Crossover from Fermi Liquid to Non-Fermi Liquid Behavior in a Solvable One-Dimensional Model

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Abstract

We consider a quantum many-body problem in one-dimension described by a Jastrow type wavefunction, characterized by an exponent $\lambda$ and a parameter $\gamma$. In the limit $\gamma = 0$ the model becomes identical to the well known $1/r^2$ pair-potential model; $\gamma$ is shown to be related to the strength of a many body correction to the $1/r^2$ interaction. Exact results for the one-particle density matrix are obtained for all $\gamma$ when $\lambda = 1$, for which the $1/r^2$ part of the interaction vanishes. We show that with increasing $\gamma$, the Fermi liquid state (at $\gamma = 0$) crosses over to distinct $\gamma$-dependent non-Fermi liquid states, characterized by effective “temperatures”.

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A special class of Jastrow type wave functions [1]

\[ \Psi(u_1, \ldots, u_N) = C \prod_{1 \leq a < b \leq N} |u_a - u_b|^{\lambda} \prod_{c=1}^{N} e^{-V(u_c)}, \]  

appear frequently in quantum many-body problems. In one dimension, many-body Hamiltonians with pair-potentials of the form \(1/r^2\) (or its periodic equivalent \(1/\sin^2 r\)), where \(r = (u_a - u_b)\), have exact ground state wave functions of the Jastrow form [2-5]. It has been proposed that variational wave functions of this type give reasonably good descriptions of models for strongly interacting fermions [6-11]. It is therefore of interest to investigate the properties of such wave functions as exactly as possible. Of particular interest is the question whether such a wave function describes how interaction can change a Fermi liquid into a non Fermi liquid state.

In the solution of the \(1/r^2\) model, the parameter \(\lambda\) in Eq. (1) is related to the strength of the \(1/r^2\) pair potential; in particular, \(\lambda = 1\) corresponds to the free fermion case. Unfortunately, the exact density matrix for this model can be obtained only for a few special values of \(\lambda\) [2,12], so the question of the nature of crossover from the free fermion to the interacting non-Fermi liquid state can not be addressed exactly in this model. In the present work, we consider a one-dimensional wave function of the above form which can be considered as a generalization of the wavefunction corresponding to the \(1/r^2\) model.

In addition to the parameter \(\lambda\), our model contains an additional parameter \(\gamma\) which we show to be related to the strength of a many body correction to the \(1/r^2\) interaction. We obtain the one-particle density matrix for this model exactly for all \(\gamma\), for the particular case of \(\lambda = 1\). Thus \(\gamma = 0\) represents the free fermion case. We show that with increasing \(\gamma\), the Fermi liquid state is destroyed by an effective non-zero “temperature” induced by interaction.
The exact density matrix of the $1/r^2$ model was obtained [2] by exploiting the analogy of the wave function with the eigenvalue distribution of random matrices [13]. The wavefunction we consider is motivated by our recent generalization of the conventional Wigner-Dyson-Mehta random matrices, for describing transport in disordered systems [14]. We consider the wavefunction given by Eq.(1) with

$$V(u) = \frac{1}{2\gamma} \left[ \sinh^{-1}[(\gamma\omega)^{1/2}u] \right]^2 + \frac{1}{2} \ln \left[ \vartheta_4 \left( \frac{\pi}{\gamma} \sinh^{-1}[(\gamma\omega)^{1/2}u]; p \right) \right],$$

characterized by a single parameter $\gamma$. Here $\vartheta_4(x;p)$ is the Jacobi Theta function [15], $p = e^{-\pi^2/\gamma}$, and $\omega$ has the dimension of $1/[\text{length}]^2$. For $\gamma = 0$, $V(u) = \frac{1}{2}\omega u^2$, and the wavefunction reduces to the well known solution of the $1/r^2$ pair potential. The case $\lambda = 1$ then represents a free Fermion problem (with the choice of Fermi statistics), with the one-particle density matrix given by $\sin[\pi Dr]/(\pi r)$, $D$ being the density of particles. The parameter $\gamma$ “deforms” the harmonic well into a weakly confining $|\ln u|^2$ term for large enough $u$, as shown in figure 1. We will show that this deformation leads to a qualitative change in the density matrix; for $\lambda = 1$ this corresponds to a change from a Fermi liquid to a non-Fermi liquid state.

In order to understand the role of the parameter $\gamma$, let us concentrate for the moment on the small $\gamma(\ll \pi^2)$ limit where the second term in Eq. (2) can be neglected, and the Hamiltonian corresponding to the above wave function has a simple form. In this limit, to leading order in $\gamma$, the Schrödinger equation (in units where $\hbar^2/2m = 1$) can be written as,

$$\frac{1}{\Psi} \sum_k \frac{\partial^2}{\partial u_k^2} \Psi = \lambda(\lambda - 1) \sum_{j \neq k} \frac{1}{(u_k - u_j)^2}$$

$$+ \omega^2 [1 + 2\gamma] \sum_k (u_k)^2 - \omega [N + \lambda N(N - 1)]$$

$$+ \frac{2}{3} \lambda \gamma \omega^2 \sum_{j \neq k} \frac{1}{u_k - u_j} [(u_k)^3 - (u_j)^3] - \frac{4}{3} \gamma \omega^3 \sum_k (u_k)^4$$

(3)
Note that for $\gamma = 0$, the Hamiltonian reduces to the $1/(u_k - u_j)^2$ pair potential, with energy $E = \omega[N + \lambda N(N - 1)]$ [2,3]. For $\lambda = 1$, with the choice of Fermi statistics, this becomes a free Fermion problem. However, for $\gamma \neq 0$, a many-body correction term survives even for $\lambda = 1$. Nevertheless, the density matrix for $\lambda = 1$ can still be found exactly for all $\gamma$. We will show that it corresponds to a non-Fermi liquid state, $\gamma$ playing the role of an effective temperature.

