ON MONOCHROMATIC ARITHMETIC PROGRESSIONS IN BINARY WORDS ASSOCIATED WITH BLOCK-COUNTING FUNCTIONS

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Abstract. Let $e_v(n)$ denote the number of occurrences of a fixed block $v$ of digits in the binary expansion of $n \in \mathbb{N}$. In this paper we study monochromatic arithmetic progressions in the class of binary words $(e_v(n) \mod 2)_{n \geq 0}$, which includes the famous Thue–Morse word $t$ and Rudin–Shapiro word $r$. We prove that the length of a monochromatic arithmetic progression of difference $d \geq 3$ starting at 0 in $r$ is at most $(d + 3)/2$, with equality for infinitely many $d$. Moreover, we compute the maximal length of a monochromatic arithmetic progression in $r$ of difference $2^k - 1$ and $2^k + 1$. For a general block $v$ we provide an upper bound on the length of a monochromatic arithmetic progression of any difference $d$. We also prove other miscellaneous results and offer a number of related problems and conjectures.

1. Introduction

Let $t = (t_n)_{n \geq 0}$ be the Thue-Morse sequence, given by $t_n = s_2(n) \mod 2$, where $s_2$ denotes the sum of binary digits. There is a vast literature devoted to the properties of the Thue-Morse sequence. For a survey concerning various appearances of $t$ in seemingly unrelated problems see Allouche and Shallit [3]. A comprehensive approach in the setting of automatic sequences is given in the monograph by the same authors [4].

The sequence $t$ can be indentified with an infinite binary word or a 2-coloring of the set $\mathbb{N}$ of nonnegative integers, namely a partition into two sets of indices $n$ such that $t_n = 0$ and $t_n = 1$, respectively. The following celebrated result of van der Waerden implies that $t$ contains arbitrarily long finite monochromatic arithmetic progressions, that is, progressions $n, n+d, \ldots, n+ld$ such that $t_n = t_{n+d} = \cdots = t_{n+ld}$.

**Theorem 1.1** (van der Waerden). A finite coloring of $\mathbb{N}$ contains arbitrarily long monochromatic arithmetic progressions.

Van der Waerden’s result does not specify where to look for a long monochromatic arithmetic progression or what its difference $d$ should be. Morgenbesser, Shallit, and Stoll [6] answered these questions for the Thue-Morse word $t$ and also gave an upper bound on the maximal length of a monochromatic arithmetic progression starting at $n = 0$. Here and in the sequel $A_t(n, d)$ denotes the maximal length of a monochromatic arithmetic progression in $t$ of difference $d$ starting at $n$, that is, $A_t(n, d) = 1 + \sup \{l \geq 1 : t_{n+ld} = t_n \}$.

The main result of [6] is the following. We remark that in the original statement the notation $f(d)$ was used instead of $A_t(0, d)$.

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**Key words and phrases.** monochromatic arithmetic progressions, Rudin–Shapiro sequence, Thue–Morse sequence, block-counting functions.

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Furthermore, for any

Moreover, of all monochromatic progressions of difference alizations to other bases was obtained by Parshina \[7\]. She showed that the lengths

Theorem 1.2 BARTOSZ SOBOLEWSKI

monochromatic arithmetic progressions.

Theorem 1.3

In the case of the Thue–Morse sequence, Parshina’s result is the following.

More precisely, define

and

starting at \[7, \text{Lemma } 1\]. In contrast to Theorem 1.2, the assumption about the progression alization, obtained by replacing the sum of binary digits \[d\]

functions. More precisely, let \(d\) the computation of

Theorem 1.4

class of Thue–Morse-like sequences).

Staynova \[1\] together with the following theorem (and other results concerning a

exposition regarding the Rudin–Shapiro sequence.

In particular, we have \(e_1(n) = s_2(n)\) so \((e_1(n) \mod 2)_{n \geq 0}\) is precisely the Thue–Morse sequence. We are going to focus most of our attention on a variant of the famous Rudin–Shapiro sequence \(r = (r_n)_{n \geq 0} = (e_{11}(n) \mod 2)_{n \geq 0}\). This sequence is usually defined over \((-1, 1)\) by \(r_n = (-1)^{e_{11}(n)}\). We refer the reader to \[4\] for an exposition regarding the Rudin–Shapiro sequence.

Similarly as before, we can treat the sequence \((c_v(n) \mod 2)_{n \geq 0}\) as an infinite word over \(\{0, 1\}\) or a 2-coloring of nonnegative integers. For \(n, d \in \mathbb{N}\) such that
$d \geq 1$, we define

$$A_v(n,d) = 1 + \sup\{ l \geq 1 : e_v(n + ld) \equiv e_v(n) \pmod{2} \},$$

$$A_v(d) = \sup_{n \in \mathbb{N}} A_v(n,d).$$

In other words, $A_v(n,d)$ is the maximal (possibly infinite) length of a monochromatic arithmetic progression in said coloring of difference $d$ and starting at $n$, while $A_v(d)$ concerns all progressions of difference $d$, regardless of the initial term. In the case of the Rudin-Shapiro sequence, namely $v = 11$, we adopt the special notation $A_r(n,d)$ and $A_r(d)$.

Our first main result is an analogue of Theorem 1.2 for the Rudin–Shapiro sequence.

**Theorem 1.5.** For all integers $d \geq 3$ we have

$$A_r(0,d) \leq \frac{1}{2}(d + 3).$$

More precisely:

(i) $A_r(0,d) = \left( \frac{d}{2} \right) + 2$ if and only if $d = 2^k - 1$ for some odd $k \geq 3$ or $d = 39$;

(ii) $A_r(0,d) = \left( \frac{d}{2} \right) + 1$ if and only if $d = 2^k + 1$ for some $k \geq 2$;

(iii) $A_r(0,d) < \frac{d}{2}$ otherwise.

The next two results give an explicit expression for $A_r(d)$ for the special cases $d = 2^k + 1$ and $d = 2^k - 1$.

**Theorem 1.6.** Let $d = 2^k + 1$ for some integer $k \geq 1$. Then

$$A_r(d) = \begin{cases} 
5 & \text{if } k = 1, \\
6 & \text{if } k = 2, \\
9 & \text{if } k = 3, \\
2^{k-1} + 2 & \text{if } k \geq 4.
\end{cases}$$

**Theorem 1.7.** Let $d = 2^k - 1$ for some integer $k \geq 1$. Then

$$A_r(d) = \begin{cases} 
4 & \text{if } k = 1, \\
5 & \text{if } k = 2, \\
9 & \text{if } k = 3, \\
10 & \text{if } k = 4, \\
2^{k-1} + 3 & \text{if } k \geq 5 \text{ is odd}, \\
2^{k-1} + 1 & \text{if } k \geq 6 \text{ is even}.
\end{cases}$$

These results, while quite precise, do not settle whether or not the values $A_v(n,d)$ and $A_v(d)$ are finite for any $n, d$. The following result shows that this is in fact the case for any block $v$ of length at least 2, and provides an upper bound in the spirit of Theorem 1.3. In its statement $\nu_2(d)$ denotes the 2-adic valuation of $d$, namely $\nu_2(d) = \sup\{ n \in \mathbb{N} : 2^n \mid d \}$.

**Theorem 1.8.** For any block $v$ of binary digits of length $|v| \geq 2$ and all positive integers $k$ we have

$$\max_{1 \leq d < 2^k} A_v(d) \leq 2^{k + |v| - \nu_2(d) - 1}.$$

We now give an outline of the contents of this paper. In Section 2 we give some general notation and terminology used throughout the paper. Section 3 is devoted to the proof of the special cases (i) and (ii) of Theorem 1.5 achieved through direct calculation. Section 4 contains the proof of the general case (iii), which is inspired by the approach used in [6] and relies on detailed binary arithmetic.
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and let Sub(\(w\))
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We now focus on the binary case \(\Sigma = \{0,1\}\). For any \(w = w_{k}w_{k-1} \cdots w_{1}w_{0} \in \{0,1\}^{+} \) the notation \([w]_{2}\) refers to the integer represented by \(w\) in base 2, namely
\[ [w]_{2} = 2^{k}w_{k} + 2^{k-1} + \cdots + 2w_{1} + w_{0}. \]
We also put \([\epsilon]_{2} = 0\). Note that adding leading zeros to \(w\) does not affect \([w]_{2}\).

Conversely, for \(n \in \mathbb{N}\) we let \((n)_{2}\) denote the canonical binary representation of \(n\), that is, the word \(w \in \{0,1\}^{*}\) without leading zeros such that \([w]_{2} = n\). The length of the binary expansion of \(n\) is denoted by \(\ell(n)\), and can be equivalently written as
\[ \ell(n) = |(n)_{2}| = 1 + \lfloor \log_{2} n \rfloor. \]
In accordance with our earlier convention, when \(v\) is a word starting with exactly \(i \geq 0\) zeros and containing at least one 1, we have
\[ e_{v}(n) = |0^{i}(n)_{2}|_{v}. \]
We also consider the negation \(\overline{w}\) of a word \(w \in \{0,1\}^{*}\), where the 1’s are swapped with 0’s.

2. Notation and terminology

We introduce some further notation and terminology related to operations on words, mainly following [4, Chapter 1]. Let \(\Sigma\) be a nonempty finite set. Then \(\Sigma^{*}\) denotes the set of all finite words (strings) consisting of the letters from \(\Sigma\), including the empty word \(\epsilon\). We also write \(\Sigma^{+} = \Sigma^{*} \setminus \{\epsilon\}\). Two words \(w, v \in \Sigma^{*}\) can be concatenated into a single word \(vw\). For \(n \in \mathbb{N}\) the notation \(w^{n}\) means the concatenation of \(n\) copies of \(w\), where \(w^{0} = \epsilon\). The length \(|w|\) of a word is the number of its letters, and for each \(l \in \mathbb{N}\) we let \(\Sigma^{l}\) denote the set of words of length \(l\). The reversal of a word \(w = w_{1}w_{2} \cdots w_{l}\) is given by \(w^{\dagger} = w_{l}w_{l-1} \cdots w_{1}\). We say that \(v\) is a subword of \(w\) if there exist words \(x,y\) such that \(w = xvy\) and let Sub(\(w\)) denote the set of subwords of \(w\). Similarly, Sub_{0}(\(w\)) is the set of length \(l\) subwords of \(w\). If \(w = v\epsilon\), then \(v\) is said to be a prefix of \(w\). If additionally \(y \neq \epsilon\), then the prefix \(v\) is called proper. Similarly, if \(w = vx\), then \(v\) is said to be a suffix of \(w\) (proper, if \(x \neq \epsilon\)). The sets of prefixes and suffixes of \(w\) are denoted by Pref(\(w\)) and Suf(\(w\)), respectively. The number of (possibly overlapping) occurrences of a nonempty word \(v\) as a subword in \(w\) is denoted by \(|w|_{v}\). The notions of subwords and prefixes naturally extend to the case where \(w\) is a right-infinite word.

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\[ [w]_{2} = 2^{k}w_{k} + 2^{k-1} + \cdots + 2w_{1} + w_{0}. \]

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\[ \ell(n) = |(n)_{2}| = 1 + \lfloor \log_{2} n \rfloor. \]
In accordance with our earlier convention, when \(v\) is a word starting with exactly \(i \geq 0\) zeros and containing at least one 1, we have
\[ e_{v}(n) = |0^{i}(n)_{2}|_{v}. \]
We also consider the negation \(\overline{w}\) of a word \(w \in \{0,1\}^{*}\), where the 1’s are swapped with 0’s.

3. Proof of Theorem 1.5 – cases (i) and (ii)

To begin, we deal with the special cases (i) and (ii) in the statement of Theorem 1.5. In the case (i) we prove a more precise result, which will be useful later when calculating \(A_{r}(d)\).

**Proposition 3.1.** If \(d = 2^{k} - 1\) for some integer \(k \geq 3\), then
\[ r_{d} = r_{2d} = \cdots = r_{2^{k-1}d}, \]
and this common value is equal to \((k - 1) \mod 2\). Moreover, if \(k\) is odd, then \(r_{(2^k-1)\ell + 1})d = k \mod 2\), and consequently \(A_r(0,d) = (d + 3)/2\).

**Proof.** Since \(e_{11}(2m) = e_{11}(m)\) for all \(m \in \mathbb{N}\), it is sufficient to show that \(e_{11}(nd) \equiv k - 1 \mod 2\) for all odd \(n < 2^{k-1}\). Fix any such \(n\) and let \(w \in \{0,1\}^{k-2}\) be such that \([w0]\) \(= n - 1\) (\(w\) may have leading zeros). Then

\[
nd = 2^{k+1}(n - 1) + (2^{k+1} - 1) - (n - 1) = [0w01\overline{w}1]\).
\]

Therefore,

\[
e_{11}(nd) = |0w0|_11 + |1\overline{w}1|_11 = |0w0|_11 + |0w0|_00.
\]

Now, for any \(u \in \{0,1\}^*\) we have

\[
|u|_11 + |u|_00 + |u|_01 + |u|_10 = |u| - 1,
\]

since both sides count the number of blocks of adjacent digits of length 2. At the same time, observe that

\[
|0w0|_11 = |0w0|_10.
\]

Combining these equalities, we obtain

\[
e_{11}(nd) = |0w0| - 1 - |0w0|_01 - |0w0|_10 = k - 1 - 2|0w0|_01 \equiv k - 1 \mod 2,
\]

as desired.

If \(k\) is odd then the values \(r_{nd}\) for \(n = 0, 1, \ldots, 2^{k-1}\) are all equal to 0. However, for \(n = (d + 3)/2 = 2^{k-1} + 1\) we have \(nd = 2^{2k-1} + 2^{k-1} - 1 = [10^k1^{k-1}]_2\), so \(e_{11}(nd) = k - 2\) and the second part of the assertion follows.

**Proposition 3.2.** If \(d = 2^k + 1\) for some integer \(k \geq 2\), then \(A_r(0,d) = (d + 1)/2\).

**Proof.** First, we prove that \(r_{nd} = 0\) for all \(n < (d + 1)/2 = 2^{k-1} + 1\). Again, it is sufficient to consider \(n odd. Letting \(w = (n)_2\), we have \(nd = [w0^m0_1]\), where \(m = k - \ell(n) - 1 > 0\). This means that

\[
e_{11}(nd) = 2e_{11}(n) \equiv 0 \mod 2.
\]

Now, for \(n = 2^{k-1} + 1\) we have \(nd = [10^k-210^{k-2}]_2\), which gives \(e_{11}(nd) = 1\). The result follows.

\[
4. \text{ Proof of Theorem 1.5} \quad \text{case (iii)}
\]

In the general case (iii) of Theorem 1.5 the relation \(r_{2m} = r_m\) allows us to restrict our attention to odd differences \(d\). We can also assume that there is a 0 in the binary expansion of \(d\), as Proposition 3.1 already covers the case \(d_2 = 1^k\). Without knowing \(d\) exactly, it seems futile to try to calculate the precise value of \(A_r(0,d)\). It is however sufficient to show that there exists a positive integer \(n < d/2\) such that \(r_{nd} = 1\).

