HILBERT SCHEMES AND $\mathcal{W}$ ALGEBRAS

WEI-PING LI$^1$, ZHENBO QIN$^2$, AND WEIQIANG WANG$^3$

Abstract. We construct geometrically the generating fields of a $\mathcal{W}$ algebra which acts irreducibly on the direct sum of the cohomology rings of the Hilbert schemes $X^{[n]}$ of $n$ points on a projective surface $X$ for all $n \geq 0$. We compute explicitly the commutators among the Fourier components of the generating fields of the $\mathcal{W}$ algebra, and identify this algebra with a $\mathcal{W}_{1+\infty}$-type algebra. A precise formula of certain Chern character operators, which is essential for the construction of the $\mathcal{W}$ algebra, is established in terms of the Heisenberg algebra generators. In addition, these Chern character operators are proved to be the zero-modes of vertex operators.

1. Introduction

Recently, a new approach has emerged in studying the geometry of the Hilbert schemes $X^{[n]}$ of $n$ points on a projective surface $X$. The starting point of this new approach is a geometric construction, due to Nakajima [Na1, Na2] and Gromov [Gro], of a Heisenberg algebra which acts irreducibly on the direct sum $\mathbb{H}_X = \bigoplus_{n \geq 0} H^*(X^{[n]})$ of the rational cohomology rings of $X^{[n]}$ for all $n \geq 0$ (this was conjectured by Vafa and Witten [VW] based on Göttsche’s formula [Got] for the Betti numbers of the Hilbert schemes $X^{[n]}$). A geometric construction of the Virasoro algebra acting on $\mathbb{H}_X$ is subsequently given by Lehn [Leh]. Intimate relations between the geometry of the Hilbert schemes and vertex operators of higher conformal weights uncovered by the authors [LQW1] provide further strong evidence toward deeper connections between these Hilbert schemes and vertex algebras. This approach has proved to be very fruitful in the study of the cohomology ring structures of the Hilbert schemes $X^{[n]}$ [Leh, LQW1, LQW2, LS1, LS2, LQW3].

On the other hand, a distinguished class of vertex algebras, which are called $\mathcal{W}$ algebras and higher-spin generalizations of the Virasoro algebra, often appears in connection with conformal field theories and with representation theory of affine Kac-Moody Lie algebras (cf. the book of E.Frenkel and Ben-Zvi [FB]). Among the $\mathcal{W}$ algebras, a well-known example is given by the so-called $\mathcal{W}_{1+\infty}$ algebra which is the central extension of the Lie algebra of differential operators on the circle (cf. [FKRW, Kac] and the references therein). We remark that the $\mathcal{W}_{1+\infty}$ algebra has an unusual feature which is in general not true for other $\mathcal{W}$ algebras: the Fourier
components of the generating fields of the $W_{1+\infty}$ algebra are closed under the Lie bracket.

One main goal of this paper is to construct geometrically the generating fields of a $\mathcal{W}$ (super)algebra, denoted by $\mathcal{W}_X$, which depends on the projective surface $X$ and acts irreducibly on $H_X$. We further identify the algebra $\mathcal{W}_X$ as an analog of the $W_{1+\infty}$ algebra in the framework of Hilbert schemes, which roughly speaking, is the $W_{1+\infty}$ algebra parametrized by the cohomology ring $H^*(X)$.[1] To that end, we will first obtain a partial description, which is of independent interest, of the cohomology ring of the Hilbert schemes $X[n]$ for a general projective surface $X$. It would be also interesting to see whether or not the appearance of the $W$ algebras in the framework of Hilbert schemes affords an explanation from string theory such as $S$-duality etc.

Let us explain in more details. For fixed $k \geq 0$ and $\alpha \in H^*(X)$, we introduced in [LQW1] certain cohomology class $G_k(\alpha,n) \in H^*(X[n])$, and then defined the Chern character operator $\mathfrak{G}_k(\alpha) \in \text{End}(H_X)$ which acts on $H^*(X[n])$ by the cup product with $G_k(\alpha,n)$ for each $n$ (also cf. [Leh]). Such an operator approach has been essential along this research direction. It was proved in [LQW1] that the classes $G_k(\alpha,n)$, where $0 \leq k < n$ and $\alpha$ runs over a linear basis of $H^*(X)$, form a set of ring generators for the cohomology ring $H^*(X[n])$. Our first main result in the present paper is to give an explicit formula for the operator $\mathfrak{G}_k(\alpha)$, at least for those $\alpha$ orthogonal to the canonical class $K$, in terms of the Heisenberg algebra generators. We emphasize that an explicit formula of this sort contains very strong information and is technically difficult to establish. For example, a partial information on the ‘leading term’ of the operator $\mathfrak{G}_k(\alpha)$ among other results was essentially responsible for establishing the ring generators statement in [LQW1] mentioned above. In particular, for a surface $X$ with numerically trivial canonical class, our precise formula for $\mathfrak{G}_k(\alpha)$ provides a complete description of the cohomology ring structure of $X[n]$ which is totally different from the one given by Lehn and Sorger [LS2]. A corollary of our results establishes a conjecture in [LS2] (which needs to be mildly modified) concerning the cohomology classes $G_k(\alpha,n)$. Another consequence is an explicit formula for certain intersection numbers which were shown earlier in [LQW3] to be independent of the surface $X$.

Our construction of the algebra $\mathcal{W}_X$ uses the commutators of the operators $\mathfrak{G}_k(\alpha)$ and the Heisenberg algebra generators. The $\mathcal{W}$ algebra $\mathcal{W}_X$ contains both the Heisenberg algebra of Nakajima-Grojnowski and the Virasoro algebra of Lehn [Leh] as subalgebras. We observe that a linear basis of $\mathcal{W}_X$ comes from the Fourier components of explicit vertex operators constructed from the Heisenberg vertex operators and, most remarkably, that the operators $\mathfrak{G}_k(\alpha)$ for $\alpha$ orthogonal to $K$ are precisely the zero-modes of these vertex operators. We compute explicitly the

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[1] Just like the usual $W_{1+\infty}$ algebra, the $W_X$ algebra has a dual nature: one may talk about a Lie algebra $\mathcal{W}_X$, or one may talk about the vertex algebra associated to a vacuum module of the Lie algebra $\mathcal{W}_X$, which is conventionally referred to as the $\mathcal{W}_X$ algebra as well. The Fourier components of the generating fields of the vertex algebra $\mathcal{W}_X$ give us a linear basis for the Lie algebra $\mathcal{W}_X$. 
commutation relation among these basis elements, which in particular implies that the algebra $\mathcal{W}_X$ is miraculously closed under the Lie bracket. Note that the commutation relations of $\mathcal{W}_X$ satisfy the transfer property as formulated in [LQW1]. The computation uses the explicit formula of the operators $\mathfrak{g}_k(\alpha)$ obtained above and a lengthy calculation by means of the operator product expansion method in the theory of vertex algebras (cf. [Bor, FB, Kac]) which incorporates the transfer property.

The commutators of two basis elements in the algebra $\mathcal{W}_X$ typically give rise to two terms, a leading term and another term which involves the Euler class $e$ of the projective surface $X$. There are no third terms due to the fact that $e^2 = 0$, and essentially for the same reason, central extension terms make appearances in the commutators of $\mathcal{W}_X$ only for the Fourier components of vertex operators of small conformal weights. We further identify these leading terms as the commutators of differential operators on the circle, and thus are justified to regard the algebra $\mathcal{W}_X$ as certain topological deformation of the $\mathcal{W}_{1+\infty}$ algebra in the framework of Hilbert schemes. In particular, for a projective surface $X$ with trivial Euler class, we have a complete identification of elements $\mathcal{W}_X$ with differential operators.

Just as the results in [LQW3], our construction of the $\mathcal{W}$ algebra admits a counterpart in term of the orbifold cohomology rings of the symmetric products of a manifold, which is worked out in [QW]. In particular, when the manifold is a point, such a counterpart specializes (with new proofs) to the construction of the $\mathcal{W}_{1+\infty}$ algebra of Lascoux and Thibon [LT] using the class functions of the symmetric groups, which in turn is an extension of a construction due to I. Frenkel and the third author [FW] of the Virasoro algebra.

The layout of the paper is as follows. In Sect. 2, starting from a commutative ring, we introduce a $\mathcal{W}$ algebra which is an analog of the $\mathcal{W}_{1+\infty}$ algebra. In Sect. 3, we quickly review some known results and constructions on the Hilbert schemes $X^{[n]}$. In Sect. 4, we establish the explicit formula for the Chern character operator $\mathfrak{g}_k(\alpha)$ in terms of Heisenberg generators. In Sect. 5, we formulate the main theorems on connections between $\mathcal{W}$ algebras and Hilbert schemes, which are proved subsequently in Sect. 6 by using the operator product expansion technique.

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2. The definition of $\mathcal{W}$ algebras

Let $A = \bigoplus_{n=0}^d A_n$ (where $A_d \cong \mathbb{C}$) be a finite-dimensional graded complex vector space. We write $|\alpha| = n$ if $\alpha \in A_n$. We define a super structure (i.e. the $\mathbb{Z}/2\mathbb{Z}$-grading) on $A$ by letting $A_0 = \bigoplus_{n \in 2\mathbb{Z}} A_n$ and $A_1 = \bigoplus_{n \in 1+2\mathbb{Z}} A_n$ (here $\mathbb{Z}_+$ stands for the set of all nonnegative integers). We assume that $A$ affords an algebra structure which is super commutative and compatible with the $\mathbb{Z}_+$-grading. We further assume that there exists a linear operator $\text{Tr} : A \to \mathbb{C}$ which is zero on the graded subspace $A_n$ unless $n$ is the top degree. This defines a supercommutative bilinear form $(-,-) : A \times A \to \mathbb{C}$ by $\langle \alpha, \beta \rangle = \text{Tr}(\alpha \beta)$. 

Let $t$ be an indeterminate and let $\partial_t = \frac{d}{dt}$. Let $\mathcal{W}_{as}$ be the associative algebra of regular differential operators on the circle $S^1$. Denote by $\mathcal{W}(A)_{as}$ the associative superalgebra $\mathcal{W}_{as} \otimes A$. It has a linear basis given by
\[ J^p_k(\alpha) = -t^{p+k}(\partial)^p \otimes \alpha, \quad p \in \mathbb{Z}_+, k \in \mathbb{Z}, \alpha \in A. \]
A different linear basis of $\mathcal{W}(A)_{as}$ is given by
\[ L^p_k(\alpha) = -t^k D^p \otimes \alpha, \quad p \in \mathbb{Z}_+, k \in \mathbb{Z}, \alpha \in A, \]
where $D = t\partial_t$. Note that $f(D)t = tf(D+1)$ for every polynomial $f(w) \in \mathbb{C}[w]$. Hence, we have $J^p_k(\alpha) = -t^k D(D-1) \cdots (D-p+1) \otimes \alpha$.

Let $\mathcal{W}(A)$ denote the Lie superalgebra obtained from the associative superalgebra $\mathcal{W}(A)_{as}$ by taking the usual super bracket of operators:
\[ [X \otimes \alpha, Y \otimes \beta] = (XY - YX) \otimes (\alpha \beta). \]
Assuming that $\alpha$ and $\beta$ are of homogenous degree, we see that $[X \otimes \alpha, Y \otimes \beta] = (-1)^{1+|\alpha| \cdot |\beta|} [Y \otimes \beta, X \otimes \alpha]$. The commutation relation in $\mathcal{W}(A)$ is given by
\[ [t^r f(D) \otimes \alpha, t^s g(D) \otimes \beta] = t^{r+s} (f(D+s)g(D) - f(D)g(D+r)) \otimes (\alpha \beta). \] (2.1)

When $A = \mathbb{C}$ (hence $d = 0$), we will simply write the Lie superalgebra $\mathcal{W}(A)$ as $\mathcal{W}$, which is the usual Lie algebra of differential operator on the circle, cf. e.g. [FGRW], [Kac]. The algebra $\mathcal{W}$ affords a universal central extension which is usually referred to as the $\mathcal{W}_{1+\infty}$ algebra. In $\mathcal{W}_{1+\infty}$, the operators $L^0_k$ generate a Heisenberg algebra, while the operators $L^1_k$ generate a Virasoro algebra.

