PULLBACK EXPONENTIAL ATTRACTORS FOR THE THREE DIMENSIONAL NON-AUTONOMOUS NAVIER-STOKES EQUATIONS WITH NONLINEAR DAMPING

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ABSTRACT. The main objective of this paper is to study the long-time behavior of solutions for the three dimensional non-autonomous Navier-Stokes equations with nonlinear damping for $r > 4$. Inspired by the methods of $\ell$-trajectories in [27], we will prove the existence of a finite dimensional pullback attractor and a pullback exponential attractor, which gives another way of considering the long-time behavior of the non-autonomous evolutionary equations.

1. Introduction. In this paper, we mainly investigate the long-time behavior of weak solutions for the following three dimensional non-autonomous Navier-Stokes equations with nonlinear damping:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p + \beta |u|^{r-2}u &= f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_t, \\
\nabla \cdot u &= 0, \quad (x,t) \in \Omega \times \mathbb{R}_t, \\
u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times \mathbb{R}_t, \\
\n\end{aligned}
\]  

(1)

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$, $\mathbb{R}_t = [\tau, +\infty)$, $\tau \in \mathbb{R}$, $r > 4$, $\nu > 0$ is the viscosity, $\beta > 0$ is a constant, $p(x,t)$ is the pressure of the fluid, $u(x,t)$ is the velocity of the fluid, $f(x,t)$ is the given external body force, $\beta |u|^{r-2}u$ is the nonlinear damping.

In the case of $\beta = 0$, problem (1) is reduced to the three dimensional incompressible Navier-Stokes equations which have been studied extensively (see

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However, the uniqueness of weak solutions and the global existence of strong solutions remain open until now. Therefore, many authors turn to consider the well-posedness and the long-time behavior of solutions of problem (1) with $\beta > 0$ (see \[4, 18, 19, 20, 33, 34, 35, 36, 40, 41\]), where the damping comes from the resistance to the motion of the flow, to which various physical phenomena such as porous media flow, drag or friction effects are related (see \[2, 3, 16, 17\]). In particular, the authors\[4\] have proved the existence of global weak solutions for $r \geq 2$, the existence of global strong solutions for $r \geq 9/2$, the uniqueness of strong solutions for $9/2 \leq r \leq 6$ in the whole space, respectively. Later, the authors \[40\] improved the results in \[4\] and obtained the existence of global strong solutions with $u_0 \in H^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ for $r \geq 4$, as well as the uniqueness of strong solutions for $4 < r \leq 6$ and $r = 4$ with sufficiently large $\beta > 0$. In \[41\], the author not only obtained the same result as \[40\] just under the assumption that $u_0 \in H^1(\mathbb{R}^3)$, but also proved that the strong solution is unique in the larger class of weak solution for any $r \geq 2$, which are significant improvements of those results established in \[4\]. In \[33, 34, 36\], the authors have established the existence of global attractors, uniform attractors and pullback attractors, respectively, in $V$ and $H^2(\Omega) \cap V$ for problem (1) by combining asymptotic a priori estimates with Sobolev compactness embedding Theorem. The existence of an exponential attractor in $V$ for problem (1) was proved by using the squeezing property in \[35\]. In \[23\], the authors have established the existence of a finite dimensional global attractor for problem (1) with $r > 4$ by the methods of $\ell$-trajectories, which extends the results established in \[33\].

The study of the long-time behavior of infinite dimensional dynamical systems or semigroups generated by autonomous partial differential equations can be usually reduced to the description of global attractor (see \[9, 29, 37\]), which has two essential drawbacks: on the one hand, the rate of attraction of the trajectories may be small and it is usually very difficult to estimate this rate in terms of the physical parameters of the problem. On the other hand, it is very sensitive to perturbations such that the global attractor can change drastically under very small perturbations of the initial dynamical system. These drawbacks obviously lead to essential difficulties in numerical simulations of global attractors and even make the global attractor unobservable in some sense. Hence, the notion of an exponential attractor was introduced in \[12\]. Until now, we mainly construct the exponential attractor of some autonomous evolution equations by the squeezing property \[12\] or the smoothing property \[13\] of the difference of two solutions. However, to ensure the finiteness of the fractal dimension of the exponential attractor in these two ways, there is an additional assumption on the Hölder continuity in time of the semigroup, which is very difficult to be proved in some situations, for example, the solutions suffer from lack of regularity.

In recent years, more attention was paid to the processes generated by the non-autonomous differential equations. In particular, the authors \[14\] have first extended the construction of exponential attractors for discrete semigroups in \[13\] to non-autonomous problems by using the concept of forwards attractors and developed an explicit algorithm for discrete evolution processes by the smoothing property of the evolution process. Moreover, they also construct an exponential attractor of the time continuous process generated by non-autonomous reaction-diffusion systems. Later, this construction has been modified in the pullback senses and the algorithm has been extended to time continuous evolution processes in \[10, 22\] based on the
existence of a fixed bounded pullback absorbing set, which leads to the boundedness of the section of the exponential pullback attractor in the past, but it may be unbounded in the future. Recently, the authors [5] proved the existence of a pullback exponential attractor for a asymptotically compact processes under significantly weak hypothesis that the process is lack of the strong regularity properties in time. Moreover, they obtained better estimates for the fractal dimension of the sections of the pullback attractor based on a family of time-dependent absorbing sets, which can even grow in the past, and establish the existence of a pullback exponential attractor whose sections are not necessarily uniformly bounded in the past.

In this paper, we mainly consider the long-time behavior of solutions for problem (1) in the case of $r > 4$. If $r \in (4, 8)$, it is very tricky to prove the uniqueness of weak solutions for the three dimensional non-autonomous Navier-Stokes equations with nonlinear damping. Therefore, we cannot define a solution process on $H$ such that the existence of pullback exponential attractors can be investigated by the theory of the infinite dimensional dynamical systems established in [10, 14, 22]. Fortunately, we know from the work of Y. Zhou [41] that any weak solutions of problem (1) for $r > 4$ will become a unique strong solution after any $t > \tau$. Inspired by the idea of the method of $\ell$-trajectories for any fixed small $\ell > 0$ proposed in [26], we will prove the existence of a pullback exponential attractor in $H$ for the three dimensional non-autonomous Navier-Stokes equations with nonlinear damping by the methods of $\ell$-trajectories. To the best of our knowledge, the method of $\ell$-trajectories is based on an observation that the limit behavior of solutions to a dynamical system in an original phase space can be equivalently captured by the limit behavior of $\ell$-trajectories which are continuous parts of solution trajectories that are para-metrized by time from an interval of the length $\ell > 0$ and it can weaken the requirements on the regularity of the solutions.

Throughout this paper, let $C$ be the generic positive constants independent of initial data and let $C(\cdot)$ be the positive constants depending on $\cdot$, let $H^s(\Omega)$ $(s \in \mathbb{N})$ and $L^q(\Omega)$ $(1 \leq q \leq \infty)$, respectively, be the usual vector or scalar Sobolev spaces and Lebesgue spaces.

2. Preliminaries. In order to study problem (1), we introduce the space of divergence-free functions defined by

$$\mathcal{V} = \{ u \in (C^\infty_c(\Omega))^3 : \nabla \cdot u = 0 \}.$$

Denote by $H$ and $V$ the closure of $\mathcal{V}$ with respect to the norms in $L^2(\Omega)$ and $H^1_0(\Omega)$, respectively. If $X$ is any Banach space, then $\| \cdot \|_X$ and $X'$ denote its norm and its dual space, respectively. Furthermore, the symbol $\langle \cdot, \cdot \rangle$ usually stands for the duality pairing between $V'$ and $V$. Let $A = -P\Delta$ be the Stokes operator, where $P$ is the Leray-Helmotz projection from $L^2(\Omega)$ onto $H$.

In what follows, we give the definition of solutions for problem (1).

**Definition 2.1.** ([23]) Assume that $u_\tau \in H$ and $f(x, t) \in L^2_{loc}(\mathbb{R}_+; H)$. For any fixed $T > \tau$, a function $(u(x, t), p(x, t))$ is called a weak solution of problem (1) on $(\tau, T)$, if

$$u(t) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^r(\tau, T; L^r(\Omega)),$$

$$u_t(t) \in L^2(\tau, T; V') + L^{\frac{n}{n-1}}(\tau, T; L^{\frac{n}{n-1}}(\Omega))$$
satisfy
\[
\int_{\Omega} u(t_1) \cdot v \, dx + \nu \int_{t_0}^{t_1} \int_{\Omega} \nabla u(t) \cdot \nabla v \, dx \, dt + \int_{t_0}^{t_1} \int_{\Omega} [(u(t) \cdot \nabla) u(t)] \cdot v \, dx \, dt
\]
\[
+ \beta \int_{t_0}^{t_1} \int_{\Omega} |u(t)|^{-2} u(t) \cdot v \, dx \, dt = \int_{t_0}^{t_1} \int_{\Omega} f(x) \cdot v \, dx \, dt + \int_{\Omega} u(t_0) \cdot v \, dx\]
for any \( v \in V \), \( t_0, t_1 \in [\tau, T] \) and \( u(x, \tau) = u_\tau(x) \) in the sense of trace.

Moreover, if \( u_\tau \in V \), a weak solution is called a strong solution of problem (1) on \((\tau, T)\), in addition, if it satisfies
\[
u(t) \in L^\infty(\tau, T; V) \cap L^2(\tau, T; H^2(\Omega)) \cap L^r(\tau, T; L^{3r}(\Omega))
\]
and
\[
\begin{cases}
u(t) \in L^{\frac{p_2-p_1}{p_2}}(\tau, T; H), & \nu(t) \in C([\tau, T]; V_u) \text{ if } r > 6; \\
u(t) \in L^2(\tau, T; H), & \nu(t) \in C([\tau, T]; V), \text{ if } r \in [3, 6],
\end{cases}
\]
where \( C([\tau, T]; V_u) \) denotes the space of weakly continuous functions valued in \( V \) defined on the interval \([\tau, T]\).

**Lemma 2.2.** ([6, 26, 27, 32]) Assume that \( p_1 \in (1, \infty] \), \( p_2 \in [1, \infty) \). Let \( X \) be a Banach space and let \( X_0, X_1 \) be separable and reflexive Banach spaces such that \( X_0 \subset \subset X \subset X_1 \). Then
\[
Y = \{ u \in L^{p_1}(0, \ell; X_0) : u' \in L^{p_2}(0, \ell; X_1) \} \subset \subset L^{p_1}(0, \ell; X),
\]
where \( \ell \) is a fixed positive constant.

**Definition 2.3.** ([24, 38]) Let \( X \) be a Banach space. A process \( \{U(t, \tau)\}_{t \geq \tau} \) is said to be norm-to-weak continuous on \( X \), if for any \( t, \tau \in \mathbb{R} \) with \( t \geq \tau \) and for every sequence \( x_n \in X \), from the condition \( x_n \to x \) strongly in \( X \), it follows that \( U(t, \tau)x_n \to U(t, \tau)x \) weakly in \( X \).

