ON ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DEGENERATE VISCOSITY AND CONSTANT STATE AT FAR FIELDS

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Abstract
In this paper, we are concerned with the Cauchy problem for one-dimensional compressible isentropic Navier-Stokes equations with density-dependent viscosity \( \mu(\rho) = \rho^\alpha (\alpha > 0) \) and pressure \( P(\rho) = \rho^\gamma (\gamma > 1) \). We will establish the global existence and asymptotic behavior of weak solutions for any \( \alpha > 0 \) and \( \gamma > 1 \) under the assumption that the density function keeps a constant state at far fields. This enlarges the ranges of \( \alpha \) and \( \gamma \) and improves the previous results presented by Jiu and Xin. As a result, in the case that \( 0 < \alpha < \frac{1}{2} \), we obtain the large time behavior of the strong solution obtained by Mellet and Vasseur when the solution has a lower bound (no vacuum).

1. INTRODUCTION
Consider the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients
\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x &= (\mu(\rho)u)_x.
\end{aligned}
\] (1.1)

Here, \( \rho(x,t) \) and \( u(x,t) \) stand for the fluid density and velocity respectively. For simplicity, the pressure term \( P(\rho) \) and the viscosity coefficient \( \mu(\rho) \) are assumed to be
\[
P(\rho) = \rho^\gamma (\gamma > 1), \quad \mu(\rho) = \rho^\alpha.
\] (1.2)
The initial data is imposed as
\[
(\rho, \rho u)|_{t=0} = (\rho_0, m_0).
\] (1.3)

When the viscosity \( \mu(\rho) \) is a positive constant, there has been a lot of investigations on the compressible Navier-Stokes equations, for smooth initial data or

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discontinuous initial data, one-dimensional or multidimensional problems (see [22, 31, 11, 29, 26, 6, 21, 8], and the references therein). However, the studies in Hoff & Serre [14], Xin [35], Liu, Xin & Yang [25] show that the compressible Navier-Stokes equations with constant viscosity coefficients behave singularly in the presence of vacuum. In [25], Liu, Xin and Yang introduced the modified compressible Navier-Stokes equations with density-dependent viscosity coefficients for isentropic fluids. Actually, in deriving the compressible Navier-Stokes equations from the Boltzmann equations by the Chapman-Enskog expansions, the viscosity depends on the temperature, and correspondingly depends on the density for isentropic cases. Meanwhile, an one-dimensional viscous Saint-Venant system for shallow water, which was derived rigorously by Gerbeau-Perthame [9], is expressed exactly as (1.1) with \( \mu(\rho) = \rho \) and \( \gamma = 2 \).

When the viscosity \( \mu(\rho) \) depends on the density, there are a large number of literatures on mathematical studies on (1.1). One-dimensional case is referred to [25, 16, 30, 17, 39, 38, 33, 23, 18, 7] and references therein. In [18], Jiu and Xin studied the global existence and large time behavior of weak solutions of the Cauchy problem to (1.1) with \( \mu(\rho) = \rho^\alpha (\alpha > 1/2) \) under some restrictions of \( \alpha \) and \( \gamma \). The vacuum or non-vacuum constant states at far fields are permitted in [18]. Recently, based on [18], Guo etc [34] studied the global existence and large time behavior of weak solutions of the Cauchy problem to (1.1) under the assumptions of \( 0 < \alpha < \frac{1}{2} \) and \( \rho_0 \in L^1(R) \). If the far fields hold different ends, the asymptotic stability of rarefaction waves for the compressible isentropic Navier-Stokes equations (1.1) with \( \mu(\rho) = \rho^\alpha (\alpha > \frac{1}{2}) \) was studied by Jiu, Wang and Xin in [19] in which the rarefaction wave itself has no vacuum, and in [20] in which the rarefaction wave connects with the vacuum. In [28], Mellet and Vasseur showed that if \( 0 < \alpha < \frac{1}{2} \) and the initial datum are regular with a positive lower bound (no vacuum), there exists a global and unique strong solution of the Cauchy problem to (1.1). However, the a priori estimates obtained in [28] depends on the time interval and hence does not yield the time-asymptotic behavior of the solutions.

In this paper, we will study the global existence and asymptotic behavior of weak solutions for any \( \alpha > 0 \) and \( \gamma > 1 \) under the assumption that the density function keeps a constant state at far fields. We will apply the similar approaches as in [19] to obtain an uniform (in time) entropy estimate (see Section 3). This type of entropy estimate was observed first in [21] for the one-dimensional case and later established in [1, 2, 4] for multi-dimensional cases. The key points in our proof are to obtain the uniform upper bound of the density and to obtain the lower bound of the density of the approximate solutions by using the uniform entropy estimate. To do that, different ranges of \( \alpha \) and \( \gamma \) will be discussed respectively and the elaborate estimates will be given. Our results relax the restrictions of \( \alpha \) and \( \gamma \) presented in [18]. In the case that \( 0 < \alpha < \frac{1}{2} \), we obtain the large time asymptotic behavior of the strong solution obtained by Mellet and Vasseur when the solution has a lower bound (no vacuum). Moreover, in the case that \( \alpha > \frac{1}{2} \), the vacuum is permitted and we study the existence and large time behavior in the framework of weak solutions.
The organization of the paper is as follows. In Section 2, we state some prelimi-
naries and main results. In Section 3, we give proofs of uniform entr opy estimates.
Based on these, lower and upper bounds of the density to the approximate solutions
will be shown. In Section 4, we give a sketch of proof of main results.

2. Preliminaries and Main results

We first give the assumptions of the initial data and the definition of weak solutions.

Define the pressure potential \( \Psi(\rho, \bar{\rho}) \) as

\[
\Psi(\rho, \bar{\rho}) = \int_{\rho}^{\bar{\rho}} \frac{p(s) - p(\bar{\rho})}{s^2} ds = \frac{1}{(\gamma - 1)\rho} [\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma - 1}(\rho - \bar{\rho})].
\]

(2.1)

We assume that there exists a constant \( \bar{\rho} \geq 0 \) such that

\[
\rho_0 \Psi(\rho_0, \bar{\rho}) \in L^1(\mathbb{R}) \quad \text{if} \quad \bar{\rho} > 0,
\]

\[
\rho_0 \in L^1(\mathbb{R}) \quad \text{if} \quad \bar{\rho} = 0.
\]

(2.2)

Moreover, we assume that the initial data satisfy

\[
\begin{align*}
\rho_0(x) > 0 & \quad \text{if} \quad 0 < \alpha \leq \frac{1}{2}, \quad \rho_0(x) \geq 0 & \quad \text{if} \quad \alpha > \frac{1}{2}; \\
(\rho_0^{\alpha - 1/2})_x & \in L^2(\mathbb{R}) \quad \text{if} \quad \alpha > 0 \quad \text{and} \quad \alpha \neq \frac{1}{2}, \\
(\log \rho_0)_x & \in L^2(\mathbb{R}) \quad \text{if} \quad \alpha = \frac{1}{2}; \\
\frac{\rho_0}{|m_0|^{2+\delta}} & \in L^1(\mathbb{R}), \quad \frac{|m_0|^{2+\delta}}{|\rho_0^{1+\delta}} \in L^1(\mathbb{R}),
\end{align*}
\]

(2.3)

where \( 0 < \delta < 1 \) is any fixed number which may be small.

**Remark 2.1.** By assumptions (2.2)-(2.3), the initial data \( \rho_0(x) \) is actually continuous and bounded. And in the case that \( 0 < \alpha \leq 1/2 \), the restriction \( \rho_0(x) > 0, x \in \mathbb{R} \) can be derived from other conditions of (2.2)-(2.3). However, in this case, the initial density still can appear vacuum at infinity, i.e., \( \lim_{|x| \to \infty} \rho_0(x) = 0 \).

The weak solutions to (1.1)-(1.3) with the far fields \( \bar{\rho} \geq 0 \) are defined as:

**Definition 2.1.** For any \( T > 0 \), a pair \( (\rho, u) \) is said to be a weak solution to (1.1)-(1.3) if

(1) \( \rho \geq 0 \ a.e., \) and

\[
\rho - \bar{\rho} \in L^\infty(0, T; L^\infty(\mathbb{R})),
\]

\[
\rho \Psi(\rho, \bar{\rho}) \in L^1(\mathbb{R}), \sqrt{\rho} u \in L^\infty(0, T; L^2(\mathbb{R})),
\]

\[
(\rho^{\alpha - \frac{1}{2}})_x \in L^\infty(0, T; L^2(\mathbb{R})) \quad \text{if} \quad 0 < \alpha \neq \frac{1}{2},
\]

(2.4)

\[
(\log \rho)_x \in L^\infty(0, T; L^2(\mathbb{R})) \quad \text{if} \quad \alpha = \frac{1}{2};
\]

(2.5)

(2) For any \( t_2 \geq t_1 \geq 0 \) and \( \xi \in C^1(\mathbb{R} \times [t_1, t_2]) \), the mass equation (1.1) holds in the following sense.
\[
\int_R \rho \zeta dx|^{t_2}_{t_1} = \int_{t_1}^{t_2} \int_R (\rho \zeta_t + \rho u \zeta_x)dxdt;
\]
(2.6)

(3) For any \( \psi \in C_0^\infty(R \times [0,T]) \), the momentum equation holds in the following sense.

\[
\int_R m_0 \psi(0,\cdot)dx + \int_{t_1}^{t_2} \int_R \left[ \sqrt{\rho} (\sqrt{\rho} u) \psi_t + (\sqrt{\rho} u)^2 + \rho^\gamma \psi_x \right] dxdt + \langle \rho^\alpha u_x, \psi_x \rangle = 0;
\]
(2.7)

where the diffusion term makes sense in the following equalities:

when \( 0 < \alpha \neq \frac{1}{2} \),
\[
\langle \rho^\alpha u_x, \psi_x \rangle = -\int_0^T \int_R \rho^{\alpha - \frac{1}{2}} \sqrt{\rho u} \psi_x dxdt - \frac{2\alpha}{2\alpha - 1} \int_0^T \int_R (\rho^{\alpha - \frac{1}{2}})_x \sqrt{\rho} u \psi dxdt,
\]
(2.8)

when \( \alpha = \frac{1}{2} \),
\[
\langle \rho^{\frac{1}{2}} u_x, \psi_x \rangle = -\int_0^T \int_R \sqrt{\rho u} \psi_x dxdt - \frac{1}{2} \int_0^T \int_R (\log \rho)_x \sqrt{\rho} u \psi dxdt.
\]
(2.9)

Before we state our main results, we review the existence results obtained in [18] as follows.

**Proposition 2.1.** ([18], \( \rho = 0 \)) Let \( \gamma > 1 \) and \( \alpha > \frac{1}{2} \). Suppose that (2.2) and (2.3) hold. If \( \bar{\rho} = 0 \), then the Cauchy problem (1.1) - (1.3) admits a global weak solution \((\rho(x,t), u(x,t))\) satisfying

\[
\rho \in C(R \times (0,T)).
\]
(2.10)

Moreover, one has

\[
\sup_{t \in [0,T]} \int_R \rho dx + \max_{(x,t) \in R \times [0,T]} \rho \leq C,
\]
(2.11)

\[
\sup_{t \in [0,T]} \int_R (|\sqrt{\rho} u|^2 + (\rho^{\alpha - \frac{1}{2}})_x^2 + \frac{1}{\gamma - 1} \rho^\gamma dx + \int_0^T \int_R [(\rho^{\frac{\gamma + \alpha - 1}{2}})_x]^2 dxdt \leq C.
\]
(2.12)

where \( C \) is an absolute constant.

