Vanishing theorems for constructible sheaves on abelian varieties over finite fields

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Abstract Let $\kappa$ be a field, finitely generated over its prime field, and let $k$ denote an algebraically closed field containing $\kappa$. For a perverse $\mathbb{Q}_\ell$-adic sheaf $K_0$ on an abelian variety $X_0$ over $\kappa$, let $K$ and $X$ denote the base field extensions of $K_0$ and $X_0$ to $k$. Then, the aim of this note is to show that the Euler–Poincare characteristic of the perverse sheaf $K$ on $X$ is a non-negative integer, i.e. $\chi(X, K) = \sum (-1)^{\nu} \dim_{\mathbb{Q}_\ell}(H^\nu(X, K)) \geq 0$. This generalizes the result of Franecki and Kapranov [9] for fields of characteristic zero. Furthermore we show that $\chi(X, K) = 0$ implies $K$ to be translation invariant. This result allows to considerably simplify the proof of the generic vanishing theorems for constructible sheaves on complex abelian varieties of [11]. Furthermore it extends these vanishing theorems to constructible sheaves on abelian varieties over finite fields.

The proof in [9] of the above estimate for the Euler–Poincare characteristic of perverse sheaves on abelian varieties over fields of characteristic zero as well as the results of [11] rely on methods from the theory of $D$-modules via the Dubson–Riemann–Roch formula for characteristic cycles. In fact, one expects a similar Riemann–Roch theorem over fields of positive characteristic, extending the results of [1] and generalizing the Grothendieck–Ogg–Shafarevich formula for the Euler–Poincare characteristic of sheaves on curves. However, in the absence of such deep results on wild ramification we will follow a different approach using methods of Gabber and Loeser [10] that are based on Ekedahl’s adic formalism.

One of the main tools in [10] for the study of perverse sheaves on tori is the Mellin transform of a perverse sheaf. This Mellin transform is a complex of sheaves on the affine spectrum of a kind of Iwasawa ring associated to the fundamental group of...
the torus. For us, the key property of the Mellin transform is the coherence of its cohomology sheaves, which is a consequence of the finiteness theorems of Ekedahl for adic sheaves \[4\]. For abelian varieties one can define the Fourier transform of a perverse sheaf in a similar way, and most of the proofs in \[10\] carry over ad hoc. Other than that, the situation for tori and abelian varieties is quite different. Whereas in the former case the non-completeness of tori leads to difficulties in defining a convolution product with nice duality properties, this is almost trivial in the case of abelian varieties. The convolution is important for the final aim to construct Tannakian categories, and both the Mellin and the Fourier transform convert the convolution product into the tensor product of complexes.

On the other hand, by fibering a torus over lower dimensional tori with one dimensional multiplicative fibers, various important results in the torus case \[10\] immediately follow from the Artin–Grothendieck vanishing theorem. The main difficulty for abelian varieties arises from the fact that analog vanishing theorems, although needed, do not follow from the Artin–Grothendieck vanishing theorem in the absence of such fibrations, but have to be proven by other means. In \[11\] for complex abelian varieties they are derived from the Dubson–Riemann–Roch theorem in the case of a simple abelian variety. In \[18,19\] the reduction to the simple case was achieved by a tedious induction on the number of simple factors together with some characteristic \(p\) argument (Cebotarev density). Improving upon that, we now give an easier and more general proof of the following key ingredient (see \[11, prop. 10.1\] and \[19\]) for these vanishing theorems.

**Main Theorem** For an abelian variety \(X_0\) over a field \(\kappa\) finitely generated field over its prime field, let \(K\) be a simple perverse sheaf on \(X\) defined over \(\kappa\). Then the Euler–Poincare characteristic satisfies \(\chi(X, K) \geq 0\). If \(\chi(X, K) = 0\), then \(K\) is translation invariant in the sense that \(T_x^*(K) \cong K\) holds for all closed points \(x\) of an abelian subvariety \(A\) of \(X\) of positive dimension.

A degeneration argument allows to reduce the proof of this theorem to the case of finite fields \(\kappa\). And, although not stated above, the theorem not only holds for finitely generated fields but also for perverse sheaves on a complex abelian variety. This again can be shown by a degeneration argument, but now via the method of Drinfeld \[8\] involving the by now proven de Jong’s conjecture. In the finite field case, by studying the support of the Fourier transform of monoidal perverse sheaves and by the crucial coherence properties, the proof is reduced to an elementary statement on the existence of nontrivial one-dimensional analytic subgroups within analytic \(\varphi\)-subvarieties of analytic tori that are endowed with a certain automorphism \(\varphi\), defined by the Frobenius. For the precise statements see Proposition 4 and the related statements preceding it. That one can apply Proposition 4 requires certain properties of the supports, that can be deduced from the Cebotarev density theorem \[18\].

As already shown in \[11\], the theorem stated above does imply the generic vanishing theorems for perverse sheaves on abelian varieties, provided one also disposes over the decomposition theorem and Hard Lefschetz theorem. To keep the range of applications flexible, in \[11\] certain classes \(P\) of \(\mathbb{Q}_\ell\)-adic perverse sheaves on abelian varieties were considered that satisfy the required properties in an axiomatic way. A typical class are the perverse sheaves of geometric origin. If the abelian variety and the perverse sheaf
Vanishing Theorem Let \( X_0 \) be an abelian variety and let \( K_0 \) be a \( \overline{\mathbb{Q}}_\ell \)-adic perverse sheaf on \( X_0 \), both defined either over the field \( \kappa \) of complex numbers or a finite field. Then for most characters \( \chi : \pi_1(X) \to \overline{\mathbb{Q}}_\ell^* \) of the etale fundamental group \( \pi_1(X) \) the etale cohomology groups \( H^v(X, K \otimes_{\overline{\mathbb{Q}}_\ell} L_\chi) \) vanish in degree \( v \neq 0 \).

Here \( L_\chi \) denotes the rank 1 local \( \overline{\mathbb{Q}}_\ell \)-adic system defined by the character \( \chi \). For the precise meaning of the notion “most” we refer to [11], but we indicate that for absolutely simple abelian varieties \( X_0 \) it just means “for almost all”.

The vanishing theorem stated above similarly holds for perverse sheaves of geometric origin. It might very well hold for arbitrary perverse sheaves on abelian varieties over arbitrary algebraically closed fields. Presently this is not known, since no specialization argument as in [8] for the characteristic zero case is known in the case of positive characteristic. But, with such generalizations in mind we also consider weaker vanishing statements where vanishing for most characters is replaced by the weaker positive characteristic. But, with such generalizations in mind we also consider weaker vanishing statements where vanishing for most characters is replaced by the weaker statement of vanishing for a generic character. See Theorem 2 and the discussion leading to it.

Axiomatic setting. Let \( k \) denote the algebraic closure of a finite field \( \kappa \) of characteristic \( p \). For an abelian variety \( X_0 \) over \( \kappa \), let \( X \) be the base extension of \( X_0 \) from \( \kappa \) to \( k \) for a fixed embedding \( \kappa \subset k \). Let \( \Lambda \) denote \( \overline{\mathbb{Q}}_\ell \) for some prime \( \ell \neq p \). On the derived category \( D^b_c(X, \Lambda) \) of \( \Lambda \)-adic sheaves with bounded constructible cohomology sheaves one has the convolution product \( K \ast L = Ra_\chi(K \boxtimes L) \) defined by the group law \( a_0 : X_0 \times X_0 \to X_0 \). The convolution product makes \( D^b_c(X, \Lambda) \) into a rigid symmetric monoidal category. The rigid dual \( K^\vee \) of \( K \in D^b_c(X, \Lambda) \) is \((-id_X)^*D(K)\) for the Verdier dual \( D(K) \) of \( K \); see [20]. By definition, a character \( \chi : \pi_1(X) \to \Lambda^* \) of the etale fundamental group \( \pi_1(X) \) of \( X \) is a continuous homomorphism with values in the group of units \( \Lambda^* \) of the ring of integers \( \mathfrak{o}_\Lambda \) of a finite extension field \( E_\Lambda \subset \Lambda \) of \( \mathbb{Q}_\ell \). Associated to a character \( \chi \) there is a smooth \( \Lambda \)-adic sheaf \( L_\chi \) on \( X \). Let \( \pi_1(X)_\ell \) denote the maximal pro-\( \ell \) quotient of \( \pi_1(X) \). Any character \( \chi \) of \( \pi_1(X) \) is the product of a character \( \chi_f \) of finite order prime to \( \ell \), and a character that factorizes over the pro-\( \ell \) quotient \( \pi_1(X)_\ell \) of \( \pi_1(X) \).

We now fix a semisimple suspended monoidal rigid subcategory \( D = D(X) \) of the triangular monoidal category \( (D^b_c(X, \Lambda), \ast) \) that is closed under retracts and that satisfies the properties formulated in [11, §5]. In other words, we furthermore assume that: \( D \) is stable under the perverse truncation functors \( p^\tau_\geq 0 \) and \( p^\tau_\leq 0 \), and hence under the perverse cohomology functors \( p^H0 = p^\tau_\geq 0 \circ p^\tau_\leq 0 \) such that \( K \cong \bigoplus_{n \in \mathbb{Z}} p^H_n(K)[-n] \) holds for \( p^H_n(K) = p^H0(K[n]) \) and all \( K \in D \). The full subcategory \( P = P(X) \) of objects in \( D \) that are perverse sheaves is semisimple. We also assume functorial hard Lefschetz isomorphisms \( p^H^{-n}(K \ast L) \cong p^H_n(K \ast L)(n) \) for all \( n \in \mathbb{Z} \) and all \( K \), \( L \in P \). We remark that the corresponding assertion in axiom (D3) of in [11, §5] should read \( K, L \in P \) instead of \( K, L \in D \). Finally we assume that \( D \) is stable under twists with the character sheaves \( L_\chi \). Then, for \( K \in D \) resp. \( K \in P \), the twist \( K_\chi := K \otimes_{\Lambda} L_\chi \) is in \( D \) resp. in \( P \).
An example is the category $D$ of all $K$ in $D^b_c(X, \Lambda)$ obtained by base extension from some objects $K_0$ in $D^b_c(X_0, \Lambda)$ with the property that $K$ decomposes into a direct sum of complex shifts of irreducible perverse sheaves on $X$. Although the convolution product $*$ on $D$, induced by the group law on $X$, makes $(D, *)$ into a rigid $\Lambda$-linear monoidal symmetric category, in general the convolution product does not preserve the subcategory $P$.

The case of finite fields $k$ is central for this paper. If occasionally we consider other fields $k$, say that are finitely generated but not finite, we tacitly may make the same axiomatic assumptions and use analogous notation.