**Lemma 2.4.** ([24, 38]) Let \( X, Y \) be two Banach spaces, and let \( X^*, Y^* \) be the dual spaces of \( X, Y \), respectively. If \( X \) is dense in \( Y \), the injection \( i : X \to Y \) is continuous and its adjoint \( i^* : Y^* \to X^* \) is dense. In addition, assume that \( \{U(t, \tau)\}_{t \geq \tau} \) is a norm-to-weak continuous process on \( Y \). Then \( \{U(t, \tau)\}_{t \geq \tau} \) is a norm-to-weak continuous process on \( X \) if and only if \( \{U(t, \tau)\}_{t \geq \tau} \) maps compact sets of \( X \) into bounded sets of \( X \) for any \( t, \tau \in \mathbb{R} \) with \( t \geq \tau \).

**Definition 2.5.** ([25]) A process \( \{U(t, \tau)\}_{t \geq \tau} \) on a Banach space \( X \) is said to be \( \tau \)-continuous, if for every \( u_0 \in X \) and every \( t \in \mathbb{R} \), the \( X \)-valued function
\[
\tau \to U(t, \tau)u_0
\]
is continuous and bounded on \((-\infty, \ell]\).

**Definition 2.6.** ([29, 37]) Let \( H \) be a separable real Hilbert space. For any non-empty compact subset \( K \subset H \), the fractal dimension of \( K \) is the number
\[
d_F(K) = \limsup_{\epsilon \to 0^+} \frac{\log(N_\epsilon(K))}{\log(\frac{1}{\epsilon})},
\]
where \( N_\epsilon(K) \) denotes the minimum number of open balls in \( H \) with radii \( \epsilon > 0 \) that are necessary to cover \( K \).
Lemma 2.7. ([27]) Let $X$ and $Y$ be two metric spaces and let $f : X \to Y$ be $\alpha$-Hölder continuous on the subset $A \subset X$. Then

$$d_F(f(A)) \leq \frac{1}{\alpha} d_F(A).$$

In particular, the fractal dimension does not increase under a Lipschitz continuous mapping.

Definition 2.8. ([5, 10, 14]) Let $\{U(t, s)\}_{t \geq s}$ be an evolution process in a metric space $X$. We call the family $\mathcal{M} = \{\mathcal{M}(t) : t \in \mathbb{R}\}$ a pullback exponential attractor for the evolution process $\{U(t, s)\}_{t \geq s}$ in $X$, if

(i) the subsets $\mathcal{M}(t) \subset X$ are non-empty and compact in $X$ for all $t \in \mathbb{R}$,
(ii) the family $\mathcal{M} = \{\mathcal{M}(t) : t \in \mathbb{R}\}$ is positively semi-invariant, that is $U(t, s)\mathcal{M}(s) \subset \mathcal{M}(t)$ for all $t \geq s$,
(iii) the fractal dimension of the sections $\mathcal{M}(t)$ in $X$ is uniformly bounded for all $t \in \mathbb{R}$,
(iv) the family $\mathcal{M} = \{\mathcal{M}(t) : t \in \mathbb{R}\}$ exponentially pullback attracts bounded subsets of $X$, that is, there exists a positive constant $\omega > 0$ such that for every bounded subset $B \subset X$ and $t \in \mathbb{R}$,

$$\lim_{s \to +\infty} e^{\omega s} \dist(U(t, t-s)B, \mathcal{M}(t)) = 0.$$

3. The existence of pullback attractors.

3.1. The well-posedness of solutions. The well-posedness of solutions for the three dimensional non-autonomous Navier-Stokes equations with nonlinear damping was obtained in [4, 41]. Here we only state the result as follows.

Theorem 3.1. Assume that $f(x, t) \in L^2_{\text{loc}}(\mathbb{R}^n; H)$ and $r \geq 2$. Then for any $u_\tau \in H$, there exists at least one weak solution $u(t) \in C(\mathbb{R}, H) \cap L^2_{\text{loc}}(\mathbb{R}; V)$ of problem (1). Furthermore, if $r > 4$ and $u_\tau \in V$, then there exists a unique global strong solution of problem (1), which depends continuously on the initial data with respect to the topology of $H$.

By Theorem 3.1, we can define a family of continuous processes $\{U(t, \tau) : -\infty < \tau \leq t < \infty\}$ in $V$ by

$$U(t, \tau)u_\tau = u(t) := u(t; \tau, u_\tau)$$

for all $t \geq \tau$, which is $(V, V)$-continuous, where $u(t)$ is the strong solution of problem (1) with $u(\tau) = u_\tau \in V$. That is, a family of mappings $U(t, \tau) : V \to V$ satisfies

$$U(\tau, \tau) = \text{id} \quad \text{(identity)},$$
$$U(t, \tau) = U(t, r)U(r, \tau)$$

for any $\tau \leq r \leq t$.

Combining Theorem 3.1 and the proof of Lemma 3.3 in [36], we can easily prove the following conclusions.

Corollary 1. Assume that $f(x, t) \in L^2_{\text{loc}}(\mathbb{R}^n; H)$. Then the process $\{U(t, \tau)\}_{t \geq \tau}$ associated with problem (1) is $\tau$-continuous.
Definition 3.2. Let \( \text{pullback attractor} \) be a sequence of \( \text{weak solution} \) for problem \((1)\) such that \( u_m(\tau) = u_{\tau,m} \). For any \( T > \tau \), if there exists a subsequence converging \((s,)\) weakly in spaces \( \{v(t) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) : v_t \in L^2(\tau, T; V') + L^\infty(\tau, T; L^\infty(\Omega))\} \) to a certain function \( u(t) \). Then \( u(t) \) is a weak solution on \([\tau, T]\) with \( u(\tau) = u_\tau \).

3.2. The existence of pullback attractors in \( \mathcal{X}_\ell \). In this subsection, we will consider the existence of pullback attractors for problem \((1)\) by using the \( \ell \)-trajectory method. From Theorem 3.1, we deduce that there may be many trajectories of problem \((1)\) starting from the same initial data \( u_\tau \in H \), but there are only one trajectory of problem \((1)\) starting from \( u(t_0) \) for any \( t_0 > \tau \). For the sake of simplicity, we denote by \( \{\chi^\beta(s, \tau; u_\tau)\}_{s \in [\tau, \tau + \ell]} \) the trajectories of problem \((1)\) starting from \( u_\tau \), for short \( \chi^\beta(s, \tau; u_\tau) \) \( (\beta \in \Gamma_{u_\tau}) \), where \( \Gamma_{u_\tau} \) is the set of indices marking trajectories starting from \( u_\tau \). In the following, we first give the mathematical framework of pullback attractor.

Definition 3.2. Let \( \ell \) be a fixed positive constant. Define

\[
\mathcal{X}_\ell = \bigcup_{\tau \in \mathbb{R}} \bigcup_{u_\tau \in H} \bigcup_{\beta \in \Gamma_{u_\tau}} \chi^\beta(s, \tau; u_\tau)
\]
equipped with the topology of \( L^2(0, \ell; H) \).

Since \( \chi^\beta(s, \tau; u_\tau) \in C([0, \ell]; H_u) \) for any \( \tau \in \mathbb{R} \), \( u_\tau \in H \) and \( \beta \in \Gamma_{u_\tau} \), it makes sense to talk about the point values of trajectories. However, it is not clear whether \( \mathcal{X}_\ell \) is closed in \( L^2(0, \ell; H) \) such that \( \mathcal{X}_\ell \) in general is not a complete metric space. In what follows, we first give the definition of some operators.

For any \( t \in [0, 1] \), we define the mapping \( e_t : \mathcal{X}_\ell \to H \) by

\[
e_t(\chi(s, \tau; u_\tau)) = \chi(s + \ell, \tau; u_\tau)
\]
for any \( \chi(s, \tau; u_\tau) \in \mathcal{X}_\ell \).

For any \( t \geq \tau \), the operators \( L(t, \tau) : \mathcal{X}_\ell \to \mathcal{X}_\ell \) are given by the relation

\[
L(t, \tau)\chi(s, \tau; u_\tau) = u(t + s - \tau, \tau; u_\tau)
\]
\[= U(t + s - \tau, \ell + \tau)e_t(\chi(s, \tau; u_\tau)) = \chi(t + s - \tau, \tau; u_\tau), \quad s \in [\tau, \tau + \ell]
\]
for any \( \chi(s, \tau; u_\tau) \in \mathcal{X}_\ell \), where \( u(s, \tau; u_\tau) \) is the weak solution of problem \((1)\) on \([\tau, \ell + \ell]\) such that \( u|_{[\tau, \tau + \ell]} = \chi(s, \tau; u_\tau) \), we can easily prove the operators \( \{L(t, \tau)\}_{t \geq \tau} \) is a process on \( \mathcal{X}_\ell \).

In what follows, let \( \mathcal{D}_\ell \) be the family of all nonempty bounded sets of \( \mathcal{X}_\ell \) and let \( \mathcal{D} \) be the family of all nonempty bounded sets of \( H \). In the following, we will perform some a priori estimates of solutions to prove the existence of pullback attractors for problem \((1)\).

Theorem 3.3. Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) satisfies

\[
\sup_{\tau \in \mathbb{R}} \int_{\tau-1}^\tau \|f(s)\|_{L^2(\Omega)}^2 \, ds < +\infty.
\]

Then there exist a positive constant \( \rho_1 \) satisfying for any \( B_\ell \in \mathcal{D}_\ell \) and a time \( \tau_1 = \tau_1(B_\ell) \geq 0 \) such that for any weak solutions of problem \((1)\) with short trajectory
\[ \chi(s, \tau; u_z) \in B_\ell, \text{ we have for any } t \in \mathbb{R}, \]
\[ \| \nabla u(t) \|_{L^2(\Omega)}^2 + \| u(t) \|_{L^r(\Omega)}^2 + \int_0^t \left( \| u(t + \zeta) \|_{L^2(\Omega)}^2 + \| u(t + \zeta) \|_{L^r(\Omega)}^{2r-2} \right) d\zeta \leq \rho_1 \]
for any \( t - \tau \geq \tau_1. \)

