THE NONABELIAN HODGE CORRESPONDENCE FOR BALANCED HERMITIAN METRICS OF HODGE-RIEMANN TYPE

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Abstract. This paper extends the nonabelian Hodge correspondence for Kähler manifolds to a larger class of hermitian metrics on complex manifolds called balanced of Hodge-Riemann type. Essentially, it grows out of a few key observations so that the known results, especially the Donaldson-Uhlenbeck-Yau theorem and Corlette’s theorem, can be applied in our setting. Though not necessarily Kähler, we show that the Sampson-Siu Theorem proving that harmonic maps are pluriharmonic remains valid for a slightly smaller class by using the known argument. Special important examples include those balanced metrics arising from multipolarizations.

1. Introduction

Let $X$ be a compact, complex manifold of dimension $n$. Recall that a hermitian metric on $X$ is called balanced if $d\omega^{n-1} = 0$, where $\omega$ is the fundamental (Kähler) $(1,1)$-form of the metric. The balanced metrics are a more restrictive class than the Gauduchon metrics, which satisfy $\partial\bar{\partial}\omega^{n-1} = 0$. Nevertheless, there are many examples of balanced, non-Kähler, metrics (cf. [18, p. 292]).

In this paper we consider a further condition. We say that a balanced metric is of Hodge-Riemann type, if it admits an expression:

\[
\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \Omega_0
\]

where $\omega_0$ (resp. $\Omega_0$) is a real (1,1) (resp. $(n-2,n-2)$), and $\Omega_0$ satisfies the Hodge-Riemann bilinear relations (see Definition 2.1 for the precise definition).

The condition of being balanced of Hodge-Riemann type seems very restrictive. However, many examples of non-Kähler metrics satisfying this property come from multipolarizations. Namely, let $\omega_0, \omega_1, \ldots, \omega_{n-2}$ be positive $(1,1)$-forms on $X$, and suppose

\[
\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \cdots \wedge \omega_{n-2}
\]

such that

- $d\omega^{n-1} = 0$;
- $d(\omega_1 \wedge \cdots \wedge \omega_{n-2}) = 0$

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(e.g. both conditions are automatic if the $\omega_i$ are Kähler). Then by a result of Timorin [22], $\omega$ is of Hodge-Riemann type. Even when the $\omega_i$ are Kähler metrics, $\omega$ is not so in general.

In this note we show that certain properties of Hermitian-Einstein metrics and equivariant harmonic maps familiar for Kähler manifolds continue to hold for balanced metrics of Hodge-Riemann type. Namely, we prove

1. a generalized Bogomolov-Miyaoka-Yau inequality for $\omega$-polystable holomorphic (and Higgs) bundles (Corollary 3.4);
2. a version of the Sampson-Siu pluriharmonicity theorem for harmonic maps to targets with nonpositive complexified sectional curvature (Theorem 4.1 and Corollary 4.3);
3. the nonabelian Hodge correspondence relating $\omega$-stable Higgs bundles with vanishing Chern classes to irreducible representations of the fundamental group (Theorem 5.1).

Let us remark that for the class of Gauduchon metrics, items (2) and (3) do not hold in general (see [2]), and the statement of item (1) cannot even be formulated.

The simple idea behind these generalizations is well explained in [21, Lemma 1.1]. Let us focus on item (3) above. Suppose $D$ be a complex connection on a vector bundle $E \to X$. A hermitian metric $h$ on $E$ gives a decomposition $D = D'' + D'$ (see (5.2)). Conversely, a Higgs bundle defines an operator $D''$, and a metric allows one to complete it to a complex connection $D$ by setting $D' = (D'')^*$. Let $F_D = D^2$ be the curvature of $D$, and $G_D = (D'')^2$ the pseudo-curvature. Flatness of $D$ is the equation $F_D = 0$, whereas $D$ arises from a Higgs bundle iff $G_D = 0$.

