Partially ordered groups and geometry of contact transformations

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November 17, 2018

Abstract

We prove, for a class of contact manifolds, that the universal cover of the group of contact diffeomorphisms carries a natural partial order. It leads to a new viewpoint on geometry and dynamics of contactomorphisms. It gives rise to invariants of contactomorphisms which generalize the classical notion of the rotation number. Our approach is based on tools of Symplectic Topology.

Dedicated to D.B. Fuchs on the occasion of his 60th birthday

1 Introduction and main results

1.1 Partially ordered groups

Let $\mathcal{D}$ be a group. A subset $C \subset \mathcal{D}$ is called a normal cone if

(i) $f \in C, g \in C \Rightarrow fg \in C$

(ii) $f \in C, h \in \mathcal{D} \Rightarrow hfh^{-1} \in C$

(iii) $1 \in C$

*Supported by the US-Israel Binational Science Foundation grant 94-00302.
Given a normal cone \( C \subset D \), one defines a relation \( f \geq g \) on \( D \) by

\[
f \geq g \text{ if } fg^{-1} \in C.
\]

Clearly this relation is reflexive (\( f \geq f \)) and transitive

\[
(f \geq g, g \geq h \Rightarrow f \geq h).
\]

If it is also anti-symmetric (\( f \geq g, g \leq f \Rightarrow f = g \)) then it is a partial order on \( D \). We call it a bi-invariant partial order induced by \( C \). Notice that the normality of the cone \( C \) implies that if \( f_1 \geq g_1 \) and \( f_2 \geq g_2 \) then \( f_1f_2 \geq g_1g_2 \).

Let us describe now a way to extract numerical invariants from a bi-invariant partial order on \( D \). An element \( f \in C \setminus \{1\} \) is called a dominant if for every \( g \in D \) there exists a number \( p \in \mathbb{N} \) such that \( f^p \geq g \). For a dominant \( f \) and any \( g \in D \) set

\[
\gamma_k(f,g) = \inf\{p \in \mathbb{Z} \mid f^p \geq g^k\}, \quad \text{where } k \in \mathbb{N}.
\]

Notice that

1. The number \( \gamma_k(f,g) \) is finite, and
2. The limit \( \gamma(f,g) = \lim_{k \to +\infty} \frac{\gamma_k}{k} \) exists.

Indeed, choose \( q \in \mathbb{N} \) such that \( f^q \geq g \). If \( f^p \geq g \) then \( g^{-k} \geq f^{-p} \), so \( f^{kq} \geq f^{-pq} \) and \( p \geq -kq \). Hence \( \gamma_k \geq -kq \) and it is finite, which proves (i). Since \( f^\gamma \geq g^\alpha \), \( f^{\gamma + \gamma} \geq g^{\alpha + \beta} \), we conclude that \( \gamma_{m+n} \leq \gamma_m + \gamma_n \), i.e., the sequence \( \gamma_k \) is subadditive. Consider now the sequence \( u_k = \gamma_k + kq \) which, as we just showed, is non-negative. Clearly it is also subadditive. This implies existence of the limit \( \lim_{k \to +\infty} \frac{u_k}{k} \), and hence of the limit (ii).

We will call \( \gamma(f,g) \) the relative growth number of \( f \) with respect to \( g \). Notice that the real number \( \gamma(f,g) \) can be of any sign, or equal to 0.

If both \( f \) and \( g \) are dominants, then \( \gamma(g,f) \) is also defined, and the following inequality holds:

\[
(1.1.A) \quad \gamma(g,f)\gamma(f,g) \geq 1.
\]

Indeed, set \( \alpha_k = \gamma_k(f,g) \) and \( \beta_k = \gamma_k(g,f) \). Then we have \( f^{\alpha_k} \geq g^k \) and \( g^{\beta_k} \geq f^k \). Hence,

\[
g^{\alpha_k\beta_k} \geq f^{\alpha_k\beta_k} \geq g^{k^2}.
\]

Since \( g \) is a dominant this implies that \( \alpha_k\beta_k \geq k^2 \). Dividing by \( k^2 \) both parts of this inequality and passing to the limit when \( k \to +\infty \) we get the required inequality 1.1.A.
1.2 The universal cover of the group of contactomorphisms

Let \((M, \xi)\) be a closed connected contact manifold with a co-oriented contact structure. Let us denote by \(\Gamma(M, \xi)\) the identity component of the group of contactomorphisms of \((M, \xi)\), and by \(\theta : \mathcal{D}(M, \xi) \to \Gamma(M, \xi)\) the universal cover of \(\Gamma(M, \xi)\) associated with the basepoint 1. Our starting observation is that \(\mathcal{D}(M, \xi)\) carries a natural normal cone. Let \((SM, \omega)\) be the symplectization of the contact manifold \((M, \xi)\). Let us remind that \(SM\) is the total space of a \(\mathbb{R}_+\)-subbundle of the cotangent bundle \(T^*M\), which is formed by contact forms compatible with the given co-orientation of \(\xi\). The symplectic structure \(\omega\) on \(SM\) is the restriction of the canonical symplectic form \(d(pdq)\) of the cotangent bundle. \(SM\) also carries a canonical conformally symplectic \(\mathbb{R}_+\)-action. Every contactomorphism \(\varphi \in \Gamma\) uniquely lifts to a \(\mathbb{R}_+\)-equivariant symplectomorphism \(\tilde{\varphi}\) of \(SM\), and conversely each \(\mathbb{R}_+\)-equivariant symplectomorphism of \(SM\) projects to a contactomorphism of \((M, \xi)\). A function \(F : SM \to \mathbb{R}\) is called a contact Hamiltonian if it is homogeneous of degree 1, that is \(F(cx) = cF(x)\) for all \(c \in \mathbb{R}_+, x \in SM\). The Hamiltonian flow, generated by a time-dependent contact Hamiltonian, is \(\mathbb{R}_+\)-equivariant, and thus defines a contact isotopy of \((M, \xi)\). Any contact isotopy \(\{\varphi_t\}\) is generated in this sense by a uniquely defined time-dependent contact Hamiltonian \(\Phi_t : SM \to \mathbb{R}\). The isotopy \(\{\varphi_t\}\) is called non-negative if \(\Phi_t \geq 0\) for all \(t\).

Let us denote the contact vector field \(\frac{d\alpha}{dt}\) by \(X_t\). A contact form \(\alpha\) with \(\xi = \{\alpha = 0\}\) can be viewed as a section \(\tilde{\alpha} : M \to SM\) of the \(\mathbb{R}_+\)-bundle \(SM \to M\), so that \(\tilde{\alpha}^*(pdq) = \alpha\). A contact Hamiltonian \(\Phi_t : SM \to M\) can be pulled back to \(M\) by the section \(\tilde{\alpha}\), and the resulting function \(\bar{\Phi}_t = \Phi_t \circ \tilde{\alpha}_t : M \to \mathbb{R}\) is also often called a contact Hamiltonian of \(\{\varphi_t\}\) with respect to \(\alpha\). \(\bar{\Phi}_t\) can be equivalently defined by the formula

\[
\bar{\Phi}_t(x) = \alpha(X_t(x)).
\]

In other words, \(\bar{\Phi}_t\) measures the transversal to \(\xi\) component of the contact vector field \(X_t\). In particular, the positivity of the deformation \(\varphi_t\) means that the vector field \(X_t\) defines the prescribed co-orientation of \(\xi\). Notice that the contact Hamiltonian with respect to \(\alpha\), which identically equals 1 defines the so-called Reeb flow of the contact form \(\alpha\). The corresponding contact vector field \(X_t\) can be characterized by the equations \(X_t \cdot d\alpha = 0\) and \(\alpha(X_t) = 1\).

Let \(C(M, \xi) \subset \mathcal{D}\) be a set of those \(f \in \mathcal{D}\), which can be represented by a
A non-negative path joining 1 with $\theta(f)$. It is easy to see that $C(M,\xi)$ is a normal cone in $\mathcal{D}(M,\xi)$. We call it the non-negative normal cone in $\mathcal{D}(M,\xi)$.

In the present paper we study the following two problems.

**Problem 1.2.A.** Let $(M,\xi)$ be a closed contact manifold with a co-oriented contact structure. Does the non-negative normal cone $C(M,\xi)$ induces a non-trivial partial order on $\mathcal{D}(M,\xi)$?

**Problem 1.2.B.** Calculate or estimate the relative growth number $\gamma(f,g)$ of a pair of contactomorphisms in geometric or dynamical terms.

The first case when the answer to Problem 1.2.A is positive is provided by the simplest contact manifold $S^1 = \mathbb{R}/\mathbb{Z}$. Its contact structure is just the field of 0-dimensional (!) tangent subspaces, and the co-orientation comes from the orientation of the circle. The group $\mathcal{D}(S^1)$ consists of all orientation-preserving diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(x + 1) = f(x) + 1$, and the normal cone $C(S^1)$ is formed by those $f$ which satisfy $f(x) \geq x$ for all $x \in \mathbb{R}$. Clearly $C(S^1)$ induces a partial order on $\mathcal{D}(S^1)$. Namely, $f \geq g$ provided $f(x) \geq g(x)$ for all $x$. For higher-dimensional manifolds Problem 1.2 requires methods of symplectic topology. The bridge between this problem and symplectic topology is given by the following criterion (see Section 2.1 below for the proof).

**Criterion 1.2.C.** The relation $\geq$ is a non-trivial partial order on $\mathcal{D}(M,\xi)$ if and only if there are no contractible loops of contactomorphisms of $(M,\xi)$ generated by a strictly positive time-periodic contact Hamiltonian.