**Proof.** Multiplying the first equation of (1) by \( u \) and integrating the resulting equality over \( \Omega \), we find
\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2(\Omega)}^2 + \nu \| \nabla u(t) \|_{L^2(\Omega)}^2 + \beta \| u(t) \|_{L^r(\Omega)}^r \leq \int_{\Omega} f(x, t) \cdot u(x, t) \, dx \\
\leq \| f(t) \|_{L^2(\Omega)} \| u(t) \|_{L^2(\Omega)} + \frac{1}{2\nu \lambda_1} \| f(t) \|_{L^2(\Omega)},
\]
which implies that
\[
\frac{d}{dt} \| u(t) \|_{L^2(\Omega)}^2 + \frac{\nu \lambda_1}{2} \| u(t) \|_{L^2(\Omega)}^2 + \frac{\nu}{2} \| \nabla u(t) \|_{L^2(\Omega)}^2 + 2\beta \| u(t) \|_{L^r(\Omega)}^r \leq \frac{1}{\nu \lambda_1} \| f(t) \|_{L^2(\Omega)}^2. \quad (2)
\]
It follows from the classical Gronwall inequality that
\[
\| u(t) \|_{L^2(\Omega)}^2 \leq \frac{1}{\ell} \int_0^\ell e^{\frac{\nu \lambda_1}{2}(s-t)} \| u(t + \zeta) \|_{L^2(\Omega)}^2 \, d\zeta \\
+ \frac{1}{\nu \lambda_1 \ell} \int_0^\ell \int_{t+\zeta}^t e^{\frac{\nu \lambda_1}{2}(s-t)} \| f(s) \|_{L^2(\Omega)}^2 \, ds \, d\zeta \\
\leq \frac{1}{\ell} e^{\frac{\nu \lambda_1}{2}(\ell-t)} \int_0^\ell \| u(t + \zeta) \|_{L^2(\Omega)}^2 \, d\zeta \\
+ \frac{1}{\nu \lambda_1} (1 + \frac{2}{\nu \lambda_1}) \sup_{t \in \mathbb{R}} \int_{t-1}^t \| f(s) \|_{L^2(\Omega)}^2 \, ds. \quad (3)
\]
Moreover, integrating (2) from \( t \) to \( t + \ell \), we obtain
\[
\frac{\nu \lambda_1}{2} \int_0^\ell \| u(t + \zeta) \|_{L^2(\Omega)}^2 \, d\zeta + \frac{\nu}{2} \int_0^\ell \| \nabla u(t + \zeta) \|_{L^2(\Omega)}^2 \, d\zeta \\
+ 2\beta \int_0^\ell \| u(t + \zeta) \|_{L^r(\Omega)}^r \, d\zeta \leq \frac{1}{\nu \lambda_1} \int_t^{t+\ell} \| f(\zeta) \|_{L^2(\Omega)}^2 \, d\zeta + \| u(t) \|_{L^2(\Omega)}^2 \\
\leq \frac{1}{\nu \lambda_1} \int_t^{t+\ell} \| f(\zeta) \|_{L^2(\Omega)}^2 \, d\zeta + \frac{1}{\ell} e^{\frac{\nu \lambda_1}{2}(\ell-t)} \int_0^\ell \| u(t + \zeta) \|_{L^2(\Omega)}^2 \, d\zeta
\]
\[ + \frac{1}{\nu_1} \left( 1 + \frac{2}{\nu_1 \lambda_1} \right) \sup_{r \in \mathbb{R}} \int_{r-1}^r \| f(s) \|^2_{L^2(\Omega)} ds \leq \frac{1}{\ell} e^{\frac{\nu_2}{\lambda_2} (\ell - t)} \int_0^\ell \| u(\tau + \zeta) \|^2_{L^2(\Omega)} d\zeta \]

\[ + \frac{1}{\nu_1} (2 + \frac{2}{\nu_1 \lambda_1}) \sup_{r \in \mathbb{R}} \int_{r-1}^r \| f(s) \|^2_{L^2(\Omega)} ds. \] (4)

Taking the \( H \) inner product of the first equation of (1) with \( Au + |u|^{r-2} u \) and using Young’s inequality, we find

\[
\frac{d}{dt} \left( \frac{1}{2} \| \nabla u(t) \|^2_{L^2(\Omega)} + \frac{1}{r} \| u(t) \|^r_{L^r(\Omega)} \right) + \left( \beta + \nu \right) \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 dx \]

\[ + \nu \| Au(t) \|^2_{L^2(\Omega)} + \frac{4(r-2)(\beta + \nu)}{r^2} \int_\Omega |\nabla u(t)|^2 dx + \beta \| u(t) \|^2_{L^{2r-2}(\Omega)} \]

\[ = \int_\Omega f(x, t) \cdot Au(t) dx + \int_\Omega f(x, t) \cdot |u(t)|^{r-2} u dx - \int_\Omega [\nabla u(t) \cdot u(t)] \cdot Au(t) dx \]

\[ \leq \| f(t) \|_{L^2(\Omega)} \| Au(t) \|_{L^2(\Omega)} + \| f(t) \|_{L^2(\Omega)} \| u(t) \|_{L^{2r-2}(\Omega)}^{r-1} \]

\[ + \| Au(t) \|_{L^2(\Omega)} \left( \int_\Omega |u(t)|^2 |\nabla u(t)|^2 dx \right)^{\frac{1}{2}} \]

\[ \leq \left( \frac{1}{\nu} + \frac{1}{2\beta} \right) \| f(t) \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| Au(t) \|^2_{L^2(\Omega)} \]

\[ + \frac{\beta}{2} \| u(t) \|^2_{L^{2r-2}(\Omega)} + \frac{1}{\nu} \int_\Omega |u(t)|^2 |\nabla u(t)|^2 dx \]

\[ \leq \left( \frac{1}{\nu} + \frac{1}{2\beta} \right) \| f(t) \|^2_{L^2(\Omega)} + \frac{\beta}{2} \| u(t) \|^2_{L^{2r-2}(\Omega)} + \frac{\beta}{2} \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 dx \]

\[ + \frac{\nu}{2} \| Au(t) \|^2_{L^2(\Omega)} + C(\beta, \nu, r) \| \nabla u(t) \|^2_{L^2(\Omega)}, \]

which implies that

\[
\frac{d}{dt} \left( \| \nabla u(t) \|^2_{L^2(\Omega)} + \frac{2}{r} \| u(t) \|^r_{L^r(\Omega)} \right) + (\beta + \nu) \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 dx \]

\[ + \nu \| Au(t) \|^2_{L^2(\Omega)} + \frac{8(r-2)(\beta + \nu)}{r^2} \int_\Omega |\nabla u(t)|^2 dx + \beta \| u(t) \|^2_{L^{2r-2}(\Omega)} \]

\[ \leq \left( \frac{2}{\nu} + \frac{1}{\beta} \right) \| f(t) \|^2_{L^2(\Omega)} + C(\beta, \nu, r) \| \nabla u(t) \|^2_{L^2(\Omega)}. \] (5)

For any \( \zeta \in (0, \ell) \), integrating inequality (5) from \( t - \zeta \) to \( t \) and integrating the resulting inequality over \( (0, \ell) \) with respect to \( \zeta \), we obtain

\[ \ell \| \nabla u(t) \|^2_{L^2(\Omega)} + \frac{2\ell}{r} \| u(t) \|^r_{L^r(\Omega)} \]

\[ \leq \left( \frac{2}{\nu} + \frac{1}{\beta} \right) \int_0^\ell \int_{t-\zeta}^t \| f(s) \|^2_{L^2(\Omega)} ds \, d\zeta + C(\beta, \nu, r) \int_0^\ell \int_{t-\zeta}^t \| \nabla u(s) \|^2_{L^2(\Omega)} ds \, d\zeta \]

\[ + \frac{2}{\nu} \int_0^\ell \| u(t - \zeta) \|^2_{L^2(\Omega)} d\zeta + \frac{2}{r} \int_0^\ell \| u(t - \zeta) \|^r_{L^r(\Omega)} d\zeta \]

\[ \leq \left( \frac{2}{\nu} + \frac{1}{\beta} \right) \ell \| f(s) \|^2_{L^2(\Omega)} ds + C(\beta, \nu, r) \ell \int_{t-\ell}^t \| \nabla u(s) \|^2_{L^2(\Omega)} ds. \]
\[ + \int_0^t \| \nabla u(t - \zeta) \|^2_{L^2(\Omega)} \, d\zeta + \frac{2}{r} \int_0^t \| u(t - \zeta) \|^r_{L^r(\Omega)} \, d\zeta \]
\[ \leq \left( \frac{2}{\nu} + \frac{1}{\beta} \right) t \int_{t-\ell}^t \| f(s) \|^2_{L^2(\Omega)} \, ds \]
\[ + \left( C(\beta, \nu, r, \ell, 1) + 1 \right) \int_{t-\ell}^t \left( \| u(s) \|^2_{L^2(\Omega)} + \frac{2}{r} \| u(s) \|^r_{L^r(\Omega)} \right) \, ds. \quad (6) \]

It follows from inequalities (4) and (6) that
\[ \| \nabla u(t) \|^2_{L^2(\Omega)} + \frac{2}{r} \| u(t) \|^r_{L^r(\Omega)} \]
\[ \leq C(\beta, \nu, r, \ell, 1) e^{\frac{\nu}{4}(2\ell + \tau - t)} \int_0^t \| u(t + \zeta) \|^2_{L^2(\Omega)} \, d\zeta \]
\[ + C(\beta, \nu, r, \ell, 1) \sup_{r \in \mathbb{R}} \int_{r-1}^r \| f(s) \|^2_{L^2(\Omega)} \, ds. \quad (7) \]

We infer from inequalities (5) and (7) that
\[ \nu \int_0^t \| Au(t + \zeta) \|^2_{L^2(\Omega)} \, d\zeta + (\beta + \nu) \int_0^t \int_{\Omega} |u(t + \zeta)|^{r-2} |\nabla u(t + \zeta)|^2 \, dx \, d\zeta \]
\[ + \frac{8(\nu - 2)(\beta + \nu)}{r^2} \int_0^t \int_{\Omega} |\nabla u(t + \zeta)|^2 \, dx \, d\zeta + \beta \int_0^t \| u(t + \zeta) \|^{2r-2}_{L^{2r-2}(\Omega)} \, d\zeta \]
\[ \leq \left( \frac{2}{\nu} + \frac{1}{\beta} \right) t \int_{t-\ell}^t \| f(t + \zeta) \|^2_{L^2(\Omega)} \, d\zeta \]
\[ + \| \nabla u(t + \zeta) \|^2_{L^2(\Omega)} \| u(t + \zeta) \|^2_{L^2(\Omega)} \]
\[ + \frac{2}{r} \| u(t + \zeta) \|^r_{L^r(\Omega)} \]
\[ \leq C(\beta, \nu, r, \ell, 1) e^{\frac{\nu}{4}(2\ell + \tau - t)} \int_0^t \| u(t + \zeta) \|^2_{L^2(\Omega)} \, d\zeta \]
\[ + C(\beta, \nu, r, \ell, 1) \sup_{r \in \mathbb{R}} \int_{r-1}^r \| f(s) \|^2_{L^2(\Omega)} \, ds. \quad (8) \]

Thanks to
\[ \| u(t) \|^2_{L^2(\Omega)} \leq \nu \| Au(t) \|^2_{L^2(\Omega)} + \left( \int_{\Omega} |u(t)|^2 |\nabla u(t)|^2 \, dx \right)^{\frac{1}{2}} \]
\[ + \| u(t) \|^{\frac{1}{r-1}}_{L^{2r-1}(\Omega)} + \| f(t) \|_{L^2(\Omega)}, \quad (9) \]
we infer from inequalities (4), (6) and (7)-(9) that there exist a positive constant \( \rho_1 \) satisfying for any \( B_\ell \in \mathcal{D}_\ell \) and a time \( \tau_1 = \tau_1(B_\ell) \geq 0 \) such that for any weak solutions of problem (1) with short trajectory \( \chi(s, \tau; u_\tau) \in B_\ell \), we have for any \( t \in \mathbb{R} \),
\[ \| \nabla u(t) \|^2_{L^2(\Omega)} + \| u(t) \|^r_{L^r(\Omega)} + \int_0^t \| u(t + \zeta) \|^2_{L^2(\Omega)} + \| u(t + \zeta) \|^{2r-2}_{L^{2r-2}(\Omega)} \]
\[ + \| u(t + \zeta) \|^2_{L^2(\Omega)} + \| u(t + \zeta) \|^r_{L^r(\Omega)} \, d\zeta \leq \rho_1 \quad (10) \]
for any \( t - \tau \geq \tau_1 \).
**Corollary 3.** Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies
\[
\sup_{r \in \mathbb{R}} \int_{r-1}^{r} \|f(s)\|_{L^2(\Omega)}^2 \, ds < +\infty.
\]
Then there exist a positive constant $\rho_1$ satisfying for any $B \in \mathcal{D}$, there exists a time $\tau'_1 = \tau'_1(B) \geq 0$ such that for any weak solutions of problem (1) with initial data $u_\tau \in B$, we have for any $t \in \mathbb{R}$,
\[
\|\nabla u(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{L^r(\Omega)}^r + \int_0^t \left(\|u(t + \zeta)\|_{L^{3r}(\Omega)}^2 + \|u(t + \zeta)\|_{L^{2r-2}(\Omega)}^{2r-2} + \|u_t(t + \zeta)\|_{L^2(\Omega)}^2 + \|u(t + \zeta)\|_{H^2(\Omega)}^2\right) \, d\zeta \leq \rho_1
\]
for any $t - \tau \geq \tau'_1$.

