Properties of gravity near the Schwarzschild radius and the cosmological redshift

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Abstract The radius of the observable region of the Universe is of the order of its Schwarzschild radius. Due to the spherical symmetry, this allows to check the properties of the gravitational force in the vicinity of the Schwarzschild radius by comparing the theoretical and observed Hubble diagram at high redshifts. This can be done in a simple model that follows from projective-invariant equations of gravitation. This paper shows that the Hubble diagram up to $z = 8$ testifies in favor of specific properties of gravity near and inside of the Schwarzschild radius.

Keywords Fundamental problems of gravitation · Modified theories of gravity · Black Holes

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1 Introduction

Properties of gravity near the Schwarzschild radius are key test of general relativity. This paper is based on the fact that the radius of the observable Universe is of the order of its Schwarzschild radius. Starting from this fact, we show that observed properties of the Hubble diagram at high redshifts testify in favor of specific properties of gravity, which follow from the projective invariant equations of gravitation. This confirms results of paper [2], where it was shown that such properties of gravity do not contradict observations near the supermassive object at the center of our galaxy.

In paper [1] has been shown that the properties of gravitation may differ significantly from those resulting from general relativity near the Schwarzschild radius and less than that. There is no an event horizon. The force acting on a freely falling test particle to a dot mass becomes repulsive near the Schwarzschild radius $r_g$, and tends to zero when the distance to the center tends to zero.

The lack of the event horizon and the weakness of the gravitational force near the center makes it possible the existence of super-massive objects without an event horizon. Such objects are candidates to the supermassive objects that exist in centers of galaxies [2, 3].

The properties of gravity near the Schwarzschild radius should be manifested in the magnitude of the velocity of galaxies at very high redshifts, and therefore, in the Hubble diagram. It allows to test gravity near the Schwarzschild radius. Fortunately, in addition to the great data from observations of supernovae Ia [4], we have now the numerous data obtained from observations of gamma-ray bursts up to $z = 8$ [5, 6, 7]. The mentioned data are not direct observations. They depend in part on the used Lambda CDM model. Therefore, their accuracy is not high. However, they can make a qualitative conclusion about the properties of gravity near the Schwarzschild radius.

2 Meaning of metric-field approach to theory of gravitation

2.1 Space-time geometry and reference system

In the early twentieth century, Henri Poincaré realized that geometry of space and time have no physical meaning by itself, without the knowledge of properties of the measuring instrument. Only the combination "geome-
try + measuring instruments" has a checked on experience meaning [11,12]. This applies also to the properties of space-time. Strictly speaking, we cannot speak about the properties of space-time without knowledge of properties of a reference frame, and we cannot speak about properties of the reference frame without knowledge of properties of space-time, because the reference frame is an instrument for investigation of space-time properties. Each of these concepts has no physical meaning in itself.

Based on the assumption that space-time in the inertial reference frames is pseudo-Euclidean, and starting from the conception of relativity of space-time, we can find a space-time metric for a certain class of non-inertial reference frames.

We believe that the concept of "frame of reference" has a physical meaning only when it is defined by some operational manner. The same can be said about a "comoving" coordinate system. This coordinate system has a physical meaning only if it is obtained by some transformation of the orthogonal coordinate system of pseudo-Euclidean space-time. Therefore, we define the non-inertial reference body frame as a system of material points which move under the influence of a force field defined in the inertial frame in Minkowski space-time. Therefore, we define the non-inertial reference body frame as a system of material points which move under the influence of a force field defined in the inertial frame in Minkowski space-time. Thus, consider a non-inertial frame of reference, reference body of which is formed by identical dot masses \( m \), moving under action of some force field \( F(x) \) given in an inertial reference frame (IRF) \( A \), where space-time is pseudo-Euclidean. The reference frame \( B \) will be here called the proper reference frame (PRF) of the force field \( F(x) \). If an observer, located in \( B \), rejects the Newtonian idea of absolute space and time, and believes that space-time is relative in the sense of Berkeley-Leibniz-Mach, then from his point of view the above dot masses are the points of his physical space. From his point of view they are at rest both in the non-relativistic and the relativistic sense. Consequently, their world lines are geodesics of space-time in the frame \( B \) so that the equality \( \delta \int ds = 0 \) holds, where \( ds \) is the line element of space-time in \( B \).

However, on the other hand, the motion of the material points in the frame \( A \) (in Minkowski space-time) is described by a Lagrangian action \( S = \int L(x, \dot{x})dt \). It follows from this fact that the line element of space-time in the frame \( B \) is of the form

\[
ds = k \ dS,
\]

where \( k \) is a constant and \( dS = L(x, \dot{x})dt \).

In the limit \( F(x) \to 0 \),

\[
S = -mc \int (1 - v^2/c^2)^{1/2} dt,
\]

where \( v \) is 3-velocity of particles of the reference body, and \( c \) is the speed of light. It follows from this fact that the constant \( k \) should be equal to \( -(mc)^{-1} \).

Consider, for example, the following PRF.

The reference body consists of noninteracting electric charges in an electromagnetic field. In a Cartesian coordinate system the action describing the motion of the particles can be written as follows [2]:

\[
S = \int \left( -mc^2(1 - v^2/c^2)^{1/2} - \frac{e}{c} A_\alpha(x) \frac{dx^\alpha}{dt} \right) dt,
\]

where \( A_\alpha \) is the 4-potential, \( e \) is charge of the particles.

