ABSTRACT. We provide a decision theoretic analysis of bandit experiments. Working within the framework of diffusion asymptotics, we define suitable notions of asymptotic Bayes and minimax risk for these experiments. For normally distributed rewards, the minimal Bayes risk can be characterized as the solution to a nonlinear second-order partial differential equation (PDE). Using a limit of experiments approach, we show that this PDE characterization also holds asymptotically under both parametric and non-parametric distribution of the rewards. The approach further describes the state variables it is asymptotically sufficient to restrict attention to, and therefore suggests a practical strategy for dimension reduction. The upshot is that we can approximate the dynamic programming problem defining the bandit experiment with a PDE which can be efficiently solved using sparse matrix routines. We derive the optimal Bayes and minimax policies from the numerical solutions to these PDEs. The proposed policies substantially dominate existing methods such as Thompson sampling. The framework can be generalized to allow for time discounting and pure exploration motives.

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1. Introduction

The multi-arm bandit problem describes an agent who needs to choose among $K$ possible treatments (arms) for a population, each corresponding to an unknown distribution of rewards. The aim is to maximize welfare, the cumulative rewards received across the population, while simultaneously learning about the effects of the actions. Compared to classical approaches like randomized trials, it enables fast learning of the optimal treatment while minimizing experimentation.

The setup describes a dynamic programming problem. However, solving this directly is typically infeasible. Instead, heuristics based on the exploration-exploitation tradeoff are commonly used, such as Thomson sampling (see, Russo et al., 2017, and references therein) and Upper Confidence Bound (UCB; Lai and Robbins, 1985) algorithms. These algorithms have been extensively studied and applied in areas ranging from news article recommendation (Chapelle and Li, 2011) to dynamic pricing (Ferreira et al., 2018), and public health interventions (Athey et al., 2021).

There is a large theoretical literature on the regret properties of bandit algorithms. Here, regret is the difference in cumulative returns from pulling the best arm and the agent’s actual cumulative return. The existing results come in two forms. The first set of results, ‘instance dependent bounds’ (Lai and Robbins, 1985; Burnetas and Katehakis, 1996), specify the rates of regret under a given set of mean rewards for each arm. Here, nature chooses a set of rewards for each arm so as to deliver the highest regret, but is not allowed to change them with the value of $n$, the number of periods of experimentation allowed. These results are of a large deviations flavor. The second, ‘minimax bounds’ (Bubeck et al., 2013), specify the minimax rates of regret, when nature is allowed to change the magnitude of the rewards with $n$. In the bandit setting the minimax rate is known be of the order $n^{-1/2}$ (see, e.g., Lattimore and Szepesvári, 2020, Chapter 9 and references therein).

Despite these advances, a number of questions still remain. How do we select amongst different algorithms with the same rates of regret? Can we quantify the performance of algorithms (as opposed to only the rate) under various notions of statistical, i.e., Bayes and minimax, risk? What are the optimal policy rules, which specify the optimal arm to pull, under these risk measures? Is it possible to efficiently compute these rules? This paper seeks to answer these questions.
The first contribution of this paper is to define a notion of asymptotic Bayes risk for bandit experiments. This is done under a suitable limiting version of the bandit problem, based on diffusion asymptotics introduced by Wager and Xu (2021) and Fan and Glynn (2021). Diffusion asymptotics consider the regime where the difference in expected rewards between the arms scales as $n^{-1/2}$, corresponding to the minimax rate. This defines the hardest instance of the bandit problem: it ensures learning about the optimal arm remains challenging even as $n$ increases and gives rise to non-trivial asymptotic risk. If the reward gap scales at a faster rate, identifying the optimal arm is straightforward asymptotically, whereas if it scales at a slower rate, there is too little difference between the arms, so the regret converges to 0 in either case. The $n^{-1/2}$ scaling thus provides a good approximation to the finite sample properties of various policies. The same scaling occurs in the decision theoretic analysis of treatment assignment rules by Hirano and Porter (2009).

Wager and Xu (2021) and Fan and Glynn (2021) study the regret properties of Thompson sampling under diffusion asymptotics, but do not address the question of optimal policies. Here, we study optimal policies under the notions of Bayes and minimax risk. We define Bayes risk using asymptotically non-negligible priors, i.e., priors which scale as $n^{-1/2}$. This serves to clarify their effect in finite samples. We show that the minimal asymptotic Bayes risk can be characterized as the solution to a second order nonlinear PDE. We first demonstrate this for normally distributed rewards. The proof makes use of the theory of viscosity solutions, see Crandall et al. (1992). This is crucial because the results of Wager and Xu (2021) and Fan and Glynn (2021) only apply to continuous policies. But the optimal policy is generically deterministic (as is evident by the fact it solves a dynamic programming problem), and hence discontinuous, so characterizing minimal Bayes risk even for Gaussian rewards requires the new techniques introduced here.

Next, using a limit of experiments approach, we show that the PDE characterization also holds asymptotically under both parametric and non-parametric distributions of the rewards. Thus, any bandit problem can be asymptotically reduced to one with normally distributed rewards. As part of this reduction we also characterize the state variables it is (asymptotically) sufficient to restrict attention to. Apart from time, only two state variables per arm are needed: the number of times the
arm has been pulled in the past, and the score process (for parametric models) or the cumulative rewards accrued from pulling the arm (for nonparametric models). These results appear to be novel and are clearly relevant for dimension reduction.

We demonstrate the equivalence of experiments by extending the posterior approximation method of Le Cam and Yang (2000, Section 6.4) to bandit experiments. The proof makes use of novel arguments involving uniform approximation of the log-likelihood with its local quadratic expansion. It also differs from the standard approach based on asymptotic representations. Asymptotic representations rely on a coupling argument to ensure the distribution of welfare from any sequence of policies is replicated by another policy in the limit experiment. But coupling is difficult to establish when the limit experiment is a continuous time stochastic process, as is the case under diffusion asymptotics. The techniques employed here are thus of independent interest for analyzing other types of sequential experiments.

The optimal Bayes policy rule can be inferred from the solution to the PDE characterizing minimal Bayes risk. In fact, this is the limiting version of the dynamic programming problem (DP) defining the bandit experiment. Even though solving the DP problem directly is infeasible, the limiting PDE can be solved very efficiently using finite difference methods and sparse matrix routines. We describe these numerical methods in detail and argue that the computational cost is low in many applications. Besides, there is usually a large gain to using the optimal Bayes rule; both theoretical and Monte-Carlo computations show that the asymptotic Bayes risk of Thompson sampling is often twice as high as that of the optimal Bayes rule.

An alternative measure of statistical risk is minimax risk. This is nothing more than Bayes risk under a least favorable prior. While it does not appear straightforward to determine the least favorable prior analytically, we provide numerical methods to compute this and the corresponding minimax optimal policy.

Our framework easily accommodates various generalizations and modifications to the bandit problem such as time discounting and best arm identification (Russo, 2016; Kasy and Sautmann, 2019). The discounted bandit problem has a rich history in economic applications, ranging from market pricing (Rothschild, 1974) to decision making in labor markets (Mortensen, 1986). For discounted problems, the optimal Bayes policy can be characterized using Gittins indices (Gittins, 1979). However,
except in simple instances, e.g., discrete state spaces, computing the Gittins index is difficult (see, Lattimore and Szepesvári, 2020, Section 35.5 for a discussion). Also, it does not apply beyond the discounted setting; the optimal Bayes policy in finite horizon settings is not an index policy (Berry and Fristedt, 1985, Chapter 6). Here we take a different route and characterize the value function directly in a PDE form.

2. DIFFUSION ASYMPTOTICS AND STATISTICAL RISK

Following Wager and Xu (2021), we start by analyzing the one-armed bandit problem (Section 6 discusses multi-armed bandits). Here, an agent needs to decide whether to pull on an arm that generates a reward with an unknown mean, as opposed to staying with an outside option that generates zero rewards. In this section, we provide a heuristic derivation of the PDE characterizing minimal Bayes risk. Formal results are provided in the next section.

Suppose the experiment concludes after \( n \) periods, where \( n \) is pre-specified. Knowledge of \( n \) is reasonable if it is the population size; indeed, the bandit setting blurs any distinction between sample and population. In other cases, it might be more reasonable to assume the agent employs discounting and allows the experiment to continue indefinitely. Our techniques can be applied equally well to both possibilities. We focus on the known \( n \) case to avoid duplication of effort, but see Section 6.3 for discounted bandits. The decision theoretic analysis employed here requires modeling all aspects of decision making including when to stop or how to discount, but our results are otherwise very broadly applicable (e.g., even to best arm identification). When \( n \) is known, the number of periods that have elapsed is a state variable, and after dividing by \( n \) will be termed ‘time’. Thus, time \( t \) proceeds from 0 and 1, and is incremented by \( 1/n \) between successive periods.

Suppose each time the arm is pulled, a reward, \( Y_i \), is drawn from the normal distribution \( \mathcal{N}(\mu_n, \sigma^2) \), where \( \mu_n := \mu/\sqrt{n} \). The scaling of the mean reward by \( \sqrt{n} \) follows Wager and Xu (2021) and Fan and Glynn (2021) and ensures the signal decays with sample size. The variance, \( \sigma^2 \), is assumed to be known. In this section and the next, we provide a detailed description of the one-armed bandit problem under such normally distributed rewards. The utility of this analysis stems from the fact that more general models - that assume either a parametric or non-parametric
distribution of rewards - reduce asymptotically to the normal setting under the limit of experiments approach, see Sections 4 and 5.

Let $A_j$ denote the action in period $j$, where $A_j = 1$ if the arm is pulled and $A_j = 0$ otherwise. Due to normality of the rewards, the only relevant state variables are the number of times the arm was pulled, $q(t) = n^{-1} \sum_{j=1}^{[nt]} \mathbb{I}(A_j = 1)$, the cumulative rewards $x(t) = n^{-1/2} \sum_{j=1}^{[nt]} \mathbb{I}(A_j = 1)Y_j$, and time $t$ (see Section 4 for a formal argument about the sufficiency of these state variables). The scaling on the cumulative rewards follows Wager and Xu (2021) and can be understood as a re-scaling of the rewards $Y_i$ by the factor $1/\sqrt{n}$. The agent chooses a policy rule $\pi(\cdot) : \mathcal{S} \to [0, 1]$ that determines the probability of pulling the arm given the current state $s := (x, q, t)$.

Wager and Xu (2021) show that under some assumptions on $\pi$, the evolution of $q(t)$ and $x(t)$ in the large $n$ limit is governed by the SDEs

$$dq(t) = \pi(s_t)dt;$$
$$dx(t) = \pi(s_t)\mu dt + \sigma \sqrt{\pi(s_t)}dW(t),$$

(2.1)

where $W(t)$ represents one-dimensional Brownian motion, and $\pi(s_t) := \pi(s(t))$. It bears emphasizing that while (2.1) is convenient for heuristics, there is no guarantee that an optimal policy $\pi(\cdot)$ possesses sufficient regularity properties for (2.1) to formally hold (however, (2.1) does hold if the policy function is Lipschitz continuous). As it turns out, our formal results, in Section 3, will not make use of (2.1).

