A statistical field approach to capital accumulation

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Abstract
This paper presents a model of capital accumulation for a large number of heterogeneous producer–consumer agents in an exchange space in which interactions depend on agents’ positions. Agents in the exchange space are subject to both attractive and repulsive forces: exchanges drive agents closer, but crowd out more distant agents. The formalism used in this paper was developed earlier by the authors and is based on statistical field theory. It allows the analytical treatment of economic models with an arbitrary number of agents, while preserving the system’s interactions and microeconomic features of the individual level. Our results show that the dynamics of capital accumulation and the agents’ positions in the exchange space are correlated. Interactions in the exchange space induce phases within the system that depend on the relative strength of the repulsive force. When the repulsive force is strong, the system is in a phase of regulated exchanges. An initial central position both favours and fastens capital accumulation in average, and high levels of initial capital drive agents towards the centre. Yet, this phase displays mild competition and a broad-based although slow improvement in exchange terms. In this phase, random shocks can redistribute capital and initiate a virtuous circle of capital accumulation. When the repulsive force is low, a phase of deregulated exchanges emerges, in which capital distribution is less homogeneous and competition among agents harshens. Increased mobility accelerates capital accumulation for high initial capital producers, whereas low initial capital producers are now evicted from the exchange space as their prices and revenues deteriorate. Thus, a threshold effect appears. Above a certain level of initial capital, agents benefit from and remain in a central position. Below this level, they remain at the periphery of the exchange space.

Keywords Path integrals · Statistical field theory · Phase transition · Capital accumulation · Exchange space · Multi-agent model · Interaction agents

JEL Classification C02 · C60 · E00 · E1

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1 Introduction

Despite its predominant role in economic modelling, the representative agent paradigm dismisses the interplay between micro- and macroscales. For instance, collective effects stemming from heterogeneous agents’ interactions, and conversely the specific impact of the whole system on agents’ dynamics, are set aside.

We argue that a large number of heterogeneous agents do not reduce to a global aggregate entity independent from its components. It should rather be seen as an environment whose characteristics largely depend on the interactions it emerges from, and that in turn impacts diversely the agents composing it. Various states of the environment may emerge that have diverging consequences on individual agents. Several branches of the literature study the interactions within a large number of agents. They inspect dynamics and equilibria arising at the collective level that are inaccessible to the representative context. However, these approaches usually rely on numerical, parameter-dependent simulations. As a consequence, and more importantly, they do not describe analytically the possible environments and their impact on individual dynamics.

An alternative approach to large systems of heterogeneous agents can be drawn from statistical field theory. Extending Kleinert’s (1989) method to economic dynamical systems, Gosselin et al. (2017, 2020) showed that the microeconomic description of a system of agents can translate into an associated field theory. Its form, although non-standard compared to physical models, can be handled using usual theoretical physics techniques. This statistical field theory describes the environment formed by an infinite number of interacting agents, from which various phases— or collective patterns—may emerge. Agents’ behaviours, how they are influenced by and interact within their environment, may then be studied. Here, field theory is an efficient modelling tool to get new insights into the micro- and macrodescriptions of a system, while keeping the relevant economic features.

Translating standard economic models into a statistical field model is a two-step process. In a first step, the usual set of optimising agents is replaced by a probabilistic description of the system. In such a setting, individual optimisation problems are discarded. Each agent is described by a time-dependent probability distribution centred around his classical optimisation path. In a second step, the individual agents’ description is replaced by a model of field theory that replicates the properties of the system when $N$, the number of agents, is large. Although approximate, this modelling is compact enough to allow an analytical treatment of the system.

The advantages of this translation are threefold. First, it preserves the agents’ main microeconomic features, such as utility, production function, etc. Heterogeneity among agents stems from their initial position in the economic space, endowments, productions, or preferences. An action functional and its partition function constitute the associated field theory which encodes the microscopic interactions of the agents. Second, a microscopic system translated in terms of field is a good tool to study the emergence of collective states: each field minimising the action provides a background field—or ground state, or phase—that encapsulates a collective state emerging from the system. The possibility of several phases reminds of multiple equilibria dynamics that could remain undetected in the context of the representative agents. Third, once
known, the phase of the system directly impacts the individuals’ dynamics. Expanding the action functional around the minimum yields the effective action of the system, which itself allows to recover the probabilistic dynamics of one or several interacting agents. Given initial conditions and interactions with others, each agent’s individual stochastic paths may be studied within a given phase. The parameters of the system are encapsulated in the form of the ground state and may drastically change the description at the individual level. Thus, the interdependence of the macro- and microlevels can be studied.

The present paper applies a simplified version of this field formalism to a model of capital accumulation for a large number of interacting heterogeneous producer–consumer agents to focus on capital dynamics. Each agent is described by his production, consumption, capital stock and position in an abstract space of exchanges. Each agent produces one differentiated good whose price is fixed by market clearing conditions. Production functions are Cobb-Douglas, and capital stocks follow the standard capital accumulation dynamic equation.

Interactions arise from exchanges and competition and depend on agents’ positions in the exchange space. Agents consume all goods but prefer goods produced by their closest neighbours. For the sake of simplicity, we assume ad-hoc love of variety consumption functions. Thus, not only does demand depend on prices, but also on the distance between consumers and producers in the exchange space. The closer the agents, the higher their propensity to exchange and the higher the demand. Moreover, the position of each agent is itself dynamic. Agents in the exchange space are subject both to attractive and repulsive forces. Exchanges drive agents closer, but beyond a certain proximity, closest agents crowd out more distant ones.

In this context, heterogeneity among agents stems from differences in initial capital, position in the exchange space and time-dependent shocks in individual dynamics.

The dynamic exchange space presented in this model allows to study the production, exchanges, market shares and capital accumulation behaviours within a large group of agents. What are the patterns of accumulation across agents? Is there a threshold effect in initial capital that induces an inequal capital accumulation? Is there a phase in which a better wealth distribution could be reached? All these questions can be addressed within our formalism.

In our model, this formalism yields the probabilistic dynamics of individual agents through the computation of so-called transition functions. Given an initial capital stock and position in the exchange space, each individual stochastic path can be found and depends on parameters such as the strengths of interaction forces, the rate of capital depreciation and the uncertainty in economic variables. We show that depending on the strengths of interaction forces, two phases of the system may appear. In a first phase, the repulsion force compensates the attractive force induced by the exchange. Capital accumulation and mobility highly depend on initial capital stock. However, this phase is stable and allows capital accumulation for most agents. The second phase appears when the attractive force is dominant. In this phase, strong attractive forces enhance exchanges and competition. In turn, highly unequal patterns of accumulation arise. High initial capital agents accumulate faster than in the first phase, whereas low initial capital agents are evicted from exchanges.
The first section reviews the literature. Section 3 describes a classical model of capital accumulation with $N$ economic agents. Section 4 translates this classical model into a probabilistic framework. Section 5 presents the field formulation associated to the model. Section 6 describes the resolution of the system and presents its two phases and the results. Section 7 interprets the results. Section 8 concludes.

2 Literature review

Several branches of the economic literature seek to replace the representative agent by a collection of heterogeneous ones.

Mean field theory formalism applied to game theory and economics is based, like our formalism, on a system of large number of agents, but the two approaches differ in many respects (for an account of mean field game theory, see, for example, Wolf et al. 2013; Lasry et al. 2008; Guéant et al. 2011, for its economic applications). Technically speaking, the notion of fields we are referring to differs from that used in mean field theory. In mean field theory, fields are probability distributions. In our formalism, they are abstract complex functions defined on the state space, analogous to second-quantised-wave functions in quantum theory.

Second and more importantly, we differ from mean field theory in the probabilistic treatment of a system. Mean field theory studies the evolution of the agents’ density in the state space—the space of economic variables. This evolution is modelled by a transport equation coupled to the Hamilton–Jacobi–Bellman equations of fully rational agents. Interactions between agents and the population as a whole can be included through the density of population. Thus, mean field theory is an intermediate scale between the macro- and microscale. It does not seek to aggregate agents, but rather to replace them by an overall probability distribution. A similar probabilistic treatment is found in heterogeneous agents new Keynesian (HANK) models, where an equilibrium probability distribution is derived from a set of optimising heterogeneous agents in a new Keynesian context (see Kaplan and Violante 2018 for an account).

On the contrary, our approach focuses on the direct interactions between agents at the microlevel. We do not look for an equilibrium probability distribution for agents, but rather build a probability density for the system of $N$ agents as a whole, ultimately translated in terms of field. Thus, the states’ space we consider is much larger than those in the previous approaches: it is the space of all paths for a large number of agents. This allows to study agents’ economic structural relations and the emergence of particular phases or collective states induced by these specific microrelations. These phases in turn impact each agent and his stochastic dynamics at the microlevel.

Third, several more specific differences may also be noticed. HANK models stress the role of an infinite number of heterogeneously-behaved consumers. The present paper rather dwells on the supply side dynamics—although heterogeneous consumers could be considered in our context (see Gosselin et al. 2020)—and models $N$ producers of differentiated goods (in the sequel $N$ will be large). Besides, our formalism does not need any assumption about agents’ rationality. Due to their large number, agents behave randomly, be they rational or partly rational.
The information-theoretic approach to economics is also close to our methodological stance (see Yang 2018). This literature considers probabilistic states around the equilibrium. It replaces the Walrasian equilibrium by a statistical notion similar to our statistical weight. However, their statistical equilibrium is derived from an entropy maximisation program, while our statistical weight is directly built from dynamic equations at the microeconomic level. The same remarks also apply to the rational inattention theory (Sims 2006) in which non-Gaussian density laws are derived from limited information and constraints, whereas our setting directly includes constraints in the probabilistic description of an agent (Gosselin et al. 2020).

The multi-agents systems economic literature, notably agent-based models (see Gaffard and Napoletano 2012) and economic networks (Jackson 2010), is related to our approach by several aspects. Both rely on numerical simulations of multi-agents systems, but deal with two distinct types of models. Agent based models deal with general macroeconomics models, whereas network models rather deal with lower scale models, such as contract theory, behaviour diffusion, information sharing or learning. In both type of settings, agents are typically defined by, and follow, various sets of rules. Some equilibria and dynamics emerge that would otherwise remain inaccessible to the representative agent set-up. The agent-based approach, like ours, does not seek to aggregate all agents, but considers the interacting system in itself. It is, however, highly numerical and model-dependent and relies on microeconomic relations—such as ad-hoc reaction functions—that may be too simplistic. On the contrary, statistical field theory accounts for the transition between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents: they are grounded in behaviours and interactions structures. Describing these structures in terms of field theory allows to study the emergence of phases at the macroscale, and in turn their impact at the individual level.

Econophysics is also related to our approach, in that it often considers the set of agents as a statistical system (for a review, see Abergel et al. 2011a, b and references therein; or Lux 2009, 2016). Kleinert (2009) uses path integrals to model the stock prices’ dynamics. However, econophysics does not apply the full potential of field theory to economic systems and rather focuses on empirical laws. Besides, the absence of microfoundations casts doubts on the robustness of these observed empirical laws. Our approach, in contrast, keeps track of usual microeconomics concepts such as utility functions, expectations and forward-looking behaviours. It includes these behaviours in the analytical treatment of multi-agents systems by translating the optimising agents’ main characteristics in terms of statistical systems.

Capital accumulation has been considered in several ways since the Solow growth model and its subsequent developments (Solow 1957; Barro and Sala-i-Martin 1995 for an account).

The approaches closest to this paper stem from economic geography (Krugman 1991) and introduce several types of producers in a geographical, core-and-periphery environment. This environment impacts the production of partly differentiated goods, such as agricultural and manufactured goods (see Fujita and Thisse 2002 for a review). Space does appear in these models but agents are static, whereas in our approach exchange positions are dynamic and impact capital accumulation. Regional industrialisation or multi-country growth models (Aghion and Durlauf 2005) study
take-off and convergence conditions, but the geographical parameters that determine
the environment—notably transportation costs and shares of immobile workers—are
fixed. On the contrary, our environment is evolutive and may be endogenised to interact
with capital accumulation.

A recent combination of evolutionary theory, complex systems and agent-based
model is also related to our purpose and allows a detailed study of capital dynamics
for a large number of producers (Dosi and Nelson 2010; Dosi et al. 2010, 2015; Dawid
et al. 2012; Ciarli et al. 2010; Mandel et al. 2010; Wolf et al. 2013). Some models
within this literature investigate how several production sectors, using neighbours’
output as inputs, interact and compete (Mandel et al. 2010; Mandel 2012; Gualdi and
Mandel 2019). Competition and capital accumulation are driven by random changes
in technology, and producers adapt their production via imitation, replacing parts
of their inputs with newer, more efficient technologies. These models are used to
simulate numerically the firms’ dynamics, track the persistence of production sectors
and compute emergent macrovariables such as total output, wages or unemployment.

This focus on the independent sectors’ evolution is close to our purpose. However,
we depart from it in numerous stances. We do not use numerical methods, and we
consider that exchanges between agents are themselves determined by the exchange
space. Moreover, we choose to ignore the evolution of technology—even though it
could be included—but rather study the impact of exchange dynamics on agents’
capital accumulation.

3 Description of the model

This section describes a standard model of capital accumulation for a large number
of agents. The usual capital dynamics and production functions are maintained, but
agents now dynamically interact through an exchange space.

3.1 Set-up

There are \( N \) consumer–producer agents that differ in their initial capital, their position
in the exchange space and the goods they produce. We will later show that depending
on the phase, these initial differences produce specific dynamics for agents.

Each agent is differentiated by his position in the exchange space, denoted \( X_i \) \((-1, 1)\). This position is a dynamic variable that interacts with the other variables of
the model. It can be seen either as a geographic or an abstract exchange position. The
central position \( X = 0 \) ensures higher exchanges.

Each agent produces a single differentiated good. Production functions are Cobb–
Douglas. The agent’s capital stock is denoted \( K_i \) \((-1, 1)\). It is a fully liquid capital whose
price is set to one. Agents are individual producers, so labour can be discarded, and
we further assume a constant technology factor \( A \), so that the individual production
function is \( AK_i^\alpha \) \((-1, 1)\). The price of each good \( P_i \) \((-1, 1)\) is determined by market-clearing

\[ P_i = K_i^{1-\alpha} \]

\[ K_i = \frac{1}{P_i} \]

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conditions. Ultimately, the agent’s income is the product of his production and price:

\[ Y_i (t) = P_i (t) AK_i^\alpha (t) \]  

(1)

Consumption behaviours could be introduced in the model through a utility function (see Gosselin et al. 2020). However, we choose to focus on the production side and assume individual love-of-variety consumption functions. Each agent consumes all the goods produced, although not in the same quantity.

We denote \( C_{ij} (t) \) the consumption of good \( j \) by agent \( i \) at time \( t \). Three factors determine this consumption. First, the quantity of each good agent \( i \) consumes is proportional to a fraction \( \kappa (0 < \kappa < 1) \) of his income. Second, this consumption is a decreasing function of the good’s relative price the agent \( i \) faces, and—third—of the distance between the producer and agent \( i \). These two last assumptions are modelled in the following way.

The consumption \( C_{ij} (t) \) is a decreasing function \( g \) of \( R_{ij} (t) \), the relative price level of the goods consumed by agent \( i \). We define \( R_{ij} (t) \) as:

\[ R_{ij} (t) = \frac{P_j (t)}{\bar{P}_i (t)} \]

It is the ratio of \( P_j (t) \), the price of good \( j \), over a general price index faced by agent \( i \), written \( \bar{P}_i (t) \). We choose a dependency of the type \( g (R_{ij}) \sim (R_{ij})^{-(1+\gamma)} \), so that \( C_{ij} (t) \) is proportional to \( (R_{ij})^{-(1+\gamma)} \).

The price level \( \bar{P}_i (t) \) is defined as:

\[ \bar{P}_i (t) = P_K^{1-\varepsilon} \left( \hat{P}_i (t) \right)^\varepsilon \]

where \( 0 < \varepsilon < 1 \). It is a weighted product of the general price of capital \( P_K \), and a price index of the goods consumed by agent \( i \), noted \( \hat{P}_i (t) \), which is position-dependent. This price index \( \hat{P}_i (t) \) is itself defined as:

\[ \hat{P}_i (t) = \frac{1}{d} \sum_j P_j (t) \exp \left( -d_{ij} (t) / d \right) \]

(2)

with:

\[ d_{ij} (t) = \left| X_i (t) - X_j (t) \right| \]

It is a weighted average of prices, where each weight is a function of the distance between the producer and the consumer. This assumption reflects the fact that agents interact mainly with their nearest neighbours. The parameter \( d \) is constant. Inside the exponential, the factor \( \frac{1}{d} \) models the fact that, in average, agents interact on an interval of length \( d \) in the exchange space. Outside the exponential, \( \frac{1}{d} \) acts as a normalisation factor.
factor. Since the price of capital is set to 1, \( \tilde{P}_i (t) = \left( \hat{P}_i (t) \right) ^{\varepsilon} \) and the ratio \( R_{i,j} (t) \) rewrites:

\[
R_{i,j} (t) = \frac{P_j (t)}{\left( \hat{P}_i (t) \right)^{\varepsilon}}
\]

The agent’s consumption also depends on his distance with the producers of the goods he consumes. We assume consumption of good \( j \) by agent \( i \) to be an exponentially decreasing function of the distance between consumer \( i \) and producer \( j \), \(| X_i (t) - X_j (t) | \). Recall that \( X_i (t) \) and \( X_j (t) \in [-1, 1] \). By analogy with the price level, we choose a dependency of the form \( \exp \left( -\frac{d_{ij} (t)}{d} \right) / d \). This decreasing factor may account for the agents’ connections: the position in the exchange space indicates the volume of exchanges an agent establishes with other agents. It may also account for transportation costs in a geographic interpretation. Finally, it also implies that an agent at the centre of the exchange space will face a higher demand for his good.

Under the previous hypotheses, consumption of good \( j \) by agent \( i \) writes:

\[
C_i^{(j)} (t) = \frac{\kappa Y_i (t)}{(R_{i,j} (t))^{1+\gamma}} \exp \left( -\frac{d_{ij} (t)}{d} \right)
\]

For the sake of simplicity, we normalise \((1 + \gamma) \varepsilon = 1\) in the sequel. The consumption function of good \( j \) by agent \( i \) therefore becomes:

\[
C_i^{(j)} (t) = \frac{\kappa Y_i (t)}{P_j^{1+\gamma} (t) \hat{P}_i (t)} \exp \left( -\frac{d_{ij} (t)}{d} \right)
\]

Remark that agent \( i \)'s propensity to consume depends explicitly on his position. More precisely, the dependence in \( X_i (t) \) of this propensity for the whole set of goods follows the pattern:

\[
\frac{1}{d} \sum_j \exp \left( -\frac{d_{ij} (t)}{d} \right)
\]

Replacing the summation by an integral, expression (5) is proportional to \( 1 - \cosh \left( \frac{X_i (t)}{d} \right) \exp \left( -\frac{1}{d} \right) \). All things equal, the agent’s propensity to consume is maximal for \( X_i (t) = 0 \), minimal for \( X_i (t) = -1 \) and \( X_i (t) = 1 \). Exchanges are a decreasing function of the distance between agents, and the marginal propensity to consume is higher at the centre than at the periphery of the exchange space.

1 Given that \( X_i \in [-1, 1] \), the normalisation factor should depend on \( X_i \). A computation shows that, in the approximation of agents uniformly distributed in space, it is equal to \( 2d \left( 1 - \cos \left( \frac{X_i}{d} \right) \right) \). This function varies slowly over \([-1, 1]\) and can be replaced by its average, \( 2d \left( 1 - \frac{\cos X_i}{\exp \left( \frac{X_i}{d} \right)} \right) \simeq d \) for \( d \) close to 1, without impairing the results’ interpretations as shown at the end of appendix 3.

2 This factor could be reintroduced without impairing the results.
The above assumptions reflect the fact that exchanges are more frequent at the centre of the exchange space. Actually, this space can be seen as an exchange scale, in which the agent’s position measures the intensity of his exchanges. Moves towards the centre or the periphery depict an improvement or a deterioration in his exchange terms, respectively. The producer exchange terms are his ability to sell his production to consumers and competitors alike at a given price within the exchange space. These terms depend on the producer capital, revenue, distance from consumers, etc.

To conclude, note that the proportionality factor $\kappa$ should be determined by optimisation of an intertemporal utility function under the constraint of future flows of expected profits. However, assuming some autonomous consumption proportional to the agent’s revenue, $\kappa$ is constant in first approximation. Agents consume an average minimal basket of goods—similar to Keynes’ autonomous consumption—given their position in the exchange space, and reinvest the full amount of their remaining income.

### 3.2 Classical description of the model

In our setting, the goods prices are determined by market clearing conditions at any time $t$. The global demand for good $i$ at time $t$ by all agents $j$, $\sum_j C_j^{(i)}(t)$, matches the production of good $i$:

$$\sum_j C_j^{(i)}(t) = AK_i^{\alpha}(t)$$

Using (1), (2) and (4), this equation can be rewritten as:

$$\frac{\kappa}{d^2} \sum_{j,k} P_j(t) K_j^{\alpha}(t) P_k(t) \exp \left( -\frac{d_{ij}(t) + d_{kj}(t)}{d} \right) = P_i^{1+\gamma}(t) K_i^{\alpha}(t) \tag{6}$$

The capital accumulation dynamics follows a standard pattern. Capital depreciates at rate $\delta$, and capital accumulation is subject to a shock $\epsilon_i(t)$. We further assume that the revenue saved is entirely reinvested in capital at a price 1. In such a setting, the capital dynamic equation becomes:

$$K_i(t + 1) = (1 - \delta) K_i(t) + Y_i(t) - \sum_j P_j(t) C_{i}^{(j)}(t) + \epsilon_i(t) \tag{7}$$

Using (4), we find:

$$K_i(t + 1) \simeq (1 - \delta) K_i(t) + Y_i(t) - \frac{\kappa}{d^2} Y_i(t) \sum_{j,k} P_k(t) \exp \left( -\frac{d_{ij}(t) + d_{ik}(t)}{d} \right) + \epsilon_i(t) \tag{8}$$

where $\epsilon_i(t)$ has variance $\sigma^2$. 

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4 Probabilistic description

4.1 Principle

Each agent dynamics is described by a path, defined by three variables $K_i(t)$, $P_i(t)$ and $X_i(t)$, from an initial to a final point. Classically, an optimal path does exist for each agent, depending on both his interactions with others and some initial conditions, up to some fluctuations. However, due to idiosyncratic uncertainties, an agent can alternatively be described by a probability density centred around the classical optimal path. This description is actually even more general than the classical path-plus-fluctuation set-up, since it can account for all types of uncertainties in agents’ behaviours, including the uncertainty in preferences (see Gosselin et al. 2020).

Similarly, when $N$ agents are involved, we consider them to randomly depart from their respective optimal path. This means that each possible dynamics for the set of agents has to be taken into account and weighted by its probability. As a consequence, the system is described by the space of all $N$ agents’ possible paths. This space is endowed with a statistical weight which computes the probability density for any configuration of $N$ arbitrary individual paths. Shocks are assumed independent and idiosyncratic. The statistical weight of the system is the product of the individual probability densities. However, it does not follow that agents are dependent. Indeed, each individual probability density explicitly includes the agent’s interactions with others—as seen in (6) and (8)—and the density probability of the system intertwines the whole set of agents.

By construction, the statistical weight of the system is centred around one—or several in case of multiple equilibria—configuration of $N$ paths that represents the classical equilibrium. Since we are working with statistical weights directly, the shape of this equilibrium is irrelevant.

Once known, the statistical weight allows to compute the transition probabilities of the system, i.e. the probabilities for any number of agents to evolve from an initial to a final state in $K_i$, $P_i$ and $X_i$ in a given time. Technically, it amounts to computing an integral over all paths between the initial and final states considered.

4.2 Translation of the model in probabilistic terms

To apply this probabilistic description to our model, we build a statistical weight for each equation. Their product is the probability description of the system. The two first variables $K$ and $P$ are standard economic variables. Their weights are derived from the equations of the model, as proposed in Gosselin et al. (2020). However, since $X$ is not a strictly standard economic variable, we will depart from our methodology and ascribe an ad-hoc form to its dynamics.

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3 Due to the infinite number of possible paths, each individual path has a null probability to exist. We therefore use the word probability density rather than probability.
4.2.1 Probabilistic description of capital dynamics

For each time \( t \), we associate a probability to the dynamic accumulation of capital, centred around the average classical capital dynamic solution.

To do so, note that (8) implies that the quantity:

\[
K_i(t + 1) - (1 - \delta) K_i(t) + Y_i(t) - \frac{\kappa}{\sigma^2} Y_i(t) \sum_{j,k} \frac{P_k(t)}{P_j(t)} \exp \left( -\frac{d_{ij}(t) + d_{ik}(t)}{d} \right)
\]

is a Gaussian random variable \( \varepsilon_i(t) \) of variance \( \sigma^2 \). In the probabilistic formulation, equation (8) is replaced by the probability density for \( K_i(t) \):

\[
\exp \left( -\frac{1}{2\sigma^2} \left( \dot{K}_i(t) + \delta K_i(t) - AP_i(t) K_i^\alpha(t) \left( 1 - \frac{\kappa}{\sigma^2} \sum_{j,k} \frac{P_k(t)}{P_j(t)} \exp \left( -\frac{d_{ij}(t) + d_{ik}(t)}{d} \right) \right) \right)^2 \right)
\]

We then sum over \( t \) in the exponential to account for the dynamics over a given time span. It associates a density probability to a path of capital accumulation over the whole time span. This models the stochastic paths \( K_i(t) \) that satisfy in average the classical dynamic accumulation equation.

Ultimately, to associate a statistical weight to the set of paths of capital accumulation for all agents, we sum over \( i \) in the exponential of (10). The statistical weight associated to the capital accumulation of the set of agents is thus:

\[
\exp \left( -\frac{1}{2\sigma^2} \sum_i \int \left( \dot{K}_i(t) + \delta K_i(t) - AP_i(t) K_i^\alpha(t) \left( 1 - \frac{\kappa}{d^2} \sum_{j,k} \frac{P_k(t)}{P_j(t)} \exp \left( -\frac{d_{ij}(t) + d_{ik}(t)}{d} \right) \right) \right)^2 dt \right)
\]

where the implicit range of integration is the chosen time span.

4.2.2 Probabilistic description of market clearing condition

Here again, the dynamics for \( P_i(t) \) can be replaced by a statistical weight derived from market clearing conditions. We assume that these market clearing conditions hold in average for a large number of agents, but that fluctuations exist for individual agents. Thus, (6) only holds up to some random noise and must be replaced for each agent \( i \) by a probability to deviate from (6):

\[
\exp \left( -\frac{1}{2\sigma_i^2} \left( P_i^{1+\gamma}(t) K_i^\alpha(t) - \kappa \sum_{j,k} P_j(t) K_j^\alpha(t) P_k(t) \exp \left( -\frac{d_{ij}(t) + d_{ik}(t)}{d} \right) \right)^2 \right)
\]
with $\sigma_1^2$ normalised to $\frac{\sigma_2^2}{\bar{A}^2}$ in the sequel, and where $\bar{A}^2$ is a constant. We consider the market clearing condition as more binding than capital accumulation, so that $\bar{A}^2 >> 1$.

As for capital, we associate a statistical weight to the set of agents’ market clearing conditions:

$$
\exp \left( -\frac{\bar{A}^2}{2\sigma_X^2} \sum_i \int \left( \frac{\dot{X}_i(t)}{2\sigma_X^2} + V_0(X_i(t)) + \sum_j V_1(d_{ij}(t)) + \sum_{j,k} V_2(d_{ij}(t), d_{ik}(t), d_{jk}(t)) \right) \right) dt
$$

(13)

### 4.2.3 Probabilistic description of the exchange space dynamics

The exchange position $X$ does not correspond to a usual economic variable. Its dynamic equation and interactions with other economic variables could be postulated, and its probabilistic description for an arbitrary number of agents then derived. However, directly choosing a probabilistic description is equivalent, simpler and faster.

We postulate three types of forces governing agents’ dynamics within the exchange space and directly write their associated statistical weights.

A first force applies to all agents and attracts them towards the centre of the exchange space. Without this force, the system would not exist or tend to disintegrate. This force could be a political or social structure ensuring the cohesion of and exchanges within the group. More broadly, it also represents the set of all factors ensuring a minimal level of exchanges for each good. In the following, we will refer to this force as the cohesion force.

We postulate a second force induced by the exchanges existing between agents. We suppose that exchanging agents create connections that will smooth their future exchanges and eventually bring them closer within the exchange space.

Finally, we postulate a third force that counterweight the second force. We suppose that a small—close and exchanging—group of agents tend to repel possible new entrants. This force can model exclusive connections such as clientelism or various degrees of market barriers.

Considering these three forces, we choose the following statistical weight for the whole set of agents:

$$
\exp \left( -\sum_i \int \left( \frac{(\dot{X}_i(t))^2}{2\sigma_X^2} + V_0(X_i(t)) + \sum_j V_1(d_{ij}(t))
\right.
\left. + \sum_{j,k} V_2(d_{ij}(t), d_{ik}(t), d_{jk}(t)) \right) dt \right)
$$

(14)

where $\sigma_X^2$ is a constant parameter measuring the inertia of $X_i$. For $\sigma_X^2 << 1$, the variable $X_i$ presents a strong inertia. For $\sigma_X^2 >> 1$, the variable $X_i$ adjusts freely.

This statistical weight describes the random dynamics of variables $X_i(t)$. The first term $\frac{(\dot{X}_i(t))^2}{\sigma_X^2}$ represents the inertia of the variable $X_i(t)$. The variation of $X_i(t)$ over
one period, measured by \( \dot{X}_i(t) \), is in average of order \( \sigma_X \): the value of \( X_i(t) \) cannot be changed instantaneously. The three other contributions represent the forces at play on each individual agent, but also on groups of various sizes.

The term \( V_0 \) is chosen to be:

\[
V_0(X_i(t)) = \frac{\kappa_0}{2\sigma_X^2} (X_i(t) - \langle X_i(t) \rangle)^2
\]

with \( \kappa_0 \ll 1 \). The function \( V_0(X_i(t)) \) is called a potential function. It describes a weak global force tending to gather all agents towards the centre of the exchange space. It represents a cohesion force.

The term \( V_1 \) represents an attraction force between agents exchanging at the individual level:

\[
V_1(d_{ij}(t)) = -\frac{\kappa_1}{2} \frac{K_i(t)K_j(t)}{\langle K \rangle_{X_i(t)} \langle K \rangle_{X_j(t)}} \exp(-\chi_1 d_{ij}(t)) \tag{15}
\]

meaning that two exchanging agents get closer in the exchange space. The brackets \( \langle K \rangle_{X_i(t)} \) and \( \langle K \rangle_{X_j(t)} \) are the average capital stock of agents at position \( X_i(t) \) and \( X_j(t) \), and \( \chi_1 \) is an exogenous parameter that measures the inverse interaction range of \( V_1 \). As (15) shows, \( V_1(d_{ij}(t)) \) is proportional to \( K_i(t)K_j(t)/\langle K \rangle_{X_i(t)} \langle K \rangle_{X_j(t)} \): the attraction force is proportional to the exchanges, revenues and consequently capital stocks of agents.

