ALGEBRAIC SYSTEMS WITH LIPSCHITZ PERTURBATIONS

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Abstract. By using variational methods, the existence of infinitely many solutions for a nonlinear algebraic system with a parameter is established in presence of a perturbed Lipschitz term. Our goal was achieved requiring an appropriate behavior of the nonlinear term $f$, either at zero or at infinity, without symmetry conditions.

1. Introduction

In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics, and conversely. The modeling/simulation of certain nonlinear problems from economics, biological neural networks, optimal control and others, enforced in a natural manner a rapid development of the theory of discrete equations.

In this paper, motivated by this large interest, we study the following algebraic system

$$Au = \lambda f(u) + h(u),$$

in which $u = (u_1, \ldots, u_n)^t \in \mathbb{R}^n$ is a column vector, $A = (a_{ij})_{n \times n}$ is a positive-definite matrix, $f(u) := (f_1(u_1), \ldots, f_n(u_n))^t$, where the functions $f_k : \mathbb{R} \to \mathbb{R}$ are assumed to be continuous for every $k \in \mathbb{Z}[1, n] := \{1, 2, \ldots, n\}$, and $\lambda$ is a positive parameter.

Moreover,

$$h(u) := (h_1(u_1), \ldots, h_n(u_n))^t,$$

where, for every $k \in \mathbb{Z}[1, n]$, the functions $h_k : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with constants $L_k \geq 0$, that is:

$$|h_k(t_1) - h_k(t_2)| \leq L_k |t_1 - t_2|,$$

for every $t_1, t_2 \in \mathbb{R}$, and $h_k(0) = 0$.

A large number of discrete problems can be formulated as special cases of the non-perturbed ($h = 0$) algebraic system, namely $(S_{A,\lambda}^f)$; see, for instance, the papers [22, 24, 25, 26, 27] and references therein.

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We also point out that the special case
\[
A := \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
0 & \ldots & -1 & 2 & -1 \\
0 & \ldots & 0 & -1 & 2 \\
\end{pmatrix}_{n \times n},
\]
has been considered in order to study the existence of nontrivial solutions
of nonlinear second-order difference equations [12, 13, 15]. Further, general
references on difference equations and their applications can be found e.g.
in [1, 10].

Here, by using variational methods, under the key assumption that
\[
L := \max_{k \in \mathbb{Z}[1,n]} L_k < \lambda_1,
\]
where \( \lambda_1 \) is the first eigenvalue of the matrix \( A \), we determine open intervals
of positive parameters such that problem \((S_{f,h}A,\lambda)\) admits either an unbounded
sequence of solutions, provided that the nonlinearity \( f \) has a suitable be-
haviour at infinity (Theorem 3.1), or a sequence of pairwise distinct solutions
that converges to zero, if a similar behaviour occurs at zero (see Theorem
3.2).

Our main tool is a recent critical point result obtained by Ricceri and
recalled here in a convenient form (see Theorem 2.1).

A special case of our results reads as follows (see Remark 4.2).

\[\text{Theorem 1.1.} \quad \text{Let } z : \mathbb{R} \to \mathbb{R} \text{ be a nonnegative and continuous function. Assume that}
\]
\[
\liminf_{t \to +\infty} \frac{\int_0^t z(\xi) d\xi}{t^2} = 0, \quad \limsup_{t \to +\infty} \frac{\int_0^t z(\xi) d\xi}{t^2} = +\infty.
\]
\[\text{Then, for each } \lambda > 0, \text{ and for every Lipschitz function } h : \mathbb{R} \to \mathbb{R} \text{ with sufficiently small constant } L_h, \text{ the following discrete problem}
\]
\[
[u(i+1,j) - 2u(i,j) + u(i-1,j)] + [u(i,j+1) - 2u(i,j) + u(i,j-1)] + \lambda z(u(i,j)) + h(u(i,j)) = 0, \quad \forall (i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]
\]
\[\text{with boundary conditions}
\]
\[
u(i,0) = u(i,n+1) = 0, \quad \forall i \in \mathbb{Z}[1,m],
\]
\[
u(0,j) = u(m+1,j) = 0, \quad \forall j \in \mathbb{Z}[1,n],
\]
\[\text{admits an unbounded sequence of solutions.}
\]

