COMPLEXES OF MODULES OVER EXCEPTIONAL LIE
SUPeralgebras $E(3,8)$ AND $E(5,10)$.

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Abstract. In this paper complexes of generalized Verma modules over the infinite-
dimensional exceptional Lie superalgebras $E(3,8)$ and $E(5,10)$ are constructed and stud-
ied.

0. Introduction.
In our papers [KR1]–[KR3] we constructed all degenerate irreducible modules over the ex-
ceptional Lie superalgebra $E(3,6)$. In the present paper we apply the same method to the
exceptional Lie superalgebras $E(3,8)$ and $E(5,10)$.

The Lie superalgebra $E(3,8)$ is strikingly similar to $E(3,6)$. In particular, as in the
case of $E(3,6)$, the maximal compact subgroup of the group of automorphisms of $E(3,8)$ is
isomorphic to the group of symmetries of the Standard Model. However, as the computer
calculations by Joris van Jeugt show, the fundamental particle contents in the $E(3,8)$ case
is completely different from that in the $E(3,6)$ case [KR2]. All the nice features of the
latter case, like the CPT symmetry, completely disappear in the former case. We believe
that the main reason behind this is that, unlike $E(3,6)$, $E(3,8)$ cannot be embedded in
$E(5,10)$, which, we believe, is the algebra of symmetries of the $SU_5$ Grand Unified Model
(the maximal compact subgroup of the automorphism group of $E(5,10)$ is $SU_5$).

However, similarity with $E(3,6)$ allows us to apply to $E(3,8)$ all the arguments from
[KR2] almost verbatim, and Figure 1 of the present paper that depicts all degenerate $E(3,8)$-
modules is almost the same as Figure 3 from [KR2] for $E(3,6)$.

The picture in the $E(5,10)$ case is quite different (see Figure 2). We believe that it depicts
all degenerate irreducible $E(5,10)$-modules, but we still do not have a proof.

1. Morphisms between generalized Verma modules.
Let $L = \oplus_{j \in \mathbb{Z}} g_j$ be a $\mathbb{Z}$-graded Lie superalgebra by finite-dimensional vector spaces. Let

$$L_\pm = \oplus_{j < 0} g_j, \quad L_+ = \oplus_{j > 0} g_j, \quad L_0 = g_0 + L_+.$$

Given a $g_0$-module $V$, we extend it to a $L_0$-module by letting $L_+$ act trivially, and define
the induced $L$-module

$$M(V) = U(L) \otimes_{U(L_0)} V.$$

If $V$ is a finite-dimensional irreducible $g_0$-module, the $L$-module $M(V)$ is called a generalized Verma module (associated to $V$), and it is called degenerate if it is not irreducible.

* Supported in part by NSF grant DMS-9970007.
Let $A$ and $B$ be two $\mathfrak{g}_0$-modules and let $\text{Hom}(A, B)$ be the $\mathfrak{g}_0$-module of linear maps from $A$ to $B$. The following proposition will be extensively used to construct morphisms between the $L$-modules $M(A)$ and $M(B)$.

**Proposition 1.1.** Let $\Phi \in M(\text{Hom}(A, B))$ be such that
\[ v \cdot \Phi = 0 \quad \text{for all } v \in L_0. \]

Then we can construct a well-defined morphism of $L$-modules
\[ \varphi : M(A) \to M(B) \]
by the rule $\varphi(u \otimes a) = u \Phi(a)$. Explicitly, write $\Phi = \sum u_m \otimes \ell_m$, where $u_m \in U(L)$, $\ell_m \in \text{Hom}(A, B)$. Then
\[ \varphi(u \otimes a) = \sum (uu_m) \otimes \ell_m(a). \]

**Proof.** We have to prove that for $v \in U(L_0)$,
\[ \varphi(uv \otimes a) = \varphi(u \otimes va), \]
in order to conclude that $\varphi$ is well-defined. Notice that condition (1.1) means
\[ \sum \{v, u_m\} \otimes \ell_m + \sum uu_m \otimes v\ell_m = 0. \]
Therefore we have:
\[ \varphi(uv \otimes a) = \sum uu_m \otimes \ell_m(a) = \sum uu_m v \otimes \ell_m(a) = \sum uu_m \otimes (v\ell_m(a)) = \sum uu_m \otimes \ell_m(va) \]
(by (1.3) and (1.2)),
\[ \sum uu_m \otimes \ell_m(va) = \varphi(u \otimes va). \]
The fact that $\varphi$ defines a morphism of $L$-modules is immediate from the definition.

**Remark 1.1.** If $L_0$ is generated by $\mathfrak{g}_0$ and a subset of $T \subset L_+$, then condition (1.1) is equivalent to
\[ g_0 \cdot \Phi = 0 \quad \text{and} \quad a \cdot \Phi = 0 \quad \text{for all } a \in T. \]
Condition (1.1a) usually gives a hint to a possible shape of $\Phi$ and is checked by general invariant-theoretical considerations. After that (1.1b) is usually checked by a direct calculation.

**Remark 1.2.** We can view $M(V)$ also as the induced $(L_- \oplus \mathfrak{g}_0)$-module: $U(L_- \oplus \mathfrak{g}_0) \otimes U(\mathfrak{g}_0) V$. Then condition (1.4a) on $\Phi = \sum u_m \otimes \ell_m$, where $u_m \in U(L_- \oplus \mathfrak{g}_0)$, $\ell_m \in \text{Hom}(A, B)$, suffices in order (1.2) to give a well-defined morphism of $(L_- \oplus \mathfrak{g}_0)$-modules. One can also replace $\mathfrak{g}_0$ by any of its subalgebras.
2. Lie superalgebras $E(3, 6)$, $E(3, 8)$ and $E(5, 10)$.

