The Kernel Pitman-Y or Process
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Abstract—In this work, we propose the kernel Pitman-Y or process (KPYP) for nonparametric clustering of data with general spatial or temporal interdependencies. The KPYP is constructed by first introducing an infinite sequence of random locations. Then, based on the stick-breaking construction of the Pitman-Y or process, we define a predictor-dependent random probability measure by considering that the discount hyperparameters of the Beta-distributed random weights (stick variables) of the process are not uniform among the weights, but controlled by a kernel function expressing the proximity between the location assigned to each weight and the given predictors.

Index Terms—Pitman-Y or process, kernel functions, unsupervised clustering

1 INTRODUCTION
Nonparametric Bayesian modeling techniques, especially Dirichlet process mixture (DPM) models, have become very popular in statistics over the last few years, for performing nonparametric density estimation [1], [2], [3]. This theory is based on the observation that an infinite number of component distributions in an ordinary finite mixture model (clustering model) tends on the limit to a Dirichlet process (DP) prior [2], [4]. Eventually, the nonparametric Bayesian inference scheme induced by a DPM model yields a posterior distribution on the proper number of model component densities (inferred clusters) [5], rather than selecting a fixed number of mixture components. Hence, the obtained nonparametric Bayesian formulation eliminates the need of doing inference (or making arbitrary choices) on the number of mixture components (clusters) necessary to represent the modeled data.

An interesting alternative to the Dirichlet process prior for nonparametric Bayesian modeling is the Pitman-Yor process (PYP) prior [6]. Pitman-Yor processes produce power-law distributions that allow for better modeling populations comprising a high number of clusters with low popularity and a low number of clusters with high popularity [7]. Indeed, the Pitman-Yor process prior can be viewed as a generalization of the Dirichlet process prior, and reduces to it for a specific selection of its parameter values. In [8], a Gaussian process-based coupled PYP method for joint segmentation of multiple images is proposed.

A different perspective to the problem of nonparametric data modeling was introduced in [9], where the authors proposed the kernel stick-breaking process (KSBP). The KSBP imposes the assumption that clustering is more probable if two feature vectors are close in a prescribed (general) space, which may be associated explicitly with the spatial or temporal position of the modeled data. This way, the KSBP is capable of exploiting available prior information regarding the spatial or temporal relations and dependencies between the modeled data.

Inspired by these advances, and motivated by the interesting properties of the PYP, in this paper we come up with a different approach towards predictor-dependent random probability measures for nonparametric Bayesian clustering. We first introduce an infinite sequence of random spatial or temporal locations. Then, based on the stick-breaking construction of the Pitman-Yor process, we define a predictor-dependent random probability measure by considering that the discount hyperparameters of the Beta-distributed random weights (stick variables) of the process are not uniform among the weights, but controlled by a kernel function expressing the proximity between the location assigned to each weight and the given predictors. The obtained random probability measure is dubbed the kernel Pitman-Y or process (KPYP) for non-parametric clustering of data with general spatial or temporal interdependencies. We empirically study the performance of the KPYP prior in unsupervised image segmentation and text-dependent speaker identification, and compare it to the kernel stick-breaking process, and the Dirichlet process prior.

The remainder of this paper is organized as follows: In Section 2, we provide a brief presentation the Pitman-Yor process, as well as the kernel stick-breaking process, and its desirable properties in clustering data with spatial or temporal dependencies. We empirically study the performance of the KPYP prior in unsupervised image segmentation and text-dependent speaker identification, and compare it to the kernel stick-breaking process, and the Dirichlet process prior.

