Ordinary Differential Equations through Dimensional Analysis.

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Abstract

In this paper we show how using D.A. it is found a simple change of variables (c.v.) that brings us to obtain differential equations simpler than the original one. In a pedagogical way (at least we try to do that) and in order to make see that each c.v. corresponds to an invariant solution (induced by a symmetry) or a particular solution, we compare (with all the tedious details, i.e. calculations) the proposed method with the Lie method. The method is checked even in odes that do not admit symmetries.

Keywords: ODEs, D.A., Symmetries.

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1 Introduction

D.A. has usually been employed in different areas (fields) such as engineering problems, fluid mechanics etc... and these problems are always described by partial differential equations (pde) (see [1]-[10]). This method (“tactic”) helps us to reduce the number of quantities that appear into an equation and to obtain ordinary differential equations (ode). We would like to point out (emphasize) that this tool is more effective if one practices the spatial discrimination, such tactic allows us to obtain better results than with the standard application of D.A. (see [11]). Knowing that D.A. works well in pde we would like to extend this method to the study of ode (the first order in this case) in a systematic way. There are in the literature previous work in this direction for ode of first order (see [12]-[13]).

The method of the Lie groups has showed as a very useful tool in order to solve nonlinear equations (nl-ode) as well as pde (see [14]-[27]). Nevertheless, when one is studying ode of first order its application results very complicated (very tedious) if one has not a computer algebra package since if one decides to look for the possible symmetries of an ode with pencil and paper this task may be turned very exhausting. However, if we know that an ode admits a concrete symmetry, then it is a trivial issue to

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find new variables which allows us to rewrite the ode in quadratures or as we will see in this paper to obtain an ode with separating variables.

Our purpose in this work is to explain, through examples, how D.A. works in order to find these changes of variables (c.v.) in a trivial way, i.e. without the knowledge of the symmetries of the ode under study. The idea is as follows. When we are studying an ode form the dimensional point of view, we must require that such ode verifies the principle of dimensional homogeneity (pdh) i.e. that each term within the equation have the same dimensions, for example, speed, or energy density. To clarify this concept we consider the following ode, \( y' = \frac{y}{x} + x \), where each term must have dimensions of \( y' \) i.e. [\( y' \)] = \( yx^{-1} \), where [] stands for dimensional equation of the quantity. As we can see, this ode does not verify the pdh, since the term \( x \) has dimensions of \( x \), i.e. \([x] = x\). In order to do that this equation verifies the pdh we need to introduce dimensional constants, in such a way that after rewrite the ode with these constants the ode now verifies such principle i.e. in this example and as we can see easily if we consider only one constant \( a \), such that \([a] = yx^{-2}\), we make that the ode \( y' = \frac{y}{x} + ax \), verifies the pdh. We would like to emphasize that this situation does not appear (arise) when one is studying physical or engineering problems since (as it is supposed) that such problems (equations) verify the pdh. We would like to emphasize that this situation does not appear (arise) when one is studying physical or engineering problems since (as it is supposed) that such problems (equations) verify the pdh and we do not need to introduce new dimensional constants, that must have physical meaning, for example, the viscosity coefficient, etc.

Precisely this dimensional constant suggests us the c.v. \((x, y(x)) \mapsto (t, u(t))\) where \((t = x, u = yx^{-2})\) in such a way that rewriting the original ode in these new variables it is obtained a new ode with separating variables: \( u + u't = 1 \). The reason is the following. We know from the Lie group theory that if it is known a symmetry of an ode then this symmetry bring us through a c.v. to obtain a simpler ode (a quadrature or a ode with separating variables) and therefore the solution is found in a closed form. To find this c.v. it is used the invariants that generate each symmetry, i.e. the first principle is that it is useful to pass to new coordinates such that one of the coordinate functions is an invariant of the group. After such transformation it often (but not always) happens that the variables separate and the equation can be solved in closed form. We must stress that taking the new dependent variables to be an invariant of the group does not guarantee the separation of variables. The choice of the independent variable is also very important.

One must note that D.A. gives us this invariant (or at least a particular solution). It is observed in our example that the solution \( y = x^2 \), (suggested by D.A.) is a particular solution for this ode, but if we study this ode form the Lie theory point of view it is obtained that such ode has the scaling symmetry \( X = x\partial_x + 2y\partial_y \) and therefore the solution \( y = x^2 \), is furthermore an invariant solution (generated by \( X \)), for this reason such c.v. works well. As we will show in the paper, the c.v. that D.A. induces is the same than the generated one by the symmetries of the ode. In order to make see this fact, we will solve each example by D.A. as well as by the Lie group technique (abusing of the trivial calculations), calculating the symmetries and their corresponding invariants and c.v..

However D.A. has a little (or rather big) drawback, it only gives us relationship of type power i.e. \( y = x^n \), with \( n \in \mathbb{R} \), and we cannot obtain relationship as \( y = (x + 1)^n \). Nevertheless, and as we will see in the examples, although D.A. does not give us an invariant solution it will be sufficient that it provides us a particular solution in order to obtain a c.v. which brings us to a simpler ode than the original one. This task will be very useful in the case of studying Riccati and Abel odes. Furthermore, this “tactic” is valid even in the case in which the ode has no symmetries but unfortunately one always founds examples in which any tactic does not work.

The paper is organized as follows: In section 2 we describe (with all the tedious and superfluous calculations), through two examples, how the proposed method works by comparing it with the Lie method in order to show which are the symmetries and their corresponding invariants. In section 3 we will show several examples beginning with a Bernoulli ode and following with two Riccati ode and three Abel ode (see [28]-[32]). We think that these kind of ode are the most difficult to solve and therefore
any simplification is welcome. In section 4 we show two Abel odes that do not admit symmetries (see [33]-[37]). In the first of them we use the D.A. to obtain a particular solution that allows us to obtain a Bernoulli ode but in the second one we are not able to find any particular solution and therefore a c.v. which brings us to obtain a simpler ode. We end with some conclusion as well as pointing out some of the limitations of the proposed tactic.

We would like to emphasize the pedagogical character of this paper (at least we try) for this reason we have abused of superfluous calculations and we have omitted some technical details since the supposed audience are mainly engineers and/or physicists but not for mathematicians.

2 The method

In this section we will explain how the D.A. works in order to solve odes through two simple examples. The main idea is to introduce dimensional constants that make the ode under study verify the principle of dimensional homogeneity. These dimensional constants help us to find c.v. which bring us to obtain ode with separating variables. These c.v. obtained through D.A. correspond to invariant solution (or at least to a particular solution) and therefore they are induced by one symmetry. We compare our tactic with the Lie one.

