Prequantum Classical Statistical Field Theory: Simulation of Probabilities of Photon Detection with the Aid of Classical Brownian Motion

Andrei Khrennikov
International Center for Mathematical Modeling in Physics, Engineering, Economics, and Cognitive Science
Linnaeus University, S-35195, Växjö-Kalmar, Sweden
Artem Bazhanov
Moscow Institute of Electronic Technology
124498, Zelenograd, Moscow, Russia Federation

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Abstract

In this paper we present results of numerical simulation based on Prequantum Classical Statistical Field Theory (PCSFT), a model with hidden variables of the field-type reproducing probabilistic predictions of quantum mechanics (QM). PCSFT is combined with measurement theory based on detectors of the threshold type. The latter describes discrete events corresponding to the continuous fields model, PCSFT. Numerical modeling demonstrated that the classical Brownian motion (the Wiener process valued in complex Hilbert space) producing clicks when approaching the detection threshold gives probabilities of detection predicted by the formalism QM (as well as PCSFT). This numerical result is important, since the transition from PCSFT to the threshold detection has a complex mathematical structure (in the framework of classical random processes) and it was modeled only approximately. We also perform numerical simulation for the PCSFT-value of the coefficient of second order coherence. Our result matches well with the prediction of quantum theory. Thus, opposite to semiclassical theory, PCSFT cannot be rejected as a consequence of measurements
of $g^{(2)}(0)$. Finally, we analyze the output of the recent experiment \cite{19} performed in NIST questioning the validity of some predictions of PCSFT.

keywords: Prequantum classical field theory; foundations of quantum mechanics; numerical simulation of discrete events; Brownian motion in complex Hilbert space; threshold detectors; coefficient of second order coherence

1 Introduction

At the very beginning Einstein’s idea that the electromagnetic field can be quantized even in vacuum, i.e., that its quanta play the fundamental role not only in the process of the energy exchange with matter\cite{1}, was not commonly accepted. In particular, Planck strongly opposed it. At the beginning (until 1920th) even Bohr was not happy with the invention of light quanta. In particular, the Bohr-Kramers-Slater theory\cite{3} was an attempt to describe the interaction of matter and electromagnetic radiation without using the notion of photon. However, very soon the majority in the quantum community accepted “photon’s existence”. Nevertheless, two of “fathers of QM”, namely Lande\cite{4,5} (in particular, this name is associated with Lande $g$-factor and the first explanation for the anomalous Zeeman effect) and Lamb\cite{6} (e.g., Lamb shift), see also Lamb and Scully\cite{7}, did not accept quantization of the electromagnetic field, they were supporters of the semi-classical model of QM: the matter is quantized, but the radiation not; the quantum-like features of radiation are exhibited only in the process of interaction with quantized matter. Here discrete events which by the orthodox interpretation of QM are identified with photons are just clicks of detectors. We also remark that the idea of photon is completely foreign to stochastic electrodynamics, see, e.g., De la Pena and Cetto, Nieuwenhuizen\cite{8,9,10}.

Recently a new approach to classical field modeling of quantum phenomena was presented\cite{11,23} in the framework of so-called Prequantum Classical Statistical Field Theory (PCSFT), a model with hidden variables of the

\footnote{In 1910 Einstein wrote: “What we understand by the theory of “light quanta” may be formulated in the following fashion: a radiation of frequency $\nu$ can be emitted or absorbed only in a well defined quantum of magnitude $h\nu$. The theoreticians have not yet even come to an agreement in regard to the following question: Can the light quanta be accounted for entirely by a characteristic of the emitting or absorbing substance, or should the electromagnetic radiation itself be assigned, besides a wave structure, such that the energy of the radiation itself is already divided in definite quanta? I believe that I have proven that this latter view should be adopted,”\cite{2}, p. 207.}
field-type reproducing probabilistic predictions of quantum mechanics. The main distinguishing feature of PCSFT is that this is a model of the purely wave type, i.e., there are no particles. PCSFT can be considered as a come-back to the early Schrödinger views: attempts to identify the wave function with a classical physical field. Later Schrödinger gave up and accepted the probabilistic interpretation of the wave function proposed by Born.

For Schrödinger, the main problem was the impossibility of the physical field interpretation of the wave function of a compound system. He pointed out that the wave function of a single electron is defined on physical space which mathematically represented as $\mathbb{R}^3$. However, the wave function of a pair of electrons is defined on the configuration space $\mathbb{R}^6$ and cannot be considered as a physical field. Pauli formulated this problem as the dilemma: *Either physical fields on unphysical spaces or virtual (probabilistic) fields on physical space*. This interpretation problem for the wave function was solved in PCSFT. Here the wave function is not directly identified with a physical field, but can be used to describe such a field mathematically. The difference in the interpretations of the wave function in QM and PCSFT can be shortly described as follows: in QM the wave function directly generates probabilities (therefore it is so convenient to consider it as a field of probabilities) and in PCSFT it also generates probabilities, but indirectly, through generation of a classical random field. The correspondence between quantum states and, so to say, subquantum random fields is based on the relation (1).

