Formation of Gapless Phases of $K^0$ Condensed Color-Flavor Locked Superconducting Quark Matter

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Abstract

Electric and color neutral solutions, and the critical conditions for the formation of gapless color superconductors, are investigated in $K^0$ condensed color-flavor locked quark matter for nonzero strange quark mass. We show that as the strange quark mass increases, gapless modes for up-strange quark pairing occur first, followed by down-strange quark pairing. The behavior of the gaps, the dispersion relations, and the thermodynamic potential are all found as functions of the strange quark mass on the basis of a Nambu Jona-Lasinio type model. To a high degree of accuracy, they are presented as relatively simple elementary functions. This allows for easy computation for any reasonable range of baryon chemical potential and strange quark mass.

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1 Introduction

Dense quark matter has been an exciting topic for theoretical study ever since it was realized that quark color superconducting phases could form the lowest energy states and could be reliably computed at very high density within QCD. For three flavors, the original color and flavor $SU(3)_{\text{color}} \times SU(3)_L \times SU(3)_R$ symmetries of QCD are broken down to the diagonal subgroup $SU(3)_{\text{color} + L + R}$ at very high baryon density [1]. Quark matter with such a particular symmetry pattern is called color-flavor locked (CFL) matter and is widely believed to be the densest phase of strongly interacting matter [2]. In the situation where the strange quark mass $m_s$ is large and/or the quark chemical potential $\mu$ (which is 1/3 of the baryon chemical potential) is small, the so-called gapless color-flavor locked phase (gCFL) has been predicted [3, 4, 5]. Similar to the gapless phase of two-flavor color superconductor [6], gCFL is triggered by mismatches between the chemical potentials for paired quasi-quarks. There the mismatch in the flavor (say $d$ and $s$) chemical potentials is characterized directly by the quantity $m_s^2/2\mu$ while that in the color (say $b$ and $g$) chemical potentials is related to $m_s^2/2\mu$ via the electric/color neutrality in CFL matter [4]. As long as $m_s^2/2\mu$ is large enough, there exist unpaired (gapless) modes for $bd - gs$ pairing, and the resulting gCFL phase becomes energetically favorable. Recently the Meissner masses for some gluons in this phase were found to be imaginary so that gCFL is actually unstable in the chromomagnetic sense [7, 8, 9]. The presence of a chromomagnetic instability is a serious problem. Hence the gapless phase of CFL matter needs to be studied further.

There is another kind of less-symmetric phase of CFL matter that was predicted when $m_s \neq 0$. At the leading order, it is convenient to regard the effect of $m_s^2/2\mu$ as an effective chemical potential associated with strangeness, namely

$$\mu_S = \frac{m_s^2}{2\mu}. \quad (1)$$

As one of the pseudo-Goldstone bosons related to excitations of the CFL superconducting quark matter (not to be confused with excitations of the QCD vacuum with the same quantum numbers), the neutral kaon mode $K^0$ has a chemical potential $\mu_{K^0} = \mu_S$. The mass of this mode, ignoring
instanton effects, is [10]

\[
m_K = \frac{\Delta}{\pi f_\pi} \sqrt{3m_u(m_s + m_d)}
\]  

(2)

where \( \Delta \) is the superconducting gap, \( m_u, m_d \) and \( m_s \) are the quark masses, and \( f_\pi \) is the pion-decay constant in the CFL matter. The latter is given by [11]

\[
f_\pi^2 = \frac{21 - 8 \ln 2}{36\pi^2} \mu^2 \approx (0.2086\mu)^2.
\]

(3)

It measures the strength of the coupling of the pion (in CFL matter) to the axial vector current. When \( \mu_S \) reaches the mass of the neutral kaon-mode, the \( K^0 \) condenses in CFL matter [10, 12]. This phase is referred to as CFL\( K^0 \).

Assuming \( m_d \ll m_s \), the condition that no \( K^0 \) condensation takes place is

\[
m_s < 3.034 \left( m_u \Delta^2 \right)^{1/3} = 44.4 \left[ \left( \frac{m_u}{5 \text{ MeV}} \right) \left( \frac{\Delta}{25 \text{ MeV}} \right)^2 \right]^{1/3} \text{ MeV}.
\]

(4)

Taking typical current-quark values of the masses, namely \( m_u = 5 \text{ MeV} \) and \( m_s = 110 \text{ MeV} \), kaon condensation will occur for reasonable values of the gap. In this paper we will generally ignore the up and down quark masses in comparison with the strange quark mass.

Starting with the ideal situation of \( m_s = 0 \) and raising \( m_s^2/2\mu \) gradually, one expects the CFL matter to be disrupted, at first by the presence of \( K^0 \) condensation, and then by the appearance of gapless modes. In the CFL matter with \( K^0 \) condensation, the formation of gapless modes has been reexamined by using an effective theory in Ref. [13, 14] and by using the NJL model in Refs. [15, 16]. It was found that gapless pairing is delayed with respect to that in the conventional CFL matter. More recently, it has been suggested that the chromomagnetic instability in the gapless phase can be resolved by the formation of the so-called kaon supercurrent state [17].

Motivated by these results, we investigate the formation of gapless modes in the CFL\( K^0 \) environment. First, we consider the electric/color neutral solution of CFL\( K^0 \) in the presence of an electron chemical potential and point out that the above-mentioned delay is a direct consequence of the deviation of the CFL\( K^0 \) neutral solution from the CFL one. Second, it is found that gapless modes for \( bu - rs \) pairing occur first, followed by \( bd - gs \) in the \( K^0 \) condensed environment. Based on this feature, the resulting gapless phase
(termed as gCFL below) is studied, including its electric/color neutrality and gap variation. The solutions can be found analytically to very good approximation, obviating the need for a complicated numerical minimization of the thermodynamic potential with respect to three chemical potentials plus three gaps. Finally, the stability of the gCFL phase is examined qualitatively; it is argued that the gCFL instability might be removed either partially or even totally.

2 Electric/Color Neutrality in Superconducting States

At asymptotically high baryon density the gap in CFL matter can be computed using weak coupling methods. The result is

$$\Delta = \frac{512\pi^4}{2^{1/3}} \left(\frac{2}{N_f}\right)^{5/2} \frac{\mu}{g^3} \exp \left[-\frac{(\pi^2 + 4)}{8}\right] \exp \left(-\frac{3\pi^2}{\sqrt{2g}}\right)$$

where $N_f$ is the number of flavors. (In this paper we consider $N_f = 3$ only.) In the asymptotic limit, $\mu \gg \Lambda_{\text{QCD}}$, it is also true that $\mu \gg m_s$ or, equivalently, $\mu_S \ll \Delta$. The only location in the known universe where superconducting quark matter might appear is in neutron stars where the baryon chemical potential is not asymptotically large. Then $\mu_S$ is not negligible, and furthermore weak coupling methods in the gauge coupling $g$ are not quantitatively reliable. Therefore different groups have used other methods to understand what happens in the non-asymptotic regime. These include: (i) variations on the Nambu Jona-Lasinio model, which allow a determination of the gaps as functions of density at the expense of one or more cut-off parameters, (ii) effective Lagrangians employing the relevant collective degrees of freedom and with coefficients matched to QCD, but which assume a given value for the gap, and (iii) so-called model independent approaches that also assume that the gap or gaps are determined by other means and that count states, degrees of freedom, and balance electric and color charges. These different approaches all tend to agree when they have a region of overlap.