The potential \( V_2 \) describes some repulsive force occurring in the interactions within small groups.

\[
V_2(d_{ij}(t), d_{ik}(t), d_{jk}(t)) = \frac{\kappa_2}{3} \exp(-\chi_2 (d_{ij}(t) + d_{ik}(t) + d_{jk}(t)))
\]

Here, we have chosen interactions within a group of three agents, but they could be generalised to \( k \) agents, with \( k \ll N \), where \( N \) is the total number of agents. The idea behind this force is that when several—more than two—agents interact, their existing interactions deter new ones, preventing large clusters. The parameter \( \chi_2 \) is exogenous and models the inverse interaction range of the repulsive interaction.

### 4.2.4 Probabilistic description of the system

Gathering (11), (13) and (14) yields the weight for \( N \) paths \( \{K_i(t), P_i(t), X_i(t)\}_{i=1,...,N} \) (\( t \) runs over the time span):

\[
\exp\left(-W(\{K_i(t), P_i(t), X_i(t)\}_{i=1,...,N})\right) = \exp\left(-\frac{1}{2\sigma^2} \sum_i \int dt \left( \dot{K}_i(t) + \delta K_i(t) \right)
- AP_i(t) K_i^\alpha(t) \left( 1 - \frac{\kappa}{d^2} \sum_{j,k} \frac{P_k(t) \exp\left(-\frac{d_{ij}(t) + d_{ik}(t)}{d} \right)}{P_j^\gamma(t)} \right)^2 \right)
\]
\[ \times \exp \left( -\frac{A^2}{2\sigma^2} \sum_i \int \left( P_{i+}^1 (t) K_i^\alpha (t) - \frac{\kappa}{d^2} \sum_{j,k} P_j (t) K_j^\alpha (t) P_k (t) \right) \right. \\
\left. \exp \left( -\frac{d_{ij} (t) + d_{kj} (t)}{d} \right) \right)^2 dt \]

\[ \times \exp \left( -\sum_i \int \left( \frac{\dot{\mathbf{x}}_i (t)^2}{2\sigma_\mathbf{x}^2} + V_0 (\mathbf{x}_i (t)) + \sum_j V_1 (d_{ij} (t)) \right. \right. \\
\left. \left. + \sum_{j,k} V_2 (d_{ij} (t), d_{ik} (t), d_{jk} (t)) \right) dt \right) \]

### 4.3 Transition functions

With the statistical weight (16) at hand, we can now compute the transition probabilities of the system, the probabilities for any number of agents to evolve from an initial state in \( K_i, P_i \) and \( X_i \) to a final one in a given time. To do so, we compute the integral of (16) over all paths between the initial and the final points considered.

Defining the path \( Z_i (s) = (K_i (s), P_i (s), X_i (s)) \) and \( Z (s)^[N] = \{Z_i (s)\}_{i=1 \ldots N} \) a set of \( N \) independent paths, the weight (16) now writes \( \exp (-W (Z (s)^[N])) \).

The transition functions \( T_t \left( (Z)^[N], (Z)^[N] \right) \) compute the probability for the \( N \) indistinguishable agents to evolve from \( N \) initial points \( Z (0)^[N] \equiv Z^[N] \) to \( N \) final points \( Z (t)^[N] \equiv (Z)^[N] \) during a time span \( t \). It is defined by:

\[ T_t \left( (Z)^[N], (Z)^[N] \right) = \frac{1}{\mathcal{N}} \int_{Z (0)^[N] = (Z)^[N]} \exp \left( -W (Z (s)^[N]) \right) DZ (s)^[N] \quad (17) \]

The integration symbol \( DZ (s)^[N] \) covers all sets of \( N \) paths constrained by \( Z (0)^[N] = (Z)^[N] \) and \( Z (t)^[N] = (Z)^[N] \). The factor:

\[ \mathcal{N} = \int \exp \left( -W (Z (s)^[N]) \right) DZ (s)^[N] \]

is a normalisation that sets to 1 the total probability defined by the weight (16).

The interpretation of (17) is straightforward: the usual dynamics of agents is replaced by the transition function (17). Rather than studying the full trajectory of one or several agents, we compute their probability to evolve from one configuration to another, the trajectory framework being valid in average.

Equation (17) can be generalised to define the transition functions \( T_t \left( (Z)^[k], (Z)^[k] \right) \) for \( k \leq N \) agents evolving from \( k \) initial points \( Z (0)^[k] \equiv Z^[k] \) to \( k \) final points.
A statistical field approach to capital accumulation

\[ Z(t)^{[k]} \equiv (Z)^{[k]} : \]

\[ T_t \left( (Z)^{[k]}, (Z)^{[k]} \right) = \frac{1}{N} \int_{Z(0)^{[k]}=(Z)^{[k]}} \exp \left( -W \left( Z(s)_{[N]} \right) \right) DZ(s)^{[N]} \]  \hspace{1cm} (18)

The difference with (17) is that only \( k \) paths are constrained by their initial and final points.

Ultimately, it will be useful to define the Laplace transform of \( T_t \left( (Z)^{[k]}, (Z)^{[k]} \right) : \)

\[ G_\alpha \left( (Z)^{[k]}, (Z)^{[k]} \right) = \int_0^\infty \exp (-\alpha t) T_t \left( (Z)^{[k]}, (Z)^{[k]} \right) dt \]  \hspace{1cm} (19)

It computes the—time-averaged—transition function for agents with random lifespan of mean \( \frac{1}{\alpha} \), up to a factor \( \frac{1}{\alpha} \).

**5 Statistical fields description of the model**

**5.1 Principle**

The statistical weight (16) encompasses the interactions between all stochastic paths of the system, and (18) or alternatively (19) computes the transition probability from an initial to a final state for \( k \) agents. Yet, this approach—which possible in some cases—is intractable for a large number of agents, since it would imply keeping track of the \( k \) agents’ probability transitions.

To overcome this hurdle, a more compact field formalism is necessary (see Gosselin et al. 2017, 2020). This formalism replaces the \( N \) agents’ trajectories by a complex valued function—the field \( \Psi \)—that only depends on a single set of variable, here \( (K, P, X) \). The statistical weight (16) is replaced by a probability density on the space \( \mathcal{H} \), defined as the space of complex valued functions of the variables \( (K, P, X) \). To define this probability density on \( \mathcal{H} \), we associate to each function \( \Psi (K, P, X) \) a statistical weight \( \exp (-S(\Psi)) \) that computes the probability density associated to a particular configuration \( \Psi (K, P, X) \). The functional \( S(\Psi) \) is called the field action, and the integral of \( \exp (-S(\Psi)) \) over \( \Psi \) is its associated partition function. The form of \( S(\Psi) \) is obtained directly from the probabilistic description of our model (16). The idea is that of a dictionary that would translate the various terms of the probabilistic description (16) in terms of their field equivalent. The technical derivation of the field action \( S(\Psi) \) is presented in appendix 1.

The field formalism preserves the essential information encoded in (16). However, it implements a change of perspective: instead of keeping track of the \( N \)-indexed agents, it describes their dynamics and interactions as a collective thread of all possible anonymous paths. This collective thread must be seen as an environment that itself conditions the dynamics of individual agents from one state to another.

The advantages of field formalism are twofold. First, it eases the computation of the transition functions (18) and (19). More importantly, it detects the collective states...
or phases encompassed in $S(\Psi)$. These collective states would remain undetectable using the probabilistic formulation (16).

5.2 Field-theoretic formulation of the model

To apply the above formalism to our model, we define the set of variables involved in the field $\Psi$ and the action $S(\Psi)$ which encodes the model in terms of field theory.

The set of variables $(K_i(t), P_i(t), X_i(t))_{i=1, \ldots, N}$ is replaced by a function $\Psi(K, P, X, \theta)$, where $(K, P, X)$ represents all possible values of capital, price and position for a non-labelled agent. The parameter $\theta$ is a counting variable and stands for time.

We define $Z = (K, P, X)$ and $d_{ij} = |X_i - X_j|$. By convention, and unless otherwise mentioned, the symbol $\int$ will refer to all the variables involved. The integration range for $(K, P)$ is $(\mathbb{R}^+)^2$, and that for $X$ is $[-1, 1]$. The action $S(\Psi)$ can be divided into two parts, the “$(K, P)$-part” and the “$X$-part” (see “Appendix 1” for derivations). The $(K, P)$-part of the field action is:

$$
S_1(\Psi) = \int \Psi^\dagger(Z, \theta) \left( -\frac{\sigma^2}{2} \nabla^2 K - \frac{\theta^2}{2} \nabla^2 \theta + \frac{1}{2\theta^2} + \alpha \right) \Psi(Z, \theta) + \int \left( \frac{(\delta K - APK\alpha(1-U_1))^2}{2\sigma^2} + \frac{\bar{A}^2(P^{1+\gamma} K\alpha + U_2)^2}{2\sigma^2} \right) |\Psi(Z, \theta)|^2
$$

with:

$$
U_1 = \frac{\kappa}{d^2} \int \frac{P_3 \exp \left( -\left( \frac{d_{12} + d_{13}}{d} \right) \right)}{P_2^\gamma} |\Psi(Z_2, \theta)|^2 |\Psi(Z_3, \theta)|^2 dZ_2 dZ_3 d\theta
$$

$$
U_2 = -\frac{\kappa}{d^2} \int P_2 (K_2)^\alpha P_3 \exp \left( -\left( \frac{d_{12} + d_{23}}{d} \right) \right) |\Psi(Z_2, \theta)|^2 |\Psi(Z_3, \theta)|^2 dZ_2 dZ_3 d\theta
$$

The operator $-\frac{\theta^2}{2} \nabla^2 \theta + \frac{1}{2\theta^2}$ describes the linear evolution of the counting variable, with $\frac{\theta^2}{2} \gg 1$ an arbitrary parameter (see “Appendix 1”). The parameter $\alpha$ encapsulates the overall time scale of the system, since $\frac{1}{\alpha} \gg 1$ represents the agents’ time horizon, as explained in Gosselin et al. (2020).

The $X$-part of the weight yields the field contribution:

$$
S_2(\Psi) = \int \Psi^\dagger(Z, \theta) \left( -\frac{\sigma^2}{2} \nabla^2 X \right) \Psi(Z, \theta) + \int V_0(X) |\Psi(Z, \theta)|^2 + \int V_1(d_{12}) |\Psi(Z_1, \theta)|^2 |\Psi(Z_2, \theta)|^2
$$
where:

\[ V_0 (X) = \frac{\kappa_0}{2 \sigma_X^2} (X - \langle X \rangle)^2 \]

\[ V_1 (|X - Y|) = -\frac{\kappa_1}{2} \frac{K \, K' \exp (-\chi_1 |X - Y|)}{(K)^2} \]

\[ V_2 (|X - Y|, |X - Z|, |Y - Z|) = \frac{\kappa_2}{3} \exp (-\chi_2 (|X - Y| + |X - Z| + |Y - Z|)) \]

and the full field action \( S (\Psi) = S_1 (\Psi) + S_2 (\Psi) \) is the sum of (20) and (22):

\[
S (\Psi) = \int \Psi^\dagger (Z, \theta) \left( -\frac{\sigma^2}{2} \nabla_k^2 - \frac{\sigma^2}{2} \nabla_X^2 - \frac{\vartheta^2}{2} \nabla_{\theta}^2 + \frac{1}{2 \vartheta^2} + \alpha \right) \Psi (Z, \theta) + \int \left( V_0 (X) + \frac{\delta K - A \, P \, K^\alpha (1 - U_1)}{2 \sigma^2} \right) \frac{A^2}{2 \sigma^2} \left( p^{1 + \gamma} K^\alpha + U_2 \right)^2 |\Psi (Z, \theta)|^2 + \int V_1 (d_{12}) |\Psi (Z_1, \theta)|^2 |\Psi (Z_2, \theta)|^2 + \int V_2 (d_{12}, d_{13}, d_{23}) |\Psi (Z_1, \theta)|^2 |\Psi (Z_2, \theta)|^2 |\Psi (Z_3, \theta)|^2
\]

### 5.3 Use of the field description: computation of transition functions

Several results can be derived from the field action \( S (\Psi) \) and its statistical weight \( \exp (-S (\Psi)) \): the transition functions, the effective action, the phases of the system describing the system’s collective background and the average quantities of the system.

The transition functions (18) and (19) can be retrieved from the field transition functions—or Green functions—of the field theory. These functions compute the probability for any number \( k \) of agents to evolve from an initial to a final state in \((K, P, X)\) and \( \theta \) within a given time span \( t \).

Considering \((Z, \theta)^{[k]}\) a set of \( k \) initial points, and \((\overline{Z}, \overline{\theta})^{[k]}\) a set of \( k \) final points, we write \( T_i \left( (Z, \theta)^{[k]}, (\overline{Z}, \overline{\theta})^{[k]} \right) \) the transition function between \((Z, \theta)^{[k]}\) and \((\overline{Z}, \overline{\theta})^{[k]}\), and \( G_{\alpha} \left( (Z, \theta)^{[k]}, (\overline{Z}, \overline{\theta})^{[k]} \right) \) its Laplace transform. Setting \((\overline{\theta})_i = 0\) and \((\overline{\theta})_i = t\) for \( i = 1, \ldots, k \), these functions reduce to (18) or (19), so that the probabilistic formalism of the transition functions is a particular case of the field formalism definition, which is why we will use the term transition function indistinctively in the sequel.

The computation of the transition functions relies on the fact that \( \exp (-S (\Psi)) \) itself represents a statistical weight for the system. Gosselin et al. (2020) showed that
\[ S(\Psi) \] can be slightly modified to define the \textit{action with source terms}:

\[ S(\Psi, J) = S(\Psi) + \int (J(Z, \theta) \Psi^\dagger(Z, \theta) + J^\dagger(Z, \theta) \Psi(Z, \theta)) d(Z, \theta) \quad (24) \]

where \( J(Z, \theta) \) is an arbitrary complex function, or auxiliary field. Introducing \( J(Z, \theta) \) in \( S(\Psi, J) \) allows to compute the transition functions by successive derivatives. Actually, we can show that, up to a normalization \( 1/\int \exp(-S(\Psi)) D\Psi D\Psi^\dagger \):

\[
G_\alpha \left( (Z, \theta)^{[k]}, (\overline{Z}, \theta)^{[k]} \right)
= \left[ \prod_{i=1}^{k} \left( \frac{\delta}{\delta J(\overline{Z}, \theta)^i} \frac{\delta}{\delta J^\dagger(\overline{Z}, \theta)^i} \right) \int \exp(-S(\Psi, J)) D\Psi D\Psi^\dagger \right]_{J=J^\dagger=0} \quad (25)
\]

The notation \( D\Psi D\Psi^\dagger \) denotes an integration over the space of functions \( \Psi(Z, \theta) \) and \( \Psi^\dagger(Z, \theta) \), i.e. an integral in an infinite-dimensional space. Actually, these integrals are formal and solely computed in simple cases. Usually, a series expansion of \( G_\alpha \left( (Z, \theta)^{[k]}, (\overline{Z}, \theta)^{[k]} \right) \) can be found using Feynman graphs techniques.

Once \( G_\alpha \left( (Z, \theta)^{[k]}, (\overline{Z}, \theta)^{[k]} \right) \) is computed, the expression of \( T_{t_i} \left( (Z, \theta)^{[k]}, (\overline{Z}, \theta)^{[k]} \right) \) can be retrieved in principle by an inverse Laplace transform. In terms of field theory, formula (25) means that the transition functions (19) are the correlation functions of the field theory with action \( S(\Psi) \).

In practice, transition functions will not be computed directly. It is useful to first find the \textit{phases} of the system. They are defined in first approximation by the field(s) \( \Psi_0(Z, \theta) \) that maximise(s) the statistical weight \( \exp(-S(\Psi)) \), i.e. minimise(s) \( S(\Psi) \). The field \( \Psi_0(Z, \theta) \) is the most likely configuration: it is a collective background field—a particular state of the system as a whole—that conditions agents’ dynamics. It is the background state in which the probability transitions and average values can be computed.

The existence of a minimum for \( S(\Psi) \) depends on the parameters of the system. For some values, only the trivial phase \( \Psi_0(Z, \theta) = 0 \) exists and amounts to a system with one type of equilibrium. For other values, non-trivial phases \( \Psi_0(Z, \theta) \neq 0 \) exist and reveal alternate equilibria that are qualitatively different from the trivial one.

The \textit{effective action}\(^4\) in a given phase is the series expansion of \( S(\Psi) \) around \( \Psi_0(Z, \theta) \). Transition functions in the phase \( \Psi_0(Z, \theta) \) can be computed in first approximation by replacing \( S(\Psi) \) in (25), with its second-order expansion around \( \Psi_0(Z, \theta) \) (i.e. its quadratic approximation), so that transition functions explicitly depend on the phase \( \Psi_0(Z, \theta) \). This approximation eases computations. Decomposing \( \Psi \) as:

\[
\Psi = \Psi_0 + \Delta \Psi
\]

\(^4\) Actually, this paper rather considers the classical effective action, an approximation which is sufficient for the computations at stake.
we can write the quadratic approximation as:

$$S (\Psi) = S (\Psi_0) + \int \Delta \Psi^\dagger (Z, \theta) O (\Psi_0 (Z, \theta)) \Delta \Psi (Z, \theta)$$  \hfill (26)

where $O (\Psi_0 (Z, \theta))$ is a differential operator of second order, similar to the two first contributions in (23) plus some potential term, and depending explicitly on the phase defined by $\Psi_0 (Z, \theta)$. We can show that in this phase the one-agent transition function is given by:

$$G_\alpha ((Z, \theta)^{[1]}, (Z, \theta)^{[1]}) = O^{-1} (\Psi_0 (Z, \theta)) \left( (Z, \theta)^{[1]}, (Z, \theta)^{[1]} \right)$$  \hfill (27)

In (27), the right-hand side $O^{-1} (\Psi_0 (Z, \theta)) \left( (Z, \theta)^{[1]}, (Z, \theta)^{[1]} \right)$ is the kernel of the inverse operator $O^{-1} (\Psi_0 (Z, \theta))$. It can be seen as the $(Z, \theta)^{[1]}$, $(Z, \theta)^{[1]}$ matrix element of $O^{-1} (\Psi_0 (Z, \theta))$.\footnote{The differential operator $O (\Psi_0 (Z, \theta))$ can be seen as an infinite-dimensional matrix indexed by the double (infinite) entries $(Z, \theta)^{[1]}$, $(Z, \theta)^{[1]}$. With this description, the kernel $O^{-1} (\Psi_0 (Z, \theta)) \left( (Z, \theta)^{[1]}, (Z, \theta)^{[1]} \right)$ is the $(Z, \theta)^{[1]}$, $(Z, \theta)^{[1]}$ element of the inverse matrix.}

In the quadratic approximation, the $k$-agents transition functions are the product of individual transition functions:

$$G_\alpha ((Z, \theta)^{[k]}, (Z, \theta)^{[k]}) = \prod_{i=1}^{k} G_\alpha ((Z, \theta)^{[1]}, (Z, \theta)^{[1]})$$  \hfill (28)

This approximation must be corrected by taking into account higher-order terms in the expansion of the action. These terms model the agents’ interactions in the environment defined by the phase. These corrections will not be considered in the following, but could be computed using Feynman graphs. Indeed, our point here is to stress the impact of the phase on individual dynamics. To do so, the mere quadratic approximation suffices.

Ultimately, we define several averages of a function $F ((Z, \theta), \Psi)$ depending on both $(Z, \theta)$ and $\Psi$.

The average of $F ((Z, \theta), \Psi)$ in the state $\Psi$ is given by the integral:

$$\int F ((Z, \theta), \Psi) |\Psi (Z, \theta)|^2 d (Z, \theta)$$  \hfill (29)

The—full—average of $F (\Psi (Z, \theta))$, written $\langle F (\Psi (Z, \theta)) \rangle$, is given by:

$$\langle F (\Psi (Z, \theta)) \rangle = \int F (\Psi (Z, \theta)) \exp (-S (\Psi)) D\Psi D\Psi^\dagger$$  \hfill (30)

and can be computed using a series expansion of $F ((Z, \theta), \Psi)$ and the Green functions defined by (25).
An additional quantity is the average of $F ((Z, \theta), \Psi)$ for a given $X$:

$$\langle F (\Psi (Z, \theta)) \rangle_X = \int F (\Psi (Z, \theta)) \exp (-S (\Psi)) D\Psi D\theta \Psi^\dagger$$  \hspace{1cm} (31)

where we integrate over all fields $\Psi (Z, \theta)$, with $X$ held constant.

6 Resolution, results and interpretation

This section studies the appearance of a non-trivial phase in the system described by (23). The parameters induce two potential phases. For each of these phases, we first compute and study prices and agents’ average capital as functions of their position in the exchange space. For each possible phase, we then find the quadratic part of the effective action. Ultimately, this simplified version of (23) will be used to compute the transition functions in each phase and interpret these results in terms of the agent’s dynamics.

6.1 Possibility of several phases

6.1.1 Resolution and results

The conditions for a non-trivial phase in the system are embedded in the configurations $\Psi_0 (Z, \theta)$ minimising the action. The contribution (20) to the field action is positive, so that its minimum is reached for $\Psi_0 (Z, \theta) = 0$, and a non-trivial minimum exists only if $S_2 (\Psi)$ defined in (22) has a non-trivial minimum.\hspace{1cm} ^6 \hspace{1cm} When this minimum exists, a collective configuration in the exchange space determines the overall state of the system. Actually, capital accumulation depends on distances, i.e. agents’ positions within the space: patterns of accumulation then depend on the collective dynamics reached in this exchange space.

Defining the squared norm of the potential background state $\Psi_0 (Z, \theta)$ by $\rho^2 = \int |\Psi_0 (Z, \theta)|^2$, the minimisation of $S_2 (\Psi)$ writes (see “Appendix 2”):

$$\frac{1}{2}\kappa_0^{\frac{1}{2}} - \kappa_1 \rho^2 + \kappa_2 \rho^4 = 0$$

Two possibilities thus arise. When:

$$\kappa_1^2 - 2\kappa_0^{\frac{1}{2}} \kappa_2 < 0$$

the system has only one phase, the trivial phase, with $\rho = 0$, i.e. $\Psi_0 (Z, \theta) = 0$. But for parameters such that:

$$\kappa_1^2 - 2\kappa_0^{\frac{1}{2}} \kappa_2 > 0$$  \hspace{1cm} (32)

\hspace{1cm} ^6 \hspace{1cm} This is a necessary although not sufficient condition.
a non-trivial phase may emerge. In that phase, we have:

\[ \rho \simeq \frac{\kappa_1 + \sqrt{\kappa_1^2 - 2 \kappa_0 \kappa_2}}{2\kappa_2} \]

Note that we only considered the minimisation of \( S_2 (\Psi) \), instead of that of the full action \( S_1 (\Psi) + S_2 (\Psi) \). The condition (32) is only an approximation: the value of \( \rho \) will be refined in section 5.2.2. The full minimisation of \( S_1 (\Psi) + S_2 (\Psi) \) is presented in “Appendix”.

6.1.2 Interpretation

The possibility of a non-trivial phase depends on the relative strength of the repulsive force over the attractive one. A non-trivial phase only exists for a relatively weak repulsive or strong attractive force. This describes an exchange space in which both agents’ mobility and competition are high: the local markets depicted by the points of the exchange space are open. We call phase 2 this non-trivial phase. Its existence reminds us that agents’ dynamics depend on the structure of the whole system as an environment.

The trivial phase, or phase 1 in the sequel, corresponds to a relatively strong repulsive force. It describes an economy with relatively reduced mobility of capital and/or closed markets.

6.2 Prices and average level of capital

Before computing the effective action (23) for each phase, two simplifications arise. First, due to the binding market clearing conditions, the price \( P \) in (23) is not a dynamic variable: there are no terms involving \( \nabla_P \) in (23). Thus, \( P \) can be expressed in terms of \( K, X \) and the field. We also derive the average capital stock of an agent located in \( X \) and use it to simplify the effective action into a quadratic expression for a field of two variables \( K \) and \( X \).

6.2.1 Phase 1: \( \rho = 0 \)

Resolution and results We assume small individual fluctuations around the market clearing condition, so that \( \bar{A} \gg A \). We also assume \( \sigma^2 < 1 \), so that the ratio \( \frac{\bar{A}}{A} \) measures the impact of fluctuations on the system. Consequently, in the statistical weight \( \exp(-S(\Psi)) \) with \( S(\Psi) \) defined by (23), the dominant term is the potential:

\[ \int \left( \frac{(\delta K - A P K^\alpha (1 - U_1))^2}{2\sigma^2} + \bar{A}^2 \left( P^{1+\gamma} K^\alpha + U_2 \right)^2 \right) |\Psi(Z, \theta)|^2 \] (33)

This implies that \( \exp(-S(\Psi)) \) is peaked around fields minimising (33). Because the dependence of \( \Psi \) in \( P \) is static—no gradient in \( P \) appears in \( S(\Psi) \)—we can consider
that the fields minimising (33) have the form:

\[ \Psi(Z, \theta) = \delta(P - P(K, X)) \Psi(K, X, \theta) \]  

where the function \( F(K, X) \) has to be determined. The function \( \delta(P - F(K, X)) \) is the Dirac delta function. It is non-null only if \( P = P(K, X) \), which means that the most likely fields are those of the form (34).

Classically, the interpretation is straightforward: due to market clearing conditions, the price is a function of \( (K, X) \). It is the price for the good produced by an agent at position \( X \) with capital \( K \). This price has no intrinsic dynamics and quickly adapts to the variations of other variables. And since goods are differentiated, so are their prices.

Inserting (34) into the potential terms (33), we can derive from the minimisation equation the values of both the prices \( P(K, X) \) and \( \langle K \rangle_X \)—the average value of capital—in phase \( \rho = 0 \) for a given \( X \) [see formula (31)]. The computations in “Appendix 3” yield in first approximation:

\[
P(K, X) = \frac{\left(1 + \frac{1}{2} \frac{1}{2} - \alpha \right) \left( \frac{1}{1 + \gamma (1 - \alpha)} \right) - 1}{1 + \gamma} \exp \left(- \frac{(1 - \alpha) |X|}{d (1 + \gamma (1 - \alpha))} \right) \tag{35}
\]

\[
\langle K \rangle_X = \left( \frac{A}{\delta f(X)} \right) 1 - h \left(1 + \frac{|X|}{d} \right) \exp \left(- \frac{|X|}{d} \left(1 - \frac{\cosh \frac{X}{d} \exp \left(\frac{1}{d} \right)}{\exp \left(\frac{1}{d} \right)} \right) \right) \left(1 + \frac{1}{1 + \gamma (1 - \alpha)} \right) \tag{36}
\]

where \( h \) is defined by:

\[
h = \frac{1 + \gamma (1 - \alpha)}{2 (2 - \alpha)} \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \tag{37}
\]

and:

\[
f(X) = D \exp \left(- \frac{|X|}{(1 + \gamma) d} \right)
\]

where \( D \) is given in “Appendix 3”, formula (118).

The formulas (35) and (36) will be used to compute the transition functions of the agents in the system in phase \( \rho = 0 \).

**Interpretation** The average level of capital at position \( X \) denoted \( \langle K \rangle_X \) is a function of \( X \), i.e. the position in the exchange space, and the parameter \( d \), the average distance of interaction in the exchange space. Agents’ average capital (36) decreases exponentially as a function of \( X \), their distance to the centre. Agents at \( X = 0 \) exchange more. They face a higher demand, set a higher price for their good, have higher incomes and accumulate more capital. Our results show that capital accumulation is an increasing
function of \(d\). Recall that agents exchange with all others, but that due to the exponential form of consumptions, agents exchange mainly with those on an interval of length \(d\) centred around their position. The farther agents can exchange—the better the infrastructures for instance—the higher the capital accumulation. Besides, equation (135) in “Appendix 3” shows that fluctuations in prices around the market clearing conditions reduce the fluctuations of agents’ capital around the average level of capital stocks. As could be expected, (36) shows that high capital productivity favours capital accumulation, and high rates of capital depreciation reduce capital stocks.

The above results show that the level of prices is a function of \(K\), \(X\) and parameter \(d\). The price (35) of a good produced by an agent in position \(X\) is a decreasing function of the ratio between the level of capital \(K\) and the level of average capital \(\langle K \rangle_X\) of agents at position \(X\). We retrieve the result that the higher the level of capital, the more agents produce and the lower their prices. However, this dependence to the level of capital is relative. Rather, it is the ratio of the agent’s level of capital to the average level of capital that is determinant. The price (35) is also an exponentially decreasing function of position \(X\). Given a constant level of capital, an agent at the periphery faces a lower demand and will set a lower price than at the centre of the exchange space. Ultimately, the price of a good produced is an increasing function of \(d\). The higher \(d\), the higher the volume of exchanges, and the higher the prices.

6.2.2 Phase 2: \(\rho > 0\)

Resolution and results In phase 2, the value of the background field, the price level and the agent’s average capital are computed simultaneously. To do so, we postulate a non-trivial minimum \(\Psi_0 (K, X, \theta)\) of (23). All quantities are computed for a translated field \(\Delta \Psi (K, X, \theta)\) defined by:

\[
\Psi (K, X, \theta) = \Psi_0 (K, X, \theta) + \Delta \Psi (K, X, \theta)
\]  

(38)

Computations are similar to phase 1, where the field \(\Delta \Psi (K, X, \theta)\) replaces \(\Psi (K, X, \theta)\), but includes the effect of \(\Psi_0 (Z, \theta)\).

The expressions for \(P\) and the average capital of an agent in position \(X\) are computed in “Appendix 4”. We find:

\[
P (K, X) = \frac{\left(\frac{1+\gamma (1-\alpha)}{2-\alpha} \kappa \rho \right)^{\frac{2}{(1+\gamma (1-\alpha))}} \left(\frac{K}{\langle K \rangle_X} \frac{\alpha}{1+\gamma} \exp \left(-\frac{(1-\alpha) |X|}{d (1+\gamma (1-\alpha))}\right) \right)}{\left(1-h \left(1+\frac{|X|}{d}\right) \exp \left(-\frac{|X|}{d}\right) \left(1-\cosh \frac{X}{d} \exp \left(\frac{X}{d}\right)\right)\right)^{\frac{1+\gamma}{\gamma (1-\alpha)}}}
\]  

(39)

and:

\[
\langle K \rangle_X = \left(\frac{A}{\delta} f (X) \left(1-h \left(1+\frac{|X|}{d}\right) \exp \left(-\frac{|X|}{d}\right) \left(1-\cosh \frac{X}{d} \exp \left(\frac{X}{d}\right)\right)\right)\right)^{\frac{1+\gamma}{\gamma (1-\alpha)}}
\]  

(40)
with:
\[ f(X) = D_\rho \exp \left( -\frac{|X|}{(1 + \gamma)d} \right) \]
and \( h \) defined in (37). Recall that \( \rho^2 \) is the squared norm of \( \Psi_0 \). The value of \( D_\rho \) is given in (142) and depends on \( \rho \).

