Game-theoretic semantics and partial specifications

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Abstract
We discuss partial specifications in first-order logic FO and also in a Turing-complete extension of FO. We compare the compositional and game-theoretic approaches to the systems.

1 Introduction
A natural Turing-complete extension \( \mathcal{L} \) of first-order logic FO can be obtained by extending FO by two new features: (1) operators that add new points to models and new tuples to relations, and (2) operators that enable formulae to refer to themselves. The self-referentiality operator of \( \mathcal{L} \) is based on a construct that enables looping when formulae are evaluated using game-theoretic semantics.

The reason the logic \( \mathcal{L} \) is particularly interesting lies in its simplicity and its exact behavioural correspondence with Turing machines. It is also worth noting that the new operators of \( \mathcal{L} \) nicely capture two fundamental classes of constructors that are omnipresent in the everyday practice of mathematics (but missing from FO): scenarios where fresh points are added to investigated constructions (or fresh lines are drawn, et cetera) play a central role in geometry, and recursive looping operators are found everywhere in mathematical practice, often indicated with the help of the famous three dots (...).

Adding new points to domains makes functions partial, and thus it is natural to consider partial functions in the framework of \( \mathcal{L} \). Previous work has concentrated on relational vocabularies. We develop a related semantics for \( \mathcal{L} \), also accommodating partial relation symbols into the picture. We begin by investigating a related system of first-order logic via a compositional semantics and then extend it to a game-theoretic semantics for \( \mathcal{L} \).

The logic \( \mathcal{L} \) was first introduced in [6]. Below we discuss the results of that article and also further results not covered there. Other systems that bear some degree of similarity to \( \mathcal{L} \) include for example BGS logic [1] and abstract state machines [3, 2]. Logics that modify structures include, for example, sabotage modal logic and public announcements logics. Recursive looping operators are a common feature in logics in finite model theory and verification. However, while the approach in \( \mathcal{L} \) bears a degree of similarity to the fixed point operators of the \( \mu \)-calculus, \( \mathcal{L} \) is not based on fixed points and no monotonicity requirements apply.

1
2 Partially fixed entities

Let \( \tau \) be a vocabulary, i.e., it is a collection of relation symbols, function symbols and constant symbols. Let \( \text{VAR} \) denote a countably infinite set of first-order variable symbols. Define \( \tau \)-terms in the usual way to be the smallest set of terms built from variables \( x \in \text{VAR} \) and constant symbols \( c \in \tau \) by applying the function symbols \( f \in \tau \) in the way that respects the function symbol arities, and let \( \text{FO}(\tau) \) denote the language of first-order logic when the vocabulary under investigation is \( \tau \). Now, let \( \mathfrak{M} \) be a \( \tau \)-model where the function symbols \( f \in \tau \) can be interpreted as partial functions on the model domain \( M \). A \( k \)-ary function symbol \( f \in \tau \) is interpreted so that \( f^{\mathfrak{M}} \) is a function \( f^{\mathfrak{M}} : N \to M \) where \( N \) is some subset of \( M^k \). We allow also the constant symbols to be interpreted as partial, meaning that \( c \in \tau \) can be interpreted as an element of \( \mathfrak{M} \) or it may not have an interpretation in \( \mathfrak{M} \) at all.

In addition to the model \( \mathfrak{M} \), we also need a variable assignment \( g \), which is a function \( g : V \to M \) mapping from some set \( V \subseteq \text{VAR} \) into the domain \( M \) of \( \mathfrak{M} \). Now consider a \( \tau \)-term \( t \). If \( t = f \), then \( f^{\mathfrak{M}, g} \) is the partial function \( f^{\mathfrak{M}, g} \). We may also write \( f^{\mathfrak{M}} \). When \( t = x \in \text{VAR} \), then \( x^{\mathfrak{M}, g} = g(x) \) when \( x \) belongs to the domain of the variable assignment \( g \), and otherwise we say that \( x^{\mathfrak{M}, g} = \) undefined (the variable is undefined). When \( t = c \in \tau \) is a constant symbol, then \( c^{\mathfrak{M}, g} = c^{\mathfrak{M}} \) is the element \( u \in M \) if \( \mathfrak{M} \) specifies such an element \( u \) for \( c \), and otherwise we say that \( c^{\mathfrak{M}, g} = c^{\mathfrak{M}} \) is undefined. Each of \( t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g} \) is defined and \( f^{\mathfrak{M}} \) is defined on the tuple \( (t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}) \); otherwise we say that \( t^{\mathfrak{M}, g} \) is not defined. Recall that \( g[x \mapsto a] \) is the assignment with domain \( \text{dom}(g) \cup \{x\} \) that is otherwise as \( g \) but maps \( x \) to \( a \). The semantics of first-order logic \( \text{FO}(\tau) \) is defined as follows.

\[
\mathfrak{M}, g \models^+ R(t_1, \ldots, t_n) \iff \text{the terms } t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g} \text{ are defined and and we have } (t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}) \in R^{\mathfrak{M}}
\]

\[
\mathfrak{M}, g \models^+ t_1 = t_2 \iff t_1^{\mathfrak{M}, g} \text{ and } t_2^{\mathfrak{M}, g} \text{ are defined and } t_1^{\mathfrak{M}, g} = t_2^{\mathfrak{M}, g}
\]

\[
\mathfrak{M}, g \models^+ \neg \varphi \iff \mathfrak{M}, g \models^+ \varphi
\]

\[
\mathfrak{M}, g \models^+ \varphi \land \psi \iff \mathfrak{M}, g \models^+ \varphi \text{ and } \mathfrak{M}, g \models^+ \varphi
\]

\[
\mathfrak{M}, g \models^+ \exists x \varphi \iff \text{there exists } a \in M \text{ such that } \mathfrak{M}, g[x \mapsto a] \models^+ \varphi
\]

\[
\mathfrak{M}, g \models^+ R(t_1, \ldots, t_n) \iff \text{the terms } t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g} \text{ are defined and and we have } (t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}) \notin R^{\mathfrak{M}}
\]

Note that for a constant symbol \( c \), we can write \( c^{\mathfrak{M}} \) as well as \( c^{\mathfrak{M}, g} \).

