On the invariant spectrum on \( \mathbb{P}^1 \)

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Abstract

Motivated by the work of Abreu and Freitas \cite{1}, we study the invariant spectrum of the Laplace operator associated to hermitian line bundles endowed with invariant metrics over \( \mathbb{P}^1 \).

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1 Introduction

Let \( \mathbb{P}^1 \) be the complex projective line and \( \omega \) a smooth and normalized Kähler form on \( \mathbb{P}^1 \). We denote by \( \lambda_1(\omega) \) the first eigenvalue of the Laplace operator defined by \( \omega \) and acting on smooth functions on \( \mathbb{P}^1 \). In \cite{2}, Hersch showed that

\[ \lambda_1(\omega) \leq 2. \]

In \cite{1}, Abreu and Freitas studied the invariant spectrum of invariant metrics on \( \mathbb{P}^1 \). Their goal was to analyze this type of inequality in the invariant setting. If we denote by \( 0 = \lambda_0(\omega) < \lambda_1(\omega) < \ldots \) the invariant eigenvalues of the Laplace operator defined by \( \omega \). Their first result shows that there is no general analogue of Hersch’s theorem, see \cite{1} theorem 1. Nevertheless, when they consider the class of invariant metrics that are isometric to a surface of revolution in \( \mathbb{R}^3 \), they gave optimal upper bounds for the invariant eigenvalues associated to this class. Their second result is the following theorem

**Theorem 1.1.** \cite{1} theorem 2] Within the class of smooth, invariant and normalized Kähler form \( \omega \) on \( \mathbb{P}^1 \) and corresponding to a surface of revolution in \( \mathbb{R}^3 \), we have

\[ \lambda_j(\omega) < \frac{\xi_j^2}{2} \quad \forall j \in \mathbb{N}_{\geq 1} \]

where \( \xi_j \) is the \( \frac{1}{2}(j + 1) \)th positive zero of the Bessel function \( J_0 \) if \( j \) is odd, and the \( \frac{1}{2}j \)th positive zero of \( J'_0 \) if \( j \) is even. These bounds are optimal.

Using symplectic coordinates they attached to any smooth and invariant Kähler form \( \omega \) a smooth function \( \mathcal{F} \in \mathcal{C}^\infty([0, 1]) \), positive on \( ]0, 1[ \) and satisfying \( \mathcal{F}(1) = 0 \) and \( \mathcal{F}'(-1) = -2 = -\mathcal{F}'(1) \). When \( \omega \) corresponds to a surface of revolution in \( \mathbb{R}^3 \) they established that these functions satisfy the following inequality

\[ \mathcal{F}(x) < \mathcal{F}_{\text{max}}(x), \quad \forall x \in [0, 1]. \]
By monotonicity principle, they deduced that \( \lambda_j(\omega) < \lambda_j(\mathcal{F}_{\text{max}}) \) for any \( j \in \mathbb{N}_{\geq 1} \), where \( \lambda_j(\mathcal{F}_{\text{max}}) \), \( j = 1, \ldots \) correspond to the invariant spectrum of a problem limit defined by \( \mathcal{F}_{\text{max}} \).

Let \( \omega \) be a smooth, invariant and normalized volume form on \( \mathbb{P}^1 \) and \( h \) a smooth and invariant hermitian metric on the holomorphic line bundle \( \mathcal{O}(m) \) over \( \mathbb{P}^1 \) (\( m \in \mathbb{N} \)). The problem of this paper is the study of the invariant spectrum of the Laplace operator \( \Delta_{\omega,h} \) defined by \( \omega \) and \( h \), and acting on the space of smooth functions with coefficients in \( \mathcal{O}(m) \). We denote by \( \lambda_0(\omega,h) = 0 < \lambda_1(\omega,h) < \lambda_2(\omega,h) < \ldots \) the invariant eigenvalues of \( \Delta_{\omega,h} \).

Our main result is the theorem 1.3 which gives optimal upper bounds for the invariant eigenvalues \( \lambda_j(\omega,h) \) for \( j = 1, \ldots \), when \( h \) and \( g \) satisfy an inequality of the same type as \( \Omega \). This theorem generalizes the result of Abreu and Freitas (see [1,1]).

First, we present a slight generalization of the symplectic coordinates formalism to a large class of singular Kähler metrics on \( \mathbb{P}^1 \). Let \( \omega \) be a continuous and invariant volume form on \( \mathbb{P}^1 \) such that \( f_{\omega}^1 \omega = 1 \). There exists \( \| \cdot \|_\omega \), a hermitian and invariant metric of class \( C^2 \) on \( \mathcal{O}(1) \) such that \( \omega = c_1(\mathcal{O}(1), \| \cdot \|_\omega) \). We denote by \( \Psi_\omega \) the function on \( \mathbb{C} \) given by \( \Psi_\omega(z) := -\frac{1}{\pi} \log \| 1 \|_\omega(z), \forall z \in \mathbb{C} \) and we set \( \tilde{F}_\omega(u) := \log \| 1 \|_\omega(\exp(-u)), \forall u \in \mathbb{R} \), where \( 1 \) corresponds to the global section \( x_0^m \), and \( \exp(-\cdot) \) is the following morphism \( \mathbb{R} \rightarrow \mathbb{P}^1, u \mapsto [1 : e^{-u}] \). Recall that we have a diffeomorphism \( \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}^* \), given by \((u, \theta) \mapsto e^{-u}e^{i\theta} \). Then on \( \mathbb{C}^* \) we have

\[
\omega_{\mathbb{C}^*} = -\frac{1}{2\pi} \frac{\partial^2 F_\omega}{\partial u^2}(u)du \wedge d\theta,
\frac{\partial^2 \Psi_\omega}{\partial z \partial \bar{z}}(z) = \frac{1}{2} \cdot \,^2 \frac{\partial^2 F_\omega}{\partial u^2}(u).
\] (2)

