Hydrodynamic time correlation functions in the presence of a gravitational field

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Abstract

This paper shows that the ordinary Brillouin spectrum peaks associated to scattered radiation off acoustic modes in a fluid suffer a shift in their values due to gravitational effects. The approach is based in the ordinary linearized Navier-Stokes equations for a fluid coupled to a Newtonian gravitational potential. The formalism leads to a dispersion relation that contains both gravitational and dissipative effects. It is also shown that the Brillouin peaks tend to condense into a single peak when the fluid modes approach the critical Jeans wave number.

1 Introduction

Dynamic light scattering theory \cite{1}-\cite{2} shows that photons interact with the acoustic modes in a fluid. This scattering arises from sound waves associated to statistical fluctuations in the fluid’s density and temperature. Early
theoretical work in this subject, as well as the measurements confirming the
existence of the Brillouin-Rayleigh (BR) spectra are given in Refs. [1]-[3].
The first theoretical approach to the subject is a lapidal note published by
Landau and Placzek back in 1936 [4] which has been the inspiration of most
of the work done on this problem. Recently, these ideas have been extended
into the realm of astrophysical and cosmological contexts in relation with the
discussion of the old problems on gravitational instabilities and Jeans' mass
number [5] [6]. The recognized fact that Cosmic Microwave Background Ra-
diation (CMBR) photons interact with acoustic fluid modes in certain stages
of the universe evolution suggests that Brillouin scattering is potentially im-
portant to CMBR physics.

On the other hand, entropy production plays an important role in re-
alistic descriptions of thermodynamical processes, and some discussion of
its physical sources may yield interesting results in the analysis of the time
evolution of statistical fluctuations in astrophysical systems [11]. Recent
work [10] shows indeed that the critical Jeans wave number for gravita-
tional collapse is slightly modified by dissipative effects such as viscosity.
The formalism used in that work is also useful to obtain structure factors,
namely scattering laws that describe line broadenings and frequency shifts
associated to the interaction between the incoming radiation and the fluid
modes. The mathematical tool in this case is the so-called density-density
correlation function, taken in the frequency (Fourier) domain which is ob-
tained by standard techniques [2]-[3]. The influence of the gravitational field
in the Brillouin spectrum will become apparent from the analytic form of the
structure factor.

In this paper we wish to discuss two results. Firstly, that the presence of a
gravitational field produces a shift in the effective frequencies of the Brillouin
doublet in a simple fluid. The shift could be significant for large enough
densities such as those of matter in early stages of the Universe. Secondly, the
Brillouin peaks tend to condense into a single peak as the critical Jeans wave
number is approached. This implies that at certain densities, the thermal
fluctuations and the “mechanical dissipative modes” behave more or less in
the same way, a fact which, if detectable, should become manifest in the
anisotropies of the observed CMBR. The significance of this fact has not yet
been explored.

This paper is thus divided as follows. Section two is dedicated to a review
of the linearized form the Navier-Stokes equations in the presence of a grav-
itational field. Section three is devoted to the derivation of the scattering
law and its comparison with the ordinary BR spectrum. A brief discussion
of the implications of the results here obtained is included in section four.

2 Linearized Navier-Stokes system in the presence of a gravitational field

The standard system describing fluctuations of the thermohydrodynamical
variables in a simple non-reacting fluid consisting of particles of mass \( m \) in the
presence of a gravitational field, is derived by following the tenets of Linear
Irreversible Thermodynamics (LIT) \([2] [10]\). Indeed, the so-called Navier-
Stokes-Fourier equations of hydrodynamics for a simple fluid arise from the
structure of what is now called LIT by supplementing the two conservation
equations for mass and momentum, respectively and the balance equation
for the internal energy, with additional information. Indeed, these equations
read \([2] [6]-[7]\).

\[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho u^i) = 0\]  \(\text{(1)}\)

\[\rho D \frac{u^i}{Dt} + \frac{\partial \Xi^{ij}}{\partial x^j} = F^i\]  \(\text{(2)}\)

\[\rho D \frac{\varepsilon}{Dt} + \frac{\partial Q^j}{\partial x^j} = -\Xi^{ij} \frac{\partial u_i}{\partial x^j}\]  \(\text{(3)}\)

where \(D\frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i}\).