Finally, for completeness, we just mention here that there is a vast lit-
erature on nonlinear difference equations based on fixed point and upper
and lower solution methods (see [2, 8]). For related topics see the works
[3, 6, 7, 21]. For a complete and exhaustive overview on variational methods
we refer the reader to the monographs [11, 19].
2. THE ABSTRACT SETTING

Let \((X, \| \cdot \|)\) be a finite-dimensional Banach space and let \(J_\lambda : X \to \mathbb{R}\) be a function satisfying the following structure hypothesis:

\(\Lambda\) for all \(u \in X\), \(J_\lambda(u) := \Phi(u) - \lambda \Psi(u)\) where \(\Phi, \Psi : X \to \mathbb{R}\) are two functions of class \(C^1\) on \(X\) with \(\Phi\) coercive, i.e. \(\lim_{\|u\| \to \infty} \Phi(u) = +\infty\), and \(\lambda\) is a real positive parameter.

Moreover, provided that \(r > \inf_X \Phi\), put

\[\varphi(r) := \inf_{u \in \Phi^{-1}[(-\infty, r)]} \left( \sup_{v \in \Phi^{-1}[(-\infty, r)]} \Psi(v) - \Psi(u) \right) - \frac{r - \Phi(u)}{\Phi(u)},\]

and

\[\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)+} \varphi(r).\]

Clearly, \(\gamma \geq 0\) and \(\delta \geq 0\). When \(\gamma = 0\) (or \(\delta = 0\)), in the sequel, we agree to read \(1/\gamma\) (or \(1/\delta\)) as \(+\infty\).

**Theorem 2.1.** Assuming that the condition \(\Lambda\) holds, one has

(a) If \(\gamma < +\infty\) then, for each \(\lambda \in ]0, 1/\gamma[,\) the following alternative holds:

- \((a_1)\) \(J_\lambda\) possesses a global minimum,
- or
- \((a_2)\) there is a sequence \(\{u_m\}\) of critical points (local minima) of \(J_\lambda\) such that \(\lim_{m \to \infty} \Phi(u_m) = +\infty\).

(b) If \(\delta < +\infty\) then, for each \(\lambda \in ]0, 1/\delta[,\) the following alternative holds:

- \((b_1)\) there is a global minimum of \(\Phi\) which is a local minimum of \(J_\lambda\),
- or
- \((b_2)\) there is a sequence \(\{u_m\}\) of pairwise distinct critical points (local minima) of \(J_\lambda\), with \(\lim_{m \to \infty} \Phi(u_m) = \inf_X \Phi\), which converges to a global minimum of \(\Phi\).

**Remark 2.1.** Theorem 2.1 is the finite-dimensional version of the quoted multiplicity result of Ricceri from [20].

As ambient space \(X\), consider the \(n\)-dimensional Banach space \(\mathbb{R}^n\) endowed by the norm

\[\|u\| := \left( \sum_{k=1}^n u_k^2 \right)^{1/2}.\]

Set \(\mathcal{X}_n\) be the class of all symmetric and positive-definite matrices of order \(n\). Further, we denote by \(\lambda_1, \ldots, \lambda_n\) the eigenvalues of \(A\), ordered as follows \(0 < \lambda_1 \leq \ldots \leq \lambda_n\).

It is well-known that if \(A \in \mathcal{X}_n\), then for every \(u \in X\), one has
\[
\lambda_1 \|u\|^2 \leq u^t Au \leq \lambda_n \|u\|^2, \tag{1}
\]

and

\[
\|u\|_\infty \leq \frac{1}{\sqrt{\lambda_1}} (u^t Au)^{1/2}, \tag{2}
\]

where \(\|u\|_\infty := \max_{k \in \mathbb{Z}[1,n]} |u_k|\).

Set

\[
\Phi(u) := \frac{u^t Au}{2} - \sum_{k=1}^{n} H_k(u_k), \tag{3}
\]

and

\[
\Psi(u) := \sum_{k=1}^{n} F_k(u_k), \quad J_\lambda(u) := \Phi(u) - \lambda \Psi(u), \tag{4}
\]

for every \(u \in X\), where \(H_k(t) := \int_0^t h_k(\xi)d\xi\) and \(F_k(t) := \int_0^t f_k(\xi)d\xi\), for every \((k,t) \in \mathbb{Z}[1,n] \times \mathbb{R}\).

