On small tension p-branes

Jonas Björnsson\textsuperscript{1} and Stephen Hwang\textsuperscript{2}
Department of Physics
Karlstad University
SE-651 88 Karlstad, Sweden

Abstract

This paper deals with p-branes with small but non-zero tension. We prove the existence of canonical transformations, within a perturbation theory, that link specific geometries of p-branes to solvable theories, namely string-like and particle-like theories. The specific shapes correspond to stretched configurations. For configurations linked to string-like theories one will upon quantization get a critical dimension of \((25+p)\).

1 Introduction and framework

The theories that describe relativistic \(p\)-branes are known to be quite complicated. For \(p = 0, 1\) one can solve the equations of motion in a flat background, but for \(p \geq 2\) one cannot do this. For \(p = 2\) i.e. membranes, one can instead make a reduction \[1, 2, 3\] which yields a maximally supersymmetric Matrix-theory \[4, 5, 6\]. This theory has many interesting features and is conjectured to give all microscopict degrees of freedom for M-theory \[7, 8, 9, 10\]. But, for \(p \geq 3\) one does not know any such reduction.

In this paper we follow another path. We are interested in \(p\)-branes with small tension and specific geometries. These geometries correspond to stretched \(p\)-branes for which \(p-1\), or \(p\), are large dimensions. For such configurations we will show that one may set up a perturbation theory around a solvable model and which makes it possible to solve the equations of motion. The stretched configurations are connected to the zero tension limit. For the string, the zero tension limit was first discussed in \[11\]. Furthermore, such limits have been discussed for D\(p\)-branes as well, \[12, 13, 14\], in which tensile non-interacting strings arise. The tensionless limit of \(p\)-branes are interesting in the same way as the tensionless limit of string theory, being relevant for a high energy description of the theory.

Here, we will generalize the results of \[15, 16, 17\] to hold also for \(p\)-branes. We will show that \(p\)-branes with small tension and stretched geometries can, in general, be described by perturbing free tensile string- or particle-like theories. They differ from regular string and particle theories since the embedding fields depend on \(p+1\) world-hypervolume parameters.

Our main result is to prove that one can, within perturbation theory, solve exactly the theory by canonically transforming to a free theory. A consequence of our result is that one can canonically

\begin{footnotesize}
\begin{itemize}
\item[\textsuperscript{1}] jonas.bjornsson@kau.se
\item[\textsuperscript{2}] stephen.hwang@kau.se
\end{itemize}
\end{footnotesize}
relate a \( p \)-brane and a \((p-b)\)-brane-like theory for arbitrary \( b \). These relations hold when the tensions are small and the branes are stretched.

Our starting point is the Dirac action \cite{13} for the bosonic \( p \)-brane in flat space-time. This theory has \( p+1 \) constraints due to the reparametrization invariance. The constraints found from this action are

\[
\phi_0 = \frac{1}{2} \left[ \mathcal{P}^2 + T_p^2 \det(h_{ab}) \right] \approx 0, \\
\phi_a = \mathcal{P}_U \partial_a X^U \approx 0, \\
\tag{1.1}
\]

where \( \mathcal{P}_U \) are the canonical momenta, \( U = 0, \ldots, d-1 \) is the spacetime index, \( a, b = 1, \ldots, p \) are the space-like directions of the \( p \)-brane and \( h_{ab} \equiv \partial_a X \partial_b X^U \). The constraints are first class and satisfy the Poisson bracket algebra

\[
\{\phi_a(\xi), \phi_b(\xi')\} = \phi_b(\xi) \partial_a(\xi - \xi') + \phi_a(\xi') \partial_b(\xi - \xi'), \\
\{\phi_a(\xi), \phi_0(\xi')\} = [\phi_0(\xi) + \phi_0(\xi')] \partial_a(\xi - \xi'), \\
\{\phi_0(\xi), \phi_0(\xi')\} = T_p^2 \epsilon^{a_1 \cdots a_p} \epsilon_{b_1 \cdots b_p} \left[ \phi_a h_{a_2}^{b_2} \cdots h_{a_p}^{b_p}(\xi) + (\xi \rightarrow \xi') \right] \partial_{a_1} \partial(\xi - \xi'). \\
\tag{1.2}
\]

One could also define a BRST-charge for the theory which yields that it is a rank \( p \) theory \cite{19}.

