A Harmonic Space Approach to Spherically Symmetric Quantum Gravity

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Abstract

After dimensional reduction the stationary spherically symmetric sector of Einstein’s gravity is identified with an SL(2,\mathbb{R})/SO(2) Sigma model coupled to a one dimensional gravitational remnant. The space of classical solutions consists of a one parameter family interpolating between the Schwarzschild and the Taub-NUT solution. A Dirac Quantization of this system is performed and the observables – the Schwarzschild mass and the Taub-NUT charge operator – are shown to be self-adjoint operators with a continuous spectrum ranging from $-\infty$ to $\infty$. The Hilbert space is constructed explicitly using a harmonic space approach.

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1 Introduction

The quantization of Einstein’s gravity is one of the outstanding and most challenging areas in theoretical physics. There have been many attempts at a solution of this problem classified by Isham in his articles about quantum gravity [17], [18]. One can start with the classical theory of general relativity and apply some kind of quantization algorithm. General relativity can be considered as a low-energy limit of another theory like superstring theory for example, one can try to force some principal ideas of a quantum field theory to be compatible with general relativity or the fundament of quantum gravity is seen to be some radically new perspective which leads in certain limiting situations to the classical theory of general relativity. Attempts in this direction are quantum groups or the noncommutative geometry invented by Connes. Nonperturbative approaches to the problem became of great importance since it is known that quantum gravity is non-renormalizable.

This paper originates from a conservative point of view. Starting from the classical theory of general relativity one achieves a quantization via a canonical scheme, the ADM quantization. For full general relativity this is still an unsolved problem, but freezing all but a few of the infinitely many degrees of freedom of the gravitational field solutions for special sectors of Einstein’s vacuum theory are within reach. It is hoped, that at least some relevant features of the quantum theory are preserved. Even restricting oneself to a particularly symmetric case, that is spherical symmetry – as it is done here – the space of classical solutions contains a black hole, the Schwarzschild solution, whose special importance is explained now. There are no particles in the sense of ordinary field theory. Even the concept of solitons has to be modified as shown in [3], [6]. One has to allow for singularities in the solutions leading to the suggestion that black holes are the proper particles of general relativity: coupling a point particle to its own gravitational field turns it into a black hole [11]. Furthermore classical black holes are stable under local perturbations [8], and they have remarkable uniqueness properties. This is why they are considered to be the fundamental pieces of a quantum theory of gravitation and the investigation of the spherically symmetric sector of Einstein’s theory of gravitation is seen in different a light [15], [7], [6]. As in conventional gauge theory, where one tries to keep manifest Poincaré invariance even in the process of gauge fixing, this sector of Einstein’s gravity has a global symmetry group, the $SL(2,\mathbb{R})$, which will be preserved even at a quantum level. The unitary irreducible representations play a key role in the process of quantization. The major advantage of this model is that the quantum theory is explicitely known.
One of the most important developments in field theory during the last two decades was the discovery of the quantum mechanical instability of black holes due to Hawking radiation. Most of the calculations of black hole radiation involve matter fields quantized in front of a classical background. After the coupling of matter fields this model opens the possibility to take a closer look at Hawking radiation quantum-mechanically. But this is considered to be a project of future research.

This paper mainly splits into a classical and a quantum part. In the beginning the spherically symmetric sector of Einstein’s gravity is introduced and a Kaluza-Klein like reduction is performed [5]. This treatment is rather concise, but a more comprehensive and more general exposition can be found in [5]. It turns out, that gravity reduced from four to three dimensions in its dual representation is equivalent to a SL(2,IR)/SO(2) nonlinear coset sigma model coupled to three dimensional gravity. Using this coset space structure all stationary spherically symmetric solutions of Einstein’s equations with spherical symmetry are constructed in [12]: there are the Schwarzschild and the Taub-NUT solution and a one-parameter family connecting these two [31]. The parameters of the theory are hidden in the sigma model currents: the Schwarzschild mass $m$ and the Taub-NUT charge $l$. In the following the problem is treated as a constraint system. The Hamiltonian is the only constraint which survives the process of dimensional reduction. The components of the sigma model currents consequently turn out to be observables in the sense of Dirac and form an sl(2,IR) algebra.

During quantization the classical phase space is turned into a Hilbert space on which the operators - the former functions on phase space - act. In general it is a difficult problem to identify the Hilbert space. On the other hand it is the Hilbert space, which is of fundamental importance to make assertions about the operators to be considered physical. The implementation of the classical SL(2,IR) symmetry at a quantum level leads to a solution of this problem. The eigenfunctions of the invariant differential operator on the group span the Hilbert space as is shown by the application of a generalized Plancherel theorem. The measure on the Hilbert space is derived quite naturally from the Haar measure on the group. Once knowing the Hilbert space explicitly one can investigate the self-adjointness and the spectrum of certain differential operators. Both the mass operator and the charge operator are self-adjoint and their spectra are purely continuous. Neither the spectrum of the mass operator nor the spectrum of the charge operator are bounded. That is: negative Taub-NUT charges as well as negative masses belong to the spectrum, too.

The most challenging feature of this model is that bases of the Hilbert space are at
and. Therefore the model serves as a kind of test laboratory for several aspects of quantum gravity. But these topics will be reserved for a forthcoming publication.

This method is not limited to the situation here. It is applicable to various other models occurring in conformal field theory, string theory, quantum cosmology or supergravity.

2 The Classical Theory

This classical part starts with the application of a Kaluza-Klein reduction \[1\] to a four dimensional Einstein space \(M_4\) with signature \((+,-,+,+)\) having a one parameter Abelian isometry group \[12\] which acts freely and corresponds to time translations. That is there exists a timelike Killing vector field \(K^M\) describing the action of the Lie algebra of the isometry group. In a coordinate basis the metric \(g_{MN}\) of \(M_4\) decomposes into a metric \(h_{mn}\) on the remaining three spatial dimensions, a scalar field \(\tau\) and the vector field \(B_m\):

\[
g_{MN} = \begin{pmatrix} -\frac{1}{\tau}h_{mn} + \tau B_mB_n & -\tau B_n \\ -\tau B_m & \tau \end{pmatrix}.
\]

The extra factor \(\tau\) assures that diffeomorphisms of the special form \(t \mapsto t + \Lambda(x^m)\) act as gauge transformations \(B_m \mapsto B_m + \partial_m \Lambda\) on the vector field \(B_m\) \[6\]. Capital letters \(M,N\) vary from one to four and small ones from one to three. Plugging this Ansatz for the four dimensional metric into the Einstein Hilbert action

\[
S = -\frac{1}{2} \int d^4x \sqrt{-g} R
\]

leads to the Lagrangian

\[
\mathcal{L}^{(4,3)} = -\frac{1}{2} \sqrt{-g} R = \sqrt{h} \left( -\frac{1}{2} R + \frac{\tau^2}{8} F_{mn}F^{mn} - \frac{1}{4\tau^2} \partial^m \tau \partial_m \tau \right).
\]

\(F_{mn}\) denotes the field strenght of \(B_m\), defined by \(F_{mn} = \partial_mB_n - \partial_n B_m\). \(R\), \(R^{(3)}\) are the scalar curvatures corresponding to the four dimensional metric \(g_{MN}\) and the three dimensional metric \(h_{mn}\), respectively. \(\sqrt{-g}\) and \(\sqrt{h}\) are the square roots of the determinants of the metrics as usual.

Under the specified conditions dimensional reduction of four dimensional pure gravity to three dimensions leads to gravity coupled to an abelian vector field and a scalar.
The equations of motion derived from (2) can be interpreted as integrability condition for the vector field $B_m$:

$$\tau^2 F_{mn} = \epsilon_{mnp} \partial^p \omega.$$ 

Elimination of the field strength in terms of the gravito-magnetic potential $\omega$ leads to a dual Lagrangian density

$$L^D = \sqrt{h} \left( -\frac{1}{2} (3) R + \frac{h_{mn}}{4 \tau^2} (\partial_m \tau \partial_n \tau + \partial_m \omega \partial_n \omega) \right),$$

which can be rewritten as

$$L^D = \sqrt{h} \left( -\frac{1}{2} (3) R + \frac{h_{mn}}{8} \text{Tr} \left( \chi^{-1} \partial_m \chi \chi^{-1} \partial_n \chi \right) \right),$$

where

$$\chi = \begin{pmatrix} \tau + \frac{\omega^2}{\tau} & \frac{\omega}{\tau} \\ \frac{\omega}{\tau} & \frac{1}{\tau} \end{pmatrix}.$$ 

That is four dimensional reduced gravity in its dual representation is equivalent to an SL(2,$\mathbb{R}$)/SO(2) sigma model coupled to three dimensional gravity. The matrix $\chi$ is an element of the Riemannian symmetric space SL(2,$\mathbb{R}$)/SO(2). The fields contained in $\chi$ have a physical interpretation: The norm $\tau$ of the Killing vector plays the role of a gravitational potential, and $\omega$ is the so called gravito-magnetic or NUT potential.

For the derivation of the equations of motion one starts from (3). $R_{MN} = 0$ is equivalent to the set of equations

$$(3) R_{mn} = \frac{1}{2} \text{Tr} \left( \chi^{-1} \partial_m \chi \chi^{-1} \partial_n \chi \right) \quad (4a)$$

$$D^m \left( \chi^{-1} \partial_m \chi \right) = 0 \quad (4b)$$

Imposing an additional SO(3) symmetry on the remaining three spatial dimensions means to constrain the involved fields in such a way that they depend on one spatial dimension only: $f, \tau$ and $\omega$ are functions of $\rho$. In this stationary case the NUT potential can be considered as a kind of “magnetic potential” in analogy to Maxwell’s theory.

