Smeared heat-kernel coefficients on the ball and generalized cone

J.S.Dowker*, Klaus Kirsten†

*Department of Theoretical Physics,
The University of Manchester, Manchester, England

†Universität Leipzig, Institut für Theoretische Physik,
Augustusplatz 10, 04109 Leipzig, Germany

Abstract

We consider smeared zeta functions and heat-kernel coefficients on the bounded, generalized cone in arbitrary dimensions. The specific case of a ball is analysed in detail and used to restrict the form of the heat-kernel coefficients $A_n$ on smooth manifolds with boundary. Supplemented by conformal transformation techniques, it is used to provide an effective scheme for the calculation of the $A_n$. As an application, the complete $A_{5/2}$ coefficient is given.
1. Introduction

The coefficients $A_n$ in the small-time asymptotic expansion of the heat-kernel corresponding to a Laplacian-like operator on smooth manifolds (possibly with a boundary) play important roles both in quantum field theory and pure mathematics.

Many schemes for their evaluation have been developed which may be divided roughly into “direct” [1–7] and “indirect” [8–12]. For manifolds without boundary, the $A_n$ are determined by algebraic equations and their computation can be, and has been, done by computer. In principle the only things needed are the coincidence limits of the geodesic distance and of its derivatives at two points. Methods employing one or another variation on this scheme can be termed direct. The “indirect” method has been developed most systematically by Branson and Gilkey [8]. Conformal transformation techniques give relations between the numerical multipliers in the heat-kernel coefficients. However, on its own, this method is unable to determine the coefficients fully. Additional information is needed, coming from other functorial relations or special case calculations [8]. Given a subset of numerical coefficients the method then provides all required information with relative ease. The aim of the present article is to give a special case calculation containing enough information which, when supplemented by the methods of [8], leads to a very effective scheme for the evaluation of at least a substantial part of any $A_n$.

In a previous article [13] (see also [14]) we considered the non-smeared heat-kernel of the Laplacian with Dirichlet or Robin boundary conditions on the $(d+1)$-dimensional bounded cone. Here we generalize this work to include a smearing function. This is an essential step in elucidating the form of the coefficients in the presence of boundaries and is also vital when conformal properties are being analyzed, particularly those of the functional determinant.

The resulting restrictions are more informative than others available and are used to determine the first few heat-kernel coefficients on an arbitrary smooth manifold with boundary. The complete pure boundary coefficient $A_{5/2}$ is given showing the practicability of our approach for high orders [15].

The paper is organized as follows. In section 2 a method is developed for the calculation of the smeared heat-kernel coefficients on the generalized cone. Both Dirichlet and Robin boundary conditions are treated. This information is used in section 3 to put restrictions on the general form of the coefficients. Supplemented by additional relations [8], the coefficients $A_0, ..., A_{5/2}$, are (re)considered and fully determined. The Conclusion summarizes our main results and suggests extensions.
2. Smeared $\zeta$–function on the generalized cone

Our immediate objective is the determination of the smeared heat-kernel coefficients on the $(d + 1)$-dimensional bounded generalized cone $\mathcal{M} = I \times \mathcal{N}$ with the hyperspherical metric cf [16]

$$ds^2 = dr^2 + r^2 d\Sigma^2,$$

where $d\Sigma^2$ is the metric on the manifold $\mathcal{N}$, and $r$ runs from 0 to 1. $\mathcal{N}$ will be referred to as the base of the cone. If it has no boundary then it is the boundary of $\mathcal{M}$ with extrinsic curvature $K^a_b = \delta^a_b$.

We consider the Laplacian on $\mathcal{M}$,

$$\Delta_{\mathcal{M}} = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\mathcal{N}},$$

with Dirichlet or Robin boundary conditions. The nonzero eigenmodes of $\Delta_{\mathcal{M}}$ that are finite at the origin have eigenvalues $-\alpha^2$ and are of the form

$$\frac{J_\nu(\alpha r)}{r^{(d-1)/2}} Y(\Omega),$$

where the harmonics on $\mathcal{N}$ satisfy

$$\Delta_{\mathcal{N}} Y(\Omega) = -\lambda^2 Y(\Omega)$$

and

$$\nu^2 = \lambda^2 + (d - 1)^2 / 4.$$  

One can also add the coupling $-\xi R$ to $\Delta_{\mathcal{M}}$, changing the following analysis slightly [13].

In order to deal with the smeared $\zeta$–function we parallel the analysis presented in [13] and refer to it for further details. We consider both Dirichlet,

$$J_\nu(\alpha) = 0,$$

and Robin boundary conditions,

$$\left(1 - \frac{D}{2} - S\right) J_\nu(\alpha) + \alpha J'_\nu(\alpha) = 0,$$

where $D = d + 1$. Pure Neumann conditions correspond to $S = 0$. 


Our main interest is in the calculation of the boundary terms of the heat-kernel expansion. A convenient way of handling these is to introduce smeared, integrated quantities, e.g. the heat-kernel

\[ K(F; \tau) = \int dx F(x)K(x, x, \tau) \]  

and its Mellin transform, the \( \zeta \)-function,

\[ \zeta(F; s) = \sum_\alpha \int dx F(x)\phi(x)\phi^*(x) \frac{1}{\alpha^{2s}} \]  

in terms of eigenfunctions, \( \phi \), and eigenvalues, \( -\alpha^2 \). In addition, the base \( \zeta \)-function

\[ \zeta_\mathcal{N}(s) = \sum_\nu d(\nu)\nu^{-2s} \]  

will turn out to be very useful. Here, \( d(\nu) \) is the ‘angular’ degeneracy.

On the generalised cone, the eigenfunctions (3) are products of Bessel functions and ‘spherical’, i.e. base, harmonics. If we smear in the radial coordinate only, then in (9) the integration over the base yields exactly the same degeneracies as in the unsmeared case, i.e. the \( d(\nu) \), and the contour expression for the \( \zeta \)-function on \( \mathcal{M} \) reads (we treat Dirichlet scalars first)

\[ \zeta(F; s) = \sum \int \frac{dk}{2\pi i}k^{-2s}\int_0^1 dr F(r)\bar{J}_\nu^2(kr)r d(\nu)\frac{\partial}{\partial k}\ln J_\nu(k) \]  

where the bar stands for normalised, \( \bar{J}_\nu(\alpha r) = \sqrt{2} J_\nu(\alpha r)/J'_\nu(\alpha) \).

The boundary parts of the coefficients contain normal derivatives, \( F_{r...r} \), of \( F \) and we choose for \( F \) a polynomial in \( r^2 \) (why \( r^2 \) will be clear soon) that contains sufficient independent derivatives to pick out the relevant contributions. For example, in \( A_1 \), since there is only one normal derivative \( F_r \), it is sufficient to take

\[ F(r) = f_0 + f_1 r^2 \]  

and to use

\[ F(1) = f_0 + f_1 \quad F_r(1) = 2f_1 \]  

in order to identify the boundary terms.

To explain our method more precisely, we continue with this simple example, (12), and afterwards generalize to an arbitrary polynomial. Using

\[ \int_0^1 dr \ r^3 \bar{J}_\nu^2(\alpha r) = \frac{2}\frac{\nu^2 - 1}{\alpha^2} + \frac{1}{3} \]  

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and substituting (12) into (11) we obtain two contributions,

\[
\zeta_M(F; s) \equiv (f_0 + \frac{1}{3} f_1) \zeta_M(s) + \frac{2}{3} f_1 \sum (\nu^2 - 1) d(\nu) \int \frac{dk}{2\pi i} k^{-2(s+1)} \frac{\partial}{\partial k} \ln J_\nu(k). \tag{15}
\]

Here, \(\zeta_M(s)\) is defined to be \(\zeta_M(1; s)\) and is known from our previous analysis [13]. Also the second line in (15) may be immediately given by direct comparison with our previous calculation. The contour integral is the same as previously apart from replacing \(s \rightarrow s + 1\). For the second term this is already all we need. The first term contains a factor \(\nu^2\), raising the argument of the base zeta function by one. (See equation (20)).

