TASI Lectures on Perturbative String Theory and Ramond-Ramond Flux

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Abstract

These lectures provide an introduction to perturbative string theory and its construction on spaces with background Ramond flux. Traditional covariant quantization of the string and its connection with vertex operators and conformal invariance of the worldsheet theory are reviewed. A supersymmetric covariant quantization of the superstring in six and ten spacetime dimensions is discussed. Correlation functions are computed with these variables. Applications to strings in anti-de Sitter backgrounds with Ramond flux are analyzed. Based on lectures presented at the Theoretical Advanced Study Institute TASI 2001, June 3-29, 2001.
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1 Introduction

Recently, the promise of solving strong coupling Yang-Mills theory by considering the Type IIB superstring on anti de Sitter space (AdS) via the AdS/CFT correspondence has led to a renewed interest in perturbative string theory and its formulation on background curved spaces. The first lecture reviews the derivation of the physical spectrum and scattering amplitudes in the old covariant quantization for open and closed bosonic string theory, with attention given to the structures that will require modification when the background spacetime is curved. The second lecture reviews various worldsheet formulations for the superstring, including the Berkovits-Vafa-Witten variables which provide a manifestly supersymmetric and covariant quantization in six dimensions. In the third lecture, these worldsheet fields are used to solve the string constraints on the vertex operators for the Type IIB superstring on $AdS_3 \times S^3 \times K3$ with background Ramond flux. A short section on computing correlation functions using these fields has been added in 4.3.

2 Old Covariant Quantization

We review the traditional quantization [1]-[3] of the open and closed bosonic strings and point to the steps that need generalization to accommodate strings with background Ramond fields.

2.1 Open Bosonic String

We introduce the Fubini-Veneziano fields:

$$X^\mu(z) = q^\mu - ip^\mu \ln z + i \sum_{n \neq 0} \frac{a^\mu_n}{n} z^{-n}$$

(1)

where

$$[a^\mu_m, a^\nu_n] = \eta^{\mu\nu} m \delta_{m,-n}; \quad [q^\mu, p^\nu] = i\eta^{\mu\nu}; \quad [q^\mu, a^\nu_n] = 0, n \neq 0; \quad n \in \mathbb{Z}.$$  

(2)
These commutators are the Lorentz covariant quantization conditions. Here $p^\mu \equiv a_0^\mu$. The fields $X^\mu(z)$ are restricted to appear in an exponential or as a derivative, since they do not exist rigorously as quantum fields which have a well-defined scaling dimension. The metric is space-like, $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$, $\mu = 0, 1, \ldots, d - 1$; and the $a_n^\mu$ satisfy the hermiticity relations $a_n^\mu = a_n^{-\mu}$. In flat spacetime, momentum is conserved and it is convenient when quantizing to use a basis of momentum eigenstates, $|k\rangle = e^{ikq}|0\rangle$, $p^\mu|k\rangle = k|k\rangle$, to represent $[q^\mu, p^\nu] = i\eta^{\mu\nu}$, where the vacuum state $|0\rangle$ satisfies $a_n^\mu|0\rangle = 0$, $n \geq 0$.

When the spacetime contains $AdS$, the isometry group no longer contains translations, so we lose momentum conservation and will just work in position space.

A string is a one-dimensional object that moves through spacetime and is governed by an action that describes the area of the worldsheet. Its trajectory $x^\mu(\sigma, \tau)$ describes the position of the string in space and time. $\tau$ is the evolution parameter $-\infty < \tau < \infty$, and $\sigma$ is the spatial coordinate labelling points along the string; $0 \leq \sigma \leq \pi$ for the open string, which is topologically an infinite strip; and $0 \leq \sigma \leq 2\pi$ for the closed string which is topologically an infinite cylinder with periodicity condition $x^\mu(\sigma, \tau) = x^\mu(\sigma + 2\pi, \tau)$. $g^{\alpha\beta}(\sigma, \tau)$ is the two-dimensional metric, $\alpha, \beta = 0, 1$. The action for the bosonic string is

$$S_2 = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau [\sqrt{|g|} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}(x) + e^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu}(x) + \alpha' \sqrt{|g|} R \phi(x)] . \quad (3)$$

The Regge slope $\alpha'$ has dimensions of length squared $[L]^2$; the string tension is defined as $T = \frac{1}{2\pi\alpha'}$. The low energy limit of the string theory is an effective point field theory. This corresponds to the leading term in an expansion in the external momentum times $\sqrt{\alpha'}$. In the zero Regge slope limit, i.e. the infinite string tension limit, the interactions of the vectors and tensors are those of Yang-Mills bosons and gravitons. In the other limit, the zero tension limit, one expects an infinite number of massless particles. It is conjectured for strings on $AdS$ that both limits may be simple field theories, since interacting massless particles of spin higher than 2 appear to be consistent in an Einstein
 spacetime, whereas they are not in flat space.

For general $G_{\mu\nu}(x)$, from a 2d point of view, $S_2$ is a nontrivial interacting field theory: a conformally invariant non-linear sigma model. To quantize in flat spacetime, we choose $G_{\mu\nu}(x) = \eta_{\mu\nu}$, and the other background fields to vanish: the two form field potential $B_{\mu\nu}$ and the dilaton $\phi$. This choice reduces $S_2$ to a free worldsheet theory. As a result, the correlation functions even at tree level in the string loop expansion are exact in $\alpha'$. The Type IIB superstring on $AdS_5 \times S^5$ requires additional background fields besides the curved metric $G_{\mu\nu}(x)$. In this latter case, $S_2$ is a nontrivial worldsheet theory where the string tree level amplitudes will appear as an expansion in $\alpha'$.

The string has two parameters: the length scale $\sqrt{2\alpha'}$ and the dimensionless string coupling constant $g$. These are both related to the dilaton vacuum expectation value $\langle 0 | \phi | 0 \rangle$, and thus the string has no free dimensionless (nor dimensionful) parameters. The gravitational coupling is $\kappa \equiv \sqrt{8\pi G}$, Newton’s constant is $G = \frac{\hbar c}{m_{\text{PLANCK}}^2}$; the Planck mass is $m_{\text{PLANCK}} = (\frac{\hbar c}{G})^{\frac{1}{2}} \sim 2.2 \times 10^{-5}\text{gm} \sim 1.2 \times 10^{19}\text{GeV}$; the Planck length is $l_{\text{PLANCK}} = (\frac{\hbar c}{G})^{\frac{1}{2}} \sim 1.6 \times 10^{-33}\text{cm}$. The Planck time is $t_{\text{PLANCK}} = (\frac{\hbar c}{G})^{\frac{3}{2}} \sim 10^{-43}\text{sec}$.

At these scales, the effects of stringiness will be important, whereas at larger distance scales or lower energies, an ordinary point quantum field theory QFT can be used as an effective theory. In flat space and for $\langle 0 | \phi | 0 \rangle = 0$, from identification of the graviton vertices we find $\kappa = \frac{1}{2}g\sqrt{2\alpha'}$, and from the Yang-Mill vertices, $g_{ym} = g$. Then $\alpha' \sim \frac{1}{m_{\text{PLANCK}}} \frac{1}{\kappa} \frac{1}{g} \frac{1}{G} \frac{1}{c^2} \kappa^2 \sim 16\pi G/g^2$. Thus the value of the universal Regge slope parameter $\alpha'$ is given in terms of Newton’s constant $G$ and the Yang-Mills coupling $g$. In particular since $g$ is of order 1, $\alpha'$ is of order the Planck length (squared). We see that the scale of the entire unified string theory is set by the Planck mass. This scale does not appear to be associated with the secret of any symmetry breaking, as does the scale $\sqrt{\lambda/3a} = \sqrt{-2m^2} \equiv m_H$ given by the Higgs mass $m_H \sim 250\text{Gev}$. Discussions of the gauge hierarchy problem, i.e. why is $m/m_{\text{PLANCK}}$ so small, what sets the ratio of $m$ to $m_{\text{PLANCK}}$? and what is the origin of mass, i.e. how is $m \neq 0$? are beyond the scope of these lectures.
The worldsheet action $S_2$ has two-dimensional general coordinate invariance $g^{\alpha\beta} \to \partial\Lambda^\alpha \partial\Lambda^\beta g^{\gamma\delta} \sim g^{\alpha\beta} + f^\gamma \partial_\gamma g^{\alpha\beta} + \partial_\gamma f^\beta g^{\gamma\alpha} + \partial_\gamma f^\alpha g^{\gamma\beta}$ and $x^\mu \to x^\mu + f^\alpha \partial_\alpha x^\mu$; and local Weyl rescaling $g^{\alpha\beta} \to \Lambda(\sigma, \tau) g^{\alpha\beta}$ and $x^\mu \to x^\mu$. These symmetries allow one to make the covariant gauge choice $g^{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1, 1)$. This results in the remnant constraint equations $(\partial_\sigma x^\mu \pm \partial_\tau x^\mu)^2 = 0$ which are equivalent to $\partial_\sigma x \cdot \partial_\tau x = 0$ and $\partial_\sigma x \cdot \partial_\sigma x + \partial_\tau x \cdot \partial_\tau x = 0$.

The equations of motion are $\partial^2 x^\mu - \partial^2 x^\mu = 0$.

The open string boundary conditions are $\partial_\sigma x^\mu(\sigma, \tau) = 0$ at $\sigma = 0, \pi$. The general solution of the equations of motion with these string boundary conditions is

$$x^\mu(\sigma, \tau) = q^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{a_n}{n} e^{-in\tau} \cos n\sigma$$

$$= q^\mu + p^\mu \tau + \frac{i}{2} \sum_{n \neq 0} \frac{a_n}{n} (e^{-in(\tau + \sigma)} + e^{-in(\tau - \sigma)})$$

$$= \frac{1}{2} X^\mu(e^{i(\tau + \sigma)}) + \frac{1}{2} X^\mu(e^{i(\tau - \sigma)}). \quad (4)$$

In covariant gauge, we implement the constraint equations by observing that

$$(\partial_\sigma x^\mu \pm \partial_\tau x^\mu)^2 = z^2 L(z), \quad z = e^{i(\tau \pm \sigma)} \quad (5)$$

where

$$L(z) = \frac{1}{2} : a(z) \cdot a(z) := \sum_n L_n z^{-n-2} \quad (6)$$

and $a^\mu(z) \equiv i \frac{dX^\mu(z)}{dz} = \sum_n a_n z^{-n-1}$. So the expectation value of the constraints vanish (as $\hbar \to 0$) for the physical state conditions in covariant gauge in the old covariant quantization:

$$L_0 |\psi\rangle = |\psi\rangle \quad (7)$$

$$L_n |\psi\rangle = 0 \quad \text{for} \ n > 0. \quad (8)$$

Furthermore (7) is the mass shell condition $p^2 = -m^2$, where $\frac{1}{2} m^2 \equiv N - 1$ so that $L_0 = \frac{1}{2} p^2 + N = 1$. Here $N = \sum_{n=1}^{\infty} a_{-n} \cdot a_n$. That is to say a physical state $|\psi\rangle$ has momentum $k$ which takes on a specific value corresponding to the $N^{th}$ excited level, $\alpha' k^2 = 1 - N$. The physical state conditions (7,8) can be shown to eliminate ghosts, i.e. negative norm states when $d = 26$. This result is known as the No-Ghost theorem [2, 3].
If instead, we had considered light-cone gauge, we would have observed that in the constraints $(\partial_\sigma x^\mu \pm \partial_\tau x^\mu)^2 = 0$ there is still residual gauge invariance to choose the light-cone gauge where $x^+(\sigma, \tau) = q^+ + p^+ \tau$, and the equations of motion are $(\partial^2_\sigma - \partial^2_\tau) x^i = 0$, $1 \leq i \leq d - 2$.

### 2.2 Locality of Worldsheet Fields

A conformal field theory $\mathcal{H}$ is a Hilbert space of states $H$, such as the space of finite occupation number states in a Fock space, together with a set of vertex operators $V(\psi, z)$, i.e. conformal fields which are in one to one correspondence with the states $\psi \in \mathcal{F}(H)$, where $\mathcal{F}(H)$ is a dense subspace of the Hilbert space $H$ of states.

$$\mathcal{H} = (H, \{V(\psi, z) : \psi \in \mathcal{F}(H)\}). \quad (9)$$

The conformal field theory requires that the vertex operators $V(\psi, z)$ form a system of mutually local fields, where $\lim_{z \to 0} V(\psi, z)|0\rangle = \psi$ for each field. That is to say the conformal fields $V(\psi, z)$ acting on the vacuum create asymptotic “in” states $\psi = V(\psi, 0)|0\rangle$ with conformal weight $h_\psi$, $L_0 \psi = h_\psi \psi$ (recall that $z = 0$ is $i\tau = t = -\infty$) on the cylinder. There is a one to one correspondence between the fields and the states in the Hilbert space they create at $z = 0$. Locality implies the s-t duality relation $V(\psi, z)V(\phi, \zeta) = V(V(\psi, z - \zeta)\phi, \zeta)$ which provides a precise version of the operator product expansion [9]. In particular locality requires

$$V(\psi, z)V(\phi, \zeta) \sim V(\phi, \zeta)V(\psi, z) \quad (10)$$

where the left side is defined for $|z| > |\zeta|$, the right side for $|\zeta| > |z| >$ and $\sim$ denotes analytic continuation. Locality ensures well defined scattering amplitudes. We shall also assume that the theory has a hermitian structure, in the sense that there is a definition of conjugation on the states, $\psi \mapsto \overline{\psi}$, an antilinear map with $V(\overline{\psi}, z) = z^{-2h}V(e^{zL_1}\psi, 1/z^*)^\dagger$. 

