On the Relationship between D’Angelo $q$-Type and Catlin $q$-Type

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Abstract We establish inequalities relating two measurements of the order of contact of $q$-dimensional complex varieties with a real hypersurface.

Keywords Orders of contact · D’Angelo finite $q$-type · Catlin finite $q$-type · Finite type domains in $\mathbb{C}^n$ · Pseudoconvexity

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1 Introduction

The study of the order of contact of complex varieties with the boundary of a domain in $\mathbb{C}^n$ stems from the investigation of the subellipticity of the $\bar{\partial}$-Neumann problem. Kohn proved in 1979 in [16] that for a pseudoconvex domain in $\mathbb{C}^n$ with real-analytic boundary the subellipticity of the $\bar{\partial}$-Neumann problem for $(p, q)$ forms is equivalent to the property that all holomorphic varieties of complex dimension $q$ have finite order of contact with the boundary of the domain. D’Angelo introduced a quantitative measure, written $\Delta_q$, for this order of contact. D’Angelo fleshed out its more important
properties culminating with openness and finite determination, which he established in 1982 in [10]. Meanwhile, Catlin extended Kohn’s result to smooth pseudoconvex domains in [2,3], and [4]. The notion of finite order of contact of holomorphic varieties of complex dimension $q$ with the boundary of the domain that he defined in [4] and showed is equivalent to the subellipticity of the $\bar{\partial}$-Neumann problem for $(p, q)$ forms for smooth pseudoconvex domains is not the same as D’Angelo’s notion. The two trivially agree for $q = 1$, i.e., for holomorphic curves, but for $q > 1$ Catlin merely expressed the hope that they might be shown to equal each other. Catlin’s notion is what became known as Catlin $q$-type, $D_q$. In 1999 in a joint survey paper by D’Angelo and Kohn [13], it was claimed that these two notions ought to be simultaneously finite.

In [4] Catlin also proved a lower bound for subelliptic gain in the $\bar{\partial}$-Neumann problem

$$\epsilon \geq \tau^{-n^2} \epsilon^n,$$

which holds for any smooth pseudoconvex domain in $\mathbb{C}^n$ and is exponential in $\tau = D_q$, his notion of contact of holomorphic varieties of complex dimension $q$ with the boundary of the domain. Apart from Catlin’s result, there are a number of either sharp or effective bounds for subelliptic gain for $(0, 1)$ forms, i.e., when $q = 1$, in terms of $\Delta_1 = D_1$; see [6,15,17], and [7]. Any other such result for $q > 1$ obtained in terms of D’Angelo’s more standardly used notion of $q$-type would have to be compared against Catlin’s benchmark estimate. Herein lies the significance of our work in this paper as we relate $\Delta_q$ to $D_q$ for $q > 1$, thus enabling this type of comparison.

Both $\Delta_q$ and $D_q$ can also be defined for ideals $\mathcal{I}$ in the ring $\mathcal{O}_{x_0}$ of germs of holomorphic functions at $x_0$. Comparing $\Delta_q$ with $D_q$ for such an ideal is much simpler, so we first prove a result of this nature:

**Theorem 1.1** Let $\mathcal{I}$ be an ideal of germs of holomorphic functions at $x_0$. Then for $1 \leq q \leq n$,

$$D_q(\mathcal{I}, x_0) \leq \Delta_q(\mathcal{I}, x_0) \leq \left( D_q(\mathcal{I}, x_0) \right)^{n-q+1}.$$

Theorem 1.1 is consistent with the simple result that $\Delta_n(\mathcal{I}, x_0) = D_n(\mathcal{I}, x_0)$ for any ideal $\mathcal{I}$ of germs of holomorphic functions in $n$ variables.

Our main result is the following:

**Theorem 1.2** Let $\Omega$ in $\mathbb{C}^n$ be a domain with $C^\infty$ boundary. Let $x_0 \in b\Omega$ be a point on the boundary of the domain, and let $1 \leq q < n$.

(i) $D_q(b\Omega, x_0) \leq \Delta_q(b\Omega, x_0)$;

(ii) If $\Delta_q(b\Omega, x_0) < \infty$ and the domain is $q$-positive at $x_0$ (the $q$ version of D’Angelo’s property $P$), then

$$\Delta_q(b\Omega, x_0) \leq 2 \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q}.$$
In particular, if \( b\Omega \) is pseudoconvex at \( x_0 \) and \( \Delta_q(b\Omega, x_0) < \infty \), then

\[
D_q(b\Omega, x_0) \leq \Delta_q(b\Omega, x_0) \leq 2 \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q}.
\]

Since \( \Delta_q(b\Omega, x_0) = D_q(b\Omega, x_0) \) for \( q = 1 \), inequality (i) is sharp. By definition, \( D_q(b\Omega, x_0) \geq 2 \), and

\[
2 \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q} = D_q(b\Omega, x_0) \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q-1},
\]

So Theorem 1.2 (ii) is not sharp. It is, however, the best result that can be obtained given our method. An example illustrating this point will be provided in Sect. 4. We exclude the value \( q = n \) because \( b\Omega \) has real dimension \( 2n - 1 \), so looking at its order of contact with an \( n \)-dimensional complex variety does not make sense. It is also known that subellipticity with exponent \( \epsilon = 1 \) holds at all boundary points for \((p, n)\) forms. The reader may consult p. 83 of [16]. The last part of Theorem 1.2 follows because a pseudoconvex domain where \( \Delta_q(b\Omega, x_0) < \infty \) satisfies \( q \)-positivity at \( x_0 \), a generalization of D’Angelo’s property \( P \) for \( q > 1 \). We are deliberately avoiding the terminology property \( P \) here in order to be consistent with D’Angelo’s usage in [12]. D’Angelo introduced property \( P \) in [10] for a notion of positivity more general than pseudoconvexity. Shortly afterward, Catlin introduced Property \( (P) \) in [5], which has since become a standard notion in several complex variables. Details can be found in [1,5], and [18]. The two names are similar enough to create confusion, so D’Angelo suppressed the term property \( P \) in subsequent work, a practice we are following here by employing \( q \)-positivity instead. We would also like to note that our method of proving Theorem 1.2 (ii) breaks down completely in the absence of \( q \)-positivity, and we have no examples on which we could even formulate a conjecture as to whether \( \Delta_q \) are \( D_q \) remain simultaneously finite.

The paper is organized as follows: Sect. 2 defines D’Angelo \( q \)-type and outlines a number of its properties. D’Angelo’s property \( P \) is also defined here along with \( q \)-positivity, its \( q \) version for \( q > 1 \). Section 3 is devoted to the Catlin \( q \)-type. The two notions are then related to each other in Sect. 4, where Theorems 1.1 and 1.2 are also proven.

2 D’Angelo \( q \)-Type and \( q \)-Positivity

Starting with [8], D’Angelo introduced various numerical functions that measure the maximum order of contact of holomorphic varieties of complex dimension \( q \) with a real hypersurface \( M \) in \( \mathbb{C}^n \) such as the boundary of a domain; see [12].

