FREENESS CHARACTERIZATIONS ON FREE CHAOS SPACES

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Abstract. This paper deals with characterizing the freeness and asymptotic freeness of free multiple integrals with respect to a free Brownian motion or a free Poisson process. We obtain three characterizations of freeness, in terms of contraction operators, covariance conditions, and free Malliavin gradients. We show how these characterizations can be used in order to obtain limit theorems, transfer principles, and asymptotic properties of converging sequences.

1. Introduction

A classical result in probability theory asserts that one can decompose any functional of a Brownian motion $W$ as an infinite sum of multiple integrals. That is, to any square integrable random variable $F$ measurable with respect to $W$, one can associate a unique sequence of symmetric and square integrable kernels $\{f_n : n \geq 0\}$ such that

$$ F = \sum_{n=0}^{\infty} I_n^W(f_n). $$

The set of all multiple Wiener-Itô integrals of the form $I_n^W(f_n)$, the so-called $n$-th Wiener chaos of $W$, thus plays a fundamental role in modern stochastic analysis. Analysing its many rigid properties (notably those related to independence and normal approximation) has become a subject in its own right, and has grown into a mature and widely applicable mathematical theory.

Among the most striking results about Wiener chaos are the following two theorems, which will play a central role in the present paper. The first one characterizes independence of multiple Wiener-Itô integrals.

**Theorem 1.1** (Üstünel and Zakai [21], 1989). Let $n, m$ be natural numbers and let $f \in L^2(\mathbb{R}^n_+)$ and $g \in L^2(\mathbb{R}^m_+)$ be symmetric functions. Then $I_n^W(f)$ and $I_m^W(g)$ are independent if and only if, for almost all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1} \in \mathbb{R}_+$,

$$ \int_0^{\infty} f(x_1, \ldots, x_{n-1}, u)g(y_1, \ldots, y_{m-1}, u)du = 0. $$

The second result is nowadays one of the most central tools of analysis on Wiener chaos, as it represents a drastic simplification with respect to the method of moments for the normal approximation of sequences of multiple Wiener-Itô integrals.

**Theorem 1.2** (Nualart and Peccati [17], 2005). A unit-variance sequence in a Wiener chaos of fixed order converges in law to the standard Gaussian distribution if and only if the corresponding sequence of fourth moments converges to three.

2010 Mathematics Subject Classification. 46L54, 68H07, 60H30.

Key words and phrases. Free probability, Wigner integrals, free Poisson integrals, free Malliavin calculus, characterization of freeness, free Fourth Moment Theorems.
Since its introduction by Voiculescu in the eighties in order to solve some long-standing conjectures about von Neumann algebras of free groups, free probability theory has become a vivid and powerful branch of mathematics, with many applications (including signal processing, channel capacity estimation and nuclear physics) and deep connections with other mathematical fields (like operator algebra, theory of random matrices or combinatorics). Free probability has many parallels with the usual probability theory (hence its name), and the study of these links often brings a new point of view which may then enrich the theory of both worlds (classical and free).

Starting from the free independence property, a genuine stochastic calculus with respect to the free Brownian motion (the free analogue of the classical Brownian motion) has emerged within the last twenty years, following the route paved by the seminal paper of Biane and Speicher \cite{2}. In particular, a common property of the classical and free settings is the possibility of expanding the space as a sum of free chaos, giving rise to the so-called Wigner chaos. By their very construction, these free chaos play in the free world a similar role as Wiener chaos in the classical setting. It is thus natural to investigate the similarities and differences between these two mathematical objects. For instance, do we have an analogue of Theorem 1.2 in the free world? The answer is yes, and is given by the following theorem taken from \cite{9}.

**Theorem 1.3** (Kemp et. al \cite{9}, 2012). *A unit-variance sequence in a Wigner chaos of fixed order converges in law to the semicircular distribution if and only if the corresponding sequence of fourth moments converges to 2.*

Shortly after the publication of \cite{9}, many other results in the spirit of Theorem 1.3 have been added to the literature, including the following ones (the list is not exhaustive).

In \cite{14}, it is shown that component-wise convergence to the semicircular distribution is equivalent to joint convergence, thus extending to the free probability setting a seminal result by Peccati and Tudor (see also \cite{18}).

In \cite{13}, a non-central counterpart of Theorem 1.3 is provided. More precisely, it is shown that any adequately rescaled sequence \( \{ F_n : n \geq 0 \} \) of self-adjoint operators living inside a fixed Wigner chaos of even order converges in distribution to a centered free Poisson random variable with rate \( \lambda > 0 \) if and only if \( \varphi(F^4_n) - 2 \varphi(F^3_n) \to 2\lambda^2 - \lambda \) (where \( \varphi \) is the relevant tracial state).

In \cite{15}, convergence in law of any sequence belonging to the second Wigner chaos is characterized by means of the convergence of only a finite number of cumulants.

In \cite{7}, making use of heavy combinatorics it is shown that any adequately rescaled sequence \( \{ F_n : n \geq 0 \} \) of self-adjoint operators living inside a fixed Wigner chaos converges in distribution to the tetilla law \( T \) if and only if \( \varphi(F^4_n) \to \varphi(T^4) \) and \( \varphi(F^6_n) \to \varphi(T^6) \) (where \( \varphi \) is the relevant tracial state). Note that this finding is not an extension of a result known in the classical probability theory, as the existence of such a result in the classical setting is still an open problem.

In \cite{6}, a class of sufficient conditions, ensuring that a sequence of multiple integrals with respect to a free Poisson measure converges to a semicircular limit, is established, thus providing an analog of Theorem 1.3 in the context of free Poisson chaos.

In \cite{3}, a fourth moment type condition is given, for an element of a free Poisson chaos of arbitrary order to converge to a free centered Poisson distribution.
In [1], an estimate for the Kolmogorov distance between a freely infinitely divisible distribution and the semicircle distribution is given, in terms of the difference between the fourth moment and two.

In [4], a multidimensional counterpart of the aforementioned central limit theorem on the free Poisson chaos is given.

In [5], a quantitative version of Theorem 1.3 is derived, using free stochastic analysis as well as a new biproduct formula for bi-integrals.

In the present paper, our main goal is to provide characterizations of free independence on the Wigner and free Poisson chaos, as well as investigate the similarities and dissimilarities between classical and free chaos, as far as (possibly asymptotic) independence properties are concerned.

Our first set of investigations yields a characterization of freeness on the Wigner and free Poisson chaos, in terms of contractions, covariances, or free Malliavin gradient, thus providing a suitable extension of Theorem 1.1 (and related results) to the free setting. Most of our results turn out to be similar to the classical setting, with the notable exception of the characterization of freeness in terms of the free Malliavin gradient, this last fact illustrating a fundamental difference between the classical and the free cases.

Our second set of investigations is concerned again with the independence property, but this time in an asymptotic context. Here, the problem is to find what conditions are to be imposed on limits of multiple integrals to be free. Surprisingly, and following an idea of Nourdin and Rosiński originally developed in the classical setting, one can actually use the independence results obtained in the asymptotic context to provide a new proof of Theorem 1.3.

The remainder of this paper is organized as follows: Section 2 contains a short introduction to free probability theory, with a special emphasis to the material needed for the rest of the paper. Section 3 is devoted to the characterization of freeness on the Wigner and free Poisson chaos, in terms of contractions, covariances, or free Malliavin gradient. This section also provides several lemmas which will be used to prove our main results in the following sections. In Section 4, we study different characterizations of asymptotic freeness, in several contexts. We devote Section 5 to the study of transfer principles between classical and free chaos. In Section 6, we provide new proofs for fourth moment type theorems in the free setting, using mainly the results developed in Section 4. Finally, Section 7 contains auxiliary results that are used throughout the paper.

2. Preliminaries

2.1. Elements of free probability. In the following, a short introduction to free probability theory is provided. For a thorough and complete treatment, see [11], [22] and [8]. Let \((\mathcal{A}, \varphi)\) be a tracial \(W^\ast\)-probability space, that is \(\mathcal{A}\) is a von Neumann algebra with involution \(*\) and \(\varphi: \mathcal{A} \to \mathbb{C}\) is a unital linear functional assumed to be weakly continuous, positive (meaning that \(\varphi(X) \geq 0\) whenever \(X\) is a non-negative element of \(\mathcal{A}\)), faithful (meaning that \(\varphi(XX^*) = 0 \Rightarrow X = 0\) for every \(X \in \mathcal{A}\)) and tracial (meaning that \(\varphi(XY) = \varphi(YX)\) for all \(X, Y \in \mathcal{A}\)). The self-adjoint elements of \(\mathcal{A}\) will be referred to as random variables. Given a random variable \(X \in \mathcal{A}\), the law of \(X\) is defined to be the unique Borel measure on \(\mathbb{R}\) having the same moments as \(X\) (see [11, Proposition 3.13]). The non-commutative space \(L^2(\mathcal{A}, \varphi)\) denotes the completion of \(\mathcal{A}\) with respect to the norm \(\|X\|_2 = \sqrt{\varphi(XX^*)}\).
Recall the definition of freeness (see [11, Definition 5.3] and [11, Remarks 5.4] or [20, Definition 2.5.18]) for a collection of non-commutative random variables living on an appropriate non-commutative probability space $(\mathcal{A}, \varphi)$.

**Definition 2.1.** A collection of random variables $X_1, \ldots, X_n$ on $(\mathcal{A}, \varphi)$ is said to be free if

$$\varphi(\{P_1(X_{i_1}) - \varphi(P_1(X_{i_1}))\} \cdots \{P_m(X_{i_m}) - \varphi(P_m(X_{i_m}))\}) = 0$$

whenever $P_1, \ldots, P_m$ are polynomials and $i_1, \ldots, i_m \in \{1, \ldots, n\}$ are indices with no two adjacent $i_j$ equal.

Let $X \in \mathcal{A}$. The $k$-th moment of $X$ is given by the quantity $\varphi(X^k)$, $k \in \mathbb{N}_0$. Now assume that $X$ is a self-adjoint bounded element of $\mathcal{A}$ (in other words, $X$ is a bounded random variable), and write $\rho(X) = \|X\| \in [0, \infty)$ to indicate the spectral radius of $X$.

**Definition 2.2.** The law (or spectral measure) of $X$ is defined as the unique Borel probability measure $\mu_X$ on the real line such that $\int_{\mathbb{R}} P(t) \, d\mu_X(t) = \varphi(P(X))$ for every polynomial $P \in \mathbb{R}[X]$. A consequence of this definition is that $\mu_X$ has support in $[-\rho(X), \rho(X)]$.

The existence and uniqueness of $\mu_X$ in such a general framework are proved e.g. in [20, Theorem 2.5.8] (see also [11, Proposition 3.13]). Note that, since $\mu_X$ has compact support, the measure $\mu_X$ is completely determined by the sequence $\{\varphi(X^k) : k \geq 1\}$.

Let $\{X_n : n \geq 1\}$ be a sequence of non-commutative random variables, each possibly belonging to a different non-commutative probability space $(\mathcal{A}_n, \varphi_n)$.

**Definition 2.3.** The sequence $\{X_n : n \geq 1\}$ is said to converge in distribution to a limiting non-commutative random variable $X_\infty$ (defined on $(\mathcal{A}_\infty, \varphi_\infty)$), if $\varphi_n(P(X_n)) \overset{n \to \infty}{\longrightarrow} \varphi_\infty(P(X_\infty))$ for every polynomial $P \in \mathbb{R}[X]$.

If $X_n, X_\infty$ are bounded (and therefore the spectral measures $\mu_{X_n}, \mu_{X_\infty}$ are well-defined), this last relation is equivalent to saying that

$$\int_{\mathbb{R}} P(t) \mu_{X_n}(dt) \overset{n \to \infty}{\longrightarrow} \int_{\mathbb{R}} P(t) \mu_{X_\infty}(dt).$$

An application of the method of moments yields immediately that, in this case, one has also that $\mu_{X_n}$ weakly converges to $\mu_{X_\infty}$, that is $\mu_{X_n}(f) \overset{n \to \infty}{\longrightarrow} \mu_{X_\infty}(f)$, for every $f : \mathbb{R} \to \mathbb{R}$ bounded and continuous (note that no additional uniform boundedness assumption is needed).

Let us now define the two main processes we will deal with in this paper, namely the free Brownian motion and the free Poisson process.

**Definition 2.4.** 1. The centered semicircular distribution with variance $t > 0$, denoted by $S(0,t)$, is the probability distribution given by

$$S(0,t)(dx) = (2t)^{-1} \sqrt{4t - x^2} \mathbf{1}_{[-2\sqrt{2t},2\sqrt{t}]}(x)dx.$$ 

2. A free Brownian motion $S$ consists of: (i) a filtration $\{\mathcal{A}_t : t \geq 0\}$ of von Neumann sub-algebras of $\mathcal{A}$ (in particular, $\mathcal{A}_s \subset \mathcal{A}_t$ for $0 \leq s < t$), (ii) a collection $S = \{S_t : t \geq 0\}$ of self-adjoint operators in $\mathcal{A}$ such that: (a) $S_0 = 0$ and $S_t \in \mathcal{A}_t$ for all $t \geq 0$, (b) for all $t \geq 0$, $S_t$ has a semicircular distribution with mean zero.
and variance \( t \), and (c) for all \( 0 \leq u < t \), the increment \( S_t - S_u \) is free with respect to \( \mathcal{A}_u \), and has a semicircular distribution with mean zero and variance \( t - u \).

