Reduction of SISO H-infinity Output Feedback Control Problem✩

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Abstract

We consider the linear matrix inequality (LMI) problem of $H_\infty$ output feedback control problem for a generalized plant whose control input, measured output, disturbance input, and controlled output are scalar. We provide an explicit form of the optimal value. This form is the unification of some results in the literature of $H_\infty$ performance limitation analysis. To obtain the form of the optimal value, we focus on the non-uniqueness of perpendicular matrices, which appear in the LMI problem. We use the null vectors of invariant zeros associated with the dynamical system for the expression of the perpendicular matrices. This expression enables us to reduce and simplify the LMI problem. Our approach uses some well-known fundamental tools, e.g., the Schur complement, Lyapunov equation, Sylvester equation, and matrix completion. We use these techniques for the simplification of the LMI problem. Also, we investigate the structure of dual feasible solutions and reduce the size of the dual. This reduction is called a facial reduction in the literature of convex optimization.

Keywords: Linear matrix inequality, $H_\infty$ control, invariant zeros, dual problem, facial reduction

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1. Introduction

The importance of $H_\infty$ control problems in robust control was first pointed out by Zames [30]. To design $H_\infty$ controllers, [5] proposed an approach via

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algebraic Riccati equations and inequalities, which works fine under some assumptions on a given generalized plant. Subsequently, an approach by using linear matrix inequalities (LMIs) are proposed in [12, 16, 22, 18]. Both these approaches enable us to design a controller that internally stabilizes the closed-loop system and makes its $H_\infty$ norm lower than a priori given bound.

In contrast, we derive an explicit form of the infimum $H_\infty$ norm in $H_\infty$ optimal controller synthesis problems without any care for controller construction. To that end, we focus on the LMI optimization problem associated with the $H_\infty$ optimal controller synthesis. Because this minimization is formulated as the infimum, it has no guarantee to have any optimal solutions. In other words, some of the variables in the LMI problem may go to infinity when the objective value approaches its optimal value. Then we may encounter numerical difficulties in such cases. Even if one can construct a controller from a computed solution of the LMI problem, it may be fragile to small changes in the parameters of the controller.

The infimum $H_\infty$ norm is often analytically computed. Such approaches are proposed in the literature of $H_\infty$ performance limitation analysis. For instance, [4] provided the $H_\infty$ performance limitations of sensitivity and complementary sensitivity functions for MIMO linear time-invariant systems. In particular, the Nevanlinna-Pick interpolation was used. After obtaining the infimum, one can compute the desired controller whose $H_\infty$ norm is close to this infimum by applying the existing Riccati, or LMI approaches.

**Contribution**

We deal with a generalized plant whose input, output, disturbance and controlled output are scalar. The plant is formulated as follows.

\[
\begin{align*}
\dot{x} &= Ax + b_1 w + b_2 u \\
z &= c_1^T x + d_{11} w + d_{12} u \\
y &= c_2^T x + d_{21} w, \\
\end{align*}
\]

where $A \in \mathbb{R}^{n \times n}$, $b_i, c_i \in \mathbb{R}^n$ and $d_{ij} \in \mathbb{R}$. In this paper, we refer to (1) as a generalized plant for SISO $H_\infty$ control problem. The contribution of this paper is to provide an explicit form of the optimal value for the well-known LMI problem in relation to the SISO $H_\infty$ output feedback control problem. The main result can be summarized in the following theorem that is obtained by combining Theorems 4, 5, 6 and 7 in this paper.

**Theorem 1.** Let $\gamma^*$ be the optimal value of the LMI problem obtained from $H_\infty$ output feedback control for (1) by applying the elimination of variable method. The transfer matrix $G(s)$ for (1) is denoted by

\[
G(s) = \begin{pmatrix}
G_{zw}(s) & G_{zu}(s) \\
G_{yw}(s) & G_{yu}(s)
\end{pmatrix}.
\]

Moreover, $\lambda_1, \ldots, \lambda_{m_1}$ (resp. $\omega_1, \ldots, \omega_{m_2}$) denote invariant zeros on the imaginary axis via the realization $(A, b_2, c_1^T, d_{12})$ of $G_{zw}$ (resp. $(A^T, c_2, b_1^T, d_{21})$ of
We assume that all of \(\lambda_1, \ldots, \lambda_{m_1}, \omega_1, \ldots, \omega_{m_2}\) and their complex conjugates are not eigenvalues of \(A\).

1. If \(d_{12} \neq 0\) and \(d_{21} \neq 0\), then \(\gamma^*\) is equal to

\[
\max \left\{ \hat{\gamma}, |G_{zw}(\lambda_j)| \ (j = 1, \ldots, m_1), |G_{zw}(\omega_j)| \ (j = 1, \ldots, m_2) \right\},
\]

where \(\hat{\gamma}\) is the maximum eigenvalue of a symmetric matrix defined by unstable invariant zeros in \(G_{zu}\) and \(G_{yw}\) and their associated vectors. (See (37) for the definition of the matrix.)

2. If \(d_{12} = 0\) or \(d_{21} = 0\), then \(\gamma^*\) is equal to

\[
\max \left\{ \hat{\gamma}, |G_{zw}(\lambda_j)| \ (j = 1, \ldots, m_1), |G_{zw}(\omega_j)| \ (j = 1, \ldots, m_2), |G_{zw}(\infty)| \right\},
\]

where \(G_{zw}(\infty)\) is the value of the transfer function \(G_{zw}\) at infinity.

Here \(|G_{zw}(\lambda_j)|\) (resp. \(|G_{zw}(\omega_j)|\)) is vanished from the above expressions of \(\gamma^*\) if the realization of \(G_{zu}\) (resp. \(G_{yw}\)) has no invariant zeros on the imaginary axis.

In general, the notion of invariant zeros is defined for the realization or state-space representation, not the transfer function. However, for the sake of brevity, we call an invariant zero of the realization \((A, b_2, c_1^T, d_{12})\) (resp. \((A^T, c_2, b_1^T, d_{21})\) an invariant zero of \(G_{zu}\) (resp. \(G_{yw}\)) throughout this paper.

We give remarks on Theorem 1.

**Remark 1.**

1. Although we have assumed that all invariant zeros \(\lambda_j\) and \(\omega_j\) on the imaginary axis are not eigenvalues of \(A\), we can remove this assumption. Then we can describe \(\gamma^*\) by the null vectors associated with invariant zeros \(\lambda_j\) and \(\omega_j\), instead of \(G_{zw}(\lambda_j)\) and \(G_{zw}(\omega_j)\) in \(\gamma^*\). See Theorem 6.

2. The \(H_\infty\) control problem for (11) is the problem of finding a controller \(K(s)\) which minimizes the \(H_\infty\) norm of the closed-loop \(G_{cl}(s, K)\) obtained by connecting \(K(s)\) with (11). Mathematically, this problem can be formulated as follows:

\[
\gamma^* = \inf_{K \in K} \sup_{s \in \sqrt{-1}\mathbb{R}} \sigma_{\max} (G_{cl}(s, K)),
\]

where \(\sqrt{-1}\) indicates the imaginary unit,

\[
G_{cl}(s, K) := G_{zw}(s) + G_{zu}(s)K(s)(1 - G_{yu}(s)K(s))^{-1}G_{yw}(s)
\]

and \(K\) is the set of rational functions on \(s\) which stabilize \(G(s)\) internally. We see that when \(\lambda\) is an invariant zero on the imaginary axis of \(G_{zu}\) (resp. \(G_{yw}\)) of (11), the value of the transfer function \(G_{cl}(\lambda, K)\) is \(G_{zw}(\lambda)\).

In fact, we have \(G_{zu}(\lambda) = 0\) (resp. \(G_{yw}(\lambda) = 0\)) because \(\lambda\) is not an eigenvalue of \(A\). The detail will be provided in Lemma 1. Therefore, Theorem 1 contains the values of \(G_{zw}\) at zeros of \(G_{zu}\) and \(G_{yw}\) over the imaginary axis.
3. The performance index $\gamma^*$ is greater than or equal to $|G_{zw}(\infty)| = |d_{11}|$ when at least one of $d_{12}$ and $d_{21}$ is zero. To see this, let $K(s)$ be the transfer function of a controller with a realization $(A_K, b_K, c_T K, d_K)$. We see from $G_{cl}(\infty, K) = d_{11} + d_{12}d_Kd_{21}$ that we can reduce the effect of the feedthrough term of (1) by the choice of $d_K$ when $d_{12} \neq 0$ and $d_{21} \neq 0$. Otherwise, we cannot reduce it because the feedthrough term of the closed-loop system is $d_{11}$, which is independent in the choice of the parameter $d_K$. Therefore the performance index $\gamma^*$ is greater than or equal to $|G_{zw}(\infty)| = |d_{11}|$ when at least one of $d_{12}$ and $d_{21}$ is zero.

To prove the main result, Theorem 1, we consider the following cases:

1. Both $d_{12}$ and $d_{21}$ are nonzero, and all invariant zeros in $G_{zu}$ and $G_{yw}$ are unstable, but not on the imaginary axis.
2. Both $d_{12}$ and $d_{21}$ are nonzero, and at least one of the invariant zeros in $G_{zu}$ or $G_{yw}$ is stable, but all unstable invariant zeros are not on the imaginary axis.
3. Both $d_{12}$ and $d_{21}$ are nonzero, and at least one of the invariant zeros in $G_{zu}$ or $G_{yw}$ exists on the imaginary axis.
4. At least one of $d_{12}$ and $d_{21}$ is zero.

All possible generalized plant of the form (1) are exactly one of these cases. For simplicity, we will assume in this paper that all invariant zeros are real and distinct from each other. In the analysis of Case 1 we use the Schur complement and the Lyapunov equation. On the other hand, we use not only these mathematical tools but also a technique of the matrix completion problem in Lemma 11 and the dual of the resulting LMI problem in the analysis of Cases 2, 3 and 4.

Why do we deal with the dual problem?

The reason is that we can reduce the dual problem and some techniques developed in Case 1 are available. In all of Cases 2, 3 and 4 the dual problem is feasible, but not strictly feasible. We exploit this property of the dual of the resulting LMI problem in these cases. More precisely, the dual problem of all these cases is formulated as follows.

$$\sup \left\{ L_0 \bullet X : L_j \bullet X = b_j \ (j = 1, \ldots, m), X \in S^n_+ \right\}, \quad (3)$$

where $L_0, L_1, \ldots, L_m$ are $n \times n$ symmetric matrices, $b_1, \ldots, b_m \in \mathbb{R}$ and $L_j \bullet X = \text{Tr}(L_jX)$ for $j = 0, 1, \ldots, m$. Then (3) has no interior feasible solutions, i.e., no positive definite solutions in (3). Hence there exists an orthogonal matrix $P \in \mathbb{R}^{n \times n}$ and a positive integer $r$ such that any dual feasible solution $X$ has the form of

$$X = P \begin{pmatrix} \hat{X} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{pmatrix} P^T \quad (4)$$
for some $\tilde{X} \in S_r^+$. In general, it is difficult to find the nonsingular matrix $P$ in the form (4) from (3). It, however, is relatively easy to compute such a matrix $P$ in all Cases 2 to 4.

Using (4), we can reduce the LMI problem that corresponds to the dual (3). Substituting this form (4) to (3), we obtain the following problem whose optimal value is equal to that of (3).

$$\sup \left\{ \tilde{L}_0 \cdot \tilde{X} : \tilde{L}_j \cdot \tilde{X} = b_j \ (j = 1, \ldots, m), \tilde{X} \in S_r^+ \right\}, \quad (5)$$

where the coefficient matrix $\tilde{L}_j \in S_r$ is a square submatrix of the matrix $P^T L_j P$ for all $j = 0, 1, \ldots, m$. Clearly the size of the positive semidefinite matrix in the LMI problem of (5) is smaller than the size of the original LMI problem. After reducing the LMI problem, some techniques developed in Case 1 are available to the reduced LMI problem.

This type of reduction is called facial reduction in the literature of the theory of convex optimization. The facial reduction was proposed in [3]. In general, the strong duality for convex optimization requires a constrained qualification. Otherwise, the strong duality may fail, i.e., no optimal solutions and/or a positive duality gap. By applying the facial reduction to such convex optimization problems, the reduced problems always satisfy a constrained qualification, and thus the strong duality holds. The facial reduction was already applied in the literature of control theory, e.g., $H_2$ analysis in [1] and $H_\infty$ state feedback control in [27, 28].

Related work

This study is inspired by [4], which deals with MIMO systems. We will obtain the same result for SISO systems to [4] in this study. The work [4] used a mathematical tool in complex analysis, while our result is obtained to analysis the LMI problem and its dual.

Furthermore, this study unifies some of the existing work [8, 9, 10]. The work [8] obtained a lower bound of the $H_\infty$ performance limitations of $(1 + PK)^{-1} P$, where $P$ and $K$ are transfer functions of a SISO linear time-invariant system and a controller, respectively. This lower bound was obtained from a detailed analysis of the resulting LMI problem. The exactness of the lower bound was proved in [10] by using a property in the dual problem. This technique was also used in [4], which deals with the $H_\infty$ performance limitations of sensitivity and complementary sensitivity functions for a SISO linear time-invariant system. The dual problems play an essential role in both studies. In this study, we extend the analysis obtained in [10] and provide the performance limitation for a more general SISO $H_\infty$ output feedback control problem. The analysis in [8] for the dual problems can be regarded as facial reduction.

The work [14] reformulated the resulting LMI problem by using Kronecker canonical form (KCF) (a.k.a. Weierstrass form in this study) obtained from a given generalized plant. This reformulation separates variables in the LMI problem into bounded and unbounded variables. The unbounded variables are
removed because they make no effect on the minimum value of the problem. In contrast, the bounded variables remain in the LMI problem. As a result, the size of the problem is reduced, and the numerical performance was improved.

In comparison with facial reduction, the reduction via KCF deals with the LMI problem, while facial reduction deals with the dual problem. In other words, the reduction via KCF has a dual relation to facial reduction. In fact, one can construct an LMI problem whose dual corresponds to the dual problem reduced via facial reduction. Then one can see that the variables removed in the reduction via KCF also vanish in the LMI problem. In this sense, we can regard facial reduction as the dual approach of the reduction via KCF.

[21] developed a variant of KCF for a given generalized plant and focused on the Riccati equations and inequalities obtained from the plant. A simplification of the Riccati equations and inequalities associated with the plant was provided with using this variant.

Organization of this paper

The purpose of this paper is to prove Theorem 1. For this, we consider the four cases, 1 to 4. These cases are discussed in Sections 4 to 7, respectively. Invariant zeros play an essential role in reducing the LMI problem. Section 2 devotes the introduction of the concept of invariant zeros and their mathematical formulation. We also present the LMI formulation of $H_\infty$ output feedback control in Section 3. We also focus on the non-uniqueness of perpendicular matrices, which appear in the LMI problem of $H_\infty$ output feedback control. For the perpendicular matrices, we use the null vectors associated with invariant zeros in the SISO dynamical system. We can see some existing results related to $H_\infty$ limitation analysis by using Theorem 1 in Section 8. We give a conclusion of this paper in Section 9. We introduce other mathematical tools and proofs of some lemmas for proving Theorem 1 in Appendices.

Notation and symbols

We introduce some notation and symbols used in this paper. Let $\mathbb{C}$ be the set of complex numbers. For $\lambda \in \mathbb{C}$, $\Re(\lambda)$ (resp. $\Im(\lambda)$) denotes the real (resp. imaginary) part of $\lambda$. We partition $\mathbb{C}$ into

$$\mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \},$$
$$\mathbb{C}_- = \{ \lambda \in \mathbb{C} : \Re(\lambda) < 0 \}$$
and
$$\mathbb{C}_0 = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \}.$$

Let $\mathbb{S}^n$, $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ be the sets of $n \times n$ symmetric matrices, $n \times n$ positive semidefinite matrices and $n \times n$ positive definite matrices. For $A, B \in \mathbb{S}^n$, $A \succeq B$ denotes $A - B \in \mathbb{S}_+^n$. We define $A \bullet B = \text{Tr}(AB^T) = \sum_{k,t=1}^n A_{kt}B_{kt}$. We define $\He(M) = M + M^T$ for any square matrix $M$.

We denote by $\sigma_{\text{max}}(A)$ the maximum singular value of a matrix $A$. In addition, for a square matrix $A$, we denote by $\lambda_{\text{max}}(A)$ the maximum eigenvalue of $A$ when $A$ is symmetric.

For a given matrix $G \in \mathbb{R}^{n \times m}$ with rank $r$, $G^\perp$ denotes an $n \times (n - r)$ matrix which satisfies $G^T G^\perp = O_{m \times (n-r)}$ and $(G^\perp)^T G^\perp \in \mathbb{S}_{++}^{n-r}$. We call $G^\perp$ a perpendicular matrix of $G$ throughout this paper. In general, $G^\perp$ is not unique for a given matrix $G$. $G^\perp T$ stands for the transpose of $G^\perp$ in this paper.
2. Preliminaries

In this section, we review the definition and several properties of the invariant zeros of SISO LTI systems. Let us consider the following SISO LTI system $G$ described by

$$
\begin{align*}
\dot{x} &= Ax + bu \\
y &= c^T x + du,
\end{align*}
$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. The transfer function of the system $G$ is given by $G(s) = c^T (sI_n - A)^{-1} b + d$. We say that $\lambda \in \mathbb{C}$ is an invariant zero of (6) if

$$
\operatorname{rank} \begin{pmatrix} A - \lambda I_n & b \\ c^T & d \end{pmatrix} < n + 1. 
$$

In addition, we say that an invariant zero $\lambda$ is stable if the real part of $\lambda$ is negative, i.e., $\lambda \in \mathbb{C}_-$. Otherwise, we say that the invariant zero $\lambda$ is unstable.

We first provide some fundamental facts on the invariant zeros.

**Lemma 1.**

1. If $\lambda \in \mathbb{C}$ is an invariant zero of (6) and if $\lambda \not\in \lambda(A)$ then $G(\lambda) = 0$. Here $\lambda(A)$ denotes the set of all the eigenvalues of the matrix $A$.

2. $\lambda \in \mathbb{C}$ is an invariant zero of (6) if and only if there exists $(\hat{v}_L)^T \in \mathbb{C}^{n+1} \setminus \{0\}$ such that

$$
(\hat{v}_L^T \hat{v}_L) \begin{pmatrix} A & b \\ c^T & d \end{pmatrix} = \lambda (\hat{v}_L^T \hat{0}).
$$

Similarly, $\lambda \in \mathbb{C}$ is an invariant zero of (6) if and only if there exists $(\hat{v}_R)^T \in \mathbb{C}^{n+1} \setminus \{0\}$ such that

$$
\begin{pmatrix} A & b \\ c^T & d \end{pmatrix} \begin{pmatrix} v_R^T \\ \hat{v}_R \end{pmatrix} = \lambda \begin{pmatrix} v_R^T \\ \hat{0} \end{pmatrix}.
$$

3. If $(A, b)$ in (6) is controllable then $\hat{v}_L \in \mathbb{C}$ in (8) is nonzero. Similarly, if $(A, b)$ in (6) is stabilizable and if $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_0$, then $\hat{v}_L \in \mathbb{C}$ is nonzero.

4. If $(A, c^T)$ in (6) is observable then $\hat{v}_R \in \mathbb{C}$ in (9) is nonzero. Similarly, if $(A, c^T)$ in (6) is detectable and if $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_0$, then $\hat{v}_R \in \mathbb{C}$ is nonzero.

**Proof.** The validity of the assertion 1 readily follows since for $\lambda \not\in \lambda(A)$ we have

$$
\operatorname{rank} \begin{pmatrix} A - \lambda I_n & b \\ c^T & d \end{pmatrix} = \operatorname{rank} \begin{pmatrix} A - \lambda I_n & 0 \\ c^T (\lambda I_n - A)^{-1} b + d. \end{pmatrix}
= \operatorname{rank} \begin{pmatrix} A - \lambda I_n & 0 \\ c^T & G(\lambda) \end{pmatrix}.
$$

The validity of the assertion 2 is obvious. For the proof of the assertion 3 we first consider the case where $(A, b)$ is controllable, i.e., $\operatorname{rank} (A - s I_n, b) = n$. Then
n (∀s ∈ ℂ). Suppose ̂v_L = 0 for contradiction. Then it follows from (8) that
v_L^r A = λv_L^r and v_L^r b = 0. This equation contradicts the controllability of (A, b). Therefore ̂v_L ≠ 0. We next consider the case where (A, b) is stabilizable, i.e., rank (A - sI_n b) = n (∀s ∈ ℂ₊ ∪ ℂ₀). Suppose ̂v_L = 0 for contradiction. Then it follows from (8) that v_L^r A = λv_L^r and v_L^r b = 0 for λ ∈ ℂ₊ ∪ ℂ₀. This equation contradicts the stabilizability of (A, b). Therefore again ̂v_L ≠ 0. The assertion 4 can be proved similarly to the proof of the assertion 3.

In the following, we call (v_L^r ̂v_L) ∈ ℂⁿ⁺¹ that satisfies (8) the left null vector associated with the invariant zero λ of G. Similarly, we call (v_R^r ̂v_R) ∈ ℂⁿ⁺¹ that satisfies (9) the right null vector associated with the invariant zero λ of G.

As we see in the next theorem, invariant zeros in (6) plays an essential role in a canonical quasi-diagonal form of the Rosenbrock system matrix
\( A, b \) of (6). The canonical form is known as the Weierstrass form in [17, eq. (3.19)]. This theorem follows from [13, Theorem 3 in Chapter XII].

