Large deviations for a stochastic Landau-Lifshitz equation, extended version

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Abstract
We study a stochastic Landau-Lifshitz equation on a bounded interval and with finite dimensional noise; this could be a simple model of magnetization in a needle-shaped domain in magnetic media. After showing that a unique, regular solution exists, we obtain a large deviation principle for small noise asymptotics of solutions using the weak convergence method. We then apply the large deviation principle to show that small noise in the field can cause magnetization reversal and also to show the importance of the shape anisotropy parameter for reducing the disturbance of the magnetization caused by small noise in the field.

1 Introduction
In this paragraph we give a brief description of magnetic media to motivate the stochastic Landau-Lifshitz model of magnetization we study in this paper. For more information on magnetic media, a readable text is [18]. Magnetic memory is made up of small single-domain ferromagnetic particles. Two types of magnetic media are commonly used. Thin film media, used in hard disks, are made of magnetic alloys 10 to 50 nm thick, with single-domain grains of a similar size. Particulate media, used in tape, are made of single-domain, needle-shaped particles of iron or chromium oxide about 300 to 700 nm long and about 50 nm in diameter. Each magnetic domain has two stable states of opposite magnetization and these states are used to represent the bits 0 and 1. For particulate media, the shape anisotropy energy is important for the stability of these states; shape anisotropy energy is minimized when the magnetization points along the needle’s axis. It is observed that, at higher temperatures, the magnetization can spontaneously reverse, causing loss of data. To investigate thermally induced magnetization reversal, it is natural to use a stochastic model of the magnetization.

Kohn, Reznikoff and Vanden-Eijnden [13] modelled the magnetization, m, in a thin film. They assumed that m is uniform in the piece of material at all times and has constant Euclidean norm, for convenience $|m(t)|_{\mathbb{R}^3} = 1$ for all $t \geq 0$ (this means that the magnetization
is always saturated). They started from the equation for the zero temperature dynamics:

\[ \dot{m} = m \times g - \alpha m \times (m \times g), \]  

(1)

where \( \times \) denotes the vector cross product in \( \mathbb{R}^3 \), \( \alpha \) is a positive real parameter and \( g \) is the effective field, \( g := -\frac{\partial E}{\partial m} \), \( E \) being the total magnetic energy, which is a function of \( m \) and the applied field. The first term on the right hand side of equation (1) is called the gyromagnetic term and the second term on the right hand side is called the damping term.

The aim of Kohn, Reznikoff and Vanden-Eijnden was to study the combined effects of a pulsing applied field and random thermal fluctuations in the field, modelled using standard white noise in \( \mathbb{R}^3 \). Their stochastic model was

\[ \dot{m} = m \times (g + \varepsilon^{\frac{1}{2}} \sqrt{\frac{2\alpha_{1}}{1+\alpha}} W) - \alpha m \times (m \times (g + \varepsilon^{\frac{1}{2}} \sqrt{\frac{2\alpha_{2}}{1+\alpha}} W)), \]  

(2)

where \( \varepsilon \) represents dimensionless temperature and \( W \) is a Wiener process in \( \mathbb{R}^3 \) which is the integrator in a Stratonovich integral. Kohn, Reznikoff and Vanden-Eijnden made a detailed computational and theoretical (using large deviations theory) study of the behaviour of the solution. At the end of their paper, they remarked that little is known about the behaviour of the solutions of stochastic Landau-Lifshitz equations which do not assume that the magnetization is uniform on the space domain.

Magnetic domains in thin film media are three dimensional grains. For this reason, Brzeźniak, Goldys and Jegaraj [3] studied the stochastic Landau-Lifshitz equation:

\[ du = [u(t) \times \Delta u(t) - \alpha u(t) \times (u(t) \times \Delta u(t))] + (u(t) \times h) \times h] dt + u(t) \times h dW(t), \]  

(3)

which is a simplified model of the magnetization, \( u \), in a bounded three-dimensional piece of material as a function of space and time. Here the term \( \Delta u \) is the field due to the exchange energy, from interaction of atomic spins. For simplicity, other magnetic energies are ignored and noise is introduced in the gyromagnetic term only, giving rise to a Stratonovich integral; the integrator is a real-valued Wiener process, \( W \), and \( h \) is a fixed \( \mathbb{R}^3 \)-valued function on the space domain (the same argument would hold with a Wiener process in \( \mathbb{R}^n \) and functions \( h_1, \ldots, h_n \), for any \( n \in \mathbb{N} \)). Existence of a weak martingale solution of equation (3) was shown, however, uniqueness was not shown.

In this article, we study a stochastic Landau-Lifshitz equation on a bounded interval of the real line. It can be thought of as a simple model of the magnetization in a needle-shaped magnetic domain in particulate media. For simplicity, we only take account of the fields due to the exchange energy and the shape anisotropy energy. Working with a one dimensional space domain means that we can prove strong results even with noise in both the gyromagnetic and damping terms: existence and uniqueness of solutions, as well as further regularity of solutions and a large deviation principle for small noise asymptotics. The large deviation principle enables us to show that noise in the field causes magnetization reversal; it also allows us to estimate the effect of the shape anisotropy parameter in reducing the disturbance caused by noise. The results we obtain provide a foundation for the computational study of stochastic Landau-Lifshitz models with one dimensional space
applied magnetic field is taken to be $H$. Maps:

For magnetization by $z(t)$, we specify the appropriate spaces. Let $\Omega$ be a bounded interval of the real line. We use the notation $H^2 := L^2(\Lambda; \mathbb{R}^3)$, $H^1 := H^1(\Lambda; \mathbb{R}^3)$ and $H^2 := H^2(\Lambda; \mathbb{R}^3)$ for spaces of functions mapping $\Lambda$ into $\mathbb{R}^3$. Norms are denoted by $| \cdot |$ and a pair of angle brackets, $\langle \cdot, \cdot \rangle$, denotes an inner product or pairing of elements from a space and its dual; we use subscripts on norms and angle brackets to specify the appropriate spaces.

The space domain for our stochastic Landau-Lifshitz equation is $\Lambda$. Fix the time horizon $T \in (0, \infty)$ and the positive damping parameter $\alpha \in (0, \infty)$. The function $u_0 \in H^1$, the initial magnetization such that $|u_0(x)|_{R^3} = 1$ for all $x \in \Lambda$ (this means that the initial magnetization is saturated). Fix the function $h \in H^1$, which describes space dependence of the noise. Let $\varepsilon \in [0, 1]$ be a parameter which will go to zero when we study small noise asymptotics. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(W(t))_{t \in [0,T]}$ be a real-valued Wiener process defined on $(\Omega, \mathcal{F}, P)$. Define the filtration generated by $(W(t))_{t \in [0,T]}$: $\mathcal{G}^2_t := \sigma(W(r) : r \in [0, t])$ for each $t \in [0, T]$; let $(\mathcal{G}_t)_{t \in [0,T]}$ be the usual augmented filtration. For each $t \in [0, T]$ define the evaluation map on continuous functions $z_t : C([0, T]; \mathbb{R}) \to \mathbb{R}$ by $z_t(u) := u(t)$ for all $u \in C([0, T]; \mathbb{R})$; define the filtration generated by evaluation maps: $\mathcal{H}^0_t := \sigma(z_r : r \in [0, t])$ for each $t \in [0, T]$. Let $F : [0, T] \times C([0, T]; \mathbb{R}) \to \mathbb{R}$ be a $(\mathcal{H}^0_t)$-predictable function such that for some positive real number $M$,

$$\int_0^T F^2(t, u) dt \leq M \text{ for all } u \in C([0, T]; \mathbb{R}).$$

(4)

For magnetization $y = (y_1, y_2, y_3)^T \in H^1$, the total magnetic energy in the absence of an applied magnetic field is taken to be

$$E(y) := \frac{1}{2} \int_{\Lambda} |Dy(x)|_{R^3}^2 \, dx + \frac{1}{2} \beta \int_{\Lambda} (y_3^2(x) + y_3^2(x)) \, dx,$$

where the first term on the right hand side is the exchange energy, the second term is a simplified form of the shape anisotropy energy and $\beta$ is a positive real constant. Guided by (2), we arrive at the following formal stochastic initial value problem for the magnetization $(y(t))_{t \in [0,T]}$:

$$\dot{y}(t) = y(t) \times (\Delta y(t) - \beta(0, y_2(t), y_3(t))^T + \varepsilon^2 h \dot{W}(t))$$

$$- \alpha y(t) \times (y(t) \times (\Delta y(t) - \beta(0, y_2(t), y_3(t))^T + \varepsilon^2 h \dot{W}(t)))$$

(5)

subject to the boundary condition $(Dy(t))(x) = 0$ for all $x \in \partial \Lambda$ (the endpoints of the interval $\Lambda$) and for all $t \in (0, T]$. When $\varepsilon = 0$, the problem is deterministic and has been investigated by Carbou and Fabrie [5], who even considered two and three dimensional domains. When $\varepsilon \in (0, 1]$, the problem is formulated rigorously using a stochastic integral equation, with Stratonovich integrals. We follow Duan and Millet [9] and Chueshov and
Millet and introduce into the equation additional ‘control terms’ involving the function $F$ which satisfies . No physical significance is attached to these control terms (to obtain the model of the physical system, just set $F$ to zero), but having them in the equation enables us to find the small noise asymptotics of solutions of the model of the physical system. Except in section where we use a large deviation principle to study the dynamics of the magnetization, we omit the shape anisotropy field $-\beta(0, y_2, y_3)^T$ from the stochastic integral equation; including it (or any field of the form $L(y)$, where $L$ is a bounded linear operator on $\mathbb{H}^1$) does not complicate the proofs of existence, uniqueness, regularity and small noise asymptotics of solutions, but it makes equations longer. Here then is the stochastic integral equation we study until section


definite and symmetric and compact and that the fractional power Hilbert space in a weak sense, as described below. We define the linear operator $A$

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In this equation, the terms involving $\varepsilon$ (the Itô integrals on the right hand side and the following Bochner integral) constitute the Stratonovich integrals $\varepsilon\frac{1}{2}\int_0^t (y(s) \times h) \circ dW(s)$ and $-\alpha\varepsilon\frac{1}{2}\int_0^t y(s) \times (y(s) \times h) \circ dW(s)$ (see page 342) for the definition of Stratonovich integral in the case of a continuous, real-valued semimartingale integrand; the definition in the case of a Hilbert space-valued integrand follows by considering the result of applying an arbitrary continuous linear functional to the integrand). The last integral on the right hand side of (6) represents the control terms. When proving existence and uniqueness of solutions, we interpret the boundary condition and the expressions $y \times \Delta y$ and $y \times (y \times \Delta y)$ in a weak sense, as described below. We define the linear operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by

\[
D(A) := \{u \in \mathbb{H}^2 : (Du)(x) = 0 \text{ for } x \in \partial A\} \quad \text{and} \quad Au := -\Delta u \text{ for } u \in D(A).
\]

We also define the operator $A_1u := Au + u$ for $u \in D(A)$; one can show that $A_1^{-1}$ is positive definite and symmetric and compact and that the fractional power Hilbert space $D(A_1^{\frac{1}{2}})$ with the graph norm is the same as $\mathbb{H}^1$. Although functions in $D(A)$ satisfy the desired boundary condition of our problem, we wish to allow $\mathbb{H}^1$-valued solutions of equation (6). Suppose that $v$, $w$, and $z$ are elements of $\mathbb{H}^1$; then we interpret the expressions $\Delta v$, $w \times \Delta v$ and $z \times (w \times \Delta v)$ as elements of the dual space $(\mathbb{H}^1)'$ of $\mathbb{H}^1$:

\[
\langle \Delta v, \phi \rangle_{\mathbb{H}^1} := -\langle Dv, D\phi \rangle_{\mathbb{H}} \forall \phi \in \mathbb{H}^1,
\]

\[
\langle w \times \Delta v, \phi \rangle_{\mathbb{H}^1} := -\langle D(\phi \times w), Dv \rangle_{\mathbb{H}} \forall \phi \in \mathbb{H}^1 \quad \text{and}
\]

\[
\langle z \times (w \times \Delta v), \phi \rangle_{\mathbb{H}^1} := -\langle D((\phi \times z) \times w), Dw \rangle_{\mathbb{H}} \forall \phi \in \mathbb{H}^1.
\]

\[
y(t) = u_0 + \int_0^t [(y(s) \times \Delta y(s)) - \alpha y(s) \times (y(s) \times \Delta y(s))] \, ds
\]

\[
+ \varepsilon\frac{1}{2}\int_0^t (y(s) \times h) \, dW(s) - \alpha\varepsilon\frac{1}{2}\int_0^t y(s) \times (y(s) \times h) \, dW(s)
\]

\[
+ \frac{1}{2}\varepsilon\int_0^t [(y(s) \times h) \times h - \alpha(y(s) \times (y(s) \times h))] \, ds
\]

\[
= -\alpha\{y(s) \times ((y(s) \times h) \times h) - \alpha(y(s) \times (y(s) \times h)) \times (y(s) \times h)
\]

\[
+ \alpha((y(s) \times (y(s) \times h)) \times h) \times y(s)) \} \, ds
\]

\[
+ \int_0^t [(y(s) \times h)F(s, W) - \alpha y(s) \times (y(s) \times h)F(s, W)] \, ds, \quad t \in [0, T]. \quad (6)
\]
If \( v \) actually belongs to \( D(A) \) then these formulae are a consequence of the divergence theorem. The maps \( y \in \mathbb{H}^1 \mapsto y \times \Delta y \in (\mathbb{H}^1)' \) and \( y \in \mathbb{H}^1 \mapsto y \times (y \times \Delta y) \in (\mathbb{H}^1)' \) are locally Lipschitz continuous and one can make sense of equation (1) for a solution \((y(t))_{t \in [0,T]}\) which is a measurable process with bounded paths in \( \mathbb{H}^1 \).

We need a few more definitions before presenting our theorems.

Let \( e_1, e_2, e_3, \ldots \) be an orthonormal basis of \( \mathbb{H} \) composed of eigenvectors of \( A \). For each \( n \in \mathbb{N} \), let \( \mathbb{H}_n \) be the linear span of \( \{e_1, \ldots, e_n\} \) and let

\[
\pi_n : u \in \mathbb{H} \mapsto \sum_{i=1}^n \langle u, e_i \rangle_{\mathbb{H}} e_i \in \mathbb{H}_n
\]

be orthogonal projection onto \( \mathbb{H}_n \).

For any positive real number \( \beta \), we write \( X^\beta \) for the fractional power Hilbert space \( D(A^\beta_1) \) with the graph norm and \( X^{-\beta} \) denotes the dual space of \( X^\beta \).

We write \( C \) for a positive real constant whose actual value may vary from line to line and we include an argument list, \( C(a_1, \ldots, a_m) \), if we wish to emphasize that the constant depends only on the values of the arguments \( a_1 \) to \( a_m \).

We now present a theorem on existence of a weak martingale solution of (6).

**Theorem 1** (Existence of a weak martingale solution in \( \mathbb{H}^1 \)). There exists a probability space \((\Omega', \mathcal{F}', P')\) with a filtration \((\mathcal{F}'_t)_{t \in [0,T]}\) on which are defined a \((\mathcal{F}'_t)\)-Wiener process \((W'(t))_{t \in [0,T]}\) and a \((\mathcal{F}'_t)\)-adapted process \((y'(t))_{t \in [0,T]}\) such that:

1. paths of \( y' \) are continuous in \( X^\beta \) for all \( \beta < \frac{1}{2} \) and are bounded in \( X^{\frac{1}{2}} \);
2. \( E' \left[ \sup_{t \in [0,T]} |y'(t)|_{\mathbb{H}^1}^{16} \right] \leq C(T, \alpha, M, u_0, h) \);
3. \( E' \left[ \left( \int_0^T |y'(s) \times \Delta y'(s)|_{\mathbb{H}}^2 \, ds \right)^2 \right] \leq C(T, \alpha, M, u_0, h) \);
4. \( |y'(t)(x)|_{\mathbb{H}^3} = 1 \) for all \( x \in \Lambda \) and for all \( t \in [0,T], P'\)-almost everywhere;
5. \( y'(t) = u_0 + \int_0^t y'(s) \times \Delta y'(s) \, ds - \alpha \int_0^t y'(s) \times (y'(s) \times \Delta y'(s)) \, ds 
+ \epsilon \int_0^t y'(s) \times h \, dW'(s) - \alpha \epsilon \frac{1}{2} \int_0^t (y'(s) \times (y'(s) \times h)) \, dW'(s) 
+ \frac{1}{2} \epsilon \int_0^t \left[ (y'(s) \times h) \times h - \alpha (y'(s) \times (y'(s) \times h)) \times h 
- \alpha \{ (y'(s) \times (y'(s) \times h)) \times h \} \right] \, ds 
+ \int_0^t (y'(s) \times h) F(s, W') \, ds - \alpha \int_0^t (y'(s) \times (y'(s) \times h)) F(s, W') \, ds 
\) for all \( t \in [0,T], P'\)-almost everywhere.
Notice that in Theorem 1, \( y' \) is an \( H^1 \)-valued process, hence the expressions \( y'(s) \times \Delta y'(s) \) and \( y'(s) \times (y'(s) \times \Delta y'(s)) \) in items 3 and 5 are interpreted in the sense of (8).

The proof of Theorem 1 is essentially the same as the proof of the existence theorem in [3], which was inspired by the pioneering work of Flandoli and Gątarek on the Navier-Stokes equation [10]. The steps are as follows.

1. We find a sequence of processes \( (y_n : [0, T] \times \Omega \to H_n)_{n \in \mathbb{N}} \) such that, for each \( n \), \( y_n \) solves a stochastic differential equation in \( H_n \) which approximates equation (6).

2. We obtain uniform (in \( n \)) bounds for the functions \( y_n \) and \( y_n \times \Delta y_n \) as random variables in appropriate path spaces. These bounds enable us to show that the sequence of laws \( (L(y_n, W))_{n \in \mathbb{N}} \) on \( C([0, T]; X^{-\frac{1}{2}}) \times C([0, T]; \mathbb{R}) \) is tight.

3. We use Skorohod’s coupling theorem to obtain a probability space \( (\Omega', F', \mathbb{P}') \) on which is defined a sequence of random variables \( ((y'_n, W'_n))_{n \in \mathbb{N}} \) which converges pointwise in \( C([0, T]; X^{-\frac{1}{2}}) \times C([0, T]; \mathbb{R}) \) to a limit \( (y', W') \).

4. We show that \( (y', W') \) has the properties listed in Theorem 1.

Steps 1 and 2 are in section 2. Step 3 and part of step 4 are in section 3. Step 4 is completed in sections 4, 5 and 6.

To obtain a large deviation principle for small noise asymptotics of solutions using the weak convergence method of Budhiraja and Dupuis [4, Theorem 4.4], we firstly need to establish the existence of a strong solution of equation (6); by ‘strong solution’ we mean a solution which is a measurable function of the given Wiener process \( W \) on the probability space \( (\Omega, F, (G_t), P) \). In Theorem 12 in section 8, we show uniqueness in law of weak martingale solutions of equation (6) with paths in \( S := C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{H}^1) \); we also show the existence of a measurable function \( J : C([0, T]; \mathbb{R}) \to S \) which maps the Wiener process \( W \) to a \((G_t)\)-adapted solution, \( y := J(W) \), of (6). This theorem is adapted from the well known theorem of Yamada and Watanabe for stochastic differential equations and the proof is the same, except that we work with paths in \( S \).

In section 9, we show that the solution, \( y \), of (6) has paths in the more regular space \( C([0, T]; H^1) \cap L^2(0, T; X^1) \). The main result is the estimate in Theorem 15:

\[
E \int_0^T |A_1 y(t)_{H^1}^2| dt \leq C(\alpha, T, M, u_0, h).
\]

This implies that, \( P \)-almost everywhere, \( y(t) \) lies in \( D(A) \) and hence satisfies the boundary conditions of our problem for almost every \( t \in (0, T] \).

The existence, uniqueness and regularity results come together in section 10 where we use the weak convergence method to obtain a large deviation principle for small noise asymptotics of solutions of equation (6) with \( F \) set to zero (recall that when \( F \) is zero, the equation models the physical system). Theorem 17 states that, as \( \varepsilon \in (0, 1] \) goes to zero, the laws of the solutions on the path space \( C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1) \) satisfy a large deviation principle with rate function \( I \) defined in (70). The estimates in Theorem 1 and
Theorem 15 are the key to the proof that the two conditions of Budhiraja and Dupuis (the conditions are stated in Statements 1 and 2, section 10) are satisfied. The proof that these two conditions are satisfied starts in Theorem 17 and ends in Lemma 23, but it is simpler than the corresponding proofs in [6] and [9]; for example, we do not need to partition the time interval $[0, T]$ into small subintervals.

In section 11, we apply the large deviation principle to a simple stochastic model of magnetization in a needle-shaped domain. We show that small noise in the field causes magnetization reversal with positive probability. We also obtain an estimate which shows the importance of the shape anisotropy parameter, $\beta$, for reducing the disturbance of the magnetization caused by small noise in the field.

## 2 Finite dimensional approximations and uniform bounds

In this section we work with processes $y_n : [0, T] \times \Omega \to \mathbb{H}_n$, $n \in \mathbb{N}$, which solve stochastic differential equations approximating equation (6). We obtain uniform bounds for certain functions of $y_n$ and conclude that the sequence of laws $(\mathcal{L}(y_n, W))_{n \in \mathbb{N}}$ on $C([0, T]; X^{1/2}) \times C([0, T]; \mathbb{R})$ is tight.

Let $\psi : \mathbb{R} \to [0, 1]$ be a function in $C^\infty_c(\mathbb{R})$ (a smooth function with compact support) such that $\psi(x) = 1$ for all $x \in [0, |h|_{L^\infty} + 1]$ and the support of $\psi$ is contained in $(-1, |h|_{L^\infty} + 2)$. For each $n \in \mathbb{N}$, let $y_n : [0, T] \times \Omega \to \mathbb{H}_n$ be the solution of the stochastic integral equation
In equation (10), the symbol \( \cdot \) denotes multiplication of a real scalar and a vector in \( \mathbb{H}_n \) and only appears because we had to split integrands across two lines. The \( \psi \) factors in the integrands help us to find bounds for the processes \( y_n \) which are uniform in \( n \) and, at the end of the proof of Theorem 1 in section 6, the \( \psi \) factors conveniently disappear (that is, they take the value 1 everywhere) in the equation satisfied by our limiting process, \( y' \).

Notice that the integrands of the Lebesgue integrals in (10) have the forms \( b_1(y_n(s)) \) or \( b_2(y_n(s))F(s, W) \) and the integrands of the Itô integrals have the form \( \kappa(y_n(s)) \), where \( b_1 \) and \( b_2 \) and \( \kappa \) are locally Lipschitz continuous functions from \( \mathbb{H}_n \) into \( \mathbb{H}_n \) and

\[
\langle b_1(u), u \rangle_{\mathbb{H}_n} \leq C(1 + |u|^2_{\mathbb{H}_n}) \quad \forall u \in \mathbb{H}_n \quad \text{and}
\]

\[
\langle b_2(u), u \rangle_{\mathbb{H}_n} = 0 \quad \forall u \in \mathbb{H}_n \quad \text{and}
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|\kappa(u)|^2_{\mathbb{H}_n} \leq C(1 + |u|^2_{\mathbb{H}_n}) \quad \forall u \in \mathbb{H}_n.
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\]

\[
|\kappa(u)|^2_{\mathbb{H}_n} \leq C(1 + |u|^2_{\mathbb{H}_n}) \quad \forall u \in \mathbb{H}_n.
\]
Consequently, the proof of [7, Theorem 10.6] can be used to show that equation (10) has a unique continuous, \((\mathcal{G}_t)\)-adapted solution \((y_n(t))_{t \in [0,T]}\).

Lemmas 2 and 3 below list uniform bounds for the processes \(y_n, n \in \mathbb{N}\).

**Lemma 2.** We have for each \(n \in \mathbb{N}\):

(i) \(|y_n(t)|_H = |\pi_n u_0|_H\) for all \(t \in [0,T]\), \(P\)-almost everywhere.

For each \(p \in [2, \infty)\) there exists a constant \(C(\alpha, T, M, p, u_0, h)\) (we emphasize that this constant depends on the function \(F\) only through \(M\) and does not depend on \(n\) or \(\varepsilon \in [0,1]\)) such that

(ii) \(E[\sup_{t \in [0,T]} |y_n(t)|^{2p}_H] \leq C(\alpha, T, M, p, u_0, h)\) and

(iii) \(E[(\int_0^T |y_n(s) \times \Delta y_n(s)|_H^2 ds)^p] \leq C(\alpha, T, M, p, u_0, h)\) and

(iv) \(E[(\int_0^T |y_n(s) \times (y_n(s) \times \Delta y_n(s))|_H^3 ds)^2] \leq C(\alpha, T, M, p, u_0, h)\).

**Proof.** Let \(n \in \mathbb{N}\). Result (i) follows from the Itô’s formula representation of \(|y_n(t)|^2_H\); in this representation, the integrand of the Itô integral vanishes, as do the integrands of Lebesgue integrals involving \(\Delta y_n\) and \(F\). The remaining Lebesgue integrals cancel. All that remains is:

\[ |y_n(t)|^2_H = |\pi_n u_0|^2_H \quad \forall t \in [0,T], \quad P\text{-almost everywhere.} \quad (11) \]

To obtain inequalities (ii), (iii) and (iv) in Lemma 2, we apply the function \(u \in \mathbb{H}_n \mapsto |Du|^2_H = -(\Delta u, u)_H\) to the process \(y_n\). By Itô’s formula and equality (11), we have
\[ \begin{align*}
|y_n(t)|^2_{\mathbb{H}} &= |\pi_n u_0|^2_{\mathbb{H}} - 2\varepsilon \int_0^t \left( \langle \Delta y_n(s), \pi_n(y_n(s) \times h) \rangle_{\mathbb{H}} dW(s) \\
&- \alpha \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot \\
&\langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} dW(s) \right) \\
&- 2 \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times \Delta y_n(s)) \rangle_{\mathbb{H}} ds \\
&+ 2\alpha \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times \Delta y_n(s))) \rangle_{\mathbb{H}} ds \\
&- 2 \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times h) \rangle_{\mathbb{H}} F(s, W) ds \\
&+ 2\alpha \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} F(s, W) ds \\
&- \varepsilon \left[ \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot \\
&\langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} ds \right] \\
&+ \varepsilon \alpha \left[ \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot \\
&\langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} ds \right] \\
&+ \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times (\pi_n(y_n(s) \times h) \times h)) \rangle_{\mathbb{H}} ds \\
&- \alpha \int_0^t \psi^2(|y_n(s)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot \\
&\langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} ds \\
&- \alpha \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times (\pi_n(y_n(s) \times (y_n(s) \times h)) \times h)) \rangle_{\mathbb{H}} ds \right] \\
&- \varepsilon \left[ \int_0^t \langle \Delta(\pi_n(y_n(s) \times h)), \pi_n(y_n(s) \times h) \rangle_{\mathbb{H}} ds \right] \\
&- 2\alpha \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot \\
&\langle \Delta(\pi_n(y_n(s) \times h)), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} ds \right] \\
&+ \alpha^2 \int_0^t \psi^2(|y_n(s)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot \\
&\langle \Delta(\pi_n(y_n(s) \times (y_n(s) \times h))), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} ds \right] (12)
\end{align*} \]
for all $t \in [0, T]$, $P$-almost everywhere. We now number the integrals on the right hand side of equality (12) in order from one to fifteen. We will estimate integrals one to fifteen in terms of $\int_t^0 \|y_n(s) \times \Delta y_n(s)\|_H^2 \, ds$ and $\int_t^0 \sup_{r \in [0,s]} \|y_n(r)\|_{H^1}^2 \, ds$. We remark that, since all norms on $H_n$ are equivalent and $|y_n(t)|_H = |\pi_n u_0|_H$ for all $t \in [0, T]$, all the functions of $y_n$ we encounter in integrands are bounded almost everywhere on $[0, T] \times \Omega$; hence all Itô and Lebesgue integrals behave nicely.

**Estimating integrals 1 and 2.** Let $p \in [2, \infty)$. We have for each $t \in [0, T]$:

$$E \left[ \sup_{r \in [0,t]} \left| \int_0^r \langle \Delta y_n(s), y_n(s) \times h \rangle_H \, dW(s) \right|^p \right]$$

$$= E \left[ \sup_{r \in [0,t]} \left| \int_0^r \langle D y_n(s), y_n(s) \times D h \rangle_H \, dW(s) \right|^p \right]$$

$$\leq C(p) E \left[ \left( \int_0^t \|y_n(s)\|_{H^1}^4 \|D h\|_{H^1}^2 \, ds \right)^{\frac{p}{2}} \right]$$

$$\leq C(p, T, h) E \left[ \int_0^t \sup_{r \in [0,s]} \|y_n(r)\|_{H^1}^{2p} \, ds \right], \quad (13)$$

where we have used the Burkholder-Davis-Gundy inequality and Jensen’s inequality.

Similarly, for integral two we have:

$$E \left[ \sup_{r \in [0,t]} \left| \int_0^r \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \langle D y_n(s), y_n(s) \times D (y_n(s) \times h) \rangle_H \, dW(s) \right|^p \right]$$

$$\leq C(p) E \left[ \left( \int_0^t \psi^2(|y_n(s)|_{L^\infty}) \|y_n(s)\|_{H^1}^4 \|y_n(s)\|_{L^\infty}^2 \|h\|_{H^1}^2 \, ds \right)^{\frac{p}{2}} \right]$$

$$\leq C(p, T, h) E \left[ \int_0^t \sup_{r \in [0,s]} \|y_n(r)\|_{H^1}^{2p} \, ds \right], \quad (14)$$

for each $t \in [0, T]$; here we used the fact that $\psi(|y_n(s)|_{L^\infty})|y_n(s)|_{L^\infty}$ is bounded by $|h|_{L^\infty} + 2$.

**Integral 3.** The integrand of this integral vanishes.

**Integral 4.** This integral is

$$\int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times \Delta y_n(s))) \rangle_H \, ds = - \int_0^t \|y_n(s) \times \Delta y_n(s)\|_{H^1}^2 \, ds$$

for all $t \in [0, T]$. The negative sign is important. This term (times $2\alpha$) serves to absorb positive multiples of $\int_0^t \|y_n(s) \times \Delta y_n(s)\|_{H^1}^2 \, ds$ that arise in our estimates of the other integrals on the right hand side of equality (12).
Estimating integrals 5 and 6. Let \( \eta \) be a positive number, to be chosen later. Integrals five and six are estimated in essentially the same way. We estimate integral six as follows:

\[
\left| \int_0^t \langle \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle H F(s, W) \, ds \right|
\]

\[
\leq \int_0^t |y_n(s) \times \Delta y_n(s)|_H |y_n(s)|_H |h|_{L^\infty} |F(s, W)| \, ds
\]

\[
\leq C(u_0, h) \left( \frac{\eta^2}{2} \int_0^t |y_n(s) \times \Delta y_n(s)|^2_H \, ds + \frac{1}{2\eta^2} \int_0^t F^2(s, W) \, ds \right)
\]

\[
\leq C(u_0, h) \left( \frac{\eta^2}{2} \int_0^t |y_n(s) \times \Delta y_n(s)|^2_H \, ds + \frac{M}{2\eta^2} \right) \forall t \in [0, T], \ P\text{-almost everywhere.}
\]

Here we used equality (11). An estimate of the same form holds for integral five.