Recall some standard notation:

\[ W_n = \left\{ \sum_{j=1}^{n} P_j(x) \partial_j \mid P_j \in \mathbb{C}[[x_1, \ldots, x_n]] \right\}, \partial_i \equiv \partial/\partial x_i \]

denotes the Lie algebra of formal vector fields in $n$ indeterminates:

\[ S_n = \left\{ D = \sum_{i=1}^{n} P_i \partial_i \mid div D = \sum_{i} \partial_i P_i = 0 \right\} \]

denotes the Lie subalgebra of divergenceless formal vector fields; $\Omega^k(n)$ denotes the associative algebra of formal differential forms of degree $k$ in $n$ indeterminates, $\Omega^k_{\text{cl}}(n)$ denoted the subspace of closed forms.

The Lie algebra $W_n$ acts on $\Omega^k(n)$ via Lie derivative $D \to L_D$. Given $\lambda \in \mathbb{C}$ one can twist this action:

\[ D \omega = L_D \omega + \lambda(\text{div}D) \omega . \]

The $W_n$-module thus obtained is denoted by $\Omega^k(n)^{\lambda}$. Recall the following obvious isomorphism of $W_n$-modules

\[ \Omega^0(n) \cong \Omega^{n}(n)^{-1}, \quad (2.1) \]

and the following slightly less obvious isomorphism of $W_n$-modules

\[ W_n \cong \Omega^{n-1}(n)^{-1}. \quad (2.2) \]

The latter is obtained by mapping a vector field $D \in W_n$ to the $(n-1)$-form $\iota_D(dx_1 \wedge \ldots \wedge dx_n)$.

Note that $(2.2)$ induces an isomorphism of $S_n$-modules:

\[ S_n \cong \Omega^{n-1}_{\text{cl}}(n). \quad (2.3) \]

Recall that the Lie superalgebra $E(5, 10) = E(5, 10)_0 + E(5, 10)_1$ is constructed as follows

\[ E(5, 10)_0 = S_5, \quad E(5, 10)_1 = \Omega^2_{\text{cl}}(5), \]

$E(5, 10)_0$ acts on $E(5, 10)_1$ via the Lie derivative, and $[\omega_2, \omega_2'] = \omega_2 \wedge \omega_2' \in \Omega^4_{\text{cl}}(5) = S_5$ (see $(2.3)$) for $\omega_2, \omega_2' \in E(S, 10)_1$.

Next, recall the construction of the Lie superalgebras $E^\rho := E(3, 6)$ and $E^d := E(3, 8)$

\[ E^\rho_0 = E^d_0 = W_3 + sl_2(\Omega^0(3)) \quad (\text{the natural semidirect sum}) ; \]
\[ E^\rho_1 = \Omega^1(3)^{-1/2} \otimes \mathbb{C}^2, \quad E^d_1 = (\Omega^0(3)^{-1/2} \otimes \mathbb{C}^2) + (\Omega^2(3)^{-1/2} \otimes \mathbb{C}^2). \]

The action of the even on the odd parts is defined via the Lie derivative and the multiplication of a function and a differential form. The bracket of two odd elements is defined by using the identifications $(2.3)$ and $(2.2)$ as follows. For $\omega_i, \omega'_i \in \Omega^i(3)$ and $v, v' \in \mathbb{C}^2$ one defines the following bracket of two elements from $E^\rho_1$:

\[ [\omega_1 \otimes v, \omega'_1 \otimes v'] = -(\omega_1 \wedge \omega'_1) \otimes (v \wedge v') - (d\omega_1 \wedge \omega'_1 + \omega_1 \wedge d\omega'_1) \otimes (v \cdot v'), \quad (2.4) \]

and the following bracket of two elements from $E^d_1$:

\[ [\omega_0 \otimes v, \omega'_0 \otimes v'] = 0, \quad [\omega_0 \otimes v, \omega'_0 \otimes v'] = -(d\omega_0 \wedge d\omega'_0) \otimes (v \wedge v'), \]
\[ [\omega_0 \otimes v, \omega_2 \otimes v'] = -(\omega_0 \wedge \omega_2) \otimes (v \wedge v') - (d\omega_0 \wedge \omega_2 - \omega_0 \wedge \omega_2) \otimes (v \cdot v'). \quad (2.5) \]
Recall also an embedding of $E^\flat$ in $E(5, 10)$ \cite{CK}, \cite{KR2}. For that let $z_+ = x_4$, $z_- = x_5$, \( \partial_+ = \partial_1 \), \( \partial_- = \partial_5 \), and let \( \epsilon^+, \epsilon^- \) denote the standard basis of \( \mathbb{C}^2 \). Then \( E^\flat_0 \) is embedded in $E(5, 10)_0 = S_5$ by \((D \in W_3, a, b, c \in \Omega^0(3))\):

\[
(2.6) \quad D \mapsto D - \frac{1}{2}(\text{div}D)(z_+ \partial_+ + z_- \partial_-), \quad \left( \begin{array}{ccc}
 a & b \\
 c & -a
\end{array} \right) \mapsto a(z_+ \partial_+ - z_- \partial_-) + b z_+ \partial_- + c z_- \partial_+,
\]
and \( E^\flat_1 \) is embedded in $E(5, 10)_1 = \Omega^2_{c\ell}(5)$ by \((f \in \Omega^0(3))\):

\[
(2.7) \quad f dx_i \otimes \epsilon^+ \mapsto z_+ dx_i \wedge df + f dx_i \wedge dz_+.
\]

Introduce the following subalgebras \( S^\flat_0 \subset E^\flat \) and \( S^\flat_1 \subset E^\flat_1 \):

\[
S^\flat_0 = S^\flat_0 = W_3 + \mathbb{C} \otimes sl_2(\mathbb{C}),
S^\flat_1 = \Omega^1_{c\ell}(3)^{-1/2} \otimes \mathbb{C}^2, \quad S^\flat_1 = \Omega^0(3)^{-1/2} \otimes \mathbb{C}^2.
\]

**Proposition 2.1.** The map \( S^\flat \rightarrow S^\flat_0 \), which is identical on \( S^\flat_0 \) and sends \( f \otimes v \in S^\flat_1 \) to \( df \otimes v \in S^\flat_0 \) is a surjective homomorphism of Lie superalgebras with 2-dimensional central kernel \( \mathbb{C} \otimes \mathbb{C}^2 \subset S^\flat_1 \).