It is convenient to parametrize the metric $h_{mn}$ by polar coordinates

$$h_{mn} = \begin{pmatrix} N^2(\rho) & 0 & f^2(\rho) \\ 0 & f^2(\rho) \sin^2 \theta \end{pmatrix}, \quad m, n = \rho, \theta, \phi$$

$$m, n = \rho, \theta, \phi$$
After substitution of this metric into the Lagrangian one obtains
\[ \mathcal{L}^D = N \left[ \frac{f'^2}{N^2} + 1 - \frac{f^2}{4N^2\tau^2} \left( \tau'^2 + \omega'^2 \right) \right]. \tag{6} \]

The prime \( ' \) denotes the derivative with respect to \( \rho \). \( N \) is the “lapse” function. “lapse” is set in quotation marks, because it usually refers to a timelike direction whereas the lapse indicates spacelike propagation.

For stationary spherically symmetric gravity \( \chi^{-1} \chi' \) is unequal to zero and simplifies to
\[
(3) R_{22} = (3) R_{33} = \left[ \frac{f''}{f} + \left( \frac{f'}{f} \right)^2 - \frac{1}{f^2} \right] = 0 \tag{7}
\]
\[
(3) R_{11} = -\frac{2f''}{f} = \frac{1}{4} \text{Tr}(\chi^{-1} \chi')^2
\]
\[
(f^2 \chi^{-1} \chi')' = 0.
\]
The lapse function refers to a gauge degree of freedom and was set equal to one meanwhile. Dobiasch and Maison \cite{12} calculated the solution of these equations of motion. They found
\[
f^2(\rho) = R^2 - a^2, \quad R = \rho - b. \tag{8}
\]
\( a, b \) are constants of integration and will be interpreted later. It turns out that \( \chi \) is of the form
\[
\chi = \chi_0 \ e^{t(\rho) \mu}, \quad \text{with} \quad \chi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t(\rho) = -\int_{\rho}^{\infty} f^{-2}(s) ds.
\]
The most general matrix \( \mu \) can be written as
\[
\mu = \sin \psi \ \mu_1 + \cos \psi \ \mu_2, \quad \text{with} \quad \mu_1 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \quad \text{and} \quad \mu_2 = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}.
\]
Using this information the classical fields \( \omega \) and \( \tau \) are derived:
\[
\tau = \frac{R^2 - a^2}{R^2 + a^2 + 2 \cos \psi R a} \tag{9a}
\]
\[
\omega = \frac{2 \sin \psi Ra}{R^2 + a^2 + 2 \cos \psi R a} \tag{9b}
\]
For the sake of completeness we also give the four dimensional metric in terms of the one dimensional data:
\[
g_{MN} = \begin{pmatrix}
\frac{-1}{\tau} & \frac{f^2}{\tau} & 0 \\
\frac{f^2}{\tau} & \frac{-f^2}{\tau} \sin^2 \theta + \frac{f^4}{\tau^3} \cos^2 \theta \omega'^2 & \frac{f^2}{\tau} \cos \theta \omega' \\
0 & \frac{f^2}{\tau} \cos \theta \omega' & \tau
\end{pmatrix},
\]
$M, N = \rho, \theta, \phi, t$. The solution corresponding to $\sin \psi = 0$ turns out to be the Schwarzschild solution, which is the gravitational field outside a spherically symmetric mass distribution. It is a static solution of Einstein’s equations. For $\cos \psi = 0$ only the $\mu_1$ part remains and it is a Taub-NUT solution. The geometrical difference between the Schwarzschild and the Taub-NUT solution is, that the latter admits an isometry group whose orbits are on 3-spheres, whereas in the case of the Schwarzschild solution they are 2-spheres. The “result” is the Taub-NUT time being a circle. Another interpretation is given in terms of the difference between staticity and stationarity: In the static case there are two symmetries: a time translation symmetry and a time reflection symmetry. For Taub-NUT the fields being time translation invariant fail to be time reflection invariant: the neighbouring orbits of the Killing vector fields twist around each other! Both solutions are asymptotically flat, but only the Schwarzschild solution is also asymptotically Minkowski.

Although it should be clear from the structure of the Lagrangian (3) it is stressed at this point once more, that the space of solutions contain more than the Schwarzschild solution. The attempts to quantize spherically symmetric gravity [9], [10], [21], [23], [27] essentially refer to the static truncation $GL(1)$ of this $SL(2, \mathbb{R})/SO(2)$ sigma model.

In order to stress the one to one correspondence between the observables and the initial data it is shown (see [12]) how to use the coset space structure to obtain (9)

The invariant line element of the coset is

$$ds^2 = \frac{1}{4} \text{Tr}(\chi^{-1}d\chi)^2 = \frac{1}{2\tau^2}(d\tau^2 + d\omega^2).$$

The geodesic motion is defined by

$$\frac{d^2\Phi^i}{d\sigma^2} + \Gamma^i_{jk} \frac{d\Phi^j}{d\sigma} \frac{d\Phi^k}{d\sigma} = 0.$$

Here $\Phi^i$ are the coordinates $\tau$ and $\omega$, $\Gamma^i_{jk}$ are the Christoffel symbols with respect to the metric defined by the line element (10) above. This leads to the following system of differential equations

$$\ddot{\tau} - \frac{\dot{\tau}^2}{\tau} + \frac{\dot{\omega}^2}{\tau} = 0,$$

$$\ddot{\omega} - \frac{2\dot{\tau}\dot{\omega}}{\tau} = 0.$$

The dot $\dot{}$ denotes differentiation with respect to $\sigma$. In order to guarantee that $\lim_{\rho \to \infty} \chi_0$ is equal to the identity, one chooses the boundary values $\tau(\sigma = 0) = 1$ for the gravitational potential and $\omega(\sigma = 0) = 0$ in the case of the gravito-magnetic
potential. To relate the geodesic variable somewhat more directly to the original \( \rho \)-variable, we cite the solution for \( a^2 > 0 \) and \( a = 0 \) from [12]. If \( a^2 > 0 \) the function \( f^2(\rho) \) has two simple zeros at \( \rho = b \pm a \) causing

\[
\sigma(\rho) = \frac{1}{a} \ln \left( \frac{\rho - b - a}{\rho - b + a} \right),
\]

which tends to \(-\infty\) at \( \rho = b - a \). Therefore \( \sigma = 0 \) corresponds to infinite value of \( \rho \) with the interpretation that at \( \sigma = 0 \) a geodesic starts at \( (\tau_0, \omega_0) \) and approaches some other value \( (\tau_h, \omega_h) \) at \( \sigma = \infty \) which marks the position of the horizon of a black hole. \( a = 0 \) means Schwarzschild mass \( m \) and Taub-NUT charge \( l \) equal to zero and for this reason represents flat space.

The system of differential equations (11) is easy to solve:

\[
\begin{align*}
\omega(\sigma) & = \cosh \beta \tanh(a\sigma + \beta) - \sinh \beta, \\
\tau(\sigma) & = \frac{\cosh \beta}{\cosh(a\sigma + \beta)}.
\end{align*}
\]

(12) (13)

The boundary conditions have to be taken into account.

To summarize: each geodesic corresponds to a solution with a required asymptotic behaviour, namely \( (\tau_0, \omega_0) = (1, 0) \). It is uniquely determined by its “velocity”

\[
(\dot{\tau}_0, \dot{\omega}_0) = (-a \tanh \beta, \frac{a}{\cosh \beta}) = (m, l).
\]

After this excursion the Lagrangian (3) or (6), respectively is investigated once more. It is invariant under \( \text{SL}(2,\mathbb{R}) \) transformations \( z \mapsto az + b, \quad cz + d \), \( z = \omega + i\tau, \quad ad - bc = 1 \).

The corresponding Noether currents \( J \) are

\[
J = \begin{pmatrix} -J^0 & J^+ \\ J^- & J^0 \end{pmatrix} := \frac{f^2}{2N^2} \chi^{-1} \chi' = \begin{pmatrix} -\frac{\tau'}{\tau} - \frac{\omega \omega'}{\tau^2} & \frac{\omega'}{\tau^2} \\ \omega' - \frac{2\omega \tau'}{\tau} - \frac{\omega^2 \omega'}{\tau^2} + \frac{\tau'}{\tau} & \frac{\omega \omega'}{\tau^2} + \frac{\tau'}{\tau} \end{pmatrix}
\]

As \( \chi \) is a symmetric matrix only two components of the current matrix are linearly independent. Imposing proper boundary conditions at infinity on the sigma model fields there exists an asymptotic multipole expansion of \( \chi \).

\[
\chi \sim \sum_{n=0}^{\infty} \rho^{-n} \chi_n(\theta, \phi).
\]

A suitable choice of coordinates asymptotically leads to

\[
\omega = \frac{l}{\rho} + O\left(\frac{1}{\rho^2}\right) \quad \text{and} \quad \tau = 1 - \frac{2m}{\rho} + O\left(\frac{1}{\rho^2}\right).
\]
One is interested in particular in the $1/\rho$ term in the expansion of $\chi$ because it contains the parameters, the Schwarzschild mass $m$ and the Taub-NUT charge $l$, which are the entries of the matrix of global charges

$$Q = \frac{1}{4\pi} \int_{\partial S^2} J_{\rho} \, d\Sigma^\rho = \chi_0^{-1} \chi_1.$$ 

By integration over an infinitely large sphere $Q$ is calculated to be

$$Q = \begin{pmatrix} m & l \\ l & -m \end{pmatrix}.$$ 

The identification of the Schwarzschild mass $m$ and the Taub-NUT charge $l$ is obtained by comparison with the standard form of the Schwarzschild and the Taub-NUT solution in the literature \[20\]. The $J^0$ component of the current is related to the Schwarzschild mass $m$, and the $J^+$ and $J^-$ components lead to the Taub-NUT charge $l$.

