LQP: The Dynamic Logic of Quantum Information
Alexandru Baltag* and Sonja Smets†

Abstract
The main contribution of this paper is the introduction of a dynamic logic formalism for reasoning about information flow in composite quantum systems. This builds on our previous work on a complete quantum dynamic logic for single systems. Here we extend that work to a sound (but not necessarily complete) logic for composite systems, which brings together ideas from the quantum logic tradition with concepts from (dynamic) modal logic and from quantum computation. This Logic of Quantum Programs (LQP) is capable of expressing important features of quantum measurements and unitary evolutions of multi-partite states, as well as giving logical characterizations to various forms of entanglement (for example, the Bell states, the GHZ states etc.). We present a finitary syntax, a relational semantics and a sound proof system for this logic. As applications, we use our system to give formal correctness proofs for the Teleportation protocol and for a standard Quantum Secret Sharing protocol; a whole range of other quantum circuits and programs, including other well-known protocols (for example, superdense coding, entanglement swapping, logic-gate teleportation etc.), can be similarly verified using our logic.

1 Introduction
As a natural extension of Hoare Logic, Propositional Dynamic Logic (PDL) is an important tool for the logical study of programs, especially by providing a basis for program verification. In the context of recent advances in quantum programming, it is natural to look for a quantum version of PDL, which could play the

*Oxford University Computing Laboratory, UK
†Vrije Universiteit Brussel, Flanders’ Fund for Scientific Research Post-doc, Belgium
The same role in proving correctness for quantum programs that classical PDL (and Hoare logic) played for classical programs.

The search for such a 'quantum PDL' has been one of the main objectives of our previous investigation into the logic of quantum information flow. In a series of presentations [Baltag 2004] and papers [Baltag and Smets 2005a, Baltag and Smets 2004, Baltag and Smets 2005b], we have proposed several logical systems: in [Baltag and Smets 2005a] we focused on single systems and presented two equivalent complete axiomatizations for a Logic of Quantum Actions, LQA, which allows actions such as measurements and unitary evolutions, but no entanglements. The completeness result was obtained with respect to infinite-dimensional classical Hilbert spaces, as models for single quantum systems. The challenge of providing a similar axiomatization for compound systems was taken up in [Baltag and Smets 2004], where a first proposal for a logic of multi-partite quantum systems was sketched.

In this paper we elaborate further, simplify and improve on the work outlined in [Baltag and Smets 2004], and develop a full-fledged Logic of Quantum Programs LQP. This includes:

1. A simple finitary syntax for a modal language, based on a minor variation of classical PDL, with dynamic modalities corresponding to (weakest preconditions of) quantum programs.

2. A relational semantics for this logic, in terms of quantum states and quantum actions over a finite-dimensional Hilbert space.

3. A sound (but not necessarily complete) proof system, which includes ax-

A single system is just an isolated physical system; the possible states of such a system are represented in quantum mechanics as rays in some Hilbert space. By contrast, a composite (also called compound, or multi-partite) system is one that we can think of as being composed of two (or more) distinct physical (sub)systems. The corresponding Hilbert space is the tensor product of each of the spaces associated to the subsystems. So, in a sense, single systems subsume composite systems (since any tensor product of Hilbert spaces is just another Hilbert space). However, treating a system as being composite amounts to having a more detailed complex theory of the system (compared with treating it as a single system) - a theory that captures the specific features arising from being a composite structure, in addition to the general features of any physical system. A 'logic' for compound systems will thus be a richer logic than one for single systems.

But note the difference between our logic LQP and the approach with a similar name in [Brunet and Jorrand 2003]: our dynamic logic goes much further in capturing essential properties of quantum systems and quantum programs, as well as in recovering the ideas of traditional quantum logic (see e.g. [Dalla Chiara and Giuntini 2002, Dalla Chiara et al 2004, Goldblatt 1974]).

Unlike the case of infinite-dimensional single systems, for which a complete logic was given in Baltag and Smets [2005a], the problem of finding a complete proof system for the logic LQP is still open.
ions to handle separation, locality and entanglement.

4. Formal proofs (in our proof system LQP) of non-trivial computational properties of compound quantum systems.

5. An analysis (with a formal correctness proof) of the Teleportation and quantum secret sharing protocols.

More generally, the strength of LQP lies in the fact that it can provide fully formal correctness proofs for a whole class of quantum circuits and protocols, a class that includes logic-gate teleportation, superdense coding and entanglement swapping, as well as more complex circuits built using quantum gates and measurements.

The logic introduced here brings together a number of ideas from several fields: theoretical foundations of quantum mechanics, operational quantum logic, dynamic modal logic, spatial logic and quantum computation. In the rest of this section we give an overview of the main concepts underlying the logic LQP.

The first fundamental idea of our approach connects two independent lines of research. The first is the long tradition in the logical-algebraic foundations of quantum mechanics, which, in particular, has produced various ‘dynamic’ interpretation of quantum logic (QL) in \cite{Daniel 1982, Daniel 1989, Faure et al 1995, Amira et al 1998, Coecke 2000, Coecke et al 2001, Coecke Moore Smets 2004, Coecke and Smets 2004, Coecke et al 1999, Smets 2001, Smets 2005}. The second line is the work on modal ‘action’ logics in Computer Science, the main example being Dynamic Logic (PDL) and its relatives (Hoare logic, but also dynamic interpretations of basic modal logics as languages for ‘processes’ or labelled transition systems, for example, Hennesey-Milner logic).

We stress the fact that, until our recent work \cite{Baltag and Smets 2005a}, these two traditions were not only independent, but did not even share a common language. The use of the word ‘dynamic’ in the QL tradition did not have much in common with ‘dynamic’ logic; QL aimed for an algebraic axiomatisation of quantum systems based on the non-distributive lattice of ‘quantum properties’, structure obtained by abstracting away from the lattice of projectors in a Hilbert space $\mathcal{H}$ (or, equivalently, the lattice of closed linear subspaces of $\mathcal{H}$); the goal was to obtain representation theorems for these logical structures with respect to Hilbert spaces, thus allowing one to claim a ‘rational’, ‘logical’ (or ‘operational’) reconstruction of quantum mechanics.\footnote{This goal was partially realised in \cite{Piron 1964} and \cite{Piron 1976}, and later improved on in \cite{Soler 1995} and \cite{Mayet 1998}, and related work.} In this context, the ‘dynamic’ twist has to do with the addition of features belonging to physical dynamics to the standard (static) QL description:
• First, a ‘dynamic’ interpretation was given to the main structure (the lattice of properties) and the logical connectives (quantum implication and quantum disjunction) of $QL$: in, for example, [Smets 2001], [Coecke and Smets 2004] and [Coecke Moore Smets 2004] (and partially anticipated in [Hardegree 1975, Hardegree 1979] and [Beltrametti and Cassinelli 1977]) the quantum-logical connectives are interpreted dynamically, as expressing potential causality (that is, what in computer science is known as weakest preconditions).

• Second, some of the researchers in $QL$ went on to incorporate ‘true’ physical dynamics, that is, Schrödinger flows (unitary evolutions), into the algebra as operators on the underlying lattice; the resulting structure is a quantale of ‘quantum actions’, which was introduced and investigated in [Coecke et al 2001] and [Coecke et al 1999]. In contrast, the modal logician’s (and the computer scientist’s) use of ‘dynamics’ refers to modelling processes as labelled transition systems (Kripke models), in which the possible ‘actions’ are represented as binary relations between possible states, and the natural descriptive language is modal, having dynamic modalities to express weakest preconditions (ensuring given post-conditions after specific actions). Thus, the ‘first fundamental idea’ of our logic, an idea first presented in [Baltag 2004] and [Smets 2004] and published in [Baltag and Smets 2005a], is to connect these two traditions by giving a quantum (re)interpretation of Dynamic Logic, in which both (projective) measurements and unitary evolutions are treated as modal actions, and to use this formalism in order to improve on the known representation theorems in $QL$. In this quantum reinterpretation, the ‘test’ actions $\varphi ?$ of $PDL$ (which are used to capture conditional programs in dynamic logic) are to be read as ‘successful measurements’ of a quantum property $\varphi$ (that is, as projectors in a Hilbert space over the subspace generated by the set of states satisfying $\varphi$), while the other basic actions of $PDL$ are taken to be quantum gates (i.e. unitary operators on a Hilbert space). As shown in [Baltag 2004, Smets 2004, Baltag and Smets 2005a], this immediately allows us to re-capture in our (Boolean) logic all the power of traditional (non-Boolean) Quantum Logic: the ‘quantum disjunction’ (expressing superpositions), the ‘quantum negation’ (the so-called ‘orthocomplement’ $\varphi \sim$, which expresses the necessary failure of a measurement) and the ‘quantum implication’ (the so-called ‘Sasaki hook’ $\varphi \rightarrow \psi$, which captures causality in quantum measurements) are all expressible

5See, for example, [Harel et al 2000] for an introduction to dynamic modalities $[\pi] \psi$ describing weakest preconditions ensuring (the satisfaction of some post-condition) $\psi$ after the execution of action $\pi$. 

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using quantum-dynamic modalities $[\varphi?]\psi$ (which capture weakest preconditions of quantum measurements). In other words: in our logic (unlike other logical approaches to quantum systems), all the non-classical ‘quantum’ effects are captured using a non-classical ‘logical dynamics’, while keeping the classical, Boolean structure of the underlying propositional logic of ‘static’ properties.

The second fundamental idea of our approach was originally outlined in [Baltag and Smets 2004], and consists of adding spatial features to dynamic logic, in order to capture relevant properties of multi-partite (that is, compound) quantum systems (for example, separation, locality, entanglement). For this, we use a finite set $N$ of indices to denote the most basic ‘parts’ (qubits) of the system, and use sets of indices $I \subseteq N$ to denote all the (possibly compound) subsystems; we have special propositional constants $1, 0, +$, and so on, to express the fact that qubits are in the state $|1\rangle, |0\rangle$ or $|+\rangle$, and so on; we use a basic propositional formula $\top_I$ to express ‘separation’ (the fact that qubits in the subsystem $I$ are separated from the rest); and we have a basic program $\top_I$, denoting a non-determined (that is, randomly chosen) local transformation (affecting only the qubits in the subsystem $I$). These ingredients are enough to define all the relevant spatial features we need, and in particular to define the notion of (local) component $\varphi_I$ of a (global) property $\varphi$, the notion of (I-)local property $I(\varphi)$ (that is, $\varphi$ is a property of the separated $I$-subsystem) and the notion of (I-)local program $P(\pi)$ (that is, $\pi$ is a program affecting only the $I$-subsystem).

The third fundamental idea that underlies our approach comes from [Coecke 2000] and [Coecke 2004], and was further elaborated in a category-theoretical setting in [Abramsky and Coecke 2005]: this is a computational understanding of entanglement, in which an entangled state is seen as a ‘static’ encoding of a program. Mathematically, this comes from the simple observation that a tensor product $H_i \otimes H_j$ of two Hilbert spaces is canonically isomorphic to the space $H_i \to H_j$ of all linear maps between the two spaces. But, as noted in [Coecke 2000] [Coecke 2004], this isomorphism has a physical meaning: the entangled state $\pi_{ij}$, which ‘encodes’ (via the above isomorphism) the linear map $\pi : H_i \to H_j$, has the property that any successful measurement of its $i$-th qubit (resulting in some local output-state $q_i$) induces a correlative collapse of the $j$-th

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\textsuperscript{6}Indeed, it turns out that a quantum implication $\phi S \psi$ is simply equivalent to the weakest precondition $[\varphi?]\psi$. In quantum logic, this dynamic view can be traced back to the analysis of the Sasaki hook as a Stalnaker conditional presented in [Hardegree 1995] [Hardegree 1999] and is reflected on in, for example, [Beltrametti and Cassinelli 1979] and [Sernadas 2001].

\textsuperscript{7}This can be compared with the exogenous quantum logic approach in [Mateus and Sernadas 2004], which makes use of general modal operators to separate subsystems.
qubit, whose local output-state (after the collapse) is computed by the map $\pi$ (that is, it is given by $\pi(q)_j$). So the above isomorphism captures the correlations between possible results of potential local measurements (on the two qubits). We use this idea to define formulas $\pi_{ij}$ that characterize such specifically entangled states (by using weakest preconditions to express potential behavior under possible measurements). The fundamental correlation given by the above isomorphism is then stated as our ‘Entanglement Axiom’, which plays a central role in our system.

This combination of quantum-dynamic and spatial logic is what allows us to give a logical characterization of Bell states and of various quantum gates, and to prove from our axioms for highly non-trivial properties of quantum information flow (such as the ‘Teleportation Property, the ‘Agreement Property’, the ‘Entanglement Preparation’ and ‘Entanglement Composition’ lemmas etc.).

It is well-known that PDL, and its fragment the Hoare Logic, are among the main logical formalisms used in program verification of classical programs, i.e. in checking that a given (classical) program is correct (in the sense of meeting the required specifications). It is thus natural to expect our quantum dynamic logic to play a significant role in the formal verification of quantum programs. In this paper, we partially fulfill this expectation by giving a fully axiomatic correctness proof for the Teleportation protocol and for a Quantum Secret Sharing protocol; more details, and similar proofs for other quantum programs (Logic-Gate Teleportation, Super-Dense Coding, Entanglement Swapping, and so on) can be found in [Akatov 2005]. More generally, our logic can be used for the formal verification of a whole range of quantum programs, including all the circuits covered by the ‘entanglement networks’ approach in [Coecke 2004].

