Testing a dissipative kinetic k-essence model

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In this work, we present a study of a purely kinetic k-essence model, characterized basically by
a parameter α in presence of a bulk dissipative term, whose relationship between viscous pressure
Π and energy density ρ of the background follows a polytropic type law Π ∝ ρ^{λ+1/2}, where λ, in
principle, is a parameter without restrictions. Analytical solutions for the energy density of the
k-essence field are found in two specific cases: λ = 1/2 and λ = (1 − α)/2α, and then we show
that these solutions possess the same functional form than the non-viscous counterpart. Finally,
both approach are contrasted with observational data from type Ia supernova, and the most recent
Hubble parameter measurements, and therefore, the best values for the parameters of the theory
are founds.

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I. INTRODUCTION

At present, the scientific community dedicated to the
study of the universe have deep and intriguing questions
unanswered. One of the most fascinating corresponds to
what we know as dark energy (DE) [1,2], a component
designed to explain the current acceleration in the ex-
pansion of the universe. In its simplest form, this can be
described by a perfect fluid with constant energy density,
which leads to the useful Λ – cold dark matter (ΛCDM)
model, the simplest model that fits a varied set of ob-
servational data. However, this model has a high de-
pendence to initial conditions that makes it unnatural
in many ways. For example, the current value for ΩΛ
and ΩDM are of the same order of magnitude, a fact
highly improbable, because the dark matter (DM) con-
tribution decreases with a−3, with a(t) the scale factor,
meanwhile the cosmological constant remains constant.
This problem in particular is known as the cosmic co-
incidence problem. It is for this reason that many of
the most sophisticated experiments and instruments have
been put in place; as the Dark Energy Survey (DES) [3],
the Baryon Oscillation Spectroscopic Survey (BOSS) [4],
and the upcoming Large Synoptic Survey Telescope
(LSST) [5] to mention some, all of them trying to find
new insights into the nature of dark energy. In this con-
text, the most natural way to understand the acceleration
of the universe, is to assume the existence of a dynamical
cosmological constant, or a theoretical model with a
dynamical equation of state parameter (p/ρ = w(z)). The
source of this dynamical dark energy could be both, a new
field component filling the universe, as a quintessence
scalar field [6,15], or it can be produced by modifying
gravity [16,22]. In this work, the so-called k-essence
model [23,24] is used, which is a type of dynamical cos-
mological constant model, but where the source of its
dynamics comes from a non trivial kinetic term, as op-
posite to the case of a typical quintessence model where
the source is a different scalar field potential, and then
put it into the test with current observational data from
both, type Ia supernovae [25], and the most update Hub-
ble parameter measurements [26].

Besides, if we focus, for example, in the dark sector as
a whole, it has been proved that the division of this sec-
tor into DM and DE is merely conventional since exist a
degeneracy between both components, resulting from the
fact that gravity only measures the total energy tensor
[27] (see also [28–34]). So, in the lack of a well confirmed
detection (nongravitational) of the DM only the overall
properties of the dark sector can be inferred from cosmo-
ological data, at the background and perturbative level.
This results has driven the research to explore alternative
models which consider a single fluid that behaves both
as DE and DM, the called unified DM models (UDM).
So this fluid must drive both the accelerated expansion
of the Universe at late times and the formation of struc-
tures (see [35] for a review of these models). Of course,
a small speed of sound should be an essential character-
istic of a viable unified model in order to do not impede
the structure formation and to have a ISW effect signal
compatible with CMB observation [36,37].

In this present work we will consider UDM models de-
derived in the framework of k-essence fields, common in
effective field theories arising from string theory and in
particular in D-branes models [42,46]. This generalization
of the canonical scalar fields models can give rise to
new dynamics not possible in quintessence. In the con-
text of cosmology, k-essence was first studied as a model
for inflation (k-inflation) [47]. K-essence models has also
addressed the problems of a dynamical DE [48,49] and
the coincidence problem [50,51]. For example, a partic-
ular case is the Generalized Chaplygin gas (GCG) which

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appears as the simplest tachyon field model, introducing in \[52\], with a constant potential. Moreover, k fields leads to a new Chaplygin gases. Within the models investigated in order to unify DE and DM are the GCG \[53, 60\] and those known as purely kinetic models \[61, 62\]. The unification of DE, DM and inflation has been addressed in \[63, 64\].