This criterion can be checked for a class of contact manifolds. For instance, for the standard contact structure on $\mathbb{R}P^{2n+1}$ this is an immediate consequence of Givental’s non-linear Maslov index theory [G] (see 1.3.C below). In Section 2 we derive the absence of contractible loops generated by strictly positive time-periodic Hamiltonians from the Lagrangian intersection theory along the lines of [P1, Lemma 3.B]. This enables us to get the positive answer to Problem 1.2.A for spaces of co-oriented contact elements of certain manifolds, as well as some prequantization spaces (see Section 1.3 below for precise formulations). In Section 1.8 we give a reformulation of Problem 1.2.A in the language of symplectic fibrations. Some potential generalizations of our results are discussed in Section 1.9.
As far as Problem 1.2.B is concerned, the simplest case of the circle $S^1$ indicates that the relative growth can be considered as a contact generalization of the notion of the rotation number of a diffeomorphism of the circle (see Section 1.6 below). Our main results (see Theorem 1.6.E and its proof in Section 3.4 below) deal with the case when $M$ is the space $\mathbb{P}_+T^*\mathbb{T}^n$ of contact elements of the torus $\mathbb{T}^n$. Here we relate the relative growth to the stable Gromov-Federer norm and the Mather minimal action. In order to calculate or estimate the relative growth number for $\mathbb{P}_+T^*\mathbb{T}^n$ we use the theory of symplectic and contact shapes developed in [S] and [E1]. This theory provides us with new invariants of contactomorphisms which turn out to be useful in the study of our partial order. The details of this approach are described in Section 3. Finally, the relative growth gives rise to a canonical partially ordered metric space associated with a contact manifold. This construction is presented in 1.7 below. Notice also that in some simple cases (see Example 1.6.C below) the computation of the invariant $\gamma(f,g)$ is straightforward if the positive answer to Problem 1.2.A is known.

1.3 Main examples

Here we list examples of contact manifolds $(M,\xi)$ for which we can prove that the non-negative normal cone induces the non-trivial partial order on $\mathcal{D}(M,\xi)$.

1.3.A. Spaces of co-oriented contact elements. Let us recall that the space of co-oriented contact elements, or, in other words, the positive projectivization of a cotangent bundle $\mathbb{P}_+T^*Y$ of any smooth manifold $Y$ carries a canonical contact structure whose symplectization coincides with the cotangent bundle $T^*Y$ with the deleted 0-section. The next result is proved in 2.4 below.

**Theorem 1.3.B.** If a closed manifold $Y$ admits a non-singular closed 1-form then the non-negative normal cone induces the non-trivial partial order on $\mathcal{D}(\mathbb{P}_+T^*Y)$.

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$^1$Let $E$ be a real vector space. We say that two vectors $e_1, e_2 \in E$ are equivalent if there exists $\lambda > 0$ such that $e_1 = \lambda e_2$. The set of all equivalence classes (or, in other words, the set of all oriented lines in $E$) is called the positive projectivization of $E$ and denoted by $\mathbb{P}_+E$. The definition extends in an obvious way to vector bundles.
1.3.C. **Prequantization spaces.** Given a closed symplectic manifold $(W, \Omega)$ with the integral cohomology class $[\Omega]$, consider a principal $S^1$-bundle $QW \to W$ whose first Chern class equals $[\Omega]$. This bundle admits an $S^1$-connection whose curvature form equals $\Omega$. The distribution of the horizontal spaces of this connection is an $S^1$-invariant contact structure on $QW$ transversal to the fibers. This contact manifold is called a prequantization space of $(W, \Omega)$. Note that a given manifold $(W, \Omega)$ may admit different (in any reasonable category) prequantization spaces. We refer to [Ki] for the survey on prequantization.

**Theorem 1.3.D.** Suppose that $(W, \Omega)$ has a closed Lagrangian submanifold $L$ with the following properties:

- the connection on $QW$ is flat over $L$ (the Bohr-Sommerfeld condition);
- the relative homotopy group $\pi_2(M, L)$ vanishes.

Then the non-negative normal cone induces the non-trivial partial order on $\mathcal{D}(M, \xi)$.

For instance such a $L$ exists when $(W, \Omega)$ is the standard symplectic torus $(\mathbb{R}^{2n}, dp \wedge dq)/\mathbb{Z}^{2n}$.

1.3.E. **Real projective space.** The standard contact $\mathbb{RP}^{2n+1}$ is a particular case of the prequantization space $QW$. Namely take $W = \mathbb{CP}^n$ and let $\Omega$ be the Fubini-Studi form normalized in such a way that its integral over the projective line equals 2. Since $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$ this situation is not covered by our previous result. Nevertheless the relation $\geq$ is a genuine partial order on $\mathcal{D}(\mathbb{RP}^{2n+1})$. This follows from [G]. In [G] Givental introduces an invariant $m(f)$ of an element $f \in \mathcal{D}(\mathbb{RP}^{2n+1})$ called the asymptotic non-linear Maslov index. Represent $f$ as the time one map of a Hamiltonian flow $\{f_t\}$ generated by a time periodic contact Hamiltonian $F$. Intuitively speaking the number $m(f)$ is defined as the density in $\mathbb{R}_+$ of the set of periods of certain closed orbits of the flow $\{f_t\}$. It is proved in [G] that $m(f) > 0$ provided $F$ is strictly positive. On the other hand, $m(1) = 0$. Combining this with 1.2.C above we get the partial order on $\mathcal{D}(\mathbb{RP}^{2n+1})$.

1.4 Dominants

Here we discuss the notion of dominants (see Section 1.1 above) in the context of contactomorphisms. In turns out that the group $\mathcal{D}(M, \xi)$ admits a
natural class of dominants. Let us denote by $C^+(M, \xi) \subset C(M, \xi) \subset D(M, \xi)$ the set of all elements which can be generated by a strictly positive contact Hamiltonian.

**Proposition 1.4.A.** Any $f \in C^+(M, \xi)$ is a dominant.

This is an immediate consequence of the following elementary statement.

**Proposition 1.4.B.** Assume that $f, g \in D(M, \xi)$. Then $f \geq g$ if and only if these elements can be generated by contact Hamiltonians $F$ and $G$ with $F \geq G$. Moreover if $f \geq g$ then the Hamiltonians $F$ and $G$ can be chosen to be time-periodic.

**Proof:**

1) Assume $f \geq g$. Then $g f^{-1}$ can be represented by a path $h_t$ generated by a non-positive contact Hamiltonian $H$. Suppose that $g_t$ generated by $G$ represents $g$. Then set $f_t = h_t^{-1} g_t$. This path represents $f$ and is generated by $F(x, t) = -H(h_t x, t) + G(h_t x, t)$. Clearly $F \geq G$. The periodicity can be achieved by a suitable time reparametrization near $t = 0$ and $t = 1$.

2) Assume that $f_t, g_t$ represent $f$ and $g$ and are generated by $F$ and $G$ respectively. Then $g_t^{-1} f_t$ is generated by a non-negative Hamiltonian provided $F \geq G$.

\[\Box\]

**1.5 Calculation of the relative growth**

In view of Proposition 1.4.A for every $f \in C^+(M, \xi)$ and $g \in D(M, \xi)$ one can define the relative growth $\gamma(f, g)$. The calculation of the relative growth seems to be a non-trivial problem. In Section 3 below we present an approach to this problem in the case when $(M, \xi)$ is the space of co-oriented contact elements to the $n$-dimensional torus. The approach is based on the theory of symplectic/contact shape developed in [3], [11]. Here is a sample result. Besides an elementary observation in Example 1.6.C this is the only class of multi-dimensional examples where we can precisely calculate the relative growth number.

Let $(p, q)$ be the canonical coordinates on $T^*\mathbb{T}^n$, and let $F(p), G(p)$ be two contact Hamiltonians on $T^*\mathbb{T}^n \setminus \{\text{zero section}\} = S(\mathbb{P}_+ T^*\mathbb{T}^n)$. Assume that $F(p) > 0$ for all $p \neq 0$, and $G(p)$ is strictly positive for some $p \neq 0$. Then one has (see Section 3.3 below for the proof)
Theorem 1.5.A. Let \( f, g \in \mathcal{D}(M, \xi) \) be elements generated by \( F \) and \( G \) respectively. Then \( \gamma(f, g) = \max_{p \neq 0} \frac{G(p)}{F(p)} \).

1.6 Relative growth as the generalized rotation number

Here we present two specifications of the relative growth which can be considered as an extension of the classical notion of the rotation number of a circle diffeomorphism. Before going into details let us explain the main motivation. It is a classical dynamical idea to measure the speed of rotation of the trajectories of a flow around a given cycle in the manifold. Such a measurement can be performed rigorously and proved to be useful in various situations including the circle diffeomorphisms and Hamiltonian dynamics. In particular it is closely related to asymptotical properties of the set of periods of certain closed orbits of the flow. We take a different point of view and consider a flow as a curve on the group of diffeomorphisms. Our suggestion is to measure the rotation of this curve around a cycle in the group! Using the notion of relative growth we can rigorously implement this idea for the group of contactomorphisms. As we will see in some examples below, the results of both measurements (the one we suggest and the classical one) are closely related to each other.

Here are precise definitions. Let \( \Pi \subset \mathcal{D}(M, \xi) \) be the (full) lift of \( 1 \in \Gamma(M, \xi) \), which is identified with the fundamental group \( \pi_1(\Gamma(M, \xi), 1) \). Every element \( f \in C^+(M, \xi) \) gives rise to a function \( f \rightarrow \gamma(f, e) \) defined on \( \Pi \). Let us mention two properties of this function:

\[
\begin{align*}
\gamma(f, e_1 e_2) & \leq \gamma(f, e_1) + \gamma(f, e_2) \quad \text{for all} \quad e_1, e_2 \in \Pi. \\
\end{align*}
\]

The first one is obvious. In order to prove the second property note that the group \( \Pi \) is abelian. Thus the inequalities

\[
\begin{align*}
f^{\gamma_n(f, e_1)} & \geq e_1^n, \quad f^{\gamma_n(f, e_2)} \geq e_2^n \\
\end{align*}
\]

imply

\[
f^{\gamma_n(f, e_1) + \gamma_n(f, e_2)} \geq (e_1 e_2)^n,
\]

See [P4] for some applications of this viewpoint in the context of Hofer’s geometry.
so that
\[ \gamma_n(f, e_1 e_2) \leq \gamma_n(f, e_1) + \gamma_n(f, e_2), \]
and the claim follows.

Here is a cousin of the above construction. Set \( \Pi^+ = \Pi \cap C^+(M, \xi) \). Every element \( e \in \Pi^+ \) gives rise to a function \( f \to \gamma(e, f) \) on \( \mathcal{D}(M, \xi) \). It follows from 1.1.A that if \( f \in C^+ \) and \( e \in \Pi^+ \) then \( \gamma(f, e) \gamma(e, f) \geq 1 \).

The next examples 1.6.B–1.6.D clarify the dynamical meaning of the functions \( \gamma(f, e) \) and \( \gamma(e, f) \).

