Mean dimension theory in symbolic dynamics for
finitely generated amenable groups

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Abstract. In this paper, we mainly elucidate a close relationship between the topological entropy and mean dimension theory for actions of polynomial growth groups. We show that metric mean dimension and mean Hausdorff dimension of subshifts with respect to the lower rank subgroup are equal to its topological entropy multiplied by the growth rate of the subgroup. Meanwhile, we also prove the above result holds for the rate distortion dimension of subshifts with respect to the lower rank subgroup and measure entropy. Furthermore, some relevant examples are indicated.

Keywords and phrases: subshift, metric mean dimension, mean Hausdorff dimension, rate distortion dimension, polynomial growth groups.

1 Introduction

Let \((X, G)\) be a \(G\)-action topological dynamical system, where \(X\) is a compact Hausdorff space and \(G\) a topological group. Throughout this paper, \(G\) is a finitely generated amenable groups. An important dynamical quantity of a shift is its entropy, which roughly measures the exponential growth rate of its projections on finite sets.

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2010 Mathematics Subject Classification: 37B40, 37C85.
For the case $G = \mathbb{N}$, we consider the one-sided infinite product $A^\mathbb{N}$ with the shift map $\sigma : A^\mathbb{N} \to A^\mathbb{N}$ defined by

$$\sigma((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}.$$ 

Define a metric which is compatible with the product topology on $A^\mathbb{N}$ as follows: for every $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in A^\mathbb{N},$

$$d(x, y) = 2^{-\min\{n | x_n \neq y_n\}}.$$ 

Let $\mathcal{X}$ be a closed invariant subset of $A^\mathbb{N}$. Furstenberg proved the following relationship among entropy, Hausdorff and Minkowski dimensions of $\mathcal{X}$ with respect to $d$ [9, Proposition III.1]:

$$\dim_H(\mathcal{X}, d) = \dim_M(\mathcal{X}, d) = h_{\text{top}}(\mathcal{X}, \sigma),$$

where $h_{\text{top}}(\mathcal{X}, \sigma)$ is the topological entropy of $(\mathcal{X}, \sigma)$. Simpson [27] generalized the above results to $\mathbb{Z}^k$ action and more general result for amenable group action appears in [8]. For more relevant studies one may refer to [3, 20].

Mean dimension is a conjugacy invariant of topological dynamical systems which was introduced by Gromov [21]. This is a dynamical version of topological dimension and it counts how many parameters per iterate we need to describe an orbit in the dynamical systems. This invariant has several applications which cannot be touched within the framework of topological entropy, see [25, 17, 18]. In particularly, it has many applications to embedding problem whether a dynamical system can be embedded into another or not, see for instance [14, 13, 22, 23, 24].

It is well known that the concepts of entropy and dimension are closely connected. So it is natural to except we can approach to mean dimension from the entropy theory viewpoint. The first attempt of such an approach was made by Lindenstrauss and Weiss [14]. They introduced the notion of metric mean dimension, which is a dynamical analogue of Minkowski dimension [14], and they proved that metric mean dimension is an upper bound of the mean dimension. It allowed them to establish the relationship between the mean dimension and the topological entropy of dynamical systems. Namely, each system with finite topological entropy has zero metric mean dimension and zero mean dimension. Lindenstrauss and Tsukamoto in [15] established a variational principle between the metric mean dimension and the rate distortion function under a mild condition on the metric $d$ (called tame growth of covering numbers, for this definition see [15]). Inspired by the classic variational principle of entropy, they [16] also considered a measure-theoretic notion of mean dimension-rate distortion dimension, which was first introduced by Kawabata and Dembo in [11] and proved a corresponding variational principle for mean dimension. In order to link the measure theoretic aspect of mean dimension theory, they introduced the mean Hausdorff dimension in [16], which is a dynamical analogue of Hausdorff dimension.
Recently, Shinoda and Tsukamoto [26] generalized Furstenberg’s result in [9] to $\mathbb{Z}^2$ action which involves metric mean dimension, mean Hausdorff dimension and rate distortion dimension. In this paper, by adopting the method of [26] and [8], we are going to prove the relationship between mean dimension quantities (metric mean dimension, mean Hausdorff dimension and rate distortion dimension) and entropy, which generalize the result of [26] to actions of polynomial growth groups. The main difficulty in carrying out this generalization is that we need a Vitali type covering lemma. To this aim we apply a more general covering lemma developed by Lindentrauss [12].

The paper is organized as follows. In section 2, we review basic definitions of finitely generated amenable groups and mean dimension theory. Meanwhile, we state our main results. In section 3, we introduce covering lemma and give the proof of Theorem 2.1. In section 4, we present the notion of rate distortion dimension and prove the Theorem 2.2 by following Shinoda and Tsukamoto’s technical line. In section 5, we give some examples to illustrate our main theorem.

2 Preliminaries

In this section, we review some of the standard concepts and results on finitely generate amenable groups, metric mean dimension and mean Hausdorff dimension. Finally, we state our main results.

2.1 Finitely generated amenable groups

Let $G$ be an infinite discrete countable group. Let $F(G)$ denote the set of all finite non-empty subset of $G$. For $K, F \in F(G)$, let $KF \in F(G)$, let $KF = \{st : s \in K, t \in F\}$ and $KF\Delta F = (KF\setminus F) \cup (F\setminus KF)$. A group $G$ is called amenable if for each $K \in F(G)$ and $\delta > 0$, there exists $F \in F(G)$ such that $|KF\Delta F| < \delta|F|$, where $|\cdot|$ is the counting measure.

Let $K \in F(G)$ and $\delta > 0$. A finite subset $A \in F(G)$ is called $(K, \delta)$-invariant if

$$\frac{|B(A, K)|}{|A|} < \delta,$$

where $B(A, K)$, the $K$-boundary of $A$, is defined by

$$B(A, K) = \{g \in G : Kg \cap A \neq \emptyset \text{ and } Kg \cap (G \setminus A) \neq \emptyset\}.$$

Another equivalent condition for the sequence of finite subset $\{F_n\}$ of $G$ to be a F\'olner sequence is that $\{F_n\}$ becomes more and more invariant, i.e., for each $K \in F(G)$ and $\delta > 0$, $F_n$ is $(K, \delta)$-invariant when $n$ is large enough. A group $G$ is amenable group if and only if $G$ admits a F\'olner sequence $\{F_n\}$. For more details and properties of the
amenable group, one is referred to [19] [6]. Let $\epsilon \in (0, 1)$. $A_1, A_2, \ldots, A_k \in F(G)$ are said to be $\epsilon$-disjoint if there exist mutually disjoint $A'_i \subset A_i$ such that $|A'_i| \geq (1 - \epsilon)|A_i|$ for $1 \leq i \leq k$. Recall that a Fölner sequence $\{F_n\}$ in $G$ is said to be tempered if there exists a constant $C > 0$ which is independent of $n$ such that

$$| \bigcup_{k<n} F_{k}^{-1}F_n | \leq C |F_n|, \text{ for any } n. \quad (2.1)$$

Let $G$ be a finitely generated amenable group with a symmetric generating set $S$. Recall that a generating set is called symmetric if together with any $s \in S$ it contains $s^{-1}$. The $S$-word-length $\ell_S(g)$ of an element $g \in G$ is the minimal integer $n \geq 0$ such that $g$ can be expressed as a product of $n$ elements in $S$, that is,

$$\ell_S(g) = \min \{ n \geq 0 : g = s_1 \cdots s_n : s_i \in S, 1 \leq i \leq n \}.$$  

It immediately follows from the definition that for $g \in G$ one has $\ell_S(g) = 0$ if and only if $g = 1_G$. Define the metric $d_S$ on $G : d_S(g, h) = \ell_S(g^{-1}h)$. It is obvious that the metric $d_S$ is invariant by left multiplication.

