Homogenization of the elasticity problem with periodically located cracks

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Abstract. In this paper we study the nonlinear problem of elasticity with periodically located cracks. On the edges of these cracks non-penetration conditions are given, which leads to a variational inequality. The period of distribution of cracks, as well as their sizes, depends on the small parameter.

In this work the corresponding variational problem and its homogenized problems are presented. We derive the necessary estimates for the theorem about convergence of the solutions of the variational inequality to the solution of the homogenized problem. For the first corrector of the classical asymptotic expansion we construct a penalty equation and a linear iterative equation in integral form. The convergence theorems for corresponding problems are formulated.

1. Introduction
In this work we consider the elasticity problem on a domain with periodically located cracks. It is supposed, that opposite edges of cracks can not penetrate into each other. The period of distribution of cracks, as well as their sizes, depends on the small parameter $\varepsilon \ll 1$. The problem is non-linear, it results in a variational inequality on some convex closed set of admissible displacements. The behavior of the solution $u^\varepsilon$ is determined by the first two terms $u^0, u^1$ of the asymptotic expansion, where $u^0(x)$ is the solution of the elasticity problem in a domain without cracks, and the first corrector $u^1(x,y)$ is the solution of the variational inequality for a given function $u^0(x)$ on a periodicity cell (see [1, §6 chapter VI]). The convergence of the $u_\varepsilon$ to the $u^0(x)$ in $L^2-$sense was proved in [2]. In this paper we analyze complete convergence, using the periodic unfolding method. The basics of this method are given in [3], as well as its applications in [4] - [6]. The main ideas of this method are the introduction of the unfolding operator and the decomposing of function to micro-macro parts. The macroscopic part does not capture the oscillations of order $\varepsilon$, while the microscopic part is designed to do so. The unfolding operator doubles the dimension of spaces and the weak convergence of unfolding functions in $L^2-$sense is equivalent to the two-scale convergence, which simplifies the proofs.

For the local problem on a cell with one crack we construct a penalty problem. We formulate the theorem about convergence of the solutions of the penalty problem to the solution of the problem on the cell, when the small regularization parameter tends to zero. Also we present the iteration equation, which approximate the nonlinear penalty problem. The rate of convergence is given in the Theorem 3. The results can be applied for numerical implementation.

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2. The periodic unfolding operator in cracked domain

Let consider a bounded domain \( \Omega \subset \mathbb{R}^2 \) with Lipschitz boundary \( \partial \Omega \). In the following \( Y = (0,1)^2 \) is a unit cell. Let the crack \( \Gamma \) is a smooth curve, strictly included in \( Y \), and \( Y_{\Gamma} \) is open set \( Y \setminus \Gamma \).

Denote \( \Xi_\varepsilon = \{ z \in Z^2 : \varepsilon(\tilde{Y} + z) \subset \Omega \} \) and \( \widehat{\Omega}_\varepsilon = \{ \bigcup_{z \in \Xi_\varepsilon}(\varepsilon z + \varepsilon Y) \} \) is the largest union of \( \varepsilon(z + \tilde{Y}) \) cells, included in \( \Omega \), while \( \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon \) is the subset of \( \Omega \), containing parts from \( \varepsilon(z + \tilde{Y}) \) cells, intersecting the boundary \( \partial \Omega \) (see Fig.2).

Then the union of all the cracks is denoted \( \Gamma_\varepsilon = \bigcup_{z \in \Xi_\varepsilon} \varepsilon(\Gamma + z) \), strictly included in \( \Omega \). So, the set \( \Gamma_\varepsilon \) can be considered as a system of periodically located cracks (with the period \( \varepsilon Y \)).

Denote \( \Omega_\varepsilon \) a cracked domain \( \Omega_\varepsilon = \Omega \cap \{ x : \frac{x}{\varepsilon} \in Y \} \). The boundary of \( \Omega_\varepsilon \) consists of an outer boundary \( \partial \Omega \) and an inner boundary \( \Gamma_\varepsilon \) with two edges \( \Gamma_\varepsilon^+ \), \( \Gamma_\varepsilon^- \). Assume, that \( \partial \Omega \) does not intersect with the system \( \Gamma_\varepsilon \) (see Fig.1).