**Integral 7.** This integral is

\[
\int_0^t \langle \Delta y_n(s), \pi_n(\pi_n(y_n(s) \times h) \times h) \rangle_H \, ds
\]

\[
= - \int_0^t \langle D_y(n(s), D(\pi_n(y_n(s) \times h))) \times h + \pi_n(y_n(s) \times h) \times Dh \rangle_H \, ds,
\]

which is bounded by

\[
C \int_0^t |y_n(s)|_{H^1} |y_n(s) \times h|_{H^1} |h|_{H^1} \, ds \leq C(h) \int_0^t \sup_{r \in [0,s]} |y_n(r)|^2_{H^1} \, ds \text{ for all } t \in [0, T].
\]

**Integral 8.** This integral is

\[
- \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot
\]

\[
\langle D_y(n(s), D(\pi_n(y_n(s) \times (y_n(s) \times h))) \times h + \pi_n(y_n(s) \times (y_n(s) \times h)) \times Dh \rangle_H \, ds
\]

which is bounded by

\[
C(h) \int_0^t \psi(|y_n(s)|_{L^\infty}) |y_n(s) \times (y_n(s) \times h)|_{H^1} \, ds \leq C(h) \int_0^t \sup_{r \in [0,s]} |y_n(s)|^2_{H^1} \, ds \text{ for all } t \in [0, T]; \text{ here the fact that } \psi(|y_n(s)|_{L^\infty}) |y_n(s)|_{L^\infty} \text{ is bounded by } |h|_{L^\infty} + 2 \text{ is again useful.}
\]

**Integral 9.** This integral is

\[
\int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot
\]

\[
\langle D_y(n(s), (y_n(s) \times h) \times D(\pi_n(y_n(s) \times h))) + D(y_n(s) \times h) \times (\pi_n(y_n(s) \times h)) \rangle \, ds
\]

When we expand this into a sum of two integrals in the obvious way, we find that the integrals are bounded by

\[
\int_0^t \psi(|y_n(s)|_{L^\infty}) |y_n(s)|_{H^1} |y_n(s)|_{L^\infty} |h|_{L^\infty} |y_n(s) \times h|_H \, ds \text{ and }
\]

\[
\int_0^t \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) |y_n(s)|_{H^1} |\pi_n(y_n(s) \times h)|_{L^\infty} |y_n(s) \times h|_H \, ds,
\]

respectively. The sum is bounded by \( C(h) \int_0^t \sup_{r \in [0,s]} |y_n(r)|^2_{H^1} \, ds \), for each \( t \in [0, T] \). Here we used the fact that \( \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) |\pi_n(y_n(s) \times h)|_{L^\infty} \text{ is bounded by } |h|_{L^\infty} + 2. \)
Integral 10. This integral is \( \int_0^t (\pi_n(y_n(s) \times h) \times h, y_n(s) \times \Delta y_n(s))_{\mathbb{H}} \, ds \) which is bounded by
\[
\int_0^t |h|_{L^\infty}^2 |y_n(s)|_{\mathbb{H}} |y_n(s) \times \Delta y_n(s)|_{\mathbb{H}} \, ds \leq C(h, u_0) \left( \frac{T}{2\eta^2} + \frac{\eta^2}{2} \right) \int_0^t |y_n(s) \times \Delta y_n(s)|_{\mathbb{H}}^2 \, ds
\]
for all \( t \in [0, T] \), \( P \)-almost everywhere; here we used equality (11) and \( \eta \) is a constant to be chosen later.

Integral 11. This integral is
\[
- \int_0^t \psi^2(|y_n(s)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot
\]
\[
\langle D(\pi_n(y_n(s) \times h), D(\pi_n(y_n(s) \times (y_n(s) \times h))) \rangle_{\mathbb{H}} \, ds.
\]
When we expand into the sum of two integrals, the first integral is bounded by
\[
\int_0^t \psi^2(|y_n(s)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times h)|_{L^\infty}) |\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty} \, ds.
\]
Both integrals are bounded by \( C(h) \int_0^t \sup_{r \in [0, t]} |y_n(r)|_{\mathbb{H}}^2 \, ds \), where we used the fact that the expressions \( \psi(|y_n(s)|_{L^\infty}) |y_n(s)|_{L^\infty} \) and \( \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) |\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty} \) are bounded by \( |h|_{L^\infty} + 2 \).

Integral 12. This integral is \( -\int_0^t \langle y_n(s) \times \Delta y_n(s), \pi_n(y_n(s) \times (y_n(s) \times h)) \rangle_{\mathbb{H}} \, ds \). It is bounded by
\[
C \int_0^t |y_n(s) \times \Delta y_n(s)|_{\mathbb{H}} |y_n(s)|_{\mathbb{H}}^2 |y_n(s)|_{\mathbb{H}}^2 \, ds
\]
\[
\leq \frac{C\eta^2}{2} \int_0^t |y_n(s) \times \Delta y_n(s)|_{\mathbb{H}}^2 \, ds + \frac{C(h, u_0)}{2\eta^2} \int_0^t \sup_{r \in [0, s]} |y_n(r)|_{\mathbb{H}}^2 \, ds
\]
for all \( t \in [0, T] \), \( P \)-almost everywhere. Here \( \eta \) is a constant to be chosen later and we used equality (11).

Integral 13. This integral is \( -\int_0^t |D(\pi_n(y_n(s) \times h))|_{\mathbb{H}}^2 \, ds \) which is bounded by
\[
\int_0^t |y_n(s) \times h|_{\mathbb{H}}^2 \, ds \leq C(h) \int_0^t \sup_{r \in [0, s]} |y_n(r)|_{\mathbb{H}}^2 \, ds
\]
for all \( t \in [0, T] \).

Integral 14. This integral is
\[
- \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \cdot
\]
\[
\langle D(\pi_n(y_n(s) \times h)), D(\pi_n(y_n(s) \times (y_n(s) \times h))) \rangle_{\mathbb{H}} \, ds.
\]
which is bounded by
\[
\int_0^t \psi(|y_n(s)|_{L^\infty}) |y_n(s) \times h|_{\mathbb{H}}^2 |y_n(s) \times (y_n(s) \times h)|_{\mathbb{H}}^2 \, ds \leq C(h) \int_0^t \sup_{r \in [0, s]} |y_n(r)|_{\mathbb{H}}^2 \, ds
\]
for all \( t \in [0, T] \).

Integral 15. This integral is bounded by
\[ f_t \psi^2(|y_n(s)|_{L^\infty}) |y_n(s) \times (y_n(s) \times h)|_{L^\infty}^2 \, ds \leq C(h) \int_0^t \sup_{r \in [0, s]} |y_n(r)|_{H^1}^2 \, ds \] for all \( t \in [0, T] \).

Our estimates for integrals three to fifteen are substituted into equality \((12)\) to obtain:

\[ |y_n(t)|_{H^1}^2 \leq |\pi_n u_0|_{H^1}^2 + 2 \sup_{s \in [0, t]} \left| \int_0^s (\Delta y_n(r), \pi_n(y_n(r) \times h))_H \, dW(r) \right| + 2 \alpha \sup_{s \in [0, t]} \left| \int_0^s (\Delta y_n(r), \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \right| \tag{15} \]

for all \( t \in [0, T] \), \( P \)-almost everywhere; here \( \eta \) is an arbitrary positive real constant. We now choose \( \eta \) to be a positive number which is sufficiently small to make the coefficient of \( \int_0^t |y_n(s) \times \Delta y_n(s)|_{H^1}^2 \, ds \) in inequality \((15)\), \((C(\alpha, u_0, h) \eta^2 - 2\alpha)\), negative: for definiteness, suppose that \( C(\alpha, u_0, h) \eta^2 - 2\alpha = -\alpha \). Now we omit the term \( -\alpha \int_0^t |y_n(s) \times \Delta y_n(s)|_{H^1}^2 \, ds \) on the right hand side of inequality \((15)\), write \( \sup_{r \in [0, s]} |y_n(r)|_{H^1}^2 \) in place of \( |y_n(t)|_{H^1}^2 \) on the left hand side, raise both sides to the power \( p \) and take expectations to obtain:

\[ E \left[ \sup_{r \in [0, T]} |y_n(r)|_{H^1}^{2p} \right] \leq C(\alpha, T, M, p, u_0, h) + C(\alpha, T, p, u_0, h) \int_0^t E \left[ \sup_{r \in [0, s]} |y_n(r)|_{H^1}^{2p} \right] \, ds \]

for all \( t \in [0, T] \); here the estimates in \((13)\) and \((14)\) were used. Therefore, by Gronwall’s lemma,

\[ E \left[ \sup_{s \in [0, T]} |y_n(s)|_{H^1}^{2p} \right] \leq C(\alpha, T, M, p, u_0, h) e^{C(\alpha, T, p, u_0, h) T}, \]

which is (ii) in the statement of Lemma 2.

We return to inequality \((15)\) and shift \( \alpha \int_0^t |y_n(s) \times \Delta y_n(s)|_{H^1}^2 \, ds \) from the right to the left hand side, where it replaces \( |y_n(t)|_{H^1}^2 \). We then raise both sides to the power \( p \). This yields:

\[ \alpha^p E \left[ \left( \int_0^T |y_n(s) \times \Delta y_n(s)|_{H^1}^2 \, ds \right)^p \right] \leq C(\alpha, T, M, p, u_0, h) + C(\alpha, p, T, h, u_0) E \left[ \sup_{r \in [0, T]} |y_n(r)|_{H^1}^{2p} \right], \]

which implies (iii) in the statement of Lemma 2.

To obtain inequality (iv), we write

\[ E[\left( \int_0^T |y_n(s) \times (y_n(s) \times \Delta y_n(s))|_{H^1}^2 \, ds \right)^{\frac{p}{2}}] \leq C(p) E[\sup_{t \in [0, T]} |y_n(t)|_{H^1}^2 (\int_0^T |y_n(s) \times \Delta y_n(s)|_{H^1}^2 \, ds)^{\frac{p}{2}}] \]

and apply the Cauchy-Schwarz inequality.

This completes the proof of Lemma 2. \[\square\]
The next lemma gives a uniform estimate for the processes $y_n$ as random variables in a fractional Sobolev space of paths in $\mathbb{H}$. The significance of this is that, while paths in such a space may not be regular enough to have a weak derivative, Flandoli and Gątarek [10, Theorem 2.2] showed that one can choose the space to have a compact embedding into $C([0, T]; X^{-\frac{1}{2}})$.

**Lemma 3.** For each $\beta \in (0, \frac{1}{2})$ and $p \in [2, \infty)$ we have:

$$\sup_{n \in \mathbb{N}} E \left[ |y_n|^p_{W^{\beta, p}(0, T; \mathbb{H})} \right] \leq C(\alpha, T, p, \beta, M, h, u_0).$$

**Proof.** Let $n \in \mathbb{N}$. We denote the twelve integrals appearing on the right hand side of equation (10) as $y_n^1(t), \ldots, y_n^{12}(t)$; then equation (10) becomes

$$y_n(t) = \pi_n u_0 + \sum_{i=1}^{12} C_i(\varepsilon, \alpha) y_i^j(t)$$

for all $t \in [0, T]$, where $C_i(\varepsilon, \alpha)$ is the constant coefficient of the $i$th integral and is bounded by $\max\{1, \alpha, \frac{1}{2} \alpha^2\}$ for $\varepsilon \in [0, 1]$.

Firstly we consider the stochastic integrals $y_n^3$ and $y_n^4$. Let $\beta \in (0, \frac{1}{2})$ and let $p \in [2, \infty)$. Flandoli and Gątarek [10, Lemma 2.1] showed that the $L^p$ norm of the stochastic integral as a random variable in $W^{\beta, p}(0, T; \mathbb{H})$ can be estimated in terms of the $L^p((0, T) \times \Omega; \mathbb{H})$ norm of the integrand. We have the two inequalities

$$E \int_0^T |\pi_n(y_n(s) \times h)|_{\mathbb{H}}^p \, ds \leq C(T, p, h, u_0)$$

and

$$E \int_0^T \psi(\pi_n(y_n(s) \times (y_n(s) \times h)))_{L^\infty} |\pi_n(y_n(s) \times (y_n(s) \times h))|_{\mathbb{H}}^p \, ds \leq C(T, p, h),$$

which follow from equality (11) and the definition of $\psi$, respectively. These bounds imply that

$$E[|y_n^3|_{W^{\beta, p}(0, T; \mathbb{H})}] \leq C(T, p, \beta, h, u_0) \quad \text{and}$$

$$E[|y_n^4|_{W^{\beta, p}(0, T; \mathbb{H})}] \leq C(T, p, \beta, h).$$

For each $i = 1, 2, 5, \ldots, 12$, we can use Lemma 2 to show that

$$E[|y_n^i|_{W^{1,2}(0, T; \mathbb{H})}] \leq C(\alpha, T, p, M, h, u_0);$$

since $W^{1,2}(0, T; \mathbb{H})$ is continuously embedded into $W^{\beta, p}(0, T; \mathbb{H})$ (by [17, Corollary 19]), we also have

$$E[|y_n^i|_{W^{\beta, p}(0, T; \mathbb{H})}] \leq C(\alpha, T, p, \beta, M, h, u_0).$$

The uniform bounds (17), (18) and (19) together with equality (16) yield:

$$E[|y_n|^p_{W^{\beta, p}(0, T; \mathbb{H})}] \leq C(\alpha, T, p, \beta, M, h, u_0).$$

$\square$
Corollary 4. The sequence of laws \( \{ \mathcal{L}(y_n) : n \in \mathbb{N} \} \) on \( C([0,T];X^{-\frac{1}{2}}) \) is tight.

Proof. Choose \( \beta \in (0, \frac{1}{2}) \) and \( p \in (2, \infty) \) satisfying \( \beta p > 1 \). By [10] Theorem 2.2, \( W^{\beta,p}(0, T; \mathbb{H}) \) is compactly embedded into \( C([0,T];X^{-\frac{1}{2}}) \). Thanks to Lemma 3 given \( \eta > 0 \) we choose \( r \in (0, \infty) \) so that

\[
P\{ |y_n|_{W^{\beta,p}(0,T;\mathbb{H})} \geq r \} \leq \frac{1}{r^p} E[|y_n|_{W^{\beta,p}(0,T;\mathbb{H})}^p] \\
\leq \frac{C(\alpha,T,p,\beta,M,h,u_0)}{r^p} \leq \eta \quad \forall n \in \mathbb{N}.
\]

Since the ball of radius \( r \) in \( W^{\beta,p}(0, T; \mathbb{H}) \) has compact closure in \( C([0,T];X^{-\frac{1}{2}}) \), we are done. \( \square \)

3 Pointwise convergence of a sequence of approximations and some properties of the limit

In this section we obtain, via Skorohod’s theorem, a probability space \((\Omega', \mathcal{F}', P')\) on which is defined a random variable \((y', W')\) in \( C([0,T];X^{-\frac{1}{2}}) \times C([0,T];\mathbb{R}) \). We shall establish some basic properties of \( y' \) which will be used in the following sections to show that \((y', W')\) is a weak martingale solution of our stochastic Landau-Lifshitz equation.

By Corollary 3 and Prohorov’s theorem, we can find a subsequence of \((y_n)\) (also denoted by \((y_n)\) to simplify notation) such that the random variables \((y_n, W)\) in \( C([0,T];X^{-\frac{1}{2}}) \times C([0,T];\mathbb{R}) \) converge in distribution. We call the limiting distribution \( \mu \).

Proposition 5. There exists a probability space \((\Omega', \mathcal{F}', P')\) and there exists a sequence \((y'_n, W'_n)\) of \( C([0,T];X^{-\frac{1}{2}}) \times C([0,T];\mathbb{R}) \)-valued random variables defined on \((\Omega', \mathcal{F}', P')\) such that the laws of \((y_n, W)\) and \((y'_n, W'_n)\) are equal for each \( n \in \mathbb{N} \) and \((y'_n, W'_n)\) converges pointwise in \( C([0,T];X^{-\frac{1}{2}}) \times C([0,T];\mathbb{R}), P'-\text{almost everywhere, to a limit } (y', W') \) with distribution \( \mu \).

Proof. This result follows immediately from Skorohod’s theorem (see, for example, [11] Theorem 4.30] because \( C([0,T];X^{-\frac{1}{2}}) \times C([0,T];\mathbb{R}) \) is a separable metric space. \( \square \)

Remark 1. For each \( n \in \mathbb{N} \), the Banach space \( C([0,T];\mathbb{H}_n) \) is continuously embedded in \( C([0,T];X^{-\frac{1}{2}}) \). Therefore, by a well known result of Kuratowski (see, for example, [19] Theorem 1.1 in chapter 1)], the Borel subsets of \( C([0,T];\mathbb{H}_n) \) are also Borel subsets of \( C([0,T];X^{-\frac{1}{2}}) \).

For each \( n \in \mathbb{N} \), we have \( P'\{y'_n \in C([0,T];\mathbb{H}_n)\} = P\{y_n \in C([0,T];\mathbb{H}_n)\} = 1; \) hence we may assume that \( y'_n \) is a random element in \( C([0,T];\mathbb{H}_n) \) and that the laws of \( y'_n \) and \( y_n \) on \( C([0,T];\mathbb{H}_n) \) are equal.
As a consequence of Remark \[1\] we have the following estimates involving the sequence \((y_n)\), which follow from those in Lemma \[2\] involving \((y_n)\): for each \(n \in \mathbb{N}\)

\[
\sup_{t \in [0,T]} |y_n(t)|_H \leq |u_0|_H \quad P' \text{ a.e. and} \quad (21)
\]

\[
E'[\sup_{t \in [0,T]} |y_n(t)|^6_{16}] \leq C(\alpha, T, M, u_0, h) \quad (22)
\]

\[
E' \left[ \left( \int_0^T |y_n(t) \times \Delta y_n(t)|^2_H \, dt \right)^2 \right] \leq C(\alpha, T, M, u_0, h) \quad (23)
\]

\[
E' \int_0^T |y_n(t) \times (y_n(t) \times \Delta y_n(t))|_H^2 \, dt \leq C(\alpha, T, M, u_0, h); \quad (24)
\]

as usual, \(C(\alpha, T, M, u_0, h)\) denotes a constant depending only on its arguments and, in particular, not on \(n\) or \(\varepsilon \in [0, 1]\). The estimates \([21]\) to \([24]\) and the pointwise convergence of \(y_n\) to \(y'\) in \(C([0,T]; X^{-\frac{1}{2}})\) yield some properties of \(y'\) which will be useful in the next section.

**Lemma 6.** The process \(y'\) has the following properties:

(i) \(\sup_{t \in [0,T]} |y'(t)|_H \leq |u_0|_H\) \(P'\)-almost everywhere.

(ii) \(E'[\sup_{t \in [0,T]} |y'(t)|_{16}] \leq C(\alpha, T, M, u_0, h)\).

(iii) For \(P'\)-almost every \(\omega \in \Omega'\), the path \(y'(\omega)\) is continuous in \(H^1\) with the weak topology and continuous in \(X^\gamma\) with the norm topology for all \(\gamma < \frac{1}{2}\).

(iv) \(E'[\sup_{t \in [0,T]} |y_n(t) - y'(t)|_H^2] \to 0\) as \(n \to \infty\).

(v) The sequence \((y_n' \times \Delta y_n')\) converges weakly in \(L^2([0,T] \times \Omega'; H)\) to \(y' \times \Delta y'\) (more precisely, the limit equals \(y' \times \Delta y'\) almost everywhere on \([0,T] \times \Omega'\)). In particular, we have \(E'[\int_0^T |y'(s) \times \Delta y'(s)|_H^2 \, ds]^2 \leq C(\alpha, T, M, u_0, h)\).

(vi) The sequence \((y_n' \times (y_n' \times \Delta y_n'))\) converges weakly in \(L^2([0,T] \times \Omega'; H)\) to \(y' \times (y' \times \Delta y')\). In particular, we have \(E' \int_0^T |y'(s) \times (y'(s) \times \Delta y'(s))|_H^2 \, ds \leq C(\alpha, T, M, u_0, h)\).

**Proof.** Firstly, recall that if \((S, d)\) is a metric space and \(f : S \to [0, \infty)\) is a lower semicontinuous function then whenever \((x_n)\) is a sequence in \(S\) that converges to a limit \(x \in S\), we have \(f(x) \leq \liminf_{n \to \infty} f(x_n)\).

**Proof of (i).** We extend the definition of \(|\cdot|_H\) to the domain \(X^{-\frac{1}{2}}\) by setting \(|u|_H := \infty\) if \(u \in X^{-\frac{1}{2}}\setminus H\). Since the embedding of \(H\) into \(X^{-\frac{1}{2}}\) is compact, the extended function \(|\cdot|_H : X^{-\frac{1}{2}} \to [0, \infty]\) is lower semicontinuous. We claim that the function \(u \in C([0,T]; X^{-\frac{1}{2}}) \mapsto \sup_{t \in [0,T]} |u(t)|_H\) is also lower semicontinuous. To see this, take a sequence \((f_n)\) from \(C([0,T]; X^{-\frac{1}{2}})\) such that \(\sup_{t \in [0,T]} |f_n(t)|_H \leq r < \infty\) for all \(n \in \mathbb{N}\) and \(f_n\) converges in \(C([0,T]; X^{-\frac{1}{2}})\) to a limit \(f\). For each \(t \in [0,T]\), \(f_n(t)\) converges to \(f(t)\) in \(X^{-\frac{1}{2}}\) and we have \(|f(t)|_H \leq \liminf_{n \to \infty} |f_n(t)|_H \leq r\); hence \(\sup_{t \in [0,T]} |f(t)|_H \leq r\). We conclude that \(\{u \in C([0,T]; X^{-\frac{1}{2}}) : \sup_{t \in [0,T]} |u(t)|_H \leq r\}\) is closed in \(C([0,T]; X^{-\frac{1}{2}})\), which proves the claim.
Since \( y_n' \) converges to \( y' \) in \( C([0, T]; X^{-\frac{1}{2}}) \), \( P' \)-almost everywhere, we have

\[
\sup_{t \in [0, T]} |y'(t)|_{H} \leq \liminf_{n \to \infty} \sup_{t \in [0, T]} |y_n'(t)|_{H} \\
\leq |u_0|_{H} \quad P' \text{ almost everywhere,}
\]

where (21) gives the second inequality.

**Proof of (ii).** This proof is similar to that of (i). We extend the definition of the norm \( | \cdot |_{\Omega} \) to the domain \( X^{-\frac{1}{2}} \) by setting \( |u|_{\Omega} := \infty \) if \( u \in X^{-\frac{1}{2}} \setminus \mathbb{H}^1 \). Then the function \( u \in C([0, T]; X^{-\frac{1}{2}}) \mapsto \sup_{t \in [0, T]} |u(t)|_{\Omega} \in [0, \infty] \) is lower semicontinuous. Using this fact, Fatou’s lemma and (22), we have

\[
E'[\sup_{t \in [0, T]} |y'(t)|_{\Omega}^{16}] \leq E'[\liminf_{n \to \infty} \sup_{t \in [0, T]} |y_n'(t)|_{\Omega}^{16}] \\
\leq \liminf_{n \to \infty} E'[\sup_{t \in [0, T]} |y_n'(t)|_{\Omega}^{16}] \\
\leq C(\alpha, T, M, u_0, h).
\]

**Proof of (iii).** Take \( \omega' \in \Omega' \) such that \( \sup_{t \in [0, T]} |y'(t)(\omega')|_{\Omega} < \infty \). Let \((s_n)\) be a sequence from \([0, T] \) converging to \( t \in [0, T] \). Since the sequence \((y'(s_n)(\omega'))\) is bounded in \( \mathbb{H}^1 \), there is a subsequence \((n_k)\) such that \( y'(s_{n_k})(\omega') \) converges weakly in \( \mathbb{H}^1 \) to a point \( z \). Since \( \mathbb{H}^1 \) is compactly embedded in \( X^{-\frac{1}{2}} \), \( y'(s_{n_k})(\omega') \) converges to \( z \) and also to \( y'(t)(\omega') \) in the norm topology of \( X^{-\frac{1}{2}} \) as \( k \) goes to infinity. Thus \( z = y'(t)(\omega') \). Since the limit is the same for every weakly convergent subsequence, the whole sequence \((y'(s_n)(\omega'))\) converges to \( y'(t)(\omega') \) weakly in \( \mathbb{H}^1 \). It follows that for each \( \gamma < \frac{1}{2}, \) \( y'(s_n)(\omega') \) converges to \( y'(t)(\omega') \) in the norm topology of \( X^{-\frac{1}{2}} \) as \( n \) goes to infinity.

**Proof of (iv).** In this proof we use the estimate:

\[
|u|_{\mathbb{H}} = \langle A_{\frac{3}{2}} u, A_{\frac{3}{2}} u \rangle_{\Omega} \leq |u|_{X^{-\frac{1}{2}}} \quad |u|_{X^{-\frac{1}{2}}} \quad \text{for all } u \in \mathbb{H}^1.
\]

We have

\[
E'[\sup_{t \in [0, T]} |y_n'(t) - y'(t)|_{\Omega}^{\frac{8}{3}}] \\
\leq C E'[\sup_{t \in [0, T]} |y_n'(t)|_{\Omega}^{\frac{4}{3}} + \sup_{t \in [0, T]} |y'(t)|_{\Omega}^{\frac{4}{3}}(\sup_{t \in [0, T]} |y_n'(t) - y'(t)|_{X^{-\frac{1}{2}}}^{\frac{4}{3}})] \\
\leq C(\alpha, T, M, u_0, h)(E'[\sup_{t \in [0, T]} |y_n'(t) - y'(t)|_{X^{-\frac{1}{2}}}^{\frac{8}{3}}])^{\frac{3}{2}} \\
\to 0 \text{ as } n \to \infty;
\]

here the second inequality is obtained using the Cauchy-Schwarz inequality and the bounds in (22) and (ii), while the last line uses Lebesgue’s dominated convergence theorem.

**Proof of (v).** Firstly we prove convergence of the sequence \((Dy_n')\). The uniform bound in (22) implies that there is a subsequence \((n_k)\) such that \( Dy_{n_k}' \) converges weakly in \( L^2([0, T] \times \Omega'; \mathbb{H}) \). We claim that the limit is \( Dy' \). To show this, take \( V : ([0, T] \times \Omega', \mathcal{B}_{[0, T]} \otimes \mathcal{F}') \to \mathbb{H} \)
an arbitrary simple function with values in $C^\infty_c(\Lambda; \mathbb{R}^3)$. Such simple functions are dense in $L^2([0, T] \times \Omega'; H)$. We have

$$E' \int_0^T \langle Dy'_{n_k}(s), V(s) \rangle_{H} ds = -E' \int_0^T \langle y'_{n_k}(s), DV(s) \rangle_{H} ds$$

$$\rightarrow -E' \int_0^T \langle y'(s), DV(s) \rangle_{H} ds \text{ by (iv)}$$

$$\rightarrow E' \int_0^T \langle Dy'(s), V(s) \rangle_{H} ds.$$

Since the limit, $Dy'$, is the same for any weakly convergent subsequence of $(Dy'_{n_k})$, the entire sequence must converge to $Dy'$ weakly in $L^2([0, T] \times \Omega'; H)$. This fact will now be used to prove (v).

By (23), some subsequence of $(y'_{n_k} \times \Delta y'_{n_k})$ converges weakly in $L^2([0, T] \times \Omega'; H)$ to a limit which we call $Y$. To simplify notation, we denote the convergent subsequence by $(y'_{n_k} \times \Delta y'_{n_k})$. Take $V : ([0, T] \times \Omega', B_{[0, T]} \otimes F') \to H^1$ an arbitrary simple function with values in $X^1$. Such simple functions are dense in $L^2([0, T] \times \Omega'; H^1)$. We have

$$E' \int_0^T \langle y'_{n_k}(s) \times \Delta y'_{n_k}(s), V(s) \rangle_{H} ds$$

$$= -E' \int_0^T \langle DV(s) \times y'_{n_k}(s), Dy'_{n_k}(s) \rangle_{H} ds$$

$$\rightarrow -E' \int_0^T \langle DV(s) \times y'(s), Dy'(s) \rangle_{H} ds \text{ as } n \to \infty \text{ (using (iv))}$$

$$\rightarrow E' \int_0^T X^{-\frac{1}{2}} \langle y'(s) \times \Delta y'(s), V(s) \rangle \chi_{ X^\frac{1}{2} } ds \text{ by definition of } y' \times \Delta y'$$

$$\rightarrow E' \int_0^T X^{-\frac{1}{2}} \langle Y(s), V(s) \rangle \chi_{ X^\frac{1}{2} } ds \text{ by definition of } Y.$$

The function $y' \times \Delta y'$ belongs to $L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})$ (because of the estimate $|u \times \Delta u|_{X^{-\frac{1}{2}}} \leq C |u|^2_{H^1} \forall u \in H^1$), therefore $Y$ and $y' \times \Delta y'$ are equal as elements of $L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})$.

It remains to show that

$$E'[\int_0^T |y'(s) \times \Delta y'(s)|^2_{H} ds]^2 \leq C(\alpha, T, M, u_0, h).$$

(25)

By the Banach-Saks theorem, there is a subsequence $(n_k)$ such that the sequence of means $(\frac{1}{N} \sum_{k=1}^N y'_{n_k} \times \Delta y'_{n_k})$ converges in the norm topology of $L^2([0, T] \times \Omega'; H)$ to $y' \times \Delta y'$. A subsequence of this sequence of means also converges pointwise in $L^2(0, T; H)$ to $y' \times \Delta y'$, $P^\epsilon$-almost everywhere. By (23), we have

$$E'[\int_0^T \frac{1}{N} \sum_{k=1}^N y'_{n_k}(s) \times \Delta y'_{n_k}(s)|^2_H ds]^2] \leq C(\alpha, T, M, u_0, h)$$

for all $N \in \mathbb{N}$ and (25) follows from Fatou’s lemma.
Proof of (vi). By (24), some subsequence of \((y'_n \times (y'_n \times \Delta y'_n))\) (we denote the subsequence in the same way, to simplify notation) converges weakly in \(L^2([0,T] \times \Omega'; \mathbb{H})\) to a limit which we call \(Z\). We will show that \(Z = y' \times (y' \times \Delta y')\) almost everywhere on \([0,T] \times \Omega'\).

Observe that the function \(y' \times (y' \times \Delta y')\) belongs to \(L^2([0,T] \times \Omega'; X^{-\frac{1}{2}})\) because of the estimate:

\[
|u \times (u \times \Delta u)|_{X^{-\frac{1}{2}}} \leq C|u|_H^3 \quad \forall u \in \mathbb{H}^1.
\]

Let \(V : ([0,T] \times \Omega', \mathcal{B}_{[0,T]} \otimes \mathcal{F}') \to \mathbb{H}^1\) be an arbitrary simple function in \(\mathbb{H}^1\); such simple functions are dense in \(L^2([0,T] \times \Omega'; \mathbb{H}^1)\). We have

\[
E' \int_0^T \langle y'_n(s) \times (y'_n(s) \times \Delta y'_n(s)), V(s) \rangle_{\mathbb{H}} \, ds
= \quad E' \int_0^T \langle y'_n(s) \times \Delta y'_n(s), V(s) \times y'_n(s) \rangle_{\mathbb{H}} \, ds
\]

\[
\Rightarrow \quad E' \int_0^T X^{-\frac{1}{2}} \langle y'(s) \times \Delta y'(s), V(s) \times y'(s) \rangle_{X^{\frac{1}{2}}} \, ds
\]

\[
= \quad E' \int_0^T X^{-\frac{1}{2}} \langle y'(s) \times (y'(s) \times \Delta y'(s)), V(s) \rangle_{X^{\frac{1}{2}}} \, ds,
\]

where the convergence in the second line is a consequence of (iv) and (v). It follows that \(Z\) and \(y' \times (y' \times \Delta y')\) are equal as elements of \(L^2([0,T] \times \Omega'; X^{-\frac{1}{2}})\).