In order to describe the method further, it is necessary to use some notation introduced in [13]. For the calculation of the heat-kernel coefficients, a split of the zeta-function into two parts is very useful. One part contains all the relevant contributions and comes from the uniform asymptotic expansion of the Bessel function \(I_\nu(k)\). As has been shown in [13] these are the only contributions to the heat-kernel coefficients. Explicitly, for \(\nu \rightarrow \infty\) with \(z = k/\nu\) fixed one has,

\[
I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi \nu}} \frac{e^{\nu \eta}}{(1 + z^2)^{1/4}} \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right], \tag{16}
\]

with \(t = 1/\sqrt{1 + z^2}\) and \(\eta = \sqrt{1 + z^2} + \ln (z/(1 + \sqrt{1 + z^2}))\). Any required number of \(u_k(t)\) polynomials can be obtained via the recursion relation given in [17,18]. In addition, we need the cumulant expansion

\[
\ln \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right] \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n}, \tag{17}
\]

where the \(D_n\) have the polynomial structure

\[
D_n(t) = \sum_{b=0}^{n} x_{n,b} t^{n+2b}. \tag{18}
\]

The second part of the split, named \(Z(s)\) in [13] and accordingly \(Z(F; s)\) here, is analytic and is of no relevance to the construction of the coefficients.

By adding and subtracting \(L\) leading terms of the asymptotic expansion, (17), and performing the same steps as described in [13] one finds the aforementioned split

\[
\zeta_M(F; s) = Z(F; s) + \sum_{i=-1}^{L} A_i(F; s), \tag{19}
\]
with the definitions

\[ A_{-1}(F; s) = \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s + 1)} \zeta_N(s - 1/2) \left[ f_0 + \frac{1}{3} f_1 + \frac{2}{3} f_1 \frac{s - 1/2}{s + 1} \right] - \frac{2}{3} f_1 \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 2)} \zeta_N(s + 1/2) \]

\[ A_0(F; s) = -\frac{1}{4} \zeta_N(s) [f_0 + f_1] - \frac{1}{4} \zeta_N'(s + 1) f_1 \]

\[ A_i(F; s) = -\frac{1}{\Gamma(s)} \zeta_N(s + i/2) \sum_{b=0}^{i} x_{i,b} \frac{\Gamma(s + b + i/2)}{\Gamma(b + i/2)} \left[ f_0 + \frac{1}{3} f_1 + \frac{2}{3} f_1 \frac{s + b + i/2}{s} \right] \]

\[ - \frac{2}{3} f_1 \zeta_N(s + 1 + i/2) \sum_{b=0}^{i} x_{i,b} \frac{\Gamma(s + 1 + b + i/2)}{\Gamma(s + 1) \Gamma(b + i/2)} \]

(20)

As is apparent in (20), base contributions are separated from radial ones. This enables the heat-kernel coefficients of the Laplacian on the manifold \( \mathcal{M} \) to be written in terms of those on \( \mathcal{N} \).

In the next section we discuss the restrictions our calculation places on the general form of the heat-kernel coefficients. It is known, for example, that the coefficient \( A_2 \) contains the third normal derivative of the smearing function \( F \) and the higher coefficients involve correspondingly higher derivatives. It is thus obvious that the \( F(r) \) employed earlier (12) will not be general enough to discuss coefficients beyond \( A_{3/2} \). In order to apply our technique to all higher coefficients, at least in principle, we consider the polynomial

\[ F(r) = \sum_{n=0}^{N} f_n r^{2n}. \]  

(21)

This leads to normalization integrals of the type

\[ S[1 + 2p] = \int_0^1 dr J_{\nu}^2(\alpha r) r^{1+2p}, \]

(22)

which can be treated using Schafheitlin’s reduction formula [19]. Writing this formula for the case when \( \alpha \) is a zero of the Bessel function, \( J_\nu(\alpha) = 0 \), one has

\[ \int_0^1 dr J_{\nu}^2(\alpha r) r^{\mu+2} = \frac{\mu + 1}{\mu + 2} \frac{(\nu^2 - (\mu + 1)^2/4)}{\alpha^2} \int_0^1 dr J_{\nu}^2(\alpha r) r^\mu + \frac{1}{\mu + 2}. \]

(23)
This can be iterated down to the standard normalisation value, \( \mu = 1 \), and is the origin of (14). In order to use this formula, which is our essential technical novelty, we see that it is necessary to have a polynomial in \( r^2 \).

Schafheitlin’s formula gives the recursion for the normalization integrals (22),

\[
S[1 + 2p] = \frac{2p}{2p + 1} \frac{\nu^2 - p^2}{\alpha^2} S[2p - 1] + \frac{1}{2p + 1}
\]

so that \( S[1 + 2p] \) has the following form,

\[
S[1 + 2p] = \sum_{m=0}^{p} \left( \frac{\nu}{\alpha} \right)^{2m} \sum_{l=0}^{m} \gamma_{ml}^p \nu^{-2l}
\]

with the numerical coefficients \( \gamma_{ml}^p \) being easily determined recursively.

As seen in the treatment of the function \( F(r) \) in equation (12), using the same rules of replacement, the \( A_i(s) \) read, after some rearrangement,

\[
A_{-1}(F; s) = \frac{1}{4\sqrt{\pi}} \sum_{l=0}^{N} \left[ \sum_{m=l}^{N} L_{m,l}^{(N)} \frac{\Gamma(s - 1/2 + m)}{\Gamma(s + 1 + m)} \right] \zeta_N(s - 1/2 + l)
\]

\[
A_0(F; s) = -\frac{1}{4} \sum_{l=0}^{N} \left[ \sum_{m=l}^{N} L_{m,l}^{(N)} \right] \zeta_N(s + l)
\]

\[
A_i(F; s) = -\sum_{l=0}^{N} \left[ \sum_{m=l}^{N} L_{m,l}^{(N)} \sum_{b=0}^{i} x_{i,b} \frac{\Gamma(s + b + i/2 + m)}{\Gamma(s + m)\Gamma(b + i/2)} \right] \zeta_N(s + l + i/2)
\]

where the linear form in the \( f_p \) is defined by

\[
L_{m,l}^{(N)} = \sum_{p=m}^{N} \gamma_{ml}^p f_p.
\]

For Dirichlet boundary conditions, these formulas provide the generalization of our formalism [13] to the radially smeared case. This is enough for our purposes because the general forms of the heat-kernel coefficients contain only normal derivatives and these are radial derivatives on the generalized cone.

In the special case of the \( D \)-ball, the residues of the poles of the base (i.e. sphere) \( \zeta \)-function are given in terms of Bernoulli polynomials and the ball coefficients are then efficiently evaluated by machine [13]. One could equally well take the torus as the base manifold, but the information obtained differs only slightly.
We now turn to Robin boundary conditions. It is possible to proceed in the same way as for Dirichlet but complications arise and the situation is sufficiently different so as to warrant a separate treatment.

