REALIZATION OF THE FRACTIONAL LAPLACIAN WITH NONLOCAL EXTERIOR CONDITIONS VIA FORMS METHOD

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Abstract. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded open set with a Lipschitz continuous boundary. In the first part of the paper, using the method of bilinear forms, we give a rigorous characterization of the realization in $L^2(\Omega)$ of the fractional Laplace operator $(-\Delta)^s$ ($0 < s < 1$) with the nonlocal Neumann and Robin exterior conditions. Contrarily to the classical local case $s = 1$, it turns out that the nonlocal (Robin and Neumann) exterior conditions are incorporated in the form domain. We show that each of the above operators generates a strongly continuous submarkovian semigroup which is also ultracontractive. In the second part, we show that the semigroup corresponding to the nonlocal Robin exterior condition is always sandwiched between the fractional Dirichlet semigroup and the fractional Neumann semigroup.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded open set with a Lipschitz continuous boundary $\partial \Omega$. The aim of the present paper is to give a rigorous characterization of the realization in $L^2(\Omega)$ of the fractional Laplace operator $(-\Delta)^s$ ($0 < s < 1$) with the nonlocal Neumann and Robin exterior conditions by using the method of bilinear forms. Here, the operator $(-\Delta)^s$ is given formally by the following singular integral:

$$(-\Delta)^s u(x) := C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n,$$

where $C_{n,s}$ is a normalization constant depending on $n$ and $s$ only. We refer to Section 2 for a rigorous definition of $(-\Delta)^s$ and the class of functions for which the singular integral exists.

It is nowadays well-known that the realizations in $L^2(\Omega)$ of the Laplace operator $(-\Delta)$ with the Neumann boundary condition, $\partial_\nu u = 0$ on $\partial \Omega$, and the Robin boundary condition, $\partial_\nu u + \gamma u = 0$ on $\partial \Omega$, are the selfadjoint operators on $L^2(\Omega)$ associated with the closed bilinear forms

$$a^N(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in D(a^N) = W^{1,2}(\Omega), \quad (1.1)$$

and

$$a^R(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} \gamma uv \, d\sigma, \quad u, v \in D(a^R) = W^{1,2}(\Omega), \quad (1.2)$$

respectively. Here, $\gamma \in L^\infty(\partial \Omega)$ is a non-negative given function. We refer to [2, 3] and the references therein for more details on this topic.

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The situation is more delicate and challenging in the case of \((-\Delta)^s\). To be more precise, we have the following difficulties.

- Firstly, let \(u\) be a given function defined on \(\Omega\), since \((-\Delta)^s\) is a nonlocal operator, in order to know \((-\Delta)^su\) in \(\Omega\), it is necessary to know \(u\) outside \(\Omega\), that is, in \(\mathbb{R}^n \setminus \Omega\). Under some conditions, functions in some Sobolev spaces defined in \(\Omega\) can be extended to all \(\mathbb{R}^n\), but if the extension is not unique, one would not have a well-posed problem, since \((-\Delta)^s\) of the extended function will not be independent of the extension.

- Secondly, since we would like to define an operator which is a realization in \(L^2(\Omega)\), then we must start with a function \(u\) defined in \(\Omega\). For this operator to be a realization of \((-\Delta)^s\), then we must find a suitable unique extension \(\tilde{u}\) of \(u\) to all \(\mathbb{R}^n\) so that \((-\Delta)^s\tilde{u}\) is unique and well-defined.

- Thirdly, for elliptic problems associated with \((-\Delta)^s\) to be well-posed, the condition must not be prescribed on the boundary \(\partial\Omega\), but instead in \(\mathbb{R}^n \setminus \Omega\). We shall call such condition an exterior condition. This shows that the condition must be given in term of the extension \(\tilde{u}\) instead of \(u\).

- Finally, it turns out that the operator playing the role for \((-\Delta)^s\) that the normal derivative does for \(\Delta\) is also a nonlocal operator. Therefore, we have to deal with a double non-locality.

For functions \(u, v \in W_{\Omega}^{s,2}\) (see Section 2 for the definition and more details on this space), we let

\[
E(u, v) := \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dy \, dx \\
+ C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dy \, dx.
\]

In the present paper, we have obtained the following specific results.

(i) Firstly, for a function \(u \in L^2(\Omega)\), we define its extension \(u_N\) to all \(\mathbb{R}^n\) as follows:

\[
u_N(x) = \begin{cases} 
  u(x) & \text{if } x \in \Omega, \\
  \frac{1}{\rho(x)} \int_\Omega \frac{u(y)}{|x - y|^{n+2s}} \, dy & \text{if } x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where the function \(\rho\) is given by

\[
\rho(x) = \int_\Omega \frac{1}{|x - y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n \setminus \Omega.
\]

Our first main result (Theorem 3.8) shows that the realization in \(L^2(\Omega)\) of \((-\Delta)^s\) with the nonlocal Neumann exterior condition is the selfadjoint operator \(A_N\) associated with the closed, symmetric and densely defined bilinear form \(a_N : D(a_N) \times D(a_N) \to \mathbb{R}\) given by

\[
D(a_N) = \{ u \in L^2(\Omega), u_N \in W_{\Omega}^{s,2} \} \quad \text{and} \quad a_N(u, v) = E(u_N, v_N).
\]

The nonlocal Neumann exterior condition is characterized by

\[
\mathcal{N}^s u_N = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega},
\]

where the operator \(\mathcal{N}^s\) is defined for a function \(v \in W_{\Omega}^{s,2}\) by
\[ N^s v(x) = C_{n,s} \int_{\Omega} \frac{v(x) - v(y)}{|x - y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}. \]

We show that the extension \( u_N \) satisfies the exterior condition (1.3), so that, contrarily to the local case \((s = 1)\) where the Neumann boundary condition did not appear in \( D(a^N) \) (see (1.1)), for the fractional case, the nonlocal Neumann exterior condition is incorporated in the form domain \( D(a_N) \). We also prove that \( -A_N \) generates a submarkovian semigroup \( T_N \) on \( L^2(\Omega) \) which is also ultracontractive in the sense that it maps \( L^1(\Omega) \) into \( L^\infty(\Omega) \).

(ii) Our second main result (Theorem 3.15) concerns the nonlocal Robin exterior condition. For this, let \( \beta \in L^\infty(\mathbb{R}^n \setminus \Omega) \) be a non-negative given function. For a function \( u \in L^2(\Omega) \), we define its extension \( u_R \) as follows:

\[
 u_R(x) = \begin{cases} 
 u(x) & \text{if } x \in \Omega, \\
 C_{n,s} \rho(x) + \beta(x) \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} \, dy & \text{if } x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

Then the realization in \( L^2(\Omega) \) of \((\Delta)^s\) with the nonlocal Robin exterior condition is the selfadjoint operator \( A_R \) associated with the closed, symmetric and densely defined bilinear form \( a_R : D(a_R) \times D(a_R) \to \mathbb{R} \) given by

\[
 D(a_R) = \left\{ u \in L^2(\Omega), \ u_R \in W^{s,2}_\Omega, \ \int_{\mathbb{R}^n \setminus \Omega} \beta(x) u_R^2(x) \, dx < \infty \right\} \quad \text{and} \quad a_R(u,v) = \mathcal{E}(u_R,v_R).
\]

The nonlocal Robin exterior condition is characterized by

\[ N^s u_R + \beta u_R = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}. \quad (1.4) \]

Here also, we have that the extension \( u_R \) satisfies the exterior condition (1.4), so that the nonlocal Robin exterior condition is incorporated in the form domain \( D(a_R) \) which is not the case for \( D(a^R) \) (see (1.2)). We prove that \( -A_R \) generates a submarkovian semigroup \( T_R \) on \( L^2(\Omega) \) which is also ultracontractive.

(iii) Our third main result (Theorem 4.3) shows that the semigroup \( T_R \) is always sandwiched between the semigroup \( T_D \) on \( L^2(\Omega) \) generated by the realization of \((\Delta)^s\) in \( L^2(\Omega) \) with zero Dirichlet exterior condition \( \bar{u} = 0 \) in \( \mathbb{R}^n \setminus \Omega \) and the semigroup \( T_N \). That is, we have

\[ 0 \leq T_D \leq T_R \leq T_N, \]

in the sense of (2.11) below.

We mention that the case of the regional fractional Laplacian \((\Delta)^s_\Omega \), defined formally by

\[ (\Delta)^s_\Omega u(x) := C_{n,s} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \quad x \in \Omega, \]

which is more similar to the classical local case, has been investigated in [18, 31] and their references. For this case, the associated normal derivative is a local operator.

Another novelty of the present paper is that contrarily to the local case \( s = 1 \), or the regional fractional Laplace case, where the proofs of the submarkovian property and the domination of the
semigroups are standard, for the fractional case $0 < s < 1$ investigated here, the proofs of the above mentioned results required a careful analysis of the associated bilinear forms.

Fractional order operators (in particular the fractional Laplacian) have recently emerged as a modeling alternative in various branches of science. They usually describe anomalous diffusion. A number of stochastic models for explaining anomalous diffusion have been introduced in the literature; among them we quote the fractional Brownian motion; the continuous time random walk; the Lévy flights; the Schneider gray Brownian motion; and more generally, random walk models based on evolution equations of single and distributed fractional order in space (see e.g. [14, 20, 25, 30]). In general, a fractional diffusion operator corresponds to a diverging jump length variance in the random walk. In the literature, the fractional Laplace operator is known as the $s$-stable Lévy process.

The rest of the paper is structured as follows. In Section 2 we introduce the function spaces needed to study our problem and we recall some well-known results on Dirichlet forms and domination of semigroups that are needed throughout the paper. In Section 3 we give a rigorous characterization of the realizations in $L^2(\Omega)$ of $(-\Delta)^s$ with the three exterior conditions (Dirichlet, Neumann and Robin). We show that each of these operators generates a submarkovian semigroup which is also ultracontractive. The result concerning the domination of the semigroups is contained in Section 4. We conclude the paper by given some open problems in Section 5.

2. FUNCTIONAL SETUP AND PRELIMINARIES

Here we introduce the function spaces needed to investigate our problem, give a rigorous definition of $(-\Delta)^s$, and recall some known results on semigroups theory.

2.1. Fractional order Sobolev spaces and the fractional Laplacian. Unless otherwise stated, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is an arbitrary bounded open set and $0 < s < 1$. Let

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy < \infty \right\},$$

be endowed with the norm

$$\|u\|_{W^{s,2}(\Omega)} := \left( \int_{\Omega} |u|^2 \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}},$$

and set

$$W^{s,2}_0(\Omega) = \mathcal{D}(\Omega)^{W^{s,2}(\Omega)},$$

where $\mathcal{D}(\Omega)$ denotes the space of all continuously infinitely differentiable functions with compact support in $\Omega$.

