Effect of intersubsystem coupling on the geometric phase in a bipartite system

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The influence of intersubsystem coupling on the cyclic adiabatic geometric phase in bipartite systems is investigated. We examine the geometric phase effects for two uniaxially coupled spin--\(\frac{1}{2}\) particles, both driven by a slowly rotating magnetic field. It is demonstrated that the relation between the geometric phase and the solid angle enclosed by the magnetic field is broken by the spin-spin coupling, in particular leading to a quenching effect on the geometric phase in the strong coupling limit.

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The geometric phase, originally conceived by Berry\textsuperscript{1} for cyclic adiabatic evolution of pure quantal states, has been extensively studied\textsuperscript{2,3,4,5,6} and generalized, for example to nonadiabatic evolution\textsuperscript{7,8}, mixed states\textsuperscript{9,10}, and open systems\textsuperscript{11}. The appearance of the geometric phase in composite entangled systems with no intersubsystem coupling has been analyzed for a pair of entangled spins in a time-independent uniform magnetic field\textsuperscript{8} and for the case of entangled spin pairs in a rotating magnetic field\textsuperscript{3}; it has also attracted interest in connection to polarization-entangled photon pairs\textsuperscript{12}, the topology of the SO(3) rotation group\textsuperscript{13}, Bell’s theorem\textsuperscript{14}, as well as in relation to the mixed state geometric phases of the subsystems\textsuperscript{15}. The importance of geometric phases to robust control of quantal systems, such as in fault tolerant quantum computation\textsuperscript{14}, has triggered extension of the geometric phase for composite systems to interacting subsystems. This topic has been addressed for two isolated interacting spins\textsuperscript{16}, for systems entangled with a quantized driving field\textsuperscript{17}, in relation to unitary representations of quantum channels\textsuperscript{18}, and more recently for a pair of interacting spin--\(\frac{1}{2}\) particles with one of the spins driven by a slowly rotating magnetic field\textsuperscript{19}. Concern about the effect of interaction on the geometric phase may also arise in the application of the geometric phase to systems with intra-variable couplings, such as, e.g., spin-orbit coupling in atomic systems, where the entanglement among a distinguished set of observables becomes an attractive issue in quantum information processing\textsuperscript{13,18}.

In this work, we develop the theory geometric phases of bipartite systems with intersubsystem coupling undergoing adiabatic cyclic evolution. We examine the effect of intersubsystem coupling on the pure state geometric phase of the composite system, as well as on the geometric phases associated with the reduced density operators of the subsystems. We calculate and analyze the geometric phases in the case of two uniaxially interacting spin--\(\frac{1}{2}\) (qubit) systems with the same magnetic dipole moment and driven by a slowly precessing magnetic field; a case of relevance to, e.g., entanglement creation by adiabatic passage techniques\textsuperscript{20} as well as to NMR quantum computation\textsuperscript{21,22}. Finally, we briefly discuss a possible extension of this analysis to systems with intra-variable coupling.

Let a quantal system \(S\) be exposed to the Hamiltonian \(H(Q), Q\) being some external control parameters. Suppose that the Hilbert space \(\mathcal{H}\) of \(S\) is \(N\) dimensional and that \(Q\) varies around a closed path \(\mathcal{C}: t \in [0,T] \rightarrow Q(t)\) in parameter space, so that \(H(Q_T) = H(Q_0)\). Expansion of the solution of the Schrödinger equation in the instantaneous eigenstates \(|n(Q_t)\rangle\) of \(H(Q_t)\) yields

\[
|\Psi(t)\rangle = \sum_{n=1}^{N} c_n(t)|n(Q_t)\rangle. \tag{1}
\]

If \(T\) is large enough, the adiabatic theorem\textsuperscript{22} entails that transitions between the instantaneous energy eigenstates are negligible, making \(|c_n|\) approximately time-independent, so that the final state reads (\(\hbar = 1\) from now on)

\[
|\Psi(T)\rangle = \sum_{n=1}^{N} |c_n|e^{-i\int_0^T E_n(t)dt} e^{i\gamma_n[C]}|n(Q_0)\rangle, \tag{2}
\]

where \(E_n(t)\) and \(\gamma_n[C]\) are the instantaneous nondegenerate energy eigenvalue and the cyclic adiabatic geometric phase, respectively, associated with the \(n\)th energy eigenstate.