Finally, we mention here some of the limitations of our approach, which arise from our purely qualitative, logic-based view of quantum information. The quantitative aspects are thus neglected: in our presentation, we follow the operational quantum logic tradition, as in, for example, [Jauch 1968] and [Jauch and Piron 1969], by abstracting away from complex numbers, ‘phases’ and probabilities. As customary in quantum logic, we identify the ‘states’ of a physical system with rays in a Hilbert space, rather than with unitary vectors, and consequently, our programs will be ‘phase-free’. This is a serious limitation, as phase aspects are important in quantum computation; there are ways to re-introduce (relative) phases in our approach, but this gives rise to a much more complicated logic, and so we will leave this development for future work. Similarly, although our dynamic logic cannot

\footnote{Indeed, one may claim that any quantum circuit in which probabilities do not play an essential role can, in principle, be verified using our logic (or some trivial extension obtained by adding constants for other relevant states and logic gates).}

\footnote{A ray is a one-dimensional linear subspace.}
express probabilities, but only ‘possibilities’ (via the dynamic modalities, which capture the system’s potential behavior under possible actions), there exist natural extensions of this setting to a probabilistic modal logic. One of our projects is to work out the full details of this setting, developing a proof system for probabilistic $LQP$.

2 Preliminaries: quantum frames

In this section we organize Hilbert spaces as relational structures, called quantum frames (also called quantum transition systems in [Baltag and Smets 2005a]). We first study the quantum frames of single quantum systems, then we consider systems compound system, that is, the quantum frames corresponding to the tensor products of Hilbert spaces, which represent physical systems that can be thought of as being composed of parts (subsystems). In this latter case we restrict our attention to systems composed of finitely many ‘qubits’.

2.1 Single-system quantum frames

A modal frame is a set of states, together with a family of binary relations between states. A (generalised) PDL frame is a modal frame $(\Sigma, \{S\rightarrow_t\}_{S \in \mathcal{L}}, \{a\rightarrow\}_{a \in \mathcal{A}})$, in which the relations on the set of states $\Sigma$ are of two types: the first, called tests and denoted by $S\rightarrow_t$, are labelled with subsets $S$ of $\Sigma$, coming from a given family $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$ of sets, called testable properties; the others, called actions, are labelled with action labels $a$ from a given set $\mathcal{A}$.

Given a PDL frame, there exists a standard way to give a semantics to the usual language of Propositional Dynamic Logic. Classical PDL can be considered as a special case of such a logic, in which tests are given by classical tests: $s \xrightarrow{S} t$ if and only if $s = t \in S$. Observe that classical tests, if executable, do not change the current state.

In the context of quantum systems, a natural idea is to replace classical tests by ‘quantum tests’, given by quantum measurements. Such tests will obviously change the state of the system. To model them, we introduce a special kind of PDL frames: quantum frames. The tests are essentially given by projectors in a Hilbert space, while the other basic actions are given by unitary evolutions. In [Baltag and Smets 2005a], we considered PDL with this non-standard semantics, having essentially the same truth clauses as in the classical case, but interpreted in quantum frames. What we obtained was a ‘quantum PDL’, in which the traditional
(orthomodular) ‘quantum logic’ could be embedded as a fragment (corresponding to the negation-free, test only part of quantum PDL). In this paper, we extend the syntax of this logic to deal with subsystems and entanglements.

Recall that a Hilbert space $\mathcal{H}$ is a complex vector space with an inner product $\langle - \mid - \rangle$, which is complete in the induced metric. The adjoint (or Hermitian conjugate) of a linear map $F : \mathcal{H} \to \mathcal{H}$ is the unique linear map $F^\dagger : \mathcal{H} \to \mathcal{H}$ s.t. $\langle x \mid F(y) \rangle = \langle F^\dagger(x) \mid y \rangle$, for all $x,y \in \mathcal{H}$. For any closed linear subspace $W \subseteq \mathcal{H}$, the projector $P_W : \mathcal{H} \to \mathcal{H}$ onto $W$ is given by: $P_W(u + v) = u$, for all $u \in W, v \in W^\perp$. Projectors are linear, idempotent ($P \circ P = P$) and self-adjoint ($P^\dagger = P$). A unitary transformation is a linear map $U$ on $\mathcal{H}$ s.t. $U \circ U^\dagger = U^\dagger \circ U = id$, where $id$ is the identity on $\mathcal{H}$. Unitary operators preserve inner products.

In Quantum Mechanics, projectors are used to represent (successful) measurements. A measurement is in fact a set of projectors (over mutually orthogonal subspaces); but, whenever a measurement is successfully performed, only one of the projectors is ‘actualised’: the outcome is given by that particular projector. In Quantum Mechanics, unitary transformations represent reversible evolutions of a system. In Quantum Computation, they correspond to quantum-logical gates.

**Quantum frames**

Given a Hilbert space $\mathcal{H}$, the following steps construct a Quantum (PDL) Frame

$$\Sigma(\mathcal{H}) := (\Sigma, \{S \mapsto \} s \in \mathcal{L}, \{U \mapsto \} U \in \mathcal{U})$$

1. Let $\Sigma$ be the set of one dimensional subspaces of $\mathcal{H}$, called the set of states. We denote a state $s = \overline{x}$ of $\mathcal{H}$ using any of the non-zero vectors $x \in \mathcal{H}$ that generate it, as a subspace. Note that any two vectors that differ only in phase (that is, $x = \lambda y$, with $\lambda \in \mathbb{C}$ with $|\lambda| = 1$) will generate the same state $\overline{x} = \overline{y} \in \Sigma$.

2. We call two states $s$ and $t$ in $\Sigma$ orthogonal, and write $s \perp t$, if every two vectors $x \in s, y \in t$ are orthogonal, that is, if $\forall x \in s \forall y \in t \langle x \mid y \rangle = 0$. Equivalently, we can state that $s \perp t$ iff $\exists x \in s, y \in t$ with $x \neq 0, y \neq 0$ and $\langle x \mid y \rangle = 0$. We put $S^\bot := \{t \in \Sigma \mid t \perp s \text{ for all } s \in S\}$; and we denote by $\overline{S} = S^{\perp \perp} := (S^\bot)^\bot$ the biorthogonal closure of $S$. In particular, for a singleton $\{x\}$, we just write $\overline{x}$ for $\{x\}$, which agrees with the notation $\overline{x}$ used above to denote the state generated by $x$. 

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10See, for example, see e.g. [Dalla Chiara and Giuntini 2002](#cite-10) [Dalla Chiara et al 2004](#cite-11) and [Goldblatt 1974](#cite-12).
3. A set of states $S \subseteq \Sigma$ is called a (quantum) testable property iff it is biorthogonally closed, i.e. if $\overline{S} = S$. (Note that $S \subseteq \overline{S}$ is always the case.) We use $\mathcal{L} \subseteq P(\Sigma)$ to denote the family of all quantum testable properties. All the other sets $S \in P(\Sigma) \setminus \mathcal{L}$ are called non-testable properties.

4. There is a natural bijective correspondence between the family $\mathcal{L}$ of all testable properties and the family $\mathcal{W}$ of all closed linear subspaces $W$ of $\mathcal{H}$, the bijection being given by $S \mapsto \overline{W} =: \bigcup S$. Observe that, under this correspondence, the image of the biorthogonal closure $\overline{S}$ of any arbitrary set $S \subseteq \Sigma$ is the closed linear subspace $\bigcup S \subseteq \mathcal{H}$ generated by the union $\bigcup S$ of all states in $S$.

5. For each testable property $S \in \mathcal{L}$, there exists a partial map $S^? : \Sigma \rightarrow \Sigma$, called a quantum test. If $W = \overline{W} = \bigcup S$ is the corresponding subspace of $\mathcal{H}$, then the quantum test is the map induced on states by the projector $P_W$ onto the subspace $W$. In other words, it is given by:

$$
S^?(x) := \begin{cases}
P_W(x) \in \Sigma, & \text{if } x \not\in \overline{S}, \text{ i.e. if } P_W(x) \neq 0 \\
\text{undefined}, & \text{otherwise}
\end{cases}
$$

We use $S^? \subseteq \Sigma \times \Sigma$ to denote the binary relation corresponding to the partial map $S^?$ that is given by: $s S^? t$ if and only if $S^?(s) = t$. So we have a family of binary relations indexed by the testable properties $S \in \mathcal{L}$.

6. For each unitary transformation $U$ on $\mathcal{H}$, consider the corresponding binary relation $U^\rightarrow \subseteq \Sigma \times \Sigma$, given by: $s U^\rightarrow t$ if and only if $U(x) = y$ for some non-zero vectors $x \in s, y \in t$. So we obtain a family of binary relations indexed by the unitary transformations $U \in \mathcal{U}$ (where $\mathcal{U}$ is the set of unitary transformations on $\mathcal{H}$).

So a quantum frame is just a PDL frame built on top of a given Hilbert space $\mathcal{H}$, by taking one-dimensional subspaces as ‘states’, projectors as ‘tests’ and unitary evolutions as ‘actions’. Our notion of ‘state’ in this paper is closely connected to the way quantum logicians approach quantum systems. As mentioned in the introduction, this imposes some limits to our approach - mainly that we will not be able to express phase-related properties.

**Operators on states, adjoints and generalised tests**

To generalise the notation we introduced earlier, observe that every linear operator $F : \mathcal{H} \rightarrow \mathcal{H}$ induces a partial map $F^? : \Sigma \rightarrow \Sigma$ on states (that is, subspaces),
given by $F(\tau) = \overline{F(x)}$, if $F(x) \neq 0$ (and undefined, otherwise). (Note that linearity ensures that this map on states is well-defined.) In particular, every map $F : \Sigma \to \Sigma$ obtained in this way has an adjoint $F^\dagger : \Sigma \to \Sigma$, defined as the map on states induced by the adjoint of the linear operator $F$ on $\mathcal{H}$. Observe that, for unitary transformations $U$, the adjoint is the inverse: $U^\dagger = U^{-1}$. Also, one can naturally generalise quantum tests to arbitrary, possibly non-testable properties, $S \subseteq \Sigma$, by putting: $S^\dagger := S^\dagger$. So we identify a test of a ‘non-testable’ property $S$ with the quantum test of its biorthogonal closure. Observe that $S^\dagger = \overline{S}$ (since projectors are self-adjoint).

**Measurement (non-orthogonality) relation**

For all $s, t \in \Sigma$, let $s \to t$ if and only if $s \rightarrow^S t$ for some property $S \in \mathcal{L}$. In other words, $s \to t$ means that one can reach state $t$ by doing some measurement on state $s$. An important observation is that the measurement relation is the same as non-orthogonality: $s \to t$ iff $s \not\perp t$.

**Quantum actions**

A quantum action is any relation $R \subseteq \Sigma \times \Sigma$ that can be written as an arbitrary union $R = \bigcup_i F_i$ of linear maps $F_i : \Sigma \to \Sigma$. The family of quantum actions forms a complete lattice (with inclusion), having set-theoretic union $R \cup R'$ as supremum. Notice also that this family is closed under relational composition

$$R; R' := \{(s, t) \in \Sigma \times \Sigma : \exists w \in \Sigma(s, w) \in R, (w, t) \in R'\}$$

and iteration $R^n := \bigcup_{k \geq 0} R^n$ (where $R^n = R; R; \cdots; R$ is a composition of $n$ terms). Quantum actions are a relational (input-output) representation of quantum programs. Indeed, in our dynamic logic we will interpret (the dynamic modalities for) quantum programs as (weakest preconditions of) quantum actions.

**Weakest precondition, image, strongest post-condition and measurement modalities**

For any property $T \subseteq \Sigma$ and any quantum action $R \subseteq \Sigma \times \Sigma$, let

$$[R]T := \{s \in \Sigma : \forall t \in \Sigma(sRt \Rightarrow t \in T)\} \text{ and } \langle R \rangle T := \Sigma \setminus ([R](\Sigma \setminus T)).$$

Similarly, put

$$R(T) := \{s \in \Sigma : \exists t \in T \text{ such that } tRs\}.$$
We also put $R[T] := \overline{R(T)}$ for the biorthogonal closure of the image. Finally, put $\Box T := \{s \in \Sigma : \forall t(s \rightarrow t \Rightarrow t \in T)\}$ and $\Diamond T := \Sigma \setminus (\Box (\Sigma \setminus T))$.

Observe that $[R]T$ expresses the weakest precondition for the ‘program’ $R$ and post-condition $T$. In particular, $[S?]T$ expresses the weakest precondition ensuring the satisfaction of property $T$ in any state after the system passes a quantum test of property $S$. Similarly, $\langle S? \rangle T$ means that one can perform a quantum test of property $S$ on the current state, ending up in a state having property $T$. $R(T)$ is the image of $T$ via $R$, which is in fact the strongest property (among all properties in $\mathcal{P}(\Sigma \times \Sigma)$) ensured to hold after applying program $R$ if a precondition $T$ holds at the input-state. This is the ‘strongest postcondition’ in an absolute sense. However, the strongest testable postcondition (ensured to hold after running $R$ if precondition $T$ holds at the input state) is given by $R[T]$. $\Box T$ means that property $T$ will hold after any measurement (quantum test) performed on the current state. Finally, $\Diamond T$ means that property $T$ is potentially satisfied, in the sense that one can do some quantum test to reach a state with property $T$.

Lemma 1. For every property $S \subseteq \Sigma$, we have $S^\bot = [S?]\emptyset = \Sigma \setminus \Diamond S$ and $\overline{S} = \Box \Diamond S$.

Proposition 1. For every property $S \subseteq \Sigma$, if $T \in \mathcal{L}$ (in other words, is testable), then $\Box S, S^\bot, [S?]T \in \mathcal{L}$ (are testable), and more generally $[R]T \in \mathcal{L}$, for every quantum relation $R$. For every state $s \in \Sigma$, we have $\{s\} \in \mathcal{L}$, that is, ‘states are testable’.

Proposition 2. A property $S \subseteq \Sigma$ is testable if and only if any of the following equivalent conditions hold:

- $S = \overline{S}$;
- $\exists T \in \Sigma$ such that $S = T^\bot$;
- $\exists T \in \Sigma$ such that $S = \Box T$.

Quantum joins
The family $\mathcal{L}$ of testable properties is a complete lattice with respect to inclusion, having as its meet set-intersection $S \cap T$, and as its join the biorthogonal closure of set-union $S \sqcup T := \overline{S \sqcup T}$, called the quantum join of $S$ and $T$. For any arbitrary property $S \subseteq \Sigma$, we have $\overline{S} = \bigcup\{\{s\} : s \in S\} = \bigcap\{T \in \mathcal{L} : S \subseteq T\}$, so the biorthogonal closure of $S$ is the strongest testable property implied by (the property) $S$. 