Let \( \tilde{F}_\omega \) be the Legendre-Fenchel transform associated to \( F_\omega \) that is the function given on \( \mathbb{R} \) by

\[
\tilde{F}_\omega(x) = \inf_{u \in \mathbb{R}} (x \cdot u - F_\omega(u)),
\]

One shows that \( \tilde{F}_\omega \) is concave and \( \tilde{F}_\omega(x) \) is finite if and only if \( x \in [0,1] \). We claim that \( \tilde{F}_\omega \) is a function of class \( C^2 \) on \( [0,1] \). Indeed, since \( F_\omega \) is \( C^2 \) then the function \( \theta : u \mapsto \frac{\partial F_\omega}{\partial u} \) defines a \( C^1 \)-diffeomorphism from \( \mathbb{R} \) onto its image, (we will show that the image is necessarily equal to \( [0,1] \)). Let \( x \in [0,1] \), since \( F_\omega \) is strictly concave then there exists a unique element \( G_\omega(x) \in \mathbb{R} \) such that \( \tilde{F}_\omega(x) = xG_\omega(x) - F_\omega(G_\omega(x)) \). Moreover, since \( F_\omega \) is \( C^2 \) then \( x = \frac{\partial F_\omega}{\partial u}(G_\omega(x)) \). Thus \( G_\omega \) is the inverse function of \( \frac{\partial F_\omega}{\partial u}(\cdot) \). In particular we deduce that \( 0,1 \) is included in the image of \( \partial \). By differentiating the previous identity we obtain \( \frac{\partial^2 F_\omega}{\partial u^2}(x) = G_\omega(x) \) for any \( x \in [0,1] \). Since \( f_{\omega}^1 \omega = 1 \), and by using \( \mathbb{R} \) we deduce that \( \frac{\partial F_\omega}{\partial u}(\cdot) \) \( (\infty \rightarrow 1) \) and \( \frac{\partial^2 F_\omega}{\partial u^2}(\cdot) \) \( (\rightarrow 0) \) are finite. We conclude that \( \frac{\partial F_\omega}{\partial u}(\cdot) \) \( \rightarrow 1 \) and \( \theta \) is a \( C^1 \)-diffeomorphism between \( \mathbb{R} \) and \( [0,1] \). We can then consider the following change of coordinates \( x = \frac{\partial F_\omega}{\partial u}(u) \). We set

\[
g_\omega(x) := -G_\omega(x) \quad \text{and} \quad \mathcal{T}_\omega(x) := \frac{1}{g_\omega(x)} x \in [0,1].
\]

One checks that \( \mathcal{T}_\omega(x) = -\frac{\partial F_\omega}{\partial u}(G_\omega(x)), \forall x \in [0,1] \). We denote by \( \mathcal{T}_{\text{can}} \) the continuous function on \( [0,1] \) given by \( \mathcal{T}_{\text{can}}(x) = 2 \min(x,1-x), \forall x \in [0,1] \) and \( \mathcal{G} \) the set of continuous functions \( \mathcal{T} \) on \( [0,1] \), positive on \( [0,1] \) and such that \( \mathcal{T} = \mathcal{T}_{\text{can}} + \mathcal{O}(\mathcal{T}_{\text{can}}) \) near the boundary of \( [0,1] \). Using the following transformation \( x \mapsto \frac{1}{2} \mathcal{T}(2x-1) \) one sees that the functions \( \mathcal{T} \) satisfying the conditions of the theorem belong to \( \mathcal{G} \) and \( \mathcal{T}_{\text{can}} \) corresponds to \( \mathcal{T}_{\text{max}} \) via this transformation. We prove the following result

**Theorem 1.2.** (see theorem (2.7)) For any \( \mathcal{T} \in \mathcal{G} \), there exists a continuous, normalized and invariant volume form \( \omega \) such that \( \mathcal{T}_\omega = \mathcal{T} \).

In particular, we show that \( \mathcal{T}_{\text{can}} \) corresponds to a natural volume form defined by the combinatorics of \( \mathbb{P}^1 \), (see example (2.2)).

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We set \( \overline{h}_\omega \) the function on \([0, 1]\) given by
\[
\overline{h}_\omega(x) := h(1, 1)(e^{-G_\omega(x)}), \forall x \in [0, 1].
\]
We show that \( \overline{h}_\omega \) is a continuous function on \([0, 1]\), positive on \([0, 1]\) such that \( \lim_{x \to 1^-} \frac{\overline{h}_\omega(x)}{1-x} \) exists and positive. Moreover, we prove that any function \( h \) satisfying the previous conditions defines a continuous and invariant hermitian metric on \( \mathcal{O}(m) \), see corollary 2.3.

**Theorem 1.3.** Let \( m \in \mathbb{N} \). Let \( \omega \) and \( h \) as before. We suppose that \( 1 < \frac{\overline{h}(x)}{\min(1, 2(1-x))^m} < \frac{\underline{h}(x)}{\underline{h}_\omega(x)}, \forall x \in [0, 1] \). Then
\[
\lambda_j(\omega, h) < \frac{\xi_{m,j}^2}{2}, \forall j \in \mathbb{N}_{\geq 1},
\]
where \( \xi_{m,j} \) is a zero of the function \( \frac{d}{dz}(z^{-m}J_0J_m) \) such that \( 0 < \frac{\xi_{m,1}^2}{2} < \frac{\xi_{m,2}^2}{2} < \ldots \) and \( J_n \) is the Bessel function of order \( n \). Moreover, these bounds are optimal.

The proof of this result is a combination of theorems 2.2 and 3.3.

