Structural instability of friction-induced vibration by characteristic polynomial plane applied to brake squeal

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Abstract

Complex eigenvalue analysis has generally been applied to squeal improvement of automotive brake systems in recent years. Discrimination of the occurrence of unstable vibration by modal coupling has become common in brake design. However, the generation mechanism is not fully understood. In particular, the transition of the eigenvalue and the bifurcation phenomenon accompanying the change of the equation of motion are difficult to predict quantitatively. One reason is that a degree of instability can be expressed by real parts of eigenvalues, but the difficulty of the coupling cannot be determined by real parts of eigenvalues. In this report, to obtain a stability index before the coupling, characteristic polynomials are the subject of this research. For the structural instability problem of a two-degree-of-freedom system, which involves a typical friction-induced vibration, the positions of the eigenvalues are geometrically shown on a complex response surface using a real part of the characteristic polynomial. The characteristic polynomial of the two-degree-of-freedom system becomes a complex quartic function; as a result, a saddle-shaped response surface that intersects the complex space appears. Without damping, when the surface and a zero plane of the complex space intersect on an imaginary axis, uncoupled eigenvalues appear. When the surface does not intersect on the imaginary axis, the eigenvalues become complex, and unstable vibration occurs. In a parameter study, a friction coefficient raises the surface monotonously, while mass and stiffness change the eigenvalues in the same way as curve veering, and damping breaks the symmetry of the surface and shifts the surface to the stable side. The vertex in the imaginary axis cross section of the surface is a point at which the eigenvalues reach the coupling. This point serves as a guideline for the stability before the coupling. Consequently, it is useful for evaluating the transition of the eigenvalues.

Keywords: Brake squeal, Friction-induced vibration, Modal coupling, Complex eigenvalue, Characteristic polynomial

1. Introduction

Brake squeal generated by friction-induced vibration is one of the most important issues in brake development because it significantly reduces the comfortability of cars. Brake squeal is a noise generated by the disk brake or drum brake, and it usually occurs at 1 to 16 kHz (Papinniemi et al., 2001). The disk brake generates a braking torque by the frictional force between the pad and the disk rotor during braking. Self-excited vibration may occur between the pad and rotor resulting from various factors, such as fluctuations in frictional force. When the disk rotor vibrates in the normal direction, air is shaken and appears as noise at a frequency close to the natural frequency of the disk rotor. The possibility of brake squeal depends on the surrounding environment and conditions of use and has a complex influence on tribological and structural aspects (Eriksson et al., 1999).

In recent years, the number of hybrid electric vehicles and electric vehicles has increased because of environmental regulations, and vehicles equipped with internal combustion engines are being modified to reduce weight, which affects
fuel efficiency. The quietness and light-weight of vehicles are contradictory phenomena with noise and vibration of disk brakes; therefore, the suppression of brake squeal is even more difficult. A considerable amount of work has been conducted on analysis of the brake squeal mechanism. However, even today, brake squeal has not disappeared, so studies on friction materials and braking devices have been conducted. Detailed reviews have also been published (Kinkaid et al., 2003).

Recently, transient brake squeal analysis for prediction has been evaluated (Oberst and Lai, 2015). Nevertheless, at this time, complex eigenvalue analysis is widely used to estimate brake squeal (Milner, 1978). This is a method for solving structural instability problems caused by coupled vibrations of two-degree-of-freedom systems including frictional forces (Milner, 1978). Complex eigenvalue analysis is widely practiced, estimating several dozens of natural frequencies of the brake assembly at once in the audible range frequency by the finite-element method (Liu et al., 2007). Meanwhile, the transition of eigenvalues leading to unstable vibrations is unclear, and the influence of equations of motion parameters on eigenvalues has been studied (Sinou and Jézéquel, 2007).

Complex eigenvalue analysis is a useful tool that can predict several types of self-excited oscillations. However, if a parameter study is not performed, it is impossible to predict how the eigenvalue changes and branches, and it is not possible to establish a guideline for suppressing noise in advance. Therefore, the degree of eigenvalue transition because of the friction coefficient, mass, and rigidity is not known unless it is calculated. In the initial design phase of brake calipers, trial and error are repeated, which is a major development issue.

In the structural instability problem, which is a typical cause of brake squeal, each mode of a two-degree-of-freedom system is coupled when the friction coefficient increases (Huang et al., 2006). Then, the eigenvalues first approach each other in the direction of the imaginary part and then repel in the direction of the real part. Even in stable two-degree-of-freedom vibrations, a phenomenon called “curve veering” occurs, the eigenvalues approach each other in the imaginary part direction, and then they repel in the eigenvalue imaginary part direction (Du Bois, 2009). Mode coupling and curve veering are known to show similar behavior, but the correlation between the two is not clear.

The purpose of this study is to explain the mechanism of such eigenvalue transitions when coupled. Focusing on the characteristic polynomial, which is the left side of the eigenvalue equation, this study considers that the eigenvalue exists at the intersection of the curved surface representing the real part of the characteristic polynomial drawn on a complex plain and the zero plane. Based on this idea, instead of directly calculating the eigenvalue, the reason the eigenvalue is determined is explained geometrically. A target of this survey is a transition of eigenvalues of a model with low degrees of freedom. The complex solution of the characteristic equation is plotted in a three-dimensional space to obtain a response surface that geometrically visualizes the location of the solution. Parameter studies explain the transition ways of complex solutions from changes in response surfaces. This research is conducted before and after the eigenvalue analysis and describes the advanced study of robust parameters with high stability, the evaluation of eigenvalue analysis results, and the technical study utilized for an improvement proposal for brake systems.

2. Current methods for the stability criterion of brake squeal

In the widely used finite-element method (FEM), the following factors cause self-excited vibration caused by friction force (Bajer et al., 2003).

