Mixed characteristic discontinuous Galerkin approach for perfect gas dynamics modeling

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Abstract. The numerical method based on the discontinuous Galerkin (DG) approach for perfect inviscid compressible gas dynamics modelling is developed. The approximate Riemann solvers in combination with gas dynamics equations system characteristic properties are used to calculate the numerical fluxes on the cells interfaces and inside the cells. A number of approximate Riemann solvers are considered. The solution characteristic decomposition is used in the similar to PPML way. The developed algorithm is tested using problems with smooth and discontinuous solutions. Both one-dimensional and two-dimensional test problems are considered. The method provides accurate discontinuity resolution and ability to compute gas-dynamic instabilities with minimal artificial distortion.

1. Introduction
The equations of gas dynamics are used in a wide range of modelling problems — from the circulatory system to the study of plasma flows in astrophysical objects. The model considered in this paper describes a perfect inviscid compressible gas [1, 2]. Such models can be applied to simulating both high-speed flows, that occur around an aircraft moving at supersonic speeds and experiments devoted to the study of a thermonuclear fusion. Due to complexity of such problems and lack of exact solutions it is necessary to use numerical methods for Euler equations. Among these methods high-order schemes are most important since they provide high quality resolution of discontinuities arising in gas problems, i.e shock waves, rarefaction waves, contact discontinuity [3, 4].

High-order schemes used for actual gasdynamics problems usually based on
• using approximate solutions of the Riemann problem [2, 5],
• using a characteristic features of hyperbolic equations [6, 7].

As an examples of first type methods schemes based on the discontinuous Galerkin method [8, 9] have been especially popular in the numerical simulation of fluid and gas flows in recent years. The advantages of the method are determined by its high resolution capabilities based on the polynomial approximation of the solution of an arbitrary order within each mesh cell and the assumption about the solution having discontinuities at the cell boundaries. The schemes of the second type, e.g. [7, 10, 11], give accurate and high order approximation of smooth solutions.

The purpose of this article is to combine these two approaches, develop a numerical method for gas dynamics equations and analyze the results obtained.
The article structure is the following. In Section 2 the governing equations and basic designations are collected. The numerical scheme is considered in detail in Section 3. Section 4 is devoted to numerical algorithm testing using 1D and 2D test problems. Conclusions to the article are given in Section 5.

2. Governing equations
The two-dimensional Euler equations for a compressible inviscid gas can be written as [2]:

$$\frac{\partial U}{\partial t} + \nabla \cdot \hat{F} = 0,$$

where $U$ is a vector of the conservative variables, $\hat{F}$ is flux tensor, $\rho$ is density, $u, v$ are projections of the velocity vector onto the $x$ and $y$ axis respectively, $p$ is pressure, $E = \rho \varepsilon + \frac{\rho (u^2 + v^2)}{2}$ is total energy per unit mass.

The system (1) should be completed by an equation of state for a perfect inviscid gas $p = (\gamma - 1)\rho \varepsilon$, where $\varepsilon$ is a specific internal energy.

The system (1) can also be written in primitive variables [3]:

$$\frac{\partial V}{\partial t} + A \frac{\partial V}{\partial x} + B \frac{\partial V}{\partial y} = 0,$$

where $V = (\rho, u, v, p)^T$.

Lets define $A, B$ as follows:

$$A = M^{-1} \frac{\partial F}{\partial U} M, \quad B = M^{-1} \frac{\partial G}{\partial U} M,$$

where $M$ is a transition matrix

$$M = \frac{\partial U}{\partial V} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ \rho & 0 & 0 & 0 \\ \frac{u^2 + v^2}{2} & \rho u & \rho v & \frac{1}{\gamma - 1} \end{pmatrix}.$$

Matrix $A$ can be decomposed as $A = R_x \Lambda_x L_x$, where $R_x$ is a matrix, which columns are the right eigenvectors of $A$, $L_x$ is a matrix, which rows are the left eigenvalues of $A$ and $\Lambda_x$ is a diagonal matrix of the eigenvalues.