For $g \in G$ and $n \in \mathbb{N}$, we denote by

$$B_S^G(g, n) = \{ h \in G : d_S(g, h) \leq n \},$$

the ball of radius $n$ in $G$ centered at the element $g \in G$. When $g = 1_G$ we have $B_S^G(1_G, n) = \{ h \in G : \ell_S(h) \leq n \}$ and we simply write $B_S^G(n)$ instead of $B_S^G(1_G, n)$. Also, when there is no ambiguity on the group $G$, we omit the subscript $G$ and we simply write $B_S(g, n)$ and $B_S(n)$ instead of $B_S^G(g, n)$ and $B_S^G(n)$. The growth function of $G$ relative to $S$ is a function $\gamma_S : \mathbb{N} \to \mathbb{N}$ defined by $\gamma_S(n) = |B_S(n)| = | \{ g \in G : \ell_S(g) \leq n \} |$.

**Remark 2.1.** [7] Let $G_1$ and $G_2$ be two finitely generated amenable groups. Then the direct product $G_1 \times G_2$ is also a finitely generated amenable group.

**Definition 2.1.** Let $G$ be a finitely generated group of polynomial growth with a symmetric generating set $S$ if there exists constants $d, A, B > 0$ such that

$$A n^d \leq \gamma_S(n) \leq B n^d$$

for all $n \in \mathbb{N}$. We denote by $\deg(G) = d$ the degree of the polynomial growth of $G$.

In this paper, we consider a finitely generated group of polynomial growth. Polynomial growth groups are amenable. We will review the definitions of metric mean dimension [4] and introduce the mean Hausdorff dimension of amenable group actions in subsection 2.2.
2.2 Metric mean dimension and mean Hausdorff dimension

Let \( G \) be a countable discrete amenable group. Let \( (\mathcal{X}, G) \) be a \( G \)-system with \( d \). For \( \epsilon > 0 \), we define \( \#(\mathcal{X}, d, \epsilon) \) as the minimum natural number \( n \) such that \( \mathcal{X} \) can be covered by open sets \( U_1, \cdots, U_n \) with \( \text{diam}(U_i) < \epsilon \) for \( 1 \leq i \leq n \). For \( F \in \mathcal{F}(G) \), define metric \( d_F \) on \( \mathcal{X} \) by

\[
d_F(x, y) = \max_{g \in F} d(gx, gy).
\]

For \( s \geq 0 \) and \( \epsilon > 0 \), we define \( H^*_\epsilon(\mathcal{X}, d) \) as

\[
H^*_\epsilon(\mathcal{X}, d) = \inf \left\{ \sum_{n=1}^{\infty} (\text{diam}E_n)^s \mid \mathcal{X} = \bigcup_n E_n \text{ with diam}E_n < \epsilon \text{ for all } n \geq 1 \right\}.
\]

We set

\[
\dim_H(\mathcal{X}, d, \epsilon) = \sup \{ s \geq 0 \mid H^*_\epsilon(\mathcal{X}, d) \geq 1 \}.
\]

The Hausdorff dimension \( \dim_H(\mathcal{X}, d) \) is given by

\[
\dim_H(\mathcal{X}, d) = \lim_{\epsilon \to 0} \dim_H(\mathcal{X}, d, \epsilon).
\]

We define the upper and lower mean Hausdorff dimension. Let \( \{F_n\} \) be a \( \text{F} \text{-}\)\( \Phi \text{-} \)sequence in \( G \), we can define

\[
\overline{\text{mdim}}_H(\mathcal{X}, \{F_n\}, d) = \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} \frac{\dim_H(\mathcal{X}, d_{F_n}, \epsilon)}{|F_n|} \right),
\]

\[
\underline{\text{mdim}}_H(\mathcal{X}, \{F_n\}, d) = \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} \frac{\dim_H(\mathcal{X}, d_{F_n}, \epsilon)}{|F_n|} \right).
\]

When these two quantities are equal to each other, we denote the common value by \( \text{mdim}_H(\mathcal{X}, \{F_n\}, d) \). For any \( \epsilon > 0 \), we define

\[
S(\mathcal{X}, G, d, \epsilon) = \lim_{n \to \infty} \frac{1}{|F_n|} \log \#(\mathcal{X}, d_{F_n}, \epsilon).
\]

The limit always exists and does not depend on the choice of the \( \text{F} \text{-}\)\( \Phi \text{-} \)sequence \( \{F_n\} \). The upper and lower metric mean dimension is then defined by

\[
\overline{\text{mdim}}_M(\mathcal{X}, G, d) = \limsup_{\epsilon \to 0} \frac{S(\mathcal{X}, G, d, \epsilon)}{|\log \epsilon|},
\]

\[
\underline{\text{mdim}}_M(\mathcal{X}, G, d) = \liminf_{\epsilon \to 0} \frac{S(\mathcal{X}, G, d, \epsilon)}{|\log \epsilon|}.
\]

When the upper and lower limits coincide, we denote the common value by \( \text{midm}_M(\mathcal{X}, G, d) \).

The following result is the dynamical analogue of the fact that Minkowski dimension no less than Hausdorff dimension.
Proposition 2.1. Let \( \{F_n\} \) be a Følner sequence, then
\[
\overline{\text{mdim}}_H(\mathcal{X}, \{F_n\}, d) \leq \text{mdim}_M(\mathcal{X}, G, d).
\]

Proof. Let \( \{F_n\} \) be a the Følner sequence in \( G \). For \( n \geq 0, \, \epsilon > 0 \), choose an open cover \( \mathcal{X} = U_1 \cup \cdots \cup U_m \) with \( \text{diam}(U_i, d_{F_n}) \leq \epsilon \) and \( m = \#(\mathcal{X}, d_{F_n}, \epsilon) \). We have
\[
H^\epsilon_\ell(\mathcal{X}, d_{F_n}) \leq m\epsilon^\ell.
\]

If \( s > \log m/\log(1/\epsilon) \), then \( H^s(\mathcal{X}, d_{F_n}) < 1 \). This shows
\[
\dim_H(\mathcal{X}, d_{F_n}, \epsilon) \leq \frac{\log \#(\mathcal{X}, d_{F_n}, \epsilon)}{\log(1/\epsilon)}.
\]
Divide this by \( |F_n| \) and take limits with respect to \( n \) and then \( \epsilon \). It follows that
\[
\overline{\text{mdim}}_H(\mathcal{X}, \{F_n\}, d) \leq \text{mdim}_M(\mathcal{X}, G, d).
\]

2.3 Statement of the main results

Now we state the main theorems. Let \( G_1 \) and \( G_2 \) be finitely generated groups of polynomial growth. Then direct product \( G = G_1 \times G_2 \) is also finitely generated of polynomial growth. Let \( S_1 \) and \( S_2 \) be finite symmetric generating subsets of \( G_1 \) and \( G_2 \). Then the set
\[
S = (S_1 \times \{1_{G_1}\}) \cup (\{1_{G_2}\} \times S_2)
\]
is a finite symmetric generating subset of \( G \). We denote by \( \text{deg}(G_1) \) and \( \text{deg}(G_2) \) the degrees of the polynomial growth of \( G_1 \) and \( G_2 \), respectively (e.g., \( \text{deg}(\mathbb{Z}^k) = k \)). Set \( S = \{s_1, \ldots, s_m\} \). Next we defines a order in \( S \) which formalize through the following construction: given \( s_i, s_j \in S \) we say that \( s_i < s_j \) if \( i < j \). Hence \( s_1 < s_2 < \cdots < s_m \).

