Cross-sections of multibrot sets

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Dedicated to David Minda on the occasion of his retirement

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Abstract We identify the intersection of the multibrot set of $z^d + c$ with the rays $\mathbb{R}^+\omega$, where $\omega^{d-1} = \pm 1$.

Keywords Mandelbrot set · Multibrot set

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1 Introduction

Let $d$ be an integer with $d \geq 2$. Given $c \in \mathbb{C}$, we define

$$p_c(z) := z^d + c \quad \text{and} \quad p_c^n := p_c \circ \cdots \circ p_c \quad (n \text{ times}).$$

The corresponding generalized Mandelbrot set, or multibrot set, is defined by

$$M_d := \left\{ c \in \mathbb{C} : \sup_{n \geq 0} |p_c^n(0)| < \infty \right\}.$$

Of course $M_2$ is just the classical Mandelbrot set. Computer-generated images of $M_3$ and $M_4$ are pictured in Figure 1. Multibrot sets have been extensively studied in the literature. Schleicher’s article [5] contains a wealth of background material on them.

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We mention here some elementary properties of multibrot sets. First of all, they exhibit \((d-1)\)-fold rotational invariance, namely
\[
M_d = \omega M_d \quad (\omega \in \mathbb{C}, \, \omega^{d-1} = 1).
\]
Indeed, for these \(\omega\), writing \(\phi(z) := \omega z\), we have \(\phi^{-1} \circ p_c \circ \phi = p_c / \omega\), so \(p_c^n(0)\) remains bounded if and only if \(p_c^n(0)\) does. (In fact, the rotations in (1) are the only rotational symmetries of \(M_d\). The paper of Lau and Schleicher [1] contains an elementary proof of this fact.)

Also, writing \(D(0, r)\) for the closed disk with center 0 and radius \(r\), we have the inclusions
\[
D(0, \alpha(d)) \subset M_d \subset D(0, \beta(d)),
\]
where
\[
\alpha(d) := (d-1)d^{-d/(d-1)} \quad \text{and} \quad \beta(d) := 2^{1/(d-1)}.
\]
The first inclusion follows from the fact that, if \(|c| \leq \alpha(d)\), then the closed disk \(D(0, d^{-1/(d-1)})\) is mapped into itself by \(p_c\), and consequently the sequence \(p_c^n(0)\) is bounded. For the second inclusion, we observe that, if \(|c| > \beta(d)\), then by induction \(|p_c^n(0)| \geq (2d)^n(|c|^{d-2}|c|)\) for all \(n \geq 0\), and the right-hand side of this inequality tends to infinity with \(n\).

When \(d\) is odd, we have
\[
M_d \cap \mathbb{R} = [-\alpha(d), \alpha(d)]. \tag{2}
\]
This equality was conjectured by Parisé and Rochon in [3], and proved by them in [4]. Also, when \(d\) is even, we have
\[
M_d \cap \mathbb{R} = [-\beta(d), \alpha(d)]. \tag{3}
\]
This equality was also conjectured in [3], and subsequently proved in [2]. When \(d = 2\), it reduces to the well-known equality \(M_2 \cap \mathbb{R} = [-2, \frac{1}{4}]\).
By virtue of the rotation-invariance property (1), the equalities (2) and (3) yield information about the intersection of $M_d$ with certain rays emanating from zero. Indeed, if $\omega d^{-1} = 1$, then

$$M_d \cap \mathbb{R}^+ \omega = \{ t \omega : 0 \leq t \leq \alpha(d) \},$$

and if $\omega d^{-1} = -1$ and $d$ is even, then

$$M_d \cap \mathbb{R}^+ \omega = \{ t \omega : 0 \leq t \leq \beta(d) \}.$$  

This leaves open the case when $\omega d^{-1} = -1$ and $d$ is odd. The purpose of this note is to fill the gap. The following theorem is our main result.

