SINGULARITIES OF RATIONAL CURVES ON K3 SURFACES

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Abstract. We proved that every rational curve in the primitive class of a general K3 surface is nodal.

1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem 1.1. For \(g \geq 3\), all rational curves in the linear system \(|O_S(1)|\) on a general primitive K3 surface \(S\) in \(\mathbb{P}^n\) are nodal.

The motivation to study this problem has been explained in [C]. Basically, we want to justify the beautiful formula of Yau and Zaslow [Y-Z], which counts the number of rational curves in \(|O_S(1)|\) on a primitive K3 surface \(S \subset \mathbb{P}^n\). The primary consequence of Theorem 1.1 is that the formula of Yau and Zaslow actually gives the number of rational curves in \(|O_S(1)|\) on a general K3 surface \(S \subset \mathbb{P}^n\).

It has been proved in [C] that Theorem 1.1 is true for \(g \leq 9\) and \(g = 11\) by degenerating a general K3 surface to a trigonal K3 surface. However, for \(g\) large, we have to further degenerate a trigonal K3 surface. The complexities involved in this process prevent us carrying out the proof for any \(g\). Although here we still use a degeneration argument, our approach is entirely different. Here is a rough sketch of the proof.

We start with the degeneration of a K3 surface to the union of two rational surfaces. Let \(X \to \Delta\) be a family of K3 surfaces of genus \(g\) over the disk \(\Delta\) whose central fiber \(X_0 = R = R_1 \cup R_2\) is the union of two rational surfaces \(R_1\) and \(R_2\) which meet transversely along an elliptic curve \(E = R_1 \cap R_2\). We may choose \(R_i\) to be \(\mathbb{P}^1 \times \mathbb{P}^1\) if \(g\) is odd and choose \(R_i\) to be \(\mathbb{F}_1\) if \(g\) is even for \(i = 1, 2\). Let us consider the case that \(g = 2k + 1\) is odd. We may construct \(X\) in such a way that the limit of primitive line bundles \(O_{X_t}(1)\) on \(X_t\) is the line bundle \(O_R(1)\) on \(X_0 = R\), whose restriction to each \(R_i \cong \mathbb{P}^1 \times \mathbb{P}^1\) is the line bundle of type \((1, k)\). So if we have a family \(\Upsilon \subset X\) of rational curves

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over $\Delta$, whose general fiber $\Upsilon_t$ is a rational curve in the linear series $|\mathcal{O}_{X_t}(1)|$ for each $t$, the central fiber $\Upsilon_0$ will be a curve in the linear series $|\mathcal{O}_{R}(1)|$ and hence $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$ where $\Sigma_i$ is a curve of type $(1, k)$ on $R_i$. Our trivial observation is that $\Upsilon_t$ is nodal if $\Upsilon_0$ is nodal. However, $\Upsilon_0$ could fail to be nodal where

1. it has a reduced (i.e. isolated) singularity other than a node; or
2. it is nonreduced.

The first case turns out much easier to handle than the second. This is basically due to the fact that each $\Sigma_i$ is a curve of type $(1, k)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. So all the isolated singularities of $\Sigma_i$ are nodes. If $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$ has an isolated singularity other than a node, it must be one of the intersections between $\Sigma_1$ and $\Sigma_2$ on $E$. The deformation of such singularities has been studied in [C]. With a bit more care, we are able to show that these singularities deform to nodes on the general fiber $\Upsilon_t$. However, if $\Upsilon_t$ is a rational curve in a multiple of the primitive class, $\Sigma_i$ might have isolated singularities other than nodes which have to be taken care of. This is one of the major obstacles to generalize Theorem 1.1 to all rational curves on K3 surfaces.

To handle the second case, i.e., to handle the nonreduced components of $\Upsilon_0$, we first divide them into three types, which we will call Type I, II or III chain (see Sec. 2), respectively. The deformation of $\Upsilon_0$ along a Type I chain is studied in Sec. 3. The basic technique used there is to normalize the total family along the Type I chain after a suitable base change. The deformation of $\Upsilon_0$ along a Type II chain is studied in Sec. 4, where we build our argument upon a lower bound estimation on the $\delta$-invariant of $\Upsilon_t$ in the neighborhood of a Type II chain. The deformation of $\Upsilon_0$ along a Type III chain is studied in Sec. 5. This turns out to be the hardest case among the three. A two-stage degeneration is used, First, we degenerate a general K3 surface to an elliptic K3 (see Sec. 5); and then we degenerate an elliptic K3 to the union of two rational surfaces described above. The degeneration of a K3 surface to an elliptic K3 is also an important step in Bryan and Leung’s work [B-L], although the elliptic K3 surfaces they used are different from the ones we use.

As a side note, there have been several progresses made on the enumeration problems on K3 surfaces following Yau and Zaslow’s work. A. Beauville pointed out that the numbers Yau and Zaslow obtained are the numbers of rational curves in $|\mathcal{O}_S(1)|$ with each curve counted with certain multiplicity [B], the multiplicity of a rational curve only depends on its singularities and is 1 if the curve is nodal. He gave an algebraic definition of the multiplicity. Later B. Fantechi, L. Göttsc...
and D. Straten proved that the multiplicity assigned by Beauville to a rational curve is positive. Recently, J. Bryan and N.C. Leung obtained Yau-Zaslow’s formula via a completely different approach \cite{B-L}.

**Conventions.**
1. Throughout the paper, we will work exclusively over \( \mathbb{C} \).
2. Here a general K3 surface \( S \) in \( \mathbb{P}^g \) refers to a general primitive K3 surface in \( \mathbb{P}^g \), where the number \( g \) is called the genus of \( S \) by convention.
3. Since we are working over \( \mathbb{C} \), we will use analytic geometry whenever possible. Hence we will use analytic neighborhoods of points instead of Zariski open neighborhoods in most cases, while you may always replace them by formal or etale neighborhoods.

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2. **Degeneration of K3 Surfaces and Limiting Rational Curves**

2.1. **Degeneration of K3 surfaces.** V. Kulikov classified all the possible degenerations of K3 surfaces in \cite{K}. Here we only need one of the simplest cases: the degeneration of K3 surfaces into a union of two rational scrolls as used in \cite{CLM}. This is also the degeneration used in the proof of the existence of rational curves on K3 surfaces in \cite{C}.

Following the notations in \cite{CLM}, let \( R = R_1 \cup R_2 \) be the union of two rational surfaces \( R_1 \) and \( R_2 \), which meet transversely along a smooth elliptic curve \( E = R_1 \cap R_2 \). For our purpose, we only need the cases that either \( R_1, R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) or \( R_1, R_2 \cong \mathbb{F}_1 \), where \( \mathbb{F}_1 \) is the rational ruled surface \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \) over \( \mathbb{P}^1 \).

We may represent such \( R \) by the tuple \((E, i_1, i_2)\) where \( i_1 : E \to R_1 \) and \( i_2 : E \to R_2 \) are the embeddings of \( E \) to \( R_1 \) and \( R_2 \), respectively. Two unions \( R \) and \( R' \) represented by \((E, i_1, i_2)\) and \((E', i'_1, i'_2)\) are isomorphic if and only if \( E \cong E' \) and there exist isomorphisms \( \phi : E' \to E, \phi_1 : R_1 \to R_1 \) and \( \phi_2 : R_2 \to R_2 \) such that \( i'_1 = \phi_1 \circ i_1 \circ \phi \) and \( i'_2 = \phi_2 \circ i_2 \circ \phi \). Then it is not hard to see that such \( R' \)'s form an irreducible moduli space of dimension 4.

Let \( C_i \) and \( F_i \) be two generators of \( \text{Pic}(R_i) \) with \( C_i \cdot F_i = 0, F_i^2 = 0 \) and \( C_i^2 = 0 \) if \( R_i \cong \mathbb{P}^1 \times \mathbb{P}^1 \); \( C_i^2 = -1 \) if \( R_i \cong \mathbb{F}_1 \), for \( i = 1, 2 \). We
use the notation $\mathcal{O}_R(aC + bF)$ to denote the line bundle on $R$ whose restrictions to $R_i$ are $\mathcal{O}_{R_i}(aC_i + bF_i)$, if such line bundle exists.

It is not hard to see that the dualizing sheaf $\omega_R$ of $R$ is trivial and $H^1(R, \mathbb{Z}) = 0$. So it is expected that a general deformation of $R$, say $X \to \Delta$ with $X_0 = R$, is a family of K3 surfaces. But since a general $R$ is not projective, the general fiber of $X$ is not algebraic. It is easy to see that $R$ is projective if and only if there exists two ample line bundles $L_1 \in \text{Pic} R_1$ and $L_2 \in \text{Pic} R_2$ such that $L_1|_E = L_2|_E$. One obvious choice of $L_i$ is $L_i = \mathcal{O}_{R_i}(C_i + kF_i)$. So we are considering the unions $R$ of the following type.

Let $R = R_1 \cup R_2$ be a union of rational surfaces described as above, which further satisfies $\mathcal{O}_E(C_1 + kF_1) = \mathcal{O}_E(C_2 + kF_2)$ for some $k \geq 1$ if $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $k \geq 2$ if $R_i \cong \mathbb{F}_1$. We will call such $R$ a union of scrolls of genus $g$, where $g = 2k + 1$ if $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $g = 2k$ if $R_i \cong \mathbb{F}_1$.

Notice that the relation $\mathcal{O}_E(C_1 + kF_1) = \mathcal{O}_E(C_2 + kF_2)$ imposes one extra condition on the tuple $(E, i_1, i_2)$ which represents $R$. So the moduli space of unions of scrolls of fixed genus $g$ has dimension 3. With a little extra effort, one can see that the moduli space is irreducible.

We are also interested in a special union of scrolls $R = R_1 \cup R_2$ which satisfies $\mathcal{O}_E(C_1) = \mathcal{O}_E(C_2)$ and $\mathcal{O}_E(F_1) = \mathcal{O}_E(F_2)$. We will call such $R$ a degenerated (or special) union of scrolls. It follows from the similar argument as before that degenerated unions of scrolls form an irreducible moduli space of dimension 2.

It was proved in [CLM] that a general union of scrolls $R$ of genus $g$ lies on the boundary of a complete family of K3 surfaces of genus $g$. The construction is straightforward to carry out by embedding $R$ to $\mathbb{P}^g$ by the complete linear series $|\mathcal{O}_R(C + kF)|$. Then $R$ lies on the component of the Hilbert scheme whose general point is a primitive K3 surfaces in $\mathbb{P}^g$.

However, we need a little bit more. We want to find a complete family of K3 surfaces of genus $g$ whose boundary also contains degenerated unions of scrolls. The previous construction fails since $\mathcal{O}_R(C + kF)$ is ample but not very ample on a degenerated union of scrolls $R$: the morphism $R \to \mathbb{P}^g$ given by $|\mathcal{O}_R(C + kF)|$ maps $R$ to a double scroll. The remedy to this situation is trivial. Instead of using $|\mathcal{O}_R(C + kF)|$, we embed $R$ to a projective space by the complete linear series $|\mathcal{O}_R(l(C + kF))|$ for some large $l$. Actually, it is enough to take $l = 2$. Namely, we embed $R$ to $\mathbb{P}^{2g-3}$ by $|\mathcal{O}_R(2C + 2kF)|$. And we can show that $R$ lies on the component of the Hilbert scheme whose general point is a K3 surface of genus $g$ in $\mathbb{P}^{2g-3}$. The argument for
this statement is identical to that in [CLM, Sec. 2.2] and we will only formulate it in the following proposition without the proof.

**Proposition 2.1.** Let $R$ be a union of scrolls of genus $g$ which is embedded into $\mathbb{P}^{4g-3}$ by $|O_R(2C + 2kF)|$, where $k = \lfloor g/2 \rfloor$. Let $N_R$ be the normal bundle of $R \subset \mathbb{P}^{4g-3}$ and $T^1_R = \mathcal{E}xt^1(\Omega_R, \mathcal{O}_R)$. Then

1. $H^1(N_R) = 0$;
2. $\dim H^0(N_R) = \dim \mathbb{P}GL(4g - 2) + 19$;
3. the natural map from $H^0(N_R)$ to $H^0(T^1_R)$ is surjective;
4. $R$ is represented by a smooth point in the component $H_g$ of the Hilbert scheme whose general point is a K3 surface of genus $g$ in $\mathbb{P}^{4g-3}$.

Let $\mathcal{R}_g \subset H_g$ be the locus in $H_g$ consisting of points representing unions of scrolls and let $\mathcal{R}_g^0 \subset \mathcal{R}_g$ be the locus consisting of points representing degenerated unions of scrolls. Every automorphism of a union of scrolls $R = R_1 \cup R_2$ induces an automorphism of $R_1$ which maps the double curve $E = R_1 \cap R_2$ to itself. Obviously, there are only finitely many automorphisms of $R_1$ with this property. So the automorphism group of $R$ is finite. Therefore, $\dim \mathcal{R}_g = \dim \mathcal{R}_g^0 + 1 = \dim \mathbb{P}GL(4g - 2) + 3$.

Let $\mathcal{H}_g$ be the blowup of $H_g$ along the closure of $\mathcal{R}_g$ and let $\mathcal{R}_g$ and $\mathcal{R}_g^0$ be the total transforms of $\mathcal{R}_g$ and $\mathcal{R}_g^0$ under the map $\mathcal{H}_g \to H_g$, respectively.

For any $[R] \in \mathcal{R}_g$, it is not hard to see that the tangent space $T_{\mathcal{R}_g,[R]}$ of $\mathcal{R}_g$ at $[R]$ lies inside the kernel of the surjection $T_{\mathcal{H}_g,[R]} = H^0(N_R) \to H^0(T^1_R)$. On the other hand, $\dim \mathcal{R}_g = \dim H^0(N_R) - \dim H^0(T^1_R)$. Therefore, $\mathcal{R}_g$ is smooth everywhere. Hence $\mathcal{H}_g$ is smooth in the neighborhood of $\mathcal{R}_g$.

Let $\mathcal{S}_g = \{(\left[ X \right], p) : p \in X \} \subset H_g \times \mathbb{P}^{4g-3}$ be the universal family over $H_g$ and $\tilde{\mathcal{S}}_g = \mathcal{S}_g \times_{\mathcal{H}_g} \mathcal{H}_g$.

It is not hard to see that every point of $\tilde{\mathcal{R}}_g$ can be uniquely represented by a pair $([R], s)$ with $[R] \in \mathcal{R}_g$ and $s \in \mathbb{P}H^0(T^1_R)$, which we will write as $[R^s]$. Notice that $T^1_R$ is a sheaf supported along $E$ whose restriction to $E$ is the line bundle $N_{E/R_1} \otimes N_{E/R_2}$, where $N_{E/R_1}$ and $N_{E/R_2}$ are the normal bundle of $E$ in $R_1$ and $R_2$, respectively. A geometric interpretation of this blowup process can be put as follows. Let $\pi : \Delta \to \mathcal{H}_g$ be a morphism from the disk $\Delta$ to $\mathcal{H}_g$, where $\pi(0) = [R^s] \in \mathcal{R}_g$ and $\pi(\Delta) \not\subset \mathcal{R}_g$. Then the corresponding family $X = \tilde{\mathcal{S}}_g \times_{\mathcal{H}_g} \Delta \to \Delta$ has sixteen rational double points which are the vanishing loci of $s$. 
Remark 2.1. Here we use the term “rational double points” in a broad sense. Let \( X \to \Delta \) be a one-parameter family of surfaces over the disk \( \Delta \). A point \( p \in X \) is called a rational double point of the family \( X \) over \( \Delta \) if \( X \) is locally isomorphic to \( \text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha z) \) at \( p \), where \( t \) parameterizes the disk \( \Delta \) and \( \alpha \) is a positive integer. If \( \alpha = 1 \), \( p \) is a rational double point of the threefold \( X \) in the usual sense, which can be resolved by blowing up \( X \) at \( p \) and then blowing down along one of the rulings of the exceptional divisor \( \mathbb{P}^1 \times \mathbb{P}^1 \). If \( \alpha > 1 \), we may resolve \( p \) in the same way as in [G-H, Appendix C, p. 39]. But we will take a more direct approach here by choosing an neighborhood \( U \) of \( p \) and simply letting \( \tilde{U} \subset U \times \mathbb{P}^1 \) be defined by
\[
\frac{x}{t^\alpha} = \frac{z}{y} = \frac{Z_0}{Z_1}
\] (2.1)
where \((Z_0, Z_1)\) are the homogeneous coordinates of \( \mathbb{P}^1 \). It is trivial to glue \( \tilde{U} \) with \( X \setminus \{ p \} \) and arrive at a “resolution” \( \tilde{X} \) of \( X \) (rigorously, \( \tilde{X} \) is not a desingularization of \( X \) since \( \tilde{X} \) is still singular in dimension one; but now \( \tilde{X} \) is isomorphic to \( \text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha) \) everywhere along its singular locus, which is all we need). Of course, we may switch \( x \) and \( y \) in (2.1) and arrive at another resolution of \( X \). Obviously, this corresponds to the “flip” phenomenon of the resolutions of an ordinary three-fold double point.

Let \( \mathcal{W}_g \) be the incidence correspondence
\[
\mathcal{W}_g = \{ ([S], [C]) : [S] \in \tilde{H}_g \text{ general,} \\
C \text{ is a rational curve in the primitive class of} \ S \} \\
\subset \tilde{H}_g \times |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)|,
\]
where we construct \( \mathcal{W}_g \) as a subvariety of \( \tilde{H}_g \times |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)| \) by identifying \([C]\) with \( 2C \in |\mathcal{O}_S(2)| = |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)| \) (let \( \mathcal{O}_S(1) \) be the primitive line bundle on \( S \)). Let \( \tilde{\mathcal{W}}_g \) be the closure of \( \mathcal{W}_g \) in \( \tilde{H}_g \times |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)| \).

Obviously, Theorem \( \square \) is equivalent to the following statement: for every point \( ([S], [C]) \) on the fiber of \( \tilde{\mathcal{W}}_g \to \tilde{H}_g \) over a general point \([S] \in \tilde{H}_g\), \( C \) is a nodal curve. Obviously, it suffices to verify this statement for every irreducible component of \( \tilde{\mathcal{W}}_g \) which dominates \( \tilde{H}_g \).

So, from now on, we will pretend that \( \tilde{\mathcal{W}}_g \) is irreducible and dominates \( \tilde{H}_g \).

It is obvious that the map \( \tilde{\mathcal{W}}_g \to \tilde{H}_g \) is generically finite. One important step to prove Theorem \( \square \) is to figure out what kind of curves \([C]\) lie on the fiber \((\tilde{\mathcal{W}}_g)_{[R]} \) of \( \tilde{\mathcal{W}}_g \to \tilde{H}_g \) over a point \([R] \in \tilde{R}_g\). Of course, \( C \in |\mathcal{O}_R(C + kF)| \). Besides, \( C \) must be the limit of rational
curves on K3 surfaces. More precisely, \( ([R^s], [C]) \in (\tilde{W}_g)_{[R^s]} \) if and only if there exist a family \( X \) of K3 surfaces over \( \Delta \) and a family \( \Upsilon \subset X \) of rational curves where \( X_0 = R \), \( X \) has sixteen rational double points which are the vanishing loci of \( s \), \( \Upsilon_0 = C \) and \( \Upsilon_t \) is a rational curve in the primitive class of \( X_t \). It turns out that there are only finitely many curves \( C \) in \( |O_R(C + kF)| \) with this property.

**Definition 2.1.** Let \( R \) be a union of scrolls and \( s \in \mathbb{P}H^0(T^1_R) \). A curve \( C \in |O_R(C + kF)| \) with \( ([R^s], [C]) \in \tilde{W}_g \) is called a limiting rational curve on \( R^s \). When there is no possibility for confusion, we will abbreviate \( R^s \) to \( R \) and simply call \( C \) a limiting rational curve on \( R \).

2.2. Classifications of limiting rational curves. Let \( \pi : \Delta \rightarrow \tilde{H}_g \) be a morphism from the disk \( \Delta \) to \( \tilde{H}_g \) where \( \pi(0) = [R^s] \in \tilde{R}_g \) and \( \pi(t) \) represents a general K3 surface for \( t \in \Delta \) general.

Let \( X = \tilde{S}_g \times_{\tilde{H}_g} \Delta \rightarrow \Delta \) be the one parameter family of K3 surfaces corresponding to \( \pi \). Obviously, \( X \) has exactly 16 rational double points \( p_1, p_2, ..., p_{16} \) lying on the double curve \( E = R_1 \cap R_2 \), where \( \{p_1, p_2, ..., p_{16}\} \) is the vanishing locus of \( s \in \mathbb{P}H^0(T^1_R) \). For a general choice of \( s, p_1, p_2, ..., p_{16} \) satisfy the only relation

\[
\mathcal{O}_E(p_1 + p_2 + ... + p_{16}) = N_{E/R_1} \otimes N_{E/R_2}
\]

where \( N_{E/R_i} = \mathcal{O}_E(2C_i + 2F_i) \) if \( R_i \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and \( N_{E/R_i} = \mathcal{O}_E(2C_i + 3F_i) \) if \( R_i \cong \mathbb{P}^1 \).

Let \( \Upsilon_t \subset X \) be a family of curves over \( \Delta \) whose general fiber \( \Upsilon_t \) is a rational curve in the primitive class of \( X_t \) for each \( t \). Let \( \tilde{\Upsilon} \) be the nodal reduction of \( \Upsilon \). Namely, \( \tilde{\Upsilon} \) is a family of stable maps to \( X \) such that \( \tilde{\Upsilon} \rightarrow X \) factors through \( \Upsilon \) and the general fiber \( \tilde{\Upsilon}_t \) of \( \tilde{\Upsilon} \) is the normalization of the general fiber \( \Upsilon_t \) of \( \Upsilon \). Notice that such \( \Upsilon \) and \( \tilde{\Upsilon} \) exist after a base change.

One of the key lemmas we use to classify limiting rational curves on \( X_0 \) is Lemma 2.2 in \[C\]. We need a slightly stronger version, which is formulated and proved as follows.

**Lemma 2.1.** Let \( X \subset \Delta^n_{x_1x_2...x_n} \times \Delta_t \) \( (n \geq 3) \) be the hypersurface cut out by \( x_1x_2 = t^\alpha \) for some \( \alpha > 0 \), where \( \Delta^n_{x_1x_2...x_n} \) is the \( n \)-dimensional polydisk parameterized by \( (x_1, x_2, ..., x_n) \) and \( \Delta_t \) is the disk parameterized by \( t \). Let \( X_0 \) be the central fiber of \( X \) over \( \Delta_t \), let \( X_0 = R_1 \cup R_2 \) where \( R_1 = \{x_1 = t = 0\} \) and \( R_2 = \{x_2 = t = 0\} \) and let \( E = R_1 \cap R_2 \). Let \( S \) be a flat family of analytic curves over \( \Delta_t \) and \( \pi : S \rightarrow X \) be a proper morphism preserving the base \( \Delta_t \). Suppose that the image \( \pi(S_0) \) of the central fiber \( S_0 \) of \( S \) meets \( E \) properly along a 0-dimensional
scheme supported at the origin. Let \( S_0 = \Gamma_1 \cup \Gamma_2 \) with \( \pi(\Gamma_1) \subset R_1 \) and \( \pi(\Gamma_2) \subset R_2 \). Then \( \pi(\Gamma_1) \cdot R_2 = \pi(\Gamma_2) \cdot R_1 \) holds on \( \{t = 0\} \cong \Delta^n_{x_1, x_2, \ldots, x_n} \).

**Proof.** Without the loss of generality, we may assume that \( S \) is irreducible and smooth; otherwise, we apply the following argument to each irreducible component of its desingularization. Let \( \eta: X \to X' = \{x_1x_2 = t\} \subset \Delta^n_{x_1, x_2, \ldots, x_n} \times \Delta_t \) be the morphism sending \((x_1, x_2, \ldots, x_n, t)\) to \((x_1, x_2, \ldots, x_n, t^\alpha)\) and let \( p = \eta \circ \pi \). Obviously, it suffices to show that \( p_*(\Gamma_1) \cdot R_2 = p_*(\Gamma_2) \cdot R_1 \) on \( X' \).

Since \( \Gamma_1 \cdot p^*(R_1 + R_2) = 0 \) and \((\Gamma_1 + \Gamma_2) \cdot p^*(R_1) = 0\) on \( S \),

\[
\Gamma_1 \cdot p^*(R_2) = \Gamma_2 \cdot p^*(R_1).
\]

Therefore, \( p_*(\Gamma_1) \cdot R_2 = p_*(\Gamma_2) \cdot R_1 \) by the projection formula. \( \Box \)

The following proposition deals with the case that \( X \) has a rational double point at the origin, which can be viewed as a corollary of Lemma 2.1.

**Corollary 2.1.** Let \( X, R_1, R_2, E, \pi, S \) and \( S_0 \) be defined as in Lemma 2.1 except that \( X \) is cut out by \( x_1x_2 = t^\alpha x_3 \) for some \( \alpha > 0 \). Suppose that \( \pi(S_0) \) contains an irreducible component \( \Gamma_1 \subset R_1 \) such that the tangent cones of \( \Gamma_1 \) and \( E \) at the origin \( p \) do not meet properly in the tangent space of \( R_1 \) at \( p \). Then \( \pi(S_0) \) also contains a component \( \Gamma_2 \subset R_2 \).