7
2.3 Virasoro Algebra

One of the conformal fields is the Virasoro current

\[ L(z) = V(\psi = \frac{1}{2} a_{-1} \cdot a_{-1}|0\rangle, z) = \sum_n L_n z^{-n-2}. \]  

(11)

It satisfies the operator product expansion

\[ L(z)L(\zeta) = \frac{c}{2}(z - \zeta)^{-4} + 2L(\zeta)(z - \zeta)^{-2} + \frac{dL(\zeta)}{d\zeta}(z - \zeta)^{-1} \]

(12)

which can be reexpressed as either of

\[ [L_n, L(\zeta)] = 2(n + 1)\zeta^n L(\zeta) + \zeta^{n+1} \frac{dL(\zeta)}{d\zeta} + \zeta^{n-2} \frac{c}{12} (n^3 - n) \]

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m}. \]  

(13)

Vertex operators for physical states are primary fields. In this case \( \phi \) is a highest weight state for the Virasoro algebra, or primary state, \( V(\phi, \zeta) \) is a primary field and

\[ L(z)V(\phi, \zeta) = h_\phi (z - \zeta)^{-2} V(\phi, \zeta) + (z - \zeta)^{-1} \frac{dV}{d\zeta} (\phi, \zeta). \]

(14)

Therefore in covariant gauge, the vertex operators for physical states have to satisfy

\[ [L_n, V(\psi, z)] = z^{n+1} \frac{d}{dz} V(\psi, z) + (n + 1) z^n V(\psi, z) \]

(15)

for all \( n \), and

\[ \lim_{z \to 0} V(\psi, z)|0 \rangle = \psi. \]

(16)

This follows from \( V(z) = \sum_r V_r z^{-r-h} \); so that \([L_n, V_r] = (-r + n(h-1)) V_{n+r} \).

Since \( V_r|0 \rangle = 0 \) for \( r > -h \), and \( \psi = V(0)|0 \rangle = V_{-h}|0 \rangle \), then \( L_n \psi = [L_n, V_{-h}]|0 \rangle = 0 \) for \( n \geq 1 \), and physical fields are primary fields of conformal dimension \( h = 1 \). (Note that if (13) holds only for \( n = 0, \pm 1 \) then \( V(\phi, \zeta) \) is a quasi-primary field.)
2.4 Mass Spectrum and Tree Level Amplitudes

We now consider the first few mass levels. For $N = 0$, there is one state $\psi = |k\rangle$ with $k^2 = 2$. The vertex operator for this state is

\[ V(k, z) = \exp\{ik \cdot X(z)\} = \exp\{ik \cdot X_<(z)\} e^{ik \cdot z} \exp\{ik \cdot X_>(z)\} \]  

(17)

where \( X_<(z) = i \sum_{n > 0} a_n^\mu z^{-n} \). Since \( z^{L_0} V(k, 1) z^{-L_0} = z V(k, z) \), the four point open string tree amplitude for these tachyonic scalars is

\[ A_4 = \alpha' \int_0^1 dz \langle 0; -k_1 | V(k_2, 1) V(k_3, z) | 0; k_4 \rangle \]

\[ = \alpha' (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3 + k_4) \int_0^1 dz z^{k_1 \cdot k_4} (1 - z)^{k_2 \cdot k_3} \]

\[ = \alpha' (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3 + k_4) B(-1 - \frac{1}{2}s, -1 - \frac{1}{2}t) \]

\[ = \alpha' (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3 + k_4) B(-1 - \alpha's, -1 - \alpha't) \]  

(18)

Here the Mandelstam variables are \( s = -(k_1 + k_2)^2 \), \( t = -(k_2 + k_3)^2 \), and \( u = -(k_1 + k_3)^2 \); the overall factor of \( \alpha' \) is due to the propagator. As mentioned earlier, this string amplitude is exact in \( \alpha' \), reflecting the fact the worldsheet theory has free operator products \( a^\mu(z) a^\nu(\zeta) = \eta^{\mu\nu}(z - \zeta)^{-2} \). Here \( a^\mu(z) = \sum a_n z^{-n-1} \). The three point open string tree amplitude for these tachyonic scalars is \( A_3 = \langle 0; -k_1 | V(k_2, 1) | 0; k_3 \rangle = (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3 + k_4) \).

The first excited level, \( N = 1 \), contains the vector states \( \psi = \epsilon \cdot a_{-1}|k\rangle \) with \( k^2 = 0 \). In order to satisfy \( L_n \psi = 0 \) for \( n > 0 \), (in particular \( L_1 \psi = 0 \)) we must have \( \epsilon \cdot k = 0 \). The vertex operator for this state is

\[ V(k, \epsilon, z) = \epsilon \cdot a(z) e^{ik \cdot X(z)}. \]  

(19)

The three point open string tree amplitude for these massless vectors is

\[ A_3 = \langle -k_1 | \epsilon_1 \cdot a_1 V(k_2, \epsilon_2, 1) \epsilon_3 \cdot a_{-1}|k_3 \rangle \]

\[ = (2\pi)^{26} \delta^{26} (k_1 + k_2 + k_3) \sqrt{2\alpha'} (\epsilon_1^a \epsilon_2^\mu \epsilon_3^\lambda t_{a\mu\lambda}(k_i) + 2\alpha' (\epsilon_1^a \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1)), \]

(20)

where

\[ t_{a\mu\lambda}(k_i) \equiv k_{2a} \eta_{\mu\lambda} + k_{3a} \eta_{\lambda\alpha} + k_{1\lambda} \eta_{a\mu}, \]

(21)
and we have recovered the dependence on the dimensional parameter $\alpha'$ by dimensional analysis, i.e. $k \to \sqrt{2\alpha'k}$. The four point open string tree amplitude for these massless vectors has tachyon poles and is given by

$$A_4 = \int_0^1 dz \left(-k_1|\epsilon_1 \cdot a_1 V(k_2, \epsilon_2, 1)V(k_3, \epsilon_3, z)\epsilon_4 \cdot a_{-1}|k_4\right)$$

$$= \int_0^1 dz \cdot z^{-\alpha'} \left(-k_1 - k_2|\epsilon_1 \cdot a_1 \epsilon_2 \cdot a(1)e^{-k_2} \sum_{n<0} \frac{a_n^{(2)} \cdot z^{-n}}{a_n} e^{-k_2} \sum_{n>0} \frac{a_n^{(2)} \cdot z^{-n}}{a_n} \epsilon_4 \cdot a(z) \epsilon_4 \cdot a_{-1}|k_4\right)$$

$$= (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3 + k_4) \int_0^1 dz \cdot z^{-\alpha'} (1 - z)^{-\alpha'}$$

$$\cdot \left[\epsilon_1 \cdot \epsilon_2 \epsilon_4 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_2 \epsilon_4 \cdot k_3 \{\epsilon_3 \cdot k_4 - \epsilon_3 \cdot k_2 z(1 - z)^{-1}\} - \epsilon_3 \cdot \epsilon_4 \epsilon_1 \cdot k_2 \{\epsilon_2 \cdot k_1 + \epsilon_2 \cdot k_3 z(1 - z)^{-1}\} - \epsilon_1 \cdot k_2 \epsilon_4 \cdot k_3 \{\epsilon_2 \cdot \epsilon_3 z(1 - z)^{-2} - \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_4\} + (\epsilon_3 \cdot k_4 \epsilon_2 \cdot k_3 + \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2) z(1 - z)^{-1} - \epsilon_3 \cdot k_2 \epsilon_2 \cdot k_3 z^2(1 - z)^{-2}\} \right].$$

(22)

### 2.5 Closed Bosonic String

The closed string satisfies the same equations of motion $\partial^2_x x^\mu - \partial^2_t x^\mu = 0$ but is topologically a cylinder with boundary condition $x^\mu(\sigma, \tau) = x^\mu(\sigma + 2\pi, \tau)$. The general solution is

$$x^\mu(\sigma, \tau) = q^\mu + p^\mu \tau + \frac{i}{2} \sum_{n \neq 0} \left\{a_n^{(L)} e^{-i(n + \sigma)} + a_n^{(R)} e^{i(n - \sigma)}\right\}$$

$$= \frac{1}{2} X_L^\mu(e^{i(\tau + \sigma)}) + \frac{1}{2} X_R^\mu(e^{i(\tau - \sigma)})$$

(23)

where $X_L^\mu(z) = q^\mu - i p_L^\mu \ln z + i \sum_{n \neq 0} \frac{a_n^{(L)}}{n} z^{-n}$, $X_R^\mu(z) = q^\mu - i p_R^\mu \ln z + i \sum_{n \neq 0} \frac{a_n^{(R)}}{n} z^{-n}$, and $p_L = p_R = p$. The covariant quantization conditions are

$$[a_m^{(L)}, a_n^{(L)}] = [a_m^{(R)}, a_n^{(R)}] = \eta^{\mu\nu} m \delta_{m,-n}; \quad [a^\mu, p^\nu] = \frac{i}{2} \eta^{\mu\nu}$$

(24)

$$[q^\mu, a_n^{(L)}] = [q^\mu, a_n^{(R)}] = 0, \quad n \neq 0$$

(25)

$$[a_m^{(L)}, a_n^{(R)}] = 0; \quad a_0^{(L)} \equiv a_0^{(R)} \equiv p^\mu.$$  

(26)

The physical state conditions are

$$\mathcal{L}_0^L |\psi\rangle = |\psi\rangle, \quad \mathcal{L}_n^L |\psi\rangle = 0, \quad \text{for } n > 0,$$

(27)
\[ L^R_0 |\psi\rangle = |\psi\rangle, \quad L^L_n |\psi\rangle = 0, \quad \text{for } n > 0. \] (28)

The mass shell condition is \( p^2 = -m^2 \) where \( m^2 = N_L + N_R - 2 \) and \( N_L = N_R \). \( L^R_0, L^L_n \) are two commuting Virasoro algebras, both with \( c = d = 26 \). In this quantization of the closed string in flat spacetime, the theory is seen to be a tensor product of left and right copies of the open string case. One can form vertices for the closed string by taking the tensor products of open string conformal fields for the left- and right movers, and using the variable \( z \) for the left vertices and \( \bar{z} \) with the right vertices. Here we have defined an euclidean world sheet metric \( i\tau \equiv t \), so that \( z = e^{t e^{i\sigma}}, \bar{z} = e^{t e^{-i\sigma}} \), which maps the cylinder traced by the moving string onto the complex plane \([7]\). Time ordering is radial ordering; \( t = \) constant hypersurfaces are circles concentric about the origin of the \( z-\)plane. In fact the local operator product relations will extend naturally to arbitrary Riemann surfaces with local conformal coordinates\( z, \bar{z} \). Since for closed strings, \((L^L_0 + L^R_0) |\psi\rangle = 2 |\psi\rangle \), the tachyon \( |k\rangle \equiv e^{2k \cdot q} |0\rangle \), \( k^2 = 2 \) has vertex operator
\[
V(k, z, \bar{z}) =: \exp\{ik \cdot X_L(z)\} :: \exp\{ik \cdot X_R(\bar{z})\} : \] (29)

When the spacetime is not flat, the conformal fields in general will not factor into left times right, although the two copies of the Virasoro algebra will remain holomorphic (and antiholomorphic) \([12]\).

The four point closed string tree amplitude for the tachyonic scalars in flat spacetime is again exact in \( \alpha' \):
\[
A_4 = \frac{\alpha'}{2\pi} \int d^2z \left\langle -k_1 | V(k_2, 1, 1) z \bar{z} V(k_3, z, \bar{z}) | k_4 \right\rangle \\
= \frac{\alpha'}{2\pi} (2\pi)^{26} \delta^{26}(k_1 + k_2 + k_3 + k_4) \int d^2z \frac{|z|^{2k_3-k_1} |1 - z|^{2k_2-k_3}}{\Gamma(-1-\alpha'/s)\Gamma(1-1-\alpha'/t)\Gamma(1-\alpha'/u)\Gamma(2+\alpha'/s)\Gamma(2+\alpha'/t)\Gamma(2+\alpha'/u)} \] (30)

which is totally symmetric under the exchange of \( s, t, u \). The massless level \( N_L = N_R = 1 \) contains the states \( \psi = e^L \cdot a^L_{-1} |k\rangle \) with \( k^2 = 0 \), and \( e^L \cdot k = e^R \cdot k = 0 \), forming the spin two graviton, the 2-form antisymmetric tensor, and the dilaton.

The preceeding analysis of physical state conditions can be reexpressed in an equivalent formulation using BRST cohomology \([7]\), which includes both
the “matter” sector described above and a BRST “ghost” sector with equal and opposite central charge. The worldsheet variables used to formulate strings in curved spacetime with Ramond flux, do not exhibit such a ‘matter times ghost’ factorization \[10, 12, 20\], although they still define physical state conditions as a version of cohomology.

