Note About Equivalence of $F(\tilde{R})$ and Scalar Tensor Hořava-Lifshitz Gravities

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ABSTRACT: In this note we study the relation between $F(\tilde{R})$ and scalar tensor Hořava-Lifshitz gravity. We find that due to the broken diffeomorphism invariance corresponding scalar tensor theory has more complicated form than in case of the full diffeomorphism invariant $F(R)$ theory of gravity. We also show that in the low energy limit this theory flows to the relativistic scalar tensor theory of gravity.

KEYWORDS: Hořava-Lifshitz gravity, F(R) gravity
1. Introduction and Summary

In 2009 Petr Hořava formulated new proposal of quantum theory of gravity (now known as Hořava-Lifshitz gravity (HL gravity) that is power counting renormalizable\(^1\) that is also expected that it reduces to General Relativity in the infrared (IR) limit. The HL gravity is based on an idea that the Lorentz symmetry is restored in IR limit of given theory while it is absent in its high energy regime. For that reason Hořava considered systems whose scaling at short distances exhibits a strong anisotropy between space and time,

\[ x' = l x, \quad t' = l^z t. \]  

(1.1)

In \((D+1)\) dimensional space-time in order to have power counting renormalizable theory requires that \(z \geq D\). It turns out however that the symmetry group of given theory is reduced from the full diffeomorphism invariance of General Relativity to the foliation preserving diffeomorphism

\[ x'^i = x^i + \zeta^i(t, x), \quad t' = t + f(t). \]  

(1.2)

The HL gravity was then generalized to the case of \(F(\tilde{R})\) HL gravities in series of papers\(^2\). \(F(\tilde{R})\) HL gravity can be considered as natural generalization of covariant \(F(R)\) gravity. Current interest to \(F(R)\) gravity is caused by several important reasons. First of all, it is known that such theory may give the unified description of the early-time inflation and late-time acceleration (for a review, see\(^3\)). Moreover, the whole sequence of the universe evolution epochs: Inflation, radiation/matter dominance and dark energy may be obtained within such theory. The remaining freedom in the choice of \(F(R)\) function could be used for fitting the theory with observational data. Second, it is known that higher derivatives gravity (like \(R^2\)-gravity, for a review, see\(^4\)) has better ultraviolet behavior than conventional General Relativity. Third, modified gravity is pretending also to be the gravitational alternative for Dark Matter. Fourth, it is expected that consistent quantum

\(^1\)For review and extensive list of references, see\(^{1, 8, 8, 7}\).

\(^2\)For further study in given direction, see\(^{14, 15}\), and for review, see\(^{16}\).
gravity emerging from string/M-theory should be different from General Relativity. Hence, it should be modified by fundamental theory. Of course, all these reasons remain to be the same also for the HL gravity.

It is well known that the $F(R)$ gravity is equivalent to the scalar tensor theory of gravity\(^3\). The scalar tensor theory of gravity has relatively simple form corresponding to the General Relativity action coupled with the scalar field with specified form of the potential term. In particular, the presence of the additional scalar mode in $F(R)$ theory of gravity is clearly seen in the scalar tensor description of the $F(R)$ theory of gravity. Further, the properties of this scalar mode can be transparently studied in this formulation as well. On the other hand it turns out that it is sometimes useful to use the original form of $F(R)$ theories of gravity, as for example for the analysis of the cosmological solutions.

Successes of the equivalence between $F(R)$ gravity and scalar tensor theory of gravity naturally implies the question whether there exists similar equivalence between $F(\tilde{R})$ HL gravity and scalar tensor version of the HL gravity. Examples of the scalar tensor HL gravities were introduced in \([21, 22, 23, 24, 25]\) where the authors analyzed the cosmological consequences of HL gravities\(^4\). The goal of this note is to understand the relation between these scalar tensor HL gravities and $F(\tilde{R})$ gravities. Our procedure is similar as in case of $F(R)$ theories of gravity. We start with the $F(\tilde{R})$ HL gravity action and introduce two auxiliary scalar fields in order to rewrite it into Jordan-like form. Then we use the anisotropic conformal transformation of the metric components in order to map this form of the action to the action where the kinetic term has the canonical form. By canonical form of the kinetic term we mean that it has the same form as the kinetic term in HL gravity that is formulated in ADM formalism\(^5\), for review, see \([28, 29]\). Now due to the fact that $F(\tilde{R})$ gravity is not fully diffeomorphism invariant we find that the resulting theory takes more general form of the scalar tensor theory. We also show that this theory flows to the relativistic scalar tensor theory of gravity in the low energy limit.

