A NOTE ON KAKEYA SETS OF HORIZONTAL AND $SL(2)$ LINES

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ABSTRACT. We consider unions of $SL(2)$ lines in $\mathbb{R}^3$. These are lines of the form

$$L = (a, b, 0) + \text{span}(c, d, 1),$$

where $ad - bc = 1$. We show that if $\mathcal{L}$ is a Kakeya set of $SL(2)$ lines, then the union $\cup \mathcal{L}$ has Hausdorff dimension 3. This answers a question of Wang and Zahl.

The $SL(2)$ lines can be identified with horizontal lines in the first Heisenberg group, and we obtain the main result as a corollary of a more general statement concerning unions of horizontal lines. This statement is established via a point-line duality principle between horizontal and conical lines in $\mathbb{R}^3$, combined with recent work on restricted families of projections to planes, due to Gan, Guo, Guth, Harris, Maldague, and Wang.

Our result also has a corollary for Nikodym sets associated with horizontal lines, which answers a special case of a question of Kim.

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1. INTRODUCTION

The purpose of this note is to study the Hausdorff dimension of unions of $SL(2)$ lines in $\mathbb{R}^3$. Here is the definition of $SL(2)$ lines, following [9]:

**Definition 1.1** ($\mathcal{L}_{SL(2)}$). The family $\mathcal{L}_{SL(2)}$ consists of the following lines $L \subset \mathbb{R}^3$. Either $L$ is a line contained in the $xy$-plane, and $0 \in L$, or then

$$L := L_{\alpha, \beta, \gamma, \delta} := (\alpha, \beta, 0) + \text{span}(\gamma, \delta, 1),$$

where $\alpha \delta - \beta \gamma = 1$.

We also use the following notation. If $\mathcal{L}$ is any family of lines in $\mathbb{R}^3$, we write $\text{dir}(\mathcal{L}) := \{e \in S^2 : \ell || \text{span}(e) \text{ for some } \ell \in \mathcal{L}\}$. Here is the main result of the note:

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Theorem 1.2. Let $\mathcal{L} \subset \mathcal{L}_{SL(2)}$ be a set with $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$. Then

$$\dim_{H}(\cup \mathcal{L}) = 3.$$  

Remark 1.3. Theorem 1.2 answers a question posed by Wang and Zahl in [9, Section 1.2]. This question was motivated by earlier work of Katz and Zahl [4]. Theorem 1.2 continues to hold if the full lines in $\mathcal{L}$ are replaced by line segments of positive length. We will discuss this briefly below (3.2).

We have been informed that Katz, Wu, and Zahl have also proved Theorem 1.2 independently, using a different method.

The $SL(2)$ lines are essentially (up to a change in coordinates) the same as *horizontal lines in the first Heisenberg group* $\mathbb{H} = (\mathbb{R}^3, *)$, viewed as subsets of $\mathbb{R}^3$ (see Proposition 2.1). We will infer Theorem 1.2 from a more general statement concerning unions of these horizontal lines, Theorem 1.5 below. We first need to define the concepts properly.

The family of all horizontal lines is denoted $\mathcal{L}(\mathbb{H})$. The “Heisenberg” definition of these lines is the following. Let $\Pi_0 := \{(x, y, 0) : x, y \in \mathbb{R}\}$ be the $xy$-plane, and for $p \in \mathbb{R}^3$, let $\Pi_p := p * \Pi_0$ be the left translate of $\Pi_0$ by the Heisenberg group product

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)).$$

Then, $\mathcal{L}(\mathbb{H})$ consists of all the lines in $\Pi_p$ (for every $p \in \mathbb{R}^3$) which contain the point $p$.