**Notation.** Throughout this article, we use $j$ to refer to the time periods and $i$ to the pulls, i.e., $j = a$ refers to the $a$-th time period while $i = a$ refers to the $a$-th pull of the arm. We also use $y_i := \{Y_k\}_{k=1}^i$ to denote the sequence of rewards after $i$ pulls of the arm. This is the so-called stack of rewards representation (Lattimore and Szepesvári, 2020, Section 4.6). Note that $\{Y_k\}_{k=1}^n$ are iid.

**2.1. Payoff and loss functions.** In bandit experiments, loss is typically defined as cumulative payoffs, where the payoff is 0 when the experiment concludes (i.e., when $t = 1$). A payoff function describes the payoff due to action $A$ and a reward $Y$ when the true parameter is $\mu$. We focus on the regret payoff

$$R(Y, A, \mu) = \frac{Y}{\sqrt{n}} \{\mathbb{1}(\mu \geq 0) - A\}.$$  

(2.2)
It is the difference in rewards between the optimal action, $\mathbb{1}\{\mu \geq 0\}$, and action $A$. Clearly, regret is just a rescaling of welfare payoff $W(Y, A, \mu) = -YA/\sqrt{n}$. These payoffs are equivalent for evaluating policies under Bayes risk. However the behavior under minimax risk is very different. Under the welfare payoff, the minimax policy is trivial and excessively pessimistic: the bandit should never pull the arm. By contrast, the minimax risk under regret payoff is non-trivial. For this reason, we focus exclusively on regret (as does most of the literature on bandit experiments).

Recently, there has been interest in best arm identification (e.g., Kasy and Sautmann, 2019), where an agent experiments until a set time $t = 1$ to determine the optimal arm. Here, the statistical loss is the regret from choosing an arm for full scale implementation at the end of experimentation. Such a loss function does not feature an ‘exploitation’ motive. We discuss best arm identification in Section 6.

2.2. Bayes risk. We now define a notion of asymptotic Bayes risk in bandit experiments.

2.2.1. Priors and posteriors. Suppose the agent places a prior over $\mu$, given by $\mu \sim m_0$. When the current state is $s \equiv (x, q, t)$, the posterior density of $\mu$ is

$$p(\mu|s) \propto p_q(x|\mu) \cdot m_0(\mu); \quad p_q(\cdot|\mu) \equiv N(\cdot|q\mu, q\sigma^2),$$

where $N(\cdot|\mu, \sigma^2)$ is the normal density with mean $\mu$ and variance $\sigma^2$. The above expression is valid exactly for all $n$ (under Gaussian rewards).

An important property of the the posterior is that it depends only on the $\lfloor nq \rfloor$ iid realizations of the rewards, $y_{nq}$, and is not affected by the past values of the actions (nor by past values of $q$). Lemma 1 shows that this property holds generally, and is not limited to Gaussian rewards. Under a deterministic policy, past actions and $q(t)$ are just deterministic functions of $y_{nq}$, while for randomized policies, these would be functions of $y_{nq}$ and an exogenous random process. Hence, they do not provide any additional information about $\mu$ beyond what $y_{nq}$ already provides.

Since the prior is applied on the local parameter $\mu$, it is asymptotically non-negligible. The rationale for employing non-negligible priors is two-fold: First, it provides a better approximation to finite sample properties by clarifying the effect of the prior on policy comparisons. Second, it enables us to characterize minimax
risk as Bayes risk under a least favorable prior (see Section 2.3). The least favorable prior is non-negligible.

In practice, we are typically provided with a prior, \( \rho_0 \), on the unscaled mean \( \mu_n = \mu / \sqrt{n} \). To apply the methods here, one needs to convert this to a prior, \( m_0(\cdot) = \rho_0(\cdot / \sqrt{n}) \), on \( \mu \). To illustrate, suppose the agent places a Gaussian prior \( \mu \sim \mathcal{N}(\mu_0, \nu^2) \). When the current state is \( s \equiv (x, q, t) \), the posterior distribution of \( \mu_n \) is

\[
\mu_n | s \sim \mathcal{N} \left( \frac{\sigma^{-2} \sqrt{n} x + \sqrt{n} \nu^{-2} \mu_0}{n \sigma^{-2} q + n \nu^{-2}}, \frac{1}{n \sigma^{-2} q + n \nu^{-2}} \right).
\]

Equivalently, the posterior distribution of the scaled mean reward \( \mu := \sqrt{n} \mu_n \) is

\[
\mu | s \sim \mathcal{N} \left( \frac{\sigma^{-2} x + \nu^{-2} \mu_0}{\sigma^{-2} q + \nu^{-2}}, \frac{1}{\sigma^{-2} q + \nu^{-2}} \right) := p(\mu | s).
\]

2.2.2. PDE characterization of Bayes and minimal Bayes risk. We define Bayes risk, \( V_\pi(s) \), at state \( s \) under a policy \( \pi \) as the expected cumulative regret in the diffusion regime, where the expectation is taken conditional on all information until state \( s \). The ex-ante expected (asymptotic) Bayes risk of a policy \( \pi \) is then \( V_\pi(0) \equiv V_\pi(s_0) \) where \( s_0 := (0, 0, 0) \) is the initial state. This is the expected cumulative regret at the outset of the experiment. It can thus be used for ranking candidate policies.

If the policy is Lipschitz continuous, \( V_\pi(s) \) is given by the stochastic integral

\[
V_\pi(s) = \mathbb{E} \left[ \int_t^1 (\mu \mathbb{I}(\mu \geq 0) - \mu \pi(s_i)) \, dt \bigg| s \right].
\]

However this representation is not formally justified for general measurable policies. This is an issue because the optimal Bayes policy is generically deterministic and hence, discontinuous. It is rather more convenient to characterize \( V_\pi(s) \) using PDE methods. We now informally derive such a PDE characterization of \( V_\pi(s) \). The formal justification is provided in the next section.

In a short time period \( \Delta t \) following state \( s \), the change to \( q \) and \( x \) is approximately (henceforth we use \( \pi \) as a shorthand for \( \pi(s) \))

\[
\Delta q \approx \pi \Delta t; \quad \Delta x \approx \pi \mu \Delta t + \sigma \sqrt{\pi} \Delta W(t).
\]

For a given \( \mu \), the regret, \( \Delta V_{\pi, \mu} \), accrued within this time period is the difference between the rewards, \( \mathbb{I}(\mu \geq 0)(\mu \Delta t + \sigma \Delta W(t)) \) and \( \Delta x \), generated, respectively,
under the infeasible optimal policy $\mathbb{I}(\mu \geq 0)$ and the given policy $\pi$:

$$\Delta V_{\pi,\mu} \approx (\mu \mathbb{I}(\mu \geq 0) - \mu \pi) \Delta t + \sigma \left( \mathbb{I}(\mu \geq 0) - \sqrt{\pi} \right) \Delta W(t).$$  \hspace{1cm} (2.4)

The recursive nature of the problem implies

$$V_\pi(s) = \mathbb{E} [\Delta V_{\pi,\mu} + V_\pi(x + \Delta x, q + \Delta q, t + \Delta t) | s],$$

with the terminal condition $V_\pi(s) = 0$ if $t = 1$.

Now, for the right-hand side of (2.5), the approximation (2.4) implies

$$\mathbb{E} [\Delta V_{\pi,\mu} | s] \approx \left( \mathbb{E}_{\mu|s}[\mu \mathbb{I}(\mu \geq 0)] - \mathbb{E}_{\mu|s}[\mu \pi] \right) \cdot \Delta t = \left( \mu^+(s) - \mu(s) \pi \right) \cdot \Delta t,$$

where $\mu^+(s) := \mathbb{E}_{\mu|s}[\max\{\mu, 0\}]$ and $\mu(s) := \mathbb{E}_{\mu|s}[\mu]$. For Gaussian priors, explicit expressions for these quantities can be given using (2.3):

$$\mu^+(s) := \mu(s) \left\{ 1 - \Phi \left( -\frac{\mu(s)}{\sigma(s)} \right) \right\} + \sigma(s) \phi \left( -\frac{\mu(s)}{\sigma(s)} \right);$$

$$\mu(s) := \frac{\sigma^{-2}(s) + \nu^{-2} \mu_0}{\sigma^{-2}q + \nu^{-2}}; \quad \sigma(s) := \left( \sigma^{-2}q + \nu^{-2} \right)^{-1/2}.$$

Next, Ito’s lemma implies

$$\mathbb{E} [V_\pi(x + \Delta x, q + \Delta q, t + \Delta t) - V_\pi(s) | s] \approx \left\{ \partial_t V_\pi + \pi \partial_q V_\pi + \mu(s) \partial_x V_\pi + \frac{1}{2} \pi \sigma^2 \partial_x^2 V_\pi \right\} \Delta t.$$

Thus, subtracting $V_\pi(s)$ from both sides of (2.5) and dividing by $\Delta t$, we obtain the following characterization of $V_\pi$:

$$\partial_t V_\pi + \mu^+(s) + \pi(s) \{-\mu(s) + L[V_\pi](s)\} = 0 \text{ if } t < 1 \hspace{1cm} (2.6)$$

$$V_\pi(s) = 0 \text{ if } t = 1.$$

Here, $L[\cdot]$ denotes the infinitesimal generator

$$L[f] := \partial_q f + \mu(s) \partial_x f + \frac{1}{2} \sigma^2 \partial_x^2 f.$$

PDE (2.6) only involves the known quantity $\sigma^2$. Hence, it can be solved for any candidate policy function $\pi$.

We can also characterize the minimal Bayes risk $V^*_\pi(s) := \inf_{\pi(\cdot) \in \Pi} V_\pi(s)$, where $\Pi$ denotes the class of all measurable policy rules. By the dynamic programming
principle, the minimal Bayes risk satisfies

\[ V^*(s) = \inf_{\pi \in [0,1]} \mathbb{E} [\Delta V_{\mu,\pi} + V^*(x + \Delta x, q + \Delta q, t + \Delta t)\mid s], \]

for any small time increment \( \Delta t \), together with the boundary condition \( V^*(x, q, 1) = 0 \ \forall \ x, q \). Then, by similar heuristic arguments as those leading to (2.6), we obtain

\[ \partial_t V^* + \mu^+(s) + \min \{-\mu(s) + L[V^*](s), 0\} = 0 \text{ if } t < 1, \]

\[ V^*(s) = 0 \text{ if } t = 1. \quad (2.7) \]

As with PDE (2.6), PDE (2.7) can be solved using knowledge only of \( \sigma^2 \). We can thus characterize the minimal ex-ante expected Bayes risk as \( V^*(0) := V^*(s_0) \).

PDE (2.7) encapsulates the exploration-exploitation tradeoff. The regret payoff can be minimized to 0 when the posterior distribution collapses to a point, in which case one chooses the optimal arm with certainty. In PDE (2.7), this is reflected in the fact the (instantaneous) regret payoffs are either \( \mu^+(s) - \mu(s) \) or \( \mu^+(s) \), both of which are always greater than 0, but \( \min\{\mu^+(s) - \mu(s), \mu^+(s)\} \to 0 \) as \( q \) increases.

The agent thus faces a tradeoff between exploration, i.e., pulling the arm enough times to increase the value of \( q \) and thereby reduce \( \min\{\mu^+(s) - \mu(s), \mu^+(s)\} \) in the future, and exploitation, i.e., choosing the optimal arm, \( \mathbb{I}\{\mu(s) \geq 0\} \), at the present.