PCSFT was completed with measurement theory based on detectors of the threshold type [20]–[23]. The latter describes *discrete events corresponding to the continuous fields model*, PCSFT. Numerical modeling presented in this paper demonstrated that the classical Brownian motion (the Wiener process valued in complex Hilbert space) producing clicks when approaching the detection threshold gives probabilities of detection predicted by the formalism QM (as well as PCSFT). This numerical result is important, since the transition from PCSFT to the threshold detection has a complex mathematical structure (in the framework of classical random processes) and it was modeled only approximately.

The main idea behind PCSFT [11] is that the quantum density operator $\rho$ can be obtained from theory of classical random fields as normalization by the trace of the covariance operator $B$ of a subquantum field $\phi(s)$:

$$\rho = \frac{B}{\text{Tr}B}. \quad (1)$$

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2So, operationally it is easier to apply the formalism of QM than the formalism of PCSFT. In the latter determination of probabilities contains additional steps which are mathematically nontrivial. However, the formal operational interpretation of QM induces numerous mysteries and exoticisms which are absent in PCSFT.
This field can be considered as classical field representation of a quantum system. Thus the basic notion of QM, the notion of the quantum state, lost its fundamental value. In PCSFT it is an emergent notion, it is reduced to the well known notion of classical probability theory, density operator.

We remark that in PCSFT all random fields have zero average. Even for such fields, in general the covariance $B$ does not determine the random field $\phi(s)$ uniquely. However, if one restricts the class of subquantum fields to the Gaussian fields, then the correspondence between fields and covariance operators is one-to-one. However, even for Gaussian subquantum fields the correspondence \[1\] is not one-to-one, since it involves the normalization constant $\sigma^2 = \text{Tr}B$. Its probabilistic meaning is the dispersion of the subquantum random field; its physical meaning is the power of the field. (Thus in PCSFT it is possible to define the power of a “photon”.)

The next step \[20\]–\[23\] was creation of measurement theory corresponding to PCSFT and describing transition from continuous subquantum fields to discrete events, “clicks of detectors.” We proceed under the assumption that all detections are detections of the threshold type. In our model a detector is an operational entity $D$ such that a click is produced when the energy of the random field interacting with $D$ approaches the detection threshold $E_d$. It is easy to present physical models of classical devices interacting with the electromagnetic field in this way. We can speculate that even quantum detectors work in this way. We proceed by considering a black box $D$ which produces a “click” when the energy level of the random field $\phi(s)$ becomes larger than the detection threshold $E_d$. It was shown \[20\]–\[23\] that (surprisingly) discrete events, clicks, produced by such operational entities can reproduce the quantum probabilities of detection.

We also perform numerical simulation for the PCSFT-value of the coefficient of second order coherence \[24\]. Our result matches well with the prediction of quantum theory \[24\]. Thus, opposite to semiclassical theory, PCSFT cannot be rejected as a consequence of measurements of $g^{(2)}(0)$ \[25\]–\[27\] (also known as experiments confirming the existing of photons). Finally, we analyze the output of the recent experiment \[1\] performed in NIST questioning the validity of some predictions of PCSFT \[21\]–\[23\].

Modeling of discrete events has the justification function for the theoretical model of PCSFT. It also provides a computer model of real processes, cf. De Raedt and Michielsen et al \[28\]–\[31\] (in these papers extended modeling of quantum phenomena in terms of discrete events was performed, from the double slit experiment to Bell’s inequality).

In this paper we restrict considerations to classical modeling of “finite-dimensional QM”, i.e., the state space of subquantum fields is finite dimensional. Therefore we shall call such random fields (in accordance with the
2 Clicks as the results of interaction of threshold detectors with classical stochastic processes

In this section we briefly repeat the detection scheme presented in [20]–[23] and derivation of detection probabilities corresponding to this scheme as well as their matching with probabilities given by QM, under the assumption that the correspondence between PCSFT and QM is based on (1). We repeat that our mathematical considerations are done at the “physical level of rigorousness”, see Remark 1.

2.1 Single channel detection scheme

We consider a threshold type detector $D$ with the threshold $E_d$. It interacts with a stochastic process $\phi(s; \omega)$, where $s$ is time and $\omega$ is a chance parameter describing randomness. For a moment, we consider a $C$-valued stochastic process, a complex stochastic process, signal.

