Penrose Tilings as Jammed Solids

Olaf Stenull and T. C. Lubensky
Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104, USA
(Dated: June 25, 2014)

Penrose tilings form lattices, exhibiting 5-fold symmetry and isotropic elasticity, with inhomogeneous coordination much like that of the force networks in jammed systems. Under periodic boundary conditions, their average coordination is exactly four. We study the elastic and vibrational properties of rational approximants to these lattices as a function of unit-cell size $N_S$ and find that they have of order $\sqrt{N_S}$ zero modes and states of self stress and yet all their elastic moduli vanish. In their generic form obtained by randomizing site positions, their elastic and vibrational properties are similar to those of particulate systems at jamming with a nonzero bulk modulus, vanishing shear modulus, and a flat density of states.

Lattices of sites connected by central-force springs provide useful models of mechanical systems as diverse as bridges, elastic solids, and granular packings. Their stability depends critically on their average coordination $z = 2N_B/N_S$, where $N_B$ is the number of bonds and $N_S$ the number of sites in the lattice. Finite lattices with $z = z_c = 2d-(d(d+1))/N_S$ and lattices under periodic boundary conditions with $z = z_c = 2d$ are what we will call Maxwell lattices [1,2]. They are critical networks that lie at the boundary between being mechanically stable and mechanically unstable, and they control the phonon structure and elasticity of nearby stable lattices in which $z$ increases above $z_c$ [3, 4]. Maxwell Lattices and their generalizations to include bond bending play an important role in engineering structures [3, 4], rigidity percolation [7, 8], the glass transition [9, 10], biopolymer gels [11], and randomly packed spheres near the jamming transition [12, 13]. The latter systems are macroscopically isotropic and are characterized by a nonzero bulk modulus and a vanishing shear modulus, which increases with $z > z_c$. Small unit-cell periodic Maxwell lattices have also been studied [14–17] but, though like jamming systems they are characterized by lengths and inverse frequencies that diverge as $z \to z^+_c$, none exhibits a nonzero bulk and vanishing shear modulus at $z = z_c$.

Here we introduce the 5-fold symmetric quasi-crystalline Penrose tiling [21], constructed from rhombic tiles as shown in Fig. 1 with local coordination number ranging from three to as high as ten with an average of four, as an elastically isotropic Maxwell lattice, which can be approached via a series of rational approximants with increasing $N_S$, each of which is a Maxwell lattice. We study the elastic and vibrational properties of rational approximants as the Penrose limit is approached. All elastic moduli in all of these approximants, like those at the rigidity percolation threshold [5, 8], are zero, and their phonon spectra, like those of the square and kagome lattices, have of order $\sqrt{N_S}$ zero modes. When site positions are randomized (rendering the lattice “generic” [22, 24]), the properties of these approximants are almost identical to those of jammed packings of increasing size [22]: their bulk modulus is nonzero for all $N_S$, their shear moduli approach zero as $N_S \to \infty$, their integrated density of states is linear in frequency, and they exhibit quasi-localized modes [26]. Thus, generic Penrose tilings provide a model system, whose construction is unhampered by dependence on numerical protocols [27] or need for precise equilibration [25, 27], that are in some sense in the same “universality class” as jammed systems even though their detailed site-bond topology is quite different from that of jammed systems and their method of preparation does not guarantee a nonzero bulk modulus. We will argue that the shear modulus must vanish in any isotropic Maxwell lattice with $B > 0$.

We create our Penrose networks using the standard projection procedure [28] from the five-dimensional hypercube $Z^5$ onto the 2-dimensional physical space. Proper choice of the orientation of physical space within hyperspace leads to the truly quasiperiodic rhombus tiling. This orientation can be expressed in terms of the golden ratio $\tau$. Approximating $\tau$ by ratios $\tau_n$ of succes-
sive Fibonacci numbers, \( \tau_1 = 1/1, \tau_2 = 2/1, \tau_3 = 3/2 \), and so on, produces periodic approximants, composed of 4-sided rhombohedral tiles arranged in rectangular unit cells of increasing size, that approach the 5-fold Penrose tiling as \( n \rightarrow \infty \). The Euler relation \( N_S - N_B + N_F = 0 \) for a torus, where \( N_F \) is the number of faces (or plaquettes), ensures that \( z = 4 \) under periodic boundary conditions for any tiling with four-sided tiles. This follows because each bond is shared by two plaquettes, implying \( N_B = 2N_F \) and \( N_S = N_F = N_B/2 \). The number of sites, \( N_S^g \), and bonds, \( N_B^g \), in the unit cell of the \( n \)th periodic approximant obey the recursion relation

\[
N_{S,B}^{n+1} = 3 N_{S,B}^n - N_{S,B}^{n-1}
\]

with \( N_S^1 = N_B^1/2 = 30 \) and \( N_S^2 = N_B^2/2 = 80 \).