1) Negative friction velocity gradient $d\frac{\mathbf{u}}{dv}$ (one degree of freedom)
2) Structural instability (two-degree-of-freedom coupling)
3) Frictional damping

Structural instability, which is considered to be the most typical brake noise factor, is targeted in this study. This is a phenomenon called “modal coupling” that occurs when two vibration modes interfere with each other under the influence of frictional force. This unstable vibration can be solved by complex eigenvalue analysis (Milner, 1978). Equation (1) is an equation of motion considering the frictional force (Trichês Júnior et al., 2008).

$$[\mathbf{M}]\ddot{\mathbf{u}} + [\mathbf{C}]{\dot{\mathbf{u}}} + [\mathbf{K}][\mathbf{u}] = \{\mathbf{F}\}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are mass, damping, and stiffness matrices, respectively, and $\mathbf{u}$ is the generalized displacement vector. Friction function $\mathbf{F}$ is contributed to by the variable friction force at a pad rotor interface; it is shown in Eq. (2)
as a displacement proportional term.

\[ F = [K_f][u] \]  

(2)

\( K_f \) is a friction stiffness matrix proportional to friction coefficient \( \mu \) and the contact stiffness between the brake pad and rotor. Because the frictional force is proportional to the contact stiffness, it is collected in a stiffness matrix, as shown in Eq. (3). Consequently, the equation deformation causes the stiffness matrix to become asymmetric and causes instability.

\[ [M][\ddot{u}] + [C][\dot{u}] + [K - K_f][u] = 0 \]  

(3)

In this section, the current status of stability determination using the widely used FEM. Several elements, which are solid elements for parts of a brake assembly and spring elements that simulate contact stiffness between parts, are formed into an FEM model (Fig. 1). The transformation of the equation of motion is treated as an eigenvalue problem to obtain eigenvalue matrix \( p \) and eigenvector matrix \( \Phi \) in Eq. (4). Here, \( p \) is a diagonal matrix having each eigenvalue in the diagonal term, as in Eq. (5), and \( \{\Phi\} \) is a matrix in which each eigenvector is arranged as in Eq. (6).

Several types of structural damping are often added to the stiffness matrix, as in Eq. (7). In the stiffness matrix \( K_{\text{total}} \), the friction component \( \alpha_f K_f \), in which the stiffness \( K_f \) is multiplied by a factor \( \alpha \) that adjusts the initial value of the friction coefficient used in the parameter study, is considered.

\[ \{p^2[M] + p[C] + [K_{\text{total}}]\}\{\Phi\} = 0 \]  

(4)

\[ p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \]  

(5)

\[ \{\Phi\} = [\{\Phi_1\} \quad \{\Phi_2\} \quad \ldots \quad \{\Phi_n\}] \]  

(6)

\[ [K_{\text{total}}] = [K](1 + iG) + \Sigma K_e G_e + \alpha_f K_f \]  

(7)

\( p \) : complex eigenvalue
\( \Phi \) : complex natural modes
\( n \) : number of degrees of freedom
\( M \) : mass matrix
\( C \) : viscous damping matrix
\( \dot{G} \) : overall structural damping coefficient
\( G_e \) : structural damping value from material property
\( \alpha_f \) : direct matrix input grid coefficient
\( K_e \) : element stiffness matrix

Because the eigenvalues representing the convergence or divergence are complex numbers, complex eigenvalue analysis obtains root plots as shown in Fig. 2. If real parts of eigenvalues are positive, they indicate instability. The increase in the coefficient of friction causes instability with the real part of a specific eigenvalue being positive. At the same time, the mode in which the real part is negative is a paired mode that leads to coupling. The utilization of the analysis predicts the occurrence of brake squeal in actual brake caliper designs, leading to an easy decision about whether the brake is stable or unstable. However, this kind of analysis does not lead an understanding of why particular eigenvalues become unstable. Therefore, the mechanism of instability occurrence using the low-degree-of-freedom model is investigated in the next section.
3. Structural instability problem for a model with low degrees of freedom

Figure 3 depicts a low-degree-of-freedom model for investigating the factor of transition of eigenvalues. This model has a caliper and a rotor, simulates a pad represented by distributed stiffness and distributed damping, and generates a friction force $f$ from the pad. The structure has two degrees of freedom of translation in the vertical direction $x$ and rotation $\theta$, and spring dampers are provided on the rotational leading side and the trailing side of the pressing parts. The contact surface has distributed spring dampers, and the portion without the distributed spring corresponds to a slit. This model is characterized by mode coupling with a minimum degree of freedom with friction.
The parameters of each part are as listed in Table 1. Experimental modal analysis for an actual brake identifies the parameters (Inoue et al., 2009) (Tanabe et al., 2009). Values that may cause instability are substituted into the parameters using the Routh Hurwitz stability determination described later.

Table 1 Parameters of the model with two degrees of freedom. They are based on an actual mass and dimensions of a brake system; some values such as springs and dampers have been identified from experimental values. The parameters are arbitrarily fine-tuned to cause instability.

| Symbol | Description | Value | Unit |
|--------|-------------|-------|------|
| 𝑀     | Mass of the caliper | 2.1   | kg   |
| 𝐽     | Moment of inertia of the caliper | 2.64×10⁻³ | kgm² |
| 𝐿     | Width of the caliper | 0.12  | m    |
| 𝑡     | Distance from the center of gravity to the caliper surface | 1.50×10⁻² | m   |
| 𝑎_𝐿   | Distance of center to stiffness of leading side | 5.00×10⁻² | m   |
| 𝑎_𝑇   | Distance of center to stiffness of trailing side | 5.00×10⁻² | m   |
| 𝑐     | Distributed damping coefficient | 260   | Ns/m² |
| 𝑐_𝐿   | Damping coefficient of leading side | 91.8  | Ns/m  |
| 𝑐_𝑇   | Damping coefficient of trailing side | 91.8  | Ns/m  |
| 𝑘     | Stiffness of distributed contact constraint | 5.00×10⁹ | N/m² |
| 𝑘_𝐿   | Stiffness of leading side | 3.00×10⁷ | N/m  |
| 𝑘_𝑇   | Stiffness of trailing side | 0.90×10⁷ | N/m  |
| 𝑙     | Width of cut slit | 5.00×10⁻³ | m   |
| 𝑦     | Position of cut slit | 0.02  | m    |
| 𝜇     | Friction coefficient | Optional | - |

