G-constellations and the maximal resolution of a quotient surface singularity

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Abstract. For a finite subgroup $G$ of $\text{GL}(2, \mathbb{C})$, we consider the moduli space $\mathcal{M}_\theta$ of $G$-constellations. It depends on the stability parameter $\theta$ and if $\theta$ is generic it is a resolution of singularities of $\mathbb{C}^2/G$. In this paper, we show that a resolution $Y$ of $\mathbb{C}^2/G$ is isomorphic to $\mathcal{M}_\theta$ for some generic $\theta$ if and only if $Y$ is dominated by the maximal resolution under the assumption that $G$ is abelian or small.

1. Introduction

The moduli spaces of $G$-constellations (on an affine space) are introduced in [CI04]. It is a generalization of the Hilbert scheme of $G$-orbits, which is denoted by $G$-Hilb. The moduli space depends on some stability parameter $\theta$ and the moduli space of $\theta$-stable $G$-constellations is denoted by $\mathcal{M}_\theta$. If $G$ is a subgroup of $\text{SL}(n, \mathbb{C})$ acting on $\mathbb{C}^n$ and $n \leq 3$, then $\mathcal{M}_\theta$ is a crepant resolution of $\mathbb{C}^n/G$ for a generic stability parameter $\theta$. The main result of [CI04] is that for a finite abelian subgroup $G \subset \text{SL}(3, \mathbb{C})$ and for a projective crepant resolution $Y \to \mathbb{C}^3/G$, there is a generic stability parameter $\theta$ such that $Y \cong \mathcal{M}_\theta$. See [Kĉ14], [NdCS17], [Jun16] and [Jun18] for related results.

The purpose of this paper is to consider the case where $G$ is a finite subgroup of $\text{GL}(2, \mathbb{C})$. In this case, $G$-Hilb($\mathbb{C}^2$) is the minimal resolution of $\mathbb{C}^2/G$ by [Ish02] but $\mathcal{M}_\theta$ is a resolution which may not be minimal for generic $\theta$ (as we see in this paper). Then what is the condition for a resolution $Y \to \mathbb{C}^2$ to be isomorphic to some $\mathcal{M}_\theta$? One important observation is that there is a fully faithful functor (see Theorem 3)

$$D^b(\text{coh } \mathcal{M}_\theta) \hookrightarrow D^b(\text{coh}^G \mathbb{C}^2)$$

between the derived categories. According to the DK hypothesis [Kaw18], the inclusion of derived categories should be related with inequalities of canonical divisors. Then it is natural to ask if the following is true: $Y$ is isomorphic to $\mathcal{M}_\theta$ for some $\theta$ if and only if $Y$ is between the minimal and the maximal.

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resolutions (see Conjecture 4), where the maximal resolution means the unique maximal one satisfying the inequality as in [KSB88]. The main result of this paper is the following. Recall that $G$ is said to be small if it contains no pseudo reflection.

**Theorem 1** (= Theorem 7). Let $G \subset \text{GL}(2, \mathbb{C})$ be a finite small subgroup and let $X = \mathbb{C}^2/G$ be the quotient singularity. Then a resolution of singularities $Y \to X$ is isomorphic to $\mathcal{M}_0$ for some 0 if and only if $Y$ is dominated by the maximal resolution.

Conjecture 4 is a conjecture for general (not necessarily small) finite subgroups where the maximal resolution is defined for the pair of the quotient variety $\mathbb{C}^2/G$ and the associated boundary divisor. The “only if” part of the conjecture is proved in Proposition 1 by using the embedding of $G$ into $\text{SL}(3, \mathbb{C})$ and the fact that the moduli space of $G$-constellations for $G \subset \text{SL}(3, \mathbb{C})$ is a crepant resolution of $\mathbb{C}^3/G$. We can show that the conjecture is true if $G$ is abelian (Theorem 5) by using the result of [CI04]. The idea in the non-abelian case of Theorem 1 is to use iterated construction of moduli spaces as in [IIDC13] and reduce the problem to the abelian group case. Namely, let $N$ be the cyclic group generated by $-I$, which is a normal subgroup of every non-abelian finite small subgroup. We consider $G/N$-constellations on the moduli space of $N$-constellations in §7. In order to do such iterated constructions, we define $G$-constellations on a general variety and consider their stability parameters in §6. A key to the proof of Theorem 1 is the description of the space of stability parameters for $G/N$-constellations on the moduli space of $N$-constellations, which is done in §8.1. The proof of Theorem 1 is completed in §8.2.

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2. **$G$-constellations on $\mathbb{C}^n$**

2.1. **Definitions.** Let $V = \mathbb{C}^n$ be an affine space and $G \subset \text{GL}(V)$ a finite subgroup.

**Definition 1.** A $G$-constellation on $V$ is a $G$-equivariant coherent sheaf $E$ on $V$ such that $H^0(E)$ is isomorphic to the regular representation of $G$ as a $\mathbb{C}[G]$-module.
Let $R(G) = \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Z}\rho$ be the representation ring of $G$, where $\text{Irr}(G)$ denotes the set of irreducible representations of $G$. The parameter space of stability conditions of $G$-constellations is the $\mathbb{Q}$-vector space

$\Theta = \{ \theta \in \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \},$

where $\mathbb{C}[G]$ is regarded as the regular representation of $G$. The definition of the stability is based on the stability of quiver representations [Kin94]:

**Definition 2.** A $G$-constellation $E$ is $\theta$-stable (or $\theta$-semistable) if every proper $G$-equivariant coherent subsheaf $0 \subset F \subset E$ satisfies $\theta(H^0(F)) > 0$ (or $\theta(H^0(F)) \geq 0$). Here the representation space $H^0(F)$ of $G$ is regarded as an element of $R(G)$.

By virtue of King [Kin94], there is a fine moduli scheme $\mathcal{M}_{\theta} = \mathcal{M}_{\theta}(V)$ of $\theta$-stable $G$-constellations on $V$.

**Definition 3.** We say that a parameter $\theta \in \Theta$ is generic if a $\theta$-semistable $G$-constellation is always $\theta$-stable.

There is a morphism $\tau : \mathcal{M}_{\theta}(V) \to V/G$ which sends a $G$-constellation to its support. It is a projective morphism if $\theta$ is generic (see [CI04, Proposition 2.2]).

### 2.2. Results of [CI04]

In this subsection, we recall results from [CI04]. Suppose $V = \mathbb{C}^3$ and let $G \subset \text{SL}(V)$ be a finite abelian subgroup. For a generic parameter $\theta \in \Theta$, the morphism

$\tau : \mathcal{M}_{\theta} \to \mathbb{C}^3/G$

is a projective crepant resolution and we have a Fourier-Mukai transform

$\Phi_{\theta} : D^b(\text{coh } \mathcal{M}_{\theta}) \xrightarrow{\sim} D^b(\text{coh}^G(\mathbb{C}^3)).$

Here for a variety $Y$, coh $Y$ denotes the category of coherent sheaves on $Y$ and if $Y$ is acted on by a finite group $G$, coh$^G(Y)$ denotes the category of $G$-equivariant coherent sheaves on $Y$. The subset of $\Theta$ consisting of generic parameters is divided into chambers; the moduli space $\mathcal{M}_{\theta}$ and the equivalence $\Phi_{\theta}$ depend only on the chamber to which $\theta$ belongs. Thus we write $\mathcal{M}_C$ and $\Phi_C$ instead of $\mathcal{M}_{\theta}$ and $\Phi_{\theta}$ where $C$ is the chamber that contains $\theta$. We write

$\varphi_C : K(\text{coh}_0 \mathcal{M}_C) \to K(\text{coh}^G_0(\mathbb{C}^3))$

for the induced isomorphism of the Grothendieck groups of the full subcategories $\text{coh}_0 \mathcal{M}_{\theta}$ and $\text{coh}^G_0(\mathbb{C}^3)$ consisting of sheaves supported on the sub-
sets \( \tau^{-1}(0) \) and on \( \{0\} \) respectively. Since \( K(\text{coh}_0^G(\mathbb{C}^3)) \) has a basis consisting of skyscraper sheaves \( \mathcal{O}_0 \otimes \rho \) with \( \rho \in \text{Irr}(G) \), it is naturally identified with \( R(G) \).

The dual of \( \phi_C \) is regarded as the map
\[
\phi_C^* : K(\text{coh}^G(\mathbb{C}^3)) \to K(\text{coh}^\mathcal{M}_0)
\]
between the Grothendieck groups of the categories of sheaves without restrictions on the supports. Then \( K(\text{coh}^G(\mathbb{C}^3)) \) is identified with \( \text{Hom}(R(G), \mathbb{Z}) \) and \( \phi_C^* \) induces an isomorphism
\[
\Theta \cong F^1K(\text{coh}^\mathcal{M}_0)_\mathbb{Q},
\]
where \( F^1K(\text{coh}^\mathcal{M}_0) \) is the subgroup consisting of the classes of objects whose supports are at least of codimension \( i \).

