Higher Dimensional Multiparameter Unitary and Nonunitary Braid Matrices: Even Dimensions

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Abstract

A class of $(2n)^2 \times (2n)^2$ multiparameter braid matrices are presented for all $n$ ($n \geq 1$).
Apart from the spectral parameter $\theta$, they depend on $2n^2$ free parameters $m_{ij}^{(\pm)}$, $i, j = 1, \ldots, n$.
For real parameters the matrices $R(\theta)$ are nonunitary. For purely imaginary parameters they became unitary.
Thus a unification is achieved with odd dimensional multiparameter solutions presented before.

1 Introduction

Higher dimensional unitary braid matrices have been studied in two recent papers \cite{1,2}.
Their simultaneous relevance to topological and quantum entanglements (as discussed, for example, in Ref. \cite{3}) was a major motivation. In Ref. \cite{2} quite different classes were presented for odd and even dimensional matrices.
There, the even dimensional $(2n)^2 \times (2n)^2$ braid matrices have no free parameter (apart from the spectral parameter $\theta$ after Baxterization) where as the $(2n+1)^2 \times (2n+1)^2$ matrices have $2n (n+2)$ free parameters $\left(m_{ij}^{(\pm)}\right)$. Here we unify the two cases by presenting multiparameter solutions

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for even dimensions. We obtain first the general case for this class and then show how to implement unitarity.

2 Constructions (Even dimensions)

The braid equation is, in standard notations, in presence of a spectral parameter $\theta$,

$$
\hat{R}_{12} (\theta) \hat{R}_{23} (\theta + \theta') \hat{R}_{12} (\theta') = \hat{R}_{23} (\theta') \hat{R}_{12} (\theta + \theta') \hat{R}_{23} (\theta),
$$

(2.1)

where $\hat{R}_{12} = \hat{R} \otimes I$ and $\hat{R}_{23} = I \otimes \hat{R}$. We present below a simple class of multiparameter solutions for $(2n)^2 \times (2n)^2$ ($n \geq 1$) braid matrices $\hat{R} (\theta)$. They are analogous to the odd dimensional solutions presented before [4]. Unitarity constraints can be implemented as in sec. 5 of Ref. [2]. Thus, for this class, one obtains a unified approach for multiparameter odd and even dimensional solutions.

Define the projectors

$$
P^{(\epsilon)}_{ij} = \frac{1}{2} \left\{ (ii) \otimes (jj) + (\bar{i}i) \otimes (\bar{j}j) + \epsilon [(ii) \otimes (\bar{j}j) + (\bar{i}i) \otimes (jj)] \right\},
$$

(2.2)

where $i, j \in \{1, \ldots, n\}$, $\bar{i} = 2n + 1 - i$, $\bar{j} = 2n + 1 - j$ and $\epsilon = \pm$. They provide a complete basis satisfying

$$
P^{(\epsilon)}_{ij} P^{(\epsilon')}_{kl} = \delta_{ik} \delta_{jl} \delta_{\epsilon \epsilon'} P^{(\epsilon)}_{ij}, \quad \sum_{\epsilon = \pm} \sum_{i,j=1}^{n} P^{(\epsilon)}_{ij} = I_{(2n)^2 \times (2n)^2}.
$$

(2.3)

Anticipating the basic constraints essential for odd dimension [4] we directly postulate the form

$$
\hat{R} (\theta) = \sum_{\epsilon = \pm} \sum_{i,j=1}^{n} e^{m^{(\epsilon)}_{ij} \theta} \left( P^{(\epsilon)}_{ij} + P^{(\epsilon)}_{\bar{i} \bar{j}} \right).
$$

(2.4)

The proof that it satisfies the braid equation (2.1) proceeds in close analogy to the equations from (A9) to (A17) of Ref. [4]. Here we have directly implemented the constraint

$$
m^{(\epsilon)}_{ij} = m^{(\epsilon)}_{\bar{i} \bar{j}}.
$$

(2.5)

It is instructive to study explicitly the simplest cases.

Case 1: $N = 2$ ($n = 1$) (Here $i = 1$, $\bar{i} = 2$ and similarly for $j$).

$$
\begin{bmatrix}
a_+ & 0 & 0 & a_-
0 & a_+ & a_- & 0
0 & a_- & a_+ & 0
a_- & 0 & 0 & a_+
\end{bmatrix},
$$

(2.6)

with

$$
a_{\pm} = \frac{1}{2} \left( e^{m^{(\epsilon)}_{i1} \theta} \pm e^{m^{(\epsilon)}_{i\bar{1}} \theta} \right).
$$