Let
\[
B_0 = \left\{ u \in V \cap L^r(\Omega) : \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^r(\Omega)}^r \leq \rho_1 \right\},
\]
then $B_0 \in \mathcal{D}$. For any $t \in \mathbb{R}$, we infer from Corollary 3 that there exists a time $\tau_0 = \tau_0(B_0) \geq 0$ such that for any initial data $u_\tau \in B_0$ and any $\tau \leq t - \tau_0$, we have
\[
U(t, \tau)B_0 \subset B_0.
\]

For any $t \in \mathbb{R}$, define
\[
B_1(t) = \bigcup_{\tau \in [t - \tau_0, t]} \{ U(t, \tau)u_\tau : u_\tau \in B_0 \},
\]
and
\[
B_2(t) = \overline{B_1(t)}^H
\]
and
\[
B_2(t) = \{ \chi \in X : \ell_0(\chi) \in B_2(t) \}.
\]
From Corollary 3 and the proof of Theorem 3.3, we deduce
\[
U(t, \tau)B_1(\tau) \subset \overline{B_1(t)}^H
\]
for any $\tau \leq t$ and for any $t \in \mathbb{R}$, $B_1(t) \in \mathcal{D}$. Moreover, we have the following conclusion.

**Proposition 1.** Assume that for any $t \in \mathbb{R}$, $B_1(t) \in \mathcal{D}$ defined above. Then for any $t \in \mathbb{R}$,
\[
B_2(t) = \overline{B_1(t)}^H \in \mathcal{D}
\]
and
\[
U(t, \tau)B_2(\tau) \subset B_2(t)
\]
for any $t \geq \tau$.

**Proof.** For any $\tau \in \mathbb{R}$, from the definition of $B_2(t)$, we infer that for any $u_\tau \in B_2(t)$, there exists a sequence $\{u_{n, \tau}\}_{n=1}^\infty \subset B_1(\tau)$ such that
\[
u_{n, \tau} \to u_\tau \text{ in } H, \text{ as } n \to \infty.
\]
We infer from (5) that for any $\tau \in \mathbb{R}$ and any $t \geq \tau$,
\[
\|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u(t)\|_{L^r(\Omega)}^r \leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \int_\tau^t \|f(s)\|_{L^2(\Omega)}^2 \, ds + \|\nabla u(\tau)\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u(\tau)\|_{L^r(\Omega)}^r \right) e^{C(\beta, \nu, r)(t-\tau)}
\]
\[
\leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \sup_{r \in \mathbb{R}} \int_{t-1}^t \|f(s)\|_{L^2(\Omega)}^2 \, ds + \|\nabla u(\tau)\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u(\tau)\|_{L^r(\Omega)}^r \right) e^{C(\beta, \nu, r)(t-\tau)}.
\]

Therefore, for any $u_\tau \in B_0$ and any $\tau \in [t-\tau_0, t]$, we have
\[
\|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u(t)\|_{L^r(\Omega)}^r \leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \tau_0 \sup_{r \in \mathbb{R}} \int_{t-1}^t \|f(s)\|_{L^2(\Omega)}^2 \, ds + \rho_1 \right) e^{C(\beta, \nu, r)\tau_0},
\]
which implies that for any fixed $t \in \mathbb{R}$, $B_1(t) \subset \{u \in H : \|\nabla u\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u\|_{L^r(\Omega)}^r \leq \rho(\tau_0)\}$. Thus, the sequence $\{u_{n, \tau}\}_{n=1}^{\infty}$ is uniformly bounded in $V \cap L^r(\Omega)$ and $V \cap L^r(\Omega)$ is a reflexive Banach space, we deduce that there exist some $v \in V \cap L^r(\Omega)$ and a subsequence $\{u_{n_j, \tau}\}_{j=1}^{\infty}$ of $\{u_{n, \tau}\}_{n=1}^{\infty}$ such that
\[
u_{n_j, \tau} \rightharpoonup v \text{ in } V \cap L^r(\Omega), \text{ as } j \to \infty.
\]

Therefore, it follows from the lower semi-continuity of norms $\|\cdot\|_{L^2(\Omega)}$ as well as $\|\cdot\|_{L^r(\Omega)}$ and the compactness of $V \cap L^r(\Omega) \subset H$ that
\[
\nu = \nu_\tau
\]
and
\[
\|\nabla \nu_\tau\|_{L^2(\Omega)}^2 + \frac{2}{r} \|\nu_\tau\|_{L^r(\Omega)}^r \leq \liminf_{j \to \infty} \left(\|\nabla u_{n_j, \tau}\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u_{n_j, \tau}\|_{L^r(\Omega)}^r \right) \leq \rho(\tau_0),
\]
which entails that for any $t \in \mathbb{R}$, $B_2(t) \subset \{u \in H : \|\nabla u\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u\|_{L^r(\Omega)}^r \leq \rho(\tau_0)\}$ and $B_2(t) = B_1(t \frac{1}{H}) \in D$.

For any fixed $\tau \in \mathbb{R}$ and any fixed $t > \tau$, we conclude from the definition of $B_2(\tau)$, we infer that for any $u_\tau \in B_2(\tau)$, there exists a sequence $\{u_{n, \tau}\}_{n=1}^{\infty} \subset B_1(\tau)$ such that
\[
u_{n, \tau} \to \nu_\tau \text{ in } H, \text{ as } n \to \infty.
\]

We conclude from Theorem 3.1 and the fact that the set $B_1(\tau)$ is a subset of $V \cap L^r(\Omega)$ for any $\tau \in \mathbb{R}$ that
\[
u(t, \tau)u_{n, \tau} \to \nu(t, \tau)u_\tau \text{ in } H, \text{ as } n \to \infty.
\]

From $\{U(t, \tau)u_{n, \tau}\}_{n=1}^{\infty} \subset B_1(\tau)$, we obtain $U(t, \tau)u_\tau \in B_2(t)$. Therefore, we deduce that
\[
u(t, \tau)B_2(\tau) \subset B_2(t)
\]
for any $t \geq \tau$. 

□
Let
\[ Y_1 = \{ \chi \in X_\ell : \chi \in L^2(0, \ell; V), \chi_\ell \in L^1(0, \ell; (H^2(\Omega) \cap V)) \} \]
and
\[ Y_2 = \{ \chi \in X_\ell : \chi \in L^2(0, \ell; H^2(\Omega) \cap V), \chi_\ell \in L^2(0, \ell; H) \}, \]
respectively, equipped with the following norms
\[ \| \chi \|_{Y_1} = \left\{ \int_0^\ell \| \nabla \chi(t) \|^2_{L^2(\Omega)} \, dt + \left( \int_0^\ell \| \chi(t) \|_{(H^2(\Omega) \cap V)} \, dt \right)^2 \right\}^{\frac{1}{2}}. \]

Define \( \hat{B}_1^t = \{ B_1^t(t) : t \in \mathbb{R} \} \) and \( \hat{B}_2^t = \{ B_2^t(t) : t \in \mathbb{R} \} \), where
\[ B_1^t(t) = \left\{ \chi \in X_\ell : \int_0^\ell \| \chi(t + r) \|^2 \, dr + \left( \int_0^\ell \| \chi(t + r) \|_{(H^2(\Omega) \cap V)} \, dr \right) \leq \rho_1 \right\} \]
and
\[ B_2^t(t) = \left\{ \chi \in X_\ell : \int_0^\ell \| \chi(t + r) \|^2_{H^2(\Omega) \cap V} \, dr + \int_0^\ell \| \chi(t + r) \|^2 \, dr \leq \rho_1 \right\}. \]

From Proposition 1 and Theorem 3.3, we know that \( L(t, \tau)B_1^0(\tau) \subset B_1^0(t) \) for any \( t \geq \tau \) as well as \( L(t, \tau)B_2^0(\tau) \subset B_1^0(t) \) and \( L(t, \tau)B_2^0(\tau) \subset B_2^0(t) \) for any \( t - \tau \geq \tau_1 \).

**Lemma 3.4.** Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) satisfies
\[ \sup_{r \in \mathbb{R}} \int_{r-1}^{r} \| f(s) \|^2_{L^2(\Omega)} \, ds < +\infty. \]
Then \( L(t, \tau)B_0^0\|B_0^0(L^2(0, \ell; H)) \subset B_0^t(\tau) \subset B_0^t(t) \) for any \( t \geq \tau \).

**Proof.** Thanks to \( L(t, \tau)B_0^0(\tau) \subset B_0^t(t) \) for any \( t \geq \tau \), it is enough to prove that for any \( t \in \mathbb{R}, \)
\[ B_0^t(t) \subset B_0^t(\tau) \subset B_0^t(t). \]
For any fixed \( \tau \in \mathbb{R} \) and any \( \chi_0 \in B_0^t(\tau) \), there exists a sequence of short trajectories \( \chi_n(s, \tau; u_n, r) \in B_0^t(\tau) \) such that \( \chi_n(s, \tau; u_n, r) \to \chi_0 \) in \( L^2(0, \ell; H) \). Since \( \epsilon_0(\chi_n(s, \tau; u_n, r)) \in B_2(\tau) \) for any \( n \in \mathbb{N} \), there exists a subsequence \( \{ \epsilon_n(\chi_n(s, \tau; u_n, r)) \}_{n=1}^\infty \) of \( \{ \epsilon_0(\chi_n(s, \tau; u_n, r)) \}_{n=1}^\infty \) and \( u_r \in H \) such that \( \epsilon_0(\chi_n(s, \tau; u_n, r)) \to u_r \) in \( H \). From the proof of the existence of weak solutions for problem (1), we deduce that for any \( S > \tau \), there exists a subsequence converging (\( * \)-) weakly in spaces \( \{ v(t) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) : v_0 \in L^2(\tau, T; V) + L^\infty(\tau, T; L^\infty(\Omega)) \} \) to a certain function \( u(t) \) with \( u(\tau) = u_r \). Therefore, we obtain \( \chi_0 \in X_\ell \) from Corollary 2. It remains to show that \( \epsilon_0(\chi_0) = u_r \in B_2(\tau) \). Since for any \( \tau \in \mathbb{R} \), \( B_2(\tau) \) is closed in \( H \), we obtain \( \epsilon_0(\chi_0) = u_r \in B_2(\tau) \). Therefore, we obtain \( \chi_0 \in B_0^t(\tau) \). 
\[ \square \]
Lemma 3.5. Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) satisfies
\[
\sup_{r \in \mathbb{R}} \int_{r-1}^{r} \| f(s) \|^2_{L^2(\Omega)} \, ds < +\infty.
\]
Then for any \( \tau \in \mathbb{R} \), the mapping \( L(t, \tau) : X_{\ell} \to X_{\ell} \) is locally Lipschitz continuous on \( B^0_\ell(\tau) \) for all \( t \geq \tau \).