![Figure 1. The domain with periodically located cracks](image1)

![Figure 2. The domain \( \widehat{\Omega}_\varepsilon \)](image2)

In the periodic setting, almost every point \( z \in \mathbb{R}^N \) can be written as \( z = [z]_Y + \{ z \}_Y \), \( [z]_Y \in Z^2 \), \( \{ z \}_Y \in Y \). Then, for almost every \( z \in \mathbb{R}^2 \), there exists a unique element in \( \mathbb{R}^2 \), denotes by \( [x]_\varepsilon \), \( \{ x \}_\varepsilon \), such that \( x - \varepsilon \frac{x}{\varepsilon} \) is unique element in \( Y \), where \( \{ x \}_\varepsilon \in Y \).

**Definition 1.** Let \( p \in [1, +\infty) \). For all \( \phi \in L^p(\Omega_\varepsilon) \), the unfolding operator \( T_\varepsilon \in L^p(\Omega \times Y_{\Gamma}) \) is defined as follows:

\[
T_\varepsilon(\phi)(x, y) = \phi\left(x + \varepsilon \frac{x}{\varepsilon} + \varepsilon y\right) \quad \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y_{\Gamma}, \quad T_\varepsilon(\phi)(x, y) = 0 \quad \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y_{\Gamma}.
\]

The main properties of the unfolding operator for the fixed domain, proved in [3], can be adapted for a domain with cracks.

3. Problem on equilibrium of an elastic body with periodically located cracks

In our model it is supposed, that the edges of the cracks can not penetrate each other. Therefore nonlinear boundary condition is given in the form of the following inequality: \( [u]_{\Gamma} = [u_\varepsilon] \geq 0 \) on \( \Gamma \), where \( [u] = [u]^+ - [u]^- \) denotes the jump of the vector \( u \) on \( \Gamma \), and \( [u]^\pm \) correspond to the values \( u \) on the edges \( \Gamma^\pm \).

The equation of state is \( \sigma_{ij} = a_{ijkl} \epsilon_{kl}(u) \), where \( \epsilon(u) = \{ \epsilon_{kl}(u) \} \) is the strain tensor, \( \epsilon_{kl}(u) = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) \), \( \sigma_{ij} \) is a stress tensor, \( i, j, k, l = 1, 2 \); the tensor of elastic modulus \( a_{ijkl} \) has the usual properties of symmetry and positive definiteness: \( a_{ijkl} = a_{jikl} = a_{klij} \), \( a_{ijkl} \xi_{kl} \xi_{ij} \geq C_0 |\xi|^2 \) \( \forall \xi_{ij} = \xi_{ji} \), \( C_0 = \text{const} > 0, i, j, k, l = 1, 2 \). Here and below repeated indices are summed. For derivative we use the notation \( \frac{\partial u}{\partial x_i} = u_i \).

We define a convex set of the admissible displacement \( K_\varepsilon = \{ v \in H^1_0(\Omega_\varepsilon) \mid [v_\varepsilon] \geq 0 \text{ on } \Gamma_\varepsilon \} \), where \( H^1_0(\Omega_\varepsilon) = \{ v = (v_1, v_2) \in H^1(\Omega_\varepsilon) \mid v = 0 \text{ on } \partial \Omega \} \). There is only one solution \( u^\varepsilon \) in \( K_\varepsilon \) of the variational inequality.
\[ u^\varepsilon \in K_\varepsilon, \quad \int_{\Omega_\varepsilon} \sigma_{ij}(u^\varepsilon) e_{ij}(v - u^\varepsilon) \geq \int_{\Omega_\varepsilon} f(v - u^\varepsilon) \quad \forall v \in K_\varepsilon. \] (1)

The strong formulation of the static problem (1) is the following: Find \( u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2) \) and \( \sigma_{ij}(u^\varepsilon) \), \( i, j = 1, 2 \):

\[-\sigma_{ij,j} = f, \quad i, j = 1, 2 \quad \text{in} \quad \Omega_\varepsilon, \]

\[ u^\varepsilon = 0, \quad \text{on} \quad \partial \Omega, \]

\[ [u^\varepsilon_p] \geq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u^\varepsilon] = 0 \quad \text{on} \quad \Gamma_\varepsilon, \]

\[ \sigma_\nu \leq 0, \quad \sigma_\tau = 0 \quad \text{on} \quad \Gamma_\varepsilon^\pm, \]

where \( f = (f_1, f_2) \in L^2(\Omega) \) are given external forces, \( \sigma_\nu, \sigma_\tau \) are normal and tangent components of the stress tensor at the boundary respectively, \( \sigma_\nu = \sigma_{ij,ij} \nu_1, \sigma_\tau = \sigma_\nu - \sigma_{ij,j} \nu_1 \), \( \sigma_{ij,ij} = (\nu_2, \nu_2) \). The models of the form (2) - (5) were studied in the works [1], [2], [7] and so on.

