Quantisation of gauged $SL(2, \mathbb{R})$ WZNW theories

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Canonical quantisation of the free-field zero modes $q, p$ on a half-line $p > 0$ provides for WZNW coset theories self-adjoint vertex operators on account of hidden symmetries generated by an $S$-matrix.

1 Introduction

The cosets of the $SL(2, \mathbb{R})$ WZNW model, the Liouville theory as well as the $SL(2, \mathbb{R})/\mathbb{R}_+$ and $SL(2, \mathbb{R})/U(1)$ black hole models, are classically completely described by gauge invariant Hamiltonian reduction. This is reviewed in [1]. A canonical quantisation can be performed in the same way as it has been done for the Liouville theory in [2, 3]. Here one uses the general solution of the coset as a canonical transformation between the non-linear coset fields and free fields, replaces the Poisson brackets of the canonical free fields by commutators, normal orders non-linear expressions in the free fields, and avoids ensuing anomalies by deforming the composite operators of the coset theories [2–4].

Our aim is to construct vertex operators for the calculation of coset correlation functions. However the coset dynamics restricts the free-field zero modes $q, p$ to the half-line $p > 0$ where the coordinate operator $\hat{q} = i\hbar \partial_p$ is not self-adjoint. We can expect that the vertex operators which contain $e^{\pm \gamma \hat{q}}$, nevertheless, act self-adjointly on $L^2(\mathbb{R}_+)$ due to symmetries provided by $S$-matrix transformations of the respective coset theory.

Quantum Liouville theory is the simplest example, and it proves to be fundamental for an understanding of the quantisation of all the other WZNW cosets. It is therefore appropriate to restrict this short contribution exclusively to the Liouville theory. Since its zero mode structure is basically Liouville particle dynamics, we shall give first a full description of the particle model. Guided by this structure we then describe the problem for the deformed field theoretical situation [5], which refers to the Liouville oscillator vacuum only. It might be worth mentioning here that the Liouville $S$-matrix corresponds to a particular Möbius transformation, which therefore leaves the Liouville field invariant. The various cosets differ even in this respect by technical peculiarities only.

2 Liouville particle dynamics

The Liouville vacuum configurations are described by a homogeneous field $\partial_\sigma \phi(\tau, \sigma) = 0$ which becomes a time dependent coordinate $\phi(\tau, \sigma) = Q(\tau)$. The Liouville equation

$$\phi_{\tau\tau}(\tau, \sigma) - \phi_{\sigma\sigma}(\tau, \sigma) + \frac{4m^2}{\gamma} e^{2\gamma \phi(\tau, \sigma)} = 0$$

(1)

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reduces then to the mechanical model
\[ \ddot{Q}(\tau) + \frac{4m^2}{\gamma} e^{2\gamma Q(\tau)} = 0. \] (2)

Its Hamiltonian
\[ H = \frac{1}{4\pi} \left( P^2 + 4\omega^2 e^{2\gamma Q} \right) \] (3)
is given by the canonically conjugate variables \( Q, P \) and the parameter \( \omega = 2\pi m / \gamma \). The particle vertex function \( V(\tau) = e^{-\gamma Q(\tau)} \) satisfies the linear equation
\[ \ddot{V}(\tau) = \gamma \frac{2}{\pi} H V(\tau), \] (4)
which can be integrated easily, and it provides the general solution of (2)
\[ e^{-\gamma Q(\tau)} = e^{-\gamma (q + p\tau / 2\pi)} + \frac{\omega^2}{p^2} e^{\gamma (q + p\tau / 2\pi)}, \] (5)
with the (zero mode) integration constants \( q \) and
\[ p = \sqrt{4\pi H} > 0. \] (6)

The particle phase space coordinates \( P, Q \) at \( \tau = 0 \) and \( p, q \) are related by a canonical and invertible map of the plane \( (P, Q) \) onto the half-plane \( (p, q) \)
\[ e^{-\gamma Q} = e^{-\gamma q} + \frac{\omega^2}{p^2} e^{\gamma q}, \quad P = -p \tanh \left( \gamma q + \log \frac{\omega}{p} \right). \] (7)

Due to this half-plane situation and the non-linear character of this map a self-adjoint quantum realisation of the vertex (5) becomes non-trivial.