The energy of the signal $\phi(s; \omega)$ is given by $E(s; \omega) = |\phi(s; \omega)|^2$ (hence, the stochastic process has the physical dimension $\sim \sqrt{\text{energy}}$). A threshold detector clicks at the first moment of time $\delta = \delta(\omega)$, when the energy of the signal $E$ exceeds the threshold:

$$E(\delta(\omega), \omega) \geq E_d.$$  

(2)

After this event the detector’s state updates and $D$ is ready to interact with the next pulse. (In reality there is also a “dead time” period when $D$ cannot interact with a new commen pulse. In this paper, in particular, in coming numerical simulation we ignore this “technicality”.) It is assumed that there is a source $S$ of pulses and each pulse in the process of interaction with $D$ is transformed into a signal. Realizations of this signal (inside $D$) are represented by the stochastic process $\phi(s; \omega)$ under consideration. Thus $\phi(s; \omega)$ is not the original signal, say $\phi_0(s; \omega)$, emitted by $S$, but the result of its interaction with $D$. It is natural to assume that the basic probabilistic characteristics of $\phi(s; \omega)$ are determined by the corresponding characteristics of $\phi_0(s; \omega)$. In particular, we shall proceed under the assumption:

**In=Out(Prob)** The process emitted by $S$, the input for $D$, and the process $\phi(s; \omega)$ have the same average (in fact, zero average) and that their covariances coincide.
This, although completely classical, picture of detection matches well Bohr’s views on quantum measurements [32]–[35]. He emphasized many times that the whole experimental arrangement has to be taken into account. And we stress the role of the detector changing the form of the signal.

Our argument leading to taking into account the transformer role of $D$ is not straightforward and it was not presented in so much details in the previous publications on PCSFT and its measurement model [21], [23]. Therefore now we spend some time to clarify treatment of $D$ as not only a device for registration of signal’s overcoming the threshold $E_d$, but also as transformer of signal’s stochastic features. Since we assume $\text{In} = \text{Out(Prob)}$ and since, as we shall see, the detection probabilities are expressed solely in terms of the covariance operator, it seems that the model can be simplified by identifying the signal $\phi(s; \omega)$ producing clicks on the basis of the decision rule (2) with the source output $\phi_0(s; \omega)$. However, as we shall see, the temporal-spatial form of the trajectories of stochastic process $\phi(s; \omega)$ do not match with the “pulse form” of signals emitted by $S$. Therefore to match with the pulse, wave-packet, picture of emitted signals and at the same time to match with quantum detection probabilities, we have to endow $D$ not only with the registration feature, see (2), but even with a transformation feature, $\phi_0 \rightarrow \phi$.

In the mathematical model the detection moment is defined as the first hitting time

$$\delta(\omega) = \inf\{s \geq 0 : \mathcal{E}(\delta(\omega), \omega) \geq E_d\}.$$  

(3)

Up to now, we have proceeded with arbitrary stochastic signals. To get detection probabilities matching with the formalism of QM, we have to select stochastic processes of a special class. For a moment, we cannot describe mathematically the class of processes generating quantum detection probabilities for the correspondence (1) between theory of classical stochastic processes and QM, by taking into account $\text{In} = \text{Out(Prob)}$. However, we found a few simple examples of Gaussian processes which produce such a matching [21], [23]. The simplest one is the Brownian motion. Thus, after arriving into a threshold type detector the classical stochastic process $\phi(s; \omega)$ behaves inside this detector as the Brownian motion in the space of (complex) fields.

Thus the process $\phi(s, \omega)$ is the Wiener process: the Gaussian process having zero average, $E\phi(s, \omega) = 0$, and covariance $E\phi(s_1, \omega)\phi(s_2, \omega) = \min(s_1, s_2)\sigma^2$; we can find average of its energy $E\mathcal{E}(s, \omega) = \sigma^2 s$. We find that the coefficient $\sigma^2 = \frac{E\mathcal{E}(s, \omega)}{s}$ has the physical dimension of power.

We are interested in average of the moments of the $\mathcal{E}_d$-threshold detection for the energy of the Brownian motion. Since moments of detection are defined formally as hitting times, we can apply theory of hitting times for
the Wiener process, see e.g. Shyryaev: $\tilde{\delta} \equiv E\delta = \xi_d^2$ or
\begin{equation}
\frac{1}{\delta} = \frac{\sigma^2}{E_d}.
\end{equation}

Hence, during a long period of time $T$ such a detector clicks $N_{\sigma}$-times, where
\begin{equation}
N_{\sigma} \approx T \frac{1}{\delta} = \frac{\sigma^2 T}{E_d}.
\end{equation}

**Remark 1.** This formula determining the number of clicks in the channel by using the average detection time is approximate; in general, this is sufficiently rough approximation. Therefore coming numerical simulation, section 3, plays the important role in justification of the presented threshold detection scheme.