Let us choose a partial gauge by fixing one of the space-like reparametrization invariances. We do this in the same way as in \cite{15} by gauging one of the space-like parameters of the hypervolume to be proportional to one of the space-like directions in space-time. Let us for simplicity choose the \( p \)'th variable and the \( d-1 \) direction, \( \chi_1 \equiv X^{D-1} - k \xi^p \approx 0 \). This will yield the remaining constraints

\[
\phi_0 = \frac{1}{2} \left[ \mathcal{P}^2 + T_p^2 k^2 \det(h'_{a'b'}) + \frac{1}{k^2} (\mathcal{P} \partial_p X)^2 + T_p^2 \det(h'_{ab}) \right] \approx 0, \\
\phi_{a'} = \mathcal{P}_a \partial_{a'} X^\mu \approx 0, \\
\tag{1.3}
\]

where \( h'_{ab} \equiv \partial_{a'} X^{\mu} \partial_b X_{\mu} \), \( a', b' = 1, \ldots, p-1 \) and where \( k \) is a constant. Computing the Poisson brackets between the constraints yields that they form a closed Poisson bracket algebra. Let us study the algebra and constraints by assuming that \( k \) is large and \( T_p \) small such that \( T_p^2 k^2 = \mathcal{T}_{p-1}^2 \) is fixed and finite. We have then that the two first terms of \( \phi_0 \) and \( \phi_{a'} \) describe a regular \((p-1)\)-brane. Thus, \( \mathcal{T}_{p-1} \) can be interpreted as the tension for a \((p-1)\)-brane-like theory, which has one extra world-hypervolume dependence compared to a regular \((p-1)\)-brane. The resulting theory is, therefore, a \((p-1)\)-brane-like theory with a non-trivial perturbation.

In order to solve the \((p-1)\)-brane theory we need to make further simplifications in order to relate it to a solvable theory. We will in the next sections discuss two possibilities, one in which the \( p \)-brane has a shape such that may be related to string-like theory and one in which it may be related to a particle-like theory.

\section{A perturbed bosonic string-like theory for stretched \( p \)-branes}

Let us fix \((p-1)\)-constraints by

\[
\chi_t = X^{D-t} - k \xi^{p+1-t} \approx 0, \\
\tag{2.1}
\]

