Matrix product solutions to the $G_2$ reflection equation

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We study the $G_2$ reflection equation for the three particles in $1 + 1$ dimension that undergo a special scattering/reflections described by the Pappus theorem. It is a sixth order equation and serves as a natural $G_2$ analogue of the Yang–Baxter and the reflection equations corresponding to the cubic and the quartic Coxeter relations of Type $A$ and $BC$, respectively. We construct matrix product solutions to the $G_2$ reflection equation by exploiting a connection to the representation theory of the quantized coordinate ring $A_q(G_2)$.

Keywords: Yang–Baxter equation, tetrahedron equation, $G_2$ reflection equation, quantized coordinate ring.

1. Introduction

The Yang–Baxter and the reflection equations

\[
R_{12}(\alpha_1)R_{13}(\alpha_1 + \alpha_2)R_{23}(\alpha_2) = R_{23}(\alpha_2)R_{13}(\alpha_1 + \alpha_2)R_{12}(\alpha_1),
\]

\[
R_{12}(\alpha_1)K_2(\alpha_1 + \alpha_2)R_{21}(\alpha_1 + 2\alpha_2)K_1(\alpha_2) = K_1(\alpha_2)R_{12}(\alpha_1 + 2\alpha_2)K_2(\alpha_1 + \alpha_2)R_{21}(\alpha_1)
\]

(1)

are the fundamental structure in quantum integrable systems in the bulk [1] and at the boundary [2–4]. They are the Yang–Baxterizations (spectral parameter dependent versions) of the cubic and the quartic Coxeter relations for the simple reflections $s_1, s_2$ of the root systems of $A_2$ and $B_2/C_2$:

\[
s_1s_2s_1 = s_2s_1s_2,
\]

\[
\Delta_+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\},
\]

\[
s_1s_2s_1s_2 = s_2s_1s_2s_1,
\]

\[
\Delta_+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.
\]

Herein, $\alpha_1, \alpha_2$ are the simple roots and $\Delta_+$ denotes the set of positive roots which formally correspond to the spectral parameters. They are so ordered that the $k$th one from the left is $s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ with $i_k = 1$ ($k$: odd) and $i_k = 2$ ($k$: even). For simplicity, we assume $R \in \text{End}(V \otimes V)$ and $K \in \text{End} V$ for some vector space $V$.

It is natural and by now classic to extend the ‘factorization’ condition like (1) to more general root systems [2]. In this article, we study the $G_2$ case of the form

\[
R_{12}(\alpha_1)G_{132}(\alpha_1 + \alpha_2)R_{23}(2\alpha_1 + 3\alpha_2)G_{213}(\alpha_1 + 2\alpha_2)R_{31}(\alpha_1 + 3\alpha_2)G_{321}(\alpha_2) = G_{231}(\alpha_2)R_{13}(\alpha_1 + 3\alpha_2)G_{123}(\alpha_1 + 2\alpha_2)R_{32}(2\alpha_1 + 3\alpha_2)G_{312}(\alpha_1 + \alpha_2)R_{21}(\alpha_1),
\]

(2)
where \( R \) is a solution to the Yang–Baxter equation by itself and \( G \in \text{End}(V \otimes V \otimes V) \) is the characteristic operator in the \( G_2 \) theory. It is a Yang–Baxterization of the \( G_2 \) Coxeter relation \( s_1s_2s_1s_2s_1 = s_2s_1s_2s_1s_2 \) with the spectral parameters corresponding to the positive roots. See (12) and (13).

Although the equation (2) was not written down explicitly in [2], it was explained to the author by Cherednik [5] that the \( G_2 \) factorization condition is depicted by a three particle scattering diagram corresponding to (2) and it is related to the geometry of the Desargues-Pappus theorem.\(^1\) The equation (2) for generic symbols \( R \) and \( G \) without assuming a tensor structure on their representation space (i.e. without indices) has appeared as a defining relation of the root algebra of type \( G_2 \) [6, Section 2]. In this article, we call (2) the \( G_2 \) reflection equation for simplicity.

The purpose of this article is to construct families of solutions to the \( G_2 \) reflection equation with \( V = (\mathbb{C}^2)^{\otimes n} \) for any positive integer \( n \). Our approach is based on the three-dimensional (3D) integrability developed in [7–10] for the Yang–Baxter equation and in [11] for the reflection equation. The most essential idea of it is to embark on a quantization or a 3D version of the \( G_2 \) reflection equation. We introduce the quantized \( G_2 \) reflection equation

\[
(L_{12}L_{132}L_{23}L_{231}L_{321}) \circ \mathcal{F} = \mathcal{F} \circ (J_{123}L_{123}L_{232}J_{312}L_{21}),
\]

which is a \( G_2 \) reflection equation (without spectral parameters) up to conjugation by a certain operator \( \mathcal{F} \) acting on an auxiliary \( q \)-boson Fock space. Our finding (Theorem 4.1) is that with a suitable choice of the quantized scattering amplitude \( L \) and \( J \), the equation (3) coincides exactly with the intertwining relation [12, equation (28)] of the \( A_q(G_2) \) modules labelled by the longest element of the Weyl group [13]. The \( \mathcal{F} \) corresponds to the intertwiner. Here \( A_q(\mathfrak{g}) \), for a finite dimensional classical simple Lie algebra \( \mathfrak{g} \) in general, denotes a Hopf subalgebra of the dual \( U_q(\mathfrak{g})^* \) called quantized coordinate ring. It has been studied from a variety of aspects. See [12–21] for example.

In short, we obtain a solution to the quantized \( G_2 \) reflection equation (3). It offers a bonus; the equation/solution can be concatenated along the \( q \)-boson Fock space for arbitrary \( n \) times. The piled \( n \) layers of the \( 1+1 \) dimensional scattering diagrams can be viewed as a 3D lattice system in which adjacent layers may be interchanged locally according to (3) without changing the total statistical weight, a feature roughly referred to as 3D integrability. Anyway, to the \( n \)-concatenation of the quantized \( G_2 \) reflection equation, one can insert the spectral parameters and evaluate the intertwiner \( \mathcal{F} \) away appropriately. It brings us back to the original \( G_2 \) reflection equation, thereby producing a solution to it for each \( n \). Actually there are two such recipes called trace reduction and boundary vector reduction. They lead to the solutions \((R^u(z), G^u(z))\) and \((R^{bv}(z), G^{bv}(z))\),\(^2\) respectively. By the construction they possess the matrix product structure containing \( n \)-product of \( L \)'s or \( J \)'s\(^3\)

\[
R^u(z) = \varrho^u(z) \text{Tr}(z^bL \cdots L), \quad G^u(z) = \kappa^u(z) \text{Tr}(z^bJ \cdots J),
\]

\[
R^{bv}(z) = \varrho^{bv}(z) \langle \xi | z^bL \cdots L | \xi \rangle, \quad G^{bv}(z) = \kappa^{bv}(z) \langle \xi | z^bJ \cdots J | \xi \rangle.
\]

where the trace and the sandwich \( \langle \xi | (\cdots) | \xi \rangle \) are taken over a \( q \)-boson Fock space. The detail will be explained in later sections. The solutions are trigonometric in the spectral parameter.\(^4\) In fact \( R^u(z) \) and \( R^{bv}(z) \) turn out to be the quantum \( R \) matrices [14, 22] for the antisymmetric tensor representations of

\(^{1}\) This is partly described in [2, p982] and will be detailed in Section 2.2.

\(^{2}\) This latter solution assumes the relation (77) yet to be proved.

\(^{3}\) \( \varrho^u(z), \varrho^{bv}(z), \kappa^u(z) \) and \( \kappa^{bv}(z) \) are scalars given in (68) and (87).

\(^{4}\) 'Trigonometric' means rational in \( z \) in (4) which corresponds to the exponential of the spectral parameters in (2).
$U_p(A^{(1)}_{n-1})$ and the spin representation of $U_p(D^{(2)}_{n+1})$ with $p^2 = -q^{-3}$. This part of the results is contained in the earlier works [7, 8].

This article may be viewed as a continuation of [7, 8] and [11] where analogous results were obtained for the Yang–Baxter and the reflection equations, respectively. To explore applications of the $G_2$ reflection equation is a future problem. For instance to architect commuting transfer matrices based on the $G_2$ reflection is an interesting issue.

The article is organized as follows. In Section 2, we explain the interpretation of the $G_2$ reflection equation in terms of a special three particle scattering following [2, 5]. The characteristic feature is the operator $G$ which encodes the simultaneous reflection of one of the particles at the boundary and scattering of the other two. The world lines of these particles form a configuration matching the classical Pappus theorem.

In Section 3, we formulate the quantized $G_2$ reflection equation by promoting $R$ and $G$ in (2) to the $q$-boson valued $L$ and $J$. The $L$ matrix (19) appeared first in [7]. The $q$-boson valued amplitude $J$ (24)–(31) is new. It has been designed deliberately to validate Theorem 4.1. It does not split into the product of operators representing the single particle reflection and the two particle scattering. See (34).

In Section 4, after recalling basic facts on the representation theory of $A_q(G_2)$ [13], we state our key observation in Theorem 4.1. It identifies the quantized $G_2$ reflection equation with the intertwining relation between certain $A_q(G_2)$ modules.

In Section 5, we review the reduction of the tetrahedron equation (cf. [23]) to the Yang–Baxter equation following [7, 8, 11]. This construction has been illustrated in many literatures recently, e.g. [9, 10], so we keep the description brief. A slightly more detailed exposition is available in [11, Appendix B].

In Section 6, we explain that the analogous reduction works perfectly also for the quantized $G_2$ reflection equation. They lead to two families of solutions $(R^e(z), G^e(z))$ and $(R^h(z), G^h(z))$ to the $G_2$ reflection equation (2), where the latter is yet based on the conjectural relation (77). The role of the intertwiner $\mathcal{F}$ is curious. Although, it is complicated and no closed formula is known, it does not give rise to a difficulty since the reduction procedure just eliminates it. Nevertheless $\mathcal{F}$ essentially controls the construction behind the scene in that it specifies precisely how the $L$ and $J$ are to be combined, how the spectral parameters should be arranged and what kind of boundary vectors are acceptable. These are essential legacy of $\mathcal{F}$.

Section 7 is a summary. Appendix A describes the precise correspondence between the quantized $G_2$ reflection equation and the intertwining relation (44) of the $A_q(G_2)$ modules. Appendix B contains explicit forms of $(R^e(z), G^e(z))$ and $(R^h(z), G^h(z))$ for small $n$.

