A wave-vortex decomposition for rotating Boussinesq flows in bounded domains

Jeffrey J. Early, Northwest Research Associates, USA
M. Pascale Lelong, Northwest Research Associates, USA
Miles A. Sundermeyer, University of Massachusetts Dartmouth, USA

Abstract

A rotating Boussinesq model bounded in the vertical and periodic in the horizontal is shown to be exactly separable into its linear wave and vortex solutions using the horizontal velocities and density anomaly at any given moment in time. The decomposition extends the familiar wave-vortex decomposition for triply periodic, constant stratification cases to vertically bounded flows with arbitrary stratification.

This work has not yet been peer-reviewed and is provided by the contributing author(s) as a means to ensure timely dissemination of scholarly and technical work on a noncommercial basis. Copyright and all rights therein are maintained by the author(s) or by other copyright owners. It is understood that all persons copying this information will adhere to the terms and constraints invoked by each author’s copyright. This work may not be reposted without explicit permission of the copyright owner.

1 Introduction

A typical approach to understanding and interpreting complex nonlinear fluid flows is to project observational or numerical data onto relevant linear solutions. Linear decomposition into inertia-gravity wave and geostrophic (vortex) solutions has been used to interpret ocean data (Lien & Müller, 1992; Bühler et al., 2014; Lien & Sanford, 2019) and analyze numerical simulations of Boussinesq flows in triply periodic domains with constant stratification (Bartello, 1995; Smith & Waleffe, 2002; Waite & Bartello, 2006; Hernandez-Duenas et al., 2014). Such a decomposition splits the flow into two inertia-gravity wave solutions characterized by nonzero horizontal divergence, with frequencies that lie between the Coriolis, $f_0$, and the buoyancy frequency, $N$; and a zero-frequency, divergent-free geostrophic solution, which accounts for all linear potential vorticity (PV).

The wave-vortex decomposition is a linear transformation that projects the variables $(u, v, \rho)$ onto an equivalent representation $(A_+, A_-, A_0)$ of two wave and one vortex mode without loss of information. Because the transformation uses vertical eigenmodes that guarantee the continuity equation is satisfied, the vertical velocity, $w$, is redundant and not needed in the transformation. The inverse transformation can recover both $w$ and pressure from the wave-vortex components $(A_+, A_-, A_0)$. Aside from being an alternative and compact representation of the dynamical variables, the wave-vortex decomposition has a number of applications.

Applying the wave-vortex decomposition to the time evolution of a nonlinear flow allows for a precise measure of wave nonlinearity. Because linear wave and vortex modes by definition have constant amplitude and phase, any variation in either indicates nonlinearity. A perfectly linear wave model will show no changes in amplitude and phase, while a non-linear wavelike process will have amplitude and phase that become decorrelated with time. Thus, while the linear projection is complete in the sense that all variance in horizontal velocity and density anomaly projects onto a wave or vortex solution, the interpretation of the components as representations of wave and geostrophic components is ultimately problem-specific.

Another diagnostic application for the wave-vortex decomposition is assessing energy transfers between modes. Just as energy transfers can be computed across wavenumbers (e.g., Arbic et al., 2012), the decomposition can be applied to the nonlinear terms in the equations of motion to assess energy transfers between wave and vortex modes at different scales.

Finally, the nonlinear equations of motion can be projected onto the wave-vortex modes to create a series of reduced-interactions models, as has been done for triply-periodic domains (Hernandez-Duenas et al., 2014). These models are reduced versions of the equations of motions that restrict interactions between certain modes. For example, restricting interactions between only PV modes results in the quasigeostrophic equations, while restricting interactions between only wave-modes results in a wave turbulence model (Lvov & Yokoyama, 2009).

With an eye toward these applications, this manuscript...
presents and extension of the wave-vortex decomposition to flows with non-constant stratification in vertically bounded domains. This problem is significantly more challenging than the triply-periodic constant stratification case since it involves solving an eigenvalue problem (EVP) to obtain the vertical dependence rather than Fourier mode expansions in the three spatial directions. We begin with the linearized equations of motion and their solutions in sections 2 and 3. Section 4 details how the vertical modes must be projected, while section 5 shows the decomposition itself. Finally, we show how the results are simplified for constant stratification and detail the numerical implementation of the projection in appendix A and B.

2 Background

The linearized, unforced, inviscid equations of motion for fluid velocity \( u(x, y, z, t) \), \( v(x, y, z, t) \), \( w(x, y, z, t) \), on an \( f \)-plane are

\[
\begin{align*}
\frac{\partial t}{u} - f_0 v &= -\frac{1}{\rho_0} \frac{\partial z}{p} \quad (1a) \\
\frac{\partial t}{v} + f_0 u &= -\frac{1}{\rho_0} \frac{\partial z}{p} \quad (1b) \\
\frac{\partial t}{w} &= -\frac{1}{\rho_0} \frac{\partial z}{p} - g \frac{\rho}{\rho_0} \quad (1c) \\
\frac{\partial z}{u} + \frac{\partial z}{v} + \frac{\partial z}{w} &= 0 \quad (1d) \\
\frac{\partial t}{\partial y} + w &\frac{\partial z}{\partial t} = 0. \quad (1e)
\end{align*}
\]

Here \( p(x, y, z, t) \) and \( \rho(x, y, z, t) \) are perturbation pressure and density, respectively, defined such perturbation total pressure \( p_{tot}(x, y, z, t) = p_0(z) + p(x, y, z, t) \) and total density \( \rho_{tot}(x, y, z, t) = \rho_0 + \bar{\rho}(z) + \rho(x, y, z, t) \) where \( \partial z p_0(z) = -g (\rho_0 + \bar{\rho}(z)) \). All variables in (1a–1e) are functions of \( x, y, z, t \), except \( \bar{\rho} \), which is only a function of \( z \). We use the usual definition of buoyancy frequency, \( N^2(z) \equiv -\frac{\partial}{\partial z} \bar{\rho} \). Throughout this manuscript we use the linear approximation to isopycnal displacement \( \eta \equiv -\rho/\bar{\rho} \) rather than density anomaly. With this notation (1e) becomes \( w = \partial_t \eta \) and (1c) can be similarly rewritten.

The boundaries are periodic in the horizontal, \( (x, y) \), and bounded in the vertical, \( z \). The lower boundary is assumed flat at \( z = -D \) with free-slip and \( w(-D) = 0 \), and no density anomaly, \( \rho(-D) = 0 \). Similarly, the upper boundary is taken to be a free-slip rigid-lid with \( w(0) = 0 \) and also no density anomaly, \( \rho(0) = 0 \). The lack of density anomalies at the boundaries is somewhat restrictive and will be addressed in future work.