Proof. For any fixed \( \tau \in \mathbb{R} \) and any \( \chi^1, \chi^2 \in B^0_\ell(\tau) \), for any \( t > \tau + \ell \), let \( u_1(t) = L(t, \tau) \chi^1, u_2(t) = L(t, \tau) \chi^2 \) and let \( u = u_1 - u_2 \), then \( u \) satisfies the following equation
\[
\frac{\partial u}{\partial t} + \nu Au + P[(u \cdot \nabla)u] + Pu \cdot \nabla u + \beta P[|u_1|^{-2}u_1 - |u_2|^{-2}u_2] = 0. \quad (11)
\]
It follows from the proof of Lemma 3.6 in [23] that
\[
\begin{aligned}
\frac{d}{dt} \| u(t) \|^2_{L^2(\Omega)} + \nu \| \nabla u(t) \|^2_{L^2(\Omega)} &+ C(r) \| u(t) \|_{L^r(\Omega)} \\
&\leq C \left( \| u_2(t) \|^2_{L^{3r}(\Omega)} + \| u_2(t) \|_{L^r(\Omega)}^{2r} \right) \| u(t) \|^2_{L^2(\Omega)} \\
&\leq C \left( 1 + \| u_2(t) \|_{L^{3r}(\Omega)} \right) \| u(t) \|^2_{L^2(\Omega)} \quad (12)
\end{aligned}
\]
for any \( r \geq 3 \).

Let \( s \in (0, \ell) \) and integrating (12) from \( \tau + s \) to \( t + s \), we obtain
\[
\begin{aligned}
\| u(t + s) \|^2_{L^2(\Omega)} &\leq C \int_{\tau + s}^{t + s} \left( 1 + \| u_2(\zeta) \|_{L^{3r}(\Omega)} \right) \| u(\zeta) \|^2_{L^2(\Omega)} d\zeta + \| u(\tau + s) \|^2_{L^2(\Omega)} \quad (13).
\end{aligned}
\]
From the classical Gronwall inequality, we deduce
\[
\begin{aligned}
\| u(t + s) \|^2_{L^2(\Omega)} &\leq \| u(\tau + s) \|^2_{L^2(\Omega)} \exp \left( C \int_{\tau + s}^{t + s} \left( 1 + \| u_2(\zeta) \|_{L^{3r}(\Omega)} \right) d\zeta \right) \\
&\leq \| u(\tau + s) \|^2_{L^2(\Omega)} \exp \left( C \int_{\tau}^{t + \ell} \left( 1 + \| u_2(\zeta) \|_{L^{3r}(\Omega)} \right) d\zeta \right). \quad (14)
\end{aligned}
\]
Integrating (14) with respect to \( s \) for \( 0 \) to \( \ell \), we obtain
\[
\begin{aligned}
\int_{0}^{\ell} \| u(t + s) \|^2_{L^2(\Omega)} \, ds &\leq \exp \left( C \int_{\tau}^{t + \ell} \left( 1 + \| u_2(\zeta) \|_{L^{3r}(\Omega)} \right) d\zeta \right) \int_{0}^{\ell} \| u(\tau + s) \|^2_{L^2(\Omega)} \, ds. \quad (15)
\end{aligned}
\]
Taking the \( H \) inner product of the first equation of (1) with \( Au \) and using Young inequality, we find
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|^2_{L^2(\Omega)} + \nu \| Au(t) \|^2_{L^2(\Omega)} &+ \beta \int_{\Omega} |u(t)|^{-2} \| \nabla u(t) \|^2 \, dx \\
&+ \frac{4(r - 2)\beta}{r^2} \int_{\Omega} |\nabla u(t)|^2 \, dx \\
&= \int_{\Omega} f(x, t) \cdot Au(t) \, dx - \int_{\Omega} [(u(t) \cdot \nabla)u(t)] \cdot Au(t) \, dx.
\end{aligned}
\]
We deduce from the Sobolev embedding inequality
\[ \|f(t)\|_{L^2(\Omega)} \|Au(t)\|_{L^2(\Omega)} + \|Au(t)\|_{L^2(\Omega)} \left( \int_\Omega |u(t)|^2 |\nabla u(t)|^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{\nu} \|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|Au(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega |u(t)|^2 |\nabla u(t)|^2 \, dx \]
\[ \leq \frac{1}{\nu} \|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|Au(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega |u(t)|^2 |\nabla u(t)|^2 \, dx \]
\[ + C(\beta, \nu, r) \|\nabla u(t)\|_{L^2(\Omega)}^2, \]
which implies that
\[ \frac{d}{dt} \|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{8(r - 2)\beta}{r^2} \int_\Omega |\nabla u(t)|^2 \, dx \]
\[ \leq \frac{2}{\nu} \|f(t)\|_{L^2(\Omega)}^2 + C(\beta, \nu, r) \|\nabla u(t)\|_{L^2(\Omega)}^2. \]  

We infer from the classical Gronwall inequality that
\[ \|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{8(r - 2)\beta}{r^2} \int_\Omega |\nabla u(t)|^2 \, dx \]
\[ \leq \left( \frac{2(t - \tau)}{\nu} \sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(\zeta)\|_{L^2(\Omega)}^2 \, d\zeta + \|\nabla u_r\|_{L^2(\Omega)}^2 \right) e^{C(\beta, \nu, r)(t-\tau)}. \]

We deduce from the Sobolev embedding inequality \( \|u\|_{L^5(\Omega)} \leq \lambda_0 \|\nabla u\|_{L^2(\Omega)} \) for any \( u \in V \) and some \( \lambda_0 > 0 \) that
\[ \int_\tau^t \|u(s)\|_{L^{5\prime}(\Omega)}^2 \, ds \]
\[ \leq \frac{\lambda_0^2 r^2}{8(r - 2)\beta} \left( \frac{2(t - \tau)}{\nu} \sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(\zeta)\|_{L^2(\Omega)}^2 \, d\zeta + \|\nabla u_r\|_{L^2(\Omega)}^2 \right) e^{C(\beta, \nu, r)(t-\tau)}. \]

Since \( e_0(\chi^2) \) is uniformly bounded in \( V \cap L^\ast(\Omega) \) for any \( \chi^2 \in B^{\ell}_0(\tau) \), we conclude from (18) that
\[ \mathcal{M}_t(t, \tau) = e^{C(t+\ell)(1+\|u_2(t)\|_{L^{3\prime}(\Omega)})} \int_{r-1}^t \]
\[ \leq e^{C(t+\ell)(\|Q(t, \tau)\|_{L^2(\Omega)} + \|\nabla u_r\|_{L^2(\Omega)}^2)} e^{C(\beta, \nu, r)(t+\ell-\tau)} \]
\[ \leq e^{C(t+\ell)(\|Q(t, \tau)\|_{L^2(\Omega)} + \|\nabla u_r\|_{L^2(\Omega)}^2)} e^{C(\beta, \nu, r)(t+\ell-\tau)} \]
\[ Q(t, \tau) = \frac{2(t + \ell - \tau)}{\nu} \sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(\zeta)\|_{L^2(\Omega)}^2 \, d\zeta. \]

Therefore, the mapping \( L_\ell : X_\ell \rightarrow X_\ell \) is locally Lipschitz continuous on \( B^{\ell}_0(\tau) \) for all \( t \geq \tau \). \( \Box \)

We can immediately obtain the existence of a pullback attractor in \( X_\ell \) stated as follows.

**Theorem 3.6.** Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) satisfies
\[ \sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(s)\|_{L^2(\Omega)}^2 \, ds < +\infty. \]
Then the process \( \{L(t, \tau)\}_{t \geq \tau} \) generated by problem (1) possesses a pullback attractor 
\( \mathcal{A}_t = \{A_t(t) : t \in \mathbb{R}\} \) in \( X_t \) and \( e_1(A_t(t - \ell)) \) is included in \( B_2(t) \) for any \( t \in \mathbb{R} \), where

\[
e_1(A_t(t - \ell)) = \{e_1(\chi) : \chi \in A_t(t - \ell)\}
\]

for any \( t \in \mathbb{R} \).

From Theorem 3.3, Lemma 2.2, Lemma 2.4 and Lemma 3.4, thanks to the compactness of \( H^2(\Omega) \cap V \subseteq L^p(\Omega) \) for any \( p > 2 \), we conclude that the process 
\( \{L(t, \tau)\}_{t \geq \tau} \) is norm-to-weak continuity on \( L^2(0, \ell; L^p(\Omega)) \) and \( B_2^p = \{B^p_2(t) : t \in \mathbb{R}\} \) is a family of compact pullback absorbing sets in \( L^2(0, \ell; L^p(\Omega)) \), where

\[
B^p_2(t) = \left\{ \chi \in X_t : \int_0^\ell \|\chi(t + r)\|^2_{H^2(\Omega) \cap V} \, dr + \int_0^\ell \|\chi(t + r)\|^2_2 \, dr \leq \rho_1 \right\}.
\]

We immediately obtain the following conclusion.

**Corollary 4.** Assume that \( f \in L^2_{loc}(\mathbb{R}; H) \) satisfies

\[
\sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(s)\|^2_{L^2(\Omega)} \, ds < +\infty.
\]

Then the process \( \{L(t, \tau)\}_{t \geq \tau} \) generated by problem (1) possesses a pullback attractor
\( \mathcal{A}_t^p = \{A^p_t(t) : t \in \mathbb{R}\} \) in \( L^2(0, \ell; L^p(\Omega)) \) for any \( p > 2 \).

4. The existence of pullback exponential attractors. In this section, we construct the pullback exponential attractors of problem (1) by combining the methods of \( \ell \)-trajectories and the smoothing property of the process \( \{L(t, \tau)\}_{t \geq \tau} \).

4.1. The existence of pullback exponential attractors in \( X_t \). In this subsection, we prove the smooth property of the process \( \{L(t, \tau)\}_{t \geq \tau} \) to construct the pullback exponential attractor \( \mathcal{E}_t = \{E_t(t) : t \in \mathbb{R}\} \).

**Theorem 4.1.** Assume that \( f \in L^2_{loc}(\mathbb{R}; H) \) satisfies

\[
\sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(s)\|^2_{L^2(\Omega)} \, ds < +\infty.
\]

For any fixed \( \tau \in \mathbb{R} \), let \( \chi^1 \) and \( \chi^2 \) be two short trajectories belonging to \( B^p_0(\tau) \). Then there exists a positive constant \( \kappa \) independent of \( t \) such that for arbitrary \( t \geq \tau + \ell \), we have

\[
\|L(t, \tau)\chi^1 - L(t, \tau)\chi^2\|^2_{L^2(\Omega)} \leq \kappa (1 + M_\tau(t, \tau)) \int_0^\ell \|\chi^1(t + r) - \chi^2(t + r)\|^2_2 \, dr,
\]

where \( M_\tau(t, \tau) \) is given in (19).

