An enlargement of some symplectic objects

This note is dedicated to the celebration of the Anniversary of Professor Augustin Banyaga

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Abstract

The study of algebraic properties of groups of transformations of a manifold gives rise to an interplay between different areas of mathematics such as topology, geometry, and dynamical systems. Especially, in this paper, we point out some interplays between topology, geometry, and dynamical systems which are underlying to the group of symplectic homeomorphisms. The latter situation can occur when one thinks of the following question: Is there a $C^0$–flux geometry which is underlying to the group of strong symplectic homeomorphisms so that Fathi’s Poincaré duality theorem still holds true? We discuss on some possible answers of the above preoccupation, and we point out various enlargements of some symplectic results and invariants. We leave several open questions and conjecture.

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1 Introduction

According to Oh-Müller [28], the automorphism group of the $C^0$–symplectic topology is the closure of the group $\text{Symp}(M,\omega)$ of all symplectomorphisms of a symplectic manifold $(M,\omega)$ in the group $\text{Homeo}(M)$ of all homeomorphisms of $M$ endowed with the $C^0$–topology. That group, denoted $\text{Sympeo}(M,\omega)$ has been called group of all symplectic homeomorphisms:

$$\text{Sympeo}(M,\omega) = \overline{\text{Symp}(M,\omega)} \subset \text{Homeo}(M).$$

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This definition has been motivated by the following celebrated rigidity theorem due to Eliashberg [16] and Gromov [18].

**Theorem 1.1.** ([16], [18]) The group $\text{Symp}(M, \omega)$ of all symplectomorphisms of a symplectic manifold $(M, \omega)$ is $C^0$ closed inside the group of diffeomorphisms over $M$.

Oh-Müller’s showed that any symplectic homeomorphism preserves the Liouville measure [28]. Furthermore, a result due to Oh-Müller showed that the group of Hamiltonian homeomorphisms is contained in the kernel of Fathi’s mass flow, and more this group is a proper subgroup in the identity component of $\text{Symp}(M, \omega)$. More recently, Buhovsky [10] observed that Eliashberg-Gromov $C^0$–rigidity follows from Oh-Müller theorem on the uniqueness of topological Hamiltonians of topological Hamiltonian systems [28]. Recently, based on the above celebrated rigidity result, Banyaga [4, 6] defined two classes of symplectic homeomorphisms named the strong symplectic homeomorphisms in the $L^\infty$–context and the strong symplectic homeomorphisms in the $L^{1,\infty}$–context.

It is proved in Banyaga-Tchuiaga [8] that the nature of any strong symplectic homeomorphism does not depend on the choice of the $L^\infty$–topology or the $L^{1,\infty}$–topology. The latter uniqueness result is used in a crucial way in [31], [9]. Especially, this result motivated the introduction and the study of strong symplectic isotopies [31], [7]. To understand the statement of the main result of this paper, we will need the following definition from [9, 31].

**Definition 1.2.** ([9], [31]) A continuous family $(\gamma_t)$ of homeomorphisms of $M$ with $\gamma_0 = \text{id}$ is called a strong symplectic isotopy (or ssympeotopy) if there exists a sequence $\Phi_i = (\phi^i_t)$ of symplectic isotopies which converges uniformly to $(\gamma_t)$ and such that the sequence of symplectic vector fields $(\dot{\phi}^i_t)$ is Cauchy in the $L^\infty$–norm.

A result due to Banyaga-Tchuiaga [9] shows that in the above definition, the limit $M(\gamma) = \lim_{i} (\dot{\phi}^i_t)$ is independent of the choice of the sequence of symplectic isotopies $\Phi_i = (\phi^i_t)$, and the authors called the latter limit the "generator" of the ssympeotopy $(\gamma_t)$. Then, it follows from Banyaga-Tchuiaga [9] that any generator of strong symplectic isotopy is of the form $(U, H)$ where the functions $(t,x) \mapsto U_t(x)$ and $t \mapsto H_t$ are continuous, and for each $t$, $H_t$ is a smooth harmonic $1$–form for some Riemannian metric $g$ on $M$. Here is the main result from Banyaga-Tchuiaga [9].

**Theorem 1.3.** ([9]) Any strong symplectic isotopy $(\gamma_t)$ determines a unique generator $(U, H)$.

This theorem generalizes the uniqueness theorem of generating functions for continuous Hamiltonian flows from Viterbo [34], [33], and Buhovsky-Seyfaddini [11]. More recently, using the above uniqueness result of generators of ssympeotopies, Banyaga pointed out that any smooth homeomorphisms is a Hamiltonian diffeomorphisms, a theorem on Hamiltonian rigidity. However, in the presence of a positive symplectic displacement energy from Banyaga-Hurtubise-Spaeth [7], we point out the following converse of Theorem 1.3.
**Theorem 1.4.** Any generator \((U, \mathcal{H})\) corresponds to a unique strong symplectic isotopy \((\gamma_t)\).

The above result generalizes a result from Oh (Theorem 3.1, [27]) in the study of topological Hamiltonian dynamics. The lines of the proof of Theorem 1.2 in [32] seem to suggest that another proof of Theorem 1.4 can be pointed out without appealing to the positivity of the symplectic displacement energy. But, this other way to prove Theorem 1.4 is a little longer than the proof involving the symplectic displacement energy.

On the other hand, in smooth symplectic geometry, it is known that on any closed connected symplectic manifold \((M, \omega)\), various construction involving symplectic paths (for example: Hofer metric, symplectic flux etc...) are based on the fact that there is a one-to-one correspondence between the space of such paths and the space of smooth families of smooth symplectic vector fields. Furthermore, according to Banyaga-Tchuiaga [8], via Hodge decomposition theorem of differential forms one can establish that there is a one-to-one correspondence between the space of smooth families of smooth symplectic vector fields and the space consisting of all the pairs \((V, K)\) where \(V\) is a smooth family of normalized function and \(K\) is a smooth family of smooth harmonic 1–form for some Riemannian metric \(g\) on \(M\). The latter pairs are then called the smooth generators of symplectic paths. It is proved in Banyaga-Tchuiaga [8] there is a group isomorphism between the space of symplectic paths and the set consisting of their generators. The set of smooth generators of symplectic paths seems to have an interesting impact in the study of symplectic dynamics. This can be justify by the following fact. A result of [32] (Theorem 1.2, [32], [7]) shows in particular the impact of smooths generators of symplectic paths in the determination of the symplectic nature of a given map \(\rho : M \to M\). For instance, in the \(C^0\) case, the uniqueness results of Theorem 1.3 and Theorem 1.4 imply that the set of strong symplectic isotopies is one-to-one with the space of theirs generators. This suggests to think of the following questions:

- Is there a \(C^0\)–flux geometry which is underlying to the group of strong symplectic homeomorphisms?
- Is there a \(C^0\)–Hofer-like geometry which is underlying to the group of strong symplectic homeomorphisms?

Of course a motivation of the first question can be find in Banyaga’s affirmative from [6] starting that "the flux geometry seems to exist on the group of strong symplectic homeomorphisms of any closed connected symplectic manifold", while the second question is motivated in party by the construction dues to Oh [28] of Hofer’s metric for Hamiltonian homeomorphisms.

The goal of this paper is to investigate the above questions.

We start the main results of this paper by observing the following well-known result from flux geometry which is due to Banyaga [2, 3]: any isotopy whose flux is sufficiently small in \(H^1(M, \mathbb{R})\) (the first de Rham cohomological group with real coefficients) can be deformed onto a Hamiltonian isotopy relatively to fixed extremities (see [2, 3]). This follows from the discreteness of flux group
dues to Ono [29]. However, in the $C^0$ case, such a result is not known yet. Our intuition seems to suggest that a key ingredient of such a problem lies in Fathi’s Poincaré duality theorem [17]. This leads to the following result.

**Theorem 1.5.** (Homotopic Fathi’s duality theorem) Let $(M, \omega)$ be any closed connected symplectic manifold. Let $\xi$ be any strong symplectic isotopy whose Fathi’s mass flow is trivial. If $(\Phi_i)$ is any sequence of symplectic isotopies that converges to $\xi$ with respect to the $C^0$–symplectic topology. Then, one can extract a subsequence $(\nu_i)$ of the sequence $(\Phi_i)$ such that for each $i$, the isotopy $\nu_i$ is homotopic to a Hamiltonian isotopy $\Psi_i$ relatively to fixed extremities. Furthermore, the sequence of Hamiltonian isotopies $(\Psi_i)$ converges with respect to the $C^0$–symplectic topology, and its limit denoted by $\mu$ is a continuous Hamiltonian flow which is homotopic with fixed extremities to $\xi$.

In particular, Theorem 1.5 implies that any sympleotopies whose Fathi’s mass flow is trivial admits its extremities in the group of Hamiltonian homeomorphisms.

Some avatars of Theorem 1.5 are underlying to the following facts: Let $\xi$ be any strong symplectic isotopy. Let $(\Phi_i)$ be any sequence of symplectic isotopies that converges to $\xi$ with respect to the $C^0$–symplectic topology. Then the following questions point out:

1. If $\xi$ is a loop at the identity, then can one deforms the above sequence $(\Phi_i)$ into a sequence of symplectic isotopies $(\Psi_i)$ with $\Psi_i(1) = \text{id}$ for all $i$ so that the latter still converges to $\xi$ with respect to the $C^0$–symplectic topology?

2. If $\xi$ is homotopic to the constant path identity, then can one extracts a subsequence $(\nu_i)$ of the sequence $(\Phi_i)$ such that for each $i$, the isotopy $\nu_i$ is homotopic to the constant path identity?

It is not too hard to see that each of the above question falls within the ambit of uniform approximation of (volume-preserving) homeomorphisms by (volume-preserving) diffeomorphisms. Such problems had been studied in Eliashberg [10], Gromov [18], Müller [24], Munkres [26], and Connell [14, 15]. The following result gives an affirmative answer of question (1).

**Theorem 1.6.** (Weinstein’s deformation) Let $(M, \omega)$ be any closed connected symplectic manifold. Let $\xi$ be any strong symplectic isotopy which is a loop at the identity. Then, any sequence of symplectic isotopies $(\Phi_i)$ that converges to $\xi$ with respect to the $C^0$–symplectic topology can be deformed onto a sequence of symplectic isotopies $(\Psi_i)$ with $\Psi_i(1) = \text{id}$ for all $i$ so that the sequence $(\Psi_i)$ still converges to $\xi$ with respect to the $C^0$–symplectic topology.

In particular, we will see that the above result implies that the well-known flux group $\Gamma$ is rigid with respect to the $C^0$–symplectic topology. Furthermore, the following theorem point out the $C^0$–counter partner of the well-known flux homomorphism.

**Theorem 1.7.** (Topological flux homomorphism) Let $(M, \omega)$ be any closed connected symplectic manifold. Then, there exists a group homomorphism from
the group of strong symplectic homeomorphisms onto a quotient of $H^1(M, \mathbb{R})$ by a discrete group; which is similar to that given by Banyaga in the smooth case, and whose kernel coincides with the group of Hamiltonian homeomorphisms.

As a consequence of Theorem 1.7 we point out the following results.

**Theorem 1.8.** Let $(M, \omega)$ be any closed connected symplectic manifold. Then, any strong symplectic isotopy on the group of Hamiltonian homeomorphisms is a continuous Hamiltonian flow.

**Theorem 1.9.** (Topological Banyaga’s theorem I ) Let $(M, \omega)$ be any closed connected symplectic manifold. Then, the group of Hamiltonian homeomorphisms is path connected, and locally connected.

Note that Oh-Müller [28] proved that the group of Hamiltonian homeomorphisms is locally path connected, then locally connected. But, in the present paper we give a different proof.

**Theorem 1.10.** (Weak topological Weinstein’s theorem) Let $(M, \omega)$ be a closed connected symplectic manifold. Then, the group of strong symplectic homeomorphisms is locally path connected.

The following result contributes to the comprehension of strong symplectic isotopies, and then to the study of symplectic homeomorphisms.

**Theorem 1.11.** Let $\beta = (\beta_t)_{t \in [0,1]}$ be a strong symplectic isotopy. Assume that $\beta = (\beta_t)_{t \in [0,1]}$ is a 1-parameter group (i.e: $\beta_{t+s} = \beta_t \circ \beta_s$ $\forall s, t \in [0,1]$ such that $(s+t)$ lies in $[0,1]$). Then, its generator $(F, \lambda)$ is time-independent.

Theorem 1.11 shows in particular that any strong symplectic isotopy which is a 1-parameter group decomposes as composition of a smooth harmonic flow and a continuous Hamiltonian flow.

Actually, we focus on the construction of a $C^0$–Hofer-like norm for strong symplectic homeomorphisms. In fact the following results points out the existence and some studies of the $C^0$–analogue of the well-known Banyaga’s Hofer-like norm found in [4].

**Theorem 1.12.** (Topological Banyaga’s norm ) Let $(M, \omega)$ be any closed connected symplectic manifold. Denote by $\mathcal{S}{\text{Sym}}{\text{peo}}(M, \omega)$ the group of strong symplectic homeomorphisms endowed with the topology that induces its structure of topological group. Then, there exists a positive continuous map, denoted by $\|\|_{SHL}$ defined from $\mathcal{S}{\text{Sym}}{\text{peo}}(M, \omega)$ onto $\mathbb{R} \cup \{+\infty\}$.
Theorem 1.13. Let \((M, \omega)\) be any closed connected symplectic manifold. Given two strong symplectic homeomorphisms \(h, f\) we have:

1. \(\|h\|_{SHL} = 0\) if \(h = \text{id}\),
2. \(\|h\|_{SHL} = \|h^{-1}\|_{SHL}\),
3. \(\|h \circ f\|_{SHL} \leq \|h\|_{SHL} + \|f\|_{SHL}\).

Theorem 1.13 and Theorem 1.12 are similar to a result which Oh [27] proved for Hamiltonian homeomorphisms. In particular, if the manifold \(M\) is simply connected, then the norm \(\|\cdot\|_{SHL}\) reduces to the norm \(\|\cdot\|_{Oh}\) constructed by Oh on the space of all Hamiltonian homeomorphisms [28]. On the other hand, it follows from the decomposition theorem of strong symplectic homeomorphisms that the group of Hamiltonian homeomorphisms is strictly contained in the group of strong symplectic homeomorphisms. So, it is important to know whether the norm \(\|\cdot\|_{SHL}\) is an extension of Oh’s norm or not. This leads to the following result.

Theorem 1.14. (Topological Banyaga’s conjecture) The norm \(\|\cdot\|_{SHL}\) restricted to the group of Hamiltonian homeomorphisms is equivalent to Oh’s norm.

This result is motivated in party by a conjecture from Banyaga [4] which Buss-Leclercq [12] proved. The latter conjecture stated that the restriction of Banyaga’s Hofer-like norm to the group of Hamiltonian diffeomorphisms is equivalent to Hofer’s norm.