Stable bulk matter must be electrically neutral and must have zero net color. Perfect CFL matter with zero quark masses satisfies these requirements without the need for electrons. Alford and Rajagopal [19] addressed this issue
in the context of a nonzero strange quark mass by using an essentially model-independent approach. They introduced chemical potentials \( \mu_e, \mu_3 \) and \( \mu_8 \) for the electrons (or negative of the electric charge \( Q \)) and for the color generators \( T_3 \) and \( 2T_8/\sqrt{3} \), respectively. (The factor of \( 2/\sqrt{3} \) arises because they used a non-standard form of the generator \( T_8 \) which was more convenient for their purposes.) Different flavors and colors of quarks then have different chemical potentials; see Table I. The thermodynamic potential \( \Omega \) must satisfy

\[
\frac{\partial \Omega}{\partial \mu_e} = \frac{\partial \Omega}{\partial \mu_3} = \frac{\partial \Omega}{\partial \mu_8} = 0.
\]

They found the solution

\[
\mu_3 = \mu_e, \quad \mu_8 = \frac{1}{2} \mu_e - \mu_S
\]

to first order in \( \mu_\tilde{Q}/\mu \). (The electron chemical potential is not determined by this calculation.) This solution is equivalent to requiring that the quarks that should pair have the same “Fermi momentum” to the extent that such a quantity has meaning in superconducting matter. To order \( m_s^4 \) (and to zero order in \( g^2 \)) the difference in thermodynamic potential between the CFL phase and ordinary unpaired quark matter is

\[
\Omega_{\text{neutral}}^{\text{CFL}} = \Omega_{\text{neutral}}^{\text{unpaired}} + 3\frac{\mu^2}{4\pi^2} \left( \mu_S^2 - 4\Delta^2 \right).
\]

The CFL phase is preferred over the unpaired phase if the gap is large enough, namely if \( \frac{1}{2} \mu_S < \Delta \). Another important feature of CFL is that its thermodynamic potential is independent of

\[
\mu_{\tilde{Q}} = -\frac{4}{9} \left( \mu_e + \mu_3 + \frac{1}{2} \mu_8 \right),
\]

since none of the CFL pairings break the rotation \( \tilde{Q} = Q - T_3 - \frac{1}{\sqrt{3}} T_8 \). In this sense, CFL matter is not merely an electric insulator [20] but also a \( \tilde{Q} \) insulator [19]. Enforcing neutrality at higher order requires that no electrons be present so that \( \mu_e = 0 \).

As mentioned in the introduction, kaon condensation will occur when the kaon chemical potential equals its mass. (We must keep in mind that familiar names like kaon refer to collective excitations of the color superconducting state and not to excitations of the QCD vacuum.) Since we are taking the up
and down quark masses to be zero (actually only $m_u = 0$ is required in this regard) kaon condensation will always occur, and CFL$K^0$ matter should have a lower free energy than CFL. In the presence of $K^0$ condensation, the ground state of Nambu-Goldstone bosons takes non-trivial values. By treating the gauge fields as dynamical degrees of freedom, Kryjevski has shown that there exist nonzero expectation values for the gluon fields in CFL$K^0$ matter [21]. He found that electric and color neutrality leads to constant time components of two color-diagonal gluon fields, which turn out to be

$$gA_3^0 = \frac{m_u^2}{4\mu}, \quad gA_8^0 = \frac{m_s^2}{4\sqrt{3}\mu}.$$  (10)

Constant values of the time-components of gauge fields play the mathematical role of shifts in the chemical potentials. In this case it corresponds to

$$\mu_3 = -\frac{1}{2}\mu_S, \quad \mu_8 = -\frac{1}{4}\mu_S$$  (11)

when the condition $\mu_e = 0$ is used. (Neutral kaon condensation does not require the presence of electrons for electric neutrality.) These results were confirmed by Kryjevski and Yamada [14], as were the results of Alford and Rajagopal. A comparison of the effective potentials for the quarks in the CFK and CFL$K^0$ states are given in Table I. The thermodynamic potential in the CFL$K^0$ state is lower than that of the CFL state (including the strange quark mass effects, electric and color neutrality) by an amount $\frac{1}{2}f_\pi^2\mu_S^2$ [10, 14]. Together with (8) this means that

$$\Omega_{\text{neutral}}^{\text{CFK}} = \Omega_{\text{neutral}}^{\text{unpaired}} + \frac{3\mu^2}{4\pi^2} \left(\mu_S^2 - 4\Delta^2\right) - \frac{1}{2}f_\pi^2\mu_S^2.$$  (12)

The condition for pairing is now somewhat relaxed to $0.4224\mu_S < \Delta$ as opposed to $0.5\mu_S < \Delta$.

3 Dispersion Relations

The modes of excitation in superconducting quark matter are sometimes labeled by the quark degrees of freedom and sometimes by the baryon degrees of freedom. The correspondence is related to the pairing ansatz

$$\langle \bar{q}_i^\alpha C\gamma_5 q_j^\beta \rangle \sim \Delta_I \epsilon^{\alpha\beta1} \epsilon_{ij1} + \Delta_{II} \epsilon^{\alpha\beta2} \epsilon_{ij2} + \Delta_{III} \epsilon^{\alpha\beta3} \epsilon_{ij3}.$$  (13)
(Here \((i, j)\) and \((\alpha, \beta)\) denote the flavor indices \((u, d, s)\) and the color indices \((r, g, b)\), respectively, and the gap parameters \(\Delta_I, \Delta_{II}\) and \(\Delta_{III}\) are approximately equal to \(\Delta_0\) in the absence of gapless phenomena.) For example, a linear combination of \(\langle ru - gd \rangle bu\) and \(\langle gu - rd \rangle bu\), where the angular brackets denote a pairing, represent an object with baryon number 1, electric charge 1, and zero net color, in other words a proton. Hence a proton is associated with \(bu\) quarks. Similarly the neutron, \(\Sigma^+, \Sigma^-, \Xi^0\) and \(\Xi^-\) are associated with the \(bd, gu, rd, gs\) and \(rs\) quarks, respectively. A \(\Lambda^0, \Lambda^8\), or \(\Sigma^0\) would be represented by some linear combination of \(ru, gd\) and \(bs\) quarks. These associations are given in Table IV.

In the CFL phase with massless up, down and strange quarks the three gaps are equal: \(\Delta_I = \Delta_{II} = \Delta_{III} = \Delta_0\). The dispersion relations for the energy as a function of momentum take the following forms. There are 8 modes with energy

\[
E = \sqrt{(p - \mu)^2 + \Delta_0^2}
\]  

and 1 mode with energy

\[
E = \sqrt{(p - \mu)^2 + (2\Delta_0)^2}.
\]

The mode with gap \(2\Delta_0\) is a mixture of \(ru, gd\) and \(bs\) quarks.

In the CFL phase with strange quarks of nonzero mass the dispersion relations are more complicated because of the necessity of nonzero effective chemical potentials that ensure color neutrality. These are found as the zeros of a determinant of a \(9 \times 9\) matrix [4]. There are a pair of modes associated with each of the pairings \(bd - gs, bu - rs\) and \(gu - rd\) described by the gaps \(\Delta_I, \Delta_{II}\) and \(\Delta_{III}\), respectively. We shall refer to these as the I, II and III channels for brevity. In this phase all three gaps are equal to a common value denoted by \(\Delta\), which in general is not equal to \(\Delta_0\) when the strange quark mass is nonzero. The pair of dispersion relations for channel I can be written as

\[
E = \sqrt{(p - \bar{\mu})^2 + \Delta^2} \pm \mu_S.
\]