For the given reference frame

\[
ds = d\sigma - \frac{e}{mc} A_\alpha dx^\alpha,
\]

where \( d\sigma \) is the line element of space-time in the IRF. It is a Finslerian metric.

Of course, such a frame of reference is not similar to the accelerated reference frame formed by neutral particles. However, this does not prevent to its theoretical analysis, assuming that the reference body is formed by identical ions. They can be regarded as an atomic clocks that are almost unaffected by accelerations.

Based on the fact that the clocks measure the length of its own world line, one can find the time interval in such PRFs. Evidently,

\[
dT = \frac{0}{0} T - \frac{e}{mc} A_\alpha dx^\alpha,
\]

where \( dT = ds/c \) and \( \frac{0}{0} T = d\sigma/c \) are proper time intervals in the PRF and IRF, correspondingly.

The following two cases are of interest:

1. The reference body consists of noninteracting electric charges in a constant homogeneous electric field \( E \) directed along the axis \( x \). According to (4)

\[
dT/dt = 1 - \frac{e}{mc^2} \varphi = 1 + \frac{e}{mc^2} E x,
\]

where \( \varphi = A_0 \), and \( E \) is the electric field strength. Because the electric force \( eE = maw \), where \( w \) is the acceleration with respect to the IFR, this result is equivalent to the well known one:

\[
dT/dt = 1 + \frac{wx}{c^2}.
\]

This result shows that difference between clock in IRF and PRF is not a kinematic effect, and is caused by a force field.

\[1\) As usual, in this paper Greek letters run from 0 to 3, and Latin - from 1 to 3.\]
The same result is obtained for the PRF of the homogeneous field of the Earth, reference body of which is formed by particles free falling in the field. The reason is that the replacement $\nu \varphi$ by the gravitational potential leads to the same equation of the motion of test particles as the equations for charges.

2. The reference body consists of noninteracting electric charges in a constant homogeneous magnetic field $H$ directed along the axis $z$.

It follows from the Stokes theorem that in this case the modulus $A$ of the potential $A_i$ of a particle at the distance $r$ from the center of the orbit of the reference body is equal $A = H r/2$, which shows that $A$ is a modulus of the 3-vector $A = \frac{1}{2} B \times r$, and the $A$ is directed tangentially to the orbit circle. For this reason, according to (1),

$$
\frac{dT}{dt} = 1 - \frac{e}{mc^2} \frac{H r^2 \omega}{2} = 1 - \frac{F_i r}{2 mc^2},
$$

where $\omega = d\varphi/dt$ is the angular velocity of the body reference, and $F_i = e H r \omega/c$ is the Lorentz force. Since the centrifugal force $F_c = F_i = m V c$, where $V c$ is the centrifugal acceleration, we arrive at the conclusion that this result is equivalent to the well known one:

$$
\frac{dT}{dt} = 1 - \frac{\omega^2 r^2}{2 c^2}.
$$

This result coincides with the result for the rotating disk due to the fact that the motion of points on the disk can be described by a similar Lagrangian.

Let us consider another important example.

In papers [10, 11] has been proved that streamlines of any perfect isentropic relativistic fluid are geodesic lines in a Riemannian space-time.

In more detail, the motion of macroscopically small elements (“particles”) of the fluid can be considered in two ways.

In Minkowski space-time, where $ds^2 = \eta_{\alpha \beta} dx^\alpha dx^\beta$ is the line element and $\eta_{\alpha \beta}(x)$ is the metric tensor, this motion can be described by the following Lagrangian

$$
L = -mc^2 \left( G_{\alpha \beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right)^{1/2} dt,
$$

where $G_{\alpha \beta} = \chi^2 \eta_{\alpha \beta}$,

$$
\chi = \frac{\vartheta}{n mc^2} = 1 + \frac{\varepsilon}{mc^2} + \frac{P}{mc^2},
$$

$\vartheta$ is the enthalpy per unit volume, $\varepsilon$ is the fluid density energy, $m$ is the mass of the fluid “particle”, and $c$ is the speed of light.

Equations of the motion of the fluid element that arise from this Lagrangian are the standard equations of the field velocities of an relativistic isentropic fluid.

On the other hand, in a co-moving reference frame this motion can be described (in the same coordinate system) as the motion along the geodesics of the Riemannian space, the line element of which has the form

$$
ds^2 = G_{\alpha \beta} dx^\alpha dx^\beta.
$$

Indeed, in this case, for an observer located in this frame time is measured by the length of its world line. When using $s$ as a parameter of the length it is easy to see that the Lagrange equations give the standard equations of a geodesic in the Riemannian space-time with the metric tensor $G_{\alpha \beta}$.

Thus, there are reasons to believe that in PRFs a force field manifests itself as a curvature of space-time. More precisely, a field $F(x)$ can be considered in two ways: a) as a force field in an IRF in the Minkowski space-time, and b) as a curvature of space-time in PRFs according to [11].

It is difficult to verify our conclusion directly, but in the case of the gravitational field this conclusion leads to observable physical consequences since space-time is a bi-metric, and the existence of a flat metric should have an impact on the field equations. In addition, we live in the expanding Universe, and so we have the opportunity to study the phenomenon in a PRF of the gravitational field produced by matter of the Universe.