We shall first give the classical definition of order of contact for \( q = 1 \), holomorphic curves. Let \( r \) be a defining function for the real hypersurface \( M \) in \( \mathbb{C}^n \). Let \( C = C(m, p) \) be the set of all germs of holomorphic curves

\[
\varphi : (U, 0) \to (\mathbb{C}^m, p),
\]
where $U$ is some neighborhood of the origin in $\mathbb{C}^1$ and $\varphi(0) = p$. For all $t \in U$, $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))$, where $\varphi_j(t)$ is holomorphic for every $j$ with $1 \leq j \leq m$. For each component $\varphi_j$, the order of vanishing at the origin $\text{ord}_0 \varphi$ is the order of the first non-vanishing derivative of $\varphi_j$, i.e., $s \in \mathbb{N}$ such that

$$
\frac{d}{dt} \varphi_j(t) = \cdots = \frac{d^{s-1}}{dt^{s-1}} \varphi_j(t) = 0,
$$

but $\frac{d^s}{dt^s} \varphi_j(t) \neq 0$. We set $\text{ord}_0 \varphi = \min_{1 \leq j \leq m} \text{ord}_0 \varphi_j$. Consider $\varphi^*r$, the pullback of $r$ to $\varphi$, and let $\text{ord}_0 \varphi^*r$ be the order of the first non-vanishing derivative at the origin of $\varphi^*r$ viewed as a function of $t$.

**Definition 2.1** Let $M$ be a real hypersurface in $\mathbb{C}^n$, and let $r$ be a defining function for $M$. The D’Angelo $1$-type at $x_0 \in M$ is given by

$$
\Delta_1(M, x_0) = \sup_{\varphi \in \mathcal{C}(n, x_0)} \frac{\text{ord}_0 \varphi^*r}{\text{ord}_0 \varphi}.
$$

If $\Delta_1(M, x_0)$ is finite, we call $x_0$ a point of finite D’Angelo $1$-type.

When holomorphic varieties have complex dimension greater than $1$, there is no longer just one natural definition of their order of contact with a real hypersurface in $\mathbb{C}^n$ as not every holomorphic variety of dimension $q \geq 2$ has a local parameterization. Following D’Angelo in [10], one approach is to reduce this case to computing $\Delta_1(\tilde{M}, x_0)$ for a related hypersurface $\tilde{M}$ sitting in a different $\mathbb{C}^m$ such that the holomorphic varieties of dimension $q$ generically become holomorphic curves in the new ambient space. Let $\phi : \mathbb{C}^{n-q+1} \to \mathbb{C}^n$ be any linear embedding of $\mathbb{C}^{n-q+1}$ into $\mathbb{C}^n$. For generic choices of $\phi$, the pullback $\phi^*M$ will be a hypersurface in $\mathbb{C}^{n-q+1}$. We can thus define $\Delta_q(M, x_0)$ as follows:

**Definition 2.2** Let $M$ be a real hypersurface in $\mathbb{C}^n$, and let $r$ be a defining function for $M$. The D’Angelo $q$-type at $x_0 \in M$ is given by

$$
\Delta_q(M, x_0) = \inf_{\phi} \sup_{\varphi \in \mathcal{C}(n-q+1, x_0)} \frac{\text{ord}_0 \varphi^*\phi^*r}{\text{ord}_0 \varphi} = \inf_{\phi} \Delta_1(\phi^*r, x_0),
$$

where $\phi : \mathbb{C}^{n-q+1} \to \mathbb{C}^n$ is any linear embedding of $\mathbb{C}^{n-q+1}$ into $\mathbb{C}^n$ and we have identified $x_0$ with $\phi^{-1}(x_0)$. If $\Delta_q(M, x_0)$ is finite, we call $x_0$ a point of finite D’Angelo $q$-type.

**Theorem 2.3** Let $M$ be a smooth real hypersurface in $\mathbb{C}^n$.

(i) $\Delta_q(M, x_0)$ is well defined, i.e., independent of the defining function $r$ chosen for $M$.

(ii) $\Delta_q(M, x_0)$ is not upper semi-continuous in general; see [9].
(iii) Let $\Delta_q(M, x_0)$ be finite at some $x_0 \in M$. Then there exists a neighborhood $V$ of $x_0$ on which

$$\Delta_q(M, x) \leq 2(\Delta_q(M, x_0))^{n-q}.$$ 

(iv) The function $\Delta_q(M, x_0)$ is finitely determined. In other words, if $\Delta_q(M, x_0)$ is finite, then there exists an integer $k$ such that $\Delta_q(M, x_0) = \Delta_q(M', x_0)$ for $M'$ a hypersurface defined by any $r'$ that has the same $k$-jet at $x_0$ as the defining function $r$ of $M$.

Remarks (1) Part (iii) is Theorem 6.2 from p. 634 of [10] and implies the set of points of finite $q$-type is open. Note that the result holds independently of pseudoconvexity.

(2) Part (iv) is Proposition 14 from p. 88 of [11], whose proof implies that if $t = \Delta_q(M, x_0) < \infty$, then we can let $k = \lceil t \rceil$, the ceiling of $t$, i.e., the least integer greater than or equal to $t$.

For the purpose of relating $\Delta_q(b\Omega, x_0)$ with $Dq(b\Omega, x_0)$, we will need to show that $\Delta_q(b\Omega, x_0)$ is generic with respect to the choices of linear embeddings $\phi : C^{n-q+1} \rightarrow C^n$. In fact, linear embeddings $\phi : C^{n-q+1} \rightarrow C^n$ are in one-to-one correspondence with non-degenerate sets of $q-1$ linear forms $\{w_1, \ldots, w_{q-1}\}$ in $O_{x_0}$, the local ring of holomorphic germs in $n$ variables at $x_0 \in C^n$. The zero set of $\{w_1, \ldots, w_{q-1}\}$ is locally the image of the embedding $\phi$.

Restating the embedding $\phi$ as a non-degenerate set of linear forms points to the necessity of having a notion of type that applies to an ideal rather than just a hypersurface, which is what has been defined so far. Indeed, D’Angelo makes the following definition on p. 86 of [11]:

Definition 2.4 Let $C^\infty_{x_0}$ be the ring of smooth germs at $x_0 \in C^n$ and let $\mathcal{I}$ be an ideal in $C^\infty_{x_0}$,

$$\Delta_1(\mathcal{I}, x_0) = \sup_{\phi \in C(n, x_0)} \inf_{g \in \mathcal{I}} \frac{\text{ord}_0 \phi^* g}{\text{ord}_0 \phi}.$$ 

Remark If $M$ is a real hypersurface in $C^n$ and $x_0 \in M$, let $\mathcal{I}(M)$ be the ideal of smooth germs in $C^\infty_{x_0}$ that vanish on the germ of $M$ at $x_0$. Then $\Delta_1(\mathcal{I}(M), x_0) = \Delta_1(M, x_0)$ because the infimum in Definition 2.4 is realized by a defining function of $M$, which has order 1 at $x_0$.

Now we can give an equivalent definition to Definition 2.2 that was first stated by D’Angelo on the bottom of p. 86 of [11]:

Definition 2.5 Let $M$ be a real hypersurface in $C^n$, and let $x_0 \in M$. The D’Angelo $q$-type at $x_0 \in M$ is given by

$$\Delta_q(M, x_0) = \inf_{\{w, \ldots, w_{q-1}\}} \Delta_1(\mathcal{I}(M), w, \ldots, w_{q-1}, x_0),$$
where \(\{w_1, \ldots, w_{q-1}\}\) is a non-degenerate set of linear forms in \(\mathcal{O}_{x_0}\), \((\mathcal{I}(M), w_1, \ldots, w_{q-1})\) is the ideal in \(\mathcal{C}_{x_0}\) generated by \(\mathcal{I}(M)\), \(w_1, \ldots, w_{q-1}\), and the infimum is taken over all such non-degenerate sets \(\{w_1, \ldots, w_{q-1}\}\) of linear forms in \(\mathcal{O}_{x_0}\).