These results are similar to phase 1. The patterns of (39) and (40) as functions of \( K, X \) and \( d \) are similar to (35) and (36) and differ only in magnitudes. This is due to the norm of the fundamental state \( \Psi_0 \), \( \rho^2 \), a parameter specific to phase 2. The effective action in phase 2 is (49), where \( \Psi_1(K_2, X_2, \theta) \) is replaced by \( \Psi_1(K, X, \theta) + \Delta \Psi(K, X, \theta) \). The effective action thus depends explicitly on \( \rho^2 \).

The parameter \( \rho \) had already been proxied to provide a qualitative condition for the existence of a non-trivial phase. This approximation can now be refined by solving the equation for the state \( \Psi_1(K, X, \theta) \). “Appendix 4” shows that, for \( \chi_1 \ll 1 \) and \( \chi_2 \ll 1 \):

\[
\rho^2 = \frac{\kappa_1 + \sqrt{\kappa_1^2 - 2\kappa_2 \left( 2\alpha + \frac{1}{\vartheta^2} + \sqrt{\kappa_0} + \sqrt{\delta^2 + \frac{A^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}} \right)}}{2\kappa_2} \tag{41}
\]

with \( U = 1 - \langle U_1 \rangle \) defined by formulas (21) and (30):

\[
U = 1 - \langle U_1 \rangle \simeq 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( -\frac{1}{2d} \right) \left( 1 - \frac{\exp \left( -\frac{1}{2d} \right)}{2} \right) \tag{42}
\]

Moreover, at the second order in \( \chi_1 \) and \( \chi_2 \), the precise form of \( \Psi_0(K, X, \theta) \) is given by:

\[
\Psi_0(K, X, \theta) = \rho N \Psi_0^{(1)}(X) \Psi_0^{(2)}(K)
\]

where:

\[
\Psi_0^{(2)}(K) = \exp \left( -\frac{\omega (K - \langle K \rangle X)^2}{2\sigma_X^2} \right) \tag{43}
\]

\[
\Psi_0^{(1)}(X) = \exp \left( -\frac{\omega_X (X - \delta X)^2}{2\sigma_X^2} \right) H(X) + \exp \left( -\frac{\omega_X (X + \delta X)^2}{2\sigma_X^2} \right) H(-X) \tag{44}
\]

\[
\omega = \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}} \tag{45}
\]

\[
\omega_X = \sqrt{\kappa_0 \left( 1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\kappa_1}{2} \frac{K}{\langle K \rangle X} \rho^2 + 2\kappa_2 \chi_2^2 \rho^4 \right) \right)} \tag{46}
\]

Note that the functions \( \Psi_0^{(1)} \) and \( \Psi_0^{(2)} \) each depend on the two variables \( X \) and \( K \), but the dependency of \( \Psi_0^{(1)} \) in \( K \) and \( \Psi_0^{(2)} \) in \( X \) is of second order in \( \chi_1 \) and \( \chi_2 \), which justifies our notations.
\[ \delta X = \frac{\sigma^2}{\kappa_0} \left( -\chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \] (47)

\[ N \simeq \frac{\sqrt{\omega \omega X}}{2\pi \sigma_X \sigma} \] (48)

and \( H(X) \) is the Heaviside function.

**Interpretation** The prices (39) are similar to (35), and the interpretation of phase 1 remains valid. However, in phase 2, the prices (39) depend on the norm \( \rho \) of the background state \( \Psi_0 (K, X, \theta) \). Equation (41) shows that \( \rho \) is a decreasing function of \( r = \kappa_2 / \kappa_1 \). This ratio is the relative magnitude of the repulsive force with respect to the attractive one. As a consequence, in phase 2, \( P(K, X) \) is an increasing function of \( r \). The higher the repulsive force, the lower the competition between agents and the higher the prices. We can also compare (39) with (35). To do so, we use the fact that the second phase only exists if [see (41)]:

\[ \kappa_1^2 - 2\kappa_2 \left( 2\alpha + \frac{1}{\vartheta^2} + \sqrt{\kappa_0} + \sqrt{\delta^2 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}} \right) > 0 \]

For interaction parameters of relatively small magnitude, \( \kappa_1 \) is of order \( \sqrt{\kappa_2} \) and \( \rho \) is of order \( \frac{1}{\sqrt{\kappa_2}} > 1 \). As a consequence, \( P(K, X) \) is lower in phase 2 than in phase 1, and phase 2 describes a system with increased competition.

### 6.3 Quadratic effective actions

Once the prices and the average level of capital found for each phase, we can rewrite (23) as a functional of the field \( \Psi(K, X, \theta) \). We find:

\[ S(\Psi) = \int \Psi^\dagger (K, X, \theta) \left( -\frac{\sigma^2}{2} \nabla^2_k - \frac{\sigma^2}{2} \nabla^2_X - \frac{\vartheta^2}{2} \nabla^2_\theta + \frac{\omega^2}{2\sigma^2} (K - \langle K \rangle_X)^2 + \frac{1}{2\vartheta^2} + \alpha \right) \Psi(K, X, \theta) \]

\[ + \int (V_0(X)) |\Psi(K, X, \theta)|^2 + \int V_1(d_{12}) |\Psi(K_1, X_1, \theta)|^2 |\Psi(K_2, X_2, \theta)|^2 \]

\[ + \int V_2(d_{12}, d_{13}, d_{23}) |\Psi(K, X, \theta)|^2 |\Psi(K_2, X_2, \theta)|^2 |\Psi(K_3, X_3, \theta)|^2 \]

(49)

where \( \omega \) has been defined in (45) and \( X_1 = X \) in \( d_{12} \) and \( d_{12} \).

In Eq. (49), we have replaced \( P \) as a function of \( K, X \) and the field \( \Psi(Z, \theta) \). In each phase, the agents’ transition probabilities can be computed by finding the quadratic approximation of (49).
The results are phase-dependent for two reasons: first, the parameter \( \langle K \rangle_X \) in (49) is itself phase-dependent. Moreover, the quadratic expansion of (49) is performed around 0 in phase 1, and around \( \Psi_0 (K, X, \theta) \) in phase 2, which yields results that are qualitatively different.

To perform the computations, we first simplify in each phase the \( X \)-dependent part of the action by replacing the two last terms in (49) by their averages defined in (30). It amounts to considering the variable \( X \) as varying more slowly than \( K \): capital evolves faster than positions do in the exchange space. As before, the computations are carried out up to second order in \( \chi_1 \) and \( \chi_2 \).

6.3.1 Phase 1: \( \rho = 0 \)

**Resolution and results** Using the effective action (49), the \( X \)-dependent part of the action \( S_2 (\Psi) \) can be evaluated by replacing the interaction terms by their average in phase 1. The computations of appendix 5 lead to an approximation of the overall quadratic action:

\[
S (\Psi) = \Psi^\dagger (K, X, \theta) \left( -\frac{\sigma^2}{2} \nabla^2 K - \frac{\vartheta^2}{2} \nabla^2 \theta - \frac{\sigma_X^2}{2} \nabla^2 X + \frac{\omega^2}{2\sigma^2} (K - \langle K \rangle_X)^2 + \frac{\omega_X}{2} X^2 + \alpha_X \right) \Psi (K, X, \theta) \tag{50}
\]

with:

\[
\omega_X = \kappa_0 + \left( \frac{\kappa_1}{2} \frac{K}{\langle K \rangle_X} (\chi_1 - \chi_1^2) - \frac{\kappa_2}{3} \left( 2\chi_1^2 - 5\chi_2^2 \right) \right) \sigma_X^2
\]

\[
\alpha_X = \alpha + \frac{1}{2\vartheta^2} - \frac{\kappa_1}{2} \frac{K}{\langle K \rangle_X} \left( 1 - \frac{\chi_1}{2} + \frac{1}{6} \chi_1^2 \right) + \frac{\kappa_2}{3} \left( 1 - \frac{3}{2} \chi_2 + \frac{4}{3} \chi_2^2 \right) \tag{51}
\]

**Interpretation** Equation (50) describes what would seem to be a system of three independent variables with quadratic potential. The transition functions of such a system are Gaussian with drift and describe a stochastic path fluctuating around a trend. However, on closer inspection, the variables in Eq. (50) influence each other non-linearly through the \( X \)-dependent average capital \( \langle K \rangle_X \). When an agent moves in the exchange space, both his neighbourhood and the average capital \( \langle K \rangle_X \) in this neighbourhood are impacted. The consequences of these interactions between variables are studied in section 5.4.

6.3.2 Phase 2: \( \rho > 0 \)

**Resolution and results** As in the trivial phase, we approximate the \( X \)-dependent part of the action \( S_2 (\Psi) \) in the effective action (49) by replacing the interaction terms by their expectations. Here, contrary to phase 1, expectations are modified by the background field \( \Psi_0 \). Actually, as in (38), we set:

\[
\Psi (K, X, \theta) = \Psi_0 (K, X, \theta) + \Delta \Psi (K, X, \theta)
\]
The effective action is written in terms of \( \Delta \Psi (K, X, \theta) \) by an expansion of (49) around \( \Psi_0 (K, X, \theta) \). To do so, we decompose \( \Delta \Psi = \Delta' \Psi + \frac{(\delta \rho)^2}{2 \rho^2} \Psi_0 \), where \( \Delta' \Psi \) is a variation orthogonal to \( \Psi_0 \), and \( \frac{(\delta \rho)^2}{2 \rho^2} \) is an infinitesimal variation of the norm, so that \( |\Psi_0 + \Delta \Psi|^2 = \rho^2 + (\delta \rho)^2 + |\Delta' \Psi|^2 \).

The full quadratic action for the system in phase 2 is computed in appendix 5. We find:

\[
S(\Psi) = \int \Delta' \Psi^\dagger (K, X, \theta) \left( -\frac{\sigma^2}{2} \nabla^2 K - \frac{\theta^2}{2} \nabla^2 \theta - \frac{\sigma_X^2}{2} \nabla^2 X + \frac{\omega^2}{2 \sigma^2} (K - \langle K \rangle)^2 + \alpha_X \right) \Delta' \Psi (K, X, \theta)
+ \int \Delta' \Psi^\dagger (K, X, \theta) \left( \frac{\alpha_X^2}{2 \sigma_X^2} \left( X - \frac{sgn(X) \sigma_X^2}{\kappa_0} \left( -\chi_1 \kappa_1 \frac{K}{\langle K \rangle} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right)^2 \Delta' \Psi^\dagger (K, X, \theta)
+ \left( 2 \kappa_2 \rho^2 - \kappa_1 \right) \rho^2 \left| \int \Delta \Psi (K, X, \theta) \Psi_0 (K, X, \theta) \right|^2
\]

where the sign function \( sgn(X) \) is equal to 1 for \( X > 0 \), and \(-1\) otherwise. The value of \( \rho^2 \) has been defined in (41), and the other parameters are given by:

\[
\omega = \sqrt{\delta^2 + \frac{A^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}}
\]

\[
\omega_X = \sqrt{\kappa_0 \left( 1 + \frac{2 \sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{2} \frac{K}{\langle K \rangle} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right)}
\]

\[
\alpha_X = -\frac{1}{2} \kappa_0 \sqrt{1 + \frac{2 \sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{2} \kappa_1 \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right)} - \sqrt{\frac{\delta^2 + \frac{A^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}}{2}}
\]

The value of \( U_1 \) is given in (42).

**Interpretation** The interpretations of the first phase apply, with one important difference: the impact of the agent’s position in the exchange space is stronger in this phase than in phase 1. This is apparent from the second term of (52), which is a direct interaction term between \( X \) and \( K \). It actually magnifies the interactions between variables, which stresses the importance of the position in the exchange space for the dynamics of capital.

### 6.4 Transition probabilities and agents’ dynamics

As in formulas (26) and (27), agents’ transition probabilities can be computed by inverting the operators defining the quadratic actions (50) in phase 1 and (52) in phase 2.
6.4.1 Phase 1: ρ = 0

Resolution and results In phase 1, the transition functions are found by averaging the mutual influences between variables in (50) (see “Appendix 5”). The results are the following:

Consider an agent moving from \((K, X)\) to \((K', X')\), and set:

\[
\langle K \rangle = \frac{\langle K \rangle_X + \langle K \rangle_Y}{2}
\]  

(54)

The quantity \(\langle K \rangle\) is the average capital for the set of agents between \(X\) and \(X'\). This is the average influence of \(X\) on capital stocks within the dynamics from \((K, X)\) to \((K', X')\).

Similarly, we define the average values for the parameters \(\omega_X\) and \(\alpha_X\) along a capital path from \(K\) to \(K'\):

\[
\bar{\omega}_X = \omega_0 + \left(\frac{\kappa_1}{4} \left( \frac{K}{\langle K \rangle_X} + \frac{K'}{\langle K \rangle_Y} \right) \left( \chi_1 - \chi_1^2 \right) - \frac{\kappa_2}{3} \left( 2\chi_2 - 5\chi_2^2 \right) \right) \sigma_X^2
\]

\[
\bar{\alpha}_X = \alpha + \frac{1}{2\bar{\sigma}^2} - \frac{\kappa_1}{4} \left( \frac{K}{\langle K \rangle_X} + \frac{K'}{\langle K \rangle_Y} \right) \left( 1 - \frac{\chi_1}{2} + \frac{1}{3} \chi_1^2 \right)
\]

\[
+ \frac{\kappa_2}{3} \left( 1 - \frac{3}{2} \chi_2 + \frac{4}{3} \chi_2^2 \right)
\]

(55)

Using (50), (54) and (55), an agent’s transition probability in the first phase is given by:

\[
G(K, K', P, P', X, X', \theta, \theta')
= \exp \left( -\left[ \frac{\delta (K - \langle K \rangle_X)^2}{2\bar{\sigma}^2} \right]_{(K, X)} (K', X') \right) \times \sqrt{\frac{\omega_X}{2\pi \sigma_X^2 \sinh (\omega \theta' - \theta)}}
\]

\[
\times \exp \left( -\bar{\omega}_X \left( \left( X^2 + (X')^2 \right) \cosh (\omega \theta' - \theta) - 2XXX' \right) \right)
\]

\[
\times \sqrt{\frac{\omega/2\pi \theta^2}{\sinh (\omega \theta' - \theta)}}
\]

\[
\times \exp \left( -\omega \left( \left( (K - \langle K \rangle)^2 + (K' - \langle K \rangle)^2 \right) \cosh (\omega \theta' - \theta) - 2(K - \langle K \rangle)(K' - \langle K \rangle) \right) \right)
\]

\[
\times \delta (P - P(K, X)) \times \delta (P' - P(K', X')) H(\theta' - \theta)
\]

(56)

where \(\delta (u)\) is the Dirac function—the function which is null for all \(u \neq 0\) and peaked at \(u = 0\). The value of \(\omega\) is given in (45). The Heaviside function \(H(u)\) is equal to 1 when \(u \geq 0\), and null otherwise.
Equation (56) computes the probability for an agent starting at \((K, P, X, \theta)\) to reach \((K', P', X', \theta')\) during a time span \((\theta' - \theta)\). It also allows to compute the probability for \(k\) agents with initial state \((K_i, P_i, X_i, \theta)\) to reach the state \((K'_i, P'_i, X'_i, \theta')\). Neglecting interactions, i.e. the two last terms in (49), it is the mere product of \(k\) copies of formula (56) \([\text{see (28)}]\):

\[
\prod_{i=1}^{k} G \left( K_i, K'_i, P_i, P'_i, X_i, X'_i, \theta, \theta' \right)
\]

This probability is a product of independent probabilities. Indeed, in this set-up, interactions have been absorbed in the Green function parameters. This illustrates that the transition functions of agents are mainly shaped by the global environment, rather than by some specific interactions.

**Interpretation** Formula (56) highlights the main features of individual agents’ dynamics.

Consider an agent evolving from \((K, X)\) at time \(\theta\) to \((K', X')\) at time \(\theta'\). This final point is random, but given the system’s parameters, some values of \((K', X')\) are more likely, so that average dynamic patterns appear.

We inspect each of the terms of Eq. (56).

The first exponential in (56) merely translates that agents’ capital stocks at \(X\) are bounded around the average stock \(\langle K \rangle_X\).

The second exponential in (56) describes the dynamics on \(X\). It has the form of a stochastic harmonic oscillator, except that the frequency \(\bar{\omega}_X\) is not constant and satisfies (55). This exponential can be rewritten:

\[
\exp \left( -\bar{\omega}_X \left( \frac{\left( \left( X - X' \right)^2 \right) + \left( \cosh \left( \bar{\omega}_X \left( \theta' - \theta \right) \right) - 1 \right) \left( X^2 + \left( X' \right)^2 \right)}{2\sigma_X^2 \sinh \left( \bar{\omega}_X \left( \theta' - \theta \right) \right)} \right) \right)
\]

The first term in the exponential \(\left( X - X' \right)^2\) represents a Brownian random walk: any agent starting at \(X'\) could move randomly with a standard deviation of \(\sqrt{\frac{\sinh \left( \bar{\omega}_X \left( \theta' - \theta \right) \right)}{\bar{\omega}_X}}\).

The additional term, however, \(\left( \cosh \left( \bar{\omega}_X \left( \theta' - \theta \right) \right) - 1 \right) \left( X^2 + \left( X' \right)^2 \right)\), favours a final \(X\) closer to the centre \(X = 0\): the cohesion force drives agents towards the centre of the exchange space and the parameter \(\bar{\omega}_X\) is a rough estimator of the speed of this move.

Actually, for \(\bar{\omega}_X << 1\), the weight (57) is similar to a Brownian motion. There is no trend in the agent’s dynamics. The driving force is weak, and the convergence very slow. For \(\bar{\omega}_X >> 1\) and \(\sigma_X^2 < 1\) on the contrary, the configuration \(X = X' = 0\) is quickly reached as \(\theta' - \theta\) increases.

Equation (55) shows that the speed \(\bar{\omega}_X\) depends on the agent’s initial capital. As expected, this speed also depends positively on \(\kappa_0\), the magnitude of the cohesion force, and negatively on \(\kappa_2\), the magnitude of the repulsive force.
More importantly, the magnitude $\kappa_1$ of the attractive force is dampened by the ratio of the agent’s current capital stock $K$ to $\langle K \rangle_X$. Recall that $\langle K \rangle_X$ is the average capital stock at the agent’s current—transitory—exchange position $X$. Since $\langle K \rangle_X$ increases as the agent moves towards the centre of the exchange space, i.e. $X \to 0$, the closer the agent gets to the centre, the slower he moves. A low initial capital and a sufficiently strong repulsive force may prevent the agent to ever reach the centre of the exchange space. The attractive force is thus unequal among agents. It favours those endowed with a higher initial capital.

The third exponential in (56) determines the dynamics of $K$. It can be rewritten:

$$\exp \left( -\frac{\omega \left( \left( (K - K')^2 + (\cosh (\omega (\theta' - \theta)) - 1) \left( (K - \langle K \rangle)^2 + (K' - \langle K \rangle)^2 \right) \right) \right)}{2\sigma^2 \sinh (\omega (\theta' - \theta))} \right)$$

where $\langle K \rangle$ is defined by (54). For a relatively stable $X$, the capital stock of an agent is in average driven towards the average stock associated to the agent’s exchange position. The quadratic term $(K - K')^2$ represents a random Brownian walk around this trend.

Equation (36) shows that, as long as an agent moves towards the centre, his average capital stock will increase in average. As a matter of fact, the two dynamics of $X$ and $K$ interact and can create a virtuous circle. A move towards the centre tends to increase capital, which in turn speeds up the move towards the centre. This positive interaction only stops when the repulsive force is stronger than the cohesion and the attractive forces. Given (55), this may occur when the initial capital is insufficient. Moreover, if an adverse shock reduces the stock of capital, the agent can be driven back towards the periphery. To sum up, in phase 1 a global move of agents towards the centre may exist, although agents with a higher initial capital are favoured in this move.

The fourth term in Eq. (56) describes the market clearing conditions given $X$ and $K$. The Dirac function in the last line of (56) implements (35). For a given capital $K$, prices are higher towards the centre of the exchange space, and for a given position $X$ prices are a decreasing function of $K$.

### 6.4.2 Phase 2: $\rho > 0$

**Resolution and results** Similarly to phase 1, the transition function in phase $\rho \neq 0$ is derived from the form of the effective action (52). The details are given in appendix 5. We find:

$$G \left( K, K', P, P', X, X', \theta, \theta' \right) = \exp \left( -\frac{\delta (K - \langle K \rangle_X)^2}{2\sigma^2} \right) \times G_{K \rightarrow K'} \left( X, X' \right) \times \sqrt{\frac{\omega/2\pi \sigma^2}{\sinh (\omega (\theta' - \theta))}}$$

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\[
\times \exp \left( -\frac{\omega \left( (K - \langle K \rangle)^2 + (K' - \langle K \rangle)^2 \right) \cosh (\omega (\theta' - \theta)) - 2 (K - \langle K \rangle) (K' - \langle K \rangle)}{2\sigma^2 \sinh (\omega (\theta' - \theta))} \right) \\
\times \delta (P - P (K, X)) \\
\times \delta (P' - P (K', X')) H (\theta' - \theta)
\] (59)

where:

\[
G_K (X, X') = \tilde{G} (X - \delta X, X' - \delta X) H (X) H (X') \\
+ \tilde{G} (X + \delta X, X' + \delta X) H (-X) H (-X') \\
+ \tilde{G} (X - \delta X, X' + \delta X) H (X) H (-X') \\
+ \tilde{G} (X + \delta X, X' - \delta X) H (-X) H (X')
\] (60)

and:

\[
\tilde{G} (X, X', \theta, \theta') \\
= \sqrt{\frac{\tilde{\omega}_X}{2\pi \sigma_X^2 \sinh (\tilde{\omega}_X (\theta' - \theta))}} \times \exp \left( -\frac{\tilde{\omega}_X}{2\sigma_X^2 \sinh (\tilde{\omega}_X (\theta' - \theta))} \right) \left( (X^2 + (X')^2) \cosh (\tilde{\omega}_X (\theta' - \theta)) - 2XX' \right)
\]

\[
= \sqrt{\frac{\tilde{\omega}_X}{2\pi \sigma_X^2 \sinh (\tilde{\omega}_X (\theta' - \theta))}} \times \exp \left( -\frac{\tilde{\omega}_X/2\sigma_X^2}{\sinh (\tilde{\omega}_X (\theta' - \theta))} \right) \left( (X - X')^2 + (\cosh (\tilde{\omega}_X (\theta' - \theta)) - 1) (X^2 + (X')^2) \right)
\] (61)

with:

\[
\tilde{\omega}_X = \sqrt{\kappa_0 \left( 1 + \frac{\sigma_X^2}{\kappa_0} \left( -\frac{\kappa_1}{2} \lambda_1^2 \left( \frac{K}{\langle K \rangle} + \frac{K'}{\langle K \rangle} \right) \rho^2 + 4\kappa_2 \lambda_2^2 \rho^4 \right) \right)}
\]

where \( H (X) \) is the Heaviside function defined by \( H (X) = 1 \) for \( X > 0 \), and 0 otherwise. As in phase 1—and for the same reasons—the definition of \( \langle K \rangle \) is given by (54). The quantity \( \delta X \) has been defined in (47).

**Interpretation** Here again, the first exponential in (59) merely translates that agents’ capital stocks at \( X \) are bounded to be around the average stock \( \langle K \rangle_X \).

The second exponential in formula (59) describes the dynamics on \( X \). Given the formulas (59) and (60), this dynamics is different from that of phase 1. Equation (52) shows that an individual \( X \)-dependent attraction point appears for each agent, along with \( X = 0 \). This point depends on both the agent’s initial position and capital.
The Green function \( G_{K+K'} (X, X') \) in (60) commands the dynamics of the exchange position. Assume that \( X \) and \( X' \) are positive. Using (61), we have:

\[
G_{K+K'} (X, X') = \frac{\hat{\omega}_X}{\sqrt{2\pi\sigma_X^2 \sinh (\hat{\omega}_X (\theta' - \theta))}} \times \exp \left( -\frac{\hat{\omega}_X^2/2\pi\sigma_X^2}{\sinh (\hat{\omega}_X (\theta' - \theta))} \left( (X - X')^2 + (\cosh (\hat{\omega}_X (\theta - \theta')) - 1) \left( (X - \delta X)^2 + (X' - \delta X)^2 \right) \right) \right)
\]

(62)

As in phase 1, the quadratic term \((X - X')^2\) represents a random Brownian walk around the starting point \(X'\). The second quadratic term:

\[
(\cosh (\hat{\omega}_X (\theta - \theta')) - 1) \left( (X - \delta X)^2 + (X' - \delta X)^2 \right)
\]

drives the position \(X\) towards the position \(\delta X\). Thus, the attraction point is shifted from 0 to \(\delta X\).

For a large capital, Eq. (47) shows that \(\delta X < 0\). This means that the attraction point appears on the agent’s opposite side of the centre \(X = 0\). This \(X\)-dependent attraction point quickly drives the agent towards \(X = 0\). The agent does not cross \(X = 0\) in average when it is reached. If he were to cross it randomly, he would be driven back towards \(X = 0\). Actually, when \(X' < 0\) and \(X < 0\)—i.e. when an agent has crossed the centre \(X = 0\)—Eqs. (60) and (61) show that the agent is driven back towards the variable attraction point \(\delta X > 0\), which in turn drives back the agent towards 0. This oscillatory process reveals that \(X = 0\) is indeed an attractive point.

For a low level of initial capital, the dynamics is different. The value of \(\delta X\) is negative, which means that the \(X\)-dependent attraction point appears on the initial position side. The agent will be driven towards this point, even if he was closer to the centre \(X = 0\). This implies an eviction mechanism from the centre for low capital agents. As a consequence, the dynamics for capital accumulation are different for the two phases.

Actually, the patterns of these dynamics may seem similar at first sight, since the third exponential in (59) and (56) is identical. Indeed, up to some random fluctuations, agents are driven towards the average capital of their current exchange position. However, since capital accumulation and exchange positions are interacting, the dynamics of \(X\) in phase 2 impacts capital accumulation. In phase 1, a positive shock on capital could drive the agent towards the centre, in turn increasing his capital stock. Here, due to the presence of the \(X\)-dependent attraction point, a small positive capital shock does not initiate a move towards the centre. An agent with low capital will be driven back towards his variable attraction point, possibly farther from the centre.

The fourth term in Eq. (59) describes the market clearing conditions given \(X\) and \(K\). Actually, the Dirac function in the last line of (59) implements (39). For a given
level of capital $K$, prices are higher towards the centre of the exchange space, and for a given position in the exchange space $X$, prices are a decreasing function of $K$.

Finally, a direct consequence of the non-trivial phase cannot be read from the Green function (59), but can be tracked back to the last term of (52):

\[
\left(2\kappa_2 \rho^2 - \kappa_1 \right) \rho^2 \left| \int \Delta \Psi(K, X, \theta) \Psi_0(K, X, \theta) \right|^2
\]

The effective action in phase 2 includes a positive term, since $\rho^2$ is now greater than $\frac{\kappa_1}{2\kappa_2}$. This term induces a barrier between agents in position $X$ (see “Appendix 6” for a derivation of this result in a general context). Agents with an initial capital stock below $\langle K \rangle_X$ will find it difficult to accumulate above $\langle K \rangle_X$ and improve their exchange position, while agents with $K > \langle K \rangle_X$ might remain above this threshold. Therefore, in phase 2, a threshold effect appears at each point on the exchange space. Agents with high initial capital will more likely overcome barriers and accumulate, while the others will be evicted from positions for which their initial stock of capital is too low.

### 7 Synthesis and discussion

Beside classical economic variables, this model considers an exchange space in which agents evolve. This space can be seen both as a geographical space—national markets or individual sectors for instance—and as a scale of exchange terms—a measure of the agents’ connections density.

Three forces characterise this exchange space. The sole force that attracts all agents towards the centre of the exchange space is the cohesion force. The attractive and repulsive forces merely bring agents together or apart on the $X$ axis, respectively.

Exchanges are proportional to the distance between agents: the attractive forces smooth exchanges, whereas the repulsive force deters competition by preventing new or more distant agents to establish or strengthen exchanges with consumers. These two—attractive and repulsive—forces always exist within a market, yet our work stresses that it is $r = \kappa_2/\kappa_1$, the relative magnitude of the repulsive force with respect to the attractive force that matters. This ratio commands the occurrence of the two phases of the dynamics.

In phase 1, $r$ is high and exchanges are limited. Markets are protected or exchanges are prevented for other reasons, such as distance, lack of infrastructure, regulation, etc. The cohesion force homogenises agents and improves exchange terms among them in average. This impacts all agents even though it favours those with higher initial capital. This phase can be seen as one of regulated exchanges. On the contrary, in phase 2, the repulsive force is weaker and $r$ is low. Agents experience higher mobility within the exchange space. This phase can be seen as one of non-regulated exchanges. Note, however, that the notion of regulated or non-regulated markets is relative here, since it depends on the relative strengths of the two main forces in the exchange space.

The two phases present similarities. In both, producer prices and agents’ average capital stock decrease exponentially as they move away from the centre. Peripheric agents produce less and sell at lower prices than agents at the centre of the space.
Yet dynamically the two phases are different. Phase 1, with its regulated exchanges, displays exchange stability, lower competition and a broad-based although slow improvement in exchange terms. In this phase, random shocks can redistribute capital and initiate a virtuous circle of capital accumulation.

In phase 2, under non-regulated exchanges, a low repulsive force induces both greater exchange instability and greater competition among agents. Increased mobility favours capital accumulation for high initial capital producers. Low initial capital producers are evicted from the exchange space as their prices and revenues deteriorate. This is due to an indirect effect: increased exchanges induce greater competition and increased inequality in wealth distribution. Indeed, a lower repulsive force opens markets to agents with higher capital. Selling their products at a relatively low price, these agents gain market shares over their competitors. In our setting, new trade relations compensate for the income loss of low prices induced by a higher production. This favours capital accumulation and will eventually improve the position in the exchange space, ultimately leading to a higher price.

In short, new high-capital entrants to a market experience higher demands, higher prices, higher revenues and in turn higher capital accumulation. This mechanism, repeated over time, may drive them along the exchange space to a dominant position. On the contrary, producers with low capital on their home market are evicted. To them, market liberalisation induces more competitors, lower demands, market shares and income losses that will in turn dampen their capital accumulation, drive them towards the periphery and ultimately evict them from their own market. Such mechanisms are at play within sectors where liberalisation favours a high concentration of capital. The direct investments of major foreign agricultural producers in Africa evicted local, low-capital producers that eventually sold their work force to their previous competitors and led others to migrate (Malet 2018).

Phase 2—of unregulated exchanges—displays a seemingly counter-intuitive result: except under mild market liberalisation—\( r \) within a certain range—average capital is lower than in the regulated phase 1. Indeed, the standard wisdom of international trade predicts that open markets induce an increase in global welfare through higher competition, lower prices and better resource allocation. In our model, openness and competition also lower prices but, since consumers are also producers, the consumer gains do not necessarily compensate the producer losses induced by an increased competition. Again, only high initial capital agents may produce at a lower price, gain market shares and increase their income. The others are evicted. Actually, bouts of trade liberalisation during the nineteenth century did not decisively led to higher overall capital accumulation and free trade may have at times coincided with crises (see Bairoch 1999). Our model therefore nuances the benefits of free trade. We find that a high disparity in capital—and consequently in revenues—does not imply an improvement in average wealth. The disparity in capital accumulation induces a fall in global demand and a lower average wealth. And although our results are derived under a constant-technology assumption, they would probably stand under a non-constant one.

