Application of the inhomogeneous Kibble-Zurek mechanism to quench dynamics from Mott-insulator to superfluid in a finite system

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We apply the theory of inhomogeneous Kibble-Zurek mechanism to understand quench dynamics from the Mott insulator to the superfluid in a cold Bose gases confined in both a two-dimensional optical lattice and a harmonic trap. The local quench time and the freeze-out region associated with the nonadiabatic transition take a nontrivial positional dependence due to the Mott-lobe structure of the ground state phase diagram of the Bose-Hubbard model. We demonstrate that the quench dynamics through the time-dependent Gutzwiller simulations, revealing inhomogeneous properties of the growth of the superfluid order parameter. The inhomogeneous Kibble-Zurek theory is applicable for the shallow harmonic trap.

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I. INTRODUCTION

Ultracold atomic gases are versatile testing beds for studying many-body quantum phenomena under isolated, clean, and highly controllable environments [1]. In this system, nonequilibrium quantum dynamics under a sudden quench of the system parameters, called quantum quench, is one of the major topics, providing many challenging problems of many-body quantum dynamics that are difficult to be solved by existing theoretical treatments [2, 3].

The Kibble-Zurek mechanism (KZM) is a well-known theory that describes the nonequilibrium process of the second order phase transition from a symmetric phase to a symmetry-breaking one, predicting a density of topological defects generated by a rapid quench of the parameters which induce the phase transition [4, 5]. The KZM has been studied for decades in various condensed matter systems [6, 7]. Later, the theory has been extended to quantum phase transitions [8] and investigated in cold Bose gases [9–13]. In the case of a Bose gas in an optical lattice, the Mott insulator (MI) and the superfluid (SF) can exist as the ground state phase, which depends on a depth of an optical lattice, strength of interatomic interactions, and particle fillings [14, 15]. The experiments of Refs. [10, 12] reported the quench dynamics from the MI to the SF phase, where the applicability of the KZM has been also discussed. The theoretical works of the KZM in the MI-SF transition has been reported by several authors [16–21].

In this work, we consider the quench dynamics of the Bose-Hubbard model (BHM) in the presence of a harmonic confinement. The impact of the harmonic potential, which makes the system inhomogeneous, to the KZM in the MI-SF transition has not been considered seriously so far. The harmonic potential gives rise to the spatially dependent chemical potential, resulting in the core-shell structure of the SF and MI domains [14, 22]. The inhomogeneous nature of the equilibration dynamics from the SF to the MI has been studied by some experiments [23–25]. In this case, the presence of the Mott shell, formed by a fast equilibration process, prevents the escape of the central excess SF component to the outside, leading to an anomalously long time scale of the global equilibration [26–28]. For the discussion on a quench from the MI to the SF, the previous theoretical studies [17, 19] did not take into account an effect of a harmonic trap. On the other hands, the experiment by Chen et al. [10] demonstrated the quantum quench in the presence of a harmonic trap, so that considering the inhomogeneous effect is necessary to understand precisely the experimental observations.

Here, we apply the theory of inhomogeneous Kibble-Zurek mechanism (IKZM) proposed by Ref. [29] to study the quench dynamics of the MI-SF transition in two-dimensional (2D) lattice system with a harmonic confinement. A recent work shows that, even for the inhomogeneous system, the universality of the quench dynamics can be seen in the quantum dynamics of the one-dimensional Ising model [30]. We find that the presence of the Mott lobe in the phase diagram of the BHM provides new features of the IKZM, where the “local” quench dynamics can be seen in the quantum dynamics of the one-dimensional Ising model [30]. Employing the time-dependent Gutzwiller methods [17, 19], we simulate the quench dynamics from the MI to the SF and formation of quantized vortices in the SF order parameter. We find that the quench dynamics is strongly dependent on the frequency of the harmonic trap, where the IKZM is valid for the system in a shallow harmonic trap, while the growth of the SF component is rather adiabatic for a steep harmonic trap. When the initial MI has a wedding cake structure, long-lived extra vortices are generated at the interface between the MI domain with different filling factors. This implies the difficulty to extract the vortices purely created through the KZM in experiments.

