On the global $W^{2,q}$ regularity for nonlinear $N$–systems of the $p$-Laplacian type in $n$ space variables

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Abstract

We consider the Dirichlet boundary value problem for nonlinear $N$-systems of partial differential equations with $p$-growth, $1 < p < 2$, in the $n$—dimensional case. For clearness, we confine ourselves to a particularly representative case, the well known $p$-laplacian system.

We are interested in regularity results, up to the boundary, for the second order derivatives of the solution. We prove $W^{2,q}$-global regularity results, for arbitrarily large values of $q$. In turn, the regularity achieved implies the Hölder continuity of the gradient of the solution. It is worth noting that we cover the singular case $\mu = 0$. See Theorem 2.1 below.

Keywords: $p$-Laplacian systems, regularity up to the boundary, full regularity.

1 Introduction

We are concerned with the regularity problem for solutions of nonlinear systems of partial differential equations with $p$-structure, $p \in (1, 2]$, under Dirichlet boundary conditions. In order to emphasize the main ideas we confine ourselves to the following representative case, where $\mu \geq 0$ is a fixed constant:

\begin{equation}
\begin{cases}
- \nabla \cdot \left( (\mu + |\nabla u|)^{p-2} \nabla u \right) = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\end{cases}
\end{equation}

The vector field $u = (u_1(x), \ldots, u_N(x))$, $N > 1$, is defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. When $\mu = 0$, the system (1.1) is the well-known $p$-Laplacian system.

It is worth noting that our interests concern global (up to the boundary), full regularity for the second derivatives of the solutions. Our results also hold in the singular case $\mu = 0$. For any bounded and sufficiently smooth domain $\Omega$, we prove $W^{2,q}(\Omega)$ regularity, for any $q \geq 2$. Therefore, we get, as a by product, the $\alpha$-Hölder continuity, up to the boundary, of the gradient of the solution, for any $\alpha < 1$. The results are obtained for $p$ belonging to intervals $[C, 2]$, where $C$ are suitable constants, whose expression may be explicitly calculated. In particular, if $\Omega$ is convex, solutions belong to $W^{2,2}(\Omega)$ for any $1 < p \leq 2$.

As usual, weak solutions are defined as follows (for notation and more precise statements see the sequel).

Definition 1.1. Assume that $f \in W^{-1, p'}(\Omega)$. We say that $u$ is a weak solution of problem (1.1) if $u \in W^{1,p}(\Omega)$ satisfies

\begin{equation}
\int_{\Omega} (\mu + |\nabla u|)^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \cdot \varphi \, dx,
\end{equation}

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for all \( \varphi \in W_0^{1,p}(\Omega) \).

It is immediate to verify that, if \( \mu > 0 \), sufficiently regular weak solutions to the problem (1.1) satisfy

\[
\Delta u - (p - 2) \frac{\nabla u \cdot \nabla \nabla u \cdot \nabla u}{(\mu + |\nabla u|)|\nabla u|} = f(\mu + |\nabla u|)^{2-p}.
\]

Here, and in the following, we use the notation \( \nabla u \cdot \nabla \nabla u \cdot \nabla u \) to indicate the vector whose \( i \)th component is \( \nabla u \cdot (\partial_i \nabla u) \partial_j u_i = (\partial_l u_k) (\partial_l^j u_k) (\partial_j u_i) \).

In the sequel we start by proving the existence of a (unique) strong solution

\( u \in W_0^{1,p}(\Omega) \cap W^{2,q}(\Omega) \),

of problem (1.3), under homogeneous Dirichlet boundary conditions, in the case \( \mu > 0 \) (see Theorem 3.1). Clearly, \( u \) solves (1.2). Furthermore, we prove that the \( W^{2,q}(\Omega) \) norms of the above strong solutions are uniformly bounded with respect to \( \mu \). This allows us, by passing to the limit as \( \mu \to 0 \), to extend the \( W^{2,q}(\Omega) \) regularity result to weak solutions of problem (1.1) in the case \( \mu = 0 \) (see Theorem 2.1).