With the center of gravity as the origin, the vertical translation direction as 𝑥, the horizontal direction as 𝑦, and the rotational angle as 𝜃, the following equation of motion is derived. The translational force 𝐹_𝐿,𝑥 by the stiffness of the leading side is shown in Eq. (8), the force 𝐹_𝑇,𝑥 by the stiffness of the trailing side is in Eq. (9), and the force 𝐹_𝑥 by the contact stiffness is in Eq. (10).

\[
F_{LX} = k_Lx + k_L a_L \sin \theta
\]  

(8)
\[ F_{kTx} = k_T x - k_T a_T \sin \theta \]  
\[ F_{kx} = \int_{-L/2}^{L/2} \left( k_0 x + k(y \sin \theta) \right) dy - \int_{y=-L/2}^{y=L/2} [k_0 x + k(y \sin \theta)] dy 
= (kLx + 0) - (klx - kyl\theta) \]  
(9)

Assuming \( \theta \) is small enough, the contact force \( F_{kx} \) is obtained by integrating over the section of the contact surface. The leading side damping force \( F_{clx} \), the trailing side damping force \( F_{cTx} \), and the contact portion damping force \( F_{cEx} \) are similarly expressed as in Eq. (11) through Eq. (13).

\[ F_{clx} = cx_\dot{} + cl \dot{} a_L \sin \theta \]  
\[ F_{cTx} = ct_\dot{} - ct a_T \sin \dot{\theta} \]  
\[ F_{cEx} = \int_{-L/2}^{L/2} [c_\dot{}(x + c(y \sin \theta))] dy - \int_{y=-L/2}^{y=L/2} [c_\dot{} x + c(y \sin \theta)] dy 
= (cLx + 0) - (clx - cly \dot{\theta}) \]  
(10)

Thus, an equation of motion in the vertical translation direction is given by Eq. (14).

\[ M \ddot{x} + F_{kTx} + F_{kx} + F_{clx} + F_{cTx} + F_{cEx} = 0 \]  
(14)

By substituting Eq. (8) through Eq. (13) into Eq. (14), Eq. (15) is obtained.

\[ M \ddot{x} + (k_L + k_T + kl - kl) x + (a_L k_L - a_T k_T + kyl\theta) \theta 
+ (c_L + c_T + cl - cl) \dot{x} + (a_L c_L - a_T c_T + c_T \theta) \ddot{\theta} = 0 \]  
(15)

Regarding the degree of freedom in the rotation direction, the torque \( T_{kL\theta} \) by the leading side stiffness, the torque \( T_{kT\theta} \) on the trailing side stiffness, and the torque \( T_{k\theta} \) at the contact surface are expressed by Eq. (16) through Eq. (18).

\[ T_{kL\theta} = a_L k_L x + a_L + (c_L (a_L \sin \theta)) \]  
\[ T_{kT\theta} = -a_T k_T x - a_T (a_T \sin \theta) \]  
\[ T_{k\theta} = \int_{-L/2}^{L/2} [y k_\dot{} x + yk(y \sin \theta)] dy - \int_{y=-L/2}^{y=L/2} [yk_\dot{} x + yk(y \sin \theta)] dy 
- \int_{-L/2}^{L/2} (\mu k_\dot{} x + \mu k(y \sin \theta) h) dy - \int_{y=-L/2}^{y=L/2} (\mu k_\dot{} x + \mu k(y \sin \theta) h) dy 
= (0 + kl^2 \theta / 12) - (kl \dot{y} x + kl^2 \theta + kl^2 \theta / 12) - (\mu kLx + 0) - (\mu kLx + \mu kL \dot{y} \theta) \]  
(18)

Similarly, the leading side damping torque \( T_{cl\theta} \), the trailing side damping torque \( T_{cT\theta} \), and the contact damping torque \( T_{c\theta} \) are expressed as in Eq. (19) through Eq. (21).

\[ T_{cl\theta} = a_L c_L x + a_L c_L (a_L \sin \theta) \]  
\[ T_{cT\theta} = -a_T c_T x - a_T c_T (a_T \sin \theta) \]  
\[ T_{c\theta} = \int_{-L/2}^{L/2} [y c_\dot{}(x + c(y \sin \theta))] dy - \int_{y=-L/2}^{y=L/2} [y c_\dot{}(x + c(y \sin \theta))] dy 
- \int_{-L/2}^{L/2} (\mu c_\dot{} x + \mu c(y \sin \theta) h) dy - \int_{y=-L/2}^{y=L/2} (\mu c_\dot{} x + \mu c(y \sin \theta) h) dy 
= (0 + cl^2 \theta / 12) - (cl \dot{y} x + cl^2 \theta + cl^2 \theta / 12) - (\mu clLx + 0) - (\mu clLx + \mu cl \dot{y} \theta) \]  
(21)

Thus, an equation of motion in the rotational direction is given by Eq. (22).

\[ J \ddot{\theta} + T_{kL\theta} + T_{kT\theta} + T_{k\theta} + T_{cl\theta} + T_{cT\theta} + T_{c\theta} = 0 \]  
(22)
By substituting Eq. (16) through Eq. (21) into Eq. (22), Eq. (23) is obtained.