Here, \(\rho(x^j, t), u^i(x^j, t)\) and \(\varepsilon(x^j, t)\) are the local density, velocity and
internal energy, respectively, \(\Xi^{ij}\) the momentum current (or stress tensor) and
\(Q^j\) the heat flux. All indices run from 1 to 3. In general, for isotropic fluids,
\(\Xi^{ij} = p\delta^{ij} + \tau^{ij}\), where \(p\) is the local hydrostatic pressure and \(\tau^{ij}\)
the viscous tensor (\(\delta^{ij}\) is the unitary dyadic). Notice that Eqs. (1-3) contain
fifteen unknowns (including the pressure) and there are only five equations,
so the system is not well determined. If we arbitrarily choose to describe
the states of the fluid through the set of variables \(\rho(x^j, t), u^i(x^j, t), T(x^j, t)\),
where \(T\) is the local temperature, we need nine dynamic equations of state
(or constitutive equations) relating \(\Xi^{ij}\) and \(Q^j\) to the state variables plus
two local equations of state \(p = p(\rho, T)\) and \(\varepsilon = \varepsilon(\rho, T)\). According to the
tenets of LIT we choose the so-called linear constitutive laws, namely

\[ \tau^{ij} = -2\eta \sigma^{ij} - \zeta \delta^{ij} \]  (4)

\[ Q^j = -\kappa \delta^{ij} \frac{\partial T}{\partial x^i} \]  (5)

with \( \eta \) and \( \zeta \), the shear and bulk viscosities, respectively, \( \sigma^{ij} \) the symmetrical traceless part of the velocity gradient, \( \theta = \frac{\partial u^i}{\partial x^i} \) and \( \kappa \) being the thermal conductivity.

Eqs. (4) and (5) are the well known constitutive equations of Navier-Stokes and Fourier, respectively. Substitution of these equations into Eqs. (1-3) yields a set of second order in space, first order in time non-linear coupled set of partial differential equations for the chosen variables \( \rho, u^i \) and \( T \). This set, which the reader may seek in the literature [2]-[7], is the so called Navier-Stokes-Fourier system of hydrodynamic equations. The non-linearities appearing in such equations have two sources, the inertial terms \( u^i \frac{\partial}{\partial x^i} \) arising from the hydrodynamic time derivatives, plus quadratic terms in the gradients of velocity arising from Eqs. (4) and (5). Moreover, it should be mentioned that this set of equations is consistent with the second law of thermodynamics, the Clausius uncompensated heat, or entropy production, is strictly positive definite.

Nevertheless, for the purpose of this paper, this set of equations is too complicated. In order to deal with fluctuations around the equilibrium state, one assures that for any of two state variables, call them \( X(x^j, t) \) one can write that,

\[ X(x^j, t) = X_o + \delta X(x^j, t) \]  (6)

where \( X_o \) is the equilibrium value of \( X \) and \( \delta X \) the corresponding fluctuation. Neglecting all terms of order \( (\delta X(x^j, t))^2 \) and higher in the NSF non-linear set one finds the linearized NSF equations of hydrodynamics [2]-[7],

\[ \frac{\partial}{\partial t} (\delta \rho) + \rho \theta = 0 \]  (7)

\[ \rho_o \frac{\partial u_k}{\partial t} = -\frac{1}{\rho_o \kappa_T} \frac{\partial}{\partial x^k} (\delta \rho) - \frac{\beta}{\kappa_T} \frac{\partial}{\partial x^k} (\delta T) + 2\eta \nabla^2 u_k - \left( \frac{2}{3} \eta - \zeta \right) \frac{\partial}{\partial x^k} (\theta) + F_k \]  (8)

\[ \frac{\partial}{\partial t} (\delta T) = D_T \nabla^2 \delta T - \frac{\beta T}{\rho_o C_p \kappa_T} \theta \]  (9)
since \( u_{k,0} = 0 \), \( u_k = \delta u_k \) and \( \kappa_T = \frac{\beta}{C_\rho} \) are the thermal diffusivity. \( C_p \) and \( C_v \) are the specific heats at constant pressure and constant volume, respectively.

Note now that from Eq. (8), \( u_k \text{Trans} \equiv \text{Rot}(u_k) \) uncouples from the hydrodynamic modes, whence, one formally arrives at the desired set, namely:

\[
\frac{\partial}{\partial t} (\delta \rho) + \rho_o \theta = 0 \tag{10}
\]

\[
\rho_o \frac{\partial \theta}{\partial t} = -\frac{1}{\rho_o \kappa_T} \nabla^2 \delta \rho - \frac{\beta}{\kappa_T} \nabla^2 \delta T + D_v \nabla^2 \delta \theta - \nabla^2 \delta \varphi \tag{11}
\]

\[
\frac{\partial}{\partial t} (\delta T) = D_T \nabla^2 \delta T - \frac{\beta T_o}{\rho_o C_p \kappa_T} \theta \tag{12}
\]