The regularity issue for systems like (1.1) has received substantial attention, mostly concerned with the scalar case \( (N = 1) \), and with \( C^{1,\alpha}_{\text{loc}} \)-regularity. Here and in the following, by local regularity we mean interior regularity. The pioneering result dates back to Ural’tseva [25], where, for \( p > 2 \) and \( N = 1 \), the author proves \( C^{1,\beta}_{\text{loc}} \)-regularity for a suitable exponent \( \beta \). Still in the case \( N = 1 \) we recall the following contributions. In [23] the author proves \( W^{2,p}_{\text{loc}} \)-regularity for any \( p < 2 \), and also \( W^{2,2}_{\text{loc}} \)-regularity, for \( p > 2 \). In [10], for \( p > 2 \), the author proves \( C^{1,\beta}_{\text{loc}} \)-regularity up to the boundary, in \( \Omega \subset \mathbb{R}^n \). In [19] the author shows, for any \( p \in (1,2) \), \( W^{2,2} \cap C^{1,\alpha}_{\text{loc}} \)-regularity up to the boundary, in \( \Omega \subset \mathbb{R}^2 \).

For systems (solutions are \( N \)-dimensional vector fields, \( N > 1 \)), we recall [1] for \( p \in (1,2) \), [11] and [24] for \( p > 2 \), and [13] for any \( p > 1 \). The results proved in papers [1], [11] and [24] are local. Moreover all these papers deal only with homogeneous systems and the techniques, sometimes quite involved, seem not to be directly applicable to the non-homogeneous setting. In particular, [1] is the only paper in which the \( L^2_{\text{loc}} \)-regularity of second derivatives is considered. The results below are, in the non-scalar case, the first regularity results up to the boundary, for the second derivatives of solutions.

For related results and for an extensive bibliography we also refer to papers [2], [5], [6], [7], [9], [10], [17], [20], [21] and references therein.

We observe that we do not consider a more general dependence on \( \nabla u \), as for instance \( \varphi(\nabla u|) \nabla u \), under suitable assumptions on the scalar function \( \varphi \), just to emphasize the core aspects of the results and to avoid additional technicalities. For the same reason we avoid the introduction of lower order terms. Note that another, very similar, representative case can be obtained with the regular term \( (\mu + |\nabla u|^2)^{p-2} \) in place of \( (\mu + |\nabla u|)^{p-2} \) in (1.1). This latter function is only Lipschitz continuous, hence in this case it seems not possible to get stronger regularity results. Finally one could also extend the results to non-homogeneous Dirichlet boundary conditions, if the boundary data belongs to a suitable \( W^{2,q} \)-space.
Remark 1.1. Different, more intricate, proofs of Theorem 2.1 and its corollaries were given in [5], in the particular case $N = n = 3$. In [5] we also consider the case where $\nabla u$ is replaced by $D u = \frac{1}{2} (\nabla u + \nabla u^T)$, and $p \in (1, +\infty)$.

2 Notation and statement of the main results

Throughout this paper we denote by $\Omega$ a bounded $n$-dimensional domain, $n \geq 3$, with smooth boundary, which we assume of class $C^2$, and we consider the usual homogeneous Dirichlet boundary conditions

\begin{equation}
\tag{2.1}
u_{\partial \Omega} = 0.
\end{equation}

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $m$ nonnegative integer and $p \in (1, +\infty)$, we denote the usual Lebesgue and Sobolev spaces, with the standard norms $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_{W^{m,p}(\Omega)}$, respectively. We usually denote the above norms by $\| \cdot \|_p$ and $\| \cdot \|_{m,p}$ when the domain is clear. Further, we set $\| \cdot \|_{-1,p'}$ the strong dual of $W^{1,p}(\Omega)$ with norm $\| \cdot \|_{-1,p'}$.

In notation concerning norms and functional spaces, we do not distinguish between scalar and vector fields. For instance $L^p(\Omega; \mathbb{R}^N) = [L^p(\Omega)]^N, N > 1$, is simply $L^p(\Omega)$.

We use the summation convention on repeated indexes. For any given pair of matrices $B$ and $C$ in $\mathbb{R}^{N \times n}$ (linear space of $N \times n$-matrices), we write $B \cdot C \equiv B_{ij}C_{ij}$.

We denote by the symbols $c, c_1, c_2, \ldots$, positive constants that may depend on $\mu$; by capital letters, $C, C_1, C_2, \ldots$, we denote positive constants independent of $\mu \geq 0$ (eventually, $\mu$ bounded from above). The same symbol $c$ or $C$ may denote different constants, even in the same equation.

