Analysis of cerebral electrical propagation generated by deep brain stimulation using finite elements

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Abstract. The surgical intervention called deep brain stimulation, is a stereotactic surgery that allows the control of some motor functions in the human body. For its execution it is necessary to make use of software engineering and an understanding of the operation and electrical activation in the brain. Based on the finite element method, this document will provide a solution to the Laplace differential equation that models the electrical potential generated during deep brain stimulation. To achieve this objective, we will initially present the theoretical foundations of the finite element method that include the approximation of Galerkin and the Max Milgram and Poincaré theorems; then assuming that the model of the human head is isotropic and uniform, the brain regions will be modeled with the characteristics of the environment (external skin of the skull, skull and brain) in such a way that the attenuation, direction and value of the potential can be visualized as it changes material and moves along the surfaces that make up the human head model. The simulations obtained using the finite element method allowed determining which areas of the brain are involved by the interaction of potential sources, observing that bioelectric sources originate electrochemical activity in brain cells that are inactive in such a way that they generate controlled movements in the human body.

1. Introduction

The deep brain stimulation, or simply DBS for its initials, works by targeting one of several areas of the brain that are part of the movement disorders in the human body. By introducing a filament (electrode) very precisely within the desired area (where the pathology is related to the area of the brain), it is possible to inject a small amount of current, which allows nerve cells to be deactivated. This electrode is powered by a battery that is installed under the skin, which allows the supply of small pulses of electricity during the day. The regions that are commonly of medical interest are: subthalamic nucleus (STN), the ventralis intermedius (VIM) of the thalamus and pedunculopontine nucleus (PPN) for Parkinson's disease, on the other hand, for pathology of essential tremor and tremor of multiple sclerosis is studied in the thalamus region.

Precision neurosurgery is a super specialty neurosurgical practice that provides expertise in the treatment of a wide range of disorders of the brain and spine [1]. Once the region of the brain to be stimulated has been detected, a pair of electrodes are placed that generate a current intensity that enables these inactive neurons to be turned on. To determine the location of the electrodes and the intensity with which it is desired to attack the brain regions, it is necessary to solve a mathematical model obtained by means of partial differential equations. This differential equation is difficult to solve analytically, which is why numerical analysis elements are used, such as the finite element method, which allows finding...
the value and distribution of the potential variable in different geometries and properties of the medium with a high degree of accuracy, through the boundary conditions to be established [2].

2. **Finite element method**

An advantage of the finite element method over the finite difference method is the ease with which the boundary conditions of the problem are applied. Many physical problems have boundary conditions that involve derivatives and irregular domains, these boundary conditions are difficult to handle with techniques such as finite divided differences because each condition at the border implies a derivative that must be approximated by a quotient of differences in the points of a mesh (discretized domain) and in an irregular form of the border make the approximations indeterminate [3]; consider the following partial differential Equation (1).

\[
\frac{\partial}{\partial x} \left( \frac{\partial u(x,y)}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u(x,y)}{\partial y} \right) = 0, \tag{1}
\]

with \((x,y) \in \Omega\) and with boundary conditions \(u(x_0,y_0) = C_1\) and \(u(x,y) = g(x,y)\). It has already been briefly indicated how a mesh is constructed (Figure 1(a)) by means of triangles like the one in Figure 1 for domains \(\Omega \subset \mathbb{R}^2\). The boundary conditions are evaluated on each element (Figure 1(b)), then, \(\Omega\) is fragmented into subdomains \(\Omega_1, \Omega_2, \Omega_3 \ldots, \Omega_e\) in such a way that each element is consecutive with each element (triangle after triangle) [4,5].

![Figure 1.](image)

Figure 1. (a) Triangular elements and (b) Domain divided into triangular elements.

Linear figure functions are interpolation polynomials that allow to know the value of the variable of interest and are given as Equation (2).

\[
\begin{bmatrix}
\phi_1 \\
\phi_i \\
\phi_k
\end{bmatrix} =
\begin{bmatrix}
1 & x_i & y_i \\
1 & x_j & y_j \\
1 & x_k & y_k
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix}, \tag{2}
\]

where \(\phi_1, \phi_i, \phi_k\) are the values of the variable of interest in each node \((i \leftrightarrow 1, j \leftrightarrow 2, k \leftrightarrow 3)\). From the system of equations proposed in Equation (2) we can obtain the values of \(\alpha_1, \alpha_2, \alpha_3\) function of the values of \(\phi_1, \phi_i, \phi_k\), Equation (3).