Hence we can consider the order in \( G \). For \( g, g' \in G \), we call \( g < g' \) if \( \ell(g) < \ell(g') \). If \( \ell(g) = \ell(g') = n \), then there exist \( s_1, \ldots, s_n \) and \( s'_1, \ldots, s'_n \) such that
\[
g = s_1 \cdots s_n, \quad g' = s'_1 \cdots s'_n.
\]

Take \( k = \min \{i : s_i \neq s'_i\} \). When \( s_k < s'_k \), we denote by \( g < g' \), otherwise, \( g > g' \). Then we can arrange the elements in the group. Let \( G = (g_n)_{n=0}^\infty \) be an enumeration of \( G \) according to the order such that \( \ell(1_G) = \ell_S(g_0) \leq \ell_S(g_1) \leq \ell_S(g_2) \cdots \).

We can define a metric \( d \) on \( A^G \) by the following:
\[
d(x, y) = 2^{-\min \{|g_n|\infty : x_{g_n} \neq y_{g_n}\}}, \quad (2.3)
\]
where \( |g_n|\infty = \max \{\ell_{S_1}(g_{n,1}), \ell_{S_2}(g_{n,2})\} \) and \( g = (g_{n,1}, g_{n,2}) \). A closed \( G \)-invariant subset \( \mathcal{X} \) of \( A^G \) is called a subshift of \( A^G \).
Theorem 2.1. Let $\mathcal{X} \subset A^G$ be a subshift. Suppose that $\deg(G_2) = 1$. Then

(1) $\underline{\text{mdim}}_M(\mathcal{X}, G_1, d) \leq c_1 \cdot h_{\text{top}}(\mathcal{X}, G)$, where $c_1 = \limsup_{n \to \infty} \frac{|B_{g_2}(n)|}{n}$.

(2) $\underline{\text{mdim}}_H(\mathcal{X}, \{B_{g_1}(n)\}, d) \geq c_2 \cdot h_{\text{top}}(\mathcal{X}, G)$, where $c_2 = \liminf_{n \to \infty} \frac{|B_{g_1}(n)|}{n}$.

In particular, if $c_1 = c_2 = c$, we have

$$\underline{\text{mdim}}_H(\mathcal{X}, \{B_{g_1}(n)\}, d) = \underline{\text{mdim}}_M(\mathcal{X}, G_1, d) = c \cdot h_{\text{top}}(\mathcal{X}, G).$$

Theorem 2.2. If $\mu$ is a Borel probability measure on $\mathcal{X}$ invariant under both $\sigma_1$ and $\sigma_2$ and $c_1 = c_2 = c$, then

$$\underline{\text{rdim}}(\mathcal{X}, \sigma_1, \{B_{g_1}(n)\}, d, \mu) = c \cdot h_\mu(\mathcal{X}, G).$$

Remark 2.2. (1) It is not clear whether the value of $c$ exists for general amenable groups. Hence we only consider the polynomial growth groups in this paper.

(2) The reason for imposing the condition $\deg(G_2) = 1$ is that if $\deg(G_2) > 1$, we don’t know whether the value of $c$ exists.

3 Proof of Theorem 2.1

For a finite alphabet $A$ and a finitely generated group of polynomial growth $G$, the full $G$-shift over $A$ is the set $A^G$, which is viewed as a compact topological space with the discrete product topology. Consider the shift action on the product space $A^G$:

$$g'(x_g)_{g \in G} = (x_{g'g})_{g \in G}, \text{ for all } g' \in G \text{ and } (x_g)_{g \in G} \in A^G.$$ 

We define the shifts $\sigma_1$ and $\sigma_2$ on $A^G$ by $(\sigma_1.g)x)_{(g_1, g_2)} = x_{(g_1g_2, h)}$ and $(\sigma_2.h)x)_{(g_1, g_2)} = x_{(g_1, g_2, h)}$ for all $g \in G_1$, $h \in G_2$, $(g_1, g_2) \in G$. Let $\mathcal{X}$ be a subshift. For $E \subset G$, let $\pi_E : \mathcal{X} \to A^E$ denote the canonical projection map, that is, the map defined by $\pi_E(x) := x|_E$ for all $x \in \mathcal{X}$, where $x|_E$ denote the restriction of $x : G \to A$ to $E \subset G$. For each finite set $F \subset G$ and $\omega \in A^F$, a subset $C \subset \mathcal{X}$ is called a cylinder over $F$ if there exists $x \in \mathcal{X}$ such that $C$ is equal to the set of all $x \in \mathcal{X}$ with $\pi_F(x) = \pi_F(y)$. The proof falls naturally into two steps.

Step 1: $\underline{\text{mdim}}_M(\mathcal{X}, G_1, d) \leq c_1 \cdot h_{\text{top}}(\mathcal{X}, G)$.

Proof. For $\epsilon > 0$, choose $M > 0$ such that $2^{-M} < \epsilon < 2^{-M+1}$. For each a natural number $N > 0$, then

$$\#(\mathcal{X}, g_{B_{g_1}(N)}^r, \epsilon) \leq |\pi(B_{g_1}(M)B_{g_1}(N)B_{g_2}(M))(\mathcal{X})|,$$
where \( d_{B_{S_1}(N)}^2(x, y) = \max_{g \in B_{S_1}(N)} d(\sigma_{1,g}x, \sigma_{1,g}y) \). Since \( M - 1 \leq \log(1/\epsilon) \leq M \),

\[
\lim_{M \to \infty} \limsup_{\epsilon \to 0} \left( \lim_{N \to \infty} \frac{\log \#(X, d_{B_{S_1}(N)}^2, \epsilon)}{|B_{S_1}(N)| \log(1/\epsilon)} \right)
\leq \lim_{M \to \infty} \left( \lim_{N \to \infty} \frac{|\pi(B_{S_1}(M)B_{S_1}(N) \times B_{S_2}(M))|\langle X \rangle|}{|B_{S_1}(N)| (M - 1)} \right)
\leq \lim_{M \to \infty} \left( \lim_{N \to \infty} \frac{|\pi(B_{S_1}(M)B_{S_1}(N) \times B_{S_2}(M))|\langle X \rangle|}{|B_{S_1}(N)| |B_{S_2}(M)|} \times \frac{|B_{S_1}(M)B_{S_1}(N)| |B_{S_2}(M)|}{|B_{S_1}(N)| (M - 1)} \right)
\]

Since \( \{B_{S_1}(N)\} \) is a Følner sequence, then \( \lim_{N \to \infty} \frac{|B_{S_1}(M)B_{S_1}(N)|}{|B_{S_1}(N)|} = 1 \). Note that
\[
\limsup_{M \to \infty} \frac{|B_{S_2}(M)|}{M} = c_1. \text{ We can get the desired result.} \]

We next give the following covering lemma which was proved by Linedentrauss \[12\]. This lemma is crucial in the proof of step 2 of Theorem \[27\].