**Proof.** It is straightforward using (2.4) and (2.3). \( \square \)

This proposition is probably the main reason for a remarkable similarity between representation theories of \( E^\flat \) and \( E^\flat_0 \). We shall stress this similarity in our notation and develop representation theory of \( E^\flat \) along the same lines as that of \( E^\flat_0 \), done in \cite{KR1}, \cite{KR2}. Sometimes we shall drop the superscript \( \flat \) or \( \sharp \) when the situation is the same.

Recall that \( E^\flat \) carries a unique irreducible consistent \( \mathbb{Z} \)-gradation. It has depth 2, and it is defined by:

\[
(2.8) \quad \deg x_i = -\deg \partial_i = 2, \quad \deg \partial_i = -1, \quad \deg \epsilon^\pm = 0, \quad \deg sl_2(\mathbb{C}) = 0.
\]

The “non-positive” part of this \( \mathbb{Z} \)-gradation is as follows:

\[
\begin{align*}
g_{-2} &= \langle \partial_i | i = 1, 2, 3 \rangle, \quad g_{-1} = \langle d^\flat_i := \epsilon^a \otimes dx_i = dx_i \wedge dz_a | i = 1, 2, 3, a = +, - \rangle, \\
g_0 &= sl_3(\mathbb{C}) \oplus sl_2(\mathbb{C}) \oplus CY,
\end{align*}
\]

where

\[
\begin{align*}
(2.9) \quad sl_3(\mathbb{C}) &= \langle h_1 = x_1 \partial_1 - x_2 \partial_2, h_2 = x_2 \partial_2 - x_3 \partial_3, e_1 = x_1 \partial_2, \quad e_2 = x_2 \partial_3, e_{12} = x_1 \partial_3, f_1 = x_2 \partial_1, f_2 = x_3 \partial_2, f_{12} = x_3 \partial_1 \rangle, \\
(2.10) \quad sl_2(\mathbb{C}) &= \langle h_3 = z_+ \partial_+ - z_- \partial_-, e_3 = z_+ \partial_-, f_3 = z_- \partial_+ \rangle, \\
(2.11) \quad Y &= \frac{4}{3} \sum_{i} x_i \partial_i - (z_+ \partial_+ + z_- \partial_-).
\end{align*}
\]

The eigenspace decomposition of \( \text{ad}(3Y) \) coincides with the consistent \( \mathbb{Z} \)-grading of \( E^\flat \).

We fix the Cartan subalgebra \( \mathcal{H} = \langle h_1, h_2, h_3, Y \rangle \) and the Borel subalgebra \( \mathcal{B} = \mathcal{H} + \langle e_i, \ i = 1, 2, 3, \ e_{12} \rangle \) of \( g_0 \). Then \( f_0 := d^\flat_1 \) is the highest weight vector of the (irreducible) \( g_0 \)-module \( g_{-1} \), the vectors

\[
e_0 := x_3 d^-_3 \quad \text{and} \quad e_0 := x_3 d^-_3 - x_2 d^-_3 + 2 z_- dx_2 \wedge dx_3
\]
are all the lowest weight vectors of the \( g_0 \)-module \( g_1 \), and one has:

\[
(2.12) \quad [e_0', f_0] = f_2,
\]

\[
(2.13) \quad [e_0', f_0] = \frac{2}{3}h_1 + \frac{1}{3}h_2 - h_3 - Y =: h_0^b,
\]

so that

\[
h_0^b = - x_2 \partial_2 - x_3 \partial_3 + 2 z_- \partial_-
\]

The following relations are also important to keep in mind:

\[
(2.14) \quad [e_0', d_1^\pm] = f_2, \quad [e_0', d_2^\pm] = - f_{12}, \quad [e_0', d_3^\pm] = 0, \quad [e_0', d_i^-] = 0,
\]

\[
(2.15) \quad [d_i^+, d_j^+] = 0, \quad [d_i^+, d_j^-] + [d_j^+, d_i^-] = 0.
\]

Recall that \( g_0 \) along with the elements \( f_0, e_0^\pm, e_0' \) generate the Lie superalgebra \( E^\natural [\text{CK}] \).

The Lie superalgebra \( E^\natural \) carries a unique consistent irreducible \( \mathbb{Z} \)-gradation of depth 3:

\[
E^\natural = \oplus_{j \geq -3} g_j.
\]

It is defined by:

\[
(2.16) \quad \deg x_i = - \deg \partial_i = \deg dx_i = 2, \quad \deg e^\pm = -3, \quad \deg s\ell_2(\mathbb{C}) = 0.
\]

In view of Proposition 2.1 and the above \( E^\natural \)-notation, we introduce the following \( E^\natural \)-notation:

\[
(2.17) \quad d^\pm := 1 \otimes e^\pm, \quad d_i^\pm := x_i \otimes e^\pm, \quad e_0^\pm := \frac{1}{2}x_3^2 \otimes e^\pm, \quad f_0 := d_1^+,
\]

\[
\epsilon_0^\pm := -(dx_2 \wedge dx_3) \otimes \epsilon^-, \quad h_0^\natural := \frac{2}{3}h_1 + \frac{1}{3}h_2 - \frac{1}{2}h_3 + \frac{1}{2}Y.
\]

If, in analogy with \( E^\natural \), we denote \( e^a = dz_a \), then \( g_{-3} = (dz_+, dz_-) \cdot g_{-2}, g_{-1} \) and \( g_0 \) are the same as for \( E^\natural \) (except that now \( [g_{-1}, g_{-2}] \neq 0 \)), and the relations (2.12), (2.13), (2.14), (2.15) still hold, but the formula for \( h_0^\natural \) is different:

\[
(2.18) \quad h_0^\natural = \frac{2}{3}h_1 + \frac{1}{3}h_2 - \frac{1}{2}h_3 + \frac{1}{2}Y = x_1 \partial_1 - z_- \partial_+.
\]

