BOUNDDED GEODESIC IMAGE THEOREM VIA BICORN CURVES

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Abstract. We give a uniform bound of the bounded geodesic image theorem for the closed oriented surfaces. The proof utilizes the bicorn curves introduced by Przytycki and Sisto [13]. With the uniformly bounded Hausdorff distance of the bicorn paths and 1-slimness of the bicorn curve triangles, we are able to show the bound is 44 for both non-annular and annular subsurfaces. In a particular case when the distance between a geodesic and an essential boundary component of subsurface (or core if it is annular) is \( \geq 18 \), then the bound can be as small as 3, which is comparable to the bound 4 in the motivating examples by Masur and Minsky [11], and is same as the bound given by Webb [18] for non-annular subsurfaces.

1. Introduction

Let \( S = S_{g,b} \) be a compact oriented surface with genus \( g \) and \( b \) boundary components, the complexity of the surface is \( \xi(S) = 3g + b - 3 \). A simple closed curve on the surface \( S \) is essential if it does not bound a disk or an annulus. A simple properly embedded arc on the surface \( S \) is essential if it does not cut off a disk.

1.1. Curve complex. Suppose that \( \xi(S) \geq 2 \), the arc and curve graph \( \mathcal{AC}(S) \) is the graph whose vertices are the isotopy classes of essential properly embedded arcs and essential curves on the surface \( S \), where the isotopy allows the endpoints of the arcs move around in the corresponding boundary components. Two vertices can be connected by an edge in \( \mathcal{AC}(S) \) if they can be realized disjointly. The curve graph \( \mathcal{C}(S) \) is the full subgraph of \( \mathcal{AC}(S) \) whose vertices are the isotopy classes of essential curves. The curve graph is the 1-skeleton of the curve complex introduced by Harvey [7] with the aim to study the mapping class group. \( \mathcal{C}_0(S) \) and \( \mathcal{AC}_0(S) \) are used to denote the vertices of \( \mathcal{C}(S) \) and \( \mathcal{AC}(S) \), respectively.

For the surfaces with \( \xi(S) \leq 1 \), we need to modify the definition slightly. If \( \xi(S) = 1 \), the surface \( S \) is either a one-holed torus \( S_{1,1} \) or a four-holed sphere \( S_{0,4} \). In both cases, two vertices are connected by an edge if some of their curve representatives intersect exactly once for \( S_{1,1} \) or twice for \( S_{0,4} \). The resulting curve graphs are isomorphic to the Farey graph. If \( \xi(S) = 0 \), then \( S = S_1 \) is a torus or \( S = S_{0,3} \) is a pair of pants. Similar to the once-punctured torus, the curve graph of torus is isomorphic to the Farey graph if two curve representatives intersect exactly once are joined by an edge. The curve graph of a pair of pants is empty. If \( S = S_{0,2} \) is an annulus, we define the vertices of \( \mathcal{C}(S) \) to be the essential arcs up to isotopy fixing the boundary component pointwise, and two vertices can be joined by an edge if they can be realized with disjoint interiors. In this case, we let \( \mathcal{C}(S) = \mathcal{AC}(S) \).

For any two vertices \( x \) and \( y \) in the \( \mathcal{C}_0(S) \), the distance \( d_{\mathcal{C}(S)}(x,y) \) between \( x \) and \( y \) is the minimal number of edges in \( \mathcal{C}(S) \) joining \( x \) and \( y \). A geodesic in the curve

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Lemma 1.3. \( C \) for \( \alpha, \beta \) so that 
\[ |i - j| = 3.1 \]

Theorem 1.2. (Bounded Geodesic Image Theorem, Masur-Minsky \([11]\) Theorem 1.1) Let \( S \) be a compact oriented surface with \( \xi(S) \geq 1 \), the curve graph \( C(S) \) is a \( \delta \)-hyperbolic metric space with infinite diameter for some \( \delta \), where \( \delta \) depends on the surface.

Alternative proofs were given by Bowditch \([2]\) and Hamenstädt \([6]\). Moreover, the hyperbolicity constant \( \delta \) can be chosen to be independent of the surface. The existence of such uniform constant has been proved independently by Aougab \([1]\), Bowditch \([3]\), Clay-Rafi-Schleimer \([5]\), Hensel-Przytycki-Webb \([8]\). Przytycki and Sisto \([13]\) also proved it for the closed oriented surfaces.

