Research Article

Weighted $A$-Statistical Convergence for Sequences of Positive Linear Operators

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We introduce the notion of weighted $A$-statistical convergence of a sequence, where $A$ represents the nonnegative regular matrix. We also prove the Korovkin approximation theorem by using the notion of weighted $A$-statistical convergence. Further, we give a rate of weighted $A$-statistical convergence and apply the classical Bernstein polynomial to construct an illustrative example in support of our result.

1. Background, Notations, and Preliminaries

We begin this paper by recalling the definition of natural (or asymptotic) density as follows. Suppose that $E \subseteq \mathbb{N} := \{1, 2, \ldots \}$ and $E_n = \{k \leq n : k \in E\}$. Then

$$\delta(E) = \lim_{n} \frac{1}{n} |E_n|$$

is called the natural density of $E$ provided that the limit exists, where $|\cdot|$ represents the number of elements in the enclosed set.

The term “statistical convergence” was first presented by Fast [1] which is a generalization of the concept of ordinary convergence. Actually, a root of the notion of statistical convergence can be detected by Zygmund [2] (also, see [3]), where he used the term “almost convergence” which turned out to be equivalent to the concept of statistical convergence. The notion of Fast was further investigated by Schoenberg [4], Šalát [5], Fridy [6], and Conner [7].

The following notion is due to Fast [1]. A sequence $x = (x_k)$ is said to be statistically convergent to $L$ if $\delta(K_\epsilon) = 0$ for every $\epsilon > 0$, where

$$K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}.$$  

Equivalently,

$$\lim_n n^{-1} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$  

In symbol, we will write $S\lim x = L$. We remark that every convergent sequence is statistically convergent but not conversely.

Let $X$ and $Y$ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix. If for each $x = (x_k)$ in $X$ the series

$$A_n x = \sum_k a_{nk} x_k = \sum_{k=1}^\infty a_{nk} x_k$$

converges for each $n \in \mathbb{N}$ and the sequence $A x = (A_n x)$ belongs to $Y$, then we say that matrix $A$ maps $X$ into $Y$. By the symbol $(X, Y)$ we denote the set of all matrices which map $X$ into $Y$.

A matrix $A$ (or a matrix map $A$) is called regular if $A \in (c, c)$, where the symbol $c$ denotes the spaces of all convergent sequences and

$$\lim_n A_n x = \lim_k x_k$$

for all $x \in c$. The well-known Silverman-Toeplitz theorem (see [8]) asserts that $A = (a_{nk})$ is regular if and only if

(i) $\lim_k a_{nk} = 0$ for each $k$;
(ii) \( \lim_n \sum_k a_{n,k} = 1; \)
(iii) \( \sup_n \sum_k |a_{n,k}| < \infty. \)

Kolk [9] extended the definition of statistical convergence with the help of nonnegative regular matrix \( A = (a_{n,k}) \) calling it \( A \)-statistical convergence. The definition of \( A \)-statistical convergence is given by Kolk as follows. For any nonnegative regular matrix \( A \), we say that a sequence is \( A \)-statistically convergent to \( L \) provided that for every \( \epsilon > 0 \) we have

\[
\lim_n \sum_{k : |x_k - L| \leq \epsilon} a_{n,k} = L. \tag{6}
\]

In 2009, the concept of weighted statistical convergence was defined and studied by Karakaya and Chishti [10] and further modified by Mursaleen et al. [11] in 2012. In 2013, Belen and Mohiuddine [12] presented a generalization of this notion through de la Vallée-Poussin mean. Quite recently, Esi [13] defined and studied the notion statistical summability through de la Vallée-Poussin mean in probabilistic normed spaces.

Let \( p = (p_k) \) be a sequence of nonnegative numbers such that \( p_0 > 0 \) and

\[
P_n = \sum_{k=0}^n p_k \to \infty \quad \text{as} \quad n \to \infty. \tag{7}
\]

Let

\[
t_n = \frac{1}{P_n} \sum_{k=0}^n p_k x_k, \quad n = 0, 1, 2, \ldots. \tag{8}
\]

We say that the sequence \( x = (x_k) \) is \((N, p_n)\)-summable to \( L \) if \( \lim_{n \to \infty} t_n = L \).

