Abstract

It is known that a non-local complex fermion can be written in terms of two Majorana fermions. We exploit this fact to explain the system of two Majorana zero modes bound to a vortex and an anti-vortex, on the surface of a topological insulator in contact with an s-wave superconductor, as a non-local complex fermion. Although the current density of a single zero mode vanishes, by starting with a wave packet consisted of the positive and negative energy complex fermions, we specify that a time-dependent oscillatory motion emerges in the system. We also show that the amplitude and frequency of the oscillations depend on the relative distance of those two zero modes. Therefore, the observation of this oscillatory motion can be considered as a signature of the Majorana zero modes. Also, as the frequency of such an oscillatory motion depends on the distance between the two zero modes, it can be adjusted to bring this frequency within the resolution of observations. Furthermore, we indicate that the predicted oscillatory current is the reminiscent of the zitterbewegung effect.

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1 Introduction

Unlike the standard solutions of the Dirac equation, i.e. the electron and positron, a solution of the Majorana equation is its own antiparticle. While such a particle has not been detected in particle physics, in solid states physics a pair of localized Majorana zero-energy modes (MZMs) have been predicted to reside at the core of vortices in the superconductor-topological insulator (STI) system [1–3]. Among different setups to realize MZMs, one can mention a p-wave superconductor with a non-relativistic kinetic term and a vortex order parameter [4]. From theoretical point of view, it is predictable that MZMs will have very important role as qubits in the topological quantum computing in future [5]. The susceptibility of qubit systems to decoherence, as the principal obstacle in realizing a scalable quantum computer, has led some efforts to detect MZMs emerging in topologically non-trivial superconducting phases in order to build fault-tolerance quantum computing. Indeed, in such a quantum computer, information is stored in a system of two MZMs located faraway from each other and thus, would be more protected against local perturbations that may cause quantum decoherence [6]. Also, the adiabatic braiding of MZMs can be used to perform qubit operations, while their fusions provide the means of qubit read out [7]. The enthusiasm about MZMs has led to many theoretical works on this subject in the literature, see, e.g. Refs. [10–15].

On the other hand, it is known that Majorana’s original work [16] was not restricted only to MZMs, which are static solutions of the Majorana equation. His approach was quite general and he introduced whole field covering the entire energy-range, called Majorana fermions (MFs), which are solutions of the Dirac equation in a specific representation. MFs are their own anti-particles and include the entire energy-range, and are the topic of research in particle physics context. One of the main candidate of being a MF is neutrino [17], and in rare nuclear decays, several highly sensitive experiments around the world are going on to search for any evidence to illustrate that neutrinos are MFs [18]. If neutrinos are MFs, the dilemma of smallness of their masses can then be explained via the see-saw mechanism [19–22], which predicts that there are two kinds of neutrinos. The light left-handed neutrinos and the heavy right-handed ones, where the higher the mass of the right-handed neutrinos, the lower the mass of the left-handed ones. In this respect, recently it has also been claimed that neutrinos can be regarded as the Bogoliubov quasi-particles (Bqp)s [23]. Nevertheless, Majorana neutrinos have been elusive so far in the particle physics experiment.

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1The mass of right-handed neutrinos can be as high as $10^{15}$ GeV in the see-saw scenario [18].
Despite the fact that MZMs are much studied on the theoretical side and the scientific excitement around those is on a par with gravitational waves and the Higgs boson, the issue of their detection has some difficulties. Several groups have reported the detection of Majorana bound states (MBSs) in nanowires through a measurement of a zero bias peak in tunneling spectroscopy experiments, see, e.g., Ref. [28]. However, the research community is still skeptical towards experimental findings in this regard [10,15]. The fact that MFs are their own anti-particles and charge-less makes those elusive for unambiguous detection in experiments, and subtle schemes are needed for indirect but conclusive signatures of their presence. In this vein, experiments are redesigned to probe other MZMs properties such as their particle-hole symmetry and spin, which might lead to clear results [29,32]. There are also many proposals to detect MFs based on interferometric structures, for example using a two-terminal Mach-Zehnder setup [33,34], or via measuring the energy of the bound state [3]. Moreover, neither the braiding nor the fusion of vortices has been realized in the laboratory, and a variety of theoretical plans have been proposed to demonstrate the appearance of non-Abelian anyons in a topological superconductor [35].

