Lagrangian reductive structures on gauge-natural bundles

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Abstract

A reductive structure is associated here with Lagrangian canonically defined conserved quantities on gauge-natural bundles. Infinitesimal parametrized transformations defined by the gauge-natural lift of infinitesimal principal automorphisms induce a variational sequence such that the generalized Jacobi morphism is naturally self-adjoint. As a consequence, its kernel defines a reductive split structure on the relevant underlying principal bundle.

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1 Introduction

Since jet spaces and formal derivatives form a natural geometric framework for the representation of partial differential equations, various differential-geometric formulations of calculus of variations on jet spaces have been proposed (see, e.g. [1, 16, 21, 22, 23, 27, 33, 35, 37, 39]). In most of these formulations the differential operator transforming Lagrangians to Euler–Lagrange expressions is nothing but a sheaf morphism of a certain differential sheaf sequence, thus providing two related frames [41]: infinite order variational bicomplexes and Krupka’s finite order variational sequences. We work within the framework of finite order variational sequences on those very important geometric constructions called gauge-natural bundles [8, 21] by considering variational derivatives of gauge-natural invariant Lagrangians of arbitrary order in the general case of \( n \) independent variables and \( m \) unknown functions. As well known, following
Noether’s theory [26], from invariance properties of the Lagrangian the existence of suitable conserved currents and identities can be deduced. Within such a picture generalized Bergmann–Bianchi identities [4] are conditions for a Noether conserved current to be not only closed but also the global divergence of a tensor density called a superpotential. First in [29] and then in a series of papers [14, 15, 28, 30, 31, 42] we proposed an approach to deal with the problem of canonical covariance and uniqueness of conserved quantities which uses variational derivatives taken with respect to the class of (generalized) variation vector fields being Lie derivatives of sections of bundles by gauge-natural lifts of infinitesimal principal automorphisms. Such variational derivatives can be suitably interpreted as vertical differentials [13].

In their stemming paper on the Hamilton–Cartan formalism [16], under certain assumptions on admissible variations, Goldschmidt and Sternberg found that the Jacobi morphism is self-adjoint along solutions of the Euler–Lagrange equations in first order Lagrangian field theory. Within the geometric framework of jets of fibered manifolds they proved that the Hessian morphism, which is nothing but the second variation of the action integral of a Lagrangian, is in fact a symmetric bilinear morphism. Their proof is based on the fact that variations are chosen to be vanishing on the boundary of the integration manifold, so that integrals of divergences vanish on the boundary by virtue of Stoke’s theorem. The Hessian morphism is in fact symmetric up to a term which is the integral of a total divergence and vanishes following standard arguments in calculus of variations. As an immediate consequence of the symmetry properties of the Hessian, the latter being the integral of the contraction of the Jacobi morphism with a variation vector field, the Jacobi morphism itself is self-adjoint. This result was used for an important application of the Morse index Theorem [34].

In [13] a quotient second variational derivative, generalizing invariantly the classical Hessian morphism up to horizontal differentials, was described as the vertical differential of the Euler–Lagrange morphism generalizing the classical Jacobi morphism which thus turns out to be self-adjoint along solutions of the Euler-Lagrange equations.

In this paper, we use intrinsic linearity properties of the gauge-natural lift functor to first prove a Lemma stating in full generality the self-adjointness of the generalized gauge-natural Jacobi morphism defined as a generalized Euler–Lagrange type morphism in a finite order variational sequence on an extended space. As one of the relevant consequence of this Lemma, we prove that the kernel of the Jacobi morphism defines a split structure on the relevant underlying principal bundle and that such a structure is also reductive.

It is now remarkable that we are framing our investigations within a formulation of the calculus of variations on fibered manifolds, the variational sequence, which is completely differential and free from the use of integrals such as the integral of action. Variational objects such as Euler–Lagrange equations – and thus all higher degree generalizations such as Bergmann-Bianchi identities, generalized Noether identities, generalized Jacobi equations – are obtained as quotient morphisms of the exterior differential operator acting on sheaves of
differential forms. In this case the integration by parts of the action is substituted by global decomposition formulae of well characterized vertical morphism (see e.g. [20, 37, 39] and the review in [41]). The problem of dealing with local divergences can be solved by using the intrinsic properties of the variational sequence itself and the very nature of variation vector fields we choose (vertical parts of gauge-natural lifts).