As in the \( E^\natural \) case, the elements \( e_i, f_i, h_i \) for \( i = 0, 1, 2, 3 \) along with \( e_0', d_0^\pm \) generate \( E^\natural \), the elements \( e_0^\pm \) and \( e_0' \) are all lowest weight vectors of the \( g_0 \)-module \( g_1 \), and \( g_0 \) along with \( e_0' \) and \( e_0^\pm \) generate the subalgebra \( \oplus_{j \geq 0} g_j [\text{CK}] \). Thus, by Remark 1.1, condition (1.4b) is equivalent to

\[
(2.19) \quad e_0' \cdot \Phi = 0 \quad \text{and} \quad e_0^\pm \cdot \Phi = 0.
\]

3. Complexes of degenerate Verma modules over \( E(3, 6) \).

Let \( W \) be a finite-dimensional symplectic vector space and let \( H \) be the corresponding Heisenberg algebra

\[
H = T(W)/(v \cdot w - w \cdot v - (v, w) \cdot 1)
\]

where \( T(W) \) denotes the tensor algebra over \( W \) and \( (\cdot, \cdot) \) is the non-degenerate symplectic form on \( W \). Given two transversal Lagrangian subspaces \( L, L' \subset W \), we have a canonical isomorphism of symplectic spaces: \( W = L + L' \simeq L \oplus L' \), and we can canonically identify the symmetric algebra \( S(L) \) with the factor of \( H \) by the left ideal generated by \( L' \):

\[
(3.1) \quad V_L := H/(L') \simeq S(L)1_L, \quad \text{where} \ 1_L = 1 + L'.
\]

One thus acquires an \( H \)-module structure on \( S(L) \).
We construct a symplectic space \( W \) by taking \( x_1, x_2, x_3, z_+, z_-, \partial_1, \partial_2, \partial_3, \partial_+, \partial_- \) as a basis with the first half being dual to the second half:

\[
(\partial_i, x_j) = -(x_j, \partial_i) = \delta_{ij}, \quad (\partial_a, z_b) = -(z_b, \partial_a) = \delta_{a,b}, \quad \text{all other pairings zero.}
\]

In general the decomposition \( W = L + L' \cong L \oplus L^* \) provides the canonical maps:

\[
\text{End}(L) \xrightarrow{\sim} L \otimes L^* \xrightarrow{\sim} L \cdot L' \hookrightarrow H,
\]

which induce a Lie algebra homomorphism: \( \mathfrak{g}(L) \to H_{\text{Lie}} \), where the Lie algebra structure is defined by the usual commutator.

We consider the following subspaces of \( W \):

\[
(3.3) \quad L_A = L'_D = \langle x_i, z_a \rangle, \quad L_B = L'_C = \langle x_i, \partial_a \rangle, \quad L_C = L'_B = \langle \partial_c, z_a \rangle, \quad L_D = L'_A = \langle \partial_i, \partial_a \rangle,
\]

where \( i = 1, 2, 3 \), \( a = +, - \). Note that these are the only \( \mathfrak{g}_0 \)-invariant Lagrangian subspaces of \( W \).

As formulae \([2.3] - [2.11]\) determine the inclusion \( \mathfrak{g}_0 \hookrightarrow \mathfrak{g}(V_X) \), where \( X = A, B, C \) or \( D \), we get a Lie algebra monomorphism:

\[
(3.4) \quad \mathfrak{g}_0 \hookrightarrow \mathfrak{g}(V_X) \hookrightarrow H_{\text{Lie}}.
\]

Thus we get a \( \mathfrak{g}_0 \)-action on \( V_X \). Let us notice that by \((3.3)\)

\[
(3.5) \quad Y \mapsto Y^b = 3 \left( \sum_i x_i \partial_i \right) - \left( \sum_a z_a \partial_a \right)
\]

and

\[
Y^b 1_A = 0, \quad Y^b 1_B = 2 1_B, \quad Y^b 1_C = -2 1_C, \quad Y^b 1_D = 0,
\]

as it should be for \( E^p = E(3, 6) \) (see [KR2]).

In the \( E(3, 8) \) case we modify the \( \mathfrak{g}_0 \)-action on \( V_X \) leaving it the same for \( s\ell_3(\mathbb{C}) \oplus s\ell_2(\mathbb{C}) \subset \mathfrak{g}_0 \), but letting

\[
(3.6) \quad Y \mapsto Y^d = -\frac{4}{3} \left( \sum_i x_i \partial_i \right) + \left( \sum_a z_a \partial_a \right) = Y^p + 2T, \quad \text{where } T = -\sum_i x_i \partial_i + \sum_a z_a \partial_a.
\]

We have got:

\[
Y x_i^p z_a^r 1_A = -\left( \frac{4}{3} p + r \right) x_i^p z_a^r 1_A,
\]

\[
Y x_i^p \partial_a^r 1_B = \left( \frac{4}{3} p - r - 2 \right) x_i^p \partial_a^r 1_B,
\]

\[
Y \partial_i^p z_a^r 1_C = \left( \frac{4}{3} q + r + 4 \right) \partial_i^p z_a^r 1_C,
\]

\[
Y \partial_i^p \partial_a^r 1_D = \left( \frac{4}{3} r - 2 \right) \partial_i^p \partial_a^r 1_D.
\]

Let \( F(p, q; r; y) \) denote the finite-dimensional irreducible \( \mathfrak{g}_0 \)-module with highest weight \( (p, q; r; y) \), where \( p, q, r \in \mathbb{Z}_+, \ y \in \mathbb{C} \), and let \( M(p, q; r; y) = M(F(p, q; r; y)) \) be the corresponding generalized Verma module over \( E(3, 8) \). This module has a unique irreducible quotient denoted by \( I(p, q; r; y) \). The latter module is called degenerate if the former is.

We announce below a classification of all degenerate irreducible \( E(3, 8) \)-modules.
Theorem 3.1. All irreducible degenerate $E(3,8)$-modules $I(p, q; r, y)$ are as follows $(p, q, r \in \mathbb{Z}^+_0)$:

- type $A$: $I(p, 0; y_A)$, $y_A = -\frac{1}{3}p + r$;
- type $B$: $I(p, 0; y_B)$, $y_B = -\frac{1}{3}p - r - 2$;
- type $C$: $I(0, q; y_C)$, $y_C = \frac{1}{4}q + r + 4$;
- type $D$: $I(0, q; y_D)$, $y_D = \frac{1}{2}q - r + 2$, and $(q, r) \neq (0, 0)$.

