QPOs: Einstein’s gravity non-linear resonances

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Summary. There is strong evidence that the observed kHz Quasi Periodic Oscillations (QPOs) in the X-ray flux of neutron star and black hole sources in LMXRBs are linked to Einstein’s General Relativity. Abramowicz&Kluźniak (2001) suggested a non-linear resonance model to explain the QPOs origin; here we summarize their idea and the development of a mathematical toy-model which begins to throw light on the nature of Einstein’s gravity non-linear oscillations.

1 Introduction

QPOs are highly coherent Lorentzian peaks observed in the X-ray power spectra of compact objects in LMXRBs. They cover a wide range of frequencies (from mHz to kHz) and they show up alone, in pairs or in triplets. In the present review we focus on kHz QPOs which occur in pairs (the so called twin peak QPOs). Many models have been proposed in order to understand kHz QPOs: some of them involve orbital motions (e.g. [14], [8], but there are also models that are based on accretion disk oscillations (e.g. [17], [7],[13],[9]). Here we present semi-analytical results, connected in particular with the Kluźniak&Abramowicz model.

2 QPOs and General Relativity

High frequency (kHz) QPOs lie in the range of orbital frequencies of circular geodesics just few Schwarzschild radii outside the central source. Moreover the frequencies scale with $1/M$, where $M$ is the mass of the central object [10]. These two facts support the strong gravity orbital oscillations models. Consider a test particle rotating around a compact source: the radial epicyclic frequency of planar motion and the vertical epicyclic frequency of nearly off-plane motion are respectively defined as

$$\omega^2_r = \left( \frac{1}{2g_{rr}} \frac{\partial^2 U_{eff}}{\partial r^2} \right)_{r_0, \pi/2}, \quad \omega^2_z = \left( \frac{1}{2g_{\theta\theta}} \frac{\partial^2 U_{eff}}{\partial \theta^2} \right)_{r_0, \pi/2}, \quad (1)$$
where $U_{eff} = g^{tt} + l g^{t\phi} + l^2 g^{\phi\phi}$ is the effective potential and $(r, \theta, \phi)$ are spherical coordinates. The eigenfrequencies depend only on the metric of the system, hence on strong gravity itself. In Newtonian gravity there is degeneracy between these eigenfrequencies and the Keplerian frequency (all three frequencies are equal) while in General Relativity this degeneracy is broken and as a consequence two or three different characteristic frequencies are present, opening the possibility of internal resonances. As a consequence, while Newtonian orbits are all close, in GR they do not close after one loop (this is for the same reason as the well-known advance of the perihelion of Mercury).

For a spherically symmetric gravitating fluid body (a better model of the accretion disk) these are the frequencies at which the center of mass (initially on a circular geodesic) oscillates.

### 3 Kluźniak-Abramowicz resonance model

QPOs often occur in pairs, and the centroid frequencies of these pairs are in rational ratio (e.g. [15]): these features suggested that high frequency QPOs are a phenomenon due to non-linear resonance, and that there may be an analogy between the radial and vertical oscillations in a Shakura-Sunyaev disk and the motion of a pendulum with oscillating point of suspension ([4]). Since in GR $\omega_r < \omega_z$ the first allowed resonance would appear for $\omega_z : \omega_r = 3 : 2$ and it would be the strongest.

In all four microquasars which exhibit double peaks, the ratio of the two frequencies is $3 : 2$, as well as in many neutron star sources. Moreover combinations of frequencies and subharmonics have been detected: these are all signatures of non-linear resonance and they confirm the validity of the model.

#### 3.1 Toy model

A mathematical toy-model for the Kluźniak-Abramowicz resonance idea for QPOs was recently developed [1]. It describes the QPOs phenomenon in terms of two coupled non-linear forced oscillators,

\begin{align*}
\ddot{\delta}r + \omega_r^2 \delta r &= F(\delta r, \delta z, \dot{\delta} r, \dot{\delta} z) + C \cos(\omega_0 t) + N_r(t), \\
\ddot{\delta}z + \omega_z^2 \delta z &= G(\delta r, \delta z, \dot{\delta} r, \dot{\delta} z) + D \cos(\omega_0 t) + N_z(t),
\end{align*}

where $F$ and $G$ are polynomials of second or higher degree (obtained in terms of expansion of the deviations from a Keplerian flow). The $\cos(\omega_0 t)$ terms represent an external forcing: they are mostly important in the case of NS, where $\omega_0$ can be the NS spin frequency. $N_r$ and $N_z$ describe the stochastic noise due to the Magneto-Rotational Instability.

