Four-Poles Parameter of an Elliptical Cavity Having the Outlet on the Body

Yuya Nishimura, and Sohei Nishimura

Abstract—This paper deals with the computation of the four-poles parameter of a thin elliptic cylinder in which the output is fitted to the side that is perpendicular to the input side. The four-poles parameter is based on the sound pressure calculated by solving the wave equations, with the assumption that the loss can be ignored. The four-poles parameter is widely used to estimate the noise characteristic for the acoustic system which are composed of several elements of various cross-sectional areas, various shape connected in series.

Index Terms—Four-Poles Parameter, Higher-Order Mode, Wave Equation, Mathieu Function.

I. INTRODUCTION

Analysis based on the wave equation is the most effective method for determining the characteristics of an acoustic system [1]-[12]. It's been nearly a century, noise control has grown into a fully-fledged science using very advanced techniques based on the wave equation and producing a thousand scientific papers each year. From the general solution of the wave equation, the sound pressure propagating inside the analysis model can be easily obtained by applying the boundary conditions. As a result, the input and output sound pressure could be obtained related to these volume velocities.

However, the drawback of this method is not suitable for complex structures. Instead, the transfer matrix comprised of the four-poles parameters characterizing the relationship between the upstream and downstream pressure and velocities is widely used in the analysis and design of silencing systems [13]-[16].

This paper deals with the computation of the four-poles parameter of a thin elliptic cylinder in which the output is fitted to the side that is perpendicular to the input side [17] as shown in Fig. 1. Following the development of the theory base on reference [17] in section 2, the calculation of the four-poles parameter is presented in section 3.

II. THE SOUND PROPAGATION IN THE ELLIPTICAL CAVITY

A. The solution of the wave equation in elliptical coordinates

Three-dimensional wave equation in elliptical coordinate \((\xi, \eta, z)\) is given as

\[
\frac{2}{q^2 (\cosh 2\zeta - \cos 2\eta)} \left[ \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0 \right] \quad (1)
\]

where \(q\) is the distance between the foci and the origin, \(\Phi\) be the complex effective value of velocity potential, \(c\) is the sound velocity, \(k = \alpha/c\) (\(\alpha\) : angular velocity).

Then the general solution of Equation (1) becomes

\[
\Phi = \left( A_0 \exp(\mu z) + B_0 \exp(-\mu z) \right) \left[ \sum_{m=0}^{\infty} C_m \overline{C}_m (\xi, \eta) \overline{c}_m (\eta, s) + \sum_{m=0}^{\infty} S_m \overline{S}_m (\xi, \eta) \overline{s}_m (\eta, s) \right]
\]

B. Model of analysis and boundary conditions

The model of analysis is shown in Fig. 2. The elliptical cavity of eccentricity \(e_0\) and length \(L\) which has an elliptical piston-driven of eccentricity \(e_1\) fitted at the center of the input. The output which has an opening area \(u \times h\) is located on the cavity and its center is at the \(\ell\) position from the input.

Fig. 1. The elliptic cavity

Fig. 2. Model analysis of an elliptic cavity
Since the distribution of $\Phi$ is symmetric about the major and the minor axis, $\Phi$ must be even and periodic in $\eta$, hence Equation (2) becomes

$$\Phi = (A_q \exp(\mu z) + B_q \exp(-\mu z))$$

$$\sum_{m=0}^{\infty} C_{2m} \text{Ce}_{2m}(\xi, s) \text{ce}_{2m}(\eta, s)$$

(4)

The even Mathieu and even modified Mathieu function are defined as [17]

$$\text{Ce}_{2m}(\xi, s) = \text{ce}_{2m}(j\xi, s) = \sum_{r=0}^{\infty} A_{2m}^{(2m)} \cosh 2r\xi$$

(5)

where $A_{2m}^{(2m)}$ are constants.

Let $V_z$ and $V_\xi$ are the velocity components in the $z$ and $\xi$ directions, $V_1$ and $V_0$ are the driving velocity at the input and the output piston, respectively. The boundary conditions are as follows:

1. At the input $V_z = \frac{\partial \Phi}{\partial z} |_{z=0} = V_1 F_1(\xi, \eta)$
2. At the output $V_\xi = -\frac{\partial \Phi}{\partial \xi} |_{\xi=\xi_0} = V_0 F_0(\eta, z)$
3. At the back surface $V_z = \frac{\partial \Phi}{\partial z} |_{z=L} = 0$

(6) (7) (8)

where $F_1(\xi, \eta) = 1$ at the piston and $F_1(\xi, \eta) = 0$ elsewhere. Similarly, $F_0(\eta, z) = 1$ at the piston and $F_0(\eta, z) = 0$ elsewhere.