4 Identifying \((y', W')\) as a solution: the drift terms of the equation

The aim of this and the next two sections is to use the results of the previous section to show that the random element \((y', W') : (\Omega', \mathcal{F}', P') \to C([0,T]; X^{-\frac{1}{2}}) \times C([0,T]; \mathbb{R})\) solves our initial value problem; that is, we want to show that the equality in Theorem 1(5) holds. Along the way, we shall show that \((W'(t))_{t \in [0,T]}\) is a Wiener process on \((\Omega', \mathcal{F}', P')\) and that the integrals on the right hand side of the equality all make sense.
For each \( n \in \mathbb{N} \), define the process \( (M_n(t) : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{H}, \mathcal{B}_\mathbb{H}))_{t \in [0,T]} \) by:

\[
M_n(t) := y_n(t) - \pi_n u_0 - \int_0^t \pi_n(y_n(s) \times \Delta y_n(s)) \, ds + \alpha \int_0^t \pi_n(y_n(s) \times (y_n(s) \times \Delta y_n(s))) \, ds
- \frac{1}{2} \varepsilon \left[ \int_0^t \pi_n(y_n(s) \times h) \, ds \right]
- \alpha \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \pi_n(y_n(s) \times (y_n(s) \times h)) \, ds
+ \frac{1}{2} \varepsilon \alpha \left[ \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \right.
\left. \pi_n(y_n(s) \times (y_n(s) \times h)) \, ds \right]
- \alpha \int_0^t \psi^2(|y_n(s)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi^2(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \pi_n(y_n(s) \times (y_n(s) \times h)) \, ds
- \alpha \int_0^t \pi_n(y_n(s) \times (y_n(s) \times h)) \, ds
- \int_0^t \pi_n(y_n(s) \times h) F(s, W) \, ds + \alpha \int_0^t \pi_n(y_n(s) \times (y_n(s) \times h)) F(s, W) \, ds
\]

(26)

for all \( t \in [0, T] \). From equality (10), the process \( (M_n(t))_{t \in [0,T]} \) is also a stochastic integral:

\[
M_n(t) = \varepsilon \frac{1}{2} \left[ \int_0^t \pi_n(y_n(s) \times h) \, dW(s) \right]
- \alpha \int_0^t \psi(|y_n(s)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \pi_n(y_n(s) \times (y_n(s) \times h)) \, dW(s)
\]

(27)

for all \( t \in [0, T] \), \( P \)-almost everywhere.
For each \( n \in \mathbb{N} \), define the process \((M'_n(t) : (\Omega', \mathcal{F}', P') \to (\mathbb{H}, \mathcal{B}_\mathbb{H}))_{t \in [0,T]}\) by:

\[
M'_n(t) := y'_n(t) - \pi_n u_0 - \int_0^t \pi_n (y'_n(s) \times \Delta y'_n(s)) \, ds + \alpha \int_0^t \pi_n (y'_n(s) \times (y'_n(s) \times \Delta y'_n(s))) \, ds
- \frac{1}{2} \varepsilon \left[ \int_0^t \pi_n (\pi_n (y'_n(s) \times h) \times h) \, ds \\
- \alpha \int_0^t \psi(|y'_n(s)|_{L^\infty}) \psi(|\pi_n (y'_n(s) \times h)|_{L^\infty}) \, ds \right] \cdot \pi_n (\pi_n (y'_n(s) \times (y'_n(s) \times h))) \times \pi_n (y'_n(s) \times (y'_n(s) \times h)) \, ds
+ \frac{1}{2} \varepsilon \alpha \left[ \int_0^t \psi(|y'_n(s)|_{L^\infty}) \psi(|\pi_n (y'_n(s) \times h)|_{L^\infty}) \psi(|\pi_n (y'_n(s) \times (y'_n(s) \times h))|_{L^\infty}) \, ds \\
- \alpha \int_0^t \pi_n (\pi_n (y'_n(s) \times (y'_n(s) \times h))) \times (y'_n(s) \times h) \, ds \\
- \alpha \int_0^t \pi_n (\pi_n (y'_n(s) \times (y'_n(s) \times h))) \times (y'_n(s) \times h) \, ds \right]
- \int_0^t \pi_n (y'_n(s) \times h) F(s, W'_n) \, ds + \alpha \int_0^t \pi_n (y'_n(s) \times (y'_n(s) \times h)) F(s, W'_n) \, ds
\]

for all \( t \in [0,T] \). Recall that for each natural number \( n \), the distributions of the random elements

\[
(y_n, W) : (\Omega, \mathcal{F}, P) \to C([0,T]; \mathbb{H}_n) \times C([0,T]; \mathbb{R}) \quad \text{and} \quad (y'_n, W'_n) : (\Omega', \mathcal{F}', P') \to C([0,T]; \mathbb{H}_n) \times C([0,T]; \mathbb{R})
\]

are equal. Since the expressions on the right hand sides of equalities \((26)\) and \((28)\) are the same measurable function of \((y_n, W)\) and \((y'_n, W'_n)\), respectively, the random variables \(M_n(t)\) and \(M'_n(t)\) have the same distribution on \(\mathbb{H}\). This fact and the Itô integral representation in \((27)\) will be used in the next section to show that \(M'_n(t)\) also has an Itô integral representation. The main result of this section is the following lemma, which examines convergence of the Bochner integral representation on the right hand side of \((28)\) as \( n \) goes to infinity.

**Lemma 7.** For each \( t \in (0,T) \), \( M'_n(t) \) converges weakly in \( L^2(\Omega'; \mathbb{H}^{-\frac{1}{2}}) \) as \( n \) goes to
infinity, to the limit

\[ M'(t) := y'(t) - u_0 - \int_0^t (y'(s) \times \Delta y'(s)) \, ds + \alpha \int_0^t (y'(s) \times (y'(s) \times \Delta y'(s))) \, ds + \frac{1}{2} \varepsilon \left[ \int_0^t ((y'(s) \times h) \times h) \, ds \right. \\
- \alpha \int_0^t \psi(|y'(s)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}). \\
\left. + \int_0^t (y'(s) \times ((y'(s) \times h) \times h)) \, ds \right] \\
- \alpha \int_0^t \psi^2(|y'(s)|_{L^\infty}) \psi^2(|y'(s) \times h|_{L^\infty}) \psi^2(|y'(s) \times (y'(s) \times h)|_{L^\infty}). \\
\left. - \alpha \int_0^t (y'(s) \times ((y'(s) \times h) \times h)) \, ds \right] \\
- \int_0^t (y'(s) \times h) F(s, W') \, ds + \alpha \int_0^t (y'(s) \times (y'(s) \times h)) F(s, W') \, ds. \quad (29) \]

**Proof.** For each \( n \in \mathbb{N} \) and \( t \in [0, T] \) we write

\[ M_n'(t) = y_n'(t) - \pi_n u_0 + \sum_{i=1}^{10} C_i \int_0^t v_{n,i}(s) \, ds, \]

where \( v_{n,i} : [0, T] \times \Omega' \to \mathbb{H} \) is the integrand of the \( i \)th integral on the right hand side of (28) and \( C_i \) is the constant coefficient of the integral. The estimates in (21) to (24) imply that each \( v_{n,i} \) belongs to \( L^2([0, T] \times \Omega'; \mathbb{H}) \). We also write

\[ M'(t) = y'(t) - u_0 + \sum_{i=1}^{10} C_i \int_0^t v_i(s) \, ds \quad \text{for all } t \in [0, T], \]

where \( v_i : [0, T] \times \Omega' \to X^{-\frac{1}{2}} \) is the integrand of the \( i \)th integral on the right hand side of equality (24). The estimates in Lemma 6 imply that each \( v_i \) belongs to \( L^2([0, T] \times \Omega'; X^{-\frac{1}{2}}) \). Let \( t \in (0, T] \). We want to show that, for each \( i \in \{1, \ldots, 10\} \), \( \int_0^t v_{n,i}(s) \, ds \) converges to \( \int_0^t v_i(s) \, ds \) weakly in \( L^2(\Omega'; X^{-\frac{1}{2}}) \) as \( n \) goes to infinity. For any function \( U \) in \( L^2(\Omega'; X^\frac{1}{2}) \), we have

\[ E_{X^{-\frac{1}{2}}} \left( \int_0^t v_{n,i}(s) \, ds, U \right)_{X^{\frac{1}{2}}} = E_{X^{-\frac{1}{2}}} \left( \int_0^t v_{n,i}(s) \, ds, U \right)_{X^{\frac{1}{2}}} ds \]

23
and the same equality holds with \( v_i \) in place of \( v_{n,i} \). Since the function \((s, \omega') \in [0, T] \times \Omega' \mapsto 1_{(0, t]}(s) U(\omega')\) belongs to \(L^2([0, T] \times \Omega'; X^{\frac{1}{2}})\), it suffices to show that \( v_{n,i} \) converges to \( v_i \) weakly in \( L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})\). The next lemma will help us to show this.

**Lemma 8.** We have the following convergences as \( n \) goes to \( \infty \):

1. \( E' \left[ \sup_{s \in [0, T]} |\pi_n(y'_n(s) \times (y'(s) \times h)) - y'(s) \times (y'(s) \times h)|_H^4 \right] \to 0; \)

2. \( E' \left[ \sup_{s \in [0, T]} |\psi(|\pi_n(y'_n(s) \times (y'(s) \times h))|_{L^\infty}) - \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty})|^4 \right] \to 0; \)

3. \( E' \left[ \sup_{s \in [0, T]} |\pi_n(y'_n(s) \times h) - y'(s) \times h|_H^4 \right] \to 0; \)

4. \( E' \left[ \sup_{s \in [0, T]} |\psi(|\pi_n(y'_n(s) \times h)|_{L^\infty}) - \psi(|y'(s) \times h|_{L^\infty})|^4 \right] \to 0; \)

5. \( E' \left[ \sup_{s \in [0, T]} |\psi(|y'_n(s)|_{L^\infty}) - \psi(|y'(s)|_{L^\infty})|^4 \right] \to 0. \)

**Proof.** We only give proofs of statements (1.) and (2.) here; proofs of statements (3.), (4.) and (5.) are similar to these. We will use estimates from section 3 and also the following interpolation estimate (see, for example, [1] Theorem 5.8):

\[
|u|_{L^\infty} \leq C |u|_H^{\frac{3}{4}} |u|_{H^1}^{\frac{1}{4}} \quad \forall u \in H^1. \tag{30}
\]

**Proof of statement (1.).** For each \( s \in [0, T] \) we have

\[
|\pi_n(y'_n(s) \times (y'_n(s) \times h)) - y'(s) \times (y'(s) \times h)|_H \\
\leq |y'_n(s) \times (y'_n(s) \times h) - y'(s) \times (y'(s) \times h)|_H + |(I_H - \pi_n)(y'(s) \times (y'(s) \times h))|_H. \tag{31}
\]

The first term on the right hand side of inequality (31) is estimated as follows:

\[
|y'_n(s) \times (y'_n(s) \times h) - y'(s) \times (y'(s) \times h)|_H \\
\leq |(y'_n(s) - y'(s)) \times (y'_n(s) \times h)|_H + |y'(s) \times ((y'_n(s) - y'(s)) \times h)|_H \\
\leq C |y'_n(s) - y'(s)|_H (|y'_n(s)|_{H^1} + |y'(s)|_{H^1}) |h|_{L^\infty}.
\]

This yields

\[
\sup_{s \in [0, T]} |y'_n(s) \times (y'_n(s) \times h) - y'(s) \times (y'(s) \times h)|_H^4 \\
\leq C(h) \sup_{s \in [0, T]} |y'_n(s) - y'(s)|_{H^1}^4 \left( \sup_{s \in [0, T]} |y'_n(s)|_{H^1}^4 + \sup_{s \in [0, T]} |y'(s)|_{H^1}^4 \right)
\]

24
and, by the Cauchy-Schwarz inequality, inequality \(22\) and Lemma 6 parts (ii) and (iv), we have
\[
E'[\sup_{s \in [0,T]} |y_n'(s) \times (y_n'(s) \times h) - y'(s) \times (y'(s) \times h)|_{H}^2] \\
\leq C(\alpha, T, M, u_0, h)(E'[\sup_{s \in [0,T]} |y_n'(s) - y'(s)|_{H}^2])^{\frac{1}{2}} \\
\to 0 \text{ as } n \to \infty.
\]

(32)

Now we consider the second term on the right hand side of inequality \(31\). The process \((y'(t) \times (y'(t) \times h))_{t \in [0,T]}\) has uniformly continuous paths in \(H\), therefore, by Lebesgue’s dominated convergence theorem, we have
\[
E'[\sup_{s \in [0,T]} |J_{\Omega}(y'(s) \times (y'(s) \times h))|_{H}^2] \to 0
\]
as \(n\) goes to infinity. This proves (1.).

Proof of (2.). Denote the derivative of \(\psi\) by \(\psi'\). For each \(s \in [0,T]\) we have
\[
|\psi(\pi_n(y_n'(s) \times (y_n'(s) \times h)))_{L^\infty}) - \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty})|
\leq \sup_{r \in \mathbb{R}} |\psi'(r)||\pi_n(y_n'(s) \times (y_n'(s) \times h)) - y'(s) \times (y'(s) \times h)|_{L^\infty}
\leq C \sup_{r \in \mathbb{R}} |\psi'(r)||\pi_n(y_n'(s) \times (y_n'(s) \times h)) - y'(s) \times (y'(s) \times h)|_{H}^{\frac{1}{2}}
\]

where the second inequality uses the interpolation inequality \(30\). Thus, we have
\[
E'[\sup_{s \in [0,T]} |\psi(\pi_n(y_n'(s) \times (y_n'(s) \times h)))_{L^\infty}) - \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty})|^4]
\leq C(\psi, h) \left( E'[\sup_{s \in [0,T]} |\pi_n(y_n'(s) \times (y_n'(s) \times h)) - y'(s) \times (y'(s) \times h)|_{H}^2]^{\frac{1}{2}} \right). \\
\{ \left( E'[\sup_{s \in [0,T]} |y_n'(s)|_{H}^2]^{\frac{1}{2}} \right) + \left( E'[\sup_{s \in [0,T]} |y'(s)|_{H}^2]^{\frac{1}{2}} \right) \}
\to 0 \text{ as } n \to \infty \text{ by (1)}.;
\]
on the right hand side of the above inequality, the two Cauchy-Schwarz factors are written over two lines and \(\cdot\) denotes their product. This proves (2.).

We will split the task of showing that \(v_{n,i}\) converges to \(v_i\) weakly in \(L^2([0,T] \times \Omega'; X^{-\frac{1}{2}})\) into three cases: terms involving \(\Delta\) \((i = 1, 2)\), terms involving \(F\) \((i = 9, 10)\) and the other terms \((i = 3, \ldots, 8)\).

Convergence of \(v_{n,i}\) to \(v_i\) for \(i = 1, 2\). By Lemma 6(v), \(v_{n,1} = \pi_n(y_n' \times \Delta y_n')\) converges weakly in \(L^2([0,T] \times \Omega'; X^{-\frac{1}{2}})\) to \(v_1 = y' \times \Delta y'\).
By Lemma 6(vi), $v_{n,2} = \pi_n(y'_n \times (y'_n \times \Delta y'_n))$ converges weakly in $L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})$ to $v_2 = y' \times (y' \times \Delta y')$.

**Convergence of $v_{n,i}$ to $v_i$ for $i = 9, 10$.** We only prove convergence of $v_{n,10}$ to $v_{10}$ in $L^2([0, T] \times \Omega'; X^{-\frac{3}{2}})$. The proof for $i = 9$ is similar.

Let $\eta > 0$. Let $w$ denote Wiener measure on $(C([0, T]; \mathbb{R}), \mathcal{B}_{C([0, T]; \mathbb{R}))}$. Recall that 

$$\int_0^T F^2(s, z) \, ds \leq M$$

for every $z \in C([0, T]; \mathbb{R})$. Choose a bounded, measurable function $F : [0, T] \times C([0, T]; \mathbb{R}) \to \mathbb{R}$ such that

$$\begin{align*}
a) \quad & \hat{F}(t, \cdot) \text{ is continuous on } C([0, T]; \mathbb{R}) \text{ for each fixed } t \in [0, T] \\
b) \quad & \int_{C([0, T]; \mathbb{R})} \int_0^T |F(s, z) - \hat{F}(s, z)|^2 \, ds \, dw(z) < \eta^2. \quad (34)
\end{align*}$$

Such a function $\hat{F}$ exists because we can firstly approximate the $(\mathcal{H}_0^2)$-predictable function $F$ in $L^2([0, T] \times C([0, T]; \mathbb{R})$, $ds \times w; \mathbb{R})$ by an elementary process which is a linear combination of indicators of predictable rectangles $1_{(b,c) \times B}$, where $0 \leq b < c \leq T$ and $B \in \mathcal{H}_0^2$. We can then approximate each indicator $1_B$ by $1_G$, for a suitable open set $G \supset B$ and we can approximate the lower semicontinuous function $1_G$ from below by a continuous function $\phi : C([0, T]; \mathbb{R}) \to [0, 1]$.

By Lemma 3(1.), we have

$$\begin{align*}
E' \int_0^T & \pi_n(y'_n(s) \times (y'_n(s) \times h)) - y'(s) \times (y'(s) \times h) \right)^2 \left(\mathbb{X}^{-\frac{1}{2}} \right) F^2(s, W'_n) \, ds \\
& \leq ME' \left( \sup_{t \in [0, T]} |\pi_n(y'_n(t) \times (y'_n(t) \times h)) - y'(s) \times (y'(s) \times h) \right)^2 \left(\mathbb{X}^{-\frac{1}{2}} \right) \to 0 \text{ as } n \to \infty.
\end{align*}$$

We also have

$$\begin{align*}
E' \int_0^T & |y'(s) \times (y'(s) \times h)|^2 \left(\mathbb{X}^{-\frac{1}{2}} \right) \left( F(s, W'_n) - \hat{F}(s, W'_n) \right)^2 \, ds \\
& \leq C|u_0|^2_{\mathbb{H}}|h|^2_{L^\infty} E' \int_0^T \left( F(s, W'_n) \right)^2 \, ds \\
& \leq C|u_0|^2_{\mathbb{H}}|h|^2_{L^\infty} \eta^2, \text{ by (34)};
\end{align*}$$

here we used the estimate: $|u \times (v \times w)| \right(\mathbb{X}^{-\frac{1}{2}} \right) \leq C|u|_{\mathbb{H}}|v|_{\mathbb{H}}|w|_{L^\infty}$ $\forall u, v \in \mathbb{H}$ and $w \in L^\infty$.

We also have

$$\begin{align*}
E' \int_0^T & |y'(s) \times (y'(s) \times h)|^2 \left(\mathbb{X}^{-\frac{1}{2}} \right) \left( \hat{F}(s, W'_n) - \hat{F}(s, W'_n) \right)^2 \, ds \\
& \leq C|u_0|^2_{\mathbb{H}}|h|^2_{L^\infty} E' \int_0^T \left( \hat{F}(s, W'_n) - \hat{F}(s, W'_n) \right)^2 \, ds \to 0 \text{ as } n \to \infty
\end{align*}$$

by Lebesgue’s dominated convergence theorem. Here the continuity of $\hat{F}(s, \cdot)$ ensures pointwise convergence of the integrand to zero.
Finally we have
\[
E' \int_0^T |y'(s) \times (y'(s) \times h)|^2_{X^{-\frac{1}{2}}} (\hat{F}(s, W') - F(s, W'))^2 \, ds \\
\leq C|u_0|_{H^1}^2 |h|_{L^\infty}^2 \eta^2 \quad \text{by (34)}.
\]

Since $\eta > 0$ is arbitrary, we are done. Notice that we implicitly used the fact that the common distribution of the processes $W'_n$ and $W'$ is Wiener measure; this was established when we invoked Skorohod’s theorem in Proposition 5.

**Convergence of $v_{n,i}$ to $v_i$ for $i = 3, \ldots, 8$.** The proofs of convergence of $v_{n,i}$ to $v_i$ for $i = 3, \ldots, 8$ use the results in Lemma 6 and Lemma 8, and are similar to each other. We shall only outline the details for $i = 7$, namely that
\[
\psi^2(|y'_n|_{L^\infty}) \psi^2(|\pi_n(y'_n \times h)|_{L^\infty}) \psi^2(|\pi_n(y'_n \times (y'_n \times h))|_{L^\infty}) \pi_n(y'_n \times (y'_n \times h)) \times (y'_n \times h)
\]
converges in $L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})$ to
\[
\psi^2(|y'|_{L^\infty}) \psi^2(|\pi_{y'}(y' \times h)|_{L^\infty}) \psi^2(|\pi_{y'}(y' \times (y' \times h))|_{L^\infty}) \pi_{y'}(y' \times (y' \times h)) \times (y' \times h).
\]

Firstly, observe that $\pi_n(y'_n \times (y'_n \times h)) \times (y'_n \times h))$ converges to $(\pi' \times (y' \times h)) \times (y' \times h)$ in $L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})$ because of Lemma 8 and Lemma 9 and the following three estimates which hold $P'$-almost everywhere:
\[
|\pi_n(y'_n \times (y'_n \times h)) - (y'_n \times (y'_n \times h)) \times (y'_n \times h)|_{L^\infty} \\
\leq C|u_0|_{H^1}|h|_{L^\infty} \sup_{r \in [0, T]} |\pi_n(y'_n(r) \times (y'_n(r) \times h)) - (y'_n \times (y'_n \times h))|_{H^1} \quad \forall s \in [0, T]
\]
and
\[
|(y'(s) \times (y'(s) \times h)) \times ((y'_n(s) - y'(s)) \times h)|_{X^{-\frac{1}{2}}} \\
\leq C|u_0|_{H^1}|h|_{L^\infty}^2 \sup_{r \in [0, T]} |y'(r)|_{H^1} \sup_{r \in [0, T]} |y'_n(r) - y'(r)|_{H^1} \quad \forall s \in [0, T]
\]
and, providing an integrable dominating function,
\[
|\pi_n - I_{H^1}((y'(s) \times (y'(s) \times h)) \times (y'_n \times h))|_{X^{-\frac{1}{2}}} \\
\leq C|u_0|_{H^1}^2 |h|_{L^\infty}^2 \sup_{r \in [0, T]} |y'(r)|_{H^1} \quad \forall s \in [0, T].
\]

We also have that $\psi^2(|y'_n|_{L^\infty}) \psi^2(|\pi_n(y'_n \times h)|_{L^\infty}) \psi^2(|\pi_n(y'_n \times (y'_n \times h))|_{L^\infty}) \pi_n(y'_n \times (y'_n \times h)) \times (y'_n \times h)$ converges in $L^4([0, T] \times \Omega'; \mathbb{R})$ because of estimates such as
\[
|\psi^2(|\pi_n(y'_n \times (y'_n \times h))|_{L^\infty}) \psi^2(|\pi_n(y'_n \times (y'_n \times h))|_{L^\infty}) - \psi^2(|y'(s) \times (y'(s) \times h)|_{L^\infty})| \\
\leq 2 \sup_{r \in [0, T]} |\psi(|\pi_n(y'_n \times (y'_n \times h))|_{L^\infty}) - \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty})| \forall s \in [0, T]
\]
and Lemma 8. It is straightforward to combine these convergence results to show that $v_{n,7}$ converges to $\nu_{\tau}$ in $L^2([0, T] \times \Omega'; X^{-\frac{1}{2}})$.

This completes the proof of Lemma 7. ∎

27
5 Identifying \((y', W')\) as a solution: the Itô integrals of the equation

In this section we show that \(W'\) is a Wiener process on \((\Omega, \mathcal{F}, P')\) and is a suitable integrator for functions of \(y'\). We then express \(M'\) as an Itô integral with respect to the integrator \(W'\). When this result is combined with the definition of \(M'\) from the previous section, we obtain the stochastic integral equation satisfied by \((y', W')\).

**Lemma 9.** i) For each \(n \in \mathbb{N}\), \((W_n'(t))_{t \in [0, T]}\) is a Brownian motion on \((\Omega, \mathcal{F}, P')\) and if \(0 \leq s < t \leq T\) then \(W_n'(t) - W_n'(s)\) is independent of the \(\sigma\)-algebra \(\sigma(W_n'(r), y_n'(r) : r \in [0, s])\). Likewise, \((W'(t))_{t \in [0, T]}\) is a Brownian motion on \((\Omega', \mathcal{F}', P')\) and if \(0 \leq s < t \leq T\) then \(W'(t) - W'(s)\) is independent of the \(\sigma\)-algebra \(\sigma(W'(r), y'(r) : r \in [0, s])\).

ii) For each \(t \in (0, T]\) we have

\[
M'(t) = \varepsilon \frac{1}{2} \int_0^t y'(s) \times h \, dW'(s) - \frac{\alpha \varepsilon^2}{2} \int_0^t \psi(\|y'(s)\|_{L^\infty}) \psi(\|y'(s) \times h\|_{L^\infty}) \cdot \psi(y'(s) \times (y'(s) \times h)) \, dW'(s) \quad (35)
\]

\(P'\)-almost everywhere.

**Proof.** Proof of (i). For each \(n \in \mathbb{N}\), the random variable \(W_n' : (\Omega, \mathcal{F}, P) \rightarrow C([0, T]; \mathbb{R})\) has the same distribution as \(W : (\Omega, \mathcal{F}, P) \rightarrow C([0, T]; \mathbb{R})\) and the sequence \((W_n')\) converges pointwise \(P'\)-almost everywhere to \(W'\). Therefore, for arbitrary \(k \in \mathbb{N}\) and \(0 \leq t_1 \leq \ldots \leq t_k \leq T\) and \((\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k\), we have

\[
E[\exp(i \sum_{j=1}^k \lambda_j W(t_j))] = E'[\exp(i \sum_{j=1}^k \lambda_j W_n'(t_j))] \quad \forall n \in \mathbb{N}
\]

\[
\rightarrow E'[\exp(i \sum_{j=1}^k \lambda_j W'(t_j))] \quad \text{as } n \rightarrow \infty.
\]

Thus, for each \(n \in \mathbb{N}\), \(W_n'\) and \(W'\) have continuous paths and the same finite-dimensional distributions as \(W\), which implies that \(W_n'\) and \(W'\) are Brownian motions on \((\Omega', \mathcal{F}', P')\).

Let \(0 \leq s < t \leq T\). Let \(n \in \mathbb{N}\). Recall that the process \((y_n(r))_{r \in [0, T]}\) is adapted to the filtration \((\mathcal{G}_r)_{r \in [0, T]}\) generated by \((W(r))_{r \in [0, T]}\) and augmented in the usual way. We define the \(\pi\)-system:

\[
\mathcal{P}_n := \{ \{y_n(r_1) \in B_1\} \cap \ldots \cap \{y_n(r_k) \in B_k\} \cap \{W(r_1) \in C_1\} \cap \ldots \cap \{W(r_k) \in C_k\} : k \in \mathbb{N}\text{ and } 0 \leq r_1 < \ldots < r_k \leq s \text{ and } B_1, \ldots, B_k \text{ are open sets of } X^{-\frac{1}{2}} \text{ and } C_1, \ldots, C_k \text{ are open sets of } \mathbb{R}\}.
\]

This \(\pi\)-system generates the \(\sigma\)-algebra \(\sigma(y_n(r), W(r) : r \in [0, s])\), which is contained in \(\mathcal{G}_s\) and independent of \(W(t) - W(s)\). Let \(\mathcal{P}_n'\) and \(\mathcal{P}'\) be the corresponding \(\pi\)-systems generating \(\sigma(y_n'(r), W_n'(r) : r \in [0, s])\) and \(\sigma(y'(r), W'(r) : r \in [0, s])\), respectively. We want to show that \(\mathcal{P}_n'\) and \(W_n'(t) - W_n'(s)\) are independent and that \(\mathcal{P}'\) and \(W'(t) - W'(s)\) are independent.
Let \( k \in \mathbb{N} \) and let \( 0 \leq r_1 < \cdots < r_k \leq s \). Instead of working with indicators of open subsets \( B_j \subset X^{-T} \) or \( C_j \subset \mathbb{R} \), we can work with continuous and bounded functions. This is because the indicator of an open set is lower semicontinuous and there is an increasing sequence of non-negative continuous functions that converges pointwise to it. Let \( \phi_1, \ldots, \phi_k \) be continuous and bounded real-valued functions on \( X^{-T} \) and let \( \nu_1, \ldots, \nu_k, \nu_{k+1} \) be continuous and bounded real-valued functions on \( \mathbb{R} \). We have:

\[
E[\phi_1(y_n(r_1)) \cdots \phi_k(y_n(r_k)) \nu_1(W(r_1)) \cdots \nu_k(W(r_k)) \nu_{k+1}(W(t) - W(s))] = E[\phi_1(y_n(r_1)) \cdots \phi_k(y_n(r_k)) \nu_1(W(r_1)) \cdots \nu_k(W(r_k))] E[\nu_{k+1}(W(t) - W(s))]. \tag{36}
\]

Because the distribution of \((y'_n, W'_n)\) is the same as that of \((y_n, W)\), we can replace \( E \) by \( E' \), \( y_n \) by \( y'_n \) and \( W \) by \( W'_n \) in equality (36) to obtain:

\[
E'[\phi_1(y'_n(r_1)) \cdots \phi_k(y'_n(r_k)) \nu_1(W'_n(r_1)) \cdots \nu_k(W'_n(r_k)) \nu_{k+1}(W'_n(t) - W'_n(s))] = E'[\phi_1(y'_n(r_1)) \cdots \phi_k(y'_n(r_k)) \nu_1(W'_n(r_1)) \cdots \nu_k(W'_n(r_k))] E'[\nu_{k+1}(W'_n(t) - W'_n(s))]. \tag{37}
\]

Letting \( n \) go to infinity in this equality, we obtain:

\[
E'[\phi_1(y'(r_1)) \cdots \phi_k(y'(r_k)) \nu_1(W'(r_1)) \cdots \nu_k(W'(r_k)) \nu_{k+1}(W'(t) - W'(s))] = E'[\phi_1(y'(r_1)) \cdots \phi_k(y'(r_k)) \nu_1(W'(r_1)) \cdots \nu_k(W'(r_k))] E'[\nu_{k+1}(W'(t) - W'(s))]. \tag{38}
\]

Equality (37) implies that \( \mathcal{P}'_n \) and \( W'_n(t) - W'_n(s) \) are independent while equality (38) implies that \( \mathcal{P}' \) and \( W'(t) - W'(s) \) are independent.

**Proof of (ii).** Fix \( t \in (0, T] \). Let \( n \in \mathbb{N} \). We firstly show that

\[
M'_n(t) = \varepsilon^{\frac{1}{2}} \int_0^t \nu_n(y'_n(s) \times h) \, dW'_n(s)
- \varepsilon^{\frac{1}{2}} \alpha \int_0^t \psi(|\pi_n(y'_n(s) \times h)|_{L^\infty})\psi(|y'_n(s)|_{L^\infty}) \cdot 
\nu_n(y'_n(s) \times h) \, dW'_n(s) \quad \text{P'} \, \text{a.e.} \tag{39}
\]

Define the partitions of \([0, T] \): \( \{s^n_m := \frac{jT}{2m} : j = 0, 1, \ldots, 2^m \} \) for \( m \in \mathbb{N} \). For each \( m \in \mathbb{N} \) the random variables in \( X^{-\frac{T}{2}} \):

\[
M_n(t) - \varepsilon^{\frac{1}{2}} \sum_{j=0}^{2^m-1} \pi_n(y_n(s^n_j \times h)) (W(t \wedge s^n_{j+1}) - W(t \wedge s^n_j)) 
+ \varepsilon^{\frac{1}{2}} \alpha \sum_{j=0}^{2^m-1} \psi(|\pi_n(y_n(s^n_j \times h)|_{L^\infty})\psi(|y_n(s^n_j)|_{L^\infty}) \cdot 
\pi_n(y_n(s^n_j \times h)) (W(t \wedge s^n_{j+1}) - W(t \wedge s^n_j)) \tag{40}
\]
and

\[
M_n'(t) - \frac{\varepsilon}{\alpha} \sum_{j=0}^{2^m-1} \pi_n(y_n'(s_j^m) \times h)(W_n'(t \wedge s_{j+1}^m) - W_n'(t \wedge s_j^m)) + \frac{\varepsilon}{\alpha} \sum_{j=0}^{2^m-1} \psi(|\pi_n(y_n'(s_j^m) \times h)(y_n'(s_j^m) \times h)\|_{L^\infty}) \psi(|\pi_n(y_n'(s_j^m) \times h)|_{L^\infty}) \psi(|y_n'(s_j^m)|_{L^\infty}) \cdot \
\pi_n(y_n'(s_j^m) \times (y_n'(s_j^m) \times h))(W_n'(t \wedge s_{j+1}^m) - W_n'(t \wedge s_j^m))
\]

(41)

have the same distribution because they are obtained by applying the same measurable mapping to \((y_n, W)\) and \((y_n', W_n')\), respectively, and these latter two random elements in \(C([0, T]; \mathbb{H}_n) \times C([0, T]; \mathbb{R})\) have the same distribution. Notice that the sums in (40) and (41) are Itô integrals of elementary processes in \(X^{-\frac{1}{2}}\) on the partition \(\{s_j^m : j = 0, \ldots, 2^m\}\). By Itô’s isometry, as \(m\) goes to infinity, the random variable in (40) converges in \(L^2(\Omega, \mathcal{F}, P; X^{-\frac{1}{2}})\) to

\[
M_n(t) - \frac{\varepsilon}{\alpha} \int_0^t \pi_n(y_n(s) \times h) \, dW(s)
\]

\[
+ \frac{\varepsilon}{\alpha} \int_0^t \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))\|_{L^\infty}) \psi(|\pi_n(y_n(s) \times h)|_{L^\infty}) \psi(|y_n(s)|_{L^\infty}) \cdot \
\pi_n(y_n(s) \times (y_n(s) \times h)) \, dW_n'(s) = 0 \quad P \text{-a.e.,}
\]

(42)

where equality to zero follows from equality (27). Similarly, the random variable in (41) converges in \(L^2(\Omega', \mathcal{F}', P'; X^{-\frac{1}{2}})\) to

\[
M_n'(t) - \frac{\varepsilon}{\alpha} \int_0^t \pi_n(y_n'(s) \times h) \, dW_n'(s)
\]

\[
+ \frac{\varepsilon}{\alpha} \int_0^t \psi(|\pi_n(y_n'(s) \times (y_n(s) \times h))\|_{L^\infty}) \psi(|\pi_n(y_n'(s) \times h)|_{L^\infty}) \psi(|y_n'(s)|_{L^\infty}) \cdot \
\pi_n(y_n'(s) \times (y_n'(s) \times h)) \, dW_n'(s)
\]

and, since this limit has the same distribution as that in (42), it is zero \(P'\)-almost everywhere. This proves (33).