**Definition 2.5.** 1. The free Poisson distribution with rate \( \lambda > 0 \), denoted by \( P(\lambda) \), is the probability distribution defined as follows: (i) if \( \lambda \in (0, 1] \), then \( P(\lambda) = (1 - \lambda)\delta_0 + \lambda \tilde{\nu} \), and (ii) if \( \lambda > 1 \), then \( P(\lambda) = \tilde{\nu} \), where \( \delta_0 \) stands for the Dirac mass at 0. Here,

\[
\tilde{\nu}(dx) = (2\pi)^{-1} \sqrt{4\lambda - (x - 1)^2} \mathbb{1}_{[1 - \sqrt{\lambda}, 1 + \sqrt{\lambda}]}(x) dx.
\]

2. A free Poisson process \( \hat{N} \) consists of: (i) a filtration \( \{\mathcal{F}_t : t \geq 0\} \) of von Neumann sub-algebras of \( \mathcal{A} \) (in particular, \( \mathcal{A}_s \subset \mathcal{A}_t \) for \( 0 \leq s < t \)), (ii) a collection \( \mathcal{N} = \{N_t : t \geq 0\} \) of self-adjoint operators in \( \mathcal{A}_+ \) (\( \mathcal{A}_+ \) denotes the cone of positive operators in \( \mathcal{A} \)) such that: (a) \( N_0 = 0 \) and \( N_t \in \mathcal{A}_+ \) for all \( t \geq 0 \), (b) for all \( t \geq 0 \), \( N_t \) has a free Poisson distribution with rate \( t \), and (c) for all \( 0 \leq u < t \), the increment \( N_t - N_u \) is free with respect to \( \mathcal{A}_u \), and has a free Poisson distribution with rate \( t - u \). \( \hat{N} \) will denote the collection of random variables \( \hat{N} = \{\tilde{N}_t = N_t - t1 : t \geq 0\} \), where \( 1 \) stands for the unit of \( \mathcal{A} \). \( \hat{N} \) will be referred to as a compensated free Poisson process.

**Remark 2.6.** In the sequel, \( \mathcal{M} \) will stand for either the free Brownian motion \( S \) or the free Poisson process \( \hat{N} \).

We continue with some definitions that will play a crucial role in the rest of the paper. For every integer \( n \geq 1 \), the space \( L^2(\mathbb{R}_+^n; \mathbb{C}) = L^2(\mathbb{R}_+^n) \) denotes the collection of all complex-valued functions on \( \mathbb{R}_+^n \) that are square-integrable with respect to the Lebesgue measure on \( \mathbb{R}_+^n \).

**Definition 2.7.** Let \( n \) be a natural number and let \( f \) be a function in \( L^2(\mathbb{R}_+^n) \).

1. The adjoint of \( f \) is the function \( f^*(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)} \).
2. The function \( f \) is called mirror-symmetric if \( f = f^* \), i.e., if

\[
f(t_1, \ldots, t_3) = \overline{f(t_n, \ldots, t_1)}
\]

for almost all \( (t_1, \ldots, t_3) \in \mathbb{R}_+^n \) with respect to the product Lebesgue measure.
3. The function \( f \) is called (fully) symmetric if it is real-valued and, for any permutation \( \sigma \) in the symmetric group \( \mathfrak{S}_n \), it holds that \( f(t_1, \ldots, t_n) = f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}) \) for almost all \( (t_1, \ldots, t_n) \in \mathbb{R}_+^n \) with respect to the product Lebesgue measure.

**Definition 2.8.** Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}_+^n) \) and \( g \in L^2(\mathbb{R}_+^m) \). Let \( p \leq n \wedge m \) be a natural number. The \( p \)-th nested contraction \( f \overset{p}{\prec} g \) of \( f \) and \( g \) is the \( L^2(\mathbb{R}_+^{n+m-2p}) \) function defined by nested integration of the middle \( p \) variables in \( f \otimes g \):

\[
(f \overset{p}{\prec} g)(t_1, \ldots, t_{n+m-2p}) = \int_{\mathbb{R}_+^p} f(t_1, \ldots, t_{n-p}, s_1, \ldots, s_p) \times g(s_p, \ldots, s_1, t_{n-p+1}, \ldots, t_{n+m-2p}) ds_1 \cdots ds_p.
\]

In the case where \( p = 0 \), the function \( f \overset{0}{\prec} g \) is just given by \( f \otimes g \). Similarly, the \( p \)-th star contraction \( f \overset{p}{\ast} g \) of \( f \) and \( g \) is the \( L^2(\mathbb{R}_+^{n+m-2p+1}) \) function defined...
by nested integration of the middle \( p - 1 \) variables and identification of the first non-integrated variable in \( f \otimes g \):

\[
(f \ast_p^{p-1} g)(t_1, \ldots, t_{n+m-2p+1}) = \int_{n-m}^{p-1} f(t_1, \ldots, t_{n-p+1}, s_1, \ldots, s_{p-1})
\times g(s_{p-1}, \ldots, s_1, t_{n-p+1}, \ldots, t_{n+m-2p+1})ds_1 \cdots ds_{p-1}.
\]

For \( f \in L^2(\mathbb{R}_+^n) \), we denote by \( I^S_n(f) \) the multiple Wigner integral of \( f \) with respect to the free Brownian motion as introduced in [2]. The space \( L^2(S, \varphi) = \{I^S_n(f): f \in L^2(\mathbb{R}_+^n), n \geq 0\} \) is a unital \( \ast \)-algebra, with product rule given, for any \( n, m \geq 1 \), \( f, g \in L^2(\mathbb{R}_+^n) \), \( g \in L^2(\mathbb{R}_+^m) \), by

\[
(I^S_n(f)I^S_m(g)) = \sum_{p=0}^{n+m} I^S_{n+m-2p}(f \ast_p g)
\]

and involution \( I^S_n(f)^* = I^S_n(f^*) \). For a proof of this formula, see [2].

Similarly, we can define free Poisson multiple integrals with respect to \( \hat{N} \) (these integrals were studied in depth in [6], and we refer to this reference for details). The space \( L^2(S, \varphi) = \{I^S_n(f): f \in L^2(\mathbb{R}_+^n), n \geq 0\} \) is a unital \( \ast \)-algebra, with product rule given, for any \( n, m \geq 1 \), \( f, g \in L^2(\mathbb{R}_+^n) \), \( g \in L^2(\mathbb{R}_+^m) \), by

\[
(I^S_n(f)I^S_m(g)) = \sum_{p=0}^{n+m} I^S_{n+m-2p}(f \ast_p g)
\]

and involution \( I^S_n(f)^* = I^S_n(f^*) \). For a proof of this formula, see [6].

Furthermore, as is well-known, both Wigner and free Poisson multiple integrals of different orders are orthogonal in \( L^2(S, \varphi) \), whereas for two integrals of the same order, the Wigner isometry holds:

\[
\varphi(I^S_n(f)I^S_m(g)^*) = \langle f, g \rangle_{L^2(S, \varphi)}.
\]

\textbf{Remark 2.9.} 1. Observe that it follows from the definition of the involution on the algebras \( L^2(S, \varphi) \) and \( L^2(S, \varphi) \) that operators of the type \( I^S_n(f) \) are self-adjoint if and only if \( f \) is mirror-symmetric.

2. In what follows, we will use the notation \( I^S_n, I^S_n, I^W_n \) and \( I^I_n \) to denote multiple Wigner integrals, multiple free Poisson integrals, multiple Wiener integrals, and multiple classical Poisson integrals, respectively.

\section{Bi-integrals and free gradient operator.} In this particular subsection, we only focus on the Wigner case, as the tools we are about to introduce do not exist in the context of free Poisson processes.

Let \( (\mathcal{A}, \varphi) \) be a \( W^\ast \)-probability space. An \( \mathcal{A} \otimes \mathcal{A} \)-valued stochastic process \( t \mapsto U_t \) is called a bi-process. For \( p \geq 1 \), \( U \) is an element of \( \mathcal{B}_p \), the space of \( L^p \)-bi-processes, if its norm

\[
\|U\|_\mathcal{B}_p^2 = \int_0^\infty \|U_t\|^2_{L^p(\mathcal{A} \otimes \mathcal{A}, \varphi \otimes \varphi)} dt
\]

is finite.

Let \( n, m \) be two positive integers and \( f = g \otimes h \in L^2(\mathbb{R}_+^n) \otimes L^2(\mathbb{R}_+^m) \). Then, the Wigner bi-integral \( [I^S_n \otimes I^S_m](f) \) is defined as

\[
[I^S_n \otimes I^S_m](f) = I^S_n(g) \otimes I^S_m(h).
\]
This definition is extended linearly to generic elements \( f \in L^2 (\mathbb{R}_n^m) \otimes L^2 (\mathbb{R}_n^m) \cong L^2 (\mathbb{R}_{n+m}^n) \), where the symbol \( \cong \) denotes an isomorphic identification. From the Wigner isometry for multiple integrals, we obtain the so-called Wigner bisometry: for \( f \in L^2 (\mathbb{R}_n^m) \otimes L^2 (\mathbb{R}_n^m) \) and \( g \in L^2 (\mathbb{R}_n^m) \otimes L^2 (\mathbb{R}_n^m) \) it holds that

\[
\varphi \otimes \varphi (|I_n^S \otimes I_m^S|)(f) = \begin{cases} \langle f, g \rangle_{L^2(\mathbb{R}_n^m) \otimes L^2(\mathbb{R}_n^m)} & \text{if } n = n' \text{ and } m = m', \\ 0 & \text{otherwise}. \end{cases}
\]

A crucial tool in the analysis of Wigner integrals is the product formula (1), and a biproduct formula for bi-integrals was recently obtained in [5], which will be a crucial tool in the sequel. It makes use of a new type of contraction, referred to in [5] as bicontractions, defined as follows. Let \( n_1, n_2, m_2 \) be positive integers. Let \( f \in L^2 (\mathbb{R}_n^{m_1}) \otimes L^2 (\mathbb{R}_n^{m_1}) \cong L^2 (\mathbb{R}_{n+m_1}^{n_1}) \) and \( g \in L^2 (\mathbb{R}_n^{m_2}) \otimes L^2 (\mathbb{R}_n^{m_2}) \cong L^2 (\mathbb{R}_{n+m_2}^{n_1}) \) and let \( p \leq n_1 \land n_2, r \leq m_1 \land m_2 \) be natural numbers. The \((p, r)\)-bicontraction \( f \overset{p, r}{\sim} g \) is the \( L^2 (\mathbb{R}_n^{n_1+n_2-2p}) \otimes L^2 (\mathbb{R}_n^{n_1+m_2-2r}) \cong L^2 (\mathbb{R}_n^{n_1+n_2+m_2-2p-2r}) \) function defined by

\[
f \overset{p, r}{\sim} g (t_1, \ldots, t_{n_1+n_2+m_1+m_2-2p-2r}) = \int_{\mathbb{R}_n^{p+r}} f(t_1, \ldots, t_{n_1-p}, s_p, \ldots, s_1, y_1, \ldots, y_r, \\
 t_{n_1+n_2+m_2-2p-r+1}, \ldots, t_{n_1+n_2+m_1+m_2-2p-2r}) \times g (s_1, \ldots, s_p, t_{n_1-p+1}, \ldots, t_{n_1+n_2+m_2-2p-r}, y_r, \ldots, y_1) ds_1 \cdots ds_p dy_1 \cdots dy_r.
\]

Remark 2.10. Observe that these bicontractions have the following properties (for a proof, see [5]). For \( n_1, m_1, n_2, m_2 \in \mathbb{N} \), let \( f \in L^2 (\mathbb{R}_n^{n_1}) \otimes L^2 (\mathbb{R}_n^{m_1}) \cong L^2 (\mathbb{R}_{n+m_1}^{n_1}) \) and \( g \in L^2 (\mathbb{R}_n^{n_2}) \otimes L^2 (\mathbb{R}_n^{m_2}) \cong L^2 (\mathbb{R}_{n+m_2}^{n_1}) \) be fully symmetric functions. Furthermore, let \( p \leq n_1 \land n_2, r \leq m_1 \land m_2 \) be natural numbers such that \( p + r = p' + r' \). Then, the following holds.

1. \( f \overset{p, r}{\sim} g = f \overset{p', r'}{\sim} g \).
2. \( f \overset{p, r}{\sim} g = f \overset{p, r'}{\sim} g \).
3. \( \|f \overset{p, r}{\sim} g\|_{L^2(\mathbb{R}_n^{n_1+n_2-2p}) \otimes L^2(\mathbb{R}_n^{n_1+m_2-2r})} = \|f \overset{p, r'}{\sim} g\|_{L^2(\mathbb{R}_n^{n_1+n_2+m_2-2p-2r})} \).
4. \( f \overset{n_1, m_1}{\sim} = \|f\|_{L^2(\mathbb{R}_n^{n_1}) \otimes L^2(\mathbb{R}_n^{m_1})} 1 \otimes 1 \), which is a constant in \( L^2 (\mathbb{R}_n^{n_1}) \otimes L^2 (\mathbb{R}_n^{m_1}) \).

We introduce \( \sharp \) to be the associative action of \( \mathcal{A} \otimes \mathcal{A}^{\text{op}} \) (where \( \mathcal{A}^{\text{op}} \) denotes the opposite algebra) on \( \mathcal{A} \otimes \mathcal{A} \), as

\[
(A \otimes B) \sharp (C \otimes D) = (AC) \otimes (DB).
\]

Furthermore, we also write \( \sharp \) to denote the action of \( \mathcal{A} \otimes L^2 (\mathbb{R}_n) \otimes \mathcal{A}^{\text{op}} \) on \( \mathcal{A} \otimes L^2 (\mathbb{R}_n) \otimes \mathcal{A} \), as

\[
(A \otimes f \otimes B) \sharp (C \otimes g \otimes D) = (AC) \otimes fg \otimes (DB).
\]

Using the bicontractions definition, the biproduct formula for Wigner bi-integrals proved in [5] can be stated as follows.
Proposition 2.11 (Bourguin and Campese [5], 2017). For $n_1, m_1, n_2, m_2 \in \mathbb{N}$, let $f \in L^2 \left( \mathbb{R}^+ \right) \otimes L^2 \left( \mathbb{R}^m \right) \cong L^2 \left( \mathbb{R}^{n_1+m_1} \right)$ and $g \in L^2 \left( \mathbb{R}^n \right) \otimes L^2 \left( \mathbb{R}^n \right) \cong L^2 \left( \mathbb{R}^{n_2+m_2} \right)$. Then it holds that

$$\left| I_{n_1} \otimes I_{m_2} \right| \left( f \right) \| 2 \left| I_{m_2} \otimes I_{m_2} \right| \left( g \right) = \sum_{p=0}^{n_1} \sum_{r=0}^{m_1} \left[ I_{n_1+m_2-2p} \otimes I_{m_1+m_2-2r} \right] \left( f \right).$$

Finally, the free gradient operator $\nabla : L^2 \left( \mathcal{S}, \varphi \right) \rightarrow \mathcal{B}_2$ is a densely-defined and closable operator whose action on Wigner integrals is given by

$$\nabla \mathcal{I}^S_n(f) = \sum_{k=1}^{n} \left[ I_{k-1} \otimes I_{n-k} \right] \left( f^{(k)} \right),$$

where $f^{(k)}(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{k-1}, t, x_k, \ldots, x_{n-1})$ is viewed as an element of $L^2 \left( \mathbb{R}^+ \otimes \mathbb{R}^{n-k} \right)$. We also define the pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{B}_2 \times \mathcal{B}_2$ and $L^2 \left( \mathcal{A} \otimes \mathcal{A}, \varphi \otimes \varphi \right)$ to be

$$\langle U, V \rangle = \int_{\mathbb{R}^+} U_s V_s^* \, ds.$$

3. Characterizations of Freeness

In this section, we are interested in providing several characterizations of freeness between two multiple integrals. We will derive those characterizations in terms of contractions, covariances and free Malliavin gradients respectively.