Throughout the article, we assume that $q$ is generic and use the following notation:

$$(z; q)_m = \prod_{k=1}^{m} (1 - z q^{k-1}), \quad (q)_m = (q; q)_m.$$ 

$\theta(\text{true}) = 1, \quad \theta(\text{false}) = 0, \quad e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n \quad (1 \leq j \leq n).$

2. $G_2$ reflection equation for three particle scattering

2.1 The $G_2$ reflection equation

Let $V$ be a vector space and consider the operators

$$R(z) \in \text{End}(V \otimes V), \quad G(z) \in \text{End}(V \otimes V \otimes V)$$

Equation (5)
depending on the spectral parameter $z$. We assume that $R(z)$ satisfies the Yang–Baxter equation:

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x) \in \text{End}(V \otimes V \otimes V).$$

(6)

By the $G_2$ reflection equation we mean the following in $\text{End}(V \otimes V \otimes V)$:

$$R_{12}(x)G_{132}(xy)R_{23}(x^2y^3)G_{213}(xy^2)R_{31}(xy^3)G_{321}(y)$$

$$= G_{231}(y)R_{13}(xy^3)G_{123}(x^2y^3)G_{312}(xy)R_{21}(x).$$

(7)

To explain the notation, write temporarily as $R(z) = \sum r_i^{(1)} \otimes r_i^{(2)}$ and $G(z) = \sum g_i^{(1)} \otimes g_i^{(2)} \otimes g_i^{(3)}$ with some sums over $l$. Then

$$R_{12}(z) = \sum r_i^{(1)} \otimes r_i^{(2)} \otimes 1,$$

$$R_{13}(z) = \sum r_i^{(1)} \otimes 1 \otimes r_i^{(2)},$$

$$R_{23}(z) = \sum 1 \otimes r_i^{(1)} \otimes r_i^{(2)},$$

$$R_{21}(z) = \sum r_i^{(2)} \otimes r_i^{(1)} \otimes 1,$$

$$R_{31}(z) = \sum r_i^{(2)} \otimes 1 \otimes r_i^{(1)},$$

$$R_{32}(z) = \sum 1 \otimes r_i^{(2)} \otimes r_i^{(1)},$$

$$G_{ijk}(z) = \sum g_i^{(1)} \otimes g_j^{(2)} \otimes g_k^{(3)}.$$  

(8)

2.2 Scattering diagram; Pappus configuration

Let us describe the special three particle scattering related to the $G_2$ reflection equation. This is due to [2, 5]. Consider the three particles 1, 2, 3 coming from $A_1, A_2, A_3$ and being reflected by the boundary at $O_1, O_2, O_3$, respectively (see Fig. 1). The bottom horizontal line is the boundary which may also be viewed as the time axis. The vertical direction corresponds to the one-dimensional space. Each line carries $V$ which specifies an internal degrees of the freedom of a particle. So a three particle state at a time is described by an element in $V \otimes V \otimes V$.

One can arrange the three particle world-lines so that the two particle scattering $P_1, P_2, P_3$ happen exactly at the same instant as the boundary reflection $O_1, O_2, O_3$ of the other particle, respectively. This is non-trivial. For instance, suppose there were only particles 2 and 3. They already determine the reflecting points $O_2, O_3$ and the intersection $P_1$ (and $Q_1$) and its projection $O_1$ onto the boundary. Let $P_2, P_3$ be the points on the world-lines of particle 3 and 2 whose projection are $O_2$ and $O_3$, respectively. In order to be able to draw the world line for the last particle 1, the three points $P_2, P_3$ and $O_1$ must be collinear. This is guaranteed by a special case of the Pappus theorem from the 4th century.

One can state it more symmetrically just by starting from $P_1, P_2$ and their projection $O_1, O_2$ onto the boundary. Let $P_1', P_2'$ be the mirror image of $P_1, P_2$ with respect to the boundary. Then the three intersections $P_1' O_2 \cap O_1 P_2, P_1 P_2' \cap P_1' P_2$ and $O_1 P_2' \cap P_1' O_2$ are collinear; in fact they are $P_3, O_3$ and the mirror image of $P_3$.

Let us call the so arranged scattering diagram a Pappus configuration. The reflection at $O_i$ with the simultaneous two particle scattering at $P_i$ will be referred to as a special three particle event ($i = 1, 2, 3$). Up to a translation in the horizontal direction, a Pappus configuration is parameterized by three real numbers. For instance one can specify it by the length of the segment $O_1 O_2$ and the (dual) reflection

---

5 Although these expansions do not specify $r_i^{(a)}$, $g_i^{(a)}$ uniquely, it suffices to make (8) unambiguous.
angles $\angle P_3O_2O_3$ and $\angle P_3O_1O_3$. Set

$$u = \angle P_3O_2O_3, \quad w = \angle P_2O_3O_2, \quad v = \angle P_3O_1O_3,$$

$$\theta_1 = \angle A_2Q_3A_1, \quad \theta_2 = \angle A_3P_2A_1, \quad \theta_3 = \angle A_1Q_1O_2,$$

$$\theta_4 = \angle A_1P_3O_2, \quad \theta_5 = \angle A_3Q_2O_3, \quad \theta_6 = \angle O_2P_1O_3. \quad (9)$$

Then it is elementary to see

$$\tan w = \tan u + \tan v, \quad (10)$$

$$\theta_1 = u - v, \quad \theta_2 = w - v, \quad \theta_3 = u + w, \quad \theta_4 = u + v, \quad \theta_5 = v + w, \quad \theta_6 = w - u. \quad (11)$$

We formally consider the infinitesimal angles hence replace (10) by $w = u + v$. In such a treatment, a Pappus configuration is labelled only by the two angles $u$ and $v$. By a further substitution $u = \alpha_1 + \alpha_2$ and $v = \alpha_2$, (11) becomes

$$\theta_1 = \alpha_1, \quad \theta_2 = \alpha_1 + \alpha_2, \quad \theta_3 = 2\alpha_1 + 3\alpha_2, \quad \theta_4 = \alpha_1 + 2\alpha_2, \quad \theta_5 = \alpha_1 + 3\alpha_2, \quad \theta_6 = \alpha_2. \quad (12)$$

Regard the symbols $\alpha_1, \alpha_2$ formally as the simple roots of $G_2$. They are transformed by the simple reflections $s_1, s_2$ of the Weyl group as

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad s_2(\alpha_1) = \alpha_1 + 3\alpha_2, \quad s_2(\alpha_2) = -\alpha_2.$$

Thus we find

$$\theta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad (i_1, i_2, i_3, i_4, i_5, i_6) = (1, 2, 1, 2, 1, 2), \quad (13)$$

and $\{\theta_1, \ldots, \theta_6\}$ yields the set of the positive roots of $G_2$. 

Fig. 1. Scattering diagram for the RHS of (7).
Fig. 2. Scattering diagram for the LHS of (7).

The RHS of the $G_2$ reflection equation (7) is obtained by attaching $R(e^{θk})$ to the two particle scattering at $Q_i$ and $G(e^{θk})$ to the special three particle event at $P_iO_i$ if it is the $k$th event starting from the left in Figure 1. Setting $e^u = x$ and $e^v = y$, the assignment reads

\[
R_{21}(x) \text{ : two particle scattering at } Q_3, \\
G_{312}(xy) \text{ : special three particle event at } P_2O_2, \\
R_{32}(x^2y^3) \text{ : two particle scattering at } Q_1, \\
G_{123}(xy^2) \text{ : special three particle event at } P_3O_3, \\
R_{13}(xy^3) \text{ : two particle scattering at } Q_2, \\
G_{231}(y) \text{ : special three particle event at } P_1O_1.
\]

The indices for each operator correspond to the ordering of the relevant particles before the process. For instance just before the special three particle event at $P_2O_2$, the incoming particles are 3,1,2 from the top to the bottom, which is encoded in $G_{312}(xy)$. The LHS of the $G_2$ reflection equation (7) represents the Pappus configuration in which the time ordering of the processes are reversed (see Fig. 2).

3. Quantized $G_2$ reflection equation

3.1 $q$-bosons

Let $F_q = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ and $F_q^* = \bigoplus_{m \geq 0} \mathbb{C}\langle m|$ be the Fock space and its dual equipped with the inner product $\langle m|m'\rangle = (q^2)^m \delta_{m,m'}$. We define the $q$-boson operators $a^+, a^-, k$ on them by

\[
\begin{align*}
a^+|m\rangle &= |m+1\rangle, \quad a^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad k|m\rangle = q^{m+\frac{1}{2}}|m\rangle, \\
\langle m|a^- &= \langle m+1|, \quad \langle m|a^+ = \langle m-1|(1-q^{2m}), \quad \langle m|k = \langle m|q^{m+\frac{1}{2}}.
\end{align*}
\]  

(14)
They satisfy \((m|X|m') = (m|(X|m'))\). Let \(F_{q^3}, F^*_{q^3}\) and \(A^+, A^-, K\) denote the same objects with \(q\) replaced by \(q^3\). Namely,

\[
A^+ |m\rangle = |m + 1\rangle, \quad A^- |m\rangle = (1 - q^{6m})|m - 1\rangle, \quad K |m\rangle = q^{3m + \frac{3}{2}} |m\rangle, \quad \langle m|A^- = |m + 1\rangle, \quad \langle m|A^+ = (m - 1)(1 - q^{6m}), \quad \langle m|K = \langle m|q^{3m + \frac{3}{2}}.
\]

The inner product in \(F_{q^3}\) is given by \(\langle m|m'\rangle = (q^6)^{m,m'}\) differing from the \(F_q\) case. However, we write the base vectors as \(\langle m|, |m\rangle\) either for \(F^*_{q^3}, F_{q^3}\) or \(F_{q^3}, F_q\) since their distinction will always be evident from the context. Note the \(q\)-boson commutation relations

\[
ka^\pm = qa^\pm k, \quad a^\pm a^\mp = 1 - q^{\mp 1}k^2, \quad K A^\pm = q^{3\pm}A^\pm K, \quad A^\pm A^\mp = 1 - q^{\mp 3}K^2. \tag{16}
\]

We will also use the number operator \(h\) defined by

\[
h |m\rangle = |m\rangle, \quad \langle m|h = \langle m|m \tag{17}
\]

either for \(F_q\) or \(F_{q^3}\). One may regard \(k = q^{h + \frac{1}{2}}\) and \(K = q^{3h + \frac{3}{2}}\). The extra 1/2 in the spectrum of \(\log_q k\) is the zero point energy, which simplifies many forthcoming formulas.

### 3.2 \(q\)-boson valued \(L\) matrix

Set \(V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \simeq \mathbb{C}^2\). This should not be confused with \(V\) in (5). In fact they will be related as \(V = V^\otimes m\) later. (See around (53).) We introduce the \(q\)-boson valued \(L\) matrix by

\[
L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma, \delta \in \{0, 1\}} v_\gamma \otimes v_\delta \otimes L^{\gamma, \delta}_{\alpha, \beta} |m\rangle, \quad \tag{18}
\]

\[
L = (L^{\gamma, \delta}_{\alpha, \beta}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & K & A^- & 0 \\
0 & A^+ & -K & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{End}(V \otimes V \otimes F_{q^3}). \tag{19}
\]

We attach a diagram to each component \(L^{\gamma, \delta}_{\alpha, \beta} \in \text{End}(F_{q^3})\) as follows\(^6\):

\[
\begin{array}{cccccc}
\delta & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & K & -K & A^+ & A^- \\
\end{array}
\]

---

\(^6\) The vertices here should be distinguished from those in Figs 1 and 2 since \(V\) and \(V\) are different.
The other configurations are to be understood as zero. So \( L \) may be regarded as defining a \( q \)-boson valued six vertex model in which the latter relation of (16) plays the role of ‘free-fermion’ condition. See equation (10.16.5)\( |d=0 \) in [1]. Explicitly we have

\[
L(v_0 \otimes v_0 \otimes |m\rangle) = v_0 \otimes v_0 \otimes |m\rangle, \quad L(v_1 \otimes v_1 \otimes |m\rangle) = v_1 \otimes v_1 \otimes |m\rangle,
\]

\[
L(v_0 \otimes v_1 \otimes |m\rangle) = v_0 \otimes v_1 \otimes K|m\rangle + v_1 \otimes v_0 \otimes A^+|m\rangle = q^{3m+\frac{3}{2}}v_0 \otimes v_1 \otimes |m\rangle + v_1 \otimes v_0 \otimes |m+1\rangle,
\]

\[
L(v_1 \otimes v_0 \otimes |m\rangle) = v_0 \otimes v_1 \otimes A^-|m\rangle - v_1 \otimes v_0 \otimes K|m\rangle = (1 - q^{6m})v_0 \otimes v_1 \otimes |m-1\rangle - q^{3m+\frac{3}{2}}v_1 \otimes v_0 \otimes |m\rangle.
\]

Note the obvious properties

\[
L^{\gamma,\delta}_{\alpha,\beta} = 0 \quad \text{unless} \quad \alpha + \beta = \gamma + \delta, \quad (20)
\]

\[
hL^{\gamma,\delta}_{\alpha,\beta} = L^{\gamma,\delta}_{\alpha,\beta}(h + \beta - \delta), \quad (21)
\]

which will be referred to as weight conservation. Up to conventional difference, the \( L \) matrix (18) appeared in [7]. See also [8, 11].