1.2. Subsurface projection. A subsurface \( Y \) is an essential subsurface of \( X \), if \( Y \) is a compact, connected, oriented, proper subsurface such that each component of \( \partial Y - \partial X \) is essential in \( X \). Suppose \( \xi(Y) \geq 1 \), we define a map \( \pi_{Y} : C_{0}(X) \to \mathcal{P}(\mathcal{AC}_{0}(Y)) \), where \( \mathcal{P}(\mathcal{AC}_{0}(Y)) \) is the power set of \( \mathcal{AC}_{0}(Y) \). Take any \( \alpha \) in \( C_{0}(X) \), then consider the representative of \( \alpha \) such that it intersects \( Y \) minimally, \( \pi_{Y}(\alpha) \) is the set of all isotopy classes of \( \alpha \cap Y \) relative to the boundary of \( Y \). \( \pi_{A}(\alpha) \) is empty if \( \alpha \) can be realized disjointly from \( Y \). We say \( \alpha \) cuts \( Y \) if \( \alpha \cap Y \neq \emptyset \), and \( \alpha \) misses \( Y \) if \( \alpha \cap Y = \emptyset \).

There is a natural way to send the arcs back to the curves in \( C_{0}(Y) \). We can define \( \pi_{0} : \mathcal{AC}_{0}(Y) \to C_{0}(Y) \) as follows. If \( \alpha \) is in the \( Y \), then \( \pi_{A}(\alpha) = \alpha \) in \( \mathcal{AC}_{0}(Y) \) and \( \pi_{0}(\pi_{A}(\alpha)) = \alpha \) in \( C_{0}(Y) \). Otherwise, \( \pi_{A}(\alpha) = \{\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\} \) is a collection of isotopy classes of essential properly embedded arcs in \( Y \). The set \( \pi_{0}(\pi_{A}(\alpha)) \) is the isotopy classes of the essential components of \( \partial N(\alpha_{i} \cup \partial Y) \) in \( Y \), where \( N(\alpha_{i} \cup Y) \) is a regular neighborhood of \( \alpha_{i} \cup \partial Y \) in \( Y \). The composition \( \pi_{0} \circ \pi_{A} = \pi_{Y} : C_{0}(X) \to C_{0}(Y) \) is called the subsurface projection.

If \( Y \) is an annulus, an alternative definition is needed. Let \( X \) be endowed with a complete hyperbolic metric with finite area and \( Y \subset X \) is an essential annulus. Let \( p_{Y} : \tilde{X}_{Y} \to X \) be the annular covering map such that \( Y \) lifts homeomorphically to \( Y' \subset \tilde{X}_{Y} \) and \( \tilde{X}_{Y} \approx \text{interior}(Y) \). The compactification of \( \tilde{X}_{Y} \) with the hyperbolic metric induced from \( X \) is a closed annulus and it is denoted by \( \overline{X}_{Y} \). If a curve \( \alpha \) cuts \( Y \), the subsurface projection \( \pi_{Y}(\alpha) \) is the set of arcs of the full preimage \( \tilde{\alpha} = p_{Y}^{-1}(\alpha) \) that connect the two boundary components of \( \overline{X}_{Y} \). If a curve \( \alpha \) misses \( Y \), then \( \pi_{Y}(\alpha) = \emptyset \). Suppose that \( v \) is the core of \( Y \), then we also use \( \pi_{v} \) to denote the projection and \( d_{n} \) to denote the distance. For more details about the subsurface projection and bounded geodesic image theorem, see the paper of Masur and Minsky \([11]\).

Theorem 1.2. (Bounded Geodesic Image Theorem, Masur-Minsky \([11]\) Theorem 3.1) Let \( Y \) be a subsurface of \( X \) with \( \xi(Y) \neq 0 \) and let \( \Gamma = (\gamma_{i})_{i \in I} \) be a geodesic in \( C(X) \). If each \( \gamma_{i} \) cuts \( Y \), then there is a constant \( M \) depending only on the surface so that \( d_{Y}(\Gamma) \leq M \).