We hope that our result can be useful for further analysis of the properties of $F(\tilde{R})$ HL gravities. For example, the scalar tensor form of $F(\tilde{R})$ theory can be useful for the analysis of the fluctuations around cosmological solutions of $F(\tilde{R})$ HL gravities. We hope to return to these problems in future.

This paper is organized as follows. In the next section (2) we introduce $F(\tilde{R})$ HL theories of gravity and map them to generalized scalar tensor theories of gravity. We also demonstrate that these theories flow to standard scalar tensor theories of gravity in its low energy region. In Appendix (A) we review the standard equivalence between $F(R)$ theory of gravity and scalar tensor theory of gravity. We perform this analysis in the ADM formalism in order to compare this result with the analysis performed in the main body of the paper.

\section{2. $F(\tilde{R})$ HL Gravity in Einstein Frame}

We begin this section with the review of basic properties of $F(\tilde{R})$ HL gravity. Our con-

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\(3\)For review and extensive list of references, see \([11, 12, 13, 16]\).

\(4\)For further study, see \([31, 32, 33]\).
vension is as follows. We consider $D + 1$ dimensional manifold $\mathcal{M}$ with the coordinates $x^\mu, \mu = 0, \ldots, D$ and where $x^\mu = (t, \mathbf{x}) = (x^1, \ldots, x^D)$. We presume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-, +, \ldots,)$. Suppose that $\mathcal{M}$ can be foliated by a family of space-like surfaces $\Sigma_t$ defined by $t = x^0$. Let $g_{ij}, i, j = 1, \ldots, D$ denote the metric on $\Sigma_t$ with inverse $g^{ij}$ so that $g_{ij}g^{jk} = \delta^i_k$. We further introduce the operator $\nabla_i$ that is covariant derivative defined with the metric $g_{ij}$. We introduce the future-pointing unit normal vector $n^\mu$ to the surface $\Sigma_t$. In ADM variables one has $n^0 = \sqrt{-\hat{g}^{00}}, n^i = -\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables we write the components of the metric $\hat{g}_{\mu\nu}$ as

\[
\begin{align*}
\dot{g}_{00} &= -N^2 + N_i g^{ij}N_j, \quad \dot{g}_{0i} = N_i, \quad \dot{g}_{ij} = g_{ij}, \\
\dot{g}^{00} &= -\frac{1}{N^2}, \quad \dot{g}^{0i} = \frac{N^i}{N^2}, \quad \dot{g}^{ij} = g^{ij} - \frac{N^iN^j}{N^2}.
\end{align*}
\] (2.1)

We further define the extrinsic derivative

\[K_{ij} = \frac{1}{2N}(\partial_\lambda g_{ij} - \nabla_i N_j - \nabla_j N_i).\] (2.2)

The general formulation of Hořava-Lifshitz $F(\tilde{R})$ gravity was presented in series of papers in [14, 15]. The action introduced in [14] takes the form

\[S_{F(\tilde{R})} = \zeta^2 \int dt d^Dx \sqrt{\hat{g}} N F(\tilde{R}),\] (2.3)

where

\[\tilde{R} = K_{ij}G^{ijkl}K_{kl} + \frac{2\mu}{\sqrt{-\hat{g}}} \partial_\mu \left( \sqrt{-\hat{g}}n^\mu K \right) - \frac{2\mu}{\sqrt{-\hat{g}}} \partial_i \left( \sqrt{-\hat{g}}g^{ij} \partial_j N \right) - V(g),\] (2.4)

where $\mu$ is constant, $K = K_{ij}g^{ji}$ and where the generalized de Witt metric $G^{ijkl}$ is defined as

\[G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl},\] (2.5)

where $\lambda$ is real constant that is believed that it flows to 1 in its low energy limit. More precisely we presume that $F(R)$ gravity is recovered in the limit $\lambda \to 1, \mu \to 1$ and $\zeta^2 \to (16\pi G)^2$. Finally $V(g)$ depends on $g_{ij}$ and its covariant derivatives whose explicit form was suggested in [10]. Our goal is to map the action (2.3) to the scalar tensor form of HL gravity. The first step is to rewrite the action (2.3) into an equivalent form

\[S_{F(\tilde{R})} = \zeta^2 \int dt d^Dx \left( \sqrt{\hat{g}} N B(K_{ij}G^{ijkl}K_{kl} - V(g) - A) + \sqrt{\hat{g}} N F(A) - 2\mu \sqrt{\hat{g}} N \nabla_n BK + 2\mu \partial_i B \sqrt{\hat{g}} g^{ij} \partial_j N \right),\] (2.6)

\[\text{For further study in given direction, see [13, 14], and for review, see [10].}\]
where
\[ \nabla_n X = \frac{1}{N} (\partial_t X - N^i \partial_i X) . \] (2.7)

