ON IDENTITIES IN THOMPSON’S GROUP

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1. Introduction

Thompson’s group $F$ is the group of all piecewise linear orientation-preserving homeomorphisms of the unit interval $[0,1]$ with finitely many breakpoints such that:

1. all slopes are powers of 2;
2. all breakpoints are in $\mathbb{Z}[\frac{1}{2}]$, the ring of dyadic rational numbers.

The group $F$ is described by the following presentation:

$$F = \langle x_0, x_1, x_2 \cdots | x_i^{-1}x_kx_i = x_{k+1} (k > i) \rangle.$$  

One can show that the group $F$ can also be given by the following presentation

$$F = \langle x_0, x_1 | [x_0x_1^{-1}, x_0^{-1}x_1x_0] = [x_0x_1^{-1}, x_0^{-2}x_1x_0^2] = 1 \rangle.$$  

The Thompson’s group $F$ posses many interesting properties, among which we shall only mention a few. We refer the reader to [1] and references there for a detailed account.

Lemma 1. If $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ and $0 = y_0 < y_1 < y_2 < \cdots y_n = 1$ are two partitions of $[0,1]$ consisting of dyadic rational numbers, then there exists a piecewise linear homeomorphism $f$ of the unit interval $[0,1]$ such that $f(x_i) = y_i$ for $i = 0,\ldots,n$, and $f$ is an element of $F$.

We shall make use of the following results on the Thompson’s group by $F$, which can be found in [1].

Theorem 4.1, [1]. The derived subgroup $[F,F]$ of $F$ consists of all elements in $F$ which are trivial in neighborhoods of 0 and 1. Furthermore, $F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}$

Theorem 4.3, [1]. Every proper quotient group of $F$ is Abelian.

Theorem 4.6, [1]. The submonoid of $F$ generated by $x_0, x_1, x_1^{-1}$ is the free product of the submonoid generated by $x_0$ and the subgroup generated by $x_1$.

Corollary 4.7, [1]. Thompson’s Group $F$ has exponential growth.

Theorem 4.8, [1]. Every non-Abelian subgroup of $F$ contains a free Abelian subgroup of infinite rank.

Corollary 4.9, [1]. Thompson’s group $F$ does not contain a non-Abelian free group.

The aim of this paper is to prove that Thompson’s group $F$ is close to a free group in that it does not satisfy an identity.

Theorem 2. For any natural numbers $n$ and $k$ there exist $n$ elements $a_1,\ldots,a_n$ in Thompson’s group $F$, such that no relation involving $n$ variables and of length less than $k$ is satisfied by $a_1,\ldots,a_n$. 

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In particular, if $n = 2$ we get the following

**Corollary 3.** For any positive integer $k$ there exists a pair of elements $a, b$ in Thompson’s group $F$, such that no relation of length less than $k$ is satisfied by $a, b$.

Our proof of Theorem 2 consists of the following three lemmas

**Lemma 4.** Let $G$ be a finitely generated group. For any positive integer $n$ the following two conditions are equivalent:

1. for any positive integer $k$ there exist $n$ elements $a_{k,1}, \ldots, a_{k,n}$ of $G$ such that the tuple $a_{k,1}, \ldots, a_{k,n}$ does not satisfy any relation of length less than $k$;
2. the group $G$ satisfies no identity in $n$ variables.

**Lemma 5.** Let $G$ be a group. If the group $G$ satisfies an identity $f(x_1, \ldots, x_n)$ in $n$ variables, then the group $G$ satisfies an identity $g$ in 2 variables.

**Proof.** Let $f$ be an identity on $G$. Set $g(x, y) = f(w_1(x, y), w_2(x, y), \ldots, w_n(x, y))$, where the words $w_1(x, y), w_2(x, y), \ldots, w_n(x, y)$ generate the free subgroup of rank $n$ in the free group $F(x, y)$. It follows that $g(x, y)$ is a nontrivial identity. \[\square\]

**Lemma 6.** Thompson’s group $F$ does not satisfy an identity.

### 2. Some examples and definitions

Let $x_0$ and $x_1$ be the following elements of the group $F$:

$$x_0(x) = \begin{cases} x/2, & 0 \leq x \leq 1/2 \\ x - 1/4, & 1/2 \leq x \leq 3/4 \\ 2x - 1, & 3/4 \leq x \leq 1 \end{cases}$$

$$x_1(x) = \begin{cases} x, & 0 \leq x \leq 1/2 \\ x/2 + 1/4, & 1/2 \leq x \leq 3/4 \\ x - 1/8, & 3/4 \leq x \leq 7/8 \\ 2x - 1, & 7/8 \leq x \leq 1 \end{cases}$$