If a classical (i.e., a twice continuously differentiable) solution, \( V^*(\cdot) \), to PDE (2.7) exists, the optimal Bayes policy is given by \( \pi^*(s) = \mathbb{I}\{L[V^*](s) \leq \mu(s)\} \). However, existence of a classical solution is in general impossible, and formally showing existence of a measurable \( \pi^*(\cdot) \) is therefore difficult. Nevertheless, one can construct measurable policies whose Bayes welfare is arbitrarily close to \( V^*(\cdot) \), and this is all that is needed in practice. One such construction is provided in Section 3.3. It is also possible to discern some properties of the optimal policy: First, it has to be a retirement policy, i.e., if the agent did not pull the arm at some time \( t \), she will not do so at any other point in the future.\(^1\) Second, it has to be weakly increasing in \( x \).

These two properties imply \( \pi^* \) is of the form \( \pi^*(s) = \mathbb{I}\{x > f(q,t)\} \), where \( f(\cdot) \) is non-decreasing in \( t \).

\(^1\)To see why, consider a policy, \( \pi_1 \), that stops pulling the arm at \( (x, q, t_1) \) and restarts it at \( (x, q, t_1 + \delta) \), \( \delta > 0 \). In the interim, \( x, q \) are unchanged, so the posterior expectation of \( \mu \) stays the same. Thus a policy \( \pi_2 \) that replicates the actions of \( \pi_1 \) from \( t + \delta \) onwards, but starts them instead from \( t + \delta/2 \) achieves at least weakly lower risk as it has an additional time of \( \delta/2 \) at the end where it can continue choosing optimal actions.
2.3. **Minimax risk.** Following Wald (1945), we define minimax risk as the value of a two player zero-sum game played between nature and the agent. Nature’s action consists of choosing a prior, $m_0 \in \mathcal{P}$, over $\mu$, while the agent chooses the policy rule $\pi$. The minimax risk $\bar{V}^*$ is defined as

$$\bar{V}^* = \sup_{m_0 \in \mathcal{P}} \inf_{\pi \in \Pi} V_\pi(0; m_0),$$

(2.8)

where $V_\pi(0; m_0)$ and $V^*(0; m_0)$ denote the ex-ante Bayes risk under a policy $\pi$, and the minimal Bayes risk, when the prior is $m_0$. The equilibrium action of nature is termed the least-favorable prior, and that of the agent, the minimax policy. Under a minimax theorem, the sup and inf operations in (2.8) can be interchanged, so that

$$\inf_{\pi \in \Pi} \sup_{m_0 \in \mathcal{P}} V_\pi(0; m_0) = \sup_{\pi \in \Pi} \inf_{m_0 \in \mathcal{P}} V_\pi^*(0; m_0),$$

where $V_\pi^*(0; \mu)$ denotes the frequentist risk of a policy $\pi$ when the local parameter is $\mu$. The last term, $\inf_{\pi \in \Pi} \sup_\mu V_\pi(0; \mu)$, is the more common definition of minimax risk, though whether the minimax theorem holds in the bandit context is an open question. In any event, the inequality $\inf_{\pi \in \Pi} \sup_\mu V_\pi(0; \mu) \geq \sup_{m_0 \in \mathcal{P}} \inf_{\pi \in \Pi} V_\pi(0; m_0)$ always holds, so $\bar{V}^*$ gives a lower bound on minimax risk even under this definition.

Equation (2.8) implies that the problem of computing minimax risk reduces to that of computing Bayes risk under the least favorable prior. In the (static) treatment assignment problem, e.g., Hirano and Porter (2009), the least favorable prior has a symmetric two point support. We conjecture, and verify numerically by solving the two player game in Section 7, that the least favorable prior for one-armed bandits also involves only two support points at $\{\mu, \bar{\mu}\}$, with $\mu < 0$ and $\bar{\mu} > 0$. Intuitively, both low and high values of $|\mu|$ are associated with low risk, the former by definition, and the latter because the agent quickly learns to always pull or never pull the arm. Hence, the risk is maximized at a unique $\mu$ for $\mu < 0$, and similarly for $\mu > 0$. It also suffices to compute the least favorable prior, $m_0^*(\cdot)$, under $\sigma = 1$; due to the scale invariance of Brownian motion, the least favorable prior under any other $\sigma$ will be $m_0^*(\cdot/\sigma)$, i.e., we need only multiply the support points of $m_0^*(\cdot)$ by $\sigma$. We also expect $m_0^*(\cdot)$ to be asymmetric, with $|\mu| > |\bar{\mu}|$ and $m_0^*(\mu) > m_0^*(\bar{\mu})$, in order to respond optimally to the exploration motive of the agent. Symmetry will be regained under multi-armed bandits if all arms have the same reward variances.
3. Formal properties under gaussian rewards

3.1. Existence and uniqueness of PDE solutions. Equation (2.7) describes a second-order nonlinear second-order PDE. It is well known that such PDEs do not admit classical, i.e., twice continuously differentiable solutions. Instead, the relevant weak solution concept is that of a viscosity solution (Crandall et al. 1992).

**Theorem 1.** *(Barles and Jakobsen, 2007, Theorem A.1)* Suppose $\mu^+(\cdot), \mu(\cdot)$ are $\gamma$-Hölder continuous for some $\gamma > 0$. Then there exists a unique, $\gamma$-Hölder continuous viscosity solution to PDE (2.7).

3.2. Convergence to the PDE solution. In Section 2, we provided a heuristic derivation of PDE (2.7). For a formal result, one would need to prove that a discrete analogue, $V_n^*(\cdot)$, of $V^*(\cdot)$, defined for a fixed $n$, converges to $V^*(\cdot)$ as $n \to \infty$. Define $I_n = \mathbb{I}\{t \leq 1 - 1/n\}$ and $Y_i$ as the $i$-th realization of the rewards (corresponding to the $i$-th pull of the arm). Let $V_n^*(\cdot)$ denote the solution to the recursive equation

\[
V_n^*(x, q, t) = \min_{\pi \in [0, 1]} \mathbb{E} \left[ \frac{\mu^+(s) - \pi \mu(s)}{n} + I_n \cdot V_n^* \left( x + \frac{A_{\pi} Y_{nq+1}}{\sqrt{n}}, q + \frac{A_{\pi}}{n}, t + \frac{1}{n} \right) \bigg| s \right];
\]

if $t < 1$, \hspace{1cm} \hspace{1cm} (3.1)

\[
V_n^*(x, q, 1) = 0.
\]

In (3.2), $A_{\pi} \sim \text{Bernoulli}(\pi)$ independent of $s$, and the expectation is a joint one over $\mu, Y_{nq+1}, A_{\pi}$ given $s := (x, q, t)$. Existence of a unique $V_n^*(\cdot)$ follows by backward induction. Clearly, $V_n^*(\cdot)$ defines the minimal Bayes risk in the fixed $n$ setting. One can thus interpret (3.2) as a discrete approximation to PDE (2.7). As such, it falls under the abstract framework, devised by Barles and Souganidis (1991), for showing convergence to viscosity solutions. An application of their techniques proves the following result (all proofs are in Appendix A): Denote $\varpi(s) := \min \{\mu^+(s) - \mu(s), \mu^+(s)\}$.

**Theorem 2.** Suppose $\mu^+(\cdot), \mu(\cdot)$ are $\gamma$-Hölder continuous, $\sup_s \varpi(s) < \infty$ and the prior $m_0$ is such that $\mathbb{E}[|\mu|^3|s] < \infty$ at each $s$. Then, as $n \to \infty$, $V_n^*(\cdot)$ converges locally uniformly to $V^*(\cdot)$, the unique viscosity solution of PDE (2.7).

It is easy to verify all the assumptions for Theorem 2 under Gaussian priors. Note also that the theorem is proved without appealing to (2.1) as in Wager and Xu (2021).
3.2.1. Implied for policy choice. Theorem 2 implies that the asymptotic risk of any policy is larger than $V^*(0)$. To quantify how close various policies come to this limit, we can use the following result: For any sequentially measurable policy $\pi(\cdot) : S \to [0,1]$, define $V_{\pi,n}(\cdot)$ as the solution to

$$V_{\pi,n}(x, q, t) = \mathbb{E} \left[ \mu^+ + \mu(\cdot) - \pi \mu(\cdot) \right] + \mathbb{I}_n \cdot V_{\pi,n} \left( x + \frac{A_{\pi}(s)Y_{n+1}}{\sqrt{n}}, q + \frac{A_{\pi}(s)}{n}, t + \frac{1}{n} \right) \,| \, s \right] ;$$

if $t < 1,$

$$V_{\pi,n}(x, q, 1) = 0$$

(3.3)  

(3.4)

where $A_{\pi}(s) \sim \text{Bernoulli}(\pi(s))$.

Corollary 1. Suppose $\mu^+(\cdot), \mu(\cdot), \pi(\cdot)$ are Hölder continuous, $\sup_s \varpi(s) < \infty$, and the prior $m_0$ is such that $\mathbb{E}[(\mu)^3 | s] < \infty$ at each $s$. Then, as $n \to \infty$, $V_{\pi,n}(\cdot)$ converges locally uniformly to $V_{\pi}(\cdot)$, the unique viscosity solution of PDE (2.6).

The proof of the above is analogous to that of Theorem 2, and is omitted. The restriction to Hölder continuous $\pi(\cdot)$ is a drawback as it disallows discontinuous policies such as UCB. The theory of viscosity solutions is well developed for continuous PDE coefficients, but similar results for discontinuous coefficients are harder to come by. We conjecture, however, that this assumption is not necessary.

In any event, Thompson sampling rule $\pi_{ts} = \mathbb{E} [\mathbb{I}\{\mu \geq 0}\,|\, x, q]$, does satisfy the requirements of Theorem 1. In Section 7, we compute $V^*(0)$ and $V_{\pi}(0)$ for Thompson sampling under various priors and parameter values for $\sigma^2$. The numerical results show that the asymptotic risk under Thompson sampling is much higher than $V^*(0)$, sometimes by a factor of 2. This implies that Thompson sampling is often substantially sub-optimal.

3.2.2. Rates of convergence. We can obtain an upper bound on the rate of convergence of $V_n^*$ to $V^*$ under the additional assumption $\sup_s \mu^+(s) < \infty$. This is satisfied if the prior has a compact support. Under this assumption, an application of Barles and Jakobsen (2007, Theorem 3.1) shows that $|V_n^* - V^*| \lesssim n^{-1/14}$, with a faster one-sided rate $V^* - V_n^* \lesssim n^{-1/6}$. A formal argument is provided in Appendix B. This result is not expected to be sharp; the work of Krylov (2005) suggests a $n^{-1/2}$ rate may be possible.
3.3. **Piece-wise constant policies and batched bandits.** While we are not able to characterize the optimal Bayes policy in closed form, it is possible to construct measurable policies whose Bayes risk is arbitrarily close to $V^*(\cdot)$. One way to do so is using piece-wise constant policies. In fact, a bandit experiment with such a policy is equivalent to a batched bandit experiment, where the data is forced to be considered in batches. The results in this section thus give an upper bound on the welfare loss due to batching. In Appendix D, we argue that the optimal piece-wise constant policy for the general $K$-armed bandit problem can be feasibly computed even if $K$ is moderately large. Hence, batched bandits are also useful as a device for approximating the (fully) optimal policy in those settings.