Now, let us focus on the case where \(S\) has a natural bipartite decomposition in terms of subsystems \(S_a\) and \(S_b\) with corresponding Hilbert spaces \(\mathcal{H}_a\) and \(\mathcal{H}_b\). For such \(S\), the energy eigenvectors may be put on Schmidt form

\[
|n(Q,g)\rangle = \sum_{k=1}^{N} \sqrt{p_k^n(Q,g)} |a_k^n(Q,g)\rangle \otimes |b_k^n(Q,g)\rangle, \tag{3}
\]

where the Schmidt vectors \(|a_k^n(Q,g)\rangle \otimes |b_k^n(Q,g)\rangle\) are characterized by \(|a_k^n(Q,g)\rangle |a_l^n(Q,g)\rangle = \delta_{kl}\) and \(|b_k^n(Q,g)\rangle |b_l^n(Q,g)\rangle = \delta_{kl}\), \(g\) is some set of fixed coupling parameters, and \(N = \min\{\dim \mathcal{H}_a, \dim \mathcal{H}_b\}\). The
Schmidt decomposition is unique provided the nonvanishing coefficients \( p_k^{(n)} \) are nondegenerate, i.e., \( p_k^{(n)} \neq p_l^{(n)} \), \( \forall k, l \). The geometric phase associated with the path \( C \) in parameter space may be written as

\[
\gamma_{ab}^{(n)}[C; g] = \gamma_{n}[C] = i \oint_C dQ \cdot \langle n(Q, g)|\nabla_Q|n(Q, g)\rangle \\
= \sum_k \left( \bar{\Gamma}_{a;k}^{(n)}[C; g] + \bar{\Gamma}_{b;k}^{(n)}[C; g] \right),
\]

where we have used that the \( p_k^{(n)} \)'s sum up to unity. Here,

\[
\bar{\Gamma}_{\xi;k}^{(n)}[C; g] = i \oint_C dQ \cdot \langle \xi_k^{(n)}(Q, g)|\nabla_Q\xi_k^{(n)}(Q, g)\rangle
\]

are geometric phases of the weighted Schmidt vectors \( \xi_k^{(n)}(Q, g) \) pertaining to subsystem \( S_{\xi=a,b} \).

Next, let us introduce the concept of nontransition eigenstates. These are defined as energy eigenstates where the \( p_k \)'s are time-independent. For vanishing intersubsystem coupling, i.e., when \( g = 0 \), only such states occur since the time evolution operator then takes the bi-local form \( U_{ab} = U_a \otimes U_b \), which exactly preserves the Schmidt coefficients \( p_k^{(n)} \). On the other hand, transitions usually occur in the presence of coupling and the nontransition condition is in general only valid for specific paths. Closed paths of this kind are rare but have been found and studied for a spin-spin interaction model in Ref. [7].

Suppose that there exists a nontransition state tracing out a closed path \( D \) in parameter space. For such a path, we may compute the cyclic geometric phase as

\[
\gamma_{ab}^{(n)}[D; g] = i \sum_k p_k^{(n)}(Q_0, g) \left( \Gamma_{a;k}^{(n)}[D; g] + \Gamma_{b;k}^{(n)}[D; g] \right),
\]

where

\[
\Gamma_{\xi;k}^{(n)}[D; g] = i \oint_D dQ \cdot \langle \xi_k^{(n)}(Q, g)|\nabla_Q\xi_k^{(n)}(Q, g)\rangle \\
= [p_k^{(n)}(Q, g)]^{-1} \bar{\Gamma}_{\xi;k}^{(n)}[D; g]
\]

with \( \xi = a, b \), constitute the one-particle geometric phases of the Schmidt vectors. The nontransition property makes it natural to extend Ref. [8] and define mixed state geometric phases for the two interacting subsystems as

\[
\gamma_{\xi}^{(n)}[D; g] = \arg \sum_k p_k^{(n)}(Q_0, g) \exp \left( i \Gamma_{\xi;k}^{(n)}[D; g] \right).
\]

Thus, the geometric phases of the subsystems are taken as the average of phase factors pertaining to the nontransition eigenstates of the reduced density operators, weighted by the corresponding eigenvalues.