3 Various Formulations of Superstring World-sheet Fields

3.1 Ramond-Neveu-Schwarz (RNS)

This a Lorentz covariant but not manifestly supersymmetric quantization. The Neveu-Schwarz (NS) fields \( b^\mu(z) = \sum_s b^\mu_s z^{-s-\frac{1}{2}} \) have \( \{ b^\mu_r, b^\nu_s \} = \eta^{\mu\nu} \delta_{r,-s} \), where \( r, s \in \mathbb{Z} + \frac{1}{2} \) and \( b^\mu_{s\dagger} = b^{\mu}_{-s} \). They are used to construct the super Virasoro generators \( G(z) = a(z) \cdot b(z) \) and \( L(z) = \frac{1}{2} : a(z) \cdot a(z) : + \frac{1}{2} : \frac{db(z)}{dz} \cdot b(z) : \) which satisfy the \( N=1 \) super Virasoro algebra

\[ [L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^2 - m) \delta_{m,-n} \]
\[ \{ G_r, G_s \} = 2 L_{r+s} + \frac{c}{12} (r^2 - \frac{1}{4}) \delta_{r,-s} \]
\[ [L_n, G_s] = \left( \frac{n^2}{2} - s \right) G_{n+s} . \] (31)

The No-Ghost theorem selects \( c = \frac{3}{2} d = 15 \). The physical state conditions in the \( \mathcal{F}_2 \)-picture are \( L_0 |\psi\rangle = \frac{1}{2} |\psi\rangle \), \( L_n |\psi\rangle = 0 \) for \( n > 0 \), \( G_s |\psi\rangle = 0 \) for \( s > 0 \). In a superconformal field theory the states are in one-to-one correspondence with a conformal superfield \( V(\psi, z, \vartheta) = V_0(\psi, z) + \vartheta V_1(\psi, z) \), where \( \vartheta \) is a fermionic coordinate, the supersymmetry partner of \( z \). \( V_0(\psi, z) \) and \( V_1(\psi, z) \) are called the lower and upper components of the superfield respectively.

\[ \lim_{z \to 0} V_0(\psi, z) |0\rangle = |\psi\rangle , \quad \lim_{z \to 0} V_1(\psi, z) |0\rangle = G_{-\frac{1}{2}} |\psi\rangle . \] (32)

In covariant gauge, the vertex operators for physical states have to satisfy

\[ [L_n, V_0(\psi, z)] = z^{n+1} \frac{d}{dz} V_0(\psi, z) + h(n+1) z^n V_0(\psi, z) \]
\[ [L_n, V_1(\psi, z)] = z^{n+1} \frac{d}{dz} V_1(\psi, z) + (h+\frac{1}{2})(n+1) z^n V_1(\psi, z) \]
\[ \{ G_s, V_0(\psi, z) \}_\pm = z^{s+\frac{1}{2}} V_1(\psi, z) \]
\[ [G_s, V_1(\psi, z)]_z = z^{s+\frac{1}{2}} \frac{dV_0(\psi, z)}{dz} + 2h(s + \frac{1}{2})z^{s-\frac{1}{2}}V_0(\psi, z) \quad (33) \]

for all \( n \) and \( s \), for \( h = \frac{1}{2} \). At level \( N = 0 \), there is one state \( \psi = |k\rangle \) with \( k^2 = 1 \). Its vertex operator has components

\[ V_0(k, z) = \exp\{ik \cdot X(z)\}, \quad V_1(k, z) = \sqrt{2\alpha'} k \cdot b(z) \exp\{ik \cdot X(z)\} \quad (34) \]

Unlike the bosonic case, the three point amplitude for the Neveu-Schwarz tachyon vanishes: \( A_3 = \langle -k_1 | V_1(k_2, 1) | k_3 \rangle = 0 \).

The massless vector is at level \( N = \frac{1}{2} \), \( \psi = \epsilon \cdot b_{-\frac{1}{2}} |k\rangle \) with \( k^2 = 0 \). To satisfy \( L_n \psi = 0 \) for \( n > 0 \), \( G_s \psi = 0 \) for \( s > 0 \) (in particular \( G_{-\frac{1}{2}} \psi = 0 \)) we must have \( \epsilon \cdot k = 0 \). Its vertex operator has components

\[ V_0(k, \epsilon, z) = \epsilon \cdot b(z) e^{ik \cdot X(z)}, \quad V_1(k, \epsilon, z) = \left\{ \sqrt{2\alpha'} k \cdot b(z) \epsilon \cdot b(z) + \epsilon \cdot a(z) \right\} \exp\{ik \cdot X(z)\} \quad (35) \]

For this state, \( G_{-\frac{1}{2}} \psi = (k \cdot b_{-\frac{1}{2}} \epsilon \cdot b_{-\frac{1}{2}} + \epsilon \cdot a_{-1}) |k\rangle \). The three point open string tree amplitude for these massless vectors is

\[ A_3 = \langle -k_1 | \epsilon_1 \cdot b_{-\frac{1}{2}} V_1(k_2, \epsilon_2, 1) \epsilon_3 \cdot b_{-\frac{1}{2}} | k_3 \rangle \\
= (2\pi)^{10} \delta^{10}(k_1 + k_2 + k_3) \sqrt{2\alpha'} \epsilon_1^\alpha \epsilon_2^\beta \epsilon_3^\gamma t_{\alpha\beta\gamma}(k_i) \quad (36) \]

We see that the Neveu-Schwarz computation of \( A_3 \) has no \( \alpha' \) corrections, in contrast with (20). Non-renormalization theorems for the superstring often prevent \( \alpha' \) corrections to the tree level three point functions, although not to four point functions. For both expressions, the zero slope limit reduces to the conventional three gluon field theory coupling: \( \lim_{\alpha' \to 0} A_3 \frac{1}{\sqrt{2\alpha'}} = \frac{(2\pi)^d \delta^{(4)}(\sum_i k_i) \epsilon_1^\alpha \epsilon_2^\beta \epsilon_3^\gamma t_{\alpha\beta\gamma}(k_i)}{\sqrt{2\alpha'}} \).

The space of states for the open superstring, \( H \), is obtained by starting with the states of the untwisted Neveu-Schwarz theory, \( H \), introduced above, then adding in keeping only the subspace of each defined by \( \theta = 1 \), with \( \theta^2 = 1 \). The states of the untwisted theory are generated by the action of \( d \) infinite sets of half-integrally moded oscillators, \( b_\mu^n \), \( 0 \leq \mu \leq d - 1 \) (together with the integrally moded oscillators, \( a_\mu^n \)), on the vacuum state, \( |0\rangle \). The twisted sector is obtained from the action of \( d \) infinite sets of integrally
moded oscillators, $d_n^a$, (together with the integrally moded oscillators, $a_n^a$), on the twisted ground states which form a $2d/2$ irreducible representation, $\mathcal{X}$, of the gamma matrix Clifford algebra, $\{\gamma^\mu\}$. The involution $\theta$ is defined on the untwisted space $\mathcal{H}$ by $\theta|0\rangle = (-1)|0\rangle$, $\theta b_n^a \theta^{-1} = -b_n^a$, and on the twisted space, $\mathcal{H}_T$, by $\theta|0\rangle_T = \pm|0\rangle_T$, $\theta d_n^a \theta^{-1} = -d_n^a$, where $\mathcal{X} = |0\rangle_T + |0\rangle_T$. Whenever $d$ is even we can define $\gamma^{d+1} \equiv \gamma^1 \gamma^2 \ldots \gamma^d$ which satisfies $\{\gamma^{d+1}, \gamma^\mu\} = 0$, $(\gamma^{d+1})^2 = 1$. The operators $\frac{1}{2}(1 \pm \gamma^{d+1}(-1)^{\sum_{n=0}^{d-n}d_n})$ are chirality projection operators; they project onto spinors of definite chirality. A spinor of definite chirality is called a Weyl spinor; and the restriction to spinors of one chirality or the other is called a Weyl condition. In the Neveu-Schwarz sector, $\theta \equiv (-1)(-1)^{\sum_{n>0} b_n^a b_n^a}$, and in the Ramond sector, $\theta \equiv \gamma^{d+1}(-1)^{\sum_{n>0} d_n^a d_n^a}$. The Ramond Fock space splits into two $SO(d)$ invariant subspaces, according to the eigenvalue of $\theta$, the ground states being denoted by $|0\rangle_T^{\pm}$. The $\theta = 1$ subspace is thus a projection onto the odd $b$ sector, and onto chiral fermions in the Ramond sector, and is known as the Gliozzi-Scherk-Olive (GSO) projection. The worldsheet fermion fields are $\psi^\mu(z)$ in either the Neveu-Schwarz or Ramond representation $\psi^\mu(z) = b^\mu(z)$ or $d^\mu(z) = \sum_n d_n^\mu z^{-n-\frac{1}{2}}$, representing the vertices for emitting the massless vector state from a Neveu-Schwarz or Ramond line, respectively. The ground state $|a\rangle$ of the Ramond sector of the superstring is a spacetime fermion which is in one-to-one correspondence with a worldsheet field $S_a(z)$ called a spin field:

$$|a\rangle = \lim_{z \to 0} S_a(z) |0\rangle .$$

The spin fields are non-local with respect to the ordinary superconformal fields $\psi^\mu(z)$:

$$\psi^\mu(z) S^a(\zeta) \sim (z - \zeta)^{-\frac{1}{2}} \gamma^\mu^a S^b(\zeta)$$
$$\psi^\mu(z) \psi^\nu(\zeta) \sim (z - \zeta)^{-1}$$
$$a^\mu(z) a^\nu(\zeta) \sim (z - \zeta)^{-2}, \quad a^\mu(z) \psi^\nu(\zeta) \sim 0, \quad a^\mu(z) S^a(\zeta) \sim 0$$

(38)

due to the non-meromorphic structure of the operator product of $\psi$ with $S$. It follows that the worldsheet supercurrents $G(z)$ are not local with respect to the spin fields. It follows that in the presence of background Ramond fields,
the superconformal invariance of the worldsheet action $S_2$ would appear to be violated: the supersymmetrization of the bosonic Polyakov action (3) contains terms of the form
\[
S_2 = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{|g|} \left( \partial_\alpha x^\mu \partial_\beta x^\nu + i \bar{\psi}_\mu \gamma^\alpha \partial_\alpha \psi^\nu \right) G_{\mu\nu}(x)
\]
and is superconformally invariant, called the guiding principle of the RNS description of perturbative superstrings. When a background Ramond field $B_{\mu\nu}$ is required, one might try to add to $S_2$ a term such as
\[
\int d\sigma d\tau S_a^L S_b^R [\gamma^\mu, \gamma^\nu]_{ab} B_{\mu\nu}(x)
\]
but the presence of spin fields jeopardizes the superconformal invariance. Several efforts to answer this problem have been made, some of which we cite here [30, 31, 32, 33, 12, 20]. One formulation [12, 20] which survives quantization dispenses with spin fields altogether. It is discussed in sections 3.3 and 4.

An extension of the RNS superstring to include BRST ghost fields recasts the physical state conditions as cohomology. In this reformulation, in addition to the left and right moving “matter” fields $X^\mu(z, \bar{z}), \psi_L^\mu(z), \psi_R^\mu(\bar{z})$ combining to give central charge $c = 15$, there are “ghost” fields $b_L(z), c_L(z), \beta_L(z), \gamma_L(z)$ and $b_R(\bar{z}), c_R(\bar{z}), \beta_R(\bar{z}), \gamma_R(\bar{z})$ contributing $c = -15$. Left and right-moving BRST charges each have the structure $Q \sim c(L_m + \frac{1}{2} L_g) + \gamma \left( G_m + \frac{1}{2} G_g \right)$ and the left and right Virasoro generators have zero central charge $L \sim L_m + L_g$.

### 3.2 Green-Schwarz (GS)

This is a supersymmetric but not Lorentz covariant quantization. The open string worldsheet fields are $S_a(z), a^\mu(z)$ and satisfy meromorphic operator products
\[
S^a(z) S^b(\zeta) \sim (z - \zeta)^{-1} c^{ab}, \quad a^i(z) a^j(\zeta) \sim (z - \zeta)^{-2} \delta^i_j, \quad a^i(z) S^a(\zeta) \sim 0
\]
(41)
since the coordinates are limited to the light-cone $1 \leq i, j \leq 8$, and $1 \leq a, b \leq 8$. Here the RNS spin field $S^a(z)$ has been promoted to a fundamental
worldsheet variable, whereas in the \textit{RNS} case in fact it is expressible as a combination of the $\psi$ fields, $S \sim e^{b \cdot d}$. The Green-Schwarz formulation of the superstring dispenses with the need to sum over different spin structures (related to the NS and R sectors) in the one-loop string amplitudes.

### 3.3 Berkovits-Vafa-Witten (BVW)

This is a covariant and supersymmetric quantization in six spacetime dimensions. It has been applied primarily to compactifications of the Type IIB string either in the “flat” case $R^6 \times K3$, or the “curved” case $AdS_3 \times S^3 \times K3$. \textit{BVW} provides a partially covariant quantization of the Green-Schwarz superstring. Eight of the sixteen supersymmetries are manifest, in the sense that they act geometrically on the target space of the worldsheet sigma model. In addition, there are no worldsheet spin fields and so can more easily incorporate Ramond-Ramond background fields.