2 Jets of gauge-natural bundles and Euler–Lagrange type morphisms

We recall some basic definitions and results from the theory of jet spaces with the main aim of stating the notation. Let $\pi : Y \to X$ be a fibered manifold, with $\dim X = n$ and $\dim Y = n + m$. For $s \geq q \geq 0$ integers we deal with the $s$-jet space $J_s Y$ of $s$-jet prolongations of (local) sections of $\pi$; in particular, we set $J_0 Y = Y$. We recall that there are the natural fiberings $\pi^*_s : J_s Y \to J_q Y$, $s \geq q$, $\pi^* : J_s Y \to X$, and, among these, the affine fiberings $\pi^*_{s-1}$. We denote by $V Y$ the vector subbundle of the tangent bundle $T Y$ of vectors on $Y$ which are vertical with respect to the fibering $\pi$.

For $s \geq 1$, we consider the following natural splitting induced by the natural contact structure on jets bundles (see e.g. [40]):

$$J_s Y \times_{J_{s-1} Y} T^* J_{s-1} Y = \left( J_s Y \times_{J_{s-1} Y} T^* X \right) \oplus C^*_s [Y],$$

(1)

where $C^*_s [Y] = Im \partial_s^* \oplus J_{s-1} Y \times V^* J_{s-1} Y \to J_s Y \times T^* J_{s-1} Y$.

A vector field $\xi$ on $Y$ is said to be vertical if it takes values in $V Y$. A vertical vector field can be prolonged to a vertical vector field $j_k \xi : J_k Y \to V J_k Y$.

The vector field $j_k \xi$ is characterized by the fact that its flow is the natural prolongation of the flow of $\xi$. Given a vector field $\Xi : J_k Y \to T J_k Y$, the splitting (1) yields $\Xi \circ \pi^*_{s+1} = \Xi_H + \Xi_V$. We shall call $\Xi_H$ and $\Xi_V$ the horizontal and the vertical part of $\Xi$, respectively. As well known, the above splitting induces also a decomposition of the exterior differential on $Y$, $(\pi^*_{s+1})^* \circ d = d_H + d_V$, where $d_H$ and $d_V$ are called the horizontal and vertical differential, respectively. We must stress that such decompositions always rise the order of the objects.

Let $P \to X$ be a principal bundle with structure group $G$. Let $r \leq k$ be integers and $W^{(r,k)} P = J_r P \times L_k (X)$, where $L_k (X)$ is the bundle of $k$-frames in $X$ [8, 21]. $W^{(r,k)} n \to G = G^r_n \odot GL_n (n)$ the semidirect product with respect to the action of $GL_n (n)$ on $G^r_n$ given by jet composition and $GL_n (n)$ is the group of $k$-frames in $R^n$. Here we denote by $G^r_n$ the space of $(r, n)$-velocities on $G$ [21]. The bundle $W^{(r,k)} n \to P$ is a principal bundle over $X$ with structure group $W^{(r,k)} n \to G$. Let $F$ be a manifold and $\zeta : W^{(r,k)} n \to G \times F \to F$ be a left action of $W^{(r,k)} n \to G$ on $F$. There is a naturally defined right action of $W^{(r,k)} n \to G$ on $W^{(r,k)} P \times F$ so that we have in the standard way the associated gauge-natural
bundle of order \((r, k)\): \(Y_\zeta \cong W^{(r,k)} P \times_\zeta F\). All our considerations shall refer to \(Y\) as a gauge-natural bundle as just defined.

Denote now by \(A^{(r,k)}\) the sheaf of right invariant vector fields on \(W^{(r,k)} P\). A functorial map \(\mathcal{O}\) is defined which lifts any right–invariant local automorphism \((\Phi, \phi)\) of the principal bundle \(W^{(r,k)} P\) to a unique local automorphism \((\Phi_\zeta, \phi)\) of the associated bundle \(Y_\zeta\). Its infinitesimal version defines the gauge-natural lift in the following way:

\[
\mathcal{O} : Y_\zeta \times A^{(r,k)} \to TY_\zeta : (y, \Xi) \mapsto \hat{\Xi}(y),
\]

where, for any \(y \in Y_\zeta\), one sets: \(\hat{\Xi}(y) = \frac{d}{dt}[(\Phi_\zeta(t))(y)]_{t=0}\), and \(\Phi_\zeta\) denotes the (local) flow corresponding to the gauge-natural lift of \(\Phi_t\). Such a functor defines a class of parametrized contact transformations.

This mapping fulfills the following properties (see [21]): \(\mathcal{O}\) is linear over \(id_{Y_\zeta}\); we have \(T\pi_\zeta \circ \mathcal{O} = id_{TX} \circ \pi_{(r,k)}\), where \(\pi_{(r,k)}\) is the natural projection \(Y_\zeta \times X \to A^{(r,k)} \to TX\); for any pair \((\tilde{\Lambda}, \tilde{\Xi}) \in A^{(r,k)}\), we have \(\mathcal{O}([\tilde{\Lambda}, \tilde{\Xi}] = [(\mathcal{O}(\tilde{\Lambda}), \mathcal{O}(\tilde{\Xi}))]\).