2 Preliminaries

Let \((M, \omega)\) be a smooth closed connected manifold of dimension \(2n\), equipped with a symplectic form \(\omega\). That is, the 2–form \(\omega\) is closed and nondegenerate, i.e. \(\omega^n \neq 0\), and in particular induces an orientation of \(M\). A diffeomorphism \(\phi : M \to M\) is called symplectic if it preserves the symplectic form \(\omega\), and this is equivalent to \(\phi^* \omega = \omega\). We denote the group of symplectic diffeomorphisms by \(\text{Symp}(M, \omega)\), and the subgroup of symplectic diffeomorphisms isotopic to the identity inside \(\text{Symp}(M, \omega)\) by \(\text{Symp}(M, \omega)_0\). We equip the group \(\text{Symp}(M, \omega)\) with the \(C^\infty\) compact-open topology [19]. An isotopy \(\Phi = \{\phi_t\}_{t \in [0, 1]}\) is a symplectic isotopy if \(\phi_t^* \omega = \omega\) for each \(t\). \(\Phi\) is symplectic if and only if the smooth family of vector fields \(Z_t = \frac{d}{dt} \phi_t \circ \phi_t^{-1}\) forms a family of symplectic vector fields, meaning the Lie derivative of \(\omega\) along \(Z_t\) is trivial. A symplectic isotopy \(\Psi = \{\psi_t\}_{t \in [0, 1]}\) is called Hamiltonian if the interior derivative of \(\omega\) along \(Z_t\) is exact, meaning that there exists a smooth function \(F : [0, 1] \times M \to \mathbb{R}\) such that the interior derivative of \(\omega\) along \(Z_t\) coincides with the exact 1–form \(dF_t\). We denote by \(\text{Iso}(M, \omega)\) the group of all symplectic isotopies, and by \(\mathcal{N}([0, 1] \times M, \mathbb{R})\) we denote the vector space of smooth functions \(F : [0, 1] \times M \to \mathbb{R}\) satisfying \(\int_M F_t \omega^n = 0\). On the other hand, apart of Hamiltonian isotopies, Banyaga [4] characterizes other class of symplectic isotopies called harmonic isotopies. Indeed, a symplectic isotopy \(\Theta = \{\theta_t\}_{t \in [0, 1]}\) is said to be harmonic if the Lie derivative of \(\omega\) along \(X_t = \frac{d}{dt} \circ \theta_t^{-1}\) is harmonic, meaning that for a given Riemann’s metric \(g\) on \(M\) there exists a smooth family of
harmonic 1–forms $\mathcal{H} = (\mathcal{H}_t)_{t \in [0,1]}$ such that the interior derivative of $\omega$ along $Z_t$ coincides with the harmonic 1–form $\mathcal{H}_t$. For more convenience, in the rest of this paper we fix a Riemann’s metric $g$ on $M$, and denote by $\text{harm}(M, g)$, the space of harmonic 1–forms on $M$. It follows from Hodge’s theory that $\text{harm}(M, g)$ is a finite dimensional vector space over $\mathbb{R}$ whose dimension is the first Betti number $b_1$ of the manifold $M$ \cite{[35]}. We equip $\text{harm}(M, g)$ with the Euclidean norm defined as follows:

$$|H| := \sum_{i=1}^{b_1} |\lambda_i|,$$

with $H = \sum_{i=1}^{b_1} \lambda_i b_i$ where $(b_i)_{1 \leq i \leq b_1}$ is a basis of the space $\text{harm}(M, g)$. The above norm on the space $\text{harm}(M, g)$ is equivalent to the $L^2$–norm due to Hodge, denoted by $\|\cdot\|_{L^2}$, and defined as follows: for all $\alpha \in \text{harm}(M, g)$,

$$\|\alpha\|_{L^2} := \int_M \alpha \wedge \ast \alpha,$$

where $\ast$ represents de Rham’s star operator with respect to the Riemannian metric $g$. We denote by $\mathfrak{P}harm(M, g)$ the space of smooth maps $\mathcal{H} : [0,1] \rightarrow \text{harm}(M, g)$.

### 2.1 The group $\mathfrak{T}(M, \omega, g)$ (Banyaga-Tchuiaga), \cite{[8]}

We recall here the outlines of construction of the group $\mathfrak{T}(M, \omega, g)$ introduced in \cite{[8]}. Let $\Phi = \{\phi_t\}_{t \in [0,1]}$ be a symplectic isotopy. As we saw earlier, the Lie derivative of $\omega$ along $Y_t = \frac{d\phi_t}{dt} \circ (\phi_t)^{-1}$ is a closed 1–form, hence it follows from Hodge’s theory that the 1–form $\iota(Y_t) \omega$ decomposes into the sum of an exact 1–form $dU_t^\Phi$ and a harmonic 1–form $\mathcal{H}_t^\Phi$ \cite{[20]}. Denote by $U$ the smooth function $U^\Phi = (U_t^\Phi)_{t \in [0,1]}$ normalized, and by $\mathcal{H}$ the smooth family of harmonic 1–forms $\mathcal{H}^\Phi = (\mathcal{H}_t^\Phi)_{t \in [0,1]}$. In \cite{[8]}, the authors denoted by $\mathfrak{T}(M, \omega, g)$ the Cartesian product $\mathcal{N}([0,1] \times M, \mathbb{R}) \times \mathfrak{P}harm(M, g)$, and endowed it with a group structure which makes the following bijection

$$\text{Iso}(M, \omega) \rightarrow \mathfrak{T}(M, \omega, g), \Phi \mapsto (U, \mathcal{H})$$

a group isomorphism. For short, in the rest of this paper, exceptionally if mention is made to the contrary we will denote any symplectic isotopy by $\phi_{(U, \mathcal{H})}$ to mean that its image by the above bijection is $(U, \mathcal{H})$. In particular, any symplectic isotopy of the form $\phi_{(0, \mathcal{H})}$ is considered to be a Hamiltonian isotopy, while any symplectic isotopy of the form $\phi_{(U, 0)}$ is considered to be a Hamiltonian isotopy. Then, the product in $\mathfrak{T}(M, \omega, g)$ is given by,

$$(U, \mathcal{H}) \rtimes (V, \mathcal{K}) = (U + V \circ \phi_{(U, \mathcal{H})}^{-1} + \Delta(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1}), \mathcal{H} + \mathcal{K}), \quad (2.1)$$

where $\Delta(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1})$ is the function $\Delta(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1}) := \int_0^1 \mathcal{K}(s \phi_{(U, \mathcal{H})}^{-1}) \circ \phi_{(U, \mathcal{H})}^{-1} ds$ normalized, and $\phi_{(U, \mathcal{H})}^{-1} := (\phi_{(U, \mathcal{H})})^{-1}$ for all $t$, while the inverse of $(U, \mathcal{H})$, denoted by $(U, \mathcal{H})$ is given by

$$(U, \mathcal{H}) = (-U \circ \phi_{(U, \mathcal{H})} - \Delta(\mathcal{H}, \phi_{(U, \mathcal{H})}), -\mathcal{H}). \quad (2.2)$$

We refer the readers to \cite{[8], [31]} for more details.
2.2 Banyaga’s Topologies [8], [31]

In this subsection, we reformulate the topology introduced by Banyaga [4] on the space of symplectic vector fields. Let \(X\) be a symplectic vector field. The 1–form \(i_X \omega\) can be decomposed in a unique way as the sum of a harmonic 1–form \(H_X\) with an exact 1–form \(dU_X\). The function \(U_X\) is given by \(U_X = \delta G(i_X \omega)\), where \(\delta\) is the codifferential operator and \(G\) is the Green operator [35]. In regard of the above decomposition of symplectic vector fields, we will denote any symplectic vector field \(X\) by \(X(U, H)\) to mean that Hodge’s decomposition of \(i(X) \omega\) gives \(dU + H\) with \(U\) normalized. According to Banyaga, the above decomposition of symplectic vectors gives rise to an intrinsic norm on the space of symplectic vector fields, defined by

\[
\|X(U, H)\| = |H| + \text{osc}(U),
\]

where \(\text{osc}(U) = \max_{x \in M} U(x) - \min_{x \in M} U(x)\), and \(|\cdot|\) represents the Euclidean norm on the space of harmonic 1–forms that we introduced in the beginning. The norm \(\|\cdot\|\) gives rise to a norm defined on the space of 1–parameter symplectic vectors fields as follows. Let \((Y_t)\) be a smooth family of symplectic vector fields, we have:

\[
\|(Y_t)\|_\infty = \max_t \|Y_t\|.
\]

The above norm is called the \(L^\infty\)– Banyaga’s norm of the family of symplectic vectors fields \((Y_t)\). For instance, the above norm induces a distance on the space \(\mathfrak{T}(M, \omega, g)\) as follows: For all \((U, H), (V, K) \in \mathfrak{T}(M, \omega, g), \)

\[
D^2((U, H), (V, K)) = \frac{\|X(U, H)_t - X(V, K)_t\|_\infty + \|X(V, K)_t - X(U, H)_t\|_\infty}{2},
\]

where for each \(t\), \((U, H)_t = (U_t, H_t)\). Therefore, the \(L^\infty\)–topology on the space \(\mathfrak{T}(M, \omega, g)\) is the one induced by the metric \(D^2\).

2.3 Banyaga’s Hofer-like norms [4]

For any symplectic isotopy \(\Phi = \phi(U, H)\), Banyaga [4] defined respectively its \(L^{(1, \infty)}\)–length and its \(L^\infty\)–length as follows,

\[
l^{(1, \infty)}(\Phi) = \int_0^1 \text{osc}(U_t) + |H_t| dt
\]

and

\[
l^\infty(\Phi) = \max_{t \in [0,1]} (\text{osc}(U_t) + |H_t|).
\]

Let \(\phi \in \text{Symp}(M, \omega)_0\), using the above lengths Banyaga [4] defined respectively the \(L^{(1, \infty)}\)–energy and \(L^\infty\)–energy of \(\phi\) by,

\[
e_0(\phi) = \inf(l^{(1, \infty)}(\Phi)), \text{ and } e^\infty(\phi) = \inf(l^\infty(\Phi)),
\]

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where the infimum is taken over all symplectic isotopies \( \Phi \) that connects \( \phi \) to the identity. The \( L^{(1,\infty)} \)-Hofer-like norm and the \( L^{\infty} \)-Hofer-like norm of \( \phi \) are respectively defined as follows,

\[
\|\phi\|_{HL} = \frac{e_0(\phi) + e_0(\phi^{-1})}{2}
\]

and

\[
\|\phi\|_{\infty, HL} = \frac{e_0^\infty(\phi) + e_0^\infty(\phi^{-1})}{2}.
\]

These norms generalize the Hofer’s norm of Hamiltonian diffeomorphisms in the following sense: if the manifold is simply connected, each of these norms reduces to Hofer’s norm. Moreover, a result which Buss-Leclecq [12] proved states that the restriction of Banyaga’s Hofer-like norm to the group of Hamiltonian diffeomorphisms is equivalent to Hofer’s norm, meaning Banyaga’s Hofer-like norm extends the Hofer’s norm to a right-invariant norm on the group of symplectic diffeomorphisms isotopic to the identity. More recently, it was proved in [32] that the following equality holds true \( \|\phi\|_{HL} = \|\phi\|_{\infty, HL} \). This points out the uniqueness of Banyaga’s Hofer-like geometry, and then generalizes a result on the uniqueness of Hofer’s geometry which Polterovich [30] (Lemma 5.1.C, [30]) proved.

2.4 Displacement energy (Banyaga-Hurtubise-Spaeth)

Definition 2.1. ([7]) The symplectic displacement energy \( e_S(A) \) of a nonempty set \( A \subset M \) is:

\[
e_S(A) = \inf \{\|\phi\|_{HL} | \phi \in \text{Symp}(M, \omega)_0, \phi(A) \cap A = \emptyset\}.
\]

Theorem 2.2. ([7]) For any nonempty open set \( A \subset M \), \( e_S(A) \) is a strict positive number.

In particular, the equality \( \|\cdot\|_{HL} = \|\cdot\|_{\infty, HL} \) implies that the above definition of positive symplectic displacement energy does not depend on the choice of Banyaga’s Hofer-like norm. We will see later how one extends the above displacement symplectic energy by the mean of strong symplectic homeomorphisms. This is supported by the uniqueness result from Banyaga-Tchuiaga [8].

2.5 The \( C^0 \)-topology

Let \( \text{Homeo}(M) \) be the group of homeomorphisms of \( M \) endowed with the \( C^0 \)-compact-open topology. The above topology coincides with the one induced by the following bi-invariant metric

\[
d_0(f, h) = \max(d_{C^0}(f, h), d_{C^0}(f^{-1}, h^{-1})) \tag{2.4}
\]

with

\[
d_{C^0}(f, h) = \sup_{x \in M} d(h(x), f(x)),
\]

where \( d \) is a distance on \( M \) induced by the Riemannian metric \( g \). On the space \( \mathcal{P}(\text{Homeo}(M), id) \) of continuous paths \( g : [0, 1] \to \text{Homeo}(M) \) such that
\( \rho(0) = id \), we consider the \( C^0 \)-topology as the topology induced by the following metric
\[
\bar{d}(\lambda, \mu) = \max_{t \in [0,1]} d_0(\lambda(t), \mu(t)). \tag{2.5}
\]

### 2.6 Topological symplectic isotopies (Banyaga-Hurtubise-Spaeth [7], Oh-Müller [28], Tchuiaga [32, 31])

The introduction of topological symplectic isotopies can be motivated by the following result found in [32].

**Theorem 2.3.** ([32]) Let \((M, \omega)\) be a closed connected symplectic manifold. Let \( \Phi_i = (\phi_i^t) \) be a sequence of symplectic isotopies generated by the sequence of smooth generators \((\{U^i, \mathcal{H}^i\})_i \), let \( \Psi = (\psi^i) \) be another symplectic path generated by \((U, \mathcal{H})\), and \( \rho : M \to M \) be a map. If the sequence \( \phi_i^t \) converges uniformly to \( \rho \), and the sequence \((\{U^i, \mathcal{H}^i\})_i \) converges to \((U, \mathcal{H})\) with respect to the \( L^\infty \)-topology. Then we must have \( \rho = \psi^1 \), i.e., \( \rho \) is symplectic.

To put Theorem 1.2 found in [32] into further prospective, observe that it seems to suggest to think of the following situation: If in Theorem 2.3 the sequence of generators \((\{U^i, \mathcal{H}^i\})_i \) is only \( L^\infty \)-Cauchy, then

- What can we say about the nature of \( \rho \)?
- Can \( \rho \) be viewed as the time-one map of some continuous path \( \lambda \)?

The \( L^{(1, \infty)} \) version of Theorem 2.3 was recently pointed out by the mean of a positive symplectic displacement energy (see [7]). The above facts justify the following definitions.

**Definition 2.4.** ([9]) A continuous map \( \xi : [0, 1] \to \text{Homeo}(M) \) with \( \xi(0) = id \) is called strong symplectic isotopy if there exists a \( D^2 \)-Cauchy (see [7]). The above facts justify the following definitions.