The algebra $\mathcal{W}(A)$ has a natural weight filtration
\[ \mathcal{W}(A)^0 \subset \mathcal{W}(A)^1 \subset \mathcal{W}(A)^2 \subset \ldots \subset \mathcal{W}(A) \]
by letting the weight of $L^p_k(\alpha)$ be $p$. Clearly $[\mathcal{W}(A)^p, \mathcal{W}(A)^q] \subset \mathcal{W}(A)^{p+q-1}$. We will denote the associated graded algebra by $\mathcal{GW}(A)$. The leading term (according to the weight filtration) of $J^p_k(\alpha)$ is just $L^p_k(\alpha)$, and so $J^p_k(\alpha)$ and $L^p_k(\alpha)$ give rise to the same element in the graded algebra $\mathcal{GW}(A)$ which is denoted by $\mathcal{L}^p_k(\alpha)$. We easily derive the following commutation relation from (2.1):
\[ [\mathcal{L}^p_m(\alpha), \mathcal{L}^q_n(\beta)] = (qm - pn) \cdot \mathcal{L}^{p+q-1}_{m+n}(\alpha \beta). \] (2.2)

The $\mathcal{W}$ (super)algebra $\widehat{\mathcal{W}}(A)$ is a central extension of the Lie superalgebra $\mathcal{GW}(A)$ by a one-dimensional center with a specified generator $C$:
\[ 0 \longrightarrow \mathbb{C} C \longrightarrow \widehat{\mathcal{W}}(A) \longrightarrow \mathcal{GW}(A) \longrightarrow 0, \]
such that the commutators in $\widehat{\mathcal{W}}(A) = \mathcal{GW}(A) + \mathbb{C} C$ are given by:
\[ [\mathcal{L}^p_m(\alpha), \mathcal{L}^q_n(\beta)] = \begin{cases} m \delta_{m,-n} \mathrm{Tr}(\alpha \beta) \cdot C, & \text{if } p = q = 0, \\ (qm - pn) \cdot \mathcal{L}^{p+q-1}_{m+n}(\alpha \beta), & \text{otherwise.} \end{cases} \] (2.2)
In the above, we have used $\delta_{a,b}$ to denote 1 if $a = b$ and 0 otherwise.

In this paper, we will be mainly interested in the $\mathcal{W}$ (super)algebra $\widehat{\mathcal{W}}(A)$ when $A$ is (a subring of) the cohomology ring $H^*(X)$ of a projective surface $X$, with the
trace being defined by \( \text{Tr}(\alpha) = -\int_X \alpha \) for \( \alpha \in H^*(X) \). Here and below \( H^*(X) \) always denote the cohomology group/ring of \( X \) with rational coefficient.

3. Basics on Hilbert schemes of points on surfaces

Let \( X \) be a smooth projective complex surface with the canonical class \( K \) and the Euler class \( e \), and \( X^{[n]} \) be the Hilbert scheme of points in \( X \). An element in \( X^{[n]} \) is represented by a length-\( n \) 0-dimensional closed subscheme \( \xi \) of \( X \). For \( \xi \in X^{[n]} \), let \( I_\xi \) be the corresponding sheaf of ideals. It is well known that \( X^{[n]} \) is smooth. Sending an element in \( X^{[n]} \) to its support in the symmetric product \( \text{Sym}^n(X) \), we obtain the Hilbert-Chow morphism \( \pi_n : X^{[n]} \to \text{Sym}^n(X) \), which is a resolution of singularities. Define the universal codimension-2 subscheme:

\[
Z_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \text{Supp}(\xi)\} \subset X^{[n]} \times X.
\]

Denote by \( p_1 \) and \( p_2 \) the projections of \( X^{[n]} \times X \) to \( X^{[n]} \) and \( X \) respectively. Let

\[
\mathbb{H}_X = \oplus_{n=0}^{\infty} H^*(X^{[n]})
\]

be the direct sum of total cohomology groups of the Hilbert schemes \( X^{[n]} \).

For \( m \geq 0 \) and \( n > 0 \), let \( Q^{[m,n]} = \emptyset \) and define \( Q^{[m+n,m]} \) to be the closed subset:

\[
\{(\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{x\}\}.
\]

We recall Nakajima’s definition of the Heisenberg operators \([\text{Nai}]\). Let \( n > 0 \). The linear operator \( a_{-n}(\alpha) \in \text{End}(\mathbb{H}_X) \) with \( \alpha \in H^*(X) \) is defined by

\[
a_{-n}(\alpha)(a) = \tilde{p}_1*([Q^{[m+n,m]}] \cdot \tilde{p}^*\alpha \cdot \tilde{p}_2* a)
\]

for \( a \in H^*(X^{[m]}) \), where \( \tilde{p}_1, \tilde{p}, \tilde{p}_2 \) are the projections of \( X^{[m+n]} \times X \times X^{[m]} \) to \( X^{[m+n]}, X, X^{[m]} \) respectively. Define \( a_n(\alpha) \in \text{End}(\mathbb{H}_X) \) to be \((-1)^n\) times the operator obtained from the definition of \( a_{-n}(\alpha) \) by switching the roles of \( \tilde{p}_1 \) and \( \tilde{p}_2 \). We often refer to \( a_{-n}(\alpha) \) (resp. \( a_n(\alpha) \)) as the creation (resp. annihilation) operator. We also set \( a_0(\alpha) = 0 \).

For \( n > 0 \) and a homogeneous class \( \gamma \in H^*(X) \), let \( |\gamma| = s \) if \( \gamma \in H^s(X) \), and let \( G_i(\gamma, n) \) be the homogeneous component in \( H^{|\gamma|+2i}(X^{[n]}) \) of

\[
G(\gamma, n) = p_1*(\text{ch}(\mathcal{O}_{Z_n}) \cdot p_2*\text{td}(X) \cdot p_2* \gamma) \in H^*(X^{[n]})
\]

where \( \text{ch}(\mathcal{O}_{Z_n}) \) denotes the Chern character of the structure sheaf \( \mathcal{O}_{Z_n} \) and \( \text{td}(X) \) denotes the Todd class. Here and below we omit the Poincaré duality used to switch a homology class to a cohomology class and vice versa. We extend the notion \( G_i(\gamma, n) \) linearly to an arbitrary class \( \gamma \in H^*(X) \). We also set \( G(\gamma, 0) = 0 \).

The Chern character operator \( \mathfrak{S}_i(\gamma) \in \text{End}(\mathbb{H}_X) \) is defined to be the operator acting on the component \( H^*(X^{[n]}) \) by the cup product with \( G_i(\gamma, n) \). It was proved in \([\text{LQW1}]\) that the cohomology ring of \( X^{[n]} \) is generated by the classes \( G_i(\gamma, n) \) where \( 0 \leq i < n \) and \( \gamma \) runs over a linear basis of \( H^*(X) \). Let \( \mathfrak{d} = \mathfrak{S}_1(1_X) \) where \( 1_X \) is the fundamental cohomology class of \( X \). The operator \( \mathfrak{d} \) was first introduced in \([\text{Le}]\). For a linear operator \( \mathfrak{f} \in \text{End}(\mathbb{H}_X) \), define its derivative \( \mathfrak{f}' \) by \( \mathfrak{f}' = [\mathfrak{d}, \mathfrak{f}] \). The higher derivative \( \mathfrak{f}^{(k)} \) is defined inductively by \( \mathfrak{f}^{(k)} = [\mathfrak{d}, \mathfrak{f}^{(k-1)}] \).
Let \( a_m, a_{m_2} \) be \( a_m a_{m_2} \) when \( m_1 \leq m_2 \) and \( a_{m_2} a_{m_1} \) when \( m_1 > m_2 \). For \( n \in \mathbb{Z} \), define a linear map \( \mathcal{L}_n : H^*(X) \to \text{End}(\mathbb{H}_X) \) by \( \mathcal{L}_n = -\frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : \tau_{2s} \).

Here for \( k \geq 1 \), \( \tau_{k*} : H^*(X) \to H^*(X^k) \) is the linear map induced by the diagonal embedding \( \tau_k : X \to X^k \), and \( a_{m_1} \cdots a_{m_k} (\tau_k(\alpha)) \) denotes \( \sum_j a_{m_1}(\alpha_{j,1}) \cdots a_{m_k}(\alpha_{j,k}) \) when \( \tau_k \alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k} \) via the Künneth decomposition of \( H^*(X^k) \).

The following is a combination of various theorems from \([\text{Na}2, \text{Gro}, \text{Leh}, \text{LQW1}]\). Our notations and convention of signs are consistent with \([\text{LQW3}]\).

**Theorem 3.1.** Let \( k \geq 0, n, m \in \mathbb{Z} \) and \( \alpha, \beta \in H^*(X) \). Then,

(i) The operators \( a_n(\alpha) \) satisfy a Heisenberg algebra commutation relation:

\[
[a_n(\alpha), a_n(\beta)] = -m \delta_{m,-n} \int_X (\alpha \beta) \cdot \text{Id}_{\mathbb{H}_X}.
\]

The space \( \mathbb{H}_X \) is an irreducible module over the Heisenberg algebra generated by the operators \( a_n(\alpha) \) with a highest weight vector \( |0\rangle = 1 \in H^0(X^0) \equiv \mathbb{Q} \).

(ii) \( [\mathcal{L}_n(\alpha), a_n(\beta)] = -n \cdot a_{m+n}(\alpha \beta) \);

(iii) \( a'_n(\alpha) = n \cdot \mathcal{L}_n(\alpha) - 2 \frac{n(|n|-1)}{m} \cdot a_n(K \alpha) \);

(iv) The operators \( \mathcal{L}_n(\alpha) \) satisfy the Virasoro algebra commutation relation:

\[
[\mathcal{L}_m(\alpha), \mathcal{L}_n(\beta)] = (m-n) \cdot \mathcal{L}_{m+n}(\alpha \beta) + \frac{m^3-m}{12} \delta_{m,-n} \int_X (\alpha \beta) \cdot \text{Id}_{\mathbb{H}_X}.
\]

(v) \( [\mathcal{G}_k(\alpha), a_{-1}(\beta)] = \frac{1}{k!} \cdot a^{(k)}_{-1}(\alpha \beta) \).

The Lie brackets in the above theorem are understood in the super sense according to the parity of the cohomology degrees of the cohomology classes involved. Also, it follows from Theorem [3.1] (i) that the space \( \mathbb{H}_X \) is linearly spanned by all the Heisenberg monomials \( a_{n_1}(\alpha_1) \cdots a_{n_k}(\alpha_k) \cdot |0\rangle \) where \( k \geq 0 \) and \( n_1, \ldots, n_k < 0 \).

We will need later on the following lemma proved in \([\text{LQW3}]\).

**Lemma 3.2.** Let \( k, s \geq 1, n_1, \ldots, n_k, m_1, \ldots, m_s \in \mathbb{Z} \), and \( \alpha, \beta \in H^*(X) \).

(i) The commutator \([a_{n_1} \cdots a_{n_k} (\tau_{k*} \alpha), a_{m_1} \cdots a_{m_s} (\tau_{s*} \beta)]\) is equal to

\[
-\sum_{t=1}^{k} \sum_{j=1}^{s} n_t \delta_{n_t,-m_j} \cdot \left( \prod_{t=1}^{j-1} a_{m_t} \prod_{1 \leq u \leq k, u \neq t} a_{n_u} \prod_{t=j+1}^{s} a_{m_t} \right) \tau_{(k+s-2)*}(\alpha \beta).
\]

(ii) The derivative \((a_{n_1} \cdots a_{n_k} (\tau_{k*} \alpha))'\) is equal to

\[
-\sum_{j=1}^{k} \frac{n_j}{2} \sum_{m_1+m_2=n_j} a_{n_1} \cdots a_{n_{j-1}} : a_{m_1} a_{m_2} : a_{n_{j+1}} \cdots a_{n_k} (\tau_{(k+1)*} \alpha)
\]

\[
-\sum_{j=1}^{k} \frac{n_j(n_j-1)}{2} \cdot a_{n_1} \cdots a_{n_k} (\tau_{k*}(K \alpha)).
\]
(iii) Let \( j \) satisfy \( 1 \leq j < k \). Then, \( a_{n_1} \cdots a_{n_k}(\tau_k \alpha) \) is equal to

\[
\left( \prod_{1 \leq s < j} a_{n_s} \cdot a_{n_{j+1}} a_{n_j} \cdot \prod_{j+1 < s \leq k} a_{n_s} \right) (\tau_k \alpha) - n_j \delta_{n_j, n_{j+1}} \prod_{1 \leq s \leq k} a_{n_s}(\tau_{(k-2)}(ea))
\]

In the lemma above and throughout the paper, the products of Heisenberg operators are understood in the increasing order of the parametrizing indices from the left to the right, e.g., \( \prod_{1 \leq s < j} a_{n_s} = a_{n_1} a_{n_2} \cdots a_{n_{j-1}} \) and \( \prod_{1 \leq u \leq k, u \neq t} a_{n_u} = a_{n_1} \cdots a_{n_{t-1}} a_{n_{t+1}} \cdots a_{n_k} \).