**Proof.** We may resolve the double point of \( X \) as in Remark 2.1. Let \( \tilde{X} \subset X \times \mathbb{P}^1 \) be given by

\[
\frac{x_1}{t^\alpha} = \frac{x_3}{x_2} = \frac{Z_0}{Z_1}
\]

where \((Z_0, Z_1)\) are the homogeneous coordinates of \( \mathbb{P}^1 \). Let \( \tilde{S} = S \times_X \tilde{X} \). Obviously, \( \tilde{X}_0 = \tilde{R}_1 \cup \tilde{R}_2 \), where \( \tilde{R}_1 \) is the blowup of \( R_1 \) along the subscheme \( \{x_2 = x_3 = 0\} \). Obviously, \( \tilde{R}_1 \) and \( \tilde{R}_2 \) meet along \( \tilde{E} = \{x_1 = x_2 = z = t = 0\} \) (let \( z = Z_1/Z_0 \)) and \( \tilde{X} \) is locally defined by

\[
(2.3) \quad x_1z = t^\alpha \text{ and } x_2 = zx_3
\]

in the neighborhood of \( \tilde{E} \). Let \( \tilde{p} \in \tilde{E} \) be the point \( \{x_1 = x_2 = \ldots = x_n = z = t = 0\} \) and let \( \tilde{\Gamma}_1 \subset \tilde{S}_0 \) be the proper transform of \( \Gamma_1 \) under the map \( \tilde{R}_1 \to R_1 \). The assumption that the tangent cones of \( \Gamma_1 \) and \( E \) at \( p \) do not meet properly implies that their proper transforms \( \tilde{\Gamma}_1 \) and \( \tilde{E} \) under the blowup \( \tilde{R}_1 \to R_1 \) still meet at a point on the exception divisor, i.e., they meet at \( \tilde{p} \). Now by (2.3), \( \tilde{X} \) is locally defined by \( x_1z = t^\alpha \) in \( \Delta^n_{x_1, x_2, \ldots, x_n} \times \Delta_t \) at \( \tilde{p} \). And \( \tilde{\pi}(\tilde{S}_0) \) has a component \( \tilde{\Gamma}_1 \)
passing through $\tilde{p}$ and lying on $\tilde{R}_1$, where $\tilde{\pi}: \tilde{S} \to \tilde{X}$ is the map induced by $\pi$. So by Lemma 2.1, $\tilde{\pi}(\tilde{S}_0)$ contains a component $\Gamma_2 \subset R_2$, i.e., $\pi(S_0)$ contains the component $\Gamma_2 \subset R_2$. 

Let $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$ with $\Sigma_i \subset R_i$ for $i = 1, 2$. And let $\Gamma_i \subset \Sigma_i$ be the irreducible component of $\Sigma_i$ in $|C_i + k_iF_i|$ for some $k_i \leq k$ ($i = 1, 2$).

Let $r_{ij}$ be all the points on $E$ satisfying $\mathcal{O}_E(2r_{ij}) = \mathcal{O}_E(F_1)$ for $i = 1, 2$ and $j = 1, 2, 3, 4$. Notice that if $R$ is a degenerated union of scroll, we have $\{r_{1j}\} = \{r_{2j}\}$. So we will simply let $r_{1j} = r_{2j}$ for $j = 1, 2, 3, 4$ if $[R] \in \mathcal{R}_g$.

For two points $p$ and $q$ on $E$ satisfying $\mathcal{O}_E(p + q) = \mathcal{O}_E(F_1)$ or $\mathcal{O}_E(F_2)$, we use the notation $\overline{pq}$ to denote the curve in $|F_1|$ or $|F_2|$ passing through $p$ and $q$. If $R$ is general in $\mathcal{R}_g$, there is no ambiguity; otherwise, we use $\overline{pq}^{(1)}$ and $\overline{pq}^{(2)}$ to distinguish on which of $R_1$ and $R_2$ this curve lies. And we use $\overline{rij}$ to denote the curve in $|F_i|$ passing through $r_{ij}$.

Since there are exactly two components on $\tilde{\Upsilon}_0$ dominates $\Gamma_1$ and $\Gamma_2$, respectively, we will continue to use $\Gamma_1$ and $\Gamma_2$ to denote these two components on $\tilde{\Upsilon}_0$.

**Definition 2.2.** Let $C$ be a stable curve. The “dual graph” $G$ of $C$ is a graph constructed by representing each component $A$ of $C$ by the vertex $[A]$ and drawing an edge between two vertices $[A]$ and $[B]$ if the corresponding curves $A$ and $B$ meet at a point. We allow multiple edges between $[A]$ and $[B]$ if they meet at more than one point; and if $A$ has a node, we will draw a loop around $[A]$. Let $\deg([A])$ denote the degree of the vertex $[A]$ in $G$.

A sequence of components $C_1, C_2, ..., C_n$ of $C$ form a chain in $G$ if there is an edge between $[C_i]$ and $[C_{i+1}]$ for $i = 1, 2, ..., n - 1$.

First, we will show that there are only finitely many limiting rational curves on $X_0$. Namely, there are only finitely many possible configurations for $\Upsilon_0$.

**Proposition 2.2.** Let $[R^s] \in \mathcal{R}_g^0$ with $s \in \mathbb{P}H^0(T_R^1)$ general. Then there are only finitely many limiting rational curves on $R^s$. Namely, the fibration of $\mathcal{W}_g \to \mathcal{H}_g$ is finite over $[R^s]$.

**Proof.** Since $[R^s] \in \mathcal{R}_g^0$, $\mathcal{O}_E(C_1) = \mathcal{O}_E(C_2) = \mathcal{O}_E(C)$ and $\mathcal{O}_E(F_1) = \mathcal{O}_E(F_2) = \mathcal{O}_E(F)$. It is not hard to see that if $\Sigma_1$ contains a component $\overline{pq}^{(1)}$, $\Sigma_2$ must contain $\overline{pq}^{(2)}$ with the same multiplicity and vice versa. Therefore, we necessarily have $\Gamma_1 \cap E = \Gamma_2 \cap E$.

Let $q_i$ be the point on $E$ such that $\mathcal{O}_E(p_i + q_i) = \mathcal{O}_E(F)$ for $i = 1, 2, ..., 16$. Let $r_j = r_{1j} = r_{2j}$ for $j = 1, 2, 3, 4$. 


By (2.2), the only relation among \( p_1, p_2, \ldots, p_{16} \) is \( \mathcal{O}_E(p_1 + p_2 + \ldots + p_{16}) = \mathcal{O}_E(4C + 4F) \) or \( \mathcal{O}_E(4C + 6F) \). Therefore, the subgroup of \( \text{Pic}(E) \) generated by \( p_l \), \( q_l \) and \( r_j \) is

\[
\oplus_{i=1}^{15} \mathbb{Z}p_l \oplus \mathbb{Z}r_1 \oplus \mathbb{Z}(4C) \oplus (\mathbb{Z} \oplus \mathbb{Z}) \]

where \( \mathbb{Z} \oplus \mathbb{Z} \) is the subgroup consisting of \( r_j - r_1 \) for \( j = 1, 2, 3, 4 \). This group clearly does not contain the divisor \( C \) when \( p_1, p_2, \ldots, p_{15} \) and \( E \) are general. Therefore, \( \Gamma_i \) must meet \( E \) at (at least) one point other than \( p_l, q_l \) and \( r_j \) for \( l = 1, 2, \ldots, 16 \) and \( j = 1, 2, 3, 4 \).

Suppose that \( p \in \Gamma_i \cap E \) and \( p \not\in \{p_1, p_2, \ldots, p_{16}, q_1, q_2, \ldots, q_{16}\} \). Let \( q \) be the point on \( E \) such that \( \mathcal{O}_E(p + q) = \mathcal{O}_E(F) \). We claim that on \( \tilde{\Upsilon}_0 \), the curves \( \Gamma_1 \) and \( \Gamma_2 \) are joined by a chain of curves which either contract to one of the two points \( p \) and \( q \) or dominate one of the two curves \( \overline{pq}^{(1)} \) and \( \overline{pq}^{(2)} \).

Let \( D_1, D_2, \ldots, D_\gamma \) be the components of \( \tilde{\Upsilon}_0 \) which either contract to one of \( p \) and \( q \) or dominate one of \( \overline{pq}^{(i)} \). Let \( G \) be the dual graph of \( \Gamma_1 \cup \Gamma_2 \cup D_1 \cup D_2 \cup \ldots \cup D_\gamma \) with each curve \( A \) being represented by the vertex \( [A] \).

If \( p \not\in \{r_1, r_2, r_3, r_4\} \), then \( \deg([\Gamma_i]) \geq 1 \) and \( \deg([D_j]) \geq 2 \) for \( i = 1, 2 \) and \( j = 1, 2, \ldots, \gamma \) by Lemma 2.1. So \( G \) either contains a cycle, which is impossible, or \( G \) is connected, which implies our claim.

If \( p = q = r_j \), then \( \Gamma_i \) must meet \( E \) transversely at \( r_j \); otherwise, \( \Gamma_i \) will meet \( \overline{r_{ij}} \) at \( r_j \) with multiplicity at least 2, which is impossible. Locally at \( r_j \), \( \tilde{\Upsilon}_0 \) consists of \( \Gamma_1 \cup \Gamma_2 \) and a multiple of \( \overline{r_{ij}} \cup \overline{r_{2j}} \). Notice that the local intersection number between \( \Gamma_i \) and \( E \) at \( r_j \) is 1 (odd) and the local intersection number between \( \overline{r_{ij}} \) and \( E \) at \( r_j \) is 2 (even). Therefore, \( [\Gamma_1] \) and \( [\Gamma_2] \) must lie on the same connected component of \( G \) by Lemma 2.1, which also implies our claim.

In conclusion, \( \Gamma_i \) meet \( E \) at (at least) one point other than \( p_l, q_l \) and \( r_j \). If \( \Gamma_1 \) meet \( E \) at \( p \) and \( p \not\in \{p_1, p_2, \ldots, p_{16}, q_1, q_2, \ldots, q_{16}\} \), then \( \Gamma_1 \) and \( \Gamma_2 \) are joined by a chain of curves on \( \tilde{\Upsilon}_0 \) whose image lie in \( \overline{pq}^{(1)} \cup \overline{pq}^{(2)} \), where \( q \in E \) is the point satisfying that \( \mathcal{O}_E(p + q) = \mathcal{O}_E(F) \). So it is not hard to see that \( \Gamma_i \) cannot meet \( E \) at more than one point other than \( p_l \) and \( q_l \); otherwise, \( \Gamma_1 \) and \( \Gamma_2 \) will be joined by two different chains of curves. In conclusion, \( \Gamma_i \) meets \( E \) at exactly one point \( p \) other than \( p_l \) and \( q_l \) for \( l = 1, 2, \ldots, 16 \) and \( p \neq r_j \) for \( j = 1, 2, 3, 4 \). Obviously, there are at most finitely many curves in \( [C_i + k_i F_i] \) \((k_i \leq k) \) with this property.

It remains to show that there are only finitely many possible configurations for a component \( \overline{pq}^{(i)} \subset \Sigma_i \). Actually, we claim that if \( \overline{pq}^{(i)} \subset \Sigma_i \), then
1. \( p, q \in \{ p_1, p_2, ..., p_{16}, q_1, q_2, ..., q_{16} \} \); OR
2. \( p = q \in \{ r_1, r_2, r_3, r_4 \} \); OR
3. \( p \) or \( q \) lies on \( \Gamma_i \).

Suppose that \( \overline{pq}^{(i)} \subset \Sigma_i \), \( p, q \notin \{ p_1, p_2, ..., p_{16}, q_1, q_2, ..., q_{16}, r_1, ..., r_4 \} \) and \( p, q \notin \Gamma_i \). Let \( D_1, D_2, ..., D_\gamma \) be the components of \( \Gamma_0 \) which either contract to one of \( p \) and \( q \) or dominate one of \( \overline{pq}^{(i)} \). Let \( G \) be the dual graph of \( D_1 \cup D_2 \cup ... \cup D_\gamma \) with each curve \( A \) represented by the vertex \([A]\). Since \( p \neq q \), \( p, q \notin \{ p_1, p_2, ..., p_{16} \} \) and \( \Gamma_i \) does not pass through \( p \) and \( q \), \( \deg([D_j]) \geq 2 \) for \( j = 1, 2, ..., \gamma \) by Lemma 2.1. So \( G \) must contain a cycle, which is a contradiction.

In summary, there are at most finitely many possible configurations for \( \Sigma_0 \). We divide the \( \overline{pq} \) into three types.

**Definition 2.3.** For each point \( p \in \Sigma_i \cap E \) \((i = 1, 2)\), we use \( m_i(p) \) to denote the local intersection multiplicity between \( \Gamma_i \) and \( E \) at \( p \) (let \( m_i(p) = 0 \) if \( \Gamma_i \) does not pass through \( p \)).

For each \( \overline{pq} \subset \Sigma_0 \), let \( \mu(\overline{pq}) \) denote the multiplicity of \( \overline{pq} \) in \( \Sigma_0 \).

**Definition 2.4.** An \( F \)-chain on \( R \) is a union of \( m \) distinct curves \( C = q_0q_1q_2...q_{m-1}q_m \), where, as the notation suggests, \( q_{m+l} \) is a curve in either \( |F_1| \) or \( |F_2| \). We call \( m \) the length of \( C \). If \( m = 0 \), we let \( C \) be the point \( q_0 \).

A maximal \( F \)-chain \( C \) in \( \Sigma_0 \) is an \( F \)-chain \( C \subset \Sigma_0 \) and it is maximal in the sense that there does not exist an \( F \)-chain \( C' \subset \Sigma_0 \) containing \( C \) as a proper subset.

Since we will always deal with maximal \( F \)-chains in \( \Sigma_0 \), we will simply call them \( F \)-chains in \( \Sigma_0 \).

We divide the \( F \)-chains in \( \Sigma_0 \) into three types.

1. A chain \( C \) is called a **Type I** chain if \( p_l \notin C \) for \( l = 1, 2, ..., 16 \) and \( r_{ij} \notin C \) for \( i = 1, 2 \) and \( j = 1, 2, 3, 4 \).
2. A chain \( C \) is called a **Type II** chain if \( p_l \in C \) for some \( 1 \leq l \leq 12 \).
3. A chain \( C \) is called a **Type III** chain if \( r_{ij} \in C \) for some \( 1 \leq i \leq 2 \) and \( 1 \leq j \leq 4 \).

Notice that a Type I or II chain could consist of a single point.

**Remark 2.2.** Since \( \overline{\Sigma}_0 \subset \overline{\Sigma}_g \), Proposition 2.2 implies that the fibration of \( \overline{\Sigma}_g \rightarrow \overline{\Sigma}_g \) is finite over a general point \( [R^*] \in \overline{\Sigma}_g \).

In the rest of this section, we will assume that the choice of \( R^* \) in the construction of \( X \) is general in \( \overline{\Sigma}_g \).
Proposition 2.3. There is exactly one Type I chain in \( \Upsilon_0 \), whose length is even. Let \( \overline{q_0q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{2l-1}q_{2l}} \subset \Upsilon_0 \) be the Type I chain and assume that \( \overline{q_0q_1} \subset R_1 \) without the loss of generality. Then

\[
\mu(\overline{q_0q_1}) = \mu(\overline{q_1q_2}) = ... = \mu(\overline{q_{2l-1}q_{2l}}) = \beta = m_2(q_0) = m_1(q_{2l}) = \beta
\]

\[
m_1(q_0) = m_1(q_1) = ... = m_1(q_{2l-1}) = m_2(q_1) = m_2(q_2) = ... = m_2(q_{2l}) = 0
\]

for some \( \beta > 0 \). There are exactly \( 2l \) components \( D_1, D_2, ..., D_{2l} \) on \( \tilde{\Upsilon}_0 \) where each \( D_i \) dominates \( \overline{q_{i-1}q_i} \) with a degree \( \beta \) map totally ramified at \( q_{i-1} \) and \( q_i \). The components \( \Gamma_2, D_1, D_2, ..., D_{2l}, \Gamma_1 \) form a chain in the dual graph of \( \tilde{\Upsilon}_0 \).

Proof. Obviously, every point in \( C \cap E \) for a Type II or Type III chain \( C \), as an element in the Picard group \( \text{Pic}(E) \), lies in the subgroup generated by \( p_i \) (\( i = 1, 2, ..., 16 \)) and \( r_{ij} \) (\( i = 1, 2 \) and \( j = 1, 2, 3, 4 \)), which, by (2.2), is

\[
\left( \bigoplus_{i=1}^{15} \mathbb{Z} p_i \right) \oplus \left( \bigoplus_{i=1}^{2} \mathbb{Z} r_{i1} \right) \oplus \mathbb{Z}(2C_1 + 2C_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\oplus 2}
\]

where two copies of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) are the subgroups consisting of \( r_{ij} - r_{i1} \) (\( j = 1, 2, 3, 4 \)) for \( i = 1, 2 \), respectively. Obviously, \( C_1 + kF_1 = C_2 + kF_2 \) does not lie in this group, i.e., it is not generated by \( p_i \) and \( r_{ij} \), for \( R \) and \( s \) general. Therefore there is at least one Type I chain.

Let \( \overline{q_0q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m} \) form a Type I chain in \( \Upsilon_0 \). It is not hard to see that \( q_0, q_m \in \Gamma_1 \cup \Gamma_2 \).

We let \( D_1, D_2, ..., D_r \) be the components on \( \tilde{\Upsilon}_0 \) dominating the curves \( \overline{q_0q_1}, \overline{q_1q_2}, ..., \overline{q_{m-1}q_m} \) and \( H_1, H_2, ..., H_\mu \) be the components contracting to the points \( q_0, q_1, q_2, ..., q_m \).

Let \( G \) be the dual graph of \( \Gamma_1, D_1, D_2, ..., D_\gamma, \Gamma_2, H_1, H_2, ..., H_\mu \) with each curve \( A \) being represented by the vertex \( [A] \). Obviously, \( G \) contains no circuit and is consequently a “forest” (a disjoint union of trees). By Lemma 7.1, \( \text{deg}([D_i]) \geq 2 \) for \( i = 1, 2, ..., \gamma \) and \( \text{deg}([\Gamma_1]) + \text{deg}([\Gamma_2]) \geq 2 \). And since \( \text{deg}([H_j]) \geq 3 \), \( G \) has at least \( \gamma + 1 + 3\mu/2 \) edges with \( \gamma + 2 + \mu \) vertices. On the other hand, \( G \) has at most \( \gamma + 1 + \mu \) edges since it is a forest. So we must have

1. \( \mu = 0 \), i.e., there are no components of \( \tilde{\Upsilon}_0 \) contracting to the points \( q_0, q_1, q_2, ..., q_m \);
2. \( G \) is a tree;
3. \( \text{deg}([D_i]) = 2 \) for \( i = 1, 2, ..., \gamma \);
4. \( \text{deg}([\Gamma_1]) = \text{deg}([\Gamma_2]) = 1 \).

And since we assume that \( \overline{q_0q_1} \subset R_1 \), we must have \( q_0 \in \Gamma_2 \) and \( q_m \in \Gamma_1 \). Hence \( m \) must be even, say \( m = 2l \).
The picture of \( G \) is very clear now. The vertices of \( G \) form a chain after some ordering. Without the loss of generality, we may assume that \([\Gamma_2], [D_1], [D_2], ..., [D_l], [\Gamma_1]\) form a chain in that order. Obviously, \( D_1 \) must dominate \( q_0q_1 \) and there is no other \( D_i \) \((i \neq 1)\) dominating \( q_0q_1 \); otherwise, \( \deg([\Gamma_2]) \geq 2 \) by Lemma 2.1. Similarly, \( D_2 \) must dominate \( q_1q_2 \) and there is no other \( D_i \) dominating \( q_1q_2 \); otherwise, \( \deg([D_1]) \geq 3 \) by Lemma 2.1. This line of argument goes on and finally shows that \( \gamma = 2l \) and each \( D_i \) dominates \( q_i-1q_i \) for \( i = 1, 2, ..., 2l \). Also, the map from \( D_i \) to \( q_i-1q_i \) must be totally ramified at \( q_{i-1} \) and \( q_i \); otherwise, \( \deg([D_i]) \geq 3 \) by Lemma 2.1. Since \( D_i \) is rational, the map from \( D_i \) to \( q_i-1q_i \) is only ramified at \( q_{i-1} \) and \( q_i \).

Let \( \beta = m_2(q_0) \). Obviously, \( \Gamma_2 \) does not pass through \( q_1, q_2, ..., q_{2l} \) and \( \Gamma_1 \) does not pass through \( q_0, q_1, ..., q_{2l-1} \); otherwise, \( \deg([\Gamma_2]) \geq 2 \) or \( \deg([\Gamma_1]) \geq 2 \). So \( m_2(q_1) = m_2(q_2) = ... = m_2(q_{2l}) = m_1(q_0) = m_1(q_1) = ... = m_1(q_{2l-1}) = 0 \). Hence each \( q_i-1q_i \) have multiplicity exactly \( \beta \) in \( \Upsilon_0 \) for \( i = 1, 2, ..., 2l \). Therefore, the map from \( D_i \) to \( q_i-1q_i \) has degree \( \beta \). This also implies that \( m_1(q_{2l}) = \beta \).

Hence \( \Gamma_1 \) and \( \Gamma_2 \) are joined by a chain of curves \( D_1 \cup D_2 \cup ... \cup D_{2l} \) on \( \tilde{\Upsilon}_0 \) whose images are contained in \( q_0q_1 \cup q_1q_2 \cup ... \cup q_{2l-1}q_{2l} \). Therefore, there is only one Type I chain in \( \Upsilon_0 \); otherwise, \( \Gamma_1 \) and \( \Gamma_2 \) will be joined by two different chains of curves on \( \tilde{\Upsilon}_0 \).

To study the behavior of a family \( \Upsilon \) of curves near a subscheme \( S \subset \Upsilon_0 \), it is usually very convenient to take an analytic neighborhood \( U \) of \( S \) in \( \Upsilon \) and study \( U \) instead of \( \Upsilon \) (or alternatively, study the formal completion of \( \Upsilon \) along \( S \)). Notice that even if \( \Upsilon \) is irreducible, \( U \) could be reducible after a base change. We call a component of \( U \) which is irreducible under any base changes “a locally irreducible component of \( \Upsilon \) around \( S \)”. And if \( U \) is irreducible under any base changes, we call \( \Upsilon \) is locally irreducible around \( S \).

Here is a trivial remark. Let \( S \subset \Upsilon_0 \) be a closed subscheme of \( \Upsilon_0 \) and \( \tilde{S} \) be the total transform of \( S \) under the map \( \tilde{\Upsilon} \rightarrow \Upsilon \). Then \( \Upsilon \) is locally irreducible around \( S \) if and only if \( \tilde{S} \) is connected.

Let us write a Type II chain in the form \( \cup_{i=-1}^{m-1}q_iq_{i+1} \), where \( q_0 \in \{p_1, p_2, ..., p_{16}\} \).

**Proposition 2.4.** Let \( \cup_{i=-1}^{m-1}q_iq_{i+1} \) be a Type II chain in \( \Upsilon_0 \) with \( q_0 \in \{p_1, p_2, ..., p_{16}\} \). Without the loss of generality, assume that \( q_0 = p_1 \) and \( q_0q_1 \subset R_1 \).

Let \( \Upsilon \) be a locally irreducible component of \( \Upsilon \) around \( \cup_{i=-1}^{m-1}q_iq_{i+1} \). Then

1. \( \mu(q_{i-1}q_{i-1}) \leq \mu(q_{i-1}q_{i-2}) \leq ... \leq \mu(q_{-1}q_0) \) and \( \mu(q_0q_1) \geq \mu(q_1q_2) \geq ... \geq \mu(q_{m-1}q_m) \);
2. $m_1(q_0), m_2(q_0) \leq 1$;
3. for each $q_j$ ($j \neq 0$), either $m_1(q_j) = 0$ or $m_2(q_j) = 0$;
4. $Y_0$ is one of the following
   (a) $Y_0 = \Delta(w)$ where $w$ is the intersection between $\Gamma_1 \cup \Gamma_2$ and $q_iq_{i+1}$ other than the points $q_i$ and $q_{i+1}$ for some $i$ and $\Delta(w) \subset \Gamma_1 \cup \Gamma_2$ is a disk centered at $w$ regarded as an analytic neighborhood of $w$ on $\Gamma_1$ or $\Gamma_2$;
   (b) $Y_0 = \Delta(q_0)$ where $\Delta(q_0) \subset \Gamma_1 \cup \Gamma_2$ is a disk centered at $q_0$ regarded as an analytic neighborhood of $q_0$ on $\Gamma_1$ or $\Gamma_2$;
   (c) $(Y_0)_{\text{red}} = \bigcup_{i=0}^{n-1} q_iq_{i+1} \cup \Delta(q_n)$ for some $n > 0$, where $\Delta(q_n)$ is a disk centered at $q_n$ regarded as an analytic neighborhood of $q_n$ on $\Gamma_1$ or $\Gamma_2$, $\Delta(q_n)$ and $q_{n-1}q_n$ lie on the different $R_j$'s and $\overline{q_iq_{i+1}}$ has multiplicity $E \cdot \Delta(q_n)$ in $Y_0$;
   (d) $(Y_0)_{\text{red}} = \bigcup_{i=0}^{n-1} q_iq_{i+1} \cup \Delta(q_{-n})$ for some $n > 0$, where $\Delta(q_{-n})$ is a disk centered at $q_{-n}$ regarded as an analytic neighborhood of $q_{-n}$ on $\Gamma_1$ or $\Gamma_2$, $\Delta(q_{-n})$ and $q_{-n}q_{-n+1}$ lie on the different $R_j$'s and $\overline{q_iq_{i+1}}$ has multiplicity $E \cdot \Delta(q_{-n})$ in $Y_0$.

Remark 2.3. The statements in Proposition 2.4 may need some further explanation. Let $U$ be an analytic neighborhood of $\bigcup_{i=1}^{m} q_iq_{i+1}$ in $\overline{\Upsilon}$. Then the central fiber $U_0$ of $U$ consists of $\bigcup_{i=1}^{m} q_iq_{i+1}$ plus a few disks which are “pieces” of $\Gamma_1$ and $\Gamma_2$. If $\Gamma_i$ meets $q_iq_{i+1}$ at $w \neq q_j, q_{j+1}$, there is a disk $\Delta(w) \subset \Gamma_i$ centered at $w$ on $U_0$; if $\Gamma_i$ passes through $q_j$, there is a disk $\Delta(q_j) \subset \Gamma_i$ centered at $q_j$ on $U_0$. Let $\tilde{U} = U \times_{\overline{\Upsilon}} \overline{\Upsilon}$ be the nodal reduction of $U$. Each locally irreducible component $Y$ of $U$ corresponds to a connected component of the dual graph of $\tilde{U}_0$ and vice versa.