**Proof.** For any fixed \( \tau \in \mathbb{R} \) and any \( \chi^1, \chi^2 \in B^p_0(\tau) \), for any \( t > \tau + \ell \), let \( u_1(t) = L(t, \tau)\chi^1, u_2(t) = L(t, \tau)\chi^2 \) and let \( u = u_1 - u_2 \), we conclude from inequality (12) that

\[
\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} + \nu \|
abla u(t)\|^2_{L^2(\Omega)} + C(r)\|u(t)\|^2_{L^r(\Omega)} \leq \mathbb{L}(t)\|u(t)\|^2_{L^2(\Omega)},
\]

where

\[
L(t) = C \left( 1 + \|u_2(t)\|^2_{L^2(\Omega)} \right).
\]
For any $t \geq \tau + \ell$, integrating (20) from $t - s$ to $t + \ell$ with $s \in [0, \frac{\ell}{2}]$, we conclude
\[
\|u(t + \ell)\|_{L^2(\Omega)}^2 + \nu \int_{t-s}^{t+\ell} \|\nabla u(\zeta)\|_{L^2(\Omega)}^2 d\zeta + C(r) \int_{t-s}^{t+\ell} \|u(\zeta)\|_{L^r(\Omega)}^r d\zeta
\leq \int_{t-s}^{t+\ell} \mathbb{L}(\zeta)\|u(\zeta)\|_{L^2(\Omega)}^2 d\zeta + \|u(t-s)\|_{L^2(\Omega)}^2.
\]
It follows from the classical Gronwall inequality that
\[
\|u(t + \ell)\|_{L^2(\Omega)}^2 + \nu \int_{t-s}^{t+\ell} \|\nabla u(\zeta)\|_{L^2(\Omega)}^2 d\zeta + C(r) \int_{t-s}^{t+\ell} \|u(\zeta)\|_{L^r(\Omega)}^r d\zeta
\leq \exp\left( \int_{t-s}^{t+\ell} \mathbb{L}(\zeta) d\zeta \right) \|u(t-s)\|_{L^2(\Omega)}^2.
\]
(21)
For any $t \geq \tau + \ell$ and any $s \in [0, \frac{\ell}{2}]$, integrating (20) from $\tau + s$ to $t - s$, we obtain
\[
\|u(t-s)\|_{L^2(\Omega)}^2 \leq \int_{\tau+s}^{t-s} \mathbb{L}(r)\|u(r)\|_{L^2(\Omega)}^2 dr + \|u(\tau+s)\|_{L^2(\Omega)}^2.
\]
We deduce from the classical Gronwall inequality that
\[
\|u(t-s)\|_{L^2(\Omega)}^2 \leq C(r) \int_{\tau+s}^{t-s} \mathbb{L}(r)\|u(\tau+s)\|_{L^2(\Omega)}^2 dr + \|u(\tau+s)\|_{L^2(\Omega)}^2.
\]
(22)
Combining (21) with (22), we obtain
\[
\nu \int_0^\ell \|\nabla u(t + \zeta)\|_{L^2(\Omega)}^2 d\zeta + C(r) \int_0^\ell \|u(t + \zeta)\|_{L^r(\Omega)}^r d\zeta
\leq \exp\left( \int_{\tau}^{t+\ell} \mathbb{L}(\zeta) d\zeta \right) \|u(\tau+s)\|_{L^2(\Omega)}^2 + \mathcal{M}_f(t, \tau)\|u(\tau+s)\|_{L^2(\Omega)}^2.
\]
Integrating the above inequality over $0, \frac{\ell}{2}$ with respect to $s$, we obtain
\[
\nu \int_0^\ell \|\nabla u(t + \zeta)\|_{L^2(\Omega)}^2 d\zeta + C(r) \int_0^\ell \|u(t + \zeta)\|_{L^r(\Omega)}^r d\zeta
\leq \frac{2\mathcal{M}_f(t, \tau)}{\ell} \int_0^{\frac{\ell}{2}} \|u(\tau+s)\|_{L^2(\Omega)}^2 ds.
\]
Since $\mathcal{M}_f(t, \tau)$ is bounded depending on $t - \tau$, we obtain
\[
\nu \int_0^\ell \|\nabla u(t + \zeta)\|_{L^2(\Omega)}^2 d\zeta + C(r) \int_0^\ell \|u(t + \zeta)\|_{L^r(\Omega)}^r d\zeta
\leq \frac{2\mathcal{M}_f(t, \tau)}{\ell} \int_0^{\ell} \|u(\tau+s)\|_{L^2(\Omega)}^2 ds.
\]
(23)
For any $v \in V \cap L^r(\Omega)$, we infer from Hölder inequality and Lemma 2.2 in [39] that
\[
\langle u(t), v \rangle \leq \nu \|\nabla u(t)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} \|u\|_{L^r(\Omega)} \|\nabla v\|_{L^2(\Omega)}
+ \|u_2\|_{L^2(\Omega)} \|u\|_{L^r(\Omega)} \|\nabla v\|_{L^2(\Omega)} + C_2 \beta \int_\Omega (|u_1| + |u_2|)^{r-2} |u| v dx
\leq C \|\nabla u_1\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \nu \|\nabla u(t)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}
\leq \frac{2\mathcal{M}_f(t, \tau)}{\ell} \int_0^{\ell} \|u(\tau+s)\|_{L^2(\Omega)}^2 ds.
It follows from the proof of Proposition 1 and inequality (2) that
\[ \text{such that} \]
Then for any \( \ell \), we obtain
\[ \text{Theorem 4.2.} \]
Assume that \( B \) is a pullback absorbing set.
Proof. From Theorem 3.3, we know that there exists a \( D_t \)-pullback absorbing set
\[ B_0(t) = \{ B_0(t) : t \in \mathbb{R} \} \] in \( X \) satisfying \( L(t, \tau)B_0(t) \subset B_0(t) \) for any \( t \geq \tau \) and \( A_t(\tau) \subset B_0(\tau) \) for any \( \tau \in \mathbb{R} \). Therefore, we infer from Theorem 4.1 that for any
$k \in \mathbb{Z}$, there exists some time $t_1 > 0$ such that the mapping \( L(kt_1, (k-1)t_1) : B_0^0((k-1)t_1) \rightarrow B_0^0((k-1)t_1) \) enjoys the smoothing property
\[
\| L(kt_1, (k-1)t_1)x_1 - L(kt_1, (k-1)t_1)x_2 \|_{Y_1} \leq K\|x_1 - x_2\|_{X}
\] (27)
and
\[
L(kt_1, (k-1)t_1)B_0^0((k-1)t_1) \subset B_1^1(kt_1)
\] (28)
for any $x_1, x_2 \in B_0^0((k-1)t_1)$, where
\[
K^2 = \kappa(1 + M_0(kt_1, (k-1)t_1))
\]
\[
= \kappa + \kappa \exp \left( C(t_1 + \ell) + \frac{C\lambda_0^2}{8(r-2)\beta} \right) \chi
\]
\[
\left( \frac{2(t_1 + \ell)}{\nu} \sup_{\tau \in \mathbb{R}} \int_{r-1}^{r} \| f(\xi) \|_{L^2(\Omega)} d\xi + \varrho(\tau_0) \right) e^{C(\beta, \nu, r)(t_1 + \ell)}
\]
is a fixed positive constant.

For any natural number $k \in \mathbb{Z}$, $B_0^0(kt_1)$ is uniformly bounded in $X$, which implies that there exists a positive constant $R$ and $\chi_k \in B_0^0(kt_1)$ such that $B_0^0(kt_1) \subset B_X(\chi_k; R)$ for all $k \in \mathbb{Z}$, denote by $W^0(k) = \{ \chi_k \}$. Moreover, for any $\eta \in (0, \frac{1}{2})$, we can choose some elements $\eta_1, \eta_2, \ldots, \eta_N \in Y_1$ such that
\[
B_Y(0; 1) \subset \bigcup_{j=1}^{N} B_X(\eta_j; \frac{\eta}{K}),
\]
where $N = N^X_\frac{\eta}{K} (B_Y(0; 1))$.

We infer from inequality (27) that for any $\chi \in B_X(\chi_k-1; R)$,
\[
\| L(kt_1, (k-1)t_1)\chi - L(kt_1, (k-1)t_1)\chi_{k-1} \|_{Y_1}
\]
\[
\leq K\|\chi - \chi_{k-1}\|_{X} \leq KR,
\]
i.e.,
\[
L(kt_1, (k-1)t_1)B_X(\chi_{k-1}; R) \subset B_Y(0; 1) \subset \bigcup_{j=1}^{N} B_X(\eta_j; \frac{\eta}{K}),
\]
which implies that for any $\chi \in B_Y(0; 1)$, we have
\[
\frac{\chi - L(kt_1, (k-1)t_1)\chi_{k-1}}{KR} \in B_Y(0; 1) \subset \bigcup_{j=1}^{N} B_X(\eta_j; \frac{\eta}{K}),
\]
and
\[
B_Y(L(kt_1, (k-1)t_1)\chi_{k-1}; KR)
\]
\[
\subset \bigcup_{j=1}^{N} B_X(L(kt_1, (k-1)t_1)\chi_{k-1} + KR\eta_j; R\eta).
\]
which yields that there exist $z_1, z_2, \ldots, z_N \in L(kt_1, (k-1)t_1)B_0^0((k-1)t_1)$ and $y_1, y_2, \ldots, y_N \in B_0^0((k-1)t_1)$ such that
\[
L(kt_1, (k-1)t_1)B_0^0((k-1)t_1)
\]
\[
= L(kt_1, (k-1)t_1) \left( B_X(\chi_{k-1}; R) \cap B_0^0((k-1)t_1) \right)
\]
\[
\subset \bigcup_{j=1}^{N} B_X(z_j; 2\eta R)
\]
and

\[ L(kt_1, (k - 1)t_1)y_j = z_j \]

for any \( j = 1, 2, \ldots, N \). Denoting the new set of centres by \( W^1(k) \), it follows

\[ L(kt_1, (k - 1)t_1)B_0^1((k - 1)t_1) \subset \bigcup_{\chi \in W^1(k)} B_X(\chi; 2\eta R) \]

with \( W^1(k) \subset L(kt_1, (k - 1)t_1)B_0^1((k - 1)t_1) \) and \( \sharp W^1(k) \leq N \).

