BIGRADED BETTI NUMBERS OF SOME SIMPLE POLYTOPES

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Abstract. The bigraded Betti numbers $\beta^{-i,2j}(P)$ of a simple polytope $P$ are the dimensions of the bigraded components of the Tor groups of the face ring $k[P]$. The numbers $\beta^{-i,2j}(P)$ reflect the combinatorial structure of $P$ as well as the topology of the corresponding moment-angle manifold $\mathcal{Z}_P$, and therefore they find numerous applications in combinatorial commutative algebra and toric topology. Here we calculate some bigraded Betti numbers of the type $\beta^{-i,2(i+1)}$ for associahedra, and relate the calculation of the bigraded Betti numbers for truncation polytopes to the topology of their moment-angle manifolds. These two series of simple polytopes provide conjectural extrema for the values of $\beta^{-i,2j}(P)$ among all simple polytopes $P$ with the fixed dimension and number of facets.

1. Introduction

We consider simple convex $n$-dimensional polytopes $P$ in the Euclidean space $\mathbb{R}^n$ with scalar product $\langle \cdot, \cdot \rangle$. Such a polytope $P$ can be defined as an intersection of $m$ halfspaces:

\begin{equation}
P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle + b_i \geq 0 \quad \text{for } i = 1, \ldots, m \},
\end{equation}

where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. We assume that the hyperplanes defined by the equations $\langle a_i, x \rangle + b_i = 0$ are in general position, that is, at most $n$ of them meet at a single point. We also assume that there are no redundant inequalities in (1.1), that is, no inequality can be removed from (1.1) without changing $P$. Then $P$ has exactly $m$ facets given by

$$F_i = \{ x \in P : \langle a_i, x \rangle + b_i = 0 \}, \quad \text{for } i = 1, \ldots, m.$$

Let $A_P$ be the $m \times n$ matrix of row vectors $a_i$, and let $b_P$ be the column vector of scalars $b_i \in \mathbb{R}$. Then we can write (1.1) as

$$P = \{ x \in \mathbb{R}^n : A_P x + b_P \geq 0 \},$$

and consider the affine map

$$i_P : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad i_P(x) = A_P x + b_P.$$

It embeds $P$ into

$$\mathbb{R}_\geq^m = \{ y \in \mathbb{R}^m : y_i \geq 0 \quad \text{for } i = 1, \ldots, m \}.$$
Following [3, Constr. 7.8], we define the space $\mathcal{Z}_P$ from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Z}_P & \xrightarrow{iz} & \mathbb{C}^m \\
\downarrow & & \downarrow \mu \\
P & \xrightarrow{ip} & \mathbb{R}^{m+}_{\geq 0}
\end{array}
$$

(1.2)

where $\mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2)$. The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

$$
\mathbb{T}^m = \{ z \in \mathbb{C}^m : |z_i| = 1 \text{ for } i = 1, \ldots, m \}
$$
on $\mathbb{C}^m$. Therefore, $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient $P$, and $iz$ is a $\mathbb{T}^m$-equivariant embedding.

By [3, Lemma 7.2], $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$, called the moment-angle manifold corresponding to $P$.

Denote by $K_P$ the boundary $\partial P^*$ of the dual simplicial polytope. It can be viewed as a simplicial complex on the set $[m] = \{1, \ldots, m\}$, whose simplices are subsets $\{i_1, \ldots, i_k\}$. The face ring (also known as the Stanley–Reisner ring) of a simplicial complex $K$ on $[m]$ is the quotient ring

$$
\mathbb{k}[K] = \mathbb{k}[v_1, \ldots, v_m]/\mathcal{I}_K
$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_k}$ for which $\{i_1, \ldots, i_k\}$ is not a simplex in $K$. We refer to $\mathcal{I}_K$ as the Stanley–Reisner ideal of $K$.

Let $\mathbb{k}$ be a field, let $\mathbb{k}[v_1, \ldots, v_m]$ be the graded polynomial algebra on $m$ variables, $\deg(v_i) = 2$, and let $\Lambda[u_1, \ldots, u_m]$ be the exterior algebra, $\deg(u_i) = 1$. The face ring (also known as the Stanley–Reisner ring) of a simplicial complex $K$ on $[m]$ is the quotient ring

$$
\mathbb{k}[K] = \mathbb{k}[v_1, \ldots, v_m]/\mathcal{I}_K
$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_k}$ for which $\{i_1, \ldots, i_k\}$ is not a simplex in $K$. We refer to $\mathcal{I}_K$ as the Stanley–Reisner ideal of $K$.

Note that $\mathbb{k}[K]$ is a module over $\mathbb{k}[v_1, \ldots, v_m]$ via the quotient projection. The dimensions of the bigraded components of the Tor-groups,

$$
\beta^{-i,2j}(K) := \dim_{\mathbb{k}} \text{Tor}^i_{\mathbb{k}[v_1, \ldots, v_m]}(\mathbb{k}[K], \mathbb{k}), \quad 0 \leq i, j \leq m,
$$

are known as the bigraded Betti numbers of $\mathbb{k}[K]$, see [8] and [3, §3.3]. They are important invariants of the combinatorial structure of $K$. We denote

$$
\beta^{-i,2j}(P) := \beta^{-i,2j}(K_P).
$$

The Tor-groups and the bigraded Betti numbers acquire a topological interpretation by means of the following result on the cohomology of $\mathcal{Z}_P$:

**Theorem 1.1** ([3, Theorem 8.6] or [6, Theorem 4.7]). The cohomology algebra of the moment-angle manifold $\mathcal{Z}_P$ is given by the isomorphisms

$$
H^*(\mathcal{Z}_P; \mathbb{k}) \cong \text{Tor}_{\mathbb{k}[v_1, \ldots, v_m]}(\mathbb{k}[K_P], \mathbb{k})
$$

$$
\cong H[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{k}[K_P], d],
$$

where the latter algebra is the cohomology of the differential bigraded algebra whose bigrading and differential are defined by

$$
bideg u_i = (-1, 2), \quad bideg v_i = (0, 2); \quad du_i = v_i, \quad dv_i = 0.\]
Therefore, cohomology of $Z_P$ acquires a bigrading and the topological Betti numbers $b^q(Z_P) = \dim_k H^q(Z_P; k)$ satisfy
\begin{equation}
\label{eq:3}
b^q(Z_P) = \sum_{-i+2j=q} \beta^{-i,2j}(P).
\end{equation}

Poincaré duality in cohomology of $Z_P$ respects the bigrading:

**Theorem 1.2** ([3, Theorem 8.18]). The following formula holds:
\[
\beta^{-i,2j}(P) = \beta^{-(m-n)+i,2(m-j)}(P).
\]

From now on we shall drop the coefficient field $k$ from the notation of (co)homology groups. Given a subset $I \subset [m]$, we denote by $K_I$ the corresponding full subcomplex of $K$ (the restriction of $K$ to $I$). The following classical result can be also obtained as a corollary of Theorem 1.1:

**Theorem 1.3** (Hochster, see [3, Cor. 8.8]). Let $K = K_P$. We have:
\[
\beta^{-i,2j}(P) = \sum_{J \subset [m], |J| = j} \dim \tilde{H}^{j-i}(K_J).
\]

We also introduce the following subset in the boundary of $P$:
\begin{equation}
\label{eq:4}P_I = \bigcup_{i \in I} F_i \subset P.
\end{equation}

Note that if $K = K_P$ then $K_I$ is a deformation retract of $P_I$ for any $I$. The following is a direct corollary of Theorem 1.3.

**Corollary 1.4.** We have
\[
\beta^{-i,2(i+1)}(P) = \sum_{I \subset [m], |I| = i+1} (cc(P_I) - 1),
\]
where $cc(P_I)$ is the number of connected components of the space $P_I$.

The structure of this paper is as follows. Calculations for Stasheff polytopes (also known as associahedra) are given in Section 2. In Section 3 we calculate the bigraded Betti numbers of truncation polytopes (iterated vertex cuts of simplices) completely. These calculations were first made in [10] using a similar but slightly different method; an alternative combinatorial argument was given in [4]. We also compare the calculations of the Betti numbers with the known description of the diffeomorphism type of $Z_P$ for truncation polytopes [1].

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## 2. Stasheff polytopes

Stasheff polytopes, also known as associahedra, were introduced as combinatorial objects in the work of Stasheff on higher associativity [9]. Explicit convex realizations of Stasheff polytopes were found later by Milnor and others, see [2] for details.

We denote the $n$-dimensional Stasheff polytope by $As^n$. The $i$-dimensional faces of $As^n$ ($0 \leq i \leq n - 1$) bijectively correspond to the sets of $n-i$ pairwise
nonintersecting diagonals in an \((n+3)\)-gon \(G_{n+3}\). (We assume that diagonals having a common vertex are nonintersecting.) A face \(H\) belongs to a face \(H'\) if and only if the set of diagonals corresponding to \(H\) contains the set of diagonals corresponding to \(H'\).

In particular, vertices of \(\text{As}^n\) correspond to complete triangulations of \(G_{n+3}\) by its diagonals, and facets of \(\text{As}^n\) correspond to diagonals of \(G_{n+3}\). We therefore identify the set of diagonals in \(G_{n+3}\) with the set of facets \(\{F_1, \ldots, F_m\}\) of \(\text{As}^n\), and identify both sets with \([m]\) when it is convenient. Note that \(m = \frac{n(n+3)}{2}\).

We shall need a convex realization of \(\text{As}^n\) from [2, Lecture II, Th. 5.1]:

**Theorem 2.1.** \(\text{As}^n\) can be identified with the intersection of the parallelepiped

\[
\{\mathbf{y} \in \mathbb{R}^n : 0 \leq y_j \leq j(n+1-j) \text{ for } 1 \leq j \leq n\}
\]

with the halfspaces

\[
\{\mathbf{y} \in \mathbb{R}^n : y_j - y_k + (j-k)k \geq 0 \text{ for } 1 \leq k < j \leq n\}
\]

**Proposition 2.2.** We have:

\[
b_3(\mathcal{Z}_{\text{As}^n}) = \beta^{-1,4}(\text{As}^n) = \binom{n+3}{4}.
\]

**Proof.** The number \(\beta^{-1,4}(P)\) is equal to the number of monomials \(v_i v_j\) in the Stanley–Reisner ideal of \(P\) [3, §3.3], or to the number of pairs of disjoint facets of \(P\). In the case \(P = \text{As}^n\) the latter number is equal to the number of pairs of intersecting diagonals in the \((n+3)\)-gon \(G_{n+3}\), see [2, Lecture II, Cor 6.2]. It remains to note that, for any 4-element subset of vertices of \(G_{n+3}\) there is a unique pair of intersecting diagonals whose endpoints are these 4 vertices. \(\square\)

**Remark.** The above calculation can be also made using the general formula \(\beta^{-1,4}(P) = \left(\begin{array}{c} \frac{n}{2} \\ i \end{array}\right) - f_i\), see [3, Lemma 8.13], where \(f_i\) is the number of \((n-i-1)\)-faces of \(P\). The numbers \(f_i\) for \(\text{As}^n\) are well-known, see [2, Lecture II].

In what follows, we assume that there are no multiple intersection points of the diagonals of \(G_{n+3}\), which can be achieved by a small perturbation of the vertices. We choose a cyclic order of vertices of \(G_{n+3}\), so that 2 consequent vertices are joined by an edge. We refer to the diagonals of \(G_{n+3}\) joining the \(i\)th and the \((i + 2)\)th vertices (modulo \(n + 3\)), for \(i = 1, \ldots, n + 3\) as short; other diagonals are long.

