ON DISCONTINUOUS DIRAC OPERATOR WITH EIGENPARAMETER DEPENDENT BOUNDARY AND TWO TRANSMISSION CONDITIONS

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Abstract. In this paper, we consider a discontinuous Dirac operator with eigenparameter dependent both boundary and two transmission conditions. We introduce a suitable Hilbert space formulation and get some properties of eigenvalues and eigenfunctions. Then, we investigate Green’s function, resolvent operator and some uniqueness theorems by using Weyl function and some spectral data.

1. Introduction

Inverse problems of spectral analysis recover operators by their spectral data. Fundamental and vast studies about the classical Sturm-Liouville, Dirac operators, Schrödinger equation and hyperbolic equations are well studied (see [1-7] and references therein).

Studies where eigenvalue dependent appears not only in the differential equation but also in the boundary conditions have increased in recent years (see [8-16] and corresponding bibliography cited therein). Moreover, boundary conditions which depend linearly and nonlinearly on the spectral parameter are considered in [8,16-20] and [21-27] respectively. Furthermore, boundary value problems with transmission conditions are also increasingly studied. These types of studies introduce qualitative changes in the exploration. Direct and inverse problems for Sturm-Liouville and Dirac operators with transmission conditions are investigated in some papers (see [7, 28-31] and references therein). Then, differential equations with the spectral parameter and transmission conditions arise in heat, mechanics, mass transfer problems, in diffraction problems and in various physical transfer problems (see [18, 28, 32-39] and corresponding bibliography).

More recently, some boundary value problems with eigenparameter in boundary and transmission conditions are spread out to the case of two, more than two or a finite number of transmission in [40-44] and references therein.

The presented paper deals with the discontinuous Dirac operator with eigenparameter dependent both boundary and two transmission conditions. The aim of the present paper is to obtain the asymptotic formulae of eigenvalues, eigenfunctions, to construct Green’s function, resolvent operator and to prove some uniqueness theorems. Especially, some parameters of the considered problem can be determined by Weyl function and some spectral data.

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We consider a discontinuous boundary value problem $L$ with function $\rho(x)$:

1. \[ ly := \rho(x)By'(x) + \Omega(x)y(x) = \lambda y(x), \quad x \in [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b] = \Lambda \]

where $\rho(x) = \begin{cases} \rho_1^{-1}, & a \leq x < \xi_1 \\ \rho_2^{-1}, & \xi_1 < x < \xi_2 \\ \rho_3^{-1}, & \xi_2 < x \leq b \end{cases}$ and $\rho_1$, $\rho_2$, and $\rho_3$ are given positive real numbers; $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}$, $p(x), q(x), r(x) \in L_2[\Lambda, \mathbb{R}]$; $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $\lambda \in \mathbb{C}$ is a complex spectral parameter; boundary conditions at the endpoints

2. \[ l_1y := \lambda (\alpha'_1y_1(a) - \alpha'_2y_2(a)) - (\alpha_1y_1(a) - \alpha_2y_2(a)) = 0 \\
3. \[ l_2y := \lambda (\gamma'_1y_1(b) - \gamma'_2y_2(b)) + (\gamma_1y_1(b) - \gamma_2y_2(b)) = 0 \]

with transmission conditions at two points $x = \xi_1, x = \xi_2$

4. \[ l_3y := y_1(\xi_1 + 0) - \alpha_3y_1(\xi_1 - 0) = 0 \\
5. \[ l_4y := y_2(\xi_1 + 0) - (\alpha_4 + \lambda)y_1(\xi_1 - 0) - \alpha_3^{-1}y_2(\xi_1 - 0) = 0 \\
6. \[ l_5y := y_1(\xi_2 + 0) - \alpha_5y_1(\xi_2 - 0) = 0 \\
7. \[ l_6y := y_2(\xi_2 + 0) - (\alpha_6 + \lambda)y_1(\xi_2 - 0) - \alpha_5^{-1}y_2(\xi_2 - 0) = 0 \]

where $\alpha_i, \gamma_i, \gamma'_i (i = 1, 0, j = 1, 2)$ are real numbers $\alpha_3 > 0, \alpha_5 > 0$ and 

\[ d_1 = \left| \begin{array}{cc} \alpha_1 & \alpha'_1 \\ \alpha_2 & \alpha'_2 \end{array} \right| > 0, \quad d_2 = \left| \begin{array}{cc} \gamma_1 & \gamma'_1 \\ \gamma_2 & \gamma'_2 \end{array} \right| > 0. \]

2. Operator Formulation and Properties of Spectrum

In this section, we present the inner product in the Hilbert Space $H := L_2(\Lambda) \oplus L_2(\Lambda) \oplus \mathbb{C}^3$ and operator $T$ defined on $H$ such that (1)-(7) can be regarded as the eigenvalue problem of operator $T$. We define an inner product in $H$ by

\[ \langle F, G \rangle := \rho^{-1}(x) \int_a^b (f_1(x)\overline{g_1}(x) + f_2(x)\overline{g_2}(x)) \, dx + \alpha_3 f_1(\xi_1 - 0)\overline{g_1}(\xi_1 - 0) + \alpha_5 f_1(\xi_2 - 0)\overline{g_2}(\xi_2 - 0) + \frac{1}{d_1}r_{\gamma_1} + \frac{1}{d_2}s_{\gamma_2} \]

for 

\[ F = \begin{pmatrix} f(x) \\ r \\ s \\ f_1(\xi_1 - 0) \\ f_1(\xi_2 - 0) \end{pmatrix} \in H, \quad G = \begin{pmatrix} g(x) \\ \gamma_1 \\ s_1 \\ g_1(\xi_1 - 0) \\ g_1(\xi_2 - 0) \end{pmatrix} \in H, \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \]

\[ g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \quad r = \alpha'_1f_1(a) - \alpha'_2f_2(a), \quad s = \gamma'_1f_1(b) - \gamma'_2f_2(b), \]

\[ r_1 = \alpha'_1g_1(a) - \alpha'_2g_2(a), \quad s_1 = \gamma'_1g_1(b) - \gamma'_2g_2(b) \]

Consider the operator $T$ defined by the domain
\[ D(T) = \{ F \in H : f(x) \in AC([a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]), I f \in L_2(\Lambda) \oplus L_2(\Lambda), I_3 f = l_5 f = 0 \} \]
such that
\[
T F := (f, \alpha_1 f_1(a) - \alpha_2 f_2(a), - (\gamma_1 f_1(b) - \gamma_2 f_2(b)), f_2(\xi_1 - 0) - \alpha_1 f_1(\xi_1 - 0), f_2(\xi_2 + 0) - \alpha_6 f_1(\xi_2 - 0) - \alpha_5^{-1} f_2(\xi_2 - 0))^T
\]
for
\[
F = (f, \alpha'_1 f_1(a) - \alpha'_2 f_2(a), \alpha'_1 f_1(b) - \gamma'_2 f_2(b), f_1(\xi_1 - 0), f_2(\xi_1 - 0))^T \in D(T).
\]

Thus, we can rewrite the considered problem (1)-(7) in the operator form as
\[ TF = \lambda F, \text{ i.e., the problem (1)-(7) can be considered as an eigenvalue problem of operator } T. \]

We define the solutions
\[
\varphi(x, \lambda) = \begin{cases} 
\varphi_1(x, \lambda), & x \in [a, \xi_1) \\
\varphi_2(x, \lambda), & x \in (\xi_1, \xi_2) \\
\varphi_3(x, \lambda), & x \in (\xi_2, b]
\end{cases}, \quad \psi(x, \lambda) = \begin{cases} 
\psi_1(x, \lambda), & x \in [a, \xi_1) \\
\psi_2(x, \lambda), & x \in (\xi_1, \xi_2) \\
\psi_3(x, \lambda), & x \in (\xi_2, b]
\end{cases}
\]
\[
\varphi_1(x, \lambda) = (\varphi_{11}(x, \lambda), \varphi_{12}(x, \lambda))^T, \quad \varphi_2(x, \lambda) = (\varphi_{21}(x, \lambda), \varphi_{22}(x, \lambda))^T, \\
\varphi_3(x, \lambda) = (\varphi_{31}(x, \lambda), \varphi_{32}(x, \lambda))^T
\]
and
\[
\psi_1(x, \lambda) = (\psi_{11}(x, \lambda), \psi_{12}(x, \lambda))^T, \quad \psi_2(x, \lambda) = (\psi_{21}(x, \lambda), \psi_{22}(x, \lambda))^T, \\
\psi_3(x, \lambda) = (\psi_{31}(x, \lambda), \psi_{32}(x, \lambda))^T
\]
of equation (1) by the initial conditions
\[
\varphi_{11}(a, \lambda) = \lambda \alpha'_1 - \alpha_2, \quad \varphi_{12}(a, \lambda) = \lambda \alpha'_2 - \alpha_1
\]
\[
\varphi_{21}(\xi_1, \lambda) = \alpha_3 \varphi_{11}(\xi_1, \lambda), \quad \varphi_{22}(\xi_1, \lambda) = (\alpha_4 + \lambda) \varphi_{11}(\xi_1, \lambda) + \alpha_3^{-1} \varphi_{12}(\xi_1, \lambda)
\]
\[
\varphi_{31}(\xi_2, \lambda) = \alpha_5 \varphi_{21}(\xi_2, \lambda), \quad \varphi_{32}(\xi_2, \lambda) = (\alpha_6 + \lambda) \varphi_{21}(\xi_2, \lambda) + \alpha_5^{-1} \varphi_{22}(\xi_2, \lambda)
\]
and similarly,
\[
\psi_{31}(b, \lambda) = \lambda \gamma'_2 + \gamma_2, \quad \psi_{32}(b, \lambda) = \lambda \gamma'_1 + \gamma_1
\]
\[
\psi_{21}(\xi_2, \lambda) = \frac{\psi_{31}(\xi_2, \lambda)}{\alpha_5}, \quad \psi_{22}(\xi_2, \lambda) = \alpha_5 \psi_{32}(\xi_2, \lambda) - (\alpha_6 + \lambda) \psi_{31}(\xi_2, \lambda)
\]
\[
\psi_{11}(\xi_2, \lambda) = \frac{\psi_{21}(\xi_1, \lambda)}{\alpha_3}, \quad \psi_{12}(\xi_2, \lambda) = \alpha_3 \psi_{22}(\xi_1, \lambda) - (\alpha_4 + \lambda) \psi_{21}(\xi_1, \lambda)
\]
respectively.