Recall that in Lemma 7 we showed that \(M_n'(t)\), defined as a Bochner integral, converges weakly in \(L^2(\Omega', \mathcal{F}', P'; X^{-\frac{1}{2}})\) to \(M'(t)\). Therefore, to prove (33) it suffices now to show that \(M_n'(t)\), expressed as an Itô integral, converges in \(L^2(\Omega', \mathcal{F}', P'; X^{-\frac{1}{2}})\) to

\[
\varepsilon \int_0^t (y'(s) \times h) \, dW'(s)
\]

\[
- \frac{\varepsilon}{\alpha} \int_0^t \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \cdot \
(y'(s) \times (y'(s) \times h)) \, dW'(s)
\]

(43)
as \( n \) goes to infinity. We shall only prove that the second Itô integral on the right hand side of (39),
\[
\int_0^t \psi(|\pi_n(y_n(s) \times (y_n(s) \times h))|_{L^\infty}) \psi(|\pi_n(y_n'(s) \times h)|_{L^\infty}) \psi(|y_n'(s)|_{L^\infty}) \cdot 
\]
\[
\pi_n(y_n'(s) \times (y_n'(s) \times h)) \, dW'_n(s), \tag{44}
\]
converges in \( L^2(\Omega', \mathcal{F}', P'; X^{-\frac{1}{2}}) \) to the second Itô integral of the expression in (43):
\[
\int_0^t \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \cdot 
\]
\[
(y'(s) \times (y'(s) \times h)) \, dW'(s). \tag{45}
\]
The proof that the first Itô integral on the right hand side of (39) converges to the first Itô integral in (43) is essentially the same.

Let \( \eta > 0 \). Lemma (iii) implies that \( y' \) has continuous paths in \( L^\infty(\Lambda; \mathbb{R}^3) \). It follows that the process \( (\psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) y'(s) \times (y'(s) \times h))_{s \in [0, T]} \) has continuous paths in \( X^{-\frac{1}{2}} \) and, furthermore, it is bounded on \( [0, T] \times \Omega' \); hence we can approximate it in \( L^2([0, T] \times \Omega'; X^{-\frac{1}{2}}) \) by elementary processes on the partitions \( \{s_j^n: j = 0, \ldots, 2^n\} \). Let us fix \( m \in \mathbb{N} \) such that the expectation
\[
E' \int_0^T \int_{\Omega'} \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) (y'(s) \times (y'(s) \times h))
\]
\[
- \sum_{j=0}^{2^n-1} 1_{[s_j^n, s_{j+1}^n]}(s) \psi(|y'(s_{j}^m) \times (y'(s_{j}^m) \times h)|_{L^\infty}) \psi(|y'(s_{j}^m) \times h|_{L^\infty}) \psi(|y'(s_{j}^m)|_{L^\infty}) \cdot 
\]
\[
(y'(s_{j}^m) \times (y'(s_{j}^m) \times h)) \, ds
\]
is less than \( \eta^2/4 \).

We will now show that for all sufficiently large \( n \), the \( L^2(\Omega', \mathcal{F}', P'; X^{-\frac{1}{2}}) \)-distance between the integrals in (44) and (45) is less than \( \eta \). The proof is obtained by combining three facts which we list below.

**Fact 1.** Our choice of \( m \) ensures that for all sufficiently large \( n \in \mathbb{N} \) there is good approximation of the Itô integral involving \( y'_n \) and \( W'_n \) in (44) by that of the corresponding
elementary process on the partition \( \{ s^m_j : j = 0, \ldots, 2^m \} \). We have, by Itô’s isometry:

\[
\begin{align*}
(E' & \left[ \int_0^t \left\{ \psi(|\pi_n(y'_n(s) \times (y'_n(s) \times h))|_{L^\infty} \psi(|\pi_n(y'_n(s) \times h)|_{L^\infty}) \psi(|y'_n(s)|_{L^\infty}) \cdot \right. \\
& \left. \pi_n(y'_n(s) \times (y'_n(s) \times h)) \right. \\
& \left. - \sum_{j=0}^{2^m-1} 1(s^m_j, s^m_{j+1}) \psi(|\pi_n(y'_n(s^m_j) \times (y'_n(s^m_j) \times h))|_{L^\infty}) \psi(|\pi_n(y'_n(s^m_j) \times h)|_{L^\infty}) \cdot \\
& \left. \psi(|y'_n(s^m_j)|_{L^\infty}) \pi_n(y'_n(s^m_j) \times h) \bigg] \\
& \bigg) \right) \right)^{1/2} \\
& \leq \left( E' \int_0^t \left| \psi(|\pi_n(y'_n(s) \times (y'_n(s) \times h))|_{L^\infty} \psi(|\pi_n(y'_n(s) \times h)|_{L^\infty}) \psi(|y'_n(s)|_{L^\infty}) \cdot \\
& \pi_n(y'_n(s) \times (y'_n(s) \times h)) \right. \\
& \left. - \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \cdot \\
& \left. y'(s) \times (y'(s) \times h) \right) \left( \frac{2}{X - \frac{1}{2}} ds \right)^{1/2} \\
& + \left( E' \int_0^t \left| \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) \cdot \\
& \left( y'(s) \times (y'(s) \times h) \right) \right) \left( \frac{2}{X - \frac{1}{2}} ds \right)^{1/2} \\
& + \left( E' \int_0^t \sum_{j=0}^{2^m-1} 1(s^m_j, s^m_{j+1}) \psi(|\pi_n(y'_n(s^m_j) \times (y'_n(s^m_j) \times h))|_{L^\infty}) \psi(|\pi_n(y'_n(s^m_j) \times h)|_{L^\infty}) \cdot \\
& \psi(|y'_n(s^m_j)|_{L^\infty}) y'_n(s^m_j) \times (y'(s^m_j) \times h) \right) \left( \frac{2}{X - \frac{1}{2}} ds \right)^{1/2} \\
& \left( y'(s^m_j) \times (y'_n(s^m_j) \times h) \right) \left( \frac{2}{X - \frac{1}{2}} ds \right)^{1/2} \\
& - \sum_{j=0}^{2^m-1} 1(s^m_j, s^m_{j+1}) \psi(|\pi_n(y'_n(s^m_j) \times (y'_n(s^m_j) \times h))|_{L^\infty}) \psi(|\pi_n(y'_n(s^m_j) \times h)|_{L^\infty}) \cdot \\
& \psi(|y'_n(s^m_j)|_{L^\infty}) \pi_n(y'_n(s^m_j) \times (y'_n(s^m_j) \times h)) \right) \left( \frac{2}{X - \frac{1}{2}} ds \right)^{1/2},
\end{align*}
\]

which is less than \( \eta/2 \) for all sufficiently large \( n \); this is because the second term on the right hand side is less than \( \eta/2 \) (by our choice of \( m \)), while the first and third terms on the right hand side converge to zero as \( n \) goes to infinity, by Lemma 8.

**Fact 2.** With the partition \( \{ s^m_j : j = 0, \ldots, 2^m \} \) fixed, the Itô integral with respect to \( W'_n \) of the elementary process involving \( y'_n \) converges as \( n \) goes to infinity to the Itô integral.
with respect to $W'$ of the elementary process involving $y'$. We have

$$E' \left[ \sum_{j=0}^{2^m-1} \psi(|\pi_n(y'_n(s_j^m) \times (y'_n(s_j^m) \times h))|_{L^\infty}) \psi(|\pi_n(y'_n(s_j^m) \times h)|_{L^\infty}) \psi(|y'_n(s_j^m)|_{L^\infty}) \cdot \right. $$

$$\pi_n(y'_n(s_j^m) \times (y'_n(s_j^m) \times h))(W'_n(t \wedge s_j^m) - W'_n(t \wedge s_j^m))$$

$$- \sum_{j=0}^{2^m-1} \psi(|y'(s_j^m) \times (y'(s_j^m) \times h)|_{L^\infty}) \psi(|y'(s_j^m) \times h|_{L^\infty}) \psi(|y'(s_j^m)|_{L^\infty}) \cdot$$

$$y'(s_j^m) \times (y'(s_j^m) \times h)(W'(t \wedge s_j^m) - W'(t \wedge s_j^m))^2 \left| \left. \frac{2}{X - \frac{1}{2}} \right| \right]$$

$$\leq 2^m \sum_{j=0}^{2^m-1} E' \left[ \psi(|\pi_n(y'_n(s_j^m) \times (y'_n(s_j^m) \times h))|_{L^\infty}) \psi(|\pi_n(y'_n(s_j^m) \times h)|_{L^\infty}) \psi(|y'_n(s_j^m)|_{L^\infty}) \cdot \right. $$

$$\pi_n(y'_n(s_j^m) \times (y'_n(s_j^m) \times h))(W'_n(t \wedge s_j^m) - W'_n(t \wedge s_j^m))$$

$$- \psi(|y'(s_j^m) \times (y'(s_j^m) \times h)|_{L^\infty}) \psi(|y'(s_j^m) \times h|_{L^\infty}) \psi(|y'(s_j^m)|_{L^\infty}) \cdot$$

$$y'(s_j^m) \times (y'(s_j^m) \times h)(W'(t \wedge s_j^m) - W'(t \wedge s_j^m))^2 \left| \left. \frac{2}{X - \frac{1}{2}} \right| \right]$$

$$\to 0 \quad \text{as } n \to \infty$$

by Lemma 8 and pointwise convergence of $W'_n$ to $W'$ in $C([0,T]; \mathbb{R})$ and uniform integrability of the sequence of random variables \{$(W'_n(t \wedge s_j^m) - W'_n(t \wedge s_j^m) - W'(t \wedge s_j^m))$ : $n \in \mathbb{N}$\} for $j = 0, \ldots, 2^m - 1$.

**Fact 3.** We chose $m$ to ensure that the distance between the Itô integral involving $y'$ and $W'$ in (45) and the Itô integral of the corresponding elementary process on the partition \{$s_j^m : j = 0, \ldots, 2^m$\} is less than $\eta/2$:

$$\left( E' \left[ \int_0^t \left\{ \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) y'(s) \times (y'(s) \times h) \right. \right. $$

$$- \sum_{j=0}^{2^m-1} \mathbb{1}_{(s_j^m, s_{j+1}^m)}(s) \psi(|y'(s_j^m) \times (y'(s_j^m) \times h)|_{L^\infty}) \psi(|y'(s_j^m) \times h|_{L^\infty}) \psi(|y'(s_j^m)|_{L^\infty}) \cdot$$

$$\left. y'(s_j^m) \times (y'(s_j^m) \times h) \right\} \right. \right. \left. dW'(s) \right|_{X - \frac{1}{2}} \right)$$

$$< \eta/2.$$

From these three facts and the triangle inequality we have that the $L^2(\Omega', \mathcal{F}', P'; X^{-\frac{1}{2}})$-distance between the integrals in (44) and (45) is less than $\eta$ for all sufficiently large $n$. \(\square\)

### 6 End of the proof of Theorem 1

In this section we complete the proof of Theorem 1 by showing that the values of $y'(t)$ are functions with Euclidean norm unity everywhere on $\Lambda$ for all $t \in [0,T]$ and that $(y', W')$
satisfies the equation in statement (5) of the theorem.

The stochastic integral equation satisfied by \((y', W')\) is obtained by equating the expressions for \(M'(t)\) in Lemma 7 and Lemma 9. The equation is:

\[
y'(t) = u_0 + \int_0^t y'(s) \times \Delta y'(s) \, ds - \alpha \int_0^t y'(s) \times (y'(s) \times \Delta y'(s)) \, ds
\]

\[
+ \varepsilon \int_0^t y'(s) \times h \, dW'(s)
\]

\[
- \alpha \varepsilon \int_0^t \psi(|y'(s)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \cdot
\]

\[
(y'(s) \times (y'(s) \times h)) \times h \, ds
\]

\[
+ \frac{1}{2} \varepsilon \int_0^t (y'(s) \times ((y'(s) \times h) \times h)) \, ds
\]

\[
- \frac{1}{2} \alpha \varepsilon \int_0^t \psi^2(|y'(s)|_{L^\infty}) \psi^2(|y'(s) \times h|_{L^\infty}) \psi^2(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \cdot
\]

\[
((y'(s) \times (y'(s) \times h)) \times (y'(s) \times h)) \, ds
\]

\[
- \alpha \varepsilon \int_0^t (y'(s) \times ((y'(s) \times (y'(s) \times h)) \times h)) \, ds
\]

\[
+ \int_0^t (y'(s) \times h) F(s, W') \, ds - \alpha \int_0^t (y'(s) \times (y'(s) \times h)) F(s, W') \, ds
\]

(46)

for all \(t \in [0, T]\), \(P^\varepsilon\)-almost everywhere.

Recall that we chose the initial condition \(u_0\) to be a function in \(H^1\) such that \(|u_0(x)|_{L^3} = 1\) for all \(x \in \Lambda\). Now we shall show that \(|y'(t)(x)|_{L^3} = 1\) for all \(x \in \Lambda\) and for all \(t \in [0, T]\), \(P^\varepsilon\)-almost everywhere. There exists a countable collection \(\{(\phi_n : n \in \mathbb{N}\}\) of smooth real-valued functions with compact support on \(\Lambda\) such that for any non-negative, integrable functions \(u\) and \(v\) on \(\Lambda\), \(\int_\Lambda \phi_n(x) u(x) \, dx = \int_\Lambda \phi_n(x) v(x) \, dx\) for all \(n \in \mathbb{N}\) implies that \(u = v\) almost everywhere on \(\Lambda\). Take \(n \in \mathbb{N}\) and consider the map \(u \in H \mapsto (\phi_n, u)_H = \int_\Lambda \phi_n(x) |u(x)|^2 \, dx\). When we apply this map to the process \(y'\), Itô’s formula becomes:

\[
\int_\Lambda \phi_n(x) |y'(t)(x)|_{L^3}^2 \, dx = \int_\Lambda \phi_n(x) |u_0(x)|^2 \, dx
\]

(47)

for all \(t \in [0, T]\), \(P^\varepsilon\)-almost everywhere. Since \(n \in \mathbb{N}\) is arbitrary and \(y'\) takes values in \(H^1\), we conclude that \(|y'(t)|_{L^3} = |u_0|_{L^3}\) everywhere on \(\Lambda\) for all \(t \in [0, T]\), \(P^\varepsilon\)-almost everywhere.

In equality (46), we have \(\psi(|y'(s) \times (y'(s) \times h)|_{L^\infty}) \psi(|y'(s) \times h|_{L^\infty}) \psi(|y'(s)|_{L^\infty}) = 1\) for all \(s \in [0, T]\), \(P^\varepsilon\)-almost everywhere; this is because the cutoff function \(\psi : \mathbb{R} \to [0, 1]\) has the
property \( \psi(x) = 1 \) for all \( x \in [0, |h|_{L^\infty} + 1] \). Therefore \((y', W')\) satisfies the equation in statement (5) of Theorem 1.

### 7 Pathwise uniqueness of weak martingale solutions

In this section we show that weak martingale solutions of equation (6) are pathwise unique. The main result is in Theorem 11.

Theorem 11 requires Lemma 10 which says that if \( u \) is an element of \( H^1 \) and \( |u|_{R^3} = 1 \) almost everywhere on \( \Lambda \) then

\[
u \times (u \times \Delta u) = -|Du|_{R^3}^2 u - \Delta u\] as elements of \((H^1)'\),

where the first term on the right hand side of (48) is defined by

\[
\langle H^1 \rangle \langle |Dw|^2_{R^3} z, \phi \rangle_{H^1} := \int_{\Lambda} |Dw(y)|^2_{R^3} \langle z(y), \phi(y) \rangle_{R^3} \, dy \quad \forall w, z \text{ and } \phi \in H^1.
\]

The proof of (48) uses the elementary identity: \( a \times (b \times c) = \langle a, c \rangle_{R^3} b - \langle a, b \rangle_{R^3} c \), for all vectors \( a, b \) and \( c \) in \( R^3 \).

**Lemma 10.** Let \( u \) be an element of \( H^1 \) such that

\[
|u(x)|_{R^3} = 1 \quad \text{for all } x \in \Lambda.
\]

Then we have

\[
u \times (u \times \Delta u) = -|Du|_{R^3}^2 u - \Delta u
\]
as elements of \((H^1)'\).

**Proof.** The condition (49) implies that \( \langle u, Du \rangle_{R^3} = \frac{1}{2} D(|u|_{R^3}^2) = 0 \) almost everywhere on \( \Lambda \) and thus for any \( \phi \in H^1 \):

\[
\langle H^1 \rangle \langle u \times (u \times \Delta u), \phi \rangle_{H^1} = \langle u \times D(\phi \times u), Du \rangle_{H^1} = \langle |u|_{R^3}^2 D\phi, Du \rangle_{H} - \langle \langle u, D\phi \rangle_{R^3} u, Du \rangle_{H} + \langle \langle u, Du \rangle_{R^3} \phi, Du \rangle_{H} - \langle \langle u, \phi \rangle_{R^3} Du, Du \rangle_{H} = \langle H^1 \rangle \langle -\Delta u, \phi \rangle_{H^1} + \langle H^1 \rangle \langle -|Du|_{R^3}^2 u, \phi \rangle_{H^1}.
\]

The setting of the following theorem is a probability space \((\Omega, F, (F_t)_{t \in [0,T]}, P)\) with filtration \((F_t)\) and \((W(t))_{t \in [0,T]}\) is a real-valued \((F_t)\)-Wiener process defined on this space.

**Theorem 11** (Pathwise uniqueness). Let \( u_1, u_2 : [0, T] \times \Omega \to H \) be measurable and \((F_t)\)-adapted functions such that, for each \( i \), paths of \( u_i \) lie in \( C([0, T]; H) \cap L^3(0, T; H^1) \) and \( u_i \)
satisfies the equation

\[ u_i(t) = u_0 + \int_0^t u_i(s) \times \Delta u_i(s) \, ds - \alpha \int_0^t u_i(s) \times (u_i(s) \times \Delta u_i(s)) \, ds \\
+ \varepsilon^{1\over 2} \int_0^t u_i(s) \times h \, dW(s) - \alpha \varepsilon^{1\over 2} \int_0^t u_i(s) \times (u_i(s) \times h) \, dW(s) \\
+ \frac{1}{2}\varepsilon \int_0^t [u_i(s) \times h - \alpha(u_i(s) \times (u_i(s) \times h))] \times h \\
- \alpha(u_i(s) \times ((u_i(s) \times h) \times h) - \alpha((u_i(s) \times (u_i(s) \times h)) \times (u_i(s) \times h))) \times h) \, ds \\
+ \int_0^t (u_i(s) \times h) F(s,W) \, ds - \alpha \int_0^t (u_i(s) \times (u_i(s) \times h)) F(s,W) \, ds \tag{50} \]

for all \( t \in [0,T], \ P\text{-almost everywhere.} \) Then we have:

1. \( |u_i(t)(x)|_{\mathbb{R}^3} = 1 \) for almost every \( x \in \Lambda \) for all \( t \in [0,T], \ P\text{-almost everywhere}; \)
2. \( u_1(\cdot,\omega) = u_2(\cdot,\omega) \) for \( P\text{-almost every } \omega \in \Omega. \)

**Proof.** We will not give the details of the proof of item (1.); it is similar to the proof in section \[ \Box \] As in that proof, one can use Itô’s formula to show that \( \langle \phi_n u_i(t), u_i(t) \rangle_H = \langle \phi_n u_0, u_0 \rangle_H \) for all \( t \in [0,T], \ P\text{-almost everywhere, for each } n \in \mathbb{N}. \) The difference is that in this theorem the hypotheses are only about the paths of \( u_i. \) Therefore a stopping time such as \( \theta_m := \inf\{t \in [0,T] : \int_0^t |u_i(s)|^2_H \, ds \geq m\} \land T \) is useful and one can show that \( \langle \phi_n u_i(t \land \theta_m), u_i(t \land \theta_m) \rangle_H = \langle \phi_n u_0, u_0 \rangle_H \) for all \( t \in [0,T], \ P\text{-almost everywhere. When writing Itô’s formula, one can account for the fact that functions such as } u_i \times \Delta u_i \text{ are } X^{-\frac{1}{2}}\text{-valued and not necessarily } H\text{-valued in the same way as in the proof of item (2.) below.}

**Proof of (2.).** By our hypothesis on the paths of \( u_i, \) the function \( u_i \) takes values in \( H^1 \) almost everywhere on \( [0,T] \times \Omega. \) We implicitly replace \( u_i \) by \( 1_{[0,T]}(u_i)u_i \) on the right hand side of equality \([\Box]\) and then each integrand function on the right hand side of \( (\Box) \) is a measurable and \( (F_t)\text{-adapted process in } X^{-\frac{1}{2}} \) whose paths lie in \( L^2(0,T; X^{-\frac{1}{2}}). \) Thus all the integrals on the right hand side of \( (\Box) \) make sense and define continuous processes in \( X^{-\frac{1}{2}}. \)

Define the process \( v(t) := u_1(t) - u_2(t), \ t \in [0,T]. \) We will write down Itô’s formula for \( |v(t)|^2_H \) and use Gronwall’s lemma to show that \( E[\sup_{t \in [0,T]} |v(t)|^4_H] = 0. \)
From equality (50) and equality (48) we have, \( P \)-almost everywhere, for all \( t \in [0,T] \):

\[
v(t) = \alpha \int_0^t \Delta v(s) \, ds + \alpha \int_0^t |Du_1(s)|_{\mathbb{R}^3}^2 v(s) \, ds \\
+ \alpha \int_0^t (|Du_1(s)|_{\mathbb{R}^3}^3 - |Du_2(s)|_{\mathbb{R}^3}^3) (|Du_1(s)|_{\mathbb{R}^3} + |Du_2(s)|_{\mathbb{R}^3}) u_2(s) \, ds \\
+ \int_0^t v(s) \times \Delta u_1(s) \, ds + \int_0^t u_2(s) \times \Delta v(s) \, ds \\
+ \frac{1}{2} \epsilon \int_0^t (v(s) \times \tilde{h}) \times \tilde{h} \, ds \\
- \frac{1}{2} \epsilon \alpha \left[ \int_0^t (v(s) \times (u_1(s) \times \tilde{h})) \times \tilde{h} \, ds + \int_0^t (u_2(s) \times (v(s) \times \tilde{h})) \times \tilde{h} \, ds \right] \\
- \frac{1}{2} \epsilon \alpha \left[ \int_0^t v(s) \times ((u_1(s) \times \tilde{h}) \times \tilde{h}) \, ds + \int_0^t u_2(s) \times ((v(s) \times \tilde{h}) \times \tilde{h}) \, ds \right] \\
+ \frac{1}{2} \epsilon \alpha^2 \left[ \int_0^t (v(s) \times (u_1(s) \times \tilde{h})) \times (u_1(s) \times \tilde{h}) \, ds \\
+ \int_0^t (u_2(s) \times (v(s) \times \tilde{h})) \times (u_1(s) \times \tilde{h}) \, ds \\
+ \int_0^t (u_2(s) \times (u_1(s) \times \tilde{h}) \times \tilde{h}) \, ds \right] \\
+ \int_0^t (v(s) \times \tilde{h}) F(s,W) \, ds \\
- \alpha \left[ \int_0^t v(s) \times (u_1(s) \times \tilde{h}) F(s,W) \, ds + \int_0^t u_2(s) \times (v(s) \times \tilde{h}) F(s,W) \, ds \right] \\
+ \epsilon^\frac{1}{2} \int_0^t v(s) \times h \, dW(s) \\
- \epsilon^\frac{1}{2} \alpha \left[ \int_0^t v(s) \times (u_1(s) \times \tilde{h}) dW(s) + \int_0^t u_2(s) \times (v(s) \times \tilde{h}) dW(s) \right]
\]

(51)

here \( z_i : [0,T] \times \Omega \rightarrow X^{-\frac{1}{2}}, \ i = 1, \ldots, 22, \) is the \( i \)th integrand function on the right hand side of equality \( (51) \) and \( C_i \) is the constant coefficient of the \( i \)th integral. Our hypothesis on the paths of \( u_1 \) and \( u_2 \) implies that each \( z_i \) is a measurable and \((\mathcal{F}_t)\)-adapted process.
in $X^{-\frac{1}{2}}$ and has paths in $L^2(0,T;X^{-\frac{1}{2}})$.

For each $n \in \mathbb{N}$, let $\pi_n : X^{-\frac{1}{2}} \to \mathbb{H}_n$ be the orthogonal projection of $X^{-\frac{1}{2}}$ onto $\mathbb{H}_n$. For some constants $\gamma_j$, \{\gamma_j e_j : j \in \mathbb{N}\} is an orthonormal basis of $X^{-\frac{1}{2}}$, thus this definition of $\pi_n$ is an extension of the definition in [11] to the larger domain $X^{-\frac{1}{2}}$. Because some of the functions $z_i$ may not be $\mathbb{H}$-valued, we firstly write down Itô’s formula for $|\pi_n v(t)|^2_H$; this is straightforward because the functions $\pi_n z_i$ take values in $\mathbb{H}_n$ and the standard Itô’s formula applies. For each $n \in \mathbb{N}$ we have

$$\pi_n v(t) = \sum_{i=1}^{19} C_i \int_0^t \pi_n z_i(s) \, ds + \sum_{i=20}^{22} C_i \int_0^t \pi_n z_i(s) \, dW(s)$$

for all $t \in [0,T]$, $P$-almost everywhere; we also have, by Itô:

$$|\pi_n v(t)|^2_H = \sum_{i=1}^{19} 2C_i \int_0^t X^{-\frac{1}{2}} \langle z_i(s), \pi_n v(s) \rangle_{X^{\frac{1}{2}}} \, ds + \sum_{i=20}^{22} 2C_i \int_0^t \langle z_i(s), \pi_n v(s) \rangle_{\mathbb{H}} \, dW(s)$$

$$+ \int_0^t \sum_{i=20}^{22} C_i \pi_n z_i(s) \, dW$$

for all $t \in [0,T]$, $P$-almost everywhere.

Consider a fixed $t \in [0,T]$. We take limits as $n$ goes to infinity in equality (52) to get Itô’s formula for $|v(t)|^2_H$. As $n$ goes to $\infty$, $|\pi_n v(t)|^2_H$ converges to $|v(t)|^2_H$ pointwise on $\Omega$.

For $P$-almost every $\omega \in \Omega$, $v(\cdot, \omega)$ belongs to $L^2(0,T;\mathbb{H}^1)$ and $z_i(\cdot, \omega)$ belongs to $L^2(0,T;X^{-\frac{1}{2}})$ for all $i = 1, \ldots, 19$; therefore, $P$-almost everywhere, we have

$$\int_0^t X^{-\frac{1}{2}} \langle z_i(s), \pi_n v(s) \rangle_{X^{\frac{1}{2}}} \, ds \to \int_0^t X^{-\frac{1}{2}} \langle z_i(s), v(s) \rangle_{X^{\frac{1}{2}}} \, ds$$

as $n \to \infty$

for all $i = 1, \ldots, 19$.

For $P$-almost every $\omega \in \Omega$, the paths $u_1(\cdot, \omega)$ and $u_2(\cdot, \omega)$ belong to $C([0,T];\mathbb{H})$ and are bounded in $L^\infty(\Lambda;\mathbb{R}^3)$, which implies that $z_i(\cdot, \omega)$ also belongs to $C([0,T];\mathbb{H})$ for $i = 20, 21$ and 22; thus

$$\int_0^t \sum_{i=20}^{22} C_i \pi_n z_i(s, \omega) \, dW \to \int_0^t \sum_{i=20}^{22} C_i z_i(s, \omega) \, dW$$

as $n \to \infty$.

We also have for $i = 20, 21$ and 22:

$$E \int_0^T \langle (H - \pi_n) v(s), z_i(s) \rangle_{\mathbb{H}}^2 \, ds \to 0 \text{ as } n \to \infty,$$

by Lebesgue’s dominated convergence theorem, thus the Itô integral $\int_0^t \langle \pi_n v(s), z_i(s) \rangle_{\mathbb{H}} \, dW(s)$ converges in $L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ to $\int_0^t \langle v(s), z_i(s) \rangle_{\mathbb{H}} \, dW(s)$.
We conclude that
\[ |v(t)|_{\mathbb{H}}^2 = \sum_{i=1}^{19} 2C_i \int_0^t \langle x^{-\frac{1}{2}}(s), v(s) \rangle_{\mathbb{H}} \, ds + \sum_{i=20}^{22} 2C_i \int_0^t \langle v(s), z_i(s) \rangle_{\mathbb{H}} \, dW(s) \]
\[ + \int_0^t \sum_{i=20}^{22} C_i z_i(s) \, ds \]
(53)
P-almost everywhere. All the terms in equality (53) are continuous, hence the equality holds for all \( t \in [0, T] \), P-almost everywhere.

We need some estimates for the integrands \( x^{-\frac{1}{2}}(z_i, v)_{\mathbb{H}} \) on the right hand side of equality (53) in terms of \( |v|_{\mathbb{H}}^2 \). In particular, we need to consider \( x^{-\frac{1}{2}}(z_i, v)_{\mathbb{H}} \) for \( i = 1, 2, 3, 5 \) and 19; for all other values of \( i \), \( x^{-\frac{1}{2}}(z_i, v)_{\mathbb{H}} \) either vanishes or, because \( \sup_{r \in [0, T]} |u_i(r)|_{L^\infty} = 1 \), P-almost everywhere, satisfies the inequality:
\[ |x^{-\frac{1}{2}}(z_i, v)_{\mathbb{H}}| \leq C |v|_{\mathbb{H}}^2 \quad \forall s \in [0, T], \text{ P-almost everywhere.} \] (54)

All constants are denoted by \( C \).

*Estimate for \( x^{-\frac{1}{2}}(z_1(s), v(s))_{\mathbb{H}} \).* We have
\[ x^{-\frac{1}{2}}(\Delta v(s), v(s))_{\mathbb{H}} = -|Dv(s)|_{\mathbb{H}}^2 \] (55)
for all \( s \in [0, T] \). The negative sign means that this term can absorb the \( |Dv(s)|_{\mathbb{H}}^2 \) terms coming from estimates of the other integrands.

*Estimate for \( x^{-\frac{1}{2}}(z_2(s), v(s))_{\mathbb{H}} \).* We have
\[ |x^{-\frac{1}{2}}(Du_1(s)|_{\mathbb{H}}^2 v(s), v(s))_{\mathbb{H}}| \leq |v(s)|_{\mathbb{H}}^2 |u_1(s)|_{\mathbb{H}}^2 + C |v(s)|_{\mathbb{H}} \|Du_1(s)|_{\mathbb{H}}^2 |u_1(s)|_{\mathbb{H}}^2 \]
\[ \leq C |v(s)|_{\mathbb{H}} \|u_1(s)|_{\mathbb{H}}^2 \|1 + \frac{1}{\eta} \|u_1(s)|_{\mathbb{H}}^2 \| + C \eta^2 |Dv(s)|_{\mathbb{H}}^2 \] (56)
for all \( s \in [0, T] \); here \( \eta \) is a small positive real number, to be chosen later. In the second inequality we used the interpolation estimate (30).

