ON THE CLASSICAL LIMIT OF QUANTUM MECHANICS I.

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Abstract. This paper is devoted to the study of the classical limit of quantum mechanics. In more detail we will elaborate on a method introduced by Hepp in 1974 for studying the asymptotic behavior of quantum expectations in the limit as Plank’s constant ($\hbar$) tends to zero. Our goal is to allow for unbounded observables which are (non-commutative) polynomial functions of the position and momentum operators. This is in contrast to Hepp’s original paper where the observables were, roughly speaking, required to be bounded functions of the position and momentum operators. As expected the leading order contributions of the quantum expectations come from evaluating the observables along the classical trajectories while the next order contributions are computed by evolving the $\hbar = 1$ observables by a linear canonical transformations which is determined by the second order pieces of the quantum mechanical Hamiltonian.

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1. INTRODUCTION

In the limit where Plank’s constant ($\hbar$) tends to zero, quantum mechanics is supposed to reduce to the laws of classical mechanics and their connection was first shown by P. Ehrenfest in [5]. There is in fact a very large literature devoted in one way or another to this theme. Although it is not our intent nor within our ability to review this large literature here, nevertheless the interested reader can find more information by searching for terms like, correspondence principle, WKB approximation, pseudo-differential operators, micro-local analysis, Moyal brackets, star products, deformation quantization, Gaussian wave packet and stationary phase approximation in the context of Feynmann path integrals to name a few. Also, [12, 6, 19, 21, 26, 14] may introduce readers a broad background on the subject of semi-classical limit in one aspect or another. In this paper we wish to concentrate on a formulation and a method to understand the classical limit of quantum mechanics which was introduced by Hepp [15] in 1974.

This paper is an elaboration on Hepp’s method to allow for unbounded observables which was motivated by Rodnianski and Schlein’s [23] treatment of the mean field dynamics associated to Bose Einstein condensation. In fact, some of the ideas in [7, 8, 9, 10, 11, 20, 1, 23] and [3] already appeared in Hepp’s [15] paper. In order to emphasize the main ideas and to not be needlessly encumbered by more complicated notation we will restrict our attention to systems with only one degree of freedom. Before summarizing the main results of this paper, we first need to introduce some notation. [See section 2 below for more details on the basic setup used in this paper.]

1.1. Basic Setup. Let $\alpha_0 = (\xi + i\pi)/\sqrt{2} \in \mathbb{C}$ ($\mathbb{C} \cong T^*\mathbb{R}$ is to be thought of as phase space), $H(\theta, \theta^*)$ be a symmetric [see Notation 2.8] non-commutative polynomial in two indeterminates, $\{\theta, \theta^*\}$, $H^{\text{cl}}(z) := H(z, \bar{z})$ for all $z \in \mathbb{C}$ be the symbol of $H$. [By Remark 2.15 below, we know $H^{\text{cl}}$ is real valued.] A differentiable function, $\alpha(t) \in \mathbb{C}$, is said to satisfy Hamilton’s equations of motion with an initial condition $\alpha_0 \in \mathbb{C}$ if

$$i \dot{\alpha}(t) = \left( \frac{\partial}{\partial \alpha} H^{\text{cl}} \right)(\alpha(t)) \quad \text{and} \quad \alpha(0) = \alpha_0. \quad (1.1)$$

[See Section 2.1 where we recall that Eq. (1.1) is equivalent to the standard real form of Hamilton’s equations of motion.] Further, let $\Phi(t, \alpha_0) = \alpha(t)$ (where $\alpha(t)$ is the solution to Eq. (1.1)) be the flow associated to Eq. (1.1) and $\Phi'(t, \alpha_0): \mathbb{C} \to \mathbb{C}$
be the real-linear differential of this flow relative to its starting point, i.e. for all $z \in \mathbb{C}$ let

$$\Phi'(t, \alpha_0) z := \frac{d}{ds}\big|_{s=0} \Phi(t, \alpha_0 + sz).$$

As $z \rightarrow \Phi'(t, \alpha_0) z$ is a real-linear function of $z$, for each $\alpha_0 \in \mathbb{C}$ there exists unique complex valued functions $\gamma(t)$ and $\delta(t)$ such that

$$\Phi'(t, \alpha_0) z = \gamma(t) z + \delta(t)\overline{z}.$$  

(1.3)

where $\gamma(0) = 1$ and $\delta(0) = 0$.

We now turn to the quantum mechanical setup. Let $L^2(m) := L^2(\mathbb{R}, m)$ be the Hilbert space of square integrable complex valued functions on $\mathbb{R}$ relative to Lebesgue measure, $m$. The inner product on $L^2(m)$ is taken to be

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} \, dm(x) \quad \forall f, g \in L^2(m)$$

(1.4)

and the corresponding norm is $\|f\| = \|f\|_2 = \sqrt{\langle f, f \rangle}$. [Note that we are using the mathematics convention that $\langle f, g \rangle$ is linear in the first variable and conjugate linear in the second.] We say $A$ is an operator on $L^2(m)$ if $A$ is a linear (possibly unbounded) operator from a dense subspace, $D(A)$, to $L^2(m)$. As usual if $A$ is closable, then its adjoint, $A^\dagger$, also has a dense domain and $A^{\dagger\dagger} = \overline{A}$ where $\overline{A}$ is the closure of $A$.

**Notation 1.1.** As is customary, let $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset L^2(m)$ denote Schwartz space of smooth rapidly decreasing complex valued functions on $\mathbb{R}$.

**Definition 1.2** (Formal Adjoint). If $A$ is a closable operator on $L^2(m)$ such that $D(A) = \mathcal{S}$ and $\mathcal{S} \subset D(A^\dagger)$, then we define the formal adjoint of $A$ to be the operator, $A^\dagger := A^\dagger|_{\mathcal{S}}$. Thus $A^\dagger$ is the unique operator with $D(A^\dagger) = \mathcal{S}$ such that $\langle Af, g \rangle = \langle f, A^\dagger g \rangle$ for all $f, g \in \mathcal{S}$.

**Definition 1.3** (Annihilation and Creation operators). For $\hbar > 0$, let $a_\hbar$ be the **annihilation operator** acting on $L^2(m)$ defined so that $D(a_\hbar) = \mathcal{S}$ and

$$\langle a_\hbar f, g \rangle := \int_{\mathbb{R}} \frac{\hbar}{2} (xf(x) + \partial_x f(x)) \overline{g(x)} \, dm(x) \quad \forall f, g \in \mathcal{S}.$$ 

(1.5)

The corresponding **creation operator** is $a_\hbar^{\dagger}$ - the formal adjoint of $a_\hbar$, i.e.

$$\langle a_\hbar^{\dagger} f, g \rangle := \int_{\mathbb{R}} \frac{\hbar}{2} (xf(x) - \partial_x f(x)) \overline{g(x)} \, dm(x) \quad \forall f, g \in \mathcal{S}.$$ 

(1.6)

We write $a$ and $a^{\dagger}$ for $a_1$ and $a_1^{\dagger}$ respectively when $\hbar = 1$.

Notice that both the creation $\left(a_\hbar^{\dagger}\right)$ and annihilation ($a_\hbar$) operators preserve $\mathcal{S}$ and satisfy the canonical commutation relations (CCRs),

$$\left[ a_\hbar, a_\hbar^{\dagger} \right] = \hbar I|_{\mathcal{S}}.$$  

(1.7)

For each $t \in \mathbb{R}$ and $\alpha_0 \in \mathbb{C}$ we also define two operators, $a(t, \alpha_0)$ and $a^{\dagger}(t, \alpha_0)$ acting on $\mathcal{S}$ by,

$$a(t, \alpha_0) = \gamma(t) a + \delta(t) a^{\dagger}$$

(1.8)

and

$$a^{\dagger}(t, \alpha_0) = \overline{\gamma(t)} a^{\dagger} + \overline{\delta(t)} a,$$ 

(1.9)
where $\gamma(t)$ and $\delta(t)$ are determined as in Eq. (1.3). Because we are going to fix $\alpha_0 \in \mathbb{C}$ once and for all in this paper we will simply write $a(t)$ and $a^\dagger(t)$ for $a(t, \alpha_0)$ and $a^\dagger(t, \alpha_0)$ respectively. These operators still satisfy the CCRs, indeed making use of Eq. (2.12) below we find,

$$[a(t), a^\dagger(t)] = [\gamma(t) a^\dagger + \delta(t) a, \gamma(t) a + \delta(t) a^\dagger] = \left(\left|\gamma(t)^2 - |\delta(t)|^2\right| I = I. \right.$$  \hspace{1cm} (1.10)

This result also may be deduced from Theorem 5.13 below.

**Definition 1.4** (Harmonic Oscillator Hamiltonian). The Harmonic Oscillator Hamiltonian is the self-adjoint operator on $L^2(m)$ defined by

$$\mathcal{H}_\hbar := a^\dagger a = \hbar^2 \partial_x^2 + \hbar^2 \partial_x \partial_x.$$  \hspace{1cm} (1.11)

As above we write $\mathcal{N}$ for $\mathcal{N}_1$ and refer to $\mathcal{N}$ as the Number operator.

**Remark 1.5.** The operator, $\mathcal{H}_\hbar$, is self-adjoint by a well know theorem of Von Neumann (see for example Theorem 3.24, p. 275 in [17]). It is also standard and well known (or see Corollary 3.26 below) that

$$D(a^\dagger a^\dagger) = D(a^\dagger a) = D\left(\mathcal{H}_\hbar^{1/2}\right) = D(\partial_x) \cap D(M_x).$$

**Definition 1.6** (Weyl Operators). For $\alpha := (\xi + i\pi)/\sqrt{2} \in \mathbb{C}$ as in Eq. (2.1), define the unitary Weyl Operator $U(\alpha)$ on $L^2(m)$ by

$$U(\alpha) = e^{i(a^\dagger a - a^\dagger a)} = e^{i(\pi M_x - \frac{1}{2} \partial_x)}.$$  \hspace{1cm} (1.12)

More generally, if $\hbar > 0$, let

$$U_\hbar(\alpha) = U\left(\frac{\alpha}{\sqrt{\hbar}}\right) = \exp\left(\frac{1}{\hbar} (\alpha \cdot a^\dagger \alpha - \alpha \cdot a a^\dagger)\right).$$  \hspace{1cm} (1.13)

The symmetric operator, $i \left(\alpha \cdot a^\dagger \alpha^\dagger - \alpha \cdot a a^\dagger\right)$, can be shown to be essentially self adjoint on $S$ by the same methods used to show $\frac{1}{2} \partial_x$ is essentially self adjoint on $C_0^\infty(\mathbb{R})$ in [12, Proposition 9.29]. Hence the Weyl operators, $U_\hbar(\alpha)$, are well defined unitary operators by Stone’s theorem. Alternatively, see Proposition 2.4 below for an explicit description of $U_\hbar(\alpha)$.

**Definition 1.7.** Given an operator $A$ on $L^2(m)$ let

$$\langle A \rangle_\psi := \langle A\psi, \psi \rangle$$

denote the expectation of $A$ relative to a normalized state $\psi \in D(A)$. The variance of $A$ relative to a normalized state $\psi \in D(A^2)$ is then defined as

$$\text{Var}_\psi(A) := \langle A^2 \rangle_\psi - \langle A \rangle_\psi^2.$$  

From Corollary 3.10 below; if $\psi \in S$ is a normalized state and $P(\theta, \theta^*)$ is a non-commutative polynomial in two variables $\{\theta, \theta^*\}$, then

$$\langle P(a^\dagger a) \rangle_{U_\hbar(\alpha)\psi} = P(\alpha, \alpha) + O\left(\sqrt{\hbar}\right)$$

$$\text{Var}_{U_\hbar(\alpha)\psi}(P(a^\dagger a)) = O\left(\sqrt{\hbar}\right).$$
Consequently, $U_h(\alpha) \psi$ is a state which is concentrated in phase space near the $\alpha$ and are therefore reasonable quantum mechanical approximations of the classical state $\alpha$.

**Definition 1.8 (Non-Commutative Laws).** If $A_1, \ldots, A_k$ are operators on $L^2(m)$ having a common dense domain $D$ such that $A_j D \subset D, D \subset D \left(A_j^*\right)$, and $A_j^* D \subset D$ for $1 \leq j \leq k$, then for a unit vector, $\psi \in D$, and a non-commutative polynomial, $$P := P(\theta_1, \ldots, \theta_k, \theta_1^*, \ldots, \theta_k^*)$$ in $2k$ indeterminants, we let $$\mu(P) := \langle P(A_1, \ldots, A_k, A_1^*, \ldots, A_k^*) \psi, \psi \rangle.$$ The linear functional, $\mu$, on the linear space of non-commutative polynomials in $2k$ variables is referred to as the law of $(A_1, \ldots, A_k)$ relative to $\psi$ and we will in the sequel denote $\mu$ by $\text{Law}_\psi(A_1, \ldots, A_k)$.

### 1.2. Main results

**Theorem 1.16** and Corollaries 1.18 and 1.20 below on the convergence of correlation functions are the main results of this paper. [The proofs of these results will be given Section 9]. The results of this paper will be proved under the Assumption 1 described below. First we need a little more notation.

**Definition 1.9.** Let $S$ be a dense subspace of a Hilbert space $K$ and $A$ be an operator on $K$. We say $A$ is symmetric on $S$ provided, $S \subseteq D(A)$ and $A|S \subseteq A|_S^*$, i.e. $\langle Af, g \rangle = \langle f, Ag \rangle$ for all $f, g \in S$.

We now introduce three different partial ordering on symmetric operators on a Hilbert space.

**Notation 1.10.** Let $S$ be a dense subspace of a Hilbert space, $K$, and $A$ and $B$ be two densely defined operators on $K$.

1. We write $A \preceq_S B$ if both $A$ and $B$ are symmetric on $S$ and $$\langle A\psi, \psi \rangle_K \preceq \langle B\psi, \psi \rangle_K \text{ for all } \psi \in S.$$  
2. We write $A \preceq B$ if $A \preceq_D(B)$, i.e. $D(B) \subset D(A)$, $A$ and $B$ are both symmetric on $D(B)$, and $$\langle A\psi, \psi \rangle_K \preceq \langle B\psi, \psi \rangle_K \text{ for all } \psi \in D(B).$$  
3. If $A$ and $B$ are non-negative (i.e. $0 \preceq A$ and $0 \preceq B$) self-adjoint operators on a Hilbert space $K$, then we say $A \preceq B$ if and only if $D\left(\sqrt{B} \right) \subseteq D\left(\sqrt{A}\right)$ and $$\|\sqrt{A}\psi\| \leq \|\sqrt{B}\psi\| \text{ for all } \psi \in D\left(\sqrt{B}\right).$$

Interested readers may read Section 10.3 of [24] to learn more properties and relations among these different partial orderings. Let us now record the main assumptions which will be needed for the main theorems in this paper. In this assumption, $\mathbb{R} \langle \theta, \theta^* \rangle$ denotes the subspace of non-commutative polynomials with real coefficients, see Subsection 2.3.

**Assumption 1.** We say $H(\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle$ satisfies Assumption 1. if, $H$ is symmetric (see Definition 2.10), $d = \deg_\theta H \geq 2$ (see Notation 2.8) is even and $H_h := H(h_{1/2}, \hat{a}_h, \hat{a}_h^*)$ satisfies; there exists constants $C > 0$, $C_\beta > 0$ for $\beta \geq 0$, and $1 \geq \eta > 0$ such that for all $h \in (0, \eta)$,
(1) $H_\hbar$ is self-adjoint and $H_\hbar + C \geq I$, and

(2) for all $\beta \geq 0$,

$$N_\hbar^\beta \preceq C_\beta (H_\hbar + C)^\beta.$$  \hfill (1.14)

The next Proposition provides a simple class of example $H \in \mathbb{R} \langle \theta, \theta^* \rangle$ satisfying Assumption 1 whose infinite dimensional analogues feature in some of the papers involving Bose-Einstein condensation, see for example, \cite{1, 23}.

**Proposition 1.11** ($\langle \theta^* \theta \rangle$ – examples). Let $p (x) \in \mathbb{R} [x]$ (the polynomials in $x$ with real coefficients) and suppose \( \deg (p) \geq 1 \) and the leading order coefficient is positive. Then $H (\theta, \theta^*) = p (\theta^* \theta) \in \mathbb{R} (\theta, \theta^*)$ will satisfy the hypothesis of Assumption 1.

**Proof.** First we will show

$$H_\hbar = p \left( a_\hbar^\dagger a_\hbar \right) = p (N_\hbar).$$

We know that $p (N_\hbar)$ is self-adjoint and by Corollaries 3.17 and 3.30 we have

$$p (N_\hbar) = p \left( a_\hbar^\dagger a_\hbar \right) \subset p \left( a_\hbar^\dagger a_\hbar \right).$$

Taking adjoint of this inclusion implies

$$\overline{p \left( a_\hbar^\dagger a_\hbar \right) }^\dagger = p \left( a_\hbar^\dagger a_\hbar \right) \subset p (N_\hbar)^* = p (N_\hbar).$$

However, since $p \left( a_\hbar^\dagger a_\hbar \right)$ is symmetric we also have

$$p \left( a_\hbar^\dagger a_\hbar \right) \subset p \left( a_\hbar^\dagger a_\hbar \right)^* = p \left( a_\hbar^\dagger a_\hbar \right) \subset p (N_\hbar),$$

which implies

$$p \left( a_\hbar^\dagger a_\hbar \right) \subset p (N_\hbar).$$

Since there exists $C > 0$ and $C_\beta$ for any $\beta \geq 0$ such that $x \leq C_\beta (p (x) + C)$ for $x \geq 0$, it follows by the spectral theorem that $H_\hbar$ satisfies Eq. (1.14).

The next example provides a much broader class of $H \in \mathbb{R} \langle \theta, \theta^* \rangle$ satisfying Assumption 1 while the corresponding operators, $H_\hbar$, no longer typically commute with the number operator.

**Example 1.12** (Example Hamiltonians). Let $m \geq 1$, $b_k \in \mathbb{R} [x]$ for $0 \leq k \leq m$, and

$$H (\theta, \theta^*) := \sum_{k=0}^m \frac{(-1)^k}{2^k} (\theta - \theta^*)^k b_k \left( \frac{1}{\sqrt{2}} (\theta + \theta^*) \right) (\theta - \theta^*)^k.$$ \hfill (1.15)

With the use of Eqs. (1.15) and (1.16), it follows

$$H_\hbar = \sum_{k=0}^m h^k \partial_x^k M_{b_k} (\sqrt{\tau_x}) \partial_x^k \text{ on } S$$ \hfill (1.16)

If

(1) each $b_k (x)$ is an even polynomial in $x$ with positive leading order coefficient, and $b_m > 0$, and

(2) $\deg_x (b_0) \geq 2$ and $\deg_x (b_k) \leq \deg_x (b_{k-1})$ for $1 \leq k \leq m$,
then by Corollary 1.10 in [4], \( H(\theta, \theta^*) \) satisfies Assumption 1. In particular, if \( m > 0 \) and \( V \in \mathbb{R}[x] \) such that \( \deg_x V = 2\mathbb{N} \) such that \( \lim_{x \to \infty} V(x) = \infty \), then
\[
H(\theta, \theta^*) = -\frac{m}{2} \left( \frac{\theta - \theta^*}{\sqrt{2}} \right)^2 + V \left( \frac{1}{\sqrt{2}} (\theta + \theta^*) \right)
\]
and
\[
H(a_h, a_h^\dagger) = -\frac{1}{2} \hbar m \hat{\partial}_x^2 + V \left( \sqrt{\hbar} x \right)
\]
satisfies Assumption 1.3.

**Remark 1.13.** The essential self-adjointness of \( H(a_h, a_h^\dagger) \) in Eq.
(1.18) and all of its non-negative integer powers on \( S \) may be deduced using results in Kato [18] and Chernoff [2]. This fact along with the Eq. (1.13) restricted to hold on \( S \) and for \( \beta \in \mathbb{N} \) could be combined together to prove Eq. (1.14) for all \( \beta \geq 0 \) as is explained in Lemma 6.13 in [4].

Using Theorem A.1 of [4], for any symmetric noncommutative polynomial, \( H(\theta, \theta^*) \in \mathbb{R}(\theta, \theta^*) \), there exists polynomials, \( b_i \left( \sqrt{\hbar} x \right) \in \mathbb{R} \left( \sqrt{\hbar} x \right) \), (polynomials in \( \sqrt{\hbar} \) and \( x \) with real coefficients), such that
\[
H(a_h, a_h^\dagger) = \sum_{k=0}^m \hbar^k \partial_x^k (\sqrt{\hbar} x) \partial_x^{-k} = H(a_h, a_h^\dagger) \text{ on } S.
\]

If it so happens that these \( b_k \left( \sqrt{\hbar} x \right) \) satisfy the assumptions of Corollary 1.10 of [4], then Assumption 1 will hold for this \( H \).

**Example 1.14.** Let
\[
H(\theta, \theta^*) = \theta^4 + \theta^4 - \frac{7}{8} (\theta - \theta^*) (\theta + \theta^*)^2 (\theta - \theta^*) \in \mathbb{R}(\theta, \theta^*).
\]
By using product rule repeatedly with Eqs. (1.5) and (1.6), it follows that
\[
H(a_h, a_h^\dagger) = \hbar^2 \partial_x^2 b_2 \left( \sqrt{\hbar} x \right) \partial_x^2 - \hbar \partial_x b_1 \left( \sqrt{\hbar} x \right) \partial_x + b_0 \left( \sqrt{\hbar} x \right)
\]
where
\[
b_0 \left( \sqrt{\hbar} x \right) = \frac{1}{2} x^4 + \frac{3 \hbar^2}{2}, \ b_1 \left( \sqrt{\hbar} x \right) = \frac{1}{2} x^2, \text{ and } b_2 \left( \sqrt{\hbar} x \right) = \frac{1}{2}.
\]
These polynomials satisfy the assumptions of Corollary 1.10 of [4] and therefore \( H(\theta, \theta^*) \) in Eq. (1.19) satisfies Assumption 1.3.

**Notation 1.15.** Given a non-commutative polynomial
\[
P(\{\theta_i, \theta_i^*\}_{i=1}^n) := P(\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*) \in \mathbb{C} \langle \theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^* \rangle,
\]
in \( 2n \) indeterminants,
\[
\Lambda_n := \{\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*\},
\]
let \( p_{\min} \) denote the minimum degree among all non-constant monomials terms appearing in \( P(\{\theta_i, \theta_i^*\}_{i=1}^n) \). In more detail there is a constant, \( P_0 \in \mathbb{C} \), such that \( P(\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*) - P_0 \) may be written as a linear combination in words in the alphabet, \( \Lambda_n \), which have length no smaller than \( p_{\min} \).

**Theorem 1.16.** Suppose \( H(\theta, \theta^*) \in \mathbb{R}(\theta, \theta^*) \), \( d = \deg_{\theta} H > 0 \) and \( 1 \geq \eta > 0 \) satisfy Assumptions 1.3 \( \alpha_0 \in \mathbb{C} \), \( \psi \in S \) is an \( L^2(m) \) - normalized state and then let;
(1) α(t) ∈ C be the solution (which exists for all time by Proposition 3.8) to Hamilton’s (classical) equations of motion (1.14).

(2) a(t) = a(t, α(t)) be the annihilation operator on L²(m) as in Eq. (1.13), and

(3) A_h(t) denote a_h in the Heisenberg picture, i.e.

\[ A_h(t) := e^{iH_nt/h}a_h e^{-iH_nt/h}. \]  

(1.22)

If \( \{ t_i \}_{i=1}^{n} \subset \mathbb{R} \) and \( P(\{ \theta_i, \theta_i^* \}_{i=1}^{n}) \in C(\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*) \) is a non-commutative polynomial in 2n - indeterminants, then for \( 0 < h < \eta \), we have

\[ \langle P(\{ A_h(t_i) - \alpha(t_i), A_h^*(t_i) - \bar{\alpha}(t_i) \}_{i=1}^{n}) \rangle_{U_h(\alpha)} \psi \]

\[ = \langle P(\{ \sqrt{h} a(t_i), \sqrt{h} a^*(t_i) \}_{i=1}^{n}) \rangle_{\psi} + O\left(\frac{\eta^{\min+1}}{h}\right). \]  

(1.23)

Remark 1.17. The left member of Eq. (1.23) is well defined because: 1) \( U_h(\alpha) \mathcal{S} = \mathcal{S} \) (see Proposition 2.4) and 2) \( e^{iH_h/t}a_h e^{-iH_h/t} \) both preserve \( \mathcal{S} \) for all \( t \in \mathbb{R} \).

This theorem is a variant of the results in Hepp [15] which now allows for unbounded observables. It should be emphasized that the operators, \( a(t) \), are constructed using only knowledge of solutions to the classical ordinary differential equations of motions while the construction of \( A_h(t) \) requires knowledge of the quantum mechanical evolution. As an easy consequence of Theorem 1.16 we may conclude that

\[ \text{Law}_{U_h(\alpha)} \psi(\{ A_h(t_i) \}_{i=1}^{n}) \cong \text{Law}_{\psi}(\{ \alpha(t_i) + \sqrt{h} a(t_i) \}_{i=1}^{n}) \text{ for } 0 < h \ll 1. \]  

(1.24)

The precise meaning of Eq. (1.24) is given in the following corollary.

Corollary 1.18. If we assume the same conditions and notations as in Theorem 1.16 then (for \( 0 < h < \eta \))

\[ \langle P(\{ A_h(t_i), A_h^*(t_i) \}_{i=1}^{n}) \rangle_{U_h(\alpha)} \psi \]

\[ = \langle P(\{ \alpha(t_i) + \sqrt{h} a(t_i), \alpha^*(t_i) + \sqrt{h} a^*(t_i) \}_{i=1}^{n}) \rangle_{\psi} + O(h). \]  

(1.25)

By expanding out the right side of Eq. (1.25), it follows that

\[ \langle P(\{ A_h(t_i), A_h^*(t_i) \}_{i=1}^{n}) \rangle_{U_h(\alpha)} \psi \]

\[ = P(\{ \alpha(t_i), \alpha^*(t_i) \}_{i=1}^{n}) + \sqrt{h} \langle P(\{ \alpha(t_i), a(t_i) \}_{i=1}^{n}) \rangle_{\psi} + O(h) \]  

(1.26)

where \( P(\{ \alpha(t_i), \theta_i \}_{i=1}^{n}) \) is a degree one homogeneous polynomial of \( \{ \theta_i, \theta_i^* \}_{i=1}^{n} \) with coefficients depending smoothly on \( \{ \alpha(t_i) \}_{i=1}^{n} \). Equation (1.26) states that the quantum expectation values,

\[ \langle P(\{ A_h(t_i), A_h^*(t_i) \}_{i=1}^{n}) \rangle_{U_h(\alpha)} \psi, \]  

(1.27)

closely track the corresponding classical values \( P(\{ \alpha(t_i), \alpha(t_i) \}_{i=1}^{n}) \). The \( \sqrt{h} \) term in Eq. (1.26) represent the first quantum corrections (or fluctuations) beyond the leading order classical behavior.
Remark 1.19. If both \( H (\theta, \theta^*) \), \( \tilde{H} (\theta, \theta^*) \) \( \in \mathbb{R} (\theta, \theta^*) \) both satisfy Assumption 1 and are such that \( H^3 (\alpha) := H (\alpha, \alpha) \) and \( \tilde{H}^3 (\alpha) := \tilde{H} (\alpha, \alpha) \) are equal modulo a constant, then Eq. (1.23) also holds with the \( A_h (t_i) \) and \( A^\dagger_h (t_i) \) appearing on the left side of this equation being replaced by

\[
e^{i\tilde{H}_h t_i / \hbar} a_h e^{-iH_h t_i / \hbar} \quad \text{where} \quad \tilde{H}_h := \tilde{H} (a_h, a_h^\dagger).
\]

In other words, if we view \( H \) and \( \tilde{H} \) as two "quantizations" of \( H^3 \), then the quantum expectations relative to \( H \) and \( \tilde{H} \) agree up to order \( \sqrt{\hbar} \).

Corollary 1.20. Under the same conditions in Theorem 1.16, we let \( \psi_h = U_h (\alpha_0) \psi \). As \( \hbar \to 0^+ \), we have

\[
\left\langle P \left( \left\{ A_h (t_i), A^\dagger_h (t_i) \right\}_{i=1}^n \right) \right\rangle_{\psi_h} \to \left\langle P \left( \left\{ \alpha_i (t), \bar{\alpha}_i (t) \right\}_{i=1}^n \right) \right\rangle_{\psi}.
\]

We abbreviate this convergence by saying

\[
\text{Law}_{\psi_h} \left( \left\{ A_h (t_i) - \alpha (t_i), \frac{A^\dagger_h (t_i) - \bar{\alpha} (t_i)}{\sqrt{\hbar}} \right\}_{i=1}^n \right) \to \text{Law}_\psi \left( \left\{ a (t_i), a^\dagger (t_i) \right\}_{i=1}^n \right).
\]

1.3. Comparison with Hepp. The primary difference between our results and Hepp’s results in [13] is that we allow for non-bounded (polynomial in \( a_h \) and \( a_h^\dagger \)) observables where as Hepp’s “observables” are unitary operators of the form

\[
U_h (z) = \exp \left( z a_h - \bar{z} a_h^\dagger \right) \quad \text{for} \quad z \in \mathbb{C}.
\]

As these observables are bounded operators, Hepp is able to prove his results under less restrictive assumptions than those in Assumption 1 of this paper. For the most part Hepp primarily works with Hamiltonian operators in the Schrödinger form of Eq. (1.17) where the potential function, \( V \), is not necessarily restricted to be a polynomial function. [Hepp does however allude to being able to allow for more general Hamiltonian operators which are not necessarily of the Schrödinger form in Eq. (1.17).] The analogue of Corollary 1.20 (for \( n = 1 \)) in Hepp [13], is his Theorem 2.1 which states; if \( z \in \mathbb{C} \) and \( \psi \in L^2 (\mathbb{R}) \), then

\[
\lim_{\hbar \to 0} \left\langle \exp \left( z \frac{a_h - \alpha (t)}{\sqrt{\hbar}} - \bar{z} \frac{a_h^\dagger - \bar{\alpha} (t)}{\sqrt{\hbar}} \right) \right\rangle_{\psi_{h(t)}} = \left\langle \exp \left( za (t) - \bar{z} a^\dagger (t) \right) \right\rangle_{\psi},
\]

where \( \psi_h (t) := e^{-iH_h t / \hbar} U_h (\alpha_0) \psi \).

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2. Background and Setup

In this section we will expand on the basic setup described above and recall some basic facts that will be needed throughout the paper.
2.1. Classical Setup. In this paper, we take configuration space to be \( \mathbb{R} \) so that our classical state space is \( T^* \mathbb{R} \cong \mathbb{R}^2 \). [Extensions to higher and to infinite dimensions will be considered elsewhere.] Following Hepp [15], we identify \( T^* \mathbb{R} \) with \( \mathbb{C} \) via

\[
T^* \mathbb{R} \ni (\xi, \pi) \mapsto \alpha := \frac{1}{\sqrt{2}}(\xi + i\pi). \tag{2.1}
\]

Taking in account the “\( \sqrt{2} \)” above, we set

\[
\frac{\partial}{\partial \alpha} := \frac{1}{\sqrt{2}}(\partial_\xi - i\partial_\pi) \quad \text{and} \quad \frac{\partial}{\partial \bar{\alpha}} := \frac{1}{\sqrt{2}}(\partial_\xi + i\partial_\pi)
\]

so that \( \frac{\partial}{\partial \alpha} \alpha = 1 = \frac{\partial}{\partial \bar{\alpha}} \bar{\alpha} \) and \( \frac{\partial}{\partial \bar{\alpha}} \alpha = 0 = \frac{\partial}{\partial \alpha} \bar{\alpha} \). As usual given a smooth real valued function \( H^\text{cl}(\xi, \pi), \) on \( T^* \mathbb{R} \) we say \( (\xi(t), \pi(t)) \) solves Hamilton’s equations of motion provided,

\[
\dot{\xi}(t) = H^\text{cl}_\pi(\xi(t), \pi(t)) \quad \text{and} \quad \dot{\pi}(t) = -H^\text{cl}_\xi(\xi(t), \pi(t)) \tag{2.2}
\]

where \( H^\text{cl}_\pi := \partial H^\text{cl}/\partial \pi \) and \( H^\text{cl}_\xi := \partial H^\text{cl}/\partial \xi \). A simple verifications shows; if

\[
\alpha(t) := \frac{1}{\sqrt{2}}(\xi(t) + i\pi(t)),
\]

then \( (\xi(t), \pi(t)) \) solves Hamilton’s Eqs. (2.2) iff \( \alpha(t) \) satisfies

\[
i\dot{\alpha}(t) = \left( \frac{\partial}{\partial \bar{\alpha}} \bar{H}^\text{cl} \right)(\alpha(t)) \tag{2.3}
\]

where

\[
\bar{H}^\text{cl}(\alpha) := H^\text{cl}(\xi, \pi) \quad \text{where} \quad \alpha = \frac{1}{\sqrt{2}}(\xi + i\pi) \in \mathbb{C}.
\]

In the future we will identify \( \bar{H}^\text{cl} \) with \( H^\text{cl} \) and drop the tilde from our notation.