2 THEORETICAL BACKGROUND
2.1 The Pitman-Yor Process
Dirichlet process (DP) models were first introduced by Ferguson [11]. A DP is characterized by a base distribution \( G_0 \) and a positive scalar \( \alpha \), usually referred to as the
innovation parameter, and is denoted as \( DP(\alpha, G_0) \). Essentially, a DP is a distribution placed over a distribution. Let us suppose we randomly draw a sample distribution \( G \) from a DP, and, subsequently, we independently draw \( M \) random variables \( \{\Theta^*_m\}_{m=1}^M \) from \( G \):

\[
G|\alpha, G_0 \sim DP(\alpha, G_0)
\]

(1)

\[
\Theta^*_m|G \sim G, \quad m = 1, \ldots, M
\]

(2)

Integrating out \( G \), the joint distribution of the variables \( \{\Theta^*_m\}_{m=1}^M \) can be shown to exhibit a clustering effect. Specifically, given the first \( M-1 \) samples of \( G \), \( \{\Theta^*_m\}_{m=1}^{M-1} \), it can be shown that a new sample \( \Theta^*_M \) is either (a) drawn from the base distribution \( G_0 \) with probability \( \frac{\alpha + d}{\alpha + M - 1} \), or (b) is selected from the existing draws, according to a multinomial allocation, with probabilities proportional to the number of the previous draws with the same allocation \[12\]. Let \( \{c_c\}_{c=1}^\infty \) be the set of distinct values taken by the variables \( \{\Theta^*_m\}_{m=1}^{M-1} \). Denoting as \( f_c^{M-1} \) the number of values in \( \{\Theta^*_m\}_{m=1}^{M-1} \) that equal to \( \Theta^*_c \), the distribution of \( \Theta^*_M \) given \( \{\Theta^*_m\}_{m=1}^{M-1} \) can be shown to be of the form \[12\]

\[
p(\Theta^*_M|\{\Theta^*_m\}_{m=1}^{M-1}, \alpha, G_0) = \frac{\alpha}{\alpha + M - 1} G_0 + \sum_{c=1}^C \frac{f_c^{M-1} - d}{\alpha + M - 1} \delta_{\Theta^*_c}
\]

(3)

where \( \delta_{\Theta^*_c} \) denotes the distribution concentrated at a single point \( \Theta^*_c \).

The Pitman-Yor process \[6\] functions similar to the Dirichlet process. Let us suppose we randomly draw a sample distribution \( G \) from a PYP, and, subsequently, we independently draw \( M \) random variables \( \{\Theta^*_m\}_{m=1}^M \) from \( G \):

\[
G|d, \alpha, G_0 \sim PY(d, \alpha, G_0)
\]

(4)

\[
\Theta^*_m|G \sim G, \quad m = 1, \ldots, M
\]

(5)

where \( d \in [0,1) \) is the discount parameter of the Pitman-Yor process, \( \alpha > -d \) is its innovation parameter, and \( G_0 \) the base distribution. Integrating out \( G \), similar to Eq. (3), we now yield

\[
p(\Theta^*_M|\{\Theta^*_m\}_{m=1}^M, d, \alpha, G_0) = \frac{\alpha + dC}{\alpha + M - 1} G_0 + \sum_{c=1}^C \frac{f_c^{M-1} - d}{\alpha + M - 1} \delta_{\Theta^*_c}
\]

(6)

As we observe, the PYP yields an expression for \( p(\Theta^*_M|\{\Theta^*_m\}_{m=1}^M, G_0) \) quite similar to that of the DP, also possessing the rich-gets-richer clustering property, i.e., the more samples have been assigned to a draw from \( G_0 \), the more likely subsequent samples will be assigned to the same draw. Further, the more we draw from \( G_0 \), the more likely a new sample will again be assigned to a new draw from \( G_0 \). These two effects together produce a power-law distribution where many unique \( \Theta^*_m \) values are observed, most of them rarely \[6\]. In particular, for \( d > 0 \), the number of unique values scales as \( \mathcal{O}(\alpha M^d) \), where \( M \) is the total number of draws. Note also that, for \( d = 0 \), the Pitman-Yor process reduces to the Dirichlet process, in which case the number of unique values grows more slowly at \( \mathcal{O}(\alpha \log M) \) \[13\].