**Lemma 3.1.** \[12\] For any \( \delta \in (0, 1/100) \), \( C > 0 \) and finite \( D \subset G \), let \( M \in \mathbb{N} \) be sufficiently large (depending only on \( \delta, C \) and \( D \)). Let \( F_{i,j} \) be an array of a finite subsets of \( G \) where \( i = 1, \ldots, M \) and \( j = 1, \ldots, \ell_i \), such that

- For every \( i, \overline{F}_{i,*} = \{F_{i,j}\}_{j=1}^{\ell_i} \) satisfies
  \[
  |\bigcup_{k' < k} F_{i,k'}^{-1}F_{i,k}| \leq C|F_{i,k}|, \text{ for } k = 2, \ldots, \ell_i.
  \]
  Denote \( F_{i,*} = \bigcup \overline{F}_{i,*} \).
- The finite set sequence \( F_{i,*} \) satisfy that for every \( 1 \leq i \leq M \) and every \( 1 \leq k \leq \ell_i \),
  \[
  |\bigcup_{i' \leq i} DF_{i',*}^{-1}F_{i,k}| \leq (1 + \delta)|F_{i,k}|.
  \]

Assume that \( A_{i,j} \) is another array of finite subset of \( G \) with \( F_{i,j}A_{i,j} \subset F \) for some finite subset \( F \) of \( G \). Let \( A_{i,*} = \bigcup_j A_{i,j} \) and

\[
\alpha = \min_{1 \leq i \leq M} |DA_{i,*}|/|F|.
\]

Then the collection of subsets of \( F \),
\[
\overline{F} = \{F_{i,j}a : 1 \leq i \leq M, 1 \leq j \leq \ell_i \text{ and } a \in A_{i,j}\}
\]
has a subfamily \( \mathcal{F} \) that is 10\( \delta^{1/4} \)-disjoint such that
\[
|\bigcup \mathcal{F}| \geq (\alpha - \delta^{1/4})|F|.
\]
Step 2: \( \text{mdim}_H(\mathcal{X}, \{B_{S_1}(N)\}, d) \geq c_2 \cdot h_{top}(\mathcal{X}, G) \).

Proof. Set \( s = c_2 \cdot h_{top}(\mathcal{X}, G) \), \( h = h_{top}(\mathcal{X}, G) \). We suppose \( \text{mdim}_H(\mathcal{X}, \{B_{S_1}(n)\}, d) < s \). We would like to get a contradiction. Take \( \epsilon \) such that \( s\epsilon - \frac{\epsilon^2}{2} < 1 \) and

\[
\text{mdim}_H(\mathcal{X}, \{B_{S_1}(n)\}, d) < s - (h + 2)\epsilon.
\]

Let \( D = \{e_G\} \subset G \) and \( C > 0 \) be the constant in the tempered condition for the Følner sequence \( \{B_S(n)\} \). We choose \( 0 < \delta < \min \{1/100, \epsilon\} \) small enough and a natural number \( M \) satisfying the following conditions:

\[
H(\delta) + \delta \log M < \epsilon^3/4c_2, \quad |A|^\delta < 2^{\epsilon^3/4c_2}, \quad 1/(1 - 10\delta^{1/4}) < 1 + \epsilon^2. \tag{3.1}
\]

Here \( H(\delta) = -\delta \log \delta - (1 - \delta) \log(1 - \delta) \). (Recall that the base of the logarithm is two.) Take \( M \approx \log(C/\delta^2) \) to satisfy the requirement of Lemma 3.1 corresponding to \( \delta, D \) and \( C \). Then we can choose a sufficiently small \( \delta \) satisfying the second and third conditions.

We choose a natural number \( r_0 \) such that

\[
r_0 > \frac{1}{\delta(1 - 10\delta^{1/4})}. \tag{3.2}
\]

and

\[
(s - (h + 2)\epsilon)r < (s - (h + 1)\epsilon)(r - 1), \quad |B_{S_2}(r)| > r(c_2 - \epsilon) \tag{3.3}
\]

for every \( r \geq r_0 \).

From \( \text{mdim}_H(\mathcal{X}, \{B_{S_1}(N)\}, d) < s - (h + 2)\epsilon \), for each \( i = 1, 2, \ldots, M \), we can find \( N_i > 0 \) satisfying

\[
\frac{1}{|B_{S_1}(N_i)|} \dim_H(\mathcal{X}, d_{B_{S_1}(N_i)}, 2^{-r_0}) < s - (h + 2)\epsilon.
\]

Also, we let the sequence \( \{N_i\} \) be increasing. This implies that there exists a covering \( \mathcal{X} = \bigcup_{j=1}^l E_j \) satisfying

\[
\text{diam}(E_j, d_{B_{S_1}(N_i)}, 2^{-r_0}) < 2^{-r_0}(1 \leq j \leq l_i) \sum_{j=1}^{l_i} (\text{diam}(E_j, d_{B_{S_1}(N_i)})(s - (h + 2)\epsilon)|B_{S_1}(N_i)|) < 1.
\]

Set \( 2^{-r_0} = \text{diam}(E_j, d_{A_{N_i}}, r_0^j) \). Then \( r_0^j \) is a natural number with \( r_0^j > r_0 \). Choose \( x_j^i \in E_j^i \), let \( C_j^i = \pi_{F_{n_i,j}}^{-1}(\pi_{F_{n_i,j}}(x_j^i)) \) be a cylinder over

\[
F_{n_i,j} := \{B_{S_1}(r_j^i - 1)B_{S_1}(N_i) \times B_{S_2}(r_j^i - 1)\}.
\]
Then $E^j_i \subset C^j_i$ and $X = \cup_{j=1}^{l_i} C^j_i$. For each $1 \leq i \leq M$, we can get $\{F_{n,1}, \ldots, F_{n,l_i}\}$. Without loss of generality, let $r^i_1 < r^i_2 < \cdots < r^i_{l_i}$. Then let $\{F_{i,1}, \ldots, F_{i,l_i}\}$ in Lemma 3.1 be as

$$\{F_{i,1}, \ldots, F_{i,l_i}\} = \{F_{n,1}, \ldots, F_{n,l_i}\}.$$ 

Since $|B_{S_2}(r)| > r(c_2 - \epsilon)$ and $s - (h + 1)\epsilon < (s - \epsilon)\frac{c_2 - \epsilon}{c_2}$, we have

$$|F_{i,j}| = |B_{S_1}(r^i_j - 1)B_{S_1}(N_i) \times B_{S_2}(r^i_j - 1)| \geq |B_{S_1}(N_i)||B_{S_2}(r^i_j - 1)|,$$

and

$$\frac{1}{c_2}(s - \epsilon)|F_{i,j}| \geq \frac{1}{c_2}(s - \epsilon)|B_{S_1}((N_i)||B_{S_2}(r^i_j - 1)|$$

$$> (s - \epsilon)|B_{S_1}((N_i)|\left(\frac{(r^i_j - 1)(c_2 - \epsilon)}{c_2}\right)$$

$$> (s - (h + 1)\epsilon)|B_{S_1}(N_i)|(r^i_j - 1)$$

$$> (s - (h + 2)\epsilon)|B_{S_1}(N_i)|r^i_j. \quad \text{(by 3.1)}$$

Hence

$$2^{-\frac{1}{c_2}(s-\epsilon)|F_{i,j}|} < 2^{-(s-(h+2)\epsilon)||B_{S_1}(N_i)||r^i_j} = \text{diam}(E^i_j, d^\sigma_{B_{S_1}(N_i)})^{(s-(h+2)\epsilon)||B_{S_1}(N_i)||}.$$ 

It follows that

$$\sum_{j=1}^{l_i} 2^{-\frac{1}{c_2}(s-\epsilon)|F_{i,j}|} \leq \sum_{j=1}^{l_i} \left(\text{diam}(E^i_j, d^\sigma_{A_{n,1}})^{(s-(h+2)\epsilon)||B_{S_1}(N_i)||}\right) < 1. \quad (3.4)$$

For any $x \in X$ and sufficiently large $N$ (independent on $x$), let

$$A_{i,j} = \{a \in B_S(N) : F_{i,j}a \subset B_S(N) \text{ and } \sigma^a x \in C^j_i\}.$$ 