On \( \mathcal{M}_C \) there are tautological bundles \( \mathcal{R}_\rho \) for irreducible representations \( \rho \) such that \( \bigoplus_\rho \mathcal{R}_\rho \otimes_{\mathcal{E}} \mathcal{R}_\rho \) has a structure of the universal \( G \)-constellation. For \( \theta \in C \),
\[
\mathcal{L}_C(\theta) := \bigotimes_\rho (\det \mathcal{R}_\rho)^{\otimes l(\rho)}
\]
is the (fractional) ample line bundle on \( \mathcal{M}_\theta \) obtained by the GIT construction. It coincides with the class
\[
[\phi_C^*(\theta)] \in F^1K(\text{coh} \mathcal{M}_C)_\mathbb{Q} / F^2K(\text{coh} \mathcal{M}_C)_\mathbb{Q} \cong \text{Pic}(\mathcal{M}_C)_\mathbb{Q} \tag{2.1}
\]
as in [CI04, § 5.1]. Hence \( [\phi_C^*(\theta)] \in \text{Amp}(\mathcal{M}_C) \) where \( \text{Amp}(\mathcal{M}_C) \) is the ample cone considered in \( \text{Pic}(\mathcal{M}_C)_\mathbb{Q} \). The main theorem of [CI04] and the argument in [CI04, § 8] show the following:

**Theorem 2 ([CI04]).** For any projective crepant resolution \( Y \to \mathbb{C}^3/G \) and a class \( l \in \text{Amp}(Y) \), there exist a chamber \( C \) with \( Y \cong \mathcal{M}_C \) and a parameter \( \theta \in C \) satisfying \( l = [\phi_C^*(\theta)] \).

**Proof.** The existence of a chamber \( C \) such that \( Y \cong \mathcal{M}_C \) is [CI04, Theorem 1.1]. Moreover, [CI04, Proposition 8.2] ensures that we can find a chamber \( C \) and a parameter \( \theta \in \bar{C} \) with \( l = [\phi_C^*(\theta)] \). Suppose \( \theta \in \bar{C} \setminus C \). We have to see we can perturb \( \theta \) in the fiber of \( p \circ \phi_C^* \) so that \( \theta \) is in some chamber, where
\[
p : F^1K(\text{coh} \mathcal{M}_C)_\mathbb{Q} \to \text{Pic}(\mathcal{M}_C)_\mathbb{Q}
\]
is the projection. Here recall that a wall of the chamber \( C \) is either the pre-image of a wall of the ample cone by \( p \circ \phi_C^* \) (type I or III) or does not contain a fiber of \( p \circ \phi_C^* \) (type 0); see [CI04, Theorem 5.9]. In our case, \( p \circ \phi_C^*(\theta) = l \).
is ample and therefore \( \theta \) is on walls of type 0. Since the images of adjacent chambers in \( F^1 K(\mathcal{H}\mathcal{M}) \) are related as in [CI04, (8.2) or (8.3)], we can perturb \( \theta \) in the fiber of \( p \circ \phi^* \) and go out of walls.

### 2.3. \( G \)-constellations on \( \mathbb{C}^2 \)

**Theorem 3.** If \( \theta \) is generic, then the moduli space \( M_0 \) is a resolution of singularities of \( \mathbb{C}^2 / G \). Moreover, the universal family of \( G \)-constellations defines a fully faithful functor

\[
\Phi_0 : D^b(\text{coh } M_0) \rightarrow D^b(\text{coh }^G \mathbb{C}^2).
\]

**Proof.** This is essentially Theorem 1.3 in the first arXiv version of [BKR01]. We have the inequality

\[
\dim M_0 \times (\mathbb{C}^2 / G) \cdot M_0 \leq \dim \mathbb{C}^2
\]

which is sharper than the assumption in [BKR01]. This allows us to apply the argument of [BKR01] (without using the triviality of the Serre functors) to show that \( \Phi_0 \) is fully faithful and that \( M_0 \) is smooth and connected (see [Ish02, Theorem 6.2]).

The problem we consider is to characterize the resolutions \( Y \) such that \( Y \cong M_0 \) for some generic \( \theta \).

### 3. The maximal resolution

Let \( G \) be a finite subgroup of \( \text{GL}(2, \mathbb{C}) \), which is not necessarily small, i.e., the action may not be free on \( \mathbb{C}^2 \setminus \{0\} \). Then the quotient variety \( X = \mathbb{C}^2 / G \) is equipped with a boundary divisor \( B \) determined by the equality \( \pi^*(K_X + B) = K_{\mathbb{C}^2} \). More precisely, \( B \) is expressed as

\[
B = \sum_j \frac{m_j - 1}{m_j} B_j,
\]

where \( B_j \subseteq X \) is the image of a one-dimensional linear subspace whose pointwise stabilizer subgroup \( G_j \subseteq G \) is cyclic of order \( m_j \). Note that \( G \) is small if and only if \( B = 0 \). Let \( \tau : Y \rightarrow X \) be a resolution of singularities and write

\[
K_Y + \tau_*^{-1} B \equiv \tau^*(K_X + B) + \sum_i a_i E_i,
\]

where \( E_i \) are the exceptional divisors and \( a_i \in \mathbb{Q} \). Recall that \( (X, B) \) is a KLT pair ([KM98, Proposition 5.20]), which implies \( a_i > -1 \) for all \( i \). Then among
the resolutions $Y$ which satisfy $a_i \leq 0$ for all $i$, there is a unique maximal one, as in [KSB88] (see also [Kaw18, Theorem 17]). It is called the maximal resolution of $(X, B)$ and we denote it by $Y_{\text{max}}$.

Notice that the system of inequalities $a_i \leq 0$ is an inequality between canonical divisors. According to the DK-hypothesis [Kaw18], the inequality should correspond to the embedding of derived categories in Theorem 3 with $Y = \mathcal{M}_\theta$. Thus we make the following conjecture:

**Conjecture 4.** Let $G \subset \text{GL}(2, \mathbb{C})$ be a finite subgroup and consider the quotient $X = \mathbb{C}^2/G$ with the boundary divisor $B$. For any resolution of singularities $Y \to X$, there is a generic $\theta \in \Theta$ with $Y \cong \mathcal{M}_\theta$ if and only if there is a morphism $Y_{\text{max}} \to Y$ over $X$. Here $Y_{\text{max}}$ is the maximal resolution of $(X, B)$.

4. “Only if” part

In this section, we show the “only if” part of Conjecture 4. Embed $\text{GL}(2, \mathbb{C})$ into $\text{SL}(3, \mathbb{C})$ by sending a matrix $A \in \text{GL}(2, \mathbb{C})$ to $\left( \begin{array}{cc} A & 0 \\ 0 & \det(A)^{-1} \end{array} \right)$.

Then for $\theta \in \Theta$, we can consider the moduli space $\mathcal{M}_\theta(\mathbb{C}^3)$ of $\theta$-stable $G$-constellations on $\mathbb{C}^3$ with respect to the action of $G$ on $\mathbb{C}^3$.

**Lemma 1.** For any $\theta \in \Theta$, there is a closed embedding $\mathcal{M}_\theta \hookrightarrow \mathcal{M}_\theta(\mathbb{C}^3)$ which fits into the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_\theta & \hookrightarrow & \mathcal{M}_\theta(\mathbb{C}^3) \\
\downarrow & & \downarrow \\
\mathbb{C}^2/G & \hookrightarrow & \mathbb{C}^3/G.
\end{array}
$$

Moreover, if $\theta$ is generic for $G$-constellations on $\mathbb{C}^3$, then the vertical arrows are projective and hence are resolutions of singularities.

**Proof.** Recall that the universal family of $G$-constellations on $\mathbb{C}^3$ is given by the tautological bundles $\{ \mathcal{R}_\rho \}_{\rho \in \text{Irr} \ G}$ and the $G$-equivariant morphism

$$
\bigoplus_{\rho} \mathcal{R}_\rho \otimes_{\mathbb{C}} \rho \to \mathbb{C}^3 \otimes \left( \bigoplus_{\rho} \mathcal{R}_\rho \otimes_{\mathbb{C}} \rho \right).
$$

(4.1)

If $\rho_{\text{nat}}$ denotes the representation given by $G \subset \text{GL}(2, \mathbb{C})$, then $\mathbb{C}^3$ above is $\rho_{\text{nat}} \oplus \det \rho_{\text{nat}}^*$. Taking the third coordinate of $\mathbb{C}^3$ in (4.1) we obtain a morphism

$$
z_\rho : \mathcal{R}_\rho \to \mathcal{R}_\rho \otimes_{\det \rho_{\text{nat}}}$$
for each \( r \). It is straightforward that the scheme theoretic intersection of the zero loci of \( z_r \)'s is isomorphic to \( \mathcal{M}_\theta \). Hence \( \mathcal{M}_\theta \) is a closed subscheme of \( \mathcal{M}_\theta(\mathbb{C}^3) \). Moreover, we can see that the composite \( \mathcal{M}_\theta \hookrightarrow \mathcal{M}_\theta(\mathbb{C}^3) \to \mathbb{C}^3/G \) factors through \( \mathbb{C}^2/G \). If \( \theta \) is generic for \( G \)-constellations on \( \mathbb{C}^3 \), then it is also generic for \( G \)-constellations on \( \mathbb{C}^2 \), from which the projectivities of the vertical arrows follow.

Now let us prove the “only if” part.

**PROPOSITION 1.** If \( \theta \) is generic, then there is a morphism \( Y_{\text{max}} \to \mathcal{M}_\theta \) over \( X \).