Note that the commutator in Lemma 3.2 (i) satisfies the transfer property, that is, it depends only on the cup product \( (\alpha \beta) \). Such a property was first formulated and emphasized in \([LQW1]\), and will also be incorporated into the formulation of the operator product expansions used later in this paper.

4. Chern character operators

In this section, we determine the operators \( a_n^{(k)}(\alpha) \) and \( \mathcal{E}_k(\alpha) \) when the cohomology class \( \alpha \) is orthogonal to the canonical class \( K \). In addition, we obtain general expressions of \( a_n^{(k)}(\alpha) \) and \( \mathcal{E}_k(\alpha) \) when \( \alpha \in H^*(X) \) is arbitrary.

4.1. Formulas for the derivatives of Heisenberg operators.

**Definition 4.1.** Let \( X \) be a smooth projective surface.

(i) Let \( \alpha \in H^*(X) \), and \( \lambda = (\cdots (-2)^{m-2}(-1)^{m-1}1^{m_1}2^{m_2} \cdots) \) be a generalized partition of the integer \( n = \sum_i i m_i \) whose part \( i \in \mathbb{Z} \) has multiplicity \( m_i \). Define \( \ell(\lambda) = \sum_i m_i, |\lambda| = \sum_i im_i = n, s(\lambda) = \sum_i i^2 m_i, \lambda^! = \prod_i m_i! \), and

\[
a_{\lambda}(\tau_\alpha) = \left( \prod_i (a_i)^{m_i} \right) (\tau_{\ell(\lambda)^!} \alpha)
\]

where the product \( \prod_i (a_i)^{m_i} \) is understood to be \( \cdots a_{-2}^{m_{-2}} a_{-1}^{m_{-1}} a_1^{m_1} a_2^{m_2} \cdots \). Let \( -\lambda \) be the generalized partition whose multiplicity of \( i \in \mathbb{Z} \) is \( m_i \).

(ii) A generalized partition becomes a partition in the usual sense if the multiplicity \( m_i = 0 \) for every \( i < 0 \). A partition \( \lambda \) of \( n \) is denoted by \( \lambda \vdash n \).

(iii) Define \( \mathfrak{A}_X = \{ \alpha \in H^*(X)|K \alpha = 0 \} \) which is an ideal in the ring \( H^*(X) \).

Our starting point is the following theorem.

**Theorem 4.2.** Let \( k \geq 0, n \in \mathbb{Z} \), and \( \alpha \in \mathfrak{A}_X \). Then, \( a_n^{(k)}(\alpha) \) is equal to

\[
(-n)^k k! \left( \sum_{\ell(\lambda) = k+1, |\lambda| = n} \frac{1}{\lambda!} a_{\lambda}(\tau_\alpha) - \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{s(\lambda) - 1}{24 \lambda!} a_{\lambda}(\tau_\alpha(e\alpha)) \right).
\]

**(Proof.** Use induction on \( k \). When \( k = 0 \) or \( 1 \), the theorem is trivially true. Next, assume that the theorem is true for \( a_n^{(k)}(\alpha) \), i.e., \( a_n^{(k)}(\alpha) \) is given by (4.1). By
Lemma 3.2 (ii) and (iii), we conclude from (4.1) that $a^{(k+1)}_n(\alpha)$ is of the form
\[
\sum_{\ell(\lambda)=k+2, |\lambda|=n} \tilde{f}_1(\lambda) a_\lambda(\tau_s \alpha) + \sum_{\ell(\lambda)=k, |\lambda|=n} \tilde{f}_e(\lambda) a_\lambda(\tau_s(e\alpha)).
\]
Fix a generalized partition $\lambda = (\cdots (-2)^{m_2}(-1)^{m_1}m_2 \cdot 2 \cdots)$ with $|\lambda| = n$. In the following, we compute the coefficients $\tilde{f}_1(\lambda)$ and $\tilde{f}_e(\lambda)$ separately.

(i) We start with the computation of $\tilde{f}_1(\lambda)$. Note that $\ell(\lambda) = k + 2$. Going from $a^{(k)}_n(\alpha)$ to $a^{(k+1)}_n(\alpha) = (a^{(k)}_n(\alpha))'$, we see that those terms in $a^{(k)}_n(\alpha)$ whose derivatives contain the term $a_\lambda(\tau_s \alpha)$ in $a^{(k+1)}_n(\alpha)$ are of the form $a_{\lambda_{i,j}}(\tau_s \alpha)$ where $i \leq j$, $m_i \geq 1$, $m_j \geq 1$, and $\lambda_{i,j}$ stands for the generalized partition obtained from $\lambda$ by subtracting 1 from the multiplicities of $i$ and $j$ and by adding 1 to the multiplicity of $(i+j)$. For fixed $i$ and $j$, $\{a_{\lambda_{i,j}}(\tau_s \alpha)\}'$ is equal to
\[
\left(-\frac{i+j}{2}\right) (m_{i+j} + 1)(2 - \delta_{i,j}) a_\lambda(\tau_s \alpha)
\]
\[
+ \sum_{\ell(\mu) = k+2, |\mu|=n} d_\mu a_\mu(\tau_s \alpha) + \sum_{\ell(\mu) = k, |\mu|=n} d_\mu a_\mu(\tau_s(e\alpha))
\]
for some constant $d_\mu$. By (4.1), the coefficient of $a_{\lambda_{i,j}}(\tau_s \alpha)$ in $a^{(k)}_n(\alpha)$ is
\[
\frac{(-n)^k k!}{(\lambda_{i,j})!} = \frac{(-n)^k k!}{\lambda^l} \cdot \frac{m_i(m_j - \delta_{i,j})}{m_{i+j} + 1}.
\]
Combining all of these, we see that the coefficient of $a_\lambda(\tau_s \alpha)$ in $a^{(k+1)}_n(\alpha)$ is
\[
\sum_{i \leq j} \frac{(-n)^k k!}{\lambda^l} \cdot \frac{m_i(m_j - \delta_{i,j})}{m_{i+j} + 1} \left(-\frac{i+j}{2}\right) (m_{i+j} + 1)(2 - \delta_{i,j})
\]
\[
= \frac{(-n)^k k!}{\lambda^l} \left(-\sum_{i<j} m_i m_j (i+j) - \sum_i m_i (m_i - 1)i\right)
\]
\[
= \frac{(-n)^k k!}{\lambda^l} \left(-\frac{1}{2} \sum_{i,j} m_i m_j (i+j) + \sum_i m_i i\right) = \frac{(-n)^{k+1}(k + 1)!}{\lambda^l}
\]
where we have used $\sum_i m_i = \ell(\lambda) = k + 2$ and $\sum_i im_i = |\lambda| = n$ in the last step.

(ii) Now we compute $\tilde{f}_e(\lambda)$. In this case, $\ell(\lambda) = k$. We shall prove that the coefficient of $a_\lambda(\tau_s(e\alpha))$ in $a^{(k+1)}_n(\alpha)$ is equal to $\frac{(-n)^{k+1}(k + 1)!}{\lambda^l} c$ where
\[
c = \frac{1 - s(\lambda)}{24} = \frac{1 - \sum_i i^2 m_i}{24}.
\]
In view of Lemma 3.2 (ii) and (iii), there are exactly two sources contributing to the term $a_\lambda(\tau_s(e\alpha))$ in $a^{(k+1)}_n(\alpha)$ from the derivatives of the terms in $a^{(k)}_n(\alpha)$. In the following, we handle these two different sources separately.
The first is similar to (i) above, and comes from the derivatives \(\{a_{\lambda,i,j}(\tau_s(e\alpha))\}'\) where \(i \leq j, m_i \geq 1, m_j \geq 1\). For fixed \(i, j\), \(\{a_{\lambda,i,j}(\tau_s(e\alpha))\}'\) is equal to

\[
\left(-\frac{i + j}{2}\right)(m_{i+j} + 1)(2 - \delta_{i,j})a_\lambda(\tau_s(e\alpha)) + \sum_{\mu \neq \lambda, \ell(\mu) = k, |\mu| = n} d_\mu a_\mu(\tau_s(e\alpha))
\]

for some constant \(d_\mu\). By (1.1), the coefficient of \(a_{\lambda,i,j}(\tau_s(e\alpha))\) in \(a_n^{(k)}(\alpha)\) is equal to

\[
\frac{(-n)^kk!}{(\lambda_{i,j})^k} \cdot \frac{1 - s(\lambda_{i,j})}{24} = \frac{(-n)^kk!}{\lambda^!} \cdot \frac{m_i(m_j - \delta_{i,j})}{m_{i+j} + 1} \cdot \left(c - \frac{ij}{12}\right).
\]

Combining all of these and noting \(\sum_i m_i = k\) and \(\sum_i im_i = n\), we see that the total contribution of the first source to the coefficient of \(a_\lambda(\tau_s(e\alpha))\) in \(a_n^{(k+1)}(\alpha)\) is

\[
\sum_{i \leq j} \frac{(-n)^kk!}{\lambda^!} \cdot \frac{m_i(m_j - \delta_{i,j})}{m_{i+j} + 1} \cdot \left(c - \frac{ij}{12}\right)(m_{i+j} + 1)(2 - \delta_{i,j})
\]

\[
= \frac{(-n)^kk!}{\lambda^!} \left(nc - knc + \frac{n}{12} \sum_i i^2m_i - \frac{1}{12} \sum_i i^3m_i\right).
\]

The second source comes from the derivatives \(\{a_{\tilde{\lambda},i,j}(\tau_s(e\alpha))\}'\) where \(i \neq 0, 0 < 2j \leq |i|, m_i \geq 1, and \tilde{\lambda}_{i,j}\) is defined as follows. There are two cases depending on \(i > 0\) or \(i < 0\). For \(i > 0\), define \(\tilde{\lambda}_{i,j}\) to be the generalized partition obtained from \(\lambda\) by subtracting 1 from the multiplicity of \(i\) and by adding 1 to the multiplicities of \(j\) and \((i - j)\). Then, \(\{a_{\tilde{\lambda},i,j}(\tau_s(e\alpha))\}'\) contains a term \((\ldots a_j \ldots a_{i-j} \ldots)\) which contains a term \((\ldots a_j \ldots a_{i-j} \ldots)\) after expanding the derivative \(a'_{i-j}\). When we switch \(a_j\) with \(a_{-j}\), we get the term \(a_\lambda(\tau_s(e\alpha))\) in \(a_n^{(k+1)}(\alpha)\). For \(i < 0\), define \(\tilde{\lambda}_{i,j}\) to be the generalized partition obtained from \(\lambda\) by subtracting 1 from the multiplicity of \(i\) and by adding 1 to the multiplicities of \(-j\) and \((i + j)\). Then \(\{a_{\tilde{\lambda},i,j}(\tau_s(e\alpha))\}'\) contains a term \((\ldots a'_{i+j} \ldots a_{-j} \ldots)\) which contains a term \((\ldots a_{i+j} \ldots a_{-j} \ldots)\) after expanding \(a'_{i+j}\). Switching \(a_j\) with \(a_{-j}\), we get \(a_\lambda(\tau_s(e\alpha))\) in \(a_n^{(k+1)}(\alpha)\).