\[
J \ddot{\theta} + (a_Lk_L - a_T k_T + kL \dot{y} - \mu k h L + \mu \dot{h} l) x + (a_Lc_L - a_T c_T + \dot{c} l \dot{y} - \mu \dot{c} h L + \mu \ddot{h} l) x
+ (a_L^2k_L + a_T^2 k_T + \ddot{c}L^2 / 12 - \dddot{y}^2 / 12 - \ddot{k}L / 12 - \mu \dot{k} h \dot{y}) \theta
+ (a_L^2c_L + a_T^2 c_T + \dddot{c}L^2 / 12 - \dddot{c}l^2 / 12 - \mu \dot{c} h \ddot{y}^2) \dot{\theta} = 0
\]  

Equation (24) is an integrated equation of motion of this model. Here, \( \{u\} \) is a column vector having a translational displacement \( x \) and a rotation angle \( \theta \), as expressed by Eq. (25). \( M, C, \) and \( K \) are \( 2 \times 2 \) matrices, as in Eq. (26); each element is as shown in Eq. (27) through Eq. (34). Similar to Eq. (3), the effect of friction is considered in the damping matrix and the stiffness matrix. Eigenvalue transition is discussed using this model.

\[
M \{\ddot{u}\} + C \{\dot{u}\} + K \{u\} = 0 \tag{24}
\]

\[\{u\} = \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{25}\]

\[
M = \begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{26}\]

\[c_{11} = c_L + c_T + \dddot{c}(L - l) \tag{27}\]

\[c_{12} = a_Lc_L - a_T c_T + \dot{c} y l \tag{28}\]

\[c_{21} = a_Lc_L - a_T c_T + \dddot{c} y l - \mu \dot{c} h (L - l) \tag{29}\]

\[c_{22} = a_L^2c_L + a_T^2 c_T + \dddot{c}L^2 / 12 - \dddot{y}^2 l - \ddot{l}^2 / 12 \tag{30}\]

\[k_{11} = k_L + k_T + \ddot{k}(L - l) \tag{31}\]

\[k_{12} = a_Lk_L - a_T k_T + \ddot{k} y l \tag{32}\]

\[k_{21} = a_Lk_L - a_T k_T + \ddot{k} y l - \mu \dot{k} h (L - l) \tag{33}\]

\[k_{22} = a_L^2k_L + a_T^2 k_T + \dddot{k}L^2 / 12 - \dddot{y}^2 l - \ddot{l}^2 / 12 \tag{34}\]

### 3.1 Stability determination by the Routh Hurwitz criterion

In this section, stability evaluation is performed on the above equation of motion of the two-degree-of-freedom system. First, the Routh Hurwitz discriminant conditional equation is applied to determine whether the characteristic roots are stable without directly obtaining the characteristic roots. The displacement vector is shown in Eq. (35), where \( U \) is amplitude. Symbol \( s \) is the Laplace operator, and \( t \) is time.

\[
\{u\} = \{U\}e^{st} \tag{35}\]

The equation of motion in Eq. (24) is converted into characteristic equation Eq. (36) by substituting Eq. (35).

\[
MJs^4 + (c_{11}J + c_{22}M)s^3 + (c_{11}c_{22} - c_{12}c_{22} + k_{11}J + k_{22}M)s^2
+ (c_{11}k_{22} + c_{22}k_{11} - c_{12}k_{21} - c_{21}k_{12})s + (k_{11}k_{22} - k_{12}k_{22}) = 0 \tag{36}\]

Each coefficient is defined as Eq. (37) through Eq. (41).

\[a_4 = MJ \tag{37}\]

\[a_3 = c_{11}J + c_{22}M \tag{38}\]

\[a_2 = c_{11}c_{22} - c_{12}c_{22} + k_{11}J + k_{22}M \tag{39}\]

\[a_1 = c_{11}k_{22} + c_{22}k_{11} - c_{12}k_{21} - c_{21}k_{12} \tag{40}\]

\[a_0 = k_{11}k_{22} - k_{12}k_{22} \tag{41}\]

The model becomes stable when Eq. (42) and Eq. (43) hold.
The criterion shows a relationship of stability with the support springs shown in Fig. 4. When the leading-side spring stiffness $k_L$ and the trailing-side spring stiffness $k_T$ are set as the coordinate axes, instability occurs in a specific region. Moreover, the area is expanded by increasing the friction coefficient $\mu$. In Eq. (43), coefficients $\alpha_3$, $\alpha_2$, $\alpha_1$, and $\alpha_0$ including a friction coefficient exist, and this is a result of expanding the region. However, if such a stability discriminant is used, it becomes possible to find an unstable parameter, but this result does not lead to the causes of the self-excited vibration, such as $du/dv$.

Fig. 4 Stability judgment results by the Routh–Hurwitz equation. The horizontal axis is the stiffness of trailing side $k_T$, and the vertical axis is $k_L$. The model becomes unstable only when $k_L$ is larger than $k_T$. In addition, several results of friction coefficient differences are shown, and the increase in the friction coefficient enlarges the unstable region.

### 3.2 Complex eigenvalue analysis for a low-degree-of-freedom system

Stability discrimination clarifies whether instability occurs. Nevertheless, it does not indicate physical causes of the friction-induced vibration. Therefore, the complex eigenvalue analysis widely practiced today is examined. To facilitate the evaluation, Eq. (44), excluding the damping factor for suppressing the destabilization, is considered.

$$M \{\ddot{u}\} + K \{u\} = 0$$  \hspace{1cm} (44)

A solution form, Eq. (45), is assumed as a general solution of second-order linear ordinary differential equations.

$$u = U e^{\lambda t}$$  \hspace{1cm} (45)

Here, $\lambda$ is an eigenvalue. As in Eq. (46), $\lambda$ is a complex number consisting of a real part $R$ indicating increase or decrease of vibration and an imaginary part $I$ indicating the speed of vibration, using imaginary unit $j$.

$$\lambda = R + Ij$$  \hspace{1cm} (46)