**Definition 1** Let \(\gamma\) be a (local) section of \(Y_\zeta\), \(\hat{\Xi}\) \(\in A^{(r,k)}\) and \(\hat{\Xi}\) its gauge-natural lift. Following [21] we define the generalized Lie derivative of \(\gamma\) along the vector field \(\hat{\Xi}\) to be the (local) section \(L_{\hat{\Xi}} \gamma : X \to VY_\zeta\), given by \(L_{\hat{\Xi}} \gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma\).

The Lie derivative operator acting on sections of gauge-natural bundles is an homomorphism of Lie algebras; furthermore, for any vector field \(\hat{\Xi} \in A^{(r,k)}\), the mapping \(\gamma \mapsto L_{\hat{\Xi}} \gamma\) is a first–order quasilinear differential operator and for any local section \(\gamma\) of \(Y_\zeta\), the mapping \(\hat{\Xi} \mapsto L_{\hat{\Xi}} \gamma\) is a linear differential operator. Moreover, we can regard \(L_{\hat{\Xi}} : J_1 Y_\zeta \to VY_\zeta\) as a morphism over the basis \(X\) and by using the canonical isomorphisms \(VJ_1 Y_\zeta \cong J_1 VY_\zeta\) for all \(s\), we have \(L_{\hat{\Xi}}[j_s \gamma] = j_s[L_{\hat{\Xi}} \gamma]\), for any (local) section \(\gamma\) of \(Y_\zeta\) and any (local) vector field \(\hat{\Xi} \in A^{(r,k)}\). We remark that, for any gauge-natural lift, the fundamental relation holds true: \(\hat{\Xi}_\nu \doteq (\mathcal{O}(\hat{\Xi}))_\nu = -L_{\hat{\Xi}}\).

The splitting [11] induces splittings in the spaces of forms [10]; here and in the sequel we implicitly use identifications between spaces of forms and spaces of bundle morphisms which are standard in the calculus of variations (see, e.g. [20, 21, 22]).

For \(s \geq 0\), we consider the standard sheaves \(\Lambda^p_s\) of \(p\)-forms on \(J_s Y\). For \(0 \leq q \leq s\), we consider the sheaves \(\mathcal{H}^p_{(s,q)}\) and \(\mathcal{H}^p_s\) of horizontal forms with respect to the projections \(\pi_q^s\) and \(\pi_0^s\), respectively. For \(0 \leq q < s\), we consider the subsheaves \(\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}\) and \(\mathcal{C}^p_{(s,q)} \subset \mathcal{C}^p_{(s,q)}\) of contact forms, i.e. horizontal forms valued into \(C^*_s[Y]\) (they have the property of vanishing along any section of the gauge-natural bundle). According to [23, 10], the fibered splitting [10] yields the sheaf splitting \(\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^p_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}\), which restricts to the inclusion \(\Lambda^p_s \subset \bigoplus_{t=0}^p \mathcal{C}^p_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}\), where \(\mathcal{H}^p_{s+1} \doteq h(\Lambda^p_s)\) for \(0 < p \leq n\) and the map \(h\) is defined to be the restriction to \(\Lambda^p_s\) of the projection of the above splitting onto the non–trivial summand with the highest value of \(t\).
Let \( \eta \in C^1_s(\mathcal{Y}) \setminus C^1_{(s,0)} \setminus \mathcal{H}^{n,h}_{s+1} \); then there is a unique morphism

\[
K_\eta \in C^1_{(2s,0)} \otimes C^1_{(2s,0)} \setminus \mathcal{H}^{n,h}_{2s+1}
\]
such that, for all \( \Xi : Y \to VY \), \( C^1_1(j_{2s}\Xi \otimes K_\eta) = E_{j_s,\Xi | \eta} \), where \( C^1_1 \) stands for tensor contraction on the first factor and \( j \) denotes inner product and \( E_{j_s,\Xi | \eta} = (\pi^{s+1}_{s+1})^{*}j_s,\Xi | \eta + F_{j_s,\Xi | \eta} \) (with \( F_{j_s,\Xi | \eta} \) a local divergence) is a uniquely defined global section of \( C^1_{(2s,0)} \setminus \mathcal{H}^{n,h}_{2s+1} \) (see \[40\]).

By an abuse of notation, let us denote by \( J \) the sheaf generated by the presheaf \( d \ker h \) the sheaf generated by the presheaf \( d \ker h \) in the standard way. We set \( \Theta^*_s \equiv \ker h + d \ker h \). We have that

\[
0 \to R_Y \to V_s^* \to X_s^* \to 0
\]
represented by Vitolo in \[40\]. A section \( E_{d^{*}} \equiv E_{n}(\lambda) \in V^{n+1}_s \) is the generalized higher order Euler–Lagrange type morphism associated with \( \lambda \). The morphism \( K_\eta \) previously introduced can be integrated by parts to provide a representation of the generalized Jacobi morphism associated with \( \lambda \) \[24\], which can then be seen to be a generalized higher degree Euler–Lagrange type morphism.

