CODING THEORY

On q-ary Codes with Two Distances $d$ and $d + 1$

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Abstract—We consider $q$-ary block codes with exactly two distances: $d$ and $d + 1$. Several constructions of such codes are given. In the linear case, we show that all codes can be obtained by a simple modification of linear equidistant codes. Upper bounds for the maximum cardinality of such codes are derived. Tables of lower and upper bounds for small $q$ and $n$ are presented.

Key words: two-distance codes, equidistant codes, bounds for codes.

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1. INTRODUCTION

Let $Q = \{0, 1, \ldots, q - 1\}$. Any subset $C \subseteq Q^n$ is called a code and is denoted by $(n, N, d)_q$, i.e., a code of length $n$, cardinality $N = |C|$, and minimum (Hamming) distance $d$. For linear codes we use the notation $[n, k, d]_q$ (i.e., $N = q^k$). An $(n, N, d)_q$ code $C$ is equidistant if for any two distinct codewords $x$ and $y$ we have $d(x, y) = d$, where $d(x, y)$ is the (Hamming) distance between $x$ and $y$. A code $C$ is constant-weight, denoted by $(n, N, w, d)_q$, if every codeword $c$ is of weight $\text{wt}(c) = w$.

We consider codes with only two distances, $d$ and $d + 1$. One of our purposes is to see how we can increase the cardinality of equidistant codes allowing one more value for the closest distance between codewords. As we can see, admitting one more distance essentially increases the variety of codes in comparison with equidistant codes. Such codes might be of interest as codes with almost constant energy under the amplitude-phase modulation (since they are almost equidistant). As we will observe, such codes are often connected to equidistant codes. In particular, we show that all linear codes of such type can be obtained from linear equidistant codes. At the same time, we are not aware of any investigations of codes with two consecutive distances.

Denote by $(n, N, \{d, d + 1\})_q$ an $(n, N, d)_q$ code $C \subseteq Q^n$ with the following property: for any two distinct codewords $x$ and $y$ from $C$ we have $d(x, y) \in \{d, d + 1\}$. We are interested in constructions, classification results, and bounds on the maximal possible size of $(n, N, \{d, d + 1\})_q$ codes. We will show here that linear $q$-ary codes with two distances $d$ and $d + 1$ are completely known and can be obtained by a simple modification of linear equidistant codes. Preliminary results of this paper (namely, the binary case and a conjecture for $q \geq 3$) were announced in [1]. Independently of us, this result was obtained by a different method in [2].

2. PRELIMINARY RESULTS

We recall the following classical Johnson bound for size $N_q(n, d, w)$ of a $q$-ary constant-weight $(n, N, w, d)_q$ code [3]:

$$N_q(n, d, w) \leq \frac{(q - 1)dn}{qw^2 - (q - 1)(2w - d)n} \quad (1)$$
if \(qw^2 > (q-1)(2w-d)n\).

**Definition 1.** A balanced incomplete block (BIB) design \(B(v, k, \lambda)\) is an incidence structure \((X, B)\), where \(X = \{x_1, \ldots, x_v\}\) is a set of elements and \(B = \{B_1, \ldots, B_b\}\) a collection of \(k\)-sets \(B_i\) of elements (called blocks) such that every two distinct elements of \(X\) are contained in exactly \(\lambda \geq 0\) blocks of \(B\) (here \(1 \leq k \leq v-1\)).

Other two parameters of a \(B(v, k, \lambda)\)-design are \(b = |B|\) (the number of blocks) and \(r\) (the number of blocks containing one fixed element):

\[
r = \lambda \frac{v-1}{k-1}, \quad b = \lambda \frac{v(v-1)}{k(k-1)} \quad \text{if} \quad \lambda > 0
\]

\((\lambda = 0\) corresponds to the case \(k = 1\) and hence \(b = rv\)).

A \(B(v, k, \lambda)\)-design is completely described by its incidence matrix \(A = [a_{i,j}]\), where \(a_{i,j} = 1\) if \(a_i \in B_j\) and \(a_{i,j} = 0\) otherwise. Thus, \(A\) is a binary \(v \times b\) matrix with columns of weight \(k\) such that any two distinct rows contain exactly \(\lambda\) common nonzero positions.

A \(B(v, k, \lambda)\) design is resolvable (denoted by \(RB(v, k, \lambda)\)) if its incidence matrix \(A\) is of the form

\[
A = \left[ A_1 \mid \ldots \mid A_r \right],
\]

where for any \(i \in \{1, \ldots, r\}\) every row of \(A_i\) is of weight 1. A \(B(v, k, \lambda)\) design is \(m\)-nearly resolvable (\(NRB_m(v, k, \lambda)\)-design) if its incidence matrix \(A\) can be represented as follows [4]:

\[
A = \left[ A_1 \mid \ldots \mid A_n \right], \quad n = \frac{bk}{v-m},
\]

such that the following properties hold:

1. Every submatrix \(A_j\) of size \(v \times \frac{v-m}{k}\) consists of rows of weight 1 with the exception of \(m\) zero rows whose indices belong to a set \(V_j\), \(|V_j| = m, V_j \subset \{1,2,\ldots,v\}\);
2. The sets \(V_1, \ldots, V_n\) (as a collection of \(n\) blocks of size \(m\)) induce a \(B(v, m, \xi)\)-design (which we call the accompanying design) for some \(\xi\).

We need the following result from [4, 5].

**Theorem 1.** Any \(m\)-nearly resolvable \(NRB_m(v, k, \lambda)\)-design induces a \(q\)-ary equidistant constant-weight \((n, N, w, d)_q\) code \(C\) with parameters \(q = (v-m+k)/k\), \(N = v\),

\[
n = \frac{\lambda v(v-1)}{(k-1)(v-m)}, \quad w = \frac{\lambda(v-1)}{k-1}, \quad d = \frac{\lambda(v+m-k)}{k-1},
\]

meeting the Johnson bound (1) with an additional property that its \(n\) blocks of size \(m = (n-w)N/n\) formed by indices of zero positions define a \(B(v, m, \xi)\)-design.

Recall the following wide class of \(q\)-ary equidistant codes constructed in [6].