C. Average sound pressure of inlet and outlet

According to the boundary conditions Equations (6)-(8), $\Phi$ can be determined as

$$\Phi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + G_{2m,i} U_o$$

$$\times \cosh \mu_{2m,i}(L - z) \text{Ce}_{2m}(\xi, s_{2m,i}) \text{ce}_{2m}(\eta, s_{2m,i})$$

(9)

The sound pressure propagates in the cavity corresponding to $\Phi$ becomes

$$P(\xi, \eta, z) = jk \rho c \Phi$$

$$= jk \rho c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + G_{2m,i} U_o$$

$$\times \cosh \mu_{2m,i}(L - z) \text{Ce}_{2m}(\xi, s_{2m,i}) \text{ce}_{2m}(\eta, s_{2m,i})$$

(10)

where $\rho$ is density, $U_i = V_z S_i$ and $U_o = V_0 S_o$ are volume velocity at input and output, the other symbols are constants defined as following

$$\mu_{2m,i} = \frac{1}{a_w} \sqrt{\lambda_{2m,i}^2 - (ka_w)^2}$$

(11)

The average sound pressure acting on the input piston can be expressed as

$$\bar{P}_i = jk \rho c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{Q_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + R_{2m,i} U_o$$

$$\left( \cosh \mu_{2m,i}(L - L) \sinh(\mu_{2m,i} h/2) \right)$$

(12)

where

$$Q_{2m,i} = H_{2m,i} \text{Ce}_{2m}(\xi_w, s_{2m,i}) \text{ce}_{2m}(\eta, s_{2m,i})$$

$$R_{2m,i} = G_{2m,i} \text{Ce}_{2m}(\xi_w, s_{2m,i}) \text{ce}_{2m}(\eta, s_{2m,i})$$

(13)

$$\chi_{2m,i} = \frac{a_w}{S_o} \int_{\eta_0}^{\eta} \sqrt{1 - e^{-2 \eta} \cos^2 \eta \text{ce}_{2m}(\eta, s_{2m,i})} d\eta$$

(14)

D. The computation of four-poles parameter

The four-poles parameter $A$, $B$, $C$, and $D$ are defined as follows [18]

$$A = \frac{\bar{P}_i}{\bar{P}_o} \bigg|_{\eta=0}$$

$$B = \frac{\bar{P}_i}{\bar{P}_o} \bigg|_{\eta=0}$$

(15)

$$C = \frac{U_i}{\bar{P}_o} \bigg|_{\eta=0}$$

(16)

$$D = \frac{U_o}{\bar{P}_o} \bigg|_{\eta=0}$$

(17)

(18)

(19)

(20)

(21)

(22)

(23)

(24)
The parameter A and C can be obtained by setting $U_o = 0$ in Equation (15) and Equation (18), respectively.

\[
A = \left| \frac{\overline{P}}{P_e} \right| = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mu_{m,n} \cos \mu_{m,n} L}{Q_{m,n}} \cosh \mu_{m,n} (L - L) \sinh \mu_{m,n} h / 2
\]

\[
C = \left| \frac{U_i}{P_e} \right| = \frac{1}{j \kappa \rho} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\mu_{m,n}^2 \sin \mu_{m,n} L}{Q_{m,n}} \cosh \mu_{m,n} (L - L) \sinh \mu_{m,n} h / 2
\]

when $\overline{P}_o$ of Equation (22) becomes zero, we have

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{Q_{2m,n}}{\mu_{2m,n} \sin \mu_{2m,n} L} \left( U_1 + R_{2m,n} U_o \right) = 0
\]

Thus, the parameter D is

\[
D = \frac{U_i}{U_o} \bigg|_{x=0} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{R_{2m,n}}{Q_{2m,n}} \mu_{m,n} \sin \mu_{m,n} L
\]

Similarly, from Equation (21) we can obtain parameter B as follows

\[
B = \frac{P_i}{P_e} \bigg|_{x=0} = j \kappa \rho \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{D_{m,n}}{Q_{m,n}} R_{m,n} + E_{m,n} \right) \cosh \mu_{m,n} L
\]

**III. APPLICATION**

Consider the case of the ventilation system having two elements connected as shown in Fig.3. The element-1 is an elliptical cavity as mentioned above and the element-2 is a rectangular cubic that has a dimension of $a_2 \times b_2 \times L_2$, respectively. Let A, B, C, and D are the four-poles parameter of element-1 and $A_2$, $B_2$, $C_2$, and $D_2$ are the four-pole parameters of element-2 then the four-pole parameters of the ventilation system becomes

\[
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{bmatrix}
\]

where

\[
\begin{align*}
\mathbf{H}_{m,n}^a &= \frac{1}{S^a} \int_{a_2} \cos \left( \frac{m \pi x}{a_2} \right) \cos \left( \frac{n \pi y}{b_2} \right) dxdy \\
\mathbf{H}_{m,n}^b &= \frac{1}{S^b} \int_{b_2} \cos \left( \frac{m \pi x}{a_1} \right) \cos \left( \frac{n \pi y}{b_1} \right) dxdy \\
\mu_{m,n} &= \sqrt{\left( \frac{m \pi}{a_1} \right)^2 + \left( \frac{n \pi}{b_1} \right)^2} - k^2
\end{align*}
\]

\[
\mu_{m,n} = \sqrt{\left( \frac{m \pi}{a_1} \right)^2 + \left( \frac{n \pi}{b_1} \right)^2} - k^2
\]