In what follows, we assume that the sets \( W^m(k) \subset L(kt_1, (k - m)t_1)B_0^m((k - m)t_1) \subset B_0^m(kt_1) \) are already constructed for all \( m \leq n \), which satisfies

\[ L(kt_1, (k - m)t_1)B_0^m((k - m)t_1) \subset \bigcup_{\chi \in W^m(k)} B_X(\chi; (2\eta)^m R) \]

and

\[ \sharp W^m(k) \leq N^m. \]

In order to construct a covering of

\[ L(kt_1, (k - n - 1)t_1)B_0^m((k - n - 1)t_1) \]

let \( \chi \in W^n(k - 1) \), we proceed as before and use the covering of the unit ball \( B_Y(0, 1) \) by \( \frac{N}{K} \) balls in \( X \) to conclude

\[ B_Y(L(kt_1, (k - 1)t_1)\chi; (2\eta)^n KR) \]

\[ \subset \bigcup_{j=1}^N B_X(L(kt_1, (k - 1)t_1)\chi + (2\eta)^n KR\eta_j; (2\eta)^n R\eta), \]

which entails that

\[ B_Y(L(kt_1, (k - 1)t_1)\chi; (2\eta)^n KR) \subset \bigcup_{j=1}^N B_X(L(kt_1, (k - 1)t_1)y_j^{\chi}; (2\eta)^{n+1} R) \]

for some \( y_1^{\chi}, \ldots, y_N^{\chi} \in L((k-1)t_1, (k-n-1)t_1)B_0^m((k-n-1)t_1) \). Constructing in the same way such a covering by balls with radius \( (2\eta)^{n+1} R \) in \( X \) for every \( \chi \in W^n(k - 1) \), we obtain a covering of the set \( L(kt_1, (k - n - 1)t_1)B_0^n((k - n - 1)t_1) \) and denote the new set of centres by \( W^{n+1}(k) \), which yields \( \sharp W^{n+1}(k) \leq N^2 W^n(k - 1) \leq N^{n+1} \) and \( W^{n+1}(k) \subset L(kt_1, (k - n - 1)t_1)B_0^m((k - n - 1)t_1) \) as well as

\[ L(kt_1, (k - n - 1)t_1)B_0^m((k - n - 1)t_1) \subset \bigcup_{\chi \in W^{n+1}(k)} B_X(\chi; (2\eta)^{n+1} R), \]
In order to obtain the existence of the pullback exponential attractor, for any \( k \in \mathbb{Z} \) and any \( n \in \mathbb{N} \), we define

\[
    E^0(k) = W^0(k) = \{ \chi_k \},
    \]

\[
    E^1(k) = L(kt_1, (k - 1)t_1)E^0(k - 1) \cup W^1(k),
    \]

\[
    \vdots
    \]

\[
    E^n(k) = L(kt_1, (k - 1)t_1)E^{n-1}(k - 1) \cup W^n(k) = \bigcup_{j=0}^{n} L(kt_1, (k - j)t_1)W^{n-j}(k - j).
    \]

From the fact that \( L(t, \tau)B^0_0(\tau) \subset B^0_0(t) \) for any \( t \geq \tau \), we conclude for any \( k \in \mathbb{Z} \),

\[
    L(kt_1, (k - n)t_1)B^k_0((k - n)t_1) \subset L(kt_1, (k - m)t_1)B^k_0((k - m)t_1)
    \]

for any \( n, m \in \mathbb{N} \) with \( n \geq m \). Moreover, for any \( k \in \mathbb{Z} \), the family of sets \( E^n(k)(n \in \mathbb{N}) \), satisfy the following properties

(i) \( L(kt_1, (k - 1)t_1)E^n(k - 1) \subset E^{n+1}(k) \), \( E^0(k) = W^0(k) \subset B_0^0(kt_1) \), \( E^n(k) \subset L(kt_1, (k - n)t_1)B^k_0((k - n)t_1) \subset B^k_0(kt_1) \),

(ii) \( \exists E^n(k) \leq \sum_{i=0}^{n} N_i \leq (n + 1)N^n \),

(iii) \( L(kt_1, (k - n)t_1)B^k_0((k - n)t_1) \subset \bigcup_{\chi \in W^n(k)} B_X(\chi; (2\eta)^nR) \)

\[
    \subset \bigcup_{\chi \in E^n(k)} B_X(\chi; (2\eta)^nR).
    \]

For any \( k \in \mathbb{Z} \), define

\[
    \tilde{\mathcal{M}}_t(k) = \bigcup_{n=0}^{\infty} E^n(k).
    \]

In what follows, we will prove that for any \( k \in \mathbb{Z} \), the set \( \tilde{\mathcal{M}}_t(k) \) is pre-compact, finite-dimensionality and positively semi-invariant with respect to the process \( \{L(mt_1, nt_1) : m \geq n \} \).

First of all, for any \( k \in \mathbb{Z} \) and any \( m \in \mathbb{N} \), it follows from the property (i) that

\[
    L((m + k)t_1, kt_1)\tilde{\mathcal{M}}_t(k) = \bigcup_{n=0}^{\infty} L((m + k)t_1, kt_1)E^n(k) \subset \bigcup_{n=0}^{\infty} E^{n+m}(m + k)
    \]

\[
    = \bigcup_{n=m}^{\infty} E^n(m + k) \subset \bigcup_{n=0}^{\infty} E^n(m + k) = \tilde{\mathcal{M}}_t(m + k).
    \]

Furthermore, for any \( k \in \mathbb{Z} \), since \( E^n(k) \subset L(kt_1, (k - n)t_1)B^k_0((k - n)t_1) \subset L(kt_1, (k - m)t_1)B^k_0((k - m)t_1) \) for any \( n \geq m \), we deduce

\[
    \tilde{\mathcal{M}}_t(k) = \bigcup_{n=0}^{\infty} E^n(k) \subset \bigcup_{n=0}^{m} E^n(k) \cup \bigcup_{n=m+1}^{\infty} E^n(k)
    \]

\[
    \subset \bigcup_{n=0}^{m} E^n(k) \cup L(kt_1, (k - m)t_1)B^k_0((k - m)t_1).
    \]
We infer from properties (ii) and (iii) that for any \( k \in \mathbb{Z} \),
\[
\mathcal{Z} \left( \bigcup_{n=0}^{m} E^n(k) \right) = \sum_{n=0}^{m} \mathcal{Z} E^n(k) \\
\leq (m+1) \mathcal{Z} E^m(k) \leq (m+1)^2 N^m
\]
and
\[
L(kt_1, (k-m)t_1) B_0^\ell((k-m)t_1) \subset \bigcup_{\chi \in W^m(k)} B_X(\chi; (2\eta)^m R).
\]
For any \( \epsilon > 0 \), there exists some positive integer \( m \) sufficiently large such that
\[
(2\eta)^m R \leq \epsilon < (2\eta)^{m-1} R.
\]
Therefore, for any \( k \in \mathbb{Z} \), we can estimate the number of \( \epsilon \)-balls needed to cover \( \mathcal{M}_\ell(k) \) as follows
\[
N^X_\epsilon(\mathcal{M}_\ell(k)) \leq \mathcal{Z} \left( \bigcup_{n=0}^{m} E^n(k) \right) + \mathcal{Z} W^m(k) \\
\leq (m+1)^2 N^m + N^m \leq 2(m+1)^2 N^m,
\]
which implies that for any \( k \in \mathbb{Z} \), there exists a finite number of \( \epsilon \)-net to cover \( \mathcal{M}_\ell(k) \). Therefore, \( \mathcal{M}_\ell(k) \) is a pre-compact subset of \( B_0^\ell(k) \) in \( X \) for any \( k \in \mathbb{Z} \).

For any fixed \( k \in \mathbb{Z} \), we conclude the fractal dimension of the set \( \mathcal{M}_\ell(k) \),
\[
dim_F(\mathcal{M}_\ell(k)) = \limsup_{\epsilon \to 0^+} \frac{\ln(N^X_\epsilon(\mathcal{M}_\ell(k)))}{-\ln \epsilon} \\
\leq \log_2(N(1))(N(1)) = \log_2(N(1)(N(1)).
\]

In what follows, we will prove that for any \( k \in \mathbb{Z} \), the set \( \mathcal{M}_\ell(k) \) exponentially attracts all bouned subsets of \( X \). For any bounded subset \( B^\ell \) of \( X_\ell \), we infer from Theorem 3.3 that for any \( k \in \mathbb{Z} \), there exists some \( t_2 = t_2(B^\ell) > 0 \) such that
\[
L(kt_1, k \tau) B^\ell \subset B_0^\ell(kt_1) \quad \text{for any} \quad kt_1 - \tau \geq t_2,
\]
which implies that there exists some natural number \( n_0 \in \mathbb{N} \) with \( n_0 t_1 \geq t_2 \) such that \( L(kt_1, (k-n)t_1) B^\ell \subset B_0^\ell(kt_1) \) for any \( n \geq n_0 \). Therefore, if \( n \geq n_0 + 1 \), we obtain
\[
\text{dist}_X(L(kt_1, (k-n)t_1) B^\ell, \mathcal{M}_\ell(k))
\]
\[
\leq \text{dist}_X(L(kt_1, (k-n+n_0)t_1) L((k-n+n_0)t_1, (k-n)t_1) B^\ell, \bigcup_{n=0}^{\infty} E^n(k))
\]
\[
\leq \text{dist}_X(L(kt_1, (k-n+n_0)t_1) B_0^\ell((k-n+n_0)t_1), \bigcup_{n=0}^{\infty} E^n(k))
\]
\[
\leq \text{dist}_X(L(kt_1, (k-n+n_0)t_1) B_0^\ell((k-n+n_0)t_1), E^{n-n_0}(k))
\]
\[
\leq (2\eta)^{n-n_0} R
\]
\[
= (2\eta)^{n-n_0} Re^{-\ln(\frac{1}{\eta})^n}.
\]

To obtain the existence of a pullback exponential attractor for the continuous time process \( \{L(t, \tau)\}_{t \geq \tau} \), we define
\[
\tilde{\mathcal{E}}_\epsilon(t) := L(t, kt_1) \mathcal{M}_\ell(k), \quad \text{for} \quad t \in [kt_1, (k+1)t_1).
\]
From Lemma 3.4, we know that $\tilde{B}_0^t$ is a family of closed subsets of $X$. For any $t \in \mathbb{R}$, let $\mathcal{E}_t(t)$ be the closure of $\tilde{\mathcal{E}}_t(t)$ in $X$.

Due to the Lipschitz-continuity of the process, the sets $\tilde{\mathcal{E}}_t(t)$ are compact in $X$. Moreover, we deduce from Lemma 2.7 that the same (uniform) bound on the fractal dimension of the sections $\mathcal{E}_t(t)$,

$$
dim_F(\mathcal{E}_t(t)) = \dim_F(\tilde{\mathcal{E}}_t(t)) = \dim_F(L(t, kt_1)\tilde{\mathcal{M}}_t(k))$$

$$\leq \dim_F(\tilde{\mathcal{M}}_t(k)) \leq \log \frac{1}{\eta} (N^X_0 (B_{Y_1}(0;1))), \text{ for } t \in [kt_1, (k+1)t_1].$$

In the following, we will prove that the sets $\{\mathcal{E}_t(t) : t \in \mathbb{R}\}$ is positively semi-invariant.