From the above point of view, gravity can be considered as a true field in the Minkowski space-time where the Lagrangian, describing the motion of test particles in this field, has the form

$$
L = -mc [g_{\alpha \beta}(x) \dot{x}^\alpha \dot{x}^\beta]^{1/2},
$$

where $g_{\alpha \beta}(x)$ is a tensor field in the the Minkowski space-time.\footnote{This field can in principle be formed by another function $\psi(x)$, which is some traditional characteristic of field in the Minkowski space-time [14].}

In this case, according to [11] the line element of space-time in PRFs is given by

$$
ds^2 = g_{\alpha \beta}(x) dx^\alpha dx^\beta,
$$

that is, space-time in PRFs of gravitational field is a Riemannian with non-zero curvature, where $g_{\alpha \beta}$ is the metric tensor.

Thus, we suppose that gravity can be considered as a field in an inertial reference frame in Minkowski space-time, and a space-time curvature in the proper reference frames.
2.2 Projective invariance

An other starting point of the theory is based on the observation that the equations of the motion of test particles (geodesic lines) are invariant under some group of the continuous transformations - geodesic (projective) mappings of the Riemannian space-time \[12\], in any fixed coordinate system.

A diffeomorphism between two pseudo-Riemannian spaces \( V_n \) and \( \tilde{V}_n \), with a metric tensor \( g \) and \( \tilde{g} \), respectively, is called geodesic if it is geodesic-preserving, that is, when it maps any geodesic of \( V_n \) into an geodesic of \( \tilde{V}_n \) again.

A necessary and sufficient condition for existence of a geodesic mapping between \( V_n \) and \( \tilde{V}_n \) is that the equations

\[
\tilde{\Gamma}^\alpha_{\beta\gamma}(x) = \Gamma^\alpha_{\beta\gamma}(x) + \delta^\alpha_{\beta}\phi(x) + \delta^\alpha_{\gamma}\phi(x)_{\beta}
\]

(10)

are satisfied, where \( \Gamma^\alpha_{\beta\gamma}(x) \) and \( \tilde{\Gamma}^\alpha_{\beta\gamma}(x) \) are components of the Christoffel symbols in \( V_n \) and \( \tilde{V}_n \), respectively.

The condition (10) is equivalent of the following Levi-Civita equations

\[
\tilde{g}_{\alpha\beta\gamma} = 2\phi_{\gamma}\tilde{g}_{\alpha\beta} + \phi_{\alpha}\tilde{g}_{\beta\gamma} + \phi_{\beta}\tilde{g}_{\alpha\gamma},
\]

where the semicolon denotes a covariant derivative in \( V_n \), \( \phi_{\alpha} \) is some gradient-like vector, i.e. \( \phi_{\alpha} = \partial \phi / \partial x^\alpha \).

For example, if \( \Gamma^\alpha_{\beta\gamma} \) are Christoffel symbols, then under using time \( t = dx^\beta / c \) as a parameter, the differential equations of a geodesic line are of the form

\[
\ddot{x}^\alpha + (\Gamma^\alpha_{\beta\gamma} - c^{-1} \tilde{\Gamma}^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma) \dot{x}^\beta \dot{x}^\gamma = 0
\]

(11)

where \( \dot{x}^\alpha = dx^\alpha / dt \), \( \ddot{x} = d\dot{x} / dt \). It easily to verify that these equations are invariant under the mapping (10) of the Christoffel symbols in any coordinate system.

Since \( \phi_{\alpha} \) is a gradient function, (10) are very similar to a generalization of the gauge transformations of 4-potentials in the classical electrodynamics.

In a supplement we give two example of this fact. Namely, it is shown that in any coordinate system the FWR and the Schwarzschild metrics have a continuous set of geodesically equivalent metrics.

It is obviously, all the connection coefficients, which are connected by such transformation, are physically equivalent. They describe the same gravitational field, because a classical field is defined by properties of motion of test particles. (Just as the 4-potentials in classical electrodynamics, connected by a gauge transformation \( A_\beta \rightarrow A_\beta + \partial_\beta \phi(x) \)).

The projective transformations of the connection coefficients induce transformation of the metric tensor, the curvature tensor and the Ricci tensor. This is a reason that the classical Einstein’s equations are not invariant under projective mappings of Riemannian spaces \[13,15\] which obviously should play a role of gauge transformations.

It follows from (10) that \( \tilde{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma \). This is a reason why geodesic invariance is not manifest itself in non-relativistic theory. It is important only in relativistic theory of gravitation.

Simple geodesic-invariant generalization of Einstein’s equations has examined in \[11\]. These equations are bimetric, i.e. these equations contain the Minkowski metric tensor \( \eta_{\alpha\beta}(x) \), and \( g_{\alpha\beta}(x) \). In cosmology this means that we have two possibilities. Either we are considering expanding Universe locally from the viewpoint of an observer in an inertial reference frame where space-time is supposed to be Minkowskian, either from the viewpoint of an observer in a co-moving frame, in which space-time is Riemannian. Both possibilities are locally in principle equivalent, if they are described by some appropriate differential equations for finding the functions \( g_{\alpha\beta}(x) \).

The fact that the used usually geometrical characteristics of space-time are not projective-invariant (like 4-potentials in electrodynamics are also not gradient-invariant) means that they cannot be considered as observable characteristics of gravitational field. The problem is to find the geometric objects which are invariant under geodesic mappings of Riemannian spaces. Two such objects are well known. These are the Weyl tensor and also the Thomas symbols \[12\], which are some natural geodesically invariant generalization of the Christoffel’s symbols.