### 2.2 Complex Wiener process as a class of stochastic processes

The notion of the complex Wiener process is more complicated than it was formally presented in the previous section. In fact, each $b = \sigma^2$ determines a class of real Wiener processes valued in $\mathbb{R}^2$ and, hence, a class of stochastic processes valued in $\mathbb{C}$. One may say that the terminology used in the previous section was misleading and it might be better from the very beginning to speak about a class of stochastic processes. However, there is a point in the aforementioned terminology. All processes from the class of stochastic processes determined by a “complex Wiener process” determine the same probability distribution of clicks of detectors. Hence, they can be considered as equivalent from the operational viewpoint. (We remark that they are not equivalent from the viewpoint of theory of stochastic processes.) This operational approach is closer to quantum formalism (if the latter is also interpreted as an operational formalism), see section 2.4.

Now we discuss the multi-process structure of the complex Wiener process. We start with the trivial remark that in general the easiest way to construct a complex valued process $\xi(t)$ is to use the combination of two real processes, $\xi_1(s), \xi_2(s)$, namely, set $\xi(s) = \xi_1(s) + i\xi_2(s)$. If the processes $\xi_1(s), \xi_2(s)$ are independent, then the dispersion of $\xi(t)$ equals to the sum of dispersions of $\xi_1(t)$ and $\xi_2(t)$.

Now consider an arbitrary factorization of $b = \sigma^2$ in the form $b = c\bar{c}$, where $c \in \mathbb{C}, c = k_1 + ik_2, k_j \in \mathbb{R}$. Consider now the standard $\mathbb{R}^2$-valued Wiener process $w(s) = (w_1(s), w_2(s))$, where $w_1(s), w_2(s)$ are...
independent one dimensional Wiener processes: \( Ew_j^2(s) = 1, j = 1, 2 \), and \( Ew_1(s)w_2(s) = 0 \). We now scale it by setting \( w'(s) = \frac{1}{\sqrt{2}}w(s) \), i.e., now \( E(w'_j(s))^2 = 1/2, j = 1, 2 \). This scaling is needed, since a complex process is combined of two real components and its total dispersion is the sum of components’ dispersions. We set \( W(s) = w'_1(s) + iw'_2(s) \). We call this stochastic process the standard complex Wiener process. (This is the concrete complex valued stochastic process determined by a pair of real Wiener processes.) Its dispersion \( E|W(s)|^2 = E(w'_1(s))^2 + (w'_2(s))^2 = 1 \). Now to obtain a \( b \)-process, we simply scale \( W(s) : \phi(s) = cW(s) \). Then \( E|\phi(s)|^2 = b \). It is interesting to consider the real representation of this process:

\[
\phi(s) = [k_1w'_1(s) - k_2w'_2(s)] + i[k_2w'_1(s) + k_1w'_2(s)].
\]  

(6)

In fact, the above consideration represented the algorithm of construction of stochastic processes which we used for numerical simulation. However, in the applications to the quantum measurement problem we need vector valued processes, see section 2.6 for analogous considerations in the multi-dimensional case.

Thus the object which we call complex Wiener process is a class of real valued two dimensional Wiener processes. They are not arbitrary; they have very special structure, see (6). This structure is a consequence of symplectic invariance of real processes representing complex processes with the aid of scaling, see section 2.6 and the paper [11]. We shall discuss possible physical interpretation of this non-uniqueness in section 2.6.

### 2.3 Multi-channel detection scheme

Consider now a stochastic process \( \phi(s; \omega) \) valued in the \( m \)-dimensional complex Hilbert space \( H \).

Let \( (e_j) \) be an orthonormal basis in \( H \). The vector-valued stochastic process \( \phi(s, \omega) \) can be expanded with respect to this basis

\[
\phi(s; \omega) = \sum_j \phi_j(s; \omega)e_j,
\]  

(7)

where \( \phi_j(s; \omega) = \langle \phi(s; \omega)|e_j \rangle \). In accordance with our detection scheme, section 2.1, the process \( \phi(s; \omega) \) is mathematical representation of the vector of physical signals constructed in the following way. The source emits the random signal \( \phi_0(s; \omega) \in H \). (For example, in the case of consideration of only polarization degrees of freedom \( H \) is the two dimensional Hilbert space; here we neglect the spatial degrees of freedom.) This process is split, e.g.,
by polarization beam splitter into components propagating in disjoint channels, \( j = 1, 2, ..., m \). Here \( \phi_0(s; \omega) = \sum_j \phi_{0j}(s, \omega) \epsilon_j \). We now assume that there is a threshold detector in each channel, \( D_1, ..., D_m \). Then each component \( \phi_{0j}(s; \omega) \) through interaction with \( D_j \) is transformed into a signal mathematically represented by the stochastic process \( \phi_j(s; \omega) \). The vector with these components can be represented as the \( H \)-valued stochastic process \( \phi(s; \omega) \), see [7]. And we proceed under the vector-form of the assumption \textbf{In}=\textbf{Out}(\textbf{Prob}), section 2.1.