**Proposition 2.2.** ([18], \( \rho > 0 \)) Let \( \alpha \) and \( \gamma \) satisfy

\[
\gamma > 1, \quad \frac{1}{2} < \alpha \leq \frac{3}{2} \quad \text{or} \quad \gamma \geq 2\alpha - 1, \quad \alpha > \frac{3}{2}.
\]
(2.13)

Suppose that (2.2) and (2.3) hold. If \( \bar{\rho} > 0 \), then the Cauchy problem (1.1) - (1.3) admits a global weak solution \((\rho(x,t), u(x,t))\) satisfying

\[
\rho \in C(R \times (0,T)).
\]
(2.14)
Moreover, one has

\[
\sup_{t \in [0,T]} \int_R |\rho - \bar{\rho}|^2 dx + \max_{(x,t) \in R \times [0,T]} \rho \leq C, \quad (2.15)
\]
\[
\sup_{t \in [0,T]} \int_R \left( |\sqrt{\rho} u|^2 + (\rho^{\alpha - \frac{1}{2}})_x^2 + \frac{1}{\gamma - 1} (\rho^\gamma - (\bar{\rho})^\gamma - \gamma (\rho)_{\gamma - 1} (\rho - \bar{\rho})) \right) dx \quad (2.16)
\]
\[
+ \int_0^T \int_R \left( (\rho^{\frac{\alpha + 1}{2}})_x^2 + \Lambda(x,t)^2 \right) dx dt \leq C, \quad (2.17)
\]

where \(C\) is an absolute constant.

Remark 2.2. It should be noted that in Proposition 2.2, the restrictions of \(\gamma\) and \(\alpha\) (2.13) are different from ones presented in Theorem 2.2 in [18]. This is due to that in [18], instead of \(\rho - \bar{\rho} \in L^\infty(0,\infty; L^1(R))\), one should use the fact that \(\rho - \bar{\rho} \in L^\infty(0,\infty; L^2(R))\) which follows from the estimate of (2.16).

Our main results are as follows.

**Theorem 2.1.** Let \(\gamma > 1\), \(0 < \alpha \leq \frac{1}{2}\) and assume that (2.2) – (2.3) hold. Then for any \(T > 0\), the Cauchy problem (1.1) – (1.3) admits a global weak solution \((\rho(x,t), u(x,t))\) in \(R \times (0,T)\) satisfying

1. \(\rho \in C(R \times (0,T)), \quad \rho(x,t) \geq 0, \quad (x,t) \in R \times (0,T)\); \quad (2.18)

2. \(\sup_{t \in [0,T]} \int_R |\rho - \bar{\rho}|^2 dx + \max_{(x,t) \in R \times [0,T]} \rho \leq C\), if \(\bar{\rho} > 0\), \quad (2.19)

3. \(\sup_{t \in [0,T]} \int_R \rho dx + \max_{(x,t) \in R \times [0,T]} \rho \leq C\), if \(\bar{\rho} = 0\); \quad (2.20)

3. When \(0 < \alpha < \frac{1}{2}\), one has

\[
\sup_{t \in [0,T]} \int_R \left( |\sqrt{\rho} u|^2 + (\rho^{\alpha - 1/2})_x^2 + \frac{1}{\gamma - 1} (\rho^\gamma - (\bar{\rho})^\gamma - \gamma (\rho)_{\gamma - 1} (\rho - \bar{\rho})) \right) dx
\]
\[
+ \int_0^T \int_R \left( (\rho^{\frac{\alpha + 1}{2}})_x^2 + \Lambda(x,t)^2 \right) dx dt \leq C, \quad (2.21)
\]

where \(C\) is an absolute constant which only depends on the initial data, and \(\Lambda(x,t) \in L^2(R \times (0,T))\) is a function which satisfies

\[
\int_0^T \int_R \Lambda \psi dx dt = - \int_0^T \int_R \rho^{\alpha - 1/2} \sqrt{\rho} \psi_x dx dt - \frac{2\alpha}{2\alpha - 1} \int_0^T \int_R \rho^{\alpha - 1/2} \sqrt{\rho} u \psi dx dt;
\]
When $\alpha = \frac{1}{2}$, one has
\[
\sup_{t \in [0, T]} \int_R \left( (\sqrt{\rho} u)^2 + (\log \rho)_x^2 + \frac{1}{\gamma - 1} (\rho^\gamma - (\bar{\rho})^\gamma - \gamma(\bar{\rho})^{\gamma - 1} (\rho - \bar{\rho})) \right) dx \\
+ \int_0^T \int_R \left( \left( (\rho^{\frac{\gamma - 1}{2}})_x^2 + \Lambda(x,t)^2 \right) \right) dx dt \leq C,
\]
where $C$ is an absolute constant which just depends on the initial data, and $\Lambda(x,t) \in L^2(R \times (0,T))$ is a function which satisfies
\[
\int_0^T \int_R \Lambda \psi dx dt = - \int_0^T \int_R \sqrt{\rho} u \psi_x dx dt - \frac{1}{2} \int_0^T \int_R (\log \rho)_x \sqrt{\rho} u \psi dx dt.
\]

**Theorem 2.2.** Let $\alpha$ and $\gamma$ satisfy
\[
\alpha > \frac{1}{2}, \quad \gamma > 1.
\]
Suppose that (2.2) - (2.3) hold. If $\bar{\rho} > 0$, then for any $T > 0$, the Cauchy problem (1.1) - (1.3) admits a global weak solution $(\rho(x,t), u(x,t))$ in $R \times (0,T)$ satisfying
\[
(1) \quad \rho \in C(R \times (0,T)), \quad \rho(x,t) \geq 0, \quad (x,t) \in R \times (0,T);
\]
\[
(2) \quad \sup_{t \in [0,T]} \int_R |\rho - \bar{\rho}|^2 dx + \max_{(x,t) \in R \times [0,T]} \rho \leq C;
\]
\[
(3) \quad \sup_{t \in [0,T]} \int_R \left( (\sqrt{\rho} u)^2 + (\rho^{\alpha - 1/2})_x^2 + \frac{1}{\gamma - 1} (\rho^\gamma - (\bar{\rho})^\gamma - \gamma(\bar{\rho})^{\gamma - 1} (\rho - \bar{\rho})) \right) dx \\
+ \int_0^T \int_R \left( (\rho^{\frac{\gamma + \alpha - 1}{2}})_x^2 + \Lambda(x,t)^2 \right) dx dt \leq C,
\]
where $C$ is an absolute constant which only depends on the initial data, and $\Lambda(x,t) \in L^2(R \times (0,T))$ is a function which satisfies
\[
\int_0^T \int_R \Lambda \psi dx dt = - \int_0^T \int_R \rho^{\alpha - 1/2} \sqrt{\rho} u \psi_x dx dt - \frac{2\alpha}{2\alpha - 1} \int_0^T \int_R \rho^{\alpha - 1/2}_x \sqrt{\rho} u \psi dx dt.
\]

**Remark 2.3.** Under assumptions of Theorem 2.2, the case $\bar{\rho} = 0$ has been dealt with in Proposition 2.1.

The following is about the large time behavior of a weak solution.

**Theorem 2.3.** Suppose that $(\rho(x,t), u(x,t))$ is a weak solution of the Cauchy problem (1.1) - (1.3) satisfying (2.18) - (2.22) or (2.23) - (2.26). Then
\[
\lim_{t \to +\infty} \sup_{x \in R} |\rho - \bar{\rho}| = 0.
\]
Remark 2.4. In [28], Mellet and Vasseur proved that if the initial data is away from
the vacuum (has a positive lower bound) and $0 < \alpha < \frac{1}{2}$, the Cauchy problem (1.1)-(1.3) has a unique
global strong solution which is defined on $[0, T]$ for any $T > 0$. In comparison with [28], our results hold uniform
estimates on $T$ and in the case that $\bar{\rho} = 0$ the vacuum at the infinity is permitted. Moreover, by Theorem 2.3, the large
time behavior of the solutions of the strong solution can be obtained.

Based on Theorem 2.3, it is easy to obtain

Theorem 2.4. Suppose that the assumptions of Theorem 2.2 hold. Let $(\rho(x, t), u(x, t))$ be a weak solution of the Cauchy problem
(1.1)−(1.3) satisfying (2.24)−(2.26). Then for any $0 < \rho_1 < \bar{\rho}$, there exists a time $T_0$ such that

$$0 < \rho_1 \leq \rho(x, t) \leq C, \quad (x, t) \in \mathbb{R} \times [T_0, +\infty),$$

where $C$ is a constant same as in (2.25). Moreover, for $t \geq T_0$, the weak solution becomes a unique strong solution to (1.1)−(1.3), satisfying

$$\rho - \bar{\rho} \in L^\infty(T_0, t; H^1(R)), \quad \rho_t \in L^\infty(T_0, t; L^2(R)),$$

$$u \in L^2(T_0, t; H^2(R)), \quad u_t \in L^2(T_0, t; L^2(R))$$

and

$$\sup_{x \in \mathbb{R}} |\rho - \bar{\rho}| + \|\rho - \bar{\rho}\|_{L^p(R)} + \|u\|_{L^2(R)} \to 0,$$

as $t \to +\infty$, where $2 < p \leq +\infty$.

Remark 2.5. Theorem 2.4 shows that if $\bar{\rho} > 0$, the vacuum will vanish in finite
time and the weak solution will become the strong one after that. Similar to [18, 23],
we can obtain some results on the blow-up phenomena of the solutions when the
vacuum states vanish, which can be referred to [18, 23] for more details.

3. A Priori Estimates

In this section, we will construct approximate solutions and obtain a priori estimates of the approximate solutions to the Cauchy problem (1.1)−(1.3). Two cases
will be considered respectively: $0 < \alpha < \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Case I. $0 < \alpha < \frac{1}{2}$.

For any given $M > 0$, we construct the smooth approximation solution of (1.1)−
(1.3) on the cutoff domain $\Omega^M = \{ x \in \mathbb{R} \mid -M < x < M \}$. Consider the initial
condition

$$(\rho, \rho u)(x, 0) = (\rho_0, m_0),$$

and the boundary condition

$$u(x, t)|_{x=\pm M} = 0,$$
Proof. It follows from (1.1) and (3.8) that
\[
\frac{\rho_0}{\alpha-1/2} \to \frac{\rho_0}{\alpha-1/2} \text{ in } L^2(\Omega^M) \quad \text{if } 0 < \alpha < \frac{1}{2},
\]
where the initial data \(\rho_0, m_0\) are smooth functions satisfying
\[
\begin{align*}
\rho_0 &\to \rho_0 \text{ in } L^1(\Omega^M) \cap L^\infty(\Omega^M), \\
(\rho_0^{\alpha-1/2})_x &\to (\rho_0^{\alpha-1/2})_x \text{ in } L^2(\Omega^M) \quad \text{if } 0 < \alpha < \frac{1}{2}, \\
(\log \rho_0)_x &\to (\log \rho_0)_x \text{ in } L^2(\Omega^M) \quad \text{if } \alpha = \frac{1}{2}, \\
(m_0^2)(\rho_0)^{-1} &\to (m_0^2)(\rho_0)^{-1}, \quad \text{and} \\
(m_0^{2+\delta})(\rho_0)^{-1-\delta} &\to (m_0^{2+\delta})(\rho_0)^{-1-\delta} \text{ in } L^1(\Omega^M),
\end{align*}
\]
as \(\epsilon \to 0\). Here \(\delta > 0\), and there exists a constant \(C_0\) which does not depend on \(\epsilon\) such that
\[
\rho_0 \geq C_0 \epsilon^{1/(2\alpha-2\theta)}.
\]

We note that the initial data can be regularized in an usual way (see [19] for instance).

