Pseudo prolate spheroidal functions

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Abstract—Let $D_T$ and $B_\Omega$ denote the operators which cut the time content outside $T$ and the frequency content outside $\Omega$, respectively. The prolate spheroidal functions are the eigenfunctions of the operator $P_{T,\Omega} = D_T B_\Omega D_T$. With the aim of formulating in precise mathematical terms the notion of Nyquist rate, Landau and Pollack have shown that, asymptotically, the number of such functions with eigenvalue close to one is $\approx \frac{|T|}{|\Omega|}$. We have recently revisited this problem with a new approach: instead of counting the number of eigenfunctions with eigenvalue close to one, we count the maximum number of orthogonal $\epsilon$-pseudo-eigenfunctions with $\epsilon$-pseudo-eigenvalue one. Precisely, we count how many orthogonal functions have a maximum of energy $\epsilon$ outside the domain $T \times \Omega$, in the sense that $\|P_{T,\Omega} f - f\|_2 \leq \epsilon$. We have recently discovered that the sharp asymptotic number is $\approx (1 - \epsilon)^{-1} \frac{|T| |\Omega|}{2\pi \epsilon}$. The proof involves an explicit construction of the pseudo-eigenfunctions of $P_{T,\Omega}$. When $T$ and $\Omega$ are intervals we call them pseudo prolate spheroidal functions. In this paper we explain how they are constructed.

I. INTRODUCTION

A. Slepian’s bandwith paradox

In his 1974 Shannon lecture, whose written version appeared in [21], David Slepian stated the following paradoxal dilemma:

"It is easy to argue that real signals must be bandlimited. It is also easy to argue that they cannot be so."

Such a dilemma (we will call it the bandwith paradox) reflects a mainstay of quantitative physical sciences: the gap between observations and models of the real world. On the one side, it is reasonable to accept that, for any measuring instrument, there is a finite cutoff above which the instrument would not be able to measure the frequencies of a signal. Hence, it can be argued that all signals are bandlimited. On the other side, bandlimited signals are represented by analytic functions. This does not allow the function to vanish in any real interval, leading to the unrealistic model where signals cannot start or stop, but must go on forever. Hence, it can be argued that no signal is bandlimited.

The heuristics of the previous paragraph are already enough to change our mindset: instead of supports one should think of essential supports. Then the question arises of what is the dimension of the set of such functions. Since in reality we are not dealing with finite dimensional sets, we need to resort to an approximated notion of dimension. For instance, Landau and Pollack [18] considered, as a notion of dimension, the minimal number $N(\epsilon)$ of functions required to approximate any essentially time-band limited function in the $L_2$ norm up to an error $\epsilon$. Based on such considerations, two solutions of the bandwith paradox have been offered, one by Landau and Pollack, the other by Slepian. We will give a brief account of the two approaches and suggest a new one, based on a line of research initiated in [1].

We note in passing that, besides the solution of the bandwith paradox, some of the above heuristics played a fundamental role in the papers [11] and [10], which spearheaded the modern theory of Compressed Sensing, where an understanding of the deep mathematical reasons behind the sparsity-promoting properties of $l_1$ minimization has been achieved [5].

B. Landau-Pollack solution: prolate spheroidal functions

Let $D_T$ and $B_\Omega$ denote the operators which cut the time content outside $T$ and the frequency content outside $\Omega$, respectively. In a nowadays classical paper [18], whose purpose was to examine the true in the engineering intuition that there are approximately $|T| |\Omega| / 2\pi$ independent signals of bandwidth $\Omega$ concentrated on an interval of length $T$, Landau and Pollack have considered the eigenvalue problem associated with the positive self-adjoint operator $P_{T,\Omega} = D_T B_\Omega D_T$. When $T$ and
The eigenfunctions of $P_{T, \Omega}$ are the prolate spheroidal functions $\{\phi_j\}_{j=0}^{\infty}$. They provide the best known dictionary for approximating essentially time and band limited functions in the $L_2$ norm and their properties are still object of current investigation. The approach to the bandwith paradox based on prolate spheroidal functions relies on the peculiar investigation \cite{19}. The discussion in 7, pag. 23 and the recent book \cite{14}, (2) is essentially supported in the time- and bandlimited region $\Omega$. Within mathematical signal analysis (see, for instance the discussion in \cite{7}, pag. 23 and the recent book \cite{14}), (2) is viewed as a mathematical formulation of the Nyquist rate, the fact that a time- and bandlimited region $T \times \Omega$ corresponds to $|T||\Omega|/2\pi$ degrees of freedom.