**Theorem 2.** Let \(p\) be a prime and let \(s, \ell,\) and \(h\) be any positive integers. Then there exists an equidistant \((n, N, d)_q\) code with parameters

\[
q = p^{sh}, \quad n = \frac{p^{s(h+\ell)} - 1}{p^s - 1}, \quad N = p^{s(h+\ell)}, \quad d = p^{h+\ell}(p^{sh} - 1)\frac{p^s - 1}{p^s - 1}.
\]

**Definition 2** [7]. Let \(G\) be an abelian group of order \(q\) written additively. A square matrix \(D\) with elements from \(G\) of order \(q\mu\) is called a difference matrix and denoted by \(D(q, \mu)\) if the componentwise difference of any two different rows of \(D\) contains any element of \(G\) exactly \(\mu\) times.

Clearly, a matrix \(D(q, \mu)\) induces an equidistant \((q\mu - 1, q\mu, \mu(q-1))_q\) code [6].
3. CONSTRUCTIONS

3.1. Combinatorial Constructions

Denote by \( W_q(n) \) a ball of radius 1 centered at the zero vector, i.e., \( W_q(n) = \{ x \in Q^n : \text{wt}(x) \leq 1 \} \).

Construction 1a. The ball \( W_q(n) \) is an \( (n, (q - 1)n + 1, \{1, 2\})_q \) code.

Construction 1b. Parity checking (modulo 2) of Construction 1a gives rise to an \( (n + 1, (q - 1)n + 1, \{2, 3\})_q \) code, which we denote by \( W_q^*(n + 1) \). For any codeword \( (0 \ldots 0a0\ldots 0) \) from \( W_q(n) \) we form the codeword \( (0 \ldots 0a0\ldots 0 \mid a) \) from \( W_q^*(n + 1) \).

Construction 2. An equidistant \( (n, N, d)_q \) code \( C \) produces two \( (n', N, \{d', d' + 1\})_q \) codes, namely, the \( (n - 1, N, \{d - 1, d\})_q \) code \( C_1 \) obtained by deleting (any) position from \( C \), and the \( (n + 1, N, \{d, d + 1\})_q \) code \( C_2 \) obtained by adding one position to \( C \).

Combining Constructions 1a and 1b with Construction 2, we obtain the following two constructions.

Construction 3a. An equidistant \( (n_1, N_1, d)_q \) code and \( W_{q_2}(n_2) = (n_2, N_2, \{1, 2\}) \) give an \( (n, N, \{d + 1, d + 2\})_q \) code with parameters

\[
q = \max\{q_1, q_2\}, \quad n = n_1 + n_2, \quad N = \min\{N_1, N_2\}.
\]

Construction 3b. An equidistant \( (n_1, N_1, d)_q \) code and \( W_{q_2}^*(n_2) = (n_2, N_2, \{2, 3\}) \) give an \( (n, N, \{d + 2, d + 3\})_q \) code with parameters

\[
q = \max\{q_1, q_2\}, \quad n = n_1 + n_2, \quad N = \min\{N_1, N_2\}.
\]

Construction 4. If there exist \( r \) mutually orthogonal Latin squares of order \( q \), then there exists a family of \( (s + 2, q^2, \{s + 1, s + 2\})_q \) codes \( C_s \), where \( s = 0, 1, \ldots, r \). If \( q \) is a prime power, the code \( C_s \) is linear.

Combining Constructions 2 and 4, we obtain the following.

Construction 5. For any prime power \( q \) there exists a family of (linear) \( [n, 2, \{d, d + 1\}]_q \) codes with parameters

\[
n = s(q + 1) + r, \quad d = sq + r - 1, \quad s \geq 1, \quad r = 1, \ldots, q + 1.
\]

Construction 6. If there exists a difference matrix \( D(q, \mu) \), then there exist \( (n, N, \{d, d + 1\})_q \) codes with parameters:

\[
n = q\mu - 2, \quad N = q\mu, \quad d = (q - 1)\mu - 1,
\]

\[
n = q\mu, \quad N = q\mu, \quad d = (q - 1)\mu.
\]

The well-known equidistant \( [4, 2, 3]_3 \) code \( C_1 \) and the \( [2, 2, \{1, 2\}]_3 \) code \( C_2 \) (Construction 4) give by Construction 3 a \( [6, 2, \{4, 5\}]_3 \) code \( C \), which is linear but its parameters are not optimal. Using the \( [3, 2, \{2, 3\}]_3 \) code \( C_3 \) (Construction 4) gives by Construction 3 a (linear) \( [7, 2, \{5, 6\}]_3 \) code, which has one word less than the optimal code.

The equidistant \( (13, 27, 9)_3 \) code (Theorem 2) implies by Construction 2 a \( (14, 27, \{9, 10\})_3 \) code, which is better than the random \( (14, 18, \{9, 10\})_3 \) code, and also a \( (12, 27, \{8, 9\})_3 \) code, which meets the upper bound (the best found random code has cardinality 18).

The difference matrix \( D(4, 3) \) (see [8]) without the trivial column is an optimal equidistant \( (11, 12, 8)_3 \) code. The difference matrix \( D(3, 4) \) (see [8]) without the trivial column is an equidistant \( (11, 12, 9)_4 \) code.
The well-known equidistant $(5,16,4)_4$ code $C_1$ and a $(5,16,\{1,2\})_4$ code $C_2$ (Construction 1) give by Construction 3 a $(10,16,\{5,6\})_4$ code (not good—there is a random $(10,20,\{5,6\})_4$ code). Twofold repetition of the $(5,16,4)_4$ code $C_1$ gives an optimal $(10,16,8)_4$ code.

The equidistant $(6,9,5)_4$ code $[4]$ implies by twofold repetition a $(12,9,\{10,11\})_4$ code (better than the random code). The equidistant $(21,64,16)_4$ code $[6]$ implies $(22,64,\{16,17\})_4$ and $(20,64,\{15,16\})_4$ codes by Construction 2. The equidistant $(9,10,8)_5$ code $[5]$ implies $(8,10,\{7,8\})_5$ and $(10,10,\{8,9\})_5$ codes by Construction 2. By Construction 5 we obtain the following family of $(n,N,\{d,d+1\})_5$ codes:

$$n = 9 + s, \quad N = 10, \quad d = 8 + s - 1, \quad s = 0,1,\ldots,6.$$  

In particular, for $s = 0$ we obtain an optimal $(9,10,8)_5$ code and for $s \geq 2$ all resulting codes are new. By Construction 1 this equidistant $(9,10,8)_5$ code implies an $(11,9,\{9,10\})_5$ code.