Let $\Delta t$ denote a small time increment, and $T_{\Delta t} := \{t_1, \ldots, t_K\}$ a set of grid points for time, where $t_1 = 0$, $t_K = 1$ and $t_k - t_{k-1} = \Delta t$ for all $k$. The optimal piece-wise constant policy

$$
\pi^*_{\Delta t} : \mathcal{X} \times Q \times T_{\Delta t} \mapsto \{0, 1\}
$$

is allowed to change only at time points on the grid $T_{\Delta t}$. In particular, suppose that $x = x_k$ and $q = q_k$ at the grid point $t = t_k$. Then one computes $\pi^*_{\Delta t}(x_k, q_k, t_k) \in \{0, 1\}$ and holds this policy value fixed until the next time point $t_{k+1}$. Define $V^*_{\Delta t,k}(x, q)$ as the Bayes risk, in the diffusion regime, at state $(x, q, t_{K-k})$ under $\pi^*_{\Delta t}(\cdot)$. We then have the following recursion for $V^*_{\Delta t,k}(x, q)$:

$$
V^*_{\Delta t,k+1}(x, q) = \min \left\{ S_{\Delta t} \left[ V^*_{\Delta t,k} \right] (x, q), V^*_{\Delta t,k}(x, q) + \Delta t \cdot \mu^+(x, q) \right\}, \ k = 0, \ldots, K - 1, \ V^*_{\Delta t,0}(x, q) = 0,
$$

where the operator $S_{\Delta t}[\phi](x, q)$ denotes the solution at $(x, q, \Delta t)$ of the linear second order PDE

$$
-\partial_t f(s) + \mu^+(s) - \mu(s) + L[f](s) = 0, \text{ if } t > 0; \quad f = \phi, \text{ if } t = 0.
$$

The following theorem assures that $V^*_{\Delta t,k}(\cdot)$ can be made arbitrarily close to $V^*(\cdot, \cdot, t_{K-k})$ by letting $\Delta t \to 0$. 

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Theorem 3. (Jakobsen et al., 2019, Theorem 2.1) Suppose that $\mu^+(\cdot), \mu(\cdot)$ are Lipschitz continuous. Then there exists $C < \infty$ that depends only on the Lipschitz constants of $\mu^+(\cdot), \mu(\cdot)$ such that $0 \leq \max_k \left\{ V^*_{\Delta t,k}(\cdot) - V^*(\cdot, t_{K-k}) \right\} \leq C(\Delta t)^{1/4}$ uniformly over $\mathcal{X} \times \mathcal{Q}$.

To reiterate, $\pi^\star t(\cdot)$ is not required to converge to some measurable $\pi^\star(\cdot)$ as $\Delta t \to 0$. Still, we can employ $\pi^\star t(\cdot)$ in the fixed $n$ setting: to apply, one simply sets $t = \lfloor i/n \rfloor$, where $i$ is the current period. The following theorem asserts that employing $\pi^\star t(\cdot)$ in this manner results in a Bayes risk that is arbitrarily close to $V^*(0)$.

Theorem 4. Suppose $\mu^+(\cdot), \mu(\cdot)$ are Lipschitz continuous and $\sup_s \mu^+(s) < \infty$. Then, for any fixed $\Delta t$, $\lim_{n \to \infty} \left| V^*_{\pi^\star t,n}(0) - V^*(0) \right| \leq C(\Delta t)^{1/4}$.

4. General parametric models

We now characterize the minimal Bayes risk under general parametric settings. Let $\{P_\theta : \theta \in \mathbb{R}\}$ denote the set of candidate probability measures for the distribution of $Y$. We start by assuming $\theta$ to be scalar to reduce the notational overhead, but the vector case (discussed in Section 4.4) does not otherwise present any new conceptual difficulties. The mean rewards are given by $\mu(\theta) \equiv \mathbb{E}_{P_\theta}[X]$. As in Hirano and Porter (2009), we focus on local perturbations of the form $\{\theta_{n,h} \equiv \theta_0 + h/\sqrt{n} : h \in \mathbb{R}\}$, where $\theta_0$ satisfies $\mu(\theta_0) = 0$. This induces diffusion asymptotics. Indeed, under these perturbations, $\mu_n(h) := \mu(\theta_{n,h}) \approx \frac{\mu_0 h}{\sqrt{n}}$, where $\mu_0 := \mu'(0)$. Thus, the mean rewards are scaled by $1/\sqrt{n}$, just as in Section 2. If instead, $\mu(\theta_0) \neq 0$, the asymptotic risk is 0 under all the policies considered here, including Thompson sampling and the PDE based proposals here. Focusing on $\mu(\theta_0) = 0$ thus ensures that we are comparing policies under the hardest instances of the bandit problem where the asymptotic risk is non-trivial.

Let $\nu$ denote a dominating measure for $\{P_\theta : \theta \in \mathbb{R}\}$, and define $p_\theta = dP_\theta/d\nu$ (in the sequel, we shorten the Radon-Nikodym derivative $dP/d\nu$ to just $dP$). Also, let $P_{n,\theta}$ denote the joint probability measure for the sequence of rewards $Y_1, \ldots, Y_n$. We assume $\{P_\theta : \theta \in \mathbb{R}\}$ is differentiable in quadratic mean, i.e., there exists $\psi(\cdot) \in L^2(P_{\theta_0})$ - usually $\partial_\theta \ln p_\theta$ - such that

$$
\int \left[ \sqrt{p_{\theta_0 + h}} - \sqrt{p_{\theta_0}} - \frac{1}{2} h \psi \sqrt{p_{\theta_0}} \right]^2 d\nu = o(|h|^2).
$$

(4.1)
The information matrix is $I = \mathbb{E}_{\theta_0} [\psi^2]$ and we set $\sigma^2 = I^{-1}$.

For each $q \in [0, 1]$, define

$$x_{nq} := \frac{\sigma^2}{\sqrt{n}} \sum_{k=1}^{[nq]} \psi(Y_k) \overset{d}{\rightarrow} N(0, q\sigma^2),$$

as the (normalized) score process. In Appendix B, we show that (4.1) implies the sequentially locally asymptotically normal (SLAN) property: for any given $h \in \mathbb{R}$

$$\sum_{i=1}^{[nq]} \ln \frac{dp_{\theta_0 + h/\sqrt{n}}}{dp_{\theta_0}} = \frac{h}{\sigma^2} x_{nq} - \frac{q}{2\sigma^2} h^2 + o_{P_{n,\theta_0}}(1), \text{ uniformly over } q. \quad (4.2)$$

The SLAN property is crucial for the results on asymptotic reduction to the normal rewards setting. An abstract version of it appears as an assumption for analyzing sequential experiments in Le Cam (1986, Chapter 13). Combining it with Le Cam’s third lemma for Polish-spaced valued random variables (Le Cam, 1986, p. 90) gives an equation for the evolution of $x$:

$$x_{n} \quad \begin{array}{c} \overset{d}{\rightarrow} \end{array} \quad h \cdot + \sigma W(\bullet). \quad (4.3)$$

### 4.1. Heuristics.

We provide informal arguments illustrating how the posterior density of $h$ given all information until time $t$ can be approximately characterized using just two state variables: the number of times the arm has been pulled $q(t) := n^{-1} \sum_{j=1}^{[nt]} \mathbb{I}(A_j = 1)$, and the score process $x(t) := x_{nq(t)}$. This is useful for dimension reduction, and for obtaining a PDE characterization of minimal Bayes risk.

Let $\{\mathcal{F}_t\}_t$ denote the filtration generated by the set of all actions and rewards until time $t$. In particular, $\mathcal{F}_t$ is the sigma-algebra generated by $\xi_t \equiv \{\{A_j\}_{j=1}^{[nt]}, \{Y_i\}_{i=1}^{[nq(t)]}\}$. Suppose the agent places a prior $m_0$ over $h$. By Lemma 1, the posterior distribution, $p(\cdot | \mathcal{F}_t)$, of $h$ depends only on $y_{nq(t)} = \{Y_k\}_{k=1}^{[nq(t)]}$, so

$$p_n(h | \mathcal{F}_t) = p_n(h | y_{nq(t)}) \propto \prod_{i=1}^{[nq(t)]} p_{\theta_0 + h/\sqrt{n}}(Y_i) \cdot m_0(h). \quad (4.4)$$

The SLAN property (4.2) suggests that, uniformly over all possible $q$, the bracketed term in (4.4) can be approximated by the density of the ‘tilted’ measure $\Lambda_{nq,h}(y_{nq})$, defined as

$$d\Lambda_{nq,h}(y_{nq}) = \exp \left\{ \frac{1}{\sigma^2} hx_{nq} - \frac{q}{2\sigma^2} h^2 \right\} dP_{nq,\theta_0}(y_{nq}). \quad (4.5)$$

This result is not needed for the subsequent theory.
where $dP_{nq,0}(y_{nq}) := \prod_{i=1}^{[nq]} p_{0i}(Y_i)$. While $\Lambda_{nq,h}$ is not probability measure, it does integrate to 1 asymptotically. Let $\tilde{S}_{nq}$ be the measure corresponding to the density $d\tilde{S}_{nq}(y_{nq},h) = d\Lambda_{nq,h}(y_{nq}) \cdot m_0(h)$. From this joint density, one can define a conditional probability density of $h$ given $y_{nq}$, denoted by $\tilde{p}_n(h|y_{nq})$; formally this is defined via the disintegration $d\tilde{S}_{nq}(y_{nq},h) = \tilde{p}_n(h|y_{nq}) \cdot d\tilde{P}_n(y_{nq})$, where $\tilde{P}_n(y_{nq}) := \int \tilde{S}_{nq}(y_{nq},h) dh$. Such a conditional probability density always exists, see, e.g., Le Cam and Yang (2000, p. 136). It can be characterized as

$$\tilde{p}_n(h|y_{nq}) \equiv \tilde{p}_n(h|x_{nq},q) \propto \Lambda_{nq,h}(y_{nq}) \cdot m_0(h)$$

$$\propto \tilde{p}_q(x_{nq}) h \cdot m_0(h); \quad \tilde{p}_q(\cdot|h) \equiv \mathcal{N}(\cdot|qh,q\sigma^2) \quad (4.6)$$

The density $\tilde{p}_n(\cdot|y_{nq})$ is a close approximation to $p(\cdot|y_{nq})$; building on Le Cam and Yang (2000, Proposition 6.4.4), we show in Appendix B that the total variation distance between $p_n(\cdot|y_{nq})$ and $\tilde{p}_n(\cdot|y_{nq})$ converges to 0 uniformly over $q \in [0,1]$.

The approximation of $p_n(\cdot|y_{nq})$ by $\tilde{p}_n(\cdot|x_{nq},q)$ implies all relevant information about $h$ is contained in just $(x_{nq},q)$. It is thus asymptotically sufficient to restrict the state variables to $s = (x,q,t)$, and the policy class, $\Pi$, to the set of all functions that map $s$ to $[0,1]$ (time is still a state variable due to the boundary condition). Knowing additional state variables will not reduce asymptotic Bayes risk. Another benefit of the approximation is that $\tilde{p}_n(\cdot|x_{nq},q)$ can often be obtained with minimal computation. For instance, under Gaussian priors, (4.6) implies the density $\tilde{p}_n(\cdot|x_{nq},q)$ is also Gaussian with a similar form as in (2.3).