3.1. Characterization in terms of contractions. Recall the well-known characterization of independence of multiple Wiener-Itô integrals by Üstünel and Zakai [21] in terms of the first contraction of the associated kernels.

Theorem 3.1 (Üstünel and Zakai [21], 1989). Let $n, m$ be natural numbers and let $f \in L^2 \left( \mathbb{R}^n \right)$ and $g \in L^2 \left( \mathbb{R}^m \right)$ be symmetric functions. Then, $I_n^W(f)$ and $I_m^W(g)$ are independent if and only if $f \otimes_1 g = 0$ almost everywhere.

Remark 3.2. In the context of a multiple Wiener-Itô integral $I_n^W(f)$, note that one can always assume without loss of generality that the kernel $f$ is symmetric, as $I_n^W(f) = I_n^W(\hat{f})$, where $\hat{f}$ denotes the symmetrization of the function $f$ given by

$$\hat{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma(1), \ldots, x_{\sigma(n)}),$$

with $\mathfrak{S}_n$ the symmetric group of $\{1, \ldots, n\}$.

A natural question is to ask whether or not the characterization of independence of Üstünel and Zakai has a counterpart in the free setting. It turns out that a similar characterization of freeness holds on both the Wigner and the free Poisson space, which is the first result of this paper.

Theorem 3.3. Let $n, m$ be natural numbers and let $f \in L^2 \left( \mathbb{R}^+ \right)$ and $g \in L^2 \left( \mathbb{R}^+ \right)$ be symmetric functions. Then,

(i) $I_n^S(f)$ and $I_n^S(g)$ are free if and only if $f \overset{\mathcal{S}_1}{\sim} g = 0$ almost everywhere.

(ii) $I_n^N(f)$ and $I_n^N(g)$ are free if and only if $f \overset{\mathcal{N}_1}{\sim} g = 0$ almost everywhere.
Proof. First, assume that $I_n^m(f)$ and $I_m^m(g)$ are free. Then, by Definition 2.1, it holds that, in particular
\[
\varphi \left[ I_n^m(f)^2 - \varphi \left( I_n^m(f)^2 \right) \right] \left[ I_m^m(g)^2 - \varphi \left( I_m^m(g)^2 \right) \right] = \varphi \left( I_n^m(f)^2 I_m^m(g)^2 - \varphi \left( I_n^m(f)^2 \right) \varphi \left( I_m^m(g)^2 \right) \right) = 0.
\]
Observe that
\[
\varphi \left( I_n^m(f)^2 I_m^m(g)^2 \right) = \sum_{p=0}^{n} \sum_{r=0}^{m} \varphi \left( I_{2n-2p}^r \left( f \overset{p}{\ast} f \right) I_{2n-2r}^m \left( g \overset{r}{\ast} g \right) \right) + \mathbbm{1}_{\{m=N\}} \sum_{p=0}^{N} \sum_{r=1}^{m} \varphi \left( I_{2n-2p}^1 \left( f \overset{p}{\ast} f \right) I_{2r}^1 \left( g \overset{r}{\ast} g \right) \right) = \sum_{p=0}^{n} \sum_{r=0}^{m} \varphi \left( I_{2p}^1 \left( f \overset{p}{\ast} f \right) I_{2r}^1 \left( g \overset{r}{\ast} g \right) \right) + \mathbbm{1}_{\{m=N\}} \sum_{p=0}^{m} \sum_{r=0}^{m-1} \varphi \left( I_{2p+1}^1 \left( f \overset{p}{\ast} f \right) I_{2r+1}^1 \left( g \overset{r}{\ast} g \right) \right).
\]
Using the isometry property (3), we get
\[
\varphi \left( I_n^m(f)^2 I_m^m(g)^2 \right) = \sum_{p=0}^{n \wedge m} \left\langle f \overset{n-p}{\ast} f, g \overset{m-p}{\ast} g \right\rangle_{L^2(\mathbb{R}^2_\gamma)} + \mathbbm{1}_{\{m=N\}} \sum_{p=0}^{(n \wedge m)-1} \left\langle f \overset{n-p}{\ast} f, g \overset{m-p-1}{\ast} g \right\rangle_{L^2(\mathbb{R}^{2+1}_\gamma)} = \sum_{p=0}^{n \wedge m} \left\| f \overset{p}{\ast} g \right\|_{L^2(\mathbb{R}^{n+m-2p}_\gamma)}^2 + \mathbbm{1}_{\{m=N\}} \sum_{p=1}^{n \wedge m} \left\| f \overset{p-1}{\ast} g \right\|_{L^2(\mathbb{R}^{n+m-2p+1}_\gamma)}^2 = \left\| f \right\|_{L^2(\mathbb{R}^2_\gamma)}^2 \left\| g \right\|_{L^2(\mathbb{R}^m_\gamma)}^2 + \sum_{p=1}^{n \wedge m} \left\| f \overset{p}{\ast} g \right\|_{L^2(\mathbb{R}^{n+m-2p+1}_\gamma)}^2.
\]
Recalling that $\varphi \left( I_n^m(f)^2 \right) = \left\| f \right\|_{L^2(\mathbb{R}^2_\gamma)}^2$ and $\varphi \left( I_m^m(g)^2 \right) = \left\| g \right\|_{L^2(\mathbb{R}^m_\gamma)}^2$ yields
\[
\varphi \left( I_n^m(f)^2 I_m^m(g)^2 \right) - \varphi \left( I_n^m(f)^2 \right) \varphi \left( I_m^m(g)^2 \right) = \sum_{p=1}^{n \wedge m} \left\| f \overset{p}{\ast} g \right\|_{L^2(\mathbb{R}^{n+m-2p}_\gamma)}^2 + \mathbbm{1}_{\{m=N\}} \sum_{p=1}^{n \wedge m} \left\| f \overset{p-1}{\ast} g \right\|_{L^2(\mathbb{R}^{n+m-2p+1}_\gamma)}^2.
\]
As the left-hand side of the above equality is zero, the fact that $f \overset{1}{\sim} g = 0$ a.e. in the Wigner case and $f \overset{0}{\ast} 0 = 0$ a.e. in the free Poisson case follows.

Conversely, assume that $f \overset{1}{\sim} g = 0$ a.e. in the Wigner case and that $f \overset{0}{\ast} 0 = 0$ a.e. in the free Poisson case. According to Definition 2.1 together with the linearity of
the functional \( \varphi \), we must prove that, for any natural number \( \ell \) and for any natural numbers \( k_1, \ldots, k_{2\ell} \),

\[
\varphi \left( \left[ I_{n}^{2\ell} (f)^{k_1} - \varphi \left( I_{n}^{2\ell} (f)^{k_1} \right) \right] \left[ I_{m}^{2\ell} (g)^{k_2} - \varphi \left( I_{m}^{2\ell} (g)^{k_2} \right) \right] \right.
\]

\[
\ldots \left[ I_{n}^{2\ell} (f)^{k_{2\ell-1}} - \varphi \left( I_{n}^{2\ell} (f)^{k_{2\ell-1}} \right) \right] \left[ I_{m}^{2\ell} (g)^{k_{2\ell}} - \varphi \left( I_{m}^{2\ell} (g)^{k_{2\ell}} \right) \right] \right)
\]

\[= 0.\]

**Remark 3.4.** Observe that we only consider an even number of powers \( k \). This comes from the tracial property of the functional \( \varphi \) together with the condition that no two adjacent indices \( i_j \) can be equal in Definition 2.1. Indeed, if we consider an odd number of powers \( k \), we would have

\[
\varphi \left( \left[ I_{n}^{2\ell} (f)^{k_1} - \varphi \left( I_{n}^{2\ell} (f)^{k_1} \right) \right] \left[ I_{m}^{2\ell} (g)^{k_2} - \varphi \left( I_{m}^{2\ell} (g)^{k_2} \right) \right] \right.
\]

\[
\ldots \left[ I_{n}^{2\ell} (f)^{k_{2\ell+1}} - \varphi \left( I_{n}^{2\ell} (f)^{k_{2\ell+1}} \right) \right] \right)
\]

\[= \varphi \left( \left[ I_{n}^{2\ell} (f)^{k_{2\ell+1}} - \varphi \left( I_{n}^{2\ell} (f)^{k_{2\ell+1}} \right) \right] \left[ I_{m}^{2\ell} (f)^{k_1} - \varphi \left( I_{m}^{2\ell} (f)^{k_1} \right) \right] \right.
\]

\[
\left[ I_{m}^{2\ell} (g)^{k_{2\ell}} - \varphi \left( I_{m}^{2\ell} (g)^{k_{2\ell}} \right) \right] \ldots \left[ I_{m}^{2\ell} (g)^{k_{2\ell}} - \varphi \left( I_{m}^{2\ell} (g)^{k_{2\ell}} \right) \right] \right),
\]

where the first two indices would be the same in the framework of Definition 2.1.

Let \( q < k \) be two non-negative integers. For \( 0 \leq q \leq k-1 \), define the multisets \( S^k_q = \{1, \ldots, 1, 0, \ldots, 0\} \) where the element 1 has multiplicity \( q \) and the element 0 has multiplicity \( k-q-1 \). Such a set is sometimes denoted \( \{(1, q), (0, k-q-1)\} \).

We denote the group of permutations of the multiset \( S^k_q \) by \( \mathcal{S}^k_q \) and its cardinality is given by the multinomial coefficient \( \binom{k}{q, k-q-1} = \frac{(k-1)!}{q!(k-q-1)!} = \binom{k-1}{q} \). Observe that in the definition of the group of permutations of a multiset, each permutation yields a different ordering of the elements of the multiset, which is why the cardinality of \( \mathcal{S}^k_q \) is \( \binom{k-1}{q} \) and not \( \binom{k-1}{q} \). Using the Wigner and free Poisson product formulas along with Equation (4.1) in [13] and Lemma 4.1 in [3], we can write

\[
I_{n}^{2\ell} (f)^k = \varphi \left( I_{n}^{2\ell} (f)^k \right) + \sum_{r=1}^{kn} I_{m}^{2\ell} (a_r(f)) + \mathbb{1}_{\{\mathbb{m} = \mathbb{N}\}} \sum_{r=1}^{kn} I_{r}^{2\ell} (b_r(f)),
\]

where

\[
a_r(f) = \sum_{(p_1, \ldots, p_{k-1}) \in A_r} \left( p_1 \circ f p_2 \cdots f \right) \right) P_{k-1} \right)
\]

with

\[
A_r = \left\{ (p_1, \ldots, p_{k-1}) \in \{0, 1, \ldots, n\}^{k-1} : k n - 2 \sum_{i=1}^{k-1} p_i = r \right\}
\]

and where

\[
b_r(f) = \sum_{q=1}^{k-1} \sum_{\pi \in \mathcal{S}^k_q} \sum_{(p_1, \ldots, p_{k-1}) \in B_{r,q}} \left( p_1 \circ f p_2 \cdots f \right) \right) P_{k-1} \right) \right)
\]

\[
\cdots p_{k-1} = \pi(k-1) f
\]
with, for each $q = 1, \ldots, k - 1$ and each $\pi \in \mathfrak{S}_q$, 

$$B_{r,q}^\pi = \left\{ (p_1, \ldots, p_{k-1}) \in \bigotimes_{s=1}^{k-1} \{\pi(s), \ldots, n\} : kn + q - 2 \sum_{i} p_i = r \right\}.$$ 

We get that 

$$\left[ I_{m}^{2n} (f)^{k_1} - \varphi \left( I_{n}^{2n} (f)^{k_1} \right) \right] \left[ I_{m}^{2n} (g)^{k_2} - \varphi \left( I_{m}^{2n} (g)^{k_2} \right) \right] \ldots \left[ I_{m}^{2n} (f)^{k_{2\ell-1}} - \varphi \left( I_{m}^{2n} (f)^{k_{2\ell-1}} \right) \right] \left[ I_{m}^{2n} (g)^{k_{2\ell}} - \varphi \left( I_{m}^{2n} (g)^{k_{2\ell}} \right) \right]$$

$$= \sum_{r_1=1}^{k_1} \ldots \sum_{r_{2\ell-1}=1}^{k_{2\ell-1}} \sum_{r_{2\ell}=1}^{k_{2\ell}} \left( a_{r_1} (f) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_1} (f) \right) \cdots \left( a_{r_{2\ell-1}} (f) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_{2\ell-1}} (f) \right)$$

$$\left( a_{r_{2\ell}} (g) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_2} (g) \right) \cdots \left( a_{r_{2\ell}} (g) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_2} (g) \right).$$