We refer to intersection points of diagonals inside \(G_{n+3}\) as distinguished points. A diagonal segment joining two distinguished points is called a distinguished segment. Finally, a distinguished triangle is a triangle whose vertices are distinguished points and whose edges are distinguished segments.

**Theorem 2.3.** We have:

\[
b_4(\mathcal{Z}_{\text{As}^n}) = \beta^{-2,6}(\text{As}^n) = 5\binom{n + 4}{6}.
\]
Proof. We need to calculate the number of generators in the 4th cohomology group of $H[\Lambda[u_1, \ldots, u_m] \times k[As^n], d]$, see Theorem 1.1 (note that here $m = \binom{n+3}{2} m$ is the number of diagonals in $G_{n+3}$). This group is generated by the cohomology classes of cocycles of the type $u_i u_j v_k$, where $i \neq j$ and $u_i v_k$, $u_j v_k$ are 3-cocycles. These 3-cocycles correspond to the pairs $\{i, k\}$ and $\{j, k\}$ of intersecting diagonals in $G_{n+3}$, or to a pair of distinguished points on the $k$th diagonal. It follows that every cocycle $u_i u_j v_k$ is represented by a distinguished segment. The identity

$$d(u_i u_j u_k) = u_i u_j v_k - u_i v_j u_k + v_i u_j u_k$$

implies that the cohomology classes represented by the cocycles in the right hand side are linearly dependent. Every such identity corresponds to a distinguished triangle.

We therefore obtain that $\beta^{-2,6}(As^n) = S_{n+3} - T_{n+3}$ where $S_{n+3}$ is the number of distinguished segments and $T_{n+3}$ is the number of distinguished triangles inside $G_{n+3}$. These numbers are calculated in the next three lemmas.

Lemma 2.4. The number of distinguished triangles in $G_{n+3}$ is given by

$$T_{n+3} = \binom{n + 3}{6}$$

Proof. We note that there is only one distinguished triangle in a hexagon (see Fig. 1); and therefore every 6 vertices of $G_{n+3}$ contribute one distinguished triangle.

![Figure 1.](image)

Given a diagonal $d$ of $G_{n+3}$, denote by $p(d)$ the number of distinguished points on $d$. We define the length of $d$ as the smallest of the numbers of vertices of $G_{n+3}$ in the open halfplanes defined by $d$. Therefore, short diagonals have length 1 and all diagonals have length $\leq \frac{n+1}{2}$. We refer to diagonals of maximal length simply as maximal. Obviously $p(d)$ depends only on the length of $d$, and we denote by $p(j)$ the number of distinguished points on a diagonal of length $j$.

Lemma 2.5. If $n = 2k - 1$ is odd, then

$$S_{n+3} = \frac{n + 3}{2} \sum_{l=1}^{k-1} \left( 4l^2 k^2 - 2k(2l^3 + l) \right) + \frac{n + 3}{4} k^2 (k^2 - 1).$$
If \( n = 2k - 2 \) is even, then
\[
S_{n+3} = \frac{n+3}{2} \sum_{l=1}^{k-1} \left( 4l^2k^2 - 2k(2l^3 + 2l^2 + l) + (l^4 + 2l^3 + 2l^2 + l) \right).
\]

Proof. First assume that \( n = 2k - 1 \). Then
\[
S_{n+3} = \sum_d \frac{p(d)(p(d) - 1)}{2} = (n + 3) \left( \sum_{j=1}^{n+1} \frac{p(j)(p(j) - 1)}{2} \right) - \left( \frac{n+3}{2} \right) p(\frac{n+1}{2})p(\frac{n+1}{2}) - 1,
\]
since the number of distinguished segments on the maximal diagonals is counted in the sum twice.

We denote by \( v \) the \((n+3)\)th vertex of \( G_{n+3} \) and numerate the diagonals coming from \( v \) by their lengths. We denote by \( c(i, j) \) the number of intersection points of the \( j \)th diagonal coming from \( v \) with the diagonals from the \( i \)th vertex, for \( 1 \leq i \leq j \leq \frac{n+1}{2} \), and set \( c(i, j) = 0 \) for \( i > j \). Then we have
\[
(2.1) \quad p(j) = \sum_{i=1}^{n+1} c(i, j),
\]
To compute \( c(i, j) \) we note that
\[
c(1, 1) = n; \quad c(i, j - 1) = c(i, j) + 1 \quad \text{for} \quad 1 \leq i < j \leq \frac{n+1}{2}; \quad c(i + 1, j + 1) = c(i, j) - 1 \quad \text{for} \quad 1 \leq i < j \leq \frac{n+1}{2}.
\]
It follows that
\[
(2.2) \quad c(i, j) = c(1, j - i + 1) - (i - 1) = c(1, 1) - (j - i) - (i - 1) = n - j + 1,
\]
for \( i \leq j \). Note that \( c(i, j) \) does not depend on \( i \). Substituting this in (2.1) and then substituting the resulting expression for \( p(j) \) in the sum for \( S_{n+3} \) above we obtain the required formula.

The case \( n = 2k - 2 \) is similar. The only difference is that there are two maximal diagonals coming from every vertex of \( G_{n+3} \), so that no subtraction is needed in the sum for \( S_{n+3} \).

**Lemma 2.6.** The number of distinguished segments is given by
\[
S_{n+3} = (n + 3) \binom{n+3}{5}.
\]
Proof. This follows from Lemma 2.5 by summation using the following formulae for the sums \( \Sigma_n \) of the \( n \)th powers of the first \((k-1)\) natural numbers:
\[
\begin{align*}
\Sigma_1 &= \frac{k(k-1)}{2}, & \Sigma_2 &= \frac{k(k-1)(2k-1)}{6}, \\
\Sigma_3 &= \frac{k^2(k-1)^2}{4}, & \Sigma_4 &= \frac{k(k-1)(2k-1)(3k^2 - 3k - 1)}{30}.
\end{align*}
\]
Now Theorem 2.3 follows from Lemma 2.5 and Lemma 2.6. □

The following fact follows from the description of the combinatorial structure of $A_s^n$ (see also [2, Lecture II, Cor. 6.2]):

**Proposition 2.7.** Two facets $F_1$ and $F_2$ of the polytope $A_s^n$ do not intersect if and only if the corresponding diagonals $d_1$ and $d_2$ of the polygon $G_{n+3}$ intersect (in a distinguished point).