The following estimate is a key one to prove the main theorem which is based on the energy and entropy estimates.

**Lemma 3.1.** Let
\[
\gamma > 1, \quad 0 < \alpha < 1/2.
\]
Assume that \((\rho_\epsilon, u_\epsilon)\) is the smooth solution of (1.1) with \(\rho_\epsilon > 0\). Then for any \(T > 0\), the following estimate holds:
\[
\sup_{t \in [0,T]} \int_{\Omega} \rho_\epsilon |u_\epsilon|^2 + \left[ \left( \frac{\rho_\epsilon^{\alpha-1/2}}{\alpha-1/2} \right)_x \right]^2 + \rho_\epsilon \psi(\rho_\epsilon, \bar{\rho}) \leq C,
\]
where \(C\) is an universal constant independent of \(\epsilon\) and \(T\).

**Proof.** It follows from (1.1) that
\[
\rho_\epsilon u_{\epsilon t} + \rho_\epsilon u_\epsilon u_{\epsilon x} + (\rho_\epsilon \gamma)_x = (\rho_\epsilon^\alpha u_{\epsilon x})_x.
\]
Multiply (3.7) by \(u_\epsilon\) to get
\[
\left( \frac{\rho_\epsilon |u_\epsilon|^2}{2} \right)_t + \left( \frac{\rho_\epsilon u_\epsilon^3}{2} \right)_x + \rho_\epsilon^\alpha (u_{\epsilon x})^2 + (P(\rho_\epsilon))_x u_\epsilon = (\rho_\epsilon^\alpha u_{\epsilon x})_x = 0
\]
In view of (2.2), we have
\[
(\rho_\epsilon \psi(\rho_\epsilon, \bar{\rho}))_t + (\rho_\epsilon u_\epsilon \psi(\rho_\epsilon, \bar{\rho}))_x + u_{\epsilon x}(P(\rho_\epsilon) - P(\bar{\rho})) = 0.
\]
It follows from (3.8) and (3.9) that
\[
(\frac{\rho_\epsilon |u_\epsilon|^2}{2} + \rho_\epsilon \psi(\rho_\epsilon, \bar{\rho}))_t + H_1 x + \rho_\epsilon^\alpha (u_{\epsilon x})^2 = 0,
\]
where
\[
H_1 = \frac{\rho_\epsilon u_\epsilon^3}{2} + \rho_\epsilon u_\epsilon \psi(\rho_\epsilon, \bar{\rho}) + u_\epsilon(P(\rho_\epsilon) - P(\bar{\rho})) - \rho_\epsilon^\alpha u_{\epsilon x}u_{\epsilon x}.
\]
Since
\[
(\rho_\epsilon^\alpha u_{\epsilon x})_x = -\rho_\epsilon(\frac{\rho_\epsilon^{\alpha-1}}{\alpha-1})_{xt} - \rho_\epsilon u_\epsilon(\frac{\rho_\epsilon^{\alpha-1}}{\alpha-1})_{xx},
\]
(3.12) can be rewritten as
\[ \rho \epsilon u_{x,t} + \rho \epsilon u_{x}u_{x} + (\rho \epsilon)_{x} = -\rho \epsilon \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{xt} - \rho \epsilon u_{x} \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{xx} \] (3.12)

Multiplying (3.12) by \( \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \), we have
\[ \left( \frac{\rho \epsilon^{\alpha-1}}{2} \right)_{t} + \rho \epsilon u_{x} \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} + (\rho \epsilon)_{x} \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} + \rho \epsilon u_{x} (\frac{\rho \epsilon^{\alpha-1}}{\alpha-1})_{x} \]
\[ - u_{t} (\rho \epsilon^{\alpha-1})_{x} + \rho \epsilon u_{x} (\frac{\rho \epsilon^{\alpha-1}}{\alpha-1})_{x} + \rho \epsilon u_{x} (\frac{\rho \epsilon^{\alpha-1}}{\alpha-1})_{x} = 0. \] (3.13)

Multiplying (3.12) by \( u_{x} \) and adding up to (3.13), we obtain
\[ \left\{ \frac{1}{2} \rho \epsilon \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right]^{2} \right\}_{t} + \left\{ \frac{1}{2} \rho \epsilon u_{x} \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right] \right\}_{x} + u_{x} (\rho \epsilon)_{x} \]
\[ + (\frac{\rho \epsilon^{\alpha-1}}{\alpha-1})_{x} (\rho \epsilon)_{x} = 0. \] (3.14)

From (3.13) and (3.14), we get
\[ \left\{ \frac{1}{2} \rho \epsilon \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right]^{2} + \rho \epsilon \Psi (\rho \epsilon, \bar{\rho}) \right\}_{t} + \left\{ \frac{1}{2} \rho \epsilon u_{x} \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right] \right\}_{x} + \rho \epsilon u_{x} \Psi (\rho \epsilon, \bar{\rho}) \]
\[ + u_{x} (P(\rho \epsilon) - P(\bar{\rho})) \right\}_{x} + (\frac{\rho \epsilon^{\alpha-1}}{\alpha-1})_{x} (P(\rho \epsilon))_{x} = 0. \] (3.15)

Now we deal with the last term of the left hand side of (3.15). Since
\[ \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} = \rho \epsilon^{\alpha-2} \rho \epsilon_{x}, \] (3.16)

we have
\[ (\frac{\rho \epsilon^{\alpha-1}}{\alpha-1})_{x} (P(\rho \epsilon))_{x} = \rho \epsilon^{\alpha-2} \rho \epsilon_{x} (P(\rho \epsilon))_{x} = \frac{4 \gamma}{(\gamma + \alpha - 1)^{2}} \left( \rho \epsilon^{\alpha+\gamma-1} - \rho \epsilon_{x}^{\alpha+\gamma-1} \right)^{2}. \] (3.17)

It follows from (3.15) and (3.17) that
\[ \left\{ \frac{1}{2} \rho \epsilon \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right]^{2} + \rho \epsilon \Psi (\rho \epsilon, \bar{\rho}) \right\}_{t} + H_{2x} + \frac{4 \gamma}{(\gamma + \alpha - 1)^{2}} \left( \rho \epsilon^{\alpha+\gamma-1} - \rho \epsilon_{x}^{\alpha+\gamma-1} \right)^{2} = 0, \] (3.18)

where
\[ H_{2}(x, t) = \frac{1}{2} \rho \epsilon u_{x} \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right] + \rho \epsilon u_{x} \Psi (\rho \epsilon, \bar{\rho}) + u_{x} (P(\rho \epsilon) - P(\bar{\rho})). \] (3.19)

Multiplying (3.19) by \( \alpha \) and then adding up to (3.10), we have
\[ \left\{ \frac{\alpha}{2} \rho \epsilon \left[ u_{x} + \left( \frac{\rho \epsilon^{\alpha-1}}{\alpha-1} \right)_{x} \right]^{2} + \rho \epsilon u_{x}^{2} \right\}_{2} + (\alpha + 1) \rho \epsilon \Psi (\rho \epsilon, \bar{\rho}) \right\}_{t} + [\alpha H_{2} + H_{1}] \]
\[ + \rho \epsilon \left( \frac{\alpha}{2} \rho \epsilon u_{x}^{2} \right) + \frac{4 \gamma}{(\gamma + \alpha - 1)^{2}} \left( \rho \epsilon^{\alpha+\gamma-1} - \rho \epsilon_{x}^{\alpha+\gamma-1} \right)^{2} = 0, \] (3.20)
Integrating (3.20) over $[0, t] \times R$ with respect to $x$, $t$ gives

$$\sup_{t \in [0, T]} \int_R \{\rho \varepsilon |u|^2 + [(\frac{\rho^{\alpha-\frac{1}{2}}}{\alpha-\frac{1}{2}})_{x}]^2 + \rho \varepsilon \Psi(\rho, \tilde{\rho})\}dx + \int_0^T \int_R \rho \varepsilon |u|^2 dx dt$$

$$+ \int_0^T \int_R [(\rho^{\alpha-\frac{1}{2}} - \tilde{\rho}^{\alpha-\frac{1}{2}})_{x}]^2 dx dt \leq C.$$ (3.21)

The proof of the lemma is finished. \hfill \Box

Based on Lemma 3.1, we have

**Lemma 3.2.** Let $0 < \alpha < 1/2$. Assume that $(\rho, u)$ is the smooth solution of (1.1) with $\rho > 0$. Then there exist two absolute constants $C, \tilde{C}$ and a constant $C(M)$ depending on $M$ such that

$$0 < \tilde{C} \leq \rho \leq C, \text{ if } \tilde{\rho} > 0;$$

$$0 < C(M) \leq \rho \leq C, \text{ if } \tilde{\rho} = 0.$$ (3.22)

**Proof.** In the case that $\tilde{\rho} > 0$, from Lemma 3.1, we have $\rho \varepsilon \Psi(\rho, \tilde{\rho}) \in L^\infty(0, T; L^1(R))$ and $(\rho^{\alpha-\frac{1}{2}})_{x} \in L^\infty(0, T; L^2(R))$. Applying Lemma 5.3 in [24], we can get

$$(\rho - \tilde{\rho})_{1\{|\rho - \tilde{\rho}| \leq \frac{\varepsilon}{2}\}} \in L^2(R) \text{ and } (\rho - \tilde{\rho})_{1\{|\rho - \tilde{\rho}| > \frac{\varepsilon}{2}\}} \in L^\gamma(R).$$ (3.24)

Since

$$\rho \varepsilon^{\gamma} - \tilde{\rho}^{\gamma} - \gamma \tilde{\rho}^{\gamma-1}(\rho - \tilde{\rho}) \geq C(\rho - \tilde{\rho})^2 \text{ if } \gamma \geq 2,$$ (3.25)

thanks to (3.21), we have

$$\sup_{[0, T]} \int_R |\rho - \tilde{\rho}|^2 dx \leq C, \text{ if } \gamma \geq 2,$$ (3.26)

where $C$ is an universal constant independent of $\varepsilon$ and $T$.