The depth integrated energetics of the flow are

\[
\begin{align*}
\text{HKE} &\equiv \frac{1}{2} \int_0^D (u^2 + v^2) \, dz, \\
\text{VKE} &\equiv \frac{1}{2} \int_0^D w^2 \, dz, \\
\text{PE} &\equiv \frac{1}{2} \int_0^D N^2 \eta^2 \, dz
\end{align*}
\]

where HKE, VKE, and PE are the horizontal kinetic energy, vertical kinetic energy and potential energy per unit mass, respectively. The other conserved quantity of interest is the quasi-geostrophic potential vorticity (PV),

\[
\text{PV} \equiv \partial_z v - \partial_y u - f_0 \partial_z \eta \quad (3)
\]

which can be directly derived from the linear equations (1a–1e), or found as the linear approximation to the available potential vorticity (APV) as defined by \( \text{Wagner & Young} \) (2015).

It is noteworthy that linearized Ertel PV does not correspond to a useful quantity in this model – it is neither conserved nor zero for the internal gravity wave solutions. Linearized Ertel PV is

\[
\text{Ertel PV} \equiv \frac{\left( \bar{\zeta} + \bar{\kappa} f_0 \right) \cdot \nabla \rho_{tot}}{\rho_{tot}} \quad (4)
\]

where \( \bar{\zeta} \) is vorticity and \( \bar{\zeta}^z = \partial_z v - \partial_y u \) is its vertical component. Notable is that linear Ertel PV per (5) does not equal the conserved PV quantity (3). Applying the total derivative to (5) results in

\[
\frac{d}{dt} \left( \text{Ertel PV} \right) \approx 1 \rho_0 \left( \bar{\rho} \partial_t \bar{\zeta}^z + f_0 \partial_t \bar{\rho} + f_0 \bar{w} \partial_z \bar{\rho} \right) \quad (6)
\]

which is not a conservation equation, but a balance between three terms: local changes in the vertical component of vorticity, local changes in the vertical gradient of the density anomaly, and the vertical advection of the background density gradient. The connection between (6) and the conserved PV (3) is found using the thermodynamic equation (1c) and re-arranging, which reproduces equation (3).

\[
\frac{d}{dt} \left( \text{Ertel PV} \right) \approx \frac{\partial \rho}{\partial \rho_0} \frac{\partial}{\partial t} \left( \bar{\rho} \left( \bar{\zeta}^z + f_0 \partial_z \rho \left( \rho/\rho_0 \right) \right) \right) \quad (7)
\]

up to a scaling factor. The key difference between quasi-geostrophic PV (3) and linear Ertel PV (5) is that the latter neglects vertical advection of the background density gradient. In the present context then, APV as defined in \( \text{Wagner & Young} \) (2015) appears to be the relevant conserved quantity.
3 Wave-vortex solutions

Solutions to (10a)-(10d) are assumed to take the separable form
\[ f(x, y, z, t) = \sum_{jkl} \frac{1}{2} \tilde{f}_{jkl}(t) e^{i(kx+ly)} F_{jkl}(z) + c.c. \] (8)
for \( u, v, p, \) and
\[ g(x, y, z, t) = \sum_{jkl} \frac{1}{2} \tilde{g}_{jkl}(t) e^{i(kx+ly)} G_{jkl}(z) + c.c. \] (9)
for \( w \) and \( \eta \). This presumes a Fourier basis satisfying the periodic boundary conditions in \( x, y, \) and real-valued functions \( F_{jkl}(z), G_{jkl}(z) \) satisfying the vertical boundary problem. The summation is over all wavenumbers \( k, l \), but also over eigenmodes \( j \) from the bases \( \{F_{jkl}(z)\} \) and \( \{G_{jkl}(z)\} \), indicated with subscripts to emphasize their dependence on wavenumbers \( k \) and \( l \). The coefficients \( \tilde{f}_{jkl}(t), \tilde{g}_{jkl}(t) \) are complex, encoding both amplitude and phase. The “c.c.” refers to the complex conjugate, which contains half the power of the real valued solutions, but no new information. Although the wave-vortex decomposition is performed at fixed time in the time domain \( \tilde{f}_{jkl}(t) \), it is useful to express solutions in the frequency domain, in which case we denote the variables with \( \hat{\cdot} \), e.g., \( \tilde{f}_{jkl}(t) = \hat{f}_{jkl} e^{i\omega t} \). Finally, we will often drop the subscripts \( jkl \) entirely, and simply work with the coefficients at a fixed \( j, k, \) and \( l \).

Using the thermodynamic equation (11c) to replace \( w \) with \( \eta \), solutions to (10a)-(10d) must satisfy
\[ i\omega \hat{u} - f_0 \hat{v} = -i(\frac{\hat{p}}{\rho_0}) \] (10a)
\[ i\omega \hat{v} + f_0 \hat{u} = -i(\frac{\hat{p}}{\rho_0}) \] (10b)
\[ (N^2 - \omega^2) \hat{G} = -\frac{\hat{p}}{\rho_0} \partial_z F \] (10c)
\[ (ik \hat{u} + il \hat{v}) F = -i\omega \hat{\eta} \partial_z G. \] (10d)

Geostrophic solutions, \( \omega = 0, k^2 + l^2 > 0 \)

Geostrophic solutions have no time variation, and the thermodynamic equation therefore implies that \( w = 0 \). Assuming non-zero horizontal wavenumber \( k^2 + l^2 > 0 \), the equations of motion (10a)-(10d) reduce to
\[ -f_0 \hat{v} = -i(\frac{\hat{p}}{\rho_0}) \] (11a)
\[ f_0 \hat{u} = -i(\frac{\hat{p}}{\rho_0}) \] (11b)
\[ N^2 \hat{G} = -\frac{\hat{p}}{\rho_0} \partial_z F_g \] (11c)
\[ (ik \hat{u} + il \hat{v}) F = 0 \] (11d)
where \( F_g(z), G_g(z) \) denote the geostrophic vertical structure functions. The only equation of consequence for the vertical structure is (11c). With no vertical velocity, the rigid lid boundary conditions place no constraint on \( F_g(z), G_g(z) \). The decision to disallow density anomalies at the boundaries implies that \( G_g(z) \) is an odd function, and therefore \( F_g(z) \) is an even function. Although formally gravity \( g \) does not enter into (11) without a free surface, it is still convenient to set the separation constant in (11c) such that \( \hat{p} = \rho_0 \hat{\eta} \) and \( N^2 G_g = -\hat{\eta} \partial_z F_g \), with \( G_g(0) = G_g(-D) = 0 \) at the boundaries. This allows the amplitude of the solution to be expressed in terms of sea-surface height, analogous to typical notation for geostrophic motions.