At this point, observe that the assumptions that $f \stackrel{\lambda}{\longrightarrow} g$ is $0$ a.s in the Wigner case and $f \stackrel{\lambda}{\longrightarrow} g$ is $0$ a.s in the free Poisson case imply, by Lemma 7.1 and Lemma 7.2 respectively, that for any given $i = 1, \ldots, 2\ell - 1$, the contractions between $(a_{r_1} (f) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_1} (f))$ and $(a_{r_{i+1}} (g) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_{i+1}} (g))$ resulting from using the appropriate product formula iteratively will all be zero a.e. except for the ones of order zero corresponding to the tensor product operation (it is the only contraction that can be non-zero under both the Wigner and free Poisson case assumptions). Hence, keeping only the non-zero terms in the above expression yields 

$$\left[ I_{m}^{2n} (f)^{k_1} - \varphi \left( I_{n}^{2n} (f)^{k_1} \right) \right] \left[ I_{m}^{2n} (g)^{k_2} - \varphi \left( I_{m}^{2n} (g)^{k_2} \right) \right] \ldots \left[ I_{m}^{2n} (f)^{k_{2\ell-1}} - \varphi \left( I_{m}^{2n} (f)^{k_{2\ell-1}} \right) \right] \left[ I_{m}^{2n} (g)^{k_{2\ell}} - \varphi \left( I_{m}^{2n} (g)^{k_{2\ell}} \right) \right]$$

$$= \sum_{r_1=1}^{k_1} \ldots \sum_{r_{2\ell-1}=1}^{k_{2\ell-1}} \sum_{r_{2\ell}=1}^{k_{2\ell}} \left( a_{r_1} (f) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_1} (f) \right) \cdots \left( a_{r_{2\ell-1}} (f) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_{2\ell-1}} (f) \right)$$

$$\cdots \left( a_{r_{2\ell}} (g) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_2} (g) \right) \cdots \left( a_{r_{2\ell}} (g) + I_{\{\mathfrak{g} = \mathfrak{N}\}} b_{r_2} (g) \right).$$

As the quantity $r_1 + \cdots + r_{2\ell}$ is strictly positive, applying $\varphi$ to the above expression yields 

$$\varphi \left[ I_{m}^{2n} (f)^{k_1} - \varphi \left( I_{n}^{2n} (f)^{k_1} \right) \right] \left[ I_{m}^{2n} (g)^{k_2} - \varphi \left( I_{m}^{2n} (g)^{k_2} \right) \right] \ldots \left[ I_{m}^{2n} (f)^{k_{2\ell-1}} - \varphi \left( I_{m}^{2n} (f)^{k_{2\ell-1}} \right) \right] \left[ I_{m}^{2n} (g)^{k_{2\ell}} - \varphi \left( I_{m}^{2n} (g)^{k_{2\ell}} \right) \right] = 0,$$

which is the desired result.

Observe that the above characterization of freeness is stated and proven for symmetric kernels only. A natural question is whether or not this characterization continues to hold in the more general case of a mirror-symmetric kernel. We provide a negative answer to this question, proving that our characterization is exhaustive.
Concretely, we will exhibit two mirror-symmetric kernels $f, g \in L^2([0, 2]^3)$ such that $\|f \otimes g\|_{L^2([0, 2]^3)} = 0$ but $I_S^S(f)$ and $I_S^S(g)$ are not free.

Indeed, consider $f = \mathbb{1}_{[0, 1] \times [0, 2] \times [0, 1]}$ and $g = \mathbb{1}_{[1, 2] \times [0, 2] \times [1, 2]}$. It is readily checked that $f \otimes g = 0$. On the other hand, using the product formula (1) iteratively, we can write

$$I_S^S(f)^7 = \sum_{(r_1, \ldots, r_6) \in C} I_{21}^S(2) \left( (((((f \otimes f) \otimes f) \otimes f) \otimes f) \otimes f) \otimes f \right)$$

$$I_S^S(g)^7 = \sum_{(r_1, \ldots, r_6) \in C} I_{21}^S(2) \left( (((((g \otimes g) \otimes g) \otimes g) \otimes g) \otimes g \right).$$

where

$$C = \{(r_1, \ldots, r_6) \in \{0, 1, 2, 3\}^6 : r_2 \leq 6 - 2r_1, \quad r_3 \leq 9 - 2r_1 - 2r_2, \ldots, r_6 \leq 18 - 2r_1 - \ldots - 2r_5 \}.$$

Using the Wigner isometry (3), we deduce that $\varphi(I_S^S(f)^7) = 0$ and $\varphi(I_S^S(g)^7) = 0$, as well as (the functions $f$ and $g$ being positive)

$$\varphi(I_S^S(f)^7 I_S^S(g)^7) \geq \left( (((((f \otimes f) \otimes f) \otimes f) \otimes f) \otimes f, \right. \left. (((((g \otimes g) \otimes g) \otimes g) \otimes g) \otimes g \right) \right)_{L^2([0, 2]^3)}$$

$$= 32 \neq 0.$$

Consequently, according to the definition of freeness given in Definition 2.1, $I_S^S(f)$ and $I_S^S(g)$ are not free.

**Remark 3.5.** The same counterexample would also yield the same conclusion in the free Poisson case (replacing the Wigner integrals by free Poisson ones) as it is also the case that $f \star_1^\pi g = 0$ and as the first part of the free Poisson product formula (2) is the same as the Wigner product formula used above.

However, even if establishing a characterization of freeness in terms of contractions in the mirror-symmetric case is not possible, we can still give a sufficient condition for freeness, which is the object of the following result.

**Theorem 3.6.** Let $n, m$ be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions.

(i) If dealing with Wigner integrals, assume that $f^{(\sigma)} \hat{\otimes} g^{(\pi)} = 0$ almost everywhere for all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$, where

$$f^{(\sigma)}(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad x_1, \ldots, x_n \in \mathbb{R}_+,$$

and a similar definition for $g^{(\pi)}$. Then, $I_n^S(f)$ and $I_m^S(g)$ are free.

(ii) If dealing with free Poisson integrals, assume that $f^{(\sigma)} \star_1^\sigma g^{(\pi)} = 0$ almost everywhere for all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$. Then, one has that $I_n^N(f)$ and $I_m^N(g)$ are free.

**Proof.** Apply the same strategy as in the proof of Theorem 3.3 with the stronger assumptions. □
3.2. Characterization in terms of covariances. The next result is a free analog of [19, Corollary 5.2] by Rosiński and Samorodnitsky, which is itself a consequence of Theorem 3.1 by Üstünel and Zakai.

Corollary 3.7. Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}_+^n) \) and \( g \in L^2(\mathbb{R}_+^m) \) be symmetric functions. Then, \( I_n^{2n}(f) \) and \( I_m^{2m}(g) \) are free if and only if their squares are uncorrelated, i.e., if and only if
\[
\text{Cov} \left( I_n^{2n}(f)^2, I_m^{2m}(g)^2 \right) = 0.
\]

Proof. First, assume that \( I_n^{2n}(f) \) and \( I_m^{2m}(g) \) are free. Then, by Definition 2.1, it holds that
\[
\varphi \left( \left[ I_n^{2n}(f)^2 - \varphi \left( I_n^{2n}(f)^2 \right) \right] \left[ I_m^{2m}(g)^2 - \varphi \left( I_m^{2m}(g)^2 \right) \right] \right) = \varphi \left( I_n^{2n}(f)^2 I_m^{2m}(g)^2 \right) - \varphi \left( I_n^{2n}(f)^2 \right) \varphi \left( I_m^{2m}(g)^2 \right) = 0.
\]
As \( \text{Cov} \left( I_n^{2n}(f)^2, I_m^{2m}(g)^2 \right) = \varphi \left( I_n^{2n}(f)^2 I_m^{2m}(g)^2 \right) - \varphi \left( I_n^{2n}(f)^2 \right) \varphi \left( I_m^{2m}(g)^2 \right) \), the desired conclusion follows.

Conversely, assume that \( \text{Cov} \left( I_n^{2n}(f)^2, I_m^{2m}(g)^2 \right) = 0 \). Using (8), it holds that
\[
\text{Cov} \left( I_n^{2n}(f)^2, I_m^{2m}(g)^2 \right) = \sum_{p=1}^{n \wedge m} \left\| f \ast_p^0 g \right\|_{L^2(\mathbb{R}_+^{n+m-2p+1})}^2 + \mathbb{1}_{\{2n=2m\}} \sum_{p=1}^{n \wedge m} \left\| f \ast_p^{p-1} g \right\|_{L^2(\mathbb{R}_+^{n+m-2p+1})}^2,
\]
which implies that all the contraction norms appearing on the right-hand side of the above equality are zero. In particular, \( \left\| f \ast_1^0 g \right\|_{L^2(\mathbb{R}_+^{n+m-2})} = 0 \) in the Wigner case and \( \left\| f \ast_0^0 g \right\|_{L^2(\mathbb{R}_+^{n+m-1})} = 0 \) in the free Poisson case, which, by Theorem 3.3 implies that \( I_n^{2n}(f) \) and \( I_m^{2m}(g) \) are free. \( \square \)

3.3. Characterization in terms of free Malliavin gradients. In the context of Wiener integrals, Üstünel and Zakai proved in [21, Proposition 2] that a necessary condition for two Wiener integrals \( I_n^W(f) \) and \( I_m^W(g) \) to be independent was that the inner product of their Malliavin derivatives was zero almost surely. More precisely, their statement reads as follows.

Theorem 3.8 (Üstünel and Zakai [21], 1989). A necessary condition for the independence of \( I_n^W(f) \) and \( I_m^W(g) \) is
\[
\langle D_{I_n^W(f)} \rangle, \langle D_{I_m^W(g)} \rangle \rangle_{L^2(\mathbb{R}_+)} = 0 \quad \text{a.s.}
\]
(9)

However, they were also able to show that this condition is not sufficient and hence cannot provide a proper characterization of independence of Wiener integrals. The technical reason for this is that this condition implies that only the symmetrization of the first contraction of \( f \) and \( g \) be zero almost everywhere, which in turns does not necessarily imply that the first contraction itself be zero almost everywhere. As the latter is an equivalent statement to independence, the sufficiency of (9) fails.
In the free case, a free version of the Malliavin calculus (with respect to the free Brownian motion) has been developed by Biane and Speicher in [2], and it is a natural question to ask whether it can be used to provide a characterization of freeness for Wigner integrals.

**Remark 3.9.** In this subsection, we only focus on Wigner integrals and not on the free Poisson case. The reason for this is that there is no free Malliavin calculus available for free Poisson random measures, which is what would be needed to explore similar statements in the free Poisson case.

The following result is the main result of this subsection, which is a characterization of freeness in terms of the free gradient operator for Wigner integrals with symmetric kernels. It is worth noting that, as opposed to the case of Wiener integrals studied by Üstünel and Zakai, we are able to provide a positive answer to the question of characterizing freeness in terms of free gradient, which illustrates a fundamental difference between the classical case and the free case.

**Theorem 3.10.** Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}_+^n) \) and \( g \in L^2(\mathbb{R}_+^m) \) be symmetric functions. Then, \( I_n^S(f) \) and \( I_m^S(g) \) are free if and only if

\[
\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle = 0 \quad a.s.
\]

where the notation \( \langle \cdot, \cdot \rangle \) is defined in (7).

**Proof.** In the following we will use the shorthand \( f_s^{(k)} \) to denote the function given by

\[
f_s^{(k)}(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{k-1}, s, x_{k+1}, \ldots, x_n)
\]

Applying the definition of the action of \( \nabla \) on Wigner integrals, we get that

\[
\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle = \int_{\mathbb{R}_+} (\nabla_s I_n^S(f)) \overline{g(I_m^S(g))} ds
\]

\[
= \sum_{k=1}^n \sum_{q=1}^m \int_{\mathbb{R}_+} [I_{k-1}^S \otimes I_{n-k}^S] (f_s^{(k)}) \overline{I_{q-1}^S \otimes I_{m-q}^S} \left( g_s^{(q)} \right) ds
\]

where the last equality follows from the full symmetry of the function \( g \). The biproduct formula (6) yields

\[
\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle
\]

\[
= \sum_{k=1}^n \sum_{q=1}^m \int_{\mathbb{R}_+} (k \land q - 1) \land (n-k) \land (m-q) \left( f_s^{(k)} p_r \overline{g_s^{(q)}} \right) ds,
\]

and by using a Fubini argument, it follows that

\[
\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle
\]

\[
= \sum_{k=1}^n \sum_{q=1}^m \sum_{p=0}^{(k \land q) - 1} \sum_{r=0}^{(n-k) \land (m-q)} \left( I_{k+q-2-2p}^S \otimes I_{n+m-k-q-2r}^S \right) \left( f_s^{(k)} p_r \overline{g_s^{(q)}} \right) ds.
\]
The full symmetry of $f$ and $g$ implies that $f^{(k)}_s = f^{(n)}_s$ for every $1 \leq k \leq n$ and $g^{(q)}_s = g^{(1)}_s$ for every $1 \leq q \leq m$. Hence, using Remark (2.10), we get

$$\int_{R_+} f^{(k)}_s \overset{p,r}{\rightsquigarrow} g^{(q)}_s \, ds = f^{p+r+1}_s \overset{p}{\rightsquigarrow} g,$$

so that we finally get

$$\langle \nabla I^n_S(f), \nabla I_m^S(g) \rangle = \sum_{k=1}^{n} \sum_{q=1}^{m} \sum_{p=0}^{(k\wedge q)-1} \sum_{r=0}^{(n-k)\wedge (m-q)} [k^{q-2} + m^{k-q-2} + \cdots + f^{n\wedge m}_s \overset{p+r+1}{\rightsquigarrow} g^u_s \overset{n}{\rightsquigarrow} g^u_s] \nabla I^n_S(f), \nabla I_m^S(g) \rangle^2.$$

Using the Wigner bisometry (4), we see that the quantity

$$\varphi \otimes \varphi \left( |\langle \nabla I^n_S(f), \nabla I_m^S(g) \rangle|^2 \right)$$

is just a sum with strictly positive coefficients only involving the contraction norms

$$\left\| f^{1}_s \overset{k}{\rightsquigarrow} g \right\|^2_{L^2(R^n_+)} \cdot \left\| f^{2}_s \overset{m}{\rightsquigarrow} g \right\|^2_{L^2(R^n_+)} \cdots \left\| f^{n\wedge m}_s \overset{u}{\rightsquigarrow} g \right\|^2_{L^2(R^n_+)}.$$

Formally, we have an equality of the type

$$\varphi \otimes \varphi \left( |\langle \nabla I^n_S(f), \nabla I_m^S(g) \rangle|^2 \right) = \sum_{u=1}^{n\wedge m} c_u \left\| f^{u}_s \overset{n\wedge m}{\rightsquigarrow} g \right\|^2_{L^2(R^n_+)}.$$

Now assume that $I^n_S(f)$ and $I_m^S(g)$ are free. By Theorem 3.3, this is equivalent to $f^{1}_s \overset{1}{\rightsquigarrow} g = 0$ almost everywhere, which by Lemma 7.1 implies that $f^{p}_s \overset{p}{\rightsquigarrow} g = 0$ almost everywhere for all $1 \leq p \leq n \wedge m$. Using (12), we get (10).

