Defining of three-dimensional acceleration and inertial mass leading to the simple form \( F = MA \) of relativistic motion equation

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Newton second law of dynamics is a law of motion but also a useful definition of force \((F = MA)\) or inertial mass \((M = F/A)\), assuming a definition of acceleration and parallelism of force and acceleration. In the special theory of relativity, out of these three only the description of force \((F = dp/dt)\) does not raise doubts. The greatest problems are posed by mass, which may be invariant rest mass or relativistic mass or even directional mass like longitudinal mass. This results from breaking the assumption of parallelism of force and standard acceleration. It turns out that these issues disappear if the relativistic acceleration \(A\) is defined as a relativistic velocity subtraction formula. This basic fact is obscured by some subtlety related to the calculation of the relativistic differential of velocity. It is based on the direction of force rather than on transformation to a resting system. The reference to a non-resting system generates a (seemingly) different velocity subtraction formula. Thus, the relativistic three-dimensional acceleration is neither rest acceleration, nor four-acceleration, nor standard acceleration. As a consequence, inertial mass in any direction of the force has the same value as relativistic mass. In other words, the concepts of transverse mass and longitudinal mass, which depend on velocity, have been unified. In this work a full relativistic equation is derived for the motion of a body with variable mass whose form confirmed the previously introduced definitions. In addition, these definitions are in line with the general version of the principle of mass and energy equivalence. The work presents a detailed review and discussion of different approaches to the subject in relation to original historical and contemporary texts. On this basis, a proposal is made for consistent definition of relativistic quantities associated with velocity change.

Key words: relativistic acceleration, velocity addition, differential of velocity, relativistic inertial mass, mass variable, correspondence rules, mass-energy equivalence

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INTRODUCTION

The special theory of relativity (SR) is well grounded both theoretically and experimentally. However, within the dynamics of SR, there are some interpretation issues that are not full agreed by physicists. Disputes about these issues, although they sometimes sound explicitly or implicitly, are often ignored and reduced to philosophy, not to mathematics or experiment. Proponents of a particular convention and interpretation are convinced that it is the only right one. In addition, the same physicists claim that there is no collision in predicting the results of experiments under the alternative conventions. If these conventions are to be physically equivalent, then how is one of them correct and the other not? This logical inconsistency shows that in SR there are issues that have not been resolved until now.

The most famous problematic issue is the topic of relativistic (total) mass versus rest mass \((M vs m)\). The physicists of elementary particles try to treat mass as a constant parameter describing a given particle. At the same time, they display carelessness in relation to the inertial mass in the mechanics. Einstein’s demand for a good definition of mass (depending on velocity) has not been implemented and is misinterpreted. This problem affects the interpretation and even the mathematical form of the famous principle of mass and energy equivalence \((E_0 = mc^2 \text{ vs } E = Mc^2 \text{ or } E = \gamma mc^2)\). Thus, in analogy to the weak and strong principle of equivalence of inertial and gravitational mass, can be speak of weak-resting \((E_0 = mc^2)\) and strong-general \((E = Mc^2)\) version of the principle of mass equivalence and energy. The strong-general version, which takes into account the kinetic and potential energy (binding energy) is not physically equivalent to the formula \(E = \gamma mc^2\) or the formula \(E = \sqrt{(mc)^2 + (pc)^2}\). Of course, binding energy can also be included in \(E_0\), but this does not change the fact that \(E\) is more general. Since the principle of energy conservation applies, all forms of energy should have an equal contribution to the mass of the system.

A somewhat less spectacular but closely related issue is the dispute over the most general formula of the second law of dynamics. It is claimed that the simple product form of this law known from the school \(\mathbf{F} = m\mathbf{a}\) (or \(\mathbf{F} = MA\)) is less general than the form with derivative of momentum respect to time (rate form \(\mathbf{F} = d\mathbf{p}/dt\)). Is this really true? To answer this question should be analyzed the mass variable in the Newton and SR mechanics. In the case of SR, the difficulty is at least double, because should be considered a variable rest mass \(m(t)\), which further complicates the variability of the relativistic mass or other mass \(M(t, \mathbf{v})\). Furthermore, it is not clear what the simple product form of the ordinary motion equation in SR is \((\mathbf{F} \neq m\mathbf{a} \text{ and } M = ? , A = ?)\). What is more, the
full general relativistic equation of motion with variable rest mass is not known yet (for \( \mathbf{F} \parallel \mathbf{v} \)).

Another problem of SR is the three-dimensional or four-dimensional duality of physical vector quantities (e.g. \( \mathbf{v}, \mathbf{F} \) versus \( \nu^a, f^\mu \)). This dualism boils down to the equality of the spatial part of the four-vector and the three-vector practically only for position and momentum. However, already for velocity or force, spatial parts of the dual quantities differ by the Lorentz factor \( (\vec{v} = \gamma \vec{v}, \vec{f} = \gamma \vec{F}) \). Formulating SR in Minkowski’s four-dimensional space using four-vector is a widely recognized elegant geometric method. However, one cannot forget that SR is a physical theory (it is part of physics) and should refer as closely as possible to observable physical reality. Unfortunately, the nature of this physical reality is at least seemingly three-dimensional, not directly four-dimensional. The fourth dimension, which is time, is not perceived as a spatial dimension, and SR does not explain its temporary limitation to a specific local value called the present. SR also does not have the time arrow highlight. In view of the above, relativistic physics can and even should be formulated in three-dimensional terms. This state of affairs is confirmed by quantum mechanics. It turns out that relativistic quantum mechanics forces the choice of three-dimensional space in a significant distinction from the time coordinate. This is evidenced by the lack of a time operator in virtually all quantum mechanics, while three-dimensional position operators exist. The situation is even more vivid in quantum gravity. Well, the Wheeler-DeWitt equation describing quantum spacetime does not contain the fourth dimension at all, the notion of time, is not perceived as a spatial dimension, and SR does not explain its temporary limitation to a specific local value called the present. SR also does not have the time arrow highlight. In view of the above, relativistic physics can and even should be formulated in three-dimensional terms. This state of affairs is confirmed by quantum mechanics. It turns out that relativistic quantum mechanics forces the choice of three-dimensional space in a significant distinction from the time coordinate. This is evidenced by the lack of a time operator in virtually all quantum mechanics, while three-dimensional position operators exist. The situation is even more vivid in quantum gravity. Well, the Wheeler-DeWitt equation describing quantum spacetime does not contain the fourth dimension at all, the notion of time, is not perceived as a spatial dimension, and SR does not explain its temporary limitation to a specific local value called the present. SR also does not have the time arrow highlight. In view of the above, relativistic physics can and even should be formulated in three-dimensional terms. This state of affairs is confirmed by quantum mechanics. It turns out that relativistic quantum mechanics forces the choice of three-dimensional space in a significant distinction from the time coordinate. This is evidenced by the lack of a time operator in virtually all quantum mechanics, while three-dimensional position operators exist. The situation is even more vivid in quantum gravity.
Moving force is the product of the mass-value of a body into the acceleration inducted in that body \([15]\).

A naive application of the rate form for a body with mass variable leads to a wrong equation:

\[
F = \frac{d(mv)}{dt} = m \frac{dv}{dt} + \frac{dm}{dt} \mathbf{v} \quad \text{(incorrect).} \tag{3}
\]

This equation is inconsistent with the Galilean transformation \([16]\). The correct equation for the movement of a body with mass variable was not discovered until the 19th century:

\[
F_{\text{ext}} \frac{dp_{\text{sys}}}{dt} = \frac{d(mv + \delta m \mathbf{u})}{dt} = m \frac{dv}{dt} - \frac{dm}{dt} (\mathbf{u} - \mathbf{v}), \tag{4}
\]

where: \(F_{\text{ext}}\) - external force, \(p_{\text{sys}}\) - momentum of system, \(\delta m\) - small attaching (or detaching) mass, \(\mathbf{u}\) - velocity of \(\delta m\). The earliest known publication with the formula \([1]\) is the work from 1812 by von Buquoy \([12]\), continued in 1815 \([13]\). The problem of mass variable was also studied by a well-known scientist Poisson \([22]\) in 1819. However, the equation \([1]\) or \([4]\) is best known as Meshchersky’s equation published in 1903 \([45, 46]\). The second law of mass variable motion in the rate form \([1]\) can be put in a simple form:

\[
ma = F_{\text{tot}} = F_{\text{ext}} + F_{\text{th}} = F_{\text{ext}} + \frac{dm}{dt} (\mathbf{u} - \mathbf{v}), \tag{5}
\]

where: \(F_{\text{tot}}\) - total force, \(F_{\text{th}}\) - thrust force (or braking). This also applies to the thrust force expression if we use the third law of dynamics:

\[
F_{\text{th}} = -F_{\delta m} = -\lim_{\Delta t \to 0} \frac{\delta m (\mathbf{u} - \mathbf{v})}{\Delta t}, \tag{6}
\]

where: \(F_{\delta m}\) - force acting on \(\delta m = -\Delta m, \Delta m\) - mass increment (implicitly negative), \(\Delta t\) - time increment. According to Johnson \([33]\), the problem of mass variable was also solved by Moore in 1813 \([51]\).