The paper is organized as follows. Section II describes a brief review of the KZM for both homogeneous and inhomogeneous situations. In Sec. III, we introduce the BHM and apply the IKZM to describe the quantum quench from a MI to a SF phase in a harmonic trap potential. In Sec. IV, we demonstrate the quench dy-
namics by using the Gutzwiller mean field equation and verify the prediction of the IKZM. Section V devotes to the conclusion.

II. SUMMARY OF INHOMOGENEOUS KIBBLE-ZUREK MECHANISM

The KZM is a theory describing the physical picture of nonequilibrium dynamics and topological defect formations in the system undergoing a rapid second order phase transition. Here, we briefly review the Kibble-Zurek (KZ) theory and its extended version to inhomogeneous systems introduced by Del Campo et al. [29].

A. KZM in homogeneous systems

We suppose that the phase transition is driven by the time-dependent controllable parameter $T(t) = T_c(1 + t/\tau_Q)$, where $T_c$ represents the critical point and $\tau_Q$ provides a time scale of a quench. Here, we assume that the symmetry-preserved (symmetry-broken) phase exists at $T < T_c$ ($T > T_c$) and the system passes the critical point at $t = 0$ through the linear ramp of $T(t)$. In the vicinity of the transition point, the relaxation time $\tau$ and the correlation length $\xi$ in equilibrium diverge as

$$\tau(\epsilon) = \frac{\tau_0}{|\epsilon|^z}, \quad \xi(\epsilon) = \frac{\xi_0}{|\epsilon|^\nu},$$

where $\tau_0$ and $\xi_0$ are typical scales of time and length, respectively, and $\nu$ and $z$ the critical exponents of a system. The time-dependent dimensionless parameter $\epsilon = [T(t) - T_c]/T_c = t/\tau_Q$ describes the deviation from the critical point.

The dynamics of the phase transition is effectively divided into adiabatic and nonadiabatic regimes, by comparing the velocity with which the correlation length would have to increase to maintain its equilibrium value $v_\xi = \dot{\xi} = (d\xi/d\epsilon)\dot{\epsilon}$ with the propagation speed $s = \xi/\tau$ of the fluctuation. Under the condition $v_\xi = s$ at $t = \dot{t}$, the dynamics can be regarded as adiabatic (nonadiabatic) for $t > |\dot{t}|$ ($t < |\dot{t}|$). The time $t$ is given by $\dot{t} = (\tau_0\tau_Q^{z\nu})^{1/(1+z\nu)}$. During the time interval $-\dot{t} \leq t \leq \dot{t}$, referred to as a “frozen region”, the ordered phase develops heterogeneously in the space, and the mismatch of the phases of the order parameters leaves the phase defects. The size of the generated domains of the ordered phase can be estimated as $\hat{\xi} = \xi(\epsilon(\dot{t}))$ and the defect density as $n_{\text{def}} \sim \hat{\xi}^{d-D}$, where $D$ and $d$ are the dimensions of the space and defect, respectively. Thus, we have the power-law relation for $\hat{\xi}$ and $n_{\text{def}}$ with respect to the quench rate $\tau_Q$ as

$$\hat{\xi} = \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{\nu/(1+z\nu)},$$

$$n_{\text{def}} = \frac{1}{\xi_0^{D-d}} \left(\frac{\tau_Q}{\tau_0}\right)^{(D-d)\nu/(1+z\nu)},$$

Suppose that the time-dependent parameter $T(t)$ passes the critical point at $t = t_F$. Then, the time $t_F$ should be determined locally as $t_F = T_c(r)$; the condition $\epsilon(t_F(r), r) = 0$ gives the transition point at the radial position $r$. When the controllable parameter changes as $T(t) = T_c(0)(1 + t/\tau_Q)$, predetermined by $T_c(0)$ at $r = 0$, we can define the local quench time as

$$\tau_Q(r) = \frac{T_c(r)}{T_c(0)}\tau_Q,$$

and get the relation $t_F(r) = \tau_Q(r) - \tau_c$. By using $\tau_Q(r)$, Eq. (4) can be rewritten as

$$\epsilon(t, r) = \frac{t + \tau_Q - t_Q(r)}{\tau_Q(r)} = \frac{t - t_F(r)}{\tau_Q(r)}.$$

As a result, $\hat{\tau} = \tau(\epsilon(\dot{t}))$ and $\hat{\xi} = \xi(\epsilon(\dot{t}))$ in the KZM also have a radial dependence through the replacement from the global quench time to the local one $T_c \rightarrow \tau_Q(r)$.

