A Conducting Checkerboard

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1 Problem

Some biological systems consist of two “phases” of nearly square fiber bundles of differing thermal and electrical conductivities. Consider a circular region of radius $a$ near a corner of such a system as shown below.

Phase 1, with electrical conductivity $\sigma_1$, occupies the “bowtie” region of angle $\pm \alpha$, while phase 2, with conductivity $\sigma_2 \ll \sigma_1$, occupies the remaining region.

Deduce the approximate form of lines of current density $j$ when a background electric field is applied along the symmetry axis of phase 1. What is the effective conductivity $\sigma$ of the system, defined by the relation $I = \sigma \Delta \phi$ between the total current $I$ and the potential difference $\Delta \phi$ across the system?

It suffices to consider the case that the boundary arc ($r = a, |\theta| < \alpha$) is held at electric potential $\phi = 1$, while the arc ($r = a, \pi - \alpha < |\theta| < \pi$) is held at electric potential $\phi = -1$, and no current flows across the remainder of the boundary.

Hint: When $\sigma_2 \ll \sigma_1$, the electric potential is well described by the leading term of a series expansion.

2 Solution

The series expansion approach is unsuccessful in treating the full problem of a “checkerboard” array of two phases if those phases meet in sharp corners as shown above. However, an analytic form for the electric potential of a two-phase (and also a four-phase) checkerboard can be obtained using conformal mapping of certain elliptic functions [1]. If the regions of one phase are completely surrounded by the other phase, rather lengthy series expansions
for the potential can be given \[2\]. The present problem is based on work by Grimvall \[3\] and Keller \[4\].

In the steady state, the electric field obeys \( \nabla \times \mathbf{E} = 0 \), so that \( \mathbf{E} \) can be deduced from a scalar potential \( \phi \) via \( \mathbf{E} = -\nabla \phi \). The steady current density obeys \( \nabla \cdot \mathbf{j} = 0 \), and is related to the electric field by Ohm’s law, \( \mathbf{j} = \sigma \mathbf{E} \). Hence, within regions of uniform conductivity, \( \nabla \cdot \mathbf{E} = 0 \) and \( \nabla^2 \phi = 0 \). Thus, we seek solutions to Laplace’s equations in the four regions of uniform conductivity, subject to the stated boundary conditions at the outer radius, as well as the matching conditions that \( \phi, E_{\parallel}, \) and \( j_{\perp} \) are continuous at the boundaries between the regions.

We analyze this two-dimensional problem in a cylindrical coordinate system \((r, \theta)\) with origin at the corner between the phases and \( \theta = 0 \) along the radius vector that bisects the region whose potential is unity at \( r = a \). The four regions of uniform conductivity are labeled \( I, II, III \) and \( IV \) as shown below.

Since \( j_{\perp} = j_r = \sigma \mathbf{E}_r = -\sigma \partial \phi / \partial r \) at the outer boundary, the boundary conditions at \( r = a \) can be written

\[
\begin{align*}
\phi_I(r = a) &= 1, \\
\frac{\partial \phi_{II}(r = a)}{\partial r} &= \frac{\partial \phi_{IV}(r = a)}{\partial r} = 0, \\
\phi_{III}(r = a) &= -1.
\end{align*}
\]

Likewise, the condition that \( j_{\perp} = j_{\theta} = \sigma \mathbf{E}_\theta = -(\sigma / r) \partial \phi / \partial \theta \) is continuous at the boundaries between the regions can be written

\[
\begin{align*}
\sigma_1 \frac{\partial \phi_I(\theta = \alpha)}{\partial \theta} &= \sigma_2 \frac{\partial \phi_{II}(\theta = \alpha)}{\partial \theta}, \\
\sigma_1 \frac{\partial \phi_{III}(\theta = \pi - \alpha)}{\partial \theta} &= \sigma_2 \frac{\partial \phi_{II}(\theta = \pi - \alpha)}{\partial \theta},
\end{align*}
\]

\( \text{etc.} \)

From the symmetry of the problem we see that

\[
\phi(-\theta) = \phi(\theta),
\]
\[ \phi(\pi - \theta) = -\phi(\theta), \]  
(7)

and in particular \( \phi(r = 0) = 0 = \phi(\theta = \pm \pi/2) \).

We recall that two-dimensional solutions to Laplace’s equations in cylindrical coordinates involve sums of products of \( r^{\pm k} \) and \( e^{\pm ik\theta} \), where \( k \) is the separation constant that in general can take on a sequence of values. Since the potential is zero at the origin, the radial function is only \( r^k \). The symmetry condition (6) suggests that the angular functions for region I be written as \( \cos k\theta \), while the symmetry condition (7) suggests that we use \( \sin k(\pi/2 - |\theta|) \) in regions II and IV and \( \cos k(\pi - \theta) \) in region III. That is, we consider the series expansions

\[ \phi_I = \sum A_k r^k \cos k\theta, \]  
(8)
\[ \phi_{II} = \phi_{IV} = \sum B_k r^k \sin \left( \frac{\pi}{2} - |\theta| \right), \]  
(9)
\[ \phi_{III} = -\sum A_k r^k \cos (\pi - \theta). \]  
(10)