The presented considerations show that in non-relativistic mechanics the rate form equation is not more general than the simple form. On the contrary, \([3]\) is false, and the equations \([5]\) and \([6]\) in the simple form are true. And the rate form \([4]\) describes a closed system with constant mass, not solely a body with mass variable.

**II. LORENTZ INVESTIGATION 1899, 1904**

As a result of analyses of different reference systems for the movement of a charge in an electromagnetic field, in 1899 Lorentz described the dependence of inertial mass on velocity and direction \([16, 17]\):

\[
\mu_{||} = \frac{F_{||}}{a_{||}} = \gamma^3 m, \quad \mu_{\perp} = \frac{F_{\perp}}{a_{\perp}} = \gamma m, \tag{7}
\]

where: \(\mu_{||}\) - longitudinal mass, \(\mu_{\perp}\) - transversal mass, \(\gamma = 1/(1 - v^2/c^2)^{1/2}\) - Lorentz factor, \(c\) - speed of light, \(F_{||}\) - component of vector parallel to velocity (for force), \(a_{\perp}\) - component of vector perpendicular to velocity (for acceleration). These formulas were presented by Lorentz in a text at the end of his work from 1899 and then included in his work from 1904 \([16]\) in a form similar to that of mass vector. In both works Lorentz also considered additional scaling factor of time-space coordinates (dilatation), which in this work equal 1. Surprisingly, correct formulas \([7]\) were obtained by Lorentz on the basis of the not perfectly explicit and clear transformation \([15, 16]\):

\[
x' = \gamma x, \quad t' = \frac{1}{\gamma} t - \gamma \frac{v}{c^2} x \quad \text{(unclear),} \tag{8}
\]

where: \(x', t'\) - position and time in a moving system; \(x, t\) - position and time in an initial system, and \(x = x - vt\) is implicitly the position in a moving system calculated according to the Galileo transformation (originally denoted by \(x\)) \([15]\). Interestingly, the original transformation denoted by Lorentz \((x\) without \(\gamma\)) with accuracy of the following inversion \(x \to -ct', x' \to -ct, t \to x/c, t' \to x/c\) is in accordance with transformation given by Tangherlini in 1958 \([83]\), Mansouri and Sexl in 1977 \([49]\) and some other contemporary independent scholars. It turns out that such a modified transformation can be explained within SR by means of appropriate clocks synchronization, in this case external synchronization \([43]\). The Lorentz transformation was simplified (implicitly equivalent) and clearly recorded on 5th June 1905 by Poincare \([71]\), who gave it a form very close to its contemporary version:

\[
x' = \gamma (x - vt), \quad t' = \gamma \left( t - \frac{v}{c^2} x \right), \quad y' = y, \quad z' = z, \tag{9}
\]

where: \(y, z, y', z'\) - coordinates along the directions perpendicular to velocity. Poincare originally used units like \(c = 1\), and additionally, just like Lorentz, he considered scaling (dilatation), which is omitted here.

**III. KAUFMANN AND ABRAHAM 1901, 1902**

In 1897 Searle \([81]\) calculated the energy \(W\) of magnetic field generated by a sphere or an ellipsoid in uniform movement. This energy was not proportional to velocity squared, which suggested the existence of rest energy and so-called electromagnetic mass which depends on velocity. The simplest way to calculate mass from energy was the formula used by Kaufmann in 1901 \([34]\):

\[
\mu = \frac{1}{\gamma} \frac{dW}{d\gamma} = \left( \mu_{||} = \frac{1}{\gamma} \frac{dE_k}{d\gamma} \right), \tag{10}
\]

where: \(\mu\) - electron mass (longitudinal, electromagnetic), \(E_k\) - kinetic energy.
It was only later when it turned out that the said formula determined the so-called longitudinal mass. The terms of longitudinal mass and transverse mass were introduced using momentum (of electromagnetic field) by Abraham in 1902 [1]:

\[
\mu_\parallel = \frac{dp}{d\nu}, \quad \mu_\perp = \frac{p}{\nu}.
\]

(11)

He also gave other formulae based on the Lagrangian function \( L \):

\[
\mu_\parallel = \frac{d^2L}{d\nu^2}, \quad \mu_\perp = \frac{1}{\nu} \frac{dL}{d\nu},
\]

which was recognised in the next work by Abraham [2] from the same year. It should be noted that Abraham apart from Lagrangian in \( L \) distinguished in \( E \) electrostatic energy (resting energy) and magnetic energy (simplified kinetic energy of motion). It was somewhat related to the longitudinal and transverse mass, but also generated an additional division into electromagnetic mass and specific mass. However, as part of the dynamics of the SR, this division does not seem necessary here.

Kaufmann mass calculated from Searl energy and Abraham masses, which were calculated in a similar way, satisfies the inequality:

\[
\mu_\parallel \leq \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}, \quad \mu_\perp \leq \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{5}{2}}} \quad \text{(incorrect)}.
\]

(13)

The original equalities were more complex and contained logarithms. The simpler inequities given here are correct in relation to the equalities of Abraham, but they contradict the correct formulae of Lorentz [7]. Abraham derived formulae with equalities in its first publication [1] and finally concluded them in his third work from 1903 [3].

In 1901 Kaufmann [34] was experimenting with the dependence of mass on velocity for radium beta radiation (tab. 1). He measured specific charge \( \frac{e}{M} \) of fast electrons on the basis of trajectory deviation in parallel electric and magnetic fields. The symbol \( M \) refers here to apparent dynamic (transverse) mass (F/a for F \( \perp \nu \)). Electric field acted on the electron over half of its trajectory. Next, the electron was deviated by the magnetic field and registered on a photographic plate. The main measuring device was slightly over 4 cm long. An analysis of the photographic plate required the use of a microscope micrometer. Unfortunately, in 1901 Kaufmann miscalculated the radius of curvature of electrons trajectory. He corrected the mistakes in his next work in 1902 [35], in which he additionally rescaled the data, which brought his results closer to Abraham theory. The intervention in the data was considerable enough to lead Kaufmann to remove one of the five experimental points (the one with the highest velocity).

| Original results | Compiled results | Recomp. results |
|------------------|------------------|-----------------|
| Kaufmann 1901    | Koczan 2019      | h' = 1.873 cm   |
| \( z_0 \) [cm]   | \( y_0 \) [cm]   | \( v \) [10^8 \text{ m/s}] |
| 0.271            | 0.0621           | 2.82            |
| 0.348            | 0.0839           | 2.68            |
| 0.461            | 0.1175           | 2.53            |
| 0.576            | 0.1565           | 2.37            |
| 0.688            | 0.198           | 2.24            |

\( M/e \) [10^12 kg/mol]
| 16.14            | 2.97             | 15.29           |
| 13.32            | 2.82             | 12.63           |
| 10.79            | 2.67             | 10.23           |
| 9.39             | 2.50             | 8.89            |
| 8.51             | 2.36             | 8.06            |

\( x_1 \) [cm] \( x_2 \) [cm] \( \delta \) [cm] \( U \) [V] \( B \) [T] \( h \) [cm]
| 2.07             | 2               | 0.1525          | 6750 | 0.0299 | 1.775 |

FIG. 1: The dependence of ratio of dynamic mass and electron charge on its velocity on the basis of the first measurements by Kaufmann from 1901 [34]. Five experimental points are blurred into sections depending on how far the electric field goes beyond the plates of the capacitor. The calculation of the left ends of the sections neglects the electric field outside the capacitor, and the right ones assume its bilateral exit by 1/3 of the distance between the plates and the aperture with the hole for electrons. Correct calculations did not require the use of relativistic mechanics and knowledge of physical constants.

Between 1902 and 1906 Kaufman continued his research, but because of the lack of determining magnetic field (among others) he did not solve the problem [36–39].