3 Jacobi equations and reductive split structure

Let \( \lambda \) be a Lagrangian and consider \( \hat{\Xi}_V \) as a variation vector field. Let us set \( \chi(\lambda, \hat{\Xi}_V) \equiv C^1_1(\hat{\Xi}_V \otimes K_{h \mathcal{L}_s, \hat{\Xi}_V } \lambda) \equiv E_{j_s,\hat{\Xi}_V | \eta} \). Because of linearity properties of \( K_{h \mathcal{L}_s, \hat{\Xi}_V } \lambda \), and by using a global decomposition formula due to Kolár \[20\], we can decompose the morphism defined above as \( \chi(\lambda, \hat{\Xi}_V) = E_{\chi(\lambda, \hat{\Xi}_V), \hat{\Xi}_V | \eta} = F_{\chi(\lambda, \hat{\Xi}_V)} \). Where \( F_{\chi(\lambda, \hat{\Xi}_V)} \) is a local horizontal differential which can be globalized by fixing of a connection; however we do not fix any connection \( a \) \( \text{priori} \) here.

**Definition 2** We call the morphism \( J(\lambda, \hat{\Xi}_V) \equiv E_{\chi(\lambda, \hat{\Xi}_V)} \), the gauge-natural generalized Jacobi morphism associated with the Lagrangian \( \lambda \) and the variation vector field \( \hat{\Xi}_V \). We call the morphism \( J_\lambda(\lambda, \hat{\Xi}_V) \equiv \hat{\Xi}_V E_{\eta}(\hat{\Xi}_V E_{\eta}(\lambda)) \) the gauge-natural Hessian morphism associated with \( \lambda \).

The morphism \( J(\lambda, \hat{\Xi}_V) \) is a linear morphism with respect to the projection \( J_{d \mathcal{L}_s Y_C} \times V J_{d \mathcal{L}_s A^{(r,k)}_C} \to J_{d \mathcal{L}_s Y_C} \). Such a morphisms has been also represented on finite order variational sequence modulo horizontal differentials \[13\] and thereby proved to be self-adjoint along solutions of the Euler–Lagrange equations, a result already well known for first order field theories \[10\]. By resorting to the relation with the Hessian morphism \[30\], we shall prove here the same property in finite order variational sequences on gauge-natural bundles without quotienting out horizontal differentials. As in the case of first order theories, we have in fact the following.
Lemma 1. The Hessian and thus the Jacobi morphism are symmetric self-adjoint morphisms.

Proof. Since \( \delta^2 \omega \lambda = \mathcal{L}_{\Xi} \mathcal{L}_{\Xi} \lambda = \hat{\Xi} V | \mathcal{E}_n (\hat{\Xi} V | \mathcal{E}_n (\lambda) ) \), we have \( \delta^2 \omega \lambda = \delta^2 \omega \lambda \); furthermore, being also \( \delta^2 \omega \lambda = \mathcal{E}_n (\hat{\Xi} V | h(d\delta \lambda)) \) (see the proof given in [29]), then \( \delta^2 \omega \lambda \) is self-adjoint. Furthermore, we have

\[
\mathcal{J}(\lambda, \hat{\Xi} V) = E_{\lambda} (\mathcal{E}_n (\hat{\Xi} V | h(d\delta \lambda)) = \delta^2 \omega \lambda. \quad (3)
\]

The Jacobi morphism \( \mathcal{J}(\lambda, \hat{\Xi} V) \) can be interpreted as an endomorphism of \( J_{2s} V A^{(r,k)} \). In the following we concentrate on some geometric aspects of the space \( \mathcal{R} \cong \ker \mathcal{J}(\lambda, \hat{\Xi} V) \). Such a kernel defines generalized gauge-natural Jacobi equations [29], the solutions of which we call generalized Jacobi vector fields. It characterizes canonical covariant conserved quantities. In fact, given \( [\alpha] \in V^n_\mathcal{R} \), since the variational Lie derivative [12] of classes of forms can be represented the variational sequence, we have the corresponding version of the First Noether Theorem:

\[
\mathcal{L}_{j} X [\alpha] = \omega(\lambda, \hat{\Xi} V) + d_H (j_2 \hat{\Xi} V | p_{dv h(\alpha)} + \xi | h(\alpha)), \quad (4)
\]

where we put \( \omega(\lambda, \hat{\Xi} V) = \hat{\Xi} V | \mathcal{E}_n (\lambda) = - \mathcal{E}_n (\lambda) \).