We set $\partial_i u = \frac{\partial u}{\partial x^i}, \partial^2_{ij} u = \frac{\partial^2 u}{\partial x^i \partial x^j}$. Moreover we set $(\nabla u)_{ij} = \partial_j u_i$. We denote by $D^2 u$ the set of all the second partial derivatives of $u$. Moreover we set

\begin{equation}
\tag{2.2}
|D^2 u|^2 := \sum_{i=1}^N \sum_{j,h=1}^n \left| \partial^2_{jh} u_i \right|^2.
\end{equation}

Let us introduce the definition of weak solution of problem (1.1) – (2.1).

**Definition 2.1.** Assume that $f \in W^{-1,p'}(\Omega)$. We say that $u$ is a weak solution of problem (1.1) – (2.1), if $u \in W^{1,p}(\Omega)$ satisfies

\begin{equation}
\tag{2.3}
\int_\Omega \left( \mu + |\nabla u|^p \right)^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \varphi \, dx,
\end{equation}

for all $\varphi \in W^{1,p}(\Omega)$.

We recall that the existence and uniqueness of a weak solution can be obtained by appealing to the theory of monotone operators, following J.-L. Lions [18].

Before stating our main results, let us recall two well known inequalities for the Laplace operator. The first, namely

\begin{equation}
\tag{2.4}
\| D^2 v \| \leq C_1 \| \Delta v \|,
\end{equation}

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holds for any function \( v \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \), with \( C_1 = C_1(\Omega) \). Note that \( C_1 = 1 \) if \( \Omega \) is convex. For details we refer to [15] (Chapter I, estimate (20)).

The second kind of estimates which we are going to use says that

\[
\|D^2 v\|_q \leq C_2 \|\Delta v\|_q,
\]

for \( v \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \), \( q \geq 2 \), where the constant \( C_2 \) depends on \( q \) and \( \Omega \). It relies on standard estimates for solution of the Dirichlet problem for the Poisson equation. Actually, there are two constants \( K_1 \) and \( K_2 \), independent of \( q \), such that

\[
K_1 q \leq C_2 \leq K_2 q.
\]

Similarly, one has

\[
\|v\|_{2,q} \leq C \|\Delta v\|_q,
\]

where the constant \( C \) depends on \( q \) and \( \Omega \). For further details we refer to [14] and [26].

We set

\[
r(q) = \begin{cases} \frac{nq}{n(p-1)+q(2-p)} & \text{if } q \in [2, n], \\ q & \text{if } q \geq n. \end{cases}
\]

Note that \( r(q) > q \) for any \( q < n \). Clearly, in (2.8), \( r(q) = n \) in both cases.

**Theorem 2.1.** Let \( p \in (1, 2] \) satisfy \( (2-p)C_2 < 1 \), where \( C_2 \) is given by (2.5). Assume that \( \mu \geq 0 \). Let \( f \in L^{r(q)}(\Omega) \) for some \( q \geq 2 \), \( q \neq n \), and let \( u \) be the unique weak solution of problem (1.1). Then \( u \) belongs to \( W^{2,q}(\Omega) \). Moreover, the following estimate holds

\[
\|u\|_{2,q} \leq C \left( \|f\|_q + \|f\|_{r(q)}^{1/r(q)} \right).
\]

**Corollary 2.1.** Let \( p, \mu \) and \( f \) be as in Theorem 2.1. Then, if \( q > n \), the weak solution of problem (1.1) belongs to \( C^{1,\alpha}(\Omega) \), for \( \alpha = 1 - \frac{\mu}{q} \).

In particular, when \( q = 2 \), one has the following corollary.

**Corollary 2.2.** Let \( p \in (1, 2] \) satisfy \( (2-p)C_1 < 1 \), where \( C_1 \) is given by (2.4). Assume that \( \mu \geq 0 \). Let \( f \in L^{r(2)}(\Omega) \) and let \( u \) be the unique weak solution of problem (1.1). Then \( u \) belongs to \( W^{2,2}(\Omega) \). Moreover, there is a constant \( C \) such that

\[
\|u\|_{2,2} \leq C \left( \|f\| + \|f\|_{r(2)}^{1/r(2)} \right).
\]

If \( \Omega \) is convex the result holds for any \( 1 < p \leq 2 \).

It is worth noting that in the limit case \( p = 2 \), when system (1.1) reduces to the Poisson equations, we recover exactly the well known result

\[
\|u\|_{2,q} \leq C \|f\|_q,
\]

since \( r(q) = q \) for \( p = 2 \).