The measurements results by Kaufmann were analysed...
in 1904 by Lorentz [36]. However, his evaluation was not exhaustive and unambiguous, for as he wrote: *I have not found time for calculating the other tables in Kaufmann’s paper*. Lorentz analysed tables III and IV from [36] and tables II and III from [37]. At the end of his analysis, Lorentz wrote: *We may expect a satisfactory agreement with my formulae*. In 1906 Kaufmann’s works [39] (table VI, VII) were analysed by Planck [70]. Despite the line of thinking drawn with his previous article [69], Planck had to admit a slight advantage of Abraham theory: *It can be seen that the latter are closer to the sphere theory than to the relative theory*. Still, because of insufficient precision of the measurements, Planck did not consider the Kaufmann’s conclusion of his critic of Lorentz and Abraham theories as final.

The subtlety of distinguishing Lorentz theory from Abraham theory is best evident from the diagram [2] drawn on the basis of the first ever Kaufmann table [34]. Contrary to the analyzes by Lorentz and Planck, the diagram takes into account the calculations of the ratio of mass and charge from the experiment. Also, the points on the graph do not use any physical constants values $m$, $e$ and $c$ and everything is calculated from the data of experiment. Surprisingly, Kaufmanns raw data fit better with Lorentz theory. However, this conformity fails if we assume that electric field is slightly outside the capacitor area ($h < h' < x_1$).

Experimental confirmation of the superiority of Lorentz theory over Abraham theory came with the experiments by Butcherer in 1908 [11], who used perpendicular, not parallel, magnetic and electric fields [91]. It is worth adding that Bucherer in 1904 [10], as well as Langevin in 1905 [12] gave yet another model of electromagnetic mass, which one was not confirmed. In the Bucherer-Langevin model, the transverse mass was $\gamma^2/3m$. In practice, their formula did not differ much from the Abraham formula, and for $v = 0.9650c$ both formulae will be equal. While the Abraham model was based on an invariably spherical electron, the Bucherer-Langevin model assumed sphere deformation during motion that retained volume [32]. However, according to SR, the sphere should flatten in the direction of movement, and its volume should then decrease. The smaller size of the charge is the greater energy of the electromagnetic field around it, which may explain why the theory with the largest (transverse) mass has been experimentally confirmed. Contrary to appearances, Lorentz did not explicitly derive his theory from full calculations of energy of the flattened sphere, but from transformational rules (somewhat like Einstein later).

IV. EINSTEIN AND PLANCK 1905-1907

On 30 June 1905, Albert Einstein published his work [18] where, based on the postulate of constant speed of light, he derived the correct transformation of coordinates in the form of [9]. Based on that he determined the transformation of the component of velocity parallel to the motion of frame (relativistic velocity subtraction formula):

$$\gamma (u - v)_{rel} = u \oplus v = u' = \frac{u - v}{1 - uv/c^2},$$

(14)

where: $u$ - body velocity, $v$ - boost velocity, $u'$ - body velocity in a new frame. Originally, Einstein wrote the law for addition, not subtraction of velocity. He also proposed a general expression for the value of resultant velocity vector $|u \oplus v|$. For further applications, consider here the (seemingly) equivalent vector form of this velocity subtraction:

$$u \oplus v = u' = \frac{u - v + \frac{\gamma q}{\sqrt{v^2 + 1}}(u \times v) \times v/c^2}{1 - uv/c^2},$$

(15)

where Lorentz factor $\gamma_v$ depends on $v$. This formula is called Einstein’s relativistic law of velocity composition (subtraction or addition). In fact, Einstein, specifying this law only for the velocity value, determined it with the accuracy of rotation. Thus, in a sense, Einstein left a gap here for Thomas-Wigner rotation. Formula (15) is a vector generalization of the formula (14), but $|u \oplus v| \neq u_t \oplus v_i$. Even so, using the same $\oplus$ symbol for action on vectors and components (or values) should not lead to misunderstandings.

Another important element of the work [18] was the derivation of relativistic equation of motion for a charge in electromagnetic field:

$$m \gamma^2 a_{\parallel} = qE_{\parallel}, \quad m \gamma^2 a_{\perp} = qE_{\perp} \times v \times B,$$

(16)

where: $E$ - intensity of electric field, $B$ - induction of magnetic field, $q$ - electric charge. The method for derivation those equations, as well as their form, referred to the components of force vector in a resting frame of a body. So Einstein explicitly determined longitudinal and transverse mass as follows:

$$\mu_{\parallel} = \frac{F'_{\parallel}}{a_{\parallel}} = \gamma^3 m, \quad \mu_{\perp} = \frac{F'_{\perp}}{a_{\perp}} = \gamma^2 m \quad (unclear),$$

(17)

where: $F'_{\parallel}, F'_{\perp}$ - values of force components in a rest frame, $\mu_{\perp}$ - Einstein’s transverse mass. Wanting to reduce the difference between longitudinal and transverse mass, Einstein overstated the transverse mass by gamma factor. He was referring there to the rest value of force, at the same time using not rest acceleration (see [7]). The factor $\gamma^2$ does occur with mass in the perpendicular component of four-force and in alternative energy based on four-force instead of force [63], but Lorentz proposal [7] was more correct.

Einstein even devoted one of his theoretical works from 1906 to the possibility of experimental measuring the ratio of transverse and longitudinal mass [20]. Unfortunately, Einstein used the ordinary and wrong formula
with longitudinal mass (like Bucherer in 1904 [10]):

$$E_k = \frac{\mu_\parallel v^2}{2} \left( \approx \frac{mv^2}{2} + \frac{3}{4} \frac{nv^4}{c^2} + \ldots \right) \text{ (incorrect).} \quad (18)$$

Despite that and [17] and [7], the presented theory of masses was called by Einstein as Lorentz-Einstein theory [20], which is somewhat justified [90]. However, writing the formula [15] is surprising because already in 1905 Einstein gave the correct expression for work $W$ of load acceleration (equals kinetic energy) [18]:

$$W = mc^2(\gamma - 1) \approx \frac{mv^2}{2} + \frac{3}{8} \frac{nv^4}{c^2} + \ldots \quad (19)$$

Because of the facts presented above Einstein never accepted the idea of mass depending on velocity [24]. Evidence of that is a letter to Barnett from 1948 whose fragment can be found in Okun’s work [60], and its corrected translation in discussion [76]. Many critics of relativistic mass refer to this letter. But they forget that from the beginning Einstein was having problems defining mass and he was searching for the right definition. Surprisingly, he did not find it in the mass-energy relation nor in the work from 1905 [19], nor in the work from 1907 [21], where he included the following equations (see also [20]):

$$m = \frac{E_0}{c^2}, \quad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad (20)$$

where: $m$ - rest mass, $E_0$ - rest energy, $E$ - total energy of the body.

Another step of the evolution of the relativistic equation of motion was rewriting Einstein equation [16] by Planck in 1906 [69] in the form:

$$m\gamma a = q(E + v \times B) - q(vE)v/c^2. \quad (21)$$

Because of the last element in this equation, it did not have the simple form, so Planck gave his equation the rate form:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\gamma \mathbf{v})}{dt}. \quad (22)$$

This equation could be presented seemingly in the simple form:

$$\mathbf{F} = m \frac{d(\gamma \mathbf{v})}{dt} = m \frac{d\vec{v}}{dt}, \quad (23)$$

where $\vec{v}$ is the three-dimensional part of the four-velocity. But from the perspective of 3D such an equation does not have correctly separated mass and acceleration (in the sense of [2] or [7]). In the following sections we will also see that this apparently simple form does not translate into an equation with a mass variable.

### V. Mass of Tolman-Lewis 1909-1912

Until that time, mass in the relativistic theory had been mostly electromagnetic mass, and its transformation principles resulted from Maxwell equations and the formula for Lorentz force of motion equations. It was only in 1909 when Lewis and Tolman generalized the Einstein’s relation of equivalency of rest mass and energy [20] onto a case of motion [11]:

$$M = \frac{E}{c^2}, \quad (24)$$

where: $M$ - relativistic mass also described by [25]. Originally, the authors used $m$ instead of $M$ and rest mass was denoted with $m_0$.