Periodic approximants (Fig. 1) shows the unit cell of the \((1/2)\)-approximant as an example) allow the natural application of periodic boundary conditions. We study the first eight periodic approximants ranging from \( \tau_8 = 34/21 \), which have 30 to 25840 sites. Assuming that the bonds are harmonic central-force springs, the total elastic energy of the so-obtained Penrose network with unit spring constant is

\[
E = \frac{1}{2} \sum_b s_b^2, \quad s_b = C_{bi} u_i,
\]

where the sum runs over all bonds, \( s_b \) is the stretch of bond \( b \), \( u_i \) the elastic displacement of site \( i \), \( C_{bi} \) the corresponding component of the compatibility matrix \( C \) [29].

Along with the compatibility matrix, the equilibrium matrix \( Q = C^T \) relating bond tensions \( t_b \) to site forces \( f_i \) via \( Q_{ib} t_b = f_i \) plays an important role in determining the nature of modes, particularly zero modes, in a lattice. The full vibrational spectrum is determined by the dynamical matrix \( D = QC \) (for site mass \( m = 1 \)). A zero mode is one that changes the positions of sites but not the length of bonds, and the total number \( N_0 \) of zero modes is simply the dimension of the nullspace of \( C \). A set of bond tensions that do not impose forces on sites is called a state of self stress [29], and the dimension of the null space of \( Q \) is the number \( S \) of states of self-stress. The rank-nullity theorem of linear algebra implies the generalized Maxwell relation [29]

\[
N_0 = 2N_S - N_B + S.
\]

Because \( N_B = 2N_S \), the 5-fold Penrose lattice and each of its rational approximants has \( N_0 = S \). This general relation is born out by our actual calculations of eigenmodes and states of self stress. Our results for \( N_0 \) and \( S \) as functions of \( N_S \) are compiled in Fig. 2. We find that \( N_0 \) and \( S \) roughly scale as \( \sqrt{N_S} \), as is the case in small-unit-cell periodic lattices such as the square and kagome lattices [13], whose states of self-stress arise from straight lines of bonds.

![Fig. 2. (color online) Number of zero modes and states of self-stress. The (red solid) line \( 0.7N_S^{1/2} \) is a guide to the eye.](image)

With modern computers, it is fairly straightforward to find the set of \( S \) vectors that provide a basis for the states of self stress. Finding a particular orthonormal basis whose elements provide a simple geometric visualization of the origins of the self stress is less so. In a careful study of the 8 states of self stress for the \((2/1)\)-approximant, we were able to show that six of the states are ladder configurations, three of which are shown in Fig. 1 and two were more uniformly spread throughout the cell. Unlike the states of self stress in the square and kagome lattice, whose tensions are all of the same sign, these exhibit both positive and negative tension, an important property as we shall soon see.

To study the mode frequencies and their spatial distribution, we diagonalize the dynamical matrix \( D \) to obtain the frequency \( \omega_m \) and normalized polarization vector \( p_{mi} \) of each site \( i \) in mode \( m \). From these quantities, we extract the integrated density of vibrational states \( \rho_{int}(\omega) \) and the participation ratio

\[
P(\omega_m) = \frac{(\sum_i |p_{mi}|^2)^2}{N_S \sum_i |p_{mi}|^2}
\]

which ranges from 0 when a mode is fully localized to 1 when a mode is fully extended. Fig. 3 shows our results for the \((13/8)\)-approximant. The results for the other approximants are similar. Rather than exhibiting Debye behavior \( (\rho_{int} \sim \omega^2) \) at small \( \omega \), \( \rho_{int}(\omega) \sim \omega \), indicating a flat non-integrated density of states as has been observed at the jamming transition [30] as well as in the square and kagome lattices [14]. \( \rho_{int}(\omega) \) remains approximately linear in \( \omega \) over the entire frequency range with deviations resembling a devil’s staircase [31]. The participation ratio, Fig. 3(b) shows that the zero modes, except for a few that include the trivial modes (rigid translations of the network) are localized or quasi localized. At intermediate frequencies, the modes are fairly extended, and at higher frequencies they become more localized, again as is the case near the jamming transition [30].