These solutions are entire functions of \( \lambda \) for each fixed \( x \in [a, b] \) and satisfy the relation
\[
\psi(x, \lambda_n) = \kappa_n \varphi(x, \lambda_n) \text{ for each eigenvalue } \lambda_n \text{ where}
\]
\[
\kappa_n = \frac{\alpha'_1 \psi_{11}(a, \lambda_n) - \alpha'_2 \psi_{12}(a, \lambda_n)}{d_1}.
\]

**Lemma 1** \( T \) is a self-adjoint operator. Therefore, all eigenvalues and eigenfunctions of the problem (1)-(7) are real and two eigenfunctions \( \varphi(x, \lambda_1) = (\varphi_1(x, \lambda_1), \varphi_2(x, \lambda_1))^T \) and
\[
\varphi(x, \lambda_2) = (\varphi_1(x, \lambda_2), \varphi_2(x, \lambda_2))^T \text{ corresponding to different eigenvalues } \lambda_1 \text{ and } \lambda_2
\]
are orthogonal in the sense of
\[
\rho^{-1}(x) \int_a^b [\varphi_1(x, \lambda_1) \varphi_1(x, \lambda_2) + \varphi_2(x, \lambda_1) \varphi_2(x, \lambda_2)] \, dx \\
+ \alpha_3 \varphi_1(\xi_1 - 0, \lambda_1) \varphi_1(\xi_1 - 0, \lambda_2) + \alpha_5 \varphi_1(\xi_2 - 0, \lambda_1) \varphi_1(\xi_2 - 0, \lambda_2)
\]
+ \frac{1}{d_1} (\alpha_1' \psi_{11}(a, \lambda_1) - \alpha_2' \psi_{12}(a, \lambda_1)) (\alpha_1' \psi_{11}(a, \lambda_2) - \alpha_2' \psi_{12}(a, \lambda_2))
+ \frac{1}{d_2} (\gamma_1' \psi_{31}(b, \lambda_1) - \gamma_2' \psi_{32}(b, \lambda_1)) (\gamma_1' \psi_{31}(b, \lambda_2) - \gamma_2' \psi_{32}(b, \lambda_2)) = 0.

Lemma 2 The following integral equations and asymptotic behaviours hold:
\varphi_{11}(x, \lambda) = -(\lambda \alpha_1' - \alpha_1) \sin \lambda \rho_1(x - a) + (\lambda \alpha_2' - \alpha_2) \cos \lambda \rho_1(x - a)
+ \int_a^x [p(t) \sin \lambda \rho_1(x - t) + q(t) \cos \lambda \rho_1(x - t)] \rho_1 \varphi_{11}(t, \lambda) dt
+ \int_a^x [q(t) \sin \lambda \rho_1(x - t) + r(t) \cos \lambda \rho_1(x - t)] \rho_1 \varphi_{12}(t, \lambda) dt
= -(\lambda \alpha_1' - \alpha_1) \sin \lambda \rho_1(x - a) + (\lambda \alpha_2' - \alpha_2) \cos \lambda \rho_1(x - a) + o(|\lambda| e^{1 \lambda \rho_1})
\varphi_{12}(x, \lambda) = (\lambda \alpha_1' - \alpha_1) \cos \lambda \rho_1(x - a) + (\lambda \alpha_2' - \alpha_2) \sin \lambda \rho_1(x - a)
+ \int_a^x [-p(t) \cos \lambda \rho_1(x - t) + q(t) \sin \lambda \rho_1(x - t)] \rho_1 \varphi_{11}(t, \lambda) dt
+ \int_a^x [-q(t) \cos \lambda \rho_1(x - t) + r(t) \sin \lambda \rho_1(x - t)] \rho_1 \varphi_{12}(t, \lambda) dt
= (\lambda \alpha_1' - \alpha_1) \cos \lambda \rho_1(x - a) + (\lambda \alpha_2' - \alpha_2) \sin \lambda \rho_1(x - a) + o(|\lambda| e^{1 \lambda \rho_1})
\varphi_{21}(x, \lambda) = \alpha_3 \varphi_{11}(\xi_1, \lambda) \cos \lambda \rho_2(x - \xi_1)
- \left( (\alpha_4 + \lambda) \varphi_{11}(\xi_1, \lambda) + \frac{1}{\alpha_3} \varphi_{12}(\xi_1, \lambda) \right) \sin \lambda \rho_2(x - \xi_1)
+ \int_{\xi_1}^x [p(t) \sin \lambda \rho_2(x - t) + q(t) \cos \lambda \rho_2(x - t)] \rho_2 \varphi_{21}(t, \lambda) dt
+ \int_{\xi_1}^x [q(t) \sin \lambda \rho_2(x - t) + r(t) \cos \lambda \rho_2(x - t)] \rho_2 \varphi_{22}(t, \lambda) dt
= (\alpha_4 + \lambda) \left[ (\lambda \alpha_1' - \alpha_1) \sin \lambda \rho_1(\xi_1 - a) \sin \lambda \rho_2(x - \xi_1)
- (\lambda \alpha_2' - \alpha_2) \cos \lambda \rho_1(\xi_1 - a) \sin \lambda \rho_2(x - \xi_1) \right] + o(|\lambda|^2 e^{1 \lambda \rho_2})
\varphi_{22}(x, \lambda) = \alpha_3 \varphi_{11}(\xi_1, \lambda) \sin \lambda \rho_2(x - \xi_1)
+ \left( (\alpha_4 + \lambda) \varphi_{11}(\xi_1, \lambda) + \frac{1}{\alpha_3} \varphi_{12}(\xi_1, \lambda) \right) \cos \lambda \rho_2(x - \xi_1)
+ \int_{\xi_1}^x [-p(t) \cos \lambda \rho_2(x - t) + q(t) \sin \lambda \rho_2(x - t)] \rho_2 \varphi_{21}(t, \lambda) dt
+ \int_{\xi_1}^x [-q(t) \cos \lambda \rho_2(x - t) + r(t) \sin \lambda \rho_2(x - t)] \rho_2 \varphi_{22}(t, \lambda) dt
= -(\alpha_4 + \lambda) \left[ (\lambda \alpha_1' - \alpha_1) \sin \lambda \rho_1(\xi_1 - a) \cos \lambda \rho_2(x - \xi_1)
- (\lambda \alpha_2' - \alpha_2) \cos \lambda \rho_1(\xi_1 - a) \cos \lambda \rho_2(x - \xi_1) \right] + o(|\lambda|^2 e^{1 \lambda \rho_2})
\varphi_{31}(x, \lambda) = \alpha_3 \varphi_{21}(\xi_2, \lambda) \cos \lambda \rho_3(x - \xi_2)
\[-\left(\frac{1}{\alpha_5} \varphi_{22}(\xi_2, \lambda) + (\alpha_6 + \lambda) \varphi_{21}(\xi_2, \lambda)\right) \sin \lambda \rho_3 (x - \xi_2)\]

\[+ \int\limits_{\xi_2}^x [p(t) \sin \lambda \rho_3(x - t) + q(t) \cos \lambda \rho_3(x - t)] \rho_3 \varphi_{31}(t, \lambda) dt\]

\[+ \int\limits_{\xi_2}^x [q(t) \sin \lambda \rho_3(x - t) + r(t) \cos \lambda \rho_3(x - t)] \rho_3 \varphi_{32}(t, \lambda) dt\]

\[= (\alpha_4 + \lambda) (\alpha_6 + \lambda) \left[ - (\lambda \alpha_1' - \alpha_1) \sin \lambda \rho_1 (\xi_1 - a) \sin \lambda \rho_2 (\xi_2 - \xi_1) + (\lambda \alpha_2' - \alpha_2) \cos \lambda \rho_1 (\xi_1 - a) \sin \lambda \rho_2 (\xi_2 - \xi_1) \right] \sin \lambda \rho_3 (x - \xi_2)\]

