Decidability and \( NP \)-completeness for some languages which extend Boolean Set Theory (\( BST \)).

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Abstract

We prove the decidability for a class of languages which extend \( BST \) and \( NP \)-completeness for a subclass of them. The languages \( BST \) extended with unordered cartesian product (\( BST\odot \)), \( BST \) extended with ordered cartesian product (\( BST\times \)) and \( BST \) extended with powerset (\( BSTP \)) fall in this last subclass.

Introduction

We strengthen the technique showed in [CU21], in order to reach the decidability for languages, which extend \( BST \) by allowing the use of literals of the type \( C(x_1, \ldots, x_n) = x \), where \( C \) belongs to a special class of set operators, called \( H \)-operators.

The principle aim of the present article consists in proving the following two theorems.

**Theorem 1.** All the theories obtained extending \( BST \) by allowing the use of an \( H \)-operator \( C \) are decidable.

For a subclass of these operators, the injective \( H \)-operators, we prove \( NP \)-completeness.

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Theorem 2. All the theories obtained extending BST by allowing the use of an injective $H$-operator $C$ are $NP$-complete.

These theorems have quite interesting applications.

The well-celebrated Hilbert’s Tenth problem (HTP, for short)\cite{Hilbert-02}, posed by David Hilbert at the beginning of last century, asks for a uniform procedure that can determine in a finite number of steps whether any given Diophantine polynomial equation with integral coefficients is solvable in integers.

In 1970, it was shown that no algorithmic procedure exists for the solvability problem of generic polynomial Diophantine equations, as a result of the combined efforts of M. Davis, H. Putnam, J. Robinson, and Y. Matiyasevich (DPRM theorem, see \cite{Rob56, DPR61, Mat70}).

In the early eighties, D. Cantone asked whether the decision problems for the theories $MLS\times$ and $MLS\otimes$ can be reducible to HTP.

The decision problem for $MLS\times$ (resp., $MLS\otimes$) can be somehow regarded as set-theoretic counterparts of HTP, where the union of disjoint sets and the Cartesian product (resp., unordered Cartesian product of disjoint sets) play the roles of integer addition and multiplication, respectively. Indeed, $|s \cup t| = |s| + |t|$, for any disjoint sets $s$ and $t$, and $|s \times t| = |s| \cdot |t|$, for any sets $s$ and $t$ (whereas $|s \otimes t| = |s| \cdot |t|$, for any disjoint sets $s$ and $t$).

When $MLS\otimes$ and $MLS\times$ are extended with the two-place predicate $|\cdot| \leq |\cdot|$ for cardinality comparison, where $|s| \leq |t|$ holds if and only if the cardinality of $s$ does not exceed that of $t$, the satisfiability problem for such extensions become undecidable, since HTP would be reducible to each of them, as proved in \cite{CCP90}.

In the above cited reduction to HTP, membership operator plays no role \cite{CU18}, then, by extending $BST\otimes$ or $BST\times$ with the two-place predicate $|\cdot| \leq |\cdot|$ for cardinality comparison, you get again a problem reducible to HTP.

Therefore, the real set-theoretic counterpart of HTP is actually $BST\otimes$ or $BST\times$.

\footnote{1} MLS\times is the acronym for MultiLevel Syllogistic (MLS) extended with the Cartesian product operator $\times$. We recall that MLS is the quantifier-free fragment of set theory involving the Boolean set operators $\cup$, $\cap$, and $\setminus$, and the equality and membership predicates.

Analogously, MLS\otimes denotes the extension of MLS with the unordered Cartesian product operator $\otimes$, namely the set operator defined by $s \otimes t := \{\{u, v\} \mid u \in s \land v \in t\}$.

\footnote{2} BST\otimes and BST\times can be obtained, respectively, from MLS\otimes and MLS\times simply by dropping membership predicates.
Among injective $H$-operators there are $\otimes$ and $\times$, so they are $NP$-complete, which in turns implies that, in both cases, cardinality constraints are essential in order to reach undecidability.

Notice that there are satisfiable $BST \otimes$ and $BST \times$ formulas that admit only infinite models.

Nevertheless, we prove in [CU14] that even theories which force a model to be infinite can be proven to be decidable by using the small witness-model property, which is a way to finitely represent the infinity.

Hence, the decidability of $BST \otimes$ or $BST \times$ cannot be proven through a small model property. In [CU21] it is proven that $BST \otimes$ is decidable by way of an algorithmic representation. Our general results prove, by-product, that $BST \times$ behaves in the same way.

In view of the above observation, to be more precise, the set-theoretic counterpart to HTP is the finite satisfiability problem for $BST \otimes$ or $BST \times$, namely the problem of establishing algorithmically for any $BST \otimes(BST \times)$-formula whether it admits a finite model or not.

In [CU21] we use a special class of algorithms to decide this problem. The same proof runs for finite satisfiability of $BST C$.

Another application of our theorems drives to the $NP$-completeness of $BST P$ which is MLSP, where membership literals are dropped. As far as we know, there is only a double exponential algorithm for such a decision problem [CFS85]. This implies that it seems that the introduction of membership literal drastically increases the complexity of the decision problem.

As shown in [Schw78, CCS90], the finite satisfiability property for the extension of MLS with cardinality comparison, namely the two-place predicate $| \cdot | \leq | \cdot |$ for cardinality comparison, where $|s| \leq |t|$ holds if and only if the cardinality of $s$ does not exceed that of $t$, can be reduced to purely existential Presburger arithmetic, which is known to be NP-complete (see [Sca84]).