**PROOF.** Putting \( Y = \mathcal{M}_\theta \), we show that \( a_i \leq 0 \) for all \( i \) in (3.1). Embed \( G \) into \( \text{SL}(3, \mathbb{C}) \) and consider \( U := \mathcal{M}_\theta(\mathbb{C}^3) \), the moduli space of \( \theta \)-stable \( G \)-constellations on \( \mathbb{C}^3 \). Here, we may assume that \( \theta \) is generic for \( G \)-constellations on \( \mathbb{C}^3 \) by slightly perturbing \( \theta \) if necessary. Then \( U \) is a crepant resolution of \( \mathbb{C}^3/G \) containing \( Y \) by Lemma 1 and therefore we have

\[
K_Y \cong \mathcal{O}_U(Y)|_Y. \tag{4.2}
\]

Let \( z \) be the coordinate function of \( \mathbb{C}^3 \) such that \( \mathbb{C}^2 \subset \mathbb{C}^3 \) is defined by \( z = 0 \). Then \( z^n \) is invariant under the action of \( G \) where \( n \) is the order of \( G \). We claim that the principal divisor \( (z^n) \) on \( U \) is of the form

\[
(z^n) = nY + \sum_j \frac{n(m_j - 1)}{m_j} B'_j + \sum_k d_k D_k \tag{4.3}
\]

where \( B'_j, D_k \subset U \) are prime divisors such that \( B'_j \cap Y = \tau^{-1}_j B_j \) and \( D_k \cap Y \) is contained in the exceptional locus of \( Y \to \mathbb{C}^2/G \) (or empty). This is saying that there exists an exceptional prime divisor \( B'_j \) of \( U \to \mathbb{C}^3/G \) lying over \( B_j \) with \( B'_j \cap Y = \tau^{-1}_j B_j \) and that its coefficient in \( (z^n) \) is \( \frac{n(m_j - 1)}{m_j} \). We may check this over the complete local ring \( \hat{\mathcal{O}}_{\mathbb{C}^3/G,P} \) at a point \( P \in B_j \setminus \{0\} \). Since \( G_j \) is the stabilizer subgroup of a point of \( \mathbb{C}^3 \) lying over \( P \), there is an isomorphism of complete local rings:

\[
\hat{\mathcal{O}}_{\mathbb{C}^3/G,P} \cong \hat{\mathcal{O}}_{\mathbb{C}^2/G_j,0}.
\]

Let \( \tilde{B}_j \) be a line in \( \mathbb{C}^2 \) mapped to \( B_j \) and take a \( G_j \)-invariant linear subspace \( \tilde{B}_j^\perp \) of \( \mathbb{C}^3 \) such that

\[
\mathbb{C}^3 = \tilde{B}_j \times \tilde{B}_j^\perp.
\]

Then \( G_j \cong \mathbb{Z}/m_j \mathbb{Z} \) is a subgroup of \( \{1\} \times \text{SL}(\tilde{B}_j^\perp) \) and therefore we have

\[
\mathbb{C}^3/G_j \cong \tilde{B}_j \times (\tilde{B}_j^\perp/G_j),
\]
where $\tilde{B}_j^+ / G_j$ is a rational double point of type $A_{m_j - 1}$. Thus we can see that on the crepant resolution

$$U \times_{(\mathbb{C}^3/G)} \text{Spec} \hat{O}_{\mathbb{C}^3/G,P} \to \text{Spec} \hat{O}_{\mathbb{C}^3/G,P} \cong \text{Spec} \hat{O}_{\mathbb{C}^3/G,[0]}$$

there is a prime divisor $\tilde{B}_j'$ with desired properties such that the coefficient of $\tilde{B}_j'$ in the divisor $(z^m)$ is $m_j - 1$. Since $m_j$ divides $n$, this proves (4.3).

From (4.2) and (4.3), we obtain

$$K_Y + \tau_*^{-1} B \equiv - \sum \frac{d_k}{n} (D_k \cap Y).$$

Here, note that $z^n$ is a regular function and therefore the coefficients in (4.3) are all non-negative. Especially, we have $d_k \geq 0$ for all $k$. This proves the assertion since $K_Y + B \in \text{Pic}(Y) \otimes \mathbb{Q} = 0$ in (3.1).

5. Abelian group case

Let $G \subset \text{GL}(2, \mathbb{C})$ be a finite abelian subgroup of order $n$. As in the previous section, we embed $G \subset \text{GL}(2, \mathbb{C})$ into $\text{SL}(3, \mathbb{C})$.

**Theorem 5.** Conjecture 4 is true if $G$ is abelian.

**Proof.** It is sufficient to prove the “if” part by Proposition 1. Let $Y \to X = \mathbb{C}^2 / G$ be a resolution which is dominated by $Y_{\text{max}}$. By Proposition 2 below, there is a projective crepant resolution $U \to \mathbb{C}^3 / G$ such that $Y \subset U$. Then [CI04] ensures that there is a generic parameter $\theta$ such that $U \cong U_{\theta}(\mathbb{C}^3)$. Then $U_{\theta}(\mathbb{C}^2)$ is isomorphic to $Y$ by Lemma 1.

Before stating the proposition, we need some notation. We diagonalize $G$ and write

$$g = \text{diag}(\zeta_n^{a_g}, \zeta_n^{b_g})$$

for $g \in G$ where $\zeta_n$ is a primitive $n$-th root of unity. Put

$$N_2 := \mathbb{Z}^2 + \sum_{g \in G} \mathbb{Z} \cdot \frac{1}{n} (a_g, b_g),$$

$$N_3 := \mathbb{Z}^3 + \sum_{g \in G} \mathbb{Z} \cdot \frac{1}{n} (a_g, b_g, -a_g - b_g)$$

which are the lattices of one-parameter subgroups for the toric varieties $\mathbb{C}^2 / G$ and $\mathbb{C}^3 / G$ respectively. The *junior simplex* $\Delta \subset (N_3)_\mathbb{R}$ is the triangle with vertices $e_1, e_2, e_3$ where $\{e_1, e_2, e_3\}$ is the basis of $\mathbb{Z}^3$ with $e_1, e_2 \in \mathbb{Z}^2$. A
crepant resolution $U$ corresponds to a basic triangulation of $\Delta$. For a basic triangulation $\Sigma$ of $\Delta$, let $U_{\Sigma}$ be the corresponding crepant resolution.

Consider the natural projection

$$p_{12} : N_3 \to N_2$$

and put $\Delta' := p_{12}(\Delta) \cong \Delta$. Let $e_i' \in (\mathbb{R}_{\geq 0})e_i \cap N_2$ be the primitive vector and write $e_i = m_ie_i'$ for $i = 1, 2$. If $B_i \subset \mathbb{C}^2/G$ denote the divisor corresponding to $e_i'$, then

$$B := \frac{m_1 - 1}{m_1}B_1 + \frac{m_2 - 1}{m_2}B_2$$

is the boundary divisor for the quotient $\mathbb{C}^2/G$. A resolution $Y$ of $\mathbb{C}^2/G$ is given by choosing primitive vectors $v_0, v_1, \ldots, v_s$ of $(\mathbb{Z}_{\geq 0})^2 \cap N_2$ such that $v_0 = e_1'$, $v_s = e_2'$ and \{v_{i-1}, v_i\} is a basis of $N_2$ for $i = 1, \ldots, s$. If $E_i$ denotes the exceptional divisor corresponding to $v_i$ for $i = 1, \ldots, s - 1$, then the discrepancy $a_i$ of $E_i$ for the pair $(X, B)$ is $\alpha_i + \beta_i - 1$ where $v_i = (\alpha_i, \beta_i)$. Therefore, $Y$ is dominated by the maximal resolution $Y_{\text{max}}$ of $(X, B)$ if and only if all of $v_1, \ldots, v_{s-1}$ are in $\Delta'$.

Let $G_{(1,0)} \subset G$ be the stabilizer subgroup of $(1, 0) \in \mathbb{C}^2 = \mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$. Then $G_{(1,0)}$ acts on $\{1\} \times \mathbb{C}^2 \cong \mathbb{C}^2$ as a subgroup of $\text{SL}(2)$ and the quotient $((\{1\} \times \mathbb{C}^2))/G_{(1,0)}$ is a closed subvariety of $\mathbb{C}^3/G$. Let

$$W \to ((\{1\} \times \mathbb{C}^2))/G_{(1,0)}$$

be the minimal resolution. Notice that $W$ is contained in any crepant resolution $U$ of $\mathbb{C}^3/G$ since $((\{1\} \times \mathbb{C}^2))/G_{(1,0)} \subset \mathbb{C}^3/G$ is transversal to the one-dimensional stratum $(\mathbb{C}^\times \times \{(0, 0)\})/G$. Now we prove the following proposition. The surjectivity of the ample cones will be used in the proof of the main theorem.

**Proposition 2.** Let $Y \to \mathbb{C}^2/G$ be a resolution dominated by $Y_{\text{max}}$. Then there is a projective crepant resolution $U = U_{\Sigma} : \mathbb{C}^3/G$ containing $Y$ such that the restriction map $\text{Amp}(U) \to \text{Amp}(W)$ of the ample cones is surjective.

**Proof.** Since $Y$ is dominated by $Y_{\text{max}}$, it is defined by primitive vectors $v_0, v_1, \ldots, v_s \in \Delta' \cap N_2$. Let $w_j \in \Delta \cap N_2$ be the unique lift of $v_i$. For a basic triangulation $\Sigma$ of $\Delta$, $U = U_{\Sigma}$ contains $Y$ if and only if the points connected to $e_3$ in $\Sigma$ are exactly $w_0, \ldots, w_s$.

We prove the assertion by the induction on the order $|G|$ of $G$. If $|G| = 1$, then there is nothing to prove. We consider the number

$$v := \#(\{w_0, \ldots, w_{s-1}\} \setminus \{e_1\}) \geq 0.$$
If \( v = 0 \), then \( s \) must be 1 and \( w_0 = e_1 \) is a primitive vector. Especially, \( \{e_1, v_1\} \) is a basis of \( N_2 \). In this case, \( A \) has a unique basic triangulation \( \Sigma \) and \( U_\Sigma \cong W \times \mathbb{C} \). Hence the restriction map \( \text{Amp}(U_\Sigma) \to \text{Amp}(W) \) is an isomorphism.