The family $\mathcal{L}(\mathbb{H})$ is a 3-dimensional submanifold of the full (4-dimensional) family of lines in $\mathbb{R}^3$. In fact, the definition above of horizontal lines will not be used in the note: rather, we focus attention on the following parametrised subset of $\mathcal{L}(\mathbb{H})$:

$$\mathcal{L}'(\mathbb{H}) = \{\ell_{(a, b, c)} : (a, b, c) \in \mathbb{R}^3\},$$

where

$$\ell_{(a, b, c)} = \{(as + b, s, \frac{1}{2}s + c) : s \in \mathbb{R}\}.$$ 

The subset $\mathcal{L}'(\mathbb{H})$ consists of all elements of $\mathcal{L}(\mathbb{H})$, except for those contained in some translate of the plane $\mathbb{V}_0 := \{(x, 0, t) : x, t \in \mathbb{R}\}$. By definition, every set $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$ can be written as

$$\mathcal{L} = \ell(P) := \{\ell_{(a, b, c)} : (a, b, c) \in P\}$$

for some set $P \subset \mathbb{R}^3$. This identification of $\mathcal{L}'(\mathbb{H})$ with $\mathbb{R}^3$ allows us to transport notions like “Borel set” and “dimension” from $\mathbb{R}^3$ to corresponding notions for subsets of $\mathcal{L}'(\mathbb{H})$.

**Definition 1.4.** Let $\mathcal{L} = \ell(P) \subset \mathcal{L}'(\mathbb{H})$. We say that $\mathcal{L}$ is a Borel set if $P \subset \mathbb{R}^3$ is a Borel set. We define $\dim_{H} \mathcal{L} := \dim_{H} P$, where “$\dim_{H} P$” refers to the Euclidean Hausdorff dimension of $P \subset \mathbb{R}^3$.

Now we can state our main result about unions of horizontal lines:

**Theorem 1.5.** Let $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$. Then,

$$\dim_{H}(\cup \mathcal{L}) = \min\{\dim_{H} \mathcal{L} + 1, 3\}.$$ 

Here “$\dim_{H}(\cup \mathcal{L})$” is the Euclidean Hausdorff dimension of the union $\cup \mathcal{L} := \bigcup_{\ell \in \mathcal{L}} \ell$.

The following corollary for horizontal lines is equivalent to Theorem 1.2:

**Corollary 1.6.** Let $\mathcal{L} \subset \mathcal{L}(\mathbb{H})$ with $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$. Then,

$$\dim_{H}(\cup \mathcal{L}) = 3.$$
Remark 1.7. Theorem 1.5 and Corollary 1.6 continue to hold if full lines are replaced by line segments of positive length; see the discussion below (3.2). Thus, if $L \subset L'(\mathbb{H})$, and every line $\ell \in L$ contains a segment $I(\ell) \subset \ell$ of positive length, then

$$\dim_H \left( \bigcup_{\ell \in L} I(\ell) \right) = \min \{ \dim_H \, L + 1, 3 \}. \quad (1.8)$$

1.1. Nikodym sets associated with horizontal lines. Theorem 1.5 easily yields information about the dimension of Nikodym sets associated with horizontal lines. A set $N \subset \mathbb{R}^3$ is called an $L(\mathbb{H})$-Nikodym set if for every $p \in \mathbb{R}^3$ (or more generally every $p \in \mathbb{R}^3$ in a measurable set of positive measure $\Omega \subset \mathbb{R}^3$) there exists a line $\ell_p \in L(\mathbb{H})$ containing $p$ such that $N$ contains a line segment $I_p \subset \ell_p$ of positive length.

Corollary 1.9. Every $L(\mathbb{H})$-set $N \subset \mathbb{R}^3$ has $\dim_H N = 3$.

It is well-known that bounds for Kakeya sets yield bounds for Nikodym sets: we only repeat the standard details below for the reader’s convenience. For a similar argument in the case of classical Kakeya and Nikodym sets, see [8, Section 11.3].

Proof of Corollary 1.9. We may assume without loss of generality that all the lines $\ell_p \in L(\mathbb{H})$ appearing in the definition of “$N$” lie in $L'(\mathbb{H})$. Namely, if this is true for a positive measure subset of the points $p \in \Omega$, we simply replace $\Omega$ by that subset. If this fails for Lebesgue almost every point $p \in \Omega$, then we apply a rotation $R$ of, say, $10^\circ$ around the $t$-axis to the objects $\Omega$, $N$, and the lines $\ell_p$, $p \in \Omega$. Rotations around the $t$-axis preserve $L(\mathbb{H})$, and the measure and dimension of $\Omega$ and $N$. After this procedure, we moreover have $\ell_p \in L'(\mathbb{H})$ for a.e. $p \in R(\Omega)$.