As usual, \( \lambda \) is defined a gauge-natural invariant Lagrangian if the gauge-natural lift \( (\hat{\Xi}, \xi) \) of any vector field \( \hat{\Xi} \in \mathcal{A}^{(r,k)} \) is a symmetry for \( \lambda \), i.e. if \( \mathcal{L}_{j+1} \Xi = 0 \). In this case, as an immediate consequence we have that \( \omega(\lambda, \hat{\Xi} V) = d_H (j_2 \hat{\Xi} V | p_{dv \lambda} + \xi | \lambda) \). The generalized Bergmann–Bianchi morphism [4] \( \beta(\lambda, \hat{\Xi} V) = E_{\omega(\lambda, \hat{\Xi} V)} \), which is nothing but the Euler–Lagrange morphism associated with the new Lagrangian \( \omega(\lambda, \hat{\Xi} V) \) defined on the fibered manifold \( J_2 \mathcal{X}_\mathcal{C} \times \mathcal{V} J_2 \mathcal{A}^{(r,k)} \to \mathcal{X} \). We proved that the generalized Bergmann–Bianchi morphism is canonically vanishing along \( \mathcal{R} \). This fact characterizes canonical covariant conserved Noether currents [28].

Notice that since \( \lambda \) is gauge-natural invariant then \( \mathcal{L}_{j+1} \hat{\Xi} V | \mathcal{L}_{j+1} \hat{\Xi} V | \lambda = \mathcal{L}_{j+1} \hat{\Xi} V | \lambda \). However, we remark that the Lagrangian \( \omega \) is not gauge-natural invariant unless either \( \hat{\Xi} H, \hat{\Xi} V = 0 \), or such a commutator is the gauge-natural lift of some infinitesimal principal automorphism.

Nevertheless, along the kernel of the gauge-natural generalized gauge-natural Jacobi morphism we have that \( \mathcal{L}_{j+1} \hat{\Xi} V | \mathcal{L}_{j+1} \hat{\Xi} V | \lambda \equiv 0 \). Hence Bergmann–Bianchi identities are equivalent to the invariance condition \( \mathcal{L}_{j+1} \hat{\Xi} V | \mathcal{L}_{j+1} \hat{\Xi} V | \lambda \equiv 0 \) and can be suitably interpreted as Noether identities associated with the invariance properties of the higher degree Euler–Lagrange morphism \( \mathcal{E}_n (\omega) [31] \). As a consequence [13] [14] there exists a covariant n-form \( \mathcal{H}(\lambda, \mathcal{R}) \) which can be interpreted as a Hamiltonian form for \( \omega(\lambda, \mathcal{R}) \) on the Legendre bundle \( \Pi \equiv \mathcal{V}^*(J_2 \mathcal{Y}_\mathcal{C} \times \mathcal{V} J_2 \mathcal{A}^{(r,k)}) \Lambda (n \lambda) T^* \mathcal{X} \). Let then \( \Omega \) be the multisimplectic form on the corresponding homogeneous Legendre bundle \( \mathcal{Z} \equiv T^*(J_2 \mathcal{Y}_\mathcal{C} \times \mathcal{X}) \).
\( V_A^{(r,k)} \wedge \Lambda^{n-1} T^* X \). Every Hamiltonian form \( \mathcal{H} \) admits a Hamiltonian connection \( \gamma_\mathcal{H}(\lambda, \mathfrak{R}) \) such that the Hamilton equations \( \gamma_\mathcal{H}(\lambda, \mathfrak{R})|\Omega = d\mathcal{H}(\lambda, \mathfrak{R}) \) hold true \([24]\). In \([30]\) we proved that the Hamilton equations for the Hamiltonian connection form \( \gamma_\mathcal{H}(\lambda, \mathfrak{R}) \) coincide with the kernel of the generalized gauge-natural Jacobi morphism. As a consequence \( \mathfrak{R} \) is characterized as a vector subbundle \([10, 15, 30, 42]\).

**Theorem 1** The kernel \( \mathfrak{R} \) defines a reductive structure on \( W^{(r+4s, k+4s)} P \).

**Proof.** Being the Jacobi morphism self-adjoint its cokernel coincides with the cokernel of the adjoint morphism, thus we have that \( \dim \mathfrak{R} = \dim \text{Coker} \mathcal{J} \).

If we further consider that \( \mathfrak{R} \) is of constant rank because, as we just recalled, it is the kernel of a Hamiltonian operator, we are able to define the split structure on \( (VW^{(r+4s, k+4s)} P) / W^{(r+4s, k+4s)} G \), given by \( \mathfrak{R} \oplus \text{Im} \mathcal{J} \).