Combining the $NP$-completeness of the pure existential presburger arithmetic with the $NP$-completeness of $BST \otimes$ (or equivalently $BST \times$), we can argue that undecidability of HTP arises from an interaction between ordered or unordered cartesian product and cardinal inequalities.
1 Extensions of BST with $C$ operator

$\text{BST}^C$ is the quantifier-free fragment of set theory consisting of the propositional closure of atoms of the following types:

$$x = y \cup z, \quad x = y \setminus z, \quad x = C(x_1 \cdots x_n), \quad x \neq y$$

where $x, y, z, x_1 \cdots x_n$ stand for set variables and $C$ is an unordered (ordered) operator on sets, that is a map $C(x_1 \cdots x_n) : \text{Sets} \otimes \cdots \otimes \text{Sets} \to \text{Sets}$ (if ordered, $C : \text{Sets} \times \cdots \times \text{Sets} \to \text{Sets}$).

For any $\text{BST}^C$-formula $\Phi$, we denote by $\text{Vars}(\Phi)$ the collection of set variables occurring in it.

The semantics of $\text{BST}^C$ follows exactly the definition of many other languages already treated (see for example [CU18, CU21]). Other operators and relators of $\text{BST}^C$ are interpreted according to their usual semantics as well as satisfiability by partitions and normalization of $\text{BST}^C$-formulae [CU21].

You can find all the following definitions in previous articles as [CU14, CU18, CU21]. For reader’s convenience we briefly resume them in the following section.

1.1 Satisfiability by partitions

A partition is a collection of pairwise disjoint non-null sets, called the blocks of the partition. The union $\bigcup \Sigma$ of a partition $\Sigma$ is its domain.

Let $V$ be a finite collection of set variables and $\Sigma$ a partition. Also, let $\mathcal{I} : V \to \text{pow}(\Sigma)$ be any map. In a very natural way, the map $\mathcal{I}$ induces a set assignment $M_\mathcal{I}$ over $V$ defined by:

$$M_\mathcal{I} v := \bigcup \mathcal{I}(v), \quad \text{for } v \in V.$$

We refer to the map $\mathcal{I}$ (or to the pair $(\Sigma, \mathcal{I})$, when we want to emphasize the partition $\Sigma$) as a partition assignment.

**Definition 1.** Let $\Sigma$ be a partition and $\mathcal{I} : V \to \text{pow}(\Sigma)$ be a partition assignment over a finite collection $V$ of set variables. Given a $\text{BST}^C$-formula $\Phi$ such that $\text{Vars}(\Phi) \subseteq V$, we say that $\mathcal{I}$ satisfies $\Phi$, and write $\mathcal{I} \models \Phi$, when the set assignment $M_\mathcal{I}$ induced by $\mathcal{I}$ satisfies $\Phi$ (equivalently, one may say that $\Sigma$ satisfies $\Phi$ via the map $\mathcal{I}$, and write $\Sigma/\mathcal{I} \models \Phi$, if we want to emphasize the partition $\Sigma$). We say that $\Sigma$ satisfies $\Phi$, and write $\Sigma \models \Phi$, if $\Sigma$ satisfies $\Phi$ via some map $\mathcal{I} : V \to \text{pow}(\Sigma)$.

The following result can be proved immediately.
Lemma 1. If a BSTC-formula is satisfied by a partition $\Sigma$, then it is satisfied by any partition $\Sigma'$ that includes $\Sigma$ as a subset, namely such that $\Sigma \subseteq \Sigma'$.

Plainly, a BSTC-formula $\Phi$ satisfied by some partition is satisfied by a set assignment. Indeed, if $\Sigma \models \Phi$, then $\Sigma/I \models \Phi$ for some map $I : V \to \text{pow}(\Sigma)$, and therefore $M_I \models \Phi$. The converse holds too. In fact, let us assume that $M \models \Phi$, for some set assignment $M$ over the collection $V = \text{Vars}(\Phi)$ of the set variables occurring in $\Phi$, and let $\Sigma_M$ be the \textit{Venn partition} induced by $M$, namely

$$\Sigma_M := \left\{ \bigcap MV' \setminus \bigcup M(V \setminus V') \mid \emptyset \neq V' \subseteq V \right\} \setminus \{ \emptyset \}.$$

Let $I_M : V \to \text{pow}(\Sigma_M)$ be the map defined by

$$I_M(v) := \{ \sigma \in \Sigma_M \mid \sigma \subseteq Mv \}, \quad \text{for } v \in V.$$

It is an easy matter to check that the set assignment induced by $I_M$ is just $M$. Thus $\Sigma_M/I_M \models \Phi$, and therefore $\Sigma_M \models \Phi$, proving that $\Phi$ is satisfied by some partition, in fact by the Venn partition induced by $M$, whose size is at most $2^{|V|} - 1$.

Thus, the notion of satisfiability by set assignments and that of satisfiability by partitions coincide.

As a by-product of Lemma 1 and the above considerations, we also have:

Lemma 2. Every BSTC-formula $\Phi$ with $n$ distinct variables is satisfiable if and only if it is satisfied by some partition with $2^n - 1$ blocks.

Regarding the literals of the type $\{\cup, \setminus\}$ the assignment models a literal whenever, applied to the partition, verifies the literal [CU18, CU21].