Example 1 Solve the homogeneous equation

\[ (x^2 + y^2) \, dx = 2xydy. \]  

Solution. In order to solve it we will use three different ways, the traditional, the dimensional and the Lie method.

Traditional method. Making the c.v. \( u = y/x \) we have:

\[ u' = \frac{1 - u^2}{2ux} \implies \frac{2udu}{1 - u^2} = \frac{dx}{x} \implies \ln(1 - u^2) = \ln x \implies y^2 = x^2 + x, \]  

Dimensional Analysis. We go next to consider eq. (1), written as follows

\[ y' = \frac{(a^2x^2 + y^2)}{2xy}, \]  

where the dimensional constant \( a \), makes the ode verify the dimensional principle of homogeneity (d.p.h.) if

\[ [a] = \left[ \frac{y}{x} \right] = X^{-1}Y, \]  

where \([\cdot]\) stands for the dimensional equation of the quantity \( \cdot \).

Applying the Pi theorem we obtain the dimensionless variables that help us to simplify the original ode. Therefore taking into account the following dimensional matrix we take

\[
\begin{array}{ccc}
y & x & a \\
x & 0 & 1 \\
y & 1 & 0 & 1 \\
\end{array}
\implies \pi_1 = \frac{ax}{y} \implies y = ax,
\]  

as we will see later this solution is a particular solution of this ode. We would like to emphasize that as we have only needed one constant then the equation is scale-invariant in such a way that the generator of this group is \( X = x\partial_x + y\partial_y \) as it is observed from eq. (5). This fact will be probed studying this equation under the Lie group tactic, see below.
In this way the new variables are \((t, u(t))\):
\[
\begin{bmatrix}
t = x, \\
u(t) = \frac{x}{y}
\end{bmatrix}, \quad \implies \begin{bmatrix}
x = t, \\
y = a \frac{t}{u(t)}
\end{bmatrix},
\]
this change of variables brings us to rewrite eq. (3) as follows:
\[
\frac{u'}{u(1-u^2)} = \frac{1}{2t} \quad \implies \quad \frac{u}{\sqrt{1-u^2}} = \frac{1}{2} \ln t,
\]
and hence
\[
y^2 = a^2 x^2 + C_1 x.
\]
As we can see in this trivial example, the D.A. induces a c.v. which helps us to obtain an ode simpler than the original one.

We can also think in the following way
\[
ay' = b \frac{x}{2y} + a \frac{y}{2x},
\]
where
\[
[a] = x, \quad [b] = y^2 x^{-1},
\]
and hence
\[
\begin{bmatrix}
t = \frac{x}{a}, \\
u(t) = \frac{y^2}{bx}
\end{bmatrix} \implies \begin{bmatrix}
x = at, \\
y = \sqrt{abt(u(t))}
\end{bmatrix},
\]
therefore eq. (9) yields
\[
u' = 1 \implies u = t + C_1,
\]
in this way we obtain the solution
\[
y^2 = \frac{b}{a} x^2 + C_1 x,
\]
once we have obtained the solution, the constants \(a, b\) are making equal to 1, i.e. \(a = b = 1\).

In the same way we may consider the following change of variables
\[
\begin{bmatrix}
t = \frac{x}{a}, \\
u(t) = \frac{y}{\sqrt{bx}}
\end{bmatrix} \implies \begin{bmatrix}
x = at, \\
y = u(t) \sqrt{abt}
\end{bmatrix},
\]
which brings us to the following ode
\[
2u'u = 1 \implies u^2 - t + C_1 = 0,
\]
and therefore we obtain again the solution (13). As we will see below all these c.v. are generated by their corresponding (respective) symmetry.

As we can see, this last change of variables is better than the first and the tactic is the same: to introduce dimensional constants that make the equation dimensional homogeneous.

**Lie Method.** In order to find the symmetry generator of a ode
\[
y' = f(x, y),
\]
we need to solve the following pde
\[
\eta_x + (\eta_y - \xi_x)f - \xi_y f^2 - \xi(x, y)f_x - \eta(x, y)f_y = 0,
\]
where
\[ f_x = \frac{df}{dx}, \quad f_y = \frac{df}{dy}. \] (18)

In this case we have to solve
\[ \eta_x + \frac{1}{2} \left( \frac{\eta_y - \xi_x}{xy} \right) \left( x^2 + y^2 \right) - \frac{1}{4} \frac{\xi_y (x^2 + y^2)^2}{x^2 y^2} - \xi_x \left( \frac{1}{y} - \frac{x^2 + y^2}{2 x^2 y} \right) - \eta_y \left( \frac{1}{x} - \frac{x^2 + y^2}{2 x y^2} \right) = 0, \] (19)

we find that \((X = \xi \partial_x + \eta \partial_y)\):
\[
\begin{align*}
X_1 &= x \frac{\partial}{\partial y}, & X_2 &= \left( -\frac{x^2 + y^2}{2 y} + \frac{y}{2} \right) \partial_y, & X_3 &= x \partial_x + y \partial_y, \\
X_4 &= \partial_x + \frac{y}{2 x} \partial_y, & X_5 &= \left( x^2 + y^2 \right) \partial_x + 2 x y \partial_y,
\end{align*}
\] (20)

observing that \(X_5\) is a trivial symmetry.

Each symmetry induces a change of variable (canonical variables) which are obtained through the following formula
\[ X t = 0, \quad X u = 1. \] (21)

In this way the change of variables that induces the field \(X_4 = \partial_x + \frac{y}{2 x} \partial_y\) is:
\[
\begin{align*}
(t = \frac{y}{\sqrt{x}}, u(t) = x) &\Rightarrow \left( x = u(t), y = t \sqrt{u(t)} \right),
\end{align*}
\] (22)

in such a way that eq. (1) yields:
\[ u' = 2t \Rightarrow u(t) = t^2 + C_1, \] (23)

and therefore
\[ x = \frac{y^2}{x} + C_1, \] (24)

which is the same solution. The c.v. induced by this symmetry is similar to the obtained one in (14).

The c.v. that induces the field \(X_1 = \frac{x}{2 y} \partial_y\) is the following one
\[
\begin{align*}
(t = x, \quad u(t) = \frac{y^2}{x}) &\Rightarrow \left( x = t, \quad y = \sqrt{u(t) t} \right),
\end{align*}
\] (25)

therefore eq.(1) yields
\[ u' = b^2, \] (26)

this c.v. is the same than the obtained one in (11).

