MOVING SURFACES BY NON-CONCAVE CURVATURE FUNCTIONS

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ABSTRACT. A convex surface contracting by a strictly monotone, homogeneous degree one function of its principal curvatures remains smooth until it contracts to a point in finite time, and is asymptotically spherical in shape. No assumptions are made on the concavity of the speed as a function of principal curvatures. We also discuss motion by functions homogeneous of degree greater than 1 in the principal curvatures.

1. INTRODUCTION

Several authors have considered convex hypersurfaces contracting by homogeneous degree one symmetric functions of their principal curvatures: Huisken [H] proved that the mean curvature flow contracts such hypersurfaces to a point in finite time while making their shape spherical. Chow proved similar results for n-dimensional hypersurfaces moving by the n-th root of Gauss curvature [C] and (with additional convexity assumptions on the initial data) by the square root of scalar curvature [C2]. The author treated a large family of such equations [A, A5], satisfying some requirements of concavity of the speed in the principal curvatures.

Since the result in [A] holds for both concave and convex functions of the principal curvatures, it seems possible that these convexity assumptions could be significantly weakened. This paper confirms that no concavity assumptions are needed for surfaces moving in space:

**Theorem 1.** Let F be a smooth, symmetric, homogeneous degree 1 function F defined on the positive quadrant in \( \mathbb{R}^2 \), with strictly positive derivative in each argument, normalized to have \( F(1,1) = 1 \). Then for any smooth, strictly convex surface \( M_0 = x_0(S^2) \subset \mathbb{R}^3 \) there is a unique family of smooth, strictly convex surfaces \( \{ M_t = x_t(S^2) \}_{0 \leq t < T} \) satisfying

\[
\frac{\partial x}{\partial t}(z,t) = -F(\kappa_1(z,t), \kappa_2(z,t))v(z,t)
\]

where \( v(z,t) \) is the outward normal and \( \kappa_1(z,t) \) and \( \kappa_2(z,t) \) are the principal curvatures of \( M_t \). \( M_t \) converges uniformly to a point \( p \in \mathbb{R}^3 \) as \( t \) approaches \( T \), while the rescaled maps

\[
\frac{x_t - p}{\sqrt{T-t}}
\]

converge smoothly to an embedding \( \tilde{x}_T \) with image equal to the unit sphere about the origin.

The proof of this result follows the same basic framework as the papers mentioned above: The crucial step is to obtain a bound on the ‘pinching ratio’, which is the supremum over the surface of the ratio of largest to smallest principal curvatures at each point. This is obtained using a maximum principle argument. In the previous papers the convexity assumptions on the speed came into the argument at two points: First, to obtain terms of a favourable sign in
the argument to control the pinching ratio, and second to allow the application of the second derivative Hölder estimates of Krylov [K]. In the two-dimensional case the author recently proved second derivative Hölder estimates which apply without any assumption of concavity [A2]. The main new estimate of this paper gives a bound on the pinching ratio without requiring any concavity of the speed function, thus removing all concavity requirements from the proof.

The last section of the paper discusses flows in which the speed function is homogeneous of degree greater than 1 in the principal curvatures: In particular, it is shown that any parabolic flow with speed homogeneous of degree \( \alpha > 1 \) preserves pinching ratios which are less than or equal to a critical value \( r_0(\alpha) \) which depends only on \( \alpha \). On the other hand there are surfaces with pinching ratio as close to \( r_0(\alpha) \) as desired, for which the pinching ratio becomes worse under the flow.

2. NOTATION AND PRELIMINARY RESULTS

Suppose that the initial surface \( M_0 \) is given by a smooth embedding \( x_0 : S^2 \to \mathbb{R}^3 \). The aim is to construct a smooth family of embeddings \( x : S^2 \times [0, T) \to \mathbb{R}^3 \) satisfying the evolution equation.

The existence of a smooth solution for a short time is guaranteed since the flow is equivalent to a scalar, strictly parabolic equation (for example one can write the evolving surfaces as graphs over a sphere).