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} = \frac{1}{2A^2}
\begin{bmatrix}
x_i(y_k - x_k y_j) + x_k(y_i - x_i y_j) + y(x_i y_j - x_j y_i)
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_i \\
\phi_k
\end{bmatrix}. \tag{3}
\]
The constants $\alpha_1$, $\alpha_2$, $\alpha_3$ will depend on the values of the variable of interest in each element and the geometry (area of each element) and location of each triangular element within the domain, with Equation (4).

$$2A^e = (x_iy_j - x_jy_i) + (x_ky_1 - x_1y_k) + (x_jy_k - x_ky_j).$$ (4)

It is possible to write the numerical solution as a linear combination of polynomial functions of degree 1 of triangular shape as expressed below in the Equation (5).

$$\phi = N_i(x,y)\phi_i + N_j(x,y)\phi_j + N_k(x,y)\phi_k,$$ (5)

where triangular functions are given by the Equation (6) to Equation (8).

$$N^e_i(x,y) = \frac{1}{2A^e}[(x_jy_k - x_ky_j) + (y_j - y_k)x + (x_k - x_j)y].$$ (6)

$$N^e_j(x,y) = \frac{1}{2A^e}[(x_ky_1 - x_1y_k) + (y_k - y_1)x + (x_1 - x_k)y].$$ (7)

$$N^e_k(x,y) = \frac{1}{2A^e}[(x_iy_j - x_jy_i) + (y_i - y_j)x + (x_j - x_i)y].$$ (8)

It should be noted that, the value of $N^e_i$ in node 1 can be obtained by substituting $x = x_i$ and $y = y_i$ in Equation (6), and take values equal to zero in nodes 2 and 3 and all other nodes that do not make part of the element $e$ [6]; the gradients of the variable $\phi$ are given by the Equation (9).

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{bmatrix}.$$ (9)

The value of $\frac{\partial N_i}{\partial x} = \frac{y_j - y_k}{2A^e}$ and the value of $\frac{\partial N_i}{\partial y} = \frac{x_k - x_j}{2A^e}$, hence $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ can be written as in Equation (10).

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \frac{1}{2A^e} \begin{bmatrix} (y_j - y_k) & (y_k - y_1) & (y_k - y_j) \\ (y_k - y_j) & (y_j - y_k) & (x_j - x_i) \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{bmatrix}.$$ (10)

To solve the differential Equation (1) it is possible to apply the Galerkin method which consists in finding a functional that represents the numerical solution of the equation in each node (element) product of the discretization. According to the proposal in [7,8], it is possible to write a solution as a linear combination of the form Equation (11).

$$V(x,y) = \sum_{i=1}^{N} y_i \phi_i(x,y) \in \Omega,$$ (11)

with the approximation of the Galerkin method, a linear system of equations of the form $[\alpha] \vec{y} = \vec{b}$ is obtained and the internal product between the functions of interpolation (functional) is defined as the integral, Equation (12).

$$I(u(x,y)) = \iint_{\Omega} \left\{ \frac{1}{2} \left[ \left( \frac{\partial u(x,y)}{\partial x} \right)^2 + \left( \frac{\partial u(x,y)}{\partial y} \right)^2 \right] + f(x,y)u(x,y) \right\} dxdy.$$ (12)
In Equation (11) the following considerations must be taken: minimize the functional \( I(u(x, y)) \) over all the test functions \( \gamma \phi_i(x, y) \) twice continuously differentiable on the borders of each element; the terms \( \gamma_1, \gamma_2, ..., \gamma_N \) are used to guarantee boundary conditions and are used to minimize the solution, that is, the approximate solution is equal to the analytical solution on the boundary and satisfies the initial conditions and is obtained by optimizing \( \frac{\partial I}{\partial \gamma_i} = 0 \), for each \( i = 1, 2, ..., N \) [6, 7].