\footnote{We will henceforth skip the prime on \( h_{ab} \).}
where \( l = 1, \ldots, p - 1 \). The constraints left are
\[
\{ \phi_0(\xi), \phi_0(\xi') \} = \{ \phi_1(\xi), \phi_1(\xi') \} = 0 \]
\[
= \phi_0(\xi) + \phi_0(\xi') \}
\[
\phi_1 = \mathcal{P}_\mu \partial_\xi X^\mu \approx 0, \quad (2.2)
\]
where \( h_{a_i} = \partial_{a_i} X^\mu \partial_{b_i} X^\mu \) and \( \epsilon_{b_1, \ldots, b_p} = 1 \). The constraints satisfy a closed Poisson bracket algebra
\[
\{ \phi_0(\xi), \phi_0(\xi') \} = \{ \phi_1(\xi), \phi_1(\xi') \} = 0 \]
\[
\phi_1 = \mathcal{P}_\mu \partial_\xi X^\mu \approx 0, \quad (2.2)
\]
Eqs. (2.2) and (2.3) are exact expressions for the \( p \)-brane i.e. they hold without any assumptions being made. The two remaining constraints are first-class, corresponding to the two remaining reparametrizations, of which one is time-like. We will in the rest of the section assume \( k \) to be large and \( T_p \) small such that \( T_p^2 k^{2(p-1)} \equiv \bar{T}_1^2 \) is fixed, even in the limit \( k \to \infty \) and \( T_p \to 0 \). Then \( \bar{T}_1 \) can be interpreted as the tension of a string-like theory with trivial dependence on the other world-hypervolume parameters. The constraints are of the form \( \phi_0 = \frac{1}{2} \left[ \mathcal{P}^2 + \bar{T}_1^2 \left( \partial_\xi X^\mu \right)^2 \right] + g(\ldots) \) and \( \phi_1 = \mathcal{P}_\mu \partial_\xi X^\mu \) where \( g \ll 1 \), so that the additional dependence on the world-hypervolume can be treated perturbatively. As \( X^{D-l} = k^l \xi^{p+1-l} \), we see that \( X^{D-l} \) will be large, so that the perturbation theory assumes brane-shapes with \( p - 1 \) large dimensions. We will proceed as in [16] solving the theory by successive canonical transformations. First, we gauge fix completely by
\[
\chi_0 = \mathcal{P}^+ - 1 \approx 0
\]
\[
\chi_1 = X^+ - \xi^0 \approx 0, \quad (4.4)
\]
where we have defined lightcone coordinates by \( A^{\pm} \equiv \frac{1}{\sqrt{2}} (A^1 \pm A^0) \). Furthermore, we have set \( \bar{T}_1 = 1 \). The Hamiltonian we take as the momentum in the \( \mathcal{P}^- \) direction,
\[
\mathcal{H} = - \int d^p \xi \mathcal{P}^-.
\]
To determine \( \mathcal{P}^- \) one uses eq. (2.4) in eq. (2.2). One then follows the steps in [16], solving the unperturbed theory, i.e. the string-like theory. The unperturbed Hamiltonian is of the form
\[
H_0 = \frac{1}{2} \sum_{(a),m} \left( \alpha_m^{(-a)} \dot{\alpha}_m^{(a)} + \alpha_m^{(-a)} \dot{\alpha}_m^{(a)} \right), \quad (2.6)
\]
where \((a) = (I; n_i)\) and \((-a) = (I; -n_i)\). \(n_i\) comes from the the dependence on the additional world-hypervolume parameters. As the unperturbed Hamiltonian is of the same form as in [16], one can use the results in the paper to show that to any order in perturbation theory there exists a canonical transformation which maps the perturbed Hamiltonian to the unperturbed one. As an example, a generic term to any finite order has the form

\[
H_N = \sum_r \sum_{j=0}^M q^{(-a_1)} \cdots q^{(-a_j)} H^{(a_1), \ldots, (a_j)}_{(r)},
\]

where \(H_{(r)}\) has modenumer \(r\) in the \(\xi^1\)-direction. The part of the infinitesimal canonical transformation which solves the equation

\[
\{H_0, G_N\} = -H_N,
\]

needed to transform away \(H_N\) is

\[
G_N = \sum_r \sum_{j=0}^M \sum_{k=0}^j \left\{ k! \binom{j}{k} (2\alpha_0^{(a_1)}) \cdots (2\alpha_0^{(a_k)}) q^{(a_{k+1})} \cdots q^{(a_j)} \right. \\
\times \left. H^{(a_1), \ldots, (a_j)}_{(r)} \binom{j}{k} \right\}
\]

\[
\times \sum_{j=0}^M \sum_{k=0}^j \left\{ k! \binom{j}{k} (2\alpha_0^{(a_1)}) \cdots (2\alpha_0^{(a_k)}) q^{(a_{k+1})} \cdots q^{(a_j)} H^{(a_1), \ldots, (a_j)}_{(0)} \right\},
\]

where \(K \equiv k_I q^I / (2k_I \alpha_0^I)\) and \(k_I\) is some fixed vector such that \(k_I \alpha_0^I \neq 0\). This shows, by construction, that one can find, to all finite orders, canonical transformations that map the perturbed Hamiltonian to the unperturbed string-like one. We could here also have taken another path by defining a BRST charge for the constraints in eq. (2.2). By using results on the BRST cohomology [17], one can again prove the canonical equivalence. The end result is the same but not as explicit as above.