To address these advances, we investigate the current density of MBSs in the STI system. To perform such a task, we exploit the fact that one cannot talk about the state of a single MZM since it contains only half a fermion. The only physical observables are the fermionic occupation numbers, which consist of two MZMs [36,37]. By utilizing a fermionic field made of two MBSs bound to a vortex and an anti-vortex, we calculate the current density of the field and specify that the interference of the positive and negative energy parts of it leads to a time-dependent oscillatory motion in the system. We also show that the amplitude and frequency of the oscillations depend on the distance between those MZMs.

This issue is somehow similar to the zitterbewegung (ZBW) – a trembling/quiivering motion – which was introduced by Schrödinger [38,40]. Such a phenomenon has been studied, for instance, for a wave-packet made of positive and negative energy electrons [41], an electron in the presence of an external magnetic field in commutative space [42] and separately for the non-commutative phase-space [43,44] and references therein. In this respect, in Ref. [41], it has been shown that the current produced by the ZBW actually causes the intrinsic magnetic moment of a Dirac fermion particle and hence, the total magnetic moment of electron is produced by both the orbital and the intrinsic angular momenta with the correct gyromagnetic g factor. Thus, theoretical understanding of the ZBW may shed light on the nature of spin of elementary particles, see, e.g., Refs. [45–49]. However, despite increasing evidence that ZBW is real and, in principle, observable (e.g., in a Bose-Einstein condensate [50], in crystalline solids [51], semiconductors [52], graphene [53] and in silicene [54]), it has always been challenged and become an unpleasant aspect [55], since Foldy and Wouthuysen (FW) observed that, in absence of external fields, the ZBW can be avoided via the transformation employed by them [56]. Indeed, the FW transformation is a unitary transformation that transforms the Dirac Hamiltonian for a localized free particle into a (block-diagonalized) Hamiltonian in which positive and negative electron energies are decoupled, see, e.g., Refs. [57,58]. In this regard, as the ZBW goes hand in hand with the existence of negative and positive energy solutions, it has been claimed to be a frame-dependent concept, and hence not observable. Nevertheless, there have been plenty of works devoted to this issue even since then, see, e.g., Refs. [42,62,66] and references therein. However, in the case of electrons and positrons, the ultrahigh frequency of ZBW is $f_{\text{ZBW}} = \frac{2mc^2}{h} \sim 10^{20} \text{Hz}$ and amplitude given by the Compton wavelength $\lambda_c \sim 10^{-13} m$, whose direct measurement is still beyond experimental capabilities, and indeed, the lack of empirical evidence is due to the transient nature of wave-packet ZBW.

The outline of the work is as follows. In the next section, we briefly review the Dirac equation, the ZBW phenomenon and the FW transformation. Sec. 3 is devoted to the physics governing the surface of topological insulator in the proximity of an s-wave superconductor, wherein its corresponding Hamiltonian is studied. We indicate that in this system the ZBW of the relevant surface excitations can avoid the FW objection. Furthermore, we study the symmetries of the Hamiltonian and show that only the gauge symmetry is conserved when the superconducting pairing is present in the system. In Sec. 4, we right down the zero-energy solutions of the STI system Hamiltonian and indicate that a wave-packet consisted of two MBSs accommodates a non-vanishing current density whose spatial components represent the ZBW while its frequency can be adjusted for detection. Finally, we conclude the summary of the results in Conclusions.