Here \(\bar{\mu} \equiv \mu - \frac{1}{3}\mu_S\) is the average chemical potential of the pair of quarks involved and \(\mu_S\) is one-half of the difference of their chemical potentials; see Table II. The same is true in channel II. The pair of dispersion relations for channel III are

\[
E = \sqrt{(p - \bar{\mu})^2 + \Delta^2}.
\]
The other three modes are linear combinations of $ru$, $gd$ and $bs$ quarks. There are 2 modes with energy

$$E = \sqrt{(p - \bar{\mu})^2 + \Delta^2} \quad (18)$$

and 1 mode with energy

$$E = \sqrt{(p - \bar{\mu})^2 + (2\Delta)^2}. \quad (19)$$

In the CFL$K^0$ phase it is sometimes convenient to use the notation of baryons instead of quarks. The translation between the two is given in Table IV. The dispersion relations are different from the CFL matter not only because of the difference in effective chemical potentials but also because of flavor required by the symmetries of QCD [13, 14]. There is a pair of modes in channel I with energy given by

$$E = \sqrt{(p - \bar{\mu})^2 + \Delta^2 \pm \frac{1}{2} \mu_s}. \quad (20)$$

There is a pair of modes in channel II with energy given by

$$E = \sqrt{(p - \left(\bar{\mu} - \frac{1}{4} \mu_s\right))^2 + \Delta^2 \pm \frac{3}{4} \mu_s}. \quad (21)$$

There is a pair of modes in channel III with energy given by

$$E = \sqrt{(p - \left(\bar{\mu} + \frac{1}{4} \mu_s\right))^2 + \Delta^2 \pm \frac{1}{4} \mu_s}. \quad (22)$$

The neutral baryons $n$, $\Xi^0$, $\Sigma^0$, $\Lambda^0$ and $\Lambda^8$ (or $bd$, $gs$ and $ru$, $gd$, $bs$) mix among themselves to produce the following dispersion relations. There is a pair of modes with energies

$$E = \sqrt{(p - \bar{\mu})^2 + \Delta^2}, \quad (23)$$

and 1 mode with energy

$$E = \sqrt{(p - \bar{\mu})^2 + (2\Delta)^2}. \quad (24)$$
4 Transition to a Gapless Phase

As pointed out in Refs. [4, 6], the gapless phase arises when the energy of one of the collective modes becomes zero. As one goes deeper into that phase there will be a finite window of momentum for which the energy of the mode would be negative. This is the blocking region. One needs to find a solution to the combined set of electric/color neutrality conditions and gap equations. When gapless modes appear, the gaps for various pairings separate from each other and their values should be solved from three gap equations, namely \( \partial \Omega / \partial \Delta_I = \partial \Omega / \partial \Delta_{II} = \partial \Omega / \partial \Delta_{III} = 0 \). If only one mode becomes gapless, it is natural to focus on variations of that gap from the original value \( \Delta_0 \) while ignoring variations in the other two gaps. If two modes become gapless simultaneously then we should consider variations of two gaps while ignoring variations of the third.

The gapless phase will arise when the mismatch between the chemical potentials, \( \delta \mu \), of two quarks exceeds the gap. These mismatches are displayed in Table II for both the CFL and the CFL\( K^0 \) phases. See also the previous section. The first mode to go gapless in CFL occurs when \( \mu_S = \Delta_0 \) and is associated with both \( \Delta_I \) and \( \Delta_{II} \). The first mode to go gapless in CFL\( K^0 \) occurs when \( \mu_S = 4 \Delta_0 / 3 \) and is associated with \( \Delta_{II} \). We will focus on gCFL\( K^0 \) since we already know that CFL\( K^0 \) is favored over CFL when \( \mu_S > 0 \).

In Nambu Jona-Lasinio type models the thermodynamic potential is written as

\[
\Omega = \Omega_0 + \frac{\Delta_I^2 + \Delta_{II}^2 + \Delta_{III}^2}{G} - \sum_{j=1}^{18} \int d^3p \frac{1}{(2 \pi)^3} |E_j(p)| - \frac{\mu_e^4}{12 \pi^2}, \quad (25)
\]

where \( \Omega_0 \) is a constant adjusted to make the energy of the true vacuum zero. The constant \( G \) is a coupling constant. The sum runs over 3 flavors, 3 colors, and particle plus antiparticle; spin is already included. The energies for the particles are generically written as

\[
E_j(p) = \sqrt{(p - \bar{\mu}_j)^2 + \Delta_j^2 + \delta \mu_j}, \quad (26)
\]

The chemical potentials \( \bar{\mu}_j \) and \( \delta \mu_j \) are linear combinations of \( \mu, \mu_3, \mu_8 \) and \( \mu_e \). Anti-particles have chemical potentials opposite to particles. The momentum integrals are quartically divergent; usually they are regulated with
a hard momentum cut-off $\Lambda$. In some sense this approximates asymptotic freedom of QCD in the context of this model. The exact expression for the integral with such a cut-off is given in the Appendix for completeness. The approximate expression which includes terms that go as positive powers of $\Lambda$ or which remain finite as $\Lambda \to \infty$ is also given in the appendix. There are several terms that are odd in the chemical potentials. These terms cancel when adding particles and anti-particles, so in this sense the inclusion of anti-particles simplifies the analysis. Keeping only those terms which are finite as $\Lambda \to \infty$ gives us

$$\Omega = \Omega_0 - \frac{9}{4\pi^2} \Lambda^4 - \frac{\mu_e^4}{12\pi^2} + \frac{\Delta_i^2 + \Delta_{ii}^2 + \Delta_{iii}^2}{G}$$

$$-\frac{1}{4\pi^2} \sum_{j=1}^{9} \left[ \Lambda^2 \Delta_j^2 + \frac{1}{2} \Delta_j^2 \left( 4\bar{\mu}_j^2 - \Delta_j^2 \right) \ln \left( \frac{2\Lambda}{\Delta_j} \right) + \frac{1}{3} \bar{\mu}_j^4 - 2\bar{\mu}_j^2 \Delta_j^2 + \frac{1}{8} \Delta_j^4 \right].$$

(27)

The term proportional to $\Lambda^4$ is absorbed into the vacuum energy. The terms quadratic and logarithmic in $\Lambda$ are usually accommodated as follows: One sets the strange quark mass to zero and solves for the common gap $\Delta_0$ at some reference chemical potential $\mu$ for some fixed values of $G$ and $\Lambda$. A change in $\Lambda$ is accompanied by a corresponding change in $G$ such that the gap does not change. Generally it is found that the results are not sensitive to the specific choice of $\Lambda$ so long as it is large compared to $\mu$.

4.1 Ideal CFL

Consider the ideal CFL matter with massless quarks. There are 8 modes with gap $\Delta$ and one mode with gap $2\Delta$. These gaps are predicted to be very small compared to the chemical potential so the terms in the thermodynamic potential of order $\Delta^4$ are normally dropped. The $\Omega_0$ and the $G$ get renormalized as

$$\Omega_0 - \frac{9\Lambda^4}{4\pi^2} \to \Omega_0$$

$$\frac{1}{G} \to \frac{1}{G}. \quad (28)$$

For convenience we also define

$$\Lambda' \equiv 2^{2/3} e^{-1} \Lambda. \quad (29)$$
Then the thermodynamic potential can be written as
\[
\Omega = \Omega_0 - \frac{9\mu^4}{12\pi^2} + \frac{3\Delta^2}{G} - \frac{6}{\pi^2}\mu^2\Delta^2 \ln\left(\frac{\Lambda'}{\Delta}\right).
\] (30)

The solutions to the gap equation, \( \partial\Omega/\partial\Delta = 0 \), are either \( \Delta = 0 \) or \( \Delta = \Delta_0 \) with the latter determined by
\[
\frac{1}{G} = \frac{\mu^2}{\pi^2} \left[ 2 \ln\left(\frac{\Lambda'}{\Delta_0}\right) - 1 \right].
\] (31)

The oft-used requirement that a change in \( \Lambda' \) should result in a change in \( G \) such that \( \Delta_0 \) be left unchanged means that
\[
G = \frac{G_0}{1 + (2\mu^2 G_0/\pi^2) \ln(\Lambda'/\Lambda'_0)}
\] (32)
So in this case we can trade the pair \( G, \Lambda' \) for one parameter, namely, \( \Delta_0 \). Then we obtain the elegant formula
\[
\Omega = \Omega_0 - \frac{9\mu^4}{12\pi^2} + \frac{3}{\pi^2}\mu^2\Delta^2 \left[ 2 \ln\left(\frac{\Delta}{\Delta_0}\right) - 1 \right].
\] (33)
This obviously has a minimum at \( \Delta = \Delta_0 \).