The lower and upper weighted densities of \( E \subseteq \mathbb{N} \) are defined by

\[
\delta_{N}^-(E) = \liminf_n \frac{1}{P_n} \left| \{k \leq P_n : k \in E\} \right|, \tag{9}
\]

\[
\delta_{N}^+(E) = \limsup_n \frac{1}{P_n} \left| \{k \leq P_n : k \in E\} \right|, \tag{10}
\]

respectively. We say that \( E \) has weighted density, denoted by \( \delta_{N}^-(E) \), if the limits of both of the above densities exist and are equal; that is, one writes

\[
\delta_{N}^-(E) = \lim_n \frac{1}{P_n} \left| \{k \leq P_n : k \in E\} \right|. \tag{11}
\]

The sequence \( x = (x_k) \) is said to be weighted statistically convergent (or \( S_{N}^{-}\)-convergent) to \( L \) if, for every \( \epsilon > 0 \), the set \( \{k \in \mathbb{N} : p_k |x_k - L| \geq \epsilon\} \) has weighted density zero; that is,

\[
\lim_n \frac{1}{P_n} \left| \{k \leq P_n : p_k |x_k - L| \geq \epsilon\} \right| = 0. \tag{12}
\]

In this case we write \( L = S_{N}^{-}\)-limit \( x \).

**Remark 1.** If \( p_k = 1 \) for all \( k \), then \((N, p_n)\)-summable is reduced to \((C, 1)\)-summable (or Cesàro summable) and weighted statistical convergence is reduced to statistical convergence.

On the other hand, let us recall that \( C[a, b] \) is the space of all functions \( f \) continuous on \([a, b]\). We know that \( C[a, b] \) is a Banach space with norm

\[
\|f\|_{C} := \sup_{x \in [a, b]} |f(x)|, \quad f \in C[a, b]. \tag{13}
\]

Suppose that \( L \) is a linear operator from \( C[a, b] \) into \( C[a, b] \). It is clear that if \( f \geq 0 \) implies \( Lf \geq 0 \), then the linear operator \( L \) is positive on \( C[a, b] \). We denote the value of \( Lf \) at a point \( x \in [a, b] \) by \( L(f; x) \). The classical Korovkin approximation theorem states the following [14].

**Theorem 2.** Let \( (T_n) \) be a sequence of positive linear operators from \( C[a, b] \) into \( C[a, b] \). Then,

\[
\lim_n \left\| T_n(f; x) - f(x) \right\|_{\infty} = 0, \tag{14}
\]

for all \( f \in C[a, b] \) if and only if

\[
\lim_n \left\| T_n(f_i; x) - f_i(x) \right\|_{\infty} = 0, \tag{15}
\]

where \( f_i(x) = x^i \) and \( i = 0, 1, 2 \).

Many mathematicians extended the Korovkin-type approximation theorems by using various test functions in several setups, including Banach spaces, abstract Banach lattices, function spaces, and Banach algebras. Firstly, Gadjiev and Orhan [15] established classical Korovkin theorem through statistical convergence and display an interesting example in support of our result. Recently, Korovkin-type theorems have been obtained by Mohiuddine [16] for almost convergence. Korovkin-type theorems were also obtained in [17] for \( \lambda \)-statistical convergence. The authors of [18] established these types of approximation theorem in weighted \( L_p \) spaces, where \( 1 \leq p < \infty \), through \( A \)-summability which is stronger than ordinary convergence. For these types of approximation theorems and related concepts, one can be referred to [19–27] and references therein.

### 2. Korovkin-Type Theorems by Weighted \( A \)-Statistical Convergence

Kolk [9] introduced the notion of \( A \)-statistical convergence by considering nonnegative regular matrix \( A \) instead of Cesàro matrix in the definition of statistical convergence due to Fast. Inspired from the work of Kolk, we introduce the notion of weighted \( A \)-statistical convergence of a sequence and then we establish some Korovkin-type theorems by using this notion.