We again remark that the scheme world be essentially simpler and it would lead to the same result, if we omit the detector-transformation step and operate directly with process emitted by the source as belonging to the class generating quantum detection probabilities (e.g., the Wiener processes), cf. with discussion in section 2.4. However, we prefer the above more complex scheme, since it is convenient to use for description of propagation in space-time (which is, in fact, absent in this paper, but possible []) one type of stochastic processes and in detectors as another type.

We also assume that all detectors have the same threshold \( E_d > 0 \).

To each detector we apply the above detection scheme. The crucial point is that the detectors work totally independently from each other. An event in one detector, a click, has no influence on the physical processes in other detectors. If in some detector \( D_j \) a click is generated, then for this detector the experimental trial is finished and a new trial is started. What is about other detectors? As was pointed out, they work without “paying any attention” to such an event in \( D_j \). Another detector \( D_i, i \neq j \), continues its interaction with signal’s component \( \phi_i(s; \omega) \) until signal’s energy will overcome its detection threshold. Thus events in detectors, clicks, are not sharply coupled events in the source, namely, emission of pulses. It can happen that the clicks are produced simultaneously in a few detectors. In this case we register all them. In numerical simulation we cannot work with the notion “simultaneously”; there will be always assumed the presence of a nontrivial time window used to identify clicks as “simultaneously occurring.”

### 2.4 Wiener process valued in complex Hilbert space

Suppose now that \( \phi(s, \omega) \) is the Wiener process valued in \( H \). This process is determined by the covariance operator \( \hat{B} : H \rightarrow H \). Any covariance operator is Hermitian, positive, and trace-class and vice versa. The complex Wiener process is characterized by Hermitian covariance operator. We have, for \( y \in H, E\langle y, \phi(s, \omega) \rangle = 0 \), and, for \( y_j \in H, j = 1, 2, E\langle y_1, \phi(s_1, \omega) \rangle\langle \phi(s_2, \omega), y_2 \rangle = \min(s_1, s_2)\langle By_1, y_2 \rangle \). The latter is the covariance function of the stochastic process; in the operator form: \( B(s_1, s_2) = \min(s_1, s_2)B \). We note that the
dispersion of the $H$-valued Wiener process (at the instant of time $s$) is given by $\Sigma^2(s) = E\|\phi(s, \omega)\|^2 = s\text{TrB}$. The quantity $\mathcal{E}(s, \omega) = \|\phi(s, \omega)\|^2$ is the total energy of the Brownian motion signal at the instant of time $s$. Hence, the quantity

$$\Sigma^2 = \frac{\Sigma^2(s)}{s} = \text{TrB} \tag{8}$$

is the average power of this random signal.

We also remark that by normalization of the covariance function for the fixed $s$ by the dispersion we obtain the operator,

$$\rho = B(s, s)/\Sigma^2(s) = B/\text{TrB}, \tag{9}$$

which formally has all properties of the density operator used in quantum theory to represent quantum states. Its matrix elements have the form $\rho_{ij} = b_{ij}/\Sigma^2$. These are dimensionless quantities.

The relation (9) is basic for PCSFT:

Each classical random process generates a quantum state (in general mixed) which is given by the normalized covariance operator of the process.

Consider components $\phi_j(s, \omega)$ of the vector valued signal $\phi(s, \omega)$. Then

$$E\phi_i(s, \omega)\phi_j(s, \omega) = \min(s_1, s_2)\langle Be_i, e_j \rangle = b_{ij}.$$  

In particular, $\sigma^2_j(s) \equiv E\mathcal{E}_j(s, \omega) \equiv E|\phi_j(s, \omega)|^2 = sb_{jj}$. This is the average energy of the $j$th component at the instant of time $s$. We also consider the average powers of components $\sigma^2_j = \frac{\sigma^2_j(s)}{s} = b_{jj}$. We remark that the average power of the total signal is equal to the sum of the average powers of its components. $\Sigma^2 = \sum_j \sigma^2_j$.