The geostrophic solution, or vortex solution, is given by,
\[
\begin{bmatrix}
  u_g \\
  v_g \\
  \eta_g
\end{bmatrix} = \frac{A_0}{2} \begin{bmatrix}
  -i\frac{\hat{p}}{f_0} F_g(z) \\
  i\frac{\hat{p}}{f_0} kF_g(z) \\
  G_g(z)
\end{bmatrix} e^{i\theta_0} + c.c.
\] (12)
where \( \hat{A}_0 \) is a complex valued amplitude containing the phase information, and \( \theta_0 = kx + ly \).

As a consequence of only having one constraint connecting \( F_g(z) \) and \( G_g(z) \), there is no preferred set of vertical basis functions for the geostrophic solution. Any complete basis satisfying the boundary conditions can be used to represent the geostrophic solution. However, near-geostrophic theories with a different choice of scalings, such as quasi-geostrophy (QG; e.g., see Pedlosky, 1987), have nonzero vertical velocities and therefore still require that three-dimensional continuity be satisfied. To maintain continuity we take (10d) and set the separation constant to \( h \), such that \( F(z) = h \partial_z G(z) \) for all \( z \). This additional requirement, combined with the hydrostatic vertical momentum condition \( N^2 G = -\hat{\eta} \partial_z F \), results in two Sturm-Liouville eigenvalue problems for hydrostatic (HS) vertical modes,
\[ \frac{d^2 G_{\text{HS}}}{dz^2} = -\frac{N^2}{gh} \frac{G_{\text{HS}}}{3} \] (13)
with boundary conditions $G^{\text{HS}}(0) = G^{\text{HS}}(-D) = 0$ or,
\[
\frac{d}{dz} \left( \frac{1}{N^2} \frac{dF^{\text{HS}}}{dz} \right) = -\frac{1}{gh_j} F^{\text{HS}}(j) \tag{14}
\]
with $\partial_z F^{\text{HS}}(0) = \partial_z F^{\text{HS}}(-D) = 0$ where $j$ is the mode number and eigenvalue $h_j$ is the equivalent depth. It follows directly from Sturm-Liouville theory that the vertical modes resulting from the HS EVPs satisfy the orthogonality conditions
\[
\frac{1}{g} \int_{-D}^{0} N^2(z) G^{\text{HS}}(j) G^{\text{HS}}(i) \, dz = \delta_{ij}, \tag{15}
\]
and
\[
\int_{-D}^{0} F^{\text{HS}}(i) F^{\text{HS}}(j) \, dz = h_i \delta_{ij} \tag{16}
\]
where we have implicitly normalized the amplitude of the modes. The $\frac{1}{g}$ normalization in (15) arises naturally when using a free-surface boundary condition, and is kept here for consistency.

The importance of the $\{G^{\text{HS}}(z)\}$ and $\{F^{\text{HS}}(z)\}$ bases are twofold. First, Sturm-Liouville theory guarantees that they are complete, and therefore capable of representing any function that satisfies their respective boundary conditions. Second, the specific relationship between these modes is such that both continuity and the linearized vertical momentum equation are satisfied. In practice, this means that they often reflect the vertical structure of various linear solutions. It is in this sense that $\{G^{\text{HS}}(z)\}$ and $\{F^{\text{HS}}(z)\}$ are ‘preferred’ bases for representing certain flows, including quasigeostrophy and hydrostatic linear internal waves.

The horizontal kinetic energy and potential energy of the geostrophic solution (12) as a function of depth are found by averaging over time and horizontally, including the energy from the complex conjugate,
\[
\text{HKE}_g = \frac{\hat{A}_0}{4} \frac{g^2}{f_0^2} K^2 \int_{-D}^{0} F^2_g(z) \, dz \tag{17}
\]
\[
\text{PE}_g = \frac{\hat{A}_0}{4} \frac{g^2}{f_0^2} \int_{-D}^{0} N^2(z) G^2_g(z) \, dz, \tag{18}
\]
where $K^2 = k^2 + l^2$. Vertical kinetic energy is identically zero. If we use the hydrostatic normal modes $F^{\text{HS}}$, $G^{\text{HS}}$ then depth-integrated horizontal kinetic energy reduces to
\[
\text{HKE}_g = \frac{\hat{A}_0}{4} \frac{g^2 h_j}{f_0^2} K^2 \text{ and depth-integrated potential energy reduces to}
\]
\[
\text{PE}_g = \frac{\hat{A}_0}{4} \frac{g^2}{f_0^2}.
\]
The linearized potential vorticity is,
\[
\text{PV}_g = -\frac{\hat{A}_0 g}{2f_0} \left( K^2 F_g(z) - \frac{d}{dz} \left( \frac{f_0^2}{N^2} \frac{dF_g(z)}{dz} \right) \right) e^{i\theta_0} + \text{c.c.} \tag{19}
\]
as is traditionally written, or simply
\[
\text{PV}_g = -\frac{\hat{A}_0 g}{2f_0} \left( g h K^2 + f_0^2 \right) F_g(z) e^{i\theta_0} + \text{c.c.} \tag{20}
\]
after using (16) to rewrite $F_g$. These expressions are exactly the potential vorticity identified in the quasi-geostrophic potential vorticity equation. In contrast, the Ertel PV is,
\[
\text{Ertel PV}_g = \frac{\hat{\rho}_0}{\rho_0} \left[ -\frac{\hat{A}_0 f_0}{2} \left( \partial_z \ln \hat{\rho}_z \right) G(z) e^{i\theta_0} + f_0 \right] + \text{c.c.} \tag{21}
\]
which does not correctly account for changes in the density gradient (see also Wagner & Young [2015]).

Under rigid lid conditions, there also exists a barotropic mode ($j = 0$) where $F^{\text{HS}}(z) = \text{const}$ with no associated buoyancy anomaly, $G^{\text{HS}}(z) = 0$. This case will be handled separately in the decomposition.