By convention the unit normal \( \nu \) points in the outward direction for a convex surface \( M \), and the principal curvatures are positive. Choose local coordinates \( y^1, y^2 \) about any point \( z \) such that the tangent vectors \( e_i = \frac{\partial x}{\partial y^i} \) are orthonormal at \( z \). Then the principal curvatures are the eigenvalues of the second fundamental form, which is the symmetric bilinear form defined by

\[
  h_{ij} = -\left( \frac{\partial^2 x}{\partial y^i \partial y^j} , \nu \right) = \left( \frac{\partial \nu}{\partial y^i} , \frac{\partial x}{\partial y^j} \right).
\]

The covariant derivatives of a tangent vector field \( X \) on \( M \) in the direction of the tangent vector \( e_j \) is given by the expression

\[
  \nabla_{e_j} X = \pi \left( \frac{\partial X}{\partial y^j} \right),
\]

where \( \pi \) is the orthogonal projection on the tangent space of \( M \). The covariant derivative of the second fundamental form is the tensor defined by

\[
  \nabla_i h_{jk} = \frac{\partial h_{jk}}{\partial y^i} - h(e_j, \nabla_{e_i} e_k) - h(\nabla_{e_i} e_j, e_k).
\]

The Codazzi identity says that this is totally symmetric.

The function \( F \) is assumed to be a smooth symmetric function of the principal curvatures, and so can also be written as a smooth function of the elementary symmetric functions of the principal curvatures [G], hence also of the components of the second fundamental form. We remark that a Theorem of Schwarz [Schw] guarantees that conversely, any smooth \( SO(n) \)-invariant function of the components of the second fundamental form can be written as a smooth symmetric function of the principal curvatures. Define \( \dot{F}^{ij} = \frac{\partial \dot{F}}{\partial h_{ij}} \) and \( \ddot{F}^{klmn} = \frac{\partial^2 \dot{F}}{\partial h_{kl} \partial h_{mn}} \). The monotonicity of \( F \) as a function of each principal curvature implies that \( \dot{F} \) is a positive definite symmetric matrix.

Suppose \( G \) is any other symmetric function of the principal curvatures, and define \( \dot{G}^{ij} \) and \( \ddot{G}^{klmn} \) to be the first and second derivatives of \( G \) with respect to the components of the second
fundamental form. It was shown in [A] that if $F$ is homogeneous of degree 1 then $G$ evolves according to the following evolution equation:

$$
\frac{\partial G}{\partial t} = \tilde{F}^{ij} \nabla_i \nabla_j G + (\tilde{G}^{ij} \tilde{F}^{klmn} - \tilde{F}^{ij} \tilde{G}^{klmn}) \nabla_i h_{kl} \nabla_j h_{mn} + \tilde{G}^{ij} h_{ij} \tilde{F}^{kl} g_{mn} h_{km} h_{ln}
$$

(2)

where the sum is over repeated indices. In particular,

$$
\frac{\partial F}{\partial t} = \tilde{F}^{ij} \nabla_i \nabla_j F + \tilde{F}^{ij} g_{kj} h_{jk} h_{jl}.
$$

(3)

The parabolic maximum principle therefore implies that the minimum of $F$ over the surface $M_t$ is non-decreasing in time as long as the solution remains smooth and convex.

3. Estimate on the Pinching Ratio

In this section the main estimate of the paper is proved by applying the maximum principle to the evolution equation for the quantity

$$
G = \frac{(\kappa_2 - \kappa_1)^2}{(\kappa_2 + \kappa_1)^2} = \frac{2|\nabla h|^2 - (\text{tr} h)^2}{(\text{tr} h)^2}.
$$

(4)

Since $G$ is homogeneous of degree zero, the Euler relation gives $\tilde{G}^{ij} h_{ij} = 0$, and the last term in equation (2) vanishes.