First, locally at $w = \Gamma_i \cap q_iq_{j+1}$ with $w \neq q_j, q_{j+1}$, our statements about $Y_0$ show that $\Gamma_i$ ($\Gamma_i \supset \Delta(w)$) is not joined to any component dominating $q_iq_{j+1}$ by a chain of curves contracting to $w$ on $\overline{\Upsilon}_0$. In a more intuitive language, $\Gamma_i$ is “separated” from $\overline{q_iq_{j+1}}$ at $w$ after the nodal reduction.

Second, locally at $q_0$, our statements about $Y_0$ show that all branches of $Y_0$ at $q_0$ are separated from each other after the nodal reduction. By that we mean among the components dominating $\Gamma_1, \Gamma_2, q_{-n}q_0$ or $\overline{q_0q_i}$ on $\overline{\Upsilon}_0$, no two are joined by a chain of curves contracting to $q_0$.

Third, locally at $q_n$ ($n \neq 0$), suppose that $\Gamma_i$ passes through $q_n$ and $Y$ is the locally irreducible component of $U$ containing the disk $\Delta(q_n)$. Then our statements about $Y_0$ show that $q_nq_{n+1}$ (if $n < 0$) or $q_{n-1}q_n$ (if $n > 0$) must lie on $R_{3-i}$. This also implies that $\Gamma_1$ and $\Gamma_2$ cannot both pass through $q_n$ for any $n \neq 0$. Let $\tilde{Y} = Y \times_{\overline{\Upsilon}} \overline{\Upsilon}$. Then $\tilde{Y}_0$ is
the union of the components of $\widetilde{U}_0$ which form a connected component of the dual graph of $U_0$ corresponding to $Y$. By Proposition 2.4, $\bar{Y}_0$ consists of $\Delta(q_n)$ and curves over $\cup_{i=0}^{n+1}q_{-i}q_i$ (if $n < 0$) or $\cup_{i=1}^{n}q_{-i}q_i$ (if $n > 0$) and each $q_{-1}q_i$ is dominated by the components of $\bar{Y}_0$ through maps whose total degree is $E \cdot \Delta(q_n)$. Later, we will prove in Theorem 4.1 that $E \cdot \Delta(q_n) = 1$, i.e., if $\Gamma_i$ passes through $q_n$, $\Gamma_i$ and $E$ must meet transversely at $q_n$.

**Proof of Proposition 2.4.** Let us first prove the statements concerning $Y_0$. The rest will follow more or less immediately.

Obviously, there is at most one disk $\Delta \subset Y_0$ with $\Delta \subset \Gamma_1 \cup \Gamma_2$ by Proposition 2.3; otherwise, either $\Gamma_i$ is joined to itself by a chain of curves over $\cup_{j=1}^{m-1}q_jq_{j+1}$ for some $i$ or $\Gamma_1$ and $\Gamma_2$ are joined by a chain of curves over $\cup_{j=-1}^{m-1}q_jq_{j+1}$, but by Proposition 2.3, $\Gamma_1$ and $\Gamma_2$ are already joined by the curves over a Type I chain.

If $q_{i-1}q_{i+1} \subset Y_0$ for some $i \geq 0$, by Lemma 2.1, either $q_{i+1}q_{i+2} \subset Y_0$ or there is a disk $\Delta(q_{i+1}) \subset \Gamma_i \cup \Gamma_2$ such that $\Delta(q_{i+1}) \subset Y_0$ and $\Delta(q_{i+1})$ and $q_{i-1}q_{i+1}$ lie on the different $R_j$’s. If $q_{i+1}q_{i+2} \not\subset Y_0$, we are done; otherwise, we apply the same argument to $q_{i+1}q_{i+2}$ again. And eventually, this sequence of curves will end up at some disk $\Delta(q_n) \subset Y_0$ for some $n > i$.

Obviously, $\frac{q_nq_{n+1}}{q_nq_{n+1}} \not\subset Y_0$; otherwise, we may continue to apply the above argument to show that there exists another disk $\Delta(q_{n'}) \subset Y_0$ for some $n' > n$. On the other hand, since $\frac{q_{i-1}q_i}{q_{i-1}q_i} \subset Y_0$, we have either $\frac{q_{i-1}q_i}{q_{i-1}q_i} \subset Y_0$ or $\Delta(q_{i-1}) \subset Y_0$ by Lemma 2.1. Since $\Delta(q_n) \subset Y_0$, we necessarily have $q_{i-1}q_i \subset Y_0$. Apply the same argument to $q_{i-1}q_i$ and we obtain that $q_{i-2}q_{i-1} \subset Y_0$. So eventually, we have $q_{i-1}q_i, q_{i-2}q_{i-1}, \ldots, q_{0} \subset Y_0$. It is impossible that $\frac{q_{i-1}q_i}{q_{i-1}q_i} \subset Y_0$; otherwise, we may apply the same line of argument to show subsequently that $\frac{q_{i-1}q_i}{q_{i-2}q_{i-1}}, \ldots, \frac{q_{i-1}q_t}{q_{i-1}} \subset Y_0$ and eventually $\Delta(q_{i-1}) \subset Y_0$. So $\frac{q_{i-1}q_i}{q_{i-1}q_i} \subset Y_0$ and hence $(Y_0)_{\text{red}} = \cup_{j=0}^{n-1}q_{ij}q_{j+1} \cup \Delta(q_n)$. By Lemma 2.1, all $q_{ij}q_{j+1}$ in this sequence has the same multiplicity $\Delta(q_n) \cdot E$ in $Y_0$. So if $q_{ij}q_{j+1} \subset Y_0$ for some $i \geq 0$, we will necessarily end up in case (c).

The same argument shows if $q_{ij}q_{j+1} \subset Y_0$ for some $i < 0$, we will end up in case (d).

It is obvious that if there is no $\frac{q_{ij}q_{j+1}}{q_{ij}q_{j+1}} \subset Y_0$, we will necessarily end up in case (a) or (b).

So if $q_{ij}q_{j+1} \subset Y_0$ for $i \geq 0$, we necessarily have $q_{ij}q_{j+1} \subset Y_0$ for any $0 \leq j \leq i$ and $q_{ij}q_{j+1}$ has the same multiplicity in $Y_0$ as $q_{ij}q_{j+1}$. Therefore, $\mu(q_{ij}q_{j}) \geq \mu(q_{ij}q_{j+1}) \geq \ldots \geq \geq \mu(q_{0}q_{m+1}q_{m})$. Similarly, $\mu(q_{i-1}q_{i-1}q_{i+2}) \leq \mu(q_{i-1}q_{i-1}q_{i+2}) \leq \mu(q_{i-1}q_{i-1})$. 


It follows from Corollary 2.1 that $m_1(q_0) \leq 1$ and $m_2(q_0) \leq 1$. Otherwise, suppose that $m_1(q_0) > 1$. Let $Y$ be the locally irreducible component of $\Upsilon$ around $\cup_{i=-l}^{l}q_{i+l}$ such that $\Delta(q_0) \subset \Gamma_1$ and $\Delta(q_0) \subset Y$. By Corollary 2.1 and the fact that $\Delta(q_0)$ is the only “piece” of $\Gamma_1 \cup \Gamma_2$ in $Y_0$, we must have $q_{-\ell} q_0 \subset Y_0$, which contradicts our results on possible $Y_0$’s.

Finally, it is impossible that $m_1(q_n) > 0$ and $m_2(q_n) > 0$ for $n \neq 0$. Otherwise, suppose that both $\Gamma_1$ and $\Gamma_2$ pass through $q_n$ for some $n > 0$. Since $\Gamma_1$ passes through $q_n$, there exists a locally irreducible component $Y$ of $\Upsilon$ around $\cup_{i=-l}^{l}q_{i+l}$ such that $(Y_0)_{\text{red}} = \cup_{i=0}^{m-1}q_{i+l} \cup \Delta(q_n)$ where $\Delta(q_n) \subset \Gamma_1$ and $\Delta(q_n)$ and $q_{n-1} q_n$ lie on the different $R_j$’s. So $q_{n-1} q_n$ lies on $R_1$. The same argument shows that $q_{n-1} q_n$ lies on $R_1$ since $\Gamma_2$ passes through $q_n$. Contradiction.

As Proposition 2.3 and 2.4 for Type I and Type II chains, we have a similar statement for Type III chains. However, we do not really need it in our proof. So we will state the proposition without a proof. Interested readers could follow the same line of argument as in the Proposition 2.3 and 2.4 and give a proof themselves.

**Proposition 2.5.** A type III chain in $\Upsilon_0$ containing the point $r_{ij}$ must also contain the curve $r_{ij}$.

Let $\overline{r_{ij} q_1} \cup \overline{q_1 q_2} \cup \ldots \cup \overline{q_{m-1} q_m}$ be a Type III chain in $\Upsilon_0$. Assume that $r_{ij} = r_{11}$ without the loss of generality.

Let $Y$ be a locally irreducible component of $\Upsilon$ around $\overline{r_{11}} \cup \overline{r_{11} q_1} \cup \overline{q_1 q_2} \cup \ldots \cup \overline{q_{m-1} q_m}$. Let $q_0 = r_{11}$.

Then

1. $2\mu(\overline{r_{11}}) \geq \mu(\overline{r_{11} q_1}) \geq \mu(\overline{q_1 q_2}) \geq \ldots \geq \mu(\overline{q_{m-1} q_m})$;
2. $m_i(q_j)$ are even for all $i$ and $j$;
3. $m_i(q_j) = 0$ if $i + j$ is odd;
4. $Y_0$ is one of the following
   (a) $Y_0 = \Delta(w)$ where $w$ is one of the intersections between $\Gamma_1 \cup \Gamma_2$ and $\overline{r_{ij}} \cup \overline{r_{ij} q_1} \cup \overline{q_1 q_2} \cup \ldots \cup \overline{q_{m-1} q_m}$ other than the points $r_{11}, q_1, q_2, \ldots, q_m$ and $\Delta(w)$ is an analytic neighborhood of $w$ on $\Gamma_1$ or $\Gamma_2$;
   (b) $(Y_0)_{\text{red}} = \overline{r_{11}} \cup \overline{r_{11} q_1} \cup \overline{q_1 q_2} \cup \ldots \cup \overline{q_{n-1} q_n} \cup \Delta(q_n)$ for some $n \geq 0$, where $\Delta(q_n)$ is an analytic neighborhood of $q_n$ on $\Gamma_1$ or $\Gamma_2$, $\Delta(q_n)$ and $q_{n-1} q_n$ lie on the different $R_j$’s, $q_{n-1} q_n$ has multiplicity $\Delta(q_n) \cdot E$ in $Y_0$ and $\overline{r_{11}}$ has multiplicity $(\Delta(q_n) \cdot E)/2$ in $Y_0$. 


3. Deformation around a Type I Chain

In this section, we will study the behavior of $\Upsilon_t$ at the neighborhood of a Type I chain in order to show that $\Upsilon_t$ has only nodes as singularities in the neighborhood of a Type I chain.

For a one-parameter family of curves $S$ over $\Delta$ and a reduced subscheme $B \subset S_0$, we use the notation $\delta(S_t, B)$ to denote the total $\delta$-invariant of the general fiber $S_t$ in a neighborhood of $B$.

The main theorem of this section is

**Theorem 3.1.** Let $\bigcup_{i=1}^{2l} \overline{q_{i-1}q_i}$ be the Type I chain in $\Upsilon_0$. Assume that $q_0q_l \subset R_1$. Then

1. $\delta(\Upsilon_t, \bigcup_{i=1}^{2l} \overline{q_{i-1}q_i}) = (2l + 1)\beta - 1$, where $\beta = m_2(q_0) = m_1(q_{2l})$;
2. $\Upsilon_t$ has exactly $\beta$ nodes in the neighborhood of each point $w_j$ for $j = 1, 2, \ldots, 2l$, where $w_j$ is the intersection between $\Gamma_1 \cup \Gamma_2$ and $\overline{q_{j-1}q_j}$ other than the points $q_{j-1}$ and $q_j$;
3. $\Upsilon_t$ has exactly $\beta - 1$ nodes in the neighborhood of $q_l$.

**Remark 3.1.** Notice that

$$\delta(\Upsilon_t, \bigcup_{i=1}^{2l} \overline{q_{i-1}q_i}) = (2l + 1)\beta - 1$$

$$= \sum_{i=0}^{2l} \text{Intsc}(\Sigma_1, E; q_i) - 1$$

$$= \sum_{i=0}^{2l} \text{Intsc}(\Sigma_2, E; q_i) - 1,$$

where $\text{Intsc}(A, B; C)$ denotes the local intersection multiplicities between $A$ and $B$ along $C$.

The main obstacle here is that $\Upsilon_0$ is nonreduced along $\bigcup_{i=1}^{2l} \overline{q_{i-1}q_i}$ if $\beta > 1$. To study the deformation of a nonreduced curve, we introduce a method called “patching technique”. Actually, this is a very commonly used method in deformation theory. To study the deformation of a projective variety or compact complex manifold $M$, we cover $M$ by affine or analytic open sets, study the deformation of each piece separately and then “patch” these deformations together. We will show how this can be done for a nonreduced curve.

3.1. Deformation of a Nonreduced Planary Curve.

**Definition 3.1.** Let $C$ be a nonreduced scheme and $C_{\text{red}}$ be its reduced subscheme. Let $I$ be the ideal sheaf of $C_{\text{red}}$ in $C$. Then $N_{C_{\text{red}}} = \mathcal{H}om(I/I^2, O_{C_{\text{red}}})$ is a coherent sheaf over $C_{\text{red}}$, which we will call the “intrinsic normal sheaf” of $C$. 
Definition 3.2. We call a nonreduced curve $C$ “planary” if it can be locally embedded to the 2-dimensional polydisk $\Delta^2_{xy}$ everywhere. We say $C$ has irreducible support if $C_{\text{red}}$ is irreducible.

If $C$ is a planary nonreduced curve with irreducible support, we can cover $C$ with analytic open sets $U_\alpha$ such that each

$$U_\alpha \cong \text{Spec} \mathbb{C}[[x, y]]/(f^m(x, y)),$$

where $f(x, y) = 0$ defines $C_{\text{red}}$ in $U_\alpha$ and $m$ is called the multiplicity of $C$. And it is obvious in this case that the intrinsic normal sheaf $N_{C_{\text{red}}}$ of $C$ is a line bundle over $C_{\text{red}}$.

Obviously, $C$ is planary if it lies on a smooth surface. If $C$ lies on a smooth surface $S$ and has irreducible support, its intrinsic normal sheaf is simply the normal sheaf $N_{C_{\text{red}}/S}$ of $C_{\text{red}} \subset S$.

Now let $C$ be a nonreduced planary curve with irreducible support and assume that $C_{\text{red}}$ is smooth. Then $C$ is locally isomorphic to $\text{Spec} \mathbb{C}[[x, y]]/(y^m)$, where $m$ is the multiplicity of $C$. Let $C$ be a one-parameter family of curves over disk $\Delta$ whose central fiber is $C$. Cover $C$ with open sets $U_\alpha$ of $C$ such that $U_\alpha \cong \text{Spec} \mathbb{C}[[x, y, t]]/(F_\alpha(x, y, t))$ where $F_\alpha(x, y, t)$ can be put into the form

$$F_\alpha(x, y, t) = y^m + \sum_{0 < i \leq j < m-2} f_{\alpha ij}(x) t^i y^j.$$

For each $\alpha$, we let

$$\gamma(\alpha) = \min \left\{ \frac{i}{m-j} : f_{\alpha ij}(x) \neq 0 \right\}.$$ 

Alternatively, we may define $\gamma(\alpha)$ as the largest number such that

$$t^{m\gamma(\alpha)} | F_\alpha(x, t^{\gamma(\alpha)} y, t).$$

After a base change, we may assume that $\gamma(\alpha) \in \mathbb{Z}$ for each $\alpha$. Then we may put $F_\alpha(x, y, t)$ into the form

$$F_\alpha(x, y, t) = y^m + \sum_{j=0}^{m-2} t^{(m-j)\gamma(\alpha)} \phi_{\alpha j}(x, t)y^j$$

where $\phi_0(x, 0), \phi_1(x, 0), ..., \phi_{m-2}(x, 0)$ are not all zeros by the choice of $\gamma(\alpha)$.

Next, we will “patch” $U_\alpha$ together. We need the following “patching” lemma.
Lemma 3.1 (Patching Lemma). Let $\Sigma_1, \Sigma_2 \subset \Delta_{xy}^2 \times \Delta_t$ be two families of curves over disk $\Delta_t$. Suppose that $\Sigma_i$ $(i = 1, 2)$ is cut out by

$$y^m + \sum_{j=0}^{m-2} t^{(m-j)}\gamma_j(x, t)y^j = 0$$

in $\Delta_{xy}^2 \times \Delta_t$, where $\gamma_i \in \mathbb{Z}$, $\gamma_i > 0$, and $\phi_i,0(x, 0), \phi_i,1(x, 0), ..., \phi_i,m-2(x, 0)$ are not all zero. If there is an isomorphism between $\Sigma_1$ and $\Sigma_2$ which preserves the base and induces the identity map on the central fibers, then $\gamma_1 = \gamma_2$ and $\phi_{1j}(x, 0) = \phi_{2j}(x, 0)$ for $j = 0, 1, ..., m - 2$.

Lemma 3.1 is more or less obvious and we will leave its proof to the readers. It follows from Lemma 3.1 that all $\gamma(\alpha)$ are equal and for each $j$, $\{\phi_{\alpha j}(x, 0)\}$ defines a global section of the line bundle $\mathcal{N}_{C_{\text{red}}}^{\oplus (m-j)}$, i.e., there exists $s \in H^0(\mathcal{N}_{C_{\text{red}}}^{\oplus (m-j)})$ such that $s|_{U_\alpha} = \phi_{\alpha j}(x, 0)$, where $\mathcal{N}_{C_{\text{red}}}$ is the intrinsic normal sheaf of $C$.

Let $\gamma = \gamma(\alpha)$. Then $\{y = t^\gamma = 0\}$ is a well-defined closed subscheme of $C$ supported at the central fiber. Let $\tilde{C}$ be the blowup of $C$ along the subscheme $\{y = t^\gamma = 0\}$. It is not hard to see that $\tilde{C}_0$ is a curve in the linear series $|O_{\mathbb{P}}(m)|$ on $\mathbb{P} = \mathbb{P}(O_{C_{\text{red}}} \oplus \mathcal{N}_{C_{\text{red}}})$. Now $\tilde{C}_0$ is “less” nonreduced than $C$, i.e., each component of $\tilde{C}_0$ has multiplicity strictly less than $m$ in $\tilde{C}_0$ due to our choice of $\gamma$. This makes $\tilde{C}$ easier to investigate than $C$.

3.2. Outline of the Proof of Theorem 3.1. Let $I_j$ be the nonreduced component of $\Sigma_0$ supported on $\frac{q_{j-1}q_j}{\gamma_j}$ with multiplicity $\beta$. Obviously, $I_j$ is a nonreduced planar curve with trivial intrinsic normal sheaf.

The proof of Theorem 3.1 is carried out by repeatedly blowing up $\Sigma$ along $\bigcup I_j$. To be precise, we will construct a sequence of families

$$(3.1) \quad \Sigma^{(l)} \to \Sigma^{(l-1)} \to ... \to \Sigma^{(0)} = \Sigma$$

where

1. the morphisms $\Sigma^{(j)} \to \Sigma^{(j-1)}$ are isomorphisms on the general fibers for $j = 1, 2, ..., l$;
2. on the central fiber, $\Sigma_0^{(j)}$ contains $\tilde{I}_1 \cup \tilde{I}_2 \cup ... \cup \tilde{I}_{j-1} \cup \tilde{I}_j \cup I_{j+1} \cup ... \cup I_{2l-j} \cup \tilde{I}_{2l-j+1} \cup \tilde{I}_{2l-j+2} \cup ... \cup \tilde{I}_{2l-1} \cup \tilde{I}_{2l}$ for $j = 0, 1, 2, ..., l$, where $\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_{2l}$ are the curves dominating $(I_1)_{\text{red}} = \frac{q_{j-1}q_j}{\gamma_1}, (I_2)_{\text{red}} = \frac{q_{j-1}q_j}{\gamma_2}, ..., (I_{2l})_{\text{red}} = \frac{q_{2l-1}q_{2l}}{\gamma_{2l}}$, respectively; recall from Proposition 2.3 that $I_j$ dominates $\frac{q_{j-1}q_j}{\gamma_j}$ with a degree $\beta$ map totally ramified at $q_{j-1}$ and $q_j$;
3. let \( w_{j1}, w_{j2}, \ldots, w_{j\beta} \) be the points over \( w_i \) in the map \( \tilde{I}_j \to (I_j)_{\text{red}} \) for \( j = 1, 2, \ldots, 2l \); there is a contractible curve \( H_j \) meeting \( \tilde{I}_j \) transversely at \( w_{j1}, w_{j2}, \ldots, w_{j\beta} \) and meeting \( \Gamma_1 \) or \( \Gamma_2 \) transversely at another point;

4. each pair of curves \( \tilde{I}_j \) and \( \tilde{I}_{j+1} \) meet transversely at a point over \( q_j \), which we still denote by \( q_j \), except that \( \tilde{I}_l \) and \( \tilde{I}_{l+1} \) meet at the point \( q_l \) with multiplicity \( \beta \) on \( \Upsilon \); more precisely, \( \Upsilon = \text{Spec} \mathbb{C}[y, z, t]/(y + z^\beta + O(t) - t^\alpha) \) in an analytic neighborhood of \( q_l \), where we use the notation \( O(f_1, f_2, \ldots, f_n) \) to denote an element generated by \( f_1, f_2, \ldots, f_n \) in some ring (it should be clear from the context which ring we are talking about).

It is obvious from the above description that \( \Upsilon \) has exactly \( \beta \) nodes in the neighborhoods of \( w_j \). The statement that \( \Upsilon \) has exactly \( \beta - 1 \) nodes in the neighborhood of \( q_l \) follows from a theorem of L. Caporaso and J. Harris on the deformation of tacnodes \([CH, \text{Lemma 4.1}]\) and the fact that the curves \( \tilde{I}_l \) and \( \tilde{I}_{l+1} \) meet transversely on the nodal reduction \( \tilde{\Upsilon} \).

So the proof of Theorem \([3.1]\) boils down to the construction of the blowup sequence \([3.1]\).

### 3.3. Construction of the Blowup Sequence

We will do the construction inductively. For that purpose, we will work on an analytic neighborhood of \( \Upsilon \) around \((\cup I_j)_{\text{red}}\), which we will still refer to by \( \Upsilon \) and can be described as follows.

First, \( \Upsilon \) is a one-parameter family of curves over disk \( \Delta_t \) with irreducible general fibers \( \Upsilon_t \). Second, \( \Upsilon_0 = I_1 \cup I_2 \cup \ldots \cup I_{2l} \cup \Delta(q_0) \cup \Delta(q_2) \cup \Delta(w_1) \cup \Delta(w_2) \cup \ldots \cup \Delta(w_{2l}) \), where \( I_j \cong \mathbb{P}^1 \times \text{Spec} \mathbb{C}[z]/(z^\beta) \) and \( \Delta(q_0), \Delta(q_2) \) and \( \Delta(w_i) \) are disks centered at \( q_0, q_2 \) and \( w_i \), regarded as the analytic neighborhoods of \( q_0, q_2 \) and \( w_i \) on \( \Gamma_1 \) or \( \Gamma_2 \). Finally, these curves are “patched” up in the following way:

1. \( q_j = (I_j)_{\text{red}} \cap (I_{j+1})_{\text{red}} \) and

\[
\Upsilon \cong \text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha, z^\beta + O(t))
\]

in the neighborhoods of \( q_j \) for \( j = 1, \ldots, 2l - 1 \),

2. \( q_0 = \Delta(q_0) \cap (I_1)_{\text{red}}, q_2 = \Delta(q_2) \cap (I_2)_{\text{red}} \) and

\[
\Upsilon \cong \text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha, y - z^\beta + O(t))
\]

in the neighborhoods of \( q_0 \) and \( q_2 \);

3. \( w_j = \Delta(w_j) \cap (I_j)_{\text{red}} \) and

\[
\Upsilon \cong \text{Spec} \mathbb{C}[[x, z, t]]/(xz^\beta + O(t))
\]

in the neighborhoods of \( w_j \) for \( j = 1, 2, \ldots, 2l \).
We also know that the nodal reduction $\tilde{\Upsilon}$ of $\Upsilon$ has the properties described in Proposition 2.3 with $\Delta(q_0), \Delta(q_{2l})$ and $\Delta(w_i)$ regarded as “pieces” of $\Gamma_1$ or $\Gamma_2$.

By (3.4), $\Upsilon$ can be locally embedded into $\Delta^2_{xz} \times \Delta_t$ at $w_j$. After applying some automorphism of $\Delta^2_{xz} \times \Delta_t$ which preserves the base and the central fiber, we may put its local defining equation at $w_j$ into the form

$$xz^\beta + t^{m_{ij}} z^{\beta-1} + \sum_{i=2}^{\beta} t^{m_{ij}} f_{ij}(x, t) z^{\beta-i} = 0$$

where $f_{ij}(x, t) \in \mathbb{C}[[x, t]]$, $f_{ij}(x, 0) \neq 0$ and we put $m_{ij} = \infty$ if the corresponding term $z^{\beta-1}$ or $f_{ij}(x, t) z^{\beta-i}$ does not appear in the defining equation. Let

$$\gamma_j = \min \left\{ \frac{m_{ij}}{i} : 1 \leq i \leq \beta \right\} \quad \text{and} \quad \gamma = \min \left\{ \frac{\alpha}{\beta}, \gamma_1, \gamma_2, ..., \gamma_{2l} \right\}.$$

Obviously, we may assume that $\gamma, \gamma_j \in \mathbb{Z}$ for $j = 1, 2, ..., 2l$ after a base change.