**Lemma 2.5.** Let \((\phi(U^i, \mathcal{H}^i), (U^i, \mathcal{H}^i))\) be a \( (D^2 + \bar{d}) \)-Cauchy sequence. Then the following holds true,
\[
\max_t \text{osc}(\Delta_t(\mathcal{H}^i - \mathcal{H}^{i-1}, \phi(U^i, \mathcal{H}^{i-1}))) \to 0, \quad i \to \infty.
\]

**Proof of Lemma 2.5.** The proof of Lemma 2.5 is a verbatim repetition of the proof of Lemma 3.4 found in [8].

**Corollary 2.6.** Let \( \Phi_i = (\phi_i^t) \) be a sequence of symplectic isotopies which is Cauchy with respect to the metric \( \bar{d} \) such that the sequence of \( 1 \)-parameter family of symplectic vectors field \( X_i^1 := (\phi_i^t) \) is Cauchy in the norm \( \| \cdot \|^\infty \). Then, the sequence of \( 1 \)-parameter family of symplectic vector fields \( Y_i^1 := -\phi_i^{-1}(X_i^1) \) is Cauchy in the norm \( \| \cdot \|^\infty \).

**Proof of Corollary 2.6.** Put \( \Phi_i = \phi(U^i, \mathcal{H}^i) \). By definition of \( Y_i^1 \), we have
\[
\|Y_i^1 - Y_i^{1+1}\|^\infty = \max_t \text{osc}(U^i_t \circ \phi_i^t - U^{i+1}_t \circ \phi_i^{t+1} - \Delta_t(\mathcal{H}^i, \Phi_i) + \Delta_t(\mathcal{H}^{i+1}, \Phi_{i+1})).
\]
To achieve the proof it remains to show that \( \max_t |\mathcal{H}^t - \mathcal{H}^{t+1}| \)

\[
\leq \max_t \text{osc}(U_t^i \circ \phi_t^i - U_{t+1}^i \circ \phi_{t+1}^i) + \max_t \text{osc}(U_t^i \circ \phi_{t+1}^i - U_{t+1}^i \circ \phi_{t+1}^i)
\]

\[+ \max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_i) - \Delta_t(\mathcal{H}^{t+1}, \Phi_{t+1})) + \max_t |\mathcal{H}^t - \mathcal{H}^{t+1}|.\]

To achieve the proof it remains to show that \( \max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_i) - \Delta_t(\mathcal{H}^{t+1}, \Phi_{t+1})) \)
tends to zero when \( i \to \infty \). Compute,

\[
\max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_i) - \Delta_t(\mathcal{H}^{t+1}, \Phi_{t+1})) \leq \max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_i) - \Delta_t(\mathcal{H}^i, \Phi_{t+1}))
\]

\[+ \max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_{t+1}) - \Delta_t(\mathcal{H}^{t+1}, \Phi_{t+1})),\]

and derive from Lemma 2.5 that

\[
\max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_{t+1}) - \Delta_t(\mathcal{H}^{t+1}, \Phi_{t+1})) \to 0, i \to \infty.
\]

Corollary 2.10 found in [32] implies that

\[
\max_t \text{osc}(\Delta_t(\mathcal{H}^i, \Phi_{t+1}) - \Delta_t(\mathcal{H}^{t+1}, \Phi_{t+1})) \to 0, i \to \infty.
\]

This achieves the proof. ♦

According to Corollary 2.6 definition 2.4 is equivalent to the following definition.

**Definition 2.7.** A continuous map \( \xi : [0, 1] \to \text{Homeo}(M) \) with \( \xi(0) = \text{id} \) is called strong symplectic isotopy if there exists a sequence of symplectic isotopies \( \Phi_j = (\phi_j^i) \) such that,

- \( d(\Phi_j, \xi) \to 0, j \to \infty \),
- \( (\phi_j^i) \) is Cauchy in \( ||.||| \).

**Question (a)**

Is any continuous path \( \gamma : t \to \gamma_t \) in the group of strong symplectic homeomorphisms a strong symplectic isotopy?

### 2.6.1 Generators of sympleotopies

Let \( \mathcal{N}^0([0, 1] \times M, \mathbb{R}) \) be the completion of the metric space \( \mathcal{N}([0, 1] \times M, \mathbb{R}) \) with respect to the \( L^\infty \) Hofer norm, and \( \mathfrak{P}^0 \text{harm}(M, g) \) be the completion of the metric space \( \mathfrak{P} \text{harm}(M, g) \) with respect to the metric \( D^2 \). Now, consider the product space \( J^0(M, \omega, g) =: \mathcal{N}^0([0, 1] \times M, \mathbb{R}) \times \mathfrak{P}^0 \text{harm}(M, g) \), and the following inclusion map \( i_0 : \Sigma(M, \omega, g) \to J^0(M, \omega, g) \). The map \( i_0 \) is uniformly continuous with respect of the topology induced by the metric \( D^2 \) on the space \( \Sigma(M, \omega, g) \), and the natural topology of the complete metric space \( J^0(M, \omega, g) \). Next, set \( L(M, \omega, g) = \text{image}(i_0) \), and consider \( \Sigma(M, \omega, g)_c \) to be the closure of the image \( L(M, \omega, g) \) inside the complete metric space \( J^0(M, \omega, g) \). That is, \( \Sigma(M, \omega, g)_c \) consists of pairs \( (U, H) \) where the functions \( (t, x) \to U_t(x) \) and
Let $t \mapsto \mathcal{H}_t$ are continuous, and for each $t$, $\mathcal{H}_t$ lies in $\text{harm}(M, g)$ such that there exists a Cauchy sequence $\{(U^j, \mathcal{H}^j)\} \subset L(M, \omega, g)$ that converges to $(U, \mathcal{H})$ inside $J^0(M, \omega, g)$.

Observe that the sequence $(F_j, \lambda_j)$ in definition (2.4) converges necessarily in the complete metric space $\mathcal{F}(\mathcal{M}, \omega, g)$. The latter limit has been called a "generator" of the strong symplectic isotopy $\xi$ [9]. In particular, if the manifold is simply connected, then the above definition reduces to the definition of continuous Hamiltonian flows, and the set of "generator" reduces to the set of corresponding generating functions [27], [34]. However, the following question is still open:

Notice that in the above definition of strong symplectic isotopy, if the sequence $(F_j, \lambda_j)$ is such that $F_j = 0$ for all $j$, then one obtains the definition of topological harmonic isotopy introduced in [31]. The following questions can be found in [31].

**Question (a-1)**
Is the set of all topological harmonic flows introduced in [31] strictly contains the space of smooth harmonic isotopies?

**Question (a-2)**
How should we thing of the intersection of the set of all topological harmonic flows with the space of all topological Hamiltonian flows?

It is not too hard to see that Theorem 1.8 (or the uniqueness results of Theorem 1.3 and Theorem 1.4) of the present paper implies that the intersection of the set of all topological harmonic flows introduced in [31] intersects the space of all topological Hamiltonian flows, and the latter intersection contains a single element which is the constant path identity.

### 2.6.2 Group structure of $\text{GSSympeo}(M, \omega, g)$, [9]

**Definition 2.8.** ([9]) The set $G\text{SSympeo}(M, \omega, g)$ is defined to be the space consisting of the pairs $(\xi, (U, \mathcal{H}))$ where $\xi$ is a strong symplectic isotopy generated by $(U, \mathcal{H})$.

The group structure on the space $G\text{SSympeo}(M, \omega, g)$ is defined as follows: For all $(\xi, (F, \lambda)), (\delta, (L, \theta)) \in G\text{SSympeo}(M, \omega, g)$, their product is given by,

$$(\xi, (F, \lambda)) \ast (\delta, (L, \theta)) = (\xi \circ \delta, (F + L \circ \xi^{-1} + \Delta^0(\theta, \xi^{-1}), \lambda + \theta)),$$

and the inverse of the element $(\xi, (F, \lambda))$, denoted by $(\xi, (F, \lambda))^{-1}$ is defined as follows,

$$(\xi, (F, \lambda))^{-1} = (\xi^{-1}, (-F \circ \xi - \Delta^0(\lambda, \xi), -\lambda)),$$

where

$$\Delta^0(\theta, \xi^{-1}) := \lim_{L \to \infty} \Delta(\theta_1, (\phi_{(F, \lambda)}^{-1})), \quad (2.6)$$
\[ \Delta^0(\lambda, \xi) := \lim_{L \to \infty} \Delta(\lambda_i, \phi(F, \lambda_i)), \]  

(2.7)

\[ \lim_{L \to \infty} (F, \lambda_i) = (F, \lambda), \lim_{L \to \infty} (L_i, \theta_i) = (L, \theta). \] 

The following remark justifies the existence of the functions \( \Delta^0(\theta, \xi^{-1}) \) and \( \Delta^0(\lambda, \xi) \).

**Remark 2.9.** Note that it follows from Corollary 2.10 found in [32] that any sequence of smooth family of smooth functions of the form \( \{ \Delta_t(H, \Psi) = \int_0^t (H_i)(\dot{\psi}_t^i) \circ \psi_t^i ds \} \) converges inside the complete metric space \( \mathcal{N}^0([0, 1] \times M, \mathbb{R}) \) provided that the sequence of symplectic isotopies \( \psi_t^i \) is Cauchy in \( d \), and the sequence of smooth harmonic 1-forms \( (H_i) \) is Cauchy in the \( L^\infty \)-norm. This justifies the existence and the continuity of the functions \( \Delta^0(\theta, \xi^{-1}) \) and \( \Delta^0(\lambda, \xi) \) defined in relation (2.6) and relation (2.6). Thus the group structure on the space \( GSSympeo(M, \omega, g) \) is well defined because of Corollary 2.10 from [32].

On the other hand, it is not too hard to check the following fact. Let \( K = (K_t) \) be a smooth family of closed 1-forms. If \( \Phi = (\phi_t) \) and \( \Psi = (\psi_t) \) are two symplectic isotopies which are homotopic relatively to fixed extremities, then we have

\[ \int_M \Delta_1(K, \Phi) \omega^n = \int_M \Delta_1(K, \Psi) \omega^n. \]

**Question (b)**

Let \( \gamma \) and \( \beta \) be two \( C^0 \)-limits of sequences of symplectic isotopies. Is \( \gamma \) homotopic to \( \beta \) relatively to fixed extremities imply

\[ \int_M \Delta_1(K, \gamma) \omega^n = \int_M \Delta_1(K, \beta) \omega^n? \]

The answer of the above question is yes if \( \gamma \) and \( \beta \) are \( C^0 \)-limits of two sequences of Hamiltonian isotopies. In fact the latter case follows directly from Fatou’s lemma of measure theory. But, what can one says about the general case?

**Definition 2.10.** The \( C^0 \)-symplectic topology on the space \( GSSympeo(M, \omega, g) \) is by definition the topology induced by the inclusion \( GSSympeo(M, \omega, g) \subset \mathcal{P}(Homeo(M), id) \times \Sigma(M, \omega, g). \)

For short, we will often write "the \((C^0 + L^\infty)\)-topology" to mean the \( C^0 \)-symplectic topology on the space \( GSSympeo(M, \omega, g) \). It is proved in [9] that \( GSSympeo(M, \omega, g) \) is a topological group.

**Definition 2.11.** The symplectic topological on the space \( SSympeo(M, \omega) \) of all strong symplectic homeomorphisms is the strongest topology on it which makes the following evaluation map becomes continuous

\[ ev : GSSympeo(M, \omega, g) \to SSympeo(M, \omega), \]

\[ (\xi, (F, \lambda)) \mapsto \xi(1). \]

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According to the above definitions, in the rest of this paper, the topology on the space $GSSympeo(M, \omega, g)$ is called the $(C^0 + L^\infty)$-topology, i.e. for any $(\lambda, (U, \mathcal{H})) \in GSSympeo(M, \omega, g)$, we will often write

$$(\lambda, (U, \mathcal{H})) = \lim_{C^0 + L^\infty} (\phi(U, \mathcal{H}))$$

to mean that $(\lambda, (U, \mathcal{H}))$ is the limit in the space $GSSympeo(M, \omega, g)$ of a sequence of symplectic isotopies $(\phi(U, \mathcal{H}), (U^i, \mathcal{H}^i))$.

Theorem 2.12 is equivalent to the following result.

**Theorem 2.12.** Any generator corresponds to a unique strong symplectic isotopy, i.e. if $(\gamma, (U, \mathcal{H})), (\xi, (U, \mathcal{H})) \in GSSympeo(M, \omega, g)$. Then we must have $\gamma = \xi$.

### 2.7 Proofs of Theorem 1.1 and Theorem 1.11

**Proof of Theorem 2.12** Let $(\gamma, (U, \mathcal{H}))$ and $(\xi, (U, \mathcal{H}))$ be two elements of $GSSympeo(M, \omega, g)$. By definition of the group $GSSympeo(M, \omega, g)$, there exists $(\phi(U_i, \mathcal{H}_i), (U_i, \mathcal{H}_i))$ and $(\phi(V_i, K_i), (V_i, K_i))$ two sequences of symplectic isotopies such that

$$(\gamma, (U, \mathcal{H})) = \lim_{C^0 + L^\infty} (\phi(U, \mathcal{H})),$$

and

$$(\xi, (U, \mathcal{H})) = \lim_{C^0 + L^\infty} (\phi(V, K)).$$

Assume that $\gamma \neq \xi$, i.e. there exists $s_0 \in [0, 1]$ such that $\gamma(s_0) \neq \xi(s_0)$. Since the map $\gamma^{-1}(s_0) \circ \xi(s_0) \in Homeo(M)$, from the identity

$$\gamma^{-1}(s_0) \circ \xi(s_0) \neq id,$$

one derives that there exists a closed ball $B$ which is entirely moved by $\gamma^{-1}(s_0) \circ \xi(s_0)$. From the compactness of $B$, and the uniform convergence of the sequence $\phi(U_i, \mathcal{H}_i) \circ \phi(V_i, K_i)$ to $\gamma^{-1} \circ \xi$, we derive that

$$(\phi(U_i, \mathcal{H}_i) \circ \phi(V_i, K_i))(B) \cap (B) = \emptyset,$$

for all sufficiently large $i$. Relation (2.8) implies that,

$$0 < \varepsilon_S(B) \leq L^\infty(\phi(U_i, \mathcal{H}_i) \circ \phi(V_i, K_i)),$$

for all sufficiently large $i$ where $\varepsilon_S$ is the symplectic displacement energy from Banyaga-Hurtubise-Spaeth [7], and $L^\infty(A)$ represents the $L^\infty$ length functional of symplectic isotopies [4]. According to a result from [31] (Lemma 3.1), the right hand side of the above estimates tends to zero when $i$ goes to infinity. This contradicts the positivity of the symplectic displacement energy from Banyaga-Hurtubise-Spaeth [7]. This achieves the proof.□

Then, according to the above uniqueness results of strong symplectic isotopies and their generators vise versa, we will denote any strong symplectic isotopy $\lambda$ by $\lambda(U, \mathcal{H})$ to mean that $\lambda$ is generated by $(U, \mathcal{H})$, or equivalently $\lambda(U, \mathcal{H})$ is
Lemma 2.13. Composition of smooth harmonic flow and a continuous Hamiltonian flow?

Is a $\mathbf{1}+$-plectic isotopy which is a composition of smooth harmonic flow and a continuous Hamiltonian flow in the sense of $\mathbf{1}+$-fixed $t \mapsto \phi_{(F, \lambda)}(s)$, the sequence of symplectic isotopies. Observe that for each given $s \in [0, 1]$, the sequence of symplectic maps $\Psi_s^i(t) = \phi_{(F, \lambda)}(s) \circ (\phi_{(F, \lambda)}(s))^{-1}$ converges in $d$ to $\beta_{(F, \lambda)}$.