Write the Robin condition (7), as

$$G_\nu(\alpha) = \alpha J'_\nu(\alpha) + u J_\nu(\alpha) = 0$$

The normalisation is

$$\int_0^1 J^2_\nu(\alpha r) r \, dr = \frac{1}{2\alpha^2} (\alpha^2 - \nu^2 + u^2) J^2_\nu(\alpha)$$ (27)

and the normalised Schafheitlin formula reads

$$\int_0^1 dr \bar{J}^2_\nu(\alpha r) r^{\mu+2} = \frac{\mu + 1}{\mu + 2} \left( \frac{\nu^2 - (\mu + 1)^2/4}{\alpha^2} \right) \int_0^1 dr \bar{J}^2_\nu(\alpha r) r^\mu$$

$$+ \frac{1}{\mu + 2} \left( 1 + \frac{(\mu + 1)(u + \frac{1}{2}(\mu + 1))}{\alpha^2 - \nu^2 + u^2} \right).$$ (28)

Continuing as in the Dirichlet case, and defining $S[1 + 2p]$ as in (22), we find the reduction formula

$$S[1 + 2p] = \frac{2p}{2p + 1} \frac{\nu^2 - p^2}{\alpha^2} S[2p - 1]$$

$$+ \frac{1}{2p + 1} \left( 1 + \frac{2p(u + p)}{\alpha^2 + u^2 - \nu^2} \right).$$ (29)

Explicitly this gives the following form

$$S[1 + 2p] = \sum_{m=0}^{p} \left( \frac{\nu}{\alpha} \right)^{2m} \sum_{l=0}^{m} \gamma_m^p \nu^{-2l}$$

$$+ \frac{1}{\alpha^2 + u^2 - \nu^2} \sum_{m=0}^{p-1} \left( \frac{\nu}{\alpha} \right)^{2m} \sum_{l=0}^{n} \delta_m^p \nu^{-2l}$$ (30)

where the $\gamma_m^p$ are the same as in (25) and the $\delta_m^p$ are also easily determined by machine.

For the $\zeta$–function we have

$$\zeta^{\text{Rob}}(F; s) = \sum \int_\gamma \frac{dk}{2\pi i} k^{-2s} \int_0^1 dr F(r) \bar{J}^2_\nu(kr) r d(\nu) \frac{\partial}{\partial k} \ln G_\nu(k)$$ (31)
where the contour $\gamma$ has to be chosen so as to enclose the zeros of only $G_\nu(k)$. Thus the poles of $S[1 + 2p]$, located at $k = \pm \sqrt{\nu^2 - u^2}$, must be outside the contour. It is important to locate the contour properly because, when deforming it to the imaginary axis, contributions from the pole at $k = \sqrt{\nu^2 - u^2}$ arise.

As a result, apart from contributions identical to (26), with the usual changes between Dirichlet and Robin boundary conditions [13], we have the extra pieces

$$
\zeta^p(F; s) = \frac{\sin \pi s}{\pi} \sum_{\nu} d(\nu) \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta^p_{ml} \nu^{-2l} \int_{m/\nu}^{\infty} dz \frac{[(z\nu)^2 - m^2]^{-s}}{u^2 - \nu^2(1 + z^2)} z^{-2m} \partial \nu \ln (uI_{\nu}(z\nu) + z\nu'_{\nu}(\nu z))
$$

$$
\zeta^p_{\text{shift}}(F; s) = -\frac{1}{2} \sum_{\nu} d(\nu) \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta^p_{ml} \nu^{2m-2l} (\nu^2 - u^2)^{-s-m-1/2} \partial \ln (kJ_{\nu}(k) + uJ'_{\nu}(\nu z)) |_{k=\sqrt{\nu^2 - u^2}}
$$

(32)

the last one arising on moving the contour over the pole at $k = \sqrt{\nu^2 - u^2}$. These are the contributions additional to those in the Dirichlet case. The index $p$ refers to the fact that these are the contributions coming from the power $r^p$ in (21). In order to obtain the full zeta function, the $\sum_{p=0}^{N} f_p \zeta^p$ has to be done.

Looking at (32), we first define the asymptotic contributions $A^p_{i,\delta}(F; s)$ in the same manner as before by taking the different terms in the asymptotic expansion of the argument of the logarithm. We illustrate the calculation by dealing with

$$
A^p_{-1,\delta}(F; s) = \frac{\sin \pi s}{\pi} \sum_{\nu} d(\nu) \sum_{m=0}^{p-1} \sum_{l=0}^{m} (-1)^m \delta^p_{ml} \nu^{1-2l} \int_{m/\nu}^{\infty} dz \frac{[(z\nu)^2 - m^2]^{-s}}{u^2 - \nu^2(1 + z^2)} z^{-2m+1} (1 + z^2)^{1/2}.
$$

Using the expansion for small $u$,

$$
\frac{1}{u^2 - \nu^2(1 + z^2)} = -\sum_{i=0}^{\infty} \frac{u^{2i}}{(\nu^2)^{i+1}(1 + z^2)^{i+1}}
$$

one arrives at

$$
A^p_{-1,\delta}(F; s) = -\frac{\sin \pi s}{\pi} \sum_{i=0}^{\infty} u^{2i} \sum_{\nu} d(\nu) \sum_{m=0}^{p-1} \sum_{l=0}^{m} (-1)^m \delta^p_{ml} \nu^{-1-2i-2l} \int_{m/\nu}^{\infty} dz \frac{[(z\nu)^2 - m^2]^{-s}}{(1 + z^2)^{i+1/2} z^{2m+1}}.
$$

(34)
At this point we can continue as in previous articles by realizing that the above integrals are representations of a hypergeometric function [20]. With the help of their Mellin-Barnes representation [20] the \( \nu \)-summation can be done, yielding the base \( \zeta \)-function, and in the massless case our final result reads

\[
A_{p,\delta}^{-1}(F; s) = \frac{1}{2\Gamma(s)} \sum_{i=0}^{\infty} u^{2i} \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta_{ml} \frac{\Gamma(-s-m)\Gamma(s+i+m+1/2)}{\Gamma(-s+1)\Gamma(i+1/2)} \zeta_N(s+i+l+1/2).
\]

In the same way one obtains for the other \( A_{i,\delta}(s) \),

\[
A_{0,\delta}^p(F; s) = -\frac{1}{4\Gamma(s)} \sum_{i=0}^{\infty} u^{2i} \frac{\sum_{m=0}^{p-1} \sum_{l=0}^{m} (-1)^m \delta_{ml}}{\Gamma(s+i+m+1)\Gamma(1-s-m)} \frac{\Gamma(s+i+l+1)}{\Gamma(1-s)} \zeta_N(s+i+l+1)
\]

\[
A_{n,\delta}^p(F; s) = \frac{1}{2\Gamma(s)} \sum_{i=0}^{\infty} u^{2i} \sum_{m=0}^{n-1} \sum_{l=0}^{m} (-1)^m \delta_{ml} \sum_{b=0}^{n} x_{n,b} (n+2b) \frac{n}{\Gamma(1-s-m)\Gamma(s+i+n/2+b+m+1)} \frac{\Gamma(n+m+1)}{\Gamma(1-s)\Gamma(i+n/2+b+2)} \zeta_N(s+i+l+1+n/2)
\]

These forms are well suited for machine evaluation and the residues relevant for the heat-kernel expansion are thereby quickly determined.

The remaining task is to deal with \( \zeta_{\text{shift}}^p(F; s) \) defined in (33). To get the relevant residues we need the asymptotic behaviour of \( J_\nu \), information on which can be found in Abramowitz and Stegun [17]. Ultimately, as a practical application, we want to restrict the general form of the \( A_{5/2}^p \) coefficient and so, restricting the calculation to the order necessary for this coefficient, we arrive at

\[
\zeta_{\text{shift}}^p(F; s) = -\frac{1}{2} \sum_{m=0}^{p-1} \sum_{l=0}^{m} \delta_{ml} \times \\
(u\zeta_N(s+l+1) + u^3(s+m+1)\zeta_N(s+l+2) + ...)
\]

after some algebra. As mentioned, the dots indicate contributions having their rightmost pole to the left of \( s = (D - 5)/2 \).