**2 The invariant metrics on \( \mathbb{P}^1 \)**

Let \( \mathbb{P}^1 \) be the complex projective line and we denote by \([x_0 : x_1]\) the homogenous coordinate and \( z = x_1/x_0 \) the affine coordinate over the open subset \( \mathbb{C} = \{x_0 \neq 0\} \). Let \( \mathbb{C}^* \) be the complex torus acting on \( \mathbb{P}^1 \) as a toric manifold and \( \mathbb{S}^1 \) the compact sub-torus in \( \mathbb{C}^* \).

Let \( m \in \mathbb{N} \). Let \( \| \cdot \| \) be a continuous hermitian metric on the line bundle \( \mathcal{O}(m) \) over \( \mathbb{P}^1 \), and we suppose that \( \| \cdot \| \) is invariant under the action of \( \mathbb{S}^1 \). To the metric \( \| \cdot \| \) we associate a continuous function \( F_{\| \cdot \|} \) defined on \( \mathbb{R} \) as follows
\[
F_{\| \cdot \|}(u) = \log \|1\|(\exp(-u)), \quad \forall u \in \mathbb{R}
\]
For instance, let \( \| \cdot \|_{m,\infty} \) be the following continuous hermitian metric on \( \mathcal{O}(m) \) given by
\[
\|s\|_{m,\infty}(z) = \frac{|s(z)|}{\max(1,|z|)^m}, \quad \forall z \in \mathbb{C}
\]
where \( s \) is a local holomorphic section of \( \mathcal{O}(m) \). Then we have \( F_{m,\infty}(u) := F_{\| \cdot \|_{m,\infty}}(u) = m \min(0, u), \forall u \in \mathbb{R} \). One can establish that there exists a bijection between the set of continuous hermitian and invariant metrics on \( \mathcal{O}(m) \) and the set of continuous functions \( F \) on \( \mathbb{R} \) such that the function \( \mathbb{C}^* \to \mathbb{R}, z \mapsto F(-\log|z|) - F_{m,\infty}(-\log|z|) \) extends to a bounded continuous function on \( \mathbb{P}^1 \).

**Example 2.1.** Following \( \mathbb{F}^1 \) and \( \mathbb{F} \), the Fubini-Study form \( \omega_{FS} \) is viewed as the "canonical" Kähler metric on \( \mathbb{P}^1 \) which is compatible with the standard moment map on \( \mathbb{P}^1 \). We set \( F_0 = F_{\omega_{FS}} \). For any \( u \in \mathbb{R} \), we have
\[
F_0(u) = -\frac{1}{2} \log(1 + e^{-2u}), \frac{\partial F_0}{\partial u}(u) = \frac{e^{-2u}}{1 + e^{-2u}}, \frac{\partial^2 F_0}{\partial u^2}(u) = -\frac{2e^{-2u}}{(1 + e^{-2u})^2},
\]
and for any \( x \in [0, 1] \)
\[
G_0(x) = \frac{x}{1-x}, \overline{G}_0(x) = 2x(1-x).
\]
Example 2.2. The second example is a singular volume form defined by the combinatorial structure of $\mathbb{P}^1$. Notice that $T\mathbb{P}^1$ is isomorphic to $\mathcal{O}(2)$ then the metric $\| \cdot \|_{2,\infty}$ induces a continuous volume form $\omega_{\text{can}}$ on $\mathbb{P}^1$. This form is given on $\mathbb{C}$ as follows

$$\omega_{\text{can}} = \frac{i}{4\pi} \frac{dz \wedge d\bar{z}}{\max(1,|z|)^4}.$$ 

One checks that $\int_{\mathbb{P}^1} \omega_{\text{can}} = 1$. We consider the following hermitian continuous metric $\| \cdot \|_{\text{can}}$ on $\mathcal{O}(1)$ defined as follows

$$\|s\|_{\text{can}}^2(z) = \frac{|s(z)|^2}{\max(1,|z|)^2} \exp(-k(z)) \quad \forall z \in \mathbb{C},$$

where $s$ is a local holomorphic section of $\mathcal{O}(1)$ and $k(z) = \frac{1}{2} \min(|z|^2, \frac{1}{|z|^2})$, $\forall z \in \mathbb{C}$. We have the following result

**Proposition 2.3.** The metric $\| \cdot \|_{\text{can}}$ is positive (i.e the current $c_1(\mathcal{O}(1), \| \cdot \|_{\text{can}})$ is positive) and $c_1(\mathcal{O}(1), \| \cdot \|_{1,\infty}) = \omega_{\text{can}}$

**Proof.** We have the following equality of currents

$$c_1(\mathcal{O}(1), \| \cdot \|_{\text{can}}) = c_1(\mathcal{O}(1), \| \cdot \|_{1,\infty}) + [dd^c k]$$

From § corollaire 6.3.5, we have $c_1(\mathcal{O}(1), \| \cdot \|_{1,\infty}) = \delta_{\mathbb{S}^1}$ (the current of integration on $\mathbb{S}^1$). Let $f$ be a smooth function on $\mathbb{P}^1$. We have

$$[dd^c k](f) = \int_{\mathbb{P}^1} k \, dd^c f$$

$$= \frac{1}{2} \int_{|z| \leq 1} |z|^2 \, dd^c f + \frac{1}{2} \int_{|z| \geq 1} |z|^{-2} \, dd^c f$$

$$= \frac{1}{2} \int_{|z| \leq 1} f \, dd^c |z|^2 + \frac{1}{2} \int_{|z| \geq 1} (dd^c f - f \cdot dd^c |z|^2) + \frac{1}{2} \int_{|z| \geq 1} f \, dd^c |z|^{-2} - \frac{1}{2} \int_{|z| \leq 1} (dd^c f + f \cdot dd^c |z|^2)$$

by Stokes

$$= \frac{1}{2} \int_{|z| \leq 1} f \, dd^c |z|^2 + \frac{1}{2} \int_{|z| \geq 1} f \, dd^c |z|^{-2} - \int_{\mathbb{S}^1} f \, dd^c |z|^2.$$ 

Therefore,

$$\int_{\mathbb{P}^1} f c_1(\mathcal{O}(1), \| \cdot \|_{\text{can}}) = \frac{i}{4\pi} \int_{|z| \leq 1} f \, dz \wedge d\bar{z} + \frac{i}{4\pi} \int_{|z| \geq 1} f \, \frac{dz \wedge d\bar{z}}{|z|^2} = \int_{\mathbb{P}^1} f \, \omega_{\text{can}},$$

which concludes the proof of the proposition. 