3.3 \( q \)-boson valued \( J \) matrix

Besides the \( L \), we need another \( q \)-boson valued matrix \( J \) which encodes a characteristic feature of the \( G_2 \) scattering. It is defined by

\[
J(v_\alpha \otimes v_\beta \otimes v_\gamma \otimes |m\rangle) = \sum_{\lambda,\mu,v \in \{0,1\}} v_\lambda \otimes v_\mu \otimes v_\nu \otimes J^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma}|m\rangle, \quad (22)
\]

\[
J = (J^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma}) \in \text{End}(V \otimes V \otimes V \otimes F_q). \quad (23)
\]

Each component \( J^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma} \in \text{End}(F_q) \) is depicted by a 90°-degrees rotated special three particle event\(^7\)

\[
J^{\lambda,\mu,\nu}_{\alpha,\beta,\gamma} = \begin{array}{ccc}
\mu & \lambda & \nu \\
\alpha & \beta & \gamma
\end{array}
\]

\[\]

\(^7\) Note, however, again that the lines here carry \( V \) whereas those in Figs 1 and 2 do \( V \). The boundary line is omitted here.
We choose the operator $J_{\alpha,\beta,\gamma}^{\mu,v} \in \text{End}(F_q)$ concretely as follows:

\begin{align*}
\begin{array}{cccc}
\begin{array}{cc}
 a & a \\
 a & a \\
\end{array} & \begin{array}{cc}
 a & a \\
 a & a \\
\end{array} & \begin{array}{cc}
 a & a \\
 a & a \\
\end{array} & \begin{array}{cc}
 a & a \\
 a & a \\
\end{array}
\end{array}
\quad \quad (a = 0, 1)
\end{align*}

\begin{align*}
\begin{array}{cccc}
\begin{array}{cc}
 a^+ & k \\
a & a \\
\end{array} & \begin{array}{cc}
 k & -k \\
 a & a \\
\end{array} & \begin{array}{cc}
 a^- & k \\
a & a \\
\end{array}
\end{array}
\quad \quad (25)
\end{align*}

\begin{align*}
\begin{array}{cccc}
\begin{array}{cc}
 1 & 0 \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
 1 & 0 \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
 1 & 0 \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
 1 & 0 \\
 0 & 1 \\
\end{array}
\end{array}
\quad \quad (26)
\end{align*}

\begin{align*}
\begin{array}{cccc}
\begin{array}{cc}
-u_2 a^+ k & r^{-1} u_2 u_4 s \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
 k^2 & -u_4 a^- k \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
 -u_3 a^+ k & r^{-1} u_3 u_4 s \\
 1 & 0 \\
\end{array} & \begin{array}{cc}
 0 & 1 \\
 1 & 0 \\
\end{array}
\end{array}
\quad \quad (27)
\end{align*}

\begin{align*}
\begin{array}{cccc}
\begin{array}{cc}
(a^+)^2 & u_4 a^+ \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
-u_3 a^+ k & r^{-1} u_3 u_4 s \\
 0 & 1 \\
\end{array} & \begin{array}{cc}
 u_2 a^- \\
 0 & 1 \\
\end{array}
\end{array}
\quad \quad (28)
\end{align*}

\begin{align*}
\begin{array}{cccc}
\begin{array}{cc}
 r^{-1} u_1 u_2 s & u_2 a^- \\
 1 & 0 \\
\end{array} & \begin{array}{cc}
 -u_1 a^- k & (a^-)^2 \\
 1 & 0 \\
\end{array} & \begin{array}{cc}
 1 & 0 \\
 1 & 0 \\
\end{array} & \begin{array}{cc}
 1 & 0 \\
 1 & 0 \\
\end{array}
\end{array}
\quad \quad (29)
\end{align*}
Here $u_1,u_2,u_3,u_4$ are parameters satisfying
\[ u_1u_2 + u_3u_4 = r := q + q^{-1}. \]  
(30)

The operator $s \in \text{End}(F_q)$ is defined by
\[ s = a^-a^+-q^{-1}k^2 = 1 - rk^2. \]  
(31)

All the $J^{\lambda,\mu,v}_{\alpha,\beta,\gamma}$'s not contained in the above list is zero. The weight conservation properties analogous to (20) and (21) hold:
\[ J^{\lambda,\mu,v}_{\alpha,\beta,\gamma} = 0 \text{ unless } \alpha + \beta = \lambda + \mu, \]  
(32)
\[ hJ^{\lambda,\mu,v}_{\alpha,\beta,\gamma} = J^{\lambda,\mu,v}_{\alpha,\beta,\gamma}(h + 1 - \gamma - \mu - v). \]  
(33)

As an illustration we have
\[
J(v_1 \otimes v_0 \otimes v_0 \otimes |m\rangle) \\
= v_1 \otimes v_0 \otimes v_0 \otimes J^{100}_{100} |m\rangle + v_1 \otimes v_0 \otimes v_1 \otimes J^{101}_{100} |m\rangle \\
+ v_0 \otimes v_1 \otimes v_0 \otimes J^{010}_{100} |m\rangle + v_0 \otimes v_1 \otimes v_1 \otimes J^{011}_{100} |m\rangle \\
= -u_2q^{m+\frac{1}{2}}v_1 \otimes v_0 \otimes v_0 \otimes |m+1\rangle + r^{-1}u_2u_4(1 - rq^{2m+1})v_1 \otimes v_0 \otimes v_1 \otimes |m\rangle \\
+ r^{-1}u_1u_2(1 - rq^{2m+1})v_0 \otimes v_1 \otimes v_0 \otimes |m\rangle + u_2q^{m-\frac{1}{2}}(1 - q^{2m})v_0 \otimes v_1 \otimes v_1 \otimes |m-1\rangle.
\]

The three particle diagram reduces to a direct product of two particle scattering and one particle boundary reflection if the dotted line were absent. Although it is not the case, the operator $J$ almost splits into such a product as
\[
J^{\lambda,\mu,v}_{\alpha,\beta,\gamma} = d\mathcal{L}^{\lambda,\mu}_{\alpha,\beta}K^v_{\gamma} + c\theta(\alpha + \gamma = \mu + v = 1)\text{id} \]  
(34)

for some constants $c,d$. Here $\mathcal{L}$ denotes (19)$_{\lambda\rightarrow a^\pm, K\rightarrow k}$ and $K^v_{\gamma}$ are the $q$-boson valued $K$ matrix introduced in [11, eq.(9)].

### 3.4 Quantized $G_2$ reflection equation

Given $L$ and $J$ in Section 3.2 and 3.3, consider the $G_2$ reflection equation (7)$_{R \rightarrow L, G \rightarrow J}$ that holds up to conjugation by an element $\mathcal{F} \in \text{End}(F_{q_3} \otimes F_q \otimes F_{q_3} \otimes F_q \otimes F_{q_3} \otimes F_q)$:
\[
(L_{12}J_{132}L_{23}J_{213}L_{31}J_{321}) \circ \mathcal{F} = \mathcal{F} \circ (J_{231}L_{132}L_{32}J_{312}L_{21}). \]  
(35)

This is an equality of linear operators on $V \otimes V \otimes V \otimes F_{q_3} \otimes F_q \otimes F_{q_3} \otimes F_q \otimes F_{q_3} \otimes F_q$, where the superscripts are just temporal labels for explanation. If they are all exhibited (35) reads as
\[
L_{124}J_{1325}L_{236}J_{2137}L_{318}J_{3219}\mathcal{F}_{456789} = \mathcal{F}_{456789}J_{2319}L_{138}J_{1237}L_{326}J_{3123}L_{214}. \]  
(36)
We fix the normalization of $\mathcal{F}$ by
\[
\mathcal{F}(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle.
\] (37)

To explain the notation in (35) and (36), write $L$ (18) and $J$ (22) as $L = \sum L_{ij}^{(1)} \otimes L_{ij}^{(2)} \otimes L_{ij}^{(3)}$ and $J = \sum J_{ij}^{(1)} \otimes J_{ij}^{(2)} \otimes J_{ij}^{(3)} \otimes J_{ij}^{(4)}$ similarly to (8), where $\sum$ means $\sum_l$. Then
\[
\begin{align*}
L_{ij4} &= \sum L_{ij}^{(i)} \otimes L_{ij}^{(j)} \otimes 1 \otimes L_{ij}^{(3)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad ((i,j) = (1, 2), (2, 1)),
L_{ij6} &= \sum 1 \otimes L_{ij}^{(i-1)} \otimes L_{ij}^{(j-1)} \otimes 1 \otimes 1 \otimes L_{ij}^{(3)} \otimes 1 \otimes 1 \otimes 1 \quad ((i,j) = (2, 3), (3, 2)),
L_{ij8} &= \sum L_{ij}^{(i)} \otimes 1 \otimes L_{ij}^{(j)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes L_{ij}^{(3)} \otimes 1 \quad ((i,j) = (1, 3), (3, 1), 1' = 1, 3' = 2),
J_{ijk5} &= \sum J_{ij}^{(i)} \otimes J_{ij}^{(j)} \otimes J_{ij}^{(k)} \otimes 1 \otimes J_{ij}^{(4)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad (\{i,j,k\} = \{1, 2, 3\}),
J_{ijk6} &= \sum J_{ij}^{(i)} \otimes J_{ij}^{(j)} \otimes J_{ij}^{(k)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes J_{ij}^{(4)} \otimes 1 \otimes 1 \quad (\{i,j,k\} = \{1, 2, 3\}),
J_{ijk9} &= \sum J_{ij}^{(i)} \otimes J_{ij}^{(j)} \otimes J_{ij}^{(k)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes J_{ij}^{(4)} \quad (\{i,j,k\} = \{1, 2, 3\}).
\end{align*}
\]

Practically, one can realize these operators from (18) and (22) by putting $L_{\alpha,\beta}$ and $J_{\alpha,\beta}^{\mu,\nu}$ at appropriate tensor components with a suitable permutations of the indices $\alpha, \beta, \ldots$. The equation (35) or equivalently (36) is a $q$-boson valued $G_2$ reflection equation without a spectral parameter up to conjugation. We call them the quantized $G_2$ reflection equation in analogy with the quantized reflection equation proposed in [11] for $C_2$. It is depicted as follows.