All the relevant results for the calculation up to the \( A_{5/2}^p \) coefficient are now to hand. It would be possible to go further, if desired. (But see our cautionary note at the end.)
3. Heat-kernel coefficients on general manifolds

In this section we describe the restrictions placed on the general form of the heat-kernel coefficients by our special case evaluation. Because the case we treat has vanishing Riemann tensor and constant extrinsic curvature, it cannot, in general, determine the complete coefficient. However, supplemented by a lemma on product manifolds and using relations of the heat-kernel coefficients under conformal rescalings [8] we will develop a very effective scheme for their calculation. Although for Dirichlet and Robin conditions the coefficients are already completely known up to $A_2$, we will describe our procedure by starting with these low coefficients. We will see that the lower the coefficient the more restrictive is the special case, ball calculation. This opens up for future applications the possibility of applying our approach to spectral boundary conditions [21–23] and to boundary conditions involving tangential derivatives discussed recently in the context of the quantization of gauge fields in the presence of boundaries [24–27].

In what follows we will take the standpoint that the volume part of the coefficients is known. This is motivated by the fact that its calculation is purely algebraic and very effective schemes already exist [1,2,7]. In contrast, the boundary contributions are not determined purely algebraically and their evaluation turns out to be much more involved. It is here that our special case evaluation of the smeared coefficients on the ball gives the additional information necessary for the complete calculation of the coefficients. We will show the effectiveness of the scheme by giving all of $A_5/2$, but we first explain things in detail starting from the lower coefficients.

Some notation is needed. Here and in the following $F[M] = \int_M dx F(x)$ and $F[\partial M] = \int_{\partial M} dy F(y)$, with $dx$ and $dy$ being the Riemannian volume elements of $M$ and $\partial M$. In addition, $\;\vdash$ denotes differentiation with respect to the Levi-Civita connection of $M$ and $\;:\vdash$ covariant differentiation tangentially with respect to the Levi-Civita connection of the boundary. Finally, our sign convention is $R^i_{jk} = -\Gamma^i_{jk,l} + \Gamma^i_{jl,k} + \Gamma^i_{nk}\Gamma^k_{jl} - \Gamma^i_{nl}\Gamma^k_{jk}$ (see for example [28]). To state the general form of the coefficients define the partial differential operator

$$P = -\Delta - E$$

together with Dirichlet or Robin boundary conditions,

$$B^- \phi \equiv \phi|_{\partial M} \quad \text{and} \quad B^\dagger_S \phi \equiv (\phi;m - S\phi)|_{\partial M}.$$  

To have a uniform notation we set $S = 0$ for Dirichlet boundary conditions and write $B^\dagger_S$. Let $D_B$ be the operator defined by the appropriate boundary conditions.
If $F$ is a smooth function on $\mathcal{M}$, there is an asymptotic series as $t \to 0$ of the form

$$\text{Tr}_{L^2} \left( F e^{-tP_B} \right) \approx \sum_{n \geq 0} t^{\frac{n-m}{2}} a_n(F, P_B),$$

where the $a_n(F, P_B)$ are locally computable [29].

We now state, one by one, the general form of the coefficients and compare them with our special case evaluation. For convenience we will drop the index $B$ of the operator $P$. The coefficient $A_0$ is, by normalization,

$$A_0(F, P) = (4\pi)^{-D/2} F[\mathcal{M}].$$

The next one is

$$A_{1/2}(F, P) = \delta(4\pi)^{-d/2} F[\partial\mathcal{M}].$$

For the ball this means

$$A_{1/2}(F, P) = \delta(4\pi)^{-d/2} F(1)\text{vol}(S^d).$$

Using the relations (26) and (36)-(39) we can immediately determine $\delta$,

$$\delta = \left( -\frac{1}{4}, \frac{1}{4} \right).$$

The coefficient $A_{1/2}$ is thus given for a general manifold from the result on the ball (which was clear of course). Passing on to $A_1$, the general form is

$$A_1(F, P) = (4\pi)^{-D/2} 6^{-1} \left\{ (6FE + FR)[\mathcal{M}] + (b_0FK + b_1F_m + b_2FS)[\partial\mathcal{M}] \right\}$$

In our special case on the ball, $K^b_a = \delta^b_a$ and thus

$$A_1(F, P) = (4\pi)^{-D/2} 6^{-1} \text{vol}(S^d) \left\{ b_0 F(1)d + b_1 F'(1) + b_2 F(1)S \right\}.$$

Comparing with the results given in the previous section one finds

$$b_0 = 2; \quad b_1 = (3^-, -3^+); \quad b_2 = 12.$$

Thus our special case also gives the entire $A_1$ coefficient without any further information being needed. It is very important that the calculation can be performed for an arbitrary ball dimension, $D$, and also for a smearing function $F(r)$. This allows one just to compare polynomials in $d$ with the associated extrinsic curvature.
terms in the general expression and simply to read off the universal constants in this expression.

The idea is now clear and in the following we will state only the general expression and the restrictions found from the special case presented in the previous section. We continue with the next higher coefficient, with the general form,

$$A_{3/2}(F, P) = \frac{\delta}{96(4\pi)^{d/2}} \left( F(c_0E + c_1R + c_2R_{mm} + c_3K^2 + c_4K_{ab}K^{ab}c_7SK + c_8S^2) + F_m(c_5K + c_6S) + c_6F_{mm} \right).$$

The ball calculation immediately gives 7 of the 10 unknowns,

$$c_3 = (7^-, 13^+), \quad c_4 = (-10^-/2^+), \quad c_5 = (30^-, -6^+),$$

$$c_6 = 24, \quad c_7 = 96, \quad c_8 = 192, \quad c_9 = -96.$$

We next apply the lemma on product manifolds mentioned above [8]. Let $N^\nu(F) = F_{mn}$ be the $\nu^{th}$ normal covariant derivative. There exist local formulae $a_n(x, P)$ and $a_n,\nu(y, P)$ so that

$$A_n(F, P, B^F_S) = \{FA_n(x, P)\}[M] + \{\sum_{\nu=0}^{2n-1} N^\nu(F)A_n,\nu(y, P, B^F_S)\}[\partial M].$$

Let $M = M_1 \times M_2$ and $P = P_1 \otimes 1 + 1 \otimes P_2$ and $\partial M_2 = \emptyset$. Then

$$A_{n,\nu}(y, P, B^F_S) = \sum_{p+q=n} A_{p,\nu}(y_1, P_1, B^F_S)A_{q}(x_2, P_2).$$

For $A_{3/2}$ this means

$$A_{3/2}(y, P) = A_{3/2}(y_1, P_1)A_0(x_2, P_2) + A_{1/2}(y_1, P_1)A_1(x_2, P_2).$$

We will choose $P_1 = -\Delta_1$ and $P_2 = -\Delta_2 + E(x_2)$ with obvious notation to obtain

$$\delta^{96^{-1}}(c_0E + c_1R(M_2)) = \delta^{6^{-1}}(6E + R(M_2))$$

where we used in addition $R(M_1 \times M_2) = R(M_1) + R(M_2)$. This gives

$$c_0 = 96, \quad c_1 = 16.$$

It is seen, that the determination of $A_{3/2}$ is relatively simple, once the ball result is to hand. The lemma on product manifolds is also very easily applied and already only one of the universal constants $c_i$, namely $c_2$, is missing.
The remaining information is obtained using the relations between the heat-kernel coefficients under conformal rescaling [8],

$$\frac{d}{dc}|_{c=0} a_n \left(1, e^{-2cF}P\right) - (D - 2n)a_n(F, P) = 0 \quad (40)$$

Setting to zero the coefficients of all terms in (40) gives several relations between the universal constant $c_i$. We will need only one of them. Thus, setting to zero the coefficient of $F;_{,mm}$ gives

$$\frac{1}{2}(D - 2)c_0 - 2(D - 1)c_1 - (D - 1)c_2 - (D - 3)c_6 = 0$$

and so $c_2 = -8$ for Dirichlet and Robin boundary conditions. This completes the calculation of $A_{3/2}$.