### 2 Dirac Equation and Foldy-Wouthuysen Transformation

The Dirac Lagrangian for a free electron with mass $m$ is written as

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi,$$

where we have used the natural units $\hbar = 1 = e$, $\bar{\Psi} = \Psi^\dagger \gamma_0$ and $\gamma_\mu$s are the Dirac gamma matrices that, in the Weyl or

2Alternatively, in Refs. [59,60], it has been stated that the ZBW provides a physical interpretation for the complex phase factor in the Dirac wave function. Also, see Ref. [61], wherein it has been shown that the ZBW cannot be described by only one frequency.
chiral representations, are defined as
\[
\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \text{and} \quad \beta \equiv \gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}
\] (2)
with \(\sigma_i\)s and \(I_2\) as the \(2 \times 2\) Pauli matrices and the unit matrix, respectively. In addition in terms of the Dirac (gamma) matrices, the \(\gamma_5\) matrix is constructed as \(\gamma_5 = i\gamma_0\gamma^1\gamma^2\gamma^3\), and the matrix \(\alpha_i \equiv \gamma^0\gamma_i\) for \(i = 1, 2, 3\). Moreover, the \(\gamma^\mu\) matrices satisfy
\[
\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^\mu\nu,
\]
where \(\eta^\mu\nu\) \((\mu, \nu = 0, \ldots, 3)\) is the Minkowski metric in \((1 + 3)\) dimensions with the signature \(-2\). Now, the Dirac Hamiltonian is
\[
H \equiv \alpha \cdot p + \beta m
\] (3)
and the corresponding equation of motion is the celebrated Dirac equation [67]
\[
i\frac{\partial}{\partial t}\Psi = (\gamma_0\gamma^1\gamma^2\gamma^3 + \beta m)\Psi.
\] (4)

By defining the right-handed and left-handed projection operators \(P_+ = (1 + \gamma_5)/2\) and \(P_- = (1 - \gamma_5)/2\) that project \(\Psi\) to its right- and left-handed components respectively and assuming \(\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}\), the Dirac equation becomes
\[
\sigma \cdot p \psi_R + m \psi_L = \varepsilon \psi_R \quad (5)
\]
\[
-\sigma \cdot p \psi_L + m \psi_R = \varepsilon \psi_L \quad (6)
\]
with the eigenvalues \(\varepsilon = \pm \sqrt{p^2 + m^2}\). The formal similarity of this equation with the one attained from the Bardeen-Cooper-Schrieffer (BCS) Hamiltonian of superconductivity [68], led Nambu to transport the BCS theory to the physics of strong interactions [69,70]. For a Dirac particle, ZBW arises when one computes the time-dependence of the position operator in the Heisenberg picture, namely
\[
\frac{\partial x_k(t)}{\partial t} = i[H, x_k] = \alpha_i,
\]
where \(x_k(t)\) is the position operator at time \(t\). However, to obtain the value of the ZBW velocity, one needs to calculate the expectation value \(\langle \alpha \rangle\) while using a wave-packet consisted of both positive and negative energy solutions of the Dirac equation, see, e.g., Refs. [42–44] and references therein.

Nevertheless, FW first introduced the \(4 \times 4\) unitary transformation operator
\[
U = I_4 \cos \vartheta + \beta \alpha \cdot \hat{p} \sin \vartheta,
\] (7)
where \(\hat{p}\) is the unit vector in momentum space and \(\vartheta\) is an arbitrary angle. Then, they acted it on a fermion eigenket,
\[
\Psi \rightarrow \Psi' = U\Psi,
\]
and simultaneously on the free-fermion Dirac Hamiltonian operator in the Dirac-Pauli representation [3] in the bi-unitary fashion as
\[
H \rightarrow H' \equiv UHU^{-1} = U(\alpha \cdot p + \beta m)U^{-1}.
\]
Using the commutativity properties of the Dirac matrices, this new Hamiltonian reads
\[
H' = (\alpha \cdot p + \beta m)(\cos 2\vartheta - \beta \alpha \cdot \hat{p} \sin 2\vartheta),
\] (8)
and hence, one gets
\[
H' = \alpha \cdot p \left(\cos 2\vartheta - \frac{m}{|p|} \sin 2\vartheta\right) + \beta(m \cos 2\vartheta + |p| \sin 2\vartheta).
\]
However, by choosing
\[
\tan 2\vartheta = \frac{|p|}{m},
\]
this Hamiltonian reduces to
\[
H' = \beta \sqrt{m^2 + |p|^2},
\]
which is the Dirac Hamiltonian in the Newton-Wigner\cite{58,71} representation. Now, the commutator $[x_i(t), H']$ is equal to the group velocity $v_g$, and accordingly, it has been claimed that the ZBW motion is a representation-dependent concept, which vanishes in the Newton-Wigner representation.