The \textit{BVW} worldsheet fields are $X^m, \theta^a, \bar{\theta}^{\dot{a}}$, the conjugate fermions $p^a, \bar{p}^{\dot{a}}$ for $1 \leq m \leq 6; 1 \leq a \leq 4$, and two additional worldsheet bosons $\rho, \sigma, \bar{\rho}, \bar{\sigma}$. These describe the $d = 6$ part of the Type IIB string. The $K3$ part is described by the standard \textit{RNS} description of a $T^4/Z_2$ orbifold. The $\theta^a$’s are ordinary conformal fields, not spin fields. In flat space, the worldsheet variables are holomorphic and satisfy free operator products relations including

$$p_a(z)\theta^b(\zeta) \sim (z - \zeta)^{-1}\delta^{ab}.$$  \hspace{1cm} (42)

In curved space, the worldsheet fields are no longer holomorphic, and the worldsheet action becomes a sigma model with the supergroup $PSU(2|2)$ as target, which is no longer a free conformal field theory nor a Wess-Zumino-Witten (WZW) model.

A ten dimensional version of these variables has appeared recently \cite{20}-\cite{24}. In flat spacetime, field redefinitions give back the \textit{RNS} formalism. In $AdS_5 \times S^5$, vertex operator constraint equations have been considered \cite{23}. The Berkovits variables are given by the ten-dimensional superspace variables $X^\mu(z, \bar{z}), \theta^a_L(z, \bar{z}), \theta^a_R(z, \bar{z})$, for $0 \leq \mu \leq 9, 1 \leq a \leq 16$; and the conjugate fermionic worldsheet fields $p^a_L(z, \bar{z}), p^a_R(z, \bar{z})$. There are additional worldsheet
bosons that are spacetime spinors $\lambda_R^\alpha(z, \bar{z})$, which separately satisfy $\lambda^\alpha \gamma_\mu \gamma_5 \lambda^\beta = 0$ and carry 22 degrees of freedom. The construction includes left and right-moving BRST charge operators and Virasoro generators. The contribution to the central charge is $10 + 22$ from the worldsheet bosons, and $-32$ from the worldsheet fermions. The variables do not exhibit the “matter times ghost” structure of conventional the BRST formalism. For $AdS_5 \times S^5$ spacetime, the worldsheet fields are not holomorphic.

4 Type IIB Superstrings on $AdS_3 \times S^3 \times K^3$

Compatification of the Type IIB superstring on either $R^6 \times K^3$ or $AdS_3 \times S^3 \times K^3$ yields a $d = 6$, $N = (2,0)$ theory which has sixteen supercharges. The 6d massless particle content is a supergravity multiplet and 21 tensor multiplets. In flat space the multiplets are representations of the light-cone little group $SO(4)$: $sg (3,3) + 5(3,1) + 4(3,2)$, $tensor (1,3) + 5(1,1) + 4(1,2)$. In curved space the number of physical degrees of freedom in the multiplets remains the same. The “compatification independent” 6d fields make up the supergravity multiplet and one of the tensor multiplets, they are the graviton $(3,3)$, an antisymmetric tensor $(3,1) + (1,3)$, and a scalar $(1,1)$: $g_{mn}(x), b_{mn}(x), \phi(x)$; four self-dual tensors $(3,1)$ and four scalars contained in $V^{--}_{a\bar{a}}(x), F^{++}(x), A^{--}_a(x), A^{+-}_{a\bar{a}}$; and four gravitinos $(3,2)$ and four spinors $(1,2)$ contained in $\xi^-_{ma}(x), \bar{\xi}^-_{ma}(x), \chi^a_m(x), \bar{\chi}^a_m(x)$.

The vertex operator which describes these states is given in terms of the (compatification independent) worldsheet fields $X^m, \theta^a, \bar{\theta}^\alpha$ by the superfield

$$V_{1,1} = \theta^a \bar{\theta}^\alpha V^{--}_{a\bar{a}} + \theta^a \theta^b \bar{\theta}^\beta \bar{\sigma}^m_{ab} \bar{\epsilon}^\alpha_{ma} + \theta^a \bar{\theta}^\alpha \bar{\theta}^\beta \sigma^m_{ab} \bar{\epsilon}^\alpha_{ma} + \theta^a \theta^b \bar{\theta}^\beta \sigma_{ab}^m (g_{mn} + b_{mn} + \bar{g}_{mn} \phi) + \theta^a (\bar{\theta}^3)_a A^{--}_{a\bar{a}} + (\theta^3)_a \bar{\theta}^\alpha A^{+-}_{a\bar{a}} + \theta^a \theta^b (\bar{\theta}^3)_a \sigma^m_{ab} \bar{\chi}^{+\bar{a}}_m + (\theta^3)_a \theta^a \bar{\theta}^\beta \sigma^m_{ab} \chi^{+\bar{a}}_m + (\theta^3)_a (\bar{\theta}^3)_a F^{++}_{a\bar{a}}.$$

(43)

The string constraint equations which select the physical states are generated in this formalism by a topological $N = 4$ superVirasoro algebra we will discuss in the next section. For flat spacetime, the constraints will result in
that all the above 6d fields satisfy $\partial^m \partial_n \phi = 0$ and

$$
\begin{align*}
\partial^m g_{mn} &= -\partial_n \phi, \quad \partial^m b_{mn} = 0, \quad \partial^m \chi^+_m = \partial^m \chi^-_m = 0 \\
\partial_{ab} \chi^+_m &= \partial_{ab} \chi^-_m = 0, \quad \partial_{cb} F^{\pm \pm \bar{a}} = \partial_{cb} F^{\pm \pm ba} = 0,
\end{align*}
$$

where

$$
F^{++-\bar{a}} = \partial^\bar{a} \Lambda^+_{\bar{a}} - \Lambda^a_{\bar{a}}, \quad F^{-+-\bar{a}} = \partial^a \Lambda_{\bar{a}}^- - \Lambda^a_{\bar{a}}^-, \quad F^{--\bar{a}} = \partial^a \partial^\bar{a} V_{\bar{a}}^-
$$

$$
\chi^+_a = \partial^a \chi^+_{mb}, \quad \chi^-_a = \partial^a \chi^-_{mb}.
$$

These are equivalent to the equations of motion for $D = 6, N = (2, 0)$ supergravity \[13\] with one tensor multiplet expanded around the six-dimensional Minkowski metric.

In the curved case $AdS_3 \times S^3$, the constraints will result in a different set of equations of motion for the 6d fields. We give the answer here for the bosonic 6d fields, and show the derivation in section 4.2. The six-dimensional metric field $g_{rs}$, the dilaton $\phi$, and the two-form $b_{rs}$ satisfy

$$
\frac{1}{2} D^p D_p b_{rs} = -\frac{1}{2} (\sigma_r \sigma^p \sigma^q)_{ab} \delta^{ab} D_p \left[ g_{qs} + \tilde{g}_{qs} \phi \right] + \frac{1}{2} (\sigma_s \sigma^p \sigma^q)_{ab} \delta^{ab} D_p \left[ g_{qr} + \tilde{g}_{qr} \phi \right] - R_{\tau rs \lambda} b^{\tau \lambda} - \frac{1}{2} R_{\tau \sigma} b_{rs} - \frac{1}{2} R_{\tau \sigma} b_{\tau \sigma}
$$

$$+ \frac{1}{4} F_{\text{asy}}^{++gh} \sigma^a_{\tau\lambda} \delta_{ab} \delta_{ac} \delta_{df}.
$$

$$
(46)
$$

$$
\frac{1}{2} D^p D_p \left( g_{rs} + \tilde{g}_{rs} \phi \right) = -\frac{1}{2} (\sigma_r \sigma^p \sigma^q)_{ab} \delta^{ab} D_p b_{qs} + \frac{1}{2} (\sigma_s \sigma^p \sigma^q)_{ab} \delta^{ab} D_p b_{qr} - R_{\tau rs \lambda} (g^{\tau \lambda} + \tilde{g}^{\tau \lambda} \phi) - \frac{1}{2} R^{\tau \sigma} (g_{rs} + \tilde{g}_{rs} \phi)
$$

$$- \frac{1}{2} R^{\tau \sigma} (g_{\tau \sigma} + \tilde{g}_{\tau \sigma} \phi) + \frac{1}{4} F_{\text{sym}}^{++gh} \sigma_{rs \sigma} \delta^{ab}.
$$

$$
(47)
$$

This is the curved space version of the flat space zero Laplacian condition $\partial^p \partial_p b_{rs} = \partial^p \partial_p g_{rs} = \partial^p \partial_p \phi = 0$.

Four self-dual tensor and scalar pairs come from the string bispinor fields $F^{++a}, V_{ab}^-, A_{ac}^+, A_{ac}^+$. From the string constraint equations they satisfy

$$
\sigma^a_{da} D_p F_{\text{asy}}^{++ab} = 0
$$

$$
(48)
$$

$$
\frac{1}{4} \left[ \delta^{Ba} \sigma^r_{ga} D_r F_{\text{sym}}^{++gH} - \delta^{Ha} \sigma^r_{ga} D_r F_{\text{sym}}^{++gB} \right] = -\frac{1}{4} \epsilon^{BH} cd F_{\text{asy}}^{++cd}.
$$

$$
(49)
$$
These can be shown [13] to be equivalent to the linearized supergravity equations [16] for the supergravity multiplet and one tensor multiplet of the $d = 6$, $N = (2,0)$ theory expanded around the $AdS_3 \times S^3$ metric and a self-dual three-form, by using the following field identifications: the vertex operator components in terms of the supergravity fields $g^{prs}, g^6_{prs}, h_{rs}, \phi^i$, $1 \leq i \leq 5$, (and $2 \leq I \leq 5$) are

\begin{align*}
H^{prs} & \equiv g^6_{prs} + 2 g^1_{prs} + B^I g^I_{prs} \\
g_{rs} & \equiv h_{rs} - \frac{i}{6} g^6_{rs} h^\lambda_{\lambda} \\
\phi & = -\frac{1}{3} h^\lambda_{\lambda} \\
F_{++ab}^{sym} &= \frac{2}{3} (\sigma_p \sigma_r \sigma_s)^{ab} B^I g^I_{prs} + \delta^{ab} \phi^{++} \\
F_{asy}^{++ab} &= \sigma^{pa} D_p \phi^{++} \\
\phi^{++} &= 4 C^I \phi^I \\
F_{sym}^{+-ab} &= \frac{2}{3} (\sigma_p \sigma_r \sigma_s)^{ab} B^I g^I_{prs} + \delta^{ab} \phi^{+-} \\
F_{asy}^{+-ab} &= \sigma^{pa} D_p \phi^{+-} \\
\phi^{+-} &= 4 C^I \phi^I
\end{align*}

which follows from choosing the graviton trace $h^\lambda_{\lambda}$ to satisfy $\phi^I - h^\lambda_{\lambda} \equiv -2 C^I \phi^I$. Here $H^{prs} \equiv \partial_p b_{rs} + \partial_r b_{sp} + \partial_s b_{pr}$. The combinations $C^I \phi^I$ and $B^I g^I_{prs}$ reflect the $SO(4)_R$ symmetry of the $D = 6, N = (2,0)$ theory on $AdS_3 \times S^3$. We relabel $C^I = C^I_{++}, B^I = B^I_{++}$. To define the remaining string components in terms of supergravity fields, we consider linearly independent quantities $C^I_\ell \phi^I, B^I_\ell g^I_{prs}, \ell = \ldots, -+,-,--$.

\begin{align*}
F_{sym}^{+-ab} &= \frac{2}{3} (\sigma_p \sigma_r \sigma_s)^{ab} B^I_{--} g^I_{prs} + \delta^{ab} \phi^{+-} \\
F_{asy}^{+-ab} &= \sigma^{pa} D_p \phi^{+-} \\
\phi^{+-} &= 4 C^I_{--} \phi^I \\
F_{sym}^{--ab} &= \frac{2}{3} (\sigma_p \sigma_r \sigma_s)^{ab} B^I_{--} g^I_{prs} + \delta^{ab} \phi^{--} \\
F_{asy}^{--ab} &= \sigma^{pa} D_p \phi^{--} \\
\phi^{--} &= 4 C^I_{++} \phi^I
\end{align*}

$V^{--}_{ab}$ is given in terms of the fourth tensor/scalar pair $C^I_{--} \phi^I, B^I_{--} g^I_{mnp}$ through

\begin{align*}
D^p D_p V^{--}_{cd} &= -\delta^{gh} \sigma^{cp}_{ch} D_p V^{--}_{gd} + \delta^{gh} \sigma^{dp}_{dh} D_p V^{--}_{cg} + \frac{1}{2} \epsilon^{gh}_{cd} V^{--}_{gh} = -8 \sigma^{m}_{ce} \sigma^{p}_{dp} \delta^{ef} g_{mn}.
\end{align*}
4.1 Topological Strings