Lemma 3. Let $\Sigma$ be a partition and let $I : V \to \text{pow}(\Sigma)$ be a partition assignment over a (finite) set of variables $V$. Then, for all $x, y, z \in V$ and $\ast \in \{\cup, \setminus\}$, we have:

(a) $I \models x = y \ast z \iff I(x) = I(y) \ast I(z)$,

(b) $I \models x \neq y \iff I(x) \neq I(y)$.

\[3\text{Hence, we have:}
- (\forall \sigma \in \Sigma_M)(\forall v \in V)(\sigma \cap Mv = \emptyset \lor \sigma \subseteq Mv),
- (\forall \sigma, \sigma' \in \Sigma_M)((\forall v \in V)(\sigma \subseteq Mv \leftrightarrow \sigma' \subseteq Mv) \leftrightarrow \sigma = \sigma'), \text{ and}
- \bigcup \Sigma = \bigcup MV.\]
2 \( C \)-graphs

An unordered constructor on a partition \( \Sigma \) is a map \( C : \text{pow}(\Sigma) \to \text{Sets} \).
An ordered constructor on a partition \( \Sigma \) is a map \( C : \text{Seq}(\Sigma) \to \text{Sets} \), where \( \text{Seq}(\Sigma) \) are finite ordered sequences \( < \sigma_1 \cdots \sigma_k > \) of blocks \( \sigma_i \in \Sigma \).

**Definition 2.** Let \( \Sigma \) be a partition of sets. A set constructor \( C : \text{pow}(\Sigma) \to \text{Sets} \) with input an ordered or unordered subset of \( \Sigma \) is **disjoint** if for all \( X, Y \subset \Sigma \) \( X \neq Y \), \( C(X) \cap C(Y) = \emptyset \) (if ordered \( X, Y \in \text{Seq}(\Sigma) \)).

From now on our constructors are supposed to be disjoint.

Let \( C \) be an unordered (ordered) operator. If for any set of variables \( V \), any partition \( \Sigma \) and partition assignment \( I : V \to \text{pow}(\Sigma) \) (\( I : V \to \text{Seq}(\Sigma) \)) there exist a function \( Q : V \otimes \cdots \otimes V \to \text{pow}(\text{pow}(\Sigma)) \) (\( Q : V \times \cdots \times V \to \text{pow}(\text{Seq}(\Sigma)) \)) and a unordered (ordered) constructor \( C \) such that

\[
\bigcup_{N \in Q(x_1,\ldots,x_n)} C(N) = C\left(\bigcup_{1}^{n} I(x_1) \cdots \bigcup_{1}^{n} I(x_n)\right)
\]

then \( C \) is an unordered (ordered) operator with constructors \((C, Q)\).

We briefly indicate with \( Q \) the generalized union of the image of \( Q \).

**Definition 3.** A subset \( \Sigma^* \) of a partition \( \Sigma \) is a **\( C \)-subpartition** of \( \Sigma \) if

\[
\bigcup \Sigma^* = \bigcup C[B], \text{ for some } B \subseteq Q.
\]

We denote by \( \Sigma_C \) the \( \subseteq \)-maximal \( C \)-subpartition of \( \Sigma \) and we refer to its elements as the **\( C \)-blocks** of \( \Sigma \). We also denote by \( \Pi_C \) the subset of \( \text{pow}(\Sigma) \) such that \( \bigcup \Sigma_C = \bigcup C[\Pi_C] \) and we refer to its elements as **\( C \)-upblocks** (‘upblocks’ for subsets of blocks which generates through the constructor \( C \) the partition \( \Sigma_C \)).

**Definition 4 (\( C \)-graphs).** A **\( C \)-graph** \( \mathcal{G} \) is a directed bipartite graph whose set of vertices comprises two disjoint parts: a set of **places** \( \mathcal{P} \) and a set of **nodes** \( \mathcal{N} \), where \( \mathcal{N} = \text{pow}(\mathcal{P}) \) (if \( C \) is ordered \( \mathcal{N} = \text{Seq}(\mathcal{P}) \)). The edges issuing from each place \( q \) are exactly all pairs \( \langle q, B \rangle \) such that \( q \in B \in \mathcal{N} \): these are called **membership edges**. The remaining edges of \( \mathcal{G} \), called **distribution** or **saturation edges**, go from nodes to places. When there is an edge \( \langle B, q \rangle \) from a node \( B \) to a place \( q \), we say that \( q \) is a **target** of \( B \). The map \( \mathcal{T} \) over \( \mathcal{N} \) defined by

\[
\mathcal{T}(B) := \{ q \in \mathcal{P} \mid q \text{ is a target of } B \}, \text{ for } B \in \mathcal{N},
\]

is the **target map** of \( \mathcal{G} \). The size of \( \mathcal{G} \) is defined as the cardinality of its set of places \( \mathcal{P} \). Plainly, a \( C \)-graph \( \mathcal{G} \) is fully characterized by its target map \( \mathcal{T} \), since the sets of nodes and of places of \( \mathcal{G} \) are expressible as \( \text{dom}(\mathcal{T}) \).
and \( \bigcup \text{dom}(T) \), respectively. When convenient, we shall explicitly write \( G = (\mathcal{P}, \mathcal{N}, T) \) for a \( C \)-graph with set of places \( \mathcal{P} \), set of nodes \( \mathcal{N} \), and target map \( T \) (in order to simplify the graph, if \( N \in \mathcal{N} \) has not outgoing arrows we drop \( N \)).

To better grasp the rationale behind the definition just stated of \( C \)-graphs, it is helpful to illustrate how to construct the \( C \)-graph \( G_\Sigma \) induced by a given partition \( \Sigma \).