Now, consider two qubits \( a \) and \( b \) as represented by a pair of spin-\( \frac{1}{2} \) particles coupled by a uniaxial exchange interaction in the \( z \) direction. In the presence of a time-dependent external magnetic field \( B(t) = B_0 \hat{n}(t) \) with the unit vector \( \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), the Hamiltonian of this system reads

\[
H(t) = J S^z_a S^z_b + \mu B(t) \cdot (S_a + S_b)
\]

with \( \mu \) the magnetic dipole moment assumed to be equal for the two spins. The first part of the Hamiltonian describes the exchange interaction (spin-spin coupling) with coupling constant \( J \), \( \mu \) is the gyromagnetic ratio, and \( S_\xi = (S^x_\xi, S^y_\xi, S^z_\xi) \) is the \( \xi \)th spin operator \( (\xi = a, b) \).

An explicit physical scenario for this model could be NMR experiments on \( ^{13}C \)-labeled trichloroethylene in which the nuclear spins of the two \( ^{13}C \) nuclei could act as two spin-\( \frac{1}{2} \) systems with nearly the same magnetic dipole moment (the chemical shift of the two nuclei for this system is typically a fraction \( 10^{-5} \) of their precession frequency [21]). In terms of the orthonormalized total spin eigenstates \( |S; M\rangle \), \( S = 0, 1 \) and \( M = -S, \ldots, S \), in the \( z \) direction and in units of \( \mu B_0 \), the Hamiltonian Eq. (9) can be expressed in the block-matrix form [10]

\[
H(t) = \begin{pmatrix} H_c(t) & 0 \\ 0 & -g \end{pmatrix},
\]

where

\[
H_c(t) = \begin{pmatrix} g - \cos \theta & \frac{1}{\sqrt{2}} \sin \theta e^{i \phi} \\ \frac{1}{\sqrt{2}} \sin \theta e^{-i \phi} & g + \cos \theta \end{pmatrix}
\]

with rescaled coupling constant \( g = \frac{j}{\mu B_0} \) that may be controlled by changing the magnitude of the external magnetic field. Thus, the spin singlet \( |0, 0\rangle \) is decoupled from the triplet states \( |1, M\rangle \), \( M = -1, 0, 1 \). Therefore, in contrast to the case where only one of the subsystems interacts with \( B(t) \) [12], the adiabatic geometric phase acquired by the singlet state vanishes here.

First, let us consider the case of vanishing spin-spin coupling characterized by \( g = 0 \). In this case, the total spin in the \( \hat{n} \) direction commutes with \( H(t) \) so that the adiabatic geometric phases may be expressed in terms of the total spin projection quantum number \( M \) along the magnetic field and the solid angle \( \Omega \) enclosed by the path \( \mathcal{D} \) in parameter space. For \( M = \pm 1 \) we obtain

\[
\gamma_{ab}^{(\pm)}[\mathcal{D}; 0] = \mp \Omega,
\]

\[
\gamma_{a}^{(\pm)}[\mathcal{D}; 0] = \gamma_{b}^{(\pm)}[\mathcal{D}; 0] = \mp \frac{\Omega}{2}.
\]

where the former follows from the standard Berry formula \( -M \Omega \) [1]. The \( M = 0 \) eigenstates are two-fold degenerate and \( |\Psi^{(0)}\rangle = \alpha|1, 0\rangle + \beta|0, 0\rangle \) for any complex numbers \( \alpha \) and \( \beta \) is an energy eigenstate. While \( \gamma_{ab}^{(0)}[\mathcal{D}; 0] \) vanishes when taking \( |\Psi^{(0)}\rangle \) around \( \mathcal{D} \), the corresponding mixed state geometric phase for the two subsystems become

