Stochastic integrability of heat-kernel bounds for random walks in a balanced random environment

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Abstract

We consider random walks in a balanced i.i.d. random environment in \( \mathbb{Z}^d \) for \( d \geq 2 \) and the corresponding discrete non-divergence form difference operators. We first obtain an exponential integrability of the heat kernel bounds. We then prove the optimal diffusive decay of the semigroup generated by the heat kernel for \( d \geq 3 \). As a consequence, we deduce a functional central limit theorem for the environment viewed from the particle.

1 Introduction

In this article we consider random walks in a balanced i.i.d. random environment in \( \mathbb{Z}^d \) for \( d \geq 2 \).

1.1 Settings

Let \( S_{d \times d} \) be the set of \( d \times d \) positive-definite diagonal matrices. A map

\[ \omega : \mathbb{Z}^d \rightarrow S_{d \times d} \]

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is called an environment. Denote the set of all environments by \( \Omega \) and let \( \mathbb{P} \) be a probability measure on \( \Omega \) so that

\[
\{ \omega(x) = \text{diag}[\omega_1(x), \ldots, \omega_d(x)], x \in \mathbb{Z}^d \}
\]

are i.i.d. under \( \mathbb{P} \). Expectation with respect to \( \mathbb{P} \) is denoted by \( \mathbb{E} \) or \( E_\phi \).

Let \( \{e_1, \ldots, e_d\} \) be the canonical basis for \( \mathbb{R}^d \). For a given function \( u : \mathbb{Z}^d \to \mathbb{R} \) and \( \omega \in \Omega \), define the non-divergence form difference operator

\[
\text{tr}(\omega(x)\nabla^2 u) = \sum_{i=1}^{d} \omega_i(x)[u(x + e_i) + u(x - e_i) - 2u(x)],
\]

where \( \nabla^2 = \text{diag}[\nabla^2_{1}, \ldots, \nabla^2_{d}] \), and \( \nabla^2_i u(x) = u(x + e_i) + u(x - e_i) - 2u(x) \).

For \( r > 0 \), \( y \in \mathbb{R}^d \) we let

\[
\mathbb{B}_r(y) = \{ x \in \mathbb{R}^d : |x - y| < r \}, \quad \mathbb{B}_r = \mathbb{B}_r(y) \cap \mathbb{Z}^d
\]

denote the continuous and discrete balls with center \( y \) and radius \( r \), respectively. When \( y = 0 \), we also write \( \mathcal{B}_r = \mathcal{B}_r(0) \) and \( \mathbb{B}_r = \mathbb{B}_r(0) \). For any \( B \subset \mathbb{Z}^d \), its discrete boundary is the set

\[
\partial B := \{ z \in \mathbb{Z}^d \setminus B : \text{dist}(z, x) = 1 \text{ for some } x \in B \}.
\]

Let \( \bar{B} = B \cup \partial B \). Note that with abuse of notation, whenever confusion does not occur, we also use \( \partial A \) and \( \bar{A} \) to denote the usual continuous boundary and closure of \( A \subset \mathbb{R}^d \), respectively.

For \( x \in \mathbb{Z}^d \), a spatial shift \( \theta_x : \Omega \to \Omega \) is defined by

\[
(\theta_x \omega)(\cdot) = \omega(x + \cdot).
\]

In a random environment \( \omega \in \Omega \), we consider the discrete elliptic Dirichlet problem

\[
\begin{cases}
\frac{1}{2} \text{tr}(\omega(x)\nabla^2 u(x)) = \frac{1}{R^2} f \left( \frac{x}{R} \right) \psi(\theta_x \omega) & x \in B_R, \\
u(x) = g \left( \frac{x}{|x|} \right) & x \in \partial B_R,
\end{cases}
\]

where \( f \in \mathbb{R}^{\mathcal{B}_1}, g \in \mathbb{R}^{\partial \mathcal{B}_1} \), are functions with nice enough regularity properties and \( \psi \in \mathbb{R}^{\Omega} \) is bounded and satisfies suitable measurability conditions. Stochastic homogenization studies (for \( \mathbb{P} \)-almost all \( \omega \)) the convergence of \( u \) to the solution \( \bar{u} \) of a deterministic effective equation

\[
\begin{cases}
\frac{1}{2} \text{tr}(\bar{a}D^2 \bar{u}) = f \bar{\psi} & \text{in } \mathcal{B}_1, \\
\bar{u} = g & \text{on } \partial \mathcal{B}_1,
\end{cases}
\]

as \( R \to \infty \). Here \( D^2 \bar{u} \) denotes the Hessian of \( \bar{u} \) and \( \bar{a} = \bar{a}(\mathbb{P}) \in \mathbb{S}_{d \times d} \) and \( \bar{\psi} = \bar{\psi}(\mathbb{P}, \psi) \in \mathbb{R} \) are deterministic and do not depend on the realization of the random environment (see the statement of Proposition \([\ldots]\) for formulas for \( \bar{a} \) and \( \bar{\psi} \)).
The difference equation (2) is used to describe random walks in a random environment (RWRE) in \( \mathbb{Z} \). To be specific, we set

\[
\omega(x, x \pm e_i) := \frac{\omega_i(x)}{2\text{tr}(\omega(x))} \quad \text{for } i = 1, \ldots, d,
\]

and \( \omega(x, y) = 0 \) if \( |x - y| \neq 1 \). Namely, we normalize \( \omega \) to get a transition probability. We remark that the configuration of \( \{\omega(x, y) : x, y \in \mathbb{Z} \} \) is also called a balanced environment in the literature [26, 22, 6].

**Definition 1.** For a fixed \( \omega \in \Omega \), the random walk \( (X_n)_{n \geq 0} \) in the environment \( \omega \) is a Markov chain in \( \mathbb{Z} \) with transition probability \( P^x_\omega \) specified by

\[
P^x_\omega (X_{n+1} = z | X_n = y) = \omega(y, z).
\]

The expectation with respect to \( P^x_\omega \) is written as \( E^x_\omega \). When the starting point of the random walk is 0, we sometimes omit the superscript and simply write \( P_\omega \) and \( E_\omega \), respectively. Notice that for random walks \( (X_n) \) in an environment \( \omega \),

\[
\overline{\omega}^i = \theta X_i \omega \in \Omega, \quad i \geq 0,
\]

is also a Markov chain, called the environment viewed from the particle process. With abuse of notation, we enlarge our probability space so that \( P_\omega \) still denotes the joint law of the random walks and \( (\overline{\omega}^i)_{i \geq 0} \).

We also consider the continuous-time RWRE \( (Y_t) \) on \( \mathbb{Z} \).

**Definition 2.** Let \( (Y_t)_{t \geq 0} \) be the Markov process on \( \mathbb{Z} \) with generator

\[
L_\omega u(x) = \sum_y \omega(x, y)[u(y) - u(x)] = \frac{1}{2\text{tr}(\omega(x))}\text{tr}(\omega(x) \nabla^2 u).
\]

With abuse of notation, we also let \( P^x_\omega \) denote the quenched law of \( (Y_t) \). If there is no ambiguity from the context, we also write, for \( x, y \in \mathbb{Z} \), \( n \in \mathbb{Z} \), \( t \in \mathbb{R} \), the transition kernels of the discrete and continuous time walks as

\[
p^x_n(x, y) = P^x_\omega(X_n = y), \quad \text{and} \quad p^x_t(x, y) = P^x_\omega(Y_t = y),
\]

respectively.

Both the discrete- and continuous-time RWRE share the same trajectory, and their behavior are very much the same. The solutions to the Dirichlet problem can be characterized using the discrete-time RWRE, whereas for the transition kernels it is easier to manipulate the continuous-time case where the derivatives have less cumbersome notation compared to theirs discrete counterparts. Hence we will use both \( (X_n) \) and \( (Y_t) \) in our paper for convenience.
1.2 Main assumptions

We assume the following points throughout the paper.

(A1) \( \{ \omega(x), x \in \mathbb{Z}^d \} \) are i.i.d. under the probability measure \( \mathbb{P} \).

(A2) \( \frac{\omega}{\text{tr} \omega} \geq 2 \kappa I \) for \( \mathbb{P} \)-almost every \( \omega \) and some constant \( \kappa \in (0, \frac{1}{2d}] \).

(A3) \( \psi \) is a measurable function of the environment with the property that \( \{ \psi(\theta_x \omega) : x \in \mathbb{Z}^d \} \) are i.i.d. under \( \mathbb{P} \).

In this paper, we use \( c, C \) to denote positive constants which may change from line to line but that only depend on the dimension \( d \) and the ellipticity constant \( \kappa \) unless otherwise stated. We write \( A \lesssim B \) if \( A \leq CB \), and \( A \asymp B \) if both \( A \lesssim B \) and \( A \gtrsim B \) hold.

1.3 Earlier results in the literature

The following quenched central limit theorem (QCLT) was proved by Lawler \[26\], which is a discrete version of Papanicolaou, Varadhan \[29\].

**Theorem A.** Assume (A2) and that law \( \mathbb{P} \) of the environment is ergodic under spatial shifts \( \{ \theta_x : x \in \mathbb{Z}^d \} \). Then

(i) There exists a probability measure \( \mathbb{Q} \approx \mathbb{P} \) such that \( (\overline{\omega}_n) \) is an ergodic (with respect to time shifts) sequence under law \( \mathbb{Q} \times P_\omega \).

(ii) For \( \mathbb{P} \)-almost every \( \omega \), the rescaled path \( X_n/n \) converges weakly (under law \( P_\omega \)) to a Brownian motion with covariance matrix \( \overline{\mathbb{E}}_\omega(\omega/\text{tr} \omega) > 0 \).

QCLT for the balanced RWRE in static environments under weaker ellipticity assumptions can be found at \[22, 6\]. For dynamic balanced random environment, QCLT was established in \[14\] and finer results concerning the local limit theorem and heat kernel estimates was obtained at \[13\]. We refer to Zeitouni’s lecture notes \[31\] for a comprehensive account of results and challenges in RWRE.

Denote the Radon-Nikodym derivative of \( \mathbb{Q} \) with respect to \( \mathbb{P} \) as

\[
\rho(\omega) = \frac{d\mathbb{Q}}{d\mathbb{P}}.
\]

For any \( x \in \mathbb{Z}^d \) and finite set \( A \subseteq \mathbb{Z}^d \), we write

\[
\rho_\omega(x) := \rho(\theta_x \omega) \quad \text{and} \quad \rho_\omega(A) = \sum_{x \in A} \rho_\omega(x).
\]

It is known that (see, e.g., \[13\]) for \( \mathbb{P} \)-almost all \( \omega \), the measure \( \rho_\omega(\cdot) \) on \( \mathbb{Z}^d \) is the unique (up to a multiplicative constant) invariant measure for the RWRE \( (X_n)_{n \geq 0} \). In this sense, \( \mathbb{Q} \) is the \textit{steady state} for the environmental process. To investigate the long term behavior of the RWRE and the homogenization of the corresponding diffusion equations, it is crucial to characterize the invariant measure \( \rho_\omega \).
Now that $\mathcal{Q}$ is the limiting ergodic measure of the environment, it is expected that, as $t \to \infty$, $\psi(\vec{o}_t) \to E_\mathcal{Q}[\psi]$ almost surely for $\psi \in L^1(\mathcal{P})$. (Recall the process $\vec{o}_t$ of the environment as viewed from the particle in (6).) When the balanced environment satisfies a finite range of dependence and $\psi$ is an $L^\infty(\mathcal{P})$ local function, it is shown in [19, Theorem 1.2] that, with overwhelming $\mathbb{P}$-probability, the average $t^{-1} \int_0^t \psi(\omega_s)ds$ converges to $E_\mathcal{Q}[\psi]$ at an algebraic speed $t^{-\alpha}$. For time-reversible random walks in random environment, Kipnis and Varadhan [24] proved that the process of the environment as viewed from the particle has diffusive behavior. Further, in the reversible setting where the walk is generated by divergence form operators $\nabla \cdot a(\omega)\nabla$, algebraic rates for the decay of $E_\mathcal{Q}[\psi(\vec{o}_t)]$ are obtained in [23,18,11].

One of our goals in this paper is to prove a diffusive behavior for the environmental process, i.e., a CLT for $\frac{1}{\sqrt{t}} \int_0^t \psi(\omega_s) - E_\mathcal{Q}[\psi]ds$ and to investigate the decay rate of $E_\mathcal{Q}[\psi(\vec{o}_t)] - E_\mathcal{Q}[\psi]$.

As an important feature of the non-divergence form model, $\rho_\omega$ does not have deterministic upper and (nonzero) lower bounds. Moreover, the heat kernel $\rho_\omega^t(\cdot,\cdot)$ is not expected to have deterministic Gaussian bounds. For ergodic balanced random environments, the following stochastic bounds for the invariant measure $\rho_\omega$ and the heat kernel were proved in [26,16,13].

For $r \geq 0$, $t > 0$, denote by

$$f(r,t) = \frac{r^2}{r \lor t} + r \log \left( \frac{r}{t} \lor 1 \right), \quad r \geq 0, t > 0. \quad (9)$$

**Theorem B.** Assume (A2) and that the environment is ergodic under spatial shifts $\{\theta_x : x \in \mathbb{Z}^d\}$. There exists a constant $p = p(d,\kappa) > 0$ such that

(i) $\mathbb{E}[\rho^{d/(d-1)}] < \infty, \quad \mathbb{E}[\rho^{-p}] < \infty;$

(ii) $\mathbb{P}$-almost surely, for any $r > 0$,

$$\rho_\omega(B_r) \geq C \rho_\omega(B_{2r});$$

(iii) $\mathbb{P}$-almost surely, for all $x \in \mathbb{Z}^d$, $t > 0$,

$$\frac{c \rho_\omega(0)}{\rho_\omega(B_{\sqrt{t}})} e^{-c|x|^2/t} \leq \rho_\omega^t(x,0) \leq \frac{C \rho_\omega(0)}{\rho_\omega(B_{\sqrt{t}})} e^{-c\|x\|^2/t}. \quad (10)$$

Moreover, for $x \in \mathbb{Z}^d$, $t > 0$,

$$\|\rho_\omega^t(0,x)\|_{L^{d+1/d}(\mathcal{P})} \leq \frac{C}{(t+1)^{d/2}} e^{-c\|x\|^2/t},$$

$$\|\rho_\omega^t(0,x)\|_{L^{-p}(\mathcal{P})} \geq \frac{c}{(t+1)^{d/2}} e^{-c\|x\|^2/t}. $$
The positive moment bound in (i) was obtained by Lawler [26], and the negative moment bound in (i) is a special case of the bound proved by Deuschel and the first named author [13] Theorem 11 in the more general time-dependent ergodic environment setting. The volume-doubling property (ii) was proved by Fabes and Stroock [16] Lemma 2.0. The heat kernel bounds together with its integrability in the form of (iii) were obtained in [13] Theorem 11 for in the more general dynamic ergodic setting.

