Online Data-Enabled Predictive Control

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Abstract

We develop an online data-enabled predictive (ODeePC) control method for optimal control of unknown systems, building on the recently proposed DeePC [1]. Our proposed ODeePC method leverages a primal-dual algorithm with real-time measurement feedback to iteratively compute the corresponding real-time optimal control policy as system conditions change. Specifically, the algorithm: a) records data from the unknown system and updates the underlying primal-dual algorithm dynamically, b) can track changes in the system’s operating point and adjust the control inputs, and c) is computationally efficient as it employs a fast Fourier transform-based algorithm, enabling the fast computation of the product of a non-square Hankel matrix with a vector. We provide theoretical guarantees regarding the asymptotic behavior of ODeePC, and we demonstrate its performance through a power system application.

Key words: Data-driven control, model predictive control, online optimization

1 Introduction

In the context of smart critical infrastructures and complex dynamic systems, applications of optimal control abound [2]. Traditional optimal control of these systems relies on accurate system models, with model predictive control (MPC) [3–7] being a classic approach. Many complex physical systems however, such as large power systems, manifest complex and hard-to-model uncertain dynamics that complicate their study and analysis. Therefore, data-based modeling and control techniques have become increasingly popular in the context of such systems due to data abundance [8, 9]. Such approaches offer an attractive alternative to classic optimal control as they are independent of any analytical system models.

System identification algorithms are exploited in many control approaches for producing approximate models when an analytical system description might not be available or might be too difficult to obtain [10–12]. Other data-based control approaches exploit different types of learning algorithms at various stages of the control design process to generate information about the model. Representative publications of this line of work are [13–18], [19], and [20]; the most prominent approaches are based on reinforcement learning and MPC using Gaussian processes. All these approaches rely on state-space model representation of the underlying dynamic system.

Data-driven optimal control approaches can be useful even when a certain (simplified) model of the system exists. In such a case, data-driven approaches can help refine the model, by capturing exogenous disturbances and other model details not accounted for in the existing model [19].

Recently, a data-enabled predictive control algorithm (DeePC) [1] was proposed as an alternative approach that is free of any parametric system model representation. DeePC, rather, uses the control inputs and plant outputs of a dynamic system to learn the system’s behavior and compute a predictive control policy directly. Despite its accuracy and efficacy for small systems (systems with few states), however, this control approach, lacks scalability as its implementation to large systems (i.e., systems with many states) can be computationally burdensome. This is partially because of the large...
dataset required to accurately represent the underlying system.

**Contributions.** In this paper, we develop an online data-enabled predictive controller (ODeePC) by building on the recently proposed DeePC. Our main contributions are highlighted as follows.

- We appropriately design ODeePC so that it has the following properties: a) it can record data through measurements from the unknown system in real-time and it can update the underlying optimization problem accordingly; b) it can track changes in the system’s operating point and adjust the control inputs dynamically; and c) it is computationally efficient. The algorithm design leverages the online time-varying optimization methods with measurement feedback [21, 22].
- To enable online implementation of ODeePC, we devise a computationally efficient algorithm for a fast computation of the product of a non-square block Hankel matrix with a vector. The algorithm exploits properties of circulant matrices and fast Fourier transform (FFT) to carry out the assigned computations efficiently. We also derive the complexity of this algorithm and prove that it precisely computes the desired product.
- Lastly, we establish theoretical guarantees regarding the asymptotic behavior of ODeePC and illustrate its effectiveness through numerical studies.

The rest of this paper is structured as follows. In Section 2, we present the notation we use throughout the paper and some preliminaries. In Section 3, we review the classic MPC and the recently proposed DeePC [1]. Section 4 includes the main results surrounding the development of the proposed ODeePC algorithm and the analysis of its performance. We validate the performance of ODeePC through a numerical example in Section 5. Finally, in Section 6, we conclude the paper with some remarks.

## 2 Notation and Preliminaries

### 2.1 Notation

Let $\mathbb{R}$ denote the set of real numbers; $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{> 0}$ respectively, denote the set of nonnegative integers. Given a matrix $A$, $A^\top$ denotes its transpose, and $A > (\prec) 0$ denotes that $A$ is positive (negative) definite. The matrix $I_n \in \mathbb{R}^{n \times n}$ is the $n \times n$ identity matrix. For $x \in \mathbb{R}^n$, $\|x\|_2$ denotes its Euclidean norm, $\text{diag}\{x\}$ is a diagonal matrix with the elements of $x$ on the main diagonal. Further, given a set $\mathcal{X} \subset \mathbb{R}^n$, $\text{Proj}_\mathcal{X}\{x\}$ denotes the projection of $x$ onto $\mathcal{X}$. Given a $Q > 0$, we define $\|v\|^2_2 = v^\top Q v$. We use subscript $t$ to denote a vector value at time $t$; and $v_{i,t}$ to denote the $i^{th}$ entry of the vector $v_t$ at time $t$. We denote the smallest eigenvalue of $Q \in \mathbb{R}^{n \times n}$ by $\lambda_{\text{min}}(Q)$.

A shift matrix is defined as $S_{m,n} = \begin{bmatrix} 0 & I_{m-1} \\ 0 & 0 \end{bmatrix} \otimes I_n$ and $S_{m,n} \in \mathbb{R}^{mn \times mn}$. For any matrix $A \in \mathbb{R}^{mn \times N}$, $S_{m,n} A$ results in shifting the elements of $A$ upward by $n$ positions.

