Self-adjoint Operators as Functions I
Lattices, Galois Connections and Spectral Order

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Abstract

Observables of a quantum system, described by self-adjoint operators in a von Neumann algebra or affiliated with it in the unbounded case, form a conditionally complete lattice when equipped with the spectral order. Using this order-theoretic structure, we develop a new perspective on quantum observables.

In this first paper, we show that self-adjoint operators affiliated with a von Neumann algebra $\mathcal{N}$ can equivalently be described as certain real-valued functions on the projection lattice $\mathcal{P}(\mathcal{N})$ of the algebra, which we call $q$-observable functions. Bounded self-adjoint operators correspond to $q$-observable functions with compact image on non-zero projections. These functions, originally defined in a similar form by de Groote in [17], are most naturally seen as adjoints (in the categorical sense) of spectral families. We show how they relate to the daseinisation mapping from the topos approach to quantum theory [14]. Moreover, the $q$-observable functions form a conditionally complete lattice which is shown to be order-isomorphic to the lattice of self-adjoint operators with respect to the spectral order.

In a subsequent paper [8], an interpretation of $q$-observable functions in terms of quantum probability theory will be given, and using results from the topos approach to quantum theory, a joint sample space for all quantum observables will be provided.

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1 Introduction

It is well-known that the self-adjoint operators representing observables of a quantum system form a real vector space. Yet, while adding self-adjoint operators and multiplying them by real numbers are mathematically natural operations, it is much less clear what these operations mean physically. For example, what physical interpretation is attached to the sum of position and momentum? And to \(-\pi\) times momentum?

The real vector space of bounded self-adjoint operators describing the physical quantities of a quantum system can be regarded as the self-adjoint part of a complex operator algebra; in particular, we will focus on von Neumann algebras here. In order to include unbounded self-adjoint operators, we will consider operators affiliated with a given von Neumann algebra.

Crucially, we emphasise the order structure on the set of self-adjoint operators in (or affiliated with) a von Neumann algebra over the linear structure. Thus, we consider the partial order on projections, and more importantly, the spectral order \([23, 16]\) on self-adjoint operators. The latter, which may be less well-known, generalises the partial order on projections and differs from the usual linear order on self-adjoint operators if the algebra is nonabelian. We provide some order-theoretic background in section 2.

Let \(\hat{A}\) be a self-adjoint operator, contained in a von Neumann algebra \(\mathcal{N}\), or affiliated with it if \(\hat{A}\) is unbounded. Let \(\mathcal{P}(\mathcal{N})\) denote the lattice of projections in \(\mathcal{N}\). The key observation is that the spectral family \(\hat{E}^{\hat{A}}: \mathbb{R} \rightarrow \mathcal{P}(\mathcal{N})\) (1.1) of \(\hat{A}\) is a meet-preserving map between meet-semilattices (technically, we will use the extended reals \(\mathbb{R}\) instead of \(\mathbb{R}\) to obtain preservation of all meets). This implies that \(\hat{E}^{\hat{A}}\) has a left adjoint \(o^{\hat{A}}\) in the sense of category theory such that \(\hat{E}^{\hat{A}}\) and \(o^{\hat{A}}\) form a Galois connection. \(o^{\hat{A}}\) is a real-valued function on the projections in the von Neumann algebra \(\mathcal{N}\).

We call \(o^{\hat{A}}\) the \textit{q-observable function} of \(\hat{A}\). This function is determined uniquely by \(\hat{E}^{\hat{A}}\), and hence by \(\hat{A}\), and conversely determines them uniquely. These functions were first considered by de Groote [15], and independently (as far we are aware) by Comman [2]. Neither of these authors used the definition via Galois connections. In section 3, we will show that the image of \(o^{\hat{A}}\) on non-zero projections equals the spectrum of \(\hat{A}\). Moreover, we will characterise \(q\)-observable functions abstractly without any reference to operators.
In section 4, we consider how q-observable functions behave under extension and restriction of their domain resp. codomain, and show that they form a conditionally complete lattice isomorphic to the lattice of self-adjoint operators affiliated with \( N \) with respect to the spectral order. Moreover, we show that there exists a limited form of ‘function calculus’: for suitable monotone functions \( f : \mathbb{R} \to \mathbb{R} \), it holds that \( o(f(\hat{A})) = f(o(\hat{A})) \).

In section 5, it is shown that in addition to q-observable functions there exists a second kind of functions associated with self-adjoint operators affiliated with a von Neumann algebra \( N \), called q-antonymous functions. The two kinds of functions relate to the maps called outer and inner daseinisation of self-adjoint operators, which are approximations with respect to the spectral order and play a key role in the topos approach to quantum theory [10, 11, 12, 13, 14, 7]; see also [19, 25].

Section 6 provides physical interpretations of the mathematical results from previous sections and gives a short outlook on the second paper, “Self-Adjoint operators as Functions II: Quantum Probability” [8].

2 Some mathematical preliminaries

In this section, we briefly present some definitions and results, mostly from order and lattice theory (see e.g. [5, 3]), that will be used in the rest of the paper.

We assume some knowledge of the basics of the theory of von Neumann algebras, such as the lattice structure on projections, right-continuous and left-continuous spectral families of bounded and unbounded self-adjoint operators, unbounded operators affiliated with a von Neumann algebra, and the spectral theorem. Standard references are [21, 26]. All Hilbert spaces are assumed to be complex.

A partially ordered set (poset) \((P, \leq)\) is a set \( P \) with a binary relation \( \leq \), the order, that is reflexive, antisymmetric and transitive. An element \( \bot_P \in P \) such that \( \bot_P \leq a \) for all elements \( a \in P \) is called a bottom element. If \( P \) has a bottom element, it is necessarily unique. An element \( \top_P \) such that \( a \leq \top_P \) for all \( a \in P \) is called a top element. If \( P \) has a top element, it is unique.

A meet-semilattice is a poset \( P \) such that any two elements \( a, b \) have a meet (greatest lower bound) \( a \land b \) in \( P \), that is,

\[
\forall c \in P : c \leq a, b \quad \iff \quad c \leq a \land b. \tag{2.1}
\]

A meet-semilattice is called complete if every family \((a_i)_{i \in I}\) of elements in \( P \) has a greatest lower bound in \( P \), denoted by \( \bigwedge_{i \in I} a_i \). In a complete meet-semilattice, the empty family has a meet \( \bigwedge \emptyset \), which is the top element of
$P$. Also, the family containing all elements of $P$ has a meet $\bigwedge_{a \in P} a$, which is the bottom element of $P$.

A **join-semilattice** is a poset $P$ such that any two elements $a, b$ have a join (least upper bound) $a \lor b$ in $P$, that is,

$$\forall c \in P : a, b \leq c \iff a \lor b \leq c.$$  \hspace{1cm} (2.2)

A join-semilattice is called **complete** if every family $(a_i)_{i \in I}$ of elements in $P$ has a least upper bound in $P$, denoted by $\bigvee_{i \in I} a_i$. In a complete join-semilattice, the empty family has a join $\bigwedge \emptyset$, which is the bottom element of $P$. Also, the family containing all elements of $P$ has a join $\bigvee_{a \in P} a$, which is the top element of $P$.

A poset $P$ that is both a meet-semilattice and a join-semilattice is called a **lattice**. A lattice is **complete** if it is complete as a meet- and a join-semilattice. If $P$ is a complete meet-semilattice, joins can be defined in $P$ by

$$\forall (a_i)_{i \in I} \subseteq P : \bigvee_{i \in I} a_i := \bigwedge \{ b \in P \mid \forall i \in I : a_i \leq b \},$$  \hspace{1cm} (2.3)

that is, the least upper bound of the family $(a_i)_{i \in I}$ is the greatest lower bound of all $b \in P$ that are greater than all the $a_i$. Conversely, in a complete join-semilattice, meets can be defined in terms of joins.

A lattice $P$ is **distributive** if, for all $a, b, c \in P$,

$$a \land (b \lor c) = (a \land b) \lor (a \land c),$$ \hspace{1cm} (2.4)

$$a \lor (b \land c) = (a \lor b) \land (a \lor c).$$ \hspace{1cm} (2.5)

In categorical terms, a poset is a category $P$ with at most one arrow between any two objects $a, b$ (expressing the fact that $a \leq b$). Such a category $P$ is a meet-semilattice if binary products exist, and a join-semilattice if binary coproducts exist. Completeness corresponds to existence of all products resp. coproducts. The bottom element is a (necessarily unique) initial object, given by the empty join and the meet over all elements of $P$. The top element is a (necessarily unique) terminal object, given by the empty meet and the join over all elements of $P$.

**Example 1** The natural numbers $\mathbb{N}$ with the usual order are a poset. There is a bottom element $\bot_{\mathbb{N}} = 0$, but no top element. $(\mathbb{N}, \leq)$ is both a meet-semilattice, with meets given by minima, and a join-semilattice, with joins given by maxima, but $(\mathbb{N}, \leq)$ is not complete: neither the empty meet nor the join over all of $\mathbb{N}$ exists.

The real numbers $\mathbb{R}$ with their usual order are a poset with no bottom and no top element. $(\mathbb{R}, \leq)$ is a meet-semilattice, with minima as meets, and a join-semilattice, with maxima as joins, but it is not complete.
But $\mathbb{R}$ is ‘almost’ a complete meet-semilattice: any family $(r_i)_{i \in I} \subset \mathbb{R}$ that has a lower bound has a greatest lower bound, given by the infimum $\inf_{i \in I} r_i$. This means that $\mathbb{R}$ is a conditionally (or boundedly) complete meet-semilattice. It can be made into a complete meet-semilattice by adding a bottom element $-\infty$ (which is the meet over all elements of $\mathbb{R}$) and a top element $\infty$ (which is the empty meet). The extended reals

$$\mathbb{R} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

form a complete meet-semilattice (with infima as meets) that will play an important role later on. Similarly, if a family $(r_i)_{i \in I}$ has an upper bound, it has a least upper bound, given by the supremum $\sup_{i \in I} r_i$. This means that $\mathbb{R}$ is a conditionally (or boundedly) complete join-semilattice. The extended reals $\mathbb{R}$ form a complete join-semilattice (with suprema as joins) and hence a complete lattice.

Both $(\mathbb{N}, \leq)$ and $(\mathbb{R}, \leq)$ are totally ordered: for any pair $(a, b)$ of elements, either $a \leq b$ or $b \leq a$ (or both if $a = b$).

**Remark 1** Of course, $\mathbb{R}$ is a compactification of $\mathbb{R}$. Our emphasis in this article is on order-theoretic aspects, so we will not use topological arguments and structure explicitly (though many arguments could be phrased in topological terms).

**Example 2** Let $S$ be a set. The power set $P(S)$, consisting of all subsets of $S$ with inclusion as partial order, is a complete lattice. Meets are given by intersections, joins by unions. The bottom element in $(P(S), \subseteq)$ is the empty subset, the top element is $S$. Moreover, $(P(S), \subseteq)$ is a distributive lattice.

Let $\mathcal{N}$ be a von Neumann algebra. The projections in $\mathcal{N}$ form a complete lattice $\mathcal{P}(\mathcal{N})$ with respect to the order

$$\forall \hat{P}, \hat{Q} \in \mathcal{P}(\mathcal{N}) : \hat{P} \leq \hat{Q} : \iff \hat{P} \hat{Q} = \hat{Q} \hat{P} = \hat{P}.$$  \hspace{1cm} (2.7)

The lattice $(\mathcal{P}(\mathcal{N}), \leq)$ is distributive if and only if the von Neumann algebra $\mathcal{N}$ is abelian.

The lattices $(P(S), \subseteq)$ and $(\mathcal{P}(\mathcal{N}), \leq)$ have additional structure: they both have complements. In $(P(S), \subseteq)$, the complement of a subset $T \subseteq S$ is the set-theoretic complement $S \setminus T$. Together with unions and intersections, this makes $(P(S), \subseteq)$ into a complete Boolean algebra. In the projection lattice $(\mathcal{P}(\mathcal{N}), \leq)$, the complement $\hat{P}$ of a projection is $\hat{1} - \hat{P}$. Together with meets and joins, this makes $(\mathcal{P}(\mathcal{N}), \leq)$ into a complete orthomodular lattice, which is a complete Boolean lattice if and only if $\mathcal{N}$ is abelian.
Let \((P, \leq)\) and \((Q, \sqsubseteq)\) be two posets. A map \(f : P \to Q\) is called **monotone (order-preserving)** if
\[
\forall a, b \in P : a \leq b \implies f(a) \sqsubseteq f(b).
\] (2.8)

A pair of monotone maps \((f : P \to Q, g : Q \to P)\) is called a **Galois connection** if
\[
\forall a \in P \forall x \in Q : f(a) \sqsubseteq x \iff a \leq g(x).
\] (2.9)

Categorically, a monotone map is a functor from \(P\) to \(Q\), and a Galois connection is a pair of adjoint functors \([22]\), where \(f\) is the left adjoint and \(g\) the right adjoint. Left adjoints preserve joins (least upper bounds), and right adjoints preserve meets (greatest lower bounds). If a functor \(f : P \to Q\) has a right adjoint \(g : Q \to P\), then \(g\) is unique; likewise, if a functor \(g : Q \to P\) has a left adjoint \(f : P \to Q\), then \(f\) is unique.