We can also consider partially defined relations. A partially defined relation \( R^{\mathfrak{M}} \) specifies (1) a set of tuples \( (u_1, \ldots, u_n) \) in the relation and (2) a set of tuples \( (v_1, \ldots, v_n) \)
not in the relation. Thus some tuples may not be specified either way (neither positive nor negative) and thus it is undefined whether such a tuple belongs to the relation. In such a framework, we get the modified clauses

\[ M, g \models^+ R(t_1, \ldots, t_n) \iff \text{the terms are defined and} \]
\[ R^{2^n} \text{ is defined on } (t_1^{2^n, g}, \ldots, t_n^{2^n, g}) \]
and we have \( (t_1^{2^n, g}, \ldots, t_n^{2^n, g}) \in R^{2^n} \)

\[ M, g \models^- R(t_1, \ldots, t_n) \iff \text{the terms } t_1^{2^n, g}, \ldots, t_n^{2^n, g} \text{ are defined and} \]
\[ R^{2^n} \text{ is defined on } (t_1^{2^n, g}, \ldots, t_n^{2^n, g}) \]
and we have \( (t_1^{2^n, g}, \ldots, t_n^{2^n, g}) \notin R^{2^n} \)

It is also possible to add a contradictory negation \( \sim \) into the picture. We can let, to give a possibility,

\[ M, g \models^+ \sim \varphi \iff \text{not } M, g \models^+ \varphi \]
\[ M, g \models^- \sim \varphi \iff \text{not } M, g \models^- \varphi. \]

We also define the determinacy operator \( d \) such that

\[ M, g \models^+ d \varphi \iff M, g \models^+ \varphi \text{ or } M, g \models^- \varphi \]
\[ M, g \models^- d \varphi \iff \text{(not } M, g \models^+ \varphi) \text{ and (not } M, g \models^- \varphi) \]

We note that further generalizations of indeterminate values come naturally. For example, one could consider \( k \)-ary function symbols that are defined on some \((k + 1)\)-tuple \((u_1, \ldots, u_{k+1})\) such that \( f^{2^n}(u, \ldots, u_k) \) is defined to not equal \( u_{k+1} \). However, we shall not consider such extensions here.

### 3 The logic \( \mathcal{L} \)

We then consider the Turing complete logic \( \mathcal{L} \), or computation logic CL, sometimes also called computation game logic. Let us first consider a relational vocabulary with no constant or function symbols. Let \( \mathcal{L} \) denote the language that extends the syntax of first-order logic by the following formula construction rules:

1. \( \varphi \mapsto Ix \varphi \)
2. \( \varphi \mapsto I_{R(x_1, \ldots, x_n)} \varphi \)
3. \( \varphi \mapsto Dx \varphi \)
4. \( \varphi \mapsto D_{R(x_1, \ldots, x_n)} \varphi \)
5. \( C_i \) is an atomic formula (for each \( i \in \mathbb{N} \))
6. \( \varphi \mapsto C_i \varphi \)
7. We also allow atoms \( X(x_1, \ldots, x_n) \) where \( X \) is a relational symbol not in the vocabulary considered. These symbols are analogous to tape symbols. The simplest way to treat \( X \) is to consider it a relation symbol interpreted initially as the empty \( n \)-ary relation \( \emptyset \).
Intuitively, a formula of type $I x \varphi(x)$ states that it is possible to insert a fresh, isolated element $u$ into the domain of the current model so that the resulting new model satisfies $\varphi(u)$. The fresh element $u$ being isolated means that $u$ is disconnected from the original model; the relations of the original model are not altered in any way by the operator $I x$, so $u$ does not become part of any relational tuple at the moment of insertion.

A formula of type $I_R(x_1, \ldots, x_n) \varphi(x_1, \ldots, x_n)$ states that it is possible to insert a tuple $(u_1, \ldots, u_n)$ into the relation $R$ so that $\varphi(u_1, \ldots, u_n)$ holds in the obtained model. The tuple $(u_1, \ldots, u_n)$ is a sequence of elements in the original model, so this time the domain of the model is not altered. Instead, the $n$-ary relation $R$ obtains a new tuple via the insertion.

A formula of type $D x \varphi$ states that it is possible to delete an element $u$ named $x$ from the domain of the current model so that the resulting new model satisfies $\varphi$. All tuples that contain $u$ are of course also deleted from the related relations.

A formula of type $D_R(x_1, \ldots, x_n) \varphi(x_1, \ldots, x_n)$ states that we can delete a tuple $(u_1, \ldots, u_n)$ named $(x_1, \ldots, x_n)$ from the relation $R$ so that $\varphi$ holds in the obtained model. The new atomic formulae $C_i$ can be regarded as variables ranging over formulae, so a formula $C_i \varphi$ can be considered to be a pointer to (or the name of) some other formula. The formulae $C_i \varphi$ could intuitively be given the following reading: the claim $C_i$, which states that $\varphi$, holds. Thus the formula $C_i \varphi$ is both naming $\varphi$ to be called $C_i$ and claiming that $\varphi$ holds. Importantly, the formula $\varphi$ can contain $C_i$ as an atomic formula. This leads to self-reference.