For $\delta \in (0, \tilde{\rho})$, we have $|\rho| \leq \tilde{\rho} + \delta$ if $|\rho - \tilde{\rho}| \leq \delta$, and hence (3.22) holds true. If $|\rho - \tilde{\rho}| \geq \delta$, we can prove that there exists a constant $C = C(\delta)$ such that

$$|\rho^s - \tilde{\rho}^s| \leq C|\rho - \tilde{\rho}|^s, \quad s > 0.$$ (3.27)

In fact, from the facts

$$\frac{|\rho^s - \tilde{\rho}^s|}{|\rho - \tilde{\rho}|^s} \to 1, \text{ as } \rho \to \infty; \quad \frac{|\rho^s - \tilde{\rho}^s|}{|\rho - \tilde{\rho}|^s} \to 1, \text{ as } \rho \to 0,$$

there exist $\tilde{\rho}_1, \tilde{\rho}_2$ satisfying $0 < \tilde{\rho}_1 < \tilde{\rho}_2 < \infty$ such that

$$|\rho^s - \tilde{\rho}^s| \leq 2|\rho - \tilde{\rho}|^s, \quad \rho \in [0, \tilde{\rho}_1] \cup [\tilde{\rho}_2, \infty).$$

If $\rho \in [\tilde{\rho}_1, \tilde{\rho}_2]$, $|\rho - \tilde{\rho}| \geq \delta$, we have

$$|\rho^s - \tilde{\rho}^s| \leq C|\rho - \tilde{\rho}|^s,$$

where $C$ depends on $\delta, \tilde{\rho}_1, \tilde{\rho}_2$. Thus (3.27) holds true.
It follows from (3.26)-(3.27) that, for \( \gamma \geq 2 \),

\[
|\rho_\epsilon - \bar{\rho}|^2 \leq \int_R (\rho_\epsilon - \bar{\rho})^2 dx + \int_R |2(\rho_\epsilon - \bar{\rho})(\rho_\epsilon - \bar{\rho})_x| dx
\]

\[
= \int_R (\rho_\epsilon - \bar{\rho})^2 dx + \int_R |2(\rho_\epsilon - \bar{\rho})\rho_\epsilon^{\frac{\alpha}{2}} - \alpha \frac{\rho_\epsilon^{\alpha - \frac{1}{2}}}{2} x| dx
\]

\[
\leq C + C(\int_R (\rho_\epsilon - \bar{\rho})^2 (\rho_\epsilon^{\frac{\alpha}{2}} - \bar{\rho}^{\frac{\alpha}{2}} - \alpha - \frac{1}{2}) dx + \int_R (\rho_\epsilon - \bar{\rho})^2 \bar{\rho}^{\frac{\alpha}{2}} dx)^{\frac{1}{2}}
\]

\[
\leq C + C(\int_R |\rho_\epsilon - \bar{\rho}|^{5-2\alpha} 1_{|\rho_\epsilon - \bar{\rho}| \geq \delta} dx)^{\frac{1}{2}}
\]

\[
\leq C + C \sup_{x \in R} |\rho_\epsilon - \bar{\rho}|^{\frac{\alpha}{2}} (\int_R (\rho_\epsilon - \bar{\rho})^2 dx)^{\frac{1}{2}}
\]

\[
\leq C + C \sup_{x \in R} |\rho_\epsilon - \bar{\rho}|^{\frac{\alpha}{2}} - \alpha - \frac{1}{2}
\]

By Young’s inequality and the condition \( 0 < \alpha < \frac{1}{2} \), we get

\[
|\rho_\epsilon - \bar{\rho}|^2 \leq C, \text{ i.e. } |\rho_\epsilon| \leq C
\]

for \( \gamma \geq 2 \).

Now we consider the case \( 1 < \gamma < 2 \). It follows from (3.26) and (3.27) that

\[
|\rho_\epsilon - \bar{\rho}|^2 \leq \int \{ |\rho_\epsilon - \bar{\rho}| > \frac{\gamma}{2} \} |\rho_\epsilon - \bar{\rho}|^2 dx + \int \{ |\rho_\epsilon - \bar{\rho}| \leq \frac{\gamma}{2} \} |\rho_\epsilon - \bar{\rho}|^2 dx
\]

\[
+ \int \{ |\rho_\epsilon - \bar{\rho}| \leq \frac{\gamma}{2} \} |\rho_\epsilon - \bar{\rho}|^2 dx + \int_R |2(\rho_\epsilon - \bar{\rho})(\rho_\epsilon - \bar{\rho})_x| dx
\]

\[
\leq C \sup \{ |\rho_\epsilon - \bar{\rho}| > \frac{\gamma}{2} \} |\rho_\epsilon - \bar{\rho}|^2 - \gamma + C \int_R |2(\rho_\epsilon - \bar{\rho})\rho_\epsilon^{\frac{\alpha}{2}} - \alpha \frac{\rho_\epsilon^{\alpha - \frac{1}{2}}}{2} x| dx
\]

\[
\leq C \sup \{ |\rho_\epsilon - \bar{\rho}| > \frac{\gamma}{2} \} |\rho_\epsilon - \bar{\rho}|^2 - \gamma + C \int \{ |\rho_\epsilon - \bar{\rho}| \geq \frac{\gamma}{2} \} \cup \{ |\rho_\epsilon - \bar{\rho}| \leq \frac{\gamma}{2} \} |\rho_\epsilon - \bar{\rho}|^2 (\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) dx
\]

\[
+ \int \{ |\rho_\epsilon - \bar{\rho}| > \frac{\gamma}{2} \} (\rho_\epsilon - \bar{\rho})^2 (\bar{\rho}^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) dx + \int \{ |\rho_\epsilon - \bar{\rho}| \leq \frac{\gamma}{2} \} (\rho_\epsilon - \bar{\rho})^2 (\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}) dx
\]
\[ \leq C \sup_{x \in \{ |\rho_\epsilon - \bar{\rho}| > 2^{-\frac{\gamma}{2}} \}} |\rho_\epsilon - \bar{\rho}|^{2-\gamma} + C + C \left[ \int_{\{ |\rho_\epsilon - \bar{\rho}| > 2^{-\frac{\gamma}{2}} \}} |\rho_\epsilon - \bar{\rho}|^{5-2\alpha} \, dx + \int_{\{ |\rho_\epsilon - \bar{\rho}| > 2^{-\frac{\gamma}{2}} \}} (\rho_\epsilon - \bar{\rho})^2 \, dx \right]^{\frac{1}{2}} \]

\[ \leq C + C \sup_{x \in \{ |\rho_\epsilon - \bar{\rho}| > 2^{-\frac{\gamma}{2}} \}} |\rho_\epsilon - \bar{\rho}|^{2-\gamma} + C \left[ \sup_{x \in \{ |\rho_\epsilon - \bar{\rho}| > 2^{-\frac{\gamma}{2}} \}} |\rho_\epsilon - \bar{\rho}|^{5-2\alpha} + \sup_{x \in \{ |\rho_\epsilon - \bar{\rho}| > 2^{-\frac{\gamma}{2}} \}} |\rho_\epsilon - \bar{\rho}|^{2-\gamma} \right]^{\frac{1}{2}} \]

\[ \leq C + C \sup_{x \in \mathbb{R}} |\rho_\epsilon - \bar{\rho}|^{2-\gamma} + C \left[ \sup_{x \in \mathbb{R}} |\rho_\epsilon - \bar{\rho}|^{5-2\alpha} + \sup_{x \in \mathbb{R}} |\rho_\epsilon - \bar{\rho}|^{2-\gamma} \right]^{\frac{1}{2}}. \]  

(3.30)

By Young’s inequality and the condition \(0 < \alpha < \frac{1}{2}\), we get

\[ |\rho_\epsilon - \bar{\rho}|^2 \leq C, \text{ i.e. } |\rho_\epsilon| \leq C \]  

(3.31) for \(1 < \gamma < 2\). (3.29) and (3.31) yield the uniform upper bound in (3.22).

Now we prove the positive lower bound estimate of \(\rho_\epsilon\) in (3.22). Noticing that \(\lim_{\rho_\epsilon \to 0} \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) = (\bar{\rho})^\gamma\), we can obtain that \(\rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho})\) has a positive lower bound on \([0, 1/2 \bar{\rho}]\). Since \(\rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho})\) is bounded in \(L^2_\mathbb{T}(L^1)\), there exists \(C_1 = C_1(T) > 0\) such that for all \(t \in [0, T]\),

\[ \text{meas}\{x \in R | \rho_\epsilon(x, t) \leq \frac{1}{2} \bar{\rho}\} \leq \frac{1}{\inf_{\rho_\epsilon \in [0, \frac{1}{2} \bar{\rho}]} \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) \int_{\{x \in R | \rho_\epsilon \leq \frac{1}{2} \bar{\rho}\}} \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) \, dx \leq C_1. \]  

(3.32)

Therefore for each \(x_0 \in R\), there exists \(N = N(T) > 0\) big enough such that

\[ \int_{|x-x_0| \leq N} \rho_\epsilon(x, t) \, dx \geq \int_{\{x \in R | \rho_\epsilon \leq \frac{1}{2} \bar{\rho}\} \cap \{x \in R | |x-x_0| \leq N\}} \rho_\epsilon \, dx \]

\[ \geq \frac{1}{2} \bar{\rho} \text{meas}\{\{x - x_0| \leq N\} \cap \{x \in R | \rho_\epsilon(x, t) > \frac{1}{2} \bar{\rho}\}\} \]

\[ = \frac{1}{2} \bar{\rho} \text{meas}\{\{x - x_0| \leq N\} \cap \{x \in R | \rho_\epsilon(x, t) \leq \frac{1}{2} \bar{\rho}\}\} \]

\[ \geq \frac{1}{2} \bar{\rho}(2N - C_1) > 0, \quad t \in [0, T]. \]  

(3.33)

Using the continuity of \(\rho_\epsilon\), there exists \(x_1 \in [x_0 - N, x_0 + N]\) such that

\[ \rho_\epsilon(x_1, t) = \int_{|x-x_0| \leq M} \rho_\epsilon(x, t) \, dx \geq \frac{1}{2} \bar{\rho}(2N - C_1). \]  

(3.34)

Then it follows from Lemma 3.3 that

\[ \rho_\epsilon^{\alpha - \frac{1}{2}}(x_0, t) = \rho_\epsilon^{\alpha - \frac{1}{2}}(x_1, t) + \int_{x_1}^{x_0} (\rho_\epsilon^{\alpha - \frac{1}{2}})(x, t) \, dx \]

\[ \leq \left[ \frac{1}{2} \bar{\rho}(2N - C_1) \right]^{\alpha - \frac{1}{2}} + \| (\rho_\epsilon^{\alpha - \frac{1}{2}})(x, t) \|_{L^2(R)} |x_1 - x_0|^\frac{1}{2} \]  

\[ \leq \left[ \frac{1}{2} \bar{\rho}(2N - C_1) \right]^{\alpha - \frac{1}{2}} + C N^\frac{1}{2}. \]  

(3.35)

Since \(0 < \alpha < \frac{1}{2}\), for any \(x_0 \in R\) and all \(t \in [0, T]\), we have

\[ \rho_\epsilon(x_0, t) \geq \left\{ \left[ \frac{1}{2} \bar{\rho}(2N - C_1) \right]^{\alpha - \frac{1}{2}} + C N^\frac{1}{2} \right\}^{\frac{2}{2\alpha-1}} := C(T). \]  

(3.36)

Up to now, we have proved (3.22).
To prove (3.23), by Lemma 3.3, we have

\[
\rho_\gamma \leq \int_R \rho_\gamma^\gamma dx + \int_R |2\rho_\gamma^{\gamma-1}\rho_\gamma x| dx \leq C + \int_R |2\rho_\gamma^{\gamma-1}\rho_\gamma^{\frac{3}{2}-\alpha}\rho_\gamma^{\alpha-\frac{3}{2}}\rho_\gamma x| dx
\]

\[
\leq C + \int_R |2\rho_\gamma^{\gamma+\frac{1}{2}-\alpha}(\rho_\gamma^{\alpha-\frac{3}{2}})_{x}| dx \leq C + \left( \int_R \rho_\gamma^{2\gamma+1-2\alpha} dx \right)^{\frac{1}{2}} \left( \int_R (\rho_\gamma^{\alpha-\frac{1}{2}})^2 dx \right)^{\frac{1}{2}}
\]

\[
\leq C + C \left( \int_R \rho_\gamma^\gamma dx \right)^{\frac{1}{2}} \sup_{x \in R} \rho_\gamma^{\frac{\alpha+1-2\alpha}{\alpha}} \leq C + C \sup_{x \in R} \rho_\gamma^{\frac{\alpha+1-2\alpha}{\alpha}}.
\]

(3.37)

Applying Young’s inequality and the condition \(0 < \alpha < \frac{1}{2}\), we obtain \(\rho_\gamma \leq C\).

To get the lower bound of \(\rho_\gamma\) in (3.23), we use the Lagrangian coordinates as follows:

\[
\xi = \int_{-M}^x \rho_\gamma(y,t) dy \quad \tau = t
\]

where \(x \in [-M,M], t > 0\), and \(\xi \in \Omega_L = (0,L) = (0, \int_{-M}^M \rho_\gamma(y,t) dy) = (0, \int_{-M}^M \rho_{0\gamma}(y) dy)\).