4. Homogenization process

Below we give a result related to the unfolding method. Let \( u_\varepsilon \) be a solution of the problem (1).

Definition 2. Let be \( \omega \) an open set, strictly included in \( \Omega \). For every \( \varepsilon \leq \frac{dist(\omega, \partial \Omega)}{4} \) and for every \( \forall \phi \in L^p(\Omega_\varepsilon) \), \( p \in [1, +\infty] \), we define \( Q_\varepsilon \in W^{1,\infty}(\omega; L^p(\Gamma_\varepsilon)) \):

- for \( \xi \in \mathbb{Z} \), \( Q_\varepsilon(\phi) = T_\varepsilon(\phi) \) \( \forall \phi \) \( \in \mathbb{Y} \), \( \forall y \in \mathbb{Y} \),

- for \( x \in \omega \), \( Q_\varepsilon(\phi)(x, y) \) is the \( Q_1 \) interpolate of \( Q_\varepsilon(\phi) \) at the vertices of the cell \( \varepsilon \frac{1}{2}(\varepsilon^2) + \varepsilon \mathbb{Y} \times \{y\} \) for \( y \in \mathbb{Y} \).

Lemma. Suppose \( \omega \in \Omega \). One has

\[ \|Q_\varepsilon(e(u^\varepsilon))\|_{L^2(\omega; H^1(\mathbb{Y}))} \leq C\|f\|_{L^2(\Omega)} \quad \text{and} \quad \|Q_\varepsilon(e(u^\varepsilon)) - T_\varepsilon(e(u^\varepsilon))\|_{L^2(\omega; H^1(\mathbb{Y}))} \leq C\varepsilon\|f\|_{L^2(\Omega)}. \] (6)

The constants do not depend on \( \varepsilon \) (it depends on \( \omega \)).

Proof. For any function \( \phi \in L^2(\Omega_\varepsilon) \) with support strictly included in \( \Omega \) and for \( \varepsilon \) small enough denote \( \Delta_k(\phi) = \phi(x + \varepsilon k) - \phi \), where \( k = e_i, \quad i = 1, 2 \). Then \( \Delta_k(\phi) \in L^2(\Omega_\varepsilon) \). Let \( dist(\omega, \partial \Omega) = \delta > 0 \). There exists a function \( \rho_\omega \in D(\Omega) \) such that \( 0 \leq \rho_\omega(x) \leq 1 \) \( \forall x \in \Omega \) and \( \rho_\omega = 0 \), when \( dist(x, \partial \Omega) \leq \frac{\delta}{4} \), \( \rho_\omega = 1 \), when \( dist(x, \partial \Omega) \geq \frac{\delta}{2} \), \( x \in \Omega \). Function \( \rho_\omega \) satisfies:

\[ \left\| \frac{\partial \rho_\omega}{\partial x_i} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\delta}, \quad \left\| \frac{\partial \rho_\omega}{\partial x_i} \right\|_{L^\infty(\Omega)} \leq \frac{C\varepsilon}{\delta^2}, \] (7)

where \( i, j = 1, 2 \), the constant \( C \) depends on \( \delta \).

Let \( v \in K_\varepsilon \). In (1) choose as a test function \( v^\varepsilon = u^\varepsilon + \rho_\omega(v - \rho_\omega u^\varepsilon) \in K_\varepsilon \). By entering the notations \( \tilde{u}^\varepsilon = \rho_\omega u^\varepsilon \) and \( \tilde{f}^\varepsilon = \rho_\omega f^\varepsilon \) one can obtains

\[ a^\varepsilon(\tilde{u}^\varepsilon, v - \tilde{u}^\varepsilon) \geq (\tilde{f}, v - \tilde{u}^\varepsilon) + b_\varepsilon(u^\varepsilon, v - \tilde{u}^\varepsilon) \quad \forall v \in K_\varepsilon \] (8)