Standard arguments show that \(J_\lambda \in C^1(X, \mathbb{R})\), as well as that critical points of \(J_\lambda\) are exactly the solutions of problem \((S_{f,h}^{A,\lambda})\); see, for instance, the paper [23].

**Lemma 2.1.** Set

\[
L := \max_{k \in \mathbb{Z}[1,n]} L_k < \lambda_1. \tag{5}
\]

Then the functional \(\Phi\) is coercive.

**Proof.** Bearing in mind (1), since \(h_k\) is a Lipschitz continuous function (for every \(k \in \mathbb{Z}[1,n]\)) with constant \(L_k \geq 0\) and \(h_k(0) = 0\), we have

\[
\Phi(u) \geq \frac{\lambda_1}{2} \|u\|^2 - \sum_{k=1}^{n} |H_k(u_k)| \geq \frac{1}{2} \|u\|^2 - \sum_{k=1}^{n} \left( \int_0^{u_k} |h_k(t)|dt \right)
\]

\[
\geq \frac{\lambda_1}{2} \|u\|^2 - \sum_{k=1}^{n} \int_0^{u_k} |t|dt = \frac{1}{2} \|u\|^2 - \frac{L}{2} \sum_{k=1}^{n} u_k^2
\]

\[
= \left( \frac{\lambda_1 - L}{2} \right) \|u\|^2.
\]

Hence, by (5), the above relation implies that the functional \(\Phi\) is coercive. \(\Box\)
3. Main results

Set
\[ A_\infty := \liminf_{t \to +\infty} \frac{\sum_{k=1}^{n} \max_{|\xi| \leq t} F_k(\xi)}{t^2}, \quad \text{and} \quad B_\infty := \limsup_{t \to +\infty} \frac{\sum_{k=1}^{n} F_k(t)}{t^2}. \]

From now on we shall assume that the functions \( h_k : \mathbb{R} \to \mathbb{R} \), for every \( k \in \mathbb{Z}[1,n] \), are Lipschitz continuous with constants \( L_k \geq 0 \) such that condition (5) holds.

**Theorem 3.1.** Let \( A \in \mathfrak{X}_n \) and assume that the following inequality holds
\[
(h_{\lambda_k}^L) \quad \frac{\lambda_1 - L}{\Tr(A) + 2 \sum_{i<j} a_{ij} + nL} B_\infty < 1 - L \frac{\lambda_1 - L}{2B_\infty}.
\]
Then, for each
\[
\lambda \in \left[ \frac{\Tr(A) + 2 \sum_{i<j} a_{ij} + nL}{2B_\infty}, \frac{\lambda_1 - L}{2A_\infty} \right],
\]
problem \((S_{A,A,\lambda}^{f,h})\) admits an unbounded sequence of solutions.

**Proof.** Fix \( \lambda \) as in the assertion of the theorem and put \( \Phi, \Psi, J_\lambda \) as in (3) and (4). Since the critical points of \( J_\lambda \) are the solutions of problem \((S_{A,A,\lambda}^{f,h})\), our aim is to apply Theorem 2.1 part (a) to function \( J_\lambda \). Clearly \((\Lambda)\) holds.

Therefore, our conclusion follows provided that \( \gamma < +\infty \) as well as that \( J_\lambda \) turns out to be unbounded from below. To this end, let \( \{c_m\} \) be a real sequence such that \( \lim_{m \to \infty} c_m = +\infty \) and
\[
\lim_{m \to \infty} \frac{\sum_{k=1}^{n} \max_{|\xi| \leq c_m} F_k(\xi)}{c_m^2} = A_\infty,
\]
Write
\[
r_m := \frac{\lambda_1 - L}{2} c_m^2,
\]
for every \( m \in \mathbb{N} \).

Since, owing to (2), it follows that
\[
\{v \in X : v^t Av < 2r_m\} \subset \{v \in X : |v_k| \leq c_m \ \forall k \in \mathbb{Z}[1,n]\},
\]
and we obtain
\[
\varphi(r_m) \leq \frac{\sup_{v^t Av < 2r_m} \sum_{k=1}^{n} F_k(v_k)}{r_m} \leq \frac{\sum_{k=1}^{n} \max_{|t| \leq c_m} F_k(t)}{r_m} = \frac{2}{\lambda_1 - L} \frac{\sum_{k=1}^{n} \max_{|t| \leq c_m} F_k(t)}{c_m^2}.
\]
Hence, it follows that
\[
\gamma \leq \lim_{m \to \infty} \varphi(r_m) \leq \frac{2}{\lambda_1 - L} A_\infty < \frac{1}{\lambda} < +\infty.
\]
Now, we verify that $J_\lambda$ is unbounded from below. First, assume that $B^\infty = +\infty$. Accordingly, fix such $M$ that