Our model has one last implication. In our setting, capital must be renewed to produce: when an agent cannot do so, he disappears as a capital-endowed producer. Over time, this may leave the agent with the sole capital he may renew, his labour.
force. Thus, the non-regulated phase 2 of our model also describes the evolution of a society with a large number of small producers towards a society with few capitalistic producers and a large number of workers: it accounts for a de-homogenisation of society.

Our formalism provides a good approach to finding the phases of a system. More importantly, it details how these phases heterogeneously impact individual agents depending on their initial conditions. To do so, the demand side has been deliberately simplified, and some standard features of macroeconomics—such as consumers optimisation, government spending, monetary policy, price stickiness—were dismissed. However, they could be reintroduced in the formalism (see Gosselin et al. 2020). In particular, we may expect public policies to play a central role in the emergence of some phases.

8 Conclusion

In this paper, we have developed a model of capital accumulation with a large number of heterogeneous agents. This model keeps some features of the classical economic models—a standard equation of capital accumulation, production function and market clearing condition. However, it includes an exchange space that dynamically interacts with capital accumulation. Besides, the classical description of the model is replaced by field theory techniques. Our results show that, depending on the parameters of the model, the system displays various phases, each describing different accumulation processes. Capital accumulation is not necessarily favoured by greater market liberalisation. Besides, capital accumulation highly depends on each agent’s initial conditions and shows that capital dynamics cannot be reduced to simple aggregates. Various dynamic patterns appear depending on agents’ initial conditions. In particular, our model highlights that dynamic divergences may split a society of atomic producers into two groups. One group accumulates capital at the expense of the other until producers of the second group are left with the only capital they may renew, their labour.

Our model shows the advantages of field theory modelling in economics. It leads to a finer description of the agents’ dynamics, both at a global level and individual level, the two descriptions interacting constantly. Our future work will extend this approach to other fields of economics.

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Appendices

Appendix 1. translation from probabilistic description to field theory

This appendix summarises the most useful steps of the method developed in Gosselin et al. (2017, 2020), to switch from the probabilistic description of the model to the field-theoretic formalism. By convention and unless otherwise mentioned, the symbol $\int$ refers to all the variables involved.

Principle

For a large number of agents, the system described by (16) involves a large number of variables $K_i(t)$, $P_i(t)$ and $X_i(t)$ that are difficult to handle. To overcome this difficulty, we consider the space $H$ of complex functions defined on the space of a single agent’s actions. The space $H$ describes the collective behaviour of the system. Each function $/\Psi_1$ of $H$ encodes a particular state of the system. We then associate to each function $/\Psi_1$ of $H$ a statistical weight, i.e. a probability describing the state encoded in $/\Psi_1$. This probability is written $\exp(-S(/\Psi_1))$, where $S(/\Psi_1)$ is a functional, i.e. the function of the function $/\Psi_1$. The form of $S(/\Psi_1)$ is derived directly from the form of (16).

The present paper’s statistical weight is a variation of the set up presented in Gosselin et al. (2020), in which a general weight describing interactions between individual agents was written, accounting for the present paper notations:

$$\sum_i \int_0^T \left( \frac{1}{2\sigma^2} \int_0^T \left( \frac{d}{ds} Z_i(s) \right)^2 + V_1 \left( Z_i(s) \right) \right) ds + \frac{1}{2} \sum_i \int_0^T \left( \frac{d}{dt} Z_i(t) - H \left( Z_i(t) \right) \right)^2 dt + \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int_0^T \int_0^T V_k \left( \frac{Z_{i_1}^{(i_1)}, \ldots, Z_{i_k}^{(i_k)}}{\xi^2} \right) ds_1 \ldots ds_k$$

where $Z_i(s)$ describes the position of agent $i$ in a space of arbitrary economic variables. In the present paper $Z_i = (K_i, P_i, X_i)$.

We showed that the field action functional:

$$S(/\Psi) = \int \left( /\Psi^\dagger \left( -\frac{\sigma^2}{2} \nabla^2 + V_1 \left( Z \right) + \alpha \right) /\Psi \left( Z \right) \right) dZ - \frac{1}{2} \sum_i \int /\Psi^\dagger \left( \eta^2 \nabla^2 + \nabla \cdot H \left( Z \right) \right) /\Psi \left( Z \right) dZ + \frac{1}{2} \sum_i \int /\Psi^\dagger \left( \nabla \cdot H \left( Z \right) \right) /\Psi \left( Z \right) dZ$$
contains the same information about the system, where $\alpha$ is the parameter arising in the Laplace transform of the statistical weight (63), and $\Psi^\dagger (Z)$ denotes the complex conjugate of $\Psi (Z)$. The operator $\nabla$ is the gradient operator, a vector whose $i$-th coordinate is the first derivative $\frac{\partial}{\partial Z_i}$: $\nabla = \left( \frac{\partial}{\partial Z_i} \right)$. The operator $\nabla^2$ denotes the Laplacian:

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial Z_i^2}$$

where the sum runs over the coordinates $Z_i$ of the vector $Z$.

However, the translation defined by (63) and (64) does not straightforwardly apply to the present paper and has to be adapted. Actually, the interactions between different agents in the global statistical weight (16)—the terms involving sums over different agents $j, k, ...$—are local in the time variable, i.e. the quantities involved in these terms are considered simultaneously. On the contrary, interactions are not local in (63).

The introduction of local interactions can be done by introducing a counting variable $\Theta_i(s)$ for each agent $i$. This variable is roughly equal to $s$, up to some random fluctuation. This amounts to introducing a dynamic variable $\theta$ in the field formalism to account for $\Theta_i$. Thus, the field $\Psi (Z)$ is replaced by a function $\Psi (Z, \theta)$.

A statistical weight has to be introduced for the counting variable $\Theta_i(s)$:

$$\exp \left( - \int \left( \frac{\dot{\Theta}_i(s) - 1}{2 \vartheta^2} \right)^2 \right)$$

where $\vartheta^2 << 1$. This ensures that $\Theta_i(s) \simeq s$, up to an initial constant. This constant can be discarded if we consider $\Theta_i(0) = 0$ in all transition functions. The field counterpart of this particular weight is:

$$\int \exp \left( - \int \left( \frac{\dot{\Theta}_i(s) - 1}{2 \vartheta^2} \right)^2 \right) \mathcal{D}\Theta_i(s)$$

$$\rightarrow \exp \left( -\Psi^\dagger (Z, \theta) \nabla_\theta \left( \frac{\vartheta^2}{2} \nabla_\theta - 1 \right) \Psi (Z, \theta) \right)$$

Local interactions can then be included. Assume a two-agent interaction of the type:

$$\sum_{i,j} \int V (Z_i(t), Z_j(t)) \, dt$$

(65)

where $V$ is an arbitrary function. We replace the time $t$ in $V (Z_i(t), Z_j(t))$ by two independent parameters $t_i$ and $t_j$. We impose the equality of the counting variables...
associated to these parameters: $\theta_j (t_j) - \theta_i (t_i) = 0$. Consequently, the potential rewrites:

$$\sum_{i,j} \int V (Z_i (t_i), Z_j (t_j)) \delta (\theta_j (t_j) - \theta_i (t_i)) dt_i dt_j$$

(66)

In this form, the translation from (63) to (64) can be used and yields a field-theoretic potential:

$$\int \Psi (Z_1, \theta) \Psi (Z_2, \theta) V (Z_1, Z_2) \Psi^\dagger (Z_1, \theta) \Psi^\dagger (Z_2, \theta) dZ_1 dZ_2 d\theta$$

This can be generalised straightforwardly for $k$ interacting agents’ potentials:

$$\sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int_0^T \int_0^T V_k (Z_{i_1}^{l_{i_1}}, \ldots, Z_{i_k}^{l_{i_k}}) \frac{d\ell_1 \ldots d\ell_k}{\xi^2}$$

(67)

that translate into the field-theoretic version:

$$\frac{1}{\xi^2} \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int \Psi (Z_1, \theta) \ldots \Psi (Z_k, \theta) V_k (Z_1 \ldots Z_k) \Psi^\dagger (Z_1, \theta) \ldots \Psi^\dagger (Z_k, \theta) dZ_1 \ldots dZ_k d\theta$$

(68)

Translation of the model in terms of field theory

We can apply the results of the previous paragraph to the model, to translate (16) into its field theory counterpart. The statistical weights of the system can be divided into two parts, a first one for the position $X$ in the exchange space, and a second one for $K$ and $P$. We will deal separately with these two parts. In the sequel we set $Z = (K, P, X)$ and $Z_i = (K_i, P_i, X_i)$ for any index $i$.

Translation of (14) The $X$-part (14) of the statistical weight directly fits into the formalism defined by (63) and (64), (65), (66), (67) and (68). As a consequence the—log—weight (14):

$$- \sum_i \int \left( \frac{\dot{X}_i (t)^2}{2\sigma_X^2} + V_0 (X_i (t)) + \sum_j V_1 (d_{ij} (t)) + \sum_{j,k} V_2 (d_{ij} (t), d_{ik} (t), d_{jk} (t)) \right) dt$$

(69)

has the following equivalent in terms of field:

$$S_2 (\Psi) = \int \Psi^\dagger (Z, \theta) \left( - \frac{\sigma_X^2}{2} \nabla_X^2 \right) \Psi (Z, \theta) + \int V_0 (X) |\Psi (Z, \theta)|^2$$

$$+ \int V_1 (d_{12}) |\Psi (Z_1, \theta)|^2 |\Psi (Z_2, \theta)|^2$$
\[
+ \int V_2 (d_{12}, d_{13}, d_{23}) |\Psi (Z_1, \theta)|^2 |\Psi (Z_2, \theta)|^2 |\Psi (Z_3, \theta)|^2
\] (70)

as reported in the text.

**Translation of (12) and (13)** The statistical weight for the price plus the capital part is the product of (12) with (13):

\[
\exp \left( - \int \left( \frac{1}{2 \sigma^2} \left( K_i (t) + \delta K_i (t) - A P_i (t) K_i^\alpha (t) + U_{1i} \right)^2 + \frac{\tilde{A}^2}{2 \sigma^2} \left( P_i^{1+\gamma} (t) K_i^\alpha (t) - U_{2i} \right)^2 \right) dt \right)
\] (71)

with:

\[
U_{1i} = \frac{\kappa}{d^2} A P_i (t) K_i^\alpha (t) \sum_{j,k} P_k (t) P_j \left( \frac{X_i (t) - X_j (t)}{d} - \frac{X_i (t) - X_k (t)}{d} \right)
\]

\[
U_{2i} = \frac{\kappa}{d^2} \sum_{j,k} P_j (t) K_j^\alpha (t) P_k (t) \left( \frac{X_i (t) - X_j (t)}{d} - \frac{X_k (t) - X_j (t)}{d} \right)
\]

We will now show that the associated field action writes:

\[
\int \Psi^\dagger (Z, \theta) \left( -\frac{\sigma^2}{2} \nabla^2 K - \frac{(\delta K - A P K^\alpha (1 - U_1))^2}{2 \sigma^2} + \left( P^{1+\gamma} K^\alpha + U_2 \right)^2 \right.

\left. - \frac{\sigma^2}{2} \nabla^2 X - \frac{\vartheta^2}{2} \nabla^2 \vartheta + \frac{1}{2 \vartheta^2} + \alpha \right) \Psi (Z, \theta)
\] (72)

with:

\[
U_1 (Z, \theta) = \frac{\kappa}{d^2} \int \frac{P_3 \exp \left( - \left( \frac{d_{12} + d_{13}}{d} \right) \right)}{P_2} |\Psi (Z_2, \theta)|^2 |\Psi (Z_3, \theta)|^2 dZ_2 dZ_3
\] (73)

\[
U_2 (Z, \theta) = -\frac{\kappa}{d^2} \int P_2 (K_2)^\alpha P_3 \exp \left( - \left( \frac{d_{12} + d_{23}}{d} \right) \right) |\Psi (Z_2, \theta)|^2 dZ_2 dZ_3 d\theta
\] (74)

where we set \( X_1 = X \) in \( d_{12} \) and \( d_{13} \).

To prove (72), we decompose (71) into its two components.

**Translation of the second part of (71)** Using (65) and (66), the second part of (71):

\[
\exp \left( - \int \frac{\tilde{A}^2}{2 \sigma^2} \left( P_i^{1+\gamma} (t) K_i^\alpha (t) - U_{2i} \right)^2 dt \right)
\]
has a direct equivalent in terms of field theory:

$$\int \Psi^\dagger (K, X, \theta) \left( \left( P^{1+\gamma} K^\alpha + U_2 \right)^2 \right) \Psi (K, X, \theta)$$  \hspace{1cm} (75)

**Translation of the first part of (71)** The first part of (71) is given by:

$$\exp \left( -\frac{1}{2 \sigma^2} \int \left( \left( \dot{K}_i (t) + \delta K_i (t) - A P_i (t) K_i^\alpha (t) + U_{1i} \right)^2 \right) dt \right)$$  \hspace{1cm} (76)

To find the field equivalent of (76), we rewrite the above expression in a more convenient form.

First, note that (76) can be written in a general form:

$$\exp \left( -\int \frac{\left( A \dot{Z}_i (t) + G (Z_i (t)) - \sum_{(j, n)} V (Z_i (t), (Z (t))_{(j, n)}) \right)^2}{2 \sigma^2} \right)$$  \hspace{1cm} (77)

where $Z_i$ is a vector variable of arbitrary dimension $l$, $(j, n)$ is a sequence of $n$ indices $(j_1, j_2, ..., j_n)$ and $(Z (t))_{(j, n)} = (Z_{j_1} (t), Z_{j_2} (t), ...Z_{j_n} (t))$ where $n$ is given (in this paper $n = 2$ for $U_1$). The matrix $A$ is of dimension $1 \times l$. Actually, we recover (76) when we set:

$$Z_i (t) = (K_i (t), P_i (t), X_i (t))^t$$

$$G (Z_i (t)) = \delta K_i (t) - A P_i (t) K_i^\alpha (t)$$

$$V (Z_i (t), Z_j (t), Z_j (t)) = -\frac{\kappa}{d^2} A P_i (t) K_i^\alpha (t) \frac{P_k (t)}{P_j (t)}$$

$$\exp \left( -\frac{|X_i (t) - X_j (t)|}{d} - \frac{|X_i (t) - X_k (t)|}{d} \right)$$

$$A = \left( \begin{array}{c} 1 \ 0 \ 0 \end{array} \right)$$  \hspace{1cm} (78)

In the sequel, $\dot{Z}_i (t)$ will stand for $A \dot{Z}_i (t)$. This is done for the sake of simplicity and does not impair the argument.

Second, as explained in the translation from (65) to (66), we can replace the time $t$ in $V (Z_i (t), (Z (t))_{(j, n)})$ by the independent parameters $t_i$ and $t_{(n)} = (t_1, ..., t_n)$. We write:

$$(Z (t_{(n)}))_{(j, n)} = (Z_{j_1} (t_1), Z_{j_2} (t_2), ...Z_{j_n} (t_n))$$

We then impose the equality of the counting variables associated to these parameters: $\theta_{j_l} (t_l) - \theta_i (t_i) = 0$ for $l = 1, ..., n$. Consequently, the weight (77) rewrites:
where we set:

\[
\delta \left( \left[ \theta(t_l) \right]_{(j,n)} - \theta_i(t_i) \right) = \prod_{l=1}^{n} \delta \left( \left[ \theta_j(t_l) \right] - \theta_i(t_i) \right)
\]

To simplify notations and account for possible generalisations, we replace the potential in (79):

\[
V \left( Z_i(t_i) , (Z(t_l))_{(j,n)} \right) \delta \left( \left[ \theta(t_l) \right]_{(j,n)} - \theta_i(t_i) \right)
\]

by a more general form, also denoted \( V \) for the sake of simplicity:

\[
\delta \left( \left[ \theta(t_l) \right]_{(j,n)} - \theta_i(t_i) \right) = \prod_{l=1}^{n} \delta \left( \left[ \theta_j(t_l) \right] - \theta_i(t_i) \right)
\]

where we set:

\[
Y_i(t_l) = (Z_i(t_l), \theta_l(t_l))
\]

\[
(Y(t_l))_{(j,n)} = (Y_{j_l}(t_1), \theta_{j_l}(t_1), Y_{j_2}(t_2), \theta_{j_2}(t_2) ...)
\]

so that the \( \delta \) factor in the potential is replaced by an arbitrary function of the counting variables \( \theta_i(t_i) \) and \( (\theta(t_l))_{(j,n)} \).

Third, we modify (77) by introducing an auxiliary variable \( \tilde{Z}_i(t_i) \), equal to \( \sum_{(j,n)} V \left( Y_i(t_l), (Y(t_l))_{(j,n)} \right) \), up to a random error of square deviation \( \sigma_2^2 \ll 1 \). Ultimately, the weight (77) becomes:

\[
\exp \left( - \frac{1}{2\sigma^2} \left( \sum_{(j,n)} V \left( Y_i(t_l), (Y(t_l))_{(j,n)} \right) \right)^2 \right)
\]

\[
\simeq \exp \left( - \frac{1}{2\sigma^2} \left( (\tilde{Z}_i(t_i) + G(Z_i(t_i)) - \sum_{(j,n)} V \left( Y_i(t_l), (Y(t_l))_{(j,n)} \right) \right)^2 \right)
\]

\[
= \exp \left( - \frac{1}{2\sigma^2} \left( (\tilde{Z}_i(t_i) + G(Z_i(t_i)) - \tilde{Z}_i(t_i)) \right)^2 \right)
\]
Actually, the condition $\sigma_2^2 \ll 1$ ensures that states with $\tilde{Z}_i (t_i) - \sum_{(j,n)} \int V \left( Y_i (t_i), (Y (t_{(n)}))_{(j,n)} \right) \neq 0$ have a negligible probability.

We also assume that:

$$\sigma_2^2 \ll \sigma^2$$

The techniques of the previous paragraphs apply, since the weight (80) has the form required. The corresponding field theory action writes:

$$\int \psi^\dagger (Y, \tilde{Z}) \left( -\nabla_\theta \left( \frac{\sigma^2}{2} \nabla_\theta - 1 \right) - \frac{1}{2} \nabla_Z \left( \sigma^2 \nabla_Z + 2 \left( G (Z) - \tilde{Z} \right) \right) 
+ \frac{G' (Z)}{2} - \frac{1}{2} \sigma^2 \nabla_Z^2 - \frac{1}{2} \omega Z \right) \right) \psi (Y, \tilde{Z})$$

$$+ \int \frac{\tilde{Z}^2_2}{2\sigma_2^2} |\psi (Y, \tilde{Z})|^2 - \int \frac{\tilde{Z} V \left( (Y, \tilde{Z}), (Y, \tilde{Z}^\prime)_{(n)} \right)}{\sigma_2^2} |\psi (Y, \tilde{Z})|^2 |\psi (Y, \tilde{Z}^\prime)_{(n)}|^2$$

$$+ \int \frac{V \left( (Y, \tilde{Z}), (Y, \tilde{Z}^\prime)_{(n)} \right) V \left( (Y, \tilde{Z}), (Y^\prime, \tilde{Z}^\prime)_{(n)} \right) |\psi (Y, \tilde{Z})|^2 |\psi (Y, \tilde{Z}^\prime)_{(n)}|^2 |\psi (Y^\prime, \tilde{Z}^\prime)_{(n)}|^2 \right)$$(81)

where $\left( Y^\prime, \tilde{Z}^\prime \right)_{(n)}$ is a multiplet of variables $\left( Y^\prime_1, \tilde{Z}^\prime_1, Y^\prime_2, \tilde{Z}^\prime_2, \ldots \right)$ and:

$$\left| \psi \left( Y^\prime, \tilde{Z}^\prime \right) \right|^2_{(n)} = \prod_{l=1}^n \left| \psi \left( Y^\prime_l, \tilde{Z}^\prime_l \right) \right|^2$$

This action can be further simplified by applying the following transformation on the field:

$$\psi (Y, \tilde{Z}) = \exp \left( - \int \frac{G (Z) - \tilde{Z}}{\sigma^2} \right) \tilde{\psi} (Y, \tilde{Z})$$

$$\psi^\dagger (Y, \tilde{Z}) = \exp \left( \int \frac{G (Z) - \tilde{Z}}{\sigma^2} \right) \tilde{\psi}^\dagger (Y, \tilde{Z})$$ (82)

and by using a change of notation for the sake of simplicity:

$$\tilde{\psi} (Y, \tilde{Z}) \rightarrow \psi (Y, \tilde{Z})$$
Equation (81) can then be replaced by the following action:

\[
\psi^\dagger(Y, \tilde{Z}) \rightarrow \psi^\dagger(Y, \tilde{Z})
\]

For any potential satisfying \( V\left(\left(Y, \tilde{Z}\right), \left(Y', \tilde{Z}'\right)_{(n)}\right) = V\left(\left(Y', \tilde{Z}'\right)_{(n)}\right) \) —which is the case considered in this paper, (83) simplifies as:

\[
\int \psi^\dagger(Y, \tilde{Z}) \left(-\nabla_{\tilde{Z}} \left(\frac{\partial^2}{2} \nabla_{\tilde{Z}} - 1\right) - \frac{\sigma^2}{2} \nabla_{\tilde{Z}}^2 + \frac{(G(Z) - \tilde{Z})^2}{2\sigma^2} - \frac{1}{2} \epsilon^2 \nabla_{\tilde{Z}}^2 - \frac{1}{2} \omega^2 \tilde{Z}^2 \right) \psi(Y, \tilde{Z})
\]

\[
+ \int \frac{\tilde{Z}^2}{2\sigma^2} |\psi(Y, \tilde{Z})|^2 - \int \frac{\tilde{Z} \nabla \left(\left(Y, \tilde{Z}\right), \left(Y', \tilde{Z}'\right)_{(n)}\right)}{\sigma^2} |\psi(Y, \tilde{Z})|^2 |\psi(Y', \tilde{Z}')_{(n)}|^2
\]

\[
+ \int \frac{V\left(\left(Y, \tilde{Z}\right), \left(Y', \tilde{Z}'\right)_{(n)}\right)}{2\sigma^2} V\left(\left(Y, \tilde{Z}\right), \left(Y', \tilde{Z}''\right)_{(n)}\right) |\psi(Y, \tilde{Z})|^2 |\psi(Y', \tilde{Z}')_{(n)}|^2 |\psi(Y'', \tilde{Z}'')_{(n)}|^2
\]

This formula expresses (79), and as a consequence also (80), in terms of field theory.

However, it further simplifies since, given our assumptions, \( \epsilon^2 \ll 1, \sigma^2 \ll 1 \). We thus consider that: \( \epsilon^2 \nabla_{\tilde{Z}}^2 \rightarrow 0 \), which implies that the condition:

\[
0 = \int \psi^\dagger(Y, \tilde{Z}) \left(\tilde{Z} - \int V\left(Y', (Y')_{(n)}\right) |\psi(Y', \tilde{Z}')_{(n)}|^2 \right) \psi(Y, \tilde{Z})
\]

imposed by \( \sigma^2 \ll 1 \) in (84) is obtained for a function of the type:

\[
\psi\left(Y, \tilde{Z}\right) = \delta\left(\tilde{Z} - \int V\left(Y', (Y')_{(n)}\right) |\psi(Y')_{(n)}|^2 \right) \psi(Y)
\]

The function \( \delta\left(\tilde{Z} - \int V\left(Y', (Y')_{(n)}\right) |\psi(Y')_{(n)}|^2 \right) \) can be interpreted as a Gaussian function of norm equal to 1, and peaked around \( \int V\left(Y', (Y')_{(n)}\right) |\psi(Y')_{(n)}|^2 \).
Expression (84) can thus be written for fields of the form (85). In fact, Eq. (85) implies that the term arising in (84):

$$\int \bar{\Psi} \left( Y, \tilde{Z} \right) \frac{\left( G(Z) - \tilde{Z} \right)^2}{2\sigma^2} \Psi \left( Y, \tilde{Z} \right)$$

(86)
can be neglected. Actually:

$$\int \bar{\Psi} \left( Y, \tilde{Z} \right) \left( G(Z) - \tilde{Z} \right)^2 \Psi \left( Y, \tilde{Z} \right)$$

$$= \int \bar{\Psi} \left( Y, \tilde{Z} \right) \left( G(Z) - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \Psi \left( Y, \tilde{Z} \right)$$

$$- 2 \int \bar{\Psi} \left( Y, \tilde{Z} \right) \left( G(Z) - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right)$$

$$\times \left( \tilde{Z} - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \Psi \left( Y, \tilde{Z} \right)$$

$$+ \int \bar{\Psi} \left( Y, \tilde{Z} \right) \left( \tilde{Z} - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \Psi \left( Y, \tilde{Z} \right)$$

(87)
Now remark that the last quantity in (87) has a negligible norm. Actually:

$$\left| \int \bar{\Psi} \left( Y, \tilde{Z} \right) \left( G(Z) - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \right|$$

$$\leq \left\| \bar{\Psi} \left( Y, \tilde{Z} \right) \left( G(Z) - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \right\|$$

$$\times \left\| \left( \tilde{Z} - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \Psi \left( Y, \tilde{Z} \right) \right\|$$

(88)
and the norm of the last factor in (88) is close to zero, since:

$$\left\| \left( \tilde{Z} - \int V \left( Y, \left( Y' \right)_{(n)} \right) \left| \Psi \left( Y', \tilde{Z}' \right) \right|_{(n)}^2 \right) \Psi \left( Y, \tilde{Z} \right) \right\|^2$$
\[ = \int \Psi^\dagger (Y, \tilde{Z}) \left( \tilde{Z} - \int V (Y, (Y')_{(n)}) \right) \left| \Psi (Y', \tilde{Z}') \right|_{(n)}^2 \right)^2 \Psi (Y, \tilde{Z}) \simeq 0 \]

(89)

Ultimately, Eqs. (88) and (89) imply that (86) is negligible.

As a consequence, the potential (84) simplifies as:

\[
\int \Psi^\dagger (Y, \tilde{Z}) \left( -\nabla_{\theta} \left( \frac{\vartheta^2}{2} \nabla_{\theta} - 1 \right) - \frac{\sigma^2}{2} \nabla_{Z}^2 \right) \Psi (Y, \tilde{Z}) \\
+ \int \Psi^\dagger (Y, \tilde{Z}) \left( \frac{G (Z) - \int V (Y, (Y')_{(n)}) \left| \Psi (Y', \tilde{Z}') \right|_{(n)}^2}{2\sigma^2} \right)^2 \Psi (Y, \tilde{Z}) \\
= \int \Psi^\dagger (Y) \left( -\frac{\sigma^2}{2} \nabla_{Z}^2 + \frac{G (Z) - \int V (Y, (Y')_{(n)}) \left| \Psi (Y') \right|_{(n)}^2}{2\sigma^2} \right) \Psi (Y)
\]

The above equation is the field equivalent of (80). We can now come back to the initial variables by letting \( Y = (Z, \theta) \). This ultimately yields the following field-theoretic translation of (77):

\[
\int \Psi^\dagger (Z, \theta) \left( -\nabla_{\theta} \left( \frac{\vartheta^2}{2} \nabla_{\theta} - 1 \right) - \frac{\sigma^2}{2} \nabla_{Z}^2 \right) \\
+ \frac{\left( G (Z) - \int V (Z, (Z')_{(n)}) \delta (\theta - \theta') \left| \Psi (Z', \theta') \right|_{(n)}^2 {\right)}^2}{2\sigma^2} \Psi (Z, \theta) \\
= \int \Psi^\dagger (Z, \theta) \left( -\nabla_{\theta} \left( \frac{\vartheta^2}{2} \nabla_{\theta} - 1 \right) - \frac{\sigma^2}{2} \nabla_{Z}^2 \right) \\
+ \frac{\left( G (Z) - \int V (Z, (Z')_{(n)}) \left| \Psi (Z', \theta') \right|_{(n)}^2 \right)^2}{2\sigma^2} \Psi (Z, \theta) \quad (90)
\]

Translation of (71) We now gather (75) and (90) to obtain the field equivalent of (71).

We perform a change of variable in the counting variable:

\[ \Psi (Z, \theta) = \exp \left( \frac{\vartheta}{\vartheta^2} \right) \bar{\Psi} (Z, \theta) \]

\[ \Psi^\dagger (Z, \theta) = \exp \left( \frac{\vartheta}{\vartheta^2} \right) \bar{\Psi}^\dagger (Z, \theta) \]
and again reset:

\[
\tilde{\Psi} (Z, \theta) \rightarrow \Psi (Z, \theta)
\]

\[
\tilde{\Psi}^\dagger (Z, \theta) \rightarrow \Psi^\dagger (Z, \theta)
\]

Given our choices for \( G \) and \( V \) in (78), we obtain (72).

**Green functions**

To conclude this section, we include the contribution of the change of variable (82) to the computation of the Green functions. Because of the change of variable, the source terms must be included from the beginning. This leads to the following action plus source terms:

\[
\int \Psi^\dagger (Z, \tilde{Z}) \left( -\frac{\partial^2}{2} \nabla_\theta^2 + \frac{1}{2\theta^2} - \frac{\sigma^2}{2} \nabla_Z^2 \right) \Psi (Z, \tilde{Z})
\]

\[
+ \int \Psi^\dagger (Z, \tilde{Z}) \left( \frac{G (Z) - \int V (Z, (Z'))_{(n)}}{\sigma^2} \right) |\Psi (Z', \tilde{Z}')|_{(n)}^2 \right) \Psi (Z, \tilde{Z})
\]

\[
+ \int J (Z) \exp \left( -\int Z \left( \frac{G (Z) - \tilde{Z}}{\sigma^2} \right) \right) \Psi (Z, \tilde{Z}) + J^\dagger (Z) \exp \left( \int Z \left( \frac{G (Z) - \tilde{Z}}{\sigma^2} \right) \right) \Psi^\dagger (Z)
\]

Given (85), (78) and (73), the value of \( \tilde{Z} \) is:

\[
\tilde{Z} (Z) = \int V (Z, (Z')_{(n)}) |\Psi (Z')|_{(n)}^2
\]

\[
= -U_1 (Z)
\]

Consequently, the action including the source terms writes:

\[
\int \Psi^\dagger (Z) \left( -\frac{\partial^2}{2} \nabla_\theta^2 + \frac{1}{2\theta^2} - \frac{\sigma^2}{2} \nabla_Z^2 \right) \Psi (Z)
\]

\[
+ \int \Psi^\dagger (Z) \left( \frac{G (Z) - \int V (Z, (Z')_{(n)}) |\Psi (Z')|_{(n)}^2} {2\sigma^2} \right) \Psi (Z)
\]

\[
+ \int J (Z) \exp \left( -\int Z \left( \frac{G (Z) + U_1 (Z)}{\sigma^2} \right) \right) \Psi (Z)
\]
\[ + \int J^\dagger (Z) \exp \left( \int^Z \frac{(G(Z) + U_1(Z))}{\sigma^2} \right) \Psi^\dagger (Z) \]

The Green functions are then computed through the following formulas:

\[
G(Z, Y) = \left\{ \exp \left( -\left( \int^Z \frac{(G(Z) + U_1(Z))}{\sigma^2} \right) \right) \Psi(Z) \Psi^\dagger (Y) \right. \\
\left. \times \exp \left( \left( \int^Y \frac{(G(Z) + U_1(Z))}{\sigma^2} \right) \right) \right\} \Psi(Z) \Psi^\dagger (Y)
\]

Appendix 2. Condition for a non-trivial phase

We find the conditions under which a non-trivial phase appears for the system by inspecting the configurations \( \Psi_0 (Z, \theta) \) minimising the action.