Suppose \( v > 0 \). Let \( w \in \{w_0, \ldots, w_{v-1}\} \backslash \{e_1\} \) be a point such that the coefficient of \( e_3 \) in \( w \) is the smallest. Then \( w \) determines a star subdivision of \( A : A = \bigcup_{i=1}^3 A_i \) where \( A_1, A_2, A_3 \) are the triangles \( we_2e_3, we_1e_3, we_1e_2 \) respectively. Note that either \( A_2 \) or \( A_3 \) may be degenerate, in which case we simply ignore the degenerate one in the sequel. This subdivision of \( A \), which is denoted by \( \Sigma_0 \), determines a projective crepant birational morphism \( U_{\Sigma_0} \to \mathbb{C}^3/G \) where \( U_{\Sigma_0} \) is a toric variety with at most Gorenstein quotient singularities. The choice of \( w \) implies that \( w_0, \ldots, w_v \) are in \( A_1 \cup A_2 \). Hence by the induction hypothesis, there are basic triangulations \( \Sigma_1, \Sigma_2 \) of \( A_1 \) and \( A_2 \) respectively, which satisfy the following conditions: in \( \Sigma_1 \cup \Sigma_2 \), the vertices connected to \( e_3 \) are exactly \( w_0, \ldots, w_v \), the map \( \text{Amp}(U_{\Sigma_1}) \to \text{Amp}(W) \) is surjective and \( \text{Amp}(U_{\Sigma_2}) \) is non-empty. We choose an arbitrary basic triangulation \( \Sigma_3 \) of \( A_3 \) with non-empty \( \text{Amp}(U_{\Sigma_3}) \). Combining the triangulations \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \) together, we obtain a basic triangulation of \( A \) such that \( U_\Sigma \supset Y \). Since \( A = \bigcup_{i=1}^3 A_i \) is a star subdivision, we see that \( U_\Sigma \to U_{\Sigma_0} \) is a projective morphism and the map \( \text{Amp}(U_\Sigma) \to \text{Amp}(U_{\Sigma_0}) \) is surjective. Therefore, the morphism \( U_\Sigma \to \mathbb{C}^3/G \) is also projective and \( \text{Amp}(U_\Sigma) \to \text{Amp}(W) \) is surjective.

6. \( G \)-constellations on a variety

In the case of \( G \)-constellations for non-abelian \( G \subset GL(2, \mathbb{C}) \), we shall use the iterated construction of moduli spaces for a normal subgroup of \( G \) as in [INdC13]. In order to do so, we have to consider \( G \)-constellations on a variety, rather than an affine space. Especially, the space of stability parameters will be larger than the affine case in general.

Suppose \( U \) is a quasi projective variety of finite type over \( \mathbb{C} \) and \( G \) is a finite group acting on \( U \). Let \( \text{coh}^G(U) \) be the abelian category of \( G \)-equivariant coherent sheaves on \( U \) and \( \text{coh}^{\text{cpt}}_G(U) \) its subcategory consisting of sheaves whose supports are proper over \( \mathbb{C} \). The corresponding Grothendieck groups are denoted by \( K(\text{coh}^G(U)) \) and \( K(\text{coh}^{\text{cpt}}_G(U)) \) respectively. We also consider the perfect derived category \( \text{Perf}^G(U) \) of \( G \)-equivariant perfect complexes and its Grothendieck group \( K(\text{Perf}^G(U)) \). For \( \alpha \in K(\text{Perf}^G(U)) \) and \( \beta \in K(\text{coh}^{\text{cpt}}_G(U)) \), we write

\[
\chi(\alpha, \beta) := \sum_i (-1)^i \dim \text{Ext}^i_{\text{Perf}^G(U)}(\alpha, \beta)^G.
\]
Let \( \text{coh}^{G}_{0,\dim}(U) \) be the subcategory of \( \text{coh}^{G}_{\text{pt}}(U) \) consisting of sheaves with 0-dimensional support. We define the stability condition of objects in \( \text{coh}^{G}_{0,\dim}(U) \).

**Definition 4.** Fix a class \( \xi \in K(\text{Perf}^{G}(U)) \). An object \( E \in \text{coh}^{G}_{0,\dim}(U) \) is said to be \( \xi \)-stable (or \( \xi \)-semistable) if \( \chi(\xi; E) = 0 \) and if for every non-trivial \( G \)-equivariant subsheaf \( F \) of \( E \), \( \chi(\xi; [F]) > 0 \) (or \( \chi(\xi; [F]) \geq 0 \)).

In the case where \( U = \mathbb{C}^{N} \) is an affine space with a linear \( G \)-action, \( K(\text{Perf}^{G}(U)) = K(\text{coh}^{G}(U)) \) is isomorphic to (the dual of) the representation ring \( R(G) \) and the definition coincides with the (\( \mathbb{Z} \)-valued) one in §2.1.

We have a well-defined function \( \text{rank} : K(\text{Perf}^{G}(U)) \to \mathbb{Z} \) which extends the rank of a locally free sheaf. Put

\[
K(\text{Perf}^{G}(U))^{0} := \{ \xi \in K(\text{Perf}^{G}(U)) | \text{rank} \xi = 0 \}.
\]

**Definition 5.** A \( G \)-constellation on \( U \) is a \( G \)-equivariant coherent sheaf \( E \) on \( U \) with finite support such that \( H^{0}(E) \) is isomorphic to the regular representation of \( G \) as a representation of \( G \) and \( \chi(\xi; E) = 0 \) for any \( \xi \in K(\text{Perf}^{G}(U))^{0} \).

For any \( \xi \in K(\text{Perf}^{G}(U))^{0} \), we can discuss the \( \xi \)-(semi)stabilities of \( G \)-constellations on \( U \) according to Definition 4. Since the multiplication by a positive integer does not change the stability condition, we may replace \( K(\text{Perf}^{G}(U))^{0} \) by \( K(\text{Perf}^{G}(U))_{\mathbb{Q}}^{0} \).

**Remark 1.** In general, there may exist an object \( E \) supported on several fixed points such that \( H^{0}(E) \cong R(G) \) but \( \chi(\xi; E) \neq 0 \) for some \( \xi \in K(\text{Perf}^{G}(U))^{0} \). Definition 5 excludes such cases.

**Remark 2.** If \( U \) is smooth, then \( K(\text{Perf}^{G}(U)) \) coincides with \( K(\text{coh}^{G}(U)) \) and we write \( K(\text{coh}^{G}(U))^{0} \) instead of \( K(\text{Perf}^{G}(U))^{0} \).

Now we define the moduli functors of \( G \)-constellations:

**Definition 6.** Fix a class \( \xi \in K(\text{Perf}^{G}(U))_{\mathbb{Q}}^{0} \). Then the moduli functor for the \( \xi \)-stable \( G \)-constellations on \( U \) is defined to be the functor

\[
S \mapsto \{ \text{flat families of } \xi \text{-stable } G \text{-constellations parameterized by } S \}/\sim
\]

for a locally noetherian scheme \( S \) over \( \mathbb{C} \) where \( E_{S} \sim F_{S} \) for flat families \( E_{S} \) and \( F_{S} \) means that there is a line bundle \( L \) on \( S \) such that \( E_{S} \cong F_{S} \otimes L \).

**Remark 3.** We show the existence of the moduli scheme in a very special case in Theorem 6. We do not discuss the existence problem in a general case in this paper.
7. Iterated construction of moduli spaces

In this section, let $V$ denote either $\mathbb{C}^2$ or $\mathbb{C}^3$ and consider a finite subgroup $G \subset \text{GL}(V)$ with a normal subgroup $N$ of $G$ such that $N \subset \text{SL}(V)$. Let

$$\theta^N : R(N) \to \mathbb{Z}$$

be a generic stability parameter for $N$-constellations on $V$, which is fixed by the conjugate action of $G$ on $R(N)$. Put $Y_N = \mathcal{M}_{\mu^N}(V)$ and $\tilde{G} = G/N$. Since $N \subset \text{SL}(V)$ and $\text{dim} V \leq 3$, there is an equivalence

$$\Phi : D^b(\text{coh} \tilde{G}(Y_N)) \cong D^b(\text{coh}^G(V))$$

(7.1)

as in [IU15, Theorem 4.1] defined by

$$\Phi(-) = \mathbb{R}(p_Y)_*((p_{YN})^*(-) \otimes \mathcal{U})$$

where $p_Y$, $p_{YN}$ are the projections of $YN \times V$ and $\mathcal{U}$ is the universal family of $N$-constellations.

**Lemma 2.** Let $\mathcal{E}$ be a $\tilde{G}$-equivariant coherent sheaf on $YN$ with finite support. Then $\mathcal{E}$ is a $\tilde{G}$-constellation on $YN$ if and only if $\Phi(\mathcal{E})$ is a $G$-constellation on $V$. In this case, $\Phi(\mathcal{E})$ is $\theta^N$-semistable.

**Proof.** By the definition of $\Phi$, we can see that $\Phi(\mathcal{E})$ is a 0-dimensional sheaf. Since $\Phi$ is an equivalence, we have $\chi(\xi, \mathcal{E}) = \chi(\Phi(\xi), \Phi(\mathcal{E}))$. Moreover, we can see rank $\xi = \text{rank} \Phi(\xi)$ for any $\xi \in K(\text{coh}^G(Y_N))$. Therefore, if $\mathcal{E}$ is a $\tilde{G}$-constellation, $\chi(\xi, \Phi(\mathcal{E})) = 0$ for any $\xi \in K(\text{coh}^G(V))^0$. This implies that $H^0(\Phi(\mathcal{E}))$ is a multiple of the regular representation $\mathbb{C}[\tilde{G}]$. If we regard $\mathcal{E}$ as an object of $\text{coh}(YN)$, it is an Artinian sheaf of length $|\tilde{G}|$ and therefore $\Phi(\mathcal{E})$ as an object of $\text{coh}^N(V)$ has a filtration of length $|\tilde{G}|$ whose factors are $\theta^N$-stable $N$-constellations. Therefore, $\Phi(\mathcal{E})$ is $\theta^N$-semistable and $H^0(\Phi(\mathcal{E}))$ as a representation of $N$ is the direct sum of $|\tilde{G}|$ copies of the regular representation of $N$. This implies that $H^0(\Phi(\mathcal{E})) \cong \mathbb{C}[G]$ and therefore $\Phi(\mathcal{E})$ is a $G$-constellation. The converse is proved in the same way.