Consider a point $(z, t)$ where $G$ attains a spatial maximum at some time $t$ in the interval of existence of the smooth convex solution of the flow. Consider the second term in Equation (2) at such a point. This is a quadratic term in the components of $\nabla h$. The Codazzi identity implies that there are four distinct components of $\nabla h$, so this term is defined by a $4 \times 4$ matrix. However, at the maximum point the first derivatives of $G$ vanish, so two components of $\nabla h$ can be eliminated, leaving only a $2 \times 2$ matrix. It will turn out that in an orthonormal basis diagonalising the second fundamental form at some point and time, this matrix is also diagonal and can be easily understood.

The computation of these terms will require results from [A] which describe the components of $\tilde{F}$ and $\tilde{G}$ in an orthonormal frame in which $h_{ij} = \text{diag}(\kappa_1, \kappa_2)$:

$$
\tilde{F}^{11,11} = \frac{\partial^2 F}{\partial \kappa_1^2},
\tilde{F}^{11,22} = \tilde{F}^{22,11} = \frac{\partial^2 F}{\partial \kappa_1 \partial \kappa_2},
\tilde{F}^{22,22} = \frac{\partial^2 F}{\partial \kappa_2^2},
\tilde{F}^{12,12} = \tilde{F}^{21,21} = \frac{\partial^2 F}{\partial \kappa_1^2} - \frac{\partial^2 F}{\partial \kappa_2^2}.
$$

The last of these identities is to be interpreted as a limit if $\kappa_1 = \kappa_2$. It follows that the terms in the evolution equation for $G$ are as follows in a frame diagonalising the second fundamental
form:

\[ Q = \left( G^{ij} F^{klmn} - F^{ij} G^{klmn} \right) \nabla_j h_{ll} \nabla_j h_{mn} \]

\[ = \left( \frac{\partial G}{\partial K_1} \frac{\partial^2 F}{\partial K_1^2} - \frac{\partial F}{\partial K_1} \frac{\partial^2 G}{\partial K_1^2} \right) (\nabla_1 h_{11})^2 + \left( \frac{\partial G}{\partial K_1} \frac{\partial^2 F}{\partial K_2^2} - \frac{\partial F}{\partial K_1} \frac{\partial^2 G}{\partial K_2^2} \right) (\nabla_1 h_{22})^2 \]

\[ + 2 \left( \frac{\partial G}{\partial K_1} \frac{\partial^2 F}{\partial K_1 \partial K_2} - \frac{\partial F}{\partial K_1} \frac{\partial^2 G}{\partial K_1 \partial K_2} \right) \nabla_1 h_{11} \nabla_1 h_{22} \]

\[ + \left( \frac{\partial G}{\partial K_2} \frac{\partial^2 F}{\partial K_1^2} - \frac{\partial F}{\partial K_2} \frac{\partial^2 G}{\partial K_1^2} \right) (\nabla_2 h_{11})^2 + \left( \frac{\partial G}{\partial K_2} \frac{\partial^2 F}{\partial K_2^2} - \frac{\partial F}{\partial K_2} \frac{\partial^2 G}{\partial K_2^2} \right) (\nabla_2 h_{22})^2 \]

\[ + 2 \left( \frac{\partial G}{\partial K_1} \frac{\partial^2 F}{\partial K_1 \partial K_2} - \frac{\partial F}{\partial K_1} \frac{\partial^2 G}{\partial K_1 \partial K_2} \right) \nabla_2 h_{11} \nabla_2 h_{22} \]

\[ + 2 \frac{\partial G}{\partial K_1} \frac{\partial^2 F}{\partial K_1 \partial K_2} \left( \nabla_1 h_{11} \right)^2 + 2 \frac{\partial G}{\partial K_2} \frac{\partial^2 F}{\partial K_1 \partial K_2} \left( \nabla_2 h_{11} \right)^2 \]

At a maximum point of \( G \), \( G \) is non-zero (otherwise \( M_t \) is a sphere and the proof is trivial) and it be can assumed without loss of generality that \( K_2 > K_1 \). The gradient conditions on \( G \) then give two equations (since \( G \) does not vanish except when \( G = 0 \)):

\[ \nabla_1 h_{11} = -\frac{\partial G}{\partial K_1} \nabla_1 h_{22}; \quad \nabla_2 h_{22} = -\frac{\partial G}{\partial K_2} \nabla_2 h_{11}. \]