However, it should be noted that the FW transformation is based on the Newton-Wigner work\cite{71}, which argues that a state, localized at a certain point, after a translation becomes orthogonal to all un-displaced states localized at that point. Whereas we will show that the main peculiarity of the present work is that the complex fermionic state is constructed out of two separated MBSs. On the other hand, it has long been argued that MFs are non-local (correlations incompatible with a local hidden variable theory) in nature\cite{72–74}. Therefore, the Newton-Wigner argument and the FW transformation would not be relevant to this case, and one can expect that the ZBW motion may not vanish there.

3 Topological Insulator Superconductor System

The superconductivity can be induced into the surface of a topological insulator in proximity of an s-wave superconductor. The Hamiltonian density of such a two dimensional system is written as\cite{2}

$$
\mathcal{H} = \psi^* \sigma \cdot p \psi - \mu (\psi^* \psi + \psi^* \psi) + \Delta \psi^* \psi^* + \Delta^* \psi \psi,
$$

(9)

where $\psi^*$ denotes complex conjugate of $\psi$, $\mu$ is the chemical potential, $p = p_x \pm ip_y$. Also, the order parameter is a scalar as

$$
\Delta (r) = v(r) e^{i\phi},
$$

(10)

where $v(r)$ is a real scalar function of the distance and $\phi$ is the polar angle.

This Hamiltonian can be written in two-component matrix notation as

$$
i \partial_t \psi = \left(\sigma \cdot p - \mu\right) \psi + \Delta i \sigma^2 \psi^*,
$$

(11)

with

$$
\psi = \begin{pmatrix} \psi_+ \\ \psi_-
\end{pmatrix},
$$

(12)

where $\sigma$ represents the two Pauli matrices ($\sigma^1, \sigma^2$). Although a static solution of Eq. (11) can easily be found, to study the time evolution of eigenstates, one needs to use two copies of (9) to get the Hamiltonian density as (in units $v_F = 1$, where $v_F$ is the Fermi velocity)

$$
\mathcal{H} = \frac{1}{2} \Psi^\dagger \begin{pmatrix} \sigma \cdot p - \mu & \Delta \\ -\sigma \cdot p + \mu & -\Delta^*
\end{pmatrix} \Psi = \frac{1}{2} \Psi^\dagger H \Psi,
$$

(13)

whose eigenvalues are

$$
E = \pm \sqrt{(p \pm \mu)^2 + \Delta^2}
$$

(14)

and its eigenstates are solutions of the Schrödinger equation

$$
H \Psi = E \Psi.
$$

(15)

The Dirac gamma matrices\cite{2} can also be used to write the Hamiltonian of the system as

$$
H = \alpha^j p_j + \beta \Delta,
$$

(16)

where $j = 1, 2$ and the matrix $\Delta$ is defined as $\Delta \equiv v(r) e^{i\gamma_5\phi}$.

In the superconductivity context, the Hamiltonian $\mathcal{H}$ is known as the Bogoliubov-de Gennes Hamiltonian, and is similar to the Dirac Hamiltonian in particle physics applications\cite{2}. An important point is that, due to the existence of $\gamma_5$ in Hamiltonian (16), the FW procedure is not applicable in the usual manner. However, it can be brought to the form which is block-diagonal with respect to negative and positive energy solutions by first performing a simple canonical transformation to remove the ‘odd’ parts of the mass term and then applying the power series FW transformation to the resulting Hamiltonian\cite{57,75}. Once again, we emphasis that the FW transformation would not be relevant to the present work due to the non-local nature of the system under study.