### 4.2 CFL with \( m_s \neq 0 \)

Next we consider CFL with \( \mu_s \neq 0 \). The thermodynamic potential now takes the form
\[
\Omega = \Omega_0 - \frac{3}{4\pi^2}\bar{\mu}^4 + \frac{3\Delta^2}{\pi^2} \left[ 2\mu^2 \ln\left(\frac{\Lambda'}{\Delta_0}\right) - 2\bar{\mu}^2 \ln\left(\frac{\Lambda'}{\Delta}\right) - \mu^2 \right],
\] (34)
where, as before \( \bar{\mu} \equiv \mu - \frac{1}{3}\mu_s \). The solution to the gap equation is either \( \Delta = 0 \) or the solution to the equation
\[
\bar{\mu}^2 \ln\left(\frac{\Delta}{\Lambda'}\right) = \mu^2 \ln\left(\frac{\Delta_0}{\Lambda'}\right) + \frac{1}{18}(6\mu - \mu_s)\mu_s.
\] (35)
When \( \mu_s \neq 0 \) the dependence on the cut-off no longer drops out. In the limit that \( \mu_s \ll \mu \) the solution to the above equation is approximately
\[
\Delta = \Delta_0 \left\{ 1 - \frac{\mu_s}{3\mu} \left[ 2 \ln\left(\frac{\Lambda'}{\Delta_0}\right) - 1 \right] \right\}.
\] (36)
Hence the gap decreases linearly with $\mu_S/\mu$.

Label the solution to the gap equation (35) as $\Delta_{CFL}(\mu_S)$. This is a definite function of $\mu_S$. In terms of it the thermodynamic potential may be written

$$\Omega = \Omega_0 - \frac{3\bar{\mu}^4}{4\pi^2} + \frac{3}{\pi^2}\mu^2\Delta^2 \left[ 2\ln \left( \frac{\Delta}{\Delta_{CFL}(\mu_S)} \right) - 1 \right].$$

(37)

This obviously has a minimum at $\Delta = \Delta_{CFL}(\mu_S)$; the dependence on $\Lambda'$ is implicit through the solution to (35).

Let us compare with Alford, Kouvaris, and Rajagopal [5] who solved the gap equation numerically without approximation. They used $\Delta_0 = 25$ MeV, $\mu = 500$ MeV, and $\Lambda = 800$ MeV ($\Lambda' = 467.18$ MeV). In the CFL phase the first gapless mode appears when $\mu_S = \Delta$. From eq. (36) we find that this occurs when $\Delta = \mu_S \approx 23.1$ MeV. This is consistent with their Fig. 1.

### 4.3 Gapless CFL

As $\mu_S$ increases, eventually one or more of the quark pairings will have a mismatch of chemical potentials that becomes as large as the gap. This stress breaks apart some of the pairs and reduces that value of the gap. Let us write the dispersion relation of quasi-quarks that first become gapless in the form

$$E(p) = \sqrt{(p - \bar{\mu})^2 + \Delta^2} - \delta\mu.$$

(38)

The excitation energy for quasi-quarks becomes zero at the momenta

$$p_\pm = \bar{\mu} \pm \sqrt{\delta\mu^2 - \Delta^2}.$$

(39)

This is appropriate when $\delta\mu \geq \Delta$. Then there exist gapless modes in the blocking region $p \in (p_-, p_+)$. As a consequence, the gapless modes provide

$$\Omega_g(\bar{\mu}, \delta\mu, \Delta) = \int_{p_+}^{p_-} \frac{p^2 dp}{2\pi^2} \left[ \sqrt{(p - \bar{\mu})^2 + \Delta^2} - \delta\mu \right]$$

$$= \frac{\delta\mu}{24\pi^2} \sqrt{\delta\mu^2 - \Delta^2} \left( 5\Delta^2 - 2\delta\mu^2 - 12\bar{\mu}^2 \right)$$

$$+ \frac{\Delta^2}{16\pi^2} \left( 4\mu^2 - \Delta^2 \right) \ln \left( \frac{\delta\mu + \sqrt{\delta\mu^2 - \Delta^2}}{\delta\mu - \sqrt{\delta\mu^2 - \Delta^2}} \right).$$

(40)

to the total free energy. This must be added to (27). At this point Alford, Kouvaris and Rajagopal solve the equations for gCFL numerically using the
NJL model. This is the gCFL phase. The three chemical potentials, \( \mu_e, \mu_3 \) and \( \mu_8 \) evolve as a function of \( \mu_S \) differently than in the CFL phase. The three gaps are no longer equal but evolve such that \( \Delta_I < \Delta_{II} < \Delta_{III} \) with \( \Delta_I \) dropping the fastest, \( \Delta_{II} \) lagging somewhat behind, and \( \Delta_{III} \) not deviating much from \( \Delta_0 \). This is as expected. At the onset of the gapless phase, channel III has no chemical potential mismatch and so the associated gap \( \Delta_{III} \) should not change much. The channels I and II both have a mismatch equal to \( \mu_S \) and so their associated gaps should decrease; \( \Delta_I \) decreases faster than \( \Delta_{II} \) because the latter is retarded by the turn-on of the electric chemical potential. We shall go no further into the gCFL since the phase with kaon condensation has lower free energy.

4.4 CFL\( \, K^0 \)

Here we repeat the analysis of section 4.2 but with the chemical potentials and dispersion relations appropriate to the CFL\( \, K^0 \) phase. The thermodynamic potential is

\[
\Omega = \Omega_0 - \frac{3\bar{\mu}^2}{4\pi^2} \left( \bar{\mu}^2 + \frac{1}{6}\mu_S^2 \right) + \frac{3\Delta^2}{\pi^2} \left[ 2\bar{\mu}^2 \ln \left( \frac{\Lambda}{\Delta_0} \right) - 2\bar{\mu}^2 \ln \left( \frac{\Lambda'}{\Delta} \right) - \mu^2 \right],
\]

where, as before \( \bar{\mu} \equiv \mu - \frac{1}{3}\mu_S \). We have dropped terms of fourth order in \( \Delta \) and \( \mu_S \). The functional dependence on the gap is identical to the CFL case. Therefore the gap equation and its solutions are also identical. We can thus write the thermodynamic potential in the form

\[
\Omega = \Omega_0 - \frac{3\bar{\mu}^2}{4\pi^2} \left( \bar{\mu}^2 + \frac{1}{6}\mu_S^2 \right) + \frac{3\Delta^2}{\pi^2} \left[ 2\ln \left( \frac{\Delta}{\Delta_{CFL}(\mu_S)} \right) - 1 \right].
\]

4.5 Gapless CFL\( \, K^0 \)

Finally we turn our attention to the gCFL\( \, K^0 \) phase. The difference in free energy between the CFL and CFL\( \, K^0 \) phases was already quoted in (12). We now face the problem of how to construct the free energy for the gCFL\( \, K^0 \) phase. The above NJL model allows for different values of the gaps in the different channels, and it has the dynamics to determine these in terms of the coupling constant \( G \) and the momentum cut-off \( \Lambda \), but it does not allow for kaon condensation. A NJL model was constructed by Forbes [15] which does
include kaon condensation. (See also [16].) The resulting calculations involve extensive numerical computation, but only the location of the onset of the gCFL\(^0\) phase was studied. The effective field theory that describes kaon condensation assumes a common gap and that this gap is determined by other considerations. In principle an effective field theory should be constructed that makes allowance for differing gaps; this is a formidable task that has yet to be worked out. We are interested in describing the gCFL\(^0\) phase itself, at least for a small range of \(\mu_s\), and not just the location of its onset.