\[
\gamma_{a}^{(0)}[\mathcal{D}; 0] = -\gamma_{b}^{(0)}[\mathcal{D}; 0]
\]
provided $2\text{Re}(\alpha^*\beta) \neq 0$. On the other hand, when $2\text{Re}(\alpha^*\beta) = 0$, which for example occurs if the two spins are associated with indistinguishable entities, in case of which the singlet and triplet states cannot mix, the reduced density operators of the subsystems are degenerate, and the corresponding geometric phases become undefined since no direction in space is singled out by the corresponding Bloch vectors. In all other cases, we have 
\[ \gamma_{ab}^{(M)}[D; 0] = \gamma_{a}^{(M)}[D; 0] + \gamma_{b}^{(M)}[D; 0], \]
which is due to the spherical symmetry of the model in the $g = 0$ case.
For $g \neq 0$, the spherical symmetry is broken, and there is no simple relation neither between the geometric phases and the solid angle nor between the geometric phase of the composite system and those of the subsystems. Thus, to proceed we need to diagonalize $H_c(t)$ to obtain its eigenstates as $(n = -1, 0, +)$
\[
\Psi^{(n)} = e^{i\phi}A^{(n)}(\theta, g)|1; -1\rangle + B^{(n)}(\theta, g)|1; 0\rangle
+ e^{-i\phi}C^{(n)}(\theta, g)|1; 1\rangle
\] (14)
with
\begin{align*}
A^{(n)} &= \frac{1}{\sqrt{2M^{(n)}}}[X^{(n)} - \cos \theta] \sin \theta, \\
B^{(n)} &= \frac{1}{\sqrt{M^{(n)}}}(X^{(n)})^2 - \cos^2 \theta, \\
C^{(n)} &= \frac{1}{\sqrt{2M^{(n)}}}[X^{(n)} + \cos \theta] \sin \theta, \\
M^{(n)} &= (X^{(n)})^4 + (1 - 3\cos^2 \theta)(X^{(n)})^2 + \cos^2 \theta. (15)
\end{align*}

The shifted instantaneous energy eigenvalues $X^{(n)} = X^{(n)}(\theta, g) \equiv E_n - g$ of the Hamiltonian Eq. (11) are solutions of
\[ X^3 + 2gX^2 - X - 2g \cos^2 \theta = 0, \] (16)
which yields $X^{(\pm)}(\theta, g) = \pm \cos \theta$ and $X^{(0)}(\theta, g) = -2g$ in the limit of $g \to \infty$. The Schmidt coefficients read
\[
p_1^{(n)} = 1 - p_2^{(n)} = \frac{1}{2} \left( 1 + \frac{A^{(n)} + C^{(n)}}{B^{(n)}} \right) \times \sqrt{2(B^{(n)})^2 + (C^{(n)} - A^{(n)})^2} = \frac{1}{2} \left( 1 + p^{(n)}(\theta, g) \right). \] (17)

Thus, $p_1^{(n)}$ and $p_2^{(n)}$ are determined by the $\phi$ independent effective Bloch vector $p^{(n)}$ and it follows that the nontransition paths are those where $\theta$ is constant. For closed paths $D : t \in [0, T] \to (\phi_t, \theta_t) = (2\pi t/T, \theta)$ with $p^{(n)} \neq 0$ and $T$ large, we uniquely obtain the adiabatic geometric phases for the corresponding Schmidt vectors pertaining to subsystem $S_{\xi=a,b}$ as
\[ \Gamma^{(n)}_{\xi,1}[D, g] = -\Gamma^{(n)}_{\xi,2}[D, g] = -\pi \left( 1 - F^{(n)} \cos \theta \right), \] (18)
where the scale factor
\[
F^{(n)} = F^{(n)}(\theta, g) = \frac{\sin \theta}{\sqrt{(X^{(n)})^4 - 2(X^{(n)})^2 \cos^2 \theta + \cos^2 \theta}} \] (19)
comprises the effect of intersubsystem coupling on the geometric phase of the Schmidt vectors. The geometric phases of the composite system and those of the subsystems read
\[
\gamma^{(n)}_{ab}[D, g] = -2\pi r^{(n)} \left( 1 - F^{(n)} \cos \theta \right), \]
\[
\gamma^{(n)}_{\xi}[D, g] = -\arctan \left[ r^{(n)} \tan \left( \pi \left( 1 - F^{(n)} \cos \theta \right) \right) \right] \] (20)
with $\xi = a, b$.

