Anomalies in N=2 supersymmetric non-linear $\sigma$ models on compact Kähler Ricci-flat target spaces

Guy Bonneau*

May 8, 2018

Abstract

We analyse with the algebraic, regularisation independent, cohomological B.R.S. methods, the renormalisability of torsionless N=2 supersymmetric non-linear $\sigma$ models built on Kähler spaces. Surprisingly enough with respect to the common wisdom, we obtain an anomaly candidate, at least in the compact Ricci-flat case. In the compact homogeneous Kähler case, as expected, the anomaly candidate disappears.
1 Introduction

Supersymmetric non-linear $\sigma$ models in two space time dimensions have been considered for many years to describe the vacuum state of superstrings [1][2]. In particular Calabi-Yau spaces, i.e. 6 dimensional compact Kähler Ricci-flat manifolds [3], appear as good candidates in the compactification of the 10 dimensional superstring to 4 dimensional flat Minkowski space; the conformal invariance of the 2.d, N = 2 supersymmetric non-linear $\sigma$ model (the fields of which are coordinates on this compact manifold) is expected to hold to all orders of perturbation theory [4].

However explicit calculations to 4 or 5 loops [5] and, afterwards, general arguments [6] show that the $\beta$ functions may not vanish. But, as argued in my recent review [7], at least two problems obscure these analyses: first, the fact that the quantum theory is not sufficiently constrained by the Kähler Ricci-flatness requirement; second, the use of “dimensional reduction” [8] or of harmonic superspace formalism [9] in actual explicit calculations and general arguments. Then, we prefer to analyse these models using the B.R.S., algebraic, regularisation free cohomological methods.

So we address ourselves to the question of the all-order renormalisability of supersymmetric (N = 2, 4) non linear $\sigma$ models in two space time dimensions, in the same spirit of what we did with the Genova group [11] for the bosonic case. Due to the non-linearity of the second supersymmetry transformation in a general field parametrisation (coordinate system on the manifold), we shall use a gradation in the number of fields and their derivatives. As in [11], the cohomology of the lowest order B.R.S. operator will give the essential information. Leaving to other publications the detailed analysis of the N=2 supersymmetry and the N=4 case [12], the present letter gives our main results for torsionless compact Kähler Ricci-flat manifolds (i.e. special N=2 supersymmetric models) and, surprisingly enough with respect to the common wisdom [1], shows that, at least for that case, there exists a possible anomaly for global supersymmetry in 2 space-time dimensions.

2 The classical theory and the Slavnov operator

As in this letter we shall be concerned in N = 2 supersymmetric non-linear $\sigma$ models in 2.d, we may use N = 1 superfields $\phi^i(x^+, x^-, \theta^+, \theta^-)$ and consequently, in the absence of torsion, the most general N = 1 invariant action is:

$$S_{\text{inv.}} = \int d^2x d^2\theta g_{ij} [\phi] D_+ \phi^i D_- \phi^j$$

1 The regularisation through dimensional reduction suffers from algebraic inconsistencies and the quantization in harmonic superspace does not rely on firm basis, due to the presence of non-local singularities (in the harmonic superspace) [10].

2 Notice also that recent works of Brandt [13] and Dixon [14] show the existence of new non-trivial cohomologies in supersymmetric theories.

3 The quantization with N = 1 superfields was put on firm basis by Piguet and Rouet [15] who proved in particular the Quantum Action Principle in that context. Moreover, in [12] a) we show the renormalisability of N=1 supersymmetric non-linear $\sigma$ models using component fields: this justifies the use of N=1 superfields for the present analysis of extended supersymmetry. Notice also that we use light-cone coordinates.
where the supersymmetric covariant derivatives

\[ D_\pm = \frac{\partial}{\partial \theta_\pm} + i\theta_\pm \frac{\partial}{\partial x_\pm} \]

satisfy

\[ \{D_+, D_-\} = 0 , \quad D^2_\pm = i \frac{\partial}{\partial x_\pm} \equiv i \partial_\pm . \]

The tensor \( g_{ij}[\phi] \) is interpreted as a metric tensor on a Riemannian manifold \( M \). As is now well known \([17]\), \( N = 2 \) supersymmetry needs \( M \) to be a \( 2n \) dimensional Kähler manifold, \( i.e. \) there should exist a covariantly constant complex structure \( J^i_{jk}[\phi] : \)

\[ J^i_{jk} J^k_j = -\delta^i_j ; \quad \nabla_k J^i_j = 0 ; \quad i, j, k = 1, 2, ..., 2n \]

and the metric has to be hermitian with respect to the complex structure :

\[ g_{kl} J^l_l + J^l_k g_{ln} = 0 . \]