We can write (3.5) as

$$xz^\beta + \sum_{i=1}^{\beta} t^{\gamma_i} F_{ij}(x, t) z^{\beta-i} = 0$$

where $F_{ij}(x, t) \in \mathbb{C}[[x, t]]$ and $F_{ij}(x, t) = F_j(t) \in \mathbb{C}[[t]]$. Notice that $F_{ij}(x, 0) = 0$ if $\gamma_j > \gamma$.

**Claim 3.1.** We claim that

1. $F_{ij}(0, 0) = 0$ for $i = 1, 2, ..., \beta$ and $j = 1, 2, ..., 2l$ and especially, $F_{ij}(x, 0) = F_j(0) = 0$;
2. $x^{-1} F_{ij}(x, 0)$ extends to a meromorphic function $G_{ij}$ on $(I_j)_{\text{red}}$;
3. each $G_{ij}$ is holomorphic everywhere on $(I_j)_{\text{red}}$ except that $G_{\beta, 1}$ and $G_{\beta, 2l}$ have a simple pole at $q_0$ and $q_{2l}$, respectively, if $\alpha = \beta \gamma$;
4. at each $q_j$ for $j = 1, 2, ..., 2l - 1$, $G_{i,j}(q_j) = G_{i,j+1}(q_j)$.

The first statement follows from the assumption that $\Delta(w_j)$ and $I_j$ are disjoint on $\tilde{\Upsilon}_0$.

Let $U$ be an analytic neighborhood of $w_j$ on $\Upsilon$. Since $\Delta(w_j)$ and $I_j$ are disjoint on $\tilde{\Upsilon}_0$, $U$ is reducible and $U = U^{(1)} \cup U^{(2)}$ after a base change where the central fibers of $U^{(1)}$ and $U^{(2)}$ are given by $x = t = 0$ and $z^\beta = t = 0$, respectively. Correspondingly, we may factor the LHS
of (3.6) in the following way

\[(3.7) \quad xz^\beta + \sum_{i=1}^\beta t^\gamma F_{ij}(x, t)z^{\beta-i} = (x + O(t))(z^\beta + O(t)).\]

It follows immediately from (3.7) that \(F_{ij}(0, 0) = 0\) which also implies \(F_{ij}(x, 0) = F_j(0) = 0\).

It follows from Lemma 3.1 that \(x^{-1}F_{ij}(x, 0)\) can be analytically extended to a section in

\[\Gamma \left( (I_j)_{\text{red}} \setminus \{q_{j-1}, q_j\}, \mathcal{N}^{\otimes i}_j \right)\]

where \(\mathcal{N}_j\) is the intrinsic normal sheaf of \(I_j\). Of course, \(\mathcal{N}_j\) is trivial. So each \(G_{ij} = x^{-1}F_{ij}(x, 0)\) is a meromorphic function on \((I_j)_{\text{red}} \cong \mathbb{P}^1\), which is holomorphic everywhere except at the points \(q_0, q_1, q_2, ..., q_2l\).

Actually, \(G_{ij}\) can be extended over \(q_1, q_2, ..., q_{2l-1}\). By (3.2), \(\Upsilon\) can be locally embedded into \(\text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha)\) at \(q_1, q_2, ..., q_{2l-1}\) and its local defining equation at \(q_j\) can be put into the form

\[(3.8) \quad z^\beta + \sum_{i=2}^\beta t^{i\delta_j} \phi_{ij}(x, y, t)z^{\beta-i} = 0\]

where \(\delta_j > 0, \phi_{ij}(x, y, t) \in \mathbb{C}[[x, y, t]]\), for \(i = 2, 3, ..., \beta\) and \(j = 1, 2, ..., 2l - 1\) and \(\phi_{2j}(x, y, t), \phi_{3j}(x, y, t), ..., \phi_{\beta j}(x, y, t)\) do not all lie in the ideal \((xy) \subset \mathbb{C}[[x, y]]\).

By comparing (3.8) with (3.6) and applying Lemma 3.1, we have \(\delta_j = \min(\gamma_j, \gamma_{j+1})\). So

1. \(\delta_j \geq \gamma\);
2. \(\gamma_j > \gamma\) and \(\gamma_{j+1} > \gamma\) if \(\delta_j > \gamma\).

Hence if \(\delta_j > \gamma\), we have \(G_{ij} = G_{i,j+1} = 0\); otherwise, if \(\delta_j = \gamma\), it is not hard to see that \(G_{i,j}(q_j) = G_{i,j+1}(q_j) = \phi_{ij}(0, 0, 0, 0)\). Hence \(G_{ij}\) are holomorphic at \(q_j\) and \(G_{i,j}(q_j) = G_{i,j+1}(q_j)\) for \(j = 1, 2, ..., 2l - 1\).

So \(G_{ij}\)'s are holomorphic everywhere except at \(q_0\) and \(q_{2l}\). Next, we will try to find out what kind of singularities \(G_{ij}\)'s could have at \(q_0\) and \(q_{2l}\).

By (3.3), \(\Upsilon\) can be locally embedded into \(\text{Spec} \mathbb{C}[[x, z, t]]\) at \(q_0\) and \(q_{2l}\). We can put its local defining equations at \(q_j\) into the form

\[(3.9) \quad x \left( z^\beta + \sum_{i=2}^\beta t^{i\delta_j} \phi_{ij}(x, t)z^{\beta-i} \right) = t^\alpha\]

where \(\delta_j > 0, \phi_{ij}(x, t) \in \mathbb{C}[[x, t]]\) for \(i = 2, 3, ..., \beta\) and \(j = 0, 2l\) and \(\phi_{2j}(x, 0), \phi_{3j}(x, 0), ..., \phi_{\beta j}(x, 0)\) are not all zero.

By comparing (3.9) with (3.6) and applying Lemma 3.1, we have
1. $\delta_j \geq \gamma$, $\alpha/\beta \geq \gamma_1$ and $\alpha/\beta \geq \gamma_2$;
2. $G_{i,1}$ and $G_{i,2l}$ are holomorphic at $q_0$ and $q_{2l}$, respectively, for $i = 2, \ldots, \beta - 1$;
3. $G_{\beta,1}$ and $G_{\beta,2l}$ have a simple pole at $q_0$ and $q_{2l}$, respectively, if $\alpha = \beta \gamma$ and is holomorphic otherwise.

So we have justified every statement in Claim 3.1 and we are ready to construct a blowup map $\Upsilon' \to \Upsilon$ where $\Upsilon'_0 \supset \tilde{I}_1 \cup I_2 \cup \ldots \cup I_{2l-1} \cup I_{2l}$.

Notice that $z = t^{\gamma} = 0$ defines subschemes of $\Upsilon$ locally at $w_j$ by (3.4). We can extend these subschemes and patch them together to obtain a closed subscheme of $\Upsilon$ due to our choice of the number $\gamma$. This subscheme is obviously supported on $(I_1)_{\text{red}} \cup (I_2)_{\text{red}} \cup \ldots \cup (I_{2l})_{\text{red}}$. The family $\Upsilon'$ is obtained by blowing up $\Upsilon$ along this closed subscheme, which is locally cut out by $z = t^{\gamma} = 0$.

First we claim that $\alpha = \beta \gamma$. If not, we necessarily have $\alpha > \beta \gamma$. Hence each $G_{ij}$ is a holomorphic function over $I_j$ and consequently a constant. By the equality $G_{i,j-1}(q_j) = G_{i,j}(q_j)$, $G_{i1} = G_{i2} = \ldots = G_{i2l}$ and we let $G_i = G_{i1}$.

Obviously, $\gamma_1 = \gamma_2 = \ldots = \gamma_{2l} = \gamma$. Otherwise, if $\gamma_k > \gamma$ for some $k$, then $G_{ik} = 0$, which implies $G_i = G_{ij} = 0$ for each $i$ and $j$. And this is a contradiction to the choice of $\gamma$.

Let us examine the behavior of $\Upsilon'$ over the point $q_0$. Since $\gamma_1 = \gamma < \alpha/\beta$, by (3.9), $\Upsilon'_0$ consists of the curve

$$x \left( z_1^{\beta} + \sum_{i=2}^{\beta} G_i z_1^{\beta-i} \right) = 0,$$

where $z_1 = z/t^{\gamma}$. Obviously, the curve $x = 0$ dominates $\Delta(q_0)$ and the curve

$$z_1^{\beta} + \sum_{i=2}^{\beta} G_i z_1^{\beta-i} = 0$$

maps to $I_1$. Since $G_2, G_3, \ldots, G_{\beta}$ are not all zero, the LHS of (3.11) has at least two distinct roots. Hence there are at least two different components over $I_1$ at $q_0$. This contradicts the fact that on $\tilde{\Upsilon}_0$ there is a single component $\tilde{I}_1$ dominating $I_1$ with a map totally ramified at $q_0$.

Therefore, we have $\alpha = \beta \gamma$. Then $G_{ij}$ are still constants except that $G_{\beta,1}$ and $G_{\beta,2l}$ have a simple pole at $q_0$ and $q_{2l}$, respectively.

Let us examine the behavior of $\Upsilon'$ over the point $q_0$ again. Since $\alpha = \beta \gamma$, by our previous analysis, $\gamma = \gamma_1 = \alpha/\beta$. By (3.9), $\Upsilon'_0$ consists
of the curve
\[(3.12) \quad x \left( 1 + \sum_{i=2}^{\beta-1} G_{i1} z_2^i \right) + (xG_{\beta,1})(q_0) z_2^\beta = zz_2 = 0 \]

where \( z_2 = t^\gamma/z \) and \( (xG_{\beta,1})(q_0) \) reads as the value of the function \( xG_{\beta,1} \) at point \( q_0 \). Obviously, the curve \( x = z_2 = 0 \) dominates \( \Delta(q_0) \) and the irreducible curve
\[ x \left( 1 + \sum_{i=2}^{\beta-1} G_{i1} z_2^i \right) + (xG_{\beta,1})(q_0) z_2^\beta = z = 0 \]
dominates \( (I_1)_{\text{red}} \) with a degree \( \beta \) map. So \( \Upsilon_0' \) contains an irreducible curve \( \tilde{I}_1 \) dominating \( (I_1)_{\text{red}} \) with a degree \( \beta \) map. And the map \( \tilde{I}_1 \to (I_1)_{\text{red}} \) is totally ramified at \( q_0 \) and \( q_1 \).

Let us examine the behavior of \( \Upsilon' \) over the point \( q_1 \), where \( \Upsilon \) is defined by \((3.8)\) in \( \text{Spec } \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha) \). Without the loss of generality, we may assume that \( I_1 \subset \{ y = 0 \} \) locally at \( q_1 \).

Since \( G_{\beta,1} \) is a meromorphic function with a simple pole at \( q_1 \), \( G_{\beta,1} \neq 0 \) in the neighborhood of \( q_1 \). Hence we must have \( \phi_{\beta,1}(x, 0, 0) = G_{\beta,1} \) and \( \delta_1 = \gamma \) in \((3.8)\). Hence the curve \( \Upsilon_0' \) is given by
\[(3.13) \quad z_1^\beta + \sum_{i=2}^{\beta} G_{i1} z_1^{\beta-i} = xy = 0 \]
in the neighborhood of \( q_1 \), where \( z_1 = z/t^\gamma \). By \((3.13)\), we necessarily have \( G_{i1}(q_1) = 0 \) for \( i = 2, 3, \ldots, \beta \); otherwise, the map \( \tilde{I}_1 \to I_1 \) will not be totally ramified at \( q_1 \). Therefore, \( G_{i2} = G_{i2}(q_1) = G_{i1}(q_1) = 0 \) and \( \gamma_2 > \gamma \), which implies the curve in \( \Upsilon_0' \) dominating \( I_2 \) is still a nonreduced curve isomorphic to \( \mathbb{P}^1 \times \mathbb{C}[[z]]/(z^\beta) \). So we will use the same notation \( I_2 \) to denote the curve in \( \Upsilon_0' \) over \( I_2 \subset \Upsilon_0 \).

The same argument can be carried out by studying the behavior of \( \Upsilon' \) over \( q_2, \ldots, q_{2l-1} \). Finally we obtained that \( G_{ij} = 0, \gamma_j > \gamma \) and \( \Upsilon_0' \) contains \( I_j \cong \mathbb{P}^1 \times \mathbb{C}[[z]]/(z^\beta) \) for \( i = 2, 3, \ldots, \beta \) and \( j = 2, 3, \ldots, 2l-1 \). And by symmetry, \( \Upsilon_0' \) contains the irreducible curve \( \tilde{I}_{2l} \) dominating \( I_{2l} \) with a degree \( \beta \) map. So we may take \( \Upsilon^{(1)} = \Upsilon' \) in \((3.1)\).

Next, we may take a neighborhood of \( \Upsilon' \) around \( I_2 \cup I_3 \cup \ldots \cup I_{2l-1} \) and go through this procedure again. Of course, we have to check that \( \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_{2l-1}, \tilde{I}_{2l} \) are “patched” up at \( q_1, q_2, \ldots, q_{2l-1} \) as required at the beginning of the construction.

Since \( \tilde{I}_1 \) and \( \tilde{I}_{2l} \) are smooth everywhere, \( \Upsilon' \) is locally given by \((3.3)\) at \( q_1 \) and \( q_{2l-1} \). It is easy to check that \( \Upsilon' \) is locally given by \((3.2)\) at \( q_2, q_3, \ldots, q_{2l-2} \) and by \((3.4)\) at \( w_2, w_3, \ldots, w_{2l-1} \). So we may go through
the same procedure for $\Upsilon^{(1)} = \Upsilon'$ by blowing up $\Upsilon'$ along a subscheme of $\Upsilon'_0$ supported on $(I_2)_{\text{red}} \cup (I_3)_{\text{red}} \cup \cdots \cup (I_{2l-1})_{\text{red}}$. The resulting family will be $\Upsilon^{(2)}$ and the blowup sequence (3.1) is constructed inductively in this way.

This finishes the construction of the blowup sequence (3.1) with only one thing left to check. We need to check that $\tilde{I}_1$ and $\tilde{I}_{l+1}$ meet at the point $q_t$ with multiplicity $\beta$ if $l = 1$, i.e., we want to show that $\tilde{I}_1$ and $\tilde{I}_{l+1}$ meet at the point $q_t$ with multiplicity $\beta$ on $\Upsilon^{(l)}$.

Suppose that $l = 1$. By our previous argument, we still have $G_{i1} = G_{i2} = 0$ for $i = 2, 3, \ldots, \beta - 1$ and $G_{\beta,1}(q_1) = G_{\beta,2}(q_1) = 0$. Since $G_{\beta,1}$ and $G_{\beta,2}$ are meromorphic functions over $\mathbb{P}^1$ with exactly one simple pole at $q_0$ and $q_2$, respectively, each of them has a simple zero at $q_1$. Notice that $\phi_{\beta,1}(x, 0, 0)$ and $\phi_{\beta,1}(0, y, 0)$ are the localizations of $G_{\beta,1}$ and $G_{\beta,2}$ at $q_1$ in (3.8) since $\delta_1 = \gamma$. Hence $\phi_{\beta,1}(x, y, 0) = ax + by + O(x^2, xy, y^2)$ for some constants $a, b \neq 0$. Therefore, $\Upsilon'$ is cut out by

$$z_1^\beta + ax + by + O(x^2, xy, y^2, t) = 0$$

in $\text{Spec } \mathbb{C}[[x, y, z_1, t]]/(xy - t^\alpha)$ locally at $q_1$.

4. Deformation Around a Type II Chain

Our main theorem of this section is the following.

**Theorem 4.1.** Let $\bigcup_{i=1}^{m-1}\overline{q_iq_{i+1}}$ be a Type II chain in $\Upsilon_0$ and let $q_0 \in \{p_1, p_2, \ldots, p_{16}\}$. Then

$$\delta(\Upsilon_t, \bigcup_{i=-l}^{m-1}\overline{q_iq_{i+1}}) \geq \sum_{i=-l}^{m} \text{Intsc}(\Sigma_i, E; q_i).$$

If the equality holds, then

1. $|\mu(q_{i-1}q_i) - \mu(q_{i}q_{i+1})| \leq 1$ for any $i$ (let $\mu(q_iq_{i+1}) = 0$ if $i < -l$ or $i \geq m$);
2. all singularities of $\Upsilon_t$ are nodes in the neighborhood of $\bigcup_{i=-l}^{m-1}\overline{q_iq_{i+1}}$.

4.1. Some Basic Results on Curve Singularities. Most of the following results on curve singularities are well known. But we will prove them here for the lack of a definite reference.

**Proposition 4.1.** Let $C = \bigcup_{i=1}^{n}C_i$ be a reduced curve in $\Delta^2$, where $C_i$ are distinct curves in $\Delta^2$. Then

$$\delta(C) = \sum_{i=1}^{n} \delta(C_i) + \sum_{1 \leq r < s \leq n} C_r \cdot C_s,$$

where $\delta(C)$ and $\delta(C_i)$ are the $\delta$-invariants of $C$ and $C_i$ at the origin.
Proof. Let \( \widetilde{\Delta}^2 \) be the blowup of \( \Delta^2 \) at the origin and let \( \widetilde{C} \) and \( \widetilde{C}_i \) be the proper transforms of \( C \) and \( C_i \) for \( i = 1, 2, \ldots, n \). Let \( \widetilde{C} \) meet the exception divisor at points \( p_1, p_2, \ldots, p_l \) and let \( \Delta^2(p_j) \) be the neighborhood of \( p_j \) in \( \widetilde{\Delta}^2 \).

Let \( \widetilde{C}_i = \bigcup_{j=1}^l C_j^i \) where \( C_j^i \subset \Delta^2(p_j) \) (let \( C_j^i = \emptyset \) if \( \widetilde{C}_i \) does not pass through \( p_j \)). Let \( m_i \) be the multiplicity of \( C_i \) at the origin and let \( m = \sum_{i=1}^n m_i \) be the multiplicity of \( C \) at the origin. We argue by induction on \( \delta(C) \). It is obvious when \( \delta(C) = 0 \). Suppose that \( \delta(C) > 0 \). Then \( m > 1 \).

First, we have
\[
\delta(C_i) = \frac{m_i(m_i - 1)}{2} + \delta(\widetilde{C}_i) = \frac{m_i(m_i - 1)}{2} + \sum_{j=1}^l \delta(C_j^i). \tag{4.2}
\]

Second, by induction hypothesis, we have
\[
\delta(C) = \frac{m(m - 1)}{2} + \sum_{j=1}^l \delta(\bigcup_{i=1}^n C_j^i) = \frac{m(m - 1)}{2} + \sum_{j=1}^l \left( \sum_{i=1}^n \delta(C_j^i) + \sum_{1 \leq r < s \leq n} C_j^r \cdot C_j^s \right). \tag{4.3}
\]

Finally, we have
\[
C_r \cdot C_s = m_r m_s + \sum_{j=1}^l C_j^r \cdot C_j^s. \tag{4.4}
\]

Combining (4.2), (4.3) and (4.4), we obtain (1.1). \( \square \)

The next is a parameterized version of Proposition 1.1.

**Proposition 4.2.** Let \( \Upsilon = \bigcup_{i=1}^n \Upsilon^{(i)} \subset \Delta^2 \times \Delta_t \) be a reduced flat family of curves over \( \Delta_t \) where \( \Upsilon^{(i)} \) are distinct flat families of curves over \( \Delta_t \). Then
\[
\delta(\Upsilon_t) = \sum_{i=1}^n \delta(\Upsilon_t^{(i)}) + \sum_{1 \leq r < s \leq n} \Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)}, \tag{4.5}
\]

where \( \delta(\Upsilon_t) \) and \( \delta(\Upsilon_t^{(i)}) \) are the total \( \delta \)-invariants of the general fibers of \( \Upsilon \) and \( \Upsilon^{(i)} \) and the intersection between \( \Upsilon_t^{(r)} \) and \( \Upsilon_t^{(s)} \) is taken on the general fiber of \( \Delta^2 \times \Delta_t \to \Delta_t \).

**Proof.** After a base change, we may assume that each singular point of \( \Upsilon_t \) is given by a section \( p : \Delta_t \to \Delta^2 \times \Delta_t \). At each point \( p = p(t) \), we
have

\[ \delta(\Upsilon_t, p) = \sum_{i=1}^{n} \delta(\Upsilon_t^{(i)}, p) + \sum_{1 \leq r < s \leq n} (\Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)})_p \]  

by Proposition 4.1, where \( \delta(\Upsilon_t, p) \) and \( \delta(\Upsilon_t^{(i)}, p) \) are the \( \delta \)-invariants of \( \Upsilon_t \) and \( \Upsilon_t^{(i)} \) at \( p \) and \( (\Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)})_p \) is the local intersection number between \( \Upsilon_t^{(r)} \) and \( \Upsilon_t^{(s)} \) at \( p \) (take \( (\Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)})_p = 0 \) if \( \Upsilon_t^{(r)} \) and \( \Upsilon_t^{(s)} \) do not meet at \( p \)). Obviously, each intersection between \( \Upsilon_t^{(r)} \) and \( \Upsilon_t^{(s)} \) is necessarily a singularity of \( \Upsilon_t \). So summing \( (4.6) \) over all sections \( p \) of singularities yields \( (4.5) \).

**Corollary 4.1.** Let \( \Upsilon = \bigcup_{i=1}^{n} \Upsilon^{(i)} \subset \Delta^2 \times \Delta_t \) be a reduced flat family of curves over \( \Delta_t \) where \( \Upsilon^{(i)} \) are distinct flat families of curves over \( \Delta_t \). If \( \Upsilon_t^{(r)} \) and \( \Upsilon_t^{(s)} \) meet properly on the central fiber of \( \Delta^2 \times \Delta_t \to \Delta_t \) for any \( 1 \leq r < s \leq n \), then

\[ \delta(\Upsilon_t) \geq \sum_{1 \leq r < s \leq n} \Upsilon_0^{(r)} \cdot \Upsilon_0^{(s)}, \]

where the intersection between \( \Upsilon_0^{(r)} \) and \( \Upsilon_0^{(s)} \) is taken on the central fiber of \( \Delta^2 \times \Delta_t \to \Delta_t \).

**Proof.** Since \( \Upsilon_0^{(r)} \) and \( \Upsilon_0^{(s)} \) meet properly, \( \Upsilon_0^{(r)} \cdot \Upsilon_0^{(s)} = \Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)} \). Then \( (4.7) \) follows from \( (4.5) \). \( \square \)

The following is a special case of Corollary 4.1, which is directly applicable to our situation.

**Corollary 4.2.** Let \( \Upsilon \subset \Delta^2 \times \Delta_t \) be a reduced flat family of curves over \( \Delta_t \) whose central fiber \( \Upsilon_0 \) consists of \( n \) irreducible components \( \Gamma_1, \Gamma_2, ..., \Gamma_n \) with multiplicities \( \mu_1, \mu_2, ..., \mu_n \), respectively.

Let \( \pi : \tilde{\Upsilon} \to \Upsilon \) be the nodal reduction of \( \Upsilon \) and let \( \tilde{\Upsilon} = \bigcup_{i=1}^{n} \tilde{\Upsilon}^{(i)} \) where \( \tilde{\Upsilon}^{(i)} \) are the connected components of \( \tilde{\Upsilon} \). Suppose that each \( \pi(\tilde{\Upsilon}_0^{(i)}) \) is supported on \( \Gamma_j \) for some \( 1 \leq j \leq n \). Then

\[ \delta(\Upsilon_t) \geq \sum_{1 \leq r < s \leq n} \mu_r \mu_s (\Gamma_r \cdot \Gamma_s). \]
Corollary 4.2}

Proposition 4.3. Let \( \Upsilon, \Upsilon_0, \Gamma_j, \mu_j, \pi, \tilde{\Upsilon} \) and \( \tilde{\Upsilon}^{(i)} \) be defined as in Corollary 4.2 except that we further assume each \( \pi(\tilde{\Upsilon}^{(i)}_0) \) to be reduced, i.e., \( \pi(\tilde{\Upsilon}^{(i)}_0) = \Gamma_j \) for some \( j \). If
\[
\delta(\Upsilon_t) = \sum_{1 \leq r < s \leq n} \mu_r \mu_s
\]
then \( \Upsilon_t \) only has nodes as singularities.

Proof. By our assumptions on \( \tilde{\Upsilon} \), we have
\[
\Upsilon = \bigcup_{i=1}^n \bigcup_{j=1}^n \Upsilon^{(i,j)}
\]
where \( \Upsilon_0^{(i,j)} = \Gamma_i \). By Proposition 1.2,
\[
\delta(\Upsilon_t) = \sum_{i,j} \delta(\Upsilon_t^{(i,j)}) + \sum_{(r,p) < (s,q)} \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)}
\]
where we define \( (r, p) < (s, q) \) if either \( r < s \) or \( r = s \) and \( p < q \). Notice that
\[
\sum_{(r,p) < (s,q)} \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)} \geq \sum_{p,q} \sum_{r<s} \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)} \geq \sum_{r<s} \mu_r \mu_s (\Gamma_r \cdot \Gamma_s)
\]
If the equality holds, we will necessarily have that \( \delta(\Upsilon_t^{(i,j)}) = 0 \), \( \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)} = 0 \) if \( r = s \) and 1 if \( r \neq s \). This implies that \( \Upsilon_t^{(r,p)} \) and \( \Upsilon_t^{(s,q)} \) meet transversely for any \( r \neq s \) and \( \Upsilon_t^{(r,p)} \cap \Upsilon_t^{(s,q)} \neq \Upsilon_t^{(r',p')} \cap \Upsilon_t^{(s',q')} \) for any \( r < s, r' < s' \) and \( (p, q, r, s) \neq (p', q', r', s') \). Therefore, \( \Upsilon_t \) has exactly \( \sum_{r<s} \mu_r \mu_s \) nodes as singularities, which are the intersections \( \Upsilon_t^{(r,p)} \cap \Upsilon_t^{(s,q)} \) for \( r < s \).
Proposition 4.4. Let $X \subset \Delta^3_{xyz} \times \Delta_t$ be defined by $xy = t^\alpha z$ for some $\alpha > 0$ and let $X_0 = R_1 \cup R_2$ and $E = R_1 \cap R_2$, where $R_1 = \{x = t = 0\}$ and $R_2 = \{y = t = 0\}$.