A characterization of the (unconditional) distribution of the random variable \( G \) drawn from a PYP, \( PY(d, \alpha, G_0) \), is provided by the stick-breaking construction of Sethuraman \[14\]. Consider two infinite collections of independent random variables \( v_c = (v_c)_c^{\infty} \), \( \{\Theta^*_c\}_c^{\infty} \) where the \( v_c \) are drawn from a Beta distribution, and the \( \Theta^*_c \) are independently drawn from the base distribution \( G_0 \). The stick-breaking representation of \( G \) is then given by \[13\]

\[
G = \sum_{c=1}^{\infty} \pi_c(v) \delta_{\Theta^*_c}
\]

(7)

where

\[
p(v_c) = Beta(v_c|1-d, \alpha + dc)
\]

(8)

\[
v = (v_c)_c^{\infty}
\]

(9)

\[
\pi_c(v) = v_c \prod_{j=1}^{c-1} (1 - v_j) \quad \in [0,1]
\]

(10)

and

\[
\sum_{c=1}^{\infty} \pi_c(v) = 1
\]

(11)

### 2.2 The Kernel Stick-Breaking Process

An alternative to the above approaches, allowing for taking into account additional prior information regarding spatial or temporal dependencies in the modeled datasets, is the kernel stick-breaking process introduced in \[9\]. The basic notion in the formulation of the KSBP consists in the introduction of a predictor-dependent prior, which promotes clustering of adjacent data points in a prescribed (general) space.

Let us consider that the observed data points \( y \in \mathcal{Y} \) are associated with positions where measurement was taken \( x \in \mathcal{X} \), arranged on a \( D \)-dimensional lattice. For example, in cases of sequential data modeling, the observed data points \( y \) are naturally associated with an one-dimensional lattice that depicts their temporal succession, i.e. the time point these measurements were taken. In cases of computer vision applications, we might be dealing with observations \( y \) measured on different locations on a two-dimensional or three-dimensional space \( \mathcal{X} \).

To take this prior information into account, the KSBP postulates that the random process \( G \) in (1) comprises a function of the predictors \( x \) related to the observable data points \( y \), expressing their location in the prescribed space \( \mathcal{X} \). Specifically, it is assumed that

\[
G = \sum_{c=1}^{\infty} \pi_c(v(x)) \delta_{\Theta^*_c}
\]

(12)
where
\[
\varpi_c(v(x)) = v_c(x, \Gamma_c; \psi_c) \prod_{j=1}^{c-1} (1 - v_j(x, \Gamma_j; \psi_j)) \in [0, 1] 
\] (13)
\[
v(x) = (v_c(x, \Gamma_c; \psi_c))_{c=1}^{\infty} 
\] (14)
\[
v_c(x, \Gamma_c; \psi_c) = V_c k(x, \Gamma_c; \psi_c) 
\] (15)
\[
p(V_c) = \text{Beta}(V_c|1, \alpha) 
\] (16)
and \(k(x, \Gamma_c; \psi_c)\) is a kernel function centered at \(\Gamma_c\) with hyperparameter \(\psi_c\).

By selecting an appropriate form of the kernel function \(k(x, \Gamma_c; \psi_c)\), KSBP allows for obtaining prior probabilities \(\varpi_c(v(x))\) for the derived clusters that depend on the values of the predictors (spatial or temporal locations) \(x\). Indeed, the closer the location \(x\) of an observation \(y\) is to the location \(\Gamma_c\) assigned to the \(c\)th cluster, the higher the prior probability \(\varpi_c(v(x))\) becomes. Thus, the KSBP prior promotes by construction clustering of (spatially or temporally) adjacent data points. For example, a typical selection for the kernel \(k(x, \Gamma_c; \psi_c)\) is the radial basis function (RBF) kernel
\[
k(x, \Gamma_c; \psi_c) = \exp \left[ -\frac{||x - \Gamma_c||^2}{\psi_c^2} \right] 
\] (17)