**Theorem 2.** Assume \((b \choose d) ≠ 0\) or \(( ̂b \choose ̂d) ≠ 0\). Then there exist non-singular matrices \( P, Q ∈ ℂ^{(n+1)×(n+1)} \), a nonnegative integer \( r \), a Jordan matrix \( Λ ∈ ℂ^{(n-r)×(n-r)} \) and a nilpotent \( N ∈ ℂ^{(r+1)×(r+1)} \) such that
\[
P \begin{pmatrix} sI_n - A & -b \\ -c^T & -d \end{pmatrix} Q = \begin{pmatrix} sI_n-r - Λ & O \\ O & sN - I_{r+1} \end{pmatrix}.
\]
(10)
Furthermore, the followings hold.
1. All eigenvalues of Λ are invariant zeros of (6).
2. The matrix N consists of only one Jordan cell, i.e., \( N^r ≠ O \) and \( N^{r+1} = O \).
3. If \( d ≠ 0 \), then \( r = 0 \).
4. If \( d = 0 \), then \( c^T b = 0, c^T A b = 0, \ldots, c^T A^{r-2} b = 0 \) and \( c^T A^{r-1} b ≠ 0 \).

**Proof.** (10) follows from [13, Theorem 3 in Chapter XII] and Jordan decomposition. We here prove all assertions. We notice that (10) is the identity on s. Then we obtain
\[
\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} Q = P^{-1} \begin{pmatrix} I_{n-r} & O \\ O & N \end{pmatrix},
\]
(11)
\[
\begin{pmatrix} A & b \\ c^T & d \end{pmatrix} Q = P^{-1} \begin{pmatrix} Λ & O \\ O & I_{r+1} \end{pmatrix}.
\]
(12)

For the assertion 1 we denote \( P^{-1}, Q ∈ ℂ^{(n+1)×(n+1)} \) by
\[
P^{-1} = \frac{n}{1} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, Q = \frac{n}{1} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.
\]
We obtain the following equations from (11) and (12), respectively.
\[
P_{11} = Q_{11}, O = P_{21} and \begin{pmatrix} A & b \\ c^T & d \end{pmatrix} \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix} = \begin{pmatrix} P_{11} \Lambda \\ P_{21} \Lambda \end{pmatrix}.
\]
Eliminating $P_{11}$ and $P_{21}$ from those equations, we obtain

$$
\begin{pmatrix}
A & b \\
c^T & d
\end{pmatrix}
\begin{pmatrix}
Q_{11} \\
Q_{21}
\end{pmatrix}
= 
\begin{pmatrix}
Q_{11} \\
O
\end{pmatrix}
\Lambda.
$$

(13)

We see from (13) that all eigenvalues of $\Lambda$ are invariant zeros of (6).

For the assertion 2, it is sufficient to prove $\text{rank } N = r$ because $N$ is a nilpotent. This follows from (11). In fact, we obtain $\text{rank } I_n = \text{rank } I_{n-r} + \text{rank } N$ from (11) because both $P$ and $Q$ are non-singular.

For simplicity of the proof of the assertions 3 and 4, we restrict the form of $N$. As we have already seen, the rank of $N$ is $r$. For any nilpotent $N$ with rank $r$, there exists a non-singular matrix $\hat{P} \in \mathbb{C}^{(r+1) \times (r+1)}$ such that

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & 1 \\
0 & \cdots & 0
\end{pmatrix}
= \hat{P}^{-1} N \hat{P}.
$$

(14)

This is obtained from the Jordan decomposition of $N$. Thus we set $N$ as the matrix in the left-hand side of (14) and will prove the assertions 3 and 4.

For the assertion 3, we suppose to the contrary that $r > 0$. We obtain from (11) and (12)

$$
\begin{pmatrix}
A & b \\
c^T & d
\end{pmatrix}
\begin{pmatrix}
Q_{12} \\
Q_{22}
\end{pmatrix}
= 
\begin{pmatrix}
Q_{12} \\
O
\end{pmatrix}.
$$

(15)

We denote $Q_{12}$ and $Q_{22}$ by $Q_{12} = (q_1 \cdots q_{r+1})$ and $Q_{22} = (\hat{q}_1 \cdots \hat{q}_{r+1})$. Substituting $Q_{12}$ and $Q_{22}$ to (15), we then obtain

$$
\begin{pmatrix}
A & b \\
c^T & d
\end{pmatrix}
\begin{pmatrix}
0 & q_1 & \cdots & q_r \\
0 & \hat{q}_1 & \cdots & \hat{q}_r
\end{pmatrix}
= 
\begin{pmatrix}
q_1 & q_2 & \cdots & q_{r+1} \\
0 & 0 & \cdots & 0
\end{pmatrix}.
$$

(16)

We see from this equation that $q_1 = 0$ and $\hat{q}_1 d = 0$. Since we assumed $d \neq 0$, $\hat{q}_1 = 0$. This contradicts the fact that $Q$ is non-singular. Thus $r = 0$. In particular, we see that $Q_{12} = 0$ and $Q_{22} \neq 0$ if $r = 0$.

For the assertion 4, we focus on (10). Then we obtain

$$
\begin{cases}
Aq_k + b\hat{q}_k = q_{k+1} & (k = 1, 2, \ldots, r), \\
c^T q_k = 0 & (k = 1, 2, \ldots, r), \\
q_1 = 0.
\end{cases}
$$

(17)

It follows from the third equation that we have $\hat{q}_1 \neq 0$. Otherwise the matrix $Q$ has the zero column and thus $Q$ is singular.

Next, eliminating $q_k$ at the left-hand side in the first equations of (17), we obtain

$$
\hat{q}_1 A^{k-1} b + \hat{q}_2 A^{k-2} b + \cdots + \hat{q}_k b = q_{k+1}
$$

(18)
for $k = 1, \ldots, r$. We can prove $c^T b = 0$, $c^T A b = 0, \ldots, c^T A^{r-2} b = 0$ by using the induction on $k$ and the second equations of (17).

Finally, we prove $c^T A^{r-1} b \neq 0$. From (18), we have $\hat{q}_1 c^T A^{r-1} b = c^T q_{r+1}$. If $c^T q_{r+1} = 0$, then we obtain the contradiction. In fact, $c^T Q_{11} = 0$ and $c^T Q_{12} = 0$. We then obtain $(c^T 0) Q = 0$. Since $c \neq 0$, this contradicts to the fact that $Q$ is non-singular.

**Remark 2.** We can summarize Theorem 2 as follows: There exist a non-singular matrix $Q \in \mathbb{C}^{(n+1) \times (n+1)}$, a nonnegative integer $r$, a Jordan matrix $\Lambda \in \mathbb{C}^{(n-r) \times (n-r)}$ and a nilpotent $N \in \mathbb{C}^{(r+1) \times (r+1)}$ such that

$$
\begin{pmatrix}
A & b \\
0 & A
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{pmatrix}
\begin{pmatrix}
I_{n-r} & 0 \\
O & N
\end{pmatrix}
= 
\begin{pmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{pmatrix}
\begin{pmatrix}
\Lambda & O \\
O & I_{r+1}
\end{pmatrix}.
$$

We can observe the following from (19).

1. If $\lambda$ is an eigenvalue of $\Lambda$ in (19) with the (algebraic) multiplicity $m$, then we call it the invariant zero of $G$ with the multiplicity $m$. We can see that (6) has $(n-r)$ invariant zeros including their multiplicity.

In analogy to the multiplicity, we can define the geometric multiplicity $m_g$ of the invariant zero. If both multiplicity do not coincide, then we cannot select $m$ linearly independent right null vectors associated with the invariant zero. We, however, can define the generalized right null vectors $(q_{m+1}, \hat{q}_{m+1}), \ldots, (q_m, \hat{q}_m)$ in a similar manner to the generalized eigenvectors. Then all the (generalized) right null vectors $(q_1, \hat{q}_1), \ldots, (q_m, \hat{q}_m)$ are linearly independent. For instance, if $m > 1$ and $m_g = 1$, then we have one right null vector and $(m - 1)$ generalized right null vectors associated with the invariant zero $\lambda$ as follows.

$$
\begin{pmatrix}
A & b \\
0 & A
\end{pmatrix}
\begin{pmatrix}
q_1 \\
\hat{q}_1
\end{pmatrix}
= \lambda \begin{pmatrix}
q_1 \\
0
\end{pmatrix},
\begin{pmatrix}
A & b \\
0 & A
\end{pmatrix}
\begin{pmatrix}
q_k \\
\hat{q}_k
\end{pmatrix}
= \lambda \begin{pmatrix}
q_k \\
0
\end{pmatrix} + \begin{pmatrix}
q_{k-1} \\
0
\end{pmatrix} (k = 2, \ldots, m).
$$

Throughout this paper, we refer to the generalized null vector as the right null vector associated with the invariant zero $\lambda$ for brevity. We can see that the submatrix $(Q_{11} Q_{21})$ obtained by collecting these $(n-r)$ right null vectors associated with all invariant zeros of $G$.

2. Assertions 3 and 4 in Theorem 2 imply that the nonnegative integer $r$ is equal to the relative degree of (6). In addition, we say that $G$ has an infinite invariant zero if $r > 0$.

3. We see from the proof of Theorem 2 that we can take $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the first column of the submatrix $(Q_{11} Q_{21})$. Thus $Q_{11}$ is of full column rank. Otherwise we obtain a contradiction to the fact that $Q$ is non-singular.

We discussed the right generalized null vectors associated with the invariant zeros of $G$ from (19). Similarly, we can also introduce the left generalized null
vectors associated with the invariant zeros of \( G \) from (10). For this, we consider the dual dynamical system of (6).

\[
G_d : \begin{cases}
\dot{x}_d &= A^T x_d + c u_d \\
y_d &= b^T x_d + d u_d,
\end{cases}
\]

(20)

Applying Theorem 2 to (20), we then obtain the following Wierestrass form.

\[
\begin{pmatrix}
A^T & c \\
 b^T & d \\
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{21} \\
 Q_{12} & Q_{22} \\
\end{pmatrix}
\begin{pmatrix}
I_{n-r} & O \\
 O & N \\
\end{pmatrix}
= \begin{pmatrix}
Q_{11} & Q_{12} \\
 O & O \\
\end{pmatrix}
\begin{pmatrix}
A & O \\
 O & I_{r+1} \\
\end{pmatrix}.
\]

By taking the transpose, then we obtain

\[
\begin{pmatrix}
I_{n-r} & O \\
 O & N^T \\
\end{pmatrix}
\begin{pmatrix}
Q_{11}^T & Q_{21}^T \\
 Q_{12}^T & Q_{22}^T \\
\end{pmatrix}
\begin{pmatrix}
A & b \\
 c & d \\
\end{pmatrix}
= \begin{pmatrix}
A^T & O \\
 O & I_{r+1} \\
\end{pmatrix}
\begin{pmatrix}
Q_{11}^T & O \\
 Q_{12}^T & O \\
\end{pmatrix}.
\]

(21)

In an analogous way to the generalized right null vector, we can define the generalized left null vector(s) from (21). We refer to those as left null vector(s) associated with an invariant zero as well as the case of the generalized right null vector(s) throughout this paper.

We have seen the flexibility in choosing \( \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} \) in the proof of the assertions 3 and 4 in Theorem 2. We give possible forms of the submatrix \( \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} \) of \( Q \) in (19) and (21). We give a proof in Appendix C.1.

**Lemma 2.** Let \( r \) be the relative degree of (6). Then we can set the submatrix \( \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} \) of \( Q \) in (19) as follows:

\[
\begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} = \begin{cases}
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } r = 0 \\
0 & Ab \\
AB & -A^{r-2}b \\
0 & A^{r-1}b \end{pmatrix} & \text{if } r > 0.
\end{cases}
\]

(22)

Similarly, we can set the submatrix \( \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} \) of \( Q \) in (21) as follows:

\[
\begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} = \begin{cases}
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } r = 0 \\
0 & c \\
AT & -c \\
0 & AT^{r-2}c \end{pmatrix} & \text{if } r > 0.
\end{cases}
\]

(22)

3. \( H_\infty \) output feedback control problem and its reformulation

3.1. Invariant zeros of \( G_{zu} \) and \( G_{yw} \)

Applying the Laplace transform to (1), we obtain

\[
\begin{pmatrix}
Z(s) \\
Y(s)
\end{pmatrix}
= G(s)
\begin{pmatrix}
W(s) \\
U(s)
\end{pmatrix}
= \begin{pmatrix}
G_{zw}(s) & G_{zu}(s) \\
G_{yw}(s) & G_{yu}(s)
\end{pmatrix}
\begin{pmatrix}
W(s) \\
U(s)
\end{pmatrix}.
\]

(23)
Here we assume that $x(0) = 0$, and each element in $G(s)$ can be described by $A, b, c$, and $d_{ij}$ as follows:

$$
G_{zw}(s) = c_1^T(sI_n - A)^{-1}b_1 + d_{11}, \quad G_{zu}(s) = c_1^T(sI_n - A)^{-1}b_2 + d_{12}, \quad G_{yw}(s) = c_2^T(sI_n - A)^{-1}b_1 + d_{21}, \quad G_{yu}(s) = c_2^T(sI_n - A)^{-1}b_2.
$$

(24)

By following the elimination-of-variables method [12, 16], the $H_\infty$ control problem (2) for the generalized plant $G$ given by (1) can be formulated as follows:

$$
\begin{align*}
\inf_{\gamma, X, Y} & \quad \gamma \\
\text{subject to} & \quad - \begin{pmatrix} b_2 \\ d_{12} \end{pmatrix}^T \begin{pmatrix} \text{He}(AX) & Xc_1 \\ b_1^T & d_{11} \end{pmatrix} \begin{pmatrix} b_2 \\ d_{12} \end{pmatrix} \in S^{n+1}_+, \\
& \quad - \begin{pmatrix} c_2 \\ d_{21} \end{pmatrix}^T \begin{pmatrix} \text{He}(YA) & Yb_1 \\ c_1^T & d_{11} \end{pmatrix} \begin{pmatrix} c_2 \\ d_{21} \end{pmatrix} \in S^{n+1}_+, \\
& \quad X - \begin{pmatrix} -I_n \\ Y \end{pmatrix} \in S^{2n}_+.
\end{align*}
$$

(25)

It should be noted that suboptimal $H_\infty$ controllers can be reconstructed by using suboptimal solutions of this SDP. The elimination-of-variables method is to solve the above SDP by primal-dual interior-point methods or the ellipsoid methods. In the following, we denote the infimal value of the SDP (25) by $\gamma^*$, which is consistent with the notation in (24).

Before getting into the specific treatments of (25), we give a fundamental result on its duality. We prove this result in Appendix B.1 on the basis of the discussion in Appendix A.

**Theorem 3.** The following statements are equivalent:

(I) $(A, b_2)$ in (1) is stabilizable and $(A, c_2^T)$ in (1) is detectable.

(II) LMI problem (25) is strictly feasible, i.e., there exists a solution $(\gamma, X, Y)$ such that

$$
\begin{align*}
- \begin{pmatrix} b_2 \\ d_{12} \end{pmatrix}^T \begin{pmatrix} \text{He}(AX) & Xc_1 \\ b_1^T & d_{11} \end{pmatrix} \begin{pmatrix} b_2 \\ d_{12} \end{pmatrix} & \in S^{n+1}_+, \\
- \begin{pmatrix} c_2 \\ d_{21} \end{pmatrix}^T \begin{pmatrix} \text{He}(YA) & Yb_1 \\ c_1^T & d_{11} \end{pmatrix} \begin{pmatrix} c_2 \\ d_{21} \end{pmatrix} & \in S^{n+1}_+ \\
\begin{pmatrix} X \\ -I_n \\ Y \end{pmatrix} & \in S^{2n}_+.
\end{align*}
$$

In particular, if (25) is strictly feasible, then it follows from the strong duality theorem (in Theorem 3) in Appendix A that the duality gap between (25) and its dual is zero, and that the dual has an optimal solution.
The key idea in working with the SDP (25) in this paper is the treatment of the perpendicular matrices
\[
\begin{pmatrix}
  b_2 \\
  d_{12} \\
  0
\end{pmatrix}
\perp
\begin{pmatrix}
  c_2 \\
  d_{21} \\
  0
\end{pmatrix}
\perp.
\]
These perpendicular matrices are usually constructed by directly working on the vectors \( (b_2^T d_{12} 0)^T \) and \( (c_2^T d_{21} 0)^T \). In stark contrast, in this paper, we use the left and right null vectors associated with the invariant zeros of the systems \( G_{zu} \) and \( G_{yw} \) given by (24).

Recall that \( \lambda \in \mathbb{C} \) is an invariant zero of \( G_{zu} \) given by (24) if \( \lambda \) is an eigenvalue of a Jordan matrix \( \Lambda \) in the following Wierestrass form for \( G_{za} \)
\[
\begin{pmatrix}
  I_{n-r_1} & O \\
  O & N_1^T
\end{pmatrix}
\begin{pmatrix}
  S^T & f \\
  \hat{S} & \hat{f}
\end{pmatrix}
\begin{pmatrix}
  A & b_2 \\
  O & I_{r_1+1}
\end{pmatrix}
= \begin{pmatrix}
  A^T & O \\
  O & I_{r_1+1}
\end{pmatrix}
\begin{pmatrix}
  S^T & O \\
  \hat{S} & O
\end{pmatrix}.
\] (26)

Here \( r \) is nonnegative integer, \( S \in \mathbb{C}^{n \times (n-r_1)} \), \( f \in \mathbb{C}^{n-r_1} \), \( \hat{S} \in \mathbb{C}^{n \times (r_1+1)} \), \( \hat{f} \in \mathbb{C}^{r_1+1} \) and \( N_1 \in \mathbb{C}^{(r_1+1) \times (r_1+1)} \) is a nilpotent. This is obtained from (21).

In particular, it follows from Theorem 2 and Remark 2 that \( r_1 \) is equal to the relative degree of the transfer function of \( G_{zu} \) and the matrix \( \begin{pmatrix} \hat{S} \\ \hat{f}^T \end{pmatrix} \) can be set as follows.
\[
\begin{pmatrix}
  \hat{S} \\
  \hat{f}^T
\end{pmatrix}
= \begin{cases}
  \begin{pmatrix}
    0 \\
    1
  \end{pmatrix} & (r_1 = 0) \\
  \begin{pmatrix}
    c \\
    1 \\
    0 \\
    0 \\
    \vdots \\
    0 \\
    0
  \end{pmatrix} & (r_1 \neq 0)
\end{cases}
\]

Similarly, recall that \( \omega \in \mathbb{C} \) is an invariant zero of \( G_{yw} \) given by (24) if \( \omega \) is an eigenvalue of a Jordan matrix \( \Omega \) in the following “transposed version” of Wierestrass form for \( G_{yw} \)
\[
\begin{pmatrix}
  I_{n-r_2} & O \\
  O & N_2^T
\end{pmatrix}
\begin{pmatrix}
  T^T & g \\
  \hat{T}^T & \hat{g}
\end{pmatrix}
\begin{pmatrix}
  A^T & c_1 \\
  O & I_{r_2+1}
\end{pmatrix}
= \begin{pmatrix}
  \Omega^T & O \\
  O & I_{r_2+1}
\end{pmatrix}
\begin{pmatrix}
  T^T & O \\
  \hat{T}^T & O
\end{pmatrix}.
\] (27)

This is obtained from (19). Symbols \( T, \hat{T}, g, \hat{g}, N_2 \) and \( \Omega \) in (27) can be defined in a similar manner to invariant zeros of \( G_{zu} \). In particular, \( \begin{pmatrix} \hat{T} \\ \hat{g}^T \end{pmatrix} \) can be set as follows.
\[
\begin{pmatrix}
  \hat{T} \\
  \hat{g}^T
\end{pmatrix}
= \begin{cases}
  \begin{pmatrix}
    0 \\
    1
  \end{pmatrix} & (r_2 = 0) \\
  \begin{pmatrix}
    b & Ab & \cdots & A^{r-2}b & A^{r-1}b \\
    1 & 0 & \cdots & 0 & 0
  \end{pmatrix} & (r_2 \neq 0)
\end{cases}
\]

We use (26) and (27) for concise descriptions of LMI problems. Throughout the paper, we make the following assumptions on the generalized plant \( G \) given by (1) and (24):
Assumption 1.  
(a) \((A, b_2)\) is stabilizable and \((A, c_2^T)\) is detectable. 
(b) All invariant zeros of \(G_{zu}\) are real. Similarly for \(G_{yw}\).

Assumption 1(a) is quite natural in controller design since otherwise, there are no internally stabilizing controllers. On the other hand, we proceed with the discussions in each section under Assumption 1(b), but this is just for simplicity. Assumption 1(b) implies that \(\begin{pmatrix} S \\ f^T \end{pmatrix}\) in (26) and \(\begin{pmatrix} T \\ g^T \end{pmatrix}\) in (27) are real and hence facilitates the descriptions of matrix inequality conditions. Under Assumption 1 and from Theorem 2 and Remark 2, we can readily obtain the next results with respect to the invariant zeros of \(G_{zu}\) and \(G_{yw}\).

Lemma 3.  
1. Let \(r_1 (0 \leq r_1 \leq n)\) be the relative degree of \(G_{zu}\). Then \(G_{zu}\) has \((n - r_1)\) real invariant zero(s). The left null vectors \(\begin{pmatrix} S \\ f^T \end{pmatrix}\) associated with all the invariant zeros of \(G_{zu}\) is of full column rank. In particular, \(S\) is also of full column rank. Furthermore, if \(d_{12} \neq 0\), then \(S\) is square and thus is non-singular.

2. Let \(r_2 (0 \leq r_2 \leq n)\) be the relative degree of \(G_{yw}\). Then \(G_{yw}\) has \((n - r_2)\) real invariant zero(s). The right null vectors \(\begin{pmatrix} T \\ g^T \end{pmatrix}\) associated with all the invariant zeros of \(G_{yw}\) is of full column rank. In particular, \(T\) is also of full column rank. Furthermore, if \(d_{21} \neq 0\), then \(T\) is square and thus is non-singular.