Let $\Upsilon \subset X$ be a reduced flat family of curves whose central fiber $\Upsilon_0$ consists of $m+n$ irreducible components $\Gamma_1^{(1)}, \Gamma_2^{(1)}, \ldots, \Gamma_m^{(1)}, \Gamma_1^{(2)}, \Gamma_2^{(2)}, \ldots, \Gamma_n^{(2)}$ where $\Gamma_j^{(i)} \subset R_i$ and $\Gamma_j^{(i)} \neq E$. Let $\mu_{ij}$ be the multiplicity of $\Gamma_j^{(i)}$ in $\Upsilon_0$.

Let $\pi : \tilde{\Upsilon} \to \Upsilon$ be the nodal reduction of $\Upsilon$ and let $\tilde{\Upsilon} = \cup \tilde{\Upsilon}^{(k)}$ where $\tilde{\Upsilon}^{(k)}$ are the connected components of $\tilde{\Upsilon}$. Suppose that each $\pi(\tilde{\Upsilon}_0^{(k)})$ is supported on $\Gamma_j^{(i)}$ for some $i$ and $j$. Then

$$\delta(\Upsilon_t) \geq \left( \sum_j \mu_{ij} \left( \Gamma_j^{(i)} \cdot E \right) \right) \left( \sum_j \mu_{3-i,j} \right), \tag{4.10}$$

for $i = 1, 2$.

Proof. We resolve the double point $p$ in the same way as in the proof of Corollary 2.1. Let $\tilde{X} \subset X \times \mathbb{P}^1$ be defined by

$$\frac{x}{t^\alpha} = \frac{z}{y} = \frac{W_1}{W_0},$$

where $(W_0, W_1)$ are the homogeneous coordinates of $\mathbb{P}^1$. Let $\tilde{X}_0 = \tilde{R}_1 \cup \tilde{R}_2$ and $\tilde{E} = \tilde{R}_1 \cap \tilde{R}_2$, where $\tilde{R}_1$ is the blowup of $R_1$ at the origin $p$ and $\tilde{E} = \{x = y = t = W_0/W_1 = 0\}$. Let $\tilde{p} = \{x = y = z = t = W_0/W_1 = 0\}$ and let $P = \{x = y = z = t = 0\}$ be the exceptional curve of $\tilde{X} \to X$. Let $\tilde{\Gamma}_j^{(1)}$ be the proper transform of $\Gamma_j^{(1)}$ and $p_j = \tilde{\Gamma}_j^{(1)} \cap P$ for $j = 1, 2, \ldots, m$. Our assumptions on $\tilde{\Upsilon}$ guarantee $p_j \neq \tilde{p}$; otherwise, $\tilde{\Gamma}_j^{(1)}$ will pass through $\tilde{p}$ and by Lemma 2.1, each component of $\tilde{\Upsilon}_0$ that dominates $\tilde{\Gamma}_j^{(1)}$ will be joined by a chain of curves to a component dominating $\Gamma_k^{(2)}$ for some $k$.

Let $\mathcal{Y} = \Upsilon \times_X \tilde{X}$, $\tilde{\mathcal{Y}} = \tilde{\Upsilon} \times_X \tilde{X}$ and $\tilde{\pi} : \tilde{\mathcal{Y}} \to \mathcal{Y}$ be the map induced by $\pi$. By Lemma 2.1, $\mathcal{Y}_0$ contains $P$ with multiplicity

$$\sum_{j=1}^n \mu_{2j} \left( \Gamma_j^{(2)} \cdot E \right).$$

Let $\tilde{\mathcal{Y}} = \cup \tilde{\mathcal{Y}}^{(k)}$ where $\tilde{\mathcal{Y}}^{(k)} = \tilde{\Upsilon}^{(k)} \times_X \tilde{X}$.

It is not hard to see that $\tilde{\pi}(\tilde{\mathcal{Y}}^{(k)})_{\text{red}} \subset \tilde{\Gamma}_j^{(1)} \cup P$ if $\pi(\tilde{\Upsilon}_0^{(k)})_{\text{red}} \subset \Gamma_j^{(1)}$ for some $j$. By Lemma 2.1, $P \not\subset \tilde{\pi}(\tilde{\mathcal{Y}}_0^{(k)})$ if $\pi(\tilde{\Upsilon}_0^{(k)})_{\text{red}} \subset \Gamma_j^{(1)}$ for some $j$. Therefore, we may apply Corollary 4.2 to each points $p_j$ for $j =$.
\[\delta(\Upsilon_t) = \delta(\Upsilon_t) \geq \sum_{j=1}^{m} \delta(\Upsilon_t, p_j) \]
\[\geq \sum_{j=1}^{m} \left( \mu_{1j} \sum_{l=1}^{n} \mu_{2l} \left( \Gamma^{(2)}_l \cdot E \right) \right) \]
\[= \left( \sum_{l=1}^{n} \mu_{2l} \left( \Gamma^{(2)}_l \cdot E \right) \right) \left( \sum_{j=1}^{m} \mu_{1j} \right). \]

Our argument needs some trivial change if \( p_j \)'s fail to be distinct, which we will leave it to the readers.

**Proposition 4.5.** Let \( X, \Upsilon, \Gamma^{(i)}_j, \mu_{ij}, \pi, \tilde{\Upsilon} \) and \( \tilde{\Upsilon}^{(k)} \) be defined as in Proposition 4.4 except that we further assume each \( \pi(\tilde{\Upsilon}^{(k)}_0) \) to be reduced, i.e., \( \pi(\tilde{\Upsilon}^{(k)}_0) = \Gamma^{(i)}_j \) for some \( i \) and \( j \). If

\[\delta(\Upsilon_t) = \left( \sum_{j=1}^{m} \mu_{1j} \right) \left( \sum_{j=1}^{m} \mu_{1j} \right), \] (4.11)

then \( \Upsilon_t \) only has nodes as singularities.

**Proof.** The proof is the same as the proof of Proposition 4.4 except that we need to apply Proposition 4.3 to each point \( p_j \) at the last step. \( \square \)

### 4.2. Proof of Theorem 4.1

Without the loss of generality, let us assume that \( \overline{q_0q_1} \subset R_1 \). Let \( \alpha_1 = m_1(q_0), \alpha_2 = m_2(q_0) \) and \( \mu_i = \mu(q_iq_{i+1}) \) for each \( i \).

Let \( w_i \) be the intersection between \( \overline{q_iq_{i+1}} \) and \( \Gamma_1 \cup \Gamma_2 \) other than the points \( q_i \) and \( q_{i+1} \), if such intersection exists.

Basically, \( \Upsilon_t \) has singularities in the neighborhoods of \( q_0 \) and \( w_i \). Proposition 2.4 tells us the configurations of \( \tilde{\Upsilon} \) over these neighborhoods, while the series of results we have obtained in 4.4 can be used to estimate the \( \delta \)-invariants of \( \Upsilon_t \) in these neighborhoods.

By Proposition 2.4 and 4.3,

\[\delta(\Upsilon_t, q_0) \geq (\alpha_1 + \mu_0)^2 = (\alpha_2 + \mu_{-1})^2. \]

(4.12)

If \( \alpha_1 = 0 \), the point \( w_0 \) exists. By Proposition 2.4 and Corollary 4.2,

\[\delta(\Upsilon_t, w_0) \geq \mu_0 \]

(4.13)

if \( \alpha_1 = 0 \). Since \( \alpha_1 = 0 \) or \( 1 \), we may write (4.13) in the form

\[\delta(\Upsilon_t, w_0) \geq (1 - \alpha_1)\mu_0. \]

(4.14)
Similarly, we have
\[ \delta(t, w_{-1}) \geq (1 - \alpha_2) \mu_{-1}. \]

Obviously, for \( i > 0 \), the point \( w_i \) exists if \( \mu_{i-1} = \mu_i \). By Proposition 2.4 and Corollary 4.2, for \( i > 0 \),
\[ \delta(t, w_i) \geq \mu_i \]
if \( \mu_{i-1} = \mu_i \). Similarly, for \( i > 1 \),
\[ \delta(t, w_{-1}) \geq \mu_{-1} \]
if \( \mu_{i-1} = \mu_{i+1} \).

Let \( 0 \leq a_0 < a_1 < a_2 < \ldots < a_n < \ldots \) be the sequence of integers such that
\[ \mu_0 = \ldots = \mu_{a_0} > \mu_{a_0+1} = \mu_{a_0+2} = \ldots = \mu_{a_1} > \mu_{a_1+1} = \mu_{a_1+2} = \ldots = \mu_{a_2} > \ldots > \mu_{a_{n-1}+1} = \mu_{a_{n-1}+2} = \ldots = \mu_n > \ldots \]

Then by (4.16),
\[ \sum_{i>0} \delta(t, w_i) \geq a_0 \mu_0 + \sum_{i>0} (a_i - a_{i-1} - 1) \mu_{a_i}. \]

And since
\[ \sum_{i>0} \text{Intsc}(\Sigma_1, E; q_i) = (a_0 + 1) \mu_0 + \sum_{i>0} (a_i - a_{i-1} - 1) \mu_{a_i}, \]

\[ \sum_{i>0} \delta(t, w_i) - \sum_{i>0} \text{Intsc}(\Sigma_1, E; q_i) \geq - \left( \mu_0 + \sum_{i>0} \mu_{a_i} \right) \]
\[ \geq - (\mu_0 + (\mu_0 - 1) + (\mu_0 - 2) + \ldots + 1) = - \frac{\mu_0(\mu_0 + 1)}{2}. \]

Similarly,
\[ \sum_{i>1} \delta(t, w_{-i}) - \sum_{i \geq 1} \text{Intsc}(\Sigma_1, E; q_{-i}) \geq - \frac{\mu_{-1}(\mu_{-1} + 1)}{2}. \]

Combining (4.12), (4.14), \((4.15)\), \((4.13)\) and (4.20), we get
\[ \delta(t, \cup_{i=-l}^{m-1} q_i q_{i+1}) - \sum_{i=-l}^{m} \text{Intsc}(\Sigma_1, E; q_i) \]
\[ \geq \delta(t, q_0) + \left( \sum_{i=0, -1} \delta(t, w_i) - \text{Intsc}(\Sigma_1, E; q_0) \right) \]
\[
\begin{align*}
&+ \left( \sum_{i>0} \delta(\Upsilon_t, w_i) - \sum_{i>0} \Intsc(\Sigma_1, E; q_i) \right) \\
&+ \left( \sum_{i>1} \delta(\Upsilon_t, w_{-i}) - \sum_{i\geq 1} \Intsc(\Sigma_1, E; q_{-i}) \right) \\
&\geq \left( \frac{1}{2}(\alpha_1 + \mu_0)^2 + \frac{1}{2}(\alpha_2 + \mu_{-1})^2 \right) \\
&+ \left( (1 - \alpha_1)\mu_0 + (1 - \alpha_2)\mu_{-1} - \frac{1}{2}(\alpha_1 + \mu_0) - \frac{1}{2}(\alpha_2 + \mu_{-1}) \right) \\
&- \frac{\mu_0(\mu_0 + 1)}{2} - \frac{\mu_{-1}(\mu_{-1} + 1)}{2} \\
&= \frac{1}{2}(\alpha_1^2 - \alpha_1) + \frac{1}{2}(\alpha_2^2 - \alpha_2) = 0.
\end{align*}
\]

Suppose that the equality in (4.21) holds. Then the equalities in (4.19) and (4.20) have to hold, which implies \(|\mu_{i-1} - \mu_i| = 0\) or 1 for any \(i\). Under this condition, it is not hard to see by Proposition 2.4 that each component of \(\tilde{\Upsilon}_0\) dominating a component \(q_iq_{i+1} \subset \cup_{i=-l}^{m-1} q_iq_{i+1}\) must dominate \(q_iq_{i+1}\) with a degree one map. So we may apply Proposition 4.3 to \(\Upsilon\) at \(w_i\) and apply Proposition 4.5 to \(\Upsilon\subset X\) at \(q_0\) to conclude that \(\Upsilon_t\) only has nodes as singularities in the neighborhood of \(w_i\) and \(q_0\). On the other hand, since the equality in (4.21) holds, \(\Upsilon_t\) does not have singularities anywhere else in the neighborhood of \(\cup_{i=-l}^{m-1} q_iq_{i+1}\). Therefore, \(\Upsilon_t\) only has nodes as singularities in the neighborhood of \(\cup_{i=-l}^{m-1} q_iq_{i+1}\) if the equality in (4.21) holds.

5. Deformation Around a Type III Chain

The deformation of \(\Upsilon\) around a Type III chain is much more complicated than the other two cases. Instead of attempting a direct analysis as we did in the previous two sections, we will employ a completely different approach. Our strategy can be briefly described as follows.

We are trying to move \([R^s]\) around in \(\hat{\mathcal{H}}_g\), or equivalently, apply a monodromy action to \([R^s]\) such that a limiting rational curve \(\Upsilon_0\) on \(R^s\) will become another limiting rational curve \(\Upsilon'_0\) and meanwhile a Type III chain in \(\Upsilon_0\) will be “transformed” to a Type II chain in \(\Upsilon'_0\).

Our main theorem of this section is the following.
Theorem 5.1. Let $r_{ij} \cup \cdots \cup r_{ij} q_1 \cup \cdots \cup q_{m-1} q_m$ be a Type III chain in $\Upsilon_0$. Then
\[
\delta(\Upsilon_t, \overline{r_{ij} q_1} \cup \cdots \cup \overline{q_{m-1} q_m}) \geq \sum_{l=0}^{m} \text{Intsc}(\Sigma_1, E; q_l)
\]
(let $q_0 = r_{ij}$) and if the equality holds, all singularities of $\Upsilon_t$ are nodes in the neighborhood of $r_{ij} \cup \cdots \cup q_{m-1} q_m$.

5.1. Notations and Definitions. In order to make our ideas precise, we need to introduce a few new objects.

Since we assume that $\tilde{W}_g$ is irreducible, $C$ has the same number of singularities for all general $([S], [C]) \in \tilde{W}_g$. Let $C$ have exactly $\beta$ singular points for a general $([S], [C]) \in \tilde{W}_g$.

Let $Z_g$ be the incidence correspondence
\[
Z_g = \{(([S], [C]), w_1, w_2, \ldots, w_\beta, \delta_1, \delta_2, \ldots, \delta_\beta, \nu_1, \nu_2, \ldots, \nu_\beta) : ([S], [C]) \in W_g \text{ general,} \\
C_{\text{sing}} = \{w_1, w_2, \ldots, w_\beta\}, \\
\delta_i \text{ is } \delta\text{-invariant of } C \text{ at } w_i, \\
\nu_i = 1 \text{ if } w_i \text{ is a node of } C; \nu_i = 0 \text{ if it is not} \}
\subset \tilde{W}_g \times (\mathbb{P}^{4g-3})^\beta \times \mathbb{Z}^{2\beta}.
\]
Again, we first define $Z_g$ for general K3 surfaces $S$ and then take its closure $\tilde{Z}_g$ in $\tilde{W}_g \times (\mathbb{P}^{4g-3})^\beta \times \mathbb{Z}^{2\beta}$.

Notice that the last $2\beta$ coordinates $(\delta_1, \delta_2, \ldots, \delta_\beta, \nu_1, \nu_2, \ldots, \nu_\beta)$ are the same throughout an irreducible component of $\tilde{Z}_g$. Theorem 1.1 is equivalent to the statement that $\nu_1 = \nu_2 = \cdots = \nu_\beta = 1$ on each irreducible component of $\tilde{Z}_g$ which dominates $\tilde{H}_g$. Without the loss of generality, let us assume that $\tilde{Z}_g$ is irreducible and dominates $\tilde{H}_g$.

Definition 5.1. Let $([S], [C], w_1, w_2, \ldots, w_\beta) \in \tilde{Z}_g$. We will call $w_i$’s "limiting singularities" of $C$.

We observe that

Proposition 5.1. Let $[R^s] \in \tilde{R}_g$ general and let $\Upsilon_0$ be a limiting rational curve on $R^s$. Then each limiting singularity of $\Upsilon_0$ lies on an $F$-chain in $\Upsilon_0$.

Proof. Notice that $\Upsilon_0$ is smooth outside of its $F$-chains. \hfill \Box

The same holds for a general $[R^s] \in \tilde{R}_g^0$.

Proposition 5.2. Let $[R^s] \in \tilde{R}_g^0$ general and let $\Upsilon_0$ be a limiting rational curve on $R^s$. Then each limiting singularity of $\Upsilon_0$ is either
a point on a component $pq \subset \Upsilon_0, pq \in |O_R(F)|$ or an point among $\Gamma_1 \cap E = \Gamma_2 \cap E$.

Proof. Again, the reason for this fact is trivially that $\Upsilon_0$ is smooth outside of the locus described.

Let us consider the K3 surfaces $S$ with Picard lattice

$$(5.1) \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

or

$$(5.2) \begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix}$$

or equivalent, K3 surfaces whose Picard groups are generated by two divisors $C$ and $F$ with $C \cdot F = 2, F^2 = 0$ and $C^2 = 0$ or $C^2 = -2$. For lacking a good name for such surfaces, we will simply call them elliptic K3’s. Although elliptic K3 surfaces usually refer to all K3 surfaces that admit an elliptic fibration, we will use the term in our context to refer to K3 surfaces with Picard lattice $(5.1)$ or $(5.2)$. It is not hard to see that such K3 surfaces $S$ can be realized as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ ramified along a curve in $|4C + 4F|$ or $|4C + 6F|$.

Notice that the divisor $C + kF$ is ample but not very ample on an elliptic K3 surface $S$. To embed $S$ into a projective space, we need to use the divisor $2C + 2kF$. Embed $S$ into $\mathbb{P}^{4g-3}$ ($g \geq 3$) by $|2C + 2kF|$ and we observe that $[S] \in \mathcal{H}_g$, where $g = 2k + 1$ if $S$ has Picard lattice $(5.1)$ and $g = 2k$ if $S$ has Picard lattice $(5.2)$.

Let $\mathcal{Y}_g \subset \mathcal{H}_g$ be the locus in $\mathcal{H}_g$ consisting of elliptic K3’s with Picard lattice $(5.1)$ or $(5.2)$ embedded into $\mathbb{P}^{4g-3}$ by $|2C + 2kF|$, where $k = \lfloor g/2 \rfloor$. And let $\mathcal{Y}_g$ be the closure of $\mathcal{Y}_g$ in $\mathcal{H}_g$, where we may regard $\mathcal{Y}_g$ as a subscheme of $\mathcal{H}_g$ since $\mathcal{Y}_g$ is disjoint from $\mathcal{R}_g$. Obviously, $\mathcal{Y}_g$ and $\mathcal{Y}_g$ are irreducible and have codimension 1 in $\mathcal{H}_g$ and $\mathcal{H}_g$, respectively. And

**Proposition 5.3.** We have $\mathcal{Y}_g \cap \mathcal{R}_g = \mathcal{R}^0_g$.

Proof. We realize $S \in \mathcal{Y}_g$ as double covers of $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ totally ramified along a curve $C \in |-2K_\mathbb{P}|$, where $K_\mathbb{P}$ is the canonical divisor of $\mathbb{P}$. We want to construct a double cover $Y$ of $X = |-2K_\mathbb{P}| \times \mathbb{P}$ whose restriction to a point $C \in |-2K_\mathbb{P}|$ is the double cover of $\mathbb{P}$ ramified along $C$.

Let $B \subset X = |-2K_\mathbb{P}| \times \mathbb{P}$ be the universal family of curves in $|-2K_\mathbb{P}|$ and let $\pi_1$ and $\pi_2$ be the projections of $X$ to $|-2K_\mathbb{P}|$ and $\mathbb{P}$, respectively. Since

$$O_X(B) = \pi_1^*O_{|-2K_\mathbb{P}|}(1) \times \pi_2^*O_{\mathbb{P}}(-2K_\mathbb{P}) \neq L^2$$
for any line bundle $L$, we cannot construct $Y$ directly as a double cover of $X$ along $B$. The remedy for this situation is trivial. We take a general $N$-dimensional linear subsystem of $|\mathcal{O}_{-2K_P}(2)|$, where $N = \dim | -2K_P |$, and use it to map $| -2K_P |$ to itself. For example, after fixing homogeneous coordinates $(Z_0, Z_1, ..., Z_N)$ of $| -2K_P |$, we may simply take the map to be sending $(Z_0, Z_1, ..., Z_N)$ to $(Z_0^2, Z_1^2, ..., Z_N^2)$. This will induce a map $f : X \to \tilde{X}$. Obviously,

$$f^*\mathcal{O}_X(B) = \pi_1^*\mathcal{O}_{-2K_P}(2) \times \pi_2^*\mathcal{O}_P(-2K_P)$$

$$= (\pi_1^*\mathcal{O}_{-2K_P}(1) \times \pi_2^*\mathcal{O}_P(-K_P))^2.$$ 

So there exists a double cover $Y$ of $X$ ramified along $f^{-1}(B)$. It is trivial to check that the fiber of $Y \to | -2K_P |$ over an irreducible $C \subseteq | -2K_P |$ is the double cover of $\mathbb{P}$ ramified along $C$, while the fiber of $Y \to | -2K_P |$ over a double curve $C = 2D$ with $D \subseteq | -K_P |$ is a surface $R \in \mathcal{R}_g$. Therefore, $\mathcal{R}_g$ lies on the closure of $\mathcal{Y}_g$ in $\mathcal{H}_g$.

Notice that both $\tilde{\mathcal{Y}}_g$ and $\tilde{\mathcal{R}}_g$ has codimension 1 in $\tilde{\mathcal{H}}_g$. Obviously, $\tilde{\mathcal{Y}}_g \cap \tilde{\mathcal{R}}_g \subset \tilde{\mathcal{R}}_g^0$. The previous argument shows that $\tilde{\mathcal{Y}}_g \cap \tilde{\mathcal{R}}_g \neq \emptyset$. And since $\tilde{\mathcal{R}}_g^0$ is irreducible with codimension 2 in $\tilde{\mathcal{H}}_g$, $\tilde{\mathcal{Y}}_g \cap \tilde{\mathcal{R}}_g = \tilde{\mathcal{R}}_g^0$. \hfill \Box

5.2. Rational Curves on Elliptic K3’s. As mentioned at the beginning of this section, our strategy is to move $[R^s]$ around in $\mathcal{H}_g$. The way we move $[R^s]$ can now be described as follows:

$$(5.3) \quad [R^s] \in \tilde{\mathcal{Y}}_g \Rightarrow [R^s] \in \tilde{\mathcal{R}}_g^0 \Rightarrow \{S\} \in \mathcal{Y}_g \Rightarrow [R^s] \in \tilde{\mathcal{R}}_g \Rightarrow [R^s] \in \tilde{\mathcal{Y}}_g.$$ 

Namely, we first degenerate a $R^s$ in $\tilde{\mathcal{R}}_g$ to a $R^s$ in $\tilde{\mathcal{R}}_g^0$, then move a $R^s$ in $\tilde{\mathcal{R}}_g^0$ to an $S$ in $\mathcal{Y}_g$ and so on.

It is clear what happens to the limiting rational curves on a $R^s$ in $\tilde{\mathcal{R}}_g$ when it degenerates to a $R^s$ in $\tilde{\mathcal{R}}_g^0$. However, we do not have much idea about what happens to the rational curves on an $S$ in $\mathcal{Y}_g$ when it degenerates to a $R^s$ in $\tilde{\mathcal{R}}_g^0$. So we will prove a collection of results regarding rational curves on an elliptic K3 and their degeneration as an elliptic K3 degenerates to a special union of scrolls.

On each $S$ in $\mathcal{Y}_g$, there is a pencil of elliptic curves in $|F|$. There are exactly 24 rational curves in $|F|$ for $S$ general. We are interested in their monodromy group as $S$ varies in $\mathcal{Y}_g$.

**Proposition 5.4.** The monodromy group of the 24 rational curves in $|F|$ as $S$ varies in $\mathcal{Y}_g$ is the full symmetric group.

**Proof.** We may realize $S$ as the double cover of $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ ramified along a curve $C \subseteq | -2K_P |$, where $K_P$ is the canonical divisor of $\mathbb{P}$. Let $\pi : S \to \mathbb{P}$ be the covering map. We use the same notation $F$
to denote the divisor on \( \mathbb{P} \) such that \( \pi^*(F) = F \) on \( S \). Obviously, the 24 rational curves in \( |\mathcal{O}_S(F)| \) correspond to the 24 curves in \( |\mathcal{O}_\mathbb{P}(F)| \) tangent to \( C \). Then the statement of the proposition is equivalent to saying that the monodromy group \( G \) of the 24 curves in \( |\mathcal{O}_\mathbb{P}(F)| \) tangent to \( C \) is the full symmetric group as \( C \) varies in \( |-2K_\mathbb{P}| \).

Following the same line of argument as in [H], we show that \( G \) is the symmetric group by arguing that

1. \( G \) is twice transitive;
2. \( G \) contains a simple transposition.

To see that \( G \) is twice transitive, let \( W \) be the incidence correspondence

\[
W = \{ (C, I_1, I_2) : C \in |-2K_\mathbb{P}|, \quad I_1 \text{ and } I_2 \text{ are two different curves in } |\mathcal{O}_\mathbb{P}(F)| \text{ tangent to } C \} \subset |-2K_\mathbb{P}| \times |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)|.
\]

Saying that \( G \) is twice transitive is equivalent to saying that \( W \) is irreducible. To see that \( W \) is irreducible, it suffices to project \( W \) to \( |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)| \). It is obvious that \( W \) dominates \( |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)| \) and \( p(W) = \{(I_1, I_2) : I_1 \neq I_2 \in |\mathcal{O}_\mathbb{P}(F)| \} \), where \( p : W \to |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)| \) is the projection. And the fibers of \( p : W \to |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)| \) are irreducible and have the same dimension everywhere. Therefore, \( W \) is irreducible and \( G \) is twice transitive.