Solutions of Eq. (15) have the general form

$$
\Psi = \begin{pmatrix} \psi \\ \psi^c
\end{pmatrix},
$$

(17)
where } \psi^c = \left( \begin{array}{c} \psi_1^c \\ -\psi_3^c \end{array} \right). \text{ In the case } E = 0, \text{ the solution is a MZM, which satisfies the pseudo-reality constraint}
\[ C \Psi^* = \Psi, \] (18)
where the charge-conjugation operator is } C = i \gamma_2 K \text{ with } K \text{ as the complex-conjugation operator. By operating the chiral projection operators } P_\pm \text{ on } \Psi, \text{ we obtain}
\[ \psi = P_+ \Psi \quad \text{and} \quad \psi^c = P_- \Psi, \] (19)
which indicates that the } \gamma_5 \text{ also relates eigenstates with positive energies to those with negative energies.

However, when } E \neq 0, \Psi \text{ does not satisfy the pseudo-reality constraint (18), and general solutions of Eq. (15) are known as Bqs and have the form of Dirac four-spinors, which correspond to spin 1/2 fermions. One may exploit these plane wave solutions, for } \mu = 0, \text{ to build the wave-packet
\[ \Phi_{\text{Bsq}}(r, t) = \int \frac{d^2 p}{(2\pi)^2} \sum_{s=1}^{2} \left[ a_s(p) \Phi_+^s(p) e^{-i(Et - p \cdot r)} + b_s^*(p) \Phi_-^s(p) e^{i(Et - p \cdot r)} \right], \] (20)
where } p = (E, \mathbf{p}), \text{ the sign } (+) \text{ stands for positive (negative) energy, and } a_s \text{ and } b^*_s \text{ are arbitrary coefficients. Also, the spinors } \Phi^s_\pm \text{ are given by
\[ \Phi_+^s = \frac{1}{2 \sqrt{\Delta(E + \Delta)}} \left( e^{-i\phi} [E + \Delta - \sigma^1 \mathbf{p}_1] \chi_1 \right) \quad \text{and} \quad \Phi_-^s = \frac{1}{2 \sqrt{\Delta(E + \Delta)}} \left( e^{i\phi} [E + \Delta + \sigma^1 \mathbf{p}_1] \chi_2 \right), \] (21)
where
\[ \chi_1 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ e^{i\phi} \end{array} \right) \quad \text{and} \quad \chi_2 = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ e^{-i\phi} \end{array} \right) \]
refer to eigenvectors of } \sigma \cdot \mathbf{p} \text{ while } \mathbf{p} = (\cos \varphi, \sin \varphi). \text{ As the spin of Bqs in topological insulator is in-plane and is perpendicularly locked to the momentum [76], it is non-chiral. Besides, the negative energy spinors are given by } C \Phi_{1,2}^s (\mathbf{-p}), \text{ and the normalization condition leads to the constraint
\[ \int \frac{d^2 p}{(2\pi)^2} \sum_{s=1}^{2} \left[ |a_s(p)|^2 + |b_s(p)|^2 \right] = 1. \] (22)

In the case } \Delta = 0, \text{ with arbitrary constants, say } \eta \text{ and } \zeta, \text{ Hamiltonian [16] is invariant under transformations
\[ \Psi \rightarrow \exp(i\eta) \Psi, \quad \bar{\Psi} \rightarrow \exp(i\eta) \bar{\Psi} \] (23)
and
\[ \Psi \rightarrow \exp \left( i\gamma_5 \zeta \right) \Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi} \exp \left( i\gamma_5 \zeta \right), \] (24)
which lead to the conserved currents } j_\mu = \bar{\Psi} \gamma_\mu \Psi \text{ and } j_{\mu 5} = \bar{\Psi} \gamma_\mu \gamma_5 \Psi. \text{ The first current is the vector current related to the gauge symmetry and the second one is the chiral current related to the } \gamma_5 \text{ symmetry. The relevant continuity equations are
\[ \partial_\mu j_\mu = 0, \] (25)
\[ \partial_\mu j_{\mu 5} = 0. \] (26)