We shall construct a thermodynamic potential from the following pieces. The first piece is the NJL form (25). Practically we will use the large cut-off limit (27) and drop the terms of order \(\Delta^4\). The second piece is the contribution of the gapless modes described by (40) which is, however, really a part of the NJL model. The third and final piece is the kaon condensation free energy. We should allow for the possibility that the values of the chemical potentials may be different once the gapless phase is entered. We take this from Kryjevski [21].

\[
\Omega_{K^0} = -f_\pi^2 \left[ \frac{1}{2} \mu_S^2 + \frac{1}{2} \left( \mu_3 + \frac{1}{2} \mu_S - \mu_e \right)^2 + \frac{2}{3} \left( \mu_8 + \frac{1}{4} \mu_S - \frac{1}{4} \mu_e \right)^2 \right]
\]  

(43)

Here we use the transcription

\[
e^A_0 \rightarrow \mu_e, \quad gA_3^0 \rightarrow -\mu_3, \quad gA_8^0 \rightarrow -\frac{2}{\sqrt{3}} \mu_S.
\]

This contribution is normalized such that it reflects the free energy difference between the CFL\(^0\) and the CFL phases, namely,

\[
\Omega_{K^0}(\mu_e = 0, \mu_3 = -\frac{1}{2} \mu_S, \mu_8 = -\frac{1}{4} \mu_S) = -\frac{1}{2} f_\pi^2 \mu_S^2.
\]

Putting everything together, our model for the CFL\(^0\)/gCFL\(^0\) matter is represented by the function

\[
\Omega_{CFL^0} = \Omega^{\text{neutral}}_{\text{unpaired}} + \frac{3}{4\pi^2} \mu^2 \mu_S^2 + \frac{\mu^2}{\pi^2} \sum_{i=I}^{III} \Delta_i^2 \left[ 2 \ln \left( \frac{\Delta_i}{\Delta_{CFL}(\mu_S)} \right) - 1 \right] - f_\pi^2 \left[ \frac{1}{2} \mu_S^2 + \frac{1}{2} \left( \mu_3 + \frac{1}{2} \mu_S - \mu_e \right)^2 + \frac{2}{3} \left( \mu_8 + \frac{1}{4} \mu_S - \frac{1}{4} \mu_e \right)^2 \right] + \sum_{i=I}^{III} \theta(\delta \mu_i^2 - \Delta_i^2) \Omega_g(\bar{\mu}_i, \delta \mu_i, \Delta_i) - \frac{\mu_e^4}{12\pi^2}.
\]  

(44)
The function $\Delta_{\text{CFL}}(\mu_S)$ is the solution to eq. (35) whose approximate solution is given by eq. (36). The average and difference of chemical potentials in the $i$’th channel, $\bar{\mu}_i$ and $\delta\mu_i$, are the functions of $\mu_e$, $\mu_3$, $\mu_8$ to be found in Tables II and III. The six variables $\mu_e$, $\mu_3$, $\mu_8$, $\Delta_I$, $\Delta_{II}$, and $\Delta_{III}$ must be determined on the basis of this function. Note that the solutions give the physical thermodynamic potential of eq. (12) when there are no gapless modes. This function allows us to study the transition from CFL$^{0}$ to gCFL$^{0}$. Note that the gapless mode in channel II is what will first drive the chemical potentials and the gap away from their values in the CFL$^{0}$ phase. Before proceeding we will make one other useful approximation; we set $\bar{\mu}_i = \bar{\mu}$. This has a minimal effect because it is the difference $\delta\mu_i$ which drives the gapless phase, not the average. In addition we use $\bar{\mu}$ in place of $\mu$ when evaluating $f_\pi$. Although $f_\pi$ has not been determined when the quark masses are nonzero, this replacement has minimal numerical impact.

Consider what happens as $\mu_S$ increases, or equivalently what happens as the baryon density decreases. Prior to the onset of the gapless modes, we have CFL$^{0}$ matter. The chemical potentials are $\mu_e = 0$, $\mu_3 = -\frac{1}{2}\mu_S$, $\mu_8 = -\frac{1}{4}\mu_S$. All three gaps are equal, $\Delta_I = \Delta_{II} = \Delta_{III} = \Delta_{\text{CFL}}(\mu_S)$, and decrease with increasing $\mu_S$. The thermodynamic potential is given by

$$\Omega_{\text{neutral}}^{\text{CFL}^0} = \Omega_{\text{unpaired}}^{\text{neutral}} + \frac{3\mu^2}{4\pi^2} \left( \mu_S^2 - 4\Delta_{\text{CFL}}^2 \right) - \frac{1}{2} f_\pi^2 \mu_S^2. \quad (45)$$

The channel II goes gapless when $\mu_S = \frac{4}{3} \Delta_{\text{CFL}}(\mu_S) \approx 1.206 \Delta_0$. The difference between the factor of 4/3 (first obtained by Kryjevski and Yamada [14]) and 1.206 (very close to the value obtained by Forbes [15]) is due to the decrease in the gap with increasing $\mu_S$. To study this phase we need to solve the equations

$$\partial \Omega_{\text{CFL}^0} / \partial \mu_3 = \partial \Omega_{\text{CFL}^0} / \partial \mu_8 = \partial \Omega_{\text{CFL}^0} / \partial \mu_e = 0 \quad (46)$$

to obtain the chemical potentials. Allowing for the possibility that any of the channels may be gapless, these can be recast in the form

$$f_\pi^2 [2\mu_S - 4\mu_e + 3\mu_3 + 2\mu_8] = \frac{3}{2} \frac{\partial \Omega_g}{\partial \delta \mu_{II}} + \frac{3}{2} \frac{\partial \Omega_g}{\partial \delta \mu_{III}} + \frac{\mu_e^3}{\pi^2} \quad (47)$$

$$f_\pi^2 [\mu_S - 2\mu_e + 2\mu_3] = \frac{1}{2} \frac{\partial \Omega_g}{\partial \delta \mu_{I}} - \frac{1}{2} \frac{\partial \Omega_g}{\partial \delta \mu_{II}} - \frac{\partial \Omega_g}{\partial \delta \mu_{III}} \quad (48)$$

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\[
f^2 [\mu_S - 2\mu_e + 4\mu_8] = -\frac{3}{2} \frac{\partial \Omega_g}{\partial \delta \mu_I} - \frac{3}{2} \frac{\partial \Omega_g}{\partial \delta \mu_{II}} \tag{49}
\]

Here we have calculated the derivatives \(\partial \delta \mu_i / \partial \mu_\alpha\) (where \(\alpha = e, 3, 8\)) by using Table II.