In 1912, Tolman introduced theoretical dependence of mass from velocity based on the relativity principle [80]:

$$M = m\gamma = \frac{m}{\sqrt{1 - v^2/c^2}} = \mu \perp. \quad (25)$$

Tolman considered the principle of momentum conservation in a perfect inelastic collision. Its derivation which regards mass as a coefficient of velocity in the expression for momentum determined transverse mass in accordance with [11]. But earlier Tolman and Lewis had stressed a more universal, i.e. [24], character of mass $M$. Contrary to the popular belief, it is not Einstein who should be considered the author of the relativistic mass, but Tolman and Lewis [29]. Einstein did not consider this concept until 1946, but he did not explicitly accept it [28].

In 1914, the ideas of Tolman and Lewis were independently confirmed in a more comprehensive and a bit forgotten work by Lorentz [47]. Lorentz used the same notations $M$ and $m$ as in this work. He also called mass $M$ as transverse mass, but he did not use longitudinal mass in his work.

Derivations such as that by Tolman for relativistic mass are propagated in several excellent textbooks, including the famous Feynmann’s lectures [25]. Unlike Tolman, Feynman presented a derivation based on elastic collision.

### VI. Actual Forms of Moving Equation

In 3D formalism, the most popular became the rate form equation by Planck [22]. This equation can be presented in the following form:

$$\mathbf{F} = m\gamma \mathbf{a} + m\gamma^3 (v\mathbf{a})v/c^2 = m\gamma \mathbf{a}_\perp + m\gamma^3 \mathbf{a}_\parallel. \quad (26)$$

Somewhat simpler is the counterpart of the equation [21]:

$$m\gamma \mathbf{a} = \mathbf{F} - (v\mathbf{F})v/c^2 = \mathbf{F}_\perp + \gamma^2 \mathbf{F}_\parallel. \quad (27)$$
In 4D formalism, the movement equation was written in 1908 by Minkowski \[53\] basically in one of the following forms:

\[
f^\mu = m \frac{d^2 x^\mu}{d\tau^2} = m \frac{d v^\mu}{d\tau} = \frac{d p^\mu}{d\tau} = m a^\mu, \tag{28}
\]

where: \(f^\mu\) - four-vector of force, \(v^\mu\) - four-vector of velocity, \(a^\mu\) - four-vector of acceleration, \(\tau\) - rest time of body. The price for the simplicity of this equation is redefining of the notions of \(f^\mu, v^\mu, a^\mu, \tau\) in relation to the 3D notions of \(\mathbf{F}, \mathbf{v}, \mathbf{a}\) and time \(t\). For example, the spatial part of the four-acceleration is expressed as follows:

\[
\ddot{a} = \frac{d \vec{a}}{d\tau} = \gamma \frac{d (\gamma \mathbf{v})}{d\tau} = \gamma^2 \mathbf{a}_\perp + \gamma^4 \mathbf{a}_\parallel. \tag{29}
\]

Thus, the last two expressions in (28) are not the rate form, or the simple form, of the equations of dynamics in the sense understood here. This is evident from the rate side as in (58).

The relativistic equation of movement of a body with variable mass was derived in 1946 by Ackeret \[4\]:

\[
F_{th} = m \gamma^3 a = \gamma \frac{dm}{dt} \mathbf{u} \odot \mathbf{v}, \tag{30}
\]

where: \(u\) - velocity of the jet gasses in external frame, \(\odot\) - relativistic subtraction of velocity in accordance with \[14\]. Originally, Ackeret did not use here the thrust force \(F_{th}\), so he reduced one factor \(\gamma\) in this equation. However, his equation is not a full equivalent of Mieszczerski formula \[23\] because it does not contain external force and it determines movement only in one spatial dimension. Similar to other researchers, Ackeret concentrated on a relativistic generalization of the Tsiołkovsky rocket equation. On the left side of the formula \[30\] there correctly occurs longitudinal mass; the expression \(\gamma dm\) on the right side can be interpreted, with precision to a sign, as relativistic mass of jet gasses.

In this section of the article it is worth emphasizing that SR has been confirmed experimentally not only in electrodynamics or mechanics, but also at the level of Lorentz invariance in particle physics. For example, the result of weak interaction research published in 2013 \[53\] and neutrino research results from 2018 \[6\] did not confirm any violations of Lorentz symmetry. Despite this, the work from 2019 \[58\] shows that in the area of SR (in the context of quantum mechanics of massive gravitons), original theoretical results are still possible.

\[\text{VII. ADLER AND OKUN OFFENSIVE 1987-1989}\]

At the beginning of the 1940s, Landau and Lifshitz released a textbook on the field theory \[41\] which included SR and GR which had no reference to relativistic mass \(M\). This trend was continued by Taylor and Wheeler \[82\] and by Ugarov \[87\] in the 1960s.

Let us assume the year 1987 of publication of Adler’s work titled ”Does Mass Really Depend on Velocity, Dad?” \[5\] as the formal beginning of open criticism of the notion of relativistic mass. Despite the fact that the work mirrors the past and contemporary views of the majority of physicists (see Meissner’s lecture \[50\]), Adler did not escape certain inconsistencies, crucial for this subject matter, in the formulas for gravitational and inertial mass. Because I belong to the supporters of the notion of relativistic mass, who are in the minority, I concentrate on the mistakes made by its critics.

Gravitational mass according to Adler had the following form:

\[
m_g = \frac{m}{\sqrt{1 + 2\Phi/c^2 - \beta^2/c^2}} \quad \text{(incorrect)}, \tag{31}
\]

where: \(m_g\) - (passive) gravitational mass, \(\Phi\) - potential of gravitation. Despite the fact that this interesting formula refers to the general theory of relativity, it is contradictory to the velocities near the velocity of light \((\Phi < 0)\). It implies diminishing of the velocity of light in gravitational field, which is contradictory to the theory of relativity (both SR and GR).

Adler described inertial mass as follows:

\[
I = \frac{\mathbf{F}}{\mathbf{a}} = \frac{m \gamma}{1 - \frac{\mathbf{v}}{c}\cos^2 \theta/c^2} \quad \text{(incorrect)}, \tag{32}
\]

where: \(I\) - inertial mass depending on the direction, \(\theta\) - the angle between force and velocity. The mistake in this formula was noticed and corrected in the same parametrization (denoted here \(\theta\)) by Nowik \[58\]:

\[
I = \frac{m \gamma^3}{\sqrt{\cos^2 \theta + \gamma^4 \sin^2 \theta}} = m \gamma \sqrt{\sin^2 \alpha + \gamma^4 \cos^2 \alpha}, \tag{33}
\]

where was introduced additional parametrization of the angle \(\alpha\) between acceleration and velocity. It is worth noting that the parametrization of this formula with the angle \(\beta\) between force and standard acceleration does not exist. We will later see how it is easiest to fix the formula \[32\] by modifying its left-hand side rather than the right-hand side as in \[58\].

In 1989, Okun became one of serious critics of the notion of relativistic mass \[60\]. Unfortunately, also he failed to avoid certain mistakes which may point to the biased nature of his article. Okun formulated two questions concerning the relation between energy and mass. First, the most correct formula should be indicated, and secondly, the formula derived by Einstein. The reader could chose from four presented below expressions, which have been paired here according to the correctness, in accordance
with the notations adopted in Okun’s work):
\[ a) \ E_0 = Mc^2, \quad b) \ E = mc^2, \quad (\text{incorrect}) \]
\[ c) \ E_0 = mc^2, \quad d) \ E = Mc^2. \]
According to the author (Okun), twice the answer was propagated the formula like (incorrect) (34) (incorrect) (35)

Another element of Okun’s work was a formula for the force of gravity, presented without any reference, which was equivalent to:
\[ F_g = -\frac{GM\epsilon E/c^2}{r^2}\left[\mathbf{r} + \mathbf{c}\mathbf{v} \times (\mathbf{r} \times \mathbf{v})/c^2\right] \quad (\text{incorrect}), \]
for \( \epsilon = 1 \), where: \( F_g \) - force of gravity, \( G \) - gravitational constant, \( M \) - source of gravitational field, \( \mathbf{r} \) - position vector. This formula for \( \epsilon = -1/2 \) and \( E = E_0 \) can be obtained from the transformation of Lorentz with a precision of second order in \( \mathbf{v} \) (obviously, it is not the proper methodology to apply here). Formula (36) was uncritically repeated by Roche (77) for \( \epsilon = 1 \), despite the fact that the author quoted the formula for \( F \) and for "\( \epsilon \)" in which \( \epsilon = -1 \), and \( E/c^2 \) was replaced with \( mc^2 \). Okun used his formula for photons, so in fact he applied approximation for low velocities to the ultrarelativistic case. Next, he based on this doubtful approximation one of the pillars of his criticism of the relativistic mass.