**Theorem 1.1.** If $\omega d^{-1} = -1$ and $d$ is odd, then

$$M_d \cap \mathbb{R}^+ \omega = \{ t \omega : 0 \leq t \leq \gamma(d) \},$$

where

$$\gamma(d) := d^{-d/(d-1)} \left( \sinh(d \xi_d) + d \sinh(\xi_d) \right), \quad (4)$$

and $\xi_d$ is the unique positive root of the equation $\cosh(d \xi_d) = d \cosh(\xi_d)$.

When $d = 3$, one can use the relation $\cosh(3x) = 4 \cosh^3 x - 3 \cosh x$ to derive the exact formula $\gamma(3) = \sqrt{32/27}$, which yields

**Corollary 1.2.** $M_3 \cap i\mathbb{R} = \{ iy : |y| \leq \sqrt{32/27} \}$.

In comparison, note that (2) gives $M_3 \cap \mathbb{R} = \{ x : |x| \leq 2/\sqrt{27} \}$. See Figure 1.

The first few values of $\alpha(d), \beta(d), \gamma(d)$ are tabulated in Table 1 for comparison.

| $d$ | $\alpha(d)$ | $\beta(d)$ | $\gamma(d)$ |
|-----|-------------|-------------|-------------|
| 2   | 0.2500000000 | 2.000000000 | 1.100917369 |
| 3   | 0.384900179  | 1.414213562 | 1.088662108 |
| 4   | 0.472470394  | 1.259921050 | 1.078336651 |
| 5   | 0.534992244  | 1.189207115 | 1.069984489 |
| 6   | 0.582355932  | 1.148698355 | 1.063192242 |
| 7   | 0.619731451  | 1.122462048 | 1.057591279 |
| 8   | 0.650122502  | 1.104089514 | 1.052904317 |
| 9   | 0.675409498  | 1.090507733 | 1.048928539 |
| 10  | 0.696837314  | 1.080059739 | 1.045514971 |
| 11  | 0.715266766  | 1.071734363 | 1.042552690 |
| 12  | 0.731314279  | 1.065041089 | 1.039957793 |

It can be shown that $\gamma(d) > 1$ for all $d$, and that

$$\gamma(d) = 2^{1/d - O((\log d)^2/d^2)} \quad \text{as} \quad d \to \infty.$$  

These statements will be justified later.
2 Proof of Theorem 1.1

In this section we suppose that $d$ is an odd integer with $d \geq 3$. If $\omega^{d-1} = -1$, then, writing $\phi(z) := \omega z$, we have $\phi^{-1} \circ p_c \circ \phi = q_c / \omega$, where

$$q_c(z) := -z^d + c.$$

Thus $M_d \cap \mathbb{R}^+ = \omega(N_d \cap \mathbb{R}^+)$, where

$$N_d := \left\{ c \in \mathbb{C} : \sup_{n \geq 0} |q_c^n(0)| < \infty \right\}.$$

We now seek to identify $N_d \cap \mathbb{R}^+$. We shall do this in two stages.

**Lemma 2.1.** Let $d$ be an odd integer with $d \geq 3$. Then

$$N_d \cap \mathbb{R}^+ = [0, \mu(d)],$$

where

$$\mu(d) := \max \left\{ a - b^d : a, b \geq 0, a^d + b^d = a + b \right\}.$$

**Proof.** Consider first the case $c \in [0, 1]$. In this case we have $q_c(0) = c$ and $q_c(c) = -c^d + c \geq 0$. Since $q_c$ is a decreasing function, it follows that $q_c([0, c]) \subset [0, c]$, and in particular that $q_c^{[n]}(0)$ is bounded. Hence $c \in N_d$ for all $c \in [0, 1]$.