**Fourier transform.** We define the Fourier transform in analogy to the Mellin transform in [10]. Since most of the arguments carry over verbatim, we restrict ourselves to give the main references from [10]. As in [10, p. 509], consider the ring $\Omega_X := \mathcal{O}_X[[\pi_1(X)]]$, a complete noetherian local ring of Krull dimension $1 + 2 \dim(X)$. For generators $\gamma_i$ of $\pi_1(X)$ define $(\mathcal{O}_X)^{\gamma_i} \cong (\mathbb{Z}/\ell)^{2 \dim(X)}$, this ring is isomorphic to the formal power series ring $\mathcal{O}_X[[t_1, \ldots, t_n]]$ in the variables $t_i = \gamma_i - 1$ for $n = 2 \dim(X)$ with coefficients in $\mathcal{O}_X$. For $\mathcal{C}(X)$, $Spec(\Lambda \otimes \mathcal{O}_X[[\pi_1(X)]])$ as in [10, 3.2], define the scheme $\mathcal{C}(X)$ as the disjoint union $\bigcup_{\chi_f} \{\chi_f\} \times \mathcal{C}(X)$, for $\chi_f$ running over the characters $\chi_f$ of $\pi_1(X)$ of finite order prime to $\ell$. By [10, A.2.2.3] the closed points of $\mathcal{C}(X)$ are the $\Lambda$-valued points of $\mathcal{C}(X)$. The $\Lambda$-valued points of the scheme $\mathcal{C}(X)$ can be identified with the “continuous” characters $\chi: \pi_1(X) \to \Lambda^*$, i.e characters in our notation. As in loc. cit. there exists a continuous character $can_X: \pi_1(X) \to \Omega_X^*$ and an associated local system $L_X$ on $X$, which is locally free of rank 1 over $\Omega_X$. For $K \in D^b_c(X, \mathcal{O}_X)$ we consider $K \otimes_{\mathcal{O}_X} L_X$ as an object in $D^b_c(X, \Omega_X)$. For the structure morphism $g: X \to Spec(k)$, following [10, p. 512 and A.1], we define the Fourier transform $\mathcal{F}: D^b_c(X, \mathcal{O}_X) \to D^b_c(\Omega_X)$ by $\mathcal{F}(K) = Rg_*\left(K \otimes_{\mathcal{O}_X} \Omega_X\right)$ analogous to the Mellin transform in loc. cit. By proposition A.1.5 of loc. cit. the functor defined by extension of scalars $- \otimes_{\mathcal{O}_X} \Omega_X$ commutes with direct images for arbitrary morphisms $f: X \to Y$ between varieties $X, Y$ over $k$. By inverting $\ell$ and passing to the direct limit over all $\mathcal{O}_X \subset \Lambda$, we easily see that $\mathcal{F}$ induces a functor from $D$ to the derived category $D^b_{coh}(\mathcal{C}(X))$ of $\mathcal{C}(X)$-module sheaf complexes with bounded coherent sheaf cohomology (see loc.cit. p. 521). The functor thus obtained

$$\mathcal{F}: (D, *) \to (D^b_{coh}(\mathcal{C}(X)), \otimes_{\Omega_X}^L)$$

is a tensor functor, since $\mathcal{F}$ commutes with the convolution product; this follows from the arguments on p. 518 of [10]. Similarly, $\mathcal{F}: (D, *) \to (D^b_{coh}(\mathcal{C}(X)), \otimes_{\Omega_X}^L)$ can be defined as in loc. cit. Furthermore, as in [10, cor. 3.3.2], the specialization $Li^*_\chi : D^b_{coh}(\mathcal{C}(X)) \to D^b_{coh}(\Lambda)$, defined by the inclusion $i_\chi: \{\chi\} \hookrightarrow \mathcal{C}(X)$ of the closed point that corresponds to the character $\chi \in \mathcal{C}(X)$, has the property

$$Li^*_\chi(\mathcal{F}(K)) = R\Gamma(X, K_X).$$

For a complex $M$ of $R$-modules and a prime ideal $p$ of $R$ the small support $supp_R(M) = \{p| k(p) \otimes^L_R M \not\cong 0\}$ is contained in the support $Supp_R(M) = \{p| M_p \not\cong 0\}$.
0]. The latter is Zariski closed in \( \text{Spec}(R) \). For a noetherian ring \( R \) and a complex \( M \) of \( R \)-modules with bounded and coherent cohomology \( R \)-modules \( H^\bullet(M) \) both supports coincide: \( \text{supp}_R(M) = \text{Supp}_R(M) \). For the regular noetherian ring \( R = \Lambda \otimes_\mathbb{k} [\pi_1(X)_\ell] \) furthermore any object \( M \) in \( D^b_{\text{coh}}(R) \cong D^b_{\text{coh}}(\mathcal{C}(X)_\ell) \) is represented by a perfect complex, i.e. a complex of finitely generated projective \( R \)-modules of finite length. Notice that \( \text{Li}_\chi^*(\mathcal{F}(K)) = k(p) \otimes^L_R \mathcal{F}(K) \) holds for the maximal ideal \( p \) of \( R \) with residue field \( k(p) = R/p \), defined by \( \chi \).

**Spectrum.** By definition, for \( K \in \mathbf{P} \) the spectrum \( \mathcal{S}(K) \subseteq \mathcal{C}(X)(\Lambda) \) is the set of characters \( \chi \) such that \( H^\bullet(X, K_\chi) \neq H^0(X, K_\chi) \). By a result of Deligne the Euler characteristic does not change if \( K \) is twisted by a character: \( \chi(X, K_\chi) = \chi(X, K) \); see e.g. [11, cor. 6.4]. Under the assumption \( \chi(X, K) = 0 \), therefore the condition \( \chi \in \mathcal{S}(K) \) is equivalent to \( H^\bullet(X, K_\chi) \neq 0 \), and hence equivalent to \( R \Gamma(X, K_\chi) \neq 0 \).

**Lemma 1** For \( K \in \mathbf{P} \) with \( \chi(X, K) = 0 \), the set of characters \( \mathcal{S}(K) \cap \mathcal{C}(X)_\ell(\Lambda) \) is the set of closed points of a Zariski closed subset of \( \mathcal{C}(X)_\ell \).

**Proof** For \( \chi(X, K) = 0 \), as explained above a character \( \chi \in \mathcal{C}(X)_\ell(\Lambda) \) is in \( \mathcal{S}(K) \) if and only \( R \Gamma(X, K_\chi) \neq 0 \) holds, or equivalently if \( \text{Li}_\chi^*(\mathcal{F}(K)) = k(p) \otimes^L_R \mathcal{F}(K) \neq 0 \). This means \( \chi \in \text{supp}_R(\mathcal{F}(K)) \). Since \( R \) is noetherian and \( k(p) \otimes^L_R \mathcal{F}(K) \) has bounded coherent cohomology, this is equivalent to the condition that \( \chi \) is contained in the Zariski closed subset \( \text{Supp}_R(\mathcal{F}(K)) \). \( \square \)

**Monoidal perverse sheaves.** Monoidal objects arise as follows: Whereas the convolution of two sheaves almost never is a sheaf, the convolution of perverse sheaf complexes under favorable conditions is a perverse sheaf complex up to some “negligible” sheaf complexes contained in the radical \( N_D \), defined in section 6 of [11]. By [11, cor. 6.3, 6.4] a semisimple complex is negligible in this sense iff all simple perverse constituents of its perverse cohomology sheaves have vanishing Euler characteristic. Therefore we mainly focus on perverse sheaves in \( \mathbf{P} \). They define a semisimple abelian subcategory of \( \mathbf{D} \). However, attached to a perverse sheaf complex \( K \), the convolution product \( K^\vee \ast K \) in general is not a perverse sheaf complex. Hence the evaluation morphism \( \text{eval}_K : K^\vee \ast K \to \delta_0 \), with values in the skyscraper sheaf \( \delta_0 \) at the origin, is not a morphism between perverse sheaves. To measure the deviation, we consider the perverse truncation functors \( p^\tau_{<v} \) on \( \mathbf{D} \) to define the degree \( v_K \) of \( K \) to be the largest integer \( v \) such that \( \text{eval}_K \circ p^\tau_{<v} \) vanishes. Then \( \text{eval}_K \) induces a nontrivial morphism of the \( v_K \)-th perverse cohomology to \( \delta_0 \)

\[
p^H_{v_K}(K^\vee \ast K)[-v_K] \to \delta_0.
\]

Since \( v_K \otimes_L = \min(v_K, v_L) \) and \( v_K[n] = v_K \), by our semisimplicity assumptions one can assume that \( K \) is a simple perverse sheaf and that \( p^H_{v_K}(K^\vee \ast K)[-v_K] \) decomposes into a finite direct sum of simple perverse sheaves. Using this, it is then not difficult to show that for simple perverse \( K \) there exists a unique simple shifted perverse direct summand

\[
\mathcal{P}_K[-v_K] \subseteq p^H_{v_K}(K^\vee \ast K)[-v_K]
\]
on which the morphism induced by \( eval_K \) is nontrivial. See [17]. The so defined perverse sheaf \( \mathcal{P}_K \) has particular properties, similar to those of ambidextrous objects in a rigid symmetric monoidal abelian category, and is called the monoidal perverse sheaf associated to the simple perverse sheaf \( K \). For a more intrinsic characterisation of monoidal perverse sheaves we refer to [17, lemma 3].

For simple objects \( K \) in \( \mathbf{P} \) this defines an integer in \([0, \dim (X)]\), the degree \( v_K \) of \( K \), and an irreducible monoidal perverse sheaf \( \mathcal{P}_K \) in \( \mathbf{P} \). By [17, lemma 1.4] the Euler–Poincare characteristic \( \chi (X, K) \) of \( K \) on \( X \) is zero if and only if \( v_K > 0 \); furthermore \( \mathcal{P}_K \cong 1 \) (unit object) holds if and only if \( v_K = 0 \). \( \mathcal{P}_K \) is called a monoid in case \( v_K > 0 \). One can show \( v_K = v_{\mathcal{P}_K} \) and that \( \mathcal{P}_K \) is the monoidal perverse sheaf associated to \( \mathcal{P}_K \) itself. Furthermore, \( K \) is negligible if and only if \( \mathcal{P}_K \) is negligible resp. if and only if \( v_K > 0 \). For details see [17, lemma 1]. Finally, we quote from [17, cor. 4] the following statement

**Convexity lemma** For monoids \( P_1, P_2 \in \mathbf{P} \) of degrees \( v_1, v_2 \) on an abelian variety \( X \) with \( P_1 \not\subseteq P_2 \) the convolution product \( L = P_1 \ast P_2 \) has degree \( v_L > (v_1 + v_2)/2 \).

**Lemma 2** If \( v_K > 0 \) holds for a simple object \( K \in \mathbf{P} \), then \( \mathcal{I}(K) = \mathcal{I}(\mathcal{P}_K) \).

**Proof** As mentioned above, \( v_K > 0 \) implies \( \chi (X, K) = 0 \). From the discussion of Lemma 1, therefore the condition \( \chi \in \mathcal{I}(K) \) is equivalent to \( R\Gamma (X, K_\chi) = 0 \) respectively \( H^\bullet (X, K_\chi) = 0 \). Now we use the split monomorphisms \( K[\pm v_K] \hookrightarrow \mathcal{P}_K \) and \( \mathcal{P}_K[\pm v_K] \hookrightarrow K \ast K^\vee \) constructed in [17] in the section “reconstructibility” preceding [17, lemma 2]. Using \((A \ast B)_\chi \cong A_\chi \ast B_\chi \), the second one implies \( H^\bullet (X, (\mathcal{P}_K)_\chi) = 0 \). Hence \( v_\mathcal{P}_K > 0 \). Indeed, we therefore see that the assertions \( H^\bullet (X, K_\chi) = 0 \) and \( H^\bullet (X, (\mathcal{P}_K)_\chi) = 0 \) are equivalent. Hence \( \mathcal{I}(K) = \mathcal{I}(\mathcal{P}_K) \).

By [11, cor. 6.4] the negligible objects define a tensor ideal (property N1). If \( v_{K_i} > 0 \) for either \( i = 1 \) or \( i = 2 \), therefore all simple constituents \( K[n] \) of \( K_1 \ast K_2 \cong \bigoplus K[n] \) satisfy \( v_K > 0 \). In general, the semisimple complexes with simple constituents of vanishing Euler–Poincare characteristic define a tensor ideal \( N_{\text{Euler}} \) in \( \mathbf{D} \). All monoids are in this tensor ideal \( N_{\text{Euler}} \). For any semisimple complex \( K \) in \( N_{\text{Euler}} \), let \( \mathcal{I}(K) \) denote the set of \( \chi \in C(X) (\Lambda) \) for which \( H^\bullet (X, K_\chi) \neq 0 \). Then \( \mathcal{I}(K \ast K') = \mathcal{I}(K) \cup \mathcal{I}(K') \), and by the Künneth formula

\[
\mathcal{I}(K \ast K') = \mathcal{I}(K) \cap \mathcal{I}(K')
\]

holds for all semisimple complexes \( K, K' \) in \( N_{\text{Euler}} \).