The following lemma follows from the arguments in [BKR01, §8]:

**Lemma 3.** Let $E$ be an $N$-equivariant coherent sheaf on $V$ with finite support such that $H^0(E)$ is isomorphic to $\mathbb{C}[N]^{\otimes s}$ for some integer $s > 0$ as a $\mathbb{C}[N]$-module. If $E$ is $\theta^N$-stable, then we have $s = 1$, i.e., $E$ is an $N$-constellation.

We compose $\theta^N$ with the restriction map $R(G) \to R(N)$ and regard it as a stability parameter for $G$-constellations as in [IINdC13, §2.2].
Lemma 4. Let $E$ be a $G$-equivariant coherent sheaf on $V$ with finite support such that $H^0(E) \cong \mathbb{Z}[G]^{\otimes s}$ for some $s$. If $E$ is $N$-semistable in $\text{coh}^G(V)$, then it is also $\theta^N$-semistable in $\text{coh}^N(V)$.

Proof. Let $\eta : R(N) \to \mathbb{Z}$ be a group homomorphism such that $\eta(\rho) > 0$ for any irreducible representation $\rho$ of $N$. We further suppose $\eta$ is invariant under the conjugate action of $G$. Then, $Z(E) := \theta^N(H^0(E)) + \sqrt{-1}\eta(H^0(E))$
defines a $G$-invariant Bridgeland stability condition [Bri07, Example 5.5] (see also [BCZ17, Lemma 7.1.3]) on $\text{coh}^N(V)_0$, the category of $N$-equivariant coherent sheaves on $V$ with 0-dimensional support. As in [BCZ17, Lemma 7.1.5], the equality $\theta^N(H^0(E)) = 0$ implies that $E$ is $\theta^N$-semistable if and only if it is semistable with respect to $Z$. Assume $E$ is not $\theta^N$-semistable and let $F \subset E$ be the first step of the Harder-Narasimhan filtration of $E$ in $\text{coh}^N(E)$ with respect to $Z$. Then the uniqueness of the HN filtration and the $G$-invariance of $Z$ imply that $F$ is invariant under the $G$-action. This means that $F$ is a subsheaf of $E$ in $\text{coh}^G(V)$, which contradicts the $\theta^N$-semistability of $E$ in $\text{coh}^G(V)$.

Proposition 3. The functor $\Phi$ induces a bijection from the set of $G$-constellations on $Y_N$ to the set of $N$-semistable $G$-constellations on $V$.

Proof. If $\mathcal{E}$ is a $G$-constellation on $Y_N$, then $\Phi(\mathcal{E})$ is a $\theta^N$-semistable $G$-constellation by Lemma 2. Conversely, suppose $E$ is a $\theta^N$-semistable $G$-constellation on $V$. By Lemma 2, it suffices to show that $\Phi^{-1}(E)$ lies in $\text{coh}^G(Y_N)$ and has a 0-dimensional support. For this purpose, we may regard $\Phi$ as an equivalence $D^b(\text{coh} Y_N) \cong D^b(\text{coh}^N(V))$. By Lemma 4, $E$ is $\theta^N$-semistable as a sheaf in $\text{coh}^N(V)$ and therefore has a filtration whose factors are $\theta^N$-stable $N$-constellations by Lemma 3. Then, $\Phi^{-1}(E)$ as an object in $D^b(\text{coh}(Y_N))$ is a sheaf with a filtration whose factors are skyscraper sheaves. This is what we needed.

Let

$$\varphi : K(\text{coh}^G(Y_N))^0 \cong K(\text{coh}^G(V))^0 \cong \Theta$$

be the isomorphism induced by $\Phi$. The following theorem generalizes [II10dC13, Theorem 2.6].

Theorem 6. Let $\theta^N : R(N) \to \mathbb{Z}$ be a generic stability condition for $N$-constellations fixed by the conjugate action of $G$ and $\xi \in K(\text{coh}^G(Y_N))^0$ be a stability parameter for $G$-constellations on $Y_N$. 


There exists a scheme $\mathcal{M}_\zeta(Y_N)$ representing the moduli functor for $\zeta$-stable $G$-constellations on $Y_N$. 

If we put 

$$\theta := m\theta^N + \varphi(\xi)$$ 

for $m \gg 0$, then $\mathcal{M}_\theta(V)$ is isomorphic to the moduli space $\mathcal{M}_\zeta(Y_N)$ of $\zeta$-stable $\overline{G}$-constellations on $Y_N$. 

**Proof.** What we prove is that $\mathcal{M}_\theta(V)$ in (2) represents the moduli functor in (1). We choose $m$ so that 

$$m > \sum_{\rho \in \text{Irr}(G)} |(\varphi(\xi))(\rho)| \dim \rho.$$ 

Then for any subsheaf $F$ of a $G$-constellation, we have $|(\varphi(\xi))(F)| < m$.

Let $\mathcal{E}$ be a $\zeta$-stable $G$-constellation on $Y_N$. Then $\Phi(\mathcal{E})$ is a $\theta^N$-semistable $G$-constellation by Proposition 3. Therefore, a subsheaf $F$ of $\Phi(\mathcal{E})$ satisfies $\theta^N(F) \geq 0$. If $\theta^N(F) > 0$, then we have $\theta(F) > 0$ by our choice of $m$. If $\theta^N(F) = 0$, then there is a subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $F = \Phi(\mathcal{F})$ as in [IINdC13, Lemma 2.6]. Then we obtain $\theta(F) = \chi(\xi, \mathcal{F}) > 0$ by the $\zeta$-stability of $\mathcal{E}$. Thus $\Phi(\mathcal{E})$ is $\theta$-stable.

Conversely, suppose $E$ is a $\theta$-stable $G$-constellation on $V$. Then it is $\theta^N$-semistable by our choice of $m$ and therefore $\mathcal{E} := \Phi^{-1}(E)$ is a $\overline{G}$-constellation by Proposition 3. For a subsheaf $\mathcal{F} \subset \mathcal{E}$, $F := \Phi(\mathcal{F})$ has a filtration as an object of $\text{coh}^N(V)$ whose factors are $N$-constellations. Therefore $F$ satisfies $\theta^N(F) = 0$ and hence we obtain $\chi(\xi, \mathcal{F}) = \theta(F) > 0$, which proves the $\zeta$-stability of $\mathcal{F}$.

Thus we have a bijection between $\zeta$-stable $\overline{G}$-constellations and $\theta$-stable $G$-constellations. To establish an isomorphism $\mathcal{M}_\theta(V) \cong \mathcal{M}_\zeta(Y_N)$, we show that for any locally noetherian scheme $S$ over $\mathbb{C}$, this bijection can be extended to a bijection between flat families of $\zeta$-stable $\overline{G}$-constellations and flat families of $\theta$-stable $G$-constellations parameterized by $S$. Let $\mathcal{U}$ be the universal $N$-constellation on $Y_N \times V$ and $\mathcal{U}_S$ be the pull back of $\mathcal{U}$ to $Y_N \times V \times S$. Then we can define a functor 

$$\Phi_S : D^b(\text{coh}^G Y_N \times S) \rightarrow D^b(\text{coh}^G V \times S)$$

by

$$\Phi_S(-) = \mathbb{R}p_{V \times S}^*(\mathcal{U}_S \otimes p_{Y_N \times S}^*(-))$$

whose quasi-inverse is given by

$$\Phi_S^{-1}(-) = ((p_{Y_N \times S})_*(\mathcal{U}_{\mathcal{U}}[\dim V] \otimes p_{V \times S}^*(-)))^N.$$
Suppose $E_S$ is a flat family of $\xi$-stable $\mathcal{G}$-constellations on $Y_N$ parameterized by $S$. Then, for any geometric point $s$ of $S$, we have $\Phi_S(E_S) \cong \Phi(\Omega_S)$ as in [Bri99, Lemma 4.1], which is a $\theta$-stable $G$-constellation on $V$. Hence the argument in [Bri99, Proposition 4.2] implies that $\Phi_S(E_S)$ is actually a flat family of $G$-constellations on $V$. Conversely, if $E_S$ is a flat family of $\theta$-stable $G$-constellations, the same argument shows that $\Phi_S^{-1}(E_S)$ is a flat family of $\xi$-stable $N$-constellations on $Y_N$.

8. The case $G \ni -I$

In this section, put $V = \mathbb{C}^2$ and assume that $G \subset \text{GL}(V)$ contains $-I$, where $I$ is the identity matrix. We put $N := \langle -I \rangle < G$ and $\mathcal{G} := G/N$. Let $\theta^N$ be any generic stability parameter for $N$-constellations (which is automatically fixed by the conjugate action of $G$ since $N$ is central) and let $Y_N = \mathcal{M}_{\theta^N}(V)$ be the moduli space of $N$-constellations on $V$, on which $\mathcal{G}$ acts naturally. Since $Y_N$ is a crepant resolution of the $A_1$ singularity $V/N$, the maximal resolution of $(Y_N/\mathcal{G}, B_N)$ coincides with the maximal resolution of $(X, B)$, where $B_N$ is the boundary divisor on $Y_N$ determined by the ramification of $Y_N \to Y_N/\mathcal{G}$.