As we saw, complete meet-semilattices are also complete join-semilattices (and vice versa), so are complete lattices. The three categories of complete meet-semilattices, complete join-semilattices and complete lattices hence have the same objects, but different kinds of arrows\(^1\) In the category of complete meet-semilattices, arrows preserve all meets, in the category of complete join-semilattices, arrows preserve all joins, and in the category of complete lattices, both.

The following theorem is called the **adjoint functor theorem for posets**:

**Theorem 1** Let \((P, \leq), (Q, \sqsubseteq)\) be complete join-semilattices, and let \(f : P \to Q\) be a monotone map. Then, \(f\) preserves all joins, i.e.,
\[
\forall (a_i)_{i \in I} \subseteq P : f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)
\] (2.10)
if and only if \(f\) has a right adjoint \(g\), given by
\[
g : Q \to P
x \mapsto \bigvee \{a \in P \mid f(a) \sqsubseteq x\}.
\] (2.11)
The right adjoint \(g\) is monotone and preserves all meets.

Conversely, if \(g : Q \to P\) is a map between complete meet-semilattices, then \(g\) has a left adjoint \(f : P \to Q\) if and only if \(g\) preserves all meets. The left adjoint \(f\) preserves all joins and is given by
\[
f : P \to Q
a \mapsto \bigwedge \{x \in Q \mid a \leq g(x)\}
\] (2.12)

\(^1\)Note that here, we consider categories with posets as objects and monotone maps as arrows, while previously, we had considered a single poset \(P\) as a category, with the elements of the poset as objects, and arrows expressing the order (i.e., if \(a, b \in P\), then there is an arrow \(a \to b\) if and only if \(a \leq b\)).
Proof. For the sake of completeness, we give a proof of this standard result.

First suppose that \( f : P \to Q \) has a right adjoint \( g : Q \to P \). Let \((a_i)_{i \in I} \subseteq P\) be an arbitrary family of elements in \( P \). Then, \( a_i \leq \bigvee_{i \in I} a_i \) for all \( i \in I \), so \( f(a_i) \sqsubseteq f \left( \bigvee_{i \in I} a_i \right) \) for all \( i \in I \) since \( f \) is monotone, and therefore

\[
\bigvee_{i \in I} f(a_i) \sqsubseteq f \left( \bigvee_{i \in I} a_i \right). \tag{2.13}
\]

Now, for all \( i \in I \), we have

\[
f(a_i) \sqsubseteq \bigvee_{i \in I} f(a_i). \tag{2.14}
\]

Since \( g \) is right-adjoint to \( f \), we have, for each \( i \in I \),

\[
a_i \leq g \left( \bigvee_{i \in I} f(a_i) \right) \tag{2.15}
\]

and so

\[
\bigvee_{i \in I} a_i \leq g \left( \bigvee_{i \in I} f(a_i) \right). \tag{2.16}
\]

Therefore, since \( f \) is left-adjoint to \( g \),

\[
f \left( \bigvee_{i \in I} a_i \right) \sqsubseteq \bigvee_{i \in I} f(a_i). \tag{2.17}
\]

Thus, by (2.13) and (2.17) we have

\[
f \left( \bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} f(a_i), \tag{2.18}
\]

for any family \((a_i)_{i \in I} \subseteq P\), so \( f \) preserves joins.

Now suppose that \( f : P \to Q \) preserves all joins. Let \( g : Q \to P \) be defined by

\[
g(x) = \bigvee \{ a \in P \mid f(a) \sqsubseteq x \} \tag{2.19}
\]

for all \( x \in Q \). Let \( a_1 \in P \) such that \( f(a_1) \sqsubseteq x \) for a given \( x \in Q \). Then

\[
a_1 \in \{ a \in P \mid f(a) \sqsubseteq x \}, \tag{2.20}
\]

so

\[
g(x) = \bigvee \{ a \in P \mid f(a) \sqsubseteq x \} \geq a_1. \tag{2.21}
\]
Conversely, suppose that \( a_1 \leq g(x) \). Then, since \( f \) is a monotone map and preserves joins,

\[
f(a_1) \subseteq f(g(x)) = f \left( \bigvee \{ a \in P \mid f(a) \subseteq x \} \right) = \bigvee \{ f(a) \in Q \mid f(a) \subseteq x \} \subseteq x,
\]

so \( f(a_1) \subseteq x \). Therefore,

\[
a \leq g(x) \iff f(a) \subseteq x \quad (2.22)
\]

for all \( a \in P \) and \( x \in Q \), so by the definition of a Galois connection \( (2.23) \), \( g \) is right-adjoint to \( f \).

The proof that a map \( q : Q \to P \) has a left adjoint \( f : P \to Q \) if and only if \( g \) preserves all meets is completely analogous.

The adjoint functor theorem for posets is a key result that will be used throughout.

Let \( \hat{A} \) be a self-adjoint operator, and let \( \mathcal{N} \) be a von Neumann algebra that contains all spectral projections of \( \hat{A} \). If \( \hat{A} \) is bounded, then \( \hat{A} \) lies in \( \mathcal{N}_{sa} \), the self-adjoint operators in \( \mathcal{N} \); if \( \hat{A} \) is unbounded, then it is affiliated with \( \mathcal{N} \). In the following, we will simply speak of self-adjoint operators affiliated with a von Neumann algebra \( \mathcal{N} \), implicitly understanding that if such an operator \( \hat{A} \) is bounded then it lies in \( \mathcal{N}_{sa} \). The set of self-adjoint operators affiliated with a von Neumann algebra \( \mathcal{N} \) is denoted \( SA(\mathcal{N}) \) (and so \( \mathcal{N}_{sa} \subset SA(\mathcal{N}) \)).

We will not consider any questions relating to domains of definition of unbounded operators in this article.

**Definition 1** The spectral order on the set \( SA(\mathcal{N}) \) of self-adjoint operators affiliated with a von Neumann algebra \( \mathcal{N} \) is defined by

\[
\forall \hat{A}, \hat{B} \in SA(\mathcal{N}) : \hat{A} \leq_{s} \hat{B} \quad \iff \quad \forall r \in \mathbb{R} : \hat{E}^{\hat{A}}_{r} \geq \hat{E}^{\hat{B}}_{r},
\]

where \( \hat{E}^{\hat{A}} = (\hat{E}^{\hat{A}}_{r})_{r \in \mathbb{R}} \) and \( \hat{E}^{\hat{B}} = (\hat{E}^{\hat{B}}_{r})_{r \in \mathbb{R}} \) are the right-continuous spectral families of \( \hat{A} \) resp. \( \hat{B} \), and where on the right-hand side, the usual order on projections is used.

Of course, the spectral order can be restricted to \( \mathcal{N}_{sa} \), the set of (bounded) self-adjoint operators in \( \mathcal{N} \). This order was originally introduced by Olson \[23\]; see also \[10\]. For some recent results on the spectral order for unbounded operators see \[24\].

With respect to the spectral order, the set \( SA(\mathcal{N}) \) (resp. \( \mathcal{N}_{sa} \)) is a conditionally complete lattice, i.e., each family \( (\hat{A}_{i})_{i \in I} \) in \( SA(\mathcal{N}) \) (resp. \( \mathcal{N}_{sa} \)) that has a lower bound has a greatest lower bound \( \bigwedge_{i \in I} \hat{A}_{i} \) in \( SA(\mathcal{N}) \) (resp. \( \mathcal{N}_{sa} \)), and if the family has an upper bound, then it has a least upper bound.
\[ \forall i \in I \hat{A}_i \text{ in } SA(\mathcal{N}) \text{ (resp. } \mathcal{N}_{sa}) \]. This is in marked contrast to the usual linear order that is given as
\[ \forall \hat{A}, \hat{B} \in SA(\mathcal{N}) : \hat{A} \leq \hat{B} \iff \hat{B} - \hat{A} \text{ is positive.} \quad (2.25) \]
As Kadison showed \cite{20}, for a nonabelian von Neumann algebra \(\mathcal{N}\) the meet \(\hat{A} \land \hat{B}\) of two self-adjoint operators in \(\mathcal{N}\) exists if and only if \(\hat{A}\) and \(\hat{B}\) are comparable with respect to the linear order, i.e., if either \(\hat{A} \leq \hat{B}\) or \(\hat{B} \leq \hat{A}\). Hence, \(\mathcal{N}_{sa}\) equipped with the linear order \(\leq\) is very far from being a lattice; Kadison called \((\mathcal{N}_{sa}, \leq)\) an anti-lattice.

Some further facts about the spectral order (for proofs see \cite{23, 16}):

(a) On projections, the spectral order and the usual linear order coincide, so the spectral order generalises the partial order on projections.

(b) On commuting operators, the spectral order and the usual linear order coincide, so for abelian von Neumann algebras, the linear order and the spectral order coincide.

(c) The spectral order is coarser than the linear order, that is, for all \(\hat{A}, \hat{B} \in SA(\mathcal{N})\), \(\hat{A} \leq_s \hat{B}\) implies \(\hat{A} \leq \hat{B}\).

(d) the spectral order does not make \(SA(\mathcal{N})\) (or \(\mathcal{N}_{sa}\)) into a vector lattice, that is, \(\hat{A} \leq_s \hat{B}\) does not necessarily imply \(\hat{A} + \hat{C} \leq_s \hat{B} + \hat{C}\).

Property (d) can be seen as an ‘incompatibility’ between the linear structure and the order structure on \(SA(\mathcal{N})\) (and on \(\mathcal{N}_{sa}\)). In this article, we will focus on the order structure provided by the spectral order.

3 Definition and basic properties of \(q\)-observable functions

Let \(\hat{A} \in SA(\mathcal{N})\) be a self-adjoint operator affiliated with a von Neumann algebra \(\mathcal{N}\). The spectral theorem (see e.g. \cite{21}) shows that there is a unique right-continuous spectral family \(\hat{E}^\hat{A} = (\hat{E}^\hat{A}_r)_{r \in \mathbb{R}}\) in \(\mathcal{P}(\mathcal{N})\) such that
\[
\hat{A} = \int_{-\infty}^{\infty} r \, d\hat{E}^\hat{A}_r. \quad (3.1)
\]
Conversely, every right-continuous spectral family \(E := (\hat{E}_r)_{r \in \mathbb{R}}\) in \(\mathcal{P}(\mathcal{N})\) determines a unique self-adjoint operator \(\hat{A}^E\) affiliated with \(\mathcal{N}\). If \((\hat{E}_r)_{r \in \mathbb{R}}\) is bounded, then \(\hat{A}^E\) is bounded and lies in \(\mathcal{N}\).

We will now show that to each right-continuous spectral family in \(\mathcal{N}\), there corresponds a real-valued function on the projections in \(\mathcal{N}\), and vice versa.
A right-continuous spectral family \( E = (\hat{E}_r)_{r \in \mathbb{R}} \) can be seen as a function

\[
E : \mathbb{R} \to \mathcal{P}(\mathcal{N})
\]

\( r \mapsto \hat{E}_r. \)

Both \( \mathbb{R} \) and \( \mathcal{P}(\mathcal{N}) \) are posets, and \( \mathcal{P}(\mathcal{N}) \) is a complete meet-semilattice with respect to the usual meet of projections. As described in Example 1, the extended reals \( \mathbb{R} \) are a complete meet-semilattice with infima as meets and a complete join-semilattice with suprema as joins.

**Definition 2** Let \( \mathcal{N} \) be a von Neumann algebra. A map

\[
E : \mathbb{R} \to \mathcal{P}(\mathcal{N})
\]

\( r \mapsto \hat{E}_r \)

such that

(a) \( \hat{E}_{-\infty} = 0 \) and \( \hat{E}_{\infty} = 1 \),

(b) \( E|_{\mathbb{R}} \) is a spectral family

is called an extended spectral family. We will use both the notations \( E : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) and \( (\hat{E}_r)_{r \in \mathbb{R}} \) for an extended spectral family.

**Remark 2** Obviously, every spectral family \( E : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) determines a unique extended spectral family \( E : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) and vice versa, so there is a canonical bijection between spectral families (defined on \( \mathbb{R} \)) and extended spectral families (defined on \( \mathbb{R} \)). If \( E : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) is right-continuous, then its extension \( E : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) is also right-continuous. In the following, we will mostly work with extended right-continuous spectral families.

The spectral theorem shows that there is a bijection between \( \mathcal{SA}(\mathcal{N}) \), the set of self-adjoint operators affiliated with a given von Neumann algebra \( \mathcal{N} \), and the set \( \mathcal{SF}(\mathbb{R}, \mathcal{P}(\mathcal{N})) \) of extended, right-continuous spectral families in \( \mathcal{P}(\mathcal{N}) \).

**Lemma 1** If we regard a right-continuous extended spectral family as a monotone function

\[
E : \mathbb{R} \to \mathcal{P}(\mathcal{N})
\]

\( r \mapsto \hat{E}_r, \)

then \( E \) preserves all meets, so it is a morphism of complete meet-semilattices. Conversely, any meet-preserving map \( E : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) with the properties...
(a) $E(-\infty) = \hat{0},$

(b) $\text{s-lim}_{r \to -\infty} E(r) = \hat{1}$ (where s-lim denotes a limit in the strong topology)

determines an extended spectral family.