We shall construct below certain $E(3,8)$-morphisms between the modules $M(p, q; r, y_X)$. This will imply that all modules $I(p, q; r, y)$ on the list are degenerate. The proof of the fact that the list is complete will be published elsewhere.

The theorem means that all the degenerate generalized Verma modules over $E(3,8)$ are in fact the direct summands of induced modules $M(V_X)$, $X = A, B, C, D$:

\[ M(V_X) = \bigoplus_{m, n \in \mathbb{Z}} M(V_X^{m,n}), \]

\[ V_X^{m,n} = \left\{ f 1_X \mid (\sum x_i \partial_i) f = mf, (\sum z_a \partial_a) f = nf \right\}, \]

(we normalize degree of $1_X$ as $(0,0)$).

We construct morphisms between these modules with the help of Proposition 3.2. As in the $E^8$ case [KR2], introduce the following operators on $M(\text{Hom}(V_X, V_X))$:

\[ \nabla = \Delta^+ \delta_+ + \Delta^- \delta_- = \delta_1 \partial_1 + \delta_2 \partial_2 + \delta_3 \partial_3, \]

\[ \Delta^\pm = \sum_{i=1}^3 d_i^\pm \otimes \partial_i, \delta_i = \sum_{a=\pm} d_i^a \otimes \partial_a. \]

Proposition 3.2. (a) The element $\nabla$ gives a well-defined morphism $M(V_X) \to M(V_X)$, $X = A, B, C, D$, by formula [1.2].
(b) $\nabla^2 = 0$.

Proof. The proof of (b) is the same as in [KR2]. In order to prove (a), we have to check conditions (1.4a) and (1.4b). It is obvious that $\Delta^\pm$ (resp. $\delta_i$) are $\mathfrak{s}\mathfrak{l}_3(\mathbb{C})$-(resp. $\mathfrak{s}\mathfrak{l}_2(\mathbb{C})$-) invariant. Using both formulas for $\nabla$, we conclude that it is $g_0$-invariant, proving (1.4a). In order to check (1.4b), first note that

\[ e_0^\nabla = (f_2 \partial_1 \partial_+ - f_1 \partial_2 \partial_+) = (x_3 \partial_2 \partial_1 - x_3 \partial_1 \partial_2) \partial_+ = 0. \]

Now

\[ e_0^\nabla = h_0^2 \partial_1 \partial_+ + f_1 \partial_2 \partial_+ + f_2 \partial_3 \partial_1 - f_3 \partial_1 \partial_- = (x_1 \partial_1 - z_+ \partial_+) \partial_+ + x_2 \partial_1 \partial_2 \partial_+ + x_3 \partial_1 \partial_3 \partial_+ - z_- \partial_1 \partial_- = (x_1 \partial_1 - z_+ \partial_+ + T + x_2 \partial_2 + x_3 \partial_3 - z_- \partial_-) \partial_+ = 0, \]

where we use (1.4a) to check that

\[ h_0^2 = \frac{1}{2}h_1 + \frac{3}{2}h_2 - \frac{1}{2}h_3 + \frac{1}{2}V^2 = x_1 \partial_1 - z_+ \partial_+ + T. \]

Let $M_{X'} = M(V_{X'})$, where:

\[ V_{A'} = \mathbb{C}[x_1, x_2, x_3]1_A = \oplus_{p \geq 0} V^{p,0}_A, \quad V_{B'} = \mathbb{C}[x_1, x_2, x_3]1_B = \oplus_{p \geq 0} V^{p,0}_B, \]

\[ V_{C'} = \mathbb{C}[\partial_1, \partial_2, \partial_3]1_C = \oplus_{q \geq 0} V^{-q,0}_C, \quad V_{D'} = \mathbb{C}[\partial_1, \partial_2, \partial_3]1_D = \oplus_{q \geq 0} V^{-q,0}_D. \]
We construct morphisms \( \nabla_2 : M_{A'} \to M_{B'} \) by extending the map defined by \( \nabla_2 \) as follows

\[ V_{A'} \xrightarrow{\sim} \mathbb{C}[x_1, x_2, x_3] \to U(L_-) \otimes \mathbb{C}[x_1, x_2, x_3] 1_B \cong U(L_-) \otimes V_{B'}, \]

We define the morphism \( \nabla_2 : M_{A'} \to M_{B'} \) by extending the map defined by \( \nabla_2 \) as follows

\[ f 1_A \mapsto f \mapsto \nabla_2 f 1_B. \]

In order to apply Proposition 1.1, we have to check conditions (1.4a) and (1.4b) for \( \nabla_2 \). As before, condition (1.4a) obviously holds. Now

\[ e_0^i \nabla_2 1_B = - \sum_i d_i^i (f_2 \partial_1 \partial_{-} - f_{12} \partial_2) \partial_i 1_B = 0 \]

because \( f_2 \partial_1 - f_{12} \partial_2 = x_3 \partial_2 \partial_1 - x_3 \partial_1 \partial_2 = 0 \). Furthermore:

\[
e_0^i \nabla_2 f 1_B = \left( - \sum_i d_i^i (h_i^1 \partial_1 + f_1 \partial_2 + f_{12} \partial_3) \partial_i - f_3 \Delta^+ \partial_1 \right) f 1_B
\]

\[ = - \left( \sum_i d_i^i (x_1 \partial_1 - z_+ \partial_+ + T + x_2 \partial_2 + x_3 \partial_3) \partial_i \partial_2 - \Delta^- \partial_1 - \Delta^+ \partial_1 \right) f 1_B. \]

As \( z_+ \partial_+ f 1_B = f z_+ \partial_+ 1_B = f \left( -1 + \partial_+ z_+ \right) 1_B = -f 1_B \), and \( f_3 f 1_B = 0 \), we conclude that

\[ -e_0^i \nabla_2 f 1_B = \sum_i d_i^i (x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 - z_+ \partial_+ + T + 1) f 1_B
\]

\[ = \sum_i d_i^i (z_+ \partial_+ + 1) f 1_B = 0. \]

Thus Proposition 1.1 applies and we get the morphism \( \nabla_2 : M_{A'} \to M_{B'} \).