In the absence of any dynamics for \( X \), and since the potential for \((K, P)\) is positive, the minimal configuration is null. The possibility of non-trivial configurations thus depends on the \( X \)-part of the action (22). Recall that we set \( Z_i = (K_i, P_i, X_i) \) for any index \( i \). Using a first approximation for \( S_2 (\Psi) \), the minimisation of (22) yields:

\[
0 = -\frac{\sigma^2}{2} \nabla^2_X \Psi(Z, \theta) + V_0 (X) \Psi(Z, \theta) + \left( \int V_1 (X - X_2) |\Psi(Z_2, \theta)|^2 \right) \Psi(Z, \theta) \\
+ \left( \int V_2 (X - X_2, X - X_3, d_{23}) |\Psi(Z_2, \theta)|^2 |\Psi(Z_3, \theta)|^2 \right) \Psi(Z, \theta)
\]

The condition for a minimum can be found by replacing some of these quantities by their averages:

\[
\langle X \rangle = 0 \\
\langle X - X_i \rangle \simeq \sqrt{\langle (X - X_i)^2 \rangle} \simeq \sqrt{2\langle X^2 \rangle} \simeq \sqrt{2\kappa_0^{-\frac{1}{4}} \sigma_X}
\]

\[
V_1 (|X - X_2|) = -\frac{\kappa_1}{2} \frac{K K'}{(K)^2} \exp \left( -\chi_1 |X - X_2| \right) \exp \left( -\chi_1 |X - X_2| \right) \\
\simeq -\frac{\kappa_1}{2} \frac{K K'}{(K)^2} \exp \left( -\chi_1 \sqrt{2\kappa_0^{-\frac{1}{4}} \sigma_X} \right) = -\frac{\bar{\kappa}_1}{2}
\]

and similarly:

\[
V_2 (X - X_2, X - X_3, d_{23}) \simeq \frac{\kappa_2}{3} \exp \left( -\chi_2 3 \sqrt{2\kappa_0^{-\frac{1}{4}} \sigma_X} \right) = \frac{\bar{\kappa}_2}{3}
\]

In the sequel, the parameters \( \chi_1, \chi_2 \) and \( \kappa_0 \) will be considered to be relatively small. Since we are only interested in finding an approximate condition for the existence of

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a non-trivial phase, we can set \( \bar{\kappa}_1 \simeq \kappa_1 \) and \( \bar{\kappa}_2 \simeq \kappa_2 \). We are thus left with:

\[
0 = -\frac{\sigma_X^2}{2} \nabla_X^2 \Psi (Z, \theta) + \frac{\kappa_0}{2\sigma_X^2} X^2 \Psi (Z, \theta)
- \kappa_1 \left( \int |\Psi (Z_2, \theta)|^2 \right) \Psi (Z, \theta) + \kappa_2 \left( \int |\Psi (Z_2, \theta)|^2 |\Psi (Z_3, \theta)|^2 \right) \Psi (Z, \theta)
\]

(91)

We assume a fundamental state of the form:

\[
\Psi_0 (Z, \theta) = \Psi_0^{(1)} (X) \Psi_0^{(2)} (K) \Psi_0^{(3)} (P)
\]

(92)

that will be justified later on. We also define:

\[
\rho^2 = \int |\Psi_0 (Z, \theta)|^2
\]

(93)

Equation (91) rewrites:

\[
0 = -\frac{\sigma_X^2}{2} \nabla_X^2 \Psi (Z, \theta) + \frac{\kappa_0}{2\sigma_X^2} X^2 \Psi (Z, \theta) - \kappa_1 \rho^2 \Psi (Z, \theta) + \kappa_2 \rho^4 \Psi (Z, \theta)
\]

(94)

This is the eigenstate equation for an harmonic oscillator in a constant potential \(-\kappa_1 \rho^2 + \kappa_2 \rho^4\). The harmonic oscillator part has eigenvalues \(\left( \frac{1}{2} + n \right) \kappa_0^{\frac{1}{2}}\) for \(n \in \mathbb{N}\). The condition for a solution of (94) with a finite norm is obtained by considering the fundamental \((n = 0)\) eigenvalue, so that (94) satisfies:

\[
\frac{1}{2} \kappa_0^{\frac{1}{2}} - \kappa_1 \rho^2 + \kappa_2 \rho^4 = 0
\]

(95)

Thus, for \(\kappa_1^2 - 2\kappa_0^{\frac{1}{2}} \kappa_2 < 0\), the system has only one—trivial—phase \(\rho = 0\), i.e. \(\Psi (Z, \theta) = 0\). However, for \(\kappa_1^2 - 2\kappa_0^{\frac{1}{2}} \kappa_2 > 0\), there is a possibility of non-trivial phase with:

\[
\rho^2 \simeq \frac{\kappa_1 + \sqrt{\kappa_1^2 - 2\kappa_0^{\frac{1}{2}} \kappa_2}}{2\kappa_2}
\]

The value of \(\rho\) will be refined in appendix 4.

As a consequence, the possibility of a non-trivial phase depends on the relative strength of the repulsive force over the attractive one. A non-trivial phase is possible only for a repulsive force that is strong enough.

**Appendix 3. Determination of prices and average capital as functions of \(\Psi\) for \(\rho = 0\)**

Let us consider the phase \(\rho = 0\), i.e. the case of the trivial background field \(|\Psi_0 (Z, \theta)|\) (see (92) and (93)). We look for a configuration satisfying the market clearing condi-

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When needed, we will set \( A \) before considering the corrections in (20) at the zeroth order in \( A \). To do so, we replace:

\[
\Psi (K, P, X, \theta) \to \delta (P - P (K, X, \theta)) \Psi (K, X, \theta)
\]  

(96)

When needed, we will set \( Z = (K, P, X, \theta) \). We will first consider the case \( \bar{A} \gg A \), before considering the corrections in \( \frac{A}{\bar{A}} \).

**Determination of prices and average capital as functions of \( \Psi \), expression of the potential. Case \( \bar{A} \to \infty \)**

**Defining equations as functions of the field**

We determine simultaneously the price \( P \) as a function of \( K \) and \( X \), and the average capital \( \langle K \rangle_X \). To do so, we consider the case where \( \bar{A} \to \infty \). We write the potential term in (20) at the zeroth order in \( \frac{A}{\bar{A}} \). Incidentally, the overall factor \( \frac{1}{2\pi^2} \) is irrelevant in this section and will be omitted.

\[
\left( \delta K - AP K^\alpha \left( 1 - \frac{\kappa}{d^2} \right) \right) \left( 1 - \frac{\kappa}{d^2} \right) \frac{p_3 \exp \left( -\frac{\left| X - X_2 \right|}{d} + \frac{\left| X - X_3 \right|}{d} \right)}{\left| \Psi (Z_2, \theta) \right|^2 \left| \Psi (Z_3, \theta) \right|^2 dZ_2 dZ_3} \right)^2
\]

(97)

with the associated constraint:

\[
P^{1+\gamma} K^\alpha = \frac{\kappa}{d^2} \int P_2 (K) K^\alpha P_3 \exp \left( -\left( \frac{\left| X - X_2 \right|}{d} + \frac{\left| X - X_3 \right|}{d} \right) \right) \times \left| \Psi (Z_2, \theta) \right|^2 \left| \Psi (Z_3, \theta) \right|^2 dZ_2 dZ_3
\]

(98)

The constraint is solved by using the field configuration (96) for a trial function \( P \) given by:

\[
P = P (K, X) = (K)^{-\alpha \gamma} f (X)
\]

(99)

Replacing this expression in the constraint (98) leads in average to the relation:

\[
(f (X))^{1+\gamma} = \frac{\kappa}{d^2} \int P_2 (K) K^\alpha P_3 \exp \left( -\left( \frac{\left| X - X_2 \right|}{d} + \frac{\left| X - X_3 \right|}{d} \right) \right) \times \left| \Psi (Z_2, \theta) \right|^2 \left| \Psi (Z_3, \theta) \right|^2 dZ_2 dZ_3
\]

(100)

The potential (97) for \( K \) then becomes:

\[
\left( \delta K - A \left( K \right) \right) \left( 1 - \frac{\kappa}{d^2} \right) \frac{\alpha \gamma \sigma f (X)}{\sqrt{\pi^2}} \left( \frac{\left( K_3 \right)^{-\alpha \gamma} f (X_3) \exp \left( -\left( \frac{\left| X - X_2 \right|}{d} + \frac{\left| X - X_3 \right|}{d} \right) \right) \left| \Psi (Z_2, \theta) \right|^2 \left| \Psi (Z_3, \theta) \right|^2 dZ_2 dZ_3 \right)^2
\]

\[
= \left( \delta K - A \left( K \right) \right) \left( 1 - \frac{\kappa}{d^2} \right) \frac{\alpha \gamma \sigma f (X)}{\sqrt{\pi^2}} \left( \frac{\left( K_3 \right)^{-\alpha \gamma} f (X_3) \exp \left( -\left( \frac{\left| X - X_2 \right|}{d} + \frac{\left| X - X_3 \right|}{d} \right) \right) \left| \Psi (K_2, X_2, \theta) \right|^2 \left| \Psi (K_3, X_3, \theta) \right|^2 dZ_2 dZ_3 \right)^2
\]

(101)
For $\alpha\gamma << 1$, $(K)^{\alpha\gamma} \simeq \langle K \rangle X^\alpha\gamma$, where $\langle K \rangle_X$ is defined by (31). Adding the kinetic part for $K$ in (20) to (101) and approximating the parenthesis in (101) by its average defined in (31), we obtain for a given $X$, the action for an Euclidian harmonic oscillator:

$$
\int \Psi^\dagger (Z, \theta) (-\sigma^2 \nabla_K^2 + \delta K - A (\langle K \rangle X^\alpha\gamma)^\alpha\gamma f (X) \times 
\times \left\{ \exp \left( -\left( \frac{|X-X_2|}{d} + \frac{|X-X_3|}{d} \right) \right) |\Psi (K, X_2, \theta)|^2 |\Psi (K, X_3, \theta)|^2 dZ_2 dZ_3 \right\} )^2 \right) \Psi (Z, \theta)
$$

whose average $\langle K \rangle_X$ satisfies:

$$
\delta \langle K \rangle_X - A (\langle K \rangle_X)^{\alpha\gamma} f (X) \times 
\times \left\{ \exp \left( -\left( \frac{|X-X_2|}{d} + \frac{|X-X_3|}{d} \right) \right) |\Psi (K, X_2, \theta)|^2 |\Psi (K, X_3, \theta)|^2 dZ_2 dZ_3 \right\} = 0
$$

(102)

with solution:

$$
\langle K \rangle_X = \left( \frac{\delta}{\sigma^2} f (X) \right)^{\frac{1+\gamma}{1+\gamma(1-\alpha)}}
\times \left( \exp \left( -\left( \frac{|X-X_2|}{d} + \frac{|X-X_3|}{d} \right) \right) |\Psi (K, X_2, \theta)|^2 |\Psi (K, X_3, \theta)|^2 dZ_2 dZ_3 \right) \right)
$$

(103)

This expression for $\langle K \rangle_X$ can then be used in (100) to identify $f (X)$:

$$
(f (X))^{1+\gamma} = \frac{\kappa}{d^2} \left( \int (K_2)^{\alpha\gamma} (K_3)^{-\alpha\gamma} f (X_3) f (X_2) \exp \left( -\left( \frac{|X-X_2|}{d} + \frac{|X_2-X_3|}{d} \right) \right) |\Psi (K, X_2, \theta)|^2 |\Psi (K, X_3, \theta)|^2 dZ_2 dZ_3 \right)
$$

(104)

and the expression for $P$ follows from (99). We can now solve the defining Eqs. (103) and (104) by using a Green function approximation.

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Expression of (103) using Green functions At the lowest order of perturbation theory, (104) can be rewritten using the Green functions associated to $|\Psi (K_2, X_2, \theta)|^2 |\Psi (K_3, X_3, \theta)|^2$. Expression (104) becomes:

\[
(f (X))^{1+\gamma} 
\simeq \frac{\kappa}{d^2} \int \{ G ((K_3, X_3, \theta) , (K_2, X_2, \theta)) G ((K_2, X_2, \theta) , (K_3, X_3, \theta)) \\
+ G ((K_3, X_3, \theta) , (K_3, X_3, \theta)) G ((K_2, X_2, \theta) , (K_2, X_2, \theta)) \}
\times (K_2)^{\alpha \gamma / (1+\gamma)} (K_3)^{-\alpha / (1+\gamma)} f (X_3) f (X_2) \exp \left(-\left|\frac{X - X_2}{d}\right| + \left|\frac{X_2 - X_3}{d}\right|\right) dZ_2 dZ_3
\]

(105)

We derive the Green functions in appendix 5. For $\theta' \simeq \theta$, the Green functions $G ((K_3, X_3) , (K_2, X_2))$ are those of harmonic oscillators for a propagation time of order $\theta^2$. Moreover, and discarding the $\theta$ dependency, we have $G ((K_3, X_3) , (K_2, X_2)) \simeq 0$ and $G ((K_2, X_2) , (K_3, X_3)) \simeq 0$ for $\theta' \simeq \theta$. We thus write (105):

\[
(f (X))^{1+\gamma} 
\simeq \frac{\kappa}{d^2} \int G ((K_3, X_3) , (K_3, X_3)) G ((K_2, X_2) , (K_2, X_2)) (K_2)^{\alpha \gamma / (1+\gamma)} (K_3)^{-\alpha / (1+\gamma)} \\
\times f (X_3) f (X_2) \exp \left(-\left|\frac{X - X_2}{d}\right| + \left|\frac{X_2 - X_3}{d}\right|\right) dZ_2 dZ_3
\]

(106)

We also assume that the dynamics for $K$ is faster than the dynamics for $X$, so that the Green functions have the form:

$$G ((K_2, X_3) , (K_2, X_2)) \simeq G_{X_2} (K_3, K_2) G (X_2, X_2)$$

This form will be justified by the formulas in appendix 5. As a consequence, the integrals over $K_2$ and $K_3$ in (106):

\[
\int G_{X_2} (K_2, K_2) (K_2)^{\alpha \gamma / (1+\gamma)} (K_3)^{-\alpha / (1+\gamma)} G_{X_3} (K_3, K_2) dK_2 dK_3
\]

compute the average of $(K_2)^{\alpha \gamma / (1+\gamma)} (K_3)^{-\alpha / (1+\gamma)}$ for states such that $X = X_2$ and $X = X_3$, respectively. We write $\left<K^{\alpha \gamma / (1+\gamma)}\right>_X$ and $\left<K^{-\alpha / (1+\gamma)}\right>_X$ these averages. Using that $\left<K^{\alpha \gamma / (1+\gamma)}\right>_X \simeq \left<K^{\alpha \gamma / (1+\gamma)}\right>_X$ and $\left<K^{-\alpha / (1+\gamma)}\right>_X \simeq \left<K^{-\alpha / (1+\gamma)}\right>_X$, it implies that:

\[
\int G ((K_3, X_3) , (K_2, X_2)) G ((K_2, X_2) , (K_2, X_2)) (K_2)^{\alpha \gamma / (1+\gamma)} (K_3)^{-\alpha / (1+\gamma)} dK_2 dK_3
\]

\[
\simeq G (X_2, X_3) G (X_3, X_2) \left<K^{\alpha \gamma / (1+\gamma)}\right>_X \left<K^{-\alpha / (1+\gamma)}\right>_X
\]

(107)
The average level of capital \((K)_X\) is computed below by identification.

As a consequence of (107), the expression (106) writes:

\[
(f(X))^{1+\gamma} \simeq \frac{\kappa}{d^2} \int G(X_2, X_2) G(X_2, X_2) \langle K \rangle_{X_2}^{\alpha / \gamma} \langle K \rangle_{X_3}^{-\alpha / \gamma} \times f(X_3) f(X_2) \exp \left( - \left( \frac{|X - X_2|}{d} + \frac{|X_2 - X_3|}{d} \right) \right) dX_2 dX_3
\]

(108)

The potential for the \(X\)-part of the action is harmonic with low frequency. We can thus assume that the integral in (108) is distributed around \(X_2 - X_3 \simeq 0\) and replace \(\exp \left( - \left( \frac{|X_2 - X_3|}{d} \right) \right)\) by \(2d\) times a delta function. As a consequence, (108) simplifies as:

\[
(f(X))^{1+\gamma} \simeq \frac{2\kappa}{d^2} \int \langle K \rangle_{X_2}^{\alpha (\gamma - 1)} (f(X_2))^2 G^2 (X_2, X_2) \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) dX_2
\]

(109)

Given the low frequency describing the \(X\)-part of the action, the positions in the exchange space can be considered homogeneously spread on \([-1, 1]\), the probability \(G(X_2, X_2)\) can be replaced by a constant density, here \(\frac{1}{2}\) in first approximation, to normalise the probability of the interval \([-1, 1]\) to 1. As a consequence, we have:

\[
(f(X))^{1+\gamma} \simeq \frac{\kappa}{2d} \int \langle K \rangle_{X_2}^{\alpha (\gamma - 1)} (f(X_2))^2 \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) dX_2
\]

(110)

Then, using (103), we can express \(\langle K \rangle_X\) as a function of \(X\):

\[
(K)_X = \left( \frac{A}{\delta} f(X) \right)^{1+\gamma \frac{1}{1+\gamma}}
\]

\[
\times \left( 1 - \frac{\kappa}{2d^2} \int \langle K \rangle_{X_2}^{\alpha / \gamma} \langle K \rangle_{X_3}^{-\alpha / \gamma} f(X_3) f(X_2) \exp \left( - \left( \frac{|X - X_2| + |X - X_3|}{d} \right) \right) \right)^{\frac{1}{1+\gamma \frac{1}{1+\gamma}}}
\]

\[
\times \left( 1 - \frac{\kappa}{2d^2} \int \langle K \rangle_{X_2}^{\alpha / \gamma} \langle K \rangle_{X_3}^{-\alpha / \gamma} f(X_3) f(X_2) \exp \left( - \left( \frac{|X - X_2| + |X - X_3|}{d} \right) \right) \right)
\]

(111)

for \(\gamma \ll 1\). The factor \(\frac{1}{4}\) normalises the probability of each interval \([-1, 1]\) to 1.

**Solving (111)**

---

8 The same result can be obtained by considering the contribution of \(\int |\Psi(K_2, X_2, \theta)|^2 |\Psi(K_3, X_3, \theta)|^2\) in any graph. It corresponds to introducing two 2-points vertices. By convolution with Gaussian propagators, they act as the identity operator in first approximation. Hence \(|\Psi(K_2, X_2, \theta)|^2\) can be replaced by \(\frac{1}{2}\) to produce in average a factor of 1 after integration.
We can solve (111) to find the average capital \( \langle K \rangle_X \). To do so, we postulate the following first approximation for \( \langle K \rangle_X \):

\[
\langle K \rangle_X = \left( \frac{A}{\delta} f(X) \left( 1 - h \left( 1 + \frac{|X|}{d} \right) \exp \left( -\frac{|X|}{d} \right) \left( 1 - \frac{\cosh \frac{X}{d}}{\exp \left( \frac{1}{d} \right)} \right) \right) \right)^{\frac{1+\gamma}{1+\gamma(1-\alpha)}}
\]

\[
\simeq \left( \frac{A}{\delta} f(X) \left( 1 - h \exp \left( -\frac{|X|}{d} \right) \left( 1 - \frac{\cosh \frac{X}{d}}{\exp \left( \frac{1}{d} \right)} \right) \right) \right)^{\frac{1+\gamma}{1+\gamma(1-\alpha)}}
\]

(112)

with \( h \) to be determined. In (111) we can, for the sake of simplicity, replace \( \langle K \rangle_X \) in the integrals by its average over \( X \), written \( \langle K \rangle_X \) (\( h < \exp \left( \frac{1}{2d} \right) \)). It is approximatively equal to:

\[
\langle K \rangle_X = \left( \frac{A}{\delta} f(X) \left( 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( -\frac{1}{2d} \right) \left( 1 - \frac{\exp \left( -\frac{1}{2d} \right)}{2} \right) \right) \right)^{\frac{1+\gamma}{1+\gamma(1-\alpha)}}
\]

(113)

Equation (113), along with (110), yields the following equation for \( f(X) \):

\[
(f(X))^{1+\gamma} \simeq \frac{\kappa}{2d} \int \langle K \rangle_{X_2}^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}} (f(X_2))^2 \exp \left( -\frac{|X - X_2|}{d} \right) \, dX_2
\]

\[
\simeq \frac{\kappa}{2d} \left( \frac{A}{\delta} \right)^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}} \int \left( 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( -\frac{1}{2d} \right) \left( 1 - \frac{\exp \left( -\frac{1}{2d} \right)}{2} \right) \right)^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}} \times (f(X_2))^{1+\gamma} \frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)} \exp \left( -\frac{|X - X_2|}{d} \right) \, dX_2
\]

(114)

We look for a solution of (114) of the form:

\[
f(X) = D \exp \left( -c \frac{|X|}{d} \right)
\]

(115)

and Eq. (114) becomes for \( \gamma \ll 1 \) :
\[
\exp\left(- (1 + \gamma) c \frac{|X|}{d}\right) 
\sim \frac{\kappa}{2d} \left(\frac{A}{\delta}\right)^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}} D^{\frac{(1-\gamma^2)(1-\alpha)}{1+\gamma(1-\alpha)}}
\times \left(1 - h \left(1 + \frac{1}{2d}\right) \exp\left(-\frac{1}{2d}\right) \left(1 - \exp\left(-\frac{1}{2d}\right)\right)\right)^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}}
\times \int \exp\left(-c \frac{(1 + \gamma) (2 - \alpha)}{1 + \gamma (1 - \alpha)} \frac{|X|}{d}\right) \exp\left(-\frac{|X - X_2|}{d}\right) dX_2
\]

(116)

The integral in (116) can be estimated as \((X > 0)\):

\[
\int_{-1}^{0} \exp\left(\frac{2aX_2}{d}\right) \exp\left(-\frac{(X - X_2)}{d}\right) + \int_{0}^{X} \exp\left(-\frac{2aX_2}{d}\right) \exp\left(-\frac{(X - X_2)}{d}\right)
\]

\[
+ \int_{X}^{1} \exp\left(-\frac{2aX_2}{d}\right) \exp\left(\frac{(X - X_2)}{d}\right)
\]

\[
= d \left(\frac{\exp\left(-\frac{X}{d}\right) \left(1 - \exp\left(-\frac{(2a+1)}{d}\right)\right)}{2a + 1}\right)
\]

\[
+ \left(\frac{\exp\left(-\frac{2aX}{d}\right) - \exp\left(-\frac{X}{d}\right)}{1 - 2a}\right) + \frac{\exp\left(-\frac{2aX}{d}\right) - \exp\left(\frac{X-(2a+1)}{d}\right)}{2a + 1}
\]

\[
\simeq d \frac{4a \exp\left(-\frac{X}{d}\right) - 2 \exp\left(-\frac{2aX}{d}\right)}{4a^2 - 1}
\]

and the identification of \(f(X)\) in (116) becomes:

\[
\exp\left(- (1 + \gamma) c \frac{|X|}{d}\right) 
\sim \frac{\kappa}{2d} \left(\frac{A}{\delta}\right)^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}} D^{\frac{(1-\gamma^2)(1-\alpha)}{1+\gamma(1-\alpha)}}
\times \left(1 - h \left(1 + \frac{1}{2d}\right) \exp\left(-\frac{1}{2d}\right) \left(1 - \exp\left(-\frac{1}{2d}\right)\right)\right)^{\frac{\alpha(\gamma-1)}{1+\gamma(1-\alpha)}}
\times \left(4a \exp\left(-\frac{X}{d}\right) - 2 \exp\left(-\frac{2aX}{d}\right)\right)
\]

(117)

We will justify below that \(2 \exp\left(-\frac{2aX}{d}\right)\) can be neglected. In this case, for \(\gamma < < 1\), one has:

\[
c \simeq \frac{1}{1 + \gamma}
\]
\[ 2a = \frac{(1 + \gamma) (2 - \alpha)}{1 + \gamma (1 - \alpha)} \frac{1}{1 + \gamma} = \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} > 1 \]

The fact that \(2a\) is greater than 1 justifies our hypothesis to neglect \(2 \exp\left(-2a \frac{X}{d}\right)\) and leads to identifying the postulated constant \(D\) in (115) by writing (117) as:

\[
\frac{\kappa}{2} \left( \frac{A}{\delta} \right)^{\frac{\alpha (\gamma - 1)}{1 + \gamma (1 - \alpha)}} D \frac{2(2 - \alpha)}{1 + \gamma (1 - \alpha)} \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \left( 1 - h \left( 1 + \frac{1}{2d} \right) \exp\left(-\frac{1}{2d} \right) \left( 1 - \frac{\exp\left(-\frac{1}{2d} \right)}{2} \right) \right) = 1
\]

whose solution is:

\[
D \simeq \left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right)^{\frac{\alpha (\gamma - 1)}{1 + \gamma (1 - \alpha)}} \frac{1 + \gamma (1 - \alpha)}{(1 - \alpha) (1 - 1/\gamma)} + \alpha (1 - \alpha) \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \exp\left(-\frac{1}{2d} \right)
\]

We ultimately rewrite (115) as:

\[
f(X) = D \exp\left(-\frac{|X|}{(1 + \gamma) d}\right) = \left( \frac{A}{\delta} \right)^{\frac{\alpha (\gamma - 1)}{1 + \gamma (1 - \alpha)}} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \exp\left(-\frac{1}{2d} \right) \left( 1 - \frac{\exp\left(-\frac{1}{2d} \right)}{2} \right) = \left( \frac{A}{\delta} \right)^{\frac{\alpha (\gamma - 1)}{1 + \gamma (1 - \alpha)}} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \exp\left(-\frac{1}{2d} \right) \left( 1 - \frac{\exp\left(-\frac{1}{2d} \right)}{2} \right)
\]

The full form of \(f(X)\) and \(\langle K \rangle_X\) is obtained by computing the constant \(h\). It is found by the identification of (112) and (111), which yields:

\[
h \left( 1 + \frac{|X|}{d} \right) \exp\left(-\frac{|X|}{d}\right) \left( 1 - \frac{\cosh \frac{X}{d}}{\frac{1}{d}} \right) = \frac{\kappa}{4d^2} \int \frac{\alpha \nu}{X_2} \left( \frac{\alpha \nu}{X_3} \right) \exp\left(-\frac{\alpha \nu}{X_3} \right) f(X_3) \left( f(X_2) \right)^{-\gamma} \times \exp\left(-\left( \frac{|X - X_2|}{d} + \frac{|X - X_3|}{d} \right) \right) dX_2 dX_3
\]

(120)
Considering $\gamma << 1$, we compute the RHS of (120) as:

$$
\frac{\kappa}{4d^2} \int \langle K \rangle_{X_2} \frac{\alpha}{\gamma} \langle K \rangle_{X_3} \left( f (X_3) (f (X_2))^{-\gamma} \exp \left( - \left( \frac{|X - X_2|}{d} + \frac{|X - X_3|}{d} \right) \right) \right) dX_2 dX_3
$$

$$
= \frac{\kappa}{4d^2} \left( \frac{A}{d} \left( 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( - \frac{1}{2d} \right) \left( 1 - \exp \left( - \frac{1}{2d} \right) \right) \right) \right) \times \int (f (X_3)) \left( (1 - \alpha + 1 + \gamma) \frac{1}{1 + \gamma (1 - \alpha)} \right) \left( (1 - \alpha + 1 + \gamma) \frac{2}{1 + \gamma (1 - \alpha)} \right) \exp \left( - \left( \frac{|X - X_2|}{d} + \frac{|X - X_3|}{d} \right) \right) dX_2 dX_3
$$

Using (119), this expression writes in first approximation:

$$
\frac{1}{4d^2} \left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha)} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right) \times \int \exp \left( - \frac{(1 - \alpha)}{1 + \gamma (1 - \alpha)} \frac{|X_3|}{d} \right) \exp \left( - \gamma \left( \frac{1 - \alpha}{1 + \gamma (1 - \alpha)} \right) \frac{|X_2|}{d} \right) \exp \left( - \left( \frac{|X - X_2|}{d} + \frac{|X - X_3|}{d} \right) \right)
$$

$$
\approx \frac{1}{4d^2} \left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha)} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right) \times \int \exp \left( - \frac{(1 - \alpha)}{1 + \gamma (1 - \alpha)} \frac{|X_3|}{d} \right) \exp \left( - \left( \frac{|X - X_2|}{d} + \frac{|X - X_3|}{d} \right) \right)
$$

$$
= \frac{1}{2d} \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha)} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \times \int \exp \left( - \frac{(1 - \alpha)}{1 + \gamma (1 - \alpha)} \frac{|X_3|}{d} \right) \exp \left( - \left( \frac{|X - X_3|}{d} \right) \right) \left( 1 - \frac{\cosh \left( \frac{X}{d} \right)}{\exp \left( \frac{X}{d} \right)} \right)
$$

(121)

where the equality:

$$
\int \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) = 2d \left( 1 - \frac{\cosh \left( \frac{X}{d} \right)}{\exp \left( \frac{X}{d} \right)} \right)
$$

(122)

has been used. We then find $h$ by computing the integral arising in (121) for $X > 0$, the general case being obtained by replacing $X$ with $|X|$.
\[ \int_{-1}^{0} \exp \left( (1 - \alpha) \frac{u}{d} - \frac{X - u}{d} \right) du + \int_{X}^{1} \exp \left( -(1 - \alpha) \frac{u}{d} + \frac{X - u}{d} \right) du \\
+ \int_{0}^{X} \exp \left( (1 - \alpha) \frac{u}{d} - \frac{X - u}{d} \right) du \\
= d \left( \frac{e_{\alpha}^{X} - 1}{\alpha} e^{-X/d} + \frac{e^{X+\alpha-2/d} - e^{X/\alpha}(\alpha-1)}{\alpha-2} - e^{-X/\alpha} \frac{1 - e^{-\alpha/d}}{\alpha-2} \right) \]

The above equation can be approximated at the zeroth order in \( \alpha \):

\[ \int \exp \left( - \frac{(1 - \alpha)}{1 + \gamma (1 - \alpha)} \frac{|X|}{d} \right) \exp \left( - \frac{|X - X|}{d} \right) \approx d \left( \frac{1}{2} e^{-X/d} - \frac{1}{2} e^{X/d} - e^{-X/2} \left( \frac{1}{2} e^{-X/2} - \frac{1}{2} \right) + \frac{X}{d} e^{-X/d} \right) \]

\[ \approx d \left( 1 + \frac{X}{d} - \frac{1}{2} e^{\frac{2X-2}{d}} \right) e^{-X/d} \] (123)

The correction terms of order \( \alpha \) are negligible for \( d \) close to 1. Moreover, for \( d \approx 1 \), formula (123) is equal to \( d \left( 1 + \frac{X}{d} \right) e^{-X/d} \) in first approximation. Restoring the absolute value \( |X| \), inserting (123) in (121) and using (120) directly yields the value of \( h \), for \( \gamma << 1 \) and \( d \) close to 1:

\[ h \approx \left( \frac{2 - \alpha}{2 (2 - \alpha)} \left( \frac{1}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \] (124)

Once \( \langle K \rangle_X \) has been found, the price can be rewritten as a function of \( \langle K \rangle_X \) using (112):

\[ P = P \left( K, X \right) \]
\[ = \left( \frac{K}{\langle K \rangle_X} \right)^{-\frac{\alpha}{1 + \gamma}} \frac{f \left( X \right)}{\left( 1 \right)^{\frac{\alpha}{1 + \gamma}}} \]
\[ = \left( \frac{K}{\langle K \rangle_X} \right)^{-\frac{\alpha}{1 + \gamma}} \left( f \left( X \right) \right)^{\frac{\alpha}{1 + \gamma (1 - \alpha)}} \]
\[ \times \left( \frac{\left( \frac{A}{\delta} \left( 1 - \left( 1 + \frac{|X|}{d} \right) \exp \left( -\frac{|X|}{d} \right) \left( 1 - \cosh \frac{X}{\delta} \exp \left( \frac{1}{\delta} \right) \right) \right) \right)^{\frac{\alpha}{1 + \gamma (1 - \alpha)}}}{\left( \frac{A}{\delta} \left( 1 - \left( 1 + \frac{|X|}{d} \right) \exp \left( -\frac{|X|}{d} \right) \left( 1 - \cosh \frac{X}{\delta} \exp \left( \frac{1}{\delta} \right) \right) \right) \right)^{\frac{\alpha}{1 + \gamma (1 - \alpha)}}} \right) \]
and thus, given (112):

\[
P(K, X) = \frac{\left(\frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa} \left(\frac{2 - \alpha}{1 + \gamma (1 - \alpha)}\right)^2 - 1\right)_{\frac{1}{1 + \gamma}} \exp\left(-\frac{(1 - \alpha) |X|}{d (1 + \gamma (1 - \alpha))}\right)}{\left(\frac{K}{\langle K \rangle_X}\right)^{\frac{\alpha}{1 + \gamma}} \left(\frac{1 - h \left(1 + \frac{|X|}{d}\right) \exp\left(-\frac{|X|}{d}\right)}{1 - h \left(1 + \frac{|X|}{d}\right) \exp\left(-\frac{|X|}{d}\right)}\right)^{\frac{1}{1 + \gamma (1 - \alpha)}}}
\]

\[
= \frac{\left(\frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa} \left(\frac{2 - \alpha}{1 + \gamma (1 - \alpha)}\right)^2 - 1\right)_{\frac{1}{1 + \gamma}} \exp\left(-\frac{(1 - \alpha) |X|}{d (1 + \gamma (1 - \alpha))}\right)}{\left(\frac{K}{\langle K \rangle_X}\right)^{\frac{\alpha}{1 + \gamma}} \left(\frac{1 - h \left(1 + \frac{|X|}{d}\right) \exp\left(-\frac{|X|}{d}\right)}{1 - h \left(1 + \frac{|X|}{d}\right) \exp\left(-\frac{|X|}{d}\right)}\right)^{\frac{1}{1 + \gamma (1 - \alpha)}}}
\]

(125)

Ultimately, the previous computations imply the following form for the potential:

\[\delta^2 (K - \langle K \rangle_X)^2\]

which justifies the assumption of harmonic oscillations.