The notation \( d_{Y}(\Gamma) := \text{diam}_{C(Y)}(\bigcup_{\gamma_{i} \in \Gamma} \pi_{Y}(\gamma_{i})) \) is used above.

Lemma 1.3. (Masur-Minsky \([11]\) Lemma 2.2) Let \( \xi(Y) \geq 1 \), and \( d_{\mathcal{AC}(Y)}(\alpha, \beta) \leq 1 \) for \( \alpha, \beta \in \mathcal{AC}(Y) \), then \( d_{C(Y)}(\pi_{0}(\alpha), \pi_{0}(\beta)) \leq 2 \).
By the Lemma 1.3, one only needs to consider the diameter of the projection \( \pi_A : C_0(X) \to \mathcal{P}(\mathcal{AC}_0(Y)) \), because \( \pi_0 : \mathcal{AC}_0(Y) \to C_0(Y) \) is 2-Lipschitz. Moreover, the projection \( \pi_A \) satisfies the 1-Lipschitz property for the surfaces with \( \xi(X) \geq 2 \).

**Lemma 1.4.** (Webb [17] Lemma 1.2) Let \( Y \) be an essential subsurface of \( X \) and let \( \gamma_1, \gamma_2 \) be curves on \( X \). Suppose that \( \gamma_1 \) cuts \( Y \), \( \gamma_2 \) cuts \( Y \) and \( \gamma_1 \) misses \( \gamma_2 \). Then \( d_{\mathcal{AC}(Y)}(\pi_A(\gamma_1), \pi_A(\gamma_2)) \leq 1 \).

Using the notation \( d_{\mathcal{AC}(Y)}(\Gamma) := \text{diam}_{\mathcal{AC}(Y)}(\bigcup_{\gamma_i \in \Gamma} \pi_A(\gamma_i)) \), we restate the theorem for closed oriented surfaces.

**Theorem 1.5.** Suppose that \( X \) is a closed oriented surface with genus \( \geq 2 \). Let \( Y \) be an essential subsurface of \( X \) with \( \xi(Y) \neq 0 \) and let \( \Gamma = (\gamma_i)_{i \in I} \) be a geodesic in \( \mathcal{C}(X) \) such that each \( \gamma_i \) cuts \( Y \), then \( d_{\mathcal{AC}(Y)}(\Gamma) \leq 44 \) for both non-annular and annular subsurfaces \( Y \).

The bounded geodesic image theorem was originally proved by Masur and Minsky [11]. With the uniform hyperbility of the curve graphs, Webb [17] proved that the constant \( M \) is independent of the surface. Later on, he gave explicit bounds in his dissertation [18] using the unicorn arcs, where the bounds is 62 for annular subsurfaces, and 50 for non-annular subsurfaces. In [17, 18], Webb remarked that an unpublished proof of Chris Leininger combined with the work of Bowditch [3] also provided a uniform bound. Our proof relies on a uniformly bounded Hausdorff distance of the bicorn paths and 1-slimness of the bicorn curve triangles. The outline of the proof is similar to Webb’s, while the proof is simple and the bound is smaller for the closed oriented surfaces. The approach can be generalized to the surfaces with boundary, possibly with larger bounds.

The paper is organized as follows. In the Section 2, we briefly describle the motivating example by Masur and Minsky. We recall the definition of bicorn curves and bicorn paths between two curves in the Section 3. In the Section 4, we obtain a uniform bound for the bounded geodesic image theorem for closed oriented surfaces.

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2. A MOTIVATING EXAMPLE

In this section, we will briefly describe a motivating example for the bounded geodesic image theorem from the Section 1.5 in [11]. The curve graph \( \mathcal{C}(S_1), \mathcal{C}(S_{1,1}) \) and \( \mathcal{C}(S_{0,4}) \) are isomorphic to the Farey graph, as roughly illustrated in the Figure 1. It can be embedded in a unit disk, in which the rational numbers and the infinity \( \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \) are located on the boundary of the unit disk as the vertices of the graph. Any two vertices \( \frac{p_1}{q_1} \) and \( \frac{p_2}{q_2} \) in lowest terms are connected by an edge if and only if the determinant \( p_1q_2 - p_2q_1 = \pm 1 \). Let \( v \) be a vertex, then the link of \( v \),
Figure 1. A typical Farey graph. Each vertex is labeled with a number in $\hat{\mathbb{Q}}$ that is a slope of the curve representative under a fixed homology basis. The figure was generated by Wolfram Mathematica, and the code snippet is available on the Stackexchange.