**Definition 3.** Let \( A = (a_{n,k}) \) be a nonnegative regular matrix. A sequence \( x = (x_k) \) of real or complex numbers is said to be weighted \( A \)-statistically convergent, denoted by \( S_{A}^{-}\)-convergent, to \( L \) if for every \( \epsilon > 0 \)

\[
\lim_n \sum_{k \in \mathbb{N} : a_{n,k} \geq \epsilon} a_{n,k} = 0, \tag{16}
\]
Theorem 5. The convergence of a sequence coincides with statistical convergence.\[E(p, \varepsilon) = \{ k \in \mathbb{N} : p_k |x_k - L| \geq \varepsilon \} \].

Remark 4. One has the following.

(i) If we take \( A = I \), where \( I \) denotes the identity matrix, then weighted \( A \)-statistical convergence of a sequence is reduced to ordinary convergence.

(ii) If we take \( A = (C, 1) \), where \((C, 1)\) denotes the Cesáro matrix of order one, then weighted \( A \)-statistical convergence of a sequence reduces to weighted statistical convergence.

(iii) If we take \( A = (C, 1) \) and \( p_k = 1 \) for all \( k \), then weighted \( A \)-statistical convergence of a sequence reduces to statistical convergence.

Note that convergent sequence implies weighted \( A \)-statistically convergent to the same value but the converse is not true in general. For example, take \( A = (C, 1) \) and \( p_k = 1 \) for all \( k \) and define a sequence \( x = (x_k) \) by

\[ x_k = \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{otherwise}, \end{cases} \]  

where \( n \in \mathbb{N} \). Then this sequence is statistically convergent to 0 but not convergent; in this case, weighted \( A \)-statistical convergence of a sequence coincides with statistical convergence.

**Theorem 5.** Let \( A = (a_{n,k}) \) be a nonnegative regular matrix. Consider a sequence of positive linear operators \( (M_k) \) from \( C[a,b] \) into itself. Then, for all \( f \in C[a,b] \) bounded on whole real line,

\[ \limsup_{k \to \infty} M_k (f, x) = f(x) \]  

if and only if

\[ \lim_{k \to \infty} M_k (f, x) = f(x), \]  

\[ \lim_{k \to \infty} M_k (v, x) = x \]  

and

\[ \lim_{k \to \infty} M_k (v^2, x) = x^2. \]  

**Proof.** Equation (20) directly follows from (19) because each of \( 1, x, x^2 \) belongs to \( C[a,b] \). Consider a function \( f \in C[a,b] \). Then there is a constant \( C > 0 \) such that \( |f(x)| \leq C \) for all \( x \in (-\infty, \infty) \). Therefore,

\[ |f(v) - f(x)| \leq 2C, \quad -\infty < v, x < \infty. \]  

Let \( \delta > 0 \) be given. By hypothesis there is a \( \delta := \delta(\varepsilon) > 0 \) such that

\[ |f(v) - f(x)| < \varepsilon \quad \forall |v - x| < \delta. \]  

Solving (21) and (22) and then substituting \( \Omega(v) = (v - x)^2 \), one obtains

\[ |f(v) - f(x)| < \varepsilon + \frac{2C}{\delta^2} \Omega, \quad \forall |v - x| < \delta. \]  

Equation (23) can also be written as

\[ -\varepsilon - \frac{2C}{\delta^2} \Omega < f(v) - f(x) < \varepsilon + \frac{2M}{\delta^2} \Omega. \]  

Operating \( M_k(1,x) \) to (24) since \( M_k(f,x) \) is linear and monotone, one obtains

\[ M_k (1, x) \left( -\varepsilon - \frac{2C}{\delta^2} \Omega \right) < M_k (f, x) - f(x) \]  

\[ < M_k (1, x) \left( \varepsilon + \frac{2C}{\delta^2} \Omega \right). \]  