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Theorem 1  The following properties hold in every quantum frame $\Sigma = \Sigma(H)$:

1. **Partial functionality**
   If $s \xrightarrow{\Sigma} t$ and $s \xrightarrow{\Sigma} v$ then $t = v$.

2. **Trivial tests**
   $\emptyset \xrightarrow{\Sigma} = \emptyset$ and $\Sigma \xrightarrow{\Sigma} = \Delta_\Sigma$, where $\Delta_\Sigma = \{(s,s) : s \in \Sigma\}$ is the identity relation on $\Sigma \times \Sigma$.

3. **Atomicity**
   States are testable, that is, $\{s\} \in L$. This is equivalent to requiring that ‘states can be distinguished by tests’, that is, if $s \neq t$ then $\exists P \in L : s \perp P, t \not\perp P$.

4. **Adequacy**
   Testing a true property does not change the state: if $s \in P$ then $s \xrightarrow{P}$.

5. **Repeatability**
   Any testable property holds after it has been successfully tested:
   if $s \xrightarrow{P} t$ then $t \in P$.

6. **Compatibility**
   If $S, T \in L$ are testable and $S?; T? = T?; S?$ then $S?; T? = (S \cap T)?$.

7. **Self-Adjointness**
   If $s \xrightarrow{P} w \rightarrow t$, then there exists some element $v \in \Sigma$ such that $t \xrightarrow{P} v \rightarrow s$.

8. **Proper superposition**
   Every two states of a quantum system can be properly superposed into a new state: $\forall s, t \in \Sigma \exists w \in \Sigma s \rightarrow w \rightarrow t$.

9. **Unitary Reversibility and Totality**
   Basic unitary evolutions are total bijective functions, having as adjoint their inverse:
   $U; U^\dagger = U^\dagger; U = id$
   where $id$ is the identity map.

10. **Orthogonality preservation**
    Basic unitary evolutions preserve (non) orthogonality: Let $s, t, s', t' \in \Sigma$ be such that $s \xrightarrow{U} s'$ and $t \xrightarrow{U} t'$; then, $s \rightarrow t$ iff $s' \rightarrow t'$.
Proofs:

1. **Partial functionality** follows from the fact that projectors correspond to partially defined maps in \( \mathcal{H} \).

2. **Trivial tests** follows from the fact that projecting on the empty space yields the empty space and that projecting on the total space does not change anything.

3. **Atomicity** follows from the fact that states are nothing but one-dimensional closed linear subspaces, that is, atoms of the lattice of all closed linear subspaces.

4. **Adequacy** follows from the fact that for every \( x \in W \) we have that \( P_W(x) = x \).

5. **Repeatability** follows from the fact that \( P_W(x) \in W \) for every \( x \in \mathcal{H} \).

6. **Compatibility** follows from the fact that if two projectors commute, that is, \( P_W \circ P_V = P_V \circ P_W \), then \( P_W \circ P_V = P_{W \cap V} \).

7. **Self-adjointness** follows from the more general Adjointness theorem stated below, together with the fact that projectors are self-adjoint (that is, \( S \dagger S = S \dagger \)).

8. **Proper superpositions** can be proved by cases:
   - If \( s \not\perp t \), that is, let \( s \rightarrow t \), then \( w = s \Rightarrow s \rightarrow s \rightarrow t \).
   - If \( s \perp t \), that is, let \( s \not\Rightarrow t \) then let \( s = \overline{x}, t = \overline{y} \) with \( x, y \in \mathcal{H} \). Take the superposition \( x + y \in \mathcal{H} \) of \( x \) and \( y \) and note that \( x + y \neq 0 \) (since from \( x + y = 0 \Rightarrow x = -y \Rightarrow s = t \) which contradicts \( s \not\perp t \)). Next observe that \( x \not\perp (x + y) \) (Indeed, suppose \( x \perp (x + y) \) then \( \langle x \mid x + y \rangle = 0 \) and then \( \langle x \mid x \rangle + (x \mid y) = 0 \); but \( x \perp y \) implies \( \langle x \mid x \rangle = 0 \). So from \( \langle x \mid x \rangle = 0 \) follows that \( x = 0 \), which yields a contradiction). Similarly, we get \( y \not\perp (x + y) \).

   Conditions 9 and 10 are immediate consequences of the definition of a unitary operator.

   Note that, as a consequence of the ‘Proper superpositions’ property, the double-box modality \( \Box \Box \) coincides with the universal modality, that is, \( \Box \Box S \neq \emptyset \) iff \( S = \Sigma \).
Theorem 2. (Adjointness) Let $F$ be a quantum map and let $s, w, t \in \Sigma$ be states: If $s \xrightarrow{F} w \xrightarrow{} t$ then there exists some state $v \in \Sigma$ such that $t \xrightarrow{F^\dagger} v \xrightarrow{} s$.

Proof. To prove this theorem we use the definition of adjointness in a Hilbert space: $\langle Fx \mid y \rangle = \langle x \mid F^\dagger y \rangle$. From this, we get the equivalence $\langle Fx \mid y \rangle = 0$ iff $\langle x, F^\dagger y \rangle = 0$; or, put another way, $Fx \perp y$ iff $x \perp F^\dagger y$. Taking the negation of both sides and using the fact that the measurement relation $s \rightarrow t$ is the same as non-orthogonality $s \not\perp t$, we obtain the equivalence: $\exists w (x \xrightarrow{F} w \xrightarrow{} y)$ iff $\exists v (y \xrightarrow{F^\dagger} v \xrightarrow{} x)$.

This proves the adjointness property. As a consequence, we have the following corollaries.

Corollary 1. For every property $P \subseteq \Sigma$ and every linear map $F$ we have

$$P \subseteq [F] \square (F^\dagger) \diamond P.$$ 

Corollary 2. If $F$ is a quantum map,

$$F^\dagger(s) = ([F]s^\perp)^\perp.$$ 

Proof. Using the fact that the negation of the measurement accessibility relation $\rightarrow$ is the orthogonality relation $\perp$, we immediately obtain from the above Adjointness theorem that

$$s \perp F^\dagger(t) \iff t \perp F(s),$$

that is,

$$s \in (F^\dagger(t))^\perp \iff F(s) \in t^\perp.$$ 

From this, we get $(F^\dagger(t))^\perp = [F]t^\perp$. Since $F^\dagger$ is a map, $F^\dagger(t)$ is a (single) state, so it is a testable property. Hence, we have $F^\dagger(t) = (F^\dagger(t))^\perp = ([F]t^\perp)^\perp$.

This result leads us to the following natural generalisation of the notion of adjoint to all quantum actions.

Adjoint of a quantum action
For every quantum action $R \subseteq \Sigma \times \Sigma$, we define a relation $R^\dagger \subseteq \Sigma \times \Sigma$ by

$$sR^\dagger t \; \text{iff} \; t \perp [R]s^\perp,$$

or, put another way,

$$R^\dagger(s) = ([R]s^\perp)^\perp.$$
Proposition 3. For all quantum actions $R, Z \subseteq \Sigma \times \Sigma$, states $s, t \in \Sigma$ and properties $S \subseteq \Sigma$, we have the following:

1. $R^\dagger$ is a quantum action.
2. If $R = F$ is a (quantum, that is, linear) map then the relational adjoint $R^\dagger$ coincides with the Hermitian adjoint $F^\dagger$ (of $F$ as linear map).
3. $s \perp R^\dagger(t)$ iff $t \perp R(s)$.
4. $(R; Z)^\dagger = Z^\dagger; R^\dagger$.
5. $(R \cup Z)^\dagger = R^\dagger \sqcup Z^\dagger$.
6. $R[S] = ([R^\dagger]S^\perp)^\perp$.

2.2 Compound-system quantum frames

In this subsection we extend the quantum frame presented above for single systems into a quantum frame for compound systems. Let $H$ be a Hilbert space of dimension 2 with basis $\{|0\rangle, |1\rangle\}$. We fix a natural number $n \geq 2$ (although later we will restrict consideration to the case $n \geq 4$), and put $N = \{1, 2, \ldots, n\}$.

Our global state space will be denoted as before by $H$, but now we assume it is an $n$-qubit state, that is, we put $H = H_n := H \otimes H \otimes \ldots \otimes H$ (n times) for the tensor product of $n$ copies of $H$. A $n$-qubit quantum frame will be the quantum frame $\Sigma := \Sigma(H)$ associated (as in the previous section) to the Hilbert space $H$.

Notation

In fact, we consider all the $n$ copies of $H$ as distinct (although isomorphic) and use $H^{(i)}$ to denote the $i$-th component of the tensor $H^{\otimes n}$. Also, for any set of indices $I \subseteq N$, we put $\mathcal{H}_I = H \otimes^{I} = \bigotimes_{i \in I} H^{(i)}$. Note that we have $\mathcal{H}_N = \mathcal{H}_n = \mathcal{H}$. We use $\epsilon_i : H \rightarrow H^{(i)}$ to denote the canonical isomorphism between $\mathcal{H}$ and $H^{(i)}$. This notation can be extended to sets $I \subseteq N$ of indices of length $|I| = k$, by putting $\epsilon_I : H^{\otimes k} \rightarrow \mathcal{H}_I$ to be the canonical isomorphism between these spaces. Similarly, for each set $I \subseteq N$, we use $\mu_I : \mathcal{H}_I \otimes \mathcal{H}_{N \setminus I} \rightarrow \mathcal{H}$ to denote the canonical isomorphism between these two spaces. For any vector $|x\rangle \in H$, we use $|x\rangle \otimes^I = \bigotimes_{i \in I} |x\rangle$ to denote the corresponding vector in $\mathcal{H}_I$ (obtained by

\footnote{We identify a map $F : \Sigma \rightarrow \Sigma$ with its graph $F \subseteq \Sigma \times \Sigma$, that is, quantum maps are special cases of quantum relations, which happen to be be partial functions. So $R = F$ means that the two sides are equal, as relations.}
tensorsing \(|I|\) copies of \(|x\rangle\). Given a set \(I \subseteq N\), we say that a state \(s \in \Sigma(\mathcal{H})\) has its \(I\)-qubits in state \(s' \in \Sigma(\mathcal{H}_I)\), and write \(s_I = s'\), if there exist vectors \(\psi \in s\), \(\psi' \in \mathcal{H}_I\) and \(\psi'' \in \mathcal{H}_{N\setminus I}\) such that \(\psi = \mu_I(\psi' \otimes \psi'')\). Note that the state \(s_I\), if it exists, is unique (having the above property). We say that the state \(s\) is \(I\)-separated iff \(s_I\) exists. In this case, \(s_I\) is called the \((I\)-local component\) (or local state) of \(s\). In particular, when \(I = \{i\}\), the local component \(s_i \in \mathcal{H}_{\{i\}} = \mathcal{H}^{(i)}\) is called the \(i\)-th coordinate of the state \(s\).

We will further use \(|+\rangle\) to denote the vector \(|0\rangle + |1\rangle\), and, similarly \(|-\rangle\) to denote \(|0\rangle - |1\rangle\). For the states generated by the vectors in a two dimensional Hilbert space we introduce the following abbreviations: \(+ := |+\rangle\), \(- := |-\rangle\), \(0 := |0\rangle\), \(1 := |1\rangle\). In order to refer to the state corresponding to a pair of qubits, we similarly delete the Dirac notation, for example, \(00 := |00\rangle = |0\rangle \otimes |0\rangle\). The Bell states will be abbreviated as follows: \(\beta_{00} := |00\rangle + |11\rangle\), \(\beta_{01} := |01\rangle + |10\rangle\), \(\beta_{10} := |00\rangle - |11\rangle\), \(\beta_{11} := |01\rangle - |10\rangle\) and \(\gamma := |00\rangle + |11\rangle\).

The following two results are well-known.

**Proposition 4.** Let \(H^{(i)}\) and \(H^{(j)}\) be two Hilbert spaces. There exists a bijective correspondence \(\psi\) between the linear maps \(F: H^{(i)} \to H^{(j)}\) and the states of \(H^{(i)} \otimes H^{(j)}\). For fixed bases \(\{\epsilon^{(i)}_\alpha\}_\alpha\) and \(\{\epsilon^{(j)}_\beta\}_\beta\) of these spaces, the correspondence \(\psi\) is maps the linear function \(F\), given by \(F(|x\rangle) = \sum_\alpha m_\alpha \epsilon^{(i)}_\alpha | x \rangle \epsilon^{(j)}_\beta\) for all \(|x\rangle \in H^{(i)}\), to the state \(\psi(F) = \sum_\alpha m_\alpha \epsilon^{(i)}_\alpha \otimes \epsilon^{(j)}_\beta\).

**Proposition 5.** Let \(\mathcal{H} = H^{\otimes n}\) and let \(W = \{x \otimes |0\rangle^{\otimes (n-1)} : x \in H\}\) be given. Any linear map \(F: \mathcal{H} \to \mathcal{H}\) induces a linear map \(F_{(1)}: H \to H\) in a canonical manner: it is defined as the unique map on \(H\) satisfying \(F_{(1)}(x) = P_W F(x \otimes |0\rangle^{\otimes (n-1)})\). Conversely, any linear map \(G: H \to H\) can be represented as \(G = F_{(1)}\) for some linear map \(F: \mathcal{H} \to \mathcal{H}\).

**Notation**
The above results allow us to specify a compound state in \(H^{(i)} \otimes H^{(j)}\) via some linear map \(F\) on \(\mathcal{H}\). Indeed, if \(F: \mathcal{H} \to \mathcal{H}\) is any such linear map, let \(F_{(1)}: H \to H\) be the map in the above proposition; this induces a corresponding map \(G_{(1)}: H^{(i)} \to H^{(j)}\), by putting \(G_{(1)} := \epsilon_j \circ F_{(1)} \circ \epsilon_i^{-1}\), where \(\epsilon_i\) is the canonical isomorphism introduced above (between \(H\) and the \(i\)-th component \(H^{(i)}\) of \(H^{\otimes n}\)).
Then we use $F_{(ij)}$ to denote the state

$$F_{(ij)} := \psi(F_{(ij)}(1))$$

given by the above mentioned bijective correspondence $\psi$ between $H^{(i)} \to H^{(j)}$ and $H^{(i)} \otimes H^{(j)}$. The following result is also known from the literature.