**Inertial oscillation solution,** $\omega \neq 0, k^2 + l^2 = 0$

This solution has no vertical velocity, density anomaly, or pressure gradients. It is simply a horizontally uniform oscillating horizontal velocity field, with no constraints on vertical structure other than the boundary conditions. In the triply periodic model used in Smith & Walffe (2002) this solution is referred to as the vertically sheared horizontal mode (VSHM), while in the bounded domain it is identified as the inertial oscillation solution,
\[
\begin{bmatrix}
  u_I \\
  v_I \\
  \eta_I
\end{bmatrix} =
\begin{bmatrix}
  U_I \sin(f_0 t + \phi_0) F_I(z) \\
  U_I \cos(f_0 t + \phi_0) F_I(z) \\
  0
\end{bmatrix}. \tag{22}
\]
Here, since there is no conjugate to $k^2 + l^2 = 0$, the amplitude is purely real. $F_I(z)$ is an arbitrary function, and can be expanded in any complete basis. This is noteworthy because it essentially leaves the boundary conditions for $F_I(z)$ unspecified, and unlike other solutions considered here, $\partial_z F_I(0)$ and $\partial_z F_I(-D)$ are not necessarily zero. Therefore one must be careful not to expand $F_I(z)$ in a basis with unnecessarily restrictive boundary conditions. That said, there is not necessarily any physical insight to be gained from this additional freedom at the boundaries, and it would certainly be reasonable to restrict the model to solutions where $\partial_z F_I(0) = \partial_z F_I(-D) = 0$.

**Wave solutions,** $\omega \neq 0, k^2 + l^2 > 0$

Similar to the geostrophic solution where we assumed that $\hat{\rho} = \rho_0 \hat{\gamma}_0$, the vertical momentum equation requires that $(N^2 - \omega^2) G = -g \partial_z F$. Combined with continuity $F = h \partial_z G$, the vertical dependence vanishes from the
problem and we are left with
\[
\begin{bmatrix}
  i\omega & -f_0 & igk \\
  f_0 & i\omega & igl \\
  kh & lh & \omega \\
\end{bmatrix}
\begin{bmatrix}
  \hat{u} \\
  \hat{v} \\
  \hat{\eta} \\
\end{bmatrix} = 
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
\end{bmatrix}.
\]
(23)

This system of equations admits the internal wave solutions when
\[
\omega = \sqrt{ghK^2 + f_0^2}.
\]
(24)
The ± wave solutions are given by,
\[
\begin{bmatrix}
  u_\pm \\
  v_\pm \\
  \eta_\pm \\
\end{bmatrix} = \frac{\hat{A}_\pm}{2} \begin{bmatrix}
  \frac{k\omega \pm if_0}{\omega f} F(z) \\
  \frac{k\omega \mp if_0}{\omega f} F(z) \\
  \mp \frac{h}{\omega} G(z) \\
\end{bmatrix} e^{i\theta_\pm} + c.c.
\]
(25)
where the horizontal phase is given by \(\theta_\pm = kx + ly \pm \omega t + \phi\) and the amplitude is chosen so that depth-integrated total energy is \(\hat{A}^2 h/2\), as will be shown below.

Combining the vertical constraints from non-hydrostatic vertical momentum \((N^2 - \omega^2)G = -g\partial_z F\) and continuity \(F = h\partial_z G\) with the dispersion relation (24) results in the \(K\)-constant, non-hydrostatic Sturm-Liouville problem,
\[
\frac{d^2 G_j}{dz^2} - K^2 G_j = -\frac{N^2 - f_0^2}{gh_j} G_j.
\]
(26)
The eigendepth \(h_j\) and eigenfrequency \(\omega_j\) are interchangeable using the dispersion relation (24) with fixed \(K\). Note that the EVP could have been written in terms of a fixed frequency \(\omega\) (with no subscript \(j\)), with eigendepth \(h_j\) and eigenfrequency \(\omega_j\) (with subscript \(j\)), but the constant frequency formulation is not relevant for the decomposition problem at fixed time.

The depth integrated energies for the \(j\)-th internal wave mode at total wavenumber \(K\) are,
\[
HK E_\pm = \hat{A}_\pm^2 \frac{1}{4} \left(1 + \frac{f_0^2}{\omega_j^2}\right) \int_{-D}^0 F_j^2(z) \, dz
\]
(27)
\[
VK E_\pm = \hat{A}_\pm^2 \frac{1}{4} K^2 h_j^2 \int_{-D}^0 G_j^2(z) \, dz
\]
(28)
\[
PE_\pm = \hat{A}_\pm^2 \frac{1}{4} K^2 h_j^2 \int_{-D}^0 N^2(z) G_j^2(z) \, dz.
\]
(29)
which sum to a depth-integrated total energy of \(\hat{A}_\pm^2 h_j/2\). The internal wave solutions have zero potential vorticity per (3), PV\(_\pm = 0\); but they do have Ertel PV per (5),
\[
\text{Ertel PV}_\pm = \frac{\hat{F}}{\rho_0} \left[ \pm \hat{A}_\pm \frac{K h_j f_0}{\omega_j} (\partial_z \ln \hat{F})(z) e^{i\theta_\pm} + f_0 \right] + c.c.
\]
(30)
again suggesting that Ertel PV may not be the appropriate quantity for this model.

### 4 Orthogonality and projection

The primary challenge that separates this wave-vortex decomposition from previous ones is dealing with the vertical modes resulting from the \(K\)-constant EVP. In a vertically periodic domain with constant stratification in \(z\), Fourier series are an appropriate basis. For a vertically bounded domain with arbitrary stratification in \(z\), the appropriate basis are the eigenmodes \(G_j\) of the \(K\)-constant EVP.

#### Orthogonality

The non-hydrostatic Sturm-Liouville problem given by \(K\)-constant EVP implies that for a given wavenumber \(K\), two modes \(G_i(z)\), \(G_j(z)\) satisfy the orthogonality condition,
\[
\frac{1}{g} \int_{-D}^0 (N^2(z) - f_0^2) G_i(z) G_j(z) \, dz = \delta_{ij}
\]
(31)
where we have implicitly normalized the modes. Unlike the hydrostatic case, there does not appear to be an equivalent Sturm-Liouville problem for the non-hydrostatic \(F_j\) modes (with constant \(K\)) and therefore no associated orthogonality condition. The closest relationship we are able to find is
\[
\int_{-D}^0 (F_i F_j + h_i h_j K^2 G_i G_j) \, dz = h_i \delta_{ij}.
\]
(32)
The difference between (16) and (32) is significant – the former can be used on any function satisfying the boundary conditions, while the latter requires a specific relationship between the dynamical variables to project on the \(F_j\) modes.