Example 2.1. If \( H(\alpha) = |\alpha|^2 + \frac{i}{2} |\alpha|^4 \), then the associated Hamiltonian equations of motion are given by

\[
i\dot{\alpha} = \frac{\partial}{\partial \bar{\alpha}} \left( \alpha \bar{\alpha} + \frac{1}{2} \alpha^2 \bar{\alpha}^2 \right) = \alpha + \alpha^2 \bar{\alpha} = \alpha + |\alpha|^2 \alpha.
\]

Proposition 2.2. Let \( z(t) := \Phi(t, 0, 0) \) be the real differential of the flow associated to Eq. (1.1) as in Eq. (1.2). Then \( z(t) \) satisfies \( z(0) = z \) and

\[
i\dot{z}(t) = u(t) \bar{z}(t) + v(t) z(t), \tag{2.4}
\]

where

\[
u(t) := \left( \frac{\partial^2}{\partial \alpha^2} \bar{H}^\text{cl} \right)(\alpha(t)) \in \mathbb{C} \quad \text{and} \quad v(t) = \left( \frac{\partial^2}{\partial \alpha \partial \bar{\alpha}} \bar{H}^\text{cl} \right)(\alpha(t)) \in \mathbb{R}. \tag{2.5}
\]

Moreover, if we express \( z(t) = \gamma(t) z + \delta(t) \bar{z} \) as in Eq. (1.3) and let

\[
\Lambda(t) := \begin{bmatrix} \gamma(t) & \delta(t) \\ \bar{\delta}(t) & \bar{\gamma}(t) \end{bmatrix}, \tag{2.6}
\]

then

\[
\det \Lambda(t) = |\gamma(t)|^2 - |\delta(t)|^2 = 1 \forall t \in \mathbb{R}
\]

\[\text{ Later } H^\text{cl} \text{ will be the symbol of a symmetric element of } H \in \mathbb{C}(\theta, \theta^*) \text{ as described in subsection } \]
and
\[ i \dot{A}(t) = \begin{bmatrix} v(t) & u(t) \\ -\bar{u}(t) & -\bar{v}(t) \end{bmatrix} A(t) \quad \text{and} \quad A(0) = I. \]  \tag{2.7}

**Proof.** First recall if \( f : \mathbb{C} \to \mathbb{C} \) is a smooth function (not analytic in general), then the real differential, \( z \to f'(\alpha) z := \frac{d}{d\alpha}|_0 f(\alpha + sz) \), of \( f \) at \( \alpha \) satisfies
\[ f'(\alpha) z = \left( \frac{\partial}{\partial \alpha} f \right)(\alpha) z + \left( \frac{\partial}{\partial \alpha} f \right)(\alpha) \bar{z}. \]  \tag{2.8}

By definition \( \Phi(t,\alpha_0) \) satisfies the differential equation,
\[ i \dot{\Phi}(t,\alpha_0) = \left( \frac{\partial}{\partial \alpha} H^c \right) (\Phi(t,\alpha_0)) \quad \text{and} \quad \Phi(0,\alpha_0) = \alpha_0. \]

Differentiating this equation relative to \( \alpha_0 \) using the chain rule along with Eq. (2.8) shows \( z(t) := \Phi^t(t,\alpha_0) z \) satisfies Eq. (2.4). The fact that \( v(t) \) is real valued follows from its definition in Eq. (2.5) and the fact that \( H^c \) is a real valued function.

Inserting the expression, \( z(t) = \gamma(t) z + \delta(t) \bar{z} \), into Eq. (2.4) one shows after a little algebra that,
\[ i \dot{\gamma}(t) z + i \dot{\delta}(t) \bar{z} = (u(t) \bar{\delta}(t) + v(t) \gamma(t)) z + (u(t) \bar{\gamma}(t) + v(t) \delta(t)) \bar{z} \]
from which we conclude that \((\gamma(t),\delta(t)) \in \mathbb{C}^2\) satisfy the equations
\begin{align*}
&I. \quad i \dot{\gamma}(t) = u(t) \bar{\delta}(t) + v(t) \gamma(t) \quad \text{and} \\
&I. \quad i \dot{\delta}(t) = u(t) \bar{\gamma}(t) + v(t) \delta(t). \tag{2.9, 2.10}
\end{align*}

Using these equations we then find;
\[ \frac{d}{dt} \left(|\gamma|^2 - |\delta|^2\right) = 2 \text{Re} \left( \dot{\gamma} \bar{\gamma} - \dot{\delta} \bar{\delta} \right) \]
\[ = 2 \text{Re} \left( -i(u\delta + v\gamma) \bar{\gamma} + i(u\bar{\gamma} + v\delta) \bar{\delta} \right) \]
\[ = 2 \text{Re} \left( -iv|\gamma|^2 + iv|\delta|^2 \right) = 0. \tag{2.11}
\]
\[ \text{Since } z(0) = z, \gamma(0) = 1 \text{ and } \delta(0) = 1 \text{ and so from Eq. (2.11) we learn} \]
\[ \left(|\gamma|^2 - |\delta|^2\right)(t) = \left(|\gamma|^2 - |\delta|^2\right)(0) = 1^2 - 0^2 = 1. \tag{2.12}
\]

Finally, Eq. (2.7) is simply the vector form of Eqs. (2.9) and (2.10). \( \square \)

**Remark 2.3.** Equation (2.4) may be thought of as the time dependent Hamiltonian flow,
\[ i \dot{z}(t) = \frac{\partial q(t,\cdot)}{\partial z}(z(t)) \]
where \( q(t,z) \in \mathbb{R} \) is the quadratic time dependent Hamiltonian defined by
\[ q(t : z) = \frac{1}{2} u(t) z^2 + \frac{1}{2} \bar{u}(t) \bar{z}^2 + v(t) \bar{z} \bar{z} \]
\[ = \frac{1}{2} \left( \frac{\partial^2}{\partial \alpha^2} H^c \right)(\alpha(t)) z^2 + \frac{1}{2} \left( \frac{\partial^2}{\partial \alpha^2} H^c \right)(\alpha(t)) \bar{z}^2 \]
\[ + \left( \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \bar{\alpha}} H^c \right)(\alpha(t)) |z|^2. \]
2.2. **Quantum Mechanical Setup.** Recall that our quantum mechanical Hilbert space is taken to be the space of Lebesgue square integrable complex valued functions on $\mathbb{R}$ ($L^2(m) := L^2(\mathbb{R}, m)$) equipped with the usual $L^2(m)$-inner product as in Eq. (1.4). To each $\hbar > 0$ ($\hbar$ is to be thought of as Planck’s constant), let

$$q_\hbar := \sqrt{\hbar}M_x \text{ and } p_\hbar := \sqrt{\frac{\hbar}{i}} \frac{d}{dx}$$

interpreted as self-adjoint operators on $L^2(m) := L^2(\mathbb{R}, m)$ with domains

$$D(q_\hbar) = \{ f \in L^2(m) : x \to xf(x) \in L^2(m) \}$$

and

$$D(p_\hbar) = D\left( \frac{d}{dx} \right) = \{ f \in L^2(m) : x \to f(x) \text{ is A.C. and } f' \in L^2(m) \}$$

where A.C. is an abbreviation of absolutely continuous. Using Corollary 3.26 below, the annihilation and creation operators in Definition 1.3 may be expressed as

$$\bar{a}_\hbar := \frac{q_\hbar + ip_\hbar}{\sqrt{2}} = \sqrt{\frac{\hbar}{2}} \left( M_x + \frac{d}{dx} \right)$$

and

$$a_\hbar^* := \frac{q_\hbar - ip_\hbar}{\sqrt{2}} = \sqrt{\frac{\hbar}{2}} \left( M_x - \frac{d}{dx} \right).$$

### 2.2.1. Weyl Operator.

**Proposition 2.4.** Let $\alpha := (\xi + i\pi) / \sqrt{2} \in \mathbb{C}$, $\hbar > 0$, and $U(\alpha)$ and $U_\hbar(\alpha)$ be as in Definition 1.6. Then

$$(U(\alpha)f)(x) = \exp\left(i\pi \left(x - \frac{1}{2} \xi \right)\right) f(x - \xi) \quad \forall f \in L^2(m),$$

$U(\alpha)S = S$,

$$U_\hbar(\alpha)^* a_\hbar U_\hbar(\alpha) = a_\hbar + \alpha,$$

$$U_\hbar(\alpha)^* a_\hbar^* U_\hbar(\alpha) = a_\hbar^* + \bar{\alpha},$$

as identities on $S$.

**Proof.** Given $f \in S$ let $F(t, x) := (U(t\alpha)f)(x)$ so that

$$\frac{\partial}{\partial t} F(t, x) = \left( i\pi x - \xi \frac{\partial}{\partial x} \right) F(t, x) \quad \text{with} \quad F(0, x) = f(x).$$

Solving this equation by the method of characteristics then gives Eq. (2.16). [Alternatively one easily verifies directly that

$$F(t, x) := \exp(it\pi(x - \frac{1}{2}t\xi))f(x - t\xi)$$

solves Eq. (2.19).] It is clear from Eq. (2.16) that $U(\alpha)S \subset S$ and $U(-\alpha)U(\alpha) = I$ for all $\alpha \in \mathbb{C}$. Therefore $S \subset U(-\alpha)S$. Replacing $\alpha$ by $-\alpha$ in this last inclusion allows us to conclude that $U(\alpha)S = S$. The formula in Eq. (2.19) also directly extends to $L^2(m)$ where it defines a unitary operator. The identities in Eqs. (2.17) and (2.18) for $\hbar = 1$ follows by simple direct calculations using Eq. (2.16). The case of general $\hbar > 0$ then follows by simple scaling arguments. ■
Remark 2.5. Another way to prove Eq. (2.17) is to integrate the identity,
\[
\frac{d}{dt}U_h(ta)^* a_h U_h(ta) = -U_h(ta)^* \left[ \frac{1}{\hbar} \left( \alpha \cdot a_h^\dagger - \bar{\alpha} \cdot a_h \right), a_h \right] U_h(ta) = \alpha,
\]
with respect to \( t \) on \( \mathcal{S} \) and the initial condition \( U(0) = I \).

**Definition 2.6.** Suppose that \( \{W(t)\}_{t \in \mathbb{R}} \) is a one parameter family of (possibly) unbounded operators on a Hilbert space \( \langle K, \langle \cdot, \cdot \rangle \rangle \). Given a dense subspace, \( D \subset K \), we say \( W(t) \) is strongly \( \| \cdot \|_K \)-norm differentiable on \( D \) if 1) \( D \subset D(W(t)) \) for all \( t \in \mathbb{R} \) and 2) for all \( \psi \in D \), \( t \to W(t) \psi \) is \( \| \cdot \|_K \)-norm differentiable. For notational simplicity we will write \( W(t) \psi \) for \( \frac{d}{dt} [W(t) \psi] \).

**Proposition 2.7.** If \( \mathbb{R} \ni t \to \alpha(t) \in \mathbb{C} \) is a \( C^1 \) function and \( \mathcal{N} := \mathcal{N}_h|_{h=1} \) the number operator defined in Eq. (2.11), then \( \{U(\alpha(t))\}_{t \in \mathbb{R}} \) is strongly \( L^2 \) \((m)\)-norm differentiable on \( D(\sqrt{\mathcal{N}}) \) as in the Definition 2.6 and for all \( f \in D(\sqrt{\mathcal{N}}) \) we have
\[
\frac{d}{dt} (U(\alpha(t)) f) = \left( \dot{\alpha}(t) a^* - \bar{\alpha}(t) a + i \text{Im} \left( \alpha(t) \bar{\alpha}(t) \right) \right) U(\alpha(t)) f
= U(\alpha(t)) \left( \dot{\alpha}(t) a^* - \bar{\alpha}(t) a - i \text{Im} \left( \alpha(t) \bar{\alpha}(t) \right) \right) f.
\]
Moreover, \( U(\alpha(t)) \) preserves \( D(\sqrt{\mathcal{N}}), C_c(\mathbb{R}) \), and \( \mathcal{S} \).

**Proof.** From Corollary 3.20 below we know \( D(\partial_x) \cap D(M_x) = D(\sqrt{\mathcal{N}}) \). Using this fact, the proposition is a straightforward verification based on Eq. (2.16). The reader not wishing to carry out these computations may find it instructive to give a formal proof based on the algebraic fact that \( e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \) where \( A \) and \( B \) are operators such that the commutator, \([A,B] := AB - BA\), commutes with both \( A \) and \( B \).}

As we do not wish to make any particular choice of quantization scheme, in this paper we will describe all operators as a non-commutative polynomial functions of \( a_h \) and \( a_h^\dagger \). This is the topic of the next subsection.

### 2.3. Non-commutative Polynomial Expansions.

**Notation 2.8.** Let \( \mathbb{C} \langle \theta, \theta^* \rangle \) be the space of non-commutative polynomials in the non-commutative indeterminates. That is to say \( \mathbb{C} \langle \theta, \theta^* \rangle \) is the vector space over \( \mathbb{C} \) whose basis consists of words in the two letter alphabet, \( \Lambda_1 = \{ \theta, \theta^* \} \), cf. Eq. (1.21). The general element, \( P(\theta, \theta^*) \), of \( \mathbb{C} \langle \theta, \theta^* \rangle \) may be written as
\[
P(\theta, \theta^*) = \sum_{k=0}^{d} \sum_{\mathbf{b} = (b_1, \ldots, b_k) \in \Lambda_1^k} c_k(\mathbf{b}) b_1 \ldots b_k, \tag{2.20}
\]
where \( d \in \mathbb{N}_0 \) and
\[
\{c_k(\mathbf{b}) : 0 \leq k \leq d \text{ and } \mathbf{b} = (b_1, \ldots, b_k) \in \Lambda_1^k \} \subset \mathbb{C}.
\]
If \( c_d : \Lambda_1^d \to \mathbb{C} \) is not the zero function, we say \( d =: \text{deg}_\theta P \) is the degree of \( P \).

It is sometimes convenient to decompose \( P(\theta, \theta^*) \) in Eq. (2.20) as
\[
P(\theta, \theta^*) = \sum_{k=0}^{d} P_k(\theta, \theta^*) \tag{2.21}
\]
where

\[ P_k(\theta, \theta^*) = \sum_{b_1, \ldots, b_k \in \Lambda} c_k(b_1, \ldots, b_k) b_1 \ldots b_k. \]  

(2.22)

Polynomials of the form in Eq. (2.22) are said to be homogeneous of degree \( k \). By convention, \( P_0 := P_0(\theta, \theta^*) \) is just an element of \( \mathbb{C} \). We endow \( \mathbb{C} \{\theta, \theta^*\} \) with its \( \ell^1 \)-norm, \( |\cdot| \), defined for \( P \) as in Eq. (2.20) by

\[ |P| := \sum_{k=0}^d |P_k| \quad \text{where} \quad |P_k| = \sum_{b=(b_1, \ldots, b_k) \in \Lambda_k^\dagger} |c_k(b)|. \]  

(2.23)

**Definition 2.10 (Symmetric Polynomials).** We say \( P \in \mathbb{C} \{\theta, \theta^*\} \) is symmetric provided \( P = P^* \).

If \( \mathcal{A} \) is any unital algebra equipped with an involution, \( \xi \to \xi^\dagger \), and \( \xi \) is any fixed element of \( \mathcal{A} \), then there exists a unique algebra homomorphism

\[ P(\theta, \theta^*) \in \mathbb{C} \{\theta, \theta^*\} \to P(\xi, \xi^\dagger) \in \mathcal{A} \]

determined by substituting \( \xi \) for \( \theta \) and \( \xi^\dagger \) for \( \theta^* \). Moreover, the homomorphism preserves involutions, i.e. \( [P(\xi, \xi^\dagger)]^\dagger = P^*(\xi, \xi^\dagger) \). The two special cases of this construction that we need here are contained in the following two definitions.

**Definition 2.11 (Classical Symbols).** The symbol (or classical) residue of \( P \in \mathbb{C} \{\theta, \theta^*\} \) is the function \( P^{cl} \in \mathbb{C}[z, \bar{z}] \) (= the commutative polynomials in \( z \) and \( \bar{z} \) with complex coefficients) defined by \( P^{cl}(\alpha) := P(\alpha, \bar{\alpha}) \) where we view \( \mathbb{C} \) as a commutative algebra with an involution given by complex conjugation.

**Definition 2.12 (Polynomial Operators).** If \( P(\theta, \theta^*) \in \mathbb{C} \{\theta, \theta^*\} \) is a non-commutative polynomial and \( h > 0 \), then \( P\left(a_h, a_h^\dagger\right) \) is a differential operator on \( L^2(m) \) whose domain is \( \mathcal{S} \). [Notice that \( P\left(a_h, a_h^\dagger\right) \) preserves \( \mathcal{S} \), i.e. \( P\left(a_h, a_h^\dagger\right) \mathcal{S} \subset \mathcal{S} \).] We further let \( P_h := P\left(a_h, a_h^\dagger\right) \) be the closure of \( P\left(a_h, a_h^\dagger\right) \). Any linear differential operator of the form \( P\left(a_h, a_h^\dagger\right) \) for some \( P(\theta, \theta^*) \in \mathbb{C} \{\theta, \theta^*\} \) will be called a polynomial operator.
We introduce the following notation in order to write out $P \left( a_h, a_h^\dagger \right)$ more explicitly.

**Notation 2.13.** For any $h > 0$ let $\Xi_h : \{ \theta, \theta^* \} \rightarrow \left\{ a_h, a_h^\dagger \right\}$ be defined by

$$\Xi_h \left( b \right) = \begin{cases} a_h & \text{if } b = \theta \\ a_h^\dagger & \text{if } b = \theta^* \end{cases} \quad \text{(2.26)}$$

In the special case where $h = 1$ we will simply denote $\Xi_1$ by $\Xi$.

With this notation if $P \in \mathbb{C} \langle \theta, \theta^* \rangle$ is as in Eq. (2.20), then $P \left( a_h, a_h^\dagger \right)$ may be written as,

$$P \left( a_h, a_h^\dagger \right) = \sum_{k=0}^{d} \sum_{b = (b_1, \ldots, b_k) \in \Lambda} c_k \left( b \right) \Xi_h \left( b_1 \right) \cdots \Xi_h \left( b_k \right) \quad \text{(2.27)}$$

or as

$$P \left( a_h, a_h^\dagger \right) = \sum_{k=0}^{d} \sum_{b = (b_1, \ldots, b_k) \in \Lambda} \hbar^{k/2} c_k \left( b \right) u_b \left( a, a^\dagger \right) \quad \text{(2.28)}$$

**Definition 2.14 (Monomial Operators).** Any linear differential operator of the form $u_b \left( a, a^\dagger \right) = \Xi_1 \left( b_1 \right) \cdots \Xi_1 \left( b_k \right)$ for some $b = (b_1, \ldots, b_k) \in \{ \theta, \theta^* \}^k$ and $k \in \mathbb{N}_0$ will be called a **monomial operator**.

**Remark 2.15.** If $H \left( \theta, \theta^* \right) \in \mathbb{C} \langle \theta, \theta^* \rangle$ is symmetric (i.e. $H = H^*$), then;

1. $H \left( a_h, a_h^\dagger \right)$ is a symmetric operator on $\mathcal{S}$ (i.e. $\left[ H \left( a_h, a_h^\dagger \right) \right]^\dagger = H \left( a_h, a_h^\dagger \right)$) for any $h > 0$ and
2. $H^{cl} \left( z \right) := H \left( z, \Xi \right)$ is a real valued function on $\mathbb{C}$.

Indeed,

$$\left[ H \left( a_h, a_h^\dagger \right) \right]^\dagger = H^* \left( a_h, a_h^\dagger \right) = H \left( a_h, a_h^\dagger \right)$$

and

$$H^{cl} \left( \alpha \right) := H \left( \alpha, \alpha \right) = H^* \left( \alpha, \alpha \right) = H \left( \alpha, \alpha \right) = H^{cl} \left( \alpha \right).$$

The main point of this paper is to show under Assumption [I] on $H$ that classical Hamiltonian dynamics associated to $H^{cl}$ determine the limiting quantum mechanical dynamics determined by $H_h := H \left( a_h, a_h^\dagger \right)$.

We have analogous definitions and statements for the non-commutative algebra, $\mathbb{C} \langle \theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^* \rangle$, of non-commuting polynomials in $2n$ – indeterminants, $\Lambda_n = \{ \theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^* \}$, as in Eq. (2.21).

**Notation 2.16.** Let $\mathbb{C} \langle x \rangle \langle \theta, \theta^* \rangle$ and $\mathbb{C} \langle \alpha, \bar{\alpha} \rangle \langle \theta, \theta^* \rangle$ denote the non-commutative polynomials in $\{ \theta, \theta^* \}$ with coefficients in the commutative polynomial rings, $\mathbb{C} \langle x \rangle$ and $\mathbb{C} \langle \alpha, \bar{\alpha} \rangle$ respectively. For $P \in \mathbb{C} \langle x \rangle \langle \theta, \theta^* \rangle$ or $P \in \mathbb{C} \langle \alpha, \bar{\alpha} \rangle \langle \theta, \theta^* \rangle$ we will write $\text{deg}_x P$ to indicate that we are computing the degree relative to $\{ \theta, \theta^* \}$ and not relative to $x$ or $\{ \alpha, \bar{\alpha} \}$. 

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For any $\alpha \in \mathbb{C}$ and $P(\theta, \theta^*) \in \mathbb{C} \langle \theta, \theta^* \rangle$ with $d = \deg_{\theta} P$, let $\{ P_k (\alpha : \theta, \theta^*) \}_{k=0}^{d} \subset \mathbb{C} [\alpha, \bar{\alpha}] \langle \theta, \theta^* \rangle$ denote the unique homogeneous polynomials in $\mathbb{C} \langle \theta, \theta^* \rangle$ with coefficients which are polynomials in $\alpha$ and $\bar{\alpha}$ such that $\deg_{\theta} P_k (\alpha : \theta, \theta^*) = k$ and

$$P(\theta + \alpha, \theta^* + \bar{\alpha}) = \sum_{k=0}^{d} P_k (\alpha : \theta, \theta^*) \tag{2.29}$$

**Example 2.17.** If

$$P(\theta, \theta^*) = \theta \theta^* \theta + \theta^* \theta \theta^*$$

then

$$P(\theta + \alpha, \theta^* + \bar{\alpha}) = (\theta + \alpha) (\theta^* + \bar{\alpha}) (\theta + \alpha) + (\theta^* + \bar{\alpha}) (\theta + \alpha) (\theta^* + \bar{\alpha})$$

$$= P_0 + P_1 + P_2 + P_{\geq 3}$$

where

$$P_0 (\alpha, \theta, \theta^*) = \alpha^2 \bar{\alpha} + \bar{\alpha}^2 \alpha = P_{\text{cl}} (\alpha)$$

$$P_1 (\alpha, \theta, \theta^*) = \left( 2 |\alpha|^2 + |\bar{\alpha}|^2 \right) \theta + \left( 2 |\alpha|^2 + |\bar{\alpha}|^2 \right) \theta^*$$

$$= \frac{\partial P_{\text{cl}}}{\partial \alpha} (\alpha) \theta + \frac{\partial P_{\text{cl}}}{\partial \bar{\alpha}} (\alpha) \theta^*$$

$$P_2 (\alpha, \theta, \theta^*) = |\alpha|^2 + \alpha \theta^* \theta + (\alpha + \bar{\alpha}) \theta^* \theta$$

$$= \frac{1}{2} \left( \frac{\partial^2 P_{\text{cl}}}{\partial \alpha^2} (\alpha) \theta^2 + \frac{\partial^2 P_{\text{cl}}}{\partial \alpha^2} (\alpha) \theta^2 \right)$$

$$= \frac{d}{dt} \big|_{t=0} \frac{d}{ds} \big|_{s=0} P(s \theta + \alpha, t \theta^* + \bar{\alpha}) \tag{2.30}$$

This example is generalized in the following theorem.

**Theorem 2.18.** Let $P(\theta, \theta^*) \in \mathbb{C} \langle \theta, \theta^* \rangle$ and $\alpha \in \mathbb{C}$, then

$$P_0 (\alpha : \theta, \theta^*) = P_{\text{cl}} (\alpha)$$

$$P_1 (\alpha : \theta, \theta^*) = \left[ \frac{\partial P_{\text{cl}}}{\partial \alpha} (\alpha) \theta + \frac{\partial P_{\text{cl}}}{\partial \bar{\alpha}} (\alpha) \theta^* \right]$$

$$P_2 (\alpha : \theta, \theta^*) = \frac{1}{2} \left( \frac{\partial^2 P_{\text{cl}}}{\partial \alpha^2} (\alpha) \theta^2 + \frac{\partial^2 P_{\text{cl}}}{\partial \alpha^2} (\alpha) \theta^2 \right)$$

$$+ \frac{d}{dt} \big|_{t=0} \frac{d}{ds} \big|_{s=0} P(s \theta + \alpha, t \theta^* + \bar{\alpha}) \tag{2.30}$$

where

$$\frac{d}{dt} \big|_{t=0} \frac{d}{ds} \big|_{s=0} P(s \theta + \alpha, t \theta^* + \bar{\alpha}) = \frac{\partial^2 P_{\text{cl}}}{\partial \alpha \partial \bar{\alpha}} (\alpha) \theta^* \theta \mod \theta^* \theta = \theta \theta^*$$

for all $\alpha \in \mathbb{C}$. So we have

$$P(\theta + \alpha, \theta^* + \bar{\alpha})$$

$$= P_{\text{cl}} (\alpha) + \left[ \frac{\partial P_{\text{cl}}}{\partial \alpha} (\alpha) \theta + \left( \frac{\partial P_{\text{cl}}}{\partial \bar{\alpha}} P_{\text{cl}} (\alpha) \theta^* \right) \right] + P_2 (\alpha : \theta, \theta^*) + P_{\geq 3} (\alpha : \theta, \theta^*) \tag{2.31}$$

where the remainder term, $P_{\geq 3}$, is a sum of homogeneous terms of degree 3 or more. Moreover if $P = P^*$, then $P_2^* = P_2$ and $P_{\geq 3}^* = P_{\geq 3}$. 
Proof. If \( p = \deg P \), then
\[
P(t\theta + \alpha, t\theta^* + \bar{\alpha}) = \sum_{k=0}^{p} t^k P_k(\alpha : \theta, \theta^*) \quad \forall \ t \in \mathbb{R},
\]
and it follows (by Taylor’s theorem) that
\[
P_k(\alpha : \theta, \theta^*) = \frac{1}{k!} \left( \frac{d}{dt} \right)^k |_{t=0} P(t\theta + \alpha, t\theta^* + \bar{\alpha}). \tag{2.32}
\]
From Eq. (2.32),
\[
P_0(\alpha : \theta, \theta^*) = P(\alpha, \bar{\alpha}) = P_{cl}(\alpha)
\]
and
\[
P_1(\alpha : \theta, \theta^*) = \left. \frac{d}{dt} \right|_{t=0} P(t\theta + \alpha, \bar{\alpha}) + \left. \frac{d}{dt} \right|_{t=0} P(\alpha, t\theta^* + \bar{\alpha}) = \frac{\partial P_{cl}}{\partial \alpha}(\alpha) \theta + \frac{\partial P_{cl}}{\partial \bar{\alpha}}(\alpha) \theta^*.
\]
Similarly from Eq. (2.32),
\[
P_2(\alpha : \theta, \theta^*) = \frac{1}{2} \left( \frac{d}{dt} \right)^2 |_{t=0} P(t\theta + \alpha, t\theta^* + \bar{\alpha})
\]
\[
= \frac{1}{2} \left( \frac{d}{dt} \right)^2 |_{t=0} \left[ P(t\theta + \alpha, \bar{\alpha}) + P(\alpha, t\theta^* + \bar{\alpha}) \right]
\]
\[
+ \left. \frac{d}{dt} \right|_{t=0} \frac{d}{ds} \bigg|_{s=0} P(s\theta + \alpha, t\theta^* + \bar{\alpha})
\]
\[
= \frac{1}{2} \left( \frac{\partial^2 P_{cl}}{\partial \alpha^2}(\alpha) \theta^2 + \frac{\partial^2 P_{cl}}{\partial \bar{\alpha}^2}(\alpha) \theta^{*2} \right)
\]
\[
+ \left. \frac{d}{dt} \right|_{t=0} \frac{d}{ds} \bigg|_{s=0} P(s\theta + \alpha, t\theta^* + \bar{\alpha})
\].

If \( P(\theta, \theta^*) \in \mathbb{C}(\theta, \theta^*) \) is symmetric, then \( P(t\theta + \alpha, t\theta^* + \bar{\alpha}) \in \mathbb{C}(\theta, \theta^*) \) is symmetric and hence from Eq. (2.32) it follows that \( P_k(\alpha : \theta, \theta^*) \in \mathbb{C}(\theta, \theta^*) \) is still symmetric and therefore so is the remainder term,
\[
P_{\geq 3}(\alpha : \theta, \theta^*) = \sum_{k=3}^{p} P_k(\alpha : \theta, \theta^*).
\]

\[\blacksquare\]

3. Polynomial Operators

3.1. Algebra of Polynomial Operators.

Notation 3.1. For \( b = (b_1, \ldots, b_k) \in \{\theta, \theta^*\}^k \), \( p(b) \), \( q(b) \), and \( \ell(b) \) be the \( \mathbb{Z} \)–valued functions defined by
\[
p(b) := \# \{ i : b_i = \theta \}, \quad q(b) := \# \{ i : b_i = \theta^* \}, \quad \text{and}
\]
\[
\ell(b) := \sum_{i=1}^{k} (1_{b_i=\theta^*} - 1_{b_i=\theta}) = q(b) - p(b). \tag{3.2}
\]
Thus \( p (b) \) (\( q (b) \)) is the number of \( \theta \)'s (\( \theta^* \)'s) in \( b \) and \( \ell (b) \) counts the excess number of \( \theta^* \)'s over \( \theta \)'s in \( b \).

**Lemma 3.2** (Normal Ordering). If \( P (\theta, \theta^*) \in \mathbb{C} \langle \theta, \theta^* \rangle \) with \( d = \deg_\theta P \), then there exists \( R (h : \theta, \theta^*) \in \mathbb{C} [h] \langle \theta, \theta^* \rangle \) (a non-commutative polynomial in \( \{ \theta, \theta^* \} \) with polynomial coefficients in \( h \)) such that \( \deg_\theta R (h : \theta, \theta^*) \leq d - 2 \) and

\[
P (a_h, a_h^\dagger) = \sum_{0 \leq k, l, k + l \leq d} \frac{1}{k! \cdot l!} \left( \frac{\partial^{k+l} P^{cl}}{\partial \alpha^k \partial \alpha^l} \right) (0) a_h^k a_h^l + h R \left( h : a_h, a_h^\dagger \right) \quad \forall \ h > 0.
\]

**Proof.** By linearity it suffices to consider the case here \( P (\theta, \theta^*) \) is a homogeneous polynomial of degree \( d \) which may be written as

\[
P (\theta, \theta^*) = \sum_{b \in \{ \theta, \theta^* \}^d} c (b) u_b (\theta, \theta^*) = \sum_{p=0}^d \sum_{b \in \{ \theta, \theta^* \}^d} 1_{p(b)=p} c (b) u_b (\theta, \theta^*).
\]

Since

\[
P (\alpha, \bar{\alpha}) = \sum_{p=0}^d \left[ \sum_{b \in \{ \theta, \theta^* \}^d} 1_{p(b)=p} c (b) \right] \alpha^p \bar{\alpha}^{d-p}
\]

it follows that

\[
\frac{1}{(d-p)! \cdot p!} \left( \frac{\partial^d p^{cl}}{\partial \alpha^d \partial \alpha^p} \right) (0) = \sum_{b \in \{ \theta, \theta^* \}^d} 1_{p(b)=p} c (b) .
\]

On the other hand, if \( b \in \{ \theta, \theta^* \}^d \) and \( p := p (b) \), then making use of the CCRs of Eq. (1.7) it is easy to show there exists \( R_b (h, \theta, \theta^*) \in \mathbb{C} [h] \langle \theta, \theta^* \rangle \) such that \( \deg_\theta R_b (h, \theta, \theta^*) \leq d - 2 \) such that

\[
u_b \left( a_h, a_h^\dagger \right) = a_h^{p(d-p)} a_h^p + h R_b \left( h, a_h, a_h^\dagger \right) .
\]

Replacing \( \theta \) by \( a_h \) and \( \theta^* \) by \( a_h^\dagger \) in Eq. (3.3) and using Eq. (3.5) we find,

\[
P (a_h, a_h^\dagger) = \sum_{p=0}^d \sum_{b \in \{ \theta, \theta^* \}^d} 1_{p(b)=p} c (b) u_b \left( a_h, a_h^\dagger \right)
\]

\[
= \sum_{p=0}^d \left[ \sum_{b \in \{ \theta, \theta^* \}^d} 1_{p(b)=p} c (b) \right] a_h^{p(d-p)} a_h^p + h \sum_{b \in \{ \theta, \theta^* \}^d} c (b) R_b \left( h, a_h, a_h^\dagger \right)
\]

\[
= \sum_{p=0}^d \frac{1}{(d-p)! \cdot p!} \left( \frac{\partial^d p^{cl}}{\partial \alpha^d \partial \alpha^p} \right) (0) a_h^{p(d-p)} a_h^p + h R \left( h, a_h, a_h^\dagger \right)
\]

where

\[
R (h, \theta, \theta^*) = \sum_{b \in \{ \theta, \theta^* \}^d} c (b) R_b (h, \theta, \theta^*) .
\]

**Corollary 3.3.** If \( P (\theta, \theta^*) \) and \( Q (\theta, \theta^*) \) are non-commutative polynomials such that \( P^{cl} = Q^{cl} \), then there exists \( R (h : \theta, \theta^*) \in \mathbb{C} [h] \langle \theta, \theta^* \rangle \) with \( \deg_\theta R (h : \theta, \theta^*) \leq \deg_\theta (P - Q) (\theta, \theta^*) - 2 \) such that

\[
P (a_h, a_h^\dagger) = Q \left( a_h, a_h^\dagger \right) + h R \left( h, a_h, a_h^\dagger \right)
\]
Proof. Apply Lemma 3.2 to the non-commutative polynomial, \( P(\theta, \theta^*) - Q(\theta, \theta^*) \).