### 2.5 Detection probabilities, the multi-channel case

Consider now a run of experiment of the duration $T$. The average number of clicks for the $j$th detector can be approximately expressed as

$$N_j \equiv N_{\sigma_j} \approx \frac{\sigma^2_j T}{\mathcal{E}_d}. \tag{10}$$

The total number of clicks is (again approximately) given by $N = \sum_j N_j \approx \frac{\Sigma^2 T}{\mathcal{E}_d}$. Hence, for the detector $D_j$, the probability of detection can be expressed as

$$P_j \approx \frac{N_j}{N} \approx \frac{\sigma^2_j}{\Sigma^2} = \rho_{jj}, \tag{11}$$
however, see Remark 1.

This is, in fact, the Born’s rule for the quantum state $\rho$ and the projection operator $\hat{C}_j = |e_j\rangle\langle e_j|$ on the vector $e_j$. For the detector $D_j$, the probability of detection can be expressed as

$$P_j = \text{Tr} \rho \hat{C}_j.$$  \hspace{1cm} (12)

### 2.6 Wiener process in complex Hilbert space as a class of vector-valued stochastic processes

Now we briefly repeat considerations of section 2.2 in the multi-dimensional case. Let $B$ be a positively defined Hermitian matrix. Consider one of its representations in the form:

$$B = CC^*,$$  \hspace{1cm} (13)

where $C$ is an arbitrary complex matrix. This $C$ can be represented as $C = K_1 + iK_2$, where $K_j, j = 1, 2$, are real matrices. In the same way we represent $B$ as $B = B_1 + iB_2$. Thus the equation (14) can be written as the system of equations:

$$B_1 = K_1K_2^* + K_2K_1^*, \quad B_1 = K_2K_1^* - K_1K_2^.$$  \hspace{1cm} (14)

Consider now the standard complex $m$ dimensional Wiener process $\mathbf{W}(s) = (W_1(s), ..., W_m(s))$, each its coordinate is one dimensional standard complex Wiener process, see section 2.2 and the they are independent. This process has the unit covariation operator:

$$E[\langle \mathbf{W}(s;\omega) | u \rangle \langle \overline{\mathbf{W}(s;\omega)} | v \rangle] = \langle u | v \rangle,$$  \hspace{1cm} (15)

where $u, v$ are two arbitrary vectors.

To obtain the $B$-process, we simply scale the standard complex Wiener process $\mathbf{W}(s)$:

$$\phi^C(s) = C\mathbf{W}(s).$$  \hspace{1cm} (16)

(In section 3 we shall use this representation to generate a complex Wiener $B$-process; one of its versions.) It is easy to check that the process $\phi^C(s)$ really has the covariance operator $B$:

$$E[\phi^C(s;\omega)\overline{\phi^C(s;\omega)}] = E[\langle \phi^C(s;\omega) | e_j \rangle \langle \overline{\phi^C(s;\omega)} | e_i \rangle] =$$

$$E[\langle \mathbf{W}(s;\omega) | C^* e_j \rangle \langle \overline{\mathbf{W}(s;\omega)} | C^* e_i \rangle] = \langle C^* e_j | C^* e_i \rangle = \langle B e_j | e_i \rangle = b_{ji},$$

see (15).
Consider now decomposition of $\phi^C(s)$ into real Wiener processes. For this, we first decompose the standard complex Wiener process $W(s) = w_1'(s) + i w'_2(s)$, where $w_j'(s), j = 1, 2$, are $1/\sqrt{2}$-scalings of two standard (independent) real Wiener processes in the $m$-dimensional space. Then

$$\phi^C(s) = [K_1 w_1'(s) - K_2 w_2'(s)] + i [K_2 w_1'(s) + K_1 w_2'(s)].$$

(17)

We remark that all processes of the form $[14]$ generate the same average of energy in each of detectors:

$$E[\phi^C(s; \omega)]^2 = b_{ii}.$$ 

(18)

And our model predicts that the statistics of clicks in detectors depends only on the (relative) average energy delivered to each detector (for a random signal in a detector which can be represented by a complex Wiener process). Therefore it is not surprising that a variety of processes $[16]$ with $[14]$ produce the same probability distribution of clicks in the detectors. One cannot distinguish these processes by using the detectors. From the PCSFT-viewpoint, the quantum formalism is an operational formalism. Here a density matrix encodes a variety of subquantum processes. Can one hope to design experiments to approach such processes? This is an open problem.

3 Numerical modeling for detection probabilities

As was pointed out in Remark 1, in our mathematical modeling of detection probabilities for the classical Wiener process interacting with the threshold detector we used rough approximations and through these approximate calculations we approached matching with the quantum formula for probabilities of the results of measurement, Born’s rule, see $[12]$. We state again that the prequantum→quantum correspondence by itself is exact, see $[1]$. However, we were not able to solve exactly the problem of theory of classical stochastic processes, the problem of expression of probabilities for the threshold detection through elements of the covariance matrix of the process. Now we would like to complete the approximate theoretical picture by numerical modeling. We produced a plenty of graphical data corresponding to such simulation.