Here, the indices 1, 2, 3 label the reflecting lines while 4, 5, 6, 7, 8, 9 are attached to the scattering/reflection events. The latter group of indices are associated with the Fock spaces, and the $q$-bosons are acting on them in the direction perpendicular to this planar diagram. If one introduces such $q$-boson arrows going from the back to the front of the diagram, the operator $\mathcal{F}_{456789}$ in the LHS (respectively RHS) corresponds to a vertex where the six arrows going toward (respectively coming from) 4, 5, 6, 7, 8, 9 intersect. In Section 6.1, we will take the concatenation of (35) for $n$ times. It corresponds to a 3D diagram involving the $n$ layers of the Pappus configurations depicted in the above.
The component of (36) corresponding to the transition \( v_i \otimes v_j \otimes v_k \mapsto v_a \otimes v_b \otimes v_c \) in \( V \otimes V \otimes V \) is given by

\[
\left( \sum L_{a_1,a_2}^{a,b} \otimes J_{\beta_1,\beta_2,\beta_3}^{\alpha_1,\alpha_2} \otimes L_{\gamma_1,\gamma_2}^{\beta_3,\gamma_1} \otimes J_{\lambda_1,\lambda_2,\lambda_3}^{\gamma_1,\beta_1,\gamma_2} \otimes L_{\mu_1,\mu_2}^{\lambda_2,\lambda_3,\mu_1} \otimes J_{\lambda_2,\lambda_3,\mu_1}^{\gamma_2,\beta_1,\gamma_1} \right) F
\]

\[
= F \left( \sum L_{j,k}^{a_2,a_1} \otimes J_{k,a_1,a_2}^{b_3,b_1,b_2} \otimes L_{\beta_3,\gamma_1}^{\beta_2,\beta_1} \otimes J_{\lambda_2,\lambda_3,\mu_1}^{\gamma_1,\beta_1,\gamma_2} \otimes L_{\mu_1,\mu_2}^{\lambda_2,\lambda_3,\mu_1} \otimes J_{\gamma_2,\beta_1,\gamma_1}^{b,c,a} \right),
\]

with the sums taken over \( \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2 \in \{0, 1\} \). The summands correspond to various diagrams with the external edges specified as

Let us illustrate the case \((a, b, c, i, j, k) = (1, 1, 1, 1, 0, 0)\). The relevant diagrams are given as follows:
The top left diagram yields the LHS while the other ones lead to the RHS. We have also shown the corresponding $q$-boson valued amplitude. As the result the quantized $G_2$ reflection equation (35) in this case becomes the following equation in $\text{End}(F_q^3 \otimes F_q \otimes F_q^3 \otimes F_q \otimes F_q^3 \otimes F_q)$:

$$
(1 \otimes k \otimes k^2 \otimes K \otimes a^-) \mathcal{F} = \mathcal{F}(A^+ \otimes a^- \otimes K \otimes k^2 \otimes K \otimes k + K \otimes (a^+)^2 \otimes A^- \otimes k^2 \otimes K \otimes k + rK \otimes k a^+ \otimes 1 \otimes k a^- \otimes K \otimes k + K \otimes k^2 \otimes A^+ \otimes (a^-)^2 \otimes K \otimes k + K \otimes k^2 \otimes K \otimes a^+ \otimes A^- \otimes k + K \otimes k^2 \otimes K \otimes k \otimes 1 \otimes a^-),
$$

(39)

where we have combined the terms with coefficients $u_1 u_2$ and $u_3 u_4$ together by means of (30).

The quantized $G_2$ reflection equation (35) is the set of $2^6 = 64$ equations like (39) corresponding to the choice of $a, b, c, i, j, k \in \{0, 1\}$ in (38). In the next section they will be identified with the intertwining relation of certain $A_q(G_2)$ modules.

4. $A_q(G_2)$ and its intertwiner

4.1 Intertwining relation of $A_q(G_2)$ modules

The quantized coordinate ring $A_q(G_2)$ is a Hopf algebra which can be realized by 49 generators $(t_{ij})_{1 \leq i, j \leq 7}$ obeying the so-called $RTT$ type quadratic relations and some additional ones. They are available in [18], which was adopted in [12, Section 3.3.3] in the form directly relevant to this article. Their concrete form
is not necessary here. What we need is the two fundamental representations \( \pi_i : A_q(G_2) \to \text{End}(F_{q_i}) \) associated with the simple reflections \( s_i \) \((i = 1, 2)\), where \( q_1 = q, q_2 = q^3 \).

\[
\begin{pmatrix}
a^- & k & 0 & 0 & 0 & 0 \\
-k & a^+ & 0 & 0 & 0 & 0 \\
0 & 0 & (a^-)^2 & r k & a^- & k^2 & 0 \\
0 & 0 & -a^+ k & s & k a^+ & 0 & 0 \\
0 & 0 & k^2 & -r k a^+ & (a^+)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & a^- & k & 0 \\
0 & 0 & 0 & 0 & 0 & -k & a^+
\end{pmatrix},
\]

\[
(\pi_1(t_{ij}))_{1 \leq i, j \leq 7} =
\]

\[
(\pi_2(t_{ij}))_{1 \leq i, j \leq 7} =
\]

Here \( r \) and \( s \) are defined in (30) and (31). These expressions are obtained from [12, equation (27)] by setting \( \mu_1 = \mu_2 = 1, \alpha_1 = q^{\frac{1}{2}}, \alpha_2 = q^{\frac{3}{2}} \).

The coproduct \( \Delta^{(k)} : A_q(G_2) \to A_q(G_2)^{\otimes k} \) takes the simple form \( \Delta^{(k)}(t_{ij}) = \sum_{1 \leq i_2, \ldots, i_k \leq 7} t_{i_2} \otimes t_{i_3} \otimes \cdots \otimes t_{i_k} \). For \( i_1, \ldots, i_k \in \{1, 2\} \), one can construct a tensor product representation by

\[
\pi_{i_1 \ldots i_k} := (\pi_{i_1} \otimes \cdots \otimes \pi_{i_k}) \circ \Delta^{(k)} : A_q(G_2) \to \text{End}(F_{q_{i_1}} \otimes \cdots \otimes F_{q_{i_k}}).
\]

According to the general theory [13], \( \pi_{i_1 \ldots i_k} \) is irreducible if and only if \( s_{i_1} \cdots s_{i_k} \) is a reduced expression of an element of the Weyl group of \( G_2 \). Moreover \( \pi_{i_1 \ldots i_k} \) and \( \pi_{j_1 \ldots j_k} \) are equivalent if \( s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k} \).

We are concerned with the two reduced expressions of the longest element \( s_2 s_1 s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2 s_1 s_2 \), the associated representations \( \pi_{212121} \) and \( \pi_{121212} \) and their isomorphism \( \pi_{212121} \simeq \pi_{121212} \).

Let \( \Phi^\vee \) be the intertwiner. Namely it is the map \( F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \to F_{q^3} \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \otimes F_q \) characterized by \( \pi_{212121} \Phi^\vee = \Phi^\vee \pi_{121212} \) up to normalization. Set \( \Phi = \Phi^\vee \circ P \), where \( P \) is a linear map reversing the order of the six-fold tensor product as \( P(x_1 \otimes x_2 \otimes \cdots \otimes x_6) = x_6 \otimes x_5 \otimes \cdots \otimes x_1 \). Then there exists the unique \( \Phi \) such that

\[
\Phi \in \text{End}(F_{q^3} \otimes F_q \otimes F_{q^3} \otimes F_q \otimes F_{q^3} \otimes F_q),
\]

\[
\pi_{212121}(g) \Phi = \Phi \pi'_{121212}(g) \quad \forall g \in A_q(G_2) \quad (\pi'_{121212} := P \pi_{121212} P),
\]

\[
\Phi([0] \otimes [0] \otimes [0] \otimes [0] \otimes [0] \otimes [0]) = [0] \otimes [0] \otimes [0] \otimes [0] \otimes [0] \otimes [0] \otimes [0].
\]

The condition (43) fixes the normalization. It suffices to impose the equation (42) for the 49 generators \( g = t_{ij} \). By using the explicit form of the coproduct \( \Delta^{(6)} \), they are expressed as

---

8 Note that unlike (14) in this article, the operator \( k \) in [12, eq.(17)] does not contain the zero point energy.
\[
\begin{align*}
\left( \sum \pi_2(t_{i,j}) \otimes \pi_1(t_{i,j}) \right) \otimes \pi_2(t_{i,j}) \otimes \pi_1(t_{i,j}) \otimes \pi_2(t_{i,j}) \otimes \pi_1(t_{i,j}) \otimes \pi_1(t_{i,j}) \Phi \\
= \Phi \left( \sum \pi_2(t_{i,j}) \otimes \pi_1(t_{i,j}) \otimes \pi_2(t_{i,j}) \otimes \pi_1(t_{i,j}) \otimes \pi_2(t_{i,j}) \otimes \pi_1(t_{i,j}) \right) (1 \leq i, j \leq 7), \quad (44)
\end{align*}
\]

where the sums are taken over \(1 \leq l_2, \ldots, l_6 \leq 7\). In this way, the intertwining relation (42) boils down to the 49 equations (44). Although the lists of \(\pi_1(t_{i,j})\), \(\pi_2(t_{i,j})\) in (40) are pretty sparse, some equations become lengthy including typically 16 terms on one side or both. We do not display them all here but present a few examples.

\(g = t_{1,1}:\)
\[\begin{align*}
(1 \otimes a^- \otimes 1 \otimes a^- \otimes 1 \otimes a^- - 1 \otimes a^- \otimes 1 \otimes k \otimes A^- \otimes k - 1 \otimes k \otimes A^- \otimes a^+ \otimes A^- \otimes k \\
- 1 \otimes k \otimes A^- \otimes k \otimes 1 \otimes a^- + 1 \otimes k \otimes K \otimes (a^-)^2 \otimes K \otimes k) \Phi
\end{align*}\]

\(g = t_{1,5}:\)
\[\begin{align*}
(1 \otimes a^- \otimes 1 \otimes k \otimes K \otimes k^2 + 1 \otimes k \otimes A^- \otimes a^+ \otimes K \otimes k^2 + 1 \otimes k \otimes K \otimes (a^-)^2 \otimes A^- \otimes k^2 \\
+ r1 \otimes k \otimes K \otimes k^2 \otimes k \otimes a^+ \otimes 1 \otimes k \otimes K \otimes k^2 \otimes A^- \otimes (a^-)^2) \Phi
\end{align*}\]

\(g = t_{1,6}:\)
\[\begin{align*}
(1 \otimes k \otimes K \otimes k^2 \otimes K \otimes a^-) \Phi
\end{align*}\]

\(g = t_{2,6}:\)
\[\begin{align*}
(A^- \otimes a^+ \otimes K \otimes k^2 \otimes K \otimes a^- + K \otimes (a^-)^2 \otimes A^+ \otimes k^2 \otimes K \otimes a^- \\
+ rK \otimes k^2 \otimes k \otimes a^+ \otimes K \otimes a^- + K \otimes k^2 \otimes A^- \otimes (a^+)^2 \otimes K \otimes a^- \\
+ K \otimes k^2 \otimes K \otimes a^+ \otimes A^- \otimes a^- - K \otimes k^2 \otimes K \otimes k \otimes 1 \otimes k) \Phi
\end{align*}\]

\(g = t_{1,7}, t_{7,1}:\)
\[\begin{align*}
[\Phi, 1 \otimes k \otimes K \otimes k^2 \otimes K \otimes k] = 0.
\end{align*}\]
4.2 Solution to the quantized $G_2$ reflection equation

Notice that (45) coincides with (39) under the identification $\mathcal{F} = \Phi$. In fact under this correspondence one can directly check that the 49 intertwining relations (44) and the 64 quantized $G_2$ reflection equations (38) are equivalent. We list the correspondence of the indices $(i,j)$ in (44) and $(a, b, c, i, j, k)$ in (38) in Appendix A. Since (43) and (37) impose the same normalization on $\Phi$ and $\mathcal{F}$, we conclude $\mathcal{F} = \Phi$. Let us summarize this result in

**Theorem 4.1** Under the normalization (37), the quantized $G_2$ reflection equation (35) with $L, J$ given in Sections 3.2 and 3.3 has the unique solution $\mathcal{F} = \Phi$ in terms of the intertwiner $\Phi$ of the $A_q(G_2)$ module characterized by (42) and (43).