We define

$$W^{s,2}_0(\overline{\Omega}) := \left\{ u \in W^{s,2}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\},$$

and we set

$$\widetilde{W}^{s,2}_0(\overline{\Omega}) = \left\{ u|_{\Omega} : u \in W^{s,2}_0(\overline{\Omega}) \right\}.$$

We also define the local fractional order Sobolev space

$$W^{s,2}_{\text{loc}}(\mathbb{R}^n \setminus \overline{\Omega}) := \left\{ u \in L^2(\mathbb{R}^n \setminus \overline{\Omega}) : u \varphi \in W^{s,2}(\mathbb{R}^n \setminus \overline{\Omega}), \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \overline{\Omega}) \right\}. \quad (2.1)$$
Remark 2.1. It is well-known that we have the following continuous embeddings:

\[
W^{s,2}_0(\Omega), \widetilde{W}^{s,2}_0(\Omega) \rightarrow \begin{cases} 
L^{\frac{2n}{n-2s}}(\Omega) & \text{if } n > 2s, \\
L^p(\Omega) & \forall p \in [1, \infty) \text{ if } n = 2s, \\
C^{0,1-\frac{n}{2s}}(\overline{\Omega}) & \text{if } n < 2s.
\end{cases}
\]  

(2.2)

In addition to (2.2), the embedding \(W^{s,2}_0(\Omega), \widetilde{W}^{s,2}_0(\Omega) \hookrightarrow L^2(\Omega)\) is compact. If \(\Omega\) has a Lipschitz continuous boundary, then (2.2) also holds with \(W^{s,2}_0(\Omega), \widetilde{W}^{s,2}_0(\Omega)\) replaced with \(W^{s,2}(\Omega)\). We refer to [21, Chapter 1] and [11] for the proof of the above results.

We have the following result.

Lemma 2.2. The following assertions hold.

(a) \(\mathcal{D}(\Omega) \subset \widetilde{W}^{s,2}_0(\Omega)\), and if \(\Omega\) has a continuous boundary, then \(\mathcal{D}(\Omega)\) is dense in \(\widetilde{W}^{s,2}_0(\Omega)\).

(b) If \(\Omega\) has a Lipschitz continuous boundary, then \(\widetilde{W}^{s,2}_0(\Omega) = W^{s,2}(\Omega)\) for \(0 < s < 1\), \(s \neq \frac{1}{2}\).

Proof. The proofs of (a) and (b) are contained in [21, Theorem 1.4.2.2] (see also [16]) and [21, Corollary 1.4.4.5], respectively. □

For more information on fractional order Sobolev spaces we refer to [11, 21, 31].

Next, we define the fractional order Sobolev type space \(W^{s,2}_\Omega := \{u : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable, } \|u\|_{W^{s,2}_\Omega} < \infty\}\), where

\[
\|u\|_{W^{s,2}_\Omega} := \left( \int_{\Omega} |u|^2 \, dx + \int_{\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \Omega)^2 \cup \{y \in \mathbb{R}^n, |y-x| > \epsilon\})} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}},
\]  

(2.3)

and

\[
\mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \Omega)^2 \cup \{y \in \mathbb{R}^n, |y-x| > \epsilon\}) := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^n \setminus \Omega)) \cup ((\mathbb{R}^n \setminus \Omega) \times \Omega).
\]

The space \(W^{s,2}_\Omega\) has been introduced in [12] to study the Neumann problem for \((-\Delta)^s\) (see (3.9)). It also appears in a more general form in [15] and has been used to study the Dirichlet problem for \((-\Delta)^s\) (see (3.3)).

The following result has been proved in [12, Proposition 3.1].

Lemma 2.3. \(W^{s,2}_\Omega\) endowed with the norm (2.3) is a Hilbert space.

To introduce the fractional Laplace operator, we set

\[
L^1_s(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable, } \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}} \, dx < \infty \right\}.
\]

For \(u \in L^1_s(\mathbb{R}^n)\) and \(\epsilon > 0\), we let

\[
(-\Delta)^s u(x) = C_{n,s} \int_{\{y \in \mathbb{R}^n, |y-x| > \epsilon\}} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n,
\]

where the normalization constant \(C_{n,s}\) is given by

\[
C_{n,s} := \frac{s^{2s} \Gamma \left( \frac{2s+n}{2} \right)}{\pi^{\frac{n}{2}} \Gamma(1-s)},
\]  

(2.4)
and $\Gamma$ is the usual Euler Gamma function. The fractional Laplacian $(-\Delta)^s$ is defined for $u \in L^1_s(\mathbb{R}^n)$ by the formula:

$$(-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_{\varepsilon} u(x), \quad x \in \mathbb{R}^n, \quad (2.5)$$

provided that the limit exists. We have that $L^1_s(\mathbb{R}^n)$ is the right space for which $v := (-\Delta)^s_{\varepsilon} u$ exists for every $\varepsilon > 0$, and $v$ being also continuous at the continuity points of $u$. For more details on the fractional Laplace operator, we refer to [4, 6, 7, 8, 11, 31] and their references.

Next, for $u \in W^{s,2}_\Omega$ we define the nonlocal normal derivative $N^s u$ of $u$ as follows:

$$N^s u(x) := C_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}. \quad (2.6)$$

Clearly, $N^s$ is a nonlocal operator and is well defined on $W^{s,2}_\Omega$ as shows the following result contained in [19, Lemma 3.2].

**Lemma 2.4.** The nonlocal normal derivative $N^s$ maps continuously $W^{s,2}_\Omega$ into $W^{s,2}_{\text{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$.

Despite the fact that $N^s$ is defined on $\mathbb{R}^n \setminus \overline{\Omega}$, it is still known as the normal derivative. This is due to its similarity with the classical normal derivative as shows the following result.

**Proposition 2.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz continuous boundary. Then the following assertions hold.

1. **The divergence theorem:** Let $u \in C^2_0(\mathbb{R}^n)$ ($u \in C^2(\mathbb{R}^n)$, $\lim_{|x| \to \infty} u(x) = 0$). Then

$$\int_{\Omega} (-\Delta)^s u \, dx = - \int_{\mathbb{R}^n \setminus \Omega} N^s u \, dx.$$

2. **The integration by parts formula:** Let $u \in W^{s,2}_\Omega$ be such that $(-\Delta)^s u \in L^2(\Omega)$. Then for every $v \in W^{s,2}_\Omega$ we have

$$\int_{\Omega} v(-\Delta)^s u \, dx = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx dy - \int_{\mathbb{R}^n \setminus \Omega} vN^s u \, dx. \quad (2.7)$$

3. **The limit as $s \uparrow 1^-$:** Let $u, v \in C^2_0(\mathbb{R}^n)$. Then

$$\lim_{s \uparrow 1^-} \int_{\mathbb{R}^n \setminus \Omega} vN^s u \, dx = \int_{\partial \Omega} v \partial_\nu u \, d\sigma.$$

**Proof.** The proofs of (a) and (c) are contained in [12, Lemma 3.2] and [12, Proposition 5.1], respectively. The proof of (b) for smooth functions can be found in [12, Lemma 3.3]. The version given here is obtained by using a density argument (see e.g. [32, Proposition 3.7]).

**Remark 2.6.** Comparing the properties (a)-(c) in Proposition 2.5 with the classical properties of the Laplacian $\Delta$, we can immediately deduce that $N^s$ plays the same role for $(-\Delta)^s$ that the classical normal derivative $\partial_\nu$ does for $\Delta$. For this reason, we call $N^s$ the nonlocal normal derivative. The name interaction operator has been also used for $N^s$ in [1, 13].
2.2. Dirichlet forms and domination of semigroups. Let $X$ be a (relatively) compact separable metric space. Let $m$ be a Radon measure on $X$ and assume that $\text{supp} |m| = X$. Notice that under our assumptions we have that $m(X) < \infty$.

We recall the following notion of energy forms, cf. \cite[Chapter 1]{17} (see also \cite[Chapter 1]{10}).

**Definition 2.7.** The form $(a, D(a))$ is said to be a Dirichlet form if the following conditions hold:

(a) $a : D(a) \times D(a) \to \mathbb{R}$, where $D(a)$ is a dense linear subspace of $L^2(X) := L^2(X, m)$.

(b) $a(\cdot, \cdot)$ is a symmetric, non-negative bilinear form.

(c) Let $\lambda > 0$ and define $a_\lambda (u, v) = a(u, v) + \lambda (u, v)_{L^2(X)}$, for $u, v \in D(a)$. The form $a$ is said to be closed, if $\{u_k\}_{k \geq 1} \subset D(a)$ with

$$a_\lambda (u_k - u_m, u_k - u_m) \to 0 \text{ as } k, m \to \infty,$$

then there exists $u \in D(a)$ such that

$$a_\lambda (u_k - u, u_k - u) \to 0 \text{ as } k \to \infty.$$

(d) For each $\epsilon > 0$, there exists a function $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that $\phi_\epsilon \in C^\infty(\mathbb{R})$, $\phi_\epsilon(t) = t$, for $t \in [0, 1]$, $-\epsilon \leq \phi_\epsilon(t) \leq 1 + \epsilon$, for all $t \in \mathbb{R}$, $0 \leq \phi_\epsilon(t) - \phi_\epsilon(\tau) \leq t - \tau$, whenever $\tau < t$, such that $u \in D(a)$ implies $\phi_\epsilon(u) \in D(a)$ and $a(\phi_\epsilon(u), \phi_\epsilon(u)) \leq a(u, u)$.

There is a one-to-one correspondence between the family of closed, symmetric, densely defined forms $(a, D(a))$ on $L^2(X)$ and the family of non-negative (definite) selfadjoint operators $A$ on $L^2(X)$ defined by

$$D(A) = \left\{ u \in D(a), \exists f \in L^2(X), a(u, v) = (f, v)_{L^2(X)} \quad \forall v \in D(a) \right\},$$

$$Au = f.$$  \hfill (2.8)

In that case the operator $-A$ generates a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ on $L^2(X)$.

**Remark 2.8.** Assume that the form $(a, D(a))$ satisfies the conditions (a)-(c) in Definition 2.7. Then Definition 2.7(d) can be replaced by the following two conditions:

(i) $u \in D(a)$ implies $|u| \in D(a)$ and $a(|u|, |u|) \leq a(u, u)$. In that case, $(e^{-tA})_{t \geq 0}$ is said to be positivity-preserving in the sense that, $u \in L^2(X)$, $u \geq 0$ implies $e^{-tA}u \geq 0$.