The degree-zero homogeneity of \( G \) implies by the Euler relation that \( K_1 \frac{\partial G}{\partial K_1} + K_2 \frac{\partial G}{\partial K_2} = 0 \). Homogeneity also implies the identity

\[ K_1^2 \frac{\partial^2 G}{\partial K_1^2} + K_2^2 \frac{\partial^2 G}{\partial K_2^2} + 2 K_1 K_2 \frac{\partial^2 G}{\partial K_1 \partial K_2} = 0. \]

Similarly, the degree 1 homogeneity of \( F \) gives the following identities:

\[ \frac{\partial^2 F}{\partial K_1^2} = -\frac{K_2}{K_1} \frac{\partial^2 F}{\partial K_2^2}; \quad \frac{\partial^2 F}{\partial K_2^2} = -\frac{K_1}{K_2} \frac{\partial^2 F}{\partial K_1^2}; \quad \frac{\partial F}{\partial K_1} + K_2 \frac{\partial F}{\partial K_2} = F. \]

Substituting these expression into the expression for \( Q \) above and applying the Codazzi symmetries \( \nabla_1 h_{12} = \nabla_2 h_{11} \) and \( \nabla_2 h_{12} = \nabla_1 h_{22} \), one finds that all of the terms involving second derivatives of \( F \) and \( G \) disappear, leaving

\[ Q = \frac{2F \frac{\partial G}{\partial K_1}}{K_2(K_2 - K_1)} (\nabla_1 h_{11})^2 + \frac{2F \frac{\partial G}{\partial K_2}}{K_1(K_2 - K_1)} (\nabla_2 h_{11})^2. \]

Now observe that \( \frac{\partial G}{\partial K_1} = -\frac{4K_1(K_2 - K_1)}{(K_1 + K_2)^3} < 0 \), and therefore \( Q \leq 0 \) and \( \frac{\partial G}{\partial t} \leq 0 \) at the maximum point. Therefore the supremum of \( G \) over \( G \) is non-increasing in time, and the pinching ratio \( r = \frac{2}{1 - \nabla G} - 1 \) is also non-increasing in time. This proves the following:

**Proposition 2.** If \( x : S^2 \times [0, T) \to \mathbb{R}^3 \) is a smooth family of convex embeddings satisfying (1), then for \( t \in (0, T) \) the pinching ratio of \( M_t = x_t(S^2) \) is no greater than the pinching ratio of \( M_0 \).

If follows from [A] Lemma 5.4 that the surfaces \( M_t \) have bounded ratio of circumradius \( r_+ \) to inradius \( r_- \): There exists \( C_0 \) such that \( r_+(M_t) \leq C_0 r_-(M_t) \) for all \( t \) in the interval of existence.
4. Regularity and Convergence

Now we discuss the proof of convergence to a point and the limit under rescaling. The key step is to derive estimates on curvature and its higher derivatives. To achieve the correct dependence of the estimates on the geometry we rescale to bring the hypersurfaces to a fixed size:

For each \( t_0 \in [0, T) \), let \( p_{t_0} \) be an centre of \( M_{t_0} \), and let \( x_{0} \) be the solution of Equation (1) defined by \( x_{0}(z, t) = r_{-}(M_{t_0})^{-1}(x(z, t_0 + r_{-}(M_{t_0})^2 t) - p_{t_0}) \) for \( t \in [-t_0/2, (T - t_0)/r_{-}^2) \). Denote the hypersurface \( x_{0}(S^2, t) \) by \( M_{t_0, t} \). Then \( M_{t_0, 0} \) lies outside the unit ball about the origin in \( \mathbb{R}^3 \), and inside the ball of radius \( C_0 + 1 \), for every \( t_0 \). By the comparison principle, \( M_{t_0, t} \) lies outside the ball of radius \( \sqrt{T - 2r} \) and inside the ball of radius \( \sqrt{(C_0 + 1)^2 - 2r} \), for each \( t \in [0, 1/2) \) in the interval of existence.