Thus, for example, $R_x, L_x, \Lambda_x$ are defined as

$$R_x = \begin{pmatrix} 1 & -\frac{\varepsilon}{\rho} & 0 & \frac{\rho}{\gamma} \\ 1 & 0 & -\frac{\varepsilon}{\rho} & 0 \\ 1 & 0 & -\frac{\varepsilon}{\rho} & 0 \\ 1 & \frac{\varepsilon}{\rho} & 0 & \frac{\rho}{\gamma} \end{pmatrix}, \quad L_x = \begin{pmatrix} 1 & 0 & -\frac{\rho}{2 \varepsilon} & \frac{1}{2 \gamma - 1} \\ 0.5 & \frac{\rho}{\gamma} & \frac{1}{2 \gamma - 1} & 0 \\ 0.5 & 0 & -\frac{\rho}{2 \varepsilon} & \frac{1}{2 \gamma - 1} \\ 0 & \frac{\rho}{2 \varepsilon} & 0 & \frac{1}{2 \gamma - 1} \end{pmatrix},$$

(4)
where $c = \sqrt{\frac{p}{\rho}}$ is the sound speed.

### 3. Numerical scheme

Let's consider the 2D computational domain $[a, b] \times [c, d]$ and time interval $[0, T]$. The initial conditions are $U(x, y, 0) = U_0(x, y)$.

We use rectangular mesh with steps $h_x = L_x/n_x, h_y = L_y/n_y$ which consists of $n_xn_y$ cells $I_{i,j}$ with centers in $x_i = h_x(i - 1/2), y_j = h_y(j - 1/2)$.

We use the discontinuous Galerkin (DG) approach [8, 9, 12] for spatial approximation of system (1) numerical solution:

$$U(x, t) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{\beta=0}^{b} C_{i,j}^\beta(t) \phi_{i,j}^\beta(x, y),$$

(6)

where $\phi_{i,j}^\beta(x, y)$ are basic functions, $C_{i,j}^\beta(t)$ is the coefficient of $\beta$ basis function in solution on the $I_{i,j}$ cell.

We use the orthogonal system of Legendre polynomials

$$\phi_{i,j}^0(x, y) = 1, \quad \phi_{i,j}^1(x, y) = \frac{2}{h}(x - x_i), \quad \phi_{i,j}^2(x, y) = \frac{2}{h}(y - y_j)$$

as the basis functions defined inside the cell.

Substituting the representation (6) into the system (1), multiplying by $\phi_{i,j}^\beta$ and integrating in time and space one can obtain

$$A^\beta(C_{i,j}^\beta - C_{i,j}^\beta) + \int_{\Gamma_{i,j}} \phi_{i,j}^\beta \left( \int_{t_j}^{t_{j+1}} \bar{F} \cdot \bar{n} dt \right) d\Gamma - \int_{I_{i,j}} \left( \int_{t_j}^{t_{j+1}} \bar{F} dt \right) \cdot \nabla \phi_{i,j}^\beta dxdy = 0,$$

(7)

where $A^\beta = \iint_{I_{i,j}} (\phi_{i,j}^\beta)^2 dxdy$.

The last integral (over the space) in the left part of (7) can be splitted:

$$\iint_{I_{i,j}} \left( \int_{t_j}^{t_{j+1}} \bar{F} dt \right) \cdot \nabla \phi_{i,j}^\beta dxdy = \iint_{I_{i,j}} \tau F_{i,j}^{n+1/2} \frac{\partial \phi_{i,j}^\beta}{\partial x} dxdy + \iint_{I_{i,j}} \tau G_{i,j}^{n+1/2} \frac{\partial \phi_{i,j}^\beta}{\partial y} dxdy,$$