Then from (2.6) we find the equation of motion for \( A \)
\[ -B + F'(A) = 0 . \] (2.8)

Assuming an existence of the inverse function \( \Psi \) defined as \( \Psi(F')(A) = A \) we would be able to determine \( A \) as a function of \( B \)
\[ A = \Psi(B) . \] (2.9)

Inserting this result into the action (2.6) we obtain
\[ S_{F(\tilde{R})} = \zeta^2 \int dt d^3x \left( \sqrt{g} NB(K_{ij}G^{ijkl}K_{kl} - V(g)) - \sqrt{g} NV(B) - 2\mu \sqrt{g} N \nabla_n B K + 2\mu \partial_i B \sqrt{g} g^{ij} \partial_j N \right) , \] (2.10)

where
\[ V(B) = B \Psi(B) - F(\Psi(B)) . \] (2.11)

Let us now consider following anisotropic Weyl transformation
\[ N' = \Omega^\omega N , \quad N'_i = \Omega^2 N_i , \quad g_{ij} = \Omega^2 g_{ij} , \] (2.12)

where \( \omega \) is free parameter whose value will be specified below. It is easy to see that the spatial connection
\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \] (2.13)
transforms under (2.12) as
\[ \Gamma'^k_{ij} = \Gamma^k_{ij} + \frac{1}{\Omega^2} (\delta^k_i \partial_j \Omega + \delta^k_j \partial_i \Omega - g^{kl} \partial_l \Omega g_{ij}) \] (2.14)

and the extrinsic curvature transforms as
\[ K'_{ij} = \Omega^{2-\omega} K_{ij} + \Omega^{1-\omega} \nabla_n \Omega g_{ij} . \] (2.15)

In the same way we find
\[ \nabla'_n B = \frac{1}{\Omega^\omega} \nabla_n B , \quad K' = \frac{K}{\Omega^\omega} + D \frac{\nabla_n \Omega}{\Omega^{1+\omega}} . \] (2.16)
Then it is easy to see that the kinetic part of the action (2.10) transforms as

\[
\zeta^2 \int dt d^Dx \sqrt{g} [BK_{ij}G^{ijkl}K_{kl} - 2\mu K \nabla_n B + \frac{2\mu}{N} g^{ij} \partial_j B \partial_i N] \rightarrow
\]

\[
\zeta^2 \int dt d^Dx \sqrt{g} \Omega^{D+\omega} \left[ \frac{1}{\Omega^{2\omega}} K_{ij} G^{ijkl} K_{kl} + \frac{2}{\Omega^{1+2\omega}} (1 - \lambda D) K \nabla_n \Omega + \frac{1}{\Omega^{2+2\omega}} \nabla_n \Omega \nabla_n \Omega (1 - \lambda D) D - 2\mu \frac{1}{\Omega^{2\omega} B} \nabla_n BK - 2\mu \frac{1}{\Omega^{1+2\omega} B} D \nabla_n B \nabla_n \Omega + \frac{2\omega \mu}{B \Omega^{2\omega} N} \partial_i B g^{ij} \partial_j \Omega + \frac{2\mu}{B \Omega^{2\omega} N} \partial_i B g^{ij} \partial_j N \right].
\]

(2.17)

Our goal is to choose \( \Omega \) in such a way so that the kinetic term takes the canonical form as in the scalar tensor HL gravity. The requirement implies following relation between \( \Omega \) and \( B \)

\[
\Omega = B^{\frac{1}{\omega - D}}
\]

(2.18)

With such a form of \( \Omega \) the expression (2.17) simplifies as

\[
\zeta^2 \int dt d^Dx \sqrt{g} \left[ K_{ij} G^{ijkl} K_{kl} + \frac{2}{B(\omega - D)}(1 - \lambda D) K \nabla_n B + \frac{(1 - \lambda D) D}{(\omega - D)^2 B^2} \nabla_n B \nabla_n B - 2\mu \frac{1}{B} \nabla_n BK - 2\mu \frac{D}{(\omega - D) B^2} \nabla_n B \nabla_n B + \frac{2\omega \mu}{(\omega - D) N} B^{\frac{2D-2}{\omega - D}} \partial_i B g^{ij} \partial_j B + \frac{2\mu}{N} B^{\frac{D+\omega-2}{\omega - D}} \partial_i B g^{ij} \partial_j N \right].
\]

(2.19)

Note that this expression is not well defined for \( \omega = z = D \) where \( z \) is the scaling dimension. This follows from the fact that the kinetic term of the HL gravity is invariant under anisotropic scaling transformation when \( \omega = D \).