#### Projection

If a dynamical variable that expands in \(G\), such as density anomaly, \(\rho(z)\), satisfies the appropriate boundary conditions, it can be written as in (9), e.g.,
\[
\rho(x, y, z, t) = \sum_{jkl} \frac{1}{g} \frac{\hat{F}}{\rho_0} \rho_{jkl}(t) G_{j}(z) e^{i(kx+ly)} + c.c.
\]
(33)
where the coefficients are recovered with
\[
\hat{\rho}_{jkl}(t) = \frac{1}{g} \frac{1}{(-D)^2} \int_{-D}^0 (N^2(z) - f_0^2) \rho(x, y, z, t) e^{-i(kx+ly)} G_{j}(z) \, dz.
\]
(34)
The projection operation (34) first requires taking a Fourier transform of the variable, then invoking the orthogonality condition (31) with $j$-th vertical mode $G_{jkl}(z)$ for wavenumber $K = \sqrt{k^2 + l^2}$. However, in order to use orthogonality condition (32) as a projection operator, dynamical variables expanded in $F$ must be added to a related dynamical variable that scales like $hG$. For example, the divergence, $\delta = \partial_z u + \partial_y v$, and vertical vorticity, $\zeta = \partial_x v - \partial_y u$, can be recovered from the wave solution (25) with,

$$\delta_j(t) = \int_{-D}^{0} (\delta(t) F_j(z) - iK^2 w(t) h_j G_j(z)) \, dz$$

$$\zeta_j(t) = \int_{-D}^{0} (\zeta(t) F_j(z) - jf_0 K^2 h_j G_j(z)) \, dz \tag{35}$$

where $\delta(z, t) = \sum \delta_j(t) F_j(z)$ and $\zeta(z, t) = \sum \zeta_j(t) F_j(z)$. However, this only works for wave solutions since the geostrophic solution does not have the same relationships between $(u, v)$ and $\eta$. It thus appears that (32) is not particularly useful in recovering solutions.

To project variables $u$ and $v$ (and also $p$) that are expanded in $F$, we instead use the relationship derived from continuity, $F_j(z) = h_j \partial_z G_j(z)$, and consider the depth-integrated quantities. That is, if

$$u(x, y, z, t) = \sum_{j,k,l} \tilde{u}_{jkl}(t) e^{i(kx + ly)} F_{jkl}(z) + c.c., \tag{36}$$

then we compute $U = \int_{-D}^{0} u \, dz'$ so that,

$$U(x, y, z, t) = \sum_{j,k,l} \tilde{u}_{jkl}(t) e^{i(kx + ly)} h_{ijkl} G_{jkl}(z) + c.c., \tag{37}$$

which can then be projected using (34) to recover $\tilde{u}_{jkl}(t)$. Notable here is that the depth-integrated quantities represented by (38) are themselves depth dependent.

As discussed in the next section, the only part of the solution that must be handled in a special manner is the barotropic $j = 0$ mode $F_0(z)$, which as previously discussed has no projection on the $G$ modes in the rigid lid case. In practice, the integration linking (37) to (38) can be performed by projecting $u$ onto either the $\{F_{0ijkl}(z)\}$ basis or a cosine basis (either of which satisfy the correct boundary conditions and have a constant/barotropic mode), integrating spectrally, and then transforming back to the spatial domain.

### 5 Wave-vortex decomposition

Per the previous discussion, the wave vortex decomposition requires integrating $(u, v)$ to get $(U, V)$, taking the Fourier transform in the horizontal of $(U, V, \eta)$, and then projecting the vertical structure at each horizontal wavenumber $k$ and $l$ onto the vertical eigenmodes found via the $K-$constant EVP. Early et al. (2020) developed a methodology for the computation and projection onto these modes. Written as a sum of individual linear solutions, and explicitly including the dependence on $j, k, l$, the three required variables are expressed as

$$U(x, y, z, t) = \sum_{j,k,l} \tilde{U}_{jkl}(t) \frac{e^{i(kx + ly)}}{2} G_{jkl}(z) + c.c. \tag{39}$$

$$V(x, y, z, t) = \sum_{j,k,l} \tilde{V}_{jkl}(t) \frac{e^{i(kx + ly)}}{2} G_{jkl}(z) + c.c. \tag{40}$$

$$\eta(x, y, z, t) = \sum_{j,k,l} \tilde{\eta}_{jkl}(t) \frac{e^{i(kx + ly)}}{2} G_{jkl}(z) + c.c. \tag{41}$$

where $\tilde{U}_{ijl}(k) = \tilde{u}_{ijl}(k) h_{ijl}$ and $\tilde{V}_{ijl}(k) = \tilde{v}_{ijl}(k) h_{ijl}$. The horizontal Fourier transform followed by the vertical projection then recovers $\tilde{U}_{ijl}(k), \tilde{V}_{ijl}(k),$ and $\tilde{\eta}_{ijl}(k)$.

#### Nonzero wavenumber solutions, $k^2 + l^2 > 0, j = 0$

When vertically integrating the horizontal velocities $u, v$ to project onto the vertical modes, the amplitude of the $j = 0$ mode must be handled separately. The $j = 0$ mode for the rigid lid boundary condition has no density anomaly, $\eta(t) = 0$, and no divergence, $\delta(t) = ik\tilde{u}(t) + il\tilde{v}(t) = 0$. This leaves only the amplitude and phase of the vorticity $\zeta(t) = ik\tilde{v}(t) - il\tilde{u}(t)$. The only valid solution is therefore the vortex solution,

$$A_0 = -i \frac{f_0}{gK^2} (k\tilde{v}(t) - l\tilde{u}(t)) \tag{42}$$

valid for all $k^2 + l^2 > 0$.