To begin with, we select a set of places \( \mathcal{P}_\Sigma \) of the same cardinality of \( \Sigma \) such that \( \mathcal{P}_\Sigma \) and \( \text{pow}(\mathcal{P}_\Sigma) \) are disjoint, and define the vertex set of \( G_\Sigma \) as the union \( \mathcal{P}_\Sigma \cup \text{pow}(\mathcal{P}_\Sigma) \). The members of \( \text{pow}(\mathcal{P}_\Sigma) \) will form the set of nodes \( \mathcal{N}_\Sigma \) of \( G_\Sigma \). Places in \( \mathcal{P}_\Sigma \) are intended to be an abstract representation of the blocks of \( \Sigma \) via a bijection \( q \mapsto q^{(*)} \) from \( \mathcal{P}_\Sigma \) onto \( \Sigma \). Likewise, nodes in \( \mathcal{N}_\Sigma \) are intended to represent the application of the constructor \( C \) to the blocks represented by their places (\( \mathcal{N}_\Sigma \subseteq Q \)) The disjoint sets \( \mathcal{P}_\Sigma \) and \( \mathcal{N}_\Sigma \) will form the parts of the bipartite graph \( G_\Sigma \) we are after. The bijection \( (\bullet) \) can be naturally extended to nodes \( B \) of \( G_\Sigma \) by putting \( B^{(*)} := \{ q^{(*)} \mid q \in B \} \).

Having defined the vertex set of \( G_\Sigma \), next we describe its edge set. The edges issuing from each place \( q \) are exactly all pairs \( \langle q, B \rangle \) such that \( q \in B \in \mathcal{N}_\Sigma \) (membership edges of \( G_\Sigma \)). The remaining edges of \( G_\Sigma \) go from nodes to places (distribution or saturation edges of \( G_\Sigma \)). Only places \( q \) corresponding to \( C \)-blocks \( q^{(*)} \) of \( \Sigma \) (hence called \( C \)-places) can have incoming edges. Likewise, only nodes \( B \) such that \( B^{(*)} \in \Pi_c \) (the candidate to be the set \( Q \) of the constructible operator) can have outgoing edges. Such nodes will be called \( Q \)-nodes. Specifically, for a \( Q \)-node \( B \) and a \( C \)-place \( q \) of \( G_\Sigma \), there is an edge \( \langle B, q \rangle \) exactly when

\[
q^{(*)} \cap C(B^{(*)}) \neq \emptyset,
\]

namely when there is some “flow” of elements built by constructor \( C \) applied to the node \( B^{(*)} \) from \( C(B^{(*)}) \) to \( q^{(*)} \) (through the edge \( \langle B, q \rangle \)). This is the sense in which a \( C \)-graph can be considered a kind of flow graph in the realm of set theory. Thus, the target map \( T_\Sigma \) of \( G_\Sigma \) is defined by

\[
T_\Sigma(B) := \{ q \in \mathcal{P}_{\Sigma,C} \mid q^{(*)} \cap C(B^{(*)}) \neq \emptyset \}, \quad \text{for } B \in \mathcal{N}_{\Sigma,C},
\]

where \( \mathcal{P}_{\Sigma,C} \) and \( \mathcal{N}_{\Sigma,C} \) denote the collections of the \( C \)-places and of the \( Q \)-nodes of \( G_\Sigma \), respectively.

Notice that each \( Q \)-node \( B \) of \( G_\Sigma \) has some target. Indeed, from \( \bigcup \Sigma_c = \bigcup C[\Pi_c] \) it follows that \( \emptyset \neq C(B^{(*)}) \subseteq \bigcup \Sigma_c \), and therefore \( T_\Sigma(B) \neq \emptyset \).
2.0.1 Accessible $C$-graphs

Definitions of source places and accessible $C$-graphs are the analogues of those in [CU21] Definition 5. A $C$-graph $G_\Sigma$ induced by a given partition $\Sigma$ is always accessible, and again the proof follows that one in the above cited article.

Only accessible $C$-graphs are relevant for our decidability purposes.

Again following [CU21] just interchanging $\otimes$ with $C$, the accessible $C$-graph induced by a partition $\Sigma$ satisfying a given BST $C$-conjunction $\Phi$ fulfills $\Phi$, according to the following definition.

**Definition 5** (Fulfillment by an accessible $C$-graph). An accessible $C$-graph $G = (P, N, T)$ fulfills a given BST $C$-conjunction $\Phi$ provided that there exists a map $\mathfrak{F}: \text{Vars}(\Phi) \rightarrow \text{pow}(P)$ (called a $G$-fulfilling map for $\Phi$) such that the following conditions are satisfied:

(a) $\mathfrak{F}(x) = \mathfrak{F}(y) \circ \mathfrak{F}(z)$, for every conjunct $x = y \circ z$ in $\Phi$, where $\circ \in \{\cup, \\}$;

(b) $\mathfrak{F}(x) \neq \mathfrak{F}(y)$, for every conjunct $x \neq y$ in $\Phi$;

(c) for every conjunct $x = C(x_1 \ldots x_n)$ in $\Phi$,

- $\emptyset \neq T(N) \subseteq \mathfrak{F}(x)$, for all $N \in Q_{x_1, \ldots, x_n}$;
- $\mathfrak{F}(x) \subseteq \bigcup_{N \in Q_{x_1, \ldots, x_n}} T[N]$;
- $\bigcup T[N \setminus Q_{x_1, \ldots, x_n}] \cap \mathfrak{F}(x) = \emptyset$.

**Lemma 4.** The accessible $C$-graph induced by a partition satisfying a given BST $C$-conjunction $\Phi$ fulfills $\Phi$.

The proof of the above lemma follows that one of [CU21] using the property of operators with constructors.

As an immediate consequence, we have:

**Corollary 1.** A satisfiable BST $C$-conjunction with $n$ variables is fulfilled by an accessible $C$-graph of size (at most) $2^n - 1$.