**Lemma 3** If for a simple perverse sheaf \( K \) in \( N_{\text{Euler}} \subset \mathbf{D} \) and a character \( \chi_f \) of order prime to \( \ell \) the scheme \( Y = \mathcal{I}(K) \cap ((\chi_f) \times C(X)_\ell) \) contains an isolated closed point \( y \), then this scheme has only this closed point \( y \) and \( K \) is a character twist of the perverse sheaf \( \delta_X := \Lambda_X[\dim (X)] \).

**Proof** We may assume \( \chi_f = 1 \) by twisting \( K \). Then \( P := \mathcal{F}(K) \) is a complex in \( D^b_{\text{cob}}(R) \). For \( \chi \) corresponding to the isolated closed point \( y \in Y \subseteq C(X)_\ell \) let \( m_Y \) be the associated maximal ideal of \( R \) with residue field \( \Lambda_y \). Then \( R\Gamma (X, K_\chi) \cong \otimes \).
Li^\ast_x(\mathcal{F}(K)) \cong P \otimes_R^L \Lambda_y. We claim: H^i(X, K_\chi) \neq 0 for some \nu with |\nu| \geq \dim(X), so K_\chi \cong \delta_X follows from the known cohomological bounds for perverse sheaves K_\chi. To find \nu, it suffices to find i \geq j + 2 \dim(X) with H^{\mu}(X, K_\chi) \neq 0 for \mu = i, j. For that replace R by its localization at m_y, a regular local ring of dimension n = 2 \dim(X). Then P \in D^{[a, b]}_{coh}(R) for some integers a \leq b, i.e. H^i(P) = 0 holds for i \neq [a, b] with cohomology supported in \{y\}, and we may assume H^a(P) \neq 0 and H^b(P) \neq 0. For any such Q \in D^{[a, b]}_{coh}(R) we claim: H^{a-n}(Q \otimes_R^L \Lambda_y) \neq 0 and H^i(Q \otimes_R^L \Lambda_y) = 0 for i < a-n. By devissage, using truncation triangles \tau^{\leq a}(Q) \to Q \to \tau^{> a}(Q) \to and shifts, this claim is easily reduced to the case where a = b = 0. Then Q represents an R-module with support in m_y and \Lambda_y is the only simple R-module with support in m_y. Hence by devissage the claim is reduced to the case where Q is quasi-isomorphic to \Lambda_y. Then Q is represented by the perfect minimal Koszul complex Kos(\Lambda_y) = (0 \to Q_{-n} \to \cdots \to Q_0 \to 0) with Q_{-n} \cong Q_0 \cong R, so our claim now follows from H^i(Q \otimes_R^L \Lambda_y) = H^i(Kos(\Lambda_y) \otimes_R \Lambda_y) with H^{−n}(Q \otimes_R^L \Lambda_y) = Tor^R_n(\Lambda_y, \Lambda_y) \cong \Lambda_y. Under the assumptions on Q above, also H^b(Q \otimes_R^L \Lambda_y) \neq 0 and H^i(Q \otimes_R^L \Lambda_y) = 0 for i > b. Again one reduces this to the case of the Koszul complex Kos(\Lambda_y). Applied to P, this shows H^i(X, K_\chi) \neq 0 and H^i(X, K_\chi) \neq 0 for i = b and j = a − 2 \dim(X).

Lemma 4 For an irreducible perverse sheaf K on X, the group \Delta_K = \{\chi | K \cong K_\chi\} is a subgroup of the group \mathcal{C}(X)(\Lambda) of all characters \chi of \pi_1(X). It is a proper subgroup unless K is a skyscraper sheaf. More precisely, let A be the abelian subvariety generated by the support of the perverse sheaf K in X and let K(A) denote the subgroup of characters in \mathcal{C}(X)(\Lambda) whose restriction to A becomes trivial. Then K(A) is a subgroup of \Delta_K and the quotient \Delta_K/K(A) is a finite group.

Proof The group \Delta_K is not changed if we replace K by a translate. So we can suppose that the irreducible subvariety Y \subseteq X defined by the closure of the support of K contains the neutral element of X. The iterated sums W_d(Y) = \sum_{i=1}^d Y obtained from the addition law of X are irreducible subvarieties of X. Hence there exists an integer r such that W_r(Y) = W_{r+1}(Y) holds by dimension reasons. Then W_r(Y) + W_r(Y) = W_r(Y), and since 0 \notin W_r(Y) we conclude that A = W_r(Y) is an abelian subvariety of X, which by definition is the abelian subvariety generated by Y in X (in the formulation of the lemma or as in [16, page 125]). Suppose K is not a skyscraper sheaf. Then Y generates an abelian subvariety A \neq 0 of X. We may replace X by this subvariety A. By [16, lemma VI.13.3] there exists an affine homomorphism h : \hat{X} \to X with finite kernel and a section \alpha : Y \to \hat{X} that is maximal in the sense of loc. cit. page 125, so that h \circ \alpha is the inclusion Y \hookrightarrow X. Obviously, we can suppose that h is a homomorphism. Since Y generates X in the sense above, the morphism h has to be surjective and hence h is an isogeny of abelian varieties of degree say C. If the inclusion \alpha : Y \hookrightarrow X is maximal, then [16, prop. VI.17.14] implies that for every separable isogeny X' \to X the pullback by \alpha is an irreducible abelian covering of Y and hence \alpha^* : H^1(X, \Lambda) \to H^1(Y, \Lambda) is injective for \Lambda = \mathbb{Z}/N\mathbb{Z} and (N, char(k)) = 1. The proof of this assertion only uses the assumptions of loc. cit. section VI.13.

For the non-maximal case, \pi_1(Y, y_0) \to \pi_1(X, y_0) then has finite cokernel of index C. There exists a Zariski open dense subset U of Y and a smooth \Lambda-adic sheaf E on
Lemma 5 Let \( \rho \) be an irreducible representation of a group \( \Gamma \) on a finite-dimensional vectorspace over \( \Lambda \), and let \( \Delta \) be a finite group of abelian characters \( \chi : \Gamma \rightarrow \Lambda^* \), defining a normal subgroup \( \Gamma' = \text{Ker}(\Delta) \) such that \( \Gamma'/\Gamma' \cong \Delta^* \). Then \( \rho \otimes \chi \cong \rho \) for all \( \chi \in \Delta \) implies \( \rho \cong \text{Ind}_{\Gamma'}^{\Gamma}(\rho') \) for some irreducible representation \( \rho' \) of \( \Gamma' \). In particular

\[ \#\Delta \leq \#\Delta \cdot \dim_{\Lambda}(\rho') = \dim_{\Lambda}(\rho). \]

Proof For the convenience of the reader we give the proof. If \( \rho \cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0) \) for some subgroup \( \Gamma' \subseteq \Gamma_0 \subseteq \Gamma \), we may replace the pair \( (\Gamma, \rho) \) by \( (\Gamma_0, \rho_0) \). Indeed, \( \rho_0 \otimes (\chi|_{\Gamma_0}) \cong \rho_0 \) for \( \chi \in \Delta \) holds. To see this: \( \rho_0 \) is a constituent of \( \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0)|_{\Gamma_0} \cong \rho|_{\Gamma_0} \), and therefore also a constituent of \( (\rho \otimes \chi)|_{\Gamma_0} \). Hence \( \rho_0 \otimes (\chi|_{\Gamma_0}) \cong \rho_0 \) by Mackey’s lemma for some \( s \in \Gamma \), with \( s \) a priori depending on \( \chi \in \Delta \). But \( s \in \Gamma_0 \), since otherwise \( \rho_0 \) could be extended to a projective representation of \( (\Gamma_0, s) \subseteq \Gamma \), and this is easily seen to contradict the irreducibility of \( \rho \cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0) \). Therefore \( s \in \Gamma_0 \), and this implies our claim: \( \rho_0 \otimes (\chi|_{\Gamma_0}) \cong \rho_0 \) for all \( \chi \in \Delta \).

Using induction in steps, without loss of generality we can therefore assume that \( \rho \cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0) \) holds for any \( \Gamma_0 \) in \( \Gamma \) such that \( \Gamma' \subseteq \Gamma_0 \neq \Gamma \). Then we have to show \( \Gamma = \Gamma' \). If \( \Gamma' \neq \Gamma \), we may now also replace the group \( \Gamma' \) by some larger group \( \Gamma_0 \) with prime index in \( \Gamma_0 \). Then there exists a character \( \chi \in \Delta \) with kernel \( \Gamma_0 \). By Mackey’s theorem and \( \rho \cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0) \), the restriction \( \rho|_{\Gamma_0} \) is an isotypic multiple \( m \cdot \rho_0 \) of some irreducible representation \( \rho_0 \) of \( \Gamma_0 \). Therefore \( (\rho_0)^\gamma \cong \rho_0 \) holds for all \( s \in \Gamma \). Hence \( \rho_0 \) can be extended to a representation of \( \Gamma \) on the representation space of \( \rho_0 \) (there is no obstruction for extending the representation since \( \Delta_0 = \Gamma/\Gamma_0 \) is a cyclic group). By Frobenius reciprocity, this extension is then isomorphic to \( \rho \); so \( m = 1 \). In other words, the restriction of \( \rho \) to \( \Gamma_0 \) is an irreducible representation of \( \Gamma_0 \), hence equal to \( \rho_0 \).

Finally, \( \rho \otimes \chi \cong \rho \) implies \( \chi \hookrightarrow \rho^\vee \otimes \rho \) (as a one dimensional constituent). Therefore \( \bigoplus_{\chi \in \Delta} \chi \hookrightarrow \rho^\vee \otimes \rho \), as representations of \( \Gamma \). Restricted to \( \Gamma_0 \), this implies \( \#\Delta \cdot 1 \hookrightarrow \rho_0^\vee \otimes \rho_0 \), since \( \rho|_{\Gamma_0} \cong \rho_0 \). But \( \text{Hom}_{\Gamma_0}(\mathbf{1}, \rho_0^\vee \otimes \rho_0) \cong \text{Hom}_{\Gamma_0}(\rho_0, \rho_0) \cong \Lambda \) since \( \rho_0 \) is irreducible. Hence \( \#\Delta_0 = [\Gamma : \Gamma_0] = 1 \). This implies \( \Gamma = \Gamma_0 \), and hence \( \Gamma = \Gamma' \).

Proof Suppose \( \dim(X) > 0 \). Then for any finite set \( \{\mathcal{P}_1, \ldots, \mathcal{P}_m\} \) of monoids in \( \mathcal{P} \), there exist characters \( \chi \in \mathcal{C}(X)_\ell \) such that \( \chi \notin \bigcup_{i=1}^{m} \mathcal{I}(\mathcal{P}_i) \).

