsilon) \), then when \( x \) runs through the set \( \tilde{E}_\lambda \), the union of the corresponding \( B^+(z, \varepsilon) \) covers \( \tilde{E}_\lambda \). Thus using Lemma 3.4, we can select a disjoint subfamily \( B^+(z_1, \varepsilon_1), B^+(z_2, \varepsilon_2), \ldots \) (which may be finite) such that
\[
\sum_h \nu_{\alpha}(B^+(z_h, \varepsilon_h)) \geq c \nu_{\alpha}(\tilde{E}_\lambda). \quad (3.20)
\]

We have
\[
\left( \int_{y \in \bigcup B^+(z_1, \varepsilon_1)} |f(y)| d\nu_{\alpha}(y) \right) = \sum_h \left( \int_{y \in B^+(z_h, \varepsilon_h)} |f(y)| d\nu_{\alpha}(y) \right),
\]
applying (3.19) and (3.20) to each of the mutually disjoint subfamily, we get
\[
\left( \int_{y \in \bigcup B^+(z_h, \varepsilon_h)} |f(y)| d\nu_{\alpha}(y) \right) > \lambda \sum_h \nu_{\alpha}(B^+(z_h, \varepsilon_h)) \geq \lambda c \nu_{\alpha}(\tilde{E}_\lambda).
\]

But since the first member of this inequality is majorized by \( \|f\|_{1, \alpha} \), we obtain
\[
\nu_{\alpha}(\tilde{E}_\lambda) \leq c \frac{\|f\|_{1, \alpha}}{\lambda},
\]
which gives that \( \tilde{M}_{\alpha} f \) is of weak type \((1, 1)\) and hence from (3.18) the same is true for \( M_{\alpha} f \).

As consequence of Theorem 3.1, we obtain the following corollary.
Corollary 3.1. If $1 < p \leq +\infty$ and $f \in L^p(\mathbb{R}^d_+, \nu_\alpha)$, then we have

$$M_\alpha f \in L^p(\mathbb{R}^d_+, \nu_\alpha) \quad \text{and} \quad \|M_\alpha f\|_{p,\alpha} \leq c \|f\|_{p,\alpha}.$$ 

Proof. Using Theorem 3.1, ([11], Corollary 21.72) and proceeding in the same manner as in the proof on Euclidean spaces (see for example Theorem 1 in [13], section 1.3), we obtain the results.

\[ \square \]

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* Department of Mathematics  
Preparatory Institute of Engineer Studies of Tunis  
1089 Monfleury Tunis, University of Tunis  
Tunisia  
E-mail address: chokri.abdelkefi@yahoo.fr

** Department of Mathematics  
Faculty of Sciences of Tunis  
1060 Tunis, University of Tunis El Manar  
Tunisia  
E-mail address: safalachoub@yahoo.com