For example the c.v. that induces the field \(X_3 = x \partial_x + y \partial_y\) is
\[
\begin{align*}
(t = y \frac{x}{x}, u(t) = \ln x) &\Rightarrow \left( x = e^{u(t)}, y = te^{u(t)} \right),
\end{align*}
\] (27)

which brings us to the following ode :
\[ u' = -\frac{2t}{t^2 - 1} \Rightarrow u(t) = -\ln (-1 + t) - \ln (1 + t) + C_1, \] (28)

now writing the solution in the original variables we take
\[ \ln x = C_1 + \ln \left( \frac{x}{(y - x)(y + x)} \right). \] (29)
Now if we consider the invariants that induce each symmetry
\[
\frac{dx}{\xi} = \frac{dy}{\eta}, \quad \mapsto \quad y' := \frac{dy}{dx} = \frac{\eta}{\xi},
\]
we take that:
\[
X_1 \mapsto I_1 = x, \quad X_2 \mapsto I_2 = x, \quad X_3 \mapsto I_3 = \frac{y}{x}, \quad X_4 \mapsto I_4 = \frac{y}{\sqrt{x}},
\]
For example the symmetry $X_3 = x \partial_x + y \partial_y$ generates the following invariant:
\[
\frac{dx}{\xi} = \frac{dy}{\eta} \implies \frac{dx}{x} = \frac{dy}{y} \implies \ln x = \ln y \implies I_3 = \frac{y}{x} \mapsto y = ax,
\]
this would be the solution that suggests us precisely the direct use of the Pi theorem (if the ode is scale invariant, as in this case, the solution obtained applying by the Pi theorem coincides with the invariant solution that generates the scale symmetry, in this case $X_3$). We see that we only obtain a particular solution, but that this is invariant, in fact if we think about the ode as a dynamical system we see that the fixed point of such equation would be precisely the solution $y = \pm x$.

Example 2 Solve the linear ode
\[
(1 - x^2) y' + xy = 1.
\]

Solution. Solution through D.A. Our first step will be to introduce dimensional constants in such a way that eq. (33) verifies the principle of dimensional homogeneity. In this way we rewrite the equation as follows
\[
(A - x^2) y' + xy = B,
\]
where
\[
[A] = X^2, \quad [B] = XY,
\]
i.e.
\[
\pi_1 = \frac{x^2}{A}, \quad \pi_2 = \frac{xy}{B},
\]
where
\[
y = \frac{1}{Bx} \varphi \left( \frac{x^2}{A} \right),
\]
being $\varphi$ a unknown function. It is possible to make the following assumption
\[
y = \frac{1}{Bx} \left( \frac{x^2}{A} \right)^n, \quad n \in \mathbb{R}.
\]
Since $y = x^{-1}$ is not a particular solution of (33) we need to combine the $\pi - monomias$ in order to find a particular solution, the simplest way is as follows:
\[
\left( t = \frac{x^2}{A}, \quad u(t) = \frac{A y}{B x} \right) \implies \left( x = \sqrt{At}, \quad y = \frac{B}{A} u(t) \sqrt{At} \right)
\]
observing that
\[
u(t) = \pi_2 \cdot \pi_1^{-1},
\]
where $y = cx$, derived from $\pi_2 \cdot \pi_1^{-1}$ is a particular solution of (33).
Remark 2.1 (Recipe) When we have two $\pi$-monomia, we have to check if they induce any particular (or invariant) solution. If they are not particular solutions then we must combine them in order to obtain such solution, this solution usually is obtained by combining them in a very simple way.

In this way our ode is written now as follows:

$$\frac{2u'}{u-1} = \frac{1}{t(t-1)} \implies u = 1 + C_1 \frac{\sqrt{t+1}}{t},$$

and in the original variables the solution of eq. (33) yields:

$$y = \frac{A}{B} \left( x \pm C_1 \sqrt{A-x^2} \right),$$

where $C_1$ is an integration constant. Now making $A = B = 1$, we have the ordinary solution to eq. (33).

Lie method. Following the standard procedure we have to solve the pde:

$$\eta_x + \frac{(\eta_y - \xi_x)(1 - xy)}{1-x^2} - \frac{\xi_y}{1-x^2} - \eta(x, y) \left(- \frac{y}{1-x^2} + 2 \frac{(1-xy)x}{(1-x^2)^2} \right) + \frac{\eta(x, y)x}{1-x^2} = 0,$$

which solutions are:

$$X_1 = \sqrt{-1+x^2} \partial_y, \quad X_2 = (-y + x) \partial_y, \quad X_3 = \left( \frac{2xy - y^2 - 1}{(x-1)(x+1)} \right) \partial_y,$$

and their corresponding invariants are:

$$X_1 \mapsto I_1 = x, \quad X_2 \mapsto I_2 = x, \quad X_3 \mapsto I_3 = x.$$

For example the symmetry $X_1$ generates the following c.v.:

$$\left( t = x, \quad u(t) = \frac{y}{\sqrt{-1+x^2}} \right) \implies \left( x = t, \quad y = u(t) \sqrt{-1+t^2} \right),$$

therefore eq. (33) is written as:

$$u' = -\frac{\sqrt{-1+t^2}}{1-2t^2+t^4},$$

which solution is:

$$u(t) = -\left( \frac{1+t^2}{4(t+1)^2} \right)^{3/2} + \left( \frac{1+t^2}{4(t-1)^2} \right)^{3/2} + C_1,$$

and hence

$$\frac{y}{\sqrt{x^2-1}} = \frac{C_1(x^2-1) + x \sqrt{x^2-1}}{x^2 - 1} \implies y = x \pm C \sqrt{x-1} \sqrt{x+1}.$$

Now, if for example we consider the symmetry $X_2$ then we have:

$$\left( t = x, \quad u(t) = -\ln(-y + x) \right) \implies \left( x = t, \quad y = \frac{te^{u(t)} - 1}{e^{u(t)}} \right),$$

therefore:

$$u' = -\frac{t}{-1+t^2},$$
finding in this way that its solution is:

\[ u(t) = -\frac{1}{2} \ln(t-1) - \frac{1}{2} \ln(t+1) + C, \]  

hence in the original variables the solution yields:

\[ -\ln(-y+x) = -\frac{1}{2} \ln(x-1) - \frac{1}{2} \ln(x+1) + C, \]

which after a simple simplification it yields:

\[ y = x \pm c \sqrt{x-1} \sqrt{x+1}, \]

as we already know. □

With these two simple examples we have tried to show how the D.A. works in order to introduce c.v. which brings us to obtain simpler odes than the original one. As we have emphasized in the recipe, the trick is to look for a particular solution. Sometimes this particular solution will be furthermore invariant solution (induced by a concrete symmetry). If this is the case, then our ode will be reduced to an ode with separating variables but if this solution is only a particular then, as we will see below, we have not any guarantee of reducing our ode to an ode with separate variables, nevertheless we will obtain a simpler ode than the original one. In the next section we will study some examples.