Note that \( x \) is fixed, so \( f(x) \) is constant number. Thus, we obtain from (25) that

\[ -\varepsilon M_k (1, x) - \frac{2C}{\delta^2} M_k (\Omega, x) \]  

\[ < M_k (f, x) - f(x) M_k (1, x) \]  

\[ < \varepsilon M_k (1, x) + \frac{2C}{\delta^2} M_k (\Omega, x). \]  

\[ (26) \]

The term \( "M_k (f,x) - f(x) M_k (1,x)" \) in (26) can also be written as

\[ M_k (f,x) - f(x) M_k (1,x) = M_k (f,x) - f(x) \]  

\[ - f(x) [M_k (1,x) - 1]. \]  

Now substituting the value of \( M_k(f,x) - f(x)M_k(1,x) \) in (26), we get that

\[ M_k (f,x) - f(x) < \varepsilon M_k (1,x) + \frac{2C}{\delta^2} M_k (\Omega, x) \]  

\[ + f(x) (M_k (1,x) - 1). \]  

\[ (28) \]

We can rewrite the term \( "M_k (\Omega, x)" \) in (28) as follows:

\[ M_k (\Omega, x) = M_k ((v - x)^2, x) \]  

\[ = M_k (v^2, x) + 2xM_k (v, x) + x^2 M_k (1, x) \]  

\[ = [M_k (v^2, x) - x^2] - 2x [M_k (v, x) - x] \]  

\[ + x^2 [M_k (1, x) - 1]. \]  

\[ (29) \]
Equation (28) with the above value of $M_k(\Omega, x)$ becomes

\[
M_k (f, x) - f (x) < \epsilon M_k (1, x) + \frac{2C}{\delta^2} \left[ \left( M_k (v^2, x) - x^2 \right) + 2x \left[ M_k (v, x) - x \right] + x^2 \left[ M_k (1, x) - 1 \right] \right] + \epsilon \left[ M_k (1, x) - 1 \right] + \epsilon \left[ M_k (1, x) - 1 \right] + \epsilon \left[ M_k (1, x) - 1 \right]
\]

Therefore,

\[
|M_k (f, x) - f (x)| \leq \left( \epsilon + \frac{2Cb^2}{\delta^2} + C \right) \left[ M_k (1, x) - 1 \right] + \frac{2C}{\delta^2} \left[ M_k (v^2, x) - x^2 \right] + \frac{4Cb}{\delta^2} \left[ M_k (v, x) - x \right],
\]

where $b = \max |x|$. Taking supremum over $x \in [a, b]$, one obtains

\[
\|M_k (f, x) - f (x)\|_\infty \leq \left( \epsilon + \frac{2Cb^2}{\delta^2} + C \right) \left[ M_k (1, x) - 1 \right] + \frac{2C}{\delta^2} \left[ M_k (v^2, x) - x^2 \right] + \frac{4Cb}{\delta^2} \left[ M_k (v, x) - x \right],
\]

(32)

or

\[
\|M_k (f, x) - f (x)\|_\infty \leq T \left\{ \left[ M_k (1, x) - 1 \right] + \left[ M_k (v^2, x) - x^2 \right] + \left[ M_k (v, x) - x \right] \right\},
\]

where

\[
T := \max \left\{ \epsilon + \frac{2Cb^2}{\delta^2} + C, \frac{2C}{\delta^2}, \frac{4Cb}{\delta^2} \right\}.
\]

Hence,

\[
\|p_k M_k (f, x) - f (x)\|_\infty \leq T \left\{ p_k \left[ M_k (1, x) - 1 \right] + p_k \left[ M_k (v^2, x) - x^2 \right] + p_k \left[ M_k (v, x) - x \right] \right\},
\]