Roughly speaking, the term \( \frac{\rho_\omega(0)}{\rho_\omega(B_i)} \) is the long term ratio between the time the RWRE visits the origin and the time it spends in the ball \( B_i \). For the special deterministic environment \( \alpha \equiv 1 \), i.e., when the RWRE is a simple random walk, we have \( \rho \equiv 1 \) and this constant is \( C_{1/2} \). However, in a random environment, one should not expect \( \frac{\rho_\omega(0)}{\rho_\omega(B_i)} \) to have deterministic bounds. Hence, to understand the bounds of the invariant measure and the heat kernel, it is crucial to obtain their stochastic integrability.

Although for general ergodic environment, positive and negative moment bounds for the invariant measure and the heat kernel were already obtained in Theorem B, it is a natural question whether better mixing properties of the environment would yield better moment bounds for these quantities.

Our another goal in this paper is to show that, in our i.i.d. balanced environment, both \( \rho_\omega \) and the heat kernel have positive and negative exponential moment bounds. In the course of our proof of better moment bounds for the invariant measure and the heat kernels, the following quantitative homogenization result for the non-divergence form operator \( L_\omega \) will be employed.

**Proposition C.** Assume (A1), (A2), (A3). Recall the measure \( \mathbb{Q} \) in Theorem A. Suppose \( g \in C^4(\partial \mathbb{B}_1) \), \( f \in C^a(\mathbb{B}_1) \) for some \( a \in (0, 1] \), and \( \psi \) is a measurable function of \( \omega(0) \) with \( ||\psi/\omega||_\infty < \infty \). For any \( q \in (0, d) \), there exist a random variable \( \mathcal{Z}_q = \mathcal{Z}_q(\omega, d, \kappa) \) with \( E[\exp(c\mathcal{Z}_q^d)] < \infty \), and a constant \( \beta = \beta(d, \kappa, q) \in (0, 1) \) such that for any \( y \in B_3 \), the solution \( u \) of

\[
\begin{cases} 
\frac{1}{R^4} \operatorname{tr}(\omega \nabla^2 u(x)) = \frac{1}{R^q} f(x/R) \psi(\theta_{x/y} \omega) & x \in B_R(y), \\
u(x) = g(\psi_{x-y}) & x \in \partial B_{3R}(y) 
\end{cases}
\]

satisfies, with \( A_1 = \|f\|_{C^0(\mathbb{B}_1)} \|\psi\|_{\operatorname{tr}(\omega)} \|g\|_{C^{0,\alpha}(\partial \mathbb{B}_1)} \),

\[
\max_{x \in B_{3R}(y)} |u(x) - \bar{u}(x/R)| \lesssim A_1 (1 + \mathcal{Z}_q R^{-q/d}) R^{-\beta}, \tag{11}
\]

where \( \bar{u} \) solves (3) with \( \bar{a} = E_\omega[\omega/\hat{\omega}] > 0 \) and \( \bar{\psi} = E_\omega[\psi/\hat{\omega}] \).

**Remark 3.** Our Proposition C in the above is a version of [19] Theorem 1.5, which can be considered as a discrete version of Armstrong, Smart [3, Theorem 1.2]. We remark that, compared to the aforementioned results in [3, 19], a difference in Proposition C is that the ball is allowed to be centered at any point \( y \in B_3R \) whereas...
the random variable $\mathcal{X}$ stays the same. This subtle feature will allow us to define a “homogenization radius” which will be useful later in our proof of the bounds for the Green functions.

The proof of Proposition C, which is a small modification of that of [19, Theorem 1.5], can be found in the Appendix.

In terms of the quantitative homogenization of non-divergence form operators in the PDE setting, Yurinski derived a second moment estimate of the homogenization error in [30] for linear elliptic case, and Caffarelli, Souganidis [10] proved a logarithmic convergence rate for the nonlinear elliptic case. Afterwards, Armstrong, Smart [3], and Lin, Smart [27] achieved an algebraic convergence rate for fully nonlinear elliptic equations, and fully nonlinear parabolic equations, respectively. Armstrong, Lin [2] obtained quantitative estimates for the approximate corrector problems.

1.4 Main results

For RWRE in an i.i.d. balanced, uniformly elliptic environment, we will establish natural bounds for both the invariant measure and the heat kernel (Theorem 4) which possess both positive and negative exponential moments, greatly improving the stochastic integrability in Theorem 4 for the ergodic setting. We then prove the optimal diffusive decay of the semigroup generated by the heat kernel for $d \geq 3$ in Theorem 5. As consequences, we obtain a functional central limit theorem (CLT) for the environment viewed from the particle process, and deduce the existence of a stationary corrector in dimension $d \geq 5$.

**Theorem 4.** Assume (A1), (A2), and $d \geq 2$. Let $s = s(d, \kappa) = 2 + \frac{1}{2\kappa} - d \geq 2$. For any $\varepsilon > 0$, there exists a random variable $\mathcal{H}(\omega) = \mathcal{H}(\omega, d, \kappa, \varepsilon) > 0$ with $\mathbb{E}[\exp(c\mathcal{H}^{-\varepsilon})] < \infty$ such that the following properties hold.

(a) For $\mathbb{P}$-almost all $\omega$,
\[
c\mathcal{H}^{-\varepsilon} \leq \rho(\omega) \leq C\mathcal{H}^{d-1}.
\]
In particular, for any $q \in (-\frac{d}{s}, \frac{d}{s-1})$, we have
\[
\mathbb{E}[\exp(c\rho^q)] < \infty.
\]

(b) Recall the function $\mathcal{H}$ in (9). For any $r \geq 1$ and $\mathbb{P}$-almost all $\omega$,
\[
c\mathcal{H}^{-\varepsilon} \leq \frac{r^d \rho(0)}{\rho(\mathcal{B}_r)} \leq C\mathcal{H}^{d-1}.
\]

(c) For any $x \in \mathbb{Z}^d$, $t > 0$, and $\mathbb{P}$-almost all $\omega$,
\[
p_t^{(\mathcal{H}, 0)}(x, 0) \leq C\mathcal{H}^{d-1}(1 + t)^{-d/2}e^{-c\mathcal{H}|x|d/2},
\]
\[
p_t^{(\mathcal{H}, 0)}(x, 0) \geq c\mathcal{H}^{-s}(1 + t)^{-d/2}e^{-C|x|^2/2}.
\]
Recall the continuous time RWRE \((Y_t)_{t \geq 0}\) in Definition \(2\). With abuse of notation, we still denote the process of the environment viewed from the particle as
\[
\tilde{\omega}^t := \theta_t \omega.
\]
Following Gloria, Neukamm, Otto \([13]\), for any measurable function \(\zeta : \Omega \to \mathbb{R}\), we define its stationary extension \(\bar{\zeta} : \mathbb{Z}^d \times \Omega \to \mathbb{R}\) as
\[
\bar{\zeta}(x) = \zeta(x; \omega) := \zeta(\theta_x \omega).
\]
Define the semigroup \(P_t, t \geq 0\), on \(\mathbb{R}^\Omega\) by
\[
P_t \zeta(\omega) = E_\omega^0[\bar{\zeta}(\tilde{\omega}^t)] = \sum_z p_t^0(0, z)\bar{\zeta}(z; \omega).
\]

The following theorem estimates the speed of decorrelation of the environmental process \(\tilde{\omega}^t\) from the original environment. It gives a rate \(t^{-d/4}\) of decay for the semigroup, which is optimal. A function \(\zeta : \Omega \to \mathbb{R}\) is said to be local if it depends only on the environment \(\{\omega(x) : x \in S\}\) in a finite set \(S \subset \mathbb{Z}^d\). Such a set \(S\) is called the support of \(\zeta\) and denoted by \(\text{Supp}(\zeta)\).

**Theorem 5.** Assume (A1), (A2), and \(d \geq 3\). For any local measurable function \(\zeta : \Omega \to \mathbb{R}\) with \(\|\zeta\|_\infty \leq 1\) and \(t \geq 0\), we have, for \(C = C(d, \kappa, \#\text{Supp}(\zeta))\),
\[
\begin{align*}
\text{Var}_{\mathbb{Q}}(P_t \zeta) &\leq C(1 + t)^{-d/2}; \\
\|P_t \zeta - E_\mathbb{Q}\zeta\|_{L^1(\mathbb{P})} + \|P_t \zeta - \mathbb{E}[P_t \zeta]\|_{L^p(\mathbb{P})} &\leq C_p(1 + t)^{-d/4} \quad \text{for all } p \in (0, 2).
\end{align*}
\]

For divergence form operators, optimal diffusive decay of the semigroup generated by the heat kernel was obtained by Gloria, Neukamm, Otto \([13]\), de Buyer, Mourrat \([11]\). Our proof of Theorem 5 follows the approach of \([13]\), which uses an Efron-Stein type inequality and the Duhamel representation formula for the vertical derivative. However, unlike the divergence form setting \([18]\), there are no deterministic Gaussian bounds for the heat kernel, and the steady state \(\mathbb{Q}\) of the environment process \((\tilde{\omega}^t)_{t \geq 0}\) is not the same as the original measure \(\mathbb{P}\). To overcome these difficulties, our heat kernel estimates and the (negative and positive) moment bounds of the Radon-Nikodym derivative \(d\mathbb{P}/d\mathbb{Q}\) in Theorem 4 play crucial roles. (Another feature of our non-divergence setting that worth mentioning is that there is no Caccioppoli estimates for \(p > 2\). See Lemma \([21]\).

As a consequence of Theorem 5 and the CLT of \([12]\), we obtain a functional CLT for an additive functional of the environmental process. Recall that \((\tilde{\omega}^t)_{t \geq 0}\) is an ergodic sequence under the law \(\mathbb{Q} \times P_\omega\) and time shifts. The following CLT says that, when \(d \geq 3\), the fluctuation around the ergodic mean is approximately Gaussian under the diffusive rescaling.
Corollary 6. Assume (A1), (A2). Let \( \mathbb{P} \)-almost all \( \omega \) and any bounded measurable local function \( \zeta \) of the environment with \( E_Q\zeta = 0 \), the \( P_\omega \) law of
\[
\frac{1}{\sqrt{t}} \int_0^t \zeta(\tilde{\sigma}^s)ds
\]
converges weakly to a Brownian motion with a deterministic diffusivity constant.

Another consequence of Theorem 5 is the existence of a stationary corrector in \( d \geq 5 \) for the non-divergence form homogenization problem (2).

Corollary 7. Assume (A1), (A2). When \( d \geq 5 \), for any bounded local measurable function \( \psi : \Omega \to \mathbb{R} \), there exists \( \hat{\psi} : \Omega \to \mathbb{R} \) such that
\[
\hat{\psi} \in L^p(\mathbb{P}) \quad \text{for all} \quad p \in (0, 2),
\]
and for \( \mathbb{P} \)-almost all \( \omega \), its stationary extension \( \tilde{\psi}(x) = \hat{\psi}(\theta_x \omega) \) solves
\[
L_\omega \tilde{\psi}(x) = \tilde{\zeta}(x) - E_Q[\zeta], \quad \text{for all} \quad x \in \mathbb{Z}^d. \quad (14)
\]

Remark 8. The corrector \( \hat{\psi}(x) \) plays a crucial role in the quantification of the homogenization error for non-divergence form operators. We remark that our Corollary 7 is a weaker version of [2, Theorem 7.1] where not only the existence of the stationary corrector was proved but a stretched exponential tail was also obtained. Specifically, [2] derived such a result using a large scale \( C^{1,1} \) estimate for the homogenization problem. Although our Corollary 7 is an immediate consequence of the optimal diffusive decay rate, it has much weaker stochastic integrability.

In the classical periodic environment setting, it is well-known that the existence of a stationary corrector implies that the optimal homogenization error of problem (2) is generically of scale \( R^{-1} \). Readers may refer to the classical books [4, 23] for the derivation of the rate in the periodic setting, and [21, 20] for discussions on the optimality of the rates.

To obtain the above results, we need to study properties of the Green functions. For \( d \geq 2, R \geq 1 \), denote the exit time from \( B_R \) of the RWRE by
\[
\tau = \tau_R = \inf \{ n \geq 0 : X_n \notin B_R \}. \quad (15)
\]

Definition 9. For \( R \geq 1, \omega \in \Omega, x \in \mathbb{Z}^d, S \subset \mathbb{Z}^d \), the Green function \( G_R(\cdot, \cdot) \) in the ball \( B_R \) for the balanced random walk is defined by
\[
G_R(x, S) = G_\omega^R(x, S) := E^x_\omega \left[ \sum_{n=0}^{\tau_R-1} \mathbb{1}_{X_n \in S} \right], \quad x \in \bar{B}_R.
\]
We also write \( G_R(x, y) := G_\omega^R(x, \{ y \}) \) and \( G_R(x) := G(x, 0) \).