The mechanical Liouville model is asymptotically a free theory and (5) gives
\[ \lim_{\tau \to -\infty} [Q(\tau) - q - \frac{p\tau}{2\pi}] = 0, \quad \lim_{\tau \to -\infty} [P(\tau) - p] = 0. \] (8)
So \( p, q \) can be interpreted as the in-variables of Liouville particle dynamics, and the half-plane condition \( p > 0 \) is indeed consistent with the positive in-momentum.

Similarly out-variables can be defined by (5) as \( \tau \to +\infty \). The in- and out-variables are related by the transformation
\[ P_{\text{out}} = -p, \quad Q_{\text{out}} = -q + \frac{2}{\gamma} \log \frac{p}{\omega}, \] (9)
which combines the reflection of \( p, q \) with a simple canonical map. It is an important observation that the general solution (5) is invariant under the transformation (9). Quantum mechanically this symmetry will be generated by the \( S \)-matrix of the particle theory. In Liouville field theory in- and out-fields [6] are related by the special Möbius transformation \( A \to -(m^2 A)^{-1}, \bar{A} \to -(m^2 \bar{A})^{-1} \), which keeps invariant the general solution
\[ \phi(\tau, \sigma) = \log \frac{A'(\tau + \sigma)\bar{A}'(\tau - \sigma)}{[1 + m^2 A(\tau + \sigma)\bar{A}(\tau - \sigma)]^2}. \] (10)
3 Vertex operator

To obtain the \( \hat{p}, \hat{q} \) operator structure of a vertex operator, we quantise first the mechanical Liouville model in the \( Q \)-representation. It will be convenient to use the notation

\[
x = \gamma Q - \log \frac{\alpha}{m} \quad \text{and} \quad \alpha = \frac{\hbar \gamma^2}{4\pi},
\]

in which the operator for the Hamiltonian (3) becomes

\[
\hat{H} = \hbar \alpha \left( -\partial_x^2 + e^{2x} \right).
\]

The eigenstates of this Hamiltonian are solutions of the Schrödinger equation

\[
-\Psi_k''(x) + e^{2x} \Psi_k(x) = k^2 \Psi_k(x).
\]

They are given by Kelvin functions \( K_{ik}(e^x) \) which are real, and \( K_{ik}(e^x) = K_{ik}(e^x) \) [7]. (To simplify formulas we use here \( k \) instead of \( p = \hbar \gamma k \).) The spectrum \( E = \hbar \alpha k^2 \) is non-degenerate and it does not contain the point \( k = 0 \). We take therefore the eigenstates \( \Psi_k(x) \) with \( k > 0 \) only, which reflects the half-plane situation quantum mechanically.

Choosing the normalisation

\[
\Psi_k(x) = d_k K_{ik}(e^x), \quad \text{with} \quad d_k = \sqrt{\frac{2}{\pi}} \frac{2^{ik}}{\Gamma(-ik)},
\]

the eigenstates will have the asymptotic behaviour

\[
\Psi_k(x) \to \frac{e^{ikx}}{\sqrt{2\pi}} + S_k \frac{e^{-ikx}}{\sqrt{2\pi}}, \quad \text{as} \quad x \to -\infty,
\]

with the reflection amplitude [8]

\[
S_k = 2^{2ik} \frac{\Gamma(ik)}{\Gamma(-ik)}.
\]

We are now able to calculate matrix elements of an arbitrary vertex operator \( e^{2b \gamma \hat{Q}(\tau)} \) between the eigenstates of (12) (Please, note that we also use the notation \( V \) for \( V_{-1} \) !)

\[
V_{2b}(k, k'; \tau) = \langle \Psi_k | e^{2b \gamma \hat{Q}(\tau)} | \Psi_{k'} \rangle.
\]