Another issue that emerges from the cosmological data is that the exotic behavior of the universal fluid can be characterized by a negative pressure and usually represented by the equation of state \( w = p/\rho \), where \( w \) lies very close to \(-1\), most probably being below \(-1\). For example, the last Planck results give \( w = -1.13^{+0.13}_{-0.14} \) and \( w = -1.090.17 \) (95\%CL) by using CMB combined with BAO and Union2.1 data \[65\], respectively, for a constant \( w \) model. In combination with SNLS3 data and \( H_0 \) measurement, the EoS for this dark component are \( w = -1.13^{+0.13}_{-0.14} \) and \( w = -1.24^{+0.18}_{-0.19} \) (2\(\sigma\)CL), respectively. The possibility of \( w < -1 \) is favored at the 2\(\sigma\) level. These results are indicating that a phantom behavior of the dark energy component can not ruled out from current cosmological data.

As it was pointed out in \[60\] dark energy with a constant EoS \( w < -1 \) leads to uncommon cosmological scenarios. First of all, there is a violation of the dominant energy condition (DEC), since \( \rho + p < 0 \). The energy density grows up to infinity in a finite time, which leads to a big rip, characterized by a scale factor blowing up in this finite time. Nevertheless, sudden future singularities are not necessarily produced by a fluids violating DEC. Solutions which develop a big rip singularity at a finite time without violate the strong-energy conditions \[63, 64\]. In the case of isotropic and homogeneous cosmologies, any dissipation process in a FRW cosmology is scalar, and therefore may be modeled as a bulk viscosity within a thermodynamical approach. The bulk viscosity introduces dissipation by only redefining the effective pressure, \( p_{\text{eff}} \), according to \( p_{\text{eff}} = p + \Pi = p - 3\zeta H \), where \( \Pi \) is the bulk viscous pressure, \( \zeta \) is the bulk viscosity coefficient and \( H \) is the Hubble parameter, and \( c = 8\pi G = 1 \) (as in all the work). Since the equation of energy balance is \( \dot{\rho} + 3H(\rho + p) = 0 \), the violation of DEC, i.e., \( \rho + p + \Pi < 0 \) implies an increasing energy density of the fluid that fills the universe, for a positive bulk viscosity coefficient. The condition \( \zeta > 0 \) guarantees a positive entropy production and, in consequence, no violation of the second law of the thermodynamics \[72\].

Some investigations have considered that the viscous pressure can drives the present acceleration of the Universe, so it can be used to eliminate the dark energy component and to formulate unified dark matter model with viscous pressure. In \[73, 74\], for example, cosmological models where the only component is a pressureless fluid with a variable and constant bulk viscosity was confronted with the observational data. Nevertheless, the bulk viscosity induces a large time variation of the gravitational potential at late times which leads to inconsistencies with the integrated Sachs-Wolfe (ISW) effect in such model \[73, 74\]. In order to overcome this problem, Velten & Schwarz \[78\] proposed a model with a viscous cold dark matter and a cosmological constant, which acts driving the accelerated expansion of the Universe. Our aim in this work is to investigate UDM models derived in the framework of k-essence fields which can also present dissipative effects.

Usually k-essence is defined as a quintessence, scalar field \( \phi \) with a non-canonical kinetic energy associated with a Lagrangian \( \mathcal{L} = -V(\phi)F(X) \). In the subsequent calculations, we shall restrict ourselves to the simple k-essence models for which the potential \( V = V_0 = \) constant. We also assume that \( V_0 = 1 \) without any loss of generality. One reason for studying k-essence is that it is possible to construct a particularly interesting class of such models in which the k-essence energy density tracks the radiation energy density during the radiation-dominated era, but then evolves toward a constant-density dark energy component during the matter-dominated era. Such a behaviour can to a certain degree solve the coincidence problem.