Proof Since the spectrum of \( R = \Lambda \otimes_{\mathcal{O}_X} \mathcal{O}_X[[x_1, \ldots, x_n]] \) is not the union of finitely many Zariski closed proper subsets for \( n = 2 \dim(X) > 0 \), it suffices that the spectrum \( \mathcal{I}(\mathcal{P})_\ell = \mathcal{I}(\mathcal{P}) \cap \mathcal{C}(X)_\gamma(\Lambda) \) of each monoid \( \mathcal{P} \) is the set of closed points of some proper Zariski closed subset of \( \mathcal{C}(X)_\ell \). We prove this by descending induction on the degree \( v_{\mathcal{P}} \). For \( v_{\mathcal{P}} = \dim(X) \) this is clear, since in this case \( \mathcal{I}(\mathcal{P}) \) is a single point [17, lemma 1]. For a given monoid \( \mathcal{P} \) and fixed \( v = v_{\mathcal{P}} < \dim(X) \),
assume our assertion is true for all monoids $\mathcal{D}$ of degree $v_\mathcal{D} > v$. By Lemma 4 there exists a character $\chi \in \mathcal{C}(X)_\ell$ such that $\mathcal{P}_\chi \not\subseteq \mathcal{P}$. Since $\mathcal{P}$ and $\mathcal{P}_\chi$ have the same degree $v = v_\mathcal{D}$, this implies that all constituents $K[m], K \in \mathcal{P}$ of $\mathcal{P} \ast \mathcal{P}_\chi$ have associated monoids $\mathcal{P}_K$ of degree $> (v_\mathcal{D} + v_\mathcal{P})/2 = v$ by the convexity lemma. Hence $\mathcal{I}(\mathcal{P} \ast \mathcal{P}_\chi)_\ell$ is contained in a proper Zariski closed subset of the spectrum $\mathcal{C}(X)_\ell$, by Lemma 2 and the induction assumption. Suppose $\mathcal{I}(\mathcal{P})_\ell$ were not contained in a proper Zariski closed subset of $\mathcal{C}(X)_\ell$. Then $\mathcal{I}(\mathcal{P})_\ell = \mathcal{C}(X)_\ell(\Lambda)$, and therefore $\mathcal{I}(\mathcal{P} \ast \mathcal{P}_\chi)_\ell = \mathcal{I}(\mathcal{P})_\ell \cap \mathcal{I}(\mathcal{P}_\chi)_\ell$. Hence $\mathcal{I}(\mathcal{P}_\chi)_\ell$ would be contained in a proper Zariski closed subset of $\mathcal{C}(X)_\ell$. Indeed, this would follow from $\mathcal{I}(\mathcal{P}_\chi)_\ell = \mathcal{I}(\mathcal{P})_\ell \cap \mathcal{I}(\mathcal{P}_\chi)_\ell = \mathcal{I}((\mathcal{P} \ast \mathcal{P}_\chi)_\ell)$ and the induction assumption. On the other hand, $\mathcal{I}(\mathcal{P}_\chi)_\ell = \chi^{-1} \mathcal{I}(\mathcal{P})_\ell = \mathcal{C}(X)_\ell(\Lambda)$. This gives a contradiction, and proves our claim for the fixed degree $v$. Now proceed by induction. 

For $K \in \mathcal{P}$ the $\ell$-spectra $\mathcal{I}(K_{\chi_f})_\ell = \mathcal{I}(K) \cap (\{\chi_f\} \times \mathcal{C}(X)_\ell(\Lambda)) \subseteq \mathcal{I}(K)$ at some given point $\chi_f$ of $\mathcal{I}(K)$ are the $\Lambda$-valued points of a Zariski closed subset of $\{\chi_f\} \times \mathcal{C}(X)_\ell$ by Lemma 1. Replacing $K$ by $K_{\chi_f}$ we may always assume $\chi_f = 1$.

**Corollary 1** For any semisimple complex $K \in \mathcal{D}$ contained in $\mathcal{N}_{Euler}$, there exists in $\mathcal{C}(X)_\ell(\Lambda)$ a character $\chi \notin \mathcal{I}(K)$.

**Proof** Since $\mathcal{I}(K) = \mathcal{I}(\mathcal{P}_K)$ for simple $K$ and $\mathcal{I}(\bigoplus_{i=1}^m K_i[n_i]) \subseteq \bigcup_{i=1}^m \mathcal{I}(K_i)$, this is an immediate consequence of Lemma 2 and Proposition 1.

**Theorem 1** For arbitrary $K \in \mathcal{P}$, the Euler–Poincare characteristic $\chi(X, K)$ is non-negative. Hence, in particular, the reductive supergroup $\mathcal{G}(K)$ attached to $K$ in [11, §7] is a reductive algebraic group over $\Lambda$.

**Proof** We may assume that $K$ is irreducible. Then, to show $\chi(X, K) \geq 0$, it is enough to show the existence of a character $\chi$ such that $H^v(X, K_\chi) = 0$ holds for all $v \neq 0$. Then $\chi(X, K) = \chi(X, K_\chi) = \dim_\Lambda(H^0(X, K_\chi))$, and the claim obviously follows from $\dim_\Lambda(H^0(X, K_\chi)) \geq 0$. So, we have to find a character $\chi \notin \mathcal{I}(K)$. Notice that by [11, cor. 6.4] the ideal $\mathcal{N}_{Euler}$ of negligible complexes in $\mathcal{D}$ satisfies the assumptions of [11, thm. 9.1]. Hence every semisimple perverse sheaf $K \in \mathcal{P}$ is an $\mathcal{N}_{Euler}$-multiplier in the sense of loc. cit, and therefore $T = \bigoplus_{v \neq 0}^PH^v(K^{*(v+1)})$ is a perverse sheaf in $\mathcal{N}_{Euler}$. As in the proof of [11, lemma 11.2(b)], one then shows for $g = \dim(X)$ that $H^\bullet(X, K_\chi) \neq H^0(X, K_\chi)$ holds if and only if $\chi \in \mathcal{I}(T)$. Hence, by Corollary 1 there exists a character $\chi \notin \mathcal{I}(T) = \mathcal{I}(K)$.

The crucial fact that $\mathcal{I}(K)$ is the spectrum $\mathcal{I}(T)$ for an object $T$ in $\mathcal{N}_{Euler}$, already exploited in the proof of the last theorem, furthermore implies.

**Theorem 2** For any $K \in \mathcal{P}$ on $X$ and any character $\chi_f$ of $\pi_1(X)$ of order prime to $\ell$, the set of characters $\chi \in \mathcal{C}(X)_\ell(\Lambda)$ for which $\chi_f \chi$ is in $\mathcal{I}(K)$ is the set of closed points of a proper Zariski closed subset of $\mathcal{C}(X)_\ell$.

For base fields $F$ of characteristic $p > 0$, the following corollary now easily follows from Theorem 1 by a specialization argument. For the case of fields $F$ of characteristic zero see [9]; but our argument could also be extended to the characteristic zero case.
Corollary 2 For \(\overline{\mathbb{Q}}_\ell\)-adic perverse sheaves \(K_0\) on abelian varieties \(X_0\) defined over a field \(F\) finitely generated over its prime field, with base extensions \(K\) resp. \(X\) to an algebraic closure of \(F\), the Euler–Poincare characteristic \(\chi(X, K)\) is non-negative.

**Translation invariance.** An irreducible perverse sheaf \(K\) on an abelian variety \(X\) will be called translation invariant if there exists an abelian subvariety \(A\) of \(X\) of dimension \(\dim(A) > 0\) such that one of the two equivalent conditions of the next lemma holds.

**Lemma 6** For an irreducible perverse sheaf \(K\) on \(X\) and an abelian subvariety \(A\) of \(X\) the following assertions are equivalent: (a) \(T^*_X(K) \cong K\) holds for all closed points of \(X\) or, resp. (b)

\[
K \cong L_X \otimes_{\Lambda_X} q^*(\tilde{K})[\dim(A)]
\]

holds for the quotient morphism \(q : X \to \tilde{X} = X/A\), some perverse sheaf \(\tilde{K}\) on \(\tilde{X}\) and some character \(\chi : \pi_1(X, 0) \to \Lambda^*\).

**Proof** \(K\) corresponds to a \(\Lambda\)-adic representation \(\phi : \pi_1(U) \to GL(n, \Lambda)\), for a suitable open Zariski dense subset \(U\) of the support \(Z\) of \(K\). If \(K\) is invariant under translations by the closed points in \(A\), the open subset \(U\) can be chosen such that \(U + A = U\). For \(\tilde{U} = U/A\) the projective morphism \(\pi_1(q|_U)\) induces an exact sequence \(\pi_1(A) \to \pi_1(\tilde{U}) \to 0\). Its first morphism \(\sigma\) is injective, since the composition \(\rho \circ \sigma\) with \(\rho : \pi_1(U) \to \pi_1(X)\) induced from \(U \hookrightarrow X\) is the injective morphism \(\pi_1(A) \to \pi_1(X)\) that is obtained from the inclusion \(A \hookrightarrow X\). Hence \(\pi_1(A)\) is a normal subgroup of \(\pi_1(U)\). We claim that \(\pi_1(A)\) is in the center of \(\pi_1(U)\). Indeed, for \(\alpha \in \pi_1(A)\) and \(\gamma \in \pi_1(U)\) there exists an \(\alpha' \in \pi_1(A)\) such that \(\gamma\alpha\gamma^{-1} = \alpha'\), and hence \(\rho(\gamma)\rho(\alpha)\rho(\gamma)^{-1} = \rho(\alpha')\). Since \(\pi_1(X) = H_1(X)\) is abelian, therefore \(\rho(\alpha) = \rho(\alpha')\). Because \(\rho \circ \sigma\) is injective, hence \(\alpha = \alpha'\). Now, since \(\pi_1(A)\) is a central subgroup of \(\pi_1(U)\), there exists a character \(\chi\) of \(\pi_1(A)\) such that \(\phi(\alpha\gamma) = \chi(\alpha)\phi(\gamma)\) holds for the irreducible representation \(\phi\) of \(\pi_1(U)\), for \(\alpha \in \pi_1(A)\) and \(\gamma \in \pi_1(U)\). Since \(\pi_1(\tilde{X})\) is a free \(\mathbb{Z}_\ell\)-module, any character \(\chi\) of \(\pi_1(A)\) with values in \(\Lambda^*\) can be extended to a character \(\chi_X\) of \(\pi_1(X)\). Thus \(\chi_X^{-1} \otimes \phi\) is an irreducible representation, which is trivial on \(\pi_1(A)\); so it defines an irreducible representation of \(\tilde{U}\) and an irreducible perverse sheaf \(\tilde{K}\) on \(\tilde{U}\) such that \(L^{-1}_{\chi_X} \otimes K = q^*(\tilde{K})[\dim(A)]\) holds on \(U\). Let \(\tilde{K}\) also denote the intermediate extension of \(\tilde{K}\) to the support \(\tilde{Z}\) of \(\tilde{K}\), which is an irreducible perverse sheaf on \(\tilde{Z}\). Since \(q : Z \to \tilde{Z}\) is a smooth morphism with connected fibers, \(q^*[\dim(A)]\) is a fully faithful functor from the category of perverse sheaves on \(\tilde{Z}\) to the category of perverse sheaves on \(Z\); see \([3, \text{prop. 4.2.5}]\). Hence \(P = q^*(\tilde{K})[\dim(A)]\) is an irreducible perverse sheaf on \(Z\). Since \(L^{-1}_{\chi_X} \otimes K\) and \(P\) are irreducible perverse sheaves on \(Z\) whose restrictions on \(U\) are nontrivial and isomorphic, the perverse sheaves \(L^{-1}_{\chi_X} \otimes K\) and \(P\) are isomorphic. This proves the nontrivial direction.

The characterization (b) of Lemma 6 implies \(\chi(X, K) = 0\) for translation invariant irreducible perverse sheaves \(K\). Over finite fields we have the following converse.

**Theorem 3** For an abelian variety \(X_0\) over a finite field \(k\), let \(K\) be a simple perverse sheaf on \(X\) defined over \(k\). If \(\chi(X, K) = 0\), then \(K\) is translation invariant.
The final aim of the proof is to study the spectrum locally in a neighborhood of a point, and in order not to loose the Galois action this specific point has to be a fixed point under the Frobenius action. We gather a couple of remarks before we start. The first remark shows the existence of Frobenius fixed points in the spectrum. The other remarks reduce the assertion of Theorem 3 to elementary statements on the geometry of nonarchimedian analytic varieties endowed with a Frobenius action that arise by localization, as formulated in Propositions 3 and 4. The key steps for this reduction to analytic geometry are explained in Remarks 2 and 3 below centering around Proposition 2. In fact, thus Theorem 3 will be reduced to Proposition 2, which in turn will follow from Proposition 3 resp. the related Proposition 4.