(2.7)
Case 2: $N = 4$ ($n = 2$) (Here $i = 1, 2$, $i = 3, 4$). In terms of $4 \times 4$ blocks ($D_{ij}$, $A_{ij}$ on the diag. and anti-diag. respectively)

\[
\begin{pmatrix}
D_{11} & 0 & 0 & A_{11} \\
0 & D_{22} & A_{22} & 0 \\
0 & A_{22} & D_{22} & 0 \\
A_{11} & 0 & 0 & D_{11}
\end{pmatrix},
\]

(2.8)

with

\[
D_{11} = D_{11} = \begin{pmatrix}
a_+ & 0 & 0 & 0 \\
0 & b_+ & 0 & 0 \\
0 & 0 & b_+ & 0 \\
0 & 0 & 0 & a_+
\end{pmatrix}, \quad D_{22} = D_{22} = \begin{pmatrix}
c_+ & 0 & 0 & 0 \\
0 & d_+ & 0 & 0 \\
0 & 0 & d_+ & 0 \\
0 & 0 & 0 & c_+
\end{pmatrix},
\]

(2.9)

and

\[
a_\pm = \frac{1}{2} \left( e^{m_{11}^{(+)} \theta} \pm e^{m_{11}^{(-)} \theta} \right), \quad b_\pm = \frac{1}{2} \left( e^{m_{12}^{(+)} \theta} \pm e^{m_{12}^{(-)} \theta} \right),
\]

\[
c_\pm = \frac{1}{2} \left( e^{m_{21}^{(+)} \theta} \pm e^{m_{21}^{(-)} \theta} \right), \quad d_\pm = \frac{1}{2} \left( e^{m_{22}^{(+)} \theta} \pm e^{m_{22}^{(-)} \theta} \right)
\]

(2.10)

We have verified, using a program, the braid equation (2.1) by inserting (2.4) for $N = 2, 4, 6, 8$. These provide direct checks for the argument indicated below (2.4). As compared to $\frac{1}{2} (N + 3)(N - 1)$ free parameters for odd [4], here for even $N$ we obtain $\frac{1}{2} N^2$ free parameters $m_{ij}^{(\pm)}$.

Let us just note that the odd dimensional solutions of Ref. [4] and the even dimensional solutions presented here can be regrouped in a single expression given by

\[
\hat{R} (\theta) = \frac{1}{2} \sum_{\varepsilon = \pm} \sum_{i,j=1}^{N} e^{m_{ij}^{(\varepsilon) \theta}} \left[ (ii) \otimes (jj) + \varepsilon (\bar{i}i) \otimes (\bar{j}j) \right],
\]

(2.11)

where

\[
m_{ij}^{(\varepsilon)} = m_{ij}^{(\varepsilon)} = m_{ij}^{(\varepsilon)} = m_{ij}^{(\varepsilon)}, \quad i, j = 1, \ldots, N \text{ and } \varepsilon = \pm 1,
\]

\[
n + 1 = n + 1 \text{ and } m_{n+1,n+1}^{(\varepsilon)} = 0 \quad (\forall \varepsilon) \quad \text{If } N \text{ is odd, i.e. } N = 2n + 1
\]

(2.12)

3 Unitarity

For all parameter real, $\hat{R} (\theta)$ is real but not unitary. Exactly as for $N$ odd, making each exponent purely imaginary, namely $\exp \left( m_{ij}^{(\pm) \theta} \right) \rightarrow \exp \left( i m_{ij}^{(\pm) \theta} \right)$, where on the right
\( m^{(\pm)}_{ij} \theta \) is now real with a coefficient \( i (i^2 = -1) \), one obtains unitarity. Now, due to the symmetry of the projectors

\[
\hat{R}(\theta)^+ = \hat{R}(-\theta) = \hat{R}(\theta)^{-1}, \quad \hat{R}(\theta)^+ \hat{R}(\theta) = I_{(2n)^2 \times (2n)^2}.
\]

(3.1)

In general, one can demonstrate that our multiparameter odd and even dimensional solutions one has a simple factorization

\[
\hat{R}(\theta_1 \pm \theta_2) = \hat{R}(\theta_1) \hat{R}(\theta_2) = \hat{R}(\theta_2) \hat{R}(\theta_1)^{\pm 1}.
\]

(3.2)

This evidently, holds for real or imaginary parameters, i.e. for nonunitary and unitary solutions. Correspondingly, the \( RTT \) relations can be expressed as follows:

\[
\left( \hat{R}(\theta) (T(\theta') \otimes I) \right) \left( (I \otimes T(\theta')) \hat{R}(\theta') \right) = \left( \hat{R}(\theta') (T(\theta') \otimes I) \right) \left( (I \otimes T(\theta)) \hat{R}(\theta) \right).
\]

(3.3)