More precisely, for fixed \(i > 0\) and \(j\), the derivative \(\{a_{\lambda,i,j}(\tau_s(e\alpha))\}'\) is

\[
j(i - j) \cdot \frac{(m_{i+j} + 1)(m_j + 1 + \delta_{i,2j})}{1 + \delta_{i,2j}} \cdot a_\lambda(\tau_s(e\alpha)) +
\]

\[
+ \sum_{\mu \neq \lambda, \ell(\mu) = k, |\mu| = n} d_\mu a_\mu(\tau_s(e\alpha)) + \sum_{\ell(\mu) = k+2, |\mu| = n} d_\mu a_\mu(\tau_s\alpha)
\]

for some constant \(d_\mu\). By (1.1), the coefficient of \(a_{\lambda,i,j}(\tau_s(e\alpha))\) in \(a_n^{(k)}(\alpha)\) is equal to

\[
\frac{(-n)^kk!}{(\lambda_{i,j})^!} = \frac{(-n)^kk!}{\lambda^!} \cdot \frac{m_i}{(m_{i+j} + 1)(m_j + 1 + \delta_{i,2j})}.
\]
It follows that the total contribution of the second source with \( i > 0 \) to the coefficient of the term \( a_\lambda^*(\tau_s(e\alpha)) \) in \( a_n^{(k+1)}(\alpha) \) is equal to

\[
\sum_{i>0} \sum_{0<j \leq i} \frac{(-n)^k k!}{\lambda^i} \cdot \frac{m_i}{(m_{i-j}+1)(m_j+1+\delta_{i,2j})} \cdot j(i-j) \cdot \frac{(m_{i-j}+1)(m_j+1+\delta_{i,2j})}{1+\delta_{i,2j}}
\]

\[
= \frac{(-n)^k k!}{\lambda^i} \sum_{i>0} m_i \sum_{0<j \leq i} j(i-j) = \frac{(-n)^k k!}{\lambda^i} \sum_{i>0} m_i \left( \frac{1}{2} \sum_{0<j<i} j(i-j) \right)
\]

\[
= \frac{(-n)^k k!}{\lambda^i} \sum_{i>0} m_i \cdot \frac{i^3 - i}{12} = \frac{(-n)^k k!}{\lambda^i} \cdot \frac{1}{12} \sum_{i>0} (i^3 m_i - i m_i).
\]

Similarly, we see that the total contribution of the second source with \( i < 0 \) to the coefficient of \( a_\lambda^*(\tau_s(e\alpha)) \) in \( a_n^{(k+1)}(\alpha) \) is equal to

\[
\frac{(-n)^k k!}{\lambda^i} \cdot \frac{1}{12} \sum_{i<0} (i^3 m_i - i m_i).
\]

Noting \( n = \sum_i i m_i \), we see from (4.2), (4.3) and (4.4) that the coefficient of the term \( a_\lambda^*(\tau_s(e\alpha)) \) in the derivative \( a_n^{(k+1)}(\alpha) \) is:

\[
\frac{(-n)^k k!}{\lambda^i} \left( nc - knc + \frac{n}{12} \sum_i i^2 m_i - \frac{1}{12} \sum_i i^3 m_i \right) + \frac{(-n)^k k!}{\lambda^i} \cdot \frac{1}{12} \sum_i (i^3 m_i - i m_i)
\]

\[
= \frac{(-n)^k k!}{\lambda^i} (nc - knc - 2nc) = (-n)^k (k+1) \frac{1}{24 \lambda^i} s(\lambda).
\]

Combining (i) and (ii), we have proved the theorem for \( a_n^{(k+1)}(\alpha) \).

\[\square\]

**Remark 4.3.** Let \( \alpha \in \mathfrak{A}_X \). From the proof of Theorem 4.2, we can read that

\[
\left( \sum_{\ell(\lambda)=k+1,|\lambda|=n} \frac{1}{\lambda^i} a_\lambda^*(\tau_s \alpha) - \sum_{\ell(\lambda)=k-1,|\lambda|=n} \frac{s(\lambda)-1}{24 \lambda^i} a_\lambda^*(\tau_s(e\alpha)) \right)'
\]

\[
= -n(k+1) \left( \sum_{\ell(\lambda)=k+2,|\lambda|=n} \frac{1}{\lambda^i} a_\lambda^*(\tau_s \alpha) - \sum_{\ell(\lambda)=k,|\lambda|=n} \frac{s(\lambda)-1}{24 \lambda^i} a_\lambda^*(\tau_s(e\alpha)) \right).
\]
Note that when \( n = 0 \), this formula is not covered by Theorem 4.2. Applying this formula twice in the second equality below, we obtain that for a fixed constant \( d \),

\[
\left( \sum_{\ell(\lambda) = k+1, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s \alpha) - \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{s(\lambda) + d}{24\lambda!} a_\lambda(\tau_s (e \alpha)) \right)’,
\]

\[
= \left( \sum_{\ell(\lambda) = k+1, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s \alpha) - \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{s(\lambda) - 1}{24\lambda!} a_\lambda(\tau_s (e \alpha)) \right)’
\]

\[
= \frac{-d + 1}{24} \left( \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s (e \alpha)) \right)’
\]

\[
= -n(k + 1) \left( \sum_{\ell(\lambda) = k+2, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s \alpha) - \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{s(\lambda) + d}{24\lambda!} a_\lambda(\tau_s (e \alpha)) \right)
\]

\[
= -\frac{n(d + 1)}{12} \cdot \sum_{\ell(\lambda) = k, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s (e \alpha)).
\]

We shall need this formula in the proof of Theorem 4.6.

Now we describe a formula of \( a_n^{(k)}(\alpha) \) for general \( \alpha \in H^*(X) \).

**Theorem 4.4.** Let \( k \geq 0 \), \( n \in \mathbb{Z} \), and \( \alpha \in H^*(X) \). Then, \( a_n^{(k)}(\alpha) \) equals

\[
(-n)^k k! \left( \sum_{\ell(\lambda) = k+1, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s \alpha) - \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{s(\lambda) - 1}{24\lambda!} a_\lambda(\tau_s (e \alpha)) \right)
\]

\[
+ \sum_{\epsilon \in \{1, e, K, K^2\}} \sum_{\ell(\lambda) = k+1-|\epsilon|/2, |\lambda| = n} \frac{f_\epsilon(\lambda)}{\lambda!} a_\lambda(\tau_s (\epsilon \alpha))
\]

where all the numbers \( f_\epsilon(\lambda) \) are independent of \( X \) and \( \alpha \).

**Proof.** We see from Lemma 3.2 (ii) and (iii) that

\[
a_n^{(k)}(\alpha) = \sum_{\epsilon \in \{1, e, K, K^2\}} \sum_{\ell(\lambda) = k+1-|\epsilon|/2, |\lambda| = n} \frac{f_\epsilon(\lambda)}{\lambda!} a_\lambda(\tau_s (\epsilon \alpha)) \quad (4.5)
\]

where all the coefficients \( f_\epsilon(\lambda) \) are independent of \( X \) and \( \alpha \).

To determine \( f_{1X}(\lambda) \) and \( f_e(\lambda) \), let \( X \) be a \( K3 \) surface and \( \alpha = 1_X \). Then, \( K\alpha = 0 \) but \( e\alpha \neq 0 \). By Theorem 4.2, \( a_n^{(k)}(\alpha) \) is equal to

\[
(-n)^k k! \left( \sum_{\ell(\lambda) = k+1, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_s \alpha) - \sum_{\ell(\lambda) = k-1, |\lambda| = n} \frac{s(\lambda) - 1}{24\lambda!} a_\lambda(\tau_s (e \alpha)) \right). \quad (4.6)
\]
Note that the set of all the Heisenberg monomials \( a_{\lambda}(\tau_s\alpha), a_{\lambda}(\tau_s(e\alpha)) \) with \( \ell(\lambda) = k + 1, |\lambda| = n, \ell(\bar{\lambda}) = k - 1, |\bar{\lambda}| = n \) is linearly independent. It follows from (4.5) and (4.6) that \( f_{1X}(\lambda) = (-n)^k k! \) and \( f_e(\lambda) = -\frac{(n)^k k!(s(\lambda) - 1)}{24} \). 

4.2. Formulas for the Chern Character operators.

**Lemma 4.5.** (i) Let \( n \geq 1, \alpha \in H^*(X) \), and \( f \in \text{End}(\mathbb{H}_X) \) with \( f' = 0 \). Then,

\[
[f, a_{-n+1}(\alpha)] = -\frac{1}{n} \cdot \{[[f, a_{-1}(1_X)]', a_{-n}(\alpha)] + [a_{-1}'(1_X), [f, a_{-n}(\alpha)]]\};
\]

(ii) For \( i = 1, 2 \), let \( f_i \in \text{End}(\mathbb{H}_X) \) such that \( f_i \cdot |0\rangle = 0 \) and \( f_i' = 0 \). If \( [f_1, a_{-1}(\alpha)] = [f_2, a_{-1}(\alpha)] \) for every \( \alpha \in H^*(X) \), then \( f_1 = f_2 \).

**Proof.** The statement (i) is proved in the Lemma 3.5 in [LQW3]. To prove (ii), we note that it suffices to show that for every \( n \geq 1 \) and \( \alpha \in H^*(X) \),

\[
[f_1, a_{-n}(\alpha)] = [f_2, a_{-n}(\alpha)]. \tag{4.7}
\]

By assumption, (4.7) holds for \( n = 1 \). So (4.7) follows from (i) and induction. \( \square \)

Our first main result in this paper is the following theorem.

**Theorem 4.6.** Let \( k \geq 0 \), and \( \alpha \in \mathfrak{A}_X \). Then, \( \Phi_k(\alpha) \) is equal to

\[
- \sum_{\ell(\lambda) = k+1, |\lambda| = 0} \frac{1}{\lambda!} a_{\lambda}(\tau_s\alpha) + \sum_{\ell(\lambda) = k, |\lambda| = 0} \frac{s(\lambda) - 2}{24\lambda!} a_{\lambda}(\tau_s(e\alpha)). \tag{4.8}
\]

**Proof.** Denote the expression (4.8) by \( f_2 \). We shall apply Lemma 4.5 (ii) to the two operators \( f_1 = \Phi_k(\alpha) \) and \( f_2 \). Note that \( \Phi_k(\alpha)|0\rangle = 0 \) and \( \Phi_k(\alpha)' = 0 \). From Theorem 3.1 (v) and Theorem 4.2, we see that

\[
[\Phi_k(\alpha), a_{-1}(\beta)] = \frac{1}{k!} \cdot a_{-1}^{(k)}(\alpha, \beta)
\]

\[
= \left( \sum_{\ell(\lambda) = k+1, |\lambda| = 1} \frac{1}{\lambda!} a_{\lambda}(\tau_s(\alpha, \beta)) - \sum_{\ell(\lambda) = k, |\lambda| = 1} \frac{s(\lambda) - 1}{24\lambda!} a_{\lambda}(\tau_s(e\alpha, \beta)) \right).
\]

Next we need to check the corresponding properties for \( f_2 \). Indeed, \( f_2 \cdot |0\rangle = 0 \) from the definition of \( f_2 \). Also, \( f_2' = 0 \) by Remark 4.3. By Lemma 3.2 (i), we obtain

\[
[f_2, a_{-1}(\beta)]
\]

\[
= \sum_{\ell(\lambda) = k+1, |\lambda| = 1} \frac{1}{\lambda!} a_{\lambda}(\tau_s(\alpha, \beta)) - \sum_{\ell(\lambda) = k, |\lambda| = 0} \frac{s(\lambda) - 2}{24(\lambda_1)!} a_{\lambda_1}(\tau_s(e\alpha, \beta))
\]

\[
= \sum_{\ell(\lambda) = k+1, |\lambda| = 1} \frac{1}{\lambda} a_{\lambda}(\tau_s(\alpha, \beta)) - \sum_{\ell(\lambda_1) = k-1, |\lambda_1| = 0} \frac{s(\lambda_1) - 1}{24(\lambda_1)!} a_{\lambda_1}(\tau_s(e\alpha, \beta))
\]

\[
= [\Phi_k(\alpha), a_{-1}(\beta)]
\]

where for a generalized partition \( \lambda \) with \( \ell(\lambda) = k \), the generalized partition \( \lambda_1 \) is obtained from \( \lambda \) by subtracting 1 from the multiplicity of 1.
Thus, we conclude from Lemma 4.3 (ii) that \( \mathfrak{G}_k(\alpha) = \hat{f}_2 \).

Now we present a formula of \( \mathfrak{G}_k(\alpha) \) for a general \( \alpha \in H^*(X) \).

**Theorem 4.7.** Let \( k \geq 0 \) and \( \alpha \in H^*(X) \). Then, \( \mathfrak{G}_k(\alpha) \) is equal to

\[
- \sum_{\ell(\lambda)=k+2,|\lambda|=0} \frac{1}{\lambda!} a_\lambda(\tau_\alpha) + \sum_{\ell(\lambda)=k,|\lambda|=0} \frac{s(\lambda) - 2}{24 \lambda!} a_\lambda(\tau_\alpha(\epsilon \alpha)) + \sum_{\epsilon \in \{K,K^2\}} \sum_{\ell(\lambda)=k+2-|\epsilon|/2,|\lambda|=0} \frac{g_\epsilon(\lambda)}{\lambda!} a_\lambda(\tau_\alpha(\epsilon \alpha))
\]

where all the numbers \( g_\epsilon(\lambda) \) are independent of \( X \) and \( \alpha \).

**Proof.** First of all, by the formula (5.6) in [LQW1], we have

\[
\mathfrak{G}_0(\alpha) = -\mathcal{L}_0(\alpha) = - \sum_{\ell(\lambda)=2,|\lambda|=0} \frac{1}{\lambda!} a_\lambda(\tau_\alpha).
\]

So our theorem is true for \( k = 0 \). In the following, we assume \( k > 0 \) and \( \alpha \neq 0 \).