Conversely, assume that

$$\langle \nabla I^n_S(f), \nabla I_m^S(g) \rangle = 0 \quad a.s.$$

Then, we have that

$$\varphi \otimes \varphi \left( |\langle \nabla I^n_S(f), \nabla I_m^S(g) \rangle|^2 \right) = 0.$$

This implies that all the norms appearing in the representation (12) are zero, and in particular that $f^{1}_s \overset{1}{\rightsquigarrow} g = 0$ almost everywhere. Using Theorem 3.3 concludes the proof.

4. Characterizations of asymptotic freeness

In the asymptotic context, the problem of interest is to find necessary and sufficient conditions for the limits in law of multiple integrals to be free. It is a much more general problem compared to before, as limits in law of multiple integrals need not be multiple integrals themselves.
4.1. Characterization in terms of contractions. In the classical case, the following result holds (see [16, Theorem 3.4]).

**Theorem 4.1** (Nourdin and Rosiński [16], 2014). Let \( n, m \) be natural numbers and let \( \{f_k: k \geq 1\} \subset L^2(\mathbb{R}_+^n) \) and \( \{g_k: k \geq 1\} \subset L^2(\mathbb{R}_+^m) \) be sequences of symmetric functions such that \( I_n^W(f_k) \xrightarrow{\text{law}} F \) and \( I_m^W(g_k) \xrightarrow{\text{law}} G \) as \( k \to \infty \), where \( F, G \) are square integrable random variables with laws determined by their moments. Then, \( F \) and \( G \) are independent if and only if \( f_k \otimes_1 g_k \xrightarrow{k \to +\infty} 0 \) a.e. for all \( p = 1, \ldots, n \land m \).

**Remark 4.2.** Observe that the limiting random variables in the above theorem need to have laws determined by their moments (a condition that we get automatically in the free setting) and that the necessary and sufficient condition for asymptotic independence is not \( f_k \otimes_1 g_k \xrightarrow{k \to +\infty} 0 \) a.e., as one could have expected in view of Theorem 3.1. This weaker condition is necessary but not sufficient in the asymptotic case, as pointed out in [16, Remark 3.2]. In the free case, the same phenomenon happens in the sense that the condition \( f_k \overset{1}{\xrightarrow{k \to +\infty}} g_k \) and \( f_k \ast_1^0 g_k \xrightarrow{k \to +\infty} 0 \) a.e. (in the free Poisson case) will prove to be necessary but not sufficient either, for the same reason. However, we need not ask anything about the moment-determinacy of the limiting random variables. In the classical case, the question of when and under what conditions this moment-determinacy condition could be removed was the object of the reference [12], which represents a substantial refinement of the results of [16].

The following result in the free case is hence rather an analog of the stronger results of [12] instead of those found in [16]. Note that here, \( F \) and \( G \) do not need to have the form of a multiple integral. This implies that sequences of multiple integrals can be used in order to prove the freeness of general random variables in \( L^2(\varphi) \) (provided these random variables admit approximating sequences of multiple integrals).

**Theorem 4.3.** Let \( n, m \) be natural numbers and let \( \{f_k: k \geq 1\} \subset L^2(\mathbb{R}_+^n) \) and \( \{g_k: k \geq 1\} \subset L^2(\mathbb{R}_+^m) \) be sequences of symmetric functions such that \( I_n^\mathfrak{M}(f_k) \xrightarrow{\text{law}} F \) and \( I_m^\mathfrak{M}(g_k) \xrightarrow{\text{law}} G \) as \( k \to \infty \), where \( F, G \) are random variables in \( L^2(\mathcal{A}, \varphi) \). Then,

(i) If \( \mathfrak{M} = S \), then \( F \) and \( G \) are free if and only if \( f_k \overset{p}{\xrightarrow{k \to +\infty}} g_k \) \( \xrightarrow{\text{a.e. for all}} \) for all \( p = 1, \ldots, n \land m \).

(ii) If \( \mathfrak{M} = \hat{N} \), then \( F \) and \( G \) are free if and only if \( f_k \overset{p}{\xrightarrow{k \to +\infty}} g_k \) \( \xrightarrow{\text{a.e. for all}} \) for all \( p = 1, \ldots, n \land m \).

**Proof.** First, assume that \( F \) and \( G \) are free. Then, it holds that \( \text{Cov}(F^2, G^2) = 0 \). Using (8) yields

\[
\text{Cov} \left( I_n^\mathfrak{M}(f_k)^2, I_m^\mathfrak{M}(g_k)^2 \right) = \sum_{p=1}^{n \land m} \left\| f \overset{p}{\to} g \right\|_{L^2(\mathbb{R}_+^{n+m-2p})}^2 + \mathbb{1}_{\left\{ \mathfrak{M} = \hat{N} \right\}} \sum_{p=1}^{n \land m} \left\| f \ast_p^0 g \right\|_{L^2(\mathbb{R}_+^{n+m-2p+1})}^2 \xrightarrow{k \to +\infty} \text{Cov}(F^2, G^2) = 0,
\]

where \( f \overset{p}{\to} g \) denotes the convergence of \( f \) to \( g \) in \( \mathbb{R}_+^{n+m-2p} \) in the sense that \( f_k \to g_k \) as \( k \to \infty \) in the space of \( \mathbb{R}_+^{n+m-2p} \).
so that for all \( p = 1, \ldots, n \land m \), \( f_k \overset{p}{\longrightarrow} g_k \; \longrightarrow \; 0 \; \text{a.e.} \) (in the Wigner case) and for all \( p = 1, \ldots, n \land m \), \( f_k \overset{p}{\longrightarrow} g_k \; \longrightarrow \; 0 \) and \( f_k \overset{p-1}{\longrightarrow} g_k \; \longrightarrow \; 0 \; \text{a.e.} \) (in the free Poisson case).

Conversely, assume that, for all \( p = 1, \ldots, n \land m \), \( f_k \overset{p}{\longrightarrow} g_k \; \longrightarrow \; 0 \; \text{a.e.} \) (in the Wigner case) or that, for all \( p = 1, \ldots, n \land m \), \( f_k \overset{p}{\longrightarrow} g_k \; \longrightarrow \; 0 \) and \( f_k \overset{p-1}{\longrightarrow} g_k \; \longrightarrow \; 0 \; \text{a.e.} \) (in the free Poisson case). As in the proof of Theorem 3.3, these conditions imply that, for any natural number \( \ell \) and for any natural numbers \( k_1, \ldots, k_{2\ell} \),

\[
\varphi \left( \left[ I^n_m (f_k)^{k_1} - \varphi \left( I^n_m (f_k)^{k_1} \right) \right] \left[ I^n_m (g_k)^{k_2} - \varphi \left( I^n_m (g_k)^{k_2} \right) \right] \right) \ldots \left[ I^n_m (f_k)^{k_{2\ell-1}} - \varphi \left( I^n_m (f_k)^{k_{2\ell-1}} \right) \right] \left[ I^n_m (g_k)^{k_{2\ell}} - \varphi \left( I^n_m (g_k)^{k_{2\ell}} \right) \right] \right) \; \longrightarrow \; 0, \\
\text{which implies that } F \text{ and } G \text{ are free as they are determined by their moments}. \]

**Remark 4.4.** Observe that the only difference between the proofs of Theorem 3.3 and Theorem 4.3 is the fact that in the non-asymptotic case, we have one additional step which states that the seemingly weaker condition \( f \overset{1}{\longrightarrow} g = 0 \) a.e. implies that, for all \( p = 1, \ldots, n \land m \), \( f \overset{p}{\longrightarrow} g = 0 \) a.e. (in the Wigner case) and that the condition \( f \overset{p}{\longrightarrow} g = 0 \) a.e. implies that, for all \( p = 1, \ldots, n \land m \), \( f \overset{p}{\longrightarrow} g = 0 \) and \( f \overset{p-1}{\longrightarrow} g = 0 \) a.e. (in the free Poisson case). Recall that these implications do not necessarily hold true asymptotically, as pointed out in [16, Remark 3.2]. For instance, the sequence \( \{ f_k : n \geq 1 \} \subset L^2([0,1]^2) \) given by

\[
f_k = \sqrt{\kappa} \sum_{i=0}^{k-1} \mathds{1}_{[\frac{i}{k}, \frac{i+1}{k}]}^2
\]
satisfies \( f_k \overset{1}{\longrightarrow} f_k \; \longrightarrow \; 0 \; \text{a.e.} \), although \( f_k \overset{2}{\longrightarrow} f_k = 1 \) for all \( k \). As we directly assume the asymptotic equivalent of the conclusions of these implications, the same arguments as in the proof of Theorem 3.3 yield the desired conclusion in the proof of Theorem 4.3.

As before with Theorem 3.6, we can give sufficient conditions for the asymptotic freeness of \( F \) and \( G \) whenever the sequences of multiple integrals have mirror-symmetric kernels instead of symmetric ones.

**Theorem 4.5.** Let \( n, m \) be natural numbers and let \( \{ f_k : k \geq 0 \} \subset L^2(\mathbb{R}_+^n) \) and \( \{ g_k : k \geq 0 \} \subset L^2(\mathbb{R}_+^m) \) be sequences of mirror-symmetric functions. Assume that, as \( k \rightarrow \infty \), \( I^n_m (f_k) \overset{\text{law}}{\longrightarrow} U \) and \( I^m_n (g_k) \overset{\text{law}}{\longrightarrow} V \).

1. If dealing with Wigner integrals, assume that, for all \( p = 1, \ldots, n \land m \) and all \( \sigma \in \mathcal{S}_n \) and \( \pi \in \mathcal{S}_m \), \( f_k^{(\sigma)} \overset{p}{\longrightarrow} g_k^{(\pi)} = 0 \) almost everywhere, where \( f_k^{(\sigma)} \) and \( g_k^{(\pi)} \) are defined as in Theorem 3.6. Then, as \( k \rightarrow \infty \),

\[
\left( I^n_m (f_k), I^m_n (g_k) \right) \overset{\text{law}}{\rightarrow} (U, V),
\]

with \( U \) and \( V \) free.

2. If dealing with free Poisson integrals, assume that, for all \( p = 1, \ldots, n \land m \) and all \( \sigma \in \mathcal{S}_n \) and \( \pi \in \mathcal{S}_m \), \( f_k^{(\sigma)} \overset{p}{\longrightarrow} g_k^{(\pi)} = 0 \) almost everywhere and \( f_k^{(\sigma)} \overset{p-1}{\longrightarrow} g_k^{(\pi)} = 0 \) almost everywhere. Then, as \( k \rightarrow \infty \),

\[
\left( I^n_m (f_k), I^m_n (g_k) \right) \overset{\text{law}}{\rightarrow} (U, V),
\]

with \( U \) and \( V \) free.
almost everywhere. Then, as $k \to \infty$,
\[
\left( R_n^N(f), R_m^N(g) \right) \xrightarrow{\text{law}} (U, V),
\]
with $U$ and $V$ free.

**Proof.** Using the exact same argument as in the proof of Theorem 3.3, we can obtain that, for any natural number $\ell$ and for any natural numbers $p_1, \ldots, p_{2\ell}$,
\[
\varphi \left( [I_n^{3n}(f)]^{p_1} - \varphi \left( [I_m^{3m}(g)]^{2p_2} - \varphi \left( [I_m^{3m}(g)]^{p_2} \right) \right) \right) \\
\cdots \left( [I_n^{3n}(f)]^{p_{2\ell - 1}} - \varphi \left( [I_m^{3m}(g)]^{2p_{2\ell}} - \varphi \left( [I_m^{3m}(g)]^{p_{2\ell}} \right) \right) \right) 
\to_{k \to +\infty} 0.
\]

We can hence deduce that $I_n^{3n}(f)$ and $I_m^{3m}(g)$ are free in the limit, so that the limits of their joint moments are determined by the moments of $U$ and $V$ separately, according to the same rule as if $U$ and $V$ were supposed free. This concludes the proof. \hfill \square

An interesting consequence of Theorem 4.5 is the following result.

**Theorem 4.6.** Let $n, m$ be natural numbers such that $n \geq m \geq 2$, $\{f_k : k \geq 0\} \subset L^2(\mathbb{R}_+^n)$ be sequences of symmetric functions, and $\{g_k : k \geq 0\} \subset L^2(\mathbb{R}_+^m)$ be sequences of mirror-symmetric functions. Set $F_k = I_n^{3n}(f_k)$ and $G_k = I_m^{3m}(g_k)$. Suppose that (note that the third condition is automatically satisfied if $n \neq m$)
\[
\varphi \left( F_k^2 \right) \to_{k \to +\infty} 1, \quad \varphi \left( G_k^2 \right) \to_{k \to +\infty} 1, \text{ and } \varphi \left( F_k G_k \right) \to_{k \to +\infty} 0.
\]

Only in the case where $\mathfrak{M} = \hat{N}$, assume additionally that the sequence of norms $\left\{ \|g_k\|_{L^2(\mathbb{R}_+^n)} : k \geq 0 \right\}$ is uniformly bounded by some constant from a certain rank up. Let $S$ denote a $S(0, 1)$ random variable and let $U$ be any self-adjoint random variable. Assume moreover that $U$ and $S$ are free. Then, as $k \to \infty$, the following two conditions are equivalent:

(i) $(F_k, G_k) \xrightarrow{\text{law}} (S, U)$;
(ii) $F_k \xrightarrow{\text{law}} S$ and $G_k \xrightarrow{\text{law}} U$.