3 Examples

In this section we will apply our pedestrian method to different odes, beginning with a Bernoulli ode. We go next to employ the method to solve Abel odes as well as Riccati odes since as anyone knows these kind of odes are truly very difficult.

Example 3 Solve the following Bernoulli ode.

\[ y' = -\frac{y}{x+1} - \frac{(x+1)y^2}{2}. \]  

Remark 3.1 Historically it was Bernoulli the first person who introduced c.v. in order to solve odes (now bearing his name). He managed to reduce this equation to a simpler linear equation.

Solution. It is observed that with

\[ [a] = X, \quad [b] = X^{-2}Y^{-1}, \]

eq. (55) yields d.h. i.e.

\[ y' = -\frac{y}{x+a} - \frac{b(x+a)y^2}{2}, \]

in this way it is obtained the following variables

\[ \left( t = \frac{x}{a} \right) \quad u(t) = \frac{1}{a^2bY} \quad \Rightarrow \quad \left( x = at, \quad y = \frac{1}{a^2btu} \right), \]

with this change of variables eq. (55) yields

\[ 2u + 2t(1+t)u' = (t+1)^2, \]
which is linear and its solution is:

\[ u = \left( \frac{t}{2} + C_1 \right) \left( 1 + \frac{1}{t} \right), \tag{60} \]

and therefore in the original variables the solution yields

\[ y = \frac{2}{(bx + C_1) (x + a)}, \tag{61} \]

in this case D.A. does not bring us to obtain a good change of variables but helps us to obtain a simpler ode than the original one. We can try now with the following c.v.

\[
\left( t = x , \quad u(t) = \frac{1}{x^2 y} \right) \Rightarrow \left( x = t , \quad y = \frac{1}{t^2 u} \right), \tag{62}
\]

which brings us to obtain a linear ode

\[ t(t+1) \left( 2u' + 4u \right) = 2t^2 u + t^2 + 2t + 1, \tag{63} \]

but with this tactic we are not able to obtain a simpler ode.

As we can see D.A. induces us to the c.v. \( \frac{1}{axy} \) but \( y = \frac{1}{x} \) or \( y = \frac{1}{x^2} \), are not particular solutions of (55). Nevertheless solutions as \( y = \frac{1}{x+1} \) or \( y = \frac{1}{(x+1)^2} \) are particular and invariant solutions for this reason these c.v. induce us to obtain an ode in separate variables, but unfortunately D.A. is unable to construct such c.v. For this reason we can only reduced our ode to a linear ode as in the theoretical case i.e. employing the theoretical method purposed by Bernoulli.

If we study this ode under the Lie group method, it is observed that eq. (55) admits the following symmetries obtained from:

\[
\begin{align*}
\eta_x + (\eta_y - \xi_x) & \left( - \frac{y}{x+1} - \frac{(x+1)y^2}{2} \right) - \xi_y \left( - \frac{y}{x+1} - \frac{(x+1)y^2}{2} \right)^2 - \\
-\xi & \left( \frac{y}{(x+1)^2} - \frac{1}{2} y^2 \right) - \eta \left( - y - xy - \frac{1}{x+1} \right) = 0
\end{align*}
\]

\[ X_1 = \frac{y^2 (x+1)}{2} \partial_y, \quad X_2 = (x+1) \partial_x - 2y \partial_y, \quad X_3 = \partial_x - \frac{y}{(x+1)} \partial_y, \quad X_4 = x \partial_x - \frac{(1+2x)y}{(x+1)} \partial_y, \tag{64}
\]

and which respective invariants are:

\[ I_1 = x, \quad I_2 = y (x+1)^2, \quad I_3 = y (x+1), \quad I_4 = yx (x+1), \]

that induces the following canonical variables. For example from \( X_1 \) :

\[
\left( t = x, \quad u(t) = - \frac{2}{y (x+a)} \right) \Rightarrow \left( x = t, \quad y = - \frac{2}{u(t+a)} \right), \tag{66}
\]

in such a way that eq. (55) yields

\[ u' = -b \Rightarrow y = bt + C_1, \tag{67} \]

and in the original variables i.e. \((x,y)\) this solution yields

\[ y = \frac{2}{(bx + C_1) (x + a)}, \tag{68} \]

as we already know. ■
Example 4. Solve the following Riccati ode

\[ y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}. \]  

(69)

Solution. It is observed that introducing the following dimensional constants eq. (69) verifies the principle of dimensional homogeneity (p.d.h.),

\[ y' = \frac{xy^2}{a} - \frac{2y}{x} - \frac{a}{x^3}, \]  

(70)

where \([a] = X^2Y\). As we can see if we need only one constant then our equation is scale invariant and therefore this constant indicates us that the generator of this symmetry is \(X = -2x\partial_x + y\partial_y\) as we will see below. Therefore we have the following variables

\[ \left( t = x, \quad u(t) = \frac{a}{yx^2} \right) \implies \left( x = t, \quad y = \frac{a}{tu^2} \right), \]  

(71)

with this new variables eq. (69) yields

\[ \frac{u'}{u^2 - 1} = \frac{1}{t} \implies \ln t + \text{arctanh}(u) + C_1 = 0, \]  

(72)

and in the original variables this solutions is written as:

\[ \ln x + \text{arctanh}\left(\frac{a}{yx^2}\right) + C_1 = 0. \]  

(73)

We would like to emphasize that in this case, as we have only one dimensional constant, we can find the particular solution \(y = ax^{-2}\) applying the Pi theorem, and that such solution (invariant particular solution) is induced by the scale symmetry.

Eq. (69) equation has the following symmetries:

\[ \eta_x + (\eta_y - \xi_x)(xy^2 - \frac{2y}{x} - \frac{1}{x^3}) - \xi_y(xy^2 - \frac{2y}{x} - \frac{1}{x^3})^2 - \xi(y^2 + \frac{2y}{x^2} + \frac{3}{x^4}) - \eta \left( 2xy - \frac{2}{x} \right) = 0 \]  

(74)

and hence

\[ X_1 = \frac{1}{x}\partial_x - \frac{2}{x^4}\partial_y, \quad X_2 = x^4(y + \frac{1}{x^2})^2\partial_y, \quad X_3 = x\partial_x - 2y\partial_y \]

\[ X_4 = \left( -\frac{1}{x^2} + y^2 x^2 \right)\partial_y, \quad X_5 = x^3\partial_x - (2 + 4x^2)\partial_y. \]  

(75)

and which corresponding invariants are:

\[ I_1 = \frac{yx^2 - 1}{x^2}, \quad I_2 = I_4 = x, \quad I_3 = yx^2, \quad I_5 = yx^2(x^2 + 1). \]  

(76)

The method of canonical variables brings us to obtain the following ode, for example, for the symmetry \(X_5\), we have

\[ \left( t = x^2(y^2 + 1), \quad u(t) = -\frac{1}{2x^2} \right), \]  

(77)

\[ u_t = \frac{1}{t^2} \implies u(t) = -\frac{1}{t} + C_1, \]  

(78)
and hence
\[-\frac{1}{2}x^2 = -\frac{1}{x^2(yx^2 + 1)} + C_1,\]  \hspace{1cm} (79)

as we already know.