**Proof.** Let $E : \mathbb{R} \to \mathcal{P}(\mathcal{N})$ be an extended spectral family, and let $(r_i)_{i \in I} \subseteq \mathbb{R}$ be an arbitrary family of elements of $\mathbb{R}$. Then

$$E(\inf_{i \in I} r_i) = \hat{E}_{\inf_{i \in I} r_i} = \bigwedge_{s > \inf_{i \in I} r_i} \hat{E}_s = \bigwedge_{i \in I} \hat{E}_{r_i},$$  \hspace{1cm} (3.5)

where the second equality is due to right-continuity of $E$ and the third is due to monotonicity.

Conversely, if we have a meet-preserving map $E : \mathbb{R} \to \mathcal{P}(\mathcal{N})$, then clearly $E$ is monotone and (writing $\hat{E}_r := E(r)$) we have

$$\forall r \in \mathbb{R} : \hat{E}_r = \hat{E}_{\inf_{s \geq r} s} = \bigwedge_{s > r} \hat{E}_s,$$  \hspace{1cm} (3.6)

so $E$ is right-continuous. Moreover,

$$\text{s-lim}_{r \to -\infty} E(r) = \bigwedge_{r \in \mathbb{R}} \hat{E}_r = \hat{E}_{\inf_{r \in \mathbb{R}} r} = E(-\infty) = \hat{0},$$  \hspace{1cm} (3.7)

where we used meet-preservation of $E$ in the second step and assumption (a) in the last step. Also by meet-preservation, we obtain

$$E(\infty) = E(\inf \emptyset) = \bigwedge \emptyset = \hat{1},$$  \hspace{1cm} (3.8)

since the empty meet is a terminal object, which in a poset is the top element. Together with assumption (b), this shows that $E : \mathbb{R} \to \mathcal{P}(\mathcal{N})$ is an extended right-continuous spectral family. \(\blacksquare\)

By the adjoint functor theorem for posets, an extended spectral family $E$ has a left adjoint

$$o^E : \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{R}$$  \hspace{1cm} (3.9)

that preserves arbitrary joins, i.e., for all families $(\hat{P}_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{N})$,

$$o^E(\bigvee_{i \in I} \hat{P}_i) = \sup_{i \in I} o^E(\hat{P}_i).$$  \hspace{1cm} (3.10)

Note that ‘adjoint’ is used here in the categorical sense, not the operator-theoretic one. (All operators that we consider are self-adjoint.)

**Definition 3** If $\hat{A}$ is a self-adjoint operator affiliated with a von Neumann algebra $\mathcal{N}$ and $E^A = (E^A_r)_{r \in \mathbb{R}}$ is its extended right-continuous spectral family, then the left adjoint $o^{E^A}$ of $E^A$ is denoted $o^A : \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{R}$ and is called the $q$-observable function associated with the self-adjoint operator $\hat{A}$. 

11
The adjoint functor theorem provides the explicit form of the function $o^E$ adjoint to an extended right-continuous spectral family $E = (E_r)_{r \in \mathbb{R}}$ in $\mathcal{P}(\mathcal{N})$: for all $\hat{P} \in \mathcal{P}(\mathcal{N})$,

$$o^E(\hat{P}) = \inf \{ r \in \mathbb{R} \mid \hat{P} \leq E_r \}. \quad (3.11)$$

If $E = \hat{E}^A$, then

$$oA(\hat{P}) = \inf \{ r \in \mathbb{R} \mid \hat{P} \leq \hat{E}r \}. \quad (3.12)$$

Remark 3 A very similar kind of functions associated with self-adjoint operators was defined by de Groote in [15], Prop. 6.2, but restricted to rank-1 projections (i.e., on projective Hilbert space $\mathcal{PH}$). He called these functions ‘observable functions’, and we adopt this naming in order to show our indebtedness to him. In [17], Def. 2.7 and following arguments, de Groote arrived at a definition very close to the one above, considering join-preserving functions from the non-zero projections to the reals. Comman considers the functions defined above in [2] and proves a number of their properties. It seems that his results were found independently from de Groote’s earlier work.

Yet, the observation that right-continuous spectral families $E$ (resp. $\hat{E}^A$) have left adjoints $o^E$ (resp. $oA$), which are exactly the $q$-observable functions, is new as far as we are aware. The definition via Galois connections is the most natural one, allows proving the properties of these functions in a direct and transparent way and opens up new perspectives. For example, we already saw that preservation of arbitrary joins is an immediate consequence of the adjoint functor theorem for posets.

We have shown so far that for every each $\hat{A} \in \text{SA}(\mathcal{N})$,

$$(oA, \hat{E}^A) \quad (3.13)$$

is a Galois connection between the complete lattices $\mathcal{P}(\mathcal{N})$ and $\mathbb{R}$. Also note that for all non-zero projections $\hat{P} \in \mathcal{P}(\mathcal{N})$,

$$oA(\hat{P}) = \inf \{ r \in \mathbb{R} \mid \hat{P} \leq E^A_r \} > -\infty. \quad (3.14)$$

If $\hat{A}$ is unbounded from below, then the image of $oA$ on non-zero projections is unbounded from below: let $r_0$ be an element in the spectrum of $\hat{A}$, then $oA(E^A_{r_0}) = r_0$. (This fact will also be used below in the proof of Lemma [2]) For the same reason, if $\hat{A}$ is unbounded from above, then the image of $oA$ is unbounded from above. In fact, there exist projections $\hat{P} \in \mathcal{P}(\mathcal{N})$ such that

$$oA(\hat{P}) = \inf \{ r \in \mathbb{R} \mid E^A_r \geq \hat{P} \} = \infty. \quad (3.15)$$
For example, the identity $\hat{1}$ is such a projection: if $\hat{A}$ is unbounded from above, then $\hat{1}$ is not in the usual spectral family of $\hat{A}$ defined over $\mathbb{R}$.

It always holds that $o^A(\hat{0}) = -\infty$, whether $\hat{A}$ is bounded or not.

**Remark 4** Since we consider extended spectral families, defined over the extended reals $\bar{\mathbb{R}}$ (with $E^A_{-\infty} = 0$ and $E^A_{\infty} = 1$), it makes sense to regard the spectrum of a self-adjoint operator $\hat{A}$ as a subset of the extended reals $\bar{\mathbb{R}}$. We include $-\infty$ in the spectrum of $\hat{A}$ if $\hat{A}$ is unbounded from below, and include $\infty$ in the spectrum if $\hat{A}$ is unbounded from above.

Of course, for any self-adjoint operator $\hat{A}$ affiliated with $\mathcal{N}$, there exist projections $\hat{P}$ such that $o^A(\hat{P})$ is some finite real number: take $\hat{P} = E^A_{r_0}$ for some $r_0 \in \text{sp} \hat{A}$ (this is always possible, since the spectrum of a self-adjoint operator is not empty), then $o^A(E^A_{r_0}) = \inf \{ r \in \bar{\mathbb{R}} \mid E^A_r \geq E^A_{r_0} \} = r_0$.

We will now characterise those functions $o : \mathcal{P}(\mathcal{N}) \to \bar{\mathbb{R}}$ that determine extended spectral families by means of a Galois connection, and hence determine self-adjoint operators affiliated with the von Neumann algebra $\mathcal{N}$.

**Definition 4** An abstract $q$-observable function on $\mathcal{P}(\mathcal{N})$ is a join-preserving function $o : \mathcal{P}(\mathcal{N}) \to \bar{\mathbb{R}}$ such that

(a) $o(\hat{0}) > -\infty$ for all $\hat{0} \neq 0$,

(b) there is a family $(\hat{P}_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{N})$ with $\bigvee_{i \in I} \hat{P}_i = \hat{1}$ such that $o(\hat{P}_i) \in \mathbb{R}$ for all $i \in I$.

The set of abstract $q$-observable functions is denoted $QO(\mathcal{P}(\mathcal{N}), \bar{\mathbb{R}})$.

Let $o$ be an abstract $q$-observable function. Note that join-preservation implies

$$o(\hat{0}) = o(\bigvee \emptyset) = \sup \emptyset = -\infty. \quad (3.16)$$

**Proposition 1** Let $\mathcal{N}$ be a von Neumann algebra. There is a bijection between the set $SF(\bar{\mathbb{R}}, \mathcal{P}(\mathcal{N}))$ of extended right-continuous spectral families in $\mathcal{P}(\mathcal{N})$ and the set $QO(\mathcal{P}(\mathcal{N}), \bar{\mathbb{R}})$ of abstract $q$-observable functions on $\mathcal{P}(\mathcal{N})$. Concretely, each abstract $q$-observable function $o$ has a right adjoint $E^o$, which is an extended right-continuous spectral family, and each extended right-continuous spectral family $E$ has a left adjoint $o^E$, which is an abstract $q$-observable function. Moreover, $E^{o^E} = E$ and $o^{E^o} = o$ for all $E \in SF(\bar{\mathbb{R}}, \mathcal{P}(\mathcal{N}))$ and all $o \in QO(\mathcal{P}(\mathcal{N}), \bar{\mathbb{R}})$. 

13
Proof. Let \( o : \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{R} \) be an abstract \( q \)-observable function. The adjoint functor theorem for posets shows that \( o \) has a right adjoint \( \hat{E}^o \), given by

\[
\hat{E}^o : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{N}) \quad r \mapsto \bigvee \{ \hat{P} \in \mathcal{P}(\mathcal{N}) \mid o(\hat{P}) \leq r \}.
\]

The map \( \hat{E}^o \) preserves arbitrary meets (greatest lower bounds), i.e., for all families \((r_i)_{i \in I} \subseteq \mathbb{R}\),

\[
\hat{E}^o(\inf_{i \in I} r_i) = \bigwedge_{i \in I} \hat{E}^o(r_i). \quad (3.18)
\]

In particular, for all \( r \in \mathbb{R} \),

\[
\hat{E}^o(r) = \hat{E}^o(\inf \{ s \in \mathbb{R} \mid r < s \}) = \bigwedge_{s > r} \hat{E}^o(s), \quad (3.19)
\]

so \( \hat{E}^o \) is right-continuous. Obviously, if \( s > r \) then \( \hat{E}^o(s) \geq \hat{E}^o(r) \), so \( \hat{E}^o \) is monotone. Join-preservation of \( o \) implies \( o(\hat{0}) = -\infty \), see (3.16). From this and \( o(\hat{P}) > 0 \) for all \( \hat{P} > 0 \) (which is condition (a) in Def. 4), we obtain

\[
\hat{E}^o(-\infty) = \bigvee \{ \hat{P} \in \mathcal{P}(\mathcal{N}) \mid o(\hat{P}) \leq -\infty \} = \hat{0}. \quad (3.20)
\]

Moreover,

\[
s\text{-lim}_{r \rightarrow -\infty} \hat{E}^o(r) = \bigwedge_{r \in \mathbb{R}} \hat{E}^o(r) \quad (3.21)
\]

\[
= \hat{E}^o(\inf_{r \in \mathbb{R}}) \quad (3.22)
\]

\[
= \hat{E}^o(-\infty) = \hat{0}, \quad (3.23)
\]

where we used that \( \hat{E}^o \) preserves all meets, the fact that \( \inf_{r \in \mathbb{R}} r = -\infty \) in \( \mathbb{R} \), and equation (3.20).

It remains to show that \( s\text{-lim}_{r \rightarrow \infty} \hat{E}^o(r) = \hat{1} \). We have

\[
s\text{-lim}_{r \rightarrow \infty} \hat{E}^o(r) = \bigvee_{r \in \mathbb{R}} \hat{E}^o(r) \quad (3.24)
\]

\[
\geq \bigvee_{r \in \{ o(\hat{P}_i) \mid i \in I \}} \hat{E}^o(r) \quad (3.25)
\]

\[
= \bigvee_{i \in I} \hat{E}^o(o(\hat{P}_i)) \quad (3.26)
\]

\[
= \bigvee_{i \in I} \bigvee \{ \hat{P} \in \mathcal{P}(\mathcal{N}) \mid o(\hat{P}) \leq o(\hat{P}_i) \} \quad (3.27)
\]

\[
\geq \bigvee_{i \in I} \hat{P}_i = \hat{1}. \quad (3.28)
\]

14
Note that $\lim_{r \to \infty} \hat{E}^o(r) = \hat{1}$ is trivial if $o(\hat{1}) \in \mathbb{R}$, since $o(\hat{1})$ is the maximum of the image of $o$ due to monotonicity, and so for all $t \in \mathbb{R}$ with $t \geq o(\hat{1})$, we have

$$\hat{E}^o_t = \bigvee \{ \hat{P} \in \mathcal{P}(\mathcal{N}) \mid o(\hat{P}) \leq t \} = \hat{1}. \quad (3.29)$$

Also, if $o(\hat{1}) \in \mathbb{R}$, condition (b) in Def. 4 is fulfilled for the family $\{ \hat{1} \}$.