assuming that \( F_k = -\frac{\partial \varphi}{\partial x_k} \), where \( \varphi \) is the gravitational potential. Also, \( D_v = \left( \frac{2}{3} \eta + \zeta \right) \frac{1}{\rho_o} \). Now, we notice that \( \frac{\partial}{\partial t} (\delta \rho) = -\rho_o \theta \) so we can reduce this set to only two equations, Eq.(11) and Eq.(12). We now perform a few minor transformations introducing the speed of sound of the fluid \( C_o^2 = \left( \frac{\partial p}{\partial \rho} \right)_s \) through the relationship

\[
\kappa_T = C_p \kappa_s = \frac{C_p}{C_v \kappa_s} \tag{13}
\]

where \( k_s \) is the adiabatic compressibility \( \left( \frac{\partial \rho}{\partial p} \right)_s \) whence, \( k_T = \frac{\gamma}{\gamma - 1} \). This leaves us finally with the set:

\[
- \frac{\partial^2 (\delta \rho)}{\partial t^2} + \frac{C_o^2}{\gamma} \nabla^2 \delta \rho + \frac{C_o^2 \beta}{\gamma} \nabla^2 \delta T + D_v \nabla^2 \left( \frac{\partial}{\partial t} (\delta \rho) \right) + \rho_o \nabla^2 \delta \varphi = 0 \tag{14}
\]

\[
\frac{\partial}{\partial t} (\delta T) - D_T \nabla^2 \delta T - \frac{\gamma - 1}{\beta} \frac{\partial}{\partial t} (\delta \rho) = 0 \tag{15}
\]

where we have used that \( C_p - C_v = \frac{\beta^2 T_o}{\rho_o^2 \kappa_T} \).

Eqs.(14-15) form a set of coupled equations for the density and temperature fluctuations in the fluid under the action of a conservative force whose nature need not to be specified for the time being. They are the basis for studying the properties of the time correlation functions of thermodynamic fluctuations. Those of the density will be of particular interest here.
3 Solution to the hydrodynamic equations

The results derived in the previous section are far from being new. Aside from the term $\nabla^2 \phi$ which arises from the presence of an external conservative force, they are identical to the ones that have been widely discussed in the literature. The question here is if the gravitational potential introduces any substantial modification in the correlation functions for the thermodynamic fluctuations. To examine this possibility we recall that, if we are considering only the fluctuations, the gravitational potential satisfies the Poisson equation:

$$\nabla^2 \delta \phi = -4\pi G \delta \rho$$  \hspace{1cm} (16)

Now, equation (15) reads

$$\nabla^2 \delta \rho = - \frac{\partial^2 (\delta \rho)}{\partial t^2} + \frac{C_o^2}{\gamma} \nabla^2 \delta \rho + \frac{C_o^2 \beta}{\gamma} \nabla^2 \delta T + D_v \nabla^2 (\frac{\partial}{\partial t} (\delta \rho)) - 4\pi G \rho_o \delta \rho = 0$$  \hspace{1cm} (17)

The introduction of the Poisson equation links fluctuations in the gravitational potential with density fluctuations. The solution to Eqs. (15) and (17) proceeds in the standard fashion. We reduce them to a set of algebraic equations by taking their Laplace-Fourier transform, choose to set the static temperature fluctuations equal to zero, and eliminate the temperature leading to an equation for $\delta \rho(\vec{k}, s)$, which is the ratio of two polynomials in $s$. In fact, one gets that

$$\frac{\delta \rho(\vec{k}, s)}{\delta \rho(\vec{k}, 0)} = \frac{s^2 + (D_v + D_t) k^2 s + k^4 D_v D_t + C_o k^2 (1 + \frac{1}{\gamma})}{(s + D_t k^2)(s^2 + D_v k^2 s + C_o^2 k^2 - 4\pi G \rho_o)}$$  \hspace{1cm} (18)

To compute $\delta \rho(\vec{k}, t)$ one must take the inverse Laplace transform of the former quantity, which demands the knowledge of the roots of the denominator, which is a cubic equation in $s$ (dispersion equation).