Next, we write the operator \( \mathfrak{G}_k(\alpha) \in \text{End}(\mathbb{H}_X) \) as a linear combination of Heisenberg monomials \( \prod_i (a_i(\alpha_{i,1}) \cdots a_i(\alpha_{i,m_i})) \). For degree reasons, \( \sum_i im_i = 0 \) and

\[
-2 \sum_i m_i + \sum_{i,j} |\alpha_{i,j}| = 2k + |\alpha|.
\]

In other words, we regard \( \mathfrak{G}_k(\alpha) \) as an element in the completion of the universal enveloping algebra of the Heisenberg algebra. We can always do so because \( \mathbb{H}_X \) is irreducible as a representation of the Heisenberg algebra. Now rewrite \( \mathfrak{G}_k(\alpha) \) as

\[
\mathfrak{G}_k(\alpha) = \sum_{|\lambda|=0} g_\lambda
\]

where for each fixed generalized partition \( \lambda = (\cdots (-2)^{m_2}(-1)^{m_1}1^{m_1}2^{m_2} \cdots) \) with \( |\lambda| = 0 \), the operator \( g_\lambda \) stands for the component in \( \mathfrak{G}_k(\alpha) \) containing all the expressions of the form \( \prod_i (a_i(\alpha_{i,1}) \cdots a_i(\alpha_{i,m_i})) \).

Fix \( n \in \mathbb{Z} \) and \( \beta \in H^*(X) \). By Lemma 3.1 (iii) and the Lemma 5.1 in [LQW3],

\[
[\mathfrak{G}_k(\alpha), a_n(\beta)] = \sum_{\epsilon \in \{1_X,K,K^2\}} \sum_{\ell(\mu)=k+1-|\epsilon|/2,|\mu|=n} d(\epsilon,\mu) a_\mu(\tau_\alpha(\epsilon \alpha \beta))
\]

where all the coefficients \( d(\epsilon,\mu) \in \mathbb{Q} \) are independent of the surface \( X \) and the cohomology classes \( \alpha, \beta \in H^*(X) \). It follows that the generalized partitions \( \lambda \) in (1.1) must satisfy \( k \leq \ell(\lambda) \leq (k+2) \). Moreover, for a fixed generalized partition \( \lambda \) with \( |\lambda| = 0 \) and \( k \leq \ell(\lambda) \leq (k+2) \), and for every \( i \) with \( m_{-i} > 0 \), we have

\[
[g_\lambda, a_i(\beta)] = \begin{cases} 
   d(1_X,\lambda_{-i}) a_{\lambda_{-i}}(\tau_\alpha(\alpha \beta)), & \ell(\lambda) = k+2 \\
   d(K,\lambda_{-i}) a_{\lambda_{-i}}(\tau_\alpha(K \alpha \beta)), & \ell(\lambda) = k+1 \\
   a_{\lambda_{-i}}(\tau_\alpha((d(\epsilon,\lambda_{-i}) + d(K^2,\lambda_{-i}) K^2) \alpha \beta)), & \ell(\lambda) = k.
\end{cases}
\]
where $\lambda_{-i}$ is the generalized partition obtained from $\lambda$ by subtracting 1 from the multiplicity of $(-i)$. Note that $d(\epsilon, \lambda_{-i})$ is independent of $X$, $\alpha$ and $\beta$.

By choosing special surfaces $X$ and $\alpha, \beta \in H^*(X)$, we now claim that

$$
\frac{d(1_X, \lambda_{-i})}{im_{-i}}, \frac{d(K, \lambda_{-i})}{im_{-i}}, \frac{d(e, \lambda_{-i})}{im_{-i}} + \frac{d(K^2, \lambda_{-i})}{im_{-i}} K^2
$$

(4.13)

are independent of $i$ as long as $m_{-i} > 0$. Indeed, to prove that the 3rd item in (4.13) is independent of $i$ as long as $m_{-i} > 0$, we may assume that $g_\lambda \neq 0$ by (4.12). Note that $|\lambda| = 0$ and $\ell(\lambda) = k$. By (4.9), $\sum_{i,j} |\alpha_{i,j}| = 4k + |\alpha|$ for every Heisenberg monomials $\prod (a_i(\alpha_{i,j}) \cdots a_n(\alpha_{i,m}))$ contained in $g_\lambda$. On the other hand, $|\alpha_{i,j}| \leq 4$ and thus $\sum_{i,j} |\alpha_{i,j}| \leq 4\ell(\lambda) = 4k$. It follows that $|\alpha| = 0$ and $|\alpha_{i,j}| = 4$ for every $i$ and $j$. So $g_\lambda = b \cdot a_\lambda(\tau_s[x])$ where $b$ is a nonzero number, and $[x] \in H^4(X)$ is the cohomology class corresponding to a point $x \in X$. Since $\alpha \neq 0$ and $|\alpha| = 0$, $\alpha = \tilde{b} \cdot 1_X$ for some nonzero number $\tilde{b}$. Choosing $\beta = 1_X$, we see from (4.12) that

$$
m_{-i} \tilde{b} \cdot a_{\lambda_{-i}}(\tau_s[x]) = [g_\lambda, a_i(1_X)] = \tilde{b} \cdot a_{\lambda_{-i}}(d(e, \lambda_{-i})e + d(K^2, \lambda_{-i})K^2).
$$

So $b[x] = \tilde{b} \cdot (d(e, \lambda_{-i})e + d(K^2, \lambda_{-i})K^2)/(im_{-i})$. Thus $\frac{d(e, \lambda_{-i})}{im_{-i}} e + \frac{d(K^2, \lambda_{-i})}{im_{-i}} K^2$ is independent of $i$ as long as $m_{-i} > 0$. Similarly, by choosing $\alpha \in H^2(X)$ with $K\alpha \neq 0$ (resp. $\alpha \in H^4(X) - \{0\}$), we see that $d(K, \lambda_{-i})/(im_{-i})$ (resp. $d(1_X, \lambda_{-i})/(im_{-i})$) is independent of $i$ as long as $m_{-i} > 0$. This proves (4.13).

Now, for an arbitrary $X$ and $\alpha \in H^*(X)$, we claim that $G_k(\alpha)$ is equal to

$$
f \overset{\text{def}}{=} \sum_{\epsilon \in \{1_X, K, K^2, e\}} \sum_{\ell(\lambda) = k + 2 - |\epsilon|/2, |\lambda| = 0} d(\epsilon, \lambda_{-i}) a_\lambda(\tau_s(\epsilon\alpha)).
$$

Indeed, for every $n \in \mathbb{Z} - \{0\}$ and $\beta \in H^*(X)$, we have

$$
[f, a_n(\beta)] = \sum_{\epsilon \in \{1_X, K, K^2, e\}} \sum_{\ell(\lambda) = k + 2 - |\epsilon|/2, |\lambda| = 0} nm_{-n} \frac{d(\epsilon, \lambda_{-i})}{im_{-i}} \cdot a_{\lambda_{-n}}(\tau_s(\epsilon\alpha\beta))
$$

$$
= \sum_{\epsilon \in \{1_X, K, K^2, e\}} \sum_{\ell(\lambda) = k + 2 - |\epsilon|/2, |\lambda| = 0} d(\epsilon, \lambda_{-n}) \cdot a_{\lambda_{-n}}(\tau_s(\epsilon\alpha\beta))
$$

$$
= \left[ G_k(\alpha), a_n(\beta) \right]
$$

where we have used (4.13) in the second equality, and (4.10) and (4.12) in the last equality. Since $H_X$ is irreducible, we see from Schur’s lemma that $(G_k(\alpha) - f)$ must be a scalar multiple of the identity operator. Recall that the bidegree of $(G_k(\alpha) - f)$ is $(0, 2k + |\alpha|)$ which is nontrivial since $k > 0$. Therefore, $(G_k(\alpha) - f) = 0$. So

$$
G_k(\alpha) = f = \sum_{\epsilon \in \{1_X, K, K^2, e\}} \sum_{\ell(\lambda) = k + 2 - |\epsilon|/2, |\lambda| = 0} \frac{d(\epsilon, \lambda_{-i})}{im_{-i}} a_\lambda(\tau_s(\epsilon\alpha)).
$$

Finally, since the numbers $d(e, \lambda_{-i})/(im_{-i})$ are independent of $X$ and $\alpha \in H^*(X)$, by using Theorem 4.6 and an argument similar to that in the proof of Theorem 4.4, we conclude that $d(1_X, \lambda_{-i})/(im_{-i}) = -1/\lambda$ and $d(e, \lambda_{-i})/(im_{-i}) = -1/\lambda$. 
(s(λ) − 2)/(24λ^t). In particular, d(e, λ_{−i})/(im_{−i}) is independent of i whenever \( m_{−i} > 0 \). So by (4.13), \( d(K^2, \lambda_{−i})/(im_{−i}) \) is also independent of i whenever \( m_{−i} > 0 \). Now denoting \( d(e, \lambda_{−i})/(im_{−i}) \) by \( g_e(\lambda)/\lambda^t \) for \( e = K \) and \( K^2 \), we have proved the theorem.

\[ \square \]

**Corollary 4.8.** Let \( n \geq 1, \ k \geq 0, \) and \( \alpha \in H^*(X) \). Then, \( G_k(\alpha, n) \) is equal to

\[
\sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1} \frac{(-1)^{[\lambda]-1}}{\lambda^t \cdot |\lambda|!} \cdot 1_{-(n-j-1)} a_{-\lambda}(\tau_\ast \alpha)[0]
\]

\[ + \sum_{0 \leq j \leq k} \frac{(-1)^{|\lambda|}}{\lambda^t \cdot |\lambda|!} \cdot \frac{|\lambda| + s(\lambda) - 2}{24} \cdot 1_{-(n-j-1)} a_{-\lambda}(\tau_\ast (e\alpha))[0] \]

\[ + \sum_{\epsilon \in (K, K^2) \atop 0 \leq j \leq k} \frac{(-1)^{|\lambda|} g_e(\lambda + (1^{j+1}))}{\lambda^t \cdot |\lambda|!} \cdot 1_{-(n-j-1)} a_{-\lambda}(\tau_\ast (e\alpha))[0] \]

where \( 1_{-(n-j-1)} \) denotes \( a_{-1}(1_X)^{n-j-1}/(n-j-1)! \) when \( (n-j-1) \geq 0 \) and is 0 when \( (n-j-1) < 0 \), the universal function \( g_e \) is from Theorem 4.7, \( \lambda + (1^{j+1}) \) is the partition obtained from \( \lambda \) by adding \( (j + 1) \) to the multiplicity of 1.

**Proof.** Let \( 1_{X^{[n]}} \) be the fundamental class of the Hilbert scheme \( X^{[n]} \). Then, we have \( 1_{X^{[n]}} = \frac{1}{n!} \cdot a_{-1}(1_X)^n[0] \). From the definition of Chern character operators, we see that \( \Phi_k(\alpha)[0] = 0 \) and \( G_k(\alpha, n) = \Phi_k(\alpha)1_{X^{[n]}} \). Also, by Theorem 4.7 and Lemma 3.2 (i), we have \([...[\Phi_k(\alpha), a_{n_1}(\alpha_1)],...], a_{n_{k+2}}(\alpha_{k+2})]\) = 0 whenever \( n_1, ..., n_{k+2} < 0 \). Combining all of these observations, we obtain

\[ G_k(\alpha, n) = \frac{1}{n!} \cdot \Phi_k(\alpha) a_{-1}(1_X)^n[0] \]

\[ = \frac{1}{n!} \cdot \sum_{u=1}^{k+1} \binom{n}{u} a_{-1}(1_X)^{n-u} \cdot [\Phi_k(\alpha), a_{-1}(1_X), \cdots, a_{-1}(1_X)]^u \]

\[ = \sum_{j=0}^{k} \frac{1}{(j+1)!} \cdot 1_{-(n-j-1)}[\Phi_k(\alpha), a_{-1}(1_X), \cdots, a_{-1}(1_X)]^{j+1} \]

Now our corollary follows immediately from Theorem 4.7. \( \square \)

**Remark 4.9.** (i) Let \( X \) be a projective surface with numerically trivial canonical class. Recall from Sect. 3 that the classes \( G_k(\alpha, n) \), where \( 0 \leq k < n \) and \( \alpha \) runs over a linear basis of \( H^*(X) \), form a set of ring generators for the cohomology ring \( H^*(X^{[n]}) \). Rewrite the cup product of \( k \) ring generators as

\[ G_{i_1}(\alpha_1, n) \cdot G_{i_2}(\alpha_2, n) \cdot ... \cdot G_{i_k}(\alpha_k, n) \]

\[ = \Phi_{i_1}(\alpha_1) \Phi_{i_2}(\alpha_1) \cdots \Phi_{i_k}(\alpha_k) \cdot \frac{1}{n!} a_{-1}(1_X)^n[0]. \]
Applying Theorem 4.6 to $\mathfrak{S}_i(\alpha_j)$ and using Heisenberg algebra commutation relation, we can express (4.13) as a linear combination of Heisenberg monomials. Such a procedure is analogous to the one in the proof of Corollary 4.8. In this way, Theorem 4.6 provides us a complete description of the ring $H^*\left(X^{[n]}\right)$. There has also been another very different description of the ring $H^*\left(X^{[n]}\right)$ in [LS2].