In exactly the same fashion we construct the morphism \( \nabla_2 : M_{C'} \to M_{D'} \) by taking \( f \in \mathbb{C}[\partial_1, \partial_2, \partial_3] \).

Thus we have proved part (a) of the following proposition; part (b) was checked in [KR2].

**Proposition 3.3.** (a) Formulae (3.9) and (3.10) define the morphisms \( \nabla_2 : M_{A'} \to M_{B'} \), \( \nabla_2 : M_{C'} \to M_{D'} \) of \( E(3,8) \)-modules.

(b) \( \nabla \nabla_2 = 0, \nabla_2 \nabla = 0 \).

Similarly we can construct morphisms with the help of the element

\[ \nabla_3 = \delta_1 \delta_2 \delta_3 = \sum_{a,b,c} d_1^a d_2^b d_3^c \partial_a \partial_b \partial_c, \quad (a, b, c = +, -). \]

Let us consider modules \( M_{X'} = M(V_{X'}) \), where

\[
V_{A'} = \mathbb{C}[z_+, z_-] 1_A = \oplus_{r \geq 0} V_{A'}^0 r, \quad V_{B'} = \mathbb{C}[\partial_+ \partial_-] 1_B = \oplus_{r \geq 0} V_{B'}^0 -r,
\]

\[
V_{C'} = \mathbb{C}[z_+, z_-] 1_C = \oplus_{r \geq 0} V_{C'}^0 r, \quad V_{D'} = \mathbb{C}[\partial_+ \partial_-] 1_D = \oplus_{r \geq 0} V_{D'}^0 -r.
\]

We construct morphisms \( \nabla_3 : M_{A'} \to M_{C'} \) and \( \nabla_3 : M_{B'} \to M_{D'} \) by extending the maps

\[ \nabla_3 : V_{A'} \to U(L_-) \otimes V_{C'}, \quad \nabla_3 : V_{B'} \to U(L_-) \otimes V_{D'}, \]

given by the left and right diagrams below:

\[
\begin{array}{ccc}
\mathbb{C}[z_a] \cdot 1_A & \xrightarrow{\sim} & \mathbb{C}[z_+, z_-] \cdot 1_B \\
\downarrow_{\nabla_3} & & \downarrow_{\nabla_3} \\
U(L_-) \otimes (\mathbb{C}[z_+, z_-] 1_C) & \xrightarrow{\sim} & U(L_-) \otimes (\mathbb{C}[\partial_+, \partial_-] 1_D).
\end{array}
\]
Here the horizontal maps are naturally $s\ell_3(\mathbb{C}) \oplus s\ell_2(\mathbb{C})$-isomorphisms, but we have to define the action of $Y$ on the target demanding the map to be a $g_0$-isomorphism. With this in mind we have to check that $Y$ commutes with $\triangledown_3$.

We shall use the Einstein summation convention argument for the vertical maps $\triangledown_3$ given by (3.12). Then for $f \in \mathbb{C}[z_+, z_-]$ we have $Y(f 1_A) = (\deg f)f 1_A$ and

$$Y(d_1^t d_2^t d_3^t \otimes \partial_a \partial_b \partial_c f 1_C) = d_1^t d_2^t d_3^t (-1 + Y^2) \partial_t \partial_b \partial_c f 1_C$$

$$= d_1^t d_2^t d_3^t \otimes \partial_a \partial_b \partial_c f (-4 + \deg f + Y^2) 1_C$$

$$= (\deg f) \cdot d_1^t d_2^t d_3^t \otimes \partial_a \partial_b \partial_c f 1_C.$$

Similarly for $f \in \mathbb{C}[\partial_+, \partial_-]$, we have $Y(f 1_B) = (- \deg f - 2)f 1_B$ and

$$Y(d_1^t d_2^t d_3^t \otimes \partial_a \partial_b \partial_c f 1_B) = d_1^t d_2^t d_3^t \otimes \partial_a \partial_b \partial_c f (-4 - \deg f + Y^2) 1_B$$

$$= (-4 - \deg f + 2)d_1^t d_2^t d_3^t \otimes \partial_a \partial_b \partial_c f 1_B,$$

where $Y^2$ defined by (3.6). Thus we get the commutativity.

One meets no problem checking $e_0^t \triangledown_3 = 0$, but we consider calculations for $e_0^t \triangledown_3$ in more detail. If $f \in \mathbb{C}[z_+, z_-]$, then

$$e_0^t(\triangledown_3 f 1_C) = h_0^t d_0^t d_3^t \otimes \partial_+ \partial_b \partial_c f 1_C - f_3 d_2^t d_3^t \otimes \partial_- \partial_b \partial_c f 1_C,$$

because $d_1^t (x_2 \partial_1) d_3^t \otimes \partial_a \partial_b \partial_c f 1_C = d_1^t d_2^t (x_3 \partial_1) \otimes \partial_a \partial_b \partial_c f 1_C = 0$. Now $f_3(d_2^t d_3^t \otimes \partial_a \partial_b \partial_c) = (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c) f_3$ and $h_0^t (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c) = (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c) (h_0^t - 2)$, where $h_0^t = \frac{2}{3} h_1 + \frac{1}{3} h_2 - \frac{1}{6} h_3 + \frac{1}{6} Y^2$, and again $Y^2$ is defined by (3.6). Therefore

$$e_0^t(\triangledown_3 f 1_C) = (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c)(h_0^t \partial_+ - 2 \partial_+ - z_+ \partial_+ \partial_-) f 1_C$$

$$= (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c)(h_0^t - z_+ \partial_+ \partial_- - 2) \partial_+ f 1_C$$

$$= (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c) (-x_2 \partial_2 - x_3 \partial_3 - 2) \partial_+ f 1_C$$

$$= (d_2^t d_3^t \otimes \partial_a \partial_b \partial_c) (-x_2 \partial_2 - x_3 \partial_3) \partial_+ f 1_C = 0.$$

The calculations in the case $f \in \mathbb{C}[\partial_+, \partial_-]$ are very much the same. So we have proved part (a) of the following proposition; part (b) was checked in [KR2].