The equidistant $(6,25,5)_5$ code implies a family of $(6 + s,25,\{5 + s - 1,5 + s\})_5$ codes, where $s = 0,1,\ldots,6$, which gives better (or new) codes for $s \geq 1$.

The well-known resolvable design $(15,35,7,3,1)$ is equivalent to an optimal equidistant $(7,15,6)_5$ code. Now using Construction 5 we obtain from this code the following codes: $n = 7 + s, N = 15, d = 6 + s - 1, s = 1,\ldots,6$.

The affine design $(16,4,1)$ implies $[5]$ the equidistant constant-weight $(16,16,15,14)_6$ code, which implies in turn the $(16,17,\{14,15\})_6$ code (by adding the zero codeword).

### 3.2. Random Codes

We use a computer program for generation of random codes by a simple heuristic algorithm. We start with a seed which contains only zero vector in the simplest version and is the best found code of length one less in the most complicated. Then the search space consists of the vectors of weight $d$ and $d + 1$. It is extracted from an initial database of all $q^n$ vectors of length $n$, which is generated (once) by standard lexicographic means. The program randomly adds suitable vectors until the resulting code is good (i.e., until it has only distances $d$ and $d + 1$). Many iterations can be implemented, but usually the best codes (found in this way) are obtained quickly. The cardinalities of such random codes are shown in Section 6 together with those of the codes obtained from constructions of this section.

The results show that this approach is good when $d = 1,2$ and when $d$ is close to $n - 1$, but it fails to find good codes with a rich structure. Probably, it is also not good for $d$ in the mid-range.

### 4. LINEAR $(n,N,\{d,d+1\})_q$ CODES

In this section we obtain classification results in the case of linear codes with distances $d$ and $d + 1$. In particular, we will see that linear codes with two distances $d$ and $d + 1$ are completely known. The next theorem was proved for the binary case in [1]. The $q$-ary case was also conjectured in [1]. Here we give a simple proof of our conjecture for the case $q \geq 2$ and $k \geq 2$ based on purely coding theoretic arguments. Simultaneously, the corresponding result for $k \geq 3$ was proved in [2] based on geometric arguments.

Let $C$ be a $q$-ary (linear) $[n,3,q^2]_q$ equidistant code of length $n = q^2 + q + 1$, distance $q^2$, and cardinality $q^3$.

**Lemma.** Assume that $C$ is the above code represented as an $n \times q^3$ matrix over $\mathbb{F}_q$ (which we denote by $[C]$). Then $[C]$ cannot be written as a concatenation of two matrices, i.e., $[C] = [C_1] \vert [C_2]$, where $[C_1]$ is an $x \times q^3$ matrix (with $x < (n - 1)/2$) which represents a linear $[x,q^3;\{d,d+1\}]_q$ code $C_1$. 

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Proof. If \( x \in C \), then clearly \( \alpha x \in C \) for all \( \alpha \in \mathbb{F}_q^* \). Thus, the \( q^3 - 1 \) nonzero codewords of \( C \) can be split into \( q^2 + q + 1 \) classes. In this way we obtain a code of classes of such elements. It is given by an \( n \times n \) matrix \( P \).

Assume that the matrix \( P \) is a concatenation of two matrices \( P_1 \) and \( P_2 \), i.e., \( P = [P_1 | P_2] \), where \( P_1 \) is an \( x \times (q^2 + q + 1) \) matrix that corresponds to the equivalence classes of an \( [x, q^3, \{d, d + 1\}] \) linear code \( C_1 \). Consequently, every word of \( C_1 \) has \( x - d \) or \( x - d - 1 \) positions with zero entries. To simplify further computations, let \( \ell = x - (d + 1) \) (since we will consider the number of zero entries of a word instead of its weight).

Since \( P \) corresponds to an equidistant code with code distance \( q^2 \), clearly \( P_2 \) corresponds to a (linear) \( [n - x, 3, q^2 - d - 1]_q \) code \( C_2 \) with two distances, \( q^2 - d - 1 \) and \( q^2 - d \). Therefore, without loss of generality we may assume that \( x \leq n/2 \). Since \( n = q^2 + q + 1 \), we assume that \( x \leq (q + 1)/2 \) and \( \ell \leq (q + 1)/2 \) (indeed, every nonzero row, as well as every column of \( P \) contains \( q + 1 \) zeros).

Since any column of \( P \) has \( q + 1 \) zeros, the matrix \( P \) contains \( n(q + 1) \) zero entries. Assume that the matrix \( P_1 \) has exactly \( \eta \) words of weight \( d + 1 \) (i.e., \( \ell \) zeros) and the remaining \( n - \eta \) words of weight \( d \) (i.e., \( \ell + 1 \) zeros). Thus, we can write

\[
\ell \eta + (\ell + 1)(n - \eta) = (q + 1)x.
\]

Solving for \( \eta \), we obtain

\[
\eta = (\ell + 1)n - (q + 1)x. \tag{4}
\]

Since \( C_1 \) is a linear code of dimension 3, for any pair of coordinate positions there exists exactly one row of \( P_1 \) with zeros at these positions. There are \( x \) coordinate positions and \( x(x - 1)/2 \) pairs of positions. On the other hand, there are \( \eta \) rows with \( \ell \) zeros (every row provides \( \ell(\ell - 1)/2 \) pairs of coordinates) and \( n - \eta \) rows with \( \ell + 1 \) zeros (every row provides \( \ell(\ell + 1)/2 \) pairs of coordinates). Thus, we obtain the following equality:

\[
\frac{x(x - 1)}{2} = \frac{\ell(\ell - 1)}{2} \eta + \frac{\ell(\ell + 1)}{2}(n - \eta). \tag{5}
\]

Our goal is to show that equality (5) cannot be valid for any \( x \) in the interval \([2, q(q + 1)/2]\). Using (4), expression (5) becomes

\[
x(x - 1) = \ell(\ell - 1)\eta + \ell(\ell + 1)n - \ell(\ell + 1)\eta
\]

\[
= \ell(\ell + 1)n - 2\ell\eta
\]

\[
= \ell(\ell + 1)n - 2\ell[(\ell + 1)n - (q + 1)x]
\]

\[
= 2\ell(q + 1)x - \ell(\ell + 1)n.
\]