Now suppose $\omega$ is a balanced metric on $X$, so that the degree and slope stability of holomorphic bundles can be defined. Given an $\omega$-slope stable Higgs bundle with $\text{ch}_1(E) = 0$, one can find a metric $h$ so that the associated connection $D$ satisfies

(1.3) $F_D \wedge \omega_0 \wedge \Omega_0 = 0$

Similarly, given a flat connection $D$ one can find a harmonic metric, meaning that

(1.4) $G_D \wedge \omega_0 \wedge \Omega_0 = 0$

Thus, under the assumptions, the forms $F_D$ and $G_D$ are “primitive”, in the sense of [21] below.

The nonabelian Hodge correspondence follows by showing that if in addition $\text{ch}_2(E) = 0$, then (1.3) implies $F_D = 0$, and on the other hand, (1.4) always implies $G_D = 0$ (as pointed out in [21, p. 17], the “pseudo-Chern class” defined by $G_D$ automatically vanishes by the flatness of $D$). Now, if we assume $\omega$ is of Hodge-Riemann type, then these conclusions hold by integrating $\text{tr}(F_D \wedge F_D)$ or $\text{tr}(G_D \wedge G_D)$ against $\Omega_0$, and using the vanishing of the Chern classes and the Hodge-Riemann bilinear relations. Thus, we see that the Kähler condition may be relaxed.

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2. Hodge-Riemann forms

In this section, we recall the notion of a Hodge-Riemann form on a polarized complex vector space, i.e. a complex space \( V \) with a constant Kähler form \( \omega_0 \). Denote \( \Lambda^{p,q} \) to be the space of constant \((p,q)\) forms over \( V \). We fix \( \Omega_0 \) to be any real \((n-p-q, n-p-q)\) form. On \( \Lambda^{p,q} \), we can define a Hermitian form as

\[
Q(\alpha, \beta) := (\sqrt{-1})^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} * (\alpha \wedge \beta \wedge \Omega_0)
\]

The space of primitive forms of degree \((p,q)\) associated to \((\Omega_0, \omega_0)\) is defined as

\[
P^{p,q} = \{ \alpha \in \Lambda^{p,q} : \alpha \wedge \omega_0 \wedge \Omega_0 = 0 \}.
\]

**Definition 2.1.** We call \( \Omega_0 \) a Hodge-Riemann form for degree \((p,q)\) with respect to \( \omega_0 \) if

1. there exists a \( Q \)-orthogonal decomposition \( \Lambda^{p,q} = \mathbb{C}\omega_0 \oplus P^{p,q} \);
2. \( Q \) is positive definite on \( P^{p,q} \).

**Remark 2.2.** It follows from the classical Hodge-Riemann relation that \( \Omega_0 = \omega_0^{n-p-q} \) is a Hodge-Riemann form (cf. [24, Thm. 6.32]).

In general, the Hodge-Riemann property of a form is difficult to verify. However, we have the following result, which has been used to get the mixed Hodge-Riemann relation (see [8]).

**Proposition 2.3** ([22, Main Theorem], see also [12]). For any constant positive \((1,1)\) forms \( \omega_1, \cdots, \omega_k \) on \((V, \omega_0)\), \( \omega_1 \wedge \cdots \wedge \omega_{n-k} \) is a Hodge-Riemann form with respect to \( \omega_0 \) for any degrees \((p,q)\) satisfying \( p + q = k \).

As a special case, this gives

**Corollary 2.4.** For any constant positive \((1,1)\) forms \( \omega_1, \cdots, \omega_{n-2} \) on \((V, \omega_0)\)

1. \( \omega_0 \wedge \cdots \wedge \omega_{n-2} \) is a strictly positive \((n-1, n-1)\) form. In particular, there exists a positive \((1,1)\) form \( \omega \) so that

\[
\frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \cdots \wedge \omega_{n-2}
\]

(cf. [18, p. 279], and also [23]).

