An extension of Calderón transfer principle to weighted spaces with some applications

Sakin Demir
Agri Ibrahim Cecen University
Faculty of Education
3rd Floor, Office C-42
04100 Ağrı, Turkey
e-mail: sakin.demir@gmail.com

Abstract
We show that Calderón’s transfer principle can be extended to the weighted spaces and we also include some applications of our results.

1 Introduction
Transfer principle plays a very important role in ergodic theory. Sometimes it is almost impossible to prove a result by only working on a dynamical system but when there is a transfer principle available for the problem we are trying to solve, it turns everything into a completely different environment.

Suppose that $X$ is a measure space which is totally $\sigma$-finite and $U^t$ is a one-parameter group of measure-preserving transformations of $X$. We will also assume that for every measurable function $f$ on $X$ the function $f(U^t x)$ is measurable in the product of $X$ with the real line. $T$ will denote an operator defined on the space of locally integrable functions on the real line with

\begin{itemize}
\item 2020 Mathematics Subject Classification: Primary 47A35, 28D05; Secondary 47A64.
\item Key words and phrases: Translation invariant operator, Weighted Inequality.
\end{itemize}
the following properties: the values of $T$ are continuous functions on the real line, $T$ is sublinear and commutes with translations, and $T$ is semilocal in the sense that there exists a positive number $\epsilon$ such that the support of $Tf$ is always contained in an $\epsilon$-neighborhood of the support of $f$.

We will associate an operator $T^\#$ on functions on $X$ with such an operator $T$ as follows:

Given a function $f$ on $X$ let

$$F(t, x) = f(U^tx).$$

If $f$ is the sum of two functions which are bounded and integrable, respectively, then $F(t, x)$ is a locally integrable function of $t$ for almost all $x$ and therefore

$$G(t, x) = T(F(t, x))$$

is a well-defined continuous function of $t$ for almost all $x$. Thus $g(x) = G(0, x)$ has a meaning and we define

$$T^\# f = g(x).$$

Recall that an operator $T$ on $L^p(X)$ is of weak type $(p, p)$ if there exists a positive constant $C$ such that

$$\mu\{x : |Tf(x)| > \lambda\} \leq \frac{C}{\lambda^p} \|f\|_p^p$$

for all $f \in L^p(X)$.

$T$ is said to be of strong type $(p, p)$ if there exists a positive constant $C$ such that

$$\|Tf\|_p \leq C\|f\|_p$$

for all $f \in L^p(X)$.

A. P. Calderón [7] proved the following results:

**Theorem 1.** Let $T_n$ be a sequence of operators as above and suppose that the operator $Sf = \sup |T_n f|$ is of strong type $(p, p)$, $1 \leq p \leq \infty$. Then the same holds for the operator $S^\# f = \sup |T^\#_n f|$ and $\|S^\#\| < \|S\|$.

**Theorem 2.** Let $T_n f = k_n * f$ where $k_n$ is bounded and has bounded support. Suppose that $Sf = \sup |T_n f|$ is of weak type $(p, p)$, $1 < p < \infty$, and that $\int k_n(t) dt$ converges and $k_n * \phi$ converges in $L^1$, as $n \to \infty$, for every infinitely differentiable $\phi$ with compact support and vanishing integral. Then $T^\#_n f$ converges almost everywhere in $L^p(X)$. 

2
2 Result

Theorem 3. Let \( w \) and \( v \) be two weight functions and let \( T_n \) be a sequence of operators as above and \( Sf = \sup |T_n f| \). Suppose that there exists a constant \( C_1 > 0 \) such that

\[
\int_{\mathbb{R}} |Sf(t)|^p w(U^t x) \, dt \leq C_1 \int_{\mathbb{R}} |f(t)|^p v(U^t x) \, dt
\]

for almost every \( x \), for all \( f \in L^1(\mathbb{R}) \), \( 1 < p < \infty \). Then we have

\[
\int_X |S^# f(x)|^p w(x) \, d\mu \leq C_1 \int_X |f(x)|^p v(x) \, d\mu
\]

for all \( f \in L^1(X) \), \( 1 < p < \infty \), where \( S^# f = \sup |T_n^# f| \).