To see that \( G \) contains a simple transposition, let \( C_0 \) be a curve in \( |-2K_\mathbb{P}| \) having exactly one node \( p \) and smooth everywhere else. If there is a family of curves \( C_t \in |-2K_\mathbb{P}| \) whose central fiber is \( C_0 \), two out of the 24 curves in \( |\mathcal{O}_\mathbb{P}(F)| \) tangent to \( C_t \) will degenerate to the curve in \( |\mathcal{O}_\mathbb{P}(F)| \) passing through the node \( p \) as \( t \to 0 \). It is easy to see that a loop around \( C_0 \) will be lifted to a simple transposition which transposes these two curves.

Let \( \pi : \Delta \to \widetilde{Y}_g \subset \widetilde{H}_g \) be a morphism from the disk \( \Delta \) to \( \widetilde{Y}_g \), where \( \pi(0) = [R^s] \in \widetilde{H}_0^g \) and \( \pi(t) \) is a general point in \( Y_g \) for \( t \in \Delta \) general. Let \( \{p_1, p_2, ..., p_{16}\} \) be the vanishing locus of \( s \).

Let \( X = \mathcal{S}_g \times \widetilde{H}_g \Delta \) be the family of K3 surfaces corresponding to \( \pi \). As before, \( X \) has sixteen rational double points \( p_1, p_2, ..., p_{16} \) lying on the double curve \( E = R_1 \cap R_2 \), where \( X_0 = R = R_1 \cup R_2 \).

Let \( r_{ij} \) be the points on \( E \) defined as before, i.e., \( \mathcal{O}_E(2r_{ij}) = \mathcal{O}_E(F_i) \).

Since \( R \in \mathcal{R}_g^0 \), \( r_{ij} = r_{2j} \). Let \( q_i \) be the point on \( E \) such that \( \mathcal{O}_E(p_i + q_i) = \mathcal{O}_E(F) \) for \( i = 1, 2, ..., 16 \).

**Proposition 5.5.** As \( X_t \) degenerates to \( X_0 = R \), the 24 rational curves in \( |\mathcal{O}_{X_t}(F)| \) on \( X_t \) behave in the following way

\[
\begin{align*}
&\text{Proposition 5.5. As } X_t \text{ degenerates to } X_0 = R, \\
&\text{the 24 rational curves in } |\mathcal{O}_{X_t}(F)| \text{ on } X_t \text{ behave in the following way.}
\end{align*}
\]
1. two of them degenerate to $\overline{r_{1j}} \cup \overline{r_{2j}}$ for each $1 \leq j \leq 4$;
2. one of them degenerates to $\overline{pq_i}^{(1)} \cup \overline{pq_i}^{(2)}$ for each $1 \leq i \leq 16$.

**Proof.** The first statement follows directly from the Caporaso-Harris theorem on tacnodes [CH, Lemma 4.1].

Let us resolve the sixteen double points $p_1, p_2, ..., p_{16}$ of $X$ by blowing up $R_1$ at these points. Let $\tilde{X}$ be the resulting family and $\tilde{X}_0 = \tilde{R}_1 \cup \tilde{R}_2$, where $\tilde{R}_1$ is the blowup of $R_1$ at the sixteen points. The total transform of $\overline{pq_i}^{(1)} \cup \overline{pq_i}^{(2)}$ consists of the proper transform $I_i$ of $\overline{pq_i}^{(1)}$, the exceptional divisor $P_i$ and $\overline{pq_i}^{(2)}$. Obviously, $I_i$ and $P_i$ meet transversely at a point $w_i$. By the same line of argument as in [C], we can deform $I_i \cup P_i \cup \overline{pq_i}^{(2)}$ to a curve on the general fiber while preserving a node in the neighborhood of $w_i$. This concludes the second statement. \[ \square \]

Let $\mathcal{Y} \subset X$ be a one-parameter family of curves whose general fiber $\mathcal{Y}_i \in |O_X(C + kF)|$ is irreducible and rational. Let $\Sigma_1, \Sigma_2, \Gamma_1, \Gamma_2$ be the components of $\mathcal{Y}_0$ defined as before.

Let $\overline{pq} = \overline{pq_i}^{(1)} \cup \overline{pq_i}^{(2)} \in |O_R(F)|$ for $p, q \in E$ and $O_E(p + q) = O_E(F)$. If $\overline{pq} \subset \mathcal{Y}_0$, from the proof of Proposition 2.2, we see that $\overline{pq}$ can only be one of the following

1. $p = p_i$ and $q = q_i$ for some $i$;
2. $p = q = r_{1j} = r_{2j}$ for some $j$; and if this is the case, $r_{1j}$ does not lie on $\Gamma_1 \cup \Gamma_2$;
3. $p, q \notin \{r_{ij}, p_i, q_i\}$ and either $p$ or $q$ lies on $\Gamma_1 \cup \Gamma_2$.

The following two propositions say that we can be more specific in this case because $\mathcal{Y}_0$ is a limit of rational curves on elliptic K3’s, while all the analysis we did in the proof of Proposition 2.2 is solely based on the fact that $\mathcal{Y}_0$ is a limit of rational curves on smooth K3 surfaces.

**Proposition 5.6.** Let $\overline{pq} = \overline{pq_i}^{(1)} \cup \overline{pq_i}^{(2)} \in |O_R(F)|$ and $\overline{pq} \subset \mathcal{Y}_0$. If $\overline{pq} = \overline{pq_i}$ for some $i$, then

1. either $p = p_i \in \Gamma_1 \cup \Gamma_2$ or $q = q_i \in \Gamma_1 \cup \Gamma_2$;
2. $\delta(\mathcal{Y}_0, \overline{pq}) = \text{Intsc}(\Sigma_1, E; p) + \text{Intsc}(\Sigma_1, E; q)$.

**Proposition 5.7.** We have $\overline{r_{1j}} \cup \overline{r_{2j}} \not\subset \mathcal{Y}_0$ for any $1 \leq j \leq 4$.

We need a few algebraic lemmas.

Let $A$ be an integral domain. We use the notation $\overline{A}$ to denote the integral closure of $A$ in the algebraic closure of its quotient field. It is well known that $\overline{\mathbb{C}[[t]]} = \mathbb{C}[t, \sqrt{t}, \sqrt{t}, ..., \sqrt{t}, ...]$.

**Lemma 5.1.** Let $A = \mathbb{C}[x, y] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$ and $W \subset A$ be

$$ W = \left\{ \sum_{i=0}^{\infty} \left( a_i(t)y^i + b_i(t)xy^i \right) : a_i(t), b_i(t) \in \overline{\mathbb{C}[[t]]} \right\}. $$
Let \( f(x, y, t) \in W \) satisfying \( f(x, y, 0) \neq 0 \). If \( f(x, y, t) \) is reducible in \( A \), then there exists \( \alpha(t) \in \mathbb{C}[t] \) such that \( \alpha(0) = 0 \) and \( f(x, \alpha(t), t) = 0 \).

**Proof.** Let \( f(x, y, t) = g(x, y, t)h(x, y, t) \) where neither of \( g(x, y, t) \) and \( h(x, y, t) \) is a unit in \( A \). Let \( \deg_x g(x, y, t) \) and \( \deg_x h(x, y, t) \) be the degrees of \( g(x, y, t) \) and \( h(x, y, t) \) as polynomials in \( x \). Obviously, since \( f(x, y, t) \in W \), at least one of \( \deg_x g(x, y, t) \) and \( \deg_x h(x, y, t) \) is zero. Assume that \( \deg_x g(x, y, t) = 0 \), i.e., \( g(x, y, t) = r(y, t) \) for some \( r(y, t) \in \mathbb{C}[y] \otimes \mathbb{C}[t] \). Since \( g(x, y, t) \) is not a unit in \( A \), \( r(0, 0) = 0 \). And \( f(x, y, 0) \neq 0 \Rightarrow r(y, 0) \neq 0 \). So using Weierstrass Preparation Theorem, we can show that there exists \( \alpha(t) \in \mathbb{C}[t] \) such that \( \alpha(0) = 0 \) and \( r(\alpha(t), t) = 0 \), which is what we want. \( \square \)

A corollary of Lemma 5.1 is

**Corollary 5.1.** Let \( A = \mathbb{C}[x, y] \otimes \mathbb{C}[t] \) and \( W \subset A \) be

\[
W = \left\{ \sum (a_i(t)y^i + b_i(t)xy^i) : a_i(t), b_i(t) \in \mathbb{C}[t] \right\}.
\]

Let \( f(x, y, t) \in W \) satisfying \( f(x, y, 0) \neq 0 \). If \( f(x, y, t) \) is irreducible in \( A \), then \( f(x, y, t) \) is also irreducible in \( \mathbb{C}[x, y] \otimes \mathbb{C}[t] \).

**Proof.** By Lemma 5.1, if \( f(x, y, t) \) is reducible in \( \mathbb{C}[x, y] \otimes \mathbb{C}[t] \), \( f(x, \alpha(t), t) = 0 \) for some \( \alpha(t) \in \mathbb{C}[t] \) and \( \alpha(0) = 0 \). On the other hand, \( f(x, y, t) = g(x, y, t)(y-\alpha(t)) + a(t)x + b(t) \) for some \( g(x, y, t) \in A \) and \( a(t), b(t) \in \mathbb{C}[t] \). So we necessarily have \( a(t) = b(t) = 0 \) and \( f(x, y, t) \) must be reducible in \( A \). Contradiction. \( \square \)

**Lemma 5.2.** Let \( C \subset \Delta^3_{xyz}/(xy = 0) = R_1 \cup R_2 \) be a reduced curve cut out by \( f(x, y, z) = 0 \) for some \( f(x, y, z) \in \mathbb{C}[[x, y, z]] \), where \( R_1 = \Delta^3_{xyz}/(x = 0) \) and \( R_2 = \Delta^3_{xyz}/(y = 0) \). Suppose that \( C \) does not contain the double curve \( E = R_1 \cap R_2 \). Then

\[
\delta(C) = \delta(C_1) + \delta(C_2) + C_1 \cdot E = \delta(C_1) + \delta(C_2) + C_2 \cdot E
\]

where \( C = C_1 \cup C_2 \), \( C_i \subset R_i \) and the intersection \( C_i \cdot E \) is taken on \( R_i \) for \( i = 1, 2 \).

**Proof.** Let \( \widetilde{\mathcal{O}}_C \) and \( \widetilde{\mathcal{O}}_{C_i} \) be the normalizations of the coordinate rings of \( C \) and \( C_i \), respectively. By the definition of \( \delta \)-invariants, \( \delta(C) = l(\widetilde{\mathcal{O}}_C/\mathcal{O}_C) \), where \( l(M) \) is the length of an \( \mathcal{O}_C \) module \( M \).

Obviously, \( \widetilde{\mathcal{O}}_C = \widetilde{\mathcal{O}}_{C_1} \times \widetilde{\mathcal{O}}_{C_2} \). Therefore,

\[
\delta(C) = l(\widetilde{\mathcal{O}}_C/\mathcal{O}_C) = l\left( (\widetilde{\mathcal{O}}_{C_1} \times \widetilde{\mathcal{O}}_{C_2})/\mathcal{O}_C \right)
\]
\[ \begin{align*}
= l \left( (\mathcal{O}_{C_1} \times \mathcal{O}_{C_2})/(\mathcal{O}_{C_1} \times \mathcal{O}_{C_2}) \right) + l((\mathcal{O}_{C_1} \times \mathcal{O}_{C_2})/\mathcal{O}) \\
= l(\mathcal{O}_{C_1}/\mathcal{O}_{C_1}) + l(\mathcal{O}_{C_2}/\mathcal{O}_{C_2}) + C_i \cdot E,
\end{align*} \]

for \( i = 1, 2 \). \hfill \Box

**Lemma 5.3.** Let \( p, q, f \in \mathbb{C}[z] \) be the polynomials in \( z \) such that \( \deg f \leq 1 \), \( \gcd(p, q) = 1 \) and \((p^2 - f q^2)^2 - q^4 \) is a perfect square in \( \mathbb{C}[z] \), where \( \gcd(p, q) \) is the greatest common divisor of \( p \) and \( q \). Then

1. both \( p \) and \( q \) are constants;
2. \( f \) is a constant if \( q \neq 0 \).

**Proof.** Let \( f_1 = f - 1 \) and \( f_2 = f + 1 \). Obviously, \( \gcd(p^2 - f_1 q^2, p^2 - f_2 q^2) = 1 \) since \( \gcd(p, q) = 1 \). And since \((p^2 - f_1 q^2)(p^2 - f_2 q^2)\) is a perfect square, both \( p^2 - f_1 q^2 \) and \( p^2 - f_2 q^2 \) are perfect squares.

It is easy to show that if one of \( p \) and \( q \) is a constant, the other is a constant too. On the other hand, if both are constants and \( q \neq 0 \), \( f \) has to be a constant too. Suppose that \( p, q \not\in \mathbb{C} \).

Solve \( p^2 - f_i q^2 = r^2 \) for \( i = 1, 2 \) and we obtain that there exist \( q_1, q_2, q_3, q_4 \in \mathbb{C}[z] \) such that \( q = q_1 q_2 = q_3 q_4 \) and

\[
p = \frac{q_1^2 + f_1 q_2^2}{2} = \frac{q_3^2 + f_1 q_4^2}{2}.
\]

Since \( q_1 q_2 = q_3 q_4 \), there exist \( s, t, u, v \in \mathbb{C}[z] \) such that \( q_1 = su, q_2 = tv, q_3 = tu \) and \( q_4 = sv \). Therefore,

\[
p = \frac{s^2 u^2 + f_1 t^2 v^2}{2} = \frac{t^2 u^2 + f_2 s^2 v^2}{2}
\]

and hence

\[
\frac{u^2 - f_1 v^2}{u^2 - f_2 v^2} = \frac{s^2}{t^2}
\]

which implies that \( (u^2 - f_1 v^2)(u^2 - f_2 v^2) \) is a perfect square.

Obviously, \( \gcd(u, v) = 1 \). Since \( q \not\in \mathbb{C} \), \( u \) and \( v \) cannot both be constants. Combining with our previous argument, neither of \( u \) and \( v \) is a constant. Therefore, \( \max(\deg u, \deg v) < \deg q \leq \max(\deg p, \deg q) \). So this procedure cannot go on forever. A contradiction. \hfill \Box

Now let us go back to the proof of Proposition 5.6 and 5.7.

Our proofs of both statements are based on the construction a blowup sequence over \( \overline{pq} \) and an induction on the multiplicity of \( \overline{pq} \) in \( \mathcal{Y}_0 \). It turns out that this process can be described more clearly if we study the behaviors of \( X \) and \( \mathcal{Y} \) in the analytic neighborhood of \( \overline{pq} \), or alternatively, study the formal completion of \( X \) and \( \mathcal{Y} \) along \( \overline{pq} \). But we will stick to the language of analytic geometry for it being more intuitive.
The following proposition is lengthy to state due to the fact that we need to give a precise description of $X$ and $\Upsilon$ in the neighborhood of $pq$. But such description is necessary for the purpose of induction.

**Proposition 5.8.** Let $X$ be a flat family of analytic surfaces over disk $\Delta_i$ whose central fiber $X_0 = R_1 \cup R_2$ where $R_i \cong \Delta \times \mathbb{P}^1$ for $i = 1, 2$. Suppose that $R_1 \cap R_2 = \Delta(p) \cup \Delta(q)$ where $\Delta(p)$ and $\Delta(q)$ are disks centering at points $p$ and $q$, respectively, and $\Delta(p)$ and $\Delta(q)$ are closed subschemes of $R_i$ $(i = 1, 2)$. Suppose that $X$ is locally given by $\text{Spec } \mathbb{C}[[x, y, z]]/(xy - t^a z^b)$ and $\text{Spec } \mathbb{C}[[x, y, z, t]]/(xy - t^a)$ at $p$ and $q$, respectively, where $a, b \in \mathbb{Z}$, $a > 0$ and $b = 0$ or 1.

Let $z \in \Gamma(\mathcal{O}_X)$ and let $z = 0$ cut out the “banana” curve $\overline{pq} = \overline{pq}^{(1)} \cup \overline{pq}^{(2)}$ on $X_0$, where $\overline{pq}^{(i)} \cong \mathbb{P}^1$, $\overline{pq}^{(i)} \subset R_i$ and each $\overline{pq}^{(i)}$ meets $\Delta(p)$ and $\Delta(q)$ transversely at $p$ and $q$, respectively, for $i = 1, 2$.

Let $\mathcal{O}_X(C)$ be a line bundle on $X$ such that the restrictions of $\mathcal{O}_X(C)$ to $\overline{pq}^{(i)}$ are $\mathcal{O}_{\mathbb{P}^1}(1)$ and let $s_1$ and $s_2$ be two global sections of $\mathcal{O}_X(C)$ which generate $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ when restricted to $\overline{pq}^{(i)}$ for $i = 1, 2$.

Let $f(s_1, s_2, z, t) \in \mathbb{C}[s_1, s_2, [z, t]]$ lie in

$$f(s_1, s_2, z, t) \in \left\{ \sum_{i=0}^{\infty} (a_i(t)s_1z^i + b_i(t)s_2z^i) : a_i(t), b_i(t) \in \mathbb{C}[[t]] \right\}$$

and $f(s_1, s_2, z, 0) \neq 0$. Suppose that $f(s_1, s_2, z, t)$ is irreducible in $\mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \mathbb{C}[[t]]$.

Let $\Upsilon \subset X$ be the subscheme of $X$ cut out by $f(s_1, s_2, z, t) = 0$. Obviously, $\Upsilon$ consists of a multiple of $\overline{pq}$ and two disks $\Gamma_1 \subset R_1$ and $\Gamma_2 \subset R_2$. Let $\mu$ be the multiplicity of $\overline{pq}$ in $\Upsilon_0$. Suppose that $\mu > 0$.

Let $\tilde{\Upsilon}$ be the nodal reduction of $\Upsilon$. Then

1. $\tilde{\Upsilon}_0$ has at most two connected components with $\Gamma_1$ and $\Gamma_2$ in each component, respectively;
2. $\tilde{\Upsilon}_0$ is connected if $b = 0$ or $p, q \not\in \Gamma_i$ for $i = 1, 2$;
3. if $\tilde{\Upsilon}_0$ has two connected components and $\Gamma_1 \neq \Gamma_2$, then

$$(5.5) \quad \delta(\Upsilon_t) = 2\mu + \Gamma_1 \cdot \Delta(p) + \Gamma_1 \cdot \Delta(q),$$

where the intersections are taken on $R_1$ (let $\Gamma_1 \cdot \Delta(p) = 0$ if $p \not\in \Gamma_1$ and $\Gamma_1 \cdot \Delta(q) = 0$ if $q \not\in \Gamma_1$).

**Proof.** Since $\overline{pq}^{(i)}$ $(i = 1, 2)$ meet $\Delta(p)$ transversely, we can choose the local coordinates $(x, y, z)$ of $X$ at $p$ such that

$$X \cong \Delta^4_{xyzt}/(xy = t^a(z^b + \alpha(t)))$$

at $p$ and the local function $z$ is exactly the restriction of the global $z$ as defined in the proposition, where $\alpha(t) \in \mathbb{C}[[t]]$ and $\alpha(0) = 0$. Of
course, if $b = 0$, we can make $\alpha(t)$ vanish. If $b = 1$, we can replace the global function $z$ by $z - \alpha(t)$, which we will call a translation on $z$. So eventually, we arrive at

\begin{equation}
X \cong \Delta_{\text{sing}}^{4}/(xy = t^{a}z^{b})
\end{equation}

at $p$ and the local coordinates $(x, y, z)$ are chosen such that the local function $z$ is the restriction of the global function $z$.

We may put the defining equation of $\Upsilon$ in the following form

\begin{equation}
f(w, z, t) = w \prod_{i=1}^{\mu} (z + a_{i}(t)) + t^{\beta}u(z, t) \prod_{j=1}^{m} (z + b_{j}(t)) = 0
\end{equation}

where $w = (c_{11}s_{1} + c_{12}s_{2})/(c_{21}s_{1} + c_{22}s_{2})$ for some

$$
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix} \in \text{SL}_2(\mathbb{C}[z, t]),
$$

$u(z, t) \in \mathbb{C}[z, t]$, $u(0, 0) \neq 0$, $a_{i}(t), b_{j}(t) \in \overline{\mathbb{C}[t]}$, $a_{i}(0) = b_{j}(0) = 0$, $\beta > 0$ and $\mu > m$. And we arrange $\{a_{i}(t)\}$ and $\{b_{j}(t)\}$ in the order that

\begin{equation}
\nu(a_{1}(t)) \leq \nu(a_{2}(t)) \leq \ldots \leq \nu(a_{\mu}(t))
\end{equation}

and

\begin{equation}
\nu(b_{1}(t)) \leq \nu(b_{2}(t)) \leq \ldots \leq \nu(b_{m}(t)),
\end{equation}

where $\nu(c(t))$ is the valuation of $c(t) \in \overline{\mathbb{C}[t]}$ and we let $\nu(0) = \infty$. If $b = 0$, we may further do a translation on $z$ and assume that

\begin{equation}
\sum_{i=1}^{\mu} a_{i}(t) = 0.
\end{equation}

It is obvious that $f(w, z, t)$ is irreducible in $\mathbb{C}[w, z] \times \mathbb{C}[z, t]$ since $f(s_{1}, s_{2}, z, t)$ is irreducible in $\mathbb{C}[s_{1}, s_{2}, [z]] \otimes \mathbb{C}[t]$.

We want to blow up $X$ along a subscheme cut out by $z = t^{\gamma} = 0$ for some $\gamma$. Let

\begin{equation}
\gamma = \min \left\{ \nu(a_{1}(t)), \frac{\beta}{\mu - m}, \gamma_{0} \right\}
\end{equation}

where

\begin{equation}
\gamma_{0} = \min_{1 \leq k \leq m} \frac{1}{\mu - m + k} \left( \beta + \sum_{j=1}^{k} \nu(b_{j}(t)) \right).
\end{equation}
Alternatively, we may define \( \gamma \) as the largest number such that
\[
t^\mu | f(w, t^\gamma z, t).
\]

After a base change, we may assume that \( \gamma \in \mathbb{Z} \).

Let \( \bar{X} \) be the blowup of \( X \) along the subscheme cut out by \( z = t^\gamma = 0 \) and let \( \bar{\mathcal{Y}} \subset \bar{X} \) be the proper transform of \( \mathcal{Y} \). The central fiber \( \bar{X}_0 \) consists of four surfaces \( R_1 \cup R_2 \cup (\bar{pq}(1) \times \mathbb{P}^1) \cup (\bar{pq}(2) \times \mathbb{P}^1) \), where \( R_i \cap (\bar{pq}(i) \times \mathbb{P}^1) = \bar{pq}(i) \) \( (i = 1, 2) \) and \( (\bar{pq}(1) \times \mathbb{P}^1) \cap (\bar{pq}(2) \times \mathbb{P}^1) = (\{p\} \times \mathbb{P}^1) \cup (\{q\} \times \mathbb{P}^1) \).

By (5.7), \( \bar{\mathcal{Y}} \) is cut out on \( \bar{X} \) by
\[
(5.12) \quad f_1(w, z_1, t) = 0
\]
where \( z_1 = z/t^\gamma \) and
\[
(5.13) \quad f_1(w, z_1, t) = t^{-\mu} f(w, t^\gamma z_1, t).
\]
Let
\[
(5.14) \quad f_1(w, z, 0) = \left(wh_1(z_1) + h_2(z_1)\right)^\prod_{i=1}^l (z_1 + \alpha_i)^{\mu_i}
\]
with \( z_1 = z/t^\gamma \), where \( h_1(z_1), h_2(z_1) \in \mathbb{C}[z_1] \), deg \( h_1(z_1) \) > deg \( h_2(z_1) \), \( wh_1(z_1) + h_2(z_1) \) is irreducible in \( \mathbb{C}[w, z_1] \), \( \alpha_1, \alpha_2, ..., \alpha_l \in \mathbb{C} \) are \( l \) distinct numbers, \( \mu_i \in \mathbb{Z}, \mu_i > 0 \) and deg \( h_1(z_1) + \sum_{i=1}^l \mu_i = \mu \).

By (5.14), we see that the central fiber \( \mathcal{Y}_0^\prime \) of \( \mathcal{Y}^\prime \) consists of components \( \Gamma_1, \Gamma_2, I_0, I_1, ..., I_l \) where

1. each \( I_j \) has two components \( I_j^{(1)} \) and \( I_j^{(2)} \) with \( I_j^{(k)} \subset \bar{pq}(k) \times \mathbb{P}^1 \) for \( j = 0, 1, ..., l \) and \( k = 1, 2 \);
2. \( I_0^{(k)} \) is cut out by \( wh_1(z_1) + h_2(z_1) = 0 \) on \( \bar{pq}(k) \times \mathbb{P}^1 \); \( I_j^{(k)} \) is cut out by \( z_1 + \alpha_j = 0 \) on \( \bar{pq}(k) \times \mathbb{P}^1 \) for \( j = 1, 2, ..., l \) and \( k = 1, 2 \) (here we regard \( (w, z) \) as the affine coordinates of \( \bar{pq}(k) \times \mathbb{P}^1 \approx \mathbb{P}^1 \times \mathbb{P}^1 \));
3. \( I_0^{(k)} \) projects to \( \bar{pq}(k) \) with a degree \( \mu_0 = \text{deg} h_1(z_1) \) map for \( k = 1, 2 \) (if \( \mu_0 = 0 \), \( I_0^{(k)} \) contracts to the point \( \Gamma_k \cap \bar{pq}(k) \));
4. \( I_j^{(k)} \) has multiplicity \( \mu_j \) in \( \mathcal{Y}_0^\prime \); \( I_j^{(1)} \) and \( I_j^{(2)} \) meet at two points \( p_j \in \{p\} \times \mathbb{P}^1 \) and \( q_j \in \{q\} \times \mathbb{P}^1 \) for \( j = 1, 2, ..., l \);
5. \( I_1, I_2, ..., I_l \) are disjoint from each other; \( I_0^{(k)} \) meets each \( I_j^{(k)} \) at exactly one point for \( j = 1, 2, ..., l \) and \( k = 1, 2 \);
6. \( I_0^{(k)} \) meets \( \Gamma_k \) at a point \( r_k \in \bar{pq}(k) \times \mathbb{P}^1 \) with coordinates \( w = 1/z_1 = 0 \) and \( r_k \notin I_j \) for \( j = 1, 2, ..., l \) and \( k = 1, 2 \).