**Proposition 3.4.** (a) Formulae (3.14) and (3.12) define the morphisms $\triangledown_3 : M_{A''} \to M_{D''}$ and $M_{B''} \to M_{D''}$ of $E(3,8)$-modules.

(b) $\triangledown \cdot \triangledown_3 = 0, \quad \triangledown_3 \cdot \triangledown = 0$.

Furthermore, there are $E(3,8)$-module morphisms

$$\triangledown_4' : M(00; 2; y_A) \to M(01; 1; y_D) \quad \text{and}$$

$$\triangledown_4'' : M(10; 0; y_A) \to M(00; 2; y_D)$$

defined by formulae (2.14) and (2.17) from [KR2], applied to $E(3,8)$. Arguments similar to those in [KR2] show that these are indeed well-defined morphisms.

Thus far we have constructed $E(3,8)$-homomorphisms $\triangledown, \triangledown_2, \triangledown_3, \triangledown_4'$, and $\triangledown_4''$ between generalized Verma modules. Note that these maps have degree 1, 2, 3 and 4, respectively with respect to the $Z$-gradation of $U(L_-)$ induced by that of $E(3,8)$.

As in the case of $E(3,6)$ [KR2], all these maps are illustrated in Figure 1. The nodes in the quadrants A,B,C,D represent generalized Verma modules $M(p,0;r;y_X)$ if $X = A$ or $B$, and $M(0,q;r;y_X)$, if $X = C$ or $D$. The plain arrows represent $\triangledown$, the dotted arrows represent $\triangledown_2$, the interrupted arrows represent $\triangledown_3$ and the bold arrows represent $\triangledown_4'$ and $\triangledown_4''$. 
Note that the generalized Verma modules $M(00; 1; y_A)$ and $M(00; 1; y_D)$ are isomorphic since $y_A = y_D = 1$. We shall identify them. This allows us to construct the $E(3, 8)$-module homomorphism

$$\tilde{\nabla} : M(00; 1; y_A) \to M(01; 2; y_D),$$

which is NOT represented in Figure 1.

Note that $I(00; 1; y_A) = I(00; 1; y_D)$ is the coadjoint $E(3, 8)$-module. It follows from the above propositions that if we remove the module $M(00; 1; y_D)$ from Figure 1, and draw $\tilde{\nabla}$ then all sequences in the modified Figure 1 become complexes. We denote by $H^p_A$, $H^p_B$, $H^q_C$, and $H^q_D$ the homology of these complexes at the position of $M(pq; r; y_X)$, $X = A, B, C, D$.

**Theorem 3.5.** (a) The kernels of all maps $\nabla$, $\nabla_2$, $\nabla_3$, $\nabla_4'$, $\nabla_4''$, $\tilde{\nabla}$ are maximal submodules.
(b) The homology $H_{\lambda}^{m,n}$ is zero except for six cases listed (as $E(3,8)$-modules) below:
\[
H_A^{0,0} = \mathbb{C}, \quad H_A^{1,1} = I(10; 0; -\frac{4}{3}),
\]
\[
H_A^{1,0} = H_D^{0,-2} = I(00; 0; -2),
\]
\[
H_D^{1,-1} = H_D^{1,-2} = I(00; 1; 1) \oplus \mathbb{C}.
\]

The proof is similar to that of the analogous $E(3,6)$-result in $[KR2]$. Note that this theorem gives the following explicit construction of all degenerate irreducible $E(3,8)$-modules:

\[
I(pq; r; yX) = M(pq; r; yX)/ \text{Ker } \nabla,
\]

where $\nabla$ is the corresponding map in the modified Figure 1.

4. Three series of degenerate Verma modules over $E(5,10)$

As in $[KR2]$ and in §3, we use for the odd elements of $E(5,10)$ the notation $d_{ij} = dx_i \wedge dx_j$ ($i, j = 1, 2, \ldots, 5$); recall that we have the following commutation relation ($f, g \in \mathbb{C}[x_1, \ldots, x_5]$):

\[
[f_{jk}, gd_{lm}] = \epsilon_{ijk\ell m} \partial_i,
\]

where $\epsilon_{ijk\ell m}$ is the sign of the permutation $ijk\ell m$ if all indices are distinct and 0 otherwise.

Recall that the Lie superalgebra $E(5,10)$ carries a unique consistent irreducible $\mathbb{Z}$-gradation $E(5,10) = \oplus_{j \geq -2} p_j$. It is defined by:

\[
\deg x_i = 2 = - \deg \partial_i, \quad \deg d_{ij} = -1.
\]

One has: $p_0 \simeq sl(5,\mathbb{C})$ and the $p_0$-modules occurring in the $L_-$ part are:

\[
p_{-1} = \langle d_{ij} \mid i, j = 1, \ldots, 5 \rangle \simeq \Lambda^2 \mathbb{C}^5,
\]

\[
p_{-2} = \langle \partial_i \mid i = 1, \ldots, 5 \rangle \simeq \mathbb{C}^{5*}.
\]

Recall also that $p_1$ consist of closed 2-forms with linear coefficients, that $p_1$ is an irreducible $p_0$-module and $p_j = p_j|_{p_0}$ for $j \geq 1$.

We take for the Borel subalgebra of $p_0$ the subalgebra of the vector fields $\langle \sum_{i \leq j} a_{ij} x_i \partial_j \rangle_{a_{ij} \in \mathbb{C}, \text{tr}(a_{ij}) = 0}$, and denote by $F(m_1, m_2, m_3, m_4)$ the finite-dimensional irreducible $p_0$-module with highest weight $(m_1, m_2, m_3, m_4)$. We let

\[
M(m_1, m_2, m_3, m_4) = M(F(m_1, m_2, m_3, m_4))
\]

denote the corresponding generalized Verma module over $E(5,10)$.