The line element $\frac{f^4}{l^2} \text{Tr}(\chi^{-1} d\chi)^2$ is constant on geodesics and assumes values greater than or equal to zero. From an algebraic point of view $\text{Tr}J^2$ is the quadratic Casimir element. Here it leads to a generalized charge conservation law: As explained above the mass $m$ and the Taub-NUT parameter $l$ are viewed as generalized charges. On the space of classical solutions the invariant line element is evaluated to be $4a^2 = m^2 + l^2$.

So far the space of classical solutions is understood completely. Now one turns to the ADM approach to Einstein’s gravity. General relativity is invariant with respect to space-time diffeomorphisms. This local symmetry relates part of the solutions stemming from the same initial conditions. In the Lagrangian formalism this results in the fact that out of ten field equations there are only six independent ones: The reparametrization invariance leads to Noether currents, the so called Bianchi identities, and in the sequel to the presence of an arbitrary function of time in the general solution of the equations of motion. A solution is mapped to a solution by a gauge transformation. Performing a Hamiltonian formalism the local symmetry transformations yield a system with constraints. This means that there are conditions on the allowed initial data which must be preserved during time evolution. This consistency requirement can lead to secondary and higher order constraints. The gauge degrees of freedom are hidden in the system of first class constraints, these are those whose Poisson bracket vanish weakly, that is on-shell.

Here a modified Hamiltonian formalism is implemented. The slicing is performed according to the $\rho$ - that is a spacelike - coordinate. How to proceed in this case can be found in \[26\], \[9\]. Due to spherical symmetry the lapse function is the only
Lagrange multiplier which survives. In other words: it expresses the invariance under \( \rho \)-reparametrization and this leads to the only primary first class constraint of the theory, the Hamiltonian constraint, which generates the gauge transformations

\[
H = \frac{1}{4} \pi^2 \tau - 1 - \frac{\tau^2}{f^2} (\pi^2 + \pi^2_\omega).
\]

In terms of the fields and their conjugate momenta \( J \) reads

\[
J = \begin{pmatrix}
-\pi \tau - \omega \pi_\omega \\
\tau^2 \pi_\omega - 2 \omega \tau \pi_\tau - \omega^2 \pi_\omega \\
\pi_\tau + \omega \pi_\omega
\end{pmatrix}.
\]

By definition an observable is a function on the constraint surface that is gauge invariant. As explained above it has weakly, i.e. on-shell, vanishing brackets with the first class constraints. Observables do not evolve in “time” and therefore there is a one to one correspondence between the observables and the initial data here: They can be identified with the space of solutions. Here the classical solutions are parametrized by the Schwarzschild mass \( m \) and the Taub-NUT charge \( l \). Therefore one expects the existence of two observables, which turn out to be “hidden” in the current matrix \( J \):

\[
\{ H, J^0 \} = 0, \quad \{ H, J^+ \} = 0, \quad \{ H, J^- \} = 0.
\]

On the other hand they generate an sl(2,IR) algebra:

\[
\{ J^+, J^0 \} = -J^+, \quad \{ J^+, J^- \} = 2J^0, \quad \{ J^0, J^- \} = -J^-.
\]

The observable \( J^0 \) measures the Schwarzschild mass, \( J^+ \) and \( J^- \) lead to the Taub-NUT charge \( l \) and \( \text{Tr} J^2 \) yields the value of the invariant line element. Because \( \text{Tr} J^2 \) commutes with all the currents

\[
\{ H, \text{Tr} J^2 \} = 0, \quad \{ \text{Tr} J^2, J^+ \} = 0, \quad \{ \text{Tr} J^2, J^- \} = 0, \quad \{ \text{Tr} J^2, J^0 \} = 0,
\]

these observables can be “measured simultaneously” (simultaneously diagonalized). A “simultaneous measurement” of the Schwarzschild mass \( m \) and the Taub-NUT charge \( l \) is not possible.

The Hamiltonian stated above is of course not the whole story \[28\], \[4\]. To develop a Hamiltonian formulation that is Poincaré invariant at infinity one needs a more precise specification of the asymptotic form of the canonical variables \[28\], \[23\]. Even more is true: in general the usual Hamiltonian in the case of an open universe does not have well defined functional derivatives and consequently such a Hamiltonian does not generate any equation of motion at all. To develop a Hamiltonian theory which is Poincaré invariant at infinity one has to subtract some boundary
terms arising as ten new constraints with arbitrary multipliers in the Hamiltonian. Physically these constraints are related to the energy, the total momentum and the angular momentum of the space-time under consideration. The surface integral leads to a specific asymptotic behaviour of the fields. But - as outlined by Regge and Teitelboim [28] - if one assures that the lapse and the shift functions have a proper asymptotic behaviour, one can consider the Hamiltonian and the Diffeomorphism constraints (the later vanish in this sector of Einstein’s gravity) without their asymptotic ends. This is what is done here.

3 Quantization

Concerning the canonical approach to quantum gravity there are two main streams to handle the problem [16]: the reduced phase space quantization and the Dirac quantization. From a “Hamiltonian point of view” the former deals with the problem by elimination of the first class constraints at an early stage. This amounts to quantize gauge invariant functions only, i.e. constants of motion. To carry out the reduced phase space quantization one must find a complete set of gauge invariant functions, which is a difficult task in general. Therefore often canonical gauge conditions are imposed to obtain the reduced phase space. After completely fixing the gauge any function of the canonical variables can be viewed as the restriction of a gauge invariant function in that gauge. Hence: once the gauge is fixed one is effectively working with gauge invariant functions. A complete set of independent gauge fixed functions provides a complete set of gauge invariant functions. This method is practicable if there are no Gribov obstructions.

Problems of that quantization scheme are that an early elimination of the gauge degrees of freedom may spoil the manifest invariance under important symmetries and in general it destroys locality in space. Furthermore it might happen that the brackets of the observables are complicated functions and the question arises how to realize them quantum mechanically. The main advantage is that only the physical degrees of freedom are quantized. Every state in the Hilbert space is a physical one.

In the so called Dirac quantization one keeps the gauge degrees of freedom. The classical phase space functions become operators acting in the Dirac representation space, which carries nonphysical information. Hence one has to select a physical subspace of gauge invariant states.

This paper proceeds along the Dirac approach. The SL(2,\mathbb{R}) symmetry of the theory
provides the key to quantize this sector of Einstein’s gravity. An outline of what happens next is given: The former phase space functions become operators. After this step a naïve approach is followed: the Wheeler-DeWitt is solved formally. For the investigation of the self-adjointness and the spectrum of the physical operators it is necessary to specify the Hilbert space by employing group theoretical arguments. Then, to justify the choice of the Hilbert space or to obtain a basis harmonic analysis is heavily used. That is what is done finally in this quantum part of the paper.

During the process of quantization the functions $J^+, J^-, J^0, \text{Tr}J^2$ and $H$ on phase space become operators on an appropriate Hilbert space. In the Schrödinger representation the fields are turned into multiplication operators and the momenta become differentiation operators $\pi \Phi \mapsto -i\partial \Phi$, where $\partial \Phi$ is an abbreviation for $\frac{\partial}{\partial \Phi}$. The generators of the currents $J^0, J^+, J^-$, the Casimir $\text{Tr}J^2$ and the Hamiltonian $H$ become:

$$\hat{J}^0 = -i \tau \partial_{\tau} - i \omega \partial_{\omega}$$
$$\hat{J}^+ = i \partial_{\omega}$$
$$\hat{J}^- = i (\tau^2 - \omega^2) \partial_{\omega} - 2i \omega \tau \partial_{\tau}$$
$$\text{Tr}J^2 = -\tau^2 (\partial_{\tau}^2 + \partial_{\omega}^2)$$
$$\hat{H} = -\frac{\partial^2}{4} + \frac{1}{2f^2} \text{Tr}J^2 - 1.$$

Applied to a wave function $\psi$ the Hamiltonian defines the so called Wheeler-DeWitt equation $\hat{H}\psi = 0$. The following commutation relations hold:

$$[\hat{H}, \hat{J}^0] = 0 \quad [\hat{H}, \hat{J}^+] = 0 \quad [\hat{H}, \hat{J}^-] = 0,$$

i.e. the current operators are observables. The Casimir operator $\text{Tr}\hat{J}^2$ commutes with the Hamiltonian and the currents

$$[\text{Tr}\hat{J}^2, \hat{H}] = 0 \quad [\text{Tr}\hat{J}^2, \hat{J}^0] = 0 \quad [\text{Tr}\hat{J}^2, \hat{J}^+] = 0 \quad [\text{Tr}\hat{J}^2, \hat{J}^-] = 0,$$

and $\hat{J}^0, \hat{J}^+, \hat{J}^-$ form an sl(2,$\mathbb{R}$) algebra:

$$[\hat{J}^+, \hat{J}^0] = -i\hat{J}^+ \quad [\hat{J}^+, \hat{J}^-] = 2i\hat{J}^0 \quad [\hat{J}^0, \hat{J}^-] = -i\hat{J}^-.$$

The Casimir operator $\text{Tr}\hat{J}^2$ is an essential part of the Wheeler-DeWitt equation. As it commutes with the current operators, $\text{Tr}\hat{J}^2$ and any one of the $\hat{J}$’s can be “measured simultaneously”, i.e. they can be simultaneously diagonalized. $\hat{J}^0, \hat{J}^+$ and $\hat{J}^-$ do not commute with each other. Therefore even formally a direct “measurement” of the Schwarzschild mass $m$ and the Taub-NUT charge $l$ is not possible.
The interpretation of the current operators as observables of the theory strongly suggests to preserve the SL(2,\(\mathbb{R}\)) symmetry at the quantum level: \(\hat{J}^0, \hat{J}^-, \hat{J}^+\) and the Casimir operator \(\hat{\text{Tr}} \hat{J}^2\) are forced to become self-adjoint operators during the process of quantization. That is, the quantum mechanical Hilbert space is built from the unitary irreducible representations of the group SL(2,\(\mathbb{R}\)). Finding a solution of the Wheeler-DeWitt equation therefore basically becomes a group theoretical problem as the equation splits into an \(f\) dependent and an \(f\) independent part, which is the Casimir operator on the group up to a constant term.