The second supersymmetry transformation writes :

\[ \delta \phi^i = J^i_j[\phi](\epsilon^+ D_+ \phi^j + \epsilon^- D_- \phi^j) . \] (2)

In the B.R.S. approach, the supersymmetry parameters \( \epsilon^\pm \) are promoted to constant, commuting Faddeev-Popov parameters \( d^\pm \) and an anticommuting classical source \( \eta_i \) for the non-linear field transformation (2) is introduced in the classical action \([3]\) :

\[ \Gamma \text{class.} = \int d^2 x d^2 \theta \left\{ g_{ij}[\phi] D_+ \phi^i D_- \phi^j + \eta_i J^i_j[\phi] (d^+ D_+ \phi^j + d^- D_- \phi^j) \right\} . \] (3)

For simplicity, no mass term is added here as we are only interested in U.V. properties. The non linear Slavnov operator is defined by

\[ S \Gamma \equiv \int d^2 x d^2 \theta \frac{\delta \Gamma}{\delta \eta_i(x, \theta)} \frac{\delta \Gamma}{\delta \phi^i(x, \theta)} \]

and we find

\[ S \Gamma \text{class.} = (d^+)^2 \int d^2 x d^2 \theta \eta_i \partial_+ \phi^k + (d^-)^2 \int d^2 x d^2 \theta \eta_i \partial_- \phi^k \]

in accordance with the supersymmetry algebra.

As is by now well known (for example see \([7]\) or \([11]\)), in the absence of a consistent regularisation that respects all the symmetries of the theory, the quantum analysis directly depends on the cohomology of the nilpotent linearized Slavnov operator :

\[ S_L = \int d^2 x d^2 \theta \left[ \frac{\delta \Gamma \text{class.}}{\delta \eta_i(x, \theta)} \frac{\delta}{\delta \phi^i(x, \theta)} + \frac{\delta \Gamma \text{class.}}{\delta \phi^i(x, \theta)} \frac{\delta}{\delta \eta_i(x, \theta)} \right] \]

\[ S^2_L = 0 \] (4)

4 Here, contrarily to our previous work where the manifold \( M \) was supposed to be an homogeneous space \([11]\), we consider renormalisability “à la Friedan” \([4]\), \( i.e. \) in the space of metrics, and analyse only the possibility of maintaining to all orders the \( N = 2 \) supersymmetry. As explained in \([3]\), in order to define unambiguously the classical action, one should add extra properties.

5 In the absence of torsion, there is a parity invariance

\[ + \rightarrow -, d^2 x \rightarrow d^2 x, d^2 \theta \rightarrow -d^2 \theta, \phi^i \rightarrow \phi^i, \eta_i \rightarrow -\eta_i . \]

Moreover, the canonical dimensions of \( [d^2 x d^2 \theta], [\phi^i], [d^\pm], [D_\pm], [\eta_i] \) are \(-1, 0, -1/2, +1/2, +1\) respectively and the Faddeev-Popov assignments \(+1\) for \( d^\pm \), \(-1\) for \( \eta_i \), \( 0 \) for the other quantities.
in the Faddeev-Popov charge +1 sector [absence of anomalies for the N = 2 supersymmetry] and 0 sector [number of physical parameters and stability of the classical action through renormalization]. Notice that the Slavnov operator (4) is unchanged under the following field and source reparametrisations:

\[ \phi^i \rightarrow \phi^i + \lambda W^i[\phi] , \quad \eta_i \rightarrow \eta_i - \lambda \eta_k W^k_{,i}[\phi], \]

where \( W^i[\phi] \) is an arbitrary function of the fields \( \phi(x, \theta) \) and a comma indicates a derivative with respect to the field \( \phi^i \). Under this change, the classical action (3) is modified

\[ \Gamma_{\text{class.}} \rightarrow \Gamma_{\text{class.}} + \lambda S_L \int d^2x d^2\theta \eta_i W^i[\phi], \]

but the Slavnov identity is left unchanged as

\[ S[\Gamma_{\text{class.}} + \lambda S_L \Delta] \equiv S\Gamma_{\text{class.}} + \lambda S_L [S_L \Delta] = S\Gamma_{\text{class.}}. \]