Using Fubini’s theorem, start by picking $y_0 \in \mathbb{R}$ such that $H^2(\Omega \cap \mathbb{W}_{y_0}) > 0$. Here $\mathbb{W}_y = \{(x, y, t) : x, t \in \mathbb{R}\}$ for $y \in \mathbb{R}$. By assumption, for every $p = (x, y_0, t) \in \Omega \cap \mathbb{W}_{y_0}$, there exists a line

$$\ell_p := \ell_{(a(p), b(p), c(p))} \in L'(\mathbb{H})$$

containing $p$ such that $N$ contains a line segment $I_p \subset \ell_p$ of positive length.

Now, note that the map $(a, b, c) \to \Psi(a, b, c) = (a y_0 + b, y_0, b y_0 + c)$ is Lipschitz, and

$$\Omega \cap \mathbb{W}_{y_0} \subset \Psi(\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\}).$$

(This is because the lines $\ell_p$ contain the points $p \in \Omega \cap \mathbb{W}_{y_0}$.) Therefore,

$$\dim_H \{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\} \geq \dim_H (\Omega \cap \mathbb{W}_{y_0}) = 2.$$

In particular, the set of lines $L := \{\ell_p : p \in \Omega\} \subset L'(\mathbb{H})$ has $\dim_H L \geq 2$ by definition. Therefore, it follows from Theorem 1.5, or to be precise (1.8), that

$$\dim_H N \geq \dim_H \left( \bigcup_{p \in \Omega} I_p \right) = 3.$$

This completes the proof. \hfill \square

Remark 1.10. Nikodym set for “restricted” families of lines were earlier considered by Kim [5]. Corollary 1.9 answers (a special case of) a question raised on [5, p. 478]. We elaborate on this a little further. The paper [5] considered general families of 2-planes $p \mapsto \Pi_a(p) \subset \mathbb{R}^3$, where $p \mapsto a(p)$ is a non-vanishing measurable vector field, and

$$p \in \Pi_a(p) \quad \text{and} \quad \text{span}(a(p)) = \Pi_a(p)^\perp.$$
One can associate Nikodym sets $N \subset \mathbb{R}^3$ to such a plane family, as follows: for every $p \in \mathbb{R}^3$, the requirement is that there exists a line $\ell_p \subset \mathbb{R}^3$ satisfying
\[ p \in \ell_p \subset \Pi_0(p), \]
and a non-trivial segment $I_p \subset N \cap \ell_p$. How small can such a Nikodym set $N \subset \mathbb{R}^3$ be? In [5], Kim approached the question via maximal function estimates, and his results depend on the properties of the vector field $\alpha$. Kim considered vector fields $\alpha$ of the form
\[ \alpha(p) = (a_{11}p_1 + a_{21}p_2, a_{12}p_1 + a_{22}p_2, -1), \quad p = (p_1, p_2, p_3) \in \mathbb{R}^3, \]
and defined the "discriminant" $D_\alpha = (a_{12} + a_{21})^2 - 4a_{11}a_{22}$. In [5, Corollary 1, p. 478], it was shown that the dimension of $N$ equals 3 if $D_\alpha \neq 0$. Right after the corollary, the question is raised, what happens in the situation $D_\alpha = 0$.

Now, recall the definition of horizontal lines $L(\mathbb{H})$: these were the lines contained in the planes $\Pi_p = p * \Pi_0$, and passing through $p$. The planes $\Pi_p$ fit in the framework of [5], choosing $\alpha(p) = (-p_2/2, p_1/2, -1)$, or $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$. In particular, $D_\alpha = 0$. Also, the $L(\mathbb{H})$-Nikodym sets defined above Corollary 1.9 are the same as the Nikodym sets of [5] associated with the planes $\Pi_p = p * \Pi_0$. Thus, Corollary 1.9 covers the special case $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ of the problem raised on [5, p. 478].