Henceforth, we shall identify $\mathcal{F}$ and $\Phi$ and write $\mathcal{F}$ to also mean the intertwiner $\Phi$. Let us quote some basic properties of $\mathcal{F}$ from [12, Section 4.4]. Set

$$\mathcal{F} ((i) \otimes (j) \otimes (k) \otimes (l) \otimes (m) \otimes (n)) = \sum_{a, b, c, d, e, f \in \mathbb{Z} \geq 0} \mathcal{F}^{abcdef}_{ijklmn} (a) \otimes (b) \otimes (c) \otimes (d) \otimes (e) \otimes (f).$$

Then the following properties are valid:

$$\mathcal{F}^{abcdef}_{ijklmn} \in \mathbb{Z}[q], \quad \mathcal{F}^{abcdef}_{ijklmn} = 0 \text{ unless } \begin{pmatrix} a + b + 2c + d + e \\ b + 3c + 2d + 3e + f \end{pmatrix} = \begin{pmatrix} i + j + 2k + l + m \\ j + 3k + 2l + 3m + n \end{pmatrix}, \quad (46)$$

$$\mathcal{F}^{-1} = \mathcal{F}, \quad \mathcal{F}^{abcdef}_{ijklmn} = \frac{(q^6)^i(q^2)^j(q^6)^k(q^2)^l(q^6)^m(q^2)^n}{(q^6)^a(q^2)^b(q^6)^c(q^2)^d(q^2)^e(q^2)^f} \mathcal{F}^{abcdef}_{ijklmn}. \quad (47)$$

Due to the latter property of (46), $\mathcal{F}$ is an infinite direct sum of finite dimensional matrices. In terms of $h$, acting as $h$ (17) on the $i$ th component from the left, it may be rephrased as the commutativity

$$[\mathcal{F}, x^{h_1}(xy)^{h_2}(x^2y^3)^{h_3}(xy)^{h_4}(xy)^{h_5}y^{h_6}] = 0, \quad (48)$$

where $x$ and $y$ are free parameters. We let $\mathcal{F}$ also act on $\langle \omega \rangle \in F^*_{q^a} \otimes F^*_{q^b} \otimes F^*_{q^c} \otimes F^*_{q^d} \otimes F^*_{q^e} \otimes F^*_{q^f}$ by $((\langle \omega |) \mathcal{F} \langle \omega \rangle) = \langle \omega | (\mathcal{F} |\omega \rangle \rangle$ for any $|\omega \rangle \in F_{q^a} \otimes F_{q^b} \otimes F_{q^c} \otimes F_{q^d} \otimes F_{q^e} \otimes F_{q^f}$.

It is possible to make a tedious computer programme to calculate $\mathcal{F}^{abcdef}_{ijklmn}$ for any given indices by using (44). However, unlike the $A_q(A_2)$ and $A_q(C_2)$ cases, an explicit general formula is yet to be constructed. At $q = 0$ $\mathcal{F}^{abcdef}_{ijklmn}$ is known to become 0 or 1, which can be determined by the ultra-discretization (tropical form) of [24, Theorem 3.1(c)].

**Example 4.2** The following is the list of all the non-zero $\mathcal{F}^{abcdef}_{ijklmn}$.

$$\begin{align*}
\mathcal{F}^{000200}_{100102} &= -q^3(1 - q^4)(1 - q^6)(1 - q^2 - q^6), \\
\mathcal{F}^{010010}_{100102} &= (1 - q^4)(1 - q^6)(1 - q^2 + q^4), \\
\mathcal{F}^{020000}_{100102} &= q^3(1 - q^6)(1 - q^4 - q^6 - 2q^8 - q^{10} - q^{12}), \\
\mathcal{F}^{010102}_{100102} &= q^4(1 + q^2 - 2q^6 - q^8 - q^{10} + q^{14} + q^{16} + q^{18}), \\
\mathcal{F}^{001010}_{100102} &= q(1 + q^2)(1 - q^6)(1 - q^2 - q^4 - q^6 + q^{10} + q^{12} + q^{14}).
\end{align*}$$
In this article, we will only need the following properties on the Fock space 

5.2 Concatenation of the tetrahedron equation

in (19) and \( q \)

5.1 Tetrahedron equation

The \( q \)-boson valued \( L \) matrix (19) is known to satisfy a version the tetrahedron equation [7]

\[
L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124} \in \text{End}(V \otimes V \otimes V \otimes F_{q^3} \otimes F_{q^3} \otimes F_{q^3}).
\] (49)

This is a Yang–Baxter equation up to conjugation by \( R \in \text{End}(F_{q^3} \otimes F_{q^3} \otimes F_{q^3}) \). Such an \( R \) is unique up to overall normalization and is known to satisfy the tetrahedron equation \( R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \) among themselves. See [25] for the approach from the representation theory of the quantized coordinate ring \( A_q(A_2) \), [7] for a quantum geometry argument and [11, Sec.3.1] for a brief guide to the background.

We let \( R \) also act on \((F_{q^3})^\otimes 3\) by \(( (a| \otimes (b| \otimes (c|)R)(i| \otimes (j|) \otimes (k|)) = (a| \otimes (b| \otimes (c|)(R(i|) \otimes (j|) \otimes (k|))).\)

In this article, we will only need the following properties

\[
R^{-1} = R, \quad [R_{123}, x^{h_1}y^{h_2}z^{h_3}] = 0, \quad (| \otimes (| \otimes (|)R = (| \otimes (| \otimes (|, \quad R(| \otimes (|) \otimes (|) = (| \otimes (|) \otimes (|), \quad \langle | \rangle = \sum_{m \geq 0} \frac{m}{(q^3)_m} F_{q^3}^{*}, \quad \langle | \rangle = \sum_{m \geq 0} \frac{m}{(q^3)_m} F_{q^3}, \quad (50)
\]

where \( x, y \) are free parameters. The relation (51) was proved in [8, Proposition 4.1].

5.2 Concatenation of the tetrahedron equation

Consider \( n \) copies of (49) in which the spaces labelled with 1, 2, 3 are replaced by 1, 2, 3, with \( i = 1, 2, \ldots, n \):

\[
(L_{1;2;4}L_{1;3;5}L_{2;3;6})R_{456} = R_{456}(L_{2;3;6}L_{1;3;5}L_{1;2;4}).
\]
Sending $\mathcal{R}_{456}$ to the left by repeatedly applying this relation, we get

\[
(L_{12,14} L_{13,15} L_{21,31} L_{21,36}) \cdots (L_{1n,2n} L_{1n,3n} L_{2n,3n}) \mathcal{R}_{456} = \mathcal{R}_{456} (L_{21,31} L_{13,15} L_{12,14}) \cdots (L_{2n,3n} L_{1n,3n} L_{1n,2n}).
\] (53)

Set $\mathbf{V} = V^\otimes n \simeq (\mathbb{C}^2)^\otimes n$ in general and $\tilde{\mathbf{V}} = \mathbf{V} \otimes \cdots \otimes \mathbf{V}$ when the label is present. The equality (53) holds in $\text{End}(\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V} \otimes F_q \otimes F_q \otimes F_q \otimes F_q)$. It is possible to rearrange it without changing the order of any two operators sharing common labels as

\[
(L_{12,14} \cdots L_{1n,2n}) (L_{1n,3n}) (L_{21,31} L_{21,36}) \mathcal{R}_{456} = \mathcal{R}_{456} (L_{21,31} L_{21,36} \cdots L_{2n,3n}) (L_{1n,3n} \cdots L_{1n,2n}).
\] (54)

Write the right relation in (50) as $\mathcal{R}_{456}^{-1} \mathcal{R}_{456}$. Multiplying this to (54) from the left we get

\[
\mathcal{R}_{456}^{-1} (L_{12,14} \cdots L_{1n,2n}) (L_{1n,3n}) (L_{21,31} L_{21,36}) \mathcal{R}_{456} = (L_{21,31} \cdots L_{2n,3n}) (L_{1n,3n} \cdots L_{1n,2n}).
\] (55)

### 5.3 Reduction to Yang–Baxter equation

The trace of (55) over $F_q \otimes F_q \otimes F_q$ gives

\[
\text{Tr}_4 (x^{h_4} L_{12,14} \cdots L_{1n,2n}) \text{Tr}_5 ((x y)^{h_5} L_{1n,3n}) \text{Tr}_6 (y^{h_6} L_{21,31} \cdots L_{2n,3n}) = \text{Tr}_6 (y^{h_6} L_{21,31} \cdots L_{2n,3n}) \text{Tr}_5 ((x y)^{h_5} L_{1n,3n}) \text{Tr}_4 (x^{h_4} L_{12,14} \cdots L_{1n,2n}).
\] (56)

Alternatively one may sandwich (55) between the bra vector $|\chi \rangle \otimes |\hat{x} \rangle \otimes |\hat{\chi} \rangle$ and the ket vector $|\chi^4 \rangle \otimes |\hat{x}^5 \rangle \otimes |\hat{\chi}^6 \rangle$. From $\mathcal{R}^{-1} = \mathcal{R}$ (50) and (51), the result becomes

\[
|\chi^4 \rangle |x^{h_4} L_{12,14} \cdots L_{1n,2n} \rangle |\hat{\chi} \rangle |\chi \rangle |\hat{x} \rangle |\chi^5 \rangle |x^{h_5} L_{1n,3n} \rangle |\hat{\chi} \rangle |\chi \rangle |\hat{x} \rangle |\chi^6 \rangle |x^{h_6} L_{21,31} \cdots L_{2n,3n} \rangle |\hat{\chi} \rangle \rangle = (|\chi^4 \rangle |x^{h_4} L_{21,31} \cdots L_{2n,3n} \rangle |\hat{\chi} \rangle |\chi \rangle |\hat{x} \rangle |\chi^5 \rangle |x^{h_5} L_{1n,3n} \rangle |\hat{\chi} \rangle |\chi \rangle |\hat{x} \rangle |\chi^6 \rangle |x^{h_6} L_{12,14} \cdots L_{1n,2n} \rangle |\hat{\chi} \rangle \rangle.
\] (57)

Set

\[
R_{12}^{au}(z) = q^u(z) \text{Tr}_4 (z^{h_4} L_{12,14} \cdots L_{1n,2n}) \in \text{End}(\mathbf{V} \otimes \mathbf{V}),
\]

\[
R_{12}^{bv}(z) = q^b(z) \langle \chi | z^{h_4} L_{12,14} \cdots L_{1n,2n} \rangle |\chi \rangle \in \text{End}(\mathbf{V} \otimes \mathbf{V}),
\]

where $a$ is a dummy label for the auxiliary Fock space $F_q^a$. The normalization factors $q^u(z)$ and $q^b(z)$ will be specified in (68). Now (56) and (57) are both stated as the Yang–Baxter equation

\[
R_{12}(xy) R_{13}(xy) R_{23}(y) = R_{23}(y) R_{13}(xy) R_{12}(x)
\] (60)
with $R(z) = R^a(z)$ and $R^{bv}(z)$. We call the above procedure to get the solutions $R^a(z)$ and $R^{bv}(z)$ of the Yang–Baxter equation from the tetrahedron equation (49) the trace reduction and the boundary vector reduction, respectively. The vectors (52) are referred to as boundary vectors.