The pinching estimate of Proposition 2 implies that the eigenvalues of \( \tilde{F} \) are bounded above and below by positive constants: By homogeneity we have \( \frac{\partial \tilde{F}}{\partial \kappa}(k_1, k_2) = \frac{\partial F}{\partial \kappa} \left( \frac{k_1}{k_1 + k_2}, \frac{k_2}{k_1 + k_2} \right) \), so the supremum and infimum are attained on the compact set \( \{(a, 1 - a) : |a - \frac{1}{2}| \leq \frac{\sqrt{2}}{2} \} \), and hence are finite and positive respectively.

Bounds above on \( F \) on the rescaled hypersurfaces follow from the argument of Tsai [1] as presented in [A, Theorem 7.5]. Together with the pinching estimate this implies a uniform upper bound on the principal curvatures of \( M_{t_0} \) for any \( t_0 \) with \( 0 \leq t \leq \frac{1}{4} \).

For any orthonormal basis for \( \mathbb{R}^3 \), and each \( t_0 \in [0, T) \) and \( t \in [0, \frac{1}{4}] \) with \( t_0 + r_{-}(t_0)^2 t < T \),

\[
M_{t_0} \cap \left\{ (x, y, z) \in \mathbb{R}^3 : z < 0, x^2 + y^2 < \frac{1}{16} \right\} = \left\{ (x, y, u_0(x, y, t)) : x^2 + y^2 < \frac{1}{16} \right\},
\]

where \(- (C_0 + 1) < u_0(x, y, t) < 0 \) and \( |Du_0|(x, y, t) \leq 4(C_0 + 1) \), and

\[
\frac{\partial u_0}{\partial t} = F(D^2 u_0, Du_0) = F \left( (g(Du_0))^{-1/2} (D^2 u_0) (g(Du_0))^{-1/2} \right),
\]

where \( g(V) = I + V^T V \). The matrix \( g \) has eigenvalues bounded below by 1 and above by \( 1 + 16(C_0 + 1)^2 \), and so the derivatives of \( \tilde{F} \) with respect to the components of \( D^2 u \) are comparable to the derivatives of \( F \), which are given by \( \frac{\partial F}{\partial \kappa} \), and Equation (5) is uniformly parabolic. The Krylov-Safonov Harnack estimate [K] gives positive lower bounds for \( F \) (hence also positive lower bounds for all principal curvatures) for \( M_{t_0} \). Theorem 5 of [A2] gives uniform Hölder bounds on the second spatial derivatives of \( u_0 \) on the same range. Finally, Schauder estimates [L, Theorem 4.9] give uniform bounds on all higher derivatives of \( u_0 \). Since the choice of basis was arbitrary, these imply uniform bounds on curvature and its higher derivatives on \( M_{t_0, t} \).

It follows that the hypersurfaces \( M_{t_0, t} \) extend to exist on \( S^3 \times [0, T] \) for every \( t_0 \in [0, T) \), and consequently \( T > t_0 + r_{-}(M_{t_0})^2/4 \) for all \( t_0 < T \), so that \( r_{-}(M_{t_0}) \leq 4(T - t_0) \) for all \( t_0 < T \), so the inradius \( r_{-}(M_t) \) (hence also the circumsradius \( r_{+}(M_t) \)) approach zero as \( t \) approaches \( T \).

Finally, the uniform bounds on the families of hypersurfaces \( \{M_{t_0, t} \} \) imply that there exists a subsequence \( t_k \to T \) such that the families \( M_{t_0, t_k} \) converge smoothly to a limiting family \( \bar{M}_t \), \( 0 \leq t \leq \frac{1}{4} \), which is again a solution of (1), on which the pinching ratio is constant in time. By the strong maximum principle applied to the evolution equation for \( G \) derived in Section 3 the limit solution is a shrinking sphere. This proves sub-ball-convergence of the rescaled solutions to a sphere, and stronger convergence can be deduced by considering the linearization of the flow about the shrinking sphere solution as in [A4, Propositions 40–41].
5. Remarks on Higher Degrees of Homogeneity

In this section the methods of the previous sections are applied to flows in which the speed is homogeneous of some degree \( \alpha > 1 \) in the principal curvatures. The conclusion is that such flows do not preserve large values of the pinching ratio. In work of the author on motion of surfaces by Gauss curvature \([A3]\), it was shown that the maximum difference between the principal curvatures does not get any larger under this flow. Thus the fact that the pinching ratio does not improve does not rule out the possibility that other curvature estimates may yield useful results.