The logic $L$ is based on game-theoretic semantics GTS which directly extends the standard GTS of FO. Recall that the GTS of FO is based on games played by Eloise (who is initially the verifier) and Abelard (initially the falsifier). In a game $G(M, g, \varphi)$, the verifier is trying to show (or verify) that $M, g \models \varphi$ and the falsifier is opposing this, i.e., the falsifier wishes to falsify the claim $M, g \models \varphi$. The players start from the position $(M, g, \varphi)$ and work their way towards the subformulae of $\varphi$. See [7] for a detailed exposition of GTS for first-order logic and [4] for some of the founding ideas behind GTS.

To deal with the logic $L$, the rules for the FO-game are extended as follows.

1. In a position $(M, g, I x \psi)$, the game is continued from a position $(M', g[x \mapsto u], \psi)$ where $M'$ is the model obtained by inserting a fresh isolated point $u$ into the domain of $M$.

2. In a position $(M, g, I_R(x_1, \ldots, x_n) \psi)$, the verifier chooses a tuple $(u_1, \ldots, u_n)$ of elements in $M$ and the game is continued from the position $(M', g[x_1 \mapsto u_1, \ldots, x_n \mapsto u_n], \psi)$ where $M'$ is the model obtained from $M$ by inserting the tuple $(u_1, \ldots, u_n)$ into the relation $R$. If the model has an empty domain (and thus no tuple can be inserted to a relation of positive arity), the game play ends and the verifier loses the play.

3. In a position $(M, g, D x \psi)$, the game is continued from a position $(M', g^*, \psi)$ where $M'$ is the model obtained by deleting the point $g(x)$ (if it exists) from the domain.
of $\mathfrak{M}$. The assignment $g^*$ is obtained from $g$ by deleting the pairs $(y, u)$ where $u = g(x)$. If $x$ does not belong to the domain of $g$, then the game ends and the verifier loses the play of the game.$^1$

4. In a position $(\mathfrak{M}, \mathcal{D}_{R(x_1, \ldots, x_n)} \psi)$, the verifier chooses a tuple $(u_1, \ldots, u_n)$ of elements in $\mathfrak{M}$ and the game is continued from the position

$$(\mathfrak{M}', g[x_1 \mapsto u_1, \ldots, x_n \mapsto u_n], \psi)$$

where $\mathfrak{M}'$ is the model obtained from $\mathfrak{M}$ by deleting the tuple $(g(x_1), \ldots, g(x_n))$ from the relation $R$. If some $x_i$ does not belong to the domain of $g$, then the game ends and the verifier loses the play of the game.$^2$

5. In a position $(\mathfrak{M}, g, C_i \psi)$, the game simply moves to the position $(\mathfrak{M}, g, \psi)$.

6. In an atomic position $(\mathfrak{M}, g, C_i \psi)$, the game moves to the position $(\mathfrak{M}, g, C_i \psi)$. Here $C_i \psi$ is a subformula of the original formula $\varphi$ that the semantic game began with. (If there are many such subformulæ, the verifier chooses which one to continue from. If there are none, the game ends and neither player wins that play of the game. Alternative conventions would be possible here as well, like the verifier losing.)

7. In a position with $\land, \neg, \exists$, the game proceeds as in first-order logic; see also [6] which discusses the logic $\mathcal{L}$. We of course need an extra flag in positions to account for the negation $\neg$, et cetera.

Just like the semantic game for first-order logic, the extended game ends if an atomic position with a first-order atom $R(x_1, \ldots, x_n)$ or $x = y$ is encountered. The winner is then decided precisely as in the FO-game.$^3$ Thus the extended game can go on forever, as for example the games for $C_i C_i$ and $C_i \neg C_i$ always will. In the case the play of the game indeed goes on forever, then that play is won by neither of the players. Note that Turing-machines exhibit precisely this kind of behaviour: they can

1. halt in an accepting state (corresponding to the verifier winning the semantic game play),
2. halt in a rejecting state (corresponding to the falsifier winning),
3. diverge (corresponding to neither of the players winning).

Indeed, there is an exact correspondence between the logic $\mathcal{L}$ and Turing machines. Let $\mathfrak{M}, g \models^+ \varphi$ (respectively, $\mathfrak{M}, g \models^- \varphi$) denote that Eloise (respectively, Abelard) has a winning strategy in the game starting from $(\mathfrak{M}, g, \varphi)$. We may drop $g$ when it is the empty assignment. Let $\text{enc}(\mathfrak{M})$ denote the encoding of the finite model $\mathfrak{M}$ according to any standard encoding scheme. Then the following theorem shows that $\mathcal{L}$ corresponds to Turing machines so that not only acceptance and rejection but even divergence of Turing computation is captured in a precise and natural way. The proof of the following theorem follows from [6].

$^1$An alternative convention would be to just ignore the deletion when it is not possible.

$^2$An alternative convention would again be to just ignore the deletion when it is not possible.

$^3$If some variable of an atom is not interpreted by $g$, then neither player wins.
Theorem 3.1. For every Turing machine $\text{TM}$, there exists a formula $\varphi \in \mathcal{L}$ such that

1. $\text{TM}$ accepts $\text{enc}(\mathcal{M})$ iff $\mathcal{M} \models^+ \varphi$
2. $\text{TM}$ rejects $\text{enc}(\mathcal{M})$ iff $\mathcal{M} \models^− \varphi$

Vice versa, for every formula $\varphi \in \mathcal{L}$, there is a Turing machine $\text{TM}$ such that the above conditions hold.