**IV. CONCLUSION**

A derivation of the four-poles parameter for the elliptical cavity having an output on the body has been presented by solving the wave equation on the assumption that the loss at the wall can be neglected. The four-poles parameter is given by Equation (26) to Equation (29) including the higher-order mode wave. The formula derived in the present study enables us to account for the characteristic of acoustic system which is composed of several elements of different form, different cross-sectional areas in series. It is also utilized to determine the characteristic of the most acoustic system, such as the muffler, to any high-frequency range.

**ACKNOWLEDGMENT**

This work was supported by JSPS KAKENHI Grant Number 18K11703 and SEI Group CSR Foundation 2019.

**REFERENCES**

[1] N. M. Cuong and J. Okda, “On fitting position of inlet and outlet pipes to elliptic cylinder muffler chambers,” J. Acoust. Soc. Jpn(E) 2, (1981) 71-78.

[2] J.G. Ih, “The reactive attenuation of rectangular plenum chambers,” J. Acoust. Soc. Am., 157(1), 93-122 (1992).

[3] S. Nishimura, T. Nishimura, and T. Yano, “Acoustic analysis of the noise reduction effect caused by the muffler having perforated pipe,” J. Acoust. Soc. Jpn., 61(10), 576-584 (2006).

[4] S. Nishimura, T. Nishimura, T. Yano, “Acoustic analysis of elliptical muffler chamber having a perforated pipe,” J. Sound Vib. 297, (2006) 761-773.

[5] Y.C. Chan, MC. Chiu, SE. Huang, “Numerical analysis of circular straight mufflers equipped with three chambers at high-order-modes,” Applied Acoustics 155 (2019) 167-179.

[6] Y. Nishimura, S. Nishimura, T. Nishimura, T. Yano, “Sound propagation in soundproofing casement windows,” Applied Acoust. 70, (2009) 1160-1167.

[7] T. Nishimura, T. Ando, T. Ikeda, “Resonance of elliptical muffler chamber having a non-uniformly perforated pipe,” Electronics and Communication in Japan, 85 (2002), 22-28.

[8] R. Glav, P. L. Regaud, M. Abom, “Study of a folded resonator including the effects of higher order modes,” Journal of Sound and Vibration, 273(2004), 777-792.

[9] T. M. Whalen, “The behavior of higher order mode shape derivatives in damaged, beam-like structures,” Journal of Sound and Vibration, 309(2008), 426-464.

[10] E. Carrera, F. Miglioretti, M. Petrolo, “Computations and evaluations of higher-order theories for free vibration analysis of beams,” Journal of Sound and Vibration, 331 (2012), 4269-4284.
[11] Y. Nishimura, N. H. Quang, S. Nishimura, T. Nishimura, T. Yano, “The acoustic design of soundproofing doors and windows,” The Open Acoustic Journal, 2010, 3, 30-37.

[12] Y. Nishimura, S. Nishimura, T. Nishimura, T. Yano “Sound propagation in soundproofing casement windows,” Journal of Applied Acoustics 2009; 70(2009); pp. 1160-1167.

[13] M. Abom, “Derivation of four-pole parameters for including higher-mode effects for expansion chamber mufflers with extended inlet and outlet,” Journal Acoustic Society of America 137(1990), 403-418.

[14] Wu TW, Zhang P, Cheng CYR. “Boundary element analysis of mufflers with an improved method for deriving the four-pole parameters,” Journal of Sound and Vibration 1998; 217(4): pp.767-79.

[15] T. Nishimura, T. Ikeda, “Four-pole-parameters for an elliptical chamber with mean flow,” Electronics and Communication in Japan, 81(1998), 1-9.

[16] A. Selamet, M. B. Xu, Lee N. T. Huff, “Analytical approach for sound attenuation in perforated dissipative silencers with inlet/outlet extensions,” Journal of the Acoustical Society of America 2005;117; 2078.

[17] S. Nishimura, Y. Nishimura, ThuLan Nguyen, “Acoustic performance of an elliptical cavity on the application for soundproof ventilation units installed in dwelling walls,” Journal of Applied Acoustics, Volume 168, November 2020, 107418, 1-9.

[18] N. W. McLachlan, “Theory and application of Mathieu Function,” Clarendon Press. Oxford, 1951.