**Proposition 6.** Let $F : \mathcal{H} \to \mathcal{H}$ be a linear map. Then the state $F_{(ij)}$ is 'entangled according to $F$'; that is, if $F_{(1)}(|x⟩) = |y⟩$ and the state of a 2-qubit system is $F_{(ij)} \in H^{(i)} \otimes H^{(j)}$, then any measurement of qubit $i$ resulting in a state $x_i$ collapses the qubit $j$ to state $y_j$.

In our axiomatic proof system, we will take (a syntactic counterpart of) this result as our central axiom, the ‘Entanglement Axiom’.

**Notation**

The notation $F_{(ij)}$ can be further extended to define a property (set of states) $F_{ij} \subseteq \Sigma = \Sigma(\mathcal{H})$, by defining it as the set of all states having the $\{i,j\}$-qubits in the state $F_{(ij)}$:

$$F_{ij} = \{ s \in \Sigma : s_{\{i,j\}} = F_{(ij)} \}$$

$$= \{ \mu_{\{i,j\}}(\psi \otimes \psi') : \psi \in F_{(ij)}, \psi' \in \mathcal{H}_{N \setminus \{i,j\}} \} \subseteq \Sigma$$

where $\mu_{\{i,j\}}$ is, as above, the canonical isomorphism between $\mathcal{H}_{\{i,j\}} \otimes \mathcal{H}_{N \setminus \{i,j\}}$. In other words, $F_{ij}$ is simply the property of an $n$-qubit compound state of having its $i$-th and $j$-th qubits (separated from the others, and) in a state that is 'entangled according to $F$'.

**Local properties and separation**

Given a set $I \subseteq N$, a property $S \subseteq \Sigma$ is $I$-local if it corresponds to a property of the subsystem formed by the qubits in $I$; in other words, if there exists some property $S' \subseteq \Sigma(\mathcal{H}_I)$ such that:

$$S = \{ s \in \Sigma : s_I \in S' \}$$

or, more explicitly, $S = \{ \mu_I(\phi \otimes \psi) : \phi \in S', \psi \in \mathcal{H}_{N \setminus I} \}$. An example is the property $F_{ij}$, which is $\{i,j\}$-local. For any $I \subseteq N$, the family of $I$-local properties forms a complete lattice (with inclusion) in which the join is given by
union $S \cup T$, the atoms correspond to local states, and the greatest element is the property

$$\top^S_I := \{s \in \Sigma : s \text{ is } I - \text{separated}\} = \bigcup\{S \subseteq \Sigma : S \text{ is } I - \text{local}\}$$

that defines separation: a state $s$ is $I$-separated iff $s \in \top^S_I$. But note that the family of $I$-local properties is not closed under complementation.

**Local Maps**

Given $I \subseteq N$, a linear map $F : \mathcal{H} \to \mathcal{H}$ is $I$-local if it ‘affects only the qubits in $I$’; in other words, if there exists a map $G : \mathcal{H}_I \to \mathcal{H}_I$ such that

$$F \circ \mu_I(\phi \otimes \psi) = \mu_I(G(\phi) \otimes \psi)$$

A map $F : \Sigma \to \Sigma$ is $I$-local if it is the map induced on $\Sigma$ by an $I$-local linear map on $\mathcal{H}$. Examples are: all the tests $S_I$ of testable $I$-local properties $S_I$; logic gates that affect only the qubits in $I$, that is, (maps on $\Sigma$ induced by) unitary transformations $U_I : \mathcal{H} \to \mathcal{H}$ such that for all $\psi \in \mathcal{H}_I$, $\psi' \in \mathcal{H}_{N-I}$, we have $U_I \circ \mu_I(\psi \otimes \psi') = \mu_I(U(\psi) \otimes \psi')$, for some $U : \mathcal{H}_I \to \mathcal{H}_I$. The family of $I$-local maps is closed under composition.

**Local actions**

An $I$-local action is a quantum action $R \subseteq \Sigma \times \Sigma$ that can be written as an arbitrary union of $I$-local maps. The family of $I$-local actions forms a complete lattice (with inclusion), in which the join is given by union $R \cup R'$, and the greatest element is the action

$$\top^{\Sigma \times \Sigma}_I := \bigcup\{F : \Sigma \to \Sigma : F \text{ is an } I - \text{local map}\}$$

**Lemma 2.** (Teleportation property). If $s$ is an $i$-separated state having its $i$-th qubit $s_i$ in the state $x \in H$, then after doing two successive bipartite measurements $\overline{G_{jk}}$? followed by $\overline{F_{ij}}$?, the $k$-th qubit ($k$-th component of) the output-state is

$$\left(\overline{T_{ij}}? \circ \overline{G_{jk}}?(s)\right)_k = G_{(1)} \circ F_{(1)}(x)$$

$^{14}$That is, possibly infinite
Lemma 3. (Entanglement Composition Lemma). The main lemma in [Coecke 2004] states (in our notation) that, given a quadruple of distinct indices $i, j, k, l$, and letting $F, G, H, U, V : H \to H$ be single-qubit linear maps (that is, 1-local transformations), we have

$$G_{jk} \circ V_k \circ U_j (F_{ij} \cap H_{kl}) \subseteq (H \circ U^\dagger \circ G \circ V \circ F)_{il}.$$  

[Coecke 2004] and [Abramsky and Coecke 2005] use these last two lemmas as the main tool in explaining teleportation, quantum gate teleportation and many other quantum protocols. We will use this work in our logical treatment of such protocols by formally proving (syntactic correspondents of) these lemmas in our axiomatic proof system and then using them to analyse teleportation and quantum secret sharing.

Observe that in the above Lemma 3, the order in which the operations $U_j$ and $V_k$ are applied is in fact irrelevant. This is a consequence of the following important property of local transformations.

Proposition 7. (Compatibility of local transformations affecting different sets of qubits). If $I \cap J = \emptyset$, $F$ is an $I$-local map and $G$ is a $J$-local map, we have

$$F \circ G = G \circ F.$$  

Another important property of local maps (on states) is given by the following proposition.

Proposition 8. (Agreement Property). Let $F_I, G_I : \Sigma \to \Sigma$ be two $I$-local maps on states, having the same domain,$^{15}$ $\text{dom}(F) = \text{dom}(G)$. Then their output-states agree on all non-$I$ qubits, that is, for all $s \in \Sigma$, $F(s)_{N \setminus I} = G(s)_{N \setminus I}$ whenever both sides of the identity exist (that is, whenever both $F(s)$ and $G(s)$ are $I$-separated.)

---

$^{15}$The domain of a map is defined by $\text{dom}(F) = \{s \in \Sigma : F(s) \text{ is defined}\}$. If $F'$ is the corresponding linear map on $H$, this means that $\text{dom}(F) = \{\psi : F' (\psi) \neq 0\}$.  

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Dynamic characterizations of main unitary transformations

It is well-known that a linear operator on a vector space in a given Hilbert space is uniquely determined by the values it takes on the vectors of an (orthonormal) basis. An important observation is that this fact is no longer ‘literally true’ when we move to ‘states’ as one-dimensional subspaces instead of vectors. The reason is that ‘phase’-aspects (or, in particular, the signs ‘+’ and ‘−’) are not ‘state’ properties in our setting. In other words, two vectors that differ only in phase, that is, \( x = \lambda y \) where \( \lambda \) is a complex number with \( |\lambda| = 1 \), belong to the same subspaces, so they correspond to the same state \( x = y \).

Example 1. (Counterexample). Consider a 2 dimensional Hilbert space in which we use \( |0\rangle \) and \( |1\rangle \) to denote the basis vectors. A transformation \( I \) is given by \( I(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle \); and a transformation \( J \) is given by \( J(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle \). Although \( I \) and \( J \) induce different operators on states, these operators map the basis states to the same images:

\[
I(0) = I(|0\rangle) = 0 = J(|0\rangle) = J(0),
\]
\[
I(1) = I(|1\rangle) = 1 = -|1\rangle = J(|1\rangle) = J(1).
\]

But of course we do distinguish the subspaces generated by different superpositions:

\[
I(+) = |0\rangle + |1\rangle = + \neq - = |0\rangle - |1\rangle = J(+).
\]

Proposition 9. A linear operator on the state space \( \Sigma(\mathcal{H}_1) \) of a 2 dimensional Hilbert space is uniquely determined by its images on the states: \( |0\rangle, |1\rangle, |+\rangle \).

Corollary 3. A linear operator on the state space \( \Sigma(\mathcal{H}_n) \) of the space \( \mathcal{H}_n \) is uniquely determined by its images on the states:

\[
\{|x\rangle_1 \otimes \ldots \otimes |x\rangle_n : |x\rangle_i \in \{|0\rangle, |1\rangle, |+\rangle\}\}
\]

In the definition of a quantum frame given above, we introduced the set \( \mathcal{U} \) as the set of unitary transformations for single systems. For compound systems the set \( \mathcal{U} \) will be extended with the kind of operators that are active on compound systems. Following the quantum computation literature, we take \( \mathcal{U} = \{X, Z, H, CNOT, \ldots\} \) where \( X \), \( Z \) and \( H \) are defined by the following table:
The transformation $CNOT$ is given by the table:

|   | 0 | 1 | + |
|---|---|---|---|
| X | 1 | 0 | + |
| Z | 0 | 1 | - |
| H | + | - | 0 |

### 3 The logic $LQP$

#### 3.1 Syntax of $LQP$

To build up the language of $LQP$, we are given the following: a natural number $n$, for which we put $N = \{1, 2, \ldots, n\}$; a set $Q$ of propositional variables; a set $C$ of propositional constants; and a set $U$ of program constants, denoting basic programs, to be interpreted as quantum gates. Each program constant $U \in U$ comes together with an index $I$, which is a sequence of distinct indices in $N$; the index gives us the set of qubits on which the quantum gate $U$ is active - when we want to make explicit the index, we write, for example, $U_I$ for an $I$-local quantum gate. In particular, for every $i, j \leq n$, we are given some special program constants $CNOT_{ij}, X_i, H_i, Z_i, \ldots \in U$. Similarly, we are given two special propositional constants $1, + \in C$, the first denoting the separated state $|1\rangle I \otimes \cdots \otimes |1\rangle$ and the second denoting the separated state $|+\rangle I \otimes \cdots \otimes |+\rangle$. The syntax of $LQP$ is an extension of the classical syntax for $PDL$, with a set of propositional formulas and a set of programs, defined by mutual induction:

$$
\varphi ::= T_I \mid p \mid c \mid \neg \varphi \mid \varphi \land \varphi \mid [\pi] \varphi
$$

$$
\pi ::= T_I \mid \varphi? \mid U \mid \pi^1 \mid \pi \cup \pi \mid \pi; \pi
$$

Here, we take $I$ to denote sequences of distinct indices in $N = \{1, 2, \ldots, n\}$. The sentence $T_I$ expresses $I$-separation: it is true iff the qubits in $I$ form a separated subsystem. So $T_I$ denotes the greatest element $\top_I^N$ of the lattice of $I$-local properties. In particular, the sentence $T_N$ denotes the ‘always true’ proposition ($\text{verum}$, usually denoted $\top$), tat is, the ‘top’ of the lattice of all properties.\(^{16}\) The constructs

\(^{16}\) Note also the distinction between the constant $1_i$ (characterising the qubit $|1\rangle I$) and the constant $T_i$ (denoting the property of being $i$-separated).
\[\neg \varphi \text{ and } \varphi \land \varphi \text{ denote classical negation and conjunction, while the construct given by dynamic modalities } [\pi] \varphi \text{ denotes the weakest precondition that ensures that property } \varphi \text{ will hold after running program } \pi.\]

On the program side: \(\top_I\) denotes the trivial \(I\)-local action \(\top^{\Sigma \times \Sigma}_I\), which acts on any given \(I\)-separated state by keeping the \(N\ \setminus I\) subsystem unchanged, while changing the \(I\) subsystem to any randomly picked \(I\) system. In other words, \(\top_I\) is the union of all \(I\)-local actions. The meaning of quantum test \(\varphi^?\), adjoint \(\pi^\dagger\), union \(\pi \cup \pi\) and composition \(\pi;\pi\) is given by the corresponding operations on quantum actions.

Notice that we did not include iteration (Kleene star) among our program constructs: this is only because we do not need it for any of the applications in this paper. Indeed, most quantum programming does not involve \textbf{while}-loops; but (as pointed in our Section 6) one can of course add iteration to our logic, if needed.

**Abbreviations in LQP**

We can enrich our basic language by introducing various abbreviations. In particular, we define the classical disjunction and classical implication in the usual way, that is, \(\varphi \lor \psi := \neg(\neg \varphi \land \neg \psi)\), \(\varphi \rightarrow \psi := \neg \varphi \lor \psi\). As in classical logic, we can introduce a constant falsum \(\bot_N := \neg \top_N\) for the ‘always false’ sentence (usually denoted \(\bot\)). In non-ambiguous contexts, we sometimes skip the subscript \(N\), and simply write \(\top\) and \(\bot\) for \(\top_N\) and \(\bot_N\). We define the classical dual of \([\pi] \varphi\) in the usual way as \(\langle \pi \rangle \varphi := \neg [\pi] \neg \varphi\); the measurement modalities \(\Box\) and \(\Diamond\) used in the quantum logic literature can be defined in LQP by putting \(\Diamond \varphi := \langle \varphi^? \rangle \top_N\) and \(\Box \varphi := \neg \Diamond \neg \varphi\). The orthocomplement is defined as \(\neg \varphi := \Box \neg \varphi\), or, equivalently, as \(\neg \varphi := [\varphi^?] \bot_N\). Using the orthocomplement, we define a binary operation for quantum join \(\varphi \sqcup \psi := \neg (\neg \varphi \land \neg \psi)\). This expresses superpositions: \(\varphi \sqcup \psi\) is true at any state which is a superposition of states satisfying \(\varphi\) or \(\psi\).