Now we come to the analysis of the potential term. We consider the SVW potential term

\[
V(g) = g_1 R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij}) + \frac{1}{\zeta^4} (g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R_j^i R_k^j R_k^i) + \frac{1}{\zeta^6} [g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk})(\nabla^i R^{jk})] + \ldots,
\]

(2.20)

where the coupling constants \( g_s, (s = 0, 1, 2, \ldots) \) are dimensionless and \ldots corresponds to the higher order terms corresponding to the fact that the critical dimension of \( D \)--dimensional HL gravity is \( z = D \). The relativistic limit in the IR requires \( g_1 = -1 \) and \( \zeta^2 = (16\pi G)^{-2} \).

\[\text{For simplicity we consider the potential term without cosmological constant contribution.}\]
We note that under transformations (2.12) the components of $D-$dimensional Ricci tensor transforms as

$$R'_{ij} = R_{ij} + 2(D - 2)\frac{1}{\Omega^2} \nabla_i \Omega (\nabla_j \Omega) - (D - 2)\frac{1}{\Omega} \nabla_i \nabla_j \Omega + (3 - D)g_{ij} \frac{\nabla_k \Omega \nabla^k \Omega}{\Omega^2} - g_{ij} \frac{\nabla_k \nabla^k \Omega}{\Omega}$$

while $D-$dimensional scalar curvature transforms as

$$R' = \Omega^{-2} \left( R - 2(D - 1)g^{ij} \nabla_i \nabla_j \Omega + (D - 1)(4 - D)\frac{\nabla_i \Omega \nabla^i \Omega}{\Omega^2} \right).$$

To proceed further it is useful to separate the contribution proportional to $R$ in (2.20) so that we rewrite the potential (2.20) in the form

$$V(g) = g_1 R + \tilde{V}(g).$$

Then the contribution proportional to $R$ given in (2.23) transforms under (2.12) as

$$-\zeta^2 g_1 \int dt d^D x \sqrt{g} N B R \rightarrow -\zeta^2 g_1 \int dt d^D x N \sqrt{g} [\frac{2\omega - 2}{\omega - D} B^{\frac{\omega + D}{2}} R + \frac{2(D - 1)}{\omega - D} B^{\frac{\omega - 2D}{2}} \partial_i g^{ij} \partial_j B + \frac{(D - 1)(4 - D)}{(\omega - D)^2} B^{\frac{\omega - 2D}{2}} \partial_i B \nabla^i B] .$$

Using this result together with (2.17) and also with $\sqrt{g} N V \rightarrow \sqrt{g} N B^{\frac{\omega + D}{2}} V(B)$ we find following form of the transformed action

$$\zeta^2 \int dt d^D x \sqrt{g} N \left[ K_{ij} G^{ijkl} K_{kl} + \frac{2}{B(\omega - D)} (1 - \mu \omega) + D(1 - \lambda) K \nabla_n B - \frac{2\mu}{B} \nabla_n B K + \frac{D}{(\omega - D)^2 B^2} (1 - 2\mu \omega + D(2\mu - \lambda)) \nabla_n B \nabla_n B + B^{\frac{2D - 2}{\omega - D}} \frac{2\omega \mu (\omega - D) - g_1 (D - 1)(4\omega - 2 - D)}{(\omega - D)^2} \partial_i B g^{ij} \partial_j B + \frac{2\mu (\omega - D) - 2g_1 (D - 1)}{(\omega - D) N} \partial_i B g^{ij} \partial_j N - g_1 B^{\frac{2\omega - 2}{\omega - D}} R(g) - \tilde{V}'(g, B) - B^{\frac{\omega + D}{2}} V(B) \right],$$