Finally, we partition \(\Lambda\) and \(\Omega\) into the parts of stable and unstable invariant zeros as follows:

\[
\Lambda = \begin{pmatrix} \Lambda_- & * \\ O_{k_1 \times (n - r_1 - k_1)} & \Lambda_+ \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} \Omega_- & * \\ O_{k_2 \times (n - r_2 - k_2)} & \Omega_+ \end{pmatrix}.
\]

(28)

Here, \(k_1\) (resp. \(k_2\)) denotes the number of unstable invariant zeros of \(G_{zu}\) (resp. \(G_{yw}\)), \(\Lambda_+\) (resp. \(\Omega_+\)) is a Jordan matrix with unstable invariant zeros of \(G_{zu}\) (resp. \(G_{yw}\)), and \(\Lambda_-\) (resp. \(\Omega_-\)) is a Jordan matrix with stable invariant zeros of \(G_{zu}\) (resp. \(G_{yw}\)). We also partition \(S, f\) and \(T, g\) conformably as follows

\[
\begin{pmatrix} S^T \\ f \end{pmatrix} \begin{pmatrix} A \\ c_1^T \end{pmatrix} = \Lambda_+^T \begin{pmatrix} S^T \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T^T \\ g \end{pmatrix} \begin{pmatrix} A^T \\ c_2^T \end{pmatrix} = \Omega_+^T \begin{pmatrix} T^T \\ 0 \end{pmatrix}
\]

(29)

where \(\circ = + \text{ or } -\). When we deal with invariant zeros on the imaginary axis explicitly, we use another partition of \(\Lambda\) and \(\Omega\), see Section 6.

3.2. Simplification of the LMI problem via invariant zeros and null vectors

We assume that \(d_{12} \neq 0\) and \(d_{21} \neq 0\) in this subsection. Since this is also assumed in Sections 4, 5 and 6 the simplification in this subsection is valid except for Section 7.
If $d_{12} \neq 0$ and $d_{21} \neq 0$, it follows from Lemma 3 that both $S$ and $T$ in (26) and (27) are non-singular. Using this property, we have
\[
\begin{pmatrix} b_2 \\ d_{12} \\ 0 \end{pmatrix} \perp \begin{pmatrix} S & 0 \\ f^T & 0 \\ 0^T & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_2 \\ d_{21} \\ 0 \end{pmatrix} \perp \begin{pmatrix} T & 0 \\ g^T & 0 \\ 0^T & 1 \end{pmatrix}.
\]

By using this fact, we have
\[
\begin{pmatrix} c_2 \\ d_{21} \\ 0 \end{pmatrix} \perp \begin{pmatrix} \text{He}(AY) & Yb_1 \\ c_1^T & d_{11} - \gamma \end{pmatrix} \begin{pmatrix} c_2 \\ d_{21} \\ 0 \end{pmatrix} \perp \begin{pmatrix} \text{He}(\Lambda^T ST X S) - \gamma f f^T \\ b_1^T S + d_{11} f f^T \end{pmatrix},
\]
\[
\begin{pmatrix} b_2 \\ d_{12} \\ 0 \end{pmatrix} \perp \begin{pmatrix} Xc_1 \\ -\gamma \\ d_{11} - \gamma \end{pmatrix} \begin{pmatrix} b_2 \\ d_{12} \\ 0 \end{pmatrix} \perp \begin{pmatrix} \text{He}(\Lambda^T ST X S) - \gamma f f^T \\ b_1^T S + d_{11} f f^T \end{pmatrix},
\]
where $\gamma$ indicates the transpose of the lower triangular part. In addition, since $S$ and $T$ are non-singular, we have
\[
\begin{pmatrix} X \\ -I_n \\ Y \end{pmatrix} \in S^{2n}_+ \iff \begin{pmatrix} ST XS & -STT \\ -T^T S & T^T Y T \end{pmatrix} \in S^{2n}._+
\]
Hence, by replacing $STX$ and $TTYT$ by $\hat{X}$ and $\hat{Y}$, respectively, the SDP (25) can be reduced to
\[
\inf \gamma \text{ subject to } -\begin{pmatrix} \text{He}(\Lambda^T \hat{X}) - \gamma f f^T \\ h_1^T \end{pmatrix} \in S^{n+1}_+ \quad \begin{pmatrix} \hat{X} \\ -J \end{pmatrix} \in S^{2n}_+ \quad \text{and} \quad -\begin{pmatrix} \text{He}(\Omega^T \hat{Y}) - \gamma gg^T \\ h_2^T \end{pmatrix} \in S^{n+1}_+ \quad \text{and} \quad \begin{pmatrix} \hat{Y} \end{pmatrix} \in S^{2n}_+,
\]
where $J := T^T S$, $h_1 := ST b_1 + d_{11} f$ and $h_2 := T^T c_1 + d_{11} g$.

In Sections 4, 5, 6 and 7 we analyze the infimal value $\gamma^*$ given by (30) separately according to the following four cases:

**Case 1** (Section 4) Both $d_{12}$ and $d_{21}$ are nonzero, and all the invariant zeros of $G_{zu}$ and $G_{yw}$ are unstable, but not on the imaginary axis.

**Case 2** (Section 5) Both $d_{12}$ and $d_{21}$ are nonzero, and at least one of the invariant zeros of $G_{zu}$ or $G_{yw}$ is stable, but all the unstable invariant zeros are not on the imaginary axis.
Case 3 (Section 6) Both \(d_{12}\) and \(d_{21}\) are nonzero, and at least one of the invariant zeros in \(G_{zu}\) or \(G_{yw}\) exists on the imaginary axis.

Case 4 (Section 7) At least one of \(d_{12}\) and \(d_{21}\) is zero. In this case, an infinite invariant zero exists in \(G_{zu}\) or \(G_{yw}\).

4. Analysis of Case 1

In this section, we assume that all invariant zeros of \(G_{zu}\) and \(G_{uw}\) given by (24) are (strictly) positive. This is represented equivalently by \(\Lambda = \Lambda_+\) and \(\Omega = \Omega_+\). Then we have \(S = S_+, f = f_+\) and \(T = T_+, g = g_+\), and thus \(J = J_+ := T_+^T S_+, h_1 = h_{1+} := S_+^{-1} b_1 + d_{11} f_+\) and \(h_2 = h_{2+} := T_+^T c_1 + d_{11} g_+\).

Under Assumption 1, Theorem 3 ensures the existence of an optimal solution to the dual of (30), while the following lemma ensures the existence of an optimal solution to (30). We give a proof of Lemma 4 in Appendix B.2.

**Lemma 4.** LMI problem (30) has an optimal solution.

In addition, we can obtain the next result with respect to LMI problem (30). We give a proof in Appendix C.2.

**Lemma 5.** The optimal value \(\gamma^*\) of LMI problem (30) is zero if and only if \(h_{1+} = 0\), \(h_{2+} = 0\) and \(J_+ = O_{n \times n}\).

Since we have already dealt with the case \(\gamma^* = 0\) explicitly in Lemma 5, we assume that the optimal value \(\gamma^*\) of (30) is (strictly) positive in the remainder of this section. Then we can apply the Schur complement to the first and second LMIs in (30) and obtain

\[
\begin{align*}
\text{He}((-\Lambda_+)^T X) + \gamma f_+ f_+^T - h_{1+} h_{1+}^T / \gamma \in S_+^n \quad &\text{and} \\
\text{He}((-\Omega_+)^T Y) + \gamma g_+ g_+^T - h_{2+} h_{2+}^T / \gamma \in S_+^n
\end{align*}
\]

for \(\gamma > 0\). It follows that LMI problem (30) can be reformulated as

\[
\begin{align*}
\inf_{\gamma, X, Y, \dot{X}, \dot{Y}} & \quad \gamma \\
\text{subject to} & \quad \text{He}((-\Lambda_+)^T X) + \gamma f_+ f_+^T - h_{1+} h_{1+}^T / \gamma - \dot{X} = O_n, \dot{X} \in S_+^n, \\
& \quad \text{He}((-\Omega_+)^T Y) + \gamma g_+ g_+^T - h_{2+} h_{2+}^T / \gamma - \dot{Y} = O_n, \dot{Y} \in S_+^n, \\
& \quad \begin{pmatrix} X & -J^T \\ -J & Y \end{pmatrix} \in S_+^{2n}, \quad \gamma > 0.
\end{align*}
\]

(31)

Since both \((-\Lambda_+)^T\) and \((-\Omega_+)^T\) are Hurwitz stable, and since the first and second equalities can be seen as the Lyapunov equations, we can solve them explicitly as follows:

\[
X = \int_0^\infty \exp(-\Lambda_+^T t) \left(\gamma f_+ f_+^T - h_{1+} h_{1+}^T / \gamma - \dot{X}\right) \exp(-\Lambda_+ t) dt, \quad (32)
\]

\[
Y = \int_0^\infty \exp(-\Omega_+^T t) \left(\gamma g_+ g_+^T - h_{2+} h_{2+}^T / \gamma - \dot{Y}\right) \exp(-\Omega_+ t) dt. \quad (33)
\]

16
We remark that $F$ is controllable under Assumption 1. This is proved in Lemma 17 of Appendix B.3.

The value $\gamma$ is positive semidefinite. On the other hand, if (36) is feasible with the objective value $\gamma$, then we can readily prove that (35) is equivalent to the next LMI problem:

$$\begin{align*}
F_+ &= \int_0^\infty \exp(-\Lambda^T_+ t) f_+ f^+_T \exp(-\Lambda_+ t) \, dt, \\
G_+ &= \int_0^\infty \exp(-\Omega^T_+ t) g_+ g^+_T \exp(-\Omega_+ t) \, dt, \\
H_{1+} &= \int_0^\infty \exp(-\Lambda^T_+ t) h_{1+} h^T_{1+} \exp(-\Lambda_+ t) \, dt, \\
H_{2+} &= \int_0^\infty \exp(-\Omega^T_+ t) h_{2+} h^T_{2+} \exp(-\Omega_+ t) \, dt.
\end{align*}$$

(34)

We remark that $F_+, G_+, H_{1+}, H_{2+}$ are positive semidefinite. In particular, $F_+$ and $G_+$ are positive definite because both pairs $(\Lambda_+, f_+)$ and $(\Omega_+, g_+)$ are controllable under Assumption 1. This is proved in Lemma 17 of Appendix B.3.

By using $F_+, G_+, H_{1+}, H_{2+} \in S^n$, we can rewrite (34) as

$$\begin{align*}
\inf_{\gamma, \hat{X}, \hat{Y}, \hat{X}, \hat{Y}} \gamma \\
\text{subject to} \quad &\hat{X} = \int_0^\infty \exp(-\Lambda^T_+ t) \hat{X} \exp(-\Lambda_+ t) \, dt, \hat{X} \in S^+_n, \\
&\hat{Y} = \int_0^\infty \exp(-\Omega^T_+ t) \hat{Y} \exp(-\Omega_+ t) \, dt, \hat{Y} \in S^+_n, \\
&\begin{pmatrix}
\gamma F_+ - \frac{1}{\gamma} H_{1+} - \hat{X} & -J^T_+ \\
-J_+ & \gamma G_+ - \frac{1}{\gamma} H_{2+} - \hat{Y}
\end{pmatrix} \in S^2_n, \gamma > 0.
\end{align*}$$

(35)

Then we can readily prove that (35) is equivalent to the next LMI problem:

$$\begin{align*}
\inf \left\{ \gamma : \begin{pmatrix}
\gamma F_+ - \frac{1}{\gamma} H_{1+} & -J^T_+ \\
-J_+ & \gamma G_+ - \frac{1}{\gamma} H_{2+}
\end{pmatrix} \in S^2_n, \gamma > 0 \right\}.
\end{align*}$$

(36)

In fact, it is clear that if (35) with the objective value $\gamma = \gamma_0$ is feasible by $(\gamma_0, \hat{X}, \hat{Y}, \hat{X}, \hat{Y})$, then (36) with $\gamma = \gamma_0$ is also feasible since $\hat{X}$ and $\hat{Y}$ are both positive semidefinite. On the other hand, if (36) is feasible with the objective value $\gamma = \gamma_0$, then (35) with $\gamma = \gamma_0$ is also feasible by $(\gamma_0, \hat{X}, \hat{Y}, \hat{X}, \hat{Y}) = (\gamma_0, O_n, O_n, O_n, O_n)$.

To summarize the results in this section, we arrive at the next theorem that is the first main result of this paper.

**Theorem 4.** Let us consider Case 1 stated at the final part of Section 3. Then the optimal value $\gamma^*$ of LMI problem (31) is equal to the maximum eigenvalue

17
of the matrix $E \in \mathbb{S}^{4n}$ defined by

$$E := \begin{pmatrix}
F_{+}^{-1/2} J_{+} T_{+} & F_{+}^{-1/2} G_{+}^{1/2} & F_{+}^{-1/2} H_{1+}^{1/2} & O \\
G_{+}^{-1/2} J_{+} T_{+} & O & O & G_{+}^{-1/2} H_{2+}^{1/2} \\
H_{1+} F_{+}^{-1/2} & O & O & O \\
H_{2+} G_{+}^{-1/2} & O & O & O
\end{pmatrix}. \quad (37)$$

Here, $F_{+}, G_{+} \in \mathbb{S}_{++}^{n}$ and $H_{1+}, H_{2+} \in \mathbb{S}_{++}^{n}$ are given by (34). Moreover, an optimal solution $(\gamma, X, Y)$ of the SDP (30) can be given explicitly by

$$\gamma = \gamma^*, \quad X = \gamma F_{+} - \frac{1}{\gamma} H_{1+}, \quad Y = \gamma G_{+} - \frac{1}{\gamma} H_{2+}.$$  

Proof. To prove $\gamma^* = \hat{\gamma} := \lambda_{\text{max}}(E)$, we use the fact that $F_{+}$ and $G_{+}$ are positive definite. By using the Schur complement, we have

$$\begin{pmatrix}
\gamma F_{+} - \frac{1}{\gamma} H_{1+} & -J_{+} T_{+} \\
-J_{+} & \gamma G_{+} - \frac{1}{\gamma} H_{2+}
\end{pmatrix} \in \mathbb{S}_{++}^{2n},$$

$$\iff \begin{pmatrix}
\gamma I_{n} - \frac{1}{\gamma} F_{+}^{-1/2} H_{1+} F_{+}^{-1/2} & * \\
* & \gamma I_{n} - \frac{1}{\gamma} G_{+}^{-1/2} H_{2+} G_{+}^{-1/2}
\end{pmatrix} \in \mathbb{S}_{++}^{2n},$$

$$\iff \begin{pmatrix}
\gamma I_{n} & * & * \\
* & \gamma I_{n} & * \\
* & * & \gamma I_{n}
\end{pmatrix} \in \mathbb{S}_{++}^{4n} \iff \gamma \geq \hat{\gamma} = \lambda_{\text{max}}(E).$$

We note that the maximum eigenvalue of $E$ is nonnegative since $E$ is indefinite. Hence the maximum eigenvalue of $E$ is nonnegative, and thus we conclude that the optimal value $\gamma^*$ of (30) is equal to the maximum eigenvalue of $E$. In addition, since (30) is equivalent to (34), an optimal solution of (30) is obtained from (32) and (33). Therefore we obtain the result. □

Before closing this section, we provide an explicit way to compute the matrix $E$ in Theorem 4. First, we compute all the invariant zeros $\lambda$ and $\omega$ of $G_{zu}$ and $G_{yw}$ and their null vectors \( \begin{pmatrix} z_s \\ f_s \end{pmatrix} \) and \( \begin{pmatrix} T_g \\ g_T \end{pmatrix} \) in (26) and (28), respectively. Collecting them, we define $\Lambda, \Omega, S, f, T$ and $g$ as in (26) and (27). We remark that we have $\Lambda_+ = \Lambda, \Omega_+ = \Omega, S_+ = S, f_+ = f, T_+ = T$ and $g_+ = g$ under this assumption. Second, we compute $J_+, h_{1+}$ and $h_{2+}$ by

$$J_+ = T_+^T S_+, \quad h_{1+} = S_+^T b_1 + d_{11} f_+, \quad h_{2+} = T_+^T c_1 + d_{11} g_+.$$  

Next, we solve the following Lyapunov equation to determine the the symmetric matrix $F_+$:

$$(-\Lambda_+^T) F_+ + F_+ (-\Lambda_+) = -f_+ f_+^T.$$  

18
Finally, we compute $F$ and $E$. Then we can obtain the matrix $W$ solution.

5. Analysis of Case 2

When dealing with Case 2 stated at the final part of Section 3, we cannot obtain Theorem 4 by a similar discussion to Section 4. The difficulty lies in the fact that we cannot represent the solutions of the Lyapunov equations with respect to $\Lambda$ and $\Omega$ since they contain negative eigenvalues in Case 2. However, we can overcome this difficulty by investigating the structure of feasible solutions of the dual of (30). We give a proof in Appendix C.3.

The following lemma provides the mathematical formulation of the dual of (30). We give a proof in Appendix C.3.

**Lemma 6.** The dual of (30) can be formulated as follows:

\[
\begin{align*}
\sup_{W_{ij}, Z_{ij}, V_{ij}} & \quad 2(h_1^T \cdot Z_{21} + h_2^T \cdot V_{21} + J \cdot W_{21}) \\
\text{subject to} & \quad f^T Z_{11} f + Z_{22} + g^T V_{11} g + V_{22} = 1, \\
& \quad W_{11} = H_{e}(AZ_{11}), W_{22} = H_{e}(\Omega V_{11}), \\
& \quad \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{21} & Z_{22} \end{pmatrix} \in S^{n+1}_{+}, \begin{pmatrix} V_{11} & V_{21} \\ V_{21} & V_{22} \end{pmatrix} \in S^{n+1}_{+}, \begin{pmatrix} W_{11} & W_{21} \\ W_{21} & W_{22} \end{pmatrix} \in S^{2n}_{+}.
\end{align*}
\]

Moreover, the duality gap between (30) and (38) is zero, and (38) has an optimal solution.

The following lemma provides the structure of solutions of (38). For this, we partition $Z_{11}, V_{11}$ and $W$ of a feasible solution $(Z, V, W)$ of (38) as follows:

\[
\begin{align*}
Z_{11} & = \begin{pmatrix} n - k_1 \\ k_1 \end{pmatrix} \begin{pmatrix} Z_{11}^1 & \left( Z_{11}^2 \right)^T \\ Z_{11}^2 & Z_{11}^3 \end{pmatrix}, \quad V_{11} = \begin{pmatrix} n - k_2 \\ k_2 \end{pmatrix} \begin{pmatrix} V_{11}^1 & \left( V_{11}^2 \right)^T \\ V_{11}^2 & V_{11}^3 \end{pmatrix}, \\
W_{11} & = \begin{pmatrix} n - k_1 \\ k_1 \end{pmatrix} \begin{pmatrix} W_{11}^1 & \left( W_{11}^2 \right)^T \\ W_{11}^2 & W_{11}^3 \end{pmatrix} \text{ and } W_{22} = \begin{pmatrix} n - k_2 \\ k_2 \end{pmatrix} \begin{pmatrix} W_{22}^1 & \left( W_{22}^2 \right)^T \\ W_{22}^2 & W_{22}^3 \end{pmatrix}.
\end{align*}
\]
Lemma 7. Any feasible solution \((Z, V, W)\) of (38) has the form of

\[
W_{11} = \begin{pmatrix} O_{(n-k_1) \times (n-k_1)} & O_{k_1 \times (n-k_1)} \\ O_{(n-k_1) \times k_1} & W_{11}^3 \end{pmatrix}, \quad Z_{11} = \begin{pmatrix} O_{(n-k_1) \times (n-k_1)} & O_{k_1 \times (n-k_1)} \\ O_{(n-k_1) \times k_1} & Z_{11}^3 \end{pmatrix},
\]

\[
W_{22} = \begin{pmatrix} O_{(n-k_2) \times (n-k_2)} & O_{k_2 \times (n-k_2)} \\ O_{(n-k_2) \times k_2} & W_{22}^3 \end{pmatrix}, \quad V_{11} = \begin{pmatrix} O_{(n-k_2) \times (n-k_2)} & O_{k_2 \times (n-k_2)} \\ O_{(n-k_2) \times k_2} & V_{11}^3 \end{pmatrix},
\]

\[
Z_{21} = (O_{1 \times (n-k_1)} Z_{21}^3), \quad W_{11}^3 = \text{He}(\Lambda_+ Z_{11}^3),
\]

\[
V_{21} = (O_{1 \times (n-k_2)} V_{21}^3) \quad \text{and} \quad W_{22}^3 = \text{He}(\Omega_+ V_{11}^3).
\]

Furthermore, it follows from the structure of \(W_{11}\) and \(W_{22}\) that we have

\[
W_{21} = \begin{pmatrix} O_{(n-k_2) \times (n-k_1)} & O_{(n-k_2) \times k_1} \\ O_{k_2 \times (n-k_1)} & W_{21}^3 \end{pmatrix}.
\]

Proof. We prove only the structure of \(Z_{11}\) and \(W_{11}\). We focus on \(W_{11} = \text{He}(\Lambda Z_{11})\), \(W_{11} \in S_+^n\) and \(Z_{11} \in S_+^n\). Then \(W_{11} = \text{He}(\Lambda Z_{11})\) is equivalently written as

\[
\begin{pmatrix} W_{11}^1 & (W_{11}^2)^T \\ W_{11}^2 & W_{11}^3 \end{pmatrix} = \text{He}\left( \begin{pmatrix} \Lambda_+ & \Lambda_- \\ \Lambda_- & \Lambda_+ \end{pmatrix} \begin{pmatrix} Z_{11}^1 & * \\ * & Z_{11}^3 \end{pmatrix} \right)
\]

\[
= \begin{pmatrix} \text{He}(\Lambda_- Z_{11}^1) & * \\ \Lambda_+ Z_{11}^2 + Z_{11}^3 \Lambda_+^T & \text{He}(\Lambda_+ Z_{11}^3) \end{pmatrix}.
\]

We remark that \(\Lambda_-\) is Hurwitz stable because all the eigenvalues of \(\Lambda_-\) are negative. Since \(W_{11}^1 = \text{He}(\Lambda_- Z_{11}^1)\) can be seen as the Lyapunov equation, and since \(W_{11}^1 \in S_+^{n-k_1}\) and \(\Lambda_-\) is Hurwitz stable, we have

\[
Z_{11}^1 = - \int_0^\infty \exp(\Lambda_- t) W_{11}^1 \exp(\Lambda_-^T t) dt.
\]