To determine the region in which the KZM may take place in a inhomogeneous system, we introduce another characteristic velocity, namely the propagation velocity of the region passing the local transition point. This can be estimated as

$$v_F = \left|\frac{dT_c(r)}{dr}\right|^{-1} = \frac{T_c(0)}{\tau_Q} \left|\frac{dT_c(r)}{dr}\right|^{-1}.$$}

In the homogeneous system, $v_F$ should be infinity. Then, the adiabaticity condition can be gained by comparing the propagation velocity $v_F$ and the sound velocity $s$ at $t = \hat{t}$, where $s(\hat{t}) \equiv \hat{s}$ is written by

$$\hat{s} = \frac{\xi}{\tau} = \frac{\dot{\xi}}{|\dot{t}|} = \frac{\xi_0}{\tau_0} \frac{\tau_0}{\tau_Q(r)} \left(\frac{\tau_Q}{\tau_0}\right)^{(1+z\nu)/(1+z\nu)}.$$
The inequality $v_F > \delta$ gives the condition in which the conventional KZM is expected, determining the spatial region for the appearance of defects through the KZM. Using a particular model, e.g., the BHM as shown below, we shall calculate the threshold value of the radius $r$ from the above condition.

### III. THE BHM AND INHOMOGENEOUS QUENCH

In this section, we introduce the BHM to describe the cold bosons in an optical lattice and apply the IKZM to study quench dynamics from the MI to the SF in an inhomogeneous situation. The inhomogeneity is included by the radial harmonic potential, which exists in typical experimental setups \[10, 12\].

#### A. BHM

We start from the 2D Bose-Hubbard hamiltonian

$$
\hat{H} = -J \sum_{\langle i,j \rangle} (\hat{b}_i^\dagger \hat{b}_j + H.c.) - \sum_j \mu(r) \hat{n}_j + \frac{U}{2} \sum_j \hat{n}_j(\hat{n}_j - 1),
$$

where $J > 0$ represents a tunneling term, $\hat{b}_j$ and $\hat{b}_j^\dagger$ the annihilation and creation operators which obey the commutation relation $[\hat{b}_j, \hat{b}_j^\dagger] = \delta_{ij}$, $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ the number of bosons at a lattice site $j = (j_x, j_y)$, and $U$ the strength of the on-site repulsion between two bosons. The sum of the first term is taken for the nearest neighbor sites $\langle i,j \rangle$. Since we consider the inhomogeneous 2D system by introducing a harmonic potential, the chemical potential $\mu$ has a radial dependance as

$$
\mu(r) = \mu(0) - \frac{1}{2} kr^2
$$

with the spring constant $k$ and the chemical potential $\mu(0)$ at the origin. The radial coordinate $r$ can be represented as $r = a_0 |j|$ with the lattice constant $a_0$. The particles are confined within the Thomas-Fermi (TF) radius given by $R_{TF} = \sqrt{2\mu(0)/k}$.

The ground state of the BHM in a homogeneous system ($k = 0$) has been well known, as seen in the standard textbook \[1\]. There are two ground state phases, namely, the SF and the MI. According to the second-order perturbative mean-field theory, the phase boundary can be given by

$$
\frac{J_c}{U} = -\frac{\mu}{U}(n/1) - (\mu/U)(2n-1) + (\mu/U)^2.
$$

### B. IKZM for BHM

Now, we consider the KZM for the BHM including a harmonic potential. Here, the transition from the MI to the SF is caused by a rapid increase of the parameter $J/U$. We assume that the on-site interaction $U$ is constant in the followings. In the previous literatures, Shimizu et al. considered the KZM of the BHM in a homogenous system by means of the time-dependent Gutzwiller simulations \[18\]. In the homogeneous situation, however, the unitary time evolution is free from the value of the chemical potential, which allows the numerical demonstration of the MI-SF transition only at the tip of the Mott lobe. Thus, two new parameter dependences of the phase transition appear in the inhomogeneous cases: (i) The transition point is dependent on the position, since the local chemical potential decreases from the center toward the outer region as shown in Fig. 1(b). (ii) The chemical potential at the center
\(\mu(0)\) can be chosen arbitrary, being determined by the total particle number in finite-size systems.