There are two particularly useful possibilities to diagonalize the Casimir operator \(\hat{\text{Tr}} \hat{J}^2\). On one hand one solves the differential equations for \(\hat{\text{Tr}} \hat{J}^2\) and the Taub-NUT charge operator \(\hat{J}^+\) simultaneously. The other possibility is to diagonalize the Schwarzschild mass operator \(\hat{J}^0\) and the Casimir operator \(\hat{\text{Tr}} \hat{J}^2\).

The former case is investigated first, i.e. the formal solutions of the following system of differential equations

\[
-\tau^2 \left( \partial^2 + \partial^2_{\omega} \right) \psi_{\lambda L}(\omega, \tau) = \lambda \psi_{\lambda L}(\omega, \tau), \quad (15a)
\]

\[
i \partial_{\omega} \psi_{\lambda L}(\omega, \tau) = L \psi_{\lambda L}(\omega, \tau). \quad (15b)
\]

are obtained. The second equation \((15b)\) is considered first. The solution is

\[
\psi_{\lambda L}(\omega, \tau) = C(\tau) e^{-iL\omega}.
\]

Substitution of this solution into the first equation yields

\[
\tau^2 \partial^2_{\tau} C(\tau) + \left( \lambda - \tau^2 L^2 \right) C(\tau) = 0.
\]

This is a differential equation of Bessel type. If \(L^2 \neq 0\), it follows that

\[
C(\tau) = \sqrt{\tau} \left( \hat{C}_1 J_k(i\tau|\tau) + \hat{C}_2 Y_k(i\tau|\tau) \right), \quad k = \frac{1}{2} \sqrt{1 - 4\lambda} \quad (16)
\]

where \(J_k\) denote the Bessel functions of the first kind and \(Y_k\) those of the second kind. Later on it will be convenient to use a different linear combination of the fundamental solutions. Therefore the following relations between different types of Bessel functions are used:

\[
J_k(iz) = e^{\frac{ik\pi}{2}} I_k(z)
\]

\[
Y_k(iz) = e^{\frac{i(k+1)\pi}{2}} I_k(z) + \frac{2}{\pi} e^{-\frac{i\pi}{2}} K_k(z)
\]

With suitably chosen constants \(C_1\) and \(C_2\) the original formal solution \((16)\) can be written as:

\[
\psi_{\lambda L}(\omega, \tau) = \psi^1_{\lambda L}(\omega, \tau) + \psi^2_{\lambda L}(\omega, \tau)
\]
with
\[
\begin{align*}
\psi^1_{\lambda L}(\omega, \tau) &= C_1 e^{-i L \omega} \sqrt{\tau} I_k(|L| \tau), \\
\psi^2_{\lambda L}(\omega, \tau) &= C_2 e^{-i L \omega} \sqrt{\tau} K_k(|L| \tau).
\end{align*}
\] (17a, 17b)

$I_k$ and $K_k$ denote the Bessel functions of imaginary argument.

Now the simultaneous diagonalization of the Casimir operator $\hat{\text{Tr}} J^2$ and the mass operator $\hat{J}^0$ is carried out. As in the former case the system of differential equations
\[
\begin{align*}
-\tau^2 (\partial^2_\tau + \partial^2_\omega) \psi_{\lambda M}(\omega, \tau) &= \lambda \psi_{\lambda M}(\omega, \tau) \quad (18a) \\
-i (\tau \partial_\tau + \omega \partial_\omega) \psi_{\lambda M}(\omega, \tau) &= M \psi_{\lambda M}(\omega, \tau) \quad (18b)
\end{align*}
\]
has to be solved. The solution of (18b) is
\[
\psi_{\lambda M}(\omega, \tau) = C(v) \omega^{|M|}, \quad (19)
\]
where $v = \frac{\omega}{\tau}$. Alternatively, $\frac{1}{v}$ could be used as independent variable. Nevertheless it turns out that (19) leads to a nice representation in terms of associated Legendre polynomials. Substituting (19) into (18a) yields
\[
-v^2 (v^2 + 1) C'' + 2mv^3 C' - \left( m(m + 1) v^2 - \lambda \right) C = 0.
\]
Here the prime $'$ denotes differentiation with respect to $v$ and $m$ is defined to be $m := iM - 1$. This differential equation is of Hypergeometric type and transformed into the Hypergeometric differential equation by
\[
C(v) = v^{\frac{1+\kappa}{2}} \eta(-v^2), \quad \kappa = \sqrt{1-4\lambda},
\]
where $\eta(\xi)$ is the solution of
\[
\xi (\xi - 1) \eta'' + [(\alpha + \beta + 1) \xi - \gamma] \eta' + \alpha \beta \eta = 0
\]
The prime $'$ denotes differentiation with respect to $\xi$, and $\alpha, \beta, \gamma$ are equal to
\[
\begin{align*}
\alpha &= \frac{1}{4} (1 + \kappa - 2m) \\
\beta &= \frac{-1}{4} (1 - \kappa + 2m) \\
\gamma &= 1 + \frac{\kappa}{2}
\end{align*}
\]
If $\lambda \neq -n^2 + 1/4$, $n \in \mathbb{Z}$, $\eta(\xi)$ is calculated to be
\[
\eta(\xi) = C_1 F(\alpha, \beta; \gamma; \xi) + C_2 \xi^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; \xi) \quad (20)
\]
(20) can be expressed by associated Legendre polynomials so that one finally obtains

\[ \psi_{\lambda M}(\omega, v) = \psi_{\lambda M}^1(\omega, v) + \psi_{\lambda M}^2(\omega, v) \]

with

\[ \psi_{\lambda M}^1(\omega, v) = C_1 \omega^{\lambda M} \sqrt{v} \left( 1 + v^2 \right)^{\frac{i\lambda}{2} + \frac{k}{2}} \frac{P_{-iM-\frac{k}{2}}}{\sqrt{1 + v^2}} \left( \frac{1}{\sqrt{1 + v^2}} \right), \quad (21a) \]

\[ \psi_{\lambda M}^2(\omega, \tau) = C_2 \omega^{\lambda M} v^{k+\frac{1}{2}} |v|^{-k} \left( 1 + v^2 \right)^{\frac{i\lambda}{2} + \frac{k}{2}} \frac{P_{-iM-\frac{k}{2}}}{\sqrt{1 + v^2}} \left( \frac{1}{\sqrt{1 + v^2}} \right), \quad (21b) \]

where \( \lambda \neq \frac{1}{4} - n^2, \quad n \in \mathbb{Z} \).

In the coordinates \( \omega \) and \( v \) the differential operators can be written in the following manner:

\[ \hat{J}_0 = -i \omega \partial_\omega \quad (22a) \]

\[ \hat{\text{Tr}}J^2 = v^2 (1 + v^2) \partial_v^2 + \omega^2 v^2 \partial_\omega^2 - 2v\omega^2 \partial_\omega \partial_v + 2v^3 \partial_v \quad (22b) \]

To solve (22) means seeking solutions of \( \hat{\text{Tr}}J^2 \psi = \lambda \psi \) of the form \( f(\omega)C(v) \).

Of course, the formal solutions of the differential equations do not yield enough structural elements to solve the problem. For a deeper understanding a Hilbert space structure is needed to single out part of the solutions of the differential equations and to show self-adjointness of the physical operators. For this one applies various techniques from functional analysis, the representation theory of \( \text{SL}(2, \mathbb{R}) \), the Plancherel theorem, and its version on the coset space \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \).

The natural choice of the Hilbert space is to take the closure of the functions on the coset space \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \) on which the physical operators can be shown to be essentially self-adjoint. What one also would like to have is a subset of functions which approximate all functions on the coset space uniformly. This is the content of the theorem by Stone and Weierstrass. It states, that given a compact metric space \( E \), which is constructed below, any subalgebra of continuous functions on that space which contains the unity and separates the points of \( E \) is dense in the Banach space of functions \( E \). One immediately recognizes that the space of \( L^2 \)-functions over the coset space measurable with respect to the Haar measure on the group \( \text{SL}(2, \mathbb{R}) \) is too small: it does not contain the constants. Furthermore: the coset space \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \) is noncompact. This problem is solved by exhausting the coset space by a sequence of compact subsets. The restriction of the solutions on each compact subset of \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \) form (by group theoretical arguments)
the required subalgebra of the Stone-Weierstrass theorem. The sequence of compact subsets converges towards $\text{SL}(2,\mathbb{R})/\text{SO}(2)$.