**Determination of prices and average capital as functions of \(\Psi\), expression of the potential. Corrections of order \(\left(\frac{A}{\bar{A}}\right)^2\)**

Having found \(\langle K \rangle_X\), the price \(P(K, X)\) and the potential for \(\bar{A} \rightarrow \infty\), we can now consider the corrections due to \(\frac{A}{\bar{A}}\):

\(\bar{A} \gg A\): once again we consider fields of the form: \(\Psi(K, P, X, \theta) \rightarrow \delta (P - P(K, X, \theta)) \Psi(K, X, \theta)\).

The factor arising in the potential term of expression (20) is now:

\[
\left(\delta K - A PK^\alpha \left(1 - \frac{\kappa}{d^2} \int P_3 \exp\left(-\left(|X - X_2| + |X - X_3|\right)\right) \frac{P_3}{P_2^\gamma} \right) |\Psi(K_2, X_2, \theta)|^2 \right) \d Z_2 \d Z_3 \right)^2
\]

\[
+ \bar{A}^2 \left( P^{1+\gamma} K^\alpha - \frac{\kappa}{d^2} \int P_2(K_2)^\alpha P_3 \exp\left(-\left(|X - X_2| + |X_2 - X_3|\right)\right) \right) |\Psi(K_2, X_2, \theta)|^2 \right) \d Z_2 \d Z_3 \right)^2
\]

\[
= \left(\delta K - A PK^\alpha U\right)^2 + \bar{A}^2 \left( P^{1+\gamma} K^\alpha - V\right)^2
\]

(126)
with:

\[ U(X) = 1 - U_1 \]
\[ = \left( 1 - \frac{\kappa}{d^2} \int \frac{P_3 \exp\left(-\left(|X - X_2| + |X - X_3|\right)\right)}{P_2^2} |\Psi(Z_2, \theta)|^2 |\Psi(Z_3, \theta)|^2 dZ_2 dZ_3 \right) \]
\[ V(X) = \frac{\kappa}{d^2} \int P_2 (K_2)^\alpha P_3 \exp\left(-\left(|X - X_2| + |X_2 - X_3|\right)\right) |\Psi(Z_2, \theta)|^2 |\Psi(Z_3, \theta)|^2 dZ_2 dZ_3 \]  

(127)

As for other quantities, we replace \( U(X) \) and \( V(X) \) in (126) by their expectations \( \langle U(X) \rangle \) and \( \langle V(X) \rangle \). The value of \( \langle U(X) \rangle \) will be needed later and is given by identifying (112) and (111):

\[ \langle U(X) \rangle = 1 - h \exp\left(-\frac{|X|}{d}\right) \left( 1 - \frac{\cosh\frac{X}{d}}{\exp\left(\frac{1}{d}\right)} \right) \]
\[ \simeq 1 - h \left( 1 + \frac{1}{2d} \right) \exp\left(-\frac{1}{2d}\right) \left( 1 - \frac{\exp\left(-\frac{1}{2d}\right)}{2} \right) \]  

(128)

In the following, for the sake of simplicity in notations, \( U \) and \( V \) will stand for \( \langle U \rangle \) and \( \langle V \rangle \).

The price \( P(K, X) \) is found through the minimisation of the potential (126). This yields the condition:

\[ - UA \left( \delta K - APK^\alpha U \right) + (1 + \gamma) P^\gamma \tilde{A}^2 \left( P^{1+\gamma} K^\alpha - V \right) = 0 \]  

(129)

At the lowest order approximation of (129) in \( \gamma \) is:

\[ - AU \left( \delta K - APK^\alpha U \right) + \tilde{A}^2 \left( P^{1+\gamma} K^\alpha - V \right) \simeq 0 \]  

(130)

We now use (130) to compute \( P^{1+\gamma} K^\alpha \) at first order in \( \frac{\tilde{A}^2}{A^2} \):

\[ P^{1+\gamma} K^\alpha = \frac{\tilde{A}^2}{A^2 U^2 + \tilde{A}^2} V + \frac{AU}{A^2 U^2 + \tilde{A}^2} \delta K \]
\[ = V + \frac{AU}{A^2 U^2 + \tilde{A}^2} (\delta K - AU V) \]

At this order, this also rewrites:

\[ P^{1+\gamma} K^\alpha = V + \frac{AU}{A^2 U^2 + \tilde{A}^2} \left( \delta K - AU P^{1+\gamma} K^\alpha \right) \]
\[ \simeq V + \frac{AU}{A^2 U^2 + \tilde{A}^2} (\delta K - AU PK^\alpha) \]  

(131)
For this value, the factor (126) becomes:

\[
(\delta K - APK^\alpha U)^2 + \tilde{A}^2 \left( P^{1+\gamma} K^\alpha - V \right)^2 \approx (\delta K - APK^\alpha U)^2 + \tilde{A}^2 A^2 U^2 \left( \frac{\delta K - APK^\alpha U}{A^2 U^2 + \tilde{A}^2} \right)^2
\]

Equations (131) and (132) can be simplified in order to find \(P(K, X)\) and \(\langle K \rangle_X\). Since we are looking for first-order corrections in \(\frac{A^2}{\tilde{A}^2}\), we can replace \(\delta K - AV\) in (131) by its lowest order approximation, that is, given our assumption \(\gamma << 1\):

\[
\delta K - AV = \delta K - AP^{1+\gamma} K^\alpha \approx \delta K - APK^\alpha
\]

To this order of approximation, we can also replace \(K\) by \(\langle K \rangle_X\). Using (102):

\[
\delta \langle K \rangle_X = AP \langle K \rangle_X^\alpha U
\]

and (112):

\[
\frac{\delta K - AU V}{V} \approx A \frac{\delta \langle K \rangle_X - AP \langle K \rangle_X^\alpha}{AP \langle K \rangle_X^\alpha} = 0
\]

Equation (131) writes at the first order in \(\frac{A^2}{\tilde{A}^2}\):

\[
P^{1+\gamma} K^\alpha = V
\]

As a consequence of (132) and (134), the equations for \(P(K, X)\) and \(\langle K \rangle_X\) with respect to the case \(\tilde{A} \rightarrow \infty\) are unchanged, as well as their solutions (112) and (125). The term (132) rewrites:

\[
\delta^2 \left( 1 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)} \right) \left( K - \frac{APK^\alpha}{\delta} \right)^2 \approx \delta^2 \left( 1 + \frac{A^2}{(A^2 U^2 + \tilde{A}^2)} \right) \left( K - \langle K \rangle_X \right)^2
\]

\[
\text{(135)}
\]

**Accounting for the normalisation factor**

At this point, the normalisation factor \(2d \left( 1 - \frac{\cosh \frac{X(t)}{\tilde{d}}}{\exp \left( \frac{X(t)}{\tilde{d}} \right)} \right)^{-1}\) for the price index \(\hat{P}_i(t)\) that was skipped for the sake of simplicity can be introduced directly without modifying the main results. An inspection of (6) and (8) shows that it amounts to replacing inside the integrals in (97) and (98) a factor \(\frac{1}{d}\) by \(2d \left( 1 - \frac{\cosh \frac{X}{\tilde{d}}}{\exp \left( \frac{X}{\tilde{d}} \right)} \right)^{-1}\)
and \( \left( 2d \left( 1 - \frac{\cosh \frac{X}{d}}{\exp \left( \frac{1}{d} \right)} \right) \right)^{-1} \), respectively. The introduction of these factors can be accounted for by keeping (97) and (98) unchanged, as well as the following computations, except for the integrals \( \int \exp \left( - \left( \frac{|X_2 - X_3|}{d} \right) \right) \) dX3 in (108), and \( \int \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) \) dX2 in (122) that have to be replaced by \( d\delta (X_2 - X_3) \) and \( d\delta \), respectively. These modifications merely affect the formula for \( \langle K \rangle_X \) and \( D \) that would then write:

\[
\langle K \rangle_X = \left( \frac{A}{\delta} f(X) \left( 1 - h \left( 1 + \frac{|X|}{d} \right) \exp \left( - \frac{|X|}{d} \right) \right) \right)^{\frac{1+\gamma}{1+\gamma(1-\alpha)}} \\
D \simeq \left( \frac{2^{-1+\gamma}(1-\alpha)}{(2-\alpha)\kappa} \left( \frac{2-\alpha}{1+\gamma(1-\alpha)} \right)^2 - 1 \right) \\
\times \left( \frac{A}{\delta} \left( 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( - \frac{1}{2d} \right) \right) \right)^{\frac{\alpha(1-\gamma)}{1+\gamma(1-\alpha)}} \left( \frac{1-\gamma^2}{1-\alpha} \right)^{\frac{1+\gamma(1-\alpha)}{1-\gamma^2(1-\alpha)}}
\]

with \( h \) still given by (124). As explained in the text, this does not modify the general interpretation for \( \langle K \rangle_X \) and the other variables, since \( 1 - \frac{\cosh \frac{X}{d}}{\exp \left( \frac{1}{d} \right)} \) decreases and varies slowly over \([-1, 1]\).

**Appendix 4. Determination of prices and average capital as functions of \( \Psi \) for \( \rho \neq 0 \)**

For a non-trivial fundamental defined by \( \rho > 0 \) [see (92) and (93)], the contribution of the background field \( \Psi_0 \) has to be added to the fluctuations. Previous computations are valid, but average values have now to be computed in the state \( \Psi_0 + \Delta \Psi \). We first consider the sole state \( \Psi_0 (K, P, X, \theta) \), in which we find the corresponding expression for the prices \( P (K, X) \). We then derive the equations for \( f(X) \) and \( \langle K \rangle_X \) and compute the form of the fundamental \( \Psi_0 \) and the value of \( \rho \). Finally, we compute the correction due to \( \Psi_0 (K, P, X, \theta) \) and find the effective action.

**Equations for \( P, f(X) \) and \( \langle K \rangle_X \) in the state \( \Psi_0 \)**

**Defining equation for \( \langle K \rangle_X \)** In the state \( \Psi_0 (K, P, X, \theta) \), the equations defining the potential \( P \) and the average capital \( \langle K \rangle_X \) are the same as in appendix 3. In the following, we proceed as in appendix 3. We first consider the case \( A \to \infty \). We will ultimately include the first-order corrections in \( \frac{A^2}{\bar{A}^2} \), since it amounts to modifying the potential (see 135).

To compute \( f(X) \), we first find the average of \( K \), denoted \( \langle K \rangle_X \). It is computed in a state \( \rho \Psi_0 (K, X) \). Given our order of approximations for \( \Psi_0 (K, X) \), this amounts to replacing \( \kappa \) by \( \kappa \rho^4 \).
Here again we set a trial function for the price:

\[ P = (K)^{-\frac{\rho}{1+\gamma}} f(X) \]  

(136)

The equation for \( f(X) \) is the same as (104):

\[
f^{1+\gamma}(X) = \frac{\kappa \rho^4}{d^2} \left\{ \int P_2 (K_2)^{\alpha} P_3 \exp \left( - \left( |X - X_2| + |X_2 - X_3| \right) \right) |\Psi_0 (Z_2, \theta)|^2 |\Psi_0 (Z_3, \theta)|^2 dZ_2 dZ_3 \right\}
\]

(137)

Equation (103) for \( \langle K \rangle_X \) is still valid in first approximation:

\[
\langle K \rangle_X = \left( \frac{A}{\delta} f(X) \right)^{1+\gamma} \frac{1}{1+\gamma(1-\alpha)} 
\left\{ 1 - \frac{\kappa \rho^4}{d^2} \int (K_2)^{\alpha \gamma} (K_3)^{1-\gamma} f(X_3)(f(X_2))^{-\gamma} \times \exp \left( - \left( \frac{|X - X_2|}{d} + \frac{|X - X_1|}{d} \right) \right) |\Psi_0 (Z_2, \theta)|^2 |\Psi_0 (Z_3, \theta)|^2 dZ_2 dZ_3 \right\}^{1+\gamma} \frac{1}{1+\gamma(1-\alpha)}
\]

(139)

Solving (139) As in appendix 3, the form of \( \langle K \rangle_X \) is postulated to solve (139):

\[
\langle K \rangle_X = \left( \frac{A}{\delta} f(X) \right) \left( 1 - h \left( 1 + \frac{|X|}{d} \right) \exp \left( - \frac{|X|}{d} \right) \left( 1 - \frac{\cosh \frac{X}{d}}{\exp \left( \frac{X}{d} \right)} \right) \right)^{1+\gamma} \frac{1}{1+\gamma(1-\alpha)}
\]

(140)
The resolution is thus similar to appendix 3, and we find:

\[ f(X) = D_\rho \exp \left( -\frac{|X|}{1 + \gamma} \right) \]

\[ = \left( \frac{A}{\delta} \left( 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( -\frac{1}{2d} \left( 1 - \exp \left( -\frac{1}{2d} \right) \right) \right) \right) \right) \]

\[ \times \left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa \rho^4} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right)^{\frac{1 + \gamma (1 - \alpha)}{(1 - \alpha)(1 - \alpha)}} \exp \left( -\frac{|X|}{d (1 + \gamma)} \right) \]

where:

\[ D_\rho = \left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa \rho^4} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right)^{\frac{\alpha(1 - \gamma)}{(1 + \gamma)(1 - \alpha)}} \exp \left( -\frac{(1 - \alpha) |X|}{d (1 + \gamma (1 - \alpha))} \right) \]

(141)

and:

\[ h = \left( \frac{1 + \gamma (1 - \alpha)}{2 (2 - \alpha)} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right) \]

(142)

Using (136) yields the expression for the price:

\[ P = \frac{\left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa \rho^4} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right)^{\frac{\alpha}{1 + \gamma}}}{\left( \frac{K}{(K)_X} \right)^{\frac{\alpha}{1 + \gamma}}} \left( \left( \frac{1 - h \left( 1 + \frac{|X|}{d} \right) \exp \left( -\frac{|X|}{d} \left( 1 - \cosh \frac{X}{d} \right) \right) \right)^{\frac{\alpha}{1 + \gamma}} \right) \]

(143)

\[ \simeq \frac{\left( \frac{1 + \gamma (1 - \alpha)}{(2 - \alpha) \kappa \rho^4} \left( \left( \frac{2 - \alpha}{1 + \gamma (1 - \alpha)} \right)^2 - 1 \right) \right)^{\frac{\alpha}{1 + \gamma}}}{\left( \frac{K}{(K)_X} \right)^{\frac{\alpha}{1 + \gamma}}} \exp \left( -\frac{(1 - \alpha) |X|}{d (1 + \gamma (1 - \alpha))} \right) \]

(144)

These results depend on \( \rho \). The next paragraph will derive the value of \( \rho \) and the expression of \( \Psi_0 \).

**Computation of \( \rho \) and \( \Psi_0 \)** The value of \( \rho \) and the form of \( \Psi_0 \) can now be computed. As explained before, this phase is possible approximatively for:

\[ \kappa_1^2 - 2 \kappa_0 \kappa_2 > 0 \]
In this phase, $\rho$ can be found approximately by taking into account only the contribution in $X$:

$$\rho \simeq \frac{\kappa_1 + \sqrt{\kappa_1^2 - 2\kappa_0 \kappa_2}}{2\kappa_2}$$

with $\frac{\partial^2 S}{\partial \rho^2} > 0$. A more precise value of $\rho$ is found by writing the equation for the state $\Psi_0(K, X)$. Discarding the dependency in $K$ to shorten the expressions, we have for the $X$-part of the action:

$$S_2(\Psi(X)) = \int \Psi^*(Z, \theta) \left( -\frac{\sigma_X^2}{2} \nabla_X^2 \right) \Psi(Z, \theta) + \frac{\kappa_0}{2\sigma_X^2} \Psi^* (X) (X - \langle X \rangle)^2 \Psi (X)$$

$$+ \int \left( \Psi^*(X) \Psi(X) \right) V_1 (|X - Y|) \left( \Psi^*(Y) \Psi(Y) \right)$$

$$+ \int \left( \Psi^*(X) \Psi(X) \right) V_2 (|X - Y|, |X - Z|, |Y - Z|) \left( \Psi^*(Y) \Psi(Y) \right)$$

$$\left( \Psi^*(Z) \Psi(Z) \right)$$

(145)

where:

$$V_1 (|X - Y|) = -\frac{\kappa_1}{2} \frac{K K' \exp(-\chi_1 |X - Y|)}{\langle K \rangle_X \langle K \rangle_Y}$$

$$V_2 (|X - Y|, |X - Z|, |Y - Z|) = \frac{\kappa_2}{3} \exp(-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|)$$

The expression for $\frac{\delta S_2(\Psi(X))}{\delta \Psi^*(X)}$:

$$-\frac{\sigma_X^2}{2} \nabla^2 \Psi(X) + \frac{\kappa_0}{2\sigma_X^2} (X - \langle X \rangle)^2 \Psi(X) + 2V_1 (|X - Y|) |\Psi(Y)|^2 \Psi(X)$$

$$+ 3V_2 (|X - Y|, |X - Z|, |Y - Z|) |\Psi(Y)|^2 |\Psi(Z)|^2 \Psi(X)$$

(146)

can be approximated by computing the potentials $V_1 (|X - Y|)$ and $V_2 (|X - Y|, |X - Z|, |Y - Z|)$ in the fundamental. As a consequence:

$$\int \kappa_1 \frac{K K' \exp(-\chi_1 |X - Y|)}{\langle K \rangle_X \langle K \rangle_Y} |\Psi(K', Y)|^2 dY dK'$$

is replaced by:

$$\int \kappa_1 \frac{K K' \exp(-\chi_1 |X - Y|)}{\langle K \rangle_X \langle K \rangle_Y} |\Psi_0(K', Y)|^2 dY dK'$$

$$\simeq \kappa_1 \exp(-\chi_1 |X|) \frac{K}{\langle K \rangle_X \rho^2}$$

$$\simeq \kappa_1 \exp(-\chi_1 |X|) \rho^2$$
where we assumed that, in first approximation, $|\Psi_0 (K', Y)|^2$ is centred around $Y = 0$. It will be justified by the formula for $\Psi_0$.

By the same token, the second part of the potential:

$$\kappa_2 \exp (-\chi_2 |X - Y| - \chi_2 |X - Z|) |\Psi_0 (K, Y)|^2 |\Psi_0 (K, Z)|^2$$

is replaced by:

$$\int \kappa_2 \exp (-\chi_2 |X - Y| - \chi_2 |X - Z|) |\Psi_0 (K, Y)|^2 |\Psi_0 (K, Z)|^2 dYdZ \\
\simeq \kappa_2 \exp (-2\chi_2 |X|) \rho^4$$

Moreover, in first approximation, for $\chi_1 << 1, \chi_2 << 1$, we can replace $X$ with its expectation $\langle X \rangle = 0$. This leads to the overall expression for $\frac{\delta S_2 (\Psi (X))}{\delta \Psi (X)}$:

$$-\frac{\sigma_X^2}{2} \nabla^2 \Psi (X) + \frac{\kappa_0}{2\sigma_X^2} X^2 \Psi (X) \\
-\kappa_1 \frac{K}{\langle K \rangle_X} \exp (-\chi_1 |X|) \Psi (K, X) \rho^2 + \kappa_2 \exp (-2\chi_2 |X|) \Psi (K, X) \rho^4 + \alpha \Psi (X)$$

Adding the $K$-part of the action to (147), the equation for the fundamental state at the lowest order in $\chi_1$ and $\chi_2$ is:

$$0 = \left(-\frac{\sigma_X^2}{2} \nabla^2 - \frac{\sigma_K^2}{2} \nabla^2 K - \frac{\sigma_\theta^2}{2} \nabla^2 \theta \right) \Psi_0 (K, X, \theta) \\
+ \left( \frac{1}{2\sigma^2} \left( \delta^2 + \frac{\hat{A}^2 A^2}{(A^2 U^2 + \hat{A}^2)^2} \right) \right) (K - \langle K \rangle_X)^2 \\
+ \frac{\kappa_0}{2\sigma_X^2} X^2 + \left( \rho^4 \kappa_2 - \rho^2 \kappa_1 + \frac{1}{2\sigma^2} + \alpha \right) \Psi_0 (K, X, \theta)$$

We will consider below the first-order corrections in $\chi_1$ and $\chi_2$ to this equation.

The Fourier transform of (148) in $\theta$ shows that the fundamental does not depend on $\theta$. We look for a fundamental eigenstate of the operator of the form $N \Psi_0 (X) \Psi_0^{(2)} (K)$. We assume $\theta < \Theta$ with $\Theta >> 1$, so that the integral over $\theta$ exists.

The normalisation factor $N$ ensures that $N \Psi_0 (X) \Psi_0^{(2)} (K)$ has norm $\rho^2$. Then $\Psi_0 (X)$ and $\Psi_0^{(2)} (K)$ can be written as the fundamental states of oscillators with eigenstates:

$$\frac{\kappa_0^2}{2} \text{ and } \frac{1}{2} \sqrt{\delta^2 + \frac{\hat{A}^2 A^2}{(A^2 U^2 + \hat{A}^2)^2}}$$
This leads to the relation:

\[ 0 = \alpha + \frac{1}{2\theta^2} + \frac{1}{2} \kappa_0^2 - \kappa_1 \rho^2 + \kappa_2 \rho^4 + \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}} \]

with:

\[ U = \left( 1 - D^{1+\gamma} \int \exp \left( -\frac{|X_3|}{1 + \gamma} \right) \exp \left( \frac{\gamma |X_2|}{1 + \gamma} \right) \exp \left( -(|X - X_2| + |X - X_3|) dX_2 dX_3 \right) \right. \]

\[ \simeq 1 - h \left( 1 + \frac{1}{2d} \right) \exp \left( -\frac{1}{2d} \right) \left( 1 - \exp \left( -\frac{1}{2d} \right) \right) \]

whose solution, with \( \frac{\partial^2}{\partial \rho^2} > 0 \), is:

\[ \rho^2 = \frac{\kappa_1 + \sqrt{\kappa_1^2 - 2\kappa_2 \left( 2\alpha + \frac{1}{\theta^2} + \sqrt{\kappa_0} + \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}} \right)}}{2\kappa_2} \]  

(149)

and the eigenstate:

\[ \Psi_0 (K, X) = N \exp \left( -\frac{\kappa_0^2 X^2}{2\sigma_X^2} \right) \exp \left( -\frac{\sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}} (K - \langle K \rangle_X)^2}{2\sigma^2} \right) \]

(150)

where \( N \) is the normalisation factor:

\[ N = \sqrt{\frac{\frac{1}{2} \kappa_0}{\sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}}}} \]

(151)

which completes the computations for \( P, \langle K \rangle_X \), the fundamental state and the potential.

We ultimately derive for later purpose the corrections to these results at the second order in \( \chi_1 \) and \( \chi_2 \). At this order, expanding the exponentials in \( \chi_1 \) and \( \chi_2 \) yields the potential part of the fundamental state’s equation (147):

\[ \frac{1}{2} \kappa_0 \frac{\kappa_0}{2\sigma_X^2} X^2 - \rho^2 \kappa_1 + \rho^4 \kappa_2 + \kappa_1 \chi_1 |X| \frac{K}{\langle K \rangle_X} \rho^2 - 2\kappa_2 \chi_2 |X| \rho^4 - \frac{\kappa_1}{2} \chi_1^2 X^2 \frac{K}{\langle K \rangle_X} \rho^2 + 2\kappa_2 \chi_2^2 X^2 \rho^4 \]
\[
\begin{align*}
&= \frac{1}{2} \kappa_0 \left( X - \text{sgn}(X) \frac{\sigma_X^2}{\kappa_0} \left( -\chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right)^2 \\
&\quad - \frac{\sigma_X^2}{2\kappa_0} \left( \chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 - 2 \chi_2 \kappa_2 \rho^4 \right)^2 \\
&\quad + \left( \frac{\kappa_1}{2} \chi_1^2 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) X^2 - \rho^2 \kappa_1 + \rho^4 \kappa_2 \\
&\quad \approx \left( \frac{\kappa_0}{2\sigma_X^2} + \left( -\frac{\kappa_1}{2} \chi_1^2 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right) \\
&\quad \left( X - \text{sgn}(X) \frac{\sigma_X^2}{\kappa_0} \left( -\chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right)^2 \\
&\quad - \frac{\sigma_X^2}{2\kappa_0} \left( \chi_1 \kappa_1 \rho^2 - 2 \chi_2 \kappa_2 \rho^4 \right)^2 - \rho^2 \kappa_1 + \rho^4 \kappa_2
\end{align*}
\]

so that the fundamental equation becomes:

\[
0 = \left( -\frac{\sigma_X^2}{2} \nabla_X^2 - \frac{\sigma_X^2}{2} \nabla_K^2 - \frac{\sigma_X^2}{2} \nabla_\theta^2 \right) \Psi_0(K, X, \theta) \\
+ \frac{\kappa_0}{2\sigma_X^2} \left( 1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\kappa_1}{2} \chi_1^2 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right) \\
\times \left( X - \text{sgn}(X) \frac{\sigma_X^2}{\kappa_0} \left( -\chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right)^2 \Psi_0(K, X, \theta) \\
+ \frac{1}{2\sigma^2} \left( \delta^2 + \frac{\tilde{A}^2 \tilde{A}^2}{(\tilde{A}^2 \tilde{U}^2 + \tilde{A}^2)^2} \right) (K - \langle K \rangle_X)^2 \Psi_0(K, X, \theta) \\
+ \left( \rho^4 \kappa_2 - \rho^2 \kappa_1 - \frac{\sigma_X^2}{2\kappa_0} \left( \chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 - 2 \chi_2 \kappa_2 \rho^4 \right)^2 + \frac{1}{2\sigma^2} + \alpha \right) \Psi_0(K, X, \theta)
\]

Equation (153) shows that the operators in \( K \) and \( X \) are intertwined. Yet, in first approximation in parameters \( \chi_1 \) and \( \chi_2 \), we can look for a fundamental state of the form \( N \Psi_0^{(1)}(X) \Psi_0^{(2)}(K) \). The expressions for \( \Psi_0^{(1)}(X) \) and \( \Psi_0^{(2)}(K) \) will depend perturbatively on \( K \) and \( X \), which justifies the notations.