Note that for \( d \geq 3 \), by [22, Theorem 1], the RWRE is transient, and so the Green function in the whole space
\[
G_\omega^R(x) := \lim_{R \to \infty} G_R(x) < \infty
\]
is well-defined for all $x \in \mathbb{Z}^2$, $\mathbb{P}$-almost surely. Whereas, when $d = 2$, the RWRE is recurrent, and thus the Green function in the whole $\mathbb{Z}^2$ is infinity. In this case, the potential kernel

$$A(x) = A^0(x) = \sum_{n=0}^{\infty} [p_n^0(0,0) - p_n^0(x,0)], \quad x \in \mathbb{Z}^2, \quad (16)$$

is well-defined. Note that $G$ and $A$ are both non-negative functions, and, for $x \in \mathbb{Z}^d$,

$$L_\omega G(x) = -\mathbb{1}_{x=0}, \quad \text{if } d \geq 3,$$

and

$$L_\omega A(x) = \mathbb{1}_{x=0}, \quad \text{if } d = 2.$$

**Theorem 10.** Assume (A1), (A2). For $r > 0$, let

$$U(r) := \begin{cases} -\log r & \text{if } d = 2, \\ r^{2-d} & \text{if } d \geq 3. \end{cases}$$

For any $\epsilon > 0$, there exists a random variable $\mathcal{H} = \mathcal{H}(\omega, d, \kappa, \varepsilon) > 0$ with $\mathbb{E}[\exp(c\mathcal{H}^{-\epsilon})] < \infty$ such that, for $\mathbb{P}$-almost surely, for all $x \in B_R$,

$$\mathcal{H}^{-s}[\mathcal{H}(\omega, d, \kappa, \varepsilon)] \leq G^\omega_R(x) \leq \mathcal{H}^{-s}[\mathcal{H}(\omega, d, \kappa, \varepsilon)]^2 - U(R + 2)],$$

where $s = s(d, \kappa) = 2 + \frac{1}{2\kappa} - d \geq 2$.

We remark that an upper bound for the Green function of the approximate corrector (which is defined in the whole $\mathbb{R}^d$) was proved by Armstrong, Lin [2, Proposition 4.1].

Our proof of the bounds of $G^\omega_R$ follows the idea of Armstrong, Lin [2, Proposition 4.1]. In Theorem 10 we apply their idea to obtain both upper and lower bounds for the Green function $G_R$ in a finite region.

**Corollary 11.** Assume (A1), (A2). Let $s$ be as in Theorem 10. For any $\epsilon > 0$, there exists a random variable $\mathcal{H} = \mathcal{H}(\omega, d, \kappa, \varepsilon) > 0$ with $\mathbb{E}[\exp(c\mathcal{H}^{-\epsilon})] < \infty$ such that, $\mathbb{P}$-almost surely, for all $x \in \mathbb{Z}^d$,

$$\mathcal{H}^{-s}(\log(|x| + 1) \leq \mathcal{H}^\omega_R(x) \leq \mathcal{H}^{-s}(\log(|x| + 1), \text{ when } d = 2;$$

$$\mathcal{H}^{-s}(1 + |x|)^{2-d} \leq \mathcal{H}^\omega_R(x) \leq \mathcal{H}^{-s}(1 + |x|)^{2-d}, \text{ when } d \geq 3.$$ 

## 2 Bounds of the Green function in a ball

By the Markov property, $G_R(x, S)$ satisfies $G_R = 0$ on $\partial B_R$ and

$$L_\omega G_R(x, S) = -\mathbb{1}_{x \in S}, \quad x \in B_R. \quad (18)$$
Our proof of the bounds of the Green function \( G_R \) (Theorem 10) follows the idea of Armstrong-Lin [2, Proposition 4.1]. The idea, which we learn from [2], is through the comparison of \( G_R \) and test functions, and it is explained as follows.

Let us call a function \( u : \mathbb{Z}^d \to \mathbb{R} \) \( \omega \)-harmonic on \( A \subset \mathbb{Z}^d \) if \( L_\omega u(y) = 0 \) for \( y \in A \). Clearly, the Green function \( G_R(x,0) \) is \( \omega \)-harmonic on \( B_R \setminus \{0\} \).

To obtain the upper bound, we construct a function \( h \) which is almost \( \omega \)-harmonic away from the origin and with (almost) zero boundary values, so that \( G_R - h \) is sub-harmonic at places that are either close to the origin or the boundary of \( B_R \). As a result, if \( (G_R - h)(x_0) = \max_{B_R \setminus \{0\}} (G_R - h) \) were positive, then the maximum principle forces the maximizer \( x_0 \) to be sufficiently far away from both the origin and the boundary. This allows enough space for the homogenization to occur around \( x_0 \), i.e., \( h \) is close to its continuous harmonic counterpart up to an algebraic error. On the other hand, by the maximal principle, the \( \omega \)-harmonic counterpart of \( G_R - h \) (which is an algebraic error away from \( G_R - h \)) cannot achieve its maximum over the ball \( B_{\lfloor\kappa/2\rfloor}(x_0) \) in the center. This would contradict the assumption that the maximizer is \( x_0 \), if we can exploit the fact that \( h \) is strictly \( \omega \)-superharmonic to give it enough room to absorb the algebraic homogenization error.

The proof of the lower bound follows similar philosophy.

Note that [2, Proposition 4.1] only deals with the Green function of the “approximate corrector” which is defined on the whole \( \mathbb{R}^d \), while our Green function corresponds to the original non-divergence form operator within a finite ball \( B_R \). Hence, in our case, the challenge also lies in the construction of test functions so that they have the desired boundary values and concavity near the discrete boundary.

The following Lemmas contain properties of some deterministic functions that will be useful in our construction of the test functions in the next subsections.

**Lemma 12.** Let \( \delta = \beta/2 \), where \( \beta = \beta(d, \kappa, q_\epsilon) \) is as defined in Proposition [C]. Define \( \tilde{\zeta}, \tilde{\xi} : (0, \infty) \to \mathbb{R} \) as

\[
\tilde{\zeta}(r) = \begin{cases} 
-(\log r) \exp(-r^{-\delta}/\delta) & d = 2 \\
r^{2-d} \exp(-r^{-\delta}/\delta) & d \geq 3,
\end{cases}
\]

\[
\tilde{\xi}(r) = \begin{cases} 
-(\log r) \exp(-r^{-\delta}/\delta) & d = 2 \\
r^{2-d} \exp(-r^{-\delta}/\delta) & d \geq 3.
\end{cases}
\]

Define two functions \( \zeta, \xi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \) as

\[
\zeta(y) = \tilde{\zeta}(|y|), \quad \text{and} \quad \xi(y) = \tilde{\xi}(|y|), \quad y \neq 0.
\]

Then, the following statements hold.

(i) \( \zeta, \xi \) are decreasing functions on \((C, \infty)\). Moreover, for \( r \geq C \),

\[
-\xi(r) \leq r^{1-d}, \quad \text{and} \quad 0.5r^{1-d} \leq -\zeta(r) \leq (d - 0.5)r^{1-d}.
\]

(ii) For \( |y| \geq C \), we have

\[
-\Delta \zeta(y) \geq |y|^{-(2+\delta)}|\zeta(y)|, \quad \text{and} \quad -\Delta \xi(y) \leq -|y|^{-(2+\delta)}|\xi(y)|.
\]
(iii) For $|y| \geq 2$ and $k \in \mathbb{N}$, there exists $C = C(k, d)$ such that

$$|D^k \zeta(y)| \leq C|y|^{-k} |\zeta(y)|,$$

and

$$|D^k \xi(y)| \leq C|y|^{-k} |\xi(y)|.$$

**Lemma 13.** There exist constants $\alpha_0 \in (0, 1)$ and $A_0 \geq 1$ depending on $\kappa$ such that, for any $\alpha \in (0, \alpha_0)$, $A \geq A_0$.

$$L_\alpha (e^{-2\alpha |x|/R}) \leq 0 \quad \text{in } B_R \setminus B_{R/2}, \quad \text{when } R \geq A_0;$$

(19)

$$L_\alpha (e^{-A|x|^2}) \geq -\mathbb{1}_{x=0}, \quad x \in \mathbb{Z}^d;$$

(20)

$$L_\alpha (e^{-A|x|^2/R^2}) > 0, \quad x \in B_R \setminus B_{R/2}, \quad \text{when } R \geq A^2.$$  

(21)

The proof of Lemma 13 is in Section A.2 of the Appendix.

### 2.1 Upper bounds of Green’s functions

Recall $\mathcal{Z}_\alpha$ in Proposition C. For any $\varepsilon \in (0, 1)$, we write

$$R_0 = R_0(\omega, d, \kappa, \varepsilon) := \mathcal{Z}_\alpha^{d/(d-\varepsilon)} + K,$$

where $K$ is a sufficiently large constant depending on $(d, \kappa)$, and denote the exit time from $B_{R_0}$ as

$$s_0 = \min\{n \geq 0 : X_n \notin B_{R_0}\}.$$  

(22)

Note that $R_0$ plays the role of a “homogenization radius” in the sense that for all $R \geq R_0$ and $y \in B_{3R}$, the upper bound in (11) can be replaced by the algebraic term $CA_1 R^{-d\alpha}$.

Let $\alpha = \alpha(d, \kappa) > 0$ be a constant to be determined in Lemma 15 and set

$$C_{a, R} := \frac{\overline{\zeta}(R/2) - \overline{\zeta}(R)}{e^{-a+2a/R} - e^{-2a}} \approx \alpha^{-1}, \quad \text{when } R \geq R_0.$$  

**Definition 14.** Let $\bar{\zeta}, \xi$ be as in Lemma 12. For any fixed $R \geq 4R_0$, we define a function $h : B_R \rightarrow [0, \infty)$ by

$$h(x) = \begin{cases} 
    h_1(x), & x \in B_{R_0}, \\
    h_2(x), & x \in B_{R/2} \setminus B_{R_0}, \\
    h_3(x), & x \in \bar{B}_R \setminus B_{R/2},
\end{cases}$$

where the functions $h_1, h_2, h_3$, are defined as below

$$h_2(y) = (\alpha^{-1} - 1)(\bar{\zeta}(R/2) - \bar{\zeta}(R)) + \zeta(y) - \bar{\zeta}(R), \quad y \in \mathbb{R}^d \setminus \{0\},$$

$$h_1(x) = E_0[h_2(X_{s_0}) + |X_{s_0} - |x||], \quad x \in \bar{B}_{R_0},$$

$$h_3(y) = R_0^{d-1} \alpha^{-1} C_{a, R} R^{2-d} [e^{-2\alpha(|y|-1)/R} - e^{-2a}], \quad y \in \mathbb{R}^d.$$  

**Lemma 15.** When $R \geq 4R_0$, there exists a constant $\alpha > 0$ such that the functions $h_1, h_2, h_3, h$ given in Definition 14 have the following properties.
(a) \( L_\omega h_1(x) = -L_\omega(|x|) \leq -\frac{1}{8} x_0 \) for \( x \in B_{R_0} \);

(b) \( h_1 = h_2 \) on \( \partial B_{R_0} \) and \( h_2 = h_3 \) on \( \partial B_{R/2} \);

(c) \( h_2 \geq h_1 \) in \( B_{R_0} \setminus B_{R_0/2} \).

(d) \( h_2 \geq h_3 \) in \( B_R \setminus B_{R/2} \), and \( h_2 \leq h_3 \) in \( B_{R/2} \setminus B_{R/2-1} \).

(e) \( L_\omega h_3 \leq 0 \) in \( B_R \setminus B_{R/2} \).

\textbf{Proof.} (a) and (b) are obvious. To see (c), note that \( h_2(x) = h_1(x) + \varphi(x) \), where \( \varphi(x) = \int_{\mathbb{R}^d} f(|y|) - f(|X_{\delta_i}|) \), with \( \delta_i \in \{0\} \setminus \{1\} \). By Lemma 16, taking \( K \) sufficiently large, \( f(r) \) is a decreasing function for \( r \in [R_0/2, R_0 + 1] \).

Next, we will prove (d). Indeed, we can write, for \( y \in \mathbb{R}^d \setminus \{0\} \),

\[ h_2(y) - h_2(0) = \int \left( \left( \frac{1}{2} - 1 \right) - \frac{1}{2} \right) \frac{1}{2} f(|y|) - f(|X_{\delta_i}|) \leq 0. \]

Hence, \( h_2 \leq h_3 \) in \( B_R \setminus B_{R/2-1} \). Item (d) then follows from the fact that \( h_2 = h_3 = 0 \) on \( \partial B_{R/2} \).

Item (e) is a consequence of (19) in Lemma 13.

\textbf{Proof of the upper bound in Theorem 10} For \( 0 < a < b \) and \( d \geq 2 \), we have an elementary inequality

\[ b - a \leq b^{d-1}(U(a) - U(b)). \tag{23} \]

When \( R \in (1, 4R_0] \), note that \( L_\omega G_R(x) \leq (R + 1 - |x|) \geq 0 \), and \( G_R = 0 \leq R + 1 - |x| \) on \( \partial B_R \). By the maximum principle, we have, for \( x \in B_R \),

\[ G_R(x) \leq R + 1 - |x| \leq (R + 2)^{d-1}[U(|x| + 1) - U(R + 2)]. \]

Hence the upper bound in Theorem 10 holds when \( R \in (1, 4R_0] \). It remains to consider the case \( R > 4R_0 \).