Note that in estimates (2.9) and (2.10), the terms \( \|f\| \) and \( \|f\|_q \) can be replaced by 1.
Remark 2.1. One could also consider the case where \( f \in L^n(\Omega) \). We omit this further case and leave it to the interested reader. In this regard we stress that our interest mostly concerns the maximal integrability of the second derivatives of the solution.

3 Proof of Theorem 2.1. The case \( \mu > 0 \).

In this section we assume that \( \mu > 0 \). Let us consider the following system

\[
- \Delta u - (p - 2) \frac{\nabla u \cdot \nabla u \cdot \nabla u}{(\mu + |\nabla u|)^2} = f (\mu + |\nabla u|)^{2-p},
\]

where we have used the notation \( \nabla u \cdot \nabla u \cdot \nabla u \) to denote the vector whose \( i \)th component is \( \nabla u \cdot (\partial_i \nabla u) \partial_i u_i = (\partial_i u_k) (\partial_i^2 u) (\partial_i u_i) \). Formally this system can be obtained from system (1.1) by computing the divergence on the left-hand side and then multiplying the equation by \( (\mu + |\nabla u|)^{2-p} \).

It is immediate to verify that if \( u \) is a sufficiently regular solution of (3.1), say \( u \in W^{2,q}(\Omega) \), then \( u \) is a weak solutions of (1.1). So, from the uniqueness of weak solutions of (1.1), it follows that to prove Theorem 2.1 under the assumption \( \mu > 0 \) it is sufficient to prove the following result for strong solutions.

Theorem 3.1. Let \( p \in (1,2] \) satisfy \( (2-p) C_2 < 1 \), where \( C_2 \) is given by (2.3). Assume that \( \mu > 0 \). Let \( f \in L^{r(q)}(\Omega) \) for some \( r > 2 \) and \( \mu \neq q \). Then, there is a strong solution \( u \in W^{2,q}(\Omega) \) of problem (3.1)–(2.1). Moreover, the following estimate holds

\[
\|u\|_{2,q} \leq C \left( \|f\|_q + \|f\|_{r(q)} \right).
\]

In the sequel we appeal to the following fixed point theorem in order to prove Theorem 3.1.

Theorem 3.2. Let \( X \) be a reflexive Banach space and \( K \) a non-empty, convex, bounded, closed subset of \( X \). Let \( F \) be a map defined in \( K \), such that \( F(K) \subset K \).

Assume that there is a Banach space \( Y \) such that:

i) \( X \subset Y \), with compact (completely continuous) immersion.

ii) If \( v_n \in K \) converges weakly in \( X \) to some \( v \in K \) then there is a subsequence \( v_m \) such that \( F(v_m) \to F(v) \) in \( Y \).

Under the above hypotheses the map \( F \) has a fixed point in \( K \).

For the proof and some comments see section 5.

In the sequel we appeal to the above theorem with \( X = W^{2,q} \) and \( Y = L^q \). Clearly, point i) in Theorem 3.2 holds.

Proof of Theorem 3.1. For any \( v \in W^{2,q} \cap W^{1,q}_0 \) we define \( C_3 = C_3(q) \) by

\[
\|\nabla v\|_{q'} \leq C_3 \|\Delta v\|_q, \text{ if } q \in (2, n),
\]

\[
\|\nabla v\|_{\infty} \leq C_3 \|\Delta v\|_q, \text{ if } q \in (n, +\infty).
\]

These estimates can be easily obtained by applying the Sobolev embeddings and then using estimate (2.7).
Define $\delta$ by
$$
\delta = 1 - (2 - p) C_2,
$$
where $C_2$ is given by (2.5), and fix a positive real $a$ by
$$
1 + 2 C_3^{2-p} a^{2-p} \leq a \delta.
$$
Note that, under our assumptions, $\delta > 0$. It is worth noting that $\delta$, $a$ and $b$ are constants of type $C$.

Define
$$
K = \{ v \in W^{2,q}(\Omega) : \| \Delta v \|_q \leq R, v = 0 \text{ on } \partial \Omega \},
$$
where
$$
R = a \left( \| f \|_q + \| f \|_{r(q)} \right).
$$
Let $f \in L^r(q)$ be given. For each $v \in K$ define $u = F(v)$ as being the solution to the linear problem

\begin{equation}
\begin{cases}
-\Delta u = (p-2) \nabla v \cdot \nabla v v \cdot \nabla v + f (\mu + \| \nabla v \|) \| \nabla v \|^{2-p}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\end{equation}

To apply Theorem 3.2 we start by showing that $F(K) \subseteq K$. Note that if the right-hand side of (3.4) belongs to $L^q(\Omega)$, from well known results on the Poisson equation, there exists a unique $u \in W^{2,q}(\Omega)$ solving the Dirichlet problem (3.4).