where $\tilde{V}'$ depends explicitly on $B$ through the relations (2.15), (2.21) and (2.22). It is important to stress that $\omega$ is free parameter whose value should be determined by requirement that in the low energy limit when we can neglect the contribution from the potential $\tilde{V}$ and when $\mu \rightarrow 1$, $\lambda \rightarrow 1$, $g_1 \rightarrow -1$ the action (2.25) flows to relativistic form of scalar tensor
theory. This requirement immediately implies that \( \omega \) should be equal to 1. Then the final form of the generalized scalar tensor HL gravity action (2.25) takes the form

\[
S = \zeta^2 \int d^Dx \sqrt{g} N \left[ K_{ij} K^{ij} - \frac{2}{B(1 - D)} \left( (1 - \mu) + D(1 - \lambda) \right) K \nabla_i B + \frac{D}{(1 - D)^2 B^2} (1 - 2\mu + D(2\mu - \lambda)) \nabla_i B \nabla_i B + \frac{2\mu(1 - D) - g_1(D - 1)(2 - D)}{(1 - D)^2 B^2} \partial_i B g^{ij} \partial_j B + \frac{2\mu(1 - D) - 2g_1(D - 1)}{(1 - D) N} \partial_i B g^{ij} \partial_j N - g_1 R(g) - \tilde{V}'(g, B) - B^{1+D} \tilde{V}(B) \right] ,
\]

(2.26)

where the potential \( \tilde{V}'(g, B) \) depends on \( B \) and \( g_{ij} \) as follows from the fact that it arises from the original potential \( \tilde{V} \) through the anisotropic conformal transformation. Explicitly, the transformation (2.12) implies following transformation rule for the spatial Ricci tensor

\[
R'_{ij} = R_{ij} + \frac{2(D - 2)}{(1 - D)^2 B^2} (\nabla_i B)(\nabla_j B) - \frac{(D - 2)}{1 - D} B \nabla_i \nabla_j B + \frac{(3 - D)}{(1 - D)^2} g^{ij} \nabla_k B \nabla^k B - \frac{1}{1 - D} g^{ij} \frac{\nabla_k \nabla^k B}{B} .
\]

(2.27)

It is also important to stress that the covariant derivative depends on \( B \) as well. As a result the potential term \( \tilde{V}' \) will give the contributions proportional to the higher order spatial derivatives (up to order \( z \)) of the field \( B \) which is the consequence of the anisotropic scaling in \( F(\tilde{R}) \) HL gravity. Note also that is convenient to formulate given theory in the canonical form when we introduce the scalar field \( \phi \) that is related to \( B \) through the relation

\[
B = e^{\Sigma \phi} , \quad \Sigma = \frac{1}{\sqrt{2}} \frac{\sqrt{D(1 - 2\mu + D(2\mu - \lambda))}}{(D - 1)} .
\]

(2.28)

The action (2.26) is the final result of our analysis. It is useful to compare this action (when we replace \( B \) with \( \phi \) given by (2.28)) with the form of the scalar tensor HL gravity

\[
S = \zeta^2 \int d^Dx \sqrt{g} N \left[ K_{ij} G^{ijkl} K_{kl} - \mathcal{V}(g) + \mathcal{L}_{\text{scal}} \right] ,
\]

(2.29)

where the Lagrangian density for the scalar field has the form

\[
\mathcal{L} = \frac{1}{2} \nabla_i \phi \nabla_i \phi - G(g^{ij} \partial_i \phi \partial_j \phi) - V(\phi) ,
\]

(2.30)

where \( V(\phi) \) is the general potential for the scalar field and where \( G \) is the polynomial in its argument up to the \( z \)-th order. We see that the structure of the action (2.26) is

\footnote{The explicit form of the scalar tensor theory written in ADM formalism is given in Appendix.}
more complicated but it can be again written as the polynomial in $g^{ij}\partial_i\phi\partial_j\phi$ where now the coefficients generally depend on $g_{ij}$ and $R_{ij}$. Finally, the action \[2.23\] contains the coupling between extrinsic curvature and the scalar field $\phi$. Note however that the low energy limit of \[2.23\] where $\lambda, \mu \to 1$, $g_1 \to -1$, $\tilde{V}' \to 0$ takes the form

$$S_{s.t.} = \frac{1}{(16\pi G)^2} \int dtdx \sqrt{g} N \left[ K_{ij} K^{ij} - K^2 - \frac{D}{(1-D)B^2} \nabla_n B \nabla_n B + \frac{D}{(1-D)B^2} \partial_i B g^{ij} \partial_j B + R(g) - B^{1+D} V(B) \right].$$

(2.31)

In Appendix we review the equivalence between $F(R)$ gravity and the scalar tensor theory written in ADM formalism and we show that the action \[2.31\] exactly coincides with the scalar tensor gravity action.