1.6.B. Diffeomorphisms of the circle. In this case the group \( \Pi \) is isomorphic to \( \mathbb{Z} \). Its generator \( e \) is represented by the loop \( x \to x + t, \quad t \in [0; 1] \). The corresponding contact Hamiltonian, viewed as a function on \( S^1 \) with the contact form \( dx \), identically equals 1. Thus \( e \in \Pi^+ \). Denote by \( \text{Rot}(f) \) the rotation number of \( f \in \mathcal{D}(S^1) \). We claim that
\[ \gamma(e, f) = \text{Rot}(f) = \gamma(f, e)^{-1}. \]

In the first equality \( f \) is arbitrary while in the second one we assume that \( f(x) > x \) for all \( x \in \mathbb{R} \).

Let us prove the second equality. The proof of the first one is absolutely similar. First assume that \( f^{\gamma_k} \geq e^k \) for some \( k \in \mathbb{N} \). Then \( f^{\gamma_k}(x) \geq x + k \) for all \( x \in \mathbb{R} \), so \( \text{Rot}(f) \geq \frac{k}{\gamma_k} \). Passing to the limit we get the inequality \( \gamma(f, e) \geq \text{Rot}(f)^{-1} \).

Let us verify the opposite inequality. Suppose that \( \text{Rot}(f) > \frac{m}{\ell} \) for some \( m, \ell \in \mathbb{N} \). We claim that \( f^\ell(x) \geq x + m \) for all \( x \in \mathbb{R} \) (see [CFS]). Indeed, if this is true for some, but not for all \( x \) then there exists \( x_0 \in \mathbb{R} \) such that \( f^\ell(x_0) = x_0 + m \). But then \( \text{Rot}(f) = \frac{m}{\ell} \), a contradiction. If \( f^\ell(x) < x + m \) for all \( x \in \mathbb{R} \) then \( \text{Rot}(f) \leq \frac{m}{\ell} \), and again we get a contradiction. The claim follows. Thus \( \gamma_m(f, e) \leq \ell \), so \( \frac{\gamma_m(f, e)}{m} \leq \frac{\ell}{m} \). Taking now a sequence \( \frac{m}{\ell} \nearrow \text{Rot}(f) \) we get that \( \gamma(f, e) \leq \frac{1}{\text{Rot}(f)} \). This completes the proof.

1.6.C. Reeb flows on prequantization spaces. Suppose that a contact manifold \((M, \xi)\) admits a contact form \( \alpha \) whose Reeb flow is 1-periodic. Let \( \varphi_t \in \mathcal{D} \) be the lift to the universal cover of this flow, and set \( e = \varphi_1 \).

All prequantization spaces defined in 1.3.C admit a contact form with this property.
If the non-negative normal cone induces a non-trivial partial order on \( \mathcal{D}(M, \xi) \) (comp. 1.3.D and 1.3.E above), then for any \( t \in \mathbb{R} \) we have

\[
\gamma(e, \varphi_t) = t.
\]

Indeed, we have \( \varphi_a \leq \varphi_b \), provided that \( a \leq b \), because \( \varphi_a \) is the time 1 map of the constant contact Hamiltonian \( a \). On the other hand, if \( t = \frac{p}{q} \) then \( (\varphi_t)^n = e^p \), and the claim follows immediately for rational numbers, and for irrational by continuity.

When \( M \) is the standard contact \( \mathbb{R}P^{2n+1} \) one can get an estimate of \( \gamma(f, e) \) and \( \gamma(e, f) \) in terms of the non-linear Maslov index \( m(f) \) (see 1.3.E above). Let \( f \in \mathcal{D}(\mathbb{R}P^{2n+1}) \) be a lift of the time 1 map of any strictly positive time-independent Hamiltonian, or, in other words, the time one map of the Reeb flow of any contact form. It follows from [G] that \( \gamma(f, e) \geq (n + 1)/m(f) \) and \( \gamma(e, f) \geq m(f)/(n + 1) \).

1.6.D. Stable norm on \( H_1(\mathbb{T}^n, \mathbb{Z}) \). Note that the torus \( \mathbb{T}^n \) acts on \( \mathbb{P}_+ T^* \mathbb{T}^n \) by shifts, thus there exists a natural monomorphism \( \pi_1(\mathbb{T}^n) \hookrightarrow \pi_1(\Gamma) \). We will identify its image with \( H_1(\mathbb{T}^n, \mathbb{R}) \).

Let \( \rho \) be a Riemannian metric on \( \mathbb{T}^n \). Denote by \( || \cdot ||_\rho \) the Gromov-Federer stable norm on \( H_1(\mathbb{T}^n, \mathbb{R}) \) associated to \( \rho \). This norm can be defined in several equivalent ways (see sections 4.C,D and especially 4.20 in [Gr2]). The simplest one is as follows. The stable norm of an integral class \( e \in H_1(\mathbb{T}^n, \mathbb{Z}) \) equals to \( \lim_{k \to +\infty} \frac{l_k}{k} \), where \( l_k \) is the minimal length of a closed geodesic in the class \( ke \).

Let \( f \in \mathcal{D}(\mathbb{P}_+ T^* \mathbb{T}^n) \) be the time-one map of the geodesic flow of \( \rho \).

**Theorem 1.6.E.** The following inequality holds:

\[
\gamma(f, e) \geq ||e||_\rho
\]

for any \( e \in H_1(\mathbb{T}^n, \mathbb{Z}) \).

We refer to Section 3.4 below for the proof and more detailed discussion.

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In 3.4 below we use another definition.
1.7 The geometry of the relative growth

Consider the following question which, in fact, has motivated this paper. The group of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold (which can be closed as well as open) carries a remarkable geometric structure, the Hofer biinvariant metric (see [H2],[HZ],[EP],[MS],[P4]). There are no natural biinvariant metrics on the group of contactomorphisms. So it seems reasonable to ask whether this group admits a geometric structure at all. The next construction should be considered as an attempt to answer this question.

A partially ordered metric space is a metric space \((Z, d)\) endowed with a partial order \(\geq\) such that for every \(a, b, c \in Z\) with \(a \geq b \geq c\) holds \(d(a, c) \geq d(b, c)\).

Let \((M, \xi)\) be a closed contact manifold. Assume that \(\geq\) is a genuine partial order on \(D(M, \xi)\). It turns out that in this situation one can associate with \((M, \xi)\) in a functorial way a partially ordered metric space \((Z(M, \xi), d, \geq)\), where the metric \(d\) comes from the relative growth \(\gamma(f, g)\).

The construction goes as follows. We will abbreviate \(C^+ = C^+(M, \xi)\) (see 1.4. above) and \(D = D(M, \xi)\). First, we formulate an elementary lemma, similar to the inequality 1.1.A above.

**Lemma 1.7.A.** For every \(f, g, h \in C^+\)

\[
\gamma(f, h) \leq \gamma(f, g)\gamma(g, h).
\]

Let us define a function \(\kappa : C^+ \times C^+ \to [0; +\infty)\) by

\[
\kappa(f, g) = \max(\log \gamma(f, g), \log \gamma(g, f)).
\]

Obviously \(\kappa\) is symmetric and vanishes on the diagonal. Further, it follows from the inequality 1.1.A that \(\kappa\) is non-negative, and from Lemma 1.7.A that \(\kappa\) satisfies the triangle inequality. Thus \(\kappa\) is a pseudo-distance. We say that

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4Here is an explanation. Consider the natural embedding \(PSL(2, \mathbb{R}) \to Diff(S^1)\). Every biinvariant metric on \(Diff(S^1)\) induces a biinvariant metric on \(PSL(2, \mathbb{R})\). But such a metric cannot generate a natural topology on \(PSL(2, \mathbb{R})\) since the conjugacy classes are non-compact!

5In fact one can perform it in a much more general context of partially ordered groups.

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two elements $f$ and $g$ in $C^+$ are equivalent if $\kappa(f, g) = 0$, and consider the corresponding quotient space $Z$ of $C^+$. The pseudo-distance $\kappa$ projects to a genuine distance function, denoted $d$, on $Z$.

Let us define now a partial order $\succeq$ on $Z$ as follows. We write $[f] \succeq [g]$ if $\gamma(f, g) \leq 1$. Then $(Z, d, \succeq)$ is a partially ordered metric space, which easily follows from Lemma 1.7.A. This completes our construction.

Let us make two remarks. First, the natural projection $C^+ \to Z$, $f \to [f]$ is clearly monotone with respect to the partial orders on $C^+$ and $Z$. Moreover, it is constant on the conjugacy classes in $C^+$. Indeed, assume that $g = h f h^{-1}$ where $f, g \in C^+$ and $h \in D$. Take a positive integer $k$ such that $f^k \geq h$ and $f^k \geq h^{-1}$. Thus for all natural numbers $n > 2k$ one has $f^{n+2k} \geq f^n \geq f^{n-2k}$. Therefore, $\gamma(f, g) = \gamma(g, f) = 1$, so $[f] = [g]$.

Second, the space $Z$ admits an action of the multiplicative semigroup of positive integers by isomorphisms, i.e. order preserving isometries. For $m \in \mathbb{N}$ define a map $\Phi_m : Z \to Z$ by $[f] \to [f^m]$. It is straightforward to check that this map is isometric and monotone with respect to $\succeq$. Of course, $\Phi_m \circ \Phi_n = \Phi_{mn}$ for all $m, n \in \mathbb{N}$.

The geometry of the space $Z(M, \xi)$ is yet to be explored. Let us give two examples.

**Example 1.7.B. The circle.** We claim that $(Z(S^1), d, \succeq)$ is simply the Euclidean line $\mathbb{R}^1$ endowed with the natural order. The natural projection $C^+ \to Z$ is given by $f \to \log \text{Rot}(f)$. This follows from the fact that for all $f, g \in C^+$

$$\gamma(f, g) = \frac{\text{Rot}(g)}{\text{Rot}(f)}. \quad (1.7.C)$$

The proof of (1.7.C) goes as follows. Recall from Example 1.6.B that there exists an element $e \in D$ which satisfies

$$\gamma(e, f)\gamma(f, e) = 1 \quad (1.7.D)$$

for all $f \in C^+$. Applying Lemma 1.7.A we get that

$$\gamma(f, g) \leq \gamma(f, e)\gamma(e, g) = \frac{\gamma(e, g)}{\gamma(e, f)}.$$

---

\textsuperscript{6}For instance, is there a meaningful description of geodesics on $Z$?
and similarly
\[
\gamma(g, f) \leq \frac{\gamma(e, f)}{\gamma(e, g)}.
\]

Multiplying these two inequalities and comparing the result with 1.1A above we get that in fact each of them must be an equality! Recall now from Example 1.6.B that \(\gamma(e, f) = \text{Rot}(f)\). This proves (1.7.C), and hence confirms our description of \(Z(S^1)\).