Our approach is a variation of the one used in [6] to prove Theorem 1.2. The analysis is split into multiple main cases, depending on the number of leading and trailing 1’s in the binary expansion of \(d\). More precisely, we write \((d)_2 = 1^a0\cdots01^b\),

where \(a, b \geq 1\) and the dots represent the remaining digits. In general, we need to consider more detailed subcases, where more digits of \((d)_2\) are specified. For fixed \(p, s \in \{0,1\}^*\) such that \(p\) starts with a 1 and \(s\) ends with a 1 we define the set

\[
D_{p,s} = \{d \in \mathbb{N} : p \in \text{Pref}((d)_2), s \in \text{Suf}((d)_2)\}.
\]

In each separate case with fixed \(p\) and \(s\), we provide positive integers \(j, k, i\) such that for all \(d \in D_{p,s}\) we have

\[
2^{\ell(d)-i}j + k < \frac{1}{2}d
\]
and

\[ e_{11}(jd) + e_{11}(kd) + e_{11}((2^\ell(d) - i + k)d) \equiv 1 \pmod{2}. \]

The condition (2) implies that at least one of \( r_{jd}, r_{kd}, r_{(2^\ell-d-i+k)d} \) is equal to 1, which paired with (1) gives

\[ A_r(0,d) \leq 2^{\ell(d) - i + k} < \frac{1}{2} d, \]

as desired. In the sequel we are going to use the simplified notation \( \ell = \ell(d) \).

We defer a more detailed explanation on how to verify that the above conditions actually hold until after Lemmas 4.1 and 4.2, which serve as a concrete illustration.

We start with the case \( a = b \geq 5 \), split into two parts depending on parity.

**Lemma 4.1.** Assume that \( a = b \geq 5 \) are odd. Then at least one of \( r_{jd}, r_{kd}, r_{(2^\ell(d) - i + k)d} \) is equal to 1, where \( k, i \) are given in Table 1.

| case | \( p \) | \( s \) | \( k \) | \( i \) |
|------|------|------|------|------|
| I    | 1^a0 | 101^a | 1    | \( a \) |
| II   | 1^a00| 0001^a| 2^a-2+1 | 2   |
| III  | 1^a00| 1001^a| 1    | \( a+1 \) |
| IV   | 1^a01| 001^a | 5    | \( a-3 \) |

**Table 1.** The case of odd \( a = b \geq 5 \).

Consequently, \( A_r(0,d) < d/2 \).

**Proof.** **Case I:** Write \( d = [1^a0u]_2 = [w10^a]_2 \), where \( u, w \in \{0,1\}^* \). Binary addition \( (2^{a-2}d + (2^\ell-d)2 \) gives

\[
1^{a-1}10u \\
+ w101^{a-1}1 \\
\hline
w111^{a-1}00u
\]

We obtain three equalities, each corresponding to a line in the above operation:

\[ e_{11}(d) = a - 1 + |u|_{11}, \]
\[ e_{11}(d) = |w|_{11} + a - 1, \]
\[ e_{11}((2^{a-2} + 1)d) = |w|_{11} + a + |u|_{11}. \]

Subtracting the first two equalities from the third one, we obtain

\[ e_{11}((2^{a-2} + 1)d) - 2e_{11}(d) = -a + 2 \equiv 1 \pmod{2}. \]

This means that

\[ A_r(0,d) \leq 2^{\ell-a} + 1 < \frac{1}{2} d, \]

since \( \ell(2^{a-2} + 1) = \ell - a + 1 < \ell - 1. \)

**Case II:** We first compute how the prefix of \( ((2^{a-2} + 1)d) \) of length \( a + 3 \) looks like:

\[
\begin{align*}
11111 & \\
+ 1^{a-2}1100\cdots & \\
\hline
10^{a-2}1c\cdots
\end{align*}
\]

where \( c \in \{0,1\} \) and \( u \) is a suffix of \( (2^{a-2} + 1)d) \). The value of \( c \) depends on whether there is a carry added directly to the left of \( u \). Now, write \( d = [w0001^a]_2 \).

For either value of \( c \), binary addition \( (2^{a-2} + 1)d) + (2^{a-2}2d) \) gives

\[
\begin{align*}
10^{a-2}1c\cdots & \\
+ w0001^{a-2}11 & \\
\hline
w0100^{a-2}c\cdots
\end{align*}
\]
and thus we get
\[ e_{11}(2^{a-2} + 1)d = 1 + |cu|_{11}, \]
\[ e_{11}(d) = |w|_{11} + a - 1, \]
\[ e_{11}(2^{\ell-2} + 2^{a-2} + 1)d) = |w|_{11} + 2c + |cu|_{11}. \]

Again, subtracting (or summing) the equalities, we get
\[ e_{11}(d) + e_{11}(kd) + e_{11}(2^{\ell-2} + 2^{a-2} + 1)d) \equiv a \equiv 1 \pmod{2}. \]

This time, the expansions \((2^{\ell-1} + 2(2^{a-2} + 1))_2\) and \((d)_2\) have equal length \(\ell\).

However, the former expansion is lexicographically smaller, which gives the desired inequality \(A_r(0, d) \leq 2^{\ell-2} + 2^{a-2} + 1 \leq d/2\).

**Case III:** Write \(d = [1^a00d]_2 = [w1001^a]_2\). Binary addition \((d)_2 + (2^{\ell-a-1}d)_2\) gives
\[
\begin{align*}
&11^{a-2}100u \\
&+ w1001^{a-2}11 \\
&\equiv w1101^{a-2}010u
\end{align*}
\]

Similarly as in the Case I, this yields \(e_{11}(2^{\ell-a-1} + 1)d) \equiv 1 \pmod{2}\), and further \(A_r(0, d) < d/2\).

**Case IV:** Binary multiplication \((5)_2 \cdot (d)_2\) gives
\[
\begin{align*}
&1^{a-2}11 \ldots \\
&+ 111^{a-2}01 \ldots, \\
&\equiv 1001^{a-2}0c\ u
\end{align*}
\]

where \(c \in \{0, 1\}\) and \(u\) is a suffix of \((5d)_2\).

Writing \(d = [w001^a]\), binary addition \((5d)_2 + (2^{\ell-a+3}d)_2\) becomes
\[
\begin{align*}
&1001^{a-4}110c\ u \\
&+ w001111^{a-4}1 \\
&\equiv w011001^{a-4}010c\ u
\end{align*}
\]

which yields \(e_{11}(5d) + e_{11}(d) + e_{11}(2^{\ell-a+3} + 5)d) \equiv 1 \pmod{2}\).

We have \(\ell(2^{\ell-a+3}+5) = \ell - a + 4 < \ell - 1\). We deduce that \(A_r(0, d) \leq 2^{\ell-a+3}+5 < d/2\) by comparing the expansions \((2(2^{\ell-a+3} + 5))_2\) and \((d)_2\) in terms of length (for \(a \geq 7\)) or lexicographically (for \(a = 5\)).

In the case of even \(a = b \geq 6\), we also need the value of \(j\) to vary.

**Lemma 4.2.** Assume that \(a = b \geq 6\) are even. Then at least one of \(r_{jd}, r_{kd}, r_{(2^e-j+k)d}\) is equal to 1, where \(j, k, i\) are given in Table 2.

| case | \(p\) | \(s\) | \(j\) | \(k\) | \(i\) |
|------|------|------|------|------|------|
| I    | 1^a0 | 001^a | 1    | 1    | \(a\) |
| II   | 1^a00 | 101^a | 2^a - 1 | 1     | \(a + 1\) |
| III  | 1^a01 | 101^a | 1    | 5    | \(a - 3\) |

Table 2. The case of even \(a = b \geq 6\).

Consequently, \(A_r(0, d) < d/2\).

**Proof.** **Case I:** Write \(d = [1^a0u]_2 = [w001^a]_2\). Then \((d)_2 + (2^{\ell-a}d)_2\) becomes
\[
\begin{align*}
&1^{a-1}10u \\
&+ w001^{a-1}1 \\
&\equiv w011^{a-1}00u
\end{align*}
\]
4.1. We again have the fact that the expansion when $p,s \leq k$, which guarantees that the exponent is nonnegative as well as $k < 2^j$ so that $\ell(2^{j-i}+k) < \ell-i + \ell(j)$. The inequality (1) is rather straightforward to check. Often we have $\ell(j) < i-1$, which immediately implies $\ell(2^{j-i}+k) < \ell-i + \ell(j)$.

The verification of the inequality $A_p(0,d) < d/2$ requires a bit more care compared to the previous cases. We have $\ell(2^{j-i}+2) = \ell$. Since $p$ and $s$ cannot overlap in $(d)_2$, we can write $(d)_2 = 1^a000101^a$ for some $x \in \{0,1\}^*$. The result follows from the fact that the expansion $2^{j-i}(2^a-1) + 2 = 1^a0^a101^a$ is lexicographically smaller than $(d)_2$ regardless of $x$.

**Case III:** The calculations are almost identical as in the Case IV of Lemma 4.1. We again have $(5d)_2 = 1001^{a-2}0c0u$, where $c \in \{0,1\}$ and $u$ is a suffix of $(5d)_2$. Writing $d = [w101^a]_2$, binary addition $(5d)_2 + (2^{j-i+1}1)2$ becomes

\[
\begin{array}{c}
1001^{a-4}11c u \\
+ w101111^{a-4}1 \\
\hline
w111001^{a-4}01c u
\end{array}
\]

which again gives $e_{11}(5d) + e_{11}(d) + e_{11}(2^{j-i+3}) + 5d) \equiv 1$ (mod 2), and consequently the assertion. □

We now discuss how the key conditions (1) and (2) can be verified in general, when $p,s$ as well as $j,k,i$ are given. In the sequel we assume that $j$ and $k$ are odd. Also, we need to have

\[i + \ell(k) \leq \min_{d \in D_{p,s}} \ell,
\]

which guarantees that the exponent $\ell - i$ in $2^{j-i}j + k$ is nonnegative as well as $k < 2^{j-i}j$ so that $\ell(2^{j-i}j + k) < \ell-i + \ell(j)$. The inequality (1) is rather straightforward to check. Often we have $\ell(j) < i-1$, which immediately implies $\ell(2^{j-i}j + k) < \ell-i + \ell(j)$. This occurs for example in case I of Lemma 4.1. On the other hand, if $\ell(j) = i-1$ then the expansions $2^{j-i}(2^{j-i}j + k)$ and $(d)_2$ have equal length and one needs to verify that the former one is lexicographically smaller. This can usually be done by comparing their prefixes, as in case II of Lemma 4.1. In the rare situation when such a comparison is inconclusive, as in case II of Lemma 4.2 one needs to consider the whole expansions, including the suffixes.

The crucial part is thus ensuring that (2) holds. This is achieved through an examination of the binary addition $2^{j-i}(jd)_2 + (kd)_2 = ((2^{j-i}j + k)d)_2$, which allows us to relate the occurrences of the block 11 in the three binary expansions. The
analysis can be reduced to the binary addition of a prefix $\pi$ of $(kd)_2$ and a suffix $\sigma$ of $(jd)_2$ (suitably shifted). This is manifested in Lemmas [1,1] and [1,2] through the irrelevance of the words $w,u$ in the calculations. Observe that it is important that the digits adjacent to $w$ and $u$ do not change when performing the mentioned binary addition. For further reference we state this observation formally as a lemma.

**Lemma 4.3.** Let $d, j, k, i$ be positive integers such that $d, j, k$ are odd and $k < 2^{\ell - i}$. Let $\pi \in \text{Pref}((kd)_2)$ and $\sigma \in \text{Suf}((jd)_2)$ be such that $|\pi|, |\sigma| \geq \max\{\ell(kd) - \ell + i, 1\}$.

Put $h = \ell - i - \ell(kd) + |\pi|$ and let $\gamma \in \{0,1\}^*$ denote the result of the binary addition $\pi + \sigma^0\gamma$, where the initial zeros of $\sigma$ (if any) are taken into account. Assume that the following conditions hold:

(a) there is no carry added to the leftmost digit of $\sigma$ in the addition $\pi + \sigma^0\gamma$;
(b) $|\pi|_1 + |\sigma|_1 + |\gamma|_1$ is odd.

Then at least one of $r_{kd}, r_{jd}, r_{2^{\ell - i}j + k}$ is equal to 1.

**Proof.** Write $kd = [\pi u]_2$ and $jd = [\sigma w]_2$ for some $u, w \in \{0,1\}^*$. Let $\pi', \sigma' \in \{0,1\}^{11 + h - 2}$ be such that $\sigma^{0|\pi - |\pi|} = \pi'c$ and $\sigma^{0h - 1} = e\sigma'$, where $c,e \in \{0,1\}$ are single digits. The assumptions on $|\pi|, |\sigma|$ guarantee that both exponents $h - 1 + |\sigma| - |\pi|$ and $h - 1$ are nonnegative. If we let $\gamma'$ denote the result of binary addition $\pi' + \sigma'$, then $\gamma = e\gamma'c$ by the assumption (a). Therefore, binary addition $\pi u + w\sigma^{0h + |\pi|} = (kd)_2 + (2^{\ell - i}j)_2$ can be written as

$$\frac{\pi'c u + w e\gamma'c u}{w e\gamma'c u}.$$  

Counting the occurrences of the block 11 in each line, we obtain

$$e_{11}(kd) = |\pi'c|_1 + |cu|_1 = |\pi|_1 + |cu|_1,$$

$$e_{11}(jd) = |we|_1 + |e\sigma'|_1 = |we|_1 + |\sigma|_1,$$

$$e_{11}((2^{h+|\pi|}j + k)d) = |we|_1 + |e\gamma'c|_1 + |cu|_1 = |we|_1 + |\gamma|_1 + |cu|_1.$$  

Subtracting the first and second equalities from the third one, we get

$$e_{11}(2^{\ell - i}j + k)d - e_{11}(kd) - e_{11}(jd) = |\gamma|_1 - |\pi|_1 - |\sigma|_1,$$

and the result follows by the assumption (b).

In practice the words $\pi, \sigma$ can be chosen in several ways (for given $p, s$ and $j, k, i$) so we stick to the convention that $\pi$ and $\sigma$ are as long as possible, based on the available digits of $p$ and $s$. In precise terms, we fix the lengths to be $|\sigma| = |s|$ and $|\pi| = |p| + \ell(kd) - \ell(k) - \ell + 1$.