The trace reduction is due to [7] and the boundary vector reduction in this article is a special case of more general ones in [8]. The solutions $R^a(z)$ and $R^{bv}(z)$ have been identified with the quantum R matrices for the antisymmetric tensor representations of $U_q(A_n^{(1)})$ and the spin representation of $U_q(D_{n+1}^{(2)})$ with $p^2 = -q^{-3}$. A concise summary of these results can be found in [11, Appendix B].

5.4 Basic properties of $R^a(z)$ and $R^{bv}(z)$

We write the base vectors of $V = V^\otimes n$ as $|\alpha\rangle = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}$ in terms of an array $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$. We warn that $|\alpha\rangle \in V$ should not be confused with the base $|m\rangle$ of a Fock space containing a single non-negative integer. Set

$$R(z)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma, \delta \in \{0, 1\}^n} R(z)^{\gamma \delta}_{\alpha \beta} |\gamma\rangle \otimes |\delta\rangle \quad (R = R^a, R^{bv}).$$

Then (58) and (59) imply the matrix product formulas as

$$R^a(z)^{\gamma \delta}_{\alpha \beta} = q^{\alpha \gamma} Tr(z^b L^\gamma_{\alpha, \beta_1} \cdots L^\gamma_{\alpha, \beta_n}), \quad (61)$$

$$R^{bv}(z)^{\gamma \delta}_{\alpha \beta} = q^{\alpha \gamma} \langle \chi | z_1^b L^\gamma_{\alpha_1, \beta_1} \cdots L^\gamma_{\alpha_n, \beta_n} | \chi \rangle, \quad (62)$$

where $L^\gamma_{\alpha, \beta}$ is given by (19) and $\text{Tr}(\cdots)$ and $\langle \chi (\cdots) | \chi \rangle$ are taken over $F_q^3$. They are evaluated by using the commutation relations (16), the formula (86) $|q \rightarrow q^3$ and

$$\text{Tr}(z^b K^r(A^+)^r(A^-)^r) = \delta_{s, s'} q^{\frac{b}{3}} (q^3; q^6)^{s}_{3r+1}.$$  

(63)

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$ set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad V_k = \bigoplus_{|\alpha| = k} \mathbb{C} |\alpha\rangle. \quad (64)$$

By the definition the direct sum decomposition $V = V_0 \oplus V_1 \oplus \cdots \oplus V_n$ holds. From (20), (21) and (52) one can show

$$R(z)^{\gamma \delta}_{\alpha \beta} = 0 \text{ unless } \alpha + \beta = \gamma + \delta \in \mathbb{Z}^n \quad (R = R^a, R^{bv}), \quad (65)$$

$$R^a(z)^{\gamma \delta}_{\alpha \beta} = 0 \text{ unless } |\alpha| = |\gamma| \text{ and } |\beta| = |\delta|. \quad (66)$$

The property (66) implies the decomposition

$$R^a(z) = \bigoplus_{0 \leq l, m \leq n} R^a_{l,m}(z), \quad R^a_{l,m}(z) \in \text{End}(V_l \otimes V_m). \quad (67)$$

$R^{bv}(z)$ corresponds to $S^{1,1}(z)$ in [11]. Since $L$ in [11] is $q^2$-boson valued, the formulas in [11, Appendix B] fit this article if $q$ there is replaced by $q^{\frac{1}{2}}$. \footnote{10}
The Yang–Baxter equation (60) with $R(z) = R^l(z)$ is valid for each subspace $V_k \otimes V_l \otimes V_m$ of $V \otimes V \otimes V$. The scalar prefactor in (58) for the summand $R^l_{ij}(z)$ in (67) may be taken as $\varphi^l_{ij}(z)$ depending on $l$ and $m$. We choose it and the one in (59) as
\begin{equation}
\varphi^l_{ij}(z) = q^{-\frac{1}{2}l-m} (1 - zq^{3l-m}), \quad \varphi^{\text{bv}}(z) = \frac{(z; q^3)_\infty (-zq^{3}; q^3)_\infty}{(z^3; q^3)_\infty}. \tag{68}
\end{equation}
They make all the matrix elements of $R^l_{ij}(z)$ and $R^{\text{bv}}(z)$ rational in $z$ and $q^\frac{1}{2}$. For example, we have
\begin{equation}
R^l_{ij}(z)(|e_{[1,l]} \rangle \otimes |e_{[1,m]} \rangle) = (-1)^{\max(l-m,0)} |e_{[1,l]} \rangle \otimes |e_{[1,m]} \rangle,
\end{equation}
\begin{equation}
R^{\text{bv}}(z)(|0 \rangle \otimes |0 \rangle) = |0 \rangle \otimes |0 \rangle,
\end{equation}
where $e_{[1,k]} = e_1 + \cdots + e_k$ and $|0 \rangle = |0, 0, \ldots, 0 \rangle$.

6. Reduction of quantized $G_2$ reflection equation

Starting from the quantized $G_2$ reflection equation (36), one can perform two kinds of reductions similar to Section 5 to construct solutions to the $G_2$ reflection equation (7) in the matrix product form. This is the main result of this article which we are going to present in this section.

6.1 Concatenation of quantized $G_2$ reflection equation

Consider $n$ copies of (36) in which the spaces labelled with 1, 2, 3 are replaced by $1, i, 2, j, 3, k$ with $i = 1, 2, \ldots, n$:
\begin{equation}
(L_{1,2,4}J_{1,3,2,5}L_{2,3,6}J_{2,1,3,7}L_{3,1,8}J_{3,2,1,9}) \mathcal{F}_{456789} = \mathcal{F}_{456789}(J_{2,3,1,9}L_{1,3,8}J_{1,2,3,7}L_{3,2,6}J_{3,1,2,5}L_{2,1,4}). \tag{70}
\end{equation}
Using (70) successively, one can bring $\mathcal{F}_{456789}$ to the left to derive
\begin{equation}
(L_{1,2,4}J_{1,3,2,5}L_{2,3,6}J_{2,1,3,7}L_{3,1,8}J_{3,2,1,9}) \cdots (L_{1n,2n,4n}J_{1n,3n,2n,5n}L_{2n,3n,6n}J_{2n,1n,3n,7n}L_{3n,1n,8n}J_{3n,2n,1n,9n}) \mathcal{F}_{456789} = \mathcal{F}_{456789}(J_{2,3,1,9}L_{1,3,8}J_{1,2,3,7}L_{3,2,6}J_{3,1,2,5}L_{2,1,4}).
\end{equation}
This can be rearranged without changing the order of operators sharing common labels as
\begin{equation}
(L_{1,2,4} \cdots L_{1n,2n,4n})(J_{1,3,2,5} \cdots J_{1n,3n,2n,5})(L_{2,3,6} \cdots L_{2n,3n,6}) \times (J_{2,1,3,7} \cdots J_{2n,1n,3n,7})(L_{3,1,8} \cdots L_{3n,1n,8})(J_{3,2,1,9} \cdots J_{3n,2n,1n,9}) \mathcal{F}_{456789} = \mathcal{F}_{456789}(J_{2,3,1,9} \cdots J_{2n,3n,1n,9})(L_{1,3,8} \cdots L_{1n,3n,8})(J_{1,2,3,7} \cdots J_{1n,2n,3n,7}) \times (L_{3,2,6} \cdots L_{3n,2n,6})(J_{3,1,2,5} \cdots J_{3n,1n,2n,5})(L_{2,1,4} \cdots L_{2n,1n,4}). \tag{71}
\end{equation}
Write (48) as
\begin{equation}
\mathcal{F}_{456789}^{-1} x h_{456789}(xy)^{h_4}(x^2 y^2)^{h_5}(x y^3)^{h_6}(x y^3)^{h_7}(x^2 y)^{h_8}(x y)^{h_9}(x y)^{h_9} = x h_{0}(x y)^{h_4}(x y)^{h_5}(x^2 y^2)^{h_6}(x y)^{h_7}(x y)^{h_8}(x y)^{h_9} \mathcal{F}_{456789}^{-1}
\end{equation}
and multiply it to (71) from the left. The result reads
\[
F^{-1}_{456789}(x^{h_4}L_{11214} \cdots L_{1n2n4})(xy)^{h_5}J_{113215} \cdots J_{1n3n2n5}(x^2y)^{h_6}L_{21316} \cdots L_{2n3n6})
\times ((x^2)^{h_7}J_{211317} \cdots J_{2n3n17})(x^3)^{h_8}L_{31118} \cdots L_{3n1n8})
\times (y^{h_9}J_{312119} \cdots J_{3n2n19})(y^{h_9}J_{412119} \cdots J_{4n2n19})F_{456789}
\]
\[
= (y^{h_9}J_{213119} \cdots J_{2n3n19})(y^{h_9}J_{313119} \cdots J_{3n3n19})(x^2)^{h_7}J_{11317} \cdots J_{1n3n7} \times ((x^2)^{h_7}L_{31236} \cdots L_{3n2n6})(x^3)^{h_6}J_{31215} \cdots J_{3n1n5})(x^{h_4}L_{211314} \cdots L_{2n1n4}).
\] (72)

6.2 Trace reduction

Taking the trace of (72) over \(F_q^3 \otimes F_q \otimes F_q^3 \otimes F_q \otimes F_q^3 \otimes F_q\), we obtain
\[
\text{Tr}_4(x^{h_4}L_{11214} \cdots L_{1n2n4})\text{Tr}_5((xy)^{h_5}J_{113215} \cdots J_{1n3n2n5})\text{Tr}_6((x^2y)^{h_6}L_{21316} \cdots L_{2n3n6})
\times \text{Tr}_7((x^2)^{h_7}J_{211317} \cdots J_{2n3n17})\text{Tr}_8((x^3)^{h_8}L_{31118} \cdots L_{3n1n8})\text{Tr}_9(y^{h_9}J_{312119} \cdots J_{3n2n19})
\]
\[
= \text{Tr}_4(y^{h_9}J_{213119} \cdots J_{2n3n19})\text{Tr}_5((x^3)^{h_8}L_{11318} \cdots L_{1n3n8})\text{Tr}_6((x^2)^{h_7}J_{11317} \cdots J_{1n3n7})
\times \text{Tr}_7((x^2)^{h_7}L_{31236} \cdots L_{3n2n6})\text{Tr}_8((x^3)^{h_6}J_{31215} \cdots J_{3n1n5})\text{Tr}_9(x^{h_4}L_{211314} \cdots L_{2n1n4}).
\] (73)

Here \(\text{Tr}_4(\cdots), \text{Tr}_6(\cdots), \text{Tr}_8(\cdots)\) are identified with \(R^u(z)\) in (58). The other factors emerging from \(J\) have the form
\[
G_{123}^{tr}(z) = \kappa^u(z)\text{Tr}_a(x^{h_a}J_{11213} \cdots J_{1n3n2n3}) \in \text{End}(V \otimes V \otimes V),
\] (74)

where \(\kappa^u(z)\) will be specified in (87). Now the relation (73) is rephrased as
\[
R_{12}^{tr}(x)G_{123}^{tr}(xy)R_{23}^{tr}(x^2y^3)G_{213}^{tr}(xy^2)R_{31}^{tr}(xy^3)G_{321}^{tr}(y)
= G_{231}^{tr}(y)R_{13}^{tr}(xy^3)G_{123}^{tr}(xy^2)R_{32}^{tr}(x^2y^3)G_{312}^{tr}(xy)R_{21}^{tr}(x).
\] (75)

Thus the pair \((R^u(z), G^u(z))\) yields a solution to the \(G_2\) reflection equation (7).