A geodesic-invariant generalization of the metric tensor can also be defined \[17\]. Such object is based on a 5-dimensional interpretation of projective mappings in homogeneous coordinates, and is of the form

\[
\tilde{g}(x)_{\alpha\beta} = g_{\alpha\beta}(x) - f_\alpha(x)f_\beta(x).
\]

(12)

In this equation

\[
f_\alpha(x) = \frac{1}{2} \frac{\partial}{\partial x^\alpha} \ln \left( \frac{g}{\eta} \right),
\]

where \( g \) and \( \eta \) are the determinants of space-time in the PRL and flat metric, correspondingly. However, in this case, geodesic mappings \( \tilde{\gamma}_{\alpha\beta}(x) \rightarrow g_{\alpha\beta}(x) \) are not arbitrary.
3 Spherically symmetric solution in infinite medium

Based on the discussion above in section 2 we consider a simple relativistic model of the homogeneous and isotropic Universe as a self-gravitating expanding dust-like medium, space-time of which is a Riemannian co-moving reference frame, and is a Pseudo-Euclidean in inertial reference frame.

The simplest geodesic-invariant generalizations of the vacuum Einstein equations are [1]:

\begin{equation}
B_{\alpha\beta\gamma}^\gamma - B_{\alpha\beta}^\delta B_{\delta\alpha}^\gamma = 0.
\end{equation}

These equations are bi-metric differential equations for the tensor

\begin{equation}
B_{\alpha\beta}^\gamma = \Pi_{\alpha\beta}^\gamma - \Pi_{\alpha\beta}^\gamma,
\end{equation}

where \(\Pi_{\alpha\beta}^\gamma\) and \(\Pi_{\alpha\beta}^\gamma\) are the Thomas symbols of Riemannian and Minkowski space-time, respectively. They are given by equations

\begin{equation}
\Pi_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma - (n + 1)^{-1} \left[ \delta_{\alpha}^{\gamma} \Gamma_{\beta\gamma}^\epsilon \delta_{\beta}^{\epsilon} \Gamma_{\gamma\alpha}^\epsilon + \delta_{\beta}^{\epsilon} \Gamma_{\gamma\epsilon}^\alpha \right],
\end{equation}

\begin{equation}
\Pi_{\alpha\beta}^\gamma = \sigma \Gamma_{\alpha\beta}^\gamma - (n + 1)^{-1} \left[ \delta_{\alpha}^{\gamma} \Gamma_{\beta\gamma}^\epsilon \delta_{\beta}^{\epsilon} \Gamma_{\gamma\alpha}^\epsilon + \delta_{\beta}^{\epsilon} \Gamma_{\gamma\epsilon}^\alpha \right],
\end{equation}

\(\sigma\Gamma_{\alpha\beta}^\gamma\) and \(\Pi_{\alpha\beta}^\gamma\) are the Christoffel symbols of the Minkowski space-time, \(\Phi_{\alpha\beta}^\gamma\) are the Christoffel symbols of the Riemannian space-time whose fundamental tensor is \(g_{\alpha\beta}\). A semi-colon denotes a covariant differentiation in flat space-time.

The Lagrangian \(L\) describing the motion of test particles in a static spherically symmetric homogeneous and isotropic dust-like medium, which is invariant under the mapping \(t \rightarrow -t\), reads:

\begin{equation}
L = -mc[A\dot{r}^2 + B(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) - c^2C]^{1/2}
\end{equation}

where \(A\), \(D\) and \(C\) are functions of the radial coordinate \(r\).

The associated line element of the space-time in PRFs is given by

\begin{equation}
ds^2 = A \, dr^2 + B[\dot{d}\theta^2 + \sin^2 \theta \, d\varphi^2] - C \,(d\lambda^0)^2.
\end{equation}

The functions \(A(r), B(r), C(r)\) should be found from the field equations [13]. Because of the projective invariance of the gravitation equations, some additional (gauge) conditions can be imposed on the Christoffel symbols. In particular, at the conditions

\begin{equation}
Q_{\alpha} = \Gamma_{\alpha\sigma}^\sigma - \Gamma_{\alpha\sigma}^\sigma = 0
\end{equation}

the gravitation equations are reduced to Einstein’s vacuum equations \(R_{\alpha\beta} = 0\). Therefore, the spherically-symmetric solution \(A(r), B(r), C(r)\) can be found as a solution of the system of the differential equations:

\begin{equation}
R_{\alpha\beta} = 0
\end{equation}

and

\begin{equation}
Q_{\alpha} = 0.
\end{equation}

It must be stressed that the condition \(Q_{\alpha} = 0\) is a tensor equation. It does not impose any conditions on the coordinate system.

Thus the classical Einstein equations are eqs. [13] at a specific gauge condition.

Let us find solutions of \(20,21\) which satisfy the conditions at infinity:

\begin{equation}
\lim_{r \rightarrow \infty} A(r) = 1, \quad \lim_{r \rightarrow \infty} (B(r)/r^2) = 1, \quad \lim_{r \rightarrow \infty} C(r) = 1.
\end{equation}

Usually such conditions at infinity are meaningful only for the spatially restricted distribution of matter. However, due to the spherical distribution of matter and specific properties of gravity, which are derived from the projective invariant equations, they are valid for any infinite homogeneous material medium.