When entering the gapless region it is sometimes useful to employ the limiting form of \(\Omega_g\) when \(\sqrt{\delta \mu^2 - \Delta^2} \ll \delta \mu, \Delta \ll \bar{\mu}\), specifically

\[
\Omega_g \approx -\frac{\bar{\mu}^2}{2\pi^2 \delta \mu} \left(\delta \mu^2 - \Delta^2\right)^{3/2}
\]

\[
\frac{\partial \Omega_g}{\partial \delta \mu} \approx -\frac{3\bar{\mu}^2}{2\pi^2} \left(\delta \mu^2 - \Delta^2\right)^{1/2}
\]

\[
\frac{\partial \Omega_g}{\partial \Delta} \approx \frac{3\bar{\mu}^2 \Delta}{2\pi^2 \delta \mu} \left(\delta \mu^2 - \Delta^2\right)^{1/2}. \tag{50}
\]

We shall henceforth use this expression. Its validity will be checked a posteriori. Using this approximation, the gap in channel \(i\) must satisfy either \(\Delta_i = 0\) or the equation

\[
\ln \left(\frac{\Delta_i}{\Delta_{CFL}}\right) + \frac{3}{8} \sqrt{\delta \mu_i^2 - \Delta_i^2} \theta \left(\delta \mu_i^2 - \Delta_i^2\right) = 0. \tag{51}
\]

We are left with solving the above set of six equations to find the behavior of \(\mu_e, \mu_3, \mu_8, \Delta_I, \Delta_{II}\) and \(\Delta_{III}\) as functions of \(\mu_S\). At this point it is already clear that \(\mu_e\) will increase from 0 and a gap will decrease from \(\Delta_{CFL}(\mu_S)\) in a closely coupled way once we enter a gapless region.

The electron chemical potential \(\mu_e\) will become nonzero once \(\frac{3}{4} \mu_S\) exceeds \(\Delta_{CFL}\). Therefore it is sensible to look for a power series expansion of the unknowns in terms of \(x \equiv \frac{3}{4} \mu_S - \Delta_{CFL}\). The results are

\[
\mu_e = x + \frac{x^3}{12} \left(\frac{1}{\bar{\mu}_e^2} - \frac{1}{\pi^2 f_{\pi}^2}\right) + \cdots
\]

\[
\mu_3 = -\frac{1}{2} \mu_S + x + \frac{x^3}{12 \bar{\mu}_e^2} + \cdots
\]

\[
2\mu_8 = -\frac{1}{2} \mu_S + x + \frac{x^3}{12} \left(\frac{1}{\bar{\mu}_e^2} + \frac{2}{\pi^2 f_{\pi}^2}\right) + \cdots
\]

\[
\Delta_I = \Delta_{CFL}
\]

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\[ \begin{align*}
\Delta_{II} &= \Delta_{CFL} - \frac{x^3}{12\mu^2} + \cdots \\
\Delta_{III} &= \Delta_{CFL} \\
\delta\mu_I &= \frac{1}{2}\mu_S - \frac{x^3}{24\pi^2f_\pi^2} + \cdots \\
\delta\mu_{II} &= \Delta_{CFL} - \frac{x^3}{12\mu^2} + \cdots \\
\delta\mu_{III} &= \Delta_{CFL} - \frac{1}{2}\mu_S - \frac{x^3}{12\mu^2} - \frac{x^3}{12\bar{\mu}^2} \left( \frac{1}{\mu^2} - \frac{1}{2\pi^2f_\pi^2} \right) + \cdots 
\end{align*} \] (52)

The gaps \( \Delta_I = \Delta_{III} = \Delta_{CFL}(\mu_S) \) are undeviated from the CFL \( K^0 \) phase. The relative order of magnitude of the corrections displayed above, before the ellipses, with \( x \approx 10 \text{ MeV} \) and \( \mu \approx 500 \text{ MeV} \), is \( 10^{-4} \), indicating that this is a very good expansion. This also supports the validity of using the approximation (50). To the order displayed in eq. (52) the important quantity \( \sqrt{\delta\mu_{II}^2 - \Delta_{II}^2} \) vanishes. It can be determined to lowest nonvanishing order to be
\[ \sqrt{\delta\mu_{II}^2 - \Delta_{II}^2} = \frac{2}{9} x^3 \bar{\mu}^3 + \cdots \] (53)

by inserting the expressions for the chemical potentials from (52) into (48) or (49).

The behavior of \( \mu_3, \mu_8 \) and \( \mu_e \) as functions of \( \mu_S \) are shown in Fig. 1 while the behavior of \( \Delta_I = \Delta_{III} = \Delta_{II} \) are shown in Fig. 2. To obtain these figures we used eqs. (35) and (52). We used the same numerical values for the parameters as in section 4.2, namely, \( \Delta_0 = 25 \text{ MeV} \), \( \mu = 500 \text{ MeV} \), and \( \Lambda = 800 \text{ MeV} \). Essentially all of the functions plotted are linear in \( \mu_S \) as is apparent from the figures. For \( \mu_S < \frac{4}{3}\Delta_{CFL} \approx 1.206\Delta_0 \approx 30.143 \text{ MeV} \) all three channels are fully gapped. For larger values of \( \mu_S \) channel II becomes gapless.

Channel I becomes gapless when \( \delta\mu_I = \Delta_{CFL} \), which occurs when
\[ \Delta_{CFL} = \frac{1}{2}\mu_S - \frac{\mu_S^3}{1536\pi^2f_\pi^2} + \cdots \]

The chemical potentials and gaps acquire shifts proportional to
\[ y = \frac{\frac{1}{4} (\mu_S - x^3/12\pi^2f_\pi^2)^2 - \Delta_{CFL}^2}{\frac{1}{2} (\mu_S - x^3/12\pi^2f_\pi^2) - (\pi^2f_\pi^2/\bar{\mu}^2)\Delta_{CFL}}. \] (54)

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Starting from (52) they are shifted as follows:

\[
\begin{align*}
\mu_e & \rightarrow \mu_e - \frac{1}{4}y + \cdots \\
\mu_3 & \rightarrow \mu_3 - \frac{3}{4}y + \cdots \\
2\mu_8 & \rightarrow 2\mu_8 + \frac{5}{4}y + \cdots \\
\Delta_I & \rightarrow \Delta_I - \frac{\pi^2 f_\pi^2}{2\mu^2}y + \cdots \\
\Delta_{II} & \rightarrow \Delta_{II} - \frac{x^2}{16\mu^2}y + \cdots \\
\Delta_{III} & \rightarrow \Delta_{III} + \cdots \\
\delta\mu_I & \rightarrow \delta\mu_I - \frac{1}{2}y + \cdots \\
\delta\mu_{II} & \rightarrow \delta\mu_{II} + \cdots \\
\delta\mu_{III} & \rightarrow \delta\mu_{III} + \frac{1}{2}y + \cdots 
\end{align*}
\]

(55)

To lowest nonvanishing order

\[\sqrt{\delta\mu_I^2 - \Delta_I^2} = \frac{4\pi^2 f_\pi^2}{3\mu^2}y\]  

(56)

Unlike the case of gapless channel II, the corrections here in channel I are not suppressed by a factor of $1/\bar{\mu}^2$. They begin to deviate rather substantially from $\Delta_{CFL}$. However, we do not need to go very deeply into the gapless channel I phase before the color superconducting matter becomes unstable to complete breakup into unpaired quark matter.

Channel I becomes gapless when $\mu_S > 2\Delta_{CFL} \approx 1.728\Delta_0 \approx 43.2$ MeV. In the region where only channel II is gapless, $\Delta_{II} < \Delta_I = \Delta_{III} = \Delta_{CFL}$, but not by much; it is not visible on the Fig. 2. In the region where both I and II are gapless, $\Delta_I$ drops much faster than $\Delta_{II}$.

With the parameters we have used, channel II becomes gapless when the strange quark mass $m_s = \sqrt{2}\mu_S > 174$ MeV, and channel I becomes gapless when $m_s > 208$ MeV.

The thermodynamic potential, as measured relative to the thermodynamic potential of neutral unpaired quark matter, in units of $3\mu^2\Delta_0^2/\pi^2$, is shown in Fig. 3. There is a smooth transition as first channel II and then channel I become gapless. Paired quark matter becomes unstable to breakup into unpaired quark matter when $\mu_S > 50.47$ MeV. Coincidentally, this is
approximately equal to $2\Delta_0$, the value one would get by ignoring the evolution of the gaps with $\mu_S$, the chemical potentials enforcing electric and color neutrality, and the formation of gapless modes.