where \( a^\varepsilon \) is a bilinear form, defined by the left hand of inequality (1) and \( b_\varepsilon \) is a bilinear form, which is equal to

\[ b_\varepsilon(u^\varepsilon, w) = \int_{\Omega_\varepsilon} a_{ijkl} \left( \frac{\partial \rho_\omega}{\partial x_i} u^\varepsilon e_{kl}(w) - e_{ij}(u^\varepsilon) \frac{\partial \rho_\omega}{\partial x_k} w^j \right) dx. \]
For a small enough $\varepsilon$ the set $(\varepsilon k + \varepsilon \Gamma) \cap \omega \subset \Omega$ and $\bar{u}^\varepsilon \cdot (\pm \varepsilon k) \in H^1_0(\Omega) \cup K_\varepsilon$. Further, substituting the $v = \bar{u}^\varepsilon \cdot (\pm \varepsilon k)$ in (8) and adding to each other the resulting inequalities we can obtain
\[
a^\varepsilon(\Delta_k \bar{u}^\varepsilon, \Delta_k \bar{u}^\varepsilon) \leq (\Delta_k \bar{f}, \Delta_k \bar{u}^\varepsilon) + b_\varepsilon(\Delta_k \bar{u}^\varepsilon, \Delta_k \bar{u}^\varepsilon)
\]
(9)
Notice, that
\[
\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\|f\|_{L^2(\Omega)}.
\]
(10)
Besides,
\[
\|\Delta_k \Delta(\bar{u}^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla(\Delta_k \bar{u}^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|e(\Delta_k \bar{u}^\varepsilon)\|_{L^2(\Omega_\varepsilon)}.
\]
(11)
Therefore, using the above estimates (7), (10) and (11) one can show that
\[
|b_\varepsilon(\Delta_k u^\varepsilon, \Delta_k \bar{u}^\varepsilon)| \leq C\varepsilon \|f\|_{L^2(\Omega)} \|e(\Delta_k \bar{u}^\varepsilon)\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^2 \|f\|_{L^2(\Omega)}^2.
\]
(12)
Further, neglecting the terms of the order $\varepsilon^2$ in (12) and using the positive definiteness of the tensor $a_{ijkl}$, we obtain the following estimate:
\[
\|e(\Delta_k \bar{u}^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|f\|_{L^2(\Omega)},
\]
(13)
where the constant $C$ does not depend on $\varepsilon$, but it depends on $\omega$.
By definition of $Q_1$-interpolate, the operator $Q_\varepsilon$ reaches its maximum at some $\varepsilon \xi$. So, the first estimate in (6) follows from the properties of the unfolding operator $T_\varepsilon$ (see [3],[4]) and the estimate (10). The second one is a consequence of the properties of the operator $T_\varepsilon$ and the estimate (13).

Let be $O$ a bounded domain in with Lipschitz boundary. Denote $W(O) = \{v \in (H^1(\Omega))^2 | \int v(x) r(x) dx = 0 \text{ for all } r \in R\}$, where $R$ is the space of rigid displacements $R = \{r = (r_1, r_2) | r(x) = a + b \wedge x, (a, b) \in \mathbb{R}^2 \times \mathbb{R}^2\}$. The space $W(O)$ is the orthogonal of $R$ in $(H^1(\Omega))^2$ for the scalar product $\langle u, v \rangle = \int e(\bar{u}) e(\bar{v}) dx + \int u v dx$.

Let $L^2(\Omega; K_{per}(Y_\Gamma)) = \{v_1 \in L^2(\Omega; H^1(\Gamma_\Gamma))^2 | v_1(x, \cdot) \in K_{per}(Y_\Gamma) \text{ for } x \in \Omega\}$, where $K_{per}(Y_\Gamma) = \{\phi \in H^1_{per}(Y_\Gamma)^2 \cap W(Y_\Gamma) | [\phi_\nu] \geq 0 \text{ on } \Gamma\}$. Now, for every open set $\omega$ with Lipschitz boundary, satisfying $\omega \subset \Omega' \subset \Omega$, due to estimates (6) and the compact embedding theorems, one obtains
\[
Q_\varepsilon(e(u^\varepsilon)) \to e(u^0) + e_y(u^1) \text{ weakly in } H^1(\omega; H^1(\Gamma_\Gamma))^2 \times 2;
\]
(14)
\[
Q_\varepsilon(e(u^\varepsilon)) \to e(u^0) + e_y(u^1) \text{ strongly in } L^2(\omega; H^1(\Gamma_\Gamma))^2 \times 2;
\]
(15)
\[
T_\varepsilon(e(u^\varepsilon)) \to e(u^0) + e_y(u^1) \text{ strongly in } L^2(\omega; H^1(\Gamma_\Gamma))^2 \times 2.
\]
(16)