In Fig. 2(b), we plot the density profile for \(\mu(0) = 1.5\) and \(J/U = 0.001\), where the phase at the center is in a deep MI with \(n = 2\). The equilibrium state is calculated by the Gutzwiller ansatz of the wave function, introduced below. There are extremely narrow regions of the SF phase between the MI domains with different filling numbers. A sudden change of \(J/U\) induces the growth of the SF region, which starts from these narrow SF regions. This can be seen from the local quench time \(\tau_Q(r)\), which is suppressed at the boundary between the MI domains as seen in the right panel of Fig. 2. However, the region in which the KZM takes place is not trivial when we compare the characteristic velocity \(v_F\) and \(s\), because the correlation length can catch up with the equilibrium value (the dynamics is adiabatic) when \(v_F < \hat{s}\), even for small \(\tau_Q\).

Let us apply the theory of the IKZM in Sec. II B to the Bose-Hubbard system. Our controllable parameter is now \(J\) in Eq. (9). From Eq. (11), the critical point \(J_c\) has a radial dependence through the chemical potential \(\mu\) as a function of \(r\), and the KZ region is strongly dependent on the value of \(\mu\). This dependence gets sharper as \(\mu\) increases, which causes a decrease in the critical radius for a relatively slow quench. Near \(\mu(0) = 1\) the KZ region becomes a maximum with respect to \(r\) and thus \(dJ_c/dr = 0\). This nontrivial radial dependence of the front velocity \(v_F\) is a characteristic feature of the Bose-Hubbard system.

Let us first consider the phase transition across the boundary of the Mott-lobe with \(n = 1\), corresponding to \(\mu(0) < U\) and Fig. 2(a). For \(\mu(0) < 0.41U \equiv \mu_1\), which gives a tip of the Mott-lobe, \(f(r)\) decreases monotonically with \(r\) and diverges at the origin. This divergence is due to the existence of a maximum of \(\mu(r)\) at the origin, which causes \(d\mu/dr|_{r=0} = 0\) and the infinite velocity \(v_F\) of the transition front [see Eq. (12)]. For \(\mu(0) > \mu_1\) there are two divergent peaks; one is at the origin and the other corresponds to the tip of the Mott lobe, at which \(J_c(r)\) becomes a maximum with respect to \(r\) and thus \(dJ_c/dr = 0\). This nontrivial radial dependence of the front velocity \(v_F\) is a characteristic feature of the Bose-Hubbard system.

Figure 3 shows the threshold radius as a function of \(\tau_Q\), obtained by seeing the crossing points of the \(f(r)\)-curve and the \(A^{-1}\)-line in Fig. 2. Here, we plot the results for the different values of \(k = 0.0025\) and \(\tilde{k} = 0.0005\). The KZ region is strongly dependent on the value of \(\mu(0)\). For \(\mu(0) < \mu_1\) the critical radius decreases monotonically with increasing \(\tau_Q\), namely the KZ region gets narrow from outside [Fig. 3(a)]. For \(\mu(0) > \mu_1\), however, there appears another non-KZ region around the intermediate region at \(r \sim 0.4R_{TF}\) [Fig. 3(b)] for \(\tau_Q/\tau_0 > 100\). This means that the KZM occurs at the central region and the ring-shaped region separated from the central KZ region for a relatively slow quench. Near \(\mu(0) \lesssim U\) the KZ regions are further shrunk to the narrow region around the center and \(r \sim 0.75R_{TF}\) as shown in Fig. 3(c). Thus, the IKZM would predict a very different behavior of the KZ dynamics from MI to SF depending on whether the central chemical potential \(\mu(0)\) is smaller or larger than \(\mu_1\).