(ii) In addition, when specialized to surfaces with numerically trivial canonical classes, our Corollary 4.8 establishes a (slightly corrected) conjectural formula in Remark 4.15, [LS2]. (A factor 2 was missing in the term involving the Euler class $e$ in the conjectural formula of Lehn-Sorger).

(iii) For a general projective surface $X$, Theorem 4.7 provides a partial description to the cohomology ring of the Hilbert scheme $X^{[n]}$.

Remark 4.10. Let $[x] \in H^4(X)$ be the cohomology class corresponding to a point $x \in X$. Assume $\sum_{i=1}^s (k_i + 2) = 2n$. By Theorem 3.1 (v), $[\mathfrak{S}_{k_i}([x]), a_{-1}([x])] = 0$ for all $1 \leq i \leq s$. Using Theorem 4.7 and Corollary 4.8 repeatedly, we see that the intersection number $\prod_{i=1}^s G_{k_i}([x], n) \in H^{4n}(X^{[n]}) \cong \mathbb{Q}$ is equal to

$$\sum_{0 \leq j_1 \leq k_1, \ldots, 0 \leq j_s \leq k_s} \prod_{i=1}^s \sum_{\lambda_j \vdash (j_1 + 1) + \cdots + (j_s + 1) = n} \frac{(-1)^{|\lambda_j| - 1}}{(\lambda_j)! |\lambda_j|!}.$$

Note that this number is independent of the projective surface $X$ (see [LQW3]).

5. Action of the $\mathcal{W}$ algebras on $\mathbb{H}_X$

In this section, we formulate the main theorems on connections between $\mathcal{W}$ algebras and Hilbert schemes. The proof of Theorem 5.3 which is very technical and uses the operator product expansion technique, will be postponed to Sect. 6.

First of all, we introduce the following definitions.

Definition 5.1. Let $X$ be a smooth projective surface.

(i) For $p \geq 0$, $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$, define $\mathfrak{W}_n(\alpha) \in \text{End}(\mathbb{H}_X)$ to be

$$p! \cdot \left( - \sum_{\ell(\lambda) = p+1, |\lambda| = n} \frac{1}{\lambda!} a_\lambda(\tau_\ast \alpha) + \sum_{\ell(\lambda) = p-1, |\lambda| = n} \frac{s(\lambda) + n^2 - 2}{24 \lambda!} a_\lambda(\tau_\ast (e_\alpha)) \right);$$

(ii) We define $\mathcal{W}_X$ to be the linear span of the identity operator $\text{Id}_{\mathbb{H}_X}$ and the operators $\mathfrak{W}_n(\alpha)$ in $\text{End}(\mathbb{H}_X)$, where $p \geq 0$, $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$.

Some of the operators $\mathfrak{W}_n(\alpha)$ can be identified with the familiar ones, by using the above definition, Theorem 4.6 and Theorem 4.2. For example, for $\alpha \in H^*(X)$, we see that $\mathfrak{W}_0(\alpha) = -a_n(\alpha)$ and $\mathfrak{W}_1(\alpha) = \mathfrak{L}_n(\alpha)$. For $\alpha \in \mathfrak{A}_X$, we obtain $\mathfrak{W}_0(\alpha) = p! \cdot \mathfrak{S}_{p-1}(\alpha)$ and $\mathfrak{W}_1(\alpha) = -a_{p-1}(\alpha)$. In general, we have the following.
Lemma 5.2. Given $p \geq 0$, $\alpha \in \mathfrak{A}_X$ and $\beta \in H^*(X)$, we have
\[
[\mathfrak{G}_p(\alpha), a_n(\beta)] = \frac{n!}{p!} \mathfrak{J}^p_n(\alpha \beta).
\]
Proof. Follows from Definition 5.1 (i), Theorem 4.6, and Lemma 3.2 (i).

By Lemma 5.2, we may regard the linear operators $\mathfrak{J}^p_n(\alpha) \in \text{End}(\mathbb{H}_X)$ being geometric since both the Chern character operators $\mathfrak{G}_p(\alpha)$ and the Heisenberg operators $a_n(\beta)$ are geometric by the constructions in Sect. 3.

We are interested in the commutation relation among the operators $\mathfrak{J}^p_n(\alpha)$. To this end, it is useful to adopt the vertex algebra language, cf. [Bo], [FL], [Kac], to our present setup. Our convention of fields is to write them in a form
\[
\phi(z) = \sum_n \phi_n z^{-n-\Delta}
\]
where $\Delta$ is the conformal weight of the field $\phi(z)$. Define
\[
\partial \phi(z) = \sum_n (-n - \Delta) \phi_n z^{-n-\Delta-1}
\]
which is called the derivative field of $\phi(z)$. We set $\phi_-(z) = \sum_{n \geq 0} \phi_n z^{-n-\Delta}$ and $\phi_+(z) = \sum_{n < 0} \phi_n z^{-n-\Delta}$. If $\psi(z)$ is another vertex operator, we define a new vertex operator, which is called the normally ordered product of $\phi(z)$ and $\psi(z)$, to be:
\[
: \phi(z) \psi(z) : = \phi_+(z) \psi(z) + (-1)^{\phi \psi} \psi(z) \phi_-(z)
\]
where $(-1)^{\phi \psi}$ is $-1$ if both $\phi(z)$ and $\psi(z)$ are odd (i.e. fermionic) fields and $1$ otherwise. Inductively we can define the normally ordered product of $k$ vertex operators $\phi_1(z), \phi_2(z), \ldots, \phi_k(z)$ from right to left by
\[
: \phi_1(z) \phi_2(z) \cdots \phi_k(z) : = : \phi_1(z) : : \phi_2(z) \cdots : : \phi_k(z) : : .
\]

For $\alpha \in H^*(X)$, we define a vertex operator (i.e. a field) $a(\alpha)(z)$ by putting
\[
a(\alpha)(z) = \sum_{n \in \mathbb{Z}} a_n(\alpha) z^{-n-1}.
\]
The field $a(z)^p : (\tau_\alpha) \in H^*(X)$ is defined to be $\sum_i a(\alpha_{i,1})(z) a(\alpha_{i,2})(z) \cdots a(\alpha_{i,p})(z)$ if we write $\tau_p(\alpha) = \sum_i \alpha_{i,1} \otimes \alpha_{i,2} \otimes \cdots \otimes \alpha_{i,p} \in H^*(X)^{\otimes p}$. Note that $a(z)^p : (\tau_\alpha)$ is of conformal weight $p$. So we rewrite $a(z)^p : (\tau_\alpha)$ componentwise as
\[
a(z)^p : (\tau_\alpha) = \sum_{m} a^p :_m (\tau_\alpha) z^{-m-p},
\]
where $a^p :_m (\tau_\alpha) \in \text{End}(\mathbb{H}_X)$ is the coefficient of $z^{-m-p}$ (i.e. the $m$-th Fourier component of the field $a(z)^p : (\tau_\alpha)$), and maps $H^*(X^{[m]})$ to $H^*(X^{[n+m]})$. Similarly, for $r \geq 1$, we can define the field $(\partial^r a(z)) a(z)^{p-1} : (\tau_\alpha)$, and define the operator $(\partial^r a) a^{p-1} :_m (\tau_\alpha)$ as the coefficient of $z^{-m-r-p}$ in $(\partial^r a(z)) a(z)^{p-1} : (\tau_\alpha)$.

We remark that when the variable $z$ or the cohomology class $\alpha$ is clear or irrelevant in the context, we shall drop $z$ or $\alpha$ from the notations of fields. For instance, we shall sometimes use $a^p : (\tau_\alpha)$ or $a^p : : (\partial^r a) a^p : (\tau_\alpha)$ or $(\partial^r a)^p : (\tau_\alpha)$, etc to stand for $a(z)^p : (\tau_\alpha)$, $(\partial^r a(z)) a(z)^p : (\tau_\alpha)$, etc respectively.
Lemma 5.3. In terms of fields, the operator $\mathfrak{F}_m^p(\alpha)$ can be rewritten as:

$$-\frac{1}{(p+1)} : a^{p+1} :_m(\tau_* \alpha) + \frac{1}{24} p(m^2 - 3m - 2p) : a^{p-1} :_m(\tau_*(\epsilon \alpha)) + \frac{1}{24} p(p-1) : (\partial^2 a) : a^{p-2} :_m(\tau_*(\epsilon \alpha)).$$

(5.1)

Proof. First of all, by the definition of $: a^{p+1} :_m(\tau_* \alpha)$, we have

$$p! \sum_{\ell(\lambda) = p+1, |\lambda| = m} \frac{1}{\lambda!} a_\lambda(\tau_*(\epsilon \alpha)) = \frac{1}{(p+1)} \sum_{i_1 + \ldots + i_{p+1} = m} : a_{i_1} \ldots a_{i_{p+1}} : (\tau_*(\epsilon \alpha)) = \frac{1}{(p+1)} : a^{p+1} :_m(\tau_* \alpha).$$

Next, recalling the definition of $s(\lambda)$, we obtain

$$p! \sum_{\ell(\lambda) = p+1, |\lambda| = m} \frac{s(\lambda) + m^2 - 2}{\lambda!} a_\lambda(\tau_*(\epsilon \alpha))$$

$$= p \sum_{i_1 + \ldots + i_{p-1} = m} \left( \sum_{b=1}^{p-1} i_b^2 + m^2 - 2 \right) : a_{i_1} \ldots a_{i_{p-1}} : (\tau_*(\epsilon \alpha))$$

$$= p \sum_{i_1 + \ldots + i_{p-1} = m} \left( \sum_{b=1}^{p-1} (i_b + 1)(i_b + 2) + m^2 - 3m - 2p \right) : a_{i_1} \ldots a_{i_{p-1}} : (\tau_*(\epsilon \alpha))$$

$$= p(p-1) : (\partial^2 a) : a^{p-2} :_m(\tau_*(\epsilon \alpha)) + p(m^2 - 3m - 2p) : a^{p-1} :_m(\tau_*(\epsilon \alpha)).$$

Now the lemma follows from the definition of the operator $\mathfrak{F}_m^p(\alpha)$. \qed

Remark 5.4. More explicitly, we can rewrite the operator $\mathfrak{F}_m^p(\alpha)$ as the $m$-th Fourier component of a vertex operator as follows:

$$\mathfrak{F}_m^p(\alpha) = -\frac{1}{(p+1)} : a^{p+1} :_m(\tau_* \alpha) + \frac{p}{24} (\partial^2 : a^{p-1} :)_m(\tau_*(\epsilon \alpha))$$

$$+ \frac{(p+1)p}{12} (\partial : a^{p-1} :)_m(\tau_*(\epsilon \alpha)) + \frac{p(p-2)}{24} : a^{p-1} :_m(\tau_*(\epsilon \alpha))$$

$$+ \frac{1}{24} p(p-1) : (\partial^2 a) : a^{p-2} :_m(\tau_*(\epsilon \alpha)).$$

In practice, (5.1) suffices for the purpose of computations in the next section. Also, since $\mathfrak{F}_0^{p+1}(\alpha) = (p+1)! \cdot \mathfrak{G}_p(\alpha)$ for $\alpha \in \Xi_X$, we conclude that the Chern character operator $\mathfrak{G}_p(\alpha)$ with $\alpha \in \Xi_X$ is the zero-mode of a vertex operator.