\[+ o(|\lambda|^3 e^{\Im \lambda |((\xi_1 - a) \rho_1 + (\xi_2 - \xi_1) \rho_2 + (x - \xi_2) \rho_3)}\)]

\[\varphi_{32}(x, \lambda) = \alpha_5 \varphi_{21}(\xi_2, \lambda) \sin \lambda \rho_3 (x - \xi_2)\]

\[+ \left(\frac{1}{\alpha_5} \varphi_{22}(\xi_2, \lambda) + (\alpha_6 + \lambda) \varphi_{21}(\xi_2, \lambda)\right) \cos \lambda \rho_3 (x - \xi_2)\]

\[+ \int\limits_{\xi_2}^x [-p(t) \cos \lambda \rho_3(x - t) + q(t) \sin \lambda \rho_3(x - t)] \rho_3 \varphi_{31}(t, \lambda) dt\]

\[+ \int\limits_{\xi_2}^x [-q(t) \cos \lambda \rho_3(x - t) + r(t) \sin \lambda \rho_3(x - t)] \rho_3 \varphi_{32}(t, \lambda) dt\]

\[= - (\alpha_4 + \lambda) (\alpha_6 + \lambda) \left[ - (\lambda \alpha_1' - \alpha_1) \sin \lambda \rho_1 (\xi_1 - a) \sin \lambda \rho_2 (\xi_2 - \xi_1) + (\lambda \alpha_2' - \alpha_2) \cos \lambda \rho_1 (\xi_1 - a) \sin \lambda \rho_2 (\xi_2 - \xi_1) \right] \cos \lambda \rho_3 (x - \xi_2)\]

\[+ o(|\lambda|^3 e^{\Im \lambda |((\xi_1 - a) \rho_1 + (\xi_2 - \xi_1) \rho_2 + (x - \xi_2) \rho_3)}\)]

**Lemma 3** The following integral equations and asymptotic behaviours hold:

\[\psi_{31}(x, \lambda) = (\lambda \gamma_2' + \gamma_2) \cos \lambda \rho_3(x - b) - (\lambda \gamma_1' + \gamma_1) \sin \lambda \rho_3(x - b)\]

\[- \int\limits_b^x [p(t) \sin \lambda \rho_3(x - t) + q(t) \cos \lambda \rho_3(x - t)] \rho_3 \psi_{31}(t, \lambda) dt\]

\[- \int\limits_x^b [q(t) \sin \lambda \rho_3(x - t) + r(t) \cos \lambda \rho_3(x - t)] \rho_3 \psi_{32}(t, \lambda) dt\]

\[= (\lambda \gamma_2' + \gamma_2) \cos \lambda \rho_3(x - b) - (\lambda \gamma_1' + \gamma_1) \sin \lambda \rho_3(x - b) + o(|\lambda| e^{\Im \lambda |(b - x) \rho_3|})\]

\[\psi_{32}(x, \lambda) = (\lambda \gamma_2' + \gamma_2) \sin \lambda \rho_3(x - b) + (\lambda \gamma_1' + \gamma_1) \cos \lambda \rho_3(x - b)\]

\[+ \int\limits_b^x [p(t) \cos \lambda \rho_3(x - t) - q(t) \sin \lambda \rho_3(x - t)] \rho_3 \psi_{31}(t, \lambda) dt\]

\[+ \int\limits_x^b [q(t) \cos \lambda \rho_3(x - t) - r(t) \sin \lambda \rho_3(x - t)] \rho_3 \psi_{32}(t, \lambda) dt\]

\[= (\lambda \gamma_2' + \gamma_2) \sin \lambda \rho_3(x - b) + (\lambda \gamma_1' + \gamma_1) \cos \lambda \rho_3(x - b) + o(|\lambda| e^{\Im \lambda |(b - x) \rho_3|})\]

\[\psi_{21}(x, \lambda) = [(\alpha_6 + \lambda) \psi_{31}(\xi_2, \lambda) - \alpha_5 \psi_{32}(\xi_2, \lambda)] \sin \lambda \rho_2(x - \xi_2)\]
\[
\begin{align*}
+ \frac{1}{\alpha_5} \psi_{31}(\xi_2, \lambda) & \cos \lambda \rho_2(x - \xi_2) - \int_{\xi_2}^{t} [p(t) \sin \lambda \rho_2(x - t) + q(t) \cos \lambda \rho_2(x - t)] \rho_2 \psi_{21}(t, \lambda) dt \\
- \int_{\xi_2}^{t} [q(t) \sin \lambda \rho_2(x - t) + r(t) \cos \lambda \rho_2(x - t)] \rho_2 \psi_{22}(t, \lambda) dt \\
= (\alpha_6 + \lambda) [(\lambda \gamma_2' + \gamma_2) \cos \lambda \rho_3(\xi_2 - b) \sin \lambda \rho_2(x - \xi_2) \\
- (\lambda \gamma_1' + \gamma_1) \sin \lambda \rho_3(\xi_2 - b) \sin \lambda \rho_2(x - \xi_2)] + o(|\lambda|^2 e^{i \text{Im} \lambda((b - \xi_2)\rho_3 + (\xi_2 - x)\rho_2)}) \\
\psi_{22}(x, \lambda) & = [- (\alpha_6 + \lambda) \psi_{31}(\xi_2, \lambda) + \alpha_5 \psi_{32}(\xi_2, \lambda)] \cos \lambda \rho_2(x - \xi_2) \\
+ \frac{1}{\alpha_5} \psi_{31}(\xi_2, \lambda) & \sin \lambda \rho_2(x - \xi_2) + \int_{\xi_2}^{t} [p(t) \cos \lambda \rho_2(x - t) - q(t) \sin \lambda \rho_2(x - t)] \rho_2 \psi_{21}(t, \lambda) dt \\
- \int_{\xi_2}^{t} [q(t) \cos \lambda \rho_2(x - t) - r(t) \sin \lambda \rho_2(x - t)] \rho_2 \psi_{22}(t, \lambda) dt \\
= -(\alpha_6 + \lambda) [(\lambda \gamma_2' + \gamma_2) \cos \lambda \rho_3(\xi_2 - b) \cos \lambda \rho_2(x - \xi_2) \\
- (\lambda \gamma_1' + \gamma_1) \sin \lambda \rho_3(\xi_2 - b) \cos \lambda \rho_2(x - \xi_2)] + o(|\lambda|^2 e^{i \text{Im} \lambda((b - \xi_2)\rho_3 + (\xi_2 - x)\rho_2)}) \\
\psi_{11}(x, \lambda) & = (\alpha_3 \psi_{22}(\xi_1, \lambda) - (\alpha_4 + \lambda) \psi_{21}(\xi_1, \lambda)) \sin \lambda \rho_1(x - \xi_1) \\
- \frac{1}{\alpha_3} \psi_{21}(\xi_1, \lambda) & \cos \lambda \rho_1(x - \xi_1) - \int_{\xi_1}^{t} [p(t) \sin \lambda \rho_1(x - t) + q(t) \cos \lambda \rho_1(x - t)] \rho_2 \psi_{11}(t, \lambda) dt \\
- \int_{\xi_1}^{t} [q(t) \sin \lambda \rho_1(x - t) + r(t) \cos \lambda \rho_1(x - t)] \rho_2 \psi_{12}(t, \lambda) dt \\
= -(\alpha_6 + \lambda) [(\lambda \gamma_2' + \gamma_2) \cos \lambda \rho_3(\xi_2 - b) \\
- (\lambda \gamma_1' + \gamma_1) \sin \lambda \rho_3(\xi_2 - b)] \sin \lambda \rho_2(\xi_1 - \xi_2) \sin \lambda \rho_1(x - \xi_1) \\
+ o(|\lambda|^3 e^{i \text{Im} \lambda((b - \xi_2)\rho_3 + (\xi_2 - \xi_1)\rho_2 + (\xi_1 - x)\rho_1)}) \\
\psi_{12}(x, \lambda) & = ((\alpha_4 + \lambda) \psi_{21}(\xi_1, \lambda) - \alpha_3 \psi_{22}(\xi_1, \lambda)) \cos \lambda \rho_1(x - \xi_1) \\
- \frac{1}{\alpha_3} \psi_{21}(\xi_1, \lambda) & \sin \lambda \rho_1(x - \xi_1) + \int_{\xi_1}^{t} [p(t) \cos \lambda \rho_1(x - t) - q(t) \sin \lambda \rho_1(x - t)] \rho_1 \psi_{11}(t, \lambda) dt \\
+ \int_{\xi_1}^{t} [q(t) \cos \lambda \rho_1(x - t) - r(t) \sin \lambda \rho_1(x - t)] \rho_1 \psi_{12}(t, \lambda) dt \\
= (\alpha_4 + \lambda) [(\lambda \gamma_2' + \gamma_2) \cos \lambda \rho_3(\xi_2 - b) \\
- (\lambda \gamma_1' + \gamma_1) \sin \lambda \rho_3(\xi_2 - b)] \sin \lambda \rho_2(\xi_1 - \xi_2) \cos \lambda \rho_1(x - \xi_1) \\
+ o(|\lambda|^3 e^{i \text{Im} \lambda((b - \xi_2)\rho_3 + (\xi_2 - \xi_1)\rho_2 + (\xi_1 - x)\rho_1)}) \\
\end{align*}

Denote
\[\Delta_1(\lambda) := W(\varphi_1, \psi_1, x) := \varphi_{11}(t) \varphi_{12} - \varphi_{12}(t) \varphi_{11}, \quad x \in \Lambda_i \quad (i = 1, 3)\]
which are independent of \(x \in \Lambda_i\) and are entire functions such that \(\Lambda_1 = [a, \xi_1]\), \(\Lambda_2 = (\xi_1, \xi_2)\), \(\Lambda_3 = (\xi_2, b)\).