We continue with the treatment of $A_2$. Its general form is [8],

$$A_2(F, P) = (4\pi)^{-D/2}2^{360 - 1/2} \left\{ F(60\Delta E + 60RE + 180E^2 + 12\Delta R + 5R^2 - 2R_{ij}R^{ij}
+ 2R_{ijkl}R^{ijkl})[M] + F(d_1R;_m + d_2R;_m + e_{12}K_{,a}^a + e_{13}K_{,ab}^{ab} + d_5E_{ab} +
+ d_6RE + d_7R_{mm}K + d_8R_{ambm}K^{ab} + d_9R_{abc}bK^{ac} + d_{10}K^3
+ d_{11}K_{ab}K^{ab}K + d_{12}K_{ab}K_c^{b}K^{ac} + d_{13}SE + d_{14}SR + d_{15}SR_{mm}
+ d_{16}SK^2 + d_{17}SK_{ab}K^{ab} + d_{18}S^2K + d_{19}S^3 + d_{20}S_{,a}^a
+ F;_{,m}(e_{11}E + e_2R + e_3R_{mm} + e_4K^2 + e_5K_{ab}K^{ab} + e_8SK + e_9S^2
+ F;_{,mm}(e_{10}K + e_{10}S) + e_7(\Delta F);_m \right\} \quad (41)$$

The ball calculation gives

$$d_{10} = (40/21^-, 40/3^+) \quad , \quad d_{11} = (-88/7^-, 8^+) \quad , \quad d_{12} = (320/21^-, 32/3^+),$$
$$d_{16} = 144 \quad , \quad d_{17} = 48 \quad , \quad d_{18} = 480 \quad , \quad d_{19} = 480$$
$$e_4 = (180/7^- , -12^+) \quad , \quad e_5 = (-60/7^- , -12^+) \quad , \quad e_6 = 24$$
$$e_7 = (30^- , -30^+) , \quad e_8 = -72 \quad , \quad e_9 = -240 \quad , \quad e_{10} = 120 \quad (42)$$

The product formula here reads

$$A_2(y, P) = A_2(y_1, P_1)A_0(x_2, P_2) - A_0(y_1, P_1)A_2(x_2, P_2)$$
$$+ A_1(y_1, P_1)A_1(x_2, P_2)$$

and leads to the universal constants,

$$d_5 = 120 \quad , \quad d_6 = 20 \quad , \quad d_{13} = 720 \quad , \quad d_{14} = 120$$
$$e_1 = (180^- , -180^+) \quad , \quad e_2 = (30^- , -30^+) \quad (43)$$
These two inputs already give 20 of the 30 unknowns, the remaining 10 are determined by the conformal rescaling (40),

$$\frac{d}{d\epsilon} |_{\epsilon=0} a_2 \left(1, e^{-2\epsilon F} P \right) - (D - 4) a_2 (F, P) = 0. \quad (44)$$

Having already evaluated many of the constants only a few more relations are required to fix the remaining ones. In the following list, we give, on the left, the term in (44) whose coefficient is equated to zero.

| Term      | Coefficient                                                                 |
|-----------|----------------------------------------------------------------------------|
| $EF_{:m}$ | $0 = -2d_1 + 60(D - 6) + d_5(D - 1) - (D - 4)e_1 - \frac{1}{2}(D - 2)d_{13}$ |
| $(\Delta F)_{:m}$ | $0 = 6(D - 6) + \frac{1}{2}(D - 2)d_1 - 2(D - 1)d_2 - (D - 4)e_7$          |
| $F_{:a}K^a$ | $0 = -4(D - 6) + (D - 4)d_3 - \frac{1}{2}(D - 2)d_5 + 2(D - 1)d_6 + d_7 + d_9$ |
| $KF_{:mm}$ | $0 = \frac{1}{2}(D - 2)d_5 - 2(D - 1)d_6 - (D - 1)d_7 - d_8 - (D - 4)e_6$ |
| $K_{ab:b}F^a$ | $0 = (D - 4)d_4 + d_8 + (D - 3)d_9 + 4(D - 6)$                              |
| $R_{mm}F_{:m}$ | $0 = (D - 1)d_7 + d_8 - 2d_9 + e_3 + 4(D - 6) - \frac{1}{2}(D - 2)d_{15}$ |
| $F_{:a}S^a$ | $0 = -\frac{1}{2}(D - 2)d_{13} + 2(D - 1)d_{14} + d_{15} + (D - 4)d_{20}$ |

From here one finds the universal constants

$$d_1 = (120^-, -24^{+}), \quad d_2 = (18^-, -42^+), \quad d_3 = 24, \quad d_4 = 0$$

$$d_7 = -4, \quad d_8 = 12, \quad d_9 = -4, \quad d_{15} = 0, \quad d_{20} = 120, \quad e_3 = 0. \quad (46)$$

This completes the evaluation of $A_2$ and we finally come to the calculation of $A_{5/2}$ which, for an arbitrary smearing function $F$, has been calculated only for a totally geodesic boundary $\partial M$. When $F = 1$, it has been determined for $M$ a domain of $\mathbb{R}^m$.

It has been shown that for a smooth, but not necessarily totally geodesic, boundary there exist universal constants such that

$$A_{5/2}(F, P) = \mp 5760^{-1}(4\pi)^{-(m-1)/2} \{ F \left( g_1 E_{:mm} + g_2 E_{:m}S + g_3 E^2 
+ g_4 E_{:a}^a + g_5 RE + 120 \Omega_{ab} \Omega^{ab} + g_6 \Delta R + g_7 R^2 + g_8 R_{ij}R^{ij} + g_9 R_{ijkl}R^{ijkl}
+ g_{10} R_{mm}E + g_{11} R_{mm}R + g_{12} RS^2 + (-360^-, 90^+) \Omega_{am} \Omega^a_m + g_{13} R_{mm}
+ g_{14} R_{mm:a}^a + g_{15} R_{mm:mm} + g_{16} R_{:m}S + g_{17} R_{mm}S^2 + g_{18} SS_{:a}^a + g_{19} S_{:a}S^a \right) \}
$$

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In this case specializing to the ball gives,