States of self stress determine the linearized macroscopic elastic energy in terms of the symmetrized strain \( u_{ij} \) [32]. Let the \( u_{ia}^\alpha, \alpha = 1, \ldots, S \) be components of the
to six independent elastic constants (in two dimensions). The generic (blue squares) Penrose network. The (red solid) lines in (a) with slope 1/2 are guides to the eye. The zero modes in (b) have been given a small nonzero frequency for displaying purposes so that they appear at the left boundary of the log-linear plot.

Orthogonalized basis vectors of the nullspace of $Q$. Then the elastic energy is

$$E = \frac{1}{2} \sum_{\alpha} \left( \sum_{b} t_{\alpha}^{b} s_{\alpha}^{b} \right)^{2} = \frac{1}{2} K_{ijkl} u_{ij} u_{kl},$$  \hspace{1cm} (5)$$

where $K_{ijkl}$ is the tensor of elastic moduli (times the cell volume) and $s_{\alpha}^{b} = \hat{e}_{b} u_{ij} e_{bj}$ is the affine stretch of bond $b$ where $\hat{e}_{b}$ is the unit vector along that bond. Since there are three independent strains, $K_{ijkl}$ can be expressed as a $3 \times 3$ elastic matrix $K$ (in the Voigt notation) with six independent components leading in general to six independent elastic constants (in two dimensions). For complete elastic stability, all three eigenvalues of the Voigt elastic matrix must be positive. From Eq. (5), it is clear that there must be at least three $(d(d + 1)/2$ in $d$-dimensions) states of self stress for complete elastic stability. With $S \propto \sqrt{N_S}$, there are more than enough states of self stress to completely stabilize elastic distortions of the non-generic Penrose approximants. Nevertheless, all elastic moduli are zero for each of them. This is because the overlap of each of their states of self-stress, which have both positive and negative tensions [Fig. 11], with the affine bond strains is zero. This is in contrast to the kagome lattice, for example, which has three states of self stress at zero wavemeter and an elastic matrix with three positive eigenvalues $[10]$

Particulate systems at the jamming transition are amorphous and generic and hence, in general, free of any but the $d$ translational zero modes (and “rattlers”, which can be removed). To model this property in Penrose tilings, we follow Ref. [9] and introduce small random site displacements to produce “generic networks” with random bond lengths and bond angles without affecting local topology. These local distortions eliminate all but the two translational zero modes and their two associated states of self-stress and thus make the generic Penrose network a more realistic model for amorphous materials such as jammed matter.