3 Particle-like theories from stretched \(p\)-branes

In this section we fix all but one of the constraints and show how one connect the theory to a particle-like theory. Fix the gauge as in eq. (2.11), but now with \(l = 1, \ldots, p\). The only constraint left is the Hamiltonian constraint which generates time-like residual reparametrizations

\[
\phi_0 = \frac{1}{2} \left\{ \mathcal{P}^2 + T_p^2 \sum_{i=0}^p \left[ \binom{p}{i} k^{2i} h_{a_1} \cdots h_{a_{p-i}} b_{p-i} \right] \right. \\
\times \left. \sum_{j_1, \ldots, j_i=1}^p \epsilon^{a_1, \ldots, a_{p-i}, j_1, \ldots, j_i} \right\} + \frac{1}{k^2} \sum_{i=2}^p (\mathcal{P}_\mu \partial_i X^\mu)^2 \approx 0
\]

(3.1)

and satisfies the algebra

\[
\{\phi_0(\xi), \phi_0(\xi')\} = \frac{2}{k^2} \sum_{i=1}^p \left[ \phi_0 \mathcal{P}_\mu \partial_i X^\mu(\xi) + \phi_0 \mathcal{P}_\mu \partial_i X^\mu(\xi') \right] \partial_i \delta(\xi - \xi').
\]

(3.2)
If one chooses $T_p \ll 1$ and $k \gg 1$ such that $T_p^2 k^{2p} = m^2$ is fixed and non-zero, the constraint is of the form $\phi_0 = \frac{1}{2} (\mathcal{P}^2 + m^2) + g (\ldots)$ where $g \equiv k^{-2} \ll 1$. Therefore, the unperturbed theory describes an infinite set of non-interacting particles. By eq. (2.1) this requires $p$ large dimensions of the brane.

We will now show that one can map the perturbed theory to the unperturbed one by canonical transformations in the same manner as in the previous section. We fix, therefore, the gauge completely by

$$\chi_0 = X^0 - \xi^0 \approx 0. \quad (3.3)$$

The Hamiltonian for the theory can be chosen to be proportional to $\mathcal{P}^0$

$$H = \int d^p \xi \mathcal{P}^0 = \int d^p \xi \sqrt{\mathcal{P}^2 + m^2 + gA} \quad (3.4)$$

where

$$A = m^2 \sum_{i=1}^{p} \left[ g^{i-1} \left( \frac{p}{i} \right) h_{a_1 b_1} \cdots h_{a_{p-i} b_{p-i}} \sum_{j_1, \ldots, j_i=1}^{p} \epsilon^{a_1, \ldots, a_{p-i}, j_1, \ldots, j_i} \right] + \sum_{i=1}^{p} (\mathcal{P}_\mu \partial_\mu X^\mu)^2 \quad (3.5)$$

Since $k \gg 1$ we have $g \ll 1$, one can expand eq. (3.4) to get

$$H = \int d^p \xi \sqrt{\mathcal{P}^2 + m^2} \sum_{j=0}^{\infty} \left( \frac{1}{2j} \right) g^j \frac{A}{\sqrt{\mathcal{P}^2 + m^2}} \quad (3.6)$$

where the unperturbed part of the Hamiltonian is $H_0 = \sqrt{\mathcal{P}^2 + m^2}$. $H_0$ satisfies

$$\{H_0, \mathcal{P}^I\} = 0 \quad \{H_0, X^I\} = -\mathcal{P}^I \frac{1}{\sqrt{\mathcal{P}^2 + m^2}} \quad (3.7)$$