3 B.R.S. cohomology of \( S_L \)

Due to the highly non-linear character of \( S_L \) (equ. (4)), it is convenient to use a “filtration” ([15], [11]) with respect to the number of fields \( \phi^i(x, \theta) \) and their derivatives. As it does not change this number, the nihilpotent lowest order part of \( S_L \), \( S^0_L \) will play a special role :

\[ S_L = S^0_L + S^1_L + S^2_L + ... \equiv S^0_L + S^r_L , \quad (S^0_L)^2 = (S^r_L)^2 = S^0_L S^r_L + S^r_L S^0_L = 0 \]

\[ S^0_L = \int d^2x d^2\theta J^i_j(0) \left\{ (d^+ D_+ \phi^i + d^- D_- \phi^i) \frac{\delta}{\delta \phi^i} + (d^+ D_+ \eta_i + d^- D_- \eta_i) \frac{\delta}{\delta \eta_i} \right\}. \]

As explained in refs. [11] and [12]a), when \( S^0_L \) has no cohomology in the Faddeev-Popov positively charged sectors, the cohomology of the complete \( S_L \) operator in the Faddeev-Popov sectors of charge 0 and +1 is isomorphic to a subspace of the one of \( S^0_L \) in the same sectors.

We now analyse the cohomology of \( S^0_L \).

3.1 \( S^0_L \) cohomology

Due to dimensions and Faddeev-Popov charge assignments, dimension zero integrated local polynomials in the Faddeev-Popov parameters, fields, sources and their derivatives have at least a Faddeev-Popov charge -1 :

\[ \Delta_{[-1]} = \int d^2x d^2\theta \eta_i V^i[\phi]. \]

Then there is no Faddeev-Popov charge -1 coboundaries, so the cohomology of \( S^0_L \) in that sector is given by the cocycle condition :

\[ S^0_L \Delta_{[-1]} = 0 \iff J^i_j(0) V^k_k = J^k_k(0) V^i_i \]

In particular, the cohomology of \( S^0_L \) in the Faddeev-Popov -1 sector restricts the dimension of the cohomology of \( S_L \) in the 0 charge sector when compared to the one of \( S^0_L \).
This condition, when expressed in a coordinate system adapted to the complex structure \( J_j^i[\phi] \) (\( J_j^\alpha = i\delta_j^\alpha, J_j^0 = -i\delta_j^\alpha, J_j^\beta = J_j^\beta = 0 \)), means that \( V^i[\phi] \) is a contravariant analytic vector: 
\[
V^\alpha = V^\alpha[\phi^\delta], \quad V^\dot{\alpha} = V^{\dot{\alpha}}[\phi^{\dot{\delta}}].
\]

Let us now turn to the Faddeev-Popov neutral charge sector:
\[
\Delta_{[0]} = \int d^2 x d^2 \theta \left\{ t_{ij}[\phi] D_+ \phi^i D_- \phi^j + \eta_i U_j^i[\phi](d^+ D_+ \phi^j + d^- D_- \phi^j) \right\}
\]

where \( t_{ij} \) is symmetric, due to parity invariance (footnote 5). Coboundaries being given by \( S^0_L \Delta_{[-1]}[\text{arbitrary } V^i(\phi)] \), the analysis of the cocycle condition \( S^0_L \Delta_{[0]} = 0 \) gives
\[
\Delta_{[0]} = \Delta^\alpha_{[0]} t_{ij}(\phi) + S^0_L \Delta_{[-1]}[V^i(\phi)]
\]

where the tensor \( t_{ij} \) which occurs in the anomalous part
\[
\Delta^\alpha_{[0]} t_{ij} = \int d^2 x d^2 \theta t_{ij}(\phi) D_+ \phi^i D_- \phi^j
\]
is constrained by:
\[
a) \quad J_j^i(0)t_{ik} + t_{ji}J_i^k(0) = 0, \\
b) \quad J_j^i(0)(t_{kt,i} - t_{it,k}) - (j \leftrightarrow k) = 0.
\]

The absence of source dependent non-trivial cohomology means that, up to a field redefinition (compare to equations (13b)), the complex structure \( J_j^i \) is left unchanged through radiative corrections. Moreover, using the same adapted coordinate system as above, condition (13a) means that the metric \( g_{ij} + \bar{h}t_{ij} \) remains hermitian, whereas (13b) expresses the covariant constancy of \( J_j^i \) with respect to the covariant derivative with a connexion corresponding to the metric \( g_{ij} + \bar{h}t_{ij} \). These are precisely the expected conditions.