**Lemma 2.8.** The number of distinguished points on a maximal diagonal of $G_{n+3}$ is given by

$$q = q(n) = \begin{cases} \frac{n(n+2)}{4}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** The case $n = 2$ is obvious. If $n$ is odd, then setting $j = \frac{n+1}{2}$ in (2.1) and using (2.2) we calculate

$$p(j) = \sum_{i=1}^{\frac{n+1}{2}} c(i, \frac{n+1}{2}) = \frac{(n+1)^2}{4}.$$ 

If $n$ is even, then the maximal diagonal has length $j = \frac{n}{2}$. It is easy to see that we have $p(j) = \sum_{i=1}^{n/2} c(i, j)$ instead of (2.1), and (2.2) still holds. Therefore,

$$p(j) = \sum_{i=1}^{\frac{n}{2}} c(i, \frac{n}{2}) = \frac{n(n+2)}{4}.$$ 

**Theorem 2.9.** Let $P = A_s^n$ be an $n$-dimensional associahedron, $n \geq 3$. The bigraded Betti numbers of $P$ satisfy

$$\beta^{-q, 2(q+1)}(P) = \begin{cases} n+3, & \text{if } n \text{ is even;} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta^{-i, 2(i+1)}(P) = 0 \text{ for } i \geq q+1;$$

where $q = q(n)$ is given in Lemma 2.8.

**Proof.** We prove the theorem by induction on $n$. The base case $n = 3$ can be seen from the tables of bigraded Betti numbers below. By Corollary 1.4, in order to calculate $\beta^{-i, 2(i+1)}(P)$, we need to find all $I \subset [m], \ |I| = i + 1$, whose corresponding $P_I$ has more than one connected component. In the case $i = q$ we shall prove that $cc(P_I) \leq 2$ for $|I| = q + 1$, and describe explicitly those $I$ for which $cc(P_I) = 2$. In the case $i > q$ we shall prove that $cc(P_I) = 1$ for $|I| = i + 1$. These statements will be proven as separate lemmas; the step of induction will follow at the end.

We numerate the vertices of $G_{n+3}$ by the integers from 1 to $n + 3$. Then every diagonal $d$ corresponds to an ordered pair $(i, j)$ of integers such that $i < j - 1$. It is convenient to view the diagonal corresponding to $(i, j)$ as the segment $[i, j]$ inside the segment $[1, n+3]$ on the real line. Then Proposition 2.7 may be reformulated as follows:
Proposition 2.10. The facets $F_1$ and $F_2$ of $P = A s^n$ do not intersect if and only if the corresponding segments $[i_1, j_1]$ and $[i_2, j_2]$ overlap, that is, $F_1 \cap F_2 = \emptyset \iff i_1 < i_2 < j_1 < j_2$ or $i_2 < i_1 < j_2 < j_1$.

Proof. Follows directly from Proposition 2.10.

Proposition 2.11. If $I = I_1 \cup I_2$ then the subsets $e(I_1)$ and $e(I_2)$ are disjoint.

Proof. Given an integer $m \in [1, n + 3]$ and a set of segments $I$, we denote by $c_I(m)$ the number of segments in $I$ that have $m$ as one of their endpoints (equivalently, the number of diagonals in $I$ with endpoint $m$). Then $0 \leq c_I(m) \leq n$.

Proposition 2.12. If $I = I_1 \cup I_2$ then there exists $m$ such that $c_I(m) \leq \frac{n + 1}{2}$.

Proof. Assume the opposite is true. Choose integers $m_1 \in e(I_1)$ and $m_2 \in e(I_2)$. Since $c_I(m_1) > \frac{n + 1}{2}$, $c_I(m_2) > \frac{n + 1}{2}$ and $e(I_1)$, $e(I_2)$ are disjoint by the previous proposition, we obtain that the total number of elements in $e(I)$ is more than $2 + \frac{n + 1}{2} + \frac{n + 1}{2} = n + 3$. A contradiction.

Lemma 2.13. We have that $cc(P_I) \leq 2$ for $|I| > l(n) = \frac{n(n + 2)}{4}$.

Proof. We prove this lemma by induction on $n$.

First let $n = 3$, and assume that the statement of the lemma fails, i.e. there is a set $I = I_1 \cup I_2 \cup I_3 \cup \ldots$ of diagonals of $G_6$, $|I| \geq 4$, such that $cc(P_I) \geq 3$. As there are only 3 long diagonals in $G_6$, there exists a short diagonal $d \in I$; assume $d \in I_1$. Since $cc(P_I) \geq 3$, every $e \in I_2$ and $f \in I_3$ intersect $d$. Hence, $e$ and $f$ meet at a vertex $A$ of $G_6$. This contradicts the fact that $e(I_2)$ and $e(I_3)$ are disjoint (see Proposition 2.11).

Now let $n > 3$ and assume that there is a set $I = I_1 \cup I_2 \cup I_3 \cup \ldots$ of diagonals of $G_{n+3}$, $|I| > \frac{n(n+2)}{4}$, with $cc(P_I) \geq 3$.

If there exists $m \in [1, n + 3]$ with $c_I(m) = 0$, then we may assume that $m$ is the first vertex, and view $I$ as a set of diagonals of $G_{n+2}$ (the segment $[2, n + 3]$ cannot belong to $I$, since otherwise $cc(P_I) = 1$). As $l(n) > l(n-1)$, the induction assumption finishes the proof of the lemma.