The Hilbert space $\mathcal{H}$ is defined to be the closure of the space of solutions of the eigenvalue equations corresponding to $\hat{\text{Tr}}J^2, \hat{\text{J}}^\dagger$ or $\hat{\text{Tr}}\overline{J}^2, \hat{\text{J}}^0$, respectively. It turns out that the K-Bessel functions (17b) constitute a basis in the case of the diagonalization of $\hat{\text{Tr}}\overline{J}^2, \hat{\text{J}}^\dagger$ and the Hilbert space is the closure of the span of these functions. For the diagonalization of $\hat{\text{Tr}}\overline{P}^2, \hat{\text{J}}^0$ the functions (21a) provide a basis. It can be shown that the Hilbert spaces are isomorphic. The scalar product is defined employing the Haar measure on the coset space dressed with a suitable damping factor, which is chosen such that the constants and enough functions to separate the points on $\text{SL}(2,\mathbb{R})/\text{SO}(2)$ belong to the Hilbert space. That is, from a technical point of view the Hilbert space theory of the continuation of symmetric operators to their closure is applied.

One realization of the coset space $\text{SL}(2,\mathbb{R})/\text{SO}(2)$ is the Poincaré upper half plane $\mathbb{H}$. The isomorphism is established now [24]. Each element $g$ of the group $\text{SL}(2,\mathbb{R})$ can be represented by a $2 \times 2$ matrix with real entries and determinant equal to one.

$$\text{SL}(2,\mathbb{R}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad - bc = 1 \right\}$$

$\mathbb{H}$ is defined to be

$$\mathbb{H} := \{ z = \omega + i\tau \in \mathbb{C}, \ \tau > 0 \}.$$

Let the $\text{SL}(2,\mathbb{R})$ act on $\mathbb{H}$ in the following manner:

$$g \circ z = \frac{az + b}{cz + d}.$$

It is easy to see, that the mapping $g \mapsto g \circ i$ from $\text{SL}(2,\mathbb{R})$ into $\mathbb{H}$ induces a bijection $\text{SL}(2,\mathbb{R})/\text{SO}(2) \to \mathbb{H}$.

As motivated above, the Hilbert space $\mathcal{H}(\mathbb{H},d\mu_n)$ is the space of square integrable functions on the upper half plane, which are measurable with respect to the $\text{SL}(2,\mathbb{R})$ invariant measure $d\mu$

$$d\mu = \frac{d\omega d\tau}{\tau^2},$$

dressed with the damping factor

$$d_n(\omega,\tau) = \frac{\tau^n}{(\omega^2 + (1 + \tau^2)^n)}, \quad \in \mathbb{N}.$$

The Hilbert space measure is defined by $d\mu_n = d_n(\omega,\tau)d\mu$. It is shown below, that $\mathcal{H}(\mathbb{H},d\mu_n)$ contains solutions of the differential equations as long as $n > 1$. Then
they are a dense subset of $\mathcal{H}(I, d\mu_n)$. With $d_{n_1}(\omega, \tau) < d_{n_2}(\omega, \tau)$ for $n_1 > n_2$ there exists a natural embedding of the $L^2$ spaces with a smaller $n$ into those with a greater $n$. In particular $\mathcal{H}(I, d\mu_0) = L^2(I)$. The scalar product $(\, , )_n$ is defined by

$$( f(\omega, \tau), g(\omega, \tau))_n = \int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\tau \frac{d_n(\omega, \tau)}{\tau^2} f(\omega, \tau)g^*(\omega, \tau)$$

Concerning (15) it is found

Lemma 1 For $1 - 4\lambda > 0$ neither $\psi^1_{\lambda L}$ nor $\psi^2_{\lambda L}$ belong to $\mathcal{H}(I, d\mu_n)$.

For $1 - 4\lambda \leq 0$ the $\psi^2_{\lambda L}$ are elements of $\mathcal{H}(I, d\mu_n)$, $n > 1$.

It is sufficient to show that

$$( \psi^1_{\lambda L}, \psi^1_{\lambda L})_n = \int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\tau \frac{d_n(\omega, \tau)}{\tau^2} |\psi^1_{\lambda L}(\omega, \tau)|^2$$

and $(\psi^2_{\lambda L}, \psi^2_{\lambda L})_n$ is finite. This is essentially proven by splitting the integrals into three parts in $\tau$, namely the asymptotic region near zero, a singularity-free part having “compact support” and the asymptotic region near infinity.

At first the case $1 - 4\lambda > 0$ is studied more closely. Near infinity $|I_0(\sqrt{2}\tau)| \sim e^\tau$ holds and therefore the integral does not exist. On the other hand $|K_0(\sqrt{2}\tau)| \geq k_0 \tau^{-\frac{1}{2}}$, which lets the norm of $\psi^2_{\lambda L}$ tend to infinity, too.

For $1 - 4\lambda = 0$ it is clear, that $I_0$ is not bounded and therefore $\psi^1_{\lambda L}$ is not either. Near zero one can approximate $K_0(x)$ by $\ln \frac{2}{x}$ and near infinity by $\sqrt{\frac{2}{x}}e^{-x}$. Splitting the integrals into the three parts mentioned above it can be shown that the norm of $\psi^2_{\lambda L}$ is finite. That the integral over $\psi^2_{\lambda L}$ remains finite for $1 - 4\lambda < 0$, too, is due to the fact that $K_0(\sqrt{2}\tau)$ can be used as an upper bound for the function $K_k$. The easiest way to see this is to use the integral representation for $K_k$.

To prove the divergence of $(\psi^1_{\lambda L}, \psi^1_{\lambda L})$ for $1 - 4\lambda < 0$ the asymptotic expansion

$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \sum_{j=0}^{\infty} (-1)^j \frac{(4\nu^2 - 1)(4\nu^2 - 3)...[4\nu^2 - (2j - 1)^2]}{8^j j! z^j}$

is employed.

Now one answers the question the norm of which part of the solutions $\psi_{\lambda M}$ of the system of differential equations (15) is finite. The parameter $\kappa$ assumes values on the positive real line $\mathbb{R}_+$ or on the positive imaginary line $i\mathbb{R}_+$.

Lemma 2 For $1 - 4\lambda > 0$ and for $1 - 4\lambda \leq 0$ the $\psi^1_{\lambda M}$ belongs to $\mathcal{H}(I, d\mu_n)$, $n > 1$. 

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This lemma is proven by rather tricky estimates of the integrals \((\psi_{1\lambda M}, \psi_{1\lambda M}^*)_n\). The first integral is reduced to one over the damping factor \(d_n\) using that by holomorphy there exists an upper bound for the Hypergeometric function on the interval \([0, 1/2]\). Concerning the second integral the existence of a lower bound in the neighbourhood of 0 is quite useful to prove divergence of the integral.

The Hilbert space is spanned by the solutions of the systems of differential equations (13) or (18), respectively. In particular they form a dense subspace of the Hilbert space. Therefore the symmetry has to be proven on the solutions of these differential equations only where the operators are diagonal. That is, by construction of the Hilbert space the current operators \(\hat{J}^0, \hat{J}^+\) and the Casimir operator \(\hat{\text{Tr}}J^2\) are already symmetric. The self-adjointness is discussed with the technique of the deficiency indices \(n_+\) and \(n_-\). Given a symmetric operator \(A\), the deficiency index \(n_{\pm}\) is defined to be the dimension of the space \(\ker(i\pm A^*)\). The operator is essentially self-adjoint iff the deficiency indices are zero. It has a self-adjoint extension, iff the deficiency indices are equal. Therefore the dimensions \(n_-\) of the kernel of \(i - \hat{J}^+\) and \(n_+\) of \(i + \hat{J}^0\) are calculated next.

**Lemma 3** The deficiency indices \(n_+\) and \(n_-\) are equal to zero, i.e. the operator \(\hat{J}^+\) is already essentially self-adjoint in \(\mathcal{H}(\mathbb{H}, d\mu_n)\).

The kernel, the dimension \(n_+\) of which is calculated, consists of the functions \(\psi_{\lambda L}\) with \(\lambda = i\) and \(L = -i\) and the ones corresponding to \(n_-\) consists of \(\psi_{\lambda L}\) with \(\lambda = -i\) and \(L = i\). The techniques to prove the preceding Lemma are applied to show the divergence of all the integrals involved.

In analogous manner one calculates the dimensions \(n_-\) of the kernel and \(i - \hat{J}^0\) and \(n_+\) of \(i + \hat{J}^0\).

**Lemma 4** The deficiency indices \(n_+\) and \(n_-\) are equal to zero. That is the operator \(\hat{J}^0\) is already essentially self-adjoint in \(\mathcal{H}(\mathbb{H}, d\mu_n)\).

\(\hat{\text{Tr}}J^2\) is shown to be essentially self-adjoint, too. With the solutions of the differential equations (13) it follows, that

**Lemma 5** The spectrum of the operators \(\hat{J}^0\) and \(\hat{J}^+\) consists of the whole real line.

To summarize: part of the formal simultaneous eigendistributions of the invariant differential operator \(\hat{\text{Tr}}J^2\) and the Taub-NUT operator \(\hat{J}^+\) as well as of \(\hat{\text{Tr}}J^2\) and the
physical “mass-operator” \( \hat{J}^0 \) are shown to belong to \( \mathcal{H}(\mathbb{H}, d\mu_n) \). They are exactly the functions which are needed to decompose a function on the Poincaré upper half plane into irreducible parts. This is discussed in the group theoretical context below. The operators \( \hat{J}^+ \) and \( \hat{J}^0 \) are shown to be essentially self-adjoint. Finally no constraints on the spectra of the operators \( \hat{J}^+ \) and \( \hat{J}^0 \) arise: they consist of the whole real line. Negative Taub-NUT charges as well as negative Schwarzschild masses belong to the spectrum, too.