In view of the Lagrangian coordinates transformation we get \((\rho_\gamma^\alpha)_\xi \in L^2(\Omega_L)\) from Lemma 3.3. Let \(v = \frac{1}{\rho_\gamma}\). Then we have

\[
v \leq \int_{\Omega_L} v d\xi + \int_{\Omega_L} v^2 |p_\gamma| d\xi \leq 2M + \frac{1}{2} \max_{\xi \in \Omega_L} v + \| (\rho_\gamma^\alpha)_\xi \|_{L^1(\Omega_L)}^{\frac{1}{2}}
\]

\[
\leq 2M + C + \frac{1}{2} \max_{\xi \in \Omega_L} v,
\]

which implies \(\rho_\gamma \geq C(M) > 0\). (3.23) is proved and the proof of the lemma is finished. \(\square\)

Case II. \(\alpha \geq \frac{1}{2}\).

In the case that \(\alpha \geq \frac{1}{2}\), we construct the approximate solutions by solving

\[
\left\{
\begin{array}{l}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x = (\mu_\varepsilon(\rho) u)_x, \\
(\rho_\varepsilon, m_\varepsilon)(x, t = 0) = (\rho^0_\varepsilon, m^0_\varepsilon)
\end{array}
\right.
\]

(3.39)

with \(\mu_\varepsilon(\rho) = \rho^\alpha + \varepsilon \rho^\theta, \varepsilon > 0, \theta \in (0, \frac{1}{2})\). The initial values \((\rho^0_\varepsilon, m^0_\varepsilon)\) are regularized in the same way as in (3.3) and (3.4).

For any fixed \(T > 0\) and for any fixed \(\varepsilon > 0\), there exists a unique smooth approximate solution to (3.39) in the region \((x, t) \in R \times (0, T)\). We refer to [28] for the wellposedness of the global strong solution to the approximate system (3.39).

Then we have

Lemma 3.3. Let

\[
\gamma > 1, \quad \alpha > 1/2.
\]

(3.40)
Proof of Lemma 3.4. We first prove the upper bound for restriction (2.13) divided into the following cases.

Assume that $(\rho_\epsilon, u_\epsilon)$ is the smooth solution of (1.1) with $\rho_\epsilon > 0$. Then for any $T > 0$, the following estimate holds:

$$
\sup_{t \in [0, T]} \int_R \left\{ |\rho_\epsilon| u_\epsilon|^2 + [\left( \frac{\rho_\epsilon - \bar{\rho}}{\alpha - \frac{1}{2}} \right)_x]_\alpha^2 + \epsilon^2 \left( \frac{\rho_\epsilon - \bar{\rho}}{\theta - \frac{1}{2}} \right)_x^2 + \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) \right\}(x, t) \, dx 
+ \int_0^T \int_R \left\{ (\rho_\epsilon^\alpha + \epsilon \rho_\epsilon^\theta)(u_\epsilon)_x^2 + \left( (\rho_\epsilon^{\alpha + \gamma - 1}_\epsilon - \bar{\rho}^{\alpha + \gamma - 1}_\epsilon)_x \right)^2 
+ \left( (\rho_\epsilon^{\alpha + \gamma - 1}_\epsilon - \bar{\rho}^{\alpha + \gamma - 1}_\epsilon)_x \right)^2 \right\}(x, t) \, dx \, dt \leq C,
$$

(3.41)

where $C$ is an universal constant independent of $\epsilon$ and $T$.

The proof is similar to that of Lemma 3.1 (see also Lemma 3.6 in [18]) and we omit the details here.

Based on this lemma, we have

**Lemma 3.4.** Let $\alpha$, $\gamma$ satisfy (3.40) and $\bar{\rho} > 0$. Assume that $(\rho_\epsilon, u_\epsilon)$ is the smooth solution of (1.1) with $\rho_\epsilon > 0$. Then there exist an absolute constant $C$ and a constant $C(\epsilon, T)$ such that

$$
0 < C(\epsilon, T) \leq \rho_\epsilon \leq C.
$$

(3.42)

**Remark 3.1.** If $\bar{\rho} = 0$, under the assumption (3.40), the estimate (3.42) has been proved in [18]. If $\bar{\rho} > 0$, the estimate (3.42) has also been proved in [18] under the restriction (2.13).

**Proof of Lemma 3.4.** We first prove the upper bound for $\rho_\epsilon(x, t)$. The proof is divided into the following cases.

If $\frac{1}{2} < \alpha \leq \frac{\gamma + 1}{2}$, it follows from (3.24), (3.27) and the entropy estimate (3.41) that

$$
|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq \int_{-\infty}^x (|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|_x^2) \, dx 
\leq \int_{-\infty}^x 2(\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})(\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})_x \, dx 
\leq \int_R (\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})_x^2 \, dx + \int_R |\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|_x^2 \, dx 
\leq C + \int_R (\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})_x^2 \, dx + \int_{\{|\rho_\epsilon - \bar{\rho}| < \frac{\epsilon}{2}\}} \rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}} \, dx 
\leq C + I_1 + I_2.
$$

(3.43)

Note that when $|\rho_\epsilon - \bar{\rho}| < \frac{\epsilon}{2}$, that is, $\frac{\epsilon}{2} < \rho_\epsilon < \frac{3\epsilon}{2}$, one has

$$
|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq |\rho_\epsilon - \bar{\rho}|^2 \leq \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}).
$$

Hence,

$$
I_1 \leq \int_R \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) \, dx \leq C.
$$

(3.44)
It follows from (3.27) that
\[ I_2 \leq \int_{\{|\rho_e - \bar{\rho}| \geq \frac{2}{3}\}} |\rho_e - \bar{\rho}|^{2(\alpha - \frac{1}{2})} dx \leq \int_{\{|\rho_e - \bar{\rho}| \geq \frac{2}{3}\}} C|\rho_e - \bar{\rho}|^{\gamma} dx \leq \int_{R} \rho_e \Psi(\rho_e, \bar{\rho}) dx \leq C, \]
(3.45)
if \( \frac{1}{2} < \alpha \leq \frac{2+1}{2} \). In view of (3.43)-(3.45), we get
\[ |\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq C. \]
(3.46)
Therefore \( \rho_e \) has upper bound in the case that \( \frac{1}{2} < \alpha \leq \frac{2+1}{2} \).

If \( \alpha > \frac{1}{2}, 1 < \gamma < 2\alpha - 1 \), the proof is divided into the following subcases.

When \( \alpha > \frac{1}{2}, \gamma \geq 2 \), it is easy to get that
\[ \rho_e^{\gamma} - \bar{\rho}^{\gamma} - \gamma \bar{\rho}^{\gamma-1}(\rho_e - \bar{\rho}) \geq C(\rho_e - \bar{\rho})^2 \quad \text{if} \quad \rho \geq 0. \]
(3.47)
It follows from (3.41) that
\[ \sup_{[0, T]} \int_{R} |\rho_e - \bar{\rho}|^2 dx \leq C, \quad \text{if} \quad \gamma \geq 2, \]
(3.48)
where \( C \) is an universal constant independent of \( \epsilon \) and \( T \). Using (3.27) and (3.48), for \( \gamma \geq 2 \), we have
\[ |\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq \int_{-\infty}^{x} (|\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2) dx \]
\[ \leq \int_{-\infty}^{x} 2(\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})(\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}})_{x} dx \leq C + \int_{R} |\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \]
\[ \leq C + \sup_{x \in R} |\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \frac{2}{\alpha - \frac{3}{2}} \int_{R} |\rho_e - \bar{\rho}|^2 dx \]
\[ \leq C + C \sup_{x \in R} |\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \frac{2}{\alpha - \frac{3}{2}}. \]
(3.49)
By Young’s inequality and the condition \( \alpha > \frac{1}{2} \), we get
\[ |\rho_e^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq C, \quad \text{i.e.} \quad |\rho_e| \leq C. \]
(3.50)
When \( \frac{1}{2} < \alpha < \frac{3}{2}, 1 < \gamma < 2 \), it follows from (3.24), (3.27) and (3.30) that
\[ |\rho_e - \bar{\rho}| \]
\[ \leq \int_{\{|\rho_e - \bar{\rho}| < \frac{2}{3}\}} |\rho_e - \bar{\rho}|^{\gamma} dx \sup_{\{|\rho_e - \bar{\rho}| < \frac{2}{3}\}} |\rho_e - \bar{\rho}|^{2-\gamma} + \int_{\{|\rho_e - \bar{\rho}| \leq \frac{2}{3}\}} |\rho_e - \bar{\rho}|^2 dx \]
\[ + \int_{R} |2(\rho_e - \bar{\rho})(\rho_e - \bar{\rho})_x| dx \]
\[ \leq C + C \sup_{x \in R} |\rho_e - \bar{\rho}|^{2-\gamma} + C(\sup_{x \in R} |\rho_e - \bar{\rho}|^{5-2\alpha-\gamma} + \sup_{x \in R} |\rho_e - \bar{\rho}|^{3-2\alpha}) \]
\[ + \sup_{x \in R} |\rho_e - \bar{\rho}|^{2-\gamma} \frac{1}{\gamma}. \]
(3.51)
By Young’s inequality and the condition $\frac{1}{2} < \alpha < \frac{3}{2}$ and $1 < \gamma < 2$, we can obtain

$$|\rho_\epsilon - \bar{\rho}|^2 \leq C, \quad \text{i.e. } |\rho_\epsilon| \leq C. \quad (3.52)$$

When $\alpha \geq \frac{3}{2}$, $1 < \gamma < 2$, it follows from (3.24) and (3.27) that

$$|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq \int_{-\infty}^{x} (|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2) dx \leq C + \int_{-\infty}^{x} (|\rho_\epsilon - \bar{\rho}|^2)^{1} dx$$

$$\leq C + C(\int_{R} (|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2)^{1} \{ |\rho_\epsilon - \bar{\rho}| \leq \frac{\alpha}{2} \} dx + \int_{R} (|\rho_\epsilon - \bar{\rho}|^2)^{1} \{ |\rho_\epsilon - \bar{\rho}| \geq \frac{\alpha}{2} \} dx$$

$$\leq C + \sup_{x \in R} |\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \int_{R} (|\rho_\epsilon - \bar{\rho}|^2)^{1} \{ |\rho_\epsilon - \bar{\rho}| \leq \frac{\alpha}{2} \} dx$$

$$+ \sup_{x \in R} |\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \int_{R} |\rho_\epsilon - \bar{\rho}|^2 \{ |\rho_\epsilon - \bar{\rho}| \geq \frac{\alpha}{2} \} dx$$

$$\leq C + C \sup_{x \in R} |\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 + C \sup_{x \in R} |\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2. \quad (3.53)$$

By Young’s inequality and the condition $\alpha \geq \frac{3}{2}$ and $1 < \gamma < 2$, we get

$$|\rho_\epsilon^{\alpha - \frac{1}{2}} - \bar{\rho}^{\alpha - \frac{1}{2}}|^2 \leq C, \quad \text{i.e. } |\rho_\epsilon| \leq C. \quad (3.54)$$

Combining (3.50), (3.52) with (3.54), we get the uniform upper bound of $\rho_\epsilon$ for any $\alpha > \frac{1}{2}$ and $\gamma > 1$.

Next, we prove the positive lower bound estimate of $\rho_\epsilon$.