The origin of the constraints is an $N = 4$ twisted superconformal algebra. In this section, we review how the superstring can be reformulated as an $N = 4$ topological string theory, and show how this formalism gives a description of the superstring with manifest $d = 6$ spacetime supersymmetry. That is to say, the spectrum of the superstring can be identified with the states surviving a set of $N = 4$ constraints. We begin this subsection by remembering how the bosonic string can be reorganized as an $N = 2$ topological string \[10, 12\]. An $N = 2$ topological string has a twisted $N = 2$ superconformal algebra

\[
\begin{align*}
\hat{T}(z)\hat{T}(\zeta) &= (z - \zeta)^{-2}2\hat{T}(\zeta) + (z - \zeta)^{-1}\partial\hat{T}(\zeta), \\
\hat{T}(z)G^+(\zeta) &= (z - \zeta)^{-2}G^+(\zeta) + (z - \zeta)^{-1}\partial G^+(\zeta), \\
\hat{T}(z)G^-(\zeta) &= (z - \zeta)^{-2}2G^-(\zeta) + (z - \zeta)^{-1}\partial G^-(\zeta), \\
G^+(z)G^-(\zeta) &= (z - \zeta)^{-3}\frac{\mathfrak{e}}{3} + (z - \zeta)^{-2}J(\zeta) + (z - \zeta)^{-1}\hat{T}(\zeta), \\
G^+(z)G^+(\zeta) &= 0, \\
G^-(z)G^-(\zeta) &= 0, \\
\hat{T}(z)J(\zeta) &= (z - \zeta)^{-3}\left(-\frac{\mathfrak{e}}{3}\right) + (z - \zeta)^{-2}J(\zeta) + (z - \zeta)^{-1}\partial J(\zeta), \\
J(z)J(\zeta) &= (z - \zeta)^{-2}\frac{\mathfrak{e}}{3}, \\
J(z)G^+(\zeta) &= (z - \zeta)^{-1}G^+(\zeta), \\
J(z)G^-(\zeta) &= -(z - \zeta)^{-1}G^-(\zeta).
\end{align*}
\]

(56)

is related to the generators of the (untwisted) $N = 2$ superconformal algebra:

\[
\begin{align*}
L(z)L(\zeta) &= (z - \zeta)^{-4}\frac{\mathfrak{e}}{2} + (z - \zeta)^{-2}2L(\zeta) + (z - \zeta)^{-1}\partial L(\zeta), \\
L(z)G^+(\zeta) &= (z - \zeta)^{-2}\frac{3\mathfrak{e}}{2}G^+(\zeta) + (z - \zeta)^{-1}\partial G^+(\zeta), \\
G^+(z)G^-(\zeta) &= (z - \zeta)^{-3}\frac{\mathfrak{e}}{3} + (z - \zeta)^{-2}J(\zeta) + (z - \zeta)^{-1}(L(\zeta) + \frac{1}{2}\partial J(\zeta)), \\
G^+(z)G^+(\zeta) &= 0, \\
G^-(z)G^-(\zeta) &= 0, \\
L(z)J(\zeta) &= (z - \zeta)^{-2}J(\zeta) + (z - \zeta)^{-1}\partial J(\zeta), \\
J(z)J(\zeta) &= (z - \zeta)^{-2}\frac{\mathfrak{e}}{3}, \\
J(z)G^+(\zeta) &= \pm(z - \zeta)^{-1}G^+(\zeta).
\end{align*}
\]

(57)

where the generators differ only by the twisted Virasoro generator $\hat{T}(z) \equiv L(z) + \frac{1}{2}\partial J(z)$. Since the OPE’s (56) resemble somewhat those of the bosonic
string (58) we can define physical fields relative to $Q_0$ cohomology where $G^+(z) = \sum_n Q_n z^{-n-1}$. Physical fields correspond to chiral primary fields $\Phi^+(z)$ with ghost charge +1 and dimension 0 (they arise from operators that have ghost charge +1 and dimension $\frac{1}{2}$ before the algebra is twisted), so that $\{Q_0, \Phi^+(z)\} = 0$.

Bosonic string theory can be viewed as a two-dimensional conformal field theory with certain additional features (we concentrate on left-movers and will denote right-movers with barred notation):

$$T_{tot}(z)T_{tot}(\zeta) = (z-\zeta)^{-2} 2T_{tot}(\zeta) + (z-\zeta)^{-1} \partial T_{tot}(\zeta),$$
$$T_{tot}(z)j_{\text{BRST}}(\zeta) = (z-\zeta)^{-2} j_{\text{BRST}}(\zeta) + (z-\zeta)^{-1} \partial j_{\text{BRST}}(\zeta),$$
$$T_{tot}(z)b(\zeta) = (z-\zeta)^{-2} 2b(\zeta) + (z-\zeta)^{-1} \partial b(\zeta),$$
$$j_{\text{BRST}}(z)b(\zeta) = (z-\zeta)^{-2} (c=9) + (z-\zeta)^{-2} j_{\text{ghost}}(\zeta) + (z-\zeta)^{-1} T_{tot}(\zeta),$$
$$j_{\text{BRST}}(z)j_{\text{BRST}}(\zeta) = -j_{\text{BRST}}(\zeta)j_{\text{BRST}}(z) = \frac{2}{N} \partial \partial c(\zeta) c(\zeta) \neq 0,$$
$$b(z)b(\zeta) = 0,$$
$$T_{tot}(z)j_{\text{ghost}}(\zeta) = (z-\zeta)^{-3} (c=9) + (z-\zeta)^{-2} j_{\text{ghost}}(\zeta) + (z-\zeta)^{-1} \partial j_{\text{ghost}}(\zeta),$$
$$j_{\text{ghost}}(z)j_{\text{ghost}}(\zeta) = (z-\zeta)^{-2} (c=3),$$
$$j_{\text{ghost}}(z)j_{\text{BRST}}(\zeta) = (z-\zeta)^{-3} 4c(\zeta) + (z-\zeta)^{-2} \partial c(\zeta) + (z-\zeta)^{-1} j_{\text{BRST}}(\zeta),$$
$$j_{\text{ghost}}(z)b(\zeta) = -(z-\zeta)^{-1} b(\zeta),$$

where the generators are

$$T_{tot} = T^{N=0}_m + T^{N=0}_g = T^{N=0}_m - 2 \partial c - \partial b c$$
$$j_{\text{BRST}}(z) = c T_m + \frac{1}{2} c T_g + \frac{3}{2} \partial^2 c = c T_m - \partial b c$$
$$j_{\text{ghost}}(z) = \partial b$$

and $c(z)b(\zeta) = -b(\zeta)c(z) = (z-\zeta)^{-1} + \zeta c(z)b(\zeta) \zeta$. The normal ordering has been defined putting the annihilation operators to the right of the creation operators, where $b_n|0\rangle^{bc} = 0$ for $n \geq -1$; $c_n|0\rangle^{bc} = 0$ for $n \geq 2$. The BRST charge is $Q \equiv \frac{1}{2\pi} \int dz j_{\text{BRST}}(z) = \sum_m c_m L_m^X - \frac{1}{2} \sum_{m,n}(m-n) c_m c_n b_{n+m}$.

Then $Q |0\rangle^{bc} \otimes |\phi\rangle_X = 0$. The ghost charge is $J_0 \equiv \sum_n \zeta c_n b_{-n} \zeta$. Then $(c_1|0\rangle^{bc} \otimes |\phi\rangle_X)$ has ghost charge eigenvalue, i.e. ghost number, equal to one. The physical state conditions in the “old covariant” formalism are $(L_0^X - 1)|\phi\rangle_X = 0$, $L_n^X |\phi\rangle_X = 0$ for $n > 0$. Since $Q|\psi\rangle = 0$ implies
$(c_0 L_0^X - 1 + \sum_n c_n L_n^X) |\psi\rangle = 0$ when $|\psi\rangle = c_1 |0\rangle^bc \otimes |\phi\rangle_X$, then in the BRST formalism the physical vertex operators are defined by ghost number one fields $\Phi^+(z)$ that obey $\{Q, \Phi^+(z)\} = 0$, i.e. the OPE of $j_{BRST}(z) \Phi^+(\zeta)$ has no single pole. So here, every physical state is in one-to-one correspondence with a primary field of the Virasoro algebra of dimension 0, i.e. $\Phi^+(z) = c(z) \phi_X(z)$. Thus we have used the twisted $N = 2$ super-Virasoro algebra to define physical fields relative to $Q_0$ cohomology where $G^+(z) = \sum_n Q_n z^{-n-1}$. Physical states correspond to chiral primary fields $\Phi^+(z)$ with ghost charge +1 and dimension 0.

For $N = 2$ topological strings, $c = 9$. $N = 4$ topological strings are used when $c = 6$. From an $N = 2$ superconformal algebra with $c = 6$, we construct a topological $N = 4$ string by defining the remaining generators and twisting the $N = 4$ superconformal algebra:

$$T(z)T(\zeta) = (z - \zeta)^{-2}T(\zeta) + (z - \zeta)^{-1}\partial T(\zeta),$$
$$\tilde{T}(z)G^+(\zeta) = (z - \zeta)^{-2}G^+(\zeta) + (z - \zeta)^{-1}\partial G^+(\zeta),$$
$$\tilde{T}(z)G^-(\zeta) = (z - \zeta)^{-2}G^-(\zeta) + (z - \zeta)^{-1}\partial G^-(\zeta),$$
$$G^+(z)G^-(\zeta) = (z - \zeta)^{-3} + (z - \zeta)^{-2}J(\zeta) + (z - \zeta)^{-1}\tilde{T}(\zeta),$$
$$G^+(z)G^+(\zeta) = 0, G^-(z)G^-(\zeta) = 0,$$
\[ \tilde{G}^+(z) \tilde{G}^+(\zeta) = 0, \quad \tilde{G}^-(z) \tilde{G}^-(\zeta) = 0 \]
\[ \tilde{T}(z) \tilde{G}^+(\zeta) = (z - \zeta)^{-2} \tilde{G}^+(\zeta) + (z - \zeta)^{-1} \partial \tilde{G}^+(\zeta), \]
\[ \tilde{T}(z) \tilde{G}^-(\zeta) = (z - \zeta)^{-2} \tilde{G}^-(\zeta) + (z - \zeta)^{-1} \partial \tilde{G}^-(\zeta), \]
\[ J(z) \tilde{G}^\pm(\zeta) = \pm (z - \zeta)^{-1} \tilde{G}^\pm(\zeta). \quad (60) \]

Since the superstring can be written as an \( N = 2 \) super Virasoro algebra with \( c = 6 \), it's necessary to find additional generators making up an \( N = 4 \) topological string. In RNS variables they are

\[
\tilde{T}(z) = T_{m=1}^{N=1} + T_{g=1}^{N=1} = T_{m=1}^{N=1} - 2 \beta \partial \gamma \gamma \gamma \gamma - \frac{3}{2} \beta \partial \beta \gamma \gamma - \frac{1}{2} \beta \partial \beta \gamma \gamma \gamma \gamma
\]
\[
G^+(z) = \gamma G_m + c(T_m - \frac{3}{2} \beta \partial \gamma \gamma - \frac{1}{2} \beta \partial \beta \gamma \gamma - b \partial c + \partial (c \eta))
\]
\[
\tilde{G}^-(z) = b(ie^\phi G_m + \eta e^{2\phi} \partial b - c \partial \xi) + \xi(T_m - \frac{3}{2} \beta \partial \gamma \gamma - \frac{1}{2} \beta \partial \beta \gamma \gamma - 2 b \partial c + c \partial b) + \partial^2 \xi,
\]

with \( c(z)b(\zeta) = -b(\zeta)c(z) = (z - \zeta)^{-1} + \gamma c(z)b(\zeta) \). Also, the super-reparametrization ghosts with \( \gamma(z) \beta(\zeta) = \beta(\zeta) \gamma(z) = (z - \zeta)^{-1} + \gamma(z) \beta(\zeta) \) have been bosonized as \((\beta = ie^{-\phi} \partial \xi, \gamma = -i \eta e^\phi)\) with \( \xi(z) \eta(\zeta) = -\eta(\zeta) \xi(z) = (z - \zeta)^{-1} + \gamma(\zeta) \eta(\zeta) \), and \( \phi(z) \phi(\zeta) = -\ln(z - \zeta) + \gamma(\zeta) \phi(\zeta) \) so that \( e^{-\phi(z)e^\phi(z)} = e^{-\phi(z)+\phi(z)}(z - \zeta) \).