First, we will prove via contradiction that

\[ G_R \leq h \quad \text{in} \quad \bar{B}_R. \tag{24} \]

Assume by contradiction that (24) fails, i.e., \( \max_{\bar{B}_R} (G_R - h) > 0 \). By Lemma 15, \( L_\omega(G_R - h) \geq 0 \) in \( B_{R_0} \), and so \( \max_{\bar{B}_R} (G_R - h) \) is achieved outside of \( \bar{B}_R \). Further, note that \( (G_R - h)|_{\partial B_R} = (h_3)|_{\partial B_R} \leq 0 \). By Lemma 15, \( L_\omega(G_R - h_3) \geq 0 \) in \( B_R \setminus B_{R/2} \), and so, by the maximum principle and Lemma 15,

\[ \max_{\bar{B}_R \setminus B_{R/2}} (G_R - h) \leq \max_{\partial(B_R \setminus B_{R/2})} (G_R - h_3) \leq 0 \quad \max_{B_{R/2} \setminus B_{R_0}} (G_R - h). \]
Hence, if \( \max_{B_r} (G_R - h) > 0 \), then there exists \( x_0 \in B_{R/2} \setminus B_R \) so that
\[
(G_R - h)(x_0) = \max_{B_R} (G_R - h) > 0.
\]
Since \( x_0 \in B_{R/2} \setminus B_R \), by Lemma [15],
\[
(G_R - h_2)(x_0) \geq \max_{B_{R/2}(x_0)} (G_R - h_2),
\]
which is equivalent to
\[
(G_R - R_0^{d-1} \xi)(x_0) \geq \max_{B_{R/2}(x_0)} (G_R - R_0^{d-1} \xi).
\] (25)

Without loss of generality, assume that \( \bar{a} = I \), and set
\[
\check{\xi}(y) := \xi(y) + c|x_0|^{-(2+\delta)}|\xi(x_0)||y - x_0|^2, \quad y \in \mathbb{R}^d \setminus \{0\},
\]
where \( c > 0 \) is chosen so that (by Lemma [12]) \( \Delta \check{\xi}(y) \leq 0 \) for \( y \in B_{|x_0|/2}(x_0) \). Then (by Lemma [12]) \( |D\check{\xi}| \leq C|x_0|^{-1} |\xi(x_0)| \) in \( B_{1+0.5|x_0|}(x_0) \), and
\[
(G_R - R_0^{d-1} \check{\xi})(x_0) \leq \max_{\partial B_{|x_0|/2}(x_0)} (G_R - R_0^{d-1} \check{\xi}) + C R_0^{d-1} |x_0|^{-\delta} |\xi(x_0)|. \] (26)

Let \( \bar{v} : \mathbb{B}_1 \to \mathbb{R} \) and \( v : \mathbb{B}_{|x_0|/2}(x_0) \to \mathbb{R} \) be the solutions of (Here \( \bar{a} = I \)).

\[
\begin{align*}
\begin{cases}
\text{tr}(\bar{a} D^2 \bar{v}) = \Delta \bar{v} = 0 & x \in \mathbb{B}_1 \\
\bar{v}(x) = R_0^{d-1} \check{\xi}(x_0 + \frac{|x_0|}{2} x) & x \in \partial \mathbb{B}_1,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
L_{\omega} v(x) = 0 & x \in B_{|x_0|/2}(x_0) \\
v(x) = R_0^{d-1} \check{\xi}(x_0 + \frac{|x_0|}{2} x) & x \in \partial B_{|x_0|/2}(x_0).
\end{cases}
\end{align*}
\]

We will show that \( v \) can be controlled by \( R_0^{d-1} \check{\xi} \) both on the boundary and inside of \( B_{|x_0|/2}(x_0) \). Indeed, for \( x \in \partial B_{|x_0|/2}(x_0) \),
\[
|v(x) - R_0^{d-1} \check{\xi}(x)| = R_0^{d-1} \left| \check{\xi}(x_0 + \frac{|x_0|(x - x_0)}{2|x - x_0|}) - \check{\xi}(x) \right|
\leq R_0^{d-1} \sup_{B_{1+3|x_0|/2}(x_0)} |D\check{\xi}|
\leq C R_0^{d-1} |x_0|^{-\delta} |\xi(x_0)|. \] (27)

For \( x \in B_{|x_0|/2}(x_0) \), applying Proposition [C] to the case \( \alpha = 1 \), there exists \( \beta = \beta(d, \kappa, \epsilon) \in (0, 1) \) such that
\[
v(x) \leq \bar{v}(\frac{x - x_0}{|x_0|/2}) + C |x_0|^{-\beta} R_0^{d-1} \sup_{y \in \partial \mathbb{B}_1} |D\check{\xi}(x_0 + \frac{|x_0|}{2} y)|
\leq \bar{v}(\frac{x - x_0}{|x_0|/2}) + C R_0^{d-1} |x_0|^{-\beta} |\xi(x_0)|. \] (28)
Furthermore, using the fact that $\Delta \bar{\varphi}(x_0 + \frac{|x_0|}{2} x) \leq 0$ for $x \in \mathbb{B}_1$, we get $\bar{\varphi}(x) \leq R_0^{d-1} \bar{\varphi}(x_0 + \frac{|x_0|}{2} x)$ in $\mathbb{B}_1$. This, together with (28), yields, for $x \in B_{|x_0|/2}(x_0)$,

$$v(x) \leq R_0^{d-1} \bar{\varphi}(x) + C R_0^{d-1} |x_0|^{-\beta} |\bar{\varphi}(x_0)|.$$  \hspace{1cm} (29)

Notice that $(G_R - v)$ is an $\omega$-harmonic function on $B_{|x_0|/2}(x_0)$, and so

$$\max_{B_{|x_0|/2}(x_0)} (G_R - v) \leq \max_{\partial B_{|x_0|/2}(x_0)} (G_R - v).$$

Therefore, combining this inequality and (27), (29), we get

$$\max_{B_{|x_0|/2}(x_0)} (G_R - R_0^{d-1} \bar{\varphi}) \leq \max_{\partial B_{|x_0|/2}(x_0)} (G_R - R_0^{d-1} \bar{\varphi}) + C R_0^{d-1} |x_0|^{-\beta} |\bar{\varphi}(x_0)|$$

which contradicts (26), since $|x_0| \in (R_0, R)$, $\delta = \beta/2$ by definition in Lemma 12 and $R_0 \geq K$ is chosen to be sufficiently large. Inequality (24) is proved.

Finally, when $x \in B_{R/2} \setminus B_{R_0}$, we have $G_R(x) \leq h_2(x)$, and, by Lemma 12(3),

$$h_2(x) \leq R_0^{d-1} \alpha^{-1} \{\bar{\varphi}(|x|) - \bar{\varphi}(R)\} \leq R_0^{d-1} \int_{|x|}^{R} (-\bar{\varphi})'(r) dr \leq R_0^{d-1} \int_{|x|}^{R} r^{1-d} dr \leq R_0^{d-1} [U(|x|) - U(R)].$$  \hspace{1cm} (30)

When $x \in B_{R_0} \setminus \{0\}$,

$$G_R(x) \leq h_1(x) = E_{a_1}|h_2(X_{x_0})| + |X_{x_0}| - |x| \leq R_0^{d-1} E_{a_1}[U(|X_{x_0}|) - U(R)] + U(|x|) - U(|X_{x_0}|)$$

$$\leq R_0^{d-1} [U(|x|) - U(R)].$$

Note that, for $|x| \geq 1$, $U(|x|) - U(R) \leq U(|x| + 1) - U(R + 2)$.

When $x \in B_R \setminus B_{R/2}$,

$$G_R \leq h_3 \leq C R_0^{d-1} R^{2-d} (e^{-2\alpha(|x|-1)/R} - e^{-2\alpha}) \leq R_0^{d-1} R^{2-d} (1 - \frac{|x|-1}{R}) = R_0^{d-1} R^{1-d} (R + 1 - |x|) \leq R_0^{d-1} [U(|x| + 1) - U(R + 2)].$$

The upper bound in Theorem 10 is proved by putting $\mathcal{H} = R_0$. \hspace{1cm} $\blacksquare$
2.2 Lower bounds of Green’s functions

The proof of the lower bound of Theorem 10, which is similar to that of Theorem 10, is via comparing $G_R$ to appropriate test functions. However, unlike the Green function in the whole space, $G_R$ is defined only in bounded region, and so the test functions should be carefully designed to capture the behavior of $G_R$ near the boundary.

**Lemma 16.** Define $\tilde{\eta} : \mathbb{R} \to (0, \infty)$ as $\tilde{\eta}(r) = (1 + r^2)^{-\theta}$, where

\[ \theta := 1/(4\kappa) \geq d/2. \tag{31} \]

Define $\eta : \mathbb{R}^d \to \mathbb{R}$ as

\[ \eta(y) = \tilde{\eta}(|y|). \]

There exists a constant $C_0 = C_0(d, \kappa) > 0$ such that, for $x \in \mathbb{Z}^d$,

\[ L_{\omega\eta}(x) \geq -\mathbb{1}_{x \in B_{C_0}}. \]

The proof of Lemma 16 is in Section A.2 of the Appendix.

Let $\gamma = \gamma(\kappa) > 0$ be a large constant to be determined, and set

\[ C_{\gamma, R} := \frac{[\xi(R/2) - \tilde{\xi}(R)]R^{d-2}}{e^{\gamma/4} - e^{-\gamma}} \asymp e^{\gamma/4} \text{ when } R \geq R_0. \]

**Definition 17.** Recall $\xi, \tilde{\xi}, \eta, \theta$ be in Lemma 12 and Lemma 16. For any fixed $R \geq 4R_0$, we define three functions $\ell^i$, $i = 1, 2, 3$, as

\[ \ell_2^i(y) = R_0^{d-2-2\theta} [(y^{-2} - 1)(\xi(R/2) - \tilde{\xi}(R)) + \xi(y) - \tilde{\xi}(R)), \quad y \in \mathbb{R}^d \setminus \{0\}; \]

\[ \ell_3^i(x) = E_0^i[\ell_2^i(x_{X_0}) + \eta(x) - \eta(x_{X_0})], \quad x \in \bar{B}_R; \]

\[ \ell_3^i(y) = R_0^{d-2-2\theta} \gamma^{-2} C_{\gamma, R} R^{2-d} (e^{-\gamma}) |y|^{2} / R^2 - e^{-\gamma}), \quad y \in \mathbb{R}^d. \]

Also, for $R \geq 4R_0$, we define a function $\ell^i : \bar{B}_R \to \mathbb{R}$ by

\[ \ell^i(x) = \begin{cases} \ell_1^i(x), & x \in B_{R_0}; \\ \ell_2^i(x), & x \in B_{R/2} \setminus B_{R_0}; \\ \ell_3^i(x), & x \in \bar{B}_R \setminus B_{R/2}. \end{cases} \]

**Lemma 18.** When $R \geq 4R_0$, there exists a constant $\gamma > 0$ such that the functions $\ell_1, \ell_2, \ell_3, \ell^i$ given in Definition 17 have the following properties.

(a) $L_{\omega\ell_1} = L_{\omega\eta} \geq -\mathbb{1}_{x \in B_{C_0}}$ for $x \in B_{R_0}$;

(b) $\ell_1 = \ell_2$ on $\partial B_{R_0}$, and $\ell_2 = \ell_3$ on $\partial B_{R/2}$;

(c) $\ell_2 \leq \ell_1$ in $B_{R_0} \setminus B_{R_0/2}$. 

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(d) $\ell_2 \leq \ell_3$ in $B_R \setminus B_{R/2}$, and $\ell_2 \geq \ell_3$ in $B_{R/2} \setminus B_{0.5R-2}$.

(e) $L_\omega \ell_3 \geq 0$ in $B_R \setminus B_{R/2}$.

Proof. (ii) follows from Lemma [16] and (i) follows from definition. To see (iii), note that $\ell_1 - \ell_2 = E^* \{ f(|X_{y_0}|) - f(|x|) \}$, where $f(r) = R_0^{d-2-2\theta} \bar{\xi}(r) - \bar{\eta}(r)$. Since $R_0 \geq K$, by taking $K$ sufficiently large and by Lemma [21], we have

$$f'(r) \geq -(d - 0.5)R_0^{d-2-2\theta}r^{1-d} + 2\theta(1 + \frac{1}{r^2})^{-\theta-1}r^{d-2-2\theta} \geq (d - 0.5)(r^{d-2-2\theta} - R_0^{d-2-2\theta}) \geq 0$$

and so $f(r)$ is decreasing for $r \in [R_0/2, R_0]$. Item (ii) is proved.

Next, we will show (d). Indeed, we can write, for $y \in \mathbb{B}_R \setminus \mathbb{B}_{0.5R-2}$, with $\bar{a} \geq a$.

$$\ell_2(y) - \ell_3(y) = R_0^{d-2-2\theta}a(|y|) + A(R_0, R),$$

where $A(R_0, R)$ is a constant, and $a(r) = \bar{\xi}(r) - \bar{\eta}(r) -2 \bar{c}_{r, \gamma} R^{2-d} e^{-\gamma r^2/R^2}$. By Lemma [12],

$$a'(r) \leq -c r^{1-d} + c \gamma^{-1} c \gamma^{-1} R^{2-d} e^{-\gamma r^2/R^2} \leq c r^{1-d} \left( -1 + c \gamma^{-1} \right) < 0$$

for $r \in [0.5R-2, R]$ if $\gamma$ is chosen to be sufficiently large. Hence, $\ell_2 - \ell_3$ is radially decreasing in $\mathbb{B}_R \setminus \mathbb{B}_{0.5R-2}$. Item (d) then follows from the fact that $\ell_2 = \ell_3$ on $\partial \mathbb{B}_{R/2}$. Item (c) is a consequence of (21) in Lemma [13].

Proof of the lower bound in Theorem [10]. It suffices to show that, for $x \in B_R$,

$$G_\omega^a(x) \geq R_0^{d-2-2\theta}(U(|x| + 1) - U(R + 1)). \quad (32)$$

Recall $U(r)$ in [17]. Indeed, (32) is equivalent to the lower bound of Theorem [10] when $x \in B_{R-1}$. When $x \in B_R \setminus B_{R-1}$, $R \geq 2$, taking $y \in B_R$ with $|y| \leq |x| - 1$ and $|x - y_1| \leq 2d$, by the assumption (A2), inequality (32) yields

$$G_\omega(x) \geq P^\omega_x \{ Y_{|x-y_1|} = y \} G_\omega(y) \geq R_0^{d-2-2\theta}(U(|x|) - U(R + 1)) \geq R_0^{d-2-2\theta}(U(|x| + 1) - U(R + 2)).$$

Our proof of (32) consists of several steps. Let $C_0$ be as in Lemma [16] and recall $\tau = \tau_\omega$ in [15].