### 2.2 Preliminaries

Let us start by assuming that the system we seek to control can be represented by a discrete-time, linear, time-invariant (LTI), state-space model given by

$$
\begin{align*}
    x_{t+1} &= A x_t + B u_t \\
    y_t &= C x_t + D u_t.
\end{align*}
$$

In this representation, $t \in \mathbb{Z}_{\geq 0}$ is the discrete time index; $A \in \mathbb{R}^{n \times n}$ is the state matrix; $B \in \mathbb{R}^{n \times m}$ is the input matrix; $C \in \mathbb{R}^{p \times n}$ is the output matrix; $D \in \mathbb{R}^{p \times m}$ is the feed-forward matrix; $x_t \in \mathbb{R}^n$ is the state vector; $u_t \in \mathbb{R}^m$ is the control input vector; and $y_t \in \mathbb{R}^p$ is the output vector. Also, let $r_t \in \mathbb{R}^p$ denote the output reference vector representing the desired value of the output at time $t$.

Let $u := (u_1^\top, ... , u_T^\top)^\top \in \mathbb{R}^{mT}$ denote the vector of the control inputs during $t = 1, ..., T$, for some $T \in \mathbb{Z}_{> 0}$. The block Hankel matrix, with each column being a set of consecutive data points of length $L \leq T$, is given by

$$
\mathcal{H}_L(u) := \begin{bmatrix} u_1 & u_2 & \cdots & u_{T-L+1} \\
    u_2 & u_3 & \cdots & u_{T-L+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_L & u_{L+1} & \cdots & u_T \end{bmatrix},
$$

with $\mathcal{H}_L(u) \in \mathbb{R}^{mL \times (T-L+1)}$. A similar matrix can be constructed for $y$. It is useful at this point to introduce the following definition.

**Definition 2.1** Let $L, T \in \mathbb{Z}_{\geq 0}$ and $T \geq L$. Then, the signal $u := (u_1^\top, ... , u_T^\top)^\top$ is persistently exciting of order $L$ if $\mathcal{H}_L(u)$, as defined in (2), is of full row rank.

Roughly speaking, a persistently exciting input is a rich enough input able to excite the system so that it generates an output that is representative of its behavior.

Moving forward, we assume that $u \in \mathbb{R}^{mT}$ is a persistently exciting input data set of order $T_{\text{tot}}$. With rich enough data to describe system (1), we can construct the following block Hankel matrices associated with the input and output data, respectively

$$
U := \mathcal{H}_{T_{\text{tot}}}(u), \quad Y := \mathcal{H}_{T_{\text{tot}}}(y),
$$
where $U \in \mathbb{R}^{m_{tot} \times \kappa}$, $Y \in \mathbb{R}^{p_{tot} \times \kappa}$, and $\kappa := T - T_{\text{tot}} + 1$. Behavioral system theory [23] states that if system (1) is controllable\(^1\), then the input-output pair $(u_{\text{tot}}, y_{\text{tot}})$, where $u_{\text{tot}} \in \mathbb{R}^{m_{\text{tot}}}$ and $y_{\text{tot}} \in \mathbb{R}^{p_{\text{tot}}}$, is a trajectory of (1) if and only if there exists a $g \in \mathbb{R}^\kappa$, such that

$$
\begin{bmatrix}
U
Y
\end{bmatrix} = \begin{bmatrix} u_{\text{tot}} \\ y_{\text{tot}} \end{bmatrix}.
$$

Equation (4) indicates that any possible trajectory should be a linear combination of $\kappa$ number of trajectories (columns) embedded in $[U^T, Y^T]^T$. We can view (4) as an alternative model of (1). The main difference is that (4) is entirely constructed by data as opposed to state evolution in (1).

3 Overview of Classic MPC and DeePC

In this section, we provide an overview of the classic MPC and the recently proposed DeePC [1]. Note that the classic MPC uses a precise model (1), whereas DeePC uses the recorded control inputs and plant outputs of the underlying system to capture its behavior and compute a predictive control policy.

3.1 MPC

MPC is a receding time horizon control algorithm that computes an optimal control input $u_0$ based on a prediction of the system’s future trajectory subject to the system’s dynamics. We consider a classic MPC setting with prediction horizon $N \in \mathbb{Z}_{>0}$ formulated as

$$
\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} f(u_k, y_k), \\
\text{s. t.} & \quad x_{k+1} = Ax_k + Bu_k, \quad \forall k \in \{0, \ldots, N-1\}, \\
& \quad y_k = Cx_k + Du_k, \quad \forall k \in \{0, \ldots, N-1\}, \\
& \quad x_0 = \hat{x}_t,
\end{align*}
$$

where $x = (x_0^T, \ldots, x_N^T)^T$; $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{Y} \subseteq \mathbb{R}^p$ are, respectively, the convex and bounded constraint sets for $u_k$ and $y_k$ for all $k$; $f : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$ is a convex cost function; and $\hat{x}_t$ serves as the initial state of the system. We let $t$ denote the current time, and $k$ index the time instances of the look-ahead horizon window. Algorithm 1 [1] can be used to solve the MPC Problem (5). Usually, after the optimal control policy is computed, only the input $u_0^\star$ that corresponds to the first look-ahead window

\begin{algorithm}
\caption{MPC [1]}
\textbf{Problem data:} matrices $A, B, C, D$, current state $\hat{x}_t$, feasible input and output sets $\mathcal{U}$ and $\mathcal{Y}$, objective function $f$.
\begin{enumerate}
  \item Solve (5) for $u^\star = (u_0^T, \ldots, u_{N-1}^T)$.
  \item Apply inputs $(u_1^\star, \ldots, u_{N}^\star)^T = (u_0^T, \ldots, u_s^T)^T$ for some $s \leq N - 1$.
  \item Set $t$ to $t + s$ and update $\hat{x}_t$.
  \item Repeat.
\end{enumerate}
\end{algorithm}

is implemented in the system (or $s = 0$). In the rest of the paper, we assume $s = 0$ without loss of generality.