- **Step (2).** On the other hand, for each $i$ compute the derivative (in $t$) of the path $t \mapsto \Psi_s^i(t)$, and derive from the chain rule that at each time $t$, the tangent vector to the path $t \mapsto \Psi_s^i(t)$ coincides with the tangent vector to the path $t \mapsto \phi_{(F, \lambda)}(s)$. That is, the isotopy $t \mapsto \Psi_s^i(t)$ is generated by an element $(U_s^i, H_s^i)$ where $U_s^i(t) = F^t(s)$ and $H_s^i(t) = \lambda_s^{t+s}$ for all $t$, for each $i$. Furthermore, the sequence of generators $(U_s^i, H_s^i)$ converges in $L^\infty$ metric to $(U_s, H_s)$ where $U_s(t) = F^{t+s}$ and $H_s(t) = \lambda^{t+s}$ for each $s$.

- It follows from step (1) and step (2) that the element $(U_s, H_s)$ generates the strong symplectic isotopy $\beta_{(F, \lambda)}$. Therefore, we derive from the uniqueness result of generator of strong symplectic isotopies (Theorem 1.3) that for each $s \in [0, 1]$ we must have $\lambda^t = \lambda^{t+s}$ and $F^t(x) = F^{t+s}(x)$ for all $t \in [0, 1]$, and for all $x \in M$. This is always true for a given $s \in [0, 1]$, i.e we have $\lambda^t = \lambda^0$ and $F^t(x) = F^0(x)$ for all $t \in [0, 1]$, and for all $x \in M$. This achieves the proof.\

As we said in the beginning, Theorem 1.11 suggests that any strong symplectic isotopy which is a 1-parameter group decomposes into the composition of smooth harmonic flow and a continuous Hamiltonian flow in the sense of Oh-Müller 28.

**Question (c)**

Is 1-parameter group any strong symplectic isotopy which decomposes into the composition of smooth harmonic flow and a continuous Hamiltonian flow?

We have the following fact.

**Lemma 2.13.** Let $\lambda^t_{(U, H)}$, $t \in [0, 1]$ be any strong symplectic isotopy. For each fixed $s \in [0, 1)$, the path $t \mapsto \lambda^{t+s}_{(V, K)} := \lambda_{(U, H)}^{t+s} \circ (\lambda_{(U, H)}^s)^{-1}$ is a strong symplectic isotopy generated by $(V, K)$ where $V(t, x) = U(t + s, x)$ and $K_{t,s} = H_{(t+s)}$ for all $t \in [0, 1 - s]$, and for all $x \in M$.

**Proof of Lemma 2.13.** Assume that $\lambda_{(U, H)} = \lim_{C^{0+} L^\infty} (\phi_{(U, H)}(t))$. For each fixed $s \in [0, 1)$, consider the sequence of symplectic isotopies defined as follows $t \mapsto \phi_{(V, K)}^{(t+s)} := \phi_{(U, H)}^{(t+s)} \circ (\phi_{(U, H)}^s)^{-1}$. Computing the derivative (in $t$) of the
This completes the proof.\

In the following, by \( \sharp \) we denote the natural isomorphism induced by the symplectic form \( \omega \) from cotangent bundle \( TM^* \) to tangent bundle \( TM \).

The proof of theorem Theorem 1.5 will need the following lemma.

**Lemma 2.14. (Sequential deformation)** Let \((M,\omega)\) be any closed connected symplectic manifold. Let \((\rho_t)\) be a sequence of harmonic isotopies which is Cauchy in \( ||.||_\infty \). Consider the sequence of 2-parameter family of vectors fields \((Z_{(s,t)}^i)\) defined as,

\[
Z_{(s,t)}^i = t\dot{\rho}_{s,t}^i - s(\int_0^t (i_{\dot{\rho}_u^i} \omega) du)^i.
\]

Then,

1. For each fixed \( t \), the sequence of family of symplectic vector fields \((Z_{(s,t)}^i)\), converges in \( ||.||_\infty \), and the sequence \((G_{(s,t)}^i)\) of its generating paths converges uniformly. The latter limit is then obviously a strong symplectic isotopy.

2. For each fixed \( s \), the sequence of family of symplectic vector fields defined by \( V_{(s,t)}^i = \frac{d}{dt} G_{(s,t)}^i ((G_{(s,t)}^i)^{-1}) \) converges in \( ||.||_\infty \), and the sequence \((G_{(s,t)}^i)\) of its generating paths converges uniformly. The latter limit is then obviously a strong symplectic isotopy.

**Proof of Lemma 2.14.** Let \((\rho_t)\) be a sequence of harmonic isotopies which is Cauchy in \( ||.||_\infty \). We will process step by step.

- **Step (1).** It is not too hard to derive from the assumption that for each fixed \( t \) the sequence symplectic vector fields \((Z_{(s,t)}^i)\) is Cauchy in \( ||.||_\infty \) because for each fixed \( t \), a straightforward calculation leads to the following estimate

\[
||Z_{(s,t)}^i - Z_{(s,t)}^{i+1}||_\infty \leq 3E(||\dot{\rho}_{s,t}^i - (\dot{\rho}_{s,t}^{i+1})||_\infty,
\]

where the constant \( E \) is defined in [4]. Then, since the sequence of harmonic vectors fields \((Z_{(s,t)}^i)\) is Cauchy in \( ||.||_\infty \), one derives from 

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Lemma 5.1, that the sequence of symplectic isotopies \((G^i(t,s))\), generated by \((Z^i(s,t))\), converges in \(\bar{d}\) to a continuous family \((G(t,s))\) of symplectic diffeomorphisms. That is, for each fixed \(t\), the continuous family \((G(t,s))\) of symplectic diffeomorphisms is a strong symplectic isotopy so that the map \((s,t)\mapsto G(s,t)\) is continuous.

- Step (2). It follows from the above that for each fixed \(s\), the sequence of symplectic isotopies \((G^i(s,t))\), converges uniformly in \(s\) to the continuous family \((G(t,s))\) of symplectic diffeomorphisms. To conclude that for each fixed \(s\), the continuous family \((G(t,s))\) of symplectic diffeomorphisms is a strong symplectic isotopy, we need to prove in addition that for all fixed \(s\), the sequence of symplectic vector fields \((V^i(s,t))\) is Cauchy in \(\|\cdot\|\). This is the goal of the step below.

- Step (3). Let \(\text{symp}(M,\omega)\) be the space of symplectic vector fields. Consider \(M\) to be the space of all smooth curves, with \(c(0) = 0\) endowed with the norm \(\|\cdot\|\). Then, we equip the product space \(M \times I\) with the following distance
\[
\delta((c,s),(c',s')) = (\|c - c'|\| + (s - s')^2)^{1/2}.
\]
On the other hand, consider \(N\) to be the space of all smooth functions \(u : I \times I \to \text{symp}(M,\omega)\) endowed with the norm
\[
\|u\|_0 = \sup_{s,t} \|u(s,t)\|.
\]
We have the following smooth mappings,
\[
A_s : c(t) \mapsto tc(st) - 2s\int_0^1 (ic(u)\omega)du = U_{s,t},
\]
\[
I_s : U_{s,t} \mapsto G_{s,t},
\]
\[
\partial_t : G_{s,t} \mapsto \frac{\partial G_{s,t}}{\partial t},
\]
which induce the following Lipschitz map \(R : \mathcal{M} \times I \to N\) where \(R = \partial_t \circ I_s \circ A_s\). Observe that for each \(i\), the 2-parameter family of vector fields \(V^i_{s,t}\) is the image of the couple \((\dot{\rho}_i, s)\) by the Lipschitz continuous map \(R\). Then, it follows from the Lipschitz uniform continuity that for all fixed \(s\), the sequence of symplectic vector fields \((V^i_{s,t})\) is Cauchy in \(\|\cdot\|\) because
\[
\sup_t \|V^i_{s,t} - V^{i+1}_{s,t}\| = \delta(R((\dot{\rho}_i), s), R((\dot{\rho}^{i+1}_i), s)) \leq \kappa \|\dot{\rho}_i - \dot{\rho}^{i+1}_i\|\;
\]
(2.9)
where \(\kappa\) Lipschitz constant of the map \(R\). The right hand side of the above estimate tends to zero when \(i\) goes at infinity.

- Step (4). One derives from step (2) and step (3) that for each fixed \(s\), the continuous family \((G(t,s))\) of symplectic diffeomorphisms is a strong symplectic isotopy. This achieves the proof. ⚫
2.8 Proofs of Theorem \[1.5\] and Theorem \[1.6\]

\textit{Proof of Theorem 1.5} \ Let \(\xi\) be a continuous symplectic flow whose Fathi’s mass flow is trivial. Let \(\Phi_i\) be a sequence of symplectic isotopies that converges in \(\text{GSSympeo}(M, \omega, g)\) to \(\xi\). Set \(\Phi_i = (\phi_i^t)\), and let \(\phi_i^t = \rho_i^t \circ \alpha_i^t\) be the Hodge decomposition of \(\phi_i^t\) for each \(i\). By Hodge’s decomposition theorem of strong symplectic isotopies, one can assume that \(\xi(t) = \pi_t \circ \alpha_t\) where \(\pi_t\) is a continuous harmonic flow, and \((\alpha_t)\) is continuous Hamiltonian flow (see [31], [28]). Consider the sequence of 2-parameter family of vector fields \((\xi_t)\) defined as in Lemma 2.14.

- Step (a). Since Fathi’s mass flow of \(\xi\) is trivial, one derives from Fathi’s Poincaré duality theorem that there exists a positive sufficiently large integer \(i_0\) such that one can assume the flux of \(\Phi_i\) to be sufficiently small for all \(i > i_0\). Then, under this assumption, it follows from Banyaga \[4\] that for all \(i > i_0\), the isotopy \((\rho_i^t)\) is homotopic to the Hamiltonian isotopy \((G_{(1,t)}^i)_t\), relatively to fixed extremities, where the isotopy \((G_{(1,t)}^i)_t\) is defined as in Lemma 2.14. Furthermore, one derives from Lemma 2.14 that the sequence \((G_{(1,t)}^i)_t\) converges in \(\text{GSSympeo}(M, \omega, g)\) to a continuous Hamiltonian flow \((G_{(1,t)})_t\).

- Step (b). Now, set \((\nu_i)_{i > i_0}\). We derive from step (a) that for each \(i\) the path \(\nu_i\) is homotopic to \(G_{(1,t)}^i \circ \alpha_i^t\) relatively to fixed extremities. For instance, the sequence \((G_{(1,t)}^i \circ \alpha_i^t)_t\) converges in \(\text{GSSympeo}(M, \omega, g)\) to the continuous Hamiltonian flow \(t \mapsto G_{(1,t)} \circ \alpha_t\) with

\[
\mu(1) = \lim_{C^0} (G_{(1,t)}^i \circ \alpha_i^t) = \lim_{C^0} (\rho_i^t \circ \alpha_i^t) = \lim_{C^0} (\phi_i^t) = \xi(1).
\]

- Step (c). Again, by Lemma 2.14 we know the following:

1. for each fixed \(t\), the sequence of symplectic isotopies \((G_{(s,t)}^i)_t\) converges in \(\text{GSSympeo}(M, \omega, g)\) to a strong symplectic isotopy \((G_{(s,t)}^i)_t\),
2. for each fixed \(s\), the sequence of symplectic isotopies \((G_{(s,t)}^i)_t\) converges in \(\text{GSSympeo}(M, \omega, g)\) to a strong symplectic isotopy \((G_{(s,t)}^i)_t\).

One derives from the above that for all \((s, t) \in [0, 1] \times [0, 1]\), the \(C^0\) limit of the sequence of symplectic diffeomorphisms \((G_{(s,t)}^i)\) exists, and the latter limit lies in \(\text{SSympeo}(M, \omega)\). Put,

\[
\lim_{C^0} (G_{(s,t)}^i) = G(s, t).
\]

Then, we define a homotopy between \(\xi\) and \(\mu\) as follows.

\[
H : [0, 1] \times [0, 1] \to \text{SSympeo}(M, \omega),
\]

\[
(s, t) \mapsto G(s, t),
\]

such that \(H(0, 0) = \text{id}, H(1, 1) = \xi(1), H(0, t) = \xi(t)\) and \(H(1, t) = \mu(t)\). This completes the proof. \(\blacklozenge\)
Proof of Theorem 1.6 Since $\Phi_i$ converges in $d$ to $\xi$ which is a loop at the identity, then the sequence of time one maps $\Phi_i(1)$ converges uniformly to the constant map identity. We know by Hodge’s decomposition theorem of symplectic isotopies that for each $i$, $\Phi_i(1) = \rho_i \circ \varrho_i$ where $\rho_i$ is harmonic and $\varrho_i$ Hamiltonian. Then, the bi-invariance of the metric $d$ implies that

$$d(\rho_i^{-1}, \varrho_i) = d(\Phi_i(1), id) \to 0, i \to \infty.$$  

This suggests that the sequences $(\rho_i^{-1})$ and $(\varrho_i)$ converges in $d$ to the same limit. But, a result of [31] shows that the sequence $(\rho_i)$ always converges in $d$ to a smooth symplectic diffeomorphism $\rho$. Hence, we derive from the above that the sequence $(\varrho_i)$ converges in $d$ to a smooth symplectic diffeomorphism $\rho^{-1}$. That is, $\rho^{-1}$ is a smooth homeomorphism. Therefore, if follows from a result on Hamiltonian rigidity dues to Banyaga [5] that $\rho^{-1}$ is a smooth Hamiltonian diffeomorphism. The above statements prove that the sequence of time one maps $(\Phi_i(1))$ converges in $d$ to the constant map identity passing through smooth Hamiltonian diffeomorphisms. Therefore, there exists a n integer $i_0$ which is large such that for $i > i_0$, $\Phi_i(1)$ lies in a small $C^\infty$ neighborhood of the constant map identity in $Symp(M, \omega)$. On the other hand, by Weinstein [36] the identity component inside the group of symplectic diffeomorphisms is locally contractible, and then locally connected by smooth arcs with respect to the $C^\infty$ compact-open topology. So, for all $i > i_0$ one can assume that each $\Phi_i(1)$ can be connected to the identity through a smooth Hamiltonian flow $\beta_i$ which is sufficiently small in the $C^\infty$ sense. The latter process generates automatically a sequence $(\beta_i)$ of sufficiently small (in the $C^\infty$ sense) Hamiltonian flows. Furthermore, the sequence of Hamiltonian vector fields $X_i = (\dot{\beta}_i^t)$ is obviously Cauchy in $\|\cdot\|_\infty$ since the latter tends to 0 when $i$ goes to infinity. Now, set $\Psi_i = \Phi_i \circ \beta_i^{-1}$ for all $i > i_0$, and compute $\Psi_i(1) = id$. The sequence $(\Psi_i)$ is Cauchy in $D^2$ and converges in $d$ to $\xi$. This achieves the proof. \n
3 An enlargement of flux

3.1 Hodge and de Rham’s theorems

3.1.1 De Rham’s theorem

We denote by $C^\infty(\wedge^k T^* M)$ the space of smooth differential $k-$forms on $M$. In local coordinates $(x_1, x_2, ..., x_{2n})$, an element $\alpha \in C^\infty(\wedge^k T^* M)$ has the following expression

$$\alpha = \Sigma_{I=(i_1, i_2, ..., i_k)} a_I dx_1 \wedge dx_2 \wedge ... \wedge dx_k = \Sigma_I a_I dx_I,$$

where $a_I$ is a smooth function of $(x_1, x_2, ..., x_k)$. The exterior differentiation is a differential operator

$$d : C^\infty(\wedge^k T^* M) \to C^\infty(\wedge^{k+1} T^* M).$$

Locally,

$$d(\Sigma_I a_I dx_I) = \Sigma_I da_I \wedge dx_I.$$  

This operator satisfies $d \circ d = 0$, hence the range of $d$ is included in the kernel of $d$.  