Knowing that \( u(x, y) \rightarrow V_h(x, y) \) is a numerical approximation of the functional [8] in each subdomain, and that this produces a function \( V_h(x, y) \) defined in chunks for each subdomain then you can approximate the behavior of the numerical solution by means of the mass matrix and coupling proposals in [9] you have to Equation (13) and Equation (14).

\[
I(u(x, y)) = I(\sum_{i=1}^{N} \gamma_i \phi_i(x, y) \in \Omega),
\]

\[
I(u) = \iint_{\Omega} \left\{ \frac{1}{2} \left( \sum_{i=1}^{N} \gamma_i \frac{\partial \phi_i(x,y)}{\partial x} \right)^2 + \left( \sum_{i=1}^{N} \gamma_i \frac{\partial \phi_i(x,y)}{\partial y} \right)^2 \right\} + f(x,y) \sum_{i=1}^{N} \gamma_i \phi_i(x,y) \right\} \, dx \, dy. \tag{14}
\]

As proposed in [6, 8] and considering \( I(\sum_{i=1}^{N} \gamma_i \phi_i(x, y) \in \Omega) \) as a function of \( \gamma_1, \gamma_2, ..., \gamma_N \) to maximize or minimize it is necessary to find the partial derivatives with respect to each \( \gamma_i \), we have Equation (15).

\[
\frac{\partial I}{\partial \gamma_i} = \iint_{\Omega} \left\{ \sum_{i=1}^{N} \gamma_i \frac{\partial \phi_i(x,y)}{\partial x} \frac{\partial \phi_i(x,y)}{\partial x} + \sum_{i=1}^{N} \gamma_i \frac{\partial \phi_i(x,y)}{\partial y} \frac{\partial \phi_i(x,y)}{\partial y} \right\} + f(x,y) \sum_{i=1}^{N} \gamma_i \phi_i(x,y) \right\} \, dx \, dy. \tag{15}
\]

By rewriting the terms and equaling to zero, you get the Galerkin residual [10, 11], like the Equation (16) to Equation (18).

\[
0 = \sum_{i=1}^{N} \left[ \iint_{\Omega} \left\{ \frac{\partial \phi_i(x,y)}{\partial x} \frac{\partial \phi_i(x,y)}{\partial x} + \frac{\partial \phi_i(x,y)}{\partial y} \frac{\partial \phi_i(x,y)}{\partial y} \right\} + f(x,y) \phi_i(x,y) \right\} \, dx \, dy \right], \tag{16}
\]

where,

\[
\begin{bmatrix} \gamma_1 & \gamma_2 & ... & \gamma_N \end{bmatrix}^T,
\]

\[
\alpha_{ij} = \iint_{\Omega} \left\{ \frac{\partial \phi_i(x,y)}{\partial x} \frac{\partial \phi_j(x,y)}{\partial x} + \frac{\partial \phi_i(x,y)}{\partial y} \frac{\partial \phi_j(x,y)}{\partial y} \right\} \, dx \, dy. \tag{18}
\]

\( \beta_i = f(x,y) \phi_i(x,y) \) they depend on the boundary conditions of the problem. for each \( i = 1, 2, ..., N \) and for each \( j = 1, 2, ..., N \)

### 3. Strong formulation

To support the Galerkin approach proposed above, the Max Milgram and Poincaré Theories are enunciated to arrive at the same residual definition of Galerkin (Equation (12) to Equation (18)). The internal product between two vectors \( \vec{u} \) and \( \vec{v} \) is defined as \( \langle u, v \rangle = \int_{\Omega} u' v' \, d\Omega \) in such a way that the Lax Milgram Theorem is stated in the Hilbert space as follow [10]: the internal product in the Hilbert space can be calculated as \( a(u, v) = \langle f, v \rangle \forall v \in H \), then a functional \( J(v) \) that depends on the vector \( v \in H \) is proposed to be minimized in the Hilbert space, like this Equation (19).

\[
J(v) = \min \left\{ \frac{1}{2} a(u, v) - \langle f, v \rangle \right\} \text{ by } v \in H. \tag{19}
\]

Now, the problem that you want to solve is to find the solution of Equation (20).
−u'' + u = f. \quad (20)

To find the solution of Equation (20), it is necessary to state the internal producer between a test function \( \varphi \) that is continuous and differentiable, then it can be multiplied and integrated Equation (20), you get Equation (21).

\[- \int u'' \varphi + \int u \varphi = \int f \varphi. \quad (21)\]

Solving the integral by parts (Equation (21)) you get Equation (22).

\[-\varphi u'|\Omega + \int_\Omega u' \varphi' d\Omega + \int_\Omega u \varphi d\Omega = \int_\Omega f \varphi d\Omega. \quad (22)\]

On the other hand, to find the functional, Equation (19), to minimize guaranteeing uniqueness, you have to Equation (23).

\[J(u) = \min \left( \frac{1}{2} \int_\Omega \nabla u \nabla \varphi d\Omega - \int_\Omega f \varphi d\Omega \right). \quad (23)\]

Equation (15) and Equation (23) can be compared term to term in such a way that they can be evaluated at the boundary (boundary conditions) to approximate the proposed solution with Equation (11) [3,4].