We also introduce some notions and notations for programs: we call a program \(\pi\) deterministic if \(\pi\) is constructed without the use of non-deterministic choice \(\cup\) or of the non-deterministic program \(\top_I\). Also, we put \(\text{flip}_{ij} := CNOT_{ij}; CNOT_{ji}; CNOT_{ij}\) for the program that (given any \(\{i, j\}\)-separated input state) permutes the \(i\text{th}\) and the \(j\text{th}\) components. Finally, we put \(\text{id} := \top^?\) for the identity map.
Order, equivalence, orthogonality, I-equivalence, testability, locality, separation

We can internalize the logical equivalence, being weaker than, and I-equivalence relations between formulas, the locality and testability, and the notion of I-component by defining the following formulas:

\[
\begin{align*}
\varphi \leq \psi & := \square \square (\varphi \rightarrow \psi) \\
\varphi = \psi & := \square \square (\varphi \leftrightarrow \psi) \\
\varphi \perp \psi & := \varphi \leq \sim \psi \\
T(\varphi) & := \sim \sim \varphi \leq \varphi \\
\varphi_I & := T_I \land \{T_{N \setminus I}\} \varphi \\
\varphi =_I \psi & := \varphi \leq T_I \land \psi \leq T_I \land \varphi_I = \psi_I \\
I(\varphi) & := \varphi = \varphi_I.
\end{align*}
\]

Recall from Section 2.1 that the double-box modality coincides with the universal modality: so, indeed, \(\varphi \leq \psi\) means that \(\varphi\) is logically weaker than \(\psi\), while \(\varphi = \psi\) means the formulas are equivalent. We read \(T(\varphi)\) as saying that ‘\(\varphi\) is testable’, and \(I(\varphi)\) as ‘\(\varphi\) is I-local’. We read \(\varphi_I\) as ‘the I-component of \(\varphi\)’: a state satisfies this sentence iff (it is I-separated and) its I-subsystem is (a subsystem of some state) satisfying \(\varphi\). For \(I = \{i\}\), we write \(\varphi_i := \varphi_I\). We read \(\varphi =_I \psi\) as ‘\(\varphi\) is I-equivalent to \(\psi\)’: the meaning is that both \(\varphi\) and \(\psi\) are I-separated and have the same I-component. Finally, we say that \(\varphi\) is I-separated iff \(\varphi \leq T_I\).

Note that it obviously follows from these definitions that every I-component \(\varphi_I\) is I-local.

Special local states

We can introduce some more propositional constants (which will denote special local states), by putting: \(0_i := \sim 1_i\) and \(\neg_i := \sim +_i\).

Image and strongest post-condition

We define the strongest testable post-condition \(\pi[\varphi]\) ensured by (applying a program) \(\pi\) on (any state satisfying a given precondition) \(\varphi\), by putting

\[
\pi[\varphi] := \sim [\pi^1] \sim \varphi
\]

If \(\varphi\) is assumed to be testable and \(\pi\) is deterministic, the strongest postcondition \(\pi[\varphi]\) coincides with the image \(\pi(\varphi)\) of \(\varphi\) via \(\pi\). The definition of image of a testable property via a program \(\pi(\varphi)\) can be extended to all programs that are finite unions of deterministic programs, by putting, for all testable formulas \(\psi\):

\[
\pi(\psi) = \pi[\psi]\text{ if }\pi\text{ is deterministic, and } (\pi \cup \pi')(\psi) = \pi(\psi) \lor \pi'(\psi)\text{ otherwise.}
\]
Note the contrast with classical \( PDL \): unlike the classical version, our quantum \( PDL \) (as considered above, that is, without program converse\(^{17}\)) has enough expressive power to define strongest post-conditions (and, in a restrict context, images) using weakest preconditions! The reason is that, in some context, the notion of adjoint can replace the notion of converse. But note that converse itself is not expressible in our logic. This is a good thing since the converse of a quantum action has no physical meaning (except in the case of reversible, unitary evolutions), while the adjoint is physically meaningful.

**Notation**

For any sequence \( I \subseteq N \) of indices and any vector \( \vec{c} = (c(i))_{i \in I} \in \{0, 1, +\}^{|I|} \), we set

\[
\vec{c}_I := \bigwedge_{i \in I} c(i)_i.
\]

**The unary maps induced by a program**

In our syntax we want to capture the construction \( F_{(1)} \), by which a linear map \( F \) on \( H \otimes n \) was used to describe a unary map \( F_{(1)} \) on \( H \). For this, we put: \( 0_i ! := 0_i ? \cup (1_i ?; X_i) \), and \( 0_I ! := 0_{i_1} !; 0_{i_2} !; \ldots; 0_{i_k} ! \), where \( I = (i_1, i_2, \ldots, i_k) \). This maps any qubit in \( I \) to 0. Similarly, we put; \( 0_I ? := (0_{i_1} \land 0_{i_2} \land \ldots \land 0_{i_k}) ? \).

Finally, we define

\[
\pi_{(1)} := 0_{N \setminus \{i\}} !; \pi; 0_{N \setminus \{i\}} ?
\]

This is the map we need (which encodes a single qubit transformation). In fact, we shall only use \( \pi_{(1)} \) in the rest of this paper. We also want to consider the \( H_i \to H_j \)-version of the transformation \( \pi_{(1)} \), so we put

\[
\pi_{ij} := \text{flip}_{1i} ; \pi_{(1)} ; \text{flip}_{1j}
\]

**Local programs**

We would like to isolate **local programs**, that is, the ones that ‘affect only the qubits in a given set \( I \subseteq N \)’. For this, we define a formula \( I(\pi) \) meaning ‘program \( \pi \) is \( I \)-local’:

\[
I(\pi) := \bigwedge_{\vec{c}, \vec{d}, \vec{d}'} \left( \vec{d}_{N \setminus I} = _{N \setminus I} \pi(\vec{c}_I \land \vec{d}_{N \setminus I}) =_I \pi(\vec{c}'_I \land \vec{d}'_{N \setminus I}) \right)
\]

where the conjunction is taken over all \( \vec{c} \in \{0, 1, +\}^{|I|} \) and all \( \vec{d}, \vec{d}' \in \{0, 1, +\}^{n-|I|} \).

\(^{17}\)There also exists a version of \( PDL \) with a program converse operator \( \pi^- \), such that the accessibility relation for the converse \( \pi^- \) is defined as the converse of the accessibility relation for \( \pi \). It is obvious that this stronger logic can express the strongest post-condition of a program \( \pi \), using the existential dynamic modalities, since \( \pi(\phi) = \langle \pi^- \rangle \phi \).

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Note that this definition is a simple formal translation of the semantic clauses that express the fact that program $\pi$ acts only ‘locally’ (affecting only the $I$-subsystem, and in a way that depends only on the $I$-subsystem of the input state) on the states of the form $\vec{c}$ (with $c \in \{0, 1, +\}$). One of our axioms below (‘Determinacy of deterministic programs’) means that this clause is enough to ensure that program $\pi$ acts locally on all ($I$-separated) states.

**Entanglement according to $\pi$**

To describe states that are ‘entangled according to $\pi$’, we introduce the following formula

$$\pi_{ij} := \top \land \bigwedge_{c \in \{0, 1, +\}} \left( [c_i ?] (\pi_{ij}(c_i))_j \land (\sim c_i \rightarrow \pi_{ij}(c_i) = \bot) \right).$$

Then, as a consequence, we will have the following obvious validity:

$$c_i ? (\pi_{ij}) = \pi_{ij}(c_i)$$

for every $c_i \in \{0, 1, +\}$.

Again, note that the identity in this definition is a formal translation of the semantic clause defining ‘entanglement according to an action’, but only for the particular case of local states of the form $c_i$ (with $c \in \{1, 0, +\}$). And again, one of our axioms below (the ‘Entanglement Axiom’) ensures that the above identity holds (not only for the elements $c_i$, but) for all $i$-local states (that is, all testable $i$-local properties).

### 3.2 Semantics of $LQP$

An $LQP$-model is a multi-partite quantum frame $\Sigma = \Sigma(\mathcal{H})$ based on an $n$-dimensional Hilbert space $\mathcal{H}$, together with a valuation function, mapping each propositional variable $p$ into a set of states $\| p \| \subseteq \Sigma$. We will use the valuation map to give an interpretation $\| \varphi \| \subseteq \Sigma$ to all our formulas, in terms of quantum properties of our multi-partite frame, that is, sets of states in $\Sigma$. At the same time, we give an interpretation $\| \pi \| \subseteq \Sigma \times \Sigma$ to all our programs, in terms of quantum actions. The two interpretations are defined by mutual recursion.

**Interpretation of Programs**

| $\| \top \| =$ | $\top^{\Sigma \times \Sigma}$ | $\| \varphi ? \| =$ | $\| \varphi \| ?$ |
| $\| U \| =$ | $U$ | $\| \pi \| ^\dagger =$ | $\| \pi \| ^\dagger$ |
| $\| \pi_1 \cup \pi_2 \| =$ | $\| \pi_1 \| \cup \| \pi_2 \|$ | $\| \pi_1; \pi_2 \| =$ | $\| \pi_2 \| ; \| \pi_1 \|$ |
The interpretation $|| \pi ||$ allows us to extend the notation $\xrightarrow{\pi}$ to all programs, by putting: $s \xrightarrow{\pi} t$ iff $(s, t) \in || \pi ||$.

Interpretation of formulas
We extend the valuation $|| p ||$ from propositional variables to all formulas, by putting for the others:

$$
|| 1 || = |1\rangle^{\otimes n}
|| \varphi \wedge \psi || = || \varphi || \cap || \psi ||
|| \neg \varphi || = \Sigma \setminus || \varphi ||
|| [\pi]\varphi || = ([|| \pi ||] || \varphi ||) || T_I ||
$$

\[\begin{align*}
|| \varphi \wedge \psi || &= || \varphi || \cap || \psi || \\
|| \neg \varphi || &= \Sigma \setminus || \varphi || \\
|| [\pi]\varphi || &= ([|| \pi ||] || \varphi ||) || T_I ||
\end{align*}\]

Proposition 10. The interpretation of any testable formula is a testable property. The interpretation of an $I$-local formula (or $I$-local deterministic program) is an $I$-local property (or $I$-local linear map on states).

Lemma 4.
\[\begin{align*}
|| \sim \varphi || &= || \varphi ||^\perp \\
|| [\varphi?]\psi || &= || \varphi ||^? || \psi || \\
|| \Box \varphi || &= \Box || \varphi || \\
|| \varphi || &= || \sim \sim \varphi ||
\end{align*}\]

Proposition 11. The following are equivalent, for every formula $\varphi$:
1. $|| \varphi ||$ is testable (that is, $T(\varphi)$ is valid).
2. $\varphi$ is semantically equivalent to $\sim \sim \varphi$.
3. $\varphi$ is semantically equivalent to some formula $\Box \psi$.
4. $\varphi$ is equivalent to some formula $\sim \psi$.

Proposition 12. For deterministic programs $\pi$, the interpretation of the construct $\pi_{ij}$ is the property of ‘being entangled according to (the linear map denoted by) $\pi’$. More precisely, for deterministic $\pi$, we have

$$
|| \pi_{ij} || = || \pi ||_{ij}
$$

where we use the notation $F_{\pi_{ij}}$ introduced (for any linear map $F$) after Proposition 6 of Section 2.2.
4 Proof theory for \(LQP\)

4.1 Axioms for single systems

First, we admit all the axioms and rules of classical \(PDL\), except for the ones concerning tests \(\varphi?\) and Kleene star \(\pi^*\). In particular, we have the following rules and axioms.

**Substitution Rule.** From \(\vdash \Theta\) infer \(\vdash \Theta[p/\varphi]\)

And the ‘normality’ conditions for the dynamic modalities \([\pi]\):

**Kripke Axiom.** \(\vdash [\pi](p \rightarrow q) \rightarrow ([\pi]p \rightarrow [\pi]q)\)

**Necessitation Rule.** From \(\vdash p\) infer \(\vdash [\pi]p\)

Considering \(\square p\), we introduce the following axioms:

**Test Generalisation Rule.** If the variable \(q\) does not occur in \(\varphi\) or \(\psi\), then, from \(\vdash \varphi \rightarrow [q?]\psi\) infer \(\vdash \varphi \rightarrow \square\psi\)

**Testability Axiom.** \(\vdash \square p \rightarrow [q?]p\)

Testability can be stated in its dual form by means of \(\langle q?\rangle p \rightarrow \lozenge p\) or, equivalently, as \(\langle q?\rangle p \rightarrow \langle p?\rangle \top\). This dual formulation of Testability allows us to give a straightforward interpretation: if the property associated to \(p\) can be actualised by a measurement (yielding an output state satisfying \(p\)), then we can directly test the property \(p\) (by doing a measurement for \(p\)). The Test Generalisation Rule encodes the fact that \(\square\) is a universal quantifier over all possible measurements.