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Definition 3.1. (35) The $k^{th}$ de Rham’s cohomology group of $M$ is defined by

$$H^k_{dR}(M) = \{ \alpha \in C^\infty(\wedge^k T^* M), d\alpha = 0 \} / dC^\infty(\wedge^{k-1} T^* M).$$

These spaces are clearly diffeomorphism invariants of $M$, moreover the deep theorem of G. de Rham says that these spaces are isomorphic to the real cohomology group of $M$, they are in fact homotopy invariant of $M$.

3.1.2 The flux (Banyaga)

Denote by $Diff(M)$ the group of diffeomorphisms on $M$ endowed with the $C^\infty$ compact open topology. Let $Diff^0_0(M)$ be the identity component of $Diff(M)$ for the $C^\infty$ compact open topology. Let $\Omega$ be any closed $p$-form on $M$. Denote by $Diff^0_0(M)$ the space of all diffeomorphisms that preserve the $p$-form $\Omega$, and by $Diff^0_0(M)$ we denote the connected component by smooth arcs of the identity in $Diff^0_0(M)$. Let $\Phi = (\phi^t)$ be a smooth path in $Diff^0_0(M)$ with $\phi_0 = Id$, and let

$$\dot{\phi}_t(x) = \frac{d\phi_t}{dt}(\phi_t^{-1}(x))$$

for all $(t, x) \in [0, 1] \times M$. It was proved in [2] that

$$\sum(\Phi) = \int_0^1 (\phi_t^* (i_{\dot{\phi}_t} \Omega)) dt,$$

is a closed $p-1$ form and its cohomology class denoted by $[\sum(\Phi)] \in H^{(p-1)}(M, \mathbb{R})$ depends only on the homotopy class $\{ \Phi \}$ of the isotopy $\Phi$ relatively with fixed ends in $Diff^0_0(M)$, and the map $\{ \Phi \} \mapsto [\sum(\Phi)]$ is a surjective group homomorphism

$$Flux_{\Omega} : Diff^0_0(M) \rightarrow H^{(p-1)}(M, \mathbb{R}).$$

In case $\Omega$ is a symplectic form $\omega$, we get a homomorphism

$$Flux_{\omega} : Symp(M, \omega)_0 \rightarrow H^1(M, \mathbb{R}),$$

where $Symp(M, \omega)_0$ is the universal covering of the space $Symp(M, \omega)_0$. Denote by $\Gamma$ the image by $Flux_{\omega}$ of $\pi_1(Symp(M, \omega)_0)$. The homomorphism $Flux_{\omega}$ induces a surjective homomorphism $flux_{\omega}$ from $Symp(M, \omega)_0$ onto $H^1(M, \mathbb{R})/\Gamma$. From the above construction, Banyaga [2, 3] proved that the group of all Hamiltonian diffeomorphisms of any compact symplectic manifold $(M, \omega)$ is a simple group which coincides with the kernel of $flux_{\omega}$, a very deep result.

3.1.3 The mass flow (Fathi)

Let $\mu$ be a ”good measure” on the manifold $M$. Let $Homeo_0(M, \mu)$ denotes the identity component in the group of measure preserving homeomorphisms $Homeo_0(M, \mu)$, and $Homeo_0(M, \mu)_0$ its universal covering. For $[h] = [[h_t]] \in Homeo_0(M, \mu)$, and a continuous $f : M \rightarrow S^1$ map, we lift the homotopy $fh_t - f : M \rightarrow S^1$ to a map $\overline{fh_t - f}$ from $M$ onto $\mathbb{R}$. Fathi proved that the
integral $\int_M f h_t - fd\mu$ depends only on the homotopy class $[h]$ of $(h_t)$ and the homotopy class $\{f\}$ of $f$ in $[M, S^1] \approx H^1(M, \mathbb{Z})$, and that the map

$$\tilde{\mathcal{F}}((h_t))(f) := \int_M f h_t - f d\mu.$$ 

defines a homomorphism

$$\tilde{\mathcal{F}}: \text{Homeo}_0(M, \mu) \to \text{Hom}(H^1(M, \mathbb{Z}), \mathbb{R}) \approx H_1(M, \mathbb{R}).$$

This map induces a surjective group morphism $\mathcal{F}$ from $\text{Homeo}_0(M, \mu)$ onto a quotient of $H_1(M, \mathbb{R})$ by a discrete subgroup. This map is called the Fathi mass flow.

### 3.1.4 Poincaré duality

Let $k$ be a fixed integer. Consider the following bilinear map,

$$I_k: H^k(M, \mathbb{R}) \times H^{2n-k}(M, \mathbb{R}) \to \mathbb{R},$$

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta,$$

is well defined, that is to say $I_k(\alpha, \beta)$ doesn’t depend on the choice of representatives in the cohomology classes $[\alpha]$ or $[\beta]$ (this is an easy application of the Stokes formula). Moreover this bilinear form provides an isomorphism between $H^k(M, \mathbb{R})$ and the dual space of $H^{2n-k}(M, \mathbb{R})$. In particular, when $\alpha \in C^\infty(\wedge^k T^*M)$ is closed and satisfies $I_k(\alpha, \beta) = 0$ for all $[\beta] \in H^{2n-k}(M, \mathbb{R})$ then there exists $\theta \in C^\infty(\wedge^{k-1} T^*M)$ such that $\alpha = d\theta$.

### 3.1.5 Fathi’s duality theorem

It is showed in [17] that the flux for volume-preserving diffeomorphisms is the Poincaré dual of Fathi’s mass flow. Furthermore, following Fathi’s [17], the latter duality result can be stated as follows. Let $\sigma$ denote the canonical volume form on $S^1$ given by the orientation of the circle. Then, for any function $f: M \to S^1$, we get

$$I_1(\text{Flux}_\omega(\Phi), \left[\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma\right]) = \tilde{\mathcal{F}}(\Phi)(f)$$

### 3.1.6 The existence of $C^0$– flux geometry for strong symplectic homeomorphisms

In the section bellow, we justify the existence of $C^0$– flux geometry for strong symplectic homeomorphisms proper to the study of strong symplectic homeomorphisms.

Let $(\Phi_i)_i$ be a sequence of symplectic isotopies which converges in $\bar{d}$ to a path $\lambda$. According to Fathi’s, the assignation $\lambda \mapsto \tilde{\mathcal{F}}(\lambda)$ is continuous with the respect to the uniform topology, hence one must have $\lim_{i \to \infty} (\tilde{\mathcal{F}}(\Phi_i)) = \tilde{\mathcal{F}}(\lambda)$. This implies that $\lim_{i \to \infty} I_1(\text{Flux}_\omega(\Phi_i), \left[\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma\right]) = \tilde{\mathcal{F}}(\lambda)(f)$. In the latter equality one cannot permute the limit with the integral because the
uniform convergence in $d$ of the sequence $(\Phi_i)$, does not implies that the sequence of cohomological classes $(\text{Flux}_\omega(\Phi_i))$, converges in $H^1(M,\mathbb{R})$. For instance, consider the following continuous mappings,

$$P_{r_2}: GSSympeo(M,\omega,g) \to \mathfrak{V}^0\text{harm}(M,g).$$

$$(\xi,(F,\theta)) \mapsto \theta,$$

$$P_{r_1}: GSSympeo(M,\omega,g) \to \mathcal{P}(\text{Homeo}(M),id),$$

$$(\xi,(F,\theta)) \mapsto \xi.$$  

One also know that any symplectic isotopy $\Phi$ can be identify in a unique way as an element in $GSSympeo(M,\omega,g)$ of the form $(\Phi,(U,H))$. Then equation (3.3) can be rewritten as follows.

$$I^1_1([\int_0^1 P_{r_2}((\Phi,(U,H)))(t)dt,\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma]) = (\widetilde{\delta} \circ P_{r_1}((\Phi,(U,H)))(f), (3.3)$$

where

$$[\int_0^1 P_{r_2}((\Phi,(U,H)))(t)dt] = \int_0^1 [\mathcal{H}_t]dt = \text{Flux}_\omega(\Phi).$$

### 3.2 The mass flow for strong symplectic isotopies

Let $\lambda_{(U,H)}$ be a strong symplectic isotopy. Then, $\lambda_{(U,H)}$ is a limit in the space $GSSympeo(M,\omega,g)$ of a sequence of symplectic isotopies $(\phi_{(U',H')},(U',H'))$. According to equation (3.3), the above identification leads to the following calculation,

$$\widetilde{\delta}(\lambda_{(U,H)})(f) = \lim_{C^0+L^\infty} I_1([\int_0^1 \mathcal{H}_t]dt,\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma])$$

$$= I_1([\int_0^1 \lim C^0+L^\infty \mathcal{H}_t]dt,\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma])$$

$$= I_1(\lim_{C^0+L^\infty} \text{Flux}_\omega((\phi_{(U',H')}),(U',H'))),\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma])$$

$$= I_1([\int_0^1 \mathcal{H}_t]dt,\frac{\omega^{n-1}}{(n-1)!} \wedge f^*\sigma]).$$

Therefore, it follows from the above estimates that the mass flow for strong symplectic isotopies is given by:

$$\widetilde{\delta}(\lambda_{(U,H)})(f) = \frac{1}{(n-1)!} \int_M (\int_0^1 \mathcal{H}_t dt) \wedge \omega^{n-1} \wedge f^*\sigma$$
### 3.3 Strong flux homomorphism

The above formula for mass flow for strong symplectic isotopies suggests that there is a $C^0$–flux geometry which is underlying to the group of strong symplectic homeomorphisms, and which still agree with Fathi’s duality theorem. Consider the map,

$$\tilde{\text{Cal}}_0 : \text{GSSympeo}(M, \omega, g) \to H^1(M, \mathbb{R}),$$

$$\xi_{(F, \theta)} \mapsto \int_0^1 [\theta t] dt.$$

The map $\tilde{\text{Cal}}_0$ is a well-defined continuous surjective group homomorphism.

#### 3.3.1 Homotopic invariance of strong flux

Now, consider on the group of all strong symplectic isotopies the following equivalent relation $\sim$. Let $\xi_{(F_1, \theta_1)}$, $i = 1, 2$ be two strong symplectic isotopies. We will say that $\xi_{(F_1, \theta_1)}$ is in relation with $\xi_{(F_2, \theta_2)}$ (we denote it by $\xi_{(F_1, \theta_1)} \sim \xi_{(F_2, \theta_2)}$) if and only if $\xi_{(F_1, \theta_1)}$ is homotopic to $\xi_{(F_2, \theta_2)}$ relatively to fixed extremities in $\text{SSympeo}(M, \omega)$. Denote by $\{\xi_{(F, \theta)}\}$ the equivalent class of $\xi_{(F, \theta)}$.

**Proposition 3.2.** The map $\tilde{\text{Cal}}_0$ does not depend on the choice of a representative in an equivalent class of a strong symplectic isotopy.

**Proof of Proposition 3.2.** Let $\xi_{(F_i, \theta_i)}$, $i = 1, 2$ be two strong symplectic isotopies such that $\xi_{(F_1, \theta_1)} \sim \xi_{(F_2, \theta_2)}$. Set

$$\xi_{(F_2, \theta_2)} = \lim_{C^0 + L^\infty} (\phi(H_i, \lambda_i)),$$

$$\xi_{(F_2, \theta_2)} = \lim_{C^0 + L^\infty} (\phi(K_i, \alpha_i)).$$

This implies that the product $(\phi(H_i, \lambda_i), (H_i, \lambda_i)) * (\phi(K_i, \alpha_i), (K_i, \alpha_i))^{-1}$ converges in the topological group $\text{GSSympeo}(M, \omega, g)$ to $\xi_{(F_2, \theta_2)} \circ \xi_{(F_1, \theta_1)}^{-1}$ which is homotopic to the constant path identity. Therefore, we derive from the continuity (with respect to the $\text{(C}^0 + L^\infty) – \text{topology}$) of equation (3.3) that we must have,

$$I(\int_0^1 \theta_1^1 dt - \int_0^1 \theta_2^1 dt, \frac{\omega^{n-1}}{(n - 1)!} \wedge f^* \sigma) = 3\mathbb{R}(Id)(f) = 0,$$

which implies that the 1–form $\int_0^1 \theta_1^1 dt - \int_0^1 \theta_2^1 dt$ must be exact. But, the 1–form $\int_0^1 \theta_1^1 dt - \int_0^1 \theta_2^1 dt$ is harmonic, and by Hodge’s theory an exact harmonic 1–form of any compact Riemannian manifold $M$ is trivial. That is,

$$[\int_0^1 \theta_1^1 dt] = [\int_0^1 \theta_2^1 dt].$$

This completes the proof. ♠
3.3.2 Covering space and quotient space

Let $\tilde{\text{GSSympeo}}(M,\omega,g)$ be the quotient space of $\text{GSSympeo}(M,\omega,g)$ with respect to the above equivalent relation $\sim$. According to Proposition 3.2, we get a group homomorphism

$$\tilde{\text{Cal}}_0 : \tilde{\text{GSSympeo}}(M,\omega,g) \to H^1(M,\mathbb{R}).$$

We have the following fact. Let $\tilde{\text{SSympeo}}(M,\omega)$ be the universal covering of the space $\text{SSympeo}(M,\omega)$ with respect to the symplectic topology, i.e we have a covering projection $P : \tilde{\text{SSympeo}}(M,\omega) \rightarrow \text{SSympeo}(M,\omega)$. Let, $P_1 : \tilde{\text{GSSympeo}}(M,\omega,g) \rightarrow \text{GSSympeo}(M,\omega,g)$, be the canonical projection.

1. Observe that any element in the quotient space $\tilde{\text{GSSympeo}}(M,\omega,g)$ determines a unique homotopic class in $\tilde{\text{SSympeo}}(M,\omega)$. So, if one denotes by

$$Q : \tilde{\text{GSSympeo}}(M,\omega,g) \rightarrow \tilde{\text{SSympeo}}(M,\omega)$$

the injection mapping of $\tilde{\text{GSSympeo}}(M,\omega,g)$ onto $\tilde{\text{SSympeo}}(M,\omega)$. Then we get a map $\tilde{P} = P \circ Q$ from the space $\tilde{\text{GSSympeo}}(M,\omega,g)$ onto the space $\tilde{\text{SSympeo}}(M,\omega)$ so that the following diagram commutes

$$\begin{array}{ccc}
\tilde{\text{GSSympeo}}(M,\omega,g) & \xrightarrow{Q} & \tilde{\text{SSympeo}}(M,\omega) \\
P_1 \uparrow & & \downarrow P \\
\text{SSympeo}(M,\omega,\omega,\omega) & \xrightarrow{\tilde{ev}} & \text{SSympeo}(M,\omega,\omega,\omega). \\
\end{array}$$

i.e

$$\tilde{ev} = P \circ Q \circ P_1 = \tilde{P} \circ P_1.$$ 

2. By definition, $\tilde{P}$ is continuous, and $\tilde{P}$ is surjective because it satisfies the relation $\tilde{ev} = \tilde{P} \circ P_1$ where $\tilde{ev}$ is surjective. In fact, $\tilde{P}$ is a covering map with respect to the quotient topology on $\tilde{\text{GSSympeo}}(M,\omega,g)$ and the symplectic topology on $\tilde{\text{SSympeo}}(M,\omega)$.