We denote by $F_{\text{can}}$ the function $u \mapsto \log(\| \cdot \|_{\text{can}}(\exp(-u)))$. We have $F_{\text{can}}(u) = \min(0, u) + \frac{1}{4} \min(e^{-2u}, e^{2u})$ for any $u \in \mathbb{R}$. An easy explicit computation shows that its Legendre-Fenchel is the following function

$$\hat{F}_{\text{can}}(x) = \begin{cases} -\frac{1}{4} x \log(2x) + \frac{1}{2} x & \text{if } x \in [0, 1/2], \\ -\frac{1}{4} (1 - x) \log(2(1 - x)) + \frac{1}{2} (1 - x) & \text{if } x \in [1/2, 1], \end{cases}$$

We see that $\hat{F}_{\text{can}}$ is a $C^1$ function on $[0, 1]$ and we set $G_{\text{can}}$ the function on $[0, 1]$ given by $G_{\text{can}}(x) = \frac{dF_{\text{can}}}{dx}(x)$. We have

$$G_{\text{can}}(x) = \begin{cases} -\frac{1}{2} \log(2x) & \text{if } x \in [0, 1/2], \\ \frac{1}{2} \log(2(1 - x)) & \text{if } x \in [1/2, 1], \end{cases}$$
We notice that $G_{can}$ defines a bijection between $[0,1]$ and $\mathbb{R}$. We set $g_{can} = -G_{can}'$ and $\overline{g}_{can} = \frac{1}{g_{can}}$. We have
\[ \overline{g}_{can}(x) = 2 \min(x, 1-x) \quad \forall x \in [0,1]. \]
We have $\overline{g}_{can}(x) = \frac{|x|^2}{\max(1,|x|^2)}$ and $\mathcal{T}_{m,\infty}(x) := \|1\|_{m,\infty}^2(e^{-G_{can}(x)}) = \min(1, 2(1-x))^m$.

We denote by $\mathcal{G}$ the set of continuous functions $\overline{g}$ on $[0,1]$, positive on $]0,1[$ and such that $\overline{g}(x) = \overline{g}_{can}(x) + O(\overline{g}_{can}(x)^2)$ near the boundary of $[0,1]$.

**Remark 2.4.** In [1] §4, Abreu and Freitas constructed a class of smooth metrics $g$. They correspond to closed surfaces of revolution in $\mathbb{R}^3$ and they proved that $\mathcal{F} := \frac{1}{g}$ is a smooth functions on $[-1,1]$ satisfying $\mathcal{F}(-1) = g(0) = 0$, $\mathcal{F}(1) = 2 = -\mathcal{F}(1)$ and $\sup_{[-1,1]} |\mathcal{F}(x)| \leq 2$. (4)

Clearly $\mathcal{F} \leq \mathcal{F}_{\max}$, where $\mathcal{F}_{\max}(x) = 2(1 - |x|)$ for any $x \in [0,1]$. Let $j \in \mathbb{N}_{\geq 1}$ and $\lambda_j(g)$ the $j$-th invariant eigenvalue of the Laplace operator defined by $g$. They showed that $\lambda_j(g)$ viewed as a function with variable $g$ is bounded over the set of smooth and invariant metrics corresponding to surfaces of revolution [1] Theorem 2.

By using the following transformation $\mathcal{F}_{[0,1]}(x) := \frac{1}{\mathcal{F}}(2x - 1)$ for any $x \in [0,1]$ (In particular, $\overline{g}_{can}(x) = \frac{1}{2}\mathcal{F}_{\max}(2x - 1)$), we see that smooth functions $\mathcal{F}$ on $[-1,1]$ satisfying $\mathcal{F} \leq \mathcal{F}_{\max}$ belong to the previous transformation, to $\mathcal{G}$. As we can expect the set $\mathcal{G}$ is not reduced to functions satisfying $\mathcal{F}$. More precisely, we will prove that there exist functions $\mathcal{F} \in \mathcal{G}$ such that $\mathcal{F}(-1) = g(1) = 0, \mathcal{F}(1) = 2 = -\mathcal{F}(1)$ and $\mathcal{F} \leq \mathcal{F}_{\max}$ but $\sup_{[-1,1]} |\mathcal{F}(x)|$ can be a large real number.

**Claim 2.5.** For any $A > 0$, there exists a smooth function $\mathcal{F}_A$ on $[0,1]$ such that $\mathcal{F}_A \in \mathcal{G}$ and $\lim_{A \to \infty} \sup_{x \in [0,1]} |\mathcal{F}_A(x)| = \infty$.