For $q \in (2, n)$, by using (3.3) we have
$$
\| \nabla u \|^{2-p} f \|_q \leq \| \nabla v \|^{2-p} \| f \|_{r(q)} \leq C_3^{2-p} \| \Delta v \|^{2-p} \| f \|_{r(q)}.
$$
For $q > n$, by using (3.3) and by recalling that $r(q) = q$ if $q > n$, we have
$$
\| \nabla v \|^{2-p} f \|_q \leq \| \nabla v \|^{2-p} \| f \|_q \leq C_3^{2-p} \| \Delta v \|^{2-p} \| f \|_q.
$$
So, in both the cases,
\begin{equation}
\| \nabla v \|^{2-p} f \|_q \leq C_3^{2-p} \| \Delta v \|^{2-p} \| f \|_{r(q)}.
\end{equation}
Therefore, since the first term on the right-hand side of (3.4) obviously belongs to $L^q(\Omega)$, there exists a unique $u \in W^{2,q}(\Omega)$ solving the Dirichlet problem (3.4).

It remains to show that $u$ satisfies the estimate $\| \Delta u \|_q \leq R$. We multiply both sides of equation (3.3) by $-\Delta u |\Delta u|^{q-2}$, and integrate in $\Omega$. We get (for details see the appendix)
$$
\int_\Omega |\Delta u|^q dx \leq (2-p) \int_\Omega |D^2 v| |\Delta u|^{q-1} dx + \int_\Omega (\mu + |\nabla v|)^{2-p} |f| |\Delta u|^{q-1} dx.
$$
The Hölder’s inequality and the inequality $(\mu + |\nabla v|)^{2-p} \leq 1 + |\nabla v|^{2-p}$ yield
\begin{equation}
\| \Delta u \|_q^q \leq (2-p) \| D^2 v \|_q \| \Delta u \|^{q-1}_2 + \| f \|_q \| \Delta u \|^{q-1}_q + \| |\nabla v|^{2-p} f \|_q \| \Delta u \|^{q-1}_q,
\end{equation}
and, by dividing both sides by $\| \Delta u \|^{q-1}_q$, one has
\begin{equation}
\| \Delta u \|_q \leq (2-p) \| D^2 v \|_q + \| f \|_q + \| |\nabla v|^{2-p} f \|_q.
\end{equation}
Let us estimate the last term on the right-hand side of (3.7). Since \( v \in K \), one has

\[
\| \Delta v \|^2_{q-p} \leq a^2 - p \left( \| f \|^2_{q-p} + \| f \|^\frac{2}{r(q)} \right).
\]

Hence, from (3.5), by using (3.8) and

\[
\| f \|^2_{q-p} \| f \|_{r(q)} \leq \| f \|^q_{q} + \| f \|_{r(q)}^{\frac{1}{p}},
\]

one gets

\[
\| \nabla v \|^2_{q-p} f \|^q_{q} \leq C_{2-p}^q a^2 - p \left( \| f \|^q_{q} + 2 \| f \|_{r(q)}^{\frac{1}{p}} \right).
\]

Therefore (3.7) becomes

\[
\| \Delta u \|^q_{q} \leq \left( (2 - p) C_a + 1 + 2 C_{2-p}^q a^2 - p \right) \left( \| f \|^q_{q} + \| f \|_{r(q)}^{\frac{1}{p}} \right).
\]

where we have appealed to (2.5). Finally from the definition of \( \delta \), it readily follows that \( u \in K \). So, \( F(K) \subset K \).

To end the proof of Theorem 3.1 it is sufficient to show the following result (which corresponds to point ii) in Theorem 3.2).

**Proposition 3.1.** Let \( v_n \rightharpoonup v \) weakly in \( W^{2,q} \), where \( v_n \in K \). If \( u_n = F(v_n) \) are the solutions to the problem

\[
- \Delta u_n = (2 - p) \frac{\nabla v_n \cdot \nabla \nabla v_n \cdot \nabla v_n}{(\mu + |\nabla v_n|)} + f (\mu + |\nabla v_n|)^{2-p} + f (\mu + |\nabla v|)^{2-p},
\]

then there is a subsequence \( v_m \) of \( v_n \) such that each of the two terms in the right hand side of (3.10) converge, in the distributional sense, to the corresponding terms in equation (3.11). This verification would be quite immediate.