Let $C$ be the exceptional curve of $Y_N \to V/N$. Then the equivalence (7.1) restricts to the equivalence

$$\Phi : D^b(\text{coh}_{\mathcal{G}}(Y_N)) \cong D^b(\text{coh}_{\mathcal{G}}(V))$$

(8.1)

of full subcategories consisting of objects supported by the subsets $C \subset Y_N$ and $\{0\} \subset V$ respectively. Consider the Grothendieck groups of (8.1):

$$K(\text{coh}_{\mathcal{G}}(Y_N)) \cong K(\text{coh}_{\mathcal{G}}(V)),$$

(8.2)

where $K(\text{coh}_{\mathcal{G}}(V))$ is isomorphic to the representation ring $R(G)$ of $G$. Recall that there is a perfect pairing

$$\chi : K(\text{coh}_{\mathcal{G}}(V)) \times K(\text{coh}_{\mathcal{G}}(V)) \to \mathbb{Z}$$

defined by (6.1), which is isomorphic to

$$\chi : K(\text{coh}_{\mathcal{G}}(Y_N)) \times K(\text{coh}_{\mathcal{G}}(Y_N)) \to \mathbb{Z}$$

by $\Phi$. Let

$$F_iK(\text{coh}_{\mathcal{G}}(Y_N)) \subset K(\text{coh}_{\mathcal{G}}(Y_N))$$

be the subgroup generated by the classes of objects whose supports are at most $i$-dimensional. Then the classes of $\mathcal{G}$-constellations on $Y_N$ lie in
$F_0K\left(\text{coh}_{C}^G(Y_N)\right)$ and for a stability parameter
\[
\xi \in K\left(\text{coh}_{C}^G(Y_N)\right)_Q \cong K\left(\text{coh}_{C}^G(Y_N)\right)^*_Q,
\]
the actual stability condition depends only on its image in $F_0K\left(\text{coh}_{C}^G(Y_N)\right)^*_/Q$. In the next subsection, we investigate the structure of $F_0K\left(\text{coh}_{C}^G(Y_N)\right)$.

### 8.1. Structure of $F_0K\left(\text{coh}_{C}^G(Y_N)\right)$

In this subsection, we assume that $G$ is not abelian. Notice that $G$ acts on the exceptional curve $C \cong \mathbb{P}(V)$ through the homomorphism
\[
G \hookrightarrow \text{GL}(V) \twoheadrightarrow \text{PGL}(V)
\]
and let $Z \subset G$ be the kernel of $G \twoheadrightarrow \text{PGL}(V)$. It is the subgroup consisting of scalar matrices in $G$.

Since $G$ is non-abelian, $G/Z \subset \text{PGL}(V)$ is a polyhedral (or dihedral) group acting on $\mathbb{P}(V)$ which we regard as a (real) 2-sphere. There are three non-free $G/Z$-orbits in $C$: the projections of the vertices, edges and faces of the regular polyhedron to the sphere. These orbits are denoted by $O_1$, $O_2$ and $O_3$ respectively.

For a $G$-orbit $O \subset C$, let $\text{coh}_{O}^G(Y_N)$ denote the category of $G$-equivariant coherent sheaves supported on $O$. Then we have an equivalence
\[
\text{coh}_{O}^G(Y_N) \cong \text{coh}_{P}^G(Y_N)
\] (8.3)
where $G_P$ is the stabilizer subgroup of a point $P \in O$ and $\text{coh}_{P}^G(Y_N)$ is the category of $G_P$-equivariant coherent sheaves supported on $P$. Taking the Grothendieck groups of the both sides, we obtain
\[
K\left(\text{coh}_{O}^G(Y_N)\right) \cong R\left(\overline{G_P}\right)
\] (8.4)
where $R(\overline{G_P})$ is the representation ring of $\overline{G}$ regarded as an additive group.

Let $\overline{G_k} \subset \overline{G}$ be the stabilizer subgroup of a point in $O_k$, which is an abelian group since $Z := Z/N \subset \overline{G_k}$ is central and $\overline{G_k}/Z$ is cyclic. We consider the pushforward maps
\[
K\left(\text{coh}_{O_k}^G(Y_N)\right) \to F_0K\left(\text{coh}_{C}^G(Y_N)\right)
\] (8.5)
for $k = 1, 2, 3$. By (8.4) for $O = O_k$, these maps are regarded as maps
\[
\beta_k : R(\overline{G_k}) \to F_0K\left(\text{coh}_{C}^G(Y_N)\right).
\]
Since $Z$ is a subgroup of $\overline{G_k}$, we have the induction maps
\[
\alpha_k : R(Z) \to R(\overline{G_k}).
\]
Define a map $\alpha : R(\mathbb{Z})^\oplus 2 \to R(\mathcal{G}_1) \oplus R(\mathcal{G}_2) \oplus R(\mathcal{G}_3)$ by

$$\alpha(a, b) = (\alpha_1(a), -\alpha_2(a) + \alpha_3(b), -\alpha_3(b)).$$

The purpose of this subsection is to prove the following.

**Proposition 4.** Let $\mathcal{G}_k, \beta_k, \alpha$ be as above. Then the following is an exact sequence of additive groups:

$$0 \to R(\mathbb{Z})^\oplus 2 \xrightarrow{\alpha} R(\mathcal{G}_1) \oplus R(\mathcal{G}_2) \oplus R(\mathcal{G}_3) \xrightarrow{\beta} F_0K(\text{coh}_C(\text{Y}_N)) \to 0$$

where $\beta = (\beta_1, \beta_2, \beta_3)$.

The proof of the proposition is divided into three steps below. We first show that $\beta$ is surjective:

**Step 1.** The additive group $F_0K(\text{coh}_C(\text{Y}_N))$ is generated by sheaves supported on $O_1 \cup O_2 \cup O_3$.

**Proof.** It is obvious that $F_0K(\text{coh}_C(\text{Y}_N))$ is generated by simple objects (objects having no non-trivial subobjects). Moreover, a simple object is supported on a single orbit $O$ and is determined by an irreducible representation of the stabilizer subgroup $\mathcal{G}_P$ of a point $P \in O$ by (8.3). Therefore, it is sufficient to show that the class in $K(\text{coh}_C(\text{Y}_N))$ of a simple object $\mathcal{E}$ supported on a free $G/\mathbb{Z}$-orbit $O_f$ coincides with the class of some object $\mathcal{F}$ supported on $O_1 \cup O_2 \cup O_3$. Actually, we prove that for any $k \in \{1, 2, 3\}$ we can choose such an object $\mathcal{F}$ supported on $O_k$. Simple objects supported on the orbit $O_f$ are determined by irreducible representations of the stabilizer subgroup $\mathbb{Z} \subset \mathcal{G}$ by (8.3). To describe them, notice that $C = \mathbb{P}(V)$ carries a $G$-equivariant line bundle $\mathcal{L} = \mathcal{O}_C(1)$ on which an element $\lambda I \in \mathbb{Z}$ acts as the fiber-wise scalar multiplication by $\lambda$. On $\mathcal{L}^2$, the $G$-action is reduced to a $\mathcal{G}$-action and the induced actions of $\mathbb{Z}$ on the fibers of $\mathcal{L}^0, \mathcal{L}^2, \ldots, \mathcal{L}^{2(l-1)}$ are the irreducible representations of the cyclic group $\mathbb{Z}$, where $l$ is the order of $\mathbb{Z}$. Therefore, the simple objects supported on $O_f$ are

$$\mathcal{L}^0|_{O_f}, \mathcal{L}^2|_{O_f}, \ldots, \mathcal{L}^{2(l-1)}|_{O_f},$$

where we regard $O_f$ as a reduced subscheme. Now consider the exact sequences

$$0 \to \mathcal{L}^{2i} \otimes \mathcal{O}_C(-O_f) \to \mathcal{L}^{2i} \to \mathcal{L}^{2i}|_{O_f} \to 0$$

and

$$0 \to \mathcal{L}^{2i} \otimes \mathcal{O}_C(-n_kO_k) \to \mathcal{L}^{2i} \to \mathcal{L}^{2i}|_{n_kO_k} \to 0$$
for any $k \in \{1, 2, 3\}$ where $n_k$ is the order of $\bar{G}_k / \mathbb{Z}$. If we show $\mathcal{C}(-O_f) \cong \mathcal{C}(-n_k O_k)$ in $\text{coh} \bar{G}(Y_N)$, then we obtain

$$[\mathcal{L}^2|_{O_f}] = [\mathcal{L}^2|_{n_k O_k}]$$

in $K(\text{coh} \bar{G}(Y_N))$ for any $k$ as desired.

Finally, we show $\mathcal{C}(-O_f) \cong \mathcal{C}(-n_k O_k)$. Let $\bar{C} \cong \mathbb{P}^1$ be the quotient of $C$ by the action of $G / \mathbb{Z}$. Then both $\mathcal{C}(-O_f)$ and $\mathcal{C}(-n_k O_k)$ are the pull-backs of $\mathcal{C}(-1)$ (equipped with the trivial $\bar{G}$-action) and hence we obtain the isomorphism.