The result

\[
V_{2b}(k, k'; \tau) = \left( \frac{\alpha}{m} \right)^{2b} d_k^* d_{k'} e^{i\alpha(k^2 - k'^2)\tau} F_{2b}(k, k'),
\]

can be expressed by the integral

\[
F_{2b}(k, k') = \int_{-\infty}^{\infty} dx K_{ik}(e^x) e^{2bx} K_{ik'}(e^x).
\]

But this integral is well defined for \( b > 0 \) only, and it diverges for \( b \leq 0 \). Using the notation

\[
\kappa = \frac{k + k'}{2}, \quad \rho = \frac{k - k'}{2},
\]

\[

\]
and integration for $b > 0$ yields [8]

$$V_{2b} = \left( \frac{\alpha}{m} \right)^{2b} e^{4\alpha \rho \kappa \tau} 4^{b-1} \Gamma(\rho + i\kappa) \Gamma(\rho - i\kappa) \Gamma(b) \Gamma(b + i\kappa) \Gamma(b - i\kappa) \frac{1}{4\pi \Gamma(2\beta)}. \quad (21)$$

In order to get the vertex function for negative $b$ one needs a smooth continuation of (21) as a generalised function. Since $\kappa > |\rho|$ ambiguities can arise at $b = -n, n = 0, 1, 2, \ldots$ Near these points if $b = -n + \epsilon$ (21) behaves like

$$\frac{1}{\pi} \frac{\epsilon}{(k + k')^2 + \epsilon^2}. \quad (22)$$

One can show that this generalised function has for holomorphic test functions the following smooth continuation from positive to negative values of $\epsilon$ [5]

$$\frac{1}{\pi} \frac{\epsilon}{(k + k')^2 + \epsilon^2} + \delta(k - k' + i\epsilon) + \delta(k - k' - i\epsilon), \quad (23)$$

where the $\delta$-function with complex arguments is defined in the standard manner

$$\int_0^{+\infty} dk' \delta(k - k' \pm i\epsilon) \psi(k') = \psi(k \pm i\epsilon). \quad (24)$$

The continuation of (21) to negative values of $b$ so creates a pair of $\delta$-functions with complex arguments each time $b$ passes a negative integer value, and for $b = -|b|$ results

$$V_{-2|b|} = V_{-2|b|}^+(\kappa, \rho; \tau) \quad \text{(25)}$$

$$+ \left( \frac{\alpha}{m} \right)^{2|b|} \sum_{l=0}^{[|b|]} C_{2|b|}^l e^{-4\alpha(|b| - |l|)\kappa \tau} \frac{\Gamma(|b| + i\kappa) \Gamma(-|b| - i\kappa)}{4\Gamma(|b| + l + i\kappa) \Gamma(-|b| + l - i\kappa)} \delta[2\rho - 2i(|b| - l)]$$

$$+ C_{2|b|}^l e^{4\alpha(|b| - |l|)\kappa \tau} \frac{\Gamma(-|b| + i\kappa) \Gamma(-|b| - i\kappa)}{4\Gamma(|b| - l + i\kappa) \Gamma(|b| - l - i\kappa)} \delta[2\rho + 2i(|b| - l)]. \quad (26)$$

Here $V_{-2|b|}^+(\kappa, \rho; \tau)$ is defined by the r.h.s of (21) where $b$ is replaced by $-|b|, [|b|]$ is the integer part of $|b|,$ and

$$C_{2|b|}^l = \prod_{j=0}^{l-1} \frac{2|b| - j}{j + 1}. \quad (27)$$

In particular, the function $V_{-2|b|}^+$ vanishes at half-integer $|b|$ due to the pole of $\Gamma(-2|b|),$ and it creates at integer $|b|$ the terms proportional to $\delta(\rho),$ so that for $2|b| = n$ one has

$$V_{-n}(\kappa, \rho; \tau) = \left( \frac{\alpha}{m} \right)^n \sum_{l=0}^n C_n^l e^{-2(n - 2l)\kappa \tau} \prod_{j=0}^{l-1} \frac{1}{4\kappa^2 + (n - 2j)^2} \delta[2\rho - i(n - 2l)], \quad (28)$$

where $C_n^l$ are now binomial coefficients.