To study the connection of the Penrose network with the jamming transition in greater depth, we consider in the following not only generic Penrose networks with $N_B = 2N_S$, but also versions of them with 1 or 2 bonds removed or 1 added somewhere in the network. Δ = $N_B - 2N_S$ measures the under or over coordination relative to the Maxwell lattice. From the eigenmodes and states of self-stress, we find that $N_S = 2$ (the trivial modes) independent of $N_S$ and $\Delta$, and $S = 2 + \Delta$ in full agreement with relation $[33]$. The integrated density of states for the generic network with $\Delta = 0$ is practically indistinguishable from that of the non-generic case, see Fig. 3 (a). In particular, we once again obtain jamming-like $\rho_{int}(\omega) \sim \omega$ for small $\omega$. The participation ratio for $\Delta = 0$, of course, differs from that of the non-generic case, because there are fewer zero modes. In Fig. 3 (b), we see that all but the trivial zero modes get lifted to finite-frequency, quasi-localized modes. In the vicinity of the jamming transition, the non-trivial low-frequency modes are also quasi-localized albeit the number of quasi-localized modes relative to the total number of modes is significantly higher there $[33]$. Next, we calculate the bulk and shear moduli of the generic Penrose network. Because we expect the average generic network to exhibit isotropic elasticity in the $N_S \rightarrow \infty$ limit, we express strain in terms of its compression, pure-shear, and simple-shear components $U = (u_{xx} + u_{yy}, u_{yy} - u_{xx}, 2u_{xy})$, and we calculate the $3 \times 3$ elastic matrix $K$ in this basis for each random configuration. In the isotropic limit, this matrix becomes $\text{diag}(B, G, G)$ where $B$ is the bulk modulus, and $G$ is the isotropic shear modulus. We verify that all of the states of self stress in the network have non-zero overlap with the vector of affine strains for $\Delta = -1, 0, 1$. We then calculate the eigenvalues of $K$ for each random configuration and verify that $2 + \Delta$ of them are positive. The largest eigenvalue in all cases very quickly corresponds with increasing $N_S$ almost exactly to pure compressions, and we identify this eigenvalue with the bulk modulus. When $\Delta = -2$, all moduli are zero, when $\Delta = -1$, the bulk modulus is the only nonzero eigenvalue. When $\Delta = 0$, there is a second positive eigenvalue, $G_1$ which corresponds to some combination of pure and simple shear depending of the random configuration; and when $\Delta = 1$, there are two shear moduli $G_1$ and $G_2$. In
the $N_S \rightarrow \infty$, isotropy requires $G_1 = G_2$. Thus, when
$\Delta = 0$, $G_1$ must tend to zero with $N_S$ because $G_2$ is
identically zero. This is a general property of periodic
Maxwell lattices that approach isotropy with increasing
unit-cell size. Adding an extra bond does generally does not
cause a discontinuous change in this picture, and
if it does not, both $G_1$ and $G_2$ approach zero with $N_S$.
Figure 4 displays the averages of $B$, $G_1$, and $G_2$ over
configurations as a function of $N_S$ for small for $\Delta = -1, 0, 1$.
At small $N_S$, $B$ undergoes a change of about a factor of
10 from $\Delta = -1$ to $\Delta = 1$. As $N_s$ increases, $B$ in-
creases in all cases, reaching a plateau at large $N_S$ with
$B(\Delta = 1) \approx B(\Delta = 0)$ and $B(\Delta = -1)$ a factor of about
4 smaller. In all cases, the shear modulus approaches
zero as $1/N_S$. For completeness, we note that we also
looked at generic Penrose networks with $\Delta$ beyond 1 and
observed elastic moduli that were qualitatively the same
as for $\Delta = 1$. We also calculated the average $(\textbf{K})$ of
the elastic matrix over all configurations for each $\Delta$ and
$N_S$. Because of the nonlinear relation between $\textbf{K}$ and random
displacements, $(\textbf{K})$ generally exhibited 3 rather than the
$2 + \Delta$ positive eigenvalues exhibited by each configura-
tion of $\textbf{K}$. The difference between the average of bulk
moduli calculated from $\textbf{K}$ and the bulk modulus of the
$(\textbf{K})$ was not generally significant but that for the shear
moduli was.

Our counting based on the index theorem [Eq. 3] and
the relation between the states of self stress and the elas-
tic energy [Eq. 5] agrees with those obtained in the con-
text of jamming through requiring these packings to be
stable with respect to shape as well as volume change [27]
and through studies of finite size effects in these packings
[29, 31]. In all cases, the bond excess $\Delta = N_B - 2N$ in
2d generic lattices under periodic boundary conditions re-
quired to ensure stability with respect to volume change
only, to volume change and one shear, and to all uniform
elastic distortions is $\Delta = -1, 0, 1$, respectively. These
results are equivalent to the observation that a lattice
under periodic boundary conditions cannot be both stat-
ically and kinematically determinate [35].

Penrose tilings and their rational approximants are
Maxwell lattices that approach elastic isotropy as $N_S \rightarrow \infty$. Remarkably, in their original form, all of their elastic
moduli are zero, but in their randomized generic form,
they, like packed spheres near jamming, have nonzero
bulk moduli and shear moduli that vanish as $N_S \rightarrow \infty$ in
spite of their not being specifically constrained to sup-
port isotropic loads. Their vibration eigenmodes are also
similar to those at the jamming transition. We have
done not carried out systematic investigation of the effects of
adding an extensive number of bonds, but we expect that
doing so will have an effect similar to increasing $z$ near
jamming. Our study focussed on the systems with sin-
gle unit cells under periodic boundary conditions, i.e.,
restricting our attention to the zero-wavenumber limit of
periodically repeated unit cells. The latter has modes at
dl all wavenumbers with zero modes at each wavenumber in
the nongeneric lattices. The original Penrose tilings are
critical lattices, similar in many respects to the kagome
lattice with $\sim \sqrt{N_S}$ states of self stress that can be re-
moved by even infinitesimal displacements of sites. This
raises the possibility that controlled displacements could
lead to different topological classes [17] with associated
zero surface modes. Three-dimensional Penrose tilings
[36] are also Maxwell lattices, and they should have prop-
erties similar to their 2d cousins. Finally, we note that
quasicrystals lie at the boundary between periodic crys-
tals and glasses, and it is intriguing that slight random-
ization of position of lattice sites leads to glassy-like be-
havior.