Consider now the case $c \in [1, \infty)$. Then $q_c(0) = c$ and $q_c^{[2]}(0) = -c^d + c \leq 0$. As $q_c$ is a decreasing function, it follows that $q_c^{[2n]}(0)$ is a decreasing sequence and $q_c^{[2n+1]}(0)$ is an increasing sequence. If, further, $c \in N_d$, then $q_c^{[n]}(0)$ is bounded, and both of these subsequences converge, say $q_c^{[2n+1]}(0) \to a$ and $q_c^{[2n]}(0) \to -b$, where $a, b \geq 0$. We then have $q_c(-b) = a$ and $q_c(a) = -b$, in other words $b^d + c = a$ and $a^d + c = b$. Adding these equations gives $a^d + b^d = a + b$. Summarizing what we have proved: if $c \in N_d \cap [1, \infty)$, then $c = a - b^d$, where $a, b \geq 0$ and $a^d + b^d = a + b$. Conversely, if $c$ is of this form, then $q_c(-b) = a$ and $q_c(a) = -b$, so $[-b, a]$ is a $q_c$-invariant interval containing 0, which implies that $q_c^{[n]}(0)$ remains bounded, and hence $c \in N_d$. Combining these remarks, we have shown that

$$N_d \cap [1, \infty) = \{ a - b^d : a, b \geq 0, a^d + b^d = a + b \} \cap [1, \infty].$$

The condition that $a^d + b^d = a + b$ can be re-written as $h(a) = -h(b)$, where $h(x) := x^d - x$. Viewed this way, it is more or less clear that the right-hand side of (5) is a closed interval containing 1, so $N_d \cap [1, \infty) = [1, \mu(d)]$, where $\mu(d)$ is as defined in the statement of the lemma.

Finally, putting all of this together, we have shown that $N_d \cap \mathbb{R}^+ = [0, \mu(d)]$. 

Next we identify $\mu(d)$ more explicitly.

**Lemma 2.2.** $\mu(d) = \gamma(d)$. 

Proof. We reformulate the maximization problem defining $\mu(d)$. Set
\[
S := \{(a, b) \in \mathbb{R}^2 : a, b \geq 0\},
\]
\[
f(a, b) := a - b^d,
\]
\[
g(a, b) := d^d + b^d - a - b.
\]
We are seeking to maximize $f$ over $S \cap \{g = 0\}$. The set $S \cap \{g = 0\}$ is compact and $f$ is continuous, so the maximum is certainly attained, say at $(a_0, b_0)$. Notice also that $\nabla g \neq 0$ at every point of $S \cap \{g = 0\}$. There are two cases to consider.

Case 1: $(a_0, b_0) \in \partial S$. The condition that $g(a_0, b_0) = 0$ then implies that
\[
(a_0, b_0) = (0, 0), (0, 1) \text{ or } (1, 0).
\]
The corresponding values of $f(a_0, b_0)$ are $0, -1, 1$ respectively. Clearly we can eliminate the first two points from consideration. As for the third, we remark that the directional derivative of $f$ at $(1, 0)$ along $\{g = 0\}$ in the direction pointing into $S$ is equal to $1/\sqrt{1 + (d - 1)^2}$, which is strictly positive. So $(1, 0)$ cannot be a maximum of $f$ either.

Case 2: $(a_0, b_0) \in \text{int}(S)$. In this case, by the standard Lagrange multiplier argument, we must have $\nabla f(a_0, b_0) = \lambda \nabla g(a_0, b_0)$ for some $\lambda \in \mathbb{R}$. Writing this out explicitly, we get
\[
1 = \lambda \left(d a_0^{d-1} - 1\right),
\]
\[
- d b_0^{d-1} = \lambda \left(d b_0^{d-1} - 1\right).
\]
Dividing the second equation by the first and then simplifying, we obtain
\[
a_0 b_0 = d^{-2/(d-1)}.
\]
Thus $a_0 = d^{-1/(d-1)}e^{\xi}$ and $b_0 = d^{-1/(d-1)}e^{-\xi}$ for some $\xi \in \mathbb{R}$. With this notation, the constraint $g(a_0, b_0) = 0$ translates to $\cosh(d\xi) = d \cosh(\xi)$, and the value of $f$ at $(a_0, b_0)$ is
\[
f(a_0, b_0) = a_0 - b_0^d = \frac{a_0 - b_0}{2} + \frac{a_0^d - b_0^d}{2} = d^{-d/(d-1)}(d \sinh(\xi) + \sinh(d\xi)).
\]
There are precisely two roots of $\cosh(d\xi) = d \cosh(\xi)$, one positive and one negative. Necessarily the positive root gives rise to the maximum value of $f$, thereby showing that $\mu(d) = \gamma(d)$. \hfill \Box