Okun’s work faced some criticism in collected correspondence (76) which also contained his defence. Okun states there that he had found the formula (36) only in one textbook by Bowler (8). Moreover, referring to E. Schucking, he presents a more general formula the relativistic apple for the gravity force corresponding to (36) for small \( M_0 \). In 1998, Okun exchanged emails with M.R. Kleemans a student of B. de Wit, about it (61). Both men agreed on the issue of Bowler derivative if this is the one you mean, I agree, is not a very good derivation (Kleemans) and they both had doubts concerning the factor in the second order term.

In 1990, Okun was invited by R.H. Romer to publish in AJP in the context of Adler and Sandin’s works. This invitation resulted in a publication (62) only in 2009 which focused on algebraic aspects of the following relation:
\[ m^2 = (E/c^2)^2 - (p/c)^2 \quad \text{or} \quad E^2 = (mc^2)^2 + (pc)^2. \]
In addition, Okun once again criticized Hawking for the form of quoting the formula of the mass and energy equivalence in his new book (31).

VIII. SANDIN, PENROSE DEFENSE 1991, 2004

In 1991, Sandin publishes a work titled "Defence of Relativistic Mass" (79). It contains the idea of the following Lemma by Sandin:
\[ F = M(v)a + \frac{dM}{dt}v = \left( \frac{dM}{dt} \neq 0 \Rightarrow M \neq \frac{|F|}{|a|} \right), \]
whose seemingly trivial proof requires the assumption \( M \neq mc^2/v^2 \). This Lemma shows that the standard Adler’s definition of inertial mass (42) is not proper if the mass changes. In other words, in the light of this methodology, only transverse mass is full consistent.

A recognised contemporary relativist, R. Penrose, referred in his book "The Road to Reality" (2004) in a concise way to the concept of mass (67). He denoted the relativistic mass \( M \) as "\( \mu \)" and named it "total mass", and invariable mass \( m \) he referred to as "\( \mu \)" (like Einstein) and called it "rest mass". He described the properties of these types of mass which can be formalized in the notations used here for the isolated frame of two not interacting bodies:
\[ M \neq \mu, \ M_1 \mu M_2 = M_1 + M_2, \ M_1 + M_2 = \text{const}; \quad (39) \]
\[ m = \mu, \ m_1 \mu m_2 \neq m_1 + m_2, \ m_1 + m_2 \neq \text{const}; \quad (40) \]
where: \( \mu \) - invariant symbol, \( \mu \) - mass connection rule. As we can see, mass \( M \) and mass \( m \) have opposing properties regarding invariance, additivity and conservativity in the additive sense. Due to the lack of additivity of rest mass, the later property can be replaced with:
\[ m_1 \mu m_2 = \text{const}. \quad (41) \]
This formula, however, does not refer to strong interaction. The analysis taking into account the contribution of potential and nuclear energy to mass is included in Einstein’s short article (23) or in Brillouin’s book (9). At present, the principle of mass and energy equivalence in the context of nuclear energy and mass deficit is such a well-established experimental fact that it does not even require a reference.

IX. CORRECT DEFINITION OF RELATIVISTIC ACCELERATION

Acceleration is a derivative of velocity with respect to time. In an inertial system, we should use time from
this system. And the differential of velocity should be calculated in accordance with the relativistic law of subtracting velocity of type (14) or (15). In the plane determined by velocity \( \mathbf{v} \) and standard acceleration \( \mathbf{a} \) in the natural base of this space, we can define the relativistic acceleration as:

\[
A_\parallel = \frac{(dv_\parallel)_{rel}}{dt} = \frac{(v + dv_\parallel) \odot v}{dt} = \gamma^2 a_\parallel = \frac{a_\parallel}{\gamma^2}, \quad (42)
\]

\[
A_\perp = \frac{(dv_\perp)_{rel}}{dt} = \frac{(0 + dv_\perp) \odot 0}{dt} = a_\perp = \frac{a_\perp}{\gamma^2}, \quad (43)
\]

where: \( A_\parallel, A_\perp \) - components of relativistic acceleration, respectively parallel and perpendicular to velocity; \( a_\parallel, a_\perp \) - components of the spatial part of four-acceleration \( \overrightarrow{a} \). The law of velocity subtraction was used here in the differential in the sense of a following to velocity components, and not as a transforming to a rest frame. The obtained acceleration does not coincide with the rest ac-

\[
A = (dv_\perp)_{rel} = \frac{a_\lambda}{1 - v_\lambda^2/c^2} = \frac{a \cos \beta}{1 - v^2 \cos^2 \theta/c^2}, \quad (44)
\]

where: \( v_\lambda \) - velocity component in the direction of \( \mathbf{A} \), \( a_\lambda \) - standard acceleration component in the direction of \( \mathbf{A} \), \( \beta \) - the angle between standard and relativistic acceleration, \( \theta \) - the angle between velocity and relativistic acceleration (also force). The above relation requires proof based on the components:

\[
\frac{\text{aA}/A}{1 - (\text{vA})^2/c^2} = \frac{(\gamma^2 a_\parallel^2 + a_\perp^2)}{A^2 - v^2 \gamma^4 a_\perp^2} = \sqrt{\gamma^4 a_\parallel^2 + a_\perp^2} = A. \quad (45)
\]

The definition of acceleration \( \mathbf{A} \) uses the rule of composition velocity (14) in the direction of the velocity components into vectors \( \hat{t}, \hat{n} \) or \( \mathbf{A} \). It turns out that it can be generalised for any direction by introducing a vector rule changing only the parallel component of velocity:

\[
\mathbf{u} \odot \parallel \mathbf{v} = \frac{\mathbf{u} - \mathbf{v} + \frac{\mathbf{uv}}{c^2} (\mathbf{u} \times \mathbf{v}) \times \mathbf{v}/c^2}{1 - \mathbf{uv}/c^2}. \quad (46)
\]

A similar law of velocity composing was researched by Fernández-Guasti [24] in 2011. Still, his formula ”(1)” is dependent of the coordinate system, but no specific system was chosen. But formally the next vector representation ”(6)” is equivalent to (46). Thanks to this law the relativistic differential of velocity:

\[
(dv)_{rel} = (v + dv) \odot \parallel v, \quad (47)
\]

in the linear part takes the following form:

\[
(dv)_{\text{rel}} = d\mathbf{v} + \gamma^2 \frac{\mathbf{v}(\mathbf{v}d\mathbf{v})}{c^2} = \gamma^2 d\mathbf{v}_\parallel + d\mathbf{v}_\perp. \quad (48)
\]

Written ”rel” without italics here means the application of the principle [16], as opposed to ”rel” which refers to (14) but not to (15). The index ”rel” can be replaced by ”rel”, in this work, but not vice versa. A vector definition of relativistic three-acceleration can now be given:

\[
\mathbf{A} := \frac{(dv)_{rel}}{dt} := \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) \odot \parallel \mathbf{v}(t)}{\Delta t}, \quad (49)
\]

where \( \odot \parallel \) can be equivalently replaced by \( \odot \perp \) [70]. The new acceleration corresponds with the four-acceleration and the standard acceleration as follows:

\[
\mathbf{A} = \frac{\mathbf{a}}{\gamma^2} = \Pi^2 \mathbf{v} [\mathbf{a}] = \gamma^2 a_\parallel + a_\perp, \quad (50)
\]

where \( \Pi^2 \) means stretching (directional scaling) in the direction of velocity \( \mathbf{v} \) and scale \( \gamma^2 \). This transformation can be interpreted as double inversion of Lorentz contraction resulting from the second order differential of position \( d^2 \mathbf{r} \) which occurs in acceleration.

**FIG. 2:** Diagram showing the relations of the main acceleration vectors and their components. The new three-dimensional vector of relativistic acceleration \( \mathbf{A} \) is a parallel projection of ordinary acceleration \( \mathbf{a} \) along the velocity on the direction of force (or spatial part of the four-acceleration \( \overrightarrow{a} \)). Thanks to this, the construction of acceleration \( \mathbf{A} \) is simpler than the construction of rest acceleration \( \mathbf{a}_0 \) and acceleration \( \mathbf{a}_1 \).