However, when } \Delta \neq 0, \text{ the vector current remains conserved, although it vanishes for MZMs, whereas the chiral current is not conserved because the Hamiltonian of the system does not commute with the chiral operator } \gamma_5 \text{ due to the } \gamma_0 \text{ term in } H. \footnote{It is in order to mention that the pairing term in Hamiltonian [16] breaks the } U(1) \text{ symmetry and the charge conservation, but, as this symmetry breaking happens spontaneously, it can be restored by the Nambu-Goldstone boson [77,79].} \footnote{This issue is not necessarily for the case of a MBS consisted of a vortex and an anti-vortex as will be shown below.}
4 Current Density of Majorana Bound States

In this section, we investigate the current density, \( j_\mu = \bar{\Psi} \gamma_\mu \Psi \), of MBSs in the STI system. In this respect, we study a superconductor attached to a topological insulator in the vortex/anti-vortex background. The zero-energy mode for the vortex at the origin approximately becomes \[ \psi_0^v \approx N e^{-i\pi/4} e^{-V(r)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] (27a)

where \( N \) is the normalization constant and \( V(r) \) is a dimensionless function in the form

\[ V(r) = \frac{1}{\hbar c} \int dr' v(r'), \] (27b)

when units are recovered. By assuming \[ v(r) \rightarrow \infty \] (27c)

where \( M \) is a positive definite constant with the units of energy (when units are recovered, otherwise its unit is the inverse of length in the natural units) that represents the magnitude of the pairing potential \( |v(r)| \) with the typical value \( 80 \) of the order \( meV \), one gets

\[ \psi_0^v \approx N e^{-i\pi/4} e^{-Mr} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \] (27d)

To the same approximation, the anti-vortex at a fixed distance, say \( r = R \), leads to

\[ \psi_0^\alpha \approx N e^{i\pi/4} e^{-M|r-R|} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (27e)

Also, the corresponding 4-spinors that solve Eq. (15) at zero-energy are

\[ \Psi_0^v \approx \begin{pmatrix} N e^{-i\pi/4} e^{-Mr} \\ 0 \\ 0 \\ -Ne^{i\pi/4} e^{-Mr} \end{pmatrix} \quad \text{and} \quad \Psi_0^\alpha \approx \begin{pmatrix} 0 \\ Ne^{i\pi/4} e^{-M|r-R|} \\ Ne^{-i\pi/4} e^{-M|r-R|} \\ 0 \end{pmatrix}. \] (28)

It is known that a non-local complex fermion can be written in terms of two MFs \[ 36, 37 \] such that

\[ c = \frac{1}{2} (\Gamma_1 + i\Gamma_2) \quad \text{and} \quad c^\dagger = \frac{1}{2} (\Gamma_1 - i\Gamma_2), \] (29)

where \( c \) is the electron annihilation operator and \( \Gamma_1 \)'s are the Majorana operators. This point can be seen more clearly by inverting relation (29) as

\[ \Gamma_1 = c^\dagger + c \quad \text{and} \quad \Gamma_2 = i (c^\dagger - c), \] (30)

which are obviously hermitian operators and hence, the Majorana operators. Hence, in our system (including a vortex and an anti-vortex located at a distance far away from each other), there should also exist two fermionic bound states. One of those with positive energy and the other one with equal magnitude but with opposite sign.