Now one switches to the group theoretical point of view. There exists a canonical Hilbert space associated to the group \( \text{SL}(2, \mathbb{R}) \). The unitary irreducible characters provide a basis of this Hilbert space. There are sufficiently many of them to separate the points of the group which means that a function on the group can be approximated by its characters: one obtains a generalized Fourier transform. This is the reason why the construction of the Hilbert space above makes sense. The property that the characters separate the points of the group reminds one of the assumption of the Stone-Weierstrass theorem mentioned above. The generalised inverse Fourier transform, which is called Plancherel theorem in the mathematical literature, yields an expansion of an arbitrary function on the group as a series or an integral of functions which occur as matrix elements of unitary irreducible representations on the group. It contains the characters of the continuous and the discrete series. The formula can then be applied to elements of \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \) with the result, that only one part of the continuous series survives. The eigendistributions of the invariant differential operators are shown to appear explicitly in the inverse Fourier transform. That is the deeper reason why the spectra of the mass and the charge operator are purely continuous.

The classification of the unitary irreducible representations of \( \text{SL}(2, \mathbb{R}) \) is due to Bargman [1]:

1. the principal continuous series \( V^{j, s} \), \( j = 0, 1/2, s = 1/2 + it, t \in \mathbb{R} \). If \( j = 0, \) then \( t > 0 \), and if \( j = \frac{1}{2}, \) then \( t \geq 0 \).

2. the limit of the discrete series \( U^{1/2}, U^{-1/2} \),

3. the discrete series \( U^n, U^{-n}, n \in \mathbb{Z}/2, n > 1/2 \),

4. the complementary series \( V^\sigma, 1/2 < \sigma < 1 \) and

5. the trivial representation.

Let \( \lambda \) denote the eigenvalue of the Casimir operator. To the five cases listed above there correspond the following eigenvalues:
1. \( \lambda = s(1 - s) \),
2. \( \lambda = \frac{1}{4} \),
3. \( \lambda = n(1 - n) \),
4. \( \lambda = \sigma(1 - \sigma) \), and
5. \( \lambda = 0 \).

That is, group theory yields a restriction of the eigenvalues of the Casimir operator and some formal solution of the differential equations are excluded from the Hilbert space by group theoretical arguments, namely the case \( 1 - 4\lambda > 0 \) mentioned above.

To the representations there correspond characters, which are class functions. To evaluate and to apply them it is quite natural to factor the group by the Iwasawa decomposition into an elliptic, a hyperbolic and a parabolic part, which are denoted by \( G_{\text{ell}} \), \( G_{\text{hyp}} \) and \( G_{\text{par}} \), respectively. Now it is outlined how to proceed:

Each \( 2 \times 2 \) matrix of \( \text{SL}(2,\mathbb{R}) \) can be written as a product of a rotation matrix, the elliptic part of \( \text{SL}(2,\mathbb{R}) \), a diagonal matrix with entries either both positive or both negative, the hyperbolic part of the group, and one matrix with a Jacobi-like upper triangular form, which parametrizes the parabolic part of the group.

\[
\text{SL}(2,\mathbb{R}) = KAN
\]

with

\[
u_{g} = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}, \quad a_{\tau} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad \eta_{\omega} = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}
\]

Denote \( H := A \cup (-1) A \), then \( G_{\text{hyp}}, G_{\text{ell}} \) and \( G_{\text{par}} \) are defined by

\[
G_{\text{hyp}} := \bigcup_{g \in G} g H g^{-1}, \quad G_{\text{ell}} := \bigcup_{g \in G} g K g^{-1}, \quad G_{\text{par}} := \bigcup_{g \in G} g (\pm N) g^{-1}.
\]

Furthermore the group \( \text{SL}(2,\mathbb{R}) \) can be decomposed into disjoint parts of classes of regular elements, i.e. elements with distinct eigenvalues, \( G'_{\text{ell}}, G'_{\text{hyp}} \) and \( G'_{\text{par}} \). \( G'_{\text{par}} \) is the empty set. \( G \) may be represented as the product of \( G'_{\text{ell}}, G'_{\text{hyp}} \) and \( G'_{\text{par}} \). The series of the unitary irreducible representations correspond in a one to one manner to the factors of the Iwasawa decomposition: the discrete series to the elliptic part, the continuous one to the hyperbolic factor and the complementary series to the parabolic one. An algorithm to compute the Iwasawa decomposition for elements of one of the classical Lie groups can be found in the Appendix A.
Before dealing with the decomposition of a function on $\text{SL}(2, \mathbb{R})$ into its irreducible parts, the space of rapidly decreasing functions $S(G) = \text{SL}(2, \mathbb{R})$ has to be introduced (see [3]):

$$S(G) := \{ f \in C^\infty(G) : \sup_{g \in G} \frac{(\omega^2 + (1 + \tau)^2)^N}{\tau^N} |(D^\alpha f)(g)| < \infty, \ \forall N \}$$

The tempered distributions $S'(G)$ are the dual space of the space of the rapidly decreasing functions. They are called slowly increasing. The triplet $S(G), L^2(G, d\mu_G)$ and $S'(G)$ is called a Gel'fand triplet. $d\mu_G$ denotes the Haar measure on $\text{SL}(2, \mathbb{R})$.

It holds

$$S(G) \subset L^2(G, d\mu_G) \subset S'(G)$$

As already mentioned above, $\mathcal{H}(G, d\mu_G) = L^2(G, d\mu_G)$. $S'(G)$ contains all of the spaces $\mathcal{H}(G, d\mu_n)$. $\mathcal{H}(G, d\mu_n)$ denotes the Hilbert space of functions on $G$, which are measurable with respect to the Haar measure of $G$.

This approach is similar to the one needed for the investigation of the Hydrogen Atom. There the coset space $\text{SO}(3)/\text{SO}(2)$ which is isomorphic to the 2-sphere $S^2$ plays the key role. The generalised Fourier transform of $\text{SO}(3)$ represented on $S^2$ leads to the Spherical Harmonics. The main difference arises from the fact, that the group $\text{SL}(2, \mathbb{R})$ is no longer compact, therefore the unitary irreducible representations are no longer finite dimensional, and apart from a discrete sum there further appears a direct integral in the decomposition formula.

The generalized Fourier transform [30] establishes a topological isomorphism between the $C^\infty$ functions the space of rapidly decreasing function on the dual of the group $\hat{G}$.

**Theorem 1** For each function $f \in S(G)$

$$f(g) = \frac{1}{2\pi} \int_0^\infty dt \text{Tr} [\hat{f}(0, \frac{1}{2} + it) V_g^{0, \frac{1}{2} + it}] \ t \ tanh \ \pi t$$

$$+ \frac{1}{2\pi} \int_0^\infty dt \text{Tr} [\hat{f}(\frac{1}{2}, \frac{1}{2} + it) V_g^{\frac{1}{2}, \frac{1}{2} + it}] \ t \ coth \ \pi t$$

$$+ \frac{1}{4\pi} \sum_{n \in \frac{1}{2} \mathbb{Z}, n \geq 1} (2n - 1) \left\{ \text{Tr} [\hat{f}(n) U_g^n] + \text{Tr} [\hat{f}(-n) U_g^{-n}] \right\}.$$  \hspace{1cm} (23)

In particular:

$$f(1) = \frac{1}{2\pi} \int_0^\infty dt \left( \Theta^{0, \frac{1}{2} + it}(f) \ t \ tanh \ \pi t \right) \hspace{1cm} + \hspace{1cm} \Theta^{\frac{1}{2}, \frac{1}{2} + it}(f) \ t \ coth \ \pi t$$

$\hat{G}$ is defined to be the set of all equivalence classes of unitary irreducible representations of the group.
\[ + \frac{1}{4\pi} \sum_{n \in \frac{1}{2} \mathbb{Z}}^{n \geq 1} (2n - 1) \left[ \Theta^n(f) + \Theta^{-n}(f) \right] \]

with
\[ \Theta^{\frac{1}{2}+it}(f) = \int_\hat{G} dg f(g^{-1}) \Theta^{\frac{1}{2}+it}(g) \]

\[ \Theta^{\frac{1}{2}+it} \text{ and } \Theta^{\frac{1}{2}+it} \text{ denote the characters of the principal continuous series and } \Theta^n \text{ and } \Theta^{-n} \text{ those of the discrete series. Only the discrete and the principal continuous series contribute to the support of the Plancherel measure in } \hat{G} \text{ in the Plancherel theorem for a general group element } g \text{ of } \text{SL}(2,\mathbb{R}). \]

In Einstein’s gravity a parametrization of the coset space is given: the upper half plane. Therefore the formula has to be applied to \[ \mathbb{H} \]. One starts with a function \[ f \in \mathcal{S}(\mathbb{H}) \], which is invariant under the action of the elliptic part \[ K \] of the group. This leads to the conclusion that the irreducible characters associated with \[ K \] do not contribute to the Fourier inversion formula for such functions. Without loss of generality \[ g \] can be taken to be an regular element of the hyperbolic part of the group \[ G'_{hyp} \]. One calculates

\[ \text{Tr} \hat{f}(n) \left. U^n_{g} \right| \right. = \text{Tr} \left( \int_\hat{G} dl f_g(l) \left. U^n_{l^{-1}} \right| \right. \].

\[ f \text{ and } f_g \text{ are both left and right invariant under the action of } K. \text{ The representation } U^n_{l^{-1}} \text{ is explicitly known. It acts on holomorphic functions } s \text{ on } \mathbb{H} \text{ in the following manner } [\text{Bar47}]: \]

\[ U^n_{l^{-1}} s(z) = \frac{1}{(cz + a)^{2n}} s \left( \frac{az + b}{cz + d} \right), \quad l^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right). \]

for a rotation matrix \[ l \] the action reads

\[ U^n_{l^{-1}} s(z) = \frac{1}{(\sin \frac{\theta}{2} z + \cos \frac{\theta}{2})^{2n}} s \left( \frac{\cos \frac{\theta}{2} z - \sin \frac{\theta}{2}}{\sin \frac{\theta}{2} z + \cos \frac{\theta}{2}} \right). \]

The integrand is a holomorphic function on \[ \mathbb{H} \] and the path of integration is closed. Therefore the integral vanishes and the discrete series does not contribute to the Plancherel formula on \[ \mathbb{H} \]. The characters of the continuous series enter into the formula in two places.