For comparison purposes we will also use the vector law of velocity composing (13) into velocity differentiation:

\[
(v + dv) \odot \mathbf{v} = \gamma d\mathbf{v} + \gamma^3 \frac{\mathbf{v}(\mathbf{v}d\mathbf{v})}{c^2} = \gamma^2 d\mathbf{v}_\parallel + \gamma d\mathbf{v}_\perp. \quad (51)
\]

This differential may be used to determining the proper (rest) acceleration:

\[
a_0 = \lim_{\Delta t \to 0} \frac{\mathbf{v}(\tau + \Delta \tau) \odot \mathbf{v}(\tau)}{\Delta \tau} = \gamma^3 a_\parallel + \gamma^2 a_\perp, \quad (52)
\]
where the derivative is calculated with respect to the self time $\Delta \tau = \Delta t/\gamma$.

It is also worth considering acceleration occurring in

$$\mathbf{a}_1 := \frac{\Delta \mathbf{v}}{\Delta \tau} := \lim_{\Delta \tau \to 0} \frac{\mathbf{v}(\tau + \Delta \tau) \ominus \mathbf{v}(\tau)}{\Delta \tau}. \quad (53)$$

Despite the simplicity of the first two records, this definition has a hybrid nature, because the numerator and denominator refers to different formalisms: four-dimensional or three-dimensional. This also applies to the last record, in which the denominator refers to the rest system, and the numerator is based on the new rule of subtracting the components of the velocity, which does not consist in a full transformation to the rest system. The non-hybrid version of this record is the correct relativistic acceleration $\left(49\right)$ or proper (rest) acceleration $\left(52\right)$. In contrast, the hybrid acceleration itself is related to others as follows:

$$\mathbf{a}_1 = \gamma \mathbf{a} = \frac{\mathbf{a}}{\gamma} = \gamma^3 \mathbf{a}_\parallel + \gamma \mathbf{a}_\perp. \quad (54)$$

The relativistic velocity differential for acceleration was analysed by R˘ebilas $\left[73\right]$ in 2008. However, he used the common approach to the algebra of velocity, which led to the confusion which Einstein obtained in $\left[17\right]$. Moreover, R˘ebilas applied relativistic addition, not subtraction, of velocity, which forced a rearranged order of operations and an equation inverse of $\left(51\right)$, but equivalent. This differential, calculated according to the usual algebra of speed vectors, can also be found in Drag˘an’s studies $\left[16 \right] \left[17\right]$. However, the differential $\left(47\right)$ does not appear in the works of Rebilas, Dragan, or even in the work of Fernández-Guasti containing operation $\left(46\right)$. Ungar $\left[88 \right] \left[89\right]$ and Oziewicz $\left[64 \right] \left[66\right]$ did not present the differential approach either.

It turns out that Lorentz group (of composition velocity and rotations) has many spectacular properties. The best example is the Thomas precession described in 1926 $\left[84\right]$ in the context of centripetal acceleration in quantum mechanics. Thomas precession surprised even Einstein himself $\left[16\right]$ and, according to R˘ebilas, he surprises physicists to this day $\left[73\right]$. Indeed, the physical phenomenon of Thomas precession in the face of the mathematical effect of Thomas-Wigner rotation gives grounds for an alternative approach to subtraction of velocity vectors $\left[13 \right] \left[88 \right] \left[89\right]$. Ungar described this as a loop structure (quasi group), which he called the gyrogroup $\left[88 \right] \left[89\right]$. Oziewicz, on the other hand, describes it in terms of relativistic groupoid $\left[64\right]$.

**X. FORCE AND SIMPLE SECOND LAW $\mathbf{F}=\mathbf{M} \mathbf{a}$**

Planck determined force $\mathbf{F}$ by formula $\left(22\right)$ in analogy to Newton formula $\left(1\right)$. In Minkowski space-time, however, we have four-force $\left(28\right)$, which is expressed as follows:

$$\left(f^\mu\right) = (f^0, \mathbf{f}) = (\gamma \mathbf{F} \mathbf{v}/c, \gamma \mathbf{F}). \quad (55)$$

Thus, we should know and be able to justify which of the vectors $\mathbf{F}$ or $\mathbf{f}$ is the vector of force which occurs in the definition of work. The choice of work in the form of $\delta W = \mathbf{F} \delta \mathbf{r}$ is the standard SR, and choosing $\delta W = f^\mu \delta x^\mu = 0$. For this and other reasons (see $\left(57\right)$ for the first component) alternative energy is incorrect and inconsistent with experiment. The norm of four-force in the signature $(+,-,-,-)$ is:

$$f_\mu f^\mu = -\mathbf{F}_\parallel^2 - \gamma^2 \mathbf{F}_\perp^2. \quad (56)$$

Since the norm of four-force is equal with accuracy to the sign to the norm of three-force parallel to velocity, this can suggest that vector $\mathbf{F}$ is more important than $\mathbf{f}$. But let us consider the following four-covector of work-impulse:

$$\left(\delta W_\mu\right) = (f_\mu) \delta \tau = (\mathbf{F} \delta \mathbf{r}/c, -\mathbf{F} \delta t), \quad (57)$$

where $\delta W_0 = \delta W/c$, but $\delta W_1 = \delta W$. Changing $\mathbf{F}$ for $\mathbf{f}$ would destroy transformational laws for $\left(57\right)$.

The derivation of the second law of dynamics in a simple form will begin with the correction Adler’s formula $\left(32\right)$. It turns out that this formula should be presented in the sense of component of the force direction:

$$F = m \gamma \frac{\mathbf{a}_\parallel}{\mathbf{a}_\parallel} \frac{a_\parallel}{1 - v^2 \cos^2 \beta/c^2}. \quad (58)$$

Which can be transformed to an equation in the direction of force operation:

$$F = m \gamma \frac{a_\parallel}{1 - v^2 / c^2}, \quad (59)$$

which can be proved with the help of $\left(44\right)$ using the parallelism of $\mathbf{F}$ and $\mathbf{A}$ from $\left(26\right)$ and $\left(42\right), \left(43\right)$. Using $\left(44\right)$ we can write it in the following form:

$$\mathbf{F} = m \gamma \frac{\left(d \mathbf{v}_\parallel\right)_{rel}}{dt} = m \gamma \left(\mathbf{v}_\parallel + d \mathbf{v}_\parallel\right) \ominus \mathbf{v}_\parallel. \quad (60)$$

This equation is true for any force direction, regardless of the direction of velocity. Also true are the equations in the direction parallel and perpendicular to velocity:

$$\mathbf{F}_\parallel = m \gamma \frac{\left(d \mathbf{v}_\parallel\right)_{rel}}{dt}, \quad \mathbf{F}_\perp = m \gamma \frac{\left(d \mathbf{v}_\perp\right)_{rel}}{dt}. \quad (61)$$

It turns out that the correction of Adler formula $\left(59\right)$ will not be true for every direction. For example, in the direction $\mathbf{a}$ a slightly different relation occurs:

$$a = \frac{F_a}{m \gamma (1 + \frac{\gamma^2 v^2}{c^2})}. \quad (62)$$
However, thanks to (47) in every direction and in every orthonormal Cartesian coordinate system the motion equation is true in the following vector form:

\[ \mathbf{F} = m\gamma \frac{d\mathbf{v}}{dt} = MA. \]  

(63)

This is the title relativistic motion equation in the simple form. It is worth stressing that for the basic directions, i.e., parallel and perpendicular to velocity and parallel to force (relativistic acceleration), the equation does not go beyond the principle (14). Only other arbitrary directions require the use of the equation (46) or (70) (see below).

At the end of this section, consider the form of a motion equation that will appeal to the supporters of the rest mass as an inertial mass:

\[ \mathbf{F} = m\frac{d\gamma \mathbf{v}}{dt} = m \frac{(\mathbf{v} + d\mathbf{v}) \ominus_{\|} \mathbf{v}}{d\tau} = ma_1. \]  

(64)

However, acceleration \(a_1\) has an artificial character, which was already justified and will still be in section XIII.