(8)

where

$$F_{i,j}^{n+1/2} = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} F(x, y, t) dt, \quad G_{i,j}^{n+1/2} = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} G(x, y, t) dt$$

(9)

are the time-averaged fluxes.
3.1. Calculating the fluxes at the boundary
In order to calculate the integral along the cell boundary, we will calculate the integrals separately for each cell edge

\[
\int_{\Gamma_{i,j}} \mathbf{F} \cdot \hat{n} d\Gamma = \sum_{m=1}^{N} A_{s+1} \int_{A_s} \mathbf{F} \cdot \hat{n} dA.
\]

To obtain the integrals under the sum we will rotate the original coordinate system by an angle \( \theta \) (fig.1). The rotation matrix for the transition to the new coordinate system has the form

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Figure 1.** The initial and the rotated coordinate systems for calculating the flux at the \( A_sA_{s+1} \) edge.

For the two-dimensional case we have [5]

\[
\int_{\Gamma_{i,j}} \mathbf{F} \cdot \hat{n} d\Gamma = \sum_{m=1}^{N} A_{s+1} \int_{A_s} (\cos \theta_s \mathbf{F}(\mathbf{U}) + \sin \theta_s \mathbf{G}(\mathbf{U})) dA = \sum_{m=1}^{N} A_{s+1} \int_{A_s} T^{-1}_s \mathbf{F}(T_s \mathbf{U}) dA. \quad (10)
\]

Each integral along \( A_sA_{s+1} \) edge is calculated using the appropriate quadrature formula.

3.2. Calculating the time-averaged fluxes
Consider in details the calculation of time-averaged flux at the boundary between \( i \) and \( i+1 \) cells (see fig. 2). We use approach close to PPM and PPML [3, 13] to calculate these fluxes in quasi-1D way. The 1D illustration is shown in the fig. 2.

For the gas dynamics equations solution there is a number of characteristics passing through each point of the \( (x,t) \) space. Thus, the gas state at the boundary between two cells \( x_{i+1/2} \) is affected by the both pieces of solution from \( i \) and \( i+1 \) cells.

According to solution characteristic features we replace the integration in time in (9) with the integration in space, assuming the characteristics to be linear. So, the solution can be averaged
over every eigenvalue domain of influence (fig. 2):

\[ \nabla_{i+1/2}^{L,p} = \frac{1}{\lambda^p \tau} \int_{x_{i+1/2} - \lambda^p \tau}^{x_{i+1/2}} V(x) dx, \quad \lambda^p > 0, \]

\[ \nabla_{i+1/2}^{R,p} = \frac{1}{|\lambda^p| \tau} \int_{x_{i+1/2}}^{x_{i+1/2} + |\lambda^p| \tau} V(x) dx, \quad \lambda^p < 0. \]

For the maximum eigenvalue the averaged solution is \( \nabla_{i+1/2}^{L,1} \). We use it to compute the eigenvectors.

Now, we obtain states \( V_{i+1/2}^L \) in the left and \( V_{i+1/2}^R \) in the right side of boundary point \( x_{i+1/2} \) using formulas

\[ V_{i+1/2}^L = V_{i+1/2}^{L,1} + \sum_{p, \lambda^p > 0} r^p (P'(V_{i+1/2}^{L,1} - V_{i+1/2}^{L,1})) \]

and

\[ V_{i+1/2}^R = V_{i+1/2}^{R,p} + \sum_{p, \lambda^p < 0} r^p (P'(V_{i+1/2}^{R,p} - V_{i+1/2}^{R,p})), \]

respectively. Here \( V_{i+1/2}^{R,1} \) is solution averaged over the minimal negative eigenvalue domain of influence.

The states \( V_{i+1/2}^L, V_{i+1/2}^R \) now can be used to calculate the numerical flux.