One of the main results in this paper is about the commutator $[\mathfrak{F}_m^p(\alpha), \mathfrak{F}_n^q(\beta)]$. To state the formula, we define an integer $\Omega_{m,n}^{p,q}$ for $m, n, p, q \in \mathbb{Z}$ as follows:

$$\Omega_{m,n}^{p,q} = mp^3 n + 3mp^2 n^2 q - p^2 nq + p^2 qn^3 - 3mp^2 n^2 + pnq$$

$$+ 3mpq^n - 3mp^2 q - m^3 q^2 p - pqn^3 - mpq + m^3 pq$$

$$+ mpq^2 + 2mpn^2 - 3mpq^n - 2m^2 nq + 3m^2 nq^2 - m^2 nq^3. \quad (5.2)$$
Theorem 5.5. The vector space $\mathcal{W}_X$ is closed under the Lie bracket. More explicitly, for $m, n \in \mathbb{Z}$, and $\alpha, \beta \in H^*(X)$, we have

$$[\mathfrak{J}^p_m(\alpha), \mathfrak{J}^q_n(\beta)] = (qm - pn) \cdot \mathfrak{J}^{p+q-1}_{m+n}(\alpha \beta) - \frac{\Omega^{p,q}_{m,n}}{12} \cdot \mathfrak{J}^{p+q-3}_{m+n}(e\alpha \beta)$$

where $(p, q) \in \mathbb{Z}_+^2$ except for the unordered pairs $(0, 0), (1, 0), (2, 0)$ and $(1, 1)$. In addition, for these four exceptional cases, we have

$$[\mathfrak{J}^0_m(\alpha), \mathfrak{J}^0_n(\beta)] = -m \delta_{m,-n} \int_X (\alpha \beta) \cdot \text{Id}_{\mathbb{A}_X},$$

$$[\mathfrak{J}^1_m(\alpha), \mathfrak{J}^0_n(\beta)] = -n \cdot \mathfrak{J}^0_{m+n}(\alpha \beta),$$

$$[\mathfrak{J}^2_m(\alpha), \mathfrak{J}^0_n(\beta)] = -2n \cdot \mathfrak{J}^1_{m+n}(\alpha \beta) + \frac{m^3 - m}{6} \delta_{m,-n} \int_X (e\alpha \beta) \cdot \text{Id}_{\mathbb{A}_X},$$

$$[\mathfrak{J}^3_m(\alpha), \mathfrak{J}^0_n(\beta)] = (m - n) \cdot \mathfrak{J}^1_{m+n}(\alpha \beta) + \frac{m^3 - m}{12} \delta_{m,-n} \int_X (e\alpha \beta) \cdot \text{Id}_{\mathbb{A}_X}.$$

The proof of this theorem, which uses the operator product expansion (OPE) method in the theory of vertex algebras with some appropriate modifications, will be given in Sect. 3. Note that our algebra $\mathcal{W}_X$ contains as subalgebras the Heisenberg algebra of Nakajima-Grojnowski generated by the operators $\mathfrak{J}^0_m(\alpha) = -a_m(\alpha)$ and the Virasoro algebra of Lehn generated by the operators $\mathfrak{J}^1_m(\alpha) = \mathfrak{L}_m(\alpha)$.

Remark 5.6. Since $\mathfrak{d} = \mathfrak{g}_1(1) = \frac{1}{2} \mathfrak{g}^0_1(1)$ modulo the $K$-term (cf. Theorem 1.7, also cf. LQW 1, FW), a special case of Theorem 5.5 reads that

$$(\mathfrak{J}^p_n(\alpha))' = \frac{1}{2} [\mathfrak{J}^0_1(1), \mathfrak{J}^p_n(\alpha)] = -n \mathfrak{J}^{p+1}_n(\alpha) - \frac{(n^2 - n)p}{12} \mathfrak{J}^{p-1}_n(e\alpha)$$

for $p \geq 1$ and $\alpha \in \mathfrak{A}_X$. Note that setting $d = n^2 - 2$ in Remark 4.3 provides a totally different way of proving the same formula.

We denote by $\mathfrak{B}_X = \{\alpha \in H^*(X) | e\alpha = K\alpha = 0\}$ which is an ideal in the ring $H^*(X)$. Obviously, $\mathfrak{B}_X \subset \mathfrak{A}_X$. Denote by $\mathcal{W}^{\mathfrak{B}}_X$ the linear span of $\mathfrak{J}^p_n(\alpha)$, where $p \geq 0, n \in \mathbb{Z}$ and $\alpha \in \mathfrak{B}_X$. From the commutation relation in Theorem 5.5 we observe that $\mathcal{W}^{\mathfrak{B}}_X$ is a Lie subalgebra of the Lie (super)algebra $\mathcal{W}_X$. Recall the (super)algebra $\hat{\mathcal{W}}(A)$ introduced in Section 3 for a general ring $A$. The next theorem follows from comparing the commutator (2.2) and the ones given in Theorem 5.5.

Theorem 5.7. Let $X$ be a smooth projective surface. Then, the Lie (super)algebra $\hat{\mathcal{W}}(\mathfrak{B}_X)$ is isomorphic to the Lie (super)algebra $\mathcal{W}^{\mathfrak{B}}_X$ by sending $C \mapsto \text{Id}_{\mathbb{A}_X}$ and $\mathfrak{J}^p_n(\alpha) \mapsto \mathfrak{J}^p_n(\alpha)$, where $p \geq 0, n \in \mathbb{Z}$, and $\alpha \in \mathfrak{B}_X$.

Remark 5.8. (i) For a smooth projective surface $X$ with numerically trivial canonical class $K$ and trivial Euler class $e$, the ideal $\mathfrak{B}_X$ is the entire cohomology ring $H^*(X)$, and thus the algebra $\mathcal{W}^{\mathfrak{B}}_X$ coincides with $\mathcal{W}_X$. In addition, Theorem 5.7 implies that the field $\mathfrak{J}^p(\alpha)(z) = \sum_{n \in \mathbb{Z}} \mathfrak{J}^p_n(\alpha)z^{-n-p-1}$ is a primary field of conformal weight $(p + 1)$ with respect to the Virasoro field.
(ii) For a general projective surface \( X \), we note that the leading term in the commutators of the algebra \( \mathcal{W}_X \) given in Theorem 5.5 is precisely the \( \mathcal{W} \) algebra introduced in Sect. 2 associated to the ring \( H^*(X) \). Therefore, we are justified to regard the \( \mathcal{W} \) algebras \( \mathcal{W}_X \) and \( \mathcal{W}^\infty_\infty \) in general as certain topological deformation of the \( \mathcal{W}_1+\infty \) algebra in the framework of Hilbert schemes.

6. Proof of Theorem 5.5

6.1. Outline of the proof

We denote the three terms in (6.1) by \( \mathfrak{J}^p_m(\alpha) \), \( \mathfrak{J}^q_m(\alpha) \) and \( \mathfrak{J}^q_n(\alpha) \) respectively, that is, \( \mathfrak{J}^p_m(\alpha) = 1 \mathfrak{J}^p_m(\alpha) + 2 \mathfrak{J}^p_m(\alpha) + 3 \mathfrak{J}^p_m(\alpha) \). Note that \( [^1 \mathfrak{J}^p_m(\alpha), ^2 \mathfrak{J}^q_n(\beta)] = 0 \) \( (i, j = 2, 3) \) by the transfer property and \( e^2 = 0 \). Thus,

\[
\begin{align*}
[\mathfrak{J}^p_m(\alpha), \mathfrak{J}^q_n(\beta)] &= [^1 \mathfrak{J}^p_m(\alpha), ^1 \mathfrak{J}^q_n(\beta)] + [^1 \mathfrak{J}^p_m(\alpha), ^2 \mathfrak{J}^q_n(\beta)] + [^1 \mathfrak{J}^p_m(\alpha), ^3 \mathfrak{J}^q_n(\beta)] \\
&\quad + [^2 \mathfrak{J}^p_m(\alpha), ^1 \mathfrak{J}^q_n(\beta)] + [^3 \mathfrak{J}^p_m(\alpha), ^1 \mathfrak{J}^q_n(\beta)].
\end{align*}
\]

(6.1)

Recall that the commutation relations of Heisenberg generators can be recast equivalently in terms of the operator product expansion (OPE) as (cf. [Kac]; [FB], section 3.3):

\[
a(\alpha)(z) a(\beta)(w) \sim \frac{-\int_X (\alpha\beta)}{(z-w)^2}.
\]

Here and below \( \sim \) means that the regular terms with respect to \( (z-w) \) on the right hand side of the OPEs are omitted; it is well known the meromorphic terms carry all the information about the corresponding commutation relations.

We derive from (6.2) the following two OPEs:

\[
a(\alpha)(z) \partial a(\beta)(w) \sim \frac{-2\int_X (\alpha\beta)}{(z-w)^3}, \quad a(\alpha)(z) \partial^2 a(\beta)(w) \sim \frac{-6\int_X (\alpha\beta)}{(z-w)^4}.
\]

(6.3)

Our goal in this section is to compute the commutator \( [\mathfrak{J}^p_m(\alpha), \mathfrak{J}^q_n(\beta)] \) by using the OPE technique. To this end, we need to compute the following OPEs

\[
(\alpha ; a(z)^{p+1} : (\tau_\alpha);) (\alpha \beta), \quad (6.4)
\]

\[
(\alpha ; a(z)^{p+1} : (\tau_\alpha);) (\beta), \quad (6.5)
\]

\[
(\alpha ; a(z)^{p+1} : (\tau_\alpha);) (\partial^2 a(w)a(z)^{q-2} : (\tau_\beta)), \quad (6.6)
\]

which correspond to the first three commutators on the right-hand-side of (6.1) respectively. Then we will be able to rewrite the terms appearing in these OPEs as combinations of (the derivative fields of) the four fields \( a(z)^{p+q} : (\tau_\alpha \beta), \) \( : a(z)^{p+q-2} : (\tau_\alpha (e\beta)), \) \( : \partial^2 (a(z))a(z)^{p+q-3} : (\tau_\alpha (e\beta)), \) and \( : \partial^3 (a(z))a(z)^{p+q-3} : (\tau_\alpha (e\beta)), \) by using the following simple lemma.

Lemma 6.1. As in the previous section, write the fields \( a^r(z), \partial^r a(z), \) etc simply as \( a^r, \partial^r a, \) etc respectively. Then for \( N \geq 0 \), we have

(i) \( \partial^2 : a^N : = N : (\partial^2 a)a^{N-1} : + N(N-1) : (\partial a)^2a^{N-2} : ; \)

(ii) \( \partial^3 : a^N : = N : (\partial^3 a)a^{N-1} : + 6N^2 : (\partial a)(\partial^2 a)a^{N-2} : + 6N(\partial a)^3a^{N-3} : ; \)

\( N \geq 0 \)
The contraction of more than two pairs of fields will be zero thanks to pairs of free fields will give rise to a term depending on $e^{\alpha \beta}$ incorporating the transfer property, Lemma 3.2 (i). Next, the contraction of two formula in the vertex algebra literature (cf. [Kac]; [FB], Lemma 11.2.6, pp. 193).

Proof. Using the Leibnitz rule, we obtain

$$\partial^2 : a^N := \partial (N : (\partial a)^{-1} : ) N : (\partial^2 a) a^{-1} : + N : (\partial a)^3 a^{-1} : + 6 \left(\frac{N}{3}\right) : (\partial a)^3 a^{-3} : .$$

This proves part (i). The proofs of the remaining formulas are similar. \qed

We will calculate various OPEs by using the Wick formula. Note that the Wick formula in the vertex algebra literature (cf. [Kac]; [FB], Lemma 11.2.6, pp. 193) has to be modified in order to apply to our setup. First of all, we naturally incorporate the transfer property, Lemma 3.2 (i). Next, the contraction of two pairs of free fields will give rise to a term depending on $e^{\alpha \beta}$ (cf. Lemma 3.2 (iii)). So the contraction of more than two pairs of fields will be zero thanks to $e^2 = 0$ and the transfer property. We refer to [FB], pp. 193, for the terminology “contraction” used above.

Now we assume $p + q > 4$ in subsections 6.2, 6.3, 6.4 and 6.5.