Thus, we arrive at the following quadratic equation for \( x \):

\[
x^2 - (2\ell(q + 1) + 1)x + \ell(\ell + 1)n = 0. \tag{6}
\]

We will show that the discriminant of this equation is negative. Thus, we have to verify that

\[
(2\ell(q + 1) + 1)^2 < 4\ell(\ell + 1)n.
\]

Recalling that \( n = q^2 + q + 1 \), this is equivalent to

\[
4\ell^2(q^2 + 2q + 1) + 4\ell(q + 1) + 1 < 4\ell(\ell + 1)(q^2 + q + 1)
\]

\[
= 4\ell^2(q^2 + q + 1) + 4\ell(q^2 + q + 1).
\]

Once simplified, it becomes

\[
4\ell^2q + 1 < 4\ell q^2.
\]
Since \( \ell \leq (q+1)/2 \), the last inequality is obviously true for \( \ell \geq 1 \). Thus, we have obtained that there is no submatrix \( P_1 \) and, consequently, the linear \([q^2 + q + 1, 3, q^2]_q\) code \( C \) cannot be represented as a concatenation of two linear codes \( C_1 \) and \( C_2 \) of type \((n, N, \{d, d+1\})_q\). \( \triangle \)

**Theorem 3.** Let \( C \) be a \( q \)-ary linear \([n, k, d]_q\) code with two distances \( d \) and \( d+1 \) and \( k \geq 2 \). Then \( C \) is obtained by Construction 2 from the previous section, i.e., by deleting or adding an arbitrary column in the parity check matrix of a linear \( q \)-ary equidistant code with the following exception for the case \( k = 2 \) and \( q \geq 3 \), when \( C \) can be obtained by Construction 2 or Construction 5.

**Proof.** First consider the case \( k = 2 \). For this case we can have an \([n, 2, d]_q\) code \( C \) with two distances \( d \) and \( d+1 \) obtained also by Construction 5. Let \( C_1 \) be an equidistant \([n_1, 2, d_1]_q\) code with parameters \( n_1 = s(q+1), d_1 = sq \) and \( C_2 \) be an \([n_2, 2, d_2]_q\) code with parameters \( n_2 = r, d_2 = r - 1 \). The generator matrix \( G \) of \( C \) is of the form \( G = [G_1 | G_2] \), where \( G_1 \) and \( G_2 \) are generator matrices of the codes \( C_1 \) and \( C_2 \), which (up to equivalence) look as follows: the matrix \( G_1 = [G_0 \ldots | G_0] \) is the \( s \)-time repetition of

\[
G_0 = \begin{bmatrix}
a_0 & a_1 & a_1 & a_1 & \ldots & a_1 \\
a_1 & a_0 & a_1 & a_2 & \ldots & a_{q-1}
\end{bmatrix},
\]

where we denote \( \mathbb{F}_q = \{a_0 = 0, a_1 = 1, a_2, \ldots, a_{q-1}\} \), and the matrix \( G_2 \) is of the form

\[
G_2 = \begin{bmatrix}
a_0 & a_1 & a_1 & a_1 & \ldots & a_1 \\
a_1 & a_0 & a_1 & a_2 & \ldots & a_{r-2}
\end{bmatrix}.
\]

All these facts are commonly known and do not need any proofs. The only thing we have to say is that all elements of the second row of \( G_2 \) starting from the second position should be different and this condition is necessary and sufficient for \( G_2 \) to be a generator matrix of the code \( C_2 \).

Now we claim that any \([n, 2, d]_q\) code with two distances should be of the form described above. This is clear for the case \( n \leq q \). For larger \( n \) assume that the code \( C_1 \) of length \( q + 1 \) is not an equidistant \([q + 1, 2, q]_q\) code, i.e., it has minimum distance \( d = q - 1 \). Since its average distance is known (and equals \( q \)), we conclude that this code has three distances, namely, \( q - 1, q, \) and \( q + 1 \). Denoting by \( \alpha_w \) the number of codewords of weight \( w \) and taking into account that \( \alpha_{q-1} = \alpha_{q+1} \), we obtain that

\[
\alpha_{q-1} = \alpha_{q+1} = q - 1, \quad \alpha_q = (q - 1)^2.
\]

(7)

As we know, the \([r, 2, r-1]_q\) code \( C_2 \) has weights \( r - 1 \) and \( r \). Denoting by \( \beta_w \) the number of codewords of weight \( w \), we deduce that

\[
\beta_{r-1} = (q - 1)r, \quad \beta_r = (q - 1)(q + 1 - r).
\]

(8)

Thus, the concatenation of these two codes \( C_1 \) and \( C_2 \) would be a code \( C \) with at least three distances \( d, d + 1, \) and \( d + 2 \), where \( d \leq q + r - 1 \); i.e., we obtain a contradiction. Thus, \( C_1 \) of length \( (q + 1)s \) should be an equidistant code. Therefore, any \([n, 2, d]_q\) code \( C \) with two distances \( d \) and \( d + 1 \) is obtained by one of the two constructions, namely, Constructions 2 or 5.

Now to complete the proof we only have to show that any \([n, 3, d]_q\)-code with two distances \( d \) and \( d + 1 \) can be obtained only by Construction 2. In contrary, assume that \( C_1 \) is an \([n_1, 3, d_1]_q\) code with two distances \( d_1 \) and \( d_1 + 1 \) of length \( n_1 \) in the interval \( 2 \leq n_1 \leq q^2 + q - 1 \). This means that there is an \([n_2, 3, d_2]_q\) code \( C_2 \) (complementary to \( C_1 \)) with two distances \( d_2 \) and \( d_2 + 1 \) of length \( n_2 = q^2 + q + 1 - n_1 \). Hence, there exists a \( q \)-ary \([n, 3, q^2]_q\) equidistant code \( C \) of length \( n = q^2 + q + 1 \) which can be written as a concatenation of two codes \( C_1 \) and \( C_2 \). But by the lemma this is impossible. Since any \([n, k \geq 4, d]_q\) code with two distances \( d \) and \( d + 1 \) can be reduced (by shortening) to an \([n', 3, d]_q\) code with two distances \( d \) and \( d + 1 \), this completes the proof. \( \triangle \)
Remark 1. The codes with weights $d$ and $d+1$ which we consider are a subclass of a wide class of two-weight codes, a classical subject of algebraic coding theory. However, such codes with weights $d$ and $d+1$ were not considered in the literature before (see, for example, the comprehensive survey of Calderbank and Kantor [9]). Therefore, Theorem 3 is a classification result for the linear codes with weights $d$ and $d+1$. As is well known [9,10], a code which is dual to any linear projective two-weight code is a uniformly packed (hence, completely regular; see [11,12]) code. Two-weight codes obtained by deleting one position from linear equidistant codes induce such completely regular codes and are well known [11,12].