However, we rather prefer to prove the convergence in a topology stronger than the distributional one.
For convenience we set
\[ A(w) = \frac{\partial_1 w \partial_2 w}{(\mu + |\nabla w|) |\nabla w|}, \]
where \( \partial_1 w \) and \( \partial_2 w \) denotes any couple of arbitrary, fixed, partial derivatives.

**Lemma 3.3.** There is a subsequence \( v_m \) of \( v_k \) such that
\[ A(v_m) \to A(v), \]
strongly in \( L_t^* \), for each \( t > 1 \).

**Proof.** Since, in particular, \( v_k \to v \) in \( W^{1,q} \), it follows, by a classical result, that almost everywhere convergence of the gradient in \( \Omega \) also holds, for some \( v_m \). So, \( A(v_m) \to A(v) \), a.e. in \( \Omega \). Further, \( |A(v_m(x))|^t \leq 1 \), point-wisely. It follows, from the reflexivity of \( L_t^* \), that \( A(v_m) \) is weakly convergent in \( L_t^* \).

Due to the a.e. convergence, see [18], chap. I, Lemma 1.3, the weak limit is just \( A(v) \). So,
\[ A(v_m) \to A(v), \]
weakly in \( L_t^* \), for each finite \( t \). This last property, together with \( \|A(v_m)\|_t \to \|A(v)\|_t \) implies strong convergence, thanks to a classical theorem, see [22] (Chap.2, n. 37). The above norm-convergence follows by appealing to Lebesgue’s dominated convergence theorem.

Next, we prove that each of the two terms in the right hand side of (3.10) converge to the corresponding terms in equation (3.11). We start by the first term. Each single addend has the form \( A(v_m) \partial^2 v_m \), where \( \partial^2 w \) denotes an arbitrary, fixed, second order derivative. We prove the following result.

**Lemma 3.4.** One has
\[ A(v_m) \partial^2 v_m \to A(v) \partial^2 v \]
weakly in \( L^s \), for each \( s < q \).

**Proof.** Set \( g = A(v) \), \( g_m = A(v_m) \), \( h = \partial^2 v \), and \( h_m = \partial^2 v_m \). Clearly, \( h_m \to h \) weakly in \( L^q \). Moreover, by the previous lemma, \( g_m \to g \) strongly in \( L_t^* \), \( t = \frac{q}{q-s} \). Moreover, and \( h_m \to h \) weakly in \( L^q \).

Write
\[ g_m h_m - gh = g(h_m - h) + (g_m - g)h_m, \]
and let \( \phi \in L^{q'} \). Since \( g(x) \) is bounded it follows that \( g\phi \in L^{q'} \). So the quantity
\[ <g(h_m - h), \phi> = <h_m - h, g\phi> \]
goes to zero as \( m \to \infty \). This proves the weak convergence to zero, in \( L^{q'} \), of the first term in the right hand side of (3.12).

On the other hand, by Hölder’s inequality,
\[ \| (g_m - g) h \|_{s} \leq \| g_m - g \|_{\frac{s}{q-s}} \| h \|_{q}^{s}. \]
This proves the strong convergence to zero, in \( L^s \), of the second term in the right hand side of (3.12). In conclusion, the first term in the right hand side of (3.10) \( v_m \) converges to the first term in the right hand side of (3.11).
Finally, the convergence of the second term in the right hand side of \(3.10\) to the corresponding term in \(3.11\) holds, since
\[
| (\mu + |\nabla v_m|)^{2-p} - (\mu + |\nabla v|)^{2-p} | \leq \frac{2-p}{\mu^{p-1}} |\nabla v_m - \nabla v|.
\]
By Cauchy-Schwartz inequality
\[
\| f (\mu + |\nabla v_m|)^{2-p} - f (\mu + |\nabla v|)^{2-p} \| \leq \frac{2-p}{\mu^{p-1}} \| f \|_q \| \nabla v_m - \nabla v \|_q,
\]
and the right-hand side goes to zero thanks to the compact embedding of \(W^{2,q}\) in \(W^{1,q}\).