3.3.3 Fundamental group of $\text{SSympeo}(M,\omega)$

As far as I know, it is not proved nowhere whether any continuous path in $\tilde{\text{SSympeo}}(M,\omega)$ is a strong symplectic isotopy or not. So, we cannot identify the set of all equivalent classes in $\tilde{\text{SSympeo}}(M,\omega)$ generated by the loops at the identity as the fundamental group of $\tilde{\text{SSympeo}}(M,\omega)$ at the identity. In regard of the above fact, in this work, we write $\pi_1(\tilde{\text{SSympeo}}(M,\omega))$ to represent the equivalent classes in the equivalent relation $\sim$, whose any representative is a loop at the identity.

Question (c)

Is any isotopy in $\tilde{\text{SSympeo}}(M,\omega)$ defines a strong symplectic isotopy?
3.3.4 Topological flux group

Set
\[ \Gamma_0 = \widetilde{Cal}_0(\pi_1(SSympeo(M, \omega))). \]

We have the following result.

**Theorem 3.3.** Let \((M, \omega)\) be any closed connected symplectic manifold. Then the group \(\Gamma_0\) coincides with the well-known flux group \(\Gamma\).

**Proof of Theorem 3.3.** The inclusion \(\Gamma \subset \Gamma_0\) is obvious. To achieve our proof, it remains to prove that \(\Gamma_0 \subset \Gamma\).

Let \(\theta \in \Gamma_0\), by definition there exists \(\{\xi\} \in \pi_1(SSympeo(M, \omega))\) such that \(\theta = \widetilde{Cal}_0(\{\xi\})\). On the other hand, there exists a sequence \((\Phi_i)\) of symplectic isotopies that converges in \(\bar{\mathcal{C}}\) to the loop \(\xi\). Since \(\xi\) is a loop at the identity, by Theorem 1.6 the sequence \((\Phi_i)\) can be deformed into a sequence of loops at the identity that converges in \(\bar{\mathcal{C}}\). More precisely, it follows from Theorem 1.6 that for each \(i\), the isotopy \(\Phi_i\) is of the form \(\zeta_i \circ \beta_i\) where \(\{\zeta_i\}\) is some subsequence of \((\Phi_i)\), and \(\lim Flux_\omega(\beta_i) = 0\). Set \(\theta_i = \widetilde{Cal}_0(\{\Psi_i\})\) for all \(i\). By construction, \((\theta_i)\) defines a sequence of elements in the flux group \(\Gamma\), i.e \((\theta_i) \subset \Gamma\).

The sequence \((\Phi_i \circ \xi^{-1})\) of strong symplectic isotopies converges in \(GSSympeo(M, \omega, g)\) to the identity. Then, we derive from the continuity of the map \(\widetilde{Cal}_0\) that \(\lim_i |\widetilde{Cal}_0(\Phi_i \circ \xi^{-1})| = 0\), i.e
\[
\lim_i |\theta_i - \theta| \leq \lim_i |\theta_i - \theta| - \widetilde{Cal}_0(\beta_i)| + \lim_i |\widetilde{Cal}_0(\beta_i)| \to 0, \ i \to \infty,
\]

because \(\widetilde{Cal}_0(\Psi_i \circ \xi^{-1}) = [\theta_i] - [\theta] - \widetilde{Cal}_0(\beta_i)\), and \(\widetilde{Cal}_0(\beta_i) = Flux_\omega(\beta_i)\). Therefore, the discreteness of flux group \(\Gamma\) which Ono \[29\] proved implies that \(\Gamma\) is closed, i.e \([\theta] \in \Gamma\). Finally one concludes that the equality \(\Gamma = \Gamma_0\) holds true. \(\diamondsuit\)

**Remark 3.4.** Theorem 3.3 can be viewed as the \((C^0 + L^\infty)\)-rigidity of the flux group \(\Gamma\) because it shows that the group \(\Gamma\) stays invariant under the perturbation of the group \(\pi_1(Symp(M, \omega))\) with respect to the \((C^0 + L^\infty)\)-topology.

The homomorphism \(\widetilde{Cal}_0\) induces a surjective group homomorphism \(Cal_0\) from \(SSympeo(M, \omega)\) onto \(H^1(M, \mathbb{R})/\Gamma\) such that the following diagram commutes
\[
\begin{array}{ccc}
GSSympeo(M, \omega, g) & \xrightarrow{\widetilde{Cal}_0} & H^1(M, \mathbb{R}) \\
\tilde{P} \downarrow & & \downarrow \pi_2 \\
SSympeo(M, \omega) & \xrightarrow{Cal_0} & H^1(M, \mathbb{R})/\Gamma.
\end{array}
\]

Denote by \(\widetilde{Hameo}(M, \omega)\) (resp. \(\widetilde{Ham}(M, \omega)\)) the set consisting of all equivalent classes that admit at least a representative which is a continuous Hamiltonian flow (resp. the universal covering of the spaces \(Ham(M, \omega)\)).
Theorem 3.5. Let $(M, \omega)$ be a closed connected symplectic manifold. Then, 
\[ \{\xi\} \in \text{Hameo}(M, \omega) \iff \text{Cal}_0(\{\xi\}) = 0. \]

Proof of Theorem 3.5. Let \( \{\xi\} \in \text{GSympeo}(M, \omega, g) \) such that \( \{\xi\} \in \text{Hameo}(M, \omega) \). Since \( h = \xi(1) \in \text{Hameo}(M, \omega) \), \( \{\xi\} \) admits a representative \( \lambda \) which is a continuous Hamiltonian flow. Then, \( \text{Cal}_0(\{\xi\}) = \text{Cal}_0(\{\lambda\}) = 0 \). For the converse, assume that \( \{\xi\} \in \text{GSympeo}(M, \omega, g) \) such that \( \text{Cal}_0(\{\xi\}) = 0 \). By Fathi’s Poincaré duality theorem, the Fathi mass flow of \( \xi \) vanishes. Then, we derive from Theorem 3.6. that \( \xi \) is homotopic to a continuous Hamiltonian flow \( \mu \) relatively to fixed extremities, and the statement follows. ▽

Theorem 3.6. Let \( \{\phi_i\} \in \text{GSympeo}(M, \omega, g) \). Then \( \phi_1 \) lies in \( \text{Hameo}(M, \omega) \) if and only if \( \text{Cal}_0(\{\phi_1\}) \) lies in \( \Gamma \).

Proof of Theorem 3.6. Let \( \{\phi_i\} \in \text{GSympeo}(M, \omega, g) \). Assume that \( \phi = \phi_1 \in \text{Hameo}(M, \omega) \). One can lift any continuous Hamiltonian flow \( \psi_t \) from the identity to \( \phi = \phi_1 \) onto an element \( \{\psi_t\} \in \text{Hameo}(M, \omega) \). The strong symplectic isotopy \( (\psi_t^{-1} \circ \phi_t) \) is a loop at the identity, i.e, \( (\psi_t^{-1} \circ \phi_t) \in \pi_1(\text{SSympeo}(M, \omega)) \) with \( \text{Cal}_0(\{\psi_t^{-1} \circ \phi_t\}) = \text{Cal}_0(\{\phi_t\}) + \text{Cal}_0(\{\psi_t^{-1}\}) \). But, we also have \( \text{Cal}_0(\{\psi_t^{-1}\}) = 0 \), i.e \( \text{Cal}_0(\{\phi_t\}) = \text{Cal}_0(\{\psi_t^{-1} \circ \phi_t\}) \in \Gamma \). For the converse, let \( \{\phi_i\} \in \text{GSympeo}(M, \omega, g) \) such that \( \text{Cal}_0(\{\phi_i\}) \) lies in \( \Gamma \). Since the map \( \text{Cal}_0 \) is surjective, there exists an element \( \{X_t\} \in \pi_1(\text{SSympeo}(M, \omega)) \) such that \( \text{Cal}_0(\{\phi_i\}) = \text{Cal}_0(\{X_t\}) \), i.e \( \text{Cal}_0(\{\phi_t \circ X_t^{-1}\}) = 0 \). Therefore, we derive from Theorem 3.5. that \( \{\phi_t \circ X_t^{-1}\} \) admits a representative which is a continuous Hamiltonian flow, i.e \( \phi_1 = \phi_1 \circ X_t^{-1} \in \text{Hameo}(M, \omega) \).

3.3.5 Some proprieties of ker Cal_0

Note that a strong symplectic homeomorphism \( h \) belongs to ker Cal_0 if one can find a strong symplectic isotopy \( \gamma(\{U, \mathcal{H}\}) \) with \( \gamma(\{U, \mathcal{H}\}) = h \), and \( \text{Cal}_0(\gamma(\{U, \mathcal{H}\})) = 0 \), i.e \( \int_0^1 \mathcal{H}_s ds = 0 \). So, if \( \{\phi_i(\{U, \mathcal{H}\})\} \) is any sequence of symplectic isotopies such that \( \gamma(\{U, \mathcal{H}\}) = \lim_{i \to \infty, L_{\infty}}(\phi_i(\{U, \mathcal{H}\})) \), then one deduces from the nullity of the integral \( \int_0^1 \mathcal{H}_s ds \) for \( i \) sufficiently large the flux of \( \phi_i(\{U, \mathcal{H}\}) \) is sufficiently small. Therefore we derive from Banyaga [2,3] that the time one map \( \phi_{i}(\{U, \mathcal{H}\}) \) is Hamiltonian for \( i \) sufficiently large.

Theorem 3.7. (Topological Banyaga's theorem III) The group ker Cal_0 is path connected.

Proof of Theorem 3.7. Let \( h \) be a strong symplectic homeomorphism that belongs to ker Cal_0. That is, there exists a strong symplectic isotopy \( \gamma(\{U, \mathcal{H}\}) \) such that \( \gamma(\{U, \mathcal{H}\}) = h \) and \( \int_0^1 \mathcal{H}_s ds = 0 \).

• Step (1). Let \( \rho_t \circ \psi_t \) be the Hodge decomposition of \( \gamma(\{U, \mathcal{H}\}) \) (see [31]), and let \( \{\phi_i(\{U, \mathcal{H}\})\} \) be any sequence of symplectic isotopies such that \( \gamma(\{U, \mathcal{H}\}) = \cdots \).
Then we have 
\[ \tilde{\sigma}(t) = \phi_t(0, H^t) \] 
It is clear that \( \tilde{\sigma} \) is Hamiltonian for \( t \) sufficiently large.

- Step (2). Following Banyaga [2, 3] we deduce the following fact. Set 
  \[ \theta_i(t) = \phi_t(0, H^t) \] 
  for each \( i \), and for all \( t \). Next, for each fixed \( t \), the isotopy 
  \[ s \mapsto h_{i(t)}(s,t) = \theta_i(t) \] 
  connects \( \theta_i(t) \) to the identity. For each \( i \), consider the 
  smooth family of harmonic vector fields defined as follows : 
  \[ X_i(s,t) = \frac{d}{ds} h_{i(t)}(s,t) \circ (h_{i(t)}(s,t))^{-1}, \] 
  and set \( \alpha_i^t = \int_0^1 i_{X_i(s,t)} \omega ds = \int_0^1 H_{i(t)}(s) ds \). 
  It follows from the above that the vector field \( Y_i \) satisfying the equation 
  \[ i_{Y_i} \omega = \alpha_i^t - t \alpha_i^t \] 
  is harmonic for each \( i \), and therefore, the 1-form 
  \[ \int_0^1 i_{(X_{i(s,t)} - Y_i)} \omega ds = t \int_0^1 H_{i(t)} du \] 
  is still harmonic. For each \( i \), let \( G_i(s,t) \) be the 2-parameter family of symplectic diffeomorphisms defined by integrating 
  (in \( s \)) the family of harmonic vector fields \( Z_i(s,t) = X_{i(s,t)} - Y_i \). Then, 
  Lemma 2.14 implies that for each fixed \( t \), the sequence of family of symplectic 
  vector fields \( (Z_i(s,t)) \) converges in \( \| \| \infty \), and the sequence \( (G_i(s,t))_s \) 
  of its generating paths converges uniformly. The latter limit denoted by 
  \( (G(s,t))_s \) is then obviously a strong symplectic isotopy so that the map 
  \( (s,t) \mapsto G(s,t) \) is continuous.