Then with the notation of Lemma [1,3] we obtain

$$h = |p| - i - \ell(k) + 1.$$  

We note that even with $|\pi|$ fixed there may exist multiple possibilities for $\pi$, depending on the carries in the binary multiplication $(k)_2 \cdot (d)_2$, and each one needs to be considered separately. Taking case II of Lemma [1,1] as an example, the prefix $\pi = 10^{n-2}10cc$ of $(2^{n-2} + 1)d$ has two possible values, while $\sigma = s = 0001^a$. Also, in the addition $((2^{n-2} + 1)d)_2 + (2^{n-2}d)_2$ the suffix $\sigma$ is shifted by 2 positions to the left with respect to $\pi$, which agrees with

$$h = |p| - i - \ell(k) + 1 = a + 2 - 2 - (a - 1) + 1 = 2.$$  

In the following result we collect the cases when one of $a, b$ is at least 5 and the other at most 2. Due to the large amount of cases the proof is omitted, however verification of each case can be performed according to the above discussion, similarly as in Lemmas [1,1] and [1,2].
Lemma 4.4. Assume that $a \geq 5, b \leq 2$ or $a \leq 2, b \geq 5$. Then at least one of $r_{jd}, r_{kd}, r_{(2^l-i-j+k)d}$ is equal to 1, where $j, k, i$ are given in the tables below.

(1) $a \geq 5$ and $b = 1$

| $p$ | $s$ | $j$ | $k$ | $i$ |
|-----|-----|-----|-----|-----|
| $1^a0$ | 00001 | 1 | 1 | 3 |
| $1^a0$ | 10001 | 3 | 1 | 3 |
| $1^a0$ | 1001 | 1 | 1 | 2 |
| $1^a0$ | 000101 | 3 | 1 | 3 |
| $1^a0$ | 100101 | 1 | 1 | 4 |
| $1^a0$ | 10101 | 1 | 1 | 2 |
| $1^a0$ | 01101 | 3 | 1 | 3 |
| $1^a0$ | 0011101 | 1 | 1 | 3 |
| $1^a0$ | 1011101 | 1 | 1 | 2 |
| $1^a0$ | 111101 | 7 | 1 | 4 |

Table 3. The case of $a \geq 5$ and $b = 1$.

(2) $a \geq 5$ and $b = 2$

| $p$ | $s$ | $j$ | $k$ | $i$ |
|-----|-----|-----|-----|-----|
| $1^a0$ | 00011 | 1 | 3 | 2 |
| $1^a0$ | 10011 | 1 | 1 | 3 |
| $1^a0$ | 1011 | 1 | 1 | 2 |

Table 4. The case of $a \geq 5$ and $b = 2$.

(3) $a = 1$ and $b \geq 5$ is odd

| $p$ | $s$ | $j$ | $k$ | $i$ |
|-----|-----|-----|-----|-----|
| 1000 | 001$^b$ | 1 | 1 | 3 |
| 10010 | 001$^b$ | 1 | 3 | 3 |
| 10011 | 001$^b$ | 1 | 1 | 4 |
| 101 | 001$^b$ | 1 | 1 | 2 |
| 100 | 101$^b$ | 1 | 1 | 2 |
| 1010 | 101$^b$ | 1 | 1 | 3 |
| 10110 | 101$^b$ | 5 | 1 | 4 |
| 10111 | 101$^b$ | 1 | 1 | 4 |

Table 5. The case of $a = 1$ and $b \geq 5$ odd.

(4) $a = 1$ and $b \geq 6$ is even
We now turn to the case $a > b \geq 3$. Here, a slightly different idea is used to obtain the desired result, as the assumption (a) of Lemma 4.3 does not hold.

**Lemma 4.5.** Assume that $a > b \geq 3$. Then for either $n = 2^f - b + 1$ or $n = 2^f + 1$ we have $r_{nd} = 1$. Consequently, $A_r(0, d) < d/2$.

**Proof.** Write $d = [w01]^b = [1^a0u]$ for suitable $u, w \in \{0, 1\}^*$. If $n = 2^f + 1 + 1$, then $(n)_2 \cdot (d)_2$ yields

$$\begin{align*}
1^{b-2}1 & 1^{a-b+1}0u \\
+ w01^{b-2}1 & w101^{b-2}01^{a-b+1}0u,
\end{align*}$$

and thus

$$e_{11}(nd) = |w1|_{11} + b - 3 + a - b + |u|_{11} = |w1|_{11} + |u|_{11} + a - 3.$$  

Similarly, if $n = 2^f - b + 1$, then we obtain

$$\begin{align*}
1^{b-1}1 & 1^{a-b}0u \\
+ w01^{b-1}1 & w11^{b-1}01^{a-b}0u,
\end{align*}$$

and

$$e_{11}(nd) = |w1|_{11} + b - 1 + a - b - 1|u|_{11} = |w1|_{11} + |u|_{11} + a - 2.$$
Hence, the parity of $e_{11}(nd)$ differs depending on the choice of $n$, and the result follows. \hfill \Box

The symmetric case $b > a \geq 3$ is very similar.

**Lemma 4.6.** Assume that $b > a \geq 3$. Then for either $n = 2^{\ell-a+1} + 1$ or $n = 2^{\ell-a} + 1$ we have $r_{nd} = 1$. Consequently, $A_{v}(0, d) < d/2$.

**Proof.** The proof is analogous as in Lemma 4.5 \hfill \Box

We have left to consider the pairs $(a, b)$ with $a \leq 4, b \leq 4$, except for $(4, 3), (3, 4)$, already covered in Lemmas 4.5 and 4.6. The analysis for such pairs turns out to be extremely tedious to carry out by hand, however it can be effectively done using computer software. We now describe the iterative algorithm we have applied in order to obtain the result.

The algorithm takes as input two blocks $p, s \in \{0, 1\}^*$ satisfying the conditions:

- $p$ begins with a 1;
- $s$ ends with a 1;
- $|s| \geq |p| \geq 3$.

Roughly speaking, we set $k = 1$ and perform an exhaustive search for values $j, i$ such that the conditions (1) and (2) hold for all $d \in D_{p,s}$ (possibly with one exception). If this attempt fails, we repeat the same procedure with additional digits specified in the prefix and suffix.

More precisely, we define the following notation:

- $K_m$ – the set of pairs $(p, s)$ to consider in the $m$th iteration, $K_1 = \{(p, s)\}$;
- $S$ – the set of solutions of the form $(p, s, j, i)$, initially $S = \emptyset$;
- $E$ – the set of exceptional values $d \in D_{p,s}$ for which $A_{v}(d) \geq d/2$, initially $E = \emptyset$.

In the $m$th iteration the algorithm performs for each $(p, s) \in K_m$ the following operations.

1. If $p \in \text{Pref}(s)$ and $A_{v}(0, [s]_2) > [s]_2/2$, add $d = [s]_2$ to $E$.
2. For each odd $j = 1, 3, \ldots, 2^{|p| - 2} - 1$ repeat the following steps.
   (a) Let $\sigma$ denote the suffix of $(j)_{2} \cdot s$ of length $|s|$.
   (b) Put $h_{\max} = \begin{cases} |p| - \ell(j) - 1 & \text{if } 2^{-\ell(j)}j < 2^{-|p|}[p]_2, \\ |p| - \ell(j) - 2 & \text{otherwise.} \end{cases}$
   (c) Check whether the assumptions (a) and (b) of Lemma 4.3 hold for some $h = 1, \ldots, h_{\max}$ and $\pi = p$:
      - if yes, add $(p, s, j, i)$ to $S$, where $i = |p| - h$, and skip to step 1 for the next pair $(p, s) \in K_m$;
      - if not, continue the loop for next $j$.
3. Since no solution has been found, add to $K_{m+1}$ the pairs $(p0, 0s), (p0, 1s), (p1, 0s), (p1, 1s)$ and move to the next pair $(p, s) \in K_m$.

The goal is to obtain $K_{m+1} = \emptyset$ at the end of the $m$th iteration, in which case the algorithm terminates and returns the sets $S$ and $E$.

**Proposition 4.7.** If the algorithm terminates at input $(p, s)$, then for all $d \in D_{p,s} \setminus E$ we have $A_{v}(0, d) < \frac{1}{2}d$.

**Proof.** For any fixed pair $(p', s')$ satisfying $|s'| \geq |p'|$ we have the partition $D_{p', s'} \setminus \{[s']_2\} = D_{p', 0s'} \cup D_{p', 0s'} \cup D_{p', 1s'} \cup D_{p', 1s'}$.
where \([s']_2 \in D_{p',s'}\) if and only if \(p' \in \text{Pref}(s')\). Let \(S_m \subset K_m\) denote the set of pairs \((p', s')\) for which a solution \((j, i)\) has been found in step 2(c). If the algorithm terminates after \(M \geq 1\) iterations, we obtain

\[
D_{p,s} \setminus E = \bigcup_{m=1}^{M} \bigcup_{(p',s') \in S_m} D_{p',s'}.
\]

Hence, it remains to prove that if a solution \((j, i)\) is found for some fixed \((p', s') \in K_m\), then the desired inequality holds for all \(d' \in D_{p',s'} \setminus \{[s']_2\}\). Choose any \(d' \in D_{p',s'}\) and assume that \(d' \neq [s']_2\). First, we verify that all assumptions of Lemma 4.3 are satisfied. Observe that \(\sigma\) is precisely the suffix of \((jd)\) of length \([s']_2\). Since \(k = 1\), we have \(\ell(kd) - \ell = 0\) and the notation \(h = [p'] - i\) agrees with the lemma. Clearly, \(2^{\ell-i} = 2^{\ell-[p']+h} > k\) and \([\sigma] \geq |p'| = h > i\). Finally, the conditions (a) and (b) are checked directly by the algorithm. Therefore, by Lemma 4.3 we have

\[
A_r(0, d) \leq 2^{\ell-i}j + 1,
\]

and our task is reduced to proving the inequality \(2^{\ell-i}j + 1 < d/2\).

If \(h \leq |p'| - \ell(j) - 2\), namely, \(\ell(j) \leq i - 2\), then \(\ell(2^{\ell-i}j + 1) \leq \ell - 2\), which gives the result. Otherwise, if \(h = |p'| - \ell(j) - 1\), then we have \(\ell(j) = i - 2\), and also necessarily \(h_{\max} = |p'| - \ell(j) - 1\), which implies

\[
2^{\ell-\ell(j)}j < 2^{\ell-[p']}[p']_2 \leq d - 1.
\]

The second inequality follows from the fact that \(d\) is odd and \(d \neq |p'|_2\), which in turn is implied by \(d \neq [s']_2\) and \([s'] \geq |p'|\). Therefore, we get

\[
2^{\ell-i}j + 1 = 2^{\ell-\ell(j)-1}j + 1 < \frac{1}{2}(d + 1).
\]

Since both sides of the inequality are integers, we also obtain \(2^{\ell-i}j + 1 < d/2\), as desired. 

We implemented the algorithm in Mathematica 13 and applied it to the pairs \((p, s) = (110, 010)\) for the remaining cases \((a, b)\). The detailed results together with the code are available at the repository \([8]\). We note that the cases where \([s']_2\) need to be further split in order to have \([s] \geq |p'| \geq 3\). For example, when \((a, b) = (3, 1)\), the algorithm is actually applied separately to each of the four possible pairs \((p, s) = (1110, **01)\), where \(*\) denotes any binary digit. In most instances the algorithm successfully terminates (after at most 6 iterations). Clearly, this cannot be the case when \((p, s) = (100, 001)\), as then \(D_{p,s}\) contains infinitely many terms of the form \(d = 2^i + 1\) for which \(A_r(0, d) = (d + 1)/2\), as already proved in Proposition 3.2. Moreover, for \((p, s) = (100, 101)\) the algorithm apparently does not terminate in all “branches”. Apart from this, we also get two exceptions, namely \(E = \{5\} \) at input \((101, 101)\) and \(E = \{39\} \) at input \((100, 0111)\). We summarize the results of the computer calculations in the following Lemma.

\textbf{Lemma 4.8.} Assume that \(1 \leq a \leq 4\) and \(1 \leq b \leq 4\), excluding \((a, b) \in \{(4, 3), (3, 4)\}\). Then we have \(A_r(0, d) < d/2\) for all:

\begin{enumerate}
\item \(d \in D_{110001}\) when \((a, b) \notin \{(1, 1), (1, 3)\};
\item \(d \in D_{1000111} \setminus \{39\};
\item \(d \in D_{100001} \setminus (D_{1000001} \cup D_{1000011101} \cup \{5\})\).
\end{enumerate}

We now deal with the problematic case (iii) of Lemma 4.8

\textbf{Lemma 4.9.} For all \(d \in D_{100001} \setminus \{2^k + 1 : k \geq 3\}\) we have \(A_r(0, d) < d/2\).
Proof. Write \(d = [u001]_2 = [100w]_2\). Then \((2^{k-1} + 1)_2 \cdot (d)_2\) yields
\[
\begin{align*}
100u & + w001 \\
\text{w0110u}
\end{align*}
\]
which implies \(A_r(0, d) \leq 2^{k-1}+1\). If \(d\) is not of the form \(2^k+1\), then \(d/2 > 2^{k-1}+1\), and the result follows. \qed

Lemma 4.10. For all \(d \in D_{100000,111101}\) we have \(A_r(0, d) < d/2\).

Proof. The idea is similar as in the proof of Lemma 4.6. We show that \(r_{nd} = 1\) for \(n\) equal to either \(2^{\ell-4} + 1\) or \(2^{\ell-5} + 1\). Write \(d = [100000u]_2 = [w111101]_2\) and let \(v\) denote the result of binary addition \(w + 1\). If \(n = 2^{\ell-4} + 1\), then \((n)_2 \cdot (d)_2\) yields
\[
\begin{align*}
100000u & + w111101 \\
\text{v00010100u}
\end{align*}
\]
while for \(n = 2^{\ell-4} + 1\) we obtain
\[
\begin{align*}
100000u & + w111101 \\
\text{v00011010u}
\end{align*}
\]
The parity of \(c_{11}(nd)\) depends on the choice of \(n\), and the result follows. \qed

To conclude this section, we complete the proof of Theorem 1.5 by showing that the final case (iii) holds.

Proposition 4.11. For all integers \(d \geq 3\) not lying in the set
\[
\{2^k - 1 : k \geq 3 \text{ is odd}\} \cup \{2^k + 1 : k \geq 2\} \cup \{39\}
\]
we have \(A_r(d) < d/2\).

Proof. The lemmas proved in this section collectively show that the claim is true for all odd integers \(d \geq 3\), except for possibly \(d = 2^k - 1\) with \(k \geq 2\) even. But for such \(d\) we get \(A_r(0, d) = 1 < d/2\) as well. We can also compute that \(A_r(0, 39) = 21\), thus \(d = 39\) satisfies the equality in the case (i) of Theorem 1.5.