(ii) $0 \leq u \in D(a)$ implies $u \wedge 1 \in D(a)$ and $a(u \wedge 1, u \wedge 1) \leq a(u, u)$. In that case, $(e^{-tA})_{t \geq 0}$ is said to be $L^\infty$-contractive in the sense that, for every $t \geq 0$ and $u \in L^2(X) \cap L^\infty(X)$,

$$\|e^{-tA}u\|_{L^\infty(X)} \leq \|u\|_{L^\infty(X)}.$$

A positivity-preserving and $L^\infty$-contractive semigroup is called submarkovian.

Any selfadjoint operator $A$, that is in one-to-one correspondence with a Dirichlet form $(a, D(a))$, turns out to possess a number of good properties provided a certain Sobolev embedding theorem holds for $D(a)$ (see, e.g. \cite[10, 17]).

**Theorem 2.9.** Let $A$ be the selfadjoint operator on $L^2(X)$ associated with a Dirichlet space $(a, D(a))$ in the sense of (2.5). Assume that $D(a) \subseteq L^2(X)$ (compact embedding) and that

$$D(a) \hookrightarrow L^{2q_a}(X), \text{ for some } q_a > 1.$$  \hfill (2.9)

Then the following assertions hold.

(a) The semigroup $(e^{-tA})_{t \geq 0}$ can be extended to a contraction semigroup on $L^p(X)$ for every $p \in [1, \infty]$, and each semigroup is strongly continuous if $p \in [1, \infty)$ and bounded analytic if $p \in (1, \infty)$. Each such semigroup on $L^p(X)$ is compact for every $p \in [1, \infty]$. 


(b) The semigroup \((e^{-tA})_{t \geq 0}\) is ultracontractive in the sense that it maps \(L^1(X)\) into \(L^\infty(X)\).
More precisely, there is a constant \(C > 0\) such that for every \(f \in L^1(X) \cap L^2(X)\), we have
\[
\|e^{-tA}f\|_{L^\infty(X)} \leq Ct^{-\frac{\beta}{\alpha - 1}}\|f\|_{L^1(X)} \quad \text{for all} \quad 0 < t \leq 1.
\]  
(2.10)

(c) The operator \(A\) has a compact resolvent. Hence, it has a discrete spectrum which is an increasing sequence of real numbers \(0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots\), that converges to \(+\infty\).

Next, let \(S\) and \(T\) be two semigroups on \(L^2(X)\) and assume that \(T\) is positivity-preserving. We shall say that \(S\) is dominated by \(T\) if
\[
|S(t)f| \leq T(t)|f|, \quad \text{for all} \quad t \geq 0 \text{ and } f \in L^2(X).
\]  
(2.11)

The following domination criterion of semigroups has been obtained in [26].

**Theorem 2.10.** Let \(S\) and \(T\) be two symmetric semigroups on \(L^2(X)\). Let \((a,D(a))\) and \((b,D(b))\) be the bilinear, symmetric and closed forms associated with \(S\) and \(T\), respectively. Assume that both semigroups are positivity-preserving. Then the following assertions are equivalent.

(i) The semigroup \(S\) is dominated by the semigroup \(T\) in the sense of (2.11).

(ii) \(\bullet\) \(D(a) \subset D(b)\), and if \(0 \leq v \leq u\) with \(u \in D(a)\) and \(v \in D(b)\), then \(v \in D(a)\). That is, \(D(a)\) is an ideal in \(D(b)\).

\(\bullet\) For all \(0 \leq u, v \in D(a)\), we have \(b(u,v) \leq a(u,v)\).

For more information on domination criteria of semigroups, we refer to [27, Chapter 2.].

3. The three exterior conditions for the fractional Laplacian

Here, we introduce the realization in \(L^2(\Omega)\) of the fractional Laplace operator with Dirichlet, and the nonlocal Neumann and Robin exterior conditions. We will also give several qualitative properties of these operators. We mention that since we shall assume that \(\Omega\) has a Lipschitz continuous boundary, then integrals over \(\mathbb{R}^n \setminus \Omega\) and over \(\mathbb{R}^n \setminus \overline{\Omega}\) are the same. In addition, a.e. in \(\mathbb{R}^n \setminus \Omega\) and a.e. in \(\mathbb{R}^n \setminus \overline{\Omega}\) are also the same notions.

Throughout the remainder of the paper, for functions \(u,v \in W^{s,2}_\Omega\) we shall let
\[
\mathcal{E}(u,v) := \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dy \, dx
\]
\[
+ C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dy \, dx,
\]  
(3.1)

where \(C_{n,s}\) is the constant given in (2.4).

3.1. The Dirichlet exterior condition. Throughout this subsection, \(\Omega \subset \mathbb{R}^n\) is an arbitrary bounded open set. Let \(u \in W^{s,2}_0(\Omega)\) be such that \((-\Delta)^s u \in L^2(\Omega)\). Then the following integration by parts formula is well-known (see e.g. [11]). For every \(v \in W^{s,2}_0(\overline{\Omega})\), we have
\[
\int_{\Omega} v(-\Delta)^s u \, dx = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dy \, dx = \mathcal{E}(u,v).
\]  
(3.2)

Several authors (see e.g. [22, 23, 28, 29, 32]) have studied the Dirichlet problem for \((-\Delta)^s\), that is, the elliptic problem
\[
(-\Delta)^s u = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{in} \quad \mathbb{R}^n \setminus \Omega,
\]  
(3.3)
and the associated parabolic problems, but not in the same spirit as in the present paper. Even if this case is straightforward, for the sake of completeness and since we would like to make a comparison with the Neumann and Robin cases, we have decided to include it here.

For a function $u \in L^2(\Omega)$ we define

$$u_D(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (3.4)$$

The following result characterizes the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the zero Dirichlet exterior condition via the method of bilinear forms.

**Theorem 3.1.** Let

$$D(a_D) = \left\{ u \in L^2(\Omega) : u_D \in W^{s,2}_0(\Omega) \right\},$$

and $a_D : D(a_D) \times D(a_D) \to \mathbb{R}$ the form given by

$$a_D(u, v) = E(u_D, v_D).$$

Then $a_D$ is a densely defined, symmetric and closed bilinear form in $L^2(\Omega)$. The selfadjoint operator $A_D$ on $L^2(\Omega)$ associated with $a_D$ in the sense of (2.8) is given by

$$D(A_D) = \left\{ u \in \tilde{W}^{s,2}_0(\Omega), \, ((\Delta)^s u_D)|_{\Omega} \in L^2(\Omega) \right\},$$

$$A_Du = ((\Delta)^s u_D)|_{\Omega}. \quad (3.5)$$

**Proof.** Firstly, we notice that $D(a_D)$ endowed with the norm $\|u\|_{D(a_D)} := \|u_D\|_{W^{s,2}_0(\Omega)}$ is a Hilbert space. Hence, the form $a_D$ is closed. Since $D(\Omega) \subset \tilde{W}^{s,2}_0(\Omega)$ (by Lemma 2.2(a)), we have that $a_D$ is densely defined.

Secondly, let $B$ be the selfadjoint operator on $L^2(\Omega)$ associated with $(a_D, D(a_D))$ in the sense of (2.8). We show that $B = A_D$. Indeed, let $u \in D(B)$ and set $f := Bu$. Then by definition, $u \in D(a_D)$ and

$$a_D(u, v) = \int_{\Omega} vBu \, dx = \int_{\Omega} vf \, dx \quad (3.6)$$

for all $v \in D(a_D)$. Since $u_D, v_D \in W^{s,2}_0(\Omega)$, then using (3.2), we get that

$$a_D(u, v) = E(u_D, u_D) = \int_{\Omega} vD(-\Delta)^s u_D \, dx. \quad (3.7)$$

It follows from (3.6) and (3.7) that

$$\int_{\Omega} vBu \, dx = \int_{\Omega} v(-\Delta)^s u_D \, dx = \int_{\Omega} vf \, dx \quad (3.8)$$

for all $v \in D(a_D)$. Since $D(a_D)$ is dense in $L^2(\Omega)$, it follows from (3.8) that

$$f = Bu = (-\Delta)^s u_D \text{ in } \Omega.$$  

Hence, $u \in D(A_D)$ and $Bu = A_Du$. 

Conversely, let \( u \in D(A_D) \) and set \( f := A_D u \). Then by definition, \( u \in \tilde{W}^{s,2}_0(\Omega) \) and hence, \( u_D \in W^{s,2}_\Omega \). Thus \( u \in D(A_D) \). Since \( A_D u = f = ((-\Delta)^s u_D)|\Omega \in L^2(\Omega) \) (by definition of \( A_D \)), then using (3.2) again, we get that
\[
\int_\Omega vf \, dx = \int_\Omega v A_D u \, dx = \int_\Omega v_D(-\Delta)^s u_D \, dx = E(u_D, v_D) = a_D(u, v),
\]
for every \( v \in D(A_D) \). We have shown that \( u \in D(B) \) and \( A_D u = Bu \). The proof is finished. \( \square \)

**Remark 3.2.** The operator \( A_D \) is the realization in \( L^2(\Omega) \) of \((-\Delta)^s\) with the zero Dirichlet exterior condition.

Denote by \( T_D = (e^{-tA_D})_{t \geq 0} \) the semigroup on \( L^2(\Omega) \) generated by \(-A_D\).

**Theorem 3.3.** The semigroup \( T_D \) is positivity-preserving.

*Proof.* Notice that for \( u, v \in D(a_D) \), we have
\[
a_D(u, v) = E(u_D, v_D) = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_D(x) - u_D(y))(v_D(x) - v_D(y))}{|x-y|^{n+2s}} \, dx \, dy.
\]
Let \( u \in D(A_D) \). By Remark 3.2(i), we have to show that \( |u| \in D(A_D) \) and \( a_D(|u|, |u|) \leq a_D(u, u) \).

Indeed, using the reverse triangle inequality we get that
\[
a_D(|u|, |u|) = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u_D(x) - u_D(y)|^2}{|x-y|^{n+2s}} \, dx \, dy
\leq \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u_D(x) - u_D(y)|^2}{|x-y|^{n+2s}} \, dx \, dy
= a_D(u, u).
\]
Hence, \( |u| \in D(A_D) \) and \( a_D(|u|, |u|) \leq a_D(u, u) \). The proof is finished. \( \square \)

Other qualitative properties of the semigroup \( T_D \) will be given in Section 4.

**Remark 3.4.** Letting
\[
\kappa(x) := C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{|x-y|^{n+2s}} \, dy, \quad x \in \Omega,
\]
then the form \( a_D \) can be defined without using the extension \( u_D \). More precisely, a simple calculation shows that for \( u, v \in D(A_D) = \tilde{W}^{s,2}_0(\Omega) \), we have
\[
a_D(u, v) = \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy + \int_{\Omega} u(x)v(x)\kappa(x) \, dx.
\]

### 3.2. The nonlocal Neumann exterior condition

Throughout the remainder of the paper, \( \Omega \subset \mathbb{R}^n \) is a bounded domain with a Lipschitz continuous boundary. In [12], the authors have studied the well-posedness of the following elliptic Neumann problem:
\[
(-\Delta)^su = f \quad \text{in} \quad \Omega, \quad N^s u = g \quad \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}, \tag{3.9}
\]
and the associated parabolic problems. The eigenvalues problem associated to (3.9) with \( g = 0 \) has been investigated without describing explicitly the associated operator. We emphasize that the non-described operator in [12] is the one that we shall completely characterize in the present subsection. Problem (3.9) in another spirit has been also studied in [1].