Due to the symmetry the coth-part denoted by \[ f_2(g) \] does not contribute to the formula either. By translation all group elements are obtained from unity and

\[ \text{Tr} \left[ \hat{f}(\frac{1}{2}, \frac{1}{2} + it) V^n_{g} \frac{1}{2}+it \right] = \Theta^{\frac{1}{2}+it}(f_2) \]

(24)
\( f \) is two-sided \( K \)-invariant leading to \( f_g(g) = f_g(-g) \). The character \( \Theta^{1/2+it}(g) \) is

\[
\Theta^{1/2+it}(g) = \begin{cases} 
\frac{\cos t\tau}{|\sinh \tau|}, & g = g_0 a_g g_0^{-1} \in \mathcal{G}'_{hyp} \\
\frac{-\cos t\tau}{|\sinh \tau|}, & g = -g_0 a_g g_0^{-1} \in \mathcal{G}'_{hyp} \\
0, & g \in \mathcal{G}'_{ell} \cup \mathcal{G}_{par}
\end{cases}
\]

It follows that \( \Theta^{1/2+it}(g) = \Theta^{1/2+it}(-g) \), i.e. the character is point symmetric in \( \tau \) and (24) vanishes. That is, the coth-part does not contribute to the Plancherel theorem on the upper half plane, either.

Given a representation space, which is the Poincaré upper half plane here, it is now possible to prove, that the Plancherel theorem on \( \text{SL}(2,\mathbb{R})/\text{SO}(2) \) can be written such that the eigendistributions of the Casimir operator appear as integrand in the formula!

**Theorem 2** For \( f \in \mathcal{S}(\mathbb{H}) \),

\[
f(\omega, \tau) = \frac{1}{\pi^2} \int_{L \in \mathbb{R}} dL \int_{t \in \mathbb{R}} dt \hat{f}(L, t) e_{L, t}(\omega, \tau) t \sinh \pi t
\]

where

\[
e_{L, t}(\omega, \tau) = \exp(-iL\omega) \sqrt{\tau} K_{it}(|L|\tau), \quad \text{if} \quad L \neq 0
\]

and

\[
\hat{f}(L, t) = \int_{\mathbb{H}} d\omega d\tau \frac{f(\omega, \tau) e_{L, t}(\omega, \tau)}{\tau^2}
\]

**Proof:**

It is sufficient to show, that the tanh-part of the Plancherel formula (23), denoted by \( f_1(g) \) is equivalent to this lemma. The nuclear spectral theorem \( \mathfrak{B} \) yields

\[
f(\omega, \tau) = 2\pi \int_{L \in \mathbb{R}} dL \int d\lambda(t) \hat{f}(L, t) e_{L, t}(\omega, \tau) = 2\pi \int d\lambda(t) \int_{\hat{\omega} \in \mathbb{R}} d\hat{\omega} \int_{\hat{\tau} \in \mathbb{R}_+} d\hat{\tau} \frac{f(\hat{\omega}, \hat{\tau})}{\hat{\tau}^2} \left( \int_{L \in \mathbb{R}} dL e_{L, t}(\hat{\omega}, \hat{\tau}) e_{L, t}(\omega, \tau) \right)
\]

The inner integral can be evaluated using the calculus of residues, which leads to an orthogonality relation for the K-Bessel functions:

\[
\int_{L \in \mathbb{R}} dL e_{L, t}(\hat{\omega}, \hat{\tau}) e_{L, t}(\omega, \tau) = \delta(\omega - \hat{\omega}) \delta(\tau - \hat{\tau}) \frac{\pi}{4 \cosh t\pi}
\]
Substitution of (26) into (25) yields
\[
f(\omega, \tau) = \frac{\pi^2}{2} \int_{t \in \mathbb{R}} d\lambda(t) \int_{\tilde{\omega} \in \mathbb{R}} \frac{f(\tilde{\omega}, \tilde{\tau})}{\tilde{\tau}^2} \delta(\omega - \tilde{\omega}) \delta(\tau - \tilde{\tau}) \frac{1}{\coth \pi t}. \tag{27}
\]

On the other hand \( f_1(g) = f_{1,g}(1) \). Therefore and because \( \text{SL}(2,\mathbb{R}) \) is unimodular \( f_1(g) \) is transformed into
\[
f_1(g) = \frac{1}{2\pi} \int_0^\infty dt \Theta^0 \frac{1}{2} + it \Theta^0 \frac{1}{2} + it(g) t \tanh \pi t = \frac{1}{2\pi} \int_0^\infty dt \int_{G'} f_{1,g}(g^{-1}) \Theta^0 \frac{1}{2} + it(g) t \tanh \pi t
\]
The character is symmetric with respect to \( t \) [Sug90] and the \( \text{SL}(2,\mathbb{R}) \) acts on the upper half plane leading to
\[
f_1(h) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dt \int_{g'_{hyp}} f_h(h^{-1}) \Theta^0 \frac{1}{2} + it(h) \Theta^0 \frac{1}{2} + it(h^{-1}) t \tanh \pi t\]
\[
= \frac{1}{4\pi} \int_{-\infty}^{\infty} dt \int_{g'_{hyp}} f_h(h) \Theta^0 \frac{1}{2} + it(h) \Theta^0 \frac{1}{2} + it(h^{-1}) t \tanh \pi t
\]
\[
= \frac{1}{4\pi} \int_{-\infty}^{\infty} dt \int_{g'_{hyp}} f_h(h) \Theta^0 \frac{1}{2} + it(h) \Theta^0 \frac{1}{2} + it(h^{-1}) t \tanh \pi t \tag{28}
\]
The normed character \( (\Phi^0 \frac{1}{2} + it(e) = 1) \) is
\[
\Theta^0 \frac{1}{2} + it(g) = \begin{cases} 
\cos t \tau, & g = g_0 a \tau g_0^{-1} \in G'_{hyp} \\
0, & \text{otherwise}
\end{cases}
\]

(27) and (28) yield
\[
\lambda'(t) = \frac{1}{4\pi} \Theta^0 \frac{1}{2} + it(e) t \tanh \pi t \frac{2}{\pi^2} \cosh \pi t = \frac{t \sinh \pi t}{2\pi^3} \Theta^0 \frac{1}{2} + it(e)
\]

and therefore
\[
\lambda'(t) = \frac{1}{2\pi^3} t \sinh \pi t
\]

This concludes the proof.

\[\square\]

This Fourier inversion formula corresponds to a simultaneous diagonalization of the Casimir operator and the differential operator corresponding to the generator of the Lie algebra, which has a 1 in the upper right corner and 0’s elsewhere. The Taub-NUT charge \( L \) appears explicitly. On the algebraic level \( \hat{J}^0 \) belongs to the generator \( \text{diag}(1, -1) \).

The representation most suitable for the Schwarzschild mass \( M \) yields another Plancherel inversion formula. Given a \( f \in \mathcal{S}(\mathbb{H}) \), this function can be written as
\[
f(\omega, v) = C \int_{M \in \mathbb{R}} dM \int_{t \in \mathbb{R}} dt \hat{f}(M, t) g_{M,t}(\omega, v) d\lambda(M, t)
\]
where
\[
  g_{M,t}(\omega, \tau) = \omega^i M \sqrt{1 + v^2} \left( 1 + v^2 \right)^{\frac{M}{2} - \frac{1}{4}} P_{-iM - \frac{1}{2}}^{-\frac{1}{2} - \frac{1}{4}} \left( \frac{1}{\sqrt{1 + v^2}} \right), \text{if } M \neq 0
\]
and
\[
  \hat{f}(M, t) = \int \mathbb{H} d\omega dv f(\omega, v) \frac{g_{M,t}(\omega, v)}{v^2 \omega}.
\]
After the diagonalization of the Casimir operator the Wheeler-DeWitt equations is transformed into
\[
  \psi'' + \left( 4 - \frac{2\lambda}{f^2} \right) \psi = 0.
\]
Equation (29) can be interpreted as evolution equation in the coordinate \( f \). In contrast to \( \rho \), \( f \) has a physical meaning as \( \frac{1}{f} \) is the curvature of the 2-spheres. Hence it is also a physical parameter. Equation (29) is a second order differential equation and therefore there does not exist a positive semi-definite probability density which is invariant under \( SL(2, \mathbb{R}) \) transformations and conserved during evolution. This situation is comparable with the one in the case of the Klein-Gordon equation: a conserved current can be associated to the Klein-Gordon equation, but the zero component of the current is not positive definite. Feshbach and Villar [13] interpreted \( j_0 \) as a charge density which measures the difference between the numbers of positive and negative charges. In the one-particle case, they showed that the density carries either a positive or a negative sign and that the two degrees of freedom of the second order differential equation are identified as two possible but equivalent charge states. In [13] it is outlined that although the nonrelativistic equation only admits charges of one sign and in the relativistic generalization two signs of the charge happen to appear. The two equivalent charge states are basically obtained by transforming the Klein-Gordon equation into a symmetrized Schrödinger equation, i.e. a system of first order differential equations. Now equation (29) is investigated in two steps. At first, the summand \( -\frac{2\lambda}{f^2} \psi \) is considered to be a \( f \)-dependent perturbation and is neglected. This “free” equation can easily be solved and two equivalent states are found. The second step consists in investigation of the “perturbed” system. With a given and finite \( \lambda \) the solution approaches a the free solution in the limit \( f \to \infty \). This fixes the asymptotic behaviour of the solution. The two linear independent solutions can then again be shown to form two equivalent but independent states.