We briefly digress to discuss the level of naturalness of the features of $\mathcal{L}$. Since $\mathcal{L}$ defines precisely the recursively enumerable classes of finite models, it cannot be closed under negation (meaning complement here). Thus $\neg$ is not the classical negation. However, $\mathcal{L}$ has a very natural translation into natural language. The key is to replace truth by verification. We read $\mathcal{M} \models^+ \varphi$ as the claim that “it is verifiable that $T(\varphi)$” where $T$ is the translation from $\mathcal{L}$ into natural language defined as follows. We let $T$ map FO-atoms in the usual way to the corresponding natural language statements, so for example $T(x = y)$ simply reads “$x$ equals $y$”. The atoms $C_i$ are read as they stand, so $T(C_i) = C_i$. The FO-quantifiers translate in the standard way, so $T(\exists x \varphi) =$ there exists an $x$ such that $T(\varphi)$ and analogously for $\forall x$. Also $\vee$ and $\wedge$ translate in the standard way, so $T(\varphi \vee \psi) = T(\varphi)$ or $T(\psi)$ and analogously for $\wedge$. However, $T(\neg \psi) =$ it is falsifiable that $T(\psi)$ (or alternatively, it is refutable that $T(\psi)$). Thus negation now translates to the dual of verifiability. Concerning the insertion operators, we let $T(I x \varphi) =$ it is possible to insert a new element $x$ such that $T(\varphi)$. Similarly, we let $T(I_R(x_1, \ldots, x_n) \varphi) =$ it is possible to insert a tuple $(x_1, \ldots, x_n)$ into $R$ such that $T(\varphi)$. Finally, we let

$$T(C_i \varphi) = \text{it is possible verify the claim } C_i \text{ which states that } T(\varphi).$$

Alternatively, we can let

$$T(C_i \varphi) = \text{it is possible verify the claim named } C_i \text{ which claims that } T(\varphi).$$

Deletion operators are similar to the insertion operators.

Thereby the logic $\mathcal{L}$ can be seen as a simple Turing-complete fragment of natural language. Thus it is not just a technical logical formalism.

It is interesting to note that $\neg$ can—and perhaps should—be read as the classical negation on those fragments of $\mathcal{L}$ where the semantic games are determined (such as standard FO contexts). Furthermore, adding a generalized quantifier to $\mathcal{L}$ corresponds precisely to adding a corresponding oracle to Turing machines.

Theorem 3.1 holds also without tape predicates, given the underlying vocabulary contains at least one binary (or higher-arity) relation.

Observation 3.2. The claim of Theorem 3.1 holds for $\mathcal{L}$ without tape predicates in the scenario where the underlying vocabulary contains at least one relation symbol of arity at least two.
Proof. The argument is based on using gadgets. Consider a formula $\varphi$ that makes use of tape predicates. To eliminate these, we will write a new formula $\varphi'$. We assume, without loss of generality, that the underlying vocabulary contains a binary relation $R$. In the case there is no binary relation, we can easily modify our argument to account for that essentially by using the first two coordinate positions of some higher-arity relation to encode a binary relation.

We begin $\varphi'$ with $Iz_1$ which introduces a new domain element $m_1$ labeled by $z_1$. Similarly, we use further insertion operators to construct a fresh successor structure with the elements $m_1, \ldots, m_k$, where $k$ is the number of tape predicates used in $\varphi$. By a successor structure, we mean that $R$ acts as a successor relation over the new elements, thereby connecting the elements $m_1, \ldots, m_k$ in the given order. This way we obtain a fresh point $m_i$ for each tape predicate $i$ in $\varphi$, and we have a successor ordering (via $R$) of the fresh points. Each point $m_i$ is labeled with the corresponding variable $z_i$.

The next task is show how to construct (or model) tuples of the tape predicate $i$. This is done via the following reification technique. Let $A$ be the domain of the current model minus all the fresh points (such as $m_1, \ldots, m_k$) that we will use for encoding tuples of tape predicates. We call $A$ the proper domain of the model. An arbitrary $n$-ary tuple $(a_1, \ldots, a_n) \in A^n$ of the predicate $i$ will be encoded using a new fresh domain point $u \notin A$ (called tuple point) such that $(m_i, u) \in R$ and such that the following conditions hold.

1. For each $p \in \{1, \ldots, n\}$, there exist a fresh point $v_p \notin A$ (called coordinate point) outside the proper domain $A$ such that $(u, v_p) \in R$.
2. $(v_p, a_p) \in R$ holds for each $p \in \{1, \ldots, n\}$.
3. There is a successor order using $R$ and ordering the points $v_p$, i.e., $(v_i, v_{i+1}) \in R$ for each $i \in \{1, \ldots, n-1\}$. This is essential for distinguishing the order of the elements of the tuple $(a_1, \ldots, a_n)$ being encoded.

In summary, if the points $m_i$ are called predicate points, then a predicate point connects to a tuple point, a tuple point to a coordinate point, and finally a coordinate point to the a point of the tuple (over the proper domain) being encoded. Furthermore, we order the predicate points as well as the coordinate points by a successor order.

It is easy to see that this kind of an encoding can be used to eliminate tape predicates. The underlying language is strong enough in expressive power to deal with also the updates of the predicate encodings during semantic games.

Finally, if nullary tape predicates are used, the way to encode them is to first turn them into unary tape predicates with an appropriate translation and then encode the obtained unary predicates in the above way.

We note that the above also proves the corresponding statement for the logic $L$ as it was defined in [6]. That is, let $L_t$ be the tape-predicate-free fragment of the logic $L$ as defined [6]. Then Theorem 3.1 and similarly Theorems 5.1 and 5.2 of [6] hold for $L_t$. The same applies to other close variants of the logic, e.g., ones that treat
the semantics of variable symbols without interpretation in a slightly modified way. The above observations were essentially made already in [6], but left there officially as conjectures for the sake of lack of space.