Before characterizing our operator, we need some preparations.
Lemma 3.5. Let \( u \in W^{s,2}_\Omega \) and \((u_k)_{k \in \mathbb{N}} \subset W^{s,2}_\Omega\) be such that \( u_k \to u \) in \( W^{s,2}_\Omega \) as \( k \to \infty \). Then \( u_k \to u \) pointwise almost everywhere in \( \mathbb{R}^n \) as \( k \to \infty \).

Proof. Let \((u_k)_{k \in \mathbb{N}}\) and \( u \) be as in the statement. By definition, \( u_k \to u \) in \( L^2(\Omega) \) as \( k \to \infty \). Hence, after a subsequence if necessary, we have that \( u_k \to u \) a.e. in \( \Omega \) as \( k \to \infty \). On the other hand, we have that

\[
\int \int_{\mathbb{R}^n \setminus \Omega} \frac{|(u_k(x) - u(x)) - (u_k(y) - u(y))|^2}{|x - y|^{n+2s}} dy \, dx \to 0 \quad \text{as} \quad k \to \infty.
\]

Hence, \( (u_k(x) - u(x)) - (u_k(y) - u(y)) \to 0 \) for a.e. \( x \in \Omega \) and \( y \in \mathbb{R}^n \setminus \Omega \), as \( k \to \infty \). Therefore, \( 0 = \lim_{k \to \infty} (u_k(x) - u(x)) = \lim_{k \to \infty} (u_k(y) - u(y))\) for a.e. \( x \in \Omega \) and \( y \in \mathbb{R}^n \setminus \Omega \). The proof is finished. \( \square \)

Next, we recall that for a function \( u \in W^{s,2}_\Omega \), we have defined \( \mathcal{N}^s u \) as follows:

\[
\mathcal{N}^s u(x) = C_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.
\]

Let

\[
\rho(x) = \int_{\Omega} \frac{1}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \Omega. \tag{3.10}
\]

We have the following result.

Lemma 3.6. Let \( u \in W^{s,2}_\Omega \). Then

\[
\mathcal{N}^s u(x) = 0, \quad x \in \mathbb{R}^n \setminus \overline{\Omega},
\]

if and only if,

\[
u(x) = \frac{1}{\rho(x)} \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.
\]

Proof. Let \( u \in W^{s,2}_\Omega \). Then by definition, the identity

\[
\mathcal{N}^s u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \overline{\Omega},
\]

is equivalent to the following:

\[
0 = \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = u(x) \int_{\Omega} \frac{1}{|x - y|^{n+2s}} dy - \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy = u(x) \rho(x) - \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.
\]

This yields the claim and the proof is finished. \( \square \)

Let \( u \in L^2(\Omega) \). We denote by \( u_N \) the extension of \( u \) as follows:

\[
u_N(x) = \begin{cases} u(x) & \text{if} \ x \in \Omega, \\ \frac{1}{\rho(x)} \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy & \text{if} \ x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.11}
\]
It follows from Lemma 3.6 that if \( u_N \in W^{s,2}_\Omega \), then \( \mathcal{N}^s u_N = 0 \) a.e. in \( \mathbb{R}^n \setminus \overline{\Omega} \).

Next, we give some properties of the extension \( u_N \) that will be used later in the paper.

**Lemma 3.7.** Let \( u \in L^2(\Omega) \) and \( u_N \) its extension given in (3.11). The following assertions hold.

(a) If \( u \geq 0 \) a.e. in \( \Omega \), then \( u_N \geq 0 \) a.e. in \( \mathbb{R}^n \).

(b) If \( u = 1 \) a.e. in \( \Omega \), then \( u_N = 1 \) a.e. in \( \mathbb{R}^n \).

(c) If \( \Omega \) is of class \( C^1 \) and \( u \in C(\overline{\Omega}) \), then \( u_N \in C(\mathbb{R}^n) \).

**Proof.** Parts (a) and (b) follow directly from (3.11).

(c) Let \( u \in C(\overline{\Omega}) \). Since \( \mathcal{N}^s u_N = 0 \) a.e. in \( \mathbb{R}^n \setminus \overline{\Omega} \), it follows from [12, Proposition 5.2] that \( u_N \in C(\mathbb{R}^n) \) and the proof is finished. \( \Box \)

Now, we are ready to give a characterization of the nonlocal Neumann exterior condition.

**Theorem 3.8.** Let

\[
D(a_N) = \left\{ u \in L^2(\Omega), \, u_N \in W^{s,2}_\Omega \right\},
\]

and \( a_N : D(a_N) \times D(a_N) \to \mathbb{R} \) be defined by

\[
a_N(u, v) = \mathcal{E}(u_N, v_N).
\]

Then \( a_N \) is a closed, symmetric and densely defined bilinear form on \( L^2(\Omega) \). The selfadjoint operator \( A_N \) on \( L^2(\Omega) \) associated with \( a_N \) is given by

\[
\begin{align*}
D(A_N) &= \left\{ u \in L^2(\Omega), \, u_N \in W^{s,2}_\Omega, \, ((-\Delta)^s u_N)|_\Omega \in L^2(\Omega) \right\}, \\
A_N u &= ((-\Delta)^s u_N)|_\Omega.
\end{align*}
\]

**Proof.** Firstly, since the extension operator \( D(a_N) \to W^{s,2}_\Omega, \, u \mapsto u_N \) is linear, and \( \mathcal{E} \) is bilinear and symmetric, we can deduce that \( a_N \) is also bilinear and symmetric.

Secondly, to show that \( a_N \) is closed, we need to prove that \( D(a_N) \) endowed with the norm

\[
\|u\|_{D(a_N)}^2 = a_N(u, u) + \|u\|_{L^2(\Omega)}^2 = \mathcal{E}(u_N, u_N) + \|u\|_{L^2(\Omega)}^2,
\]

is a Hilbert space. Indeed, let \( (u_k)_{k \in \mathbb{N}} \subset D(a_N) \) be such that

\[
\|u_k - u_m\|_{L^2(\Omega)}^2 + a_N(u_k - u_m, u_k - u_m) \to 0
\]
as \( k, m \to \infty \). This is the same as

\[
\|(u_k)_N - (u_m)_N\|_{L^2(\Omega)}^2 + \mathcal{E}((u_k)_N - (u_m)_N, (u_k)_N - (u_m)_N) \to 0,
\]
as \( k, m \to \infty \). Recall that \( W^{s,2}_\Omega \) endowed with the norm given in (3.15) is a Hilbert space (see Lemma 2.3). Thus, there is a function \( v \in W^{s,2}_\Omega \) such that \( (u_k)_N \to v \) in \( W^{s,2}_\Omega \) as \( k \to \infty \). Using Lemma 2.4, we get that \( \mathcal{N}^s v = \lim_{k \to \infty} \mathcal{N}^s (u_k)_N = 0 \) a.e. in \( \mathbb{R}^n \setminus \overline{\Omega} \). Let us define \( u := v|_\Omega \). Then, Lemma 3.6 implies that \( u_N = v \). Thus,
\[
\lim_{n \to \infty} \|u - u_k\|_{L^2(\Omega)}^2 + a_N(u - u_k, u - u_k) = \lim_{n \to \infty} \|(u_k)_N - u_N\|_{L^2(\Omega)}^2 + \mathcal{E}((u_k)_N - u_N, (u_k)_N - u_N) = 0.
\]

Hence, \(D(a_N)\) is complete and we have shown that the form \(a_N\) is closed.

By Lemma 4.1 below, \(D(a_D) \subset D(a_N)\) and since \(D(a_D)\) is dense in \(L^2(\Omega)\) (by Theorem 3.1), we have that \(D(a_N)\) is dense in \(L^2(\Omega)\). Thus, \(a_N\) is densely defined.

Thirdly, let \(B\) be the selfadjoint operator on \(L^2(\Omega)\) associated with \(a_N\) in the sense of (2.8). We show that \(B = A_N\). Indeed, let \(u \in D(B)\) and set \(f := Bu\). Then by definition, \(f \in L^2(\Omega)\) and \(u \in D(a_N)\). Using the integration by parts formula (2.7), we get that

\[
\int_{\Omega} fv \, dx = a_N(u, v) = \mathcal{E}(u_N, v_N) = \int_{\Omega} v_N(-\Delta)^s u_N \, dx + \int_{\mathbb{R}^n \setminus \Omega} v_N N^s u_N \, dx = \int_{\Omega} (-\Delta)^s u_N \, dx,
\]

for every \(v \in D(a_N)\), where in the last equality, we have used that \(N^s u_N = 0\) in \(\mathbb{R}^n \setminus \overline{\Omega}\). Since \(D(a_N)\) is dense in \(L^2(\Omega)\), it follows that the identity

\[
\int_{\Omega} fv \, dx = \int_{\Omega} v(-\Delta)^s u_N \, dx
\]

holds for every \(v \in L^2(\Omega)\). This yields \((-\Delta)^s u_N)|_{\Omega} \in L^2(\Omega)\) and \(Bu = ((-\Delta)^s u_N)|_{\Omega}\). We have shown that \(D(B) \subset D(A_N)\) and \(Bu = A_N u\).

Conversely, let \(u \in D(A_N)\) and set \(f := A_N u\). Then by definition, \(u_N \in W^{s,2}_\Omega\). Thus \(u \in D(a_N)\). Since \(A_N u = f = ((-\Delta)^s u_N)|_{\Omega} \in L^2(\Omega)\) (by definition of \(A_N\)), then using (2.7) again, we get that

\[
\int_{\Omega} vf \, dx = \int_{\Omega} v(A_N u) \, dx = \int_{\Omega} v_N(-\Delta)^s u_N \, dx = \mathcal{E}(u_N, v_N) - \int_{\mathbb{R}^n \setminus \Omega} v_N N^s u_N \, dx = \mathcal{E}(u_N, v_N) = a_N(u, v),
\]

for every \(v \in D(a_N)\), where we have used that \(N^s u_N = 0\) in \(\mathbb{R}^n \setminus \overline{\Omega}\). Thus, \(u \in D(B)\) and \(A_N u = Bu\). We have shown that \(A_N = B\) and the proof is finished. \(\square\)

**Remark 3.9.** The operator \(A_N\) is the realization in \(L^2(\Omega)\) of \((-\Delta)^s\) with the nonlocal Neumann exterior condition.