When the transition includes the several Mott-lobes, the situation becomes more complicated, as shown in Fig. 2(b) and Fig. 3(d)-(f) for the case including both the \(n = 1\) and \(n = 2\) Mott-lobes. The KZ region can
be obtained similarly by considering that the velocity $v_F$ diverges both at the center and at the tips of the Mott lobes. The KZ region for the $n = 2$ MI domain is maximized when $\mu(0)$ is located near the tip of the Mott lobe, while it takes place only near the tip of the Mott lobe for the surrounding $n = 1$ MI domain. We show the KZ region for $\mu(0) = 1.5$ [Fig. 3(c)] and $\tau_Q/\tau_0 = 50$ by the shaded region in the right panel of Fig. 1.

**IV. TIME-DEPENDENT GUTZWILLER ANALYSIS OF QUENCH DYNAMICS**

To demonstrate the above prediction of the IKZM, we make numerical simulations of the quench dynamics described by the BHM with the harmonic potential. To study the real time evolution in the 2D system, we employ the time-dependent Gutzwiller method. This method is based on the mean-field approximation and is often used to explain the qualitative understanding of the experimental observations except for the 1D problems. Some studies have also used this method to study the quench dynamics relevant to the KZM.

The Gutzwiller ansatz for the many-body wave function is written as

$$|\Psi_G(t)\rangle = \prod_j \sum_n f_{j,n}(t) |n\rangle_j,$$

where $f_{j,n}(t)$ represents the complex coefficients for the number state $|n\rangle_j$ at the $j$-th site. The ansatz corresponds to a mean-field approximation by ignoring the correlation between different sites. Under the variational principle, we minimize $\langle \Psi_G | H - i\hbar \frac{d}{dt} | \Psi_G \rangle$ with respect to $f_{j,n}$ to obtain the time-dependent Gutzwiller equation.

$$i\hbar \frac{df_{j,n}}{dt} = -J \sum_{\langle i,j \rangle} \left[ \sqrt{n} \psi_i f_{j,n-1} + \sqrt{n+1} \psi_i^* f_{j,n+1} \right]$$

$$+ \left[ \mu(r)n + \frac{U}{2} n(n-1) \right] f_{j,n},$$

Here, the SF order parameter is given as

$$\psi_j = \langle \hat{b}_j \rangle = \sum_n \sqrt{n+1} f_{j,n}^* f_{j,n+1},$$

Equation (16) is solved numerically by the Crank-Nicholson method, where the Neumann boundary condition is used. We have confirmed that the total energy and the norm is conserved during the time evolution in the case of constant values of $J$ and $U$. We scaled the equation by the on-site interaction $U$, the unit of time being $\hbar/U$, which can be regarded as $\tau_0$ in the KZ theory.

In order to induce the quench from the MI to the SF phase, the hopping amplitude is varied as

$$\frac{J(t) - J_c(0)}{J_c(0)} = \frac{t}{\tau_Q},$$

with the critical hopping $J_c(0)$ at $r = 0$, determined by Eq. (11) within the perturbation theory. The initial states of the simulations are prepared in the deep MI region, where we choose the hopping amplitude as $J(t = 0) = 0.001U$. Then, the value $J(t)$ is linearly increased according to Eq. (17) to the final value $J(t = t_f) = 0.05U$ at which the whole system is in the SF phase. The linear slope of the ramping up of $J(t)$ is given by $J_c(0)/\tau_Q$.

For the initial MI state, we set the phases of $\{f_{j,n}(t = 0)\}$ fully random to mimic the quantum fluctuation in the mean-field method. However, the amplitude of the initial noise plays a crucial role to the growth time of the SF order parameter $\psi$. Thus, it is difficult to determine the observation time of the vortex number, called as the KZM time, from the $\psi$-based protocols. Furthermore, since the amplitude of the SF density grows inhomogeneously in our inhomogeneous system, we cannot determine uniquely the KZM time fitted to the whole system. Thus, we do not discuss the scaling property of the defect density with respect to the global quench time $\tau_Q$. Alternately, we discuss how the vortices nucleate from the inhomogeneous system and the difference from the homogeneous situation.