Notice that the isometry \(\Phi_n\) in this case acts on \(\mathbb{R} = Z(S^1)\) as the translation \(x \mapsto x + \log n, x \in \mathbb{R}\). In fact, our proof shows that if for a contact manifold \((M, \xi)\) there exists an element \(e \in C^+\) which satisfies (1.7.D) for all \(f \in C^+\) then \(Z(M, \xi)\) can be identified with a subset of the Euclidean line, invariant under translations \(\Phi_n\), and the natural projection is given by \([f] = \log \gamma(e, f)\). We will see in the next example that no such element exists when \(M = \mathbb{P}_+ T^* \mathbb{T}^n\) with \(n > 1\).

**Example 1.7.E.** \(\mathbb{P}_+ T^* \mathbb{T}^n\). Consider the subset \(K \subset \mathcal{D}\) which consists of the natural lifts of the time-one-maps \(\phi_F\) of autonomous contact flows generated by strictly positive contact Hamiltonians \(F = F(p)\). Intuitively speaking this is the positive part of “the maximal torus” of the group \(\mathcal{D}\). Denote by \(Z_K\) the image of \(K\) in \(Z(\mathbb{P}_+ T^* \mathbb{T}^n)\).

**Theorem 1.7.F.** The space \(Z_K\), endowed with the induced metric and partial order, is isomorphic (in the category of partially ordered metric spaces) to the linear space \(C^\infty(S^{n-1})\) of all smooth functions on the sphere \(S^{n-1}\) with the norm \(||u|| = \max |u|\) and with the natural partial order.

The conclusion is that the space \(Z(\mathbb{P}_+ T^* \mathbb{T}^n)\) contains an infinite-dimensional flat piece.

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\(\text{The natural partial order on } C^\infty(S^{n-1}) \text{ is defined as follows. We write } u \succeq v \text{ for functions } u \text{ and } v \text{ when } u(x) \geq v(x) \text{ for all } x.\)
It turns out that the existence of the partial order on the universal cover of the contactomorphisms group is closely related to the notion of fat connection on a symplectic fibration.

Let \( p : P \to S^2 \) be a symplectic fibration with the fiber \((SM,\omega)\) whose structural group is the group of \(\mathbb{R}_+\)-equivariant Hamiltonian diffeomorphisms of \(SM\) (and thus the structural group can be identified with \(\Gamma(M,\xi)\)). We call it an \(\mathbb{R}_+\)-equivariant symplectic fibration. The fibration \(P\) is endowed with the canonical fiberwise symplectic structure \(\omega_x\), \(x \in S^2\), and the canonical \(\mathbb{R}_+\)-action \((x,z) \to (x,R_c z)\) where \(x \in S^2\), \(z \in p^{-1}(x)\) and \(c \in \mathbb{R}_+\). Let \(\nu\) be a connection on \(P\) whose parallel transport maps act by equivariant symplectomorphisms. We call \(\nu\) an equivariant symplectic connection on \(P\).

The curvature \(\rho\) of \(\nu\) is a 2-form on \(S^2\) which at a point \(x \in S^2\) takes values in the Lie algebra of the group of \(\mathbb{R}_+\)-equivariant Hamiltonian diffeomorphisms of \(p^{-1}(x)\). In other words, for \(v, w \in T_x S^2\) one considers \(\rho(v, w)\) as a contact Hamiltonian on \(p^{-1}(x)\). Consider the splitting

\[ TP = T(SM) \oplus TS^2 \]

associated to connection \(\nu\). Define a 2-form \(\delta\) on \(P\) by

\[ \delta = \omega_x \oplus -\rho. \]

The form \(\delta\) is called the coupling form of the connection \(\nu\). It is known to be closed. Moreover \(R^*_c \delta = c \delta\) for all \(c \in \mathbb{R}_+\), and thus \(\delta\) is exact. Following Weinstein [W] we call the connection \(\nu\) fat if the coupling form \(\delta\) is symplectic. Note that if \(P\) admits a fat connection then the quotient space \(P/\mathbb{R}_+\) admits a contact structure which extends the contact structure on fibers \(p^{-1}(x)/\mathbb{R}_+\).

Loops of Hamiltonian diffeomorphisms are closely related to symplectic fibrations over \(S^2\) (see [Se],[P3],[P4],[LMP],[M1]). This link admits a straightforward generalization to the category of \(\mathbb{R}_+\)-equivariant symplectic fibrations. In particular, there exists a one-to-one correspondence between \(\pi_1(\Gamma(M,\xi))\) and isomorphism classes of \(\mathbb{R}_+\)-equivariant symplectic fibrations. Denote by \(P(\alpha)\) the symplectic fibration corresponding to an element \(\alpha \in \pi_1(\Gamma(M,\xi))\). Note that the fibration corresponding to the neutral element of \(\pi_1(\Gamma(M,\xi))\) is simply the trivial fibration \(SM \times S^2 \to S^2\). Further,

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8We refer to [GLS] and [MS] for basic theory of symplectic fibrations.
one can show along the lines of [P3] that an element $\alpha$ can be represented by a loop generated by a strictly positive contact Hamiltonian if and only if fibration $P(\alpha)$ admits a fat connection. Combining this with Criterion 1.2.C we get the following result.

**Proposition 1.8.A.** The relation $\geq$ on $D(M, \xi)$ is a genuine partial order if and only if the trivial $\mathbb{R}_+\text{-equivariant symplectic fibration } SM \times S^2 \to S^2$ does not admit a fat connection.

### 1.9 Possible generalizations

It is possible that Problem 1.2.A has the positive answer for all contact manifolds, but at the moment the authors do not see any tools to handle the problem in such generality. In fact, overtwisted contact structures on 3-manifolds could potentially provide counter-examples to such a general conjecture.

However, it seems that the contact homology theory, which is currently under construction by A. Givental, H. Hofer and one of the authors (see [E2] and [U]) would provide an adequate tool for establishing existence of the partial order on a large class of those contact manifolds for which the contact homology algebra is non-trivial.

The simplest closed contact manifold which potentially can be covered by this techniques, but not covered by the results of this paper, is the standard contact 3-sphere $S^3$. Let us elaborate on how the contact homology theory could be applied to Problem 1.2.A.

With each element $f \in D(M, \xi = \{\alpha = 0\})$ one can associate (see Section 3 below) a domain $V^+(f)$ in the contact manifold $\left( M \times T^*S^1, \tilde{\xi} = \{\alpha + rdt = 0\} \right)$ defined canonically up to a contact isotopy. An important property of this correspondence is that if $f \geq g$ then there exists a contact isotopy which takes $V^+(g)$ inside $V^+(f)$. Thus one needs a technique which provides non-squeezing type results for domains of the form $V^+(f)$. In symplectic topology

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9 This contact manifold is called the stabilization of $(M, \xi)$, see Section 2.2 below.
results of this type can be proved via the theory of symplectic capacities. On the other hand, the only so far known invariants of contact domains which are suitable for this job are the, so-called, contact shapes (see [E1] and Section 3 below). We employ these invariants in the current paper. Since the contact homology provide more powerful and adequate invariants of contact domains, we hope that they should allow us to settle Problem 1.2.A for a more general class of contact manifolds which includes the standard contact spheres.

It is also possible that the finite-dimensional approach of the theory of generating functions (see [EG]), employed by Givental [G] in his theory of the non-linear Maslov index can be extended to a larger class of contact manifolds. Let us mention in this context, that Mohan Bhupal informed us that he proved, using generating functions, existence of the partial order on the group \( D_0(\mathbb{R}^{2n+1}) \) of compactly supported contact transformations of the standard contact \( \mathbb{R}^{2n+1} \).

Further, one can try to attack Problem 1.2.A using its reformulation in the language of symplectic fibrations (see Section 1.8 above). In the theory of compact symplectic fibrations over \( S^2 \) there exists a powerful technique of Gromov-Witten invariants (see [Se],[LMP],[M1],[M2]) which leads to various geometric and topological consequences. It is a challenging problem to extend it to the \( \mathbb{R}_+ \)-equivariant case and to apply to the study of the partial order on \( D(M,\xi) \).

Finally, one can consider Problem 1.2.A in a more general context of Legendrian submanifolds. The binary relation \( \geq \) on the group \( D(M,\xi) \) admits a natural extension to the following homogeneous space of the group. Let \( \mathcal{L} \) be a connected component of the space of all Legendrian submanifolds of \( M \). The tangent space to \( \mathcal{L} \) at some point \( L \in \mathcal{L} \) can be canonically identified with the space \( C^\infty(L) \) of smooth functions on \( L \). Consider the field of tangent cones to \( \mathcal{L} \) formed by non-negative functions. This field is invariant under the action of the contactomorphism group. Its lift to the universal cover \( \tilde{\mathcal{L}} \) defines in the obvious way a binary relation on \( \tilde{\mathcal{L}} \). It would be interesting to decide when this is a partial order relation.

\[^{10}\text{A symplectic capacity } c \text{ is an invariant of a symplectic domain } (V,\Omega) \text{ which is monotone with respect to inclusion and satisfies } c(\lambda \Omega) = |\lambda| c(\Omega) \text{ (see [HZ],[MS]).} \]
1.10. Historical remarks.