The evolution of a curvature function \( G \) under a flow with speed \( F \) which is homogeneous of degree \( \alpha \) is as follows (compare equation (2)):

\[
\frac{\partial G}{\partial t} = F^{ij}\nabla_i \nabla_j G + (G^{ij} F_{kln} - F^{ij} G_{kln}) \nabla_i h_{kl} \nabla_j h_{mn} + G^{ij} h_{ij} F_{kln} h_{mn} - (\alpha - 1) F^{ij} G^{kl} h_{kij} h_{jln}.
\]

(6)

If \( G \) is homogeneous of degree zero, then the first term on the second line vanishes. Also, \( G^{ij} g^{kl} h_{ij} h_{jl} = \frac{\partial G}{\partial \kappa_1} \kappa_1^2 + \frac{\partial G}{\partial \kappa_2} \kappa_2^2 = \frac{\partial G}{\partial \kappa_1} \kappa_2 (\kappa_2 - \kappa_1) \geq 0 \) for \( G \) as in Section 3 above. Therefore the last term is non-positive. It remains to understand the gradient terms, as before.

The difference from the computation in Section 3 arises from a change in the Euler identities:

\[
\kappa_1^2 \frac{\partial^2 F}{\partial \kappa_1^2} + 2 \kappa_1 \frac{\partial^2 F}{\partial \kappa_1 \partial \kappa_2} + \kappa_2^2 \frac{\partial^2 F}{\partial \kappa_2^2} = \alpha (\alpha - 1) F; \quad \kappa_1 \frac{\partial F}{\partial \kappa_1} + \kappa_2 \frac{\partial F}{\partial \kappa_2} = \alpha F.
\]

These lead to the following expression for the gradient terms:

\[
Q = \left( G^{ij} F_{kln} - F^{ij} G_{kln} \right) \nabla_i h_{kl} \nabla_j h_{mn} = \frac{\partial G}{\partial \kappa_1} \left( \alpha (\alpha - 1) \frac{F}{\kappa_1^2} + 2 \alpha \frac{F}{\kappa_2 (\kappa_2 - \kappa_1)} \right) (\nabla_1 h_{22})^2 + \frac{\partial G}{\partial \kappa_2} \left( \alpha (\alpha - 1) \frac{F}{\kappa_1} - 2 \alpha \frac{F}{\kappa_1 (\kappa_2 - \kappa_1)} \right) (\nabla_2 h_{11})^2.
\]

For this to be negative the pinching ratio \( r = \kappa_2 / \kappa_1 \) must satisfy \( 2 \alpha + (\alpha - 1)(r - 1) \geq 0 \) and \( (\alpha - 1)r - (\alpha - 1) - 2 \leq 0 \). The first is always true since \( r \geq 1 \), but the second holds only if

\[
r \leq r_0(\alpha) = 1 + \frac{2}{\alpha - 1}.
\]

Thus the flow will improve pinching ratios no greater than \( r_0(\alpha) \). The argument to prove smooth convergence to a sphere then follows exactly that in \([A\&M]\). The precise result is as follows:

**Theorem 3.** Let \( F \) be a smooth function defined on the positive cone, homogeneous of degree \( \alpha > 1 \), strictly increasing in each argument, and normalized to have \( F(1,1) = 1 \). Then for any surface \( M_0 = x_0(S^2) \) which is smooth and strictly convex with pinching ratio \( r \leq r_0(\alpha) \) there exists a unique smooth solution \( x : S^2 \times [0,T) \to \mathbb{R}^3 \) of the evolution equation

\[
\frac{\partial x}{\partial t}(z,t) = -F(\kappa_1(z,t), \kappa_2(z,t)) \nu(z,t);
\]

\[
x(z,0) = x_0(z).
\]