The MPC algorithm has been proven to be effective in numerous applications, e.g., autonomous driving [24] and flight control [25], where the goal is primarily trajectory tracking. Despite that, the requirement for an accurate model description still restricts the application domain as systems whose dynamics are hard to model or unknown and cannot be considered. To this end, DeePC, which leverages measured system data instead of an accurate system model to capture the system’s behavior, overcomes these limitations and has been shown to work well for small systems. We review the DeePC control approach [1] next.

3.2 DeePC

DeePC relies on the past input/output data to construct the model shown in (4). Let $u_{ini} \in \mathbb{R}^{m_{ini}}$ and $y_{ini} \in \mathbb{R}^{p_{ini}}$ denote a given initial trajectory of the system of length $T_{ini} \in \mathbb{Z}_{>0}$ over time interval $[t - T_{ini}, \ldots, t - 1]$. Any trajectory $u = (u_0^T, \ldots, u_{N-1}^T)$ and $y = (y_0^T, \ldots, y_{N-1}^T)$ over the time interval $[t, t + N - 1]$ should satisfy

$$
\begin{bmatrix}
U
Y
\end{bmatrix} g = \begin{bmatrix} U_p \\ U_f \\ Y_p \\ Y_f \end{bmatrix} g = \begin{bmatrix} Y_{ini} \\ u \\ y \\ y_{ini} \end{bmatrix},
$$

for some $g \in \mathbb{R}^\kappa$. Note that we have $T_{\text{tot}} = T_{ini} + N$; and we have partitioned $U \in \mathbb{R}^{m_{\text{tot}} \times \kappa}$ into $U_p$ and $U_f$, with $U_p \in \mathbb{R}^{m_{ini} \times \kappa}$ and $U_f \in \mathbb{R}^{m_{N-1} \times \kappa}$. Similarly, $Y \in \mathbb{R}^{p_{\text{tot}} \times \kappa}$ is partitioned into $Y_p \in \mathbb{R}^{p_{ini} \times \kappa}$ and $Y_f \in \mathbb{R}^{p_{N-1} \times \kappa}$. The length of the initial trajectory, $T_{ini}$, should be selected large enough to ensure unique $y$ for any given $u$; cf. [23, Lemma 1]. We assume that the total number of measured input/output pairs, $T$, is large enough to construct a persistently exciting $U$ of order $T_{\text{tot}}$; cf. Definition 2.1. In the following, we use the shorthand notation $H = [U_p^T, U_f^T, Y_p^T, Y_f^T]^T$.

With the elements in place, we state the DeePC formu-
We note that in Algorithm 2, the matrix $H$ is not updated over time as the data is recorded offline and used to solve the receding horizon predictive control problem. Only the RHS elements $h = [u^{\top}_{ini}, y^{\top}_{ini}, u^{\top}, y^{\top}]$ are updated as $t$ evolves. In cases where we are dealing with time-varying or nonlinear systems, not updating the matrix $H$ over time might result in bad system representation and subsequently invalid computed control policies. We next present our proposed Online Data-enabled Predictive Control (ODeePC) which tackles these challenges.

4 Online Data-enabled Predictive Control

Below are the main aspects of the proposed algorithm; the details are given in the ensuing sections.

- It exploits available real-time data obtained from system measurements to dynamically update both the matrix $H$ and the vector $h$.
- It uses a primal-dual gradient descent algorithm to iteratively compute the optimal control policy, thus allowing the intermediate control inputs to be implemented in the system in real-time. Further, it allows real-time measured information about the system’s state to take part in the algorithm and affect the computed optimal control policy.
- It exploits a FFT-based algorithm to efficiently compute the products of block Hankel matrices with vectors.

These aspects of ODeePC will be explained in detail.

4.1 Primal-dual Algorithm for the Regularized DeePC Problem

We first design a primal-dual algorithm to iteratively solve the Lagrangian formulation of (7). Then, we appropriately modify the designed algorithm to arrive at the ODeePC algorithm’s iterative update rule.

We start with considering the following min-max optimization problem associated with (7):

$$\max_{\nu} \left( \min_{g \in \mathbb{R}^n, u \in \mathcal{U}, y \in \mathcal{Y}} L(u, y, g, \nu) \right),$$

where $\nu \in \mathbb{R}^{N\nu}$, $N\nu = (m + p)(T_{ini} + N)$, and $L$ is the regularized Lagrangian, given as:

$$L(u, y, g, \nu) = \sum_{k=0}^{N-1} f(u_k, y_k) + \frac{\epsilon_g}{2} \|g\|_2^2 + \nu^\top (Hg - h) - \frac{\epsilon_{\nu}}{2} \|\nu\|_2^2,$$

where $\epsilon_g > 0$ and $\epsilon_{\nu} > 0$ are (small) constants. The regularization terms improve the convergence rate of the gradient-based method at the expense of converging to a point that is close to but not exactly the real optimal point of (7). This particular regularization is widely used for solving convex optimization problems [21], [22]. Using the primal-dual gradient descent algorithm to solve (8), we arrive at the problem’s saddle-flow dynamics. These, are given by

$$\nu^{t+1} = \text{Proj}_H \{u^t - \alpha (\nabla_u \tilde{f}(u^{t}, y^t) - \nu^{t}_u)\}, \quad (9a)$$
$$y^{t+1} = \text{Proj}_Y \{y^t - \alpha (\nabla_y \tilde{f}(u^{t}, y^t) - \nu^{t}_y)\}, \quad (9b)$$
$$g^{t+1} = g^t - \alpha (H\nu^t - h - \epsilon_g g^t), \quad (9c)$$
$$\nu^{t+1} = \nu^t + \alpha (Hg^t - h - \epsilon_{\nu} \nu^t), \quad (9d)$$

where $\tau$ is the iteration number, $\alpha \in \mathbb{R}_+$ is the step size, and $\tilde{f}(u, y) := \sum_{k=0}^{N-1} f(u_k, y_k)$. Further, we define $\nu_u^t$ and $\nu_y^t$ as the elements of $\nu^t$ associated with the inputs and outputs, respectively. The saddle-flow dynamics (9) solve the static optimization problem (8).