**Proposition 3.4.** For all \( H \in \mathbb{C} \langle \theta, \theta^* \rangle \), there exists a polynomial, \( p_H \in \mathbb{C} [z, \bar{z}] \) such that

\[
H_2 (\alpha : a, a^\dagger) = \frac{1}{2} \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) a^2 + \frac{1}{2} \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) a^{12} + \left( \frac{\partial^2 H^{cl}}{\partial \alpha \partial \bar{\alpha}} \right) (\alpha) a^\dagger a + p_H (\alpha, \bar{\alpha}) I
\]

for all \( \alpha \in \mathbb{C} \) where \( H_2 (\alpha : \theta, \theta^*) \) is defined in Eq. (2.29).

Proof. As we have seen the structure of \( H_2 (\alpha : \theta, \theta^*) \) implies there exists \( \rho, \gamma, \delta \in \mathbb{C} [\alpha, \bar{\alpha}] \) such that

\[
2H_2 (\alpha : \theta, \theta^*) = \rho (\alpha, \bar{\alpha}) \theta^2 + \overline{\rho (\alpha, \bar{\alpha})} \theta^{*2} + \gamma (\alpha, \bar{\alpha}) \theta^* \theta + \delta (\alpha, \bar{\alpha}) \theta \theta^*.
\]

From this equation we find,

\[
2H_2 (\alpha : z, \bar{z}) = \rho (\alpha, \bar{\alpha}) z^2 + \overline{\rho (\alpha, \bar{\alpha})} \bar{z}^2 + [\gamma (\alpha, \bar{\alpha}) + \delta (\alpha, \bar{\alpha})] z \bar{z}
\]

while form Eq. (2.30) we may conclude that

\[
2H_2 (\alpha : z, \bar{z}) = \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) z^2 + \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) \bar{z}^2 + 2 \left( \frac{\partial^2 H^{cl}}{\partial \alpha \partial \bar{\alpha}} \right) (\alpha) z \bar{z}. \tag{3.5}
\]

Comparing these last two equations shows,

\[
\left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) = \rho (\alpha, \bar{\alpha}), \quad \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) = \overline{\rho (\alpha, \bar{\alpha})}, \quad \text{and}
\]

\[
\left( \frac{\partial^2 H^{cl}}{\partial \alpha \partial \bar{\alpha}} \right) (\alpha) = \frac{1}{2} [\gamma (\alpha, \bar{\alpha}) + \delta (\alpha, \bar{\alpha})].
\]

Using these last identities and the canonical commutations relations we find,

\[
2H_2 (\alpha : a, a^\dagger)
\]

\[
= \rho (\alpha, \bar{\alpha}) a^2 + \overline{\rho (\alpha, \bar{\alpha})} a^{12} + \gamma (\alpha, \bar{\alpha}) a^\dagger a + \delta (\alpha, \bar{\alpha}) a a^\dagger
\]

\[
= \rho (\alpha, \bar{\alpha}) a^2 + \overline{\rho (\alpha, \bar{\alpha})} a^{12} + [\gamma (\alpha, \bar{\alpha}) + \delta (\alpha, \bar{\alpha})] a^\dagger a + \delta (\alpha, \bar{\alpha}) I
\]

\[
= \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) a^2 + \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) a^{12} + 2 \left( \frac{\partial^2 H^{cl}}{\partial \alpha \partial \bar{\alpha}} \right) (\alpha) a^\dagger a + p_H (\alpha, \bar{\alpha}) I
\]

with \( p_H (\alpha, \bar{\alpha}) = \delta (\alpha, \bar{\alpha}) ). \)

Proposition 3.4 and the following simple commutator formulas,

\[
[a^\dagger a, a] = -a, \quad [a^{12}, a] = -2a^\dagger,
\]

\[
[a^\dagger a, a^\dagger] = a^\dagger, \quad \text{and} \quad [a^2, a^\dagger] = 2a,
\]

immediately give the following corollary.

**Corollary 3.5.** If \( H \in \mathbb{C} \langle \theta, \theta^* \rangle \) and \( \alpha \in \mathbb{C} \), then

\[
[H_2 (\alpha : a, a^\dagger), a] = - \left( \frac{\partial^2 H^{cl}}{\partial \alpha \partial \bar{\alpha}} \right) (\alpha) a - \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) a^\dagger
\]

\[
[H_2 (\alpha : a, a^\dagger), a^\dagger] = \left( \frac{\partial^2 H^{cl}}{\partial \alpha^2} \right) (\alpha) a + \left( \frac{\partial^2 H^{cl}}{\partial \alpha \partial \bar{\alpha}} \right) (\alpha) a^\dagger.
\]
3.2. Expectations and variances for translated states. The next result is a fairly easy consequence of Proposition 2.4 and the expansion of non-commutative polynomials into their homogeneous components.

**Corollary 3.6** (Concentrated states). Let \( P (\theta, \theta^*) \in \mathbb{C} \langle \theta, \theta^* \rangle, \), \( \psi \in \mathcal{S}, \) \( \hbar > 0, \) and \( \alpha \in \mathbb{C}, \) then

\[
\left\langle P \left( a_h, a_h^\dagger \right) \right\rangle_{U_h(\alpha) \psi} = P (\alpha, \bar{\alpha}) + O \left( \sqrt{\hbar} \right) \quad (3.6)
\]

\[
\text{Var}_{U_h(\alpha) \psi} \left( P \left( a_h, a_h^\dagger \right) \right) = O \left( \sqrt{\hbar} \right), \quad (3.7)
\]

and

\[
\lim_{\hbar \downarrow 0} \left\langle P \left( \frac{a_h - \alpha}{\sqrt{\hbar}}, \frac{a_h^\dagger - \bar{\alpha}}{\sqrt{\hbar}} \right) \right\rangle_{U_h(\alpha) \psi} = \left\langle P \left( a, a^\dagger \right) \right\rangle_\psi \quad (3.8)
\]

where \( \left\langle \cdot \right\rangle_{U_h(\alpha) \psi} \) is defined in Definition 4.7. [In fact, the equality in the last equation holds before taking the limit as \( \hbar \to 0. \)]

**Proof.** From Proposition 2.4 and Eq. (2.29),

\[
U_h (\alpha)^* P \left( a_h, a_h^\dagger \right) U_h (\alpha) = P \left( a_h + \alpha, a_h^\dagger + \bar{\alpha} \right) = \sum_{k=0}^{d} P_k \left( \alpha : a_h, a_h^\dagger \right) \quad (3.9)
\]

and hence

\[
\left\langle P \left( a_h, a_h^\dagger \right) \right\rangle_{U_h(\alpha) \psi} = \left\langle U_h(\alpha)^* P \left( a_h, a_h^\dagger \right) U_h(\alpha) \right\rangle_\psi = \left\langle P \left( a_h + \alpha, a_h^\dagger + \bar{\alpha} \right) \right\rangle_\psi
\]

\[
= \left\langle \sum_{k=0}^{d} P_k \left( \alpha : a_h, a_h^\dagger \right) \right\rangle_\psi = P_0 (\alpha) + \sum_{k=1}^{d} \hbar^{k/2} \left\langle P_k \left( \alpha, a, a^\dagger \right) \right\rangle_\psi
\]

from which Eq. (3.6) follows where \( P_0 (\alpha) \) is defined in Notation 2.8. Similarly, making use of the fact that \( (P^2)_0 (\alpha) = (P^2_0) (\alpha) \)

\[
\left\langle P^2 \left( a_h, a_h^\dagger \right) \right\rangle_{U_h(\alpha) \psi} = (P^2_0) (\alpha) + \sum_{k=1}^{2d} \hbar^{k/2} \left( \left\langle P^2 \right\rangle_k \left( \alpha, a, a^\dagger \right) \right)_\psi
\]

and hence

\[
\text{Var}_{U_h(\alpha) \psi} \left( P \left( a_h, a_h^\dagger \right) \right) = (P^2_0) (\alpha) + \sum_{k=1}^{2d} \hbar^{k/2} \left( \left\langle P^2 \right\rangle_k \left( \alpha, a, a^\dagger \right) \right)_\psi - \left[ \left( P_0 (\alpha) + \sum_{k=0}^{d} \hbar^{k/2} \left\langle P_k \left( \alpha, a, a^\dagger \right) \right\rangle_\psi \right) \right]^2 = O \left( \sqrt{\hbar} \right).
\]

Lastly, using Eq. (3.9) one shows,

\[
\left\langle P \left( \frac{a_h - \alpha}{\sqrt{\hbar}}, \frac{a_h^\dagger - \bar{\alpha}}{\sqrt{\hbar}} \right) \right\rangle_{U_h(\alpha) \psi} = \left\langle P \left( \frac{a_h + \alpha - \alpha}{\sqrt{\hbar}}, \frac{a_h^\dagger + \bar{\alpha} - \bar{\alpha}}{\sqrt{\hbar}} \right) \right\rangle_\psi = \left\langle P \left( a, a^\dagger \right) \right\rangle_\psi
\]

which certainly implies Eq. (3.8). \( \blacksquare \)
Remark 3.7. If $\psi \in \mathcal{S}$ and $\alpha \in \mathbb{C}$, Eqs. (3.6) and (3.7) should be interpreted to say that for small $\hbar > 0$, $U_\hbar (\alpha) \psi$ is a state which is concentrated in phase space near $\alpha$. Consequently, these are good initial states for discussing the classical ($\hbar \to 0$) limit of quantum mechanics.

The next result shows that, under Assumption [1] the classical equations of motions in Eq. (1.1) have global solutions which remain bounded in time.

**Proposition 3.8.** If $C$ and $C_1$ are the constants appearing in Eq. (1.14) of Assumption [1] $\alpha_0 \in \mathbb{C}$, and $\alpha (t) \in \mathbb{C}$ is the maximal solution of Hamilton’s ordinary differential equations (1.2), then $\alpha (t)$ is defined for all time $t$ and moreover,

$$|\alpha (t)|^2 \leq C_1 \left( H^{cl} (\alpha (0)) + C \right),$$

(3.11)

where $H^{cl} (\alpha) := H (\alpha, \alpha)$.

**Proof.** Equation (1.14) with $\beta = 1$ implies

$$\langle N_h \rangle \psi \leq C_1 \langle H_h + C \rangle \psi$$

for all $\psi \in \mathcal{S}$. (3.12)

Replacing $\psi$ by $U_\hbar (\alpha) \psi$ in Eq. (3.12) and then letting $\hbar \downarrow 0$ gives (with the aid of Corollary 3.6) the estimate,

$$|\alpha| \leq C_1 \left( H^{cl} (\alpha) + C \right)$$

(3.13)

for all $\alpha \in \mathbb{C}$. If $\alpha (t)$ solves Hamilton’s Eq. (1.1) then $H^{cl} (\alpha (t)) = H^{cl} (\alpha (0))$ for all $t$. As the level sets of $H^{cl}$ are compact because of the estimate in Eq. (3.13) there is no possibility for $\alpha (t)$ to explode and hence solutions will exist for all times $t$ and moreover must satisfy the estimate in Eq. (3.11). □

3.3. Analysis of Monomial Operators of $a$ and $a^\dagger$. In this subsection, recall that $a = a_1$ and $a^\dagger = a_1^\dagger$ as in Definition 1.3. Let

$$\Omega_0 (x) := \frac{1}{\sqrt{4\pi}} \exp \left( -\frac{1}{2} x^2 \right) \quad \text{and} \quad \Omega_n := \frac{1}{\sqrt{n!} a^n \Omega_0},$$

(3.14)

for all $n \in \mathbb{Z}$ with $n < 0$.

**Convention:** $\Omega_n \equiv 0$ for all $n \in \mathbb{Z}$ with $n < 0$.

The following theorem summarizes the basic well known and easily verified properties of these functions which essentially are all easy consequences of the canonical commutation relations, $[a, a^\dagger] = I$ on $\mathcal{S}$. We will provide a short proof of these well known results for the readers convenience.

**Theorem 3.9.** The functions $\{\Omega_n\}_{n=0}^\infty \subset \mathcal{S}$ form an orthonormal basis for $L^2 (m)$ which satisfy for all $n \in \mathbb{N}_0$,

$$a \Omega_n = \sqrt{n} \Omega_{n-1},$$

(3.15)

$$a^\dagger \Omega_n = \sqrt{n+1} \Omega_{n+1} \quad \text{and} \quad a^\dagger a \Omega_n = n \Omega_n,$$

(3.16)

(3.17)

**Proof.** First observe that $\Omega_0 (x)$ is a polynomial $(p_n (x))$ of degree $n$ times $\Omega_0 (x).$ Therefore the span of $\{\Omega_n\}_{n=0}^\infty$ are all functions of the form $p (x) \Omega_0 (x)$ where $p \in \mathbb{C} [x]$. As $\mathbb{C} [x]$ is dense in $L^2 (\Omega_0^2 (x) dx)$ it follows that $\{\Omega_n\}_{n=0}^\infty$ is total in $L^2 (m)$.

For the remaining assertions let us recall, if $A$ and $B$ are operators on some vector space (like $\mathcal{S}$) and $ad_A := [A, B]$, then $ad_A$ acts as a derivation, i.e.

$$ad_A (BC) = (ad_A) C + B \langle ad_A \rangle.$$
Combining this observation with \( ada^\dagger = I \) then shows \( ada^\dagger n = na^{n-1} \) so that
\[
a\Omega_n = a \frac{1}{\sqrt{n!}} a^n \Omega_0 = \frac{1}{\sqrt{n!}} \left( ada^\dagger n \right) \Omega_0 = \frac{n}{\sqrt{n!}} a^{(n-1)} \Omega_0 = \sqrt{n} \Omega_{n-1}
\]
which proves Eq. (3.15). Equation (3.16) is obvious from the definition of \( \{ \Omega_n \}_{n=0}^\infty \) and Eq. (3.17) follows from Eqs. (3.15) and (3.16). As \( \{ \Omega_n \}_{n=0}^\infty \) are eigenvectors of the symmetric operator \( a^\dagger a \) with distinct eigenvalues it follows that \( \langle \Omega_n, \Omega_m \rangle = 0 \) if \( m \neq n \). So it only remains to show \( \| \Omega_n \|^2 = 1 \) for all \( n \). However, taking the \( L^2(m) \)-norm of Eq. (3.16) gives
\[(n+1)\|\Omega_{n+1}\|^2 = \|a^\dagger \Omega_n\|^2 = \langle \Omega_n, aa^\dagger \Omega_n \rangle = \langle \Omega_n, (a^\dagger a + I) \Omega_n \rangle = (n+1)\|\Omega_n\|^2,
\]
i.e. \( n \to \|\Omega_n\|^2 \) is constant in \( n \). As we normalized \( \Omega_0 \) to be a unit vector, the proof is complete.

**Notation 3.10.** For \( N \in \mathbb{N}_0 \), let \( \mathcal{P}_N \) denote orthogonal projection of \( L^2(m) \) onto \( \text{span} \{ \Omega_n : 0 \leq n \leq N \} \), i.e.
\[
\mathcal{P}_N f := \sum_{n=0}^N \langle f, \Omega_n \rangle \Omega_n \text{ for all } f \in L^2(m).
\]

**Notation 3.11** (Standing Notation). For the remainder of this section let \( k, j \in \mathbb{N}, b = (b_1, \ldots, b_k) \in \{ \theta, \theta^* \}^k, q := q(b), l := \ell(b), d = (d_1, \ldots, d_j) \in \{ \theta, \theta^* \}^j \), and \( \ell(d) \) be as in Notation 3.1. We further let \( \mathcal{A} \) and \( \mathcal{D} \) be the two monomial operators,
\[
\mathcal{A} := u_b \left( a, a^\dagger \right) = \Xi(b_1) \ldots \Xi(b_k) \text{ and } \mathcal{D} := u_d \left( a, a^\dagger \right) = \Xi(d_1) \ldots \Xi(d_j).
\]

**Lemma 3.12.** To each monomial operator \( \mathcal{A} = u_b \left( a, a^\dagger \right) \) as in Notation 3.11 there exists \( c_A : \mathbb{N}_0 \to [0, \infty) \) such that
\[
\mathcal{A} \Omega_n = c_A(n) \cdot \Omega_{n+l} \text{ for all } n \in \mathbb{N}_0
\]
where (as above) \( \Omega_m := 0 \) if \( m < 0 \). Moreover, \( c_A \) satisfies \( c_A^l(n) = c_A(n-l) \) (where by convention \( c_A(n) \equiv 0 \) if \( n < 0 \)),
\[
0 \leq c_A(n) \leq (n+q)^k \text{ and } c_A(n) \asymp n^{k/2} \text{ (i.e. } \lim_{n \to \infty} \frac{c_A(n)}{n^{k/2}} = 1 \).
\]

**Proof.** Since \( a \) and \( a^\dagger \) shift \( \Omega_n \) to its adjacent \( \Omega_{n-1} \) and \( \Omega_{n+1} \) respectively from Theorem 3.9 it is easy to see that Eq. (3.20) holds for some constants \( c_A(n) \in \mathbb{R} \). Moreover a simple induction argument on \( k \) shows there exists \( \delta_i \in \mathbb{Z} \) with \( \delta_i \leq q \) such that
\[
c_A(n) = \left( \prod_{i=1}^k (n + \delta_i) \right) \geq 0.
\]
The estimate and the limit statement in Eq. (3.21) now follows directly from the Eq. (3.22).

Since \( \mathcal{A}^\dagger \Omega_n = c_{A^\dagger}(n) \Omega_{n-l} \), we find
\[
c_{A^\dagger}(n) = \langle \mathcal{A}^\dagger \Omega_n, \Omega_{n-l} \rangle = \langle \Omega_n, \mathcal{A} \Omega_{n-l} \rangle = \langle \Omega_n, c_{A}(n-l) \Omega_n \rangle = c_{A}(n-l).
\]
Example 3.13. Suppose that \( p,q \in \mathbb{N}_0, k = p+q, \ell = q-p \), and \( \mathcal{A} = a^p a^{\dagger q} \). Then

\[
\mathcal{A} \Omega_n = a^p a^{\dagger q} \Omega_n = a^p \prod_{i=1}^{q} (n + i) \cdot \Omega_{n+q} = \sqrt{\prod_{i=1}^{q} (n + i) \cdot \Omega_{n+q}} = \sqrt{\prod_{i=1}^{q} (n + i) - \prod_{j=0}^{p-1} (n + q - j) \Omega_{n+i}}
\]

where \( \prod_{i=1}^{q} (n + i) \cdot \prod_{j=0}^{p-1} (n + q - j) \leq (n + q)^{k} \).

(3.23)

Definition 3.14. For \( \beta \geq 0 \), let

\[
D_{\beta} := \left\{ f \in L^2(\mathbb{R}) : \sum_{n=0}^{\infty} |(f, \Omega_n)|^2 n^{2\beta} < \infty \right\}.
\]

We will see shortly that \( D_{\beta} = D(\mathcal{A}^\beta) \), see Example 3.19

Theorem 3.15. Let \( k = \text{deg}_g u_b (\theta, \theta^*) \), \( \mathcal{A} = u_b (a, a^\dagger) \), \( l = \ell (b) \in \mathbb{Z} \) be as in Notations 3.11 and 3.14 and \( c_\mathcal{A} (n) \) be coefficients in Lemma 3.13. Then \( \mathcal{A} \) and \( \mathcal{A}^\dagger \) are closable operators satisfying;

1. \( \bar{\mathcal{A}} = \mathcal{A}^{\dagger*} \) and \( \bar{\mathcal{A}^\dagger} = \mathcal{A}^* \) where we write \( \mathcal{A}^{\dagger*} \) for \( (\mathcal{A}^\dagger)^* \).
2. \( D(\bar{\mathcal{A}}) = D_{k/2} = D(\bar{\mathcal{A}^\dagger}) \) and if \( g \in D_{k/2} \), then

\[
\mathcal{A}^* g = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \mathcal{A}^\dagger \Omega_n = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle c_\mathcal{A} (n-l) \Omega_{n-l} \quad \text{and} \quad (3.24)
\]

\[
\mathcal{A}^{\dagger*} g = \bar{\mathcal{A}} g = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \mathcal{A} \Omega_n = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle c_\mathcal{A} (n) \Omega_{n+l} \quad \text{and} \quad (3.25)
\]

with the conventions that \( c_\mathcal{A} (n) \) and \( \Omega_n = 0 \) if \( n < 0 \).

3. The subspace,

\[
\mathcal{S}_0 := \text{span} \{ \Omega_n \}_{n=0}^{\infty} \subset \mathcal{S} \subset L^2(m)
\]

is a core of both \( \bar{\mathcal{A}} \) and \( \bar{\mathcal{A}^\dagger} \). More explicitly if \( g \in D_{k/2} \), then

\[
\bar{\mathcal{A}} g = \lim_{N \to \infty} \mathcal{A} \mathcal{P}_N g \quad \text{and} \quad \bar{\mathcal{A}^\dagger} g = \lim_{N \to \infty} \mathcal{A}^\dagger \mathcal{P}_N g
\]

where \( \mathcal{P}_N \) is the orthogonal projection operator onto \( \text{span} \{ \Omega_k \}_{k=0}^{n} \) as in Notation 3.10.

Proof. Since \( \langle \mathcal{A} f, g \rangle = \langle f, \mathcal{A}^\dagger g \rangle \) for all \( f, g \in \mathcal{S} = D(\mathcal{A}) = D(\mathcal{A}^\dagger) \), it follows that \( \mathcal{A} \subset \mathcal{A}^{\dagger*} \) and \( \mathcal{A}^\dagger \subset \mathcal{A}^* \) and therefore both \( \mathcal{A} \) and \( \mathcal{A}^\dagger \) are closable (see Theorem VIII.1 on p.252) and

\[
\bar{\mathcal{A}^\dagger} \subset \mathcal{A}^* \quad \text{and} \quad \bar{\mathcal{A}} \subset \mathcal{A}^{\dagger*}.
\]

(3.27)
If \( g \in D(A^*) \subset L^2(n) \), then from Theorem 3.9 and Lemma 3.12 we have

\[
A^*g = \sum_{n=0}^{\infty} \langle A^*g, \Omega_n \rangle \Omega_n = \sum_{n=0}^{\infty} \langle g, A\Omega_n \rangle \Omega_n
\]

\[
= \sum_{n=0}^{\infty} \langle g, c_A(n) \Omega_{n+l} \rangle \Omega_n = \sum_{n=0}^{\infty} \langle g, \Omega_{n+l} \rangle c_A(n) \Omega_n
\]

\[
= \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle c_A(n-l) \Omega_{n-l} = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle A^\dagger \Omega_n, \quad (3.28)
\]

wherein we have used the conventions stated after Eq. (3.25) repeatedly. Since, by Lemma 3.12, \( \{A^\dagger \Omega_n = c_A(n-l) \Omega_{n-l}\}_{n=0}^{\infty} \) is an orthogonal set such that

\[
\|A^\dagger \Omega_n\|_2^2 = |c_A(n-l)|^2 \approx n^k,
\]

it follows that the last sum in Eq. (3.28) is convergent if

\[
\sum_{n=0}^{\infty} |\langle g, \Omega_n \rangle|^2 n^k < \infty \iff g \in D_{k/2}.
\]

Conversely if \( g \in D_{k/2} \) and \( f \in S = D(A) \) we have,

\[
\left\langle \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle A^\dagger \Omega_n, f \right\rangle = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \left\langle A^\dagger \Omega_n, f \right\rangle = \sum_{n=0}^{\infty} \langle g, \Omega_n \rangle \left\langle \Omega_n, Af \right\rangle = \langle g, Af \rangle
\]

from which it follows that \( g \in D(A^*) \) and \( A^*g \) is given as in Eq. (3.28).

In summary, we have shown \( D(A^*) = D_{k/2} \) and \( A^*g \) is given by Eq. (3.28). Moreover, from Eq. (3.28), if \( g \in D_{k/2} \) then

\[
A^*g = \lim_{N \to \infty} \sum_{n=0}^{N} \langle g, \Omega_n \rangle A^\dagger \Omega_n = \lim_{N \to \infty} A^\dagger P_N g
\]

which implies \( g \in D(A^\dagger) \) and \( A^*g = A^\dagger g \), i.e. \( A^* \subset A^\dagger \). Combining this last assertion with the first inclusion in Eq. (3.27) implies \( A^* = A^\dagger \). This proves all of the assertions involving \( A^* \) and \( A^\dagger \). We may now complete the proof by applying these assertions with \( A = u_b(a,a^1) \) replaced by \( A^\dagger = u_{b^*}(a,a^\dagger) \) and using the facts that \( A^\dagger = A^\dagger \), \( \ell(b^*) = -\ell(b) = -l \), and \( c_{A^\dagger}(n) = c_A(n-l) \).

**Theorem 3.16.** Let \( k = deg_\theta u_b(\theta,\theta^*) \), \( j = deg_\theta u_d(\theta,\theta^*) \), \( A = u_b(a,a^1) \), \( D = u_d(a,a^1) \), \( \ell(b) \), and \( \ell(d) \) be as in Notations 3.11 and 3.1. Then;

1. \( A A^\dagger = \hat{A} \hat{A}^\dagger \)
2. \( (AD)^\dagger = D^\dagger A^* \)
3. \( \hat{A} := u_b(a,a^1) = u_b(\bar{a},a^\dagger) \), i.e. if \( A \) is a monomial operator in \( a^1 \) and \( a^\dagger \), then \( \hat{A} \) is the operator resulting from replacing \( a \) by \( \bar{a} \) and \( a^1 \) by \( a^\dagger \) everywhere in \( A \).

**Proof.** Because of the conventions described after Eq. (3.25), in the argument below it will be easier to view all sums over \( n \in \mathbb{Z} \) instead of \( n \in \mathbb{N}_0 \). We will denote all of these infinite sums simply as \( \sum_n \). We now prove each item in turn.
Since $A\mathcal{D}$ is a monomial operator of degree $k + j$ it follows from Theorem 3.15 that $D(A\mathcal{D}) = D_{(k+j)/2}$. On the other hand, $f \in D(A\mathcal{D})$ iff $f \in D(D) = D_{j/2}$ and $\overline{D}f \in D(A) = D_{k/2}$. Moreover, $\overline{D}f = D^* f \in D(A) = D_{k/2}$ iff

$$\infty > \sum_n \left| \langle \mathcal{D}f, \Omega_n \rangle \right|^2 n^k = \sum_n \left| \langle f, \mathcal{D}^\dagger \Omega_n \rangle \right|^2 n^k$$

$$= \sum_n \left| \langle f, \Omega_{n-\ell(d)} \rangle \right|^2 |c_{\mathcal{D}}(n)|^2 n^k.$$  \hspace{1cm} (3.29)

However, by Lemma 3.12 we know $|c_{\mathcal{D}}(n)|^2 \approx n^j$ and so the condition in Eq. (3.29) is the same as saying $f \in D_{(k+j)/2}$. Thus we have shown $D(A\mathcal{D}) = D(A\mathcal{D})$. Moreover, if $f \in D_{(k+j)/2}$, then by Theorem 3.15 and Lemma 3.12 we find,

$$\mathcal{A}\mathcal{D}f = \sum_n \langle \mathcal{D}f, \Omega_n \rangle \mathcal{A}\Omega_n = \sum_n \langle f, \mathcal{D}^\dagger \Omega_n \rangle \mathcal{A}\Omega_n$$

$$= \sum_n \langle f, c_{\mathcal{D}}(n-\ell(d)) \Omega_{n-\ell(d)} \rangle \mathcal{A}\Omega_n$$

$$= \sum_n \langle f, \Omega_n \rangle \mathcal{A}c_{\mathcal{D}}(n) \Omega_{n+\ell(d)}$$

$$= \sum_n \langle f, \Omega_n \rangle \mathcal{A}\mathcal{D}\Omega_n = \mathcal{A}\mathcal{D}f.$$  \hspace{1cm} (3.30)

(2) By item 1. of Theorem 3.15 and item 1. of this theorem,

$$(AD)^* = (\mathcal{A}\mathcal{D})^\dagger = D\mathcal{A}^\dagger = D\mathcal{A}^\dagger = D^* A^*.$$

(3) This follows by induction on $k = \deg \theta u_b$ making use of item 1. of Theorem 3.15 and item 1.

**Corollary 3.17** (Diagonal form of the Number Operator). If $\mathcal{N} = u(\theta^*, \theta) (\bar{a}, a^*) = a^* \bar{a}$ as in Definition 1.3, then by $\mathcal{N} = a^\dagger a$,

$$D(\mathcal{N}) = D_1 = \left\{ f \in L^2(m) : \sum_{n=0}^{\infty} n^2 |\langle f, \Omega_n \rangle|^2 < \infty \right\},$$

and for $f \in D(\mathcal{N})$,

$$\mathcal{N}f = \sum_{n=0}^{\infty} n \langle f, \Omega_n \rangle \Omega_n.$$

**Proof.** Since $\mathcal{N} = u(\theta^*, \theta) (\bar{a}, a^*)$, it follows by Theorem 3.15 that

$$\mathcal{N} = u(\theta^*, \theta) (a^\dagger) = a^\dagger a$$

and then by Theorem 3.15 that $D(\mathcal{N}) = D_1$. Moreover, by items 1 and 2 in the Theorem 3.15 if $f \in D(\mathcal{N})$, then

$$\mathcal{N}f = \sum_{n=0}^{\infty} \langle f, \Omega_n \rangle a^\dagger a \Omega_n = \sum_{n=0}^{\infty} n \langle f, \Omega_n \rangle \Omega_n.$$
Definition 3.18 (Functional Calculus for \( \mathcal{N} \)). Given a function \( G : \mathbb{N}_0 \to \mathbb{C} \) let \( G (\mathcal{N}) \) be the unique closed operator on \( L^2 (m) \) such that \( G (\mathcal{N}) \Omega_n := G (n) \Omega_n \) for all \( n \in \mathbb{N}_0 \). In more detail,

\[
D (G (\mathcal{N})) := \left\{ u \in L^2 (m) : \sum_{n=0}^{\infty} |G (n)|^2 |\langle u, \Omega_n \rangle|^2 < \infty \right\} \quad (3.31)
\]
and for \( u \in D (G (\mathcal{N})) \),

\[
G (\mathcal{N}) u := \sum_{n=0}^{\infty} G (n) \langle u, \Omega_n \rangle \Omega_n.
\]

Example 3.19. If \( \beta \geq 0 \), then \( D (\mathcal{N}^\beta) = D_\beta \) where \( D_\beta \) was defined in Definition 3.14.

Notation 3.20. If \( J \subset \mathbb{N}_0 \) and

\[
1_J (n) := \begin{cases} 1 & \text{if } n \in J \\ 0 & \text{otherwise} \end{cases},
\]
then

\[
1_J (\mathcal{N}) f = \sum_{n \in J} \langle f, \Omega_n \rangle \Omega_n \quad (3.32)
\]
is orthogonal projection onto \( \text{span} \{ \Omega_n : n \in J \} \). When \( J = \{ 0, 1, \ldots, N \} \), then \( 1_J (\mathcal{N}) \) (or also write \( 1_{\mathcal{N} \leq N} \)) is precisely the orthogonal projection operator already defined in Eq. (3.19) above.

At this point it is convenient to introduce a scale of Sobolev type norms on \( L^2 (m) \).