This same definition can also be given for an ideal \(\mathcal{I}\) in \(\mathcal{O}_{x_0}\) and is the notion that appears in the statement of Theorem 1.1:

**Definition 2.6** If \(\mathcal{I}\) is an ideal in \(\mathcal{O}_{x_0}\),

\[
\Delta_q(\mathcal{I}, x_0) = \inf_{\{w_1, \ldots, w_{q-1}\}} \Delta_1(\mathcal{I}(M), w_1, \ldots, w_{q-1}, x_0)
\]

\[
= \inf_{\{w_1, \ldots, w_{q-1}\}} \sup_{\varphi \in \mathcal{C}(n, x_0)} \inf_{g \in \mathcal{I}(M), w_1, \ldots, w_{q-1}} \frac{\text{ord}_0 \varphi^* g}{\text{ord}_0 \varphi},
\]

where \(\{w_1, \ldots, w_{q-1}\}\) is a non-degenerate set of linear forms in \(\mathcal{O}_{x_0}\), \((\mathcal{I}, w_1, \ldots, w_{q-1})\) is the ideal in \(\mathcal{O}_{x_0}\) generated by \(\mathcal{I}, w_1, \ldots, w_{q-1}\), and the infimum is taken over all such non-degenerate sets \(\{w_1, \ldots, w_{q-1}\}\) of linear forms in \(\mathcal{O}_{x_0}\).

Since working in the ring \(\mathcal{C}_{x_0}\) is not particularly easy, it would be helpful to reduce the computation of \(\Delta_q(M, x_0)\) to a computation in \(\mathcal{O}_{x_0}\), which has much better algebraic properties. Let us assume for the moment that \(\Delta_q(M, x_0) = t < \infty\), and let \(k = \lceil t \rceil\), the ceiling of \(t\). By Theorem 2.3 (iv) and Remark (2) following it, \(\Delta_q(M, x_0) = \Delta_q(M_k, x_0)\), where \(M_k\) is the real hypersurface defined by \(r_k\), the polynomial that has the same \(k\)-jet at \(x_0\) as the defining function \(r\) of \(M\). The advantage of working with \(r_k\) is that we can apply polarization to it; namely, we can give a holomorphic decomposition for \(r_k\) as

\[
r_k = \text{Re}[h] + ||f||^2 - ||g||^2,
\]

where \(||f||^2 = \sum_{j=1}^N |f_j|^2\), \(||g||^2 = \sum_{j=1}^N |g_j|^2\), and the functions \(h, f_1, \ldots, f_N, g_1, \ldots, g_N\) are all holomorphic polynomials in \(n\) variables. This idea first appeared in Sect. 3 of [10]. Furthermore, if \(\mathcal{U}(N)\) is the group of \(N \times N\) unitary matrices, then for every such unitary matrix \(U \in \mathcal{U}(N)\), we can consider the ideal of holomorphic polynomials \(\mathcal{I}(U, x_0) = (h, f - U g)\) generated by \(h\) and the \(N\) components of \(f - U g\), where \(f = (f_1, \ldots, f_N)\) and \(g = (g_1, \ldots, g_N)\). It turns out that

\[
\sup_{U \in \mathcal{U}(N)} \Delta_1(\mathcal{I}(U, x_0), x_0) \leq \Delta_1(M_k, x_0) \leq 2 \sup_{U \in \mathcal{U}(N)} \Delta_1(\mathcal{I}(U, x_0), x_0). \quad (2.1)
\]

We have used Corollary 3.7 on p. 627 of [10]. If we now combine this result with Definition 2.5 and Theorem 2.3 (iv), we obtain

\[
\inf_{\{w_1, \ldots, w_{q-1}\}} \sup_{U \in \mathcal{U}(N)} \Delta_1((\mathcal{I}(U, x_0), w_1, \ldots, w_{q-1}), x_0) \leq \Delta_q(M, x_0)
\]

\[
\leq 2 \inf_{\{w_1, \ldots, w_{q-1}\}} \sup_{U \in \mathcal{U}(N)} \Delta_1((\mathcal{I}(U, x_0), w_1, \ldots, w_{q-1}), x_0). \quad (2.2)
\]
Since $\mathcal{U}(N)$ is compact, the supremum over it would be easy to handle if $\Delta_1$ were an upper semi-continuous quantity, but it is not as shown by D’Angelo in [9]. We thus need to compare $\Delta_1$ to some other quantity computed in $\mathcal{O}_{x_0}$ that is upper semi-continuous. We will use

$$D(\mathcal{I}, x_0) = \dim_{\mathbb{C}}(\mathcal{O}_{x_0}/\mathcal{I}),$$

where $\mathcal{I}$ is an ideal of holomorphic germs at $x_0$. Here $\dim_{\mathbb{C}}(\mathcal{O}_{x_0}/\mathcal{I})$ means the dimension of $\mathcal{O}_{x_0}/\mathcal{I}$ viewed as a vector space over $\mathbb{C}$. This notion appears under different names in the literature. For example, on p. 153 of [4], Catlin calls it the multiplicity of the ideal $\mathcal{I}$.

**Proposition 2.7** Let $\mathcal{I}(\lambda)$ be an ideal in $\mathcal{O}_{x_0}$ that depends continuously on $\lambda$. Then $D(\mathcal{I}(\lambda), x_0)$ is an upper semi-continuous function of $\lambda$.

This result is part of Proposition 5.3 on p. 39 of [19] cited by D’Angelo in [10]. In our case, $\mathcal{I}(U, x_0)$ obviously depends continuously on $U$, so $D(\mathcal{I}(U, x_0), x_0)$ is upper semi-continuous on the compact set $\mathcal{U}(N)$. Thus $D(\mathcal{I}(U, x_0), x_0)$ achieves a maximum on $\mathcal{U}(N)$ because each $U \in \mathcal{U}(N)$ has an open neighborhood $V(U)$ such that

$$D(\mathcal{I}(U^{'}, x_0), x_0) \leq D(\mathcal{I}(U, x_0), x_0)$$

for every $U^{'} \in V(U)$ from the upper semi-continuity, $\{V(U)\}$ is an open cover of $\mathcal{U}(N)$, we can thus pass to a finite open subcover $\{V(U_j)\}_{1 \leq j \leq p}$, and then we can take

$$\max_{1 \leq j \leq p} D(\mathcal{I}(U_j, x_0), x_0).$$

Since we are primarily interested in the case $q > 1$, the object that appears naturally corresponding to a proper ideal $\mathcal{I}$ in $\mathcal{O}_{x_0}$ is $D((\mathcal{I}, w_1, \ldots, w_{q-1}), x_0)$, where $\{w_1, \ldots, w_{q-1}\}$ is a non-degenerate set of linear forms. It turns out that $D((\mathcal{I}, w_1, \ldots, w_{q-1}), x_0)$ is generic when we consider its value over all non-degenerate sets of linear forms $\{w_1, \ldots, w_{q-1}\}$:

**Proposition 2.8** Let $\mathcal{I}$ be a proper ideal in $\mathcal{O}_{x_0}$, and let $x_0 \in \mathbb{C}^n$.

$$\inf_{\{w_1, \ldots, w_{q-1}\}} D((\mathcal{I}, w_1, \ldots, w_{q-1}), x_0) = \text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} D((\mathcal{I}, w_1, \ldots, w_{q-1}), x_0),$$

where $\{w_1, \ldots, w_{q-1}\}$ is a non-degenerate set of linear forms in $\mathcal{O}_{x_0}$, $(\mathcal{I}, w_1, \ldots, w_{q-1})$ is the ideal in $\mathcal{O}_{x_0}$ generated by $\mathcal{I}, w_1, \ldots, w_{q-1}$, and the infimum and the generic value are both taken over all such non-degenerate sets $\{w_1, \ldots, w_{q-1}\}$ of linear forms in $\mathcal{O}_{x_0}$. In other words, the infimum is achieved and equals the generic value.