Relatively large systems with numerous states would give rise to a large matrix $H$. This would inevitably render the algorithm computationally very expensive. To see this, consider that the product of a general matrix $H \in \mathbb{R}^{n \times m}$ with a $m$-element vector (at every iteration as imposed by (9c) and (9d)), can be computed by carrying out $n \times m$ multiplications and $(n \times m - n)$ additions. One can realize that exact and real-time computation of this product and thus the implementation of the algorithm would be quite challenging. This would be even more challenging in an online setting where $u_{ini}$, $y_{ini}$ and $H$ would be frequently updated. Motivated by this,
we carefully design ODDeePC so that it is computationally efficient and practically implementable. We accomplish this by exploiting a computationally efficient algorithm that leverages FFT to compute the block Hankel matrix-vector multiplication quickly. The details will be provided in the sequel.

4.2 ODDeePC

We now present the underlying time-varying optimization problem associated with our proposed ODDeePC. We consider, the vector \( h \) in the optimization \( (8) \) to be frequently updated with the latest input-output pair, \( u_{ini} \) and \( y_{ini} \). In addition, to allow our algorithm to cope with time-varying systems, we also update \( \mathcal{H} \) at every iteration for better system characterization. The following formulation captures the time-varying properties described.

\[
\max_{\nu} \left( \min_{g} \mathcal{L}(u, y, g, \nu) \right),
\]

where \( t \) captures the time instances when the optimization problem is updated. The time varying Lagrangian is the same as \( (8) \) except that \( H \) and \( h \) are time dependent, given as

\[
\mathcal{H}^t = \begin{bmatrix} U_p^t & U_f^t \\ Y_p^t & Y_f^t \end{bmatrix}, \quad h^t = \begin{bmatrix} u_{ini}^t \\ y_{ini}^t \end{bmatrix}.
\]

Remark 4.1 (ODDeePC for linear time-varying (LTV) systems). For a LTI system, updating \( \mathcal{H} \) online using the measured data is not very meaningful as the original \( \mathcal{H} \) already captures all the properties that characterize the system; however, in order to allow ODDeePC to deal with LTV systems or nonlinear dynamical systems, frequent online updating of the matrix \( \mathcal{H} \) is necessary.

Let the variable \( \tau \) track the algorithm iteration, and assume that the control is implemented on the system every time \( N_f \) iterations have been completed. The iteration index \( \tau \) and the system update instances \( t \) are related as follows. When \( \tau \) coincides with the system update instance \( t \), then \( \tau + N_f \) would coincide with the system update instant \( t + 1 \). In our proposed ODDeePC system, all the inner-loop iterations in the interval \( (t, t + 1) \) are carried out using the update rules \( (9) \), with \( \mathcal{H} \) being fixed and given by \( \mathcal{H}^t = \begin{bmatrix} U_p^t & U_f^t & Y_p^t & Y_f^t \end{bmatrix} \). In addition, the \( u_{ini}^t \) and \( y_{ini}^t \) elements of \( h \) are, respectively, fixed at \( u_{ini}^t \) and \( y_{ini}^t \). Every \( N_f \) iterations however, starting at instant \( t + 1 \) (or \( \tau + N_f \)), these elements are updated, and ODDeePC deploys the following update rules to compute the new input-output pairs and dual variables:

\[
\begin{align*}
  u^{\tau+1} &= \text{Proj}_{\mathcal{U}}\{\hat{u}^\tau - \alpha(\nabla_u \hat{f}(\hat{u}^\tau, \hat{y}^\tau) - \hat{\nu}_u^\tau)\}, \quad (11a) \\
  y^{\tau+1} &= \text{Proj}_{\mathcal{Y}}\{\hat{y}^\tau - \alpha(\nabla_y \hat{f}(\hat{u}^\tau, \hat{y}^\tau) - \hat{\nu}_y^\tau)\}, \quad (11b) \\
  g^{\tau+1} &= g^\tau - \alpha(\mathcal{H}^{(t+1)}\hat{\nu}^\tau + \epsilon_g g^\tau), \quad (11c) \\
  \nu^{\tau+1} &= \nu^\tau + \alpha(\mathcal{H}^{(t+1)}\hat{\nu}^\tau - h^{t+1} - \epsilon_\nu \nu^\tau), \quad (11d)
\end{align*}
\]

where \( \hat{u}^\tau = S_{N,m} u^\tau \), \( \hat{y}^\tau = S_{N,p} y^\tau \), \( \hat{\nu}^\tau = [\hat{\nu}_u^\tau \hat{\nu}_y^\tau]^\top \), \( \hat{\nu}_u^\tau = S_{\tau u,m} \nu_u^\tau \), and \( \hat{\nu}_y^\tau = S_{\tau y,p} \nu_y^\tau \). Observe that the update rules \( (11) \) are different from \( (9) \). The main differences between \( (11) \) and \( (9) \) are that the variables \( u^\tau \), \( y^\tau \), \( \nu^\tau \) are updated through the shift matrices. Both \( \mathcal{H} \) and \( h \) are also updated to reflect the change of the optimization problem. The shift matrix moves \( u_{k+1}^\tau \) in the place of \( u_k^\tau \) for all \( k = 0, \cdots, N - 2 \). The other variables are updated similarly through the shift matrices.