**Proof.** Let $\rho$ be a non-zero, positive, smooth function on $\mathbb{R}$ with support in $[1/4, 3/4]$ and bounded from above by $1/8$. Let $A \geq 1$, one checks that $\rho(A(x - 1/2) + 1/2) \leq 1/2 \min(x, 1-x), \forall x \in [0,1]$. It follows that $\rho(A(x - 1/2) + 1/2) + 1 = 2\rho(x - 1/2) + 1/2 \leq 2 \min(x, 1-x)$ for any $x \in [0,1]$. We set $\mathcal{F}_A(x) = 2x - 1 + \rho(A(x - 1/2) + 1/2)$, then it is easy to see that $\mathcal{F}_A$ is smooth and belongs to $\mathcal{G}$. If we set $x_A = A(x_0 - 1/2) + 1/2$ where $x_0 \in [0,1]$ is such that $\rho(x_0) \neq 0$, then $\mathcal{F}_A(x_A) = 2 - 4x_A + 2\rho(x_0) / A \sim_{A \to \infty} A$. Thus $\lim_{A \to \infty} \sup_{x \in [0,1]} |\mathcal{F}_A(x)| = \infty$.

**Theorem 2.6.** Let $\omega$ be a smooth and invariant Kähler form on $\mathbb{P}^1$ such that $\int_{\omega} \omega = 1$. We have $\mathcal{F}_\omega \in \mathcal{G}$.

**Proof.** Recall that for any $z \in \mathbb{C}^*$, $\frac{\partial^2 \psi_{\omega}}{\partial z \partial \overline{z}}(z) = \frac{1}{4} \epsilon^{2u} \frac{\partial^2 F}{\partial u^2}(u) = \frac{1}{4} e^{2G_{\omega}(x)} \mathcal{F}_\omega(x)$ and $G_\omega$ is finite over $[0,1]$. Then $\mathcal{F}_\omega$ is positive on $[0,1]$. Moreover, since $\lim_{x \to 0} e^{2G_{\omega}(x)} = +\infty$, then $\lim_{x \to +0} e^{2G_{\omega}(x)} \mathcal{F}_\omega(x) = 2 \frac{\partial^2 \psi_{\omega}}{\partial z \partial \overline{z}}(z) = \omega_{\omega}$ which is finite and non-zero. Let $\varepsilon \in [0,1/l_\omega]$, then there exists a positive real number $\eta$ such that $(1/l - \varepsilon) e^{-2G_{\omega}(z)} \mathcal{F}_\omega(x) \leq (1/l + \varepsilon) e^{-2G_{\omega}(z)} \mathcal{F}_\omega(x) \leq (1/l + \varepsilon) x \leq \eta$. It follows that $(1/l - \varepsilon)x \leq -\frac{1}{2} \int_0^x e^{-2G_{\omega}(z)} \mathcal{F}_\omega(x) \leq (1/l + \varepsilon)x \leq \eta$. Therefore
\[ (1/l - \varepsilon)x \leq e^{-2G_{\omega}(z)} \mathcal{F}_\omega(x) \leq (1/l + \varepsilon)x, \quad \forall x \leq \eta. \]

It follows that
\[ \lim_{x \to 0} \frac{\mathcal{F}_\omega(x)}{x} = \lim_{x \to 0} e^{2G_{\omega}(x)} \frac{e^{-2G_{\omega}(x)} \mathcal{F}_\omega(x)}{x} = \mathcal{F}_\omega(0) = 2. \]

We claim that $\mathcal{F}_\omega(x)e^{-2G_{\omega}(z)} = l_\omega + O(x)$ for $0 < x \ll 1$. Indeed, since $\frac{\partial^2 \psi_{\omega}}{\partial z \partial \overline{z}}(z) = \omega_{\omega} + O(|z|^2)$ in a small open neighborhood of $z = 0$. So, $\frac{\partial^2 \psi_{\omega}}{\partial z \partial \overline{z}}(e^{-G}(z)) = \frac{l_\omega}{2} + O(e^{-2G_{\omega}(z)})$ for $0 < x \ll 1$. From [3] we deduce that $\frac{\partial^2 \psi_{\omega}}{\partial z \partial \overline{z}}(e^{-G}(z)) = \frac{l_\omega}{2} + O(x)$ for $0 < x \ll 1$. 

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Therefore, 
\[ \frac{1}{2} \frac{d e^{-2G_\omega(x)}}{dx} = g_\omega(x) e^{-2G_\omega(x)} = \frac{1}{L_\omega} + O(x). \]

Then 
\[ e^{-2G_\omega(x)} = \frac{2}{L_\omega} x + O(x^2). \] (6)

Thus, 
\[ \mathcal{F}_\omega(x) = e^{-2G_\omega(x)}(L_\omega + O(x)) = 2x + O(x^2). \]

To conclude the proof of the theorem we need to prove the following 
\[ \mathcal{F}_\omega(x) = 2(1 - x) + O((1 - x)^2) \quad \forall 0 < 1 - x < 1. \] (7)

We consider the following biholomorphic map \( \tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1, z \mapsto z^{-1} \). Then \( \tau^*\omega \) is smooth, Kähler and invariant. We claim that 
\[ F_{\tau^*\omega}(-u) = -u + F_\omega(u) \quad \forall u \in \mathbb{R}. \]

This is follows from the following equality over \( \mathbb{C}^* \{ |x| = |z||x_0| \} \). Then, for any \( x \in [0, 1] \)
\[ \tilde{F}_{\tau^*\omega}(x) = \inf_{u \in \mathbb{R}} (xu - F_{\tau^*\omega}(u)) = \inf_{u \in \mathbb{R}} (u(1 - x) - F_\omega(u)) = \tilde{F}_\omega(1 - x), \]

Thus 
\[ G_{\tau^*\omega}(x) = -G_\omega(1 - x), \] (8)

So \( \mathcal{F}_{\tau^*\omega}(x) = \mathcal{F}_\omega(1 - x) \). We conclude that the proof of (7) can be deduced from the first case. \( \square \)

**Theorem 2.7.** For any \( \mathcal{F} \in \mathcal{G} \), there exists a continuous, normalized and invariant volume form \( \omega \) such that \( \mathcal{F}_\omega = \mathcal{F} \).