5 Chromomagnetic Instabilities

Finally, we analyze qualitatively the possibility of eliminating the chromomagnetic instability that occurs in gCFL. In the gCFL case, channels I and II become gapless simultaneously at the point $(\mu_e, \mu_S) = (0, \Delta_{CFL})$. The fact that the $gs - bd$ and $rs - bu$ pairings become breached simultaneously has an important consequence. As stressed in [7, 8], this feature is responsible for the instability which occurs for $A_1$ and $A_2$ gluons. Since the self-energy for $A_1$ and $A_2$ stems from the loop diagram composed of $gs$ and $rs$ quarks, the coexistence of gapless (unpaired) $gs$ and $rs$ modes provides a singular contribution to the corresponding Meissner masses. In the gCFL$^0$ case, however, the gapless formation in channel I is delayed relative to channel II. Therefore, singularities should no longer exist in the self-energy for $A_1$, $A_2$, Meissner masses, and the corresponding instability should disappear in the gCFL$^0$ phase.

On the other hand, the feature that the gCFL location is very close to its critical line in the $\mu_e - \mu_S$ plane has been pointed out to be primarily responsible for the instability of $A_3$ and $A_8$ gluons and $A_\gamma$ photons [8]. For instance, the (almost) infinitesimal channel II gapless strength in gCFL leads to the singularities in the $A_{3,8,\gamma}$ Meissner masses [8]. In the gCFL$^0$ case, however, the location is not limited to the vicinity of the critical line. Instead, a finite region in the $\mu_e - \mu_S$ plane is available for the gCFL$^0$ existence. Thus such kind of instability might no longer appear, at least for some region in the $\mu_e - \mu_S$ plane.

Since only $A_{1,2}$ and $A_{3,8,\gamma}$ exhibit imaginary Meissner masses in gCFL, the previous-predicted instability is very likely to be removed in the gCFL$^0$ phase. Physically, it can be understood from the viewpoint that the chromomagnetic instability is eliminated by building on the proper vacuum. It has been suggested that nonzero vacuum expectation values of gluons such as $\langle A_3^0 \rangle$ and/or $\langle A_8^0 \rangle$ are helpful for removing the gCFL instability [7]. In [22], the instability in two-flavor superconductor was argued to be resolved by gluon condensation. In the present work, the nonzero $\langle A_3^0 \rangle$ and $\langle A_8^0 \rangle$
(anisotropic vacuum) derived from the kaon condensation have been attributed to the change of the electric/color neutral solution and then been considered in the study of the gapless formation. In this sense, the conjecture that gCFL$K^0$ is not chromomagnetically unstable is not very surprising. However, intrinsic links between the $p$-wave kaon supercurrent phase established in [17] and the presently discussed gCFL$K^0$ phase still remain to be clarified. Also, we cannot rule out the possibility that other instabilities, especially for $A_4$, $A_5$, $A_6$ and $A_7$ gluons, arise once again. The quantitative calculation of the gluonic Meissner masses in gCFL$K^0$ is still necessary, but this is beyond the scope of the present work.

6 Conclusion

In summary, we investigated electric/color neutrality and gapless formation in the CFL matter with $K^0$ condensation. By taking the CFL$K^0$ neutral solution into account, we clarify why the gCFL$K^0$ formation is delayed in comparison with the gCFL case. More importantly, it is found that the gapless phenomenon for down-strange pairing (channel I) is delayed while that for up-strange pairing (channel II) occurs first. The novel phase structure implies that the previously predicted gCFL instability might be removed or at least weakened. These conclusions are likely to be important for fully understanding the unconventional CFL phases in the presence of strange quarks with nonzero masses $m_S$ and eletrons $\mu_e$.

Of course, there are still some unanswered aspects in the present work. First of all, the $K^0$ condensed CFL matter has been treated as a background for the gapless formation, which is safe only if $\mu$ is not too small; otherwise, the condensation may be supressed by instanton effect [23]. The gCFL$K^0$ phase is lower in free energy than the gCFL phase. Thus, our suggested gapless phase should replace the gCFL in neutron star cores [24], ignoring any possible gCFL$K^0$ instability. Secondly, more physics involving Goldstone-mode condensation should be addressed. These include the possibilities of charged kaon and other meson condensations. For instance, if $\mu_e$ is large enough and $K^-$ condensation occurs, the gapless phenomenon for up-down pairing in channel III needs to be included. Some of the above-mentioned problems are being investigated.
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Appendix

An integral that appears frequently is

$$\int_{\Lambda}^{0} dp \ p^2 \sqrt{(p - \bar{\mu})^2 + \Delta^2} = \frac{1}{8} \bar{\mu} \left[ 6 \bar{\mu}^2 + \Delta^2 \right] \sqrt{\bar{\mu}^2 + \Delta^2}$$

$$+ \frac{1}{8} (\Lambda - \bar{\mu}) \left[ 2 (\Lambda - \bar{\mu})^2 + 4 \bar{\mu}^2 + \Delta^2 \right] \sqrt{(\Lambda - \bar{\mu})^2 + \Delta^2}$$

$$+ \frac{2}{3} \left\{ \left[ (\Lambda - \bar{\mu})^2 + \Delta^2 \right]^{3/2} - \left[ \bar{\mu}^2 + \Delta^2 \right]^{3/2} \right\}$$

$$+ \frac{1}{8} \Delta^2 \left( 4 \bar{\mu}^2 - \Delta^2 \right) \ln \left( \frac{\Lambda - \bar{\mu} + \sqrt{(\Lambda - \bar{\mu})^2 + \Delta^2}}{\sqrt{\bar{\mu}^2 + \Delta^2} - \bar{\mu}} \right)$$

The limit of this expression as $\Lambda \to \infty$ is

$$\int_{0}^{\Lambda} dp \ p^2 \sqrt{(p - \bar{\mu})^2 + \Delta^2} \to \frac{1}{4} \Lambda^4 - \frac{1}{3} \bar{\mu} \Lambda^3 + \frac{1}{4} \Delta^2 \Lambda^2 + \frac{1}{2} \bar{\mu} \Delta^2 \Lambda$$

$$+ \frac{1}{8} \Delta^2 \left( 4 \bar{\mu}^2 - \Delta^2 \right) \ln \left( \frac{2\Lambda}{\sqrt{\bar{\mu}^2 + \Delta^2} - \bar{\mu}} \right) + \frac{1}{12} \bar{\mu}^4 - \frac{1}{2} \bar{\mu}^2 \Delta^2 + \frac{1}{32} \Delta^4$$

$$+ \frac{1}{24} \bar{\mu} \left( 2 \bar{\mu}^2 - 13 \Delta^2 \right) \sqrt{\bar{\mu}^2 + \Delta^2}$$

The terms which are odd in $\bar{\mu}$ cancel when particles and anti-particles are added together.
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Table I: Effective quark chemical potentials. The third and fourth columns evaluate them in the electron-free ($\mu_e = 0$) CFL ($\mu_3 = 0, \mu_8 = -\mu_S$) and CFLK$^0$ ($\mu_3 = -\frac{1}{2}\mu_S, \mu_8 = -\frac{1}{4}\mu_S$) phases.