$\text{Link}(v)$, is a set of vertices that share an edge with $v$, which can be identified with the integers $\mathbb{Z}$. For example, if $v = \frac{0}{1}$, then the $\text{Link}(v) = \{\cdots, -\frac{1}{2}, -\frac{1}{1}, 0, \frac{1}{1}, \frac{1}{2}, \cdots\}$.

Let $u$ and $w$ be two vertices of a geodesic $h$ in the Farey graph, and a vertex $v$ of $h$ follows $u$ and followed by $w$, the subsurface projection $d_v(u,w)$ defines a distance in the $\text{Link}(v)$, where $v$ is the core of an annular subsurface. Let $h'$ be another geodesic with the same endpoints as $h$. If $d_v(u,w) \geq 5$, then $h'$ must contain the vertex $v$. In other words, if a geodesic does not contain $v$ (each vertex of the geodesic cuts $v$), then $d_v(u,w) \leq 4$ for any $u,w$ in the geodesic.

3. Bicorn curves

Przytycki and Sisto [13] introduced the bicorn curves to give a simple proof of the hyperbility of curve graphs for the closed oriented surfaces $S_{g \geq 2}$. For more applications of the bicorn curves, see [4, 9, 12, 14, 15, 16].

**Definition 3.1 (Bicorn curves).** Let $\alpha, \beta \subset S_{g \geq 2}$ be two essential simple closed curves that intersect minimally. An essential simple closed curve $\gamma$ is a *bicorn curve* between $\alpha$ and $\beta$ if either $\gamma = \alpha$, $\gamma = \beta$, or $\gamma$ is represented by the union of an arc $a \subset \alpha$ and an arc $b \subset \beta$, which we call the $\alpha$-arc and the $\beta$-arc of $\gamma$, and $a$ only intersects $b$ at the endpoints. If $\gamma = \alpha$, then its $\alpha$-arc is $\alpha$ and its $\beta$-arc is empty, similarly if $\gamma = \beta$, then its $\beta$-arc is $\beta$ and its $\alpha$-arc is empty.

Based on the orientations of two intersection points, there are two configurations of the bicorn curves illustrated in the Figure 2. If the surface is closed, any bicorn curves defined above are essential, as $\alpha$ and $\beta$ intersect minimally. The intersection number of $\alpha$ and $\beta$ is finite, so the number of bicorn curves is finite.

The collection of bicorn curves can be partially ordered. Two bicorn curves $\gamma < \gamma'$ if the $\beta$-arc of $\gamma'$ strictly contains the $\beta$-arc of $\gamma$. Take a minimal subarc $b' \subset \beta$ that
does not intersect $\alpha$ except for the endpoints. Denote the bicorn curve $\alpha_1 = a' \cap b'$ as the union of the minimal subarc $b' \subset \beta$ and the subarc $a'$ of $\alpha$ determined by the endpoints. Then, $\alpha$ intersects $\alpha_1$ at most once. One can extend the minimal subarc $b'$ to the next intersection point with $a'$. The extended subarc $b''$ of $\beta$ intersects $a'$ on the endpoint, the bicorn curve is denoted as $\alpha_2 = a'' \cup b''$. See the Figure 3. As we can see, $\alpha_1$ intersects $\alpha_2$ at most once.

Next, we extend the subarc $b''$ to the minimal subarc $b'''$ such that $b'''$ intersects $a''$ right on the endpoint, the subarc of $a''$ with bounded by the new intersection point is denoted by $a'''$. The bicorn curve $\alpha_3 = a''' \cup b'''$ intersects $\alpha_2$ at most once.