#### Nonzero wavenumber solutions, $k^2 + l^2 > 0, j > 0$

For each wavenumber $(k, l)$ and mode $j$ there are six unknowns: the amplitudes and phases of the three different solutions. We denote the complex amplitudes as $A_+, A_-$, and $A_0$, for the positive and negative wave, and geostrophic solutions, respectively. In matrix form the three linearly independent solutions from (12) and (25) at wavenumbers $k, l,$ and mode $j$ are given by

$$\begin{bmatrix}
\tilde{U}(t) \\
\tilde{V}(t) \\
\tilde{\eta}(t)
\end{bmatrix} = 
\begin{bmatrix}
\frac{\omega - il f_0}{\omega K} h_{ijkl} & \frac{\omega + il f_0}{\omega K} h_{ijkl} & -i \frac{g h}{f_0} l \\
-\omega k & \omega k & 0 \\
-\omega l & -\omega l & 1
\end{bmatrix} \begin{bmatrix}
A_+ \\
A_- \\
A_0
\end{bmatrix} \tag{43}
$$
which can be inverted to solve for $\hat{A}_+, \hat{A}_-$, and $\hat{A}_0$,
\[
\hat{A}_\pm = \frac{e^{\mp i\omega t}}{2K\hbar} \left[ \frac{k\omega \pm i f_0}{\omega K\hbar} \hat{U}(t) + \frac{lf_0}{\omega K\hbar} \hat{V}(t) \mp \frac{gK}{\omega} \hat{\eta}(t) \right]
\]
\[
\hat{A}_0 = \frac{f_0}{\omega^2} \hat{V}(t) - i \frac{k f_0}{\omega^2} \hat{V}(t) + \frac{f^2}{\omega^2} \hat{\eta}(t).
\]

There is some insight to be gained by defining depth-integrated versions of horizontal divergence and potential vorticity,
\[
\hat{\Delta}(t) = i \left( k \hat{U}(t) + l \hat{V}(t) \right),
\]
\[
\hat{\Pi}(t) = i \left( k \hat{V}(t) - l \hat{U}(t) \right) - f_0 \hat{\eta}(t).
\]

Now the solution has the form,
\[
\hat{A}_+ = \frac{e^{-i\omega t}}{2K\hbar} \left[ -i \hat{\Delta}(t) - \frac{1}{\omega} \left( f \hat{\Pi}(t) + \omega^2 \hat{\eta}(t) \right) \right]
\]
\[
\hat{A}_- = \frac{e^{i\omega t}}{2K\hbar} \left[ -i \hat{\Delta}(t) + \frac{1}{\omega} \left( f \hat{\Pi}(t) + \omega^2 \hat{\eta}(t) \right) \right]
\]
\[
\hat{A}_0 = - \frac{f}{\omega^2} \hat{\Pi}(t).
\]

Importantly, (47a)–(47c) show that the vortex solution is recovered directly from potential vorticity and the sum of the two wave solutions is recovered from the divergence of the transport. Extracting the phase information and energetics of individual wave solutions still requires additional information from vorticity and isopycnal displacement.

Zero wavenumber solutions, $k^2 + l^2 = 0$

The only $k^2 + l^2 = 0$ solution is still inertial oscillations, per (22), with simple rotation and zero isopycnal displacement, i.e.,
\[
u_f(t) = \frac{\hat{u}(t) \cos(f_0 t) - \hat{v}(t) \sin(f_0 t)}{F_1(z)}
\]
\[
u_l(t) = \frac{\hat{u}(t) \sin(f_0 t) + \hat{v}(t) \cos(f_0 t)}{F_1(z)}
\]
\[
\eta_f(t) = 0.
\]

Summary of the decomposition

A key feature of the decomposition is that the recovered coefficients (42), (47) and (48) are strictly independent of time when applied to time-dependent linear solutions. That is, the left-hand sides of these equations are time-independent, while the right-hand sides contain terms that are time-dependent. This is not a contradiction; it simply reflects the fact that for unforced inviscid flow, the amplitude and phase of the linear solutions will remain fixed for all time. Applying this methodology to nonlinear flows, the actual linearity or nonlinearity of a flow can be made precise by assessing time variation in the recovered coefficients. For example, if $\hat{A}_+$ for a given $j, k, l$ at time $t = t_0$ is exactly equal to $\hat{A}_+$ computed at time $t = t_1$, then that component of the flow was perfectly linear in the sense that the wave solution (25) exactly described its evolution.

The important takeaway when applying the above decomposition is that any time variation in the recovered coefficients, (42), (47) and (48), by definition implies nonlinearity.

6 Conclusion

There is one important caveat to the claim that the decomposition can account for all variance. In practice, numerical models are run with a finite number of grid points, $N$ – and it is not true in general that $N$ vertical modes will be resolved with $N$ grid points, even if those points are judiciously chosen. Grid points would need to be placed at (or near) the Gaussian quadrature points for the normal modes $F(z), G(z)$ at all resolved $K$, which is likely impossible for all but constant stratification. Thus, in practice, some of the model’s high mode variance will not project onto a resolved vertical mode and therefore cannot be projected onto a solution. In this case there will be a residual.

This decomposition has been fully implemented for the case of constant stratification (see Appendices A and B) and tested with output from a linear simulation with a Boussinesq model (Winters & Fuente, 2012). In bounded domains, the Winters model uses sine and cosine bases in the vertical with evenly spaced grid points. Solutions therefore project exactly onto the constant stratification vertical modes. For linear simulations, given a solution at any point in time, the decomposition presented here is able to recover the model solution at any other point in time, accounting for all variance.

The utility of this decomposition has yet to be fully explored. The ability to precisely assess linearity allows one to test the linearity assumptions of various nondimensional scales, as has been done for linear PV using the vertical Froude number in the triply periodic case (Waite & Bartello, 2006). However, it remains untested whether or how weakly nonlinear waves will project their energy onto the wave-vortex coefficients, and vice versa, and which estimates of linearity are the best indicators. Furthermore, because the projection onto wave and vortex modes is complete, it is also possible to derive reduced-interaction models, such as in Hernandez-Duenas et al. (2014), but for vertically bounded flows with variable stratification.