Calculation shows that the current density \( j_\mu = \bar{\Psi}_0^v \gamma_\mu \Psi_0^v \) vanishes for each MZM state alone. On the other hand, the complex fermionic state composed of two non-local MZMs, bound to a vortex localized at \( r = R/2 \) and an anti-vortex at \( r = -R/2 \), can be written as

\[ \Psi^{v\alpha}(r) = \Psi_0^v(r) + i \Psi_0^\alpha(r) = N e^{-M|r-R/2|} \begin{pmatrix} e^{-i(\alpha/2 + \pi/4)} \\ 0 \\ 0 \\ -e^{i(\alpha/2 + \pi/4)} \end{pmatrix} + N e^{-M|r+R/2|} \begin{pmatrix} 0 \\ i e^{i(3\alpha/4 + \pi/4)} \\ i e^{-i(3\alpha/4 + \pi/4)} \\ 0 \end{pmatrix} \] (31)

with the energy \( \varepsilon \approx e^{-MR} \), where \( \alpha \) is a constant phase originating from the mutual effect of vortex/anti-vortex on each other \[ 2, 3 \]. It should be emphasized that \( \varepsilon \) is the energy of the complex fermionic state composed of the two MZMs and not the energy of each MZM. Using MZMs properties, i.e., \( (\Psi_0^v)^c(r) = \Psi_0^\alpha(r) \) and \( (\Psi_0^\alpha)^c(r) = \Psi_0^v(r) \), the complex anti-fermionic state obviously is

\[ (\Psi^{v\alpha})^c(r) = \Psi_0^\alpha(r) - i \Psi_0^v(r) = N e^{-M|r-R/2|} \begin{pmatrix} e^{-i(\alpha/2 + \pi/4)} \\ 0 \\ 0 \\ -e^{i(\alpha/2 + \pi/4)} \end{pmatrix} - N e^{-M|r+R/2|} \begin{pmatrix} 0 \\ i e^{i(3\alpha/4 + \pi/4)} \\ i e^{-i(3\alpha/4 + \pi/4)} \\ 0 \end{pmatrix}. \] (32)
which belongs to the energy $\varepsilon \approx -e^{-MR}$.

A wave-packet consisted of both the fermionic fields can be written as

$$\Phi(r, t) \equiv \Psi^{\psi_0}(r)e^{-i\varepsilon t} + (\Psi^{\bar{\psi}})^\dagger(r)e^{i\varepsilon t}, \quad (33)$$

in which $\Psi^{\psi_0}(r)$ and $(\Psi^{\bar{\psi}})^\dagger(r)$ respectively represent the positive and negative energy complex fermionic states. Employing wave-packet $(33)$ to calculate $j_\mu = q\psi_0 r^\mu \Phi$, with $q$ as the electric charge, leads to a time-dependent current density, which originates from the interference of the positive and negative energy parts. For simplicity, we consider $R$ along the x-axis. Then, we calculate the zero component of this current density for areas far away from those two MBSs, i.e. when $r \gg R$, in two limits: for regions close to the axis that joins those (i.e., for when $\phi \to 0$) and regions close to the direction perpendicular to the axis of their connection (i.e., for when $\phi \to \pi/2$). The results represent a transient emergent electric charge described as (when units are recovered)

$$j_0^{\phi \to 0} = 2qe^{-MR/(hc)} \tanh(MR/(hc)) \cos(2\varepsilon t), \quad (34)$$

and for regions of $\phi \to \pi/2$, it becomes

$$j_0^{\phi \to \pi/2} = 0. \quad (35)$$

The interference term exhibits an oscillatory behavior along the axis joining the two MBSs, which vanishes only after averaging it in time. In other words, although the charge of a single MZM is exactly zero, when two MZMs take part to build a charged fermion, an oscillatory transient charge (and not a net charge) appears in the system. This result can give some insight to the vortex/anti-vortex system and is different from the results of Ref. [3], wherein emergence of a field of non-vanishing energy was mentioned without specifying the charge of such a field. The amplitude of oscillations fade away when $R \to 0$, as expected since this limit corresponds to the fusion of the vortex and the anti-vortex. The frequency of the oscillation is proportional to $2e^{-MR}$, which goes to zero in the limit $R \to \infty$. As it is shown in Fig. 1, due to the proportionality of the frequency and amplitude of the oscillations to distances $r$ and $R$, a small change in these distances makes those change substantially. More specifically, the amplitude is very sensitive to changes in $r$, $R$ and $M$. Hence for it to be detectable. we consider, say, $MR/(hc) \approx 1$ or less and subsequently $MR/(hc) \ll 1$ due to the $r \gg R$ condition. The values mentioned in the caption would lead to a detectable setup. On the other hand, relation $(35)$ shows that, in the direction perpendicular to the axis connecting the two MBSs, the current density vanishes identically.