Other \(LQP\)-axioms are:

- **Partial Functionality.** \(\vdash \neg [p?]q \rightarrow [p?]\neg q\)
- **Adequacy.** \(\vdash p \land q \rightarrow (p?q)\)
- **Repeatability.** \(\vdash T(p) \rightarrow [p?]p\)
- **Proper Superpositions.** \(\vdash \langle \pi\rangle \square p \rightarrow [\pi]p\)
- **Unitary Functionality.** \(\vdash \neg [U]q \leftrightarrow [U]\neg q\)
- **Unitary Bijectivity 1.** \(\vdash p \leftrightarrow [U;U^\dagger]p\)
- **Unitary Bijectivity 2.** \(\vdash p \leftrightarrow [U^\dagger;U]p\)
- **Adjointness.** \(\vdash p \rightarrow [\pi]\square (\pi^\dagger)\lozenge p\)

**Proposition 13.** Testability is closed under conjunctions, weakest preconditions; \(\square\)-sentences, orthocomplements and strongest postconditions are testable:

\[
\vdash T(p) \land T(q) \rightarrow T(p \land q)
\]

\(^{18}\)We skip the axioms for iteration \(\pi^*\) only because we chose not to include this construct in our logic. However, if one adds \(\pi^*\) to our syntax, the usual \(PDL\) axioms for iteration are still sound, so they can be added to the proof system.
⊢ \( T(p) \rightarrow T([\pi]p) \)

⊢ \( T(\Box p) \)

⊢ \( T(\sim p) \)

⊢ \( T(\pi[p]) \)

A formula \( \varphi \) is called testable if the theorem

\[ \vdash T(\varphi) \]

is provable in our system. Observe that this notion is proof-theoretic. However, the above proposition gives us a purely syntactical way to check testability:

**Corollary.** Any formula of the formula of the form \( \Box \varphi, \sim \varphi \) or \( \top \), or which can be obtained from these formulas using only conjunctions \( \phi \land \psi \) and weakest preconditions \( [\pi] \varphi \), is testable.

**Proposition 14. (Quantum logic, weak modularity or quantum modus ponens).** All the axioms and rules of traditional Quantum Logic are satisfied by our testable formulas. In particular, from our axioms one can prove ‘Quantum Modus Ponens’ \( \varphi \land [\varphi?] \psi \leq \psi \). In its turn, this rule is equivalent to the condition known in quantum logic as weak modularity, which is stated as follows: \( \varphi \land (\sim \varphi \sqcup (\varphi \land \psi)) \leq \psi \).

**Theorem 3. (Soundness and Completeness).** All the other axioms above are sound. Moreover, if we eliminate from the syntax of our logic all the special constants (both propositional constants \( \top, 1 \) and \( + \), and program constants \( \top \), \( \text{CNOT}, X, H, Z \), and so on), then there exists a complete proof system for (single-system) Hilbert spaces, which includes the above axioms.

The proof of this theorem is given in our paper [Baltag and Smets 2005a], and is based on an extension of (Mayet’s version [Mayet 1998] of Solèr’s Theorem [Solèr 1995], which is itself an extension of Piron’s Representation Theorem for Piron lattices [Piron 1964, Piron 1976, Amemiya and Araki 1967].

\[ ^{19} \text{This explains why the weakest precondition \([\varphi?] \psi \) has been taken as the basic implicational connective in traditional Quantum Logic, under the name of ‘Sasaki hook’ and denoted by \( \varphi \rightarrow \psi \).} \]

\[ ^{20} \text{In addition, the system includes two more axioms of a rather technical nature, namely Piron’s ‘Covering Law’ [Piron 1976] and ‘Mayet’s Condition’ [Mayet 1998]. See [Baltag and Smets 2005a] for details.} \]
Proposition 15. The formula $\pi[\varphi]$ expresses the *strongest testable postcondition* ensured by executing program $\pi$ on any state satisfying (precondition) $\varphi$. In other words, for every *testable* $\psi$, we have

$$\pi[\varphi] \leq \psi \iff \varphi \leq [\pi]\psi$$

Proposition 16. (Adjointness Theorem). For all *testable* formulas $\varphi, \psi$, we have

$$\varphi \perp \pi[\psi] \iff \pi^\dagger[\varphi] \perp \psi$$

4.2 Axioms for compound systems

Separation Axioms. Every state is $N$-separated; if a state is both $I$-separated and $J$-separated, then it is also $N \setminus I$-separated, $I \cup J$-separated and $I \cap J$-separated:

$$\vdash \top_N$$
and

$$\vdash \top_I \land \top_J \rightarrow \top_{N \setminus I} \land \top_{I \cup J} \land \top_{I \cap J}$$

Axioms for the trivial $I$-local program. The program $\top_I$ is the weakest $I$-local program; that is,

$$\vdash I(\pi) \rightarrow \langle \pi \rangle p \leq \langle \top_I \rangle p$$
and

$$\vdash I(\top_I)$$

As an immediate consequence, we obtain the following corollary.

Corollary 4. The formula $\top_I$ is the weakest $I$-local property; that is,

$$\vdash I(\top_I)$$
and

$$\vdash I(p) \rightarrow p \leq \top_I$$

*Proof.* By the definition of $I$-locality $I(\pi)$ of a program, it is easy to see that the identity program $id$ is $I$-local for every $I$. Applying the first part of the above axiom (for $\top_I$), we obtain $\top_I = < id > \top_I \leq < \top_I > \top_I$, from which we deduce that $\top_I = \top_I \land < \top_I > \top_I$. Applying the definition of $\varphi_I$, we conclude that $\top_I = (\top_I)_I$, and thus (by definition of $I$-locality $I(\varphi)$ of a sentence) we
derive $I(\top_I)$. The second part of the corollary follows trivially from the definition of $I(p)$.

Syntactically, we define an ‘$I$-local state’ to be any sentence $\varphi$ such that

$$\vdash I(\varphi) \land \varphi \neq \bot \land (I(p) \land \bot \neq p \rightarrow p = \varphi)$$

for some $p$ not occurring in $\varphi$. In other words, these are propositions that can be proved to be atoms of the lattice of (consistent) $I$-local properties.

**Local States Axiom.** Testable local properties are ‘local states’ (in the above sense, that is, atomic local properties): if $I \neq N$,

$$\vdash T(p) \land I(p) \land I(\varphi) \land \bot \neq q \leq p \rightarrow q = p$$

**Basic-State Testability Axiom.** Our basic local states $c_i, \pi_{ij}$ are testable and local (for the appropriate subsystem). More precisely, if $i, j \in N, c \in \{0, 1, +, -\}$ and $\pi$ is a deterministic program, then

$$\vdash T(c_i) \land I(c_i) \land T(\pi_{ij}) \land \{i, j\}(\pi_{ij})$$

As an immediate consequence of the last two axioms, all constants of the form $\vec{c}$ (with $\vec{c} \in \{0, 1, +, -\}^{I(f)}$) are (testable) $I$-local states; similarly, if $\pi$ is deterministic then $\pi_{ij}$ is a (testable) $\{i, j\}$-local state.

The following corollary is another immediate consequence.

**Corollary 5.** $\sim \top = \bot$

**Proof.** by the Adequacy axiom, we have $1 \land \top_N \leq < 1_N > > \top_N$. But $\top_N$ is the ‘always true’ sentence, so we have $\neg < 0_I ? > > \top_N = [0_I ?]_N = \sim 0_I = 1_I = 1_I \land \top_N \leq < 1_I ? > > \top_N$. From this, we get $\top_N = [< 0_I ? > > \top_N \lor < 1_I ? > > \top_N]$. By Adequacy again, we always have $\top_N \leq < 0_I ? > 0_I \leq < 0_I ? > \top_I$ (since $0_I$ is $I$-local, so $0_I \leq \top_I$) and, similarly, $\top_N \leq < 1_I ? > \top_I$. Putting these three together, we deduce $\top_N \leq (< 0_I ? > \top_I \lor < 1_I ? > \top_I)$. But by the Testability axiom (in its dual form), we have $< 0_I ? > \top_I \leq \diamond \top_I$ and, similarly, $< 1_I ? > \top_I \leq \diamond \top_I$. Hence we have $\top_N \leq (\diamond \top_I \lor \diamond \top_I) = \diamond \top_I =$ $\top_I ? > \top_N$, and thus $\sim \top_I = [\top_I ?]_N = \neg < \top_I ? > > \top_N \leq \neg \top_N = \bot_N = \bot$.

To capture the fact that the lattice of local properties is atomistic, we accept the following inference rule.

**Local Atomicity Rule.** Local properties are unions of testable local properties (that is, of local states): if $I \neq N$ and the variable $p$ does not occur in $\varphi, \psi$ or $\theta$, 

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then from $\vdash \psi \land T(p_I) \land p_I \leq \varphi \rightarrow p_I \leq \theta$

infer $\vdash \psi \land I(\varphi) \rightarrow \varphi \leq \theta$

As a consequence of the above axioms and rules, we obtain the following corollary.

**Corollary.** For $I \neq N$, every local state is testable. In other words, if $I \neq N$ and $p$ does not occur in $\varphi$, then from

$$\vdash I(\varphi) \land \varphi \neq \perp \land (I(p) \land \perp \neq p \leq \varphi \rightarrow p = \varphi)$$

we can infer

$$\vdash T(\varphi).$$

The following axioms state that $+_i$ and $-_i$ are proper superpositions of $0_i$ and $1_i$.

**Proper Superposition Axioms.**

$$\vdash +_i \rightarrow \diamond 0_i \land \diamond 1_i$$ and

$$\vdash -_i \rightarrow \diamond 0_i \land \diamond 1_i.$$ 

The next axiom expresses the above-mentioned property of linear operators on $\mathcal{H}$ of being uniquely determined by their values on all the states $|x_i \rangle_1 \otimes \cdots \otimes |x_n \rangle$, with $|x_i \rangle_i \in \{|0_i\rangle, |1_i\rangle, |+\rangle_i\}$.

**Determinacy Axiom of Deterministic Programs.** For deterministic programs $\pi, \pi'$,

$$\vdash \bigwedge_{\bar{c} \in \{0,1,\}^n} (\pi(\bar{c}) = \pi'(\bar{c}) \rightarrow \pi(p) = \pi'(p))$$

The next axiom is the central one of our system, capturing the computational essence of entanglement, as a semantic counterpart of Proposition 6 of Section 2.

**Entanglement Axiom.** If $\pi$ is deterministic and $i \neq j$, then

$$\vdash T(p_i) \rightarrow p_i ?(\pi_{ij}) = j \pi_{ij}(p_i)$$

Before presenting out next axioms, we note some consequence of the previous ones. First, as for testability, we can define a proof-theoretic notion of locality. A formula $\varphi$ is $I$-local if $\vdash I(\varphi)$ is a theorem; similarly, a program $\pi$ is $I$-local if $\vdash I(\pi)$ is a theorem.

**Proposition 17.** Any formula of the form $\varphi_I$ is always $I$-local. Any formula of the form $\overline{\varphi_{ij}}$ is $\{i, j\}$-local. If $\varphi$ and $\psi$ are $I$-local formulas and $\pi$ is an $I$-local program, then $\varphi \lor \psi$, $\varphi \land \neg \psi$ and $\varphi \land [\pi] \psi$ are $I$-local. If $\varphi$ is $I$-local and $\psi$ is $J$-local, then $\varphi \land \psi$ is $I \cup J$-local.
Proposition 18. If \( \varphi \) is a testable \( I \)-local formula, then \( \varphi^- \) is an \( I \)-local program. \( \top_I \) is \( I \)-local. If \( \pi \) and \( \pi' \) are \( I \)-local, then \( \pi \cup \pi' \) and \( \pi ; \pi' \) are \( I \)-local.

Proposition 19. Local programs act locally. In other words,
\[
\vdash I(\pi) \wedge p =_I q \rightarrow p =_{N\setminus I} \pi(p) =_I \pi(q)
\]

Proposition 20. Systems composed of identical parts are identical:
\[
\vdash p =_I q \wedge p =_J q \rightarrow p =_{I \cup J} q
\]

Proposition 21. \( \vdash p \perp q \leftrightarrow p \perp q_I \)

Proposition 22. (Dual Local Atomicity Rule). If \( I \neq N \), \( \varphi \) and \( \theta \) are \( I \)-separated, and \( p \) does not occur in \( \varphi \), \( \psi \) or \( \theta \), then, from
\[
\vdash \psi \wedge T(p_I) \wedge p_I \perp \varphi \rightarrow p_I \perp \theta
\]
infer
\[
\vdash \psi \wedge T(\varphi_I) \wedge T(\theta_I) \rightarrow \varphi =_I \theta
\]

Proof. By using the fact that \( p_I \perp q \leftrightarrow p_I \perp q_I \) and the \( I \)-locality of \( p_I \), we can rewrite the assumption as
\[
\vdash \psi \wedge T(p_I) \wedge p_I \leq (\top_I \wedge \sim \varphi_I) \rightarrow p_I \leq (\top_I \wedge \sim \theta_I)
\]

Now assume \( \psi \wedge T(\varphi_I) \wedge T(\theta_I) \). Then the formula \( \top_I \wedge \sim \varphi_I = \top_I \wedge \sim(\top_I \wedge [\varphi_I^-]_I \perp) \) is \( I \)-local (since \( \varphi_I^- \) is testable \( I \)-local, so \( \varphi^-I \) is an \( I \)-local program, we have \( \top_I \wedge [\varphi_I^-]_I \perp \) is \( I \)-local) and, similarly, \( \top_I \wedge \sim \theta_I \) is \( I \)-local. So we can apply the Local Atomicity Rule to get \( (\top_I \wedge \sim \varphi_I) \leq (\top_I \wedge \sim \theta_I) \). Applying orthocomplementation, we have \( \neg(\top_I \wedge \sim \theta_I) \leq \neg(\top_I \wedge \sim \varphi_I) \). From this we get
\[
\theta_I = \neg \neg \theta_I = \bot \sqcup \neg \neg \theta_I = \neg \neg (\top_I \wedge \sim \theta_I) \leq \neg (\top_I \wedge \sim \varphi_I) = \neg \neg (\top_I \wedge \sim \varphi_I) = \top_I \sqcup \sim \varphi_I = \top_I \sqcup \varphi_I = \varphi_I
\]

However, by the Local States Axiom, this then implies that \( \theta_I = \varphi_I \) (since both are testable \( I \)-local with \( I \neq N \), and thus they are local states). Since both \( \theta_I \) and \( \varphi_I \) are \( I \)-separated, it follows that \( \theta =_I \varphi \).
Theorem 4. (Compatibility of Programs Affecting Different Qubits). If \( I \cap J = \emptyset \) and \( \pi, \pi' \) are deterministic, then

\[ \vdash I(\pi) \land J(\pi') \rightarrow \pi; \pi'(p) = \pi'(\pi(p)) \]