2. \( \omega_1 \wedge \cdots \wedge \omega_{n-2} \) is a Hodge-Riemann form for degrees \((p,q)\) satisfying \( p + q = 2 \) with respect to \( \omega_0 \).

This combined with [20, Cor. 8.5] implies the following

**Proposition 2.5.** For any constant Kähler forms \( \omega_1, \omega_2 \) on \((V, \omega_0)\), the form

\[
\omega_1^{n-2} + \omega_1 \wedge \omega_2^{n-3} + \cdots \omega_2^{n-2}
\]

is a Hodge-Riemann form for degree \((p,q)\) with \( p + q = 2 \) with respect to \( \omega_0 \).

**Remark 2.6.** It is known that the Hodge-Riemann property is not invariant under convex linear combinations. For example, fix any two Kähler forms \( \omega_1 \) and \( \omega_2 \) on \( \mathbb{C}^4 \), \( \omega_1^2 + a \omega_2^2 \) is not a Hodge-Riemann form for degree \((1,1)\) for certain positive values of \( a \) (see [20, Rem. 9.3] and also [22, Rem. 3] for other examples).
Timorin’s result motivates the following:

**Definition 2.7.** A hermitian metric $\omega$ on a complex manifold $X$ is said to be *balanced of Hodge-Riemann type* if the following hold:

1. we have an expression
   \[
   \frac{\omega^{n-1}}{(n-1)!} = \omega_0 \wedge \Omega_0
   \]
   where $\omega_0$ is a strongly positive real $(1,1)$ form on $X$ and $\Omega_0$ is a real $(n-2,n-2)$;
2. at every point $\Omega_0$ is a Hodge-Riemann form for $(p,q)$, $p + q = 2$;
3. $\Omega_0$ and $\omega_0 \wedge \Omega_0$ are closed.

Note that (3) is equivalent to $\omega$ being balanced and $\Omega_0$ being closed.

### 3. Bogomolov-Miyaoka-Yau inequality

Below we show how the Donaldson-Uhlenbeck-Yau (resp. Hitchin-Simpson) theorem relating stability of holomorphic (resp. Higgs) bundles to the existence of Hermitian-Einstein type metrics results in a Chern class inequality. The main result is Corollary 3.4. In this section, assume $(X, \omega)$ is a compact complex Hermitian manifold that satisfies items (1) and (3) of Definition 2.7, as well as the Hodge-Riemann condition (2) for the case $(p,q) = (1,1)$.

Recall that associated to every coherent analytic sheaf $E \to X$ is a holomorphic line bundle $\det E$. The first Chern class of $E$ is by definition $c_1(E) := c_1(\det E) \in H^2(X,\mathbb{Z}) \cap H^{1,1}_\partial(X)$. We define the $\omega$-degree of $E$ by:

\[
\deg E := \int_X c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_X c_1(E) \wedge \omega_0 \wedge \Omega_0
\]

Because of the balanced condition, this is well-defined on the class of $c_1(E)$.

The *slope* of a (nonzero) torsion-free sheaf is

\[
\mu(E) = \frac{\deg E}{\text{rank } E}
\]

Then we say a holomorphic bundle $E \to X$ is $\omega$-*stable* if $\mu(S) < \mu(E)$ for every coherent subsheaf $S \subset E$ with $0 < \text{rank } S < \text{rank } E$.

A *Higgs bundle* on $X$ is a pair $(E, \theta)$, where $E \to X$ is a holomorphic bundle, $\theta$ is a holomorphic 1-form with values in $\text{End } E$, and $\theta \wedge \theta = 0$. We say that a Higgs bundle is $\omega$-stable if $\mu(S) < \mu(E)$ for every coherent subsheaf $S \subset E$ with $0 < \text{rank } S < \text{rank } E$ and $\theta(S) \subset S \otimes \Omega^1_X$, where $\Omega^1_X$ is the holomorphic cotangent sheaf of $X$. Thus, stable vector bundles are a special case of stable Higgs bundles, where $\theta \equiv 0$. Finally, we say that $(E, \theta)$ is $\omega$-*polystable* if $(E, \theta)$ splits as a direct sum of Higgs subbundles, all with the same slope.