Let
\[\Delta_3(\lambda) = \Delta(\lambda) = W(\varphi, \psi, b) = (\lambda \gamma_1' + \gamma_1) \varphi_{31}(b, \lambda) - (\lambda \gamma_2' + \gamma_2) \varphi_{32}(b, \lambda)\]
and

\[ \mu_n := \rho^{-1}(x) \int_{a}^{b} [\varphi_1^2(x, \lambda_n) + \varphi_2^2(x, \lambda_n)] dx \]

\[ + \alpha_3\varphi_1^2(\xi_1 - 0, \lambda_n) + \alpha_5\varphi_1^2(\xi_2 - 0, \lambda_n) + \frac{1}{d_1} (\alpha'_1 \varphi_11(a, \lambda_n) - \alpha'_2 \varphi_12(a, \lambda_n))^2 \]

\[ + \frac{1}{d_2} (\gamma'_1 \varphi_{31}(b, \lambda_n) - \gamma'_2 \varphi_{32}(b, \lambda_n))^2. \]

The function \( \Delta(\lambda) \) is called the characteristic function and numbers \( \{\mu_n\}_{n \in \mathbb{Z}} \) are called the normalizing constants of the problem (1)-(7).

**Lemma 4** The following equality holds for each eigenvalue \( \lambda_n \)

\[ \frac{\lambda}{\lambda_n} = -\kappa_n \mu_n. \]

**Proof** Since

\[ \rho(x)\varphi_1^2(x, \lambda_n) + p(x) \varphi_1(x, \lambda_n) + q(x) \varphi_2(x, \lambda_n) = \lambda_n \varphi_1(x, \lambda_n), \]

\[ \rho(x)\varphi_2^2(x, \lambda_n) + p(x) \varphi_1(x, \lambda_n) + q(x) \varphi_2(x, \lambda_n) = \lambda_n \varphi_2(x, \lambda_n), \]

and

\[ -\rho(x)\varphi_1^2(x, \lambda_n) + q(x) \varphi_1(x, \lambda_n) + r(x) \varphi_2(x, \lambda_n) = \lambda_n \varphi_2(x, \lambda_n), \]

\[ -\rho(x)\varphi_2^2(x, \lambda_n) + q(x) \varphi_1(x, \lambda_n) + r(x) \varphi_2(x, \lambda_n) = \lambda_n \varphi_2(x, \lambda_n), \]

we obtain

\[ \varphi_1(x, \lambda_n)\psi_2(x, \lambda) - \varphi_2(x, \lambda_n)\psi_1(x, \lambda) \left( \frac{\xi_1}{a} + \frac{\xi_2}{\xi_1} + \frac{b}{\xi_2} \right) \]

\[ = (\lambda - \lambda_n) \rho_1 \int_{\xi_1}^{\xi_2} [\psi_1(x, \lambda) \varphi_1(x, \lambda_n) + \psi_2(x, \lambda) \varphi_2(x, \lambda_n)] dx \]

\[ + (\lambda - \lambda_n) \rho_2 \int_{\xi_1}^{\xi_2} [\psi_1(x, \lambda) \varphi_1(x, \lambda_n) + \psi_2(x, \lambda) \varphi_2(x, \lambda_n)] dx \]

\[ + (\lambda - \lambda_n) \rho_3 \int_{\delta_2}^{\xi_2} [\psi_1(x, \lambda) \varphi_1(x, \lambda_n) + \psi_2(x, \lambda) \varphi_2(x, \lambda_n)] dx. \]

After that adding and subtracting \( \Delta(\lambda) \) on the left-hand side of the last equality and by using the conditions (2)-(7); to get

\[ \Delta(\lambda) - (\lambda - \lambda_n) (\alpha'_2 \varphi_2(a, \lambda_n) - \alpha'_1 \psi_1(a, \lambda_n)) + (\lambda - \lambda_n) (\gamma'_2 \varphi_{31}(b, \lambda_n) - \gamma'_1 \psi_1(b, \lambda_n)) \]

\[ + \alpha_3 (\lambda - \lambda_n) \varphi_1(\xi_1 - 0, \lambda_n) \psi_1(\xi_1 - 0, \lambda) + \alpha_5 (\lambda - \lambda_n) \varphi_1(\xi_2 - 0, \lambda_n) \psi_1(\xi_2 - 0, \lambda) \]

\[ = (\lambda - \lambda_n) \rho_1 \int_{\xi_1}^{\xi_2} [\psi_1(x, \lambda) \varphi_1(x, \lambda_n) + \psi_2(x, \lambda) \varphi_2(x, \lambda_n)] dx \]

\[ + (\lambda - \lambda_n) \rho_2 \int_{\xi_1}^{\xi_2} [\psi_1(x, \lambda) \varphi_1(x, \lambda_n) + \psi_2(x, \lambda) \varphi_2(x, \lambda_n)] dx \]

\[ + (\lambda - \lambda_n) \rho_3 \int_{\delta_2}^{\xi_2} [\psi_1(x, \lambda) \varphi_1(x, \lambda_n) + \psi_2(x, \lambda) \varphi_2(x, \lambda_n)] dx, \]

or
where

\[ \lambda_0 = \frac{n\pi}{(\xi_1 - a)\rho_1 + (\xi_2 - \xi_1)\rho_2 + (b - \xi_2)\rho_3} + \Psi_n, \quad \sup_n |\Psi_n| < \infty, \quad n = 0, 1, 2, \ldots \]

**Theorem 2** The eigenvalues \( \lambda_n \) which are located in positive side of real axis satisfy the following asymptotic behaviour;

\[ \lambda_n = \lambda_0^n + o(1), \quad n \to \infty \]

**Proof** Denote

\[ G_n := \{ \lambda : 0 \leq \text{Re} \lambda \leq \lambda_0^n - \delta, \quad |\text{Im} \lambda| \leq \lambda_0^n - \delta, \quad n = 0, 1, 2, \ldots \} \cup \{ \lambda : |\lambda| < \delta \} \]

where \( \delta \) is a sufficiently small number. The relations

\[ |\Delta_0(\lambda)| \geq C |\lambda|^4 e^{\text{Im} \lambda ((\xi_1 - a)\rho_1 + (\xi_2 - \xi_1)\rho_2 + (b - \xi_2)\rho_3)} \]

and

\[ \Delta(\lambda) - \Delta_0(\lambda) = o(\lambda)^4 e^{\text{Im} \lambda ((\xi_1 - a)\rho_1 + (\xi_2 - \xi_1)\rho_2 + (b - \xi_2)\rho_3)} \]

are valid for \( \lambda \in \partial G_n \).
Then, by Rouche’s theorem that the number of zeros of $\Delta_0(\lambda)$ coincides with the number of zeros of $\Delta(\lambda)$ in $G_n$, namely $n + 4$ zeros, $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{n+3}$. In the annulus between $G_n$ and $G_{n+1}$, $\Delta(\lambda)$ has accurately one zero, namely $k_n : k_n = \lambda_n^0 + \delta_n$, for $n \geq 1$. So, it follows that $\lambda_{n+4} = k_n$. Applying to Rouche’s theorem in $\eta_\varepsilon = \{ \lambda : |\lambda - \lambda_n^0| \leq \varepsilon \}$ for sufficiently small $\varepsilon$ and sufficiently large $n$, we get $\delta_n = o(1)$. Finally, we obtain the asymptotic formula

$$
\lambda_n = \lambda_{n-4}^0 + o(1).
$$

**Example** Let $\Omega(x) = 0$, $a = 0$, $b = \pi$, $\xi_1 = \frac{\pi}{4}$, $\xi_2 = \frac{\pi}{2}$, $\alpha_3 = \gamma_4 = 1$, $\gamma_3 = \alpha_4 = 0$, $\alpha_5 = \gamma_6 = 1$, $\gamma_5 = \alpha_6 = 0$, $\gamma'_2 = \gamma'_1 = 0$, $\gamma'_1 = 1$, $\gamma'_2 = -1$, $\alpha'_2 = \alpha'_1 = 0$, $\alpha'_1 = -1$, $\alpha'_2 = 1$.