3.1 Partially fixed entities in $\mathcal{L}$

The logic $\mathcal{L}$ has, as such, natural indeterminacy properties stemming from divergence of Turing machines and also in other ways. The truth teller formula $C_i, C_i$ is indeterminate with the philosophy that while it would not be problematic to let it be ‘true’, there is no reason to force it to be ‘true’. More particularly, there is no reason to force it verifiable. This is because it is not well-founded, so we cannot ‘dig’ the truth value for the formula from any atomic level. Such an atomic level could be considered unproblematic, like the atomic formulas in games for first-order logic, but for $C_i, C_i$ there is no such atomic ground level. We need to keep on digging, or at least we cannot reach a bottom, no matter what. Also, if the interpretation of $x$ is deleted, then $P(x)$ becomes naturally indeterminate. Even $x = x$ becomes so with the reading for atoms that it can be directly observed that $x = x$. Such a reading works well also for other first-order atoms.

However, $\mathcal{L}$ is a natural logic for considering partial functions. Indeed, when we add an isolated element to a model, then all functions become partial. Also, if we delete the interpretation of a constant symbol, a similar situation is realized. However, the game-theoretic semantics nicely facilitates the treatment of such phenomena. If the game ends up in an atom $R(t_1, \ldots, t_n)$, then the following happens.

1. The verifier wins if $t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}$ are all defined and $R^{\mathfrak{M}}$ is defined on $(t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g})$ and we have $(t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}) \in R^{\mathfrak{M}}$.

2. The falsifier wins if $t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}$ are all defined and $R^{\mathfrak{M}}$ is defined on $(t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g})$ and we have $(t_1^{\mathfrak{M}, g}, \ldots, t_n^{\mathfrak{M}, g}) \notin R^{\mathfrak{M}}$.

3. Otherwise the game ends and neither player wins the play of the game.

The treatment of equality atoms $t_1 = t_2$ are similar.

1. The verifier wins if $t_1^{\mathfrak{M}, g}$ and $t_2^{\mathfrak{M}, g}$ are defined and we have $t_1^{\mathfrak{M}, g} = t_2^{\mathfrak{M}, g}$.

2. The falsifier wins if $t_1^{\mathfrak{M}, g}$ and $t_2^{\mathfrak{M}, g}$ are defined and we have $t_1^{\mathfrak{M}, g} \neq t_2^{\mathfrak{M}, g}$.

3. Otherwise the game ends and neither player wins the play of the game.

Generally, we allow terms $t$ in all atoms, while the first take on $\mathcal{L}$ allowed first-order variables only. Note that $D$ makes a tuple $(u_1, \ldots, u_n)$ not belong to the modified relation, whether it was undefined or positively true that $(u_1, \ldots, u_n)$ is in the relation originally. Similarly $I$ positively adds the tuple, no matter the previous status of the tuple.
Finally, the semantics for first-order logic with partially defined entities given in the beginning of this document is equivalent to what we obtain for that logic from the described game-theoretic semantics.

**Theorem 3.3.** The game-theoretic semantics extends the compositional one on the first-order fragment.

Concerning $\mathcal{L}$ it is of course possible and natural to extend $\mathcal{L}$ so that it can also modify partial functions $f$ and partially constants $c$ the same way it is allowed to modify partial relation symbols. Furthermore, there are many ways to add $\sim$ to $\mathcal{L}$. Some of the natural choices lead to a logic that captures the arithmetic hierarchy. Also, we could, e.g., consider variants of $\mathcal{D}$ that only make it undefined whether a tuple belongs to a relation. Indeed, we could naturally define similar operators for all other modifications between positive, undefined and negative instances.

### 4 Operators

We now define a notion of a **modifier** (cf. [5]). This notion works well with the three-valued logic of partial definitions.

Let $V_R$ denote the full relational vocabulary consisting of countably infinitely many relation symbols $R$ for each arity $k \in \mathbb{N}$. Similarly, let $V_f$ denote the full functional vocabulary consisting of countably infinitely many function symbols of each arity $k \in \mathbb{Z}^+$. Moreover, let $V_c$ be the full constant vocabulary consisting of countably infinitely many constant symbols. Recall that $\text{VAR}$ denotes a countably infinite set of variable symbols. We note that these sets can also be considered having higher infinite numbers of symbols, but for most purposes, countably infinite sets suffice. Let $\text{SYMB}$ denote the union $V_R \cup V_f \cup V_c \cup \text{VAR}$.

A **symbol list** is a 4-tuple of the form

$$(R_1, \ldots, R_{k_R}), (f_1, \ldots, f_{k_f}), (c_1, \ldots, c_{k_c}), (y_1, \ldots, y_{k_y})$$

that specifies lists of relation symbols, function symbols, constant symbols and variables. Here $k_R, k_f, k_c, k_y \in \mathbb{N}$ are natural numbers. Generally, these can be any finite natural numbers, including zero. Nullary relation symbols are allowed. The **type** of the above symbol list is the 4-tuple $(\overline{s}_1, \overline{s}_2, k_c, k_y)$ such that the following conditions hold.

1. $\overline{s}_1$ is a tuple in $\mathbb{N}^{k_R}$ that gives the arities of the symbols in $(R_1, \ldots, R_{k_R})$.
2. Similarly, $\overline{s}_2$ is a tuple in $\mathbb{Z}^{k_f}$ that gives the arities of the symbols in $(f_1, \ldots, f_{k_f})$. 