Conversely, let $E : \mathbb{R} \to \mathcal{P}(\mathcal{N})$ be an extended right-continuous spectral family. The left adjoint

$$o^E : \mathcal{P}(\mathcal{N}) \to \mathbb{R}$$

$$\hat{P} \mapsto \inf \{ r \in \mathbb{R} \mid \hat{P} \leq E(r) \} \quad (3.30)$$

is the (concrete) $q$-observable function of $\hat{A}^E$, the self-adjoint operator affiliated with $\mathcal{N}$ that is determined by $E$. Being a left adjoint, $o^E$ preserves all joins. In particular,

$$o^E(\hat{0}) = o^E(\bigvee \emptyset) = \sup \emptyset = -\infty. \quad (3.31)$$

If $\hat{P} > \hat{0}$, then

$$o^E(\hat{P}) = \inf \{ r \in \mathbb{R} \mid \hat{P} \leq E(r) \} > -\infty. \quad (3.32)$$

Moreover, $E|\mathbb{R} = (E(r))_{r \in \mathbb{R}}$ is a family of projections with $\bigvee_{r \in \mathbb{R}} E(r) = \hat{1}$ such that

$$\forall r \in \mathbb{R} : o^E(E(r)) = \inf \{ s \in \mathbb{R} \mid E(r) \leq E(s) \} \leq r \in \mathbb{R}, \quad (3.33)$$

so condition (b) in Def. 4 is fulfilled for the family $(E(r))_{r \in \mathbb{R}}$. In fact, one can always use a countable family, e.g. pick $(E(n))_{n \in \mathbb{N}}$. If $o^E(\hat{1}) \in \mathbb{R}$, then the trivial family $\{ \hat{1} \}$ can be used.

The fact that we are using an adjunction (that is, a Galois connection between posets) implies that $E o^E = E$ and $o^E_n = o$ for all $E \in SF(\mathbb{R}, \mathcal{P}(\mathcal{N}))$ and all $o \in QO(\mathcal{P}(\mathcal{N}), \mathbb{R})$, since adjoints are unique.

This leads us to our representation of self-adjoint operators as real-valued functions:

**Theorem 2** Let $\mathcal{N}$ be a von Neumann algebra. There is a bijection between the set $SA(\mathcal{N})$ of self-adjoint operators affiliated with $\mathcal{N}$ and the set $QO(\mathcal{P}(\mathcal{N}), \mathbb{R})$ of abstract $q$-observable functions on $\mathcal{P}(\mathcal{N})$.

**Proof.** Each $\hat{A} \in SA(\mathcal{N})$ determines a unique right-continuous extended spectral family $\hat{E}^A$ in $\mathcal{P}(\mathcal{N})$ by the spectral theorem. By Prop. 1 $\hat{E}^A$
determines a unique abstract $q$-observable function $o^A = o^E^A$. Conversely, each abstract $q$-observable function $o$ determines a unique right-continuous extended spectral family $\hat{E}^o$ and hence a unique self-adjoint operator $\hat{A}^E^o$.

In other words, every abstract $q$-observable function $o$ is the $q$-observable function $o^A$ of some self-adjoint operator affiliated with $\mathcal{N}$, and every self-adjoint operator affiliated with $\mathcal{N}$ also determines an abstract $q$-observable function. Hence, we will speak simply of $q$-observable functions in the following.

Summing up, we have three bijections:

\[
SA(\mathcal{N}) \cong SF(\mathbb{R}, \mathcal{P}(\mathcal{N})), \quad \hat{A} \mapsto \hat{E}^\hat{A}, \quad \hat{E}^\hat{A} \mapsto \int_{-\infty}^{\infty} r \, d\hat{E}^\hat{A}_r; \quad (3.34)
\]

\[
SF(\mathbb{R}, \mathcal{P}(\mathcal{N})) \cong QO(\mathcal{P}(\mathcal{N}), \mathbb{R}), \quad E \mapsto o^E, \quad o^E \mapsto \hat{E}^o; \quad (3.35)
\]

\[
SA(\mathcal{N}) \cong QO(\mathcal{P}(\mathcal{N}), \mathbb{R}), \quad \hat{A} \mapsto o^\hat{A}, \quad o^\hat{A} \mapsto \int_{-\infty}^{\infty} r \, d\hat{E}^\hat{A}_r. \quad (3.36)
\]

**Lemma 2** Let $\mathcal{P}_0(\mathcal{N})$ denote the non-zero projections in a von Neumann algebra $\mathcal{N}$, and let $o^\hat{A} \in QO(\mathcal{P}(\mathcal{N}), \mathbb{R})$ be the $q$-observable function corresponding to a self-adjoint operator $\hat{A} \in SA(\mathcal{N})$. Then

\[
o^\hat{A}(\mathcal{P}_0(\mathcal{N})) = sp \hat{A}, \quad (3.37)
\]

i.e., the image of $o^\hat{A}$ on the non-zero projections is equal to the spectrum of the operator. If $\hat{A}$ is unbounded from above, then $\infty$ is in $o^\hat{A}(\mathcal{P}_0(\mathcal{N}))$. (Recall that we regard $\infty$ as an element of the spectrum of $\hat{A}$ if $\hat{A}$ is unbounded from above, since we see the spectrum of $\hat{A}$ as a subset of the extended reals $\mathbb{R}$.)

**Proof.** Let $r \in sp \hat{A}$, then

\[
o^\hat{A}(\hat{E}^\hat{A}_r) = \inf \{ s \in \mathbb{R} \mid \hat{E}^\hat{A}_s \geq \hat{E}^\hat{A}_r \}, \quad (3.38)
\]

which clearly equals $r$ by right-continuity of $\hat{E}^\hat{A}$. Conversely, if $r$ is in the image of $o^\hat{A}$, then $\hat{E}^\hat{A}_t < \hat{E}^\hat{A}_r$ for all $t < r$, so $\hat{E}^\hat{A}$ is not constant on any neighbourhood of $r$, hence $r \in sp \hat{A}$.

If $\hat{A}$ is unbounded from above, then $\hat{E}^\hat{A}_s < \hat{1}$ for all $s \in \mathbb{R}$ and $\hat{E}^\hat{A}_\infty = \hat{1}$, so $o^\hat{A}(\hat{1}) = \infty$. ■

In Remark 2.13 in [17], de Groote had shown that the image of $o^\hat{A}$ on non-zero-projections is dense in the spectrum of $\hat{A}$. We now see that in fact these two sets are equal.
Of course, if one considers all projections including $\hat{0}$, then
\[ o^{\hat{A}}(P(N)) = \text{sp} \hat{A} \cup \{-\infty\}. \tag{3.39} \]
Since a bounded self-adjoint operator has a compact spectrum, we have:

**Corollary 1** If $\hat{A}$ is a bounded self-adjoint operator in $N$, then the image $o^{\hat{A}}(P_0(N))$ of the corresponding $q$-observable function $o^{\hat{A}}$ is compact.

**Proposition 2** Let $N$ be a von Neumann algebra. There is a bijection between $N_{sa}$, the set of self-adjoint operators in $N$, and $QO^c(P(N), \mathbb{R})$, the $q$-observable functions with compact image on $P_0(N)$.

**Proof.** This follows immediately from Thm. 2, the fact that all self-adjoint operators in $N$ are bounded, and Cor. 1.

## 4 Further properties of $q$-observable functions

### 4.1 Restricting the codomain

Let $\hat{A}$ be a (bounded) self-adjoint operator in a von Neumann algebra $N$. Then the compact set $\text{sp} \hat{A}$ is a complete sublattice of $\mathbb{R}$, that is, arbitrary infima and suprema exist in $\text{sp} \hat{A}$ and coincide with those in $\mathbb{R}$. Hence, we can describe $\hat{A}$ equivalently by a join-preserving function
\[
o^{\hat{A}} : P(N) \rightarrow \text{sp} \hat{A} \cup \{-\infty\} \\
o^{\hat{A}}(P) = \inf\{r \in \text{sp} \hat{A} \cup \{-\infty\} \mid \hat{E}_r \geq \hat{P}\},
\]
given by restricting the codomain of $o^{\hat{A}} : P(N) \rightarrow \mathbb{R}$. Note that $o^{\hat{A}}(P_0(N)) = \text{sp} \hat{A}$; only the zero projection is mapped to $-\infty$. More generally, if $C$ is any compact subset of $\mathbb{R}$ that contains $\text{sp} \hat{A}$, then $C \cup \{-\infty\}$ can serve as the codomain of $o^{\hat{A}}$. Note that if we consider all self-adjoint operators in $N$ (which by definition are bounded), then $\mathbb{R} \cup \{-\infty\} \subset \mathbb{R}$ is the smallest common codomain of the associated $q$-observable functions.

### 4.2 Extending the domain

Let $\hat{A}$ be a self-adjoint operator affiliated with a von Neumann algebra $N$, and let $M \supset N$ be a von Neumann algebra containing $N$ such that the unit elements in $N$ and $M$ coincide. Then $\hat{A}$ is also affiliated with $M$. Let $o^{\hat{A}}_N$ be the $q$-observable function of $\hat{A}$ with respect to $N$, and $o^{\hat{A}}_M$ the one with respect to $M$. Then, since $P(N) \subset P(M)$, we clearly have
\[
o^{\hat{A}}_M|_{P(N)} = o^{\hat{A}}_N. \tag{4.2} \]
There is no difficulty at all in extending the $q$-observable function $o^A_N : \mathcal{P}(N) \to \mathbb{R}$ of $\hat{A}$ (seen as an element of $N$) to the $q$-observable function $o^A_M : \mathcal{P}(M) \to \mathbb{R}$ of $\hat{A}$ (seen as an element of $M$). In particular, $N$ may be abelian, while $M$ is non-abelian.

4.3 Restricting the domain – outer daseinisation of self-adjoint operators

Let $\hat{A}$ be self-adjoint and affiliated with $N$, and let $M \subset N$ be a von Neumann subalgebra such that the unit elements in $N$ and $M$ coincide. We consider the restriction of $o^A : \mathcal{P}(N) \to \mathbb{R}$ to the smaller domain $\mathcal{P}(M)$:

$$o^A|_{\mathcal{P}(M)} : \mathcal{P}(M) \to \text{sp}\hat{A} \cup \{-\infty\} \quad (4.3)$$

$
\hat{Q} \mapsto \inf\{r \in \mathbb{R} \mid \hat{Q} \leq \hat{E}^A_r\}$.

Since $\mathcal{P}(M)$ is a complete sublattice of $\mathcal{P}(N)$, the restriction $o^A|_{\mathcal{P}(M)}$ is a join-preserving function on $\mathcal{P}(M)$ with $o^A|_{\mathcal{P}(M)}(\hat{Q}) > -\infty$ for all $\hat{Q} > \hat{0}$, so it is a $q$-observable function on $\mathcal{P}(M)$. In the following, we will determine which self-adjoint operator affiliated with $M$ corresponds to this $q$-observable function.

Given a projection $\hat{P}$ in $\mathcal{P}(N)$, the projection lattice of the bigger von Neumann algebra $N$, we define an approximation to $\hat{P}$ in the smaller algebra $M$ by

$$\delta_i(\hat{P}) := \bigvee \{\hat{Q} \in \mathcal{P}(M) \mid \hat{Q} \leq \hat{P}\}, \quad (4.4)$$

that is, we take the largest projection in $M$ that is dominated by $\hat{P}$. This approximation was considered before in [15] and [11]. As will be discussed in detail in [9], the map

$$\delta_i : \mathcal{P}(N) \to \mathcal{P}(M) \quad \hat{P} \mapsto \delta_i(\hat{P})_M \quad (4.5)$$

is the right adjoint of the inclusion $i : \mathcal{P}(M) \hookrightarrow \mathcal{P}(N)$. Note that $i$ is a morphism of complete lattices and hence preserves all meets and all joins. Join-preservation implies that $i$ has a right adjoint, which is $\delta_i$. Meet-preservation implies that $i$ has a left adjoint, too. Since we will need it later, we also introduce the left adjoint of $i$:

$$\delta_o : \mathcal{P}(N) \to \mathcal{P}(M) \quad \hat{P} \mapsto \delta_o(\hat{P})_M := \bigwedge \{\hat{Q} \in \mathcal{P}(M) \mid \hat{Q} \geq \hat{P}\}. \quad (4.6)$$

Applying the approximation $\delta_i$ to all elements of the spectral family of our operator $\hat{A}$, we obtain:
**Lemma 3** Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( \mathcal{N} \), and let \( \mathcal{M} \subset \mathcal{N} \) be a von Neumann subalgebra such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. Let \( \hat{E}^A = (\hat{E}^A_r)_{r \in \mathbb{R}} \) be the extended right-continuous spectral family of \( \hat{A} \) in \( \mathcal{P}(\mathcal{N}) \). Then \((\delta^i(\hat{E}^A_r))_{r \in \mathbb{R}} \) is an extended right-continuous spectral family in \( \mathcal{P}(\mathcal{M}) \).