### 3 Proposed Approach

#### 3.1 Model Formulation

We aim to obtain a clustering algorithm which takes into account the prior information regarding the (temporal or spatial) adjacencies of the observed data in the locations space \(\mathcal{X}\), promoting clustering of data adjacent in the space \(\mathcal{X}\), and discouraging clustering of data points relatively near in the feature space \(\mathcal{Y}\) but far in the locations space \(\mathcal{X}\). For this purpose, we seek to provide a location-dependent nonparametric prior for clustering the observed data \(y\).

Motivated by the definition and the properties of the Pitman-Yor process discussed in the previous section, to effect these goals, in this work we introduce a random probability measure \(G(x)\) under which, given the first \(M - 1\) samples \(\Theta^*_{m1}, \ldots, \Theta^*_{mM-1}\) drawn from \(G\), a new sample \(\Theta^*_M\) associated with a measurement location \(x\) is distributed according to
\[
p(\Theta^*_M|x, k, \alpha, \bar{X}, G_0) = \frac{\alpha + \sum_{c=1}^{C} (1 - k(x, \bar{x}_c; \psi_c))}{\alpha + M - 1} G_0 + \frac{\sum_{c=1}^{C} f_c^{M-1} + k(x, \bar{x}_c; \psi_c) - 1}{\alpha + M - 1} \delta_{\theta_c} 
\] (18)

where \(f_c^{M-1}\) is the number of values in \(\Theta^*_{m1}, \ldots, \Theta^*_{mM-1}\) equal to \(\theta_c\), \(\Theta^*_{m1}, \ldots, \Theta^*_{mM-1}\) is the set of distinct values taken by the variables \(\Theta^*_{m1}, \ldots, \Theta^*_{mM-1}\), \(G_0\) is the employed base measure, \(\bar{x}_c\) is the location assigned to the \(c\)th cluster, \(\hat{x}_c\) is a bounded kernel function taking values in the interval \([0, 1]\), such that
\[
\lim_{x \to \hat{x}} k(x, \hat{x}; \psi) = 1 
\] (19)
\[
\lim_{\text{dist}(x, \hat{x}) \to \infty} k(x, \hat{x}; \psi) = 0 
\] (20)
\(\alpha\) is the innovation parameter of the process, conditioned to satisfy \(\alpha > 0\), and \(\text{dist}(\cdot, \cdot)\) is the distance metric used by the employed kernel function. We dub this random probability measure \(G(x)\) the kernel Pitman-Yor process, and we denote
\[
\Theta^*_m|x; G \sim G(x), \quad m = 1, \ldots, M 
\] (21)

with
\[
G(x)|k, \alpha, \bar{X}, G_0 \sim \text{KPYP}(x; k, \alpha, \bar{X}, G_0) 
\] (22)

The stick-breaking construction of the KPYP \(G(x)\) follows directly from the above definition (18), and the relevant discussions of section 2. Considering a KPYP \(G\) with cluster locations set \(\bar{X} = \{\hat{x}_c\}_{c=1}^{\infty}\), kernel function \(k(\cdot, \cdot)\) satisfying the constraints (19) and (20), and innovation parameter \(\alpha\), we have
\[
G(x) = \sum_{c=1}^{\infty} \varpi_c(v(x)) \delta_{\theta_c} 
\] (23)

where
\[
v_c(x) = \text{Beta}(k(x, \bar{x}_c; \psi_c), \alpha + [1 - k(x, \hat{x}_c; \psi_c)]) 
\] (24)
and
\[
\varpi_c(v(x)) = v_c(x) \prod_{j=1}^{c-1} (1 - v_j(x)) \in [0, 1] 
\] (25)