The generators (61) satisfy the \( (c = 6) \) twisted \( N = 4 \) superconformal algebra given in (60). Since the algebra is twisted, the Virasoro generators close with no anomaly (i.e. \( c = 0 \)) but \( c \) still appears in the rest of the algebra, such as the anomaly of the \( U(1) \) current \( J \). For the IIB superstring we have both the holomorphic \( N = 4 \) superconformal algebra (61) and another anti-holomorphic one. The holomorphic generators, when specialized for IIB compactified to 6d, and rewritten in terms of BVW worldsheet variables which display manifest 6d spacetime supersymmetry and eschew spin fields, become

\[
T = -\frac{1}{2} \partial x^m \partial x_m - p_u \partial \theta^u - \frac{1}{2} \partial \rho \partial \rho - \frac{1}{2} \partial \sigma \partial \sigma + \partial^2 (\rho + i \sigma) + T_C
\]

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\[ G^+ = -e^{-2\rho-\iota\sigma}(p)^4 + \frac{i}{2}e^{-\rho}p_\alpha p_\beta \partial x^{ab} + i\sigma (\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial \theta^a - \frac{1}{2}\partial (\rho + i\sigma)\partial (\rho + i\sigma) + \frac{1}{2}\partial^2 (\rho + i\sigma)) + G^+_C \]

\[ G^- = e^{-\iota\sigma} + G^-_C \]

\[ J = \partial (\rho + i\sigma) + J_C \]

\[ \tilde{G}^+ = e^{iH_C}(-e^{-3\rho-2\iota\sigma}(p)^4 + \frac{i}{2}e^{-2\rho-\iota\sigma}p_\alpha p_\beta \partial x^{ab} + e^{-\rho}(-\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial \theta^a - \frac{1}{2}\partial (\rho + i\sigma)\partial (\rho + i\sigma) + \frac{1}{2}\partial^2 (\rho + i\sigma)) + e^{-\rho-\iota\sigma}\tilde{G}^-_C \]

\[ J^+ = e^{\rho+i\sigma}J^+_C \]

\[ J^- = e^{-\rho-\iota\sigma}J^-_C . \]

These currents are given in terms of the left-moving bosons \( \partial x^m, \rho, \sigma \), and the left-moving fermionic worldsheet fields \( p_\alpha, \theta^\alpha \), where \( 1 \leq m \leq 6, 1 \leq a \leq 4 \).

The conformal weights of \( p_\alpha, \theta^\alpha \) are 1 and 0, respectively. We define \( p^4 = \frac{1}{24} \epsilon^{abcd}p_\alpha p_\beta p_\gamma p_\delta = p_1 p_2 p_3 p_4 \) and \( \partial x^{ab} = \partial x^m \sigma_m^{ab} \) where \( \sigma_m^{ab} \sigma_n^{ac} + \sigma_n^{ab} \sigma_m^{ac} = \eta_{mn} \delta^b_c \). Here lowered indices mean \( \sigma_{mab} \equiv \frac{1}{2} \epsilon_{abcd} \sigma_m^{cd} \). Note that \( e^\rho \) and \( e^{i\sigma} \) are worldsheet fermions. Also \( e^{\rho+i\sigma} \equiv e^\rho e^{i\sigma} = -e^{i\sigma}e^\rho \). Here \( J_C \equiv i\partial H_C, J^+_C \equiv -e^{iH_C}, J^-_C \equiv e^{-iH_C} \). Both \( \tilde{T}, G^\pm, J, J^\pm, \tilde{G}^\pm \) and the generators describing the compactification \( \tilde{T}_C, G^\pm_C, J_C, J^\pm_C, \tilde{G}^\pm_C \) satisfy the twisted \( N = 4, c = 6 \), superconformal algebra \( (54) \), i.e. both \( \tilde{T} \) and \( \tilde{T}_C \) have \( c = 0 \). However, as seen in \( (54) \) and \( (62) \), \( c \) still appears in the twisted \( N = 4 \) and \( N = 2 \) algebras; and the \( N=2 \) generators in \( (62) \) \( \tilde{T}, G^\pm \), \( J \) decompose into a \( c = 0 \) six-dimensional part and a \( c = 6 \) compactification-dependent piece. (That is to say, the uncompactified piece of the twisted \( N = 2 \) generators in \( (62) \) satisfies \( (54) \) with \( c = 0 \), not just for the Virasoro generator but also wherever \( c \) appears in \( (54) \).

The other non-vanishing OPE’s are \( x^m(z, \bar{z}) x^n(\zeta, \bar{\zeta}) = -\eta^{mn} \ln |z - \zeta| \); for the left-moving worldsheet fermion fields \( p_\alpha(z) \theta^\alpha(\zeta) = (z - \zeta)^{-1} \delta_\alpha^\beta \); and for the left-moving worldsheet bosons \( \rho(z) \rho(\zeta) = -\ln(z - \zeta) \); \( \sigma(z) \sigma(\zeta) = -\ln(z - \zeta) \). Right-movers are denoted by barred notation and have similar
OPE’s.

Both holomorphic and anti-holomorphic sets of generators are used to implement the physical state conditions on the vertex operators, a procedure \[10, 11, 12\] which results in a set of string constraint equations for flat spacetime. The notation $O_n\Phi$ denotes the pole of order $d + n$ in the OPE of $O$ with $\Phi$, when $O$ is a dimension $d$ operator. For the generators (62) since $G^+$ and $\tilde{G}^+$ are dimension one, and nilpotent, in analogy with the bosonic string, the physical $N = 4$ topological vertex operators $\Phi^+(z)$ are defined by the conditions:

$$G_0^+ \Phi^+ = 0; \quad \tilde{G}_0^+ \Phi^+ = 0; \quad (J_0 - 1)\Phi^+ = 0.$$  \hspace{1cm} (63)

These are the physical conditions for a BRST-invariant vertex operator in the standard RNS formalism for the superstring. Since $\tilde{G}_0^+ = \eta_0$, the $\tilde{G}_0^+$ cohomology is trivial, i.e. $\tilde{G}_0^+ \Phi^+ = 0$ implies $\Phi^+ = \tilde{G}_0^+ V$. So it is always possible to define a $V$ satisfying

$$\Phi^+ = \tilde{G}_0^+ V; \quad G_0^+ \tilde{G}_0^+ V = J_0 V = 0.$$  \hspace{1cm} (64)

Note that $\Phi^{++}(z)$ is a worldsheet fermion and $V(z)$ is a worldsheet boson. To describe the massless compactification independent states we introduce the $U(1)$-neutral vertex operator $V(z)$, but it is straightforward to go from $V(z)$ to the $U(1)$ charge equal to one vertex operator $\Phi(z)$, using the relationship described above. In addition to (64), one can use the gauge invariance $V \sim V + G_0^+ \Lambda + \tilde{G}_0^+ \tilde{\Lambda}$ to further require

$$G_0^+ V = \tilde{G}_0^+ V = T_0 V = 0.$$  \hspace{1cm} (65)

In this gauge the physical $U(1)$-charged vertex operators $\Phi^{++}(z)$ of the closed $N = 4$ topological string must satisfy

$$G_0^+ \Phi^{++} = \tilde{G}_0^+ \Phi^{++} = G_0^+ \Phi^{++} = \tilde{G}_0^+ \Phi^{++} = 0,$$
$$G_0^- \Phi^{++} = \tilde{G}_0^- \Phi^{++} = T_0 \Phi^{++} = (J_0 - 1) \Phi^{++} = 0,$$
$$\tilde{G}_0^- \Phi^{++} = \tilde{G}_0^- \Phi^{++} = \tilde{T}_0 \Phi^{++} = (\tilde{J}_0 - 1) \Phi^{++} = 0.$$  \hspace{1cm} (66)

Similarly the $U(1)$-neutral vertex operator $V$ defined by $\Phi^{++} = \tilde{G}_0^+ \tilde{G}_0^+ V$ must satisfy the conditions

$$G_0^+ \tilde{G}_0^+ V = \tilde{G}_0^+ \tilde{G}_0^+ V = 0,$$
\begin{align}
G_0 V = \tilde{G}_0 V = G_0 V = \tilde{G}_0 V = T_0 V = T_0 V = J_0 V = J_0 V = 0. & \quad (67)
\end{align}

The integrated form of the \textit{closed} superstring vertex operator \( \Phi^{++}(z) \) is 
\( \int d^2 z G_{-1} \Phi^{++} \). In terms of \( V \) it will be defined as
\begin{align}
U &= \int d^2 z G_{-1} \tilde{G}_0 G^{-1}_0 \tilde{G}_0 V. & \quad (68)
\end{align}

The \( N = 4 \) topological prescription \cite{[12]} for calculating superstring tree-level amplitudes is
\begin{align}
< V_1(z_1) (\tilde{G}_0^+ V_2(z_2)) (G_0^+ V_3(z_3)) \prod_{r=1}^{n} \int dz_r G_{-1} G_0^+ V(z_r) >. & \quad (69)
\end{align}

Note that since \( V \) are \( U(1) \)-neutral, the amplitude \((69)\) has operators with a total \( U(1) \) charge equal to 2. This is related to the RNS requirement that non-vanishing tree scattering amplitudes must have total superconformal ghost charge \(-2\) and total conformal ghost charge \(3\). The \textit{closed} \( N = 4 \) topological tree-level amplitudes are given by
\begin{align}
< V_1(z_1, \bar{z}_1) (\tilde{G}_0^+ \tilde{G}_0^+ V_2(z_2, \bar{z}_2))(G_0^+ G_0^+ V_3(z_3, \bar{z}_3)) \prod_{r=1}^{n} \int d^2 z_r G_{-1} G_{-1} G_0^+ G_0^+ V(z_r, \bar{z}_r) >. & \quad (70)
\end{align}

\section{4.2 String Constraint Equations}

Using \((67)\) on the general massless vertex operator
\begin{align}
V = \sum_{m, n = -\infty}^{\infty} e^{m(i\sigma + \rho) + n(i\bar{\sigma} + \bar{\rho})} V_{m,n}(x, \theta, \bar{\theta}), & \quad (71)
\end{align}
we find in flat spacetime the constraints from the left and right-moving worldsheet super Virasoro algebras to be
\begin{align}
(\nabla)^4 V_{1,n} &= \nabla_a \nabla_b \partial^{ab} V_{1,n} = 0 \\
\frac{1}{6} \epsilon^{abcd} \nabla_b \nabla_c \nabla_d V_{1,n} &= -i \nabla_b \partial^{ab} V_{0,n} \\
\nabla_a \nabla_b V_{0,n} - \frac{i}{2} \epsilon^{abcd} \partial^{cd} V_{1,n} &= 0, \quad \nabla_a V_{-1,n} = 0; & \quad (72)
\end{align}
\begin{align}
\nabla^4 V_{n,1} &= \nabla_a \nabla_b \bar{\partial}^{ab} V_{n,1} = 0 \\
\frac{1}{6} \epsilon^{abcd} \nabla_b \nabla_c \nabla_d V_{n,1} &= -i \nabla_b \bar{\partial}^{ab} V_{n,0} \\
\nabla_a \nabla_b V_{n,0} - \frac{i}{2} \epsilon^{abcd} \bar{\partial}^{cd} V_{n,1} &= 0, \quad \nabla_a V_{n,-1} = 0. & \quad (73)
\end{align}
\[ \partial^p \partial_p V_{m,n} = 0 \quad (74) \]

for \(-1 \leq m, n \leq 1\), with the notation \(\nabla_a = d/d\theta^a\), \(\nabla_{\bar{a}} = d/d\bar{\theta}^a\), \(\partial^a = -\sigma^{pab} \partial_p\). These conditions further imply \(V_{m,n} = 0\) for \(m > 1\) or \(n > 1\) or \(m < 1\) or \(n < 1\), leaving nine non-zero components. In fact, the independent degrees of freedom can be shown to reside in \(V_{11}\), and the surviving constraints yield (14).

In \(AdS_3 \times S^3\) space, we generalize [13] the flat space string constraint equations (72-74) as follows:

\[
F^4 V_{1,n} = F_a F_b K^{ab} V_{1,n} = 0 \\
\frac{1}{6} \epsilon^{abcd} F_b F_c F_d V_{1,n} = -i F_b K^{ab} V_{0,n} + 2 i F^a V_{0,n} - E^a V_{-1,n} \\
F_a F_b V_{0,n} - \frac{i}{2} \epsilon_{abcd} K^{cd} V_{-1,n} = 0, \quad F_a V_{-1,n} = 0; \quad (75)
\]

\[
\bar{F}^4 V_{n,1} = \bar{F}_a \bar{F}_b K^{\bar{a}\bar{b}} V_{n,1} = 0 \\
\frac{1}{6} \epsilon^{\bar{a}\bar{b}\bar{c}\bar{d}} \bar{F}_b \bar{F}_c \bar{F}_d V_{n,1} = -i \bar{F}_b \bar{K}^{\bar{a}\bar{b}} V_{n,0} + 2 i \bar{F}^{\bar{a}} V_{n,0} - \bar{E}^{\bar{a}} V_{n,-1} \\
\bar{F}_a \bar{F}_b V_{n,0} - \frac{i}{2} \epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \bar{K}^{\bar{c}\bar{d}} V_{n,-1} = 0, \quad \bar{F}_a V_{n,-1} = 0. \quad (76)
\]

There is also a spin zero condition constructed from the Laplacian

\[
(F_a E_a + \frac{1}{8} \epsilon_{abcd} K^{ab} K^{cd}) V_{n,m} = (\bar{F}_a \bar{E}_{\bar{a}} + \frac{1}{8} \epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \bar{K}^{\bar{a}\bar{b}} \bar{K}^{\bar{c}\bar{d}}) V_{n,m} = 0. \quad (77)
\]