Proof. This is an immediate application of the Determinacy Axiom above. By that axiom, it is enough to show the required identity for all \( p \) of the form \( \vec{c} \in \{0, 1, +\}^n \). Using the fact that \( I \cup (N \setminus (I \cup J)) \subseteq N \setminus J \) and \( J \cup (N \setminus (I \cup J)) \subseteq N \setminus I \) (since \( I \cap J = \emptyset \)) and Proposition 19 (saying that local programs ‘act locally’), we can easily show that

\[ (\pi; \pi')(\vec{c}N) = N \setminus (I \cup J)^{cN} = (\pi'; \pi)(\vec{c}N) \]

Using Proposition 20, we put these together to conclude that

\[ (\pi; \pi; \pi')(\vec{c}N) = (\pi'; \pi)(\vec{c}N) \]

Proposition 23. (Dual Entanglement). If \( \pi \) is deterministic and \( i \neq j \), then

\[ \vdash T(q_j) \rightarrow q_j?(\pi_{ij}(\vec{c}N)) \]

Proof. Assume \( T(q_j) \) and we need to show that \( q_j?(\pi_{ij}(\vec{c}N)) =_i \pi_{ij}^+(q_j) \). It is easy to see that both sides are \( i \)-separated (that is, \( \leq_T \pi_{ij} \)), and also that both \( (q_j?(\pi_{ij}(\vec{c}N))) \) and \( (\pi_{ij}^+(q_j)) \) are testable (since they are local states), so we are in the conditions of the Dual Local Atomicity Rule (Proposition 22) above. By that Proposition, to prove the above identity, it is enough to show that

\[ \vdash T(p_i) \land p_i \perp \pi_{ij}^+(q_j) \rightarrow p_i \perp q_j?(\pi_{ij}(\vec{c}N)) \]

To show this, let \( p_i \) be such that \( T(p_i) \) and \( p_i \perp \pi_{ij}^+(q_j) \). By the Adjointness Theorem, we have then \( \pi_{ij}(p_i) \perp q_j \), and thus \( q_j?(\pi_{ij}(p_i)) = \perp \). By the previous Proposition (on Compatibility of Programs on Different Qubits), we have

\[ p_i?(q_j?(\pi_{ij}(\vec{c}N))) = (p_i;p_j?q_j)(\pi_{ij}) \]
= \langle q_j?; p_i? \rangle(\pi_{ij})
= q_j?(p_i?(\pi_{ij}))
= q_j?(\pi_{ij}(p_i))
= \perp

(where we have used the Entanglement Axiom). So we get \( p_i \perp q_j?(\pi_{ij}) \). (Thus, using the Dual Local Atomicity Rule, the desired conclusion follows).

**Proposition 24. (Entanglement Preparation Lemma).**

\[ \vdash \pi_{ij}(p_i) \perp q_j \rightarrow \pi_{ij} \perp (p_i \land q_j) \]

**Proof.** From the hypothesis, we get \( q_j \perp (\pi_{ij}(p_i))_j \), and thus \((p_i \land q_j) \perp (\pi_{ij}(p_i))_j\), from which it follows that \((p_i \land q_j) \perp [p_i?(\pi_{ij}(p_i)))_j\) (using the fact that \(p_i?(p_i \land q_j) = p_i \land q_j\), by Adequacy). On the other hand, we have \(\pi_{ij} \leq [p_i?(\pi_{ij}(p_i)))_j\) (since \(p_i?(\pi_{ij}) \leq (p_i?(\pi_{ij}))_j = (\pi_{ij}(p_i))_j\), by the Entanglement Axiom), so we get \(p_i \land q_j \perp \pi_{ij}\).

**Theorem 5. (Teleportation Property).** If \(i, j, k\) are distinct indices then

\[ \vdash \langle \sigma_{jk}?; \pi_{ij}? \rangle(p_i) \equiv_k (\pi_{ij}; \sigma_{jk})(p_i) \]

**Proof.** By the same argument as above, it is enough to prove

\[ \vdash T(q_k) \land q_k \perp (\pi_{ij}; \sigma_{jk})(p_i) \rightarrow q_k \perp (\sigma_{jk}?; \pi_{ij}?)(p_i) \]

To show this, let \(q_k\) be such that \(T(q_k)\) and \(T(q_k) \perp (\pi_{ij}; \sigma_{jk})(p_i)\). Then \(q_k \perp \sigma_{jk}(\pi_{ij}(p_i))\), and, by the Adjointness Theorem, we have \(\sigma_{jk}^\dagger(q_k) \perp \pi_{ij}(p_i)\). By Dual Entanglement, it follows that \(q_k?(\sigma_{jk}) \perp \pi_{ij}(p_i)\). By the Entanglement Preparation Lemma, we have \(\pi_{ij} \perp (q_k?(\sigma_{jk}) \land p_i)\). Hence we get

\[ q_k?(\langle \sigma_{jk}?; \pi_{ij}? \rangle(p_i)) = q_k?(\pi_{ij}?)(\sigma_{jk}^\dagger(p_i)) = \pi_{ij}!(q_k?(\sigma_{jk}^\dagger(p_i))) = \pi_{ij}!(q_k?(\sigma_{jk}^\dagger(p_i))) = \pi_{ij}(q_k?(\sigma_{jk} \land p_i)) = \perp \]

(where we have used Theorem 4 on the Compatibility of Programs on Different Qubits). So we get \(q_k \perp (\sigma_{jk}?; \pi_{ij}?)(p_i)\), as desired.

**Corollary 6.** If \(i, j, k\) are distinct,

\[ \vdash \pi_{ij}?((p_i \land \sigma_{jk})) \equiv_k (\pi_{ij}; \sigma_{jk})(p_i) \]
Proof. By the Repeatability Axiom, we have \( \sigma_{jk}! \)(\( p_i \)) \leq \sigma_{jk}. \) Assuming \( \sigma_{jk}! \)(\( p_i \)) \not= \bot, \) we get \( \sigma_{jk}! \)(\( p_i \)) = \( jk \) \( \sigma_{jk} \) (since \( \sigma_{jk} \) is testable and \{\( j, k \}\}-local, and hence it is a local state) and also that \( \sigma_{jk}! \)(\( p_i \)) = \( i \) \( p_i \) (since ‘local programs act locally’, by Proposition 19). Thus, we get \( \sigma_{jk}! \)(\( p_i \)) = \( ijk \) \( p_i \wedge \sigma_{jk} \). Applying the \{\( i, j \}\}-local program \( \pi_{ij} \), we get
\[
\pi_{ij}!(\( p_i \wedge \sigma_{jk} \)) = \pi_{ij}!(\( \sigma_{jk} ! \)(\( p_i \)))
\]
\[
= (\sigma_{jk} ! ; \pi_{ij} !)(\( p_i \))
\]
\[
= k \ (\pi_{ij} ; \sigma_{jk})(\( p_i \))
\]
from which we get the desired conclusion.

By a refinement of the proof of Teleportation Property, we can prove the following proof-theoretic version of Lemma 3 in Section 2.2.

Proposition 25. (Entanglement Composition Lemma). For distinct indices \( i, j, k, l \), programs \( \pi, \pi' \), \( \pi'' \) and local \{\( 1 \}\}-programs \( \sigma_1, \rho_1 \) we have

\[
\vdash \pi_{ij} \wedge \pi'_{kl} \rightarrow \left[ \sigma_j ; \rho_k ; \pi_{jk} ! \right] \left[ \pi_1 ; \pi'' ; \rho_1 ! \right]_{il}
\]

The domain \( \text{dom}(\varphi) \) of a map \( \pi \) is defined as \( \text{dom}(\pi) := \langle \pi \rangle > T \).

Theorem 6. (Agreement Property). If two \( I \)-local maps \( \pi, \pi' \) have the same domain and they separate the input-state, then their output states agree on all non-\( I \) qubits: that is, if \( I \cap J = \emptyset \) then for all deterministic programs \( \pi, \pi' \) we have

\[
\vdash T(\pi) \wedge I(\pi) \wedge I(\pi') \wedge \text{dom}(\pi) = \text{dom}(\pi') \wedge \pi(p) \leq \top \wedge \pi'(p) \leq \top \rightarrow \pi(p) = \text{N}_{\backslash I} \pi'(p).
\]

Proof. Put \( \psi := T(\pi) \wedge I(\pi) \wedge I(\pi') \wedge \text{dom}(\pi) = \text{dom}(\pi') \wedge \pi(p) \leq \top \wedge \pi'(p) \leq \top, \) and assume that \( \psi \) is true. By definition, \( \pi(p) \) is testable (since \( \pi \) is deterministic, so \( \pi(p) = \pi[p] \approx \pi^* \approx p \), and every sentence of the form \( \approx \psi \) is testable), and the same is true for \( \pi'(p) \). So we can use the Dual Local Atomicity Rule to prove the above identity. Let now \( q_{N_{\backslash I}} \) be such that \( T(q_{N_{\backslash I}}) \) and \( q_{N_{\backslash I}} \perp \pi(p) \). Then \( (\pi; q_{N_{\backslash I}} ! ?)(p) = \bot. \) By the Compatibility of Programs on Different Qubits, we get \( (q_{N_{\backslash I}} ! ; \pi)(p) = \bot, \) that is \( p \leq [q_{N_{\backslash I}}][\pi] \perp = [q_{N_{\backslash I}}] \neg z(\pi) \top = [q_{N_{\backslash I}}] \neg \text{dom}(\pi). \) But \( \text{dom}(\pi) = \text{dom}(\pi'), \) so \( p \leq [q_{N_{\backslash I}}] \neg \text{dom}(\pi') = [q_{N_{\backslash I}}][\pi'] \perp, \) that is, \( (q_{N_{\backslash I}} ! ?)(p) = \bot. \) Working
now in reverse, we again apply the Compatibility of Programs on Different Qubits, obtaining \((\pi'; q_{N\setminus I})(p) = \bot\), that is, \(q_{N\setminus I} \perp \pi'(p)\). So we have proved that

\[\vdash \psi \land T(q_{N\setminus I}) \land q_{N\setminus I} \perp \pi(p) \rightarrow q_{N\setminus I} \perp \pi'(p)\]

By now applying the Dual Local Atomicity Rule, we get

\[\vdash \psi \rightarrow \pi(p) =_{N\setminus I} \pi'(p),\]

which is, the desired conclusion.

**Characteristic Formulas.**

In order to formulate our next axioms (which deal with special logic gates), we now give some characteristic formulas for binary states, considering two qubits indexed by \(i\) and \(j\).

| States \(\ket{00}_{ij}\) = \(\ket{0}_i \otimes \ket{0}_j\) | Characteristic Formulas \(\langle 0, ? \rangle_0 \land \langle 1, ? \rangle_0 \perp\) |
|---|---|
| Bell states: \(\beta_{xy}^{ij} = \ket{0}_i \otimes \ket{y}_j + (-1)^x \ket{1}_i \otimes \ket{\bar{y}}_j\) with \(0 = 1\) and \(\bar{1} = 0\), \(x,y \in \{0,1\}\) \(\langle 0, ? \rangle_0 \land \langle 1, ? \rangle_0 \land \langle (+, ?) \rangle_0 \land \langle (-, ?) \rangle_0\) where \((-)^x = -\) if \(x = 1\) and \((-)^x = +\) if \(x = 0\) |
| \(\gamma_{xy}^{ij} = \beta_{00}^{ij} + \beta_{01}^{ij} =\) \(\langle 00 \rangle_{ij} + \langle 01 \rangle_{ij} + \langle 10 \rangle_{ij} + \langle 11 \rangle_{ij}\) \(\langle 0, ? \rangle_0 \land \langle 1, ? \rangle_0 \land \langle (+, ?) \rangle_0 \land \langle (+, ?) \rangle_0\) |

**Locality Axiom for Quantum Gates.**

Our special quantum gates are local, affecting only the specified qubits:

\[\vdash \{i\}(X_i) \land \{i\}(Z_i) \land \{i\}(H_i) \land \{i, j\}(CNOT_{ij})\]

In addition to this, we require for \(X, Z, H\).

**Characteristic Axioms for Quantum Gates \(X\) and \(Z\).**

\[\vdash 0_i \rightarrow [X_i]1_i \quad \vdash 1_i \rightarrow [X_i]0_i \quad \vdash +_i \rightarrow [X_i]+_i\]
\[\vdash 0_i \rightarrow [Z_i]0_i \quad \vdash 1_i \rightarrow [Z_i]1_i \quad \vdash +_i \rightarrow [Z_i]-_i\]
\[\vdash 0_i \rightarrow [H_i]+_i \quad \vdash 1_i \rightarrow [H_i]-_i \quad \vdash +_i \rightarrow [H_i]0_i\]

**Notation (Bell formulas)**

For \(x,y \in \{0,1\}\) and distinct indices \(i,j \in N\), we make the abbreviations for \(\beta_{xy}^{ij} := (Z_i^x \cdot X_i^y)_{ij}\), and refer to these expressions as ‘the Bell formulas’.
Proposition 26. The Bell states $\beta_{ij}^{xy}$ are characterised by the logic Bell formulas $\beta_{ij}^{xy}$. In other words, a state satisfies one of these formulas iff it coincides with the corresponding Bell state.