Our aim is to get the vertex operator in terms of the zero mode operators $\hat{p}, \hat{q}.$ So we have to establish a connection between the $Q$-representation and the standard $p$-representation on the half-plane. Wave functions of the $p$-representation are $\psi(p) \in L^2([\mathbb{R}_+])$ and the operators $e^{\pm \gamma \hat{q}}$ can be defined as

$$e^{\pm \gamma \hat{q}} = e^{\pm (1/\gamma \partial_\rho + \lambda)}, \quad (29)$$
with some constant $\lambda$. Due to (6) the eigenstates of (12) $|\Psi_k\rangle$ can also be identified with the eigenstates of the momentum operator $\hat{p} |\Psi_k\rangle = \hbar \gamma k |\Psi_k\rangle$. This identification connects the wave functions $\psi(p)$ with wave functions of the $Q$-representation $\Psi(x)$ by

$$\psi(p) = \int_{-\infty}^{+\infty} dx \Psi^*_k(x) \Psi(x), \quad \text{for} \quad p = \hbar \gamma k.$$  

(29)

Since the eigenstates $|\Psi_k\rangle$ are complete this transformation is unitary.

Let us now consider as the simplest non-trivial example the vertex operator $\hat{V}(\tau)$. Its matrix elements correspond to the case $n = 1$ of (27)

$$V(k, k'; \tau) = \frac{m}{\alpha} \left( e^{-\alpha(k+k')\tau} \delta(k - k' - i) + \frac{e^{\alpha(k+k')\tau}}{4k k'} \delta(k - k' + i) \right).$$  

(30)

From this expression one can easily read off the zero mode operator structure

$$\hat{V}(\tau) = e^{-\gamma \hat{p} / 4\pi \tau} e^{-\gamma \hat{\bar{p}} / 4\pi \tau} + \omega^2 \frac{e^{\gamma \hat{p} / 4\pi \tau}}{\hat{p}} e^{\gamma \hat{\bar{p}} / 4\pi \tau}.$$  

(31)

This fixes the parameter $\lambda$ of (28) as $\lambda = \log(\frac{\alpha}{\hbar})$. One can show that the vertex operator $\hat{V}_{-n}$ is the $n$-th power of (31), but for positive or non-integer negative $2\hbar \hat{V}_{2n}$ is given by an infinite series of $q$-exponentials, much as in quantum Liouville field theory [2].

It remains to consider the self-adjoint action of the vertex operator (31) on $L^2(\mathbb{R}_+)$. Since asymptotically for $\tau \to \pm \infty$ only one term survives one is tempted to demand hermiticity for each term of (31) separately. But a proof of hermiticity for $e^{\gamma \hat{p}/p}$ would require very special boundary conditions on holomorphic functions $\psi(p)$ at $\text{Re} \, p = 0$, a mathematically by itself interesting but still unsolved problem [5].

However, the vertex operator (31) can be shown to become self-adjoint as a whole \footnote{We thank an anonymous referee of [5] for a simple realisation of this idea. In fact this idea was intensively discussed in [9], but the results were inconclusive.} on account of its symmetry under transformations given by the $S$-matrix

$$S = \hat{P} S(p).$$  

(32)

Here $S(p)$ is the multiplicative reflection amplitude (16) with $p = \hbar \gamma k$ and $\hat{P}$ the parity operator $\hat{P} \psi(p) = \psi(-p)$. It is easy to see that (32) replaces the in-coming first term of (31) by the out-going second one, and vice versa, so that the particle vertex operator remains invariant. That means $\hat{V}_{\text{out}}(\tau) = \hat{S} \hat{V}(\tau) \hat{S}^{-1}$ is identical to (31). This also holds in general as one can see from (21) and (25). The $\hat{S}$-matrix is just the quantum version of the symmetry transformation (9). $\hat{S}$ also maps the Hilbert space of the in-fields $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R}_-)$ for the out-fields, which for the wave functions $\psi(p) \in L^2(\mathbb{R}_+), \tilde{\psi}(p) \in L^2(\mathbb{R}_-)$ is given by $\hat{S}(\psi) = S(p) \tilde{\psi}(p)$. The last relation is defined by (29) using (14) and $\Psi^*_L(x) = d^*_L K_{-ik} = S_k \Psi^*_R(x)$.