Conditions \(20,21\) yield one equation:

\begin{equation}
B^2AC = r^4.
\end{equation}

It allows to exclude the function \(A\) from the [13]. Then the equations \(R_{11} = 0\) and \(R_{00} = 0\) take the form:

\begin{equation}
-2BC' + 2rB'C' + rBC'' = 0,
\end{equation}

\begin{equation}
-4BCB' + rCB^2 - 2BC' + 2rBB'C' + 2rBCB'' + rB^2C'' = 0
\end{equation}

Because of equality \(24\) the sum of tree terms in \(25\) is equal to zero, and we obtain a differential equation for the function \(B(r)\):

\begin{equation}
2rBB'' + rB'^2 - 4BB' = 0.
\end{equation}

A general solution of this equation can be written as

\begin{equation}
B = a(r^3 + K^3)^{2/3}
\end{equation}

where \(a\) and \(K\) are some constants.
Now the function $C$ can be found from differential equation (24) in the form

$$C'' + 2 \frac{rB' - B}{rB} C' = 0,$$

(27)

where $(rB' - B)/rB = (r^3 - K^3)/(r^4 + rK^3)$. A general solution of this equation is $C = b - \mathcal{Q}/f$ where $f = (r^3 + K^3)^{1/3}$, and $\mathcal{Q}$ is a constant. It follows from the Newtonian limit that $b = 1$ and $\mathcal{Q} = 2GM/c^2$ is the Schwarzschild radius of mass $M$.

Therefore, the solution for spherically-symmetric field is given by:

$$C = 1 - r_g/f, f = (r^3 + r_g^3)^{1/3}, B = f^2, A = f'^2, \quad (28)$$

where

$$f = (r^3 + K^3)^{1/3}, f' = df/dr. \quad (29)$$

The constant $K$ cannot be obtained from (22) or from the consideration of non-relativistic limit. However, one can use a physical argumentation. Namely, we demand that the spherically symmetric solution must not have a singularity at the center.

It is enough to set $\mathcal{K} = \mathcal{Q} = r_g$. In this case, in the coordinate system used the line element of space-time is of the form:

$$ds^2 = -\frac{f^2 dr^2}{(1 - r_g/f)} - f^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) + (1 - \frac{r_g}{f}) dx^2 + dy^2 + dz^2, \quad (30)$$

where $f = (r^3 + r_g^3)^{1/3}$. This solution has no the event horizon and no singularity in the center.

This equation formally coincides with the original Schwarzschild solution [16] of the Einstein equations. It is a particular solution of the equations at the condition $\det[g_{\alpha\beta}(x)] = 1$.

We suppose that the lack of the singularity is sufficiently important reason to investigate just this case. Especially because it is not contradict the available observations [17,2]. For this reason, the solution (28) with $\mathcal{K} = r_g$ and with our conditions at infinity will be a basis for further consideration.

At the above conditions the equations of the motion of test particles in the plane $\theta = \pi/2$ can be obtained by the law of conservation of the energy $E$ and the angular moment $J$:

$$\dot{r} \frac{\partial L}{\partial \dot{r}} - \dot{\varphi} \frac{\partial L}{\partial \dot{\varphi}} - L = E \quad \text{and} \quad \frac{\partial L}{\partial \varphi} \dot{\varphi} = J, \quad (31)$$

where $\dot{r} = dr/dt$, $\dot{\varphi} = d\varphi/dt$.

These equations are of the form

$$r^2 = c^2 C^2 \left[ 1 - \frac{x^2}{C^2/E^2} (1 + \mathcal{J}^2/x^2) \right] \quad (32)$$

where $E = E/mc$, $\mathcal{J} = J/r_g mc$. Fig. 1 shows a radial acceleration $w$ of a free test particle as the function of the distance $r = r/r_g$ from the central dot mass $M$. This magnitude is $w = v dv/dr$, where $v$ is the radial velocity. The radial acceleration $w$ (or the force $F = mw$, acting on a test mass $m$) is a finite on the all interval $0 \leq r \leq \infty$. If $r \gg r_g$, the acceleration $w = w(r)$ is the same in the Newtonian mechanics. However, near $r_g$ the acceleration change the sign and eventually, inside the Schwarzschild radius $r_g$ the acceleration tend to zero when the distance tends to zero.

Because of the peculiarity of the gravitational force $F = mw$ some peculiar supermassive objects (up to $10^{10} M\odot$ and more than that) without the event horizon can exist with the radius less than the Schwarzschild radius. Such objects are candidates to supermassive objects in galactic centers [23].

Consider now of an observer in the Minkowski space-time in the center of a homogeneous dust-like sphere with the density of the order of the Universe density $(10^{-27} \pm 10^{-28} g/cm^3)$. For him the radial acceleration of a test particle at the distance $R$ at any moment depends only on the mass $M$ inside of the sphere of the radius $R$.

This acceleration is given by the external (vacuum) solution of the eqs. (14) at $r = R$.

The radial velocity at the surface of homogeneous sphere as the function of its radius is given by

$$v(R)^2 = c^2 C(R)^2 \left( 1 - \kappa^2 C(R)/E^2 \right). \quad (32)$$

Here $v(R)$ and $C(R)$ are given by eqs. (22), where $r$ is replaced by $R$, $M = (4/3)\pi \rho R^3$ is the matter mass inside of the sphere of the radius $R$, $\rho$ is the matter density, and $r_g = (8/3)\pi \rho c^{-2} G \rho R^3$ is Schwarzschild’s radius of the matter inside of the sphere.