It follows from the positive semidefiniteness of \(Z_{11}^1\) and \(W_{11}^1\) that \(Z_{11}^1\) must be the zero matrix. Substituting this into the Lyapunov equation, we obtain \(W_{11}^1 = O\). Consequently, \(Z_{11}^2\) and \(W_{11}^3\) are also the \(k_1 \times (n-k_1)\) zero matrix because \(Z_{11} \in S_+^n\) and \(W_{11} \in S_+^n\), respectively. \(\square\)

Substituting the structure of dual solutions \((Z, V, W)\) to the first equality constraint in (38), we obtain

\[
f^T Z_{11} f + Z_{22} + g^T V_{11} g + V_{22}
\]

\[
= \begin{pmatrix} f_- & f_+ \end{pmatrix}^T \begin{pmatrix} O_{(n-k_1) \times (n-k_1)} & O_{k_1 \times (n-k_1)} \\ O_{k_1 \times (n-k_1)} & Z_{11}^3 \end{pmatrix} \begin{pmatrix} f_- \\ f_+ \end{pmatrix} + Z_{22}
\]

\[
+ \begin{pmatrix} g_- & g_+ \end{pmatrix}^T \begin{pmatrix} O_{(n-k_2) \times (n-k_2)} & O_{k_2 \times (n-k_2)} \\ O_{k_2 \times (n-k_2)} & V_{11}^3 \end{pmatrix} \begin{pmatrix} g_- \\ g_+ \end{pmatrix} + V_{22}
\]

\[
f_+^T Z_{11}^1 f_+ + Z_{22} + g_+^T V_{11}^3 g_+ + V_{22} = 1.
\]
Moreover, we have

\[
J \cdot W_{21} = \text{Trace}(T^T S W_{21}^T) = \text{Trace} \left( T^T S \begin{pmatrix} O_{(n-k_1) \times (n-k_2)} & O_{(n-k_1) \times k_2} \\ O_{k_1 \times (n-k_2)} & (W_{21}^3)^T \end{pmatrix} \right) \\
= \text{Trace}(T_+ S_+^T (W_{21}^3)^T) = J_+ \cdot W_{21}^3,
\]

\[
h_1^T \cdot Z_{21} = (h_{1-}^T - h_{1+}^T) \cdot (O_{1 \times (n-k_1)} Z_{21}^2) = h_{1+}^T \cdot Z_{21}^2, \quad \text{and}
\]

\[
h_2^T \cdot V_{21} = (h_{2-}^T - h_{2+}^T) \cdot (O_{1 \times (n-k_2)} V_{21}^2) = h_{2+}^T \cdot V_{21}^2.
\]

Therefore \((38)\) is equivalent to the following optimization problem:

\[
\left\{ \begin{array}{l}
\sup_{\hat{W}_{ij}, \hat{Z}_{ij}, \hat{V}_{ij}} \\
\text{subject to} \\
\quad f_+^T \hat{Z}_{11} + f_+ + Z_{22} + g_+^T \hat{V}_{11} + V_{22} = 1, \\
\quad \hat{W}_{11} = \text{He}(\Lambda_+ \hat{Z}_{11}), \quad \hat{W}_{22} = \text{He}(\Omega_+ \hat{V}_{11}), \\
\quad (\hat{W}_{11} \quad \hat{W}_{21}) \in S_+^{k_1 + k_2}, \\
\quad (\hat{Z}_{11} \quad \hat{Z}_{21}) \in S_+^{k_1 + 1}, \quad (\hat{V}_{11} \quad \hat{V}_{21}) \in S_+^{k_2 + 1}.
\end{array} \right. 
\]

(39)

**Remark 3.** We have successfully reduced the dual \((38)\) when at least either \(G_{zw}\) or \(G_{yw}\) has stable invariant zeros. This reduction corresponds to **facial reduction** in the literature of the optimization theory, which was proposed by Borwein and Wolkowicz in \(3\) for general convex cone programming problems. Thereafter, \(19\) and \(20\) proposed facial reduction for SDP problems.

Facial reduction for SDP problems is a finitely iterative algorithm. It works for non-strictly feasible SDP problems like dual \((38)\), and generates a strictly feasible SDP problem whose optimal value is equal to the original.

The number of minimal iterations of facial reduction is an important concept in convex analysis and is called the **degree of singularity of the SDP problem**. The degree is used for the error bound analysis of SDP feasibility problems in \(23\) and perturbation analysis of SDP in \(6\). In the proof of Lemma \(7\) we can see that the facial reduction spends only one iteration in Case \(2\). It is proved in \(20\) that the same fact holds for \(H_\infty\) output feedback control problem for MIMO dynamical system.

The next lemma provides the dual of \((39)\).

**Lemma 8.** The dual of \((39)\) can be reformulated as follows:

\[
\left\{ \begin{array}{l}
\inf_{\gamma, \hat{X}, \hat{Y}} \\
\text{subject to} \\
\quad \left( \text{He}(\Lambda_+^T \hat{X}) - \gamma f_+ f_+^T - h_{1+}^T - \gamma \right) \in S_+^{k_1 + 1}, \\
\quad \left( \text{He}(\Omega_+^T \hat{Y}) - \gamma g_+ g_+^T - h_{2+}^T - \gamma \right) \in S_+^{k_2 + 1}, \\
\quad \left( \hat{X} \quad -J_+ \right) \in S_+^{k_1 + k_2}.
\end{array} \right. 
\]

(40)
Moreover the duality gap between (40) and (39) is zero, and both (40) and (39) have optimal solutions.

Proof. We can prove by a similar manner in Lemma 6 the fact that the dual of (40) is (39). In fact, it is well-known that the dual of the dual problem is exactly the primal problem. The proof on the zero duality gap between (40) and (39) is provided in Appendix B.3. In addition, we can prove the existence of optimal solutions of (40) and (39) by similar manners to the proofs in Appendix B.2 and Appendix B.3. 

Since all eigenvalues of both Λ+ and Ω+ are positive, we obtain the same result as Theorem 4 by applying the discussion in Section 4. We summarize the result in this subsection as follows:

**Theorem 5.** Let us consider Case 2 stated at the final part of Section 3. Then the optimal value γ∗ of LMI problem (30) is equal to the maximum eigenvalue of the symmetric matrix E defined by

\[
E := \begin{pmatrix}
O & F_+^{-1/2}f_+^T G_+^{1/2} & F_+^{-1/2}H_1^1 & O \\
G_+^{1/2}J_+ F_+^{-1/2} & O & O & G_+^{1/2}H_2^1 \\
H_1^1 F_+^{-1/2} & O & O & O \\
O & H_2^1 G_+^{-1/2} & O & O
\end{pmatrix}.
\]

Here \(F_+, H_1^+ \in \mathbb{S}^{k_1}_+\) and \(G_+, H_2^+ \in \mathbb{S}^{k_2}_+\) are defined by

\[
\begin{align*}
F_+ &= \int_0^\infty \exp(-\Lambda_+^T t) f_+ f_+^T \exp(-\Lambda_+ t) \, dt, \\
G_+ &= \int_0^\infty \exp(-\Omega_+^T t) g_+ g_+^T \exp(-\Omega_+ t) \, dt, \\
H_1^+ &= \int_0^\infty \exp(-\Lambda_+^T t) h_1 h_1^T \exp(-\Lambda_+ t) \, dt, \\
H_2^+ &= \int_0^\infty \exp(-\Omega_+^T t) h_2 h_2^T \exp(-\Omega_+ t) \, dt.
\end{align*}
\]

In particular, \(F_+\) and \(G_+\) are positive definite because of Assumption 7.

Proof. All the optimal values of (30), (38), (39) and (40) are equivalent. In fact, the equivalence between (30) and (38) follows from Theorem 3. The optimal values of (38) is equal to the optimal value of (39) because we obtain (39) from (38) by investigating the structure of solutions of (38). The equivalence between the optimal values of (39) and (40) follows from Lemma 8. Finally, we can prove that the optimal value of (40) is given by \(\lambda_{\text{max}}(E)\). In fact, we can prove that if \(\gamma^* = 0\), then \(h_1 = 0\), \(h_2 = 0\) and \(J_+ = O\), and thus \(E\) is the zero matrix. Clearly, \(\gamma^* = \lambda_{\text{max}}(E)\). Otherwise, since \(\gamma^* > 0\), we can prove \(\gamma^* = \lambda_{\text{max}}(E)\) in a similar manner to the proof of Theorem 4.

We remark that the size of the matrix \(E\) in (37) is \(4n\), while in (41) the size is \(2(k_1 + k_2)\). When both of \(G_{zu}\) and \(G_{yw}\) have no stable invariant zeros, then (41) is equal to (37).

We obtain the following corollary from Theorem 5.
Corollary 1. If all the invariant zeros of $G_{zu}$ are stable, then the optimal value $\gamma^*$ of (30) is equal to
\[
\lambda_{\text{max}} \begin{pmatrix} O & G_{+}^{-1/2} \frac{H_{1+}^{1/2}}{2} \\ H_{2+}^{1/2} \frac{G_{+}^{-1/2}}{2} & O \end{pmatrix}.
\]

Similarly, if all the invariant zeros in $G_{yw}$ are stable, then the optimal value $\gamma^*$ of (30) is equal to
\[
\lambda_{\text{max}} \begin{pmatrix} O & F_{+}^{-1/2} \frac{H_{1+}^{1/2}}{2} \\ H_{1+}^{1/2} \frac{F_{+}^{-1/2}}{2} & O \end{pmatrix}.
\]

Finally, if both of $G_{zu}$ and $G_{yw}$ have no unstable zeros, then $\gamma^*$ is equal to zero.

Proof. We prove (42) only. Intuitively, (42) directly follows from Theorem 5 because $F_{+}$ and $H_{1+}$ both vanish in the present case. The proof can be made more rigorous as follows. Since all the invariant zeros in $G_{zu}$ are stable, we have $\Lambda = \Lambda_-$, $S = S_-$, and $f = f_-$. It follows from Lemma 7 that any feasible solution $(Z, V, W)$ of (38) has the form of

\[
Z = \begin{pmatrix} O_n & 0 \\ 0^T & Z_{22} \end{pmatrix}, \quad V = \begin{pmatrix} O_{n-k_2} & O_{(n-k_2) \times k_2} \\ 0 & V_{11} \end{pmatrix} \begin{pmatrix} 0 \\ V_{21} \end{pmatrix}, \quad W = \begin{pmatrix} O_{(n-k_2) \times n} & O_{n \times (n-k_2)} \\ 0 & O_{k_2 \times (n-k_2)} \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}.
\]

Substituting them into (38), it can be reformulated as

\[
\inf \left\{ \gamma : \gamma \geq 0, \quad \begin{pmatrix} \text{He}(\Omega^T \bar{Y}^T) - \gamma g + \gamma T \\ h_{2+}^T \end{pmatrix} \in S_{+}^{k_2+1}, Z_{22} \geq 0, \hat{W}_{22} \in S_{+}^{k_2} \right\}.
\]

By following a similar line to Lemma 8 we obtain the following dual problem:

\[
\lambda_{\text{max}} \begin{pmatrix} O & G_{+}^{-1/2} \frac{H_{1+}^{1/2}}{2} \\ H_{2+}^{1/2} \frac{G_{+}^{-1/2}}{2} & O \end{pmatrix}.
\]

We can prove the duality gap between (45) and (44) is zero. For (45), it follows from a similar manner in the proof of Theorem 5 that

\[
\gamma^* = \hat{\gamma} = \lambda_{\text{max}} \begin{pmatrix} O & G_{+}^{-1/2} \frac{H_{1+}^{1/2}}{2} \\ H_{2+}^{1/2} \frac{G_{+}^{-1/2}}{2} & O \end{pmatrix}.
\]
6. Analysis of Case 3

In this section, we deal with Case 3, stated in the final part of Section 3. For simplicity, we assume the following.

- We allow both $G_{zu}$ and $G_{yw}$ to have complex invariant zeros on the imaginary axis.

- All the invariant zeros on the imaginary axis are distinct from each other.

- Both $G_{zu}$ or $G_{yw}$ do not have 0 as invariant zero.

In particular, the first assumption corresponds to remove (b) of Assumption 1. Otherwise, we deal with only 0 as invariant zeros on the imaginary axis. Thus this assumption makes the discussions in Case 3 more general. Other assumptions are imposed to improve the readability. However, we emphasize that the result in Theorem 6 is still valid without assuming them.

Under (a) of Assumption 1 and these additional assumptions, we consider the case where $G_{zu}$ (resp. $G_{yw}$) has $2m_1$ (resp. $2m_2$) invariant zeros $\lambda_1, \ldots, \lambda_{m_1}$ (resp. $\omega_1, \ldots, \omega_{m_2}$) and their complex conjugates $\bar{\lambda}_1, \ldots, \bar{\lambda}_{m_1}$ (resp. $\bar{\omega}_1, \ldots, \bar{\omega}_{m_2}$) on the imaginary axis. Moreover, these invariant zeros are distinct from each other. The null vectors $(s_j^T, f_j)$ $(j = 1, \ldots, m_1)$ and $(t_j^T, g_j)$ $(j = 1, \ldots, m_2)$ associated with the invariant zeros $\lambda_j$ $(j = 1, \ldots, m_1)$ and $\omega_j$ $(j = 1, \ldots, m_2)$ can be written, respectively, by

\[
\begin{pmatrix}
  s_j \\
  f_j
\end{pmatrix} = \begin{pmatrix}
  s_j^T \\
  f_j^T
\end{pmatrix} + \sqrt{-1} \begin{pmatrix}
  s_j^T \\
  f_j^T
\end{pmatrix} (j = 1, \ldots, m_1),
\]

\[
\begin{pmatrix}
  t_j \\
  g_j
\end{pmatrix} = \begin{pmatrix}
  t_j^T \\
  g_j^T
\end{pmatrix} + \sqrt{-1} \begin{pmatrix}
  t_j^T \\
  g_j^T
\end{pmatrix} (j = 1, \ldots, m_2).
\]

Here $s_j^T$, $s_j$, $t_j^T$, and $t_j$ are in $\mathbb{R}^n$ and $f_j^T$, $f_j$, $g_j^T$, and $g_j$ are in $\mathbb{R}$. Note that $|f_j|^2 = (f_j^T)^2 + (f_j)^2$ and $|g_j|^2 = (g_j^T)^2 + (g_j)^2$ are nonzero due to Assumption 1 and 2 of Lemma 1. Then we have

\[
\begin{pmatrix}
  (s_1^T)^T \\
  (s_1)^T \\
  \vdots \\
  (s_{m_1}^T)^T \\
  (s_{m_1})^T
\end{pmatrix}
\begin{pmatrix}
  f_1^T \\
  f_1 \\
  \vdots \\
  f_{m_1}^T \\
  f_{m_1}
\end{pmatrix}
= \begin{pmatrix}
  A \\
  b_2
\end{pmatrix}
\begin{pmatrix}
  b_2^T \\
  d_{12}
\end{pmatrix}
= \begin{pmatrix}
  F(\lambda_1) \\
  \vdots \\
  F(\lambda_{m_1})
\end{pmatrix},
\]

\[
(46)
\]

\[
\begin{pmatrix}
  (t_1^T)^T \\
  (t_1)^T \\
  \vdots \\
  (t_{m_2}^T)^T \\
  (t_{m_2})^T
\end{pmatrix}
\begin{pmatrix}
  g_1^T \\
  g_1 \\
  \vdots \\
  g_{m_2}^T \\
  g_{m_2}
\end{pmatrix}
= \begin{pmatrix}
  A^T \\
  b_2^T
\end{pmatrix}
\begin{pmatrix}
  c_2 \\
  d_{21}
\end{pmatrix}
= \begin{pmatrix}
  F(\omega_1) \\
  \vdots \\
  F(\omega_{m_2})
\end{pmatrix},
\]

\[
(47)
\]

where $F(\lambda)$ is defined by

\[
F(\lambda) = \begin{pmatrix}
  0 & \Im(\lambda) \\
  -\Im(\lambda) & 0
\end{pmatrix}.
\]
Here $\Im(\lambda)$ denotes the imaginary part of $\lambda \in \mathbb{C}$. Note that $F(\lambda)$ is non-singular when $\lambda \neq 0$. For simplicity, we denote (46) and (47) by
\[
(S_0^T \ f_0) \begin{pmatrix} A & b_2 \\ c_1^T & d_{12} \end{pmatrix} = \Lambda_0^T (S_0^T \ 0) \quad \text{and} \quad (T_0^T \ g_0) \begin{pmatrix} A^T & c_2 \\ b_1^T & d_{21} \end{pmatrix} = \Omega_0^T (T_0^T \ 0).
\] (48)

We remark that the sizes of $\Lambda_0$ and $\Omega_0$ are $2m_1$ and $2m_2$, respectively. In addition, we can reformulate (26) and (27) as
\[
\begin{pmatrix} S_0^T \\ S_0^T \ f_0 \\ f_0^T \\ f_0^T \ 0^T \end{pmatrix} = \begin{pmatrix} A^T \\ A^T \ c_1^T \\ b_2^T \\ c_1^T \ d_{12} \end{pmatrix} = \begin{pmatrix} \Lambda_0^T \\ \Lambda_0^T \ A^T \\ \Omega_0^T \\ \Omega_0^T \ T^T \end{pmatrix} \begin{pmatrix} S_0^T \\ S_0^T \ 0 \\ 0 \end{pmatrix},
\]
\[
\begin{pmatrix} T_0^T \\ T_0^T \ g_0 \\ g_0^T \\ g_0^T \ 0^T \end{pmatrix} = \begin{pmatrix} A^T \\ A^T \ b_1^T \\ d_{21} \end{pmatrix} = \begin{pmatrix} \Omega_0^T \\ \Omega_0^T \ T^T \end{pmatrix} \begin{pmatrix} T_0^T \\ T_0^T \ 0 \end{pmatrix}.
\]

As we have assumed that $d_{12} \neq 0$ and $d_{21} \neq 0$, it follows from Lemma 3 that $(S_0, S), (T_0, T) \in \mathbb{R}^{n \times n}$ are non-singular, and we have
\[
\begin{pmatrix} b_2 \\ d_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} S_0 \\ f_0^T \\ 0^T \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_2 \\ d_{21} \\ 0 \end{pmatrix} = \begin{pmatrix} T_0 \\ g_0^T \\ 0^T \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

By following similar lines leading to (30), we can reformulate (26) as in
\[
\begin{cases}
\inf_{\gamma, \tilde{X}, \tilde{Y}, \tilde{h}_1, \tilde{h}_2} \gamma \\
\text{subject to} \quad - \begin{pmatrix} \text{He} \left( \begin{pmatrix} \Lambda_0^T \\ \Lambda_0^T \ A^T \end{pmatrix} \tilde{X} \right) - \gamma \begin{pmatrix} f_0 \\ f_0 \end{pmatrix} \begin{pmatrix} f_0^T \\ f_0 \end{pmatrix} \right) \tilde{h}_1 \end{pmatrix} \in \mathbb{S}^{n+1}_+, \\
- \begin{pmatrix} \text{He} \left( \begin{pmatrix} \Omega_0^T \\ \Omega_0^T \ T^T \end{pmatrix} \tilde{Y} \right) - \gamma \begin{pmatrix} g_0 \\ g_0 \end{pmatrix} \begin{pmatrix} g_0^T \\ g_0 \end{pmatrix} \right) \tilde{h}_2 \end{pmatrix} \in \mathbb{S}^{n+1}_+, \\
\begin{pmatrix} \tilde{X} & -\tilde{J}^T \\ -\tilde{J} & \tilde{Y} \end{pmatrix} \in \mathbb{S}^{2n}_{++},
\end{cases}
\]

where we define $\tilde{X}, \tilde{Y}, \tilde{h}_1$ and $\tilde{h}_2$ as follows:
\[
\begin{align*}
\tilde{X} &= \begin{pmatrix} S_0^T \\ S_0 \end{pmatrix} \begin{pmatrix} X & S_0 \end{pmatrix}, \\
\tilde{Y} &= \begin{pmatrix} T_0^T \\ T_0 \end{pmatrix} \begin{pmatrix} Y & T_0 \end{pmatrix}, \\
\tilde{h}_1 &= \begin{pmatrix} h_{10}^T \\ h_{11}^T \end{pmatrix}, \\
\tilde{h}_2 &= \begin{pmatrix} h_{20}^T \\ h_{21}^T \end{pmatrix}, \\
h_{10}^T &= \begin{pmatrix} (h_{10})_1 \\ (h_{10})_2 \\ \vdots \\ (h_{10})_{m_1} \end{pmatrix}, \\
h_{11}^T &= \begin{pmatrix} (h_{11})_1 \\ (h_{11})_2 \\ \vdots \\ (h_{11})_{m_1} \end{pmatrix} = b_{1}^T S_0 + d_{11} f^T, \\
h_{20}^T &= \begin{pmatrix} (h_{20})_1 \\ (h_{20})_2 \\ \vdots \\ (h_{20})_{m_2} \end{pmatrix} = b_{1}^T T_0 + d_{11} g^T.
\end{align*}
\]
Applying Lemma 6, its dual is formulated as follows:

\[
\begin{align*}
\text{sup} & \quad \begin{pmatrix} \hat{h}_1^T \\ f_0 f_0^T & f f^T & f f^T & f f^T \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} \hat{h}_2^T \\ h_3 \end{pmatrix} \cdot \begin{pmatrix} V \end{pmatrix} + \begin{pmatrix} j^T \end{pmatrix} \cdot \begin{pmatrix} W \end{pmatrix} \\
\text{subject to} & \quad W_{11} = \text{He} (\begin{pmatrix} A_0 & \Lambda \\ \Omega_0 & \Omega \end{pmatrix} \begin{pmatrix} Z_{11} & * \\ Z_{21} & Z_{22} \end{pmatrix}), \\
& \quad W_{22} = \text{He} (\begin{pmatrix} V_{11} & * \\ V_{21} & V_{22} \end{pmatrix}), \\
& \quad Z = \begin{pmatrix} Z_{11} & * & * \\ Z_{21} & Z_{22} & * \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \in \mathbb{S}_+^{n+1}, V = \begin{pmatrix} V_{11} & * & * \\ V_{21} & V_{22} & * \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \in \mathbb{S}_+^{n+1}.
\end{align*}
\]

This result follows from Theorem 3 because the proof on the zero duality gap between (25) and its dual in Theorem 3 is independent of the computation of the perpendicular matrices in (25). We remark that the duality gap between (49) and (50) is zero, and (50) has an optimal solution. In fact, these facts follows from Theorem 3 because the proof on the zero duality gap between (25) and its dual in Theorem 3 is independent of the computation of the perpendicular matrices in (25).