**Preliminary remarks.** (1) If $X$ is defined over $\kappa$, the Frobenius $F_\kappa$ acts on the points in $\mathcal{C}(X)(\Lambda)$ and $\mathcal{C}(X)(\Lambda)_\ell$. If $K$ is defined over $\kappa$, the Frobenius action preserves the spectrum $\mathcal{I}(K)$. A point $\chi$ in $\mathcal{C}(X)(\Lambda)$ is a fixed point under some Frobenius power $F_\kappa^m$ if and only if $\chi$ is defined over some finite field extension of $\kappa$, and this is the case if and only if $\chi$ is a torsion character of $\pi_1(X)$. Although a priori not obvious at all, if the perverse sheaf $K$ is not translation invariant, then under the conditions of Theorem 3 the spectrum $\mathcal{I}_\ell(K)$ must contain infinitely many torsion characters by [18, lemma 17]. So we can replace $K$ by a suitable twist with some torsion character $\chi$, thereby passing to a suitable finite extension field of $\kappa$ if necessary, so that $\mathcal{I}_\ell(K) = \mathcal{I}(K) \cap \mathcal{C}(X)(\Lambda)$ is nonempty and contains the trivial character $\chi_0$. The character $\chi_0$ is a fixed point of Frobenius $F_\kappa$, and by Lemma 3 we may suppose that $\chi_0$ is not an isolated point of the scheme $\mathcal{I}_\ell(K)$.

Remark (2) We may assume that $K$ is a monoidal perverse sheaf. Recall the monoidal perverse sheaf $\mathcal{P}_K$ associated to an irreducible perverse sheaf $K$ on $X$ and the corresponding degree $v_K$. By definition, up to a degree shift this perverse sheaf is a direct summand of $K^\vee \ast K$ for the convolution product of $K$ with its Tannaka dual $K^\vee = (id_X)^*(DK)$. Since $\mathcal{P}_K$ appears with multiplicity one in $p^*H^{-v_K}(K^\vee \ast K)$ by [17, lemma 1.6], it defines a pure Weil perverse sheaf on $X$ coming from a perverse sheaf $\mathcal{P}_{K_0}$ over $\kappa$. Hence we can assume $K_0(\alpha) = \mathcal{P}_{K_0}$ for a generalized Tate twist $K_0(\alpha)$ of $K_0$, defined by some $\alpha = \alpha(K_0)$ in $\Lambda^*$. Recall, $\chi(X, K) = 0$ is equivalent to $0 < v_K$. Since $v_K = v_{\mathcal{P}_K}$ holds by [17, lemma 3], this implies $\chi(K, \mathcal{P}_K) = 0$. Furthermore $\mathcal{P}_K$ is translation invariant if and only if $K$ is translation invariant; see [17, lemma 2.3]. So from now on suppose that $K$ is a monoidal perverse sheaf on $X$.

To prove Theorem 3 it then suffices to show

**Proposition 2** For an abelian variety $X_0$ over a finite field $\kappa$, any irreducible monoidal perverse sheaf $K$ on $X$ defined over $\kappa$ is of the form

$$K \cong L_X \otimes_{\Lambda_\kappa} \delta_A,$$

for an abelian subvariety $A$ of $X$ so that $v_K = \dim(A)$ and a character $\chi : \pi_1(X) \rightarrow \Lambda^*$ of finite order. Here $\delta_A$ denotes the constant perverse sheaf with support on $A$.

For the proof of Proposition 2, we may normalize $K$ as in Remark (1). We may furthermore replace $X$ by the abelian subvariety $A$ of $X$, generated by the support of the monoidal perverse sheaf $K$ in $X$. Then, by Lemma 4 we have
\[ \Delta_K := \{ \chi \mid K \cong K_\chi \} < \infty. \]

Now assuming \( X = A \), it suffices to show that \( \Delta_K < \infty \) implies \( v_K = \dim(X) \). Recall, by [17, lemma 1.2], for an irreducible monoidal perverse sheaf \( v_K = \dim(X) \) resp. \( v_K = 0 \) holds if and only if \( K \cong L_\chi \) holds for some character \( \chi \) of the fundamental group (i.e. translation invariance by \( X \)) resp. \( K \cong \delta_0 \) holds. In particular, \( v_K = 0 \) corresponds to the trivial case where the dimension \( \dim(A) \) of the support is zero. So, if Proposition 2 does not hold, there would exist a counterexample with finite \( \Delta_K \) for which \( 0 < v_K < \dim(X) \) holds. So, for the proof we will suppose that \( K \) is such a critical monoidal counterexample for which \( v_K \) is chosen maximal with respect to the property \( 0 < v_K < \dim(X) \), and then argue by contradiction.

Remark (3) Since \( \Delta_K < \infty \), we may assume that \( \mathcal{S}_\ell(K) \) does not contain an infinite abstract subgroup \( A \) of characters in \( \mathcal{G}(X)_\ell(\Lambda) \).

Indeed, otherwise we immediately get a contradiction that proves Proposition 2. This is shown as follows: Since \( A \) is a group, for \( \chi \in A \) we get \( \chi^{-1}A \cap A = A \) and hence

\[ A \subseteq \mathcal{S}(K_\chi) \cap \mathcal{S}(K). \]

Since \( K_\chi \not\cong K \) for almost all \( \chi \in A \subset \mathcal{S}(K)_\ell \), on the other hand by the maximality of the degree of \( v_K \),

\[ \mathcal{S}(K_\chi) \cap \mathcal{S}(K) = \mathcal{S}(K_\chi \ast K) \]

is a finite set for almost all \( \chi \in A \) by the convexity lemma. A contradiction.

Remark (4) Using a theorem of Drinfeld [7] and its variant for perverse sheaves [18, thm. 5], for our subsequent proof of Proposition 2 we may replace the \( \Lambda \)-adic perverse sheaf \( K \) by a \( \Lambda' \)-adic perverse sheaf \( K' \) for the algebraic closure \( \Lambda' \) of \( \mathbb{Q}_\ell \) for some other prime \( \ell' \) different from \( p \), without changing the Euler characteristic:

\[ \chi(X, K') = \chi(X, K) = 0. \]

Indeed, by the Grothendieck-Lefschetz etale fixed point formula the vanishing of the Euler characteristic of \( K \) can be expressed in terms of the functions \( f^K_m(x) \) for \( x \in X(\kappa_m) \) attached to the perverse sheaf \( K_0 \) on \( X_0 \) in [18], for all finite extension fields \( \kappa_m \) of \( \kappa \). Similarly, translation invariance of \( K \) resp. \( K' \) can also be expressed in terms of the functions \( f^K_m(x) \) resp. \( f^{K'}_m(x) \). This justifies our claim, since \( \tau(f^K_m(x)) = f^{K'}_m(x) \) holds for a suitable field isomorphism \( \tau : \Lambda \cong \Lambda' \).

Remark (5) The Frobenius substitution \( F_\kappa \) acts continuously on the Tate module \( H_1(X, \mathbb{Z}_\ell) \) of the abelian variety \( X \). Choosing a basis, we identify \( H_1(X, \mathbb{Z}_\ell) \) with \( \mathbb{Z}_\ell^n \). The associated \( \mathbb{Q}_\ell \)-linear map \( F_\kappa \) on \( H_1(X, \mathbb{Q}_\ell) \) can be diagonalized over some finite dimensional extension field \( E_\lambda \) of \( \mathbb{Q}_\ell \). Its eigenvalues \( \alpha_1, \ldots, \alpha_n \in E_\lambda^* \) are Weil numbers of weight \( w > 0 \), i.e. they are algebraic numbers and units at all nonarchimedean places outside \( p = char(k) \) with absolute value \( p^{w/2} \) at the archimedean places and \( E = \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \) is a finite extension of \( \mathbb{Q} \). So we can choose rational primes \( \ell' \) (for some large prime \( \ell' \neq p \)) in such a way that this extension splits over \( \ell' \), thereby changing the coefficient system \( \Lambda \) of \( K \) into \( \Lambda' \) via Remark (4) from above. Therefore, without restriction of generality, we can assume \( \alpha_1, \ldots, \alpha_n \in \mathbb{Z}_\ell \) so that
also $F_\kappa \in \text{Gl}(n, \mathbb{Z}_\ell)$ is diagonalized by conjugation within the group $\text{Gl}(n, \mathbb{Z}_\ell)$; look at the diagonalizing matrix in $\text{Gl}(n, E)$ and choose $\ell'$ large enough.

Remark (6) We identified $\mathcal{C}(X)_{\ell'}$ with the spectrum of the ring $R_n = \Lambda \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[t_1, \ldots, t_n]]$, viewing its closed points as characters of $\pi_1(X)$ with values in $\Lambda^*$. Thus we identify the closed points $\mathcal{F}_{\ell}(K)(\Lambda)$ of $\mathcal{F}_{\ell}(K)$ with characters in $\mathcal{C}(X)_{\ell'}(\Lambda)$.

By Remark (5) we can choose $\ell$ and coordinates $t_1, \ldots, t_n$ such that $F_\kappa(t_i) = \alpha_i \cdot t_i + P_i(t_i)$ holds for power series $P_i$ with leading degree $\geq 2$. Indeed

$$F_\kappa(\log(1 + t_i)) = \alpha_i \cdot \log(1 + t_i).$$

Notice, these are not power series in $R_n$. Instead we have to pass to the subring $A_n$ of locally convergent power series (see the appendix) in the formal completion $R_n = \Lambda[[t_1, \ldots, t_n]]$ of the ring $R_n$ with respect to the maximal ideal $(t_1, \ldots, t_n) \subset R_n$. In $A_n$ we dispose over new local parameters $x_i = \log(1 + t_i)$ of $A_n$. Since $\gamma_i = 1 + t_i$ are topological generators of $\pi_1(X)$, the natural action of $\text{Gl}(n, \mathbb{Z}_\ell)$ on $R_n$ carries over to a continuous action of $\text{Gl}(n, \mathbb{Z}_\ell)$ on $A_n \subset \hat{R}_n$ and $\hat{R}_n$, that is linear on the local parameters $x_1, \ldots, x_n$ of $A_n$ given by $x_i \mapsto \sum_{i,j} \varphi_{ij} x_j$, provided $\varphi$ is given by the matrix $(\varphi_{ij}) \in \text{Gl}(n, \mathbb{Z}_\ell)$. By Remark (5) we can therefore assume that the endomorphism $F_\kappa$ acts diagonally so that for $i = 1, \ldots, n$ the induced continuous ring automorphism $\varphi$ of $A_n$ is given by $\varphi(x_i) = \alpha_i \cdot x_i$ on the local parameters $x_i$.

Proof of the main theorem. The last Theorem 3 and Corollary 2 imply the main theorem, as stated in the introduction, for finite fields. Before we proceed with the proof of Theorem 3, let us briefly indicate how one extends the main theorem from the finite field case to the case of finitely generated fields by specialization. Let $E$ be finitely generated over its prime field, $X_0$ be a scheme of finite type over $E$ and $K_0$ a $\sigma$-adic complex on $X_0$ in $D^b_c(X_0, \sigma)$ for a finite extension ring $\sigma$ of $\mathbb{Z}_\ell$, $\ell$ different from $\text{char}(E)$. For an embedding $E \hookrightarrow \overline{E}$ into an algebraic closure of $E$, let $X$, $K$ be the extensions of $X_0$, $K_0$ to $\overline{E}$. Then following [8, section 4] we can choose a very good model $X_R$ of $X_0$ over some finitely generated ring $R$ with quotient field $E$ so that for every closed point $\nu$ of $\text{Spec}(R)$ with Henselization $R_\nu$, and geometric point $\nu$ over $\nu$ with strict henselization $R_\nu$ of $R$ at $\nu$ with embedding into $\overline{E}$ we obtain specializations $X_0$ resp. $\overline{X}$, obtained by the reduction of $X_R$ resp. $\overline{X}_u := X_R \times_{\text{Spec}(R)} \text{Spec}(R_u)$ with respect to $\nu$ and $\nu$ (notice, in loc..cit.our $\overline{X}$ is denoted $X_u$). Moreover: Associated to $K$, or finitely many such complexes on $X$, there exist categories $D_M(X, \sigma)$, $D_M(X_\nu, \sigma)$ and $D_M(\overline{X}, \sigma)$ and equivalences of categories between these categories [8, lemma 4.6] and [8, 4.9]. The precise nature of these categories is not important for us; perhaps it suffices that they are defined by finitely many perverse sheaves $M_i$ in $D^b_c(X_R, \overline{E})$ for $\overline{E}$ a finite field of characteristic $\ell$ that include an extension of $K_0 \otimes_{\mathbb{Z}_\ell} \sigma/\pi$ to $X_R$. This implies $K \in D_M(X, \sigma)$. The above equivalences of categories then relate $K$ on $X$ (over $\overline{E}$) to $K_u$ on $X_\nu$ (over $R_\nu$) resp. to $\overline{K}$ on $\overline{X}$ (over $k$), and this can be done for finitely many $K$ simultaneously (for suitably chosen $R$).