We now informally describe how the posterior approximation implies that the minimal Bayes risk is approximated by the solution to a second order PDE. We can expand the expression for regret payoffs in (2.2) as

$$R(Y,A,h) = \frac{\mu_n(h)}{\sqrt{n}} \left\{ \mathbb{I}(\mu_n(h) \geq 0) - A \right\} + \frac{\epsilon}{\sqrt{n}} \left\{ \mathbb{I}(\mu_n(h) \geq 0) - A \right\},$$

where $\epsilon := Y - \mu_n(h)$ is mean 0 conditional on $\mathcal{F}_t$. In a short time period $\Delta t$ following state $s$, the expected regret under some sequentially measurable (wrt $\mathcal{F}_t$) policy $\pi$ is approximately $\mathbb{E} [\Delta V_{\pi,h} | \mathcal{F}_t] \approx (n\mathbb{E} [R(Y,A,h)|\mathcal{F}_t]) \cdot \Delta t$. Since $\mu_n(h) \approx$

---

3By the iid realizations of the rewards, $\mathbb{E} [\epsilon | \mathcal{F}_t] = \mathbb{E} [\mathbb{E} [\epsilon | h] | \mathcal{F}_t] = 0$.

4Intuitively, $\mathbb{E} [R(Y,A,h)|\mathcal{F}_t]$ is the per period expected reward, and $n\Delta t$ periods elapse within a $\Delta t$ time interval.
\[ \dot{\mu}_0 h / \sqrt{n}, \]  the above suggests
\[
\mathbb{E} [\Delta V_{\pi, h} | \mathcal{F}_t] \approx \sqrt{n} \mathbb{E} \left[ (\mu_n(h)I(\mu_n(h) \geq 0) - \mu_n(h)\pi) | \mathcal{F}_t \right] \cdot \Delta t \\
\approx \left\{ \mathbb{E} [\max(\dot{\mu}_0 h, 0) | \mathcal{F}_t] - \mathbb{E} [\dot{\mu}_0 h\pi | \mathcal{F}_t] \right\} \cdot \Delta t 
\]  (4.7)

Setting \( s := (x, q, t) \) and \( \hat{\mathbb{E}}[\cdot | s] \) to be the expectation under \( \hat{p}_n(\cdot|y_{nq}) \equiv \hat{p}_n(\cdot|x_{nq}, q) \), the posterior approximation and (4.7) imply
\[
\mathbb{E} [\Delta V_{\pi, h} | \mathcal{F}_t] \approx \hat{\mathbb{E}} \left[ (\dot{\mu}_0 h(\hat{\mu}_0 h \geq 0) - \hat{\mu}_0 h\pi) | s_t \right] \cdot \Delta t = \left( \mu^+(s_t) - \mu(s_t)\pi \right) \cdot \Delta t 
\]  (4.8)

where \( \mu^+(s) := \dot{\mu}_0 \hat{\mathbb{E}} [hI(\hat{\mu}_0 h \geq 0) | s] \), \( \mu(s) := \hat{\mu}_0 h(s) \) and \( h(s) := \hat{\mathbb{E}} [h | s] \). These quantities can be computed using the posterior distribution (4.6). Given (4.3) and (4.8), one can derive a PDE characterization of the minimal Bayes risk \( V^*(\cdot) \) by a similar heuristic argument as in Section 2.2. Intuitively, (4.3) suggests that the evolution of the state variables \( q(t), x(t) := x_{nq(t)} \) is still governed by (2.1) with \( h \) replacing \( \mu \), so the same arguments apply. The resulting PDE is the same as (2.7), but the infinitesimal generator is now modified slightly to (the difference is that \( \partial_x f \) is multiplied by \( h(s) \) as opposed to \( \mu(s) = \dot{\mu}_0 h(s) \))
\[
L[f] := \partial_q f + h(s)\partial_x f + \frac{1}{2} \sigma^2 \partial_x^2 f. 
\]  (4.9)

4.2. Formal results. The argument above is heuristic. The formal statement is given below. Denote \( q_j = q_{t-|nj|}, x_j = x_{nq}, \mathcal{F}_j \equiv \mathcal{F}_{t-|nj|} \) and \( \pi_j = \pi(\xi_j) \). Let \( \Pi \) denote the class of all sequentially measurable (wrt \( \mathcal{F}_j \)) policies. For simplicity we suppose \( \Pi \) consists only of deterministic policies. This restriction is immaterial for minimal Bayes risk. In fact, apart from a bit of additional notation, it is straightforward to allow for randomized policies, as we argue later.

Given some deterministic policy, \( \pi(\cdot) \), the variables \( (\xi_j, \pi_j, x_j, q_j, A_j) \) are all deterministic functions of \( y_n \) (note that \( A_j = \pi_j \) for deterministic policies). Set
\[
R_n(h, \pi) := \sqrt{n} \mu_n(h) \{ I(\mu_n(h) \geq 0) - \pi \},
\]
and note that \( R_n(h, \pi_j) = \sqrt{n} \mathbb{E} [R(Y_j, A_j, h) | h] \), where \( R(Y, A, h) \) is the regret payoff (2.2). Also, take \( \mathbb{E}_{(y_n, n)}[\cdot] \) to be the expectation under the joint density \( dM_n(y_n, h) = \left\{ \prod_{i=1}^n p_{\theta_0+h/\sqrt{n}}(Y_i) \right\} \cdot m_0(h) \). Then, defining \( V_{\pi, n}(0) \) as the ex-ante expected Bayes
risk under some policy $\pi \in \Pi$ in the fixed $n$ setting, we have

$$V_{\pi,n}(0) = \mathbb{E}_{(y_n,h)} \left[ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} R(Y_j, A_j, h) \right] = \mathbb{E}_{(y_n,h)} \left[ \frac{1}{n} \sum_{j=1}^{n} R(h, \pi_j) \right],$$

where the second equality follows by the law of iterated expectations. The minimal ex-ante Bayes risk, $V^*_n(0)$, in the fixed $n$ setting is then

$$V^*_n(0) = \inf_{\pi \in \Pi} \mathbb{E}_{(y_n,h)} \left[ \frac{1}{n} \sum_{j=1}^{n} R_n(h, \pi_j) \right].$$

The main theoretical result of this section is that $V^*_n(0)$ converges to $V^*(0)$, the solution, evaluated at $s_0 = (0,0,0)$, of PDE (2.7) with the infinitesimal generator (4.9). This is shown under the following assumptions:

**Assumption 1.**

(i) The class $\{P_\theta\}$ is differentiable in quadratic mean as in (4.1).

(ii) $\mathbb{E}_{P_{\theta_0}}[\exp |\psi(Y)|] < \infty$.

(iii) There exists $\mu_0 < \infty$ such that $\sqrt{n}\mu_n(h) \equiv \sqrt{n}\mu(P_{\theta_0+h/\sqrt{n}}) = \mu_0 h + o(|h|^2)$.

(iv) The support of $m_0(\cdot)$ is a compact set $\{h : |h| \leq \Gamma\}$ for some $\Gamma < \infty$.

(v) $\mu(\cdot)$ and $\mu^+(\cdot)$ are Hölder continuous. Additionally, $\sup_s \varpi(s) \leq C < \infty$.

Assumptions 1(i), (iii) and (v) are standard. Assumption 1(ii) is restrictive, but is related to the choice of the tilted measure in (4.5). Instead of tilting the reference probability by $\exp\{z\}$ with $z = {1 \over \sigma^2} h x_n q - {q^2 \over \sigma^2} h^2$, one could instead tilt by $g(\cdot)$, say, where $g(z) = 1 + z + z^2/2 + o(z^3)$ and bounded for large $z$, e.g., $g(z) = \min\{2, \max\{1 + z + z^2/2, 0\}\}$. We do not take this route as it would complicate the proof, but in this scenario Assumption 1(ii) can be replaced with the weaker requirement $\mathbb{E}_{P_{\theta_0}}[|\psi(Y)|^3] < \infty$. Assumption 1(iv) requires the prior to have a compact support. This assumption is also employed in Le Cam and Yang (2000, Proposition 6.4.4). It suffices for deriving a lower bound on minimax risk as it was conjectured that the least favorable prior has a two point support. Still, it is possible to drop the assumption some under additional conditions, e.g., if the prior has finite $1 + \alpha$ moments, $\alpha > 0$, and Assumption 1(iii) is strengthened to $|\mu(P_{\theta_0+h})| \leq C|h|$ for all $h$. Assumptions 1(ii) and (iv) are therefore not the most general possible, but they lead to relatively transparent proofs.

The theorem below makes use of the following notation: $\Pi^S$ denotes the class of all sequentially measurable policies that are functions only of $s = (x,q,t)$, and
\( V_{n}^{\mathcal{S}^*}(0) := \min_{\pi \in \Pi^{\mathcal{S}}} \mathbb{E}_{(y_n, h)} \left[ \frac{1}{n} \sum_{j=1}^{n} R_n(h, \pi_j) \right] \). Also, \( \pi_{\Delta t}^* \) denotes the optimal piece-wise constant policy with \( \Delta t \) increments as described in Section 3.3.

**Theorem 5.** Suppose that Assumption 1 holds. Then:

(i) \( \lim_{n \to \infty} \left| V_{n}^*(0) - V_{n}^{\mathcal{S}^*}(0) \right| = 0 \).

(ii) \( \lim_{n \to \infty} V_{n}^*(0) = V^*(0) \), where \( V^*(\cdot) \) solves PDE (2.7) with the infinitesimal generator (4.9).

(iii) If, further, \( \mu(\cdot), \mu^+(\cdot) \) are Lipschitz continuous, \( \lim_{n \to \infty} | V_{n}^{\pi_{\Delta t}^*, n}(0) - V^*(0) | \lesssim \Delta t^{1/4} \) for any fixed \( \Delta t \).

Part (i) of Theorem 5 shows that it is sufficient to restrict attention to just three state variables \( s = (x, q, t) \). This is a substantial reduction in dimensionality. Part (ii) asserts that the minimal Bayes risk can be computed by solving PDE (2.7), while part (iii) implies piece-wise constant policies can attain this bound.

There is also the following result analogous to Corollary 1:

**Corollary 2.** Suppose Assumption 1 holds, and \( \pi(\cdot) \in \Pi^{\mathcal{S}} \) is Hölder continuous. Then, \( \lim_{n \to \infty} V_{\pi, n}(0) = V_{\pi}(0) \), where \( V_{\pi}(\cdot) \) is the viscosity solution of PDE (2.6).

The proof of this follows that of Theorem 5. We need to allow for randomized policies, but this only requires minor tweaks, see Appendix B for details.

4.3. Implications for Thompson sampling. The Thompson sampling rule is \( \pi_{ts}(y_{nq}) = P(h > 0|y_{nq}) \). By our posterior approximation results, see e.g., (A.10) in Appendix A, \( \sup_{y_{nq}} | \pi_{ts}(y_{nq}) - \tilde{\pi}_{ts}(x_{nq}, q) | \to 0 \), where \( \tilde{\pi}_{ts}(x, q) := \int_{h > 0} \tilde{p}_n(h|x, q) dh \) and \( \tilde{p}_n(h|x, q) \) is defined in (4.6). Under mild conditions, \( \tilde{\pi}_{ts}(x, q) \) is Lipschitz continuous, so Corollary 2 applies, and we can therefore characterize the Bayes risk of Thompson sampling using PDE (2.6).