C. Slepian’s solution: approximated dimension theorem

With a view to solving the bandwith paradox, Slepian replaced the notions of bandlimited and timelimited by more quantitative concepts, regarding signals as $\epsilon$-timelimited in $T$ if the energy of the signal outside $T$ is less than $\epsilon$ and $\epsilon$-bandlimited in $\Omega$ if the if the energy of the Fourier transform of the signal outside $\Omega$ is less than $\epsilon$. Slepian associates $\epsilon$ to the precision of measuring instruments and defines a flexible notion of $\epsilon$-approximate dimension as follows. The set $F$ of signals is said to have approximate dimension $N$ at level $\epsilon$ in the set $T$ if, for every $r \in F$, there exist $a_1, \ldots, a_N$ such that

$$\int_T \left| r(t) - \sum_{j=1}^{N} a_j \phi_j(t) \right|^2 dt < \epsilon$$

and there is no set of $N-1$ functions that approximates every element of $F$ in such a way. Slepian’s dimension theorem states that the approximated dimension $N(\Omega, rT, \epsilon, \epsilon)$ at level $\epsilon > 0$ of the set $F$, of $\epsilon$-band and timelimited functions, in the sense that $\|D_r f - f\| \leq \epsilon$ and $|B_\Omega f - f| \leq \epsilon$, satisfies

$$\lim_{r \rightarrow \infty} \frac{N(rT, \Omega, \epsilon, \epsilon)}{r} = \frac{|T||\Omega|}{2\pi}. \quad (5)$$

Slepian’s proof is also constructive. He defines a sequence of functions using the prolates and their associated eigenvalues as follows:

$$g_j(t) = \sqrt{\frac{\epsilon}{1 - \lambda_j}} \phi_j(t) + \sqrt{\frac{\epsilon}{\lambda_j(1 - \lambda_j)}} 1_{[-1,1]} \left( \frac{2t}{T} \right) \phi_j(t).$$

The $g_j$ are not complete in $F$, but Slepian has proved that they are the best sequence to use for approximating functions in $F$. 

D. Pseudospectra enters the picture

We remark that in the dimension theorems of Landau-Pollak and of Slepian, the amount of energy outside $T \times \Omega$ does not appear in the asymptotic limits \cite{3} and \cite{5}. With the aim of developing an approximation theory of almost band-limited functions where the number of degrees of freedom adjusts to the energy left outside $T \times \Omega$, we have introduced a new sequence of functions which, we call *pseudo prolate spheroidal functions*. Our research program is not fully completed, but it is reasonable to expect these functions to have good linear approximation properties of essentially band-limited functions, like those recently proved for other orthogonal systems in \cite{15}. Moreover, we also expect the increase of the degrees of freedom to have sparsity-promoting properties similar to the frame-based representations, following the intuition provided by the "dictionary example" \cite{6}: “The larger and richer is my dictionary the shorter are the phrases I compose.”

We start by reformulating Landau-Pollack’s approach in the following way: instead of counting the number of eigenfunctions $f$ satisfying $P_{T, \Omega} f = \lambda f$ which are associated with $\lambda \approx 1$, we count the number of orthogonal functions such that $P_{T, \Omega} f \approx f$, in the sense that the $L_2$ distance between $P_{T, \Omega} f$ and $f$ is smaller than a prescribed amount of energy $\epsilon$. Precisely, we assume $\|f\| = 1$ and require that

$$\|P_{T, \Omega} f - f\|^2 \leq \epsilon. \quad (6)$$

This measures the concentration of $f$ because $\epsilon$ controls the maximum amount of energy left outside $T \times \Omega$. For instance, if
\[ \| D_r f - f \| \leq \epsilon \text{ and } \| B_\Omega f - f \| \leq \epsilon, \text{ then } \| P_{T_\Omega} f - f \|^2 \leq 4\epsilon^2. \] The idea has been introduced in [11]. It is based on the concept of pseudospectra of linear operators, which has found remarkable applications in the last decade (see [12], [13], [8], the surveys [22] and [20] or the book [23]). In general, \( \lambda \) is an \( \epsilon \)-pseudoeigenvalue of \( L \) if there exists \( f \) with \( \| f \| = 1 \) such that \( \| L f - \lambda f \| \leq \epsilon \). We call \( f \) an \( \epsilon \)-pseudoeigenfunction corresponding to \( \lambda \).