**Theorem 1.** Let $u^\varepsilon$ is a solution of the problem (1).
There exists a subsequence, still denoted $\{u^\varepsilon\}$, $u^0 \in H^1_1(\Omega)^2$, and $u^1 \in L^2(\Omega; K_{per}(Y_\Gamma))$ such that the following convergences hold:
\[
T_\varepsilon(u^\varepsilon) \to u^0 \text{ strongly in } L^2(\Omega; H^1(\Gamma_\Gamma))^2,
\]
\[
T_\varepsilon(e(u^\varepsilon)) \to e(u^0) + e_y(u^1) \text{ strongly in } L^2(\Omega; H^1(\Gamma_\Gamma))^2 \times 2.
\]
Besides, the pair $(u^0, u^1) \in (H^1_0(\Omega))^2 \times L^2(\Omega; K_{per}(Y_\Gamma))$ is a unique solution of the following variational inequality problem:
\[
\int_{\Omega'} \int_{Y_\Gamma} a_{ijkl}(e_kl(u^0) + e_{y,kl}(u^1)) (e_{ij}(v - u^0) + e_{y,ij}(w - u^1)) dx dy \geq \int_{\Omega} f(v - u^0) dx
\]
$\forall (v, w) \in (H^1_0(\Omega))^2 \times L^2(\Omega; K_{per}(Y_\Gamma))$.

The proof is based on the above convergences (14)-(16) and the standart substitution of test functions.
5. Iterative method for equation with penalty

We introduce spaces $H_Y = \{v = (v_1, v_2) : v_i \in H^1(\Gamma), \quad v_i - 1 \text{ periodic by } y, \quad i = 1, 2\}$, $\tilde{H}_Y = \{v \in H_Y : \tilde{v} = 0\}$, where $\tilde{v} = \int v\,dy$, and the sets of functions on the cell $Y_{\Gamma}$:

$$K_Y = \{v \in H_Y : [v'] \geq 0 \text{ on } \Gamma\}, \quad \tilde{K}_Y = \{v \in K_Y : \tilde{v} = 0\}.$$ 

Consider functions $u^\varepsilon$ and $v = v^\varepsilon$ in the form

$$u^\varepsilon = u^0(x) + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + \ldots, \quad (17)$$

and $v^\varepsilon = v^0(x) + \varepsilon v^1(x, y) + \varepsilon^2 v^2(x, y) + \ldots, \quad y = x / \varepsilon, \quad x \in \Omega$. Then, using inequality (1), one can obtain problems for functions $u^0, u^1$ from the formal expansion (17), as it is shown in [1, §6 chapter VI]. The function $u^0$ is a solution of the following variational problem: Find $u^0 \in (H^1_0(\Omega))^2$ such that

$$\int_\Omega \int_Y a_{ijkl} \left( \frac{\partial u^k_i}{\partial x_l} + \frac{\partial u^i_k}{\partial y_l} \right) \frac{\partial}{\partial x_j} (v^0_i - u^0_i)dx\,dy - \int_\Omega f(v^0 - u^0)dx = 0 \quad \forall v \in (H^1_0(\Omega))^2.$$

This problem is equivalent to the following boundary problem: Find $u^0 \in (H^1_0(\Omega))^2$ such that

$$-\bar{\sigma}_{ij,j}(u^0) = f \quad \text{in } \Omega, \quad u^0 = 0 \quad \text{on } \partial \Omega \quad (18)$$

where $\bar{\sigma}_{ij} = \int \sigma_{ij}(u^0)dy$, $\sigma_{ij}(u^0) = a_{ijkl} \left( \frac{\partial u^k_i}{\partial x_l} + \frac{\partial u^i_k}{\partial y_l} \right)$, $i, j = 1, 2$.

Now consider a local problem on $Y$.