Let $t, s \in \mathbb{R}$ and $t \geq s$, then $s = kt_1 + s'_1$ and $t = lt_1 + t'_1$ for some $k, l \in \mathbb{Z}$, $k \leq l$ and $s'_1, t'_1 \in [0, t_1)$. If $l \geq k + 1$, we obtain

$$L(t, s)\tilde{\mathcal{E}}_t(s) = L(lt_1 + t'_1, kt_1 + s'_1)\tilde{\mathcal{E}}_t(kt_1 + s'_1)$$

$$= L(lt_1 + t'_1, kt_1 + s'_1)L(kt_1 + s'_1, kt_1)\tilde{\mathcal{M}}_t(k)$$

$$= L(lt_1 + t'_1, lt_1)L(lt_1, kt_1)\tilde{\mathcal{M}}_t(k)$$

$$\subset L(lt_1 + t'_1, lt_1)\tilde{\mathcal{M}}_t(l)$$

$$= \tilde{\mathcal{E}}_t(lt_1 + t'_1) = \tilde{\mathcal{E}}_t(t)$$

where we used the semi-invariance of the family $\{\tilde{\mathcal{M}}_t(k) : k \in \mathbb{Z}\}$ under the action of the process $\{L(mt_1, nt_1) : m \geq n\}$. On the other hand, if $l = k$, then $s = kt_1 + s'_1$ and $t = kt_1 + t'_1$ for some $s'_1, t'_1 \in [0, t_1)$ with $t'_1 \geq s'_1$ and

$$L(t, s)\tilde{\mathcal{E}}_t(s) = L(kt_1 + t'_1, kt_1 + s'_1)\tilde{\mathcal{E}}_t(kt_1 + s'_1)$$

$$= L(kt_1 + t'_1, kt_1 + s'_1)L(kt_1 + s'_1, kt_1)\tilde{\mathcal{M}}_t(k)$$

$$= L(kt_1 + t'_1, kt_1)\tilde{\mathcal{M}}_t(k) = \tilde{\mathcal{E}}_t(kt_1 + t'_1) = \tilde{\mathcal{E}}_t(t).$$

By the continuity of the process, we obtain the semi-invariance of the family $\{\mathcal{E}_t(t) : t \in \mathbb{R}\}$.

Finally, the set $\mathcal{E}_t(t)$ exponentially pullback attracts all bounded subsets of $X$ at time $t \in \mathbb{R}$. This follows immediately from the exponential pullback attracting property of the sets $\{\tilde{\mathcal{M}}_t(k) : k \in \mathbb{Z}\}$ and the Lipschitz-continuity property of the process $\{L(t, \tau) : t \geq \tau\}$. Therefore, the family $\tilde{\mathcal{E}}_t = \{\mathcal{E}_t(t) : t \in \mathbb{R}\}$ is a pullback exponential attractor for the process $\{L(t, \tau)\}_{t \geq \tau}$ in $X$. □

4.2. The existence of a pullback exponential attractor in $H$. In this subsection, we prove the existence of a pullback exponential attractor in $H$ of problem (1).

**Theorem 4.3.** Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies

$$\sup_{r \in \mathbb{R}} \int_{r-1}^r \|f(s)\|^2_{L^2(\Omega)} ds < +\infty.$$ 

Then for any fixed $\tau \in \mathbb{R}$, the mapping $e_1 : B_0^\tau(\tau - \ell) \rightarrow B_2(\tau) = e_1(B_0^\tau(\tau - \ell))$ is Lipschitz continuous. That is, for any two short trajectories $\chi^1, \chi^2 \in B_0^\ell(\tau)$, there exists a positive constant $\theta$ dependent on $\ell$ such that

$$\|e_1(\chi^1) - e_1(\chi^2)\|^2_{L^2(\Omega)} \leq \theta \int_0^\ell \|\chi^1(r) - \chi^2(r)\|^2_{L^2(\Omega)} dr.$$
Proof. For any fixed \( \tau \in \mathbb{R} \) and any \( \chi^1, \chi^2 \in B_0^\ell(\tau) \), for any \( t > \tau + \ell \), let \( u_1(t) = L(t, \tau)\chi^1 \), \( u_2(t) = L(t, \tau)\chi^2 \) and let \( u = u_1 - u_2 \). We conclude from inequality (12) that
\[
\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} + \nu \|\nabla u(t)\|^2_{L^2(\Omega)} + C(\tau)\|u(t)\|_{L^2(\Omega)} \leq L(t)\|u(t)\|^2_{L^2(\Omega)},
\]
where
\[
L(t) = C \left( 1 + \|u_2(t)\|^2_{L^2(\Omega)} \right).
\]
For any \( \tau \in \mathbb{R} \) and any \( \zeta \in (0, \ell) \), we infer from the classical Gronwall inequality that
\[
\|u(\tau + \ell)\|^2_{L^2(\Omega)} \leq \|u(\tau + \zeta)\|^2_{L^2(\Omega)} \exp(\int_{\tau+\zeta}^{\tau+\ell} L(r) \, dr)
\leq \|u(\tau + \zeta)\|^2_{L^2(\Omega)} \exp(\int_{\tau}^{\tau+\ell} L(r) \, dr).
\]
Integrating (31) over \((0, \ell)\), we obtain
\[
\|u(\tau + \ell)\|^2_{L^2(\Omega)} \leq \frac{1}{\ell} \exp(\int_{\tau}^{\tau+\ell} L(r) \, dr) \int_0^\ell \|u(\tau + \zeta)\|^2_{L^2(\Omega)} \, d\zeta.
\]
Thanks to (19), we know that
\[
\mathcal{M}_\ell(\tau, \tau) = e^{C\int_{\tau}^{\tau+\ell}(1 + \|u_2(\tau)\|^2_{L^2(\Omega)}) \, d\tau}
\leq e^{C\ell + \frac{c\beta^2}{a_\tau^2} \left( \frac{1}{\ell} \sup_{\tau \in \mathbb{R}} \int_{\tau-1}^{\tau} \|f(\zeta)\|^2_{L^2(\Omega)} \, d\zeta + g(\tau) \right)} e^{C(a_\tau, \nu, \ell)},
\]
which implies that the mapping \( e_1 : B_0^\ell(\tau - \ell) \to B_2(\tau) \) is Lipschitz continuous. \( \square \)

**Theorem 4.4.** Assume that \( f \in L^2_{loc}(\mathbb{R}; H) \) satisfies
\[
\sup_{\tau \in \mathbb{R}} \int_{\tau-1}^{\tau} \|f(s)\|^2_{L^2(\Omega)} \, ds < +\infty.
\]
Then for any \( \eta \in (0, \frac{1}{2}) \), there exists a pullback exponential attractor \( \mathcal{E} = \mathcal{E}^\eta = \{ \mathcal{E}(t) : t \in \mathbb{R} \} = \{ e_1(\mathcal{E}_t(t - \ell)) : t \in \mathbb{R} \} \) for the process \( \{ U(t, \tau) \}_{t \geq \tau} \) generated by problem (1).

**Proof.** From Lemma 2.7, Theorem 4.2 and Theorem 4.3, we know that for any \( t \in \mathbb{R} \), \( \mathcal{E}(t) = e_1(\mathcal{E}_t(t - \ell)) \) is compact and its fractal dimension is uniformly finite. As a result, \( L(t - \ell, s - \ell)\mathcal{E}_t(s - \ell) \subset \mathcal{E}_t(t - \ell) \) for any \( t \geq s \), we have
\[
U(t, s)\mathcal{E}(s) = U(t, s)e_1(\mathcal{E}_s(s - \ell))
= e_1(L(t - \ell, s - \ell)\mathcal{E}_s(s - \ell)) \subset e_1(\mathcal{E}_t(t - \ell)) = \mathcal{E}(t)
\]
for any \( t \geq s \). From the definition of \( B_2 \) and \( B_0^\ell \), we deduce that for any \( t \in \mathbb{R} \) and any bounded subset \( B \) of \( H \), there exists some time \( \bar{t} = \bar{t}(B) \) such that
\[
U(t, t - \tau)B \subset B_2(t) = e_0(B_0^\ell(t))
\]
for any \( \tau \geq \bar{t} \), which implies that there exists some natural number \( n_0 \) with \( n_0 t_1 \geq \bar{t} \) such that \( L(t, t - nt_1)B^\ell \subset B_0^\ell(t) \) for any \( n \geq n_0 \). Therefore, for any \( s \geq (n_0 + 1)t_1 \),
there exists some $k_0 \in \mathbb{N}$ and $s_1 \in [0, t_1)$ such that $s = k_0 t_1 + s_1$, we conclude from Theorem 4.3 and (29) that
\[
\text{dist}_X(U(t, t-s)B, \mathcal{E}(t)) = \text{dist}_X(U(t, t-s+\tilde{t})U(t-s+\tilde{t}, t-s)B, \mathcal{E}(t)) \\
\leq \text{dist}_X(U(t, t-s+\tilde{t})B_\ell(t-s+\tilde{t}, \mathcal{E}(t)) \\
= \text{dist}_X(U(t, t-s+\tilde{t})e_1(B_\ell^0(t-s+\tilde{t}-\ell)), e_1(\mathcal{E}_\ell(t-\ell))) \\
= \text{dist}_X(e_1(L(t-\ell, t-s+\tilde{t}-\ell)B_\ell^0(t-s+\tilde{t}-\ell)), e_1(\mathcal{E}_\ell(t-\ell))) \\
\leq \text{dist}_X[L(t-\ell, t-s+\tilde{t}-\ell)B_\ell^0(t-s+\tilde{t}-\ell), \mathcal{E}_\ell(t-\ell)],
\]
which implies that the family $\mathcal{E} = \{\mathcal{E}(t) : t \in \mathbb{R}\}$ exponentially attracts all bounded subsets of $H$ uniformly. Therefore, the family $\mathcal{E} = \{\mathcal{E}(t) : t \in \mathbb{R}\}$ is a pullback exponential attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ in $H$.

Corollary 5. Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies
\[
\sup_{r \in \mathbb{R}} \int_{r-1}^{r} \|f(s)\|^2_{L^2(\Omega)} \, ds < +\infty.
\]
Then the process $\{U(t, \tau)\}_{t \geq \tau}$ generated by problem (1) possesses a pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\} = \{e_1(A_\ell(t-\ell)) : t \in \mathbb{R}\}$, where $A_\ell(t-\ell)$ is the section of pullback attractor $\mathcal{A}_\ell = \{A_\ell(t) : t \in \mathbb{R}\}$ in $X_\ell$ for the process $\{L(t, \tau)\}_{t \geq \tau}$ generated by problem (1) obtained in Theorem 3.6.

Remark 1. Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies
\[
\sup_{r \in \mathbb{R}} \int_{r-1}^{r} \|f(s)\|^2_{L^2(\Omega)} \, ds < +\infty.
\]
Then each member $\mathcal{E}(t)$ of the pullback exponential attractor $\mathcal{E} = \{\mathcal{E}(t) : t \in \mathbb{R}\}$ in $H$ for the process $\{U(t, \tau)\}_{t \geq \tau}$ generated by problem (1) contains the section $A(t)$ of the pullback attractor established in Corollary 5.

Remark 2. Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies
\[
\sup_{r \in \mathbb{R}} \int_{r-1}^{r} \|f(s)\|^2_{L^2(\Omega)} \, ds < +\infty.
\]
If the Hölder continuity in time of the process $\{U(t, \tau)\}_{t \geq \tau}$ generated by problem (1) in $H$ can be obtained, the exponential attractor $\mathcal{M} = \{M(t) : t \in \mathbb{R}\}$ for the time continuous process $\{U(t, \tau)\}_{t \geq \tau}$ generated by problem (1) can be constructed in the usual way:
\[
\mathcal{M}_\ell(t) = \bigcup_{s \in [0, t_1]} L(t, s)\mathcal{E}_\ell(s)
\]
and
\[
M(t) = e_1(\mathcal{M}_\ell(t-\ell)).
\]

Remark 3. In the case that $r = 4$ and $\beta > \frac{1}{4p}$, after a minor modification, we can still obtain the conclusions in this paper.

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