XI. MASS VARIABLE MOTION EQUATION

Now a relativistic generalization of the Meshchersky equation (5), more general than the Ackeret equation (30), will be found. In order to do that we will use four-vector rate form of the motion equation for system of body and additional mass:

\[ f^\mu_{\text{ext}} = \frac{d}{d\tau} \left[ m(\tau) \dot{x}_\mu + \delta m(\tau) v_\mu \right], \]

(65)

where: \(f_{\text{ext}}\) - external four-force, \(m(\tau)\) - body rest mass depends on self time, \(\dot{x}_\mu\) - body four-velocity as a derivative of the position with respect to self time, \(v_\mu\) - four-velocity of little additional mass \(\delta m\). After obtaining the derivative:

\[ f^\mu_{\text{ext}} = m \ddot{x}_\mu + m \dot{x}_\mu + \delta \dot{m} v_\mu. \]  

(66)

The condition of orthogonality \(f^\mu_{\text{ext}} \dot{x}_\nu = 0\) allows to calculate:

\[ \delta \dot{m} = -mc^2/(v^\nu \dot{x}_\nu). \]  

(67)

Application of this relation in (66) will lead to a ready equation:

\[ f^\mu_{\text{ext}} = m \ddot{x}_\mu + m \dot{x}_\mu - \dot{v}_\mu c^2/(v^\nu \dot{x}_\nu). \]  

(68)

Note that the expression in square brackets is orthogonal to the four-velocity \(\dot{x}_\mu = v_\mu\). Therefore, it is worth writing this equation in the three dimensional version:

\[ \mathbf{F}_{\text{ext}} = m \frac{d(\gamma \mathbf{v})}{dt} - \gamma \frac{dm}{dt} \mathbf{u} \ominus_{\perp} \mathbf{v}. \]  

(69)

The operation \(\ominus_{\perp}\) is a new, but the simplest so far, way to subtract velocity vectors:

\[ \mathbf{u} \ominus_{\perp} \mathbf{v} = \frac{\mathbf{u} - \mathbf{v} + (\mathbf{u} \times \mathbf{v}) \times \mathbf{v}/c^2}{1 - \mathbf{u} \cdot \mathbf{v}/c^2}. \]  

(70)

This rule of velocity used for parallel velocities, similar to \(\ominus_{\|}\) (46), coincides with the original rule \(\ominus_{\perp}\) (15). All three rules work differently for perpendicular velocities, \(\ominus_{\|}\) does not change the perpendicular component, and \(\ominus_{\perp}\) changes it the most. In spite of this difference, the modified versions of the velocity subtraction formula are differentially equivalent:

\[ (\mathbf{v} + d\mathbf{v}) \ominus_{\perp} \mathbf{v} = (\mathbf{v} + d\mathbf{v}) \ominus_{\|} \mathbf{v} = (d\mathbf{v})_{\text{rel}}, \]  

(71)

where the details are presented in (48), and differ from (51). An alternative approach to algebra of velocity quasigroup (or groupoid) is not something completely new and appears in the literature on the subject [16, 24, 64, 89]. Particularly interesting is the antisymmetric operation from Dragan’s monograph containing lectures from SR [19]:

\[ \mathbf{u} \ominus_{\Lambda} \mathbf{v} = \frac{(\gamma_u^{-1} - \gamma_v^{-1})(\gamma_u \mathbf{u} - \gamma_v \mathbf{v})}{1 - \frac{\mathbf{u} \cdot \mathbf{v}}{c^2} + \gamma_u \gamma_v (1 - \frac{\mathbf{u} \cdot \mathbf{v}}{c^2})}. \]  

(Dragan’s).  

(72)

Despite the complexity and differences, it can be proved that this operation is differentially equivalent to operations \(\ominus_{\|}\) or \(\ominus_{\perp}\):  

\[ (\mathbf{v} + d\mathbf{v}) \ominus_{\Lambda} \mathbf{v} = (d\mathbf{v})_{\text{rel}}. \]  

(73)

Based on the proove, many simpler antisymmetric and differential-equivalent operations can be constructed. The simplest of them is operation that looks almost like a fourvelocity subtraction:

\[ \mathbf{u} \ominus_{\nu} \mathbf{v} = \gamma_{uv}^{-1}(\gamma_u \mathbf{u} - \gamma_v \mathbf{v}), \quad \gamma_{uv}^{-1} = \sqrt{1 - (\mathbf{u} \cdot \mathbf{v})^2/c^2}. \]  

(74)

Ungar called similar operation in his work Einstein’s co-operation [89]:

\[ \mathbf{u} \ominus_{\gamma} \mathbf{v} = \frac{\gamma_u \mathbf{u} - \gamma_v \mathbf{v}}{\gamma_u + \gamma_v}, \quad (\text{Ungar’s}) \]  

(75)

whereby the specific original multiplication \(\ominus\) has been replaced here by a simple multiplication \(\cdot\) by 2. The differential still unifies new and previous operations (except for operation \(\ominus\)):

\[ (\mathbf{v} + d\mathbf{v}) \ominus \mathbf{v} = (\mathbf{v} + d\mathbf{v}) \ominus_{\nu} \mathbf{v} = (d\mathbf{v})_{\text{rel}}. \]  

(76)

The most general mass variable motion equation (69) can be written in simple form as follows:

\[ m\gamma A = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{th}} = \mathbf{F}_{\text{ext}} + \frac{\gamma dm}{dt} \mathbf{u} \ominus_{\perp} \mathbf{v}. \]  

(77)

Operation \(\ominus_{\perp}\) here refers to finite velocities, so it cannot be replaced by another differential equivalent operation. In this way, this simplest operation stands out from the others.
XII. INERTIAL MASS DEFINITIONS

After World War II (and after the atomic bombings of Hiroshima and Nagasaki), Einstein wrote a short and rarely cited popular science article about the analysis of the generalization of the principle of mass and energy equivalence in the case of potential gravity energy and nuclear energy [23]. It was basically Einstein’s only article in which he considered mass in the general form $E/c^2$, and not only in resting form $E_0/c^2$ (23 vs 29). Unfortunately, Einstein tried to reject this concept on the basis of a negligible mass of heat energy. However, in the case of nuclear energy, its contribution to mass is already measurable. Einstein described this fact, but without explicit reference to the expression $E/c^2$ (or $E_0/c^2$).

As mentioned earlier, Einstein in his letter to Barnett from 1948 postulated the need to present a good definition of the mass dependent on the velocity. Below are given 10 equivalent but not identity definitions of inertial mass (dependent on the velocity), of which the first 3 are sufficiently general.

1. Force and relativistic acceleration ratio (for $A \neq 0$):
   \[ M := \frac{F}{A} = \frac{F}{\frac{dE}{dt}}. \] (78)
   The direction of the force is absolutely free, and the velocity subtraction rule for differential of velocity does not go beyond (14).

2. Time component of mass-momentum four-vector:
   \[ M := \frac{p_0}{v} = m\gamma c^2 \frac{\partial}{\partial t} + m\frac{\partial \gamma}{\partial x}, \] (79)
   where $\hat{p}$ is expressed here in the language of modern differential geometry (classical, not quantum). This definition of mass does not refer directly to the velocity of light $c$ and it is correct also in the Galilean spacetime.

3. Mass energy equivalent in general form:
   \[ M := \frac{E}{c^2} = \frac{p_0}{c^2} + \hat{p} = m\gamma c^2 dt + m\gamma v\, dr. \] (80)
   This definition is not the same as (79) because energy-momentum four-covector $\hat{p}$ does not exist in Galilean spacetime, which makes the correspondence with a non-relativistic theory difficult. Einstein used the formula $E_0 = mc^2$ or $E = m\gamma c^2$, but he eventually did not decide to adopt the formula $E = Mc^2$ [21][22][27][29]. While there is no rational reason to consider this formula improper [9][13][23]. Most probably, Tolman and Lewis were the first ones to propose it [44]. The general version of the mass-energy equivalence principle has such great heuristic power that, despite some subtleties, it can be studied theoretically [22], as well as planned experimental studies [80] of the phenomenon of gravitational mass deficit.

4. Momentum and velocity ratio (for $v \neq 0$):
   \[ M := \frac{p}{v}. \] (81)
   This is the simplest definition of the notion of mass, but it may not be convincing in the light of the redefinition of the formula for momentum with the factor $\gamma$. This definition was used, among others, by Abraham, Tolman and Lewis and Feynman.

5. Generalization of Abraham’s first formula:
   \[ M := \frac{dp}{(dv)_{rel}}. \] (82)
   The original formula (11) without the relativistic differential of velocity determined longitudinal mass.

6. Generalization of the Kaufmann formula:
   \[ M := \frac{1}{v} \frac{dE}{(dv)_{rel}}. \] (83)
   This definition used without the velocity subtraction formula (10) determined the longitudinal mass.

7. Generalization of Abraham’s second formula:
   \[ M := \frac{d^2 E}{(dv)^2_{rel}}. \] (84)
   In its original version [12] for Lagrangian function and without rel, this formula defines longitudinal mass.