Since $\Delta(\lambda) = \lambda^4 \sin \rho \lambda (\xi_2 - \xi_1) [\gamma'_1 \sin \rho \lambda (b - \xi_2) + \gamma'_2 \cos \rho \lambda (b - \xi_2)]$ 

$$
\times [\alpha'_2 \cos \rho \lambda (\xi_1 - a) - \alpha'_1 \sin \rho \lambda (\xi_1 - a)] 
+ o(\lambda^{4\varepsilon} \varepsilon^{4(\xi_2 - \xi_1)(b - \xi_2) + (\xi_1 - a)\rho_{\lambda}}),
$$

the eigenvalues of the boundary value problem (1)-(7), satisfy the following asymptotic formulae;

$$
\lambda_{n_1} = \frac{4(n - 4)}{\rho_1} + o(1), \quad \lambda_{n_2} = \frac{4(n - 4)}{\rho_2} + o(1), \quad \lambda_{n_3} = \frac{2(n - 4)}{\rho_3} + o(1).
$$

### 3. Construction of Green Function

In this section, we get the resolvent of the boundary-value problem (1)-(7) for $\lambda$, not an eigenvalue. Hence, we find the solution of the non-homogeneous differential equation

$$
(15) \quad \rho(x) By'(x) + \Omega(x) y(x) = \lambda y(x) + f(x), \quad x \in \Lambda
$$

which satisfies the conditions (2)-(7).

We can find the general solution of homogeneous differential equation

$$
\rho(x) By'(x) + \Omega(x) y(x) = \lambda y(x), \quad x \in \Lambda
$$

in the form

$$
U_1(x, \lambda) = \left( \begin{array}{c} c_1 \varphi_{11}(x, \lambda) + c_2 \chi_{11}(x, \lambda) \\ c_1 \varphi_{12}(x, \lambda) + c_2 \chi_{12}(x, \lambda) \end{array} \right), \quad [a, \xi_1]
$$

$$
U_2(x, \lambda) = \left( \begin{array}{c} c_3 \varphi_{21}(x, \lambda) + c_4 \chi_{21}(x, \lambda) \\ c_3 \varphi_{22}(x, \lambda) + c_4 \chi_{22}(x, \lambda) \end{array} \right), \quad (\xi_1, \xi_2)
$$

$$
U_3(x, \lambda) = \left( \begin{array}{c} c_5 \varphi_{31}(x, \lambda) + c_6 \chi_{31}(x, \lambda) \\ c_5 \varphi_{32}(x, \lambda) + c_6 \chi_{32}(x, \lambda) \end{array} \right), \quad (\xi_2, b)
$$

where $c_i, i = 1, 6$ are arbitrary constants. By using method of variation of parameters, we shall investigate the general solution of the non-homogeneous linear differential equation (15) in the form

$$
U_1(x, \lambda) = \left( \begin{array}{c} c_1(x, \lambda) \varphi_{11}(x, \lambda) + c_2(x, \lambda) \chi_{11}(x, \lambda) \\ c_1(x, \lambda) \varphi_{12}(x, \lambda) + c_2(x, \lambda) \chi_{12}(x, \lambda) \end{array} \right), \quad \text{for } x \in [a, \xi_1]
$$

$$
U_2(x, \lambda) = \left( \begin{array}{c} c_3(x, \lambda) \varphi_{21}(x, \lambda) + c_4(x, \lambda) \chi_{21}(x, \lambda) \\ c_3(x, \lambda) \varphi_{22}(x, \lambda) + c_4(x, \lambda) \chi_{22}(x, \lambda) \end{array} \right), \quad \text{for } x \in (\xi_1, \xi_2)
$$

$$
U_3(x, \lambda) = \left( \begin{array}{c} c_5(x, \lambda) \varphi_{31}(x, \lambda) + c_6(x, \lambda) \chi_{31}(x, \lambda) \\ c_5(x, \lambda) \varphi_{32}(x, \lambda) + c_6(x, \lambda) \chi_{32}(x, \lambda) \end{array} \right), \quad \text{for } x \in (\xi_2, b)
$$
where the functions $c_i (x, \lambda)$ ($i = 1, 6$) satisfy the following linear system of equations

\[
\begin{align*}
\begin{cases}
  c'_1 (x, \lambda) \varphi_{11} (x, \lambda) + c'_2 (x, \lambda) \chi_{11} (x, \lambda) = f_1 (x) \\
  c'_1 (x, \lambda) \varphi_{12} (x, \lambda) + c'_2 (x, \lambda) \chi_{12} (x, \lambda) = f_2 (x)
\end{cases}
\end{align*}
\]
for $x \in [a, \xi_1)$

\[
\begin{align*}
\begin{cases}
  c'_3 (x, \lambda) \varphi_{21} (x, \lambda) + c'_4 (x, \lambda) \chi_{21} (x, \lambda) = f_1 (x) \\
  c'_3 (x, \lambda) \varphi_{22} (x, \lambda) + c'_4 (x, \lambda) \chi_{22} (x, \lambda) = f_2 (x)
\end{cases}
\end{align*}
\]
for $x \in (\xi_1, \xi_2)$

\[
\begin{align*}
\begin{cases}
  c'_5 (x, \lambda) \varphi_{31} (x, \lambda) + c'_6 (x, \lambda) \chi_{31} (x, \lambda) = f_1 (x) \\
  c'_5 (x, \lambda) \varphi_{32} (x, \lambda) + c'_6 (x, \lambda) \chi_{32} (x, \lambda) = f_2 (x)
\end{cases}
\end{align*}
\]
for $x \in (\xi_2, b]$.

Since $\lambda$ is not an eigenvalue, each of the linear system of equations has a unique solution. Thus,

\[
\begin{vmatrix}
  \varphi_{11} (x, \lambda) & \chi_{11} (x, \lambda) \\
  \varphi_{12} (x, \lambda) & \chi_{12} (x, \lambda)
\end{vmatrix} \neq 0,
\begin{vmatrix}
  \varphi_{21} (x, \lambda) & \chi_{21} (x, \lambda) \\
  \varphi_{22} (x, \lambda) & \chi_{22} (x, \lambda)
\end{vmatrix} \neq 0,
\begin{vmatrix}
  \varphi_{31} (x, \lambda) & \chi_{31} (x, \lambda) \\
  \varphi_{32} (x, \lambda) & \chi_{32} (x, \lambda)
\end{vmatrix} \neq 0.
\]

It is obvious that

\[
c_1 (x, \lambda) = \frac{1}{\Delta_1 (\lambda)} \int_{x}^{\xi_1} \left( \chi_{11} (t, \lambda) f_2 (t) - \chi_{12} (t, \lambda) f_1 (t) \right) dt + c_1,
\]

\[
c_2 (x, \lambda) = \frac{1}{\Delta_1 (\lambda)} \int_{a}^{x} \left( \varphi_{11} (t, \lambda) f_2 (t) - \varphi_{12} (t, \lambda) f_1 (t) \right) dt + c_2,
\]

\[
c_3 (x, \lambda) = \frac{1}{\Delta_2 (\lambda)} \int_{x}^{\xi_2} \left( \chi_{21} (t, \lambda) f_2 (t) - \chi_{22} (t, \lambda) f_1 (t) \right) dt + c_3,
\]

\[
c_4 (x, \lambda) = \frac{1}{\Delta_2 (\lambda)} \int_{a}^{x} \left( \varphi_{21} (t, \lambda) f_2 (t) - \varphi_{22} (t, \lambda) f_1 (t) \right) dt + c_4,
\]

\[
c_5 (x, \lambda) = \frac{1}{\Delta_3 (\lambda)} \int_{x}^{b} \left( \chi_{31} (t, \lambda) f_2 (t) - \chi_{32} (t, \lambda) f_1 (t) \right) dt + c_5,
\]

\[
c_6 (x, \lambda) = \frac{1}{\Delta_3 (\lambda)} \int_{\xi_2}^{b} \left( \varphi_{31} (t, \lambda) f_2 (t) - \varphi_{32} (t, \lambda) f_1 (t) \right) dt + c_6
\]

where $c_i, i = 1, 6$ are arbitrary constants. Substituting these above expressions in (16), then we obtain the general solution of non-homogeneous linear differential
equation (15) in the form:

\begin{align}
\text{for } x \in [a, \xi_1), U_1 (x, \lambda) &= \frac{1}{\Delta_1 (\lambda)} \int_{x}^{\xi_1} (\chi_{11} (t, \lambda) f_2 (t) - \chi_{12} (t, \lambda) f_1 (t)) \varphi_{11} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_1 (\lambda)} \int_{x}^{\xi_1} (\chi_{11} (t, \lambda) f_2 (t) - \chi_{12} (t, \lambda) f_1 (t)) \varphi_{12} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_1 (\lambda)} \int_{x}^{\xi_1} (\varphi_{11} (t, \lambda) f_2 (t) - \varphi_{12} (t, \lambda) f_1 (t)) \chi_{11} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_1 (\lambda)} \int_{a}^{x} (\varphi_{11} (t, \lambda) f_2 (t) - \varphi_{12} (t, \lambda) f_1 (t)) \chi_{12} (x, \lambda) \, dt \\
&+ c_1 \varphi_{11} (x, \lambda) + c_2 \chi_{11} (x, \lambda) + c_1 \varphi_{12} (x, \lambda) + c_2 \chi_{12} (x, \lambda)
\end{align}