**Proof.** Let \( \mathcal{F} \in \mathcal{G} \) and we set \( g := 1/\mathcal{F} \). By hypothesis we can find two positive constants \( k \) and \( k' \) such that 
\[ k \leq \frac{\mathcal{F}(x)}{\mathcal{F}_{\text{can}}(x)} = \frac{g_{\text{can}}(x)}{g(x)} \leq k' \quad \forall x \in [0, 1]. \] (9)

We set \( G_g(x) := -\int_{1/2}^x g(s) ds \) and \( L_g(x) := \int_{1/2}^x G(s) ds \) for any \( x \in [0, 1] \). Since \( g \) is positive, then \( L_g \) is strictly concave on \( [0, 1] \) and by \( \mathbb{R} \) we can show that \( L \) is of Legendre type \( \mathbb{F} \) on \( [0, 1] \). It follows that the function \( x \mapsto \frac{\partial L_g}{\partial x} \) defines a \( C^1 \)-diffeomorphism between \( [0, 1] \) and \( \mathbb{R} \). Moreover, we can prove there exist two constants \( \alpha, \alpha' \) such that \( \alpha \leq L_g(x) \leq \alpha' \), \( \forall x \in [0, 1] \). Therefore the function \( F_g \) given on \( \mathbb{R} \) by 
\[ F_g(u) = \inf_{x \in [0, 1]} (ux - L_g(x)) \] is of class \( C^2 \) and satisfies 
\[ -\alpha' + F_{1,\infty}(u) \leq F_g(u) \leq -\alpha + F_{1,\infty}(u) \quad \forall u \in \mathbb{R}. \] (10)

Moreover, \( \frac{\partial F_g}{\partial u} \) is the inverse function of \( \frac{\partial L_g}{\partial x} \).

We consider the following differential form on \( C^* (\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}) \)
\[ \omega_g := -\frac{i}{4\pi} \frac{\partial^2 F_g}{\partial u^2} e^{2u} dz \wedge d\overline{z}. \]

---

1 Let \( C \subset \mathbb{R} \) an open convex set. A differentiable concave function \( f : C \rightarrow \mathbb{R} \) is of Legendre type if it is strictly concave and \( \lim_{u \rightarrow \infty} \| \frac{\partial f}{\partial u} (u_i) \| = \infty \) for every sequence \( (u_i)_{i \geq 1} \) converging to a point in the boundary of \( C \).
Since \( \frac{\partial^2 F_u}{\partial u^2}(u) = \frac{\partial F_u}{\partial u} \) then \( \omega_g \) is positive on \( \mathbb{C}^* \). Let \( 0 < x \ll 1 \), we have \( \frac{\partial F_u}{\partial u} = 2x + O(x^2) \). Then

\[
\frac{\partial}{\partial u}\left(e^{2u}\frac{\partial F_u}{\partial u}\right)^{-1} = O(e^{-2u}) \quad \text{and} \quad \frac{\partial}{\partial u}\left|e^{2u}\frac{\partial F_u}{\partial u}\right| = O(\frac{\partial F_u}{\partial u}) \quad \forall u \gg 1.
\]

The first equality gives \( (e^{2u}\frac{\partial F_u}{\partial u}(u))^{-1} = (e^{2u}\frac{\partial F_u}{\partial u}(v))^{-1} = O(e^{-2u} - e^{-2v}) \) for \( u, v \gg 1 \). This shows that the following limit \( l_g := \lim_{u \to \infty}(e^{2u}\frac{\partial F_u}{\partial u}(u))^{-1} \) exists and finite. The limit \( l_g \) is necessarily non-zero. Indeed, by the second equality we have \( e^{2u}\frac{\partial F_u}{\partial u}(u) \leq e^{2\sigma}\frac{\partial F_u}{\partial u}(v)e^{O(F_u(v) - F_u(u))} \) for \( u, v \gg 1 \) and from \([10]\) the RHS of the previous inequality is bounded for fixed \( v \) and \( u \gg 1 \). Therefore,

\[
-e^{2u}\frac{\partial^2 F_u}{\partial u^2}(u) = \frac{2}{l_g} + O(1) \quad \forall u \gg 1.
\]

Then the form \( \omega_g \) extends to \( \mathbb{C} \).

Let \( \overline{g} \) be the function on \([0, 1]\) given by \( \overline{g}(x) = \overline{g}(1 - x), \forall x \in [0, 1] \). Clearly \( \overline{g} \in \mathcal{G} \). We set \( g^* = 1/\overline{g} \). We have \( L_g(x) = L_{g^*}(1 - x) \) for any \( x \in [0, 1] \). Then

\[
-e^{2u}\frac{\partial^2 F_u}{\partial u^2}(u) = \frac{2}{l_{g^*}} + O(1) \quad \forall u \gg 1.
\]

As before we can show that \( F_{g^*}(-u) = -u + F_{g^*}(u), \forall u \in \mathbb{R} \). We deduce that

\[
-e^{-2u}\frac{\partial^2 F_u}{\partial u^2}(u) = \frac{2}{l_{g^*}} + O(1) \quad \forall (-u) \gg 1.
\]

We conclude that \( \omega_g \) extends to a positive, invariant and continuous (1, 1)-form on \( \mathbb{P}^1 \). We denote it also by \( \omega_g \). Finally, notice that \( \int_{[0, 1]} \omega_g = \frac{\partial F_u}{\partial u}(-\infty) - \frac{\partial F_u}{\partial u}(+\infty) = 1 \).