It is convenient to represent the elements of Thompson’s group $F$ as rectangular diagrams. For example, $x_0$ and $x_1$ can be represented as follows:

![Diagram of x0](image)

![Diagram of x1](image)

Here the upper side of the rectangle is mapped onto the lower one. For instance, take the product of $x_1$ and $x_0^{-1}$:

![Diagram of x1 and x0^{-1}](image)

Then the rectangular diagram for $x_1x_0^{-1}$ has the form:

![Diagram of x1x0^{-1}](image)
Let us construct a special partition of the unit interval \([0, 1]\). We subdivide \([0, 1]\) into infinitely many pieces of the form \([1/2^{k+1}, 1/2^k]\), \([1 - 1/2^k, 1 - 1/2^{k+1}]\), where \(k \geq 1\) is an integer. This subdivision is shown on the figure below:

\[
\begin{array}{ccccccccc}
0 & \frac{1}{2} & 1 \\
\end{array}
\]

The element \(x_0\) shifts these pieces of this subdivision to the right:

\[
\ldots, \left[\frac{1}{8}, \frac{1}{4}\right) \rightarrow \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{4}, \frac{1}{2}\right) \rightarrow \left[\frac{3}{4}, \frac{7}{8}\right), \left[\frac{3}{4}, \frac{7}{8}\right) \rightarrow \left[\frac{7}{8}, \frac{15}{16}\right), \ldots
\]

The element \(x_0^{-1}\) shifts the pieces of the subdivision to the left:

\[
\ldots, \left[\frac{1}{8}, \frac{1}{4}\right) \leftarrow \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{4}, \frac{1}{2}\right) \leftarrow \left[\frac{3}{4}, \frac{7}{8}\right), \left[\frac{3}{4}, \frac{7}{8}\right) \leftarrow \left[\frac{7}{8}, \frac{15}{16}\right), \ldots
\]

3. Proof of Lemma 4

We first prove that \((1)\) implies \((2)\). Take an arbitrary reduced word \(f\) in \(n\) variables and of length \(l\). Then there exist elements \(a_{l+1,1}, \ldots, a_{l+1,n} \in G\) such that \(f(a_{l+1,1}, \ldots, a_{l+1,n}) \neq 1\). Hence \(f\) is not an identity.

We now prove the converse. Let us assume, that the group \(G\) does not satisfy any identities. Assume the contrary, i.e. that condition \((1)\) from Lemma 4 does not hold. Then there exist natural numbers \(n\) and \(k\), such that any \(n\) elements \(a_1, \ldots, a_n \in G\) satisfy a relation of length less then \(k\). Consider the set of all such relations, that is the set of all reduced words (one can treat these words as elements of a free group) \(B_k = \{f_1, \ldots, f_{m_k}\}\) in \(n\) letters and of length less than \(k\).

Using the set \(B_k\) we now construct an identity \(h_{m_k}\), which is satisfied by the group \(G\). Set \(h_1 = f_1\). If \([h_{i-1}, f_i] = 1\) in the free group, then there is word \(w\), such that \(h_{i-1} = w^\alpha\) and \(f_i = w^\beta\) for some \(\alpha\) and \(\beta\) in \(\mathbb{Z}\). Define \(h_i = w^{\alpha \beta}\). If \([h_{i-1}, f_i] \neq 1\) in the free group, then set \(h_i = [h_{i-1}, f_i]\). Consider the word \(h_{m_k}\). For any \(n\)-tuple of elements \(a_1, \ldots, a_n \in G\) we have: \(h_{m_k}(a_1, \ldots, a_n) = 1\). It follows that \(h_{m_k}\) is an identity on \(G\) - a contradiction.

4. Identities in Thompson’s Group \(F\)

By Lemma 5 it suffices to show that \(F\) does not have an identity in two variables. Let \(w(x, y) = w_k \ldots w_2 w_1\) be an arbitrary reduced non-trivial word in \(x\) and \(y\) of length \(k\). That is \(w_i \in \{x, x^{-1}, y, y^{-1}\}, 1 \leq i \leq k\). We shall construct elements \(a\) and \(b\) of Thompson’s group \(F\), for which \(w(a, b) \neq 1\).