6.3 Boundary vector reduction

Set
\[
\langle \xi \rangle = \sum_{m \geq 0} \frac{\langle m \rangle}{(q)_m} \in F_q^* \quad \text{and} \quad |\xi\rangle = \sum_{m \geq 0} \frac{|m\rangle}{(q)_m} \in F_q,
\] (76)

which are formally the boundary vectors (52) with \(q^3\) replaced by \(q\). Supported by computer experiments we conjecture\(^{11}\)
\[
(\langle x \rangle \otimes |\xi\rangle \otimes \langle x \rangle \otimes |\xi\rangle \otimes \langle x \rangle \otimes |\xi\rangle \otimes \langle x \rangle \otimes |\xi\rangle) = (\langle x \rangle \otimes |\xi\rangle \otimes (\langle x \rangle \otimes |\xi\rangle \otimes \langle x \rangle \otimes |\xi\rangle) \otimes |\xi\rangle \otimes |\xi\rangle \otimes |\xi\rangle \otimes |\xi\rangle,\]
\[
\mathcal{F}(|\chi\rangle \otimes (|\xi\rangle \otimes |\chi\rangle \otimes |\xi\rangle \otimes |\chi\rangle \otimes |\xi\rangle \otimes |\chi\rangle \otimes |\xi\rangle) = |\chi\rangle \otimes |\xi\rangle \otimes |\chi\rangle \otimes |\xi\rangle \otimes |\chi\rangle \otimes |\xi\rangle \otimes |\chi\rangle \otimes |\xi\rangle),
\] (77)

\(^{11}\) The two relations in (77) are actually equivalent due to the right property in (47).
where $\langle \chi |$ and $| \chi \rangle$ are defined in (52). Sandwich the relation (72) between the bra vector $\langle \chi | \otimes \xi \otimes (\xi \otimes \langle \chi | \otimes (\xi \otimes (\xi \otimes | \chi \rangle \otimes | \xi \rangle \otimes | \xi \rangle \otimes | \chi \rangle \otimes | \xi \rangle$. Thanks to (77) and the left relation in (47), the result becomes

$$
\langle \chi | x^h L_{11,21} \cdots L_{1n,2n} | \chi \rangle = \langle \xi | (x^h y^z) L_{11,12} \cdots L_{1n,2n} | \xi \rangle \langle \xi | J_{11,21} \cdots J_{1n,2n} | \xi \rangle.
$$

The factors $\langle \chi | \cdots | \chi \rangle$ involving $L$ are identified with $R^{bv}(z)$ in (59). The other factors emerging from $J$ have the form

$$
G^{bv}_{123}(z) = \kappa^{bv}(z) \langle \xi | z^a h J_{11,21} \cdots J_{1n,2n} a | \xi \rangle \in \text{End}(V \otimes \overline{V} \otimes \overline{V}),
$$

where the scalar $\kappa^{bv}(z)$ will be specified in (87). In terms of (79) and (62), the relation (78) is stated as

$$
R^{bv}_{12}(x) G^{bv}_{132}(xy) R^{bv}_{23}(x^2 y^3) G^{bv}_{231}(xy^2 y^3) R^{bv}_{13}(xy) G^{bv}_{123}(xy) = G^{bv}_{231}(y) R^{bv}_{13}(xy^3) G^{bv}_{123}(x^2 y^3) R^{bv}_{32}(x^2 y^3) G^{bv}_{321}(xy) R^{bv}_{21}(x).
$$

Thus the pair $(R^{bv}(z), G^{bv}(z))$ provides another solution to the $G_2$ reflection equation (7).}

6.4 Basic properties of $G^{ur}(z)$ and $G^{bv}(z)

The construction (74) and (79) imply the matrix product formula for each element as

$$
G(z)(| \alpha \rangle \otimes | \beta \rangle \otimes | \gamma \rangle) = \sum_{\lambda, \mu, \nu \in \{0,1\}^n} G(z)^{\lambda, \mu, \nu}_{a, b, \gamma} | \lambda \rangle \otimes | \mu \rangle \otimes | \nu \rangle
$$

where $G^{ur}(z)$ is specified in (24)–(31). From (32) and (33), one can show

$$
G^{ur}(z)^{\lambda, \mu, \nu}_{a, b, \gamma} = 0 \quad \text{unless} \quad \alpha + \beta = \lambda + \mu \in \mathbb{Z}^n \quad \text{and} \quad n + | \beta | - | \gamma | = | \mu | + | \nu |
$$

or equivalently the direct sum decomposition:

$$
G^{ur}(z) = \bigoplus_{l,m,k} G^{ur}(z)_{l,m,k}, \quad G^{ur}(z)_{l,m,k} : V_l \otimes \overline{V}_m \otimes \overline{V}_k \to \bigoplus_{k'} V_{l+k+k'-n} \otimes \overline{V}_{m-k'-n} \otimes \overline{V}_{k'},
$$

where the sums extend over $l, m, k, k' \in [0, n]$ such that the indices $l + k + k' - n$ and $m - k - k' + n$ also belong to $[0, n]$. 

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The trace (82) is evaluated by means of (63) with $q^3$ replaced by $q$. The quantity $\langle \xi | (\cdots ) | \xi \rangle$ in (83) is calculated from (15) and

$$
\langle \xi | z^h (a^+)^k m | \xi \rangle = q^m z (-q ; q)_\infty \frac{(-q^{j+m+1} z ; q)_\infty}{(q^m z ; q)_\infty} \sum_{j=0}^{\infty} \frac{(w z q)_j}{(z q)_j} \left[ \prod_{r=0}^{j} z^r \right],
$$

which is easily derived by only using the elementary identity $\sum_{j=0}^{\infty} \frac{(w z q)_j}{(z q)_j} e^{j z} = \frac{1}{(z q)_\infty}$. We choose the normalization factors in (82) and (83) as

$$
\kappa^\text{tr} (z) = 1, \quad \kappa^\text{bv} (z) = \frac{(z ; q)_\infty}{(-q z ; q)_\infty}.
$$

Then all the matrix elements (82) and (83) become rational in $z$ and $q^{1/2}$. For instance we have

$$
G^e (z) (| e_{[1,l]} \rangle \otimes | e_{[1,m]} \rangle \otimes | 0 \rangle) = \frac{1}{(-q^{l+m+1} z ; q)_\infty} \frac{(z q)^{m-l+n} (q^m z)^{l+m-n} e_{[1,l]} \otimes e_{[1,m]} \otimes | 1 \rangle + \cdots (l \leq m),}
$$

$$
G^e (z) (| e_{[1,l]} \rangle \otimes | e_{[1,m]} \rangle \otimes | 1 \rangle) = \frac{(-q^m z)^{l+m-n} (z q)^{m-l+n}}{(-q^{l+m+1} z q^m z)^{l+m-n}} | e_{[1,l]} \rangle \otimes e_{[1,m]} \otimes | 0 \rangle + \cdots (l \leq m),
$$

$$
G^b (z) (| e_{[1,l]} \rangle \otimes | e_{[1,m]} \rangle \otimes | 0 \rangle) = q^{m-l+n} (z q)^{m-l+n} | e_{[1,l]} \rangle \otimes e_{[1,m]} \otimes | 1 \rangle + \cdots (l \leq m),
$$

where the symbol $e_{[1,l]}$ was defined after (69) and $| 1 \rangle = | e_{[1,l]} \rangle = | 1, 1, \ldots, 1 \rangle$.

**Remark 6.1** Using (67) and (85) it is easy to see that the both sides of (75) applied to $V_{s} \otimes V_{l} \otimes V_{u}$ generate the space $\bigoplus_{r} V_{l-s+r} \otimes V_{l-r} \otimes V_{m-u-r+l}$. There are three $R^e (z)$’s on each side of (75). One can check that changing their normalization as $R^e_{l,m} (z) \rightarrow \phi_{l,m} (z) R^e_{l,m} (z)$ depending on $l, m$ in (67) keeps (75) valid for any function $\phi_{l,m} (z)$ of the form $\phi_{l,m} (z)$. In fact the both sides acquire the common overall factor $\phi_{s+t-n} (x y^3) \phi_{t+s-n} (x^2 y^3) \phi_{s-t} (x)$ under the change. Our $\phi^e_{l,m} (z)$ (68) is of this form, hence (75) remains valid despite the “mixing” of weights (or indices) in (85).

**7. Summary**

We have studied the $G_2$ reflection equation (7) which is a natural $G_2$ analogue of the Yang–Baxter and the reflection equations corresponding to the $A_2$ and $B_2/C_2$ Coxeter relations, respectively. It describes the three particle scattering/reflections whose world-lines form a Pappus configuration. We introduced the quantized $G_2$ reflection equation (35). It is a $q$-boson valued $G_2$ reflection equation that holds up to conjugation. We gave a solution to it in Theorem 4.1 by exploiting a connection to the quantized coordinate ring $A_q (G_2)$ [12]. From the concatenation of the solution we have constructed matrix product solutions to the original $G_2$ reflection equation $(R^e (z), G^e (z))$ in (75) and $(R^b (z), G^b (z))$ in (80), where the latter assumes the conjecture (77). The $R$ and $G$ matrices are linear operators on $V \otimes V$ and $V \otimes V \otimes V$ with
\( V \simeq (C^2)^{\otimes n} \) and trigonometric in the spectral parameter. The special three particle event characteristic to the \( G_2 \) theory is encoded in \( G^\alpha(z), G^\beta(z) \) whereas the companion \( R \) matrices \( R^\alpha(z), R^\beta(z) \) for the two particle scattering are the known ones for the antisymmetric tensor representations of \( U_q(A^{(1)}_{n-1}) \) and the spin representations of \( U_q(D^{(2)}_{n+1}) \) with \( p^2 = -q^3 \).