For these finitely many $K$ the so defined specialization $K \mapsto \overline{K}$ respects perverse sheaves [8, 4.9] and isomorphism between them, commutes with the action of $\text{Gal}(\overline{E}/E_\nu)$ for $E_\nu = \text{Quot}(R_\nu)$ [8, remark 4.7], and for suitable $R$ respects forming direct image complexes for finitely many fixed morphisms [8, 6.2.3]. Considering
the structure morphism, the latter implies that we can assume that this specialization respects Euler–Poincare characteristics. Hence $\chi(K) < 0$ resp. $\chi(K) = 0$ implies $\chi(\overline{K}) < 0$ resp. $\chi(\overline{K}) = 0$. Passing to coefficients in $\Lambda = \overline{\mathbb{Q}}_\ell$, still associated perverse sheaves specialize to perverse sheaves so that simple ones specialize to simple ones. If $K$ is $Gal(\overline{E}/E)$-equivariant, then it is $Gal(\overline{E}/E_v)$-equivariant and hence $\overline{K}$ is a Weil sheaf on $\overline{X}$. Hence for absolute simple $K_0$ on $X_0$ the specialization $\overline{K}$ of $K$ is a simple perverse Weil sheaf on $\overline{X}$. So, up to a generalized Tate twist, by [13, cor. VII.8], [3, cor. 5.3.2] the perverse sheaf $\overline{K}$ is pure of weight zero and descends to a finite field. By Theorem 1 this contradicts $\chi(\overline{K}) < 0$, and shows for $\chi(K) = 0$ that $\overline{K}$ is translation invariant by an abelian subvariety $Y$ of $\overline{X}$ of positive dimension by Theorem 3. If $K$ is not translation invariant on $X$ in a similar way, there exists a finite set $S$ of closed torsion points $x$ of $X$ such that $T^*_x(K) \not\cong K$ holds for all $x$ not in $S$ [18, lemma 21]. We can assume that all points in $S$ are $N$-torsion points for some integer $N$. If we arrange the specialization procedure simultaneously for all translates $T^*_x(K)$ by $\ell N$ torsion points, then we get a contradiction since $Y$ always contains reductions $\overline{X}$ of $\ell N$-torsion points $x$ that are not in the reduction of the $N$-torsion points in $S$. Then $T^*_x(\overline{K}) \cong T^*_x(\overline{K})^* \cong \overline{K}$, but $T^*_x(K) \not\cong K$. This contradiction proves the main theorem.

This being said and with the above preliminary remarks in mind, let us start to explain the strategy for the proof of Theorem 3. As already explained in Remark 2, for Theorem 3 it suffices to prove Proposition 2.

Strategy of proof. We prove Proposition 2 by contradiction. In Proposition 3 below we construct an infinite subgroup $A$ of the spectrum $\mathcal{S}_\ell(K)$ and thus derive the desired contradiction from Remarks 2 and 3 above. As a subscheme of $\mathcal{S}(X)_\ell$, the spectrum $\mathcal{S}_\ell(K)$ of $K$ defines an ideal of the ring $R_n \cong \Lambda \otimes_{\mathbb{Q}_\ell} \mathfrak{a}_K[[\pi_1(X)_\ell]]$. The Frobenius automorphism $F_\kappa$ acts on $\pi_1(X)_\ell \cong H_1(X, \mathbb{Z}_\ell)$. This action extends to a continuous action of $F_\kappa$ by ring automorphisms of $R_n$. Since the perverse sheaf $K$ is defined over $\kappa$, its spectrum $\mathcal{S}_\ell(K)$ is invariant under the Frobenius automorphism on $\mathcal{S}(X)_\ell$. Enlarging the finite ground field $\kappa$, hence replacing the Frobenius by a suitable power, we may assume that the Frobenius stabilizes each irreducible component of $\mathcal{S}_\ell(K)$, so in particular stabilizes the irreducible component of $\mathcal{S}_\ell(K)$ that contains the trivial character. This irreducible component $Spec(R_n/I)$, defined by an ideal $I \subset R_n$, has Krull dimension $\dim(R_n/I) \geq 1$ by Remark 1 above. Since the trivial character corresponds to the maximal ideal $m = (t_1, \ldots, t_n)$ of $R_n$, we get $I \not\subset m$. By Theorem 2 we also know $I \neq 0$. Therefore

$$\{0\} \subsetneq I \subsetneq m.$$  

We say that an ideal $J \subset R_n$ is defined over $\kappa$ if $F_\kappa(J) = J$ holds, i.e. if it is stable under the action of the Frobenius $F_\kappa$. Replacing the component by its reduced subscheme, we may suppose that $I$ is a prime ideal of $R_n$ defined over $\kappa$. Then we apply the next proposition to complete the proof of Proposition 2 and Theorem 3.

**Proposition 3** For any prime ideal $I \subset R_n$ defined over $\kappa$ that contains the trivial character in its spectrum, either $I$ is the maximal ideal $m$, or there exists an infinite...
subset $A$ in the $\Lambda$-valued closed points of its spectrum, so that $A$ defines an abstract group when viewed as a subset of the group of characters $\mathcal{C}(X)_\ell(\Lambda)$.

We start with some general remarks. We assumed $I \subseteq m$. The ideals $I \subseteq m \subset R_n$ correspond one-to-one to ideals in the localization $\hat{R}_n = (R_n)_m$ of $R_n$ at the maximal ideal $m = (t_1, \ldots, t_n)$. The completion of $R_n$ or $\hat{R}_n$ at the maximal ideal $m$ is the ring of formal power series in the variables $t_1, \ldots, t_n$ over the field $\Lambda$ that contains the subring $A_n = \Lambda[[t_1, \ldots, t_n]]$ of locally convergent power series in $t_1, \ldots, t_n$, so that

$$R_n \subset \hat{R}_n \subset A_n \subset \hat{R}_n.$$ 

Since the power series $\log$ and $\exp$ are locally convergent, the coordinate substitutions $x_i = \log(1 + t_i)$ and $1 + t_i = \exp(x_i)$ for $i = 1, \ldots, n$ define an automorphism of $A_n$. Hence $A_n = \Lambda[[x_1, \ldots, x_n]]$. The ring automorphism $F_\kappa : R_n \to R_n$, corresponding to the Weil numbers $\alpha_1, \ldots, \alpha_n \in \Lambda^*$ of weight $w > 0$, extends to the completion $\hat{R}_n$ of $R_n$ and preserves the subrings $\hat{R}_n$ and $A_n$. In the coordinates $x_i$ it induces the continuous ring automorphism $\varphi : A_n \to A_n$ (and also of $\hat{R}_n$) defined by

$$\varphi(x_i) = \alpha_i \cdot x_i.$$ 

We say an ideal $J$ of $A_n$ is a $\varphi$-ideal if $\varphi(J) = J$ holds. An ideal $J \subseteq m \subset R_n$ defined over $\kappa$ extends by completion to a $\varphi$-ideal $\hat{J}$ of $\hat{R}_n$, or $\hat{J} \cap A_n$ of $A_n$. For what follows, notice these easy facts: The radical ideal of a $\varphi$-ideal is a $\varphi$-ideal. For a $\varphi$-ideal $J$ in $A_n$ its prime components are permuted by $\varphi$. If we replace the automorphism $\varphi$ by a suitable power $\varphi^k$, without restriction of generality we may assume that $\varphi$ stabilizes all irreducible components of $J$.

**Proof of Proposition 3** Suppose $I$ is generated by power series $P_j(T)$, $j \in M$ as an ideal in $R_n$. Then, as explained in the appendix, the $\varphi$-ideals $\hat{I}$ in $\hat{R}_n$ resp. $\hat{I} \cap A_n$ in $A_n$ are generated by the locally convergent power series $p_j(X) := p_j(x_1, \ldots, x_n) = P_j(\exp(x_1) - 1, \ldots, \exp(x_n) - 1)$ for $j \in M$, and $I$ is maximal in $R_n$ iff $\hat{I} \cap A_n$ is maximal in $A_n$ iff $\hat{I}$ is maximal in $\hat{R}_n$. For the proof we can assume $I \neq m$ without restriction of generality, hence dim($A_n/(\hat{I} \cap A_n)$) > 0 holds for the $\varphi$-ideal $\hat{I} \cap A_n$. $R_n$ is noetherian, hence we can assume that $M$ is finite. By the next proposition, there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ so that all $p_j(\lambda_1 \cdot x, \ldots, \lambda_n \cdot x) \in \Lambda$ are convergent and vanish for $j \in M$ and all $x \notin \Lambda$ sufficiently small, i.e. $|x| < \varepsilon$ for some $\varepsilon$. Indeed, since also the ring $A_n$ is noetherian, the $\varphi$-ideal $\hat{I} \cap A_n$ has only finitely many prime components. By replacing $\hat{I} \cap A_n$ with one of these prime components and $\varphi$ by a finite power of $\varphi$, the assumptions of Proposition 4 are satisfied. For $x \in \Lambda$ of sufficiently small absolute value $|x| < \varepsilon$ the exponential map $(\exp(\lambda_1 \cdot x), \ldots, \exp(\lambda_n \cdot x))$ converges and has values contained in the open poly disk $\{(t_1, \ldots, t_n) \in \Lambda^n \mid |t_i| < 1 \text{ for all } i = 1, \ldots, n\}$ whose points are the $\Lambda$-values closed points of $R_n$, and allows to defines the desired infinite abstract subgroup $A$ of points with the property $A \subset \text{Spec}(R_n/I)(\Lambda) \subseteq \text{Spec}(R_n)(\Lambda)$.

**Proposition 4** For a prime $\varphi$-ideal $I$ contained in the maximal ideal $m$ of $A_n$, but different from $m$, there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ not all zero, so that $I$ is in the kernel of

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the evaluation homomorphism \( \pi : A_n \to A_1 = \Lambda[[x]] \) defined by \( x_i \mapsto \lambda_i \cdot x \) for \( i = 1, \ldots, n \) such that the following compatibility conditions hold: After replacing \( \varphi \) by a suitable power, \( \alpha := \alpha_i \) does not depend on \( i \) for all indices \( i \) for which \( \lambda_i \neq 0 \) holds.