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By a **symbol set**, we mean a set that contains relation symbols, function symbols, constant symbols and variables. Thus symbol sets are unions of vocabularies and variable sets. By an **interpretation** we mean a pair \((\mathcal{M}, f)\) where \(\mathcal{M}\) is a model and \(f\) is a variable assignment mapping some set \(V \subseteq \text{VAR}\) into the domain \(M\) of \(\mathcal{M}\). If \(\tau\) is a symbol set, we let \(\text{Int}_{\tau}\) denote the class of interpretations \((\mathcal{M}, f)\) where \(\mathcal{M}\) is a model over the vocabulary \(\tau \setminus \text{VAR}\) and the domain of \(f\) is precisely \(\tau \cap \text{VAR}\). These can be called \(\tau\)-interpretations.

Let \(\sigma\) be a symbol set. A **\(\sigma\)-property triple** is a product of the form

\[
P(\text{Int}_{\sigma}) \times P(\text{Int}_{\sigma}) \times P(\text{Int}_{\sigma})
\]

where \(P\) denotes the power set (or power class) operator. An **\(\ell\)-classification** of type \((\sigma_1, \ldots, \sigma_\ell)\) is a sequence

\[
(\text{Pt}(\sigma_1), \ldots, \text{Pt}(\sigma_\ell))
\]

where each \(\text{Pt}(\sigma_i)\) is a \(\sigma_i\)-property triple. We let \(C_{(\sigma_1, \ldots, \sigma_\ell)}\) denote the class of \(\ell\)-classifications of type \((\sigma_1, \ldots, \sigma_\ell)\).

Consider symbol sets \(\tau, \sigma_1, \ldots, \sigma_\ell\), and let \(\overline{\sigma}\) denote \((\sigma_1, \ldots, \sigma_\ell)\), so we can also write \(C_{\tau}\) for \(C_{(\sigma_1, \ldots, \sigma_\ell)}\). Now, a \((\tau, \overline{\sigma})\)-**operator** is a mapping

\[
F_{\tau, \overline{\sigma}} : \text{Int}_{\tau} \rightarrow P(C_{\overline{\sigma}})
\]

that gives a class of \(\ell\)-classifications of type \(\overline{\sigma}\) for each input \(\mathcal{I} \in \text{Int}_{\tau}\) (recall indeed that \(P\) is the power set—or power class—operator). Furthermore, the following natural invariance conditions hold.

1. Suppose \(\mathcal{I} \in \text{Int}_{\tau}\) and \(\mathcal{I}' \in \text{Int}_{\tau}\) are isomorphic via an isomorphism \(f\). Then there is a one-to-one mapping \(g\) from \(F(\mathcal{I})\) to \(F(\mathcal{I}')\), where we let \(F\) denote \(F_{\tau, \overline{\sigma}}\) for simplicity.

2. For each tuple \(T = (T_1, \ldots, T_\ell) \in F(\mathcal{I})\) and the corresponding tuple \(g(T) = (T'_1, \ldots, T'_\ell) \in F(\mathcal{I}')\), and for each \(i \in \{1, \ldots, \ell\}\), there exist one-to-one correspondences \(h_1 : T_i[1] \rightarrow T'_i[1]\), \(h_2 : T_i[2] \rightarrow T'_i[2]\) and \(h_3 : T_i[3] \rightarrow T'_i[3]\), where \(X[j]\) denotes the \(j\)th member of the triple \(X\).

3. For each of these one-to-one correspondences \(h : T_i[j] \rightarrow T'_i[j]\) and for each \(\mathcal{I}''\) and \(\mathcal{I}''\) such that \(h(\mathcal{I}'') = \mathcal{I}''\), there exists an isomorphism \(r : \mathcal{I}'' \rightarrow \mathcal{I}''\).

The above conditions simply force isomorphism invariance.

Now, let \(f : \text{SYMB} \rightarrow \text{SYMB}\) be a bijection such that for each \(V \in \{V_R, V_f, V_c, \text{VAR}\}\), the restriction \(f \mid V\) is a bijection from \(V\) onto \(V\) that preserves arities of relation symbols and function symbols. We call such a bijection a **symbol name permutation**. Given a symbol name permutation \(f\), by \(f(\tau)\) we denote the symbol set \(\{f(x) \mid x \in \tau\}\) obtained from the symbol set \(\tau\) in the natural way. For \(\overline{\sigma} = (\sigma_1, \ldots, \sigma_\ell)\), we let \(f(\overline{\sigma})\) denote \((f(\sigma_1), \ldots, f(\sigma_\ell))\). An **\(f\)-isomorphism** from \((\mathcal{M}, g) \in \text{Int}_{\tau}\) to \((\mathcal{N}, h) \in \text{Int}_{f(\tau)}\) is a bijection \(p\) from the domain \(M\) of \(\mathcal{M}\) onto the domain \(N\) of \(\mathcal{N}\) such that the following conditions hold.
1. The domains of the assignments $g$ and $h$ have the same number of variables, and for each $x$ in the domain of $g$, we have $g(x) = m \in M$ if and only if $h(f(x)) = p(m)$.

2. The vocabularies of $\mathcal{M}$ and $\mathcal{N}$ are of the same type in the sense that the following conditions hold.
   
   (a) For each $k \in \mathbb{N}$, both vocabularies have the same number of relation symbols of arity $k$.
   
   (b) For each $k \in \mathbb{Z}^+$, both vocabularies have the same number of function symbols of arity $k$.
   
   (c) Both vocabularies have the same number of constant symbols.

3. The following is true for each constant symbol $c \in \tau$ and the corresponding symbol $d := f(c)$. It holds that $c^\mathcal{M}$ is defined to be $m \in M$ if and only if $d^\mathcal{N}$ is defined to be $p(m) \in N$.

4. For each arity $k$, the following is true for each relation symbol $R$ of arity $k$ and the corresponding symbol $S := f(R)$ and for each tuple $(m_1, \ldots, m_k) \in M^k$ of domain elements.