**Proof** For every non-degenerate set of linear forms $\{w_1, \ldots, w_{q-1}\}$ in $\mathcal{O}_{x_0}$, we consider a linear change of variables at $x_0$ such that $w_1, \ldots, w_{q-1}$ become the coordinate
functions $z_1, \ldots, z_{q-1}$. Let $\tilde{I}$ be the image of $I$ under this linear change of variables. Consider now

$$D\left((\tilde{I}, z_1, \ldots, z_{q-1}), x_0\right) = \dim_{\mathbb{C}} \left(\mathcal{O}_{x_0}/(\tilde{I}, z_1, \ldots, z_{q-1})\right).$$

Since the variables $z_1, \ldots, z_{q-1}$ get set to zero in the quotient $\mathcal{O}_{x_0}/(\tilde{I}, z_1, \ldots, z_{q-1})$ and the quantity $D\left((\tilde{I}, z_1, \ldots, z_{q-1}), x_0\right)$ is invariant under linear changes of variables, it follows that

$$\inf_{\{w_1, \ldots, w_{q-1}\}} D\left((\mathcal{I}, w_1, \ldots, w_{q-1}), x_0\right) = \text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} D\left((\mathcal{I}, w_1, \ldots, w_{q-1}), x_0\right),$$

as needed. \qed

Propositions 2.7 and 2.8 together make looking at

$$\sup_{U \in \mathcal{U}(N)} \inf_{\{w_1, \ldots, w_{q-1}\}} D\left((\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0\right)$$

much easier in the sense that the supremum and the infimum, which are both achieved, can be exchanged here. D’Angelo has already related this quantity to $\Delta_q(M, x_0)$ in Theorem 14 on p. 91 of [11]:

**Theorem 2.9**

$$\Delta_q(M, x_0) \leq 2 \sup_{U \in \mathcal{U}(N)} \inf_{\{w_1, \ldots, w_{q-1}\}} D\left((\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0\right) \leq 2 \left(\Delta_q(M, x_0)\right)^{n-q}.$$ 

We shall use the lower bound,

$$\frac{1}{2} \Delta_q(M, x_0) \leq \sup_{U \in \mathcal{U}(N)} \inf_{\{w_1, \ldots, w_{q-1}\}} D\left((\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0\right), \quad (2.3)$$

in the proof of Theorem 1.2.

Another ingredient in the proof of Theorem 1.2 is showing that $\Delta_q(M, x_0)$ assumes the generic value with respect to choices of non-degenerate sets of linear forms $\{w_1, \ldots, w_{q-1}\}$ in its definition:

**Proposition 2.10** Let $M$ be a real hypersurface in $\mathbb{C}^n$, and let $x_0 \in M$. The infimum in the definition of $\Delta_q(M, x_0)$ is achieved and equal to the generic value,

$$\Delta_q(M, x_0) = \inf_{\{w_1, \ldots, w_{q-1}\}} \Delta_1\left((\mathcal{I}(M), w_1, \ldots, w_{q-1}), x_0\right) = \text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} \Delta_1\left((\mathcal{I}(M), w_1, \ldots, w_{q-1}), x_0\right).$$

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where \( \{w_1, \ldots, w_{q-1}\} \) is a non-degenerate set of linear forms in \( \mathcal{O}_{x_0} \), \( \mathcal{I}(M) \), \( w_1, \ldots, w_{q-1} \) is the ideal in \( C_x^\infty \) generated by \( \mathcal{I}(M) \), \( w_1, \ldots, w_{q-1} \), and the infimum and the generic value are both taken over all such non-degenerate sets \( \{w_1, \ldots, w_{q-1}\} \) of linear forms in \( \mathcal{O}_{x_0} \).

**Proof** We consider two cases. First, if

\[
\Delta_q(M, x_0) = \inf_{\{w_1, \ldots, w_{q-1}\}} \Delta_1\left(\mathcal{I}(M), w_1, \ldots, w_{q-1}, x_0\right) = \infty,
\]

then clearly \( \Delta_1\left(\mathcal{I}(M), w_1, \ldots, w_{q-1}, x_0\right) = \infty \) for every non-degenerate set of linear forms \( \{w_1, \ldots, w_{q-1}\} \) in \( \mathcal{O}_{x_0} \), so

\[
\Delta_q(M, x_0) = \text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} \Delta_1\left(\mathcal{I}(M), w_1, \ldots, w_{q-1}, x_0\right),
\]

i.e., the infimum is achieved and equal to the generic value. Second, assume \( \Delta_q(M, x_0) = t < \infty \). One of the equivalent ways of defining \( \Delta_q(M, x_0) \) is as

\[
\Delta_q(M, x_0) = \inf_{\{w_1, \ldots, w_{q-1}\}} \sup_{\phi \in \mathcal{C}(n, x_0)} \inf_{g \in \mathcal{I}(M), \{w_1, \ldots, w_{q-1}\}} \frac{\text{ord}_0 \varphi^* g}{\text{ord}_0 \varphi}.
\]

Obviously, the supremum is realized here by curves that set \( w_1, \ldots, w_{q-1} \) to zero as these functions have the lowest order of vanishing, namely 1. It follows that

\[
\Delta_q(M, x_0) = \text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} \Delta_1\left(\mathcal{I}(M), w_1, \ldots, w_{q-1}, x_0\right),
\]

and the infimum over all non-degenerate sets \( \{w_1, \ldots, w_{q-1}\} \) of linear forms in \( \mathcal{O}_{x_0} \) is indeed achieved and equal to the generic value. \( \square \)

We shall now state as a corollary the equivalent result for \( \Delta_q(\mathcal{I}, x_0) \), which is needed for the proof of Theorem 1.1:

**Corollary 2.11** If \( \mathcal{I} \) is any ideal in \( \mathcal{O}_{x_0} \), the infimum in the definition of \( \Delta_q(\mathcal{I}, x_0) \) is achieved and equal to the generic value,

\[
\Delta_q(\mathcal{I}, x_0) = \text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} \Delta_1\left(\mathcal{I}, w_1, \ldots, w_{q-1}, x_0\right).
\]

**Proof** The proof of Proposition 2.10 applies verbatim with \( \Delta_q(\mathcal{I}, x_0) \) replacing \( \Delta_q(M, x_0) \) and \( \mathcal{I} \) replacing \( \mathcal{I}(M) \). \( \square \)

For the proof of Theorem 1.1, we also need to state part of D’Angelo’s Theorem 2.7 from p. 622 of [10]:

**Theorem 2.12** Let \( \mathcal{I} \) be an ideal of \( \mathcal{O}_{x_0} \). Then \( \Delta_1(\mathcal{I}, x_0) \leq D(\mathcal{I}, x_0) \).
Finally, we turn our attention to D’Angelo’s property P, which he defined on p. 631 of [10]:

**Definition 2.13** Let $M$ be a real hypersurface of $\mathbb{C}^n$, and let $x_0$ be a point of finite type on $M$. We suppose that $\Delta_1(M, x_0) < k$. Let $j_{k, x_0} r = r_k = \text{Re}(h) + ||f||^2 - ||g||^2$ be a holomorphic decomposition at $x_0$ of the $k$-jet of the defining function $r$ of $M$. We say that $M$ satisfies property P at $x_0$ if for every holomorphic curve $\varphi \in \mathcal{C}(n, x_0)$ for which $\varphi^* h$ vanishes, the following two conditions are satisfied:

(i) $\text{ord}_0 \varphi^* r$ is even, i.e., $\text{ord}_0 \varphi^* r = 2a$, for some $a \in \mathbb{N}$; 

(ii) $\left( \frac{d}{dt} \right)^a \left( \frac{d}{dt} \right)^a \varphi^* r(0) \neq 0$.

**Remarks**

1. Due to Theorem 2.3 (iv), the finite determination property of $\Delta_1(M, x_0)$ for all $1 \leq q < n$, this definition is independent of $k$ provided $k$ is large enough.