We derived the curved space equations (75-77) by deforming the flat space equations by requiring invariance under the \(PSU(2|2)\) transformations (78) that replace the \(d = 6\) super Poincare transformations of flat space. The Lie algebra of the supergroup \(PSU(2|2)\) contains six even elements \(K_{ab} \in SO(4)\) and eight odd \(E_a, F_a\). They generate the infinitesimal symmetry transformations of the constraint equations:

\[
\Delta_a V_{m,n} = F_a V_{m,n}, \quad \Delta_{ab} V_{m,n} = K_{ab} V_{m,n} \\
\Delta^+_a V_{1,n} = E_a V_{1,n}, \quad \Delta^+_a V_{0,n} = E_a V_{0,n} + i F_a V_{1,n}, \quad \Delta^+_a V_{-1,n} = E_a V_{-1,n} - i F_a V_{0,n}. \quad (78)
\]
We write $E_a$, $F_a$, and $K_{ab}$ for the operators that represent the left action of $e_a$, $f_a$, and $t_{ab}$ on $g$. In the above coordinates,

\[ F_a = \frac{d}{d\theta^a}, \quad K_{ab} = -\theta_a \frac{d}{d\theta^b} + \theta_b \frac{d}{d\theta^a} + t_{Lab} \]

\[ E_a = \frac{1}{2} \epsilon_{abcd} \theta^b (t^c_d - \theta^c \frac{d}{d\theta^d}) + h_{ab} \frac{d}{d\theta^b}, \]

(79)

where we have introduced an operator $t_L$ that generates the left action of $SU(2) \times SU(2)$ on $h$ alone, without acting on the $\theta$'s. Here

\[ g = g(x, \theta, \bar{\theta}) = e^{\theta^a f_a} e^{\frac{1}{2} \sigma^{p cd} x_p t_{cd} e^{\bar{\theta}^a \bar{e}_a}} = e^{\theta^a f_a} h(x) e^{\bar{\theta}^a \bar{e}_a}, \]

(80)

\[ t_{Lab} g = e^{\theta^a f_a} (-t_{ab}) h(x) e^{\bar{\theta}^a \bar{e}_a}, \]

(81)

and we found (79) by requiring $F_a g = f_a g$, $E_a g = e_a g$, $K_{ab} g = -t_{ab} g$. Similar expressions hold for the right-acting generators $\bar{K}_{\bar{a} \bar{b}}$, $\bar{E}_{\bar{a}}$, and $\bar{F}_{\bar{a}}$.

The operators $t_{ab}^L, t_{ab}^R$ describe invariant derivatives on the $SO(4)$ group manifold. These can be related to covariant derivatives $\mathcal{T}_{L}^{cd} \equiv -\sigma^{p cd} D_p, \mathcal{T}_{R}^{cd} \equiv \sigma^{p cd} D_p$, where for example, acting on a function, $\mathcal{T}_L = t_L$ and $\mathcal{T}_R = t_R$. But when acting on fields that carry vector or spinor indices, they differ so that for example on spinor indices $t_{ab}^L V_e = \mathcal{T}_{ab}^L V_e + \frac{1}{2} \delta^a_c \delta^{bc} V_c - \frac{1}{2} \delta^b_c \delta^{ac} V_c$.

In fact, for the Type IIB superstring on $AdS_3 \times S^3 \times K3$ with background Ramond flux, a sigma model \[12\] with conventional local interactions (no spin fields in the action) was found using the supergroup $PSU(2|2)$ as target, coupled to ghost fields $\rho$ and $\sigma$. The spacetime symmetry group is $PSU(2|2) \times PSU(2|2)$, acting by left and right multiplication on the group manifold, i.e. by $g \to a g b^{-1}$ where $g$ is a $PSU(2|2)$-valued field, and $a, b \in PSU(2|2)$ are the symmetry group’s Lie algebra elements. The supergroup is generated by the super Lie algebra with 12 bosonic generators forming a subalgebra $SO(4)^2$ together with 16 odd generators. Hence our model has non-maximal supersymmetry with 16 supercharges.

The $PSU(2|2)$-valued field $g$ is given in terms of $x, \theta$, and $\bar{\theta}$, which are identified as coordinates on the supergroup manifold. In addition, the Type IIB on $AdS_3 \times S^3 \times M$ has worldsheet variables describing the compactification
degrees of freedom on the four-dimensional space $M$. The vertex operators $V_{mn}(x, \theta, \bar{\theta})$ are examples of the field $g$.

To interpret the generators $E_a, F_a, K_{ab}$, we recall that in flat space, the $d = 6$ supersymmetry algebra for the left-movers is given by

$$\{q^+_a, q^-_c\} = \frac{1}{2} \epsilon_{abcd} P^{cd}$$

$$[P_{ab}, P_{cd}] = 0 = [P_{ab}, q^\pm_c] = \{q^+_a, q^+_b\} = \{q^-_a, q^-_b\}$$

(82)

where $P_{ab} \equiv \delta_{ac} \delta_{bd} P^{cd}$ and

$$q^-_a = \oint F_a(z)$$

$$q^+_a = \oint (e^{-\rho-i\sigma} F_a(z) + i E_a(z))$$

$$P^{ab} = \oint \partial x_m(z) \sigma^{mab}.$$  

(83)

(84)

In flat space we have $F_a(z) = p_a(z)$ and $E_a(z) = \frac{1}{2} \epsilon_{abcd} \theta^b(z) \partial x_m(z) \sigma^{mcd}$. We distinguish between the currents and their zero moments $E_a, F_a$ which together with $P_{ab}$ also generate the flat space supersymmetry algebra

$$[P_{ab}, P_{cd}] = 0 = [P_{ab}, F_c] = [P_{ab}, E_c],$$

$$\{E_a, F_b\} = \frac{1}{2} \epsilon_{abcd} P^{cd}, \quad \{E_a, E_b\} = \{F_a, E_b\} = \{F_a, F_b\} = 0.$$

(85)

On $AdS_3 \times S^3$, the Poincare supersymmetry algebra (86) is replaced by the $PSU(2|2)$ superalgebra

$$[K_{ab}, K_{cd}] = \delta_{ac} K_{bd} - \delta_{ad} K_{bc} - \delta_{bc} K_{ad} + \delta_{bd} K_{ac}$$

$$[K_{ab}, E_c] = \delta_{ac} E_b - \delta_{bc} E_a \quad [K_{ab}, F_c] = \delta_{ac} F_b - \delta_{bc} F_a$$

$$\{E_a, F_b\} = \frac{1}{2} \epsilon_{abcd} K^{cd} \quad \{E_a, E_b\} = \{F_a, E_b\} = \{F_a, F_b\} = 0.$$

(86)
The generators $q^\pm_a$, which generate the AdS transformations \((83)\), still have a form similar to \((83)\) but $E_a(z, \bar{z})$, $F_a(z, \bar{z})$ are no longer holomorphic and their zero moments with respect to $z$ satisfy \((86)\).

For the bosonic field components of the vertex operator the AdS constraint equations \((75-77)\) result in

\begin{align*}
\Box h^g_\alpha V_{\alpha}^{--} &= -4 \sigma^m_{ab} \sigma^n_{gh} \delta^{bh} h^g_\alpha G_{mn} \\
\Box h^g_\alpha h^b_\beta \sigma^m_{ab} \sigma^n_{gh} G_{mn} &= \frac{1}{4} \epsilon_{abc} \epsilon_{fghk} \delta^{ch} h^f_\alpha h^g_\beta F^{++ek} \\
\Box h^a_\gamma F^{++ag} &= 0, \quad \Box h^\bar{a}_g A^{-+g} = 0, \quad \Box h^g_{\bar{a}} A^{+-a} = 0 \\
\epsilon_{eacd} t^c_L h^b_\alpha A^{+-a} &= 0, \quad \epsilon_{e\bar{a}cd} t^c_R h^a_\bar{a} A^{+-\bar{b}} = 0 \\
\epsilon_{eacd} t^c_L h^\bar{a}_a F^{++ab} &= 0, \quad \epsilon_{e\bar{b}cd} t^c_R h^a_\bar{a} F^{++\bar{a}b} = 0 \\
t^{ab}_L h^g_\alpha h^b_\beta \sigma^m_{ab} \sigma^n_{gh} G_{mn} &= 0, \quad t^{ab}_R h^g_\alpha h^\bar{a}_b \sigma^m_{gh} \sigma^n_{\bar{a}b} G_{mn} = 0. 
\end{align*}

We have expanded $G_{mn} = g_{mn} + b_{mn} + \bar{g}_{mn} \phi$. The SO(4) Laplacian is

\(\Box \equiv \frac{1}{8} \epsilon_{abcd} t^a_L t^b_L t^c_L = \frac{1}{8} \epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} t^\bar{a}_R t^\bar{b}_R t^\bar{c}_R\). In order to compare this with the supergravity field theory, we can use our expressions for the group manifold invariant derivatives terms of covariant derivatives. We will also use the fact that on $AdS_3 \times S^3$ we can write the Riemann tensor and the metric tensor as

\begin{align*}
\bar{R}_{mnpq} &= \frac{1}{4} (g_{mr} \bar{R}_{np} + \bar{g}_{np} \bar{R}_{mr} - g_{nr} \bar{R}_{mp} - \bar{g}_{mp} \bar{R}_{nr}) \\
\bar{g}_{mn} &= \frac{1}{2} \sigma^m_{ab} \sigma^n_{ab}. 
\end{align*}

The sigma matrices $\sigma^{mab}$ satisfy the algebra $\sigma^{mab} \sigma^n_{ac} + \sigma^{nab} \sigma^m_{ac} = \eta^{mn} \delta^c_b$ in flat space, where $\eta^{mn}$ is the six-dimensional Minkowski metric. Sigma matrices with lowered indices are defined by $\sigma^m_{ab} = \frac{1}{2} \epsilon_{abcd} \sigma^{mcd}$, although for other quantities indices are raised and lowered with $\delta^a_b$, so we distinguish $\sigma^m_{ab}$ from $\delta^a_b \delta^b_c \sigma^{mcd}$. In curved space, $\eta_{mn}$ is replaced by the $AdS_3 \times S^3$ metric $\bar{g}_{mn}$. We then find from the string constraints that the six-dimensional string field components $g_{mn}, b_{mn}, \phi$, etc. satisfy \((16,17)\).
4.3 Correlation Functions

We will review [17] the six-dimensional three-graviton tree level amplitude in (6d) flat space, for Type IIB superstrings on $\mathbb{R}^6 \times K^3$ in the BVW formalism. It is contained in the closed string three-point function

$$< V(z_1, \bar{z}_1) (\mathcal{G}_0^+ \mathcal{G}_0^+ V(z_2, \bar{z}_2)) (\tilde{\mathcal{G}}_0^+ \tilde{\mathcal{G}}_0^+ V(z_3, \bar{z}_3)> \tag{94}$$

where the vertex operators are given by

$$V(z, \bar{z}) = e^{i\sigma(z)+\rho(z)} e^{i\bar{\sigma}(\bar{z})} \theta^a(z) \theta^b(\bar{z}) \bar{\theta}^\alpha(\bar{z}) \bar{\theta}^\beta(z) \sigma^m_{ab} \sigma^n_{\bar{a}\bar{b}} \phi_{mn}(X(z, \bar{z})) \tag{95}$$

when the field

$$\phi_{mn} = g_{mn} + b_{mn} + \bar{g}_{mn} \phi$$

satisfies the constraints we found previously $\partial^m \phi_{mn} = 0$, and $\Box \phi_{mn} = 0$. These constraints imply the gauge conditions $\partial^m b_{mn} = 0$ for the two-form, and $\partial^m g_{mn} = -\partial_n \phi$ for the traceless graviton $g_{mn}$ and dilaton $\phi$. There is a residual gauge symmetry

$$g_{mn} \rightarrow g_{mn} + \partial_m \chi_n + \partial_n \chi_m, \quad \phi \rightarrow \phi, \quad b_{mn} \rightarrow b_{mn} \tag{96}$$

with $\Box \chi_n = 0$, $\partial \cdot \chi = 0$. To evaluate (94), we extract the simple poles as

$$\mathcal{G}_0^+ \mathcal{G}_0^+ V(z, \bar{z}) = e^{i\sigma(z)+\rho(z)} e^{i\bar{\sigma}(\bar{z})} (-4) [\phi_{mn}(X) \partial X^m \bar{\partial} X^n - p_a \theta^b \sigma^m_{cb} \sigma_{pca} \bar{\partial} X^a \partial_c \phi_{mn}(X) - \bar{p}_a \bar{\theta}^b \sigma^n_{\bar{c}b} \sigma_{pca} \partial X^b \partial_c \phi_{mn}(X) + p_a \theta^b \bar{p}_b \sigma^m_{cb} \sigma_{pca} \sigma^n_{\bar{c}b} \sigma_{pca} \partial_p \partial_q \phi_{mn}(X) ] \tag{97}$$

$$\tilde{\mathcal{G}}_0^+ \tilde{\mathcal{G}}_0^+ V(z, \bar{z}) = e^{iH_C+2\rho+i\sigma} e^{i\bar{H}_C+2\bar{\rho}+i\bar{\sigma}} \theta^a \theta^b \bar{\theta}^\alpha \bar{\theta}^\beta \sigma^m_{ab} \sigma^n_{\bar{a}\bar{b}} \phi_{mn}(X). \tag{98}$$