The transformation induced by \(X_3\) (scaling symmetry) is the following one:

\[(t = yx^2, \quad u(t) = \ln(x)) \Longrightarrow \left(x = e^{u(t)}, \quad y = \frac{t}{(e^{u(t)})^2}\right),\]  \hspace{1cm} (80)

\[u_t = \frac{1}{t^2 - 1} \Longrightarrow u(t) = -\text{arctanh}(t) + C_1,\]  \hspace{1cm} (81)

therefore

\[\ln(x) = -\text{arctanh}(yx^2) + C_1,\]  \hspace{1cm} (82)

which looks a little different than the other solution. \(\blacksquare\)

**Example 5** Solve the following Riccati ode:

\[y' = \frac{y + 1}{x} + \frac{y^2}{x^3}.\]  \hspace{1cm} (83)

**Solution.** The ode verifies the p.d.h. if we introduce the following dimensional constant:

\[y' = \frac{y + a}{x} + b\frac{y^2}{x^3},\]  \hspace{1cm} (84)

where \([a] = Y, [b] = X^2Y^{-1}\). In this way it is obtained the c.v.

\[(t = \frac{ax^2}{by^2}, \quad u(t) = \frac{y}{a}) \Longrightarrow \left(x = \sqrt{abtu^2}, \quad y = au\right),\]  \hspace{1cm} (85)

and therefore eq. (83) yields

\[\frac{u}{u'} + t = \frac{ut}{(ut + t + 1)},\]  \hspace{1cm} (86)

which is a Bernoulli ode and its solution is:

\[\frac{1}{u} - \sqrt{t} \left(\arctan \sqrt{t} + C_1\right) = 0,\]  \hspace{1cm} (87)

hence in the original variables the solution is

\[\frac{a}{y} - \sqrt{\frac{ax^2}{by^2}} \left(\arctan \sqrt{\frac{ax^2}{by^2} + C_1}\right) = 0.\]  \hspace{1cm} (88)

It is observed that we have the particular solution \(y = Cx\), obtained from the relationship \(\frac{ax^2}{by^2}\).

Since this c.v. is not very good for our purposes we may now proceed as follows:

\[axy' = ay + 1 + b^2\frac{y^2}{x^2}\]  \hspace{1cm} (89)

where

\([a] = y^{-1}, \quad [b] = y^{-1}x\)  \hspace{1cm} (90)
in such a way that these constants induce the following change of variables:

\[
\begin{align*}
(t = \frac{a}{b} x, \quad u(t) = \frac{x}{by}) & \implies (x = \frac{b}{a} x, \quad y = \frac{t}{au(t)})
\end{align*}
\]

(91)

which brings us to obtain

\[
\frac{-u'}{u^2 + 1} = \frac{1}{t^2} \implies -\frac{1}{t} + \arctan(u) + C_1 = 0
\]

(92)

and hence

\[
-\frac{b}{ax} + \arctan\left(\frac{x}{by}\right) + C_1 = 0
\]

(93)

which is a better approximation. In the same way we can obtain a quadrature simply considering \(t = \frac{b}{ax}\) and \(u(t) = \frac{x}{by}\). In this way it is obtained \(u' = u^2 + 1\).

The Lie method bring us to solve the following pde

\[
\eta_x + (\eta_y - \xi_x)(\frac{y + 1}{x} + \frac{y^2}{x^3}) - \xi_y(\frac{y + 1}{x} + \frac{y^2}{x^3})^2 - \xi(\frac{y + 1}{x} - \frac{3y^2}{x^4}) - \eta(\frac{1}{x} + \frac{2y}{x^3}) = 0,
\]

(94)

and which solutions are:

\[
X_1 = \left(x + \frac{y^2}{x}\right) \partial_y, \quad X_2 = \left(\frac{(ycos\left(\frac{1}{x}\right) + xsin\left(\frac{1}{x}\right))^2}{x}\right) \partial_y
\]

\[
X_3 = \left(-\frac{\sin\left(\frac{1}{x}\right)^2(y \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) x)^2}{x(-1 + \cos\left(\frac{1}{x}\right)^2)}\right) \partial_y, \quad X_4 = x^2 \partial_x + xy \partial_y
\]

(95)

and which invariants are:

\[
I_1 = I_2 = I_3 = x, \quad I_4 = \frac{y}{x}.
\]

(96)

The symmetry \(X_1 = \left(x + \frac{y^2}{x}\right) \partial_y\) generates the following c.v.:

\[
\left(t = x, \quad u(t) = \arctan\left(\frac{y}{x}\right)\right) \implies (y = \tan(u(t)) t, \quad x = t),
\]

(97)

\[
u_t = \frac{1}{t^2} \implies u(t) = -\frac{1}{t} + C_1,
\]

(98)

and hence the solution is

\[
\arctan\left(\frac{y}{x}\right) = -\frac{1}{x} + C_1,
\]

(99)

as we already know.

We would like to emphasize that this ode is not scale invariant and nevertheless we have been able to obtain a c.v. that reduces the ode to a quadrature.

---

**Example 6** Solve the following Abel ode

\[
y' = \frac{1 - y^2}{xy} + 1
\]

(100)
Solution. Introducing the following dimensional constants, we make that eq. \((100)\) verifies the p.d.h.

\[
(xy)y' = -y^2 + bxy + a,
\]

where \([b] = yx^{-1}, [a] = y^2\). Therefore the c.v. that suggests the D.A. is the following one:

\[
\left(t = \frac{bxy}{a}, \quad u(t) = \frac{y^2}{a}\right) \implies \left(x = \frac{at}{b\sqrt{ua}}, \quad y = \sqrt{ua}\right),
\]

hence eq. \((100)\) yields

\[
u'(2 + t - u) t = 2u(1 + t - u),
\]

and its solution is:

\[
u = \frac{t^2}{-2t + 2\ln(1 + t) - C_1},
\]

in the original variables it yields:

\[-\frac{b^2x^2}{2a} = -\frac{bxy}{a} + \ln(1 + \frac{bxy}{a}) + C_1.\]

Once again we emphasize that the particular solution, \(y = Cx^{-1}\), (in this case invariant solution , see below) has been obtained from the relationship \(\frac{bxy}{a}\).