The operators in \( X \) and \( K \) are harmonic oscillators with frequencies:

\[
\kappa_0^{\frac{1}{2}} \sqrt{1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\kappa_1}{2} \chi_1^2 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right)} \quad \text{and} \quad \sqrt{\delta^2 + \frac{\tilde{A}^2 \tilde{A}^2}{(\tilde{A}^2 \tilde{U}^2 + \tilde{A}^2)^2}}
\]
The fundamental states of these operators have thus the form:

\[
\Psi_0^{(1)}(X) = \exp \left( - \frac{\omega_X}{2\sigma_X^2} \left( X - \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{(K)_X} \frac{K}{(K)_X^2} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right) \right) \right) H(X)
\]

\[
+ \exp \left( - \frac{\omega_X}{2\sigma_X^2} \left( X + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{(K)_X} \frac{K}{(K)_X^2} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right) \right) \right) H(-X)
\]

\[
\equiv \tilde{\Psi}_0^{(1)}(X - \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{(K)_X} \frac{K}{(K)_X^2} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)) H(X)
\]

\[
+ \tilde{\Psi}_0^{(1)}(X + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{(K)_X} \frac{K}{(K)_X^2} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)) H(-X)
\] (154)

with:

\[
\omega_X = \kappa_0 \sqrt{1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{2} \frac{K}{(K)_X} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)}
\]

\[
\simeq \kappa_0 \sqrt{1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{2} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)}
\] (155)

and:

\[
\Psi_0^{(2)}(K) = \exp \left( - \sqrt{\frac{\delta^2 + \frac{\tilde{A}^2}{(\tilde{A}^2 U^2 + \tilde{A}^2)^2} \rho^4}{\sigma^2}} \right) (K - (K)_X)^2
\] (156)

so that the total fundamental state in both variables writes:

\[
\Psi_0(K, X) = \rho N \left[ \tilde{\Psi}_0^{(1)}(X - \delta X) H(X) + \tilde{\Psi}_0^{(1)}(X + \delta X) H(-X) \right]
\]

\[
\times \exp \left( - \sqrt{\frac{\delta^2 + \frac{\tilde{A}^2}{(\tilde{A}^2 U^2 + \tilde{A}^2)^2} \rho^4}{\sigma^2}} \right) (K - (K)_X)^2
\] (157)

where \( N \) is a normalisation factor:

\[
N \simeq \sqrt{\frac{\kappa_0}{2\pi \sigma_X \sigma}} \sqrt{1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\chi_1}{2} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)} \sqrt{\delta^2 + \frac{\tilde{A}^2}{(\tilde{A}^2 U^2 + \tilde{A}^2)^2}}
\] (158)
for $\delta X << 1$ and:

$$
\delta X = \frac{2\sigma_X^2}{\kappa_0} \left( -\chi_1 \kappa_1 \frac{K}{\langle K \rangle_X} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right)
$$

(159)

with $\rho$ satisfying the condition:

$$
0 \simeq \left( \rho^4 \kappa_2 - \rho^2 \kappa_1 - \frac{\sigma_X^2}{2\kappa_0} \left( \chi_1 \kappa_1 \rho^2 - 2 \chi_2 \kappa_2 \rho^4 \right)^2 + \frac{1}{2} \theta^2 + \alpha \right)
$$

$$
+ \frac{1}{2} \kappa_0 \frac{1}{\sqrt{1 + \frac{2\sigma_X^2}{\kappa_0} \left( -\frac{\kappa_1}{2} \chi_1 \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right)^2 + \frac{\sqrt{\delta^2 + \frac{\Delta^2 \sigma^2}{(\Delta^2 \rho^2 + \Delta^2)^2}}}{2}}}
$$

(160)

**Contribution of $\Delta \Psi$** We have computed the contribution of the fundamental $\Psi_0$ to $\langle K \rangle_X$. We must now find the correction to $\langle K \rangle_X$ and $P$ due to $\Delta \Psi$ in the decomposition $\Psi = \Psi_0 + \Delta \Psi$. To do so, we compute the corrections to the potential in (20) due to $\Delta \Psi$. We first consider the case of $\Delta \Psi$ orthogonal to $\Psi_0$ and ultimately add the contribution for $\Delta \Psi$ proportional to $\Psi_0$.

We can replace $\Psi_0 (K_3, P_3, X_3, \theta)$ in the previous computations by $\Psi_0 (K_3, P_3, X_3, \theta) + \Delta \Psi (K_3, P_3, X_3, \theta)$ as follows.

For any quantity $(K)^{-\frac{a}{1+\gamma}}$ and $f (X)$, the expectation of the vector $\left( (K)^{-\frac{a}{1+\gamma}} f (X) \right)$ is defined by:

$$
\left( \Psi_0^\dagger (K_3, P_3, X_3, \theta) + \Delta \Psi^\dagger (K_3, P_3, X_3, \theta) \right) \left( (K)^{-\frac{a}{1+\gamma}} f (X) \right) \langle \Psi_0 (K_3, P_3, X_3, \theta) + \Delta \Psi (K_3, P_3, X_3, \theta) \rangle
$$

and is approximatively equal to:

$$
\Psi_0^\dagger (K_3, P_3, X_3, \theta) \left( (K)^{-\frac{a}{1+\gamma}} f (X) \right) \langle \Psi_0 (K_3, P_3, X_3, \theta) \rangle
$$

$$
+ \Delta \Psi^\dagger (K_3, P_3, X_3, \theta) \left( (K)^{-\frac{a}{1+\gamma}} f (X) \right) \Delta \Psi (K_3, P_3, X_3, \theta)
$$

Given their form, $(K)^{-\frac{a}{1+\gamma}}$ and $f (X)$ can be considered to be close to their average $\langle K \rangle^{-\frac{a}{1+\gamma}}$ and $f (\langle X \rangle)$. As a consequence:

$$
\left\{ \Psi_0^\dagger \left( (K)^{-\frac{a}{1+\gamma}} f (X) \right), \Delta \Psi \right\} \simeq 0
$$

(161)

for a perturbation $\Delta \Psi (K, X)$ orthogonal to $\Psi_0 (K, X)$. 
The second-order development of the first potential term—the constraint—in (20) for the state \( \Psi_0 + \Delta \Psi \) is then:

\[
\int \Psi_0^\dagger (K, X) \left( f^{1+\gamma} (X) - \frac{\kappa \rho^4}{d^2} \int P_2 (K_2)^{\alpha} P_3 \right)
\times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) |\Psi_0 (K_2, X_2)|^2 |\Psi_0 (K_3, X_3)|^2 \right)^2 \Psi_0 (K, X)
\]

\[
+ \int \Delta \Psi^\dagger (K, X) \left( f^{1+\gamma} (X) - \frac{\kappa \rho^4}{d^2} \int P_2 (K_2)^{\alpha} P_3 \right)
\times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \Delta \Psi (K, X)
\]

\[
- 2 \frac{\kappa \rho^2}{d^2} \int \Psi_0^\dagger (K, X) \left( \int \Psi_0^\dagger (K_3, X_3) \Delta \Psi^\dagger (K_2, X_2) P_2 (K_2)^{\alpha} P_3 \right)
\times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \Delta \Psi (K, X)
\]

\[
\int \Psi_0^\dagger (K_2, X_2) \Delta \Psi^\dagger (K_3, X_3) P_2 (K_2)^{\alpha} P_3 \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \right) |\Psi_0 (K_2, X_2)|^2
\]

\[
|\Psi_0 (K_3, X_3)|^2 \right) \Psi_0 (K, X)
\]

(162)

where as before, we define \( f (X) = P K^{\frac{\alpha}{1+\gamma}} \). Integrals in (162) are taken over \( Z_2 \) and \( Z_3 \), the factor \( dZ_2 dZ_3 \) being implicit. To compute (162), we define several quantities. First, the average capital in state \( \Psi_0 \):

\[
\langle K \rangle_{X,0} = \int K |\Psi_0 (K, X)|^2 dK
\]

As a consequence, in state \( \Psi_0 \), the distribution of \( K \) is centred around \( \langle K \rangle_{X,0} \). Apart from the change of notation, \( \langle K \rangle_{X,0} \) has been computed previously in formula (140).

Similarly, in the state \( \Delta \Psi, K \) is centred around:

\[
\langle K \rangle_{X,1} = \int K |\Delta \Psi (K, X)|^2 dK
\]

Recall that \( \langle K \rangle_X \) stands for the average value of \( K \) in the full state \( \Psi_0 (K, X) + \Delta \Psi (K, X) \). Given the orthogonality relation (161), the relation \( \langle K \rangle_X = \langle K \rangle_{X,0} + \langle K \rangle_{X,1} \) holds.

For both states \( \Psi_0 (K, X) \) and \( \Delta \Psi (K, X) \), we set:

\[
G_0 (X_i, X_i) = \rho^2 \Psi_0^\dagger \left( \langle K \rangle_{X_i,0} \cdot X_i \right) \Psi_0 \left( \langle K \rangle_{X_i,0} \cdot X_i \right)
\]

\[
G (X_i, X_i) = G \left( \left( \langle K \rangle_{X_i,1} \cdot X_i \right), \left( \langle K \rangle_{X_i,1} \cdot X_i \right) \right)
\]
We also define:

\[
f_0 (X) = \int PK^{-\frac{i\alpha}{\gamma\tau}} |\Psi_0 (K, X)|^2 dK \simeq \langle P \rangle_{X,0} \left( K^{-\frac{i\alpha}{\gamma\tau}} \right)_{X,0} |\Psi_0 ((K)_{X,0} , X)|^2
\]

\[
f_0^{1+\gamma} (X) = \int P^{1+\gamma} K^{-\alpha} |\Psi_0 (K, X)|^2 dK \simeq \left( P^{1+\gamma} \right)_{X,0} \left( K^{-\alpha} \right)_{X,0} |\Psi_0 ((K)_{X,0} , X)|^2
\]

and:

\[
f_1 (X) = \int PK^{-\frac{i\alpha}{\gamma\tau}} |\Delta \Psi (K, X)|^2 dK \simeq \langle P \rangle_{X,1} \left( K^{-\frac{i\alpha}{\gamma\tau}} \right)_{X,1} |\Delta \Psi ((K)_{X,1} , X)|^2
\]

\[
f_1^{1+\gamma} (X) = \int P^{1+\gamma} K^{-\alpha} |\Delta \Psi (K, X)|^2 dK \simeq \left( P^{1+\gamma} \right)_{X,1} \left( K^{-\alpha} \right)_{X,1} |\Delta \Psi ((K)_{X,1} , X)|^2
\]

Remark that \( f_0 (X) \) was computed in (141).

Given these definitions, we can compute (162) in first approximation by replacing \( K \) and \( P \) by their expectations in the states \( \Psi_0 \) and \( \Delta \Psi \), and using the Green functions. The first term is the action for the sole state \( \Psi_0 (K, X) \) can be rewritten:

\[
\int \Psi_0^\dagger (K, X) \left( (f (X))^{1+\gamma} - \frac{\kappa \rho^4}{d^2} \int G_0 (X_2, X_2) G_0 (X_3, X_3) \left( K \right)^{\alpha \dagger}_{X_2,0} \left( K \right)^{-\alpha}_{X_3,0} f_0 (X_3) f_0 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) dX_2 dX_3 \right)^2 \Psi_0 (K, X)
\]

The last term in (162) rewrites in first approximation:

\[
\frac{\kappa \rho^4}{d^2} \int G_0 (X_2, X_2) G_0 (X_3, X_3) \left( K \right)^{\alpha \dagger}_{X_2,0} \left( K \right)^{-\alpha}_{X_3,0} f_0 (X_3) f_0 (X_2) \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) dX_2 dX_3 \times \left( (f_0 (X))^{1+\gamma} - \frac{\kappa \rho^4}{d^2} \int G_0 (X_2, X_2) G_0 (X_3, X_3) \left( K \right)^{\alpha \dagger}_{X_2,0} \left( K \right)^{-\alpha}_{X_3,0} f_0 (X_3) f_0 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) dX_2 dX_3 \right) G_0 (X, X)
\]

\[
\frac{\kappa \rho^4}{d^2} \int G_0 (X_2, X_2) G_0 (X_3, X_3) \left( K \right)^{\alpha \dagger}_{X_2,0} \left( K \right)^{-\alpha}_{X_3,0} f_0 (X_3) f_0 (X_2) \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) dX_2 dX_3 \times \left( (f_0 (X))^{1+\gamma} - \frac{\kappa \rho^4}{d^2} \int G_0 (X_2, X_2) G_0 (X_3, X_3) \left( K \right)^{\alpha \dagger}_{X_2,0} \left( K \right)^{-\alpha}_{X_3,0} f_0 (X_3) f_0 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) dX_2 dX_3 \right) G_0 (X, X)
\]

(163)
We can approximate the integrals by their estimations, as we did in appendix 3. In first approximation, the exponential exp\left(-|X_2 - X_3|/d\right) is replaced by 2d\delta (X_2 - X_3) and we consider a uniform distribution for X, so that (163) is equal to:

\[-\frac{\kappa \rho^4}{2d} \int \left( (K)_{X_2,0}^{a \gamma} (K)_{X_2}^{\alpha / (1 + \gamma)} + (K)_{X_2}^{a \gamma} (K)_{X_2,0}^{\alpha / (1 + \gamma)} \right) f_1 (X_2) f_0 (X_2) dX_2 \]

\[\times \int \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) \left( (f_0 (X))^{1+\gamma} \right) \times \left( (f_0 (X))^2 \right) \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) \right) dX_2 \]

\[-\frac{\kappa \rho^4}{2d} \int \left( (K)_{X_2,0}^{a (\gamma - 1)} (K)_{X_2}^{\alpha / (1 + \gamma)} \right) f_1 (X) f_0 (X) dX_2 \]

This expression is approximately equal to 0, since the equality:

\[(f_0 (X))^{1+\gamma} - \frac{\kappa \rho^4}{2d} \int (K)_{X_2,0}^{a (\gamma - 1)} (f_0 (X))^2 \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) dX_2 \approx 0\]

holds for X (see (138)). Thus, at the second order, the term (163) becomes:

\[\int \Delta \Psi^\dagger \left( (f (X))^{1+\gamma} - \frac{\kappa \rho^4}{2d} \int G_0^2 (X_2, X_2) (K)_{X_2,0}^{a (\gamma - 1)} (f_0 (X))^2 \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) dX_2 \right) \Delta \Psi \]

(165)

with, as before:

\[f^{1+\gamma} (X) = P^{1+\gamma} K^\alpha\]

As a consequence, (162) writes:

\[\int \Psi_0^\dagger (K, X) \left( (f (X))^{1+\gamma} - \frac{\kappa \rho^4}{2d} \int G_0^2 (X_2, X_2) (K)_{X_2,0}^{a (\gamma - 1)} (f_0 (X))^2 \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) dX_2 \right)^2 \Psi_0 (K, X) \]

\[+ \int \Delta \Psi^\dagger \left( (f (X))^{1+\gamma} - \frac{\kappa \rho^4}{2d} \int G_0^2 (X_2, X_2) (K)_{X_2,0}^{a (\gamma - 1)} (f_0 (X))^2 \exp \left( - \left( \frac{|X - X_2|}{d} \right) \right) dX_2 \right) \Delta \Psi \]
where (161) has been used. The defining equation for \( f (X) \) is thus:

\[
(f (X))^{1+\gamma} = \frac{\kappa \rho^4}{2d} \int G_0^2 (X_2, X_2) \frac{x(x-1)}{x_0^2} \left( f_0 (X_2) \right)^2 \exp \left( -\frac{\kappa \rho^4}{2d} X \right) dX
\]

which is (138), and where the assumption of a uniform distribution for \( X \) has been used.

The second potential term in (20) is written:

\[
\int \Psi^\dagger (K, X) \delta K - A P K^\alpha \left( 1 - \frac{\kappa \rho^4}{d^2} \int P_2 \exp \left( -\frac{|X-X_2|+|X_1-X_3|}{P_2^\gamma} \right) \frac{|\Psi_0 (K_2, X_2)|^2 |\Psi_0 (K_3, X_3)|^2}{|\Psi_0 (K, X)|^2} \right) \bigg) \Psi_0 (K, X)
\]

Here again, the integration factor \( dZ_2 dZ_3 \) is omitted. At the second order in \( (\Delta \Psi, \Delta \Psi^\dagger) \), (168) is given by:

\[
\int \Psi_0^\dagger (K, X) \Delta \Psi (K, X) \left( \delta K - A P K^\alpha \left( 1 - \frac{\kappa \rho^4}{d^2} \int P_2 \exp \left( -\frac{|X-X_2|+|X_1-X_3|}{P_2^\gamma} \right) \frac{|\Psi_0 (K_2, X_2)|^2 |\Psi_0 (K_3, X_3)|^2}{|\Psi_0 (K, X)|^2} \right) \right) \bigg) \Psi_0 (K, X)
\]

\[
\times \exp \left( -\frac{|X-X_2|}{d} \right) dX_2 \bigg) \Psi_0 (K, X)
\]
The first term computes the potential in the state $\Psi_0$. Under our assumptions, the last term of (169) rewrites:

\[
\int \left( \int G_0 (X_3, X_3) G (X_2, X_2) \left( \langle K \rangle_{X_3,0} \right)^{\alpha_{\gamma \gamma}} \left( \langle K \rangle_{X_2,1} \right)^{-\alpha_{\gamma \gamma}} f_0 (X_3) (f_1 (X_2))^{-\gamma} \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \right) dX_2 dX_3 \\
+ \int G_0 (X_2, X_2) G (X_3, X_3) \left( \langle K \rangle_{X_3,1} \right)^{\alpha_{\gamma \gamma}} \left( \langle K \rangle_{X_2,0} \right)^{-\alpha_{\gamma \gamma}} f_1 (X_3) f_0 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \right) dX_2 dX_3 \\
\times \left( \delta \langle K \rangle_{X,0} - AP \langle K \rangle_{X,0}^{\alpha, \gamma} \right)
\]

and is null in first approximation, since:

\[
0 = \delta \langle K \rangle_{X,0} - AP \langle K \rangle_{X,0}^{\alpha, \gamma} \\
\times \left( 1 - \frac{\kappa \rho^4}{4d^2} \int \langle K \rangle_{X_2,0}^{\alpha_{\gamma \gamma}} \langle K \rangle_{X_3,0}^{-\alpha_{\gamma \gamma}} f_0 (X_3) f_1 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \right) dX_2 dX_3
\]

holds for all $X$ [see (102), applied in the state $\Psi_0$]. Thus, the second-order term of the potential in (168) becomes:

\[
\int \Delta \Psi \times (K, X) \left( \delta K - AP K^\alpha \left( 1 - \frac{\kappa \rho^4}{4d^2} \int \langle K \rangle_{X_2,0}^{\alpha_{\gamma \gamma}} \langle K \rangle_{X_3,0}^{-\alpha_{\gamma \gamma}} f_0 (X_3) f_1 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \right) \right)^2 dX_2 dX_3
\]

Moreover, as a consequence of (170), the potential (168) writes:

\[
\int (\Psi_0 + \Delta \Psi) \times (K, X) \left( \delta K - AP K^\alpha \left( 1 - \frac{\kappa \rho^4}{4d^2} \int \langle K \rangle_{X_2,0}^{\alpha_{\gamma \gamma}} \langle K \rangle_{X_3,0}^{-\alpha_{\gamma \gamma}} f_0 (X_3) f_1 (X_2) \times \exp \left( -\frac{|X - X_2| + |X_2 - X_3|}{d} \right) \right) \right)^2 dX_2 dX_3
\]

\[
\simeq \int (\Psi_0 + \Delta \Psi) \times (K, X) \left( \delta K - AP K^\alpha \left( 1 - \frac{\kappa \rho^4}{2d} \int \langle K \rangle_{X_2,0} \ (f_1 (X_2))^{1 - \gamma} \right) \right)
\]
\[
\times \exp \left( - \frac{|X - X_2|}{d} \right) dX_2 \right)^2 (\Psi_0 + \Delta \Psi)
\]

\[
= \int (\Psi_0 + \Delta \Psi) \delta^2 (K - \langle K \rangle_X)^2 (\Psi_0 + \Delta \Psi)
\]  

(172)

Equations (172) and (167) mean that \( f(X) \simeq f_0(X) \) and \( \langle K \rangle_X \simeq \langle K \rangle_{X,0} \). These two quantities have been computed previously in this appendix. The corrections due to \( \Delta \Psi(K, X) \) can be neglected in first approximation. We will use this quadratic approximation for the potential to compute the Green function in phase 2.

Ultimately, recall that the potentials (166) and (172) have been computed for \( \Delta \Psi \) orthogonal to \( \Psi_0 \). To compute the second-order potential, we have to introduce a fluctuation proportional to \( \Psi_0 \), namely \( \Delta \Psi = \frac{(\delta \rho)^2}{2\rho^2} \Psi_0 \) corresponding to a variation \( \rho^2 \rightarrow \rho^2 + (\delta \rho)^2 \). The associated variation of \( S(\Psi) \) is \( \frac{1}{2} \frac{\partial S(\Psi_0)}{\partial \rho^2} (\delta \rho)^2 \). A sufficient first approximation can be found using (95) in appendix 2. This equation states that:

\[
S(\Psi) \simeq \left( \frac{1}{2} \kappa_0^2 \rho^2 - \kappa_1 \rho^4 \right) \rho^2
\]

for \( \Psi \) proportional to \( \Psi_0 \). Thus:

\[
\frac{\partial S(\Psi_0)}{\partial \rho^2} (\delta \rho)^2 = \left( 2\kappa_2 \rho^2 - \kappa_1 \right) \rho^2 (\delta \rho)^2
\]  

(173)

since \( \frac{1}{2} \kappa_0^2 - \kappa_1 \rho^2 + \kappa_2 \rho^4 = 0 \) for \( \Psi_0 \). Writing:

\[
\Delta \Psi = \Delta' \Psi + \frac{(\delta \rho)^2}{2\rho^2} \Psi_0
\]  

(174)

where \( \Delta' \Psi \) is orthogonal to \( \Psi_0 \), we can also write (173) as:

\[
\frac{1}{2} \frac{\partial S(\Psi_0)}{\partial \rho^2} (\delta \rho)^2 = \left( 2\kappa_2 \rho^2 - \kappa_1 \right) \rho^2 \int \Delta \Psi(K, X, \theta) \Psi_0(K, X, \theta) \left| \Psi_0(K, X, \theta) \right|^2
\]  

(175)

Formulas (172) and (175) will be used to compute the quadratic action for phase 2.

**Corrections of order** \( \frac{A^2}{\bar{A}^2} \** to the potential**

We can include, as we did in the first phase, the first-order corrections in \( \frac{A^2}{\bar{A}^2} \). Since the terms appearing in the second-order expansion are (165) and (171), the potential becomes:

\[
\int \delta^2 \left( 1 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2} \right) \Delta \Psi^+ (K - \langle K \rangle_X)^2 \Delta \Psi
\]  

(176)
Appendix 5

In this section, we compute the effective quadratic action and the Green functions in both phases. Recall that, deriving the average capital level $\langle K \rangle_X$ and the price level $P$, we obtained the quadratic approximation of the potential term for the capital. In phase 1, we had (135):

$$\delta^2 \left( 1 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2} \right) (K - \langle K \rangle_X)^2$$  \hspace{1cm} (177)

In phase 2, (176) holds:

$$\int \delta^2 \left( 1 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2} \right) \Delta \Psi^\dagger (\delta K - \langle K \rangle_{X,0})^2 \Delta \Psi$$  \hspace{1cm} (178)

where $U$ stands for $\langle U \rangle$ defined in (128). For each phase, we first compute the quadratic effective action and then derive the Green function.

Case $\rho = 0$

Effective quadratic action For $\rho = 0$, using (177) and discarding the $X$ contribution, the action is:

$$\int \Psi^\dagger (K, P, X, \theta) \left( -\frac{\sigma^2}{2} \nabla^2 K - \frac{\vartheta^2}{2} \nabla^2 \theta + \frac{1}{2\sigma^2} \left( \delta^2 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2} \right) \right) \Psi (K, P, X, \theta)$$

$$\left( K - \langle K \rangle \right)^2 + \frac{1}{2\vartheta^2} + \alpha \right) \Psi (K, P, X, \theta)$$  \hspace{1cm} (179)

whose Green function is given by:

$$G_0 (K, K', \theta, \theta', t)$$

$$= \sqrt{\frac{\omega}{2\pi \sigma^2 \sinh (\omega t)}} \exp \left( \left( \frac{\omega}{2\sigma^2 \sinh (\omega t)} \right) \left( (K^2 + (K')^2) \cosh (\omega t) - 2KK' \right) \right)$$

$$\times \sqrt{\frac{1}{2\pi \vartheta^2 t}} \exp \left( -\frac{(\theta' - \theta)^2}{2\vartheta^2 t} \right)$$

with:

$$\omega^2 = \delta^2 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}$$
To include the $X$-dependent part of the action (in the sequel, the dependency of $\Psi$ in $\theta$ is understood):

$$\int \Psi^\dagger (K, X) \left( -\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{\kappa_0}{2\sigma_X^2} X^2 \right) \Psi (K, X)$$

$$- \int \frac{\kappa_1}{2} (\Psi^\dagger (K, X) \Psi (K, X)) \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle \langle K \rangle} \langle \Psi^\dagger (K', Y) \Psi (K', Y) \rangle$$

$$+ \int \frac{\kappa_2}{3} (\Psi^\dagger (K, X) \Psi (K, X)) \exp (-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) \langle \Psi (K', Y) \rangle^2 \langle \Psi (K'', Z) \rangle^2$$

(180)

and to account for its contribution to the Green function, we replace the interaction potential by its average in variables $Y$ and $Z$. As a consequence, in (180):

$$- \frac{\kappa_1}{2} |\Psi (K, X)|^2 \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle \langle K \rangle} |\Psi (K', Y)|^2$$

and:

$$\frac{\kappa_2}{3} |\Psi (K, X)|^2 \exp (-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) \langle \Psi (K', Y) \rangle^2 \langle \Psi (K'', Z) \rangle^2$$

are replaced by:

$$- \frac{\kappa_1}{2} \left\{ \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle \langle K \rangle} \langle \Psi (K', Y) \rangle^2 \right\} |\Psi (K, X)|^2$$

(181)

and:

$$\frac{\kappa_2}{3} |\Psi (K, X)|^2$$

$$\left\{ \exp (-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) \langle \Psi (K', Y) \rangle^2 \langle \Psi (K'', Z) \rangle^2 \right\},$$

(182)

respectively.

To compute (181), we use the fact that, at the lowest order in perturbation:

$$\left\{ \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle \langle K \rangle} \langle \Psi (K', Y) \rangle^2 \right\}$$

$$\simeq \int \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle \langle K \rangle} G (\langle K', Y \rangle, \langle K', Y \rangle) dK' dY$$

$$\simeq \int \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle \langle K \rangle} G_Y (K', K') G (Y, Y) dK' dY$$
The hypotheses are the same as in appendix 3, that is, $K'$ is spread around $\langle K \rangle_Y$ and the $X$-part of the Green function is approximately uniformly distributed on the interval $[-1, 1]$. The previous expression thus becomes:

$$
\left\langle \int \frac{K K' \exp (-\chi_1 |X - Y|)}{\langle K \rangle_X \langle K \rangle_Y} |\Psi (K', Y)|^2 dK' dY \right\rangle \simeq \frac{K}{2 \langle K \rangle_X} \int \exp (-\chi_1 |X - Y|) dY
$$

This last integral is computed for $\chi_1 << 1$ as:

$$
\int_{-1}^{1} \exp (- (\chi_1) |X - Y|) dY
$$

$$
= \int_{-1}^{-X} \exp (-\chi_1 |u|) du = \int_{-1}^{0} \frac{\exp (\chi_1 u)}{\sqrt{\alpha}} du + \int_{0}^{1-X} \frac{\exp (-\chi_1 u)}{\sqrt{\alpha}} du
$$

$$
= \left(2 - \exp (-\chi_1 (1 + X)) - \exp (-\chi_1 (1 - X))\right) \frac{\chi_1}{\chi_1}
$$

$$
\simeq 2 \left(1 - \exp (-\chi_1)\right) - \exp (-\chi_1) \chi_1 X^2
$$

$$
\simeq 2 - \chi_1 + \frac{1}{3} \chi_1^2 + \left(\chi_1^2 - \chi_1\right) X^2
$$

and (183) becomes:

$$
\left\langle \int \frac{K K' \exp (-\chi_1 |X - Y|)}{\langle K \rangle_X \langle K \rangle_Y} |\Psi (K', Y)|^2 dK' dY \right\rangle \simeq \frac{K}{2 \langle K \rangle_X} \left(\frac{2 (1 - \exp (-\chi_1))}{\chi_1} - \exp (-\chi_1) \chi_1 X^2\right)
$$

By the same token, we compute the contribution (182) by writing:

$$
\int \kappa_2 \left\langle \exp (-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) |\Psi (K', Y)|^2 |\Psi (K'', Z)|^2 \right\rangle dK' dK'' dY dZ
$$

$$
\simeq \int \kappa_2 \exp (-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) G (Y, Y) G (Z, Z) dY dZ
$$

$$
\simeq \int \frac{\kappa_2}{4} \exp (-\chi_2 (|X - Y| + |X - Z| + |Y - Z|)) dY dZ
$$

where we used that:

$$
G \left(\left(K', Y\right), \left(K', Y\right)\right) = G_Y \left(K', K'\right) G \left(Y, Y\right)
$$

and $\int G_Y \left(K', K'\right) dK'$ has been normalised to 1.
As a consequence, replacing \( \exp(-\chi_2 |Y - Z|) \) by \( \frac{2(1-\exp(-\chi_2))\delta(Y-Z)}{\chi_2} \) yields:

\[
\int \kappa_2 \left\{ \exp(-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) \left| \Psi(K', Y) \right|^2 \left| \Psi(K'', Z) \right|^2 \right\} dK'dK''dYdZ
\]
\[
\simeq \int \kappa_2 \frac{1 - \exp(-\chi_2)}{\chi_2} \exp(-2\chi_2 (|X - Y|)) dY
\]
\[
\simeq \kappa_2 \left( 1 - \exp(-2\chi_2) \right) \left( \frac{1 - \exp(-2\chi_2)}{\chi_2} - 2 \exp(-2\chi_2) \chi_2 X^2 \right)
\]
\[
\simeq \kappa_2 \left( 1 - \frac{3}{2} \chi_2 + \frac{4}{3} \kappa_2 \chi_2^2 \right) + \kappa_2 \left( \frac{5}{2} \chi_2^2 - \chi_2 \right) X^2 \tag{185}
\]

Under the approximations (183) and (185), the \( X \)-dependent part of the action becomes:

\[
\int \left( \Psi^\dagger(X) \left( -\frac{\sigma_X^2}{2} \nabla^2_X \right) \Psi(X) + \frac{\omega_X^2}{2\sigma_X^2} \Psi^\dagger(X) X^2 \Psi(X) + \alpha_X \Psi^\dagger(X) \Psi(X) \right) \tag{186}
\]

where:

\[
\omega_X^2 = \omega_0 + \left( \frac{\kappa_1}{2} \frac{K}{\langle K \rangle_X} \left( \chi_1 - \chi_1^2 \right) - \frac{\kappa_2}{2} \left( 2\chi_2 - 5\chi_2^2 \right) \right) \sigma_X^2 \tag{187}
\]

and:

\[
\alpha_X = \alpha + \frac{1}{2\theta^2} - \frac{\kappa_1}{2} \frac{K}{\langle K \rangle_X} \left( 1 - \frac{\chi_1}{2} + \frac{1}{6} \chi_1^2 \right) + \frac{\kappa_2}{3} \left( 1 - \frac{3}{2} \chi_2 + \frac{4}{3} \chi_2^2 \right) \tag{188}
\]

As a consequence, gathering (179) and (186), the overall second-order action becomes:

\[
\int \Psi^\dagger(K, X, \theta) \left( -\frac{\sigma^2 K^2}{2} - \frac{\theta^2 \nabla^2_\theta}{2} - \frac{\sigma_X^2 \nabla^2_X}{2} \right.
\]
\[
+ \frac{1}{2} \left( \delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2} \right) (K - \langle K \rangle)^2 + \frac{\omega_X^2 X^2}{2\sigma_X^2} + \alpha_X \right) \Psi(K, X, \theta) \tag{189}
\]

This quadratic action will be used to compute the Green functions. To do so, we must include an exponential factor induced by the change of variable (82).