We investigate a dark energy model described by an effective minimally coupled scalar field with a non-canonical kinetic term. If for a moment we neglect the part of the Lagrangian containing ordinary matter, the general action for a k-essence field \( \phi \) minimally coupled to gravity is

\[
S = S_G + S_\phi = -\int d^4x\sqrt{-g} \left( \frac{R}{2} + F(\phi, X) \right),
\]

where \( F(\phi, X) \) is an arbitrary function of \( \phi \) that represents the k-essence action and \( X = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \) is the kinetic term. We now restrict ourselves to the subclass of kinetic k-essence, with an action independent of \( \phi \)

\[
S_\phi = -\int d^4x\sqrt{-g} F(X).
\]

Unless otherwise stated, we consider \( \phi \) to be smooth on scales of interest so that \( X = \frac{1}{2}\phi^2 \geq 0 \). The energy-momentum tensor of the k-essence is obtained by varying the action \[3\] with respect to the metric, yielding

\[
T_{\mu\nu} = F_X \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} F,
\]

where the subscript \( X \) denotes differentiation with respect to \( X \). Identifying \[4\] as the energy-momentum tensor of a perfect fluid, we have the k-essence energy density \( \rho \) and pressure \( p \)

\[
\rho = F - 2XF_X, \quad p = -F.
\]
Throughout this paper, we will assume that the energy density is positive so that \( F - 2XF_X > 0 \). The equation of state for the k-essence fluid can be written as \( p = w_{\phi} \rho = (\gamma_{\phi} - 1) \rho \) with \( F > 0 \),

\[
\frac{w_{\phi}}{\rho} = \gamma_{\phi} - 1 = \frac{p}{\rho} = \frac{F}{2XF_X - F}.
\]

\[\text{(6)}\]

**II. THE K-ESSENCE MODEL WITH DISSIPATION**

The Friedman–Lemaître–Robertson–Walker (FLRW) metric for a homogeneous and isotropic flat universe is given by

\[
ds^2 = -dt^2 + a(t)^2 \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right],
\]

where \( a(t) \) is the scale factor and \( t \) represents the cosmic time. In the framework of the first order thermodynamic theory of Eckart [79], the field equations in the presence of bulk viscous stresses yield

\[
\left( \frac{\dot{a}}{a} \right)^2 = H^2 = \frac{\rho}{3},
\]

\[\text{(8)}\]

\[
\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6} (\rho + 3p_{vff}),
\]

\[\text{(9)}\]

where the effective pressure is given by

\[
p_{vff} = p + \Pi,
\]

\[\text{(10)}\]

and

\[
\Pi = -3H\zeta,
\]

\[\text{(11)}\]

is the bulk dissipative pressure and \( \zeta \) the viscosity. In what follows we will assume a power law dependence for the viscosity in terms of the the density

\[
\zeta = \zeta_0 \rho^{\lambda},
\]

\[\text{(12)}\]

where \( \zeta_0 \) is a positive semi-definite constant with dimension \( M^{1-\lambda} L^{3\lambda-1} T^{-1} \), and \( \lambda \) may take any value. For example, the most common values are \( \lambda = 1/2 \), i.e., \( \zeta \propto \rho^{1/2} \) [80 83] and \( \lambda = 1 \), i.e., \( \zeta \propto \rho \) [83 80]. These values were chosen because leads to well known analytic solutions. Therefore, the conservation equation for the fluid can be written as

\[
\dot{\rho} + 3H(\rho + p + \Pi) = 0.
\]

\[\text{(13)}\]

In this work we consider the following function \( F \) for the k-essence field [52]

\[
F(X) = \frac{1}{2\alpha - 1} [X^{\alpha} - 2\alpha \alpha_0 \sqrt{X}],
\]

\[\text{(14)}\]

where \( \alpha \) and \( \alpha_0 \) are two real constants. This generating function exhibits a transition from a power law phase to a de Sitter stage, inducing a modified Chaplygin gas. The explicit equation of state can be obtained from Eqs. (11) and (12)

\[
p = (n - 1)\rho - n \alpha_0 \rho^{\frac{n-1}{n}},
\]

\[\text{(15)}\]

where the parameter \( n \) is a function of the constant \( \alpha \), given by

\[
n = \frac{2\alpha}{2\alpha - 1}.
\]

\[\text{(16)}\]

Obviously, the range of this parameter is \( 1 > n > 0 \), if \( -\infty < \alpha < 0 \); \( 0 > n > -\infty \), if \( 0 < \alpha < 1/2 \); and \( \infty > n > 1 \), if \( 1/2 < \alpha < \infty \).