Notation 3.21 (\( \beta \)-Norms). For \( \beta \geq 0 \) and \( f \in L^2 (m) \), let

\[
\| f \|_\beta^2 := \sum_{n=0}^{\infty} |\langle f, \Omega_n \rangle|^2 (n+1)^{2\beta}.
\]

Remark 3.22. From Definition 3.18 and Notation 3.21 it is readily seen that

\[
D_\beta = D (\mathcal{N}^\beta) = \left\{ f \in L^2 (m) : \| f \|_\beta^2 < \infty \right\},
\]

\[
\| f \|_\beta^2 = \left\| (\mathcal{N} + 1)^\beta f \right\|_{L^2 (m)}^2 \forall f \in D (\mathcal{N}^\beta),
\]

\[
D (\mathcal{N}^\beta) = D ( (\mathcal{N} + 1)^\beta ) \text{ for all } \beta \geq 0, \text{ and}
\]

\[
\| \|_{\beta_1} \leq \| \|_{\beta_2} \text{ and } D (\mathcal{N}^{\beta_2}) \subseteq D (\mathcal{N}^{\beta_1}) \text{ for all } 0 \leq \beta_1 \leq \beta_2.
\]

The normed space, \( \left( D (\mathcal{N}^\beta), \| \|_\beta \right) \), is a Hilbertian space which is isomorphic to \( \ell^2 (\mathbb{N}_0, \mu_\beta) \) where \( \mu_\beta (n) := (1+n)^{2\beta} \). The isomorphism is given by the unitary map,

\[
f \in D (\mathcal{N}^\beta) \to \{ \langle f, \Omega_n \rangle \}_{n=0}^{\infty} \in \ell^2 (\mathbb{N}_0, \mu_\beta).
\]

It is well known (see for example, [25, Theorem 1]) that

\[
S = \bigcap_{n=0}^{\infty} D (\mathcal{N}^n) = \bigcap_{\beta \geq 0} D (\mathcal{N}^\beta). \quad (3.34)
\]
The inclusion $S \subset \bigcap_{n=0}^{\infty} D(\mathcal{N}^n)$ is easy to understand since if $n \in \mathbb{N}_0$, $(a^t a + I)^n$ is symmetric on $S$ and therefore if $f \in S$ we have,

$$
\|f\|_n^2 = \sum_{n=0}^{\infty} |\langle f, \Omega_n \rangle|^2 (n+1)^{2n} = \sum_{n=0}^{\infty} \left| \langle f, (a^t a + I)^n \Omega_n \rangle \right|^2
$$

$$
= \sum_{n=0}^{\infty} \left| \langle (a^t a + I)^n f, \Omega_n \rangle \right|^2 = \left\| (a^t a + I)^n f \right\|_{L^2(m)}^2 < \infty.
$$

The following related result will be useful in the sequel.

**Proposition 3.23.** The subspace $S_0$ in Eq. (3.20) is dense (and so is $S$) in $\left( D(\mathcal{N}^\beta), \|\cdot\|_\beta \right)$ for all $\beta \geq 0$. Moreover, if $f \in D(\mathcal{N}^\beta)$, then $P_N f \in S_0$ and $\|f - P_N f\|_\beta \to 0$ as $N \to \infty$.

**Proof.** If $f \in D(\mathcal{N}^\beta)$, then

$$
\sum_{n=0}^{\infty} |\langle f, \Omega_n \rangle|^2 (1 + n)^{2\beta} = \|f\|_\beta^2 < \infty
$$

and hence

$$
\|f - P_N f\|_\beta^2 = \sum_{n=N+1}^{\infty} |\langle f, \Omega_n \rangle|^2 (1 + n)^{2\beta} \to 0 \text{ as } N \to \infty.
$$

\[ \blacksquare \]

**Remark 3.24.** The zero norm, $\|\cdot\|_0$, is just a standard $L^2(m)$-norm and we will typically drop the subscript 0 and simply write $\|\cdot\|$ for $\|\cdot\|_0 = \|\cdot\|_{L^2(m)}$.

**Remark 3.25.** If $A = u_b(a,a^t)$ and $k = \deg_\vartheta u_b(\theta,\theta^*)$, then by Eq. (3.33) and the Theorem 3.15 we have

$$
D(\tilde{A}) = D_{k/2} = D \left( \mathcal{N}^{\frac{k}{2}} \right). \tag{3.35}
$$

**Corollary 3.26.** The following domain statement holds;

$$
D(\bar{a}) = D(a^*) = D \left( \mathcal{N}^{1/2} \right) = D(M_x) \cap D(\partial_x). \tag{3.36}
$$

Moreover for $f \in D(\mathcal{N}^{1/2}),$

$$
\bar{a} f = \sum_{n=1}^{\infty} \sqrt{n} \langle f, \Omega_n \rangle \Omega_{n-1} \text{ and } \tag{3.37}
$$

$$
a^* f = \sum_{n=0}^{\infty} \sqrt{n+1} \langle f, \Omega_n \rangle \Omega_{n+1}. \tag{3.38}
$$

**Proof.** $D(\bar{a}) = D(a^*) = D \left( \mathcal{N}^{1/2} \right)$ is followed by the Eq. (3.35) in the Remark 3.25. Eqs (3.37) and (3.38) are consequence from Theorem 3.15. The only new statement to prove here is that $D(\mathcal{N}^{1/2}) = D(M_x) \cap D(\partial_x)$. If $f \in D(M_x) \cap D(\partial_x)$ we have

$$
\sqrt{n} \langle f, \Omega_n \rangle = \langle f, a^t \Omega_{n-1} \rangle = \frac{1}{\sqrt{2}} \langle f, (M_x - \partial_x) \Omega_{n-1} \rangle
$$

$$
= \frac{1}{\sqrt{2}} \langle (M_x + \partial_x) f, \Omega_{n-1} \rangle
$$

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from which it follows that

$$\sum_{n=1}^{\infty} \sqrt{n} \langle f, \Omega_n \rangle^2 = \frac{1}{2} \| (M_x + \partial_x) f \|^2 < \infty$$

and therefore $f \in D(\bar{a}) = D(N^{1/2})$. Conversely if $f \in D(N^{1/2})$ and we let $f_m := \sum_{k=0}^{m} \langle f, \Omega_k \rangle \Omega_k$ for all $m \in \mathbb{N}$, then $f_m \to f$, $\bar{a} f_m \to \bar{a} f$ and $a^* f_m \to a^* f$ in $L^2$. Thus it follows that in the limit as $m \to \infty$,

$$M_x f_m = \frac{1}{\sqrt{2}} (\bar{a} + a^*) f_m \to \frac{1}{\sqrt{2}} (\bar{a} + a^*) f$$

and

$$\partial_x f_m = \frac{1}{\sqrt{2}} (\bar{a} - a^*) f_m \to \frac{1}{\sqrt{2}} (\bar{a} - a^*) f.$$

As $M_x$ and $\partial_x$ are closed operators, it follows that $f \in D(M_x) \cap D(\partial_x)$. ■

### 3.4. Operator Inequalities.

**Notation 3.27 ($\beta_1, \beta_2$ – Operator Norms).** Let $\beta_1, \beta_2 \geq 0$. If $T : D(N^{\beta_1}) \to D(N^{\beta_2})$ is a linear map, let

$$\| T \|_{\beta_1 \to \beta_2} := \sup_{0 \neq \psi \in D(N^{\beta_1})} \frac{\| T \psi \|_{\beta_2}}{\| \psi \|_{\beta_1}}.$$  \hspace{1cm} (3.39)

denote the corresponding operator norm. We say that $T$ is $\beta_1 \to \beta_2$ bounded if $\| T \|_{\beta_1 \to \beta_2} < \infty$. In the special case when $\beta_1 = \beta_2 = \beta$, let $B(D(N^{\beta}))$, $\| \|_{\beta \to \beta}$ denote the Banach space of all $\beta \to \beta$ bounded linear operators, $T : D(N^{\beta}) \to D(N^{\beta})$.

**Remark 3.28.** Let $\beta_1, \beta_2, \beta_3 \geq 0$. As usual, if $T : D(N^{\beta_1}) \to D(N^{\beta_2})$ and $S : D(N^{\beta_2}) \to D(N^{\beta_3})$ are any linear operators, then

$$\| ST \|_{\beta_1 \to \beta_3} \leq \| S \|_{\beta_2 \to \beta_3} \| T \|_{\beta_1 \to \beta_2}.$$  \hspace{1cm} (3.40)

**Proposition 3.29.** Let $k = \deg_{\theta} u_b (\theta, \theta^*)$ and $A = u_b (a, a^*)$ be as in Notations \ref{notation2} and \ref{notation3}. If $\beta \geq 0$, then $A D(N^{\beta + k/2}) \subset D(N^{\beta})$ and

$$\| \tilde{A} \|^2_{\beta + \frac{k}{2} \to \beta} \leq k^k (k + 1)^{2\beta} \leq (k + 1)^{2\beta + k}.$$  \hspace{1cm} (3.41)

Moreover,

$$\| \tilde{A} f \|_{\beta} \leq \| (N + k)^{k/2} (N + k + 1)^{\beta} f \| \forall f \in D(N^{\beta + k/2}).$$  \hspace{1cm} (3.42)

**Proof.** Let $f \in D(N^{\beta + k/2}) \subset D(N^{k/2})$ and recall from Lemma 3.12 that $c_{\tilde{A}}(n) = c_A(n - l)$ and $|c_A(n)|^2 \leq (n + k)^k$. Using these facts and the fact that
\( \bar{\mathcal{A}} = \mathcal{A}^* \) (see Theorem \[3.15\]), we find,

\[
\|\bar{\mathcal{A}} f\|_\beta^2 = \sum_n \langle \bar{\mathcal{A}} f, \Omega_n \rangle^2 (1 + n)^{2\beta} = \sum_n \langle f, \mathcal{A}^* \Omega_n \rangle^2 (1 + n)^{2\beta} 1_{n \geq 0} \\
= \sum_n \langle f, \Omega_{n+l} \rangle^2 |c_\mathcal{A} (n - l)| (1 + n)^{2\beta} 1_{n \geq 0} \\
= \sum_n \langle f, \Omega_n \rangle^2 (1 + n + l)^{2\beta} 1_{n+l \geq 0} |c_\mathcal{A} (n)|^2 \\
\leq \sum_n \langle f, \Omega_n \rangle^2 (n + k + 1)^{2\beta} (n + k)^k \\
= \| (\mathcal{N} + k)^{k/2} (\mathcal{N} + k + 1)^\beta f \|_0^2
\]

which proves Eq. \[3.42\]. Using

\[ n + a \leq a (n + 1) \text{ for } a \geq 1 \text{ and } n \in \mathbb{N}_0 \]

in Eq. \[3.43\] with \( a = k \) and \( a = k + 1 \) shows,

\[
\| \bar{\mathcal{A}} f \|_\beta^2 \leq k^k (k + 1)^{2\beta} \sum_n \langle f, \Omega_n \rangle^2 (1 + n)^{2\beta+k} = k^k (k + 1)^{2\beta} \| f \|_{\beta+k/2}^2.
\]

The previous inequality proves Eq. \[3.41\] and also \( \bar{\mathcal{A}} \mathcal{D} (\mathcal{N}^{\beta+k/2}) \subset \mathcal{D} (\mathcal{N}^\beta) \). ■

**Corollary 3.30.** If \( P (\theta, \theta^*) \in \mathcal{C} (\theta, \theta^*) \) and \( d = \deg_\theta P \), then \( D (\mathcal{N}^{d/2}) = D (P (\bar{a}, a^*)) \), \( P (\bar{a}, a^*) \subset P (a, a^*) \), and

\[
\| P (\bar{a}, a^*)\|_{\beta+d/2-\beta} \leq \sum_{k=0}^d k^{k/2} (k + 1)^{\beta} |P_k| \text{ for all } \beta \geq 0.
\]

**Proof.** The operator \( P (\bar{a}, a^*) \) is a linear combination of operators of the form \( u_b (\bar{a}, a^*) \) where \( k = \deg_\theta u_b (\theta, \theta^*) \leq d \). By Theorem \[3.15\] it follows that \( D (u_b (\bar{a}, a^*)) = D (\mathcal{N}^{k/2}) \supset D (\mathcal{N}^{d/2}) \) and hence \( D (\mathcal{N}^{d/2}) = D (P (\bar{a}, a^*)) \). Further, Proposition \[3.29\] shows

\[
\| u_b (\bar{a}, a^*)\|_{\beta+d/2-\beta} \leq \| u_b (\bar{a}, a^*)\|_{\beta+k/2-\beta} \leq k^{k/2} (k + 1)^{\beta}.
\]

This estimate, the triangle inequality, and the definition of \( |P_k| \) in Eq. \[2.24\] leads directly to the inequality in Eq. \[3.41\].

If \( f \in D (\mathcal{N}^{d/2}) \), it follows from Eq. \[3.43\] and Proposition \[3.23\] that

\[
P (\bar{a}, a^*) f = \lim_{N \to \infty} P (\bar{a}, a^*) \mathcal{P}_N f = \lim_{N \to \infty} P (a, a^*) \mathcal{P}_N f
\]

which shows \( f \in D \left( \bar{P} (a, a^*) \right) \) and \( \bar{P} (a, a^*) f = P (\bar{a}, a^*) f \). ■

**Notation 3.31.** For \( x \in \mathbb{R} \) let \( (x)_+ := \max (x, 0) \).

**Lemma 3.32.** If \( \mathcal{A} = u_b (a, a^*) \), \( k = \deg_\theta u_b (\theta, \theta^*) \), \( l = \ell (b) \in \mathbb{Z} \) are as in Notations \[3.17\] and \[3.7\] then for all \( \beta \geq 0 \) we have,

\[
(\mathcal{N} + 1)^\beta \bar{\mathcal{A}} f = \bar{\mathcal{A}} ((\mathcal{N} + l)_+ + 1)^\beta f \text{ for all } f \in D \left( \mathcal{N}^{\beta+k/2} \right).
\]
Proof. Using Proposition 3.29 and Remark 3.28 it is readily verified that the operators on both sides of Eq. (3.40) are bounded linear operators from $D\left(N^{\beta+\frac{1}{2}}\right)$ to $L^2(m)$. Since $S_0$ is dense in $D\left(N^{\beta+\frac{1}{2}}\right)$ (see Proposition 3.23) it suffices to verify Eq. (3.40) for $f = \Omega_n$ for all $n \in \mathbb{N}_0$ which is trivial. Indeed, $\bar{A}\Omega_n = c_A(n)\Omega_{n+l}$ which is zero if $n + l < 0$ and hence
\[
(N+1)^\beta \bar{A}\Omega_n = ((n+l)+1)^\beta \bar{A}\Omega_n = \bar{A}((n+l)+1)^\beta \Omega_n
\]
and hence
\[
\bar{A} = ((n+l)+1)^\beta \Omega_n.
\]

Proposition 3.33. Let $k \in \mathbb{N}$, $b \in \{\theta, \theta^*\}^k$, $\mathcal{A}$, and $\ell(b)$ be as in Notation 3.1. For any $\beta \geq 0$, it gets
\[
\left\| (N+1)^\beta , \bar{A} \right\| (N+1)^{-\beta} \varphi \leq \beta k^{k/2} \left| \ell(b) \right| (1 + |\ell(b)|)^{\beta-1} \left\| (N+1)^{k/2-1} 1_{N \geq -l} \varphi \right\| (3.47)
\]
\[
\leq \beta k^{k/2} \left| \ell(b) \right| (1 + |\ell(b)|)^{\beta-1} \left\| (N+1)^{k/2-1} \varphi \right\| (3.48)
\]
for all $\varphi \in D\left(N^{k/2}\right)$.

Proof. Let $l := \ell(b)$. By Lemma 3.32 and the identity, $\bar{A} = \bar{A}1_{N+l \geq 0}$, for all $\psi \in D\left(N^{k/2+\beta}\right)$ we have,
\[
\left[ (N+1)^\beta , \bar{A} \right] \psi = \left[ (N+1)^\beta , \bar{A} \right] (N+1)^\beta \psi
\]
\[
= \bar{A} \left[ ((N+l)+1)^\beta - (N+1)^\beta \right] \psi
\]
\[
= \bar{A}1_{N+l \geq 0} \left[ ((N+l)+1)^\beta - (N+1)^\beta \right] \psi
\]
\[
= \bar{A} \left[ (N+l+1)^\beta - (N+1)^\beta \right] 1_{N+l \geq 0} \psi
\]
\[
= \bar{A} \left[ \beta \int_0^l (N+1+r)^{\beta-1} dr \right] 1_{N+l \geq 0} \psi.
\]
Combining this equation with Eq. (3.42) of Proposition 3.29 shows,
\[
\left\| (N+1)^\beta , \bar{A} \right\| \psi \leq \left\| (N+k)^{k/2} \beta \int_0^l (N+1+r)^{\beta-1} dr \right\| 1_{N \geq -l} \psi
\]
\[
\leq \beta \int_0^l \left\| (N+k)^{k/2} (N+1+r)^{\beta-1} 1_{N \geq -l} \psi \right\| dr
\]
\[
\leq \beta k^{k/2} \int_0^l \left\| (N+1)^{k/2} (N+1+r)^{\beta-1} 1_{N \geq -l} \psi \right\| dr
\]
For $x \geq \max(0, -l)$ and $r$ between 0 and $l$, one shows
\[
(x+1+r)^{\beta-1} \leq (1 + |l|)^{\beta-1} (x+1)^{\beta-1}
\]
which combined with the previously displayed equation implies,
\[
\left\| (N+1)^{\beta} , \bar{A} \right\| \psi \leq \beta k^{k/2} (1 + |l|)^{\beta-1} |l| \left\| (N+1)^{k+\beta-1} 1_{N \geq -l} \psi \right\| (3.49)
\]
Finally, Eq. (3.37) easily follows by replacing $\psi$ by $(\mathcal{N} + 1)^{-\beta} \varphi \in D(\mathcal{N}^{k/2})$ in Eq. (3.40).

3.5. Truncated Estimates.

**Notation 3.34** (Operator Truncation). If $Q = P(a,a^1)$ is a polynomial operator on $L^2(m)$ and $M > 0$, let

$$Q_M := 1_{\mathcal{N} \leq M} Q 1_{\mathcal{N} \leq M} = \mathcal{P}_M Q \mathcal{P}_M.$$  

(3.50)

and refer to $Q_M$ as the level-$M$ truncation of $Q$. [Recall that $\mathcal{P}_M = 1_{\mathcal{N} \leq M}$ are as in Notations 3.10 and 3.20]

**Proposition 3.35.** If $k \in \mathbb{N}$, $\beta \geq 0$, $0 < M < \infty$, $b \in \{\theta, \theta^*\}^k$, $A = u_b(a,a^1)$, and $\ell(b)$ are as in Notation 3.7, then

$$\|A_M\|_{\beta \rightarrow \beta} \leq (M + k)^{k/2} (1 + |\ell(b)|)^{\beta} \leq k^{k/2} (1 + |\ell(b)|)^{\beta} (M + 1)^{k/2}. \quad (3.51)$$

Consequently if $P \in \mathcal{C}(\theta, \theta^*)$ with $d = \deg_P P$, then

$$\left\| [P(a,a^1)]_M \right\|_{\beta \rightarrow \beta} \leq \sum_{k=0}^d (M + k)^{k/2} (1 + k)^\beta |P_k| \quad (3.52)$$

which in particular implies that the map,

$$P \in \mathcal{C}(\theta, \theta^*) \rightarrow [P(a,a^1)]_M \in \left( B(D(\mathcal{N}^\beta)), \|\cdot\|_{\beta \rightarrow \beta} \right),$$

depends continuously on the coefficients of $P$.

**Proof.** With $l = \ell(b)$, we have for all $n \in \mathbb{N}_0$,

$$A_M^* \Omega_n = (\mathcal{P}_M A \mathcal{P}_M)^* \Omega_n = \mathcal{P}_M A^* \mathcal{P}_M \Omega_n$$

$$= 1_{n \leq M} \mathcal{P}_M A^1 \Omega_n = 1_{n \leq M} c_A(n-l) \mathcal{P}_M \Omega_{n-l}$$

$$= 1_{n \leq M} 1_{n-l \leq M} c_A(n-l) \Omega_{n-l}. \quad (3.53)$$

From this identity and simple estimates using Eq. (3.44) repeatedly we find, for $f \in D(\mathcal{N}^\beta)$

$$\|A_M f\|_{\beta}^2 = \sum_n |\langle A_M f, \Omega_n \rangle|^2 (1 + n)^{2\beta}$$

$$= \sum_n 1_{0 \leq n \leq M} 1_{0 \leq n-l \leq M} |\langle f, \Omega_{n-l} \rangle|^2 |c_A(n-l)|^2 (1 + n)^{2\beta}$$

$$= \sum_n 1_{0 \leq n+l \leq M} 1_{0 \leq n \leq M} |\langle f, \Omega_n \rangle|^2 |c_A(n)|^2 (1 + n + l)^{2\beta}$$

$$\leq \sum_n 1_{0 \leq n+l \leq M} 1_{0 \leq n \leq M} |\langle f, \Omega_n \rangle|^2 (k + n)^k (1 + n + |l|)^{2\beta}$$

$$\leq (M + k)^k (1 + |l|)^{2\beta} \sum_n 1_{0 \leq n+l \leq M} 1_{0 \leq n \leq M} |\langle f, \Omega_n \rangle|^2 (1 + n)^{2\beta}$$

$$\leq (M + k)^k (1 + |l(b)|)^{2\beta} \|f\|_{\beta}^2 \leq k^k (M + 1)^k (1 + |l(b)|)^{2\beta} \|f\|_{\beta}^2.$$
Theorem 3.36. Let $k \in \mathbb{N}$, $\beta \geq 0$, $b \in \{\theta, \theta^*\}^k$, and $A = u_b (a, a^t)$ be as in Notation 3.7. If $\alpha \geq \beta + k/2$, then
\[
\| \bar{A} - A_M \|_{\alpha \to \beta} \leq (M - k + 2)^{(\beta + k/2 - \alpha)} \text{ for all } M \geq k.
\] (3.54)
Consequently, if $\alpha > \beta + k/2$, then
\[
\lim_{M \to \infty} \| (\bar{A} - A_M) \varphi \|_{\beta}^2 = 0 \quad \forall \varphi \in D(\mathcal{N}^\alpha).
\] (3.55)

Proof. Let $M \geq k$. From Proposition 3.29, $\bar{A} - A_M$ is a bounded operator from $(D(\mathcal{N}^\alpha), \| \cdot \|_{\alpha})$ to $\left(D(\mathcal{N}^\beta), \| \cdot \|_{\beta}\right)$. Making use of Eq. (3.53) we find
\[
(A^t - P_M A^t P_M) \Omega_n = c_A (n - l) [1 - 1_{n \leq M} \cdot 1_{n - l \leq M}] \Omega_{n - l}
\]
\[
= c_A (n - l) 1_{n > M \wedge (M + l)} \Omega_{n - l} \quad \text{for all } n \in \mathbb{Z}.
\]
Hence, if $\varphi \in D(\mathcal{N}^\alpha) \subset D(\mathcal{N}^k \wedge 2) = D(\tilde{A})$, then
\[
\| (\bar{A} - A_M) \varphi \|_{\beta}^2 = \sum_n | \langle (\bar{A} - A_M) \varphi, \Omega_n \rangle |^2 (n + 1)^{2\beta}
\]
\[
= \sum_n | \langle \varphi, (A^t - P_M A^t P_M) \Omega_n \rangle |^2 (n + 1)^{2\beta}
\]
\[
= \sum_n 1_{n > M \wedge (M + l)} (n + 1)^{2\beta} | \langle \varphi, \Omega_{n - l} \rangle |^2 | c_A (n - l) |^2
\]
\[
= \sum_n 1_{n + l > M \wedge (M + l)} (n + l + 1)^{2\beta} | \langle \varphi, \Omega_{n} \rangle |^2 | c_A (n) |^2
\]
\[
= \sum_n \rho(n) (n + 1)^{2\alpha} | \langle \varphi, \Omega_n \rangle |^2 \leq \max_n \rho(n) \| \varphi \|_{\alpha}^2
\]
where
\[
\rho(n) := 1_{n + l > M \wedge (M + l)} (n + l + 1)^{2\beta} (n + 1)^{2\alpha} | c_A (n) |^2.
\]
This completes the proof since simple estimates using Lemma 3.12 and the fact that $n \geq M - k + 1$ shows,
\[
\rho(n) \leq k(k + 1)^{2\beta} (M - k + 2)^{2(\beta + k/2 - \alpha)}
\]

Corollary 3.37. If $P (\theta, \theta^*) \in C \{\theta, \theta^*\}$, $d = \deg P$, $\beta \geq 0$, and $\alpha \geq \beta + d/2$, then for any $M \geq d$,
\[
\| [P (a, a^t)]_M - P (\bar{a}, a^*) \|_{\alpha \to \beta} \leq \sum_{k=0}^{d} |P_k| (M - k + 2)^{\beta + k/2 - \alpha}
\]
\[
\leq (M - d + 2)^{\beta + d/2 - \alpha} |P|.
\] (3.56)

Proof. This result a simple consequence of Theorem 3.36, the triangle inequality, and the elementary estimate,
\[
(M - k + 2)^{(\beta + k/2 - \alpha)} \leq (M - d + 2)^{\beta + d/2 - \alpha} \quad \text{for } 0 \leq k \leq d.
\]
Proposition 3.38. If \( P (\theta, \theta^*) \in \mathbb{C} \langle \theta, \theta^* \rangle \) is as in Eq. (2.20) and \(|P_k|\) is as in Eq. (2.35), then for all \( \beta \geq 0\),
\[
\left\| \left[ (N + 1)^{\beta} , P \left( a, a^\dagger \right)_M \right] (N + 1)^{-\beta} \right\|_{0 \to 0}
\leq \sum_{k=1}^{d} \beta k^{k/2} (1 + k)^{\beta - 1} (M + 1)^{(k/2 - 1)} |P_k|
\]  \hspace{1cm} (3.57)
\[
\leq K(\beta, d) \cdot \sum_{k=1}^{d} (M + 1)^{(k/2 - 1)} |P_k|
\]  \hspace{1cm} (3.58)

where
\[
K(\beta, d) := \beta d^{1+d} (1 + d)^{\beta - 1}. \hspace{1cm} (3.59)
\]

Proof. If \( f \in L^2 (m), \ b \in \{\theta, \theta^*\}^k \) and \( A_b := u_b \ (a, a^\dagger) \), then by Proposition 3.33,
\[
\left\| \left[ (N + 1)^{\beta} , [A_b]_M \right] (N + 1)^{-\beta} f \right\| = \left\| \left[ (N + 1)^{\beta} , P_M A_b P_M \right] (N + 1)^{-\beta} f \right\|
\leq \beta k^{k/2} (1 + k)^{\beta - 1} \left\| (N + 1)^{k/2 - 1} P_M f \right\|
\leq \beta k^{k/2} (1 + k)^{\beta - 1} (M + 1)^{(k/2 - 1)} + \|f\|.
\]

Hence \( P \in \mathbb{C} \langle \theta, \theta^* \rangle \) with \( d = \deg \theta P \) is given as in Eq. (2.20) (so that \( P \ (a, a^\dagger) \) is as in Eq. (2.23) with \( h = 1 \)), then by the triangle inequality we find,
\[
\left\| \left[ (N + 1)^{\beta} , P \left( a, a^\dagger \right)_M \right] (N + 1)^{-\beta} \right\|_{0 \to 0}
\leq \sum_{k=1}^{d} \left\| \left[ (N + 1)^{\beta} , P_k \left( a, a^\dagger \right)_M \right] (N + 1)^{-\beta} \right\|_{0 \to 0}
\leq \sum_{k=1}^{d} \beta k^{k/2} (1 + k)^{\beta - 1} (M + 1)^{(k/2 - 1)} + |P_k|
\]
where the absence of the \( k = 0 \) term is a consequence of \( P_0 \ (a, a^\dagger)_M \) is proportional to \( P_M \) and hence commutes with \( (N + 1)^{\beta} \). \( \blacksquare \)

4. BASIC LINEAR ODE RESULTS

Notation 4.1. If \( (X, \|\cdot\|) \) is a Banach space, then \( B(X) \) is notated as a collection of bounded linear operators from \( X \) to itself and \( \|\cdot\|_{B(X)} \) is denoted as an operator norm. (e.g. \( B \left( D (N^{\beta}) \right), \|\cdot\|_{B(\beta)} \) in Notation 3.27)

Lemma 4.2 (Basic Linear ODE Theorem). Suppose that \( (X, \|\cdot\|) \) is a Banach space and \( t \to C(t) \in B(X) \) is an operator norm continuous map. Then to each \( s \in \mathbb{R} \) there exists a unique solution, \( U(t, s) \in B(X) \), to the ordinary differential equation,
\[
\frac{d}{dt} U(t, s) = C(t) U(t, s) \quad \text{with} \quad U(s, s) = I. \hspace{1cm} (4.1)
\]
Moreover, the function \((t, s) \to U(t, s) \in B(X)\) is operator norm continuously differentiable in each of its variables and \((t, s) \to \partial_t U(t, s)\) and \((t, s) \to \partial_s U(t, s)\) are operator norm continuous functions into \(B(X)\),
\[
\partial_t U(t, s) = -U(t, s)C(s) \quad \text{with} \quad U(t, t) = I, \quad \text{and}
\]
\[
U(t, s) U(s, \sigma) = U(t, \sigma) \quad \text{for all} \ s, \sigma, t \in \mathbb{R}.
\]

**Proof.** Let \(V(t)\) and \(W(t)\) in \(B(X)\) solve the ordinary differential equations,
\[
\frac{d}{dt} V(t) = C(t) V(t) \quad \text{with} \quad V(0) = I \quad \text{and}
\]
\[
\frac{d}{dt} W(t) = -W(t) C(t) \quad \text{with} \quad W(0) = I.
\]
We then have
\[
\frac{d}{dt} \left[ W(t) V(t) \right] = -W(t) C(t) V(t) + W(t) C(t) V(t) = 0
\]
so that \(W(t) V(t) = I\) for all \(t\). Moreover, \(Z(t) := V(t) W(t)\) solves the differential equation,
\[
\frac{d}{dt} Z(t) = - V(t) W(t) C(t) + C(t) V(t) W(t) = [C(t), Z(t)] \quad \text{with} \quad Z(0) = V(0) W(0) = I.
\]
The unique solution to this differential equation is \(Z(t) = I\) from which we conclude \(V(t) W(t) = I\) for all \(t \in \mathbb{R}\). In summary, we have shown \(W(t)\) and \(V(t)\) are inverses of one another. It is now easy to check that
\[
U(t, s) = V(t) V(s)^{-1} = V(t) W(s)
\]
from which all of the rest of the stated results easily follow. \(\blacksquare\)

**Proposition 4.3** (Operator Norm Bounds). Suppose that \((K, \langle \cdot, \cdot \rangle)\) is a Hilbert space, \(A\) is a self-adjoint operators on \(K\) with \(A \geq I\), and make \(D(A)\) into a Hilbert space using the inner product, \(\langle \cdot, \cdot \rangle_A\), defined by
\[
\langle \psi, \varphi \rangle_A := \langle A \psi, A \varphi \rangle \quad \text{for all} \ \varphi, \psi \in D(A).
\]
Further suppose that \(t \to C(t) \in B(K)\) see Notation 4.1 is a \(\|\cdot\|_K\)-operator norm continuous map such that \(C(t) D(A) \subset D(A)\) for all \(t\) and the map \(t \to C(t) D(A) \subset B(D(A))\) is \(\|\cdot\|_A\)-operator norm continuous. Let \(U(t, s) \in B(K)\) be as in Lemma 4.2. Then,
\begin{enumerate}
\item \(U(t, s) D(A) \subset D(A)\) for all \(s, t \in \mathbb{R}\), and
\[
U(t, s) U(s, \sigma) = U(t, \sigma).
\]
\item \(U(t, s) |_{D(A)}\) solves
\[
\frac{d}{dt} U(t, s) |_{D(A)} = C(t) |_{D(A)} U(t, s) |_{D(A)} \quad \text{with} \quad U(s, s) |_{D(A)} = I_{D(A)}
\]
where the derivative on the left side of this equation is taken relative to the operator norm on the Hilbert space, \((D(A), \langle \cdot, \cdot \rangle_A)\).
\item For all \(s, t \in \mathbb{R}\),
\[
\|U(t, s)\|_{B(K)} \leq \exp \left( \frac{1}{2} \int_s^t \|C(\tau) + C(\tau)\|_{B(K)} \ d\tau \right) \quad (4.2)
\]
\end{enumerate}
where \( \| \cdot \|_{B(K)} \) is as in Notation 4.1. Moreover, \( U(t,s) \) is unitary on \( K \) if \( C(t) \) is skew adjoint for all \( t \in \mathbb{R} \).

(4) For all \( s,t \in \mathbb{R} \),

\[
\| U(t,s) \|_{B(D(A))} \leq \exp \left( \int_s^t \left[ \frac{1}{2} \| C(\tau) + C^*(\tau) \|_{B(K)} + \|[A,C(\tau)]A^{-1}\|_{B(K)} \right] \, d\tau \right). \tag{4.3}
\]

and

\[
\| U(t,s) \|_{B(D(A))} \geq \exp \left( - \int_s^t \left[ \frac{1}{2} \| C(\tau) + C^*(\tau) \|_{B(K)} + \|[A,C(\tau)]A^{-1}\|_{B(K)} \right] \, d\tau \right). \tag{4.4}
\]

where \( \|[A,C(\tau)]A^{-1}\|_{B(K)} \) is defined to be \( \infty \) if \( [A,C(\tau)]A^{-1} \) is an unbounded operator on \( K \).