For even \(d\), we use the equality \(A_r(0, 2d) = A_r(0, d)\). We have \(A_r(0, 2^k) = A_r(0, 1) = 3\), which gives the desired inequality for any \(k \geq 2\). When \(d = 2^k d'\), where \(k \geq 1\) and \(d' \geq 3\) is odd, we get \(A_r(0, 6) = 1\) and for \(d > 6\) the inequality
\[
A_r(0, d) = A_r(0, d') \leq \frac{1}{2}(d' + 3) \leq \frac{3}{2} < \frac{d}{2}. \quad \square
\]

5. Proof of Theorem 1.6 and Theorem 1.7

In this section we consider the values \(A_r(d)\) when \(d = 2^k + 1\) or \(d = 2^k - 1\). To begin, we make a simple observation. Let \(n \in \mathbb{N}\) and write
\[
n = 2^k m + l,
\]
where \(m \in \mathbb{N}\), and \(0 \leq l < 2^k\). Adding \(d = 2^k + 1\) to \(n\) amounts to increasing both \(m\) and \(l\) by 1, unless \(l = 2^k - 1\). Similarly, adding \(d = 2^k - 1\) corresponds to increasing \(m\) and decreasing \(l\) by 1, unless \(l = 0\).

At the same time, we have
\[
r_n \equiv \begin{cases} r_m + r_l & \text{mod } 2, & \text{if } l < 2^k-1, \\ r_{2m+1} + r_l & \text{mod } 2, & \text{if } l \geq 2^k-1. \end{cases}
\]
Suppose there is a monochromatic arithmetic progression of difference $2^k + 1$ and length $a$ starting at $n$, and assume that $l + i \not\in \{2^{k-1} - 1, 2^k - 1\}$ for any $i = 0, 1, \ldots, a - 1$. As we argue later, it is sufficient to consider $r_n = 0$. Then, depending on $l$, we obtain either $r_{n+i} = r_{l+i}$ for $i = 0, 1, \ldots, a - 1$ or $r_{2(m+i)+1} = r_{2(l+i)}$ for $i = 0, 1, \ldots, a - 1$. In the same fashion, when considering a progression of difference $2^k - 1$ we arrive the equalities $r_{n+i} = r_{l+i}$ and $r_{2(m+i)+1} = r_{l+i}$. In any case, we need to study subwords of $r$ and its subsequence $(r_{2m+i})_{m \geq 0}$, as well as their reversals. In particular, we would like to show that the existence of a long progression imposes some relation on $m$ and $l$. This is conveniently done using the description of the Rudin-Shapiro sequence in terms of a substitution $\rho$, which we now recall. Let $\rho$ be defined on the alphabet $\Sigma_r = \{00, 01, 10, 11\}$ by

\[
\rho(00) = 0001, \quad \rho(01) = 0010, \quad \rho(10) = 1101, \quad \rho(11) = 1110.
\]

Letting $\rho(vw) = \rho(v)\rho(w)$ for $v, w \in \Sigma_r$, we can uniquely extend $\rho$ to all finite and infinite words with letters in $\Sigma_r$. Then $r$ can be identified with the fixed point of $\rho$ starting with 00, where each block is split into two individual letters. More precisely, for $t \in \mathbb{N}$ let $\rho^t$ be the $t$-fold composition of $\rho$ with itself, where $\rho^0$ is the identity. Then

\[
r = \lim_{t \to \infty} \rho^t(00) = 0001 \rho(01) \rho^2(01) \cdots.
\]

Here the limit means that each of the words $\rho^t(00)$ is a prefix of the infinite word $r$ and their length tends to infinity. For each $t \in \mathbb{N}$ we can also write

\[
r = \rho^t(00) \rho^t(01) \rho^t(00) \rho^t(10) \cdots.
\]

For $w \in \Sigma_r$ each block $\rho^t(w)$, treated as a word over $\{0, 1\}$, has length $2^{t+1}$ and we call it a $2^{t+1}$-aligned block. Hence, for every $t \in \mathbb{N}$ there are precisely four distinct $2^{t+1}$-aligned blocks. Note that the blocks $\rho^t(00), \rho^t(11)$ appear at even positions (counting from 0) in the representation (3), while the blocks $\rho^t(01), \rho^t(10)$ – at odd positions. Also observe that for any $w \in \Sigma_r$ we have $\rho(\overline{w}) = \overline{\rho(w)}$.

It is useful to have a similar description in terms of a substitution for the subsequence $(r_{2^{m+1}})_{m \geq 0}$. For $n \in \mathbb{N}$ we put $s_n = r_{2^{n+1}}$ and $s = (s_n)_{n \geq 0}$. We obtain a nice property linking $2^{t+1}$-aligned subwords of $r$ and $s$.

**Proposition 5.1.** The sequence $s$ is the fixed point starting with 01 of the substitution $\sigma$, defined by

\[
\sigma(01) = 0100, \quad \sigma(00) = 0111, \quad \sigma(11) = 1000, \quad \sigma(10) = 1011.
\]

Moreover, for any odd integer $t \geq 1$ we have

\[
\sigma^t(01) = \rho^t(01)^R, \quad \sigma^t(00) = \rho^t(11)^R, \quad \sigma^t(11) = \rho^t(00)^R, \quad \sigma^t(10) = \rho^t(10)^R,
\]

while for any even integer $t \geq 2$ we have

\[
\sigma^t(01) = \rho^t(10)^R, \quad \sigma^t(00) = \rho^t(00)^R, \quad \sigma^t(11) = \rho^t(11)^R, \quad \sigma^t(10) = \rho^t(01)^R.
\]

**Proof.** The word $r$ is the fixed point starting with 0001 of the following substitution, derived from $\rho$:

\[
0001 \mapsto 0001 0010, \quad 0010 \mapsto 0001 1101, \quad 1101 \mapsto 1110 0010, \quad 1110 \mapsto 1110 1101.
\]

Note that the first and third digit (counting from 0) uniquely determine each block of length 4. By extracting the digits at odd positions from the above substitution, we obtain precisely $\sigma$. But this corresponds to taking the subsequence $(r_{2^{2n+1}})_{n \geq 0}$, and thus we get the first part of the assertion.

In order to prove the identities linking $\rho$ and $\sigma$, we use induction on $t$. For $t = 1$ the result follows immediately from the definition of $\sigma$. Now let $t \geq 2$ be even. By
We can write

$$\sigma^t(01) = \sigma^{t-1}(01)\sigma^{t-1}(00) = \rho^{t-1}(01)R\rho^{t-1}(11)R$$

$$= (\rho^{t-1}(11)\rho^{t-1}(01))^R = \rho'(10)^R,$$

as claimed. The verification for other blocks and even values of $t$ is similar. \qed

We now prove a few (standard) auxiliary results concerning the subwords of the infinite words $r$ and $s$. Lemmas 5.2, 5.3 have the same statement and almost identical proof for both $r$ and $s$. Hence, in these lemmas we let $u$ denote any of $r, s$ and only provide the proof for $u = r$. The first result says that the set of subwords of $u$ is invariant with respect to negation.

**Lemma 5.2.** We have $w \in \text{Sub}(u)$ if and only if $\overline{w} \in \text{Sub}(u)$.

**Proof.** If $w \in \text{Sub}(r)$, then $w$ is a subword of $\rho^t(00)$ for some $m \geq 1$, and thus a subword of $\rho^{t+3}(11) = \rho^m(00)$. This means that $\overline{w} \in \text{Sub}(u)$. The other inclusion follows by $\overline{w} = w$. \qed

The next lemma shows that from the knowledge of a sufficiently long prefix of $u$ we can infer all the subwords of given length and their positions modulo a power of 2.

**Lemma 5.3.** Let $w \in \text{Sub}(u)$ have length $|w| \leq 2^t + 1$, where $t \geq 1$ is an integer. Then $w \in \text{Sub}(p)$, where $p$ is the prefix of $u$ of length $7 \cdot 2^t + 1$. More precisely, if $w$ appears in $u$ at some position modulo $2^t + 1$, then it appears in $p$ at the same position modulo $2^t + 1$.

**Proof.** Divide $r$ into $2^t$-aligned blocks:

$$r = \rho^{t-1}(00)\rho^{t-1}(01)\rho^{t-1}(00)\rho^{t-1}(10) \cdots .$$

Any occurrence of a subword $w$ of length at most $2^t + 1$ is contained in a single block $\rho^{t-1}(A)$ or a concatenation $\rho^{t-1}(A)\rho^{t-1}(B)$, where one of $A, B$ belongs to the set $\{00, 11\}$, and the other to $\{01, 10\}$. Hence, in order to prove both parts of the claim it is sufficient to show that each possible concatenation $\rho^{t-1}(A)\rho^{t-1}(B)$ appears within 14 initial terms of $\{4\}$. In the case $t = 1$ we can check this by inspection of the initial 2-aligned blocks:

$$00 01 00 10 00 11 01 00 01 00 10 11 10.$$

For $t \geq 2$ the result follows by applying $\rho^{t-1}$ to this word block-by-block. \qed

We now show that the positions at which the same word $w$ occurs in $u$ are unique modulo a power of 2. Note that unlike in the previous lemma, the length of $w$ is also bounded from below.

**Lemma 5.4.** Let $w \in \text{Sub}(u)$ have length $|w| \geq 2^t + 1$, where $t \geq 3$ is an integer. If $w$ is a subword of a $2^{t+1}$-aligned block in $u$, then its position modulo $2^{t+2}$ in $u$ is uniquely determined.

**Proof.** We prove the result by induction on $t$. The assumption implies that $|w| \leq 2^{t+1}$. In the base case $t = 3$, Lemma 5.3 guarantees that if $w$ occurs at some position modulo $2^3$ in $r$, then it already occurs at the same position modulo $2^3$ in the prefix of $r$ of length $7 \cdot 2^3$. A computer search shows that for each choice of $w$ there is exactly one such position.

Now let $t \geq 4$ and express $r$ as a concatenation of $2^t$-aligned blocks:

$$r = \rho^{t-1}(00)\rho^{t-1}(01) \cdots = R_0R_1 \cdots .$$

By the assumption $w$ is a subword of a $2^{t+1}$-aligned block $R_{2n}R_{2n+1}$ for some $n \in \mathbb{N}$. We can write $w = uv$, where $u \in \text{Suf}(R_{2n})$ and $v \in \text{Pref}(R_{2n+1})$. Hence, each of
$u$, $v$ is a subword of a $2^i$-aligned word and at least one of them has length greater than or equal to $2^{t-1} + 1$. In either case it follows from the inductive assumption that the position of $w$ modulo $2^{t+1}$ in $r$ is uniquely determined. This also shows that the factorization $w = uv$ is unique.

In order to determine the position of $w$ modulo $2^{t+2}$ it remains to deduce the parity of $n$. We have $R_{2n} = \rho^{-1}(A)$ and $R_{2n+1} = \rho^{-1}(B)$ for some $A \in \{01, 10\}$. Because both words $u$, $v$ are nonempty, at least the last letter of $\rho^{-1}(A)$ and the first letter of $\rho^{-1}(B)$ are known. As $\rho^{-1}(11) = \rho^{-1}(00)$ and $\rho^{-1}(10) = \rho^{-1}(01)$, this already allows us to determine $A$ and $B$. Let $C \in \{0, 1\}^2$ be such that $\rho(C) = AB$. Then $R_{2n} R_{2n+1} = \rho(C)$ and thus $C$ is the $n$th subsequent $2$-aligned block in $r$ (counting from 0). Hence, $n$ is even if $C \in \{00, 11\}$, and odd otherwise.

The following two lemmas provide some relations between the words $r$ and $s$, which are consequences of Proposition 5.1. First, we show that the subwords of $r$ are precisely the reversed subwords of $s$ and relate the positions at which $w$ and $w^R$ can appear in the respective sequences.

**Lemma 5.5.** We have $w \in \text{Sub}(r)$ if and only if $w^R \in \text{Sub}(s)$. Moreover, assume that $w$ has length $|w| \geq 2^t + 1$ and is a subword of a $2^{t+1}$-aligned block in $r$ for some integer $t \geq 3$. If $w = r_n r_{n+1} \cdots r_{n+|w|-1}$ and $w^R = s_m s_{m+1} \cdots s_{m+|w|-1}$ for some integers $n, m \geq 0$, then

$$n + m + |w| \equiv 0 \pmod{2^{t+2}}. \quad (5)$$

**Proof.** If $w \in \text{Sub}(r)$, then it is a subword of some $2^{t+1}$-aligned block in $r$. Proposition 5.1 shows that $2^{t+1}$-aligned blocks in $s$ are precisely reversals of $2^{t+1}$-aligned blocks, hence $w^R \in \text{Sub}(r)$. The converse is proved in the same fashion.

We proceed to the second part of the statement. Write $r$ and $s$ as the concatenation of $2^{t+1}$-aligned blocks:

$$r = R_0 R_1 R_2 \cdots,$$
$$s = S_0 S_1 S_2 \cdots.$$  

Let $w = r_n r_{n+1} \cdots r_{n+|w|-1} \in \text{Sub}(R_i)$ for some $i \in \mathbb{N}$. Write $n = 2^t + i + l$, where $l < 2^t + 1$ is the position of the initial letter of $w$ in $R_i$ (counting from 0). At the same time, for some $j \in \mathbb{N}$ we have $S_j = R_i^R$, and thus $w^R \in \text{Sub}(S_j)$. More precisely, we have $w^R = s_m s_{m+1} \cdots s_{m+|w|-1}$, where $m = 2^t + 1(j + 1) - l - |w|$. The relations of Proposition 5.1 imply that $i, j$ have different parity, which gives the congruence \((5)\) for these particular occurrences of $w$ and $w^R$. But Lemma 5.4 implies that the positions at which $w$ and $w^R$ appear in $r$ and $s$, respectively, are unique modulo $2^{t+2}$, and therefore \((5)\) holds in general.

The final auxiliary result shows that $r$ and $s$ have only finitely many common subwords.

**Lemma 5.6.** Assume that one of the following conditions holds:

(i) $w \in \text{Sub}(r) \cap \text{Sub}(s)$;
(ii) $w \in \text{Sub}(r)$ and $w^R \in \text{Sub}(r)$.

Then $|w| \leq 14$.

**Proof.** Due to Lemma 5.5 the conditions (i) and (ii) are equivalent. By Lemma 5.3 all possible subwords $w$ of length 15 appear at least once in the prefix of $r$ of length $7 \cdot 2^t$. A computer search of this prefix shows that there are no $w$ of length 15 (or greater) such that $w^R$ is also a subword of $r$.

We are now ready to prove Theorems 1.6 and 1.7.
Proof of Theorem 7.6. Let \( d = 2^k + 1 \) We begin with the case \( k \leq 3 \), together with \( k = 4, 5 \), where we verify the assertion by direct calculation. Consider for example \( k = 3 \). One can check that a monochromatic arithmetic progression of difference 9 and length 9 starts with \( r_{28} \). In order to verify that there exist no longer progressions we can use Lemma 5.3. Any monochromatic arithmetic progression of difference 9 and length 10 would have to be contained in a subword of \( r \) of length \((10 - 1) \cdot 9 + 1 = 82 \leq 2^7 + 1\). By Lemma 5.3, each such subword appears in the prefix of \( r \) of length \( 7 \cdot 2^8 \). But a direct search shows that there is no such progression of length 10 among these initial terms, and hence neither in \( r \).