1.2. Ingredients of the proof. The proof of Theorem 1.5 is based on two ingredients. The first one is a point-line duality between horizontal lines and conical lines in $\mathbb{R}^3$, namely translates of lines contained in the light cone $\{(x, y, t) : t^2 = x^2 + y^2\}$. This duality was formalised in our paper [2], although it was already implicit in the work [6] of Liu. Using this point-line duality, Kakeya-type problems for horizontal lines can be transformed into projection problems in $\mathbb{R}^3$. These projection problems concern "restricted" families of projections to planes in $\mathbb{R}^3$. Sharp results for such families were recently established by Gan, Guo, Guth, Harris, Maldague, and Wang [3]. This is the second key component in the proof of Theorem 1.5.

2. PROOFS CONCERNING $SL(2)$ LINES

In this section we formalise the connection between $SL(2)$ lines and horizontal lines. We also deduce our main result, Theorem 1.2, from Corollary 1.6.

Recall the $SL(2)$ lines from Definition 1.1. We write $L'_{SL(2)}$ for all the lines in $L_{SL(2)}$, except for the $x$-axis, or lines of the form $L_{\alpha, \beta, \gamma, \delta}$ with $\delta = 0$. The difference between $L_{SL(2)}$ and $L'_{SL(2)}$ is the same as the difference between $L(\mathbb{H})$ and $L'(\mathbb{H})$. Consider the map
\[ \Xi(x, y, t) := (x, y, t/2). \]
We claim that $\Xi$ maps the $SL(2)$ lines to horizontal lines. More precisely:

**Proposition 2.1.** If $L_{\alpha, \beta, \gamma, \delta} \in L'_{SL(2)}$ with $\delta \neq 0$ and $\alpha \delta - \beta \gamma = 1$, then
\[ \Xi(L_{\alpha, \beta, \gamma, \delta}) = \ell_{(a, b, c)} \in L'(\mathbb{H}), \]
where
\[ \begin{cases} a = \gamma/\delta, \\ b = 1/\delta, \\ c = -\beta/(2\delta). \end{cases} \]
Proof. Fix $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha \delta - \beta \gamma = 1$. Write $L_{\alpha, \beta, \gamma, \delta}(s) = (\alpha, \beta) + (s\gamma, s\delta, s)$. It is a straightforward computation to check that

$$\Xi(L_{\alpha, \beta, \gamma, \delta}(s)) = \ell_{(a,b,c)}(\beta + s\delta), \quad s \in \mathbb{R}. $$

Since $\delta \neq 0$ by assumption, this completes the proof. \hfill $\Box$

We are then prepared to prove Theorem 1.2.

Proof of Theorem 1.2. We may assume that $\mathcal{L} \subset \mathcal{L}_{SL(2)}^\prime$, since the directions of the lines in $\mathcal{L}_{SL(2)} \setminus \mathcal{L}_{SL(2)}^\prime$ are contained in the $\mathcal{H}^2$ null set $S^2 \cap \{(x, 0, t) : x, t \in \mathbb{R}\}$. Similarly, we may assume that $\mathcal{L}$ contains no lines in the $xy$-plane; thus every $L \in \mathcal{L}$ has the form $L = L_{\alpha, \beta, \gamma, \delta}$ for some $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha \delta - \beta \gamma = 1$.

Since $\mathcal{L} \subset \mathcal{L}_{SL(2)}^\prime$, we infer from Proposition 2.1 that $\Xi(\mathcal{L}) := \{\Xi(\ell) : \ell \in \mathcal{L}\} \subset \mathcal{L}^\prime(\mathbb{H})$. We claim that

$$\mathcal{H}^2(\text{dir}(\Xi(\mathcal{L}))) > 0. \quad (2.3)$$

According to Corollary 1.6, this will imply that

$$\dim_H(\cup \mathcal{L}) = \dim_H(\cup \Xi(\mathcal{L})) = \dim_H(\cup \Xi(\mathcal{L})) = 3,$$

and complete the proof.