$$
+ g_{20} R_{amm b} R^{ab} + g_{21} R_{mm} R_{mm} + g_{22} R_{amm b} R^{ab}_{mm} + g_{23} E S^2 + g_{24} S^4 \\
+ F_{;m} \left\{ g_{25} R_{;m} + g_{26} RS + g_{27} R_{mm} S + g_{28} S_{;a} + g_{29} E_{;m} + g_{30} ES + g_{31} S^3 \right\} \\
+ F_{;mm} \left\{ g_{32} R + g_{33} R_{mm} + g_{34} E + g_{35} S^2 \right\} + g_{36} S F_{;mm} + g_{37} S F_{;mmmm} \\
+ F \left\{ d_1 K E_{;m} + d_2 K R_{;m} + d_3 K^{ab} R_{amm b} \right\} + d_4 K S_{;b} + d_5 K^{ab} S_{;} \\
+ d_6 K^{b c} + d_7 K^{b c} S_{;} + d_8 K^{b c} S + d_9 K^{ab} S + d_{10} K^{b c} + d_{11} K^{a b} \right\} \\
+ d_{12} K^{b c} + d_{13} K^{ab c} K^{ab c} + d_{14} K^{a b c} K^{a b c} + d_{15} K^{b c} K \\
+ d_{16} K^{a b c} + d_{17} K^{a b c} + d_{18} K^{b c} + d_{19} K^{b c} \\
+ g_{38} K SE + d_{20} K S R_{mm} + g_{39} K S R + d_{21} K_{;bb} R^{ab} S + d_{22} K^{ab} S R_{amm b} \\
+ g_{40} K^2 E + g_{41} K^{ab} E + g_{42} K^2 R + g_{43} K^{ab} R + d_{23} K^2 R_{mm} \\
+ d_{24} K^{ab} R_{mm} + d_{25} K K_{;b} R^{ab} + d_{26} K^{ab} R_{amm b} + d_{27} K^{a b} K R_{b} \\
+ d_{28} K^{a b} K^{a b} R_{bmm b} + d_{29} K_{;b c d} R^{a b c d} + d_{30} K S^3 + d_{31} K^2 S^2 + d_{32} K^{ab} S^2 \\
+ d_{33} K^3 S + d_{34} K K^{a b} S + d_{35} K^{a b} K^{a b} S + d_{36} K^4 + d_{37} K^2 K^{a b} K^{a b} \\
+ d_{38} K_{;a b c} K^{a b c} + d_{39} K K_{;b c} K^{a b c} + d_{40} K^{a b c} K_{;b c} K^{a b c} + d_{41} K^{a b c} K^{a b c} + d_{42} K^2 S^2 \\
+ d_{43} K^{a b} + d_{44} K^{a b} + d_{45} K^{a b} R^{ab} + d_{46} K^{a b} R_{amm b} + d_{47} K^2 S \\
+ d_{48} K^{a b} S + d_{49} K^3 + d_{50} K K^{a b} + d_{51} K^{a b} K^{a b} + d_{52} K^4 \\
+ F_{;m} \left\{ d_{52} K S + d_{53} K^2 + d_{54} K^{a b} \right\} + d_{55} K^2 F_{;mm} \right\} [\partial M] \\
(47)
$$

In this case specializing to the ball gives,

$$
\begin{align*}
g_{24} &= 1440 & g_{31} &= -720 \\
g_{35} &= 360 & g_{36} &= -180 \\
g_{37} &= 45 & d_{30} &= 2160 \\
d_{31} &= 1080 & d_{32} &= 360 \\
d_{33} &= 885/4 & d_{34} &= 315/2 \\
d_{35} &= 150 & d_{36} &= (-65/128^{-}, 2041/128^{+}) \\
d_{37} &= (-141/32^{-}, 417/32^{+}) & d_{40} &= (-327/8^{-}, 231/8^{+}) \\
d_{42} &= -600 & d_{47} &= -705/4 \\
d_{48} &= 75/2 & d_{49} &= (495/32^{-}, -459/32^{+}) \\
d_{50} &= (-1485/16^{-}, -267/16^{+}) & d_{51} &= (225/2^{-}, 54^{+}) \\
d_{52} &= 30 & d_{53} &= (1215/16^{-}, 315/16^{+}) \\
d_{54} &= (-945/8^{-}, -645/8^{+}) & d_{55} &= (105^{-}, 30^{+})
\end{align*}
$$
and \( d_{38} + d_{39} = (1049/32^-, 1175/32^+) \).

The product formula explicitly reads

\[
A_{5/2}(y, P) = A_{5/2}(y_1, P_1)A_0(x_2, P_2) + A_{3/2}(y_1, P_1)A_1(x_2, P_2) + A_{1/2}(y_1, P_1)A_2(x_2, P_2),
\]

which gives the 22 universal constants,

\[
g_3 = 720, \quad g_5 = 240, \quad g_6 = 48, \quad g_7 = 20,
\]
\[
g_8 = -8, \quad g_9 = 8, \quad g_{10} = -120, \quad g_{11} = -20,
\]
\[
g_{12} = 480, \quad g_{23} = 2880, \quad g_{26} = -240, \quad g_{30} = -1440,
\]
\[
g_{32} = 60, \quad g_{34} = 360, \quad g_{38} = 1440, \quad g_{39} = 240
\]
\[
g_{40} = (105^-, 195^+) \quad g_{41} = (-150^-, 30^+), \quad g_{42} = (105/6^-, 195/6^+)
\]
\[
g_{43} = (-25^-, 5^+), \quad g_{44} = (450^-, -90^+), \quad g_{45} = (75^-, -15^+).
\]

All this information puts us in a very good position to use the relations between the heat kernel coefficients that result from conformal rescalings. The relevant relation reads

\[
\frac{d}{d\epsilon} |_{\epsilon=0} A_{5/2} (1, e^{-2\epsilon F} P) - (D - 5) A_{5/2}(F, P) = 0 \quad (48)
\]

Setting to zero the coefficients of all terms in (48) we obtain the equations given in (50). (They are ordered in such a way that nearly every equation immediately yields a universal constant. This was the main motivation for the given ordering.) Using the relation (50) we find

\[
g_1 = 360, \quad g_2 = -1440, \quad g_4 = 240,
\]
\[
g_{13} = 12, \quad g_{14} = 24, \quad g_{15} = 15,
\]
\[
g_{16} = -270, \quad g_{17} = 120, \quad g_{18} = 960,
\]
\[
g_{19} = 600, \quad g_{20} = -16, \quad g_{21} = -17,
\]
\[
g_{22} = -10, \quad g_{25} = (60^-, 195/2^+), \quad g_{27} = 90,
\]
\[
g_{28} = -270, \quad g_{29} = (450^-, 630^+), \quad g_{33} = 90,
\]
\[
d_1 = (450^-, -90^+), \quad d_2 = (42^-, -111/2^+), \quad d_3 = (0^-, 30^+)
\]
\[
d_4 = 240, \quad d_5 = 420, \quad d_6 = 390,
\]
\[
d_7 = 480, \quad d_8 = 420, \quad d_9 = 60,
\]
\[
d_{20} = 30, \quad d_{21} = -60, \quad d_{22} = -180
\]
Thus the equations given up to this point allow for the determination of the universal constants apart from two groups. The first group is \(d_{23}, \ldots, d_{29}, d_{38}, d_{39}, d_{41}, d_{45}, d_{46}\) and the second one, \(d_{10}, \ldots, d_{19}, d_{43}, d_{44}\). The first group is completely determined using the relations given in (51). One finds
\[
\begin{align*}
d_{23} &= (-215/16^-, -275/16^+), & d_{24} &= (-215/8^-, -275/8^+), \\
d_{25} &= (14^-, -1^+), & d_{26} &= (-49/4^-, -109/4^+), \\
d_{27} &= 16, & d_{28} &= (47/2^-, -133/2^+), \\
d_{29} &= 32, & d_{38} &= (777/32^-, 375/32^+), \\
d_{39} &= (17/2^-, 25^+), & d_{41} &= (-255/8^-, 165/8^+), \\
d_{45} &= (-30^-, -15^+), & d_{46} &= (-465/4^-, -165/4^+)
\end{align*}
\]
Finally we consider the second group mentioned above. As we will see, one needs just one more relation in addition to those obtained from equation (48), which are presented in (52). They yield
\[
\begin{align*}
d_{11} &= (58^-, 238^+), & d_{15} &= (6^-, 111^+), \\
d_{16} &= (-30^-, -15^+), & d_{19} &= (54^-, 114^+)
\end{align*}
\]