Similarly, the time-averaged solutions in the quadrature formula nodes inside the cells can be calculated. The time step \( \tau \) has to be chosen in such way that the characteristics passing through the quadrature nodes \( (x_k, t_{n+1}) \) do not leave the \( \alpha \) cell (see fig. 3). So the quadrature formula nodes should be as far as possible from the boundaries of the cell to provide the scheme stability.
Figure 3. Instability occurs at the step $\tau$, at the step $\tau'$ the scheme is stable

3.3. The approximate Riemann solver

Now two states $V^L_{i+1/2}$ and $V^R_{i+1/2}$ are obtained. Next we need to calculate the numerical flux $F^i_{i+1/2}$. For a 1D gas dynamics equations system the numerical flux can be obtained using an exact or approximate solution of the Riemann problem. In the first way is complicated and has high computational cost. We use Roe approximate Riemann solver [1].

In the Roe scheme the solution of the nonlinear problem is replaced by the solution for a linearized system with $F = \Omega V$ and the flux can be defined as

$$F = \frac{F_L + F_R}{2} - \frac{\Omega}{2} (V_R - V_L),$$

where $\Omega$ is a matrix of the linearized system. It is suggested to choose $\Omega$ so that the equation below completes

$$F_R - F_L = \Omega (V_R - V_L).$$

In this case the solution of the linear problem satisfies the conservation laws. In the Roe scheme it is proposed to take the matrix

$$\Omega = \frac{\partial F}{\partial V}(V^*),$$

where $V^* = (\rho^*, u^*, p^*)$ is an intermediate state, which is obtained with $V^L_{i+1/2}, V^R_{i+1/2}$ as follows:

$$\rho^* = \sqrt{\rho^L \rho^R}, \quad u^* = \frac{\sqrt{\rho^L u^L} + \sqrt{\rho^R u^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}}, \quad h^* = \frac{\sqrt{\rho^L h^L} + \sqrt{\rho^R h^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}},$$

where $h$ is specific enthalpy, $h = \frac{E+p}{\rho}$. The eigenvalues $\lambda^p$ are $\lambda^1 = u^* - c^*$, $\lambda^2 = u^*$, $\lambda^3 = u^* + c^*$.

So, the Roe numerical flux is

$$F^i_{i+1/2} = \frac{F_L + F_R}{2} - \frac{1}{2} \sum_p |\lambda^p(V^*)| \Delta \alpha^p_{i+1/2} p_{i+1/2}(U^*),$$

where $F^L = F(U^L_{i+1/2}), F^R = F(U^R_{i+1/2}), \Delta \alpha^p_{i+1/2} = p(U^*)(V^R_{i+1/2} - V^L_{i+1/2}).$
4. Testing the algorithm
The parallel numerical code based on the proposed method is developed. Let us consider some test problems both with smooth exact solution and with discontinuous solution.

4.1. 1D tests
The first test we consider has smooth initial conditions [11]
\[ \rho = 1 + 0.5e^{-80(x-0.5)^2}, \quad u = 0, \quad p = 1 + 0.5e^{-80(x-0.5)^2} \]
and periodical boundary conditions.

Figure 4 shows numerical solutions with number of cells \( n = 8, 16 \) for linear \((b = 1, \text{left panel})\) and quadratic \((b = 2, \text{right panel})\) approximation of the solution. The exact solution is depicted by a black line.

![Figure 4. Density distribution at the moment \( t = 0.25 \), left panel — \( b = 1 \), right panel — \( b = 2 \)](image)