When $R \in (1, 2C_0\theta^2)$, by (20), taking $A = A(\kappa) \geq 1$ sufficiently large,

$$L_\omega[G_R - (e^{-A|x|} - e^{-A\tau^2})] \leq 0 \quad \text{in } B_R.$$
Since \( G_R = 0 \geq (e^{-A|x|^2} - e^{-AR^2}) \) on \( \partial B_R \), by the maximum principle, we have
\[ G_R \geq e^{-AR^2} - e^{-AR^2} \text{ in } B_R. \]
Thus, using the inequality \( e^a \geq 1 + a \) for \( a \geq 0 \), we get, for \( x \in B_R \). (Note that \( R \gg 1 \) in this case.)
\[ G_R \geq e^{-AR^2} \left( e^{A(R^2 - |x|^2)} - 1 \right) \geq e^{-AR^2} (R^2 - |x|^2) \geq R - |x| \]
By the fact \( a \geq \log(1 + a) \), \( a \geq 0 \), we have \( R - |x| \geq \log \frac{R + 1}{|x| + 1} \). Moreover, since \( R \gg 1 \), for \( d \geq 3 \), we also have \( R - |x| \geq (|x| + 1)^{2-d} - (R + 1)^{2-d} \). Thus \( (32) \) holds for this case.

When \( R \geq 2C_0\theta^2 \), by assumption (A2), for \( x \in B_R \) and any \( y \in B_{C_0\theta^2} \),
\[ G_R(x) \geq \sum_{i=0}^{\infty} P_{s_i}^y(X_i = y, i < \tau_R) P_{s_i}^y(X_{|y|} = 0) \]
\[ \geq G_R(x,y)_{k^{|y|}} \geq G_R(x,y), \]
and so (Recall \( G_R(\cdot, \cdot) \) in Definition 9)
\[ G_R(x) \geq G_R(x, B_{C_0\theta^2}) =: H_R(x), \quad x \in B_R. \quad (33) \]
Thus it suffices to obtain the corresponding lower bound for \( H_R \) defined above.

When \( R \in (2C_0\theta^2, 4R_0] \), since (by Lemma 15) \( L_n(H_R - \eta) \leq 0 \) in \( B_R \), by the maximum principle, we have \( H_R \geq \eta - \bar{\eta}(R) \) in \( B_R \). Notice that
\[ \bar{\eta}(r_1) - \bar{\eta}(r_2) \geq R_0^{d-2-2\theta} (U(r_1) + 1) - U(r_2 + 1), \quad \forall r_1 < r_2 \leq 4R_0. \quad (34) \]
Indeed, for \( d = 2 \),
\[ \bar{\eta}(r_1) - \bar{\eta}(r_2) = \left( \frac{1+r_1^2}{1+r_2^2} \right)^{-\theta} - 1 \]
\[ \geq C R_0^{-2\theta} \log \frac{1+r_1^2}{1+r_2^2} \geq C R_0^{-2\theta} \log \frac{1+r_1}{1+r_2}, \]
where we used the fact \( a \geq \log(1 + a) \) for \( a \geq 0 \) in the second inequality. For \( d \geq 3 \), recalling that \( \theta \geq d/2 \) in \( (31) \),
\[ \bar{\eta}(r_1) - \bar{\eta}(r_2) = (1 + r_1^2)^{-\theta} - (1 + r_2^2)^{-\theta} \]
\[ \geq (1 + r_2^2)^{0.5d-1-\theta} [(1 + r_1^2)^{1-0.5d} - (1 + r_2^2)^{1-0.5d}] \]
\[ \geq R_0^{d-2-2\theta} [(1 + r_1)^{2-2d} - (1 + r_2)^{2-2d}] \]
Hence, we obtain \( H_R \geq R_0^{d-2-2\theta} (U(|x| + 1) - U(R + 1)) \) for this case.

It remains to consider the case \( R \geq 4R_0 \). To this end, we will prove
\[ (G_R \geq \ell) H_R \geq \ell \quad \text{in } \bar{B_R}. \quad (35) \]
Assume by contradiction that (35) fails, i.e., \( \max_{B_R} (\ell - H_R) > 0 \). By Lemma 18, \( \max_{B_R} (\ell - H_R) \) is achieved outside of \( B_{R_0} \). Further, note that \( (\ell - H_R)|_{\partial B_R} \leq 0 \). By Lemma 18(c, d), and the maximum principle,

\[
\max_{B_R \setminus B_{R/2}} (\ell - H_R) \leq \max_{\partial (B_R \setminus B_{R/2})} (\ell - H_R) \leq 0 \vee \max_{B_{R/2} \setminus B_{R_0}} (\ell - H_R).
\]

Hence, if \( \max_{B_R} (\ell - H_R) > 0 \), then there exists \( x_0 \in B_{R/2} \setminus B_{R_0} \) such that

\[
(\ell - H_R)(x_0) = \max_{B_R} (\ell - H_R).
\]

Since \( x_0 \in B_{R/2} \setminus B_{R_0} \), by Lemma 18(c, d),

\[
(\ell_2 - H_R)(x_0) \geq \max_{B_{R/2} \setminus B_{R_0}} (\ell_2 - H_R),
\]

which is equivalent to

\[
(R_0^{d-2-2\theta} - H_R)(x_0) \geq \max_{B_{R_0} \setminus B_{R_0/2}} (R_0^{d-2-2\theta} - H_R).
\]

Without loss of generality, assume \( \bar{a} = I \), and set, for \( y \in \mathbb{R}^d \setminus \{0\} \),

\[
\tilde{\xi}(y) := \xi(y) - c|x_0|^{-(2+\delta)}|\xi(x_0)||y - x_0|^2,
\]

where \( c > 0 \) is chosen so that (Lemma 12 ii) \(-\Delta \tilde{\xi} \leq 0\) for \(|y| \geq C\). Then

\[
(R_0^{d-2-2\theta} - H_R)(x_0) \geq \max_{\partial B_{R_0} \setminus B_{R_0/2}} (R_0^{d-2-2\theta} - H_R) + c R_0^{d-2-2\theta} |x_0|^{-\delta} |\tilde{\xi}(x_0)|. \tag{36}
\]

Next, let \( g(x) = \tilde{\xi}(x_0 + \frac{|x_0|}{2}x) \), and let \( u \) be the solution of

\[
\begin{cases}
L_0 v(x) = 0 & x \in B_{|x_0|/2}(x_0) \\
u(x) = R_0^{d-2-2\theta} g(\frac{x-x_0}{|x-x_0|}) & x \in \partial B_{|x_0|/2}(x_0).
\end{cases}
\]

Note that \( v \) is close to \( R_0^{d-2-2\theta} \tilde{\xi} \) on \( \partial B_{|x_0|/2}(x_0) \) in the sense that \(|g(\frac{x-x_0}{|x-x_0|}) - \tilde{\xi}(x)| \leq C|x_0|^{-1} |\tilde{\xi}(x_0)| \) for \( x \in \partial B_{|x_0|/2}(x_0) \). Comparing the \( L_0 \)-harmonic functions \( v \) and \( H_R \) in \( B_{|x_0|/2}(x_0) \) via the maximum principle, we have, for \( x \in B_{|x_0|/2}(x_0) \),

\[
H_R(x) + \max_{\partial B_{|x_0|/2}(x_0)} (R_0^{d-2-2\theta} \tilde{\xi} - H_R) \geq v(x) - c R_0^{d-2-2\theta} |x_0|^{-1} |\tilde{\xi}(x_0)|. \tag{37}
\]

By Proposition 1 for \( x \in B_{|x_0|/2}(x_0) \),

\[
v(x) \geq \bar{v}(\frac{x-x_0}{|x_0|/2}) - C R_0^{d-2-2\theta} |x_0|^{-\beta} g_{\mathbb{C} \setminus \{0\}}(\partial B_1) \geq \bar{v}(\frac{x-x_0}{|x_0|/2}) - C R_0^{d-2-2\theta} |x_0|^{-\beta} |\tilde{\xi}(x_0)|,
\]

(38)
where \( \bar{\nu} \) solves
\[
\begin{cases}
\text{tr}(\delta D^2 \bar{\nu}) = \Delta \bar{\nu} = 0 & x \in \mathbb{B}_1 \\
\bar{\nu}(x) = R_0^{d-2-2\theta} g(x) & x \in \partial \mathbb{B}_1.
\end{cases}
\]
Furthermore, using the fact that \( \Delta g(x) = |x_0|^2 \Delta \bar{\xi}(x_0 + \frac{|x_0|}{2} x) \geq 0 \) for \( x \in \mathbb{B}_1 \), we get \( \bar{\nu} \geq R_0^{d-2-2\theta} g \) in \( \mathbb{B}_1 \). Therefore, by (37), (38), for \( x \in B_{|x_0|/2}(x_0) \),
\[
R_0^{d-2-2\theta} g (\frac{x-x_0}{|x_0|/2}) - H_R(x) \leq \max_{\partial B_{|x_0|/2}(x_0)} (R_0^{d-2-2\theta} \bar{\xi} - H_R) + C R_0^{d-2-2\theta} |x_0|^{-\beta} |\bar{\xi}(x_0)|.
\]
Noting that \( g (\frac{x-x_0}{|x_0|/2}) = \bar{\xi}(x) \), the above inequality contradicts (36) since \( |x_0| > R_0 \geq K \), where \( K \) is chosen to be sufficiently large. Inequality (35) is proved.

When \( x \in B_{R/2} \setminus B_{R_0} \), we have \( G_R \gtrsim \ell_2 \), and, by Lemma 12(i),
\[
\ell_2 \gtrsim R_0^{d-2-2\theta} [\bar{\xi}(|x|) - \bar{\xi}(R)]
\]
\[
\gtrsim R_0^{d-2-2\theta} \int_{|x|}^R r^{d-1} \, dr
\]
\[
\gtrsim R_0^{d-2-2\theta} (U(|x|) + 1) - U(R + 1)).
\]  \( (39) \)
When \( x \in B_{R_0} \),
\[
G_R \gtrsim \ell_1 = E_\omega^x [\ell_2(X_{x_0}) + \eta(x) - \eta(X_{x_0})]
\]
\[
\gtrsim R_0^{d-2-2\theta} E_\omega^x [U(|X_{x_0}|) + 1) - U(R + 1) + U(|x|) + 1) - U(|X_{x_0}|) + 1)
\]
\[
= R_0^{d-2-2\theta} [U(|x|) + 1) - U(R + 1)].
\]  \( (35), (34) \)
Finally, when \( x \in B_R \setminus B_{R/2} \), using the inequality \( e^a \geq 1 + a \) for \( a \geq 0 \),
\[
G_R \gtrsim \ell_3 \gtrsim R_0^{d-2-2\theta} R^{2-d} \left( e^{R(1-|x|^2/R^2)} - 1 \right)
\]
\[
\gtrsim R_0^{d-2-2\theta} R^{2-d} (1 - \frac{|x|}{R}).
\]

For \( d = 2 \), note that \( 1 - \frac{|x|}{R} \asymp \frac{R}{|x|} - 1 \geq \log \frac{R}{|x|} \). For \( d = 3 \), clearly
\[
R^{2-d} (1 - \frac{|x|}{R}) \gtrsim |x|^{2-d} (1 - (\frac{|x|}{R})^{d-2}).
\]
Our proof is complete. \( \square \)

**Lemma 19.** Assume (A1), (A2). When \( d = 2 \), then for \( \mathbb{P} \)-almost all \( \omega \),
\[
\frac{c \rho(\omega)}{\log |x|} \leq \lim_{|x| \to \infty} \frac{A^\omega(x)}{\log |x|} \leq \lim_{|x| \to \infty} \frac{A^\omega(x)}{\log |x|} \leq C \rho(\omega).
\]
3 Heat kernel bounds and consequences

3.1 Integrability of \( \rho \) and the heat kernel bounds

Using bounds of the Green functions, we will obtain the exponential integrability (under \( \mathbb{P} \)) of the Radon-Nikodym derivative \( \rho(\omega) \) (defined in (8)) and the heat kernel of the RWRE.

The goal of this section is to prove Theorem 4. Recall the continuous-time RWRE in Definition 2 and its transition kernel \( p_t(x, y) \). We remark that for the time continuous random walk \( (Y_t) \), setting

\[
\tau^Y = \tau^Y_R := \inf\{t \geq 0 : Y_t \notin B_R\},
\]

the corresponding Green functions of \( (Y_t) \) can be defined similarly as

\[
\int_0^\infty p_t^0(x, S) dt, \quad d \geq 3, \quad \text{and} \quad E_{\omega}^x \left[ \int_0^{\tau^Y} 1_{Y_t \in S} dt \right], \quad d \geq 2,
\]

and they have the same values as \( G(x, S) \) and \( G_R(x) \), respectively. Thus we do not need to distinguish notations in discrete and continuous time cases and use \( \mathcal{H} \) to denote Green’s functions in both settings.

**Corollary 20.** Assume (A1), (A2) and \( d = 2 \). For any \( \varepsilon > 0 \), there exists a random variable \( \mathcal{H} = \mathcal{H}(\omega, d, \kappa, \varepsilon) > 0 \) with \( \mathbb{E}[\exp(c \mathcal{H}^{d-\varepsilon})] < \infty \) such that \( \mathbb{P} \)-almost surely, for \( R > 0 \),

\[
\int_0^R p_t^0(x, 0) dt \leq \mathcal{H}(1 + \log \frac{R+1}{|x|+1}), \quad \forall x \in B_R.
\]

**Proof.** We only consider \( R \geq 4 \). Let \( R_k := 2^{k-1} R \), and define recursively \( T_0 = 0 \), \( T_k := \min\{t \geq T_{k-1} : Y_t \notin B_{R_k}\}, \quad k \in \mathbb{N}\).