Finally, let us consider the Faddeev-Popov charge +1 sector:
\[
\Delta_{[1]} = \int d^2 x d^2 \theta \left\{ t^{[ijkl]}(d^+)^2(d^-)^2 \eta_{ij} \eta_{jk} \right\} + d^+ \eta_{ij} t_{ij}^2 \right\}
\]

where, due to the anticommuting properties of \( \eta \) and \( D_\pm \phi^i \) and to the integration by parts freedom, the tensors \( t_{ijkl}^{[ij]}, t_{ij}^{[ik]}, t_{ij}^{[jn]}, t_{ij}^{[n]}, \bar{t}_{ij}^{[ij]} \) are antisymmetric in \( i, j, k \), and \( s_{ij}^{[ij]} \) symmetric in \( i, j \).
Using the same adapted coordinate system as above, condition (17 a) means that the tensor vanishes whereas (17 b) means that it is analytic (i.e., is constrained by:

\[ \Delta_{[+1]} = \Delta_{[+1]}^a[t^{[ijk]}(\phi)] + S_L^0 \Delta_{[0]}[t_{ij}(\phi), U^j(\phi)] \]

where the antisymmetric tensor \( t^{[ijk]}(\phi) \) which occurs in the anomalous part

\[ \Delta_{[+1]}^a = \int d^2x d^2\theta t^{[ijk]}(\phi) (d^+)^2 (d^-)^2 \eta_i \eta_j \eta_k \]

is constrained by:

a) \( J^i_n(0)t^{[njk]} \) is i, j, k antisymmetric,  
b) \( J^i_n(0)t^{[njk]} = J^i_n(0)t^{[ijk]} \)

(17)

Using the same adapted coordinate system as above, condition (17 a) means that the tensor \( t^{[ijk]} \) is a pure contravariant antisymmetric tensor (i.e. \( t^{[alpha beta gamma]} \neq 0 \), the other components vanish) whereas (17 b) means that it is analytic (i.e. \( t^{[alpha beta gamma]} = t^{[alpha beta gamma]}(\phi^delta) \), \( t^{[alpha beta gamma]} = t^{[alpha beta gamma]}(\phi^delta) \)). In particular, due to the vanishing of \( t^{[alpha beta gamma]} \), such tensor cannot be a candidate for a torsion tensor on a Kähler manifold [19].

Consider the covariant tensor

\[ t^{[alpha beta gamma]} = g^{alpha beta} g^{gamma} t^{[alpha beta gamma]} \]

It satisfies \( \nabla_{delta} t^{[alpha beta gamma]} = 0 \). Then the (3-0) form

\[ \omega' = \frac{1}{3!} t^{[alpha beta gamma]} d\phi^alpha \wedge d\phi^beta \wedge d\phi^gamma \]

which satisfies \( d' \omega' = 0 \), may be shown to be harmonic if \( M \) is a compact manifold \( (21), (22) \). It is known that the number of such forms is given by the Hodge number \( h^{3,0} \) : then this number determines an upper bound for the dimension of the cohomology space of \( S_L \) in the anomaly sector.

As a first result, this proves that if the manifold \( M \) has a complex dimension smaller than 3, there is no anomaly candidate. Another special case is the compact Kähler homogeneous one ( N=2 supersymmetric extension of our previous work on the bosonic case [14]) : in such a case the Ricci tensor is positive definite [20] which forbids \( (21), (22) \) the existence of such analytic tensor \( t^{[alpha beta gamma]}(\phi^delta) \). As a consequence, the cohomology of \( S_L^0 \) - and then of \( S_L \) vanishes in the anomaly sector (for details, see ref. [22] b)).

We are now in a position to discuss the cohomology of the complete \( S_L \equiv S_L^0 + S_L^+ \) operator.