The following lemma is useful to reduce (50). We give a proof in Appendix C.4.

**Lemma 9.** Any feasible solution \((W_{ij}, Z_{ij}, V_{ij})\) has the form of

\[
\begin{align*}
Z_{11} &= \text{Diag}(z_1, z_2, \ldots, z_m), Z_{21} = O_{(n-2m_1) \times 2m_1}, \\
V_{11} &= \text{Diag}(v_1, v_2, \ldots, v_{m_2}, v_{m_2}), V_{21} = O_{(n-2m_2) \times 2m_1}, \\
(W_{11})_{ij} &= 0 (i, j = 1, \ldots, 2m_1), (W_{22})_{ij} = 0 (i, j = 1, \ldots, 2m_2), \\
(W_{21})_{ij} &= 0 (i = 1, \ldots, n, j = 1, \ldots, 2m_1 \text{ and } i = 1, \ldots, 2m_2, j = 2m_1 + 1, \ldots, n),
\end{align*}
\]

where \(\text{Diag}(a_1, \ldots, a_n)\) stands for the diagonal matrix with the diagonal elements \(a_1, \ldots, a_n\).

**Remark 4.** We have reduced the size of the matrix variable \(W\) in (50) in Lemma 9. This reduction also corresponds to the facial reduction for SDP as well as Lemma 7. In addition, we can also apply Lemma 7 when at least either \(G_{zu}\) or \(G_{uv}\) has stable invariant zeros.
By using (51), (52) and (53), we can reformulate (50) as follows:

\[
\begin{align*}
\sup & \quad \begin{pmatrix} h_{10} \\ h_{11}^T \\ h_{12}^T \\ f_0 f_0^T \\ f_0 f_0^T \\ f_0 f_0^T \\ f_0 f_0^T \\ 1 \\ 1 \end{pmatrix} \cdot Z + \begin{pmatrix} h_{20} \\ h_{21}^T \\ h_{22}^T \\ g_0 g_0^T \\ g_0 g_0^T \\ g_0 g_0^T \\ g_0 g_0^T \\ 1 \end{pmatrix} \cdot V + \left( J^T \right) \cdot \hat{W}
\end{align*}
\]

subject to

\[
\begin{align*}
W_{11} &= \text{He} (\Lambda Z_{22}), W_{22} = \text{He} (\Omega V_{22}), \hat{W} = \begin{pmatrix} \hat{W}_{11} \\ \hat{W}_{21} \\ \hat{W}_{22} \end{pmatrix} \in S_{+}^{n_0}, \\
Z &= \begin{pmatrix} Z_{11} \\ O \\ Z_{31}^T \\ Z_{32} \\ Z_{33} \end{pmatrix} \in S_{+}^{n+1}, V = \begin{pmatrix} V_{11} \\ O \\ V_{31}^T \\ V_{32} \\ V_{33} \end{pmatrix} \in S_{+}^{n+1}, \\
Z_{11} &= \text{Diag}(z_1, z_1, \ldots, z_{m_1}, z_{m_1}), z_j \in \mathbb{R} (j = 1, \ldots, m_1), \\
V_{11} &= \text{Diag}(v_1, v_1, \ldots, v_{m_2}, v_{m_2}), v_j \in \mathbb{R} (j = 1, \ldots, m_2),
\end{align*}
\]

where \( n_0 = 2n - 2(m_1 + m_2) \). Let \( \mathcal{F} \) be the feasible region of (54). The next lemma shows the dual of (54). We give a proof in Appendix C.5.

**Lemma 10.** The dual of (54) can be formulated as follows:

\[
\begin{align*}
\inf & \quad \gamma \\
\text{subject to} & \quad - \begin{pmatrix} U_{11}^X - \gamma f_0 f_0^T \\ U_{21}^X - \gamma f_0 f_0^T \\ U_{21}^Y - \gamma g_0 g_0^T \\ U_{21}^Y - \gamma g_0 g_0^T \end{pmatrix} \in S_{+}^{n+1}, \\
& \quad - \begin{pmatrix} U_{11}^Y \\ U_{21}^Y \end{pmatrix} \in S_{+}^{2m_1}, U_{11}^X \in S_{+}^{2m_2}, \quad U_{21}^X, U_{21}^Y \in \mathbb{R}^{(n-2m_1) \times 2m_1}, \quad U_{21}^Y \in \mathbb{R}^{(n-2m_2) \times 2m_2}, \\
& \quad (U_{11}^X)_{2j-1,2j-1} + (U_{11}^X)_{2j,2j} = 0 \quad (j = 1, \ldots, m_1), \\
& \quad (U_{11}^Y)_{2j-1,2j-1} + (U_{11}^Y)_{2j,2j} = 0 \quad (j = 1, \ldots, m_2).
\end{align*}
\]

Moreover, the duality gap between (55) and (54) is zero, and (54) has an optimal solution.

We focus on the first inequality constraint in (54) and can see that all the off-diagonal elements of \( U_{11}^X \) and all the elements of \( U_{21}^X \) do not appear in the other constraints in (55). Hence it is enough to compute them after finding \( \gamma \), \( \hat{X} \) and all the diagonal elements of \( U_{11}^X \). Proposition 1 shown below gives a simplification of LMI problem (55) based on this idea. For this, we use the following lemma. This lemma plays an essential role in the proof of Proposition 1 and can be directly proved in a similar manner to the proof in [28, Appendix D].

27
Lemma 11. Let $k, \ell$ be positive integers. We assume that the two matrices \[
\begin{pmatrix}
U_{11} & U_{31} \\
U_{31} & U_{33}
\end{pmatrix} \in \mathbb{S}^{k+1} \quad \text{and} \quad \begin{pmatrix}
U_{21} & U_{32}^T \\
U_{32} & U_{33}
\end{pmatrix} \in \mathbb{S}^{\ell+1}
\] are positive semidefinite. Then there exists $U_{21} \in \mathbb{R}^{\ell \times k}$ such that the matrix \[
\begin{pmatrix}
U_{11} & U_{21} & U_{31} \\
U_{21} & U_{32}^T & U_{32} \\
U_{31} & U_{32} & U_{33}
\end{pmatrix}
\] is positive semidefinite.

Proposition 1. The optimal value of (56) is equivalent to the following LMI problem:

\[
\begin{align*}
\inf_{\gamma, X, Y, u_x^i, u_y^j} \quad & \gamma \\
\text{subject to} \quad & -\left( \begin{pmatrix} \text{He}(A^T X) & \gamma & h_1 \end{pmatrix}^T \right) \in \mathbb{S}^{-2m_1+1}_+; \\
& -\left( \begin{pmatrix} \text{He}(\Omega U Y) & -\gamma & h_2 \end{pmatrix}^T \right) \in \mathbb{S}^{-2m_2+1}_+; \\
& \left( \begin{pmatrix} (f^r)^2 - u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
& \left( \begin{pmatrix} (f^r)^2 + u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
& \left( \begin{pmatrix} (g^r)^2 - u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
& \left( \begin{pmatrix} (g^r)^2 + u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
& \left( \begin{pmatrix} \text{He}(\Lambda^T X) & \gamma & h_1 \end{pmatrix}^T \right) \in \mathbb{S}^{-2m_1+1}_+; \\
& -\left( \begin{pmatrix} \text{He}(\Omega U Y) & -\gamma & h_2 \end{pmatrix}^T \right) \in \mathbb{S}^{-2m_2+1}_+; \\
& \left( \begin{pmatrix} (f^r)^2 - u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
& \left( \begin{pmatrix} (f^r)^2 + u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
& \left( \begin{pmatrix} (g^r)^2 - u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
& \left( \begin{pmatrix} (g^r)^2 + u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
\end{align*}
\]

Proof. Since the matrix in the left side of each constraint in (56) is a submatrix in the left side of each constraint in (55), any feasible solution of (55) is also feasible for (56) with the same objective value. It is sufficient to prove that any feasible solution of (56) is feasible for (55) with the same objective value. Let $(\gamma, X, Y, u_x^i, u_y^j)$ be a feasible solution of (56). Then the solution satisfies

\[
\left( \begin{pmatrix} (f^r)^2 - u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
\left( \begin{pmatrix} (f^r)^2 + u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
\left( \begin{pmatrix} (g^r)^2 - u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
\left( \begin{pmatrix} (g^r)^2 + u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
\left( \begin{pmatrix} \text{He}(\Lambda^T X) & \gamma & h_1 \end{pmatrix}^T \right) \in \mathbb{S}^{-2m_1+1}_+; \\
\left( \begin{pmatrix} \text{He}(\Omega U Y) & -\gamma & h_2 \end{pmatrix}^T \right) \in \mathbb{S}^{-2m_2+1}_+; \\
\left( \begin{pmatrix} (f^r)^2 - u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
\left( \begin{pmatrix} (f^r)^2 + u_x^i \left( -h_{10}^i \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
\left( \begin{pmatrix} (g^r)^2 - u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+; \\
\left( \begin{pmatrix} (g^r)^2 + u_y^j \left( -h_{20}^j \right) \end{pmatrix}^T \right) \in \mathbb{S}^2_+ \\
\end{align*}
\]

Applying Lemma 11 to the above matrices repeatedly, we see that there exists $v_{kj} \in \mathbb{R}$ $(j = 1, \ldots, 2m_1, k = j + 1, \ldots, 2m_1 + 1)$ such that the following matrix is positive semidefinite:

\[
\begin{pmatrix}
(v_{21})^2 & v_{31} & \cdots & v_{2m_1,1} & -(h_{10}^i)^1 \\
(v_{31})^2 & v_{32} & \cdots & v_{2m_2,1} & -(h_{10}^i)^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
v_{2m_1,1} & v_{2m_2,1} & \cdots & v_{2m_1,2m_1-1} & \cdots \\
-(h_{10}^i)^1 & -(h_{10}^i)^1 & \cdots & \cdots & -(h_{10}^i)^1 \\
\end{pmatrix}
\]

We define $U_{11}^{X}$ as follows:

\[
\begin{align*}
(U_{11}^{X})_{kj} = \begin{cases}
-\frac{u_x^{(k+1)/2}}{u_{k+1}^{(k+1)/2}} & \text{if } k = j \text{ and } k : \text{ odd number}, \\
\gamma(f_0)(f_0)_j - v_{kj} & \text{if } k = j \text{ and } k : \text{ even number}, \\
\gamma(f_0)(f_0)_j - v_{kj} & \text{if } k > j, \\
\gamma(f_0)(f_0)_j - v_{kj} & \text{o.w.}
\end{cases}
\end{align*}
\]

28
Then we have
\[
- \begin{pmatrix}
U_{11}^X - \gamma f_0 f_0^T & h_{10} \\
 h_{10}^T & -\gamma
\end{pmatrix} \in S_{+}^{2m_1+1}.
\] (57)

Similarly, there exists \(U_{11}^Y \in S^{2m_2} \) such that
\[
- \begin{pmatrix}
U_{11}^Y - \gamma g_0 g_0^T & h_{20} \\
 h_{20}^T & -\gamma
\end{pmatrix} \in S_{+}^{2m_2+1}.
\] Moreover, we have
\[
- \begin{pmatrix}
\text{He}(\Lambda^T \hat{X}) - \gamma f f^T & h_1 \\
 h_1^T & -\gamma
\end{pmatrix} \in S_{+}^{n-2m_1+1}.
\] (58)

From (57), (58) and Lemma 11, there exists \(Z_{X21} \in \mathbb{R}^{(n-2m_1) \times 2m_1} \) such that
\[
- \begin{pmatrix}
U_{11}^X - \gamma f_0 f_0^T & h_{10} \\
 h_{10}^T & -\gamma
\end{pmatrix} \in S_{+}^{2m_1+1}.
\] As well as the above, we have \(Z_{Y21} \in \mathbb{R}^{(n-2m_2) \times 2m_2} \) that satisfies
\[
- \begin{pmatrix}
U_{11}^Y - \gamma g_0 g_0^T & h_{20} \\
 h_{20}^T & -\gamma
\end{pmatrix} \in S_{+}^{2m_2+1}.
\]

We define \(U_{21}^X \) and \(U_{21}^Y \) by \(U_{21}^X = Z_{X21} + \gamma f_1 f_1^T \) and \(U_{21}^Y = Z_{Y21} + \gamma g_1 g_1^T \). Then the solution \((\gamma, \hat{X}_{22}, \hat{Y}_{22}, U_{11}^X, U_{11}^Y, U_{21}^X, U_{21}^Y)\) is feasible for (55), and thus the optimal value of (56) is equal to the optimal value of (55). □

We can simplify (56) by applying the following lemma:

**Lemma 12.** Let \(f_1, f_2, h_1, h_2, \gamma \in \mathbb{R} \). There exists \(p \in \mathbb{R} \) such that
\[
\begin{pmatrix}
\gamma f_1^2 - p & -h_1 \\
-h_1 & \gamma
\end{pmatrix} \in S_{+}^2 \quad \text{and} \quad \begin{pmatrix}
\gamma f_2^2 + p & -h_2 \\
-h_2 & \gamma
\end{pmatrix} \in S_{+}^2
\] (59)

if and only if \(\gamma \) satisfies
\[
\gamma \geq \sqrt{\frac{h_1^2 + h_2^2}{f_1^2 + f_2^2}}.
\] (60)

**Proof.** If \(\gamma = 0\), then \(h_1 = h_2 = 0\), and thus the equivalence is obvious. We assume \(\gamma > 0\). We see that (59) holds if and only if
\[
-\gamma f_2^2 + \frac{h_2^2}{\gamma} \leq p \leq \gamma f_1^2 - \frac{h_1^2}{\gamma}.
\]
We obtain (60) from this inequality. On the other hand, if (60) holds, then we define \(p\) by
\[
p = \frac{1}{2} \left( \gamma(f_1^2 - f_2^2) - \frac{h_1^2 - h_2^2}{\gamma} \right).
\] Then (59) holds by direct computation. □
Applying Lemma \[12\] to (55), it can be reformulated as follows:

\[
\begin{cases}
\inf_{\gamma, \bar{X}, \bar{Y}} \gamma \\
\text{subject to} - \left( \begin{array}{cc}
\text{He}(\Lambda^T \dot{X}) - \gamma ff^T & h_1 \\
h_1^T & -\gamma
\end{array} \right) \in S^{n-2m_1+1}_+,
- \left( \begin{array}{cc}
\text{He}(\Omega^T \dot{Y}) - \gamma gg^T & h_2 \\
h_2^T & -\gamma
\end{array} \right) \in S^{n-2m_2+1}_+,
\bar{X} - JF \bar{Y} \in S^{n_0}_+,
\end{cases}
\]

(61)

Here we used \((h_{10})^2_j + (h_{10})^2_j = |(h_{10})_j|^2\) and \((f^r)^2_j + (f^l)^2_j = |f_j|^2\) etc. for the last inequalities.

It should be noted that the optimal values of optimization problems, which appear in Section 6, that is, (49), (50), (54), (55), (56) and (61) are equal to \(\gamma^*\). In fact, the optimal value of (49) is \(\gamma^*\). Since (50) is the dual of (49), the optimal value of (50) is \(\gamma^*\). In addition, it follows from Lemma 10 that the optimal value of (55) is equal to the optimal value of (54). From Proposition 11 and Lemma 12, the optimal value of (61) is equal to the optimal values of (56) and (55). Hence all the optimal values of these optimization problems are equal to \(\gamma^*\).

From (61), we see that

\[\gamma^* = \max \left\{ \tilde{\gamma}, \max_{j=1, \ldots, m_1} \left\{ \left| \frac{(h_{10})_j}{f_j} \right| \right\}, \max_{j=1, \ldots, m_2} \left\{ \left| \frac{(h_{20})_j}{g_j} \right| \right\} \right\},\]

where \(\tilde{\gamma}\) is the optimal value of the following LMI problem

\[
\begin{cases}
\inf_{\gamma, \bar{X}, \bar{Y}} \gamma \\
\text{subject to} - \left( \begin{array}{cc}
\text{He}(\Lambda^T \dot{X}) - \gamma ff^T & h_1 \\
h_1^T & -\gamma
\end{array} \right) \in S^{n-2m_1+1}_+,
- \left( \begin{array}{cc}
\text{He}(\Omega^T \dot{Y}) - \gamma gg^T & h_2 \\
h_2^T & -\gamma
\end{array} \right) \in S^{n-2m_2+1}_+,
\bar{X} - JF \bar{Y} \in S^{n_0}_+,
\end{cases}
\]

(62)

It should be noted that if \(G_{zu}\) (resp. \(G_{yw}\)) has a stable invariant zero, then Theorem 5 is available to obtain an expression of \(\tilde{\gamma}\). Otherwise, Theorem 4 is available. Therefore, we obtain the following theorem from (61):

**Theorem 6.** Let us consider Case 3 stated at the final part of Section 5. We assume that invariant zeros \(\lambda_1, \ldots, \lambda_{m_1}\) in \(G_{zu}\) and \(\omega_1, \ldots, \omega_{m_2}\) in \(G_{yw}\) of (11) exist on the imaginary axis. Let \(\tilde{\gamma} := \lambda_{\max}(E)\) where \(E\) is given by (11) in Theorem 5. Then the optimal value \(\gamma^*\) of (19) is equal to

\[
\max \left\{ \tilde{\gamma}, \left| \frac{s_j^T b_1}{f_j} + d_{11} \right| \left( j = 1, \ldots, m_1 \right), \left| \frac{t_j^T c_1}{g_j} + d_{11} \right| \left( j = 1, \ldots, m_2 \right) \right\}.
\]
In particular, if all invariant zeros of \( G_{zu} \) and \( G_{yw} \) on the imaginary axis are not the eigenvalues of \( A \), respectively, then \( \gamma^* \) is equal to

\[
\max \{ |\gamma_j|, |G_{zw}(\lambda_j)| \ (j = 1, \ldots, m_1), |G_{zw}(\omega_j)| \ (j = 1, \ldots, m_2) \}.
\]

Proof. We have already mentioned that all the optimal values of (49), (50), (54), (55), (56) and (61) are equal to \( \gamma^* \).

We rewrite the last three constraints in (61). We have \( h_{10} = s_j^T b_1 + d_{11} f_j \) and \( h_{20} = t_j^T c_1 + d_{11} g_j \) for all \( j \). From these equations, we obtain

\[
\gamma \geq \left| \frac{s_j^T b_1}{f_j} + d_{11} \right| \quad \text{and} \quad \gamma \geq \left| \frac{t_j^T c_1}{g_j} + d_{11} \right|
\]

for all \( j \). Therefore, we obtain the desired result.

We prove the next statement. If \( \lambda_j \) is not an eigenvalue of \( A \), then we have \( s_j^T = f_j c_j^T (\lambda_j I_N - A)^{-1} \). Similarly, \( \omega_j \) is not an eigenvalue of \( A \), then we have \( t_j^T = g_j b_j^T (\omega_j I_N - A^T)^{-1} \). By using these expressions together with (48), we obtain

\[
\left| \frac{s_j^T b_1}{f_j} + d_{11} \right| = |G_{zw}(\lambda_j)| \quad \text{and} \quad \left| \frac{t_j^T c_1}{g_j} + d_{11} \right| = |G_{zw}(\omega_j)|.
\]

Therefore we obtain the desired result.

7. Analysis of Case 4

Finally, we deal with the case where at least one of the conditions \( d_{12} = 0 \) and \( d_{21} = 0 \) holds. When \( d_{12} = 0 \), for instance, the transfer function \( G_{zu} \) may be identically zero. Then the transfer function \( G_{cl}(s, K) \) of the closed-loop system is \( G_{cl}(s, K) = G_{zw}(s) \), and thus \( \gamma^* = \| G_{zw} \|_\infty \). This also holds when \( G_{yw} \) is identically zero. Therefore we consider the case where both transfer functions \( G_{zu} \) and \( G_{yw} \) are not identically zero.