We would like to acknowledge National Science Foundation awards 1658564 and 1536747 for funding this re-
The consequence is that
\[ \Delta h_j = \frac{N^2 - f^2}{g} \] and vertical wavenumber \( m_j = \frac{2\pi}{\Delta h_j} \).
Using the normalization \( A^2 = \frac{1}{g N_f} \) results in the following orthogonality conditions,
\[ \frac{1}{g} \int_{-D}^{0} (N_0^2 - f_0^2) G_i^{N_i}(z) G_j^{N_j}(z) \, dz = \delta_{ij} \] (50a)
\[ \int_{-D}^{0} F_i^{N_i}(z) F_j^{N_j}(z) \, dz = \frac{g h^2 m_i^2}{N_0^2 - f_0^2} \delta_{ij}. \] (50b)
The orthogonality conditions imply that if \( \eta(z) = \sum_n \hat{\eta}_n G_i^{N_i}(z) \) or \( u(z) = \sum_n \hat{u}_n F_i^{N_i}(z) \), then
\[ \hat{\eta}_n = \frac{N_0^2 - f_0^2}{g} \int_{-D}^{0} G_i^{N_i}(z) \eta(z) \, dz \] (51a)
\[ \hat{u}_n = \frac{N_0^2 - f_0^2}{gh^2 m_n^2} \int_{-D}^{0} F_i^{N_i}(z) u(z) \, dz. \] (51b)
The consequence is that \( \hat{u}, \hat{v} \) can be recovered directly, without integrating to get transport quantities, that is (47a)–(47c) with \( \tilde{\Delta}(t) \) replaced by \( \delta(t), \Pi(t) \) replaced by \( PV(t) \), and \( A_+ \) no longer normalized by \( h \).
\[ \hat{A}_+ = \frac{e^{-ikx}}{2K} \left[ -i \tilde{\delta}(t) - \frac{1}{\omega} \left( fPV(t) + \frac{\omega^2}{h} \tilde{\eta}(t) \right) \right] \] (52)
\[ \hat{A}_- = \frac{e^{ikx}}{2K} \left[ -i \tilde{\delta}(t) + \frac{1}{\omega} \left( fPV(t) + \frac{\omega^2}{h} \tilde{\eta}(t) \right) \right] \] (53)
\[ \hat{A}_0 = - \frac{f_0 h}{\omega^2} PV(t) \] (54)
where,
\[ \tilde{\delta}(t) = i(k \tilde{u}(t) + l \tilde{v}(t)), \]
\[ PV(t) \equiv i(k \tilde{v}(t) - l \tilde{u}(t)) - f_0 \tilde{\eta}(t)/h. \] (55)

**B Numerical implementation**

The decomposition was tested with output from a linear simulation with a rotating spectral Boussinesq model [Winters & Fuente, 2012] with constant stratification. Implementation of the methodology requires the following steps,

1. Discrete Fourier transforms of \( u, v, \) and \( \eta \) in \( x \) and \( y \).
2. A discrete cosine transform of \( u \) and \( v \) and a discrete sine transform of \( \eta \) in \( z \).
3. Computation of the wave-vortex coefficients from the transformed variables.

The last step requires careful bookkeeping to ensure that all terms are properly accounted for and not double-counted.

The domain is assumed to have \([N_x \times N_y \times (N_z + 1)]\) points, where \( N_x \) and \( N_y \) are typically in powers of 2 to take advantage of the fast Fourier transform (FFT), while \( N_z + 1 \) has 2\(^n + 1 \) points to accommodate the type-I discrete cosine transforms (DCT-I) and type-I discrete sine transforms (DST-I) used by the Winters model.

**B.1 Horizontal transformation**

The finite-length Fourier transform in a periodic domain is given by
\[ \mathcal{F}[f(x)] = \frac{1}{L} \int_{-D}^{0} f(x)e^{-ikx} \, dx \] (56)
where \( k_j = \frac{2\pi j}{L} \). For a discretized domain with points at \( x_n = n\Delta \) where \( n = [0 \ldots N - 1] \) and \( \Delta = L/N \), the discrete Fourier transform is
\[ \hat{f}(k_j) = DFT\left[f(x_n)\right] = \frac{1}{L} \sum_{n=0}^{N-1} f(x_n)e^{-ik_jx_n} \Delta. \] (57)

with wavenumbers \( k_j = \frac{2\pi j}{L} \) now limited to \( j = [0 \ldots N - 1] \). Variance is preserved following Plancherel’s theorem,
\[ \frac{1}{N} \sum_{n=0}^{N-1} |f(x_n)|^2 = \sum_{j=0}^{N-1} S(k_j) \, dk \] (58)
where \( dk = \frac{2\pi}{L} \) and \( S(k_j) = \frac{L}{2\pi} |\hat{f}(k_j)|^2 \) is defined as the spectrum.

Applying the DFT in both \( x \) and \( y \) using the usual numerical algorithms on a real value function results in a two-dimensional transformed matrix as shown in table B.1. For a real valued function the power is split between the two conjugate pairs, and therefore \( \hat{f}(k_j) \) has to be doubled to be compared to \( f_{jkl} \) in (8). The grey and pink regions in table B.1 are Hermitian conjugates of other values in the table. The Nyquist frequency \( j = N/2 \) is unresolved since the sine at the Nyquist is zero, and thus the orange regions are also ignored. Only the white regions and the cyan component at \( k = l = 0 \) contain the information for the inversion.
Table 1. Table of FFT coefficients for wavenumbers \((k,l)\). We have let \(-4 \leq k \leq 3\) and \(-4 \leq l \leq 3\), consistent with an 8x8 2D FFT. The grey shaded region shows the redundant coefficients that are determined from Hermitian conjugacy by changing the sign on the \(l\) component. The pink shaded components are Hermitian conjugate by changing the sign on the \(k\) component. The orange components are Nyquist components, and thus not fully resolved. The cyan shaded component (including \((k,l)=(-4,0), (-4,4)\) and \((0,-4))\) are self-symmetric, and therefore strictly real.

B.2 Vertical transformation

The finite-length sine and cosine transforms are given by

\[
S[g(z)] = \frac{2}{D} \int_{-D}^{D} g(z) \sin(m_j z) \, dz \tag{59}
\]

and

\[
C[f(z)] = \frac{2}{D} \int_{-D}^{D} f(z) \cos(m_j z) \, dz \tag{60}
\]

where \(m_j = \frac{4 \pi j}{2D}\). The discretized versions of these transforms, the DST-I and DCT-I used by the Winters model, are defined with points at \(z_n = n \Delta\) where \(n = 0..N_z\) and \(\Delta = D/N_z\). Note that this differs from the discretization for the \(DFT\) by including endpoints. This choice of discretization results in the discrete transforms

\[
\hat{g}(m_j) = DST [g(z_n)] = \frac{2}{D} \sum_{n=0}^{N_z} g(z_n) \sin(m_j z_n) \Delta \tag{61}
\]

and

\[
\hat{f}(m_j) = DCT [f(z_n)] = \frac{2}{D} \left( \frac{f(0)}{2} + \sum_{n=1}^{N_z-1} f(z_n) \cos(m_j z_n) + (-1)^j \frac{f(D)}{2} \right) \Delta. \tag{62}
\]

with vertical wavenumbers at \(m_j = \frac{2 \pi j}{D}\) where \(j = 0..N_z\). The sum in the \(DCT\) treats the end points separately, as they have only half the width of the other points, \(\Delta/2\). For the \(DST\) the function is zero at the endpoints, \(g(z_0) = g(z_n) = 0\), and the \(m_0\) and \(m_{N_z}\) wavenumber components have zero power.