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Footnote 7: In this Kählerian case, one firstly obtains from \( \nabla_i t^{[alpha beta gamma]} = 0 \), \( \Delta t^{[alpha beta gamma]} = g^{delta gamma} \nabla_i \nabla_i t^{[alpha beta gamma]} - [R^i_{alpha beta gamma} + \text{perms.}] \) ; on another hand, the Ricci identity gives \( g^{delta gamma} \nabla_i \nabla_i t^{[alpha beta gamma]} = -[R^i_{alpha beta gamma} + \text{perms.}] \). So \( \Delta t^{[alpha beta gamma]} = 2g^{delta gamma} \nabla_i \nabla_i t^{[alpha beta gamma]} \). Now, the manifold being compact, one may compute:

\[ (d\omega', d\omega') + (d^i \omega', d^i \omega') = (\omega', (d^i + d^delta) \omega') = (\omega', \Delta \omega') = \]

\[ = \int_M d^a 2g^{[alpha beta gamma]} [\nabla_i \nabla_i t^{[alpha beta gamma]}] = \int_M d^a 2g^{[alpha beta gamma]} [\nabla_i \nabla_i t^{[alpha beta gamma]}] \]

\[ \Rightarrow (d\omega', d\omega') + 3(d\omega', d\omega') = 0 \Rightarrow \delta \omega' = d\omega' = \Delta \omega' = 0 \quad Q.E.D. \]
3.2 $S_L$ cohomology

Thanks to the simplicity of $\Delta_{[-1]}$, the cohomology of the complete $S_L$ operator in the Faddeev-Popov charge -1 sector is easily obtained: the vector $V^i[\phi]$ should satisfy:

- $\int d^2x d^2\theta \frac{\delta S^{\text{inv.}}}{\delta \phi(x, \theta)} V^i[\phi(x, \theta)] = 0 \iff V^i[\phi]$ is a Killing vector for the metric $g_{ij}[\phi]$.

- $J^i_\phi \nabla V = \nabla J V^i[\phi] \iff V^i[\phi]$ is a contravariant vector, analytic with respect to $J^i_\phi[\phi]$.

Consider now the Faddeev-Popov neutral charge sector. We shall prove that, despite the non-vanishing $S_L^0$ cohomology in a Faddeev-Popov positively charged sector (16), the cohomology of $S_L$ is a subspace of the one of $S_L^0$, i.e. that one can always construct the cocycles for $S_L$ starting from those of $S_L^0$. Indeed, notice that $S_L^0 \Delta_{[0]}$ contains at most one source $\eta_i$; then it cannot intercept $\Delta_{[+1]}^{an}$, the cohomology of $S_L^0$ in the anomaly sector. As a consequence ([11], [12]a), there will be no obstruction in the construction of the cocycles of $S_L$ starting from those of $S_L^0$.

It may however happen that some of the so doing constructed cocycles for $S_L$ become coboundaries: this occurs when there is some cohomology for $S_L^0$ in the Faddeev-Popov charge -1 sector ([12]a), [22]). We have previously seen that this relies on the existence of Killing vectors for the metric $g_{ij}[\phi]$; this is natural as such vectors signal extra isometries that constrain the invariant action (1) or equivalently, signal the non physically relevant character of some of the

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*Let us sketch the proof. Under the filtration,

$$S_L \equiv \sum S_L^0 + S_L^1 + S_L^2 + \ldots$$

The cocycle condition $S_L \Delta_{[0]} = 0$ gives, using (14):

$$S_L^0 \Delta_{[0]}^1 = 0 \Rightarrow \Delta_{[0]}^1 = S_L^0 \Delta_{[-1]}^1.$$

At the next step,

$$S_L^0 \Delta_{[0]}^2 + S_L^1 \Delta_{[0]}^1 = 0 \Rightarrow S_L^0 [\Delta_{[0]}^2 - S_L^1 \Delta_{[-1]}^1] = 0 \Rightarrow \Delta_{[0]}^2 = \Delta_{[0]}^{an.2} + S_L^0 \Delta_{[-1]}^2 + S_L^1 \Delta_{[-1]}^1.$$

At the next step,

$$S_L^0 \Delta_{[0]}^3 + S_L^1 \Delta_{[0]}^2 + S_L^2 \Delta_{[0]}^1 = 0 \Rightarrow S_L^0 [\Delta_{[0]}^3 - S_L^1 \Delta_{[-1]}^2 - S_L^2 \Delta_{[-1]}^1] + S_L^2 \Delta_{[0]}^{an.2} = 0$$

where we have used $S_L^0 S_L^0 + S_L^1 S_L^1 + S_L^2 S_L^2 = 0$. The last equation implies, using (13)

$$S_L^0 (S_L^1 \Delta_{[0]}^{an.2}) = 0 \Rightarrow S_L^1 \Delta_{[0]}^{an.2} = \Delta_{[+1]}^{an.3} + S_L^0 \Delta_{[0]}^{an.3}$$