together with the relations
\[
\begin{align*}
2d_{10} + d_{43} &= -91, & 2d_{10} - d_{18} &= (-983/8^-, -1403/8^+), \\
2d_{14} - 3d_{18} &= (-913/4^-, -2533/4^+), & d_{13} + d_{14} &= (297/8^-, 837/8^+), \\
d_{18} - d_{44} &= (60^-, 225^+), & 2d_{12} - 2d_{17} - d_{18} &= (-7/4^-, -787/4^+), \\
2d_{12} - d_{17} &= 32.
\end{align*}
\]
This is all we can get with the relation (48). It is seen that, given \(d_{43}\) or \(d_{44}\), for example, the remaining constants can be determined. This is achieved with the equation [8]
\[
\frac{d}{de} \epsilon=0 A_{5/2} \left(e^{-2\epsilon f} F, e^{-2\epsilon f} P\right) = 0 \quad \text{for } D = 7. \tag{49}
\]
Thus, finally, one gets
\[
\begin{align*}
d_{10} &= (-413/16^-, 487/16^+), & d_{12} &= (-11/4^-, 49/4^+), \\
d_{13} &= (355/8^-, 535/8^+), & d_{14} &= (-29/4^-, 151/4^+), \\
d_{17} &= (-75/2^-, -15/2^+), & d_{18} &= (285/4^-, 945/4^+), \\
d_{43} &= (-315/8^-, -1215/8^+), & d_{44} &= 45/4,
\end{align*}
\]
which concludes the calculation of the complete \(A_{5/2}\) coefficient on a smooth manifold with boundary. All terms not displayed in the above lists have been used as checks on the computed universal constants.
4. Conclusions

In this article we have developed a technique for the calculation of smeared heat-kernel coefficients on the generalized cone. This is a generalization of our previous work [13] where we treated the $F = 1$ case only. All technical and aesthetic advantages emphasized previously are still present for arbitrary $F$. Namely, by restricting attention to the ball, and using a function $F$ as general as needed, the coefficients can be found as polynomials in the dimension of the ball. This has the advantage that the special case evaluation can easily be used to put restrictions on the general form of the heat-kernel coefficients. This idea was applied to the coefficients $A_0, ..., A_{5/2}$ for Dirichlet and Robin boundary conditions. Supplemented by a lemma on product manifolds and relations from conformal rescalings, we have shown that starting with the results of the special case, treated here for the first time, the complete coefficients are obtained very effectively.

The method is clearly capable of being applied to other situations. An example of even more complexity is the generalized boundary condition involving tangential derivatives of the field [24–27]. Up to now, for this case, we have applied the technique of special case evaluation only to the 4-dimensional ball [30]. The treatment of the $D$-dimensional ball is under consideration with the aim of finding the general form of the coefficients using the ideas presented here. Another situation of interest is the spectral boundary condition applied to spinor fields. These conditions are nonlocal and it is known that the relations obtained with conformal techniques are not sufficient for the determination of the entire coefficient [31]. However, supplemented by the ball calculation, it is possible to find at least the lower coefficients in this case too. We reserve exposition of these extensions for later.

A general word of caution, however. The evaluation of higher and higher coefficients quickly becomes prodigiously complicated, even for just the volume terms, and there is the danger of it becoming an end in itself. The question is whether there is any value in displaying an impenetrable profusion of terms, without some strong motivation. So far as the boundary terms go, we feel that with $A_{5/2}$ we probably have reached the limit of what can sensibly be calculated and displayed. Already the expressions are becoming unwieldy. Further progress in this area should, we think, be limited to extending the class of manifolds, say to those with non-smooth boundaries, and to the consideration of other fields and boundary conditions as indicated in the preceding paragraph.
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Appendix

In this appendix we list the relations resulting from the conformal property (48).

The first group of relations is

| Term | Coefficient |
|------|-------------|
| $EF_{;mm}$ | $0 = -2g_1 + (D - 2)g_3 - 2(D - 1)g_5 - (D - 1)g_{10} - (D - 5)g_{34}$ |
| $ESF_{;m}$ | $0 = -2g_2 - (D - 2)g_{23} + (D - 1)g_{38} - (D - 5)g_{30}$ |
| $SF_{;mmm}$ | $0 = \frac{1}{2}(D - 2)g_2 - 2(D - 1)g_{16} - (D - 5)g_{36}$ |
| $KSF_{;mm}$ | $0 = \frac{1}{2}(D - 2)g_2 - 2(D - 1)g_{16} + \frac{1}{2}(D - 2)g_{38} - (D - 1)d_{20}$ |
| $F \Delta R$ | $0 = \frac{1}{2}(D - 2)g_5 - (D - 4)g_6 - 4(D - 1)g_7 - Dg_8 - 4g_9 - g_{11} - g_{13} + \frac{1}{2}g_{20}$ |
| $FR_{;mm}$ | $0 = -\frac{1}{2}(D - 2)g_5 + (D - 4)g_6 + 4(D - 1)g_7 + 2(D - 1)g_8 + 8g_9 + g_{11}$ |
| $FR_{mm:a}$ | $0 = \frac{1}{2}(D - 2)g_1 - 2(D - 1)g_6 + \frac{1}{2}(D - 2)g_{10} - 2(D - 1)g_{11} - 2(D - 1)g_{13}$ |
| $F_{;mm}S^2$ | $0 = -2(D - 1)g_{12} - (D - 1)g_{17} + \frac{1}{2}(D - 2)g_{23} - (D - 5)g_{35}$ |
| $FS_{;a}S^a$ | $0 = -4(D - 1)g_{12} - 2g_{17} - (D - 3)g_{18} + 2g_{19} + (D - 2)g_{23}$ |
| $F_{;m}E_{;m}$ | $0 = -5g_1 - \frac{1}{2}(D - 2)g_2 + (D - 1)d_1 - (D - 5)g_{29}$ |
For the second one we have

\[
\begin{align*}
F_{;mm}K &= 0 = \frac{1}{2} (D-2)g_1 - 4(D-1)g_6 - 2(D-1)g_{13} - g_{15} + \frac{1}{2} (D-2)d_1 \\
&\quad - 2(D-1)d_2 + d_3 - (D-5)d_{55} \\
F_{;m}R_{;m} &= 0 = -\frac{1}{4} (D-2)g_1 + (2D-7)g_6 + (D-6)g_{13} - 2g_{15} \\
&\quad - \frac{1}{2} (D-2)g_{16} + (D-1)d_2 - \frac{1}{2} d_3 - (D-5)g_{25} \\
F_{;mm}R_{;mm} &= 0 = -(D-2)g_1 + 4(D-1)g_6 - 2(D-2)g_8 - 8g_9 + \frac{1}{2} (D-2)g_{10} \\
&\quad - 2(D-1)g_{11} + 4(D-1)g_{13} - 2(D-1)g_{21} - 2g_{22} - (D-5)g_{33} \\
F_{;m}R_{;mm}S &= 0 = -\frac{1}{2} (D-2)g_2 + 2(D-1)g_{16} - (D-2)g_{17} + (D-1)d_{20} \\
&\quad - d_{21} - d_{22} - (D-5)g_{27} \\
FK_{;a}S_{;a} &= 0 = -(D-4)d_4 - d_5 + d_6 + d_7 - d_8 - d_9 + \frac{1}{2} (D-2)g_{38} \\
&\quad - d_{20} - 2(D-1)g_{39} - d_{21} \\
FK_{;a}S &= 0 = -\frac{1}{2} (D-2)g_2 + 2(D-1)g_{16} - d_4 + d_6 - (D-4)d_8 \\
&\quad + \frac{1}{2} (D-2)g_{38} - d_{20} - 2(D-1)g_{39} - d_{21} \\
FK_{ab}S_{;a} &= 0 = -(D-2)g_2 + 4(D-1)g_{16} + 3d_5 - (D-2)d_7 + (D-2)d_9 \\
&\quad - (D-2)d_{21} + d_{22} \\
FK_{ab}S &= 0 = -d_5 + d_7 - (D-4)d_9 - (D-2)d_{21} + d_{22} \\
F_{;m}S_{;a} &= 0 = \frac{1}{2} (D-2)g_2 - 2(D-1)g_{16} - (D-2)g_{18} + (D-2)g_{19} + (D-1)d_4 \\
&\quad + d_5 - (D-1)d_6 - d_7 + (D-1)d_8 + d_9 - (D-5)g_{28}
\end{align*}
\]