*Estimate for \( x^{-\frac{1}{2}}(z_3(s), v(s))_{\mathbb{H}} \).* We have
\[ |x^{-\frac{1}{2}}(|Du_1(s)|_{\mathbb{H}}^3 |Du_2(s)|_{\mathbb{H}}^3 |v(s)|_{\mathbb{H}}| \leq |Dv(s)|_{\mathbb{H}} \|u_1(s)|_{\mathbb{H}}^2 + |u_2(s)|_{\mathbb{H}}^2 \|v(s)|_{L^\infty} \quad \forall s \in [0, T], \text{ P-a.e.,} \]
\[ \leq \frac{1}{2\eta} |v(s)|_{L^\infty} \|u_1(s)|_{\mathbb{H}}^2 + |u_2(s)|_{\mathbb{H}}^2 \| \eta |Dv(s)|_{\mathbb{H}}^2 \]
\[ \leq \frac{C}{\eta} |v(s)|_{\mathbb{H}} \|u_1(s)|_{\mathbb{H}}^2 + |u_2(s)|_{\mathbb{H}}^2 \| + \frac{1}{\eta} \|u_1(s)|_{\mathbb{H}}^2 + |u_2(s)|_{\mathbb{H}}^2 \| + C \eta |Dv(s)|_{\mathbb{H}}^2 \] (57)
for all \( s \in [0, T] \), P-almost everywhere; here we used the fact \( \sup_{r \in [0, T]} |u_2(r)|_{L^\infty} = 1 \) P-almost everywhere and we used inequality (54) to estimate \( |v(s)|_{L^\infty} \|u_i(s)|_{\mathbb{H}}^2 \).
Estimate for $X_{\cdot \hbar} (z_0(s), v(s))_{\mathcal{H}}$. We have

$$x_{\cdot \hbar} (u_2(s) \times \Delta v(s), v(s))_{\mathcal{H}}$$

= $|v(s)|_{L^\infty} |u_2(s)|_{\mathcal{H}} |Dv(s)|_{\mathcal{H}}$

$\leq \frac{1}{2 \eta} |v(s)|_{L^\infty}^2 |u_2(s)|_{\mathcal{H}}^2 + \frac{\eta}{2} |Dv(s)|_{\mathcal{H}}^2$

$\leq C \eta |v(s)|_{\mathcal{H}}^2 (|u_2(s)|_{\mathcal{H}}^2 + \frac{1}{\eta^2} |u_2(s)|_{\mathcal{H}}^4) + C \eta |Dv(s)|_{\mathcal{H}}^2$ (58)

for all $s \in [0, T]$.

Estimate for $X_{\cdot \hbar} (z_1(s), v(s))_{\mathcal{H}}$. We have

$$x_{\cdot \hbar} (u_2(s) \times (v(s) \times h) F(s, W), v(s))_{\mathcal{H}} \leq C |v(s)|_{\mathcal{H}}^2 |F(s, W)|$$ (59)

for all $s \in [0, T]$, $P$-almost everywhere.

We substitute the estimates (54) to (59) in the right hand side of (53) to obtain:

$$|v(t)|_{\mathcal{H}}^2 \leq (C(\eta + \eta^2) - 2\alpha) \int_0^t |Dv(s)|_{\mathcal{H}}^2 ds$$

$$+ C \int_0^t |v(s)|_{\mathcal{H}}^2 (1 + (1 + \frac{1}{\eta^2})(|u_1(s)|_{\mathcal{H}}^4 + |u_2(s)|_{\mathcal{H}}^4)) ds$$

$$+ C \int_0^t |v(s)|_{\mathcal{H}}^2 |F(s, W)| ds + C \int_0^t \langle v(s), u_2(s) \times (v(s) \times h) \rangle_{\mathcal{H}} dW(s)$$ (60)

for all $t \in [0, T]$, $P$-almost everywhere. We now choose $\eta$ small enough to make the coefficient of $\int_0^t |Dv(s)|_{\mathcal{H}}^2 ds$ in (60) negative. Then we have, by the Cauchy-Schwarz inequality:

$$|v(t)|_{\mathcal{H}}^2 \leq C \int_0^t |v(s)|_{\mathcal{H}}^4 ds \left(1 + \int_0^t |u_1(s)|_{\mathcal{H}}^8 ds + \int_0^t |u_2(s)|_{\mathcal{H}}^8 ds\right)$$

$$+ C \left(\int_0^t \langle v(s), u_2(s) \times (v(s) \times h) \rangle_{\mathcal{H}} dW(s)\right)^2$$

for all $t \in [0, T]$, $P$-almost everywhere.

For $n \in \mathbb{N}$, define the $(F_n)$-stopping time:

$$\tau_n := \inf\{r \in [0, T] : \int_0^r |u_1(s)|_{\mathcal{H}}^8 ds + \int_0^r |u_2(s)|_{\mathcal{H}}^8 ds \geq n\} \wedge T.$$

Then we have

$$\sup_{r \in [0, t]} |v(r \wedge \tau_n)|_{\mathcal{H}}^4$$

$$\leq C \int_0^t \sup_{r \in [0, s]} |v(r \wedge \tau_n)|_{\mathcal{H}}^4 ds \left(1 + \int_0^{t \wedge \tau_n} |u_1(s)|_{\mathcal{H}}^8 ds + \int_0^{t \wedge \tau_n} |u_2(s)|_{\mathcal{H}}^8 ds\right)$$

$$+ C \sup_{r \in [0, t]} \left|\int_0^r 1_{[0, \tau_n]}(s) \langle v(s \wedge \tau_n), u_2(s) \times (v(s \wedge \tau_n) \times h) \rangle_{\mathcal{H}} dW(s)\right|^2$$

40
for all \( t \in [0, T] \), \( P \)-almost everywhere. It follows that
\[
E\left[ \sup_{r \in [0,t]} |v(r \wedge \tau_n)|_{H^1}^4 \right] \leq C(1 + n) \int_0^t E\left[ \sup_{r \in [0,s]} |v(r \wedge \tau_n)|_{H^1}^4 \right] ds + C \int_0^t E\left[ \sup_{r \in [0,s]} |v(r \wedge \tau_n)|_{H^1}^4 \right] ds \quad \text{for all } t \in [0, T].
\]
Gronwall’s lemma then gives \( E[\sup_{r \in [0,T]} |v(r \wedge \tau_n)|_{H^1}^4] = 0 \). Since \( \tau_n(\omega) \) increases to \( T \) as \( n \) goes to infinity for each \( \omega \in \Omega \), by the monotone convergence theorem we have \( E[\sup_{r \in [0,T]} |v(r)|_{H^1}^4] = 0 \).

8 Uniqueness in law and existence of strong solutions

A fundamental theorem of Yamada and Watanabe for stochastic differential equations in \( \mathbb{R}^n \) (see, for example, [15, Theorem 17.1]) says that pathwise uniqueness and existence of weak solutions implies uniqueness in law and existence of strong solutions. In Theorem 12 we see that this statement also applies to equation (6). In this section we say ‘weak solution’ instead of ‘weak martingale solution’. To simplify notation we set
\[
S := C([0, T]; \mathbb{H}) \cap L^8(0, T; \mathbb{H}^1).
\]
This is the path space in Theorem 11 and is a separable Banach space which is continuously embedded in \( C([0, T]; \mathbb{H}) \) and in \( L^8(0, T; \mathbb{H}^1) \).

**Theorem 12.** For equation (6) we have uniqueness in law and existence of strong solutions in the following sense:

1. if \((W, y)\) and \((W', y')\) are two weak solutions of equation (6) (here \( W \) and \( W' \) are the Wiener processes, possibly defined on different probability spaces) and \( y \) and \( y' \) are \( S \)-valued, then \( y \) and \( y' \) have the same law on \( S \);

2. there exists a measurable function \( J : (C([0, T]; \mathbb{R}), \mathcal{B}_{C([0, T]; \mathbb{R})}) \to (S, \mathcal{B}_S) \) with the following property: for any probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0,T]}, \tilde{P})\), where the filtration \((\tilde{\mathcal{F}}_t)\) is such that \( \tilde{\mathcal{F}}_0 \) contains all sets in \( \tilde{\mathcal{F}} \) of \( \tilde{P} \)-measure zero, and for any \((\tilde{\mathcal{F}}_t)\)-Wiener process \((\tilde{W}(t))_{t \in [0,T]}\) defined on this space, the process \( J(\tilde{W}) \) is \((\tilde{\mathcal{F}}_t)\)-adapted and the pair \((\tilde{W}, J(\tilde{W}))\) is a weak solution of equation (6).

**Proof.** The proof is essentially the same as that of the theorem of Yamada and Watanabe in [15, Theorem 17.1], hence we will refer to that theorem for some details. Let
\[
(W, y) : (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P) \to (C([0, T]; \mathbb{R}) \times S, \mathcal{B}_{C([0, T]; \mathbb{R})} \otimes \mathcal{B}_S) \quad \text{and}
(W', y') : (\Omega', \mathcal{F}', (\mathcal{F}_t')_{t \in [0,T]}, P') \to (C([0, T]; \mathbb{R}) \times S, \mathcal{B}_{C([0, T]; \mathbb{R})} \otimes \mathcal{B}_S)
\]
be two weak solutions of equation (6). Here \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\) is a probability space with filtration \((\mathcal{F}_t)\) and \( W \) is a \((\mathcal{F}_t)\)-Wiener process and \( y \) is a \((\mathcal{F}_t)\)-adapted process with paths in \( S \) and the pair \((W, y)\) satisfies equation (6); \( W' \) and \( y' \) have analogous meanings.
Let $Q$ and $Q' : C([0, T]; \mathbb{R}) \times \mathcal{B}_S \to [0, 1]$ be probability kernels from $C([0, T]; \mathbb{R})$ to $S$ such that $Q(W, \cdot)$ is the conditional distribution of $y$ given $W$ and $Q'(W', \cdot)$ is the conditional distribution of $y'$ given $W'$. Let $\mathbf{w} : \mathcal{B}_{C([0, T]; \mathbb{R})} \to [0, 1]$ be Wiener measure.

Let $X$ be the product space $X := C([0, T]; \mathbb{R}) \times S \times S$ and we write $(\beta, x, x')$ for a typical member. We also write $\beta$ for the projection map of $X$ onto the first coordinate and we write $x$ and $x'$ for the projection maps onto the second and third coordinates, respectively. Define a probability measure $\overline{P}$ on the product $\sigma$-algebra $\mathcal{B}_{C([0, T]; \mathbb{R})} \otimes \mathcal{B}_S \otimes \mathcal{B}_S$ of $X$ by the kernel product:

$$\overline{P}(A \times B_1 \times B_2) := \int_{C([0, T]; \mathbb{R})} \int_{\mathcal{B}_S} 1_{A \times B_1 \times B_2}(\beta, x, x') dQ'(\beta, \cdot)(x') dQ(\beta, \cdot)(x) d\mathbf{w}(\beta)$$

for all $A \in \mathcal{B}_{C([0, T]; \mathbb{R})}$ and $B_1, B_2 \in \mathcal{B}_S$. For arbitrary $A \in \mathcal{B}_{C([0, T]; \mathbb{R})}$ and $B \in \mathcal{B}_S$ it is straightforward to show that

$$\overline{P}\{\beta \in A, x \in B\} = P\{W \in A, y \in B\} \text{ and } \overline{P}\{\beta \in A, x' \in B\} = P'\{W' \in A, y' \in B\};$$

hence the distribution of $(W, y)$ equals that of $(\beta, x)$ and the distribution of $(W', y')$ equals that of $(\beta, x')$. In order to prove item (1.) of the theorem we will show that $(\beta, x)$ and $(\beta, x')$ are weak solutions of equation (61) and then invoke pathwise uniqueness.

For each $t \in [0, T]$ let $\zeta_t : S \to \mathbb{R}$ be the evaluation map at time $t$, $\zeta_t(u) := u(t) \forall u \in S$; recall also that $z_t$ is the corresponding evaluation map on $C([0, T]; \mathbb{R})$. For each $t \in [0, T]$ we write $\beta_t := \zeta_t \beta$, the evaluation at time $t$ of $\beta$ and $x_t := \zeta_t x$ and $x'_t := \zeta_t x'$, the evaluations at time $t$ of $x$ and $x'$, respectively. Define a filtration for $X$ by $\mathcal{F}^0 := \sigma(\beta_t, x_t, x'_t : r \in [0, t])$, $t \in [0, T]$.

One can show that $(\beta_t)_{t \in [0, T]}$ is a $(\mathcal{F}^0)$-Wiener process using the proofs in Lemmas 17.9 and 17.12. The hardest task is showing that if $0 \leq s < t \leq T$ then $\beta_t - \beta_s$ and $\mathcal{F}^0_s$ are independent. The key result in Lemma 17.9] is that if $B$ is a subset of $S$ belonging to the $\sigma$-algebra $\sigma(\zeta_r : r \in [0, s])$ then $Q(\cdot, B)$ equals a $\mathcal{H}^0_{\zeta_s}$-measurable random variable on $C([0, T]; \mathbb{R})$, up to a set of $\mathbf{w}$-measure zero (recall that $\mathcal{H}^0_{\zeta_s} := \sigma(\zeta_r : r \in [0, s])$); $Q'$ has the same property.

By Theorem 11 item (1.), we have $|y(t)(x)|_{\mathbb{R}^3} = 1$ for almost every $\xi \in \Lambda$ for each $t \in [0, T]$, $P$-almost everywhere. Because $y$ and $x$ have the same distribution on $S$, we also have $|e(t)(x)|_{\mathbb{R}^3} = 1$ for almost every $\xi \in \Lambda$ for each $t \in [0, T]$, $P$-almost everywhere. This property of $y$ and $x$, the equality of the laws of $(W, y)$ and $(\beta, x)$ and the fact that $(W, y)$ satisfies equation (3) can be used to show that $(\beta, x)$ is a weak solution of equation (3).

For given $t \in [0, T]$, one can show that each of the Bochner integrals on the right hand side of equation (6) is a measurable function of $(W, y)$ into $X^{-\frac{1}{2}}$; also for any natural number $n$ and partition $t_k := \frac{k}{n} T$, $k = 0, 1, \ldots, 2^n$, the random variables

$$\sum_{j=0}^{2^n-1} [y(t_j) \times h - \alpha y(t_j) \times (y(t_j) \times h)](W(t \wedge t_{j+1}) - W(t \wedge t_j))$$

and

$$\sum_{j=0}^{2^n-1} [x(t_j) \times h - \alpha x(t_j) \times (x(t_j) \times h)](\beta(t \wedge t_{j+1}) - \beta(t \wedge t_j))$$
in \( X^{-\frac{1}{2}} \) have the same distribution. One then takes limits in distribution as \( n \) goes to infinity to show that, \( \overline{P} \)-almost everywhere, equality (5) holds with \( x \) in place of \( y \) and \( \beta \) in place of \( W \).

Similarly, \((\beta, x')\) is a weak solution of equation (6). By Theorem (II) item (2.), we have

\[
\overline{P}\{(\beta, x, x') \in X : x = x'\} = 1.
\]  

(62)

This implies that \( x \) and \( x' \) have the same law on \( S \) and thus \( y \) and \( y' \) have the same law on \( S \).

Now to prove item (2.) of the theorem. We must show that there is a Borel measurable function \( J : C([0, T]; \mathbb{R}) \to S \) such that for any probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P})\), where \( \tilde{\mathcal{F}}_0 \) contains all sets in \( \tilde{\mathcal{F}} \) with \( \tilde{P} \)-measure zero, and \((\tilde{\mathcal{F}}_t)\)-Wiener process \( \tilde{W} \) defined on this space, we have

- \( \zeta_s J(\tilde{W}) \) is \( \tilde{\mathcal{F}}_s \)-measurable for each \( s \in [0, T] \) and
- the laws of \((\tilde{W}, J(\tilde{W}))\) and \((W, y)\) on \( C([0, T]; \mathbb{R}) \times S \) are equal.

Set \( D := \{(x, x') \in S \times S : x = x'\} \) and define the function

\[
I(\beta) := \int_S \int_S 1_D(x, x') \, dQ'(\beta, \cdot)(x') \, dQ(\beta, \cdot)(x), \quad \beta \in C([0, T]; \mathbb{R}).
\]

By the definition of \( \overline{P} \) in (61), equality (62) implies that \( w \{ I = 1 \} = 1 \). If \( I(\beta) = 1 \) then there must be a unique element of \( S \), call it \( J(\beta) \), such that \( Q'(\beta, \{ J(\beta) \}) = Q(\beta, \{ J(\beta) \}) = 1 \). We extend the domain of \( J \) to all of \( C([0, T]; \mathbb{R}) \) by setting \( J(\beta) := 0 \) if \( \beta \in \{ I \neq 1 \} \).

Notice that for any \( \beta \) in \( \{ I = 1 \} \) and Borel subset \( B \) of \( S \), \( J(\beta) \) belongs to \( B \) if and only if \( Q(\beta, B) = 1 \). It immediately follows that \( J : C([0, T]; \mathbb{R}) \to S \) is Borel measurable. For each \( s \in [0, T] \), one can show that \( \zeta_s J(\tilde{W}) \) is \( \tilde{\mathcal{F}}_s \)-measurable by again using the fact that \( Q(\cdot, B) \) equals a \( \mathcal{H}^B \)-measurable random variable up to a set of \( w \)-measure zero when \( B \) belongs to the \( \sigma \)-algebra \( \sigma(\zeta_r : r \in [0, s]) \) (see [15, Lemma 17.9]); here we also make use of the assumption that \( \tilde{\mathcal{F}}_0 \) contains all sets in \( \tilde{\mathcal{F}} \) of \( \tilde{P} \)-measure zero.

For any \( A \in \mathcal{B}_{C([0, T]; \mathbb{R})} \) and \( B \in \mathcal{B}_S \) we have

\[
P\{W \in A, y \in B\} = \int_{C([0, T]; \mathbb{R})} 1_A(u)Q(u, B) \, dw(u) = \int_{C([0, T]; \mathbb{R})} 1_{\{I=1\}}(u)1_A(u)1_B(J(u)) \, dw(u) = \tilde{P}\{\tilde{W} \in A, J(\tilde{W}) \in B\};
\]

hence \((W, y)\) and \((\tilde{W}, J(\tilde{W}))\) have the same distribution on \( C([0, T]; \mathbb{R}) \times S \).

One can now show that \((\tilde{W}, J(\tilde{W}))\) is a weak solution of equation (6) by the same reasoning we used earlier to show that \((\beta, x)\) is a weak solution of equation (6).
9 Further regularity

In this section, let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(W(t))_{t \in [0,T]}$ be a Wiener process defined on this space and let $(W,y)$ be a weak solution of (10) such that $y$ has paths in $S$. Recall that, by Theorem 12, the distribution of $y$ is unique. Some regularity properties of $y$ are listed in Theorem 11. The main result in this section is Theorem 15 which estimates $y$ as a square integrable random element in $L^2(0,T;X^1)$. In Proposition 16 we use this estimate to show that paths of $y$ lie in $C([0,T];\mathbb{H}^1)$, $P$-almost everywhere; this improves upon the continuity property in Theorem 11. Recall the definition of the linear operator $(A, D(A))$ in (7) and $A_1 u := u + Au$ for $u \in D(A)$. The semigroup generated by $-A$, $(e^{-tA})$, is ultracontractive (see, for example, [2]); that is, for $1 \leq p \leq q \leq \infty$ and $t > 0$

$$|e^{-tA}f|_{L^q} \leq C_t \frac{C^p}{t^{\frac{1}{p} - \frac{1}{q}}} |f|_{L^p} \quad \forall f \in L^p(\Lambda; \mathbb{R}^3).$$ (63)

We start with a lemma that expresses $y$ in a form which allows us to exploit the regularizing properties of the semigroup $(e^{-tA})$. To shorten notation in this section, set

$$z(t) := (y(t) \times h) \times h - \alpha(y(t) \times (y(t) \times h)) \times h - \alpha(y(t) \times ((y(t) \times h) \times h) - \alpha(y(t) \times (y(t) \times h)) \times (y(t) \times h) + \alpha((y(t) \times (y(t) \times h)) \times h) \times y(t))$$

for all $t \in [0,T]$.

**Lemma 13.** For each $t \in [0,T]$

$$y(t) = e^{-\alpha t A} u_0 + \int_0^t e^{-\alpha(t-s)A} (y(s) \times \Delta y(s)) ds + \alpha \int_0^t e^{-\alpha(t-s)A} (|Dy(s)|^2_{\mathbb{H}^2} y(s)) ds$$

$$+ \int_0^t e^{-\alpha(t-s)A} (y(s) \times h - \alpha y(s) \times (y(s) \times h)) F(s, W) ds$$

$$+ \varepsilon^2 \int_0^t e^{-\alpha(t-s)A} (y(s) \times h - \alpha y(s) \times (y(s) \times h)) dW(s)$$

$$+ \frac{1}{2} \varepsilon \int_0^t e^{-\alpha(t-s)A} z(s) ds,$$ (64)

$P$-almost everywhere.

**Proof.** The proof of this lemma is much the same as the first part of the proof of Theorem 9.15 in [13]. We outline the idea: for each $t \in (0,T]$ and $j \in \mathbb{N}$ and $n \geq j$, one can write down Itô’s formula for the process $(\{e^{-\alpha(t-s)A} e_j, \pi_n y(s)\}_{s \in [0,t]} |\) here $\pi_n$ is orthogonal projection onto $\mathbb{H}_n$. One sets $s = t$ in this formula and obtains

$$\langle y(t), e_j \rangle_{\mathbb{H}} = \langle \text{ right hand side of (64)}, e_j \rangle_{\mathbb{H}}$$

$P$-almost everywhere. Since $j$ is arbitrary, this completes the proof.

The following lemma will be used in the proof of Theorem 15.
Lemma 14. Suppose that \( E \int_0^T |Dy(s)|_{L^4}^4 \, ds \leq C(\alpha, T, M, u_0, h) \). Then \( E \int_0^T |A_1y(s)|_{H^2}^2 \, ds \leq C(\alpha, T, M, u_0, h) \).

Proof. By the hypothesis of the lemma, we have
\[
E \int_0^T ||Dy(s)||_{L^4}^2 \, ds \leq C(\alpha, T, M, u_0, h). \tag{65}
\]

Let \( n \in \mathbb{N} \) and let \( \pi_n : X^{-\frac{1}{2}} \to \mathbb{H}_n \) be orthogonal projection onto \( \mathbb{H}_n \). To shorten notation in this proof, set \( y^n(t) := \pi_n y(t) \) for all \( t \in [0, T] \). Itô’s formula for \( |Dy^n(T)|_{H^2}^2 \) is:
\[
|Dy^n(T)|_{H^2}^2 = |D(\pi_n u_0)|_{H^2}^2 \]
\[
\quad - 2 \int_0^T \langle \Delta y^n(s), \pi_n [y(s) \times \Delta y(s) - \alpha y(s) \times (y(s) \times h)] \rangle_{H^2} \, ds \]
\[
\quad + \frac{1}{2} \varepsilon z(s) \, \left( \sigma_y(s) + (y(s) \times h) F(s, W) \right) \, ds \]
\[
\quad - \alpha(y(s) \times (y(s) \times h)) F(s, W) \rangle_{H^2} \, ds \]
\[
\quad - 2 \varepsilon \beta \int_0^T \langle \Delta y^n(s), \pi_n (y(s) \times h - \alpha y(s) \times (y(s) \times h)) \rangle_{H^2} \, dW(s) \]
\[
\quad + \varepsilon \int_0^T |D(\pi_n (y(s) \times h - \alpha y(s) \times (y(s) \times h)))|_{H^2}^2 \, ds \tag{66}
\]

\( P \)-almost everywhere. It is straightforward to show that \( \pi_n \Delta u = \Delta \pi_n u \) for all \( u \in \mathbb{H}^1 \); hence \( \pi_n \Delta y(s) = \Delta y^n(s) \) for all \( s \in [0, T] \). We use this fact and Young’s inequality to obtain from (66):
\[
2\alpha \int_0^T |\Delta y^n(s)|_{H^2}^2 \, ds \leq |u_0|_{H^1}^2 + C\eta^2 \int_0^T |\Delta y^n(s)|_{H^2}^2 \, ds
\]
\[
\quad + \frac{C}{\eta^2} \left[ \int_0^T |y(s) \times \Delta y(s)|_{H^2}^2 \, ds + \int_0^T |Dy(s)|_{L^4}^2 \, ds \right.
\]
\[
\quad \left. + \int_0^T |z(s)|_{H^1}^2 \, ds + \int_0^T |y(s) \times h|_{H^2}^2 F^2(s, W) \, ds \right]
\]
\[
\quad + \int_0^T |y(s) \times (y(s) \times h)|_{H^2}^2 F^2(s, W) \, ds \]
\[
\quad + \int_0^T |y(s) \times h - \alpha y(s) \times (y(s) \times h)|_{H^2}^2 \, dW(s) \]
\[
\quad + \int_0^T |y(s) \times h - \alpha y(s) \times (y(s) \times h)|_{H^2}^2 \, ds \tag{67}
\]

\( P \)-almost everywhere. Here \( C \) denotes a positive real constant whose value may depend only on \( \alpha \). The real constant \( \eta \) was introduced via Young’s inequality and we choose its value so that the coefficient, \( C\eta^2 \), of \( \int_0^T |\Delta y^n(s)|_{H^2}^2 \, ds \) on the right hand side of (67) equals \( \alpha \); then we
Lemma 37.8], we will estimate $E_{H}$ the Hilbert space and $X_{H}$. Hence

For each $t \in (0, T]$, we have

$$E \int_{0}^{T} |A_{1}y(t)|_{H}^{2} dt \leq C(\alpha, T, u_{0}, h).$$

**Proof.** Invoking Lemma 13, we write

$$y(t) = \sum_{k=1}^{6} y_{k}(t) \quad \text{for all} \ t \in [0, T], \ P\text{-almost everywhere;}$$

here $y_{k}(t)$ is the $k$th summand on the right hand side of 641. Lemma 14 says that it suffices to estimate $E \int_{0}^{T} |Dy_{k}(s)|_{A}^{4} ds$. Since it is convenient to work in a fractional power Hilbert space and $X_{H}$ is continuously embedded in $W^{1,4}(\Lambda; \mathbb{R}^{3})$ (see, for example, [16, Lemma 37.8]), we will estimate $E \int_{0}^{T} |A_{1}^{1/2}y_{k}(s)|_{H}^{4} ds$.

For each $t \in (0, T]$, we have

$$|A_{1}^{1/2}e^{-\alpha t A}u_{0}|_{H}^{4} \leq C(\alpha, T)|A_{1}^{1/2}e^{-\alpha t A_{1}}A_{2}^{1/2}u_{0}|_{H}^{4} \leq C(\alpha, T)t^{-\frac{4}{3}}|u_{0}|_{H^{1}}^{4}.$$

Hence $E \int_{0}^{T} |A_{1}^{1/2}y_{1}(t)|_{H}^{4} dt \leq C(\alpha, T, u_{0})$.

For each $t \in (0, T]$, we have

$$\int_{0}^{t} |A_{1}^{1/2}e^{-\alpha(t-s)A}(y(s) \times \Delta y(s))|_{H} ds \leq C(\alpha, T) \int_{0}^{t} (t-s)^{-\frac{1}{16}}|y(s) \times \Delta y(s)|_{H} ds.$$

Since the function $f(x) := 1_{(0,T)}(x)x^{-\frac{1}{16}}$ belongs to $L_{1}(\mathbb{R})$, for any function $g$ in $L_{2}(\mathbb{R})$, the convolution $f * g$ belongs to $L_{1}(\mathbb{R})$ and $|f * g|_{L_{1}(\mathbb{R})}^{4} \leq |f|_{L_{2}(\mathbb{R})}^{4}||g|_{L_{2}(\mathbb{R})}^{4}$, by Young’s convolution theorem. Hence

$$\int_{0}^{T} |A_{1}^{1/2}y_{2}(t)|_{H}^{4} dt \leq C(\alpha, T) \left( \int_{0}^{T} (t-s)^{-\frac{1}{16}}|y(s) \times \Delta y(s)|_{H} ds \right)^{4} dt \leq C(\alpha, T) \left( \int_{0}^{T} x^{-\frac{11}{12}} dx \right)^{3} \left( \int_{0}^{T} |y(s) \times \Delta y(s)|_{H}^{2} ds \right)^{2},$$

and, by Theorem 1, $E \int_{0}^{T} |A_{1}^{1/2}y_{2}(t)|_{H}^{4} dt \leq C(\alpha, M, T, u_{0}, h)$. The same argument yields $E \int_{0}^{T} |A_{1}^{1/2}y_{4}(s)|_{H}^{4} ds \leq C(\alpha, M, T, u_{0}, h)$ and $E \int_{0}^{T} |A_{1}^{1/2}y_{6}(t)|_{H}^{4} dt \leq C(\alpha, M, T, u_{0}, h).$
For all \( s \) and \( t \) such that \( 0 \leq s < t \leq T \), we have
\[
|A_1^{\frac{1}{T}} e^{-\alpha(t-s)A}(|Dy(s)|^2_{\mathbb{R}^3} y(s))|_{\mathbb{H}} \leq C(\alpha, T)(t - s)^{-\frac{1}{2}} |e^{-\frac{1}{2} \alpha(t-s)A}(|Dy(s)|^2_{\mathbb{R}^3} y(s))|_{\mathbb{H}}.
\]
By (63) with \( p = 1 \) and \( q = 2 \), we have the estimate:
\[
|e^{-\frac{1}{2} \alpha(t-s)A}(|Dy(s)|^2_{\mathbb{R}^3} y(s))|_{\mathbb{H}} \leq C(\alpha) \sup_{r \in [0,T]} |y(r)|^2_{\mathbb{H}^1} (t - s)^{-\frac{1}{4}}
\]
for all \( 0 \leq s < t \leq T \), \( P \)-almost everywhere. Hence
\[
E \int_0^T |A_1^{\frac{1}{T}} y_3(t)|^4_{\mathbb{H}} \, dt \leq C(\alpha, T) E \int_0^T \left( \sup_{r \in [0,T]} |y(r)|^2_{\mathbb{H}^1} \int_0^t (t - s)^{-\frac{1}{4}} \, ds \right)^4 \, dt 
\leq C(\alpha, M, T, u_0, h),
\]
by Theorem 1.

Lastly, we find the estimate for the stochastic convolution \( y_5(t) = \varepsilon^2 \int_0^t e^{-\alpha(t-s)A} (y(s) \times h - \alpha y(s) \times (y(s) \times h)) \, dW(s) \). We shall estimate one part of \( y_5 \). The other part of \( y_5 \) can be handled in the same way. For each \( t \in (0, T] \) we have
\[
E \left[ \left( \int_0^t A_1^{\frac{1}{T}} e^{-\alpha(t-s)A} (y(s) \times (y(s) \times h)) \, dW(s) \right)^4 \right] 
\leq C(\alpha, T) E \left[ \left( \int_0^t A_1^{\frac{1}{T}} e^{-\alpha(t-s)A} (y(s) \times (y(s) \times h)) \, ds \right)^2 \right]^2
\leq C(\alpha, T) E \left[ \left( \int_0^t (t - s)^{-\frac{1}{4}} |y(s) \times (y(s) \times h)|^2_{\mathbb{H}^1} \, ds \right)^2 \right]
\leq C(\alpha, T) E \left[ \sup_{r \in [0,T]} |y(r)|^8_{\mathbb{H}^1} \right].
\]

By Theorem 1, it follows that \( E \int_0^T |A_1^{\frac{1}{T}} y_5(t)|^4_{\mathbb{H}} \, dt \leq C(\alpha, T, M, u_0, h) \).

This completes the proof of Theorem 1.