The other cases with \( k \leq 5 \) can be proved in the same fashion. In Table 9 below we provide the initial term of a monochromatic arithmetic progression of postulated length \( A_r(d) \) and the length of a prefix of \( r \), where a longer progression would have to appear (which is not the case).

| \( k \) | \( d \) | \( A_r(2^k + 1) \) | initial term | prefix length |
|---|---|---|---|---|
| 1 | 3 | 5 | 28 | \( 7 \cdot 2^0 \) |
| 2 | 5 | 6 | 31 | \( 7 \cdot 2^0 \) |
| 3 | 9 | 9 | 43 | \( 7 \cdot 2^5 \) |
| 4 | 17 | 10 | 495 | \( 7 \cdot 2^9 \) |
| 5 | 33 | 18 | 980 | \( 7 \cdot 2^{11} \) |

Table 9. The location of monochromatic arithmetic progressions of length \( A_r(2^k + 1) \) for \( k \leq 5 \)

We now move on to the general case \( k \geq 6 \). We first exhibit a monochromatic arithmetic progression of postulated length \( A_r(d) = 2^{k-1} + 2 \). Let \( n = 2^{2k+1} - 2^k - 1 = [1^k01^k]_2 \), so that \( r_n = 0 \). Then \( n + d = 2^{2k+1} = [10^{2k+1}]_2 \), thus also \( r_{n+d} = 0 \). Furthermore, for all \( i = 0, 1, \ldots, 2^{k-1} \) we can write \( n + (i + 1)d = 2^{2k+1} + 2^i + i \) and it is easy to see that \( c_{11}(n + (i + 1)d) = 2c_{11}(i) \equiv 0 \) (mod 2). Therefore, the \( 2^{k-1} + 2 \) terms \( n, n + d, \ldots, n + (2^{k-1} + 1)d \) constitute a monochromatic arithmetic progression in \( r \). Note that it cannot be prolonged in any direction, as the binary expansions of the numbers

\[
\begin{align*}
    n - d &= 2^{2k+1} - 2^k - 2 = [1^{k-1}01^k0]_2, \\
    n + (2^{k-1} + 2)d &= 2^{2k+1} + 2^{2k-1} + 2^k + 2^{k-1} + 1 = [1010^{k-2}110^{k-2}]_2
\end{align*}
\]

both contain an odd number of occurrences of 11.

We proceed to show that \( A_r(d) \leq 2^{k-1} + 2 \) in general by giving a bound on \( A_r(n, d) \) for each \( n \). By Lemma 5.2 it is sufficient to consider only arithmetic progressions of “color” 0. Thus, assume that \( n \in \mathbb{N} \) is such that \( r_n = 0 \). As at the beginning of this section, write

\[ n = 2^k m + l, \]

where \( m \in \mathbb{N} \) and \( 0 \leq l < 2^k \). We are going consider four cases, depending on which quarter of the interval \([0, 2^k)\) the number \( l \) belongs to.

**Case 1:** \( 0 \leq l < 2^{k-2} \)

For \( i = 0, 1, \ldots, 2^{k-2} \) the blocks \( (m + i)_2 \) and \( (l + i)_2 \) in the binary expansion of \((n + i)_2\) are broken up by at least one zero, so we have

\[
    r_{n+id} \equiv r_{m+i} + r_{l+i} \quad (\text{mod } 2) .
\]

If \( A(n, d) < 2^{k-2} + 1 \), there is nothing to prove. Otherwise, we get \( r_{k+1} = r_{l+i} \) for \( i = 0, 1, \ldots, 2^{k-2} \). The word \( r_tr_{t+1} \cdots r_{t+2^k-2} \) is a subword of length \( 2^{k-2} + 1 \) of the \( 2^{k-1} \)-aligned block \( r_0r_1 \cdots r_{2^{k-1} - 1} \) so Lemma 5.4 implies \( m \equiv l \) (mod \( 2^k \)).
Letting $m = 2^k j + l$, we get $n = 2^{2k} j + (2^k + 1) l = 2^{2k} j + ld$. Comparing the binary expansions

\[(n + (2^{k-1} - l) d) = (j)_{2} 10^{k-1} 10^{k-1},\]

\[(n + (2^{k-1} - l + 1) d) = (j)_{2} 10^{k-2} 110^{k-2},\]

we see that the latter contains exactly one more occurrence of the block 11, regardless of $j$. Hence, the monochromatic arithmetic progression cannot be prolonged beyond $n + (2^{k-1} - l) d \leq n + 2^{k-1} d$, and therefore $A(n, d) \leq 2^{k-1} + 1$.

**Case II:** $2^{k-2} \leq l < 2^k - 1$

Put $n' = n + (2^{k-1} - l) d = 2^k m' + 2^{k-1}$, where $m' = m + 2^{k-1} - l + 1$. For $i = 0, 1, \ldots, 2^{k-1} - 1$ the block $(m' + i)_{2}$ in the binary expansion of $(n' + id)_{2}$ is directly followed by a 1, so we have

\[r_{n' + id} = r_{2^k + 1 + i} \pmod{2}.\]

It follows that $r_{n' + id} = 0$ as long as $r_{2^k + 1 + i} = r_{2^k - 1 + i}$. By Lemma 5.6 the former equality does not occur for some $i \leq 14$. Hence, by the assumption $k \geq 6$ we get

\[A(n, d) \leq \frac{n' - n}{d} + A(n', d) \leq 2^{k-2} + 14 < 2^{k-1}.\]

**Case III:** $2^{k-1} \leq l < 3 \cdot 2^{k-2}$

For at least $i = 0, 1, \ldots, 2^{k-2}$ we have

\[r_{n + id} = r_{2^k + 1 + i} + r_{l + i} \pmod{2}.\]

Again, due to Lemma 5.6 the right hand-side cannot be constantly equal to 0 for all $i = 0, 1, \ldots, 14$. Therefore, $A(n, d) \leq 14 < 2^{k-1}$. 

**Case IV:** $3 \cdot 2^{k-2} \leq l < 2^k$

Put $n' = n + (2^k - l) d = 2^k m'$, where $m' = 2^k - l + 1$. If we assume that $A(n, d) \geq 2^{k-1} + 1$, then $A(n', d) \geq 2^{k-2} + 1$. Arguing precisely as in case I (where $n$ is replaced by $n'$), we obtain $m' \equiv 0 \pmod{2^k}$. Hence, we can write $m' = 2^k j$, so that $n' = 2^k j$.

Consider the binary expansions:

\[(n' - d)_{2} = (j - 1)_{2} 1^{k-1} 01^k,\]

\[(n' - 2d)_{2} = (j - 1)_{2} 1^{k-2} 01^k0.\]

The former expansion contains exactly one more occurrence of the block 11, which implies $r_{n' - d} \neq r_{n' - 2d}$. In view of our assumption $r_{n} = r_{n + d} = \cdots = r_{n'} = \cdots$, the only possibility is that $n = n' - d$. But from case I we already know that $A(n', d) \leq 2^{k-1} + 1$ so $A(n, d) \leq 2^{k-1} + 2$, as claimed. 

**Proof of Theorem 1.7** For $k \leq 5$ the claim can be verified numerically, in the same way as in the previous proof. In Table 10 below we provide the initial term of a monochromatic arithmetic progression of postulated length $A_r(d)$ and the length of a prefix of $r$ which is sufficient to be checked for the existence of a longer monochromatic progression.

| $k$ | $d$ | $A_r(d)$ | initial term | prefix length |
|-----|-----|----------|--------------|--------------|
| 1   | 1   | 1        | 7            | $7 \cdot 2^4$ |
| 2   | 3   | 5        | 28           | $7 \cdot 2^5$ |
| 3   | 7   | 9        | 95           | $7 \cdot 2^7$ |
| 4   | 15  | 10       | 39           | $7 \cdot 2^9$ |
| 5   | 31  | 19       | 32           | $7 \cdot 2^{11}$ |

Table 10. The location of monochromatic arithmetic progressions of length $A_r(2^k - 1)$ for $k \leq 5$
In what follows we assume that $k \geq 6$. We now show the existence of a monochromatic arithmetic progression of difference $d$ and desired length $A_\epsilon(d)$ in the general case $k \geq 5$. Regardless of the parity of $k$, let $n = 2^k + d = [10^k]$. For each $i = 0, 1, \ldots, 2^k - 1$ we can write $(n + id)_2 = 10^q(i + 1)d_2$ for some $q \geq 1$ (depending on $i$), which implies $r_{n+id} = r_{id}$. Proposition 5.3 assures that the terms $n, n+d, \ldots, n+(2^k-1)d$ form a monochromatic arithmetic progression of “color” $(k-1) \mod 2$. Because $n + 2^k - 1 = 2^k + 2^k - 1 = 11[10^k]_2$, we also have $e_1(n + 2^k - 1) = k - 1$ and the progression can be prolonged by yet another term. This proves that $A_\epsilon(d) \geq 2^{k-1} + 1$. Moreover, if $k$ is odd we obtain a stronger inequality $A_\epsilon(d) \geq 2^{k-1} + 3$, as the monochromatic progression can be extended backwards, namely to $n - d = 2^k = [10^k]$ and $n - 2d = 2^k = 2^k + 1 = [10^k]_2$.

We now prove that there do not exist longer monochromatic progressions of difference $d$. Just like in the proof of Theorem 1.6, we can restrict our attention to progressions of “color” 0. Let $n \in \mathbb{N}$ be such that $r_n = 0$ and write $n = 2^km + l$, where $m \in \mathbb{N}$ and $0 \leq l < 2^k$. Again, we consider four cases.

**Case I:** $0 \leq l < 2^{k-2}$

We argue that the inequality $A_\epsilon(n, d) \geq 2^{k-1} + 2$ implies that $k$ is odd and also that $A_\epsilon(n, d) \leq 2^{k-1} + 3$, which is sufficient to obtain the result. Put $n' = n + (l + 1)d = 2^m + (2^k - 1)$, where $m = m + l$. By our assumption we get $A(n', d) \geq 2^{k-2} + 1$, and consequently for at least $i = 0, 1, \ldots, 2^{k-2}$ we have
\begin{equation}
0 = r_{n+id} \equiv r_{2^l(m+i)+1} + r_{2^l-i} \quad \text{(mod 2)}.
\end{equation}

By Lemma 5.5 we must have
\[0 \equiv n' + (2^l - 1 - 2^{l-2}) + (2^{l-2} - 1) \equiv m' \quad \text{(mod 2^k)}.\]

Letting $m' = 2^j$, we obtain $n' = 2^k + 2^k - 1$ and it is easy to check that $k$ must be odd, as otherwise we would have $r_{n'} \neq r_{n' - d}$.

At the same time, the number of occurrences of the pattern 11 in the expansions
\begin{align*}
(n' + 2^{k-1})d_2 &= (j)210^k1^{k-1}, \\
(n' + 2^{k-1})d_2 &= (j)210^k1^{k-2}101^{k-2}20,
\end{align*}
differs by 1, so the monochromatic progression cannot exceed $n' + 2^{k-1}d$. Therefore, we obtain $A_\epsilon(n', d) \leq 2^{k-1} + 1$. It remains to show $n \in \{n' - d, n' - 2d\}$, as then $A_\epsilon(n, d) \leq 2 + A_\epsilon(n', d) \leq 2^{k-1} + 3$.

If $n = 0$, then by definition we have $n = n' - d$. If $n > 0$, we have $j > 0$ so the indices $n' - 2d$ and $n' - 3d$ are positive. By analyzing the binary expansions
\begin{align*}
(n' - 2d)_2 &= (j - 1)210^{k-1}1, \\
(n' - 3d)_2 &= (j - 1)210^{k-1}101^{k-1}10,
\end{align*}
we deduce that $r_{n' - 2d} \neq r_{n' - 3d}$. It follows that $n \in \{n' - d, n' - 2d\}$, or else the considered arithmetic progression would not be monochromatic.

**Case II:** $2^{k-2} \leq l < 2^{k-1}$

For at least $i = 0, 1, \ldots, 2^{k-2}$ we have
\[r_{n+id} \equiv r_{e+i} + r_{l-i} \quad \text{(mod 2)}.
\]
As long as $r_{n+id} = 0$, we obtain $r_{e+i} = r_{l-i}$ and Lemma 5.6 implies that $A_\epsilon(n, d) \leq 14 < 2^{k-1}$.

**Case III:** $2^{k-1} \leq l < 3 \cdot 2^{k-2}$

The number $n'' = n + (l - 2^{k-1} + 1)d = 2^k(m + l - 2^{k-1} + 1) + 2^{k-1} - 1$ falls under case II. As a consequence, we get
\[A_\epsilon(n, d) \leq l - 2^{k-1} + 1 + A_\epsilon(n', d) \leq 2^{k-2} + 14 < 2^{k-1}.
\]

**Case IV:** $3 \cdot 2^{k-2} \leq l < 2^k$
For at least \( i = 0, 1, \ldots, 2^k - 2 \) we have
\[
r_{n+id} \equiv r_{2(m+i)+1} + r_{l-i} \pmod{2}.
\]
If we assume that \( A_v(n, d) \geq 2^k - 1 \), then Lemma 5.5 implies that we must have \( m+l \equiv -1 \pmod{2^k} \). Letting \( m = 2^k j - l - 1 \), we obtain \( n = 2^{2k}j - 2^k - (2^k - 1)l = 2^{2k}j - (l + 1)d - 1 \). But then we get
\[
(n + (l + 1 - 2^{k-1})d_2 = (j - 1)2^{10}k^{k-1},
(n + (l + 2 - 2^{k-1})d_2 = (j - 1)2^{10}k^{k-2}101^{k-2}0,
\]
and the former expansion contains exactly one more occurrence of 11. It follows that
\[
A_v(n, d) \leq l + 2 - 2^{k-1} \leq 2^k - 1 + 2 - 2^{k-1} = 2^{k-1} + 1. \quad \square
\]

6. SOME RESULTS AND CONJECTURES FOR GENERAL \( v \)

In this section we investigate the values \( A_v(0, d) \) and \( A_v(d) \) for a general nonempty block \( v \in \{0, 1\}^* \). Based on the results in Section 1 concerning the Thue–Morse and Rudin–Shapiro sequence, one may expect that \( A_v(d) \) (and consequently \( A_v(0, d) \)) is bounded from above by a linear function in \( d \).

We first consider the lengths of monochromatic progressions starting at 0, namely the values \( A_v(0, d) \). A description for \( v = 1 \) is already provided in Theorem 1.2. Also observe that the case when \( v \) is a block of zeros is rather uninteresting. Indeed, if \( v = 0^i \) for some \( i \geq 1 \), then \( e_v(2^i) = 1 + e_v(2^{i-1}d) \) so \( A_v(0, d) \leq 2^i \) for any \( d \geq 1 \).