We start with the data corresponding to dichotomous observables, expressed in terms of the two channel model. Consider the two dimensional complex Hilbert space. There is given a quantum observables, say $\hat{A}$, let $(e_1, e_2)$ be the basis consisting of its eigenvectors. Consider the Wiener process in this space with the covariance operator $B$, the subquantum process
induced as the result of interaction of “quantum systems” with detectors measuring $\hat{A}$. In the eigen-basis, the covariance operator is represented by the corresponding matrix: $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, where $b_{11}, b_{22}$ are real numbers and $\bar{b}_{12} = b_{21}$. To consider the general situation, we do not assume that $\text{Tr}B = 1$, i.e., the prequantum→ quantum map [1] is not identity, but contains nontrivial normalization by $b_{11} + b_{22}$.

We now present an example of numerical simulation. We use the scheme which was presented in section 2.6. Here with the covariance matrix has the form

$$B = \begin{pmatrix} 10 & 5 + 2i \\ 5 - 2i & 9 \end{pmatrix} = SS^*, \quad (19)$$

and we selected

$$S = \begin{pmatrix} 1 & 3i \\ 2 - 2i & i \end{pmatrix}, \quad (20)$$

Here $P_1 = 10/(10 + 9) = 10/19 \approx 0.526, P_2 = 9/(10 + 9) = 9/19 \approx 0.474$. The relative frequencies $\nu_1 = N_1/(N_1 + N_2), \nu_2 = N_2/(N_1 + N_2)$, where $N_j; j = 1, 2$, are the numbers of clicks in corresponding detectors. The results of simulation showed that the relative frequencies approach the predicted probabilities, although they fluctuate at the initial segment of series of trials.

We also present the results of the multi-dimensional simulation. We consider the covariance matrix

$$B = \begin{pmatrix} 14, & 4 - 2i & -2 - 5i & 7 - 4i \\ 4 + 2i & 12 & -7 - i & 2 \\ -2 + 5i & -7 + i & 8 & 1 + 4i \\ 7 + 4i & 2 & 1 - 4i & 6 \end{pmatrix} = SS^*, \quad (21)$$

and we selected

$$S = \begin{pmatrix} 2 - 2i & i & 1 & 2 \\ 1 & 3i & 1 & -1 \\ i & -2i & i & 1 + i \\ 2 & 0 & 1 & 1 \end{pmatrix}, \quad (22)$$

Here our model predicts the probabilities of detection: $P_1 = 14/40 = 0.35, P_2 = 12/40 = 0.3, P_3 = 8/40 = 0.2, P_4 = 6/40 = 0.15$. The results of simulation showed that the relative frequencies approach the predicted probabilities, although they fluctuate at the initial segment of series of trials.
4 Numerical modeling for the coefficient of second order coherence

We remark that quantum theory predicts [24] that, for single photon states, the coefficient of second order coherence $g^{(2)}(0) = 0$. At the same time for semiclassical models $g^{(2)}(0) \geq 1$. Therefore measurements of $g^{(2)}(0)$ played the crucial role in distinguishing the quantum and semiclassical models of micro-phenomena [24]-[27]. Such experiments were also crucial to confirm experimentally the “existence of photon” [25], [26]. Therefore it is important to find the magnitude of the coefficient of second order coherence in the framework of PCSFT (completed with the threshold model of measurement).

It is difficult to find $g^{(2)}(0)$ analytically by using theory of classical stochastic processes; in [21]-[23] there was found its estimate from above showing that at least for the special inter-relation between the detection threshold and signal’s energy this coefficient is less than 1. In this paper we use numerical simulation for this purpose. The coefficient of second order coherence is defined as

$$g^{(2)}(0) = \frac{P_{12}}{P_1 P_2},$$

(23)

where $P_j$ are the probabilities of detection in the channels $j = 1, 2$ and $P_{12}$ is the probability of the coincidence of clicks in the two detectors. In previous sections, to find $P_j$, we used the normalization by the sum of clicks in the two channels. This approach did not take into account the coincidence detections. Now we consider the normalization which is proper for calculation of $g^{(2)}(0)$ (and more generally $g^{(2)}(s), s \geq 0$). By considering the sum $N = N_1 + N_2$ we take the number of coincidence clicks $N_{12}$ twice. Therefore, for the proper normalization, we have to use $N = N_1 + N_2 - N_{12}$. Hence, in this section we set

$$P_j = \frac{N_j}{N_1 + N_2 - N_{12}}, P_{12} = \frac{N_{12}}{N_1 + N_2 - N_{12}},$$

(24)

Numerical simulations shows that for our model the quantity $N_{12}/(N_1+N_2)$ is small. Therefore the values of probabilities calculated in the previous sections change only slightly, so the graphs at Fig. ???, Fig. ?? are indistinguishable from the graphs corresponding to the frequency-probabilities given by (25).