The notion of a partially ordered group is classical (see for instance [F]). The relative growth number $\gamma(f, g)$ is a variation on the theme of order-unit norm in partially ordered Abelian groups (see [Go]). For results and references on partially ordered finite dimensional Lie groups and homogeneous spaces we refer to [HiN]. The partial order associated to the cone of non-negative Hamiltonians in the group $\text{Ham}(\mathbb{R}^{2n})$ of compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ was first described without a proof in [E3], but the first published proofs appeared in Viterbo’s paper [V] and Hofer-Zehnder’s book [HZ]. The both proofs are based on a construction of a remarkable invariant $\mathcal{I}$ of a Hamiltonian diffeomorphism $f$. It equals the symplectic action of a specially chosen critical point of $f$. The choice of the critical point is performed on the basis of a careful study of either the Floer homology of the action functional of $f$ or (in the similar vein) the Morse theoretical properties of the generating function of $f$. A crucial property of this invariant is that it vanishes provided $f \leq 1$ and is strictly positive provided $f > 1$. It would be interesting to extend the partial order on $\text{Ham}(\mathbb{R}^{2n})$ to other open symplectic manifolds. Notice that for closed symplectic manifolds no natural partial order on Ham is known. The relative growth number $\gamma(f, g)$, and the geometry and dynamics related to it, were not yet studied in the symplectic context, although it seems that the technique developed in [HZ] and [V] should allow to compute or estimate $\gamma(f, g)$ for suitable pairs of Hamiltonian diffeomorphisms.

2 Establishing the partial order

2.1. Loops of contactomorphisms

A loop of contactomorphisms is a smooth map $S^1 = \mathbb{R}/\mathbb{Z} \rightarrow \Gamma(M, \xi)$ which takes 0 to 1 $\in \Gamma$. Loops are generated by 1-periodic time dependent contact Hamiltonians. A loop is called non-negative if its contact Hamiltonian is non-negative, and strictly positive if its contact Hamiltonian is strictly positive. We start with the following elementary

\footnote{We were unable to find our $\gamma(f, g)$ in the literature.}

\footnote{In fact it is still an open problem to show that the invariants constructed in [HZ] and [V] coincide.}
Proposition 2.1.A. The relation $\geq$ on $\mathcal{D}(M, \xi)$ is a non-trivial partial order if and only if every non-negative contractible loop of contactomorphisms is the constant loop.

Proof: Assume that every contractible non-negative loop of contactomorphisms is constant. Consider an element $f \in \mathcal{D}(M, \xi)$ such that $f \geq 1$ and $f \leq 1$. Then there exist two paths $\{f'_t\}, \{f''_t\}, t \in [0; 1]$ which represent $f$ with the following properties. First, $f'_0 = f''_0 = 1$, $f'_1 = f''_1$ and the paths are homotopic with fixed end points. Second, $\{f'_t\}$ is generated by a non-negative contact Hamiltonian, and $\{f''_t\}$ is generated by a non-positive one. We have to show that then both paths are constant. Indeed, this would mean that $f \geq 1$ and $f \leq 1$ implies $f = 1$, that is the relation $\geq$ is antisymmetric.

Without loss of generality one can assume that $f'_t = f''_t = 1$ when $t$ is close to 0, and $f'_t = f''_t = f'_1$ when $t$ is close to 1. Consider the union of $\{f'_t\}$ with $\{f''_t\}$, where the second path is taken with the opposite orientation. We get a loop of contactomorphisms which is both contractible and non-negative. By our assumption such a loop must be constant. Thus $f = 1$ and our claim follows. The proof of the converse statement is analogous.

\[\Box\]

Proposition 2.1.B. If a closed contact manifold $(M, \xi)$ admits a non-constant contractible non-negative loop of contactomorphisms, then it admits a contractible strictly positive loop of contactomorphisms.

Proof: The proof is based on “ergodic” ideas from [P2]. Let $f : S^1 \to \Gamma$ be a smooth contractible loop, and $F(x, t)$ is the corresponding non-negative Hamiltonian. Since the loop is non-constant there exists $t_0$ such that $F(x, t_0) \neq 0$. In what follows we describe a three-step sequence of modifications of $\{f_t\}$ to a strictly positive contractible loop.

1) Set $g(t) = f(t + t_0)f(t_0)^{-1}$. The loop $g$ is generated by the Hamiltonian $G(x, t) = F(x, t + t_0)$. Hence $G(x, 0) \neq 0$. We assume that $G(x, 0) > 0$ for all $x \in SU$, where $SU$ is the symplectization of a domain $U \subset M$.

2) Let $\varphi_1, \ldots, \varphi_d$ be a sequence of elements of $\Gamma$ which will be chosen later on. Set

$$h(t) = g(t)\varphi_1 g(t) \cdot \ldots \cdot \varphi_d g(t) (\varphi_1 \ldots \varphi_d)^{-1}.$$ 

This is a contractible loop generated by the Hamiltonian

$$H(x, t) = G(x, t) + G(\varphi_1^{-1} g(t)^{-1} x, t) + \ldots + G(\varphi_d^{-1} g(t)^{-1} \ldots \varphi_1^{-1} g(t)^{-1} x, t),$$

18
where we write \( \tilde{\phi} \) for the symplectization of a contactomorphism \( \phi \). Set \( \psi_0 = 1, \psi_k = \varphi_1 \ldots \varphi_k \) where \( k = 1, \ldots, d \). Thus \( H(x, 0) = \sum_{k=0}^{d} G(\tilde{\psi}_k^{-1}x, 0) \).

Assume now that \( \{\varphi_k\} \) is chosen in such a way that \( \bigcup_{k=0}^{d} \psi_k(U) = M \). Then for every \( x \in M \) there exists \( k \) such that \( \psi_k^{-1}x \in U \). Thus \( H(x, 0) > 0 \) for all \( x \in SM \).

3) The last inequality implies that there exists an open arc \( \Delta \subset S^1, \Delta \ni 0 \) such that \( H(x,t) > 0 \) for all \( x \in SM, t \in \Delta \). Choose a real number \( \alpha \) and a positive integer \( m \) such that the sequence \( \{t + k\alpha\}, k = 0, \ldots, m \) meets \( \Delta \) for all \( t \in S^1 \). Consider a new loop

\[
e(t) = h(t)h(t + \alpha) \ldots h(t + m\alpha) (h(0)h(\alpha) \ldots h(m\alpha))^{-1}.
\]

Clearly it is contractible. Its Hamiltonian \( E \) is given by

\[
H(x,t) + H(\tilde{h}(t)^{-1}, t + \alpha) + H(\tilde{h}(t + \alpha)^{-1} \tilde{h}(t)^{-1} x, t + 2\alpha) + \\
\ldots + H(\tilde{h}(t + (m - 1)\alpha)^{-1} \ldots \tilde{h}(t)^{-1} x, t + m\alpha).
\]

Our choice of \( \alpha \) guarantees that for all \( x \in SM, t \in S^1 \) at least one of the summands is strictly positive. Since all other summands are non-negative, we conclude that \( E(x,t) > 0 \) for all \( x, t \). This completes the proof.

\[\blacksquare\]

Criterion 1.2.C follows immediately from 2.1.A and 2.1.B.

### 2.2 Stabilization in the contact category

We start with the following well known remark. Let \((X, \Omega)\) be a symplectic manifold endowed with a free \( \mathbb{R}_+\)-action \( x \to R_c x \) \((c \in \mathbb{R}_+)\) which admits a global slice and such that \( R_c^* \Omega = c\Omega \) for every \( c \). Then the quotient space \( X/\mathbb{R}_+ \) carries in a canonical way a contact structure \( \eta \) such that \( S(X/\mathbb{R}_+, \eta) = (X, \Omega) \).

Let \((M, \xi)\) be a contact manifold with a co-oriented contact structure. We define its stabilization with respect to dimension as

\[
\text{Stab}M = (SM \times T^* S^1, \omega + dr \wedge dt)/\mathbb{R}_+,
\]
where the action of $\mathbb{R}_+$ is the diagonal one. This action is given by

$$(x, r, t) \rightarrow (R_c x, cr, t)$$

for $c \in \mathbb{R}_+$. 

If the contact structure $\xi$ is defined by a contact 1-form $\alpha$ then the contact manifold $\text{Stab} M$ is contactomorphic to $M \times T^* S^1$ with the contact structure defined by the form $\alpha + r dt$.

Let $\tau : SM \rightarrow M$ and $\sigma : SM \times T^* S^1 \rightarrow \text{Stab} M$ be the natural projections. Denote by $Z$ the zero section $\{r = 0\}$ of $T^* S^1$. The procedure of stabilization extends naturally to two remarkable classes of submanifolds of $(M, \xi)$, pre-Lagrangian and Legendrian. Recall that $K \subset M$ is called pre-Lagrangian if there exists a Lagrangian submanifold $L \subset SM$ such that $K = \tau(L)$ and $\tau|^L : L \rightarrow K$ is a diffeomorphism. The submanifold $L$ is called a Lagrangian lift of $K$. Define the stabilization

$$\text{Stab} K = \sigma(L \times Z).$$

This definition does not depend on the choice of the particular lift $L$ of $K$. Clearly, $\text{Stab} K$ is a pre-Lagrangian submanifold of $\text{Stab} M$.

A Legendrian submanifold $\Lambda \subset M$, i.e. a maximal integral submanifold of $\xi$, can be characterized by the property that its lift $S \Lambda = \tau^{-1} \Lambda$ to $SM$ is Lagrangian. The Legendrian submanifold

$$\text{Stab} \Lambda = \sigma(S \Lambda \times Z) \subset \text{Stab} M$$

is called the stabilization of $\Lambda$.

Stabilizations arise naturally in the study of loops of contactomorphisms. Let $N \subset SM$ be a Lagrangian submanifold, and let $\{h_t\}$ be a loop of contactomorphisms of $(M, \xi)$ generated by a contact Hamiltonian $H(x, t)$. Define the suspension $\text{Susp}_h N$ as the image of the Lagrangian embedding

$$N \times S^1 \rightarrow SM \times T^* S^1,$$

$$(x, t) \rightarrow (\tilde{h}_t x, -H(\tilde{h}_t x, t), t),$$

where $\tilde{h}_t$ stands for the lift of $h_t$ to a $\mathbb{R}_+$-equivariant Hamiltonian diffeomorphism of $SM$. Let $K$ be a pre-Lagrangian submanifold of $M$. Set
Susp\textsubscript{h}K = \sigma(\text{Susp}\textsubscript{h}L), where \(L\) is a Lagrangian lift of \(K\). The submanifold \(\text{Susp}\textsubscript{h}K\) does not depend on the particular choice of lift \(L\), and is a pre-Lagrangian submanifold of \(\text{Stab}M\). Similarly, let \(\Lambda\) be a Legendrian submanifold of \(M\). Set \(\text{Susp}\textsubscript{h}\Lambda = \sigma(\text{Susp}\textsubscript{h}\Sigma\Lambda)\). Again, \(\text{Susp}\textsubscript{h}\Lambda\) is a Legendrian submanifold of \(\text{Stab}M\). Note that if \(h\) is the constant loop \(h_t \equiv 1\) then \(\text{Susp}\textsubscript{h}K = \text{Stab}K\) and \(\text{Susp}\textsubscript{h}\Lambda = \text{Stab}\Lambda\).