The equation can be expressed as an eigenvalue problem using the eigenvalue $\lambda$ and the eigenvector $\phi$ in Eq. (47).

$$(K - \lambda^2 M) \{\phi\} = 0$$  \hspace{1cm} (47)

Figure 5 shows results of the complex eigenvalue analysis of the two-degree-of-freedom model. Fig. 5(a) depicts the transition of two natural frequencies when the coefficient of friction is increased by 0.01. The frequencies have a
correlation with the eigenvalue imaginary parts. The increase in the coefficient of friction reduces the difference between the natural frequencies and causes the two natural frequencies to coincide on the way. After the coincidence, the real part of the eigenvalue changes from zero, as shown in Fig. 5(b). In other words, the two complex eigenvalues asymptotically approach in the imaginary axis direction and then repel in the real axis direction. As shown in Fig. 5(c), the absolute values of the real parts increase symmetrically as the friction coefficient increases. This three-dimensional transition shows a twisted diagram, as in Fig. 5(d). The divergent vibration has a positive real part of the eigenvalue, and the cause of instability is thought to be structural instability resulting from modal coupling of a two-degree-of-freedom system. This result reproduces a similar previous study (Flint and Hultén, 2002). Such a phenomenon has also been confirmed in experiments (Akay et al., 2009). The real part transitions that mean divergence or convergence are discussed in more detail later.

Thus, eigenvalue analysis reveals not only the stability but also the modal coupling as the cause of instability. Additionally, the degree of instability is also indicated in the real part of eigenvalue. However, before the coupling, the real part of the eigenvalue is always zero, which cannot express how stable the vibration system is. When the vibration system includes a damper, the real part of the eigenvalue is affected by the damping, however, it does not indicate the difficulty of the coupling. To estimate how stable it is, the difference between the two natural frequencies is substituted. However, many studies have shown that the coupling can easily occur despite the large difference. For the brake noise prediction, an evaluation value indicating the stability is required, not the instability.

![Fig. 5](image)

**Fig. 5**  Complex eigenvalue analysis results of the two-degree-of-freedom model. (a) Friction coefficient vs. frequency, which shows the transition of two natural frequencies. The two natural frequencies coincide with each other at a coefficient of friction of 0.15 or more. At this time, it is destabilizing. (b) Real eigenvalue vs. frequency, which shows the repelling between the two eigenvalues. (c) Friction coefficient vs. real eigenvalue. (d) Friction coefficient vs. real eigenvalue vs. frequency, which shows the twisting motion.
This behavior is presumed to be a similar phenomenon to the curve veering observed in a general vibration system (Du Bois et al., 2009). Figure 6 shows a model of a translational two-degree-of-freedom system with curve veering, and Fig. 7 indicates the evolution of the respective eigenvalues with increasing mass. From a macro point of view, the two modes appear to intersect after they approach each other, and, after that, they appear to separate from each other. However, their ways of transition are interchanged without crossing in fact. The connecting spring $s$ is set small enough, and they only change the mass $m_2$. As a result, only one of the eigenvalues is actively changed. Meanwhile, in the previous complex eigenvalue analysis, the eigenvalues are attracted to each other in a symmetrical manner. The behavior leading to the curve veering in friction-induced vibration has not been sufficiently studied so far, and the factor of this behavior is explained in the next section.

![Fig. 6 Translational two-degree-of-freedom model by Du Bois et al. The connecting spring $s$ is set small enough.](image1)

![Fig. 7 State of transition of the eigenvalues by Du Bois et al.](image2)

### 4. Evaluation by characteristic polynomial plane

#### 4.1 Geometrical representation

In a vibration system including friction, seemingly, eigenvalues shifting from stable to unstable have similarity to curve veering. Meanwhile, in a complex eigenvalue analysis by FEM with a large degree of freedom, the transition ways and amount of the eigenvalues are various, and the factors are not sufficiently considered. One of the factors is presumed to be that the margin of the coupling during the stable state is not sufficiently expressed. Therefore, instead of dealing with a characteristic equation directly to find eigenvalues, a characteristic polynomial, which is an element of a characteristic equation obtained from an equation of motion, is processed. First, a characteristic polynomial is derived for the two-degree-of-freedom model.

For the equation of motion of Eq. (24) above, the translational velocity and rotational velocity are determined as Eq. (48) and Eq. (49).

$$\dot{x} = v_x$$  \hspace{1cm} (48)

$$\dot{\theta} = v_\theta$$  \hspace{1cm} (49)

They are substituted into Eq. (24) to obtain Eq. (50).

$$\begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_\theta \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} v_x \\ v_\theta \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = 0 \hspace{1cm} (50)$$

Equation (50) is transformed as Eq. (51) to obtain the linear vector field style shown in Eq. (52). In addition, $Y$ and $A$ are as Eq. (53) and Eq. (54).

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \\ \dot{\theta} \\ \dot{v}_\theta \end{bmatrix} = \begin{bmatrix} \frac{k_{11}}{M} & \frac{1}{M} & 0 & -\frac{k_{12}}{M} \\ -\frac{c_{11}}{M} & \frac{1}{M} & -\frac{c_{12}}{M} & 0 \\ -\frac{k_{21}}{J} & 0 & \frac{1}{J} & 0 \\ \frac{k_{22}}{J} & -\frac{c_{22}}{J} & 0 & -\frac{c_{22}}{J} \end{bmatrix} \begin{bmatrix} x \\ v_x \\ \theta \\ v_\theta \end{bmatrix} \hspace{1cm} (51)$$

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$$\dot{Y} = AY$$

$$Y = \begin{bmatrix} x \\ v_x \\ \theta \\ v_\theta \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_{11}}{M} & -\frac{c_{11}}{M} & -\frac{k_{12}}{M} & -\frac{c_{12}}{M} \\ 0 & 0 & 0 & 1 \\ -\frac{k_{21}}{J} & -\frac{c_{21}}{J} & -\frac{k_{22}}{J} & -\frac{c_{22}}{J} \end{bmatrix}$$