$$M > \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2\lambda}$$

and let $\{b_m\}$ be a sequence of positive numbers, with $\lim_{m \to \infty} b_m = +\infty$, such that

$$\sum_{k=1}^{n} F_k(b_m) > M b_m^2, \quad (\forall \ m \in \mathbb{N}).$$

Thus, taking in $X$ the sequence $\{s_m\}$ which, for each $m \in \mathbb{N}$, is given by $(s_m)_k := b_m$ for every $k \in \mathbb{Z}[1,n]$, owing to (1) and noting that

$$\Phi(u) \leq \frac{u^t A u}{2} + \sum_{k=1}^{n} \left( \int_{0}^{u_k} |h_k(t)| dt \right) \leq \frac{u^t A u}{2} + \frac{L}{2} \sum_{k=1}^{n} a_k^2 = \frac{u^t A u}{2} + \frac{L}{2} ||u||^2.$$

one immediately has

$$J_\lambda(s_m) = \frac{s_m^t A s_m}{2} - \lambda \sum_{k=1}^{n} F_k(b_m) \leq \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2} b_m^2 - \lambda \sum_{k=1}^{n} F_k(b_m) < \left( \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2} - \lambda M \right) b_m^2.$$

that is, $\lim_{m \to \infty} J_\lambda(s_m) = -\infty$.

Next, assume that $B^\infty < +\infty$. Since

$$\lambda > \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2B^\infty},$$

we can fix $\varepsilon > 0$ such that

$$\varepsilon < B^\infty - \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2\lambda}.$$

Therefore, also calling $\{b_m\}$ a sequence of positive numbers such that $\lim_{m \to \infty} b_m = +\infty$ and

$$(B^\infty - \varepsilon)b_m^2 < \sum_{k=1}^{n} F_k(b_m) < (B^\infty + \varepsilon)b_m^2, \quad (\forall \ m \in \mathbb{N})$$
arguing as before and by choosing \( \{s_m\} \) in \( X \) as above, one has

\[
J_\lambda(s_m) < \left( \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2} - \lambda (B^\infty - \varepsilon) \right) b_m^2.
\]

So, \( \lim_{m \to \infty} J_\lambda(s_m) = -\infty \).

Hence, in both cases \( J_\lambda \) is unbounded from below. The proof is thus complete. \( \square \)

**Remark 3.1.** If \( f_k \) are nonnegative continuous functions, condition \((h^L_L)\) reads as follows

\[
\liminf_{t \to +\infty} \frac{\sum_{k=1}^n F_k(t)}{t^2} < \frac{\lambda_1 - L}{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL} \limsup_{t \to +\infty} \frac{\sum_{k=1}^n F_k(t)}{t^2}.
\]

Arguing as in the proof of Theorem 3.1 and applying part (b) of Theorem 2.1, we obtain the following result.

**Theorem 3.2.** Let \( A \in X_n \) and assume that the following inequality holds

\[
(h^L_L) \quad A_0 < \frac{\lambda_1 - L}{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL} B^0.
\]

Then, for each

\[
\lambda \in \left[ \frac{\text{Tr}(A) + 2 \sum_{i<j} a_{ij} + nL}{2B^0}, \frac{\lambda_1 - L}{2A_0} \right],
\]

problem \((S^f_{A,\lambda})\) admits a sequence of nontrivial solutions \( \{u_m\} \) such that

\[
\lim_{m \to \infty} \|u_m\| = \lim_{m \to \infty} \|u_m\|_\infty = 0.
\]

4. An application

In this section we consider a discrete system, namely \((E^{f,h}_\lambda)\), given as follows

\[
[u(i+1,j) - 2u(i,j) + u(i-1,j)] + [u(i,j+1) - 2u(i,j) + u(i,j-1)]
+ \lambda f((i,j), u(i,j)) + h(u(i,j)) = 0, \quad \forall (i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n],
\]

with boundary conditions

\[
u(i,0) = u(i,n+1) = 0, \quad \forall i \in \mathbb{Z}[1,m],
\]

\[
u(0,j) = u(m+1,j) = 0, \quad \forall j \in \mathbb{Z}[1,n],
\]

where \( f : \mathbb{Z}[1,m] \times \mathbb{Z}[1,n] \times \mathbb{R} \to \mathbb{R} \) denotes a continuous function, \( \lambda \) is a positive real parameter and \( h : \mathbb{R} \to \mathbb{R} \) be a Lipschitz continuous function with constant \( L_h \).