Consider a partitioning \(\beta_1 < \beta_2 < \cdots < \beta_{k+1}\) of \([0, 1]\), where \(\beta_i\) is a dyadic number. We represent such partitions of \([0, 1]\) by cells. To differentiate such presentation from rectangular diagrams we draw the left side of the rectangle as a sinuous line:

\[
\begin{array}{cccc}
0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_k & \beta_{k+1} & 1 \\
\end{array}
\]

Let \(w(a, b) = u_k \ldots u_2 u_1\), where \(u_i \in \{a, a^{-1}, b, b^{-1}\}, 1 \leq i \leq n\). Set \(u_1(\beta_1) = \beta_2, \ldots, u_k(\beta_k) = \beta_{k+1}\). Since the word \(w(a, b)\) does not contain subwords of the form \(a^\epsilon a^{-\epsilon}, b^\epsilon b^{-\epsilon}, \epsilon = \pm 1\), it follows that \(a\) and \(b\) are increasing functions on points on which they are defined.
Example. Let \( w(x, y) = y^{-1}x^{-1}yx \). Then the partially defined maps \( a \) and \( b \) have the following form:

\[
\begin{align*}
\text{a} & \quad 0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5 \quad 1 \\
\text{b} & \quad 0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5 \quad 1
\end{align*}
\]

The corresponding diagram for the word \( w(a, b) = b^{-1}a^{-1}ba \) takes the form:

\[
\begin{align*}
\text{a} & \quad 0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5 \quad 1 \\
\text{b} & \quad 0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5 \quad 1
\end{align*}
\]

We use Lemma 1 to define the functions \( a \) and \( b \) on the whole segment \([0, 1]\).

Thus we have constructed two functions \( a \) and \( b \) so that \( w(a, b)(\beta_1) = \beta_{k+1} > \beta_1 \).

Therefore \( w(a, b) \) is not the identity map, hence it does not represent the identity element in Thompson’s group \( F \). This finishes our proof of Lemma 6.

Theorem 2 follows from Lemmas 4 and 6 and the fact that Thompson’s group \( F \) is torsion-free.

Appendix A. An alternative proof of Lemma 1

Consider two adjacent points \( x_i \) and \( x_{i+1} \) of the partition \( 0 = x_0 < x_1 < x_2 < \cdots < x_n = 1 \). Define the function \( f \) on the segment \([x_i, x_{i+1}]\) as follows

\[
\begin{align*}
\text{f} & \quad x_i \quad x_{i+1} \\
\text{y_i} & \quad ? \\
\text{y_{i+1}} & \quad y_i + 1
\end{align*}
\]

Let \( x_{i+1} - x_i = c_1, y_{i+1} - y_i = c_2 \). Without loss of generality we may assume that \( c_1 < c_2 \), where \( c_1 \) and \( c_2 \) are diadic numbers, that is \( c_1 = \frac{t_1}{2^{j_1}}, c_2 = \frac{t_2}{2^{j_2}} \). Then \( z_1 = t_12^{j_2}, z_2 = t_22^{j_1} \) are integers. It follows that \( d = \frac{z_1 - 1}{z_1}c_1 = \frac{t_12^{j_2-1}}{t_12^{j_2}} \) is a diadic number and \( d < c_1 \). Define the function \( f \) on the segment \([x_i, x_i + d]\) as a linear function, whose slope equals 1:

\[
\begin{align*}
\text{f} & \quad x_i \quad x_{i+1} \\
\text{y_i} & \quad y_i \quad y_i + d \\
\text{y_{i+1}} & \quad y_i + 1
\end{align*}
\]

The ratio of lengths of the remaining pieces equals

\[
\frac{c_2 - d}{c_1 - d} = \frac{z_2 - \frac{d}{c_1} - (z_1 - 1)\frac{d}{c_2}}{c_1 - (z_1 - 1)\frac{d}{c_1} = \frac{z_2 - z_1 + 1}{1}}.
\]
Write this number as a sum of powers of 2, \( z_2 - z_1 + 1 = 2^{k_1} + \cdots + 2^{k_n} \), where \( k_1, \ldots, k_n \) are non-negative integers. Proportionally to this decomposition, we subdivide the segment \([y_i + d, y_{i+1}]\) into \( m \) pieces. Furthermore, write 1 as a sum of powers of 2, \( 1 = 2^{k'_1} + \cdots + 2^{k'_n} \), where \( k'_1, \ldots, k'_n \) are negative integers. We then subdivide proportionally the segment \([x_i + d, x_{i+1}]\) into \( m \) pieces. We use the above defined partitions and define the function \( f \) on the segment \([x_i + d, x_{i+1}]\) as a piecewise linear function. It follows that the slopes are powers of 2:

![Diagram of function f](image)

Construct the function \( f \) for all pairs \( x_i, x_{i+1} \), \( i = 0, \ldots, n \) as above. It follows that the function \( f \) is defined on the whole segment \([0, 1]\). This finishes our proof of Lemma 1.

REFERENCES

[1] J. W. Cannon, W. J. Floyd, and W. R. Parry. *Introductory notes on Richard Thompson’s groups*. Enseign. Math. (2), 42 (3-4):215-256, 1996.

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