3 Decidability of Extensions of BST with $H$-operators

We investigate operators with constructors such that for all $C$-graph there exists a set assignment to the sources that allow to fill an assigned accessible $C$-graph. As described above, places in $P$ are intended to be an abstract representation of the blocks of a partition $\Sigma$. We are going to fill these
abstract blocks with sets, whenever this is done we point out the correspondence between the abstract block and the block filled of sets via a bijection $q \mapsto q^{(*)}$ from $\mathcal{P}$ onto $\Sigma$, that is the partition we are going to create.

Assume $C$ is a set-constructor and $\mathcal{P}$ defined as before. Let $H_0$ be a set such that $H_0 = \bigcup_{q \in S_0} q^{(*)}$, $S_0 \subseteq \mathcal{P}$. Let $C(B_0) = \bigcup_{q \in S_1} q^{(*)}$, where $B_0 \subseteq S_0$ and $S_1 \subseteq \mathcal{P} \setminus S_0$.

We continue inductively by defining $C(B_i) = \bigcup_{q \in S_{i+1}} q^{(*)}$, where $B_i \subseteq S_0 \cup \cdots \cup S_i$ and $S_{i+1} \subseteq \mathcal{P} \setminus (S_0 \cup \cdots \cup S_i)$.

**Definition 6.** A set-constructor $C$ is an $H$-constructor if for any set of places $\mathcal{P}$ there exists a finite set $H_0$ of any cardinality such that

(a) for all $i$, $C(B_i) \cap H_0 = \emptyset$;

(b) for any $q \in B$, $|C(B)| \geq |q|$.

An operator $C$ with constructors $(C, Q)$ is an $H$-operator if $C$ is an $H$-constructor.

### 3.1 Construction process

Lemma 4 can be reversed, thus yielding a proof of Theorem 1.

**Lemma 5.** If a BSTC-conjunction is fulfilled by an accessible $C$-graph, then it is satisfiable.

The proof runs exactly as the analogue in [CU21] except for the use of the properties of $H$-operator $C$. Indeed, in the initialization we use the set $H_0$ to fill source places. Then, when you have to fill the remaining places on the graph, you use the constructor $C$ on the nodes $Q$. In order to be sure that you are constructing a real partition you have to be sure that all the elements created by the constructor $C$ are not created before. These elements cannot belong to $H_0$ by the property (m) of $H$-operator and cannot belong to the other already filled places by disjointness of $C$. This guarantees that what you construct is a partition.

The fact that $H_0$ can be of whatever cardinality and property (b) of $H$-operator ensure that you have enough elements to follow all the paths of the graph.

The remaining details of the proof are the same.

For the sake of completeness we report the whole proof in the Appendix.

Moreover, just substituting $\otimes$ with $C$, the proof of finite satisfiability runs in the same way.
4 NP-completeness

Consider an operator $C$ with constructor $(C, Q)$. Suppose that for any $\Sigma$ the growth of $|Q|$ in terms of $|\Sigma|$ is polynomial. In this case we call $C$ a POLYNOMIAL operator.

**Definition 7.** An $H$-operator $C$ is injective if it is polynomial and whenever $C(x_1, \ldots, x_n) = C(y_1, \ldots, y_n)$ then $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. If the operator requires an ordered set of entries then $x_i = y_i$ following the order.

We prove $NP$ completeness for such injective $H$-operators.

**[Proof of Theorem 2]**

Consider a $BST$ conjunction $\Phi$ and a model $M$ inherited by a partition assignment $I$.

We can assume that such a model is injective.

For each variable $x$ select two places $q_1^x, q_2^x \in I(x)$ in such a way if a literal $x \neq y$ appears in $\Phi$ then $I(x) \ni q_x \notin I(y)$. Denote by $P^*$ such a selection of places and $I^*$ the assignment $I$ restricted to $P^*$.

Since $C$ is an operator with constructors there exist a function $Q^*$ such that for each $C(x_1, \ldots, x_n)$ there are $Q^*(x_1, \ldots, x_n) \subseteq pow(P^*)$ (or $Seq(P^*)$ in case of ordered operator) such that

$$\bigcup_{N \in Q^*(x_1 \cdots x_n)} C(N) = C(\bigcup I^*(x_1) \cdots \bigcup I^*(x_n))$$

Consider the graph of the variables built in the following manner. For any variable $x$ such that $C(x_1, \ldots, x_n) \in \Phi$ it follows $x \mapsto \{x_1, \ldots, x_n\}$ and for any $x$ such that $C(x_1, \ldots, x_n) = x \in \Phi$ it follows $\{x_1, \ldots, x_n\} \mapsto x$. Since $M$ is a model of $\Phi$ this graph is accessible.

Then, build a $C$-graph $G^* = (P^*, N^*, T^*)$ with the selected places in the following manner. For any $q \in N$ create an arrow $q \mapsto N$ and for any $C(x_1, \ldots, x_n) = x$ we create arrows from any $N \in Q_{x_1, \ldots, x_n}$ to any place $q_x \in T^*(x)$, $N \mapsto q_x$. Since the graph of variables is accessible, $G^*$ is accessible, as well. Define $\tilde{\Phi}^*$: $\text{Vars}(\Phi) \rightarrow \text{pow}(P^*)$ of the map $I^*$, which is defined by

$$\tilde{\Phi}^*(x) := \{q \in P^* | q(x) \in I^*(x)\}, \quad \text{for } x \in \text{Vars}(\Phi).$$

We show that $P^*$ together with $\tilde{\Phi}^*$ fulfills the given $BST$-conjunction $\Phi$.

(a) holds by Lemma 2.35 [CU18].

(b) holds by the selection performed for literals of the type $x \neq y$. 