### 2.3 Stable intersections in contact manifolds

Let \((M,\xi)\) be a contact manifold (not necessarily compact), and \(K \subset M\) be a closed pre-Lagrangian submanifold. Let \(A \subset M\) be a closed submanifold which is either pre-Lagrangian or Legendrian. We say that the pair \((K, A)\) has the intersection property if for every contactomorphism \(\phi \in \Gamma(M,\xi)\) the intersection \(\phi(K) \cap A\) is non-empty. We say that \((K, A)\) has the stable intersection property if \((\text{Stab}K, \text{Stab}A)\) has the intersection property in \(\text{Stab}M\).

The detection and proof of the intersection property is one of the central problems in symplectic and contact geometry and certain techniques (Floer homology, generating functions etc.) are developed to handle it. If it is possible to prove the intersection property for \((K, A)\) then usually (but not always) the same argument allows to handle the stable intersection problem. In 2.4 below we discuss some examples of pairs with stable intersection property.

The next result links stable intersections with the partial order.

**Theorem 2.3.A.** Suppose that \((M,\xi)\) contains a pair with the stable intersection property. Then \(\geq\) is a non-trivial partial order on \(\mathcal{D}(M,\xi)\).

This is an immediate consequence of Criterion 1.2.C and the next

**Proposition 2.3.B.** Let \((K, A)\) be a pair with the stable intersection property. Let \(H(x, t)\) be a 1-periodic Hamiltonian generating a contractible loop of contactomorphisms. Then there exists a point \((x_0, t_0) \in A \times S^1\) such that the function \(H(\cdot, t_0)\) vanishes on the ray \(\tau^{-1}(x_0)\).

This statement and its proof is analogous to [P1, Lemma 3.B].

**Proof of 2.3.B:** Denote by \(h\) the loop of contactomorphisms generated by \(H\). Since \(h\) is contractible then \(\text{Susp}_hK\) is isotopic to \(\text{Susp}_1K = \text{Stab}K\)
through pre-Lagrangian submanifolds of StabM. One can easily check that
this isotopy extends to a contact isotopy of StabM. Thus the stable in-
tersection property guarantees that Susp_hK ∩ StabA is non-empty. Using
definitions of the stabilization and suspension we obtain that there exists a
point (y_0, t_0) ∈ K × S^1 such that h_{t_0}y_0 ∈ A and H(., t_0) vanishes on the ray
τ^{-1}(h_{t_0}y_0). Thus the points x_0 = h_{t_0}y_0 and t_0 are as needed. This completes
the proof.

2.4. Examples of stable intersections

Examples 2.4.A and 2.4.B below combined with Theorem 2.3.B prove Theo-

Example 2.4.A. Let Y be a closed manifold which admits a non-singular
closed 1-form. The graph K of this form in P_+T^*Y is a pre-Lagrangian
submanifold. The pair (K, K) has the stable intersection property. This is a
particular case of the Arnold conjecture proved in [H1],[LS].

Example 2.4.B. Consider the situation described in 1.3.D. Let L ⊂ W be
a Bohr-Sommerfeld Lagrangian submanifold. Denote by K its full lift to
the prequantization space QW. Then K is a pre-Lagrangian submanifold of
QW. It is foliated by flat Legendrian lifts of L. Pick up such a lift, say Λ. If
π_2(W, L) = 0 then (K, Λ) has the stable intersection property.

This example is borrowed from [EHS] (see also [Ono]) where it is proved
that (K, Λ) has the intersection property. The theory developed in [EHS] is
presented for closed contact manifolds. In order to apply it to the open man-
ifold StabM and get the stable intersection property of (K, Λ) an additional
argument is needed. We present this argument below.

A prequantization of a symplectic manifold (W, Ω) is given by the follow-
ing data:

• a principal S^1-bundle p : QW → W;

• an S^1-invariant connection ξ on QW defined by an S^1-invariant 1-form
  α with dα = p^*Ω, which integrates to 1 over fibers of the bundle.
We will denote such a prequantization by \((QW, p, \alpha)\). Fix now a sufficiently large positive integer \(l\). Consider the torus \(T^2 = T^*S^1/\mathbb{Z}\) where the action of the group \(\mathbb{Z}\) is generated by the shift \((r, t) \to (r + l, t)\), and a symplectic manifold
\[
W' = (W \times \mathbb{T}^2, \Omega \oplus dr \wedge dt).
\]
Write \(B\) for the annulus
\[
\{(r, t) \mid |r| < l'\},
\]
where \(0 < l' < l/2\). We claim that there exists a prequantization \((QW', p', \alpha')\) of \(W'\) with the following properties:

- The bundle \(p' : QW' \to W'\) over \(B\) coincides with
  \[
  QW \times B \to W \times B, \ (z, r, t) \to (p(z), r, t).
  \]

- The form \(\alpha'\) over \(B\) equals \(\alpha + r dt\).

Indeed, denote by \(z \to z + s, \ s \in S^1 = \mathbb{R}/\mathbb{Z}\) the circle action on \(QW\). The space \(QW'\) can be described explicitly as
\[
(QW \times T^*S^1)/\mathbb{Z},
\]
where the \(\mathbb{Z}\)-action is generated by the map
\[
(2.4.C) \quad (z, r, t) \to (z - lt, r + l, t),
\]
and the projection \(p'\) is given by
\[
p'(z, r, t) = (p(z), r \mod l, t \mod 1).
\]

One readily checks that \(\mathbb{Z}\)-action (2.4.C) preserves the form \(\alpha + r dt\) on \(QW \times T^*S^1\). Define the connection form \(\alpha'\) as the push-forward of \(\alpha + r dt\) to \(QW'\). The claim follows now from the fact that \(QW \times B\) is naturally embedded into the fundamental domain \(QW \times \{-l/2 < r \leq l/2\}\) of the \(\mathbb{Z}\)-action (2.4.C).

Notice now that open contact manifolds \((QW \times B, \alpha + r dt)\) exhaust the stabilization \(\text{Stab}QW\) when \(l' \to +\infty\). Thus it suffices to establish the intersection property of \((\text{Stab}K, \text{Stab}\Lambda)\) for each such manifold. In view of the above claim \(QW \times B\) is contained in \(QW' = Q(W \times \mathbb{T}^2)\). Hence, the intersection property for the pair \((\text{Stab}K, \text{Stab}\Lambda)\) in the closed contact manifold \(QW'\), which is ensured by Theorem 2.5.4 of [EHS], yields the intersection property for this pair in \(QW \times B\). This completes the proof of the stable intersection property.
3 Relative growth and shape

3.1 From paths of contactomorphisms to domains

We begin with two lemmas which are proved at the end of this section (see 3.5 below).

**Lemma 3.1.A.** If a contact diffeomorphism $f \in \mathcal{D}(M,\xi)$ can be generated by a positive Hamiltonian, then it can also be generated by a 1-periodic positive Hamiltonian.

Take any element $f \in \mathcal{D}(M,\xi)$ and represent it by a path \( \{ f_t \} \), $t \in [0; 1]$, $f_0 = 1$ generated by a 1-periodic contact Hamiltonian $F(x, t)$. Consider a symplectic manifold $N = SM \times T^*S^1$ endowed with the symplectic form $\omega + dr \wedge dt$ where $(r, t)$, $r \in \mathbb{R}$, $t \in \mathbb{R}/\mathbb{Z}$, are canonical coordinates in $T^*S^1$. Consider domains

\[
V^+(\{ f_t \}) = \{ r + F(x, t) \geq 0 \} \subset N
\]

and

\[
V^-(\{ f_t \}) = \{ r + F(x, t) \leq 0 \} \subset N.
\]

**Lemma 3.1.B.** Given two such paths \( \{ f_t \}, \{ f'_t \} \) with $f_1 = f'_1$ which are homotopic with fixed endpoints, there exists a Hamiltonian isotopy of $N$ which takes $V^+(\{ f_t \})$ to $V^+(\{ f'_t \})$ and $V^-(\{ f_t \})$ to $V^-(\{ f'_t \})$.

3.2. Shape

Set

\[
N = S(\mathbb{P}_+T^*\mathbb{T}^n) \times T^*S^1 \subset T^*\mathbb{T}^n \times T^*S^1.
\]

For a subset $V \subset N$ we define (see [S], [E1]) a subset

\[
\text{Shape}(V) \subset H^1(\mathbb{T}^n \times S^1; \mathbb{R}) = H^1(\mathbb{T}^n; \mathbb{R}) \times \mathbb{R}
\]

as the collection of all pairs $(a, b)$, such that there exists a Lagrangian embedding $\chi : \mathbb{T}^n \times S^1 \to V$ with the following properties:

(i) $\chi^* [pdq + rdt] = (a, b)$;
(ii) $\chi$ is *homologically standard* which means that the composition of $\chi$ with the natural projection $V \to \mathbb{T}^n \times S^1$ induces the identity map of $H^1(\mathbb{T}^n \times S^1; \mathbb{R})$.

Let us make identifications

$$T^n = \mathbb{R}^n/\mathbb{Z}^n, \quad T^*T^n = (\mathbb{R}^n)^* \times \mathbb{T}^n, \quad H^1(\mathbb{T}^n, \mathbb{R}) = (\mathbb{R}^n)^*.$$ 

It was proved by Gromov [Gr1] that every Lagrangian embedding $\chi$ as above must intersect the split Lagrangian torus $\{p = a, \quad r = b\}$. This result plays a crucial role below.

**Example 3.2.A (Sikorav [S])** Take a domain $U \subset ((\mathbb{R}^n)^* \setminus \{0\}) \times \mathbb{R}$, and set $V = U \times \mathbb{T}^n \times S^1$. Here $V$ is considered as a subset of $N = ((\mathbb{R}^n)^* \setminus \{0\}) \times \mathbb{T}^n \times \mathbb{R} \times S^1$. We claim that $\text{Shape}(V) = U$. Indeed, since $V$ is foliated by split Lagrangian tori we get that $U \subset \text{Shape}(V)$. The opposite inclusion is an immediate consequence of the abovementioned Lagrangian intersection result.