**Proof.** We only need to prove (ii) $\Rightarrow$ (i), so assume that (ii) holds. In order to prove that (i) holds, we are going to prove that the hypotheses of Theorem 4.5 are satisfied, from which the desired conclusion will follow. For any $p = 1, \ldots, (n - 1) \land m$, any $\sigma \in \mathcal{S}_n$ and any $\pi \in \mathcal{S}_m$, we have
\[
\left\| f_k^\sigma \overset{p}{\sim} g_k^{\pi} \right\|_{L^2(\mathbb{R}_+^{n+m-2p})}^2 = \left\| f_k^\sigma \overset{p}{\sim} g_k^{\pi} \right\|_{L^2(\mathbb{R}_+^{n+m-2p})}^2
\]
as $f_k$ is symmetric. Using a Fubini argument, we can then write
\[
\left\| f_k^\sigma \overset{p}{\sim} g_k^{\pi} \right\|_{L^2(\mathbb{R}_+^{n+m-2p})}^2 = \left\langle f_k^{n-p} f_k, g_k^{m-p} (g_k^{\pi})^* \right\rangle_{L^2(\mathbb{R}_+^m)}
\]
which by the Cauchy-Schwarz inequality yields
\[
\left\| f_k^\sigma \overset{p}{\sim} g_k^{\pi} \right\|_{L^2(\mathbb{R}_+^{n+m-2p})}^2 \leq \left\| f_k \overset{p}{\sim} f_k \right\|_{L^2(\mathbb{R}_+^{n-2p})} \left\| g_k^{\pi} \right\|_{L^2(\mathbb{R}_+^m)} \left\| (g_k^{\pi})^* \right\|_{L^2(\mathbb{R}_+^m)}
\]
\[
= \left\| f_k \overset{p}{\sim} f_k \right\|_{L^2(\mathbb{R}_+^{n-2p})} \left\| g_k^{\pi} \right\|_{L^2(\mathbb{R}_+^m)}^2 \to_{k \to +\infty} 0,
\]

where the convergence in the last line is guaranteed by the fact that $F_k$ converges in distribution to a semicircular element and by applying the free Fourth Moment Theorem 1.3 (see [9, Theorem 1.6] for the exhaustive list of equivalent conditions). This allows us to conclude whenever $m \leq n - 1$. In the case where $n = m$, we have (the first equality being due to the symmetry of $f_k$)

$$f_k^{(\sigma)} = (f_k, g_k)_{L^2(\mathbb{R}_+)} = \varphi (F_k G_k).$$

Consequently, we have

$$f_k^{(\sigma)} \overset{\text{w}}{\rightarrow} 0$$

in this case as well, so that Theorem 4.5 yields the conclusion in the case where $\mathcal{M} = S$.

If $\mathcal{M} = \tilde{N}$, then in addition to the previous conclusion, we also have, for any $p = 1, \ldots, n \wedge m$, any $\sigma \in \mathcal{S}_n$, and any $\pi \in \mathcal{S}_m$,

$$\left\| f_k^{(\sigma)} *_{p}^{-1} \pi \right\|_{L^2(\mathbb{R}_+^{n+m-2p+1})} = \left\| f_k *_{p}^{-1} \pi \right\|_{L^2(\mathbb{R}_+^{n+m-2p+1})}$$

as $f_k$ is symmetric. Using a Fubini argument, we can then write

$$\left\| f_k^{(\sigma)} *_{p}^{-1} \pi \right\|_{L^2(\mathbb{R}_+^{n+m-2p})} = \left\langle f_k *_{p}^{-1} \pi \right\rangle_{L^2(\mathbb{R}_+^{n+m-2p+1})},$$

which by the Cauchy-Schwarz inequality yields

$$\left\| f_k^{(\sigma)} *_{p}^{-1} \pi \right\|_{L^2(\mathbb{R}_+^{n+m-2p})} \leq \left\| f_k *_{p}^{-1} \pi \right\|_{L^2(\mathbb{R}_+^{n+m-2p})} \left\| g_k^{(\pi)} \right\|_{L^4(\mathbb{R}_+^m)} \left\| \left( g_k^{(\pi)} \right)^* \right\|_{L^4(\mathbb{R}_+^m)}.$$

Using the additional hypothesis that the sequence of norms $\left\{ \left\| g_k \right\|_{L^4(\mathbb{R}_+^m)} : k \geq 0 \right\}$ is uniformly bounded by some constant from a certain rank up, we can conclude that

$$\left\| f_k *_{p}^{-1} \pi \right\|_{L^2(\mathbb{R}_+^{n+m-2p})} \overset{k \to +\infty}{\longrightarrow} 0$$

by using the fact that $F_k$ converges to a semicircular element and applying the free Poisson Fourth Moment Theorem 6.1 (see [6, Theorem 4.3] for an exhaustive list of equivalent conditions).

4.2. **Characterization in terms of covariances.** Based on Theorem 4.1, Nourdin and Rosinski obtained the following result that links component-wise convergence and joint convergence of multiple integrals (see [16, Corollary 3.6]). As before, note that in the following results, the random variables $F$ and $G$ need not have the form of multiple integrals. This implies that sequences of multiple integrals can be used in order to prove the freeness of general random variables in $L^2 (\varphi)$ (provided these random variables admit approximating sequences of multiple integrals).

**Theorem 4.7.** Let $n, m$ be natural numbers and let $\{f_k : k \geq 1\} \subset L^2 (\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2 (\mathbb{R}_+^m)$ be sequences of symmetric functions such that $\overset{\text{law}}{I_n^{W} (f_k) \to} F$ and $\overset{\text{law}}{I_m^{W} (g_k) \to} G$ as $k \to \infty$, where $F, G$ are square integrable random variables with laws determined by their moments. Furthermore, if

$$\text{Cov} \left( I_n^{W} (f_k)^2, I_m^{W} (g_k)^2 \right) \overset{k \to +\infty}{\longrightarrow} 0,$$
then \( (I_n^W(f_k), I_m^W(g_k)) \overset{\text{law}}{\to} (F, G) \), as \( k \to \infty \), with \( F \) and \( G \) independent.

In the free case, we obtain the following similar result.

**Theorem 4.8.** Let \( n, m \) be natural numbers and let \( \{f_k: k \geq 1\} \subset L^2(\mathbb{R}_+^n) \) and 
\( \{g_k: k \geq 1\} \subset L^2(\mathbb{R}_+^m) \) be sequences of symmetric functions such that \( I_n^m(f_k) \overset{\text{law}}{\to} F \) 
and \( I_m^m(g_k) \overset{\text{law}}{\to} G \) as \( k \to \infty \), where \( F, G \) are random variables in \( L^2(\mathcal{A}, \varphi) \). Then, 
\( (I_n^m(f_k), I_m^m(g_k)) \overset{\text{law}}{\to} (F, G) \), as \( k \to \infty \), with \( F \) and \( G \) free if and only if 
\[ \text{Cov} \left( I_n^m(f_k)^2, I_m^m(g_k)^2 \right) \overset{k \to +\infty}{\longrightarrow} 0. \]

**Proof.** First, observe that (8) implies that the statement 
\[ \text{Cov} \left( I_n^m(f_k)^2, I_m^m(g_k)^2 \right) \overset{k \to +\infty}{\longrightarrow} 0 \]
is equivalent to the fact that, for all \( p = 1, \ldots, n \wedge m, f_k \overset{p}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \ a.e. \) 
(in the Wigner case) and that, for all \( p = 1, \ldots, n \wedge m, f_k \overset{p}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \) and 
f_k \overset{p-1}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \ a.e. \) (in the free Poisson case).

Assume that \( (I_n^m(f_k), I_m^m(g_k)) \overset{\text{law}}{\to} (F, G) \), as \( k \to \infty \), with \( F \) and \( G \) free. The free 
ness of \( F \) and \( G \) along with Theorem 4.3 ensures that, for all \( p = 1, \ldots, n \wedge m, f_k \overset{p}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \ a.e. \) (in the Wigner case) and that, for all \( p = 1, \ldots, n \wedge m, f_k \overset{p}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \) and 
f_k \overset{p-1}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \ a.e. \) (in the free Poisson case), which is 
the desired conclusion.

Conversely, assume that, for all \( p = 1, \ldots, n \wedge m, f_k \overset{p}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \ a.e. \) (in 
the Wigner case) and that, for all \( p = 1, \ldots, n \wedge m, f_k \overset{p}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \) and 
f_k \overset{p-1}{\not \sim} g_k \overset{k \to +\infty}{\longrightarrow} 0 \ a.e. \) (in the free Poisson case). Then, by Theorem 4.3 again (but 
using the converse implication this time), we obtain that \( F \) and \( G \) are free. Using 
[11, Lemma 5.13], this implies in particular that the joint moments of \( (F, G) \) are 
completely determined by the knowledge of the individual moments of \( F \) and \( G \) 
([11, Examples 5.15] provide a technique that allows one to compute mixed (joint) 
moments using the individual moments and the freeness condition). As \( I_n^m(f_k) \overset{\text{law}}{\to} F \) 
and \( I_m^m(g_k) \overset{\text{law}}{\to} G \) as \( k \to \infty \), we deduce that \( (I_n^m(f_k), I_m^m(g_k)) \overset{\text{law}}{\to} (F, G) \). 

\[ \square \]

4.3. **Characterization in terms of free Malliavin gradients.** It is also possible 
to characterize asymptotic freeness in terms of the free gradient quantity appearing 
in Theorem 3.10. We offer the following statement.

**Theorem 4.9.** Let \( n, m \) be natural numbers and let \( \{f_k: k \geq 1\} \subset L^2(\mathbb{R}_+^n) \) and 
\( \{g_k: k \geq 1\} \subset L^2(\mathbb{R}_+^m) \) be sequences of symmetric functions such that \( I_n^m(f_k) \overset{\text{law}}{\to} F \) 
and \( I_m^m(g_k) \overset{\text{law}}{\to} G \) as \( k \to \infty \), where \( F, G \) are random variables in \( L^2(\mathcal{A}, \varphi) \). Then, 
\( F \) and \( G \) are free if and only if 
\[ \left\langle \nabla I_n^m(f_k), \nabla I_m^m(g_k) \right\rangle \overset{k \to +\infty}{\longrightarrow} 0 \ in L^2(\mathcal{A} \otimes \mathcal{A}, \varphi \otimes \varphi), \]
where the notation \( \langle \cdot, \cdot \rangle \) is defined in (7).

**Proof.** Combine the representation (12) with Theorem 4.3. 
\[ \square \]
5. Transfer principles

Since the characterizations of freeness we have obtained in Section 3 involve quantities which are similar whatever the context (classical or free, Brownian or Poisson), it is natural to study possible transfer principles from one setting to another one. It is the goal of this section to study these aspects.

**Theorem 5.1.** Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}_+^n) \) and \( g \in L^2(\mathbb{R}_+^m) \) be symmetric functions. Assume that \( I_n^N(f) \) and \( I_m^N(g) \) are free. Then, \( I_n^S(f) \) and \( I_m^S(g) \) are free. However, the fact that \( I_n^S(f) \) and \( I_m^S(g) \) are free does not necessarily imply that \( I_n^N(f) \) and \( I_m^N(g) \) are free, as illustrated by Example 5.2.

**Proof.** By Theorem 3.3, if \( I_n^N(f) \) and \( I_m^N(g) \) are free, then it holds that \( f \ast_1^0 g = 0 \) a.e. Lemma 7.2 guaranties that \( f \ast_1^0 g = 0 \) a.e. implies \( f \perp g = 0 \) a.e. Using Theorem 3.3 again concludes the proof. \( \square \)

**Example 5.2.** Let \( T \) be a positive real number and let \( f, g \in L^2(\mathbb{R}_+) \) be functions defined by

\[
f(x) = x \mathbb{1}_{[0, T]}(x) \quad \text{and} \quad g(x) = \left( x^2 - \frac{3T}{4} x \right) \mathbb{1}_{[0, T]}(x).
\]

Note that

\[
f \perp g = \langle f, g \rangle_{L^2(\mathbb{R}_+)} = \int_0^T x \left( x^2 - \frac{3T}{4} x \right) dx = \int_0^T \left( x^3 - \frac{3T}{4} x^2 \right) dx = 0
\]

whereas

\[
f \ast_1^0 g(x) = f(x) \cdot g(x) = \left( x^3 - \frac{3T}{4} x^2 \right) \mathbb{1}_{[0, T]}(x) \neq 0.
\]

Hence, by Theorem 3.3, \( I_n^S(f) \) and \( I_m^S(g) \) are free but \( I_n^N(f) \) and \( I_m^N(g) \) are not free.

Based on Theorem 3.1 and Theorem 3.3, we can obtain the following transfer principles between the Wiener and Wigner chaos.

**Proposition 5.3.** Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}_+^n) \) and \( g \in L^2(\mathbb{R}_+^m) \) be symmetric functions. It holds that \( I_n^W(f) \) and \( I_m^W(g) \) are free if and only if \( I_n^W(f) \) and \( I_m^W(g) \) are independent.