In absence of turbulence, the first finding, which can be derived by using any perturbative method, is that in the approximate solution there are terms
with the denominators in the form \( n\omega_r - m\omega_z \) (with \( n \) and \( m \) being integers). These \( n \) and \( m \) cannot take any value, but they depend on the symmetry of the metric and of the perturbation: for a plane symmetric configuration one can demonstrate (e.g.,[12],[6]) that \( m = 2p \) (\( p \) integer). Due to this, the regions where \( n\omega_r = 2p\omega_z \) are candidate regions of internal resonance, and the strongest one would be for \( \omega_z : \omega_r = 3 : 2 \) (a similar reasoning can be done for the forcing term).

3.2 Correlation and anticorrelation

A well known property of weakly non-linear oscillators is the fact that the frequencies of oscillation (\( \omega_r^* \) and \( \omega_z^* \)) depend on the amplitudes, even if their value remains close to that of the eigenfrequencies (\( \omega_r \) and \( \omega_z \)). Perturbative methods help us again (e.g.,[11]): indeed one can write

\[
\begin{align*}
\omega_r^* &= \omega_r + \epsilon^2 \omega_r^{(2)} + \epsilon^3 \omega_r^{(3)} + O(\epsilon^4), \\
\omega_z^* &= \omega_z + \epsilon^2 \omega_z^{(2)} + \epsilon^3 \omega_z^{(3)} + O(\epsilon^4)
\end{align*}
\]

(4)

and find the frequencies corrections (\( \omega^{(j)}_i \)) by constraining the solution to be stationary at a given timescale (of the order of \( \epsilon^{-j} \)). For small perturbations one would get that \( \omega_z^* = A\omega_r^* + B \). In this way one can qualitatively explain the observed linear correlation between the twin frequencies in NS sources: the example of the NS source Sco X-1 was studied by [3] and [12] (see Fig. 1).

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{fig1}
  \caption{The dotted line is the least-squares best-fit to the data points (the observed kHz QPOs frequencies in Sco X-1); the thin solid line corresponds to a slope of 3 : 2 (for reference). The thick solid segment is the analytic approximation, in which the frequencies are scaled for comparison with observations.}
\end{figure}

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{fig2}
  \caption{The anticorrelation between shifts (\( B \)) and slopes (\( A \)). The points correspond to the individual Z and atoll sources listed in [2]. The best fit line goes through the point (0, 1.5) (Courtesy of Gabriel Török).}
\end{figure}
A direct consequence of the internal resonance is the fact that for different sources the coefficients $A$ and $B$ should be nearly linearly anticorrelated ([1]):

$$A = \frac{\omega_z}{\omega_r} - \frac{B}{f_0(M)}$$

where $f_0(M)$ is a function which depends on the mass of the central object. Note that at higher orders the relation is expected to slightly deviate from linearity: anyway up to now this feature fits very well the available data (see Fig.2).

### 3.3 The effect of turbulence

Accretion disks are characterized by a huge number of degrees of freedom. The turbulent processes can be assumed to have a stochastic nature. In particular, we have investigated a simplified model for the Kluźniak-Abramowicz nonlinear theory and showed that a small noise in the vertical direction can trigger coupled epicyclic oscillations (see Fig.3 and 4). On the other hand too much noise would disrupt the quasi-periodic motion [16]. This is similar to the stochastically excited p-modes in the Sun, and it may help in estimating the strength and the nature of the turbulence itself.

**Fig. 3.** Low turbulence: power spectra (upper part) and phase diagrams (lower part) for $r$ and $z$. The displacements are in units of $r_0$, the frequencies are scaled to kHz (e.g. assuming a central mass $M$ of $2M_\odot$). It does not differ much from the behavior in absence of turbulence.

**Fig. 4.** The same as in the previous figure. In this case however the turbulence is strong enough to feed the resonance: as a consequence the amplitudes of oscillation are much greater.
4 Conclusions

Non-linear parametric resonances occur everywhere in Nature: together with GR they could explain the mechanism at the basis of kHz QPOs. In this way the mass and the angular momentum (e.g.,[5]) of the central compact object could be precisely measured, but most of all Einstein’s strong gravity could be proved: a good motivation to keep on investigating on this puzzling phenomenon.

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