**Proposition 16.** Paths of \( y \) lie in \( C([0,T]; \mathbb{H}^1) \), \( P \)-almost everywhere.

**Proof.** For each \( n \in \mathbb{N} \), let \( \pi_n : \mathbb{H} \to \mathbb{H}_n \) be orthogonal projection of \( \mathbb{H} \) onto \( \mathbb{H}_n \). To shorten notation in this proof, for each \( n \in \mathbb{N} \) set \( y^n(t) := \pi_n y(t) \) for all \( t \in [0, T] \). Let \( n \)}
and $m$ be positive integers such that $n \geq m$. We have
\[
y^n(t) - y^m(t) \\
= (\pi_n - \pi_m)u_0 \\
+ \int_0^t (\pi_n - \pi_m)(y(s) \times \Delta y(s)) \, ds - \alpha \int_0^t (\pi_n - \pi_m)(y(s) \times (y(s) \times \Delta y(s))) \, ds \\
+ \varepsilon \frac{t}{2} \int_0^t (\pi_n - \pi_m)(y(s) \times h - \alpha y(s) \times (y(s) \times h)) \, dW(s) + \frac{1}{2} \varepsilon \int_0^t (\pi_n - \pi_m)z(s) \, ds \\
+ \varepsilon \int_0^t (\pi_n - \pi_m)(y(s) \times h - \alpha y(s) \times (y(s) \times h))F(s, W) \, ds
\]
for all $t \in [0, T]$, $P$-almost everywhere. We apply the map $u \in \mathbb{H}_n \mapsto |u|^2_{\mathbb{H}_1}$ to the process $(y^n(t) - y^m(t))_{t \in [0, T]}$ and obtain, by Itô’s formula:
\[
|y^n(t) - y^m(t)|^2_{\mathbb{H}_1} \\
= |(\pi_n - \pi_m)u_0|^2_{\mathbb{H}_1} \\
+ 2\varepsilon \frac{t}{2} \int_0^t (A_1(\pi_n - \pi_m)y(s), y(s) \times h - \alpha(y(s) \times (y(s) \times h)))_{\mathbb{H}} \, dW(s) \\
+ 2 \int_0^t (A_1(\pi_n - \pi_m)y(s), y(s) \times \Delta y(s) - \alpha(y(s) \times (y(s) \times \Delta y(s)))) \\
+ \frac{1}{2} \varepsilon z(s) + [y(s) \times h - \alpha y(s) \times (y(s) \times h)]F(s, W)_{\mathbb{H}} \, ds \\
+ \varepsilon \int_0^t |(\pi_n - \pi_m)(y(s) \times h - \alpha y(s) \times (y(s) \times h))|^2_{\mathbb{H}_1} \, ds
\]
for all $t \in [0, T]$, $P$-almost everywhere. From this, we have the estimate:
\[
\sup_{t \in [0, T]} |y^n(t) - y^m(t)|^2_{\mathbb{H}_1} \\
\leq |(\pi_n - \pi_m)u_0|^2_{\mathbb{H}_1} \\
+ 2\varepsilon \frac{t}{2} \sup_{t \in [0, T]} \left| \int_0^t (A_1(\pi_n - \pi_m)y(s), y(s) \times h - \alpha y(s) \times (y(s) \times h))_{\mathbb{H}} \, dW(s) \right| \\
+ 2 \int_0^T |A_1(\pi_n - \pi_m)y(s)|_{\mathbb{H}}|y(s) \times \Delta y(s) - \alpha(y(s) \times (y(s) \times \Delta y(s)))) \\
+ \frac{1}{2} \varepsilon z(s) + [y(s) \times h - \alpha y(s) \times (y(s) \times h)]F(s, W)_{\mathbb{H}} \, ds \\
+ \varepsilon \int_0^T |(\pi_n - \pi_m)(y(s) \times h - \alpha y(s) \times (y(s) \times h))|^2_{\mathbb{H}_1} \, ds
\]
$P$-almost everywhere. Taking expectations and using Doob’s inequality and the Cauchy-
Schwarz inequality, we have:

\[
E[\sup_{t\in[0,T]} |y^n(t) - y^m(t)|_{H^1}^2] \\
\leq |(\pi_n - \pi_m)u_0|_{H^1}^2 \\
+ \varepsilon \frac{1}{2} C(\alpha, u_0, h) \left( \int_0^T |(\pi_n - \pi_m)A_1 y(s)|_{H^1}^2 \, ds \right)^{\frac{1}{2}} \\
+ 2 \left( \int_0^T |(\pi_n - \pi_m)A_1 y(s)|_{H^1}^2 \, ds \right)^{\frac{1}{2}} \\
+ \left( \int_0^T |y(s) \times \Delta y(s) - \alpha y(s) \times (y(s) \times \Delta y(s))| \, ds \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \varepsilon |y(s) \times h - \alpha y(s) \times (y(s) \times h)|F(s, W)|_{H^1}^2 \, ds \\
+ \varepsilon \int_0^T |(\pi_n - \pi_m)(y(s) \times h - \alpha y(s) \times (y(s) \times h))|_{H^1}^2 \, ds,
\]

where \( \cdot \) denotes multiplication of the two square roots. Fixing \( m \) and letting \( n \) go to infinity in this inequality, the monotone convergence theorem yields:

\[
E[\sup_{t\in[0,T]} |y(t) - y^m(t)|_{H^1}^2] \\
\leq |u_0 - \pi_m u_0|_{H^1}^2 \\
+ \varepsilon \frac{1}{2} C(\alpha, u_0, h) \left( \int_0^T |I_{H^1} - \pi_m A_1 y(s)|_{H^1}^2 \, ds \right)^{\frac{1}{2}} \\
+ 2 \left( \int_0^T |I_{H^1} - \pi_m A_1 y(s)|_{H^1}^2 \, ds \right)^{\frac{1}{2}} \\
+ \left( \int_0^T |y(s) \times \Delta y(s) - \alpha y(s) \times (y(s) \times \Delta y(s))| \, ds \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \varepsilon |y(s) \times h - \alpha y(s) \times (y(s) \times h)|F(s, W)|_{H^1}^2 \, ds \\
+ \varepsilon \int_0^T |I_{H^1} - \pi_m y(s) \times h - \alpha y(s) \times (y(s) \times h))|_{H^1}^2 \, ds,
\]

where \( I_{H^1} \) is the identity operator on \( H^1 \). Thanks to Theorem 11 and the estimates in Theorem 1, the right hand side of (68) goes to zero as \( m \) goes to infinity. Hence for some subsequence \( (m_k) \), we have \( \sup_{t\in[0,T]} |y(t) - y^{m_k}(t)|_{H^1} \rightarrow 0 \) as \( k \rightarrow \infty \), \( P \)-almost everywhere. Since, for each \( k \), \( y^{m_k} \) has paths in \( C([0, T]; H^1) \), it follows that \( y \) has paths in \( C([0, T]; H^1) \), \( P \)-almost everywhere.
10 Small noise asymptotics

In this section we obtain a large deviation principle for the family of distributions of the solutions of equation (4) with $F$ identically zero and the parameter $\varepsilon \in (0, 1]$ approaching zero. Recall that when $F$ is identically zero in (3), the equation is a simplified model of the magnetization in a piece of ferromagnetic material, with random thermal disturbances in the field driven by a Wiener process multiplied by $\varepsilon^2$.

The setting in this section is as defined in section 1: $(\Omega, \mathcal{F}, P)$ is a probability space on which is defined a real-valued Wiener process $(W(t))_{t \in [0, T]}$ and $\mathcal{G}^0 := \sigma(W(r) : r \in [0, t])$ for each $t \in [0, T]$ and $(\mathcal{G}_t)_{t \in [0, T]}$ is the augmented filtration.

For each $\varepsilon \in (0, 1]$, let $J^\varepsilon : (C([0, T]; \mathbb{R}), \mathcal{B}_{C([0, T]; \mathbb{R})}) \to (S, \mathcal{B}_S)$ be the measurable function in Theorem 12 that acts on a Wiener process to give a weak solution of equation (6) with $F$ identically zero; by the results of section 9, the image under $J^\varepsilon$ of the Wiener process has paths in $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ almost surely. In the following, to ensure that we are working with random elements in $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$, we could write $1_{\{J^\varepsilon \in C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)\}}J^\varepsilon$ instead of just $J^\varepsilon$; however, for brevity, we will not write the indicator.

Define the map $\mathcal{J} : L^2(0, T; \mathbb{R}) \to C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ as follows: for each $v \in L^2(0, T; \mathbb{R})$,

$$\mathcal{J}(v) := y_v,$$

where $y_v$ satisfies the equation

$$y_v(t) = u_0 + \int_0^t y_v(s) \times \Delta y_v(s) \, ds - \alpha \int_0^t y_v(s) \times (y_v(s) \times \Delta y_v(s)) \, ds + \int_0^t (y_v(s) \times h)v(s) \, ds - \alpha \int_0^t y_v(s) \times (y_v(s) \times h)v(s) \, ds \quad \forall t \in [0, T]; \quad (69)$$

this is equation (6) with $\varepsilon = 0$ and $F(s, u) := v(s) \forall (s, u) \in [0, T] \times C([0, T]; \mathbb{R})$. By the results of section 9, $\mathcal{J}(v)$ lies in $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ for each $v \in L^2(0, T; \mathbb{R})$.

We will show that the family of laws $\{\mathcal{L}(J^\varepsilon(W)) : \varepsilon \in (0, 1]\}$ on $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ satisfies the large deviation principle with rate function $\mathcal{I} : C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1) \to [0, \infty]$ defined by

$$\mathcal{I}(u) := \inf \left\{ \frac{1}{2} \int_0^T \phi^2(s) \, ds : \phi \in L^2(0, T; \mathbb{R}) \text{ and } u = \mathcal{J}(\phi) \right\} \quad (70)$$

for all $u \in C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$, where the infimum of the empty set is taken to be infinity. We use the weak convergence method of Budhiraja and Dupuis [4]; this method was also used by Duan and Millet [3] and Chueshov and Millet [6].

For $M \in (0, \infty)$, define

$$B_M := \{\phi \in L^2(0, T; \mathbb{R}) : \int_0^T \phi^2(s) \, ds \leq M\}$$

with the weak topology of $L^2(0, T; \mathbb{R})$.

Assume for the moment that the map $\mathcal{J}$ is Borel measurable. Budhiraja and Dupuis [4] Theorem 4.4] showed that the family of laws $\{\mathcal{L}(J^\varepsilon(W)) : \varepsilon \in (0, 1]\}$ satisfies the large deviation principle with rate function $\mathcal{I}$ if the following two statements are true.

50
Statement 1. For each $R \in (0, \infty)$, the set $\{J(\phi) : \phi \in L^2(0, T; \mathbb{R}) \text{ and } \int_0^T \phi^2(s) \, ds \leq R\}$ is a compact subset of $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$.

Statement 2. If $(\varepsilon_n)$ is any sequence from $(0, 1]$ that converges to 0 and $(v_n : [0, T] \times \Omega \to \mathbb{R})$ is any sequence of $(\mathcal{G}^0_t)$-predictable processes such that for some $M \in (0, \infty)$ we have $\int_0^T v_n^2(s, \omega) \, ds \leq M \forall \omega \in \Omega$ and $\forall n \in \mathbb{N}$ and $(v_n)$ converges in distribution on $B_M$ to $v$, then $(J_{\varepsilon_0} W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^T v_n(s) \, ds)$ converges in distribution on $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ to $J(v)$.

We show that the two statements are true in the proof of Theorem 17. In the proof of Statement 2, we interpret each process $J_{\varepsilon_0} W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^T v_n(s) \, ds$ as the solution of equation (6) with $\varepsilon_n$ in place of $\varepsilon$ and an appropriate $F$. To do this, we use the fact that any $(\mathcal{G}^0_t)$-predictable process $u : [0, T] \times \Omega \to \mathbb{R}$ such that $\int_0^T u^2(t, \omega) \, dt \leq M \forall \omega \in \Omega$ can be written as

$$u(t, \omega) = F(t, W(\omega)) \quad \forall (t, \omega) \in [0, T] \times \Omega,$$

for some $(\mathcal{H}^0_t)$-predictable function $F : [0, T] \times C([0, T]; \mathbb{R}) \to \mathbb{R}$ satisfying $\int_0^T F^2(s, z) \, ds \leq M \forall s \in C([0, T]; \mathbb{R})$. Observe that, by Girsanov’s theorem (see, for example, Theorem 5.1 in chapter 3), for $\varepsilon \in (0, 1]$, the process $(\tilde{W}(t) := W(t) + \frac{1}{\sqrt{\varepsilon}} \int_0^t u(s) \, ds)_{t \in [0, T]}$ is a $(\mathcal{G}_t)$-Wiener process on a probability space $(\Omega, \mathcal{G}_T, \tilde{P})$, where $\tilde{P}$ is a certain probability measure (depending on $\varepsilon$ and $u$) that is equivalent to $P$ on $\mathcal{G}_T$; also, for $y := J^{\varepsilon}(\tilde{W})$ we have

$$\int_0^t y(s) \times (y(s) \times h) \, d\tilde{W}(s) = \int_0^t y(s) \times (y(s) \times h) \, dW(s) + \frac{1}{\sqrt{\varepsilon}} \int_0^t y(s) \times (y(s) \times h)u(s) \, ds$$

for all $t \in [0, T]$, $P$-almost everywhere and the same equality holds with $y(s) \times h$ in place of $y(s) \times (y(s) \times h)$ on both sides. Hence $y = J^{\varepsilon}(W + \frac{1}{\sqrt{\varepsilon}} \int_0^T u(s) \, ds)$ satisfies equation (6) with $F$ the function in (71).

Theorem 17. The family of laws $\{\mathcal{L}(J^{\varepsilon}(W)) : \varepsilon \in (0, 1]\} \subset C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ satisfies the large deviation principle with rate function $I$.

Proof. Most of the work in this proof is done in a number of lemmas. Firstly we describe how the lemmas can be used to prove Theorem 17. We show in Lemma 18 that if $(w_n : [0, T] \to \mathbb{R})$ is a sequence from $L^2(0, T; \mathbb{R})$ that converges weakly in $L^2(0, T; \mathbb{R})$ to $w$ then $(\mathcal{J}(w_n))$ converges to $\mathcal{J}(w)$ in $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$. This implies that $\mathcal{J}$ is Borel measurable and that Statement 1 is true.

Let $(\varepsilon_n)$ be a sequence from $(0, 1]$ that converges to 0 and let $(v_n : [0, T] \times \Omega \to \mathbb{R})$ be a sequence of $(\mathcal{G}^0_t)$-predictable processes that converges in distribution on $B_M$, for some $M \in (0, \infty)$, to $v$. We show in Lemma 20 that $(J_{\varepsilon_0} W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^T v_n(s) \, ds) - \mathcal{J}(v_n)$ converges in probability (as a sequence of random variables in $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$) to 0.

Lemma 18 implies that $(\mathcal{J}(v_n))$ converges in distribution on $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ to $\mathcal{J}(v)$; this is because $B_M$ is a separable metric space and Skorohod’s coupling theorem (see, for example, Theorem 4.30) allows us to work with a sequence of processes that converges pointwise in $B_M$ almost surely. These two convergence results imply that
$(J^n(W + \frac{1}{\sqrt{n}} \int_0^1 v_n(s) \, ds))$ converges in distribution on $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ to $J(v)$: for any uniformly continuous and bounded function $f : C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1) \rightarrow \mathbb{R}$ we have

$$\left| \int_\Omega f(J^n(W + \frac{1}{\sqrt{n}} \int_0^1 v_n(s) \, ds)) \, dP - \int_{C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)} f(x) \, d\mathcal{L}(J(v))(x) \right|$$

$$\leq \int_\Omega \left| f(J^n(W + \frac{1}{\sqrt{n}} \int_0^1 v_n(s) \, ds)) - f(J(v_n)) \right| \, dP$$

$$+ \int_{C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)} f(x) \, d\mathcal{L}(J(v_n))(x) - \int_{C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)} f(x) \, d\mathcal{L}(J(v))(x)$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty.$$ 

Thus Statement 2 is true.

It remains to prove Lemma 18 and Lemma 21.

**Lemma 18.** Suppose that $(w_n)$ is a weakly convergent sequence from $L^2(0, T; \mathbb{R})$ and the limit is $w \in L^2(0, T; \mathbb{R})$. Then $J(w_n)$ converges to $J(w)$ in $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$.

**Proof.** To simplify notation, in this proof we write $y_{w_n}$ for $J(w_n)$ and $y_w$ for $J(w)$. Set

$$M := \sup_{n \in \mathbb{N}} \int_0^T w_n^2(s) \, ds < \infty.$$  \hfill (72)

By Theorem 1 (with $\varepsilon = 0$ and $F(s, u) := w_n(s) \forall (s, u) \in [0, T] \times C([0, T]; \mathbb{R})$) and Theorem 15 and uniqueness of solutions, there is a finite constant $C(T, \alpha, M, u_0, h)$ (depending only on $T$, $\alpha$, $M$, $u_0$ and $h$) such that for all $n \in \mathbb{N}$:

$$\sup_{t \in [0, T]} |y_{w_n}(t)|_{\mathbb{H}^1} \leq C(T, \alpha, M, u_0, h), \quad \text{(73)}$$

and

$$\int_0^T |\Delta y_{w_n}(s)|_{\mathbb{H}^1}^2 \, ds \leq C(T, \alpha, M, u_0, h); \quad \text{(74)}$$

also we have

$$|y_{w_n}(t)(x)|_{\mathbb{H}^1} = 1 \forall x \in \Lambda \text{ and } \forall t \in [0, T]. \quad \text{(75)}$$

The same estimates hold with $w$ in place of $w_n$. We use these estimates in the following lemma.

**Lemma 19.** Each subsequence of $(y_{w_n})$ has a further subsequence that converges in $C([0, T]; \mathbb{H})$.

**Proof.** Since $\mathbb{H}^1$ is compactly embedded in $\mathbb{H}$, the estimate (73) implies that for each fixed $t \in [0, T]$, \{ $y_{w_n}(t) : n \in \mathbb{N}$ \} is relatively compact in $\mathbb{H}$. Hence, by a diagonal argument, there is a subsequence $(n_i)$ such that $(y_{w_{n_i}}(t))$ converges in $\mathbb{H}$ for each rational $t \in [0, T]$. We now show that \{ $y_{w_n} : n \in \mathbb{N}$ \} is a uniformly equicontinuous subset of $C([0, T]; \mathbb{H})$. We have for $0 \leq t < t' \leq T$:

$$y_{w_n}(t') - y_{w_n}(t) = \int_t^{t'} y_{w_n}(s) \times \Delta y_{w_n}(s) \, ds - \alpha \int_t^{t'} y_{w_n}(s) \times (y_{w_n}(s) \times \Delta y_{w_n}(s)) \, ds$$

$$+ \int_t^{t'} (y_{w_n}(s) \times h) w_n(s) \, ds - \alpha \int_t^{t'} y_{w_n}(s) \times (y_{w_n}(s) \times h) w_n(s) \, ds.$$
Estimating the integrals on the right hand side of the equality using (74) and (75) and the Cauchy-Schwarz inequality, we have:

\[ |y_{wn}(t') - y_{wn}(t)|_H \leq (1 + \alpha)\sqrt{t'} - t \sqrt{C(T, \alpha, M, u_0, h)} + (1 + \alpha)|u_0h|_{L^\infty}\sqrt{t'} - t\sqrt{M}. \]

It is straightforward to use the uniform equicontinuity property and pointwise convergence at rational \( t \in [0, T] \) to show that \((y_{wn})_i \) is Cauchy in \( C([0, T]; \mathbb{H}) \) and hence converges in \( C([0, T]; \mathbb{H}) \). This completes the proof of Lemma 19.

We use Lemma 19 to prove Lemma 20 which will be needed when we apply Gronwall’s lemma to estimate \( y_{wn} - y_w \).

**Lemma 20.** Let \( q : [0, T] \to \mathbb{H} \) be a measurable function and \( \int_0^T |q(s)|_H^2 \ ds < \infty \). We have

\[
\sup_{t \in [0,T]} \left| \int_0^t \langle q(s), y_{wn}(s) - y_w(s) \rangle_H (w_n(s) - w(s)) \ ds \right| \to 0 \ \text{as} \ n \to \infty.
\]

**Proof.**

Suppose, to get a contradiction, that there is an \( \epsilon > 0 \) and a subsequence \((n_i)\) such that

\[
\sup_{t \in [0,T]} \left| \int_0^t \langle q(s), y_{wn_i}(s) - y_w(s) \rangle_H (w_{n_i}(s) - w(s)) \ ds \right| \geq \epsilon \ \forall i \in \mathbb{N}. \tag{76}
\]

By Lemma 19 we may assume that \((y_{wn_i})_i \) converges in \( C([0, T]; \mathbb{H}) \) to a limit \( z \), say. We have

\[
\sup_{t \in [0,T]} \left| \int_0^t \langle q(s), y_{wn_i}(s) - y_w(s) \rangle_H (w_{n_i}(s) - w(s)) \ ds \right| \\
\leq \sup_{t \in [0,T]} \left| \int_0^t \langle q(s), y_{wn_i}(s) - z(s) \rangle_H (w_{n_i}(s) - w(s)) \ ds \right| \\
+ \sup_{t \in [0,T]} \left| \int_0^t \langle q(s), z(s) - y_w(s) \rangle_H (w_{n_i}(s) - w(s)) \ ds \right|. \tag{77}
\]

We shall show that the right hand side of (77) converges to 0 as \( i \) goes to infinity, contradicting (76). We have

\[
\sup_{t \in [0,T]} \left| \int_0^t \langle q(s), y_{wn_i}(s) - z(s) \rangle_H (w_{n_i}(s) - w(s)) \ ds \right| \\
\leq 2\sqrt{M} \sqrt{\int_0^T |q(s)|_H^2 \ ds} \sup_{r \in [0,T]} |y_{wn_i}(r) - z(r)|_H \to 0 \ \text{as} \ i \to \infty.
\]

We also have

\[
\sup_{t \in [0,T]} \left| \int_0^t \langle q(s), z(s) - y_w(s) \rangle_H (w_{n_i}(s) - w(s)) \ ds \right| \to 0 \ \text{as} \ i \to \infty.
\]
because the linear operator

\[ v \in L^2(0, T; \mathbb{R}) \mapsto \int_0^t (q(s), z(s) - y_w(s)) v(s) \, ds \in C([0, T]; \mathbb{R}) \]

is compact and \((w_n, t)\) converges to \(w\) weakly in \(L^2(0, T; \mathbb{R})\). This completes the proof of Lemma 20. \(\square\)

For each \(n \in \mathbb{N}\), we have

\[
y_w(t) - y_w(t) = \int_0^t (y_w(s) - y_w(s)) \Delta y_{w_n}(s) \, ds + \int_0^t y_w(s) \Delta (y_{w_n}(s) - y_w(s)) \, ds
\]

\[
+ \int_0^t (|Dy_w(s)|_{E^3} - |Dy_w(s)|_{E^3})(|Dy_{w_n}(s)|_{E^3} + |Dy_w(s)|_{E^3})y_{w_n}(s) \, ds
\]

\[
+ \alpha \int_0^t |Dy_w(s)|_{E^3}^2 (y_{w_n}(s) - y_w(s)) \, ds + \alpha \int_0^t \Delta (y_{w_n}(s) - y_w(s)) \, ds
\]

\[
+ \int_0^t ((y_{w_n}(s) - y_w(s)) \times h)w_n(s) \, ds + \int_0^t (y_w(s) \times h)(w_n(s) - w(s)) \, ds
\]

\[
- \alpha \left[ \int_0^t (y_{w_n}(s) - y_w(s)) \times (y_{w_n}(s) \times h)w_n(s) \, ds
\]

\[
+ \int_0^t y_w(s) \times ((y_{w_n}(s) - y_w(s)) \times h)w_n(s) \, ds
\]

\[
+ \int_0^t y_w(s) \times (y_w(s) \times h)(w_n(s) - w(s)) \, ds \right] \forall t \in [0, T]
\]

(78)

and each integrand on the right hand side of (78) is a measurable function from \([0, T]\) into \(\mathbb{H}\) (after modification on a set of measure zero) and square integrable, because of (73), (74) and (75).

Fix \(n \in \mathbb{N}\). To simplify notation, we set \(u_n := y_{w_n} - y_w\). We need the integral equation for \(|u_n(\cdot)|_{\mathbb{H}^1}^2\). Denote the sum of the integrands on the right hand side of (78) by \(\tilde{u}_n\). For each \(m \in \mathbb{N}\), we have \(\pi_m u_n(t) = \int_0^t \pi_m \tilde{u}_n(s) \, ds\) for all \(t \in [0, T]\) (here \(\pi_m : \mathbb{H} \rightarrow \mathbb{H}_m\) is orthogonal projection onto \(\mathbb{H}_m\)): it is straightforward to show from this equation that the function \(|\pi_m u_n(\cdot)|_{\mathbb{H}^1}^2\) is absolutely continuous and \(|\pi_m u_n(t)|_{\mathbb{H}^1}^2 = 2 \int_0^t \langle \tilde{u}_n(s), A_1 \pi_m u_n(s) \rangle_\mathbb{H} \, ds\) for all \(t \in [0, T]\). Taking limits as \(m\) goes to infinity for each \(t\) yields

\[
|u_n(t)|_{\mathbb{H}^1}^2 = 2 \int_0^t \langle \tilde{u}_n(s), A_1 u_n(s) \rangle_\mathbb{H} \, ds \forall t \in [0, T].
\]

(79)

Let \(N\) be an arbitrary positive integer. The functions \(s \mapsto y_w(s), s \mapsto y_w(s) \times \pi_N h\) and
\[ s \mapsto y_w(s) \times (y_w(s) \times \pi_N h) \] all belong to \( L^2(0, T; X^1) \). Therefore, from (79) we have:

\[
\begin{align*}
|u_n(t)|_{L^2}^2 &= 2 \left[ \int_0^t \langle u_n(s) \times \Delta y_w(s), -\Delta u_n(s) \rangle_{X^2} ds + \int_0^t \langle y_w(s) \times \Delta u_n(s), u_n(s) \rangle_{X^2} ds \\
& \quad + \alpha \int_0^t \langle |Dy_w(s)|_{H^3}, |Dy_w(s)|_{H^3} \rangle_{L^2} ds + |Dy_w(s)|_{H^3} u_n(s), u_n(s) - \Delta u_n(s) \rangle_{X^2} ds \\
& \quad - \alpha \int_0^t |Du_n(s)|_{H^1}^2 ds - \alpha \int_0^t |\Delta u_n(s)|_{H^1}^2 ds \\
& \quad - \int_0^t \langle u_n(s) \times h, \Delta u_n(s) \rangle_{X^2} w_n(s) ds \\
& \quad + \int_0^t \langle y_w(s) \times h, u_n(s) \rangle_{X^2} (w_n(s) - w(s)) ds \\
& \quad - \int_0^t \langle \Delta (y_w(s \times \pi_N h), u_n(s)) \rangle_{X^2} (w_n(s) - w(s)) ds \\
& \quad - \int_0^t \langle y_w(s) \times (h - \pi_N h), \Delta u_n(s) \rangle_{X^2} (w_n(s) - w(s)) ds \\
& \quad + \alpha \int_0^t \langle u_n(s) \times (y_w(s) \times h), \Delta u_n(s) \rangle_{X^2} w_n(s) ds \\
& \quad - \alpha \int_0^t \langle y_w(s) \times (u_n(s) \times h), u_n(s) \rangle_{X^2} w_n(s) ds \\
& \quad - \alpha \int_0^t \langle y_w(s) \times (y_w(s) \times h), u_n(s) \rangle_{X^2} (w_n(s) - w(s)) ds \\
& \quad + \alpha \int_0^t \langle \Delta (y_w(s) \times (y_w(s) \times \pi_N h)), u_n(s) \rangle_{X^2} (w_n(s) - w(s)) ds \\
& \quad + \alpha \int_0^t \langle y_w(s) \times (y_w(s) \times (h - \pi_N h)), \Delta u_n(s) \rangle_{X^2} (w_n(s) - w(s)) ds \right] 
\] (80)

for all \( t \in [0, T] \). In the rest of this proof, \( C \) denotes a positive real constant whose value may depend only on \( \alpha, T, M, u_0 \) and \( h \); the actual value of the constant may be different in different instances. We estimate the integrands on the right hand side of (80) using the
We choose \( \eta \) for all \( t \) has the form \( C\eta \) only on \( \alpha \) braces on the right hand side of (81) is bounded by a finite constant, estimates in (72), (73) and (74) ensure that the integral from 0 to Cauchy-Schwarz inequality and (30) and Young's inequality to obtain:

\[
|u_n(t)|^2_{H^1} + 2\alpha \int_0^t |\Delta u_n(s)|^2_{H^1} \, ds \\
\leq C\eta^2 \int_0^t |\Delta u_n(s)|^2_{H^1} \, ds \\
+ C \int_0^t |u_n(s)|^2_{H^1} \left\{ 1 + \frac{1}{\eta^2} + \frac{1}{\eta^2} |\Delta y_{w_n}(s)|^2_{H^1} \right. \\
\left. + (1 + \frac{1}{\eta^2}) |y_{w_n}(s)|_{H^1} \{ |y_{w_n}(s)|_{\mathbb{H}} + |\Delta y_{w_n}(s)|_{\mathbb{H}} \} + |y_{w}(s)|_{H^1} \{ |y_{w}(s)|_{\mathbb{H}} + |\Delta y_{w}(s)|_{\mathbb{H}} \} + \frac{1}{\eta^2} w_n^2(s) + |w_n(s)| \right\} \, ds \\
+ C|h - \pi_N h|_{H} \left( \int_0^T |\Delta u_n(s)|^2_{H^1} \, ds \right)^{\frac{1}{2}} \left( \int_0^T (w_n(s) - w(s))^2 \, ds \right)^{\frac{1}{2}} \\
+ \sup_{r \in [0,T]} \left| \int_0^r \langle y_{w}(s) \times h, u_n(s) \rangle_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \\
+ \sup_{r \in [0,T]} \left| \int_0^r \langle \Delta (y_{w}(s) \times \pi_N h), u_n(s) \rangle_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \\
+ C \sup_{r \in [0,T]} \left| \int_0^r \langle y_{w}(s) \times (y_{w}(s) \times h), u_n(s) \rangle_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \\
+ C \sup_{r \in [0,T]} \left| \int_0^r \langle \Delta (y_{w}(s) \times (y_{w}(s) \times \pi_N h)), u_n(s) \rangle_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| 
\tag{81}
\]

for all \( t \in [0,T] \). In (81), \( \eta \) is an arbitrary real constant introduced via Young’s inequality. We choose \( \eta \) so that the coefficient of \( \int_0^T |\Delta u_n(s)|^2_{H^1} \, ds \) on the right hand side of (81), which has the form \( C\eta^2 \), equals \( \alpha \). We can now apply Gronwall’s lemma to (81). Observe that the estimates in (72), (73) and (74) ensure that the integral from 0 to \( T \) of the function in curly braces on the right hand side of (81) is bounded by a finite constant, \( \gamma \), say, which depends only on \( \alpha, T, M, u_0 \) and \( h \). We can also estimate the integrals multiplying \( |h - \pi_N h|_{\mathbb{H}} \) on
the right hand side of (81) using (72) and (74). Thus Gronwall’s lemma yields:

\[
\sup_{t \in [0,T]} |u_n(t)|_{\mathbb{H}^1}^2 + \alpha \int_0^T |\Delta u_n(s)|_{\mathbb{H}}^2 \, ds \\
\leq C \left[ |h - \pi_N h|_{\mathbb{H}} + \sup_{r \in [0,T]} \left| \int_0^r (y_w(s) \times h, u_n(s))_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \\
+ \sup_{r \in [0,T]} \left| \int_0^r (\Delta (y_w(s) \times \pi_N h), u_n(s))_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \\
+ \sup_{r \in [0,T]} \left| \int_0^r (\Delta (y_w(s) \times (y_w(s) \times h), u_n(s))_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \\
+ \sup_{r \in [0,T]} \left| \int_0^r (\Delta (y_w(s) \times (y_w(s) \times \pi_N h)), u_n(s))_{\mathbb{H}} (w_n(s) - w(s)) \, ds \right| \right] e^{C\gamma T}. \tag{82}
\]

Since \( N \in \mathbb{N} \) is arbitrary and, by Lemma 20, for each \( N \) the four suprema on the right hand side of (82) converge to zero as \( n \) goes to infinity, we have \( \sup_{t \in [0,T]} |u_n(t)|_{\mathbb{H}^1}^2 + \alpha \int_0^T |\Delta u_n(s)|_{\mathbb{H}}^2 \, ds \to 0 \) as \( n \to \infty \). This completes the proof of Lemma 18.