Given a hermitian metric $h$ on $E$, let $\overline{\partial}_E + \partial_E$ denote the Chern connection of $(E,h)$, $\theta^*$ the hermitian adjoint of $\theta$ with respect to $h$. Thus, $\theta^*$ is a $(0,1)$-form with values in $\text{End } E$, satisfying $\partial_E \theta^* = 0$. We will consider the complex connection

\[
D = \overline{\partial}_E + \partial_E + \theta + \theta^*
\]

and its curvature $F_D$. To be explicit, we will write: $D = (E, \theta, h)$. 
Definition 3.1. Fix a Higgs bundle \((\mathcal{E}, \theta)\) on \(X\). A hermitian metric \(h\) on \(\mathcal{E}\) is called Hermitian-Einstein (HE) if
\[
\sqrt{-1} F(\mathcal{E}, \theta, h) \wedge \omega_0 \wedge \Omega_0 = \lambda \cdot \text{Id} \cdot \omega_0^n \wedge \Omega_0
\]
for some constant \(\lambda\).

Now we have the following generalized Donaldson-Uhlenbeck-Yau, Hitchin-Simpson theorem.

Theorem 3.2. \((\mathcal{E}, \theta)\) is \(\omega\)-polystable if and only if it admits a HE metric. Moreover, if \((\mathcal{E}, \theta)\) is \(\omega\)-stable, such a metric is unique up to scaling.

We will use the key fact that for balanced metrics, the Kähler identities hold for \((1, 0)\) and \((0, 1)\) forms ([10, Prop. 1]; see also [16, Lemma 7.1.1]).

Lemma 3.3. Given an \(n\)-dimensional hermitian manifold \((X, \omega)\) with \(d\omega^{n-1} = 0\), the following hold:
\[
\bar{\partial}^* \alpha_{0,1} = -\sqrt{-1} \Lambda \partial \alpha_{0,1}, \quad \partial^* \alpha_{1,0} = \sqrt{-1} \Lambda \bar{\partial} \alpha_{1,0}
\]
for any \((0, 1)\)-form \(\alpha_{0,1}\), and any \((1, 0)\)-form \(\alpha_{1,0}\).

Proof of Theorem 3.2. That the existence of a HE metric implies polystability is well-known. For the converse, it suffices to assume \((\mathcal{E}, \theta)\) is \(\omega\)-stable.
Since \(\omega\) is balanced, it is in particular Gauduchon, and so by the result of Li-Yau [15], generalized to Higgs bundles by Lübke-Teleman [17], there is a metric \(\tilde{h}\) such that
\[
\sqrt{-1} F(\mathcal{E}, \theta, \tilde{h}) \wedge \omega_0^{n-1} = \tilde{\lambda} \cdot \text{Id} \cdot \omega_0^n
\]
where \(\tilde{\lambda} = 2\pi \mu(\mathcal{E}) / \text{vol}(X, \omega)\). Now there is a positive function \(f\) such that
\[
\omega_0 \wedge \frac{\omega^{n-1}}{(n-1)!} = f \cdot \frac{\omega^n}{n!}
\]
Choose \(\lambda\) such that
\[
(3.2) \quad \lambda \int_X f \cdot \frac{\omega^n}{n!} = \tilde{\lambda} \text{vol}(X, \omega) = 2\pi \mu(\mathcal{E})
\]
Then we can find a function \(\varphi\) satisfying: \(\Delta_\omega \varphi = 2(\lambda f - \tilde{\lambda})\). Let \(h = e^{\varphi} \tilde{h}\). Then
\[
F_{(\mathcal{E}, \theta, h)} = F_{(\mathcal{E}, \theta, \tilde{h})} - \partial \bar{\partial} \varphi \cdot \text{Id}
\]
By Lemma 3.3, the Hodge and Dolbeault laplacians on functions are related: \(\Delta_\omega = 2\Delta_{\bar{\partial}}\). Hence,
\[
-i \bar{\partial} \partial \varphi \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \Delta_\omega \frac{\omega^n}{n!} = \lambda \omega_0 \wedge \frac{\omega^{n-1}}{(n-1)!} - \tilde{\lambda} \frac{\omega^n}{n!}
\]
The result follows.