Also, if there exists a constant \( C_2 > 0 \) such that for every \( \lambda > 0 \)

\[
\int_{\{t:|Sf(t)|>\lambda\}} w(U^t x) \, dt \leq \frac{C_2}{\lambda^p} \int_{\mathbb{R}} |f(t)|^p v(U^t x) \, dt
\]

holds for almost every \( x \), \( 1 \leq p < \infty \), and for all \( f \in L^1(\mathbb{R}) \), then for every \( \lambda > 0 \)

\[
\int_{\{x:|S^# f(x)|>\lambda\}} w(x) \, d\mu \leq \frac{C_2}{\lambda^p} \int_X |f(x)|^p v(x) \, d\mu
\]

holds for all \( f \in L^1(X) \), \( 1 \leq p < \infty \).

Proof. We adapt the argument of A. P. Calderón [7] to prove our theorem. Without loss of generality we may assume that the sequence \( T_n \) is finite, for if the theorem is established in this case, the general case follows by a passage to the limit. Under this assumption the operator \( S \) has the same properties as the operator \( T \) above. We note that

\[
F(t, U^s x) = F(t + s, x),
\]

which means that for any two given values \( t_1, t_2 \) of \( t \), \( F(t_1, x) \) and \( F(t_2, x) \) are equimeasurable functions of \( x \). On the other hand, due to translation invariance of \( S \), the function \( G(t, x) \) has the same property. In fact we have

\[
G(t, U^s x) = S(F(t, U^s x)) = S(F(t + s, x)) = G(t + s, x).
\]
Let now \( F_a(t, x) = F(t, x) \) if \( |t| < a \), \( F_a(t, x) = 0 \) otherwise, and let
\[
G_a(t, x) = S(F_a(t, x)).
\]
Since \( S \) is positive (i.e., its values are non-negative functions) and sublinear, we have
\[
G(t, x) = S(F) = S(F_{a+\epsilon} + (F - F_{a+\epsilon})) \\
\leq S(F_{a+\epsilon}) + S(F - F_{a+\epsilon})
\]
and since \( F - F_{a+\epsilon} \) has support in \( |t| > a + \epsilon \), and \( S \) is semilocal, the last term on the right vanishes for \( |t| \leq a \) for \( \epsilon \) sufficiently large, independently of \( a \). Thus we have \( G \leq G_{a+\epsilon} \) for \( |t| \leq a \). Suppose now that there exists a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}} |Sf(t)|^{p}w(U^t x) dt \leq C_1 \int_{\mathbb{R}} |f(t)|^{p}v(U^t x) dt
\]
Then since \( G(0, x) \) and \( G(t, x) \) are equimeasurable functions of \( x \), we have
\[
2 \int_X G(0, x)^{p}w(x) dx = 2 \int_{|t|<a} dt \int_X G(t, x)^{p}w(U^t x) dx \\
\leq \frac{1}{a} \int_{|t|<a} dt \int_X G_a(t, x)^{p}w(U^t x) dx \\
= \frac{1}{a} \int_X dx \int_{|t|<a} G_a(t, x)^{p}w(U^t x) dt
\]
and since \( SF_{a+\epsilon} = G_{a+\epsilon} \)
\[
\int_{|t|<a} G_{a+\epsilon}(t, x)^{p}w(U^t x) dt \leq C_1 \int_X |F_{a+\epsilon}(t, x)|^p v(U^t x) dt,
\]
whence substituting above we obtain
\[
2 \int_X G(0, x)^{p}w(x) dx \leq \frac{1}{a} C_1 \int_X dx \int |F_{a+\epsilon}(t, x)|^p v(U^t x) dt
\]
and again, since \( F(0, x) \) and \( F(t, x) \) are equimeasurable, the last integral is equal to
\[
2(a + \epsilon) \int_X |F(0, x)|^p v(x) dx
\]
and
\[
\int_X G(0, x)^p w(x) \, dx \leq \frac{1}{a} (a + \epsilon) C_1 \int_X |F(0, x)|^p v(x) \, dx
\]
\[= \frac{1}{a} (a + \epsilon) C_1 \int_X |f(x)|^p v(x) \, dx.
\]

Letting \(a\) tend to infinity, we prove the first part of our theorem.