Noticing that $\lim_{\rho_\epsilon \to 0} \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) = \bar{\rho}^\gamma$, we can obtain that $\rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho})$ has a positive lower bound on $[0, \frac{1}{2} \bar{\rho}]$. Since $\rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho})$ is bounded in $L_T^\infty(L^1)$, there exists $C_1 = C_1(T) > 0$ such that for all $t \in [0, T]$,

$$\text{meas} \{ x \in R | \rho_\epsilon(x, t) \leq \frac{1}{2} \bar{\rho} \} \leq \inf_{\rho_\epsilon \in [0, \frac{1}{2} \bar{\rho}]} \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) \int_{\{ x \in R | \rho_\epsilon \leq \frac{1}{2} \bar{\rho} \}} \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) dx \leq C_1. \quad (3.55)$$

For each $x_0 \in R$, there exists $N = N(T) > 0$ large enough such that

$$\int_{|x-x_0| \leq N} \rho_\epsilon(x, t) dx \geq \int_{|x-x_0| \leq N} \cap \{ x \in R | \rho_\epsilon(x, t) > \frac{1}{2} \bar{\rho} \} \rho_\epsilon dx$$

$$\geq \frac{1}{2} \bar{\rho} \text{meas} \{ \{ |x-x_0| \leq N \} \cap \{ x \in R | \rho_\epsilon(x, t) > \frac{1}{2} \bar{\rho} \} \}$$

$$= \frac{1}{2} \bar{\rho} \text{meas} \{ \{ |x-x_0| \leq N \} \cap \{ x \in R | \rho_\epsilon(x, t) \leq \frac{1}{2} \bar{\rho} \} \}$$

$$\geq \frac{1}{2} \bar{\rho}(2N - C_1) > 0, \quad t \in [0, T]. \quad (3.56)$$

As the continuity of $\rho_\epsilon$, there exists $x_1 \in [x_0 - N, x_0 + N]$ such that

$$\rho_\epsilon(x_1, t) = \int_{|x-x_0| \leq M} \rho_\epsilon(x, t) dx \geq \frac{1}{2} \bar{\rho}(2N - C_1). \quad (3.57)$$
Then we can get from Lemma 3.2 that

\[ \rho \theta - \frac{1}{2} \hat{\epsilon}(x_0, t) = \rho \theta - \frac{1}{2} \hat{\epsilon}(x_1, t) + \int_{x_1}^{x_0} \rho \theta \frac{\theta - \frac{1}{2}}{\theta - \frac{1}{2}} \hat{\epsilon} x(x, t) dx \]

\[ \leq \left[ \frac{1}{2} \bar{\rho}(2N - C_1) \right]^{\theta - \frac{1}{2}} + \left( \rho \theta \frac{\theta - \frac{1}{2}}{\theta - \frac{1}{2}} \hat{\epsilon} x(x, t) \right) \| L^2(R) \| x_1 - x_0 \frac{1}{2} \]

\[ \leq \left[ \frac{1}{2} \bar{\rho}(2N - C_1) \right]^{\alpha - \frac{1}{2}} + C_\epsilon N^{\frac{1}{2}}. \quad (3.58) \]

For \( \alpha > \frac{1}{2} \) and \( \gamma > 1 \), for any \( x_0 \in R \) and all \( t \in [0, T] \), we have

\[ \rho \epsilon(x_0, t) \geq \left\{ \left[ \frac{1}{2} \bar{\rho}(2N - C_1) \right]^{\alpha - \frac{1}{2}} + C_\epsilon N^{\frac{1}{2}} \right\}^{\frac{1}{2\alpha - 1}} = C(\epsilon, T). \quad (3.59) \]

The proof of the lemma is complete. \( \square \)

Similarly, when \( \alpha = \frac{1}{2} \), we can establish the following a priori estimates:

**Lemma 3.5.** Let \( \alpha = 1/2 \). Assume that \( (\rho_\epsilon, u_\epsilon) \) is smooth solution of (1.1) with \( \rho_\epsilon > 0 \). Then for any \( T > 0 \), the following estimate holds.

\[
\sup_{t \in [0, T]} \int_R \{ \rho_\epsilon |u_\epsilon|^2 + (\log \rho_\epsilon)_x^2 + \epsilon^2 (\rho_\epsilon^{\theta - \frac{1}{2}})_x^2 + \rho_\epsilon \Psi(\rho_\epsilon, \bar{\rho}) \} dx + \int_0^T \int_R \rho_\epsilon^{\frac{1}{2}} u_\epsilon^2 dx dt \\
+ \int_0^T \int_R \left[ \rho_\epsilon^{\frac{2 - \frac{1}{2}}{2}} - \bar{\rho}^{\frac{2 - \frac{1}{2}}{2}} \right]_x^2 + \epsilon \left[ (\rho_\epsilon^{\frac{\theta + 1}{2}} - \bar{\rho}^{\frac{\theta + 1}{2}})_x \right]^2 dx dt \leq C.
\]

Here \( C \) is an universal constant independent of \( \epsilon \) and \( T \).

**Lemma 3.6.** Let \( \alpha = 1/2 \). Assume that \( (\rho_\epsilon, u_\epsilon) \) is smooth solution of (1.1) with \( \rho_\epsilon > 0 \). Then there exist an absolute constant \( C \) and a positive constant \( C(\epsilon, T) \) such that

\[ 0 < C(\epsilon, T) \leq \rho_\epsilon \leq C. \quad (3.61) \]

**Proof.** Similar to the proof of lemma 3.4, we first prove the positive uniform upper bound of \( \rho_\epsilon \).
If \( \tilde{\rho} > 0 \), it follows from (3.60) that
\[
|\rho_\epsilon - \tilde{\rho}|^2 \leq \int_R (\rho_\epsilon - \tilde{\rho})^2 \, dx + \int_R |2(\rho_\epsilon - \tilde{\rho})(\rho_\epsilon - \tilde{\rho})_x| \, dx
\]
\[
= \int_R (\rho_\epsilon - \tilde{\rho})^2 \, dx + \int_R |2(\rho_\epsilon - \tilde{\rho})\rho_\epsilon(\rho_\epsilon - \tilde{\rho})_x\rho_\epsilon^{-1}| \, dx
\]
\[
= \int_R (\rho_\epsilon - \tilde{\rho})^2 \, dx + \int_R |2(\rho_\epsilon - \tilde{\rho})\rho_\epsilon(\log \rho_\epsilon)_x| \, dx
\]
\[
\leq C + C \left( \int_R (\rho_\epsilon - \tilde{\rho})^2 \rho_\epsilon^2 \, dx \right)^{\frac{1}{2}} \left( \int_R (\log \rho_\epsilon)_x^2 \, dx \right)^{\frac{1}{2}}
\]
(3.62)
\[
\leq C + C \left( \int_R (\rho_\epsilon - \tilde{\rho})^2(\rho_\epsilon - \tilde{\rho})^2 \, dx + \int_R (\rho_\epsilon - \tilde{\rho})^2 \rho_\epsilon^2 \, dx \right)^{\frac{1}{2}}
\]
\[
\leq C + C \left( \int_R (\rho_\epsilon - \tilde{\rho})^2(\rho_\epsilon - \tilde{\rho})^2 \, dx + C \right)^{\frac{1}{2}}
\]
\[
\leq C + \frac{1}{2} \sup_{x \in R} |\rho_\epsilon - \tilde{\rho}|^2.
\]

By using Young’s Inequality, we obtain that
\[
|\rho_\epsilon - \tilde{\rho}|^2 \leq C, \quad \text{i.e. } \rho_\epsilon \leq \tilde{\rho} + C.
\] (3.63)

If \( \tilde{\rho} = 0 \), it follows from (3.60) that
\[
\rho_\epsilon^\gamma \leq \int_R \rho_\epsilon^\gamma \, dx + \int_R |2\rho_\epsilon^{\gamma-1}\rho_\epsilon| \, dx \leq C + \int_R |2\rho_\epsilon^{\gamma-1}\rho_\epsilon\rho_\epsilon^{-1}\rho_\epsilon_x| \, dx
\]
\[
\leq C + \int_R |2\rho_\epsilon^{\gamma}(\log \rho_\epsilon)_x| \, dx \leq C + (\int_R \rho_\epsilon^{2\gamma} \, dx)^{\frac{1}{2}} \left( \int_R (\log \rho_\epsilon)_x^2 \, dx \right)^{\frac{1}{2}}
\] (3.64)
\[
\leq C + C \left( \int R \rho_\epsilon^\gamma \, dx \right)^{\frac{1}{2}} \sup_{x \in R} \rho_\epsilon^{\frac{\gamma}{2}} \leq C + C \left( \sup_{x \in R} \rho_\epsilon^{\gamma} \right)^{\frac{1}{2}}.
\]

By using Young’s Inequality, we obtain that \( \rho_\epsilon \leq C \).

We now prove the positive lower bound estimate of \( \rho_\epsilon \). Using (3.32)–(3.34) and (3.60), we obtain
\[
\rho_\epsilon^{-\frac{1}{2}}(x_0, t) = \rho_\epsilon^{-\frac{1}{2}}(x_1, t) + \int_{x_1}^{x_0} (\rho_\epsilon^{-\frac{1}{2}})_x(x, t) \, dx
\]
\[
\leq \left[ \frac{1}{2} \tilde{\rho}(2N - C_1) \right]^{-1} + \max_{\rho_\epsilon^{-\frac{1}{2}}}(\rho_\epsilon^{-\frac{1}{2}})_x(x, t) \left\| \rho_\epsilon^{-\frac{1}{2}} \right\|_{L^2(R)} \left| x_1 - x_0 \right|^\frac{1}{2}
\] (3.65)
\[
\leq \left[ \frac{1}{2} \tilde{\rho}(2N - C_1) \right]^{-1} + C_\epsilon N.
\]

Since \( 0 < \theta < \frac{1}{2} \), by the construction of the approximate solutions in (3.39), we have
\[
\rho_\epsilon(x_0, t) \geq C \left[ \frac{1}{2} \tilde{\rho}(2N - C_1) \right]^{-1} + C_\epsilon N \right]^{-1} := C(\epsilon, T),
\] (3.66)

for any \( x_0 \in R \) and \( t \in [0, T] \).

The proof of the lemma is finished. \( \square \)
4. Proof of the Main Results

In this section, we give the proof of the main results. The proof is completely similar to those in \[10\ 18\ 23\] and we give a sketch of proof here.

**Proof of Theorem 2.1.** Based on a priori estimates of Lemma 3.1-Lemma 3.2 and Lemma 3.5-Lemma 3.6, applying similar approaches in \[10\ 23\ 27\] and the references therein, we can obtain that for any \( T > 0 \) there exists a unique global smooth solution of \((1.1)-(1.3)\) satisfying

\[
\rho_\epsilon, \rho_{ex}, u_\epsilon, u_{ex}, u_{t\epsilon}, u_{xx}, u_{xxx} \in C^{\beta, \frac{\alpha}{2}}([-M, M] \times [0, T]), \quad 0 < \beta < 1,
\]

and \( \rho_\epsilon \geq C(\epsilon) > 0 \) in \([-M, M] \times [0, T]\) when \( 0 < \alpha \leq \frac{1}{2} \). And, the estimates in Lemma 3.1-Lemma 3.2 and Lemma 3.5-Lemma 3.6 hold for \( \{\rho_\epsilon, u_\epsilon\} \).