We obtain the following expression for $g^{(2)}(0)$:

$$g^{(2)}(0) = \frac{N_{12}(N_1 + N_2 - N_{12})}{N_1 N_2}.$$  

(25)

This formula can be applied for numerical simulation and experiment. However, there is one pitfall which is typically not taken into account in the
discussions on calculation of \( g^{(2)}(0) \) (see, however, [27]). Both in numerical simulation and experiment we cannot proceed with continuous time and the corresponding notion of coincidence “at the same instance of time.” A coincidence time window \( \tau \) has to be used. Therefore the quantities depend on this time window: \( N_1 = N_1(\tau), N_2 = N_2(\tau), N_{12} = N_{12}(\tau) \) and, hence, \( g^{(2)}(0; \tau) \). The graph for \( g^{(2)}(0; \tau) \) as the function of the time window \( \tau \) is presented at Fig. 1. We see that for small time widows the coefficient of second order coherence is strictly less than 1 and it goes to zero for \( \tau \to 0 \). This is in the complete agreement with the prediction of QM. Hence, opposite to semiclassical optics, the PCSFT cannot be rejected as the result of experimental measuring of the coefficient of second order coherence.

![Graph](image)

**Figure 1:** The graph of \( g^{(2)} \) depending on the coincidence time window \( \tau \) (for the \( B \)-Wiener process with the covariance matrix given by (19) and the threshold \( E_d = \text{Tr}B/20 \)).

Besides the graph at Fig. 1 it is also useful to present the numerical values of \( g^{(2)}(0; \tau) \) for various values of the time window \( \tau = 1, 2, \ldots, 9, 10, 20, \ldots, 50 \):

\[
g^{(2)} = 0, 0.0112465; 0, 0.0216078; 0, 0.0314473; 0, 0.039609; 0, 0.0484313; 0, 0.0574435;
0, 0.0676744; 0, 0.0779637; 0, 0.0860406; 0, 0.0972437; 0, 0.191344; 0, 0.271691; 0, 0.350341; 0, 0.428744.
\]

In classical and quantum optics not only \( g^{(2)}(0) \), but also \( g^{(2)}(s), s \geq 0 \), plays an important role [24]. In fact, our function \( g^{(2)}(0; \tau), \tau \geq 0 \), can be considered as a good approximation for \( g^{(2)}(\tau), \tau \geq 0 \). And we see again that the graph at Fig. ?? matches well with the corresponding graph from quantum theory [24].
Finally, we remark that recently a group of experimenters from NIST performed the experimental test [1] challenging the predictions of PCSFT about the dependence of the coefficient $g^{(2)}(0)$ on the inter-relation of the detection threshold and the signal energy [21–23]. They did not find dependence of the form predicted in [21–23]. There are a few objections to consider this experiment as a negative test for PCSFT (with the measurement model based on the threshold detection).

The first remark is that in [21–23] we were able only to get an estimate from above and the upper bound depends on the aforementioned parameters. This, of course, does not imply that $g^{(2)}(0)$ by itself depends on them in this way; it might be that this is just a bad estimate. Unfortunately, in the present paper we were not able to model dependence of $g^{(2)}(0)$ on the detection threshold and energy to approach the regime considered in [21–23], i.e., for example, very high levels of the detection threshold compare with $TrB$. Here the Brownian motion exceeds the detection threshold not so often and the available computational resources were not sufficient to perform such a numerical simulation (one has to use really powerful computer to collect good statistics).

Another objection to the interpretation of the results of the NIST-experiment [1] as confronting with PCSFT is that the experimenters did not use the real single photon source; they worked with so-called heralded photons. The latter technique is standard in the recent experiments on the calculation of the coefficient of second order coherence. We remark that even the original experiment of Grangier [25], [26] was done with heralded photons. Typically the coefficient calculated with the aid of heralded photons is not distinguished from the “genuine coefficient of second order coherence”, i.e., calculated with the aid of “really single photon sources”. However, here one has to careful, since these are, in fact, two different quantities, although both reflect the same quantum feature of light.

5 Concluding remarks and further studies

In this paper we confirmed with the aid of numerical simulation the theoretical predictions of the threshold detection model for PCSFT which were obtained in [21–23]. As was pointed out, in these works calculations were done at the physical level of rigorousness, quantum mechanical predictions were approached only asymptotically and without estimation of the degree of approximation. Therefore numerical simulation presented here is very important for justification of the predictions of [21–23].
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