Using the OPE’s for the ghost fields and $H_C$, we partially compute (94) by evaluating the leading singularities to find
\[ < V_1(z_1, \bar{z}_1) (G^+_0 \bar{G}^+_0 V_2(z_2, \bar{z}_2))(\tilde{G}^+_0 \bar{G}^+_0 V_3(z_3, \bar{z}_3)) > \\
= (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)^{-1}(\bar{z}_1 - \bar{z}_2)(\bar{z}_2 - \bar{z}_3)(\bar{z}_1 - \bar{z}_3)^{-1} \\
\cdot 4 < e^{iH_C(z_1)} e^{i\theta}(z_1)+2\rho(z_1) e^{i\sigma(z_1)+i\sigma(z_2)+i\sigma(z_3)} e^{iH_C(\bar{z}_1)} e^{i\theta}(\bar{z}_1)+2\rho(\bar{z}_1) e^{i\sigma(\bar{z}_1)+i\sigma(\bar{z}_2)+i\sigma(\bar{z}_3)} \\
\cdot \theta^a(z_1) \theta^b(\bar{z}_1) \bar{\theta}^a(z_1) \bar{\theta}^b(\bar{z}_1) \sigma^m_{ab} \sigma^n_{ab} \phi_{mn}(X(z_1, \bar{z}_1)) \\
\cdot [ \phi_{jk}(X(z_2, \bar{z}_2)) \partial X^j(z_2) \bar{\partial} X^k(\bar{z}_2) -p_c(z_2) \theta^f(z_2) \sigma^i_{af} \sigma^{pue} \partial X^k(z_2) \partial_p \phi_{jk}(X(z_2, \bar{z}_2)) \\
- p_c(\bar{z}_2) \bar{\theta}^f(\bar{z}_2) \sigma^j_{bf} \sigma^{qve} \bar{\partial} X^i(\bar{z}_2) \partial_q \phi_{jk}(X(z_2, \bar{z}_2)) + p_c(z_2) \theta^f(z_2) \bar{p}_c(\bar{z}_2) \bar{\theta}^f(\bar{z}_2) \sigma^i_{af} \sigma^{pve} \partial_p \phi_{jk}(X(z_2, \bar{z}_2)) ] \\
\cdot \theta^c(z_3) \theta^d(z_3) \bar{\theta}^c(\bar{z}_3) \bar{\theta}^d(\bar{z}_3) \sigma^g_{cd} \sigma^h_{cd} \phi_{gh}(X(z_3, \bar{z}_3)) > . \] (99)

Evaluating the remaining z_2, z_3 operators products, and using the SL(2, C) invariance of the amplitude to take the three points to constants z_1 \to \infty, z_1 \to 1, z_2 \to 1, z_3 \to 0, \bar{z}_3 \to 0, we find

\[ < V_1(z_1, \bar{z}_1) (G^+_0 \bar{G}^+_0 V_2(z_2, \bar{z}_2))(\tilde{G}^+_0 \bar{G}^+_0 V_3(z_3, \bar{z}_3)) > \\
= (z_2 - z_3)(\bar{z}_2 - \bar{z}_3)(\bar{z}_2 - z_3)^{-1}(\bar{z}_2 - \bar{z}_3)^{-1} \cdot 4 \\
\cdot < e^{iH_C(0)+3\rho(0)+3\iota(0)} e^{iH_C(0)+3\rho(0)+3\iota(0)} e^{i\theta}(0) e^{i\theta}(0) \sigma^m_{ab} e^{i\theta}(0) e^{i\theta}(0) \sigma^n_{ab} e^{i\theta}(0) e^{i\theta}(0) \\
\cdot \sigma^g_{cd} \sigma^h_{cd} < \phi_{mn}(X(\infty)) \phi_{jk}(X(1)) \partial^i \partial^k \phi_{gh}(X(0)) > \\
+ 2\sigma^m_{ab} (\sigma^g_{cd} \sigma^h_{cd}) \sigma^p_{ab} \sigma^q_{cd} < \phi_{mn}(X(\infty)) \partial_p \phi_{jk}(X(1)) \partial^k \phi_{gh}(X(0)) > \\
+ 2\sigma^m_{ab} (\sigma^g_{cd} \sigma^h_{cd}) \sigma^q_{ab} \sigma^p_{cd} < \phi_{mn}(X(\infty)) \partial_p \phi_{jk}(X(1)) \partial^q \phi_{gh}(X(0)) > \\
+ 4\sigma^m_{ab} (\sigma^g_{cd} \sigma^h_{cd}) \sigma^q_{ab} \sigma^p_{cd} < \phi_{mn}(X(\infty)) \partial_p \partial_q \phi_{jk}(X(1)) \phi_{gh}(X(0)) > ) \\
\text{which results in} \]

\[ = 4 \left[ \bar{g}^{mgh} g^{nh} < \phi_{mn}(x_0) \phi_{jk}(x_0) \partial^i \partial^k \phi_{gh}(x_0) > \\
- \bar{g}^{nh} (\sigma^m \sigma^p \sigma^q \sigma^h) < \phi_{mn}(x_0) \partial_p \phi_{jk}(x_0) \partial^k \phi_{gh}(x_0) > \\
- \bar{g}^{mg} (\sigma^n \sigma^p \sigma^q \sigma^h) < \phi_{mn}(x_0) \partial_p \phi_{jk}(x_0) \partial^i \phi_{gh}(x_0) > \\
+ (\sigma^m \sigma^p \sigma^q \sigma^h) < \phi_{mn}(x_0) \partial_p \partial_q \phi_{jk}(x_0) \phi_{gh}(x_0) > \right]. \] (101)

The second equality follows from the vacuum expectation value of the ghost fields, H_C and eight fermion zero modes

\[ < e^{iH_C(0)+3\rho(0)+3\iota(0)} e^{iH_C(0)+3\rho(0)+3\iota(0)} e^{i\theta}(0) e^{i\theta}(0) e^{i\theta}(0) e^{i\theta}(0) e^{i\theta}(0) e^{i\theta}(0) e^{i\theta}(0) e^{i\theta}(0) > = \frac{1}{16} \epsilon^{abcd} \epsilon^{abcd}. \] (102)
We have also used various sigma matrix identities. Since \((\sigma^m\sigma^n\sigma^p\sigma^q)_{\#} = \bar{g}^{mn}\bar{g}^{pq} + \bar{g}^{mq}\bar{g}^{np} - \bar{g}^{mp}\bar{g}^{rq}\) where in flat space \(\bar{g}_{mn} = \eta_{mn}\), and using the gauge condition \(\partial^m \phi_{mn} = 0\) once more, we find

\[
\begin{align*}
<V_1(z_1, \bar{z}_1) (G_0^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2))(\bar{G}_0^+ \bar{G}_0^+ V_3(z_3, \bar{z}_3)) > \\
= 4 \left[ \bar{g}^{mh} \bar{g}^{nh} \phi_{mn}(x_0) \partial_{m} \partial_{n} \phi_{jk}(x_0) \partial^k \phi_{gh}(x_0) \right. \\
\left. - \bar{g}^{mh} (\bar{g}^{pq} \bar{g}^{nj} - \bar{g}^{mp} \bar{g}^{qj}) \phi_{mn}(x_0) \partial_{m} \partial_{n} \phi_{jk}(x_0) \right. \\
\left. \partial^k \phi_{gh}(x_0) \right] \\
+ 12 \left[ <\phi_{m}^{\mu}(x_0) \phi_{ij}^{\nu}(x_0) \partial_{m} \partial_{i} \phi_{jk}(x_0) + 2 \phi_{n}^{\mu}(x_0) \partial_{m} \phi_{ij}^{\nu}(x_0) \partial_{j} \phi_{nk}(x_0)> \right].
\end{align*}
\]

(103)

To compare this with the supergravity field theory, we consider the Einstein-Hilbert action \(I = \int d^d x \sqrt{|g|} \left\{ -\frac{R}{2\kappa^2} \right\}\). Expanding to third order in \(\kappa\) using \(g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}\), we find the three-point interaction \(I_3\). In harmonic gauge, \(i.e.\) when \(\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h_{\rho} = 0\), and on shell \(\Box h_{\mu\nu} = 0\), the cubic coupling is given by

\[
I_3 = -\kappa \int d^d x [h_{\mu\nu} h_{\nu\rho} \partial_\mu \partial_\rho h_{\mu\sigma} + 2 h_{\mu\nu} \partial_\nu h_{\rho\sigma} \partial_\rho h_{\mu\sigma}].
\]

(104)

The gauge transformations

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu
\]

(105)

leave invariant the harmonic gauge condition and \(I_3\), given in (104), when \(\Box \xi_\mu = 0\). With this gauge symmetry, we could further choose \(h_\rho = 0\), \(\partial^\mu h_{\mu\nu} = 0\). Then \(I_3\) is the three-graviton amplitude, and it is invariant under residual gauge transformations that have \(\partial \cdot \xi = 0\).

To identify the string theory three-graviton amplitude from (103), we set \(b_{mn}\) to zero, and use the field identifications that relate the string fields \(g_{mn}, \phi\) to the supergravity field \(h_{mn}\) via \(\phi \equiv -\frac{1}{3} h_{\rho}\) and \(g_{mn} \equiv h_{mn} - \frac{1}{2} \bar{g}_{mn} h_\rho\), where \(h_{mn}\) is in harmonic gauge. Then \(\phi_{mn} = h_{mn} - \frac{1}{2} \bar{g}_{mn} h_\rho\), and from (103) the on shell string tree amplitude is

\[
-\frac{\kappa}{12} V_1(z_1, \bar{z}_1) (G_0^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2))(\bar{G}_0^+ \bar{G}_0^+ V_3(z_3, \bar{z}_3)) >
\]

33
\[\begin{align*}
&= -\kappa \int d^3x \left[ \phi^{mn}(x) \phi^{jk}(x) \partial_m \partial_n \phi_{jk}(x) + 2 \phi^{mn}(x) \partial_n \phi^{jk}(x) \partial_j \phi_{nk}(x) \right] \\
&= -K \int d^3x \left[ h^{mn} h^{jk} \partial_m \partial_n h_{jk} + 2 h^{mn} \partial_m h^{jk} \partial_j h_{nk} \right] + \kappa \int d^3x h^{mn} \partial_m h^k \partial_n h^p \\
&= I_3 + I'_3
\end{align*}\]

where \(I'_3\) is the one graviton - two dilaton amplitude, \(I_3\) is the three graviton interaction in harmonic gauge, and \(d = 6\). \(I_3\) and \(I'_3\) are invariant separately under the gauge transformation (105) with \(\xi_n = 0\) and \(\partial \cdot \xi = 0\), which corresponds to the gauge symmetry of the string field \(\phi_{mn} \rightarrow \phi_{mn} + \partial_m \xi_n + \partial_n \xi_m\). \(I_3\) by itself is also invariant under gauge transformations for which \(\partial \cdot \xi \neq 0\), and these can be used to eliminate the trace of \(h_{mn}\) in \(I_3\). In the string gauge, the trace of \(\phi_{mn}\) is related to the dilaton \(\phi_{m}^{m} = 6\phi\), so even when \(b_{mn} = 0\), (103) contains both the three graviton amplitude and the one graviton - two dilaton interaction. So it turns out we could have extracted \(I_3\) from (103) simply by setting both \(b_{mn} = 0\) and \(\phi = 0\), since then \(\phi_{mn} = g_{mn}\) and \(\partial^m g_{mn} = 0\).

Correlation functions on \(AdS_3 \times S^3\) have also been studied [17].

5 Concluding Remarks

Type IIB superstrings on \(AdS_3 \times S^3 \times K3\) can have either Neveu-Schwarz or Ramond background flux to ensure the background metric is a solution to the equations of motion. The Neveu-Schwarz case corresponds to a WZW model and has been extensively studied [40]-[52]. Since these two cases are S-dual to each other, the massless spectrum is the same, but the perturbative massive spectrum will be different. For the Type IIB superstring on \(AdS_5 \times S^5\), the flux supporting the metric can only be Ramond [26]-[29]. Thus the \(AdS_3 \times S^3\) analysis discussed in these lectures is meant as a step towards the \(AdS_5\) case.

The conjectured duality between M-theory or Type IIB string theory on anti-de Sitter (AdS) space and the conformal field theory on the boundary of AdS space [34]-[39] may be useful in giving a controlled systematic approximation for strongly coupled gauge theories. The formulation of vertex operators and string theory tree amplitudes for the IIB superstring on \(AdS_5 \times S^5\) will allow access to the dual conformal \(SU(N)\) gauge field theory \(CFT_4\) at large \(N\), but small fixed 't Hooft coupling \(x = g^{2}_{YM}N\) in the dual
correspondence, as $(g_{YM}^2 N)^{1/2} (4\pi)^{1/2} = R_{sph}/\alpha'$. Presently only the large $N$, and large ‘t Hooft coupling $x$ limit is accessible in the $CFT$, since only the supergravity limit ($\alpha' \to 0$) of the correlation functions of the AdS theory is known.

Tree level $n$-point correlation functions for $n \geq 4$ presumably have $\alpha'$ corrections, since the worldsheet theory is not a free conformal field theory. However, there may be sufficiently many symmetry currents to determine the tree level correlation functions exactly in $\alpha'$ as well. This might be possible via integrable methods for sigma models which have a supergroup manifold target space \cite{18,19} such as the $AdS_3 \times S^3$ theory.

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