Suppose now that \(S\) satisfies weighted weak type inequality. For any given \(\lambda > 0\), let \(E\) and \(\tilde{E}\) be the set of points where \(G(0, x) > \lambda\) and \(G_{a+\epsilon}(t, x) > \lambda\), respectively, and \(\tilde{E}_y\) the intersection of \(\tilde{E}\) with the set \(\{(t, x) : x = y\}\). Then we have
\[
2aw(E) \leq w(\tilde{E}) = \int_X w(\tilde{E}_x) \, dx.
\]

On the other hand, \(S\) satisfies weighted weak type inequality
\[
w(\tilde{E}_x) \leq \frac{C_2}{\lambda^p} \int |F_{a+\epsilon}(t, x)|^p v(U^t x) \, dt.
\]

By using the above inequalities and the fact that \(F(0, x)\) and \(F(t, x)\) are equimeasurable we have
\[
aw(E) \leq \frac{C_2}{\lambda^p} (a + \epsilon) \int |F(0, x)|^p v(x) \, dx
\]

When we let \(a\) tend to \(\infty\) we find the desired result. \(\square\)

Let \(P_n\) be another sequence of operators satisfying exactly the same properties as \(T_n\) does, and let \(P f = \sup |P_n f|\) and \(P^# f = \sup |P_n^# f|\), by using a very similar argument to the proof we have just presented for Theorem 3 it is easy to prove the following more general form of Theorem 3:

**Theorem 4.** Let \(w\) and \(v\) be two weight functions. Suppose that there exists a constant \(C_1 > 0\) such that
\[
\int_{\mathbb{R}} |S f(t)|^p w(U^t x) \, dt \leq C_1 \int_{\mathbb{R}} |P f(t)|^p v(U^t x) \, dt
\]
for almost every \(x\), for all \(f \in L^1(\mathbb{R})\), \(1 < p < \infty\). Then we have
\[
\int_X |S^# f(x)|^p w(x) \, d\mu \leq C_1 \int_X |P^# f(x)|^p v(x) \, d\mu
\]
for all \( f \in L^1(X), 1 < p < \infty \).

Also, if there exists a constant \( C_2 > 0 \) such that for every \( \lambda > 0 \)
\[
\int \mathcal{M}(s_{f(t)} \geq \lambda) \, w(U^t x) \, dt \leq \frac{C_2}{\lambda^p} \int_\mathbb{R} |P f(t)|^p v(U^t x) \, dt
\]
holds for almost every \( x, 1 \leq p < \infty \), and for all \( f \in L^1(\mathbb{R}) \), then for every \( \lambda > 0 \)
\[
\int \{ x : |s f(x)| > \lambda \} \, w(x) \, d\mu \leq \frac{C_2}{\lambda^p} \int_X |P f(x)|^p v(x) \, d\mu
\]
holds for all \( f \in L^1(X), 1 \leq p < \infty \).

3 Applications

We say that a non-negative measurable function \( w \in L^1_{\text{loc}}(\mathbb{R}) \) is an \( A_p \) weight for some \( 1 < p < \infty \) if
\[
\sup_I \left[ \frac{1}{|I|} \int_I w(x) \, dx \right] \left[ \frac{1}{|I|} \int_I [w(x)]^{1/(p-1)} \, dx \right]^{p-1} \leq C < \infty
\]
The supremum is taken over all intervals \( I \subset \mathbb{R} ; |I| \) denotes the measure of \( I \).

A non-negative measurable function \( w \) is an \( A_{\infty} \) weight if given an interval \( I \subset \mathbb{R} \) there exist \( \delta > 0 \) and \( \epsilon > 0 \) such that for any measurable set \( E \subset I \),
\[
|E| < \delta \cdot |I| \implies w(E) < (1 - \epsilon) \cdot w(I).
\]
Here
\[
w(E) = \int_E w(x) \, dx.
\]

We say that \( w \in A_1 \) if
\[
\frac{1}{|I|} \int_I w(x) \, dx \leq C \text{ ess inf}_I w
\]
for all intervals \( I \).