It is worth to mention the following characterization of \(D(a_N)\).

**Lemma 3.10.** Let \(D(a_N)\) be the space defined in (3.12). Then

\[
D(a_N) = \{ u|_{\Omega} : u \in W^{s,2}_\Omega \},
\]

(3.16)
Proof. Denote by $D$ the right hand side of (3.16). It is clear that $D(a_N) \subseteq D$. Now, let $v \in D$. Then $v = u|_{\Omega}$ for some $u \in W^{s,2}_\Omega$. We have to show that $\mathcal{E}(v_N, v_N) < \infty$. Calculating, we get that

$$\mathcal{E}(v_N, v_N) = \mathcal{E}(u, u) - \int_{\mathbb{R}^n \setminus \Omega} \frac{(v(x) - v_N(y))^2 - (v(x) - u(y))^2}{|x - y|^{n+2s}} \, dy \, dx$$

$$= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -\rho(y)v_N^2(y) + 2\rho(y)v_N(y)u(y) - \rho(y)u^2(y) \right) \, dy$$

$$= - C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \rho(y) \left( v_N(y) - u(y) \right)^2 \, dy \leq 0.$$ 

We have shown that $\mathcal{E}(v_N, v_N) \leq \mathcal{E}(u, u) < \infty$ and the proof is finished. $\square$

As a direct consequence of Lemma 3.10, we have the following result.

**Proposition 3.11.** Let $(a_N, D(a_N))$ be the form defined in (3.12)-(3.13). Then

$$a_N(u, u) := \mathcal{E}(u_N, u_N) = \inf \left\{ \mathcal{E}(v, v) : v \in W^{s,2}_\Omega, v|_{\Omega} = u \right\}.$$ (3.17)

In other words, for $u \in L^2(\Omega)$, we have that $u_N$ is the smallest extension in $W^{s,2}_\Omega$ with respect to the $W^{s,2}_\Omega$-norm, or equivalently, the infimum in the right hand side of (3.17) is attained at $u_N$.

Denote by $T_N = (e^{-tA_N})_{t \geq 0}$ the semigroup on $L^2(\Omega)$ generated by $-A_N$.

**Theorem 3.12.** The semigroup $T_N$ is positivity-preserving.

Proof. Let $u \in D(a_N)$. We want to show that $a_N(|u|, |u|) \leq a_N(u, u)$. Notice that for $v, w \in W^{s,2}_\Omega$, $\mathcal{E}(v, w)$ is a sum of integrals over $\Omega \times \Omega$ and over $\Omega \times \mathbb{R}^n \setminus \Omega$ (see (3.11)). Firstly, let us inspect the $\Omega \times \Omega$ part. We define

$$\mathcal{E}_\Omega(v, w) = \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{n+2s}} \, dx \, dy.$$ (3.18)

The reverse triangle inequality yields

$$\mathcal{E}_\Omega(|u|, |u|) \leq \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x)| - |u(y)|}{|x - y|^{n+2s}} \, dx \, dy$$

$$\leq \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = \mathcal{E}_\Omega(u, u).$$ (3.19)

Secondly, for the $\Omega \times \mathbb{R}^n \setminus \Omega$ part, we have that
Using (3.21), we can deduce from (3.20) that the triangle inequality yields
\[ (u_N(y) - |u|N(y)) \leq 0 \quad \text{and} \quad (u_N(y) + |u|(y)) \geq 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega. \tag{3.21} \]

Using (3.21), we can deduce from (3.20) that
\[
C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{(|u| - |u|_N(y))^2 - (u(x) - u_N(y))^2}{|x - y|^{n+2s}} dy \, dx \\
= C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^2 - 2|u(x)||u|_N(y) + |u|_N^2(y) - u^2(x) + 2u(x)u_N(y) - u_N^2(y)}{|x - y|^{n+2s}} dy \, dx \\
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2\rho(y)|u|_N^2(y) + (\rho(y)|u|_N^2(y) + 2\rho(y)u_N^2(y) - \rho(y)u_N^2(y) \right) dy \\
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \rho(y) \left( u_N(y) - |u|N(y) \right) \left( u_N(y) + |u|N(y) \right) dy. \tag{3.20} \]

The triangle inequality yields $|u_N| \leq |u|_N$. Hence, $-|u|_N \leq u_N \leq |u|_N$. This implies that
\[
(u_N(y) - |u|_N(y)) \leq 0 \quad \text{and} \quad (u_N(y) + |u|(y)) \geq 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega. \tag{3.21} \]

Using (3.21), we can deduce from (3.20) that
\[
C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{(|u| - |u|_N(y))^2 - (u(x) - u_N(y))^2}{|x - y|^{n+2s}} dy \, dx \\
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \rho(y) \left( u_N(y) - |u|N(y) \right) \left( u_N(y) + |u|N(y) \right) dy \leq 0. \tag{3.22} \]

Combining (3.19) and (3.22) we get that $a_N(|u|, |u|) \leq a_N(u, u)$. The proof is finished. \hfill \Box

3.3. **The nonlocal Robin exterior condition.** We recall that both Dirichlet and the nonlocal Neumann exterior conditions where realized by some kind of extension to $\mathbb{R}^n$ of functions defined in $\Omega$. Here also, we need to find an appropriate extension.

We start with the following result.

**Lemma 3.13.** Let $\beta \in L^\infty(\mathbb{R}^n \setminus \Omega)$ be a fixed non-negative function and $u \in W^{s,2}_\Omega$. Then
\[
\mathcal{N}^s u(x) + \beta(x)u(x) = 0, \quad x \in \mathbb{R}^n \setminus \Omega, \tag{3.23} \]

if and only if,
\[
u(x) = \frac{C_{n,s,\rho}(x)}{C_{n,s,\rho}(x) + \beta(x)} u_N(x) = \frac{C_{n,s}}{C_{n,s,\rho}(x) + \beta(x)} \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \Omega, \tag{3.24} \]

where we recall that $\rho(x)$ has been defined in (3.10) and $u_N$ is given in (3.11).

**Proof.** Let $u \in W^{s,2}_\Omega$. A simple calculation gives that the condition (3.23), that is,
\[
0 = \mathcal{N}^s u(x) + \beta(x)u(x) \\
= C_{n,s}u(x)\rho(x) - C_{n,s,\rho}(x)u_N(x) + \beta(x)u(x) \\
= \left( C_{n,s,\rho}(x) + \beta(x) \right) u(x) - C_{n,s,\rho}(x)u_N(x), \quad x \in \mathbb{R}^n \setminus \Omega,
\]
is equivalent to (3.24). The proof is finished. \hfill \Box
For a function \( u \in L^2(\Omega) \), we define its extension \( u_R \) as follows:

\[
u_R(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ \frac{C_{n,s}}{C_{n,s} \rho(x) + \beta(x)} \int_{\Omega} \frac{u(y)}{|x-y|^{n+2s}} dy & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}\]

**Remark 3.14.** Let \( u \in W^{s,2}_\Omega \). By Lemma 3.13 we have that \( u_R \) satisfies (3.23), that we call the nonlocal Robin exterior condition. We notice that if \( u \geq 0 \) a.e. in \( \Omega \), then \( u_R \geq 0 \) a.e. in \( \mathbb{R}^n \). In addition, by (3.24), we have that

\[
u_R(x) = \frac{C_{n,s} \rho(x)}{C_{n,s} \rho(x) + \beta(x)} u_N(x) \quad \text{in } \mathbb{R}^n \setminus \Omega. \tag{3.25}\]

Now we introduce the realization in \( L^2(\Omega) \) of \((-\Delta)^s\) with the nonlocal Robin exterior condition.

**Theorem 3.15.** Let

\[
D(a_R) = \left\{ u \in L^2(\Omega), \ u_R \in W^{s,2}_\Omega \text{ and } \int_{\mathbb{R}^n \setminus \Omega} \beta(x) u_R^2(x) \, dx < \infty \right\}
\]

and \( a_R : D(a_R) \times D(a_R) \to \mathbb{R} \) given by

\[
a_R(u, v) = \mathcal{E}(u_R, v_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta(x) u_R(x) v_R(x) \, dx. \tag{3.27}\]

Then \( a_R \) is a closed, symmetric and densely defined bilinear form on \( L^2(\Omega) \). The selfadjoint operator \( A_R \) associated with \( a_R \) is given by

\[
\begin{align*} 
D(A_R) &= \left\{ u \in L^2(\Omega), u_R \in W^{s,2}_\Omega, \ ((-\Delta)^s u_R)|_\Omega \in L^2(\Omega) \right\}, \\
A_R u &= ((-\Delta)^s u_R)|_\Omega.
\end{align*}
\]

**Proof.** Let \( u \in D(a_D) \). It follows from Lemma 4.1 below that \( u \in D(a_N) \). Thus the extensions \( u_N \) and \( u_R \) are well defined. Obviously, \( D(a_R) = \{ u \in L^2(\Omega) : a_R(u, u) < \infty \} \). Thus,

\[
a_R(u, u) - a_D(u, u) = C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{-2u(x)u_R(y) + u_R^2(y)}{|x-y|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n \setminus \Omega} \beta(y) u_R^2(y) \, dy \\
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2 \rho(y) u_N(y) u_R(y) + \rho(y) u_R^2(y) + \frac{\beta(y)}{C_{n,s}} u_R^2(y) \right) \, dy \\
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2 \rho(y) \frac{C_{n,s} \rho(y) + \beta(y)}{C_{n,s} \rho(y)} + \rho(y) + \frac{\beta(y)}{C_{n,s}} u_R^2(y) \right) \, dy \\
= -C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \rho(y) + \frac{\beta(y)}{C_{n,s}} \right) u_R^2(y) \, dy \leq 0,
\]

where in the third equality, we have used (3.25). This implies that \( 0 \leq a_R(u, u) \leq a_D(u, u) < \infty \). Therefore, \( D(a_D) \subset D(a_R) \) and hence, \( D(a_R) \) is dense in \( L^2(\Omega) \) (since \( D(a_D) \) is dense in \( L^2(\Omega) \) by Theorem 3.1).