(c₁) holds by construction of $G^*$.

Indeed, if $C(x_1 \cdots x_n) = x$ and $C(x_1 \cdots x_n) = y$ then, by injectivity of the model, $x = y$.

(c₂) holds by injectivity of $H$ operator. Indeed, a place of $I^*(x)$ cannot receive an arrow but from nodes as $N \in Q(x_1 \cdots x_n)$. Since $H$ is injective, \{x₁ \cdots xₙ\} = \{y₁ \cdots yₙ\} (or in the order if the operator requires ordered entries), therefore $Q(x_1 \cdots x_n)$ and $Q(y_1 \cdots y_n)$ are equal too. This means that the set of all the targets of $Q(x_1 \cdots x_n)$ contains the set of places of $x$.

(c₃) Still, even the set of targets of $Q(x_1 \cdots x_n)$ contains the set of places of $x$, it could be that other nodes, but $Q(x_1 \cdots x_n)$, could have some target in $I^*(x)$, but this happens only if $N' \in Q(y_1 \cdots y_n)$ and $C(y_1 \cdots y_n) = x$. By injectivity of $C$ they both, $C(x_1 \cdots x_n)$ and $C(y_1 \cdots y_n)$, have been created using the same set constructors which means that $Q(x_1 \cdots x_n)$ and $Q(y_1 \cdots y_n)$ are equal.

Observe that if the operator is polynomial the decision procedure showed in the previous section is NEXP-time by the fact that the cardinality of the places involved is $2^{n-1}$. In this case the cardinality of the places is polynomial in the number of variables, since the operator is polynomial by hypothesis, the cardinality of the vertices of $G^*$ is polynomial, as well. Therefore, the decision process is NP. Moreover, since BST is NP-complete and $H$-operators $C$ are injective, BSTC are NP-complete.

5 Applications

We give a list of applications of Theorem 1 and Theorem 2.

- If $C = \text{pow}$ then $C = \text{pow}^*$, where

  \[
  \text{pow}^*(S) := \{ t \subseteq \bigcup S \mid t \cap s \neq \emptyset, \text{ for every } s \in S \}
  \]

  If $\text{pow}(x) = y$, $Q(x)$ is the collection of subsets of $I(x)$, that is sets of the following type \{q₁ \cdots qₙ\} ⊆ $I(x)$.

- If $C = \otimes$ then $C = \otimes$ and $Q_{x₁,x₂}$ is the collection of sets of the following type \{q_{x₁}, q_{x₂}\} or \{q_{x₁}\} with $q_{x_i} \in I(x_i)$.

- If $C = \times$ then $C = \times$ and $Q_{x₁,x₂}$ is the collection of sets of the following type, ordered sequence \langle q_{x₁}, q_{x₂} \rangle with $q_{x_i} \in I(x_i)$.

In all the three cases, $H_0$ can be composed of sets of the same rank. Trivially there exist sets of this kind of any cardinality and together with
⊗, ×, pow∗ as constructors verify the property required by definition of injective H-operator.

Appendix

Proof of Lemma 5

Proof. Let $G = (P, N, T)$ be an accessible $C$-graph, and let us assume that $G$ fulfills a given BST $C$-conjunction $\Phi$ via the map $\Phi: \text{Vars}(\Phi) \to \text{pow}(P)$.

To each place $q \in P$, we associate a set $q^{(*)}$, initially empty. Then, by suitably exploiting the $C$-graph $G$ as a kind of flow graph, we shall show that the sets $q^{(*)}$'s can be monotonically extended by a (possibly infinite) construction process (comprising a finite initialization phase and a subsequent (possibly infinite) stabilization phase) in such a way that the following properties hold:

(P1) After each step, the sets $q^{(*)}$'s are pairwise disjoint.

(P2) At the end of the initialization phase all the $q^{(*)}$'s are nonempty (and pairwise disjoint). Thus, after each step in the subsequent stabilization phase, the sets $q^{(*)}$'s, with $q \in P$, form a partition equipollent with $P$.

(P3) After each step in the stabilization phase, the inclusion

$$q^{(*)} \subseteq \bigcup \{C(A^{(*)}) \mid A \in T^{-1}(q)\}$$

holds, for each $C$-place $q \in P_C$, where we are using the notation $B^{(*)} := \{p^{(*)} \mid p \in B\}$ for $B \in N$.

(P4) At the end of the construction process, we have

$$C(A^{(*)}) \subseteq \bigcup \{q^{(*)} \mid q \in T(A)\},$$

for each $C$-node $A \in N_C$ (namely for each node $A \in N$ such that $T(A) \neq \emptyset$).

Subsequently, we shall prove that the properties [P1], [P4] together with the conditions [a]–[c] of Definition 5 characterizing the fulfilling $C$-graph

\footnote{Should the construction process involve denumerably many steps, the final values of the $q^{(*)}$'s are to be intended as limit of the sequences of their values after each step in the stabilization phase.}
allow one to show that the partition \( \{ q^*(\bullet) \mid q \in P \} \) resulting from the above construction process satisfies our conjunction \( \Phi \).

The initialization and stabilization phases of our construction process consist of the following steps.