Clearly, $\text{Shape}(V)$ is invariant under Hamiltonian isotopies of $V$. Thus, given $f \in \mathcal{D}$ one can define subsets

$$\text{Shape}^+(f) = \text{Shape}(V^+(\{f_i\}), \\
\text{Shape}^-(f) = \text{Shape}(V^-(\{f_i\}))$$

which in view of 3.1.B do not depend on the particular choice of the path $\{f_i\}$ representing $f$. Furthermore, if $V \supset V'$ then $\text{Shape}(V) \supset \text{Shape}(V')$. Combining this with Proposition 1.4.B above we get the following

**Proposition 3.2.B.** If $f \geq g$ then $\text{Shape}^+(f) \supset \text{Shape}^+(g)$, and $\text{Shape}^-(f) \subset \text{Shape}^-(g)$.

Note also that $\text{Shape}^\pm(f)$ is invariant under the $\mathbb{R}_+$-action on $H^1(\mathbb{T}^n, \mathbb{R}) \times \mathbb{R}$.

It is convenient to extract some numerical invariants from $\text{Shape}^\pm(f)$ as follows. Given $f \in \mathcal{D}$, and $a \in H^1(\mathbb{T}^n, \mathbb{R}) \setminus \{0\}$ set

$$r_-(a, f) = -\inf\{b|(a, b) \in \text{Shape}^+(f)\}$$

and

$$r_+(a, f) = -\sup\{b|(a, b) \in \text{Shape}^-(f)\}.$$ 

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Let us list some useful properties of functions $r_+$ and $r_-$. Fix $f \in \mathcal{D}$ and $a \in H^1(\mathbb{T}^n, \mathbb{R}) \setminus \{0\}$.

\begin{enumerate}
  \item \[(a, b) \in \text{Shape}^+(f) \text{ for all } b > -r_-(a, f);
  \]
  \item \[(a, b) \in \text{Shape}^-(f) \text{ for all } b < -r_+(a, f).
  \]
  \item \[r_+(a, f) \geq r_-(a, f)\]
  \item \[(\text{Normalization}) \quad r_\pm(a, 1) = 0.\]
  \item \[(\text{Monotonicity}) \quad \text{If } f \geq g \text{ then } r_\pm(a, f) \geq r_\pm(a, g).\]
  \item \[(\text{Behaviour under iterations}) \text{ For } k \in \mathbb{N}, \]
    \[r_-(a, f^k) \geq kr_-(a, f);\]
    \[r_+(a, f^k) \leq kr_+(a, f).\]
  \item \[r_+(a, f^{-1}) = -r_-(a, f)\]
  \item \[r_\pm(c a, f) = cr_\pm(a, f) \text{ for all } c > 0\]
  \item \[r_\pm(a, h f h^{-1}) = r_\pm(a, f) \text{ for all } h \in \mathcal{D}\]
\end{enumerate}

**Proof of Properties (1) -(8):**

Denote by $R_c : \mathbb{N} \to \mathbb{N}$ the shift $r \mapsto r + c$ along the $r$-axis, where $c \in \mathbb{R}$. Suppose that $\chi : \mathbb{T}^n \times S^1 \to \{r + F(x, t) \geq 0\}$ is a homologically standard
Lagrangian embedding with $\chi^*[pdq + rdt] = (a, b)$. Then for $c > 0$, $R_c \circ \chi(T^n \times S^1) \subset \{r + F(x, t) > 0\}$ and $(R_c \circ \chi)^*[pdq + rdt] = (a, b + c)$, and hence $(a, b + c) \in \text{Shape}^+(f)$. Taking $b$ arbitrary close to $-r_-(a, f)$ we get the first statement of (1). The second one is analogous.

Recall that a homologically standard Lagrangian torus in $T^*\mathbb{T}^{n+1}$ must intersect a split torus with the same Liouville class. Since $R_c \circ \chi(T^n \times S^1) \cap \{r + F(x, t) > 0\} = \emptyset$ we get that $(a, b + c) \not\in \text{Shape}^-(f)$. This proves (3). The property 4 is an immediate corollary of Proposition 1.4.B.

Let us show (5). Assume that $f$ is generated by a 1-periodic Hamiltonian $F(x, t)$. Then $f^k$ is generated by $F_k(x, t) = kF(x, kt)$. Consider the covering

\[ \pi : T^*\mathbb{T}^n \times \mathbb{R} \times \mathbb{R} / \mathbb{Z} \to T^*\mathbb{T}^n \times \mathbb{R} \times \mathbb{R} / \mathbb{Z} 
\]

\[ (x, r, t) \mapsto (x, \frac{r}{k}, kt) . \]

Clearly, the domain $\{r + F(x, t) \geq 0\}$ lifts to $\{r + F_k(x, t) \geq 0\}$. If $\chi : \mathbb{T}^n \times S^1 \to \{r + F(x, t) \geq 0\}$ is a homologically standard Lagrangian embedding with $\chi^*[pdq + rdt] = (a, b)$ then $\chi$ lifts to a homologically standard Lagrangian embedding $\tilde{\chi}$ with $\tilde{\chi}^*[pdq + rdt] = (a, kb)$. Thus if $(a, b) \in \text{Shape}^+(f)$ then $(a, kb) \in \text{Shape}^+(f^k)$, and so $r_-(a, f^k) \geq kr_-(a, f)$. The second inequality follows in the same way.

The proof of (5) follows from the fact that $f^{-1}$ is generated by $F_{-1}(x, t) = -F(x, -t)$. The involution

\[ (x, r, t) \mapsto (x, -r, -t) \]

sends $\{r + F \geq 0\}$ to $\{r + F_{-1} \leq 0\}$, hence the result. The property 7 follows from the fact that contact Hamiltonians are homogeneous. Finally, (8) is obvious.

\[ \square \]
3.3 An application to relative growth

**Theorem 3.3.A.** Let \( f, g \in \mathcal{D} \), \( f \) is a dominant, and \( r_-(a, g) > 0 \) for some \( a \in H^1(\mathbb{T}^n, \mathbb{R}) \setminus \{0\} \). Then

\[
\gamma(f, g) \geq \frac{r_-(a, g)}{r_+(a, f)}.
\]

**Proof:** The proof is based on the properties of functions \( r_\pm \) listed in the previous section. Set \( \gamma_k = \gamma_k(f, g) \) and notice that \( \gamma_k > 0 \). Indeed, if \( \gamma_k < 0 \) then since \( f^{\gamma_k} \geq g^k \) and \( f \geq 1 \) we would have that \( g^k \leq 1 \), and

\[
0 \geq r_-(a, g) \geq kr_-(a, g) > 0,
\]
which contradicts to the assumption. Hence, \( f^{\gamma_k} \geq g^k \) implies

\[
k r_-(a, g) \leq r_-(a, g^k) \leq r_-(a, f^{\gamma_k}) \leq r_+(a, f^{\gamma_k}) \leq \gamma_k r_+(a, f),
\]
and hence

\[
\frac{\gamma_k}{k} \geq \frac{r_-(a, g)}{r_+(a, f)}.
\]
Passing to the limit when \( k \to +\infty \), we get the desired inequality.

\[\blacksquare\]

**Proof of Theorem 1.5.A :** Identify \( H^1(\mathbb{T}^n, \mathbb{R}) \) with \( (\mathbb{R}^n)^* \) where \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \). It follows from 3.2.A that \( r_\pm(p, f) = F(p) \) and \( r_\pm(p, g) = G(p) \). Thus \( \gamma(f, g) \geq \max_{p \neq 0} \frac{G(p)}{F(p)} \) in view of 3.3.A above. The converse inequality follows immediately from the fact that

\[
G(p) \leq \max_{p \neq 0} \frac{G}{F} \cdot F(p)
\]
combined with Proposition 1.4.B.

\[\blacksquare\]
3.4 Relative growth, stable norm and minimal action

In this section we prove Theorem 1.6.E above. We start from another definition of the stable norm (see [Gr2, 4.35]). Let $\rho$ be a Riemannian metric on $\mathbb{T}^n$. For a cohomology class $a \in H^1(\mathbb{T}^n, \mathbb{R})$ we set

$$\|a\|^* = \inf \left\{ \max_{x \in \mathbb{T}^n} |\alpha_x|_\rho \mid \alpha \text{ is a closed 1-form with } [\alpha] = a \right\}.$$ 

One can show that $\| \cdot \|^*$ is a norm. The dual norm $\| \cdot \|$ on $H^1(\mathbb{T}^n, \mathbb{R})$ is precisely the stable Gromov-Federer norm (see [Gr2]). More explicitly, for $e \in H^1(\mathbb{T}^n, \mathbb{R})$, we have

$$\|e\| = \max \{ \langle a, e \rangle \mid a \in H^1(\mathbb{T}^n, \mathbb{R}), \|a\|^* = 1 \}.$$ 

Proof of Theorem 1.6.E: The geodesic flow on $P_+T^*\mathbb{T}^n$ is given by the contact Hamiltonian $F(p, q) = \| p \|_\rho$. Take an element $e \in H^1(\mathbb{T}^n, \mathbb{Z})$, and fix $\varepsilon > 0$. Pick a closed 1-form $\alpha$ with $[\alpha] = a$ such that $\max_{x \in \mathbb{T}^n} \| \alpha_x \|_\rho \leq 1 + \varepsilon$ and $\langle a, e \rangle = \| e \|$. Since $\text{graph}(\alpha) \subset \{ F \leq 1 + \varepsilon \}$ we have $\text{graph}(\alpha) \times \{ r = -1 - \varepsilon \} \subset \{ r + F \leq 0 \}$. Thus $(a, -1 - \varepsilon) \in \text{Shape}^-(f)$, from which we conclude that $r_+(a, f) \leq 1 + \varepsilon$.

Recall now that $H^1(\mathbb{T}^n, \mathbb{Z})$ is identified with a subgroup of $\pi_1(\Gamma(M, \xi))$. With this identification and also identifying $(\mathbb{R}^n)^*$ with $H^1(\mathbb{T}^n, \mathbb{R})$ we get that $e$ is generated by a Hamiltonian $G(p) = \langle p, e \rangle$. In view of Example 3.2.A we have $r_\pm(a, f) = \| a, e \| = \| e \|$, and applying Theorem 3.3.A we get that

$$\gamma(f, e) \geq \frac{r_-(a, e)}{r_+(a, f)} \geq \frac{\| e \|}{1 + \varepsilon}.$$ 

Since $\varepsilon$ is arbitrarily small, this implies the desired inequality.