Set \( N_R = \max\{n \geq 0 : T_n \leq R^2\} \). Then, using the strong Markov property, a.s.,

\[
E_{\omega}^x \left[ \int_0^{R^2} 1_{Y_t = 0} dt \right] \leq E_{\omega}^x \left[ \sum_{k=1}^\infty \int_{T_{k-1}}^{T_k} 1_{Y_t = 0, T_{k-1} \leq R^2} dt \right] \leq \sum_{k=1}^\infty E_{\omega}^x \left[ G_{R_k}^\omega (Y_{T_{k-1}}) 1_{T_{k-1} \leq R^2} \right] \leq \mathcal{H} \left( G_{R_k}^\omega (x) + \sum_{k=2}^\infty P_{\omega}^x (T_{k-1} \leq R^2) \right),
\]

where we used (by Theorem 10) the fact \( G_{R_k}^\omega (Y_{T_{k-1}}) \leq \mathcal{H}, \ k \geq 2 \), in the last inequality. Note that \( (Y_t) \) is a martingale. By Hoeffding’s inequality, for \( k \geq 1 \),

\[
P_{\omega}^x (T_k \leq R^2) \leq P_{\omega}^x \left( \sup_{t \leq R^2} |Y_t| \geq R_k \right) \leq C e^{-c R_k^2/R^2} \leq C \exp(-c4^k).
\]

The conclusion follows. \( \square \)
Proof of Theorem 4: First, we will show the upper bounds in (a) and (b). To this end, for \( r \geq 1 \), we take \( x_0 \in \partial B_r \). We claim that

\[
|x_0|^{2-d} \mathcal{H}^{d-1} \geq \int_{r/2}^{r} \rho_{\omega}^0(x_0,0)dr \geq r^2 \frac{\rho_{\omega}(0)}{\rho_{\omega}(B_r)}.
\]

(41)

Indeed, the lower bound of (41) follows from integrating the lower bound of (10). For \( d = 2 \), the upper bound in (41) is a consequence of Corollary 10. When \( d \geq 3 \), the upper bound of (41) follows from Corollary 11. Note that \(|x_0| \approx r\). The upper bound in (b) is proved. The upper bound in (a) then follows from taking \( r \to \infty \) and the ergodic theorem.

To obtain the lower bound in (b), for \( r \geq 5 \), recall \( \tau_r \), and \( G_r(\cdot, \cdot) \) in Definition 9. For any fixed \( y_0 \in \partial B_{r/2} \), the function \( v(x) = G_r(y_0, x)/\rho_{\omega}(x) \) solves the adjoint equation

\[
L^*_\omega v(x) := \sum_y \omega^*(x, y)[v(y) - v(x)] = 0, \quad x \in B_r/2,
\]

(42)

where

\[
\omega^*(x, y) := \frac{\rho_{\omega}(y)\omega(y, x)}{\rho_{\omega}(x)}.
\]

(43)

Here, we used the facts that \( \sum_y \rho_{\omega}(y)\omega(y, x) = \rho_{\omega}(x) \) and \( \sum_y G_r(y_0, y)\omega(y, x) = G_r(y_0, x) \). By the Harnack inequality for the adjoint operator [13, Theorem 6], we have \( v(0) \asymp v(x) \) for all \( x \in B_{r/4} \). Hence

\[
G_r(y_0, 0) \frac{\rho_{\omega}(B_{r/4})}{\rho_{\omega}(0)} \asymp G_r(y_0, B_{r/4}).
\]

(44)

Moreover, since \( (|X_n|^2 - n) \) is a martingale under \( P_{\omega} \), by the optional stopping lemma we get \( E_{\omega}^0[|X_{\tau_r}|^2 - \tau_r] = |y_0|^2 \geq 0 \), and so

\[
G_r(y_0, B_{r/4}) \leq E_{\omega}^0[\tau_r] \leq E_{\omega}^0[|X_{\tau_r}|^2] \leq Cr^2.
\]

The above inequality, together with (44) and Theorem 10 yields

\[
\frac{\rho_{\omega}(0)}{\rho_{\omega}(B_{r/4})} \gtrsim \frac{G_r(y_0, 0)}{G_r(y_0, B_{r/4})} \gtrsim \frac{\mathcal{H}^{-s}r^{2-d}}{r^2} \gtrsim \mathcal{H}^{-s}r^{-d}.
\]

The lower bound in Theorem 4(b) follows. Letting \( r \to \infty \), we also get the lower bound in (a).

3.2 Optimal semigroup decay for \( d \geq 3 \)

The Efron-Stein inequality [47] of Boucheron, Bousquet, and Massart [8] will be used in our derivation of the variance decay for the semi-group.
Let \( \omega'(x), x \in \mathbb{Z}^d \), be independent copies of \( \omega(x), x \in \mathbb{Z}^d \). For any \( y \in \mathbb{Z}^d \), let \( \omega'_y \in \Omega \) be the environment such that

\[
\omega'_y(x) = \begin{cases} 
\omega(x) & \text{if } x \neq y, \\
\omega'(y) & \text{if } x = y.
\end{cases}
\]

That is, \( \omega'_y \) is a modification of \( \omega \) only at location \( y \). For any measurable function \( Z \) of the environment \( \omega \), we write, for \( y \in \mathbb{Z}^d \),

\[
Z'_y = Z(\omega'_y), \quad \partial'_y Z(\omega) = Z'_y - Z, \tag{45}
\]

and set

\[
V(Z) = \sum_{y \in \mathbb{Z}^d} (\partial'_y Z)^2. \tag{46}
\]

By an \( L_p \) version of Efron-Stein inequality \([8, \text{Theorem 3}]\), for \( q \geq 2 \),

\[
\mathbb{E}[|Z - \mathbb{E}[Z]|^q] \leq C q^{q/2} \mathbb{E}[V^{q/2}]. \tag{47}
\]

Following the strategy of \([18]\), our proof of the diffusive decay of the semi-group \( \{P_t\} \) will make use of the Efron-Stein type inequality \((47)\) and the Duhamel representation formula \((50)\) for the vertical derivative. Let us reemphasize that, in the non-divergence form setting, there is no deterministic Gaussian bounds for the heat kernel, and the steady state \( \mathbb{Q} \) of the environment process \((\partial'_y)_{t \geq 0}\) is not the same as the original measure \( \mathbb{P} \). To overcome these difficulties, we employ crucially the heat kernel estimates and the (negative and positive) moment bounds of the Radon-Nikodym derivative \( \frac{d\omega}{d\mathbb{Q}} \) in Theorem 4.

For any \( \zeta \in L^1(\Omega) \), we write

\[
v(t) := P_t \zeta(\omega).
\]

Then, its stationary extension \( \bar{v}(t, x) \) solves the parabolic equation

\[
\begin{aligned}
\partial_t \bar{v}(t, x) - L_{\omega} \bar{v}(t, x) &= g(t, x) & t \geq 0, x \in \mathbb{Z}^d, \\
v(0, x) &= g_0(x) & x \in \mathbb{Z}^d,
\end{aligned} \tag{48}
\]

with \( g(t, x) = 0 \) and \( g_0(x) = \zeta(x; \omega) \). In general, the solution of (48) can be represented by Duhamel’s formula

\[
\bar{v}(t, x) = \sum_z p_t^{\omega}(x, z) g_0(z) + \int_0^t p_{t-s}(x, z) g(s, z) ds. \tag{49}
\]

To apply (47), recall notations \( Z'_y \) and \( \partial'_y Z \) in (45). By enlarging the probability space, we still use \( \mathbb{P} \) to denote the joint law of \( (\omega, \omega') \). For \( y \in \mathbb{Z}^d \), applying \( \partial'_y \) to (48), we get that \( \partial'_y \bar{v} \) satisfies

\[
\begin{aligned}
\partial_t (\partial'_y \bar{v})(t, x) - L_{\omega} (\partial'_y \bar{v})(t, x) &= (\partial'_y \omega(x)) \nabla_i ^2 \bar{v}(t, x) & t \geq 0, x \in \mathbb{Z}^d, \\
(\partial'_y \bar{v})(0, x) &= \partial'_y \bar{v}(x), & x \in \mathbb{Z}^d.
\end{aligned} \tag{48}
\]

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Here we used the convention of summation over repeated integer indices. Hence, by formula (49), \( \partial_t^j \tilde{v} \) has the representation

\[
\partial_t^j \tilde{v}(t, x) = \sum_z p^0_t(x, z) \partial_y^j \tilde{\zeta}(z) + \sum_{i=1}^d \int_0^t p_i^0(x, z) (\partial_t^i \omega_i(z)) \nabla_i^2 \tilde{v}_i(s, z) ds \nonumber
\]

\[
= \sum_{z \in y + \text{Supp}(\zeta)} p^0_t(x, z) \partial_y^j \tilde{\zeta}(z) + \sum_{i=1}^d \int_0^t p_i^0(x, y) (\partial_t^i \omega_i(y)) \nabla_i^2 \tilde{v}_i(s, y) ds,
\]

(50)

where in the last equality we used the fact \( \partial_t^i \omega_i(z) = 0 \) for \( y \neq z \), and that \( \partial_y^j \tilde{\zeta}(z) = 0 \) for \( z \notin y + \text{Supp}(\zeta) \).

**Proof of Theorem 5**  
Recall the notation \( \mathcal{H} \) for \( \mathcal{H}_z := \mathcal{H} (\theta_{t\omega}) \) and \( S := \text{Supp}(\zeta) \). Without loss of generality, assume \( E_{\mathbb{Q}} \zeta = 0 \). Using (50), for any \( p > 1 \),

\[
\|V(\nu(t))\|_p = \| \sum_y (\partial_t^j \tilde{v}(t, 0))^2 \|_p \nonumber
\]

\[
\leq \| \sum_y \left( \sum_{z \in S} p^0_t(0, z + y) \right)^2 \|_p + \| \sum_y \left( \int_0^t p_i^0(0, y) K(y, s) ds \right)^2 \|_p \nonumber
\]

\[
\leq (\#S)^2 \| \sum_y p^0_t(0, y)^2 \|_p + \int_0^t \| \sum_y p_i^0(0, y)^2 K(y, s)^2 \|_p^{1/2} ds \nonumber
\]

\[
=: (\#S)^2 I + II^2,
\]

(51)

where \( \#S \) denote the cardinality of \( S \), and in the last inequality we applied the Cauchy-Schwarz inequality and Minkowski’s integral inequality to the two norms respectively. Then, by Theorem 4(c),

\[
I = \| \sum_y p^0_t(0, y)^2 \|_p \leq (1 + r)^{-d} \| \sum_z \mathcal{H}^{2(d-1)} e^{-c_b |z|_1} \|_p \nonumber
\]

\[
\leq (1 + r)^{-d} \| \sum_z e^{-c_b |z|_1} \| \mathcal{H}^{2(d-1)} \|_p \nonumber
\]

\[
\leq (1 + r)^{-d/2} \| \mathcal{H}^{2(d-1)} p \|_1^{1/p},
\]

(52)

where in the last inequality we used the translation invariance of \( \mathbb{P} \).
Further, using Theorem 4(b) again,

\[
\| \mathcal{H}^{(d-1)p} K(0,s)^2 \|_1 \leq \| \mathcal{H}^{(d-1)p} \|_{2p} \| K(0,s)^2 \|_p = C_p \| K(0,s)^2 \|_p \\
\leq C_p \sum_{i=1}^{d} \| \nabla_i^2 \bar{v}(s,0) \|_{2p}^2 \\
\leq C_p \sum_{e : |e| = 1} \| \bar{v}(s,e) - \bar{v}(s,0) \|_{1/p}^{2/p} \\
\text{Hölder } \leq C_p \| \rho_{\omega}^{-1/p} \|_{1/p} \sum_{e : |e| = 1} \| \rho_{\omega} | \bar{v}(s,e) - \bar{v}(s,0) \|_1 \|_p^{1/p} \]

(54)

where in the third inequality we used the fact \( \| \bar{v} \|_\infty \leq 1 \). Then, setting

\[ u(t) := \text{Var}_Q(v(t)), \]

by (54), Theorem 4(b) and Lemma 21, we obtain

\[
\| \mathcal{H}^{(d-1)p} K(0,s)^2 \|_1 \leq C_p \sum_e E_Q(\| \bar{v}(s,e) - \bar{v}(s,0) \|_{2/p}^{1/p}) \leq C_p (-\frac{d}{dt} u)^{1/(2p)}. \]

(55)

This inequality, together with (51), (52), (53), implies

\[
\| V(v(t)) \|_p^{1/2} \lesssim_p (1+t)^{-d/4} + \int_0^t (1+t-s)^{-d/4} (-\frac{d}{dt} u)^{1/(2p)} ds,
\]

where \( \lesssim_p \) means that the multiplicative constant depends on \( (p,d,\kappa) \).

Furthermore, by Hölder’s inequality and Theorem 4(a),

\[
u(t) \leq E_Q(\| v(t) - \mathbb{E} v(t) \|^2) \leq \| \rho_{\omega} \|_{2p} \| v(t) - \mathbb{E} v(t) \|_p \lesssim_p \| V(v(t)) \|_p
\]

where we applied (47) in the last inequality.

Therefore, we conclude that, for any \( p > 1 \),

\[
u(t)^{1/2} \lesssim_p (1+t)^{-d/4} + \int_0^t (1+t-s)^{-d/4} (-\frac{d}{dt} u(s))^{1/(2p)} ds.
\]
When $d \geq 3$, we can take $p > 1$ sufficiently close to 1 and apply \cite{Lemma 3.5}) (Under the notation of \cite{1}, we apply it to the case $\gamma = d/4$ and $\delta = 1/p^3$..) to obtain 

$$\text{Var}_{\mathcal{Q}}(v(t)) = u(t) \lesssim (1 + t)^{-d/2}.$$ 

Thus, (12) is proved. By Hölder’s inequality,

$$\|v(t)\|_1 \leq \|\rho_{o_1}^{-1}\|_1^{1/2} \|\rho_{o_2}v(t)^2\|_1^{1/2} \leq C u(t)^{1/2} \leq C(1 + t)^{-d/4}.$$ 

Then, by the triangle inequality, we get $E_{\mathcal{Q}}[(v(t) - \mathbb{E} v(t))^2] \lesssim (1 + t)^{-d/2}$. By Hölder’s inequality, for any $q > 1$,

$$\|v(t) - \mathbb{E} v(t)^{2/q}\|_1 \leq \|\rho_{o_1}^{-1/q}\|_q \|v(t) - \mathbb{E} v(t)^{2/q}\|_q \lesssim_{q} E_{\mathcal{Q}}[(v(t) - \mathbb{E} v(t))^2]^{1/q}.$$ 

Display (13) is proved.\hfill \Box

**Lemma 21.** For any bounded measurable function $\zeta \in \mathbb{R}^\Omega$,

$$\frac{d}{dt} \text{Var}_{\mathcal{Q}}(v(t)) \leq - \sum_{e : |e| = 1} E_{\mathcal{Q}} \left[ (\bar{v}(t, e) - \bar{v}(t, 0))^2 \right],$$

where $L_{o_e}$ only acts on the spatial variable.