Due to these properties one can extend the definition of the matrix element of the vertex operator from $L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$

$$\int_{0}^{\infty} dp \psi^*_L(p) \tilde{\psi}^*_R(p) \hat{V}(\tau) \psi_R(p) = \frac{1}{2} \int_{0}^{\infty} dp \psi^*_L(p) \hat{V}(\tau) \psi_R(p).$$  

(33)

Here $\psi(p) = \psi(p)$ for $p > 0$, $\tilde{\psi}(p) = \hat{S}(P) \psi(-p)$ for $p < 0$, and $\psi(p) \in L^2(\mathbb{R})$ satisfies $\hat{S} \psi(p) = \psi(p)$. Self-adjointness of the operator (31) obviously holds on $L^2(\mathbb{R})$ for holomorphic wave functions $\Psi(p)$. As a consequence, self-adjointness of $\hat{V}(\tau)$ on $L^2(\mathbb{R}_+)$ requires for $\psi(p)$ a holomorphic extension to the negative half-line so that $\psi(-p) = \psi(-p) = \hat{S}(p) \psi(p)$. Such functions are given by $\psi(p) = d^*_L f(p)$, where $d(p)$ is defined by (14) with $p = \hbar \gamma k$ and $f(p)$ is an even holomorphic function $f(-p) = f(p)$. 


4 Reduced Liouville field theory

The Liouville vertex operator which corresponds to (31) results as an oscillator vacuum matrix element \[2,5\]
\[
\hat{V}(\tau) = e^{-\left(\gamma \hat{p}/4\pi\right)\tau} e^{-\gamma \hat{q}} e^{-\left(\gamma \hat{p}/4\pi\right)\tau} + \omega_\alpha \frac{e^{\left(\gamma \hat{p}/4\pi\right)\tau}}{\hat{p}} \frac{\Gamma \left(-i \frac{\gamma p}{2\pi}\right)}{\Gamma \left(i \frac{\gamma p}{2\pi}\right)} \Gamma \left(-i \frac{\gamma p}{2\pi}\right) \Gamma \left(i \frac{\gamma p}{2\pi}\right) \frac{e^{\gamma \hat{q}}}{\hat{p}}. \tag{34}
\]

where
\[
\omega_\alpha = \frac{2\pi m_\alpha \Gamma(1+2\alpha)}{\gamma} \quad \text{and} \quad m_\alpha^2 = \frac{\sin 2\pi \alpha}{2\pi \alpha} m^2. \tag{35}
\]

Both, the parameter \(\omega_\alpha\) and the \(\Gamma\)-functions describe classical-valued deformations which are due to oscillator as well as zero mode contributions.

Asking the question whether the operator (34) is symmetric under a \(S\)-matrix transformation \(\hat{S}_L = \hat{P} S_L(p)\) which exchanges the in-coming with the deformed out-going zero mode part, we derived in [5] as an answer the deformed reflection amplitude
\[
S_L(p) = -\left(m_\alpha \Gamma(2\alpha)\right)^{-2ip/h\gamma} \frac{\Gamma \left(i \frac{\gamma p}{2\pi \alpha}\right)}{\Gamma \left(-i \frac{\gamma p}{2\pi \alpha}\right)} \frac{\Gamma \left(i \frac{\gamma p}{2\pi}\right)}{\Gamma \left(-i \frac{\gamma p}{2\pi}\right)}. \tag{36}
\]

This reflection amplitude is surprisingly identical to that derived in [10] by a symmetry of a 3-point correlation function suggested in [11] on account of path-integral considerations. This result motivates one to look for a deeper understanding of the path-integral conjectures by means of the operator approach, as investigated in [9].

For a proof of self-adjointness knowledge of the full \(S\)-matrix for Liouville field theory is required.

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