**Step 2.** $\beta \circ \alpha = 0$.

**Proof.** This is equivalent to the equality

$$\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2 = \beta_3 \circ \alpha_3.$$

We recall the isomorphism (8.4) for a free $G / \mathbb{Z}$-orbit $O_f \subset C$:

$$R(\mathbb{Z}) \cong K(\text{coh} \bar{G}_f(Y_N)).$$

Then it is sufficient to prove that $\beta_k \circ \alpha_k$ is identified with the pushforward map

$$K(\text{coh} \bar{G}_f(Y_N)) \rightarrow F_0 K(\text{coh} \bar{G}_f(Y_N)).$$

Recall that $K(\text{coh} \bar{G}_f(Y_N))$ has a basis of the form (8.6) and that their images in $K(\text{coh} \bar{G}_f(Y_N))$ satisfy (8.7). Hence the problem is reduced to showing that the map

$$K(\text{coh} \bar{G}_f(Y_N)) \rightarrow K(\text{coh} \bar{G}_f(Y_N))$$

defined by

$$[\mathcal{L}^2|_{O_f}] \mapsto [\mathcal{L}^2|_{n_k O_k}]$$

is identified with the induction map $\alpha_k$. The irreducible representation $\rho_i$ of $\mathbb{Z}$ corresponding to $[\mathcal{L}^2|_{O_f}]$ is defined by sending $[\lambda I] \in \mathbb{Z}$ to $\lambda^{2i} \in \mathbb{C}^\times$. On the other hand, we have

$$[\mathcal{L}^2|_{n_k O_k}] = \sum_{j=0}^{n_k - 1} [\mathcal{L}^2|_{-j O_k}]|_{O_k}.$$

Here $\mathcal{L}^2|_{O_k}$ corresponds to a representation of $\bar{G}_k$ whose restriction to $\mathbb{Z}$ is $\rho_i$. Moreover, $\mathcal{C}(-j O_k)|_{O_k}$ ($0 \leq j \leq n_k - 1$) correspond to the irreducible representations of the cyclic group $\bar{G}_k / \mathbb{Z}$. Thus the element of $R(\bar{G}_k)$ corresponding to $[\mathcal{L}^2|_{n_k O_k}]$ is the sum of all the irreducible representations of $\bar{G}_k$ whose restrictions to $\mathbb{Z}$ are $\rho_i$. Since $\bar{G}_k$ is an abelian group, this is the induced representation of $\rho_i$. Thus we obtain $\beta \circ \alpha = 0$.

**Step 3.** $\ker \beta = \text{Im} \alpha$. 
Proof. Notice that coker \( \alpha \) is torsion free, \( \beta \) is surjective and \( \beta \circ \alpha = 0 \). Therefore it suffices to show

\[
\text{rank } F_0 K(\text{coh}_{\mathcal{C}}(Y_N)) = \sum_{k=1}^{3} \text{rank } R(\mathcal{G}_k) - 2 \text{ rank } R(\mathbb{Z}).
\]

This follows from the following two equalities:

\[
\text{rank } F_0 K(\text{coh}_{\mathcal{C}}(Y_N)) = \text{rank } G - \text{rank } R(\mathbb{Z}) \tag{8.8}
\]

\[
\sum_{k=1}^{3} \text{rank } R(\mathcal{G}_k) = \text{rank } G + \text{rank } R(\mathbb{Z}) \tag{8.9}
\]

We first consider (8.8). The isomorphism (8.2) reduces (8.8) to the equality

\[
\text{rank } K(\text{coh}_{\mathcal{C}}(Y_N)) = \text{rank } R(G)/C_{02} \text{ rank } R(\mathbb{Z})
\]

and therefore it suffices to show that the classes

\[
[\mathcal{O}_{\mathcal{C}}], [\mathcal{O}^2], \ldots, [\mathcal{O}^{2(l-1)}]
\]

form a free basis of the quotient \( K(\text{coh}_{\mathcal{C}}(Y_N))/F_0 K(\text{coh}_{\mathcal{C}}(Y_N)) = \text{rank } \mathbb{Z} \)

where \( l := \text{rank } R(\mathbb{Z}) = |\mathbb{Z}| \).

Recall that \( \mathcal{O}^2 \cong \omega_{\mathcal{C}}^{-1} \) is a \( \mathcal{G} \)-equivariant line bundle on \( C = \mathbb{P}(V) \). Since \( \mathbb{Z} \) acts on \( C \) trivially, if we regard \( \mathcal{O}^2 \) as an object of \( \text{coh}^2(\mathcal{C}) \), we have

\[
\mathcal{O}^{2i} \cong \mathcal{O}_{\mathcal{C}}(2i) \otimes \rho_i \quad \text{in } \text{coh}^2(\mathcal{C}) \tag{8.11}
\]

where \( i = i \text{ mod } l \) and \( \rho_0, \rho_1, \ldots, \rho_{l-1} \) are the irreducible representations of the cyclic group \( \mathbb{Z} \cong \mathbb{Z}/l\mathbb{Z} \). This implies that (8.10) is linearly independent. To see that (8.10) is a generator, we show that for any object \( \mathcal{E} \in \text{coh}_{\mathcal{C}}(Y_N) \) its class \([\mathcal{E}]\) is a linear combination of (8.10) modulo \( F_0 K(\text{coh}_{\mathcal{C}}(Y_N)) \). We may assume that \( \mathcal{E} \) is a locally free sheaf on \( C \) and we use the induction on rank \( \mathcal{E} \).

If rank \( \mathcal{E} = 0 \), there is nothing to prove and we may suppose rank \( \mathcal{E} > 0 \). If we regard \( \mathcal{E} \) as an object of \( \text{coh}^2(\mathcal{C}) \), it splits as \( \mathcal{E} = \bigoplus_i \mathcal{E}_i \oplus \bigoplus_i \rho_i \) with \( \mathcal{E}_i \in \text{coh}(\mathcal{C}) \). Suppose \( \mathcal{E}_i \neq 0 \). For any integer \( m \) we have

\[
\text{Hom}_{\mathcal{C}}(\mathcal{O}^2, \mathcal{E} \otimes \mathcal{O}^{2ml}) = H^0((\mathcal{E} \otimes \mathcal{O}^{2ml-2i})^{\mathcal{G}}/\mathcal{Z}). \tag{8.12}
\]

Here, (8.11) shows

\[
(\mathcal{E} \otimes \mathcal{O}^{2ml-2i}) = \mathcal{E}_i \otimes \mathcal{O}(2ml - 2i) \neq 0
\]

and the restriction map

\[
H^0((\mathcal{E} \otimes \mathcal{O}^{2ml-2i})^{\mathcal{G}}/\mathcal{Z}) \to H^0((\mathcal{E} \otimes \mathcal{O}^{2ml-2i})^{\mathcal{G}}/\mathcal{Z}|_{\mathcal{O}_i})
\]
is surjective for a $\mathcal{G}/\mathbb{Z}$-free orbit $\mathcal{O}_f \subset \mathcal{C}$ if $m$ is sufficiently large. Since $H^0((\mathcal{E} \otimes \mathcal{L}^{2m+2})^2|_{\mathcal{O}_f})$ is a non-zero multiple of the regular representation of $\mathcal{G}/\mathbb{Z}$, its $\mathcal{G}/\mathbb{Z}$-invariant part is non-zero. Therefore, (8.12) is non-zero and hence there is a non-zero homomorphism

$$\alpha : \mathcal{L}^{2l} \to \mathcal{E} \otimes \mathcal{L}^{2lm}.$$ 

Now the induction hypothesis shows that $\text{coker } \alpha$ is a linear combination of (8.10) modulo $F_0K(\text{coh}_\mathcal{C}(Y_N))$. This shows that the class $[\mathcal{E} \otimes \mathcal{L}^{2lm}]$ is also a linear combination of (8.10) modulo $F_0K(\text{coh}_\mathcal{C}(Y_N))$. Since we have

$$[\mathcal{E}] - [\mathcal{E} \otimes \mathcal{L}^{2lm}] \in F_0K(\text{coh}_\mathcal{C}(Y_N)),$$

$[\mathcal{E}]$ is a linear combination of (8.10) modulo $F_0K(\text{coh}_\mathcal{C}(Y_N))$. Thus (8.10) is a free basis of $K(\text{coh}_\mathcal{C}(Y_N))/F_0K(\text{coh}_\mathcal{C}(Y_N))$ and therefore we have established (8.8).

Next we prove (8.9). Let $\mathbb{Z}L(\mathcal{V}) \subset \mathbb{G}L(\mathcal{V})$ be the subgroup consisting of the non-zero scalar matrices and consider the multiplication map

$$\mu : \mathbb{Z}L(\mathcal{V}) \times \mathbb{S}L(\mathcal{V}) \to \mathbb{G}L(\mathcal{V}).$$

Then the kernel of $\mu$ is a group of order 2 generated by $(-I, -I)$. We put $\mathcal{G} = \mu^{-1}(G)$ and let $H \subset \mathbb{S}L(\mathcal{V})$ be the image of $\mathcal{G}$ with respect to the second projection. For any element $(z, h) \in \mathcal{G}$, denote by $Z_G(z, h)$ and $Z_G(zh)$ the centralizers of $(z, h)$ in $\mathcal{G}$ and $zh$ in $G$ respectively. Then the restriction $\mu : Z_G(z, h) \to Z_G(zh)$ is a surjective two-to-one map and hence the number of conjugates of $(z, h)$ coincides with the number of conjugates of $zh$. Therefore, the number of conjugacy classes in $\mathcal{G}$ is twice the number of conjugacy classes in $G$. Thus we obtain

$$\text{rank } R(G) = \frac{1}{2} \text{rank } R(\mathcal{G}).$$

Moreover, since $\mathcal{G}/\mathbb{Z} \cong H$ and $\mathbb{Z}$ is central in $\mathcal{G}$, this can be written as

$$\text{rank } R(G) = \frac{1}{2} \text{rank } R(H) \times |Z| = \text{rank } R(H) \times |Z|. \quad (8.13)$$

Notice that $H$ acts on $\mathcal{V}$ and $\Pi := H/N \cong \mathcal{G}/\mathbb{Z} \subset \mathbb{P}GL(\mathcal{V})$ acts on $\mathbb{C} = \mathbb{P}(\mathcal{V})$. Since $H$ is in $\mathbb{S}L(\mathcal{V})$, the McKay correspondence for the binary polyhedral (or dihedral) group $H$ establishes

$$\sum_{k=1}^{3} |\Pi_k| = \text{rank } R(H) + 1 \quad (8.14)$$
8.2. Main theorem.

**Proposition 5.** Suppose a finite subgroup $G \subset \text{GL}(2, \mathbb{C})$ contains $-I$ and $Y \to Y_N/G$ is a resolution dominated by $Y_{\text{max}}$. Then there exists a generic stability parameter $\theta \in \Theta$ such that $\mathcal{M}_\theta \cong Y$. Especially, the maximal resolution $Y_{\text{max}}$ of $(\mathbb{C}^2/G, B)$ is isomorphic to the moduli space of $G$-constellations for some generic stability parameter $\theta$.