Algorithm 3 ODDeePC

1: Initialize \( \tau = 1 \), \( \alpha, \epsilon, u^\tau, y^\tau, g^\tau, \nu^\tau, \mathcal{H}^t, h^t \)
2: Repeat
3: if \( \text{mod}(\tau, N_f) \geq 1 \) then
4: \quad Compute \( (9) \) using Algorithm 5
5: else
6: \quad Apply control \( u_0^\tau \) to the system
7: \quad Update \( \mathcal{H}^{t+1} \) and \( h^{t+1} \) using \( u_0^\tau \) and the measured \( y_0^\tau \)
8: \quad Compute \( (11) \) using Algorithm 5
9: \quad \tau \rightarrow \tau + 1
10: end if
11: \tau \rightarrow \tau + 1

Note that Algorithm 3 incorporates Algorithm 5, which will be introduced in Section 4.4, to compute \( \mathcal{H}^{t+1} \nu^\tau \) and \( \mathcal{H}^\tau g^\tau \) in \( (9) \) and \( (11) \). The matrix-vector multiplications are the most computationally heavy part in Algorithm 3. Algorithm 5 gets around the multiplications with FFTs. We will first show the convergence of the ODDeePC and then in Section 4.4 we explain how FFT-based approach works.

4.3 Convergence of ODDeePC

In this section, we show that under mild assumptions, Algorithm 3 converges Q-linearly to a neighborhood of
the optimal point. For convenience of notation, we denote \( z^\tau = \left[ u^\tau \top \ y^\tau \top \ g^\tau \top \ \nu^\tau \top \right] \top \), and we rewrite (11) as:

\[
z^{\tau+1} = \text{Proj}_{U \times Y \times R^{n+m}} \{ z^\tau - \alpha \Psi^\tau(z^\tau) \}, \tag{12}
\]

where the time-varying gradient step is embedded in \( \Psi^\tau \).

We assume the Lipschitz continuity and monotonicity of \( \Psi^\tau \), stated formally in Assumption 1.

**Assumption 1 (Lipschitz continuity and monotonicity of the gradient).** There exists a finite constant \( \sigma_{\Psi} \in \mathbb{R}_+ \) such that \( \| \Psi^\tau(z^1) - \Psi^\tau(z^2) \| \leq \sigma_{\Psi} \| z^1 - z^2 \| \) for all \( z^1, z^2 \in U \times Y \times \mathbb{R}^{n+m} \), for all \( \tau \). In addition, \( \Psi^\tau \) is strongly monotone with constant \( \eta \).

Assumption 1 is actually equivalent to [26, Lemma 3.4]. The reason of making Assumption 1 instead of deriving [26, Lemma 3.4] for our case is that the proof of [26, Lemma 3.4] will require breaking the equality constraint on \( g \) into inequality constraints. In addition, because the gradient mapping \( \Psi^\tau \) has two different forms (dependent on the iteration step), respectively (9) and (11), we need to show the Lipschitz continuity and strong monotonicity for both forms. Those are tedious and not the focus of this paper.

We further make Assumption 2 that considers that the variation of the optimal point is bounded, enabling each time close tracking of the optimal point within a reasonable number of iterations of ODeePC.

**Assumption 2 (Bounded variation of the optimal point).** There exists a finite constant \( \sigma_z \in \mathbb{R}_+ \) such that the optimal points for the consecutive time steps \( t \) (or \( \tau \)) and \( t+1 \) (or \( \tau + N_1 \)) of optimization (10) satisfy \( \| z^{\tau+N_1} - z^{\tau} \|_2 \leq \sigma_z \).

Recall that we index \( z \) by the iteration number rather than the time instances \( t \). The optimal point of \( z \) for iterations \( \tau, \ldots, \tau + N_1 - 1 \) (or time \( t \)) stays unchanged because the associated optimization remains the same. The optimal point changes for iteration \( \tau + N_1 \) (time \( t+1 \)). With all the elements in place, we state Theorem 2 which establishes convergence of Algorithm 3.

**Theorem 4.2 (Convergence of ODeePC)** If Assumptions 1 and 2 hold, and \( \rho(\alpha) := \sqrt{1+\alpha^2\sigma_{\Psi}^2 - 2\alpha\eta} < 1 \), then Algorithm 3 has \( z^\tau \) converge Q-linearly to a neighborhood of optimal point of (10), given as:

\[
\lim_{\tau \to \infty} \| z^\tau - z^{\tau,*} \|_2 = \frac{\sigma_z}{1 - \rho(\alpha)}.
\]

The proof of Theorem 2 is similar to the proof of [22, Theorem 4]. The main element of the proof is the derivation of an inequality between \( \| z^{\tau+1} - z^{\tau+1,*} \| \) and \( \| z^\tau - z^{\tau,*} \| \).

In the proof, Assumption 1 is used for substituting the terms \( \psi(\tau, z^\star) \) by the product of some coefficient (determined by \( \sigma_{\Psi} \) and \( \eta \)) and \( \| z^\tau - z^{\tau,*} \| \). Readers are referred to [22, Theorem 4] for more details.

### 4.4 Fast Computation of Hankel Matrix-Vector Product

In this section, we describe Algorithm 5, which efficiently computes the matrix-vector multiplications in (9) and (11). We will use the fact that \( H \) in (9) and (11) is the concatenation of block Hankel matrices \( U \) and \( Y \), and we apply FFT to exploit the convolutional structure of the Hankel matrices.