The characteristic polynomial $f_\lambda(\lambda)$ is formulated as Eq. (55), where $\lambda$ is the eigenvalue; the expression is expanded to get Eq. (56).

$$f_\lambda(\lambda) = \det (\lambda I - A)$$

$$f_\lambda(\lambda) = (MJ\lambda^4 + (c_{11}J + c_{22}M)\lambda^3 + (c_{11}c_{22} - c_{12}c_{22} + k_{11}J + k_{22}M)\lambda^2$$

$$(+ (c_{11}k_{22} + c_{22}k_{11} - c_{12}k_{21} + c_{21}k_{12})\lambda + (k_{11}k_{22} - k_{12}k_{22}))/MJ$$

Next, the eigen equation is regarded as an equation in which both the real and imaginary parts of the characteristic polynomial are zero. The real part of the characteristic polynomial obtained by substituting an arbitrary complex number $\lambda_r + j\lambda_i$ into the eigenvalue $\lambda$ is as shown in Eq. (57), as the real part of the eigenvalue is $\lambda_r$ and the imaginary part of the eigenvalue is $\lambda_i$. Similarly, the imaginary part is shown in Eq. (58).

$$\text{Real } f_\lambda(\lambda) = (MJ(\lambda_r^4 - 6\lambda_r^2\lambda_i^2 + \lambda_i^4) + (c_{11}J + c_{22}M)(\lambda_r^3 - 3\lambda_r\lambda_i^2)$$

$$(+ (c_{11}c_{22} - c_{12}c_{22} + k_{11}J + k_{22}M)(\lambda_r^2 - \lambda_i^2)$$

$$(+ (c_{11}k_{22} + c_{22}k_{11} - c_{12}k_{21} + c_{21}k_{12})\lambda_r + (k_{11}k_{22} - k_{12}k_{22}))/MJ$$

$$\text{Imaginary } f_\lambda(\lambda) = j(4MJ(\lambda_r^3\lambda_i - \lambda_r\lambda_i^3) + (c_{11}J + c_{22}M)(3\lambda_r^2\lambda_i - \lambda_i^3)$$

$$+ 2\lambda_r\lambda_i(c_{11}c_{22} - c_{12}c_{22} + k_{11}J + k_{22}M)$$

$$+ \lambda_i(c_{11}k_{22} + c_{22}k_{11} - c_{12}k_{21} - c_{21}k_{12}))/MJ$$

In the following, for simplicity, the equations are treated without damping. Equation (57) is a quartic function, which becomes an even function that usually has three extreme vertices. Equation (58) becomes zero when $\lambda_r = 0$ or $\lambda_i = 0$. In the case of a stable vibration phenomenon with no damping, eigenvalues always have pure imaginary numbers. Consequently, the eigenvalues should be obtained when Eq. (57) becomes zero with $\lambda_r = 0$. To obtain $\lambda_i$, at which the real part of the characteristic polynomial becomes zero, it is geometrically illustrated to facilitate understanding of this feature. In a stable state, the response surface that is the real part of the characteristic polynomial is shown in Fig. 8. Because the characteristic polynomial is a complex quartic function, it is characterized by an axisymmetric shape having the characteristics of an even function.

Next, Fig. 9 shows overhead views of response surfaces when the friction coefficient is changed. In Fig. 9(a), the eigenvalues exist at the points where the response surface intersects with the imaginary axis. In the two-degree-of-freedom stable system, four pure imaginary eigenvalues are obtained. Attention is paid to two positive coefficients of imaginary parts of eigenvalues representing vibration among these four. These two eigenvalues appear at the valley boundary where the response surface is below the zero plane. Figure 9(b) shows the result of unstable vibration when the friction coefficient is 0.6. The valley of the response surface is above the zero plane and is not divided in the imaginary axis direction. The eigenvalues appear in the constriction, changing from a pure imaginary number to complex number.
Fig. 8  Response surface showing the real part of the characteristic polynomial. The surface is shown in a three-dimensional space in which the imaginary part and the real part of the variable to be substituted are each axis of the horizontal plane, and the vertical axis is the real part of the characteristic equation. It intersects a plane, where the real part of the characteristic polynomial, is zero. The region to be considered in the real phenomenon is the positive range of the imaginary part of the eigenvalue indicating the speed of vibration.

(a) Response surface at stable vibration with $\mu = 0$. Red points indicate eigenvalues obtained by solving the characteristic equation using eigenvalue analysis. Thus, when the damping factor is not considered, the point where the response surface intersects with the imaginary axis is the solution. (b) Response surface at unstable vibration with $\mu = 0.6$. The response surface has no solution on the imaginary axis. The solutions of the characteristic equation appear at the red points of the narrowest parts of the surface.

Fig. 9  Overhead views of response surfaces with different coefficients of friction. (a) Response surface at stable vibration with $\mu = 0$. Red points indicate eigenvalues obtained by solving the characteristic equation using eigenvalue analysis. Thus, when the damping factor is not considered, the point where the response surface intersects with the imaginary axis is the solution. (b) Response surface at unstable vibration with $\mu = 0.6$. The response surface has no solution on the imaginary axis. The solutions of the characteristic equation appear at the red points of the narrowest parts of the surface.

Figure 10 shows a perspective view and a side view of the response surface with $\mu = 0$. In Fig. 10(f), two points
where the imaginary axis intersects with the response surface having a downwardly convex valley are stable eigenvalues. Thus, the shape of the response surface, particularly the height and position of the vertex, directly determines the value of the eigenvalue.