2. Since the D’Angelo type does not depend on the coordinate system and is always greater than or equal to 2, no holomorphic curve with $\varphi^* h \not\equiv 0$ can realize the supremum. The function $h$ has order 1 at $x_0$ and can be mapped to $z_n$ via a change of coordinates. Hence holomorphic curves satisfying $\varphi^* h \not\equiv 0$ are irrelevant to the type consideration.

Let us now define the $q$ version of D’Angelo’s property P, $q$-positivity, the hypothesis that appears in Theorem 1.2.

**Definition 2.14** Let $M$ be a real hypersurface of $\mathbb{C}^n$, and let $x_0 \in M$ be such that $\Delta_q(M, x_0) < k$. Let $j_{k, x_0} r = r_k = \text{Re}(h) + ||f||^2 - ||g||^2$ be a holomorphic decomposition at $x_0$ of the $k$-jet of the defining function $r$ of $M$. We say that $M$ is $q$-positive at $x_0$ if for every holomorphic curve $\varphi \in \mathcal{C}(n, x_0)$ for which $\varphi^* h$ vanishes and such that the image of $\varphi$ locally lies in the zero locus of a non-degenerate set of linear forms $\{ w_1, \ldots, w_{q-1} \}$ at $x_0$, the following two conditions are satisfied:

(i) $\text{ord}_0 \varphi^* r$ is even, i.e., $\text{ord}_0 \varphi^* r = 2a$, for some $a \in \mathbb{N}$; 

(ii) $\left( \frac{d}{dt} \right)^a \left( \frac{d}{dt} \right)^a \varphi^* r(0) \neq 0$.

**Remark** The change here versus D’Angelo’s property P is that we ask that his conditions be satisfied only for the holomorphic curves that come into the computation of $\Delta_q(M, x_0)$.

The reason D’Angelo introduced property P is that it allowed him to prove the following result, which appears as Theorem 5.3 on p. 631 of [10]:

**Theorem 2.15** Suppose that $M$ satisfies property P at $x_0$. Then

$$\Delta_1(M, x_0) = 2 \Delta_1(\mathcal{I}(U, x_0), x_0),$$

i.e., the upper bound in Eq. (2.1) is achieved.

A pseudoconvex domain of finite D’Angelo type has property P as do the hypersurfaces corresponding to truncations of the defining function at $x_0$ of any order higher.
than the type. Before formally stating this result, we shall state a proposition that appeared as Proposition 2 on p. 138 of [12], which justifies why such a result ought to be true.

**Proposition 2.16** Suppose that $M$ is a pseudoconvex hypersurface containing the origin with local defining function $r$. Suppose further that $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ is a parameterized holomorphic curve such that the Taylor series for $\varphi^* r$ satisfies

(i) $\text{ord}_0 \varphi^* r = m$

(ii) $\left( \frac{d}{dt} \right)^a \varphi^* r(0) = 0 \quad a \leq m.$

Then the order of vanishing $m = 2k$ is even, and the coefficient of $|t|^{2k}$ in $\varphi^* r$ is positive.

**Remark** Condition (ii) is eliminating pure terms up to and including of order $m$, which is the vanishing order of $\varphi^* r$. The same is achieved via the requirement that $\varphi^* h \equiv 0$ in the definition of D’Angelo’s property $P$ as well as in the definition of $q$-positivity.

The following result appears on p. 632 of [10]:

**Proposition 2.17** Suppose $M$ is pseudoconvex near $x_0$, and that $\Delta_1(M, x_0)$ is finite. Then $M$ and $M_k$, the hypersurface corresponding to the truncation of order $k$ of the defining function at $x_0$, satisfy property $P$ at $x_0$ for all sufficiently large $k$.

It is easy to see from Proposition 2.16 that the equivalent result should hold for $q$-positivity as well:

**Proposition 2.18** Suppose $M$ is pseudoconvex near $x_0$, and that $\Delta_q(M, x_0)$ is finite. Then $M$ and $M_k$, the hypersurface corresponding to the truncation of order $k$ of the defining function at $x_0$, are $q$-positive at $x_0$ for all sufficiently large $k$.

### 3 Catlin $q$-Type

Catlin wished to avoid having to characterize the order of contact of a holomorphic variety $V^q$ of complex dimension $q$ with the boundary of the domain along the singular locus of the variety, which can be considerably more complicated when $q > 1$ than for holomorphic curves. To that end, he introduced in [4] a numerical function $D_q(M, x_0)$ that measures the order of contact of varieties $V^q$ with $M$ only along generic directions. Following D’Angelo in [10], he also defined such an order of contact $D_q(I, x_0)$ for an ideal $I$ of holomorphic germs in $O_{x_0}$.

Let $V^q$ be the germ of a holomorphic variety of complex dimension $q$ passing through $x_0$. Let $G^{n-q+1}$ be the set of all $(n-q+1)$-dimensional complex planes through $x_0$. Consider the intersection $V^q \cap S$ for $S \in G^{n-q+1}$. For a generic, thus open and dense, subset $\tilde{W}$ of $G^{n-q+1}$, $V^q \cap S$ consists of finitely many irreducible one-dimensional components $V^q_{S,k}$ for $k = 1, \ldots, P$. Let us parameterize each such germ of a curve by some open set $U_k \ni 0$ in $\mathbb{C}$. Thus, $\gamma^k_S : U_k \to V^q_{S,k},$ where $\gamma^k_S(0) = x_0$. For every holomorphic germ $f \in O_{x_0}$, consider the quantity...
\[
\tau(f, V^q \cap S) = \max_{k=1, \ldots, P} \frac{\text{ord}_0 (\gamma^k_S) \ast f}{\text{ord}_0 \gamma^k_S}
\]

Likewise, for \( r \) the defining function of a real hypersurface \( M \) in \( \mathbb{C}^n \) passing through \( x_0 \), set

\[
\tau(V^q \cap S, x_0) = \max_{k=1, \ldots, P} \frac{\text{ord}_0 (\gamma^k_S) \ast r}{\text{ord}_0 \gamma^k_S}
\]

In Sect. 3 of [4], Catlin showed \( \tau(V^q \cap S, x_0) \) assumes the same value for all \( S \) in a generic subset \( \tilde{W} \) of planes. Therefore, he defined

\[
\tau(V^q, x_0) = \text{gen.val}_{S \in \tilde{W}} \{ \tau(V^q \cap S) \}
\]

and

\[
\tau(\mathcal{I}, V^q) = \min_{f \in \mathcal{I}} \tau(f, V^q).
\]

**Definition 3.1** Let \( \mathcal{I} \) be an ideal of holomorphic germs at \( x_0 \). Then the Catlin \( q \)-type of the ideal \( \mathcal{I} \) is given by

\[
D_q(\mathcal{I}, x_0) = \sup_{V^q} \{ \tau(\mathcal{I}, V^q) \},
\]

where the supremum is taken over the set of all germs of \( q \)-dimensional holomorphic varieties \( V^q \) passing through \( x_0 \).

In the same Sect. 3 of [4], Catlin showed \( \tau(V^q \cap S, x_0) \) assumes the same value for all \( S \) in a generic subset \( \tilde{W} \) of planes, so he defined

\[
\tau(V^q, x_0) = \text{gen.val}_{S \in \tilde{W}} \{ \tau(V^q \cap S, x_0) \}.
\]

**Definition 3.2** Let \( M \) be a real hypersurface in \( \mathbb{C}^n \). The Catlin \( q \)-type at \( x_0 \in M \) is given by

\[
D_q(M, x_0) = \sup_{V^q} \{ \tau(V^q, x_0) \},
\]

where the supremum is taken over the set of all germs of \( q \)-dimensional holomorphic varieties \( V^q \) passing through \( x_0 \).