To verify (2.3), fix $L = L_{\alpha, \beta, \gamma, \delta} \in \mathcal{L}$. Then, by (3.4), we have $\Xi(L) = \ell_{(a,b,c)}$ with

$$\begin{cases} a = \gamma/\delta, \\
                   b = 1/\delta. \quad (2.4)\end{cases}$$

Since $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$, and the direction of $L_{\alpha, \beta, \gamma, \delta} = (\alpha, \beta, 0) + \text{span}(\gamma, \delta, 1)$ is determined by $\gamma$ and $\delta$, we know that

$$\mathcal{H}^2(\{(\gamma, \delta) \in \mathbb{R}^2 : L_{\alpha, \beta, \gamma, \delta} \in \mathcal{L}\}) > 0.$$

It now follows from (2.4) that also

$$\mathcal{H}^2(\{(a, b) \in \mathbb{R}^2 : \ell_{(a,b,c)} \in \Xi(\mathcal{L})\}) > 0.$$

Since the direction of $\ell_{(a,b,c)} = (b, 0, c) + \text{span}(a, 1, b/2)$ is determined by $(a, b)$, we may now infer that $\mathcal{H}^2(\text{dir}(\Xi(\mathcal{L}))) > 0$, as desired. \hfill $\Box$

3. Proofs concerning horizontal lines

We start by proving Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality, we may assume that $\mathcal{L} = \ell(P)$ is a Borel set of lines, that is, $P \subset \mathbb{R}^3$ is a Borel set. For the full details of this reduction, see [6, Section 3] or [1, Theorem 7.9]. The idea is that we can first replace $\cup \mathcal{L}$ by a $G_\delta$-set $G \supset \cup \mathcal{L}$ without affecting $\dim_H(\cup \mathcal{L})$. Then, it is easy to check that the set of parameters $P' := \{p \in \mathbb{R}^3 : \ell(p) \subset G\}$ is a Borel set with $P' \supset P$, in particular $\dim_H P' \geq \dim_H P$. Finally, writing $\mathcal{L}' := \ell(P')$, we have

$$\dim_H(\cup \mathcal{L}) = \dim_H G \geq \dim_H(\cup \mathcal{L}').$$

So, if the result is known for Borel sets of lines, it follows for $\mathcal{L}$. Write $\mathcal{L} := \ell(P)$, where $P \subset \mathbb{R}^3$ is Borel. Write also

$$K_y := \{(ay + b, \frac{b}{2}y + c) : (a, b, c) \in P\}, \quad y \in \mathbb{R},$$

and
and note that $K_y$ is a "slice" of $\cup \mathcal{L}$ with the plane $\mathbb{W}_y := \{(x, y, t) : x, t \in \mathbb{R}\}$:

$$(\cup \mathcal{L}) \cap \mathbb{W}_y \cong K_y,$$

where "$\cong$" refers to isometry. In order to prove that

$$\dim_H(\cup \mathcal{L}) \geq \min\{\dim_H \mathcal{L} + 1, 3\},$$

(3.1)

we now claim that

$$\dim_H K_y = \min\{\dim_H P, 2\} \quad \text{for a.e. } y \in \mathbb{R}. \quad (3.2)$$

If $\mathcal{L}$ consisted of line segments of positive length, and not full lines, then we would have to modify (3.2) as follows: for every $\epsilon > 0$, there exists an interval $I \subset \mathbb{R}$ of positive length such that $\dim_H K_y \geq \min\{\dim_H P - \epsilon, 2\}$ for a.e. $y \in I$. This interval would (be chosen to) consist of points $y \in \mathbb{R}$ with the property that the plane $\mathbb{W}_y$ intersects a family of segments corresponding to a $(\dim_H P - \epsilon)$-dimensional Borel subset $P' \subset P$. We refer the reader to [6, Section 3] for a very similar argument.