When \( d_{12} = 0 \), the relative degree \( r_1 \) of \( G_{zu} \) is positive and it follows from the assertion 3 in Theorem 2 and 2 of Remark 2 that we have

\[
\begin{align*}
d_{12} = 0, c_1^T b_2 & \neq 0 & \text{if } r_1 = 1, \\
d_{12} = 0, c_1^T A^r b_2 = 0 & (r = 0, \ldots, r_1 - 2) \text{ and } c_1^T A^{r_1 - 1} b_2 & \neq 0 & \text{if } r_1 > 1.
\end{align*}
\]

Similarly, when \( d_{21} = 0 \), callly zero, then the relative degree \( r_2 \) of \( G_{yw} \) is positive and we have

\[
\begin{align*}
d_{21} = 0, c_2^T b_1 & \neq 0 & \text{if } r_2 = 1, \\
d_{21} = 0, c_2^T A^r b_1 = 0 & (r = 0, \ldots, r_2 - 2) \text{ and } c_2^T A^{r_2 - 1} b_1 & \neq 0 & \text{if } r_2 > 1.
\end{align*}
\]

Combining them with \( S^T A + f^T \gamma = \Lambda^T S^T, T^T A^T + gb^T = \Omega^T T^T, S^T b_2 = 0 \) and \( T^T c_2 = 0 \), we obtain

\[
\begin{align*}
S^T A^r b_2 & = 0 & (r = 0, \ldots, r_1 - 1), \quad (63) \\
T^T (A^T)^r c_2 & = 0 & (r = 0, \ldots, r_2 - 1). \quad (64)
\end{align*}
\]
We define $P_1 \in \mathbb{R}^{n \times r_1}$, $p_1 \in \mathbb{R}^{r_1}$, $P_2 \in \mathbb{R}^{n \times r_2}$ and $p_2 \in \mathbb{R}^{r_2}$ as follows:

$\begin{pmatrix} P_1 \\ p_1^T \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } r_1 = 1, \\ \begin{pmatrix} 0 & c_1 & (A^T)c_1 \\ 1 & 0 & 0 \end{pmatrix} & \text{if } r_1 > 1, \\ \end{cases}$

Using (63) and (64), we have

$\begin{pmatrix} P_2 \\ p_2^T \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } r_2 = 1, \\ \begin{pmatrix} 0 & b_1 & A b_1 \\ 1 & 0 & 0 \end{pmatrix} & \text{if } r_2 > 1. \\ \end{cases}$

Using (63) and (64), we have

$(b_2^T (A^T) P_1)_j = 0 \quad (r = 0, 1, \ldots, r_1 - 1, j = 1, \ldots, r_1 - r), \quad (65)$

$(c_2^T (A^T) P_2)_j = 0 \quad (r = 0, 1, \ldots, r_2 - 1, j = 1, \ldots, r_2 - r). \quad (66)$

We provide the perpendicular matrices of the vectors $(b_2^T, 0, 0)^T$ and $(c_2^T, 0, 0)^T$ in Lemma 13. This lemma can be proved in a similar manner to the proof of Lemma 2.

**Lemma 13.** Suppose $d_{12} = 0$ in $G_{zu}$ and its relative degree is $r_1$. Then we have

$\begin{pmatrix} b_2 \\ 0 \\ 0 \end{pmatrix} \perp \begin{pmatrix} S & P_1 \\ f^T & p_1^T \\ 0 & 0 & 1 \end{pmatrix},$

i.e., $S^T b_2 = 0$, $P_1^T b_2 = 0$ and the following square matrix is nonsingular:

$\begin{pmatrix} b_2 & S & P_1 \\ 0 & f^T & p_1^T \\ 0 & 0 & 0 \end{pmatrix}$.

Similarly, suppose $d_{21} = 0$ in $G_{yw}$ and its relative degree is $r_2$. Then we have

$\begin{pmatrix} c_2 \\ 0 \end{pmatrix} \perp \begin{pmatrix} T & P_2 \\ g^T & p_2^T \\ 0 & 0 & 1 \end{pmatrix}$.

From Lemma 13, (25) is equivalent to the following optimization problem:

$$\inf_{\gamma, X, Y} \gamma$$

subject to $-\begin{pmatrix} \text{He}(\Lambda^T S^T XS) - \gamma f f^T \\ \text{He}(\Lambda^T S^T XS) - \gamma f f^T \\ \text{He}(\Lambda^T S^T XS) - \gamma f f^T \end{pmatrix} \in \mathbb{S}_{++}^{n+1},$

$$-\begin{pmatrix} \text{He}(\Omega^T TT^T Y) - \gamma g g^T \\ \text{He}(\Omega^T TT^T Y) - \gamma g g^T \\ \text{He}(\Omega^T TT^T Y) - \gamma g g^T \end{pmatrix} \in \mathbb{S}_{++}^{n+1},$$

$$\begin{pmatrix} X & -I_n \\ -I_n & Y \end{pmatrix} \in \mathbb{S}_{++}^{2n}.$$
where
\[ h_1 = S^T b_1 + d_{11} f, \quad h_2 = T^T c_1 + d_{11} g, \]
\[ \tilde{h}_1 = P_1^T b_1 + d_{11} p_1 = (d_{11} \quad c_1^T b_1 \quad c_1^T A b_1 \ldots \quad c_1^T A^{n-2} b_1)^T, \]
\[ \tilde{h}_2 = P_2^T c_1 + d_{11} p_2 = (d_{11} \quad b_1^T c_1 \quad b_1^T A^T c_1 \ldots \quad b_1^T (A^T)^{r-2} c_1)^T. \]
\[ \tilde{P}_1 = A^T P_1 + c_1 p_1^T = (c_1 \quad A^T c_1 \ldots \quad (A^T)^{r-1} c_1), \]
\[ \tilde{P}_2 = A P_2 + b_1 p_2^T = (b_1 \quad A b_1 \ldots \quad A^{r-1} b_1). \]

Similar properties to (65) and (66) hold in \( \tilde{P}_1 \) and \( \tilde{P}_2 \) as follows:
\[ (b_2^T (A^T)^T \tilde{P}_1)_j = 0 \quad (r = 0, \ldots, r_1 - 2, j = 1, \ldots, r_1 - r - 1), \]
\[ (c_2^T (A^T) \tilde{P}_2)_j = 0 \quad (r = 0, \ldots, r_2 - 2, j = 1, \ldots, r_2 - r - 1). \]

The following lemma provides the dual of (67). We give a proof in Appendix C.6.

**Lemma 14.** The dual of (67) can be formulated as follows:

\[
\begin{align*}
\sup & \left( \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right) \cdot Z + \left( \begin{pmatrix} h_3 \\ h_4 \end{pmatrix} \right) \cdot V + \left( \begin{array}{cc} I_n & I_n \end{array} \right) \cdot W \\
\text{subject to} \quad & W_{11} = \text{He}(S\Lambda(Z_{11}^S + Z_{21}^T P_1^T)) + \tilde{P}_1(Z_{21}^S + Z_{22} P_1^T), \quad W_{22} = \text{He}(T\Omega(V_{11}^T + V_{21}^T P_2^T)) + \tilde{P}_2(V_{21}^T + V_{22} P_2^T), \quad Z = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \in S^{n+1}_+, \quad V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \in S^{n+1}_+, \quad Z_{21} \in \mathbb{R}^{(n-r_1) \times r_1}, \quad Z_{22} \in \mathbb{R}^{r_1 \times r_1}, \quad V_{21} \in \mathbb{R}^{(n-r_2) \times r_2}, \quad V_{22} \in \mathbb{R}^{r_2 \times r_2}, \quad W_{11} \quad W_{21} \quad W_{22} \in S^n_+. 
\end{align*}
\]

Moreover, the duality gap between (67) and (70) is zero.

From the following lemma, we can reduce the size of (70) equivalently. We give a proof in Appendix C.7.

**Lemma 15.** Let \( Z_{ij}, V_{ij}, W_{ij} \) be a feasible solution of (70). Then \( Z_{21} = O_{r_1 \times (n-r_1)} \) and \( V_{21} = O_{r_2 \times (n-r_2)} \). In addition, if \( r_1 > 1 \), we have \( (Z_{22})_{kr} = 0 \) for all \( (k, r) \neq (1, 1) \) and \( (Z_{32})_k = 0 \) for \( k = 2, \ldots, r_1 \). Similarly, if \( r_2 > 1 \), we have \( (V_{22})_{kr} = 0 \) for all \( (k, r) \neq (1, 1) \) and \( (V_{32})_k = 0 \) for \( k = 2, \ldots, r_2 \).

**Remark 5.** As well as Remark 3 and 4, we have reduced the size of the matrix variables \( Z \) and \( V \) in (70) in Lemma 15. This reduction also corresponds to the facial reduction for SDP as well as Lemmas 7 and 9.
By applying Lemma 15 to (70) and substituting zeros in \((Z, V, W)\), we can reformulate it as follows.

\[
\begin{align*}
\text{sup} & \quad \begin{pmatrix} h_1 \cr d_{11} \cr f^T \end{pmatrix} \cdot Z + \begin{pmatrix} h_2 \cr d_{11} \cr f^T \end{pmatrix} \cdot V + \begin{pmatrix} I_n \cr I_n \end{pmatrix} \cdot W \\
\text{subject to} & \quad \begin{pmatrix} f^T \cr f^T \cr 1 \end{pmatrix} \cdot Z + \begin{pmatrix} g g^T \cr g \cr 1 \end{pmatrix} \cdot V = 1,
\end{align*}
\]

subject to \(W_{11} = S \text{He}(\Lambda Z_{11}) S^T, W_{22} = T \text{He}(\Omega V_{11}) T^T, \left(\begin{array}{cc} W_{11} & W_{12}^T \\ W_{21} & W_{22} \end{array}\right) \in S_+^{2n}, \)

\[
Z = \begin{pmatrix} Z_{11} & O & Z_{31}^T \\ O & Z_{22} & Z_{32}^T \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \in S_+^{n-r_1+2}, V = \begin{pmatrix} V_{11} & O & V_{12}^T \\ O & V_{22} & V_{22}^T \\ V_{31} & V_{32} & V_{32}^T \end{pmatrix} \in S_+^{n-r_2+2}.
\]

(71)

It follows from constraints on \(W_{ij}\) in (71) that there exists \(\tilde{W}_{11} \in S_+^{n-r_1}, \tilde{W}_{21} \in \mathbb{R}^{(n-r_2) \times (n-r_2)}\) and \(\tilde{W}_{22} \in S_+^{n-r_2}\) such that

\[
\left(\begin{array}{cc} \tilde{W}_{11} & \tilde{W}_{21}^T \\ \tilde{W}_{21} & \tilde{W}_{22} \end{array}\right) \in S_+^{(n-r_1)+(n-r_2)}, W_{11} = S \tilde{W}_{11} S^T, W_{22} = T \tilde{W}_{22} T^T\) and \(W_{21} = T \tilde{W}_{21} S^T,\)

and thus (71) can be reformulated as

\[
\begin{align*}
\text{sup} & \quad \begin{pmatrix} h_1 \cr d_{11} \cr f^T \end{pmatrix} \cdot Z + \begin{pmatrix} h_2 \cr d_{11} \cr f^T \end{pmatrix} \cdot V + \begin{pmatrix} T^T S & S^T T \end{pmatrix} \cdot \tilde{W} \\
\text{subject to} & \quad \begin{pmatrix} f^T \cr f^T \cr 1 \end{pmatrix} \cdot Z + \begin{pmatrix} g g^T \cr g \cr 1 \end{pmatrix} \cdot V = 1,
\end{align*}
\]

subject to \(\tilde{W}_{11} = \text{He}(\Lambda Z_{11}), \tilde{W}_{22} = \text{He}(\Omega V_{11}), \tilde{W} = \left(\begin{array}{cc} \tilde{W}_{11} & \tilde{W}_{21}^T \\ \tilde{W}_{21} & \tilde{W}_{22} \end{array}\right) \in S_+^{n_x}, \)

\[
Z = \begin{pmatrix} Z_{11} & O & Z_{31}^T \\ O & Z_{22} & Z_{32}^T \\ Z_{31} & Z_{32} & Z_{33} \end{pmatrix} \in S_+^{n-r_1+2}, V = \begin{pmatrix} V_{11} & O & V_{12}^T \\ O & V_{22} & V_{22}^T \\ V_{31} & V_{32} & V_{32}^T \end{pmatrix} \in S_+^{n-r_2+2}.
\]

(72)

where \(n_x = 2n - r_1 - r_2\). The next lemma provides the dual of (72).
Lemma 16. The dual of (72) can be formulated as follows:

\[
\begin{aligned}
\inf_{\gamma, X, Y, \xi, \eta} & \quad \gamma \\
\text{subject to} & \quad - \begin{pmatrix}
\text{He}(\Lambda^T \hat{X}) - \gamma f f^T & \xi - \gamma f & h_1 \\
\xi^T - \gamma f^T & -\gamma & d_{11} \\
\end{pmatrix} \in S^{n-r_1+2}_+, \\
& \quad - \begin{pmatrix}
\text{He}(\Omega^T \hat{Y}) - \gamma g g^T & \eta - \gamma g & h \\
\eta^T - \gamma g^T & -\gamma & d_{11} \\
\end{pmatrix} \in S^{n-r_2+2}_+, \\
& \quad \begin{pmatrix}
X & -J \\
-J & Y \\
\end{pmatrix} \in S^{n, \infty}_+, \xi \in \mathbb{R}^{n-r_1}, \eta \in \mathbb{R}^{n-r_2}.
\end{aligned}
\]

Moreover, the duality gap between (73) and (74) is zero, and (72) has an optimal solution.

Proof. We can prove by applying similar arguments in Lemmas 6 that the dual of (72) is (73). The zero duality gap between (73) and (72), and the existence of an optimal solution of (72) can be proved by a similar way in Appendix B.3.

For (73), we consider the following LMI problem:

\[
\begin{aligned}
\inf_{\gamma, X, Y} & \quad \gamma \\
\text{subject to} & \quad - \begin{pmatrix}
\text{He}(\Lambda^T \hat{X}) - \gamma f f^T & h_1 \\
\xi^T - \gamma f^T & -\gamma \\
\end{pmatrix} \in S^{n-r_1+1}_+, \quad \begin{pmatrix}
\hat{X} & -J \\
-J & \hat{Y} \\
\end{pmatrix} \in S^{n, \infty}_+, \\
& \quad - \begin{pmatrix}
\text{He}(\Omega^T \hat{Y}) - \gamma g g^T & h_2 \\
\eta^T - \gamma g^T & -\gamma \\
\end{pmatrix} \in S^{n-r_2+1}_+, \quad \begin{pmatrix}
\gamma & -d_{11} \\
-d_{11} & \gamma \\
\end{pmatrix} \in S^2_+.
\end{aligned}
\]

The following proposition holds for (73) and (74):

Proposition 2. The optimal value of (74) is equal to the optimal value of (73).

Proof. Let \( \hat{\gamma} \) and \( \hat{\gamma} \) be the optimal values of (73) and (74), respectively. Since any feasible solution of (73) is also feasible for (74) with the same objective value, we have \( \hat{\gamma} \geq \hat{\gamma} \). For this, it is sufficient to prove \( \hat{\gamma} \leq \hat{\gamma} \).

Let \((\gamma, X, Y)\) be a feasible solution for (74). If \( \gamma = 0 \), then \( d_{11} = 0, h_1 = h_2 = 0 \). In addition, as we have \( \hat{X} = O, \hat{Y} = O, J = O \). This is also feasible for (73) by taking \( \xi = \eta = 0 \). Hence \( \gamma = 0 \).

We assume that \( \gamma > 0 \). We define \( \xi \) and \( \eta \) by \( \xi = \gamma f + \frac{d_{11}}{\gamma} h_1 \) and \( \eta = \gamma g + \frac{d_{11}}{\gamma} h_2 \), respectively. It follows Lemma 11 that \((\gamma, \hat{X}, \hat{Y}, \xi, \eta)\) satisfies

\[
- \begin{pmatrix}
\text{He}(\Lambda^T \hat{X}) - \gamma f f^T & h_1 \\
\xi^T - \gamma f^T & -\gamma \\
\end{pmatrix} \in S^{n-r_1+2}_+, \\
- \begin{pmatrix}
\text{He}(\Omega^T \hat{Y}) - \gamma g g^T & h_2 \\
\eta^T - \gamma g^T & -\gamma \\
\end{pmatrix} \in S^{n-r_2+2}_+.
\]
Hence \((\gamma, \hat{X}, \hat{Y}, \xi, \eta)\) is feasible for (73) with the same objective value as \(\gamma\). Therefore we have \(\tilde{\gamma} = \hat{\gamma}\).

It should be noted that the optimal values of optimization problems, which appear in Section 7, that is, (67), (70), (71), (72), (73) and (74) are equal to \(\gamma^*\). In fact, the optimal value of (49) is \(\gamma^*\). Since (70) is the dual of (67), it follows from Lemma 14 that the optimal value of (70) is \(\gamma^*\). Applying Lemma 15, we see that the optimal value of (71) is equal to \(\gamma^*\). As we have seen, the optimal value of (72) is also equal to \(\gamma^*\). It follows from Lemma 16 that the optimal value of (73) is equal to (72). Finally, form Proposition 2, we see that the optimal value of (74) is equal to the optimal value of (73). Hence all the optimal values of these optimization problems are equal to \(\gamma^*\).

From (74), we see that
\[
\gamma^* = \max \{ \tilde{\gamma}, |d_{11}| \},
\]
where \(\tilde{\gamma}\) is the optimal value of the following LMI problem
\[
\inf_{\gamma, \hat{X}, \hat{Y}} \gamma \quad \text{subject to} \quad \begin{cases} 
- \left( \text{He}(\Lambda^T \hat{X}) - \gamma ff^T \begin{pmatrix} h_1 \\ h_1^T \\ -\gamma \end{pmatrix} \right) \in S^n_{r_1+1}, \\
- \left( \text{He}(\Omega^T \hat{Y}) - \gamma gg^T \begin{pmatrix} h_2 \\ h_2^T \\ -\gamma \end{pmatrix} \right) \in S^n_{r_2+1}, \\
\begin{pmatrix} \hat{X} \\ -J \end{pmatrix} \begin{pmatrix} -J^T \\ \hat{Y} \end{pmatrix} \in S^{n\infty}. 
\end{cases}
\]

It should be noted that if \(G_{zu}\) (resp. \(G_{yw}\)) has an invariant zero on the imaginary axis, then Theorem 6 is available to obtain an expression of \(\tilde{\gamma}\). Also, if \(G_{zu}\) (resp. \(G_{yw}\)) has a stable invariant zero, then Theorem 5 is available to obtain an expression of \(\tilde{\gamma}\). Otherwise, Theorem 4 is available. Therefore, we obtain the following theorem.

**Theorem 7.** Let us consider Case 1 stated at the final part of Section 3. We assume that invariant zeros \(\lambda_1, \ldots, \lambda_{m_1}\) in \(G_{zu}\) and \(\omega_1, \ldots, \omega_{m_2}\) in \(G_{yw}\) of (11) exist on the imaginary axis. Define \(\tilde{\gamma} := \max \{ \tilde{\gamma}, |d_{11}| \}\) where \(E\) is given by (41) in Theorem 5. Then the optimal value \(\gamma^*\) of (49) is equal to
\[
\max \left\{ \tilde{\gamma}, \left| \frac{s_j^T b_1}{f_j} \right| d_{11}, \left| \frac{t_j^T c_1}{g_j} \right| d_{11}, \left| G_{zw}(\infty) \right| \right\}.
\]

If \(G_{zu}\) (resp. \(G_{yw}\)) has no invariant zeros on the imaginary axis, then \(\left| \frac{s_j^T b_1}{f_j} \right| d_{11}\) (resp. \(\left| \frac{t_j^T c_1}{g_j} \right| d_{11}\)) is vanished from the above expression of \(\gamma^*\).

**Proof.** We have already seen that all the of optimal values of (67), (70), (71), (72), (73) and (74) are equal to \(\gamma^*\). As \(|d_{11}| = |G_{zw}(\infty)|\), we obtain the desired result. \(\square\)
8. Application of Theorem 1 to limitation analysis

We here provide an application of Theorem 1 to the limitation analysis of $H_\infty$ output feedback control for sensitivity function. This application has been already considered in [4, Theorem 5.1] via Nevanlinna-Pick interpolation. We provide the same result for SISO dynamical systems from Theorem 1.

Consider the following generalized plant.

\[
\begin{aligned}
\dot{x} &= Ax + bu \\
z &= c^T x + w \\
y &= c^T x + w
\end{aligned}
\]  

(76)

In addition to Assumption 1, we impose that (76) has no invariant zeros on the imaginary axis. Then we can rewrite (29) as follows.

\[
\begin{pmatrix} \Lambda & \Lambda \\
\Omega & \Omega \\
\end{pmatrix} 
\begin{pmatrix} A & b \\
c & 0 \\
\end{pmatrix} = 
\begin{pmatrix} S^T & 0 \\
S^T & 0 \\
\end{pmatrix},
\]

(77)

\[
\begin{pmatrix} T & g \\
T & g \\
\end{pmatrix} 
\begin{pmatrix} A & c \\
p & 1 \\
\end{pmatrix} = 
\begin{pmatrix} \Omega^T & \Omega^T \\
\Omega^T & \Omega^T \\
\end{pmatrix},
\]

(78)

Then any eigenvalue of $\Lambda$ (resp. $\Omega$) is an unstable zero (resp. pole) of (76).

Let $Z_{++}$ and $P_{++}$ be the sets of unstable zeros and poles in (76), respectively. In addition, $h_1$ and $h_2$ are rewritten by

\[
\begin{pmatrix} h^1 \\
h^2 \\
\end{pmatrix} = 
\begin{pmatrix} S^T & 0 \\
S^T & 0 \\
\end{pmatrix} + 
\begin{pmatrix} f & 0 \\
0 & 1 \\
\end{pmatrix},
\]

(79)

and thus $H_{1+} = F_+$ and $H_{2+} = O$. By using those equations, we can simplify $\gamma^*$ in Theorem 1 as follows.

\[
\gamma^* = \max \{ \hat{\gamma}, 1 \}, \text{ where } \\
\hat{\gamma} = \lambda_{\max} \left( \begin{pmatrix} O & F^{-1/2} J T G^{-1/2} I O \\
G^{-1/2} J F^{-1/2} & I O O \\
I & O O O \\
O & O O O \\
\end{pmatrix} \right) \\
= \sqrt{1 + \sigma_{\text{max}}^2(G_{++}^{-1/2} J F_{++}^{-1/2})}.
\]

In fact, since $F_+$ and $G_+$ are positive definite, we have

\[
\gamma \geq \lambda_{\max} \left( \begin{pmatrix} O & F^{-1/2} J T G^{-1/2} I O \\
G^{-1/2} J F^{-1/2} & I O O \\
I & O O O \\
O & O O O \\
\end{pmatrix} \right) \geq O
\]

\[
\iff \gamma \geq \sqrt{1 + \sigma_{\text{max}}^2(G_{++}^{-1/2} J F_{++}^{-1/2})}.
\]

37
From this result, we can obtain some results in [4, 9]:

**Example 1.** We consider the case where (1) has a solo unstable zero \( z \) with degree 1 and a solo unstable pole \( p \) with degree 1. Furthermore, assume \( d = 1 \). Then we have

\[
F_+ = \frac{f_2^2}{2z}, \quad G_+ = \frac{g_2^2}{2p} \quad \text{and} \quad J_+ = \frac{fg}{p - z}.
\]

Hence \( G_+^{1/2} J_+ F_+^{1/2} = \frac{2\sqrt{p}}{p - z} \) and \( \gamma^* = \frac{|p + z|}{|p - z|} \).