The above discussion also highlights a practical use for our results in terms of computation. For general parametric models, computing the exact posterior for \( P(h > 0|y_{nq}) \) can be demanding if the likelihood does not admit a conjugate prior, but computation of \( \tilde{p}_n(h|x, q) \) and \( \tilde{\pi}_{ts}(x, q) \) is instantaneous, especially under a normal prior (in fact, \( \tilde{p}_n(h|x, q) \) is the Laplace approximation to the posterior). Since our results imply that the (asymptotic) Bayes risk of \( \tilde{\pi}_{ts}(\cdot) \) and \( \pi_{ts}(\cdot) \) is the same, it is possible to use the former in place of the latter. Note, however, that in either case, Thompson sampling does not attain minimal Bayes risk.
4.4. **Vector valued** $\theta$. The vector case can be analyzed in the same manner as the scalar setting, so we only describe the results. The score function $\psi(\cdot)$ is now a vector of the same dimension as $\theta$. Let $\Sigma = \mathbb{E}_{P_{\theta_0}}[\psi\psi^\top]^{-1}$ denote the inverse of the information matrix, and $x(t) = n^{-1/2} \sum_{t=1}^n \Sigma \psi(Y_t)$, the normalized score process. The asymptotically sufficient state variables are still $s = (x(t), q(t), t)$.

Suppose a prior $m_0(\cdot)$ is placed on the local parameter $h$. Then the approximate posterior density is

$$
\tilde{p}_n(h|\cdot) \propto \tilde{p}_q(x_n|\cdot),
$$

where $\tilde{p}_q(x_n|\cdot) = N(qh, q\Sigma)$. Based on this approximate posterior, we define

$$
\mu(s) = \tilde{E}[\mu_0| 0 < \mu_0^\top h | s], \quad h(s) = \tilde{E}[h| s],
$$

where $\tilde{E}[\cdot| s]$ is the expectation corresponding to $\tilde{p}_n(\cdot| x, q)$ and $\mu_0 := \nabla \mu(\theta_0)$. With these definitions, the evolution of the minimal Bayes risk is still characterized by PDE (2.7), but with the infinitesimal generator now being

$$
L[f] := \partial_q f + h(s)^\top D_x f + \frac{1}{2} \text{Tr} \left[ \Sigma \cdot D_x^2 f \right].
$$

4.5. **Lower bound on minimax risk.** Theorem 5 implies a lower bound on minimax risk. Let $V_{n,\pi}(0; h)$ denote the frequentist risk, i.e., the (finite $n$) ex-ante expected risk under a sequentially measurable policy $\pi$ when the local parameter is $h$. We also make the dependence of $V_n^*(0), V^*(0)$ on the priors $m_0$ explicit by writing them as $V_n^*(0; m_0), V^*(0; m_0)$. Clearly, $\inf_{\pi \in \Pi} \sup_{|h| \leq \Gamma} V_{n,\pi}(0; h) \geq V_n^*(0; m_0)$ for any prior $m_0$ supported on $|h| \leq \Gamma$. So, Theorem 5 implies

$$
\lim_{n \to \infty} \inf_{\pi \in \Pi} \sup_{|h| \leq \Gamma} V_{n,\pi}(0; h) \geq \sup_{m_0 \in P} V^*(0; m_0),
$$

(4.12)

where $P$ is the set of all compactly supported distributions. The left hand side of (4.12) is the value of minimax risk. The right hand side of (4.12) is the minimal (asymptotic) Bayes risk under the least-favorable prior. This can be computed using our PDE characterization once the least-favorable prior is determined numerically. The requirement of compact support is not binding since the least-favorable prior has a two point support. Proving equality in (4.12), i.e., the sharpness of the lower bound, is more involved and left for future research: one would need to show

$$
\inf_{\pi \in \Pi} \sup_{|h| \leq \Gamma} V_{n,\pi}(0; h) = \inf_{\pi \in \Pi} \sup_{m_0 \in P} V_{n,\pi}(0; m_0) = \sup_{m_0 \in P} \inf_{\pi \in \Pi} V_{n,\pi}(0; m_0).
$$

The first equality is immediate, but the second requires a minimax theorem.
5. The non-parametric setting

We often do not have any a-priori information about the distribution of the rewards. This is the non-parametric setting. Let $\mathcal{P}$ denote a candidate class of probability measures with bounded variance, and dominated by some measure $\nu$. Also, let $P_0 \in \mathcal{P}$ denote some reference probability distribution. Following Van der Vaart (2000), we consider smooth one-dimensional sub-models of the form $\{P_{t,h} : t \leq \eta\}$ for some $\eta > 0$, where $h(\cdot)$ is a measurable function satisfying

$$
\int \left[ \frac{(dP_{t,h}^{1/2} - dP_0^{1/2})}{t} - \frac{1}{2}hdP_0^{1/2} \right]^2 d\nu \to 0 \text{ as } t \to 0. \quad (5.1)
$$

In analogy with the parametric setting, we compute Bayes risk under the local (i.e., local to $P_0$) sequence of probability measures $P_{1/\sqrt{n},h}$, treating $h$ as a local parameter.

It is well known, see e.g Van der Vaart (2000), that (5.1) implies $\int h dP_0 = 0$ and $\int h^2 dP_0 < \infty$. The set of all such candidate $h$ is termed the tangent space $T(P_0)$. This is a subset of the Hilbert space $L^2(P_0)$, endowed with the inner product $\langle f, g \rangle = \mathbb{E}_{P_0}[fg]$ and norm $\|f\| = \mathbb{E}_{P_0}[f^2]^{1/2}$. As in Section 4, (5.1) implies the SLAN property that for all $h \in T(P_0)$,

$$
\sum_{i=1}^{[nq]} \ln \frac{dP_{1/\sqrt{n},h}(Y_i)}{dP_0} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nq]} h(Y_i) - \frac{q}{2} \|h\|^2 + o_P(1), \text{ uniformly over } q. \quad (5.2)
$$

The mean rewards under $P$ are given by $\mu(P) = \int xdP(x)$. To obtain non-trivial risk bounds, we suppose $\mu(P_0) = 0$. The rationale for this is similar to setting $\mu(\theta_0) = 0$ in the parametric setting: it focuses attention on the hardest instances of the bandit problem. Let $\psi(x) := x - \int xdP_0(x) = x$ and $\sigma^2 := \int x^2dP_0(x)$. Then, $\psi(\cdot)$ is the influence function corresponding to $\mu$, in the sense that under some mild assumptions on $\{P_{t,h}\}$,

$$
\frac{\mu(P_{t,h}) - \mu(P_0)}{t} - \langle \psi, h \rangle = \frac{\mu(P_{t,h})}{t} - \langle \psi, h \rangle = o(t). \quad (5.3)
$$

Equation (5.3) implies $\mu(P_{1/\sqrt{n},h}) \approx \langle \psi, h \rangle / \sqrt{n}$. Clearly, this is the right scaling for diffusion asymptotics.

It is possible to select $\{\phi_1, \phi_2, \ldots\} \in T(P_0)$ in such a manner that $\{\psi/\sigma, \phi_1, \phi_2, \ldots\}$ form an orthonormal basis for the closure of $T(P_0)$; here, the division by $\sigma$ in the first component ensures $\|\psi/\sigma\|^2 = \int x^2/\sigma^2dP_0(x) = 1$. We can also choose these bases.
so they lie in $T(P_0)$, i.e., $E_{P_0}[\phi_j] = 0$ for all $j$. By the Hilbert space isometry, each $h \in T(P_0)$ is then associated with an element from the $l_2$ space of square integrable sequences, $(h_0/\sigma, h_1, \ldots)$, where $h_0 = \langle \psi, h \rangle$ and $h_k = \langle \phi_k, h \rangle$ for all $k \neq 0$.

Following Hirano and Porter (2009), we define asymptotic Bayes risk in terms of priors on the tangent space $T(P_0)$, or equivalently, in terms of priors on $l_2$. Let $(\varphi(1), \varphi(2), \ldots)$ denote an arbitrary permutation of $(1, 2, \ldots)$. We restrict attention to those priors, $\rho_0$, that are supported on a finite dimensional sub-space, 

$$H_I \equiv \left\{ h \in T(P_0) : h = \langle \psi, h \rangle \frac{\psi}{\sigma} + \sum_{k=1}^{I-1} \langle \phi_{\varphi(k)}, h \rangle \phi_{\varphi(k)} \right\}$$

of $T(P_0)$, or isometrically, on a subset of $l_2$ of finite dimension $I$. Crucially, the first component of $h \in l_2$, corresponding to $h_0 = \langle \psi, h \rangle$, is always included in the support of the prior. The restriction to finite dimensional sub-spaces follows the tradition in statistics of defining minimax risk through finite dimensional sub-models, see e.g., Van der Vaart (2000, Theorem 25.21). We further suppose that the prior can be decomposed as $\rho_0 = m_0 \times \lambda$, where $m_0$ is a prior on $h_0$ and $\lambda$ is a prior on $(h_{\varphi(1)}, h_{\varphi(2)}, \ldots)$. Recall that $\mu_n(h) := \mu(P_1/\sqrt{n}, h) \approx h_0/\sqrt{n}$. Thus $m_0$ is effectively equivalent to a prior on the scaled rewards $\sqrt{n}\mu_n$, just as in Section 2.

We do not intend to suggest the placement of priors on tangent sets as a principled method for Bayesian inference. The priors are also too restrictive for this to be realistic. Instead, similar to Hirano and Porter (2009), we suggest the use of a prior $m_0$ on $h_0$ as a way to rank policies in terms of average risk (we show that $\lambda$, the other component of the prior, does not matter asymptotically). The results also suffice to provide a lower bound on minimax risk according to the definition in Van der Vaart (2000, Theorem 25.21).