Clearly (3.1) follows from (3.2) by the "Fubini inequality" for Hausdorff measures (hence dimension), see [1, Theorem 5.8] or [7, Theorem 7.7]. To prove (3.2), we define

$$v(y) := (y, 1, 0) \quad \text{and} \quad w(y) := (0, y/2, 1), \quad y \in \mathbb{R}.$$  

Then, we note that for $y \in \mathbb{R}$ fixed, $K_y$ can be expressed as

$$K_y = \{\langle p, v(y) \rangle, \langle p, w(y) \rangle \} : p \in P \},$$

where "$\langle \cdot, \cdot \rangle$" is the Euclidean dot product. This is a "projection" of $P$ to the plane

$$V_y := \text{span}\{v(y), w(y)\},$$

but not an orthogonal projection, since $\{v(y), w(y)\}$ is not an orthonormal basis of $V_y$. Regardless,

$$\dim_H K_y = \dim_H \pi_{V_y}(P), \quad y \in \mathbb{R}, \quad (3.3)$$

because $K_y$ is an invertible linear image of $\pi_{V_y}(P)$. Let us check this carefully. First, $v(y)$ and $w(y)$ are linearly independent, so we may write (for $y \in \mathbb{R}$ fixed)

$$\begin{cases} e_1 := \frac{v(y)}{|v(y)|}, \\ e_2 := \alpha e_1 + \beta w(y), \end{cases}$$

where $\{e_1, e_2\}$ is an orthonormal basis of $V_y$. The vectors $e_1, e_2$ and the coefficients $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$ depend on $y$, but we suppress this from the notation.

With this notation, we define the invertible linear map $M_y : \mathbb{R}^2 \to \mathbb{R}^2$:

$$M_y(z_1, z_2) := (|v(y)|z_1, \frac{1}{\beta}z_2 - \frac{\alpha}{\beta}z_1), \quad (z_1, z_2) \in \mathbb{R}^2.$$  

Then, one may calculate that

$$M_y(\langle p, e_1 \rangle, \langle p, e_2 \rangle) = (\langle p, v(y) \rangle, \langle p, w(y) \rangle), \quad p \in \mathbb{R}^3, y \in \mathbb{R}.$$  

The left hand side is the $M_y$-image of the orthogonal projection $\pi_{V_y}(p)$. Therefore, $K_y$ is indeed the $M_y$-image of $\pi_{V_y}(P)$, hence (3.3) holds.

To complete the proof, we claim that

$$\dim_H \pi_{V_y}(P) = \min\{\dim_H P, 2\} \quad \text{for a.e. } y \in \mathbb{R}. \quad (3.4)$$

The idea is that $\{\pi_{V_y}\}_{y \in \mathbb{R}}$ is a 1-parameter family of orthogonal projections to planes in $\mathbb{R}^3$ which satisfies the hypotheses of [3, Corollary 1].
Which planes are the planes $V_y$? Note that

$$v(y) \times w(y) = (1,-y,y^2/2) =: e_y.$$  

Thus, $V_y = e_y \perp$. Moreover, the lines $\ell_y := \text{span}(e_y)$ are all contained in a $45^\circ$ rotated copy of the light cone

$$C := \{(x,y,t) \in \mathbb{R}^3 : t^2 = x^2 + y^2\},$$

see [2, Section 2.2] for the details. This implies that the projections $\{\pi_{V_y}\}_{y \in \mathbb{R}}$ satisfy the curvature condition [3, (1)]. In fact, up to the rotation by $45^\circ$, this family of projections is precisely the "model example" mentioned just below [3, (1)]. Therefore, (3.4) follows from [3, Corollary 1], and the proof is complete. □

We conclude the paper by proving Corollary 1.6.

**Proof of Corollary 1.6.** First, note that $H^2(\text{dir}(L \cap L'(\mathbb{H}))) > 0$. This is because $\text{dir}(L'(\mathbb{H}))$ contains all the directions on $S^2$, except for those contained in the null set $\{(x,0,t) : x,t \in \mathbb{R}\}$. Therefore, we may assume that $L \subset L'(\mathbb{H})$.

Write $L = \ell(P)$, where $P \subset \mathbb{R}^3$. Recall that

$$L = \ell(P) = \{(as+b, s(\frac{b}{2}s+c) : s \in \mathbb{R}, (a,b,c) \in P\}
= \{(b,0,c) + \text{span}(a,1,\frac{b}{2}) : (a,b,c) \in P\}.$$

Since $H^2(\text{dir}(L)) > 0$ by assumption, we see that

$$H^2(\{(a, \frac{b}{2}) : (a,b,c) \in P\}) > 0,$$

and consequently $\dim_H P \geq 2$. The claim now follows from Theorem 1.5. □

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