Let \( \mathfrak{h} \) be the Lie algebra of right-invariant vector fields on \( W^{(r+4s, k+4s)} P \) and \( \mathfrak{k} \) the Lie subalgebra of generalized Jacobi vector fields defined as solutions of generalized Jacobi equations. The Lie derivative of a solution of Euler–Lagrange equations with respect to a Jacobi vector field is again a solution of Euler–Lagrange equations. However, the Lie derivative with respect to vertical parts of the commutator between the gauge-natural lift of a Jacobi vector field and (the vertical part of) a lift not lying in \( \mathfrak{R} \) is not a solution of Euler–Lagrange equations. Thus, since \( \mathcal{J} \) is a projector and a derivation of \( \mathfrak{h} \), it is easy to see that the split structure is also reductive, being \( [\mathfrak{k}, \text{Im} \mathcal{J}] = \text{Im} \mathcal{J} \).

**Remark 1** As a consequence generalized Jacobi vector fields define a kind of reductive gauge-natural lift: we are concerned with the reduction of the structure bundle \( W^{(r,k)} P \) to a subbundle with structure group the subgroup of a differential group of the base manifold; thus recovering the geometric framework of reductive G-structures, reductive lifts and induced reductive Lie derivatives of sections, as constructed in \([17]\). Our investigations are mainly concerned with the existence of covariant canonically defined conserved currents \([28]\). More precisely, we consider Lagrangian field theories which are assumed to be invariant with respect to the action of a gauge-natural group \( W^{(r,k)}_n G \) defined as the semidirect product of a \( k \)-th order differential group of the base manifold with the group of \( r \)-th order velocities in \( G \) (see Section \(2\)). In fact, notice that the group \( \text{Diff}(X) \) is not canonically embedded into \( \text{Aut}(P) \) (see, in particular, the discussion of this aspect presented in \([11, 17, 25]\)). We denote by \( \text{Aut}(P) \) the group of all automorphisms of the underlying principal bundle \( P \), not the “gauge group” of vertical principal automorphisms, as it is sometimes done in Physics. In other words, we are faced with the following general problem: we know how fields transform corresponding to a transformation in \( P \) but we do not know how fields transform under changes of coordinates in the base manifold, so that Lie derivatives with respect to infinitesimal base transformations cannot be defined neither in a natural nor in a canonical way, at least \( a \text{ priori} \).
It is a well known fact that the covariance of the Lagrangian and thus of
the Euler-Lagrange equations does not guarantee the corresponding covariance
of Noether conserved quantities (a well known example is the Einstein energy-
momentum pseudotensor which was covariantized by Komar via the introduc-
tion of a connection). In all generality, it is a well known fact that one need the
fixing of a linear connection on the base manifold and of a principal connection
on the principal bundle to get covariant conserved quantities in gauge-natural
field theories (this is the outcome of the fact that a global Poincaré–Cartan
form can be defined only by fixing such a couple of connections [9]). However,
we showed that a canonical determination of Noether conserved quantities is
always possible on a reduced bundle of $W^{(r,k)}P$ completely determined by the
original $W_o^{(r,k)}G$-invariant variational problem, without fixing any connection
a priori. Instead connections can be characterized by means of such canonical
reduction [10].

Concerning this last point, we stress that several aspects of the geometric
formulation of field theories on bundles associated with principal bundles,
with different techniques and adopting alternative points of views, for example
stressing the rôle of the Poincaré–Cartan form, or formulated on infinite order
jet prolongations, have been the subject of a widespread research activity since
the Seventies of last Century. Among them, of relevant interest for Physics,
the study of reductions of the underlying principal bundle i.e., reductions of the
gauge group of the theory. We refer in particular to [2, 3, 5, 7, 6, 17, 18, 19, 32].
Most of such researches were essentially motivated by possible generalizations
of the Utiyama Theorem [38].

The papers [5, 7, 6] are mainly concerned with the reduction of a given vari-
atinal problem on principal bundles. The word ‘reduction’ here takes a strictly
variational meaning: one considers reduced variational problems, given by the
reduced Lagrangian on $J_1P/G$, describing so-called Euler–Poincaré equations.
Assuming the existence of a reduction of the principal bundle $P$, the corre-
sponding reduced variational problem describing Lagrange–Poincaré equations
is characterized and the reconstruction of the original variational problem from
the reduced one is expressed by means of a certain condition on the curvature
of a relevant suitably constructed principal connection. The ‘semidirect prod-
uct reduction’ considered in [7] is still of purely gauge nature, and does not
involve the semidirect action of a differential group of the base manifold on the
group of $r$-th order velocities in $G$. The papers [2, 3, 52] are rather closer to
our approach: the existence of reduced gauge transformations induced by the
variational problem itself (in particular by the existence of Noether identities)
is investigated; however only purely gauge theories are studied and the problem
of the relation of gauge transformations with diffeomorphisms of the base man-
ifold is not considered. Utiyama-like theorems in the case of principal bundles
having the structure of a gauge-natural prolongation [18, 19] are mainly purely
geometric constructions concerning naturality and functorial aspects, without
direct relation with the calculus of variations.
We stress again that the main point at the base of our paper [29] was the possibility of relating the vertical components of a gauge natural lifts with the horizontal ones: i.e. relate gauge infinitesimal transformations with infinitesimal transformations of the base manifold, a first step towards the theory of a unified field. Bundles of fields associated with the class of principal bundles obtained as gauge-natural prolongations of principal bundles [21] have a richer structure than principal bundles tout court. In fact a relation between gauge charges and energy-momentum-like conserved quantities can be obtained when considering invariant variational problems on such a subclass of principal bundles.