Remark 2. The proof of the lemma which we used above can be extended to any equidistant code $C$ satisfying the following property: for any two coordinate positions, say $i$ and $j$, $1 \leq i, j \leq n$, there exist $\lambda$ codewords $c = (c_1, \ldots, c_n)$ from $C$ with two zeros in these two positions, i.e., $c_i = c_j = 0$, where $\lambda$ is the same for any $i$ and $j$. This means, for example, that any binary nonlinear $(n,N,d) = (4m-1, 4m, 2m)$ Hadamard code cannot be represented as a concatenation of two codes $C_1$ and $C_2$ (with lengths $n_i \geq 2$) with two consecutive distances $d_1, d_1 + 1$ and $d_2, d_2 + 1$, respectively. But we cannot use that lemma for proving the corresponding theorem, where linearity is very essential. Also note that this reduction to the quadratic equation does not give any result for codes with distances $d$ and $d + \delta$ where $\delta \geq 2$. Even for the next case $\delta = 2$, the corresponding codes exist (for the case $q = 2^m$ [9]), and the solution of the quadratic equation reduces to a solution of a Diophantine equation, which is beyond the scope of this paper.

5. UPPER BOUNDS

We are interested in upper bounds for the quantity

$$A_q(n; \{d,d+1\}) = \max\{|C| : C \text{ is an } (n,|C|, \{d,d+1\}) \text{ code}\},$$

the maximum possible cardinality of a code in $Q^n$ with two distances $d$ and $d+1$.

The general Delsarte harmonic bound [10]

$$A_q(n; \{d,d+1\}) \leq 1 + (q-1)n + (q-1)^2 \binom{n}{2}$$

and the Barg–Musin improvement [13] to

$$A_q(n; \{d,d+1\}) \leq 1 + (q-1)^2 \binom{n}{2}$$

(for $(2d+1)q < 2n(q-1)+2-q$) seem to be quite general, while our situation appears very specific.

5.1. Linear Programming Bounds

For fixed $n$ and $q$, the (normalized) Krawtchouk polynomials are defined by

$$Q_i^{(n,q)}(t) = \frac{1}{r_i} K_i^{(n,q)}(d), \quad d = \frac{n(1-t)}{2}, \quad r_i = (q-1)^i \binom{n}{i},$$

where

$$K_i^{(n,q)}(d) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{d}{j} \binom{n-d}{i-j}.$$
are the (usual) Krawtchouk polynomials. If \( f(t) \in \mathbb{R}[t] \) is of degree \( m \geq 0 \), then it can be uniquely expanded as
\[
f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t).
\]

The next theorem is adapted for an estimation of \( A_q(n; \{d, d+1\}) \) from the general Delsarte linear programming bound. Proofs of such bounds are usually considered as folklore (see, for example, [10, 14]).

**Theorem 4.** Let \( n \geq q \geq 2 \), and let \( f(t) \) be a real polynomial of degree \( m \leq n \) such that
(A1) \( f(t) \leq 0 \) for \( t \in \{1 - 2d/n, 1 - 2(d+1)/n\} \);
(A2) The coefficients in the Krawtchouk expansion \( f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t) \) satisfy \( f_i \geq 0 \) for every \( i \).

Then \( A_q(n; \{d, d+1\}) \leq f(1)/f_0 \). If this bound is attained by an \( (n, N, \{d, d+1\})_q \) code \( C \) and a polynomial \( f(t) \), then \( f(1 - 2(d + i)/n) = 0 \), \( i = 0, 1 \), whenever there are points of \( C \) at distance \( d + i \), \( i = 0, 1 \), and \( f_i M_i(C) = 0 \), where
\[
M_i(C) = \sum_{x,y \in C} Q_i^{(n,q)}(1 - 2d(x,y)/n)
\]
is the \( i \)-th moment of \( C \).

Most of the upper bounds in the table below are obtained by Theorem 4 with the simplex method (i.e., we obtain the best possible bounds from Theorem 4). We describe now some cases where analytic forms of good bounds are possible.

The first degree polynomial \( f(t) = t - 1 + 2d/n \) gives the Plotkin bound, which is attained for many large \( d \). Optimization over the second degree polynomials gives the following result.

**Theorem 5.** If \( d \geq (n - 1)(q - 1)/q \), then
\[
A_q(n; \{d, d+1\}) \leq \frac{q^2 d (d + 1)}{n^2 (q - 1)^2 - n(q - 1)(2dq + q - 1) + dq^2(d + 1)}.
\]  

If this bound is attained by an \( (n, N, \{d, d+1\})_q \) code \( C \), then \( M_2(C) = 0 \) and, moreover, \( M_1(C) = 0 \) whenever \( d > (n - 1)(q - 1)/q \).

**Proof.** Consider the second degree polynomial
\[
f(t) = \left( t - 1 + \frac{2d}{n} \right) \left( t - 1 + \frac{2d + 2}{n} \right) = f_0 + f_1 Q_1^{(n,q)}(t) + f_2 Q_2^{(n,q)}(t),
\]
where
\[
f_0 = \frac{4(n^2(q - 1)^2 - n(q - 1)(2dq + q - 1) + dq^2(d + 1))}{n^2 q^2},
\]
\[
f_1 = \frac{8(q - 1)(dq - (q - 1)(n - 1))}{nq^2},
\]
\[
f_2 = \frac{4(q - 1)^2(n - 1)}{nq^2}.
\]
Condition (A1) is obviously satisfied.

The condition \( f_0 > 0 \) is equivalent to a quadratic inequality with respect to \( dq \), giving that \( n \geq q \) implies it. The condition \( f_1 \geq 0 \) is equivalent to \( dq \geq (n - 1)(q - 1) \) and \( f_2 > 0 \) is obvious. Thus, \( f(t) \) satisfies (A1) and (A2) provided that \( d \geq (n - 1)(q - 1)/q \). Now calculation of \( f(1)/f_0 \) gives the bound (9).