Continue in this way, one will be able to construct a sequence of bicorn curves $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n$, where the adjacent curves $\alpha_i, \alpha_{i+1}$ intersect at most once. Since the intersection number of $\alpha$ and $\beta$ is finite, the sequence must terminate at $\beta$, that is, $\alpha_n = \beta$. The sequence of bicorn curves constructed above is called a bicorn path. We will use $B(\alpha, \beta)$ to denote one bicorn path between $\alpha$ and $\beta$, and all the bicorn paths are denoted by $B(\alpha, \beta)$. Note that a bicorn path is not a real path, as the adjacent curves are not necessarily disjoint.

The bicorn paths $B(\alpha, \beta)$ have uniformly bounded Hausdorff distance to the geodesics between $\alpha$ and $\beta$.

**Proposition 3.2.** Let $\Gamma$ be a geodesic connecting two curves $\alpha$ and $\beta$, then a bicorn path $B(\alpha, \beta)$ between $\alpha$ and $\beta$ stay in the 14-neighborhood of $\Gamma$ in the curve graph.
In [4], a proof was given by Chang, Menasco and the author for the curves $\alpha$ and $\beta$ with coherent intersection. The proof proceeds without any change for the general cases.

4. THE PROOF

In this section, we will utilize the bicorn curves to prove the Theorem 1.5 for the closed oriented surfaces $X$ with genus $\geq 2$. With the uniformly bounded Hausdorff distance of the bicorn paths and 1-slimness of the bicorn curve triangles, the proof follows immediately from Webb’s strategy to the proof of the Theorem 3.2 in [17]. To start off, let us deal with a particular case when the geodesic is away from the curves in the subsurface.

Lemma 4.1. Suppose that $X$ is a closed oriented surface with genus $\geq 2$ and $Y$ is an essential subsurface of $X$ with $\xi(Y) \neq 0$. Let $\Gamma$ be a geodesic in $\mathcal{C}(X)$ such that each $x \in \Gamma$ cuts $Y$, and $\alpha$ be an essential boundary component of the non-annular subsurface $Y$ or the core of $Y$ if it is annular. If $d_{\mathcal{C}(X)}(x, \alpha) \geq 18$, for any vertex $x \in \Gamma$, then $d_{\mathcal{AC}(Y)}(\Gamma) \leq 3$.

Proof. Let $\gamma_i$ and $\gamma_j$ be any two distinct vertices in the geodesic $\Gamma$, we need to show that $d_{\mathcal{AC}(Y)}(\gamma_i, \gamma_j) \leq 3$. By the Proposition 3.2, each bicorn curve of $B(\gamma_i, \gamma_j)$ stays in the 14-neighborhood of the geodesic segment $[\gamma_i, \gamma_j] \subset \Gamma$. Since $d_{\mathcal{C}(X)}(x, \alpha) \geq 18$ for any $x \in \Gamma$, then $d_{\mathcal{C}(X)}(\gamma, \alpha) \geq 4$ for any bicorn curve $\gamma \in B(\gamma_i, \gamma_j)$. The minimal intersection number of a filling pair on a closed oriented surface $X$ with genus $\geq 2$ is 4, so $|\gamma \cap \alpha| \geq 4$.

Suppose that $\gamma_i$, $\gamma_j$ and $\alpha$ are in pairwise minimal intersection without any triple points. For any bicorn curve $\gamma = a \cup b$, where $a$ is $\gamma_i$-arc and $b$ is $\gamma_j$-arc, we consider the intersection points of $\alpha$ with the $a$ and $b$. One can take a minimal arc $c$ of $\alpha$ intersecting $a$ only at the two endpoints and $c$ intersects $b$ at most once (or intersecting
b only at the two endpoints and intersects a at most once). This is the key observation to prove the Lemma 2.6 [13] that the bicorn curve triangles are 1-slim, see the Figure 5 for an illustration. Recall that |γ ∩ α| = |(a ∪ b) ∩ α| ≥ 4, then either |a ∩ α| ≥ 2 or |b ∩ α| ≥ 2, so the curve surgery is allowed.

![Figure 5](image)

**Figure 5.** Either the γ_i-arc a intersects α at least twice or the γ_j-arc b intersects α at least twice. The minimal arc c ⊂ α intersects a (or b) at the two endpoints (intersection number ≤ 1) while intersecting the b (or a) at most once.