**Lemma 21.** The sequence of \( C([0,T];\mathbb{H}^1) \cap L^2(0,T; X^1) \)-valued random elements \( J^n(W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^T v_n(s) \, ds) - J(v_n) \) converges in probability to 0.

**Proof.** To simplify notation in this proof, set \( y_n := J^n(W + \frac{1}{\sqrt{\varepsilon_n}} \int_0^T v_n(s) \, ds) \) and \( u_n := J(v_n) \) for all \( n \in \mathbb{N} \).

By Theorem 11 and Theorem 15 and uniqueness in law of solutions of equation (6) (see Theorem 12), there is a finite constant \( C(T, \alpha, M, u_0, h) \), depending only on \( T, \alpha, M, u_0 \) and \( h \), such that for each \( n \in \mathbb{N} \):

\[
E \left[ \sup_{t \in [0,T]} |y_n(t)|_{\mathbb{H}^1}^{16} \right] \leq C(T, \alpha, M, u_0, h) \tag{83}
\]

and \( \sup_{t \in [0,T]} |u_n(t, \omega)|_{\mathbb{H}^1} \leq C(T, \alpha, M, u_0, h) \) \( \forall \omega \in \Omega \) \tag{84}

and \( E \int_0^T |\Delta y_n(s)|_{\mathbb{H}}^2 \, ds \leq C(T, \alpha, M, u_0, h) \tag{85} \)

and \( \int_0^T |\Delta u_n(s, \omega)|_{\mathbb{H}}^2 \, ds \leq C(T, \alpha, M, u_0, h) \) \( \forall \omega \in \Omega \); \tag{86} \)

for each \( n \in \mathbb{N} \), we also have

\(|y_n(t)(x)|_{\mathbb{H}^3} = 1 \) and \( |u_n(t)(x)|_{\mathbb{H}^3} = 1 \) \( \forall x \in \Lambda \) and \( \forall t \in [0,T] \), \( P \)-almost everywhere. \tag{87} \)

Let \( N \in (|u_0|_{\mathbb{H}^1}, \infty) \). For each \( n \in \mathbb{N} \), define the \((\mathcal{G}_t)\)-stopping time

\( \tau_n(\omega) := \inf \{ t \in [0,T] : |y_n(t, \omega)|_{\mathbb{H}^1} \geq N \} \wedge T \) \( \forall \omega \in \Omega \). \tag{88}
The key to the proof of Lemma 21 is the fact:

\[ E \left[ \sup_{t \in [0,T]} |y_n(t \land \tau_n) - u_n(t \land \tau_n)|_{H^1}^2 + \int_0^{\tau_n} |y_n(s) - u_n(s)|_{X_1}^2 \, ds \right] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty, \quad (89) \]

which follows from Lemma 22 and Lemma 23 below. Let us use (89) to prove Lemma 21.

Let \( \delta > 0 \) and \( \nu > 0 \). Choose \( N \in (|u_0|_{H^1}, \infty) \) such that \( \frac{1}{N} \sup_{n \in \mathbb{N}} E[\sup_{t \in [0,T]} |y_n(t)|_{H^1}] < \frac{\nu}{2} \); the uniform bound (83) enables us to do this. We have

\[
P\left\{ \sup_{t \in [0,T]} |y_n(t) - u_n(t)|_{H^1}^2 + \int_0^T |y_n(s) - u_n(s)|_{X_1}^2 \, ds \geq \delta \right\} 
\leq P\left\{ \sup_{t \in [0,T]} |y_n(t \land \tau_n) - u_n(t \land \tau_n)|_{H^1}^2 + \int_0^{\tau_n} |y_n(s) - u_n(s)|_{X_1}^2 \, ds \geq \delta, \, \tau_n = T \right\} 
+ P\left\{ \sup_{t \in [0,T]} |y_n(t)|_{H^1} \geq N \right\} 
\leq \frac{1}{\delta} E\left[ \sup_{t \in [0,T]} |y_n(t \land \tau_n) - u_n(t \land \tau_n)|_{H^1}^2 + \int_0^{\tau_n} |y_n(s) - u_n(s)|_{X_1}^2 \, ds \right] 
+ \frac{1}{N} E\left[ \sup_{t \in [0,T]} |y_n(t)|_{H^1} \right] 
< \nu \quad \text{for all sufficiently large} \quad n \in \mathbb{N},
\]

by our choice of \( N \) and (89).

Now we present Lemma 22 and Lemma 23 which imply (89).

**Lemma 22.** Let \( N \in (|u_0|_{H^1}, \infty) \). For \( \tau_n \) as defined in (88) we have

\[
E\left[ \sup_{t \in [0,T]} |y_n(t \land \tau_n) - u_n(t \land \tau_n)|_{H^1}^2 + \int_0^{\tau_n} |y_n(s) - u_n(s)|_{X_1}^2 \, ds \right] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
Proof. Let \( n \in \mathbb{N} \). We have

\[
y_n(t) - u_n(t) = \int_0^t (y_n(s) - u_n(s)) \times \Delta y_n(s) \, ds + \int_0^t u_n(s) \times \Delta (y_n(s) - u_n(s)) \, ds \\
+ \alpha \int_0^t (|Dy_n(s)|_{\mathbb{R}^3} - |Du_n(s)|_{\mathbb{R}^3})(|Dy_n(s)|_{\mathbb{R}^3} + |Du_n(s)|_{\mathbb{R}^3})y_n(s) \, ds \\
+ \frac{1}{\varepsilon_n} \int_0^t y_n(s) \times h \, dW(s) - \alpha \frac{1}{\varepsilon_n} \int_0^t y_n(s) \times (y_n(s) \times h) \, dW(s) \\
+ \frac{1}{2} \frac{1}{\varepsilon_n} \int_0^t z_n(s) \, ds \\
+ \int_0^t ((y_n(s) - u_n(s)) \times h)v_n(s) \, ds \\
- \alpha \int_0^t (y_n(s) - u_n(s)) \times (y_n(s) \times h)v_n(s) \, ds \\
- \alpha \int_0^t u_n(s) \times ((y_n(s) - u_n(s)) \times h)v_n(s) \, ds \quad \forall t \in [0, T], \ P\text{-a.e.,} \tag{90}
\]

where

\[
z_n(s) := (y_n(s) \times h) \times h - \alpha (y_n(s) \times (y_n(s) \times h)) \times h \\
- \alpha (y_n(s) \times ((y_n(s) \times h) \times h)) \times (y_n(s) \times h) \\
+ \alpha ((y_n(s) \times (y_n(s) \times h)) \times h) \times y_n(s) \quad \forall s \in [0, T].
\]

The estimates (83), (84), (85), (86) and (87) imply that the integrands on the right hand side of (90) have paths which are square integrable in \( \mathbb{H}, P\)-almost everywhere. Hence Itô’s
formula for $|y_n - u_n|_{H^1}$ is:

\[
|y_n(t) - u_n(t)|_{H^1}^2 \\
= 2 \int_0^t \langle y_n(s) - u_n(s), u_n(s) \times \Delta(y_n(s) - u_n(s)) \rangle \\
+ \alpha(|D^2 y_n(s)|_{H^1}^2 - |D u_n(s)|_{H^1}^2)(|D^2 y_n(s)|_{H^1}^2 + |D u_n(s)|_{H^1}^2) y_n(s) \\
+ \alpha|Du_n(s)|_{H^1}^2 (y_n(s) - u_n(s)) \\
+ \alpha \Delta(y_n(s) - u_n(s)) \\
+ \frac{1}{2} \varepsilon_n z_n(s) \\
- \alpha u_n(s) \times ((y_n(s) - u_n(s)) \times h) v_n(s))_{\mathbb{H}} ds \\
+ 2 \varepsilon_n^\frac{1}{2} \int_0^t \langle y_n(s) - u_n(s), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h) \rangle_{\mathbb{H}} dW(s) \\
+ \varepsilon_n \int_0^t |y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h)|_{H^1}^2 ds \tag{91}
\]

for all $t \in [0, T]$, $P$-almost everywhere. The plan is as follows: we firstly estimate the integrands on the right hand side of (91) using (30), (31), and (37); then, focusing on the stopped process $(y_n(t \wedge \tau_n) - u_n(t \wedge \tau_n))_{t \in [0,T]}$, we take expectations and apply Gronwall’s lemma. The expectation in the statement of Lemma 22 converges to 0 thanks to the factor $\varepsilon_n$ appearing in the estimate from Gronwall’s lemma. In the rest of the proof, $C$ denotes a positive real constant that may depend only on $\alpha$, $T$, $M$, $u_0$ or $h$; the actual value of the constant may be different from line to line.

From (91) we have

\[
|y_n(t) - u_n(t)|_{H^1}^2 \\
\leq C \int_0^t \left| y_n(s) - u_n(s) \right|_{H^1}^2 \left| y_n(s) - u_n(s) \right|_{H^1}^2 \left| u_n(s) \right|_{H^1}^2 ds \\
+ C \int_0^t \left| y_n(s) - u_n(s) \right|_{H^1}^2 \left| y_n(s) - u_n(s) \right|_{H^1}^2 \left| y_n(s) \right|_{H^1}^2 + \left| u_n(s) \right|_{H^1}^2 ds \\
+ C \int_0^t \left| y_n(s) - u_n(s) \right|_{H^1}^2 \left| y_n(s) - u_n(s) \right|_{H^1}^2 \left| u_n(s) \right|_{H^1}^2 ds \\
- 2\alpha \int_0^t |D(y_n(s) - u_n(s))|_{H^1}^2 ds \\
+ \varepsilon_n C + \frac{C}{2} \int_0^t \left| y_n(s) - u_n(s) \right|_{H^1}^2 |u_n(s)| ds \\
+ 2 \varepsilon_n^\frac{1}{2} \int_0^t \left| y_n(s) - u_n(s), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h) \right|_{\mathbb{H}} dW(s) \tag{92}
\]

for all $t \in [0, T]$, $P$-almost everywhere. We add $2\alpha \int_0^t |y_n(s) - u_n(s)|_{H^1}^2 ds$ to both sides of inequality (92) and then estimate integrands on the right hand side using Young’s inequality.
and the estimate in (84). We obtain:

\[
|y_n(t) - u_n(t)|^2_{H^1} + 2\alpha \int_0^t |y_n(s) - u_n(s)|^2_{H^1} \, ds \\
\leq C(\eta^2 + \eta^4) \int_0^t |y_n(s) - u_n(s)|^2_{H^1} \, ds \\
+ C \int_0^t |y_n(s) - u_n(s)|^2_{H^1} (\frac{1}{\eta^2} + \frac{1}{\eta^4}) + \frac{1}{\eta^r} |y_n(s)|_{H^1}^4 + |v_n(s)| + 1 \, ds \\
+ \varepsilon_n C \\
+ 2\varepsilon_n \left| \int_0^t (y_n(s) - u_n(s), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h)) \, dW(s) \right| 
\]

(93)

for all \( t \in [0, T] \), \( P \)-almost everywhere, where \( \eta \) is a positive real constant whose value we can choose. We choose \( \eta \) so that the coefficient \( C(\eta^2 + \eta^4) \) of \( \int_0^t |y_n(s) - u_n(s)|^2_{H^1} \, ds \) on the right hand side of (93) equals \( \alpha \); thus we eliminate this integral from the right hand side and leave \( \alpha \int_0^t |y_n(s) - u_n(s)|^2_{H^1} \, ds \) on the left hand side of the inequality.

We then square both sides of the inequality and stop at \( \tau_n \), yielding:

\[
|y_n(t \wedge \tau_n) - u_n(t \wedge \tau_n)|^4_{H^1} + \alpha^2 \left( \int_0^{t \wedge \tau_n} |y_n(s) - u_n(s)|^2_{H^1} \, ds \right)^2 \\
\leq C \left( 1 + \int_0^t 1_{[0, \tau_n]}(s) |y_n(s)|^8_{H^1} \, ds \right) \int_0^t 1_{[0, \tau_n]}(s) |y_n(s \wedge \tau_n) - u_n(s \wedge \tau_n)|^4_{H^1} \, ds \\
+ C \varepsilon_n^2 \\
+ C \varepsilon_n \left| \int_0^t 1_{[0, \tau_n]}(s) (y_n(s) - u_n(s), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h)) \, dW(s) \right|^2 
\]

(94)

for all \( t \in [0, T] \), \( P \)-almost everywhere. From this inequality and the definition of \( \tau_n \), we have

\[
E[\sup_{r \in [0, t]} |y_n(r \wedge \tau_n) - u_n(r \wedge \tau_n)|^4_{H^1}] \\
\leq C(1 + N^8T) \int_0^t E[\sup_{r \in [0, s]} |y_n(r \wedge \tau_n) - u_n(r \wedge \tau_n)|^4_{H^1}] \, ds \\
+ C \varepsilon_n^2 \\
+ C \varepsilon_n \left[ \sup_{r \in [0, t]} \left| \int_0^r 1_{[0, \tau_n]}(s) (y_n(s) - u_n(s), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h)) \, dW(s) \right|^2 \right] 
\]

(95)

\[
\leq C(1 + N^8T) \int_0^t E[\sup_{r \in [0, s]} |y_n(r \wedge \tau_n) - u_n(r \wedge \tau_n)|^4_{H^1}] \, ds \\
+ C \varepsilon_n^2 + C \varepsilon_n \quad \forall t \in [0, T]; 
\]

(96)

here we used Doob’s inequality and (84) to estimate the expectation of the square of the maximal function of the stochastic integral on the right hand side of (95). By Gronwall’s
lemma, we obtain from (96):

\[ E \left[ \sup_{r \in [0,T]} |y_n(r \wedge \tau_n) - u_n(r \wedge \tau_n)|^4_{H} \right] \leq C(\varepsilon^2_n + \varepsilon_n) e^{C(1+N^8)T} \to 0 \text{ as } n \to \infty. \]

Returning now to (94), we have also

\[ E \left[ \left( \int_0^{\tau_n} |y_n(s) - u_n(s)|^2_{H} \ ds \right)^2 \right] \leq C(\varepsilon^2_n + \varepsilon_n) e^{C(1+N^8)T} \to 0 \text{ as } n \to \infty. \]

This completes the proof of Lemma 22. \(\square\)

**Lemma 23.** Let \(N \in [\|u_0\|_{H^1}, \infty)\). For \(\tau_n\) as defined in (88) we have

\[ E \left[ \sup_{t \in [0,T]} |D(y_n(t \wedge \tau_n) - u_n(t \wedge \tau_n)|^2_{H} + \int_0^{\tau_n} |\Delta (y_n(s) - u_n(s))|^2_{H} \ ds \right] \to 0 \]

as \(n \to \infty\).

**Proof.** Let \(n \in \mathbb{N}\). We need Itô’s formula for the process \((|D(y_n(t) - u_n(t))|^2_{H})_{t \in [0,T]}\).

We proceed by firstly writing down Itô’s formula for \((|D(\pi_m(y_n(t) - u_n(t)))|^2_{H})\), where \(m \in \mathbb{N}\) and \(\pi_m : H \to H_m\) is orthogonal projection onto \(H_m\):

\[
|D(\pi_m(y_n(t) - u_n(t)))|^2_{H} = -2 \int_0^t \langle D\pi_m(y_n(s) - u_n(s)), (y_n(s) - u_n(s)) \times \Delta y_n(s) \rangle \]
\[
+ u_n(s) \times \Delta (y_n(s) - u_n(s)) + \alpha |D\pi_m(y_n(s) - u_n(s))|_{H^3} + |D\pi_m(y_n(s) - u_n(s))|_{H^3} y_n(s) \]
\[
+ \alpha |\Delta y_n(s) - u_n(s)|_{H^3} y_n(s) + \frac{1}{2} \varepsilon_n z_n(s) \]
\[
+ \langle (y_n(s) - u_n(s)) \times h v_n(s), y_n(s) \times h v_n(s) \rangle + \alpha \langle y_n(s) - u_n(s), y_n(s) \times h v_n(s) \rangle \]
\[
+ \alpha \langle y_n(s) - u_n(s), y_n(s) \times h v_n(s) \rangle \xi \]
\[
- 2 \varepsilon_n \int_0^t \langle D\pi_m(y_n(s) - u_n(s)), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h) \rangle_{H} \xi \]
\[
+ \varepsilon_n \int_0^t |D(\pi_m(y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h)))|^2_{H} \xi \] \hspace{1cm} (97)

for all \(t \in [0,T]\), \(P\)-almost everywhere; in this equality

\[
z_n(s) := (y_n(s) \times h) \times h - \alpha (y_n(s) \times (y_n(s) \times h)) \times h \]
\[
- \alpha (y_n(s) \times ((y_n(s) \times h) \times h) - (y_n(s) \times (y_n(s) \times h)) \times (y_n(s) \times h)) \]
\[
+ \alpha ((y_n(s) \times (y_n(s) \times h)) \times h) \times y_n(s) \}
\]

\(\forall s \in [0,T]\).
For each $t \in [0, T]$, we take pointwise limits as $m$ goes to infinity in (97) to obtain:

$$|D(y_n(t) - u_n(t))|_H^2$$

$$= -2 \int_0^t \langle \Delta(y_n(s) - u_n(s)), (y_n(s) - u_n(s)) \times \Delta y_n(s)$$

$$+ u_n(s) \times \Delta(y_n(s) - u_n(s))$$

$$+ \alpha \langle |Dy_n(s)|_{R^3} - |Du_n(s)|_{R^3} \rangle (|Dy_n(s)|_{R^3} + |Du_n(s)|_{R^3})y_n(s)$$

$$+ \alpha |Du_n(s)|_{R^3}^2 (y_n(s) - u_n(s)) + \alpha \Delta(y_n(s) - u_n(s))$$

$$+ \frac{1}{2} \varepsilon \eta \Delta u_n(s)$$

$$+ ((y_n(s) - u_n(s)) \times h)v_n(s)$$

$$- \alpha (y_n(s) - u_n(s)) \times (y_n(s) \times h)v_n(s)$$

$$- \alpha u_n(s) \times ((y_n(s) - u_n(s)) \times h)v_n(s) \rangle ds$$

$$- 2 \varepsilon \frac{n}{\eta} \int_0^t \langle \Delta(y_n(s) - u_n(s)), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h) \rangle \, dW(s)$$

$$+ \varepsilon \int_0^t |D(y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h))|_H^2 \, ds$$

(98)

$P$-almost everywhere. We add $2\alpha \int_0^t |\Delta(y_n(s) - u_n(s))|_H^2 \, ds$ to both sides of (98) and estimate the integrands on the right hand side of the resulting equality using the Cauchy-Schwarz inequality, Young’s inequality and (30); the aim is to find estimates involving $|y_n(s) - u_n(s)|_H$ or $|y_n(s) - u_n(s)|_H^3$ so that the proof can be completed using Lemma 22.

In the rest of this proof, we write $C$ to denote a positive real constant whose value may depend only on $\alpha, T, M, u_0$ and $h$; the actual value of $C$ may be different in different instances. The three main estimates we use are:

1. $\langle (\Delta(y_n(s) - u_n(s)), (y_n(s) - u_n(s)) \times \Delta y_n(s)) \rangle_H$

$$= \langle (\Delta(y_n(s) - u_n(s)), (y_n(s) - u_n(s)) \times \Delta u_n(s)) \rangle_H$$

$$\leq C\eta^2 |\Delta(y_n(s) - u_n(s))|_H^2 + C \frac{\eta}{\eta^2} |y_n(s) - u_n(s)|_H^2 |y_n(s) - u_n(s)|_H^3 |\Delta u_n(s)|_H^2;$$

notice that we chose to have $\Delta u_n(s)$ instead of $\Delta y_n(s)$ on the right hand side of this estimate.

2. $|\Delta(y_n(s) - u_n(s)), |Dy_n(s)|_{R^3} - |Du_n(s)|_{R^3} \rangle (|Dy_n(s)|_{R^3} + |Du_n(s)|_{R^3})y_n(s)|_H|

$$\leq C\eta^2 |\Delta(y_n(s) - u_n(s))|_H^2$$

$$\frac{C}{\eta^2} |y_n(s) - u_n(s)|_H^2 (|y_n(s) - u_n(s)|_H^1 + |\Delta(y_n(s) - u_n(s)))|_H^1 (|y_n(s)|_H^2 + |u_n(s)|_H^2)$$

$$\leq C\eta^2 |\Delta(y_n(s) - u_n(s))|_H^2$$

$$+ C|y_n(s) - u_n(s)|_H^2 \left( \frac{1}{\eta^2} (|y_n(s)|_H^2 + |u_n(s)|_H^2) + \frac{1}{\eta^2} (|y_n(s)|_H^4 + |u_n(s)|_H^4) \right).$$

63
3.

\[
|\langle \Delta(y_n(s) - u_n(s)), Du_n(s) \rangle |_{\mathbb{R}^1}^2 (y_n(s) - u_n(s))|_{\mathbb{H}}^2
\]

\[
\leq |\Delta(y_n(s) - u_n(s))|_{\mathbb{H}} |Du_n(s)|_{L^\infty} |y_n(s) - u_n(s)|_{L^\infty}
\]

\[
\leq C\eta^2 |\Delta(y_n(s) - u_n(s))|_{\mathbb{H}}^2
\]

\[
+ \frac{C}{\eta^2} |u_n(s)|_{\mathbb{H}^1} (|u_n(s)|_{\mathbb{H}^1} + |Du_n(s)|_{\mathbb{H}^1} |y_n(s) - u_n(s)|_{\mathbb{H}}
\]

These three estimates hold for almost every \( s \in [0, T] \), \( P \)-almost everywhere. We obtain from (98):

\[
|D(y_n(t) - u_n(t))|_{\mathbb{H}}^2 + 2\alpha \int_0^t |\Delta(y_n(s) - u_n(s))|_{\mathbb{H}}^2 \, ds
\]

\[
\leq C\eta^2 \int_0^t |\Delta(y_n(s) - u_n(s))|_{\mathbb{H}}^2 \, ds
\]

\[
+ \frac{C}{\eta^2} \sup_{r \in [0, t]} |y_n(r) - u_n(r)|_{\mathbb{H}} \sup_{r \in [0, T]} |y_n(r) - u_n(r)|_{\mathbb{H}^1}
\]

\[
+ C(\frac{1}{\eta^2} + \frac{1}{\eta^2}) \sup_{r \in [0, t]} (|y_n(r)|_{\mathbb{H}^1}^2 + |u_n(r)|_{\mathbb{H}^1}^2 + |u_n(r)|_{\mathbb{H}^1}^2) \int_0^t |y_n(s) - u_n(s)|_{\mathbb{H}}^2 \, ds
\]

\[
+ \frac{C}{\eta^2} \sup_{r \in [0, T]} |y_n(r)|_{\mathbb{H}^1}^2 \sup_{r \in [0, T]} |y_n(r) - u_n(r)|_{\mathbb{H}^1}
\]

\[
+ \frac{C}{\eta^2} \int_0^t |y_n(s) - u_n(s)|_{\mathbb{H}}^2 \, ds
\]

\[
+ \frac{C}{\eta^2} \varepsilon_n^2 \int_0^T |z_n(s)|_{\mathbb{H}}^2 \, ds
\]

\[
+ \frac{C}{\eta^2} \sup_{r \in [0, t]} |y_n(r) - u_n(r)|_{\mathbb{H}}^2
\]

\[
+ 2C\eta \sup_{r \in [0, T]} \left| \int_0^t \langle \Delta(y_n(s) - u_n(s)), y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h) \rangle_{\mathbb{H}} dW(s) \right|
\]

\[
+ \varepsilon_n \int_0^T |D(y_n(s) \times h - \alpha y_n(s) \times (y_n(s) \times h))|_{\mathbb{H}}^2 \, ds
\]

(99)

for all \( t \in [0, T] \), \( P \)-almost everywhere. We choose the constant \( \eta \) so that the coefficient \( C\eta^2 \) of \( \int_0^t |\Delta(y_n(s) - u_n(s))|_{\mathbb{H}}^2 \, ds \) on the right hand side of (98) equals \( \alpha \); this integral then cancels from the right hand side, leaving \( \alpha \int_0^t |\Delta(y_n(s) - u_n(s))|_{\mathbb{H}}^2 \, ds \) on the left hand side. We stop both sides of the resulting inequality at \( \tau_n \) (defined in (88)) and then use (83),
and Doob’s inequality to obtain:

\[
E \left[ \sup_{r \in [0,T]} |D(y_n(t \wedge \tau_n) - u_n(t \wedge \tau_n))|_{H^1}^2 + \alpha \int_0^{\tau_n} |\Delta(y_n(s) - u_n(s))|_{H^1}^2 \, ds \right] \\
\leq C \left( E \left[ \sup_{r \in [0,T]} |y_n(r \wedge \tau_n) - u_n(r \wedge \tau_n)|_{H^1}^2 \right] \right)^{\frac{1}{2}} \\
+ C (1 + N^2 + N^4) E \int_0^{\tau_n} |y_n(s) - u_n(s)|_{H^1}^2 \, ds \\
+ C E \left[ \sup_{r \in [0,T]} |y_n(r \wedge \tau_n) - u_n(r \wedge \tau_n)|_{H^1}^2 \right] \\
+ C \varepsilon_n^2 + C \varepsilon_n^4 + C \varepsilon_n \\
\rightarrow 0 \text{ as } n \to \infty, \text{ by Lemma 22}
\]

This completes the proof of Lemma 23. \( \square \)

This completes the proof of Lemma 21. \( \square \)

This completes the proof of Theorem 17. \( \square \)

11 Application to a model of a ferromagnetic needle

In this section we use the large deviation principle to investigate the dynamics of a stochastic Landau-Lifshitz model of magnetization in a needle-shaped particle. Here the shape anisotropy energy is crucial. When there is no applied field and no noise in the field, the shape anisotropy energy gives rise to two stable stationary states of opposite magnetization. We add a small noise term to the field and use the large deviation principle to show that noise induced magnetization reversal occurs and to quantify the effect of material parameters on sensitivity to noise.

The axis of the needle is represented by the interval \( \Lambda \); at each \( x \in \Lambda \), the magnetization is assumed to be constant over the cross-section of the needle. We define the total magnetic energy of the needle by

\[
E(y) := \frac{1}{2} |Dy|^2_{H^1} + \frac{1}{2} \beta \int_{\Lambda} (y_2^2(x) + y_3^2(x)) \, dx - \langle K, y \rangle_{H},
\]

(100)

where \( y = (y_1, y_2, y_3)^T \in \mathbb{H}^1 \) is the magnetization, \( \beta \) is the positive real shape anisotropy parameter and \( K \) is the externally applied magnetic field. With this magnetic energy, the deterministic Landau-Lifshitz equation becomes:

\[
\begin{align*}
\frac{dy}{dt}(t) &= y(t) \times (\Delta y(t) - \beta(0, y_2(t), y_3(t))^T + K(t)) \\
&\quad - \alpha y(t) \times (y(t) \times (\Delta y(t) - \beta(0, y_2(t), y_3(t))^T + K(t))) \\
y(0) &= u_0;
\end{align*}
\]

(101)

we assume, as before, that the initial state \( u_0 \) belongs to \( \mathbb{H}^1 \) and \( |u_0(x)|_{\mathbb{R}^3} = 1 \) for all \( x \in \Lambda \) and we also assume that the applied field \( K(t) : \Lambda \to \mathbb{R}^3 \) is constant on \( \Lambda \) at each time.
Equation (101) has nice features: the dynamics of the solution can be studied using elementary techniques and, when the externally applied field $\mathcal{K}$ is zero, the equation has two stable stationary states, $(-1,0,0)^T$ and $(1,0,0)^T$.

We now outline the structure of this example. Firstly, recall that $\mathbb{H}^1$ is continuously embedded in $C(\Lambda; \mathbb{R}^3)$ and there is a positive real constant $k$ such that

$$\sup_{x \in \Lambda} |u(x)|_{\mathbb{R}^3} \leq k |u|_{\mathbb{H}^1}^{\frac{1}{2}} |u|_{\mathbb{H}^1}^{\frac{1}{2}} \quad \text{for all} \quad u \in \mathbb{H}^1;$$

this constant appears in our results and can be taken to be $k = 2(1 \lor \frac{1}{\sqrt{l(\Lambda)}})$ (see [1, Theorem 5.8 and its proof]), where $l(\Lambda)$ is the length of the interval $\Lambda$.

In Proposition 25, we show that if the applied field $\mathcal{K}$ is zero and the initial state $u_0$ satisfies $|u_0 - \zeta|_{\mathbb{H}^1} < \frac{1}{2k^2 \sqrt{l(\Lambda)}} \frac{\alpha}{1 + 2\alpha}$, where $\zeta = (-1,0,0)^T$, then the solution $y(t)$ of (101) converges to $\zeta$ in $\mathbb{H}^1$ as $t$ goes to $\infty$. In Lemma 26, we obtain a result for nonzero $\mathcal{K}$, which is useful in the proof of Proposition 28. Let $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T \in \mathbb{R}^3$ and $|\zeta|_{\mathbb{R}^3} = 1$. Since $\frac{dy}{dt}$, the time derivative of the solution of

11.1 Stable stationary states of the deterministic equation

In this subsection, we identify stable stationary states of the deterministic equation (101) when the applied field $\mathcal{K}$ does not vary with time. In Proposition 25, we show that the two spatially uniform states $(-1,0,0)^T$ are stable stationary states when $\mathcal{K} = 0$. In Lemma 26, we obtain a result for nonzero $\mathcal{K}$, which is useful in the proof of Proposition 28.
Proof. The key fact is

\( |y(t) - \zeta|^2_{\mathbb{H}} \)

belongs to \( L^2(0, T; \mathbb{H}) \) and \( y \) belongs to \( L^2(0, T; \mathbb{X}^1) \), we have for all \( t \geq 0 \):

\[
|u_0 - \zeta|^2_{\mathbb{H}} + 2 \int_0^t \langle y(s) - \zeta, y(s) \times \Delta y(s) + y(s) \times (-\beta \left( \frac{y_2}{y_3} \right) + \mathcal{K}) - \alpha y(s) \times (y(s) \times \Delta y(s)) \rangle_{\mathbb{H}} ds
\]

\( (101) \)

and

\[
|Dy(t)|_{\mathbb{H}}^2 \leq |Dy_0|_{\mathbb{H}}^2 - 2 \int_0^t \langle \Delta y(s), y(s) \rangle_{\mathbb{H}} - \alpha y(s) \times (y(s) \times \Delta y(s)) - \alpha y(s) \times (y(s) \times (-\beta \left( \frac{y_2}{y_3} \right) + \mathcal{K})) \rangle_{\mathbb{H}} ds.
\]

\( (102) \)

Lemma 24. Let \( \zeta = (\mp 1, 0, 0)^T \). Suppose that \( u = (u_1, u_2, u_3)^T \) is an element of \( \mathbb{H}^1 \) such that \( |u(x)|_{\mathbb{R}^3} = 1 \) for all \( x \in \Lambda \) and \( |u - \zeta|_{\mathbb{H}^1} \leq \frac{1}{2k^2\sqrt{l(\Lambda)}} \). Then we have:

1. \( \frac{1-u_2^2(x)}{u_1^2(x)} + \alpha \left( \frac{1-u_2^2(x)}{u_1^2(x)} \right)^2 - \alpha u_1^2(x) \leq 0 \) for all \( x \in \Lambda \),

2. \( \langle u(x), \zeta \rangle_{\mathbb{R}^3} \geq \frac{3}{4} \) for all \( x \in \Lambda \) and

3. \( \frac{3}{8}|u(x) - \zeta|_{\mathbb{R}^3}^2 \leq |u(x) \times \zeta|_{\mathbb{R}^3}^2 \) for all \( x \in \Lambda \).