Now add to the sphere outside some spherical shell of a mass $M_1$ and radius $R_1 > R$. The mass of this shell does not change the value of $w(R)$. The velocity and acceleration at $r = R_1$ is defined at any moment by the full mass inside the external sphere and by the
radius $R_1$. Continuing to do the same one can find the dependence of the radial acceleration on $r$ around the observer for very large distances.

The mass of the homogeneous sphere is proportional to first degree of density, and to third degree of the sphere radius. The observable radius of the Universe is of order of $10^{27}$ cm that is of the order of the Schwarzschild radius or less than that. Therefore, the acceleration (or the force $maw(r)$, acting on a test body of mass $m$) decreases, and at $r \to \infty$ tends to zero. At that the functions $A(r)$, $B(r)$, $C(r)$ tend to their values in the Minkowski space. Thus, space-time in PRFs of gravitational field of a very large space distribution of a homogeneous mass is an Euclidean on infinity, i.e is in full compliance with used conditions (22).

Fig. 2 shows the radial acceleration of a test particle as the function of the radius $R$ of a homogeneous sphere with real cosmological parameters.

![Fig. 1 The acceleration of a free falling test particle (arbitrary units) near the attractive point mass](image1)

![Fig. 2 The acceleration of a test particle in the expanding Universe vs. the distance from the observer.](image2)

Two conclusions can be made from this figure.

1. If the density $\rho = 6 \cdot 10^{-30} \text{ g cm}^{-3}$, the Schwarzschild radius $r_0$ becomes more then $R$ at $R > 1.5 \cdot 10^{28}$ cm. Before this, at $R = 6 \cdot 10^{27}$ cm, the relative acceleration change the sign, the same as for the particle free falling to a dot mass. If $R > 6 \cdot 10^{27}$ cm, the acceleration is positive. Hence, for sufficiently large distance $R$ the gravitational force gives rise an acceleration of galaxies. It must be emphasized that all magnitudes has sense also at the distances less than the Schwarzschild radius of the considered masses.

2. The gravitational force, affecting the particles, tends to zero when $R$ tends to infinity. The reason of the fact is that the ratio $R/r_0$ tends to zero when $R$ tends to infinity. Consequently, the gravitational influence on galaxies at large distance $R$ cased mainly by the matter insider of the sphere of the radius $R$.

4 The Hubble diagram

A magnitude which is related with observations in the expanding Universe is the relative velocity of a distant galaxy with respect to the observer. The radial velocity $v$ is given by (23).

Proceeding from this result we will find Hubble diagram, following mainly the method being used in (18).

Let $v_0$ be a local frequency of light in the co-moving reference frame of a moving source at the distance $R$ from an observer, $v_l$ be this frequency in a local inertial frame, and $v$ be the frequency as measured by the observer in the center of the sphere with the radius $R$.

The redshift $z = (v - v_0)/v$ is caused by both the Doppler-effect and gravitational field.

The Doppler-effect is a consequence of a difference between the local frequency of the source in inertial and co-moving reference frame, and it is given by (33)

$$v_l = v_0 [(1 - v/e)(1 + v/e)]^{1/2}.$$  (33)

The gravitational redshift is caused by the matter inside of the sphere of the radius $R$. It is a consequence of the energy conservation for a photon. According to the equations of the motion of a test particle (31) the rest energy of a particle in gravitation field is given by

$$E = mc^2\sqrt{C}.$$  (34)

Therefore, the difference in two local level $E_1$ and $E_2$ of an atom the energy in the field is $\Delta E = (E_2 - E_1)\sqrt{C}$, so that

$$\nu = \nu_l\sqrt{C},$$  (35)

where we take into account that for the observer location $C = 1$. It follows from (33) and (35) that the relationship between the frequency $\nu$, as measured by the observer, and the proper frequency $v_0$ of the moving source in the gravitational field takes the form

$$\frac{\nu}{v_0} = \sqrt{\frac{1 - v/c}{1 + v/c}}.$$  (36)

This equation yields the quantity $z = v_0/v - 1$ as a function of $R$. By solving this equation numerically we obtain the dependence $R = R(z)$ of the measured distance $R$ as a function of the redshift. Therefore, the distance modulus for a remote galaxy is given by

$$\mu = 5 \log_{10}[R(z)(z + 1)] - 5$$  (37)

where $R(1 + z)$ is a bolometric distance (in pc) to the object.
If eq. (32) have to give a correct radial velocity of distant of the galaxy in the expansive Universe, it have to lead to the classic Hubble law at small distances of $R$. At this condition the Schwarzschild radius $r_g = (8/3)\pi G\rho R^3$ of the matter inside of the sphere is very small compared with $R$. For this reason $f \approx r$, and $C = 1 - r_g/R$. Therefore, at $E = 1$, we obtain from (32) that

$$v = HR,$$

where

$$H = \sqrt{(8/3)\pi G\rho}.$$

If $E \neq 1$, then equation (32) does not lead to the Hubble law since $v$ does not tend to zero when $R \to 0$. For this reason we set $E = 1$ and look for the value of the density for which a good accordance with observation data can be obtained.

The fig. 3 shows the Hubble diagram up to $z = 1.8$ obtained by eq. (37) compared with observations data obtained by supernova Ia [4]. A good agreement between theory and observation is obvious.