It follows that either one can construct a bicorn curve in B(γ_i, α) from a and c or a bicorn curve in B(γ_j, α) from b and c. Since γ_i ∈ B(γ_i, γ_j) and α can only produce bicorn curves in B(γ_i, α), and γ_j ∈ B(γ_i, γ_j) and α can only produce bicorn curves in B(γ_j, α), one can conclude that there must be two adjacent bicorn curves b_k, b_k+1 (possibly same) in B(γ_i, γ_j) such that b_k will produce a bicorn curve b'_k ∈ B(γ_i, α) and b_k+1 will produce a bicorn curve b'_{k+1} ∈ B(γ_j, α), as illustrated in the Figure 4.

By the construction of the curves b_k, b'_k, b_{k+1}, b'_{k+1}, one can find one subarc from π_A(γ_i) and another subarc from π_A(γ_j) that are disjoint from each other. More precisely, such subarcs can be chosen from π_A(δ_i) and π_A(δ_j), where δ_i belongs to the γ_i-arc of b_k that is bounded by the endpoints of one arc c ⊂ α, and δ_j belongs to the γ_j-arc of b_{k+1} that is bounded by the endpoints of another arc c ⊂ α. One issue is that the π_A(b'_k) and π_A(b'_{k+1}) might be empty. It only occurs when the arcs bounded by the endpoints of arc c ⊂ α does not have any intersection with α in the interior of the arcs and the two intersection points have opposite orientations. It implies that the b'_k (or b'_{k+1}) is the bicorn curve is disjoint from α. Then,

\[ d_{C(X)}(b_k, α) ≤ d_{C(X)}(b_k, b'_k) + d_{C(X)}(b'_k, α) ≤ 2 + 1 = 3, \]

which contradicts to \( d_{C(X)}(b_k, α) ≥ 4 \). With the 1-Lipschitz property, one will obtain

\[ d_{AC(Y)}(γ_i, γ_j) ≤ 1 + 1 + 1 = 3. \]

\[ □ \]

**Remark 4.2.** The bound in the Lemma 4.1 is comparable to the bound 4 in the motivating examples by Masur and Minsky [10]. For the non-annular subsurfaces, the bound is same as the bound given by Webb using the unicorn arcs in the Theorem 4.1.7 [18], while it is slightly better for annular subsurfaces.
Proof of the Theorem 1.5. Let $\alpha$ be an essential boundary component of the non-annular subsurface $Y$ and the core of $Y$ if it is annular. Let $I = N_{18}(\alpha) \cap \Gamma$ be the intersection (possibly empty) of the 18-neighborhood of the $\alpha$ and the geodesic $\Gamma$, then there exists a geodesic segment $g$ of length at most 36 with $I \subset g \subset \Gamma$. Denote $\gamma'_i$ and $\gamma'_j$ as the two ends of the geodesic segment $g$. Suppose at least one of $\gamma_i$ and $\gamma_j$ is not in $g$. Otherwise, since each vertex of $g$ cuts $Y$, then $d_{AC}(Y)(\gamma_i, \gamma_j) \leq 36$ follows from the Lemma 1.4.

Assume $\gamma_i$, $\gamma_j$ or both are not in the geodesic segment $g \subset \Gamma$, see the Figure 6. Using the Lemma 1.4 and the Lemma 4.1, one will have

$$d_{AC}(Y)(\gamma_i, \gamma_j) \leq d_{AC}(Y)(\gamma_i, \gamma'_i) + d_{AC}(Y)(\gamma'_i, \gamma'_j) + d_{AC}(Y)(\gamma'_j, \gamma_j) \leq 3 + 1 + 36 + 1 + 3 = 44.$$ 

□

Remark 4.3. The strategy can be applied to the surfaces with boundary, because the bicorn curves can be defined in the same manner as long as the bicorn curves are essential. Restricting to the nonseparating curves, the bicorn curves triangles have been used to prove the uniform hyperbolicity of the nonseparating curve graphs by Rasmussen [15]. Combined with the uniformly bounded Hausdorff distance of the bicorn paths on the surfaces with boundary (Corollary 2.17, Rasmussen [14]), one will be able to prove a similar result, possibly with larger bounds.

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