The exact density matrix corresponding to the wave function defined by eqns. (1) and (2) for $N \to \infty$ and for all $\gamma$, as obtained by exploiting the analogy of the present problem with the random matrix model recently constructed for describing transport in disordered systems [14], is given by

$$\rho(u, v) = f(\gamma) Q(\mu, \nu) \frac{\vartheta_1 \left( \frac{\pi \mu}{2 \gamma}; p \right)}{u - v},$$

where

$$Q(\mu, \nu) = \frac{\vartheta_1 \left( \frac{\pi \mu + \nu}{2 \gamma}; p \right)}{\vartheta_1 \left( \frac{\pi \mu}{\gamma}; p \right) \vartheta_1 \left( \frac{\pi \nu}{\gamma}; p \right)}; \mu = \sinh^{-1}(\sqrt{\gamma} \omega u), \quad \nu = \sinh^{-1}(\sqrt{\gamma} \omega v),$$

$\vartheta_1(x; p)$ is a Jacobi Theta function [15] and $f(\gamma)$ is a known function of $\gamma$. For $\gamma \ll \pi^2$, a simpler form for the density matrix is obtained

$$\rho(u, v) \approx \frac{1}{\pi} \sin \left[ \frac{\pi}{2 \gamma} \left( \sinh^{-1} \left[ (\omega \gamma)^{1/2} u \right] - \sinh^{-1} \left[ (\omega \gamma)^{1/2} v \right] \right) \right].$$

In the limit $\gamma \to 0$, in terms of the density at the origin $D_0 = \frac{1}{2}(\pi \gamma)^{1/2}$, this reduces to the free Fermion density matrix $\rho(u - v) = \sin[D_0 \pi(u - v)]/\pi(u - v)$. These oscillations are the characteristic signature of a normal Fermi system; its Fourier transform—the momentum distribution—is the familiar step function. On the other hand for increasing $\gamma$, these oscillations begin to die out, destroying the Fermi liquid behavior. The Fourier transform of $\rho(u, v)$ would involve two external momenta and this makes comparison
with the Fermi liquid momentum distribution difficult. We shall instead consider the Fourier transform of \( \rho(u,0) \), \( n_k = \int_{-\infty}^{+\infty} du \, e^{iku} \rho(u,0) \). With the change of variable \( y = \sinh^{-1}(2\gamma D_0 u) \), we have

\[
 n_k = \frac{1}{\pi} \int_0^\infty dy \coth y \left[ \sin \frac{\pi}{2\gamma} \left( \frac{|k|}{\pi D_0} \sinh y + y \right) - \sin \frac{\pi}{2\gamma} \left( \frac{|k|}{\pi D_0} \sinh y - y \right) \right].
\]

(7)

For small enough \( \gamma \) we can replace \( \sinh y \) by \( y \) within the \( \sin \) function, and \( \coth y \) by \( 1/\sinh y \). The resulting distribution is given by

\[
 n_k = n \left[ \frac{\pi}{\gamma} \left( \frac{|k|}{D_0} - \pi \right) \right] - n \left[ \frac{\pi}{\gamma} \left( \frac{|k|}{D_0} + \pi \right) \right],
\]

(8)

where \( n[x] = 1/e^x + 1 \) is the Fermi function. As expected, this reduces to the step of the free fermions when \( \gamma = 0 \). Moreover, we find from the explicit expression (8) that \( \gamma \) plays the role of an effective “temperature”.

In summary, we have considered a one-dimensional quantum many-body problem described by a Jastrow type wave function characterized by an exponent \( \lambda \) and a parameter \( \gamma \). We show that our model is a generalization of the \( 1/r^2 \) pair-potential model considered by Calogero and Sutherland, which is obtained in the limit \( \gamma = 0 \). We obtain the exact one-particle density matrix for all \( \gamma \) for the case \( \lambda = 1 \) where the \( 1/r^2 \) interaction vanishes. For \( \gamma = 0 \), and the choice of Fermi statistics this becomes a free fermion problem, and we recover the step function for the momentum distribution. For \( \gamma \neq 0 \), an interaction term survives for the case \( \lambda = 1 \) and the resulting momentum distribution is smeared out, destroying the Fermi Liquid. The explicit expression for the momentum distribution as a function of \( \gamma \) for small \( \gamma \) shows that the destruction of the Fermi liquid state occurs as increasing interaction induces an increase in the effective “temperature” in this regime.

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Figure Caption

Fig.1: $V(u)$ characterizing the model wave function considered, as given by Eq.(2), for
various values of $\gamma$. The parameter $\omega$ has been set equal to unity.

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