Therefore, we will focus on blocks \( v \) belonging to the set
\[
V = \{v \in \{0, 1\}^*: |v| \geq 2 \text{ and } v \neq 0^1\}.
\]

Proposition 6.1 below implies that for each fixed \( v \in V \) the values \( A_v(0, d) \) are arbitrarily large. More precisely, this result exhibits infinite families of \( d \) such that \( A_v(0, d) \) is “almost” linear with respect to \( d \).

**Proposition 6.1.** Let \( v \in V \) and write \( v = 0^iu0^j \), where \( i \geq 0, j \geq 0 \), and \( u \) begins and ends with a 1. Then there exist \( x, y \in \{0, 1\}^{|v| - 1} \) such that
\[
|0^ix|_v + |yx|_v + |y0^j|_v \equiv 1 \pmod{2}.
\]

Furthermore, assume that \( (x_{\text{min}}, y_{\text{min}}) \) is the lexicographically minimal solution to (8) and put
\[
C_v = \frac{|x_{\text{min}}|2}{2^{i+|v|-1}},
B_v = |y_{\text{min}}|2 - 2^iC_v.
\]

Fix \( l, m \in \mathbb{N} \), where \( m \) is odd and let \( d_k = 2^{l+j}(2^k + 1)m \) for \( k \in \mathbb{N} \). Then we have
\[
A_v(0, d_k) = \frac{C_v}{2^{l+j}}d_k + O(1),
\]
as \( k \to \infty \), where the implied constant depends only on \( v, l, m \). In particular, if \( m = 1 \), then for all \( k \geq 2^{|v| - 2} \) we have
\[
A_v(0, d_k) = \frac{C_v}{2^j}d_k + B_v.
\]

**Proof.** For (8) to hold it is necessary and sufficient that either precisely one or all three of the following congruences are satisfied:
\[
|yx|_v \equiv 1 \pmod{2},
(9) |0^ix|_v \equiv 1 \pmod{2},
(10) |y0^j|_v \equiv 1 \pmod{2}.
(11) \]
If \( v \) begins with a 0, then for \( x = 0^{i-1}u0^j \) and \( y = 1^{[v]-1} \) out of the three above congruences only (10) holds. If \( v \) ends with a 0, then for \( x = 1^{[v]-1} \) and \( y = 0^i0^0 \) only (11) holds. Finally, if \( v \) both begins and ends with a 1, then (10) and (11) can never be satisfied, and thus we can take \( gx = v0^{[v]-2} \) so that (9) holds.

Fix \( k \) such that \( 2^{k-2_v} > m \). For each positive integer \( n < 2^k/m \) let \( w_n \in \{0, 1\}^k \) satisfy \( |w_n|_2 = mn \). Write \( w_n = x_nz_ny_n \), where \( |x_n| = |y_n| = [v] - 1 \). Then we have

\[
e_v(d_kn) = |0^i w_n w_n^0^{i+j}|_v \equiv |0^i x_n|_v + |y_n x_n|_v + |y_n 0^j|_v \quad \text{(mod 2)}
\]

so \( e_v(d_kn) \equiv 1 \) (mod 2) if and only if \( (x, y) = (x_n, y_n) \) is a solution to (8).

We are now going to show that for any \( x, y \in \{0, 1\}^{[v]-1} \) there exists \( n \) such that \( x_n = x \) and \( y_n = y \). These conditions are equivalent to

(12) \[ 2^{k-[v]+1}|x|_2 \leq mn < 2^{k-[v]+1}|x|_2 + 1 \]

and

(13) \[ mn \equiv |y|_2 \quad \text{(mod } 2^{[v]-1}) \]

respectively. Since

\[
2^{k-[v]+1}|x|_2 + 1 - 2^{k-[v]+1}|x|_2 > 2^{[v]-1}m,
\]

in each congruence class modulo \( 2^{[v]-1} \) there exists \( n \) for which (12) holds. Therefore, we obtain some \( n \) simultaneously satisfying (12) and (13). Moreover, we see that minimal such \( n \) satisfies the stronger inequality

(14) \[ 2^{k-[v]+1}|x|_2 \leq mn < 2^{k-[v]+1}|x|_2 + 2^{[v]-1}m. \]

Now, if \( n = A_v(0, d_k) \), then we must have \( x_n = x_{\min} \) (though not necessarily \( y_n = y_{\min} \)). Plugging \( x = x_{\min} \), \( n = A_v(0, d_k) \), and \( 2^k = 2^{(l_j) + 1}/m - 1 \) into (14), after some manipulation we get the desired asymptotic expression for \( A_v(0, d_k) \).

If \( m = 1 \), we have \( |w_n|_2 = n \), and thus the minimal \( n \) satisfying \( e_v(d_kn) \equiv 1 \) (mod 2) is

\[
A_v(0, d_k) = n = |x_{\min} 0^{k-2[v]+2} y_{\min}|_2 = 2^{k-[v]+1}|x_{\min}|_2 + |y_{\min}|_2 = C' v / 2^i d_k + |y_{\min}|_2 - 2^{i} C_v. \]

Note that taking \( v = 11 \) and \( l = 1 \) we obtain precisely case (ii) of Theorem 1.3. As an immediate corollary, we can determine an infinite set of limit points of the sequence \( A_v(0, d)/d \) for all \( d \geq 0 \).

**Corollary 6.2.** Let \( L_v \) denote the set of limit points of the sequence \( A_v(0, d)/d \) for all \( d \geq 0 \). Then for all \( v \in V \) we have

\[
\left\{ \frac{C_v}{2^i m^2} : l, m \in \mathbb{N}, m \geq 1 \right\} \cup \{0\} \subset L_v.
\]

Based on experimental calculations, we suspect that this is in fact the entire set of limit points. Thus, we pose the following conjecture, where we also include the case \( v = 1 \) with \( C_1 = 1 \).

**Conjecture 1.** For all \( v \in V \cup \{1\} \) we have

\[
L_v = \left\{ \frac{C_v}{2^i m^2} : l, m \in \mathbb{N}, m \geq 1 \right\} \cup \{0\}.
\]

In particular,

\[
C_v = \limsup_{d \to \infty} \frac{A_v(0, d)}{d}.
\]
A typical distribution of the values $\log_2(A_v(0,d)/d)$ is shown in Figure 1 below (with $v = 11$).

As per Theorems 1.2 and 1.5, the upper limit is indeed equal to $C_v$ for at least $v \in \{1, 11\}$. For other blocks $v$, Theorem 1.8 together with the subsequent corollary (given later in this section) imply that the upper limit is finite. Moreover, according to the computations we believe that the choice of $d = d_k = 2^j(2^k + 1)$ yields the maximal values of $A_v(0,d)$ in the sense of the following conjecture. This time the blocks $v \in \{1, 11\}$ need to be excluded due to the special case $d = 2^k - 1$, where the postulated inequality does not work.

**Conjecture 2.** Let $v \in V \setminus \{11\}$ and write $v = w0^j$, where $w$ ends with a 1. Then for all $d \geq 2^{|v|+1}$ we have

$$A_v(0,d) \leq C_v d + B_v.$$  

Moreover, equality occurs if and only if $d = 2^j(2^k + 1)$ for some $k \geq 2|v| - 2$.

Based on the above discussion, it is a matter of interest to determine the constants $C_v$ and (to a lesser extent) $B_v$ for $v \in V$. From the definition we see that they are positive dyadic rational numbers, where additionally $C_v < 1$ (this inequality is strengthened in Proposition 6.4 below). Table 11 provides the values $C_v$ and $B_v$ for all $v \in V$ of length $|v| \leq 4$.

| $v$ | $01$ | $10$ | $11$ | $001$ | $010$ | $011$ | $100$ | $101$ | $110$ | $111$ | $0001$ | $0010$ | $0011$ | $0100$ |
|-----|-----|-----|-----|------|------|------|------|------|------|------|------|------|------|------|
| $C_v$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/4$ | $1/4$ | $1/4$ | $1/4$ | $1/4$ | $1/4$ | $1/8$ | $1/8$ | $1/8$ |
| $B_v$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ |

| $v$ | $0101$ | $0110$ | $0111$ | $1000$ | $1001$ | $1010$ | $1011$ | $1100$ | $1101$ | $1110$ | $1111$ |
|-----|------|------|------|------|------|------|------|------|------|------|------|
| $C_v$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ |
| $B_v$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ | $3/4$ |

**Table 11.** The values of $C_v$ and $B_v$ for $v \in V$ such that $|v| \leq 4$
In the case when \( v \) ends with a 1 the constants \( C_v \) and \( B_v \) can be quickly calculated without directly solving the congruence \([8]\), as the following proposition shows. An explicit description should also be possible to obtain when \( v \) ends with a 0, although additional complications arise in this case.

**Proposition 6.3.** Let \( v \in V \) and assume that \( v \) ends with a 1. Write \( v = ps \), where \( s \) is the lexicographically minimal proper and nonempty suffix of \( v \). Then the lexicographically minimal solution to \([8]\) is given by

\[
\begin{align*}
 x_{\text{min}} &= s0^{|p|-1}, \\
y_{\text{min}} &= \begin{cases} 0^{|s|-1}p & \text{if } p \neq 0, \\ 0^{|v|-2}1 & \text{if } p = 0. \end{cases}
\end{align*}
\]

In particular, we have \( C_v = 2^{-|s|}[s]_2 \) and \( B_v = \max\{|p|,2,1\} - C_v \).

**Proof.** Because \( v \) ends with a 1, the congruence \([11]\) has no solution \( y \in \{0,1\}^{|v|-1} \). Our task reduces to showing that \((x_{\text{min}},y_{\text{min}})\) as in the statement is the lexicographically minimal pair such that precisely one of \([9]\) and \([10]\) holds.

We first show that the pair

\[
(x_1, y_1) = (s0^{|p|-1}, 0^{|s|-1}p)
\]

is the minimal solution to \([9]\). To see this, assume that \( |yx|_v \equiv 1 \pmod{2} \) for some \( x, y \in \{0,1\}^{|v|-1} \) such that \((x,y) \leq (x_1,y_1)\) lexicographically. This implies that \( yx \) contains at least one occurrence of \( v \), hence we can write \( v = p's' \) with \( p', s' \) both nonempty, where \( x = s'q \) and \( y = rp' \) for some \( q, r \in \{0,1\}^* \). We also have \( s'q \leq s0^{|p|-1} \). Since \( s' \) ends with a 1, this implies \( |s'| \leq |s| \). Hence, we also have \( s' \leq s \), which yields \( s' = s \) by the choice of \( s \), and consequently \( x = x_1 \). This also means that \( p' = p \), and thus \( y = y_1 \), since \( y_1 \) is the lexicographically minimal word of length \(|v| - 1\) with suffix \( p \).

If \( v \) begins with a 1 (which falls under the case \( p \neq 0 \)), then \((x_1, y_1)\) is already the minimal solution to \([8]\) as \([10]\) is never satisfied. Hence, assume from now on that \( v \) begins with \( i \geq 1 \) zeros. Then the pair

\[
(x_2, y_2) = (0^{i-1}u, 0^{|v|-1})
\]

is the minimal one satisfying \([10]\).

We always have \((x_1, y_1) \leq (x_2, y_2)\) in the lexicographical sense with equality if and only if \( p \) is a block of zeros. By minimality of \( s \) we must have \( s = 0^{|v|-1}u \) in such a case so the condition \((x_1, y_1) = (x_2, y_2)\) is equivalent to \( p = 0 \). In the case \((x_1, y_1) < (x_2, y_2)\) the pair \((x_1, y_1)\) remains the minimal solution to \([8]\). Otherwise, if \((x_1, y_1) = (x_2, y_2)\), then \((x, y) = (s0^{|p|-1}, 0^{|v|-2}1) = (0^{i-1}u, 0^{|v|-2}1)\) is the successor of \((x_1, y_1)\). Observe that such pair \((x, y)\) still satisfies \([10]\). On the other hand, if \([9]\) held for such \((x, y)\) then we would have an occurrence of \( v \) in the word \( yx = 0^{|v|-2}1s0^{|p|-1} \). Since \( v \neq 1s \) and \( v \) ends with a 1, there would exist a representation \( v = p's' \), where \( s' \) is a proper prefix of \( s \) but this contradicts the minimality of \( s \).

It remains to observe that \( 2^{-|s|[s]_2} = 2^{-|x_{\text{min}}|[x]_2} = C_v \). \(\square\)

In the following proposition we provide a good upper bound on \( C_v \) for all \( v \in V \).

**Proposition 6.4.** Let \( v \in V \) and write \( v = w0^j \), where \( w \) ends with a 1. Then we have

\[
C_v \leq \begin{cases} 1/2 & \text{if } j = 0, \\ 2^{-2j} & \text{if } j \geq 1. \end{cases}
\]
Proof. First, assume that \( v \) ends with a 1. If \( v = 01^l \) or \( v = 1^l \) for some \( l \geq 1 \), then with the notation of Proposition 6.3 we have \( s = 1 \), which gives \( C_v = 1/2 \). Otherwise, \( s \) must begin with a 0 so \( C_v < 1/2 \).

Now, assume that \( v \) ends with a 0, that is, \( j \geq 1 \). Write \( w = 0^j u \), where \( u \) begins with a 1. We are going to show that the congruence (6) has a solution with \( x = 0^{j-1}10^{v_{j-1}-j-1} \), which in turn gives \( [x_{\min}]_2 \leq [x]_2 = 2^{v_{j-1}-1} \), and therefore \( C_v \leq 2^{-2j} \). We consider a few possibilities.

First, if \( 0^j x \) contains \( v \), then we must have \( u = 1 \) and \( i \geq 1 \), which gives \( |0^j x|_v = 1 \). In such a case, for \( y = 1^{v_i-1} \) the pair \((x, y)\) is a solution to (8).

Hence, assume that \( |0^j x|_v = 0 \). If \( 0^{j-1}10^j \) is not a suffix of \( v \), put \( y = 0^{j-1}w \). Then we get \( |y0^j|_v = 1 \) and \( |yx|_v = 0 \) so \((x, y)\) satisfies (8). Otherwise, if \( 0^{j-1}10^j \) is a suffix of \( v \) (which entails \( 2j + 1 \leq |v| \)), write \( v = q0^{j-1}10^j \) and put \( y = 0^{2j-1}q \). Then we get \( |y0^j|_v = 0 \) and \( |yx|_v = 1 \). This also seems to be the case when \( v = 0^j \), unlike for progressions starting at zero. In fact, for \( v = 0 \) we have the equality \( A_0(d) = A_0(d) \) for all \( d \geq 1 \), which immediately gives us a nice characterization by virtue of Theorem 1.3.