We note here that $A_{i,j}$ depends on $x$. For any $g \in B_S(N) \setminus B(B_S(N), F_{i,*})$, we have $F_{i,*}g \subset B_S(N)$. Then $F_{i,j}g \subset B_S(N)$ for each $1 \leq j \leq l_i$. Since $\{C^j_i\}_{j=1}^{l_i}$ cover $X$, there exists $C^j_i$ such that $\sigma^g x \in C^j_i$. This implies $g \in A_{i,j}$ and $B_S(N) \setminus B(B_S(N), F_{i,*}) \subset A_{i,*}$. Let $N$ be sufficiently large so that $B_S(N)$ is $(F_{i,*}, \delta)$-invariant for all $1 \leq i \leq M$, $1 \leq j \leq l_i$, then

$$\alpha = \min_{1 \leq i \leq M} |DA_{i,*}| \frac{1}{|B_S(N)|} > 1 - \delta.$$ 

We note that the array $\{F_{i,j}\}$ meet the first requirement in Lemma 3.1 because of the tempered condition of $\{B_S(N)\}$. For the second requirement, we need to choose $N_i$ large enough compared with $r^i_{l_i-1}$ for every $2 \leq i \leq M$. Now we can apply Lemma 3.1 to

$$\mathcal{F} = \{F_{i,j}a : 1 \leq i \leq M, 1 \leq j \leq l_i \text{ and } a \in A_{i,j}\},$$

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we can find a subcollection $\mathcal{F}$ that is $10\delta^{1/4}$-disjoint such that
\[
|\cup \mathcal{F}| \geq (1 - \delta - 10\delta^{1/4})|B_S(N)|. \tag{3.5}
\]
The element in $\mathcal{F}$ will be denoted by $F_{i,j}a$. Denote by $\overline{A}$ the collection of $a$'s such that $F_{i,j}a$ occurs in $\mathcal{F}$. The cardinality of $\overline{A}$ is no more than the cardinality of the subcollection $\mathcal{F}$. Then $|\overline{A}| \leq |\mathcal{F}|$. Note that $\mathcal{F}$ is $10\delta^{1/4}$-disjoint, we have
\[
\sum_{F_{i,j}a \in \mathcal{F}} |F_{i,j}a| \leq \frac{1}{1 - 10\delta^{1/4}}|\cup \mathcal{F}| \leq \frac{1}{1 - 10\delta^{1/4}}|B_S(N)|.
\]
Then
\[
|\overline{A}| \leq |\mathcal{F}| \leq \frac{1}{\min |F_{i,j}|} \cdot \frac{1}{1 - 10\delta^{1/4}}|B_S(N)| \leq \delta|B_S(N)|. \quad \text{(by (3.2))}
\]
Set
\[
D(x) = \{(a, i, j) : a \in A_{i,j} \ 1 \leq i \leq M, \ 1 \leq j \leq \ell_i \text{ such that } F_{i,j}a \in \mathcal{F}\}.
\]

By the above argument, if $N > 0$ is sufficiently large, we have the following conclusion:

1. For each $(a, i, j) \in D(x)$, we have $\sigma^a x \in C_j^i$ and $F_{i,j}a \subset \mathcal{F}$.

2. If $(a, i, j)$ and $(a', i', j')$ are two different element of $D(x)$, then $F_{i,j}a$ and $F_{i',j'}a'$ is $10\delta^{1/4}$ disjoint.

3. $|\bigcup_{(a,i,j) \in D(x)} F_{i,j}a| \geq (1 - \delta - \delta^{1/4})|B_S(N)|$.

For each $x \in \mathcal{X}$ we define $\underline{D}(x) \subset B_S(N) \times [1, M]$ as the set of $(a, i)$ such that there exists $j \in [1, \ell_i]$ with $(a, i, j) \in D(x)$. (Note that the sets $D(x)$ and $\underline{D}(x)$ depend on $N$). For simplicity of notation, we write $D(x)$ and $\underline{D}(x)$ instead of $D_N(x)$ and $\underline{D}_N(x)$.

**Claim 1.** If $N$ is sufficiently large then the number of possibilities of $\underline{D}(x)$ is bounded as follows:
\[
|\{ \underline{D}(x) | x \in \mathcal{X} \}| < 2^{(e^{3/4\alpha_3})|B_S(N)|}.
\]

**Proof.** It is well-known that
\[
\binom{n}{k} \leq 2^{nH(k/n)}.
\]

Since $|\overline{A}| \leq \delta|B_S(N)|$, then the number of possibilities of $\underline{D}(x)$ is bounded by
\[
\sum_{k=1}^{\delta|B_S(N)|} \binom{|B_S(N)|}{k} \times M^{|B_S(N)|} \leq |B_S(N)| \cdot 2^{|B_S(N)|H(\delta)} \times 2^{|B_S(N)|\delta \log M} = |B_S(N)| \cdot 2^{|B_S(N)|(H(\delta) + \delta \log M)}.
\]
We assume $H(\delta) + \delta \log M < (\epsilon^3/4c_2)$ in (3.1). Hence, if $N$ is sufficiently large then

$$|B_S(N)| \cdot 2^{|B_S(N)|/(H(\delta) + \delta \log M)} < 2^{(\epsilon^3/4c_2)|B_S(N)|}.$$ 

Take a subset $E \subset B_S(N) \times [1, M]$ such that there exists $x \in \mathcal{X}$ with $D(x) = E$. We denote by $\mathcal{X}_E$ the set of $x \in \mathcal{X}$ with $D(x) = E$. Let $E = \{(a_1, i_1), (a_2, i_2), \ldots, (a_k, i_k)\}$.

**Claim 2.**

$$|\pi_{B_S(N)}(\mathcal{X}_E)| \cdot 2^{-\frac{1}{c_2}(s-\delta) \log M} = |A|^{(\delta + \delta^{1/4})|B_S(N)|}.$$

**Proof.** For $j = (j_1, \ldots, j_k) \in [1, \ell_{i_1}] \times \cdots \times [1, \ell_{i_k}]$, we denote by $\mathcal{X}_{E,j} \subset \mathcal{X}_E$ the set of $x \in \mathcal{X}_E$ with $D(x) = \{(a_1, i_1, j_1), \ldots, (a_k, i_k, j_k)\}$. We have $\sigma^{a_{im}}(x) \in C^m_{im}$ for $x \in \mathcal{X}_{E,j}$. Therefore we have

$$|\pi_{B_S(N)}(\mathcal{X}_{E,j})| \leq |A|^{(\delta + \delta^{1/4})|B_S(N)|}.$$

Here the inequality follows from (3.5). This follows from

$$|\pi_{B_S(N)}(\mathcal{X}_E)| \cdot 2^{-\frac{1}{c_2}(s-\delta) \log M} = \sum_j |\pi_{B_S(N)}(\mathcal{X}_{E,j})| \cdot 2^{-\frac{1}{c_2}(s-\delta) \log M} \leq \sum_j |A|^{(\delta + \delta^{1/4})|B_S(N)|} 2^{-\frac{1}{c_2}(s-\delta) \log M}.$$ 

Take $j = (j_1, \ldots, j_k) \in [1, \ell_{i_1}] \times \cdots \times [1, \ell_{i_k}]$ with $\mathcal{X}_{E,j} \neq \emptyset$. Since

$$\sum_{F_{i,j,a} \in \mathcal{F}} |F_{i,j,a}| \leq \frac{1}{1 - 10\delta^{1/4}}|B_S(N)| < (1 + \epsilon^2)|B_S(N)|$$

by (3.1),

we have

$$2^{-\frac{1}{c_2}(s-\delta) \log M} \leq \prod_{m=1}^k 2^{-\frac{1}{c_2}(s-\delta)|F_{m,jm}|}.$$ 