But, thanks to the upper remark, the anomalous term $\Delta_{[+1]}^{an.3}$ cannot appear here, and finally we get

$$S_L^0 [\Delta_{[0]}^3 + \Delta_{[0]}^{an.3} - S_L^1 \Delta_{[-1]}^2 - S_L^2 \Delta_{[-1]}^1] = 0$$

so that, using (13)

$$\Delta_{[0]}^3 = -\Delta_{[0]}^{an.3} + S_L^1 \Delta_{[-1]}^2 + S_L^2 \Delta_{[-1]}^1 + S_L^0 \Delta_{[-1]}^3 + \Delta_{[0]}^{an.3}.$$

Finally, up to that order,

$$\Delta_{[0]}^3 = (S_L \Delta_{[-1]}^3)_{[0]} + \Delta_{[0]}^{an.3} \text{ e.t.c.} \quad Q.E.D.
parameters of the classical action that may be reabsorbed through a conveniently chosen field and source reparametrisation \[11\]. Up to this restriction, the cohomology in the Faddeev-Popov neutral sector is then characterized by a symmetric tensor \( t_{ij}[\phi] \) such that \( g'_{ij} = g_{ij} + h t_{ij} \) is a metric, hermitian with respect to the very complex structure \( J^j_i \) we started from, and such that \( J^j_i \) is covariantly constant with respect to the covariant derivative with connexion \( \Gamma^k_{ij}[g'_{mn}] \). This is the necessary stability of the theory which ensures that, at a given perturbative order where the Slavnov identity holds (absence of anomaly up to this order), the U.V. divergences in the Green functions may be compensated-for through the usual renormalisation algorithm and normalisation conditions \[4\]. Of course, the trivial cohomology \( S_L \Delta_{[+1]}[V^i(\phi)] \) corresponds to field and source reparametrisations according to (14).

Let us finally study the Faddeev-Popov charge +1 sector. As in this letter we restrict ourselves to compact Kähler Ricci-flat manifolds, if the Hodge number \( h^{(3,0)} = h^{(0,3)} \) does not vanish, we have a true anomaly candidate. Starting from the \( S_L^0 \) cohomology \[10\] :

\[
\Delta^a_{[+1]} = \int d^2 x d^2 \theta t^{[ij]}[\phi] (d^+) (d^-)^2 \eta_i \eta_j \eta_k ,
\]

we were able to construct the \( S_L \) cohomology in the same Faddeev-Popov sector \((13b)\) :

\[
\Delta^a_{[+1]} = \int d^2 x d^2 \theta [t^{[ij]}[\phi]] \left\{ (d^+) (d^-)^2 \eta_i \eta_j \eta_k - \frac{3}{2} d^+ d^- (\eta_i \eta_j J_{kn} (d^+ D_+ \phi^n - d^- D_- \phi^n) + 2 \eta_i J_{in} J_{jm} D_+ \phi^n D_- \phi^m) + \frac{3}{4} J_{in} J_{jm} J_{kl} (d^+ D_+ \phi^n D_+ \phi^m D_- \phi^l - d^- D_- \phi^n D_- \phi^m D_+ \phi^l) + \tilde{t}_{[nm]} (d^+ D_+ \phi^n D_+ \phi^m D_- \phi^l - d^- D_- \phi^n D_- \phi^m D_+ \phi^l) \right\} ,
\]

where \( \tilde{t}_{[ij]}[\phi] \) is related to \( t^{[ij]}[\phi] \) through (in complex coordinates) :

\[
\tilde{t}_{[\alpha \beta]} \gamma , \quad \tilde{t}_{[\alpha \beta]} \gamma \neq 0 , \quad \text{the other vanish} ;
\]

\[
\tilde{t}_{[\alpha \beta]} \gamma = - \frac{i}{4} q \partial_\gamma [g_{\alpha \bar{\alpha}} g_{\beta \bar{\beta}} K_{i \bar{i}} t^{[\alpha \beta \bar{\gamma}]}] \quad \text{where K is the Kahler potential}.
\]

As a consequence, if at a given perturbative order this anomaly appears with a non zero coefficient

\[
S_L \Gamma_{\phi^0 \text{order}} = a(h)^p \Delta^a_{[+1]} , \quad a \neq 0
\]

the N = 2 supersymmetry is broken as \( \Delta^a_{[+1]} \) cannot be reabsorbed (being a cohomology element, it is not a \( S_L \Delta_{[0]} \)) and, \( \text{a priori} \), we are no longer able toanalyse the structure of the U.V. divergences at the next perturbative order, which is the death of the theory.