Now $c_I(m) \geq 1$ for every $m \in [1, n + 3]$. Then by the argument similar to that of Proposition 2.12, there exists $m$ with $c_I(m) \leq \frac{n}{2}$. Consider 2 cases: 1. There exists $m_0 \in e(I_k)$ for some $1 \leq k \leq cc(P_I)$ with the smallest value of $c_I(m) \leq \frac{n}{2}$, such that $|I_k| > c_I(m_0)$.

We may assume that one of these $m_0$ is the first vertex. Removing from $I$ all segments with endpoint 1, we obtain a new set $I$ of segments inside $[2, n + 3]$ (the segment $[2, n + 3]$ cannot belong to $I$, as otherwise $cc(P_I) \leq 2$). We have:

$$|I| = |I| - c_I(1) > \frac{n(n + 2)}{4} - \frac{n}{3} > \frac{(n - 1)(n + 1)}{4} = l(n - 1).$$

By the induction assumption, $2 \geq cc(P_I) \geq cc(P_I) \geq 3$. A contradiction.
2. For every vertex $m_0$ with the smallest value of $c_I(m) \geq 1$ we have $|I_k| = c_I(m_0)$, where $m_0 \in e(I_k)$.

Again, we may assume that one of these $m_0$ is the first vertex $1 \in I_k$. We have $c_I(1) = 1$, as otherwise there are $\geq 2$ integer points $m$ inside $[2, n + 3]$ which belong to $e(I_k)$ and have $c_I(m) = 1$ (remember that $|I_k| = c_I(m_0)$).

Without loss of generality we may assume that $k = 1$. Then

$$|I| = 1 + |I_2| + |I_3| + \ldots \leq 1 + (1 + q(n - 1)) \leq 2 + \frac{n^2}{4} \leq \frac{n(n + 2)}{4}.$$  

The first inequality above holds since $\bar{I} = I_2 \sqcup I_3 \sqcup \ldots$ is a set of diagonals of $G_{n+3}$ (the segment $[2, n + 3]$ cannot belong to $I$, because $cc(P_I) \geq 3$), and we can apply to $\bar{I}$ the induction assumption in the proof of the main Theorem 2.9, which gives us $|\bar{I}| \leq 1 + q(n - 1)$. We get a contradiction with the assumption $|I| > \frac{n(n+2)}{4}$. \hfill \qed

**Lemma 2.14.** Assume that $I = I_1 \sqcup I_2$, $|I| \geq q + 1$, $|I_1| \geq 2$ and $|I_2| \geq 2$. Then there exists another $I'$ such that $I' = I_1' \sqcup I_2'$, $|I'| = 1$ and $|I'| > |I|$.

**Proof.** The proof is by induction on $n$. The cases $n = 3, 4, 5$ are checked by a direct computation (see also the tables at the end of this section).

Changing the numeration of vertices of $G_{n+3}$ if necessary, we may assume that the first vertex has the smallest value of $c_I(m)$. Then $c_I(1) \leq \frac{n+1}{2}$ by Proposition 2.12. Without loss of generality we may assume that $1 \not\in e(I_1)$.

We claim that the segment $[2, n + 3]$ does not belong to $I$. Indeed, in the opposite case $c_I(1) > 0$ (otherwise $cc(P_I) = 1$), $1 \in e(I_2)$, $[2, n + 3] \in I_1$. If $c_I(1) \geq 2$, then there is an integer point $m \in e(I_2)$ inside $[2, n + 3]$ with $c_I(m) = 1 < c_I(1)$, which contradicts the choice of the first vertex. Then $c_I(1) = 1$ and $[2, n + 3] \in I_1$ imply that $|I_2| = c_I(1) = 1$ which contradicts the assumption $|I_2| \geq 2$ in the lemma.

Removing from $I$ all segments with endpoint 1, we obtain a new set $\bar{I}$ of integer segments inside $[2, n + 3]$. Note that

$$(2.3) \quad |\bar{I}| = |I| - c_I(1) \geq |I| - \left[ \frac{n+1}{2} \right].$$

We want to apply the induction assumption to the set $\bar{I}$ of integer segments inside $[2, n + 3]$, viewed as diagonals in an $(n + 2)$-gon $G_{n+2}$. To do this, we need to check the assumptions of the lemma for $\bar{I}$.

First, we claim that $\bar{I} = \bar{I}_1 \sqcup \bar{I}_2$, i.e. $P_{\bar{I}}$ has exactly two connected components. Indeed, it obviously has at least two components, and the number of components cannot be more than two by Lemma 2.13, since

$$|\bar{I}| \geq |I| - \frac{n+1}{2} \geq q + 1 - \frac{n+1}{2} > \frac{(n+1)^2}{4} - \frac{n+1}{2} = l(n-1).$$

Second, $|\bar{I}_1| = |I_1| \geq 2$ and $|\bar{I}_2| \geq |\bar{I}_2| \geq 1$. If $|\bar{I}_2| = 1$ then we have either $c_I(1) = 1$ or $c_I(1) = 2$. (Indeed, if $c_I(1) = 0$ then $|I_2| = |\bar{I}_2| = 1$, which contradicts the assumption, and $c_I(1)$ cannot be more than 2 as otherwise $c_I(1)$ is not the smallest one.) Therefore, $|I_2| \leq 3$. We also have $|I_1| = |I_1| \leq p(d)$, where $d \in I_2 = \{d\}$, because $d$ intersects every diagonal from $I_1$. Due to Lemma 2.8, $p(d) \leq q(n - 1) \leq \frac{n^2}{4}$. Hence,

$$|I| = |I_1| + |I_2| \leq p(d) + 3 \leq \frac{n^2}{4} + 3 \leq \frac{(n+1)^2}{4} \leq q(n) + 1 \leq |I|$$
for \( n \geq 6 \). A contradiction. Thus, \(|\tilde{J}_2| \geq 2\).