**Conjecture 4.1.** The following is a complete list of generalized Verma modules over $E(5,10)$ $(m, n \in \mathbb{Z}_+)$:

\[
M(m, n, 0, 0), M(0, 0, m, n) \text{ and } M(m, 0, 0, n).
\]

In this section we construct three complexes of generalized $E(5,10)$ Verma modules which shows, in particular, that all modules from the list given by Conjecture 4.1 are degenerate. We let:

\[
S_A = S(\mathbb{C}^5 + \Lambda^2 \mathbb{C}^5), \quad S_B = S(\mathbb{C}^{5*} + \Lambda^2 \mathbb{C}^{5*}), \quad S_C = S(\mathbb{C}^5 + \mathbb{C}^{5*}).
\]

Denote by $x_i$ ($i = 1, \ldots, 5$) the standard basis of $\mathbb{C}^5$ and by $x_{ij} = - x_{ji}$ ($i, j = 1, \ldots, 5$) the standard basis of $\Lambda^2 \mathbb{C}^5$. Let $x_i^*$ and $x_{ij}^*$ be the dual bases of $\mathbb{C}^{5*}$ and $\Lambda^2 \mathbb{C}^{5*}$, respectively. Then $S_A$ is the polynomial algebra in 15 indeterminates $x_i$ and $x_{ij}$, $S_B$ is the
polynomial algebra in 15 indeterminates \( x_i^* \) and \( x_j^* \) and \( S_C \) is the polynomial algebra in 10 indeterminates \( x_i \) and \( x_j^* \).

Given two irreducible \( p_0 \)-modules \( E \) and \( F \), we denote by \( (E \otimes F)_{\text{high}} \) the highest irreducible component of the \( p_0 \)-module \( E \otimes F \). If \( E = \oplus_i E_i \) and \( F = \oplus_j F_j \) are direct sums of irreducible \( p_0 \)-modules, we let \( (E \otimes F)_{\text{high}} = \oplus_{i,j}(E_i \otimes F_j)_{\text{high}} \). If \( E \) and \( F \) are again irreducible \( p_0 \)-modules, then \( S(E \oplus F) = \oplus_{m,n \in \mathbb{Z}_+} S^m E \otimes S^n F \), and we let \( S_{\text{high}}(E \oplus F) = \oplus_{m,n \in \mathbb{Z}_+} (S^m E \otimes S^n F)_{\text{high}} \). We also denote by \( S_{\text{low}}(E \oplus F) \) the complement to \( S_{\text{high}}(E \oplus F) \).

It is easy to see that we have as \( p_0 \)-modules:

\[
S_{A, \text{high}} \cong \oplus_{m,n \in \mathbb{Z}_+} F(m,n,0,0),
\]
\[
S_{B, \text{high}} \cong \oplus_{m,n \in \mathbb{Z}_+} F(0,0,m,n),
\]
\[
S_{C, \text{high}} \cong \oplus_{m,n \in \mathbb{Z}_+} F(m,0,0,n).
\]

Introduce the following operators on the spaces \( M(\text{Hom}(S_X, S_X)) \), \( X = A, B \) or \( C \):

\[
\nabla_X = \sum_{i,j=1}^{5} d_{ij} \otimes \theta_{ij}^X,
\]

where

\[
\theta_{ij}^A = \frac{d}{dx_{ij}}, \quad \theta_{ij}^B = x_{ij}^*, \quad \theta_{ij}^C = x_i^* \frac{d}{dx_i} - x_j^* \frac{d}{dx_j}.
\]

It is immediate to see that \( p_0 \cdot \nabla_X = 0 \). In order to apply Proposition 4.1, we need to check that

(4.1) \( p_1 \cdot \nabla_X = 0 \).

This is indeed true in the case \( X = C \), but it is not true in the cases \( X = A \) and \( B \). In fact (4.1) applied to \( f \in S_X \), \( X = A \) or \( B \), is equivalent to the following equations, respectively \((a,b,c,d = 1, \ldots, 5)\):

\[
\left( \frac{d}{dx_{ab}} \frac{d}{dx_{cd}} - \frac{d}{dx_{ac}} \frac{d}{dx_{bd}} + \frac{d}{dx_{ad}} \frac{d}{dx_{bc}} \right) f = 0,
\]
\[
(x_{ab}^* x_{cd}^* - x_{ac}^* x_{bd}^* + x_{ad}^* x_{bc}^*) f = 0.
\]

It is not difficult to check the following lemma.

**Lemma 4.2.** (a) The subspace of \( S_A \) defined by equations (4.2) is \( S_{A, \text{high}} \).

(b) Equations (4.3) hold in \( S_B/S_B, \text{low} \).

(c) Equation \( \nabla_X^2 = 0 \) is equivalent to the system of equations \((a,b,c,d = 1, \ldots, 5)\):

\[
\theta_{ab} \theta_{cd} - \theta_{ac} \theta_{bd} + \theta_{ad} \theta_{bc} = 0.
\]

We let:

\[
V_A = S_{A, \text{high}}, \quad V_B(\text{resp. } C) = S_B(\text{resp. } C)/S_B(\text{resp. } C), \text{low}.
\]

The above discussion implies

**Proposition 4.3.** (a) The operators \( \nabla_X \) define \( E(5,10) \)-morphisms \( M(V_X) \rightarrow M(V_X) \) \((X = A, B \text{ or } C)\).

(b) \( \nabla_X^2 = 0 \) \((X = A, B \text{ or } C)\).

(c) \( \nabla_X = 0 \) iff \( X = A \) and \( n = 0 \), or \( X = C \) and \( m = 0 \).
The non-zero maps $\nabla_X$ are illustrated in Figure 2. The nodes in the quadrants $A$, $B$ and $C$ represent generalized Verma modules $M(m, n, 0, 0)$, $M(0, 0, m, n)$ and $M(m, 0, 0, n)$, respectively. The arrows represent the $E(5, 10)$-morphisms $\nabla_X$, $X = A$, $B$ or $C$ in the respective quadrants.

The following theorem summarizes our results on degenerate $E(5, 10)$-modules.

**Theorem 4.4.** (a) Each connected component in Figure 2 is a complex of $E(5, 10)$-modules. (b) The kernels of all morphisms in these complexes are irreducible maximal submodules.

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