We only give a proof of the case \( 0 < \alpha < \frac{1}{2} \). The case \( \alpha = \frac{1}{2} \) can be proved in a similar way. For any fixed \( M > 0 \), similar to \[10\ 18\ 23\], we can obtain that (up to a subsequence)

\[
\rho_\epsilon \to \rho \quad \text{in} \; C([0, T] \times [-M, M]),
\]

\[
(\rho^{\frac{\alpha}{2}}_\epsilon)_x \to (\rho^{\frac{\alpha}{2}})_x \quad \text{weakly in} \; L^2((0, T) \times [-M, M]),
\]

\[
\rho^\alpha u_\epsilon \to \Lambda \quad \text{weakly in} \; L^2((0, T) \times [-M, M]),
\]

as \( \epsilon \to 0 \), for some function \( \rho \in C([0, T] \times [-M, M]) \) and \( \Lambda \in L^2((0, T) \times [-M, M]) \) which satisfy

\[
\int_0^T \int_{-M}^M \Lambda \varphi dx dt = -\int_0^T \int_{-M}^M \rho^{\frac{\alpha}{2}} \sqrt{\rho} u \varphi dx dt - \frac{2\alpha}{2\alpha - 1} \int_0^T \int_{-M}^M (\rho^{\frac{\alpha}{2}})_x \sqrt{\rho} u \varphi dx dt.
\]

To get the convergence of the term \( \sqrt{\rho} u_\epsilon \), we apply similar approaches in \[10\ 18\ 23\ 27\]. More precisely, we have \( \rho_\epsilon u_\epsilon \) converges strongly in \( L^1([0, T] \times [-M, M]) \) and \( L^2([0, T]; L^{1+\zeta}(-M, M)) \) and almost everywhere to some function \( m(x, t) \), where \( \zeta > 0 \) is some small positive number. Also, we can prove that \( \sqrt{\rho} u_\epsilon \) converges strongly in \( L^2([0, T] \times [-M, M]) \) to \( \frac{m}{\sqrt{\rho}} \) which is defined to be zero when \( m = 0 \) and there exists a function \( u(x, t) \) such that \( m(x, t) = \rho(x, t) u(x, t) \). Moreover, we have

\[
\rho \in C(R \times (0, T)),
\]

\[
\sup_{t \in [0, T]} \int_{-M}^M |\rho - \bar{\rho}|^2 dx + \max_{(x, t) \in R \times [0, T]} \rho \leq C,
\]

\[
\sup_{t \in [0, T]} \int_{-M}^M \left( |\sqrt{\rho} u|^2 + (\rho^{\frac{\alpha}{2}})_x^2 + \frac{1}{\gamma - 1} (\rho^\gamma - (\bar{\rho})^\gamma - \gamma (\bar{\rho})^{\gamma - 1} (\rho - \bar{\rho})) \right) dx
\]

\[
+ \int_0^T \int_{-M}^M \left( (\rho^{\frac{\alpha}{2} - 1})^2 \right) dx dt \leq C,
\]

where \( C \) is an absolute constant depending on the initial data.

Using a diagonal procedure, we obtain that the above converges (up to a subsequence) remain true for any \( M > 0 \) and the existence of weak solutions of \((1.1)-(1.3)\) can be directly proved. Moreover, \((2.18)-(2.22)\) hold true due to \((4.5)-(4.7)\). The proof of the theorem is complete. \( \square \)
Using Lemma 3.3-Lemma 3.4, we can prove Theorem 2.2 in a similar way (see also [10, 23, 27]) and we omit the details here.

Now we give the proof of Theorem 2.3, which is about the asymptotic behavior of the weak solutions. We assume that the solutions are smooth enough. The rigorous proof can be obtained by using the usual regularization procedure.

Proof of Theorem 2.3. We only prove the case of $0 < \alpha < \frac{1}{2}$ in Theorem 2.3 since other cases can be proved in a similar way. The proof can be done by considering the cases $\bar{\rho} > 0$ and $\bar{\rho} = 0$ respectively.

If $\bar{\rho} > 0$, since $0 \leq \rho \leq C$, $\bar{\rho} > 0$, for some constant $C_1 > 0$, we have

$$C_1^{-1}(\rho - \bar{\rho})^2 \leq \rho \Psi(\rho, \bar{\rho}) \leq C_1(\rho - \bar{\rho})^2. \tag{4.8}$$

From Lemma 3.2 we have $|\rho^s - \bar{\rho}^s|^2 \leq C|\rho - \bar{\rho}|^2$ for any $s > 0$. Hence,

$$\int_R |\rho^s - \bar{\rho}^s|^2 dx \leq C \int_R |\rho - \bar{\rho}|^2 dx \leq C. \tag{4.9}$$

Similarly,

$$\int_R |\rho^s - \bar{\rho}^s|^{2\lambda} dx \leq C \int_R |\rho - \bar{\rho}|^{2\lambda} dx \leq C, \tag{4.10}$$

for any $s > 0, \lambda > 1$. Moreover, one has

$$\int_R |[(\rho^s - \bar{\rho}^s)^{2\lambda}]_x|^2 dx = 2s \int_R |(\rho^s - \bar{\rho}^s)^{2\lambda-1}[\rho^{s-1}\rho_x]| dx \leq \frac{2s}{|\alpha - \frac{1}{2}|}(\int_R (\rho^s - \bar{\rho}^s)^{2(2\lambda-1)}\rho^{2s+1-2\alpha} dx)^{\frac{1}{2}}(\int_R [(\rho^{\alpha - \frac{1}{2}})_x]^2 dx)^{\frac{1}{2}} \leq C. \tag{4.11}$$

By Lemma 3.40, one has

$$\int_0^T \int_R [(\rho^{\alpha + \gamma - 1} - \bar{\rho}^{\alpha + \gamma - 1})_x]^2(x, t) dt dx \leq C \tag{4.12}$$

Denote $b = \frac{\alpha + \gamma - 1}{2}$. Then

$$\int_0^T \int_R [(\rho^b - \bar{\rho}^b)_x]^2(x, t) dt dx \leq C \tag{4.13}$$

Choosing $s > b + 1$, one has

$$(\rho^s - \bar{\rho}^s)^2 = \int_{-\infty}^{x} [(\rho^s - \bar{\rho}^s)^2]_x dx = 2 \int_{-\infty}^{x} (\rho^s - \bar{\rho}^s)(\rho^s - \bar{\rho}^s)_x dx = 2s \int_{-\infty}^{x} (\rho^s - \bar{\rho}^s)(\rho^{s-1}\rho_x) dx = \frac{2s}{b} \int_{-\infty}^{x} (\rho^s - \bar{\rho}^s)[(\rho^b - \bar{\rho}^b)_x\rho^{s-b}] dx \leq \|\rho^s - \bar{\rho}^s\|_{L^2(R)} \|\rho^b - \bar{\rho}^b\|_{L^2(R)} \tag{4.14}$$

Consequently,

$$\int_0^t \sup_{x \in R} (\rho^s - \bar{\rho}^s)^4 dt \leq C \sup_{x \in R} \|\rho^s - \bar{\rho}^s\|^2_{L^2(R)} \int_0^t \|\rho^b - \bar{\rho}^b\|_{L^2(R)}^2 dt \leq C. \tag{4.15}$$
Moreover, applying (4.10), one has
\[
\int_0^t \int_R (\rho^s - \bar{\rho}^s)^4(\rho^s - \bar{\rho}^s)^{2l} dx dt \leq \int_0^t [\sup_{x \in R}(\rho^s - \bar{\rho}^s)^4] \int_R (\rho^s - \bar{\rho}^s)^{2l} dx dt
\]
\[
\leq \sup_t \int_R (\rho^s - \bar{\rho}^s)^{2l} dx \int_0^t \sup_{x \in R}(\rho^s - \bar{\rho}^s)^4 dt \leq C,
\]
where $l \geq 1$ is any real number. Hence
\[
\int_0^t \int_R (\rho^s - \bar{\rho}^s)^{4+2l} dx dt \leq C. \tag{4.17}
\]

Denote $f(t) = \int_R (\rho^s - \bar{\rho}^s)^{4+2l} dx$. Then, from (4.10) and (4.17), one has $f(t) \in L^1(0, \infty) \cap L^\infty(0, \infty)$. Furthermore, direct calculations show that
\[
\frac{d}{dt} f(t) = (4 + 2l) s \int_R (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-1} \rho_x dx
\]
\[
= -(4 + 2l) s \int_R (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-1} (\rho u)_x dx
\]
\[
= (4 + 2l)(3 + 2l) s \int_R (\rho^s - \bar{\rho}^s)^{2+2l}(\rho^s)_x \rho^{s-1} \rho u dx
\]
\[
+ (4 + 2l) s \int_R (\rho^s - \bar{\rho}^s)^{3+2l}(s - 1) \rho^{s-2} \rho_x \rho u dx
\]
\[
= (4 + 2l)(3 + 2l) s \int_R (\rho^s - \bar{\rho}^s)^{2+2l}(\rho^s)_x \rho^{s-1} \rho u dx
\]
\[
+ (4 + 2l) s (s - 1) \int_R (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-2} \rho_x \rho u dx
\]
\[
= J_1 + J_2. \tag{4.18}
\]

Now, we claim that $J_i(t) \in L^2(0, +\infty), \ (i = 1, 2)$. In fact,
\[
J_1(t) = (4 + 2l)(3 + 2l) s \int_R (\rho^s - \bar{\rho}^s)^{2+2l}(\rho^s)_x \rho^{s-1} \rho u dx
\]
\[
= (4 + 2l)(3 + 2l) s \frac{2}{b} \int_R (\rho^s - \bar{\rho}^s)^{2+2l}(\rho^b)_x \rho^{2s-b} u dx
\]
\[
= (4 + 2l)(3 + 2l) s \frac{2}{b} \int_R (\rho^s - \bar{\rho}^s)^{2+2l}(\rho^b)_x \rho^{2s-b-\frac{1}{2}} \sqrt{\rho} u dx
\]
\[
\leq C \|\sqrt{\rho} u\|_{L^2(R)} \| (\rho^b - \bar{\rho}^b)_x \|_{L^2(R)},
\]

Hence, by Lemma 3.3
\[
\int_0^t |J_1(t)|^2 dt \leq C \sup_{t \in [0,T]} \|\sqrt{\rho} u\|_{L^2(R)}^2 \int_0^t \| (\rho^b - \bar{\rho}^b)_x \|_{L^2(R)}^2 dt \leq C. \tag{4.20}
\]
Letting 

\[ J_2(t) = (4 + 2l)s(s - 1) \int_R (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-1} \rho u dx \]

\[ = (4 + 2l)(s - 1) \int_R (\rho^s - \bar{\rho}^s)^{3+2l} (\rho^b)_x \rho^{s-b} u dx \]

\[ = \frac{(4 + 2l)(s - 1)s}{b} \int_R (\rho^s - \bar{\rho}^s)^{3+2l} (\rho^b)_x \rho^{s-b} u dx \]

\[ \leq C \| \sqrt{\rho u} \|_{L^2(R)} \| (\rho^b - \bar{\rho}^b)_x \|_{L^2(R)}. \]  \tag{4.21}

Using Lemma 3.3 again, one has

\[ \int_0^t |J_2(t)|^2 dt \leq C \sup_{t \in [0,T]} \| \sqrt{\rho u} \|_{L^2(R)}^2 \int_0^t \| (\rho^b - \bar{\rho}^b)_x \|_{L^2(R)}^2 dt \leq C. \]  \tag{4.22}

Consequently,

\[ \frac{d}{dt} f(t) \in L^2(0, +\infty). \]  \tag{4.23}

Combining the obtained fact that \( f(t) \in L^1(0, \infty) \cap L^\infty(0, \infty) \), one has

\[ f(t) \to 0, \quad t \to +\infty. \]  \tag{4.24}

Letting \( m \geq 1 \) be any real number to be determined later, we have

\[ |\rho^s - \bar{\rho}^s|^m = | \int_{-\infty}^x [(\rho^s - \bar{\rho}^s)^m]_{x} dx | = \left| m \int_{-\infty}^{x} (\rho^s - \bar{\rho}^s)^{m-1}(\rho^s - \bar{\rho}^s)_x dx \right| \]

\[ = \left| m \int_{-\infty}^{x} (\rho^s - \bar{\rho}^s)^{m-1} \left[ \frac{s}{\alpha - \frac{1}{2}} (\rho^{s-\alpha+\frac{1}{2}} (\rho^{\alpha-\frac{1}{2}})_x) \right] dx \right| \]  \tag{4.25}

\[ \leq C \| (\rho^s - \bar{\rho}^s)^{m-1} \|_{L^2(R)} \| (\rho^{\alpha-\frac{1}{2}})_x \|_{L^2(R)} \]

\[ \leq C \| (\rho^s - \bar{\rho}^s)^{m-1} \|_{L^2(R)}. \]