For fixed $x$ and $u^0$ the function $u^1(x, y) \in K_Y$ satisfies the following variational inequality

$$\int_\Gamma a_{ijkl} \left( \frac{\partial u^k_i}{\partial x_l} + \frac{\partial u^i_k}{\partial y_l} \right) \frac{\partial}{\partial y_j} (w_i - u^1_i)dy \geq 0 \quad \forall w \in K_Y. \quad (19)$$

Note, that the inequality (19) includes only gradients in the $y$ variable of the functions $u^1, w$. Therefore we can add a constant to each function and the left hand of the inequality (19) does not change. So we can use spaces $\tilde{H}_Y$ and $\tilde{K}_Y$ instead of $H_Y, K_Y$.

Introduce the following bilinear and linear forms in $H_Y$:

$$a_Y(v^1, w) = \int_Y a_{ijkl} \frac{\partial v^k_i}{\partial y_l} \frac{\partial w_i}{\partial y_j} \,dy, \quad L(w) = \int_Y a_{ijkl} \frac{\partial w^0_i}{\partial x_l} \frac{\partial w_i}{\partial y_j} \,dy.$$

Then we can rewrite the variational inequality (19) in the form: Find a function $u^1(x, y) \in \tilde{K}_Y$ such that

$$a_Y(u^1, w - u^1) + L(w - u^1) \geq 0 \quad (20)$$

for all $w \in \tilde{K}_Y$. The local problem (20) has an unique solution on $\tilde{K}_Y$ (see [1, Theorem 6.1, §7 chapter VI]). Convergence of the solution $u^1$ of the variational inequality (1) to the solution $u^0$ of the homogenised problem (18) is proved in [2, Theorem 4, §2].

Let $H(Y)$ be the space $(\tilde{H}_Y)^2$, with norm $\|v\|_{H(Y)}^2 = \|v\|_{L^2(\Gamma)}^2 + \sum_{i=1}^2 \|v_i\|_{L^2(\Gamma)}^2$. Then we introduce the penalty operator $\beta(u^1) : H(Y) \to H(Y)^*$

$$\langle \beta(u^1), v \rangle = -\int_{\Gamma} [u^1_i] [v_i] \,dT.$$
where angle brackets denote the duality between the spaces $H(Y)$ and $H(Y)^*$ and lower minus sign is a negative part of a function, i.e.: $[u^1_\nu]_-$ = 0, if $[u^1_\nu]_+ \geq 0$; and $[u^1_\nu]_-$ = $-[u^1_\nu]_+$, if $[u^1_\nu]_+ < 0$.

Consider the nonlinear penalty equation with the parameter $\delta > 0$

$$a_Y(u^{1\delta}, w) + \frac{1}{\delta} \langle \beta (u^{1\delta}), w \rangle = -L(w) \quad \forall \ w \in H(Y).$$  \hspace{1cm} (21)

**Theorem 2.** Let $u^{1\delta}$ is a solution of the penalty problem (21) and $u^1$ is a solution of the problem (20). Then

$$u^{1\delta} \rightarrow u^1 \text{ strongly in } H(Y) \text{ with } \delta \rightarrow 0.$$

Let introduce scalar product $(u, w) = a_Y(u, w) + \int_\Gamma [u_\nu] [w_\nu] d\Gamma$ and equivalent norm $\|u\|^2_a = (u, u)$ in the given space $H(Y)$.

To solve the nonlinear penalty problem (21) approximately we construct the following iterative procedure for fixed $\delta$ and for a given function $u^{1\delta,0} \in H(Y)$, $n = 0, 1, 2, ..$:

$$a_Y(u^{1\delta,n+1}, w) + \frac{1}{\delta} (u^{1\delta,n+1} - u^{1\delta,n}, w) = -L(w) - \frac{1}{\delta} \langle \beta (u^{1\delta,n}), w \rangle \quad \forall \ w \in H(Y).$$  \hspace{1cm} (22)

There exists a solution $u^{1\delta,n+1} \in H(Y)$ of the problem (22).

**Theorem 3.** There is convergence $u^{1\delta,n+1} \rightarrow u^{1\delta}$ strongly in $H(Y)$ with $n \rightarrow \infty$. Moreover,

$$\|u^{1\delta,n+1} - u^{1\delta}\|^2_a \leq (1 + 2C\delta)^{-(n+1)} \|u^{1\delta,0} - u^{1\delta}\|^2_a.$$

Proofs of Theorem 2 and Theorem 3 are given in [8].

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