**Initialization phase:**

(I\(_1\)) To begin with, let \( \{ \overline{q} \mid q \in P \setminus P_C \} \) be any partition equipollent to the set \( P \setminus P_C \) of the source places of \( G \), where each block \( \overline{q} \), for \( q \in P \setminus P_C \), is filled by elements of the set \( H_0 \), that we choose of cardinality greater than \( |P|^{k+1} \) where \( k \) is the length of the longest path of the \( C \)-graph. This allow us to distribute at least \( |P|^k \) to each source place, then put

\[
q^*(\bullet) = \begin{cases} 
\overline{q} & \text{if } q \in P \setminus P_C \\
\emptyset & \text{if } q \in P_C.
\end{cases}
\]

We say that a place \( q \in P \) has already been *initialized* when \( q^*(\bullet) \neq \emptyset \). Likewise, a \( C \)-node \( A \in N_C \) has been *initialized* when its places have all been initialized. During the initialization phase, an initialized \( C \)-node \( A \in N_C \) is said to be *ready* if it has some target that has not been yet initialized.

(I\(_2\)) While there are places in \( P \) not yet initialized, pick any ready node \( A \in N \) and distribute evenly all the members of \( C(A^*(\bullet)) \) among all of its targets.

The accessibility of \( G \) guarantees that the while-loop \((I_2)\) terminates in a finite number of iterations.

Concerning property \((P1)\) we observe that since \( C \) is a disjoint constructor so whenever it applies at each distribution step to different nodes it produces disjoint sets that are to distributed to the respective targets, moreover \( C \) is an \( H \)-operator therefore its constructor \( C \) guarantees that \( C(A^*(\bullet)) \cap H_0 = \emptyset \).

At the end of the initialization phase all the \( q^*(\bullet) \)'s are nonempty, so property \((P2)\) holds. Indeed, if there were no \( C \)-places, then all places would be initialized just after step \((I_1)\) and so all the \( q^*(\bullet) \)'s would be nonempty. On the other hand, if \( |P_C| > 0 \) consider any path \( L \) of the \( C \)-graph. The first arrow of \( L \) start from a node \( A \) composed of \( P \setminus P_C \) of the source places of \( G \). Each of the places has at least \( |P|^k \) of elements. Since \( C \) is an \( H \)-operator \( C(A^*(\bullet)) \geq |q^*(\bullet)| \geq |P|^k \). Hence, each of the \( |T(A)| \leq |P_C| \) sets \( t^*(\bullet) \),
for \( t \in T(A) \), will receive at least \(|P|^{k-1}\) elements by the distribution step relative to the node \( A \).

Inductively, at the \( h \)-arrow of the path \( L \) each target has at least \(|P|^{k-h}\) elements and this guarantees that all path can be followed until the end, then at the end of the while-loop (I2) we shall have \(|q|^\bullet \neq \emptyset\), for each \( q \in P \).

**Stabilization phase:** During the stabilization phase, a \( C \)-node \( A \in \mathcal{N}_C \) is ripe if

\[
C(A^\bullet) \setminus \bigcup \{q^\bullet \mid q \in T(A)\} \neq \emptyset.
\]

We execute the following (possibly infinite) loop:

(S1) While there are ripe \( C \)-nodes, pick any of them, say \( A \in \mathcal{N} \), and distribute all the members of \( C(A^\bullet) \setminus \bigcup \{q^\bullet \mid q \in T(A)\} \) (namely the members of \( C(A^\bullet) \) that have not been distributed yet) among its targets howsoever.

The fairness condition that one must comply with is the following:

once a \( C \)-node becomes ripe during the stabilization phase, it must be picked for distribution within a finite number of iterations of the while-loop (S1).

A possible way to enforce such condition consists, for instance, in maintaining all ripe \( C \)-nodes in a queue \( Q \), picking always the \( C \)-node to be used in a distribution step from the front of \( Q \) and adding the \( C \)-nodes that have just become ripe to the back of \( Q \), provided that they are not already in \( Q \).

By induction on \( n \in \mathbb{N} \), it is not hard to show that properties (P1) and (P3) will hold just after the \( n \)-th iteration of the while-loop (S1) of the stabilization phase, and that property (P4) will hold at the end of the stabilization phase, in case of termination.

Instead, when the stabilization phase runs for denumerably many steps, the final partition \( P^\bullet \) is to be intended as the limit of the partial partitions constructed after each step of the stabilization phase. Specifically, for each place \( q \in P \), we let \( q^{(i)} \) be the value of \( q^\bullet \) just after the \( i \)-th iteration of (S1). Plainly, we have

\[
q^{(i)} \subseteq q^{(i+1)}, \quad \text{for } i \in \mathbb{N}.
\]

Then we put

\[
q^\bullet := \bigcup_{i \in \mathbb{N}} q^{(i)}, \quad \text{for } q \in P
\]
notation overloading should not be a problem).

By way of illustration, we prove that property $[\text{P4}]$ holds for the partition $\mathcal{P}(\bullet) = \{q(\bullet) \mid q \in \mathcal{P}\}$, when the $q(\bullet)$'s are defined by (2). To this purpose, let $A \in \mathcal{P}$ be such that $\mathcal{T}(A) \neq \emptyset$, and assume for contradiction that

$$C(A(\bullet)) \subseteq \bigcup \{q(\bullet) \mid q \in \mathcal{T}(A)\}.$$ 

Let $s$ be any element in $C(A(\bullet)) \setminus \bigcup \{q(\bullet) \mid q \in \mathcal{T}(A)\}$, and let $i \in \mathbb{N}$ be the smallest index such that $s \in C(A(i))$, where $A(i) := \{q(i) \mid q \in A\}$. Since $s \in C(A(i)) \setminus \bigcup \{q(i) \mid q \in \mathcal{T}(A)\}$, the node $A$ must have been ripe just after the $i$-th iteration of $[S_1]$. Therefore, by the fairness condition, the node $A$ will be picked for distribution in a finite number of steps, say $k$, after the $i$-th step, so that we have

$$C(A(i)) \subseteq C(A(i+k)) \subseteq \bigcup \{q(i+k+1) \mid q \in \mathcal{T}(A)\} \subseteq \bigcup \{q(\bullet) \mid q \in \mathcal{T}(A)\},$$

and therefore $s \in \bigcup \{q(\bullet) \mid q \in \mathcal{T}(A)\}$, which is a contradiction. Thus, property $[\text{P4}]$ holds also when the construction process takes a denumerable number of steps.