Proposition 6.5. For all positive integers \( d \) we have \( A_0(d) = A_0(d) \). Therefore, for any positive integer \( k \), we get

\[
\max_{1 \leq d \leq 2^k} A_0(d) \leq 2^k
\]

and

\[
\max_{1 \leq d \leq 2^k} A_0(d) = A_0(2^k - 1) = \begin{cases} 2k + 4, & \text{if } k \equiv 0 \pmod{2}, \\ 2^k, & \text{otherwise}. \end{cases}
\]

Proof. Fix \( d \geq 1 \) and \( n \in \mathbb{N} \). Observe that the value \( A_0(n, d) \) is finite, since

\[
e_0(2^{(n+1)d} + n) = 1 + e_0(2^{(n+1)d} + n).
\]

For any integer \( l \geq \ell(d) + \ell(n + dA_0(n, d)) \) we also get

\[
A_0(2^l + n, d) = A_0(n, d).
\]

At the same time, for each \( i = 0, 1, \ldots, A_0(d) - 1 \), we have the equality

\[
l + 1 = \ell(2^l + n + id) = e_1(2^l + n + id) + e_0(2^l + n + id).
\]

It follows that \( e_1(2^l + n + id) \mod 2 \) is constant with respect to \( i \), and therefore

\[
A_4(d) \geq A_4(2^l + n, d) \geq A_0(2^l + n, d) = A_0(n, d).
\]

Taking the supremum with respect to \( n \), we obtain \( A_4(d) \geq A_0(d) \). The reverse inequality is obtained in a similar fashion. □

We now proceed to prove Theorem 1.3, which gives a universal upper bound on \( A_0(d) \). In order to obtain this result we will need a technical lemma.

Lemma 6.6. Let \( x, y, v \in \{0, 1\}^* \) be such that \( |x| \geq 1 \), \( |y| \geq 1 \), \( |v| \geq 2 \), and the initial digit of \( x \) is different from the initial digit of \( y \). Then there exists a word \( w \in \{0, 1\}^{|v| - 1} \) such that \( |wx|_v \not\equiv |wy|_v \pmod{2} \).
Proof. If \(|x|_v \not\equiv |y|_v \pmod{2}\), it is sufficient to take any \(w\) such that \(\text{Pref}(v) \cap \text{Suf}(w) = \emptyset\). For example, we can choose \(w = 0^{v_1-1}\), if \(v\) begins with 1, and otherwise \(w = 1^{v_1-1}\). Clearly, \(|wx|_v = |x|_v\) and \(|wy|_v = |y|_v\), so our assertion holds.

The case \(|x|_v \equiv |y|_v \pmod{2}\) is a bit more complicated. Let \(s\) be the longest proper suffix of \(v\) which is simultaneously a prefix of one of \(x, y\). Since \(x, y\) begin with distinct digits, \(s\) is nonempty and is a prefix of precisely one of these words. Without loss of generality assume that this is the case for \(x\). Let \(p \in \{0, 1\}^*\) be such that \(v = ps\) and note that \(p\) is nonempty and has length \(|p| \leq |v| - 1\). We put \(w = zp\) for some \(z \in \{0, 1\}\)\(^{|v|_1 - |p|}\) (to be determined) and make the following claims, from which the result will follow:

- we can choose \(z\) in such a way that \(p\) is the longest word in \(\text{Pref}(v) \cap \text{Suf}(w)\);
- for such choice of \(z\) we have \(|wx|_v = |x|_v + 1\) and \(|wy|_v = |y|_v\).

Let \(P'\) denote the set of these \(p' \in \text{Pref}(v)\) such that \(p\) is a proper suffix of \(p'\). If \(P'\) is empty, then the first claim is already true for any \(z\). Otherwise, for each \(p' \in P'\) there are precisely \(2^{|v|_1 - |p'|}\) ways to choose \(z\) so that \(p' \in \text{Pref}(v) \cap \text{Suf}(w)\). Hence, the number of ways to choose \(z\) so that the first claim holds is

\[
2^{|v|_1 - |p|} - \sum_{p' \in P'} 2^{|v|_1 - |p'|} \geq 2^{|v|_1 - |p|} - \sum_{l=|p|+1}^{|v|_1} 2^{|v|_1 - l} \geq 1.
\]

We move on to the second claim. Clearly, we have an occurrence of \(v\) in \(wz\) which does not already appear \(x\). Moreover, for any such occurrence we have \(v = p's'\) for some nonempty \(p', s'\), where \(p' \in \text{Suf}(w)\) and \(s' \in \text{Pref}(x)\). We have \(|p'| \leq |p|\) by the earlier claim and \(|s'| \leq |s|\) by the initial assumption about \(s\). Hence, \(p' = p, s' = s\), which implies \(|wx|_v = |x|_v + 1\). Along the same lines, if \(wy\) were to contain an additional occurrence of \(v\), then again \(v = p's'\), where \(p' \in \text{Suf}(w)\) and \(s' \in \text{Pref}(y)\) are proper. Similarly, we get \(|p'| \leq |p|\) so \(|s'| \geq |s|\). But the last inequality is a contradiction due to the choice if \(s\).

Proof of Theorem 1.8. Fix \(d \geq 1\) and write \(d = 2^{\nu_2(d)} d'\). Take any \(n \in \mathbb{N}\) and let \(l = n \mod 2^{\nu_2(d)}\). We are going to show that the postulated bound holds for \(A_v(n, d)\). Let \(w \in \{0, 1\}^{|v|_1 - 1}\) be a word obtained from Lemma 6.6 applied to \(x = 0^{\ell(d)}(l)\) and \(y = (d')_2(l)\). We can find an integer \(i \in \{0, 1, \ldots, 2^{|v|_1 + \ell(d') - 1} - 1\}\) such that \(n + id = |wz|_2\) for some \(u \in \{0, 1\}^*\). Consequently, we also have \(n + (i + 1)d = |wy|_2\). Prepending \(u\) with suitably many leading zeros if necessary, by Lemma 6.6 we get the non-congruence

\[
e_v(n + (i + 1)d) \equiv |uw|_v + |wy|_v \not\equiv |uw|_v + |wx|_v \equiv e_v(n + id) \pmod{2}
\]

If \(d < 2^k\), then \(\ell(d) \leq k\) so the conclusion follows from the inequality

\[
A_v(n, d) \leq i + 1 \leq 2^{|v|_1 + \ell(d) - \nu_2(d) - 1} \leq 2^{|v|_1 - \nu_2(d) - 1},
\]

whose right-hand side is independent of \(n\).

As an immediate corollary of the theorem, we obtain that \(A_v(d)\) (and consequently \(A_v(0, d)\)) is bounded by a linear function in \(d\).

Corollary 6.7. For any block of binary digits of length \(|v| \geq 2\) and \(d \geq 1\) odd we have

\[
A_v(0, d) \leq A_v(d) \leq 2^{|v| - \nu_2(d)} d.
\]

The obtained inequalities for \(A_v(d)\) are most likely far from optimal, which is discussed in more detail in the next section.
7. Further problems and conjectures

In this section we suggest possible improvements to the results presented in this paper and pose related problems and conjectures.

We have already conjectured the form of the set of limit points of the sequence \( (A_v(0,d)/d)_{d \geq 1} \) as well as a sharp upper bound for \( A_v(0,d) \). It is natural to consider similar problems for \( A_v(d) \).

**Problem 1.** For nonempty \( v \in \{0,1\}^+ \) find the set of limit points of the sequence \( (A_v(d)/d)_{d \geq 1} \). In particular, compute
\[
E_v = \limsup_{d \to \infty} \frac{A_v(d)}{d}.
\]

**Problem 2.** Does there exist a constant \( F_v \) such that for all sufficiently large \( d \) we have
\[
A_v(d) \leq E_v d + F_v,
\]
where equality occurs for infinitely many \( d \)?

For all \( d \geq 1 \) we know that the following inequality holds:
\[
A_v(d) \leq 2^{|v|} d,
\]
which follows from Corollary 6.7 for \( |v| \geq 2 \) and from Theorem 1.3 and Proposition 6.5 for \( v \in \{0,1\} \) (with a single exception \( A_3(3) = 8 \)). This immediately implies
\[
E_v \leq 2^{|v|}.
\]

Moreover, when \( v \) is not of the form \( 0^i \) for some \( i \geq 2 \), Proposition 6.1 and again Theorem 1.3 also provide a lower bound
\[
C_v \leq E_v.
\]

Experimental computations for \( |v| \leq 3 \) and \( d \leq 2^{12} + 1 \) suggest that half of the time \( C_v \) is in fact the correct value (the values of \( A_v(d) \) are available at [8]). More precisely, it seems that if \( v \) ends with a 1, then the upper limit in question is identical for \( v \) and \( \overline{v} \), and equal to \( C_v \):
\[
E_v = E_{\overline{v}} = C_v.
\]

Furthermore, it seems that the answer to the question in Problem 2 is also affirmative, where the equality \( A_v(d) = E_v d + F_v \) holds for \( d \) of the form \( 2^k - 1 \) for \( v \in \{1,11,0,00\} \), or \( 2^k + 1 \) for other \( v \). In the case of the Thue-Morse and Rudin-Shapiro sequences this observation leads to the following conjectures.

**Conjecture 3.** For all \( d \geq 1 \) we have
\[
A_t(d) \leq d + 5.
\]
Moreover, equality occurs if and only if \( d = 2^k - 1 \) for some even \( k \geq 2 \).

**Conjecture 4.** For all \( d \geq 23 \) we have
\[
A_r(d) \leq \frac{1}{2}(d + 7).
\]
Moreover, equality occurs if and only if \( d = 2^k - 1 \) for some odd \( k \geq 5 \).

Put another way, “large” values \( A_v(d) \) are approximated well by \( A_v(0,d) \). Comparing the results stated in the introduction, this is indeed the case for the words \( v = 1, v = 11 \). Going one step further, one may also expect that the limit points of the sequences \( (A_v(d)/d)_{d \geq 1} \) and \( (A_v(0,d)/d)_{d \geq 1} \) are identical, i.e., Conjecture 1 also applies to \( A_v(d) \).
We note that precise numerical results concerning $A_v(d)$ are computationally expensive to obtain even for small $|v|$. In the final section we discuss how the values $A_v(d)$ can be effectively calculated.

Problems 1 and 2 seem quite hard to solve even for fixed $v$. We believe that it is easier to find an upper bound for $A_v(d)$ in the flavor of Theorem 1.3.

**Problem 3.** Compute $\max_{1 \leq d < 2^k} A_v(d)$ and determine $d$ for which this maximum is reached.

For example, for the Rudin-Shapiro sequence Theorem 1.6 together with Conjecture 4 (if valid) would imply the following.

**Conjecture 5.** For all integers $k \geq 5$ we have
\[
\max_{1 \leq d \leq 2^k - 2} A_v(d) \leq 2^{k-1}
\]
and
\[
A_v(2^k - 1) = \begin{cases} 2^{k-1} + 1, & \text{if } k \equiv 0 \pmod{2}, \\
2^{k-1} + 3, & \text{otherwise.}
\end{cases}
\]

We move on to some miscellaneous problems. The first one is closely related to the equality $A_0(d) = A_4(d)$ of Proposition 6.5 and also the supposed equality (15).

**Problem 4.** Describe the set of $d$ such that $A_v(d) = A_\tau(d)$.

According to our computations, when $v \notin \{00, 11, 000, 111\}$ the equality $A_v(d) = A_\tau(d)$ apparently holds for all $d \geq 1$. One might suspect that this pattern continues, namely $A_v(d) = A_\tau(d)$ for all $d$ when $v$ is not of the form $0^i$ or $1^j$. However, when $v \in \{00, 11, 000, 111\}$, the equality still holds for most $d$.

Next, recall the equality $A_4(2d) = A_4(d)$, which implies $A_0(2d) = A_0(d)$ by Proposition 6.5. It turns out that a similar relation is true for the Rudin-Shapiro sequence.

**Proposition 7.1.** For all $d \geq 1$ we have $A_v(2d) = A_v(d)$.

**Proof.** It is sufficient to show the for any fixed odd $d$ and $j \geq 1$ we have $A_v(2^j d) = A_v(d)$. First, for any $n, i \in \mathbb{N}$ we have $r_{n+i} = r_{2^n+i} = r_{2^j d}$, which yields $A_v(2^j d) \leq A_v(d)$.

Conversely, take any $n \in \mathbb{N}$ and write $n = 2^i m + l$, where $m, l \in \mathbb{N}$ and $l < 2^i$. Then for any $i \in \mathbb{N}$ we have
\[
r_{n+i} \equiv \begin{cases} r_l + r_{m+i} \pmod{2} & \text{if } l < 2^{i-1}, \\
raj + sm_{m+i} \pmod{2} & \text{if } l \geq 2^{i-1}
\end{cases}
\]
(recall that $s = (r_{2^n+i})_{n \geq 0}$). Therefore,
\[
A_v(n, 2^j d) \leq \max\{A_v(m, d), A_v(m, d)\} \leq \max\{A_v(d), A_v(d)\} = A_v(d),
\]
where we used $A_v(d) = A_v(d)$, a consequence of Lemma 5.5. The result follows by taking the supremum over $n$.

It seems interesting to ask whether a similar relation holds for any $v$.

**Problem 5.** Describe the set of $d$ such that $A_v(2d) = A_v(d)$.

Based on the calculations, $v = 01, 10$ are the only other blocks of length at most 3 such that the equality holds for all $d$. Nevertheless, in all cases we have $A_v(2d) \geq A_v(d)$ with equality for most $d$.

The same question can be asked for the values $A_v(0, d)$.

**Problem 6.** For $v$ ending with a 0 describe the set of $d$ such that $A_v(0, 2d) = A_v(0, d)$. 


We exclude $v$ ending with a 1 as then $A_v(0, 2d) = A_v(0, d)$ follows immediately from the fact that $e_v(2n) = e_v(n)$. The case when $v$ ends with a 0 seems to be nontrivial and the considered equality holds for most but not all $d$.

Most of the results in this paper concern the growth of $A_v(0, d)$ and $A_v(d)$ with particular attention paid to specific large terms. However, it is also interesting to look at the distribution of all the values $A_v(0, d)$ and $A_v(d)$.

**Problem 7.** Describe the distribution of the values $A_v(0, d)$ and $A_v(d)$.

Here it is easy to see that the similarity in behavior of large terms of the sequences $(A_v(0, d))_{d \geq 1}$ and $(A_v(d))_{d \geq 1}$ does not carry over to the whole sequences. In Figures 2 and 3 below we provide histograms for the example sequences $(A_r(0, d))_{d \geq 1}$ and $(A_r(d))_{d \geq 1}$.

**Figure 2.** The distribution of $A_r(0, d)$

**Figure 3.** The distribution of $A_r(0, d)$

The distribution of the values $A_r(0, d)$ resembles the geometric distribution with success probability $1/2$ (after proper scaling). This phenomenon may be explained by a very simple heuristic. For roughly half values $d$ we have $r_d = 1$, and thus $A_r(0, d) = 1$. For the remaining values $d$ around half of the time we should have $r_{3d} = 1$, implying $A_r(0, d) = 3$, and so on. On the other hand, the distribution of $A_r(d)$ seems approximately normal.