5.1. On posteriors and their approximations. By construction, the prior $\rho_0$ is supported on a finite-dimensional subset of the tangent space of the form $\left\{ h^\top \chi(Y_i) : h \in \mathbb{R}^I \right\}$, where $\chi := (\psi/\sigma, \phi_{\varphi(1)}, \ldots, \phi_{\varphi(I-1)})$. In what follows, we drop the permutation $\varphi$ for simplicity. Consider the posterior density, $p_n(\cdot | \mathcal{F}_t)$, of the vector $h$ given $\mathcal{F}_t$, where the filtration $\mathcal{F}_t$ is defined as in Section 4. By Lemma 1,

$$p_n(\cdot | \mathcal{F}_t) = p_n(\cdot | Y_{nt}) \propto \left\{ \prod_{i=1}^{[nt]} P_{1/\sqrt{n}, h^\top \chi(Y_i)} \right\} : \rho_0(h).$$
Here, as before, \( q(t) = n^{-1} \sum_{j=1}^{[nt]} \mathbb{I}(A_j = 1) \). Equation (5.2) suggests that \( P_{nq,h}(y_{nq}) := \prod_{i=1}^{nq} P_{1/\sqrt{n}} \cdot \chi(Y_i) \) may be approximated by the ‘tilted’ measure, \( \Lambda_{nq,h}(\cdot) \),

\[
d\Lambda_{nq,h}(y_{nq}) = \exp \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nq]} h^T \chi(Y_i) - \frac{q}{2} \|h\|^2 \right\} dP_{1/\sqrt{n},0}(y_{nq}). \tag{5.4}
\]

Define \( \chi_{nq} := n^{-1/2} \sum_{i=1}^{[nq]} \chi(Y_i) \). Then, letting \( \tilde{p}_n(\cdot|y_{nq}) \) denote the corresponding posterior density as in Section 4, we have:

\[
\tilde{p}_n(h|y_{nq}) \propto \Lambda_{nq,h}(y_{nq}) \cdot \rho_0(h) \\
\propto \tilde{p}_q(\chi_{nq}|h) \cdot \rho_0(h); \text{ where } \tilde{p}_q(\cdot|h) \equiv \mathcal{N}(\cdot|qh, qI). \tag{5.5}
\]

The approximate posterior of \( h \) depends on the \( I \) dimensional quantity \( \chi_{nq} \). However, it is possible to achieve further dimension reduction for the marginal posterior density, \( \tilde{p}_n(h_0|y_{nq}) \), of \( h_0 \). For any \( h \in T(P_0) \),

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nq]} h(Y_i) - \frac{q}{2} \|h\|^2 = \frac{h_0}{\sigma/\sqrt{n}} \sum_{i=1}^{[nq]} Y_i - \frac{q}{2\sigma^2} h_0^2 + (\text{terms independent of } h_0)
\]

where the equality follows from the Hilbert space isometry which implies \( h = (h_0/\sigma) \psi + \sum_{k=1}^I h_k \phi_k \), and \( \|h\|^2 = (h_0/\sigma)^2 + \sum_{k=1}^I h_k^2 \). So, defining \( x_i = n^{-1/2} \sum_{k=1}^I Y_k \),

\[
\tilde{p}_n(h_0|y_{nq}) \propto \exp \left\{ \frac{h_0}{\sigma} x_{nq} - \frac{q}{2\sigma^2} h_0^2 \right\} \cdot m_0(h_0) \\
\propto \tilde{p}_q(x_{nq}|h_0) \cdot m_0(h_0), \text{ where } \tilde{p}_q(\cdot|h_0) \equiv \mathcal{N}(\cdot|qh_0, q\sigma^2). \tag{5.6}
\]

In other words, one can approximate the posterior distribution of \( h_0 \) under \( \mathcal{F}_t \) by \( \tilde{p}(h_0|x_{nq(t)}, q(t)) = \tilde{p}_n(h_0|y_{nq(t)}) \propto p_{q(t)}(x_{nq(t)}|h_0) \cdot m_0(h_0) \), just as in Section 4. Since the expected reward depends only on \( h_0 \) due to (5.3), this suggests that it is sufficient, at least asymptotically, to restrict the state variables to \( s_t := (x(t), q(t), t) \).

5.2. **Formal results.** Set \( \mathbb{E}[\cdot|s] \) to be the expectation under \( \tilde{p}_n(h_0|x, q), \mu^+(s) := \mathbb{E}[h_0|\{h_0 > 0\}|s] \) and \( \mu(s) := \mathbb{E}[h_0|s] \). Equation (5.6) implies these terms are exactly the same as in Section 2.2.2. Also, set \( h(\chi_{nq}, q) := \mathbb{E}[h|\chi_{nq}, q] \) where \( \mathbb{E}[\cdot|\chi_{nq}, q] \) is the expectation under the posterior distribution \( \tilde{p}_n(h|\chi_{nq}, q) \), defined in (5.5). Let \( V^*_n(0) \) denote the minimal ex-ante Bayes risk in a fixed \( n \) setting as in Section 4.2. We show that \( V^*_n(0) \) converges to \( V^*(0) \), where \( V^*(\cdot) \) is the solution to PDE (2.7) with \( \mu^+(\cdot), \mu(\cdot) \) defined as above. This is done under the following following assumptions:
Theorem 6. \( \rho \) of the posterior in (5.5) implies this should be satisfied under mild assumptions on (2.7), it arises in the course of various PDE approximations in the proof. The formonly in requiring continuity of 
expense of more intricate proofs. Finally, Assumption 2(v) differs from Assumption 
sup \( \in \mathcal{H}_T \) \( : \mathbb{E}_{P_0} [\exp |h|] \leq \Gamma \) for some \( \Gamma < \infty \).

\( (v) \) \( \mu(\cdot) \) and \( \mu^+(\cdot) \) are Hölder continuous and \( \sup_s \varpi(s) \leq C < \infty \). Furthermore, \( h(\chi, q) \) is also Hölder continuous.

Assumption 2(iii) is a stronger version of (5.3), but is satisfied for all commonly used sub-models. For instance, if \( dP_{1/\sqrt{n}, h} := (1 + n^{-1/2}h)dP_0 \) as in Van der Vaart (2000, Example 25.16), \( \sqrt{n}\mu(P_{1/\sqrt{n}, h}) = \langle \psi, h \rangle = h_0 \). Assumption 2(iv) requires the prior to be supported on score functions with finite exponential moments. As with Assumption 1(iv), it ensures the tilt \( d\Lambda_{nq, h}(y_{nq})/dP_{1/\sqrt{n}, 0}(y_{nq}) \) in (5.4) is uniformly bounded. It is somewhat restrictive as it implies \( \mathbb{E}_{P_0}[\exp |h_0 Y|] < \infty \) for all \( h_0 \in \text{supp}(m_0) \). However, similar to Assumption 1(iv), we suspect it can be relaxed at the expense of more intricate proofs. Finally, Assumption 2(v) differs from Assumption 1(v) only in requiring continuity of \( h(\chi, q) \). While \( h(\chi, q) \) is not present in PDE (2.7), it arises in the course of various PDE approximations in the proof. The form of the posterior in (5.5) implies this should be satisfied under mild assumptions on \( \rho_0 \). It is certainly satisfied for Gaussian \( \rho_0 \).

Theorem 6. Suppose Assumption 2 holds. Then:

(i) \( \lim_{n \to \infty} V_n^*(0) = V^*(0) \), where \( V^*(\cdot) \) is the solution to PDE (2.7).

(ii) If, further, \( \mu(\cdot), \mu^+(\cdot) \) are Lipschitz continuous, \( \lim_{n \to \infty} |V_{n, \Delta t, n}(0) - V^*(0)| \lesssim \Delta t^{1/4} \) for any fixed \( \Delta t \), where \( V_{n, \Delta t, n}(0) \) is defined as in Section 3.3.

As with parametric models, Theorem 6 implies a lower bound on minimax risk. Let \( V_{n, \pi}(0; h) \) denote the (fixed \( n \)) ex-ante expected risk of a sequentially measurable policy \( \pi \) under \( P_{1/\sqrt{n}, h} \). We also make the dependence of \( V_n^*(0), V^*(0) \) on the priors \( \rho_0, m_0 \) explicit by writing them as \( V_n^*(0; \rho_0), V^*(0; m_0) \). Clearly, \( \inf_{\pi \in \Pi} \sup_{h \in \mathcal{H}_t} V_{n, \pi}(0; h) \geq V_n^*(0; \rho_0) \) for any prior \( \rho_0 \). Let \( \mathcal{P} \) is the set of all \( m_0 \) consistent with Assumption 2(iv); if \( \mathbb{E}_{P_0}[\exp |Y|] < \infty \), \( \mathcal{P} \) can be the set of compactly supported priors. So, Theorem 6 implies

\[
\sup_{I} \lim_{n \to \infty} \inf_{\pi \in \Pi} \sup_{h \in \mathcal{H}_t} V_{n, \pi}(0; h) \geq \sup_{m_0 \in \mathcal{P}} V^*(0; m_0), \quad (5.7)
\]
where the first supremum on the left hand side is taken over all finite subsets $I$ of tangent space $T(P_0)$. By Van der Vaart (2000, Theorem 25.21), the left hand side of (5.7) is the value of minimax risk. The right hand side of (5.7) is the minimal (asymptotic) Bayes risk under the least-favorable prior. Because PDE (2.7) characterizes $V^*(0;m_0)$ even in the non-parametric setting, the least-favorable prior here is the same as that under Gaussian rewards. As in Section 4, proving equality in (5.7) is more involved and left for future research.

An analogous result to Corollary 2 also holds; we skip this for brevity.

6. Generalizing the one-armed bandit problem

This section considers various generalizations of the one-armed bandit problem. For brevity, we do not formally restate the theoretical results from Sections 3 - 5 for these generalizations. Appendix C describes these results and sketches how the proofs can be modified.

6.1. Multi-armed bandits. Suppose the bandit has $K$ arms. Assume that the rewards in arm $k$ are distributed as $N(\mu_{k,n},\sigma_k^2)$ where $\mu_{k,n} := \mu_k/\sqrt{n}$ is unknown, but $\sigma_k^2$ is known. By scaling the mean rewards for all arms, we ensure they all influence risk in the diffusion limit. In the Bayesian setting, it is without loss of generality to set one of $\{\mu_k; k = 1,\ldots,K\}$ to 0.

Let $A_i \in \{1,\ldots,K\}$ denote the action of the agent in period $i$, where $A_i = k$ implies the arm $k$ is pulled. The state space is $2K$ dimensional, with the relevant state variables being $s = (x_1,q_1\ldots x_K,q_K,t)$, where one of $q_1,\ldots,q_K$ may be dropped. Here, $q_k$ denotes the fraction of times arm $k$ was pulled, $q_k(t) = \frac{1}{n-1}\sum_{j=1}^{\lfloor nt \rfloor} \mathbb{I}(A_j = k)$, and $x_k$ denotes the scaled cumulative rewards from that arm $k$: $x_k(t) = \frac{1}{n^{1/2}}\sum_{j=1}^{\lfloor nt \rfloor} \mathbb{I}(A_j = k)Y_j$. The policy rule, $\pi(\cdot) : \mathcal{S} \to [0,1]^K$, provides the probabilities of pulling the arms $1,\ldots,K$ given the current state $s$. Let $\pi_k(\cdot)$ denote the probability of pulling arm $k$ under policy $\pi(\cdot)$. By similar heuristics as in Section 2, the evolution of $q_k, x_k$ under the policy $\pi(\cdot)$ is given by the SDEs

\[
\begin{align*}
\,dq_k(t) &= \pi_k(s_t)\,dt; \\
\,dx_k(t) &= \pi_k(s_t)\mu_k\,dt + \sigma_k\sqrt{\pi_k(s_t)}\,dW_k(t),
\end{align*}
\]

where $\{W_1(t),\ldots,W_K(t)\}$ are $K$ independent Brownian motions.
The agent places a joint prior, $m_0$, on $\mu := (\mu_1, \ldots, \mu_K)$. Let $Y := (Y_1, \ldots, Y_K)$ denote the realizations of rewards in each arm. The expression for regret is

$$R(Y, A, \mu) = \frac{Y_{k^*}}{\sqrt{n}} - \sum_k \frac{Y_k}{\sqrt{n}} \mathbb{1}(A = k),$$

where $k^* = \arg \max_k \mu_k$. Let $\mathbb{E}[\cdot|s]$ denote the expectation under the posterior distribution of $\mu$ given $s$, and define $\mu(s) = (\mu_1(s), \ldots, \mu_K(s))$, where $\mu_k(s) = \mathbb{E}[\mu_k|s]$. Also, let $\mu^{\text{max}}(s) = \mathbb{E}[\max_k \mu_k|s]$. As in Section 2, it is straightforward to derive the posterior distribution of $\mu(\cdot)$, along with closed form expressions for $\mu(\cdot), \mu^{\text{max}}(\cdot)$ under Gaussian priors, though we do not show it here for brevity.