As ambient space \( X \), we consider the \( mn \)-dimensional Banach space \( \mathbb{R}^{mn} \) endowed by the norm

\[
\|u\| := \left( \sum_{k=1}^{mn} u_k^2 \right)^{1/2}.
\]
Further, if $\ell \in \mathbb{N}$, the symbol $\mathcal{M}_{\ell \times \ell}(\mathbb{R})$ stands for the linear space of all the matrices of order $\ell$ with real entries.

Let $v : \mathbb{Z}[1,m] \times \mathbb{Z}[1,n] \to \mathbb{Z}[1,mn]$ be the bijection defined by $v(i,j) := i + m(j - 1)$, for every $(i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]$.

Let us denote $w_k := u(v^{-1}(k))$ and $g_k(w_k) := f(v^{-1}(k), w_k)$, for every $k \in \mathbb{Z}[1,mn]$. With the above notations, problem $(E^{f,h}_\lambda)$ can be written as a nonlinear algebraic system of the form

$$Aw = \lambda g(w) + \tilde{h}(w),$$

where $A$ is given by

$$A := \begin{pmatrix}
D & -I_m & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-I_m & D & -I_m & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -I_m & D & -I_m & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -I_m & D & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & D \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & D \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & D \\
0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & D
\end{pmatrix} \in \mathcal{M}_{mn \times mn}(\mathbb{R}),$$

in which $D$ is defined by

$$D := \begin{pmatrix}
4 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 4
\end{pmatrix} \in \mathcal{M}_{m \times m}(\mathbb{R}),$$

$I_m \in \mathcal{M}_{m \times m}(\mathbb{R})$ is the identity matrix and $g(w) := (g_1(w_1), \ldots, g_{mn}(w_{mn}))^t$, $\tilde{h}(w) := (h(w_1), \ldots, h(w_{mn}))^t$, for every $w \in X$.

In [9], Ji and Yang studied the structure of the spectrum of the above (non-perturbed) Dirichlet problem. By their result we have that $A \in \mathcal{X}_{mn}$.

It is easy to observe that the solutions of $(E^{f,h}_\lambda)$ are the critical points of the $C^1$-functional

$$J_\lambda(w) := \frac{w^tAw}{2} - \lambda \sum_{k=1}^{mn} \int_0^{w_k} g_k(t)dt - \sum_{k=1}^{mn} \int_0^{w_k} h(t)dt, \quad \forall w \in X.$$

Denote by $\lambda_A$ the first eigenvalue of the matrix $A$. By using the above variational framework, Theorem 3.1 assumes the following form:
Theorem 4.1. Assume that $\lambda_A < L_h$, in addition to

$$\left( h^\infty \right)_\infty \liminf_{t \to +\infty} \frac{\sum_{k=1}^{mn} \max_{|t| \leq t} \int_0^t g_k(s) ds}{t^2} < \lambda_A - L_h \frac{\limsup_{t \to +\infty} \sum_{k=1}^{mn} \int_0^t g_k(s) ds}{t^2}.$$ 

Then for each $\lambda \in \left( 2 + L_h \right)(m + n) \frac{\lambda_A - L_h}{2B^\infty}, \frac{\lambda_A - L_h}{2A^\infty}$, problem $(E^{f,h}_\lambda)$ admits an unbounded sequence of solutions.

Remark 4.1. Substituting $\xi \to +\infty$ with $\xi \to 0^+$ in Theorem 4.1, the same statement as Theorem 3.2 is easily proved.

Remark 4.2. We just point out that Theorem 1.1 in Introduction directly follows by Theorem 4.1 assuming that $L_h < \lambda_A$.

Remark 4.3. We refer to the paper of Galewski and Orpel [5] for some multiplicity results on discrete partial difference equations as well as to the monograph of Cheng [4] for their discrete geometrical interpretation. See also the papers [14, 16, 17, 18] for recent contribution on discrete problems.

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