As a direct corollary of this, we have the following generalized Bogomolov-Miyaoka-Yau inequality.
Corollary 3.4. For any $\omega$-polystable rank $r$ Higgs bundle $(\mathcal{E}, \theta)$, the following inequality holds:

$$\int_X (2rc_2(\mathcal{E}) - (r - 1)c_1(\mathcal{E})^2) \wedge \Omega_0 \geq 0$$

where the equality holds if and only if $\mathcal{E}$ is projectively flat.

Proof. Eq. (3.1) implies $F(\mathcal{E}, \theta, h) - \frac{1}{r} \text{tr}(F(\mathcal{E}, \theta, h)) \cdot \text{Id} \cdot \omega_0$ is primitive. Now use the Hodge-Riemann property of $\Omega_0$.

For emphasis, we state the following version of the Donaldson-Uhlenbeck-Yau theorem for the slope stability condition defined by multipolarizations (cf. [11]).

Theorem 3.5. Suppose $X$ is a compact Kähler manifold with $(n - 1)$ Kähler forms $\omega_0, \cdots, \omega_{n-2}$. Given a holomorphic vector bundle $\mathcal{E}$ that is slope stable with respect to $[\omega_0] \cup \cdots \cup [\omega_{n-2}]$, there exists a Hermitian-Einstein metric $h$ on $\mathcal{E}$, i.e.

$$\sqrt{-1} F(\mathcal{E}, h) \wedge \omega_0 \wedge \cdots \wedge \omega_{n-2} = \lambda \cdot \text{Id} \cdot \omega_0^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}$$

for some constant $\lambda$. Moreover, such a metric is unique up to constant rescalings.

Remark 3.6. We emphasize here that the Chern connection of $(\mathcal{E}, h)$ is not a Yang-Mills connection in general.

By Corollary 3.4 and Proposition 2.3, we have the following generalization of the Bogomolov-Gieseker inequality for multipolarizations, proven in the projective case by Miyaoka [19, Cor. 4.7].

Corollary 3.7. Suppose $X$ is a compact Kähler manifold with $(n - 1)$ Kähler forms $\omega_0, \cdots, \omega_{n-2}$, and $\mathcal{E}$ is a slope stable holomorphic vector bundle over $X$ with respect to $[\omega_0] \cup \cdots \cup [\omega_{n-2}]$. Then the following holds:

$$\int_X (2rc_2(\mathcal{E}) - (r - 1)c_1^2(\mathcal{E})) \wedge \Omega_j \geq 0$$

for any $j = 0, \ldots, n - 2$. Here, $\Omega_j = \omega_0 \wedge \cdots \wedge \omega_{j-1} \wedge \omega_{j+1} \cdots \wedge \omega_{n-2}$. Moreover, the equality holds for some $j$ if and only if $\mathcal{E}$ is projectively flat.

Remark 3.8. The gauge theoretic side of the HE connections defined via multipolarizations is studied in [4].

4. The Sampson-Siu theorem

In this section, we prove

Theorem 4.1. Let $X$ be a compact complex manifold with a balanced metric of Hodge-Riemann type, and assume $\Omega_0$ is strongly positive. If $N$ is a Riemannian manifold with nonpositive complexified sectional curvature, then every harmonic map $u : X \to N$ is pluriharmonic.
In the statement of the theorem, \( \omega \) satisfies the condition of Definition 2.7, but we make the additional assumption that \( \Omega_0 \) is a strongly positive \((n-2,n-2)\)-form in the sense of [7] Ch. III, Def. 1.1.