The analysis leading to Eq. (18) clearly points out the fact that the solution to the cubic equation giving rise to the poles of the function $\delta \rho(\vec{k}, s)$, is exact. This is an improvement over the current version in the literature [2]-[3], [7] asserting that the roots are only approximate to order $k^2$. There are two immediate implications of this result, namely, Rayleigh's peak is not affected by the gravitational field within the Navier-Stokes regime and the Jeans number, thoroughly discussed in Ref. [10] cannot be affected by thermal
dissipation, the thermal conductivity will never appear in its definition, as will be pointed out below. The former result appears to be in contradiction with previous work stating that Rayleigh’s peak is modified by a constant gravitational force [8]. Yet, notice that this is a \( k^4 \) effect, which is beyond the Navier-Stokes domain. Work along this line with the Burnett and SuperBurnett equations is in progress. So, we shall leave the discussion of this subtlety for the future.

Returning to Eq. (18), the two roots of the quadratic equation in the denominator are:

\[
s_{1,2} = -\frac{D_v k^2}{2} \pm i \left[ \left( C_o^2 k^2 - 4\pi G \rho_o \right) - \frac{D_v^2 k^4}{4} \right]^{1/2}
\]  

(19)

If \( 4\pi G \rho_o = 0 \) and viscosity dominates over the term in \( k^2 \), this result reduces to the one giving rise to the standard Brillouin peaks which correspond to density fluctuations of the type [2] [7],

\[
\delta \rho (\vec{k}, t) = \delta \rho (\vec{k}, 0) \frac{1}{\gamma} e^{-D_v k^2 t} \cos [C_o k t]
\]  

(20)

which are the acoustic modes damped by the Stokes-Kirchhoff factor \( D_v \).

On the other hand, if \( 4\pi G \rho_o \neq 0 \), the threshold value for \( k \) distinguishing between damped oscillations and growing modes is given by

\[
\left( C_o^2 k^2 - 4\pi G \rho_o \right) - \frac{D_v^2 k^4}{4} = 0
\]  

(21)

or,

\[
k^2 = \frac{2C_o^2}{D_v^2} \left( 1 \pm \sqrt{1 - \frac{4\pi G \rho_o D_v^2}{C_o^4}} \right)
\]  

(22)

Eq. (22) is a generalization of Jeans wave number when dissipative effects due to viscosity are non-negligible, and is the main result of the paper. We can note that, if \( \frac{4\pi G \rho_o D_v^2}{C_o^4} \ll 1 \), and taking the – sign for the square root, we have:

\[
k^2 \approx \frac{2C_o^2}{D_v^2} \left[ 1 - 1 + \frac{1}{2} \left( \frac{4\pi G \rho_o D_v^2}{C_o^4} \right) \right]
\]  

(23)

or

\[
k^2 \approx \frac{4\pi G \rho_o}{C_o^2} = K_j^2
\]  

(24)
This is the square value of the Jeans wave number. It was derived by Jeans in 1902 and rederived in many other waves by several authors. Here we simply show that it is almost a trivial consequence of LIT. Jeans wave number has been used by cosmologists to estimate the minimum mass required to form a galaxy. This question has been discussed at length in Ref. [9].

Coming back to our main result, Eq. (18), it is clear that damped acoustic waves are associated to the case in which the dispersion equation has complex roots. The corresponding expression for the density fluctuations in the \((k, t)\) space, neglecting the last term in the square root of Eq. (19), is then given by:

\[
\delta \rho(k, t) = \delta \rho(k, 0) \left\{ \left( 1 - \frac{1}{\gamma} \right) e^{-D_T k^2 t} + \frac{1}{\gamma} e^{-\Gamma k^2 t} \cos \left[ \left(C_o^2 k^2 - 4\pi G \rho_o \right)^{1/2} t \right] \right\}
\]

(25)

Eq. (25) represents damped acoustic waves propagation with an effective frequency \((C_o^2 k^2 - 4\pi G \rho_o)^{1/2}\). When gravity is negligible, one recovers the usual frequency \(\omega = \pm C_o k\). Notice, however, that although \(G\) is a small number, it is multiplied by the density \(\rho_o\). In fact, it may turn that for a critical value of \(\rho_o\) the damped modes are replaced by growing modes (Jeans instability).

4 Structure factor and modified Brillouin spectrum

Following the standard dynamic radiation scattering theory [2]-[3], one may then construct the density-density correlation function from Eq. (25). This leads to the expression:

\[
\frac{\langle \delta \rho^*(k, 0) \delta \rho(k, t) \rangle}{\langle \delta \rho^*(k, 0) \delta \rho(k, 0) \rangle} = \left( 1 - \frac{1}{\gamma} \right) e^{-D_T k^2 t} + \frac{1}{\gamma} e^{-\Gamma k^2 t} \cos \left[ \left(C_o^2 k^2 - 4\pi G \rho_o \right)^{1/2} t \right]
\]

(26)

where the brackets \(< >\) indicate an average over an equilibrium ensemble.