We slightly abuse the notation in this subsection, and we let \( n \) and \( m \) denote the dimensions of a general Hankel matrix \( H \) so that \( H \in \mathbb{R}^{n \times m} \), unrelated to the state-space model representation (1). Given that, the main problem we are concerned with here can be described as follows.

**Problem 1** Given a \( m \)-element vector \( v \in \mathbb{R}^m \):

\[
v := \left( v_1 \ v_2 \ \cdots \ v_m \right)^\top,
\]

and a Hankel matrix \( H \in \mathbb{R}^{n \times m} \), where \( h_i \in \mathbb{R} \), \( \forall i \), defined as:

\[
H := \left( \begin{array}{cccc}
\ h_1 & h_2 & \cdots & h_{m-1} & h_m \\
\ h_2 & h_3 & \cdots & h_{m} & h_{m+1} \\
\ \vdots & \vdots & \ddots & \vdots & \vdots \\
\ h_{n-1} & h_n & \cdots & h_{n+m-3} & h_{n+m-2} \\
\ h_n & h_{n+1} & \cdots & h_{n+m-2} & h_{n+m-1}
\end{array} \right), \tag{14}
\]

compute the product \( p_H = Hv \) in a computationally efficient manner.

Building on the algorithm proposed in [27] for computing the product of a square Hankel matrix \( H \) and a vector, we propose Algorithm 4 for computing the product \( p_H = Hv \) efficiently for non-square Hankel matrices. The complexity of Algorithm 4 is introduced through the following theorem.

**Theorem 4.3** The complexity of Algorithm 4 that computes the product of a Hankel matrix \( H \in \mathbb{R}^{n \times m} \) and a vector \( v \in \mathbb{R}^m \) is \( O(\max(n, m) \log(\max(n, m))) \).

**PROOF.** Following the same steps as in [27], we first note that each FFT has a complexity \( 5(n+m-1) \log(n+m-1) \), and the pointwise multiplication has a complexity
Algorithm 4 Fast Hankel matrix-vector product

Given a vector \( v \in \mathbb{R}^m \) and a Hankel matrix \( H \in \mathbb{R}^{n \times m} \), compute the vector \( p_H = Hv \) via the following steps.

(1) Define a new \((n + m - 1)\)-element vector \( c \) as:

\[
c = \left( h_m, h_{m+1}, \ldots, h_{n+m-1}, 1, h_2, \ldots, h_{m-1} \right) ^\top
\]

Note that the vector \( c \) is the vector that fully specifies a Circulant matrix \( C \).

(2) Define a \((n + m - 1)\)-element vector \( v_c \in \mathbb{R}^{n+m-1} \) by permuting the vector \( v \) and adding \((n - 1)\) zeros so that:

\[
v_c = \left( v_m, v_{m-1}, \ldots, v_1, 0, \ldots, 0 \right) ^\top
\]

(3) Compute a \((n + m - 1)\)-element vector \( y \) as:

\[
y = \text{IFFT}(\text{FFT}(c) \circ \text{FFT}(v_c))
\]

where \( \circ \) is the Hadamard product of the two vectors, \( \text{FFT} \) is the fast Fourier transform, and \( \text{IFFT} \) is the inverse fast Fourier transform.

(4) Let \( y = \left( y_1, y_2, \ldots, y_{(n+m-2)} \right)^\top \), \( y \in \mathbb{R}^{n+m-1} \). Then the product \( p_H = Hv \) is given by:

\[
p_H = \left( y_1, y_2, \ldots, y_{n-1}, y_n \right) ^\top
\]

i.e., the subvector defined by the first \( n \) elements of the vector \( y \).

\[
6(n + m - 1) \text{ and the inverse } \text{FFT} \ 5(n + m - 1) \log(n + m - 1). \text{ By combining all of these together, we have that the complexity of the algorithm is:}
\]

\[
15(n + m - 1) \log(n + m - 1) + 6(n + m - 1). \quad (15)
\]

Equation (15) can be rewritten in Big-O notation, which is the desired \( O(m \log(m)) \).

It remains to show that the proposed algorithm precisely computes the desired product \( p_H = Hv \). This is carried out through the following theorem.

Theorem 4.4 Consider a Hankel matrix \( H \in \mathbb{R}^{n \times m} \) as defined in (14) and a vector \( v \in \mathbb{R}^n \) as defined in (13). The first \( n \) elements of the vector \( y \) obtained from the following computation:

\[
y = \text{IFFT}(\text{FFT}(c) \circ \text{FFT}(v_c)) \quad (16)
\]

where \( c \) and \( v_c \) are \((n + m - 1)\)-element vectors, defined as:

\[
c := \left( h_m, h_{m+1}, \ldots, h_{n+m-1}, 1, h_2, \ldots, h_{m-1} \right)^\top,
\]

\[
v_c := \left( v_m, v_{m-1}, \ldots, v_1, 0, \ldots, 0 \right)^\top,
\]

yields the exact product \( p_H = Hv \).

PROOF. The proof is deferred to the appendix.

Through this theorem, we have established that Algorithm 4 computes the product \( p_H = Hv \). By building on these results, we will now design an algorithm that will allow us to efficiently compute the product of a block Hankel matrix \( H \) (comprising column vectors \( h_i \)) and a vector \( v \). The main problem we seek to address can be stated as follows.