   (a) $R^\mathcal{M}$ is defined on the tuple $(m_1, \ldots, m_k)$ if and only if $S^\mathcal{N}$ is defined on the tuple $(p(m_1), \ldots, p(m_k))$.
   
   (b) $(m_1, \ldots, m_k)$ is defined to be in $R^\mathcal{M}$ if and only if $(p(m_1), \ldots, p(m_k))$ is defined to be in $S^\mathcal{N}$. Note that this, together with the above condition, implies that $(m_1, \ldots, m_k)$ is defined to not be in $R^\mathcal{M}$ if and only if $(p(m_1), \ldots, p(m_k))$ is defined to not be in $S^\mathcal{N}$.

5. For each arity $k$, the following holds for each relation symbol $q$ of arity $k$ and the corresponding symbol $r := f(q)$ and for each tuple $(m_1, \ldots, m_k) \in M^k$.

   (a) $q^\mathcal{M}$ is defined on the tuple $(m_1, \ldots, m_k)$ if and only if $r^\mathcal{N}$ is defined on the tuple $(p(m_1), \ldots, p(m_k))$.
   
   (b) $q^\mathcal{M}(m_1, \ldots, m_k)$ is defined to be $m \in M$ if and only if $r^\mathcal{N}(p(m_1), \ldots, p(m_k))$ is defined to be $p(m)$. We note that condition $a$ actually follows already from condition $b$.

Let $f$ be a symbol name permutation. Now, an $f$-renaming of a $(\tau, \sigma)$-operator is an $(f(\tau), f(\sigma))$-operator $F$ such that the following condition holds.

1. Suppose there exists an $f$-isomorphism from $\mathcal{I} \in \text{Int}_\tau$ to $\mathcal{I}' \in \text{Int}_{f(\tau)}$. Then there is a one-to-one mapping $g$ from $F(\mathcal{I})$ to $F(\mathcal{I}')$.

2. For each tuple $T = (T_1, \ldots, T_\ell) \in F(\mathcal{I})$ and the corresponding tuple $g(T) = (T_1', \ldots, T_\ell') \in F(\mathcal{I}')$, and for each $i \in \{1, \ldots, \ell\}$, there exist one-to-one correspondences $h_1 : T_i[1] \to T_i'[1]$, $h_2 : T_i[2] \to T_i'[2]$ and $h_3 : T_i[3] \to T_i'[3]$, where $X[j]$ denotes the $j$th member of the triple $X$. 

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3. For each of these one-to-one correspondences \( h : T_i \rightarrow T_i' \) fixed above and for each \( \mathcal{I}' \) and \( \mathcal{I}'' \) such that \( h(\mathcal{I}') = \mathcal{I}'' \), there exists an \( f \)-isomorphism \( r : \mathcal{I}'' \rightarrow \mathcal{I}''' \).

Let \( \mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_\ell) \) be a tuple of \( \ell \) symbol list types, so each \( \mathcal{S}_i \) is a symbol list type. A symbol list tuple \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) \) is a tuple of symbol lists \( \mathcal{P}_i \). The symbol list tuple \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) \) is of type \( \mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_\ell) \) if each \( \mathcal{P}_i \) has the type \( \mathcal{S}_i \). Now, an operator \( \mathcal{O} \) of type \( \mathcal{S} \) is a mapping that takes as input a symbol set \( \tau \) and a symbol list tuple \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) \) of type \( \mathcal{S} \), and based on this information, outputs a \((\tau, \mathcal{P})\)-operator \( \mathcal{O}_{\tau}(\mathcal{P}) \) such that the following conditions hold.

1. Note first indeed that the \((\tau, \mathcal{P})\)-operator is denoted by \( \mathcal{O}_{\tau}(\mathcal{P}) \) where we have the symbol list tuple \( \mathcal{P} \) instead of the symbol set tuple \( \mathcal{P} \). The symbol set tuple \( \mathcal{P} \) has \( \ell \) symbol sets, so we can write \( \mathcal{P} = (\sigma_1, \ldots, \sigma_\ell) \).

2. For each \( \sigma_i \), in the tuple \( \mathcal{P} = (\sigma_1, \ldots, \sigma_\ell) \), we have \( \sigma_i \subseteq \tau \cup \text{sym} \mathcal{P}_i \) where \( \text{sym} \mathcal{P}_i \) denotes the set of all symbols that appear in the tuples of the symbol list \( \mathcal{P}_i \).

Note that \( \tau \) in the input of an operator can vary freely, and so can \( \mathcal{P} \), as long as it is of type \( \mathcal{S} \). An operator of relaxed type \( \mathcal{S} \) lifts the restrictions \( \sigma_i \subseteq \tau \cup \text{sym} \mathcal{P}_i \), i.e., those restrictions do not apply. Nevertheless, we give \( \mathcal{P} \) of type \( \mathcal{S} \) (together with \( \tau \)) as a convenient syntactic input to an operator of relaxed type \( \mathcal{S} \). An operator of \( \ell \)-relaxed type allows any symbol list tuple \( \mathcal{P} \) with \( \ell \) symbol lists as an input, so \( \mathcal{P} \) does not have to be of type \( \mathcal{S} \), and again the restrictions \( \sigma_i \subseteq \tau \cup \text{sym} \mathcal{P}_i \) do not apply.

Now, let \( \tau \) and \( \tau' \) be symbol sets. Let \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) \) and \( \mathcal{P}' = (\mathcal{P}_1', \ldots, \mathcal{P}_\ell') \) be symbol list tuples. We say that \((\tau, \mathcal{P})\) and \((\tau', \mathcal{P}')\) are naming variants if there is a symbol name permutation \( f \) such that \( f(\tau), f(\mathcal{P}) = (\tau', \mathcal{P}') \), where \( f(\mathcal{P}) \) denotes the symbol list tuple obtained from \( \mathcal{P} \) by replacing every symbol \( \# \in \text{Symb} \) that appears in \( \mathcal{P} \) by the corresponding symbol \( f(\#) \). The restriction of \( f \) to the set of symbols appearing in \((\tau, \mathcal{P})\) is then the corresponding renaming bijection from \((\tau, \mathcal{P})\) to \((\tau', \mathcal{P}')\).