As in the previous approaches, the set of \( \epsilon \)-localized functions in \( T \times \Omega \) is not a linear space, making no sense to strictly talk about its dimension. However, we can count the maximal number \( \eta(\epsilon, rT, \Omega) \) of orthogonal functions satisfying (6). In [11], using an explicit construction, we have shown that, as \( r \to \infty \), the following inequalities hold:

\[
\frac{|T| |\Omega|}{2\pi} (1 + \epsilon) \leq \lim_{r \to \infty} \frac{\eta(\epsilon, rT, \Omega)}{r^d} \leq \frac{|T| |\Omega|}{2\pi} (1 - 2\epsilon)^{-1}.
\]

Recently, we have obtained the sharp version of these inequalities:

\[
\lim_{r \to \infty} \frac{\eta(\epsilon, rT, \Omega)}{r} = (1 - \epsilon)^{-1} \frac{|T| |\Omega|}{2\pi}. 
\]

Our proof of the lower inequality in (7) is also constructive. As in Slepian’s approach and as in [11], the orthogonal functions will be built in terms of the prolate spheroidal functions. We will describe the construction of the functions achieving the sharp result (7). Since they result from a pseudospectra analogue of the spectral problem defining the prolate spheroidal functions, we will call the corresponding pseudoeigenfunctions pseudo prolate spheroidal functions. Full proofs of (7) and other results will appear in [2].

II. CONSTRUCTION OF THE PSEUDO PROLATE SPHEROIDAL FUNCTIONS.

A. Time- and band-limiting operators

A description of the general set-up of [16] and [17] follows. The sets \( T \) and \( \Omega \) are general subsets of finite measure of \( \mathbb{R}^d \). Let

\[ F f (\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-i\xi t} dt \]

denote the Fourier transform of a function \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). The subspaces of \( L^2(\mathbb{R}^d) \) consisting, respectively, of the functions supported in \( T \) and of those whose Fourier transform is supported in \( \Omega \) are

\[
\mathcal{D}(T) = \{ f \in L^2(\mathbb{R}^d) : f(x) = 0, x \notin T \} \\
\mathcal{B}(\Omega) = \{ f \in L^2(\mathbb{R}^d) : F f (\xi) = 0, \xi \notin \Omega \}.
\]

Let \( D_T \) be the orthogonal projection of \( L^2(\mathbb{R}^d) \) onto \( \mathcal{D}(T) \), given explicitly by the multiplication of a characteristic function of the set \( T \) by \( f \):

\[ D_T f(t) = \chi_T(t) f(t) \]

and let \( B_\Omega \) be the orthogonal projection of \( L^2(\mathbb{R}^d) \) onto \( \mathcal{B}(\Omega) \), given explicitly as

\[ B_\Omega f = F^{-1} \chi_\Omega F f = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} h(x-y) f(y) dy, \]

where \( F h = \chi_\Omega \). The following Theorem, comprising Lemma 1 and Theorem 1 of [17] gives important information concerning the spectral problem associated with the operator \( D_{rT} B_\Omega D_{rT} \). The notation \( o(r^d) \) refers to behavior as \( r \to \infty \).

Theorem A ([17]). The operator \( D_{rT} B_\Omega D_{rT} \) is bounded by 1, self-adjoint, positive, and completely continuous. Denoting its set of eigenvalues, arranged in nonincreasing order, by \( \{ \lambda_k(r,T,\Omega) \} \), we have

\[ \sum_{k=0}^{\infty} \lambda_k(r,T,\Omega) = r^d (2\pi)^{-d} |T| |\Omega| \]

\[ \sum_{k=0}^{\infty} \lambda_k^2(r,T,\Omega) = r^d (2\pi)^{-d} |T| |\Omega| - o(r^d). \]

Moreover, given \( 0 < \gamma < 1 \), the number \( M_r(\gamma) \) of eigenvalues which are not smaller than \( \gamma \), satisfies, as \( r \to \infty \),

\[ M_r(\gamma) = (2\pi)^{-d} |T| |\Omega| r^d + o(r^d). \]