A related problem is the following.
Problem 8. Determine the values attained by the sequences \((A_v(0, d))_{d \geq 1}\) and \((A_v(d))_{d \geq 1}\).

It appears that \(A_v(0, d)\) takes on all positive odd values and finitely many even values (none if \(v\) ends with a 1). At the same time, \(A_v(d)\) attains almost all integer values in the interval \([10, 30]\) for each considered \(v\). We expect that the set of all values of \(A_v(d)\) may have asymptotic density 1.

Since the sequences \((A_v(0, d))_{d \geq 1}\) and \((A_v(d))_{d \geq 1}\) describe binary expansions of integers one may wonder whether they are 2-regular (in the sense of Allouche and Shallit [2]). We expect that the general answer is negative.

Problem 9. Are the sequences \((A_v(0, d))_{d \geq 1}\) and \((A_v(d))_{d \geq 1}\) 2-regular?

Finally, we can generalize our setting and study the values \(e_{b,v}(n)\) modulo \(m\), where \(e_{b,v}(n)\) counts the occurrences of a block \(v\) in the base-\(b\) expansion of \(n\), and \(m \geq 2\) is an integer. It is interesting to ask whether similar results hold for monochromatic arithmetic progressions in \(m\)-colorings of \(N\) induced by these sequences.

Problem 10. Generalize the study to other bases and moduli.

8. The Calculation of \(A_v(d)\)

In this section we discuss how the values \(A_v(d)\) can be calculated for a general \(v\). Let us denote \(g_v(n) = e_v(n) \mod 2\) and \(g_v = (g_v(n))_{n \geq 0}\). Our approach is an extension of the method used to calculate particular values of \(A_v(d)\) in the proofs of Theorems 1.6 and 1.7. There are certain differences depending on whether or not \(v\) contains a 1. We now describe the case when \(v\) contains a 1 and at the end of the section highlight the modifications that need to be made when \(v = 0^r\).

The first step is to determine a substitution such that \(g_v\) is its fixed point. To this end, write \(g_v\) as a concatenation of \(2^{(|v|-1)}\)-aligned blocks:

\[
g_v = G_v(0)G_v(1)G_v(2) \cdots ,
\]

where \(G_v(n) = g_v(2^{(|v|-1)}n) \cdots g_v(2^{(|v|-1)}(n+1) - 1)\) for each \(n \in \mathbb{N}\). Also let

\[
\Sigma_v = \{G_v(n) : n \in \mathbb{N}\}.
\]

In this way we can identify \(g_v\) with the infinite word \(G_v = (G_v(n))_{n \geq 0}\) over the alphabet \(\Sigma_v\). In the following proposition we give the desired description.

Proposition 8.1. For any \(v \in \{0, 1\}^*\) which contains a 1 we have the following.

(i) \(\Sigma_v = \{G_v(l), G_v(l) : 0 \leq l < 2^{(|v|-1)}\}\).

(ii) The word \(g_v\) is a fixed point of a substitution \(\gamma_v : \Sigma_v^* \to \Sigma_v^*\) defined by

\[
\gamma_v(G_v(n)) = G_v(2n)G_v(2n + 1).
\]

(iii) \(\gamma_v(G) = \gamma_v(G)\) for any \(G \in \Sigma_v\).

Proof. We start with (i). Choose \(n \in \mathbb{N}\) and write \(n = 2^{(|v|-1)}m + l\), where \(0 \leq l < 2^{(|v|-1)}\). Then for each \(j = 0, 1, \ldots, 2^{(|v|-1)} - 1\) we have

\[
g_v(2^{(|v|-1)}n + j) \equiv g_v(n) + g_v(2^{(|v|-1)}l + j) \pmod{2}.
\]

Therefore, either \(G_v(n) = G_v(l)\) or \(G_v(n) = G_v(l)\).

The other inclusion follows immediately for the blocks \(G_l\). To show that \(G_v(l) \in \Sigma_v\), it is sufficient to take any \(n \in \mathbb{N}\) such that \(n \equiv l \pmod{2^{(|v|-1)}}\) and \(g_v(n) = 1\), and use (16). A suitable choice is for example \(n = 2^{(|v|-1)}l[v] + l = [v0^{(|v|-1)}l]_2\) if \(v\) begins with a 1, and \(n = 2^{(|v|-1)}l[v] + 2^{(|v|-1)}(2^{(|v|-1)} - 1) + l = [v1^{(|v|-1)}l]_2\) otherwise.
In order to prove (ii) note that for all $j = 0, 1, \ldots, 2^{|v|} - 1$ we have
\begin{align}
  g_v(2^{|v|}n + 2j) &\equiv g_v(2^{|v|}n + j) + g_v(2j) \pmod{2}, \\
  g_v(2^{|v|}n + 2j + 1) &\equiv g_v(2^{|v|}n + j) + g_v(2j + 1) \pmod{2}.
\end{align}
If we assume that $G_v(n) = G_v(m)$, then the right-hand side of (17) and (18) remains the same after replacing $n$ by $m$, which implies $G_v(2n)G_v(2n + 1) = G_v(2m)G_v(2m + 1)$.

Part (iii) follows directly from the congruences (17) and (18).

Having this result, the substitution $\gamma_v$ can be effectively obtained by generating sufficiently many terms of $g_v$.

Now, a monochromatic arithmetic progression in $g_v$ of length $l$ and difference $d$ is contained in a subword of length $(l - 1)d + 1$. At the same time, for $|v| \geq 2$ and all $d \geq 1$ we have the upper bound
\begin{equation}
  A_v(d) \leq 2^{l(d) + |v| - \nu_2(d) - 1},
\end{equation}
which follows from Theorem 1.8. The same inequality also holds when $v = 1, 0$ and $d \neq 2^k - 1$ by Theorem 1.3 and the relations $A_k(2d) = A_k(d), A_0(d) = A_k(d)$. Since the values $A_k(2^k - 1)$ are already known, we omit this case in the further considerations. As a result, the progression of maximal length has to be contained in a subword of $g_v$ of length
\begin{equation}
  (2^{l(d) + |v| - \nu_2(d) - 1} - 1)d + 1 < 2^{2^{l(d) + |v| - \nu_2(d) - 1}}.
\end{equation}
Further, any such subword is contained in a word of the form $\gamma_v(G_vG_{n+1})$ for some $n \in \mathbb{N}$, where $s = 2l(d) - \nu_2(d)$. We state this observation as a proposition.

**Proposition 8.2.** Let $v \in \{0, 1\}^*$ contain a 1. Then for each $d \geq 1$ (except for $d = 2^k - 1$ in the case $v = 1$) the value $A_v(d)$ is equal to the maximal length of a monochromatic arithmetic progression of difference $d$ in the words $\gamma_v(GH)$, where $GH \in \text{Sub}(G_v) \cap \Sigma^2$ and $s = 2l(d) - \nu_2(d)$.

This essentially reduces our task to finding all subwords of length 2 of a fixed point of a 2-uniform morphism. Recall that a substitution $\varphi : \Sigma^* \to \Sigma^*$ is a 2-uniform morphism if $|\varphi(a)| = 2$ for all $a \in \Sigma$. The following proposition is rather standard, however we include a proof for completeness.

**Proposition 8.3.** Let $\Sigma$ be a finite alphabet. Let $a = (a_n)_{n \geq 0}$ be a fixed point of a 2-uniform morphism $\varphi : \Sigma^* \to \Sigma^*$. For $k \geq 1$ define
\begin{equation}
  S_k = \text{Sub}(\varphi^k(a_0)) \cap \Sigma^2
\end{equation}
and assume that $S_{K+1} = S_K$ for some $K$. Then we have
\begin{equation}
  \text{Sub}(a) \cap \Sigma^2 = S_K.
\end{equation}

**Proof.** Since the words $\varphi^k(a_0)$ tend to $a$, we have
\begin{equation}
  \text{Sub}(a) \cap \Sigma^2 = \bigcup_{k=1}^{\infty} S_k.
\end{equation}
Since also $S_k \subset S_{k+1}$, it is sufficient to prove that $S_{K+n+1} = S_{K+n}$ for all $n \in \mathbb{N}$. The case $n = 1$ is our assumption. Now, suppose that the claim holds for some $n$ and choose any word $w \in S_{K+n+2}$. Then $w$ is a subword of $\varphi(u)$ for some $u \in S_{K+n+1}$. But by the inductive assumption we have $u \in S_{K+n}$ and thus also $w \in S_{K+n+1}$. The result follows. □
Since there are only finitely many distinct words of length 2 we eventually get $S_{n+1} = S_k$. In our case this means that the subwords of $G_v$ of length 2 can be effectively computed by considering $\gamma_v^S(G_v)$ for subsequent $k$.

The approach presented above can be improved in a few ways from the computational point of view. Firstly, the amount calculations required to compute $A_v(d)$ according to the method outlined above depends on the length of the binary expansion of $d$, regardless of the actual value $A_v(d)$. However, based on smaller-scale calculations, one can observe that most values $A_v(d)$ are small. Therefore, it is beneficial to first filter out $d$ yielding these values. In general, assume that $f: \mathbb{N}_+ \to \mathbb{R}_+$ is some function and we would like to compute the values $A_v(d)$ satisfying the inequality $A_v(d) \leq f(d)$. By the same calculation as before, a monochromatic progression of difference $d$ and length greater than $f(d)$ would have to appear in some block $\gamma_v^S(GH)$, where $GH \in \text{Sub}(G_v) \cap \Sigma_v^2$ and $s = |\log_2(f(d))| - |v| + 1$. If for given $d$ this is not the case, then we obtain the value $A_v(d)$. This process can be repeated for the remaining $d$ using another function $f_1$, and so on. As the final bound one can use the general inequality \[19\]. In our calculations we have used $f(d) = 20$, $f_1(d) = 35$, $f_2(d) = d$ (or $f_2(d) = d + 5$ for the Thue-Morse sequence) with considerable improvement in calculation time.

Secondly, parts (i) and (iii) of Proposition 8.4 imply that $GH \in \text{Sub}(G_v) \cap \Sigma_v^2$ if and only if $\overline{GH} \in \text{Sub}(G_v) \cap \Sigma_v^2$. Since negation only affects the color but not the length of monochromatic arithmetic progressions, it is sufficient to search only one of $\gamma_v^S(GH)$, $\gamma_v^S(\overline{GH})$ for each $GH \in \text{Sub}(G_v) \cap \Sigma_v^2$. This essentially cuts the task in half.

Finally, in the case of the Thue-Morse sequence we can restrict our attention to $d$ odd by the equality $A_v(2d) = A_v(d)$. The same reduction can be done for the Rudin-Shapiro sequence by Proposition 7.1 and possibly other sequences $g_v$ (see Problem 5).

To conclude this section, we discuss the (slight) modifications that need to be made when $v = 0^i$ for some $i \geq 1$. In particular, we see that Proposition 8.4(ii) does not hold for $v = 0^i$, since for example $G_v(0) = G_v(1) = 0^{2^i-1}$ but $G_v(2 \cdot 0) = 0^{2^i-1} \neq 10^{2^i-1} = G_v(2 \cdot 1)$. In order to deal with this, we define $\gamma_v$ on the set $\Sigma_v$ with an additional element $X$, which is later mapped to the block $G_v(0)$.

**Proposition 8.4.** If $v = 0^i$ for some $i \geq 1$, then we have the following.

(i) $\Sigma_v = \{G_v(l), G_v(\overline{l}) : 0 \leq l \leq 2^{|v|}-1\}$.

(ii) Define the substitution $\gamma_v: (\Sigma_v \cup \{X\})^* \to (\Sigma_v \cup \{X\})^*$ by

$$
\gamma_v(X) = XG_v(1),
\gamma_v(G_v(n)) = G_v(2n)G_v(2n + 1), \quad n \geq 1.
$$

The word $g_v$ is the image of the fixed point of $\gamma_v$ starting with $X$ under the map $X \mapsto G_v(0)$ and $G \mapsto G$ for $G \in \Sigma_v$.

(iii) $\gamma_v(\overline{G}) = \gamma_v(G)$ for any $G \in \Sigma_v$.

**Proof.** The proof is similar as in Proposition 7.1 so we primarily highlight the differences. In part (i) we again take $n \in \mathbb{N}$ and write $n = 2^{|v|}-1m + l$, where $0 \leq l < 2^{|v|}-1$. If $l \neq 0$, then the congruence \[16\] holds and $G_v(n)$ is equal to one of $G_v(l), G_v(\overline{l})$. If $l = 0$ and $m \neq 0$, then we instead get for $j = 0, 1, \ldots 2^{|v|}$ the congruence

$$
g_v(2^{|v|}-1n + j) \equiv g_v(n) + g_v(2^{|v|}-2 + j).
$$

It follows that $G_v(n)$ is equal to one of $G_v(2^{|v|}-1), G_v(2^{|v|}-1)$. In the remaining case $n = 0$ the block $G_v(0)$ trivially belongs to the set on the right-hand side.

Conversely, we need to show that the blocks $G_v(\overline{l})$ belong to $\Sigma_v$. If $0 < l < 2^{|v|}-1$, then we have $g_v(n) = 1$ for $n = 2^{|v|} + 2^{|v|}-1 + l = [1v1(l)_{2}]_2$ and by \[16\] we get...
One can also check that $G_v(0) = 2^{|v|} = G_v(2^{|v|} + 1)$ and $G_v(2^{|v|} - 1) = G_v(2^{|v|})$, thus both blocks also belong to $\Sigma_v$.

In part (ii), if $n \geq 1$ the congruence (17) does not hold for $j = 0$, and we get two cases:

$$g_v(2^{|v|} n + 2j) \equiv \begin{cases} g_v(2^{|v|} n - 1) + 1 \pmod{2} & \text{if } j = 0, \\ g_v(2^{|v|} n + j) \pmod{2} & \text{if } 1 \leq j < 2^{|v|} - 1. \end{cases}$$

At the same time, the congruence (18) remains identical. In any case, we deduce that $\gamma_v(G)$ is well-defined for $G \in \Sigma_v$. It is also clear that $g_v$ is the image of the fixed point of $\gamma_v$ under the map given in the statement. □

The remaining part of the procedure for $v = 0^i$ stays mostly the same as before. The only difference is that after finding all length 2 subwords $GH$ in the fixed point of $\gamma_v$ and calculating $\gamma_v^s(GH)$ for suitable $s$, one needs to apply the map $X \mapsto G_v(0) = 2^i$ to the word $\gamma_v^s(XG_v(1))$. The improvements to the method discussed earlier apply as well. We also note that the case $v = 0$ does not need to be considered separately due to the equality $A_0(d) = A_1(d)$ of Proposition 6.5.

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