Fig. 10 A geometric interpretation of the eigenvalue determination by the response surface is shown in Fig. 11. As the friction coefficient increases, the height of the peaks formed by the surface gradually decreases relative to the horizontal plane where the real part of the characteristic polynomial is zero. When the saddle of the surface is lower than the horizontal plane, the eigenvalues change from the pure imaginary numbers to the complex numbers. The eigenvalue is determined by the relative height between the response surface and the horizontal plane. Here, an assumption is made that the relative height of the horizontal plane changes with reference to the response surface. Then, increasing the friction coefficient can be regarded as a decrease in the height of the horizontal surface. The pair of eigenvalues behaves like two balls rolling from two high peaks, and the eigenvalues are seemingly determined by where they touch the zero plane. In the case of unstable vibration, the ball reaches the saddle part of the response surface before reaching the zero plane, and the direction of the transition is shifted by 90° from the imaginary to real axis. They move as if repelling in the real axis direction, and as a result, the eigenvalues become complex numbers.

A geometric interpretation of the eigenvalue determination by the response surface is shown in Fig. 11. As the friction coefficient increases, the height of the peaks formed by the surface gradually decreases relative to the horizontal plane where the real part of the characteristic polynomial is zero. When the saddle of the surface is lower than the horizontal plane, the eigenvalues change from the pure imaginary numbers to the complex numbers. The eigenvalue is determined by the relative height between the response surface and the horizontal plane. Here, an assumption is made that the relative height of the horizontal plane changes with reference to the response surface. Then, increasing the friction coefficient can be regarded as a decrease in the height of the horizontal surface. The pair of eigenvalues behaves like two balls rolling from two high peaks, and the eigenvalues are seemingly determined by where they touch the zero plane. In the case of unstable vibration, the ball reaches the saddle part of the response surface before reaching the zero plane, and the direction of the transition is shifted by 90° from the imaginary to real axis. They move as if repelling in the real axis direction, and as a result, the eigenvalues become complex numbers.
When the damping coefficients are zero, the response surface is symmetric with respect to the imaginary axis and the real axis. The eigenvalues are also symmetric with respect to the imaginary axis.

Fig. 11 Geometrical interpretation by response surface. Assuming that the height of a plane where the real part of the characteristic polynomial is zero is relatively changed, the transition of the eigenvalues compared to balls is shown. When the plane is relatively lowered, the balls fall below the saddle portion of the surface. Therefore, they have the real parts of the eigenvalues.

4.2 Effects of parameter changes on response surfaces

In the previous section, the existence positions of the eigenvalues specified from the response surface were clarified. The form of the response surface determines whether the vibration system is stable or unstable. In this section, the effects of some parameters of the equation of motion on the response surfaces are evaluated, including how the parameters that make up the equation of motion, such as mass, damping, stiffness, and friction coefficient, affect the shape of the response surface. This evaluation helps you understand how the value of the eigenvalue is determined.

First, to facilitate evaluation, the damping coefficients of the vibration system are zero. Because the eigenvalues exist on the imaginary axis during stable vibration, the imaginary axis sectional view indicates the influence of each parameter. Taking damping into consideration, overhead views of response surfaces are verified to determine the influences of damping.

In Table 1, representative parameters, such as friction coefficient $\mu$, mass $M$, and stiffness $k_T$, are given. For qualitative evaluation, the response surface needs to be changed boldly. Therefore, the change width of each parameter is greatly changed numerically, ignoring general physical characteristics.

Figure 12 shows the transition of the cross sections when the coefficient of friction $\mu$ is changed. Increasing the friction coefficient $\mu$ raises the cross section in the direction of the real axis. The eigenvalues are the intersections of the cross section and the imaginary axis. When the friction coefficient is small, the separation between eigenvalues is wide. Conversely, the separation width decreases as the friction coefficient increases. This parameter study describes how a large friction coefficient causes the eigenvalues not to exist on the imaginary axis, and the eigenvalues change from a pure imaginary number to a complex number, leading to unstable vibration. From another viewpoint, when the vertex moves from the minus side to the plus side of the characteristic polynomial real part, it becomes unstable. Because the vertex position is unknown only by eigenvalue analysis, it is not known how stable the initial vibration system is with respect to changes in the friction coefficient. If there is information, such as whether the slope of the convex left and right is gentle or tight and how stable the vertex position is, there is no need to study parametrically the eigenvalue analysis results.

Figure 13 shows the transition of the cross sections when the mass $M$ is changed without friction. As the mass increases, the convex parts of the cross section shift in the direction of the origin of the imaginary axis. The shifting indicates that the increases of the modal mass decrease the absolute values of imaginary parts of the eigenvalues. In
addition, in this process, the real parts of the vertexes increase halfway, and decrease halfway. Characteristically, the cross sections have a fixed point that does not depend on the size of the mass $M$. When the apex of the convex portion coincides with the fixed point, the cross section is at the uppermost position; in that case, a slight increase in the friction coefficient causes the apex to move to the positive side of the real axis. That is, the point at which the eigenvalue separation becomes zero is a fixed point, and unstable vibration easily occurs there. In this geometrical expression, the vertex position is clear in advance. When it is desired to move efficiently the vertex position to the stable side, it can be determined whether the mass should be increased or decreased based on the positional relationship between the vertex and the fixed point. Even in a parameter study using eigenvalue analysis, the interval between eigenvalues can be expanded, but the stability margin for the interval is not known, and a measure for expanding the interval uniformly is insufficient. If this curved surface is utilized, the tolerance to destabilization of the eigenvalue separation can be obtained. The fixed point is the point where curve veering occurs, and this time the frictional force is not generated, so it appears at the zero point in the real part axis direction of the characteristic polynomial. In other words, the eigenvalues when the mass is changed are close to each other and rebound in the imaginary direction after being closest to the curve veering point. If a frictional force is applied to this vibration system, it is probable that the eigenvalue repels in the direction of the real axis, as shown in Fig. 13. Therefore, the curve veering in the stable vibration is similar to the modal coupling in the unstable vibration, except that the direction of rebound is the imaginary axis or the real axis.