**Exponential factor** The change of variable (82) modifies the Green functions and includes a factor:

\[
\exp \left( -\int \left( \frac{(\delta K - A PK^a + U_1)}{\sigma^2} \right) \right) \Psi(K, P, X, \theta)
\]

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with:

\[
U_1 = \frac{\kappa}{d^2} A P K^\alpha \int \frac{P_3 \exp \left( - \left( |X - X_2| + |X - X_3| \right) \right)}{P_2^\gamma} |\Psi (K_2, P_2, X_2, \theta)|^2 \\
\times |\Psi (K_3, P_3, X_3, \theta)|^2 \, dZ_2 \, dZ_3
\]

Given that:

\[
\int \left( A P K^\alpha - U_1 \right) \approx \int A P K^\alpha \left( 1 - \frac{\kappa}{d^2} \int \frac{P_3 \exp \left( - \left( |X_1 - X_2| + |X_1 - X_3| \right) \right)}{P_2^\gamma} |\Psi (K_2, P_2, X_2, \theta)|^2 \\
\times |\Psi (K_3, P_3, X_3, \theta)|^2 \right) \right) = \int \kappa A P K^\alpha U
\]

the exponential factor rewrites:

\[
\exp \left( - \int \frac{\left( \delta K - A P K^\alpha + V_1 \right)}{\sigma^2} \right) = \exp \left( - \int \frac{\left( \delta K - A P K^\alpha U \right)}{\sigma^2} \right) = \exp \left( - \int \frac{\left( \delta K - A (K) \frac{\partial}{\partial P} f(X) U \right)}{\sigma^2} \right)
\]

We have seen in appendix 3 that:

\[
\delta \left( K \right)_X - A \left( (K) \frac{\partial}{\partial P} f(X) U \right) = 0
\]

so that we can rewrite the term in the exponential as:

\[
\frac{\delta}{\sigma^2} \int \left( K - \left( \frac{K}{(K)_X} \frac{\partial}{\partial P} (K)_X \right) \right) \simeq \frac{\delta}{\sigma^2} \int (K - \langle K \rangle)_X \simeq \frac{\delta (K - \langle K \rangle)_X^2}{2\sigma^2} \quad (190)
\]

for \( \gamma << 1 \).

**Computation of the Green function** The action (189) is now quadratic, but the variables \( K \) and \( X \) are entangled via \( \omega \) and \( \omega_X \). To find the Green function between \( (K, X) \) and \( (K', X') \), we simplify the problem by replacing \( (K, X) \) in \( \omega \) and \( \omega_X \) by

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their average trajectory values. In first approximation, it amounts to replacing any expression $f(K, X)$ by its average $f(K, X) + f(K', X')$. We then set (189):

$$
\langle K \rangle = \frac{\langle K \rangle_X + \langle K \rangle_{X'}}{2}
$$

(191)

$$
\tilde{\omega}_X^2 = \kappa_0 + \left( \frac{\kappa_1}{4} \left( \frac{K}{\langle K \rangle_X} + \frac{K'}{\langle K \rangle_{X'}} \right) \right) \left( \chi_1 - \chi_1' \right) - \frac{\kappa_2}{3} \left( 2\chi_2 - 5\chi_2' \right) \sigma_X^2
$$

(192)

$$
\tilde{\alpha}_X = \alpha + \frac{1}{2\theta^2} - \frac{\kappa_1}{4} \left( \frac{K}{\langle K \rangle_X} + \frac{K'}{\langle K \rangle_{X'}} \right) \left( 1 - \chi_1 + \frac{1}{6} \chi_1' \right) + \frac{\kappa_2}{3} \left( 1 - 3\chi_2 + \frac{4}{3} \chi_2' \right)
$$

(193)

$$
\tilde{U} = 1 - h \exp\left( -\frac{|X| + |X'|}{2d} \right) \left( 1 - \cosh \frac{X}{d} + \cosh \frac{X'}{d} \right)
$$

(194)

The value of $h$ is given by (143). If we replace $|X|$ and $|X'|$ by their average value $\frac{1}{2}$, we have:

$$
\tilde{U} = U = 1 - h \exp\left( -\frac{1}{d} \right) \left( 1 - \frac{\cosh \frac{1}{d}}{\exp \left( \frac{1}{d} \right)} \right)
$$

$$
\tilde{\omega} \simeq \omega = \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 \tilde{U}^2 + \tilde{A}^2)^2}}
$$

(195)

Using these average values, and including the exponential factor (190), the Green function of the action (189) can be computed. It is the Laplace transform of the following temporal transition function, with parameter $\tilde{\alpha}_X \simeq \alpha$ and small coupling parameters:

$$
G(K, K', P, P', X, X', \theta, \theta', t)
$$

$$
= \exp\left( -\frac{\delta (K - \langle K \rangle_X)^2}{2\sigma^2} \right)_{(K, X)}
$$

$$
\times \frac{\tilde{\omega}/2\pi \sigma^2}{\sinh (\tilde{\omega}t)} \exp\left( -\frac{\tilde{\omega} \left( (K - \langle K \rangle)^2 + (K' - \langle K \rangle)^2 \right) \cosh (\tilde{\omega}t) - 2(K - \langle K \rangle)(K' - \langle K \rangle)}{2\sigma^2 \sinh (\tilde{\omega}t)} \right)
$$

$$
\times \frac{\tilde{\omega}_X/2\pi \sigma^2_X}{\sinh (\tilde{\omega}_X t)} \exp\left( -\frac{\tilde{\omega}_X \left( (X^2 + (X')^2 \cosh (\tilde{\omega}_X t) - 2XX' \right)}{2\sigma^2_X \sinh (\tilde{\omega}_X t)} \right)
$$

$$
\times \frac{1}{2\pi \sigma^2 t} \exp\left( -\frac{(\theta' - \theta - t)^2}{2\sigma^2 t} \right) \times \delta \left( P - D \exp\left( -\frac{|X|}{\langle K \rangle + \frac{1}{\alpha}} \right) \right) \delta \left( P' - D \exp\left( -\frac{|X'|}{\langle K' \rangle + \frac{1}{\alpha'}} \right) \right)
$$
For \( \vartheta^2 << 1 \), the variable \( t \) can be replaced by \( \theta' - \theta \), for \( \theta' > \theta \). Actually, due to the term:

\[
\exp \left( -\frac{(\theta' - \theta - t)^2}{2\vartheta^2} \right)
\]

the Green function is non-null for values of \( \theta \) and \( \theta' \) such that \( \theta' - \theta - t = 0 \). Since \( t > 0 \), this implies that the replacement is only valid for \( \theta' > \theta \); otherwise, the Green function is equal to 0. As a consequence, we can remove the time dependency in the Green function to obtain:

\[
G \left( K, K', P, P', X, X', \theta, \theta' \right) = \exp \left( -\frac{\delta (K - \langle K \rangle)^2}{2\sigma^2} \right) \left( K', X' \right)
\]

\[
\times \frac{\bar{\omega}}{2\pi \sigma^2 \sinh (\bar{\omega} (\theta' - \theta))} \exp \left( -\frac{\bar{\omega} \left( (K - \langle K \rangle)^2 + (K' - \langle K \rangle)^2 \right)}{2\sigma^2 \sinh (\bar{\omega} (\theta' - \theta))} \cosh (\bar{\omega} (\theta' - \theta) - 2(K - \langle K \rangle)(K' - \langle K \rangle)) \right)
\]

\[
\times \frac{\bar{\omega}_X}{2\pi \sigma^2 X \sinh (\bar{\omega}_X (\theta' - \theta))} \exp \left( -\frac{\bar{\omega}_X \left( (X^2 + \langle X \rangle^2 \right)}{2\sigma^2 X \sinh (\bar{\omega}_X (\theta' - \theta))} \cosh (\bar{\omega}_X (\theta' - \theta) - 2XX') \right)
\]

\[
\times \delta \left( P - D \exp \left( -\frac{|X|}{1+\vartheta} \right) \right) \delta \left( P' - D \exp \left( -\frac{|X'|}{1+\vartheta} \right) \right) H (\theta' - \theta)
\]

where \( H (\theta' - \theta) \) is the Heaviside function. Replacing \( \bar{\omega} \) by \( \omega \) (see (195)) yields the formula of the text.

Note that in (196), for \( \theta' - \theta \to 0 \), the dominant part becomes \( \frac{t^2}{2\vartheta^2} \). This means that \( t \) can replaced in average by \( \vartheta^2 \), which yields the description in terms of harmonic oscillators used in appendix 3.

**Case \( \rho \neq 0 \)**

**Effective quadratic action** For \( \rho \neq 0 \), the \( K \)-part of the second-order expansion of the action around \( \Psi_0 \) is obtained by using (178):

\[
\int \Delta' \Psi^\dagger (K, X, \theta) \left( \frac{\sigma^2}{2} \nabla^2_K - \frac{\vartheta^2}{2} \nabla^2_\theta - \frac{\vartheta^2}{2} \nabla^2_X \right) \Delta' \Psi (K, X, \theta)
\]

\[
+ \int \Delta' \Psi^\dagger (K, X, \theta) \left( \frac{1}{2\sigma^2} \left( \delta^2 + \frac{\tilde{A}^2A^2}{(A^2U^2 + \tilde{A}^2)^2} \right) (K - \langle K \rangle_X)^2 + \alpha + \frac{1}{2\vartheta^2} \right) \Delta' \Psi (K, X, \theta)
\]

where \( \Delta' \Psi \) is a variation orthogonal to \( \Psi_0 \), and \( \Delta \Psi = \Delta' \Psi + \frac{(\delta \rho)^2}{2\rho^2} \Psi_0 \) (see 174).
As explained in appendix 4, the averages are computed in state $\Psi_0$. We deal with the $X$-part of the action as in the previous case.

\[
\int \Psi^\dagger (K, X) \left( -\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{\kappa_0}{2\sigma_X^2} X^2 \right) \Psi (K, X) - \frac{k_1}{2} \int |\Psi (K, X)|^2 \frac{KK' \exp (-\chi_1 |X - Y|)}{\langle K \rangle^2} |\Psi (K', Y)|^2 \\
+ \int \frac{k_2}{3} \exp (-\chi_2 |X - Y| - \chi_2 |X - Z| - \chi_2 |Y - Z|) |\Psi (K, X)|^2 \\
|\Psi (K', Y)|^2 |\Psi (K'', Z)|^2
\]

(198)

We obtain the second-order expansion:

\[
\int \Delta' \Psi^\dagger (K, X, \theta) \frac{\delta^2 S_2 (\Psi (K, X, \theta))}{\delta \Psi (K, X, \theta) \delta \Psi^\dagger (K', X', \theta')} \Delta' \Psi (K', X', \theta')
\]

by using that in average $\langle \Delta \Psi (K, X, \theta) \Delta \Psi^\dagger (K', X', \theta') \rangle \simeq 0$ for $\theta \neq \theta'$. We are thus left with:

\[
\int \Delta' \Psi^\dagger (K, X, \theta) \frac{\delta^2 S_2 (\Psi (K, X, \theta))}{\delta \Psi (K, X, \theta) \delta \Psi^\dagger (K', X', \theta')} \Delta' \Psi (K, X, \theta)
\]

where the expression $\frac{\delta S_2 (\Psi(K,X,\theta))}{\delta \Psi^\dagger (K,X,\theta)}$ has already been computed in (146) to find the fundamental state. The second-order derivative, in the notations of (146), is then:

\[
\frac{\delta^2 S_2 (\Psi (X))}{\delta \Psi (X) \delta \Psi^\dagger (X)} = -\frac{\sigma_X^2}{2} \nabla^2 + \frac{\kappa_0}{2\sigma_X^2} (X - \langle X \rangle)^2 + 2V_1 (|X - Y|) |\Psi (Y)|^2 \\
+ 3V_2 (|X - Y|, |X - Z|, |Y - Z|) |\Psi (Y)|^2 |\Psi (Z)|^2
\]

(199)

This is the operator involved in the fundamental state equation. Using (152), we find the second-order term for the $X$-part of the action:

\[
\int \Delta' \Psi^\dagger (K, X, \theta) \left( \frac{\kappa_0}{2\sigma_X^2} \left( 1 + \frac{2\sigma_X^2}{\kappa_0} \left( \frac{-\kappa_1}{2} \chi_1 K_{(K)_{x,0}}^2 \rho^2 + 2\chi_2 \rho^4 \right) \right) \right) \\
\times \left( X - \frac{\text{sgn} (X) \sigma_X^2}{\kappa_0} \left( -\chi_1 \frac{K_{(K)_{x,0}}}{K} \rho^2 + 2\chi_2 \rho^4 \right) \right)^2 \Delta' \Psi (K, X, \theta) \\
+ \int \Delta' \Psi^\dagger (K, X, \theta) \left( \rho^4 \chi_2 - \rho^2 \chi_1 - \frac{\sigma_X^2}{2\kappa_0} \left( \chi_1 \frac{K_{(K)_{x,0}}}{K} \rho^2 - 2\chi_2 \rho^4 \right) \right)^2 \Delta' \Psi (K, X, \theta)
\]

(200)
The norm $\rho^2$ of $\Psi_0$ was derived at zeroth order in $\chi_1$ and $\chi_2$ in “Appendix 4”, Eq. (149):

$$\rho^2 = \frac{\kappa_1 + \sqrt{\kappa_1^2 - 2\kappa_2 \left(2\alpha + \frac{1}{\vartheta^2} + \sqrt{\kappa_0} + \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}}\right)}}{2\kappa_2}$$

(201)

for $\tilde{A}^2 >> A^2$. The complete action is then obtained by gathering (200), (197) and (175):

$$\int \Delta' \Psi^\dagger(K, X, \theta) \left(-\frac{\sigma^2}{2} \nabla^2 K - \frac{\vartheta^2}{2} \nabla^2 \theta - \frac{\sigma^2}{2} \nabla^2 X + \frac{\omega^2}{2\sigma^2} (K - \langle K \rangle_{X,0})^2 + \alpha_X\right) \\
\times \Delta' \Psi(K, X, \theta) + \int \Delta' \Psi^\dagger(K, X, \theta) \frac{\omega_X}{2\sigma_X} \left(X - \frac{\text{sgn}(X) \sigma_X^2}{\kappa_0} \left(-\chi_1 \kappa_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)\right)^2 \\
\times \Delta' \Psi(K, X, \theta) + \left(2\kappa_2 \rho^2 - \kappa_1\right) \rho^2 \left| \int \Delta \Psi(K, X, \theta) \Psi_0(K, X, \theta) \right|^2$$

(202)

where:

$$\omega = \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}}$$

$$\omega_X = \sqrt{\kappa_0 \left(1 + \frac{2\sigma_X^2}{\kappa_0} \left(-\frac{\kappa_1}{2} \chi_1^2 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)\right)}$$

$$\alpha_X = \alpha + \frac{1}{2\vartheta^2} + \left(-\kappa_1 \rho^2 + \kappa_2 \rho^4 - \frac{\sigma_X^2}{2\kappa_0} \left(\chi_1 \kappa_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 - 2\chi_2 \kappa_2 \rho^4 \right)\right)^2$$

(203)

Since (160) implies that:

$$0 = \alpha + \frac{1}{2\vartheta^2} + \frac{1}{2} \kappa_0 \frac{1}{\sqrt{1 + \frac{2\sigma_X^2}{\kappa_0} \left(-\frac{\kappa_1}{2} \chi_1^2 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2\chi_2 \kappa_2 \rho^4 \right)}}$$

$$-\kappa_1 \rho^2 + \kappa_2 \rho^4 - \frac{\sigma_X^2}{2\kappa_0} \left(\chi_1 \kappa_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 - 2\chi_2 \kappa_2 \rho^4 \right) + \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}} \frac{2}{2}$$
we have:

\[
\alpha \chi = -\frac{1}{2} \frac{1}{\kappa_0} \sqrt{1 + \frac{2 \sigma_X^2}{\kappa_0}} \left( -\frac{\kappa_1}{2} \chi_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2 \kappa_2 \chi_2 \rho^4 \right) \sqrt{\frac{\delta^2 + \frac{\bar{A}^2 A^2}{(A^2 U^2 + \bar{A}^2)^2}}{2}}
\]

and the second-order action (202) rewrites:

\[
\begin{align*}
\int & \Delta^{'\Psi^1} (K, X, \theta) \left( -\frac{\sigma_X^2}{2} \nabla^2 - \frac{\sigma^2}{2} \nabla^2 - \frac{\sigma_X^2}{2} \nabla^2 + \frac{\sigma^2}{2 \sigma^2} (K - \langle K \rangle_{X,0}) \right) \Delta^{'\Psi} (K, X, \theta) \\
& + \int \Delta^{'\Psi^1} (K, X, \theta) \frac{\sigma^2}{2 \sigma^2} \left( K - \frac{\sigma_X^2}{\kappa_0} (\chi_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2 \chi_2 \kappa_2 \rho^4) \right) \Delta^{'\Psi} (K, X, \theta) \\
& - \int \Delta^{'\Psi^1} (K, X, \theta) \left( \frac{1}{2} \frac{1}{\kappa_0} \sqrt{1 + \frac{2 \sigma_X^2}{\kappa_0}} \left( -\frac{\kappa_1}{2} \chi_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right) \Delta^{'\Psi} (K, X, \theta) \\
& + (2 \chi_2 \rho^2 - \kappa_1) \rho^2 \left| \int \Delta \Psi (K, X, \theta) \Psi_0 (K, X, \theta) \right|^2
\end{align*}
\]

(205)

**Computation of the Green function** The Green function is computed by including the influence of the background field. We have shown that the fundamental level of the X-part of the action has the form:

\[
\Psi_0^{(1)} (X) = N_1 \exp \left( -\frac{1}{2} \frac{1}{\kappa_0} \sqrt{1 + \frac{2 \sigma_X^2}{\kappa_0}} \left( -\frac{\kappa_1}{2} \chi_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right) \frac{(X - \delta X)^2}{2 \sigma_X^2} H (X)
\]

\[
+ N_1 \exp \left( -\frac{1}{2} \frac{1}{\kappa_0} \sqrt{1 + \frac{2 \sigma_X^2}{\kappa_0}} \left( -\frac{\kappa_1}{2} \chi_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right) \right) \frac{(X + \delta X)^2}{2 \sigma_X^2} H (-X)
\]

\[
\simeq N_1 \exp \left( -\frac{1}{2} \frac{1}{\kappa_0} (X - \delta X)^2 \right) H (X) + N_1 \exp \left( -\frac{1}{2} \frac{1}{\kappa_0} (X + \delta X)^2 \right) H (-X)
\]

\[
\equiv \Psi_0^{(1)} (X - \delta X) H (X) + \Psi_0^{(1)} (X + \delta X) H (-X)
\]

where:

\[
\delta X = \frac{\sigma_X^2}{\kappa_0} \left( -\chi_1 \frac{K}{\langle K \rangle_{X,0}} \rho^2 + 2 \chi_2 \kappa_2 \rho^4 \right)
\]
For a given value of $K$, the Green function for the $X$-part of the action is obtained by its expansion as a function of all the eigenstates $\tilde{\Psi}_n$ of the system:

$$
G_K (X, X') = \sum_n \left( \tilde{\Psi}_n (X - \delta X) H (X) + \tilde{\Psi}_n (X + \delta X) H (-X) \right) \\
\times \left( \tilde{\Psi}_n (X' - \delta X) H (X') + \tilde{\Psi}_n (X' + \delta X) H (-X') \right)
$$

$$
= \tilde{G} (X - \delta X, X' - \delta X) H (X) H (X')
+ \tilde{G} (X + \delta X, X' + \delta X) H (-X) H (-X')
+ \tilde{G} (X - \delta X, X' + \delta X) H (X) H (-X')
+ \tilde{G} (X + \delta X, X' - \delta X) H (-X) H (X')
$$

where we have defined:

$$
\tilde{G} (X, X', \theta, \theta')
\quad = \sqrt{\frac{\bar{\omega}_X}{2\pi \sigma_X^2 \sinh (\bar{\omega}_X (\theta' - \theta))}}
\times \exp \left( \left( -\frac{\bar{\omega}_X}{2\sigma_X^2 \sinh (\bar{\omega}_X (\theta' - \theta))} \right) \left( (X^2 + (X')^2) \cosh (\bar{\omega}_X (\theta' - \theta)) - 2XX' \right) \right)
$$

with:

$$
\bar{\omega}_X = \sqrt{\frac{\kappa_0}{1 + \frac{\sigma_X^2}{\kappa_0} \left( -\frac{\kappa_1}{2} \chi_1^2 \left( \frac{K}{\langle K \rangle_{X,0}} + \frac{K'}{\langle K \rangle_{X',0}} \right) \rho^2 + 4\kappa_2 \chi_2 \rho^4 \right)}}
$$

As in the first phase, we have replaced $\omega_X$ defined in (203) by $\bar{\omega}_X$, its average over the trajectory.

The exponential term associated to the change of variable is computed as in the first phase. However, it has to be evaluated in the state $\Psi_0 (K, X, \theta) + \Psi (K, X, \theta)$. “Appendix 4” showed how, in first approximation, this amounts to computing it in the
state \( \Psi_0 (K, X, \theta) \). We find again an exponential factor:

\[
\exp \left( - \frac{\delta (K - \langle K \rangle_X, X, \theta)}{2\sigma^2} \right)_{(K, X)}
\]

with \( \langle K \rangle_X \) computed in the state \( \Psi_0 (K, X, \theta) \), as in “Appendix 4”.

Ultimately, we can associate a Green function to (205), as in the first phase. It is the Laplace transform of a temporal Green function with parameter \( \alpha_X \) given in (204):

\[
G (K, K', P, P', X, X', \theta, \theta', t) = \exp \left( - \frac{\delta (K - \langle K \rangle_X, X, \theta)}{2\sigma^2} \right)_{(K, X)} \times \frac{\omega}{2\pi \sigma^2} \exp \left( - \frac{\omega ((K - \langle K \rangle)^2 + (K' - \langle K \rangle)^2) \cosh (\omega t) - 2(K - \langle K \rangle)(K' - \langle K \rangle)}{2\sigma^2 \sinh (\omega t)} \right) \times G_{K+K'} (X, X', \theta, \theta') \times \sqrt{\frac{1}{2\pi \sigma^2 t}} \exp \left( - \frac{(\theta' - \theta)^2}{2\sigma^2 t} + \frac{\theta' - \theta}{\sigma^2} \right) \times \delta \left( \frac{P - D \exp \left( - \frac{| X |}{1+\gamma} \right)}{\frac{K'}{(K)_1}} \right) \times \delta \left( \frac{P' - D \exp \left( - \frac{| X' |}{1+\gamma} \right)}{\frac{K'}{(K)_1}} \right)
\]

where, as in (195), we have used the average of \( U \) to compute \( \omega \):

\[
\omega = \sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}}
\]

\[
U = 1 - h \exp \left( - \frac{1}{d} \right) \left( 1 - \frac{\cosh \frac{1}{d}}{\exp \left( \frac{1}{d} \right)} \right)
\]

Even though \( \alpha_X < 0 \) [see (204)], the Laplace transform is well defined, since \( -\alpha_X \) is the lower bound of the terms in the exponential.\(^9\) The value of \( \omega \) is given by (194) with \( h \) defined in (143).

As in phase 1, this Green function is centred around \( t = \theta' - \theta \), so that we can replace \( t = \theta' - \theta \) in the Green function. This leads to:

\[^9\] the parameter \( -\alpha_X \) is the lowest eigenvalue associated to the evolution operator whose \( G \) is the Green function.
\[
G \left( K, K', P, P', X, X', \theta, \theta' \right) = \exp \left( - \frac{\delta \left( K - \langle K \rangle \right)^2}{2\sigma^2} \right)_{(K,X)} \left( K', X' \right) \times \sqrt{\frac{\omega}{2\pi\sigma^2}} \sinh \left( \frac{\omega}{\sigma^2} \right) \exp \left( - \frac{\omega \left( \left( K - \langle K \rangle \right)^2 + (K' - \langle K \rangle)^2 \right)}{2\sigma^2} \sinh \left( \frac{\omega}{\sigma^2} \right) - 2 \left( K - \langle K \rangle \right) (K' - \langle K \rangle) \right)_{(K,X)} \left( K', X' \right) \times \delta \left( P - \frac{D \exp \left( - \frac{|X|}{r+\gamma} \right)}{\frac{\sigma}{K'}} \right)_{(P', X', \theta, \theta')}, \delta \left( P' - \frac{D \exp \left( - \frac{|X|}{r+\gamma} \right)}{\frac{\sigma}{K'}} \right)_{(P', X', \theta, \theta')} H (\theta' - \theta)
\]

As in the case \( \rho = 0 \), we can replace \( \bar{\omega} \) by \( \omega \) to obtain the formula in the text.

**Appendix 6**

This appendix studies the emergence of a \( K \)-dependent barrier potential as described in section 5.4.2 of the text. To do so, we consider a general model:

\[
S (\Psi) = \int \Psi^\dagger (X) \left( -\sigma^2 \nabla^2_X + V \right) \Psi (X) + \int \frac{1}{2} \Psi^\dagger (X) \Psi^\dagger (Y) W (X, Y) \Psi (X) \Psi (Y)
\]

that encompasses the model studied in this paper. The field \( \Psi (X) \) depends on an arbitrary number of variables \( X \) belonging to some configuration space, and \( W (X, Y) = W (Y, X) \). We have chosen a fourth-order interaction term, but a more general choice, such as a sum of powers, would not change the result. We assume that there is a non-trivial minimum to the action \( S (\Psi) \), so that the equation:

\[
\left( -\sigma^2 \nabla^2_X + V \right) \Psi (X) + \left( \int \Psi^\dagger (Y) W (X, Y) \Psi (Y) \right) \Psi (X) = 0
\]

has a solution \( \rho \Psi_0 (X) \neq 0 \), and \( \Psi_0 \) of norm equal to 1. We show that the non-trivial vacuum implies separating the system into two sub-systems defined on two half-space of the configuration space.

Given that the field \( \Psi_0 \) minimises the action, the second-order variation of \( S (\Psi) \) is:
\[
\int \Delta \Psi^\dagger (X) \left( -\sigma_X^2 \nabla_X^2 + V \right) \Delta \Psi (X) \, dX + \Delta \Psi^\dagger (X) \\
\left( \int \rho^2 \Psi_0^\dagger (Y) W (X, Y) \Psi_0 (Y) \, dY \right) \Delta \Psi (X) \, dX \\
+ 2 \oint \Re \left( \Delta \Psi^\dagger (X) \left( \rho^2 \Psi_0^\dagger (Y) W (X, Y) \Psi_0 (X) \right) \Delta \Psi (Y) \right) \, dXdY \\
= \int \Delta \Psi^\dagger (X) \left( \left( -\sigma_X^2 \nabla_X^2 + V (X) \right) \Delta \Psi (X) + \Delta \Psi^\dagger (X) V_0 (X) \Delta \Psi (X) \right) \, dX \\
+ \int \Delta \Psi^\dagger (X) \left( W_0 (X, Y) + W_0^\dagger (X, Y) \right) \Delta \Psi (Y) \, dXdY \\
\tag{206}
\]

where:

\[
W_0 (Y, X) = \rho^2 \Psi_0^\dagger (Y) W (X, Y) \Psi_0 (X) \\
W_0^\dagger (X, Y) = W_0 (Y, X)
\]

and:

\[
V_0 (X) = \int \rho^2 \Psi_0^\dagger (Y) W (X, Y) \Psi_0 (Y)
\]

Now, assume that \( \Psi_0^{(1)} (X) \) is peaked around some \( X_0 \), which is the case for equation (60) in the text for the \( K \)-part of \( \Psi_0 \) (see (43)). Then:

\[
V_0 (X) \simeq \int \rho^2 \Psi_0^\dagger (Y) W (X, X_0) \Psi_0 (Y) = W (X, X_0) \rho^2 \\
W_0 (Y, X) \simeq \rho^2 \int \Psi_0^\dagger (X_0) W (X_0, X_0) \Psi_0 (X) \delta (X - X_0) \delta (Y - X_0)
\]

and the following contributions of the second-order variation become:

\[
\Delta \Psi^\dagger (X) V_0 (X) \Delta \Psi (X) \rightarrow \rho^2 \Delta \Psi^\dagger (X) W (X, X_0) \Delta \Psi (X) \\
\Delta \Psi^\dagger (X) \left( W_0 (X, Y) + W_0^\dagger (X, Y) \right) \Delta \Psi (Y) \rightarrow 2 \rho^2 |\Psi_0 (X)|^2 W (X_0, X_0) \Delta \Psi^\dagger (X_0) \Delta \Psi (X_0)
\]

For \( \rho^2 W (X_0, X_0) \gg 1 \), the contributions of the statistical weight due to the fields \( \Delta \Psi \) such that \( \Delta \Psi (X_0) \neq 0 \) are suppressed. Then the integrals over \( \Delta \Psi \) can be limited to contributions such that \( \Delta \Psi (X_0) = 0 \).

This means that \( \Delta \Psi \) can be decomposed into two parts:

\[
\Delta \Psi (X) = \Delta \Psi_+ (X) + \Delta \Psi_- (X)
\]

where \( \Delta \Psi_+ (X) \) and \( \Delta \Psi_- (X) \) are independent and defined on two half space \( X_\pm \), respectively. They satisfy:

\[
\Delta \Psi_\pm (X_\mp) = 0
\]
As a consequence, the action at the second order for $\Delta \Psi_\pm$ becomes:

$$
\Delta \Psi_\pm^\dagger (X) \left( -\sigma X^2 \nabla_X^2 + V (X) \right) \Delta \Psi_\pm (X) + \rho^2 \Delta \Psi_\pm^\dagger (X) W (X, X_0) \Delta \Psi_\pm (X)
$$

This action also models two independent fields with constraint $\Delta \Psi_\pm (X_\mp) = 0$. They are defined on the whole space, but constrained by a potential wall $H_\pm (X)$, this wall being defined on the space $X_\mp$. The second-order action (206) becomes:

$$
\Delta \Psi_\pm^\dagger (X) \left( -\sigma X^2 \nabla_X^2 + V (X) + H_\pm (X) \right) \Delta \Psi_\pm (X) + \rho^2 \Delta \Psi_\pm^\dagger (X) W (X, X_0) \Delta \Psi_\pm (X)
$$

This models two sets of different agents, evolving on $X_\pm$ and subject to a wall potential. Applied to our case, the results are the following. Recall that we found the fundamental state (157):

$$
\Psi_0 (K, X) = \rho N \left[ \tilde{\Psi}_0^{(1)} (X - \delta X) H (X) + \tilde{\Psi}_0^{(1)} (X + \delta X) H (-X) \right] \times \exp \left( -\frac{\sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}} (K - \langle K \rangle_X)^2}{2\sigma} \right)
$$

where $N$ and $\rho$ are given by (158) and (149) and (159), respectively. If we assume, as we did in “Appendix 3”, that:

$$
\sqrt{\delta^2 + \frac{\tilde{A}^2 A^2}{(A^2 U^2 + \tilde{A}^2)^2}} \sigma > \kappa_0 \frac{1}{2}
$$

i.e. that the $X$-variable is more spread than $K$, then $\Psi_0 (K, X)$ is peaked on the hypersurface $K = \langle K \rangle_X$. As a consequence, the space $(K, X)$ is divided into two subspaces, $S_+$, defined by $K > \langle K \rangle_X$, and $S_-$, defined by $K < \langle K \rangle_X$. These half-spaces correspond to two systems that are independent in first approximation. An agent starting in $S_+$ ($S_-$, respectively) will remain in $S_+$ ($S_-$, respectively).

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