**Example 2.** We consider the case where (1) has \( k \) unstable zero \( z_1, \ldots, z_k \) with degree 1 and \( k \) unstable poles \( p_1, \ldots, p_k \) with degree 1. This implies that both matrices \( \Lambda_+ \) and \( \Omega_+ \) are diagonal. Furthermore, assume \( d = 1 \). Then \( F_+, G_+ \in S^k \) and \( J_+ \) can be written as follows:

\[
F_+ = \begin{pmatrix} f_i f_j \end{pmatrix}_{1 \leq i, j \leq k}, \quad G_+ = \begin{pmatrix} g_i g_j \end{pmatrix}_{1 \leq i, j \leq k} \quad \text{and} \quad J_+ = \begin{pmatrix} g_i f_j \end{pmatrix}_{1 \leq i, j \leq k}.
\]

In this case, the result in this subsection (seems to) coincide to [4, Theorem 5.1]. Furthermore, in the case where (1) has \( k \) unstable zero \( z_1, \ldots, z_k \) with degree 1 and one unstable pole \( p \) with degree 1, we obtain by using Symbolic Math Toolbox [24]

\[
\sigma_{\max}(G_+^{1/2} J_+ F_+^{1/2}) = \sqrt{p(z_1 + z_2)(p^2 + z_1 z_2)} \quad \frac{1}{|p - z_1||p - z_2|},
\]

and thus \( \gamma^* = \frac{|p + z_1||p + z_2|}{|p - z_1||p - z_2|} \).

**Example 3.** We consider the case where (1) has a solo unstable zero \( z \) with degree 2 and a solo unstable poles \( p \) with degree 2. This implies that both matrices \( \Lambda_+ \) and \( \Omega_+ \) have the forms of

\[
\Lambda_+ = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix} \quad \text{and} \quad \Omega_+ = \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}.
\]

Then

\[
F_+ = \begin{pmatrix} 1 & 1/4z^2 \\ 1/4z^2 & 1/4z^3 \end{pmatrix}, \quad G_+ = \begin{pmatrix} 2p & 1/4p^2 \\ 1/4p^2 & 1/4p^3 \end{pmatrix} \quad \text{and} \quad J_+ = \begin{pmatrix} 1 & 1/2(z - p)^2 \\ 1 & 1/z - p \end{pmatrix}.
\]

In this case, we have

\[
\sigma_{\max}(G_+^{1/2} J_+ F_+^{1/2}) = 2\sqrt{p/z} \left( (p + z)^4 + \sqrt{p^4 + 14p^2 z^2 + z^4} \right).
\]

This is the same as the result in [9, Theorem 2].
9. Conclusion

We considered the LMI problem of $H_\infty$ output feedback control problem for the SISO dynamical system (1). We assumed the stabilizability of $(A, b_2)$ and the detectability of $(A, c_2^T)$. In addition to these assumptions, we impose some technical assumptions for simplicity. Then we provided an explicit form of the optimal value of the LMI problem. When all invariant zeros of $G_{zu}$ and $G_{yw}$ are in the open right half plane, the Schur complement and Lyapunov equation are useful to derive the explicit form. Otherwise, we had seen that the dual problem is not strictly feasible. Then facial reduction is applicable to reduce the size of the dual problem. As a result, the LMI problem of the reduced dual problem is also simplified.

Our explicit form of the optimal value is the unification of some results in the literature of $H_\infty$ performance limitation analysis. For instance, we had seen that we obtain the same results in [4, 9].

We considered the case of the SISO time-invariant dynamical system. It is natural to consider the case of the MIMO time-invariant dynamical system. Our analysis will be easily extended to the case of MIMO with $m_1 = m_2 = p_1 = p_2$. The Weierstrass form is still useful in the case. Otherwise, we will need to consider the Kronecker canonical forms of the transfer functions rather than their Weierstrass forms because their Rosenbrock system matrices are not square. A variant of Kronecker canonical form developed in [21, Chapter 1] may also be useful in the simplification and reduction of the LMI problem obtained from a general MIMO system. [21] provided a simplification of Riccati equations and inequalities obtained from $H_\infty$ control problem. This consideration is not straightforward, and the extension of our explicit form is future study.

Appendix A. Fundamental facts on semidefinite program

To prove the zero duality gap in some theorems and the existence of optimal solutions of some optimization problems under Assumption 1, we need to use the strong duality in Theorems 8 and 9 given below. For the statements of these two theorems, however, we first need to introduce some notation and symbols on semidefinite program.

Let us consider the LMI problem

\[ \theta_p^* = \inf \left\{ d^T y : \sum_{j=1}^{m} y_j L_{ij} - L_{i0} \in S_+^{n_i} (i = 1, \ldots, p), y \in \mathbb{R}^m \right\} \quad (A.1) \]

where $L_{i0}, \ldots, L_{im} \in \mathbb{S}^{n_i} (i = 1, \ldots, p)$ and $d \in \mathbb{R}^m$. The problem (A.1) is said to be strictly feasible if there exists $y_j (j = 1, \ldots, m)$ such that $\sum_{j=1}^{m} y_j L_{ij} - L_{i0} \in S_+^{n_i}$ for all $i = 1, \ldots, p$. On the other hand, its dual can be formulated
as follows:

\[ \theta_D^* = \sup_{X_i} \left\{ \sum_{i=1}^p L_{i0} \cdot X_i : \sum_{i=1}^p L_{ij} \cdot X_i = d_j \ (j = 1, \ldots, m), \quad X_i \in S_+^n \ (i = 1, \ldots, p) \right\}. \]  

(A.2)

The problem (A.2) is said to be strictly feasible if there exists \(X_i \in S_+^n \ (i = 1, \ldots, p)\) such that \(\sum_{i=1}^p L_{ij} \cdot X_i = d_j\) for all \(j = 1, \ldots, m\). We call the value \(\theta_D^* - \theta_P^*\) the duality gap between (A.1) and (A.2).

For every feasible solution \(y_j\) of (A.1) and \(X_i\) of (A.2), we have \(d^T y \geq \sum_{i=1}^p L_{i0} \cdot X_i\). This inequality is called the weak duality for (A.1) and (A.2). The weak duality implies \(\theta_P^* \geq \theta_D^*\), i.e., the duality gap is nonnegative. It is well-known that the duality gap between (A.1) and (A.2) is zero, i.e. \(\theta_P^* = \theta_D^*\) holds under a mild assumption. This is called the strong duality for (A.1) and (A.2). We summarize the details of the strong duality in the next theorem.

**Theorem 8.** (see e.g., [7, Theorem 2.2]) If (A.1) is strictly feasible and the optimal value is bounded below, then \(\theta_P^* = \theta_D^*\) and (A.2) has an optimal solution. Similarly, if (A.2) is strictly feasible and the optimal value is bounded above, then \(\theta_P^* = \theta_D^*\) and (A.1) has an optimal solution.

Finally, we provide a known fact on the strict feasibility of (A.1) and (A.2).

**Theorem 9.** (see e.g., [22, Lemmas 1 and 2]) For (A.1), exactly one of the following two statements is true:

1. (A.1) is strictly feasible.
2. There exist \(\hat{X}_i \in S_+^n \ (i = 1, \ldots, p)\) such that at least one of \(\hat{X}_i\) is nonzero, \(\sum_{i=1}^p L_{i0} \cdot \hat{X}_i \geq 0\) and \(\sum_{i=1}^p L_{ij} \cdot \hat{X}_i = 0\) for all \(j = 1, \ldots, m\).

In particular, if (1) holds and \(\sum_{i=1}^p L_{i0} \cdot \hat{X}_i > 0\), then (A.1) is infeasible. Similarly, for (A.2), exactly one of the following two statements is true:

1. (A.2) is strictly feasible.
2. There exists \(\hat{y} \in \mathbb{R}^m \setminus \{0\}\) such that \(\sum_{j=1}^m L_{ij} \hat{y}_j \in S_+^n\) for all \(i = 1, \ldots, p\) and \(d^T \hat{y} \leq 0\).

In particular, if (1) holds and \(d^T \hat{y} < 0\), then (A.2) is infeasible.

**Appendix B. Proofs on the strong duality**

**Appendix B.1. Proof on the zero duality gap in Theorem 3**

It is clear that the optimal value \(\gamma^*\) of (25) is nonnegative. Therefore from Theorem 3, all the assertions of Theorem 3 can be verified by proving that (I) \(\Leftrightarrow\) (II). We first prove (I) \(\Rightarrow\) (II). Since \((A, b_2)\) is stabilizable, there exists \(K \in \mathbb{R}^{1 \times n}\) and \(X_0 - I_n \in S_+^n\) such that \(- \text{He}((A + b_2 K)X_0) \in S_+^n\). Similarly,
since \((A, c_2)\) is detectable, there exists \(L \in \mathbb{R}^{n \times 1}\) and \(Y_0 - I_n \in S_{++}^n\) such that 
\[- \text{He}(Y_0(A + Lc_2^T)) \in S_{++}^n.\]
It follows that for sufficiently large \(\gamma\) we have
\[
- \begin{pmatrix}
\text{He}((A + b_2K)X_0) & X_0(c_1^T + d_{12}K)^T b_1 \\
(c_1^T + d_{12}K)X_0 & -\gamma \\
b_1^T & d_{11} & -\gamma
\end{pmatrix} 
\in S_{++}^{n+2},
\]
\[
- \begin{pmatrix}
\text{He}(Y_0(A + Lc_2^T)) & Y_0(b_1 + Ld_{21}) b_1 \\
(b_1 + Ld_{21})^TY_0 & -\gamma \\
c_1^T & d_{11} & -\gamma
\end{pmatrix} 
\in S_{++}^{n+2},
\]
\[
\begin{pmatrix}
X_0 & -I_n \\
-I_n & Y_0
\end{pmatrix} 
\in S_{++}^n.
\]
These can be restated equivalently as
\[
- \begin{pmatrix}
\text{He}(AX) & X_0c_1 \\
c_1^TX_0 & -\gamma \\
b_1^T & d_{11} & -\gamma
\end{pmatrix} + \text{He} \begin{pmatrix}
b_2 \\
d_{12} \\
KX_0 & 0 & 0
\end{pmatrix} 
\in S_{++}^{n+2},
\]
\[
- \begin{pmatrix}
\text{He}(Y_0A) & Y_0b_1 \\
b_1^TY_0 & -\gamma \\
c_1^T & d_{11} & -\gamma
\end{pmatrix} + \text{He} \begin{pmatrix}
c_2 \\
d_{21} \\
L^TY_0 & 0 & 0
\end{pmatrix} 
\in S_{++}^{n+2},
\]
\[
\begin{pmatrix}
X_0 & -I_n \\
-I_n & Y_0
\end{pmatrix} 
\in S_{++}^n.
\]
The above matrix inequalities clearly show that LMI problem \((25)\) is strictly feasible and hence (II) holds.

To prove (I)\(\iff\)(II), suppose (II) holds. Then, from Elimination Lemma \([12, 16]\), there exist \(X, Y, \gamma\) and \(F_1, F_2, F_3, G_1, G_2, G_3\) of appropriate size such that
\[
- \begin{pmatrix}
\text{He}(AX) & X_0c_1 \\
c_1^TX_0 & -\gamma \\
b_1^T & d_{11} & -\gamma
\end{pmatrix} + \text{He} \begin{pmatrix}
b_2 \\
d_{12} \\
F_1 & F_2 & F_3
\end{pmatrix} 
\in S_{++}^{n+2},
\]
\[
- \begin{pmatrix}
\text{He}(Y_0A) & Y_0b_1 \\
b_1^TY_0 & -\gamma \\
c_1^T & d_{11} & -\gamma
\end{pmatrix} + \text{He} \begin{pmatrix}
c_2 \\
d_{21} \\
G_1 & G_2 & G_3
\end{pmatrix} 
\in S_{++}^{n+2},
\]
\[
\begin{pmatrix}
X & -I_n \\
-I_n & Y
\end{pmatrix} 
\in S_{++}^n.
\]
This in particular implies that \(X \in S_{++}^n\) and \(-\text{He}((A + b_2K)X) \in S_{++}^n\) hold with \(K = F_1X^{-1}\) and hence \((A, b_2)\) is stabilizable. Similarly, we have \(Y \in S_{++}^n\) and \(-\text{He}(Y(A + Lc_2^T)) \in S_{++}^n\) hold with \(L = Y^{-1}G_1^T\) and hence \((A, c_2)\) is detectable. It follows that (I) holds, and this completes the proof.

\textit{Appendix B.2. Proof of Lemma [4]}

We will prove that the dual of \((30)\) is strictly feasible. Theorem \([3]\) and the weak duality on LMI problems introduced in \textit{Appendix A} imply that the
It is clear that we obtain no solution \((\gamma, X, Y) \in \mathbb{R} \times S^n_+ \times S^n_+\) such that
\[
\begin{pmatrix}
-\text{He}(A^T_+ X) - \gamma f_+ f^T_+ & 0 \\
0 & -\gamma
\end{pmatrix} \in S^{n+1}_+, \\
-\text{He}(\Omega^T_+ Y) - \gamma g_+ g^T_+ & 0 \\
0 & -\gamma
\end{pmatrix} \in S^{n+1}_+, \\
\gamma & \leq 0, (\gamma, X, Y) \neq (0, O_{n \times n}, O_{n \times n})
\]
(B.1)

It is clear that \(\gamma = 0\) is necessary for (B.1) being valid, and by substituting it, we obtain
\[-\text{He}(A^T_+ X) \in S^n_+, -\text{He}(\Omega^T_+ Y) \in S^n_+, (X, Y) \neq (O_{n \times n}, O_{n \times n})\) (B.2)

As both \(-\Lambda_+\) and \(-\Omega_+\) are Hurwitz stable, (B.2) has no solutions, and thus (B.1) has no solutions. It follows that the dual of (30) is strictly feasible, and hence (30) has an optimal solution.

Appendix B.3. Proof on the zero duality gap in Lemma 8

We use Theorem 9 for this proof. The condition 2 in the first part of Theorem 9 can be described by
\[
\begin{aligned}
&\begin{pmatrix}
Z_{11} & Z_{21}^T \\
Z_{21} & Z_{22}
\end{pmatrix} \in S^{k_1+1}_+, \begin{pmatrix} V_{11} & V_{21}^T \\
V_{21} & V_{22}\end{pmatrix} \in S^{k_2+1}_+, \\
&\begin{pmatrix} W_{11} & W_{21}^T \\
W_{21} & W_{22}\end{pmatrix} \in S^{k_1+k_2}_+, W_{11} = \text{He}(\Lambda_+ Z_{11}), W_{22} = \text{He}(\Omega_+ V_{11}),
\end{aligned}
\]
(B.3)

Any solution of (B.3) satisfies \(Z_{22} = V_{22} = 0\). Substituting them, we reformulate (B.3) into
\[
\begin{aligned}
&Z_{11} \in S^{k_1}_{++}, V_{11} \in S^{k_2}_{++}, \begin{pmatrix} W_{11} & W_{21}^T \\
W_{21} & W_{22}\end{pmatrix} \in S^{k_1+k_2}_+, J_+ \bullet W_{21} \geq 0, \\
&W_{11} = \text{He}(\Lambda_+ Z_{11}), W_{22} = \text{He}(\Omega_+ V_{11}), f^T_+ Z_{11} f_+ + g^T_+ V_{11} g_+ = 0
\end{aligned}
\]
(B.4)

The next lemma is useful in analyzing (B.4):

Lemma 17. 1. If \((A, b_2)\) is stabilizable, then \((\Lambda_+, f_+)\) is controllable.
2. If \((A, e^T_2)\) is detectable, then \((\Omega_+, g_+)\) is controllable.
3. If \((\Lambda_+, f_+)\) is controllable, then there does not exist any \(Z_{11} \in S^{k_1}_{++} \backslash \{O_{k_1}\}\) such that \(f^T_+ Z_{11} f_+ = 0\) and \(\text{He}(\Lambda_+ Z_{11}) \in S^{k_1}_{++}\).
4. If \((\Omega_+, g_+)\) is controllable, then there does not exist any \(V_{11} \in S^{k_2}_{++} \backslash \{O_{k_2}\}\) such that \(g^T_+ V_{11} g_+ = 0\) and \(\text{He}(\Omega_+ V_{11}) \in S^{k_2}_{++}\).
Proof. We prove 1 and 3 only because we can prove 2 and 4 by a similar manner.

To prove 1 by contradiction, suppose that there exists $\lambda > 0$ such that $v^T \Lambda_+ = \lambda v^T$ and $v^T f_+ = 0$. Then, it follows from (20) that $(v^T S_1^T)A = \lambda(v^T S_1^T)$ and $(v^T S_1^T)b_2 = 0$. This implies that $(A, b_2)$ is not stabilizable, and thus we obtain a contradiction.

Again to prove 3 by contradiction, suppose that there exists a non-zero $Z_{11} \in S^n_+$ such that $f^T Z_{11} f_+ = 0$ and $\text{He}(\Lambda_+ Z_{11}) \in S^n_+$. Then, since the latter condition can be seen as the Lyapunov equation $\text{He}(\Lambda_+ Z_{11}) = W$ by introducing $W \in S^n_+$ and since $-\Lambda_+$ is Hurwitz stable, we can solve this equation explicitly as

$$Z_{11} = \int_0^\infty \exp(-\Lambda_+ t)W \exp(-\Lambda_+^T t) dt.$$ 

It follows from $f^T Z_{11} f_+ = 0$ that $W \exp(-\Lambda_+^T t)f_+ = 0$ for all $t \geq 0$. If $W$ is the zero matrix, then $Z_{11}$ is also zero, which contradicts the assumption that $Z_{11}$ is nonzero. Hence $W$ is nonzero. This implies that there exists $p \in \mathbb{C}^{k_1} \setminus \{0\}$ such that $p^T \exp(-\Lambda_+^T t)f_+ = 0$ for all $t \geq 0$. This contradicts the controllability of $(\Lambda_+, f_+)$.

It follows from Lemma 17 and Assumption 1 that (3.3) has no nonzero solutions, and thus we see from Theorem 9 that (40) is strictly feasible. Therefore the zero duality gap between (40) and (39) holds from Theorem 8. This implies that the existence of an optimal solution of (39).

The existence of an optimal solution in (40) can be proved similarly to Lemma 4.

Appendix C. Proofs of technical lemmas

Appendix C.1. Proof of Lemma 2

Since the relative degree of the dual system (20) is equal to $r$, we prove only the first part of the Lemma 2. We note that the submatrices $Q_{11}$ and $\left(\begin{array}{cc} Q_{11} & Q_{11} \\ Q_{21} & Q_{22} \end{array}\right)$ are of full column rank.

It follows from 3 in Remark 2 that we can set $\left(\begin{array}{c} Q_{12} \\ Q_{22} \end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \end{array}\right)$ when $r = 0$. We consider the case $r > 0$. Clearly, $\left(\begin{array}{c} Q_{12} \\ Q_{22} \end{array}\right)$ in (22) satisfies (17). Thus it is sufficient to prove that $Q$ is non-singular. For this, we consider $\alpha \in \mathbb{C}^{n-r}$ and $\beta = (\beta_1, \ldots, \beta_{r+1})^T \in \mathbb{C}^{r+1}$ that satisfies

$$\left(\begin{array}{c} Q_{11} \\ Q_{21} \\ Q_{22} \end{array}\right) \left(\begin{array}{c} \alpha \\ \beta \end{array}\right) = 0.$$ 

We note $d = 0$ and thus $c^T Q_{11} = 0$ because of (13). Using this, we obtain $c^T Q_{12} \beta = 0$. Furthermore, from the definition of the relative degree, we obtain $\beta_{r+1} = 0$. Next, we see $c^T A Q_{11} = 0$ from (19). Thus we obtain $\beta_r = 0$ from $c^T A Q_{12} \beta = 0$ and $\beta_{r+1} = 0$. Applying this procedure repeatedly, we obtain

$$\left(\begin{array}{c} Q_{11} \\ Q_{21} \\ Q_{22} \end{array}\right) \left(\begin{array}{c} \alpha \\ \beta_1 \end{array}\right) = 0.$$
Since $Q_{11}$ is of full column rank, $\alpha = 0$ and $\beta_1 = 0$. Therefore, $Q$ is non-singular because $Q$ is square and all the columns are linearly independent.

**Appendix C.2. Proof of Lemma C.1**

We first prove the “if” part. Suppose $h_{1+} = 0$, $h_{2+} = 0$ and $J_+ = O_{n \times n}$. Then LMI problem (30) can be reformulated as

$$
\inf_{\gamma} \left\{ \begin{array}{l}
- \text{He}(\Lambda^T X) - \gamma f^T f^T \\
- \text{He}(\Omega^T Y) - \gamma g^T g^T \\
\end{array} \right\} \in S_{n+1}^{n+1},
$$

$$
\begin{array}{l}
0 \\
0 \\
\end{array}
$$

From this form it is very clear that $(\gamma, X, Y) = (0, O_{n \times n}, O_{n \times n})$ is an optimal solution achieving $\gamma^* = 0$.