By Lemma 2.4, the way in which \( I_0^{(k)} \) are connected to \( \Gamma_k \) \( (k = 1, 2) \) on \( \mathcal{Y}_0 \) can be described as follows.
(*) Either \( I_0^{(k)} \) and \( \Gamma_k \) are joined by curves contracting to \( r_k \) on \( \tilde{Y}_0 \) or \( I_0^{(k)} \) and \( \Gamma_{3-k} \) are joined by curves contracting to \( r_k \) on \( \tilde{Y}_0 \) for \( k = 1, 2 \) (the latter could happen when \( r_1 = r_2 \in I_0^{(1)} \cap I_0^{(2)} \)).

We will argue by induction on the pair \((\mu, b)\). We define \((\mu, b) < (\mu', b')\) if \( \mu < \mu' \) or \( \mu = \mu' \) and \( b < b' \).

\( l > 0 \)

Take a component \( I_j \) and an analytic neighborhood \( U \) of \( \tilde{X} \) around \( I_j \). Let \( Y = U \cap \tilde{Y}' \). Then \( Y \) and \( U \) have all the properties described in the proposition. For example, \( Y \) is cut out on \( U \) by \( f_2(w, z_2, t) = 0 \) where

\[
(5.15) \quad f_2(w, z_2, t) = f_1(w, z_2 - \alpha_j, t) = t^{-\mu_j} f(w, t^\gamma(z_2 - \alpha_j), t)
\]

By Lemma 5.1, \( f_2(w, z_2, t) \) is irreducible in \( \mathbb{C}[w, [z]] \otimes_{\mathbb{C}} \mathbb{C}[[t]] \) since \( f(w, z, t) \) is irreducible in \( \mathbb{C}[w, [z]] \otimes_{\mathbb{C}} \mathbb{C}[[t]] \). And in the neighborhoods of \( p_j \) and \( q_j, U \) is given by \( \text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^{(a+\gamma)} z^c) \) and \( \text{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^{(a+\gamma)}) \), respectively, where \( c = 0 \) if either \( b = 0 \) or \( \alpha_j \neq 0 \) and \( c = 1 \) otherwise. The central fiber \( Y_0 \) of \( Y \) contains \( I_j \) with multiplicity \( \mu_j \) plus two disks \( \Gamma'_1 \) and \( \Gamma'_2 \) attached to \( I_j^{(1)} \) and \( I_j^{(2)} \), respectively. Obviously, \( \Gamma'_k \) is a piece of \( I_0^{(k)} \) for \( k = 1, 2 \).

To apply the induction hypothesis, we have to check that \((\mu_j, c) < (\mu, b)\), which is an easy consequence of \((5.10)\) and the way we choose the number \( \gamma \).

Let \( \tilde{Y} \) be the nodal reduction of \( Y \). By the induction hypothesis, \( \tilde{Y}_0 \) has at most two connected components with \( \Gamma'_1 \) and \( \Gamma'_2 \) in each component. So combining with \((*)\), we see that \( \tilde{Y}_0 \) has at most two connected components with \( \Gamma_1 \) and \( \Gamma_2 \) in each component. Furthermore, as long as there exists one \( I_j \) such that the corresponding \( \tilde{Y}_0 \) is connected, \( \tilde{Y}_0 \) is connected. By the induction hypothesis, \( \tilde{Y}_0 \) is connected if \( c = 0 \) or \( I_0 \) meets \( I_j \) at points other than \( q_j \) for \( 1 \leq j \leq l \). Therefore, \( \tilde{Y}_0 \) is connected if one of the following holds:

1. \( b = 0 \);
2. \( \alpha_j \neq 0 \) for some \( 1 \leq j \leq l \);
3. \( I_0 \) meets \( I_j \) at points other than \( p_j \) and \( q_j \) for some \( 1 \leq j \leq l \).

Let us deal with the remaining case that \( b = l = 1, \alpha_1 = 0 \) and \( I_0 \) meets \( I_1 \) at \( p_1 \) or \( q_1 \). We necessarily have \( \mu_0 > 0 \) due to the way we choose the number \( \gamma \). So \( I_0^{(1)} \) and \( I_0^{(2)} \) should meet at (at least) one point \( p_0 \) on \( \{p\} \times \mathbb{P}^1 \) and one point \( q_0 \) on \( \{q\} \times \mathbb{P}^1 \).
If \( \{p_0, q_0\} \not\subset \{r_1, r_2, p_1, q_1\} \) for some \( p_0 \in I_0^{(k)} \cap \{p\} \times \mathbb{P}^1 \) and \( q_0 \in I_0^{(k)} \cap \{q\} \times \mathbb{P}^1 \), it is not hard to see by Lemma 2.1 that \( I_0^{(1)} \) and \( I_0^{(2)} \) are joined by a chain of curves contracting to \( p_0 \) or \( q_0 \) (and hence to \( p \) and \( q \)) on \( \widetilde{\mathcal{Y}}_0 \) and \( \widetilde{\mathcal{Y}}_0 \) is hence connected.

If \( \{p_0, q_0\} \subset \{r_1, r_2, p_1, q_1\} \) for any \( p_0 \in I_0^{(k)} \cap \{p\} \times \mathbb{P}^1 \) and \( q_0 \in I_0^{(k)} \cap \{q\} \times \mathbb{P}^1 \), then \( I_0^{(1)} \) and \( I_0^{(2)} \) only meet at \( p_0 \) and \( q_0 \) and

A. \( p \in \Gamma_i \) (\( i = 1, 2 \)), \( r_1 = r_2 = p_0 \) and \( q_0 = q_1 \); OR

B. \( q \in \Gamma_i \) (\( i = 1, 2 \)), \( r_1 = r_2 = q_0 \) and \( p_0 = p_1 \).

We need to prove (5.3) in both cases if we further assume that \( \widetilde{\mathcal{Y}}_0 \) has two connected components and \( \Gamma_1 \neq \Gamma_2 \). Let us work on case A and case B will follow from the same argument.

By Lemma 5.2 and the induction hypothesis,

\[
\delta(\mathcal{Y}_t) = \delta(\mathcal{Y}^t) = \delta(\mathcal{Y}^t, p_0) + \delta(\mathcal{Y}^t, I_1) = (\mu_0 + \Gamma_1 \cdot \Delta(p)) + (\mu_0 + 2\mu_1) = 2(\mu_0 + \mu_1) + \Gamma_1 \cdot \Delta(p) = 2\mu + \Gamma_1 \cdot \Delta(p).
\]

\( l = 0 \)

Since \( \mu_0 = \mu > 0 \), \( I_0^{(1)} \) and \( I_0^{(2)} \) meet at (at least) one point \( p_0 \in \{p\} \times \mathbb{P}^1 \) and one point \( q_0 \in \{q\} \times \mathbb{P}^1 \).

If \( b = 0 \) or \( q_0 \not\in \{r_1, r_2\} \) for some \( q_0 \in I_0^{(k)} \cap \{q\} \times \mathbb{P}^1 \), then by Lemma 2.1, \( I_0^{(1)} \) and \( I_0^{(2)} \) are joined by a chain of curves contracting to either \( p_0 \) or \( q_0 \) on \( \widetilde{\mathcal{Y}}_0 \) and \( \widetilde{\mathcal{Y}}_0 \) is hence connected.

If \( b = 1 \), \( q_0 = r_1 = r_2 \) and \( I_0^{(k)} \) meets \( \{p\} \times \mathbb{P}^1 \) at (at least) two different points \( p_0 \) and \( p'_0 \), then by Lemma 2.1, \( I_0^{(1)} \) and \( I_0^{(2)} \) are joined by a chain of curves contracting to either \( p_0 \) or \( p'_0 \) on \( \widetilde{\mathcal{Y}}_0 \) and \( \widetilde{\mathcal{Y}}_0 \) is hence connected.

If \( b = 1 \), \( q_0 = r_1 = r_2 \), \( I_0^{(k)} \) meets \( \{p\} \times \mathbb{P}^1 \) only at one point \( p_0 \) and \( \mu \geq 2 \), then it follows from Corollary 2.1 that \( I_0^{(1)} \) and \( I_0^{(2)} \) are joined by a chain of curves contracting to \( p_0 \) on \( \widetilde{\mathcal{Y}}_0 \) and \( \widetilde{\mathcal{Y}}_0 \) is hence connected.

So the only case left is that \( b = 1 \), \( q_0 = r_1 = r_2 \) and \( \mu = 1 \). Obviously, it follows from (\*\*) that \( \widetilde{\mathcal{Y}}_0 \) has at most two connected components with \( \Gamma_1 \) and \( \Gamma_2 \) in each component, respectively. We need to verify (5.3) if we further assume that \( \widetilde{\mathcal{Y}}_0 \) has two connected components and \( \Gamma_1 \neq \Gamma_2 \).

By Lemma 5.2,

\[
\delta(\mathcal{Y}_t) = \delta(\mathcal{Y}_t, p) + \delta(\mathcal{Y}_t, q) = 1 + (1 + \Gamma_1 \cdot \Delta(q)) = 2\mu + \Gamma_1 \cdot \Delta(q).
\]

\[\square\]

Proposition 5.6 follows more or less directly from Proposition 5.8.
Proof of Proposition 5.6. Suppose that $H^0(\mathcal{O}_X(C + kF))$ is generated by $g + 1$ global sections $Y_0, Y_1, ..., Y_g$ as a free $\mathbb{C}[t]$ module. Let $H^0(\mathcal{O}_X(F))$ be generated by two global sections $Z_0$ and $Z_1$ where $Z_1 = 0$ cuts out $\mathbb{P}q$ on $X_0$. Let $W_1, W_2 \in H^0(\mathcal{O}_X(F))$ be two global sections of $\mathcal{O}_X(C + F)$ whose restrictions to $\mathbb{P}q$ generate $H^0(\mathcal{O}_{\mathbb{P}1}(1))$. All these $Y_0, Y_1, ..., Y_g, Z_0, Z_1, W_1, W_2$ exist after a base change.

Let $s_1 = W_1/Z_0$ and $s_2 = W_2/Z_0$. When restricted to an analytic neighborhood of $\mathbb{P}q$, $s_1$ and $s_2$ are holomorphic sections of $\mathcal{O}_X(C)$ and

$$
\left\{ \sum_{i=0}^{g} a_i(t)Y_i : a_i(t) \in \mathbb{C}[t] \right\} 
\subset \left\{ \sum_{i=0}^{k} (b_i(t)s_1 Z_0^i Z_1^{k-i} + c_i(t)s_2 Z_0^i Z_1^{k-i}) : b_i(t), c_i(t) \in \mathbb{C}[t] \right\}.
$$

Therefore, $\Upsilon$ is locally cut out by $f(s_1, s_2, z, t) = 0$ as described in Proposition 5.8, where $z = Z_1/Z_0$. Since $\Upsilon_i$ is (geometrically) irreducible, it follows from Lemma 5.1 that $f(s_1, s_2, z, t)$ is irreducible in $\mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \mathbb{C}[t]$. Then by Proposition 5.8, $\Gamma_1$ and $\Gamma_2$ must pass through either $p$ or $q$; otherwise, $\Gamma_1$ and $\Gamma_2$ will be joined by a chain of curves on $\tilde{\Upsilon}_0$ whose images lie in $\mathbb{P}q$ and we have shown in the proof of Proposition 2.2 that $\Gamma_1$ and $\Gamma_2$ are joined by a chain of curves somewhere else on $\tilde{\Upsilon}_0$, which leads to a contradiction.

If $\Gamma_1$ and $\Gamma_2$ pass through $p$ or $q$, then by Proposition 5.8,

$$
\delta(\Upsilon_t, \mathbb{P}q) = 2\mu + \text{Intsc}(\Gamma_1, E; p) + \text{Intsc}(\Gamma_1, E; q) 
= \text{Intsc}(\Sigma_1, E; p) + \text{Intsc}(\Sigma_1, E; q)
$$

where $\mu$ is the multiplicity of $\mathbb{P}q$ in $\Upsilon_0$. \hfill \Box

Proposition 5.9. Let $X \subset \Delta^4_{xyzt}$ be defined by

$$
y(y + x^2 + t^a z^b) = \lambda t^c
$$

where $a, b, c \in \mathbb{Z}$, $a + b > 0$, $b = 0$ or $1$, $c > 0$, $\lambda = \lambda(x, y, z, t) \in \mathbb{C}[x, y, z, t]$ and $\lambda(0, 0, 0, 0) \neq 0$.

Let $f(w, z, t) \in \mathbb{C}[w, [z, t]]$ lie in

$$
f(w, z, t) \in \left\{ \sum_{i=0}^{\infty} (a_i(t)z^i + b_i(t)wz^i) : a_i(t), b_i(t) \in \mathbb{C}[t] \right\}
$$

and $f(w, z, 0) = wz^\mu$ for some $\mu \in \mathbb{Z}$ and $\mu \geq 0$. Suppose that $f(w, z, t)$ is irreducible in $\mathbb{C}[w, [z]] \otimes_{\mathbb{C}} \mathbb{C}[t]$. 


Let \( \Upsilon \subset X \) be a flat family of curves cut out by \( f(w, z, t) = 0 \) on \( X \), where \( w = w(x, y, z, t) \in \mathbb{C}[x, y, z, t] \) satisfying
\[
w(0, 0, 0, 0) = 0 \quad \text{and} \quad \frac{\partial w}{\partial x}(0, 0, 0, 0) \neq 0.
\]

Let \( I_1, I_2, \Gamma_1, \Gamma_2 \) be the irreducible components of \( \Upsilon_0 \) where \( I = I_1 \cup I_2 \) is cut out by \( z = 0 \) and \( \Gamma_1 \cup \Gamma_2 \) is cut out by \( w = 0 \) on \( X_0 = R_1 \cup R_2 \). Let \( \Gamma_j \subset R_j \) and \( I_j \subset R_j \) for \( j = 1, 2 \). Suppose that \( \Gamma_j \) meets \( E \) properly for \( j = 1, 2 \). Let \( \widetilde{\Upsilon} \) be the nodal reduction of \( \Upsilon \). Then either \( \Gamma_1 \) and \( \Gamma_2 \) lie on the same connected component of \( \widetilde{\Upsilon}_0 \) or the dual graph of \( \widetilde{\Upsilon}_0 \) contains a circuit.

**Proof.** If \( a = 0 \), the conclusion is more or less obvious. Notice that \( b = 1 \) if \( a = 0 \). Hence \( \Gamma_j \) meets \( E \) transversely, while \( I_j \) is tangent to \( E \) with multiplicity 2. So it follows from Lemma 2.1 that \( \Gamma_1 \) and \( \Gamma_2 \) lie on the same connected component of \( \widetilde{\Upsilon}_0 \).

It is also trivial if \( \mu = 0 \).

Suppose that \( a > 0 \) and \( \mu > 0 \). Again, we will argue by induction on \((\mu, b)\). We need to blow up \( \Upsilon \) three times along subschemes on the central fiber in order to “bring down” the pair \((\mu, b)\).

We may put \( f(w, z, t) = 0 \) in the form (5.7) and also assume \( \{a_i(t)\} \) and \( \{b_j(t)\} \) to satisfy (5.8) and (5.9) and to satisfy (5.10) if \( b = 0 \). Since \( \Gamma_i \) meets \( E \) properly, \( w(0, 0, z, 0) \neq 0 \). Let
\[
w(0, 0, z, t) = \sum_{i=1}^{\eta} (z + c_i(t))
\]
where \( \eta > 0 \), \( c_i(t) \in \mathbb{C}[[t]] \) and \( c_i(0) = 0 \). We arrange \( \{c_i(t)\} \) in the order that
\[
\nu(c_1(t)) \leq \nu(c_2(t)) \leq ... \leq \nu(c_{\eta}(t))
\]
Let
\[
(5.17) \quad \gamma = \min \left\{ \frac{a}{2\eta - b}, \frac{c}{4\eta}, \nu(c_1(t)), \nu(a_1(t)), \frac{\beta}{\mu - m + \eta}, \gamma_0 \right\},
\]
where
\[
\gamma_0 = \min_{1 \leq k \leq m} \frac{1}{\mu - m + k + \eta} \left( \beta + \sum_{j=1}^{k} \nu(b_j(t)) \right).
\]
Alternatively, \( \gamma \) can be defined as the largest number such that
\[
t^{(\mu+\eta)\gamma} \left| f(t^{\nu\gamma} x, t^{2\nu\gamma} y, t^{\nu\gamma} z, t) \right| \text{ and } t^{b\nu\gamma} \left| g(t^{\nu\gamma} x, t^{2\nu\gamma} y, t^{\nu\gamma} z, t) \right|
\]
where we let \( g(x, y, z, t) = y(y + x^2 + t^a z^b) - \lambda t^c \).
Let $\gamma \in \mathbb{Z}$ after a base change. Let $\Upsilon'$ be the blowup of $\Upsilon$ at the 0-dimensional subscheme $x = y = z = t^\gamma = 0$. Then $\Upsilon'$ is given by

$$y_1 \left( y_1 + t^\gamma x_1^2 + t^{a-(1-b)\gamma} z_1^b \right) = \lambda e^{-2\gamma}$$

and

$$f_1(w_1, z_1, t) = 0$$

where $x_1 = x/t^\gamma$, $y_1 = y/t^\gamma$, $z_1 = z/t^\gamma$,

$$w_1 = w_1(x_1, y_1, z_1, t) = t^{-\gamma}w(t^\gamma x_1, t^\gamma y_1, t^\gamma z_1, t)$$

and

$$f_1(w_1, z_1, t) = t^{-(\mu+1)\gamma} f(t^\gamma w_1, t^\gamma z_1, t).$$

By (5.18) and (5.19), we can describe the central fiber $\Upsilon'_0$ of $\Upsilon$ as follows.

The exceptional locus of $\Upsilon' \to \Upsilon$ is a reducible and nonreduced curve $F$ cut out on $\mathbb{P}^3$ by

$$\begin{cases}
y_1^2 = 0 \\
f_1(w_1(x_1, y_1, z_1, 0), z_1, 0) = 0
\end{cases}$$

where $x_1 = X_1/T_1$, $y_1 = Y_1/T_1$ and $z_1 = Z_1/T_1$ are the affine coordinates of $\mathbb{P}^3$ with corresponding homogeneous coordinates $(X_1, Y_1, Z_1, T_1)$.

It is not hard to see that $\Gamma_1 \cup \Gamma_2$ meets $F$ at the point $p$ with coordinates $\left(-\frac{\partial w}{\partial z}(0, 0, 0, 0), 0, \frac{\partial w}{\partial x}(0, 0, 0, 0), 0\right)$ and $I_1 \cup I_2$ meets $F$ at the point $q$ with coordinates $(1, 0, 0, 0)$. Obviously, $p \neq q$, i.e., the blowup $\Upsilon' \to \Upsilon$ has separated $\Gamma_1 \cup \Gamma_2$ from $I_1 \cup I_2$.

The point $p = \Gamma_k \cap F$ lies on a unique irreducible component $\Sigma$ of $F$ and $\Sigma$ has multiplicity 2 in $F$.

We may continue to use $\widetilde{\Upsilon}$ to denote the nodal reduction of $\Upsilon'$. By Lemma 2.1, each $\Gamma_i$ is joined to a component dominating $\Sigma$ on $\widetilde{\Upsilon}_0$ by a chain of curves contracting to the point $p$ for $i = 1, 2$. Therefore, in order to show that $\Gamma_1$ and $\Gamma_2$ lie on the same connected component of $\widetilde{\Upsilon}_0$, it suffices to show that

(*) all the components dominating $\Sigma$ lie on the same connected component of $\widetilde{\Upsilon}_0$.

This line of argument naturally leads to the second and the third blowups. This time we need to blow up $\Upsilon'$ along some subscheme supported along $F_{\text{red}}$. 
Let $\mathcal{V}''$ be the blowup of $\mathcal{V}'$ along the subscheme cut out by $x_1 = t^{(\eta-1)\gamma} = 0$. And let $\mathcal{V}$ be the blowup of $\mathcal{V}''$ along the subscheme cut out by $y_1 = t^{(2\eta-1)\gamma} = 0$. Notice that if $\eta = 1$, we do not need the intermediate family $\mathcal{V}''$. Finally, we obtain $\mathcal{V}$, which is given by

$$y_2 \left( y_2 + x_2^2 + t^{a-(2\eta-b)\gamma} z_1^b \right) = \lambda t^{c-4}\gamma$$

(5.21)

and

$$f_2(w_2, z_1, t) = 0$$

(5.22)

where $x_2 = x_1/t^{(\eta-1)\gamma}$, $y_2 = y_1/t^{(2\eta-1)\gamma}$,

$$w_2 = w_2(x_2, y_2, z_1, t) = \frac{1}{t^{(\eta-1)\gamma}} w_1(t^{(\eta-1)\gamma} x_2, t^{(2\eta-1)\gamma} y_2, z_1, t)$$

and

$$f_2(w_2, z_1, t) = \frac{1}{t^{(\eta-1)\gamma}} f_1(t^{(\eta-1)\gamma} w_2, z_1, t).$$

Let

$$f_2(w_2, z_1, 0) = \left( w_2 h_1(z_1) + h_2(z_1) \right) \prod_{i=1}^l (z_1 + \alpha_i)^{\mu_i} = 0$$

(5.23)

where $h_1(z_1), h_2(z_1) \in \mathbb{C}[z_1]$, $\deg h_1(z_1) > \deg h_2(z_1)$, $w_2 h_1(z_1) + h_2(z_1)$ is irreducible in $\mathbb{C}[w_2, z_1]$, $\alpha_1, \alpha_2, ..., \alpha_l \in \mathbb{C}$ are $l$ distinct numbers, $\mu_i \in \mathbb{Z}$, $\mu_i > 0$ and $\deg h_1(z_1) + \sum_{i=1}^l \mu_i = \mu$.

By (5.23), there are curves $\tilde{\Sigma} \subset \mathcal{V}_0$ and $J_i \subset \mathcal{V}_0$ where $\tilde{\Sigma}$ is cut out by $w_2 h_1(z_1) + h_2(z_1) = 0$ and $J_i$ is cut out by $z_1 + \alpha_i = 0$ for $i = 1, 2, ..., l$.

Obviously, $\tilde{\Sigma}$ dominates $\Sigma$ with a degree two map.

To be precise, $\tilde{\Sigma}$ and $J_i$ are complete curves lying on the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. However, to argue $(\ast)$, we only need to study their affine parts. So we will treat them as affine curves in $\mathbb{A}^3 = \text{Spec} \mathbb{C}[x_2, y_2, z_1]$. For example, when we talk about the intersections between these curves, we are talking about their intersections in $\mathbb{A}^3$.

$$c = 4\eta\gamma$$

We claim that $\tilde{\Sigma}$ is irreducible in this case and then $(\ast)$ will follow immediately.

By (5.21) and (5.22), $\tilde{\Sigma}$ is given by

$$\begin{cases} 
  y_2(y_2 + x_2^2 + \lambda_1 z_1^b) = \lambda_0 \\
  w_2(x_2, 0, z_1, 0) h_1(z_1) + h_2(z_1) = 0
\end{cases}$$

(5.24)

where $\lambda_1 = 0$ or $1$ and $\lambda_0 = \lambda(0, 0, 0, 0) \neq 0$. 

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Notice that \( w_2(0,0,z_1,0) \) is a degree \( \eta \) polynomial in \( z_1 \) due to the way we choose \( \gamma \). So if we solve \( w_2(x_2,0,z_1,0)h_1(z_1) + h_2(z_1) = 0 \) for \( x_2 \), we obtain \( x_2 \in \mathbb{C}(z_1) \) and \( x_2 \not\in \mathbb{C} \).

It is not hard to see that \( \tilde{\Sigma} \) is irreducible if \( (x_2^2 + \lambda_1 z_1^4)^2 + \lambda_0 \) is a perfect square in \( \mathbb{C}(z_1) \) for some \( x_2 \in \mathbb{C}(z_1) \) and \( x_2 \) nonconstant. Such \( x_2 \) does not exist by Lemma 5.3.

If \( c > 4\eta\gamma \), then each of \( \tilde{\Sigma} \) and \( J_i \) have exactly two irreducible components. Let \( \tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \) and \( J_i = J_i^{(1)} \cup J_i^{(2)} \) for \( i = 1, 2, ..., l \).

\[ c > 4\eta\gamma \quad \text{and} \quad a = (2\eta - b)\gamma \]

If \( b = 0 \), \( J_i^{(1)} \) and \( J_i^{(2)} \) meet at two points for all \( i \). If \( b = 1 \), \( J_i^{(1)} \) and \( J_i^{(2)} \) meet at a single point if and only if the corresponding \( \alpha_i = 0 \) and this point must be \( r = (x_2 = y_2 = z_1 = t = 0) \). It is not hard to see that \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \) meet at (at least) one point other than \( r \).