Of course, the speed of sound is affected by the viscous pressure, which becomes

\[
v_{sf}^2 = \frac{\partial p_{sf}}{\partial \rho} = v_{\phi}^2 - (\lambda + 1/2) \frac{\|\Pi\|}{\rho},
\]

\[\text{(17)}\]

where \( v_{\phi} \) is the speed of sound in the purely k-essence background [52], given by

\[
v_{\phi}^2 = (n - 1) \left( 1 - \frac{\alpha_0}{\rho^{1/n}} \right).
\]

\[\text{(18)}\]

From Eqs. (8 - 14), together with the EoS (15), we obtain the evolution equation for \( H \) in terms of the redshift,

\[
- a_0 \frac{dH}{dx} + a_1 H + a_2 H^{\gamma - 1} + a_3 H^{\beta - 1} = 0,
\]

\[\text{(19)}\]

where \( x = \ln(1 + z) \), and the coefficients are given by

\[
a_0 = 2, \quad a_1 = 3n, \quad a_2 = -3^{n-1} \alpha_0 n, \quad a_3 = -3^{\lambda + 1} \zeta_0,
\]

\[\text{(20)}\]

whereas the exponents reads

\[
\eta = 2 \left( \frac{n - 1}{n} \right) = \frac{1}{\alpha}, \quad \beta = 2\lambda + 1.
\]

\[\text{(21)}\]

As a first observation, we note that there are two special values that yields to well know equation without viscosity [52]: \( \beta = 2 \) (\( \lambda = 1/2 \)) and \( \beta = \eta \) (\( \lambda = 1/2 \alpha \neq 1/2 \)). These values leads to a single equation which posses a generic structure for its quadrature given by

\[
\frac{dH}{dx} = A_1 H + A_2 H^{\gamma - 1} \equiv A_1 (H + y H^{\eta - 1}),
\]

\[\text{(22)}\]

where \( y \equiv A_2 / A_1 \), and the new coefficients are given in terms of the above by the following expressions

\[
A_1 = \frac{a_1 + a_3}{a_0} = \frac{3}{2} (n - \sqrt{3} \zeta_0),
\]

\[\text{(23)}\]

\[
A_2 = \frac{a_2}{a_0} = -\frac{3^{n-1} \alpha_0 n}{2},
\]

\[\text{(24)}\]

\[
y = \frac{a_2}{a_1 + a_3} = \frac{\alpha_0 n}{3^{\frac{1}{2}} (\sqrt{3} \zeta_0 - n)},
\]

\[\text{(25)}\]
for \( \lambda = 1/2 \) (the model A), whereas
\[
A_1 = \frac{a_1}{a_0} = \frac{3}{1 - 2\lambda},
\]
\[
A_2 = \frac{a_2 + a_3}{a_0} = -\frac{3^{1+\frac{1}{n}}}{2} \left( 2\alpha_0 - 2\lambda + \sqrt{3}\zeta_0 \right),
\]
\[
y = \frac{a_2 + a_3}{a_1} = \frac{3^{1+\frac{1}{n}}}{2} \left( \sqrt{3}\zeta_0 (2\lambda - 1) - 2\alpha_0 \right).
\]
for \( \lambda \neq 1/2 \) (the model B). So, a direct integration of Eq. (22) leads to
\[
H(z) = H_0 \left[ \frac{1 + z^{2\lambda}}{1 - A_3} \right]^{\frac{1}{2}},
\]
and therefore, the energy density is given by
\[
\rho(z) = 3H_0^2 \left[ \frac{1 + z^{2\lambda}}{1 - A_3} \right]^n,
\]
where we have defined
\[
A_3 = \frac{\mathcal{R}}{1 + \mathcal{R}}, \quad \left( \mathcal{R} = \frac{y}{H_0} \right).
\]

We note that the generic expression (30) (or Eq. (29)) has the form found by Chimento (22), and obviously, these case is entirely recuperated by making \( \zeta_0 \to 0 \) and \( \lambda \to 0 \). A second observation is that, in the case \( \lambda \neq 1/2 \) and by using Eqs. (21) and (26), the expression (29) takes the form
\[
H(z) = H_0 \left[ \frac{(1 + z)^3}{1 - A_3} \right]^{-\frac{1}{2\alpha}}.
\]
Finally, there is a future singular value of the redshift, say \( z_s \), for which the Hubble function takes its zero value:
\[
z_s = A_3^{\frac{2}{3\sqrt{3}}} - 1.
\]