The dependence of the geometric phase upon the coupling constant $g$ is illustrated in Fig. (1). An interesting feature of Fig. (1) is that all geometric phases for the composite system and its subsystems tend to an integer multiple of $2\pi$ when $g \to \infty$. This limit corresponds
to the case when the second term in the Hamiltonian in Eq. 6 can be ignored. In other words, the triplet states |1; M⟩ would become the instantaneous eigenstates of the system with g → ∞, thus making the enclosed area in state space to vanish and thereby the geometric phase factors become trivial. It is worth stressing that the Hamiltonian in Eq. 6 has permutation symmetry, resulting in γ[ab](n)[D; g] = γ[ba](n)[D; g]. In fact, the mixed state geometric phases presented in Fig. 1 are for γ[ab](n)[D; g] = 2γ[ba](n)[D; g] = 2γ[ba](n)[D; g]; the sum of mixed state geometric phases of the two qubits being equal to twice the geometric phase for each subsystem. Note in particular that γ[ab](n)[D; g] ≠ γ[ba](n)[D; g] + γ[ba](n)[D; g] in general. The state of the composite system are entangled at most time when g ≠ 0, this indicates that there are at least two nonvanishing Schmidt coefficient μk and thus most the geometric phases for the Schmidt vector in Eq. 6 would have similar properties in the limit g → ∞ except that pertaining to |Ψ(0)⟩, where X(0)(0, g → ∞) → −g, and then F(0) tends to zero, consequently, Γ(0)[D, g → ∞] → π as shown by 0s lines in Fig. 1.

For transition paths C along which the polar angle θ and thereby the Schmidt coefficients vary, the geometric phase of the composite system reads

$$\gamma[ab](n)[C, g] = -\oint_C r^{ab}(n)(1 - F^{(n)} \cos \theta) d\phi.$$  

(21)

In Fig. 2, γ[ab](n)[C, g] is shown as a function of the rescaled coupling constant g for the closed path C : t ∈ [0, T] → (ϕt, θd) = (πt/T, π sin(πt/T)). With g → ∞, the phases tend to zero, in analogy with the nontransition case discussed above.

An interesting extension of above analysis is to the case of systems with intra-variable coupling. For example, one may consider an atom with electronic orbital and spin angular momentum L and S, respectively, precessing in a time-dependent magnetic field B, the Hamiltonian describing such a system reads, H = μn(t) · (L + 2S) + gL · S, the last term describes the spin orbit coupling. An analysis is expected to show that the spin-orbit coupling would affect the geometric phase of the atom in a similar way as in the intersubsystem coupling case.

In conclusion, we have analyzed the cyclic adiabatic geometric phase of bipartite systems, focusing on the effect of intersubsystem coupling. We have distinguished two different kind of evolution in regard to whether or not the Schmidt coefficients are time-dependent. The geometric phases of the subsystems naturally extends in terms of the standard mixed state geometric phase [6] in the nontransition case, i.e., when the Schmidt coefficients and thereby the eigenvalues of the corresponding reduced density operators are fixed. We have found a striking evidence for a strong influence of the intersubsystem coupling on the geometric phases for two uniaxially coupled spin–1 2 systems, such as a divergence from the standard relation to the solid angle enclosed by the driving magnetic field, leading to a quenching effect on the geometric phases in the strong coupling limit. This latter result has also been demonstrated in the transition case, where the Schmidt coefficients are changing around the curve in parameter space. Physically, the quenching effect may be viewed as a consequence of the broken spherical symmetry as expressed by the existence of a preferred direction in space singled out by the uniaxial exchange direction of the spins, making the energy eigenstates independent of the slowly rotating magnetic field.

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