Proof. It is enough to check that the formulas $\beta_{ij}^{xy}$ imply the corresponding characteristic formulas in the above table. For this, we use the Entanglement Axiom and the following (easily checked) theorems:

\[ \vdash 0_1 \leftrightarrow <Z^x_1; X^y_1 > y_1 \]
\[ \vdash 1_1 \leftrightarrow <Z^x_1; X^y_1 > \bar{y}_1 \]
\[ \vdash +_1 \rightarrow <Z^x_1; X^y_1 > (\bar{y})^x_1 \]

Generalised Bell formulas, GHZ States. As shown by the first author’s student Dmitri Akatov in his Master’s thesis [Akatov 2005], the above dynamic-logical characterisation of Bell states can be recursively extended to the so-called generalised (k-qubit) Bell states (which form an orthonormal basis for the k-qubit space), for all $k \leq n$. Here, we only mention a special case, that of the so-called GHZ state (after Greenberg, Horne and Zeilinger):

\[ \beta_{i,j,k}^{000} = |000 \rangle_{ijk} + |111 \rangle_{ijk} \]

This state, of a special significance for various quantum protocols, can be characterised by the formula

\[ \beta_{i,j,k}^{000} := <0_i ? > (0_j \land 0_k) \land <1_j ? > (1_j \land 1_k) \land <+_i ? > \beta_{ij}^{0k} \]

From this, it is obvious that we have $+_i ? (\beta_{i,j,k}^{000}) = jk \beta_{i,j}^{0k}$; but one can easily check that we also have $-_i ? (\beta_{i,j,k}^{000}) = jk \beta_{i,j}^{1k}$. Using the notation $(-)^z$ introduced above for $z = 0, 1$ (putting $(-)^2 := = \text{if} \ z = 1 \text{and} (-)^2 := + \text{if} \ z = 0$), we can summarize this as

\[ (-)^z_i ? (\beta_{i,j,k}^{000}) = jk \beta_{i,j}^{zk} \]

Characteristic Axioms for CNOT. With the above notation, we put

| $\vdash 0_i \land c_j \rightarrow [CNOT_{ij}]c_j$ | $\vdash 1_i \land 0_j \rightarrow [CNOT_{ij}]1_j$ |
| $\vdash 1_i \land 1_j \rightarrow [CNOT_{ij}]0_j$ | $\vdash 1_i \land +_j \rightarrow [CNOT_{ij}] + j$ |
| $\vdash +_i \land 0_j \rightarrow [CNOT_{ij}]\beta_{0i}^{ij}$ | $\vdash +_i \land 1_j \rightarrow [CNOT_{ij}]\beta_{1i}^{ij}$ |
| $\vdash +_i \land +_j \rightarrow [CNOT_{ij}]\gamma_i^{ij}$ where $\gamma_i^{ij} = \langle 0_i ? \rangle + j \land \langle 1_i ? \rangle + j \land \langle +_i ? \rangle + j$ |
Proposition 27. For all \( x, y \in \{0, 1\} \),
\[
\vdash (H_i; \text{CNOT}_{i,j}(x_i \land y_j)) = \beta^{ij}_{xy}
\]

Corollary 7. If \( i, j, k \) are all distinct,
\[
\vdash (\text{CNOT}_{i,j}; H_j; (x_i \land y_j)\?)(p) =_k \beta^{i,j\?}_{xy}(p)
\]

Proof. From the above Proposition and from \( H^\dagger = H, \text{CNOT}^\dagger = \text{CNOT} \),
we get \((\text{CNOT}_{i,j}; H_i)(\beta^{ij}_{xy}) = x_i \land y_i\), and thus
\[
\text{dom}(\text{CNOT}_{i,j}; H_i) = (\text{CNOT}_{i,j}; H_i; (x_i \land y_j)\?)(\top)
\]
\[
= (\beta^{ij\?}_{xy})(\top)
\]
\[
= \text{dom}(\beta^{ij\?}_{xy})
\]
The conclusion follows from this, together with the Agreement Property.

Theorem 7. All the above axioms and rules are sound for (quantum frames as-
associated to) \( n \)-dimensional Hilbert spaces of the form \( H^\otimes n \), where \( H \) is any two-
dimensional Hilbert space.

The problem of obtaining a complete proof system for this logic is still open\(^{21}\).

5 Applications: correctness of quantum programs

As applications to our logic, one can provide formal correctness proofs for a whole
range of quantum programs; one could claim that all quantum circuits and pro-
tocols in which probabilities do not play an essential role can, in principle, be
verified using our logic, or some trivial extension of this logic (obtained by intro-
ducing more basic constants for other relevant states and programs). In particular,
all the quantum programs covered by the ‘entanglement networks’ approach in
[Coecke 2004] can be treated in this logic. In his Master’s thesis [Akatov 2005],
D. Akatov has applied our logic to the verification of various other protocols, for

\(^{21}\)However, we have strong reasons to believe the above system is not complete. At least one
other sound interesting axiom (of particular significance to quantum computing) has been proposed
by the first author’s student D. Akatov in his Master’s thesis [Akatov 2005]. This is the ‘Determinacy
of States’ axiom, which captures the converse of our Entanglement axiom: any entangled state is
‘entangled according to some quantum program’ \( \pi \) (that is, it is of the form \( \pi_{i,j} \)); we chose not to
include it here, as we have not used it in this paper.
example, superdense coding, quantum secret sharing, entanglement swapping, logic gate teleportation, circuits for parallel computation of (sequential) compositions of programs using Bell base measurements. The proofs are modular, using as ingredients the main lemmas proved above: the Compatibility Theorem, the Teleportation Property, the Entanglement Composition Lemma and the Agreement Property. For simplicity, we will only consider two basic examples here: quantum teleportation and quantum secret sharing.

**Quantum teleportation**

Following [Nielsen and Chuang 2000], quantum teleportation is the name of a technique that makes it possible to ‘teleport’ (that is, move) a quantum state between two agents, even in the absence of a quantum communication channel linking the sender and the recipient. We are working in $H \otimes H \otimes H$, with $H$ being the two-dimensional (qubit) space, and so $n = 3$. There are two agents, Alice and Bob who, separated in space, each having one qubit of an entangled EPR pair, represented by $\beta^{2,3}_{00} \in H^{(2)} \otimes H^{(3)}$. In addition to her part of the EPR pair, Alice has another qubit $1$, in an unknown state $q_1$. (Note that $q_1$ is a testable 1-local property, since it is a 1-local state.) Alice wants to ‘teleport’ this unknown qubit to Bob, that is, to execute a program that will output a state satisfying $id_{13}(q_1)$.

To do this, she entangles the qubit $q_1$ with her part $q_2$ of the EPR pair, by performing first a $CNOT_{12}$ gate on the two qubits and then a Hadamard transformation $H_1$ on the first component. Then Alice measures her qubits in the standard basis, thus destroying the entanglement, so that Bob’s qubit is left in a separated state $q_3$. Though this state is unknown, the results of Alice’s measurements indicate the actions that Bob will have to perform in order to transfer his qubit from state $q_3$ into the state $id_{13}(q_1)$ (corresponding to the qubit Alice had before the protocol).

It is thus enough for Alice to send Bob two classical bits encoding the result $x_1$ of the first measurement and the result $y_2$ of the second measurement. To achieve ‘teleportation’, Bob will have to apply the $X$-gate $y$ times, then apply the $Z$ gate $x$ times.

In our syntax, the quantum program described here is

$$\pi = \bigcup_{x,y \in \{0,1\}} CNOT_{12}; \ H_1; (x_1 \land y_2)?; X_{y}; Z_{x}$$

and the validity expressing the correctness of teleportation is

$$\vdash \pi(q_1 \land \beta^{2,3}_{00}) = id_{13}(q_1)$$

22However, note that a classical communication channel is required!
To show this, observe that by applying the above Corollary to Proposition 27, in which we take \( i = 1 \), \( j = 2 \), \( k = 3 \), we get that the validity above (to be proved) is equivalent to

\[
\vdash (\beta_{xy}^{12}; X_i^y; Z_3^x)(q_1 \land \beta_{00}^{23}) = 3 id_{13}(q_1)
\]

Repeating the logical Bell formulas with their definition \( \beta_{xy}^{ij} := (Z_i^x; X_i^y; Z_3^x) \), we obtain the following equivalent validity:

\[
\vdash ((Z_i^x; X_i^y; Z_3^x)_{12}; X_i^y; Z_3^x)(q_1 \land \text{id}_{23}) = 3 \text{id}_{13}(q_1),
\]

where \( \text{id} = Z_1^0; X_1^0 \) is the identity. This last validity follows from applying the (Corollary of) Teleportation Property and the validity \( Z_i^x; X_i^y; Z_3^x = \text{id} \) (due to \( X^{-1} = X, Z^{-1} = Z \)).

**Quantum secret sharing**

As described in [Gruska 1999], this protocol realises the splitting of quantum information into a given number \( m \) of ‘shares’ (among \( m \) agents), such that the original information (the ‘secret’) can be recovered only by pooling together the information in all the shares. The protocol uses GHZ states in a similar way to teleportation. We consider here the case \( m = 3 \) as an example: suppose Alice, Bob and Charles share a GHZ triple state \( \beta_{000}^{234} \) (each ‘having’ one of the three entangled qubits, in increasing order: for example, Alice has qubit 2, and so on). In addition, Alice has another qubit 1, in an unknown state \( q \). To split this information \( q \) into three shares, Alice measures her two qubits 1 and 2 in the Bell basis, obtaining two bits \( x, y \) (corresponding to which of the four Bell states \( \beta_{xy}^{12} \) she obtained). After that, Bob measures his qubit 2 in the dual basis \( \{+, -\} \), obtaining another bit \( | - >^z \) (with \( z \in \{0, 1\} \)). Finally, Charles is given qubit 4, which is now in one of 8 possible states \( \psi(x, y; z) \) (depending on the results obtained by Alice and Bob).

To recover the original ‘secret’ \( q \) from his qubit \( \psi(x, y; z) \), Charles can now apply a local unitary transformation \( Z_z^4; X_z^4; Z_z^4 \). But notice that for this, he needs to know \( x, y \) and \( z \), that is, the three agents have to share their information in order to recover \( q \).

The quantum program described here is

\[
\pi = \bigcup_{x, y \in \{0, 1\}} \beta_{xy}^{12}; (\text{z})_{3}; Z_4^x; X_4^y; Z_4^z
\]

Footnote 2: Here we use for vectors a similar notation to the notation \( (\text{z}) \) introduced for states in the previous section, that is, \( | - >^z := | - > \) for \( z = 0 \), and \( | - >^z := | + > \) for \( z = 1 \).
To prove correctness, we need to show

\[ \vdash \pi(q_1 \land \beta_{000}^{234}) = i d_{14}(q_1) \]

To show this, we use Compatibility and the 3-locality of \((-)_{3}^{234}\) to compute

\[
(\beta_{xy}^{12}; (-)^{234}_{3}; Z_{4}^{z}; X_{4}^{y}; Z_{4}^{x})(q_1 \land \beta_{000}^{234}) = ((-)^{z}_{3}; \beta_{xy}^{12}; Z_{4}^{z}; X_{4}^{y}; Z_{4}^{x})(q_1 \land \beta_{000}^{234})
\]

\[
= (Z_{4}^{z}; X_{4}^{y}; Z_{4}^{x})(\beta_{xy}^{12}; (q_1 \land (-)^{z}_{3}((\beta_{000}^{234}))))
\]

But recall that

\[
(-)^{z}_{3}((\beta_{000}^{234}) = 24 \beta_{24}^{0} = 24 Z_{124}^{xy},
\]

so we have

\[
\pi(q_1 \land \beta_{000}^{234}) = 24 (Z_{4}^{z}; X_{4}^{y}; Z_{4}^{x})(\beta_{xy}^{12}; (q_1 \land (Z_{1}^{z})_{24}))
\]

\[
= (Z_{4}^{z}; X_{4}^{y}; Z_{4}^{x})((Z_{1}^{z}; X_{1}^{y})_{12}; (q_1 \land (Z_{1}^{z})_{24}))
\]

Applying the (Corollary of) the Teleportation Property, we get

\[
\pi(q_1 \land \beta_{000}^{234}) = 4 (Z_{1}^{z}; X_{1}^{y}; Z_{1}^{x})(((Z_{1}^{z}; X_{1}^{y})_{12}; (Z_{1}^{z})_{24})(q_1))
\]

\[
= (Z_{1}^{z}; X_{1}^{y}; Z_{1}^{x})_{14}((Z_{1}^{z}; X_{1}^{y}; Z_{1}^{x})_{14}(q_1))
\]

\[
= (Z_{1}^{z}; X_{1}^{y}; Z_{1}^{x}; X_{1}^{y}; Z_{1}^{x})_{14}(q_1)
\]

\[
= i d_{14}(q_1)
\]

**Note:** This proof can be easily generalised to the case of an \(m\)-share split (among \(m\) agents) of the secret. See [Aklov 2005](#) for details.

## 6 Conclusions and future work

We have presented here a dynamic logic for compound quantum systems, capable of expressing and proving highly non-trivial features of quantum information flow, such as entanglement and teleportation, properties of local transformations, separation, Bell states and so on. The logic is Boolean, but has modalities capturing the non-classical logical dynamics of quantum systems; in addition, it has spatial features, allowing us to express properties of subsystems of a compound quantum system. The logic comes with a simple relational semantics, in terms of quantum states and quantum actions in a Hilbert space. We have presented a sound proof system, which can be used to prove many interesting properties of quantum information, including formal correctness proofs for a whole range
of quantum protocols (we have treated teleportation and quantum secret sharing here, but there are also many others to be considered, such as superdense coding, entanglement swapping and logic-gate teleportation).

However, a number of open problems remain. While in [Baltag and Smets 2005a] we sketched a completeness result for the quantum dynamic logic of single-system quantum frames, no corresponding completeness result is known for compound systems. So the completeness problem for the logic LQP presented in this paper is still open.

In this paper we have not included iteration \((\text{Kleene star}) \pi^*\) among our operations on programs, since it was not needed in our simple quantum programming applications. But one can, of course, add iteration and consider the resulting logic, which would be useful in applications to quantum programs involving \textit{while}-loops. The usual PDL axioms for Kleene star are sound, but, again, completeness remains an open problem.

Another problem, which is of great importance for quantum computation, is extending our setting to deal with the quantitative aspects of quantum information (in particular with notions like phase and probability). Our aim in this paper was to develop a logic to reason about qualitative quantum information flow, so we have ignored the probabilistic aspects of quantum systems. There are natural ways to extend our setting, using quantum versions of probabilistic modal logic, and we plan to investigate them in future work.

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