Clearly, \( D_1(M, x_0) = D_1(M, x_0) \) as there is only one \( n \)-dimensional complex plane passing through \( x_0 \) in \( \mathbb{C}^n \).

Some more explanations are in order regarding Catlin’s construction. We have claimed \( \tau(f, V^q \cap S) \) and \( \tau(V^q \cap S, x_0) \) are constant on a generic set \( \tilde{W} \) of \( (n - q + 1) \)-dimensional complex planes \( S \) through \( x_0 \). The quantities \( \tau(f, V^q \cap S) \) and \( \tau(V^q \cap S, x_0) \) are computed by looking at the normalized vanishing orders of \( f \) and \( r \) respectively along the curves \( V^q_{S,k} \) for \( k = 1, \ldots, P \) that represent the intersection of
$V^q$ with $S$. In fact, the number of curves in the intersection, $P$ is the same for all $S \in \tilde{W}$, and furthermore, the curves $V^q_{S_a,k}$ can be smoothly parameterized via a parameter $a = (a_1, \ldots, a_N)$ for $N = (n - q + 1)(q - 1)$, the dimension of $\tilde{W}$. Proposition 3.1 (ii) on p. 140 of [4] states that as $S_a \in \tilde{W}$ varies smoothly, the intersection curves $V^q_{S_a,k}$ do as well, and their number stays constant.

In his proof of Proposition 3.1 from [4], Catlin has to remove three different sets $W_1$, $W_2$, and $W_3$ from $G^{n-q+1}$ in order to arrive at his generic set $\tilde{W}$ on which such strong conclusions hold. To the germ of the variety $V^q$, there corresponds a prime ideal $I$ in the ring $O_{x_0}$ of all germs of holomorphic functions that vanish on $V^q$. Catlin uses Gunning’s Local Parameterization Theorem from p. 16 of [14] in order to construct a set of canonical equations for $V^q$. This construction involves choosing a special set of coordinates where the generators of the ideal simultaneously satisfy the Weierstrass Preparation Theorem with respect to the variables that give the regular system of parameters that $I$ has as a prime ideal in the regular local ring $O_{x_0}$. The intersection $V^q \cap S$ is ill behaved where $V^q$ does not have pure dimension $q$ as the intersection might consist of points rather than curves as well as along the singular locus of $V^q$. To remove both, Catlin constructs a conic variety $X'$ whose defining equation consists of the product of the discriminants of the Weierstrass polynomials that give the canonical equations for $V^q$ (these discriminants capture the singular locus of $V^q$) with the additional generator that gives the non-pure-dimensional part of $V^q$. $W_1$ consists of all $(n - q + 1)$-dimensional complex planes that intersect $X'$.

Additionally, for the intersection $V^q \cap S$ to behave well, a good notion of transversality has to apply. Transversality cannot be tested well for curves, which is what $V^q \cap S$ yields generically, but it can be tested very well for points. To reduce the intersection to points, Catlin looks at the conic variety corresponding to $V^q$, which he calls $V'$. The variety $V'$ captures the tangent cone of $V^q$, exactly where singularities of $V^q$ manifest themselves as the dimension of the tangent cone jumps at a singular point. $V'$ still has dimension $q$. Consider $\tilde{V}$, the projective variety in $\mathbb{P}^{n-1}$ corresponding to $V'$. $\tilde{V}$ has dimension $q - 1$. For every $S \in G^{n-q+1}$, there corresponds a projective plane $\tilde{S}$ of dimension $n - q$ in $\mathbb{P}^{n-1}$. Generically, $\tilde{V} \cap \tilde{S}$ consists of finitely many points $\tilde{z}^1, \ldots, \tilde{z}^D$ with transverse intersections, meaning that each $\tilde{z}^i$ is a smooth point of $\tilde{V}$ and the tangent spaces satisfy $T_{\tilde{z}^i} \tilde{V} \cap T_{\tilde{z}^i} \tilde{S} = 0$ for $i = 1, \ldots, D$. Let $W_2$ be the subset of $G^{n-q+1}$ where this generic behavior does not take place.

Finally, the construction of $W_1$ involved the use of canonical equations for $V^q$. Hence variables $z_{q+1}, \ldots, z_n$ give the regular system of parameters corresponding to the pure $q$-dimensional part of the variety $V^q$. The variable $z_q$ corresponds to the additional generator that gives the non-pure-dimensional part of $V^q$. The $(n - q + 1)$-dimensional complex plane $S$ is defined by the linear equations $\sum_{j=1}^n a_j z_j = 0$ for $i = 1, \ldots, q - 1$, which need to be linearly independent. A $(q - 1) \times (q - 1)$ minor of $(a_j^i)$ should thus have full rank. On the other hand, for the intersection $V^q \cap S$ to behave well, this $(q - 1) \times (q - 1)$ minor should be exactly $(a_j^i)_{1 \leq i, j \leq q-1}$ with respect to the complementary variables $z_1, \ldots, z_{q-1}$. Therefore, Catlin sets

$$W_3 = \left\{ S \in G^{n-q+1} \mid \det(a_j^i)_{1 \leq i, j \leq q-1} = 0 \right\}.$$
We shall now state the rest of the results from [4] that play a role in the proofs of Theorems 1.1 and 1.2. The following is Catlin’s Theorem 3.7 on p. 154 of [4]:

**Theorem 3.3** Let \( I \) be an ideal in \( \mathcal{O}_{x_0} \). Then

\[
\text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} D\left((I, w_1, \ldots, w_{q-1}), x_0\right) \leq \prod_{i=q}^n D_i(I, x_0),
\]

where the generic value is computed over all non-degenerate sets \( \{w_1, \ldots, w_{q-1}\} \) of linear forms in \( \mathcal{O}_{x_0} \).

**Remark** In the context of Catlin’s definitions, the \((n - q + 1)\)-dimensional complex plane \( S \) through \( x_0 \) is precisely the zero locus of the non-degenerate set of linear forms \( \{w_1, \ldots, w_{q-1}\} \).

Since \( D_k(I, x_0) \leq D_q(I, x_0) \) for all \( k \geq q \), we obtain the following corollary to Theorem 3.3:

**Corollary 3.4** Let \( I \) be an ideal in \( \mathcal{O}_{x_0} \). Then

\[
\text{gen.val}_{\{w_1, \ldots, w_{q-1}\}} D\left((I, w_1, \ldots, w_{q-1}), x_0\right) \leq \left(D_q(I, x_0)\right)^{n-q+1}.
\]

In case \( \Delta_q(M, x_0) = t < \infty \), the truncation \( r_k \) of the defining function \( r \) of \( M \) at \( x_0 \) of order \( k = \lceil t \rceil \) has the holomorphic decomposition \( r_k = \Re(h) + \|f\|^2 - \|g\|^2 \), and \( M \) is \( q \)-positive at \( x_0 \), we would like to relate \( \tau(V^q, x_0) \) to \( \tau(I(U), V^q) \) for any unitary matrix \( U \in \mathcal{U}(N) \) and any \( q \)-dimensional complex variety \( V^q \). The reader may wish to check by going through the proof of D’Angelo’s Theorem 5.3 on p. 631 of [10], which we stated here as Theorem 2.15, that \( q \)-positivity at \( x_0 \) implies

\[
\tau(V^q, x_0) \geq 2 \tau(I(U), V^q) \quad \forall U, \forall V^q.
\]

Catlin uses this reasoning on p. 156 of [4] in order to finish the proof of his Theorem 3.4 without formally defining \( q \)-positivity. Instead, he employs this argument for a pseudoconvex domain, where we have shown that \( q \)-positivity holds if the D’Angelo \( q \)-type is finite. Catlin assumes that \( D_q(M, x_0) \) is finite instead.