**Proof.** Observe that as \( f \) and \( g \) are symmetric functions, it holds that \( f \otimes_1 g = f \perp g \). Using Theorem 3.1 and Theorem 3.3 concludes the proof. \( \square \)

**Remark 5.4.** 1. As pointed out in [19, Example 5.3], the fact that their squares are uncorrelated does not imply that \( I_n^N(f) \) and \( I_m^N(g) \) are independent and because of that, no characterization of independence for multiple Poisson integrals is available within the contraction framework that we are using here (for more precision, see Remark 2). This makes it difficult to establish any independence correspondence or transfer principles between the classical and free Poisson chaos. However, it can be pointed out that the freeness of free Poisson multiple integrals implies the freeness of the corresponding Wigner integrals and the independence of the corresponding Wiener integrals.
2. In the classical Poisson case, there is no known characterization of independence in terms of the almost sure nullity of a contraction. By using similar techniques as the ones used in the proof of Theorem 3.3 (using the definition of moment independence in place of the definition of freeness), one can prove that the condition \( f \ast_0^1 g = 0 \text{ a.e.} \) implies moment independence. However, moment independence only implies \( f \ast_1^0 g = 0 \text{ a.e.} \), which is weaker than \( f \ast_0^1 g = 0 \text{ a.e.} \). Summing up, one can prove that the condition \( f \ast_0^1 g = 0 \text{ a.e.} \) is sufficient but not necessary and that the condition \( f \ast_1^0 g = 0 \text{ a.e.} \) is necessary but not sufficient (the fact that it is not sufficient is illustrated by the counterexample provided in [19, Example 5.3]). Also pointed out in [19, Example 5.3] is the fact that the squares of multiple Poisson integrals being uncorrelated does not imply that these multiple integrals are independent. This makes it difficult to establish any independence correspondence or transfer principles between the classical and free Poisson chaos. However, it can be pointed out that the freeness of free Poisson multiple integrals implies the freeness of the corresponding Wigner integrals and the independence of the corresponding Wiener integrals.

Despite the second point of the above remark, we can still provide the following partial transfer result.

**Proposition 5.5.** Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}^+_{\mathbb{R}}) \) and \( g \in L^2(\mathbb{R}^m_+) \) be symmetric functions. It holds that \( I_n^N(f) \) and \( I_n^N(g) \) are free if and only if \( I_n^R(f) \) and \( I_n^R(g) \) are moment independent and \( f \ast_0^1 g = 0 \text{ a.e.} \).

**Proof.** Assuming that \( I_n^R(f) \) and \( I_n^R(g) \) are free, Theorem 3.3 states that \( f \ast_0^1 g = 0 \text{ a.e.} \), which, as pointed out in Remark 2, is a sufficient condition for \( I_n^R(f) \) and \( I_n^R(g) \) to be moment independent. Conversely, if it holds that \( I_n^R(f) \) and \( I_n^R(g) \) are moment independent and \( f \ast_1^0 g = 0 \text{ a.e.} \), Theorem 3.3 ensures that \( I_n^N(f) \) and \( I_n^N(g) \) are free. □

If we want to drop the condition on \( f \ast_1^0 g \), we can obtain the following implication.

**Corollary 5.6.** Let \( n, m \) be natural numbers and let \( f \in L^2(\mathbb{R}^+_{\mathbb{R}}) \) and \( g \in L^2(\mathbb{R}^m_+) \) be symmetric functions. Assume that \( I_n^N(f) \) and \( I_n^N(g) \) are free. Then, \( I_n^R(f) \) and \( I_n^R(g) \) are moment independent.

6. Fourth moment theorems with respect to free random measures

Following the ideas of [16, Section 4], we will be able to provide an alternate proof of the free fourth moment theorems proved by Kemp, Nourdin, Peccati and Speicher in [9], by Bourguin and Peccati in [6] and also to generalize the free multidimensional fourth moment theorems obtained by Nourdin, Peccati and Speicher in [14] and by Bourguin in [4].

**Theorem 6.1** (Kemp et. al [9], 2012 – Bourguin and Peccati [6], 2014). Let \( n \geq 1 \) be an integer and \( \{f_k: k \geq 1\} \) be a sequence of symmetric functions in \( L^2(\mathbb{R}^+_{\mathbb{R}}) \) such that, for each \( k \geq 1 \), \( \|f_k\|^2_{L^2(\mathbb{R}^+_{\mathbb{R}})} = 1 \). Then, as \( k \to \infty \), the two following conditions are equivalent:

(i) \( \varphi \left( I_n^{2n}(f_k)^4 \right) \to 2; \)

(ii) \( \|f_k\|^4_{L^4(\mathbb{R}^+_{\mathbb{R}})} \to 2. \)
(ii) $I^n_{2k}(f_k) \xrightarrow{\text{law}} \mathcal{S}(0, 1)$.

Proof. As free convergence in distribution is equivalent to convergence of moments, assuming that $I^n_{2k}(f_k) \xrightarrow{\text{law}} \mathcal{S}(0, 1)$ immediately implies that $\varphi \left( I^n_{2k}(f_k)^4 \right) \to 2$. Conversely, assume that $\varphi \left( I^n_{2k}(f_k)^4 \right) \to 2$. Let $\{g_k : k \geq 1\}$ be a sequence of symmetric functions in $L^2(\mathbb{R}_n^n)$ such that $I^n_{2k}(g_k)$ is a free copy of $I^n_{2k}(f_k)$. Since the sequences $\{I^n_{2k}(f_k) : k \geq 1\}$ and $\{I^n_{2k}(g_k) : k \geq 1\}$ are bounded in $L^2(\varphi)$ by assumption, they are relatively compact, so we can assume that $I^n_{2k}(f_k) \xrightarrow{\text{law}} F$ and $I^n_{2k}(g_k) \xrightarrow{\text{law}} G$ where $G$ is a free copy of $F$. It remains to prove that $F$ and $G$ are distributed according to the $\mathcal{S}(0, 1)$ distribution. Observe that

\[
\text{Cov} \left( (I^n_{2k}(f_k) + I^n_{2k}(g_k))^2, (I^n_{2k}(f_k) - I^n_{2k}(g_k))^2 \right) \\
= \varphi \left( I^n_{2k}(f_k)^4 - I^n_{2k}(f_k)^3 I^n_{2k}(g_k) - I^n_{2k}(f_k)^2 I^n_{2k}(g_k) I^n_{2k}(f_k) \right) \\
+ I^n_{2k}(f_k)^2 I^n_{2k}(g_k)^2 + I^n_{2k}(f_k) I^n_{2k}(g_k) I^n_{2k}(f_k) \\
- I^n_{2k}(f_k) I^n_{2k}(g_k) I^n_{2k}(f_k) I^n_{2k}(g_k) - I^n_{2k}(f_k) I^n_{2k}(g_k)^2 I^n_{2k}(f_k) \\
+ I^n_{2k}(f_k) I^n_{2k}(g_k)^3 + I^n_{2k}(g_k) I^n_{2k}(f_k)^3 - I^n_{2k}(g_k) I^n_{2k}(f_k)^2 I^n_{2k}(g_k) \\
- I^n_{2k}(g_k) I^n_{2k}(f_k) I^n_{2k}(g_k) I^n_{2k}(f_k) + I^n_{2k}(g_k) I^n_{2k}(f_k) I^n_{2k}(g_k)^2 \\
+ I^n_{2k}(g_k)^2 I^n_{2k}(f_k)^2 - I^n_{2k}(g_k)^2 I^n_{2k}(f_k) I^n_{2k}(g_k) - I^n_{2k}(g_k)^3 I^n_{2k}(f_k) \\
+ I^n_{2k}(g_k)^4)
\]

Using the tracial property of $\varphi$, the fact that multiple integrals are centered together with the fact that, for each $k \geq 1$, $I^n_{2k}(f_k)$ and $I^n_{2k}(g_k)$ are normalized, free and have the same law (and hence the same moments) yields

\[
\text{Cov} \left( (I^n_{2k}(f_k) + I^n_{2k}(g_k))^2, (I^n_{2k}(f_k) - I^n_{2k}(g_k))^2 \right) = 2 \left( \varphi \left( I^n_{2k}(f_k)^4 \right) - 2 \right).
\]

As $\varphi \left( I^n_{2k}(f_k)^4 \right) \to 2$ as $k \to \infty$, we get that

\[
\text{Cov} \left( (I^n_{2k}(f_k) + I^n_{2k}(g_k))^2, (I^n_{2k}(f_k) - I^n_{2k}(g_k))^2 \right) \to 0
\]
as $k \to \infty$. By Theorem 4.8, we get that, as $k \to \infty$,

\[
(I^n_{2k}(f_k) + I^n_{2k}(g_k), I^n_{2k}(f_k) - I^n_{2k}(g_k)) \to (F + G, F - G)
\]
with $F + G$ free of $F - G$. Using the free Bernstein Theorem (see e.g. [10]) ensures that both $F$ and $G$ are distributed according to the $\mathcal{S}(0, 1)$ distribution, which concludes the proof.

In the multivariate case, Nourdin, Peccati and Speicher obtained the following result in [14] in the Wigner case, whereas the free Poisson counterpart was obtained in [4].
Theorem 6.2 (Nourdin, Peccati and Speicher [14], 2013 – Bourguin [4], 2016). Let \( d \geq 2 \) and \( n_1, \ldots, n_d \) be integers, and consider a positive definite symmetric matrix \( c = \{c(i, j) : i, j = 1, \ldots, d\} \). Let \( s = (s_1, \ldots, s_d) \) be a semicircular family with covariance \( c \). Consider a sequence of random vectors

\[
\{F_k = \left( I_{n_1}^{\mathbb{R}} \left( f_k^{(1)} \right), \ldots, I_{n_d}^{\mathbb{R}} \left( f_k^{(d)} \right) \right) : k \geq 1 \}
\]

where, for each \( i = 1, \ldots, d \), \( \{f_k^{(i)} : k \geq 1\} \) is a sequence of mirror-symmetric functions in \( L^2(\mathbb{R}_+^d) \) such that, for all \( i, j = 1, \ldots, d \),

\[
\varphi \left[ I_{n_i}^{\mathbb{R}} \left( f_k^{(i)} \right) I_{n_j}^{\mathbb{R}} \left( f_k^{(j)} \right) \right] \xrightarrow[k \to \infty]{} c(i, j).
\]

Then, the following three assertions are equivalent, as \( k \to \infty \).

1. The vector \( \left( I_{n_1}^{\mathbb{R}} \left( f_k^{(1)} \right), \ldots, I_{n_d}^{\mathbb{R}} \left( f_k^{(d)} \right) \right) \) converges in distribution to \( (s_1, \ldots, s_d) \).
2. For every \( i = 1, \ldots, d \), the random variable \( I_{n_i}^{\mathbb{R}} \left( f_k^{(i)} \right) \) converges to \( s_i \).
3. For every \( i = 1, \ldots, d \), \( \varphi \left[ I_{n_i}^{\mathbb{R}} \left( f_k^{(i)} \right)^4 \right] \xrightarrow[k \to \infty]{} 2c(i, i)^2 = \varphi \left[ s_i^4 \right] \).

The upcoming result adds an equivalent condition to their result in terms of the fourth moment of the Euclidean norm of the involved random vectors when the kernels of the multiple integrals are fully symmetric.

Theorem 6.3. Let \( d \geq 2 \) and \( n_1, \ldots, n_d \) be integers, and consider a positive definite symmetric matrix \( c = \{c(i, j) : i, j = 1, \ldots, d\} \). Let \( s = (s_1, \ldots, s_d) \) be a semicircular family with covariance \( c \). Consider a sequence of random vectors

\[
\{F_k = \left( I_{n_1}^{\mathbb{R}} \left( f_k^{(1)} \right), \ldots, I_{n_d}^{\mathbb{R}} \left( f_k^{(d)} \right) \right) : k \geq 1 \}
\]

where, for each \( i = 1, \ldots, d \), \( \{f_k^{(i)} : k \geq 1\} \) is a sequence of symmetric functions in \( L^2(\mathbb{R}_+^d) \) such that, for all \( i, j = 1, \ldots, d \),

\[
\varphi \left[ I_{n_i}^{\mathbb{R}} \left( f_k^{(i)} \right) I_{n_j}^{\mathbb{R}} \left( f_k^{(j)} \right) \right] \xrightarrow[k \to \infty]{} c(i, j).
\]

Then, the following two assertions are equivalent, as \( k \to \infty \).

1. The vector \( \left( I_{n_1}^{\mathbb{R}} \left( f_k^{(1)} \right), \ldots, I_{n_d}^{\mathbb{R}} \left( f_k^{(d)} \right) \right) \) converges in distribution to \( (s_1, \ldots, s_d) \).
2. \( \varphi \left( \left\| I_{n_1}^{\mathbb{R}} \left( f_k^{(1)} \right), \ldots, I_{n_d}^{\mathbb{R}} \left( f_k^{(d)} \right) \right\|^4 \right) \xrightarrow[k \to \infty]{} \varphi \left( \left\| (s_1, \ldots, s_d) \right\|^4 \right) \),

where \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^d \).