Alternatively we can try the following c.v. Writing the original Abel ode, eq.(100) in the following form:

\[
axyy' = -\frac{y^2}{a} + xy + b,
\]

where \([a] = XY^{-1}, [b] = XY\), therefore we find the next c.v.

\[
\left(t = \frac{xy}{b}, \quad u(t) = \frac{x^2}{ab}\right) \implies \left(x = \sqrt{abu}, \quad y = \frac{bt}{\sqrt{abu}}\right),
\]

that brings us to rewrite eq. \((106)\) as follows:

\[
u' = \frac{2t}{1 + t} \implies u = 2t - 2\ln(1 + t) + C_1,
\]

and hence in the original variables \((x, y)\) we get:

\[
\frac{x^2}{ab} = 2\frac{xy}{b} - 2\ln(1 + \frac{xy}{b}) + C_1,
\]

as we already know. We can check that the following c.v. also works well

\[
\left(t = \frac{x^2}{ab}, \quad u(t) = \frac{xy}{b}\right) \implies \left(x = \sqrt{abu}, \quad y = \frac{bu}{\sqrt{abu}}\right),
\]

since in these variables eq. \((106)\) is written as

\[
2u'u = 1 + u \implies t - 2u + 2\ln(1 + u) + C_1 = 0,
\]

and therefore

\[
\frac{x^2}{ab} - 2\frac{xy}{b} + 2\ln(1 + \frac{xy}{b}) + C_1 = 0,
\]

obtaining the solution.
Applying the Lie method we need to solve the following pde:

\[
\eta_x + (\eta_y - \xi_x)(\frac{a - y^2}{xy} + 1) - \xi_y \left( \frac{a - y^2}{xy} + 1 \right)^2 + \xi \left( \frac{a - y^2}{x^2y} + \eta \left( \frac{2}{x} + \frac{a - y^2}{xy^2} \right) \right) = 0,
\]

which solution is:

\[
X_1 = -\frac{1}{x} \partial_x + \frac{y}{x^2} \partial_y \quad \implies \quad I_1 = xy.
\]

This symmetry induces the following c.v.:

\[
\left( u(t) = -\frac{x^2}{2}, \quad t = xy \right),
\]

and therefore in this variables eq. (106) is written as follows:

\[
u' = -\frac{t}{a + t} \implies u(t) = -t + a \ln(a + t) + C_1,
\]

hence in the original variables we get:

\[
-x^2 = -xy + a \ln(a + xy) + C_1,
\]

This is another example of an ode that it is not scale invariant, and nevertheless, we have been able to reduce it to a quadrature.

---

**Example 7** Solve the following Abel ode.

\[
x (y + 4) y' = y^2 + 2y + 2x.
\]

**Solution.** Following our pedestrian method, we begin by introducing dimensional constants and rewriting the odes as follows

\[
x (y + 4a) y' = y^2 + 2ay + 2bx,
\]

where \( [a] = y \), and \( [b] = x^{-1}y^2 \), therefore D.A. suggest us the following c.v. (note that \( y = \sqrt{x} \), is a particular solution of (118)))

\[
\left( t = \frac{y^2}{bx}, \quad u(t) = \frac{y}{a} \right) \implies \left( x = \frac{u^2a^2}{ct}, \quad y = ua \right),
\]

in such a way that eq. (118) yields

\[
t (t + 4) u' = t(u + 2) + 2u,
\]

which is a linear ode and its solution is:

\[
u = t + C_1 \sqrt{t(t + 4)},
\]

hence

\[
\frac{y}{a} = \frac{y^2}{bx} + C_1 \sqrt{\frac{y^2}{bx} \left( \frac{y^2}{bx} + 4 \right)},
\]

and simplifying, it is obtained the solution:

\[
\frac{bx}{a} = y + C_1 \sqrt{(y^2 + 4bx)}.
\]
Another particular solution could be found from the following relationship \( \frac{y}{x^2} \left( \frac{y}{a} \right)^{-1} \) i.e. \( y = Cx \).

But this particular solution bring us to the following c.v.

\[
(t = \frac{ay}{bx}, \quad u(t) = \frac{y}{a} \implies \left( x = \frac{ua^2}{ct}, \quad y = ua \right),
\]

which transforms eq. (118) into a Bernoulli ode,

\[
t^2 (u + 4) u' = (ut + 2t + 2) (u't - u),
\]

but this situation is not desirable since it is always more difficult to solve a Bernoulli ode than a linear ode.

Applying the Lie method, following the standard procedure, we need to solve the following pde

\[
\eta_x + \left( \eta_y - \xi_x \right) \frac{y^2 + 2y + 2x}{x(y + 4)} - \xi_y \frac{y^2 + 2y + 2x}{x^2(y + 4)^2} = 0,
\]

and which solutions are:

\[
X_1 = \left( \frac{(2y + 4 - x)(y - x)}{y + 4} \right) \partial_y, \quad X_2 = \left( \frac{(y^2 + 4x)(y - x)}{x(y + 4)} \right) \partial_y, \quad X_3 = (4x + x^2) \partial_x + x(y + 4) \partial_y
\]

where their correspond invariants are:

\[
I_1 = I_2 = x, \quad I_3 = \frac{y + 4}{x + 4}
\]

For example the c.v. that induces \( X_3 \) is:

\[
\left( t = \frac{y + 4}{4 + x}, \quad u(t) = \frac{1}{4} \ln(x) - \frac{1}{4} \ln(4 + x) \right),
\]

hence:

\[
\left( y = -\frac{4(t - 1 + e^{4u(t)})}{-1 + e^{4u(t)}}, \quad x = -\frac{4e^{4u(t)}}{-1 + e^{4u(t)}} \right),
\]

finding that:

\[
u' = \frac{t}{2(2t^2 - 3t + 1)},
\]

which solution is:

\[
u(t) = \frac{1}{2} \ln(t - 1) - \frac{1}{4} \ln(2t - 1) + C_1,
\]

therefore, in the original variables it yields:

\[
\frac{1}{4} \ln(x) - \frac{1}{4} \ln(4 + x) = \frac{1}{2} \ln\left( \frac{y + 4}{4 + x} - 1 \right) - \frac{1}{4} \ln\left( \frac{2(y + 4)}{4 + x} - 1 \right) + C_1,
\]

as we already know. ■

Example 8 Solve the Abel ode.

\[
y' = Cx^3 y^3 + Bxy^2 - A \frac{y}{x},
\]

15
Solution. If we rewrite eq. (135) introducing the following dimensional constants,

\[ y' = a^2C x^3 y^3 + aB x y^2 - A \frac{y}{x}, \]

(136)

where \( A, B, C \in \mathbb{R} \), and \([a] = X^{-2} Y^{-1}\). As we can see this ode is scale invariant since we have needed introduce only one constant. D.A. suggests us the following c.v.