### 4 Comparing \( \Delta_q \) with \( D_q \)

The proof of Theorem 1.1 comprises two results.

**Proposition 4.1** Let \( I \) be any ideal in \( \mathcal{O}_{x_0} \). For any \( 1 \leq q \leq n \), \( D_q(I, x_0) \leq \Delta_q(I, x_0) \).

**Proof** Let \( \Delta_q(I, x_0) = t < \infty \); else, the estimate is trivially true. Assume \( D_q(I, x_0) > t \). Since \( D_q(I, x_0) \) is defined as the supremum over all \( q \)-dimensional...
holomorphic varieties passing through \( x_0 \) of \( \tau(\mathcal{I}, V^q) \), there exists such a holomorphic variety \( V^q \) for which
\[
\tau(\mathcal{I}, V^q) = \min_{f \in \mathcal{I}} \tau(f, V^q) = \min_{f \in \mathcal{I}} \text{gen.val}_{S \in \tilde{W}} \{ \tau(f, V^q \cap S) \} = t' > t,
\]
but as shown in Proposition 2.11, \( \Delta_q(\mathcal{I}, x_0) \) is generic over the choice of \( S \), so the curves in \( V^q \cap S \) already enter into the computation of \( \Delta_q(\mathcal{I}, x_0) \). Therefore, \( \Delta_q(\mathcal{I}, x_0) \geq t' > t \), a contradiction.

\[ \square \]

**Proposition 4.2** Let \( \mathcal{I} \) be any ideal in \( \mathcal{O}_{x_0} \). For any \( 1 \leq q \leq n \),
\[
\Delta_q(\mathcal{I}, x_0) \leq (D_q(\mathcal{I}, x_0))^{n-q+1}.
\]

**Proof** Let \( \{ w_1, \ldots, w_{q-1} \} \) be any non-degenerate set of linear forms in \( \mathcal{O}_{x_0} \). First, we apply Theorem 2.12 to the ideal \( (\mathcal{I}, w_1, \ldots, w_{q-1}) \) to obtain
\[
\Delta_1\left( (\mathcal{I}, w_1, \ldots, w_{q-1}), x_0 \right) \leq D\left( (\mathcal{I}, w_1, \ldots, w_{q-1}), x_0 \right).
\]
Next, we take the generic value over all non-degenerate sets \( \{ w_1, \ldots, w_{q-1} \} \) of linear forms in \( \mathcal{O}_{x_0} \):
\[
\text{gen.val}_{\{ w_1, \ldots, w_{q-1} \}} \Delta_1\left( (\mathcal{I}, w_1, \ldots, w_{q-1}), x_0 \right) \leq \text{gen.val}_{\{ w_1, \ldots, w_{q-1} \}} D\left( (\mathcal{I}, w_1, \ldots, w_{q-1}), x_0 \right).
\]
By Corollary 2.11,
\[
\Delta_q(\mathcal{I}, x_0) = \text{gen.val}_{\{ w_1, \ldots, w_{q-1} \}} \Delta_1\left( (\mathcal{I}, w_1, \ldots, w_{q-1}), x_0 \right),
\]
while by Corollary 3.4,
\[
\text{gen.val}_{\{ w_1, \ldots, w_{q-1} \}} D\left( (\mathcal{I}, w_1, \ldots, w_{q-1}), x_0 \right) \leq (D_q(\mathcal{I}, x_0))^{n-q+1}.
\]

\[ \square \]

**Proof of Theorem 1.1** Proposition 4.1 proves the left-hand side inequality, while Proposition 4.2 proves the right-hand side inequality.

As mentioned in the Introduction, if \( q = n \), Theorem 1.1 implies \( \Delta_n(\mathcal{I}, x_0) = D_n(\mathcal{I}, x_0) \) for every ideal \( \mathcal{I} \) in \( \mathcal{O}_{x_0} \). From the proof of Proposition 4.2, it follows that
\[
\Delta_n(\mathcal{I}, x_0) = D_n(\mathcal{I}, x_0) = \text{gen.val}_{\{ w_1, \ldots, w_{n-1} \}} D\left( (\mathcal{I}, w_1, \ldots, w_{n-1}), x_0 \right). \tag{4.1}
\]
Applying Catlin’s Theorem 3.3 together with Eq. (4.1) allows us to give the corresponding result to Corollary 3.4 for an ideal \( \mathcal{I}(U) = (h, f - Ug) \) in \( \mathcal{O}_{x_0} \) arising from

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a holomorphic decomposition \( r_k = \text{Re}\{h\} + ||f||^2 - ||g||^2 \) of the truncation \( r_k \) of the defining function \( r \) of the real hypersurface \( M \) at \( x_0 \) when \( \Delta_q(M, x_0) = t \) and \( k = \lceil t \rceil \):

**Corollary 4.3** Let \( M \) be a real hypersurface in \( \mathbb{C}^n \), let \( r \) be a defining function for \( M \), and let \( x_0 \in M \). If \( \Delta_q(M, x_0) = t \), \( r_k = \text{Re}\{h\} + ||f||^2 - ||g||^2 \) is the holomorphic decomposition of the truncation \( r_k \) of the defining function \( r \) for \( k = \lceil t \rceil \), and \( \mathcal{I}(U) = (h, f - U_g) \) is an ideal in \( \mathcal{O}_{x_0} \) corresponding to this holomorphic decomposition for \( U \) a unitary matrix, then

\[
\text{gen.val}_{[w_1, \ldots, w_{q-1}]} D\left( (\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0 \right) \leq \left( D_q(\mathcal{I}(U), x_0) \right)^{n-q}.
\]

**Proof** \( r_k = \text{Re}\{h\} + ||f||^2 - ||g||^2 \) defines a real hypersurface in \( \mathbb{C}^n \), so the holomorphic function \( h \) must contain a term of first order; else, the gradient of \( r_k \) would be zero at \( x_0 \). Therefore, the ideal \( \mathcal{I}(U) = (h, f - U_g) \) contains an element with a term of first order. From Theorem 3.3, we know

\[
\text{gen.val}_{[w_1, \ldots, w_{q-1}]} D\left( (\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0 \right) \leq \prod_{i=q}^n D_i(\mathcal{I}(U), x_0),
\]

while Eq. (4.1) gives us

\[
\Delta_n(\mathcal{I}(U), x_0) = D_n(\mathcal{I}(U), x_0) = \text{gen.val}_{[w_1, \ldots, w_{n-1}]} D\left( (\mathcal{I}(U), w_1, \ldots, w_{n-1}), x_0 \right).
\]

The fact that \( \mathcal{I}(U) \) contains an element with a term of first order means that generically \( D\left( (\mathcal{I}(U), w_1, \ldots, w_{n-1}), x_0 \right) = 1 \), which implies \( D_n(\mathcal{I}(U), x_0) = 1 \). Since \( D_k(\mathcal{I}(U), x_0) \leq D_q(\mathcal{I}(U), x_0) \) for all \( k \geq q \), the result follows. \( \square \)

**Remark** The claim \( D_n(\mathcal{I}(U), x_0) = 1 \) appears at the top of p. 156 of [4].

Each part of Theorem 1.2 will now be proven in a separate proposition.

**Proposition 4.4** Let \( \Omega \) in \( \mathbb{C}^n \) be a domain with \( C^\infty \) boundary. Let \( x_0 \in b\Omega \) be a point on the boundary of the domain. For any \( 1 \leq q < n \), \( D_q(b\Omega, x_0) \leq \Delta_q(b\Omega, x_0) \).