The rapid quench can leave topological defects, namely vortices in our 2D system, in the SF phase through the KZM. The vortices are identified by calculating the current density $j_{kl} = -\sqrt{n_k n_l} \sin(\theta_k - \theta_l)$ between $k$ and $l$-sites, where $k(\neq l)$ represents the label of the spatial grids in the 2D space; if all $j_{kl}$ along a certain minimal loop $(k, l) \rightarrow (k+1, l) \rightarrow (k+1, l+1) \rightarrow (k, l+1) \rightarrow (k, l)$ have the same sign, a vortex exists at the inside of the loop. Due to the random phases in the initial state, there

![Figure 3](image-url)
are vortices even before the SF order parameter develops. Thus, we can say that the vortices via the KZM are survivors from the vanishing random noise of the phase.

In the following, we consider the situations with three different values of the chemical potential $\mu(0)$ at the origin, namely $\mu(0)/U = 0.4$, $\mu(0)/U = 0.9$, $\mu(0)/U = 1.5$, and demonstrate the quench dynamics from MI to SF in the harmonic trap potential.

A. $\mu(0)/U = 0.4$

In this case, the quench of $J(t)$ induces the quantum phase transition from the periphery of the system. The timing which passes through the transition point is latest at the center. To see the inhomogeneity of the transition dynamics, we represent the time development of the radial distribution of the SF density $|\psi(r)|^2 = (2\pi)^{-1} \int d\theta |\psi(r, \theta)|^2$ and the radial vortex density $n_v(r) = (2\pi)^{-1} \int d\theta n_v(r, \theta)$. Here, the vortex density is taken only for vortices within the Thomas-Fermi radius $R_{TF}$. In Fig. 4, we plot the results for $k = 0.0025$ and $k = 0.0005$ for $\tau_Q/\tau_0 = 50$. For the steep trap $k = 0.0025$, the SF component arises from the periphery of the condensate. The SF density flows from the periphery to the center, along with the transition line of the Mott-lobe. Since the MI has a strong phase fluctuation, the region within the Thomas-Fermi radius is initially filled with noisy vortices. After the quench, due to the appearance of the phase coherence in the SF phase, the vortices are disappeared from the outside to the inside smoothly, as shown in the left panel of Fig. 4(b). The mechanism of the disappearance of vortices is the vortex–anti-vortex pair annihilation.

For a shallow trap $k = 0.0005$, the dynamics is slightly different. The SF density begins to grow and the vortices are erased gradually from the outside; at a certain time, however, the transition occurs homogeneously within the radius $r_c$ smaller than $R_{TF}$. The right panel of Fig. 4(b) clearly indicates that the most of vortices due to the random noise in the MI are suddenly disappeared at $t/U = 60$ within the radius $r_c$. This region with a sudden disappearance of $n_v(r)$ can be relevant to the non-adiabatic KZM. Thus, the applicability of the IKZM theory is strongly dependent on the shallowness of the harmonic trap potential.

We find that the critical radius $r_c$ is dependent on the quench time and the trap spring constant $k$. Figure 5 shows the critical radius $r_c/R_{TF}$, extracted from the development of $n_v(r)$, as a function of the spring constant $k$ for several quench times $\tau_Q$. The critical radius $r_c/R_{TF}$ increases as $k$ decreases, as expected from the fact that the transition approaches to the homogeneous limit $r_c/R_{TF} \rightarrow 1$ for $k \rightarrow 0$. The slow quench suppresses the appearance of the KZ region, where $r_c = 0$ means that the transition is adiabatic. We also plot $r_c$ obtained from the IKZM in Sec. 4. Although the theory overestimates the KZ region, the numerical result approaches the theoretical estimation for smaller values of $k$, i.e., a shallow trap potential, and the fast quench. The difficulty of the application to IKZM in our case might be the breakdown of the local approximation of the chemical potential to evaluate the MI-SF phase diagram.

As can be seen in the time evolution of $n_v$, the vortices caused by the original phase fluctuation are almost disappeared along with the growth of the SF density. The vortices surviving after the growth of the phase coherence can be regarded as the vortices created by the KZM. However, we see that the vortices are entered from outside of the Thomas-Fermi radius even after the SF order parameter is sufficiently grown. These vortices are not relevant to the KZM.