**Proof of Theorem 4.1.** Let \( \nabla \) denote the Levi-Cività connection on \( N \). This induces a connection on \( u^*TN \). The harmonic map equation is: \( d\bar{\nabla}u = 0 \).

Since \( \omega \) is balanced, Lemma 3.3 implies that \( u \) is harmonic if and only if

\[
\langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle \wedge \Omega_0 + \langle d\bar{\nabla}d\bar{\nabla}u \wedge d\bar{\nabla}d\bar{\nabla}u \rangle \wedge \Omega_0 = 0
\]

Next, we follow the argument in [1, pp. 73-75]. Since \( \Omega_0 \) is closed,

\[
d\langle d\bar{\nabla}d\bar{\nabla}u \wedge d\bar{\nabla}u \rangle \wedge \Omega_0 = \langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle \wedge \Omega_0 + \langle d\bar{\nabla}d\bar{\nabla}u \wedge d\bar{\nabla}d\bar{\nabla}u \rangle \wedge \Omega_0
\]

and so,

\[
0 = \int_X \{(\langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle \wedge \Omega_0 + \langle d\bar{\nabla}d\bar{\nabla}u \wedge d\bar{\nabla}d\bar{\nabla}u \rangle \wedge \Omega_0\}
\]

By (4.1) and the Hodge-Riemann property, the second term is nonpositive. We claim that also

\[
\langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle \wedge \Omega_0 \leq 0
\]

Given this, it follows that \( d\bar{\nabla}d\bar{\nabla}u = 0 \); hence, the pluriharmonicity. The claim follows from the assumption on \( \Omega_0 \) and the nonpositivity of the complexified sectional curvature of \( N \). We work at a point \( x \). By definition, we know that

\[
\Omega_0 = \sum_i \mu_i \sqrt{-1} \alpha_i^1 \wedge \cdots \wedge \sqrt{-1} \alpha_i^j \wedge \alpha_{i-1}^{n-2} \wedge \alpha_{i-1}^{n-2}
\]

where \( \mu_i \geq 0 \), and \( \{\alpha_i^1, \cdots, \alpha_i^j\} \) are linearly independent \((1,0)\) forms. Denote by \( P_i \) the complex two dimensional subspace of \( TM \) where \( \alpha_j^i \vert_{P_i} = 0 \) for \( j = 1, \cdots, n-2 \). Fix \( X_i, Y_i \) so that \( \{X_i, Y_i, JX_i, JY_i\} \) form an orthogonal basis for \( P_i \). Then

\[
\langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle \wedge \Omega_0 = \sum_i \mu_i \langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle (X_i, Y_i, JX_i, JY_i) d\text{Vol}
\]

for some \( \mu_i \geq 0 \). Now as in [11 p. 75], we know

\[
\langle R_N(d\bar{\nabla}u) \wedge d\bar{\nabla}u \rangle (X_i, Y_i, JX_i, JY_i) = R_N(Z_i, W_i, W_i, Z_i) \leq 0
\]

where \( Z_i = du(X_i - JX_i) \) and \( W_i = du(W_i - JW_i) \). The claim follows. \( \square \)

**Remark 4.2.** If \( \tilde{X} \) is the universal cover of \( X \), then Theorem 4.1 remains valid for harmonic maps \( u : \tilde{X} \rightarrow N \) that are equivariant with respect to a representation \( \rho : \pi_1(X) \rightarrow \text{Iso}(N) \). These play a role in the next section. The existence of equivariant harmonic maps to nonpositively curved targets \( N \) is guaranteed if \( \rho \) is reductive (or semisimple) (see [5 9 14 13]).