Proof. The key fact is

\[
\sup_{x \in \Lambda} |u(x) - \zeta|_{\mathbb{R}^3}^2 \leq k^2|u - \zeta|_{\mathbb{H}}|u - \zeta|_{\mathbb{H}^1}, \quad \text{by } (102),
\]

\[
\leq k^2 \frac{1}{2k^2\sqrt{l(\Lambda)}} \frac{\alpha}{1 + 2\alpha} = \frac{\alpha}{1 + 2\alpha}. \quad (106)
\]

From \( (106) \), for any \( x \in \Lambda \) we have

\[
u_1^2(x) = 1 - \frac{u_2^2(x) + u_3^2(x)}{u_1^2(x)} \geq 1 - |u(x) - \zeta|_{\mathbb{R}^3} \geq \frac{1 + \alpha}{1 + 2\alpha}; \quad (107)
\]
one can use (107) and straightforward algebraic manipulations to verify that
\[ \frac{1 - u_1^2(x)}{u_1^2(x)} + \alpha \left( \frac{1 - u_1^2(x)}{u_1^2(x)} \right)^2 - \alpha u_1^2(x) \leq 0. \]

Statements 2 and 3 of Lemma 24 follow from (106) and the elementary equalities which hold for all \( x \in \Lambda \): 2\(u(x), \zeta_3 \rangle = 2 - |u(x) - \zeta_3^2|_3^2 \) and \( |u(x) \times \zeta_3^2|_3^2 = 1 - \langle u(x), \zeta_3^2 \rangle \). □

### 11.1.1 Stable stationary states for zero applied field, \( K = 0 \)

This subsection is devoted to the following proposition.

**Proposition 25.** Let the applied field \( K \) be zero. Let \( \zeta = (\mp 1, 0, 0)^T \) and let
\[
|u_0 - \zeta|_{H^1} < \frac{1}{2k^2 \sqrt{I(\Lambda)}} \frac{\alpha}{1 + 2\alpha},
\]
(108)

Then \( y(t) = (y_1(t), y_2(t), y_3(t))^T \) converges to \( \zeta \) in \( H^1 \) as \( t \) goes to \( \infty \).

**Proof.** Using some algebraic manipulation and the fact that \( \langle Dy(s), \zeta(s) \rangle_{\mathbb{R}^3} = 0 \) a.e. on \( \Lambda \) for each \( s \geq 0 \), one may simplify equations (104) and (105).

We obtain from (104):
\[
|y(t) - \zeta|^2_{H^1} = |u_0 - \zeta|^2_{H^1} - 2\alpha \int_0^t \int_\Lambda |Dy(s)|_{\mathbb{R}^3}^2 \langle y(s), \zeta \rangle_{\mathbb{R}^3} \, dx \, ds
\]
\[ - 2\alpha \beta \int_0^t \int_\Lambda \langle y(s), \zeta \rangle_{\mathbb{R}^3} |y(s) \times \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) |^2_{\mathbb{R}^3} \, dx \, ds \quad \forall t \geq 0. \]
(109)

We obtain from (105):
\[
|Dy(t)|_{\mathbb{R}^3}^2 = |Du_0|^2_{\mathbb{R}^3} - 2\alpha \int_0^t |y(s) \times \Delta y(s)|_{\mathbb{R}^3}^2 \, ds + 2\beta \int_0^t \int_\Lambda R(s) \, dx \, ds \quad \forall t \geq 0,
\]
(110)

where, for each \( s \geq 0 \),
\[
R(s) := Dy_1(s)(y_3(s)Dy_2(s) - y_2(s)Dy_3(s))
\]
\[ + \alpha (Dy_1(s))^2 - \alpha y_1^2(s)Dy_2(s)^2 \]
\[ = \frac{-y_2(s)Dy_2(s) - y_3(s)Dy_3(s)}{y_1(s)} (y_3(s)Dy_2(s) - y_2(s)Dy_3(s))
\]
\[ + \alpha (1 - y_1^2(s)) \left( \frac{y_2(s)Dy_2(s) + y_3(s)Dy_3(s)}{y_1(s)} \right)^2 - \alpha y_1^2(s) ((Dy_2(s))^2 + (Dy_3(s))^2)
\]
\[ \leq \left( \frac{1 - y_1^2(s)}{y_1^2(s)} + \alpha (1 - y_1^2(s))^2 - \alpha y_1^2(s) \right) ((Dy_2(s))^2 + (Dy_3(s))^2),
\]
(112)

by the Cauchy-Schwarz inequality.

Define \( \tau := \inf \{ t \geq 0 : |y(t) - \zeta|_{H^1} \geq \frac{1}{2k^2 \sqrt{I(\Lambda)}} \frac{\alpha}{1 + 2\alpha} \} \). By our choice of \( u_0 \), \( \tau > 0 \). For each \( s \in [0, \tau) \), \( y(s) \) satisfies the hypotheses of Lemma 24 hence \( R(s) \leq 0 \) for all \( x \in \Lambda \),

68
we have nonincreasing, it follows that \( \lim_{\tau \to \infty} \), where (115) on \( |H| \) and let the applied field be Lemma 26.

In this subsection we show that if the applied field has sufficiently large magnitude, then \( \text{Suppose, to get a contradiction, that } \tau < \infty. \) Then, from (113) and (114), we have

\[
|y(\tau) - \zeta|_{H} \leq |u_0 - \zeta|_{H} < \frac{1}{2k^2 \sqrt{t(\Lambda)}} \frac{\alpha}{1 + 2\alpha},
\]

which contradicts the definition of \( \tau \). Therefore, \( \tau = \infty \). Since (113) holds for all \( t \geq 0 \), we have \( \int_{0}^{\infty} |Dy(s)|^2_{H} ds < \infty \) and \( \int_{0}^{\infty} |y(s) - \zeta|^2_{H} ds < \infty \) and, since the integrands are nonincreasing, it follows that \( \lim_{t \to \infty} |Dy(t)|_{H} = 0 \) and \( \lim_{t \to \infty} |y(t) - \zeta|_{H} = 0 \). \qed

11.1.2 Stable stationary states for nonzero applied field, \( \mathcal{K} \neq 0 \)

In this subsection we show that if the applied field has sufficiently large magnitude, then there is a stable stationary state that is roughly in the direction of the applied field.

**Lemma 26.** Let \( \mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)^T \in \mathbb{R}^3 \) satisfy

\[
|\mathcal{H}|_{\mathbb{R}^3} > \left( \frac{4\beta + 4\alpha\beta}{3\alpha} \right) \sqrt{\frac{2\beta + 4\alpha\beta}{\alpha}}
\]

and let the applied field be

\[
\mathcal{K} := \mathcal{H} + \frac{\beta}{|\mathcal{H}|_{\mathbb{R}^3}} \begin{pmatrix} 0 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{pmatrix}.
\]

Let \( |u_0 - \frac{\mathcal{H}}{|\mathcal{H}|_{\mathbb{R}^3}}|_{H} < \frac{1}{\tau} \). Then

\[
|y(t) - \frac{\mathcal{H}}{|\mathcal{H}|_{\mathbb{R}^3}}|_{H} \leq |u_0 - \frac{\mathcal{H}}{|\mathcal{H}|_{\mathbb{R}^3}}|_{H} e^{-\frac{\tau}{2}t} \quad \forall t \geq 0,
\]

where \( \gamma := (\alpha |\mathcal{H}|_{\mathbb{R}^3} + \alpha - 2\beta - 4\alpha\beta) \wedge \left( \alpha |\mathcal{H}|_{\mathbb{R}^3} - 2\beta - 2\alpha\beta \right) \) is positive, by the condition (113) on \( |\mathcal{H}|_{\mathbb{R}^3} \). Hence, under the above hypotheses, \( y(t) \) converges in \( H^1 \) to \( \frac{\mathcal{H}}{|\mathcal{H}|_{\mathbb{R}^3}} \) as \( t \) goes to \( \infty \).

**Proof.** In the course of the proof of Lemma 26, we shall need the following elementary lemma, whose proof is omitted.

**Lemma 27.** If \( y \) and \( z \) are two points in \( \mathbb{R}^3 \) such that \( |y|_{\mathbb{R}^3} = |z|_{\mathbb{R}^3} = 1 \) and \( |y - z|_{\mathbb{R}^3} \leq 1 \) then \( \langle y, z \rangle_{\mathbb{R}^3} \geq \frac{1}{2} \) and \( \frac{\sqrt{2}}{2} |y - z|_{\mathbb{R}^3} \leq |y \times z|_{\mathbb{R}^3} \leq |y - z|_{\mathbb{R}^3} \).

69
We have, from (103) and (104) with $\zeta$ replaced by $\frac{H}{|H|_{l_3}}$:

$$|y(t) - \frac{H}{|H|_{l_3}}|_{L_3}^2 = |u_0 - \frac{H}{|H|_{l_3}}|_{L_3}^2 + 2\int_0^t \langle y(s) - \frac{H}{|H|_{l_3}}, y(s) \times (H - \beta \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds$$

$$+ 2\alpha \int_0^t \langle \Delta y(s), y(s) \times (y(s) \times \frac{H}{|H|_{l_3}}) \rangle_{L_3} ds$$

$$- 2\alpha \int_0^t \langle y(s) \times \frac{H}{|H|_{l_3}}, y(s) \times (H - \beta \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds$$

$$= |u_0 - \frac{H}{|H|_{l_3}}|_{L_3}^2 - 2\beta \int_0^t \langle y(s) - \frac{H}{|H|_{l_3}}, y(s) \times \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds$$

$$- 2\alpha \int_0^t \int_\Lambda |Dy(s)|_{L_3}^2 \langle y(s), \frac{H}{|H|_{l_3}} \rangle_{L_3} dx ds$$

$$- 2\alpha |H|_{L_3} \int_0^t |y(s)|_{L_3}^2 \frac{H}{|H|_{l_3}} ds$$

$$+ 2\alpha \beta \int_0^t \langle y(s) \times \frac{H}{|H|_{l_3}}, y(s) \times \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds \quad \forall t \geq 0.$$

From (105) we have:

$$|Dy(t)|_{L_3}^2 = |Du_0|_{L_3}^2 - 2\int_0^t \langle \Delta y(s), y(s) \times (H - \beta \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds$$

$$- \alpha y(s) \times (y(s) \times \Delta y(s))$$

$$- \alpha y(s) \times (y(s) \times (H - \beta \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds$$

$$= |Du_0|_{L_3}^2 + 2\beta \int_0^t \langle \Delta y(s), y(s) \times \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds$$

$$- 2\alpha \int_0^t |y(s) \times \Delta y(s)|_{L_3}^2 ds$$

$$- 2\alpha |H|_{L_3} \int_0^t \int_\Lambda |Dy(s)|_{L_3}^2 \langle y(s), \frac{H}{|H|_{l_3}} \rangle_{L_3} dx ds$$

$$- 2\alpha \beta \int_0^t \langle \Delta y(s), y(s) \times (y(s) \times \left( \frac{y_2(s)}{y_3(s)} - \frac{0}{y_3(s)} \right) \rangle_{L_3} ds \quad \forall t \geq 0. \quad (118)$$

Define

$$\tau_1 := \inf\{t \geq 0 : |y(t) - \frac{H}{|H|_{l_3}}|_{L^1} \geq \frac{1}{k}\}. \quad (119)$$
By our choice of \( u_0 \), \( \tau_1 \) is greater than zero. Observe that
\[
\sup_{x \in \Lambda} |y(t)(x)| - \frac{\mu}{|H|_{\mathbb{R}^3}} < 1 \quad \text{for all } t < \tau_1.
\] (120)

Hence we have, by Lemma 27,
\[
\frac{3}{4} |y(t)| - \frac{\mu}{|H|_{\mathbb{R}^3}} \leq |y(t)| \times - \frac{\mu}{|H|_{\mathbb{R}^3}} \leq |y(t)| - \frac{\mu}{|H|_{\mathbb{R}^3}} \quad \forall t < \tau_1
\] (121)
and
\[
\langle y(t)(x), \frac{\mu}{|H|_{\mathbb{R}^3}} \rangle_{\mathbb{R}^3} \geq \frac{1}{2} \quad \forall x \in \Lambda \text{ and } \forall t < \tau_1.
\] (122)

Adding equalities (117) and (118) we obtain:
\[
|y(t)| - \frac{\mu}{|H|_{\mathbb{R}^3}} \geq |y| \times - \frac{\mu}{|H|_{\mathbb{R}^3}} \leq |y| - \frac{\mu}{|H|_{\mathbb{R}^3}} \quad \forall t < \tau_1
\] (123)

where we used (120), (121) and (122). Because of hypothesis (115), the two expressions \( (\alpha + \alpha|H|_{\mathbb{R}^3} - 2\beta - 4\alpha\beta) \) and \( (\frac{3}{2}\alpha|H|_{\mathbb{R}^3} - 2\beta - 2\alpha\beta) \) on the right hand side of (124) are positive numbers.

Suppose, to get a contradiction, that \( \tau_1 < \infty \). Then, from (121), we have
\[
|y(\tau_1)| - \frac{\mu}{|H|_{\mathbb{R}^3}} < |u_0| - \frac{\mu}{|H|_{\mathbb{R}^3}} < \frac{1}{k},
\]
which contradicts the definition of \( \tau_1 \) in \( \| 119 \). Hence \( \tau_1 = \infty \). It now follows from \( \| 124 \) that

\[
\int_0^\infty |Dy(s)|^2_{\mathbb{H}} \, ds < \infty, \quad (125)
\]

\[
\int_0^\infty |y(s) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}} \, ds < \infty, \quad (126)
\]

and

\[
\int_0^\infty |y(s) \times \Delta y(s)|^2_{\mathbb{H}} \, ds < \infty, \quad (127)
\]

From \( \| 123 \) and these three inequalities, the function \( t \in [0, \infty) \mapsto |y(t) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}} \) is absolutely continuous and, for almost every \( t \in [0, \infty) \), its derivative is:

\[
(|y(\cdot) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}}\rangle'_{\mathbb{H}}(t) = -2(2 + 2\alpha |\mathcal{H}|_{\mathbb{R}^3}) \int_\Lambda |Dy(t)|^2_{\mathbb{R}^3} \langle y(t), \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} \rangle_{\mathbb{R}^3} \, dx \\
+ 2\beta \langle \Delta y(t), y(t) \times (y(t) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} \rangle_{\mathbb{H}}^2 \, ds \leq -(\alpha + \alpha |\mathcal{H}|_{\mathbb{R}^3} - 2\beta - 4\alpha \beta) |Dy(t)|^2_{\mathbb{H}} \\
- \frac{3}{2} (\alpha |\mathcal{H}|_{\mathbb{R}^3} - 2\beta - 2\alpha \beta) |y(t) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}} \\
- 2\alpha \gamma |y(t) \times \Delta y(t)|^2_{\mathbb{H}} \leq -\gamma |y(t) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}}, \quad (128)
\]

where \( \gamma := (\alpha + \alpha |\mathcal{H}|_{\mathbb{R}^3} - 2\beta - 4\alpha \beta) \wedge (\frac{3}{2} \alpha |\mathcal{H}|_{\mathbb{R}^3} - 2\beta - 2\alpha \beta) > 0 \). Multiplying both sides of \( \| 128 \) by the integrating factor \( e^{\tau t} \) yields:

\[
(e^{\gamma} |y(\cdot) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}}\rangle'_{\mathbb{H}}(t) \leq 0 \quad \text{for almost every } t \in [0, \infty)
\]

and hence we have

\[
e^{\gamma} |y(t) - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}} - |u_0 - \frac{\mathcal{H}}{\mathcal{H}_{\mathbb{H}}^3} |^2_{\mathbb{H}} \leq 0 \quad \forall t \geq 0.
\]

This proves \( \| 116 \) and completes the proof of the lemma. \( \square \)
11.2 Noise induced instability and magnetization reversal

In Proposition \[23\] we showed that the states \((\pm 1, 0, 0)^T\) are stable stationary states of the deterministic Landau-Lifshitz equation \([101]\) when the externally applied field \(K\) is zero. In this section we show that a small noise term in the field may drive the magnetization from the initial state \((-1, 0, 0)^T\) to any given \(H^1\)-ball centred at \((1, 0, 0)^T\) in any given time interval \([0, T]\). We also find an exponential upper bound for the probability that small noise in the field drives the magnetization outside a given \(H^1\)-ball centred at the initial state \((-1, 0, 0)^T\) in time interval \([0, T]\). Firstly we need a definition.

**Definition 1.** Let \(\delta\) be a given small positive real number. Suppose that the initial magnetization is \((-1, 0, 0)^T\) and that at some time \(T\) the magnetization lies in the open \(H^1\)-ball centred at \((1, 0, 0)^T\) and of radius \(\delta\). Then we say that magnetization reversal has occurred by time \(T\).

We consider a stochastic equation for the magnetization, obtained by setting \(K\) to zero and adding a three dimensional noise term to the field. Denoting the stochastic magnetization by \(Y = (Y_1, Y_2, Y_3)^T\), the equation is:

\[
\begin{align*}
\frac{dY(t)}{dt} & = Y(t) \times (\Delta Y(t) - \beta \left( \begin{array}{c} 0 \\ y_2(t) \\ y_3(t) \end{array} \right) ) dt + \varepsilon \frac{1}{2} \sum_{i=1}^{3} (Y(t) \times a_i) \times dW_i(t) \\
& \quad - \alpha Y(t) \times (Y(t) \times (\Delta Y(t) - \beta \left( \begin{array}{c} 0 \\ y_2(t) \\ y_3(t) \end{array} \right) )) dt \\
& \quad - \alpha \varepsilon \frac{1}{2} \sum_{i=1}^{3} (Y(t) \times (Y(t) \times a_i)) \times dW_i(t), \\
Y(0) & = (-1, 0, 0)^T.
\end{align*}
\]

In \((129)\), \(W_1, W_2, W_3\) are independent Brownian motions defined on probability space \((\Omega, \mathcal{F}, P)\) and \(dW_i\) denotes a Stratonovich integral with respect to \(W_i\); \(a^1, a^2, a^3\) are fixed linearly independent elements of \(\mathbb{R}^3\); the positive real parameter \(\varepsilon\) corresponds to the ‘dimensionless temperature’ parameter appearing in the stochastic differential equation \([2]\) of Kohn, Reznikoff and Vanden-Eijnden.

Fix \(T \in (0, \infty)\). There is no deterministic applied field in \((129)\) but, as we will see, the lower bound of the large deviation principle satisfied by the solutions \(Y^\varepsilon\) \((\varepsilon \in (0, 1))\) of \((129)\) implies that, for all sufficiently small positive \(\varepsilon\), the probability of magnetization reversal by time \(T\) is positive.

Firstly, we shall use Lemma \([20]\) to construct a piecewise constant (in time) deterministic applied field, \(K\), such that the solution \(y\) of \((101)\), with initial state \((-1, 0, 0)^T\), undergoes magnetization reversal by time \(T\).

Take points \(u^i = (u_1^i, u_2^i, u_3^i)^T\), \(i = 0, 1, \ldots, N\), lying on the unit sphere in \(\mathbb{R}^3\) such that \(u^0 = (-1, 0, 0)^T\) and \(u^N = (1, 0, 0)^T\) and

\[
|u^i - u^{i+1}|_{H^1} = |u^i - u^{i+1}|_{\mathbb{R}^3} \sqrt{1/|A|} < \frac{1}{k} \quad \text{for } i = 0, 1, \ldots, N - 1.
\]
Let $\eta := \min\{\frac{1}{T} - |u^i - u^{i+1}|_{H^1} : i = 1, \ldots, N - 1\} \land \frac{\delta}{T}$. Using Lemma 22, we can take the applied field to be

$$\mathcal{K}(t) := \sum_{i=0}^{N-1} 1(i \frac{T}{N}, (i+1) \frac{T}{N})(t) \left(Ru^{i+1} + \beta \left(\begin{array}{c} 0 \\
_{u_{2i+1}} \\
_{u_{3i+1}} \end{array}\right)\right), \quad t \geq 0,$$

(130)

with the positive real number $R$ chosen to ensure that, as $t$ varies from $i \frac{T}{N}$ to $(i+1) \frac{T}{N}$, $y(t)$ starts at a distance of less than $\eta$ from $u^i$ (i.e. $|y(i \frac{T}{N}) - u^i|_{H^1} < \eta$) and moves to a distance of less than $\eta$ from $u^{i+1}$ (i.e. $|y((i+1) \frac{T}{N}) - u^{i+1}|_{H^1} < \eta$). Specifically, we take $R \in (0, \infty)$ such that

$$\frac{1}{k} e^{-\frac{1}{2}|(\alpha R + \alpha - 2\alpha \beta)\wedge (2 \alpha R - 2\alpha \beta)|^2 R} < \eta.$$

For each $i = 0, 1, \ldots, N - 1$, let $\phi^{i+1} = (\phi_1^{i+1}, \phi_2^{i+1}, \phi_3^{i+1}) \in \mathbb{R}^3$ be the vector of scalar coefficients satisfying the equality

$$\phi_1^{i+1} a^1 + \phi_2^{i+1} a^2 + \phi_3^{i+1} a^3 = Ru^{i+1} + \beta \left(\begin{array}{c} 0 \\
_{u_{2i+1}} \\
_{u_{3i+1}} \end{array}\right),$$

and define

$$\phi(t) := \sum_{i=0}^{N-1} 1(i \frac{T}{N}, (i+1) \frac{T}{N})(t) \phi^{i+1}, \quad t \in [0, T].$$

(131)

We remark that the function $\phi$ depends on the chosen values of $\delta$ and $T$, the material parameters $\Lambda$, $\alpha$ and $\beta$ and the noise parameters $a^1$, $a^2$ and $a^3$.

Recall that $Y^\varepsilon$ denotes the solution of (129). By an argument very much like that leading to Theorem 17, the family of laws $\{\mathcal{L}(Y^\varepsilon) : \varepsilon \in (0, 1)\}$ on $C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ satisfies a large deviation principle. In order to define the rate function, we define the map $\mathcal{J} : L^2(0, T; \mathbb{R}^3) \to C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1)$ by

$$\mathcal{J}(\psi) := y_\psi = ((y_\psi)_1, (y_\psi)_2, (y_\psi)_3)^T \quad \text{for each } \psi = (\psi_1, \psi_2, \psi_3)^T \in L^2(0, T; \mathbb{R}^3),$$

where $y_\psi$ satisfies the equality

$$y_\psi(t) = \left(\begin{array}{c} -1 \\
0 \\
0 \end{array}\right) + \int_0^t y_\psi(s) \times (\Delta y_\psi(s) - \beta \left(\begin{array}{c} 0 \\
(y_\psi)_2(s) \\
(y_\psi)_3(s) \end{array}\right)) \, ds$$

$$- \alpha \int_0^t y_\psi(s) \times (y_\psi(s) \times (\Delta y_\psi(s) - \beta \left(\begin{array}{c} 0 \\
(y_\psi)_2(s) \\
(y_\psi)_3(s) \end{array}\right))) \, ds$$

$$+ \sum_{j=1}^{3} \int_0^t (y_\psi(s) \times a^j) \psi_j(s) \, ds$$

$$- \alpha \sum_{j=1}^{3} \int_0^t (y_\psi(s) \times a^j) \psi_j(s) \, ds \quad \forall t \in [0, T].$$

(132)
The rate function of the large deviation principle, \( \mathcal{I} : C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X) \to [0, \infty] \), is defined by:

\[
\mathcal{I}(v) := \inf \left\{ \frac{1}{2} \int_0^T |\psi(s)|^2_{\mathbb{R}^3} ds : \psi \in L^2(0, T; \mathbb{R}^3) \text{ and } v = \mathcal{J}(\psi) \right\},
\]

where the infimum of the empty set is taken to be \( \infty \).

Let \( y \) be the solution of equation (101) with \( u_0 = (-1, 0, 0)^T \) and \( K \) as defined in (130). Using the notation in (132), we have \( y = y_\phi \), for \( \phi \) defined in (131). Therefore \( \mathcal{I}(y|_{[0, T]} \leq \frac{1}{2} \int_0^T |\phi(s)|^2_{\mathbb{R}^3} ds < \infty \). Since \( y \) experiences magnetization reversal by time \( T \), paths of \( Y^\varepsilon \) which lie close to \( y \) also undergo magnetization reversal by time \( T \). In particular, by the Freidlin-Wentzell formulation of the lower bound of the large deviation principle (see, for example, [8, Proposition 12.2]), given \( \xi > 0 \), there exists an \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) we have

\[
P\{\sup_{t \in [0, T]} |Y^\varepsilon(t) - y(t)|_{\mathbb{H}^1} + (\int_0^T |Y^\varepsilon(s) - y(s)|^2_{X^1} ds) \cdot \frac{1}{\varepsilon} < \frac{\delta}{T} \}
\geq \exp \left( -\frac{1}{\varepsilon} \int_0^T |\phi(s)|^2_{\mathbb{R}^3} ds - \frac{\delta}{\varepsilon} \right).
\]

Since we have \( |y(T) - (-1, 0, 0)^T|_{\mathbb{H}^1} < \frac{\delta}{T} \), the right hand side of (134) provides a lower bound for the probability that \( Y^\varepsilon \) undergoes magnetization reversal by time \( T \). We summarize our conclusions in the following proposition.

**Proposition 28.** For all sufficiently small \( \varepsilon > 0 \), the probability that the solution \( Y^\varepsilon \) of (120) undergoes magnetization reversal by time \( T \) is bounded below by the expression on the right hand side of (134); in particular, it is positive.

We shall now use the upper bound of the large deviation principle satisfied by \( \{\mathcal{L}(Y^\varepsilon) : \varepsilon \in (0, 1)\} \) to find an exponential upper bound for the probability that small noise in the field drives the magnetization outside a given \( \mathbb{H}^1 \)-ball centred at the initial state \((-1, 0, 0)^T\) in time interval \([0, T]\). This is done in Proposition 30 below; the proof of the proposition uses Lemma 29 in Lemma 29 and Proposition 30 for \( \psi \) an arbitrary element of \( L^2(0, T; \mathbb{R}^3) \), \( y_\psi \) denotes the function in \( C([0, T]; \mathbb{H}^1) \cap L^2(0, T; X^1) \) which satisfies equality (132) and \( \tau_\psi \) is defined by \( \tau_\psi := \inf\{t \in [0, T] : |y_\psi(t) - (-1, 0, 0)^T|_{\mathbb{H}^1} \geq \frac{1}{2k^2 \sqrt{t(1 + 2\alpha)}} \} \).

**Lemma 29.** For each \( \psi \in L^2(0, T; \mathbb{R}^3) \), we have \( |Dy_\psi(t)|_{\mathbb{H}^1} = 0 \) for all \( t \in [0, \tau_\psi \land T] \).

**Proof.** Let \( \psi \in L^2(0, T; \mathbb{R}^3) \). To simplify notation in this proof, we write \( y \) instead of \( y_\psi \). Proceeding as in the derivation of (110), we obtain

\[
|Dy(t)|^2_{\mathbb{H}^1} = -2\alpha \int_0^t |y(s) \times \Delta y(s)|^2_{\mathbb{R}^3} ds + 2\beta \int_0^t \int_\Lambda R(s) \, dx \, ds
- 2\alpha \sum_{i=1}^3 \int_0^t \langle Dy(s), y(s) \times (Dy(s) \times a^i) \rangle_{\mathbb{H}^1} \psi_i(s) \, ds \quad \forall t \in [0, T], \tag{135}
\]

75
where \( R(s) \) is defined as in (111) and satisfies inequality (112). For each \( s \in [0, \tau_\psi \wedge T) \), \( y(s) \) satisfies the hypotheses of Lemma 24, thus we have \( R(s)(x) \leq 0 \) for all \( x \in \Lambda \). It follows from (135) that for all \( t \in [0, \tau_\psi \wedge T) \):

\[
|Dy(t)|^2_H \leq 2\alpha \int_0^t |Dy(s)|^2_H \sum_{i=1}^3 |a_i|^2_{H^3} |\psi_i(s)| \, ds. \tag{136}
\]

By Gronwall’s lemma applied to (136), \( |Dy(t)|^2_H = 0 \) for all \( t \in [0, \tau_\psi \wedge T) \).

**Proposition 30.** Let \( 0 < r < \rho \leq \frac{1}{2k_2 \sqrt{\lambda(\Lambda)}} \alpha \). Given \( \xi \in (0, \infty) \), there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \):

\[
P \{ \sup_{t \in [0,T]} |Y^\varepsilon(t) - (-1, 0, 0)^T|_{H^1} \geq \rho \} \leq \exp \left( \frac{1}{\varepsilon} \left( \frac{-\alpha \beta}{8 \max_{1 \leq i \leq 3} |a_i|^2_{H^3}|l(\Lambda)(1 + \alpha^2)|^2 + \xi} \right) \right). \tag{137}
\]

**Proof.** We shall use the Freidlin-Wentsell formulation of the upper bound of the large deviation principle (see, for example, [8, Proposition 12.2]) satisfied by \( \{Y^\varepsilon(t) : \varepsilon \in (0,1)\} \). Recall that \( \mathcal{I} \), defined in (133), is the rate function of the large deviation principle. Our main task is to show that the level set \( \{ \mathcal{I} \leq \frac{\alpha \beta}{8 \max_{1 \leq i \leq 3} |a_i|^2_{H^3}|l(\Lambda)(1 + \alpha^2)|^2 \} \) is contained in \( \{v \in C([0,T]; H^1) : \sup_{t \in [0,T]} |v(t) - (-1, 0, 0)^T|_{H^1} \leq r \} \). Take \( \psi \in L^2(0,T; \mathbb{R}^3) \) such that

\[
\frac{1}{2} \int_0^T |\psi(s)|^2_{H^3} \, ds \leq \frac{\alpha \beta}{8 \max_{1 \leq i \leq 3} |a_i|^2_{H^3}|l(\Lambda)(1 + \alpha^2)|^2}. \tag{138}
\]

For simplicity of notation, in this proof we write \( y \) in place of \( y_\psi \). We have for all \( t \in [0,T] \):

\[
|y(t \wedge \tau_\psi) - (-1, 0, 0)^T|_{H^1}^2 = |y(t \wedge \tau_\psi) - (-1, 0, 0)^T|_{H}^2 \quad \text{by Lemma 20}
\]

\[
= -2\alpha \int_0^{t \wedge \tau_\psi} \int_{\Lambda} |Dy(s)|^2_H \langle y(s), (-1, 0, 0)^T \rangle_{H^3} \, dx \, ds \\
- 2\alpha \beta \int_0^{t \wedge \tau_\psi} \int_{\Lambda} \langle y(s), (-1, 0, 0)^T \rangle_{H^3} |y(s) \times (1, 0, 0)^T|_{H^3}^2 \, dx \, ds \\
+ 2\alpha \beta \sum_{i=1}^3 \int_0^{t \wedge \tau_\psi} \left( \frac{1}{2} \langle y(s) \times (-1, 0, 0)^T, \frac{2}{\alpha \beta} a_i \psi_i(s) \rangle_{H^3} \right. \\
- 2\alpha \beta \sum_{i=1}^3 \int_0^{t \wedge \tau_\psi} \left( \frac{1}{2} \langle y(s) \times (-1, 0, 0)^T, \frac{2}{\beta} (y(s) \times a_i) \psi_i(s) \rangle_{H^3} \right. \\
\leq -\frac{3}{2} \alpha \beta \int_0^{t \wedge \tau_\psi} |y(s) \times (1, 0, 0)^T|_{H^3}^2 \, ds \\
+ \frac{3}{2} \alpha \beta \int_0^{t \wedge \tau_\psi} |y(s) \times (-1, 0, 0)^T|_{H^3}^2 \, ds \\
+ \frac{4}{\beta} \left( 1 + \alpha \right) |l(\Lambda)| \sum_{i=1}^3 |a_i|^2_{H^3} \int_0^{t \wedge \tau_\psi} \psi_i^2(s) \, ds, \tag{139}
\]

76
where we estimated the integrals on the right hand side of the second equality as follows: the first integral vanished thanks to Lemma 29, Lemma 24 was used for the integrand of the second integral and the Cauchy-Schwarz inequality and Young’s inequality were used for the integrands of the other integrals. Using (138) in (139), we obtain

$$|y(t \wedge \tau_\psi) - (-1,0,0)^T|_{H^1} \leq r < \frac{1}{2k^2 \sqrt{t(\Lambda)}} \frac{\alpha}{1 + 2\alpha} \quad \forall t \in [0,T].$$

From (140) and the definition of $\tau_\psi$, we conclude that $\tau_\psi > T$. Hence we have $\sup_{t \in [0,T]} |y(t) - (-1,0,0)^T|_{H^1} \leq r$.

By the Freidlin-Wentsell formulation of the upper bound of the large deviation principle, since $r < \rho$, given $\xi \in (0,\infty)$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, inequality (137) holds.

Remark. Our use of Lemma 29 in the proof of Proposition 30 means that we would have obtained the same result if we had assumed that the magnetization is uniform on the space domain at all times (as Kohn, Reznikoff and Vanden-Eijnden did in their paper [13]).

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