**Corollary 2.8.** Let \( h \) be a continuous and invariant hermitian metric on \( \mathcal{O}(m) \). Then the function \( \overline{h}_\omega \) on \([0, 1]\) by

\[
\overline{h}_\omega(x) = h(1, 1)(e^{-G_\omega(x)}), \forall x \in [0, 1],
\]

is continuous on \([0, 1]\), positive on \([0, 1]\) and the limit \( \lim_{x \to 1^-} \overline{h}_\omega(x) \) exists, finite and non-zero. Moreover, any continuous function \( \overline{h} \) verifying the previous conditions, defines a continuous and invariant hermitian metric on \( \mathcal{O}(m) \).

**Proof.** Let \( h \) be a continuous and invariant hermitian metric on \( \mathcal{O}(m) \). There exists a continuous and invariant function \( f \) on \( \mathbb{P}^1 \) such that \( h = e^f h_{m, \infty} \). Then it suffices to prove the corollary for the metric \( h_{m, \infty} \). We have \( h_{m, \infty}(x) = \min(1, e^{2mG_0(x)}) \) for any \( x \in [0, 1] \). Clearly \( h_{m, \infty} \) is continuous on \([0, 1]\). By \([3]\) and \([8]\) we deduce that \( h_{m, \infty}(x) = 1 \) for \( 0 < x < 1 \) and \( h_{m, \infty}(x) = \frac{2}{r} (1 - x)^m + O((1 - x)^{m + 1}) \).

Now, let \( \overline{h} \) be a continuous function on \([0, 1]\), positive on \([0, 1]\) and such that the limit \( \lim_{x \to 1^-} \overline{h}_\omega(x) \) exists, finite and non-zero. The function \( x \mapsto \overline{h}_\omega(x) \) is continuous and positive on \([0, 1]\). Thus it extends to a continuous, positive and invariant function on \( \mathbb{P}^1 \). Therefore, \( \overline{h} \) defines a continuous hermitian metric on \( \mathcal{O}(m) \). 

\[ \square \]
3 The invariant spectrum of the Laplace operator

Let \( m \in \mathbb{N} \) and \( \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) the space of smooth functions on \( \mathbb{P}^1 \) with coefficients in \( \mathcal{O}(m) \). Let \( \omega \) be smooth, invariant and normalized Kähler form on \( \mathbb{P}^1 \) and \( h \) an invariant smooth hermitian metric on \( \mathcal{O}(m) \). The metrics \( \omega \) and \( h \) induce a \( L^2 \)-scalar product \( (\cdot, \cdot)_{\omega,h} \) on \( \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) given as follows

\[
(s,t)_{\omega,h} = \int_{\mathbb{P}^1} h(s(x), t(x)) \omega(x),
\]

for \( s, t \in \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \). The Cauchy-Riemann operator \( \overline{\partial}_{\mathcal{O}(m)} : \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \to \mathcal{A}^{(0,1)}(\mathbb{P}^1, \mathcal{O}(m)) \) has an adjoint for the \( L^2 \)-scalar product, i.e there is a map \( \overline{\partial}_{\mathcal{O}(m)}^* : \mathcal{A}^{(0,1)}(\mathbb{P}^1, \mathcal{O}(m)) \to \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) such that \( (s, \overline{\partial}_{\mathcal{O}(m)}^* t)_{\omega,h} = (\overline{\partial}_{\mathcal{O}(m)} s, t)_{\omega,h} \) for any \( s \in \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) and \( t \in \mathcal{A}^{(0,1)}(\mathbb{P}^1, \mathcal{O}(m)) \). The operator \( \Delta_{\omega,h} := \overline{\partial}_{\mathcal{O}(m)}^* \overline{\partial}_{\mathcal{O}(m)} \) on \( \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) is called the Laplace operator. We denote by \( H_2 \) the completion of \( \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) with respect to \( \| \cdot \|_2 \) defined as follows

\[
\| s \|^2_2 = \int_{\mathbb{P}^1} h(s(s), s) + \frac{i}{2\pi} \int_{\mathbb{P}^1} h \frac{\partial s}{\partial \overline{\partial}^*} \frac{\partial s}{\partial \overline{\partial}} dz \wedge d\overline{z} \quad \forall s \in \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)).
\]

Recall that \( \Delta_{\omega,h} \) admits a maximal and positive self-adjoint extension to \( H_2 \) and has a discrete, infinite and positive spectrum.

In \([2]\), we associated to the metrics \( \omega_{\text{can}} \) and \( h_{m,\infty} \) a singular Laplace operator \( \Delta_{\mathcal{O}(m)_{\infty}} \) which extends the definition of the classical one, and we showed that this operator has the same properties as in the classical situation. More precisely, we established that \( \Delta_{\mathcal{O}(m)_{\infty}} \) admits a maximal positive and self-adjoint extension to \( H_2 \) see \([2]\) theorem 0.3] and has a discrete, infinite and positive spectrum \([2]\) theorem0.4. Moreover we computed it explicitly.

**Remark 3.1.** Following the notations of this article we set \( \Delta_{\omega_{\text{can}}, h_{m,\infty}} := 2\Delta_{\mathcal{O}(m)_{\infty}} \), since the volume form in \([2]\) was not normalized.

Let \( n \in \mathbb{Z} \) and \( J_n \) the Bessel function of order \( n \). We consider the function \( L_{m,n} \) defined on \( \mathbb{C}^* \) as follows:

\[
L_{m,n}(z) = -z^m \frac{d}{dz} \left( z^{-m} J_n(z) J_{n-m}(z) \right) \quad \forall z \in \mathbb{C}^*.
\]

We have

**Theorem 3.2.** For any \( m \in \mathbb{N} \), \( \Delta_{\omega_{\text{can}}, h_{m,\infty}} \) admits a discrete, positive and infinite spectrum, and

\[
\text{Spec}(\Delta_{\omega_{\text{can}}, h_{m,\infty}}) = \left\{ 0 \right\} \cup \left\{ \frac{\lambda^2}{4} \mid \exists n \in \mathbb{N}, L_{m,n}(\lambda) = 0 \right\}.
\]

If we denote by \( 0 < \lambda_1(\omega_{\text{can}}, h_{m,\infty}) < \lambda_2(\omega_{\text{can}}, h_{m,\infty}) < \ldots \) the invariant eigenvalues of \( \Delta_{\omega_{\text{can}}, h_{m,\infty}} \). Then the set of invariant eigenvalues of \( \Delta_{\omega_{\text{can}}, h_{m,\infty}} \) is equal to \( \left\{ \frac{\lambda^2}{4} \mid L_{m,0}(\lambda) = 0 \right\} \).