6.2. Calculation of the OPE (6.4).

We calculate the OPE (6.4) by using the Wick theorem and (6.2) as follows:

$$(a(z)^{p+1} : (\tau \alpha)) (a(w)^{q+1} : (\tau \beta))$$

$$\sim \frac{1}{(z-w)^2}$$

which by the Taylor expansion at $(z-w)$ is

$$\sim -(p+1)(q+1) : a(w)^{p+q} : (\tau \alpha \beta) \frac{1}{(z-w)^2}$$

$$- \frac{(p+1)p^2}{p+q} \partial : a(w)^{p+q} : (\tau \alpha \beta) \frac{1}{z-w}$$

$$+ 2 \left(\frac{p+1}{2}\right) \left(\frac{q+1}{2}\right) : a(w)^{p+q+2} : (\tau \alpha \beta) \frac{1}{(z-w)^4}$$

$$+ \frac{(p+1)p(p-1)(q+1)q}{2(p+q-2)} \partial : a(w)^{p+q+2} : (\tau \alpha \beta) \frac{1}{(z-w)^4}$$

$$+ \left(\frac{p+1}{3}\right) \left(\frac{q+1}{2}\right) \left[\partial : a(w) a(w)^{p+q+2} : a(w)^{q-1} : (\tau \alpha \beta) \frac{1}{(z-w)^2}\right]$$

$$+ : \partial^2 : a(w) a(w)^{p+q+2} : a(w)^{q-1} : (\tau \alpha \beta) \frac{1}{z-w} \right].$$

(6.7)
Applying Lemma 6.1 to the two terms in the square brackets in (6.7), we obtain

\[
\frac{\partial}{\partial a(w)} \left( (\tau_*(e\alpha\beta)) \right)
\]

\[
= \frac{p - 2}{(p + q - 2)(p + q - 3)} \partial^2 (\tau_*(e\alpha\beta))
\]

\[
+ \frac{q - 1}{p + q - 3} \partial^2 a(w) a(w)^{p+q-3} : (\tau_*(e\alpha\beta))
\]

(6.8)

and

\[
\frac{\partial^2}{\partial a(w)} \left( a(w)^{p-2} : (\tau_*(e\alpha\beta)) \right)
\]

\[
= \frac{(p - 2)(p - 3)}{(p + q - 2)(p + q - 3)(p + q - 4)} \partial^3 (\tau_*(e\alpha\beta))
\]

\[
+ \frac{3(p - 2)(q - 1)}{(p + q - 3)(p + q - 4)} \partial (\tau_*(e\alpha\beta))
\]

\[
- \frac{(p - q)(q - 1)}{(p + q - 3)(p + q - 4)} \partial^3 a(w) a(w)^{p+q-3} : (\tau_*(e\alpha\beta)).
\]

(6.9)

6.3. Calculation of the OPE (6.5).

We calculate the OPE (6.5) by using the Wick theorem, (6.2) and Lemma 6.1:

\[
(\tau_*(e\alpha\beta))
\]

\[
\sim (p + 1)(q - 1) : a(z)^p a(w)^q : (\tau_*(e\alpha\beta)) \frac{1}{(z - w)^2}
\]

which by the Taylor expansion at \((z - w)\) is

\[
\sim -(p + 1)(q - 1) : a(w)^{p+q-2} : (\tau_*(e\alpha\beta)) \frac{1}{(z - w)^2}
\]

\[
- \frac{(p + 1)(q - 1)}{p + q - 2} \partial (\tau_*(e\alpha\beta)) \frac{1}{z - w}.
\]

(6.10)

6.4. Calculation of the OPE (6.6).

We calculate the OPE (6.6) by using the Wick theorem, (6.2) and (6.3) as follows:

\[
(\tau_*(e\alpha\beta))
\]

\[
\sim (p + 1)(q - 2) : a(z)^p \partial^2 a(w) a(w)^{q-2} : (\tau_*(e\alpha\beta)) \frac{1}{(z - w)^2}
\]

\[
+ (p + 1) : a(z)^p a(w)^{q-2} : \frac{-6}{(z - w)^4}
\]
which by the Taylor expansion at \((z - w)\) is

\[
\sim - (p + 1)(q - 2) : \partial^2 a(w) a(w)^{p+q-3} : (\tau_* (e\alpha \beta)) \frac{1}{(z - w)^2} \\
- (p + 1)p(q - 2) : \partial a(w) \partial^2 a(w) a(w)^{p+q-4} : (\tau_* (e\alpha \beta)) \frac{1}{(z - w)} \\
- 6(p + 1) : a(w)^{p+q-2} : (\tau_* (e\alpha \beta)) \frac{1}{(z - w)^4} \\
- 6(p + 1)p : \partial a(w) a(w)^{p+q-3} : (\tau_* (e\alpha \beta)) \frac{1}{(z - w)^3} \\
- 3(p + 1)p : \partial \left( \partial a(w) a(w)^{p-1} \right) a(w)^{q-2} : (\tau_* (e\alpha \beta)) \frac{1}{(z - w)^2} \\
- (p + 1)p : \partial^2 \left( \partial a(w) a(w)^{p-1} \right) a(w)^{q-2} : (\tau_* (e\alpha \beta)) \frac{1}{z - w}. \tag{6.11}
\]

We denote by \(B_i \ (i = 1, 2)\) the coefficients of \(\frac{1}{(z - w)^i}\) on the right-hand-side of the above equation \(\text{(6.11)}\). We then rewrite them by using Lemma \(\text{[6.1]}\) as follows:

\[
B_2 = -(p + 1)(3p + q - 2) : \partial^2 a(w) a(w)^{p+q-3} : (\tau_* (e\alpha \beta)) \\
- 3(p + 1)p(p - 1) : (\partial a(w))^2 a(w)^{p+q-4} : (\tau_* (e\alpha \beta)) \\
= - \frac{3(p + 1)p(p - 1)}{(p + q - 2)(p + q - 3)} \partial^2 \left( a(w)^{p+q-2} : \right) (\tau_* (e\alpha \beta)) \\
- \frac{(p + 1)(4p + q - 3)(q - 2)}{(p + q - 3)} : \partial^2 a(w) a(w)^{p+q-3} : (\tau_* (e\alpha \beta)). \tag{6.12}
\]

and

\[
B_1 = -(p + 1)p \left[ (3p + q - 5) : \partial a(w) \partial^2 a(w) a(w)^{p+q-4} : (\tau_* (e\alpha \beta)) \\
+ (p - 1)(p - 2) : (\partial a(w))^3 a(w)^{p+q-5} : (\tau_* (e\alpha \beta)) \right] \\
= -(p + 1)p \cdot \left[ \frac{(p - 1)(p - 2)}{\prod_{i=2}^4 (p + q - i)} \partial^3 \left( a(w)^{p+q-2} : \right) (\tau_* (e\alpha \beta)) \\
+ \frac{(q - 2)(4p + q - 7)}{\prod_{i=3}^4 (p + q - i)} \partial \left( \partial^2 a(w) a(w)^{p+q-3} : \right) (\tau_* (e\alpha \beta)) \\
- \frac{2(p - 1)(p - 2)}{\prod_{i=3}^4 (p + q - i)} : \partial^3 a(w) a(w)^{p+q-3} : (\tau_* (e\alpha \beta)) \right]. \tag{6.13}
\]

6.5. **Calculation of the commutator** \([\mathfrak{H}_m^p(\alpha), \mathfrak{H}_n^q(\beta)]\) **when** \(p + q > 4\).
It follows from (5.1) and Lemma 5.3 that the commutator $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$ is given by the coefficient of $z^{-m-p-1}w^{-n-q-1}$ of the following OPE:

$$\frac{1}{(p+1)(q+1)}(: a(z)^{p+1} : (\tau_s \alpha) (: a(w)^{q+1} : (\tau_s \beta))$$

$$+ \frac{q}{24(p+1)}(2q + 3n - n^2) (: a(z)^{p+1} : (\tau_s \alpha) (: a(w)^{q-1} : (\tau_s (e\alpha))) w^{-2}$$

$$- \frac{q(q-1)}{24(p+1)} (: a(z)^{p+1} : (\tau_s \alpha) (: \partial^2 a(w)a(w)^{q-2} : (\tau_s (e\beta)))$$

$$+ \frac{p}{24(q+1)}(2p + 3m - m^2) (: a(z)^{p-1} : (\tau_s (e\alpha)) (: a(w)^{q+1} : (\tau_s \beta)) z^{-2}$$

$$- \frac{p(p-1)}{24(q+1)} (: \partial^2 a(z)a(z)^{p-2} : (\tau_s (e\alpha)) (: a(w)^{q+1} : (\tau_s \beta)).$$

Note that the contributions of the last two OPEs in the above can be obtained (up to a sign) from the preceding two OPEs by switching $p, m, z$ with $q, n, w$ simultaneously. The first three OPEs have been computed in subsections 6.2, 6.3, respectively. So the coefficient of $: (\tau_s \alpha) ($ is $0$. So : $(\tau_s (e\alpha))$ does not appear in $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$. 

We observe from the earlier computations of OPEs that there will be four possible terms in the commutator $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$, that is, $: a^{p+q} :_{m+n} (\tau_s (e\beta))$, $: a^{p+q-2} :_{m+n} (\tau_s (e\alpha \beta))$, $(\partial^2 a) a^{p+q-3} :_{m+n} (\tau_s (e\alpha \beta))$, and $(\partial^3 a) a^{p+q-3} :_{m+n} (\tau_s (e\alpha \beta))$. We can now determine the coefficients of these terms one by one from the computations of OPEs (cf. (5.1), (5.8), (5.9), (5.10), (5.11), (5.12), (5.13)).

The coefficient of $: a^{p+q} :_{m+n} (\tau_s (e\alpha \beta))$ in the commutator $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$ is:

$$\frac{1}{(p+1)(q+1)} \left[ -(p+1)(q+1)(m+p) - \frac{(p+1)p(q+1)}{p+q}(-m-n-p-q) \right]$$

where the two terms in the brackets come from the first two terms in (6.7) respectively. So the coefficient of $: a^{p+q} :_{m+n} (\tau_s (e\alpha \beta))$ in $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$ equals

$$\frac{pn - qm}{(p+q)}.$$  \hspace{1cm} (6.14)

Similarly, the coefficient of $(\partial^2 a) a^{p+q-3} :_{m+n} (\tau_s (e\alpha \beta))$ in $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$ is

$$- \frac{1}{(p+1)(q+1)}(p+1)p(p-1)(q+1)q \frac{(p-q)(q-1)}{(p+q-3)(p+q-4)}$$

$$- \frac{q(q-1)}{24(p+1)}\frac{2(p-1)(q-2)}{(p+q-3)(p+q-4)}$$

$$+ \frac{p(p-1)}{24(q+1)}\frac{2(q-1)(p-2)}{(p+q-3)(p+q-4)}$$

which is $0$. So $(\partial^3 a) a^{p+q-3} :_{m+n} (\tau_s (e\alpha \beta))$ does not appear in $[\mathfrak{Y}_m^p(\alpha), \mathfrak{Y}_n^q(\beta)]$. 

The coefficient of $: a^{p+q} :_{m+n} (\tau_s (e\alpha \beta))$ is $0$.
Finally, similar lengthy computations together with MAPLE show that the coefficient of : \((\partial^2 a) a^{p+q-3} \cdot_{m+n} (\tau_*(e\alpha\beta)) \) in \([\mathfrak{J}_m^p(\alpha), \mathfrak{J}_n^q(\beta)]\) is
\[
\frac{(pm - qm)(p + q - 1)(p + q - 2)}{24},
\]
and the coefficient of : \(a^{p+q-2} \cdot_{m+n} (\tau_*(e\alpha\beta)) \) in \([\mathfrak{J}_m^p(\alpha), \mathfrak{J}_n^q(\beta)]\) is equal to
\[
\Theta_{m,n}^{p,q} + \frac{\Omega_{m,n}^{p,q}}{12(p + q - 2)}. \tag{6.16}
\]
In the above, \(\Omega_{m,n}^{p,q}\) is given in (5.2), and \(\Theta_{m,n}^{p,q}\) is defined to be
\[
\frac{(pn - qm)(p + q - 1)}{24} (2p + 2q - 2 + 3m + 3n - (m + n)^2).
\]

Collecting all the above computations together (Eqs. (6.14), (6.15), and (6.16)) and using Definition 5.1 (i), we have established Theorem 5.5 for \(p + q > 4\).

6.6. Completion of the proof of Theorem 5.5.

For the cases of \((p, q)\) with \(p + q \leq 4\), we can proceed either directly or using OPEs as before while keeping in mind that many of the formulas can be simplified. That is, some of the equations such as (6.8), (6.9), (6.12), (6.13) are no longer needed, and in fact they do not make sense as some of the denominators would be zero. This is the reason why we need to treat these cases separately. When the unordered pair \((p, q)\) satisfies \(3 \leq p + q \leq 4\), we have checked that the commutators are given by the same formula as the one for \(p + q > 4\). Finally, when \(0 \leq p + q \leq 2\), the commutators are found to be those stated in the Theorem 5.5 (in fact, the formulas for the unordered pairs \((0, 0), (1, 0), (1, 1)\) are precisely Theorem 3.1 (i), (ii), (iv) respectively). We omit the tedious details here.

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**Department of Mathematics, HKUST, Clear Water Bay, Kowloon, Hong Kong**

*E-mail address:* mawpli@ust.hk

**Department of Mathematics, University of Missouri, Columbia, MO 65211, USA**

*E-mail address:* zq@math.missouri.edu

**Department of Mathematics, University of Virginia, Charlottesville, VA 22904**

*E-mail address:* ww9c@virginia.edu