**Proposition 1.** The stochastic process \(G(x)\) defined in (23)-(25) is a valid random probability measure.

**Proof.** We need to show that
\[
\sum_{c=1}^{\infty} \varpi_c(v(x)) = 1 
\] (26)

For this purpose, we follow an approach similar to [9]. From (25), we have
\[
1 - \sum_{c=1}^{C} \varpi_c(v(x)) = \prod_{c=1}^{C-1} [1 - v_c(x)] 
\] (27)

Then, in the limit as \(C \to \infty\), and taking logs in both sides of (27), we have
\[
\sum_{c=1}^{\infty} \varpi_c(v(x)) = 1 \text{ if and only if } \sum_{c=1}^{\infty} \log [1 - v_c(x)] = -\infty 
\] (28)

Based on Kolmogorov three series theorem, the summation on the right is over independent random variables and is equal to \(-\infty\) if and only if \(\sum_{c=1}^{\infty} \log [1 - v_c(x)] \to -\infty\). However, \(v_c(x)\) follows a Beta distribution, which means \(v_c(x) \in [0, 1]\), thus \(\log [1 - v_c(x)] \leq 0\), and hence its expectation is negative; thus, the condition is satisfied, and (26) holds true.
3.2 Relation to the KSBP

Indeed, the proposed KPYP shares some common ideas with the KSBP of [9]. The KSBP considers that

$$G(x) = \sum_{c=1}^{\infty} \varpi_c(v(x)) \delta_{\theta_c}$$  \hspace{1cm} (29)

where

$$\varpi_c(v(x)) = v_c(x) \prod_{j=1}^{c-1} (1 - v_j(x)) \in [0, 1]$$  \hspace{1cm} (30)

$$v_c(x) = V_c k(x, \hat{x}_c; \psi_c)$$  \hspace{1cm} (31)

$$p(V_c) = \text{Beta}(V_c | 1, \alpha)$$  \hspace{1cm} (32)

From this definition, we observe that there is a key difference between the KPYP and the KSBP: the KPYP multiplies stick variables sharing the same Beta prior, with the prior of each stick variable employing a different “discount hyperparameter,” defined as a bounded kernel centered at a location \(\hat{x}\) unique for each stick. This way, the KPYP controls the assignment of observations to clusters by discounting clusters the centers of which are too far from the clustered data points in the locations space \(\mathcal{X}\).

It is interesting to compute the mean and variance of the stick variables \(v_c(x)\) for these two stochastic processes, for a given observation location \(x\) and cluster center \(\hat{x}_c\). In the case of the KPYP, we have

$$\mathbb{E}[v_c(x)] = \frac{k(x, \hat{x}_c; \psi_c)}{k(x, \hat{x}_c; \psi_c) + \alpha_c}$$  \hspace{1cm} (33)

$$\mathbb{V}[v_c(x)] = \frac{k(x, \hat{x}_c; \psi_c) \alpha_c}{(k(x, \hat{x}_c; \psi_c) + \alpha_c)^2 (k(x, \hat{x}_c; \psi_c) + \alpha_c + 1)}$$  \hspace{1cm} (34)

where

$$\alpha_c \triangleq \alpha + c (1 - k(x, \hat{x}_c; \psi_c))$$  \hspace{1cm} (35)

On the contrary, for the KSBP we have

$$\mathbb{E}[v_c(x)] = \frac{k(x, \hat{x}_c; \psi_c)}{1 + \alpha}$$  \hspace{1cm} (36)

$$\mathbb{V}[v_c(x)] = \frac{k(x, \hat{x}_c; \psi_c)^2 \alpha}{(1 + \alpha)^2 (\alpha + 2)}$$  \hspace{1cm} (37)

From (33) and (36), we observe that the for a given observation location \(x\) and cluster center \(\hat{x}_c\), same increase in the value of the kernel function \(k(x, \hat{x}_c; \psi_c)\) induces a much greater increase in the expected value of the stick variable \(v_c(x)\) employed by the KPYP compared to the increase in the expectation of the stick variable \(v_c(x)\) employed by the KSBP. Hence, the predictor (location)-dependent prior probabilities of cluster assignment of the KPYP appear to vary more steeply with the employed kernel function values compared to the KSBP.