**Proof.** We clearly have \( \delta^i(\hat{E}_{-\infty}^A)_\mathcal{M} = \delta^i(0)_\mathcal{M} = 0 \) and \( \delta^i(\hat{E}^A_\infty)_\mathcal{M} = \delta^i(1)_\mathcal{M} = 1 \). If \( r < s \), then \( \hat{E}^A_r \leq \hat{E}^A_s \), so

\[
\delta^i(\hat{E}^A_r)_\mathcal{M} = \bigvee \{ \hat{Q} \in \mathcal{P}(\mathcal{M}) \mid \hat{Q} \leq \hat{E}^A_r \} \quad (4.7)
\]

\[
\leq \bigvee \{ \hat{R} \in \mathcal{P}(\mathcal{M}) \mid \hat{R} \leq \hat{E}^A_s \} = \delta^i(\hat{E}^A_s)_\mathcal{M}. \quad (4.8)
\]

Moreover, \( \delta^i : \mathcal{P}(\mathcal{N}) \to \mathcal{P}(\mathcal{M}) \) is the right adjoint of the inclusion \( i : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{N}) \) and hence preserves meets, so

\[
\bigwedge_{s > r} \delta^i(\hat{E}^A_s)_\mathcal{M} = \delta^i(\bigwedge_{s > r} \hat{E}^A_s)_\mathcal{M} = \delta^i(\hat{E}^A_r)_\mathcal{M}, \quad (4.10)
\]

that is, \((\delta^i(\hat{E}^A_r)_\mathcal{M})_{r \in \mathbb{R}} \) is right-continuous. \( \blacksquare \)

Coming back to the restriction \( o^A|_{\mathcal{P}(\mathcal{M})} \) of the \( q \)-observable function \( o^A \) of a self-adjoint operator \( \hat{A} \) affiliated with \( \mathcal{N} \), we see that for all \( \hat{Q} \in \mathcal{P}(\mathcal{M}) \),

\[
o^A|_{\mathcal{P}(\mathcal{M})}(\hat{Q}) = \inf \{ r \in \mathbb{R} \mid \hat{Q} \leq \hat{E}^A_r \} \quad (4.11)
\]

\[
= \inf \{ r \in \mathbb{R} \mid \hat{Q} \leq \delta^i(\hat{E}^A_r)_\mathcal{M} \}, \quad (4.12)
\]

since \( \hat{Q} \leq \hat{E}^A_r \) implies \( \hat{Q} \leq \delta^i(\hat{E}^A_r)_\mathcal{M} \), because \( \delta^i(\hat{E}^A_r)_\mathcal{M} \) is the largest projection in \( \mathcal{M} \) dominated by \( \hat{E}^A_r \). This shows:

**Lemma 4** The extended spectral family of the self-adjoint operator corresponding to \( o^A|_{\mathcal{P}(\mathcal{M})} \) is \((\delta^i(\hat{E}^A_r)_\mathcal{M})_{r \in \mathbb{R}} \).

We denote the self-adjoint operator affiliated with \( \mathcal{M} \) whose extended spectral family is \((\delta^i(\hat{E}^A_r)_\mathcal{M})_{r \in \mathbb{R}} \) by \( \delta^o(\hat{A})_\mathcal{M} \), that is,

\[
\delta^o(\hat{A})_\mathcal{M} := \int_{-\infty}^{\infty} r \, d(\delta^i(\hat{E}^A_r)_\mathcal{M}). \quad (4.13)
\]

The operator \( \delta^o(\hat{A})_\mathcal{M} \) is called the outer daseinisation of \( \hat{A} \) to \( \mathcal{M} \).

The following result (for the case of bounded operators) was first observed by de Groote in [18] and discussed in detail in [12, 7].
Proposition 3 Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( \mathcal{N} \), and let \( \mathcal{M} \) be a von Neumann subalgebra of \( \mathcal{N} \) such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. The self-adjoint operator \( \delta^o(\hat{A})_\mathcal{M} \) affiliated with \( \mathcal{M} \) is given by

\[
\delta^o(\hat{A})_\mathcal{M} = \bigwedge \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \succeq_s \hat{A} \},
\]

where the meet is taken with respect to the spectral order on \( SA(\mathcal{M}) \).

Proof. Clearly, \( \delta^o(\hat{A})_\mathcal{M} \succeq_s \hat{A} \), since for each \( r \in \mathbb{R} \), we have

\[
\hat{E}_r^{\delta^o(\hat{A})}_\mathcal{M} = \delta^o(\hat{E}_r^{\hat{A}})_\mathcal{M} \leq \hat{E}_r^{\hat{A}}.
\]

Let \( \hat{B} \in SA(\mathcal{M}) \) such that \( \hat{B} \succeq_s \hat{A} \). This is equivalent to

\[
\forall r \in \mathbb{R}: \hat{E}_r^{\hat{B}} \leq \delta^o(\hat{E}_r^{\hat{A}})_\mathcal{M} \leq \hat{E}_r^{\hat{A}},
\]

so \( \hat{B} \succeq_s \delta^o(\hat{A})_\mathcal{M} \). Hence, \( \delta^o(\hat{A})_\mathcal{M} \) is the greatest lower bound (meet) of the set \( \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \succeq_s \hat{A} \} \).

Hence, we have shown:

**Theorem 3** Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( \mathcal{N} \), and let \( \mathcal{M} \subset \mathcal{N} \) be a von Neumann subalgebra such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. The restriction of the \( \eta \)-observable function \( o^\hat{A} : \mathcal{P}(\mathcal{N}) \to \mathbb{R} \) to the smaller domain \( \mathcal{P}(\mathcal{M}) \) corresponds to the self-adjoint operator

\[
\delta^o(\hat{A})_\mathcal{M} = \bigwedge \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \succeq_s \hat{A} \}
\]

affiliated with \( \mathcal{M} \), that is, \( o^\hat{A}|_{\mathcal{P}(\mathcal{M})} = o^{\delta^o(\hat{A})}_\mathcal{M} \).

**Lemma 5** If \( \hat{A} \) is a bounded self-adjoint operator in a von Neumann algebra \( \mathcal{N} \) and \( \mathcal{M} \subset \mathcal{N} \) is a von Neumann subalgebra with the same unit element, then \( \delta^o(\hat{A})_\mathcal{M} \) is bounded as well and is contained in \( \mathcal{M}_{sa} \).

Proof. Let \( \mathcal{P}_0(\mathcal{M}) := \mathcal{P}(\mathcal{M}) \setminus \{\hat{0}\} \). Then \( o^\hat{A}|_{\mathcal{P}(\mathcal{M})}(\mathcal{P}_0(\mathcal{M})) \) is a subset of the compact set \( sp \hat{A} \) (since \( \mathcal{P}_0(\mathcal{M}) \subset \mathcal{P}_0(\mathcal{N}) \) and \( o^\hat{A}(\mathcal{P}_0(\mathcal{N})) = sp \hat{A} \) according to Lemma 2). Since \( o^\hat{A}|_{\mathcal{P}(\mathcal{M})} \) determines a unique self-adjoint operator \( \delta^o(\hat{A})_\mathcal{M} \), as we saw above, this operator has spectrum \( o^\hat{A}|_{\mathcal{P}(\mathcal{M})}(\mathcal{P}_0(\mathcal{M})) \subset sp \hat{A} \), so it is bounded. All spectral projections of \( \delta^o(\hat{A})_\mathcal{M} \) lie in \( \mathcal{M} \), so \( \delta^o(\hat{A})_\mathcal{M} \in \mathcal{M}_{sa} \).

**Remark 5** In [11, 12, 7] and other places, the projection \( \delta^i(\hat{P})_\mathcal{M} \) is called the inner daseinisation of the projection \( \hat{P} \) to \( \mathcal{M} \), and \( \delta^o(\hat{P})_\mathcal{M} \) is called the
outer daseinisation of $\hat{P}$ to $\mathcal{M}$. Similarly, $\delta^i(\hat{A})_{\mathcal{M}}$ (not discussed yet) and $\delta^o(\hat{A})_{\mathcal{M}}$ are called inner and outer daseinisation of the self-adjoint operator $\hat{A}$ to $\mathcal{M}$. These constructions play a central role in the topos approach to quantum theory [14].

The main motivation for the current article was to gain a better mathematical understanding of daseinisation of self-adjoint operators and the underlying operator-theoretic constructions. In standard quantum theory, physical quantities are represented by self-adjoint operators. In the topos approach, physical quantities are represented by certain natural transformations generalising functions from the state space of a system to the real numbers (see [12]). This ‘function-like’ representation of physical quantities is based on daseinisation of self-adjoint operators. As we have seen, to each self-adjoint operator $\hat{A}$ there corresponds a q-observable function $o^\hat{A} : \mathcal{P}(\mathcal{N}) \to \mathbb{R}$, and this function gives all the ‘daseinised’ (i.e., approximated) operators $\delta^o(\hat{A})_{\mathcal{M}}$ simply by restriction, $o^\hat{A}|_{\mathcal{P}(\mathcal{M})} : \mathcal{P}(\mathcal{M}) \to \mathbb{R}$. In the topos approach, one considers such approximations/restrictions to all contexts, i.e., abelian von Neumann subalgebras $V$ of $\mathcal{N}$ (where $V$ and $\mathcal{N}$ have the same unit element). Physically, each abelian subalgebra represents one classical perspective on the quantum system.

If $\mathcal{P}$ is some complete sublattice of $\mathcal{P}(\mathcal{N})$, but not necessarily the lattice of projections of a von Neumann subalgebra $\mathcal{M}$, then all arguments about restrictions made so far generalise straightforwardly. For example, one may consider a unital $C^*$-algebra $\mathcal{A}$ and define open projections in its enveloping von Neumann algebra $\mathcal{A}''$ as those projections that are suprema of nets of positive elements in $\mathcal{A}$. The open projections form a complete sublattice of the lattice of all projections in $\mathcal{A}''$, and self-adjoint operators whose spectral families consist of open projections only can be interpreted as analogues of continuous functions [1], or semicontinuous functions [4]. For a lattice-theoretic treatment of the latter case, see [2].

Lemma 6 Let $\hat{A}$ be a self-adjoint operator affiliated with $\mathcal{N}$, and let $o^\hat{A} : \mathcal{P}(\mathcal{N}) \to \mathbb{R}$ be its q-observable function. Then

$$\forall \hat{P} \in \mathcal{P}_0(\mathcal{N}) : o^\hat{A}(\hat{P}) = \inf\{r \in \mathbb{R} \mid \hat{P} \leq \hat{E}^\hat{A}_r\} = \min\{r \in \mathbb{R} \mid \hat{P} \leq \hat{E}^\hat{A}_r\}. \quad (4.18)$$

This follows directly from meet-preservation of $\hat{E}^\hat{A} : \mathbb{R} \to \mathcal{P}(\mathcal{N})$.

4.4 Representation of the lattice of self-adjoint operators

We can say more about how the spectral order relates to our constructions. In particular, the spectral order on self-adjoint operators corresponds to the pointwise order on the associated q-observable functions.
Proposition 4 Let \((SA(N), \leq_s)\) be the poset of self-adjoint operators affiliated with \(N\), equipped with the spectral order, and let \((QO(P(N), \mathbb{R}), \leq)\) be the poset of \(q\)-observable functions, equipped with the pointwise order. The map
\[
\phi : (SA(N), \leq_s) \rightarrow (QO(P(N), \mathbb{R}), \leq) \tag{4.19}
\]
\[
\hat{A} \mapsto o^\hat{A}
\]
is an order-isomorphism of conditionally complete lattices.

Proof. Let \(\hat{A}, \hat{B}\) be two self-adjoint operators affiliated with a von Neumann algebra \(N\). Then
\[
\hat{A} \leq_s \hat{B} \iff \forall r \in \mathbb{R} : \hat{E}_r^\hat{A} \geq \hat{E}_r^\hat{B}, \tag{4.20}
\]
which implies, for all \(\hat{P} \in P(N)\),
\[
o^{\hat{A}}(\hat{P}) = \inf \{ r \in \mathbb{R} | \hat{E}_r^\hat{A} \geq \hat{P} \} \leq \inf \{ t \in \mathbb{R} | \hat{E}_t^\hat{B} \geq \hat{P} \} = o^{\hat{B}}(\hat{P}). \tag{4.21}
\]
Conversely, if \(o^{\hat{A}} \leq o^{\hat{B}}\), then
\[
\forall r \in \mathbb{R} : \hat{E}_r^{o^{\hat{A}}} = \hat{E}_r^\hat{A} \geq \bigvee \{ \hat{P} \in P(N) | r \geq o^{\hat{A}}(\hat{P}) \} \geq \bigvee \{ \hat{Q} \in P(N) | r \geq o^{\hat{B}}(\hat{Q}) \} = \hat{E}_r^{o^{\hat{B}}} = \hat{E}_r^\hat{B}, \tag{4.22}
\]
so \(\hat{A} \leq_s \hat{B}\). \(\blacksquare\)

Hence, we can represent the set \((SA(N), \leq_s)\) of self-adjoint operators affiliated with the von Neumann algebra \(N\), equipped with the spectral order, faithfully by the set of real-valued functions in \((QO(P(N), \mathbb{R}), \leq)\), partially ordered under the pointwise order.