**Problem 2** Given a \( m \)-element vector \( v \in \mathbb{R}^m \):

\[
v := \left( v_1, v_2, \ldots, v_m \right)^\top, \quad (17)
\]

and a block Hankel matrix \( H \in \mathbb{R}^{(nl) \times m} \), where \( h_i \in \mathbb{R}^l \), \( l > 1 \), \( \forall i \), defined as:

\[
H := \begin{pmatrix}
    | & | & & | & & | & |
    | h_1 & h_2 & \cdots & h_{m-1} & h_m & |
    | h_2 & h_3 & \cdots & h_m & h_{m+1} & |
    | \vdots & \vdots & \ddots & \vdots & \vdots & |
    | h_{n-1} & h_n & \cdots & h_{n+m-3} & h_{n+m-2} & |
    | h_n & h_{n+1} & \cdots & h_{n+m-2} & h_{n+m-1} & |
\end{pmatrix}, \quad (18)
\]

Compute the product \( p_{BH} = Hv \) in a computationally efficient manner.

Our goal is to compute the product of a block Hankel matrix \( H \) that comprises column vectors \( h_i \in \mathbb{R}^l \), with a vector \( v \). We propose the following algorithm for solving Problem (2).

Following the steps in the proof of Theorem 4.4, one can show that application of Algorithm 5 indeed results in the precise computation of the product \( p_{BH} \) of a block Hankel matrix \( H \) and a vector \( v \).

5 Numerical Validation of ODdecPC

We consider the following predictive control problem with a linear time-varying dynamic system:

\[
\min_{x,u \in \mathcal{U}, y \in \mathcal{Y}} \sum_{k=0}^{N-1} \|y_k - r_{t+k}\|^2, \quad (19)
\]
The product \( p_{BH} \) of a vector \( v \in \mathbb{R}^m \) and a block Hankel matrix \( H \in \mathbb{R}^{(n \times 1) \times m} \) can be computed through the following steps.

1. **Initialize** \( \beta = 1 \) and define the vector: \( c \in \mathbb{R}^{l(n+m-1)} \)

\[
c = \left( h_m^T \cdots h_{n+m-1}^T h_1^T h_2^T \cdots h_{n-1}^T \right)^T
\]

2. Define a \((n + m - 1)\)-element vector \( v_c \in \mathbb{R}^{n+m-1} \) by permuting the vector \( v \) and adding \((n-1)\) zeros so that:

\[
v_c = \left( v_m \ v_{m-1} \cdots v_1 \ 0 \ \cdots \ 0 \right)^T
\]

3. While \( \beta \leq l \), construct:

\[
e^{(\beta)} = \left( h_m(\beta) \cdots h_{n+m-1}(\beta) h_1(\beta) \cdots h_{n-1}(\beta) \right)^T
\]

and apply Algorithm 4, where \( c = e^{(\beta)} \), to obtain \( y^{(\beta)} = y \in \mathbb{R}^{n+m-1} \). Update \( \beta \rightarrow \beta + 1 \).

4. Compute the product \( p_{BH} = Hv \in \mathbb{R}^{nl} \) as:

\[
p_{BH} = \left( y^{(1)}(1) \ y^{(1)}(2) \cdots y^{(1)}(n) \ y^{(0)}(n) \right)^T
\]

s. t. \[
x_{k+1} = A_t x_k + B_t u_k, \forall k \in \{0, \cdots, N-1\},
\]
\[
y_k = C x_k, \forall k \in \{0, \cdots, N-1\},
\]
\[
x_0 = \hat{x}_t,
\]

where for every time instance \( t \), \( A_t \in \mathbb{R}^{10 \times 10}, \ B_t \in \mathbb{R}^{10 \times 10}, \ C \in \mathbb{R}^{10 \times 10}, \) and \( r_t \) is the reference signal. The references \( r_t \) are changed randomly with uniform distribution between \([0, 0.1]\) for every 1000 iterations of \( t \). To apply DeePC or ODeePC to the predictive control problem defined by (19), we construct the data-based model (6) by permuting a number of iterations with a sequence of \( u_t \), which is persistently exciting of order \( 10(T_{ini} + N) \). Optimization (19) can then be formulated in the form of (10) (the objective function can be time-varying without loss of generality). The parameters of the optimization and model are given in Table 1.

| \( N_t \) | \( T_{ini} \) | \( N \) | \( \kappa \) | \( \epsilon_g \) |
|---|---|---|---|---|
| 50 | 20 | 1800 | 1651 | 0.1 |

**Table 1**

Number setup for the simulations.

To simulate a dynamic environment, we let \( A_t \) and \( B_t \) vary with \( t \). Specifically, the magnitude of each entry of \( A_t \) increases \( p \% \) of its value whenever \( t \) changes to \( t + 1 \), where \( p \) is randomly generated with a uniformly distribution in \([-0.01, 0.01]\). Similar biases are imposed on \( B_t \). With these biases, the optimization associated with DeePC becomes infeasible in less than 10 iterations.

To compare the performance of ODeePC with that of DeePC, we implemented the Gradient-DeePC algorithm which is the same as Algorithm 3 except that \( \mathcal{H}^t \) is kept unchanged over time. The results are shown in Figures 1, 2, and Figure 3. It can be seen that Gradient-DeePC starts diverging at around \( t = 6000 \); on the contrary, ODeePC converges to a near optimal point with the cost close to zero.

Another reason for the performance difference between ODeePC and Gradient-DeePC is on the element shifting step embedded in (11) when the optimization is updated from \( t \) to \( t + 1 \). Recall that ODeePC updates both \( \mathcal{H} \) and \( h \), whereas Gradient-DeePC updates \( h \) and keeps \( \mathcal{H} \) unchanged. This makes a difference because the shifting on variable \( z \) results in proper initialization of the optimization problem at \( t + 1 \) using the solution at \( t \). For the case of ODeePC, the majority of the constraints defined by \( \mathcal{H}g = h \) are kept when \( t \) is changed to \( t + 1 \), so the shifts defined in (11) result in variables that satisfy most equality constraints of the optimization problem at \( t + 1 \). On the contrary, Gradient-DeePC only shifts \( h \) when the optimization problem is updated. This implies that all the constraints of the optimization problem at \( t + 1 \) are different from the ones at \( t \). The “disconnection” between consecutive instances of the optimization problem is an additional disadvantage of Gradient-DeePC compared to ODeePC, which partially causes the divergence observed in Fig. 3.