B. Construction of the pseudo prolate spheroidal functions

Suppose (6) holds for a positive real \( \epsilon \). Let \( \sigma > 0 \) be such that \( \sigma^2 \leq \epsilon \) and let \( \mathcal{F} = \{ \phi_k \} \) be the normalized system of eigenfunctions (in the one dimension interval case they are the prolaters) of the operator \( P_{rT,\Omega} \) with eigenvalues \( \lambda_k > 1 - \sigma \). Now, given \( f \in L^2(\mathbb{R}^d) \), write

\[ f = \sum a_k \phi_k + h, \]

with \( h \in \ker(P_{rT,\Omega}) \). Then

\[ P_{rT,\Omega} f = \sum a_k \lambda_k \phi_k \]

and

\[ \| P_{rT,\Omega} f - f \|^2 = \| \sum (1 - \lambda_k) a_k \phi_k + h \|^2 \leq \sigma^2 \sum |a_k|^2 + \| h \|^2 \]

\[ = \sigma^2 \| f \|^2 + (1 - \sigma^2) \| h \|^2. \]

For the given \( \sigma > 0 \) we pick a real number \( \gamma \) such that

\[ \sigma^2 + (1 - \sigma^2) \gamma = \epsilon. \]
Writing this as $\gamma = (\varepsilon - \sigma^2)/(1 - \sigma^2)$ it's clear that $\gamma$ is a positive increasing function of $\sigma$, and $\gamma \to \varepsilon$ as $\sigma \to 0$.

Now take $n = \# F$, define $\Gamma = \{ F \}$ and let $m$ be a positive integer (its value will be made precise later). Choose $h_1, h_2, \ldots, h_m$ orthonormal functions in \text{Ker}(P_{r,T,\Omega}) and let $\Lambda$ be the space spanned by these functions. This can be done since $\text{Ker}(P_{r,T,\Omega})$ has infinite dimension, due to the inclusion $D(\mathbb{R}^d - r T) \subset \text{Ker}(P_{r,T,\Omega})$. Note that this $m$ functions together with the $n$ functions of $\mathcal{F}$ form an orthonormal basis of $\Gamma \oplus \Lambda$, since the first are orthogonal to the latter. We now define the \textit{pseudoeigenfunctions} as a second orthonormal basis of $\Gamma \oplus \Lambda$, denoted by $\{ \Phi_j \}_{j=1}^{m+n}$, with

$$\Phi_j = \psi_j + \rho_j,$$

$\psi_j \in \Gamma$ and $\rho_j \in \Lambda$ for $j \in \{1,2,\ldots,m+n\}$. The proof of the lower inequality in (7) requires the construction of the functions (12) in such a way that

$$\|\rho_j\|^2 = \frac{m}{m+n}, \quad j = 1,2,\ldots,m+n. \quad (13)$$

This will be done using a linear algebra argument detailed in the next paragraph.

Consider the automorphism $Q$ in $\Gamma \oplus \Lambda$ that maps the first basis to the functions in (12). One can see $Q$ as an orthogonal $(m+n) \times (m+n)$ matrix of the form

$$Q = [Q^\Gamma Q^\Lambda]_{(m+n) \times (m+n)},$$

where its first $n$ columns $Q^\Gamma$ map $\Gamma$ to $\Gamma \oplus \Lambda$ and the last $m$ columns $Q^\Lambda$ map $\Lambda$ to $\Gamma \oplus \Lambda$. Then, since $\|h_j\| = 1$, the condition (13) is equivalent to

$$\|Q^\Lambda_j\|^2 = \frac{m}{m+n}, \quad j = 1,2,\ldots,m+n, \quad (14)$$

where $Q^\Lambda_j$ denotes the $j$th line of $Q^\Lambda$. Let $X$ be the Discrete Fourier Transform matrix of order $m+n$, with entries

$$X_{ij} = \frac{1}{\sqrt{m+n}} \omega^{ij}, \quad i,j = 0,1,\ldots,m+n-1,$$

where $\omega = e^{-i2\pi/n}$ is the $(m+n)$th-root of the unity. Then define the $(m+n) \times (m+n)$ matrix $X'$ as

$$X'_{ij} = \Re(X_{ij}) + \Im(X_{ij}), \quad i,j = 0,1,\ldots,m+n-1. \quad (15)$$