Figure 4 shows the dependence of $v$ on $z$ up to $z = 8$.

![Fig. 3](image1.png)  
Fig. 3 The distance modulus $v$ vs. the redshift $z$ for the density $\rho = 6 \times 10^{-30}g\text{ cm}^{-3}$. The small squares denote the observation data according to [4].

![Fig. 4](image2.png)  
Fig. 4 The radial velocity vs. redshift $z$ for the density $\rho = 6 \times 10^{-30}g\text{ cm}^{-3}$.

A comparing the last two plots show good agreement between the theory and observation.

5 Conclusion

In paper [2] was shown that a specific properties of gravity in the vicinity of the Schwarzschild radius does not contradict the observational data.

In this paper we have shown that just such properties of the gravitational force allow us to understand the well known peculiarity of the Hubble diagram at large redshifts up to $z = 8$ that indicates an acceleration of the Universe expansion.

It should be noted that the model actually uses only one fitting parameter - the density.

This result shows that high redshift can be an important instrument for testing theory of gravitation.

6 Supplement

The most interesting examples of non-uniqueness of the metric of space-time in a given coordinate system due to the existence of the geodesic equivalence are FRW and Schwarzschild metrics. Here is a proof of this fact, based on papers [15][19], which shows that the metrics obtained from a given by some continuous one-parameter transformations have common geodesics.

FRW Metric

Consider the line elements of a Riemannian space-time $V$:

$$ds^2 = b(t)\, dt^2 + a(t) \, \sigma_{ik} (x^1, x^2, x^3) \, dx^i dx^k. \quad (38)$$

Geodesics of such metric are the same as the ones of the space-time $\nabla$ with the line element

$$\tilde{ds}^2 = B(t)\, dt^2 + A(t) \, \sigma_{ik} (x^1, x^2, x^3) \, dx^i dx^k. \quad (39)$$
where

\[ B(t) = \frac{b(t)}{[1 + q\, a(t)]^2}, \]  

(40)

\[ A(t) = \frac{a(t)}{1 + q\, a(t)}, \]  

(41)

and \( q \) is an arbitrary constant. The proof of this important fact is follows.

Contracting (10) with respect to \( \alpha \) and \( \beta \), we obtain

\[ T_{\beta\gamma}^\beta = \Gamma^\beta_{\beta\gamma} + (n + 1)\, \psi_\beta. \]  

Consequently,

\[ \psi_\beta = \frac{1}{2(n + 1)} \frac{\partial}{\partial x^\beta} \ln \left| \frac{\det g}{\det \eta} \right|, \]  

(42)

which shows that in the case under consideration only 0-component of \( \psi_\beta \) is other than zero.

The useful for us components of the Christoffel symbols of \( V \) are given by:

\[ T_{00}^\beta = b'(t)/2\, b, \quad T_{11}^\beta = a'(t)/2\, b(t), \]

\[ T_{10}^\beta = a'(t)/2\, a(t), \]

and the same components of \( V \) are:

\[ T_{00}^\beta = B'(t)/2\, B, \quad T_{11}^\beta = -A'(t)/2\, B(t), \]

\[ T_{10}^\beta = A'(t)/2\, A(t) \]  

(43)

Then eqs. (10) gives the following equations

\[ \frac{A'(t)}{A(t)} - \frac{a'(t)}{a(t)} = 2\, \psi_0, \]  

(44)

\[ \frac{B'(t)}{B(t)} - \frac{b'(t)}{b(t)} = 4\, \psi_0, \]  

(45)

\[ \frac{A'(t)}{B(t)} - \frac{a'(t)}{b(t)} = 0. \]  

(46)

So, \( A/a = \exp(2\int \psi(t)\, dt), \) \( B/b = \exp(4\int \psi(t)\, dt) \) where the integration constants are equal to 1 because at \( \psi(t)_0 = 0 \) the functions \( A(t) = a(t) \) and \( B(t) = b(t) \). Consequently, \( B(t)/b(t) = (A(t)/a(t))^2 \), and with \( 10 \) we obtain the differential equations

\[ A'(x) - A(t)^2 \frac{a'(t)}{a(t)} = 0, \]  

(47)

which gives (11). Now from previous equation we obtain the function \( B(t) \) in the form (40).

On the contrary, if in eq. (10) to set \( \psi_1 = 0 \) for \( i=1,2,3 \), and \( \psi_0 = -\frac{1}{2}\, a^2 b(t)/dt \), then eqs. (44), (45), and (46) are satisfied. Thus, with this choice of the co-vector field \( \psi(x)_\alpha \), the line element (48) at \( b = -1 \) is equivalent to (49). In other words, the both line elements have the same (no-parameterized) equations of motion of test particles.

Schwarzschild Metric

As another example, we show here that a static centrally symmetric metric

\[ ds^2 = b(r)\, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) - a(r)\, dt^2, \]  

(48)

(in particular, Shirtsleeve metric) in a given coordinate system is not unique. Namely, in any given coordinate system it has common geodesic lines with a metric of the form

\[ ds^2 = B(r)\, dr^2 + F(r)^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) - A(r)\, dt^2, \]  

(49)

where \( A(x), B(x) \) and \( F(x) \) are functions of \( x^\alpha \), depending on a continuous parameter.