Next, let \((u_k)_{k \in \mathbb{N}} \subset D(a_R)\) be such that

\[
a_R(u_k - u_m, u_k - u_m) + \|u_k - u_m\|^2_{L^2(\Omega)} \to 0 \quad \text{as } k, m \to \infty.
\]
This is the same as
\[ \mathcal{E}((u_k)_R - (u_m)_R, (u_k)_R - (u_m)_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta((u_k)_R - (u_m)_R)^2 \, dx + \|u_k - u_m\|_{2, \Omega}^2 \rightarrow 0, \]
as \( k, m \rightarrow \infty \). Since \( W_{\Omega}^{s, 2} \) is a Hilbert space, it follows that there exists a function \( v \in W_{\Omega}^{s, 2} \) such that \( (u_k)_R \rightarrow v \) in \( W_{\Omega}^{s, 2} \) as \( k \rightarrow \infty \). Using Lemma 2.4, we get that
\[ \mathcal{N}^s(u_k)_R \rightarrow \mathcal{N}^s v = f \text{ in } L^2(\mathbb{R}^n \setminus \Omega) \text{ as } k \rightarrow \infty. \]
This implies that \( \beta(u_k)_R \rightarrow -f \) a.e. in \( \mathbb{R}^n \setminus \Omega \) as \( k \rightarrow \infty \). Since \( (u_k)_R \) converges in \( W_{\Omega}^{s, 2} \), hence, it converges a.e. in \( \mathbb{R}^n \) (by Lemma 3.5), we have that \( \beta(u_k)_R \rightarrow \beta v \) a.e. in \( \mathbb{R}^n \setminus \Omega \) as \( k \rightarrow \infty \). Furthermore, since \( \beta((u_k)_R - u_R)^2 \rightarrow 0 \) as \( k \rightarrow \infty \), it follows that \( (u_k)_R \rightarrow g \) in \( L^2(\mathbb{R}^n \setminus \Omega, \sqrt{\beta} \, dx) \) as \( k \rightarrow \infty \). But again, the pointwise convergences shows that \( g = v \) a.e. in \( \mathbb{R}^n \setminus \Omega \). Hence, \( v \) fulfills the nonlocal Robin exterior condition (3.23) and
\[ \mathcal{E}((u_k)_R - v, (u_k)_R - v) + \|\beta((u_k)_R - v)\|_{2, (\mathbb{R}^n \setminus \Omega)}^2 \rightarrow 0 \text{ as } k \rightarrow \infty. \]
Define \( u = v|_{\Omega} \). Then \( u_R = v \) and
\[ a_R(u_k - u, u_k - u) + \|u_k - u\|_{2, \Omega}^2 \rightarrow 0 \text{ as } k \rightarrow \infty. \]
We have shown that \( a_R \) is closed.

Next, denote by \( B \) the operator associated with \( a_R \) in the sense of (2.8). Let \( u \in D(B) \) and set \( f := Bu \). Then the integration by parts formula (2.7) yields
\[ \int_{\Omega} v f \, dx = \int_{\Omega} vBu \, dx = a_R(u, v) = \mathcal{E}(u_R, u_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta u_R v_R \, dx \]
\[ = \int_{\Omega} v_R(-\Delta)^s u_R \, dx + \int_{\mathbb{R}^n \setminus \Omega} v_R \mathcal{N}^s u_R \, dx + \int_{\mathbb{R}^n \setminus \Omega} \beta u_R v_R \, dx \]
\[ = \int_{\Omega} v_R(-\Delta)^s u_R \, dx + \int_{\mathbb{R}^n \setminus \Omega} v_R \left( \mathcal{N}^s u_R + \beta u_R \right) \, dx \]
\[ = \int_{\Omega} v(-\Delta)^s u_R \, dx, \]
for every \( v \in D(a_R) \), where in the last identity we have used that \( \mathcal{N}^s u_R + \beta u_R = 0 \) a.e. in \( \mathbb{R}^n \setminus \overline{\Omega} \) (by (3.23)). Since \( D(a_R) \) is dense in \( L^2(\Omega) \), it follows that
\[ (-\Delta)^s u_R = Bu = f \in L^2(\Omega). \]
Hence, \( u \in D(A_R) \) and \( Bu = A_R u \).

Conversely, let \( u, v \in D(A_R) \). Using (2.7) again and (3.23), we get that
\[ a_R(u, v) = \mathcal{E}(u_R, v_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta(x) u_R(x) v_R(x) \, dx = \mathcal{E}(u_R, v_R) - \int_{\mathbb{R}^n \setminus \Omega} v_R \mathcal{N}^s u_R \, dx \]
\[ = \int_{\Omega} v_R(-\Delta)^s u_R \, dx = \int_{\Omega} vA_R u \, dx. \quad (3.28) \]
Taking \( v = u \) in (3.28), we can deduce that \( u \in D(B) \) and \( A_R u = B u \). The proof is finished. \( \square \)

The following is the variant of Lemma 3.10 for the form \( \mathfrak{a}_R \).

**Lemma 3.16.** Let \( D(\mathfrak{a}_R) \) be the space defined in (3.26). Then

\[
D(\mathfrak{a}_R) = \left\{ u|\Omega : u \in W^{s,2}_\Omega \text{ and } \int_{\mathbb{R}^n \setminus \Omega} \beta(x)|u(x)|^2 \, dx < \infty \right\}. \tag{3.29}
\]

**Proof.** Let \( D \) denote the right hand side of (3.29). It is clear that \( D(\mathfrak{a}_R) \subseteq D \).

Conversely, let \( v \in D \). Then \( v = u|\Omega \) for some \( u \in W^{s,2}_\Omega \) satisfying \( \int_{\mathbb{R}^n \setminus \Omega} \beta(x)|u(x)|^2 \, dx < \infty \).

We have to show that \( \mathcal{E}(v_R, v_R) < \infty \). Calculating, we get that

\[
\mathcal{E}(v_R, v_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta(y)|v_R(y)|^2 \, dy - \mathcal{E}(u, u) - \int_{\mathbb{R}^n \setminus \Omega} \beta(y)|u(y)|^2 \, dy
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{(v(x) - v_R(y))^2 - (v(x) - u(y))^2}{|x - y|^{n+2s}} \, dy \, dx
+ \int_{\mathbb{R}^n \setminus \Omega} \beta(y) \left( |u(y)|^2 - |u(y)|^2 \right) \, dy
= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \rho(y)v_N^2(y) - 2\rho(y)v_N(y)v_R(y) + 2\rho(y)v_N(y)u(y) - \rho(y)u^2(y) \right) \, dy
+ \int_{\mathbb{R}^n \setminus \Omega} \beta(y) \left( |v_R(y)|^2 - |u(y)|^2 \right) \, dy. \tag{3.30}
\]

Using the fact that (by (3.24))

\[
v_N(y) = \left( 1 + \frac{\beta(y)}{C_{n,s} \rho(y)} \right) v_R(y) \text{ for a.e. } y \in \mathbb{R}^n \setminus \Omega,
\]

we get from (3.30) that

\[
\mathcal{E}(v_R, v_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta(y)|v_R(y)|^2 \, dy - \mathcal{E}(u, u) - \int_{\mathbb{R}^n \setminus \Omega} \beta(y)|u(y)|^2 \, dy
= - C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \rho(y) \left( 1 + \frac{\beta(y)}{C_{n,s} \rho(y)} \right) \left( v_R(y) - u(y) \right)^2 \, dy \leq 0.
\]

We have shown that

\[
\mathcal{E}(v_R, v_R) + \int_{\mathbb{R}^n \setminus \Omega} \beta(x)|v_R(x)|^2 \, dx \leq \mathcal{E}(u, u) + \int_{\mathbb{R}^n \setminus \Omega} \beta(x)|u(x)|^2 \, dx < \infty.
\]

The proof is finished. \( \square \)

Here also, we have the following result as a direct consequence of Lemma 3.16.

**Proposition 3.17.** Let \( (\mathfrak{a}_R, D(\mathfrak{a}_R)) \) be the form defined in (3.26)-(3.27). Then for \( u \in D(\mathfrak{a}_R) \), we have that
\[ a_R(u, u) := \mathcal{E}(u_R, u_R) = \inf \left\{ \mathcal{E}(v) + \int_{\mathbb{R}^n \setminus \Omega} \beta(x) |v(x)|^2 \, dx : v \in W^{s,2}_\Omega, \, v|_\Omega = u \right\}. \tag{3.31} \]

In other words, if \( u \in L^2(\Omega) \), then \( u_R \) is the smallest extension in \( W^{s,2}_\Omega \) with respect to the \( W^{s,2}_\Omega \cap L^2(\mathbb{R}^n \setminus \Omega, \sqrt{\beta(x)} \, dx) \)-norm, or equivalently, the infimum in (3.31) is attained at \( u_R \).

Next, denote by \( T_R = (e^{-tA_R})_{t \geq 0} \) the semigroup on \( L^2(\Omega) \) generated by \( -A_R \).

**Theorem 3.18.** The semigroup \( T_R \) is positivity-preserving.

**Proof.** Let \( u \in D(a_R) \). We want to show that \( a_R(|u|, |u|) \leq a_R(u, u) \). As in the proof of Theorem 3.12, we have a sum of integrals over \( \Omega \times \Omega \) and over \( \Omega \times \mathbb{R}^n \setminus \Omega \). Firstly, let us inspect the \( \Omega \times \Omega \) part. Let \( \mathcal{E}_\Omega \) be as in (3.18). Then proceeding as in (3.19), we get that

\[ \mathcal{E}_\Omega(|u|, |u|) \leq \mathcal{E}_\Omega(u, u). \tag{3.32} \]

Secondly, for the \( \Omega \times \mathbb{R}^n \setminus \Omega \) part, we have

\[
\begin{align*}
  &C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{(|u|(x) - |u|R(y))^2 - (u(x) - u_R(y))^2}{|x - y|^{n+2s}} \, dy \, dx \\
  &+ \int_{\mathbb{R}^n \setminus \Omega} \beta(y)|u|^2_R - \beta(y)u_R(y)^2 \, dy \\
  &= C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^2 - 2|u(x)|u_R(y) + |u|^2_R(y) - u^2(x) + 2u(x)u_R(y) - u_R^2(y)}{|x - y|^{n+2s}} \, dy \, dx \\
  &+ \int_{\mathbb{R}^n \setminus \Omega} \left( \beta(y)|u|^2_R - \beta(y)u^2_R(y) \right) \, dy \\
  &= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2\rho(y)|u|N(y)|u|R(y) + \rho(y)|u|^2_R(y) + 2\rho(y)u_N(y)u_R(y) - \rho(y)u^2_R(y) \right) \, dy \\
  &+ \int_{\mathbb{R}^n \setminus \Omega} \left( \beta(y)|u|^2_R - \beta(y)u^2_R(y) \right) \, dy \\
  &= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2\rho(y)\left(1 + \frac{\beta(y)}{C_{n,s}\rho(y)}\right)|u|^2_R(y) + \rho(y)|u|^2_R(y) \right) \, dy \\
  &+ \int_{\mathbb{R}^n \setminus \Omega} \left( 2\rho(y)\left(1 + \frac{\beta(y)}{C_{n,s}\rho(y)}\right)u^2_R(y) - \rho(y)u^2_R(y) \right) \, dy \\
  &+ C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \frac{\beta(y)}{C_{n,s}}|u|^2_R(y) - \frac{\beta(y)}{C_{n,s}}u^2_R(y) \right) \, dy \\
  &= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -\rho(y)|u|^2_R(y) - \frac{\beta(y)}{C_{n,s}}|u|^2_R(y) + \rho(y)u^2_R(y) + \frac{\beta(y)}{C_{n,s}}u^2_R(y) \right) \, dy \\
  &= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \rho(y) + \frac{\beta(y)}{C_{n,s}} \right) \left( u_R(y) + |u|_R(y) \right) \left( u_R(y) - |u|_R(y) \right) \, dy \leq 0, \tag{3.33} \end{align*}
\]

where we have used (3.25) and the last inequality follows by using (3.21) with \( u_N \) and \( |u|_N \) replaced by \( u_R \) and \( |u|_R \), respectively. Combining (3.32) and (3.33), we get that \( a_R(|u|, |u|) \leq a_R(u, u) \). The proof is finished. \( \square \)
4. Some domination results

In this section we give some results on domination of the semigroups constructed in Section 3. First, we need some preliminary results.