Let \((SF(\mathbb{R}, P(N)), \leq_i)\) be the poset of extended spectral families in \(P(N)\), equipped with the inverse pointwise order, that is,
\[
\hat{E}_r^{\hat{A}} \leq_i \hat{E}_r^{\hat{B}} :\iff (\forall r \in \mathbb{R} : \hat{E}_r^{\hat{A}} \geq \hat{E}_r^{\hat{B}}). \tag{4.25}
\]
Then \((SF(\mathbb{R}, P(N)), \leq_i)\) is order-isomorphic to \((SA(N), \leq_s)\) by definition of the spectral order, and one easily shows:

Lemma 7 The map
\[
\phi' : (QO(P(N), \mathbb{R}), \leq) \rightarrow (SF(\mathbb{R}, P(N)), \leq_i) \tag{4.26}
\]
\[
o \mapsto \hat{E}^o \tag{4.27}
\]
is an order-isomorphism of conditionally complete lattices.
4.5 Algebraic structure

The order-isomorphism (4.19) between the self-adjoint operators affiliated with $\mathcal{N}$ and their $q$-observable functions, sending $\hat{A}$ to $o^\hat{A}$, is not a linear map. Counterexamples are easy to find: consider e.g. $\hat{A} = 1 = Q + (1 - Q)$, where $Q$ is some non-zero projection. We have, for all $\hat{P} \in \mathcal{P}_0(\mathcal{N})$,

$$o^1(\hat{P}) = \inf\{r \in \mathbb{R} \mid \hat{E}^1_r \geq P\} = 1.$$ (4.28)

Assume that $\hat{P}$ is a projection such that $\hat{P} \not\leq \hat{Q}$ and $\hat{P} \not\leq \hat{1} - \hat{Q}$, then

$$o^Q(\hat{P}) + o^{1-Q}(\hat{P}) = \inf\{r \in \mathbb{R} \mid \hat{E}^Q_r \geq \hat{P}\} + \inf\{s \in \mathbb{R} \mid \hat{E}^{1-Q}_s \geq \hat{P}\} = 1 + 1 = 2,$$ (4.29)

so $o^{Q+(1-Q)} \neq o^Q + o^{1-Q}$.

Yet, a limited amount of algebraic structure is preserved by the map $\hat{A} \mapsto o^\hat{A}$, as we will now show. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that preserves all joins (that is, suprema), including the empty join $\bigvee \emptyset = \sup \emptyset = -\infty$ and the maximal join $\bigvee \mathbb{R} = \sup \mathbb{R} = \infty$ (i.e., $f(-\infty) = -\infty$ and $f(\infty) = \infty$). Moreover, we assume that $f(\mathbb{R}) \subseteq \mathbb{R}$. By the adjoint functor theorem for posets, $f$ has a right adjoint

$$g : \mathbb{R} \to \mathbb{R}$$

$$r \mapsto \sup\{s \in \mathbb{R} \mid f(s) \leq r\}$$ (4.31)

that preserves all meets, i.e., infima. In particular, $g(-\infty) = -\infty$ and $g(\infty) = \infty$. If $f(\mathbb{R}) \subseteq \mathbb{R}$, then also $g(\mathbb{R}) \subseteq \mathbb{R}$.

**Proposition 5** Let $\hat{A}$ be a self-adjoint operator affiliated with a von Neumann algebra $\mathcal{N}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a join-preserving function such that $f(\mathbb{R}) \subseteq \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the right adjoint of $f$. For all $r \in \mathbb{R}$,

$$\hat{E}_f(\hat{A}) = \hat{E}^A_{g(r)}.$$ (4.33)

**Proof.** Let $\mathcal{B}(\mathbb{R})$ denote the Borel subsets of $\mathbb{R}$, and let $e^{f(\hat{A})} : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{N})$ be the spectral measure of $f(\hat{A})$. Then

$$\hat{E}_f(\hat{A}) = e^{f(\hat{A})}((-\infty, r]) = e^{\hat{A}}((-\infty, x])$$ (4.34)

for some $x \in \mathbb{R}$. Since $f$ is monotone, $x$ will only depend on $r$, the maximum of the interval $(-\infty, r]$, so $x = h(r)$ for some function $h : \mathbb{R} \to \mathbb{R}$. If $f$
happens to be injective, we can take $h = f^{-1}$ and thus $h(r) = f^{-1}(r)$. In general, we must take

$$h(r) = \sup\{s \in \mathbb{R} \mid f(s) \leq r\},$$  \hspace{1cm} (4.35)

that is, $h = g$, the right adjoint of $f$. Then, by join-preservation of $f$,

$$f(g(r)) = f(\sup\{s \in \mathbb{R} \mid f(s) \leq r\}) = \sup_{s : f(s) \leq r} f(s) \leq r,$$  \hspace{1cm} (4.36)

and $f(t) > r$ for any $t > g(r)$. So we obtain

$$\hat{E}_r^{f(\hat{A})} = e^{f(\hat{A})}((\infty, r]) = e^{\hat{A}((\infty, g(r)])} = \hat{E}_{g(r)}^{\hat{A}}.$$  \hspace{1cm} (4.37)

This result is interesting, because the kind of functions acting on self-adjoint operators that is considered here is characterised by an order-theoretic property (join-preservation) and not by a measure-theoretic/topological one, as is usual (i.e., being a Borel function).

We can now express the $q$-observable function of $f(\hat{A})$ in terms of the $q$-observable function of $\hat{A}$:

**Theorem 4** Let $\hat{A}$ be a self-adjoint operator affiliated with a von Neumann algebra $\mathcal{N}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a join-preserving function such that $f(\mathbb{R}) \subseteq \mathbb{R}$. Then

$$o^{f(\hat{A})} = f(o^{\hat{A}}).$$  \hspace{1cm} (4.38)

**Proof.** Let $g : \mathbb{R} \to \mathbb{R}$ denote the right adjoint of $f$. By definition, $o^{\hat{A}}(\hat{P}) = \inf\{t \in \mathbb{R} \mid \hat{E}_t^{\hat{A}} \geq \hat{P}\}$, so

$$\hat{E}_{g(r)}^{\hat{A}} \geq \hat{P} \iff g(r) \geq o^{\hat{A}}(\hat{P}).$$  \hspace{1cm} (4.39)

Moreover, $f : \mathbb{R} \to \mathbb{R}$ is the left adjoint of $g : \mathbb{R} \to \mathbb{R}$, so

$$g(r) \geq z \iff r \geq f(z).$$  \hspace{1cm} (4.40)

We obtain, for all $\hat{P} \in \mathcal{P}(\mathcal{N})$,

$$o^{f(\hat{A})}(\hat{P}) = \inf\{r \in \mathbb{R} \mid \hat{E}_r^{f(\hat{A})} \geq \hat{P}\}$$  \hspace{1cm} (4.41)

$\overset{\text{Prop. 4.39}}{=} \inf\{r \in \mathbb{R} \mid \hat{E}_{g(r)}^{\hat{A}} \geq \hat{P}\}$  \hspace{1cm} (4.42)

$\overset{(4.39)}{=} \inf\{r \in \mathbb{R} \mid g(r) \geq o^{\hat{A}}(\hat{P})\}$  \hspace{1cm} (4.43)

$\overset{(4.40)}{=} \inf\{r \in \mathbb{R} \mid r \geq f(o^{\hat{A}}(\hat{P}))\}$  \hspace{1cm} (4.44)

$= f(o^{\hat{A}}(\hat{P})).$  \hspace{1cm} (4.45)
Corollary 2. Let $t \in \mathbb{R}$ and $\hat{A} \in SA(\mathcal{N})$, then $o^{\hat{A}+t1} = o^{\hat{A}} + t$. If $s \in \mathbb{R}^+$ is a positive real number, then $o^{s\hat{A}} = so^{\hat{A}}$.

Proof. The function

$$f_1 : \mathbb{R} \rightarrow \mathbb{R}$$

$$r \mapsto r + t,$$

where $-\infty + t = -\infty$ and $\infty + t = \infty$, preserves suprema. Hence, by Thm. 4, it holds that

$$o^{\hat{A}+t1} = o^{f_1(\hat{A})} = f_1(o^{\hat{A}}) = o^{\hat{A}} + t.$$ (4.48)

Similarly,

$$f_2 : \mathbb{R} \rightarrow \mathbb{R}$$

$$r \mapsto sr$$

preserves suprema, so $o^{s\hat{A}} = so^{\hat{A}}$. ◼

5 $q$-antonymous functions

It is natural to ask how a $q$-observable function $o^{\hat{A}}$ behaves when multiplied by a negative real number. The essential issue of course is the behaviour under multiplication by $-1$, which we will consider now. This will lead us to the definition of a second kind of functions associated with self-adjoint operators, besides the $q$-observable functions.

In subsection 5.2 we will show how this new kind of functions, called $q$-antonymous functions, arises from Galois connections as well, and in subsection 5.3 the relation to the inner daseinisation of self-adjoint operators will be described.

5.1 Definition of $q$-antonymous functions

Let $\hat{A}$ be a self-adjoint operator affiliated with a von Neumann algebra $\mathcal{N}$, and let $o^{\hat{A}}$ be its $q$-observable function. Then, for all $\hat{P} \in \mathcal{P}(\mathcal{N})$,

$$-o^{\hat{A}}(\hat{P}) = -\inf\{r \in \mathbb{R} \mid \hat{P} \leq \hat{E}_{r^+}^{\hat{A}}\}$$

$$= \sup\{-r \in \mathbb{R} \mid \hat{P} \leq \hat{E}_{r^+}^{\hat{A}}\}$$

$$= \sup\{r \in \mathbb{R} \mid \hat{P} \leq \hat{E}_{-r}^{\hat{A}}\}. \quad (5.1)$$

In the following, some arguments will involve left-continuous extended spectral families $\hat{F}^{\hat{A}} : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{N})$ (where $\hat{F}^{\hat{A}}_{-\infty} = 0$ and $\hat{F}^{\hat{A}}_\infty = 1$, of course) such
that
\[ \forall r \in \mathbb{R} : \hat{F}_r^\hat{A} = \bigvee_{q < r} \hat{F}_q^\hat{A}. \quad (5.4) \]

Given an extended spectral family \( \hat{E}^\hat{A} : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \), which need not be right-continuous, one obtains an extended left-continuous spectral family by left regularisation, defined by
\[ \forall r \in \mathbb{R} : l(\hat{E}^\hat{A})_r := \bigvee_{q < r} \hat{E}_q^\hat{A}. \quad (5.5) \]

Right regularisation is defined analogously. The spectral theorem of course works both for right-continuous and left-continuous spectral families (in fact, it does not depend on continuity of the spectral family).

Let \( \hat{F}^{-\hat{A}} = (\hat{F}_r^{-\hat{A}})_{r \in \mathbb{R}} \) be the unique left-continuous extended spectral family of \(-\hat{A}\). As is well known,
\[ \forall r \in \mathbb{R} : \hat{F}_r^{-\hat{A}} = \mathbb{1} - \hat{E}_r^{-\hat{A}}. \quad (5.6) \]

Hence, we obtain from equation (5.3)
\[ -\hat{o}^\hat{A}(\hat{P}) = \sup \{ r \in \mathbb{R} \mid \hat{P} \leq \mathbb{1} - \hat{F}_r^{-\hat{A}} \}. \quad (5.7) \]

As one may have expected, \(-\hat{o}^\hat{A}\) relates to \(-\hat{A}\) (but \(-\hat{o}^\hat{A}\) is not a \(q\)-observable function).

**Definition 5** Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( \mathcal{N} \). The function
\[ a^\hat{A} : \mathcal{P}(\mathcal{N}) \longrightarrow \mathbb{R} \]
\[ \hat{P} \mapsto \sup \{ r \in \mathbb{R} \mid \hat{P} \leq \mathbb{1} - \hat{F}_r^{-\hat{A}} \} \]
is called the \(q\)-antonymous function associated with \( \hat{A} \). The set of \(q\)-antonymous functions of self-adjoint operators affiliated with \( \mathcal{N} \) is denoted \( QA(\mathcal{P}(\mathcal{N}), \mathbb{R}) \).

We already saw that there is a bijection
\[ n : QO(\mathcal{P}(\mathcal{N}), \mathbb{R}) \longrightarrow QA(\mathcal{P}(\mathcal{N}), \mathbb{R}) \]
\[ \hat{A} \mapsto -\hat{o}^\hat{A} = a^{-\hat{A}} \]

between the sets of \(q\)-observable and \(q\)-antonymous functions associated with a von Neumann algebra \( \mathcal{N} \). Thm. \( \mathbb{2} \) implies that there is a bijection between \( QA(\mathcal{P}(\mathcal{N}), \mathbb{R}) \) and \( SA(\mathcal{N}) \), the self-adjoint operators affiliated with \( \mathcal{N} \). Hence, \(q\)-antonymous functions provide a second representation of self-adjoint operators by real-valued functions.
Proposition 6  There is an order-isomorphism
\[ \gamma : SA(N) \to QA(\mathcal{P}(N), \mathbb{R}) \]  \hspace{1cm} (5.10)
\[ \hat{A} \mapsto a^{\hat{A}}. \]

**Proof.** This can be proven in a similar way as Prop. 4, or using Prop. 4, more directly as follows: for all \( \hat{A}, \hat{B} \in SA(N) \),
\[ \hat{A} \leq_s \hat{B} \iff -\hat{A} \geq_s -\hat{B} \]  \hspace{1cm} (5.11)
\[ \iff -o^{-\hat{A}} \leq -o^{-\hat{B}} \]  \hspace{1cm} (5.12)
\[ \iff a^{\hat{A}} \leq a^{\hat{B}}. \]  \hspace{1cm} (5.13)

Clearly, we have, for all \( \hat{A} \) affiliated with \( N \):
(a) \( a^{\hat{A}} : \mathcal{P}(N) \to \mathbb{R} \) is antitone (order-reversing),
(b) \( \text{im} a^{\hat{A}} = -\text{im} o^{-\hat{A}} = -(\text{sp}(\hat{A}) \cup \{-\infty\}) = \text{sp} \hat{A} \cup \{\infty\} \). In particular, \( a^{\hat{A}}(0) = \infty \) and \( a^{\hat{A}}(1) = \inf \text{sp} \hat{A} \), which is a minimum if \( \text{sp} \hat{A} \) is bounded from below, and \(-\infty\) otherwise.