We can define $K_2 \equiv \frac{1}{k_1 \mathcal{P}_0} \int d^p \xi \left( k_1 X^I \right) \sqrt{\mathcal{P}^2 + m^2}$. It satisfies $\{H_0, K_2\} = -1$, provided $k_1 \mathcal{P}_0^I \neq 0$, which we assume. We now have all the tools needed to determine the solution to all finite orders in perturbation theory. The perturbation will, to all orders, be polynomials of $X^I$ and derivatives of $X^I$. Furthermore, it will involve polynomials which have zero Poisson bracket with the Hamiltonian, $\mathcal{P}^I$, derivatives of $\mathcal{P}^I$, $1/\sqrt{\mathcal{P}^2 + m^2}$ and $(k_1 \mathcal{P}_0^I)^{-1}$. To simplify the problem, one can make a Fourier expansion of the variable dependence of the fields. Let us show the explicit solution to a generic term

$$H_N = \sum_{j=0}^{M} X(-a_1) \cdots X(-a_j) H_N^{(a_1), \ldots, (a_j)} \quad (3.8)$$

where the index $(a)$ is a collective index for $(I; n_i)$ and $(-a) = (I; -n_i)$. This term may be transformed away by a canonical transformation generated by

$$G_N = \sum_{j=0}^{M} \sum_{k=0}^{j} \frac{1}{k+1} \left( \frac{j}{k} \right) K_2^{k+1} \left( \int d^p \xi \frac{\mathcal{P}_I}{\sqrt{\mathcal{P}^2 + m^2}} \exp \left( in_1 \xi^I \right) \right) \cdots \times \left( \int d^p \xi \frac{\mathcal{P}_I}{\sqrt{\mathcal{P}^2 + m^2}} \exp \left( in_k \xi^I \right) \right) X(-a_{k+1}) \cdots X(-a_j) H_N^{(a_1), \ldots, (a_j)} \quad (3.9)$$
We have thus shown that the perturbed Hamiltonian can be mapped to the unperturbed one by successive canonical transformations.

4 Further results and quantization

In the previous sections we have shown that stretched \( p \)-branes are canonically equivalent, within a perturbation theory, to either a free string-like theory, or to a free particle-like theory. We can use this result to show the canonical equivalence between a stretched \( p \)-brane with small tension and a stretched \( (p - b) \)-brane-like theory with a small tension.

The stretched \( (p - b) \)-brane is canonically equivalent to an unperturbed string- or particle-like theory for small \( (p - b) \)-brane tension. This requires \( p - b - 1 \), or \( p - b \) for the particle case, large dimensions. This result holds clearly also if we add trivial dependence on additional world-hypervolume parameters. Thus, we can canonically link a \( (p - b) \)-brane-like theory to a string- or particle-like theory. As the stretched \( p \)-brane is canonically equivalent to the string- or particle-like theory for small tensions as well, one can use the inverse canonical transformation to show that the \( p \)-brane and the \( (p - b) \)-brane-like theories are canonically equivalent. This requires \( p - 1 \), or \( p \), large dimensions for the \( p \)-brane and \( p - b - 1 \), or \( p - b \), large dimensions for the \( (p - b) \)-brane-like theory.

Let us also briefly discuss the quantization of the \( p \)-branes and, furthermore, consider the case where the unperturbed theory is the string-like theory. The quantization procedure follows, straightforwardly, from [16]. One defines a vacuum and a normal ordering from the solutions of the free string-like theory. Then one makes the inverse infinitesimal canonical transformation, which classically is equal to the \( p \)-brane theory up to some order, and which defines a quantum theory for the \( p \)-brane perturbatively. This procedure yields, among other things, a non-trivial ordering of the operators. This ordering will imply the existence of a critical dimension coming from the one for the bosonic string. One finds the critical dimension for a consistent quantum theory of the \( p \)-brane to be \( d = 25 + p \). A further result, is the mass-spectrum, which as shown in [16], will get a constant shift compared to the string spectrum. This holds to all non-zero orders in perturbation theory.

For the case when the unperturbed theory is particle-like, we will not get a critical dimension. Thus, the two possibilities that we have treated are not equivalent at the quantum level. This is perhaps not surprising as we quantize around two different types of geometries.

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