4. Analysis and results

By means of the finite element method enunciated in section 2 and section 3, with elements of linear approximations of triangular form as observed in Figure 2, the Laplace differential equation (Equation (1)) was solved determining the potential distribution (variable \( v \)) in the three media that were taken to represent the brain regions. The approximate solution is shown in Figure 3 to Figure 5.

The brain regions were modeled with the characteristics of the environment (external skin of the skull, skull and brain) in such a way that the attenuation, direction and value of the potential could be visualized as the material changes and moves along the lines. surfaces that make up the human head model. It is assumed in principle that the model of the human head is isotropic and uniform with the following properties: External skin of the skull with an electrical conductivity \( \sigma_{\text{scalp}} = 4.5 \text{ ms/cm} \) and external radius \( R_{\text{scalp}} = 9.2 \text{ cm} \), the skull with electrical conductivity \( \sigma_{\text{skull}} = 0.056 \text{ ms/cm} \) and external radius \( R_{\text{skull}} = 8.5 \text{ cm} \), and the brain with an electrical conductivity \( \sigma_{\text{brain}} = 4.5 \text{ ms/cm} \) and external radius \( R_{\text{brain}} = 8 \text{ cm} \). The radius of the electrodes in this simulation was \( R = 0.5 \text{ cm} \) and they were located at the coordinates (0; 6.837) and (0; -6.837) with a constant potential difference equal to 0.24 V. Therefore, Figure 2 shows the location of the electrically charged electrodes. By means of these electrodes, a potential difference was generated along the aforementioned materials in such a way that this signal crossed all the media and with this we could verify the attenuation in the inner region of the brain [12].

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**Figure 2.** Head model with symmetrically positioned electrodes.
The distribution of the potential value was simulated in the Figure 3 and Figure 4 allowed us to specifically determine the brain regions where there is greater (areas of warm color) or less (areas of cold color) affectation due to the intensity of the current generated by the electrodes.

![Figure 3. Potential distribution in the YZ plane in the three regions (external, average, interior).](image1)

![Figure 4. Potential distribution in the XY plane in the three regions (external, average, interior).](image2)

In this case, it is observed that the skull is the most affected region and that the brain was the least affected. With the determination of these areas we can decide where to apply current with greater intensity in such a way that neurons are activated so that movement is generated in the human body. It can be seen that as the potential variable of Equation (1) crosses different media, a change in this variable occurred due to properties such as electrical conductivity, in such a way that it is visualized which areas receive the greatest potential to activate neurons and tissues, and thus produce movement in the human body with patients with pathologies that relate motor skills and poor mobility.

In Figure 5, the simulation allows us to see the propagation effect of the signal produced by the electrodes in the superficial region of the skull. This effect determines the attenuation of the potential variable in the middle region of the first surface (Figure 2) that models the human head.

![Figure 5. Distribution of potential in the external region.](image3)

From the results obtained in the previous simulations we can affirm that the finite element method is undoubtedly a powerful tool of numerical analysis that allows us to approximate the solution of the partial differential equations that model bioengineering problems with efficient computational response.
5. Conclusion

Deep brain stimulation has become very precise due to the implementation of computational techniques that allow us to approximate the numerical solution of the partial differential equations that model the effects of the potential difference in the human brain. This technique allows surgeons to place electrodes in virtually any area of the brain, and turn it on or off, such as a radio or a thermostat, to correct defects. According to the results shown in Figure 2 to Figure 5 it is possible to determine which areas of the brain are involved by the interaction of potential sources, bioelectric sources originate electrochemical activity in brain cells that are inactive from so that they generate controlled movements in the human body. Not only can the interaction of the skull model be studied with constant sources (electrodes with constant potential difference, Figure 3 but also with the interaction of varying sources in time. Furthermore, as studied in this document, how was the behavior of the potential in different media for an isotropic and uniform model of the human head, the behavior of time-varying sources can also be studied in applications such as electrocardiography to activate inactive regions of the heart, electromyography to improve the functioning of muscles in patients with conditions that relate to movement.

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