\begin{align}
\text{for } x \in (\xi_1, \xi_2), U_2 (x, \lambda) &= \frac{1}{\Delta_2 (\lambda)} \int_{x}^{\xi_2} (\chi_{21} (t, \lambda) f_2 (t) - \chi_{22} (t, \lambda) f_1 (t)) \varphi_{21} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_2 (\lambda)} \int_{x}^{\xi_2} (\chi_{21} (t, \lambda) f_2 (t) - \chi_{22} (t, \lambda) f_1 (t)) \varphi_{22} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_2 (\lambda)} \int_{x}^{\xi_2} (\varphi_{21} (t, \lambda) f_2 (t) - \varphi_{22} (t, \lambda) f_1 (t)) \chi_{21} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_2 (\lambda)} \int_{x}^{\xi_2} (\varphi_{21} (t, \lambda) f_2 (t) - \varphi_{22} (t, \lambda) f_1 (t)) \chi_{22} (x, \lambda) \, dt \\
&+ c_3 \varphi_{21} (x, \lambda) + c_4 \chi_{21} (x, \lambda) + c_3 \varphi_{22} (x, \lambda) + c_4 \chi_{22} (x, \lambda)
\end{align}

\begin{align}
\text{for } x \in (\xi_2, b], U_3 (x, \lambda) &= \frac{1}{\Delta_3 (\lambda)} \int_{x}^{b} (\chi_{31} (t, \lambda) f_2 (t) - \chi_{32} (t, \lambda) f_1 (t)) \varphi_{31} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_3 (\lambda)} \int_{x}^{b} (\chi_{31} (t, \lambda) f_2 (t) - \chi_{32} (t, \lambda) f_1 (t)) \varphi_{32} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_3 (\lambda)} \int_{x}^{b} (\varphi_{31} (t, \lambda) f_2 (t) - \varphi_{32} (t, \lambda) f_1 (t)) \chi_{31} (x, \lambda) \, dt \\
&+ \frac{1}{\Delta_3 (\lambda)} \int_{x}^{b} (\varphi_{31} (t, \lambda) f_2 (t) - \varphi_{32} (t, \lambda) f_1 (t)) \chi_{32} (x, \lambda) \, dt \\
&+ c_5 \varphi_{31} (x, \lambda) + c_6 \chi_{31} (x, \lambda) + c_5 \varphi_{32} (x, \lambda) + c_6 \chi_{32} (x, \lambda)
\end{align}
Therefore, we easily get that
\[ l_1(U_1) = -c_2 \Delta_1 (\lambda), \quad l_2(U_3) = c_5 \Delta_3 (\lambda). \]
Since \( \Delta_1 (\lambda) \neq 0, \ \Delta_2 (\lambda) \neq 0 \) and from boundary conditions (2)-(3), \( c_2 = 0 \) and \( c_5 = 0 \).

On the other hand, considering these results and transmission conditions (4)-(7), the following linear equation system according to the variables \( c_1, c_3, c_4, \) and \( c_6 \) are acquired:

\[ -c_1 \varphi_{21} (\xi_1 + 0) + c_3 \varphi_{21} (\xi_1 + 0) + c_4 \chi_{21} (\xi_1 + 0) \]

\[ = - \frac{1}{\Delta_2 (\lambda)} \int_{\xi_1}^{\xi_2} (\chi_{21} (t, \lambda) f_2 (t) - \chi_{22} (t, \lambda) f_1 (t)) \varphi_{21} (\xi_1 + 0, \lambda) \, dt \]

\[ + \frac{1}{\Delta_1 (\lambda)} \int_{a}^{\xi_1} (\varphi_{11} (t, \lambda) f_2 (t) - \varphi_{12} (t, \lambda) f_1 (t)) \chi_{21} (\xi_1 + 0, \lambda) \, dt \]

\[ - c_1 \varphi_{22} (\xi_1 + 0) + c_3 \varphi_{22} (\xi_1 + 0) + c_4 \chi_{22} (\xi_1 + 0) \]

\[ = - \frac{1}{\Delta_2 (\lambda)} \int_{\xi_1}^{\xi_2} (\chi_{21} (t, \lambda) f_2 (t) - \chi_{22} (t, \lambda) f_1 (t)) \varphi_{22} (\xi_1 + 0, \lambda) \, dt \]

\[ + \frac{1}{\Delta_1 (\lambda)} \int_{a}^{\xi_1} (\varphi_{11} (t, \lambda) f_2 (t) - \varphi_{12} (t, \lambda) f_1 (t)) \chi_{22} (\xi_1 + 0, \lambda) \, dt \]

\[ - c_3 \varphi_{31} (\xi_2 + 0) - c_4 \chi_{31} (\xi_2 + 0) + c_6 \chi_{31} (\xi_2 + 0) \]

\[ = - \frac{1}{\Delta_3 (\lambda)} \int_{\xi_2}^{b} (\chi_{31} (t, \lambda) f_2 (t) - \chi_{32} (t, \lambda) f_1 (t)) \varphi_{31} (\xi_2 + 0, \lambda) \, dt \]

\[ + \frac{1}{\Delta_2 (\lambda)} \int_{\xi_1}^{\xi_2} (\varphi_{21} (t, \lambda) f_2 (t) - \varphi_{22} (t, \lambda) f_1 (t)) \chi_{31} (\xi_2 + 0, \lambda) \, dt \]

\[ - c_3 \varphi_{32} (\xi_2 + 0) - c_4 \chi_{32} (\xi_2 + 0) + c_6 \chi_{32} (\xi_2 + 0) \]

\[ = - \frac{1}{\Delta_3 (\lambda)} \int_{\xi_2}^{b} (\chi_{31} (t, \lambda) f_2 (t) - \chi_{32} (t, \lambda) f_1 (t)) \varphi_{32} (\xi_2 + 0, \lambda) \, dt \]

\[ + \frac{1}{\Delta_2 (\lambda)} \int_{\xi_1}^{\xi_2} (\varphi_{21} (t, \lambda) f_2 (t) - \varphi_{22} (t, \lambda) f_1 (t)) \chi_{32} (\xi_2 + 0, \lambda) \, dt \]

Remembering the definitions of solutions \( \varphi_{ij} (x, \lambda) \) and \( \chi_{ij} (x, \lambda) \) (\( i = 2, 3, j = 1, 2 \)), the following relation is gotten for the determinant of this linear equation system:

\[
\begin{vmatrix}
-\varphi_{21} (\xi_1 + 0) & \varphi_{21} (\xi_1 + 0) & \chi_{21} (\xi_1 + 0) & 0 \\
-\varphi_{22} (\xi_1 + 0) & \varphi_{22} (\xi_1 + 0) & \chi_{22} (\xi_1 + 0) & 0 \\
0 & -\varphi_{31} (\xi_2 + 0) & -\chi_{31} (\xi_2 + 0) & \chi_{31} (\xi_2 + 0) \\
0 & -\varphi_{32} (\xi_2 + 0) & -\chi_{32} (\xi_2 + 0) & \chi_{32} (\xi_2 + 0)
\end{vmatrix} = -\Delta_2 (\lambda) \Delta_3 (\lambda).
\]
Since above determinant is different from zero, the solution of (18) is unique. When we solve system (18), we obtain the following equality for the coefficients $c_1, c_3, c_4$ and $c_6$:

\[
\begin{align*}
c_1 &= \frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{\xi_2} (\chi_{21}(t, \lambda)f_2(t) - \chi_{22}(t, \lambda)f_1(t)) \, dt \\
&\quad + \frac{1}{\Delta_3(\lambda)} \int_{\xi_2}^{b} (\chi_{31}(t, \lambda)f_2(t) - \chi_{32}(t, \lambda)f_1(t)) \, dt \\
c_3 &= \frac{1}{\Delta_3(\lambda)} \int_{\xi_2}^{b} (\chi_{31}(t, \lambda)f_2(t) - \chi_{32}(t, \lambda)f_1(t)) \, dt, \\
c_4 &= \frac{1}{\Delta_1(\lambda)} \int_{a}^{\xi_1} (\varphi_{11}(t, \lambda)f_2(t) - \varphi_{12}(t, \lambda)f_1(t)) \, dt \\
c_6 &= \frac{1}{\Delta_2(\lambda)} \int_{\xi_1}^{\xi_2} (\varphi_{21}(t, \lambda)f_2(t) - \varphi_{22}(t, \lambda)f_1(t)) \, dt \\
&\quad + \frac{1}{\Delta_2(\lambda)} \int_{\xi_2}^{b} (\varphi_{21}(t, \lambda)f_2(t) - \varphi_{22}(t, \lambda)f_1(t)) \, dt
\end{align*}
\]