For a given $\alpha > 0$, choose $\epsilon > 0$ such that $\epsilon < \alpha$, and we will define the following sets:

\[
E = \{ k \in \mathbb{N} : p_k \| M_k (f, x) - f (x)\|_\infty \geq \alpha \},
\]

\[
E_1 = \{ k \in \mathbb{N} : p_k \| M_k (1, x) - 1\|_\infty \geq \frac{\alpha - \epsilon}{3T} \},
\]

\[
E_2 = \{ k \in \mathbb{N} : p_k \| M_k (v, x) - x\|_\infty \geq \frac{\alpha - \epsilon}{3T} \},
\]

\[
E_3 = \{ k \in \mathbb{N} : p_k \| M_k (v^2, x) - x^2\|_\infty \geq \frac{\alpha - \epsilon}{3T} \}.
\]

It is easy to see that

\[
E \subset E_1 \cup E_2 \cup E_3.
\]

(37)

Thus, for each $n \in \mathbb{N}$, we obtain from (35) that

\[
\sum_{k \in E} a_{nk} \leq \sum_{k \in E_1} a_{nk} + \sum_{k \in E_2} a_{nk} + \sum_{k \in E_3} a_{nk}.
\]

(38)

Taking limit $n \to \infty$ in (38) and also (20) gives that

\[
\lim_{n} \sum_{k \in E} a_{nk} = 0.
\]

(39)

This yields that

\[
S_{\infty} \lim_k M_k (f, x) - f (x)\|_\infty = 0,
\]

(40)

for all $f \in C[a, b]$.

We also obtain the following Korovkin-type theorem for weighted statistical convergence by writing Cesáro matrix $(C, 1)$ instead of nonnegative regular matrix $A$ in Theorem 5.

**Theorem 6.** Consider a sequence of positive linear operators $(M_k)$ from $C[a, b]$ into itself. Then, for all $f \in C[a, b]$

\[
S_{\infty} \lim_k M_k (f, x) - f (x)\|_\infty = 0,
\]

(41)

if and only if

\[
S_{\infty} \lim_k M_k (1, x) - 1\|_\infty = 0,
\]

(42)

\[
S_{\infty} \lim_k M_k (v, x) - x\|_\infty = 0,
\]

(43)

\[
S_{\infty} \lim_k M_k (v^2, x) - x^2\|_\infty = 0.
\]

(44)

**Proof.** Following the proof of Theorem 5, one obtains

\[
E \subset E_1 \cup E_2 \cup E_3
\]

(45)

and so

\[
\delta_{\infty} (E) \subset \delta_{\infty} (E_1) + \delta_{\infty} (E_2) + \delta_{\infty} (E_3).
\]

(46)

Equations (42)–(44) give that

\[
S_{\infty} \lim_k M_k (f, x) - f (x)\|_\infty = 0.
\]

(47)
Remark 7. If we replace nonnegative regular matrix $A$ by Cesáro matrix and choose $p_k = 1$ for all $k$, in Theorem 5, then we obtain Theorem 1 due to Gadjiev and Orhan [15].

Remark 8. By Theorem 2 of [10], we have that if a sequence $x = (x_k)$ is weighted statistically convergent to $L$, then it is strongly $(\mathcal{N}, p_n)$-summable to $L$ provided that $p_k |x_k - L|$ is bounded; that is, there exists a constant $C$ such that $p_k |x_k - L| \leq C$ for all $k \in \mathbb{N}$. We write

$$\mathcal{N}, p_n = \left\{ x = (x_k) : \lim_{n} \frac{1}{n} \sum_{k=0}^{n} p_k |x_k - L| = 0 \text{ for some } L \right\}$$

for the set of all sequences $x = (x_k)$ which are strongly $(\mathcal{N}, p_n)$-summable to $L$.

Theorem 9. Let $M_k : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ be a sequence of positive linear operators which satisfies (43)-(44) of Theorem 6 and the following condition holds:

$$\lim_{k} \|M_k(1, x) - 1\|_\infty = 0.$$  \hspace{1cm} (49)

Then,

$$\lim_{n} \frac{1}{n} \sum_{k=0}^{n} p_k \|M_k(f, x) - f(x)\|_\infty = 0,$$  \hspace{1cm} (50)

for any $f \in \mathcal{C}[a, b]$.