An operator \( \mathcal{O} \) is renaming invariant if for any inputs \((\tau, \mathcal{P})\) and \((\tau', \mathcal{P}')\) that are naming variants with a renaming bijection \( b \) from \((\tau, \mathcal{P})\) to \((\tau', \mathcal{P}')\), the output operator \( \mathcal{O}_{\tau'}(\mathcal{P}') \) is an \( f \)-renaming of \( \mathcal{O}_{\tau}(\mathcal{P}) \) for some \( f \) that extends \( b \). Renaming invariant operators are logically quite natural, whether they are of some type \( \mathcal{S} \), or of relaxed type \( \mathcal{S} \) or just of \( \ell \)-relaxed type.

Now, consider an operator \( \mathcal{O} \) of type \( \mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_\ell) \), and let \( \mathcal{O} \) be a related symbol referring to \( \mathcal{O} \). We continue treating operators of relaxed type \( \mathcal{S} \) and \( \ell \)-relaxed type similarly, so alternatively \( \mathcal{O} \) can be an operator of one of the relaxed types. We define a semantics for formulae \( \mathcal{O}(\varphi_1, \ldots, \varphi_\ell) \), where \( \mathcal{P} \) is a symbol list tuple of type \( \mathcal{S} \) (or of any type with \( \ell \) symbol lists in the case of an operator of \( \ell \)-relaxed type). Let \((\mathfrak{M}, g)\) be a \( \tau \)-interpretation. We define that \( \mathfrak{M}, g \models^+ \mathcal{O}(\varphi_1, \ldots, \varphi_\ell) \) if and only if there exists a tuple \( C \) in \( F((\mathfrak{M}, g)) \), where \( F \) denotes \( \mathcal{O}_{\tau}(\mathcal{P}) \), such that the following conditions hold.

1. For each \( i \in \{1, \ldots, \ell\} \), let \( C_i \) denote the \( i \)th triple in \( C \), and for each \( j \in \{1, 2, 3\} \), let \( C_i[j] \) denote the \( j \)th member of \( C_i \). Then, for each \( i \in \{1, \ldots, \ell\} \), we have \( \mathfrak{M}, h \models^+ \varphi_i \) for each \((\mathfrak{M}, h) \in C_i[1] \).
2. Similarly, for each \( i \in \{1, \ldots, \ell \} \), we have \( \mathfrak{M}, h \models \neg \varphi_i \) for each \( (\mathfrak{M}, h) \in C_i[3] \).

3. For each \( i \in \{1, \ldots, \ell \} \), we have \( \mathfrak{M}, h \not\models \varphi_i \) and \( \mathfrak{M}, h \not\models \neg \varphi_i \) for each \( (\mathfrak{M}, h) \in C_i[2] \).

Note that renaming invariant modifiers are insensitive to change of names of operators. This can be natural. Further invariance conditions are also natural for various purposes. It is easy to generalize, e.g., to model sets. There the input is a set (or class) of interpretations \( (\mathfrak{M}, g) \). However, also simpler operators are natural. A simple operator is a renaming invariant operator of type \( \mathfrak{S} = (((\emptyset, \emptyset, \emptyset), \emptyset)) \) (where \( \emptyset \) is the empty tuple) which returns, given \( (\mathfrak{M}, g) \), an interpretation \( (\mathfrak{M}, h) \) that has the same symbol set as \( (\mathfrak{M}, g) \). A very simple operator is a simple operator that returns, given \( (\mathfrak{M}, g) \), an interpretation \( (\mathfrak{M}, h) \) where the assignment has not changed and the domain of \( \mathfrak{M} \) is the same as that of \( \mathfrak{M} \). Simple and very simple operators come with extensions that deal with several formulae instead of only one.

Note that the considered three-valued logic is of course just a generalization of the two-valued approach, and we can always define operators where each of the second sets \( C_i[2] \) is always empty. That covers the two-valued approach. Finally, (for example) the standard connectives and generalized quantifiers are easy to deal with via modifiers, but it is perhaps more interesting to consider ‘more dynamic’ model modifiers not relating directly to quantifier statements or plain connectives.

References

[1] Andreas Blass, Yuri Gurevich, and Saharon Shelah. Choiceless polynomial time. *Ann. Pure Appl. Log.*, 100(1-3):141–187, 1999.

[2] Egon Börger and Robert F. Stärk. *Abstract State Machines. A Method for High-Level System Design and Analysis*. Springer, 2003.

[3] Yuri Gurevich. A new thesis. Technical Report 85T-68-203, August 1985. Abstracts, American Mathematical Society.

[4] Jaakko Hintikka. *Logic, Language-games and Information: Kantian Themes in the Philosophy of Logic*. Clarendon Press, 1973.

[5] Antti Kuusisto. On Games and Computation. CoRR abs/1910.14603, 2019.

[6] Antti Kuusisto. Some turing-complete extensions of first-order logic. In Adriano Peron and Carla Piazza, editors, *Proceedings Fifth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2014, Verona, Italy, September 10-12, 2014*, volume 161 of *EPTCS*, pages 4–17, 2014.

[7] Allen L. Mann, Gabriel Sandu, and Merlijn Sevenster. *Independence-Friendly Logic - a Game-Theoretic Approach*, volume 386 of *London Mathematical Society lecture note series*. Cambridge University Press, 2011.