Moreover

$$|\pi_{B_S(N)}(\mathcal{X}_E)| \cdot 2^{-\frac{1}{c_2}(s-\delta) \log M} \leq \sum_j |A|^{(\delta + \delta^{1/4})|B_S(N)|} \prod_{m=1}^k 2^{-\frac{1}{c_2}(s-\delta)|F_{m,jm}|}.$$ 

The right-hand side isn’t more than

$$|A|^{(\delta + \delta^{1/4})|B_S(N)|} \left( \sum_{j=1}^{l_{i_1}} 2^{-\frac{1}{c_2}(s-\delta)|F_{i_1,j}|} \right) \times \cdots \times \left( \sum_{j=1}^{l_{i_k}} 2^{-\frac{1}{c_2}(s-\delta)|F_{i_k,j}|} \right).$$ 

According to (3.4), we have

$$|\pi_{B_S(N)}(\mathcal{X}_E)| \cdot 2^{-\frac{1}{c_2}(s-\delta) \log M} \leq |A|^{(\delta + \delta^{1/4})|B_S(N)|}.$$ 

\[ \Box \]
We can now proceed to prove Theorem 2.1
\[ |\pi_{B_S(N)}(X_E)| \cdot 2^{-\frac{1}{c_2}(s-\epsilon)(1+\epsilon^2)|B_S(N)|} \leq |A|^{(\delta+\delta^{1/4})|B_S(N)|} \leq 2^{|B_S(N)|\epsilon^3/4c_2}, \text{ by (3.1)} \]

If $N$ is sufficiently large, the number of choices $E \subset B_S(N) \times [1, M]$ such that $X_E \neq \emptyset$ is not greater than $2^{(\epsilon^3/4c_2)|B_S(N)|}$. Hence
\[
|\pi_{B_S(N)}(X)| \cdot 2^{-\frac{1}{c_2}(s-\epsilon)(1+\epsilon^2)|B_S(N)|} = \sum_{E \text{ with } X_E \neq \emptyset} |\pi_{B_S(N)}(X_E)| \cdot 2^{-\frac{1}{c_2}(s-\epsilon)(1+\epsilon^2)|B_S(N)|} < 2^{(\epsilon^3/4c_2)|B_S(N)|} \times 2^{(\epsilon^3/4c_2)|B_S(N)|} = 2^{(\epsilon^3/2c_2)|B_S(N)|}.
\]

Then
\[
\log \frac{|\pi_{B_S(N)}(X)|}{|B_S(N)|} < \frac{1}{c_2}(s - \epsilon + se^2 - \frac{1}{2}\epsilon^3).
\]

Letting $N \to \infty$, we have
\[
h_{top}(X, G) \leq \frac{1}{c_2}(s - \epsilon + se^2 - \frac{1}{2}\epsilon^3) < \frac{1}{c_2}s = h_{top}(X, G) \text{ (by } se - \epsilon^2/2 < 1).\]

This is a contradiction.

\[
\square
\]

4 Proof of Theorem 2.2

4.1 Mutual information

Here we prepare some basics of mutual information. Let $(\Omega, \mathbb{P})$ be a probability space. Let $\mathcal{X}$ and $\mathcal{Y}$ be measurable spaces, and let $X : \Omega \to \mathcal{X}$ and $Y : \Omega \to \mathcal{Y}$ be measurable maps. We want to define their mutual information $I(X; Y)$ as the measure of the amount of information $X$ and $Y$ share. For more details and properties of mutual information, one is referred to [5].

Case 1: Suppose $\mathcal{X}$ and $\mathcal{Y}$ are finite sets. Then we define
\[
I(X; Y) = H(X) + H(Y) - H(X, Y) = H(X) - H(X|Y).
\]

More explicitly
\[
I(X; Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \mathbb{P}(X = x, Y = y) \log \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)\mathbb{P}(Y = y)}.
\]

Here we use the convention that $0 \log(0/a) = 0$ for all $a \leq 0$. 

Case 2: In general, take measurable maps \( f : \mathcal{X} \to A \) and \( g : \mathcal{Y} \to B \) into finite sets \( A \) and \( B \). Then we can consider \( I(f \circ X; g \circ Y) \) defined by Case 1. We define \( I(X; Y) \) as the supremum of \( I(f \circ X; g \circ Y) \) over all finite-range measurable maps \( f \) and \( g \) defined on \( \mathcal{X} \) and \( \mathcal{Y} \). This definition is compatible with Case 1 when \( \mathcal{X} \) and \( \mathcal{Y} \) are finite sets.

Lemma 4.1 (Data-Processing inequality). Let \( X \) and \( Y \) be random variables taking values in measurable spaces \( \mathcal{X} \) and \( \mathcal{Y} \) respectively. If \( f : \mathcal{Y} \to \mathcal{Z} \) is a measurable map then \( I(X; f(Y)) \leq I(X; Y) \).

4.2 Rate distortion theory

We introduce rate distortion function and dimension. Let \((\mathcal{X}, G)\) be a dynamical system with a distance \( d \) on \( \mathcal{X} \). Take an invariant probability \( \mu \in M(\mathcal{X}, G) \). For a positive number \( \epsilon \) and \( F \in F(G) \), we define \( R(\mu, \epsilon, F) \) as the infimum of

\[
I(X, Y),
\]

(4.1)

\( X \) and \( Y = (Y_0, \cdots, Y_{n-1}) \) are random variables defined on some probability space \((\Omega, \mathbb{P})\) such that

- \( X \) takes values in \( \mathcal{X} \) and its law is given by \( \mu \).

- Each \( Y_g \) takes values in \( \mathcal{X} \) and \( Y = (Y_g)_{g \in F} \) approximates the process \((gX)_{g \in F}\) in the sense that

\[
\mathbb{E} \left( \frac{1}{|F|} \sum_{g \in F} d(gX, Y_g) \right) < \epsilon.
\]

(4.2)

Here \( \mathbb{E} \) is the expectation with respect to the probability measure \( \mathbb{P} \). Note that \( R(\mu, \epsilon, F) \) depends on the distance \( d \) although it is not explicitly written in the notation.

We define the rate distortion function

\[
R(\mu, (B_S(N)), \epsilon) = \limsup_{N \to \infty} \frac{R(\mu, (B_S(N)), \epsilon)}{|B_S(N)|}.
\]

The upper and lower rate distortion dimensions are defined by

\[
\overline{rdim}(\mathcal{X}, (B_S(N)), d, \mu) = \limsup_{\epsilon \to 0} \frac{R(\mu, (B_S(N)), \epsilon)}{\log(1/\epsilon)},
\]

\[
\underline{rdim}(\mathcal{X}, (B_S(N)), d, \mu) = \liminf_{\epsilon \to 0} \frac{R(\mu, (B_S(N)), \epsilon)}{\log(1/\epsilon)}.
\]
4.3 Proof of Theorem 2.2

The proof of Theorem 2.2 is divided into two steps.