The solution \(u\) obviously satisfies \(3.2\), as \(u \in K\).

\[\square\]

4 Proof of Theorem 2.1. The case \(\mu = 0\).

In the previous step we have obtained estimates on the \(L^3\)-norm of the second derivatives, uniformly in \(\mu\), \(\mu > 0\). Let us denote by \(u_\mu\) the sequence of solutions of \(1.1\) for the different values of \(\mu > 0\). We have shown that the sequence \((u_\mu)\) is uniformly bounded in \(W^{2,q}(\Omega)\). Therefore, there exists a vector field \(u \in W^{2,q}(\Omega)\) and a subsequence, which we continue to denote by \((u_\mu)\), such that \((u_\mu) \rightharpoonup u\) weakly in \(W^{2,q}(\Omega)\), and, by Rellich’s theorem, strongly in \(W^{1,s}(\Omega)\), for any \(s < q^*\) if \(q < n\). In particular \((u_\mu)\) converges to \(u\) strongly in \(W^{1,p}(\Omega)\). Let us prove that
\[
\int_\Omega (\mu + |\nabla u|)^{p-2} \nabla u \cdot \nabla \varphi \, dx = \lim_{\mu \to 0^+} \left\{ \int_\Omega (\mu + |\nabla u_\mu|)^{p-2} \nabla u_\mu \cdot \nabla \varphi \, dx \right\},
\]
for any \(\varphi \in W^{1,p}_0(\Omega)\). We recall the following well known estimate (see, for instance, \(8\))
\[
| (\mu + |A|)^{p-2} A - (\mu + |B|)^{p-2} B | \leq C \frac{|A - B|}{(\mu + |A| + |B|)^{2-p}},
\]
for any pair \(A\) and \(B\) in \(\mathbb{R}^{Nn}\), where \(C\) is a positive constant independent of \(\mu\).

By applying \(1.2\) and then Hölder’s inequality, we get
\[
\begin{align*}
\left| \int_\Omega (\mu + |\nabla u|)^{p-2} \nabla u \cdot \nabla \varphi \, dx - \int_\Omega (\mu + |\nabla u_\mu|)^{p-2} \nabla u_\mu \cdot \nabla \varphi \, dx \right| \\
\leq C \int_\Omega (\mu + |\nabla u| + |\nabla u_\mu|)^{p-2} |\nabla u - \nabla u_\mu| |\nabla \varphi| \, dx \\
\leq C \int_\Omega |\nabla u - \nabla u_\mu|^{p-1} |\nabla \varphi| \, dx \leq C \| \nabla u - \nabla u \|_{p-1} \| \nabla \varphi \|_p.
\end{align*}
\]

The right-hand side of the last inequality tends to zero, as \(\mu\) goes to zero, thanks to the strong convergence of \(u_\mu\) to \(u\) in \(W^{1,p}(\Omega)\). This proves \(1.1\). Finally, for each \(\mu > 0\), the right-hand side of \(4.1\) is equal to \(\int_\Omega f \cdot \varphi \, dx\). So, \(u\) satisfies the integral identity \((2.3)\). Hence \(u\) is a weak solution of \(1.1\), and belongs to \(W^{2,q}(\Omega)\). Finally, \(2.10\) follows since \(\| u \|_{2,q} \leq \liminf_{\mu \to 0^+} \| u_\mu \|_{2,q}\).
The Corollary 2.1 is an immediate consequence of Theorem 2.1 by using the regularity of the domain and the Sobolev embedding.

The results in Corollary 2.2 can be obtained by replacing in the proof of Theorem 2.1 hence in the proof of Theorem 3.1 the constant $C_2$ with the constant $C_1$. The last assertion in Corollary 2.2 follows from the validity of (2.4), for a smooth convex domain, with $C_1 = 1$. We omit further details.

5 The fixed point theorem. Proof and remarks.

Theorem 3.2 is a simplification of an idea introduced in reference [3] to prove existence of strong solutions to initial boundary value problems for non-linear systems of evolution equations, specially in Sobolev spaces. See the section 3, in the above reference. Successively, the method has been applied with success to many other problems, in particular to the compressible Euler equations (see [4]). Main requirements, in applications, are the reflexivity of the Banach space $X$, and its sufficiently strong topology. Shauder’s fixed point theorem is applied with respect to a quite arbitrary “container space” $Y$. Roughly speaking, the above two properties allow us to trivialize both compactness and continuity requirements, respectively. So, to apply the theorem, the main point is to show that $F(K) \subset K$, for some convex, bounded, closed subset $K$.