B. $\mu(0)/U = 0.9$

Next, we consider the case $\mu(0)/U = 0.90$, where the MI-SF transition is expected to start both at the periphery and the center, according to the phase diagram of Fig. 3. Figure 6 shows the similar plot with Fig. 4 for steep and shallow trap cases. In both case, we obtain

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{The time development of the radial SF density $|\psi(r)|^2$ (a) and the radial vortex density $n_v(r)$ (b) for $\mu(0)/U = 0.4$ and $\tau_Q/\tau_0 = 50$. The left panels and right panels correspond to $k = 0.0025$ and $k = 0.0005$, respectively. The Thomas-Fermi radius is $R_{TF}/a_0 = 18$ for (a) and $R_{TF}/a_0 = 40$ for (b).}
\end{figure}
FIG. 5. The critical radius giving the KZ region as a function of the spring constant \( \tilde{k} \) of the harmonic trap. The plots are obtained from the numerical simulations, where the global quench times are given by \( \tau_Q/\tau_0 = 50 \) (red circles), 100 (blue triangles), 200 (green squares), and 300 (black diamonds). The value of \( r_c \) is found by seeing the radial dependence of the time at which \( n_v(r) \) vanishes mostly. The dashed lines correspond to \( r_c \) given by the IKZM theory, where \( \tau_Q/\tau_0 = 50, 100, 200, 300 \) from top to bottom.

Contrary to expectations, the SF component grows only from the outside, not from the center. This is due to the fact that the central region is embedded deeply in the MI so that the trigger for the growth of the SF component from the center is strongly suppressed. Also, there is a depression of the SF density around \( r \sim 0.7R_{TF} \), where the chemical potential \( \mu(0.7R_{TF}) \) corresponds to the tip of the Mott lobe. This radial position can exhibit an anomalous behavior because the \( v_F/\tilde{s} \) diverges but the local quench time becomes the longest one. We see that, in the later stage of the dynamics, the vortices are accumulated in the ring-shaped region around \( r \sim 0.7R_{TF} \). These behaviors are not changed for different values of the global quench time \( \tau_Q \).

C. \( \mu(0)/U = 1.5 \)

Finally, we study the quench dynamics for \( \mu(0)/U = 1.5 \), where the initial state consists of a wedding cake of the \( n = 2 \) and \( n = 1 \) MI islands. As seen in Fig. 7(a), for both the shallow and steep trap case, the SF component grows from both the periphery and the boundary between the \( n = 2 \) and \( n = 1 \) MI domains, where the very small fraction of the SF density exists even in the initial state. For the steep trap, the SF order grows continuously from these boundary to the inside of the respective MI domain. A similar behavior can be seen for the shallow trap case, but a clear separation of the SF density can be seen between the inner and outer regions. From Fig. 7(b), there are small central regions that are relevant to the KZM, where most of the noisy vortices are disappeared homogeneously. In the right panel of Fig. 7(b), the transition takes place adiabatically in the surrounding \( n = 1 \) MI domain since the local quench time is longer than the inner region as seen in Fig. 1. In addition to these observations, we can see that long-lived vortices are generated from the boundary between the \( n = 2 \) and \( n = 1 \) MI. Since the SF order parameters have the different origins in the two regions, the discontinuity of the SF phase may yields the vortices. These vortices can have a long life time compared with the vortices arising from the random phase distribution; the latter are soon disappeared via pair annihilation. This fact implies a difficulty for the experimental identification of vortices purely through the KZM.

V. CONCLUSION

In conclusion, we consider the quench dynamics from the MI to the SF of bosons in an optical lattice and a harmonic trap, which is usually utilized in the cold atom ex-
experiments, and the applicability of the IKZM [29]. Due to the nontrivial radial dependence of the transition point, we can identify the region where the KZM is expected by applying the theory of the IKZM to the system of the BHM. The Gutzwiller simulations of the BHM demonstrate the rich phenomenology of the quench dynamics, where the IKZM is applicable only for the system with a shallow harmonic trap. The inhomogeneity of the system may cause the unexpected generation of the vortices, from the periphery of the Thomas-Fermi radius as well as the phase boundary between \( n = 1 \) and \( n = 2 \) MI domains. These bring the difficulty for quantitative study of the KZM in the cold atom system in OL and the harmonic trap potential. More precise simulations including precisely quantum fluctuations in the initial state, e.g., through the truncated Wigner method, remain for future work.

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