It remains to check that \( |\tilde{I}| \geq q(n-1)+1 \). If \( n \) is odd, then
\[
|\tilde{I}| \geq |I| - \frac{n+1}{2} \geq \frac{(n+1)^2}{4} + 1 - \frac{n+1}{2} = \frac{(n-1)(n+1)}{4} + 1 = q(n-1)+1.
\]

If \( n \) is even, then
\[
|\tilde{I}| \geq |I| - \frac{n}{2} \geq \frac{n(n+2)}{4} + 1 - \frac{n}{2} = \frac{n^2}{4} + 1 = q(n - 1) + 1.
\]

Now, applying the induction assumption to \( \tilde{I} \), we find a new set of integer segments \( J \) inside \([2, n + 3]\) with \( |J| > |\tilde{I}| \) and \(|\tilde{J}_1| = 1\). Then \( \tilde{J}_1 = \{d\} \), where \( d \) is a diagonal of \( G_{n+2} \). Hence, \(|J| = |\tilde{J}_1| + |\tilde{J}_2| \leq 1 + p(d)\). We have \( p(d) \leq q(n-1) \), and the equality holds if and only if \( d = d_{\text{max}} \) is a maximal diagonal in \( G_{n+2} \). Therefore, we can replace \( \tilde{J} \) by \( J' = J'_1 \cup J'_2 \), where \( J'_1 = \{d_{\text{max}}\} \) and \( J'_2 \) is the set of diagonals in \( G_{n+2} \) which intersect \( d_{\text{max}} \) at its distinguished points. Indeed, we have
\[
|J'| = 1 + q(n-1) \geq 1 + p(d) \geq |\tilde{J}| > |\tilde{I}|.
\]

Choosing \( d_{\text{max}} \) in \( G_{n+3} \) as the diagonal corresponding to the segment \([2, k]\) where \( k = \left\lceil \frac{n+2}{2} \right\rceil \) we observe that it is also a maximal diagonal for \( G_{n+3} \). Now take \( I'_1 = \{d_{\text{max}}\} \) and take \( I'_2 \) to be the union of \( J'_2 \) and all diagonals with endpoint 1 intersecting \( d_{\text{max}} \). Since the number of distinguished points on \( d_{\text{max}} \) is \( \left\lceil \frac{n+1}{2} \right\rceil \), we obtain from (2.4) and (2.3)
\[
|I'| = 1 + |I'_2| = 1 + |J'_2| + \left\lceil \frac{n+1}{2} \right\rceil = |J'| + \left\lceil \frac{n+1}{2} \right\rceil > |\tilde{I}| + \left\lceil \frac{n+1}{2} \right\rceil \geq |I|,
\]
which finishes the inductive argument. \( \square \)

Lemma 2.15. Suppose \( cc(P_I) = 2 \), \( I = I_1 \sqcup I_2 \) and \(|I| \geq q + 1\). Then either \(|I_1| = 1\) or \(|I_2| = 1\).

Proof. Assume the opposite, i.e. \(|I_1| \geq 2\) and \(|I_2| \geq 2\). By Lemma 2.14, we may find another \( I' = I'_1 \sqcup I'_2 \) such that \(|I'_1| = 1\) and \(|I'| > |I| \geq q + 1\). On the other hand \(|I'_1| = 1\) implies that \( I'_1 = \{d\} \) and \(|I'| \leq 1 + p(d) \leq 1 + q\). A contradiction. \( \square \)

Lemma 2.16. Suppose \( cc(P_I) = 2 \), \( I = I_1 \sqcup I_2 \) and \(|I| = q + 1\). Then \( I_1 \) consists of a single maximal diagonal \( d_{\text{max}} \), and \( I_2 \) consists of all diagonals of \( G_{n+3} \) which intersect \( d_{\text{max}} \).

Proof. By Lemma 2.15, we may assume that \( I_1 \) consists of a single diagonal \( d \). Then
\[
1 + q = |I| = |I_1| + |I_2| \leq 1 + p(d) \leq 1 + q,
\]
which implies that \( p(d) = q \) and \(|I_2| = p(d)|. \( \square \)

Lemma 2.17. Suppose \(|I| > q + 1\). Then \( cc(P_I) = 1\).

Proof. We have \(|I| > q + 1 \geq l(n)\). Hence, \( cc(P_I) \leq 2 \) by Lemma 2.13. Assume \( cc(P_I) = 2 \) and \( I = I_1 \sqcup I_2 \). Then \(|I_1| = 1\) by Lemma 2.15, i.e. \( I_1 = \{d\} \) and \(|I| \leq 1 + p(d) \leq 1 + q\). This contradicts the assumption \(|I| > q + 1\).
\( \square \)
Now we can finish the induction in the proof of Theorem 2.9. From Corollary 1.4 and Lemma 2.16 we obtain that the number \( \beta^{-q,2(q+1)}(P) \) is equal to the number of maximal diagonals in \( G_n+3 \). The latter equals \( n+3 \) when \( n \) is even, and \( \frac{n+3}{2} \) when \( n \) is odd. The fact that \( \beta^{-i,2(i+1)}(P) \) vanishes for \( i \geq q+1 \) follows from Corollary 1.4 and Lemma 2.17.

We also calculate the bigraded Betti numbers of \( As^n \) for \( n \leq 5 \) using software package Macaulay 2, see [5].

The tables below have \( n-1 \) rows and \( m-n-1 \) columns. The number in the intersection of the \( k \)th row and the \( l \)th column is \( \beta^{-l,2(l+k)}(As^n) \), where \( 1 \leq l \leq m-n-1 \) and \( 2 \leq l+k \leq m-2 \). The other bigraded Betti numbers are zero except for \( \beta^{0,0}(As^n) = \beta^{-(m-n),2m}(As^n) = 1 \), see [3, Ch.8]. The bigraded Betti numbers given by Theorem 2.9 are printed in bold.