Figure 14 shows the transition of the cross sections when the stiffness $k_T$ is changed. Contrary to the increase in mass, the convex portion moves away from the origin. In addition, the convex portions turn from rising to falling and have a fixed point at the highest position. Thus, mass and stiffness have similar properties, and each specific value makes the vibration system most unstable. Therefore, adjusting the mass and stiffness, so that the eigenvalue moves away from the fixed point, leads to brake squeal countermeasures. Similar to the case of mass change, the stability margin can be known by evaluating the curved surface shape.

To summarize the above, focusing on the vertices of the cross sections, the values of the real parts of the characteristic polynomial at the vertices can express not only the degree of the instability but also the stability. When the friction coefficient increases without damping, the real parts of the characteristic polynomial at the vertices change from negative to positive. The evaluation method using the real part of the characteristic polynomial is compared with the eigenvalue analysis in Fig.15. It shows the transition of the eigenvalues and the transition of the real part of the characteristic polynomial at the vertex when the friction coefficient changes. In Fig.15 (b), the vertex shows the consistent transition before and after the coupling. Therefore, the stability and the instability can be expressed relatively.

Fig. 12  Response cross sections when friction coefficient $\mu$ is increased, ignoring the real physical property and drawing 11 levels of cross section with $-5 \leq \mu \leq 5$ show a relatively monotonous change.
Fig. 13  Response cross sections when mass $M$ is increased. Drawing nine levels of cross section with $2.0 \leq M \leq 3.0$. The increase in the mass reduces the overall unevenness. Yellow points are veering points that are fixed points.

Fig. 14  Response cross sections when stiffness $k_T$ is increased. Drawing 9 levels of cross section with $10^6 \leq k_T \leq 10^9$. The increase in the stiffness expands the overall unevenness. Yellow points are veering points that are fixed points.

Fig. 15  Difference between the transition of the eigenvalues and the transition of the vertex of the real part of the characteristic polynomial. The friction coefficient increases from 0 to 0.2 in 0.05 steps. (a) Transition of the eigenvalues. The direction of the transition changes before and after the coupling. (b) The vertex rises steadily. Thus, the vertex can be regarded as a substitute for the difficulty of the coupling.
The next study is about the change of the response surface caused by the increase of the damping coefficient $c_T$. A previous study reported that the transition of eigenvalues including damping is complicated (Fritz et al., 2007). When unstable with small damping, the paired eigenvalues exist in the constriction of the response surface to cross the imaginary axis. This is equivalent to the previous case without damping in Fig. 9. Figure 16 shows an overhead view of the response surface with large damping. As the damping increases, the symmetry of the response surface with respect to the imaginary axis is lost, the surface on the negative side of the real axis becomes lower, and the area exceeding the zero plane decreases. Furthermore, the center of the constricted part shifts to the negative side of the real axis. Thus, the overhead view geometrically reveals that the real part of the eigenvalue becomes negative and stabilizes. In general, when damping is applied to a vibration system, the real part of the eigenvalue shifts to the minus side, and the real part of the eigenvalue representing the speed of vibration decreases. In the case of coupled vibration with damping, the imaginary parts of the eigenvalues take different values, and even if the eigenvalue imaginary part does not match, if friction force flows in, it becomes unstable. By evaluating this curved surface, it can be explained how a vibration system including damping becomes unstable, even if the eigenvalue imaginary part does not match. It is difficult to derive such an interpretation by eigenvalue analysis.

Fig. 16  Top view of response surface when damping coefficient is increased, with the damping coefficient $c_T = 10^3$. Red points are eigenvalues calculated from eigenvalue analysis. The symmetry of the response surface with respect to the imaginary axis is lost.

The main transitions of the above parameters are summarized in Table 2. This study geometrically reveals that the transition changes depending on whether the eigenvalues are larger or smaller than the fixed points. One of measures against brake squeal is to increase the separation of eigenvalue pairs that are coupled by changing the mass and stiffness. This is attempted for multiple-degree-of-freedom systems; however, it often fails to obtain the desired amount of expansion. The response surface provides guidance for improvement studies.
5. Conclusion

In this research, trends of transitions of eigenvalues were investigated using the complex response surface, which shows the real part of the characteristic polynomial for the friction-induced vibration in a low-degree-of-freedom system. It was revealed geometrically how each parameter constituting the equation of motion, is related to the shape of the response surface that determines the eigenvalues. The real part of the proper polynomial forms a quartic surface that is extended to complex space. The surfaces are useful for geometrically indicating the location of eigenvalues obtained by eigenvalue analysis. The shape of the curved surface is effective for estimating how the eigenvalue changes because of the change of the parameter. The stability of the vibration system can be judged by how far the vertex position of the curved surface is from the zero plane. It is also possible to estimate the ease of destabilization by grasping the vertex position and the separation between eigenvalues. In eigenvalue analysis, an eigenvalue is naturally obtained, but where the vertex position is cannot be obtained. If one compares the eigenvalues to a ball rolling on a curved surface, both mode coupling and curve veering can be regarded as equivalent phenomena. The evaluation of the result of the FEM complex eigenvalue analysis that is widely used is a difficult task that relies only on the position of the intersection of the curved surface and the zero plane, and the eigenvalue transition leading to coupled vibration is not sufficiently considered. Adopting this evaluation suggests how eigenvalues are affected by each parameter. It is hoped that this will lead to effective measures to suppress brake noise.

In the future, this research should be applied to the transition of complex eigenvalues of the FEM with many degrees of freedom. The factors are clearly understood and verified by experiments. In addition, an effective countermeasure against brake squeal of the actual brake assembly is formulated. Squeal suppression by optimization techniques for complex eigenvalue analysis has been studied (Matsushima et al., 2014) (Inoue et al., 2015); however, the technique for the optimal solution was not revealed. This evaluation should physically explain factors of optimal shapes.

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