**Proof.** We may assume $G$ is non-abelian by Theorem 5 so we may apply the results of section 8.1. If we show there exists a generic parameter $\zeta \in K(\text{coh}^G(Y_N))^0_\mathbb{Q}$ such that $\mathcal{M}_\zeta(Y_N) \cong Y$, then the assertion follows from Theorem 6.

Let $P \in C$ be a point. Since $\bar{G}$ acts on $Y_N \times \mathbb{C} = \mathcal{M}_\theta(V \times \mathbb{C})$ and $\mathbb{Z}$ fixes $(P, 0)$, $\mathbb{Z}$ acts on the Zariski tangent space $\tilde{T} := T_{(P, 0)}(Y_N \times \mathbb{C}) \cong \mathbb{C}^3$ as a subgroup of $\text{SL}(\tilde{T})$. Note that as a representation of $\mathbb{Z}$, $\tilde{T}$ is independent of the choice of the point $P$. Let $T' \subset \tilde{T}$ be the two-dimensional $\mathbb{Z}$-invariant subspace transversal to $C$; then $\mathbb{Z} \subset \text{SL}(T')$. Fix a generic stability parameter $\theta^Z \in R(\tilde{Z})^0_\mathbb{Q}$ for $\tilde{Z}$-constellations (on $\tilde{T}$) satisfying $\theta^Z(\mathcal{C}[\tilde{Z}]) = 0$. Then $W := \mathcal{M}_{\theta^Z}(T')$ is the minimal resolution of $T'/\mathbb{Z}$. The Fourier-Mukai transform $\phi^*_{\theta^Z} : R(\tilde{Z})^0_\mathbb{Q} \cong K(\text{coh}^Z(T'))^0_\mathbb{Q} \xrightarrow{\sim} K(\text{coh} W)^0_\mathbb{Q}$ sends $\theta^Z$ to an element $l_{\theta^Z}$ of $F^1 K(\text{coh} W)^0_\mathbb{Q} \cong \text{Pic}(W)^0_\mathbb{Q}$ and it lies in the ample cone $\text{Amp}(W)$ as in (2.1). (Notice that here $\dim T' = 2$ and $F^2 K(\text{coh} W) = 0$.)

Take a point $P_k$ in the orbit $O_k$ for each $k \in \{1, 2, 3\}$. We consider the tangent spaces $T_k := T_{(P_k, 0)}(Y_N \times \mathbb{C})$ and $T_k = T_{P_k}(Y_N)$. Let $R_k$ denote the complete local ring of $T_k/\bar{G}_k$ at $[0]$ which is isomorphic to the complete local
ring of \( Y_N/\mathcal{G} \) at \([P_k]\):

\[
R_k := \mathcal{O}_{T_k, [0]}(P_k) \cong \mathcal{O}_{Y_k}([P_k]/C_{138}).
\]

By this isomorphism, there is a resolution

\[
Y_k \to T_k/\mathcal{G}_k
\]

with an isomorphism

\[
Y_k \times_{(T_k/\mathcal{G}_k)} \text{Spec } R_k \cong Y \times_{(Y_N/\mathcal{G})} \text{Spec } R_k
\]

(8.16) over Spec \( R_k \). Since \( \mathcal{G}_k \) is abelian, we can apply Proposition 2 where the first factor of \( T_k \cong \mathbb{C}^2 \) is \( T_{P_k}(C) \) (so that \((1,0)\) lies in \( T_{P_k}(C) \) and \( G_{(1,0)} = \bar{Z} \)) and obtain a projective crepant resolution

\[
U_{\Sigma_k} \to \mathcal{T}_k/\mathcal{G}_k
\]

such that \( Y_k \subset U_{\Sigma_k} \) and that the restriction map \( \text{Amp}(U_{\Sigma_k}) \to \text{Amp}(W) \) is surjective. Choose a class \( l_k \in \text{Amp}(U_{\Sigma_k}) \) which is mapped to \( l_{o\xi} \in \text{Amp}(W) \) for each \( k \). Then by Theorem 2 we can find a generic stability parameter \( \theta_k \) for \( \mathcal{G}_k \)-constellations on \( \mathcal{T}_k \) such that \( \mathcal{M}_{\theta_k}(\mathcal{T}_k) \cong U_{\Sigma_k} \) and the class of \( \varphi_{\theta_k}(\theta_k) \) in \( \text{Pic}(U_{\Sigma_k})_\mathbb{Q} \) coincides with \( l_k \). Since \([\varphi_{\theta_k}(\theta_k)] = l_k \) and \( l_k \) restricts to \( l_{o\xi} \), \( \theta_k \) restricts to \( \theta_{o\xi}^2 \) on \( R(Z) \). Then Corollary 1 shows that \( (\theta_1, \theta_2, \theta_3) \) determines an element of \( F_0 K(\text{coh}\mathcal{G}(Y_N))_\mathbb{Q} \). Lift it to an element \( \xi \in K(\text{coh}\mathcal{G}(Y_N))_\mathbb{Q} \cong K(\text{coh}\mathcal{G}(O_k))_\mathbb{Q} \). Since the restriction of \( \xi \) to \( K(\text{coh}\mathcal{G}(O_k))_\mathbb{Q} \cong R(\mathcal{G}_k)^* \mathbb{Q} \) is \( \theta_k \) which is of rank 0, we have rank \( \xi = 0 \) and we can consider the moduli space \( \mathcal{M}_\xi(Y_N) \).

We claim that there is an isomorphism

\[
\mathcal{M}_\xi(Y_N) \times_{(Y_N/\mathcal{G})} \text{Spec } R_k \cong \mathcal{M}_{\theta_k}(T_k) \times_{(T_k/\mathcal{G}_k)} \text{Spec } R_k
\]

(8.17) over Spec \( R_k \). For any locally noetherian scheme \( S \) over Spec \( R_k \), an \( S \)-valued point of the left hand side of (8.17) is given by a flat family of \( \xi \)-stable \( \mathcal{G} \)-constellations on \( Y_k \) parameterized by \( S \), which is an object of \( \text{coh}\mathcal{G}(Y_N \times_{(Y_N/\mathcal{G})} S) \). Similarly, an \( S \)-valued point of the right hand side of (8.17) is given by a flat family of \( \theta_k \)-stable \( \mathcal{G}_k \)-constellations on \( T_k \) parameterized by \( S \), which is an object of \( \text{coh}\mathcal{G}_k(T_k \times_{(T_k/\mathcal{G}_k)} S) \).

Notice that

\[
Y_N \times_{(Y_N/\mathcal{G})} S \cong (Y_N \times_{(Y_N/\mathcal{G})} \text{Spec } R_k) \times_{(\text{Spec } R_k)} S
\]

\[
\cong \left( \prod_{\mathcal{O}_k} \text{Spec } \mathcal{O}_Y \right) \times_{(\text{Spec } R_k)} S
\]

\[
\cong \text{Spec } \mathcal{O}_Y \times_{(\text{Spec } R_k)} S
\]
\[ \cong \text{Spec } \hat{\mathcal{O}}_{T_k,0} \times_{\text{Spec } R_k} S \]
\[ \cong T_k \times_{(T_k/\mathcal{G}_k)} S \]

which induces an equivalence

\[ \text{coh } \tilde{G}(Y_N \times_{(Y_N/\mathcal{G})} S) \cong \text{coh } \tilde{G}_k(T_k \times_{(T_k/\mathcal{G}_k)} S) \]

(this is almost the same as (8.3)). This equivalence gives a bijection between \( S \)-valued points of the both sides of (8.17) and we obtain (8.17).

Our choice of \( \theta_k \) implies \( \mathcal{M}_{\theta_k}(T_k) \cong Y_k \) and hence (8.16) and (8.17) yield an isomorphism

\[ \mathcal{M}_{\xi}(Y_N) \times_{(Y_N/\mathcal{G})} \text{Spec } R_k \cong Y \times_{(Y_N/\mathcal{G})} \text{Spec } R_k. \]

over Spec \( R_k \). Since \( \mathcal{M}_{\xi}(Y_N) \) and \( Y \) are both isomorphic to \( Y_N/\mathcal{G} \) except over the points \([P_1],[P_2],[P_3]\), we obtain \( \mathcal{M}_{\xi}(Y_N) \cong Y \).

Recall that we say \( G \subset \text{GL}(2, \mathbb{C}) \) is small if \( G \) acts freely on \( \mathbb{C}^2 \setminus \{0\} \). The following lemma follows from the classification of small subgroups of \( \text{GL}(2, \mathbb{C}) \) but we give a proof for the reader’s sake.

**Lemma 5.** If a finite small subgroup \( G \subset \text{GL}(2, \mathbb{C}) \) is non-abelian, then it contains \( -I \) as a unique element of order 2.

**Proof.** If \( G \) is non-abelian, then its image \( G' \subset \text{PGL}(2, \mathbb{C}) \) is also non-abelian and therefore it is either a dihedral or a polyhedral group. Especially, the orders \( |G'| \) and \( |G| \) are even. Then \( G \) contains an element of order 2. If it is not \( -I \), then it fixes a line in \( \mathbb{C}^2 \), contradicting the smallness of \( G \).

**Theorem 7.** If \( G \subset \text{GL}(2, \mathbb{C}) \) is a finite small subgroup, then Conjecture 4 is true.

**Proof.** The abelian case follows from Theorem 5. Otherwise, \( G \) contains \( -I \) by the above lemma. Moreover, the minimal resolution of \( V/G \) factors through \( Y_N/\mathcal{G} \); see [Bri68]. Then the assertion follows from Proposition 5.

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