- Step (3). On the other hand, we derive from the assumption that for \( i \) 
  sufficiently large, the quantity \( \int_0^1 i_{X_{i(s,t)} - Y_i} \omega ds \) is sufficiently small, and 
  then it follows from Banyaga [2, 3] that the isotopy \( t \mapsto G_i(1,t) \) is Hamiltonian for all sufficiently large \( i \). That is, the limit \( t \mapsto G_i(1,t) \) is a 
  continuous Hamiltonian flow, i.e a continuous path in \( \ker \text{Cal}_0 \). For instance, 
  we compute \( G_i(1,t) = \lim_i \theta_i^t(1) = \rho_1 \), and deduce that the continuous path 
  \( t \mapsto G_i(1,t) \circ \psi_1 \) is a continuous path in \( \ker \text{Cal}_0 \) that connects \( G_i(1,1) \circ \psi_1 = \psi_1 \) 
  to the identity. This completes the proof. \( \blacksquare \)

**Theorem 3.8.** Any strong symplectic isotopy in \( \ker \text{Cal}_0 \) is a continuous Hamiltonian flow.

**Proof of Theorem 3.8** We will adapt a proof from Banyaga [2, 3] into our case. Let \( \sigma(U,H) \) be a strong symplectic isotopy in \( \ker \text{Cal}_0 \), and let \( (\phi(U,H))_t \) be any sequence of symplectic isotopies such that 
\[
\sigma(U,H) = \lim_{C^0 + L^\infty} (\phi(U,H)).
\]
Since \( \pi_1(\widetilde{SSympeo}(M,\omega)) \) acts on \( \widetilde{SSympeo}(M,\omega,g) \), we denote by 
\( \pi_1(\widetilde{Sympeo}(M,\omega)) \). \( \ker \text{Cal}_0 \) the set consisting of orbits of the points of \( \ker \text{Cal}_0 \). 
It is clear that \( \tilde{P} : \widetilde{SSympeo}(M,\omega,g) \to \widetilde{Sympeo}(M,\omega) \) is the covering map. 
Then we have 
\[
\tilde{P}^{-1}(\ker \text{Cal}_0) = \pi_1(\widetilde{Sympeo}(M,\omega)), \ker \text{Cal}_0.
\]
Let \( \tilde{\sigma} \) be the lifting of the path \( \sigma(U,H) \) such that \( \tilde{\sigma}(0) = \text{id}_M \). By assumption, 
we have \( \text{Cal}_0(\tilde{\sigma}(t)) \in \Gamma \) for all \( t \). A verbatim repetition of some arguments
from Banyaga [2, 3] supported by the discreteness of the flux group $\Gamma$ leads to
\[ \tilde{\sigma}(t) \in \ker C_{al_0} \text{ for all } t. \] We then derive from the lines of the proof of Theorem 3.7 that for each $t$, the homotopy class $\tilde{\sigma}(t)$ admits a representative $s \mapsto G_{s,t}$ which is a continuous Hamiltonian flow, i.e., a continuous path in $\ker C_{al_0}$ so that the map $(s, t) \mapsto G_{s,t}$ is continuous. For all fixed $t$, the following map:
\[ (u, s) \mapsto G_{(u(s-1)+1),(u-1)s+u}t \]
induces a homotopy between $s \mapsto \sigma^{(s,t)}_{(U, H)}$ and $s \mapsto G_{s,t}$. It follows from the above that $C_{al_0}(\sigma(t.s)) = \int_0^1 H_{u}du = 0$ for all $t$. On the other hand, since $\sigma(U, H) = \lim_{C^0 + L^\infty}(\phi(U, H))$, then for each $t$, we derive from the above that the strong symplectic isotopy $\Psi_t : s \mapsto \sigma^{(s,t)}_{(U, H)}$ is the $(C^0 + L^\infty)$-limit of the sequence of symplectic isotopies $\Psi^i_t : s \mapsto \phi_{(U, H)}^{(s,t)}$. For each $t$, the symplectic isotopy $\Psi_t : s \mapsto \phi_{(U, H)}^{(s,t)}$ is generated by $(V^i_t, K^i_t)$ where $V^i_t(s) = tU^i_{(s,t)}$ and $K^i_t(s)^i = t[H^i_{(s,t)}]$ for all $s$. Therefore, the fact that $C_{al_0}(\sigma(t.s)) = 0$ for all $t$ suggests that the flux of the isotopy $\Psi^i_t$ tends to zero when $i$ goes to the infinity, i.e, $\int_0^1 H^i_{(s,t)}du = 0$ for all $t$, and for $i$ sufficiently large. This implies obviously that $|H^i_t| \rightarrow 0, i \rightarrow \infty$ for all $t$. That is, $\sup_t |H^i_t| \rightarrow 0, i \rightarrow \infty$, i.e.
\[ \sup_t |H^i_t| \leq \sup_t |H^i_{t} - H^i| + \sup_t |H^i_t| \rightarrow 0, i \rightarrow \infty. \]

Finally, we have proved that the harmonic part $H$ of the generator of the strong symplectic isotopy $\sigma(U, H)$ is trivial, i.e $\sigma(U, H) = \sigma(U, 0)$ is a continuous Hamiltonian flow. $\blacksquare$

**Proposition 3.9.** $\text{Hameo}(M, \omega)$ coincides with $\ker C_{al_0}$.

**Proof of Proposition 3.9.** According to Oh-Müller [28], any Hamiltonian homeomorphism $h$ can be connected to the identity through a continuous Hamiltonian flow $\xi$, then derive that $C_{al_0}(\xi) = 0$. This implies that $h \in \ker C_{al_0}$. In fact, $h \in \ker C_{al_0}$ implies that for any equivalent class such that $\xi(1) = h$, we have $\pi_2(C_{al_0}(\xi)) = 0$, i.e $C_{al_0}(\xi) \in \Gamma$. This is equivalent to say that $h \in \text{Hameo}(M, \omega)$. $\blacksquare$

**Proof of Theorem 1.8** The proof of Theorem 1.8 is a verbatim repetition of the proof of Theorem 3.8 $\clubsuit$

**Proof of Theorem 1.7** The proof of Theorem 1.7 follows from Theorem 3.8 and the diagram (II). $\spadesuit$

**Proof of Theorem 1.9** The result of Theorem 3.7 states that $\text{Hameo}(M, \omega) = \ker C_{al_0}$ is path connected. On the other hand, the result of Theorem 3.8 implies that $\Gamma = C_{al_0}(\pi_1(\text{SSympeo}(M, \omega)))$ is discrete, and then suggests that $\text{Hameo}(M, \omega) = \ker C_{al_0}$ is locally connected $\square$

**Proof of Theorem 1.10** Since $\text{SSympeo}(M, \omega)$ is a topological group, the latter is a homogeneous space, i.e., for all $p, q \in \text{SSympeo}(M, \omega)$, there exists
a homeomorphism \( \Phi \) that sends \( p \) onto \( q \), i.e. \( \Phi(p) = q \). So, it suffices just to check a local property at a point (at the identity for example) to prove it on the entire group. On the other hand, the result of Theorem 5.8 states that any strong symplectic isotopy in \( \text{Hameo}(M,\omega) \) is a continuous Hamiltonian flow, while a result due to Oh-Müller [28] states that \( \text{Hameo}(M,\omega) \) is locally path connected. Therefore, one derives from the above statements that for every open neighbourhood \( V \subseteq \text{SSympeo}(M,\omega) \) of the constant map identity, there exists a path connected open set \( U \subset \text{Hameo}(M,\omega) \) with \( \text{id} \in U \subset V \). Now, let \( p \in \text{SSympeo}(M,\omega) \), and let \( V_p \subseteq \text{SSympeo}(M,\omega) \) be an arbitrary neighbourhood of \( p \). There exists a homeomorphism \( \Phi \) that sends the identity onto \( p \), i.e. \( \Phi(\text{id}) = p \), and then \( \Phi \) displaces some neighbourhood \( V_0 \) of the constant map identity (with \( V_0 \subset U \)) onto a neighbourhood \( U_0 \) of \( p \). But, according to Oh-Müller [28], \( V_0 \) contains a path connected neighbourhood of the identity, and the latter is displaced by \( \Phi \) onto a path connected neighbourhood of \( p \) denoted by \( O_p \) with \( O_p \subset U_0 \). That is, \( O_p \cap V_p \) is a path connected neighbourhood of \( p \) which is contained in \( V_p \). \( \blacksquare \)

4 An enlargement of Banyaga’s Hofer-like norm

We recall that the symplectic topology on \( \text{SSympeo}(M,\omega) \) is defined to be the strongest topology on it which makes the map

\[ \overline{ev} : \text{GSSympeo}(M,\omega,g) \to \text{SSympeo}(M,\omega), \]

\[ \gamma(U,\mathcal{H}) \mapsto \gamma_1 \]

becomes a surjective homomorphism in the category of topological groups with respect to the \((C^0 + L^\infty)-\)topology on the space \( \text{GSSympeo}(M,\omega,g) \).

**Definition 4.1.** For any strong symplectic isotopy \( \gamma(U,\mathcal{H}) \), we define its length by

\[ l(\gamma(U,\mathcal{H})) = \frac{l_\infty(\gamma(U,\mathcal{H})) + l_\infty(\gamma^{-1}(U,\mathcal{H}))}{2}, \]

with \( l_\infty(\gamma(U,\mathcal{H})) = \max_t(\text{osc}(U_t) + |\mathcal{H}_t|) \).

**Definition 4.2.** For any \( h \in \text{SSympeo}(M,\omega) \), we define the Hofer-like norm of \( h \) by

\[ \|h\|_{SHL} := \inf \{ l(\gamma(U,\mathcal{H})) \}, \]

where the infimum is taken over all strong symplectic isotopies \( \gamma(U,\mathcal{H}) \) with \( \gamma_1 = h \).

The above definitions make sense because of the uniqueness results (Theorem 2.12 and Theorem 1.3). This agrees with the usual definition of Hofer-like length of symplectic isotopies [4].

**Proof of Theorem 1.12.** We will process step by step.

- Step (1). Positivity is obvious. In the way that the symplectic topologies has been defined on the spaces \( \text{SSympeo}(M,\omega) \) and \( \text{GSSympeo}(M,\omega,g) \) (see [8]), the map

\[ \overline{ev} : \text{GSSympeo}(M,\omega,g) \to \mathbb{R}, \]
Step (3). Next, it follows from the remark 2.9 that:

\[ h \mapsto \|h\|_{SHL} \]

is continuous if and only if

\[ GSSympeo(M, \omega, g) \xrightarrow{\text{top}} \mathfrak{g} Sympeo(M, \omega) \xrightarrow{\|\cdot\|_{SHL}} \mathbb{R}, \]

is continuous with respect to symplectic topology on the space \( GSSympeo(M, \omega, g) \). Then, the continuity of the map

\[ \mathfrak{g} Sympeo(M, \omega) \to \mathbb{R}, \]

\[ h \mapsto \|h\|_{SHL}, \]

is proved as follows: Let \( \gamma(U_k, H_k) \) be a sequence of elements of \( GSSympeo(M, \omega, g) \) that converges to \( \gamma(U, H) \) in \( GSSympeo(M, \omega, g) \) with respect to the \( (C^0 + L^\infty) \)-topology. We always have:

\[ 2.\|\gamma(1)\|_{SHL} - \|\gamma_k(1)\|_{SHL} \leq 2.\|(\gamma_k^{-1} \circ \gamma(U, H))(1)\|_{SHL} \]

\[ \leq 2.\|\gamma(U, H)\|_{SHL}, \]

\[ \leq \max_{t \in [0,1]} \{ \text{osc}(U_k^t \circ \gamma(U_k, H_k)) - U^t \circ \gamma(U_k, H_k)) + |H^t - H_k^t| \}

\[ + \max_{t \in [0,1]} \text{osc}(\Delta_0^0(\mathcal{H}_k, \gamma(U_k, H_k)) - \Delta_0^0(\mathcal{H}, \gamma(U_k, H_k))) \]

\[ + \max_{t \in [0,1]} \text{osc}(\Delta_0^0(\mathcal{H}, \gamma(U, H)) - \Delta_0^0(\mathcal{H}_k, \gamma(U_k, H_k))). \]

- Step (2). By assumption the sequence \( \gamma(U_k, H_k) \) converges to \( \gamma(U, H) \) with respect to the \( (C^0 + L^\infty) \)-topology on the space \( GSSympeo(M, \omega, g) \), as well as the sequence \( \gamma_k^{-1} \circ \gamma(U_k, H_k) \) converges to \( \gamma(U, H) \) with respect to the \( (C^0 + L^\infty) \)-topology on the space \( GSSympeo(M, \omega, g) \) because \( GSSympeo(M, \omega, g) \) is a topological group with respect to the symplectic topology. Therefore we derive from the above statement that

\[ \max_{t \in [0,1]} \{ \text{osc}(U_k^t \circ \gamma(U_k, H_k)) - U^t \circ \gamma(U_k, H_k)) + |H^t - H_k^t| \} = \max_{t \in [0,1]} \{ \text{osc}(U_k^t - U^t) + |H^t - H_k^t| \}, \]

\[ \max_{t \in [0,1]} \{ \text{osc}(U_k^t \circ \gamma(U, H)) - U^t \circ \gamma(U, H)) + |H^t - H_k^t| \} = \max_{t \in [0,1]} \{ \text{osc}(U_k^t - U^t) + |H^t - H_k^t| \}, \]

where the right-hand sides of the above estimates tend to zero when \( i \) goes to the infinity.

- Step (3). Next, it follows from the remark 2.9 that:

\[ \max_{t \in [0,1]} \text{osc}(\Delta_0^0(\mathcal{H}, \gamma(U, H)) - \Delta_0^0(\mathcal{H}_k, \gamma(U_k, H_k))) = \max_{t \in [0,1]} \text{osc}(\Delta_0^0(\mathcal{H}_k - \mathcal{H}, \gamma(U, H))), \]

\[ \max_{t \in [0,1]} \text{osc}(\Delta_0^0(\mathcal{H}_k, \gamma(U_k, H_k)) - \Delta_0^0(\mathcal{H}, \gamma(U_k, H_k))) = \max_{t \in [0,1]} \text{osc}(\Delta_0^0(\mathcal{H}_k - \mathcal{H}, \gamma(U_k, H_k))). \]
On the other hand, by definition of $GSSympeo(M, \omega, g)$, for any fixed integer $k$, there exists a sequence of symplectic isotopies $(\phi_{(V, H), K(k,j)})$ such that

$$
\gamma(U_k, H_k) = \lim_{C^0 + L^\infty} ((\phi_{(V, H), K(k,j)})).
$$

Similarly, there exists a sequence of symplectic isotopies $(\phi_{(V', H'), U})$ which does not depend on $k$ such that

$$
\gamma(U, H) = \lim_{C^0 + L^\infty} (\phi_{(V', H')}).
$$

We compute,

$$
\Delta^0(\mathcal{H}_k - \mathcal{H}, \gamma(U_k, H_k)) = \lim_{L \to \infty} \{ \int_0^L (K_i^{k,j}) - \mathcal{H}_i^k) (\phi_{(V, H')}^{s}) \circ \phi_{(V', H')}^{s} ds \}
$$

$$
\Delta^0(\mathcal{H} - \mathcal{H}_k, \gamma(U_k, H_k)) = \lim_{L \to \infty} \{ \int_0^L (\mathcal{H}_i^k - K_i^{k,j}) (\phi_{(V, H), K(k,j)}^{s}) \circ \phi_{(V', H')}^{s} ds \}.
$$

- Step (4). Since both sequences $(\phi_{(V, H)})_i$ and $(\phi_{(V, H), K(k,j)})_j$ are Cauchy in $d$, and we have $\max_{i \in [0,1]} |K_i^k - K_i^{k,j}| \to 0$ when $i > k$ and $k \to \infty$. Under the above assumption a result from [32] (Corollary 2.10) implies that

$$
\max_{t \in [0,1]} \text{osc}(\mathcal{H}_k - \mathcal{H}, \gamma(U_k, H_k)) \to 0, k \to \infty,
$$

$$
\max_{t \in [0,1]} \text{osc}(\mathcal{H} - \mathcal{H}_k, \gamma(U, H)) \to 0, k \to \infty.
$$

- Step (5). Finally, the results of step (2) and step (4) show that

$$
|\gamma_{(U_k, H_k)}^{-1} \circ \gamma(U, H)| \to 0, k \to \infty.
$$

This completes the proof. \(\diamondsuit\)

**Proof of Theorem 1.13** The item (2) follows from the definition of the map $\|\cdot\|_{S_HL}$. For (1), we adapt the proof of the nondegeneracy that given Oh [27] into our general case. We will process step by step.

- Step (a). Suppose that $h \neq id$. Then $h$ displaces a small nonempty compact ball $B$ of positive symplectic displacement energy $e_{S}(B) > 0$. For such a ball $B$, put $\delta = e_{S}(B) > 0$. By the characterization of the infimum, one can find a strong symplectic isotopy $\gamma(U, H)$ with $\gamma_{(U, H)}^{-1} = h$, such that

$$
\|h\|_{S_HL} > l(\gamma(U, H)) - \frac{\delta}{4}.
$$

- Step (b). On the other hand, it follows from the definition of the topological group $GSSympeo(M, \omega, g)$ that there exists a sequence $(\phi_{(F, \lambda)})$ that converges to $\gamma(U, H)$ with respect to the $(C^0 + L^\infty)$—topology. So, we can find an integer $i_0$ with $(\phi_{(F, \lambda)})$ is sufficiently close to $\gamma(U, H)$
resp. \( \phi^{-1}_{(F_0, \lambda_0)} \) sufficiently close to \( \gamma^{-1}_{(U, \mathcal{H})} \) with respect to the \((C^0 + L^\infty)\)-topology so that

\[
l(\gamma_{(U, \mathcal{H})}) - l(\phi_{(F_0, \lambda_0)}) = \frac{\delta}{4},
\]

and such that \( \phi = \phi^{-1}_{(F_0, \lambda_0)} \) still displaces \( B \).