As a matter of example let us in fact consider the Lie derivative of spinor fields. It is possible to recover the Kosmann lift (which is not a natural lift) in terms of the reductive lift induced by the kernel of the generalized gauge-natural Jacobi morphism. The Jacobi equations for the well known Einstein–(Cartan)–Dirac Lagrangian just implies that – if $\tilde{\Xi}_{\nu}^{a} = \tilde{\Xi}_{b}^{a} - \omega_{b\mu}^{a} \xi^{\mu}$ is the vertical part of $\tilde{\Xi}$ with respect to the spin connection $\omega$ and the corresponding superpotential for the Noether current is given by $\nu(\lambda, \tilde{\Xi}) = -\frac{1}{2} \bar{\Xi}_{\nu}^{ab} \epsilon_{ab}$ – then we obtain $\tilde{\Xi}_{\nu}^{ab} = -\tilde{\nabla}^{[a} \xi^{b]}$, i.e. the well known Kosmann lift, where $\tilde{\nabla}$ is the covariant derivative with respect to the standard transposed connection on the bundle of spin-tetrads. On the other hand, the Lie derivative of spinor fields can be expressed in terms of the horizontal part of $\tilde{\Xi}$ with respect to the spinor-connection $\tilde{\Xi}_{h}^{a}$: consequently we obtain a constraint on the corresponding connection $\tilde{\omega}$ on the spinor bundle, as well as on the superpotential corresponding to $\tilde{\Xi}_{\nu}^{ab}$. In this way we characterize a unique canonical superpotential invariant with respect to the reduced group. Here we used a quite standard notation [9]; for further details we refer to [10, 31, 42].

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References

[1] I. M. Anderson, J. Pohjanpelto: The cohomology of invariant variational bicomplexes, in Geometric and algebraic structures in differential equations, Acta Appl. Math. 41 (1-3) (1995) 3–19.

[2] D. Bashkirov, G. Giachetta, L. Mangiarotti, G. Sardanashvily: Noether’s second theorem for BRST symmetries, J. Math. Phys. 46 (5) (2005) 053517, 23 pp..

[3] D. Bashkirov, G. Giachetta, L. Mangiarotti, G. Sardanashvily: Noether’s second theorem in a general setting: reducible gauge theories, J. Phys. A 38 (23) (2005) 5329–5344.

[4] P.G. Bergmann: Non-Linear Field Theories, Phys. Rev. 75 (4)(1949) 680–685.
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[5] M. Castrillón López, P.L. García Pérez, T.S. Ratiu: Euler-Poincaré reduction on principal bundles, Lett. Math. Phys. 58 (2) (2001) 167–180.

[6] M. Castrillón López, J.E. Marsden: Some remarks on Lagrangian and Poisson reduction for field theories, J. Geom. Phys. 48 (1) (2003) 52–83.

[7] M. Castrillón López, T.S. Ratiu: Reduction in principal bundles: covariant Lagrange-Poincaré equations, Comm. Math. Phys. 236 (2) (2003) 223–250.

[8] D.J. Eck: Gauge-natural bundles and generalized gauge theories, Mem. Amer. Math. Soc. 247 (1981) 1–48.

[9] L. Fatibene, M. Francaviglia: Natural and gauge natural formalism for classical field theories. A geometric perspective including spinors and gauge theories; Kluwer Academic Publishers, Dordrecht, 2003.

[10] M. Ferraris, M. Francaviglia, M. Palese, E. Winterroth: Canonical connections in gauge-natural field theories, submitted.

[11] M. Ferraris, M. Francaviglia, M. Raiteri: Conserved Quantities from the Equations of Motion (with applications to natural and gauge natural theories of gravitation) Class. Quant. Grav. 20 (2003) 4043–4066.

[12] M. Francaviglia, M. Palese, R. Vitolo: Symmetries in Finite Order Variational Sequences, Czech. Math. J. 52(127) (2002) 197–213.

[13] M. Francaviglia, M. Palese, R. Vitolo: The Hessian and Jacobi Morphisms for Higher Order Calculus of Variations, Diff. Geom. Appl. 22 (1) (2005) 105–120.