We next prove the “only if” part. If $\gamma^* = 0$, it follows from Lemma C.1 that LMI problem (30) has an optimal solution $(0, X, Y)$. Then $(X, Y)$ satisfies

$$
- \left( \begin{array}{c}
\text{He}(\Lambda^T X) \\
h^T_{1+} \\
\end{array} \right) \in S_{n+1}^{n+1}, - \left( \begin{array}{c}
\text{He}(\Omega^T Y) \\
h^T_{2+} \\
\end{array} \right) \in S_{n+1}^{n+1}, \left( \begin{array}{c}
X \\
-J \\
Y \\
\end{array} \right) \in S_{n+1}^{2n}.
$$

From these matrix inequalities we readily obtain $h_{1+} = 0$, $h_{2+} = 0$. In addition, we obtain the following equations.

$$
\begin{array}{l}
\text{He}((-\Lambda^T)X) = X, X \in S_{n+1}^{n}, X \in S_{n+1}^{n}; \\
\text{He}((-\Omega^T)Y) = Y, Y \in S_{n+1}^{n}, Y \in S_{n+1}^{n}.
\end{array}
$$

These equations can be seen as the Lyapunov equations, and thus we have

$$
X = - \int_0^{+\infty} \exp(-\Lambda t) \hat{X} \exp(-\Lambda t) dt \quad \text{and} \quad Y = - \int_0^{+\infty} \exp(-\Omega t) \hat{Y} \exp(-\Omega t) dt.
$$

Therefore $X = O_{n \times n}$ and $Y = O_{n \times n}$ because $\hat{X}, \hat{Y} \in S_{n+1}^{n}$ and $X, Y \in S_{n+1}$. Then $J_+$ must be the zero matrix. This completes the proof.

**Appendix C.3. Proof of Lemma C.2**

We define the Lagrange function $L$ for (30) as follows:

$$
L(\gamma, \hat{X}, \hat{Y}, Z, V, W) = \gamma + Z \bigg( \text{He}(\Lambda^T \hat{X}) - \gamma f^T f^T h^T_{1+} - \gamma \bigg) - W \bigg( \hat{X} - J^T \hat{Y} \bigg)
$$

$$
+ V \bigg( \text{He}(\Omega^T \hat{Y}) - \gamma g^T g^T h^T_{2+} - \gamma \bigg).
$$

The Lagrange function $L$ can be reformulated as follows:

$$
L(\gamma, \hat{X}, \hat{Y}, Z, V, W) = 2(Z_{21} \bullet h^T_{1+} + V_{21} \bullet h^T_{2+} + W_{21} \bullet J) + \hat{X} \bullet (\text{He}(\Lambda Z_{11}) - W_{11}) + \hat{Y} \bullet (\text{He}(\Omega V_{11}) - W_{22}) + \gamma(1 - f^T Z_{11} f - Z_{22} - g^T V_{11} g - V_{22}).
$$
Then for any \((Z, V, W) \in \mathbb{S}_+^{n+1} \times \mathbb{S}_+^{n+1} \times \mathbb{S}_+^{2n}\), we consider the following Lagrange relaxation problem:

\[
\inf \left\{ L(\gamma, \hat{X}, \hat{Y}, Z, V, W) : \gamma \in \mathbb{R}, \hat{X}, \hat{Y} \in \mathbb{S}^n \right\}
\]

\[
= \left\{ \begin{array}{ll}
2(Z_{21} \cdot h_1^T + V_{21} \cdot h_2^T + W_{21} \cdot J) & \text{if } f^T Z_{11} f + Z_{22} + g^T V_{11} g + V_{22} = 1, \\
-\infty & \text{o.w.}
\end{array} \right.
\]

Hence we obtain (51). The first equation of (53) is proved from the positive values, the unique solution of the Sylvester equation is

\[
Z = \text{He}(\Lambda Z_{11}), W = \text{He}(\Omega V_{11}),
\]

Theorem 3 that the duality gap between (30) and (38) is zero, and (38) has an optimal solution. This completes the proof.

**Appendix C.4. Proof of Lemma 3**

First of all, we prove (51) and the first equation in (53). Any feasible solution \((W_{ij}, Z_{ij}, V_{ij})\) satisfies

\[
W_{11} = \text{He}\left( \begin{pmatrix} \Lambda_0 & Z_{21} \\ -Z_{21} & \Lambda \end{pmatrix} \right) = \begin{pmatrix} \text{He}(\Lambda_0 Z_{11}) & \Lambda_0 Z_{21}^T + Z_{21}^T \Lambda^T \\ \Lambda Z_{21} + Z_{21} \Lambda_0^T & \text{He}(\Lambda Z_{22}) \end{pmatrix}.
\]

For simplicity, we assume \(m_1 = 2\). Other cases can be proved by a similar manner. Then \(\Lambda_0\) and \(Z_{11}\) can be written by

\[
\Lambda_0 = \begin{pmatrix} F(\lambda_1) & \Lambda_0 Z_{11} \\ F(\lambda_2) & \Lambda_0 Z_{21} \end{pmatrix}, Z_{11} = \begin{pmatrix} Z_{11}^T \\ Z_{21}^T \end{pmatrix} \text{ and } Z_{11} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}.
\]

Hence we have

\[
\text{He}(\Lambda_0 Z_{11}) = \begin{pmatrix} \text{He}(F(\lambda_1) Z_{11}^T) & * \\ (F(\lambda_1) Z_{11}^T + Z_{11} F(\lambda_1)^T) \text{He}(F(\lambda_2) Z_{11}^T) \end{pmatrix},
\]

\[
\text{He}(F(\lambda_1) Z_{11}) = \begin{pmatrix} 2\text{Im}(\lambda_1) a_2 & \text{Im}(\lambda_1)(a_3 - a_1) \\ \text{Im}(\lambda_1)(a_3 - a_1) & -2\text{Im}(\lambda_1) a_2 \end{pmatrix}.
\]

Since (C.1) is positive semidefinite, the diagonal elements in (C.2) must be nonnegative, and thus \(a_2 = 0\) and \(a_1 = a_3\). This implies that (C.2) is the zero matrix. Therefore we obtain \(F(\lambda_2) Z_{11}^T + Z_{11} F(\lambda_1)^T = O_{2 \times 2}\) from the positive semidefiniteness of (C.1). The first equation can be regarded as the Sylvester equation on \(Z_{11}^T\). As we have assumed that invariant zeros on the imaginary axis are distinct from each other, we have \(\lambda_1 \neq \lambda_2\), and thus \(F(\lambda_1)\) has no common eigenvalues with \(-F(\lambda_2)^T\). Thus \(Z_{11}^T\) the zero matrix, see [15, Theorem 2.4.4.1] for the existence and uniqueness of the solution in the Sylvester equation. Similarly, we obtain \(Z_{11}^3\) is diagonal with nonnegative elements.

Also since \(\text{He}(\Lambda_0 Z_{11}) = O_{4 \times 4}\), the matrix \(\Lambda Z_{21} + Z_{21} \Lambda_0^T\) is zero. This equation is also the Sylvester equation. Since \(\Lambda\) and \(-\Lambda_0^T\) have no common eigenvalues, the unique solution of the Sylvester equation is \(Z_{21} = O_{(n-2m_1) \times 2m_1}\). Hence we obtain (51). The first equation of (53) is proved from the positive semidefiniteness of \(W\) and \(\text{He}(\Lambda_0 Z_{11}) = O\). By following similar lines, the rest equations (52) and those in (53) can be proved.
Appendix C.5. Proof of Lemma

We prove by a similar manner to Lemma that the Lagrange dual of \(55\) is \(54\). We define the Lagrange function \(L\) for \(55\):

\[
L(\gamma, \hat{X}, \hat{Y}, U_{ij}, U_{ij}^Y, Z, V, \hat{W}, z, v)
= \gamma + Z \cdot \begin{pmatrix}
(U_{11}^X) + \gamma f_0 f_0^T \quad (U_{21}^X) + \gamma g_0 g_0^T \\
(U_{11}^Y) + \gamma g_0 g_0^T \quad \text{He}(\Lambda^T \hat{X}) - \gamma f f^T
\end{pmatrix}
+ V \cdot \begin{pmatrix}
(U_{21}^Y) + \gamma g_0 g_0^T \quad \text{He}(\Omega^T \hat{Y}) - \gamma g g^T
\end{pmatrix}
+ \sum_{j=1}^{m_1} z_j ((U_{11}^X)_{2j-1,2j-1} + (U_{11}^Y)_{2j,2j}) + \sum_{j=1}^{m_2} v_j ((U_{11}^Y)_{2j-1,2j-1} + (U_{11}^Y)_{2j,2j})
= \begin{pmatrix}
h_1^T + h_1^T \cdot Z_{31} + h_1^T \cdot Z_{32} + h_2^T \cdot V_{31} + h_2^T \cdot V_{32} + J \cdot \hat{W}_{21}
+ \hat{X} \cdot (\text{He}(AZ_{22}) - \hat{W}_{11}) + \hat{Y} \cdot (\text{He}(\Omega V_{22}) - \hat{W}_{22})
+ \gamma \left(1 - (f_0 f_0^T) (Z_{21} \quad Z_{22}) (f_0 f_0^T) - Z_{33} - (g_0 g_0^T) (V_{21} \quad V_{22}) (g_0 g_0^T) - V_{33}\right)
+ \sum_{i \neq j} (U_{11}^X)_{ij} (Z_{11})_{ij} + \sum_{i \neq j} (U_{11}^Y)_{ij} (V_{11})_{ij} + \sum_{j=1}^{m_1} (U_{11}^Y)_{2j-1,2j-1} ((Z_{11})_{2j-1,2j-1} + z_j)
+ \sum_{j=1}^{m_2} (U_{11}^Y)_{2j,2j} ((Z_{11})_{2j,2j} + z_j) + \sum_{j=1}^{m_2} (U_{11}^Y)_{2j-1,2j-1} ((V_{11})_{2j-1,2j-1} + v_j)
+ \sum_{j=1}^{m_2} (U_{11}^Y)_{2j,2j} ((V_{11})_{2j,2j} + v_j) + 2(U_{21}^X \cdot Z_{21} + U_{21}^Y \cdot V_{21}) \end{pmatrix}
\]

Then for any \((Z, V, W, z, v) \in S_{+}^{n+1} \times S_{+}^{n+1} \times S_{+}^{n_0} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}\), the Lagrange relaxation problem can be formulated as

\[
\inf \left\{ L(\gamma, \hat{X}, \hat{Y}, U_{ij}, U_{ij}^Y, Z, V, \hat{W}, z, v) : \gamma \in \mathbb{R}, \hat{X}, \hat{Y} \in S_{+}^{n-2m_1}, Y \in S_{+}^{n-2m_2}, U_{11}^X \in \mathbb{R}^{2m_1}, U_{11}^Y \in \mathbb{R}^{2m_2}, U_{21}^X \in \mathbb{R}^{(n-2m_1) \times 2m_1}, U_{21}^Y \in \mathbb{R}^{(n-2m_2) \times 2m_2} \right\}
\]

\[
= \begin{cases} 
2 \left(h_{10}^T \cdot Z_{31} + h_1^T \cdot Z_{32} + h_2^T \cdot V_{31} + h_2^T \cdot V_{32} + J \cdot \hat{W}_{21}\right) & \text{if } (Z, V, \hat{W}, z, v) \in \mathcal{F} \\
-\infty & \text{o.w.}
\end{cases}
\]

As it is easy to obtain the dual \(54\) from \(\mathcal{F}\), we omit the detail. The zero duality gap and the existence of an optimal solution of \(54\) can be proved similarly to Appendix B.3.
Appendix C.6. Proof of Lemma 14

We define the Lagrange function \( L \) for (67):

\[
L(\gamma, X, Y, Z, V, W) = \gamma + Z \begin{pmatrix} \text{He}(\Lambda^TX^T XS) - \gamma f^Tf^T & \text{He}(\tilde{P}_1^T XP_1) - \gamma p_1p_1^T \\ \text{He}(\tilde{P}_1^T XS + P_1^T XSA - \gamma p_1f^T) & \tilde{h}_1^{T} \end{pmatrix} + V \begin{pmatrix} \text{He}(\tilde{P}_2^T Y^T T) - \gamma g^Tg^T & \text{He}(\tilde{N}_1^T YP_2) - \gamma p_2p_2^T \\ \text{He}(\tilde{P}_2^T Y^T T - \gamma g^Tg^T) & \tilde{h}_2^{T} \end{pmatrix} - W \begin{pmatrix} X - I_n & -I_n \\ Y & \end{pmatrix}.
\]

By applying a similar discussion in Lemma 6, we can obtain the Lagrange relaxation problem for \((Z, V, W) \in S_+^{n+1} \times S_+^{n+1} \times S_+^{2n}\) and the Lagrange dual problem. We see that the Lagrange dual is equivalent to (70). The zero duality gap between (67) and (70) follows from Theorem 3. In fact, the proof is independent of the computation of the perpendicular matrices in (25). In addition, it follows from Assumption 1 and Theorem 3 that (70) has an optimal solution.

Appendix C.7. Proof of Lemma 13

We prove only the statement on \(Z_{ij}\) because we can also prove the statement on \(V_{ij}\) in a similar manner. Since we use equations (63), (65) and (68) in this proof, we rewrite here.

\[
S^T A^r b_2 = 0 \quad (r = 0, \ldots, r_1 - 1)
\]
\[
(P_1^T A^r b_2)_{j} = 0 \quad (r = 0, \ldots, r_1 - 1, j = 1, \ldots, r_1 - r)
\]
\[
(\tilde{P}_1^T A^r b_2)_{j} = 0 \quad (r = 0, \ldots, r_1 - 2, j = 1, \ldots, r_1 - r - 1).
\]

We focus on the following constraint of (70):

\[
W_{11} = \text{He}(SA(Z_{11}S^T + Z_{21}P_1^T) + \tilde{P}_1(Z_{21}S^T + Z_{22}P_2^T)).
\]

In addition, we use the positive semidefiniteness of \(Z_{22}\) and \(W_{11}\), and the following well-known facts.
**Fact 1.** If there exists nonzero \( v \in \mathbb{R}^n \) such that \( v^T W_{11} v = 0 \), then \( W_{11} v = 0 \).

**Fact 2.** Let \( k \in \{1, \ldots, r \} \). If \((Z_{22})_{kk} = 0\), then \((Z_{22})_{kj} = (Z_{22})_{jk} = 0\) for all \( j = 1, \ldots, r \).

First of all, it follows from (65) and (68) that we have \( S^T b_2 = 0 \) and \( P_1^T b_2 = 0 \), and thus \( b_2^T W_{11} b_2 = 0 \). \( W_{11} b_2 = 0 \) follows from Fact 1.

If \( r_1 = 1 \), we have \( P_1 = 0 \) and \( \bar{P}_1 = c \), and thus \( W_{11} = \text{He}(S\Lambda Z_{11} + c_1 Z_{11}^T S^T) \). Multiplying \( b_2^T \) from the left side, we obtain \( b_2^T W_{11} = (b_2^T c_1) Z_{21}^T S^T = 0 \). This implies \((Z_{21})_{r_1,j} = 0\) for all \( j = 1, \ldots, n - r_1 \) because \( S \) is of full column rank and \( c_1^T b_2 \neq 0 \). This is the proof of the case \( r_1 = 1 \).

Secondly, we consider the case \( r_1 > 1 \). We prove the statement on \( Z_{22} \) by induction. For this, we first prove \((Z_{22})_{r_1,r_1} = (Z_{22})_{r_2,r_2} = 0\) for all \( r = 1, \ldots, r_1 \).

By Fact 2 and (65), respectively are equal to

\[
\begin{pmatrix} r_1 - 1 & 1 \\ 0 & b_2^T (A^T)^{r_1-1} c_1 \end{pmatrix} = (b_2^T (A^T)^{r_1-1} c_1) e_{r_1}, \quad \text{(C.3)}
\]

where \( e_{r_1} \in \mathbb{R}^{r_1} \) is the \( r_1 \)th unit vector. It follows from \( W_{11} b_2 = 0 \), \( S^T b_2 = 0 \), \( P_1^T b_2 = 0 \) and \( (C.3) \) that we have \( W_{11} b_2 = (S Z_{21} + P_1 Z_{22})(b_2^T (A^T)^{r_1-1} c_1) e_{r_1} = 0 \). Multiplying \((A b_2)^T\) into the left side of the above equation, we obtain \((b_2^T (A^T)^{r_1-1} c_1)^2 e_{r_1}^T Z_{22} e_{r_1} = 0\) from \((C.3)\). Hence it follows from Fact 2 that \((Z_{21})_{1,r} = (Z_{22})_{r_{r_1}} = 0\) for all \( r = 1, \ldots, r_1 \). Moreover, \((Z_{21})_{r,r} = 0\) for all \( r = 1, \ldots, n - r_1 \) and \((Z_{22})_{r_{r_1}} = 0\) because we have \( b_2^T (A^T)^{r_1-1} c_1 \neq 0 \) and the positive semidefiniteness of \( Z \).

Thirdly, we fix \( 1 \leq r \leq r_1 - 2 \) arbitrary, and assume that \( Z_{21} \) and \( Z_{22} \) form

\[
Z_{21}^T = \begin{pmatrix} \tilde{Z}_{21}^T & O_{(n-r_1) \times r} \end{pmatrix} \quad \text{and} \quad Z_{22} = \begin{pmatrix} (r_1 - r) & (r_1 - r) \\ r & 0 \end{pmatrix} \begin{pmatrix} \tilde{Z}_{22} & O_{(r_1-r) \times (r_1-r)} \\ O_{r \times (r_1-r)} & O_{r \times r} \end{pmatrix}.
\]

Then it follows from \( S^T A^T b_2 = 0 \) from (63) that we have

\[
(A^T b_2)^T \text{He}(S A (Z_{11} S^T + Z_{21}^T P_1^T) + \bar{P}_1 Z_{21}^T S^T)(A^T b_2) = 0.
\]

In addition, we see from (62) that the first \((r_1-r)\) elements of the vector \( P_1^T A^T b_2 \) are zero. Hence, it follows from the structure of \( Z_{22} \) that we have

\[
(A^T b_2)^T W_{11} (A^T b_2) = \text{He}((A^T b_2)^T \bar{P}_1 Z_{22} Z_{21} P_1^T (A^T b_2)) = 0. \quad \text{(C.4)}
\]

\( W_{11} (A^T b_2) = 0 \) follows from Fact 1 (63), (65) and this equation imply that

\[
0 = W_{11} (A^T b_2) = (S Z_{21}^T + P_1 Z_{22}) \bar{P}_1 (A^T b_2). \quad \text{(C.5)}
\]

In addition, we have from (63) and (68),

\[
\begin{pmatrix} (r_1 - r - 1) & 1 \\ 0 & c_1^T A^T c_{r_1-1} b_2 \end{pmatrix} = \begin{pmatrix} (r_1 - r - 1) & 1 \\ 0 & c_1^T A^T c_{r_1-1} b_2 \end{pmatrix}.
\]
Multiplying \((\mathbf{A}^{r+1} \mathbf{b}_2)^T\) from the left side of (C.5), we obtain
\[
0 = b_2^T (\mathbf{A}^r)^{r+1} P_1 Z_{22} \hat{P}_1^T (\mathbf{A}^r \mathbf{b}_2) = (Z_{22})_{(r_1 - r), (r_1 - r)} (c_1^T \mathbf{A}^{r_1-1} \mathbf{b}_2)^2.
\]
Therefore from Fact 2, we obtain \((Z_{22})_{(r_1 - r), j} = (Z_{22})_{j, (r_1 - r)} = 0\) for all \(j = 1, \ldots, r_1 - r\). This means that \((Z_{22})_{ij} = 0\) for all \(1 \leq i, j \leq r_1\) except for \((i, j) = (1, 1)\), \((Z_{21})_{ij} = 0\) for all \(2 \leq i \leq r_1, 1 \leq j \leq n - r_1\) and \((Z_{32})_{r} = 0\) for \(r = 2, \ldots, r_1\).

Finally, we prove that \((Z_{21})_{ij} = 0\) for all \(j = 1, \ldots, n - r_1\). As (63) and (65) hold for \(r = r_1 - 1\), we have \((\mathbf{A}^{r_1-1} \mathbf{b}_2)^T W_{11} (\mathbf{A}^{r_1-1} \mathbf{b}_2) = 0\), and thus \(W_{11} (\mathbf{A}^{r_1-1} \mathbf{b}_2) = 0\). We have already known from the induction that
\[
Z_{21}^T (\tilde{\mathbf{z}} \ O_{(n-r_1) \times (r_1 - 1)}) \text{ and } Z_{22} = \begin{pmatrix}
0 \\
O_{(r_1 - 1) \times (r_1 - 1)}
\end{pmatrix}
\]
for some \(\tilde{\mathbf{z}} \in \mathbb{R}^{n-r_1}\) and \(\mathbf{z} \in \mathbb{R}\). We substitute them to \(W_{11} (\mathbf{A}^{r_1-1} \mathbf{b}_2) = 0\). Then as we have
\[
Z_{21}^T P_1^T = (\tilde{\mathbf{z}} \ O_{(n-r_1) \times (r_1 - 1)}) \begin{pmatrix} 0 \\ * \end{pmatrix} = O_{(n-r_1) \times n},
\]
\[
Z_{21}^T \hat{P}_1^T = (\tilde{\mathbf{z}} \ O_{(n-r_1) \times (r_1 - 1)}) \begin{pmatrix} c_1^T \\ * \end{pmatrix} = \tilde{\mathbf{z}} c_1^T,
\]
\[
Z_{22}^T P_1^T = \begin{pmatrix} \mathbf{z} \\ 0 \end{pmatrix} O_{(r_1 - 1) \times (r_1 - 1)} \begin{pmatrix} 0 \\ * \end{pmatrix} = O_{r_1 \times n},
\]
we obtain
\[
W_{11} (\mathbf{A}^{r_1-1} \mathbf{b}_2) = (S \Lambda Z_{21}^T P_1^T + S Z_{21}^T \hat{P}_1^T + \hat{P}_1 Z_{22} P_1^T + P_1 Z_{22} \hat{P}_1^T) (\mathbf{A}^{r_1-1} \mathbf{b}_2)
\]
\[
= (c_1^T \mathbf{A}^{r_1-1} \mathbf{b}_2) S \tilde{\mathbf{z}}^T = 0.
\]
As \(S\) is of full column rank and \(c_1^T \mathbf{A}^{r_1-1} \mathbf{b}_2 \neq 0\), we have \(\tilde{\mathbf{z}} = 0\). Therefore \(Z_{21} = O\).

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