Let \( G \) be the dual graph of the following components of \( \tilde{\mathcal{Y}}_0 \) (let \( \tilde{\mathcal{Y}} \) be the nodal reduction of \( \mathcal{Y} \))

1. \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \);
2. the components dominating \( J_i^{(1)} \) or \( J_i^{(2)} \) for \( J_i^{(1)} \) and \( J_i^{(2)} \) that meet at two points;
3. the contractible components which contract to a point in \( (\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2) \cup (J_i^{(1)} \cap J_i^{(2)}) \cup (J_i^{(2)} \cap J_i^{(2)}) \cup ... \cup (J_i^{(1)} \cap J_i^{(2)}) \{r\} \).

Then Lemma 2.1 tells us that \( \deg(\tilde{\Sigma}_k) \geq 1 \) for \( k = 1, 2 \) and all the other vertices of \( G \) has degree at least two. Therefore, either \([\tilde{\Sigma}_1] \) and \([\tilde{\Sigma}_2] \) lie on the same component of \( G \) or \( G \) contains a circuit.

Let \( \mu_0 = \deg h_1(z_1) \).

\[ c > 4\eta\gamma, \quad a > (2\eta - b)\gamma \quad \text{and} \quad \mu_0 > 0 \quad \text{or} \quad l > 1 \]

Let \( p_1 \in \tilde{\Sigma}_1 \cap \tilde{\Sigma}_2 \). If no \( J_i \) passes through \( p_1 \), then (*) follows directly from Lemma 2.1. Otherwise suppose that \( p_1 \in J_1 \). By (5.21) and (5.22), \( \mathcal{Y} \) is locally defined by

\[ y \left( y + x^2 + t^{a - (2\eta - b)\gamma} z^{b'} \right) = \lambda t^{c-4r\gamma} \]

and

\[ f_3(w_2, z, t) = 0 \]

at \( p_1 \), where \( f_3(w_2, z, t) = f_2(w_2, z - \alpha_1, t), b' = 0 \) if \( b = 0 \) or \( \alpha_1 \neq 0 \) and \( b' = 1 \) otherwise. Notice that \( f_3(w_2, z, t) \) is irreducible in \( \mathbb{C}[w_2, [z]] \otimes_{\mathbb{C}} \mathbb{C}[[t]] \) by Lemma 5.1 and \( (\mu_1, b') < (\mu, b) \). Apply the induction hypothesis and we are done.
Notice that this case happens only if \( b = 1 \) due to (5.10). Following the argument of the previous case, we observe that \( b' = 0 \) in (5.25). So \((\mu, b') < (\mu, b)\) and the induction hypothesis still applies.

We necessarily have \( \gamma = \nu(c_1(t)) \) in this case. So there exists \( p_1 \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 \) such that \( p_1 \not\in J_1 \). Then (*) follows immediately from Lemma 2.1.

**Proposition 5.10.** Let \( X \) be a flat family of analytic surfaces over disk \( \Delta_i \), whose central fiber \( X_0 = R_1 \cup R_2 \) where \( R_i \cong \Delta \times \mathbb{P}^1 \) for \( i = 1, 2 \). Suppose that \( R_1 \cap R_2 = \Delta(r) \) where \( \Delta(r) \) is a disk centering at point \( r \) and \( \Delta(r) \) is a closed subscheme of \( R_i \) (\( i = 1, 2 \)). Suppose that \( X \) is locally defined by (5.10) at \( r \) when embedded to \( \Delta_{\text{glob}} \).

Let \( z \in \mathcal{O}_X \) and let \( z = 0 \) cut out the curve \( \overline{\tau} = \overline{\tau}^{(1)} \cup \overline{\tau}^{(2)} \) on \( X_0 \), where \( \overline{\tau}^{(i)} \cong \mathbb{P}^1 \), \( \overline{\tau}^{(i)} \subset R_i \) and each \( \tau^{(i)} \) meets \( \Delta(r) \) at \( r \) with multiplicity 2, for \( i = 1, 2 \).

Let \( \mathcal{O}_X(C) \) be a line bundle on \( X \) such that the restrictions of \( \mathcal{O}_X(C) \) to \( \tau^{(i)} \) are \( \mathcal{O}_{\mathbb{P}^1(1)} \) and let \( s_1 \) and \( s_2 \) be two global sections of \( \mathcal{O}_X(C) \) which generate \( \mathcal{H}^0(\mathcal{O}_{\mathbb{P}^1(1)}) \) when restricted to \( \tau^{(i)} \) for \( i = 1, 2 \).

Let \( f(s_1, s_2, z, t) \in \mathbb{C}[s_1, s_2, [z, t]] \) lie in

\[
 f(s_1, s_2, z, t) \in \left\{ \sum_{i=0}^{\infty} (a_i(t)s_1z^i + b_i(t)s_2z^i) : a_i(t), b_i(t) \in \mathbb{C}[[t]] \right\}
\]

and \( f(s_1, s_2, z, 0) \neq 0 \). Suppose that \( f(s_1, s_2, z, t) \) is irreducible in \( \mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \mathbb{C}[[t]] \).

Let \( \Upsilon \subset X \) be the subscheme of \( X \) cut out by \( f(s_1, s_2, z, t) = 0 \). Obviously, \( \Upsilon_0 \) consists of a multiple of \( \tau \) and two disks \( \Gamma_1 \) and \( \Gamma_2 \). Let \( \mu \) be the multiplicity of \( \tau \) in \( \Upsilon_0 \). Suppose that \( \mu > 0 \) and \( \Gamma_i \) meets \( \Delta(r) \) properly if \( \Gamma_i \) passes through \( r \) for \( i = 1, 2 \).

Let \( \Upsilon \) be the nodal reduction of \( \Upsilon \). Then either \( \Gamma_1 \) and \( \Gamma_2 \) lie on the same connected component of \( \Upsilon_0 \) or the dual graph of \( \Upsilon_0 \) contains a circuit.

**Proof.** Our argument proceeds almost identically to that for Proposition 5.8.

Just as in the proof of Proposition 5.8, we can choose local coordinates \((x, y, z, t)\) of \( X \) at \( r \) such that \( X \) is defined by (5.10) at \( r \) and the local function \( z \) is the restriction of the global function \( z \).
We may put the defining equation in the form (5.7) and also assume \( \{a_i(t)\} \) and \( \{b_j(t)\} \) to satisfy (5.8) and (5.9) and to satisfy (5.10) if \( b = 0 \).

Let \( \gamma \) be the number defined by (5.11) and let \( \gamma \in \mathbb{Z} \) after a base change.

The case that \( \Gamma_i \) passes through \( r \) has been covered by Proposition 5.9. Suppose that \( \Gamma_i \) does not pass through \( r \) for \( i = 1, 2 \). Again, we will argue by induction on \( (\mu, b) \).

Let \( \tilde{X} \) be the blowup of \( X \) along the subscheme cut out by \( z = t^\gamma = 0 \) and \( \Upsilon' \subset \tilde{X} \) be the proper transform of \( \Upsilon \). The central fiber \( \tilde{X}_0' \) consists of four surfaces \( R_1 \cup R_2 \cup (\bar{\tau}^{(1)} \times \mathbb{P}^1) \cup (\bar{\tau}^{(2)} \times \mathbb{P}^1) \), where \( R_i \cap (\bar{\tau}^{(i)} \times \mathbb{P}^1) = \bar{\tau}^{(i)} (i = 1, 2) \) and \( (\bar{\tau}^{(1)} \times \mathbb{P}^1) \cap (\bar{\tau}^{(2)} \times \mathbb{P}^1) = \{r\} \times \mathbb{P}^1 \).

We have the same defining equations (5.12) for \( \Upsilon' \) and (5.14) for \( \Upsilon_0' \). Let \( I_0 = I_0^{(1)} \cup I_0^{(2)} \), \( I_1 = I_1^{(1)} \cup I_1^{(2)} \), \( I_2 = I_2^{(1)} \cup I_2^{(2)} \subset \Upsilon_0' \) be the components of \( \Upsilon_0' \) defined in the same way as in the proof of Proposition 5.8. Let \( p_j = I_j^{(1)} \cap I_j^{(2)} \).

Let \( r_k \) be the intersection between \( \Gamma_k \) and \( I_0^{(k)} \) with coordinates \( w = 1/z_1 = 0 \) on \( \bar{\tau}^{(k)} \times \mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \). By Lemma 2.1.

\[ (*) \hspace{1em} I_0^{(k)} \text{ and } \Gamma_k \text{ are joined by curves contracting to } r_k \text{ on } \tilde{\Upsilon}_0 \text{ for } k = 1, 2. \]

\[ l > 0 \]

Take a component \( I_j \) and an analytic neighborhood \( U \) of \( \tilde{X} \) around \( I_j \). Let \( Y = U \cap \Upsilon' \). Then \( Y \) and \( U \) have all the properties described in the proposition. And in the neighborhood of \( p_j \), \( U \) is given by \( y(y + x^2 + t^{a_j+\gamma}z b^\gamma) = t^{\epsilon} \), where \( b^\gamma = 0 \) if either \( b = 0 \) or \( \alpha_j \neq 0 \) and \( b^\gamma = 1 \) otherwise. The central fiber \( Y_0 \) of \( Y \) contains \( I_j \) with multiplicity \( \mu_j \) plus two disks \( \Gamma_1 \) and \( \Gamma_2 \) attached to \( I_j^{(1)} \) and \( I_j^{(2)} \), respectively.

Obviously, \( \Gamma_k \) is a piece of \( I_0^{(k)} \) for \( k = 1, 2 \).

To apply the induction hypothesis, we have to check that \( (\mu_j, b^\gamma) < (\mu, b) \), which is an easy consequence of (5.10) and the way we choose the number \( \gamma \). Also we need to check that \( \Gamma_k \) meets \( \{r\} \times \mathbb{P}^1 \) properly if \( \Gamma_k \) passes through \( p_j \). This is trivially true because \( \{r\} \times \mathbb{P}^1 \not\subset I_0 \). So the induction hypothesis applies. Combining with \((*)\), we obtain the statement of the proposition.

\[ l = 0 \]

Obviously, \( I_0^{(1)} \) and \( I_0^{(2)} \) meet at some point \( p \in \{r\} \times \mathbb{P}^1 \). Since \( \Gamma_k \) does not pass through \( r \), \( p \neq r_k \) for \( k = 1, 2 \). Therefore, \( I_0^{(1)} \) and \( I_0^{(2)} \) are joined by curves contracting to \( p \) on \( \tilde{\Upsilon}_0 \). Combining with \((*)\),
we conclude that $\Gamma_1$ and $\Gamma_2$ lie on the same connected component of $\widetilde{\Upsilon}_0$. \hfill \Box

Proposition 5.7 follows directly from Proposition 5.10. We will leave the details to the readers.

5.3. Proof of Theorem 5.1. Before we proceed, we would like to raise a simple question.

Let $\pi : X \rightarrow Y$ be a proper and dominant map between two irreducible varieties $X$ and $Y$. Assume that $Y$ is normal. Let $q \in Y$ and $p \in \pi^{-1}(q)$ be a point on the fiber over $q$. Is it true that for any analytic neighborhood $U$ of $p$, $\pi(U)$ contains a neighborhood $V$ of $q$?

In general, this is not true. For example, we may take $X$ to be the blowup of $Y = \mathbb{P}^2$ at a point $p$ and $q$ to be a point on the exceptional divisor. The statement is true for $\pi$ finite. More generally, by using Stein factorization, we have the following.

**Proposition 5.11 (Open Mapping Principle).** Let $\pi : X \rightarrow Y$ be a proper and dominant map between two irreducible varieties $X$ and $Y$. Assume that $Y$ is normal. Let $q \in Y$ and let $W$ be a connected component of $\pi^{-1}(q)$. Then for any analytic neighborhood $U$ of $W$, $\pi(U)$ contains a neighborhood $V$ of $q$.

Suppose that for a general point $[R^s] \in \widetilde{R}_g$, there is a point $$([R^s], [\Upsilon_0], w_1, w_2, ..., w_\beta, \delta_1, \delta_2, ..., \delta_\beta, \nu_1, \nu_2, ..., \nu_\beta) \in \widetilde{Z}_g$$
lying on the fiber of $\widetilde{Z}_g \rightarrow \widetilde{H}_g$ over $[R^s]$ such that $\Upsilon_0$ contains a Type III chain $r_{ij} \cup r_{ij}^q \cup q_1 \cup q_2 \cup ... \cup q_{m-1}q_m$. Let $W_0$ be the index set such that $$\{w_1, w_2, ..., w_\beta\} \cap (r_{ij} \cup r_{ij}^q \cup q_1 \cup q_2 \cup ... \cup q_{m-1}q_m) = \{w_i : i \in W_0\}.$$ Theorem 5.1 is equivalent to saying that $\sum_{i \in W_0} \delta_i \geq \sum_{i=0}^m \operatorname{Intsc}(\Sigma_1, E; q_i)$ (let $q_0 = r_{ij}$) and if the equality holds, $\nu_i = 1$ for $i \in W_0$.

As indicated in (5.3), we will move $R^s$ in four steps.

**Step 1.** $[R^s] \in \widetilde{R}_g \Rightarrow [R^s] \in \widetilde{R}_g^0$

When a $R^s$ in $\widetilde{R}_g$ degenerates to a general $R^s$ in $\widetilde{R}_g^0$, the points $r_{ij} = q_0, q_1, q_2, ..., q_m$ on the Type III chain “collapse” to the point $r_{ij} = r_{2j}$. At the same time,

$$\sum_{i=0}^m \operatorname{Intsc}(\Sigma_1, E; q_i) = \operatorname{Intsc}(\Sigma_1, E; r_{1j})$$

where the $\Sigma_1$ on the LHS refers to the component of $\Upsilon_0$ lying on the old $R^s$ and the $\Sigma_1$ on the RHS lies on the new $R^s$. 
Meanwhile, the limiting singularities of the old \( R^s \) which lie on the Type III chain will degenerate to the points on \( \overline{r_{1j}} \cup \overline{r_{2j}} \). So for \([R^s] \in \tilde{\mathcal{R}}_0^g\), there is a point 

\[([R^s], [\Upsilon_0], w_1, w_2, ..., w_\beta, ...) \in \tilde{\mathcal{Z}}_g\]

lying on the fiber of \( \tilde{\mathcal{Z}}_g \to \tilde{\mathcal{H}}_g \) over \([R^s]\) such that \( \Upsilon_0 \) contains \( \overline{r_{1j}} \cup \overline{r_{2j}} \) with multiplicity \( \mu \) and 

\[\{w_1, w_2, ..., w_\beta\} \cap (\overline{r_{1j}} \cup \overline{r_{2j}}) = \{w_i : i \in W_0\}\].

In the proof of Proposition 2.2, we have shown that \( \Gamma_1 \cup \Gamma_2 \) does not pass through \( r_{1j} = r_{2j} \) on the new \( R^s \) in \( \tilde{\mathcal{R}}_0^g \). So by (5.27),

\[2\mu = \sum_{l=0}^m \mbox{Intsc}(\Sigma_1, E; q_l)\].

**Step 2.** \([R^s] \in \tilde{\mathcal{R}}_0^g \Rightarrow [S] \in \mathcal{Y}_g\)

Let us consider the fiber \((\tilde{\mathcal{Z}}_g)_{[R^s]} \) of \( \tilde{\mathcal{Z}}_g \to \tilde{\mathcal{H}}_g \) over a point \([R^s] \in \tilde{\mathcal{R}}_0^g\). Let the point 

\[(\tilde{\mathcal{Z}}_g)_{[R^s]}, w_1, w_2, ..., w_\beta, \delta_1, \delta_2, ..., \delta_\beta, \nu_1, \nu_2, ..., \nu_\beta)\]

with the property described above lie on a connected component \( Z \) of \((\tilde{\mathcal{Z}}_g)_{[R^s]}\). Then for any point 

\[(\tilde{\mathcal{Z}}_g)_{[R^s]}, w'_1, w'_2, ..., w'_\beta, \delta'_1, \delta'_2, ..., \delta'_\beta, \nu'_1, \nu'_2, ..., \nu'_\beta)\]

lying on the same component \( Z \), we necessarily have that

1. \( \Upsilon'_0 \) contains \( \overline{r_{1j}} \cup \overline{r_{2j}} \) with multiplicity \( \mu \);
2. it follows from Proposition 5.2 that 

\[\{w'_1, w'_2, ..., w'_\beta\} \cap (\overline{r_{1j}} \cup \overline{r_{2j}}) = \{w'_i : i \in W_0\}\].

If we take an analytic neighborhood \( U \) of \( Z \) and project it to \( \tilde{\mathcal{H}}_g \), by Proposition 5.11, the image \( \pi(U) \) will contain a neighborhood \( V \) of \([R^s] \in \tilde{\mathcal{R}}_0^g \), where \( \pi \) is the map \( \tilde{\mathcal{Z}}_g \to \tilde{\mathcal{H}}_g \). By Proposition 5.3, \( V \) contains general points of \( \mathcal{Y}_g \).

For a general point \([S] \in \mathcal{Y}_g \cap V \), the fiber of \( U \to \pi(U) \) over \([S]\) consists of points 

\[([S], [D], w_1, w_2, ..., w_\beta, ...)\]

where

1. by Proposition 5.3 and 5.6, \( D \) contains two connected components \( D_1, D_2 \in \mathcal{O}_S(F) \) with multiplicities \( \mu_1 \) and \( \mu_2 \), respectively, where \( \mu_1 + \mu_2 = \mu \);
2. \[ \{w_1, w_2, ..., w_\beta\} \cap (D_1 \cup D_2) = \{w_i : i \in W_0\} \]

Basically, if we have a family of surfaces \(S_t\) approach \(R^s\), by Proposition 5.4, the corresponding family of curves \(D_t \subset S_t\) will have components in \(|\mathcal{O}_S(F)|\) with total multiplicities \(\mu\) degenerating to \(\overline{\tau}_1 \cup \overline{\tau}_2\). And by Proposition 5.3, there are exactly two rational curves in \(D_1, D_2 \in |\mathcal{O}_S(F)|\) in the neighborhood of \(\overline{\tau}_1 \cup \overline{\tau}_2\). So \(D\) will contain \(D_1\) and \(D_2\) with a total multiplicity \(\mu\). Let \(G\) be the irreducible component of \(D\) in \(|\mathcal{O}_S(C + k'F)|\) for some \(k'\).

Let \(W_1\) and \(W_2\) be the index sets such that \(\{w_1, w_2, ..., w_\beta\} \cap D_j = \{w_i : i \in W_j\}\) for \(j = 1, 2\).

Of course, \(W_0 = W_1 \cup W_2\) and \(W_1 \cap W_2 = \emptyset\).

Step 3. \([S] \in \mathcal{V}_g \Rightarrow [R^s] \in \tilde{\mathcal{R}}^0_g\)

If we degenerate an elliptic K3 surface \(S\) in \(\mathcal{V}_g\) to a general \(R^s\) in \(\tilde{\mathcal{R}}^0_g\) with the corresponding \(D \subset S\), described as above, degenerating to \(Y_0 \subset R\), by Proposition 5.3 and 5.4, we can make \(D_1\) degenerate to \(\overline{\rho}_1q_1 = \overline{\rho}_1q_1^{(1)} \cup \overline{\rho}_1q_1^{(2)}\) and make \(D_2\) degenerate to \(\overline{\rho}_2q_2 = \overline{\rho}_2q_2^{(1)} \cup \overline{\rho}_2q_2^{(2)}\). So \(Y_0\) will contain \(\overline{\rho}_j\) with multiplicity at least \(\mu_j\) for \(j = 1, 2\). Let \(Y_0\) contain \(\overline{\rho}_j\) with multiplicity \(\mu_j + \mu_j'\). Namely, \(G_t \subset S_t\) will degenerate to a curve on \(R\) containing \(\overline{\rho}_j\) with multiplicity \(\mu_j\).

Let \(W_1'\) and \(W_2'\) be the index sets such that \(\{w_1, w_2, ..., w_\beta\} \cap \overline{\rho}_j = \{w_i : i \in W_j\} \cup \{w_i : i \in W'_j\}\) for \(j = 1, 2\)

where \(([R^s], [Y_0], w_1, w_2, ..., w_\beta, ...) \in \tilde{Z}_g\) is the corresponding limit when \(S\) degenerates to \(R^s\).

More intuitively, those \(w_i\)'s for \(i \in W_1' \cup W_2'\) are the extra limiting singularities we get due to the extra multiplicities \(\mu_1'\) and \(\mu_2'\). We have to control the extra \(\delta\)-invariant

\[
\sum_{i \in W_1' \cup W_2'} \delta_i.
\]

By Proposition 5.3 and Lemma 5.2,

\[
\sum_{i \in W_1' \cup W_2'} \delta_i = \delta(G_t, \overline{\rho}_1q_1) + \delta(G_t, \overline{\rho}_2q_2)
\]

\[(5.29)\]

\[
= 2\mu_1' + \text{Intsc}(\Gamma_1, E; p_1) + \text{Intsc}(\Gamma_1, E; q_1) + 2\mu_2' + \text{Intsc}(\Gamma_1, E; p_2) + \text{Intsc}(\Gamma_1, E; q_2).
\]

Step 4. \(\tilde{\mathcal{R}}^0_g \Rightarrow [R^s] \in [R^s] \in \tilde{\mathcal{R}}_g\)
By using the similar argument to that used in Step 2, we can show that for a \([(R^s)'] \in \tilde{R}_g\) lying in a neighborhood of \([R^s] \in \tilde{R}_g^0\), there is a point
\[
\left([(R^s)'], [\mathcal{Y}'_0], w'_1, w'_2, \ldots, w'_\beta, \ldots\right) \in \tilde{Z}_g
\]
where \(\mathcal{Y}'_0\) contains two Type II chains \(C_1\) and \(C_2\) such that

1. \(C_1\) and \(C_2\) degenerate to \(\overline{p_1q_1}\) and \(\overline{p_2q_2}\), respectively, when \((R^s)'\) degenerates to \(R^s\);

2. \(\sum q \in C_j \cap E \text{Intsc}(\Sigma'_1, E; q) = \text{Intsc}(\Sigma_1, E; p_j) + \text{Intsc}(\Gamma_1, E; q_j)\)
   for \(j = 1, 2\), where \(\Sigma'_1\) is the component of \(\mathcal{Y}'_0\) defined as usual;

3. \(\{w'_1, w'_2, \ldots, w'_\beta\} \cap C_j = \{w_i : i \in W_j \cup W'_j\}\)
   for \(j = 1, 2\).

So applying Theorem 4.1 to \(\mathcal{Y}'_0 \subset (R^s)'\), we have
\[
(5.31) \quad \sum_{i \in W_j \cup W'_j} \delta_i \geq \sum_{q \in C_j \cap E} \text{Intsc}(\Sigma'_1, E; q),
\]
for \(j = 1, 2\). Combining with (5.29) and (5.30), we have
\[
(5.32) \quad \sum_{i \in W_1} \delta_i + \sum_{i \in W_2} \delta_i \geq 2(\mu_1 + \mu_2) = 2\mu.
\]
If the equality in (5.32) holds, the equalities in (5.31) have to hold. So by Theorem 4.1, \(\nu_i = 1\) for \(i \in W_1 \cup W_2 \cup W'_1 \cup W'_2\). Combining with (5.28), we have proved Theorem 5.1.

5.4. **Completion of the Proof of Theorem 1.1.** With Theorem 3.1, 4.1 and 5.1 in place, Theorem 1.1 is more or less obvious.

Let \(C_0, C_1, \ldots, C_l \subset \mathcal{Y}_0\) be all the \(F\)-chains in \(\mathcal{Y}_0\) where \(C_0\) is the Type I chain. It follows from Theorem 3.1, 4.1 and 5.1 that
\[
\delta(\mathcal{Y}_t) = \sum_{i=0}^l \delta(\mathcal{Y}_t, C_i)
\]
\[
\geq \sum_{i=0}^l \sum_{q \in C_i \cap E} \text{Intsc}(\Sigma_1, E; q) - 1
\]
\[
= \Sigma_1 \cdot E - 1 = g.
\]
Obviously, $\delta(\Upsilon_t) = g$. Hence we must have
\[
\delta(\Upsilon_t, C_i) = \sum_{q \in C_i \cap E} \text{Intsc}(\Sigma_1, E; q)
\]
for $i = 1, 2, \ldots, l$. Consequently, by Theorem 4.1 and 5.1, $\Upsilon_t$ only has nodes as singularities in the neighborhood of $C_i$ for $i = 1, 2, \ldots, l$. And by Theorem 3.1, $\Upsilon_t$ only has nodes as singularities in the neighborhood of $C_0$. Therefore, all the singularities of $\Upsilon_t$ are nodes.

REFERENCES

[B] Beauville A., Counting Rational Curves on K3 Surfaces, preprint alg-geom/9701019 (1997).

[B-L] Bryan J. and Leung N.C., The Enumerative Geometry of K3 surfaces and Modular Forms, preprint alg-geom/9711031.

[C] Chen X., Rational Curves on K3 Surfaces, preprint math.AG/9804075, to appear on J. Alg. Geom (1998).

[CH] Caporaso L. and Harris J., Counting plane curves of any genus, preprint alg-geom/9608023 (1996).

[CLM] Ciliberto C., Lopez A. and Miranda R., Projective Degenerations of K3 Surfaces, Guassian Maps, and Fano Threefolds, Invent. Math. 114, 641-667 (1993). Also: alg-geom/9311002.

[FGS] Fantechi B., Göttsche L. and Straten D., Euler number of the compactified Jacobian and multiplicity of rational curves, preprint alg-geom/9708012.

[G-H] Griffith P. and Harris J., On the Noether-Lefschetz Theorem and Some Remarks on Codimension-two Cycles, Math. Ann. 271, 31-51 (1985).

[H] Harris J., Galois Group of Enumerative Problems, Duke Math J. 46(4), 685-724 (1979).

[K] Kulikov V., Degenerations of K3 Surfaces and Enriques Surfaces, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 5, 1008-1042, 1199.

[Y-Z] Yau S.T. and Zaslow E., BPS States, String Duality, and Nodal Curves on K3, Nuclear Physics B, 471(3), 503-512 (1996). Also: hep-th/9512121.

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