With these definitions of the transform, Plancherel’s Theorem states that,

\[
\frac{1}{N_z} \sum_{n=1}^{N_z} |g(x_n)|^2 = \sum_{j=1}^{N_z-1} S(m_j) \Delta \tag{63}
\]

with \(S(m_j) = \frac{D}{2\pi} |\hat{g}(m_j)|^2\) and

\[
\frac{1}{N_z} \left( \frac{|f(0)|^2}{2} + \sum_{n=1}^{N_z-1} |f(z_n)|^2 + \frac{|f(D)|^2}{2} \right) = \left( \frac{S(m_0)}{2} + \sum_{j=1}^{N_z-1} S(m_j) + 2S(m_{N_z}) \right) \Delta \tag{64}
\]

with \(S(m_j) = \frac{D^2}{2\pi} |\hat{f}(m_j)|^2\) where \(dm = \frac{\Delta}{D}\). The variance of the \(m_0\) wavenumber is notably a factor of 2 larger than the variance of a constant function.

B.3 Wave-vortex coefficients

Applying the discrete transformations exactly as defined above to \(u, v\) and \(\eta\) results in the following matrices,

\[
\hat{u}_{k,l} = DST_z [DFT_y [DFT_x [u(x, y, z)]]] \tag{65a}
\]

\[
\hat{v}_{k,l} = DCT_z [DFT_y [DFT_x [v(x, y, z)]]] \tag{65b}
\]

\[
\hat{\eta}_{k,l} = DST_z [DFT_y [DFT_x [\eta(x, y, z)]]]. \tag{65c}
\]

B.3.1 Coefficients, \(k^2 + l^2 > 0, j = 0\)

Starting with the \(j = 0\) mode, the discrete transforms give nonzero coefficient functions for \(\hat{u}_{k,0}\) and \(\hat{v}_{k,0}\) but zero for \(\hat{\eta}_{k,0}\). The \(DCT\) inflates the power by a factor of two, but the \(DFT\) returns only half the power of the real-value function. The result is that,

\[
\hat{A}_0 = -i \frac{f_0}{gK^2} (k\hat{v}_{k,0} - l\hat{u}_{k,0}) \tag{66}
\]

exactly as written before.

B.3.2 Coefficients, \(k^2 + l^2 > 0, j > 0\)

The projection operations as defined in \([51]\) can be related to the sine and cosine transformations in \([59]\) and \([60]\) by a
scaling,

\[ \bar{u}_{kj} = \sqrt{D \left( N_0^2 - f_0^2 \right)} \hat{u}_{kj} \]  
\[ \bar{v}_{kj} = \sqrt{D \left( N_0^2 - f_0^2 \right)} \hat{v}_{kj} \]  
\[ \bar{\eta}_{kj} = \sqrt{D \left( N_0^2 - f_0^2 \right)} \hat{\eta}_{kj}. \]  

The wave-vortex coefficients are then recovered with,

\[ \hat{A}_\pm = \frac{e^{\pm i\omega t}}{2} \left[ \frac{k \omega \mp ilf_0}{\omega K} \hat{u}_{kj} \pm \frac{l \omega \mp ikf_0}{\omega K} \hat{v}_{kj} \pm \frac{gK}{\omega} \hat{\eta}_{kj} \right] \]  
\[ \hat{A}_0 = \frac{ihf_0}{\omega^2} \hat{u}_{kj} - \frac{ihf_0}{\omega^2} \hat{v}_{kj} + \frac{f_0^2}{\omega^2} \hat{\eta}_{kj}. \]

References

ARBIC, BRIAN K., SCOTT, ROBERT B., FLIERL, G.R., MORTEN, ANDREW J., RICHMAN, JAMES G. & SHRIVER, JAY F. 2012 Nonlinear Cascades of Surface Oceanic Geostrophic Kinetic Energy in the Frequency Domain*. Journal of Physical Oceanography 42 (9), 1577 – 1600.

BARTELLO, P. 1995 Geostrophic adjustment and inverse cascades in rotating stratified turbulence. Journal of the atmospheric sciences 52 (24), 4410 – 4428.

BÜHLER, OLIVER, CALLIES, JÖRN & FERRARI, RAFFAELE 2014 Wave vortex decomposition of one-dimensional ship-track data. Journal of Fluid Mechanics 756, 1007 – 1026.

EARLY, JEFFREY J., LELONG, M. PASCALE & SMITH, K. SHAFER 2020 Fast and accurate computation of vertical modes. Journal of Advances in Modeling Earth Systems, arXiv: 1910.14615.

HERNÁNDEZ-DUENAS, GERARDO, SMITH, LESLIE M & STECHMANN, SAMUEL N 2014 Investigation of Boussinesq dynamics using intermediate models based on wavevortical interactions. Journal of Fluid Mechanics 747, 247–287.

LIEN, REN-CHIEH & MÜLLER, PETER 1992 Normal-Mode Decomposition of Small-Scale Oceanic Motions. Journal of Physical Oceanography 22 (12), 1583–1595.

LIEN, REN-CHIEH & SANFORD, THOMAS B. 2019 Small-Scale Potential Vorticity in the Upper Ocean Thermocline Small-Scale Potential Vorticity in the Upper Ocean Thermocline. Journal of Physical Oceanography 49 (7), 1845–1872.

LVOV, YURI V. & YOKOYAMA, NAOTO 2009 Nonlinear wave-wave interactions in stratified flows: Direct numerical simulations. Physica D: Nonlinear Phenomena 238 (8), 803–815.

PEDLOSKY, JOSEPH 1987 Geophysical fluid dynamics.

SMITH, LESLIE M. & WALEFFE, FABIAN 2002 Generation of slow large scales in forced rotating stratified turbulence. Journal of Fluid Mechanics pp. 145–168.

WAGNER, G. L. & YOUNG, W. R. 2015 Available potential vorticity and wave-averaged quasi-geostrophic flow. Journal of Fluid Mechanics 785, 401–424.

WAITE, MICHAEL L. & BARTELLO, PETER 2006 The transition from geostrophic to stratified turbulence. Journal of Fluid Mechanics 568, 89–108.

WINTERS, KRAIG B. & FUENTE, ALBERTO DE LA 2012 Modelling rotating stratified flows at laboratory-scale using spectrally-based DNS. Ocean Modelling 49-50 (C), 47 – 59.