**Proof** We prove this for the one-dimensional cases \( \dim(A_n/I) = 1 \) first. We remark that if \( \pi \) exists in a one-dimensional case, then \( I = \text{Kern}(\pi) \) by dimension reasons. Since the case \( n = 1 \) is trivial, assume \( n = 2 \). Then \( I \neq \{0\} \) and \( I \neq m \), and \( 0 \leq I \) implies \( ht(I) \geq 1 \) and \( I \subseteq m \) implies \( \dim(A_2/I) \geq 1 \). Because for prime ideals \( I \) the inequality \( ht(I) + \dim(A_2/I) \leq \dim(A_2) \) holds, hence \( I \) is a prime ideal of height 1. Since \( A_2 \) is factorial (Lemma 7), the prime ideal \( I \) is a principal ideal of \( A_2 \) (Lemma 8) generated by some power series \( f(x_1, x_2) \in A_2 \). Since \( I \) is a \( \varphi \)-ideal,

\[
f(\alpha_1 \cdot x_1, \alpha_2 \cdot x) = u(x_1, x_2) \cdot f(x_1, x_2)
\]

holds for some unit \( u(x_1, x_2) \in A_2 \). In the completion \( \hat{R}_2 \) of \( A_2 \) we can find another unit \( h(X) \in 1 + (x_1, \ldots, x_2) \subset \hat{R}_2 \) and some constant \( c \in \Lambda^* \) so that \( u = c \cdot h/ h^\varphi \) holds. We construct this unit \( h(X) \) in \( \hat{R}_2 \) for \( h_v = \prod_{i=1}^{v} (1 + zi) \) with \( zi \in m^i \) as an infinite product \( h = \lim_{v \to \infty} h_v \), requiring \( h_v/h_v^\varphi = uc^{-1} + y_v \) for \( y_v \in m^{v+1} \). We define \( z_v \) recursively by solving the equation \( \varphi(z_v) - z_v = cy_{v-1}u^{-1} \) modulo \( m^{v+1} \), using that \( \varphi - id \) acts on \( m^\varphi/m^{\varphi+v} \) by an invertible \( \Lambda \)-linear map. So the formal power series \( g(X) = f(X) \cdot h(X) \) is invariant under \( \varphi \), in the sense that the following identity \( g(\alpha_1 \cdot x_1, \alpha_2 \cdot x_2) = c \cdot g(x_1, x_2) \) holds. This implies that \( c \) is finite product of \( \alpha_1 \) and \( \alpha_2 \) and hence a Weil number of weight \( k \cdot w \) for some integer \( k \), so \( g(x_1, x_2) \) must be a homogeneous polynomial of degree \( k \) in \( x_1, x_2 \). The ideals \( (f(X)) \subset A_2 \) and \( (g(X)) \subset A_2 \) become equal in the completion \( \hat{R}_2 \). Since \( A_2 \) is a Zariski ring, this implies that \( f(X) \) and \( g(X) \) generate the same ideal in \( A_2 \). Hence \( I = (g(X)) \) and because \( g(X) \) is homogeneous, we have \( g(X) = \prod_j (\gamma_j'x_1 - \gamma_jx_2) \) for certain \( \gamma_j, \gamma_j' \in \Lambda \). Since \( I \neq 0 \) is prime, the polynomial \( g(X) = \gamma' \cdot x_1 - \gamma \cdot x_2 \neq 0 \) must be linear. Without restriction of generality we can assume \( \gamma \neq 0 \). Then \( \lambda_1 = 1, \lambda_2 = \gamma'/\gamma \) define the required substitution \( \pi \).

For \( n \geq 3 \) we use induction. If \( I \subset (x_2, \ldots, x_n) \subset A_n \), the assertion is clear. Indeed \( m' = (x_2, \ldots, x_n) \) is a prime ideal of \( A_{n-1} \) and \( A_n/m' \) and \( A_n/I \) both have dimension 1, hence \( I = m' \). Put \( \lambda_1 = 1 \) and \( \lambda_i = 0 \) for \( i \geq 2 \). So we can assume that \( I \) contains an element \( p(X) \) not in \( m' \). Up to a scalar in \( \Lambda \), then \( p(x_1, 0, \ldots, 0) = x_1^a + \sum_{i > a} b_i \cdot x_1^i \) for some \( a \in \mathbb{N} \) and \( b_i \in \Lambda \).

By the Weierstraß preparation theorem we have \( p(X) = (x_1^a + \sum_{i=0}^{a-1} c_i(x_2, \ldots, x_n) \cdot x_1^i) \cdot u(X) \) for a unit \( u(X) \in A_n \) with coefficients \( c_i(X') \in A_{n-1} \) by Lemma 7. We can therefore assume

\[
p(X) := x_1^a + \sum_{i=0}^{a-1} c_i(x_2, \ldots, x_n) \cdot x_1^i \in I.
\]

Intersecting with the subring \( A_{n-1} \subset A_n \) analytically generated by \( x_2, \ldots, x_n \), we now define the prime ideal \( J := I \cap A_{n-1} \subset m' \subset A_{n-1} \). It is a \( \varphi \)-ideal, for the
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Restriction of \( \varphi \) to \( A_{n-1} \). Notice, \( A_{n-1}/J \hookrightarrow A_n/I \) is a finite injective ring extension. Indeed, \( A_n \) is finite over the subring \( S = \Lambda[[p(X), x_2, \ldots, x_n]] \) by Lemma 9 and \( S/(S \cap I) \hookrightarrow A_n/I \) is a finite injective extension of rings. Furthermore \( S \cap I = \Lambda[[p(X), x_2, \ldots, x_n]] \cap I = p(X) \cdot S + (A_{n-1} \cap I) \), hence \( S/(S \cap I) \cong A_{n-1}/J \). Since for finite injective ring extensions the Krull dimension remains the same, this implies \( \dim(A_{n-1}/J) = 1 \). So, by the induction assumption, the prime ideal \( J \) is the kernel of a surjective ring homomorphism \( \pi' : A_{n-1} \twoheadrightarrow A_1 = \Lambda[[x]] \) given by \( x_i \mapsto \lambda_i' \cdot x \) for \( i = 2, \ldots, n \) for some \( \lambda_i' \) satisfying the compatibility conditions for \( \alpha_2, \ldots, \alpha_n \). Consider the diagram

\[
\begin{array}{ccc}
A_n & \xrightarrow{\pi} & A_2 = \Lambda[[x_1, x]] \\
\downarrow & & \downarrow \\
S = \Lambda[[p(X), x_2, \ldots, x_n]] & \xrightarrow{\pi'} & A_1 = \Lambda[[x]]
\end{array}
\]

for the corresponding evaluation map \( \pi : x_i \mapsto \lambda_i' \cdot x \) for \( i = 2, \ldots, n \) and \( \pi(x_1) = x_1 \). Notice, \( g(x_1, x) \in \pi(I) \) for \( g \in A_2 \) holds iff \( g(x_1, x) = f(x_1, \lambda_2' \cdot x, \ldots, \lambda_n' \cdot x) \) holds for some \( f(X) \in I \). Hence \( g(\alpha_1 \cdot x_1, \alpha \cdot x) = f(\alpha_1 \cdot x_1, \alpha \lambda_2' \cdot x, \ldots, \alpha \lambda_n' \cdot x) = f(\alpha_1 \cdot x_1, \alpha_2 \lambda_2' \cdot x, \ldots, \alpha_n \lambda_n' \cdot x) \) is also in \( \pi(I) \), using that \( f^\varphi \in I \) holds for \( f \in I \) and a \( \varphi \)-ideal \( I \). In other words, \( \pi(I) \) is a \( \varphi \)-ideal in \( A_2 \) where \( \varphi \) acts by the eigenvalues \( \alpha_1 \) resp. \( \alpha \) on \( x_1 \) resp. \( x \). Since \( p(X) \in I \) maps to \( q(x_1, x) \neq 0 \), the image ideal is not zero \( \pi(I) \neq 0 \). A power series \( f(X) = \sum_{i=0}^\infty c_i(X') \cdot x_1^i \in A_n \) is in \( K := \text{Kern}(\pi : A_n \twoheadrightarrow A_2) \) if and only if \( c_i \in J \) holds for all \( i \), hence \( K = A_n \cdot J \cap A_n = A_n \cdot J \subseteq I \). We obtain

\[
\begin{array}{ccc}
\pi(I)^\varphi & \longrightarrow & A_2 \\
\downarrow & & \downarrow \\
\pi(I \cap S) = (q)^\varphi & \longrightarrow & \pi(S)
\end{array}
\]

and \( \pi(I \cap S) = \pi((I + K) \cap (S + K)) = \pi(I \cap (S + K)) = \pi((I \cap S) + K) = \pi(I \cap S) \). Since \( A_n \) is finite over \( S \), \( \pi'(A_{n-1}) \cong \pi(S/I \cap S) = \pi(S)/\pi(I \cap S) = \pi(S)/\pi(S) \cap \pi(I) \hookrightarrow A_2/\pi(I) \) is finite and injective again. Since the dimension of \( \pi'(A_{n-1}) \cong A_1 \) is one, the dimension of \( A_2/\pi(I) \) is one and therefore the induction assumption applies. Hence \( \pi(I) \) is the kernel of a homomorphism \( \tilde{\pi} : A_2 \rightarrow \Lambda[[y]] \) defined by \( x_1 \mapsto \lambda_1 \cdot y \) and \( x \mapsto \tilde{\lambda} \cdot y \), satisfying the compatibility condition for \( \alpha_1 \) and \( \alpha \). Because the \( x_1 \)-regular power series \( g \) is contained in \( \pi(I) \), this implies \( \tilde{\lambda} \neq 0 \).

The composition

\[
\begin{array}{ccc}
A_n & \xrightarrow{\pi} & A_2 \xrightarrow{\pi'} \Lambda[[y]]
\end{array}
\]

defines the desired specialization \( x_1 \mapsto \lambda_1 \cdot y \) and \( x_i \mapsto \lambda_i \cdot y \) with \( \lambda_i = \tilde{\lambda} \lambda_i' \) for \( i = 2, \ldots, n \). It is easy to see that the compatibility conditions for the \( \alpha_1, \ldots, \alpha_n \) are
satisfied. For \( \lambda_1 = 0 \) there is nothing to show, and \( \lambda_1 \neq 0 \) implies \( \alpha_1 = \alpha \). Since \( \alpha_i = \alpha \) for all \( i \geq 2 \) with \( \lambda_i \neq 0 \), this completes the proof of the one-dimensional case of Proposition 4.

For the higher dimensional situation, we use both induction on \( n \) and then for fixed \( n \) induction on \( r = \dim(A_n/I) \). Suppose \( I \neq m \) is a prime \( \varphi \)-ideal in \( A_n \), now of dimension \( r = \dim(A_n/I) > 1 \). Consider the intersection of \( \Spec(A_n/I) \) with the hypersurface \( \Spec(A_n/P) \equiv \Spec(A_{n-1}) \) defined by the \( \varphi \)-ideal \( P = (x_1) \subset A_n \). Then either \( P \subseteq I \), i.e. \( \Spec(A_n/I) \) is contained in the hypersurface \( \Spec(A_n/P) \), and we may replace \( A_n \) by \( A_n/P = A_{n-1} \) and \( I \) by \( I/P \) to conclude by induction on \( n \) using \( \dim((A_n/P)/(I/P)) = \dim(A_n/I) \neq 0 \). Or the intersection of \( \Spec(A_n/I) \) with \( \Spec(A_n/P) \), given by the \( \varphi \)-ideal \( (I, x_1) \subseteq m \subset A_n \), has the property \( I \neq (I, x_1) \). Since \( I \) is prime, \( A_n/I \) is a domain. The image \( \overline{x}_1 \) of \( x_1 \) in \( A_n/I \) therefore is not a zero divisor nor a unit. Since \( A_n/I \) is local and noetherian, therefore \( \dim(A_n/(I, x_1)) = \dim((A_n/I)/\overline{x}_1(A_n/I)) = \dim(A_n/I) - 1 \) by [2, 11.18]. We then find a minimal prime ideal of \( A_n/(I, x_1) \) whose preimage \( I' \) in \( A_n \) satisfies \( \dim(A_n/I') = \dim(A_n/I) - 1 = r - 1 > 0 \), hence \( I' \neq m \). Since the prime ideal \( I' \) defines one of the finitely many irreducible components of the spectrum of the \( \varphi \)-ideal \( (I, x_1) \), replacing \( \varphi \) with some power we can assume that \( I' \) is a \( \varphi \)-ideal of \( A_n \).

Hence there exists \( \pi' : A_n/I' \to A_1 \) by the induction assumption on \( r \), so that finally \( \pi : A_n/I \to A_n/I' \to A_1 \) defines the desired evaluation morphism.

Appendix

Let \( A_n = \Lambda[[t_1, \ldots, t_n]] \) denote the ring of power series in the variables \( t_1, \ldots, t_n \) with coefficients in some finite extension field \( E_\lambda \) of \( \mathbb{Q}_\ell \) that are convergent in \( \Lambda \) with some positive radius \( r \) of convergence with respect to the nonarchimedean norm on \( \Lambda \), with \( r \) and \( E_\lambda \) depending on the power series.

Lemma 7 The Weierstraß preparation theorem holds for \( A_n \). The ring \( A_n \) is a regular noetherian local ring of Krull dimension \( n \), hence \( A_n \) is a normal factorial domain.