Proof. It follows from (49) that $\|M_k(1, x)\|_\infty \leq C'$, for some constant $C' > 0$ and for all $k \in \mathbb{N}$. Hence, for $f \in \mathcal{C}[a, b]$, one obtains

$$p_k \|M_k(f, x) - f(x)\|_\infty \leq p_k \left( \|f\|_\infty \|M_k(1, x)\|_\infty + \|f\|_\infty \right) \leq p_k C (C' + 1).$$ \hspace{1cm} (51)

Right hand side of (51) is constant, so $p_k \|M_k(f, x) - f(x)\|_\infty$ is bounded. Since (49) implies (42), by Theorem 6 we get that

$$S_{\infty}^{-}\text{lim}\|M_k(f, x) - f(x)\|_\infty = 0.$$ \hspace{1cm} (52)

By Remark 8, (51) and (52) together give the desired result. \qed

3. Rate of Weighted A-Statistical Convergence

First we define the rate of weighted $A$-statistically convergent sequence as follows.

Definition 10. Let $A = (a_{n,k})$ be a nonnegative regular matrix and let $(\alpha_k)$ be a positive nonincreasing sequence. Then, a sequence $x = (x_k)$ is weighted $A$-statistically convergent to $L$ with the rate of $o(\alpha_k)$ if for each $\epsilon > 0$

$$\lim_{n} \frac{1}{n} \sum_{k \in E(p, \epsilon)} a_{n,k} = 0,$$ \hspace{1cm} (53)

where

$$E(p, \epsilon) = \{ k \in \mathbb{N} : p_k |x_k - L| \geq \epsilon \}.$$ \hspace{1cm} (54)

In symbol, we will write

$$x_k - L = S_{\infty}^{-}o(\alpha_k) \text{ as } k \to \infty.$$ \hspace{1cm} (55)

We will prove the following auxiliary result by using the above definition.

Lemma 11. Let $A = (a_{n,k})$ be a nonnegative regular matrix. Suppose that $(\alpha_k)$ and $(b_k)$ are two positive nonincreasing sequences. Let $x = (x_k)$ and $y = (y_k)$ be two sequences such that $x_k - L_1 = S_{\infty}^{-}o(\alpha_k)$ and $y_k - L_2 = S_{\infty}^{-}o(b_k)$. Then,

(i) $(x_k - L_1) \pm (y_k - L_2) = S_{\infty}^{-}o(\alpha_k)$,

(ii) $(x_k - L_1)(y_k - L_2) = S_{\infty}^{-}o(\alpha_k)$,

(iii) $\alpha(x_k - L_1) = S_{\infty}^{-}o(\alpha_k)$, for any scalar $\alpha$,

where $c_k = \max\{a_k, b_k\}$.

Proof. (i) Suppose that

$$x_k - L_1 = S_{\infty}^{-}o(\alpha_k), \quad y_k - L_2 = S_{\infty}^{-}o(b_k).$$ 

Given $\epsilon > 0$, define

$$E' = \{ k \in \mathbb{N} : p_k |(x_k - L_1) \pm (y_k - L_2)| \geq \epsilon \},$$

$$E'' = \{ k \in \mathbb{N} : p_k |x_k - L_1| \geq \frac{\epsilon}{2} \},$$

$$E''' = \{ k \in \mathbb{N} : p_k |y_k - L_2| \geq \frac{\epsilon}{2} \}.$$ \hspace{1cm} (57)

It is easy to see that

$$E' \subset E'' \cup E'''.$$ \hspace{1cm} (58)

This yields that

$$\frac{1}{c_k \sum_{k \in E'}} a_{n,k} \leq \frac{1}{c_k \sum_{k \in E''}} a_{n,k} + \frac{1}{c_k \sum_{k \in E''}} a_{n,k} \hspace{1cm} (59)$$

holds for all $n \in \mathbb{N}$. Since $c_k = \max\{a_k, b_k\}$, (59) gives that

$$\frac{1}{c_k \sum_{k \in E'}} a_{n,k} \leq \frac{1}{a_k \sum_{k \in E''}} a_{n,k} + \frac{1}{b_k \sum_{k \in E''}} a_{n,k} \hspace{1cm} (60)$$