We are restricted to the realistic values for the future singularity, so we expect that \(-1 < z_s < 0 \). Thus, this condition impose that \( y > 0 \), which implies that \( n < 1/\sqrt{3}\zeta_0 \) if \( \lambda = 1/2 \), and \( \lambda > 1/2 + \alpha_0 / (\sqrt{3}\zeta_0) \) if \( \alpha = (2\lambda + 1)^{-1} \). In this context, notice that \( \lambda = 1 \) (i.e. \( \alpha = 1/3 \)) leads to the condition 2 \( \alpha_0 < \sqrt{3}\zeta_0 \).

This kind of future singularity corresponds to a novel type, because although both the Hubble parameter (29) and the energy density (30) vanish at this redshift.

### III. OBSERVATIONAL CONSTRAINTS

In this section we use observational data to put some constraints in the free parameters of the models. We use type Ia supernova data, specifically the Union 2 data set and the most recent Hubble parameter \( H(z) \) measurements compiled in (87), consisting in 28 data points expanding a range in redshift \( 0.015 < z < 2.3 \).

The comoving distance from the observer to redshift \( z \), in a flat universe, is given by
\[
r(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')},
\]
where \( E(z) = H(z)/H_0 \). The SNIa data give the luminosity distance \( d_L(z) = (1 + z)r(z) \). Notice that the procedure we follow differ from those used by Bandyopadhyay et al. (89). In this work the authors define an intermediate parametrization for the luminosity distance as a function of two parameters \( \alpha, \beta \) which after the fitting is related to the physical parameters of the model. Here we constrain directly the physical parameters of the model.

We fit the SNIa with the cosmological model by minimizing the \( \chi^2 \) value defined by
\[
\chi^2_{SNIa} = \sum_{i=1}^{557} \frac{[\mu(z_i) - \mu_{obs}(z_i)]^2}{\sigma_{\mu i}^2},
\]
where \( \mu(z) \equiv 5 \log_{10}[d_L(z)/\text{Mpc}] + 25 \) is the theoretical value of the distance modulus, and \( \mu_{obs} \) is the corresponding observed one.

From (30) we can write down explicitly
\[
E(z) = \left[ \frac{(1 + z)^{2\lambda}}{1 - A_3} \right]^{-\frac{1}{2\alpha}}.
\]
This form of the solution enable us to test both models at the same time by reinterpreting the constants values. The best fit values using both SNIa and \( H(z) \) data leads to an \( \chi^2_{red} \approx 0.96 \), and \( A_1 = 1.50 \pm 0.15 \), \( \eta = -0.29 \pm 0.19 \), and \( A_3 = -2.9 \pm 0.5 \).

For the case \( \lambda = 1/2 \), the free parameters are three: \( A_1 \), the parameter that changes with the model, \( \eta \), which is defined in (21), and \( A_3 \), defined in (31). Straightforward calculations lead to \( n = 0.87 \pm 0.07 \), \( \zeta_0 = -0.075 \pm 0.069 \), and \( \alpha_0 = (2.96 \pm 2.6) \times 10^{-4} \). Note that for the case \( \lambda = 1/2 \) and since we have taken \( G = 1/8\pi \) and \( e = 1 \), it is straightforward to see that parameter \( \zeta_0 \) is dimensionless.