It was established by Bangert see ([Ba], Sect. 2.C) that the stable norm is related to Mather’s minimal action [Ma]. Namely, assume that $\mu$ is an invariant Borel probability measure of the geodesic flow on $T\mathbb{T}^n$. Define its action

$$A(\mu) = \frac{1}{2} \int_{T^M} |v|^2_\rho d\mu(v).$$ 

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Denote by $\mathcal{M}$ the set of all measures with $A(\mu) < \infty$. The rotation number $R(\mu)$ is an element from $H_1(T^n, \mathbb{R})$ which satisfies $\langle [\alpha], R(\mu) \rangle = \int \alpha d\mu$, where a closed 1-form is considered as a function on $TM$. Define Mather’s minimal action

$$
\beta : H_1(T^n, \mathbb{R}) \rightarrow \mathbb{R}
$$

by the formula

$$
\beta(e) = \inf \{ A(\mu) \mid \mu \in \mathcal{M} \text{ and } R(\mu) = e \}.
$$

Bangert proved that $\beta(e) = \frac{1}{2} \| e \|_p^2$. Thus our result above implies that $\gamma(f, e) \geq \sqrt{2}\beta(e)$. This inequality is not a specific feature of geodesic flows. In fact it remains true for any $f \in \mathcal{D}(\mathbb{R}^+ T^* T^n)$ generated by a positive time-independent contact Hamiltonian $F(p, q)$ whose square $F^2$ is strictly convex with respect to $p$-variable. This is an easy consequence of [CIPP, Cor. 1]. It is unclear however whether an inequality of this type remains true if one considers time-one maps of time-dependent Hamiltonians. Further there is a strong feeling that the quantity $\gamma(f, e)$ is related to invariant pre-Lagrangian tori of $f$ (compare with Siburg’s theory [Si] which links together invariant tori, Mather’s minimal action and Hofer’s geometry).

### 3.5 Proof of lemmas 3.1.A and 3.1.B

**Proof of Lemma 3.1.A:** Let $f_t, t \in [0, 1]$, be a path of contactomorphisms connecting 1 with $f$, and $\tilde{f}_t : SM \rightarrow SM$ the symplectization of this path. Let $F(x, t), t \in [0, 1], x \in SM$, be a positive homogeneous Hamiltonian which generates the path $\tilde{f}_t$. We will assume that $F(x, t)$ is extended for $t \in [0, 2]$ as a positive Hamiltonian, set $G(x, t) = F(x, t+1), t \in [0, 1]$, and denote by $g_t : M \rightarrow M$ and $\tilde{g}_t : SM \rightarrow SM$ the contact and symplectic isotopies defined by the Hamiltonian $G(x, t)$. Take the product $S_2M = SM \times SM$ with the symplectic structure $\Omega = (\omega) \oplus \omega$ and consider Lagrangian graphs

$$
\Gamma_t = \{(x, \tilde{f}_t(x)) \mid x \in SM\} \quad \text{and} \quad \Delta_t = \{(x, \tilde{g}_t(x)) \mid x \in SM\}
$$

of symplectomorphisms $f_t$ and $g_t$. Notice that these graphs are invariant with respect to the diagonal action of $\mathbb{R}_+$ on $(S_2M, \Omega)$. Furthermore, the $\mathbb{R}_+$-homogeneous Hamiltonian functions $\tilde{F}(x, y, t) = F(y, t)$ and $\tilde{G}(x, y, t) = \ldots$
\(G(y, t), (x, y) \in S_2M, t \in [0, 1]\), generate symplectic isotopies \(\text{Id} \times \tilde{f}_t : SM \to SM\) and \(\text{Id} \times \tilde{g}_t : SM \to SM\) which move the diagonal \(\Gamma_0 = \Delta_0\) to \(\Gamma_t\) and \(\Delta_t\), respectively. In particular, we see that the Lagrangian isotopies \(\Gamma_t\) and \(\Delta_t\) are generated by positive (homogeneous) Hamiltonian functions. The converse is also true: if \(B_t\) is a homogeneous Lagrangian graphical (with respect to the splitting \(S_2M = SM \times SM\)) isotopy, which is generated by a family of positive homogeneous functions \(K_t : \Gamma_t \to \mathbb{R}\) then the corresponding family of \(\mathbb{R}_+\)-equivariant symplectomorphisms of \(SM\) is generated by a positive homogeneous (time-dependent) Hamiltonian on \(SM\).

Next, observe that there exists a \(\mathbb{R}_+\)-invariant neighborhood \(U\) of the diagonal \(\Gamma_0\) which is equivariantly symplectomorphic to a \(\mathbb{R}_+\)-invariant neighborhood of the 0-section in the cotangent bundle \(T^*(SM)\). Here we canonically extend the action of \(\mathbb{R}_+\) on \(SM\) to a conformally symplectic action on \(T^*(SM)\) as follows. Write \(R_c\) for the action \(q \to cq\), where \(q \in SM\) and \(c \in \mathbb{R}_+\). Then the extended action in the canonical coordinates \((p, q)\) on \(T^*(SM)\) is given by

\[(p, q) \to (c(R_c^*)^{-1}p, R_c(q)).\]

There exists a small \(\delta > 0\) such that for \(t \leq \delta\) the Lagrangian submanifolds \(\Gamma_t\) and \(\Delta_t\) are contained in \(U\), and hence we can identify them with Lagrangian submanifolds of \(T^*(SM)\), still denoted by \(\Gamma_t\) and \(\Delta_t\). These submanifolds are \(\mathbb{R}_+\)-invariant, and hence are defined by homogeneous generating functions \(S_t : SM \to \mathbb{R}\) and \(T_t : SM \to \mathbb{R}\), respectively. Notice that the Lagrangian isotopies \(\Gamma_t\) and \(\Delta_t\), \(t \in [0, \delta]\) are generated by positive Hamiltonian functions

\[A_t(p, q) = \frac{\partial S_t}{\partial t}(q)\quad \text{and} \quad B_t(p, q) = \frac{\partial T_t}{\partial t}(q),\]

and hence we have \(\frac{\partial S_t}{\partial t}(q), \frac{\partial T_t}{\partial t}(q) > 0\) for all \(q \in SM, t \in [0, \delta]\). Let us recall that \(\Gamma_0\) and \(\Delta_0\) (viewed as submanifolds of the cotangent bundle of \(SM\)) coincide with the zero-section in \(T^*(SM)\). Thus \(S_0 \equiv T_0 \equiv 0\). The inequalities above imply that for \(t \in (0, \delta]\) the functions \(S_t\) and \(T_t\) are positive. Further, for a sufficiently small positive \(\varepsilon < \frac{\delta}{2}\) we have \(T_\varepsilon < S_{\frac{\delta}{2}}\). Therefore, one can find a smooth family of \(\mathbb{R}_+\)-homogeneous functions \(U_t, t \in [0, \delta]\), on \(SM\) which coincides with \(T_t\) for \(t \in [0, \varepsilon]\) and with \(S_t\) for \(t \in [\frac{\delta}{2}, \delta]\), and such that \(\frac{\partial U}{\partial t} > 0\) for all \(t \in [0, 1]\). If \(\delta\) have been chosen small enough then one can guarantee, in addition, that the Lagrangian submanifolds of \(T^*(SM)\) generated by the functions \(U_t\), and viewed in a neighborhood of the diagonal in \(S_2M\), are graphical with respect to the splitting \(S_2M = SM \times SM\), and
hence correspond to $\mathbb{R}_+\text{-equivariant} \text{ symplectomorphisms } \tilde{h}_t : SM \to SM$. It remains to observe that the family $\tilde{h}_t, t \in [0, \delta]$, together with the family $\tilde{f}_t, t \in [\delta, 1]$, define a smooth path of $\mathbb{R}_+\text{-equivariant} \text{ symplectomorphisms } SM \to SM$ which is generated by a time-periodic positive Hamiltonian. 

**Proof of Lemma 3.1.B:** Let $h_{t,s}$ be a family of homogeneous symplectomorphisms of $SM$ such that

- for a fixed $s$, the symplectomorphism $h_{t,s}$ is generated by a 1-periodic Hamiltonian $F(x, t, s)$;
- $h_{0,s} \equiv 1$, $h_{1,s} \equiv f$;
- for a fixed $t$, $h_{t,s}$ is generated by $G(x, t, s)$.

We assume without loss of generality that $h_{t+1,s} = h_{t,s} \circ f$ for all $s$, and hence $G$ is 1-periodic in $t$. We also have $\frac{\partial F}{\partial s} = \frac{\partial G}{\partial t} - \{F, G\}$. Consider $G(x, t, s)$ as a Hamiltonian on $SM \times T^*S^1(x, r, t)$, depending on time $s$. Thus

$$\frac{dr}{ds} = -\frac{\partial G}{\partial t}, \quad \frac{dt}{ds} = 0, \quad \frac{dx}{ds} = \text{grad} G_{t,s}.$$ 

We claim that this Hamiltonian flow takes \{r + F(x, t, 0) = 0\} to \{r + F(x, t, s) = 0\}, in other words $r(s) + F(x(s), t, s) = 0$. Indeed, differentiating by $s$ one gets

$$-\frac{\partial G}{\partial t} + \{F, G\} + \frac{\partial F}{\partial s} = 0.$$ 

In fact, the Hamiltonian isotopy in question is given explicitly by

$$(x, r, t) \mapsto \left(h_{t,s}h_{t,0}^{-1}x, \ r - \int_0^s \frac{\partial G_{t,u}}{\partial t} \left(h_{t,u}(h_{t,0}^{-1}(x))\right) du, \ t \right).$$

Note that it is homogeneous with respect to $\mathbb{R}_+$-action on $SM \times T^*S^1$:

$$(x, r, t) \mapsto (cx, cr, t).$$

**Acknowledgments.** We thank Miguel Abreu and Ana Cannas da Silva, the organizers of the Symplectic Geometry conference at Lisboa in June 1999 for the opportunity to present results of this paper. We are grateful to Alexander Givental for illuminating discussions concerning the non-linear Maslov index. A part of the manuscript has been written while the second author was visiting IHES (Bures-sur-Yvette). He thanks IHES for the hospitality.
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