Theorem 4.1 combined with [7] Prop. III.1.11] implies

**Corollary 4.3.** Suppose \( X \) is a compact complex manifold with a balanced metric \( \omega \) of the form \((1,2)\), where \( \omega_i \) are positive \((1,1)\)-forms and

\[
d(\omega_1 \wedge \cdots \wedge \omega_{n-2}) = 0
\]

If \( N \) is a Riemannian manifold with nonpositive complexified sectional curvature, then every harmonic map \( u : X \rightarrow N \) is pluriharmonic.
5. The nonabelian Hodge correspondence

The goal of this section is to prove the following generalization of [21, Cor. 1.3].

**Theorem 5.1.** Suppose \( X \) is a compact complex manifold and \( \omega \) is a balanced metric of Hodge-Riemann type on \( X \). Then the nonabelian Hodge correspondence holds over \((X, \omega)\). More precisely, we have a 1-1 correspondence between

1. semisimple flat bundles on \( X \), and
2. isomorphism classes of \( \omega \)-polystable Higgs bundles \((E, \theta)\) with \( \text{ch}_1(E) \cup [\omega^{n-1}] = 0 \) and \( \text{ch}_2(E) \cup [\Omega_0] = 0 \).

**Remark 5.2.**
- When \( \omega_0 = \omega, \Omega_0 = \omega^{n-2}/(n-1)! \), then by our assumptions \((X, \omega)\) is Kähler. Then Theorem 5.1 reduces to the well known nonabelian Hodge correspondence for compact Kähler manifolds.
- There exist many examples where \( \omega \) is not a Kähler metric, even when the underlying manifold \( X \) is Kähler, or even projective algebraic. For example, take \( X \) to be projective with \( [\omega_i], i = 0, \ldots, n-2 \), all ample classes, and take \( \omega \) as in (1.2). Then the class \( \omega^{n-1} \) represents a point in the interior of the cone of movable curves (see [3]).
- Notice that if \( \Omega_0 = \omega^{n-2} \), then \( d\Omega_0 = 0 \) implies \( dw_1 = 0 \), and the manifold is Kähler. However, closedness of \( \Omega_0 = \omega_1 \wedge \cdots \wedge \omega_{n-2} \), for different \( \omega_i \), can occur in the non-Kähler setting. One might expect that this will provide new insights for the study the non-Kähler complex manifolds, since the results obtained here already put restrictions on complex manifolds admitting such structures.

**Proof of Theorem 5.1.** The proof, of course, closely follows the lines of the classical theorem, taking care to avoid the Kähler condition.

First, assume \((E, \theta)\) is an \( \omega \)-polystable Higgs bundle. If \( \text{ch}_1(E) \cup [\omega^{n-1}] = 0 \), then by Theorem 3.2 there is a hermitian metric \( h \) on \( E \) such that (1.3) is satisfied for

\[
D = \partial_E + \overline{\partial}_E + \theta + \theta^* \tag{5.1}
\]

(note that \( \lambda = 0 \) by (3.2)). Hence, \( F_D \) is primitive. Moreover, \( \sqrt{-1} F_D \) is of type \((1,1)\) and hermitian. Since \( \text{ch}_2(E) \cup [\Omega_0] = 0 \), we have

\[
0 = \int_X \text{tr}(F_D \wedge F_D) \wedge \Omega_0 \tag{5.2}
\]

Since \( \Omega_0 \) is a Hodge-Riemann form, we conclude that \( D \) is a flat connection, which is necessarily semisimple.

Now let \( D \) be a semisimple flat connection on \( E \). By Corlette’s theorem (see Remark 4.2) there exists a harmonic metric, which has the following consequence. Decomposing into type we can express \( D \) as in (5.1), where \( \theta \in \Omega^{1,0}(X, \text{End } E) \), and \( \theta + \theta^* \) is essentially \( du \) for an equivariant harmonic map \( u : \tilde{X} \to \text{GL}(n, \mathbb{C})/\text{U}(n) \). Let

\[
D'' = \overline{\partial}_E + \theta \quad , \quad D' = \partial_E + \theta^* \tag{5.3}
\]

...
We wish to prove that $G_D = (D'')^2 = 0$, for then $\bar{\partial}_E$ is integrable, $\bar{\partial}_E \theta = 0$, and $\theta \wedge \theta = 0$; i.e. $(\bar{\partial}_E, \theta)$ is a Higgs bundle.