(50)

For the second one we have

\[
\begin{align*}
\text{Term} & & \text{Coefficient} \\
F_{;mm}K_{ab}K^{ab} &= 0 = -(D-2)g_1 + 4(D-1)g_6 + 4(D-1)g_{13} + 2g_{15} + d_3 + \frac{1}{2} (D-2)g_{41} \\
&\quad - 2(D-1)g_{43} - (D-1)d_{24} - d_{27} + d_{28} - (D-5)d_{54} \\
F_{;mm}K^2 &= 0 = -2(D-1)g_6 + \frac{1}{2} (D-2)d_1 - 2(D-1)d_2 + (D-2)g_{40} \\
&\quad - 2(D-1)g_{42} - (D-1)d_{23} - d_{25} + d_{26} - (D-5)d_{53} \\
F_{;m}KR &= 0 = \frac{1}{2} (D-2)g_5 - 2g_6 - 4(D-1)g_7 - 2g_8 - g_{11} - 2d_2 - \frac{1}{2} (m-2)g_{39} \\
&\quad + 2(D-1)g_{42} + 2g_{43} + d_{25} - (D-5)g_{45}
\end{align*}
\]
\[ F_{,m}K R_{mm} = 0 = \frac{1}{2} (D - 2) g_1 + \frac{1}{2} (D - 2) g_{10} - 2 (D - 1) g_11 - 2 (D - 1) g_{13} + 4 g_{15} + g_{20} - 2 g_{21} - \frac{1}{2} (D - 2) d_1 + 2 (m - 1) d_2 + d_3 - \frac{1}{2} (D - 2) d_{20} + 2 (D - 1) d_{23} + 2 d_{24} - d_{25} - d_{26} - (D - 5) d_{41} \]

\[ F_{,m}K_{ab} R_{ab}^b = 0 = -\frac{1}{2} (D - 2) g_1 + 2 (D - 1) g_6 - 2 (D - 2) g_8 - 8 g_9 + 2 (D - 1) g_{13} - 4 g_{15} + g_{20} - d_3 - \frac{1}{2} (D - 2) d_{21} + (D - 1) d_{25} + 2 d_{27} + 2 d_{29} - (D - 5) d_{45} \]

\[ F_{,m}K_{ab} K_{c}^{bc} K_{c}^a = 0 = (D - 2) g_1 - 4 (D - 1) g_6 - 4 (D - 1) g_{13} - 2 g_{15} - d_3 - (D - 2) d_{27} + d_{28} + 2 d_{29} - \frac{1}{2} (D - 2) d_{35} + (D - 1) d_{39} + 4 d_{40} - (D - 5) d_{51} \]

\[ FR_{ac} K_{b}^c K_{b}^a = 0 = -2 (D - 2) g_8 - 8 g_9 + 4 g_{15} + g_{20} + 2 d_3 + 4 d_{13} + 4 d_{14} - 4 d_{19} - (D - 2) d_{27} + d_{28} + 2 d_{29} \]  

Finally, the third group:

| Term | Coefficient |
|------|-------------|
| \(FK_{,b}K_{x}^{b}\) | 0 = 2 (D - 1) g_6 - 4 g_{15} - (D - 2) g_{20} + 2 g_{22} - \frac{1}{2} (D - 2) d_1 + 2 (D - 1) d_2 + 2 d_{10} + d_{11} - (D - 3) d_{15} - d_{16} - d_{18} + (D - 2) g_{40} - 4 (D - 1) g_{42} - 2 d_{23} - 2 d_{25} |
| \(FK_{,ab} K_{x}^{b}\) | 0 = 2 (D - 2) g_1 - 4 (D - 1) g_6 - 8 (D - 1) g_{13} + (4 D - 6) g_{20} - 8 g_{22} - (D - 2) d_1 + 4 (m - 1) d_2 - (D - 3) d_{11} + 2 d_{12} - 2 d_{14} + 2 d_{16} - 2 d_{17} + 2 d_{18} - 2 (D - 2) d_{25} + 2 d_{26} - 4 d_{29} |
| \(FK_{,ab} K_{c}^{ab}\) | 0 = (D - 2) g_1 - 4 (D - 1) g_6 - 4 (D - 1) g_{13} - 2 g_{15} + (D - 2) g_{20} - 2 g_{22} - 3 d_3 + 2 d_{13} + 2 d_{14} - (D - 3) d_{19} + (D - 2) g_{41} - 4 (D - 1) g_{43} - 2 d_{24} - 2 d_{27} |
| \(FKK_{,ab}^{ab}\) | 0 = 4 (D - 2) g_8 + 16 g_9 - 4 g_{15} - D g_{20} + 2 g_{22} + d_{11} + 2 d_{12} - (D - 4) d_{16} - 2 d_{17} - d_{18} - (D - 2) d_{25} + d_{26} - 2 d_{29} |
\[ F_{m} K_{a}^{a} = 0 = \frac{3}{2} (D - 2) g_{1} + 4 (D - 1) g_{6} - 4 (D - 2) g_{8} - 16 g_{9} + 6 (D - 1) g_{13} \]
\[ \quad + \frac{1}{2} (D - 2) d_{1} - 2 (D - 1) d_{2} - d_{3} - \frac{1}{2} (D - 2) d_{4} + \frac{1}{2} (D - 2) d_{6} \]
\[ \quad - \frac{1}{2} (D - 2) d_{8} - 2 (D - 1) d_{10} \]
\[ \quad - d_{11} - 2 d_{13} + 2 (D - 1) d_{15} + d_{16} + d_{18} + 2 d_{19} - (D - 5) d_{43} \]
\[ F_{m} K_{a}^{b} K_{b}^{c} = 0 = \frac{1}{2} (D - 2) g_{1} - 2 (D - 1) g_{6} + 4 (D - 2) g_{8} + 16 g_{9} - 2 (D - 1) g_{13} \]
\[ \quad + 2 g_{15} + 2 d_{3} - \frac{1}{2} (D - 2) d_{5} + \frac{1}{2} (D - 2) d_{7} - \frac{1}{2} (D - 2) d_{9} \]
\[ \quad - (D - 1) d_{11} - 2 d_{12} - 2 d_{14} + (D - 1) d_{16} + 2 d_{17} + (D - 1) d_{18} \]
\[ \quad - (D - 5) d_{44} \]
\[ F K_{a}^{b} K_{b}^{c} = 0 = (4 - 3 D) g_{20} + 6 g_{22} - 2 d_{3} - 2 (D - 2) d_{12} - 4 d_{13} - 2 d_{14} \]
\[ \quad + (D + 1) d_{17} + 4 d_{19} - (D - 2) d_{27} + d_{28} + 2 d_{29} \]
\[ F K_{a}^{b} K_{b}^{b} = 0 = 2 (D - 2) g_{1} - 4 (D - 1) g_{6} - 2 (D - 2) g_{8} - 8 g_{9} - 8 (D - 1) g_{13} \]
\[ \quad + (4 D - 5) g_{20} - 8 g_{22} - (D - 2) d_{1} + 4 (D - 1) d_{2} - (D - 2) d_{11} \]
\[ \quad - 2 d_{14} + (D - 2) d_{16} + 3 d_{18} - (D - 2) d_{25} + d_{26} - 2 d_{29} \]

(52)

This completes the list of relations used for the calculation of the \( A_{5/2} \) coefficient.

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