Moreover, we have:

**Lemma 8** Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( N \), and let \( o^{\hat{A}} \) and \( a^{\hat{A}} \) be its q-observable resp. q-antonymous function. Then
\[ \forall \hat{P} \in \mathcal{P}(N) \setminus \{\hat{0}, \hat{1}\} : a^{\hat{A}}(\hat{P}) \leq o^{\hat{A}}(\hat{P}). \]  \hspace{1cm} (5.15)

**Proof.** Let \( \hat{E}^{\hat{A}} \) denote the right-continuous spectral family of \( \hat{A} \) and \( \hat{F}^{\hat{A}} \) the left-continuous one. Observe that
\[ a^{\hat{A}}(\hat{P}) = \sup\{r \in \mathbb{R} \mid \hat{P} \leq \hat{1} - \hat{F}^{\hat{A}}_r\} \]  \hspace{1cm} (5.16)
\[ = \sup\{r \in \mathbb{R} \mid \hat{P} \leq \hat{1} - \hat{E}^{\hat{A}}_r\}. \]  \hspace{1cm} (5.17)

Let \( r \in \mathbb{R} \) be such that \( \hat{P} \leq \hat{E}^{\hat{A}}_r \), then \( \hat{P} \not\leq \hat{1} - \hat{E}^{\hat{A}}_r \), so \( r > a^{\hat{A}}(\hat{P}) \). Since
\[ o^{\hat{A}}(\hat{P}) = \inf\{r \in \mathbb{R} \mid \hat{P} \leq \hat{1} - \hat{E}^{\hat{A}}_r\}, \]  \hspace{1cm} (5.18)
it follows that \( o^{\hat{A}}(\hat{P}) \geq a^{\hat{A}}(\hat{P}). \)  \hspace{1cm} ■

**Remark 6** Antonymous functions were first introduced by one of us (AD) in a similar form in [6], where some of their properties were proven. Yet, the natural definition from a Galois connection, to be introduced in the next subsection, was missing. De Groote was the first to point out [17] that \(-o^A = a^{-A}\) (for a similar kind of functions).
5.2 \( q \)-antonymous functions from an adjunction

Just like \( q \)-observable functions, \( q \)-antonymous functions arise from an adjunction. In order to argue with monotone functions and their adjoints as in the rest of this article, we reverse the order on \( \mathcal{P}(\mathcal{N}) \), the domain of \( \hat{a}^A \) (and hence the codomain of a tentative adjoint) for now. We later switch back from \( \mathcal{P}(\mathcal{N})^{op} \) to \( \mathcal{P}(\mathcal{N}) \).

Meets and joins in \( \mathcal{P}(\mathcal{N})^{op} \) will be denoted \( \tilde{\wedge} \) and \( \tilde{\vee} \), respectively. The partial order in \( \mathcal{P}(\mathcal{N})^{op} \) is written \( \tilde{\leq} \). With \( \tilde{\wedge} \), \( \tilde{\vee} \) and \( \tilde{\leq} \) denoting the operations in \( \mathcal{P}(\mathcal{N}) \), we of course have the correspondences \( \tilde{\wedge} \leftrightarrow \tilde{\vee} \), \( \tilde{\vee} \leftrightarrow \tilde{\wedge} \), and \( \tilde{\leq} \leftrightarrow \tilde{\geq} \).

Reinterpreting \( \hat{a}^A : \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{R} \) as a function from \( \mathcal{P}(\mathcal{N})^{op} \), we have the monotone function

\[
\tilde{a}^A : \mathcal{P}(\mathcal{N})^{op} \rightarrow \tilde{\mathbb{R}} \\
\hat{P} \mapsto \sup \{ r \in \mathbb{R} \mid \hat{P} \gtrless \hat{1} - \hat{\mathcal{F}}^A_r \}.
\]

**Lemma 9** \( \tilde{a}^A : \mathcal{P}(\mathcal{N})^{op} \rightarrow \tilde{\mathbb{R}} \) preserves all meets.

**Proof.** Let \( (\hat{P}_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{N})^{op} \), then

\[
\tilde{a}^A(\bigwedge_{i \in I} \hat{P}_i) = \sup \{ r \in \mathbb{R} \mid \bigwedge_{i \in I} \hat{P}_i \gtrless \hat{1} - \hat{\mathcal{F}}^A_r \} \\
= \sup \{ r \in \mathbb{R} \mid \forall i \in I : \hat{P}_i \gtrless \hat{1} - \hat{\mathcal{F}}^A_r \} \\
= \inf \sup \{ r \in \mathbb{R} \mid \hat{P}_i \gtrless \hat{1} - \hat{\mathcal{F}}^A_r \} \\
= \inf_{i \in I} \tilde{a}^A(\hat{P}_i).
\]

Hence, \( \tilde{a}^A : \mathcal{P}(\mathcal{N})^{op} \rightarrow \tilde{\mathbb{R}} \) has a left adjoint

\[
\hat{G}^A : \tilde{\mathbb{R}} \rightarrow \mathcal{P}(\mathcal{N})^{op} \\
r \mapsto \tilde{\bigwedge} \{ \hat{P} \in \mathcal{P}(\mathcal{N})^{op} \mid r \leq \tilde{a}^A(\hat{P}) \}.
\]

**Lemma 10** The left adjoint \( \hat{G}^A : \tilde{\mathbb{R}} \rightarrow \mathcal{P}(\mathcal{N})^{op} \) is given by \( \hat{G}^A_r = \hat{1} - \hat{\mathcal{F}}^A_r \) for all \( r \in \tilde{\mathbb{R}} \), where \( \hat{\mathcal{F}}^A = (\hat{\mathcal{F}}^A_r)_{r \in \tilde{\mathbb{R}}} \) is the left-continuous extended spectral family of \( \hat{A} \).

**Proof.** \( (\hat{G}^A, \tilde{a}^A) \) is a Galois connection, with \( \hat{G}^A \) the left adjoint and \( \tilde{a}^A \) the right adjoint, so

\[
\forall \hat{P} \in \mathcal{P}(\mathcal{N})^{op} \forall r \in \tilde{\mathbb{R}} : \quad \hat{G}^A_r \gtrless \hat{P} \iff r \leq \tilde{a}^A(\hat{P}).
\]
Clearly, we have
\[ \forall \hat{P} \in \mathcal{P}(\mathcal{N})^{\text{op}} \forall r \in \mathbb{R} : \quad \hat{1} - \hat{F}_r^\hat{A} \preceq \hat{P} \iff r \leq \sup \{ s \in \mathbb{R} \mid \hat{P} \succeq \hat{1} - \hat{F}_s^\hat{A} \}. \]
(5.26)

Since left adjoints are unique, it follows
\[ \forall r \in \mathbb{R} : \quad \hat{G}_r^\hat{A} = \hat{1} - \hat{F}_r^\hat{A}. \]
(5.27)

We see that \( \tilde{a}^\hat{A} \) (and thus also \( a^\hat{A} \)) determines \( \hat{G}^\hat{A}, \hat{F}^\hat{A} \) and hence \( \hat{A} \) uniquely, and conversely it is uniquely determined by them. Most results from the previous sections on \( q \)-observable functions can be transferred straightforwardly to \( q \)-antonymous functions. For example,
\[ a^\hat{A}(\mathcal{P}_0(\mathcal{N})) = \text{sp} \hat{A}, \]
(5.28)

which is proven in a similar way to Lemma 3. It is also possible to give an abstract characterisation of \( q \)-antonymous functions in analogy to Def. 4, without any reference to self-adjoint operators. More directly, using Def. 4, Prop. 1 and the bijection (5.9), we see that each function of the form
\[ a = -o : \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{R}, \]
where \( o \) is an (abstract) \( q \)-observable function, is an abstract \( q \)-antonymous function.

As mentioned, \( \tilde{a}^\hat{A} : \mathcal{P}(\mathcal{N})^{\text{op}} \rightarrow \mathbb{R} \) is monotone, while \( a^\hat{A} : \mathcal{P}(\mathcal{N}) \rightarrow \mathbb{R} \) is antitone (order-reversing). \( \tilde{a}^\hat{A} \) preserves meets by Lemma 3, so \( a^\hat{A} \) maps joins to meets: let \((\hat{P}_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{N})\), then
\[ a^\hat{A}(\bigvee_{i \in I} \hat{P}_i) = \tilde{a}^\hat{A}(\bigwedge_{i \in I} \hat{P}_i) = \inf_{i \in I} \tilde{a}^\hat{A}(\hat{P}_i) = \inf_{i \in I} a^\hat{A}(\hat{P}_i). \]
(5.29)

### 5.3 Restrictions and inner daseinisation of self-adjoint operators

Let \( \hat{A} \) be a self-adjoint operator \( \hat{A} \) affiliated with a von Neumann algebra \( \mathcal{N} \), and let \( \mathcal{M} \subset \mathcal{N} \) be a von Neumann subalgebra such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. We saw in Thm. 3 that the restriction of the \( q \)-observable function \( a^\hat{A} \) of \( \hat{A} \) to the projection lattice \( \mathcal{P}(\mathcal{M}) \subset \mathcal{P}(\mathcal{N}) \) of \( \mathcal{M} \) is the \( q \)-observable function of the self-adjoint operator
\[ \delta^o(\hat{A})_\mathcal{M} = \bigwedge \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{A} \leq_s \hat{B} \}, \]
(5.30)
affiliated with \( \mathcal{M} \), called the outer daseinisation of \( \hat{A} \) to \( \mathcal{M} \). (Recall that the meet is taken with respect to the spectral order on \( SA(\mathcal{M}) \).)

It is natural to ask if there is a function from \( \mathcal{P}(\mathcal{N}) \) to \( \overline{\mathbb{R}} \) such that its restriction to \( \mathcal{P}(\mathcal{M}) \) corresponds to the inner daseinisation
\[ \delta^i(\hat{A})_\mathcal{M} = \bigvee \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \leq_s \hat{A} \} \]
(5.31)
of \( \hat{A} \) to \( \mathcal{M} \). We will show that the \( q \)-observable function \( a^{\hat{A}} : \mathcal{P}(\mathcal{N}) \to \overline{\mathbb{R}} \) plays this role.

Recall that the outer daseinisation of projections (which is needed in the definition of the inner daseinisation of self-adjoint operators) is given by (4.6),

\[
\delta^o(\hat{P})_{\mathcal{M}} = \bigwedge \{ \hat{Q} \in \mathcal{P}(\mathcal{M}) \mid \hat{P} \leq \hat{Q} \},
\]

where \( \hat{P} \in \mathcal{P}(\mathcal{N}) \) and \( \mathcal{M} \subset \mathcal{N} \) is a von Neumann subalgebra such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. One observes:

\[
\forall \hat{P} \in \mathcal{P}(\mathcal{N}) : \delta^i(\hat{1} - \hat{P})_{\mathcal{M}} = \bigvee \{ \hat{Q} \in \mathcal{P}(\mathcal{M}) \mid \hat{Q} \leq \hat{1} - \hat{P} \} = \hat{1} - \bigvee \delta^o(\hat{P})_{\mathcal{M}}.
\]

The restriction \( a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})} \) of the \( q \)-antonymous function \( a^{\hat{A}} \) of a self-adjoint operator \( \hat{A} \) affiliated with \( \mathcal{N} \) to the projections \( \mathcal{P}(\mathcal{M}) \) in the subalgebra \( \mathcal{M} \subset \mathcal{N} \) is given for all \( \hat{Q} \in \mathcal{P}(\mathcal{M}) \) by

\[
a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})}(\hat{Q}) = \text{sup} \{ r \in \overline{\mathbb{R}} \mid \hat{Q} \leq \hat{1} - \hat{F}^A_r \} = \text{sup} \{ r \in \overline{\mathbb{R}} \mid \hat{Q} \leq \delta^i(\hat{1} - \hat{F}^A_r)_{\mathcal{M}} \} = \text{sup} \{ r \in \overline{\mathbb{R}} \mid \hat{Q} \leq \hat{1} - \delta^o(\hat{F}^A_r)_{\mathcal{M}} \},
\]

since \( \hat{Q} \leq \hat{1} - \hat{F}^A_r \) implies \( \hat{Q} \leq \delta^i(\hat{1} - \hat{F}^A_r)_{\mathcal{M}} \), because \( \delta^i(\hat{1} - \hat{F}^A_r)_{\mathcal{M}} \) is the largest projection in \( \mathcal{M} \) dominated by \( \hat{1} - \hat{F}^A_r \).

Using Lemma 110, we can read off the left-continuous spectral family of the self-adjoint operator corresponding to \( a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})} \):

\[
F^{a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})}} : \overline{\mathbb{R}} \rightarrow \mathcal{P}(\mathcal{N})
\]

\[
r \mapsto \delta^o(\hat{F}^A_r)_{\mathcal{M}}.
\]

This is indeed a left-continuous spectral family, since \( \hat{F}^A_r \) is left-continuous and \( \delta^o : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{M}) \), being the left adjoint of the inclusion \( i : \mathcal{P}(\mathcal{M}) \hookrightarrow \mathcal{P}(\mathcal{N}) \), preserves joins. Clearly, \( F^{a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})}}(-\infty) = 0 \) and \( F^{a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})}}(\infty) = \hat{1} \).