**Step 1:** \( \text{rdim}(\mathcal{X}, \sigma_1, \{B_{S_1}(N)\}, d, \mu) \leq c \cdot h_\mu(\mathcal{X}, G) \).

*Proof.** Let \( X \) be a random variable taking values in \( \mathcal{X} \) and obeying \( \mu \). Let \( 0 < \epsilon < 1 \) and choose \( M > 0 \) such that \( 2^{-M} < \epsilon \leq 2^{-M+1} \). Given \( N > 0 \) and for every point \( x \in \pi_{B_{S_1}(N)B_{S_2}(M)}(\mathcal{X}) \) we take \( q(x) \in \mathcal{X} \) satisfying \( \pi_{B_{S_1}(N)B_{S_2}(M)}(q(x)) = x \). Let \( X' = q(\pi_{B_{S_1}(N)B_{S_2}(M)}(\mathcal{X})) \) and \( Y = (\sigma_{1,g}X')_{g \in B_{S_1}(N)} \), we conclude that

\[
\frac{1}{|B_{S_1}(N)|} \sum_{g \in B_{S_1}(N)} d(\sigma_{1,g}X, Y_g) = \frac{1}{|B_{S_1}(N)|} \sum_{g \in B_{S_1}(N)} d(\sigma_{1,g}X, \sigma_{1,g}X') \leq 2^{-M} < \epsilon,
\]

and

\[
I(X;Y) \leq H(Y) = H(X') = H\left( (X_{g'})_{g' \in B_{S_1}(N)B_{S_1}(N)B_{S_2}(M)} \right).
\]

This yields that

\[
R_\mu(\{B_{S_1}(N)\}, \epsilon) \leq \frac{I(X;Y)}{|B_{S_1}(N)|} \leq \frac{H\left( (X_{g'})_{g' \in B_{S_1}(N)B_{S_1}(N)B_{S_2}(M)} \right)}{|B_{S_1}(N)|},
\]

and

\[
\frac{R_\mu(\{B_{S_1}(N)\}, \epsilon)}{\log(1/\epsilon)} \leq \frac{H\left( (X_{g'})_{g' \in B_{S_1}(N)B_{S_1}(N)B_{S_2}(M)} \right)}{|B_{S_1}(N)|} \times \frac{|B_{S_1}(M)B_{S_1}(N) \times B_{S_2}(M)|}{|B_{S_1}(N)|(M - 1)}.
\]

Letting \( N \to \infty \) and then take \( \epsilon \to 0 \). Note that \( \{B_{S_1}(N)\} \) is a Følner sequence, we get

\[
\lim_{N \to \infty} \frac{|B_{S_1}(M)B_{S_1}(N)|}{|B_{S_1}(N)|} = 1.
\]

Since \( \lim_{M \to \infty} \frac{|B_{S_1}(M)|}{M - 1} = c \), then

\[
\text{rdim}(\mathcal{X}, \sigma_1, \{B_{S_1}(N)\}, d, \mu) \leq c \cdot h_\mu(\mathcal{X}, G).
\]

\( \square \)

For the proof of Theorem 2.2 we need the following lemma. The proof of this idea is adapted from [13 14].

**Lemma 4.2.** Let \( N \geq 1 \) and \( B \) a finite set. Let \( X = (X_g)_{g \in B_{S_1}(N)} \) and \( Y = (Y_g)_{g \in B_{S_1}(N)} \) be random variables taking values in \( B^{B_{S_1}(N)} \) (namely, each \( X_g \) and \( Y_g \) takes values in \( B \)) such that for some \( 0 < \delta < 1/2 \)

\[
\mathbb{E}(\# \{ g \in B_{S_1}(N) : X_g \neq Y_g \}) < \delta |B_{S_1}(N)|.
\]

Then

\[
I(X;Y) > H(X) - |B_{S_1}(N)||H(\delta) - \delta|B_{S_1}(N)|\log |B|.
\]
Proof. Let \( Z_g = 1_{\{X_g \neq Y_g\}} \) and \( Z = \{ g \in B_{S_1}(N) | X_g \neq Y_g \} \). We can identity \( Z \) with \((Z_g)_{g \in B_{S_1}(N)}\) and hence

\[
H(Z) \leq \sum_{g \in B_{S_1}(N)} H(Z_g) = \sum_{g \in B_{S_1}(N)} H(\mathbb{E}Z_g)
\]

\[
\leq |B_{S_1}(N)| H \left( \frac{1}{B_{S_1}(N)} \sum_{g \in B_{S_1}(N)} \mathbb{E}Z_g \right) < |B_{S_1}(N)| H(\delta).
\]

Then \( H(Z) < |B_{S_1}(N)| H(\delta) \). We expand \( H(X, Z|Y) \) in two ways:

\[
H(X, Z|Y) = H(X|Y) + H(Z|X, Y) = H(Z|Y) + H(X|Y, Z).
\]

This, we conclude that

\[
H(X|Y) = H(Z|Y) + H(Z|Y, Z) < |B_{S_1}(N)| H(\delta) + H(X|Y, Z).
\]

Noticing that

\[
H(X|Y, Z) = \sum_{E \subset B_{S_1}(N)} \mathbb{P}(Z = E) H(X|Y, Z = E).
\]

For \( Y \) and the condition \( Z = E \), the possibilities of \( X \) is at most \( |B|^{|E|} \). So \( H(X|Y, Z = E) \leq |E| \log |B| \) and

\[
H(X|Y, Z) \leq \sum_{E \subset B_{S_1}(N)} |E| \mathbb{P}(Z = E) \log |B|
\]

\[
= \mathbb{E}|Z| \cdot \log |B|
\]

\[
\leq \delta|B_{S_1}(N)| \log |B|.
\]

Thus \( H(X|Y) < |B_{S_1}(N)| H(\delta) + \delta|B_{S_1}(N)| \log |B| \) and

\[
I(X; Y) > H(X) - |B_{S_1}(N)| H(\delta) - \delta|B_{S_1}(N)| \log |B|.
\]

\[
\square
\]

Step 2: \( \text{rdim}(\mathcal{X}, \sigma_1, d, \{B_{S_1}(N)\}, \mu) \geq c \cdot h_\mu(\mathcal{X}, G) \).

Proof. Let \( X \) be a random variable taking values in \( \mathcal{X} \) with \( \text{Law}(X) = \mu \). Given \( 0 < \epsilon < \delta < 1/2, N > 0 \) and let \( Y = (Y_g)_{g \in B_{S_1}(N)} \) be a random variable taking values in \( \mathcal{X}^{B_{S_1}(N)} \) and satisfying

\[
\mathbb{E} \left( \frac{1}{|B_{S_1}(N)|} \sum_{g \in B_{S_1}(N)} d(\sigma_{1, g} X, Y_g) \right) < \epsilon.
\]
We will estimate the lower bound of $I(X; Y)$. Choose $M \geq 0$ with $\delta 2^{-M-1} < \epsilon < \delta 2^{-M}$.

For $g \in B_{S_1}(N)$, set

$$X'_g = \pi_{\{g\} \times B_{S_2}(M)}(X) = (X_{(g, g_2)})_{g_2 \in B_{S_2}(M)},$$

$$Y'_g = \pi_{\{1_{G_1}\} \times B_{S_2}(M)}(Y) = ((Y'_g)(1_{G_1}, g_2))_{g_2 \in B_{S_2}(M)}.$$  

If $X'_g \neq Y'_g$ for some $g$ then $d(\sigma_{1,g}X, Y_g) \geq 2^{-M}$. Therefore $\mathbb{E}d(\sigma_{1,g}X, Y_g) \geq 2^{-M} \mathbb{P}(X'_g \neq Y'_g)$ and

$$\mathbb{E}(\# \{g \in B_{S_1}(N) : X'_g \neq Y'_g\}) = \sum_{g \in B_{S_1}(N)} \mathbb{P}(X'_g \neq Y'_g) \leq 2^M \mathbb{E}\left(\sum_{g \in B_{S_1}(N)} d(\sigma_{1,g}X, Y_g)\right) < 2^M \epsilon |B_{S_1}(N)| \leq \delta |B_{S_1}(N)|.$$