Next, we show that the final partition $\mathcal{P}(\bullet) = \{q(\bullet) \mid q \in \mathcal{P}\}$ satisfies $\Phi$. In particular, we prove that the partition assignment $\mathcal{I} : \text{Vars}(\Phi) \rightarrow \text{pow}(\mathcal{P}(\bullet))$ defined by

$$\mathcal{I}(x) := \{q(\bullet) \mid q \in \mathcal{F}(x)\}, \quad \text{for } x \in \text{Vars}(\Phi),$$

satisfies $\Phi$, where we recall that $\mathcal{F}$ is the $\mathcal{G}$-fulfilling map for $\Phi$.

Since $\mathcal{F}$ is a $\mathcal{G}$-fulfilling map for $\Phi$, then

- for every literal $x = y \ast z$ in $\Phi$, with $\ast \in \{\cup, \setminus\}$, we have $\mathcal{F}(x) = \mathcal{F}(y) \ast \mathcal{F}(z)$, so that $\mathcal{I}(x) = \mathcal{I}(y) \ast \mathcal{I}(z)$ holds; and

- for every literal $x \neq y$ in $\Phi$, we have $\mathcal{F}(x) \neq \mathcal{F}(y)$, so that $\mathcal{I}(x) \neq \mathcal{I}(y)$ holds.

Thus, by Lemma 3, the partition assignment $\mathcal{I}$ satisfies all Boolean literals in $\Phi$ of types

$$x = y \cup z, \quad x = y \setminus z, \quad x \neq y.$$ 

Next, let $x = C(x_1 \cdots x_n)$ be a conjunct of $\Phi$. We prove separately that
the following inclusions hold:

\[ \bigcup \mathcal{J}(x) \subseteq C(\bigcup \mathcal{J}(x_1), \cdots, \bigcup \mathcal{J}(x_n)) \quad (3) \]
\[ C(\bigcup \mathcal{J}(x_1), \cdots, \bigcup \mathcal{J}(x_n)) \subseteq \bigcup \mathcal{J}(x). \quad (4) \]

We recall that \( C \) is an operator with constructor then

\[ \bigcup_{N \in Q(x_1 \cdots x_n)} C(N^\bullet) = C(\bigcup \mathcal{J}(x_1) \cdots \bigcup \mathcal{J}(x_n)) \]

Concerning (3), let \( q^\bullet \subseteq \bigcup \mathcal{J}(x) \). Then \( q^\bullet \in \mathcal{J}(x) \), so that \( q \in \mathcal{F} \). By \( (P3) \) \( q \) cannot be a source place. Hence, by (P3), we have:

\[ q^\bullet \subseteq \bigcup \{ C(A^\bullet) \mid A \in \mathcal{T}^{-1}(q) \}. \]

By definition \( \mathcal{F} \) and \( \mathcal{T}^{-1}(q) \subseteq Q(x_1 \cdots x_n) \) therefore

\[ q^\bullet \subseteq \bigcup_{N \in Q(x_1 \cdots x_n)} C(N^\bullet). \]

which in turn implies

\[ \bigcup_{q \in \mathcal{J}(x)} q^\bullet \subseteq C(\bigcup \mathcal{J}(x_1) \cdots \bigcup \mathcal{J}(x_n)). \]

and finally implies \( \mathcal{J} \)

Concerning the inclusion \( \mathcal{J} \), let \( s \in C(\bigcup \mathcal{J}(x_1), \cdots, \bigcup \mathcal{J}(x_n)) \). Hence, \( s \in C(N^\bullet) \), for some \( N \in Q(x_1 \cdots x_n) \).

From \( (P4) \) we have \( \emptyset \neq \mathcal{T}(N) \subseteq \mathcal{F}(x) \). Thus, by \( (P4) \)

\[ C(N^\bullet) \subseteq \bigcup \{ q^\bullet \mid q \in \mathcal{T}(N) \} \]
\[ \subseteq \bigcup \{ q^\bullet \mid q \in \mathcal{F}(x) \} \]
\[ = \bigcup \mathcal{J}(x), \]

and therefore \( s \in \bigcup \mathcal{J}(x) \), proving \( \mathcal{J} \) by the arbitrariness of \( s \in C(\bigcup \mathcal{J}(x_1), \cdots, \bigcup \mathcal{J}(x_n)) \).

Hence, the partition assignment \( \mathcal{J} \) satisfies also all the literals in \( \Phi \) of the form \( x = C(x_1, \cdots, x_n) \), and in turn the final partition \( \mathcal{P}^\bullet \) satisfies the conjunction \( \Phi \).

By combining Lemmas 4 and 5 and Corollary 1 we obtain:

**Theorem 3.** A BSTC-conjunction with \( n \) variables is satisfiable if and only if it is fulfilled by an accessible C-graph of size (at most) \( 2^n - 1 \).
The preceding theorem is at the base of the following trivial decision procedure for $\text{BSTC}$:

\begin{verbatim}
procedure BSTC-satisfiability-test(Φ);
1. n := |Vars(Φ)|;
2. for each C-graph G with $2^n - 1$ places do
3.   if G is accessible and fulfills Φ then
4.     return “Φ is satisfiable”;
5. return “Φ is unsatisfiable”;
end procedure;
\end{verbatim}

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