The Christoffel symbols for (48) is given by

\[ \Gamma^r_{rt} = \frac{1}{2} \frac{b'(r)}{b(r)}, \quad \Gamma^\theta_\theta = \frac{1}{r}, \quad \Gamma^\phi_\phi = \frac{1}{b(r)}, \quad \Gamma^t_{rt} = \frac{1}{2} \frac{a'(r)}{a(r)}, \quad \Gamma^r_{\theta\phi} = -\frac{r}{b(r)}, \]  

(50)

\[ \Gamma^r_{\phi\phi} = \frac{\cos \theta}{\sin \theta}, \quad \Gamma^\theta_\phi = -\frac{r}{b(r)} \sin^2 \theta, \quad \Gamma^\phi_\phi = -\sin \theta \cos \theta, \quad \Gamma^t_{tt} = \frac{1}{2} \frac{a'(r)}{b(r)}. \]  

(51)

The Christoffel symbols for (49) are:

\[ \Gamma^r_{rr} = \frac{1}{2} \frac{B'(r)}{B(r)}, \quad \Gamma^\theta_\theta = \frac{F'(r)}{F(r)}, \quad \Gamma^\phi_\phi = \frac{F'(r)}{F(r)}, \quad \Gamma^t_{rt} = \frac{1}{2} \frac{A'(r)}{A(r)}, \quad \Gamma^r_{\theta\theta} = \frac{-F(r)}{B(r)} F'(r), \]  

(52)

\[ \Gamma^r_{\phi\phi} = \frac{\cos \theta}{\sin \theta}, \quad \Gamma^\theta_\phi = -\frac{F(r)}{F'(r)} F^2(r) \sin Ap J^2 \theta \sin \theta, \quad \Gamma^\phi_\phi = -\sin \theta \cos \theta, \quad \Gamma^t_{tt} = \frac{1}{2} \frac{A'(r)}{B(r)}. \]  

(53)

where a prime here and later denotes a derivative with respect to \( r \).
In view of this, Levi-Chevita equations (10) yields:

\[ \frac{B'(r)}{B(r)} - \frac{b'(r)}{b(r)} = 4\psi_r(r), \quad F(r)F'(r)b(r) - rB(r) = 0, \]

\[ A'(r)b(r) - a'(r)B(r) = 0, \quad (54) \]

\[ \frac{F'(r)}{F(r)} - \frac{1}{r} = \psi_r(r), \quad \frac{A'(r)}{A(r)} - \frac{a'(r)}{a(r)} = 2\psi_r(r), \]

\[ \psi_\theta(r) = \psi_\phi(r) = \psi_t(r) = 0 \quad (55) \]

According to (42) the function \( \psi_r(r) \) can be written as

\[ \psi_r = \partial \ln \chi / \partial r, \]

where

\[ \chi(r) = \left( \frac{g}{g_0} \right)^{1/2(n+1)}. \]

Consequently,

\[ B = b\chi^4; \quad A = a\chi^2; \quad F = \chi; \]

\[ A' = a'\chi^4; \quad (F^2)' = 2r\chi^4; \]

Formulas for the function \( F(r) \) are compatible only if the functions \( \chi(r) \) are the solution of the differential equations

\[ r\chi'(r) + \chi(r) - \chi(r)^3 = 0 \]

which yields

\[ \chi(r) = (1 + kr^2)^{1/2}, \]

where \( k \) is an arbitrary constant.

As a result, formulas which express the \( A(r), B(r), \) and \( F(r) \) by \( a(r) \) and \( b(r) \) are given by

\[ A(r) = \frac{a(r)}{1 + kr^2}, \quad B(r) = \frac{b(r)}{(1 + kr^2)^2}, \]

\[ F(r) = \frac{r}{(1 + kr^2)^{1/2}}. \quad (56) \]

References

1. L. Verozub, Ann.Phys.(Berlin) , 16 , (2008), 28
2. L. Verozub, Astr. Nachr.355 (2006), 355
3. L. Verozub, Nuovo.Cim. , 123. (2008), 1653
4. A. Riess, A. et al., ApJ, 607 (2004) , 65
5. M. Cardone, S. Caozziello, M. Dainotti, Mon. R. Soc. , 400 , (2009), 775
6. B. Schaefer, ApJ , 660 (2007), 16
7. M. Demianski, C. Rubano, Mon. R. Soc. 411 (2011), 1213
8. L. Verozub, Grav. and Cosm. , 17,(2011), 308
9. L. Landau & E. Lifshitz, The Classical Theory of Field, (Addison -Wesley, Massachusetts), 1971
10. L. Verozub, Int. J. Mod.Phys.D , 17 (2008), 337
11. L. Verozub, Grav. and Cosm., 19. (2013), 124
12. L. Eisenhart, (1984) Riemannian geometry (Princeton Univ. Press)
13. A. Petrov,(1969) Einstein Spaces , Pergamon Press.
14. W. Thirring, Ann. Phys., 16,(1961) 96
15. Mikeš J., Kiosak V., Vanšurová A., (2008) Geodesic mappings, Olomouc
16. Abrams L. Can. J. Phys., 67, (1989), 919
17. L. Verozub, & A. Kochetov, Grav.and Cosm., 6 (2000), 24
18. Ia. Zeldovich & I. Novikov, Relativistic Astrophysics, v. 2, (University of Chicago Press), 1971
19. L. Verozub, e-print arXiv.org: 1101.5536