[14] M. Francaviglia, M. Palese, E. Winterroth: Generalized Bianchi identities in gauge-natural field theories and the curvature of variational principles, Rep. Math. Phys. 56 (1) (2005) 11–22.

[15] M. Francaviglia, M. Palese, E. Winterroth: Second variational derivative of gauge-natural invariant Lagrangians and conservation laws, Proc. IX Int. Conf. Diff. Geom. Appl., Prague 2004, J. Bures et al. eds.; (Charles University, 2005) 591–604.

[16] H. Goldschmidt, S. Sternberg: The Hamilton–Cartan Formalism in the Calculus of Variations, Ann. Inst. Fourier, Grenoble 23 (1) (1973) 203–267.

[17] M. Godina, P. Matteucci: Reductive G-structures and Lie derivatives, J. Geom. Phys. 47 (1) (2003) 66–86.

[18] J. Janyška: Higher order Utiyama-like theorem, Rep. Math. Phys. 58 (1) (2006) 93–118.

[19] J. Janyška: Higher-order Utiyama invariant interaction, Rep. Math. Phys. 59 (1) (2007) 63–81.

[20] I. Kolár: A Geometrical Version of the Higher Order Hamilton Formalism in Fibred Manifolds, J. Geom. Phys., 1 (2) (1984) 127–137.

[21] I. Kolář, P.W. Michor, J. Slovák: Natural Operations in Differential Geometry, (Springer–Verlag, N.Y., 1993).

[22] D. Krupka: Some Geometric Aspects of Variational Problems in Fibred Manifolds, Folia Fac. Sci. Nat. UJEP Brunensis 14, J. E. Purkyně Univ. (Brno, 1973) 1–65.

[23] D. Krupka: Variational Sequences on Finite Order Jet Spaces, Proc. Diff. Geom. and its Appl. (Brno, 1989); J. Janyška, D. Krupka eds.; World Scientific (Singapore, 1990) 236–254.
[24] L. Mangiarotti, G. Sardanashvily: Connections in Classical and Quantum Field Theory, (World Scientific, Singapore, 2000).

[25] P. Matteucci: Einstein-Dirac theory on gauge-natural bundles, Rep. Math. Phys. 52 (1) (2003) 115–139.

[26] E. Noether: Invariante Variationsprobleme, Nachr. Ges. Wiss. Göt., Math. Phys. Kl. 11 (1918) 235–257.

[27] R. Palais: Foundations of global non–linear analysis, Benjamin, 1968.

[28] M. Palese, E. Winterroth: Covariant gauge-natural conservation laws, Rep. Math. Phys. 54 (3) (2004) 349–364; erratum in arXiv:math-ph/0406009.

[29] M. Palese, E. Winterroth: Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles, Arch. Math. (Brno) 41 (3) (2005) 289–310.

[30] M. Palese, E. Winterroth: The relation between the Jacobi morphism and the Hessian in gauge-natural field theories, Theoret. Math. Phys. 152 (2) (2007) 1191–1200.

[31] M. Palese, E. Winterroth: Noether identities in Einstein–Dirac theory and the Lie derivative of spinor fields, Proc. Diff Geom. Appl., Olomouc (2007), World Scientific 2008, in print.

[32] G. Sardanashvily: Reduction of principal superbundles, Higgs superfields, and supermetric, arXiv:hep-th/0609070 preprint.

[33] D.J. Saunders: The Geometry of Jet Bundles, Cambridge Univ. Press (Cambridge, 1989).

[34] S. Smale: On the Morse index theorem, J. Math. Mech., 14 (1965) 1049–1055.

[35] F. Takens: A global version of the inverse problem of the calculus of variations, J. Diff. Geom. 14 (1979) 543–562.

[36] A. Trautman: Noether equations and conservation laws, Comm. Math. Phys. 6 (1967) 248–261.

[37] W. M. Tulczyjew: The Lagrange Complex, Bull. Soc. Math. France, 105 (1977) 419–431.

[38] R. Utiyama: Invariant Theoretical Interpretation of Interaction, Phys. Rev. 101 (5) (1956) 1597–1607.

[39] A. M. Vinogradov: On the algebro–geometric foundations of Lagrangian field theory, Soviet Math. Dokl. 18 (1977) 1200–1204.

[40] R. Vitolo: Finite Order Lagrangian Bicomplexes, Math. Proc. Camb. Phil. Soc. 125 (1) (1999) 321–333.

[41] R. Vitolo: Variational sequences, in Handbook of Global Analysis, D. Krupka, D.J. Saunders eds., Elsevier 2007.

[42] E.H.K. Winterroth: Variational derivatives of gauge-natural invariant Lagrangians and conservation laws, PhD thesis University of Torino, 2007.