It is well known and can easily be seen that \( w \in A_{\infty} \) implies \( w \in A_p \) if \( 1 < p < \infty \).
Let $f$ be a locally integrable function defined on $\mathbb{R}$ and consider the square function

$$ S_{\mathbb{R}}f(x) = \left( \sum_{n=-\infty}^{\infty} \left| \frac{1}{2^n} \int_{0}^{2^n} f(x-t) \, dt - \frac{1}{2^{n-1}} \int_{0}^{2^{n-1}} f(x-t) \, dt \right|^2 \right)^{1/2} $$

It is proven in A. De La Torre and J. L. Torrea [13] that $S_{\mathbb{R}}f$ satisfies the weak type weighted inequality for $w \in A_p$ with $1 \leq p < \infty$ and it satisfies the weighted strong type inequality for $w \in A_p$ with $1 < p < \infty$.

Let now $\mathcal{X} = (X, \mathcal{B}, \mu)$ be a dynamical system and $\{U_t : -\infty < t < \infty\}$ be a one-parameter ergodic measure preserving flow on $X$.

Let us now define the ergodic square function as

$$ Sf(x) = \left( \sum_{n=-\infty}^{\infty} \left| \frac{1}{2^n} \int_{0}^{2^n} f(U_t x) \, dt - \frac{1}{2^{n-1}} \int_{0}^{2^{n-1}} f(U_t x) \, dt \right|^2 \right)^{1/2} $$

for a locally integrable function $f$ defined on $X$.

We say that $w$ satisfies condition $A'_p$ if there exists a constant $M > 0$ such that for a.e. $x \in X$

$$ \sup_I \left[ \frac{1}{|I|} \int_I w(U_t x) \, dt \right] \left[ \frac{1}{|I|} \int_I [w(U_t x)]^{-1/(p-1)} \, dt \right]^{p-1} \leq M $$

The supremum is taken over all intervals $I \subset \mathbb{R}$; $|I|$ denotes the measure of $I$.

Here $w \in A'_1$ if

$$ \frac{1}{|I|} \int_I w(U_t x) \, dt \leq C \text{ ess inf}_I w(U_t x) $$

for all intervals $I$.

When we apply our result to the above mentioned weighted weak type and strong type inequalities for $S_{\mathbb{R}}f$, we see that $Sf$ satisfies the weak type weighted inequality for $w \in A'_p$ with $1 \leq p < \infty$ and it satisfies the weighted strong type inequality for $w \in A'_p$ with $1 < p < \infty$.

Let $f$ be a locally integrable function on $\mathbb{R}$, the Hardy-Littlewood maximal function $Mf$ is defined by

$$ Mf(t) = \sup_s \frac{1}{s} \int_0^s |f(t + u)| \, du. $$
B. Muckenhoupt [17] has proved that $Mf$ maps $L^p(w)$ to $L^p(w)$ for $1 < p < \infty$ when $w$ satisfies the $A_p$ condition.

Consider now the ergodic maximal function with continuous parameter defined by

$$ M^\#: f(x) = \sup_n \frac{1}{n} \int_0^n |f(U_t x)| dt $$

where $f$ is a locally integrable function on $X$.

When we apply our result to the above mentioned result of B. Muckenhoupt [17], we see that the ergodic maximal function $M^\# f$ maps $L^p(w)$ to $L^p(w)$ for $1 < p < \infty$ when $w$ satisfies the $A'_p$ condition.

Similarly when we apply the second part of our result to the weighted weak type inequality of B. Muckenhoupt [17] for the Hardy-Littlewood maximal function $Mf$ we see that there exists a constant $C > 0$ such that for all $\lambda > 0$ and for all $f \in L^1(X)$

$$ \int_{\{x : M^\# f(x) > \lambda\}} w(x) \, d\mu \leq \frac{C}{\lambda} \int_X f(x) w(x) \, d\mu $$

when $w$ satisfies the $A'_1$ condition.

Consider now the Hilbert transform on $\mathbb{R}$ defined by

$$ Hf(x) = \lim_{\epsilon \to 0^+} \int_{\epsilon \leq |t| \leq 1/\epsilon} \frac{f(t + u)}{u} \, du $$

for $f \in L^1(\mathbb{R})$.