3.3 Variational Bayesian Inference

Inference for nonparametric models can be conducted under a Bayesian setting, typically by means of variational Bayes (e.g., [15]), or Monte Carlo techniques (e.g., [16]). Here, we prefer a variational Bayesian approach, due to its better computational costs. For this purpose, we additionally impose a Gamma prior over the innovation parameter \(\alpha\), with

$$p(\alpha) = \mathcal{G}(\alpha | \eta_1, \eta_2).$$  \hspace{1cm} (38)

Let us consider a set of observations \(Y = \{y_n\}_{n=1}^N\) with corresponding locations \(X = \{x_n\}_{n=1}^N\). We postulate for our observed data a likelihood function of the form

$$p(y_n | z_n = c) = p(y_n | \theta_c)$$  \hspace{1cm} (39)

where the hidden variables \(z_n\) are defined such that \(z_n = c\) if the \(n\)th data point is considered to be derived from the \(c\)th cluster. We impose a multinomial prior over the hidden variables \(z_n\), with

$$p(z_n = c | x_n) = \varpi_c(v(x_n))$$  \hspace{1cm} (40)

where the \(\varpi_c(v(x))\) are given by (25), with the prior over the \(v_c(x)\) given by (24). We also impose a suitable conjugate exponential prior over the likelihood parameters \(\theta_c\).

Our variational Bayesian inference formalism consists in derivation of a family of variational posterior distributions \(q(.)\) which approximate the true posterior distribution over \(\{z_n\}_{n=1}^N, \{v(x_n)\}_{n=1}^N, \{\theta_c\}_{c=1}^C\), and the innovation parameter \(\alpha\). Apparently, under this infinite dimensional setting, Bayesian inference is not tractable. For this reason, we fix a value \(C\) and we let the variational posterior over the \(v_c(x)\) have the property \(q(v_c(x) = 1) = 1, \forall x \in \mathcal{X}\), i.e. we set \(\varpi_c(v(x))\) equal to zero for \(c > C\), \(\forall x \in \mathcal{X}\).

Let \(W = \{\alpha, \{z_n\}_{n=1}^N, \{(v_c(x_n))_{c=1}^C\}_{n=1}^N, \{\theta_c\}_{c=1}^C\}\) be the set of the parameters of our truncated model over which a prior distribution has been imposed, and \(\Xi\) be the set of the hyperparameters of the model, comprising the \(\{\psi_c\}_{c=1}^C\) and the hyperparameters of the priors imposed over the innovation parameter \(\alpha\) and the likelihood parameters \(\theta_c\) of the model. Variational Bayesian inference consists in derivation of an approximate posterior \(q(W)\) by maximization (in an iterative fashion) of the variational free energy

$$\mathcal{L}(q) = \int dW q(W) \log \frac{p(X, Y, W | \Xi)}{q(W)}$$  \hspace{1cm} (41)

Having considered a conjugate exponential prior configuration, the variational posterior \(q(W)\) is expected to take the same functional form as the prior, \(p(W)\) [17].
The variational free energy of our model reads

\[ \mathcal{L}(q) = \int dq(\alpha) \left\{ \log \frac{p(\alpha|\eta_1, \eta_2)}{q(\alpha)} \right. \]

\[ + \sum_{c=1}^{C-1} \sum_{n=1}^{N} \left[ \int dv_c(x_n) q(v_c(x_n)) \log \frac{p(v_c(x_n)|\alpha)}{q(v_c(x_n))} \right] \]

\[ + \sum_{c=1}^{C} \int d\theta_c q(\theta_c) \log \frac{p(\theta_c)}{q(\theta_c)} + \sum_{c=1}^{C} \sum_{n=1}^{N} q(z_n = c) \]

\[ \times \left\{ \int dv(x_n) q(v(x_n)) \log p(z_n = c|x_n) \right. \]

\[ \left. - \log p(z_n = c) + \int d\theta_c q(\theta_c) \log p(y_n|\theta_c) \right\} \]