Proof. Let \( U(t,s) \) be as in Lemma 4.2 when \( X = K \) and \( U_A(t,s) \) be as in Lemma 4.2 when \( X = D(A) \). Further suppose that \( \psi_0 \in D(A) \) and let \( \psi(t) := U(t,s)\psi_0 \) and \( \psi_A(t) := U_A(t,s)\psi_0 \). We now prove each item in turn.

(1) Since \( \psi(t) \) and \( \psi_A(t) \) both solve the differential equation (in the \( K \)-norm)
[Note: \( \| \cdot \|_A \geq \| \cdot \|_K \)]

\[
\dot{\varphi}(t) = C(t)\varphi(t) \text{ with } \varphi(s) = \psi_0, \tag{4.5}
\]

it follows by the uniqueness of solutions to ODE that

\[
U(t,s)\psi_0 = \psi(t) = \psi_A(t) = U_A(t,s)\psi_0 \in D(A).
\]

The results of items 1. and 2. now easily follow.

(2) It is well known and easily verified that \( U(t,s) \) is unitary on \( K \) if \( C(t) \) is skew adjoint. The estimate in Eq. (4.2) is a special case of the estimate in Eq. (4.3) when \( A = I \) so it suffices to prove the latter estimate.

(3) With \( \psi(t) = U(t,s)\psi_0 = U_A(t,s)\psi_0 \in D(A) \) as above we have,

\[
\frac{d}{dt} \| \psi \|_A^2 = 2 \Re \langle C\psi, \psi \rangle_A = 2 \Re \langle AC\psi, A\psi \rangle
\]

\[
= 2 \Re \langle [CA\psi, A\psi] + \langle [A,C] \psi, A\psi \rangle \rangle
\]

\[
= \langle (C + C^*) A\psi, A\psi \rangle + 2 \Re \langle [A,C] A^{-1} A\psi, A\psi \rangle
\]

and therefore,

\[
\frac{d}{dt} \| \psi \|_A^2 \leq \left( \| C + C^* \|_{B(K)} + 2 \|[A,C] A^{-1}\|_{B(K)} \right) \| \psi \|_A^2.
\]

This last inequality may be integrated to find,

\[
\left( \frac{\| \psi(t) \|_A^2}{\| \psi_0 \|_A^2} \right)^{\pm 1} \leq \exp \left( \int_s^t \left[ \| C(\tau) + C^*(\tau) \|_{B(K)} + 2 \|[A,C(\tau)] A^{-1}\|_{B(K)} \right] \, d\tau \right)
\]

from which Eqs. (4.3) and (4.4) easily follow.

\[\blacksquare\]
4.1. **Truncated Evolutions.** Now suppose that \( P(t : \theta, \theta^*) \in \mathbb{C} \langle \theta, \theta^* \rangle \) with \( \deg \beta P(t : \theta, \theta^*) = d \in \mathbb{N} \) is a one parameter family of **symmetric** non-commutative polynomials whose coefficients depend continuously on \( t \). In more detail we may write \( P(t : \theta, \theta^*) \) as:

\[
P(t : \theta, \theta^*) = \sum_{k=0}^{d} P_k(t : \theta, \theta^*) \quad \text{where}
\]

\[
P_k(t : \theta, \theta^*) = \sum_{b \in \{\theta, \theta^*\}^k} c_k(t, b) u_b(\theta, \theta^*)
\]

and all coefficients, \( t \to c_k(t, b) \) are continuous in \( t \). Let \( Q(t) = P(t : a, a^\dagger) \) and for any \( M > 0 \) let \( Q_M(t) = P_M Q(t) P_M \) be the truncation of \( Q(t) \) as in Notation \[3.31\].

Applying Lemma \[4.2\] with \( C \) exists \((M > 3.34)\) and again are continuous functions of \( \beta, t, s \) (3.21) and Again are continuous functions of \( (t, s) \) in this topology and satisfy

\[
i\frac{d}{dt} U^M(t, s) \varphi = Q_M(t) U^M(t, s) \varphi
\]

(4.10)

\[
i\frac{d}{ds} U^M(t, s) \varphi = -U^M(t, s) Q_M(s) \varphi.
\]

(4.11)

(4) If \( \beta \geq 0 \) and \( t, s \in \mathbb{R} \), then with \( K(\beta, d) < \infty \) as in Eq. \[3.59\] we have

\[
\|U^M(t, s)\|_{\beta \to \beta} \leq \exp \left( K(\beta, d) \sum_{k=1}^{d} (M + 1)^{(k/2 - 1)} \int_{J_{st}} |P_k(\tau, \theta, \theta^*)| d\tau \right),
\]

(4.12)

where \( J_{st} = [\min (s, t), \max (s, t)] \), and \( \|\cdot\|_{\beta \to \beta} \) is as in Notation \[3.27\].

Remark 4.5. Taking \( t = \sigma \) in Eq. \[4.9\] and using the fact that \( U^M(t, s) \) is unitary on \( L^2(m) \), it follows that

\[
U^M(t, s)^{-1} = U^M(s, t) = U^M(t, s)^*.
\]

(4.13)

Remark 4.6. From the item 3 of the Theorem and Eq. \[3.31\], we can conclude that \( U^M(t, s) S = S \).
The continuity of $U^M$ in the item 1. and the identity in Eq. (4.9) both follow from Lemma 4.2. Since $Q_M(t)^* = Q_M(t)$ it follows that $C(t) := -iQ_M(t)$ is skew-adjoint and so the unitary property in the first item is a consequence of item 3. of Proposition 4.3. The remaining item 3. and 4. follow from Proposition 4.3 with $A : = (N + I)^β$ and $C(t) := -iQ_M(t)$. The hypothesis that $C(t) D(A) \subset D(A)$ and $t \to C(t) \in B(D(A))$ is $\|\cdot\|_β$-operator norm continuous in $t$ has been verified in Proposition 3.35. Moreover, from Eq. (3.58) of Proposition 3.38 we know

$$\| [A, C(\tau)] A^{-1} \|_{B(L^2(\mathbb{R}))} \leq K(\beta, d) \sum_{k=1}^d (M + 1)^{(k/2 - 1)_+} \|P_k(\tau, \theta, \theta^*)\|.$$  \hspace{1cm} (4.12)

Equation (4.12) now follows directly from Eq. (4.3) and the fact that $\|\cdot\|_β$-operator norm continuous in $t$ has been verified in Proposition 3.35. Moreover, from Eq. (3.58) of Proposition 3.38 we know

$$\| [A, C(\tau)] A^{-1} \|_{B(L^2(\mathbb{R}))} \leq K(\beta, d) \sum_{k=1}^d (M + 1)^{(k/2 - 1)_+} \|P_k(\tau, \theta, \theta^*)\|.$$  \hspace{1cm} (4.13)

The opposite inclusion is then deduced using $U^M(t, s)^{-1} = U^M(s, t)$ which follows from Eq. (4.9).

**Corollary 4.7.** Recall $P(t : \theta, \theta^*)$ as in Eq. (4.6). Let $h > 0$, $M > 0$, $U^M_h(t, s)$ denotes the solution to the ordinary differential equation,

$$ih \frac{d}{dt} U^M_h(t, s) = \left[ P \left( t : a_h, a_h^\dagger \right) \right] M U^M_h(t, s) \text{ with } U^M_h(s, s) = I,$$

If $\beta \geq 0$ and $s, t \in \mathbb{R}$, then

$$\| U^M_h(t, s) \|_{\beta \to \beta} \leq e^{K(\beta, d) \sum_{k=1}^d h^{(k/2 - 1)_{+}} (M + 1)^{(k/2 - 1)_{+}} \int_{t, s} |P_k(\tau, \theta, \theta^*)| d\tau}, \hspace{1cm} (4.14)$$

where $K(\beta, d) < \infty$ is as in Eq. (3.59). In particular if $P_t(t : \theta, \theta^*) \equiv 0$, $\eta \in (0, 1]$, and $0 < h \leq \eta \leq 1$, then

$$\| U^M_h(t, s) \|_{\beta \to \beta} \leq e^{K(\beta, d) (hM + 1)^{(k/2 - 1)_+} \sum_{k=1}^d \int_{t, s} |P_k(\tau, \theta, \theta^*)| d\tau}. \hspace{1cm} (4.15)$$

**Proof.** Since

$$\frac{1}{h} P_k \left( t : a_h, a_h^\dagger \right) = \frac{1}{h} h^{k/2} P_k \left( t : a, a^\dagger \right) = h^{k/2 - 1} P_k \left( t : a, a^\dagger \right),$$

Eq. (4.14) follows from Theorem 4.4 after making the replacement,

$$P(t : \theta, \theta^*) \rightarrow \sum_{k=0}^d h^{k/2 - 1} P_k \left( t : \theta, \theta^* \right).$$

Equation (4.15) then follows from Eq. (4.14) since for $2 \leq k \leq d$ and $0 < h \leq \eta \leq 1$,

$$h^{k/2 - 1} (M + 1)^{(k/2 - 1)_+} = (hM + h)^{(k/2 - 1)_+} \leq (hM + 1)^{(k/2 - 1)_+}.$$

**5. Quadratically Generated Unitary Groups**

Let $P(t : \theta, \theta^*) \in \mathbb{C}(\theta, \theta^*)$ be a continuously varying one parameter family of symmetric polynomials with $d = \deg P(t : \theta, \theta^*) \leq 2$. Then $Q(t) := P(t : a, a^\dagger)$ may be decomposed as;

$$Q(t) = \sum_{j=0}^6 c_j(t) \mathcal{A}^{(j)} \hspace{1cm} (5.1)$$

where $\mathcal{A}^{(j)}$ is a monomial in $a$ and $a^\dagger$ of degree no bigger than 2 and $c_j(\cdot)$ is continuous for each $0 \leq j \leq 6$ and $\mathcal{A}^{(0)} = 1$ by convention. The main goal of this
chapter is to record the relevant information we need about solving the following
time dependent Schrödinger equation:

\[ i\dot{\psi}(t) = Q(t)\psi(t) \text{ with } \psi(s) = \varphi, \quad (5.2) \]

where \( s \in \mathbb{R} \) and \( \varphi \in D(N) \) and the derivative is taken in \( L^2(m) \).

**Theorem 5.1 (Uniqueness of Solutions).** If \( \mathbb{R} \ni t \rightarrow \psi(t) \in D(N) \) solves Eq. \( (5.2) \) then \( \|\psi(t)\| = \|\varphi\| \) for all \( t \in \mathbb{R} \). Moreover, there is at most one solution to Eq. \( (5.2) \).

**Proof.** If \( \psi(t) \) solves Eq. \( (5.2) \), then because \( Q(t) \) is symmetric on \( D(N) \),

\[ \frac{d}{dt} \|\psi(t)\|^2 = 2\text{Re} \langle \dot{\psi}(t), \psi(t) \rangle = 2\text{Re} \langle -iQ(t)\psi(t), \psi(t) \rangle = 0. \]

Therefore it follows that \( \|\psi(t)\|^2 = \|\psi(s)\|^2 = \|\varphi\|^2 \) which proves the isometry property and because the equation \( (5.2) \) is linear this also proves uniqueness of solutions. \( \blacksquare \)

Theorem \( 5.5 \) below (among other things) guarantees the existence of solutions to Eq. \( (5.2) \). This result may be in fact be viewed as an aspect of the well known metaplectic representation. Nevertheless, we will provide a full proof as we need some detailed bounds on the solutions to Eq. \( (5.2) \).

In order to prove existence to Eq. \( (5.2) \) we are going to construct the evolution operator \( U(t,s) \) associated to Eq. \( (5.2) \) as a limit of the truncated evolution operators, \( U^M(t,s) \), defined by Eq. \( (4.8) \) with \( Q_M(t) = P_M Q(t) P_M \) where \( Q(t) \) is as in Eq. \( (5.1) \). The next estimate provides uniform bounds on \( U^M(t,s) \).

**Corollary 5.2 (Uniform Bounds).** Continuing the notation above if \( \beta \geq 0, -\infty < S < T < \infty, \) and \( M \in \mathbb{N} \), then

\[ \|U^M(t,s)\|_{\beta \rightarrow \beta} \leq \exp(K(\beta,S,T,P)|t-s|) \text{ for all } S < s, t \leq T \quad (5.3) \]

where

\[ K(\beta,S,T,P) = \beta 4 \cdot 3^{\beta-1} \sum_{j=1}^{6} \max_{\tau \in [S,T]} |c_j(\tau)| < \infty. \quad (5.4) \]

**Proof.** This result follows directly from Theorem \( 4.4 \) and the assumed continuity of the coefficients of \( P(t : \theta, \theta^*) \) along with the assumption that \( d = \text{deg}_\beta P(t : \theta, \theta^*) \leq 2 \).

The next proposition will be a key ingredient in the proof of Proposition \( 5.4 \) below which guarantees that \( \lim_{M \rightarrow \infty} U^M(t,s) \) exists.

**Proposition 5.3.** If \( \beta \in \mathbb{R} \) and \( \psi \in D(N^{\beta+1}) \), then for all \( -\infty < S < T < \infty \)

\[ \lim_{M \rightarrow \infty} \sup_{K < \infty} \sup_{S \leq \tau \leq T} \left\| \frac{Q(\tau)}{Q_M(\tau)} - Q_M(\tau) \right\| U^K(\tau,s) \psi \|_\beta = 0 \text{ and } (5.5) \]

\[ \lim_{M \rightarrow \infty} \sup_{K < \infty} \sup_{S \leq \tau \leq T} \left\| U^K(\tau,s) \frac{Q(s)}{Q_M(s)} - Q_M(s) \right\| \psi \|_\beta = 0. \quad (5.6) \]

**Proof.** Let us express \( Q(t) \) as in Eq. \( (5.1) \). Since

\[ Q_M(t) = \sum_{j=0}^{6} c_j(t) A_M^{(j)} \quad (5.7) \]
where $\mathcal{A}^{(j)}$ is the truncation of $\mathcal{A}^{(j)}$ as in Notation 3.34 to complete the proof it suffices to show,

\[
\lim_{M \to \infty} \sup_{K < \infty} \sup_{s, t, \tau \leq T} \left\| [\bar{A} - A_M] U^K (\tau, s) \psi \right\|_\beta = 0 \tag{5.8}
\]

\[
\lim_{M \to \infty} \sup_{K < \infty} \sup_{s, t, \tau \leq T} \left\| U^K (\tau, s) [\bar{A} - A_M] \psi \right\|_\beta = 0 \tag{5.9}
\]

where $\mathcal{A}$ is a monomial in $a$ and $a^\dagger$ with degree 2 or less.

According to Theorem 3.36 and Corollary 5.2 if $\psi \in D (N^\alpha)$ with $\alpha \geq \beta + 1$, then

\[
\left\| [\bar{A} - A_M] U^K (\tau, s) \psi \right\|_\beta \leq \left\| [\bar{A} - A_M] U^K (\tau, s) \right\|_{\alpha \to \beta} \left\| \psi \right\|_\alpha
\]

\[
\leq \left\| [\bar{A} - A_M] \right\|_{\alpha \to \beta} \left\| U^K (\tau, s) \right\|_{\alpha \to \alpha} \left\| \psi \right\|_\alpha
\]

\[
\leq C (\alpha, \beta, S, T, P) (M + 1)^{\beta + 1 - \alpha} \left\| \psi \right\|_\alpha \tag{5.10}
\]

and

\[
\left\| U^K (\tau, s) [\bar{A} - A_M] \psi \right\|_\beta \leq \left\| U^K (\tau, s) \right\|_{\beta \to \beta} \left\| [\bar{A} - A_M] \psi \right\|_\beta
\]

\[
\leq \left\| U^K (\tau, s) \right\|_{\beta \to \beta} \left\| [\bar{A} - A_M] \right\|_{\alpha \to \beta} \left\| \psi \right\|_\alpha
\]

\[
\leq \tilde{C} (\alpha, \beta, S, T, P) (M + 1)^{\beta + 1 - \alpha} \left\| \psi \right\|_\alpha \tag{5.11}
\]

from which Eqs. (5.8) and (5.9) follow if $\psi \in D (N^\alpha)$ with $\alpha > \beta + 1$.

The general case, $\alpha = \beta + 1$, follows by a standard "3ε" argument, the uniform (in $M > 0$) estimates in Eq. (5.10) and (5.11) and the density of $S_0 \subset S \subset D (N^{\beta+1})$ from Proposition 3.23.

Proposition 5.4. If $\beta \geq 0$, $-\infty < S < T < \infty$ and $\psi \in D (N^\beta)$, then it follows that

\[
\lim_{M, K \to \infty} \sup_{S \leq s, t \leq T} \left\| [U^K (t, s) - U^M (t, s)] \right\|_\beta = 0 \tag{5.12}
\]

Proof. By item 3 in Theorem 4.4 we have

\[
i \frac{d}{dt} \left[ U^M (s, t) U^K (t, s) \right] = U^M (s, t) [Q_K (t) - Q_M (t)] U^K (t, s) \tag{5.13}
\]

in the sense of $\left\| \cdot \right\|_\beta$-operator norm. Integrating the identity Eq. (5.13) gives

\[
U^M (s, t) U^K (t, s) = I - i \int_s^t U^M (s, \tau) [Q_K (\tau) - Q_M (\tau)] U^K (\tau, s) d\tau. \tag{5.14}
\]

Using Eq. (5.9) in Theorem 4.4 and multiplying this identity by $U^M (t, s)$ then shows,

\[
U^K (t, s) - U^M (t, s) = -i \int_s^t U^M (t, \tau) [Q_K (\tau) - Q_M (\tau)] U^K (\tau, s) d\tau.
\]
Corollary 5.2 along with a standard density argument shows Eq. (5.12) holds for \( \psi \) in Proposition 5.3. This proves Eq. (5.12) for \( \psi \) in Theorem 5.5.

Let \( \theta \) be as above, i.e. \( \theta \) is a symmetric non-commutative polynomial of \( \{ \theta, \theta^* \} \) of \( \deg \theta \leq 2 \) and having coefficients depending continuously on \( t \in \mathbb{R} \). Then there exists a unique strongly continuous family of unitary operators \( \{ U(t,s) \}_{t,s \in \mathbb{R}} \) on \( L^2(m) \) such that for all \( \varphi \in D(N) \), \( \psi(t) := U(t,s) \varphi \) solves Eq. (5.12). Furthermore \( \{ U(t,s) \}_{t,s \in \mathbb{R}} \) satisfies the following properties:

1. For all \( s, t, \tau \in \mathbb{R} \) we have
   \[
   U(t,s) = U(t,\tau) U(\tau,s). \tag{5.15}
   \]

2. For all \( \beta \geq 0 \) and \( s, t \in \mathbb{R} \), \( U(t,s) D(N^\beta) = D(N^\beta) \) and \( (t,s) \rightarrow U(t,s) \varphi \) are jointly \( \| \cdot \|_\beta \)-norm continuous for all \( \varphi \in D(N^\beta) \).

3. If \( -\infty < S < T < \infty \), then
   \[
   C(\beta, S, T) := \sup_{S \leq t, s \leq T} \| U(t,s) \|_{\beta \rightarrow \beta} < \infty. \tag{5.16}
   \]

4. For \( \beta \geq 0 \) and \( \varphi \in D\left(N^{\beta+1}\right) \), \( t \rightarrow U(t,s) \varphi \) and \( s \rightarrow U(t,s) \varphi \) are strongly \( \| \cdot \|_\beta \)-differentiable (see Definition 2.6) and satisfy
   \[
   \frac{id}{dt} U(t,s) \varphi = \bar{Q}(t) U(t,s) \varphi \quad \text{with} \quad U(s,s) \varphi = \varphi \tag{5.17}
   \]
   and
   \[
   \frac{id}{ds} U(t,s) \varphi = -U(t,s) \bar{Q}(s) \varphi \quad \text{with} \quad U(s,s) \varphi = \varphi \tag{5.18}
   \]
   where the derivatives are taken relative to the \( \beta \)–norm, \( \| \cdot \|_\beta \).

**Proof.** Item 1. Let \( \varphi \in D(N^\beta) \). From Proposition 5.4 we know that \( L_\varphi(t,s) := \lim_{M \to \infty} U^M(t,s) \varphi \) exists locally uniformly in \( (t,s) \) in the \( \beta \)–norm.
and therefore \((t,s) \rightarrow L_\varphi (t,s) \in D (\mathcal{N}^\beta)\) is \(\beta\) - norm continuous jointly in \((t,s)\).

In particular, this observation with \(\beta = 0\) allows us to define

\[
U (t,s) = s - \lim_{M \to \infty} U^M (t,s)
\]

where the limit is taken in the strong \(L^2 (m)\) - operator topology. Since the operator product is continuous under strong convergence, by taking the strong limit of Eq. (4.19) shows the first equality in Eq. (5.15) holds.

Indeed, since (see Eq. (5.1)) in particular, this observation with \(\beta = 0\) allows us to define

\[
U (t,s) = s - \lim_{M \to \infty} U^M (t,s)
\]

where the limit is taken in the strong \(L^2 (m)\) - operator topology. Since the operator product is continuous under strong convergence, by taking the strong limit of Eq. (4.19) shows the first equality in Eq. (5.15) holds. By taking \(s = t\) in Eq. (5.15) we conclude that \(U (t,s)\) is invertible and hence is unitary on \(L^2 (m)\) as it is already known to be an isometry because it is the strong limit of unitary operators. This proves the item 1. of the theorem.

**Items 2.** As we have just seen, for any \(\varphi \in D (\mathcal{N}^\beta)\) we know that \((t,s) \rightarrow U (t,s) \varphi = L_\varphi (t,s) \in D (\mathcal{N}^\beta)\) is \(\| \cdot \|_\beta\) - continuous which proves item 2. Along the way we have shown \(U (t,s) D (\mathcal{N}^\beta) \subset D (\mathcal{N}^\beta)\) and equality then follows using Eq. (5.16).

**Item 3** follows by the Eq. (5.3) in Corollary 5.2 where the bounds are independent of \(M\).

So it only remains to prove item 4. of the theorem. We begin with proving the following claim.

**Claim.** If \(\varphi \in D (\mathcal{N}^\beta + 1)\), then

\[
Q_M (\tau) U^M (\tau,s) \varphi \rightarrow \bar{Q} (\tau) U (\tau,s) \varphi \quad \text{as} \quad M \to \infty \quad \text{and} \quad (5.19)
\]

\[
U^M (\tau,s) Q_M (s) \varphi \rightarrow U (\tau,s) \bar{Q} (s) \varphi \quad \text{as} \quad M \to \infty \quad (5.20)
\]

locally uniformly in \((\tau,s)\) in the \(\| \cdot \|_\beta\) - topology.

**Proof of the claim.** Using \(\sup_{\tau \in [S,T]} \|\bar{Q} (\tau)\|_{\beta+1 \to \beta} < \infty\) (see Corollary 3.30) and the simple estimate,

\[
\| Q_M (\tau) U^M (\tau,s) \varphi - \bar{Q} (\tau) U (\tau,s) \varphi \|_\beta \\
\leq \| [Q_M (\tau) - \bar{Q} (\tau)] U^M (\tau,s) \varphi \|_\beta + \| \bar{Q} (\tau) \{ U^M (\tau,s) - U (\tau,s) \} \varphi \|_\beta \\
\leq \| [Q_M (\tau) - \bar{Q} (\tau)] U^M (\tau,s) \varphi \|_\beta + \| \bar{Q} (\tau) \|_{\beta+1 \to \beta} \| U^M (\tau,s) - U (\tau,s) \|_\beta \| \varphi \|_{\beta+1} ,
\]

the local uniform convergence in Eq. (5.19) is now a consequence of Propositions 5.3 and 5.4. The local uniform convergence in Eq. (5.20) holds by the same methods now based on the simple estimate,

\[
\| U^M (\tau,s) Q_M (\tau) \varphi - U (\tau,s) \bar{Q} (\tau) \varphi \|_\beta \\
\leq \| U^M (\tau,s) [Q_M (\tau) - \bar{Q} (\tau)] \varphi \|_\beta + \| U^M (\tau,s) - U (\tau,s) \| \bar{Q} (\tau) \varphi \|_\beta
\]

(5.21)

along with Propositions 5.3 and 5.4. Indeed, since (see Eq. (5.1)) \(\bar{Q}(t)\varphi = \sum_{j=0}^6 c_j (t) \hat{A}^{(j)} \in D (\mathcal{N}^\beta)\) where each \(c_j (t)\) is continuous in \(t\), the latter term in Eq. (5.21) is estimated by a sum of 7 terms resulting from the estimates in Proposition 5.4 with \(\psi = \hat{A}^{(j)} \varphi\) for \(0 \leq j \leq 6\). This completes the proof of the claim.
Item 4. By integrating Eqs. (4.10) and (4.11) on $t$ we find,

$$U^M (t, s) \varphi = \varphi - i \int_{s}^{t} Q_M (\tau) U^M (\tau, s) \varphi d\tau$$

and

$$U^M (t, s) \varphi = \varphi + i \int_{t}^{s} U^M (t, \sigma) Q_M (\sigma) \varphi d\sigma$$

where the integrands are $\|\cdot\|_\beta$ – continuous and the integrals are taken relative to the $\|\cdot\|_\beta$ – topology. As a consequence of the above claim, we may let $M \rightarrow \infty$ in Eqs. (5.22) and (5.23) to find

$$U (t, s) \varphi = \varphi - i \int_{s}^{t} \bar{Q} (\tau) U (\tau, s) \varphi d\tau$$

and

$$U (t, s) \varphi = \varphi + i \int_{t}^{s} U (t, \sigma) \bar{Q} (\sigma) \varphi d\sigma$$

where again the integrands are $\|\cdot\|_\beta$ – continuous and the integrals are taken relative to the $\|\cdot\|_\beta$ – topology. Equations (5.17) and (5.18) follow directly from the previously displayed equations along with the fundamental theorem of calculus. ■

Remark 5.6. By taking $t = s$ in Eq. (5.15) and using the fact that $U (t, s)$ is unitary on $L^2 (m)$, it follows that

$$U (t, \tau)^{-1} = U (\tau, t) = U^* (t, \tau),$$

where $U^* (t, \tau)$ is the $L^2 (m)$ – adjoint of $U (\tau, t)$. Also observe from Item 2. of Theorem 5.5 and Eq. (3.34) that

$$U (t, s) \mathcal{S} = \mathcal{S}$$

for all $s, t \in \mathbb{R}$.

Remark 5.7. Recall that if $X$ is a Banach space, $\psi (h) \in X$, $T (h) \in B (X)$ for $0 < |h| < 1$, and $\psi (h) \rightarrow \psi \in X$ and $T (h) \xrightarrow{h \rightarrow 0} T \in B (X)$ as $h \rightarrow 0$, then $T (h) \psi (h) \rightarrow T \psi$ as $h \rightarrow 0$.

Theorem 5.8. Let $Q (t)$ and $U (t, s)$ be as in Theorem 5.5 and set $W (t) := U (t, 0)$. If $\varphi \in \mathcal{S}$, $R \in \mathbb{C} (\theta, \theta^*)$, and $\mathcal{R} := R (a, a^*)$, then

$$\frac{d}{dt} W (t)^* \mathcal{R} W (t) \varphi = i W (t)^* [Q (t), \mathcal{R}] W (t) \varphi$$

where the derivative may be taken relative to the $\|\cdot\|_\beta$ – topology for any $\beta \geq 0$.

Proof. Let $d = \deg \theta R$, $\psi (t) = \mathcal{R} W (t) \varphi$ and

$$f (t) := W (t)^* \mathcal{R} W (t) \varphi = W (t)^* \psi (t) = U (0, t) \psi (t).$$

In the proof we will write $\|\cdot\|_\beta \frac{d}{dt} \psi (t)$ to indicate that we are taking the derivative relative to the $\beta$ – norm topology.

Using the result of Theorem 5.5 and the fact that $\|\mathcal{R}\|_{\beta + d/2 \rightarrow \beta} < \infty$ (Corollary 3.30) it easily follows that

$$\|\cdot\|_\beta \frac{d}{dt} \psi (t) = -i \mathcal{R} Q (t) W (t) \varphi.$$  (5.26)

Combining this assertion with Remark 5.4 and the $\beta$ – norm strong continuity of $W (t)^*$ (again Theorem 5.5) we may conclude that

$$\|\cdot\|_\beta \lim_{h \rightarrow 0} W (t + h)^* \frac{\psi (t + h) - \psi (t)}{h} = W (t)^* \dot{\psi} (t) = -i W (t)^* \mathcal{R} Q (t) W (t) \varphi.$$
Hence, as
\[
\frac{f(t+h) - f(t)}{h} = W(t+h)^* \frac{\psi(t+h) - \psi(t)}{h} + \frac{W(t+h)^* - W(t)^*}{h} \psi(t),
\]
we may conclude
\[
\|\cdot\|_\beta \frac{d}{dt} f(t) = \|\cdot\|_\beta \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}
\]
\[
= -iW(t)^* RQ(t) W(t) \varphi + W^* \psi(t)
\]
\[
= -iW(t)^* RQ(t) W(t) \varphi + iW(t)^* Q(t) RW(t) \varphi
\]
which completes the proof. ■

5.1. Consequences of Theorem 5.5.

**Notation 5.9.** Let \( H \in C \langle \theta, \theta^* \rangle \) be a symmetric non-commutative polynomial in \( \theta \) and \( \theta^* \). Let \( \alpha \in C \) and \( H_2(\alpha : \theta, \theta^*) \) as in Eq. (2.24) be the degree 2 homogeneous component of \( H(\theta + \alpha, \theta^* + \tilde{\alpha}) \). From Remark \( 2.13 \) and Theorem \( 2.18 \), \( H^{cl}(\alpha) \) is real-valued and \( H_3(\alpha : \theta, \theta^*) \) is still symmetric.

**Corollary 5.10.** Let \( H \in C \langle \theta, \theta^* \rangle \) be a symmetric non-commutative polynomial in \( \theta \) and \( \theta^* \), \( H_2(\alpha : \theta, \theta^*) \) be as in Notation \( 5.9 \) and suppose that \( \mathbb{R} \ni t \to \alpha(t) \in C \) is a given continuous function. Then there exists a unique one parameter strongly continuous family of unitary operators \( \{W_0(t)\}_{t \in \mathbb{R}} \) on \( L^2(m) \) such that (with \( W_0^*(t) \) being the \( L^2 \)-adjoint of \( W_0(t) \));

1. \( W_0(t) S = S \) and \( W_0^*(t) S = S \).
2. \( W_0(t) D(N^\beta) = D(N^\beta), W_0(t)^* D(N^\beta) = D(N^\beta) \), and for all \( 0 \leq T < \infty \), there exists \( C_{T,\beta} = C_{T,\beta}(\alpha) < \infty \) such that
\[
\sup_{|t| \leq T} \|W_0(t)\|_{\beta \to \beta} \vee \|W_0(t)^*\|_{\beta \to \beta} \leq C_{T,\beta}. \tag{5.27}
\]
3. The maps \( t \to W_0(t) \psi \) and \( t \to W_0^*(t) \psi \) are \( \|\cdot\|_\beta \)-norm continuous for all \( \psi \in D(N^\beta) \).
4. For each \( \beta \geq 0 \) and \( \psi \in D(N^{\beta+1}) \);
\[
i \left( \|\cdot\|_\beta \frac{\partial}{\partial t} \right) W_0(t) \psi = H_2(\alpha(t) : a, a^t) W_0(t) \psi \quad \text{with} \quad W_0(0) \psi = \psi \tag{5.28}
\]
and
\[
- i \left( \|\cdot\|_\beta \frac{\partial}{\partial t} \right) W_0(t)^* \psi = W_0(t)^* H_2(\alpha(t) : a, a^t) \psi \quad \text{with} \quad W_0(0)^* \psi = \psi. \tag{5.29}
\]

[In Eqs. (5.28) and (5.29), one may replace \( H_2(\alpha(t) : a, a^t) \) by \( H_2(\alpha(t) : \overline{a}, a^* \) as both operators are equal on \( D(N) \) by Corollary \( 3.30 \)].