It is proven in R. Hunt et al [16] that $Hf$ maps $L^p(w)$ to $L^p(w)$ for $1 < p < \infty$ when $w$ satisfies the $A_p$ condition and there exists a constant $C > 0$ such that for all $\lambda > 0$ and for all $f \in L^1(\mathbb{R})$

$$ \int_{\{x : Hf(x) > \lambda\}} w(x) \, d\mu \leq \frac{C}{\lambda} \int_{\mathbb{R}} f(x) w(x) \, d\mu $$

when $w$ satisfies the $A_1$ condition.

When we apply our result to these inequalities we see that the ergodic Hilbert transform defined as

$$ H^\# f(x) = \lim_{\epsilon \to 0^+} \int_{\epsilon \leq |t| \leq 1/\epsilon} \frac{f(U_t x)}{t} \, dt $$
for $f \in L^1(X)$ maps $L^p(w)$ to $L^p(w)$ for $1 < p < \infty$ when $w$ satisfies the $A'_p$ condition and there exists a constant $C > 0$ such that for all $\lambda > 0$ and for all $f \in L^1(X)$

$$\int_{\{x : H^f(x) > \lambda\}} w(x) \, d\mu \leq \frac{C}{\lambda} \int_X f(x) w(x) \, d\mu$$

when $w$ satisfies the $A'_1$ condition.

References

[1] H. Aimer, *On weighted inequalities for ergodic operators*, Studia Math. 82 (1985) 265-269.

[2] E. Atencia and A. De La Torre, *A dominated ergodic estimate for $L_p$ space with weights*, Studia Math. 74 (1982) 35-47.

[3] N. Asmar, A. Berkon and T. A. Gillespie, *Transference of strong type maximal inequalities by separation-preserving representations*, Amer. J. Math. 113 (1) (1991) 47-74.

[4] N. H. Asmar and S. J. Montgomery-Smith, *Transference in spaces of measures*, J. Func. Analysis 165 (1999) 1-23.

[5] E. Atencia and F. J. Martin-Reyes, *The maximal ergodic Hilbert transform with weights*, Pacific J. of Math. 108 (2) (1983) 257-263.

[6] E. Atencia and F. J. Martin-Reyes, *Weak type inequalities for the maximal ergodic function and the maximal Hilbert transform in weighted spaces*, Studia Math. 78 (1984) 231-244.

[7] A. P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. USA 59 (1968) 349-353. Proc. Nat. Acad. Sci. USA 59 (1968) 349-353.

[8] R. R. Coifman and G. Weiss, *Transference methods in analysis*, AMS Regional Conf. Series in Math. No. 1 May 31-June 4, 1976. Proc. Nat. Acad. Sci. USA 59 (1968) 349-353.
[9] O. Blasco and P. Villarroya, *Transference of vector-valued multipliers on weighted $L^p$-spaces*, Canad. J. Math. 63 (3) (2013) 510-543.

[10] O. Blasco and P. Villarroya, *Transference of bilinear multiplier operators on Lorentz spaces*, Illinois. J. Math. 47 (4) (2003) 1327-1343.

[11] S. Demir, *A generalization of Calderón transfer principle*, J. Comp. & Math. Sci. 9 (5) (2018) 325-329.

[12] S. Demir, *An extension of Caldeón Transfer Principle and its application to ergodic maximal function*, Asian J. of Mathematical Sciences, 4 (2) (2020) 15-18.

[13] A. De La Torre and J. L. Torrea, *One-sided discrete square function*, Studia Math. 156 (3) (2003) 243-260.

[14] C. Finet and P. Wantiez, *Transfer principles and ergodic theory in Orlicz spaces*, Note di Matematica 25 (1) (2005/2006) 167-189.

[15] D. Kosz, *Sharp constants in inequalities admitting the Calderón transference principle*, Ergodic Th. & Dyn. Sys. 44 (6) (2024) 1597-1608. https://doi.org/10.1017/etds.2023.59.

[16] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for conjugate function and Hilbert transform*, Trans. AMS, 176 (1973) 227-251.

[17] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. AMS, 165 (1972) 207-226.