In order to estimate the scheme order of accuracy we carried out several calculations with \( n = 8, 16, 32, 64 \) and calculated the error of approximate solution \( \tilde{u} \) using formula
\[
E = \|u - \tilde{u}\|_{L_2} = \left( \int_a^b (u - \tilde{u})^2 \, dx \right)^{1/2},
\]
where \( u \) is exact solution. The results of calculations are presented in table 1

| \( i \) | \( n \) | \( b = 1 \) | \( \frac{E_i}{E_{i+1}} \) | \( \log_2 \frac{E_i}{E_{i+1}} \) | \( b = 2 \) | \( \frac{E_i}{E_{i+1}} \) | \( \log_2 \frac{E_i}{E_{i+1}} \) |
|---|---|---|---|---|---|---|---|
| 1 | 8 | 0.016 | | | 0.0032 | | |
| 2 | 16 | 0.0043 | 3.9 | 1.96 | 0.0006 | 5.11 | 2.35 |
| 3 | 32 | 0.001 | 4 | 1.98 | 0.00007 | 7.96 | 3 |
| 4 | 64 | 0.0002 | 4.8 | 2.26 | 0.00001 | 4.57 | 2.19 |

Note that although the accuracy orders for the linear and quadratic basis in the cell are similar, the absolute error is significantly less in the second case.
The second test is Sod problem [5]. It is the case of Riemann problem with initial discontinuity at \( x_p = 0.5 \) point and with initial conditions

\[
\rho^L = 1, \ u^L = 0, \ p^L = 1,
\]
\[
\rho^R = 0.125, \ u^R = 0, \ p^R = 0.1.
\]

Figure 5 shows the numerical solutions for density with spatial steps \( h = 0.02 \) and \( h = 0.0025 \). The exact solution is depicted by a red line. The method gives high resolution of discontinuities, e.g. the shock wave is resolved in two cells and contact discontinuity in five cells. At the same time as a rule no limiter is needed. The method gives the numerical solution only with a small artificial oscillations or without one at all. The oscillations noticeable especially in the left panel of fig.5 in front of contact discontinuity are linked to heating (entropy) error which is intrinsic for Roe solver based methods. On the fine mesh (right panel of fig.5) this entropy decreasing is also visible but occupies the same number of cells as on coarse mesh.

4.2. 2D test — cylindrical Sod problem

The cylindrical Sod test problem is solved in the square domain \([0, 1] \times [0, 1]\). The initial conditions consist of the region inside a circle with radius \( R = 0.2 \) centered at \((1/2, 1/2)\) and the region outside. The two gas states out and inside a circle are chosen to be

\[
\rho^{in} = 1, \ u^{in} = 0, \ v^{in} = 0, \ p^{in} = 1,
\]
\[
\rho^{out} = 0.125, \ u^{out} = 0, \ v^{out} = 0, \ p^{out} = 0.1.
\]

The boundary conditions model the impenetrable wall.

The ending time of calculations is \( T = 0.25 \). Spatial quadrilateral mesh contains 150 cells in each direction. The figure 6 shows a density distribution in computational domain and in x-axis section. The solution contains a circular shock wave travelling away from the center, a contact surface travelling in the same direction and a circular rarefaction wave.

As in 1D case the calculations show high resolution of discontinuities. The high order of accuracy provides the ability of method to maintain gas instabilities. The figure 7 shows the Richtmeyer — Meshkov instability in the solution after multiple reflections of shock wave from the boundary of computational domain.
Figure 6. Density distribution for the cylindrical Sod problem: full domain (left panel) and x-axis section (right panel)

Figure 7. The illustration of Richtmeyer — Meshkov instability development in cylindrical Sod problem solution
5. Conclusions
The numerical method based on the discontinuous Galerkin approach and the characteristic properties of the Euler equations is proposed. The developed method eliminates the standard for DG schemes Runge — Kutta time integration. Using the averaging the solution over the characteristic domain of influence all calculations at every time step are performed in one stage. It significantly reduces the computational complexity of the algorithm and leads to high accuracy and resolution of the method. The method is tested using 1D problems (with linear and quadratic basis functions) both with smooth solution and with discontinuous one. It is also tested using 2D problems. The algorithm is able to resolve discontinuities without using any limiters or only with rare use of it. Low dissipation of method leads to it high abilities to maintain the solutions with gas instabilities.

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