|   | effective chemical potential | CFL | CFLK$^0$ |
|---|------------------------------|-----|----------|
| $ru$ | $\mu - \frac{2}{3}\mu_e + \frac{1}{3}\mu_3 + \frac{1}{3}\mu_8$ | $\mu - \frac{1}{3}\mu_S$ | $\mu - \frac{1}{3}\mu_S$ |
| $gd$ | $\mu + \frac{1}{3}\mu_e - \frac{1}{2}\mu_3 + \frac{1}{3}\mu_8$ | $\mu - \frac{1}{3}\mu_S$ | $\mu + \frac{1}{6}\mu_S$ |
| $bs$ | $\mu + \frac{1}{3}\mu_e - \frac{2}{3}\mu_8 - \mu_S$ | $\mu - \frac{1}{3}\mu_S$ | $\mu - \frac{5}{6}\mu_S$ |
| $rd$ | $\mu + \frac{1}{3}\mu_e + \frac{1}{2}\mu_3 + \frac{1}{3}\mu_8$ | $\mu - \frac{1}{3}\mu_S$ | $\mu - \frac{1}{3}\mu_S$ |
| $gu$ | $\mu - \frac{2}{3}\mu_e - \frac{1}{2}\mu_3 + \frac{1}{3}\mu_8$ | $\mu - \frac{1}{3}\mu_S$ | $\mu + \frac{1}{6}\mu_S$ |
| $rs$ | $\mu + \frac{1}{3}\mu_e + \frac{1}{2}\mu_3 + \frac{1}{3}\mu_8 - \mu_S$ | $\mu - \frac{4}{3}\mu_S$ | $\mu - \frac{1}{3}\mu_S$ |
| $bu$ | $\mu - \frac{2}{3}\mu_e - \frac{2}{3}\mu_8$ | $\mu + \frac{2}{3}\mu_S$ | $\mu + \frac{1}{6}\mu_S$ |
| $gs$ | $\mu + \frac{1}{3}\mu_e - \frac{1}{2}\mu_3 + \frac{3}{4}\mu_8 - \mu_S$ | $\mu - \frac{4}{3}\mu_S$ | $\mu - \frac{5}{6}\mu_S$ |
| $bd$ | $\mu + \frac{1}{3}\mu_e - \frac{2}{3}\mu_8$ | $\mu + \frac{2}{3}\mu_S$ | $\mu + \frac{1}{6}\mu_S$ |
Table II: Differences of effective quark chemical potentials relevant to pairing, defined as \( \delta \mu = \frac{1}{2} (\mu_q1 - \mu_q2) \) for the pair \( q1-q2 \). The third and fourth columns evaluate them in the electron-free \( (\mu_e = 0) \) CFL \( (\mu_3 = 0, \mu_8 = -\mu_S) \) and CFL\(^0 \) \( (\mu_3 = -\frac{1}{2}\mu_S, \mu_8 = -\frac{1}{4}\mu_S) \) phases.

|       | \( \delta \mu \)                      | CFL | CFL\(^0 \) |
|-------|--------------------------------------|-----|------------|
| \( ru - gd \) | \( \frac{1}{2} (-\mu_e + \mu_3) \) | 0   | \( -\frac{1}{7}\mu_S \) |
| \( ru - bs \) | \( \frac{1}{2} (-\mu_e + \frac{1}{2}\mu_3 + \mu_8 + \mu_S) \) | 0   | \( \frac{1}{4}\mu_S \) |
| \( gd - bs \) | \( \frac{1}{2} (-\frac{1}{2}\mu_3 + \mu_8 + \mu_S) \) | 0   | \( \frac{1}{2}\mu_S \) |
| \( bd - gs \) | \( \frac{1}{2} (\frac{1}{2}\mu_3 - \mu_8 + \mu_S) \) | \( \mu_S \) | \( \frac{1}{2}\mu_S \) |
| \( bu - rs \) | \( \frac{1}{2} (-\mu_e - \frac{1}{2}\mu_3 - \mu_8 + \mu_S) \) | \( \mu_S \) | \( \frac{3}{4}\mu_S \) |
| \( gu - rd \) | \( \frac{1}{2} (-\mu_e - \mu_3) \) | 0   | \( \frac{1}{4}\mu_S \) |

Table III: Averages of effective quark chemical potentials relevant to pairing. The third and fourth columns evaluate them in the electron-free \( (\mu_e = 0) \) CFL \( (\mu_3 = 0, \mu_8 = -\mu_S) \) and CFL\(^0 \) \( (\mu_3 = -\frac{1}{2}\mu_S, \mu_8 = -\frac{1}{4}\mu_S) \) phases.

|       | average                      | CFL | CFL\(^0 \) |
|-------|------------------------------|-----|------------|
| \( ru - gd \) | \( \mu - \frac{1}{6}\mu_e + \frac{1}{7}\mu_8 \) | \( \mu - \frac{1}{7}\mu_S \) | \( \mu - \frac{1}{12}\mu_S \) |
| \( ru - bs \) | \( \mu - \frac{1}{6}\mu_e + \frac{1}{7}\mu_3 - \frac{1}{8}\mu_8 - \frac{1}{3}\mu_S \) | \( \mu - \frac{1}{7}\mu_S \) | \( \mu - \frac{1}{12}\mu_S \) |
| \( gd - bs \) | \( \mu + \frac{1}{3}\mu_e - \frac{1}{7}\mu_3 - \frac{1}{7}\mu_8 - \frac{1}{3}\mu_S \) | \( \mu - \frac{1}{7}\mu_S \) | \( \mu - \frac{1}{12}\mu_S \) |
| \( bd - gs \) | \( \mu + \frac{1}{3}\mu_e - \frac{1}{7}\mu_3 - \frac{1}{7}\mu_8 - \frac{1}{3}\mu_S \) | \( \mu - \frac{1}{7}\mu_S \) | \( \mu - \frac{1}{12}\mu_S \) |
| \( bu - rs \) | \( \mu - \frac{1}{6}\mu_e + \frac{1}{7}\mu_3 - \frac{1}{7}\mu_8 - \frac{1}{2}\mu_S \) | \( \mu - \frac{1}{7}\mu_S \) | \( \mu - \frac{1}{12}\mu_S \) |
| \( gu - rd \) | \( \mu - \frac{1}{6}\mu_e + \frac{1}{3}\mu_8 \) | \( \mu - \frac{1}{3}\mu_S \) | \( \mu - \frac{1}{12}\mu_S \) |
Table IV: Correspondence between quark and baryon descriptions with the condensate form \( \langle q_i^\alpha C\gamma_5 q_j^\beta \rangle \sim \Delta_I \epsilon^{\alpha\beta_1} \epsilon_{ij1} + \Delta_{II} \epsilon^{\alpha\beta_2} \epsilon_{ij2} + \Delta_{III} \epsilon^{\alpha\beta_3} \epsilon_{ij3} \).

| baryon label | \( \sim \) |
|--------------|-------------|
| ru           | mixture of \( \Sigma^0, \Lambda^0, \Lambda^8 \) |
| gd           | mixture of \( \Sigma^0, \Lambda^0, \Lambda^8 \) |
| bs           | mixture of \( \Sigma^0, \Lambda^0, \Lambda^8 \) |
| gu           | \( \Sigma^+ \) |
| rs           | \( \Xi^- \) |
| bu           | \( p \) |
| gs           | \( \Xi^0 \) |
| bd           | \( n \) |
Figure 1: Chemical potentials as functions of $\mu_S$. The numerical parameters chosen are $\mu = 500$ MeV, $\Delta_0 = 25$ MeV, and $\Lambda = 800$ MeV. The curves are terminated when the paired phase becomes unstable to decay to the unpaired phase.
Figure 2: The gaps as functions of $\mu_S$. The numerical parameters chosen are $\mu = 500$ MeV, $\Delta_0 = 25$ MeV, and $\Lambda = 800$ MeV. The curves are terminated when the paired phase becomes unstable to decay to the unpaired phase.
Figure 3: The difference in thermodynamic potentials between the paired and unpaired phases as a function of $\mu_s$. The numerical parameters chosen are $\mu = 500$ MeV, $\Delta_0 = 25$ MeV, and $\Lambda = 800$ MeV. The transition occurs when $\mu_s \approx 50.47$ MeV.