Let us conclude our result. We show that the action for the scalar tensor formulation of $F(\tilde{R})$ HL gravity is much more complicated than in case of the $F(R)$ gravity. In other words there is no straightforward correspondence between $F(\tilde{R})$ and scalar tensor HL gravities written in their simplest form. For that reason we mean that it is more natural to study $F(\tilde{R})$ HL gravities directly without reference to their scalar tensor images.

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A. Appendix: Equivalence between $F(R)$ Gravity and Scalar Tensor Theory written in ADM Formalism

In this Appendix we review the well known equivalence between $D + 1$ dimensional $F(R)$ theory of gravity and corresponding scalar tensor theory\(^8\). We perform this analysis when the $F(R)$ action is formulated in ADM formalism in order to see the relation with the result derived in the main body of the paper. For that reason we consider following form of $F(R)$ gravity action

$$S_{F(R)} = \frac{1}{(16\pi G)^2} \int dtdx \left[ \sqrt{g} N B (K_{ij} g^{ijkl} K_{kl} - R) - \sqrt{g} N V(B) - 2\sqrt{g} N \nabla_n B K + 2\partial_i B \sqrt{g} g^{ij} \partial_j N \right],$$

(A.1)

where we implicitly integrated out the scalar field $A$ so that the potential $V(B)$ takes the same form as in \[2.11\]. As usual in order to find Einstein frame form of the action \[A.1\] we perform the conformal rescaling of metric components

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

(A.2)

that in $D + 1$ decomposition takes the form

$$N' = \Omega N, \quad N'_i = \Omega^2 N_i, \quad g_{ij} = \Omega^2 g_{ij}$$

(A.3)

\(^8\)For nice discussion, see [24, 27].
which is the special case of the transformation (2.12) for $\omega = 1$. Following the same analysis as in previous section and choosing $\Omega = B^{1-D}$ we easily find that the kinetic term of the $F(R)$ gravity action (A.1) transforms as

$$
\frac{1}{(16\pi G)^2} \int dt d^Dx \sqrt{\tilde{g}} N \left[ K_{ij}K^{ij} - K^2 - 2K \nabla_n B \right] \to 
\frac{1}{(16\pi G)^2} \int dt d^Dx \sqrt{\tilde{g}} N \left[ K_{ij}K^{ij} - K^2 - \frac{D}{1 - D} \frac{1}{B^2} \nabla_n B \nabla_n B \right]
$$

(A.4)

while the potential term transforms as

$$
\frac{1}{(16\pi G)^2} \int dt d^Dx \left[ -\sqrt{\tilde{g}} NR + 2\partial_i B \sqrt{\tilde{g}} g^{ij} \partial_j N - \sqrt{\tilde{g}} N V(B) \right] \to 
\frac{1}{(16\pi G)^2} \int dt d^Dx \left[ -\sqrt{g} R(g) + \frac{D}{1 - D} \sqrt{\tilde{g}} N \frac{1}{B^2} \nabla_i B \nabla^i B - \sqrt{\tilde{g}} N B^{1+D} V(B) \right].
$$

(A.5)

Collecting (A.4) and (A.5) together we obtain following form of the scalar tensor theory of gravity action

$$
\frac{1}{(16\pi G)^2} \int dt d^Dx \sqrt{\tilde{g}} N \left[ K_{ij}K^{ij} - K^2 - R(g) - \frac{D}{1 - D} \frac{1}{B^2} \nabla_n B \nabla_n B + \frac{D}{1 - D} \frac{1}{B^2} \nabla_i B \nabla^i B - B^{1+D} V(B) \right].
$$

(A.6)

In order to find more familiar form of the action (A.6) it is convenient to perform the substitution

$$
B = \exp \frac{1}{\sqrt{2}} \frac{\sqrt{D - 1}}{\sqrt{D}} \phi.
$$

(A.7)

Note that (2.28) reduces to (A.7) in the limit when $\mu \to 1$, $\lambda \to 1$. Using (A.7) we can rewrite the action (A.6) into the covariant form (24, 27)

$$
\frac{1}{(16\pi G)^2} \int d^{(D+1)}x \sqrt{-\hat{g}} \left[ R(\hat{g}) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V'(\phi) \right],
$$

(A.8)

where

$$
V'(\phi) = \exp \left( \frac{1 + D}{\sqrt{2} \sqrt{(D - 1)D}} \phi \right) V(\phi).
$$

(A.9)
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