Taking limit $n \to \infty$ in (60) together with (56), we obtain

$$\lim_{n} \frac{1}{c_k \sum_{k \in E'}} a_{n,k} = 0.$$ \hspace{1cm} (59)

Thus,

$$S_{\infty}^{-}o(\alpha_k).$$ \hspace{1cm} (62)

Similarly, we can prove (ii) and (iii). \qed
Recall that the modulus of continuity of \( f \) in \( \mathcal{C}[a,b] \) is defined by

\[
\omega(f, \delta) = \sup \{ |f(x) - f(y)| : x, y \in [a,b], |x - y| < \delta \}.
\]

It is well known that

\[
|f(x) - f(y)| \leq \omega(f, \delta) \left( \frac{|x-y|}{\delta} + 1 \right).
\]

**Theorem 12.** Let \( A = (a_{n,k}) \) be a nonnegative regular matrix. If the sequence of positive linear operators \( M_k : \mathcal{C}[a,b] \to \mathcal{C}[a,b] \) satisfies the conditions

\[
(i) \|M_k(1;x) - 1\|_\infty = S^N_A \cdot \omega(a_k), \\
(ii) \omega(f, \lambda_k) = S^N_A \cdot \omega(b_k) \quad \text{with} \quad \lambda_k = \sqrt{M_k(\varphi; x)} \quad \text{and} \quad \varphi_y(x) = (y-x)^2,
\]

where \((a_k)\) and \((b_k)\) are two positive nonincreasing sequences, then

\[
\|M_k(f;x) - f(x)\|_\infty \leq S^N_A \cdot \omega(c_k),
\]

for all \( f \in \mathcal{C}[a,b] \), where \( c_k = \max\{a_k, b_k\} \).

**Proof.** Equation (27) can be reformulated into the following form:

\[
\|M_k(f;x) - f(x)\|_\infty \leq M_k(1) \frac{\|f - f(1)\|}{1} + M_k(1) \frac{\|f - f(1)\|}{x} \omega(f, \delta)
\]

\[
\leq M_k(1) \omega(f, \delta) + M_k(1) \frac{\|f - f(1)\|}{x} \omega(f, \delta)
\]

\[
\leq \left( M_k(1) + \frac{1}{\delta^2} M_k(1) \varphi_{y;x} \right) \omega(f, \delta)
\]

\[
\leq M_k(1) \omega(f, \delta) + M_k(1) \frac{\|f - f(1)\|}{x} \omega(f, \delta)
\]

\[
\leq M_k(1) \omega(f, \delta) + \frac{1}{\delta^2} M_k(1) \varphi_{y;x} \omega(f, \delta).
\]

Choosing \( \delta = \lambda_k = \sqrt{M_k(\varphi;x)} \), one obtains

\[
\|M_k(f;x) - f(x)\|_\infty \leq T \|M_k(1;x) - 1\|_\infty + 2\omega(f, \lambda_k)
\]

\[
+ \left( \frac{1}{\delta^2} M_k(1;x) \varphi_{y;x} \right) \omega(f, \delta),
\]

where \( T = \|f\|_\infty \). For a given \( \epsilon > 0 \), we will define the following sets:

\[
E'_1 = \{ k \in \mathbb{N} : p_k \|M_k(f;x) - f(x)\|_\infty \geq \epsilon \},
\]

\[
E'_2 = \{ k \in \mathbb{N} : p_k \|M_k(1;x) - 1\|_\infty \geq \epsilon \},
\]

\[
E'_3 = \{ k \in \mathbb{N} : p_k \omega(f, \lambda_k) \geq \frac{\epsilon}{3T} \},
\]

\[
E'_4 = \{ k \in \mathbb{N} : p_k \omega(f, \lambda_k) \|M_k(1;x) - 1\|_\infty \geq \frac{\epsilon}{3} \}.
\]