The conditions for attaining (9) follow from the general conditions of Theorem 4. △
The bound (9) is attained in some cases. In particular, \( A_q(n; \{d,d + 1\}) \leq q^2 \) for \( d = n - 1 \), which is attained for \( (q,n) = (3,3), (3,4), (4,5), \) and \( (5,6) \). Further, we have \( A_2(7; \{4,5\}) = A_2(7; \{3,4\}) = 8, A_2(10; \{5,6\}) = 12, A_3(12; \{8,9\}) = A_3(13, \{9,10\}) = 27 \) by (9). The cases of attaining (9) are marked by \( d^2 \) in the tables below.

Furthermore, if the bound (9) is attained by some code \( C \) and \( d > (n-1)(q-1)/q \) (i.e., \( f_1 > 0 \)), then \( M_1(C) = M_2(C) = 0 \). Thus, \( C \) is an orthogonal array of strength 2. In particular, the cardinality of \( C \) is divisible by \( q^2 \). This argument implies improvements of (9) by one giving the exact values \( A_2(12, \{5, 6\}) = A_2(12, \{6, 7\}) = A_2(13, \{6, 7\}) = 13 \) and the bounds \( 13 \leq A_3(6, \{4, 5\}) \leq 14 \). These cases are marked by \( n \) in the tables. One more interesting case is \( A_3(7, \{4, 5\}) = 15 \), where (9) is attained in the case \( d = (n-1)(q-1)/q \).

Further bounds can be obtained by some ad hoc polynomials. For example, the polynomial

\[
f(t) = 1 + (q-1)n\binom{n+q}{(n(q-1)+1)/q}f(t)
\]

gives \( A_q(n, \{1,2\}) = (q-1)n+1 \) (see Construction 1a) whenever \( q \) divides \( n - 1 \). In particular, we obtain \( A_2(n, \{1,2\}) = n+1 \) for odd \( n \). Similarly, the polynomial

\[
f(t) = 1 + \frac{n+2}{2}Q_{n/2}^{(n,2)}(t) + \frac{n}{2}Q_{1+n/2}^{(n,2)}(t),
\]

where \( n \) is even, gives \( A_2(n, \{1,2\}) \leq f(1)/f_0 = n + 2 \). If this bound is attained, then the linear programming slackness conditions of Theorem 4 give the equations \( M_{n/2} = M_{1+n/2} = 0 \) which, together with the trivial equation \( A_d(x) + A_{d+1}(x) = |C| - 1 \), allow us to compute (by MAPLE) the distance distributions of attaining (two-distance) codes. Here \( A_{d+i}(x) = |\{y \in C : d(x,y) = d+i\}| \), \( i = 0,1, \ldots \), \( x \in C \). Since this distance distribution is not integral, we obtain a contradiction. Thus, both polynomials prove that \( A_2(n, \{1,2\}) = n+1 \) (attained by Construction 1a). Such cases are marked with \( a \) in the tables.

Further careful examination of conditions for attaining the linear programming bounds could probably lead to other improvements in the tables.

### 5.2. Bounds via Spherical Codes

Codes from \( Q^n \) are naturally mapped to the sphere \( S^{(q-1)n-1} \). We first map bijectively the alphabet symbols \( 0, 1, \ldots, q - 1 \) to the vertices of the regular simplex in \( q - 1 \) dimensions and then map the codewords of a \( q \)-ary code \( C \subset Q^n \) coordinatewise to \( \mathbb{R}^{(q-1)n} \). It is easy to see that all vectors have the same norm, and after a normalization we obtain a spherical code on \( S^{(q-1)n-1} \).

This spherical code has cardinality \( |C| \) and maximal inner product \( 1 - 2d(q-1)n \) (equivalently, squared minimum distance \( 2dq/(q-1)n \)). Clearly, the \( q \)-ary codes with distances \( d \) and \( d+1 \) are mapped to the spherical \( 2 \)-distance codes with squared distances \( 2dq/(q-1)n \) and \( 2(d+1)q/(q-1)n \). This relation implies the following upper bound for \( A_q(n, \{d,d + 1\}) \).

**Theorem 6.** If \( d > \sqrt{2(q-1)n-1}/2 \), then

\[
A_q(n, \{d,d + 1\}) \leq 2(q-1)n + 1.
\]

**Proof.** Larman, Rogers, and Seidel [15] proved that if the cardinality of a two-distance set in \( \mathbb{R}^n \) with distances \( a \) and \( b \), \( a < b \), is greater than \( 2n + 3 \), then the ratio \( a^2/b^2 \) equals \( (k-1)/k \), where \( k \) is a positive integer satisfying \( 2 \leq k \leq (\sqrt{2n+1}+1)/2 \). The restriction \( 2n + 3 \) was moved to \( 2n + 1 \) in [16].