The surfaces \( M_t = x_t(S^2) \) converge to a point \( p \in \mathbb{R}^3 \) as \( t \to T \), and the rescaled hypersurfaces \( M_t^{\frac{1}{((1+\alpha)(T-t))^{(1+\alpha)}}} \) converge in \( C^\infty \) to the unit sphere about the origin.
Next we show that this result cannot be improved, by constructing examples of smooth, strictly convex surfaces for which the pinching ratio becomes larger, for any flow with a speed homogeneous of degree \( \alpha > 1 \). Remarkably, the particular surface which provides a counterexample depends only on the degree of homogeneity \( \alpha \).

Consider surfaces given by rotating the graph \( y = u(x) \) about the \( x \) axis. The principal curvatures of this surface are given in terms of \( u \) by

\[
\kappa_1 = \frac{1}{u \sqrt{1 + (u')^2}}
\]

in the direction within the \( y - z \) plane, and

\[
\kappa_2 = -\frac{u''}{(1 + (u')^2)^{3/2}}
\]

in the direction along the \( x \) axis. Therefore the ratio of principal curvatures is equal to

\[
r = -\frac{uu''}{1 + (u')^2}.
\]

The ratio of principal curvatures can be prescribed as a function of distance from the axis: Let \( f : \mathbb{R} \to \mathbb{R} \) be any smooth positive even function with \( f(0) = 1 \). Then it is necessary to solve the following ordinary differential equation with initial data of the form \( u(0) = U, u'(0) = 1 \):

\[
f(u) = -\frac{uu''}{(1 + (u')^2)}.
\]

This can be integrated to give

\[
(u')^2 = \exp \left\{ 2 \int_u^t \frac{f(z)}{z} \, dz \right\} - 1,
\]

which can in turn be integrated to give a smooth solution on \([0, L)\) with \( u(x) \sim C \sqrt{L^2 - x^2} \) as \( x \) approaches \( L \). Extending this to be even in \( x \) gives a smooth, strictly convex surface with ratio of principal curvatures equal to \( f(u) \) where \( u \) is the distance from the axis of rotation.

Now choose \( f(u) \) to be a smooth function with \( f(u) = r_1 > 1 \) for \( u \geq u_0 \) and \( f(u) \in (1, r_1) \) for \( 0 < u < u_0 \). If \( U > u_0 \), then this defines a smooth, strictly convex surface with pinching ratio equal to \( r_1 \), with this pinching ratio attained on an open annular region away from the ‘poles’ of the surface. In this region, \( \nabla G = 0 \) and \( \nabla \nabla G = 0 \), where \( \nabla \) is the covariant derivative and \( G \) is as given in section 3. A direct calculation also shows that \( \nabla_1 h_{22} = 0 \) everywhere, while

\[
|\nabla_2 h_{11}|^2 = \frac{(u')^2(1 + (u')^2)}{u^4} (f(u) - 1)^2
\]

which is certainly not identically zero on this annular region. Therefore in the evolution equation \( 6 \) for \( G \), the first term is zero, the second term is positive, and the last term is negative. To get further one can write out these terms explicitly in terms of \( u \): At points with \( r = r_1 \),

\[
\frac{\partial G}{\partial t} = \frac{F \partial G}{\kappa_2} (r_1 - 1) \frac{\alpha ((\alpha - 1)(r_1 - 1) - 2)(u')^2(1 + (u')^2)^3 - (\alpha - 1)r_1^2}{r_1 u^2(1 + (u')^2)^2}
\]

But now by choosing \( u_0 \) sufficiently close to zero while keeping \( U \) fixed, it can be guaranteed that there are points achieving the pinching ratio which have \( u' \) as large as desired. As long as
$r_1 > r_0(\alpha)$, the positive first term in the bracket dominates the negative second term, so that $G$ is strictly increasing, and the pinching ratio becomes larger for small positive times.

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