8. Thrust mass implies with (30) or (77):
   \[ M_{dm} := \gamma |dm|. \] (85)
   It can be see that a change in the rest mass leads to relativistic mass regardless of the direction and value of jet velocity with the Lorentz factor for body velocity.

9. A simple definition consistent with the Sandin lemma:
   \[ M(v) = \text{const} \land a \neq 0 \Rightarrow M := \frac{F}{a} = \frac{F_A}{a}. \] (86)
   Inertial mass favours transverse mass as it does not change during measuring (defining).
10. Average longitudinal mass:

\[ M := \frac{1}{v} \int_0^v \mu_{||}(v) dv. \tag{87} \]

This definition allows us to easily calculate acceleration time with a constant force:

\[ F \int_0^t dt = \int_0^v \mu_{||}(v)dv \Rightarrow t = \frac{Mv}{F}, \tag{88} \]

which approaches \( \infty \) for \( v \to c \). And if we make the mistake of averaging the relativistic mass (instead of the longitudinal mass):

\[ F \int_0^T dt = \int_0^c M(v)dv \quad \text{(incorrect)}, \tag{89} \]

this time will turn out to be finite \[28\]:

\[ T = \frac{\pi mc}{2F} \quad \text{(incorrect).} \tag{90} \]

The formula \eqref{eq:89} can be easily corrected by changing \( dv \) for \((dv)_{\text{rel.}}\). This observation allows for rewrite \eqref{eq:87} as a self-consistent formula for averaging one type of mass:

\[ M(v) = \frac{1}{v} \int_0^v M(u)(du)_{\text{rel.}}. \tag{91} \]

All the definitions lead to the same formula \( M = m\gamma \). In some cases it is obvious and in others it requires elementary differentiation or integration.

The time has come to present the final interpretation of mass \( M \) depending on the velocity. First and foremost, it is not self or rest mass but the mass expressing the dynamic relation of motion of a body in relation to the spacetime. As postulated by Mach, physical mass is not only an immanent property of a body but it also determines its relations with the environment. In this case, it is the relation connected with motion, which is relative. Mach criticized the substantial definition of mass by Newton and in 1883 he wrote: \textit{The true definition of mass can be deduced only from the dynamical relations of bodies} \[48\]. An increase of mass along with velocity is similar to the contraction of length, but it operates in the opposite direction. In a rest frame, a body has a particular rest length and rest mass. In a frame moving fast relative to a body, the instantaneous length of the body is smaller and the inertial mass is higher. Contraction of Lorentz length is considered to be an experimental and theoretical fact, but the increase of mass has many opponents among physicists (e.g. \[5, 50\], \[58\]).

Supporters of the relativistic mass are rather an older generation of relativists (e.g. \[9\], \[25\], \[47\]). Mass growth can also be compared to time dilation. It is often acknowledged that the relativistic mass results from time dilation and is denied its own sense. The matter, however, is not obvious, because the acceleration determining the inertial mass is the second (and not the first) derivative of displacement over time. Time is subject to dilation, and displacement is subject to a form of contraction, so with the second derivative it is difficult to say that only the first effect decides. These controversies arose from the lack of a solid definition of inertial mass based on the relativistic law of velocity subtraction, ergo on the lack of a correct manner of determining acceleration. The lack of these definitions entailed numerous, and not always correct, speculations. The dependence of mass on velocity has been demonstrated experimentally by Kaufmann four years before Einstein announced his theory of relativity. The fact of this dependence was undeniable, and only its detailed formula was debatable (Lorentz’s formula versus Abraham’s formula). In later years the analogous experiments \[11\] were repeated but the results of the measuring points were presented in the form of rest mass \( m = m_0(v) = \text{const} \). In the author’s opinion, such an approach is inconsistent with the historic Kaufmann and Newtons point of view (in the sense of its simple form), although it was convenient in the verification of the theory.

This work concerns only special relativity (SR), so it is not attempting to prove that \( M \) is a passive or active gravitational mass, because it would require the use of general relativity (GR). This does not mean, however, that this is not true, which is partly suggested by the work from 2019 \[92\], contrary to some previously cited works. Similarly, the relativistic mass of photons has not been discussed here, because it would require the use of quantum mechanics and the formulae of wave-particle duality \( E = h\nu \) and \( p = h/\lambda \). The use of these formulae for photons can be found in another author’s work \[10\].

### XIII. Spacetime-Space Corerespondence

Let’s consider in Minkowski space the physical quantity: four-vector \((q^\mu) = (q^0, \vec{q})\) or scalar \(q\). This quantity corresponds to the three dimensional vector \(Q\) or a semiscalar \(Q\). If the following relation occurs:

\[ \vec{q} = \gamma^n Q \quad \text{or} \quad q = \gamma^n Q \tag{92} \]

we will say that the given quantity has the rank \(n\). The table \[14\] presents six most important mechanical quantities (three kinetic and three dynamic), which meet the aforementioned condition.

We can see that without introducing a new definition of relativistic acceleration \(\vec{A}\) and relativistic mass \(M\), the table \[14\] would not be so clear. Velocity is the first derivative with respect to time, so it has the rank 1. By analogy, acceleration has the rank 2. Force is a derivative of momentum (rank 0) with respect to time, so it has rank 1. The consequence of the relation of momentum
haps this problem will be explained by quantum gravity [78].

why time is not observed as a geometrical dimension. Per-
spacetime quantities. Theory of relativity does not explain
needed to define three-dimensional observables based on
observable physical world is three-dimensional, so rules are
TABLE II: Correspondence of 4D and 3D quantities. The
mass. The first one is based on the scalar
form (69) and even the simple form (77). Operations
⊖
must be replaced with operation ⊖ (46) or operation
⊖
(14) in the sense of following to velocity components and
not as a transforming to a rest frame.

But for any give direction, the vector operation ⊖ (15)
must be replaced with operation ⊖ || (16) or operation ⊖ \perp (70). The last operation was derived with a completely
independent method, which increases the importance of
those differential equivalent operations (78). This indepen-
dent method is in fact the derivation of a general rel-
ativistic equation (in SR) of body motion with a variable
rest mass in the covariant form (65), three dimensional
form (69) and even the simple form (77). Operations ⊖ ||
and ⊖ \perp turned out to be differentially-equivalent to the
antisymmetric operation ⊖ \wedge found in the literature. A
simplified differential-equivalent version ⊖ \wedge of the found
operation was also given. Ultimately, it turned out that
all considered alternative operations ⊖ ||, ⊖ \perp, ⊖ \wedge, ⊖ \vee are
differentially-equivalent to operation \boxslash similar to Ein-
stein’s cooperation in the theory of gyrogroup.

The consequence of the correct definition of relativistic
three dimensional acceleration, motion equation in the
simple form and of using the relativistic velocity
subtraction formula is clarification, generalisation and
unification of numerous formulae for inertial mass (rela-
tivistic mass). Five of the ten definition mass formulae
[78, 82, 83, 84, 91] given in section XII are based on
the notion of relativistic differential velocity. In these
definitions, it is not necessary to replace the standard
subtraction velocity ⊖ (14) with other alternative operations
considered (and it would be equivalent anyway). In
other words, in the direction parallel to the velocity, all
the laws of velocity subtraction considered in the work
are differential-equivalent, without exceptions. Similarly,
the defined inertial mass meets the general principle of
mass and energy equivalence \( E = M c^2 \) without excep-
tion for kinetic energy. In addition, it has been shown
at the level of dimensional analysis and transformation
laws that the time component of the four-momentum is
strictly speaking the mass \( p^t = M \), and not energy or its
other conversion equivalent (e. g. \( E / c \)). This fact also
applies to Galileo’s spacetime, where \( p^t = M = m \).
The energy in Minkowski spacetime is strictly the time com-
ponent of the four-momentum covector \( p_t = E \), which
has no equivalent in Galileo’s spacetime. Analysis of the
equation describing relativistic rockets showed that jet
gases have a relativistic mass with a Lorentz factor for
the rocket. And the velocity of rejected gases itself is
taken into account by the operation ⊖ \perp of subtracting it
from the velocity of the rocket.

The final confirmation of the correctness of the adopted
definitions is the simplification of the correspondence be-
tween the 4D spacetime and 3D space quantities de-
scribed in the previous section. An important element
of the justification of this correspondence was previously
introduced work-impulse four-covector \( \delta W_m \) (57).

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