![Fig. 1. Evolution of costs over iterations for ODeePC and Gradient-DeePC algorithms.](image1)

![Fig. 2. Tracking performance of one entry of the output y.](image2)
We also compare the computational times between Algorithm 5 and the direct matrix vector multiplication method. In our rather moderate-scale example, Algorithm 5 takes 0.51 second on average with $N_f = 50$ inner gradient iterations, whereas the average time for the direct multiplication is approximately 1.1 seconds. Both simulations are done on a desktop with 3.5-GHz CPU and 16-GB RAM. Algorithm 5 slashes the computational time in half, and the advantage expands for larger scale systems.

6 Conclusion and future work

In this paper, we presented an online data-enabled predictive (ODeePC) control method for optimal control of unknown systems, building on the recently proposed DeePC [1]. Our proposed ODeePC method leverages a primal-dual algorithm with real-time measurement feedback and recorded system data to compute the optimal control policy in real-time as system conditions change. ODeePC can generate control inputs dynamically, tracking changes in the system operating point while manifesting high computational efficiency. We prove that ODeePC’s iterative update rule converges to a neighborhood of the optimal control policy. In future work, we will explore ways to leverage information about the physical laws that govern the dynamics of the system in our data-driven method, to construct a more reliable system model and thus have a more robust overall control method.

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Thus, we also have that:

One can additionally verify that:

Using (.2), we can express the product of the Hankel matrix

One can easily verify that indeed:

Our proof is constructive. We first reduce the Hankel matrix-vector product into an equivalent Toeplitz matrix-vector product and eventually into a circulant matrix-vector product. We then show that the last product can be computed efficiently using FFT.

First, we multiply the Hankel matrix \( H \in \mathbb{R}^{n \times m} \) by a matrix \( \Pi \in \mathbb{R}^{m \times m} \) to obtain a Toeplitz matrix \( T \in \mathbb{R}^{n \times m} \). The matrix \( \Pi \) is required to have the following structure:

\[
\Pi = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\] (1)

One can easily verify that indeed:

\[
H \cdot \Pi = T.
\] (2)

Using (.2), we can express the product of the Hankel matrix \( H \) with the vector \( v \) as:

\[
H \cdot v = T \cdot \Pi^{-1} \cdot v.
\] (3)

One can additionally verify that:

\[
T \cdot \Pi = H.
\] (4)

Thus, we also have that:

\[
H \cdot v = T \cdot \Pi \cdot v = T \cdot v_p.
\] (5)

where \( v_p = \Pi \cdot v, \) \( v_p \in \mathbb{R}^m \). This vector has the same elements as \( v \) but sorted in reverse order:

\[
v_p := \left( v_m \ v_{m-1} \ \cdots \ v_1 \right)^T.
\] (6)

So far, we have shown that the product \( H \cdot v \) is equivalent to the product \( T \cdot v_p \). The next step is to embed the Toeplitz matrix \( T \) into a larger circulant matrix \( C \in \mathbb{R}^{(n+m-1) \times (n+m-1)} \) whose product with a vector can be computed efficiently. We construct the matrix \( C \) as follows:

\[
C = \begin{pmatrix}
T & * \\
* & *
\end{pmatrix}.
\] (7)

We emphasize here that, a \( n \times m \) Toeplitz matrix \( T \) should be embedded in a \( (n+m-1) \times (n+m-1) \) circulant matrix \( C \) with the matrix \( T \) being on its upper left block. This is because the distinct elements of the \( (n+m-1) \) diagonals of the Toeplitz matrix \( T \) are those that define the vector \( c \), which characterizes the circulant matrix \( C \) and precisely matches its first column. Moving forward, in light of (.7), the product \( T \cdot v_p \) can be expressed as a function of the circulant matrix \( C \) as follows:

\[
T \cdot v_p = \left( I_n \ 0_{m-1} \right) \cdot C \cdot v_c,
\] (8)

where the vector \( v_c \) is defined as:

\[
v_c = \left( v_p \ \ 0_{n-1} \right).
\] (9)

We know that the circulant matrix \( C \) has the nice property of being diagonalized by the FFT matrix \( F \). That is:

\[
C = F^{-1} \Lambda F.
\] (10)

Hence, the product \( C \cdot v_c \) can be computed as:

\[
C \cdot v_c = F^{-1} \Lambda F v_c,
\] (11)

which can be written as:

\[
C \cdot v_c = F^{-1} \Lambda \hat{v}_c,
\] (12)

where \( \hat{v}_c \) is the discrete Fourier transform of \( v_c \), and \( \Lambda = \text{diag}(F c) = \text{diag}(C) \) is a diagonal matrix with the elements of the discrete fourier transform of the vector \( c \), \( C \), on its diagonal. From (.8), it is easy to conclude that \( H \cdot v \) can be obtained as the first \( n \) elements of the Hadamard product of the vectors \( C \) and \( \hat{v}_c \). Note that \( c \) is the vector that fully specifies the circulant matrix \( C \) and corresponds to its first column. That completes the proof.
Algorithm 4 can be used to efficiently compute the product \( p_H \) of a non-square Hankel matrix \( H \in \mathbb{R}^{n \times m} \) with a vector \( v \in \mathbb{R}^m \). In this analysis, we computed the complexity of the proposed algorithm and proved that it carries out the appropriate computation by exploiting FFT.