Flatness of $D$ implies,

1. $\partial E \theta = 0$;
2. $(\bar{\partial}_E \theta)^* = -\bar{\partial}_E \theta$;
3. $\partial^2 E + \frac{1}{2} [\theta, \theta] = 0$;

By (4.1), we have $(\bar{\partial}_E \theta - (\bar{\partial}_E \theta)^*) \wedge \omega^{n-1} = 0$, and combining this with (2) we have

$$(5.3) \quad \bar{\partial}_E \theta \wedge \omega^{n-1} = \bar{\partial}_E \theta \wedge \omega_0 \wedge \Omega_0 = 0$$

i.e. $G_D$ is primitive (note that we have only used the balanced condition for this part).

To prove that $G_D = 0$, we argue as in the proof of Theorem 4.1 (see also [5, proof of Thm. 5.1]). We have:

$$d \text{tr}(\bar{\partial}_E \theta \wedge \theta^*) \wedge \Omega_0 = \partial \text{tr}(\bar{\partial}_E \theta \wedge \theta^*) \wedge \Omega_0$$

$$= \text{tr}(\partial E \bar{\partial}_E \theta \wedge \theta^*) \wedge \Omega_0 + \text{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0$$

$$= -\frac{1}{2} \text{tr}([\theta, \theta^*] \wedge \theta) \wedge \Omega_0 + \text{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0$$

$$= -\frac{1}{4} \text{tr}(\theta \wedge [\theta, \theta]^*) \wedge \Omega_0 + \text{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0$$

so integrating,

$$0 = -\frac{1}{4} \int_X \text{tr}(\theta \wedge [\theta, \theta]^*) \wedge \Omega_0 + \int_X \text{tr}(\bar{\partial}_E \theta \wedge (\bar{\partial}_E \theta)^*) \wedge \Omega_0$$

By the Hodge-Riemann property of $\Omega_0$, both terms on the right hand side are nonpositive, and hence vanish. We conclude that $\bar{\partial}_E \theta = 0$, and $[\theta, \theta] = 0$. By (3) above, $\bar{\partial}_E$ is integrable, and this completes the proof.

6. Rigidity of representations of fundamental groups

For the sake of completeness, in this last section we point out that two important results of Corlette and Simpson generalize to our setting. Let $G_R$ be a simple real algebraic group acting by isometries on the irreducible bounded symmetric domain $G_R/K$. We assume $(X, \omega)$ is a compact complex manifold with a balanced metric of Hodge-Riemann type. Let $P$ be the principle $G_R$ bundle with structure group reduced to $K$. As in [6], one can associate a volume $\text{vol}(P)$ to $P$ by defining it as a power of the first Chern class of $P$ up to a conformal factor. Now the following generalizes [6, Thm. 0.1].

**Theorem 6.1.** Suppose $P$ is flat with $\text{vol}(P) \neq 0$ and $G_R/K$ is not of the form $U(n, 1)/U(n) \times U(1)$ or $SO(2n + 1, 2)/SO(2n + 1) \oplus O(2)$. Then the monodromy homomorphism of the fundamental group of $X$ into $G_R$ is locally rigid as a homomorphism of the fundamental group of $X$ into the complexification of $G_R$.

The argument follows by replacing [6, Prop. 2.4] with argument in the proof of Theorem 5.1 to get the holomorphic property of the harmonic sections. This is the only place where the Kähler assumption is needed in [6]. Simpson’s argument in the Kähler case also gives (see [21]).
Theorem 6.2. Suppose $\rho : \pi_1(X) \to \text{GL}(n, \mathbb{C})$ is a locally rigid representation of the fundamental group of $X$. Then the associated flat vector bundle is the underlying vector bundle of a complex variation of Hodge structure.

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