Proof Any substitution \( t_i \mapsto \lambda_i \cdot t_i \) for \( \lambda_i \in \Lambda^* \) defines an automorphism of \( A_n \). For the proof of the Weierstraß preparation theorem suppose given \( G \in \mathfrak{a}_n \) and a \( t_1 \)-regular \( F \in \mathfrak{a}_n \) such that \( F(t_1, 0, \ldots, 0) = c \cdot t_1^a \) plus terms of higher order. We have to show the existence of \( U \in \mathfrak{a}_n \) and \( R_0, \ldots, R_{a-1} \in \mathfrak{a}_{n-1} = \Lambda[[t_2, \ldots, t_n]] =: \Lambda[[T']] \) such that \( G(T) = U(T) \cdot F(T) + \sum_{i=0}^{a-1} R_i(T') \cdot t_1^i \). We may assume that \( F \) is not a unit in \( \mathfrak{a}_n \) and that \( G(0, \ldots, 0) \) has absolute value \( \leq 1 \). Then, by a suitable substitution \( T \mapsto \lambda \cdot T \), we can assume \( F, G \in \mathfrak{a}_n[[t_1, \ldots, t_n]] \subset \mathfrak{a}_n \) for some subring \( \mathfrak{a}_n \) of \( \Lambda \) that is finite over \( \mathbb{Z}_\ell \) with maximal ideal \( m_\lambda \). Replacing \( F \) by \( c^{-1-a} F(ct_1, c^{a+1} t_2, \ldots, c^{a+1} t_n) \) for some nonzero \( c \in \mathfrak{a}_n \), we may assume \( F(t_1, 0, \ldots, 0) = t_1^a + \sum_{i \geq 0} c_i(T') \cdot t_1^i \) for \( c_i(T') \in \mathfrak{a}_{n-1} \) and \( c^{-1-a} F(ct_1, c^{a+1} t_2, \ldots, c^{a+1} t_n) \in \mathfrak{a}_n[[t_1, \ldots, t_n]] \). Again, by replacing \( F(T) \) with \( b^{-a} F(bt_1, b^{a+1} t_2, \ldots, b^{a+1} t_n) \) for some \( b \in m_\lambda \), we can assume \( F(T) \equiv t_1^a \) modulo \( (m_1, t_2, \ldots, t_n) \). Since \( G(bct_1, c(bc)^a t_2, \ldots, c(bc)^a t_n) \) is in \( \mathfrak{a}_n[[t_1, \ldots, t_n]] \) and \( b^a c^{1+a} \) is a unit in \( \mathfrak{a}_n \), the proof of the preparation theorem is reduced to [10, prop. A.2.1(i)], i.e. the assertion that the Weierstraß preparation theorem holds for \( \mathfrak{a}_n[[t_1, \ldots, t_n]] \). Since, up to a linear coordinate change, any nontrivial
ideal $I$ of $A_n$ contains a $t_1$-regular element $F(T) = t_1^a + c_1(T')t_1^{a-1} + \cdots + c_a(T')$ with coefficients $c_v \in \Lambda[[t_2, \ldots, t_n]] \subseteq A_{n-1}$, by the Weierstraß preparation theorem then $F$ and $I' = I \cap \{A_{n-1}t_1^{a-1} + \cdots + A_{n-1}\}$ generate $I$. To show that $A_n$ is noetherian we can assume $A_{n-1}$ to be noetherian by induction, so $I'$ is a finitely generated $A_{n-1}$-module and its generators together with $F$ generate $I$ as an $A_n$-module. This proves that $A_n$ is noetherian. It is easy to see that for any powers series $F(T) \in A_n$ with $F(0) \neq 0$ the formal power series $1/F(T)$ again has positive radius of convergency. Hence $A_n$ is a local ring with maximal ideal $m = \langle t_1, \ldots, t_n \rangle$. Since $\hat{A}_n$ is isomorphic to the regular ring $\hat{R}_n$ of formal powers series over $\Lambda$, $\hat{A}_n$ is a regular local ring. □

The regular noetherian ring $R_n = \Lambda \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[t_1, \ldots, t_n]]$ is a subring of the ring $A_n$. The completions $\hat{A}_n$ resp. $\hat{R}_n$ of $A_n$ resp. $R_n$ with respect to $m = \langle t_1, \ldots, t_n \rangle$ coincide with the formal power series ring $\Lambda[[t_1, \ldots, t_n]]$. Any power series $P(t_1, \ldots, t_n) \in R_n$ with $P(0, \ldots, 0) \neq 0$ is a unit in $A_n$. Hence the localisation $\hat{R}_n = (R_n)_m$ of $R_n$ with respect to its maximal ideal $m = \langle t_1, \ldots, t_n \rangle$ is a local noetherian subring of $A_n$. Both local rings $\hat{R}_n$ and $A_n$ have the same completion (for their maximal ideals), namely the formal power series ring $\hat{R}_n$. The ideals $I$ in $m \subset R_n$ correspond one-to-one to ideals in $\hat{R}_n$. Since Noetherian local rings are Zariski rings [21, p. 264], any ideal $I$ of a noetherian local ring $R$ can be recovered from its completion $\hat{I}$ by intersection $I = \hat{I} \cap R$, and furthermore $\hat{I} = I \cdot \hat{R}$ holds. See [21, VIII, §2, thm. 5, cor. 2] resp. [21, VIII, §4, thm. 8], and for the definition of Zariski rings [21, VIII §4]. For $I \subset R_n$ this applies for both $\hat{R}_n \cdot I \subset \hat{R}_n$ and $A_n \cdot I \subset A_n$, hence $I$ can be recovered from $\hat{R}_n \cdot I$, which can be recovered from $\hat{R}_n \cdot I$ or $A_n \cdot I$. In particular, for any ideal $I \subset R_n$ contained in $\langle t_1, \ldots, t_n \rangle$ the ideal $\hat{I} = I \cdot A_n = \hat{I} \cap A_n$ generated by $\hat{I}$ in $A_n$ is maximal resp. zero if and only if $\hat{I}$ is maximal resp. zero in $A_n$. Similarly, two ideals $J, J'$ in $A_n$ are equal iff $J \cdot \hat{R}_n$ and $J' \cdot \hat{R}_n$ are equal.

**Lemma 8** A normal noetherian domain $R$ is factorial if and only if every prime ideal $I$ of height $ht(I) = 1$ is a principal ideal.

**Proof** [Bourbaki 7.3, no. 2, thm. 1] or [15, thm. 3.2.5.3, p. 6 cor.]. □

**Lemma 9** Let $p(X)$ be a $x_1$-regular homogenous polynomial $x_1^a + \sum_{v < a} c_v(X') \cdot x_1^v$ of degree $a > 0$ in $A_n = \Lambda[[x_1, x_2, \ldots, x_n]]$ with coefficients $c_v(X')$ in $A_{n-1} = \Lambda[[x_2, \ldots, x_n]]$ for $i = 0, \ldots, a - 1$ and with $c_0(X')$ in $\langle x_2, \ldots, x_n \rangle$. Then $A_n$ is a finite ring extension of its subring $\Lambda[[p(X), x_2, \ldots, x_n]]$.

**Proof** Any $g(X)$ in $\Lambda[[X]]$ can be written in the form $g(X) = u(X) \cdot p(X) + \sum_{i=0}^{a-1} r_i(X') \cdot x_1^i$ by the Weierstraß preparation theorem (Lemma 7). If we apply this iteratively for $u(X)$ instead of $g(X)$ and continue, we obtain formal power series $f_i(y_1, \ldots, y_n) \in \Lambda[[y_1, \ldots, y_n]]$ so that $g(X) = \sum_{i=0}^{a-1} f_i(p(X), x_2, \ldots, x_n) \cdot x_1^i$ holds in $\Lambda[[x_1, \ldots, x_n]]$. To prove $\Lambda[[X]] = x_1^{a-1} \cdot \Lambda[[p, X']] + x_1^{a-2} \Lambda[[p, X']] + \cdots + \Lambda[[p, X']]$ it suffices to show $f_i \in \Lambda[[Y]]$. If $g(X)$ and $p(X)$ both have integral coefficients in $\mathbb{Z}_l$ with $p(X) \equiv x_1^0$ modulo $(m_1, x_2, \ldots, x_n)$, then all $f_i(Y)$ have integral coefficients by the Weierstraß preparation theorem for the ring $\mathbb{Z}_l[[x_1, \ldots, x_n]]$. The general case can be easily reduced to this by the method used in the proof of Lemma 7. We may assume $g(0) = 0$ and replace $g(X)$ by $g(c \cdot x_1, c^a \cdot x_2, \ldots, c^a \cdot x_n)$.
and \( p(X) \) by \( \bar{p}(X) = c^{-a} \cdot p(c \cdot x_1, c^a \cdot x_2, \ldots, c^a x_n) \) to show \( f_i(y_1, \ldots, y_n) = c^{-i} \cdot g_i(c^{-a} y_1, y_2, \ldots, y_n) \) for certain \( g_i \in \mathfrak{o}_A[[x_1, \ldots, x_n]] \). Hence all \( f_i(y_1, \ldots, y_n) \) are locally convergent. \( \square \)

References

1. Abbes, A., Saito, T.: Ramification and cleanliness. Tohoku Math J Centennial Issue 63(4), 775–853 (2011)
2. Atiyah, M.F., Macdonald, I.G.: Introduction to Commutative Algebra. Addison Wesley (1969)
3. Beilinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. In: Analyse et topologie sur les espaces singuliers (I), asterisque 100, SMF (1982)
4. Ekedahl, T.: On the adic formalism. In: The Grothendieck Festschrift, vol. II, Progr. Math., vol. 87, pp. 197–218. Birkhäuser, Boston (1990)
5. Deligne, P.: La conjecture de Weil, II. Inst. Hautes Etudes Sci. Publ. Math. 52, 137–252 (1980)
6. Deligne, P.: Finitude de l’extensions de \( \mathbb{Q} \) engendree par les traces de Frobenius, en caracteristique finie. Moscow Math. J. 12 (2012)
7. Drinfeld, V.: On a conjecture of Deligne. Moscow Math. J. 12 (2012)
8. Drinfeld, V.: On a conjecture of Kashiwara. Math. Res. Lett. 8, 713–728 (2001)
9. Franek, J., Kapranov, M.: The Gauss map and a noncompact Riemann–Roch formula for constructible sheaves on semiabelian varieties. Duke Math. J. 104(1), 171–180 (2000)
10. Gabber, O., Loeser, F.: Faisceaux pervers \( \ell \)-adiques sur un tore. Duke Math. J. 83(3), 501–606 (1996)
11. Krämer, T., Weissauer, R.: Vanishing Theorems for constructible sheaves on abelian varieties. J. Algebr. Geom. 24, 531–568 (2015)
12. Krämer, T., Weissauer, R.: On the Tannaka group attached to the Theta divisor of a generic principally polarized abelian variety. Math. Z. arXiv:1309.3754 (2015, to appear)
13. Lafforgue, L.: Champs de Drinfeld et correspondance de Langlands. Invent. Math. 147(1), 1–241 (2002)
14. Laumon, G.: Letter to Gabber and Loeser (22/12/91)
15. Samuel, P.: Lectures on Unique Factorization Domains. Tata Institute, Bombay (1964)
16. Serre, J.P.: Algebraic Groups and Class Fields. Springer, Berlin (1988)
17. Weissauer, R.: On the rigidity of BN-sheaves. arXiv:1204.1929
18. Weissauer, R.: Why certain Tannaka groups attached to abelian varieties are almost connected. arXiv:1207.4039
19. Weissauer, R.: Degenerate perverse sheaves on abelian varieties. arXiv:1204.2247
20. Weissauer, R.: A remark on the rigidity of BN-sheaves. arXiv:1111.6095 (2011)
21. Zariski, O., Samuel, P.: Commutative Algebra, vol. II. Springer, Berlin (1960)