\[ (t = x, u(t) = ax^2 y) \implies (x = t, y = \frac{u}{at^2}), \]

(137)

in such a way that eq. (135) is written in the following form:

\[ tu' = u (u^2 + u + 1), \]

(138)

and its solution is:

\[ \ln t + \frac{1}{2} \ln (u^2 + u + 1) + \frac{\sqrt{3}}{3} \arctan \left( \left( \frac{3}{2} u + \frac{1}{3} \right) \sqrt{3} \right) - \ln u + C_1 = 0, \]

(139)

therefore in the original variables it yields:

\[ \ln x + \frac{1}{2} \ln \left( (ax^2 y)^2 + ax^2 y + 1 \right) + \frac{\sqrt{3}}{3} \arctan \left( \left( \frac{3}{2} ax^2 y + \frac{1}{3} \right) \sqrt{3} \right) - \ln (ax^2 y) + C_1 = 0. \]

(140)

In second place, we study eq. (135)

\[ y' = C x^3 y^3 + B x y^2 - A \frac{y}{x}, \]

(141)

with respect to the dimensional base \( B = \{T\} \). This ode verifies the principle of dimensional homogeneity with respect to this dimensional base. Note that \([y] = \left[ \frac{1}{T^2} \right] = T^2\), and \([x] = \left[ y \right] = T^{-1} \) hence \([y'] = T^3\). Therefore rewriting the equation in a dimensionless way we find that \( y \propto x^{-2} \).

But if we study this equation with respect to the dimensional base \( B = \{X, Y\} \), we need to introduce new dimensional constants that make the equation verify the principle of dimensional homogeneity

\[ y' = \alpha C x^3 y^3 + \beta B x y^2 - A \frac{y}{x}, \]

(142)

where \([\alpha^{1/2}] = [\beta] = X^{-2} Y^{-1}\), hence

\[
\begin{array}{c|cc}
X & y & \beta \\
Y & 0 & -2 \\
-1 & 1 & 0
\end{array}
\implies y \propto \frac{\beta}{x^2},
\]

(143)

As we can see we have obtained the same solution than in the case of the invariant solution. This is because the invariant solution that induces a scaling symmetry is the same as the obtained one through the Pi theorem.

This ode admits the following symmetry (scale-invariant)

\[ X = x \partial_x - 2y \partial_y, \implies I = x^2 y \]

(144)

which is a scaling symmetry and it induces the following change of variables,

\[ r = x^2 y, \quad s(r) = \ln(x), \implies x = e^{s(r)}, \quad y = \frac{r}{e^{2s(r)}}, \]

(145)
which brings us to obtain the next ode in quadratures

\[ s' = \frac{1}{r (Cr^2 + Br + 2 - A)}, \]  

and which solution is:

\[ s(r) = -\ln r + \frac{1}{2} \ln \left( \frac{Cr^2 + Br + 2 - A}{A - 2} \right) - \frac{\text{Barctanh} \left( \frac{2Cr + B}{\sqrt{B^2 + 4C(A - 2)}} \right)}{(A - 2) \sqrt{B^2 + 4C(A - 2)}} + C_1, \]

and hence in the original variables \((x, y)\):

\[ \ln x = -\ln \left( \frac{x^2 y}{A - 2} \right) + \frac{1}{2} \ln \left( \frac{Cx^4 y^2 + Bx^2 y + 2 - A}{A - 2} \right) - \frac{\text{Barctanh} \left( \frac{2Cx^2 y + B}{\sqrt{B^2 + 4C(A - 2)}} \right)}{(A - 2) \sqrt{B^2 + 4C(A - 2)}} + C_1, \]

which is the most general solution for this ode.

### 4 Pathological cases.

In this section we will present two examples of odes that do not admit symmetries (Lie point symmetries). Nevertheless in the first of them D.A. helps us to obtain a simple c.v. that will bring us to obtain a simpler ode (through a particular solution). In the second case we will show that unfortunately sometimes one finds odes that at this time have no solution, or at least we do not know how to solve them.

**Example 9** Solve the Abel ode

\[ (x^2 y + x^5 - x) y' = xy^2 - (x^4 + 1) y. \]  

**Solution.** Following our pedestrian method we begin by introducing dimensional constants

\[ ax^2 y' + bx^5 y' - cxy' = dxy^2 - ex^4 y + fy \]

where

\[ [a] = [d] = x^{-2} y^{-1}, \quad [b] = [e] = x^{-5}, \quad [c] = [f] = x^{-1}, \]

therefore \([c^5] = [x^5] = [b] = [e]\), having only two dimensional constants. Since \(y = x^{-2}\) is not a particular solution of eq.\((149)\) then we look for a combination between the monomias finding in this way that

\[ \pi_1 = \frac{1}{cx}, \quad \pi_2 = \frac{ax y}{c} \]

where \(y = x^{-1}\) is a particular solution of \((149)\). The c.v. that induces the D.A. is the following one:

\[ (t = xy, \quad u(t) = x) \implies \left( x = u, \quad y = \frac{t}{u} \right), \]

hence using these new variables eq. \((149)\) is written now as:

\[ 2u't(t + 1) + u \left( 1 - t - u^4 \right) = 0, \]
which is a Bernoulli ode and its solution is
\[ u = \sqrt{t \sqrt{-2t - 2 \ln(t - 1) + C_1}} \]  
(155)

undoing the c.v. we find that the solution to eq. (149) is:
\[ x = \sqrt{xy \sqrt{-2xy - 2 \ln(xy - 1) + C_1}} \]  
(156)

Applying the Lie method we see that the pde to solve is:
\[
\eta_x + (\eta_x - \xi_x) y \frac{y (xy - x^4 - 1)}{x(x+y^4-1)} - \xi_y y^2 \frac{(xy - x^4 - 1)^2}{x(x+y^4-1)^2} - \\
-\xi \left( y \frac{y - 4x^3}{x(x+y^4-1)} - y \frac{y (xy - x^4 - 1)}{x^2(x+y^4-1)} - y \frac{y (xy - x^4 - 1)(y + 4x^3)}{x(x+y^4-1)^2} \right) - \\
-\eta \left( \frac{xy - x^4 - 1}{x(x+y^4-1)} + y \frac{y}{(x+y^4-1)} - y \frac{y (xy - x^4 - 1)}{(x+y^4-1)^2} \right) = 0. 
\]  
(157)

but in this case we have not found any solution, i.e. eq. (149) does not admit symmetries. Nevertheless, one always may try to find, as if by magic, any c.v. that brings to find a simpler ode.