The potential must be continuous at the boundaries between the regions, which requires

\[ A_k \cos k\alpha = B_k \sin k \left( \frac{\pi}{2} - \alpha \right). \]  
(11)

The normal component of the current density is also continuous across these boundaries, so eq. (11) tells us that

\[ \sigma_1 A_k \sin k\alpha = \sigma_2 B_k \cos k \left( \frac{\pi}{2} - \alpha \right). \]  
(12)

On dividing eq. (12) by eq. (11) we find that

\[ \tan k\alpha = \frac{\sigma_2}{\sigma_1} \cot k \left( \frac{\pi}{2} - \alpha \right). \]  
(13)

There is an infinite set of solutions to this transcendental equation. When \( \sigma_2/\sigma_1 \ll 1 \) we expect that only the first term in the expansions (8)-(9) will be important, and in this case we expect that both \( k\alpha \) and \( k(\pi/2 - \alpha) \) are small. Then eq. (13) can be approximated as

\[ k\alpha \approx \frac{\sigma_2/\sigma_1}{k(\pi/2 - \alpha)}, \]  
(14)

and hence

\[ k^2 \approx \frac{\sigma_2/\sigma_1}{\alpha(\pi/2 - \alpha)} \ll 1. \]  
(15)

Equation (11) also tells us that for small \( k\alpha \),

\[ A_k \approx B_k k \left( \frac{\pi}{2} - \alpha \right). \]  
(16)

Since we now approximate \( \phi_I \) by the single term \( A_k r^k \cos k\theta \approx A_k r^k \), the boundary condition (11) at \( r = a \) implies that

\[ A_k \approx \frac{1}{a^k}, \]  
(17)
and eq. (16) then gives
\[ B_k \approx \frac{1}{ka^k(\frac{\pi}{2} - \alpha)} \gg A_k. \]  
(18)

The boundary condition (2) now becomes
\[ 0 = kBk^a\theta \approx k(\frac{\pi}{2} - \theta) \approx \frac{A}{a} \frac{\pi}{2}, \]  
(19)

which is approximately satisfied for small \( k \).

So we accept the first terms of eqs. (8)-(10) as our solution, with \( k, A, B \) given by eqs. (15), (17) and (18).

In region I the electric field is given by
\[ E_r = -\frac{\partial \phi_I}{\partial r} \approx -k \frac{r^{k-1}}{a^k} \cos k\theta \approx -k \frac{r^{k-1}}{a^k}, \]  
(20)
\[ E_\theta = -\frac{1}{r} \frac{\partial \phi_I}{\partial \theta} \approx \frac{k \frac{r}{k-1} a^{k}}{\sin k\theta} \approx k^2 \frac{r^{k-1}}{a^k}. \]  
(21)

Thus, in region I, \( E_\theta/E_r \approx k\theta \ll 1 \), so the electric field, and the current density, is nearly radial. In region II the electric field is given by
\[ E_r = -\frac{\partial \phi_{II}}{\partial r} \approx -k \frac{r^{k-1}}{ka(k^{(\pi/2) - \alpha})} \sin k^\left(\frac{\pi}{2} - \theta\right) \approx -k \frac{r^{k-1}}{ka^k(\pi/2 - \alpha)}, \]  
(22)
\[ E_\theta = -\frac{1}{r} \frac{\partial \phi_{II}}{\partial \theta} \approx \frac{k \frac{r^{k-1}}{k^{(\pi/2) - \alpha}}}{\cos k^\left(\frac{\pi}{2} - \theta\right)} \approx \frac{r^{k-1}}{a^{k}(\pi/2 - \alpha)}. \]  
(23)

Thus, in region II, \( E_r/E_\theta \approx k(\pi/2 - \theta) \ll 1 \), so the electric field, and the current density, is almost purely azimuthal.

The current density \( j \) follows the lines of the electric field \( E \), and therefore behaves as sketched below:

\[ j_\perp = 0 \]
\[ j_\parallel \]
\[ \phi = -1 \]
\[ \phi = 1 \]

The total current can be evaluated by integrating the current density at \( r = a \) in region I:
\[ I = 2a \int_0^\alpha j_r d\theta = 2a \sigma_1 \int_0^\alpha E_r(r = a) d\theta \approx -2k \sigma_1 \int_0^\alpha d\theta = -2k \sigma_1 \alpha = -2 \sqrt{\sigma_1 \sigma_2 \alpha}. \]  
(24)
In the present problem the total potential difference $\Delta \phi$ is -2, so the effective conductivity is

$$\sigma = \frac{I}{\Delta \phi} = \sqrt{\frac{\sigma_1 \sigma_2 \alpha}{\frac{\pi}{2} - \alpha}}.$$  

(25)

For a square checkerboard, $\alpha = \pi/4$, and the effective conductivity is $\sigma = \sqrt{\sigma_1 \sigma_2}$. It turns out that this result is independent of the ratio $\sigma_2/\sigma_1$, and holds not only for the corner region studied here but for the entire checkerboard array [5].

### References

[1] R.V. Craster and Yu.V. Obnosov, Checkerboard composites with separated phases, J. Math. Phys. 42, 5379 (2001).

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