Proof. The implication \((i) \Rightarrow (ii)\) is a consequence of the continuous mapping theorem. It remains to prove that \((ii) \Rightarrow (i)\). By the same argument as in the proof of Theorem 6.1, we assume that \( F_k \xrightarrow{law} F \) and we have to show that \( F \) is a semicircular family with covariance \( c \). We are going to prove instead that \((ii)\) actually implies that, for every \( i = 1, \ldots, d \), the random variable \( I_{n_i}^{\mathbb{R}} \left( f_k^{(i)} \right) \) converges to \( s_i \), which will imply that \( F \) is a semicircular family with covariance \( c \) by Theorem 6.2 (by the fact that component-wise and joint convergence are equivalent). Let \( s' \) be a free copy of the semicircular family \( s \). By the free Bernstein
theorem (see e.g. [10]) and the fact that \( s \) and \( s' \) have the same law, the random vectors \( s + s' \) and \( s - s' \) are free as well. Using this fact along with Lemma 7.3 yields

\[
\frac{1}{2} \text{Cov} \left( \|s + s'\|^2, \|s + s'\|^2 \right) = \varphi \left( \|s\|^4 \right) - \varphi \left( \|s\|^2 \right)^2 - \sum_{i,j=1}^{d} \text{Cov} \left( s_i, s_j \right)^2 = 0,
\]

so that

\[
\varphi \left( \|s\|^4 \right) = \varphi \left( \|s\|^2 \right)^2 + \sum_{i,j=1}^{d} \text{Cov} \left( s_i, s_j \right)^2 = \sum_{i,j=1}^{d} \left( c(i, i)c(j, j) + c(i, j)^2 \right).
\]

For every \( k \geq 1 \), take \( G_k \) to be a free copy of \( F_k \) converging in law to a limit \( G \), free from \( F \). Using Lemma 7.3 again for \( F_k \) and \( G_k \) yields

\[
\frac{1}{2} \text{Cov} \left( \|F_k + G_k\|^2, \|F_k - G_k\|^2 \right) = \varphi \left( \|F_k\|^4 \right) - \varphi \left( \|F_k\|^2 \right)^2 - \sum_{i,j=1}^{d} \text{Cov} \left( F_{k, i}, F_{k, j} \right)^2,
\]

so that, using (13),

\[
\frac{1}{2} \text{Cov} \left( \|F_k + G_k\|^2, \|F_k - G_k\|^2 \right) = \varphi \left( \|F_k\|^4 \right) - \varphi \left( \|F_k\|^2 \right)^2 + \sum_{i,j=1}^{d} \left( c(i, i)c(j, j) - \varphi (F_{k, i}) \varphi (F_{k, j}) + c(i, j)^2 - \text{Cov} \left( F_{k, i}, F_{k, j} \right)^2 \right).
\]

Using the assumptions, we get that

\[
\text{Cov} \left( \|F_k + G_k\|^2, \|F_k - G_k\|^2 \right) \xrightarrow{k \to +\infty} 0,
\]

which by Theorem 4.8, the fact that

\[
\text{Cov} \left( \|F_k + G_k\|^2, \|F_k - G_k\|^2 \right) = \sum_{i,j=1}^{d} \text{Cov} \left( (F_{k, i} + G_{k, i})^2, (F_{k, j} - G_{k, j})^2 \right)
\]

\[
\geq \text{Cov} \left( (F_{k, i} + G_{k, i})^2, (F_{k, j} - G_{k, j})^2 \right)
\]

and the fact that, by (8), \( \text{Cov} \left( (F_{k, i} + G_{k, i})^2, (F_{k, j} - G_{k, j})^2 \right) \geq 0 \) implies that \( F + G \) and \( F - G \) are free. Recalling that \( F \) and \( G \) are free as well, using the free Bernstein theorem on each coordinates of these vectors implies that, for each \( i = 1, \ldots, d \), \( F_i \text{ law } s_i \), which concludes the proof. \( \square \)

7. Auxiliary results

This last section contains auxiliary results that have been used throughout the text. The first two lemmas, 7.1 and 7.2, have been indeed used along the proof of Theorem 3.3, whereas Lemma 7.3 contains a crucial formula for the proof of Theorem 6.2.

**Lemma 7.1.** Let \( n, m \) be natural numbers and let \( f \in L^2 \left( \mathbb{R}^n_+ \right) \) and \( g \in L^2 \left( \mathbb{R}^m_+ \right) \) be mirror-symmetric functions. Assume furthermore that \( f \perp g = 0 \) almost everywhere. Then, for all \( p = 1, \ldots, n \wedge m \), it holds that \( f \overset{p}{\perp} g = 0 \) almost everywhere.
Proof. Observe that, for any \( p = 1, \ldots, n \wedge m \),
\[
\begin{align*}
f \overset{p}{\sim} g (t_1, \ldots, t_{n+m-2p}) &= \int_{\mathbb{R}_+^n} f (t_1, \ldots, t_{n-p}, s_p, \ldots, s_1) g (s_1, \ldots, s_p, t_{n-p+1}, \ldots, t_{n+m-2p}) \, ds_1 \cdots ds_p \\
&= \int_{\mathbb{R}_+^{p-1}} \left( \int_{\mathbb{R}_+} f (t_1, \ldots, t_{n-p}, s_p, \ldots, s_1) g (s_1, \ldots, s_p, t_{n-p+1}, \ldots, t_{n+m-2p}) \, ds_1 \right) ds_2 \cdots ds_{p-1} \\
&= \int_{\mathbb{R}_+^{p-1}} f \overset{1}{\sim} g (t_1, \ldots, t_{n-p}, s_p, \ldots, s_2, s_1, s_2, \ldots, s_p, t_{n-p+1}, \ldots, t_{n+m-2p}) \, ds_2 \cdots ds_{p-1}.
\end{align*}
\]
Using the assumption that \( f \overset{1}{\sim} g = 0 \) a.e., we get \( f \overset{p}{\sim} g = 0 \) a.e., which concludes the proof. \( \Box \)

**Lemma 7.2.** Let \( n, m \) be natural numbers and let \( f \in L^2 (\mathbb{R}_+^n) \) and \( g \in L^2 (\mathbb{R}_+^m) \) be mirror-symmetric functions. Assume furthermore that \( f \overset{0}{\ast} g = 0 \) almost everywhere. Then, for all \( p = 1, \ldots, n \wedge m \) and all \( r = 2, \ldots, n \wedge m \), it holds that \( f \overset{p}{\sim} g = 0 \) and \( f \overset{r-1}{\ast} g = 0 \) almost everywhere.

Proof. Observe that, for any \( p = 1, \ldots, n \wedge m \),
\[
\begin{align*}
f \overset{p}{\sim} g (t_1, \ldots, t_{n+m-2p}) &= \int_{\mathbb{R}_+^n} f (t_1, \ldots, t_{n-p}, s_p, \ldots, s_1) g (s_1, \ldots, s_p, t_{n-p+1}, \ldots, t_{n+m-2p}) \, ds_1 \cdots ds_p \\
&= \int_{\mathbb{R}_+^{p-1}} f \overset{0}{\ast} g (t_1, \ldots, t_{n-p}, s_p, \ldots, s_1, s_2, \ldots, s_p, t_{n-p+1}, \ldots, t_{n+m-2p}) \, ds_1 \cdots ds_{p-1}.
\end{align*}
\]
Similarly, it holds that, for any \( r = 2, \ldots, n \wedge m \),
\[
\begin{align*}
f \overset{r-1}{\ast} g (t_1, \ldots, t_{n+m-2r+1}) &= \int_{\mathbb{R}_+^{n-1}} f (t_1, \ldots, t_{n-r+1}, s_{r-1}, \ldots, s_1) g (s_1, \ldots, s_{r-1}, t_{n-r+1}, \ldots, t_{n+m-2r+1}) \, ds_1 \cdots ds_{r-1} \\
&= \int_{\mathbb{R}_+^{n-1}} f \overset{0}{\ast} g (t_1, \ldots, t_{n-r+1}, s_{r-1}, \ldots, s_1, s_2, \ldots, s_{r-1}, t_{n-r+1}, \ldots, t_{n+m-2r+1}) \, ds_1 \cdots ds_{r-1}.
\end{align*}
\]
Using the assumption that \( f \overset{0}{\ast} g = 0 \) a.e. concludes the proof. \( \Box \)

**Lemma 7.3.** Assume that \( F = (F_1, \ldots, F_d) \) is a centered random vector with moments of all orders. Let \( G \) be a free copy of \( F \). Then, it holds that
\[
\frac{1}{2} \text{Cov} \left( \| F + G \|^2 , \| F - G \|^2 \right) = \varphi \left( \| F \|^4 \right) - \varphi \left( \| F \|^2 \right)^2 - \sum_{i,j=1}^{d} \text{Cov} (F_i, F_j)^2,
\]
where \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^d \).
Proof. It holds that

\[ \text{Cov} \left( \|F + G\|^2, \|F - G\|^2 \right) = \sum_{i,j=1}^{d} \left[ \varphi \left( (F_i + G_i)^2 (F_j - G_j)^2 \right) - \varphi \left( (F_i + G_i)^2 \right) \varphi \left( (F_j - G_j)^2 \right) \right]. \]

On one hand,

\[ \varphi \left( (F_i + G_i)^2 (F_j - G_j)^2 \right) = \varphi \left( F_i^2 F_j^2 \right) - \varphi \left( F_i^2 G_j F_j \right) - \varphi \left( F_i G_j G_j \right) + \varphi \left( F_i^2 G_j^2 \right) + \varphi \left( F_i G_i G_j F_j \right) - \varphi \left( G_i G_i F_j F_j \right) + \varphi \left( G_i G_j G_j \right) + \varphi \left( G_i^2 G_j^2 \right). \]

Using the fact that \( F \) and \( G \) are free along with the definition of freeness yields

\[ \varphi \left( (F_i + G_i)^2 (F_j - G_j)^2 \right) = \varphi \left( F_i^2 F_j^2 \right) + \varphi \left( F_i^2 \right) \varphi \left( G_j^2 \right) - 2 \varphi \left( F_i F_j \right) \varphi \left( G_i G_j \right) + \varphi \left( G_i^2 \right) \varphi \left( F_j^2 \right) + \varphi \left( G_i^2 G_j^2 \right). \]

Remark 7.4. It is interesting to notice at this point that in the classical case, we would have obtained the same expression but with a constant 4 instead of the constant 2. This difference can be explained by two terms for which independence and freeness do not yield the same quantity. Indeed, in the classical case, the commutativity and independence definition would have given us \( \mathbb{E} (F_i G_i F_j G_j) = \mathbb{E} (G_i F_i G_j F_j) = \mathbb{E} (F_i F_j) \mathbb{E} (G_i G_j) \), hence adding two additional covariance products to the two first ones, whereas the non-commutativity and the freeness definition yield \( \varphi \left( F_i G_i F_i F_j G_j \right) = \varphi \left( G_i F_i G_j F_j \right) = 0 \) in the free case. The rest of the terms happen to be the same in both the classical case and in the free case because of the tracial property of \( \varphi \) and the similarities between classical independence and freeness.

On the other hand, we have

\[ \varphi \left( (F_i + G_i)^2 \right) \varphi \left( (F_j - G_j)^2 \right) = \varphi \left( F_i^2 \right) \varphi \left( F_j^2 \right) + \varphi \left( G_i^2 \right) \varphi \left( G_j^2 \right), \]

so that

\[ \text{Cov} \left( \|F + G\|^2, \|F - G\|^2 \right) = \sum_{i,j=1}^{d} \left[ \varphi \left( F_i^2 F_j^2 \right) + \varphi \left( G_i^2 G_j^2 \right) - \varphi \left( F_i^2 \right) \varphi \left( F_j^2 \right) - \varphi \left( G_i^2 \right) \varphi \left( G_j^2 \right) - 2 \varphi \left( F_i F_j \right) \varphi \left( G_i G_j \right) \right] \]

\[ = 2 \varphi \left( \|F\|^4 \right) - 2 \varphi \left( \|F\|^2 \right)^2 - 2 \sum_{i,j=1}^{d} \text{Cov} \left( F_i, F_j \right)^2 \]

by using the fact that \( F \) and \( G \) have the same law. \( \square \)
References

[1] AREZMENDI, O., AND JARAMILLO, A. Convergence of the fourth moment and infinite divisibility: quantitative estimates. *Electron. Commun. Probab.* 19 (2014).

[2] BLASZCZUK, P., AND SPEICHER, R. Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory Related Fields* 112, 3 (1998), 373–409.

[3] BOURGUIN, S. Poisson convergence on the free Poisson algebra. *Bernoulli* 21, 4 (2015), 2139–2156.

[4] BOURGUIN, S. Vector-valued semicircular limits on the free Poisson chaos. *Electron. Commun. Probab.* 21, 55 (2016), 1–11.

[5] BOURGUIN, S., AND CAMPESI, S. Free quantitative fourth moment theorems on Wigner space. *To appear in Int. Math. Res. Not. IMRN* (2017+).

[6] BOURGUIN, S., AND SPEICHER, R. Semicircular limits on the free Poisson chaos: counterexamples to a transfer principle. *J. Funct. Anal.* 267, 4 (2014), 963–997.

[7] DEYA, A., AND NOURDIN, I. Convergence of Wigner integrals to the tetilla law. *ALEA Lat. Am. J. Probab. Math. Stat.* 9 (2012), 101–127.

[8] HEAI, F., AND PETZ, D. The semicircle law, free random variables and entropy, vol. 77 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[9] KEMP, T., NOURDIN, I., PECCATI, G., AND SPEICHER, R. Wigner chaos and the fourth moment. *Ann. Probab.* 40, 4 (2012), 1577–1635.

[10] NIC, A. $R$-transforms of free joint distributions and non-crossing partitions. *J. Funct. Anal.* 135, 2 (1996), 271–296.

[11] NIC, A., AND SPEICHER, R. *Lectures on the combinatorics of free probability*, vol. 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[12] NOURDIN, I., NUALART, D., AND PECCATI, G. Strong asymptotic independence on Wiener chaos. *Proc. Amer. Math. Soc.* 144, 2 (2016), 875–886.

[13] NOURDIN, I., AND PECCATI, G. Poisson approximations on the free Wigner chaos. *Ann. Probab.* 41, 4 (2013), 2709–2723.

[14] NOURDIN, I., PECCATI, G., AND SPEICHER, R. Multi-dimensional Semicircular Limits on the Free Wigner Chaos. In *Seminar on Stochastic Analysis, Random Fields and Applications VII*, R. C. Dalang, M. Dozzi, and F. Russo, Eds., no. 67 in *Progress in Probability*. Springer Basel, Jan. 2013, pp. 211–221.

[15] NOURDIN, I., AND POLY, G. Convergence in law in the second Wiener/Wigner chaos. *Electron. Commun. Probab.* 17 (2012), no. 36, 12.

[16] NOURDIN, I., AND ROSIŃSKI, J. Asymptotic independence of multiple Wiener-Itô integrals and the resulting limit laws. *Ann. Probab.* 42, 2 (2014), 497–526.

[17] NUALART, D., AND PECCATI, G. Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* 33, 1 (2005), 177–193.

[18] PECCATI, G., AND TUDOR, C. A. Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII*, vol. 1857 of *Lecture Notes in Math.* Springer, Berlin, 2005, pp. 247–262.

[19] ROSIŃSKI, J., AND SAMORODNITSKY, G. Product formula, tails and independence of multiple stable integrals. In *Advances in stochastic inequalities (Atlanta, GA, 1997)*, vol. 234 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 169–194.

[20] TAO, T. *Topics in random matrix theory*, vol. 132 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[21] USTUNEL, A. S., AND ZAKAI, M. On Independence and Conditioning On Wiener Space. *Ann. Probab.* 17, 4 (Oct. 1989), 1441–1453.

[22] VOICULESCU, D. V., DYKEMA, K. J., AND NIC, A. *Free random variables*, vol. 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.