**Proof.** Without loss of generality, assume $\mathbb{E}[v(t)] = 0$. Then

$$\frac{d}{dt} E_{\mathcal{Q}}[v(t)^2] = 2 E_{\mathcal{Q}}[v(t)L_{o_e}\bar{v}(t, 0)]$$

$$= E_{\mathcal{Q}} \left[ L_{o_e}(\bar{v}(t, 0)^2) - \sum_{e : |e| = 1} o(0, e)(\bar{v}(t, e) - \bar{v}(t, 0))^2 \right]$$

$$= - \sum_{e : |e| = 1} E_{\mathcal{Q}} \left[ o(0, e)(\bar{v}(t, e) - \bar{v}(t, 0))^2 \right]$$

where in the last equality we used the fact that (since $o'(v)$ is stationary under $Q \times P_{o_e}$) for any $f \in L^1(Q)$, $E_{\mathcal{Q}}[L_{o_e}f(0; o)] = 0$. The lemma follows by the uniform ellipticity assumption.\hfill \Box

**3.3 Proof of Corollary 7: Existence of a stationary corrector for $d \geq 5$**

**Proof of Corollary 7:** Without loss of generality, assume $E_{\mathcal{Q}}[\zeta] = 0$. By Theorem 5 since $\int_{0}^{\infty} (1 + t)^{-d/4} dt < \infty$ when $d \geq 5$, the limit

$$\phi(o) = \lim_{t \to \infty} \phi_t(o) : = \lim_{t \to \infty} \int_{0}^{t} (P_s \zeta) ds$$

exists in $L^p(\mathcal{P})$ form any $p \in (0, 2)$. By (48), $\phi_t(x)$ satisfies

$$L_{o_e} \phi_t(x) = \zeta(\theta_x o) - P_t \zeta(\theta_x o), \quad x \in \mathbb{Z}^d.$$ 

Note that by Theorem 5 $\lim_{t \to \infty} P_t \zeta = 0$ in $L^p(\mathcal{P})$, $\forall p \in (0, 2)$. Therefore, taking $L^p(\mathcal{P})$ limits as $t \to \infty$, we conclude that $\phi$ satisfies $L_{o_e} \phi(x) = \zeta(x), x \in \mathbb{Z}^d$.\hfill \Box
A Appendix

Define the parabolic operator $\mathcal{L}_u$ as
$$
\mathcal{L}_u u(x, t) = \sum_{y: y \sim x} \omega(x, y)[u(y, t) - u(x, t)] - \partial_t u(x, t)
$$
for every function $u : \mathbb{Z}^d \times \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable in $t$.

**Theorem A.1.** ([13, Proposition 5]) Assume $\omega_{\text{tr}} > 2\kappa I$ for some $\kappa > 0$ and $R > 0$. Any non-negative function $u$ with $\mathcal{L}_u u = 0$ in $B_{2R} \times (0, 4R^2)$ satisfies
$$
\sup_{B_{R(2R^2)}} u \leq C \inf_{B_{R(3R^2, 4R^2)}} u.
$$

**Corollary A.2.** Assume $\omega_{\text{tr}} > 2\kappa I$ for some $\kappa > 0$ and $(x_0, t_0) \in \mathbb{Z}^d \times \mathbb{R}$. There exists $\gamma = \gamma(d, \kappa) \in (0, 1)$ such that any non-negative function $u$ with $\mathcal{L}_u u = 0$ in $B_R(x_0) \times (t_0 - R^2, t_0)$, $R > 0$, satisfies
$$
|u(\hat{x}) - u(\hat{y})| \leq C \left( \frac{R}{r} \right)^\gamma \sup_{B_r(x_0) \times (t_0 - r^2, t_0)} u
$$
for all $\hat{x}, \hat{y} \in B_r(x_0) \times (t_0 - r^2, t_0)$ and $r \in (0, R)$.

### A.1 Proof of Proposition C

Before giving a proof, recall that by [19, Proposition 2.1], for any $p \in (0, d)$, there exists $\delta_p$ depending on $(d, \kappa, p)$ such that for any $R > 0$, the solution $\phi : \bar{B}_R \rightarrow \mathbb{R}$ of
$$
\begin{cases}
\mathcal{L}_o \phi = \bar{a} - a & \text{in } B_R, \\
\phi = 0 & \text{on } \partial B_R
\end{cases}
\tag{57}
$$
satisfies
$$
\mathbb{P}(\max_{B_R} |\phi| \geq CR^{2-\delta_p}) \leq C \exp(-cR^p).
\tag{58}
$$

Set $\delta := \delta_1$. For $q \in (0, d)$, let $\gamma = \gamma(d, \kappa, q)$ be the constant
$$
\gamma = \min \left\{ \frac{d - q}{d(1 + \delta)} \cdot \frac{1}{2} \right\}
\tag{59}
$$
and set
$$
R_0 := R^\gamma, \quad \sigma = \min\{n \geq 0 : X_n - X_0 \notin B_{R_0}\}.
$$

Let
$$
\omega_0 = \frac{\omega_{\text{tr}(\omega)}}{\text{tr}(\omega)} , \quad \psi_0 = \psi_0(\omega) = \frac{\psi}{\text{tr}(\omega)}.
$$

Following [19, Definition 4.1], we define bad points.

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Definition 22. Let $\delta = \delta(d, \kappa)$ be as above. We say that a point is good (and otherwise bad) if for any $\zeta(\omega) \in \{ \psi_0, \omega_0 \}$,

$$\left| E_\omega \left[ \sum_{i=0}^{\sigma-1} (E_Q [\zeta - \zeta(\tilde{\omega})]) \right] \right| \leq C \| \zeta \|_{0} R_0^{2-\delta}.$$  

Note that by (58), $P(x \text{ is bad}) \leq Ce^{-cR_0}$.

We will give first the proof for the special case $g \in C^{2,\alpha}(\partial B_1)$. It is a small modification of the proof of [19, Theorem 1.5].

Proof of Proposition C for the case $f \in C^4(B_1), g \in C^{2,\alpha}(\partial B_1)$ and $y = 0$: Note that if $g \in C^{2,\alpha}(\partial B_1)$, it can be extended to be a function $\tilde{g} \in C^{2,\alpha}(\mathbb{B}_2)$ such that

$$|\tilde{g}|_{2,\alpha;B_2} \leq C|g|_{2,\alpha;\partial B_1}.$$  

By [17, Theorem 6.6],

$$|\bar{u}|_{2,\alpha,B_1} \leq |f|_{0,\alpha;B_1} \||w|_{tr(\omega_0)}||_{\infty} + |g|_{2,\alpha;\partial B_1} =: A$$  

(60)

Step 1. Set $\bar{u}_R(x) = \bar{u}(x/R)$ for $x \in \mathbb{B}_R$. We will show that in $B_R$, $\bar{u}_R$ is very close to the solution $\hat{u} : B_R \to \mathbb{R}$ of

$$\left\{ \begin{array}{ll}
L_\omega \hat{u} = \frac{\epsilon}{\omega_0} \text{tr}[\omega_0 D^2 \hat{u}_R] & \text{in } B_R \\
\hat{u} = \tilde{g}(\frac{x}{|x|}) & \text{on } \partial B_R.
\end{array} \right.$$  

To this end, define $u_+, u_-$ by $u_+(x) = \tilde{g}(\frac{x}{R+1}) \pm CA \frac{(R+1)^2 - |x|^2}{R^2}$, $x \in \mathbb{B}_{R+1}$. Here $A$ is as defined in (61). Then, for $x \in \mathbb{B}_{R+1}$, taking $C$ large enough,

$$\text{tr}[\bar{u} D^2 (u_+ - \bar{u}_R + 1)] \leq \frac{C}{R^2} (|g|_{2,\alpha;B_1} + |f|_{0;B_1} - CA) \leq 0.$$  

and similarly $\text{tr}[\bar{u}(u_- - \bar{u}_R + 1)] \geq 0$. The comparison principle then yields

$$u_- \leq \bar{u}_R \leq u_+ \quad \text{in } \mathbb{B}_{R+1}.$$  

In particular, for $x \in \partial B_R$, $|\bar{u}_R(x) - \tilde{g}(\frac{x}{R+1})| \leq A \frac{(R+1)^2 - |x|^2}{R^2} \leq \frac{A}{R}$ and so

$$\max_{\partial B_R} |\hat{u} - \bar{u}_R| = \max_{x \in \partial B_R} |g(\frac{x}{|x|}) - \bar{u}_R(x)| \leq \frac{A}{R}$$  

(61)

Moreover, noting that $D^2 \tilde{u}_R(x) = R^{-2} D^2 \tilde{u}(\frac{x}{R})$, in $B_R$,

$$|L_\omega (\hat{u} - \bar{u}_R + 1)| = \text{tr}[\omega_0 (D^2 \hat{u}_R - V^2 \bar{u}_R + 1)]$$  

$$\leq \text{tr}[\omega_0 (D^2 \hat{u}_R - D^2 \bar{u}_R + 1)] + |\text{tr}[\omega_0 (D^2 \tilde{u}_R + V^2 \bar{u}_R + 1)]|$$  

$$\leq R^{-2-a} |\bar{u}|_{2,\alpha;B_1} \lesssim AR^{-2-a}.$$  

(62)

Hence, by the ABP maximum principle [19, Lemmas 2.3 and 2.4], (61) and (62) imply $\max_{B_R} |\hat{u} - \bar{u}_R| \lesssim AR^{-a}$, and so, by (60),

$$\max_{B_R} |\hat{u} - \bar{u}_R| \lesssim AR^{-a}.$$  

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Step 2. Let $\omega = u - \hat{u}$. Then $u$ solves

$$
\begin{align*}
L_\omega u &= \frac{1}{2} \frac{\partial}{\partial R^2} |(\hat{u} - \omega_0)D^2\hat{u}_R| + \frac{1}{R^2} f(\frac{\hat{u}}{R})(\psi_0 - \hat{\psi}) & \text{in } B_R, \\
\| u \| &= 0 & \text{on } \partial B_R.
\end{align*}
$$

Note that, for $x \in B_{R-R_0}$ and $y \in B_{R_0}(x)$,

$$
|D^2\hat{u}_R(x) - D^2\hat{u}_R(y)| \lesssim \frac{R_0^2}{R^2} [\hat{u}]_{2,\alpha;B_1} \lesssim AR^{-2}(\frac{R_0}{R})^\alpha,
$$

and

$$
|f(\frac{x}{R}) - f(\frac{y}{R})| \lesssim \left( \frac{R_0}{R} \right)^\alpha [f]_{\alpha;B_1}.
$$

Hence, if $x \in B_{R-R_0}$ is a good point, setting

$$
\tilde{\omega}_0^i := \omega(X_i) \quad \text{and} \quad \psi_0^i := \psi(\theta_i, \omega),
$$

and noting that $E_\omega^x[\sigma] \lesssim (R_0 + 1)^2$, we have

$$
E_\omega^x[v(X_\sigma) - v(x)]
$$

and

$$
= E_\omega^x \left[ \frac{\partial}{\partial R^2} \sum_{i=0}^{\sigma-1} |(\tilde{\omega}_0^i - \tilde{a})| D^2\hat{u}_R(X_i)| + \frac{1}{R^2} f(\frac{x}{R})(\hat{\psi} - \hat{\psi}_0^i) \right]
$$

$$
\lesssim \frac{1}{R^2} \left[ E_\omega^x \left[ \sum_{i=0}^{\sigma-1} |(\tilde{\omega}_0^i - \tilde{a})| D^2\hat{u}_R(X_i)| + \frac{1}{R^2} f(\frac{x}{R}) E_\omega^x \left[ \sum_{i=0}^{\sigma-1} |\hat{\psi} - \hat{\psi}_0^i| \right] + A \frac{R_0^2}{R^2} E_\omega^x[\sigma] \right]
$$

$$
\lesssim \frac{1}{R^2} \left[ E_\omega^x \left[ \sum_{i=0}^{\sigma-1} |(\tilde{\omega}_0^i - \tilde{a})| |\hat{u}|_{2,\alpha;B_1} + \frac{1}{R^2} [f]_{0;\alpha} |E_\omega^x| \left[ \sum_{i=0}^{\sigma-1} |\hat{\psi} - \hat{\psi}_0^i| \right] + A \left( \frac{R_0}{R} \right)^{2+\alpha} \right]
$$

$$
\lesssim AR^{-2}(\frac{R_0}{R})^\alpha + A \left( \frac{R_0}{R} \right)^{2+\alpha} \lesssim AR^{-2}\alpha R_0^2.
$$