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9 As previously mentioned (subsection 3.1), a reparametrisation of the sources and fields has been used to compensate for the change \( U^i_j[\phi] \) in the original complex structure \( J^j_i \).
4 Concluding remarks

In this letter we have analysed the cohomology of the B.R.S. operator associated to $N = 2$ supersymmetry in a $N = 1$ superfield formalism. We have found an anomaly candidate for torsionless models built on compact Kähler Ricci-flat target spaces with a non vanishing Hodge number $h^{3,0} = h^{0,3}$. Calabi-Yau manifolds (3 complex dimensional case) where $h^{3,0} = 1$ are interesting examples due to their possible relevance for superstring theories. Of course, as no explicit metric is at hand, one can hardly compute the anomaly coefficient.

This anomaly in global extended supersymmetry is a surprise with respect to common wisdom (but see other unexpected cohomologies in supersymmetric theories, in the recent works of Brandt [13] and Dixon [14]) and the fact that if we have chosen, from the very beginning, a coordinate system adapted to the complex structure, the second supersymmetry will be linear and there will be no need for sources $\eta_i$. However, as known from chiral symmetry, even a linearly realised transformation can lead to anomalies; moreover, here the linear susy transformations do not correspond to an ordinary group but rather to a supergroup where, contrarily to ordinary compact groups, no general theorems exists: then there is no obvious contradiction. This emphasizes the special structure of the supersymmetry algebra.

Of course, our analysis casts some doubts on the validity of the previous claims on U.V. properties of N=2 supersymmetric non linear $\sigma$ models: there, the possible occurrence at 4-loops order of (infinite) counterterms non-vanishing on-shell, even for Kähler Ricci-flat manifolds, did not “disturb” the complex structure; on the other hand, we have found a possible “instability” of the second supersymmetry, which confirms that there are some difficulties in the regularisation of supersymmetry by dimensional reduction assumed as well in explicit perturbative calculations than in finiteness “proofs” or higher order counterterms analysis. We would like to emphasize the difference between Faddeev-Popov charge cohomology which describes the stability of the classical action against radiative corrections (the usual “infinite” counterterms) and which offers no surprise, and the anomaly sector which describes the “stability” of the symmetry (the finite renormalisations which are needed, in presence of a regularisation that does not respect the symmetries of the theory, to restore the Ward identities): of course, when at a given perturbative order the Slavnov (or Ward) identities are spoiled, at the next order, the analysis of the structure of the divergences is no longer under control. In particular, the Calabi-Yau uniqueness theorem for the metric supposes that one stays in the same cohomology class for the Kähler form, a fact which is not certain in the absence of a regularisation that respects the $N=2$ supersymmetry (the possible anomaly we found expresses the impossibility to find a regularisation that respects all the symmetries of these theories).

Of course, if one has added from the very beginning extra geometrical (or physical!) constraints that would fix the classical action, we bet that our anomaly candidate would disappear: as previously mentioned, this is the case when the manifold is a compact homogeneous Kähler space; moreover we have also been able to prove that, if one enforces $\mathbb{N}=4$ supersymmetry (HyperKähler manifolds), there is no possible tensor $t^{ijk}[\phi]$ and then no anomaly ($[12]$ b)). Our final conjecture is that the requirement of conformal invariance of the theory would be sufficient to rule out a possible anomaly. We hope to be able to report on that subject in a near future.

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10 As $\det |g| = 1$, a representative of $t^{[\alpha\beta\gamma]}$ is the constant antisymmetric tensor $\epsilon^{[\alpha\beta\gamma]}$ (with $\epsilon^{123} = +1$).
11 In the appendix A of ref. [11], it is proven that any linearly realised symmetry corresponding to a compact group of transformations can be implemented to all orders of perturbation theory.
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