**Proof.** The stated results follow from Theorem 5.5 and Remark 5.6 with \( Q(t) := H_2(\alpha(t) : a, a^t) \) after setting \( W_0(t) = U(t,0) \) in which case that \( W_0(t)^* = U(t,0)^* = U(0,t) \). ■

**Corollary 5.11.** If \( \alpha \in C \), \( U(\alpha) \) is as in Definition 1.6 and \( U(\alpha)^* \) is the \( L^2(m) \)-adjoint of \( U(\alpha) \), then for any \( \beta \geq 0 \);

1. \( U(\alpha) S = S \) and \( U(\alpha)^* S = S \) (also seen in Proposition 2.4),
2. \( U(\alpha) D(N^\beta) = D(N^\beta) \) and \( U(\alpha)^* D(N^\beta) = D(N^\beta) \), and
(3) the following operator norm bounds hold,
\[
\|U (\alpha)\|_{\beta \rightarrow \beta} \vee \|U (\alpha)^*\|_{\beta \rightarrow \beta} \leq \exp \left(8\beta \cdot 3^{|\beta|-1} |\alpha| \right).
\]

**Proof.** Let \( \alpha (t) = ta \),
\[
H (t : \theta, \theta^*) = \dot{\alpha} (t) \theta^* - \overline{\alpha (t) \theta} + i \text{Im} \left( \alpha (t) \overline{\alpha (t)} \right) = \alpha \theta^* - \overline{\alpha \theta}
\]
so that
\[
Q (t) = \alpha a^\dagger - \overline{\alpha a} + i \text{Im} (ta) = \alpha a^\dagger - \overline{\alpha a}.
\]
By Proposition 2.7 if \( \varphi \in D (\mathcal{N}) \), \( \psi (t) := U (ta) U (sa)^* \varphi \), then \( \psi \) satisfies Eq. (5.2) and therefore items 1. and 2. follow Theorem 5.5 and Remark 5.6. To get \( \varphi \) by Proposition 2.7, if \( \alpha \) is necessarily real valued by Remark 2.15.) Further the maps
\[
i \left( \|\|_{\beta} \cdot \frac{\partial}{\partial t} \right) U (\alpha (t)) \psi = \overline{Q (t) U (\alpha (t)) \psi}
\]
and
\[
-i \left( \|\|_{\beta} \cdot \frac{\partial}{\partial t} \right) U (\alpha (t))^* \psi = U (\alpha (t))^* \overline{Q (t) \psi}.
\]
**Proof.** Let
\[
H (t : \theta, \theta^*) := \dot{\alpha} (t) \theta^* - \overline{\alpha (t) \theta} + i \text{Im} \left( \alpha (t) \overline{\alpha (t)} \right)
\]
so that \( Q (t) = H (t : a, a^\dagger) \). By Proposition 2.7 if \( \varphi \in D (\mathcal{N}) \), \( \psi (t) := U (\alpha (t)) U (\alpha (s))^* \varphi \), then \( \psi \) satisfies Eq. (5.2) and therefore the corollary again follows from Theorem 5.5 and Remark 5.6. 

**Theorem 5.13** (Properties of \( a (t) \)). Let \( H \in \mathbb{C} \left( \theta, \theta^* \right) \) be symmetric and \( H^{cl} \in \mathbb{C} [z, \overline{z}] \) be the symbol of \( H \), \( H^{cl} \) is necessarily real valued by Remark 2.13.) Further suppose that \( a (t) \in \mathbb{C} \) satisfying Hamilton’s equations of motion (see Eq. (2.3) has global solutions, \( a (t) \) and \( a^\dagger (t) \) are the operators on \( S \) as described in Eqs. (1.8),
and (1.9), and \( W_0(t) \) is the unitary operator in Corollary 5.10. Then for all \( t \in \mathbb{R} \) the following identities hold:

\[
\begin{align*}
W_0(t)^*aW_0(t) &= a(t), \quad W_0(t)^* a^\dagger W_0(t) = a^\dagger(t), \\
W_0(t)^* a^\dagger W_0(t) &= a(t), \quad W_0(t)^* a^* W_0(t) = a^*(t), \\
W_0(t)^* a^* W_0(t) &= a^\dagger(t), \\
D\left(a(t)\right) &= D\left(a^*(t)\right), \\
a^\dagger(t) &= a(t), \\
a(t) &= \gamma(t) a^\dagger + \delta(t) a^*, \quad \text{and} \\
a^*(t) &= \delta^*(t) a + \gamma^*(t) a^*, 
\end{align*}
\]

where the closures and adjoints are taken relative to the \( L^2(m) \)-inner product.

**Proof.** Recall from Proposition 2.2 that

\[
\nu(t) := \frac{\partial^2 \mathcal{H}^cl}{\partial \alpha \partial \alpha^*}(\alpha(t)) \in \mathbb{R} \quad \text{and} \quad u(t) := \frac{\partial^2 \mathcal{H}^cl}{\partial \alpha} (\alpha(t)) \in \mathbb{C}.
\]

With this notation, the commutator formulas in Corollary 3.5 with \( \alpha = \alpha(t) \) may be written as,

\[
\left[ H_2 \left( \alpha(t) : a, a^\dagger \right), a \right] = -v(t) a - u(t)(\alpha(t)) a^\dagger, \\
\left[ H_2 \left( \alpha(t) : a, a^\dagger \right), a^\dagger \right] = \bar{u}(t) a + v(t) a^\dagger.
\]

For \( \varphi \in \mathcal{S} \), let

\[
\psi(t) := W_0(t)^* a W_0(t) \varphi \quad \text{and} \quad \psi^\dagger(t) := W_0(t)^* a^\dagger W_0(t) \varphi.
\]

From Theorem 5.8 with \( W(t) = W_0(t) \), \( Q(t) = H_2 \left( \alpha(t) : a, a^\dagger \right) \), and \( \mathcal{R} = a \) and \( \mathcal{R} = a^\dagger \), we find

\[
\begin{align*}
i \frac{d}{dt} \psi(t) &= W_0(t)^* \left[ v(t) a + u(t)(\alpha(t)) a^\dagger \right] W_0(t) \varphi \\
&= v(t) \psi(t) + u(t) \psi^\dagger(t) \\
i \frac{d}{dt} \psi^\dagger(t) &= -W_0(t)^* \left[ \bar{u}(t) a + v(t) a^\dagger \right] W_0(t) \varphi \\
&= -\bar{u}(t) \psi(t) + v(t) \psi^\dagger(t).
\end{align*}
\]

In other words,

\[
i \frac{d}{dt} \begin{bmatrix} \psi(t) \\ \psi^\dagger(t) \end{bmatrix} = \begin{bmatrix} v(t) & u(t) \\ -\bar{u}(t) & -\bar{v}(t) \end{bmatrix} \begin{bmatrix} \psi(t) \\ \psi^\dagger(t) \end{bmatrix} \in L^2(m) \times L^2(m).
\]

This linear differential equation has a unique solution which, using Proposition 2.2 is given by

\[
\begin{bmatrix} \psi(t) \\ \psi^\dagger(t) \end{bmatrix} = \Lambda(t) \begin{bmatrix} \psi(0) \\ \psi^\dagger(0) \end{bmatrix} = \Lambda(t) \begin{bmatrix} a \varphi \\ a^\dagger \varphi \end{bmatrix}
\]

where \( \Lambda(t) \) is the \( 2 \times 2 \) matrix given in Eq. (2.6). This completes the proof of Eq. (5.33) since

\[
\begin{bmatrix} W_0(t)^* a W_0(t) \varphi \\ W_0(t)^* a^\dagger W_0(t) \varphi \end{bmatrix} = \begin{bmatrix} \psi(t) \\ \psi^\dagger(t) \end{bmatrix} \quad \text{and} \quad \Lambda(t) \begin{bmatrix} a \varphi \\ a^\dagger \varphi \end{bmatrix} = \begin{bmatrix} a(t) \varphi \\ a^\dagger(t) \varphi \end{bmatrix}.
\]
The statements in Eqs. (5.34), (5.35) and (5.36) are easy consequences of the fact that \( W_0(t) \) is a unitary operator on \( L^2(\mathcal{N}) \) which preserves \( D(\mathcal{N}) \) (see Corollary 5.10). Using Eqs. (5.34) and (5.35) along with Theorem 3.15, we find
\[
\overline{a t} (t) = W_0(t) a \overline{t} W_0(t) = W_0(t) a^* W_0(t) = a(t)^*
\]
which gives Eq. (5.37).

If \( \varphi \in D(\mathcal{N}) \), using item 3. of Theorem 3.15 and the formula for \( a(t) \) and \( a^t(t) \) in Eqs. (1.8) and (1.9) we find
\[
\lim_{M \to \infty} a(t) \mathcal{P}_M \varphi = \lim_{M \to \infty} \left[ \gamma(t) a \mathcal{P}_M \varphi + \delta(t) a^t \mathcal{P}_M \varphi \right] = \gamma(t) \mathcal{P} \varphi + \delta(t) a^* \varphi
\]
and
\[
\lim_{M \to \infty} a^t(t) \mathcal{P}_M \varphi = \lim_{M \to \infty} \left[ \delta(t) a \mathcal{P}_M \varphi + \gamma(t) a^t \mathcal{P}_M \varphi \right] = \delta(t) \mathcal{P} \varphi + \gamma(t) a^* \varphi.
\]
The above two equations along with Corollary 3.30 show Eqs. (5.38) and (5.39).

6. Bounds on the Quantum Evolution

Throughout this section and the rest of the paper, let \( H \in \mathbb{R} (\theta, \theta^*) \) be a non-commutative polynomial satisfying Assumption 1. Before getting to the proof of the main theorems we need to address some domain issues. Recall as in Assumption 1 we let \( H_\beta := H(a_h, a_h^t) \).

The following abstract proposition (Stone’s theorem) is a routine application of the spectral theorem, see [22, p.265] for details.

**Proposition 6.1.** Supposed \( H \) is a self-adjoint operator on a separable Hilbert space, \( \mathcal{K} \), and there is a \( C \in \mathbb{R} \) and \( \varepsilon > 0 \) such that \( H + CI \geq \varepsilon I \). For any \( \beta \geq 0 \) let \( \| \cdot \|_{(H + CI)^\beta} \geq \varepsilon \| \cdot \|_{\mathcal{K}} \) be the Hilbertian norm on \( D((H + CI)^\beta) \) defined by,
\[
\| f \|_{(H + CI)^\beta} = \| (H + CI)^\beta f \|_{\mathcal{K}} \quad \forall \ f \in D((H + CI)^\beta).
\]
Then for all \( t \in \mathbb{R} \) and \( \beta \geq 0 \),
\[
e^{-itH} D((H + CI)^\beta) = D((H + CI)^\beta) \quad \text{and} \quad \| e^{-itH} \psi \|_{(H + CI)^\beta} = \| \psi \|_{(H + CI)^\beta} \quad \forall \ \psi \in D((H + CI)^\beta).
\]
Moreover, if \( \beta \geq 0 \) and \( \varphi \in D((H + CI)^{\beta+1}) \), then
\[
\| \cdot \|_{(H + CI)^\beta} \frac{d}{dt} e^{-itH} \varphi = -iHe^{-itH} \varphi = -ie^{-itH} H \varphi.
\]

In this section we are going to show, as a consequence of Proposition 6.3 below, that
\[
e^{iHt/h} \varphi e^{-iHt/h} S \quad \text{and} \quad e^{iHt/h} a^* e^{-iHt/h} S \subseteq S. \quad (6.1)
\]

**Lemma 6.2.** For any unbounded operator \( T \) and constant \( C \in \mathbb{R} \), then for any \( n \in \mathbb{N}_0 \),
\[
D((T + C)^n) = D(T^n).
\]
**Proof.** We first show by induction that $D((T + C)^n) \subset D(T^n)$ for all $n \in \mathbb{N}$. The case $n = 1$ is trivial. Then the induction step is
\[
 f \in D\left((T + C)^{n+1}\right) \implies f \in D((T + C)^n) \text{ and } (T + C)^n f \in D(T + C)
\]
\[
 \implies f \in D((T + C)^n) \text{ and } (T + C)^n f \in D(T)
\]
\[
 \implies f \in D(T^n) \text{ and } (T + C)^n f \in D(T)
\]
But
\[
 (T + C)^n f = T^n f + \sum_{k=0}^{n-1} \binom{n}{k} C^{n-k} T^k f = T^n f + g
\]
where $g \in D(T)$ and hence
\[
 T^n f = (T + C)^n f - g \in D(T) \implies f \in D\left(T^{n+1}\right).
\]
finishing the inductive step.

To finish the proof, we replace $T$ by $T - C$ above to learn
\[
 D(T^n) = D((T - C + C)^n) \subset D((T - C)^n)
\]
and then replace $C$ by $-C$ to find $D(T^n) \subset D((T + C)^n)$.

**Proposition 6.3.** Let $H(\theta, \theta^*)$ and $\eta > 0$ be as in Assumption \[1\] then $\exp(-iH_t) S$ leaves $S$ invariant and more explicitly, it is $\exp(-iH_t) S = S$ for all $t \in \mathbb{R}$. **Proof.** The fact that $S \subseteq H^n$ for all $n \in \mathbb{N}$ along with Eq. \[1.14\] in the Assumption \[1\] and Eq. \[3.56\], we learn that
\[
 S(\mathbb{R}) = \bigcap_{n=1}^{\infty} D(H^n) \subseteq \bigcap_{n=1}^{\infty} D(N^n) = S(\mathbb{R})
\]
This shows $S(\mathbb{R}) = \bigcap_{n=1}^{\infty} D(H^n)$ and this finishes the proof since, see Proposition \[6.1\] $\exp(-iH_t) S$ leaves $\bigcap_{n=1}^{\infty} D(H^n)$ invariant, i.e., $\exp(-iH_t) S \subseteq S$ for all $t \in \mathbb{R}$. By multiplying $\exp(iH_t)$ on both sides, we yield $S \subseteq \exp(iH_t) S$. Therefore, $\exp(-iH_t) S = S$ is resulted if we replacing $t$ to $-t$.

**Lemma 6.4.** If $P \subseteq (\theta, \theta^*)$, $\delta := \deg_0 P \in N_0$, and $C(P) := \sum_{k=0}^{\delta} z_k | P_k | k^{1/2}$, then
\[
 \| P(\bar{a}, a^*) \| \leq C(P) \| (I + N_h)^{\delta/2} \psi \| \quad \forall \ 0 < h \leq 1 \quad \text{and} \quad \psi \in D\left(N^{\delta/2}\right).
\]
**Proof.** Let $P_k$ be the degree $k$ homogeneous component of $P$ as in Eq. \[2.22\]. Then according to Corollary \[3.30\] with $\beta = 0$ and $d = k$ we have,
\[
 \| P_k(\bar{a}, a^*) \| = \| P_k(\bar{a}, a^*) \| \leq | P_k | k^{1/2} h^{1/2} \| \psi \|_{k/2}
\]
\[
 = | P_k | k^{1/2} h^{1/2} \| (I + N_h)^{1/2} \| \| \psi \|
\]
\[
 = | P_k | k^{1/2} \| (hI + N_h)^{1/2} \| \| \psi \|
\]
\[
 \leq | P_k | k^{1/2} \| (I + N_h)^{1/2} \| \| \psi \| \leq | P_k | k^{1/2} \| (I + N_h)^{1/2} \| \| \psi \|.
\]
Summing this inequality on $k$ using $P = \sum_{k=0}^{\delta} P_k$ and the triangle inequality leads directly to Eq. \[6.2\].
The next important result may be found in Heinz [13], also see Kato [16, Theorem 2] and [24, Proposition 10.14, p.232].

**Theorem 6.5** (Löwner-Heinz inequality). Let \(A\) and \(B\) be non-negative self-adjoint operators on a Hilbert space. If \(A \leq B\) (see Notation 1.10), then \(A^r \leq B^r\) for \(0 \leq r \leq 1\).

**Corollary 6.6.** Let \(H (\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle\), \(1 > \eta > 0\), and \(C\) be as in Assumption 7 and set \(\tilde{C} := C + 1\). Then for each \(\beta \geq 0\), there exists constants \(\tilde{C}_\beta < \infty\) and \(\tilde{D}_\beta < \infty\) such that, for all \(0 \leq \theta < \eta\),

\[
\begin{align*}
    (\mathcal{N}_h + I)^\beta &\leq \tilde{C}_\beta \left( H_h + \tilde{C} \right)^\beta \quad \text{and} \quad (\mathcal{N}_h + I)^\beta \leq \tilde{D}_\beta (\mathcal{N}_h + I)^{\beta d/2}.
\end{align*}
\]

**Proof.** Using the simple estimate,

\[
    (x + 1)^\beta \leq 2^{(\beta - 1)} (x^\beta + 1) \quad \forall \ x, \beta \geq 0,
\]

along with Eq. (6.4) implies,

\[
    (\mathcal{N}_h + I)^\beta \leq 2^{(\beta - 1)} \left( \mathcal{N}_h^\beta + I \right) \leq 2^{(\beta - 1)} \left( C_\beta (H_h + C)^\beta + I \right)
\]

\[
    \leq 2^{(\beta - 1)} C_\beta (H_h + C + I)^\beta,
\]

wherein we have assumed \(C_\beta \geq 1\) without loss of generality. Lemma 10.10 of [24, p.230] asserts, if \(A\) and \(B\) are non-negative self-adjoint operators and \(A \leq B\), then \(A \leq \tilde{B}\). Therefore we can deduce from Eq. (6.6) that

\[
    (\mathcal{N}_h + I)^\beta \leq 2^{(\beta - 1)} C_\beta (H_h + C + I)^\beta
\]

which gives Eq. (6.3).

We now turn to the proof of Eq. (6.4). For \(n \in \mathbb{N}\), let \(P^{(n)} \in \mathbb{C} \langle \theta, \theta^* \rangle\) be defined by

\[
P^{(n)} (\theta, \theta^*) := \left( H (\theta, \theta^*) + \tilde{C} \right)^n
\]

so that \(\deg P^{(n)} = dn\) and for \(\psi \in D (\mathcal{N}^{dn/2})\), we have

\[
    \left( H_h + \tilde{C} \right)^n \psi = P^{(n)} (\tilde{a}_h, \tilde{a}_h^*) \psi.
\]

With these observations, we may apply Lemma 6.3 to find for any \(0 < h < \eta \leq 1\) that

\[
    \left\| \left( H_h + \tilde{C} \right)^n \psi \right\| \leq C \left( P^{(n)} \right) \left\| (I + \mathcal{N}_h)^{dn/2} \psi \right\| \quad \forall \ \psi \in D (\mathcal{N}^{dn/2}).
\]

The last displayed equation is equivalent (see Notation 1.10) to the operator inequality,

\[
    \left( H_h + \tilde{C} \right)^{2n} \leq C \left( P^{(2n)} \right) (I + \mathcal{N}_h)^{dn}.
\]

Hence if \(0 \leq \beta \leq 2n\), we may apply the Löwner-Heinz inequality (Theorem 6.5) with \(r = \beta/2n\) to conclude

\[
    \left( H_h + \tilde{C} \right)^\beta \leq C \left( P^{(n)} \right)^{\beta/2n} (I + \mathcal{N}_h)^{\beta d/2}.
\]

As \(n \in \mathbb{N}\) was arbitrary, the proof is complete. \(\blacksquare\)
Theorem 6.7. Let $H(\theta, \theta^*) \in \mathbb{R}(\theta, \theta^*)$, $d = \deg_{\theta} H$, and $1 > \eta > 0$ be as in Assumption 7 and suppose $0 < h < \eta \leq 1$.

1. If $\beta \geq 0$ then
   \[ e^{-iH_{ht}/h} D \left( \mathcal{N}^{\beta d/2} \right) \subseteq D \left( \mathcal{N}^{\beta} \right) \]  \hspace{1cm} (6.7)
   and there exists $C_{\beta} < \infty$ such that
   \[ \left\| e^{-iH_{ht}/h} \right\|_{\beta d/2 \rightarrow \beta} \leq C_{\beta} h^{-\beta} \text{ for all } t \in \mathbb{R}. \]  \hspace{1cm} (6.8)

2. If $\beta \geq 0$ and $\psi \in D \left( \mathcal{N}^{(\beta+1)d/2} \right) \subseteq D \left( H_{ht}^{\beta+1} \right)$, then
   \[ e^{-iH_{ht}/h} \psi, H_{ht} e^{-iH_{ht}/h} \psi, \text{ and } e^{-iH_{ht}/h} H_{ht} \psi \]
   are all in $D \left( \mathcal{N}^{\beta} \right)$ for all $t \in \mathbb{R}$ and moreover,
   \[ i h \left( \frac{d}{dt} \beta \right) e^{-iH_{ht}/h} \psi = H_{ht} e^{-iH_{ht}/h} \psi = e^{-iH_{ht}/h} H_{ht} \psi, \]
   \hspace{1cm} (6.9)
   where, as before, $\frac{d}{dt} \beta$ indicates the derivative is taken in $\beta$-norm topology.

Proof. If $\beta \geq 0$, it follows from Corollary 6.6 (with $\beta$ replaced by $2\beta$) that
   \[ D \left( \mathcal{N}^{\beta d/2} \right) = D \left( \mathcal{N}^{\beta d/2}_h \right) \subseteq D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right) \subseteq D \left( \mathcal{N}^{\beta}_h \right) \subseteq D \left( \mathcal{N}^{\beta} \right) \]  \hspace{1cm} (6.10)

and
   \[ \| \psi \|_{(N_{h+1})^\beta} \leq \sqrt{\bar{C}_{2\beta}} \| \psi \|_{(H_{ht} + \tilde{C})^\beta} \forall \psi \in D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right). \]

Moreover if $0 < h < \eta \leq 1$, a simple calculus inequality shows
   \[ h^\beta \| \psi \|_\beta = h^\beta \| \psi \|_{(N_{h+1})^\beta} \leq \| \psi \|_{(N_{h+1})^\beta} \]

and hence
   \[ \| \psi \|_\beta \leq h^{-\beta} \sqrt{\bar{C}_{2\beta}} \| \psi \|_{(H_{ht} + \tilde{C})^\beta} \forall \psi \in D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right). \]  \hspace{1cm} (6.11)

From Proposition 6.1 we know for all $t \in \mathbb{R}$ that
   \[ e^{-iH_{ht}/h} D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right) = D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right) \]
   and
   \[ \left\| e^{-iH_{ht}/h} \psi \right\|_{(H_{ht} + \tilde{C})^\beta} = \| \psi \|_{(H_{ht} + \tilde{C})^\beta}. \]

Combining these statements with Eqs. (6.10) and (6.11) respectively shows,
   \[ e^{-iH_{ht}/h} D \left( \mathcal{N}^{\beta d/2} \right) \subseteq e^{-iH_{ht}/h} D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right) \subseteq D \left( \mathcal{N}^{\beta} \right). \]

Moreover, if $\varphi \in D \left( \mathcal{N}^{\beta d/2} \right) \subseteq D \left( \left( H_{ht} + \tilde{C} \right)^{\beta} \right)$, then
   \[ \left\| e^{-iH_{ht}/h} \varphi \right\|_\beta \leq h^{-\beta} \sqrt{\bar{C}_{2\beta}} \left\| e^{-iH_{ht}/h} \varphi \right\|_{(H_{ht} + \tilde{C})^\beta} = h^{-\beta} \sqrt{\bar{C}_{2\beta}} \| \varphi \|_{(H_{ht} + \tilde{C})^\beta}. \]
   \[ \text{However, from Eq. (6.4) (again with } \beta \to 2\beta \text{) we also know} \]
   \[ \| \varphi \|_{(H_{ht} + \tilde{C})^\beta} \leq \sqrt{\bar{D}_{2\beta}} \cdot \| \varphi \|_{(N_{h+1})^\beta d/2} \leq \sqrt{\bar{D}_{2\beta}} \cdot \| \varphi \|_{(N_{h+1})^\beta d/2}. \]

Combining the last two displayed equations proves the estimate in Eq. (6.8) with $C_\beta := \sqrt{C_{2\beta} \cdot \bar{D}_{2\beta}}$.

If we now further assume that $\psi \in D\left(\mathcal{N}^{(\beta+1)d/2}\right)$, then $\psi \in D\left(H^{\beta+1}_h\right)$ by Eq. (6.10) then, by Proposition 6.1 it follows that

$$H_h e^{-iH_h t/\hbar} \psi = e^{-iH_h t/\hbar} H_h \psi \in D\left((H_h + \bar{C})^\beta\right) \subset D\left(\mathcal{N}^\beta\right)$$

and

$$i\hbar \left(\|H^\beta_h \cdot \frac{d}{dt}\right) \psi(t) = H_h \psi(t) = e^{-iH_h t/\hbar} H_h \psi_0.$$

(6.12)

Owing to Eq. (6.11) the $\beta$-norm is weaker than $\|\cdot\|_{H^\beta_h}$ norm and hence Eq. (6.12) directly implies the weaker Eq. (6.9).

7. A Key One Parameter Family of Unitary Operators

In this section (except for Lemma 7.2) we will always suppose that $H(\theta, \theta^*)$ and $1 \geq \eta > 0$ are as in Assumption 1, $\alpha_0 \in \mathbb{C}$, and $\alpha(t)$ denotes the solution to Hamilton’s classical equations (1.1) of motion with $\alpha(0) = \alpha_0$. From Corollary 6.6, $U_h(\alpha_0) \psi$ is a state on $L^2(m)$ which has position and momentum concentrated at $\xi_0 + i\pi_0 = \sqrt{2}\alpha_0$ in the limit as $h \downarrow 0$. Thus if quantum mechanics is to limit to classical mechanics as $h \downarrow 0$, one should expect that the quantum evolution, $\psi_h(t) := e^{-iH_h t/\hbar} U_h(\alpha_0) \psi$, of the state, $U_h(\alpha_0) \psi$, should be concentrated near $\alpha(t)$ in phase space as $h \downarrow 0$. One possible candidate for these approximate states would be $U_h(\alpha(t)) \psi$ or more generally any state of the form, $U_h(\alpha(t)) W_0(t) \psi$, where $\{W_0(t) : t \in \mathbb{R}\}$ are unitary operators on $L^2(m)$ which preserve $\mathcal{S}$. All states of this form concentrate their position and momentum expectations near $\sqrt{2}\alpha(t)$, see Remark 5.7. These remarks then motivate us to consider the one parameter family of unitary operators $V_h(t)$ defined by,

$$V_h(t) := U_h (-\alpha(t)) e^{-iH_h t/\hbar} U_h(\alpha_0) = U_h(\alpha(t))^* e^{-iH_h t/\hbar} U_h(\alpha_0).$$

(7.1)

Because of Propositions 2.4 and 6.3, we know $V_h(t) \mathcal{S} = \mathcal{S}$ for all $0 < h < \eta$ and in particular, $V_h(t) \mathcal{S} = \mathcal{S} \subset D\left(P(\alpha, a^1)\right)$ for any $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$. The main point of this section is to study the basic properties of this family of unitary operators with an eye towards showing that $\lim_{h \downarrow 0} V_h(t)$ exists (modulo a phase factor). Our first task is to differentiate $V_h(t)$ for which we will need the following differentiation lemma.

Lemma 7.1 (Product Rule). Let $P(\theta, \theta^*) \in \mathbb{C}\langle\theta, \theta^*\rangle$, $k := \deg_P P(\theta, \theta^*) \in \mathbb{N}_0$, and $P := P(a, a^1)$. Suppose that $U(t)$ and $T(t)$ are unitary operators on $L^2(m)$ which preserve $\mathcal{S}$. We further assume;

(1) for each $\varphi \in \mathcal{S}$, $t \rightarrow U(t) \varphi$ and $t \rightarrow T(t) \varphi$ are $\|\cdot\|_\beta$ - differentiable for all $\beta \geq 0$. We denote the derivative by $\dot{U}(t) \varphi$ and $\dot{T}(t) \varphi$ respectively. Notice that $\dot{U}(t) \varphi$ and $\dot{T}(t) \varphi$ are all in $\cap_{\beta \geq 0} D(\mathcal{N}^\beta) = \mathcal{S}$, see Eq. (3.34) for the last equality, i.e. $\dot{U}(t)$ and $\dot{T}(t)$ preserves $\mathcal{S}$.

(2) For each $\beta \geq 0$ there exists $\alpha \geq 0$ and $\varepsilon > 0$ such that

$$K := \sup_{|\Delta| \leq \varepsilon} \|U(t+\Delta)\|_{\alpha \rightarrow \beta} < \infty.$$
Then for any \( \beta \geq 0 \),
\[
\|\|_\beta \frac{d}{dt} [U(t) P T(t) \varphi] = \dot{U}(t) P T(t) \varphi + U(t) P \dot{T}(t) \varphi. \tag{7.2}
\]

**Proof.** Let \( \varphi \in S \) and then define \( \varphi(t) = U(t) P T(t) \varphi \). To shorten notation let \( \Delta f \) denote \( f(t + \Delta) - f(t) \). We then have,
\[
\frac{\Delta \varphi}{\Delta} = \left[ U(t + \Delta) P \frac{\Delta T}{\Delta} + \frac{\Delta U}{\Delta} P T(t) \right] \varphi
\]
and so
\[
\frac{\Delta \varphi}{\Delta} - U(t) P \dot{T}(t) \varphi - \dot{U}(t) P T(t) \varphi
\]
\[
= U(t + \Delta) P \left[ \frac{\Delta T}{\Delta} - \dot{T}(t) \right] \varphi + [\Delta U] P \dot{T}(t) \varphi + \left[ \frac{\Delta U}{\Delta} - \dot{U}(t) \right] P T(t) \varphi. \tag{7.3}
\]

Using the assumptions of the theorem it follows that for each \( \beta < \infty \), since \( PT(t) \varphi \in S \), we may conclude that
\[
\left\| [\Delta U] P \dot{T}(t) \varphi \right\|_\beta \to 0 \text{ as } \Delta \to 0, \text{ and}
\]
\[
\left\| \left[ \frac{\Delta U}{\Delta} - \dot{U}(t) \right] P T(t) \varphi \right\|_\beta \to 0 \text{ as } \Delta \to 0.
\]

Furthermore, using the assumptions along with Eq. (3.41) in the Proposition 3.29 it follows that when \( \Delta \to 0 \),
\[
\left\| U(t + \Delta) P \left[ \frac{\Delta T}{\Delta} - \dot{T}(t) \right] \varphi \right\|_\beta
\]
\[
\leq \|U(t + \Delta)\|_{\alpha \to \beta} \|P\|_{\alpha \to \beta + \frac{1}{2}} \left\| \left[ \frac{\Delta T}{\Delta} - \dot{T}(t) \right] \varphi \right\|_{\alpha + \frac{1}{2}} \to 0.
\]
which combined with Eq. (7.3) shows \( \varphi(t) = U(t) P T(t) \varphi \) is \( \|\|_\beta \) - differentiable and the derivative is given as in Eq. (7.2).

**Lemma 7.2.** If \( \alpha : \mathbb{R} \to \mathbb{C} \) is any \( C^1 \) function and \( V_h(t) \) is defined as in Eq. (7.7), then for all \( \psi \in S, t \to V_h(t) \psi \) and \( t \to V^*_h(t) \psi \) are \( \|\|_\beta \) - norm differentiable for all \( \beta < \infty \) and moreover,
\[
\frac{d}{dt} V_h(t) \psi = \Gamma_h(t) V_h(t) \psi \text{ and} \tag{7.4}
\]
\[
\frac{d}{dt} V^*_h(t) \psi = -V^*_h(t) \Gamma_h(t) \psi \tag{7.5}
\]

where
\[
\Gamma_h(t) := \frac{1}{\hbar} \left( \hat{\alpha}(t)a_h - \hat{\alpha}(t) a^\dagger_h + i \text{Im} \left( \alpha(t) \hat{\alpha}(t) \right) - i H \left( a_h + \alpha(t), a^\dagger_h + \hat{\alpha}(t) \right) \right). \tag{7.6}
\]

**Proof.** Let \( U(t) := U_h(-\alpha(t)) = U \left( -\alpha(t) / \sqrt{\hbar} \right) \), \( T(t) := e^{-i H t / \hbar} \) and \( \varphi := U_h(\alpha_0) \psi \). From Propositions 2.4 and 2.7 we know \( U(t) S = S \) and
\[
i \frac{d}{dt} U(t) f = Q(t) U(t) f \text{ for } f \in S. \tag{7.7}
\]
where
\[
Q(t) = i \left( -\frac{\dot{\alpha}(t)}{\sqrt{\hbar}} a^\dagger + \frac{\overline{\dot{\alpha}}(t)}{\sqrt{\hbar}} a \right) - \frac{i}{\hbar} \text{Im} \left( \alpha(t) \overline{\dot{\alpha}}(t) \right).
\] (7.8)

As \(Q(t)\) is linear in \(a\) and \(a^\dagger\), we may apply Corollaries 5.11 and 5.12 in order to conclude that \(U(t)\) satisfies the hypothesis in Lemma 7.1. Moreover, by Proposition 6.3 and the item 2 in Theorem 6.7, we also know that \(T(t) \mathcal{S} = \mathcal{S}\) and it satisfies the hypothesis of Lemma 7.1. Therefore by taking \(P(\theta, \theta^*) = 1\) (so \(P = I\)) in Lemma 7.1 we learn
\[
\frac{d}{dt} V_h(t) \psi = \dot{U}(t)T(t) \varphi + U(t) \dot{T}(t) \varphi
\]
\[
= \left[ -\dot{\alpha}(t) a^\dagger + \frac{\overline{\dot{\alpha}}(t)}{\sqrt{\hbar}} a \right] + \frac{i}{\hbar} \text{Im} \left( \alpha(t) \overline{\dot{\alpha}}(t) \right) U(t) T(t) \varphi
\]
\[
+ U(t) \frac{H_h}{i\hbar} T(t) \varphi
\]
\[
= \frac{1}{\hbar} \left[ -\dot{\alpha}(t) a^\dagger + \frac{\overline{\dot{\alpha}}(t)}{\sqrt{\hbar}} a \right] V_h(t) \psi
\]
\[
+ U_h(-\alpha(t)) \left( \hbar \frac{H_h}{i\hbar} U_h(\alpha(t)) U_h(-\alpha(t)) T(t) \varphi
\]
\[
= \Gamma_h(t) V_h(t) \psi,
\]
wherein the last equality we have used Proposition 2.4 to conclude,
\[
U_h(-\alpha(t)) H \left( a_h, a_h^\dagger \right) U_h(\alpha(t)) = H \left( a_h + \alpha(t), a_h^\dagger + \overline{\dot{\alpha}}(t) \right).
\]

This completes the proof of Eq. (7.4). We now turn to the proof of Eq. (7.5).