In our situation \( a^2/b^2 = d/(d+1) = (k-1)/k \) holds, whence we conclude that \( d = k - 1 \) has to belong to the interval \( [1, (\sqrt{2(q-1)n-1})/2] \). In other words, there exist no \( q \)-ary codes with
Table 1. Bounds for $q = 2$.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 7   | 7–10 | 8$^{d_2}$ | 8$^{d_2}$ | 2$^*$ | 2$^*$ |     |     |     |     |     |     |     |     |     |     |     |
| 8   | 9$^a$ | 8–12 | 8–10 | 8–10 | 4$^*$ | 2$^*$ | 2$^*$ |     |     |     |     |     |     |     |     |     |
| 9   | 10 | 9–14 | 8–16 | 8–10 | 6$^*$ | 4$^*$ | 2$^*$ | 2$^*$ |     |     |     |     |     |     |     |     |
| 10  | 11$^a$ | 10–16 | 8–16 | 10–16 | 12$^{d_2}$ | 6$^{d_2}$ | 2–3 | 2$^*$ | 2$^*$ |     |     |     |     |     |     |     |
| 11  | 12 | 11–18 | 8–19 | 10–20 | 12$^{d_2}$ | 12$^{d_2}$ | 4$^*$ | 2$^*$ | 2$^*$ | 2$^*$ |     |     |     |     |     |
| 12  | 13$^a$ | 12–20 | 8–25 | 10–21 | 13$^a$ | 13$^n$ | 4$^*$ | 4$^*$ | 2$^*$ | 2$^*$ | 2$^*$ |     |     |     |     |
| 13  | 14 | 13–22 | 8–26 | 10–27 | 13–19 | 13$^n$ | 8$^{d_2}$ | 4$^*$ | 2$^*$ | 2$^*$ | 2$^*$ | 2$^*$ |     |     |     |
| 14  | 15$^a$ | 14–24 | 8–29$^{d_6}$ | 10–29$^{d_6}$ | 14–27 | 14–19 | 16$^{d_2}$ | 8$^*$ | 4$^*$ | 2–3 | 2$^*$ | 2$^*$ | 2$^*$ |     |     |
| 15  | 16 | 15–26 | 8–31$^{d_6}$ | 11–31$^{d_6}$ | 14–29 | 14–30 | 16 | 16$^*$ | 4$^*$ | 4$^*$ | 2$^*$ | 2$^*$ | 2$^*$ |     |
| 16  | 17$^a$ | 16–28 | 8–33$^{d_6}$ | 11–33$^{d_6}$ | 14–33$^{d_6}$ | 15–33$^{d_6}$ | 16–18 | 16–18 | 6$^*$ | 4$^*$ | 2$^*$ | 2$^*$ | 2$^*$ |     |
| 17  | 18 | 17–30 | 9–35$^{d_6}$ | 12–35$^{d_6}$ | 14–35$^{d_6}$ | 15–35$^{d_6}$ | 17–22 | 16–18 | 10$^*$ | 6$^*$ | 4$^*$ | 2$^*$ | 2$^*$ | 2$^*$ |
| 18  | 19$^a$ | 18–32 | 9–37$^{d_6}$ | 12–37$^{d_6}$ | 14–37$^{d_6}$ | 15–37$^{d_6}$ | 17–35 | 18–22 | 20$^*$ | 10 | 4$^*$ | 2–4 | 2$^*$ | 2$^*$ | 2$^*$ |

Table 2. Bounds for $q = 3$.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 3   | 9  |     |     |     |     |     |     |     |     |     |     |     |     |
| 4   | 9$^a$ | 9  |     |     |     |     |     |     |     |     |     |     |     |
| 5   | 11–13 | 9–17 | 11–13 | 6  |     |     |     |     |     |     |     |     |     |
| 6   | 13–15 | 11–18 | 11–16 | 13–14$^a$ | 4  |     |     |     |     |     |     |     |     |
| 7   | 15$^a$ | 13–27 | 11–27 | 15$^{d_2}$ | 10  | 3  |     |     |     |     |     |     |     |
| 8   | 17–19 | 15–31 | 11–30 | 15–31 | 18–19 | 9  |     |     |     |     |     |     |     |
| 9   | 19–21 | 17–33 | 11–37$^{d_6}$ | 15–36 | 18–25 | 18–21 | 6  |     |     |     |     |     |     |
| 10  | 21$^a$ | 19–45 | 11–41$^{d_6}$ | 15–41$^{d_6}$ | 18–41$^{d_6}$ | 18–21 | 13–14 | 3  |     |     |     |     |     |
| 11  | 23–25 | 21–45 | 11–45$^{d_6}$ | 15–45$^{d_6}$ | 18–45 | 18–45 | 18–25 | 12$^*$ | 4$^*$ | 3$^*$ |     |     |
| 12  | 25–27 | 23–51 | 12–49$^{d_6}$ | 15–49$^{d_6}$ | 18–49$^{d_6}$ | 18–49$^{d_6}$ | 18–30 | 27$^{d_2}$ | 9$^*$ | 4$^*$ | 3$^*$ |     |
| 13  | 27$^a$ | 25–63 | 13–53$^{d_6}$ | 15–53$^{d_6}$ | 18–53$^{d_6}$ | 18–53$^{d_6}$ | 18–53$^{d_6}$ | 18–27 | 27$^{d_2}$ | 6$^*$ | 3$^*$ | 3$^*$ |     |
| 14  | 29–31 | 27–63 | 14–57$^{d_6}$ | 15–57$^{d_6}$ | 18–57$^{d_6}$ | 18–57$^{d_6}$ | 18–57$^{d_6}$ | 18–45 | 27–31 | 12–13 | 6$^*$ | 3$^*$ | 3$^*$ |

distances $d$ and $d + 1$ and cardinality greater than $2(q - 1)n + 1$ whenever $d > \left(\frac{\sqrt{2}(q - 1)n - 1}{2}\right)$; i.e., we have $A_q(n, \{d, d + 1\}) \leq 2(q - 1)n + 1$. △

The bound of Theorem 6 is usually better than the simplex method for a large enough $n$ and a middle range $d$. The first time where this happens is $(n, d) = (13, 4)$ for $q = 2$, $(9, 3)$ for $q = 3$, $(8, 3)$ for $q = 4$, and $(7, 4)$ for $q = 5$.

We did not find linear programming or semidefinite programming bounds for spherical codes to imply good bounds for our codes.

6. TABLES

The tables below are for $q = 2, 3, 4, 5$. Horizontally we give $d$, vertically $n$. The lower bounds show the best of the computer generated random codes and the constructions from Section 3. All our random codes are available upon request.

The upper bounds are taken from the best of the linear programming bound obtained by the simplex method (unmarked), ad hoc approaches as in Section 5.1 (marked with $d_2$, $n$, and $a$, respectively), the corresponding best known upper bound on $A_q(n, d)$ [17] (marked with *), and the bound of Theorem 6 (marked with $d_6$).
Table 3. Bounds for \(q = 4\).

| \(n\) | 1 \(d\) | 2 \(d\) | 3 \(d\) | 4 \(d\) | 5 \(d\) | 6 \(d\) | 7 \(d\) | 8 \(d\) | 9 \(d\) | 10 \(d\) | 11 \(d\) |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 5    | 16\(^a\) | 16–25 | 16    | 16*   | 9*    |       |       |       |       |       |       |
| 6    | 19–22 | 16–37 | 16–37 | 18–22 | 9*    |       |       |       |       |       |       |
| 7    | 22–26 | 19–41 | 16–43 | 18–41 | 21–26 | 8*    |       |       |       |       |       |
| 8    | 25–28 | 22–50 | 16–49\(16\) | 18–49\(16\) | 21–32 | 19–28 | 5*    |       |       |       |       |
| 9    | 28\(^a\) | 25–67 | 16–86 | 18–55\(16\) | 21–55\(16\) | 19–28 | 15–20* | 5*    |       |       |       |
| 10   | 31–34 | 28–72 | 16–90 | 18–61\(16\) | 20–61\(16\) | 19–61\(16\) | 21–34 | 16*   | 5*    |       |       |
| 11   | 34–38 | 31–78 | 16–134 | 18–67\(16\) | 21–67\(16\) | 19–67\(16\) | 20–56 | 22–38 | 12*   | 4*    |       |
| 12   | 37–40 | 34–97 | 18–152 | 18–73\(16\) | 21–73\(16\) | 19–73\(16\) | 20–73\(16\) | 22–43 | 21–40 | 9*    | 4*    |