One can check that this matrix is orthogonal (more details will be given in (2)). Now we finally define $Q$ as a permutation of the columns of $X'$, depending on the parity of $m$. If $m$ is even, we choose the last $m/2$ columns of $Q$ to be the $1$ to $m/2$ and the last $m/2$ columns of $X'$. This leads to

$$\| Q_j^\Lambda \|^2 = \sum_{k=n+1}^{m+n} Q_{jk}^2$$

$$= \frac{1}{m+n} \sum_{k=1}^{m/2} (a_{jk} + b_{jk})^2 + (a_{jk} - b_{jk})^2$$

$$= \frac{1}{m+n} \sum_{k=1}^{m/2} (a_{jk} + b_{jk})^2 + (a_{jk} - b_{jk})^2$$

$$= \frac{1}{m+n} \sum_{k=1}^{m/2} 2a_{jk}^2 + 2b_{jk}^2$$

$$= \frac{m}{m+n},$$

since $a_{jk}^2 + b_{jk}^2 = 1$, thus $Q$ satisfies (14). If $m$ is odd, we add to this columns the column $0$, which has all entries equal to $1/\sqrt{m+n}$, the additional calculations in this case are trivial.

We have finally proved that there are $m+n$ functions as in (12) which verify (13). Since $\psi_j$ are linear combinations of elements of $\mathcal{F} = \{ \phi_k \}$, and $\rho_j \in \text{Ker}(P_{r,T,\Omega})$, (12) is a representation of the form (8). We can now apply (10) and (13) to obtain

$$\| P_{r,T,\Omega} \Phi_j - \Phi_j \|^2 \leq \sigma^2 \| \Phi_j \|^2 + (1 - \sigma^2) \| \rho_j \|^2$$

$$= \sigma^2 + (1 - \sigma^2) \frac{m}{m+n} \quad (16)$$

We now choose $m$ so that (16) is at most $\varepsilon$, or equivalently,

$$\frac{m}{m+n} \leq \gamma. \quad \text{Clearly, this happens if and only if } m \leq \frac{n}{1-\gamma}. \quad \text{Choosing the biggest $m$ which verifies this condition, leads to } m \geq \frac{n}{1-\gamma} - 1.$$ We now use Theorem A (the fact that $n = \# F = r^d (2\pi)^{-d} |T| |\Omega| + o(r^d))$ and this last inequality

$$\# \{ \Phi_j \}_{j=1}^{m+n} = m + n$$

$$\geq n \left( \frac{\gamma}{1-\gamma} + 1 \right) - 1$$

$$= \left( \frac{1}{1-\gamma} \right) n - 1$$

$$= (1-\gamma)^{-1} (r^d(2\pi)^{-d} |T| |\Omega| + o(r^d)) - 1$$

$$= (1-\gamma)^{-1} (r^d(2\pi)^{-d} |T| |\Omega| + o(r^d)),$$

since $1 = o(r^d)$. We have obtained by construction the pseudo prolate spheroidal functions $\{ \Phi_j \}_{j=1}^{m+n}$. They are also orthonormal and verify (6).

The lower inequality in (7) is now a simple consequence of this construction. Denote by $M^-(r,T,\Omega,\varepsilon)$ the minimum number of orthonormal functions satisfying (6). Then,

$$M^-(r,T,\Omega,\varepsilon) \geq \# \left( \bigcup_{j=1}^{\#(\Phi_j)} \{ \Phi_j \}_{j=1}^{\#(\Phi_j)} \right)$$

$$\geq (1-\gamma)^{-1} r^d (2\pi)^{-d} |T| |\Omega| + o(r^d).$$
Finally, take $\sigma \to 0$, so that $\gamma \to \epsilon$ to yield
\[
M^{-}(rT, \Omega, \epsilon) \geq \# \left[ \bigcup_{i=1}^{d} \{ \Phi_{j}^{(i)} \}_{j=1}^{n+1} \right] \\
\geq (1 - \epsilon)^{-1} r^{d} (2\pi)^{-d} |T| |\Omega| + o(r^{d}).
\]

Remark 1: The above construction applies to several settings where properties similar to Theorem A are available, as in [3] and [9].

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