The “free” differential equation \( \psi''_0 \) has the solution \( \psi_0(f) = C_1 e^{2if} + C_2 e^{-2if} \). It follows that
\[
  \psi_0^* \partial_f \psi_0 - \psi_0 \partial_f \psi_0^* = \frac{i}{4} (C_1^2 - C_2^2) = \frac{i}{4} (C_1^2 - C_2^2)
\]
with $\phi_0 = C_1 e^{2i\bar{f}}$ and $\chi_0 = C_2 e^{-2i\bar{f}}$. It is not difficult to solve the differential equation (29). The solutions with the proper asymptotic behaviour turn out to be the Bessel functions of third kind $H_{\nu}^{(1)}$ and $H_{\nu}^{(2)}$:

$$\psi(f) = C_1 \sqrt{f}H_{\nu}^{(1)}(2f) + C_2 \sqrt{f}H_{\nu}^{(2)}(2f), \quad \nu = \frac{1}{2}\sqrt{1 + 8\lambda}.$$

Using that the complex conjugate of $H_{\nu}^{(1)}$ is $H_{\nu}^{(2)}$, the density is calculated to be

$$\psi \partial_f \psi - \psi \partial_f \psi = 2C_1^2 f \left[ H_{\nu}^{(2)}(2f) \partial_f H_{\nu}^{(1)}(2f) - H_{\nu}^{(1)}(2f) \partial_f H_{\nu}^{(2)}(2f) \right]$$

$$+ 2C_2^2 f \left[ H_{\nu}^{(1)}(2f) \partial_f H_{\nu}^{(2)}(2f) - H_{\nu}^{(2)}(2f) \partial_f H_{\nu}^{(1)}(2f) \right]$$

$$= \frac{8i}{\pi} (C_1^2 - C_2^2).$$

The last equality uses the Wronskian of $H_{\nu}^{(1)}$ and $H_{\nu}^{(2)}$. Two “equivalent” states in this case are

$$\phi = \frac{1}{2} \left( \partial_f \psi + i\psi \right), \quad \text{and} \quad \chi = \frac{1}{2} \left( \partial_f \psi - i\psi \right).$$

In these new variables the density $\rho$ reads $\rho = 2i(\chi \chi^* - \phi \phi^*)$.

The choice of the equivalent states is not unique. Define $\chi$ and $\phi$ by

$$\begin{pmatrix} \psi \\ \partial_f \psi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}.$$

As long as $a_{12}a_{21} - a_{22}a_{11} = 0$ the states $\chi$ and $\phi$ are equivalent in the sense that $\rho$ is the difference of two positive semi-definite densities. One part of the freedom can be fixed by normalisation.

Yet the physical meaning of this discrete symmetry (“charge conjugation”) is not clear.

There is a nice relation between the quantization of the four dimensional reduced spherically symmetric gravity in its dual representation and a $\text{SL}(2,\mathbb{R})$ WZNW model in the point particle version, which is further reduced to a Liouville theory [Ful96]. There are also some important differences which lead to a deeper understanding of the quantization of the system dealt with in this paper. Performing the reduction to Liouville theory Fülöp ended with a Hamiltonian system with constraints, which correspond to initial values. Choosing these initial values the Hamiltonian of the latter system has exactly the form of the remaining part of the Wheeler-DeWitt equation (29), where the initial values take the place of the eigenvalue of the Casimir operator. The Liouville Hamiltonian is interpreted to be
the Hamiltonian of a relativistic particle which moves in a potential. Formally, there are three cases to be distinguished depending on the sign of the constant $\lambda$ for the initial values. If $\lambda > 0$, the particle is affected by an infinitely high potential barrier, when it travels in the negative $f$-direction. For $\lambda = 0$ the particle is free, and for $\lambda < 0$ an infinitely deep potential valley attracts the particle towards the negative $f$-direction. This interpretation is in agreement with the group theoretical facts. $\lambda > 0$ corresponds to $\lambda = s(1 - s) = \frac{1}{4} + t^2$ and these are the values of the continuous series, which lead to scattering states. $\lambda < 0$ means $\lambda = n(1 - n), n \in \mathbb{Z}/2, n > 1$, that is, it is an element of the discrete series of representation theory. Bound states are obtained. Here $\lambda \geq 0$ holds, because the Casimir operator can be shown to be the square of the Taub-NUT charge and the Schwarzschild mass operator. If the Casimir operator on the group would no longer be positive semi-definite, one would expect the discrete series of representations to play a role. The spectra of the physical would have a discrete part. However, in the case of stationary spherically symmetric gravity this argumentation is certainly not more than a consistency check: here the coordinate $f$ has no intrinsic group theoretical meaning in contrary to the situation which occurs in the Liouville model. It is a “gravitational remnant”.

Another important difference is that Fülöp considers the momenta of the particle to be fundamental. In particular they become hermitian operators. Here, as the physical meaning is contained in the currents, there is no reason for the momenta to become hermitian operators. Moreover it turns out that starting with hermitian momenta the currents are non-hermitian. On the other hand, the conditions that $\hat{J}^0, \hat{J}^+$ and $\hat{J}^-$ are hermitian defines these operators to be Lie derivatives. This also fixes the operator ordering of the Laplacian and therefore of the Hamiltonian, too.

4 Results and Discussion

After proper dimensional reduction the spherically symmetric sector of Einstein’s vacuum theory can be identified with a SL(2,IR)/SO(2) sigma model coupled to a gravitational remnant which belongs to a gauge degree of freedom. The “true” dynamical degrees of freedom are entirely hidden in the sigma model. In addition to the Schwarzschild mass $m$ another parameter of the classical space of solutions – the Taub-NUT charge $l$ – shows up. The invariance of the Lagrangian under the group SL(2,IR) offers the key to quantize this part of gravity. In a modified Hamiltonian formalism the invariant differential operator on the group – the Casimir operator – appears quite naturally in the Hamiltonian constraint known as Wheeler-DeWitt equation. It is possible to diagonalize the Casimir operator and the mass operator or
the Taub-NUT charge operator simultaneously. The part of the eigendistributions not increasing more rapidly then a polynomial yields a spectral decomposition of the Laplacian. In the case of the mass operator one finds essentially associated Legendre polynomials, and in the case of the charge operator there appear $K$-Bessel functions. It was emphasized that the parametrization of the fields or in other words the chosen representation space plays quite an important role. The Plancherel formula of $\text{SL}(2,\mathbb{R})$ which allows to decompose each function on the group into irreducible parts can be applied to the coset space $\text{SL}(2,\mathbb{R})/\text{SO}(2)$. In the case of group $\text{SL}(2,\mathbb{R})$ the continuous and the discrete series support the Plancherel measure whereas on the coset space one is left with one part of the continuous series. In the literature were noted some difficulties with the self-adjointness of the observables. This may be due to the restriction to the quantization of the Schwarzschild solution only, which corresponds to a diagonal matrix $\chi$. Then one is forced to use the invariant measure of $\mathbb{R}_+$, instead of the Plancherel measure on $\text{SL}(2,\mathbb{R})$. Probably the results of this paper indicate that sectors of gravity can only be quantized consistently, if one takes into account the whole classical space of solutions belonging to it.

The group theoretical methods are not limited to the situation here. They are applicable to various other models in conformal field theory, string theory, quantum cosmology and supergravity. A list of models can be found in [6].

5 Appendix A: Some remarks about the Iwasawa decomposition of a general group $G$

For an arbitrary group element of one of the classical Lie groups, which is written as a matrix $M$, with $\det M \neq 0$, there exists an algorithm for the Iwasawa decomposition. Two numerical methods can be used, the Householder transformation or the Gram-Schmidt orthonormalisation method. Basically they work as follows [29]: Multiplying $M$ from the left by a unitary matrix $Q$, an upper triangular matrix $R$

\[
R = \begin{pmatrix}
  r_{11} & \cdots & r_{1n} \\
  & \ddots & \vdots \\
  0 & & r_{nn}
\end{pmatrix}
\]
is obtained (QR decomposition). $R$ is further decomposed into a diagonal matrix $A$ and a strictly upper triangular matrix $N$ with ones on the diagonal:

$$A = \begin{pmatrix} r_{11} & 0 \\ \vdots & \ddots \\ 0 & \cdots & r_{nn} \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & \frac{r_{12}}{r_{11}} & \cdots & \frac{r_{1n}}{r_{11}} \\ \frac{r_{11}}{\cdot} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \frac{r_{n-1n}}{\cdot} \\ 0 & \cdots & 1 & \frac{r_{n-1n}}{r_{n-1n-1}} \end{pmatrix}.$$  

The Gram-Schmidt orthonormalisation method directly yields the factor $K = (Q^\dagger)^*$. In the case of the Householder transformation one should remind that $Q$ is the product of reflections with $\det Q = -1$. Therefore if the rank of $M$ is even one has to multiply one row or one column by $-1$ in order to get a rotation. For computational convenience the parametrization should be chosen according to the factors of the Iwasawa decomposition.

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