Since the exponent in (30) reduces to \( 2\lambda = 3 \), in the case \( \lambda \neq 1/2 \) the free parameters reduce to \( A_3 \) and \( \eta \). For this reason, is not possible to invert the equations completely, because this model is described by three parameters, \( \zeta_0, \alpha_0 \) and \( \lambda \). In fact, from the best fit, we can write down directly the value for \( \lambda = -0.65 \pm 0.08 \). The other two parameters are tightly related through the relation
\[
\alpha_0 = -\frac{y}{3\lambda^{-1/2}} + 2\sqrt{3}(2\lambda - 1)/\zeta_0.
\]
Because the best value for parameter \( y \) is large compared with the second term in the right hand side (for reasonable positive values of \( \zeta_0 \)), the value for \( \alpha_0 \) is largely better constrained than \( \zeta_0 \).
In order to make manifest the quality of the fit of our models, in Figure (1) we show the theoretical curves of best fit for each model together with the observational data of $H(z)$. There we show the 28 data points measurements of the Hubble parameter together with the best theoretical fit. We have to notice that although the lines does not seems to follow the observational points very well, this is because the best fit model was computed using both SNIa data and $H(z)$ measurements, and the first data set statistically weighs more than the second one, just because of the number of data in each case. We also display in figure (2) the confidence level contours for the parameters $\eta$ and $A_3$ at one and two $\sigma$, and in figure (3) the confidence contours for the model B parameters.

**FIG. 1:** Using the values of the best fit for each model, here we display the theoretical curve of each model along the observational data for $H(z)$. The continuous line is model A, and the dashed line describe model B. We have adopted $h = 0.673$ from the Planck Collaboration [88].

In both cases, because the analysis was performed without imposing external priors on the parameters, we found a preference for nearly zero to negative values for the viscosity constant $\zeta_0$.

Despite the strange results – a negative value for the viscosity constant – after put in tension our solutions with the data, we have confident that such a analysis can be done in the first place for any other analytical solution that can be obtained in the future. Of course, we do not expect to find that just our special (analytical) solutions be the best fit to the data immediately. Cosmology has entered into the era of precision cosmology, and with it, the possibility to rule out effectively a particular cosmological model.

**IV. CONCLUSIONS**

We have analysed the general relations for a model of k-essence generated by the function $F(X) = \frac{1}{2\alpha-1}(X^{\alpha} - 2\alpha\sqrt{X})$ (proposed by Chimento [52]), when a dissipative pressure $\Pi \propto \rho^{\lambda+1/2}$ is included. We found a family of analytical solutions in two special cases: $\lambda = 1/2$ and $\lambda = (1 - \alpha)/2\alpha$ (with $\alpha \neq 1/2$), which coming from a similar differential equations and posses the same structure that the non-viscous case (compare, for example, Eq. (69) in reference [52] with Eq. (61)).

Also, a quick observation of Eq. (17) shows that, depending on the value of $\lambda$, the speed of sound may be greater ($\lambda < -1/2$), equal ($\lambda = -1/2$) or less ($\lambda > -1/2$) than the speed of sound without viscosity. Obviously, a well behaved fluid requires $\lambda \geq -1/2$, which corresponds to a consistency relation for $\lambda$.

As a light of observational data, we confront both analytical solution with measurements of $H(z)$ and supernovas. The best fit yields the following values for the parameters: $\chi^2_{red} \simeq 0.96$, $A_1 = 1.50 \pm 0.15$, $\eta = -0.29 \pm 0.19$, and $A_3 = -2.9 \pm 0.5$. Therefore, we obtain for the model A that $n = 0.87 \pm 0.07$, $\zeta_0 = -0.075 \pm 0.069$, and $\alpha_0 = (2.96 \pm 2.6) \times 10^{-4}$, while for the model B we obtain $\lambda = -0.65 \pm 0.08$, and the other two parameters are tightly related through Eq. (57). So, both model present a controversy with the physical meaning ($\zeta_0 < 0$ in the model A and $\lambda < -1/2$ in the model B).

This work can be improved in many ways. On one hand, we can attempt an alternative way to obtain analytical solutions, a possibility we are already studying using a novel technique proposed to solve complex differential equations [90–95]. In this case, we have the possibility to consider $\lambda$ a free parameter, enhancing the parameter space to find a best fit with data.

Certainly a more realistic model would also be interesting to study. In this work we have considered a UDM model assuming nothing else but a k-essence field is present. We can add explicitly a dark matter term and/or a radiation component. We are interested in testing if adding these terms would alleviate our concerns about the sign of the viscosity coefficient.

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FIG. 2: Here we display the 68.27% and 95.45% confidence regions for the parameters $A_1$, $A_3$ and $\eta$ for model A.
FIG. 3: Here we display the 68.27% and 95.45% confidence regions for the parameters $A_3$ and $\eta$ for model B.