We thank C. P. Goodrich and A. J. Liu for helpful
discussions. This work was supported by the NSF under
grants DMR-1104707 and DMR-1120901.

[1] J. C. Maxwell, Philosophical Magazine 27, 294 (1864).
[2] The term isostatic is often incorrectly used to describe
any system with $z = z_c$. Finite Isostatic lattices have $z = z_c$
and no states of self stress - See Ref. 29 for example.
There is no universally accepted definition of isostatic
lattices with periodic boundary conditions, but one
such as the non-generic Penrose tilings with many states
of self-stress is surely not isostatic, though the generic
lattices might reasonably be labelled isostatic.
[3] M. Wyart, S. R. Nagel, and T. A. Witten, Euro-
phys. Lett. 72, 486 (2005).
[4] M. Wyart, Annales De Physique 30, 1 (2005).
[5] J. Heyman, The Science of Structural Engineering
(Cengage Learning, Stamford CT, 2005).
[6] A. Kassimali, Structural Analysis (Cengage Learning,
Stamford CT, 2005).
[7] S. Feng and P. N. Sen, Phys. Rev. Lett. 52, 216 (1984).
[8] D. J. Jacobs and M. F. Thorpe, Phys. Rev. Lett. 75, 4051
(1995).
[9] J. C. Dyre, Rev. Mod. Phys. 78, 953 (2006).
[10] L. Berthier and G. Biroli, Rev. Mod. Phys. 83, 587
C. P. Broedersz and F. C. MacKintosh, Reviews of Modern Physics, ??, ?? (2014).

A. J. Liu and S. R. Nagel, “The jamming transition and the marginally jammed solid,” in Annual Review of Condensed Matter Physics, Vol 1 (2010) pp. 347–369.

A. J. Liu and S. R. Nagel, Soft Matter 6, 2869 (2010).

A. Souslov, A. J. Liu, and T. C. Lubensky, Phys. Rev. Lett. 103, 205503 (2009).

X. M. Mao, N. Xu, and T. C. Lubensky, Phys. Rev. Lett. 104, 085504 (2010).

K. Sun, A. Souslov, X. M. Mao, and T. C. Lubensky, PNAS 109, 12369 (2012).

C. L. Kane and T. C. Lubensky, Nature Physics 10, 39 (2014).

D. Levine and P. J. Steinhardt, Phys. Rev. Lett. 53, 2477 (1984).

P. J. Steinhardt and S. Ostlund, The Physics of Quasicrystals (World Scientific, Singapore, 1987).

D. DiVincenzo and P. J. Steinhardt, Quasicrystals - The State of the Art (World Scientific, Singapore, 1991).

P. Roger, Bull. Inst. Math Appl. 10, 266 (1974).

There have been a number of studies of the excitation spectrum of quasicrystals, but most either concentrate on scalar (electronic) excitations or on lattices with $z > 2d$. We have not found a reference that treats vector phonons in a lattice with $z = 2d$. For a review, see for example, M. Quilichini and T. Janssen, Rev. Mod. Phys. 69, 277 (1997).

E. Guyon, S. Roux, A. Hansen, D. Bideau, J. P. Trodaced, and H. Crapo, Reports on Progress in Physics 53, 373 (1990).

D. J. Jacobs and M. F. Thorpe, Phys. Review E 53, 3682 (1996).

C. P. Goodrich, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. 109, 095704 (2012).

C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Review E 68, 011306 (2003).

S. Dagois-Bohy, B. P. Tighe, J. Simon, S. Henkes, and M. van Hecke, Phys. Rev. Lett. 109, 095703 (2012).

N. D. Bruijn, Proc. K. Ned. Acad. Wet. Ser. A 43, 39 (1981).

C. R. Calladine, Int. J. Solids and Structures 14, 161 (1978).

L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. 95, 098301 (2005).

P. Bak, Reports on Progress in Physics 45, 587 (1982).

S. Pellegrino, Int. J. Solids and Structures 30, 3025 (1993).

N. Xu, V. Vitelli, A. J. Liu, and S. R. Nagel, EPL 90, 56001 (2010).

C. P. Goodrich, S. Dagois-Bohy, B. P. Tighe, M. van Hecke, L. A. J., and S. R. Nagel, xxx xxx, xxx (2014).

S. D. Guest and J. W. Hutchinson, J. Mech. and Phys. of Solids 51, 383 (2003).

D. Levine and P. J. Steinhardt, Phys. Rev. B 34, 596 (1986).