Now let \(U(t) = U_h^* (\alpha_0) e^{iH_h t/\hbar}\) and \(T(t) := U_h(\alpha(t))\) and observe by taking adjoint of Eq. (7.1) that
\[
V_h^*(t) := U_h^* (\alpha_0) e^{iH_h t/\hbar} U_h(\alpha(t)) = U(t) T(t).
\]

Working as above, we again easily show that both \(U(t)\) and \(T(t)\) satisfy the hypothesis of Lemma 7.1 and moreover by replacing \(\dot{\alpha}\) by \(-\alpha\) in Eq. (7.8) we know
\[
\frac{d}{dt} T(t) \psi = T(t) \left[ i \left( \frac{\dot{\alpha}(t)}{\sqrt{\hbar}} a^\dagger - \frac{\overline{\dot{\alpha}}(t)}{\sqrt{\hbar}} a \right) + \frac{i}{\hbar} \text{Im} \left( \alpha(t) \overline{\dot{\alpha}}(t) \right) \right] \psi.
\]

We now apply Lemma 7.1 with \(P(\theta, \theta^*) = 1\) and \(\varphi = \psi\) along with some basic algebraic manipulations to show Eq. (7.5) is also valid.

Specializing our choice of \(\alpha(t)\) in Lemma 7.2 leads to the following important result.

**Theorem 7.3.** Let \(\Gamma_h(t)\) be as in Eq. (7.6). If \(\alpha(t)\) satisfies Hamilton’s equations of motion (Eq. (7.1)), \(V_h(t)\) is defined as in Eq. (7.1), then
\[
\Gamma_h(t) = \frac{i}{\hbar} \text{Im} \left( \alpha(t) \overline{\dot{\alpha}}(t) \right) - \frac{i}{\hbar} H^{cl}(\alpha(t))
\]
\[
- iH_2(\alpha(t) : a, a^\dagger) - \frac{i}{\hbar} H_{\geq 3}(\alpha(t) : a_h, a_h^\dagger),
\] (7.9)
on \(\mathcal{S}\) where \(H^{cl}\), \(H_2\) and \(H_{\geq 3}\) are as in Eq. (2.31) by replacing \(P\) by \(H\).
Proof. From the expansion of $H(\theta + \alpha, \theta^* + \bar{\alpha})$ described in Eq. (2.24) and Theorem 2.18 we have
\[
H \left( a_h + \alpha(t), a_h^\dagger + \bar{\alpha}(t) \right) = H(\alpha(t)) + \left( \frac{\partial H \cl}{\partial \alpha} \right)(\alpha(t)) a_h + \left( \frac{\partial H \cl}{\partial \bar{\alpha}} \right)(\alpha(t)) a_h^\dagger + H_2 \left( \alpha(t) : a_h, a_h^\dagger \right).
\]
(7.10)

So if $\alpha(t)$ satisfies Hamilton’s equations of motion,
\[
i\dot{\alpha}(t) = \left( \frac{\partial}{\partial \bar{\alpha}} H \cl \right)(\alpha(t)) \quad \text{with} \quad \alpha(0) = \alpha_0,
\]
(7.11)
it follows using Eq. (7.10) in Eq. (7.6) that we may cancel all the terms linear in $a_h$ or $a_h^\dagger$ in which case $\Gamma_\hbar(t)$ in Eq. (7.6) may be written as in Eq. (7.9).

In order to remove a (non-essential) highly oscillatory phase factor\(^2\) from $V_\hbar(t)$ let
\[
f(t) := \int_0^t \left( H(\alpha(\tau)) - \text{Im} \left( \alpha(\tau) \alpha^\dagger(\tau) \right) \right) d\tau
\]
(7.12)
and then define
\[
W_\hbar(t) = e^{i\Phi(t)} V_\hbar(t) = e^{i\Phi(t)} U_\hbar(-\alpha(t)) e^{-iH_\hbar t/\hbar} U_\hbar(\alpha_0) .
\]
(7.13)

More generally for $s, t \in \mathbb{R}$, let
\[
W_\hbar(t, s) = W_\hbar(t) W_\hbar^\ast(s) = e^{i[f(t) - f(s)]} U_\hbar(-\alpha(t)) e^{-iH_\hbar(t-s)/\hbar} U_\hbar(\alpha(s)).
\]
(7.14)

Proposition 7.4. Let $H(\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle$ and $\eta > 0$ satisfy Assumption \(\text{H}\)\(d = \deg_H, \text{and} W_\hbar(t, s) \text{ be as in Eq. (7.13). Then}
\[
W_\hbar(t, s) D \left( \mathcal{N}^{3/2} \right) \subseteq D(\mathcal{N}^3) \quad \forall \, s, t \in \mathbb{R} \quad \text{and} \quad \beta \geq 0.
\]
(7.15)
Moreover, we have $W_\hbar(t, s) \mathcal{S} = \mathcal{S}$ for all $s, t \in \mathbb{R}$.

Proof. Eq. (7.15) is a direct consequence from $U_\hbar(\alpha(\cdot)) \mathcal{N}^3 = \mathcal{N}^3$ in Corollary 5.11 and $e^{-iH_\hbar t/\hbar} D \left( \mathcal{N}^{3/2} \right) \subseteq D(\mathcal{N}^3)$ from the item 1 in Theorem 6.7. Then, by Eq. (3.34), it follows that $W_\hbar(t, s) \mathcal{S} \subseteq \mathcal{S}$. By multiplying $W_\hbar(t, s)^{-1} = W_\hbar(s, t)$ on both sides of the last inclusion, we can conclude that $W_\hbar(t, s) \mathcal{S} = \mathcal{S}$.

Definition 7.5. For $\hbar > 0$ and $t \in \mathbb{R}$, $L_\hbar(t)$ be the operator on $\mathcal{S}$ defined as,
\[
L_\hbar(t) = \frac{1}{\hbar} \left( H \left( a_h + \alpha(t), a_h^\dagger + \bar{\alpha}(t) \right) - H(\alpha(t)) - H_1 \left( \alpha(t) : a_h, a_h^\dagger \right) \right)
\]
(7.16)
\[
= H_2 \left( \alpha(t) : a, a^\dagger \right) + \frac{1}{\hbar} H_{\geq 3} \left( \alpha(t) : a_h, a_h^\dagger \right).
\]

Theorem 7.6. Both $t \to W_\hbar(t, s)$ and $s \to W_\hbar(t, s)$ are strongly continuous on $L^2(m)$. Moreover, if $\psi \in \mathcal{S}$ and $\beta \geq 0$, then
\[
i \left( \| \cdot \|_{\beta - \partial_\beta} \right) W_\hbar(t, s) \psi = L_\hbar(t) W_\hbar(t, s) \psi, \quad \text{and}
\]
(7.17)
\[
i \left( \| \cdot \|_{\beta - \partial_\beta} \right) W_\hbar(t, s) \psi = -W_\hbar(t, s) L_\hbar(s) \psi.
\]
(7.18)

\(^2\)As usual in quantum mechanics, the overall phase factor will not affect the expected values of observables and so we may safely ignore it in this introductory description.
The strong continuity of $W_h (t,s)$ in $s$ and in $t$ follows from the strong continuity of both $U \left( \alpha \left( t \right) \right)$ and $e^{-iH_{\pm}/t}$, see Corollary 5.10 and Proposition 6.1. The derivative formulas in Eqs. (7.17) and (7.18) follow directly from Lemma 7.2 and Theorem 7.3 along with the additional term coming from the product rule involving the added scalar factor, $e^{i \left[f(t) - f(s) \right]}$.

For the rest of the paper the following notation will be in force.

Notation 7.7. Let $\alpha_0 \in C$, $H \left( \theta , \theta^* \right) \in \mathbb{R} (\theta, \theta^*)$ satisfy the Assumption 1, $t \rightarrow \alpha \left( t \right)$ solve the Hamiltonian’s equation Eq. (1.1) with $\alpha \left( 0 \right) = \alpha_0$, and $H_2 \left( \alpha \left( \tau \right) : \theta , \theta^* \right)$ be the degree 2 homogeneous component of $H \left( \theta + \alpha \left( \tau \right) , \theta^* + \overline{\alpha} \left( \tau \right) \right)$ as in Proposition 3.4. Further let

$$ W_0 \left( t,s \right) := W_0 \left( t \right) W_0^* \left( s \right) \tag{7.19} $$

where $W_0 \left( t \right)$ is the unique one parameter strongly continuous family of unitary operators satisfying,

$$ i \frac{\partial}{\partial t} W_0 \left( t \right) = H_2 \left( \alpha \left( t \right) : a, a^\dagger \right) W_0 \left( t \right) \quad \text{with} \quad W_0 \left( 0 \right) = I \tag{7.20} $$

as described in Corollary 5.10.

Remark 7.8. Since

$$ i \frac{\partial}{\partial t} W_0 \left( t \right) = \frac{\partial}{\partial t} \left( \alpha \left( t \right) : a, a^\dagger \right) \tag{7.24} $$

where

$$ \frac{\partial}{\partial t} \left( \alpha \left( t \right) : a, a^\dagger \right) = i\sqrt{\hbar} \sum_{l \geq 2} h^{(l-3)/2} H_l \left( \alpha \left( t \right) : a, a^\dagger \right) , \tag{7.25} $$

it follows that $L_\hbar \left( t \right)$ in Eq. (7.16) satisfies,

$$ \lim_{\hbar \downarrow 0} L_\hbar \left( t \right) \psi = H_2 \left( \alpha \left( t \right) : a, a^\dagger \right) \psi \quad \text{for all} \quad \psi \in S. \tag{7.26} $$

From this observation it is reasonable to expect $W_\hbar \left( t \right) \rightarrow W_0 \left( t \right)$ where $W_0 \left( t \right)$ is as in Notation 7.4. This is in fact the key content of this paper, see Theorem 9.3 below. To complete the proof we will still need a fair number of preliminary results.

### 7.1. Crude Bounds on $W_\hbar$.

**Theorem 7.9.** Suppose that $H \left( \theta , \theta^* \right) \in \mathbb{R} (\theta, \theta^*)$ and $0 < \hbar < \eta \leq 1$ satisfy Assumption 1, $d = \deg_\theta H$, and $W_\hbar \left( t,s \right)$ is as in Eq. (7.16). Then for all $\beta \geq 0$, there exists $C_{\beta, \hbar} < \infty$ depending only on $\beta \geq 0$ and $H$ such that, for all $s,t \in \mathbb{R}$,

$$ W_\hbar \left( t,s \right) D \left( \mathcal{N}^{\beta d/2} \right) \subset D \left( \mathcal{N}^\beta \right) \quad \text{and} \quad \left\| \mathcal{N}^\beta W_\hbar \left( t,s \right) \psi \right\| \lesssim \hbar^{-\beta} C_{\beta, \hbar} \left\| \psi \right\|. \tag{7.21} $$

(This bound is crude in the sense that $\hbar^{-\beta} C_{\beta, \hbar} \uparrow \infty$ as $\hbar \downarrow 0$. We will do much better later in Theorem 9.1.)

**Proof.** Let $\beta \geq 0$. From Proposition 7.4 it follows that $W_\hbar \left( t,s \right) D \left( \mathcal{N}^{\beta d/2} \right) \subset D \left( \mathcal{N}^\beta \right)$. Moreover,

$$ \left\| \mathcal{N}^\beta W_\hbar \left( t,s \right) \psi \right\| \lesssim \left\| W_\hbar \left( t,s \right) \psi \right\|_{\beta} $$

$$ = \left\| U_\hbar \left( -\alpha \left( t \right) \right) e^{-iH_{\pm}/t} U_\hbar \left( \alpha \left( s \right) \right) \psi \right\|_{\beta} $$

$$ \leq \left\| U_\hbar \left( \alpha \left( t \right) \right) \right\|_{\beta \rightarrow \beta} \left\| e^{-iH_{\pm}/t} \right\|_{\beta d/2 \rightarrow \beta} \left\| U_\hbar \left( \alpha \left( s \right) \right) \right\|_{\beta d/2 \rightarrow \beta} \left\| \psi \right\|_{\beta d/2}. $$

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Note that $\kappa := \sup_{t \in \mathbb{R}} |\alpha(t)| < \infty$ from Proposition 3.8, then by the Corollary 5.11 there exists a constant $C = C(\beta, d, \kappa)$ such that
\[ \sup_{t \in \mathbb{R}} \| U_h^t(\alpha(t)) \|_{\beta \to \beta} \forall \sup_{s \in \mathbb{R}} \| U_h^s(\alpha(s)) \|_{\beta d/2 \to \beta d/2} \leq C(\beta, d, \kappa). \]

Then, combining all above inequalities along with Eq. 6.8 in Theorem 6.7, we have
\[ \| N^2W_h(t, s) \psi \| \leq C_{\beta, H}h^{-\beta} \| \psi \|_{\beta d/2} \]
and therefore, Eq. 6.21 follows immediately. □

8. ASYMPTOTICS OF THE TRUNCATED EVOLUTIONS

As in Section 7, we assume that $H(\theta, \theta^*) \in \mathbb{R}(\theta, \theta^*)$ and $\eta > 0$ are as in Assumption 1, $\alpha_0 \in C$, and $\alpha(t)$ denotes the solution to Eq. 1.1 with $\alpha(0) = \alpha_0$. Further let $L_h(t)$ be as in Eq. (7.16), i.e.
\[ L_h(t) = \sum_{k=2}^{d} h^{k-1} H_k(\alpha(t) : a, a^*) \].

**Definition 8.1** (Truncated Evolutions). For $0 \leq M < \infty$ and $0 < h < \infty$, let $L^M_h(t) = \mathcal{P}_M L_h(t) \mathcal{P}_M$ be the level $M$ truncation of $L_h(t)$ (see Notation 3.34) and let $W^M_h(t, s)$ be the associated truncated evolution defined to be the solution to the ordinary differential equation,
\[ i \frac{d}{dt} W^M_h(t, s) = L^M_h(t) W^M_h(t, s) \text{ with } W^M(s, s) = I \] as in Section 4.1. We further let $W^M_0(t) = W^M_h(t, 0)$.

From the results of Theorem 4.4 with $Q_M(t) = L^M_h(t)$ and $U^M(t, s) = W^M_h(t, s)$, we know that $W^M_h(t, s)$ is unitary on $L^2(m)$ and
\[ W^M_h(t, s) = W^M_h(t, 0) W^M_h(0, s) = W^M_0(t) W^M_0(s)^* \]
and in particular, $W^M_0(t)^* = W^M_0(0, t)$.

**Proposition 8.2.** Suppose that $H(\theta, \theta^*) \in \mathbb{R}(\theta, \theta^*)$ and $\eta > 0$ satisfy Assumption 4.4 and $d = \deg_H H > 0 \in 2\mathbb{N}$, and further let $W_h(t, s)$, $W_0(t, s)$, and $W^M_h(t, s)$ be as in Eq. (7.14), Notation 7.7, and Definition 8.1 respectively. If $\psi \in D(\mathcal{N}^{\frac{1}{2}})$ and $0 < h < \eta$, then
\[ W_h(t, s) \psi - W^M_h(t, s) \psi = i \int_s^t W_h(t, \tau) \left[ L_h^M(\tau) - L_h(\tau) \right] W^M_h(\tau, s) \psi d\tau \] (8.3)
and
\[ W_h(t, s) \psi - W_0(t, s) \psi = i \int_s^t W_h(t, \tau) \left[ H(\alpha(\tau) : \bar{a}, a^*) - L_h(\tau) \right] W_0(\tau, s) \psi d\tau \] (8.4)
where $L_h$ and $H(\alpha(\tau) : \bar{a}, a^*)$ are as in Eqs. 7.16 and 7.20 and $L^M_h(\tau) = \mathcal{P}_M L_h(\tau) \mathcal{P}_M$ as in Definition 8.1. [The integrands in Eqs. 8.3 and 8.4 are $L^2(m)$-norm continuous functions of $\tau$ and therefore the integrals above are well defined.]
Proof. Let \( B \left( D \left( \mathcal{N}^{\frac{d}{2}} \right), L^2 (m) \right) \) denote the space of bounded linear operators from \( D \left( \mathcal{N}^{\frac{d}{2}} \right) \) to \( L^2 (m) \). The integrals in Eq. (8.3) and (8.4) may be interpreted as \( L^2 (m) \) -- valued Riemann integrals because their integrands are \( L^2 (m) \)-- continuous functions of \( \tau \). This is a consequence of the observations that both

\[
F (\tau) := W_h (t, \tau) \left[ L_k^M (\tau) - \overline{L_k^M (\tau)} \right] W_k^M (\tau, s) \quad \text{and}
\]

\[
G (\tau) := W_h (t, \tau) \left[ H_2 (\alpha (\tau) : \bar{a}, a^*) - \overline{L_h (\tau)} \right] W_0 (\tau, s)
\]

are strongly continuous \( B \left( D \left( \mathcal{N}^{\frac{d}{2}} \right), L^2 (m) \right) \)-- valued functions of \( \tau \). To verify this assertion recall that;

1. \( \tau \to W_k^M (\tau, s) \) is \( \|d/2\to d/2\) continuous by Item 3. of Theorem 4.4 and \( \tau \to W_0 (\tau, s) \) \( \psi \) is \( \|d\) -- continuous by Corollary 5.10.

2. Both \( L_k^M (\tau) - \overline{L_k^M (\tau)} \) and \( H_2 (\alpha (\tau) : \bar{a}, a^*) - \overline{L_h (\tau)} \) are easily seen to be strongly continuous as functions of \( \tau \) with values in \( B \left( D \left( \mathcal{N}^{\frac{d}{2}} \right), L^2 (m) \right) \) by using Corollary 3.30 and noting that the coefficients of the four operators depend continuously on \( \tau \).

3. The map, \( \tau \to W_h (t, \tau) \) is strongly continuous on \( L^2 (m) \) by Theorem 7.0.

As strong continuity is preserved under operator products, it follows that both \( F (\tau) \) and \( G (\tau) \) are strongly continuous.

By Remark 4.6 and Proposition 7.4 we know that \( W_k^M (t, s) S = S \) and \( W_k (t, s) S = S \). Moreover, from item 3. of Theorem 4.4 and Theorem 7.0 if \( \varphi \in S \), then both \( t \to W_k^M (t, s) \varphi \) and \( t \to W_k (t, s) \varphi \) and are \( \|d\)-differentiable for \( \beta \geq 0 \). Since \( W_h (t, s) \) is unitary (see Eq. (7.11)), it follows that \( \sup_{t,s \in \mathbb{R}} \|W_h (t, s)\|_{0 \to 0} = 1 \). Therefore, by applying Lemma 7.1 with \( U (\tau) = W_h (t, \tau) \), \( P (\theta, \theta^*) = 1 \), and \( T (\tau) = W_k^M (\tau, s) \) while making use of Eqs. (7.18) and (8.2) to find,

\[
i \frac{d}{d\tau} W_h (t, \tau) W_k^M (\tau, s) \varphi = F (\tau) \varphi.
\]

A similar argument using Corollary 5.10 in place of Theorem 4.4 shows,

\[
i \frac{d}{d\tau} W_h (t, \tau) W_0 (\tau, s) \varphi = G (\tau) \varphi.
\]

Equations (8.3) and (8.4) now follow for \( \psi = \varphi \in S \) by integrating the last two displayed equations and making use of the fundamental theorem of calculus.

By the uniform boundedness principle (or by direct estimates already provided), it follows that

\[
\sup_{\tau \in J_{s,t}} \|F (\tau)\|_{\frac{d}{2} \to 0} < \infty \quad \text{and} \quad \sup_{\tau \in J_{s,t}} \|G (\tau)\|_{\frac{d}{2} \to 0} < \infty,
\]

where \( J_{s,t} := [\min (s, t), \max (s, t)] \). Because of these observation and the fact that \( S \) is dense in \( D \left( \mathcal{N}^{\frac{d}{2}} \right) \), it follows that by a standard “\( \varepsilon/3 \) -- argument” that Eqs. (8.3) and (8.4) are valid for all \( \psi \in D \left( \mathcal{N}^{\frac{d}{2}} \right) \). □

**Theorem 8.3.** Let \( 0 < \eta \leq 1 \), \( H (\theta, \theta^*) \in \mathbb{R} \langle \theta, \theta^* \rangle \) be a polynomial of degree \( d \) satisfying Assumption 4 and \( d \geq 2 \) be an even number. Then for all \( \beta \geq d/2 \) and
\( -\infty < S < T < \infty \), there exists a constant, \( K(\beta, \alpha_0, H, S, T) < \infty \) such that
\[
\sup_{S < s, t < T} \left\| W_h(t, s) - W_h^{h^{-1}}(t, s) \right\|_{\beta \to 0} \leq K(\beta, \alpha_0, H, S, T) h^{\beta - 1} \forall 0 < h < \eta. \tag{8.5}
\]

**Proof.** Since \( W_h(t, s) \) and \( W_h^{h^{-1}}(t, s) \) are unitary from Theorem 4.4 and Eq. (7.14) and \( \|\cdot\|_{\beta} \geq \|\cdot\|_{0} \) in Remark 3.22 it follows
\[
\sup_{S < s, t < T} \left\| W_h(t, s) - W_h^{h^{-1}}(t, s) \right\|_{\beta \to 0} \leq 1, \tag{8.6}
\]
and hence Eq. (8.5) holds if \( \eta \wedge d^{-1} \leq h < \eta \). The remaining thing to show is Eq. (8.5) still holds for \( 0 < h < \eta \wedge d^{-1} \).

Let \( \psi \in D(\mathcal{N}') \subset D(\mathcal{N}^{d/2}) \). Taking the \( L^2(m) \) norm of Eq. (8.3) implies,
\[
\left\| \left[ W_h(t, s) - W_h^M(t, s) \right] \psi \right\| \leq \int_{J_{s,t}} \left\| W_h(t, \tau) \left( L_h^M(\tau) - L_h(\tau) \right) W_h^M(\tau, s) \psi \right\| d\tau,
\]
where
\[
\left\| W_h(t, \tau) \left( L_h^M(\tau) - L_h(\tau) \right) W_h^M(\tau, s) \psi \right\|
= \left\| \left( L_h^M(\tau) - L_h(\tau) \right) W_h^M(\tau, s) \psi \right\|
\leq \left\| L_h^M(\tau) - L_h(\tau) \right\|_{\beta \to 0} \left\| W_h^M(\tau, s) \right\|_{\beta \to \beta} \left\| \psi \right\|_{\beta}.
\]
In order to simplify this estimate further, let
\[
P(h, t : \theta, \theta^*) = \sum_{k=2}^{d} h^{k-1} H_k(\alpha(t) : \theta, \theta^*),
\]
in which case, \( L_h(t) = P(h, t : a, a^t) \). It follows from Corollary 3.37 with \( \beta = 0 \) and \( \alpha \to \beta \) that (for \( M \geq d \))
\[
\left\| L_h^M(\tau) - L_h(\tau) \right\|_{\beta \to 0} \leq \sum_{k=2}^{d} h^{k-1} |H_k(\alpha(t) : \theta, \theta^*)| (M - k + 2)^{k/2 - \beta}
\leq K(\alpha_0, H) h^{-1} \sum_{k=2}^{d} (hM - kh + 2h)^{k/2} (M - k + 2)^{-\beta}
\]
and from Eq. (4.11) that
\[
\left\| W_h^M(\tau, s) \right\|_{\beta \to \beta} \leq e^{K(\beta, d)(hM + 1) h^{\beta - 1} \sum_{k=2}^{d} \int_{J_{s,t}} \left| h^{\beta - 1} H_k(\alpha(\sigma) : \theta, \theta^*) \right| d\sigma}
\leq e^{K(\beta, d, H)(hM + 1) h^{\beta - 1} |t-s|}.
\]
Thus reducing to the case where \( M = h^{-1} \) (i.e. \( M_h = 1 \)) we see there exists \( K(\beta, \alpha_0, H, S, T) < \infty \) such that
\[
\left\| L_h^{-1}(\tau) - L_h(\tau) \right\|_{\beta \to 0} \left\| W_h^{h^{-1}}(\tau, s) \right\|_{\beta \to \beta} \leq \tilde{K}(\beta, \alpha_0, H, S, T) h^{\beta - 1}
\]
which combined with Eqs. (8.7) and (8.8) implies Eq. (8.5) with \( K(\beta, \alpha_0, H, S, T) = \tilde{K}(\beta, \alpha_0, H, S, T) [T - S] \).
9. PROOF OF THE MAIN THEOREMS

The next theorem combines the crude bound in Theorem 7.9 with the asymptotics of the truncated evolutions in Theorem 8.3 in order to give a much improved version of Theorem 7.9.

**Theorem 9.1 (N – Sobolev Boundedness of \( W_h(t) \)).** Suppose that \( H(\theta, \theta^*) \in \mathbb{R}(\theta, \theta^*) \) and \( \eta > 0 \) satisfy Assumption 7, \( d = \deg H > 0 \in 2\mathbb{N} \), and \( W_h(t, s) \) and \( W_h(t) \) be as in Eqs. (7.12) and (7.13) respectively. Then for each \( \beta \geq 0 \), \( -\infty < S < T < \infty \), there exists \( \tilde{K}_\beta(S, T) < \infty \) such that for all \( \psi \in D(\mathcal{N}^{(2\beta+1)d}) \), all \( 0 < h < \eta \leq 1 \), and all \( S \leq s, t \leq T \) we have

\[ \| \mathcal{N}^\beta W_h(t, s) \psi \| \leq K_\beta(S, T) \| \psi \|_{(2\beta+1)d}, \tag{9.1} \]

and

\[ \sup_{S \leq s, t \leq T} \| W_h(t, s) \|_{(2\beta+1)d \rightarrow \beta} \leq \tilde{K}_\beta(S, T), \tag{9.2} \]

where

\[ \tilde{K}_\beta(S, T) := (1 + K_\beta(S, T)) 2^{(\beta-1)} \]

In particular this estimate implies, for \( 0 < h < \eta \leq 1 \),

\[ \sup_{S \leq t \leq T} \left( \| W_h(t) \|_{(2\beta+1)d \rightarrow \beta} \vee \| W_h^* (t) \|_{(2\beta+1)d \rightarrow \beta} \right) \leq \tilde{K}_\beta(S, T). \tag{9.4} \]

[The bound in Eq. (9.2) improves on the crude bound in Eq. (8.3) in that the bound now does not blow up as \( h \downarrow 0 \).]

**Remark 9.2.** The bound in Eq. (9.1) is not tight in that the index, \( (2\beta+1)d \), of the norm on the right side of this equation is not claimed to be optimal.

**Proof.** The case \( \beta = 0 \) is a trivial and so we now assume \( \beta > 0 \). If \( \psi \in D(\mathcal{N}^{(2\beta+1)d}) \), then by Proposition 7.3 \( W_h(t, s) \psi \in D(\mathcal{N}^{(2\beta+1)d}) \). Some simple algebra then shows

\[ \langle W_h(t, s) \psi, \mathcal{N}^{2\beta} W_h(t, s) \psi \rangle = A + B, \]

where

\[ A := \langle W_h^{-1}(t, s) \psi, \mathcal{N}^{2\beta} W_h^{-1}(t, s) \psi \rangle \quad \text{and} \]

\[ B := \left[ \langle W_h^{-1}(t, s) - W_h^{-1}(t, s) \rangle \psi, \mathcal{N}^{2\beta} W_h(t, s) \psi \right] + \left[ \mathcal{N}^{2\beta} W_h^{-1}(t, s) \psi \right] \left[ W_h(t, s) - W_h^{-1}(t, s) \right] \psi. \]

The \( |B| \) term is bounded by the following two terms.

\[ |B| \leq \left\| \left[ W_h(t, s) - W_h^{-1}(t, s) \right] \psi \right\| \cdot \left\| \mathcal{N}^{2\beta} W_h(t, s) \psi \right\| \]

\[ + \left\| \left[ W_h(t, s) - W_h^{-1}(t, s) \right] \psi \right\| \cdot \left\| \mathcal{N}^{2\beta} W_h^{-1}(t, s) \psi \right\|. \]

Therefore, using Eq. (4.15) in Corollary 4.7, Theorem 8.3 with \( \beta \) replaced by \( \frac{\beta}{2} + 2\beta \), and Theorem 7.9, it follows that

\[ |B| \leq \left\| \left[ W_h(t, s) - W_h^{-1}(t, s) \right] \psi \right\| \cdot \left( \left\| \mathcal{N}^{2\beta} W_h(t, s) \psi \right\| + \left\| \mathcal{N}^{2\beta} W_h^{-1}(t, s) \psi \right\| \right) \]

\[ \leq C h^{\beta+\frac{\beta}{2}-1} \left\| \psi \right\|_{\frac{\beta}{2}+2\beta} \cdot \left( h^{-2\beta} \left\| \mathcal{N} + I \right\|_{2\beta} \right) + \left\| \mathcal{N} + I \right\|_{2\beta} \}

\[ \leq C h^{\frac{\beta}{2}-1} \left\| \psi \right\|_{\frac{\beta}{2}+2\beta} \left( \left\| \psi \right\|_{\beta + h^{2\beta} \left\| \psi \right\|_{2\beta}} \right) \]

\[ \leq C h^{\frac{\beta}{2}-1} \left\| \psi \right\|^2_{(2\beta+1)d} < \infty \quad \text{for all} \quad S \leq s, t \leq T \quad \text{and} \quad 0 < h < \eta. \tag{9.5} \]
In the last inequality we have used, \( \frac{d}{2} + 2\beta \leq (2\beta + 1)d \) when \( \beta > 0 \) and \( d \geq 2 \). Corollary 4.7 directly implies there exists \( C > 0 \) such that

\[
|A| = \left\| \Lambda^\beta W_h^{-1} (t, s) \psi \right\|^2_{\beta} \leq C \left\| \psi \right\|_{(2\beta + 1)d}^2 \leq C \left\| \psi \right\|^2_{(2\beta + 1)d}
\]

for all \( S \leq s, t \leq T \) and therefore, we get

\[
\left\| \Lambda^\beta W_h (t, s) \psi \right\|^2 = \left\langle W_h (t, s) \psi, \Lambda^{2\beta} W_h (t, s) \psi \right\rangle \leq (K_\beta (S, T))^2 \left\| \psi \right\|^2_{(2\beta + 1)d} \quad (9.6)
\]

for an appropriate constant \( K_\beta (S, T) \). Equation (9.1) is proved and Eq. (9.2) is a consequence of Eq. (9.1) and the inequality in Eq. (6.5). Equation (9.2) also implies Eq. (9.3) because \( W_h (t) = W_h (t, 0) \) and \( W_h^* (t) = W_h (0, t) \). \( \Box \)

**Theorem 9.3.** Suppose that \( H (\theta, \theta^*) \in \mathbb{R} (\theta, \theta^*) \) and \( 0 < \eta \leq 1 \) satisfy Assumptions 7. Let \( d = \deg_{\theta} H \in 2\mathbb{N} \), \( W_h (t, s) \), and \( W_0 (t, s) \) be as in Eq. (7.3) and Notation 7.7 respectively. Then \( W_h (t, s) \to W_0 (t, s) \) as \( h \downarrow 0 \). Moreover for all \( \beta \geq 0 \) and \( -\infty < S < T < \infty \) there exists \( K = K_\beta (S, T) < \infty \) such that, for \( 0 < h < \eta \leq 1 \),

\[
\sup_{S \leq s, t \leq T} \left\| \Lambda^\beta (W_0 (t, s) - W_h (t, s)) \psi \right\| \leq K \sqrt{h} \left\| \psi \right\|_{\frac{4}{2}(4\beta + 3)} \forall \psi \in D \left( \Lambda^{\frac{\beta}{2}(4\beta + 3)} \right) \quad (9.7)
\]

and, with \( \tilde{K} := (1 + K)^{2(\beta - 1)} \),

\[
\sup_{s, t \in [S, T]} \left\| W_0 (t, s) - W_h (t, s) \right\|_{\frac{4}{2}(4\beta + 3) - \beta} \leq \tilde{K} \sqrt{h}. \quad (9.8)
\]

In particular, for \( 0 < h < \eta \leq 1 \),

\[
\sup_{S \leq s, t \leq T} \left\| W_0 (t) - W_h (t) \right\|_{\frac{4}{2}(4\beta + 3) - \beta} \forall \left\| W_0^* (t) - W_h^* (t) \right\|_{\frac{4}{2}(4\beta + 3) - \beta} \leq \tilde{K} \sqrt{h}. \quad (9.9)
\]

**Proof.** The claimed strong convergence now follows from Eq. (9.7) with \( \beta = 0 \) along with a standard density argument. To simplify notation, let

\[
p = d (2\beta + 1) \quad \text{and} \quad q = \frac{d}{2} (4\beta + 3) = p + \frac{d}{2}
\]

If \( \psi \in D \left( \Lambda^{\frac{\beta}{2}} \right) \subseteq D \left( \Lambda \right) \), then by Eq. (8.4) in Proposition 8.2, Eq. (7.16), and Corollary 3.30

\[
W_h (t, s) \psi - W_0 (t, s) \psi = i \int_s^t W_h (t, \tau) \left[ H_2 (\alpha (\tau) : \bar{a}, a^*) - \bar{L}_h (\tau) \right] W_0 (\tau, s) \psi d\tau
\]

\[
= i \int_s^t W_h (t, \tau) \left[ \frac{1}{h} H_{\geq 3} \left( \alpha (\tau) : \bar{a}, a_h^* \right) \right] W_0 (\tau, s) \psi d\tau.
\]

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Then, by using theorem [9.1] we find for all $0 < h < \eta \leq 1$ and $S \leq s, t \leq T$ (with $d = \deg_0 H$) that

\[
\|(W_h (t, s) - W_0 (t, s)) \psi\|_\beta \\
\leq \int_{J_{s, t}} \left\| W_h (t, \tau) \left( \frac{1}{h} H_{\geq 3} (\alpha (\tau) : \bar{a}_h, a_h^*) \right) W_0 (\tau, s) \psi \right\|_\beta d\tau \\
\leq \int_S \|W_h (t, \tau)\|_{p-\beta} \left\| \left( \frac{1}{h} H_{\geq 3} (\alpha (\tau) : \bar{a}_h, a_h^*) \right) W_0 (\tau, s) \psi \right\|_p d\tau \\
\leq K \int_S \left\| \frac{1}{h} H_{\geq 3} (\alpha (\tau) : \bar{a}_h, a_h^*) \right\|_{q-p} \|W_0 (t, \tau)\|_{q-q} \|\psi\|_q d\tau \\
\leq K \sqrt{h} \int_S \left\| H_{\geq 3} (\alpha (\tau), \sqrt{h} : \bar{a}, a^*) \right\|_{q-q} \|W_0 (t, \tau)\|_{q-q} d\tau < \infty
\]

(9.10)

where $H_{\geq 3} (\alpha (\tau), \sqrt{h} : \theta, \theta^*) \in \mathbb{R} [\alpha (\tau), \sqrt{h} (\theta, \theta^*)]$ is a polynomial in $(\alpha (\tau), \sqrt{h} : \theta, \theta^*)$ which is a sum of terms homogeneous of degree three or more in the $\{\theta, \theta^*\}$-grading. By Eq. (6.5) in Corollary 3.30 and Eq. (5.27) in Corollary 5.10, (9.10) can be used to prove the main theorems of this paper, namely Theorem 9.1.