We denote the self-adjoint operator affiliated with \( \mathcal{M} \) whose extended left-continuous spectral family is \( (\delta^o(\hat{F}^A_r)_{\mathcal{M}})_{r \in \overline{\mathbb{R}}} \) by \( \delta^i(\hat{A})_{\mathcal{M}} \), that is,

\[
\delta^i(\hat{A})_{\mathcal{M}} := \int_{-\infty}^{\infty} r d(\delta^o(\hat{F}^A_r)_{\mathcal{M}}).
\]

The operator \( \delta^i(\hat{A})_{\mathcal{M}} \) is called the inner daseinisation of \( \hat{A} \) to \( \mathcal{M} \).
Proposition 7 Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( \mathcal{N} \), and let \( \mathcal{M} \) be a von Neumann subalgebra of \( \mathcal{N} \) such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. The self-adjoint operator \( \delta^i(\hat{A})_\mathcal{M} \) affiliated with \( \mathcal{M} \) is given by

\[
\delta^i(\hat{A})_\mathcal{M} = \bigvee \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \leq_s \hat{A} \},
\]

where the join is taken with respect to the spectral order on \( SA(\mathcal{M}) \).

Proof. We first note that for self-adjoint operators \( \hat{B}, \hat{C} \) affiliated with a von Neumann algebra \( \mathcal{N} \), it holds that

\[
\hat{B} \leq_s \hat{C} \iff \forall r \in \mathbb{R}: \hat{F}_r^{\hat{B}} \geq \hat{F}_r^{\hat{C}},
\]

where we use the left-continuous spectral families of \( \hat{B} \) and \( \hat{C} \). This is immediate from (2.24) and (5.5).

We have \( \delta^i(\hat{A})_\mathcal{M} \leq_s \hat{A} \), since for each \( r \in \mathbb{R} \), it holds that

\[
\hat{F}_r^{\delta^i(\hat{A})_\mathcal{M}} = \delta^o(\hat{F}_r^\hat{A})_\mathcal{M} \geq \hat{F}_r^\hat{A}.
\]

Let \( \hat{B} \in SA(\mathcal{M}) \) such that \( \hat{B} \leq_s \hat{A} \). This is equivalent to

\[
\forall r \in \mathbb{R}: \hat{F}_r^{\hat{B}} \geq \delta^o(\hat{F}_r^\hat{A})_\mathcal{M} \geq \hat{F}_r^\hat{A},
\]

so \( \hat{B} \leq_s \delta^i(\hat{A})_\mathcal{M} \). Hence, \( \delta^i(\hat{A})_\mathcal{M} \) is the least upper bound (join) of the set \( \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \leq_s \hat{A} \} \).

Hence, we have shown:

Theorem 5 Let \( \hat{A} \) be a self-adjoint operator affiliated with a von Neumann algebra \( \mathcal{N} \), and let \( \mathcal{M} \subset \mathcal{N} \) be a von Neumann subalgebra such that the unit elements in \( \mathcal{N} \) and \( \mathcal{M} \) coincide. The restriction of the \( q \)-antonymous function \( a^\hat{A} : \mathcal{P}(\mathcal{N}) \to \mathbb{R} \) to the smaller domain \( \mathcal{P}(\mathcal{M}) \) corresponds to the self-adjoint operator

\[
\delta^i(\hat{A})_\mathcal{M} = \bigvee \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \leq_s \hat{A} \}
\]

affiliated with \( \mathcal{M} \), that is, \( a^{\hat{A}}|_{\mathcal{P}(\mathcal{M})} = a^{\delta^i(\hat{A})_\mathcal{M}} \).

This should be compared with Thm. 3. Taken together, these two theorems show that the inner and outer daseinisation of a self-adjoint operator \( \hat{A} \), which are approximations with respect to the spectral order, can simply be obtained by restricting the \( q \)-antonymous resp. \( q \)-observable function of \( \hat{A} \).

6 Physical interpretation and outlook

We have shown that to each self-adjoint operator \( \hat{A} \) affiliated with a von Neumann algebra \( \mathcal{N} \), one can associate a \( q \)-observable function \( o^{\hat{A}} : \mathcal{P}(\mathcal{N}) \to \mathbb{R} \).
\( \mathbb{R} \), which is the left adjoint of the (extended) spectral family \( \hat{E}^A : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) of \( \hat{A} \). Conversely, by Thm. 2 each self-adjoint operator affiliated with \( \mathcal{N} \) arises from an abstract \( q \)-observable function, i.e., a join-preserving function \( o : \mathcal{P}(\mathcal{N}) \to \mathbb{R} \) such that

\[
\begin{align*}
(\text{a}) & \quad o(\hat{P}) > -\infty \text{ for all } \hat{P} > 0, \\
(\text{b}) & \quad \text{there is a family } (\hat{P}_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{N}) \text{ with } \bigvee_{i \in I} \hat{P}_i = 1 \text{ such that } o(\hat{P}_i) \in \mathbb{R} \text{ for all } i \in I.
\end{align*}
\]

The \( q \)-observable function corresponding to \( \hat{A} \) is denoted \( o^A \). It is easy to show that the image of \( o^A \) on non-zero projections is the spectrum of \( \hat{A} \), see Lemma 2. If \( \hat{A} \) is bounded from above, then the family in condition (b) can be chosen to be \( \{1\} \) trivially.

This gives a new, non-standard way of representing physical quantities in quantum theory, which are usually described by self-adjoint operators in von Neumann algebras, or – for physical quantities like position and momentum which have unbounded spectra – by self-adjoint operators affiliated with a von Neumann algebra.

The spectral order makes \( SA(\mathcal{N}) \), the set of self-adjoint operators affiliated with \( \mathcal{N} \), a conditionally complete lattice \( (SA(\mathcal{N}), \leq_s) \). We saw in Prop. 4 that this lattice is isomorphic to the conditionally complete lattice \( (QO(\mathcal{P}(\mathcal{N}), \mathbb{R}), \leq) \) of \( q \)-observable functions with the pointwise order. This shows that \( q \)-observable functions faithfully represent self-adjoint operators and their order structure, as provided by the spectral order (which differs from the linear order if and only if the von Neumann algebra is nonabelian).

A limited ‘function calculus’ applies to \( q \)-observable functions: as long as we consider join-preserving (and hence monotone) functions \( f : \mathbb{R} \to \mathbb{R} \), we have \( o^{f(A)} = f(o^A) \), see Thm. 4. Physically, such functions \( f \) can be interpreted as rescalings of the spectrum of a physical quantity. For such rescalings, it makes sense to preserve the order on \( \mathbb{R} \): if measurements of a physical quantity produce outcomes \( a \) and \( b \) (which are real numbers in the spectrum of the self-adjoint operator \( \hat{A} \)) such that \( a < b \), and afterwards some rescaling \( f : \mathbb{R} \to \mathbb{R} \) is applied, then it is sensible to demand \( f(a) \leq f(b) \).

The question how \( q \)-observable functions behave under multiplication by \(-1\) led us to the definition of \( q \)-antonymous functions, which correspond bijectively to \( q \)-observable functions by \( -o^A = a^{-\hat{A}} \). This also implies that \( q \)-antonymous functions correspond bijectively to self-adjoint operators and that \( \hat{A} \leq_s \hat{B} \) is equivalent to \( a^A \leq a^B \) in the pointwise order. \( q \)-antonymous functions arise as left adjoints from adjunctions, too, but since they are antitone functions, the corresponding right adjoint must be antitone as well.

‘Opposite’ spectral families \( \hat{G}^A \), given by \( \hat{G}^A_r = \hat{1} - \hat{F}_r^A \) for all \( r \in \mathbb{R} \), turn
out to be the right concept.

The initial motivation for this article was twofold: on the one hand, we wanted to elucidate some of de Groote’s results [15, 17, 18] by providing an order-theoretic perspective and by using Galois connections systematically, while on the other hand, we aimed to gain a deeper understanding (and generalisation to unbounded operators) of the daseinisation maps from the topos approach to quantum theory [11, 12, 7].

Outer daseinisation of a self-adjoint operator \( \hat{A} \) affiliated with a von Neumann algebra \( \mathcal{N} \) to a von Neumann subalgebra \( \mathcal{M} \subset \mathcal{N} \) (such that the unit elements in \( \mathcal{M} \) and \( \mathcal{N} \) coincide) is approximation ‘from above’ with respect to the spectral order, that is, \( \delta^o(\hat{A})_{\mathcal{M}} = \bigwedge \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{A} \leq_s \hat{B} \} \). In the topos approach, this kind of approximation is considered for all abelian subalgebras \( \mathcal{A} \subset \mathcal{N} \).

Thm. \( \boxed{3} \) shows that \( o^A|_{\mathcal{P}(\mathcal{M})} = o^{\delta^o(\hat{A})_{\mathcal{M}}} \), that is, the \( q \)-observable function of the outer daseinisation of \( \hat{A} \) to \( \mathcal{M} \) is given by restriction of the \( q \)-observable function of \( \hat{A} \) to the smaller domain \( \mathcal{P}(\mathcal{M}) \). Analogously, we have \( a^A|_{\mathcal{P}(\mathcal{M})} = a^{\delta^i(\hat{A})_{\mathcal{M}}} \), that is, the \( q \)-antonymous function of the inner daseinisation of \( \hat{A} \) to \( \mathcal{M} \) is given by restriction of the \( q \)-antonymous function of \( \hat{A} \) to the smaller domain \( \mathcal{P}(\mathcal{M}) \); see Thm. \( \boxed{5} \). In this way, the \( q \)-observable function \( o^\hat{A} \) encodes all approximations \( \delta^o(\hat{A})_{\mathcal{M}} \), for \( \mathcal{M} \) varying over the von Neumann subalgebras of \( \mathcal{N} \) that share the unit element with \( \mathcal{N} \), while the \( q \)-antonymous function \( a^\hat{A} \) encodes all approximations \( \delta^i(\hat{A})_{\mathcal{M}} = \bigvee \{ \hat{B} \in SA(\mathcal{M}) \mid \hat{B} \leq_s \hat{A} \} \).

Physically, the inner and outer daseinisation of a self-adjoint operator \( \hat{A} \) to a von Neumann subalgebra \( \mathcal{M} \subset \mathcal{N} \) can be seen as approximations of (the physical quantity described by) \( \hat{A} \) to the subalgebra \( \mathcal{M} \). Outer daseinisation is approximation ‘from above’, while inner daseinisation is approximation ‘from below’ with respect to the spectral order. The more familiar linear order, in which \( \hat{A} \leq \hat{B} \) if and only if \( \hat{B} - \hat{A} \) is positive, does not give a lattice structure on the self-adjoint operators. (In fact, Kadison famously called \((\mathcal{N}_{sa}, \leq)\) an anti-lattice [20] .) Hence, the approximations from above and from below could not be defined with respect to the linear order, since the relevant meets and joins do not exist in general. Moreover, the approximation with respect to the spectral order guarantees [12, 14] that

\[
\text{sp}(\delta^o(\hat{A})_{\mathcal{M}}) \subseteq \text{sp} \hat{A}, \quad \text{sp}(\delta^i(\hat{A})_{\mathcal{M}}) \subseteq \text{sp} \hat{A},
\]

which is sensible physically: the approximations \( \delta^o(\hat{A})_{\mathcal{M}} \) and \( \delta^i(\hat{A})_{\mathcal{M}} \) have a spectrum that is a subset of the operator \( \hat{A} \). In physical terms, the approximations with respect to the spectral order introduce degeneracy, but they do not shift the spectral values.

Often, \( \mathcal{M} \) is taken to be abelian and thus represents a measurement context, since commuting self-adjoint operators correspond to compatible, i.e.,
co-measurable physical quantities. The representation of physical quantities in the topos approach to quantum theory is based on this idea of approximation \[12, 14\], with \( \mathcal{M} \) varying over the abelian von Neumann subalgebras of \( \mathcal{N} \) that share the unit element with \( \mathcal{N} \).

**Outlook.** In the second paper \[8\], entitled “Self-adjoint Operators as Functions II: Quantum Probability”, we will interpret \( q \)-observable functions in the light of quantum probability theory. It will be shown that \( \sigma^A : \mathcal{P}(\mathcal{N}) \to \mathbb{R} \) is the generalised quantile function of the projection-valued spectral measure \( e^A : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{N}) \) given by a self-adjoint operator \( \hat{A} \), where \( \mathcal{B}(\mathbb{R}) \) denotes the Borel subsets of the real line. In this perspective, the spectral family \( \hat{E}^A : \mathbb{R} \to \mathcal{P}(\mathcal{N}) \) takes the role of the cumulative distribution function of the measure \( e^A \).

Using some results from the topos approach to quantum theory, it will be shown that the so-called *spectral presheaf* \( \Sigma \) of a von Neumann algebra can serve as a joint sample space for all quantum observables (seen as random variables). This is possible despite no-go theorems like the Kochen-Specker theorem, since \( \Sigma \) is a generalised set — in fact, an object in a presheaf topos — with a complete bi-Heyting algebra of measurable subobjects, generalising the classical situation of a set with a \( \sigma \)-algebra of measurable subsets. We will show how the Born rule arises in this new formulation.

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