- Step (c). It follows from the definition of Banyaga’s Hofer-like norm \( \| \cdot \|_{HL} \) that

\[
l(\phi_{(F_0, \lambda_0)}) - \frac{2\delta}{3} \geq \| \phi \|_{HL} - \frac{\delta}{2},
\]

and by definition of the symplectic displacement energy \( e_S \), we have

\[
\| \phi \|_{HL} - \frac{\delta}{2} \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0.
\]

- Step (d). The statements of step (a), step (b) and step (c) imply that

\[
\| h \|_{SHL} > l(\gamma_{(U, \mathcal{H})}) - \frac{2\delta}{3} \geq \| \phi \|_{HL} - \frac{\delta}{2} > 0.
\]

The item (3) follows from the continuity of the map \( \| \cdot \|_{SHL} \) closely the proof triangle inequality of Banyaga’s Hofer-like norm. This completes the proof. ★

**Proof of Theorem 1.14** By construction, we always have

\[
\| \cdot \|_{SHL} \leq \| \cdot \|_{Oh}.
\]

To complete the proof, we need to show that there exists a positive finite constant \( A_0 \) such that

\[
\| \cdot \|_{Oh} \leq A_0 \| \cdot \|_{SHL}.
\]

or equivalently, via the sequential criterion, it suffices to prove that any sequence of Hamiltonian homeomorphisms converging to the constant map identity for \( \| \cdot \|_{SHL} \), converges to the constant map identity for Oh’s norm \( \| \cdot \|_{Oh} \). The proof we give here heavily relies ideas of Oh [27] and Banyaga [4] used in the proof of the nondegeneracy of their norms. Let \( \psi^i \) be a sequence of Hamiltonian homeomorphisms that converges to the identity with respect to the norm \( \| \cdot \|_{SHL} \). For each \( i \), and any \( \epsilon > 0 \) there exists a strong symplectic isotopy \( \psi_{(U^{i, \epsilon}, \mathcal{H}^{i, \epsilon})} \) such that \( \psi^i_{(U^{i, \epsilon}, \mathcal{H}^{i, \epsilon})} = \psi^i \), and \( l(\psi_{(U^{i, \epsilon}, \mathcal{H}^{i, \epsilon})}) < \| \psi^i \|_{SHL} + \epsilon \). On the other hand, for a fixed \( i \), there exists a sequence of symplectic isotopies \( (\phi_{(V_{i,j}, \mathcal{K}_{i,j})}) \) such that

\[
\psi_{(U^{i, \epsilon}, \mathcal{H}^{i, \epsilon})} = \lim_{C^0+L^\infty} (\phi_{(V_{i,j}, \mathcal{K}_{i,j})}).
\]

In particular, one can find an integer \( j_0 \) with \( \phi_{(V_{i,j_0}, \mathcal{K}_{i,j_0})} \) sufficiently close to \( \psi_{(U^{i, \epsilon}, \mathcal{H}^{i, \epsilon})} \) with respect to the \((C^0 + L^\infty)\)-topology so that

\[
l(\psi_{(U^{i, \epsilon}, \mathcal{H}^{i, \epsilon})}) - l(\phi_{(V_{i,j_0}, \mathcal{K}_{i,j_0})}) = \frac{\epsilon}{4}.
\]

Observe that when \( i \) tends to infinity, Banyaga’s length of the symplectic isotopy \( \phi_{(V_{i,j_0}, \mathcal{K}_{i,j_0})} \) can be considered as being sufficiently small. Thus, it comes
from Banyaga [4] that the time-one map of such an isotopy is a Hamiltonian diffeomorphism. So, we can assume without losing generality that \( \phi^1_{(V_{i,j},\kappa_{i,j})} \) is Hamiltonian for \( i \) sufficiently large. Now, we derive from the above statements that

\[
l(\psi(U_{i,\epsilon}, H_{i,\epsilon})) > \| \phi^1_{(V_{i,j},\kappa_{i,j})} \|_{HL} - \frac{\epsilon}{4}.
\]

Since for \( i \) sufficiently large, \( \phi^1_{(V_{i,j},\kappa_{i,j})} \) is Hamiltonian, it follows from a result from Buss-Leclercq [12] that there exists a positive finite constant \( D \) which does not depend on \( i \) such that

\[
\frac{1}{D} \| \phi^1_{(V_{i,j},\kappa_{i,j})} \|_H \leq \| \phi^1_{(V_{i,j},\kappa_{i,j})} \|_{SHL},
\]

where \( \| \cdot \|_H \) represents the Hofer norm of Hamiltonian diffeomorphisms. The above statements imply that for \( i \) sufficiently large,

\[
\| \psi^i \|_{SHL} + \epsilon > \frac{1}{D} \| \phi^1_{(V_{i,j},\kappa_{i,j})} \|_H - \frac{\epsilon}{4},
\]

At this level, we use an argument coming from Müller’s thesis stating that the restriction of Oh’s norm to the group of Hamiltonian diffeomorphisms is bounded from above by Hofer’s norm. Therefore, we derive from the above get

\[
\| \psi^i \|_{SHL} + \epsilon > \frac{1}{D} \| \phi^1_{(V_{i,j},\kappa_{i,j})} \|_{Oh} - \frac{\epsilon}{4},
\]

for \( i \) sufficiently large. Passing to the limit in the latter estimate, one obtains

\[
\lim_{i \to \infty} \| \psi^i \|_{SHL} \geq \frac{5\epsilon}{4} \geq \frac{1}{D} \lim_{j_0 \to \infty} \| \phi^1_{(V_{i,j},\kappa_{i,j})} \|_{Oh} = \frac{1}{D} \lim_{i \to \infty} \| \psi^i \|_{Oh},
\]

for all \( \epsilon \). Finally, we have proved that for all positive real number \( \delta \) (replacing \( \epsilon \) by \( \frac{4\delta}{5D} \)), we have

\[
\delta \geq \lim_{i \to \infty} \| \psi^i \|_{Oh}.
\]

This completes the proof. \( \square \)

5 An enlargement of Symplectic displacement energy

**Definition 5.1.** The strong symplectic displacement energy \( E_S(B) \) of a non empty compact set \( B \subset M \) is:

\[
E_S(B) = \inf \{ \| h \|_{SHL} \mid h \in \text{Sympeo}(M, \omega), h(B) \cap B = \emptyset \}.
\]

**Lemma 5.2.** For any non empty compact set \( B \subset M \), \( E_S(B) \) is a strict positive number.

**Proof of Lemma 5.2.** Let \( \epsilon > 0 \). By definition of \( E_S(B) \) there exists a strong symplectic isotopy \( \psi_{(F,\lambda)} \) such that \( \psi^1_{(F,\lambda)} = h \), and \( E_S(B) > \| h \|_{SHL} - \epsilon \). On the other hand, there exists a sequence of symplectic isotopies \( \{ \phi_{(F,\lambda_j)} \} \) that
converges to $\psi(F, \lambda')$ with respect to the $\left( C^0 + L^\infty \right)$-topology so that $\phi^i_{(F, \lambda_i)}$ still displaces $B$ for $i$ sufficiently large. It follows from Theorem 2.2 that

$$\|\phi^i_{(F, \lambda_i)}\|_{HL} \geq e_S(B) > 0,$$

for $i$ sufficiently large. Since $\epsilon > 0$ is arbitrary, the continuity of the map $\| \cdot \|_{SHL}$ imposes that

$$\|h\|_{SHL} - \epsilon \geq \|\phi^i_{(F, \lambda_i)}\|_{SHL} = \|\phi^i_{(F, \lambda_i)}\|_{HL},$$

for $i$ sufficiently large. Therefore,

$$E_S(B) > \|h\|_{SHL} - \epsilon \geq e_S(B) > 0. \quad \square$$

5.1 $L^{(1, \infty)}$-norm for strong symplectic isotopies

For any strong symplectic isotopy $\gamma(U, H)$, we define its interpolation length by

$$l^{(1, \infty)}(\gamma(U, H)) = \frac{l_0(\gamma(U, H)) + l_0(\gamma^{-1}(U, H))}{2},$$

with

$$l_\infty(\gamma(U, H)) = \int_0^1 (\text{osc}(U_t) + |H_t|)dt.$$

Therefore, for any $h \in SSympeo(M, \omega)$, we define the $L^{(1, \infty)}$ Hofer-like norm of $h$ by

$$\|h\|^{(1, \infty)}_{SHL} = \inf\{l^{(1, \infty)}(\gamma(U, H))\}, \quad (5.1)$$

where the infimum is taken over all strong symplectic isotopies $\gamma(U, H)$ with $\gamma^1(U, H) = h$.

Conjecture

Let $(M, \omega)$ be a closed connected symplectic manifold. The equality

$$\|h\|_{SHL} = \|h\|^{(1, \infty)}_{SHL},$$

holds true.

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References

[1] R. Abraham and J. Robbin, : *Transversal mappings and flow*, W. A. Benjamin, Inc., New York-Amsterdam, 39(1967).

[2] A. Banyaga, : *Sur la structure de difféomorphismes qui préservent une forme symplectique*, comment. Math. Helv. 53 (1978) pp 174 – 2227.

[3] A. Banyaga, : *The Structure of classical diffeomorphisms groups*, Mathematics and its applications vol 400. Kluwer Academic Publisher’s Goup, Dordrescht, The Netherlands (1997).

[4] A. Banyaga, : *A Hofer-like metric on the group of symplectic diffeomorphisms*, Contemp. Math. Amer. Math. Soc. RI. Vol 512 (2010) pp 1 – 24.

[5] A. Banyaga, : *An enlargement of the group of Hamiltonian homeomorphisms*, (Preprint) (2014)

[6] A. Banyaga, : Erratum To : *On the group of strong symplectic homeomorphisms*, Cubo. A Mathematical Journal. Vol 14 3(2012) pp 57 – 59.

[7] A. Banyaga, E. Hurtubise, and P. Spaeth : *On the symplectic displacement energy*, preprint (2013).

[8] A. Banyaga and S. Tchuiaga, : *The group of strong symplectic homeomorphisms in $L^\infty$-metric*, Adv. Geom. DOI 10.1515 / advgeom-2013-0041 14 (2014) pp 523–539.

[9] A. Banyaga and S. Tchuiaga : *Uniqueness of generators of symplectoopies*, preprint (2013).

[10] L. Buhovsky, *Variation on Eliashberg-Gromov theorem I, $C^0$– Symplectic Topology and Dynamical Systems*, (2014), IBS Center for Geometry and Physics, Korea

[11] L. Buhovsky and S. Seyfaddini *Uniqueness of generating Hamiltonian for continuous Hamiltonian flows*, J. Symplectic. Geom 11( 2010), no. 1, 37-52.

[12] G. Bus and R. Leclercq : *Pseudo-distance on symplectomorphisms groups and application to the flux theory*, Math, Z (2012) 272: 1001 - 1022.

[13] E. Calabi, : *On the group of automorphisms of a symplectic manifold*, Problem in analysis (Lectures at the Sympos. in honor of Salomon Bochner), Princeton, N.J., (1969), Princeton Univ. Press, Princeton, N.J., (1970) 1 – 26.

[14] E. H. Connell, *Approximating stable homeomorphisms by piecewise linear ones*, Ann. of Math. 78 (1963), no. 2, 326–338.

[15] E. H. Connell, *Stable homeomorphisms can be approximated by piecewise linear ones*, Bull. Amer. Math. Soc. 69 (1963), 87–90.

[16] Y. Eliashberg, : *A theorem on the structure of wave fronts and its application in symplectic topology*, Funct. Anal. and Its Applications(1987)21 227 – 232.
[17] A. Fathi, : Structure of the group of homeomorphisms preserving a good measure on a compact manifold, Ann. Scient. Ec. Norm. Sup., 13(1980) 45 – 93.

[18] M. Gromov, : Pseudoholomorphic cuves in symplectic manifold, Inent. Math. 82(1985) 307 – 347.

[19] M. Hirsch, : Differential Topology, Graduate Texts in Mathematics, no. 33, Springer Verlag, New York–Heidelberg. 3 (1976) corrected reprint (1994).

[20] H. Hofer, : On the topological properties of symplectic maps, Proc. Royal Soc. Edinburgh 115A(1990), pp 25 – 38.

[21] H. Hofer and E. Zehnder, : Symplectic invariants and hamiltonian dynamics,

[22] F. Lalonde, D. McDuff, : The geometry of symplectic energy, Ann. of Math. 141(1995), 711-727.

[23] S. M"uller, : The group of Hamiltonian homeomorphisms in the $L^\infty$-norm, T. Korean Math. Soc. 45(2008), 1769-1784.

[24] S. M"uller, : Approximation of volume-preserving homeomorphisms by volume-preserving diffeomorphisms Preprint (2010).

[25] S. M"uller and P. Spaeth, : Topological contact dynamics II, Arxiv, 21 May 2012 (1203 – 4655).

[26] J. R. Munkres, Obstructions to the Smoothing of piecewise-differentiable Homeomorphisms, Bull. Am. Math. Soc. 65 (1959), 332–334.

[27] Y-G. Oh : The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows, submitted to the proceedings of the 2007 AMS-IMS-SIAM Summer Research Conference, Snowbird, Utah (2007).

[28] Y-G. Oh and S. M"uller, : The group of Hamiltonian homeomorphisms and $C^0$-symplectic topology, J. Symp. Geometry 5(2007) 167 – 225.

[29] K. Ono, : Floer-Novikov cohomology and the flux conjecture, Geom. Funct. Anal. 16(2006) no 5 981 – 1020.

[30] L. Polterovich, : The Geometry of the Group of Symplectic Diffeomorphism, Lecture in Mathematics ETH Zürich, Birkhäuser Verlag, Basel-Boston (2001).

[31] S. Tchuiaga, : Some Structures of the Group of Strong Symplectic Homeomorphisms, Global Journal of Advanced Research on Classical and Modern Geometry. Vol.2, Issue 1, pp.

[32] S. Tchuiaga, : On symplectic dynamics, preprint (2013)

[33] C. Viterbo, : Symplectic topology as the geometry of generating functions, Math.Annalen 292(1992) 687 – 710.
[34] C. Viterbo, : On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonian flows, Internat. Math. Res. Notices, vol (2006), Article ID 34028, 9 pages. Erratum, ibid Vol 2006 article ID 38784, 4 pages.

[35] F. Warner, : Foundation of differentiable manifolds and Lie groups, Scott., Foresman and Co., London, (1971).

[36] A. Weinstein, : Symplectic manifolds and their lagrangian submanifolds, Advance in Maths. 6 (1971) 329 – 345.

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