For comparison one may note that the real unitary braid matrix for all \( N = 2n \) presented in [2] can be written as

\[
\hat{R}(z) = \left( \frac{1 - iz}{1 + iz} \right)^{1/2} P_+ + \left( \frac{1 + iz}{1 - iz} \right)^{1/2} P_-,
\]

(3.4)

where

\[
z = \tanh(\theta), \quad \left( \frac{1 \mp iz}{1 \pm iz} \right)^{1/2} \equiv e^{\pm i\phi},
\]

(3.5)

say, giving phases for the coefficients and

\[
P_\pm = \frac{1}{2} (I \otimes I \pm iK \otimes J).
\]

(3.6)

\( K, J \) being given by (2.2) of Ref. [2]. \( P_\pm \) can be expressed as sums of the projectors of the type

\[
Q^{(\epsilon)}_{ij} = \frac{1}{2} \left\{ (ii) \otimes (jj) + (\bar{i}i) \otimes (\bar{j}j) + e\cdot(-1)\bar{j} [(ii) \otimes (\bar{j}j) - (\bar{i}i) \otimes (jj)] \right\}
\]

(3.7)

defining analogously \( Q^{(\epsilon)}_{ij} \) (with \( j \rightarrow \bar{j}, \bar{j} \rightarrow j \) in \( Q^{(\epsilon)}_{ij} \)). The imaginary factor \( i \) in \( Q^{(\epsilon)}_{ij} \), \( Q^{(\epsilon)}_{ij} \) and the phases cancel giving a real \( \hat{R}(z) \),

\[
\hat{R}(z) = I \otimes I + zK \otimes J.
\]

(3.8)

Due to the summing up of \( Q^{(\epsilon)}_{ij} \) into \( P_\pm \) the effective number of projectors does not increase with \( N \), nor the number of parameters. Here we have presented a class of solutions where the number of free parameters increase as \( N^2 \). For this case, one can prove that

\[
\hat{R}(z_1) \hat{R}(z_2) = \hat{R}(z_3),
\]

(3.9)

where

\[
z_3 = \frac{z_1 + z_2}{1 - z_1 z_2} \neq \tanh (\theta_1 + \theta_2), \quad (z_1 z_2 \neq 1).
\]

(3.10)
4 \( \theta \)-Expansion

In section 5 of Ref. [4] exponentiation and \( \theta \)-expansion of \( \hat{R}(\theta) \) was presented for odd dimension. We present below a brief analogous treatment for even dimensions. One have

\[
e^{m_{ij}^{(e)} \theta} \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) = \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( m_{ij}^{(e)} \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) \right)^{k} \theta^{k}. \tag{4.1}
\]

Defining

\[
X = \sum_{\epsilon=\pm} \sum_{i,j=1}^{n} m_{ij}^{(e)} \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) \implies X^{n} = \sum_{\epsilon=\pm} \sum_{i,j=1}^{n} \left( m_{ij}^{(e)} \right)^{n} \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) \tag{4.2}
\]
due to the orthogonality of the projectors \( \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) \) for different sets of indices. Now from (2.4), due to the completeness (2.3),

\[
\hat{R}(\theta) = \sum_{\epsilon=\pm} \sum_{i,j=1}^{n} e^{m_{ij}^{(e)} \theta} \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right) = I + \sum_{k=1}^{\infty} \frac{1}{k!} X^{k} \theta^{k} = e^{\theta X}. \tag{4.3}
\]

Hence the braid equation (2.1) reduces to

\[
e^{\theta X_{12} e^{(\theta + \theta') X_{23}} e^{\theta' X_{12}}} = e^{\theta' X_{23} e^{(\theta + \theta') X_{12}}} e^{\theta X_{23}}, \tag{4.4}
\]

where \( X_{12} = X \otimes I \) and \( X_{23} = I \otimes X \). Expanding both sides and comparing coefficients of \( \theta^{a} (\theta + \theta')^{b} \theta^{c} \) one obtains a sequence of relations involving \( X_{12}, X_{23} \). Some have been pointed out in section 5 of Ref. [4]. There would be parallel features here.

After implementing unitarity as in section 3 one can define (with \( i \) as above (3.1))

\[
X = i \sum_{i,j,\epsilon} m_{ij}^{(e)} \left( P_{ij}^{(e)} + P_{ij}^{(e)} \right). \tag{4.5}
\]

One then proceeds as above.