As a result, if we substitute the coefficients $c_i : (i = 1, 3, 4, 6)$ in (17), then we get the resolvent $U(x, \lambda)$ as follows:

\begin{equation}
U(x, \lambda) = \frac{\chi(x, \lambda)}{\Delta_1(\lambda)} \int_{a}^{x} (f_2\varphi_{11} - f_1\varphi_{12}) \, dt + \frac{\varphi(x, \lambda)}{\Delta_1(\lambda)} \int_{x}^{b} (f_2\chi_{11} - f_1\chi_{12}) \, dt, \quad i = 1, 3
\end{equation}

such that

\[
\varphi(x, \lambda) = \begin{cases}
\begin{pmatrix} \varphi_{11}(x, \lambda) \\ \varphi_{12}(x, \lambda) \end{pmatrix}, & x \in [a, \xi_1) \\
\begin{pmatrix} \varphi_{21}(x, \lambda) \\ \varphi_{22}(x, \lambda) \end{pmatrix}, & x \in (\xi_1, \xi_2) \\
\begin{pmatrix} \varphi_{31}(x, \lambda) \\ \varphi_{32}(x, \lambda) \end{pmatrix}, & x \in (\xi_2, b]
\end{cases}
\]

\[
\chi(x, \lambda) = \begin{cases}
\begin{pmatrix} \chi_{11}(x, \lambda) \\ \chi_{12}(x, \lambda) \end{pmatrix}, & x \in [a, \xi_1) \\
\begin{pmatrix} \chi_{21}(x, \lambda) \\ \chi_{22}(x, \lambda) \end{pmatrix}, & x \in (\xi_1, \xi_2) \\
\begin{pmatrix} \chi_{31}(x, \lambda) \\ \chi_{32}(x, \lambda) \end{pmatrix}, & x \in (\xi_2, b]
\end{cases}
\]

We can easily find Green’s function from the resolvent (19) as follows:

\[
G(x, t; \lambda) = \begin{cases}
\begin{pmatrix} \chi(x, \lambda) \\ \Delta_1(\lambda) \end{pmatrix}, & a \leq t \leq x \leq b, x \neq \xi_1, \xi_2, \ t \neq \xi_1, \xi_2 \\
\begin{pmatrix} \varphi(x, \lambda) \\ \Delta_1(\lambda) \end{pmatrix}, & a \leq t \leq x \leq b, x \neq \xi_1, \xi_2, \ t \neq \xi_1, \xi_2
\end{cases}
\]

We can rewrite the formula (19) in the following form

\[
U(x, \lambda) = \int_{a}^{b} G(x, t; \lambda) f(t) \, dt \quad \text{such that} \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}
\]

Now, we define the resolvent operator
\[ R(T, \lambda) := (T - \lambda I)^{-1} : H_1 \rightarrow H_1. \]

It is easy to see that operator equation \((T - \lambda I)Y = F, F \in H_1\) is equivalent to boundary value problem \((15), (2)-(7)\) where \(\lambda\) is not an eigenvalue, \(Y = (y_1(x), y_2(x), y_3, y_4, y_5, y_6)^T\), such that \(y_3 = \alpha'_1 y_1(a) - \alpha'_2 y_2(a)\), \(y_4 = \gamma'_1 y_1(b) - \gamma'_2 y_2(b)\), \(y_5 = y_1(\xi_1 - 0), y_6 = y_1(\xi_2 - 0)\) and \(F = (f_1(x), f_2(x), z_3, z_4, z_5, z_6)^T\) where \(z_3 = z_4 = z_5 = z_6 = 0\).

### 4. Inverse Problems

In this section, we study the inverse problems for the reconstruction of boundary value problem \((1)-(7)\) by Weyl function and spectral data.

We consider the boundary value problem \(\tilde{L}\) which has the same form with \(L\) but with different coefficients \(\tilde{\Omega}(\tilde{\lambda})\), \(\tilde{\alpha}, \tilde{\gamma}, \tilde{\alpha}'_j, \tilde{\gamma}'_j; j = 1, 2\) such that \(\tilde{\Omega}(\tilde{x}) = \begin{pmatrix} \tilde{p}(x) & q(x) \\ q(x) & \tilde{r}(x) \end{pmatrix} \).

If a certain symbol \(\tilde{\sigma}\) denotes an object related to \(L\), then the symbol \(\tilde{\sigma}\) denotes the corresponding object related to \(\tilde{L}\).

Let \(\Phi(x, \lambda)\) be a solution of equation \((1)\) which satisfies the condition \((\lambda \alpha'_2 - \alpha_2) \Phi_2(a, \lambda) - (\lambda \alpha'_1 - \alpha_1) \Phi_1(a, \lambda) = 1\) and the transmissions \((4)-(7)\).

Assume that the function \(\phi(x, \lambda) = \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix}^T\) is the solution of equation \((1)\) that satisfies the conditions \(\phi_1(a, \lambda) = d_1^{-1} \alpha'_2, \phi_2(a, \lambda) = d_1^{-1} \alpha'_1\) and the transmission conditions \((4)-(7)\).

Since \(W[\varphi, \phi] = 1\), the functions \(\phi\) and \(\varphi\) are linearly independent. Therefore, the function \(\psi(x, \lambda)\) can be represented by

\[ \psi(x, \lambda) = d_1^{-1} (\alpha'_1 \psi_1(a, \lambda) - \alpha'_2 \psi_2(a, \lambda)) \varphi(x, \lambda) + \Delta(\lambda) \phi(x, \lambda) \]

or

\[ \Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)} = \phi(x, \lambda) + \frac{\alpha'_1 \psi_1(a, \lambda) - \alpha'_2 \psi_2(a, \lambda)}{d_1 \Delta(\lambda)} \varphi(x, \lambda) \]

which is called the Weyl solution and

\[ \frac{\alpha'_1 \psi_1(a, \lambda) - \alpha'_2 \psi_2(a, \lambda)}{d_1 \Delta(\lambda)} = M(\lambda) = d_1^{-1} (\alpha'_1 \Phi_1(a, \lambda) - \alpha'_2 \Phi_2(a, \lambda)) \]

is called the Weyl function.

**Lemma 6** The following representation is true:

\[ M(\lambda) = \sum_{n=-\infty}^{\infty} \frac{1}{\mu_n (\lambda_n - \lambda)}. \]

**Proof** Weyl function \(M(\lambda)\) is meromorphic function with respect to \(\lambda\) which has simple poles at \(\lambda_n\). Therefore, we calculate

\[ \Re s M(\lambda) = \frac{\alpha'_1 \psi_1(a, \lambda_n) - \alpha'_2 \psi_2(a, \lambda_n)}{d_1 \Delta(\lambda_n)}, \]

Since \(\kappa_n = \frac{\alpha'_1 \psi_1(a, \lambda_n) - \alpha'_2 \psi_2(a, \lambda_n)}{d_1}\) and \(\Delta(\lambda_n) = -\kappa_n \mu_n\),

\[ \Re s M(\lambda) = -\frac{1}{\mu_n}. \]
Let $\Gamma_n = \{ \lambda : |\lambda| \leq |\lambda_0^*| + \varepsilon \}$ where $\varepsilon$ is a sufficiently small number. Consider the contour integral $I_n(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{M(\mu)}{\mu - \lambda} d\mu$, $\lambda \in \text{int} \Gamma_n$.

Since $\Delta(\lambda) \geq C_\delta \lambda^4 e^{\text{Im} \lambda(|(\xi_1-a)\rho_1 + (\xi_2-\xi_1)\rho_2 + (b-\xi_2)\rho_3|)}$ and $M(\lambda) = \frac{\alpha_1^* \psi_1(a, \lambda) - \alpha_2^* \psi_2(a, \lambda)}{d_1 \Delta(\lambda)}$, \[|M(\lambda)| \leq \frac{C_\delta}{|\lambda|}\] for $\lambda \in F_\delta = \{ \lambda : |\lambda - \lambda_n| \geq \delta, n = 0, \pm 1, \ldots \}$, where $\delta$ is a sufficiently small number.

Thus, $\lim_{n \to \infty} I_n(\lambda) = 0$. Then, the residue theorem yields

\[M(\lambda) = \sum_{n=-\infty}^{\infty} \frac{1}{\mu_n (\lambda_n - \lambda)}.\]

**Theorem 3** If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$, i.e., $\Omega(x) = \tilde{\Omega}(x)$, a.e. and $\alpha_j = \tilde{\alpha}_j$, $\gamma_j = \tilde{\gamma}_j$, $\alpha'_j = \tilde{\alpha}'_j$, $\gamma'_j = \tilde{\gamma}'_j$, $j = 1, 2$.

**Proof** We introduce a matrix $P(x, \lambda) = [P_{kj}(x, \lambda)]_{k,j=1,2}$ by the formula

\[
P(x, \lambda) = \begin{pmatrix}
\tilde{\varphi}_1 & \tilde{\Phi}_1 \\
\tilde{\varphi}_2 & \tilde{\Phi}_2
\end{pmatrix} = \begin{pmatrix}
\varphi_1 & \Phi_1 \\
\varphi_2 & \Phi_2
\end{pmatrix}
\]

or

\[
P(11)(x, \lambda) P(12)(x, \lambda) \\
P(21)(x, \lambda) P(22)(x, \lambda)
\]

where $\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)}$ and $W(\tilde{\Phi}, \tilde{\varphi}) = 1$. Thus, we find

\[
P_{11}(x, \lambda) = \varphi_1(x, \lambda) \tilde{\psi}_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda) \varphi_1(x, \lambda) + \left( \tilde{M}(\lambda) - M(\lambda) \right) \varphi_1(x, \lambda) \tilde{\varphi}_2(x, \lambda)
\]

\[
P_{12}(x, \lambda) = -\varphi_1(x, \lambda) \tilde{\psi}_1(x, \lambda) + \tilde{\varphi}_1(x, \lambda) \varphi_1(x, \lambda)
\]

\[
P_{21}(x, \lambda) = \varphi_2(x, \lambda) \tilde{\psi}_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda) \varphi_2(x, \lambda)
\]

\[
P_{22}(x, \lambda) = -\varphi_2(x, \lambda) \tilde{\psi}_1(x, \lambda) + \tilde{\varphi}_1(x, \lambda) \varphi_2(x, \lambda)
\]

Thus, if $M(\lambda) = \tilde{M}(\lambda)$ then the functions $P_{ij}(x, \lambda) (i, j = 1, 2)$ are entire in $\lambda$ for each fixed $x$. Moreover, since asymptotic behaviours of $\varphi_1(x, \lambda)$, $\tilde{\varphi}_1(x, \lambda)$, $\psi_1(x, \lambda)$, $\tilde{\psi}_1(x, \lambda)$ and $|\Delta(\lambda)| \geq C_\delta |\lambda|^4 e^{\text{Im} \lambda(|(\xi_1-a)\rho_1 + (\xi_2-\xi_1)\rho_2 + (b-\xi_2)\rho_3|)}$ in $F_\delta \cap \tilde{F}_\delta$, we can
easily see that functions $P_{ij}(x,\lambda)$ are bounded with respect to $\lambda$. As a result, these functions don’t depend on $\lambda$.