1. \( n = 2, m = 5 \).

\[
\begin{array}{cccccc}
5 & 5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

2. \( n = 3, m = 9 \).

\[
\begin{array}{cccccc}
15 & 35 & 24 & 3 & 0 & 0 \\
0 & 3 & 24 & 35 & 15 & 0 \\
\end{array}
\]

3. \( n = 4, m = 14 \).

\[
\begin{array}{cccccccc}
35 & 140 & 217 & 154 & 49 & 7 & 0 & 0 \\
0 & 28 & 266 & 784 & 1094 & 784 & 266 & 28 & 0 \\
0 & 0 & 0 & 7 & 49 & 154 & 217 & 140 & 35 \\
\end{array}
\]

4. \( n = 5, m = 20 \).

\[
\begin{array}{cccccccc}
70 & 420 & 1089 & 1544 & 1300 & 680 & 226 & 44 & 4 & 0 & 0 \\
0 & 144 & 1796 & 8332 & 20924 & 32309 & 32184 & 20798 & 8480 & 2053 & 264 \\
0 & 0 & 12 & 264 & 2053 & 8480 & 20798 & 32184 & 32309 & 20924 & 8332 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 44 & 226 & 680 & 1300 & 1544 \\
\end{array}
\]

The topology of moment-angle manifolds \( Z_P \) corresponding to associahedra is far from being well understood even in the case when \( P \) is 3-dimensional. In this case the cohomology ring \( H^*(Z_P) \) has nontrivial triple Massey products by a result of Baskakov (see [3, §8.4] or [6, §5.3]), which implies that \( Z_P \) is not formal in the sense of rational homotopy theory.

3. Truncation polytopes

Let \( P \) be a simple \( n \)-polytope and \( v \in P \) its vertex. Choose a hyperplane \( H \) such that \( H \) separates \( v \) from the other vertices and \( v \) belongs to the positive halfspace \( H_\geq \) determined by \( H \). Then \( P \cap H_\geq \) is an \( n \)-simplex, and \( P \cap H_\leq \) is a simple polytope, which we refer to as a vertex cut of \( P \). When the choice of the cut vertex is clear or irrelevant we use the notation \( vc(P) \).

We also use the notation \( vc^k(P) \) for a polytope obtained from \( P \) by iterating the vertex cut operation \( k \) times.

As an example of this procedure, we consider the polytope \( vc^k(\Delta^n) \), where \( \Delta^n \) is an \( n \)-simplex, \( n \geq 2 \). We refer to \( vc^k(\Delta^n) \) as a truncation polytope; it has \( m = n + k + 1 \) facets. Note that the combinatorial type of \( vc^k(\Delta^n) \) depends on the choice of the cut vertices if \( k \geq 3 \), however we shall not reflect this in the notation.
Simplicial polytopes dual to $vc^k(\Delta^n)$ are known as stacked polytopes. They can be obtained from $\Delta^n$ by iteratively adding pyramids over facets.

The Betti numbers for stacked polytopes were calculated in [10], but the grading used there was different. We include this result below, with a proof that uses a slightly different argument and our ‘topological’ grading and notation:

**Theorem 3.1.** Let $P = vc^k(\Delta^n)$ be a truncation polytope. Then for $n \geq 3$ the bigraded Betti numbers are given by the following formulæ:

$$
\beta^{-i,2(i+1)}(P) = i \binom{k+1}{i+1}, \\
\beta^{-i,2(i+n-1)}(P) = (k+1-i) \binom{k+1}{k+2-i}, \\
\beta^{-i,2j}(P) = 0, \text{ for } i+1 < j < i+n-1.
$$

The other bigraded Betti numbers are also zero, except for $\beta^0,0(P) = \beta^{-m-n,2m}(P) = 1$.

**Remark.** The first of the above formulæ was proved in [4] combinatorially.

**Proof.** We start by analysing the behavior of bigraded Betti numbers under a single vertex cut. Let $P$ be an arbitrary simple polytope and $P' = vc(P)$. We denote by $Q$ and $Q'$ the dual simplicial polytopes respectively, and denote by $K$ and $K'$ their boundary simplicial complexes. Then $Q'$ is obtained by adding a pyramid with vertex $v$ over a facet $F$ of $Q$. We also denote by $V$, $V'$ and $V(F)$ the vertex sets of $Q$, $Q'$ and $F$ respectively, so that $V' = V \cup v$.

The proof of the first formula is based on the following lemma:

**Lemma 3.2.** Let $P$ be a simple $n$-polytope with $m$ facets and $P' = vc(P)$. Then

$$
\beta^{-i,2(i+1)}(P') = \binom{m-n}{i} + \beta^{-i,2(i-1)}(P) + \beta^{-i,2(i+1)}(P).
$$

**Proof.** Applying Theorem 1.3 for $j = i+1$, we obtain:

$$
\beta^{-i,2(i+1)}(P') = \sum_{W \subseteq V' : |W| = i+1} \dim \tilde{H}_0(K'_W) = \sum_{W \subseteq V', v \in W : |W| = i+1} \dim \tilde{H}_0(K'_W) + \sum_{W \subseteq V', v \notin W : |W| = i+1} \dim \tilde{H}_0(K'_W).
$$

Sum (3.2) above is $\beta^{-i,2(i+1)}(P)$ by Theorem 1.3.