We have started with multiparameter case. For

\[
\hat{R}(z) = I \otimes I + zK \otimes J = \left( \frac{1 - i z}{1 + i z} \right)^{1/2} P_{+} + \left( \frac{1 + i z}{1 - i z} \right)^{1/2} P_{-} \equiv e^{i \phi} P_{+} + e^{-i \phi} P_{-}. \tag{4.6}
\]

(see the discussion following (3.3))

\[
X = i (P_{+} - P_{-}) = -K \otimes J \tag{4.7}
\]

and

\[
\hat{R}(z) = \hat{R}(\varphi) = e^{i \phi X}, \tag{4.8}
\]

where \( e^{i \varphi} = \left( \frac{1 - \tanh \theta}{1 + \tanh \theta} \right)^{1/2} \). By inserting this \( X \) in (4.4), one can develop in \( \varphi \) as explained before.
5 Discussion

For the multiparameter solutions presented in sections 2 and 3 one can study $\hat{R}$TT relations, transfer matrices, Hamiltonians and factorizable $S$-matrices in a closely analogous fashion to that for odd dimensions [5]. They are beyond the scope of this paper, limited essentially to construction of multiparameter $(2n)^2 \times (2n)^2$ braid matrices (nonunitary and unitary).

Beyond the unification presented there is a basic difference between odd and even dimensional cases. For the $(2n+1)^2 \times (2n+1)^2$ braid matrices with a basis of our ”nested sequence” of projectors our multiparameter solutions are the most general ones. The presence of the central element 1 in $\hat{R}(\theta)$ imposes the simple exponential solutions for the coefficients of all other projectors. This has been emphasized in appendix A of Ref. [4], ("Solving the braid equation"). But for even dimension this not the case. The class of solutions presented here is only one possibility. Already for the $4 \times 4$ case the intensively studied 6- and 8-vertex solutions can be canonically expressed on our basis (sections 6 and 7 of Ref. [6]). The multidimensional generalization of the 6-vertex matrix presented in Ref. [7] (citing previous sources) remains restricted to a single parameter $\gamma$. Are there authentic multiparameter generalizations of 6- and 8-vertex solutions to $(2n)^2 \times (2n)^2$ matrices for $n > 1$? We intend to explore such possibilities elsewhere.

We point out moreover that a pure imaginary spectral parameter ($\theta \rightarrow i\theta$) renders the 6-vertex and 8-vertex braid matrices unitary. This particularly evident form the respective canonical forms ((6.5) for 6-vertex and (7.2) with (7.6), (7.7) for 8-vertex of Ref. [6]) where the normalization factors are also suitably adapted. The coefficient of each real symmetric projector is evidently inverted under conjugation ($i\theta \rightarrow -i\theta$). Hence

$$\hat{R}^+(\theta) \hat{R}(\theta) = \hat{R}(-\theta) \hat{R}(\theta) = I.$$  \hspace{1cm} (5.1)

Now one no longer has statistical models with real, non-negative Boltzmann weights. But the unitary matrices become relevant concerning entanglement. Such parametrizations of entangled states will be studied in a following paper.

The unitary $(2n)^2 \times (2n)^2$ braid matrices generate entangled quantum states with one difference as compared to the odd dimensional case. In the last section of Ref. [2] it was pointed out that the product of pure states $|0\rangle |0\rangle$ conserved its status under action of $(2n+1)^2 \times (2n+1)^2$ unitary matrix. For the present case there is no such exceptional state.

As already pointed out for odd dimensions (see section 5, Ref. [2]), even dimensional, unitary, multiparameter braid matrices are also periodic or quasiperiodic in $\theta$ accordingly as the $m$’s are mutually commensurate or not.

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References

[1] Y. Zhang and M.L. Ge, *GHZ States, Almost-Complex Structure and Yang–Baxter Equation (I)*, quant-ph/0701244.

[2] B. Abdesselam, A. Chakrabarti, V.K. Dobrev and S.G. Mihov, *Higher Dimensional Unitary Braid Matrices: Construction, Associated Structures and Entanglements*, Jour. Math. Phys. 48 (2007) 053508.

[3] L.H. Kauffman and S.J. Lomonaco Jr., *Braiding operators are universal quantum gates*, New. J. Phys. 6 (2004) 34. quant-ph/0401094.

[4] A. Chakrabarti, *A nested sequence of projectors and corresponding braid matrices \( \hat{R}(\theta) \): (1) Odd dimensions*, Jour. Math. Phys. 46 (2005) 063508. math.QA/0401207.

[5] B. Abdesselam and A. Chakrabarti, *A nested sequence of projectors: (2) Multiparameter multistate statistical models, Hamiltonians, S-matrices*, Jour. Math. Phys. 47(2006) 053508. math.QA/0601584

[6] A. Chakrabarti, *Canonical factorization and diagonalization of Baxterized braid matrices: Explicit constructions and applications*, Jour. Math. Phys. 44 (2003) 5320.

[7] H.J. De Vega, *Yang-Baxter algebras, integrable theories and quantum groups*, Int. Jour. Mod. Phys. A vol. 4 (1989) 2371.