Choosing \( 2(m - 1) = 4 + 2l \), one has

\[ \sup_{x \in R} |\rho^s - \bar{\rho}^s|^m \leq C f^{\frac{1}{2}}(t) \to 0, \quad t \to 0. \]  \tag{4.26}

Therefore, \( \lim_{t \to +\infty} \sup_{x \in R} |\rho^s - \bar{\rho}^s| = 0 \). Using the fact that

\[ |\rho - \bar{\rho}|^s = |\rho - \bar{\rho}|^s 1_{0 \leq \rho \leq \frac{\bar{\rho}}{2}} + |\rho - \bar{\rho}|^s 1_{\rho > \frac{\bar{\rho}}{2}} \leq C |\rho^s - \bar{\rho}^s|^s 1_{0 \leq \rho \leq \frac{\bar{\rho}}{2}} + C |\rho^s - \bar{\rho}^s|^s 1_{\rho > \frac{\bar{\rho}}{2}}. \]

Hence,

\[ \sup_{x \in R} |\rho - \bar{\rho}|^s \leq C \sup_{x \in R} |\rho^s - \bar{\rho}^s|^s + C \sup_{x \in R} |\rho^s - \bar{\rho}^s|^s \to 0, \quad t \to +\infty, \]

which implies that

\[ \lim_{t \to +\infty} \sup_{x \in R} |\rho - \bar{\rho}| = 0. \]
If $\bar{\rho} = 0$, from Lemma 3.2, we have that $\rho^s \leq C \bar{\rho}$ for any $s > \frac{\gamma}{2}$. Hence,

$$\int_R \rho^s dx \leq C \int_R \rho^{\gamma} dx \leq C. \tag{4.27}$$

Similarly,

$$\int_R (\rho^s)^{2\lambda} dx \leq \int_R \rho^{2s\lambda} dx \leq C \int_R \rho^{\gamma\lambda} dx \leq C, \tag{4.28}$$

for any $s > \frac{\gamma}{2}, \lambda > 1$. Moreover, one has

$$\int_R |(\rho^{2s\lambda})_x| dx = 2\lambda s \int_R |\rho^{2s\lambda-1}\rho_x| dx = 2\lambda s \int_R |\rho^{2s\lambda-1} \rho^{\frac{3}{2}-\alpha} \rho^{\alpha-\frac{3}{2}} \rho_x| dx$$

$$\leq \frac{2s\lambda}{|\alpha - \frac{1}{2}|} (\int_R \rho^{4s\lambda-2\alpha+1} dx)^{\frac{1}{2}} (\int_R |(\rho^{\alpha-\frac{1}{2}})_x|^2 dx)^{\frac{1}{2}} \leq C. \tag{4.29}$$

Denote $b = \frac{\alpha+\gamma-1}{2}$. Then we have

$$\int_0^T \int_R [(\rho^b)_x^2] (x,t) dx dt \leq C, \tag{4.30}$$

by Lemma 3.3 and $\bar{\rho} = 0$. Choosing $s > b + \frac{\gamma}{2}$, one has

$$\rho^s = \int_{-\infty}^x (\rho^s)_x dx = 2s \int_{-\infty}^x \rho^{2s\lambda-1} \rho_x dx = \frac{2s}{b} \int_{-\infty}^x \rho^{2s\lambda-\alpha} \rho_x^{\alpha} (\rho^b)_x dx$$

$$= \frac{2s}{b} (\int_{-\infty}^x \rho^{2s\lambda+1-(\alpha+\gamma)} dx)^{\frac{1}{2}} (\int_{-\infty}^x (\rho^b)_x^2 dx)^{\frac{1}{2}} \leq C \|\rho\|_{L^\gamma(R)} \|\rho^b\|_{L^2(R)}. \tag{4.31}$$

Consequently,

$$\int_0^t \sup_{x \in R} \rho^s dx \leq C \sup_{x \in R} \rho^2_{L^\gamma(R)} \int_0^t \|\rho^b\|_{L^2(R)}^2 dt \leq C. \tag{4.32}$$

Moreover, applying (4.28), one has

$$\int_0^t \int_R (\rho^s)^{4+2l} dx dt \leq \int_0^t (\sup_{x \in R} \rho^s) \int_R \rho^{2s} dx dt \leq \sup_{t} \int_R \rho^{2s} dx \int_0^t \sup_{x \in R} \rho^s dt \leq C, \tag{4.33}$$

where $l \geq 1$ is any real number.

Denote $f(t) = \int_R (\rho^s)^{4+2l} dx$. Then, from (4.28) and (4.33), one has $f(t) \in L^1(0, \infty) \cap L^\infty(0, \infty)$. The left is same as in Case 1 ($\bar{\rho} > 0$).

The proof of the theorem is finished. □

The proof of Theorem 2.4 is completely same as in [23, 18, 19] and we omit it here.

**References**

[1] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.*, 238(1-2) (2003) 211-223.

[2] D. Bresch and B. Desjardins, Quelques modeles diffusifs capillaires de type Korteweg, *C. R. Acad. Sci.* , Paris, section mecanique 332 (11)(2004) 881-886.
[3] D. Bresch and B. Desjardins, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models, *J. Math. Pures Appl.* 86 (2006), 362-368.

[4] D. Bresch, B. Desjardins, Chi-Kun Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations* 28(3-4) (2003), 843-868.

[5] D. Bresch, B. Desjardins, D. Gerard-Varet, On compressible Navier-Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.* 87 (2006), 362-368.

[6] D. Bresch, B. Desjardins, Chi-Kun Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations* 28(3-4) (2003), 843-868.

[7] C. S. Dou, Q. S. Jiu, A remark on free boundary problem of 1-D compressible Navier-Stokes equations with density-dependent viscosity, *Math. Meth. Appl. Sci.* 33 (2010), 103-116.

[8] E. Feireisl, A. Novotný and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations of isentropic compressible fluids, *J. Math. Fluid Mech.* 3 (2001) 358-392.

[9] J.F. Gerbeau, B. Perthame. Derivation of Viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation, Discrete and Continuous Dynamical Systems, *Ser. B, Vol. 1, Num. 1* (2001) 89-102.

[10] Z. Guo, Q. Jiu, Z. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *SIAM J. Math. Anal.* 39 (2008) 1402-1427.

[11] D. Hoff, Global existence of 1D compressible isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc. 303* (1) (1987) 169-181.

[12] D. Hoff, Strong convergence to global solutions for multidimensional flows of compressible viscous fluids with polytropic equations of state and discontinuous initial data, *Arch. Rat. Mech. Anal.* 132 (1995) 1-14.

[13] D. Hoff, The zero-Mach limit of compressible flows, *Comm. Math. Phys.* 192 (1998) 543-554.

[14] D. Hoff, D. Serre, The failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow, *SIAM J. Appl. Math.* 51 (1991) 887-898.

[15] D. Hoff, J. Smoller, Non-formation of vacuum states for compressible Navier-Stokes equations, *Comm. Math. Phys.* 216 (2001) no. 2, 255-276.

[16] S. Jiang, Global smooth solutions of the equations of a viscous, heat-conducting one-dimensional gas with density-dependent viscosity, *Math. Nachr.* 190 (1998) 169-183.

[17] S. Jiang, Z. P. Xin and P. Zhang, Global weak solutions to 1D compressible isentropy Navier-Stokes with density-dependent viscosity, *Methods and Applications of Analysis 12* (3) (2005) 239-252.

[18] Q. Jiu, Z. P. Xin, The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients, *Kinet. Relat. Models* 1 (2) (2008) 313-330.

[19] Q. Jiu, Y. Wang, Z.P. Xin, Stability of Rarefaction Waves to the 1D Compressible Navier-Stokes Equations with Density-dependent Viscosity, *Comm. Partial Differential Equations* 36 (2011) 602-634.

[20] Q. Jiu, Y. Wang, Z.P. Xin, Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum, arXiv:1202.1382v1.

[21] J. I. Kanel, A model system of equations for the one-dimensional motion of a gas. (Russian) *Differencial’nye Uravnenija* 4 (1968) 721-734.

[22] A. V. Kazhikhov, V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* 41 (1977) 273-282; translated from *Prikl. Mat. Meh.* 41 (1977) 282-291.

[23] H. L., Li, J. Li, Z. P. Xin, Vanishing of Vacuum States and Blow-up Phenomena of the Compressible Navier-Stokes Equations, Comm. Math. Phys. 281(2) (2008) 401-444.

[24] P. L. Lions, *Mathematical Topics in Fluid Dynamics 2, Compressible Models*, Oxford Science Publication, Oxford, 1998.
[25] T. P. Liu, Z. P. Xin, T. Yang, Vacuum states of compressible flow, *Discrete Contin. Dynam. Systems* 4(1)(1998) 1-32.

[26] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* 20(1) (1980) 67-104.

[27] A. Mellet and A. Vasseur, On the isentropic compressible Navier-Stokes equation, *Comm. Partial Differential Equations* 32(2007) 431-452.

[28] A. Mellet and A. Vasseur, Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations, *SIAM J. Math. Anal.* 39(2008) No. 4, 1344-1365.

[29] J. Nash, Le probleme de Cauchy pour Les equations differentielles d’un fluids general, *Bulletin de la S. M. F.* 90 (1962) 487-497.

[30] M. Okada, S. Matusu-Necasova, T. Makino, Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity, *Ann. Univ. Ferrara Sez. VII (N.S.)* 48 (2002) 1-20.

[31] D. Serre, Sur l’équation monodimensionnelle d’un fluide visqueux, compressible et conducteur de chaleur, *C. R. Acad. Sci. Paris Sér. I* 303 (1986) 703-706.

[32] V. A. Vaigant and A. V. Kazhikhov, On existence of global solutions to the two-dimensional Navier-Stokes equations for a compressible viscous fluid, *Siberian J. Math.* 36 (1995) 1283-1316.

[33] S. W. Vong, T. Yang, C. J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum II, *J. Differential Equations* 192 (2003) 475-501.

[34] J. Wei, L. He, Z. Guo, A Remark on the Cauchy Problem of 1D Compressible Navier-Stokes Equations with Density-dependent Viscosity Coefficients, Accepted by Acta Mathematicae Applicatae Sinica, English Series, 2011.

[35] Z. P. Xin, Blow-up of smooth solution to the compressible Navier-Stokes equations with compact density, *Comm. Pure Appl. Math.* 51 (1998) 229-240.

[36] T. Yang, Z. A. Yao, C. J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, *Comm. Partial Differential Equations* 26 (5-6) (2001) 965-981.

[37] T. Yang, H. J. Zhao, A vacuum problem for the one-dimensional Compressible Navier-Stokes equations with density-dependent viscosity, *J. Differential Equations* 184 (1)(2002) 163-184.

[38] T. Yang, C. J. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum, *Comm. Math. Phys.* 230 (2)(2002) 329-363.

[39] T. Yang, Z. A. Yao, C. J. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum, *Comm. Partial Differential Equations* 26 (5-6)(2001) 965-981.