The Brillouin specific intensity of the scattered light due to its interaction with the acoustic modes of a fluid is obtained by Fourier’s transform of Eq. (26). The ensuing expression reads:
\[ S_{\rho \rho}(k, \omega) = \frac{1}{\pi} \left( 1 - \frac{1}{\gamma} \frac{D_{\rho}}{D_{\omega}^2 k^4} + \frac{1}{\gamma} \left( \frac{\Gamma}{\omega - (C_o^2 k^2 - 4\pi G \rho_o)^{1/2}} \right)^2 + \Gamma^2 k^4 \right) \]

(27)

The last two terms in Eq. (27) correspond to the Brillouin doublet and deserve some attention. This doublet reflects a shift in frequency governed by the speed of sound of the fluid, the wave number of the incoming radiation, and the effect of gravity. This last effect can be roughly estimated by performing a binomial expansion of the square root included in the effective wave frequency. Indeed, let \( \omega_p \), the modified frequency be defined as:

\[
\omega_p = \pm \left( C_o^2 k^2 - 4\pi G \rho_o \right)^{1/2} \simeq C_o k \left( 1 + \frac{2\pi G \rho_o}{C_o^2 k^2} \right)
\]

(28)
to first order in \( G \rho_o \). Thus, if we now call \( |\Delta \nu_p| = \frac{1}{2\pi} |\omega - \omega_p| \) we see that

\[
|\Delta \nu_p| = \frac{G \rho_o}{C_o k}
\]

(29)
and the relative change in the frequency Brillouin peaks is, roughly:

\[
\frac{|\Delta \nu_p|}{C_o k} = \left( \frac{G \rho_o c^2}{4\pi^2 C_o^2} \right) \frac{1}{\nu}
\]

(30)
if using \( k = \frac{2\pi}{\lambda} = \frac{2\pi}{c} \nu \). Written in this way the frequency exhibits a much subtler dependency on other variables such as the temperature, which implicitly appears in \( C_o \).

5 Final remarks

A rather interesting result appears in Eq. (27) if \( k = k_J = \frac{4\pi G \rho_o}{C_o^2} \), the Jeans wave number. In that case, \( \delta \rho(k, t) \) is a simple decaying exponential with time, and the Brillouin-Rayleigh spectra reduces to the ordinary Rayleigh peak superposed to a similar one arising from the collapse of the Brillouin peaks, suggesting that the Stokes-Kirchhoff dissipative factor behaves in a way similar to the thermal fluctuations. This collapse of the Brillouin peaks would cause a line broadening in scattering situations and would reflect a
competition between entropy fluctuations \( (D_T) \) and mechanical fluctuations \( (D_v) \). Whether or not this result is of any relevance at all remains to be tested.

A final remark concerns Eq. (29). It is well-known that for typical electromagnetic radiation, the frequencies are rather high, even radio frequencies. This is not the case for gravitational waves, whose wavelengths are very long. One could then speculate about the possibility of having gravitational radiation Brillouin scattered by cosmological matter. This would enhance the relative change in frequency of the Brillouin peaks.

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References

[1] R.M. Mountain, Rev. Mod. Phys. 38, 205 (1966);
[2] B.J. Berne and R. Pecora, Dynamic Light Scattering (Dover, N.Y., 2000);
[3] J.P. Boon and S. Yip, Molecular Hydrodynamics (Dover, N.Y., 1991).
[4] L. Landau and G. Placzek, Phys. Zeit. Sow. 5, 172 (1934). Also included in, The Collected Papers of L.D. Landau, ed. D. ter Haar (Pergamon Press, London, 1962).
[5] A. Sandoval-Villalbazo and R. Maartens (2001) astro-ph/0105323
[6] S.R. de Groot and P. Mazur, (1984) Non-Equilibrium Thermodynamics, Dover, N.Y., USA.
[7] L.S. García-Colín, Termodinámica de los Procesos Irreversibles (UAM-Iztapalapa, México D.F.1990) (in spanish).
[8] P.M. Sergh, R. Schmitz and J.V. Sengers, Physica A 195, 31 (1993).
[9] E.W. Kolb and M.S. Turner, (1990) The Early Universe Addison-Wesley, Reading, MA., USA.
[10] A. Sandoval-Villalbazo and L.S. García-Colín, Class. Quantum. Grav. 19 (2002) 2171 astro-ph/0110295
[11] S. Weinberg. Ap. J. 168, 175 (1971).
[12] A. Sandoval-Villalbazo; Physica A 313, 456 (2002).