Applying Lemma 4.2 to $X'_g$ and $Y'_g$ with $B = A^{B_{S_2}(M)}$:

$$I\left(\left(X'_g\right)_{g \in B_{S_1}(N)}; \left(Y'_g\right)_{g \in B_{S_1}(N)}\right) > H((X'_g)_{g \in B_{S_1}(N)}) - |B_{S_1}(N)|H(\delta) - \delta |B_{S_1}(N)||B_{S_2}(M)| \log |A|.$$

According to the data-processing inequality (Lemma 4.1),

$$I(X; Y) \geq I((X'_g)_{g \in B_{S_1}(N)}; (Y'_g)_{g \in B_{S_1}(N)}).$$

Then

$$\frac{I(X; Y)}{|B_{S_1}(N)|} \geq \frac{H\left(\left(X'_g\right)_{g \in B_{S_1}(N)} \times B_{S_2}(M)\right)}{|B_{S_1}(N)|} - H(\delta) - \delta |B_{S_2}(M)| \log |A|.$$

This holds for any $N > 0$. So

$$R_\mu(\{B_{S_1}(N)\}, \epsilon) \geq \inf_N \frac{H\left(\left(X'_g\right)_{g \in B_{S_1}(N)} \times B_{S_2}(M)\right)}{|B_{S_1}(N)|} - H(\delta) - \delta |B_{S_2}(M)| \log |A|$$

$$= \lim_{N \to \infty} \frac{H\left(\left(X'_g\right)_{g \in B_{S_1}(N)} \times B_{S_2}(M)\right)}{|B_{S_1}(N)|} - H(\delta) - \delta |B_{S_2}(M)| \log |A|.$$  

We divide this by $\log(1/\epsilon)$ and take the limit $\epsilon \to 0$. Since $\log(1/\epsilon) < \log(1/\delta) + (M+1)$ (here $\delta$ has been fixed) and $\lim_{M \to \infty} \frac{|B_{S_2}(M)|}{M - 1} = c$, we obtain

$$\limsup_{N \to \infty} \frac{H\left(\left(X'_g\right)_{g \in B_{S_1}(N)} \times B_{S_2}(M)\right)}{|B_{S_1}(N)||B_{S_2}(M)|} = c \cdot h_\mu(\mathcal{X}, G) - c\delta \log |A|.$$  

Here we have used

$$h_\mu(\mathcal{X}, G) = \lim_{N, M \to \infty} \frac{H\left(\left(X'_g\right)_{g \in B_{S_1}(N)} \times B_{S_2}(M)\right)}{|B_{S_1}(N)||B_{S_2}(M)|}$$

Letting $\delta \to 0$, we have $\limsup_{N \to \infty} \frac{H\left(\left(X'_g\right)_{g \in B_{S_1}(N)} \times B_{S_2}(M)\right)}{|B_{S_1}(N)||B_{S_2}(M)|} = c \cdot h_\mu(\mathcal{X}, G)$.
5 Examples

In this section, we consider the following examples to illustrate our main theorem for the case of \( c_1 = c_2 = c \). (see [7] for more details)

**Example 5.1.** Let \( G_1 = \mathbb{Z}^d \), \( S_1 = \{(1,0,\ldots,0), \ldots, (0,\ldots,1), (0,0,\ldots,-1)\} \). Let \( G_2 = \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \) and \( S_2 = \{(1,0), (-1,0), (0,1)\} \). Then the ball radius \( r \) centered at the element \( (n,0) \) is represented in Fig 1. We deduce that \( \gamma_{S_2}(n) = (2n+1)+2(n-1)+1 = 4n \). Take \( G = G_1 \times G_2 \), by Theorem 2.1, we can have

\[
\dim_H(\mathcal{X}, \{B_{S_1}(n)\}, d) = \dim_M(\mathcal{X}, G_1, d) = 4h_{top}(\mathcal{X}, G).
\]

![Figure 1](image1)

Recall infinite dihedral group, that is, the group of isometries of the real line \( \mathbb{R} \) generated by reflections \( r : \mathbb{R} \rightarrow \mathbb{R} \) and \( s : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
\begin{align*}
    r(x) &= -x \quad \text{(symmetry with respect to 0)} \\
    s(x) &= 1 - x \quad \text{(symmetry with respect to 1/2)}
\end{align*}
\]

for all \( x \in \mathbb{R} \). Note that \( r^2 = s^2 = 1_G \).

**Example 5.2.** Let \( G_1 = \mathbb{Z}^d \), \( S_1 = \{(1,0,\ldots,0), \ldots, (0,\ldots,1), (0,0,\ldots,-1)\} \). Let \( G_2 \) be the infinite dihedral group and \( S_2 = \{r,s\} \). Then the ball of radius \( n \) centered at the element \( g \in G_2 \) is represented in Fig 2. It follows that \( \gamma_{S_2}(n) = 2n + 1 \). Take \( G = G_1 \times G_2 \). By Theorem 2.1, thus \( \dim_H(\mathcal{X}, \{B_{S_1}(n)\}, d) = \dim_M(\mathcal{X}, G_1, d) = 2 \cdot h_{top}(\mathcal{X}, G) \).

![Figure 2](image2)

Let \( G \) be a group. The lower central series of \( G \) is the sequence \( (C^i(G))_{i \geq 0} \) of subgroup of \( G \) defined by \( C^0(G) = G \) and \( C^{i+1} = [C^i(G), G] \) for all \( i \geq 0 \). Here \( [h,k] := hkh^{-1}k^{-1} \) for \( h, k \in G \). An easy induction shows that \( C^i(G) \) is normal in \( G \) and that \( G^{i+1}(G) \subset C^i(G) \) for all \( i \). The group \( G \) is said to be nilpotent if there
is an integer $i \geq 0$ such that $C^i(G) = \{1_G\}$. The group $G$ is said to be nilpotent if there is an integer $i \geq 0$ such that $C^i(G) = \{1_G\}$. The smallest integer $i \geq 0$ such that $C^i(G) = \{1_G\}$ is then called the nilpotency degree of $G$. Every nilpotent group is amenable.

1972, H. Bass [2] showed that the growth of a nilpotent group $G$ with finite symmetric generating subset $S$ is exactly polynomial in the sense that there are positive constants $C_1$ and $C_2$ such that $C_1 n^d \leq \gamma_S(n) \leq C_2 n^d$, for all $n \geq 1$, where $d = d(G)$ is an integer which can be computed explicitly from the lower central series of $G$.

**Example 5.3.** Let $G_1$ be a nilpotent group with finite symmetric generating subset $S_1$ and $\deg(G_1) = d$. Let $G_2$ be the infinite dihedral group and $S_2 = \{r, s\}$. Set $G = G_1 \times G_2$. Then

$$\text{mdim}_H(\mathcal{X}, \{B_{S_1}(n)\}, d) = \text{mdim}_M(\mathcal{X}, G_1, d) = 2 \cdot h_{\text{top}}(\mathcal{X}, G).$$

Similarly, the above examples hold for Theorem [22]

**Acknowledgements.** The first author was supported by the Postgraduate Research Innovation Program of Jiangsu Province (KYCX201162). The first and second author were supported by NNSF of China (11671208 and 11431012). The third author was supported by NNSF of China (11971236,11601235), NSF of Jiangsu Province (BK20161014), NSF of the Jiangsu Higher Education Institutions of China (16KJD110003), China Postdoctoral Science Foundation (2016M591873), and China Postdoctoral Science Special Foundation (2017T100384). The work was also funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. We would like to express our gratitude to Tianyuan Mathematical Center in Southwest China, Sichuan University and Southwest Jiaotong University for their support and hospitality.

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