Example 10 Try to solve the Abel ode
\[ y' = -Ax^2 y^3 - By^2 - \frac{y}{x}, \]  
(158)

where \( A \) and \( B \) are dimensional constant.

Solution. In this occasion we already have the dimensional constants
\[ [A] = x^{-3} y^{-2}, \quad [B] = y^{-1} x^{-1}, \]  
(159)
in such a way that all the terms of the equation has dimensions of \( \frac{y}{x} \). We check if one of the dimensional relationship induces any particular solution finding that this is not the case. Therefore we go next to look for any trivial combination between them but we are not able to find any particular solution. Nevertheless we try to obtain any result with the following c.v.:
\[
\left( t = \frac{B^2}{Ax}, \quad u(t) = \frac{1}{Byx} \right) \implies \left( x = \frac{B^2}{At}, \quad y = \frac{At}{B^3 u(t)} \right) 
\]  
(160)

which bring us to obtain the next ode:
\[ tl^2 u' u + tu + 1 = 0, \]  
(161)
but unfortunately we have not advanced.

Another try is the following one:

\[
(t = \frac{1}{B y x}, \ u(t) = \frac{B^2}{A x}) \implies \left( x = \frac{B^2}{A u(t)}, \ y = \frac{A u(t)}{B^3 t} \right) \tag{162}
\]

and hence:

\[
t u^2 + u' \left(1 + u t \right) = 0, \tag{163}
\]

but as we supposed these attempts do not simplify our ode.

Now we change the strategy and we are going to suppose that the ode has dimension of \( y \). In this case we need to introduce the following dimensional constants in order to make eq. (158) verify the p.d.h.

\[
ay' = -bx^2y^3 - cy^2 - \frac{y^2 x}{x}, \tag{164}
\]

where \([a] = x, [b] = y^{-2} x^{-2}\) and \([c] = y^{-1}\), which bring us to the following c.v.

\[
(t = x, \ u(t) = \frac{1}{y}) \implies \left( x = t, \ y = \frac{1}{u(t)} \right) \tag{165}
\]

therefore we obtain this new ode

\[
u' u t - 2u^2 - t^5 - t^2 u = 0. \tag{166}
\]

but as in the above tactic we have not advanced. This c.v. is precisely the suggested one by the theoretic methods.

Following the Lie method we have to solve the pde:

\[
\eta_x + (\eta_y - \xi_x) \left( -Ax^2 y^3 - By^2 - \frac{y}{x} \right) - \xi_y \left( -Ax^2 y^3 - By^2 - \frac{y}{x} \right)^2 - \\
-\xi \left( \frac{1}{x^2} y - 2Axy^3 \right) - \eta \left( -2By - \frac{1}{x} - 3Ax^2 y^2 \right) = 0, \tag{167}
\]

but this ode does not admit any symmetry.

To end, we would like to show that the theoretical method does not work either. For this purpose we follow step by step the method beginning with a generic Abel ode of first order written as follows:

\[
y' = f_3 y^3 + f_2 y^2 + f_1 y + f_0, \tag{168}
\]

where in our case:

\[
f_3 = -x^2, \ f_2 = -1, \ f_1 = -\frac{1}{x}, \ f_0 = 0. \tag{169}
\]

The c.v. suggested by the theoretical method is the following one

\[
(t = x, \ u(t) = \frac{1}{y}) \implies \left( x = t, \ y = \frac{1}{u(t)} \right), \tag{170}
\]

(note that this c.v. is the same than the suggested one by D.A. (165)). In this way our ode is now rewritten as:

\[
u' u = h_2 u^2 + h_1 u + h_0, \tag{171}
\]

where

\[
h_2 = -f_1, \ h_1 = -f_2, \ h_0 = -f_3, \tag{172}
\]
finding therefore that in our case we have:

\[ u' u = \frac{u^2}{t} + u + t^2, \]  

(173)

which is an Abel of second order (this ode has no solution). To try to find a solution of this ode we make the following c.v.

\[ (r = t, \quad s(r) = u(t)E) \implies \left( t = r, \quad u(t) = \frac{s(r)}{E} \right), \]  

(174)

obtaining this new ode:

\[ ss' = F_1 s + F_0, \]  

(175)

where

\[ E = \exp(-\int h_2), \quad F_1 = h_1 E, \quad F_0 = h_0 E^2, \]  

(176)

hence

\[ ss' = \frac{s}{r} + 1, \]  

(177)

but unfortunately we do not know how to solve this apparently simple ode.

As we have seen this ode seems very pathological since none of the followed tactics have helped us to obtain any solution. ■

### 5 Conclusions and discussion.

We have seen how writing the odes in such a way that they verify the pdh i.e. introducing dimensional constants, we can obtain in a trivial way c.v. that bring us to obtain simpler ode than the original and therefore their integration is immediate. Furthermore, we have tried to show that these c.v. are not obtained as if by magic but that they correspond to invariant solutions or to particular solutions and therefore they are generated by the symmetries that admit the ode.

Nevertheless, the D.A. has strong limitations. For example D.A. is unable (at least at this time we do not know how to do it) to solve the following simple linear ode

\[ y' = \left( x^3 + \frac{1}{x} + 3 \right) y + \left( 3x^2 - \frac{1}{x^2} \right), \]  

(178)

for this reason one must not put all his confidence in this “tactic”. This is one of the greater inconvenience that presents the proposed method. But as we have noticed in the introduction, we think that our pedestrian method continues having validity at least when one is studying ode derived from engineering problems or physical problems etc... where, as it is supposed, such odes must verify the pdh in such a way that for example the ode \( y' = x + 1 \), lacks of any sense (physical sense, since we cannot add a number to a physical quantity).

Nevertheless, and in spite of the limitations that we have not avoided to show, we continued believing in the kindness (goodness) of the method and that it can be applied to obtain solutions (at least particular solutions and in concrete invariant solutions) to more complicated equations like the following
ones:

\[
y'' = \frac{y'^2}{y} - ay^2, \quad [a] = y^{-1}x^{-2} \implies y = \frac{1}{x^2}, \quad (179)
\]

\[
y''' = \frac{a}{y^3}, \quad [a] = y^4x^{-3} \implies y = x^{3/4}, \quad (180)
\]

\[
y''' = -ayy'', \quad [a] = y^{-1}x^{-1} \implies y = x^{-1}, \quad (181)
\]

\[
y'' = \frac{y'}{x} + \frac{a}{2} \frac{y'^2}{x^2}, \quad [a] = y^{-1}x \implies y = x, \quad (182)
\]

\[
y''' = \frac{\left(\frac{y''}{y'}\right)^2}{y'(1+ay')}, \quad [a] = y^{-1}x \implies y = x, \quad (183)
\]

but these are questions that we will approach in a forthcoming paper.

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