**Lemma 4.1.** We have that $D(a_D) \subset D(a_N)$.

**Proof.** Let $u \in D(a_D)$ and $u_D$ given by (3.4). By definition, $u_D \in W^{s,2}_\Omega$. We have to show that $u_N \in W^{s,2}_\Omega$. That is, we have to prove that

$$E(u_N,u_N) = \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dy \, dx + C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{(u(x) - u_N(y))^2}{|x-y|^{n+2s}} dy \, dx < \infty.$$  \hfill (4.1)

Since $u \in D(a_D)$, we have that

$$E(u_N,u_N) = \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dy \, dx + C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{u^2(x)}{|x-y|^{n+2s}} dy \, dx < \infty.$$  \hfill (4.1)

Obviously, $E(u_N,u_N) \geq 0$. Thus, $E(u_N,u_N) - E(u_D,u_D) > -\infty$. On the other hand, we have that

$$E(u_N,u_N) - E(u_D,u_D) = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{-2u(x)u_N(y) + u_N^2(y)}{|x-y|^{n+2s}} dy \, dx$$

$$= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2u_N(y) \int_\Omega \frac{u(x)}{|x-y|^{n+2s}} dx \right) dy$$

$$+ C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{u_N^2(y)}{|x-y|^{n+2s}} dx \, dy$$

$$= C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2u_N(y)\rho(y)u_N(y) + u_N^2(y)\rho(y) \right) dy$$

$$= - C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} u_N^2(y)\rho(y) dy \leq 0,$$  \hfill (4.1)

where we have used the fact that $\rho$ is a non-negative function. The estimate (4.1) implies that

$$0 \leq a_N(u,u) = E(u_N,u_N) \leq E(u_D,u_D) = a_D(u,u) < \infty.$$  \hfill (4.1)

The proof is finished. \hfill $\square$

Next, we consider the semigroups $T_D$ and $T_N$.

**Theorem 4.2.** We have that

$$0 \leq T_D \leq T_N,$$  \hfill (4.2)

in the sense of (2.11).

**Proof.** We have already proved in Theorems 3.3 and 3.12 that $T_D$ and $T_N$ are positivity-preserving. Hence, we have to verify if the conditions in Theorem 2.10(ii) are satisfied.
Step 1: Recall that $D(a_D) \subset D(a_N)$ by Lemma 4.1. Next, let $u \in D(a_D)$ and $v \in D(a_N)$ be such that $0 \leq v \leq u$. Then,
\[
a_D(v, v) = \frac{C_{n,s}}{2} \int_{\Omega} \int_{\Omega} (v(x) - v(y))^2 \frac{dy}{|x - y|^{n+2s}} + C_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{v^2(x)}{|x - y|^{n+2s}} \frac{dy}{|x - y|^{n+2s}} dx d\Omega
\leq a_N(v, v) + C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{v^2(x)}{|x - y|^{n+2s}} \frac{dy}{|x - y|^{n+2s}} dx
\leq a_N(v, v) + C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{u^2(x)}{|x - y|^{n+2s}} \frac{dy}{|x - y|^{n+2s}} dx
\leq a_N(v, v) + a_D(u, u) < \infty.
\]
Hence, $v \in D(a_D)$ and we have shown that $D(a_D)$ is an ideal in $D(a_N)$.

Step 2: Let $0 \leq u, v \in D(a_D)$. Then $0 \leq u_N, v_N$ by Lemma 3.7(a). Calculating, we get that
\[
a_N(u, v) - a_D(u, v) = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{-u(x)v_N(y) - u_N(y)v(x) + u_N(y)v_N(y)}{|x - y|^{n+2s}} \frac{dy}{|x - y|^{n+2s}} dx
\]
\[
= -C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{\rho(y)u_N(y)v_N(y)}{|x - y|^{n+2s}} dy \leq 0.
\]
Thus, $a_N(u, v) \leq a_D(u, v)$.

Step 3: Finally, it follows from Theorem 2.10 that (4.2) holds and the proof is finished.

Now, we consider the semigroups $T_D$, $T_R$ and $T_N$.

**Theorem 4.3.** We have that
\[
0 \leq T_D \leq T_R \leq T_N,
\] (4.3)
in the sense of (2.11).

**Proof.** We prove the result in three steps.

**Step 1:** We show that
\[
0 \leq T_D \leq T_R.
\] (4.4)
We have already proved in Theorems 3.3 and 3.18 that $T_D$ and $T_R$ are positivity-preserving. We claim that $D(a_D)$ is an ideal in $D(a_R)$. Indeed, $D(a_D) \subset D(a_R)$ by Theorem 3.15. Proceeding exactly as in the proof of Theorem 4.2, we can easily deduce that if $u \in D(a_D)$ and $v \in D(a_R)$ are such that $0 \leq v \leq u$, then $v \in D(a_D)$. The claim is proved.

Next, let $0 \leq u, v \in D(a_D)$. Then $0 \leq u_R, v_R$ by Remark 3.14. Calculating and using (3.25), we get that
\[ a_R(u, v) - a_D(u, v) \]
\[ = C_{n,s} \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{-u(x)v_R(y) - u_R(y)v(x) + u_R(y)v_R(y)}{|x - y|^{n+2s}} \, dy \, dx + \int_{\mathbb{R}^n \setminus \Omega} \beta(y)u_R(y)v_R(y) \, dy \]
\[ = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -\rho(y)u_N(y)v_R(y) - \rho v_N(y)u_R(y) + \rho(y)u_R(y)v_R(y) + \frac{\beta(y)}{C_{n,s}} u_R(y)v_R(y) \right) \, dy \]
\[ = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -2\rho(y) \frac{C_{n,s}\rho(y)}{\beta(y)} u_R(y) + \frac{\beta(y)}{C_{n,s}} u_R(y)v_R(y) \right) \, dy \]
\[ = -C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \rho(y)u_R(y)v_R(y) + \frac{\beta(y)}{C_{n,s}} u_R(y)v_R(y) \right) \, dy \leq 0. \]

Hence, \( a_R(u, v) \leq a_D(u, v) \). It then follows from Theorem 2.10 that (4.4) holds.

**Step 2:** Here, we show that

\[ 0 \leq T_R \leq T_N. \quad (4.5) \]

Firstly, we claim that \( D(a_R) \) is an ideal in \( D(a_N) \). Indeed, let \( u \in D(a_R) \). Then calculating and using (3.25) again, we get that

\[ a_N(u, u) - a_R(u, u) \]
\[ = C_{n,s} \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{(u(x) - u_N(y))^2}{|x - y|^{n+2s}} \, dy \, dx - \int_{\mathbb{R}^n \setminus \Omega} \beta(y)u_R^2(y) \, dy \]
\[ = C_{n,s} \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{\beta(y)}{C_{n,s}\rho(y)} u_R(y) \, dy \, dx - \int_{\mathbb{R}^n \setminus \Omega} \beta(y)u_R^2(y) \, dy \]
\[ = C_{n,s} \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} -\frac{2\rho(y)u_N(y)}{C_{n,s}\rho(y)} \beta(y) u_R(y) + \frac{\beta^2(y)}{C_{n,s}^2\rho^2(y)} u_R^2(y) \, dy \, dx - \int_{\mathbb{R}^n \setminus \Omega} \beta(y)u_R^2(y) \, dy \]
\[ = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -\frac{2\rho(y)u_N(y)}{C_{n,s}\rho(y)} \beta(y) u_R(y) + \frac{\beta^2(y)}{C_{n,s}^2\rho^2(y)} u_R^2(y) \right) \, dy \]
\[ = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( -\frac{2\beta(y)}{C_{n,s}} \frac{1}{C_{n,s}\rho(y)} + \frac{\beta(y)}{C_{n,s}^2\rho(y)} + \frac{\beta^2(y)}{C_{n,s}^2\rho(y)} \right) u_R^2(y) \, dy \]
\[ = -C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \frac{\beta^2(y)}{C_{n,s}^2\rho(y)} + \frac{\beta(y)}{C_{n,s}} \right) u_R^2(y) \, dy \leq 0. \quad (4.6) \]

Therefore, \( 0 \leq a_N(u, u) \leq a_R(u, u) \) which implies that \( u \in D(a_N) \) and hence, \( D(a_R) \subset D(a_N) \). Next, let \( u \in D(a_R) \) and \( v \in D(a_N) \) be such that \( 0 \leq v \leq u \). We have to show that \( v \in D(a_R) \). It follows from (4.6) that
\[ a_R(v, v) = E(v_N, v_N) + C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \frac{\beta^2(y)}{C_{n,s}^2 \rho(y)} + \frac{\beta(y)}{C_{n,s}} \right) v_R^2(y) \, dy \]
\[ \leq a_N(v, v) + a_R(u, u) < \infty. \]

Thus, \( v \in D(a_R) \) and the proof of the claim is complete.

Secondly, let \( 0 \leq u, v \in D(a_R) \). Then a similar calculation yields

\[ a_N(u, v) - a_R(u, v) = -C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \left( \frac{\beta^2(y)}{C_{n,s}^2 \rho(y)} + \frac{\beta(y)}{C_{n,s}} \right) u_R(y) v_R(y) \, dy \leq 0, \]

where we have used that \( u_R, v_R \geq 0 \) by Remark 3.14 (since \( u, v \geq 0 \)). Thus, \( a_N(u, v) \leq a_R(u, v) \).

It then follows from Theorem 2.10 that (4.5) holds.