Let $\tau_R = \min\{n \geq 0 : X_n \notin B_R \}$ and set

$$
\tau(x) = v(x) + C_1 AR^{-2-\alpha} E_\omega^x[\tau_R].
$$

Then, for any good point $x \in B_{R-R_0}$, by choosing $C_1$ big enough,

$$
E_\omega^x[w(X_\sigma) - v(x)] = E_\omega^x[v(X_\sigma) - v(x)] - C_1 AR^{-2-\alpha} E_\omega^x[\sigma] < 0,
$$

where we used the fact that $E_\omega^x[\sigma] \geq R_0^2$. This implies

$$
\partial w(x); B_R = \emptyset \quad \text{for any good point } x \in B_{R-R_0},
$$

where $\partial w(x; B_R)$ denote the sub-differential set of $w$ at $x$ with respect to $B_R$. For the definition of the sub-differential set and the ABP inequality, we refer to [19, Definition 2.2, Lemmas 2.3 and 2.4]. Next, we will apply the ABP inequality to bound $|v|$ from the above.
By [19, Lemma 2.4], since
\[ L_\omega u = L_\omega v - C_1 AR^{-2 - \alpha} \lesssim AR^{-2}, \]
we know that \(|\partial u(x; B_R)| \lesssim A^d R^{-2d}\) for \(x \in B_R\). Let
\[ \mathcal{B}_R = \mathcal{B}_R(\omega, \gamma) := \# \text{bad points in } B_{R-R_0}, \]
where \#\(S\) denotes the cardinality of a set \(S\). Display (63) then yields
\[ |\partial u(B_R)| \lesssim [\mathcal{B}_R + \#(B_R \setminus B_{R-R_0})]A^d R^{-2d} \lesssim (\mathcal{B}_R + R^{d-1+\gamma})A^d R^{-2d}. \]
Hence, by the ABP inequality [19, Lemma 2.4],
\[ \min_{\mathcal{B}_R} w \geq -C R |\partial u(B_R)|^{1/d} \geq -A(R^{-1} \mathcal{B}_R^{1/d} + R^{-(1-\gamma)/d}). \]
Therefore, noting that \(\max_{x \in B_R} E_\omega^x[\tau_R] \leq (R + 1)^2\) and choosing \(\delta < 1/d\),
\[ \min_{\mathcal{B}_R} v \geq \min_{\mathcal{B}_R} w - C A R^{-\alpha} \gtrsim -A(R^{-1} \mathcal{B}_R^{1/d} + R^{-\alpha}). \]
Similar bound for \(\min_{\mathcal{B}_R} (-v)\) can be obtained by substituting \(f, g\) by \(-f, -g\) in the problem. Therefore
\[ \max_{\mathcal{B}_R} |v| \lesssim A(R^{-1} \mathcal{B}_R^{1/d} + R^{-\alpha}). \]
Step 3. Combining results in Steps [1] and [2] we get
\[ \max_{\mathcal{B}_R} |u - \bar{u}_R| \lesssim A(R^{-1} \mathcal{B}_R^{1/d} + R^{-\alpha}). \]
It is shown in Step 6 of [19, Proof of Theorem 1.5] that, with
\[ \mathcal{X} = \mathcal{X}(\omega) := \max_{R \geq 1} R^{-\gamma} \mathcal{B}_R^{1/d}, \quad (64) \]
we have \(E[\exp(c\mathcal{X}^d)] < \infty\). Therefore, recalling the values of \(\gamma, A\) in (59), (60), we conclude that
\[ \max_{\mathcal{B}_R} |u - \bar{u}_R| \lesssim R^{-\alpha}(1 + R^{-\alpha} \mathcal{X})(|f|_{1,0;B_1} \|\psi_0\|_\infty + |g|_{2,0;B_1}). \quad (65) \]

\[ \square \]

In what follows, we will relax the regularity of \(g\) to be \(C^{0,\beta}(\partial B_1)\).
Proof of Proposition $C$. First, we consider the case $y = 0$. The function $g \in C^{0,\alpha}(\partial \mathcal{B}_1)$ can be extended into $\mathbb{R}^d$ so that $g \in C^{0,\alpha}(\mathbb{R}^d)$ and

$$|g|_{0,\alpha, \partial \mathcal{B}_1} \leq C|g|_{0,\alpha, \partial \mathcal{B}_1}.$$ 

We can further obtain a smooth perturbation of it. To this end, let $\rho \in C^\infty(\mathbb{R}^d)$ be a mollifier supported on $\mathcal{B}_1$ with $\int_{\mathcal{B}_1} \rho \, dx = 1$. For $h \in (0, 1)$, set $\rho_h(x) := h^{-d} \rho\left(\frac{x}{h}\right)$, and let $g_h = \rho_h \ast g$. That is, $g_h(x) = \int_{\mathbb{R}^d} \rho_h(x - z)g(z) \, dz$. Then $g_h$ satisfies

$$\begin{cases}
|g - g_h|_{0,\alpha, \partial \mathcal{B}_1} \leq Ch^\alpha|g|_{0,\alpha, \partial \mathcal{B}_1} \\
|g_h|_{2,\alpha, \partial \mathcal{B}_1} \leq C h^{-2}|g|_{0,\alpha, \partial \mathcal{B}_1}.
\end{cases}$$

Next, for $h \in (0, 1)$, let $v : \mathcal{B}_R \to \mathbb{R}$ and $\tilde{v} : \mathcal{B}_1 \to \mathbb{R}$ be solutions of

$$\begin{cases}
\frac{1}{2} \text{tr}(\omega \nabla^2 v) = \frac{1}{R^d} f\left(\frac{x}{R}\right) \psi(\theta \omega) & \text{in } \mathcal{B}_R \\
v(x) = g_h\left(\frac{x}{R}\right) & \text{for } x \in \partial \mathcal{B}_R
\end{cases}$$

and

$$\begin{cases}
\text{tr}(\tilde{a} D^2 \tilde{v}) = f \tilde{\psi} & \text{in } \mathcal{B}_1 \\
\tilde{v} = g_h & \text{on } \partial \mathcal{B}_1.
\end{cases}$$

Then, $\max_{\mathcal{B}_1} |\tilde{u} - \tilde{v}| \leq \max_{\partial \mathcal{B}_1} |g - g_h| \leq h^\alpha|g|_{0,\alpha, \partial \mathcal{B}_1}$, and

$$\max_{\mathcal{B}_R} |u - v| \leq \max_{\partial \mathcal{B}_R} |g\left(\frac{x}{R}\right) - g_h\left(\frac{x}{R}\right)| \leq Ch^\alpha|g|_{0,\alpha, \partial \mathcal{B}_1}.$$ 

Moreover, by \eqref{est}, with $A_1 = \|f\|_{C^{0,\alpha}(\mathcal{B}_1)} \|\frac{\psi}{\psi(0)}\|_\infty + |g|_{C^{0,\alpha}(\partial \mathcal{B}_1)}$ as in Proposition $C$

$$\max_{x \in \mathcal{B}_R} |v(x) - \tilde{v}\left(\frac{x}{R}\right)| \leq R^{-\alpha \delta} \left(1 + R^{-q/d} \mathcal{X} \right) \left(\|f\|_{0,\alpha, \partial \mathcal{B}_1} \|\frac{\psi}{\psi(0)}\|_\infty + |g_h|_{2,\alpha, \partial \mathcal{B}_1}\right)$$

$$\leq A_1 h^{-2} R^{-\alpha \delta} \left(1 + R^{-q/d} \mathcal{X} \right).$$

Notice that up to an additive constant, we may assume that $\inf_{\partial \mathcal{B}_1} g = 0$, so that $|g|_{0,\alpha, \partial \mathcal{B}_1} \leq C|g|_{0,\alpha, \partial \mathcal{B}_1}$. Therefore, putting $h = R^{-\alpha \delta / 3}$, by the triangle inequality,

$$\max_{x \in \mathcal{B}_R} |u - \tilde{u}\left(\frac{x}{R}\right)| \lesssim A_1 R^{-\alpha \delta / 3} \left(1 + R^{-q/d} \mathcal{X} \right). \quad \text{(66)}$$

We proved Proposition $C$ for the case $y = 0$ with $\beta = \gamma \delta / 3$.

Finally, for any $y \in B_{3\delta}$, it follows from \eqref{est} that

$$\max_{x \in B_{R}(y)} |\tilde{u}\left(\frac{x}{R}\right) - u(x)| \leq A_1 R^{-\alpha \delta} \left(1 + \mathcal{X}(\theta \omega) R^{-q/d} \right).$$

Let

$$\mathcal{B}_R(y) = \mathcal{B}_R(\theta \omega, y) := \# \text{bad points in } B_{R - R_0}(y).$$

Observe that $\mathcal{B}_R(y) \leq \mathcal{B}_{4R}(0)$. Thus, recalling the definition of $\mathcal{X}$ in \eqref{est},

$$\mathcal{X}(\theta \omega) \leq \max_{R \geq 1} R^{-\tau} \mathcal{B}_{\frac{1}{d}}^{1/d} \leq 4^\tau \mathcal{X}.$$ 

Our proof of Proposition $C$ is complete. \qed
A.2 Proofs of Lemmas 16 and 13

Proof of Lemma 16 By direct computation, for any $i = 1, \ldots, d$, $x \in \mathbb{Z}^d$,
\[ \partial_i^3 \eta(x) = \theta(1 + |x|^2)^{-3-2}[4 \theta + 1]x_i^2 - 2|x|^2 - 2, \]
\[ |\partial_i^3 \eta(x)| \leq C \theta^3 |x|(1 + |x|^2)^{-3-2}. \]
Moreover, noting that for any $y \in \mathbb{R}^d$ and $x \notin B_\theta$ with $|y - x| \leq 1$,
\[ |\partial_i^3 \eta(y)| \leq C \theta^3 \frac{1 + \theta}{\theta} |x|(1 + |x|^2)^{-3-2} \leq C \theta^3 |x|(1 + |x|^2)^{-3-2} \]
and $|\nabla_i^3 \eta(x) - \partial_i^3 \eta(x)| \leq C \sup_{y: |y - x| \leq 1} |\partial_i^3 \eta(y)|$, we have, for a sufficiently large constant $C_0(\kappa) > 1$ and $x \notin B_{C, \theta^2}$,
\[ L_{\alpha}(x) \geq \sum_{i=1}^d \frac{\omega_i(x)}{2tr \omega(x)} \left[ \partial_i^3 \eta(x) - C \theta^3 |x|(1 + |x|^2)^{-3-2} \right] \]
\[ \geq \theta(1 + |x|^2)^{-2-2}[4 \kappa (\theta + 1) |x|^2 - |x|^2 - 1 - C \theta^2 |x|] > 0. \]
On the other hand, for $x \in B_{C, \theta^2}$, clearly $L_{\alpha}(x) \geq -\eta(x) \geq -1$.

The lemma is proved.

Proof of Lemma 13 Computations show that, for $x \neq 0$, $i = 1, \ldots, d$,
\[ \partial_i^3 e^{-2a|x|/R} = \left( \frac{-2a}{|x|} + \frac{2a x_i^2}{|x|^2} + \frac{4a^2 x_i^2}{R|x|^2} \right) R^{-1} e^{-2a|x|/R}, \]
\[ \partial_i^3 e^{-2a|x|/R} = \left( \frac{6a}{R} + \frac{3}{|x|} \frac{4a x_i^2}{R^2 |x|^2} - \frac{4a^2 x_i^2}{R^2 |x|^2} - \frac{3x_i^2}{|x|^2} \right) \frac{2ax_i}{R|x|^2} e^{-2a|x|/R}. \]
Note that, for $i = 1, \ldots, d$, $x \in B_R \setminus B_{R/2}$,
\[ \left| \left( \partial_i^3 - \frac{1}{2} \nabla_i^2 \right) e^{-2a|x|/R} \right| \leq C \sup_{y: |y - x| \leq 1} \left| \partial_i^3 e^{-2a|x|/R} \right| \leq \frac{C \alpha}{R^5} e^{-2a|x|/R}, \]
and so
\[ \left| \sum_{i=1}^d \omega(x, x + e_i) (\partial_i^3 - \frac{1}{2} \nabla_i^2) e^{-2a|x|/R} \right| \leq \frac{C \alpha}{R^5} e^{-2a|x|/R}. \]

Further, by taking $K > 0$ sufficiently large (note $R \geq K$), and choosing $\alpha > 0$ to be sufficiently small, we have, for $x \in B_R \setminus B_{R/2}$,
\[ \sum_{i=1}^d \omega(x, x + e_i) \partial_i^3 e^{-2a|x|/R} \]
\[ = \sum_{i=1}^d \omega(x, x + e_i) \left( \frac{-2a}{|x|} + \frac{2a x_i^2}{|x|^2} + \frac{4a^2 x_i^2}{R|x|^2} \right) R^{-1} e^{-2a|x|/R} \]
\[ \leq \left( \frac{-a}{|x|} + \frac{1-2ax_i^2}{|x|^2} + \frac{2(1-2ax_i^2)}{R|x|^2} \right) R^{-1} e^{-2a|x|/R} \]
\[ \leq C(-1 + C \alpha) \frac{\omega}{R^5} e^{-2a|x|/R} \leq -\frac{C \alpha}{R^5} e^{-2a|x|/R}. \]
This, together with (68), implies,
\[ L_0 e^{-2\alpha |x|/R} \leq -\frac{C_a}{R} e^{-2\alpha |x|/R}, \quad \text{for } x \in B_R \setminus B_{R/2}. \]
Display (19) is proved.

To prove (20), note that when \( x = 0 \), \( L_0 (e^{-A|x|^2}) = e^{-A} - 1 > -1 \). When \( x \in \mathbb{Z}^2 \setminus \{0\} \), choosing \( A > 0 \) sufficiently large,
\[ L_0 (e^{-A|x|^2}) = e^{-A|x|^2} \left[ \sum_{i=1}^{d} \omega(x, x + e_i)(e^{2Ax_i-A} + e^{-2Ax_i-A}) - 1 \right] \geq e^{-A|x|^2} [\kappa e^{2A-A} - 1] > 0. \]
Display (20) is proved.

It remains to prove (21). Using the inequalities \( e^a + e^{-a} \geq 2 + a^2, e^a \geq 1 + a \), we get, by taking \( A \) sufficiently large, for \( x \in B_R \setminus B_{R/2} \),
\[ L_0 (e^{-A|x|^2/R^2}) = e^{-A|x|^2/R^2} \sum_{i=1}^{d} \omega(x, x + e_i) \left[ e^{-A(1+2\lambda_i)/R^2} + e^{-A(1-2\lambda_i)/R^2} - 2 \right] \geq e^{-A|x|^2/R^2} \sum_{i=1}^{d} \omega(x, x + e_i) \left[ e^{-A/R^2} (2 + 4A^2\lambda_i^2/R^4) - 2 \right] \geq \frac{A}{R^2} e^{-A|x|^2/R^2} \sum_{i=1}^{d} \omega(x, x + e_i) \left[ \frac{4A^2\lambda_i^2}{R^2} (1 - \frac{A}{R^2}) - 1 \right] \geq \frac{A}{R^2} e^{-A|x|^2/R^2} \left[ \frac{2A^2|x|^2}{R^2} - 1 \right] > 0. \]

Our proof is complete. \( \square \)

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