As yet another feature of MBSs, using the interference of the positive and negative energy components of wave-packet $(33)$ and after making some calculations, the spatial current density components, for both $\phi \to 0$ and $\phi \to \pi/2$, are

$$j_1 = 2qe^{-2MR} \cos \alpha \text{sech}(MR) \sin(2\varepsilon t),$$

$$j_2 = 2qe^{-2MR} \sin \alpha \text{sech}(MR) \sin(2\varepsilon t). \quad (36)$$

These results describe an oscillatory behavior in the $x - y$ plane and, in principle, should also be detectable for values of the energy $\varepsilon$ in the range of experimental resolution.

As stated above, the two MBSs lead to a non-local complex fermion along with its anti-particle. Accordingly, using wave-packet $(33)$, one can calculate the expectation value $\langle \alpha \rangle$, which corresponds to the ZBW velocity of a complex
fermion when both positive and negative energies are involved. The result of such a calculation coincides with the spatial component of the current density given in relations (36). In the other word, relations (36) describe the ZBW motion of the complex fermion built out of the two MBSs. Also, as the involved complex fermion is non-local in nature, the FW transformation would not be relevant to this case. Besides, one of the main obstacles regarding the detection of the ZBW of electron (i.e., the ultrahigh frequency of the oscillations) would be absent in the MBSs setup because the frequency can be adjusted. However it should be mentioned, that if a system of two MZMs bound to two vortices instead of a vortex and an anti-vortex, then no oscillatory pattern will emerge in the system. It is interesting to note that these results might be applicable to the one dimensional models as well [6].

5 Conclusions

The push to fabricate topologically protected qubits using MZMs (which are predicted to emerge as localized zero-energy bound states in topological superconductors) is one of the most appealing research topics in quantum condensed matter physics. However, to create MZMs in practice it needs the combination of cutting-edge fields such as the nanotechnology, superconductivity, the device engineering and materials science. In other words, being charge-less and its own anti-particle makes it difficult to take a MZM out of the sample for private inspection. These problems have led to redesign experiments in such a way that to probe other features of MZMs in indirect detection, for example via measuring the energy of MBS systems.

In this work, we have taken advantage of locating two MBSs at a distance from each other in order to be able properly describe a non-local complex fermionic state. Accordingly, we have computed the current density of a wave-packet consisted of the complex fermionic state along with its anti-particle with opposite energy. We have managed to specify that although MZMs break the $U(1)$ gauge symmetry in general, a non-vanishing conserved oscillatory transient charge emerges, as shown in Figure 1, which vanishes in the direction perpendicular to the axis joining the two MBSs. The frequency of this oscillation and its amplitude are proportional to the distance between the two MBSs. Hence, by adjusting this distance, such a current density would be detectable. Moreover, the spatial components of the current density exhibit an oscillatory behavior with the same property, which is also a reminiscent of the ZBW effect. In addition, this oscillatory behavior should also be detectable for values of energy in the range of experimental resolution.

The ZBW phenomenon is a long sought effect which has eluded the experimental observation so far due to its extremely high frequency. In this regard, it has even been doubted to be physical due to the FW transformation, which eliminates negative energy components in electron wave functions. In this work, we have demonstrated that the emerged ZBW motion originates from the interference of the positive and negative energy parts of the fermionic states constructed from the two non-local MBSs (correlations incompatible with a local hidden variable theory). Therefore, the FW transformation would not be relevant to this case. Furthermore, those measurements that indicate the existence of the current oscillations of the MBSs in the vortex and anti-vortex background, on the surface of topological insulators in contact with an s-wave superconductor, would be a verdict confirmation of the existence of the MZMs and the ZBW phenomenon at the same time and point to an existing direction to help unravel the mystery of ZBW.

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