Proof of Theorem 3.2. Obviously $K$ is convex, bounded, and pre-compact in $Y$.

Let $y_n \in K$ converge to some $y$ in the $Y$ norm. We start by showing that $K$ is closed, hence compact, in $Y$, and that the sequence $y_n \to y$ weakly in $X$. Since $K$ is $X$-bounded, and $X$ is reflexive, there is a subsequence $y_m$ which is $X$-weakly convergent to some $u \in X$. Since the immersion $X \subset Y$ is continuous, $y_m$ is also weakly convergent to $u$ in $Y$. Since, by assumption, this sequence is strongly convergent in $Y$ to $y$, it follows that $u = y$. Further, since convex sets in Banach spaces are weakly closed if and only if they are strongly closed, it follows that $y \in K$. So, $K$ is $Y$-closed. Further, from the uniqueness of the limit $y$, we deduce that the whole sequence $y_n$ converges weakly in $X$ to $y$.

Finally, to prove that $F(y_n) \to F(y)$ strongly in $Y$ it is sufficient to show, by using standard arguments, that any subsequence $y_k$ contains a subsequence $y_m$ such that $F(y_m) \to F(y)$ strongly in $Y$. Obviously, $y_k \to y$ weakly in $X$. By assumption ii), there is a subsequence $y_m$ such that $F(y_m) \to F(y)$ strongly in $Y$. This shows that the map $F$ is continuous on $K$ with respect to the $Y$ topology. So, Schauder’s fixed point theorem guarantees the existence of, at least, one fixed point $y_0 \in K$, $F(y_0) = y_0$.

6 Appendix

Our aim is to prove the estimate

$$|I| := |\nabla v \cdot (\partial_j \nabla v) (\partial_j v_i) \Delta v_i| \leq |\nabla v|^2 |D^2 v| |\Delta v|.$$ 

In the sequel, for the reader’s convenience, we avoid the summation convention.
We recall that
\[(D^2 v_k)^2 := \sum_{j,h=1}^n |\partial_{jh}^2 v_k|^2 \quad \text{and} \quad |D^2 v|^2 := \sum_{k=1}^N (D^2 v_k)^2 = \sum_{k=1,j,h=1}^n |\partial_{jh}^2 v_k|^2 .\]

We introduce the \(n\)-vector \(b\) and \(N\)-vector \(w\), whose components are defined as follows
\[b_j := (\partial_j v) \cdot \Delta v, \quad w^2_k := \sum_{j,h=1}^n (\partial_h v_k)b_j^2 .\]

The modulus of vector \(b\) satisfies the following estimate:
\[|b|^2 = \sum_{j=1}^n b_j^2 \leq \sum_{j=1}^n |\partial_j v|^2 |\Delta v|^2 = |\Delta v|^2 \sum_{j=1}^n (\partial_j v_i)^2 = |\Delta v|^2 |\nabla v|^2 .\]

Hence
\[(6.1) \quad w^2_k = \sum_{h=1}^n (\partial_h v_k)^2 \sum_{j=1}^n b_j^2 = |\nabla v_k|^2 |\Delta v|^2 |\nabla v|^2 .\]

Moreover
\[|I| = \left| \sum_{k=1}^N \sum_{j,h=1}^n (\partial_h v_k)(\partial_{jh}^2 v_k) b_j \right| \leq \sum_{k=1}^N \left| \sum_{j,h=1}^n (\partial_{jh}^2 v_k)(\partial_h v_k) b_j \right| \leq \sum_{k=1}^N \left( \sum_{j,h=1}^n (\partial_{jh}^2 v_k)^2 \right)^{1/2} \left( \sum_{j,h=1}^n (\partial_h v_k)^2 \right)^{1/2},\]

where, in the last step, we have used that, for any pair of tensors \(A\) and \(B\), there holds \(|A \cdot B| \leq |A||B|\). Hence, by the above notations and estimate (6.1), we get
\[|I| \leq \sum_{k=1}^N |D^2 v_k||w_k| \leq |\Delta v||\nabla v| \sum_{k=1}^N |D^2 v_k||\nabla v_k| \leq |\Delta v||\nabla v| \sqrt{\sum_{k=1}^N |D^2 v_k|^2} \sqrt{\sum_{k=1}^N |\nabla v_k|^2} = |\Delta v||\nabla v|^2 |D^2 v| ,\]

which is our thesis.

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