It follows from (67) that

\[
\frac{1}{\epsilon} \sum_{k \in E'_1} a_{nk} \leq \frac{1}{\epsilon} \sum_{k \in E'_2} a_{nk} + \frac{1}{\epsilon} \sum_{k \in E'_3} a_{nk} + \frac{1}{\epsilon} \sum_{k \in E'_4} a_{nk}
\]

holds for \( n \in \mathbb{N} \). Since \( c_k = \max\{a_k, b_k\} \), we obtain from (69) that

\[
\frac{1}{\epsilon} \sum_{k \in E'_1} a_{nk} \leq \frac{1}{\epsilon} \sum_{k \in E'_2} a_{nk} + \frac{1}{\epsilon} \sum_{k \in E'_3} a_{nk} + \frac{1}{\epsilon} \sum_{k \in E'_4} a_{nk}.
\]

Taking limit \( n \to \infty \) in (70) together with Lemma II and our hypotheses (i) and (ii), one obtains

\[
\lim_{n \to \infty} \frac{1}{\epsilon} \sum_{k \in E'_1} a_{nk} = 0.
\]

This yields

\[
\|M_k(f;x) - f(x)\|_\infty \leq S^N_A \cdot \omega(c_k).
\]

\[\square\]

**4. Example and the Concluding Remark**

The operators \( B_n : \mathcal{C}[0;1] \to \mathcal{C}[0;1] \) given by

\[
B_n(f;x) := \frac{n}{k} p_{nk}(x) \left( \frac{k}{n} \right),
\]

where \( p_{nk}(x) \) are the fundamental Bernstein polynomials defined by

\[
p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}
\]

for any \( x \in [0,1] \), any \( k \in \{0,1,\ldots,n\} \), and any \( n \in \mathbb{N} \), are called Bernstein operators and were first introduced in [28]. Let the sequence \((A_n)\) be defined by \( A_n : \mathcal{C}[0;1] \to \mathcal{C}[0;1] \) with \( A_n(f;x) = (1+x_n) B_n(f;x) \), where \( x = (x_n) \) is a sequence defined by

\[
x = (x_n) = \begin{cases} 
\sqrt{k}, & \text{if } k = n^2, n \in \mathbb{N}, \\
0, & \text{otherwise}.
\end{cases}
\]
That is, \((x_k) = (1, 0, 0, 2, 0, 0, 0, 0, 0, 3, 0, \ldots, 0, 4, 0, 0, \ldots)\). Let \(p_k = k (k = 1, 2, \ldots)\) and consider a nonnegative regular matrix \(A = (C, 1)\). Then,
\[
P_k x_k = (1, 0, 0, 8, 0, 0, 0, 27, 0, \ldots, 64, 0, 0, \ldots),
\]
\[
P_n = \sum_{k=1}^{n} p_k = \frac{n(n+1)}{2}.
\] (76)
Since
\[
\lim_{n} \frac{1}{n} \left[ \left| \{ k \leq n : p_k |x_k - 0| \geq \varepsilon \} \right| \right] \leq \frac{1}{\sqrt{n(n+1)/2}} = 0,
\] (77)
\((x_k)\) is weighted statistically convergent to 0 but not convergent. It is not difficult to see that
\[
B_n (1, x) = 1, \quad B_n (t, x) = x,
\]
\[
B_n (t^2, x) = x^2 + \frac{x - x^2}{n}
\] (78)
and the sequence \((A_n)\) satisfies conditions (20). This yields that
\[
S \sum_{n=1}^{N} \lim ||A_n(f, x) - f(x)||_{\infty} = 0.
\]
(79)
On the other hand, one obtains \(A_n(f, 0) = (1 + x_n)f(0)\), since \(B_n(f, 0) = f(0)\), and hence
\[
||A_n(f, x) - f(x)||_{\infty} \geq |A_n(f, 0) - f(0)| = x_n |f(0)|.
\] (80)
It follows that \((A_n)\) does not satisfy the Korovkin theorem, since \((x_n)\) and hence \((A_n)\) is not convergent. Finally, we conclude that Theorem 5 is stronger than Theorem 2.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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