For sum (3.1) we have: in $W$ there are $i$ ‘old’ vertices and one new vertex $v$. Therefore, the number of connected components of $K'_W$ (which is by 1 greater than the dimension of $\tilde{H}_0(K'_W)$) either remains the same (if $W \cap F \neq \emptyset$) or increases by 1 (if $W \cap F = \emptyset$, in which case the new component is the new vertex $v$). The number of subsets $W$ of the latter type is equal to the
number of ways to choose $i$ vertices from the $m - n$ ‘old’ vertices that do not lie in $F$. Sum (3.1) is therefore given by

$$\sum_{W \subset V, |W| = i} \dim \tilde{H}_0(K_W) + \binom{m - n}{i} = \beta^{-(i-1),2}(P) + \binom{m - n}{i},$$

where we used Theorem 1.3 again. □

Now the first formula of Theorem 3.1 follows by induction on the number of cut vertices, using the fact that $\beta^{-(i-1),2(i+1)}(\Delta^n) = 0$ for all $i$ and Lemma 3.2.

The second formula follows from the bigraded Poincare duality, see Theorem 1.2.

The proof of the third formula relies on the following lemma.

Lemma 3.3. Let $P$ be a truncation polytope, $K$ the boundary complex of the dual simplicial polytope, $V$ the vertex set of $K$, and $W$ a nonempty proper subset of $V$. Then

$$\tilde{H}_i(K_W) = 0 \quad \text{for} \quad i \neq 0, n - 2.$$

Proof. The proof is by induction on the number $m = |V|$ of vertices of $K$. If $m = n + 1$, then $P$ is an $n$-simplex, and $K_W$ is contractible for every proper subset $W \subset V$.

To make the induction step we consider $V' = V \cup v$ and $V(F)$ as in the beginning of the proof of Theorem 3.1. Assume the statement is proved for $V$ and let $W$ be a proper subset of $V'$.

We consider the following 5 cases.

Case 1: $v \in W$, $W \cap V(F) \neq \emptyset$.

If $V(F) \subset W$, then $K'_W$ is a subdivision of $K_W - \{v\}$. It follows that $\tilde{H}_i(K'_W) \simeq \tilde{H}_i(K_W - \{v\})$.

If $W \cap V(F) \neq V(F)$, then we have

$$K'_W = K_W - \{v\} \cup K'_{W \cap V(F) \cup \{v\}}, \quad K_W - \{v\} \cap K'_{W \cap V(F) \cup \{v\}} = K_W \cap V(F),$$

and both $K_W \cap V(F)$ and $K'_{W \cap V(F) \cup \{v\}}$ are contractible. From the Mayer–Vietoris exact sequence we again obtain $\tilde{H}_i(K'_W) \cong \tilde{H}_i(K_W - \{v\})$.

Case 2: $v \in W$, $W \cap V(F) = \emptyset$.

In this case it is easy to see that $K'_W = K_W - \{v\} \sqcup \{v\}$. It follows that

$$\tilde{H}_i(K'_W) \cong \begin{cases} \tilde{H}_i(K_W - \{v\}) \oplus k, & \text{for} \ i = 0; \\ \tilde{H}_i(K_W - \{v\}), & \text{for} \ i > 0. \end{cases}$$

Case 3: $W = V' - \{v\} = V$.

Then $K'_W$ is a triangulated $(n - 1)$-disk and therefore contractible.

Case 4: $v \notin W$, $V(F) \subset W$, $W \neq V$.

We have

$$K_W = K'_W \cup F, \quad K'_W \cap F = \partial F,$$
where $\partial F$ is the boundary of the facet $F$. Since $\partial F$ is a triangulated $(n-2)$-sphere and $F$ is a triangulated $(n-1)$-disk, the Mayer–Vietoris homology sequence implies that

$$\tilde{H}_i(K'_W) \cong \begin{cases} \tilde{H}_i(K_W), & \text{for } i < n-2; \\ \tilde{H}_i(K_W) \oplus k, & \text{for } i = n-2. \end{cases}$$

**Case 5:** $v \notin W$, $V(F) \subset W$. In this case we have $K'_W \cong K_W$.

In all cases we obtain

$$\tilde{H}_i(K'_W) \cong \tilde{H}_i(K_W - \{v\}) = 0 \quad \text{for } 0 < i < n-2,$$

which finishes the proof by induction. □

Now the third formula of Theorem 3.1 follows from Theorem 1.3 and Lemma 3.3.

The last statement of Theorem 3.1 follows from [3, Cor. 8.19]. □

For the sake of completeness we include the calculation of the bigraded Betti numbers in the case $n = 2$, that is, when $P$ is a polygon.

**Proposition 3.4.** If $P = vc^k(\Delta^2)$ is an $(k+3)$-gon, then

$$\beta^{-i,2(i+1)}(P) = i \left( \frac{k+1}{i+1} \right) + (k+1-i) \left( \frac{k+1}{k+2-i} \right),$$
$$\beta^{0,0}(P) = \beta^{-(k+1),2(k+3)}(P) = 1,$$
$$\beta^{-i,2j}(P) = 0, \quad \text{otherwise.}$$

**Proof.** This calculation was done in [3, Example 8.21]. It can be also obtained by a Mayer–Vietoris argument as in the proof of Theorem 3.1. □

**Corollary 3.5.** The bigraded Betti numbers of truncation polytopes $P = vc^k(\Delta^n)$ depend only on the dimension and the number of facets of $P$ and do not depend on its combinatorial type. Moreover the numbers $\beta^{-i,2(i+1)}$ do not depend on the dimension $n$.

The topological type of the corresponding moment-angle manifold $Z_P$ is described as follows:

**Theorem 3.6** (see [1, Theorem 6.3]). Let $P = vc^k(\Delta^n)$ be a truncation polytope. Then the corresponding moment-angle manifold $Z_P$ is diffeomorphic to the connected sum of sphere products:

$$\bigoplus_{j=1}^{k} \left( S^{j+2} \times S^{2n+k-j-1} \right)^{\#j^{(k+1)}},$$

where $X^{\#k}$ denotes the connected sum of $k$ copies of $X$.

It is easy to see that the Betti numbers of the connected sum above agree with the bigraded Betti numbers of $P$, see (1.3).
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