(42)

### 3.4 Variational Posteriors

Let us denote as \( \langle \cdot \rangle \) the posterior expectation of a quantity. We have

\[ q(v_c(x_n)) = \text{Beta}(v_c(x_n)|\hat{\beta}_{c,n}, \hat{\beta}_{c,n}) \]

(43)

where

\[ \hat{\beta}_{c,n} = k(x_n, \hat{x}_c; \psi_c) + \sum_{m:z_m = c} q(z_m = c) \]

(44)

\[ \hat{\beta}_{c,n} = \langle \alpha \rangle + c \left[ 1 - k(x_n, \hat{x}_c; \psi_c) \right] \]

(45)

and

\[ q(\alpha) = \Gamma(\alpha|\hat{\eta}_1, \hat{\eta}_2) \]

(46)

where

\[ \hat{\eta}_1 = \eta_1 + N(C - 1) \]

(47)

\[ \hat{\eta}_2 = \eta_2 - \sum_{c=1}^{C-1} \sum_{n=1}^{N} \left[ \psi(\hat{\beta}_{c,n}) - \psi(\hat{\beta}_{c,n} + \hat{\beta}_{c,n}) \right] \]

(48)

\( \psi(.) \) denotes the Digamma function, and

\[ \langle \alpha \rangle = \frac{\hat{\eta}_1}{\hat{\eta}_2} \]

(49)

Further, the cluster assignment variables yield

\[ q(z_{nc} = 1) \propto \exp \left\{ \langle \log \varphi_c(v(x_n)) \rangle \right\} \exp \langle \varphi_{nc} \rangle \]

(50)

where

\[ \langle \log \varphi_c(v(x_n)) \rangle = \sum_{c' = 1}^{c-1} \left( \langle \log (1 - v_{c'}(x_n)) \rangle + \langle \log v_c(x_n) \rangle \right) \]

(51)

\[ \varphi_{nc} = \langle \log p(y_n|\theta_c) \rangle q(\theta_c) \]

(52)

and

\[ \langle \log v_c(x_n) \rangle = \psi(\hat{\beta}_{c,n}) - \psi(\hat{\beta}_{c,n} + \hat{\beta}_{c,n}) \]

(53)

\[ \langle \log (1 - v_c(x_n)) \rangle = \psi(\hat{\beta}_{c,n} - \psi(\hat{\beta}_{c,n} + \hat{\beta}_{c,n}) \]

(54)

Regarding the parameters \( \theta_c \), we obtain

\[ \log q(\theta_c) \propto \log p(\theta_c) + \sum_{n=1}^{N} q(z_n = c) \log p(y_n|\theta_c) \]

(55)

Finally, regarding the model hyperparameters \( \Xi \), we obtain the hyperparameters of the employed kernel functions \( \psi_c \) by maximization of the lower bound \( \mathcal{L}(q) \), and we heuristically select the values of the rest.

### 3.5 Learning the cluster locations \( \hat{x}_c \)

Regarding determination of the locations assigned to the obtained clusters, \( \hat{x}_c \), these can be obtained by either random selection or maximization of the variational free energy \( \mathcal{L}(q) \) over them. The latter procedure can be conducted by means of any appropriate iterative maximization algorithm; here, we employ the popular L-BFGS algorithm \(^{[13]} \) for this purpose. Both random \( \hat{x}_c \) selection and estimation by means of variational free energy optimization, using the L-BFGS algorithm, shall be evaluated in the experimental section of our paper.

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