**Proof.** See \([2]\) theorems 0.6, 0.7 \(\square\)

Since \( \mathcal{O}(m) \) is endowed with its global sections, an element \( \xi \in \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) can be written in the following form \( f \otimes 1 \) where \( f = \sum_{j=0}^{m} f_j z^j \) and \( f_j \) are smooth functions on \( \mathbb{P}^1 \). Then an invariant element in \( \mathcal{A}^{(0,0)}(\mathbb{P}^1, \mathcal{O}(m)) \) corresponds to a smooth function \( f \) of the previous form invariant under the action of \( S^1 \). To this function \( f \) we associate a smooth function \( \phi_f \) on \( [0,1] \) as follows \( \phi_f(x) = f(\exp(-G(x))) \),
∀ \lambda \in [0,1]. We set \( \mathcal{H}_\omega(x) := h(1,1)(\exp(-G_\omega(x))) \) for any \( x \in [0,1] \) and we consider the following norm on \( \mathcal{C}^\infty([0,1]) \)
\[
\|\varphi\|_2^2 = \int_0^1 \mathcal{H}_\omega(x)|\varphi(x)|^2 \, dx + \int_0^1 \mathcal{H}_\omega(x)|\varphi'(x)|^2 \, dx.
\]

One checks easily that \( \|\delta\|_2 = \|f \otimes 1\|_2 \). We set \( K_{\omega,h} = \{ \varphi \in \mathcal{C}^\infty([0,1]) \mid \|\varphi\| < \infty \} \) and we denote by \( H^2_2 \) the completion of \( K_{\omega,h} \) with respect to \( \|\cdot\|_2 \). This completion doesn’t depend on the choice of the metrics, since it is the restriction of \( \|\cdot\|_2 \) to the space of invariant elements in \( H_2 \) which doesn’t depend on \( \omega \) and \( h \).

We set \( R_{\omega,h} \)
\[
R_{\omega,h}(\varphi) := \frac{\int_0^1 \mathcal{H}_\omega(x)|\varphi(x)|^2 \, dx}{\int_0^1 \mathcal{H}_\omega(x)|\varphi(x)|^2 \, dx} \quad \forall \varphi \in K_{\omega,h} \setminus \{0\}.
\]

We denote by \( 0 = \lambda_0(\omega,h) < \lambda_1(\omega,h) < \lambda_1(\omega,h) \ldots \) the invariant eigenvalues of \( \Delta_{\omega,h} \) then by the Min-Max principle,
\[
\lambda_j(\omega,h) = \inf_{\varphi \in K_{\omega,h,j} \setminus \{0\}} R_{\omega,h}(\varphi) \quad \text{for } j \in \mathbb{N}_{\geq 1}.
\]

where \( K_{\omega,h,j} \) is the orthogonal to the subspace of \( H^2_2 \) spanned by the eigenfunctions associated to \( \lambda_k(\omega,h) \) for \( k = 0, \ldots, j - 1 \).

**Theorem 3.3.** Suppose that \( 1 < \frac{1}{\mathcal{H}_m} < \frac{1}{\mathcal{H}_m} \). Then
\[
\lambda_j(\omega,h) \leq \lambda_j(\omega_{\text{can}},h_{m,\infty}) \quad \forall j \in \mathbb{N}_{\geq 1}.
\]

**Proof.** Suppose that \( 1 < \frac{1}{\mathcal{H}_m} < \frac{1}{\mathcal{H}_m} \). Then,
\[
R_{\omega,h}(\varphi) \leq \frac{\int_0^1 \mathcal{H}_{m,\infty}(x)|\varphi(x)|^2 \, dx}{\int_0^1 \mathcal{H}_{m,\infty}(x)|\varphi(x)|^2 \, dx} \quad \forall \varphi \in K \setminus \{0\}.
\]

Notice that
\[
R_{\omega_{\text{can}},h_{m,\infty}}(\varphi) = \frac{\int_0^1 \mathcal{H}_{m,\infty}(x)|\varphi(x)|^2 \, dx}{\int_0^1 \mathcal{H}_{m,\infty}(x)|\varphi(x)|^2 \, dx}, \quad \forall \varphi \in K \setminus \{0\} \quad \text{(see the notations of 2.2)}
\]

By the monotonicity principle and the theorem 3.2, we obtain
\[
\lambda_j(\omega,h) \leq \lambda_j(\omega_{\text{can}},h_{m,\infty}) \quad \forall j \in \mathbb{N}_{\geq 1}.
\]

In particular, when \( m = 0 \) and \( h = h_{0,\infty} \) is the constant metric on \( \mathcal{O} \), the theorem becomes
\[
\lambda_j(\omega,h) \leq \lambda_j(\omega_{\text{can}},h_{0,\infty}) \quad \forall j \in \mathbb{N}_{\geq 1}
\]
and since \( \{\lambda_k(\omega_{\text{can}},h_{0,\infty}), k \in \mathbb{N}_{\geq 1}\} = \left\{ \frac{\alpha^2}{2} \mid J_0(\alpha)J'_0(\alpha) = 0 \right\} \). Then we recover the result of [1] theorem 2.
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