Remark. Clearly $f(1, 0) = 1$. The treatment of Case 1 above shows that $f$ does not attain its maximum over $S \cap \{g = 0\}$ at $(1, 0)$, and so $\mu(d) > 1$. This shows that $\gamma(d) > 1$, thereby justifying a statement made in the introduction.

Proof of Theorem 1.1 Combining the various results already obtained in this section, we have
\[
M_d \cap \mathbb{R}^+ \omega = \omega(N_\omega \cap \mathbb{R}^+) = \omega[0, \mu(d)] = \omega[0, \gamma(d)].
\]
This concludes the proof of Theorem 1.1. \hfill \Box
3 An asymptotic formula for $\gamma(d)$.

Our aim is to justify the following statement made in the introduction.

**Proposition 3.1.** If $\gamma$ is defined as in (4), then

$$\gamma(d) = 2^{1/d + O((\log d)^2/d^2)} \quad \text{as } d \to \infty. \quad (6)$$

There is no need to suppose that $d$ is an integer here.

**Proof.** We begin by deriving an asymptotic formula for $\xi_d$ as $d \to \infty$. On the one hand, since

$$e^{d \xi} \geq \cosh(d \xi) = d \cosh(\xi) \geq d,$$

we certainly have $\xi_d \geq (\log d)/d$. On the other hand, since the unimodal function $(\cosh x)/x$ takes the same values at $\xi_d$ and $d \xi$, we must have $\xi_d \leq \eta \leq d \xi$, where $\eta$ is the point at which $(\cosh x)/x$ assumes its minimum. Thus

$$e^{d \xi} \leq \cosh(d \xi) = d \cosh(\xi) \leq d \cosh \eta,$$

whence

$$\xi_d = \log \frac{d}{d} + O\left(\frac{1}{d}\right).$$

This is not yet precise enough. Substituting into the equation $\cosh(d \xi) = d \cosh(\xi_d)$, we obtain

$$e^{d \xi} = d + O\left(\frac{(\log d)^2}{d}\right),$$

whence

$$\xi_d = \frac{\log(2d)}{d} + O\left(\frac{(\log d)^2}{d^3}\right).$$

This is good enough for our needs.

We now estimate $\gamma(d)$ as $d \to \infty$. First of all, we have

$$d \sinh(\xi_d) = d \xi_d + O(d \xi_d) = \log(2d) + O\left(\frac{(\log d)^3}{d^3}\right).$$

Also

$$\sinh(d \xi_d) = \sinh\left(\log(2d) + O\left(\frac{(\log d)^2}{d^2}\right)\right) = d + O\left(\frac{(\log d)^2}{d}\right).$$

Hence

$$\log \gamma(d) = \log \left(\frac{d \sinh(\xi_d) + \sinh(d \xi_d)}{d - 1}\right) = \frac{d}{d - 1} \log d$$

$$= \log d + \log(2d) + O\left(\frac{(\log d)^2}{d}\right) = \log d + \frac{\log(2d)}{d} + O\left(\frac{(\log d)^2}{d^2}\right)$$

$$= \log d + \frac{\log(2d)}{d} + O\left(\frac{(\log d)^2}{d^2}\right).$$

Finally, taking exponentials of both sides, we get (6). \qed
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