**Proof** Modulo notational changes, the same proof as for Proposition 4.1 applies here, but we give it again for completeness. If \( \Delta_q(b\Omega, x_0) = \infty \), then the estimate is obviously true. We thus restrict ourselves to the case when \( \Delta_q(b\Omega, x_0) = t < \infty \). Assume the estimate is false, i.e., \( D_q(b\Omega, x_0) > t \). Since \( D_q(M, x_0) \) is defined as the supremum over all \( q \)-dimensional holomorphic varieties passing through \( x_0 \) of \( \tau(V^q, x_0) \), there exists such a holomorphic variety \( V^q \) for which

\[
\tau(V^q, x_0) = \text{gen.val} \left\{ \tau(V^q \cap S, x_0) \right\} = t' > t,
\]

but as we have shown in Proposition 2.10, \( \Delta_q(b\Omega, x_0) \) is generic over the choice of \( S \), so the curves in \( V^q \cap S \) enter into the computation of \( \Delta_q(b\Omega, x_0) \). Therefore,
\[ \Delta_q(b\Omega, x_0) \geq t' > t, \]

which is obviously a contradiction. \[ \square \]

**Proposition 4.5** Let \( \Omega \) in \( C^n \) be a domain with \( C^\infty \) boundary. Let \( x_0 \in b\Omega \) be a point on the boundary of the domain. For any \( 1 \leq q < n \), if the domain is \( q \)-positive at \( x_0 \), then

\[ \Delta_q(b\Omega, x_0) \leq 2 \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q}. \]

**Proof** Since the domain is \( q \)-positive at \( x_0 \), as shown at the end of Sect. 3, we obtain

\[ \tau(V^q, x_0) \geq 2 \tau(\mathcal{I}(U), V^q) \ \forall \ U, \ \forall V^q. \]

Therefore,

\[ D_q(b\Omega, x_0) = \sup_{V^q} \{ \tau(V^q, x_0) \} \geq 2 \sup_{V^q} \{ \tau(\mathcal{I}(U), V^q) \} = 2 D_q(I(U), x_0) \ \forall \ U. \]

In other words,

\[ \frac{D_q(b\Omega, x_0)}{2} \geq D_q(I(U), x_0) \ \forall \ U \]

and

\[ \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q} \geq \left( D_q(I(U), x_0) \right)^{n-q} \ \forall \ U. \]

We can now take the supremum on the right over all unitary matrices \( U \in \mathcal{U}(N) \) and use Corollary 4.3 to obtain

\[ \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q} \geq \sup_{U \in \mathcal{U}(N)} \left( D_q(I(U), x_0) \right)^{n-q} \]

\[ \geq \sup_{U \in \mathcal{U}(N)} \operatorname{gen.val}_{[w_1, \ldots, w_{q-1}]} D \left( (\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0 \right). \]

By Proposition 2.8 and Eq. (2.3),

\[ \left( \frac{D_q(b\Omega, x_0)}{2} \right)^{n-q} \geq \sup_{U \in \mathcal{U}(N)} \inf_{[w_1, \ldots, w_{q-1}]} D \left( (\mathcal{I}(U), w_1, \ldots, w_{q-1}), x_0 \right) \]

\[ \geq \frac{1}{2} \Delta_q(b\Omega, x_0). \]

\[ \square \]
Proof of Theorem 1.2 We put together the results of Proposition 4.5 with Proposition 4.4 and the fact mentioned in Sect. 2 that a pseudoconvex domain is $q$-positive when $\Delta_q(b\Omega, x_0) < \infty$, namely Proposition 2.18.

Let us now address the issue of sharpness for Theorem 1.2 (ii) via an example. Corollary 4.3, a consequence of Catlin’s Theorem 3.3, is an essential part of the proof of Proposition 4.5 and is responsible for the jump in power that destroys any chance for this type of proof to yield the sharp estimate when $q = 1$. Yet, a result like Corollary 4.3 cannot be avoided because it relates the only truly well-behaved quantity in the problem $D \left( (I(U), w_1, \ldots, w_{q-1}), x_0 \right)$, which is upper semi-continuous with respect to $U$ and generic over the choice of $\{w_1, \ldots, w_{q-1}\}$, to $D_q$. Note that upper semi-continuity is necessary because it allows one to handle the supremum over all unitary matrices produced by polarization, exactly the technique that reduces the problem from the ill-behaved local ring $C_{x_0}^\infty$ to $\mathcal{O}_{x_0}$. We shall now show by example that Corollary 4.3 and Theorem 3.3 are both sharp. Let $r = Re\{z_3\} + |z_1|^2 + |z_2|^2$ and $n = 3$. Since $g = 0$ in the holomorphic decomposition, no unitary matrices appear in its corresponding holomorphic ideal $I = (z_3, z_1^3, z_2^3)$. $\Delta_1(I, 0) = D_1(I, 0) = 3$ and $D(I, 0) = 3^2 = 9$, so $D(I, 0) = (D_1(I, 0))^2$ for $q = 1$. To show Theorem 3.3 is also sharp, we drop $z_3$ from this ideal, and consider $I = (z_1^3, z_2^3)$ for $n = 2$. Once again, $\Delta_1(I, 0) = D_1(I, 0) = 3$, and $D(I, 0) = 9$. By Eq. (4.1) above, $D_2(I, 0) = gen.val_{\{w\}} D \left( (I, w), 0 \right) = 3$, so when $q = 1$

$D(I, 0) = 9 = D_1(I, 0) \cdot D_2(I, 0)$.

For comparison, see Example 2.14.1 on p. 624 of [10] used by D’Angelo to show the inequalities in his Theorem 2.7 were sharp.

We now compute $\Delta_q$ and $D_q$ for $q = 1, 2, 3$ for three ideals of germs of holomorphic functions of three variables. Trivially, $\Delta_1 = D_1$ and $\Delta_3 = D_3$:

(a) $I = (z_1^3 + z_2^3 - z_3^3)$. Here $\mathcal{V}(I)$ is a surface, hence $\Delta_1(I, 0) = D_1(I, 0) = \infty$ and $\Delta_2(I, 0) = D_2(I, 0) = \infty$, while $\Delta_3(I, 0) = D_3(I, 0) = gen.val_{\{w_1, w_2\}} D \left( (I, w_1, w_2), 0 \right) = 3$.

(b) $I = (z_1^3 + z_2^3 - z_3^3, (z_1 - z_3)^m)$, where $m \in \mathbb{N}$, $m > 3$. Here $\mathcal{V}(I)$ is a curve, hence $\Delta_1(I, 0) = D_1(I, 0) = \infty$. $\Delta_2(I, 0) = D_2(I, 0) = m > 3$ is obtained by using $\mathcal{V}((z_1^3 + z_2^3 - z_3^3))$, the variety corresponding to the ideal $\tilde{I} = (z_1^3 + z_2^3 - z_3^3)$. $\Delta_3(I, 0) = D_3(I, 0) = gen.val_{\{w_1, w_2\}} D \left( (I, w_1, w_2), 0 \right) = 3$.

(c) Let $I$ be any ideal generated by homogeneous polynomials in $z_1, z_2$, and $z_3$ all of degree $p$ satisfying that $\mathcal{V}(I) = \{0\}$. In this case, $\Delta_1(I, 0) = D_1(I, 0) = p$, $\Delta_2(I, 0) = D_2(I, 0) = p$, while $\Delta_3(I, 0) = D_3(I, 0) = gen.val_{\{w_1, w_2\}} D \left( (I, w_1, w_2), 0 \right) = p$.

In these examples $\Delta_2 = D_2$ as well, but it is the authors’ hope that a future investigation will reveal whether equality holds in general or is merely an artifact here of the difficulty of computing $D_q$. 

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