Here, we denote $\bar{F}_\delta = \{ \lambda : |\lambda - \lambda_n| \geq \delta, \ n = 0, \pm 1, \pm 2, \ldots \}$ where $n$ is sufficiently small number, $\lambda_n$ are eigenvalues of the problem $\bar{L}$.

From (25),

$$P_{11}(x, \lambda) - 1 = \frac{\tilde{\psi}_2(x, \lambda) (\varphi_1(x, \lambda) - \tilde{\varphi}_1(x, \lambda)) - \tilde{\varphi}_2(x, \lambda)}{\Delta(\lambda)} \left( \frac{\psi_1(x, \lambda) - \tilde{\psi}_1(x, \lambda)}{\Delta(\lambda)} \right)$$

$$P_{12}(x, \lambda) = \frac{\psi_1(x, \lambda) (\tilde{\varphi}_1(x, \lambda) - \varphi_1(x, \lambda))}{\Delta(\lambda)} + \varphi_1(x, \lambda) \left( \frac{\psi_1(x, \lambda) - \tilde{\psi}_1(x, \lambda)}{\Delta(\lambda)} \right)$$

$$P_{21}(x, \lambda) = \varphi_2(x, \lambda) \left( \frac{\tilde{\psi}_2(x, \lambda) - \psi_2(x, \lambda)}{\Delta(\lambda)} \right) + \psi_2(x, \lambda) \left( \frac{\varphi_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda)}{\Delta(\lambda)} \right)$$

$$P_{22}(x, \lambda) - 1 = \frac{\psi_2(x, \lambda) (\tilde{\psi}_1(x, \lambda) - \varphi_1(x, \lambda)) + \varphi_2(x, \lambda) \left( \frac{\psi_1(x, \lambda) - \tilde{\psi}_1(x, \lambda)}{\Delta(\lambda)} \right)}{\Delta(\lambda)}$$

It follows from the representations of solutions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$,

$$\lim_{\lambda \to \infty} \frac{\tilde{\psi}_2(x, \lambda) (\varphi_1(x, \lambda) - \tilde{\varphi}_1(x, \lambda))}{\Delta(\lambda)} = 0$$

$$\text{and} \quad \lim_{\lambda \to \infty} \frac{\tilde{\varphi}_2(x, \lambda)}{\Delta(\lambda)} \left( \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x, \lambda)}{\Delta(\lambda)} \right) = 0$$

for all $x$ in $\Lambda$. Thus, $\lim_{\lambda \to \infty} (P_{11}(x, \lambda) - 1) = 0$ is valid uniformly with respect to $x$.

So we have $P_{11}(x, \lambda) \equiv 1$ and similarly $P_{12}(x, \lambda) \equiv 0$, $P_{21}(x, \lambda) \equiv 0$, $P_{22}(x, \lambda) \equiv 1$.

From (23), we obtain $\varphi_1(x, \lambda) \equiv \tilde{\varphi}_1(x, \lambda)$, $\Phi_1 \equiv \tilde{\Phi}_1$, $\varphi_2(x, \lambda) \equiv \tilde{\varphi}_2(x, \lambda)$ and $\Phi_2 \equiv \tilde{\Phi}_2$ for all $x$ and $\lambda$. Moreover, from $\Phi(x, \lambda) = \frac{\psi_2(x, \lambda)}{\psi_1(x, \lambda)}$, we get

$$\frac{\tilde{\psi}_2(x, \lambda) (\varphi_1(x, \lambda) - \tilde{\varphi}_1(x, \lambda))}{\psi_1(x, \lambda)}$$

Hence, $\Omega(x) = \tilde{\Omega}(x)$, i.e., $p(x) = \tilde{p}(x)$, $r(x) = \tilde{r}(x)$ almost everywhere. On the other hand, since \( \left( \begin{array}{c} \varphi_{11}(a, \lambda) \\ \varphi_{12}(a, \lambda) \end{array} \right) = \left( \begin{array}{c} \lambda \alpha_2 - \alpha_2 \\ \lambda \alpha_1 - \alpha_1 \end{array} \right) \), \( \left( \begin{array}{c} \psi_{11}(b, \lambda) \\ \psi_{12}(b, \lambda) \end{array} \right) = \left( \begin{array}{c} \gamma_1 \tilde{\gamma}_2 + \tilde{\gamma}_2 \\ \gamma_1 + \tilde{\gamma}_1 \end{array} \right) \), we get easily that $\alpha'_2 = \tilde{\alpha}'_2$, $\alpha_2 = \tilde{\alpha}_2$, $\alpha'_1 = \tilde{\alpha}'_1$, $\alpha_1 = \tilde{\alpha}_1$ and $\gamma'_2 = \tilde{\gamma}'_2$, $\gamma_2 = \tilde{\gamma}_2$, $\gamma'_1 = \tilde{\gamma}'_1$, $\gamma_1 = \tilde{\gamma}_1$. Therefore, $L \equiv \tilde{L}$.

**Theorem 4** If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for all $n$, then $L \equiv \tilde{L}$, i.e., $\Omega(x) = \tilde{\Omega}(x)$, a.e., $\alpha_j = \tilde{\alpha}_j$, $\gamma_i = \tilde{\gamma}_i$, $\alpha'_j = \tilde{\alpha}'_j$, $\gamma'_j = \tilde{\gamma}'_j$, $j = 1, 2$. Hence, the problem (1)-(7) is uniquely determined by spectral data \( \{\lambda_n, \mu_n\} \).

**Proof** If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for all $n$, then $M(\lambda) = \tilde{M}(\lambda)$ by Lemma 6. Therefore, we get $L = \tilde{L}$ by Theorem 3.

Let us consider the boundary-value problem $L_1$ with the condition $\alpha'_1 y_1(a, \lambda) - \alpha'_2 y_2(a, \lambda) = 0$ instead of the condition (2) in $L$. Let $\{\tau_n\}_{n \in \mathbb{Z}}$ be the eigenvalues of the problem $L_1$. It is clear that $\tau_n$ are zeros of $\Delta_1(\tau) := \alpha'_1 \psi_1(a, \tau) - \alpha'_2 \psi_2(a, \tau)$ which is characteristic function of $L_1$.

**Theorem 5** If $\lambda_n = \tilde{\lambda}_n$ and $\tau_n = \tilde{\tau}_n$ for all $n$, then $L(\Omega, \gamma_i, \gamma'_i) = \tilde{L}(\tilde{\Omega}, \tilde{\gamma}_i, \tilde{\gamma}'_i)$, $j = 1, 2$.

Hence, the problem $L$ is uniquely determined by the sequences $\{\lambda_n\}$ and $\{\tau_n\}$ except coefficients $\alpha_j$ and $\alpha'_j$. 

Proof Since the characteristic functions $\Delta(\lambda)$ and $\Delta_1(\tau)$ are entire of order 1, functions $\Delta(\lambda)$ and $\Delta_1(\tau)$ are uniquely determined up to multiplicative constant with their zeros by Hadamard’s factorization theorem [46]

$$\Delta(\lambda) = C \prod_{n=-\infty}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right),$$

$$\Delta_1(\tau) = C_1 \prod_{n=-\infty}^{\infty} \left( 1 - \frac{\tau}{\tau_n} \right),$$

where $C$ and $C_1$ are constants depend on $\{\lambda_n\}$ and $\{\tau_n\}$, respectively. When $\lambda_n = \lambda_n$ and $\tau_n = \tau_n$ for all $n$, $\Delta(\lambda) \equiv \hat{\Delta}(\lambda)$ and $\Delta_1(\tau) \equiv \hat{\Delta}_1(\tau)$. Hence, $\alpha'_1 \psi_1(a, \tau) - \alpha'_2 \psi_2(a, \tau) = \alpha'_1 \hat{\psi}_1(a, \tau) - \alpha'_2 \hat{\psi}_2(a, \tau)$. As a result, we get $M(\lambda) = \hat{M}(\lambda)$ by (21). So, the proof is completed by Theorem 3.

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