Step 3: Finally, (4.3) follows from (4.4) and (4.5). The proof is finished. \( \square \)

Remark 4.4. For an arbitrary regular Borel measure \( \mu \) on \( \mathbb{R}^n \setminus \Omega \), one would like to define the following form:

\[ a_\mu(u, v) = a_N(u, v) + \int_{\mathbb{R}^n \setminus \Omega} u_N v_N \, d\mu, \quad (4.7) \]

with

\[ D(a_\mu) := \left\{ u \in D(a_N) \cap C(\overline{\Omega}) : \int_{\mathbb{R}^n \setminus \Omega} |u_N|^2 \, d\mu < \infty \right\}, \]

where we recall that by Lemma 3.7 under the assumption that \( \Omega \) is of class \( C^1 \), if \( u \in C(\overline{\Omega}) \), then \( u_N \in C(\mathbb{R}^n) \). Let us assume that the form \( a_\mu \) is closable in \( L^2(\Omega) \) and denote its closure again by \( a_\mu \). Let \( T_\mu \) be the associated semigroup. Then we have the following situation.

(a) It is clear that \( 0 \leq T_\mu \leq T_N \). But the domination \( T_D \leq T_\mu \) is not true in general. Indeed, let \( u \in D(a_D) \). Then calculating, we get that

\[ a_D(u, u) - a_\mu(u, u) = C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \rho(y) u_N^2(y) \, dy - \int_{\mathbb{R}^n \setminus \Omega} u_N^2(y) \, d\mu. \]

Hence, \( T_D \leq T_\mu \) if and only if

\[ \int_{\mathbb{R}^n \setminus \Omega} u_N^2(y) \, d\mu \leq C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \rho(y) u_N^2(y) \, dy, \quad (4.8) \]

for every \( u \in D(a_D) \). The estimate (4.8) fails, for example, if one takes \( d\mu = 2C_{n,s} \rho(y) \, dy \).

(b) On the other hand, we have that for every \( u, v \in D(a_R) \),

\[ a_R(u, v) = a_N(u, v) + C_{n,s} \int_{\mathbb{R}^n \setminus \Omega} u_N(y) v_N(y) \frac{\beta(y) \rho(y)}{C_{n,s} \rho(y) + \beta(y)} \, dy, \]

and

\[ \rho(y) - \frac{\beta(y) \rho(y)}{C_{n,s} \rho(y) + \beta(y)} = \frac{C_{n,s} \rho^2(y)}{C_{n,s} \rho(y) + \beta(y)} \geq 0. \]
In that case, taking the measure $\mu$ as follows:

$$d\mu = \frac{C_{n,s}\beta(y)\rho(y)}{C_{n,s}\rho(y) + \beta(y)} dy,$$

we get that $a_R = a_\mu$.

Next, we show some contractivity properties of the three semigroups.

**Theorem 4.5.** The semigroups $T_D$, $T_R$ and $T_N$ are submarkovian.

**Proof.** Since $T_D$, $T_R$ and $T_N$ and positivity-preserving, it suffices to that $T_D$, $T_R$ and $T_N$ are $L^\infty$-contractive. We prove the theorem in two steps.

**Step 1:** We claim that $T_N$ is $L^\infty$-contractive. By [31, Lemma 2.7], we have that for every $0 \leq f \in W_{s,2}^\Omega$,

$$\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f, f). \quad (4.9)$$

Next, let $0 \leq u \in D(a_N)$. Then $u_N \in W_{s,2}^\Omega$ and by (4.9) we have

$$\mathcal{E}(u_N \wedge 1, u_N \wedge 1) \leq \mathcal{E}(u_N, u_N) = a_N(u, u). \quad (4.10)$$

We want to show that

$$a_N(u \wedge 1, u \wedge 1) = \mathcal{E}((u \wedge 1)_N, (u \wedge 1)_N) \leq \mathcal{E}(u_N \wedge 1, u_N \wedge 1).$$

Observe that $u_N \wedge 1 = (u \wedge 1)_N$ in $\Omega$ (but not in $\mathbb{R}^n \setminus \Omega$). Calculating we get that
Theorem 4.6. The following assertions hold.

(a) There is a constant $C > 0$ such that

$$\max \left\{ \|T_D(t)\|_{L^1(\Omega), L^\infty(\Omega)}, \|T_R(t)\|_{L^1(\Omega), L^\infty(\Omega)} \right\} \leq Ct^{-\frac{n}{2s}}, \quad \forall t > 0. \tag{4.13}$$

(b) There is a constant $C > 0$ such that

$$\|T_N(t)\|_{L^1(\Omega), L^\infty(\Omega)} \leq Ct^{-\frac{n}{2s} + \epsilon'}, \quad \forall t > 0. \tag{4.14}$$

Proof. Recall that $D(a_D) = \tilde{W}_0^{s,2}(\Omega)$. It follows from (3.16) and (3.29) that the continuous embedding $D(a_N), D(a_R) \hookrightarrow W^{s,2}(\Omega)$ holds. In addition, by Theorem 4.5 we have that $(a_D, D(a_D))$, $(a_R, D(a_R))$ and $(a_N, D(a_N))$ are Dirichlet forms on $L^2(\Omega)$. Hence, using Remark 2.1 we can deduce that $(a_D, D(a_D))$, $(a_R, D(a_R))$ and $(a_N, D(a_N))$ satisfy all the hypotheses in Theorem 2.9 with $q_\alpha = \frac{n}{n-2s} > 1$. We have shown that the operators $A_D, A_R, A_N$, and the semigroups $T_D, T_R,$
Thus, the estimate (4.13) follows from Theorem 2.9 together with the fact that the first eigenvalues of $T_D$ and $T_R$ are strictly positive. The estimate (4.14) also follows from Theorem 2.9 and the fact that for $T_N$, its first eigenvalue is zero, since the constant function $1 \in D(a_N)$ and $a_N(1,1) = 0$. The proof is finished. □

We conclude the paper by giving some open problems.

5. Open problems

In this section we give some interesting open problems related to the three Dirichlet forms and semigroups we have investigated in the previous sections.

1. **Kernel estimates of the semigroups $T_R$ and $T_N$.** It follows from Theorem 2.9 that each of the semigroups $T_D$, $T_R$ and $T_N$ is given by a kernel $K(t, \cdot, \cdot)$ which belongs to $L^\infty(\Omega \times \Omega)$.

It has been shown in [5, 9] that there are two constants $0 < C_1 \leq C_2$ such that for a.e $x, y \in \Omega$ and $t > 0$,

$$C_1 t^{-\frac{N}{2s}} \left(1 + |x - y| t^{-\frac{s}{2}}\right)^{-(N+2s)} \leq K(t, x, y) \leq C_2 t^{-\frac{N}{2s}} \left(1 + |x - y| t^{-\frac{s}{2}}\right)^{-(N+2s)}. \quad (5.1)$$

What is the corresponding estimate for the kernels $K_R(t, \cdot, \cdot)$ and $K_N(t, \cdot, \cdot)$?

2. **Analyticity on $L^1$ of the semigroups $T_R$ and $T_N$.** By Theorem 2.9 again, the semigroups $T_D$, $T_R$ and $T_N$ are analytic on $L^p(\Omega)$ for every $1 < p < \infty$. Very recently, using the estimate (5.1), it has been shown in [24] that the semigroup $T_D$ is analytic on $L^1(\Omega)$.

The analyticity on $L^1(\Omega)$ of $T_R$ and $T_N$ remains an open problem.

3. **Sandwiched semigroups.** The relative capacity $\text{Cap}_\Omega$ has been defined in [2, 3] for an arbitrary set $A \subset \Omega$ by

$$\text{Cap}_\Omega(A) := \inf \left\{ \|u\|_{W^{1,2}(\Omega)} : \exists O \text{ relatively open, } A \subset O, \ u \geq 1 \text{ a.e. in } O \right\}.$$

Let $\eta$ be a regular Borel measure on $\partial \Omega$. Assume that $\eta$ is absolutely continuous with respect to $\text{Cap}_\Omega$ in the sense that $\text{Cap}_\Omega(B) = 0$ $\Rightarrow$ $\eta(B) = 0$ for any Borel set $B \subset \partial \Omega$. Let

$$\text{Cap}_\Omega(B) = 0 \Rightarrow \eta(B) = 0 \text{ for any Borel set } B \subset \partial \Omega. \quad (5.2)$$

Let

$$D(a^\eta) := \left\{ u \in W^{1,2}(\Omega) : \int_{\partial \Omega} |\tilde{u}|^2 \, d\eta < \infty \right\},$$

where $\tilde{u}$ denotes the relative quasi-continuous version of $u$, and define the closed bilinear form $a^\eta : D(a^\eta) \times D(a^\eta) \to \mathbb{R}$ on $L^2(\Omega)$ by

$$a^\eta(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} \tilde{u} \tilde{v} \, d\eta,$$

It has been shown in [2] that the semigroup $T^\eta$ associated with the form $a^\eta$ always satisfies

$$0 \leq T^D \leq T^\eta \leq T^N,$$
in the sense of (2.11), where $T^D$ is the semigroup on $L^2(\Omega)$ associated with the form

$$a^D(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in D(a^D) = W^{1,2}_0(\Omega),$$

and $T^N$ is the semigroup on $L^2(\Omega)$ associated with $(a^N, D(a^N))$ (see (1.1)). Conversely, they have also proved that any symmetric semigroup on $L^2(\Omega)$ associated with a regular and local Dirichlet form (see e.g. [17, Chapter 1] for the definition of a regular and local form), and sandwiched between $T^D$ and $T^N$, is always given by $T^\eta$ for some regular Borel measure $\eta$ on $\partial \Omega$ satisfying (5.2).

For the nonlocal case, we have seen in Remark 4.4(a) that for the form $a_\mu$ (recall that here, $\mu$ is a regular Borel measure on $\mathbb{R}^n \setminus \Omega$) given in (1.7), the associated semigroup $T_\mu$ is not always sandwiched between $T_D$ and $T_N$. Of course in this case, we have shown that $T_\mu \leq T_N$, but the domination $T_D \leq T_\mu$ is not always true.

In addition, consider the form $a : D(a_D) \times D(a_D) \to \mathbb{R}$ given by

$$a(u, v) = \frac{1}{2} \left( a_D(u_D, v_D) + a_N(u_N, v_N) \right).$$

It is easy to see that $a$ is symmetric, closed and densely defined. Let $A$ be the selfadjoint operator on $L^2(\Omega)$ associated with $a$ and $T$ the associated semigroup. Then one can easily show that we have the domination $0 \leq T_D \leq T \leq T_N$. But it is not clear if $A$ is a realization of $(-\Delta)^s$. Therefore a natural question arises.

Let $(a_\mu, D(a_\mu))$ be the form defined in Remark 4.4. Let $T$ be a symmetric semigroup on $L^2(\Omega)$ satisfying $0 \leq T_D \leq T \leq T_N$ and $(a, D(a))$ the closed bilinear form on $L^2(\Omega)$ associated with $T$. Under which conditions on $(a, D(a))$ is $a = a_\mu$?

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