Table 4. Bounds for \(q = 5\).

| \(n\) | 1 \(d\) | 2 \(d\) | 3 \(d\) | 4 \(d\) | 5 \(d\) | 6 \(d\) | 7 \(d\) | 8 \(d\) | 9 \(d\) |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 5    | 25    | 25–30 | 25–30 | 19–25 |       |       |       |       |       |
| 6    | 25\(^a\) | 25–51 | 25–51 | 19–25 | 15–25* |       |       |       |       |
| 7    | 29–34 | 25–66 | 25–81 | 19–57\(16\) | 25–34 | 12–15* |       |       |       |
| 8    | 33–40 | 29–75 | 25–88 | 19–65\(16\) | 22–65\(16\) | 26–40 | 10*   |       |       |
| 9    | 37–43 | 33–83 | 25–130 | 21–73\(16\) | 22–73\(16\) | 26–65 | 25–43 | 8–10* |       |
| 10   | 41–45 | 37–114 | 25–177 | 21–81\(16\) | 22–81\(16\) | 26–81\(16\) | 25–49 | 25–45 | 7*    |

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REFERENCES

1. Boyvalenkov, P., Delchev, K., Zinoviev, D.V., and Zinoviev, V.A., Codes with Two Distances: \(d\) and \(d+1\), in Proc. 16th Int. Workshop on Algebraic and Combinatorial Coding Theory (ACCT-XVI), Svetlogorsk, Russia, Sept. 2–8, 2018, pp. 40–45. Available at https://www.dropbox.com/s/h7u89lh8vyirww9/Proceedings\%20final.pdf?dl=0.

2. Landjev, I., Rousseva, A., and Storme, L., On Linear Codes of Almost Constant Weight and the Related Arcs, C. R. Acad. Bulgare Sci., 2019, vol. 72, no. 12, pp. 1626–1633.

3. Bassalygo, L.A., New Upper Bounds for Error Correcting Codes, Probl. Peredachi Inf., 1965, vol. 1, no. 4, pp. 41–44 [Probl. Inf. Transm. (Engl. Transl.), 1965, vol. 1, no. 4, pp. 32–35].

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4. Bassalygo, L.A., Zinoviev, V.A., and Lebedev, V.S., On $m$-Near-Resolvable Block Designs and $q$-ary Constant-Weight Codes, *Probl. Peredachi Inf.*, 2018, vol. 54, no. 3, pp. 54–61 [Probl. Inf. Transm. (Engl. Transl.), 2018, vol. 54, no. 3, pp. 245–252].

5. Bassalygo, L.A. and Zinoviev, V.A., Remark on Balanced Incomplete Block Designs, Near-Resolvable Block Designs, and $q$-ary Constant-Weight Codes, *Probl. Peredachi Inf.*, 2017, vol. 53, no. 1, pp. 55–59 [Probl. Inf. Transm. (Engl. Transl.), 2017, vol. 53, no. 1, pp. 51–54].

6. Semakov, N.V., Zinoviev, V.A., and Zaitsev, G.V., A Class of Maximum Equidistant Codes, *Probl. Peredachi Inf.*, 1969, vol. 5, no. 2, pp. 84–87 [Probl. Inf. Transm. (Engl. Transl.), 1969, vol. 5, no. 2, pp. 65–68].

7. Beth, T., Jungnickel, D., and Lenz, H., *Design Theory*, Cambridge: Cambridge Univ. Press, 1986.

8. Bogdanova, G.T., Zinoviev, V.A., and Todorov, T.J., On the Construction of $q$-ary Equidistant Codes, *Probl. Peredachi Inf.*, 2007, vol. 43, no. 4, pp. 13–36 [Probl. Inf. Transm. (Engl. Transl.), 2007, vol. 43, no. 4, pp. 280–302].

9. Calderbank, R. and Kantor, W.M., The Geometry of Two-Weight Codes, *Bull. London Math. Soc.*, 1986, vol. 18, no. 2, pp. 97–122.

10. Delsarte, P., An Algebraic Approach to the Association Schemes of Coding Theory, *Philips Res. Rep. Suppl.*, 1973, no. 10.

11. Semakov, N.V., Zinoviev, V.A., and Zaitsev, G.V., Uniformly Packed Codes, *Probl. Peredachi Inf.*, 1971, vol. 7, no. 1, pp. 38–50 [Probl. Inf. Transm. (Engl. Transl.), 1971, vol. 7, no. 1, pp. 30–39].

12. Borges, J., Rifà, J., and Zinoviev, V.A., On Completely Regular Codes, *Probl. Peredachi Inf.*, 2019, vol. 55, no. 1, pp. 3–50 [Probl. Inf. Transm. (Engl. Transl.), 2019, vol. 55, no. 1, pp. 1–45].

13. Barg, A. and Musin, O., Bounds on Sets with Few Distances, *J. Combin. Theory Ser. A*, 2011, vol. 118, no. 4, pp. 1465–1474.

14. Levenshtein, V.I., Krawtchouk Polynomials and Universal Bounds for Codes and Designs in Hamming Spaces, *IEEE Trans. Inform. Theory*, 1995, vol. 41, no. 5, pp. 1303–1321.

15. Larman, D.G., Rogers, C.A., and Seidel, J.J., On Two-Distance Sets in Euclidean Space, *Bull. London Math. Soc.*, 1977, vol. 9, no. 3, pp. 261–267.

16. Neumaier, A., Distance Matrices, Dimension, and Conference Graphs, *Nederl. Akad. Wetensch. Indag. Math.*, 1981, vol. 43, no. 4, pp. 385–391.

17. Brouwer, A.E., Tables of Bounds for $q$-ary Codes. Published electronically at [www.win.tue.nl/~aeb/](http://www.win.tue.nl/~aeb/).