Proof of Theorem 9.1. We now finish this paper by showing that Eqs. (9.4) and (9.9) can be used to prove the main theorems of this paper, namely Theorem 1.10 and Corollaries 1.18 and 1.20. For the rest of Section 9 we always assume that $H \in \mathbb{R} (\theta, \theta^*)$ and $1 \geq \eta > 0$ satisfy Assumption [1]. $d = \deg_0 H > 0 \in 2\mathbb{N}$, $W_h (t)$ is defined as in Eq. (1.13), and $W_0 (t)$ is as in Notation 7.7.

Notation 9.4. For $h \geq 0$, let

\[
a (h : t) := W_h^* (t) a W_h (t) \quad \text{and} \quad a^\dagger (h : t) := W_h^* (t) a^\dagger W_h (t)
\]

(9.11)
as operator on $\mathcal{S}$. It should be noted that under Assumption [1] we have $a^\dagger (h : t) = a (h : t)^\dagger$ for $0 \leq h < \eta$.

According to Theorem 5.13 if $a (t)$ and $a^\dagger (t)$ are as in Eqs. (1.8) and (1.9) respectively then satisfies,

\[
a (t) = W_0^* (t) a W_0 (t) = a (0 : t) \quad \text{and} \\
a^\dagger (t) = W_0^* (t) a^\dagger W_0 (t) = a^\dagger (0 : t)
\]

(9.12)

(9.13)
as operators on $\mathcal{S}$. For this reason we will typically write $a (t)$ and $a^\dagger (t)$ for $a (0 : t)$ and $a (0 : t)$ respectively.
By Proposition [2.4] and Eq. (7.13), the operator \( A_h (t) \) defined in Eq. (1.22) satisfies,

\[
U_h^* (\alpha_0) A_h (t) U_h (\alpha_0) = U_h^* (\alpha_0) e^{itH_h / \hbar} a_{\hbar} e^{-itH_h / \hbar} U_h (\alpha_0) \\
= W^*_h (t) (\alpha_h + \alpha (t)) W_h (t) \\
= \alpha (t) + \sqrt{\hbar} W^*_h (t) a W_h (t) \\
= \alpha (t) + \sqrt{\hbar} a (h : t) \text{ on } S.
\]

(9.14)

**Notation 9.5.** For \( t \in \mathbb{R} \) and \( 0 \leq h < \eta \), let

\[
B_\theta (h : t) := a (h : t) = W^*_h (t) a W_h (t) \quad \text{and} \quad B_{\theta^*} (h : t) := a (h : t)^* = W^*_h (t) a^* W_h (t).
\]

When \( h = 0 \) we will denote \( B_\theta (0 : t) \) more simply as \( B_\theta (t) \) for \( b \in \{ \theta, \theta^* \} \).

**Lemma 9.6.** Let \( \eta > 0 \) and \( d > 0 \in 2\mathbb{N} \) be as in Theorem [7.7] \( b \in \{ \theta, \theta^* \} \), \( t \in [S, T] \), and \( B_\theta (h : t) \) be as in Notation [7.7]. Then, for any \( \beta \geq 0 \), there exists a constant \( C (\beta, S, T) > 0 \) such that

\[
\sup_{t \in [S, T]} \max_{b \in \{ \theta, \theta^* \}} \| B_\theta (h : t) \|_{\theta (\beta) \to \beta} \leq C (\beta, S, T) \quad \text{for } 0 < h < \eta
\]

(9.15)

where \( g (\beta) = 4d^2 \beta + 2d (d + 1) \).

**Proof.** For definiteness, suppose that \( b = \theta^* \) as the case \( b = \theta \) is proved analogously. If \( q = (2\beta + 1) \) and

\[
p = \left[ 2 \left( q + \frac{1}{2} \right) + 1 \right] d = 4d^2 \beta + 2d (d + 1),
\]

then

\[
\| B_\theta (h : t) \|_{p \to \beta} \leq \| W^*_h (t) \|_{q \to \beta} \| a^* \|_{q + \frac{1}{2} \to \eta} \| W_h (t) \|_{p \to q + \frac{1}{2} \to \eta}
\]

which combined with the estimates in Eqs. (3.41) and (9.4) gives the estimate in Eq. (9.15). □

**Lemma 9.7.** Let \( \beta \geq 0 \), \( b \in \{ \theta, \theta^* \} \), \( -\infty < S < T < \infty \), \( \eta > 0 \), and \( d > 0 \in 2\mathbb{N} \) be the same as Lemma [7.6]. Then there exists a constant \( C (\beta, S, T) > 0 \) such that

\[
\sup_{t \in [S, T]} \| B_\theta (h : t) - B_\theta (t) \|_{r (\beta) \to \beta} \leq C (\beta, S, T) \sqrt{\hbar} \quad \text{for } 0 < h < \eta
\]

(9.16)

where \( r (\beta) = (4d^2) \beta + (3d + 2) \).

**Proof.** Let us suppose that \( b = \theta \) as the proof for \( b = \theta^* \) is very similar. Given \( p \geq \beta \) (to be chosen later) we have,

\[
\| B_\theta (h : t) - B_\theta (t) \|_{p \to \beta} \\
= \| W^*_h (t) \overline{a W_h (t)} - W_0^* (t) \overline{a W_0 (t)} \|_{p \to \beta} \\
\leq \| [W^*_h (t) - W_0^* (t)] \overline{a W_h (t)} \|_{p \to \beta} + \| W_0^* (t) \overline{a [W_h (t) - W_0 (t)]} \|_{p \to \beta}.
\]

(9.17)

Using Eqs. (3.41), (9.4), and (9.9), there exists a constant \( C_1 := C_1 (\beta, S, T) \) such that the first term will become

\[
\| [W^*_h (t) - W_0^* (t)] \overline{a W_h (t)} \|_{p_1 \to \beta} \\
\leq \| [W^*_h (t) - W_0^* (t)] \|_{q_1 \to \beta} \| \overline{a} \|_{q_1 + \frac{1}{2} \to q_1} \| W_h (t) \|_{p_1 \to q_1 + \frac{1}{2}} \leq C_1 \sqrt{\hbar}
\]

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where

\[ q_1 = \frac{d}{2} (4\beta + 3) \text{ and } p_1 = \left( 2 \left( q_1 + \frac{1}{2} \right) + 1 \right) d = (4d^2) \beta + (3d + 2) d. \]

Likewise, using Eqs. (3.41), (5.27) and (9.9), there exists a constant \( C_2 := C_2 (\beta, S, T) \) such that the second term will become

\[
\| W^*_0 (t) \bar{a} [W_h (t) - W_0 (t)] \|_{p_2 \to \beta} \\
\leq \| W^*_0 (t) \|_{q_2 \to \beta} \| \bar{a} \|_{q_2 \to \beta} \| W_h (t) - W_0 (t) \|_{p_2 \to q_2 + \frac{1}{2}} \leq C_2 \sqrt{h}
\]

where

\[ q_2 = \beta \text{ and } p_2 = \frac{d}{2} \left( 4 \left( q_2 + \frac{1}{2} \right) + 3 \right) = (2d) \beta + \frac{5d}{2}. \]

Since \( d \geq 2 \) and \( \beta > 0 \), it follows that \( p_2 \leq p_1 \) and so taking \( p = p_1 \) in Eq. (9.17) and making use of the previous estimates proves Eq. (9.16). \( \square \)

**Notation 9.8.** For \( n \in \mathbb{N} \), let \( d = \text{deg}_y H > 0 \) and

\[ \sigma_n := (4d^2) \ 2d \ (d + 1) \ (4d^2)^n - 1 + (3d + 2) d. \quad (9.18) \]

**Lemma 9.9.** Let \( S, T, d \) and \( \eta \) be the same as Lemma 9.6 and \( \sigma_n \) be as in Notation 9.8 for \( n \in \mathbb{N} \). Then there exists \( C_n (S, T) < \infty \) such that for any \( b = (b_1, \ldots, b_n) \in \{ \theta, \theta^* \}^n \), \( 0 \leq h < \eta \), and \( (t_1, \ldots, t_n) \in [S, T] \) we have

\[ \| B_1 (h) \ldots B_n (h) - B_1 \ldots B_n \|_{\sigma_n \to 0} \leq C_n (S, T) \sqrt{h}, \quad (9.19) \]

where \( B_i (h) := B_{b_i} (h : t_i) \) and \( B_i := B_i (0) = B_{b_i} (t_i) \) for \( 1 \leq i \leq n \), see Notation 9.6.

**Proof.** By a telescoping series arguments,

\[
B_1 (h) \ldots B_n (h) - B_1 \ldots B_n \\
= \sum_{i=1}^{n} \ [B_1 (h) \ldots B_i (h) B_{i+1} \ldots B_n - B_1 (h) \ldots B_{i-1} (h) B_i \ldots B_n] \\
= \sum_{i=1}^{n} B_1 (h) \ldots B_{i-1} (h) [B_i (h) - B_i] B_{i+1} \ldots B_n
\]

and therefore

\[
\| B_1 (h) \ldots B_n (h) - B_1 \ldots B_n \|_{\sigma_n \to 0} \\
\leq \sum_{i=1}^{n} \| B_1 (h) \ldots B_{i-1} (h) [B_i (h) - B_i] B_{i+1} \ldots B_n \|_{\sigma_n \to 0}. \quad (9.20)
\]

To finish the proof it suffices to show for \( 1 \leq i \leq n \) that

\[
\| B_1 (h) \ldots B_{i-1} (h) [B_i (h) - B_i] B_{i+1} \ldots B_n \|_{\sigma_n \to 0} \leq C \sqrt{h}.
\]

Now

\[
\| B_1 (h) \ldots B_{i-1} (h) [B_i (h) - B_i] B_{i+1} \ldots B_n \|_{\sigma_n \to 0} \\
\leq \| B_1 (h) \ldots B_{i-1} (h) \|_{\nu \to 0} \| B_i (h) - B_i \|_{\nu \to \nu} \| B_{i+1} \ldots B_n \|_{\sigma_n \to \nu}
\]
where we will choose all $\sigma_n, u, v \geq 0$ appropriately. First off if $\beta \geq 0$ and $A = \pi$ or $a^*$, then (see Proposition 5.29) $A : D (N^{\beta+\frac{1}{2}}) \to D (N^\beta)$ and (see Corollary 5.10) $W_0 (t) : N^{\beta} \to N^\beta$ are bounded operators and therefore,

$$\|B_{i+1} \ldots B_n\|_{\sigma_n \to u} < \infty \text{ if } \sigma_n = u + \frac{1}{2} (n - i) .$$

(9.21)

Also, with $r (v)$ as in Lemma 9.7, there exists $C$ such that, for $0 < h < \eta$,

$$\|B_i (h) - B_i\|_{u \to v} \leq C \sqrt{h} \text{ if } u = r (v) .$$

(9.22)

Using Lemma 9.6 there exists $C > 0$ such that, for $0 < h < \eta$,

$$\|B_1 (h) \ldots B_{i-1} (h)\|_{v \to 0} \leq C$$

provided that

$$v = g^{i-1} (0) = 2d (d + 1) \frac{(4d^2)^i - 1}{4d^2 - 1}. \quad (9.23)$$

If we let $1 \leq i \leq n$ and

$$\sigma_n (i) = r \left( g^{i-1} (0) \right) + \frac{1}{2} (n - i)$$

$$= (4d^2) 2d (d + 1) \frac{(4d^2)^i - 1}{4d^2 - 1} + (3d + 2) d + \frac{1}{2} (n - i) ,$$

then by the above bounds it follows that

$$\|B_1 (h) \ldots B_{i-1} (h) [B_i (h) - B_i] B_{i+1} \ldots B_n\|_{\sigma_n (i) \to 0} < \infty. \quad (9.24)$$

One shows $\sigma_n (i)$ is increasing in $i$ and therefore $\max_{1 \leq i \leq n} \sigma_n (i) = \sigma_n (n) = \sigma_n$ where $\sigma_n$ is as in Notation 9.8 Equation (9.19) now follows from Eqs. (9.20) and (9.21) with $\sigma_n (i)$ increased to $\sigma_n$. \hfill \Box

We finish the proof of Theorem 1.16 with Lemma 9.9.

**Proof of Theorem 1.16.** Note that we have already shown that $A_h (t_i)$ and $A_h^\dagger (t_i)$ preserve $S$ from Eq. (6.1) and $U_h (\alpha_0) S = S$ and $U_h (\alpha_0)^* S = S$ from Proposition 2.4. To show Eq. (1.23), for $\psi \in S$, we have

$$\begin{align*}
\left\langle P \left( \left\{ A_h (t_i) - \alpha (t_i), A_h^\dagger (t_i) - \pi (t_i) \right\}_{i=1}^n \right) \right\rangle_{U_h (\alpha_0) \psi} \\
= \left\langle P \left( \left\{ U_h^\ast (\alpha_0) A_h (t_i) U_h (\alpha_0) - \alpha (t_i), U_h^\ast (\alpha_0) A_h^\dagger (t_i) U_h (\alpha_0) - \pi (t_i) \right\}_{i=1}^n \right) \right\rangle_{\psi} \\
= \left\langle P \left( \left\{ \sqrt{ha} (h : t_i), \sqrt{ha}^\dagger (h : t_i) \right\}_{i=1}^n \right) \right\rangle_{\psi} \quad (9.25)
\end{align*}$$

where $\langle \cdot \rangle_{\psi}$ is defined in Definition 1.17 and the last step is asserted by Eq. (9.14). Supposed $p = \text{deg} (P (\{ \theta_i, \theta_i^* \}_{i=1}^n))$ and $p_{\min}$ is then minimum degree of each non-constant term in $P (\{ \theta_i, \theta_i^* \}_{i=1}^n)$. As $p = 0$ is a trivial case, we assume $p > 0$. Then, it follows

$$P (\{ \theta_i, \theta_i^* \}_{i=1}^n) = P_0 + \sum_{k=p_{\min}}^p P_k (\{ \theta_i, \theta_i^* \}_{i=1}^n) \quad (9.26)$$

where $P_0 \in \mathbb{C}$ and

$$P_k (\{ \theta_i, \theta_i^* \}_{i=1}^n) = \sum_{b_1, \ldots, b_k \in \{ \theta_i, \theta_i^* \}_{i=1}^n} c (b_1, \ldots, b_k) b_1 \ldots b_k$$
is a homogeneous polynomial of \( \{\theta_i, \theta_i^*\}_{i=1}^n \) with degree \( k \). Plugging Eq. (9.26) into Eq. (9.25) gives,

\[
\left\langle P \left( \{ \sqrt{\theta} a (h : t_i) , \sqrt{\theta} a^\dagger (h : t_i) \}_{i=1}^n \right) \right\rangle_{\psi} = P_0 + \sum_{k=p_{\min}}^p h^{\frac{k}{2}} \left\langle P_k \left( \{ a (h : t_i) , a^\dagger (h : t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( \sqrt{\theta} \right). \tag{9.27}
\]

wherein we have used the fact that \( P_k \) is a homogeneous polynomial of degree \( k \) in \( \{\theta_i, \theta_i^*\}_{i=1}^n \). By Lemma 9.9, for \( 0 < h < \eta \), we have

\[
\left\| P_k \left( \{ a (h : t_i) , a^\dagger (h : t_i) \}_{i=1}^n \right) \right\| = \left\| P_k \left( \{ a (t_i) , a^\dagger (t_i) \}_{i=1}^n \right) \right\| + O \left( \sqrt{\theta} \right).
\]

Therefore, for \( k \geq 1 \), we have

\[
h^{\frac{k}{2}} \left\langle P_k \left( \{ a (h : t_i) , a^\dagger (h : t_i) \}_{i=1}^n \right) \right\rangle_{\psi} = h^{\frac{k}{2}} \left\langle P_k \left( \{ a (t_i) , a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( h^{\frac{k}{2}-\frac{1}{2}} \right). \tag{9.28}
\]

Applying Eq. (9.28) to Eq. (9.27), we have

\[
\left\langle P \left( \{ \sqrt{\theta} a (h : t_i) , \sqrt{\theta} a^\dagger (h : t_i) \}_{i=1}^n \right) \right\rangle_{\psi} = P_0 + \sum_{k=p_{\min}}^p h^{\frac{k}{2}} \left\langle P_k \left( \{ a (t_i) , a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( h^{\frac{p_{\min}}{2}} \right)
\]

\[
= \left\langle P \left( \{ \sqrt{\theta} a (t_i) , \sqrt{\theta} a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( h^{\frac{p_{\min}}{2}-\frac{1}{2}} \right).
\]

Therefore, Eq. (1.23) follows immediately. \( \square \)

9.2. Proof of Corollary 1.18 Let \( P \left( \{ \theta_i, \theta_i^*\}_{i=1}^n \right) \subseteq C \langle \{ \theta_i, \theta_i^*\}_{i=1}^n \rangle \) be a non-commutative polynomial, \( \psi \in S \) and \( \{ t_1, \ldots, t_n \} \subseteq \mathbb{R} \). With out loss of generality, we may assume \( \deg (P) \geq 1 \). We define, (see Notation 2.16),

\[
\tilde{P} \left( \{ \alpha (t_i) : \theta_i, \theta_i^*\}_{i=1}^n \right) = P \left( \{ \alpha (t_i) + \alpha (t_i) , \theta_i^* + \alpha (t_i) \}_{i=1}^n \right)
\]

\[
\in \mathbb{C} \left[ \{ \alpha (t_i) , \alpha (t_i) \}_{i=1}^n \right] \langle \{ \theta_i , \theta_i^*\}_{i=1}^n \rangle.
\]

Note that \( \deg \left( \tilde{P} \right) = \deg (P) \) (see Notation 2.16) and \( p_{\min} \geq 1 \) because \( \deg \left( \tilde{P} \right) \geq 1 \). By Theorem 1.16, for \( 0 < h < \eta \), we have

\[
\left\langle P \left( \{ A_h (t_i) , A_h^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{U_h(\alpha (t)) \psi} = \left\langle \tilde{P} \left( \{ \alpha (t_i) : A_h (t_i) - \alpha (t_i) , A_h^\dagger (t_i) - \alpha (t_i) \}_{i=1}^n \right) \right\rangle_{U_h(\alpha (t)) \psi}
\]

\[
= \left\langle \tilde{P} \left( \{ \alpha (t_i) : \sqrt{\theta} a (t_i) , \sqrt{\theta} a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( h^{\frac{p_{\min}}{2}} \right)
\]

\[
= \left\langle P \left( \{ \alpha (t_i) + \sqrt{\theta} a (t_i) , \sqrt{\theta} a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( h^{\frac{p_{\min}}{2}} \right)
\]

\[
= \left\langle P \left( \{ \alpha (t_i) + \sqrt{\theta} a (t_i), \alpha (t_i) + \sqrt{\theta} a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O \left( h^{\frac{p_{\min}}{2}} \right).
\]

The last equality is because \( p_{\min} \) is at least \( 1 \). Therefore, Eq. (1.25) follows.
9.3. Proof of Corollary 1.20 By Eqs. (1.28) and (1.29) in Definition 1.3, the term \( \langle P_k \left( \{ \alpha (t_i) : a (t_i), a^\dagger (t_i) \}_i^n \right) \rangle_\psi \) in Eq. (1.26) is bounded independent of \( \hbar \) for \( \psi \in S \). Therefore, by setting \( \hbar \to 0 \) in Eq. (1.29), Eq. (1.28) follows. To show Eq. (1.29), let \( p_{\text{min}} \) be the minimum degree of all non constant terms in \( P \left( \{ \theta_i, \theta_i^\dagger \}_i^n \right) \). We assume \( p_{\text{min}} \geq 1 \) as usual. Otherwise, it means \( P \) is a constant polynomial which is a trivial case in Eq. (1.29). With the same notations as in Eq. (9.26), we have

\[
P \left( \{ \theta_i, \theta_i^\dagger \}_i^n \right) = P_0 + \sum_{k=p_{\text{min}}}^p P_k \left( \{ \theta_i, \theta_i^\dagger \}_i^n \right).
\]

Then, we apply Eq. (1.23) on each term \( P_k \) where \( k \geq 1 \), and get

\[
\langle P_k \left( \{ A_h (t_i) - \alpha (t_i), A_h^\dagger (t_i) - \alpha^\dagger (t_i) \}_i^n \right) \rangle_{U_h (a_0) \psi} = \langle P_k \left( \{ \sqrt{\hbar} a (t_i), \sqrt{\hbar} a^\dagger (t_i) \}_i^n \right) \rangle_{\psi} + O \left( \hbar^{\frac{k+1}{2}} \right).
\]

By applying Eq. (9.29), we have

\[
\langle P \left( \left\{ \frac{A_h (t_i) - \alpha (t_i), A_h^\dagger (t_i) - \alpha^\dagger (t_i) \}}{\sqrt{\hbar}} \right\}_i^n \right) \rangle_{U_h (a_0) \psi} = P_0 + \sum_{k=p_{\text{min}}}^p \frac{1}{\hbar\frac{k}{2}} \langle P_k \left( \{ a (t_i), a^\dagger (t_i) \}_i^n \right) \rangle_{U_h (a_0) \psi} = P_0 + \sum_{k=p_{\text{min}}}^p \langle P_k \left( \{ a (t_i), a^\dagger (t_i) \}_i^n \right) \rangle_{\psi} + O \left( \hbar^{\frac{k}{2}} \right).
\]

Eq. (1.29) follows.

10. Appendix: Main Theorems in terms of the standard CCRs

Let

\[
\hat{a}_h = \frac{1}{\sqrt{2}} \left( M_x + \hbar \frac{d}{dx} \right) \text{ and } \hat{a}_h^\dagger = \frac{1}{\sqrt{2}} \left( M_x - \hbar \frac{d}{dx} \right)
\]

(as an operator on \( S \)) be the more standard representation for the annihilation and creation operators form of the CCRs used in the physics literature. We will reformulate Theorem 1.16 Corollaries 1.18 and 1.20 in the standard CCRs. The following lemma (whose proof is left to the reader) implements the equivalence of our representation of the canonical commutation relations (CCRs) to the standard representation of the CCRs.

Lemma 10.1. For \( \rho > 0 \), let \( S_\rho : L^2 (\mathbb{R}) \to L^2 (\mathbb{R}) \) be the unitary map defined by

\[
(S_\rho f) (x) := \sqrt{\rho} f (\rho x) \text{ for } x \in \mathbb{R}.
\]

Then \( S_\rho S = S \) and it follows that

\[
\hat{a}_h = S_{h^{-1/2}} a_h S_{h^{1/2}} \text{ and } \hat{a}_h^\dagger = S_{h^{-1/2}} a_h^\dagger S_{h^{1/2}}
\]
\textbf{Definition 10.2.} For \( h > 0 \) and \( \alpha := (\xi + i\pi)/\sqrt{2} \), let
\[ \hat{U}_h (\alpha) = \exp \left( \frac{1}{\hbar} \left( \alpha \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \right) \right) \]
be the unitary operator on \( L^2(\mathbb{R}) \) which implements translation by \( (\xi, \pi) \) in phase space.

Using the more standard representation of the CCRs instead, we have an immediate corollary from Theorem \textbf{10.16}.

\textbf{Theorem 10.3.} Suppose \( H (\theta, \theta^*) \in \mathbb{R} (\theta, \theta^*) \) is a non-commutative polynomial in two indeterminates, \( d = \text{deg} H > 0 \) and \( 0 < \eta \leq 1 \) satisfying the same assumptions in Theorem \textbf{10.16}. Let \( \hat{H}_h := H (\hat{a}_h, \hat{a}_h^\dagger) \). We define
\[ \hat{A}_h (t) := e^{i \hat{H}_h t / \hbar} \hat{a}_h e^{-i \hat{H}_h t / \hbar} \]
denote \( \hat{a}_h \) in the Heisenberg picture. Furthermore for all \( \psi \in S, \alpha_0 \in \mathbb{C}, 0 < \hbar < \eta \), real numbers \( \{ t_i \}_{i=1}^n \subset \mathbb{R} \), and non-commutative polynomial, \( P (\{ \theta_i, \theta_i^* \}_{i=1}^n) \in \mathbb{C} \langle \{ \theta_i, \theta_i^* \}_{i=1}^n \rangle \), in \( 2n \)-indeterminants where \( p_{\text{min}} \) be the minimum degree of all non constant terms in \( P (\{ \theta_i, \theta_i^* \}_{i=1}^n) \), the following weak limits (in the sense of non-commutative probability) hold;
\[ \left\langle P \left( \{ \hat{A}_h (t_i) - \alpha (t_i) , \hat{A}_h^\dagger (t_i) - \overline{\alpha} (t_i) \}_{i=1}^n \right) \right\rangle_{\hat{U}_h (\alpha_0) S \hskip -1pt \psi} \]
\[ = \left\langle P \left( \{ \sqrt{\hbar} a (t_i) , \sqrt{\hbar} a^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\psi} + O (\hbar^{p_{\text{min}}+1}). \]  \hspace{1cm} (10.1)
where \( a (t) \) and \( a^\dagger (t) \) are as in Eqs. \textbf{(10.8)} and \textbf{(10.9)}.

\textbf{Proof.} By the Lemma \textbf{10.1}, we have
\[ A_h (t) = S_{h^{1/2}} \hat{A}_h (t) S_{h^{-1/2}} \text{ and } \hat{U}_h (\alpha_0) = S_{h^{1/2}} \hat{U}_h (\alpha_0) S_{h^{-1/2}} \]
on \( S \). Therefore,
\[ \left\langle P \left( \{ \hat{A}_h (t_i) - \alpha (t_i) , \hat{A}_h^\dagger (t_i) - \overline{\alpha} (t_i) \}_{i=1}^n \right) \right\rangle_{\hat{U}_h (\alpha_0) S \hskip -1pt \psi} \]
\[ = \left\langle P \left( \{ S_{h^{1/2}} \hat{A}_h (t_i) S_{h^{-1/2}} - \alpha (t_i) , S_{h^{1/2}} \hat{A}_h^\dagger (t_i) S_{h^{-1/2}} - \overline{\alpha} (t_i) \}_{i=1}^n \right) \right\rangle_{S_{h^{1/2}} \hat{U}_h (\alpha_0) S_{h^{-1/2}} \psi} \]
\[ = \left\langle P \left( \{ A_h (t_i) - \alpha (t_i) , A_h^\dagger (t_i) - \overline{\alpha} (t_i) \}_{i=1}^n \right) \right\rangle_{\hat{U}_h (\alpha_0) \psi}. \]
Then, Eq.\textbf{(10.1)} follows by applying Eq. \textbf{(1.23)}. \hspace{1cm} \blacksquare

Likewise we can show two corollaries of Theorem \textbf{10.3} below which behave like Corollaries \textbf{1.18} and \textbf{1.20}.

\textbf{Corollary 10.4.} Under the same notations and assumptions in Theorem \textbf{10.3} then, for \( 0 < \hbar < \eta \), we have
\[ \left\langle P \left( \{ \hat{A}_h (t_i) , \hat{A}_h^\dagger (t_i) \}_{i=1}^n \right) \right\rangle_{\hat{U}_h (\alpha_0) S \hskip -1pt \psi} \]
\[ = \left\langle P \left( \{ \alpha (t_i) + \sqrt{\hbar} a (t_i) , \overline{\alpha} (t_i) + \sqrt{\hbar} a^\dagger (t_i) \} \right) \right\rangle_{\psi} + O (\hbar). \]  \hspace{1cm} (10.2)
Proof. It is a similar proof as Theorem 10.3. Using Lemma 10.1 we can conclude
\[
\left\langle \mathcal{P} \left( \left\{ \hat{A}_h(t_i), \hat{A}_h^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{\hat{U}_h(\alpha_0)S_{\hbar^{-1/2}}\psi} = \left\langle \mathcal{P} \left( \left\{ A_h(t_i), \hat{A}_h^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{U_h(\alpha_0)\psi}.
\]
Then, the rest of the proof is simply to apply Eq. (1.25) and hence, Eq. (10.2) follows.

Corollary 10.5. Under the same notations and assumptions in Theorem 10.3, let \( \hat{\psi}_h = \hat{U}_h(\alpha_0)S_{\hbar^{-1/2}}\psi \). As \( \hbar \to 0^+ \), we have
\[
\left\langle \mathcal{P} \left( \left\{ \hat{A}_h(t_i), \hat{A}_h^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{\hat{\psi}_h} \to \left\langle \mathcal{P} \left( \left\{ \alpha(t_i), \hat{\alpha}^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{\psi}.
\]
and
\[
\left\langle \mathcal{P} \left( \left\{ \frac{\hat{A}_h(t_i) - \alpha(t_i)}{\sqrt{\hbar}}, \frac{\hat{A}_h^\dagger(t_i) - \hat{\alpha}(t_i)}{\sqrt{\hbar}} \right\}_{i=1}^n \right) \right\rangle_{\hat{\psi}_h} \to \left\langle \mathcal{P} \left( \left\{ a(t_i), \hat{a}^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{\psi}.
\]
We abbreviate this convergence by saying
\[
\text{Law}_{\hat{\psi}_h} \left( \left\{ \frac{\hat{A}_h(t_i) - \alpha(t_i)}{\sqrt{\hbar}}, \frac{\hat{A}_h^\dagger(t_i) - \hat{\alpha}(t_i)}{\sqrt{\hbar}} \right\}_{i=1}^n \right) \to \text{Law}_\psi \left( \left\{ a(t_i), \hat{a}^\dagger(t_i) \right\}_{i=1}^n \right).
\]

Proof. Similar to the proof in Theorem 10.3, by using Lemma 10.1 we have
\[
\left\langle \mathcal{P} \left( \left\{ \hat{A}_h(t_i), \hat{A}_h^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{\hat{\psi}_h} = \left\langle \mathcal{P} \left( \left\{ A_h(t_i), \hat{A}_h^\dagger(t_i) \right\}_{i=1}^n \right) \right\rangle_{U_h(\alpha_0)\psi},
\]
and
\[
\left\langle \mathcal{P} \left( \left\{ \frac{\hat{A}_h(t_i) - \alpha(t_i)}{\sqrt{\hbar}}, \frac{\hat{A}_h^\dagger(t_i) - \hat{\alpha}(t_i)}{\sqrt{\hbar}} \right\}_{i=1}^n \right) \right\rangle_{\hat{\psi}_h} = \left\langle \mathcal{P} \left( \left\{ \frac{A_h(t_i) - \alpha(t_i)}{\sqrt{\hbar}}, \frac{\hat{A}_h^\dagger(t_i) - \hat{\alpha}(t_i)}{\sqrt{\hbar}} \right\}_{i=1}^n \right) \right\rangle_{U_h(\alpha_0)\psi}.
\]
Therefore, the corollary is a direct consequence of Corollary 1.20.

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