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Appendix

A. Correspondence between (38) and (44)

We indicate the equivalence of (38) with \((a, b, c, i, j, k) \in \{0, 1\}^6 \) and (44) with \((i, j) \to (i', j') \in \{1, \ldots, 7\}^2 \) as \((abcijk; ij')\). In this notation the example treated in (39) and (45) is \((111100; 16)\). There is no need to take linear combinations etc of the equations and the equivalence literally means the same equation up to an overall constant.

\[
\begin{align*}
(000000; 77), & \quad (000001; 74), \quad (000010; 75), \quad (000011; 72), \quad (000100; 76), \quad (000101; 73), \\
(000110; 74), & \quad (000111; 71), \quad (001000; 47), \quad (001001; 44), \quad (001010; 45), \quad (001011; 42), \\
(001100; 46), & \quad (001101; 43), \quad (001110; 44), \quad (001111; 41), \quad (010000; 57), \quad (010001; 54), \\
(010010; 55), & \quad (010011; 52), \quad (010100; 56), \quad (010101; 53), \quad (010110; 54), \quad (010111; 51), \\
(011000; 27), & \quad (011001; 24), \quad (011010; 25), \quad (011011; 22), \quad (011100; 26), \quad (011101; 23), \\
(011110; 24), & \quad (011111; 21), \quad (100000; 67), \quad (100001; 64), \quad (100010; 65), \quad (100011; 62), \\
(100100; 66), & \quad (100101; 63), \quad (100110; 64), \quad (100111; 61), \quad (101000; 37), \quad (101001; 34), \\
(101010; 35), & \quad (101011; 32), \quad (101100; 36), \quad (101101; 33), \quad (101110; 34), \quad (101111; 31), \\
(110000; 47), & \quad (110001; 44), \quad (110010; 45), \quad (110011; 42), \quad (110100; 46), \quad (110101; 43), \\
(110110; 44), & \quad (110111; 41), \quad (111000; 17), \quad (111001; 14), \quad (111010; 15), \quad (111011; 12), \\
(111100; 16), & \quad (111101; 13), \quad (111110; 14), \quad (111111; 11).
\end{align*}
\]

Note for instance that \((000001; 74)\) and \((000110; 74)\) imply that not all of the quantized reflection equations (38) are independent.

B. Examples

Recall the notation \(|\alpha\rangle \in V = V^{\otimes n} \simeq (C^2)^{\otimes n}\) declared in the beginning of Section 5.4. We write \(|\alpha\rangle \otimes |\beta\rangle \in V \otimes V \) and \(|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \in V \otimes V \otimes V \) with \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n\) etc as \(|\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n\rangle\) and \(|\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_n, \gamma_1 \ldots \gamma_n\rangle\), respectively.
B.1 $R^u(z)$ and $G^u(z)$ with $n = 1$

$R^u(z)$ is given by $[i,j] \mapsto (-1)^{i-j}[i,j]$ $(i,j = 0, 1)$, which is almost the identity. On the other hand $G^u(z)$ is non-trivial even for $n = 1$. Its action on $V^\otimes 3 = V^\otimes 3$ is given by

$$
|0, 0, 0\rangle \mapsto \frac{q^\frac{1}{2}|0, 0, 1\rangle}{1 - qz}, \quad |0, 0, 1\rangle \mapsto -\frac{q^\frac{1}{2}|0, 0, 0\rangle}{1 - qz}, \quad |0, 1, 0\rangle \mapsto \frac{q|0, 1, 1\rangle}{1 - q^2z},
$$

$$
|0, 1, 1\rangle \mapsto -\frac{u_1u_3(q^2 - z)|0, 1, 0\rangle}{r(1 - z)(1 - q^2z)} - \frac{u_3u_4(q^2 - z)|1, 0, 1\rangle}{r(1 - z)(1 - q^2z)},
$$

$$
|1, 0, 0\rangle \mapsto -\frac{u_1u_2(q^2 - z)|0, 1, 0\rangle}{r(1 - z)(1 - q^2z)} - \frac{u_2u_4(q^2 - z)|1, 0, 1\rangle}{r(1 - z)(1 - q^2z)},
$$

$$
|1, 0, 1\rangle \mapsto q|1, 0, 0\rangle, \quad |1, 0, 0\rangle \mapsto q^\frac{1}{2}|1, 1, 1\rangle, \quad |1, 1, 1\rangle \mapsto -\frac{q^\frac{1}{2}|1, 1, 0\rangle}{1 - qz},
$$

where $r, u_1, u_2, u_3, u_4$ are to obey (30). The two kinds of the denominators $1 - qz$ and $1 - q^2z$ originate in $j_{0,0,0}^1 = k$ and $j_{0,1,0}^1 = k^2$.

B.2 $R^u(z)$ and $G^u(z)$ with $n = 2$

$R^u_{i,m}(z)$ $(0 \leq l, m \leq 2)$ is the identity except $R^u_{1,0}(z) = -\text{id}$, $R^u_{2,1}(z) = -\text{id}$ and $R^u_{1,1}(z)$. The last one $R^u_{1,1}(z)$ is given by

$$
|ij, ij\rangle \mapsto |ij, ij\rangle \quad (i = 1 - j = 0, 1),
$$

$$
|01, 10\rangle \mapsto -\frac{q^3(1 - z)|01, 10\rangle}{1 - q^6z} + \frac{(1 - q^6)|01, 01\rangle}{1 - q^6z},
$$

$$
|10, 01\rangle \mapsto \frac{(1 - q^6)|01, 10\rangle}{1 - q^6z} - \frac{q^3(1 - z)|01, 01\rangle}{1 - q^6z},
$$

which is a six-vertex. As for $G^u(z)$, it is too lengthy to present all the data. So we give just a few examples.

$$
|00, 00, 00\rangle \mapsto \frac{q|00, 00, 11\rangle}{1 - q^2z}, \quad |00, 00, 01\rangle \mapsto \frac{(1 - q^2z)|00, 00, 01\rangle}{(1 - z)(1 - q^2z)} - \frac{q|00, 00, 10\rangle}{1 - q^2z},
$$

$$
|00, 10, 11\rangle \mapsto \frac{q^2u_1u_3(q - z)|00, 10, 00\rangle}{r(1 - qz)(1 - q^2z)} - \frac{q^2(1 - q^2)u_3z|00, 00, 01\rangle}{(1 - qz)(1 - q^2z)} + \frac{q^2u_4u_3(q - z)|10, 00, 10\rangle}{r(1 - qz)(1 - q^2z)},
$$
\[ \begin{align*}
|10, 01, 01\rangle & \mapsto \frac{u_2^2u_2u_3(q^4 + z - 2q^2z - 2q^4z + q^6z + q^2z^2)\langle 00, 11, 00\rangle}{r^2(1 - z)(1 - q^2z)(1 - q^4z)} \\
& + \frac{u_1u_2u_3u_4(q^4 + z - 2q^2z - 2q^4z + q^6z + q^2z^2)\langle 01, 10, 01\rangle}{r^2(1 - z)(1 - q^2z)(1 - q^4z)} \\
& - \frac{q(1 - q^2)u_2u_3|01, 10, 01\rangle}{(1 - q^2z)(1 - q^4z)} \\
& - \frac{q(1 - q^2)u_2u_3z|10, 01, 01\rangle}{(1 - q^2z)(1 - q^4z)} \\
& + \frac{u_1u_2u_3u_4(q^4 + z - 2q^2z - 2q^4z + q^6z + q^2z^2)\langle 10, 01, 10\rangle}{r^2(1 - z)(1 - q^2z)(1 - q^4z)} \\
& + \frac{u_2u_3u_4^2(q^4 + z - 2q^2z - 2q^4z + q^6z + q^2z^2)\langle 11, 00, 11\rangle}{r^2(1 - z)(1 - q^2z)(1 - q^4z)}.
\end{align*} \]

B.3 \( R^{bn}(z) \) and \( G^{bn}(z) \) with \( n = 1 \)

\( R^{bn}(z) \) reduces to another six-vertex model:

\[ \begin{align*}
|i, i\rangle & \mapsto |i, i\rangle \quad (i = 0, 1), \\
|0, 1\rangle & \mapsto \frac{q^\frac{3}{2}(1 - z)|0, 0\rangle}{1 + q^2z} + \frac{(1 + q^3)z|1, 0\rangle}{1 + q^3z}, \\
|1, 0\rangle & \mapsto \frac{(1 + q^3)|0, 1\rangle}{1 + q^3z} - \frac{q^\frac{3}{2}(1 - z)|1, 0\rangle}{1 + q^3z}.
\end{align*} \]

\( G^{bn}(z) \) is given by

\[ \begin{align*}
|0, 0, 0\rangle & \mapsto \frac{(1 + q)z|0, 0, 0\rangle}{1 + qz} + \frac{q^\frac{1}{2}(1 - z)|0, 0, 1\rangle}{1 + qz}, \\
|0, 0, 1\rangle & \mapsto \frac{-q^\frac{1}{2}(1 - z)|0, 0, 0\rangle}{1 + qz} + \frac{(1 + q)|0, 0, 1\rangle}{1 + qz}, \\
|0, 1, 1\rangle & \mapsto \frac{q^\frac{3}{2}(1 + q)u_1(1 - z)z|0, 1, 0\rangle}{(1 + qz)(1 + q^2z)} + \frac{q(1 - z)(1 - qz)|0, 1, 1\rangle}{(1 + qz)(1 + q^2z)} \\
& + \frac{(1 + q)(1 + q^2z^2)|1, 0, 0\rangle}{(1 + qz)(1 + q^2z)} + \frac{q^2(1 + q)u_4(1 - z)z|1, 0, 1\rangle}{(1 + qz)(1 + q^2z)}, \\
|0, 1, 1\rangle & \mapsto \frac{u_3(-q^2 + z + 2qz + 2q^2z + q^2z^2)|0, 1, 0\rangle + u_4|1, 0, 1\rangle}{r(1 + qz)(1 + q^2z)} \\
& + \frac{q^\frac{3}{2}(1 + q)u_3(1 - z)(|0, 1, 1\rangle - z|1, 0, 0\rangle)}{(1 + qz)(1 + q^2z)}, \\
|1, 0, 0\rangle & \mapsto \frac{u_2(-q^2 + z + 2qz + 2q^2z + q^2z - q^2z^2)|0, 1, 0\rangle + u_4|1, 0, 1\rangle}{r(1 + qz)(1 + q^2z)} \\
& + \frac{q^\frac{3}{2}(1 + q)u_2(1 - z)(|0, 1, 1\rangle - z|1, 0, 0\rangle)}{(1 + qz)(1 + q^2z)},
\end{align*} \]
\[ |1, 0, 1\rangle \mapsto -\frac{q^3(1 + q)u_1(1 - z)|0, 1, 0\rangle}{(1 + qz)(1 + q^2z)} + \frac{(1 + q)(1 + q^2)|0, 1, 1\rangle}{(1 + qz)(1 + q^2z)}, \]

\[ |1, 1, 0\rangle \mapsto \frac{(1 + q)z|1, 1, 0\rangle}{1 + qz} + \frac{q^2(1 - z)|1, 1, 1\rangle}{1 + qz}, \quad |1, 1, 1\rangle \mapsto -\frac{q^2(1 - z)|1, 1, 0\rangle}{1 + qz} + \frac{(1 + q)|1, 1, 1\rangle}{1 + qz}. \]

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