For each $k$, denote by $L_k[\cdot]$ the infinitesimal generator

$$L_k[f](s) := \partial_{q_k} f + \mu_k(s) \partial_{x_k} f + \frac{1}{2} \sigma_k^2 \partial_{x_k}^2 f.$$

Because one of $q_1, \ldots, q_K$ is redundant, we may take, e.g., $\partial_{q_1} f = 0$. In what follows, we assume this implicitly, and also that $s$ does not include $q_1$. The minimal risk function, $V^*(\cdot)$, can be obtained as the solution to the PDE

$$\partial_t V^* + \mu^{\text{max}}(s) + \min_k \{-\mu_k(s) + L_k[V^*](s)\} = 0 \text{ if } t < 1;$$

$$V^* = 0 \text{ if } t = 1. \quad (6.2)$$

The assumption of Gaussian rewards is not necessary. Let $P_\theta := P^{(1)}_\theta \times \cdots \times P^{(K)}_\theta$ denote the probability distribution over the vector of outcomes in $K$ arms. It is without loss of generality to assume the distributions across arms are independent of each other (as we only observe the outcome from a single arm in any period). The parameter $\theta$ may have some components that are shared across all the arms. Suppose the agent place a prior $m_0$ on the local parameter $h$ indexing $\theta_h = \theta_0 + h/\sqrt{n}$, where $\theta_0$ is such that $\mathbb{E}_{P_{\theta_0}^{(k)}}[Y_k] = 0$ for all $k$. Then, we can apply the techniques from Sections 4 and 5 to show the following: (1) The (asymptotically) sufficient state variables are $(x_1, q_1, \ldots, x_K, q_K, t)$, where $x_k(t) := \sigma_k^2 n^{-1/2} \sum_{i=1}^{[nq_k(t)]} \psi_k(Y_{ik})$, $\psi_k(\cdot)$ is the score function corresponding to $P_{\theta_0}^{(k)}$ and $\sigma_k^2 = \left(\mathbb{E}_{P_{\theta_0}^{(k)}}[\psi_k^2]\right)^{-1}$. (2) The posterior is closely approximated by $p(h|s) \propto \prod_k \tilde{p}_{\theta_h}(x_k|h) \times m_0(h)$, where $\tilde{p}_{\theta_h}(\cdot|h) \equiv \mathcal{N}(q_k h, q_k \sigma_k^2)$. (3) The same PDE characterization (6.2) holds under parametric and non-parametric distributions of rewards. See Appendix C for further details. We conjecture that the least favorable prior has a $K$ point support over $\mu$. 

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6.2. **Best arm identification.** Best arm identification describes a class of sequential experiments in which the agent is allowed to experiment among $K$ arms of a bandit until a set time $t = 1$ (corresponding to $n$ time periods). At the end of the experimentation phase, an arm is selected for final implementation. Statistical loss is determined by expected payoffs during the implementation phase, but not on payoffs generated during experimentation, i.e., there is no exploitation motive. In the Gaussian setting, it is sufficient to use the same state variables $s = (x_1, q_1, \ldots x_K, q_K, t)$ as in $K$ armed bandits, and drop one of $q_1, \ldots, q_K$.

Let $\mu := (\mu_1, \ldots, \mu_K)$ denote the mean rewards of each arm, and $\pi(I) \in \{1, \ldots, K\}$ the action of the agent in the implementation phase. Following the best arm identification literature, see, e.g., Kasy and Sautmann (2019), we take the loss function to be expected regret in the implementation phase (also known as “simple regret”)

$$L(\pi(I), \mu) = \max_k \mu_k - \sum_k \mu_k \mathbb{I}(\pi(I) = k).$$

Suppose that the state variable at the end of experimentation is $s$. The Bayes risk of policy $\pi(I)$ given $s$ is

$$V_{\pi(I)}(s) = \mathbb{E} \left[ L(\pi(I), \mu) \mid s \right] = \mu^{\max}(s) - \sum_k \mu_k(s) \mathbb{I}(\pi(I) = k).$$

Hence the optimal Bayes policy is $\pi(I) = \arg \max_k \mu_k(s)$ and the minimal Bayes risk at the end of the experimentation phase is $V^*(s) = \mu^{\max}(s) - \max_k \mu_k(s)$ when $t = 1$. This determines the boundary condition at time $t = 1$.

For ‘interior states’ with $t < 1$, the evolution of the state variables in the diffusion limit is still (heuristically) determined by (6.1). Hence, the recursion

$$V^*(s) = \inf_{\pi \in [0,1]} \mathbb{E} \left[ V^*(x + \Delta x, q + \Delta q, t + \Delta t) \mid s \right]$$

must hold for any small time increment $\Delta t$, where $\Delta x$ and $\Delta q$ are the same as in Section 2.2. Thus, by similar heuristics as in that section, $V^*(\cdot)$ satisfies

$$\partial_t V^* + \min_k L_k[V^*](s) = 0 \text{ if } t < 1; \quad V^*(s) = \varpi(s) \text{ if } t = 1,$$

where $\varpi(s) := \mu^{\max}(s) - \max_k \mu_k(s)$. We require $\varpi(s)$ to be Hölder continuous and $\sup_s \varpi(s) < \infty$. All previous theoretical results continue to apply with minor modifications to the statements and the proofs; for details, see Appendix C.
6.2.1. Minimax optimal policy for two arms. When there are only two arms, it is shown in Adusumilli (2022a) that the minimax optimal policy assigns a fixed proportion, $\gamma^* = \sigma_1/(\sigma_1 + \sigma_2)$, of units to arm 1 and the rest to arm 2. This policy is independent of the data, so there is no adaptation to previous outcomes. After the experimentation phase, arm 1 is chosen for implementation if and only if $\sigma_1^{-1}x_1(1) - \sigma_2^{-1}x_2(1) \geq 0$. The corresponding least favorable prior has a two point support at the points $(\sigma_1 \Delta^*, -\sigma_0 \Delta^*), (\sigma_1 \Delta^*, \sigma_0 \Delta^*)$, where $\Delta^* = \arg\max_\delta \delta \Phi(-\delta) \approx 0.756$.

6.3. Discounting. Our methods also apply to bandit problems without a definite end point under discounted rewards. Suppose the rewards in successive periods are discounted by $e^{-\beta/n}$ for some $\beta > 0$. Here, $n$ is to be interpreted as a scaling of the discount factor. It is the number of periods of experimentation in unit time when the agent experiments in regular time increments and intends to discount rewards by the fraction $e^{-\beta}$ after $\Delta t = 1$. Discounting ensures the cumulative regret is finite.

In the Gaussian setting with one arm, the relevant state variables are $s := (x, q)$, where $x, q$ are defined in the same manner as before, but $q$ can now take values above 1 (it is the number of times the arm is pulled divided by $n$).

Note that discounting changes the considerations of the agent, who will now be impatient to start ‘exploitation’ sooner as future rewards are discounted. Popular bandit algorithms such as Thompson sampling do not naturally admit discounting and will therefore be substantially sub-optimal.

The counterpart of PDE (2.7) for discounted rewards is

$$\beta V^* - \mu^+(s) - \min \{-\mu(s) + L[V^*](s), 0\} = 0.$$  \hspace{1cm} (6.5)

Note that PDE (6.5) does not need a boundary condition. All the previous theoretical results continue to apply to discounted bandits (see Appendix C).

7. Computation and simulations

7.1. Numerical solutions using finite difference methods. PDE (2.7) can be solved efficiently using “upwind” finite-difference (FD) schemes. Appendix D describes the computational procedure in detail. We remark here on a few properties: (1) The solution can be computed very efficiently using sparse matrix routines, e.g., for our results below on one-armed bandits, the solutions take 5-10 minutes. (2) The
Note: The default parameter values are $\mu_0 = 0$, $\nu = 50$ and $\sigma = 5$.

**Figure 7.1.** Risk of Thompson sampling vs minimal Bayes risk

batched bandit setting (Section 3.3) suggests a feasible approach for approximating the optimal policy under moderately many arms. The optimal batched policy is solved using backward in time; the computational advantage is that in going from each grid point $t_k$ to $t_{k-1}$, PDE (6.2) can be de-coupled into $K$ linear PDEs that can be solved efficiently using FD methods or by Monte-Carlo using the Feynman-Kac formula. The computational cost of this approach scales linearly with the number of arms, alleviating the curse of dimensionality. (3) The discounted bandit problem can also be solved much faster than the one with a fixed number of periods.

### 7.2. Optimal Bayes policy and Thompson sampling under normal priors.

The Thompson Sampling rule is $\pi_{ts}(s) = P(\mu \geq 0|s)$, where $P(\cdot|s)$ denotes the posterior probability of $\mu$. Its asymptotic Bayes risk can be computed by solving PDE (2.6). Figure 7.1 compares the Bayes risk of Thompson sampling with the minimal Bayes risk from solving PDE (2.7), for different values of the prior variance $\nu^2$, and $\sigma^2$. In all cases, the prior is a normal with mean 0. Bayes risk decreases with prior variance $\nu^2$ but increases with $\sigma^2$. Thompson sampling is inferior to the optimal Bayes policy across all parameter values and substantially so; its Bayes risk is typically twice as high.

Figure 7.2 plots the optimal policy rule as a function of $x, q$ at various points in time for the parameter values ($\mu_0 = 0$, $\nu^2 = 1600$, $\sigma^2 = 25$). Clearly, the policy rule satisfies the properties identified in Section 2.2.2; it is of the form $\pi^*(s) = \mathbb{I}\{x \geq f(q, t)\}$, where $f(\cdot)$ is increasing in $t$. For comparison, we also plot a heatmap of the
policy rule, $\pi_{ts}(\cdot)$, under Thompson sampling. Note that $\pi_{ts}(\cdot)$ does not vary with time. Compared to the optimal rule, Thompson sampling is inferior in two regards: First, with a sizable probability, it can fail to pull the arm in regions, e.g., $x \geq 0$, where pulling the arm is always optimal (and similarly it sometimes pulls the arms in regions where it is not optimal to do so). Second, it does not change with time. Together, these properties explain the inferior performance of Thompson sampling.

7.3. **Simulations.** The risk comparisons in the previous section were based on analytical results. To assess their validity in finite samples, we ran Monte Carlo simulations with a normal prior $\mu \sim N(0, 50^2)$ and $\sigma = 5$, for various values of $n$ (the number of Monte-Carlo draws was 5000). Figure 7.3 displays the results, along with the interquartile range for the realizations of cumulative regret. The analytical expressions provide a very good approximation to finite sample Bayes risk, even for relatively small $n$. The optimal Bayes policy retains its dominance over Thompson sampling even in small samples. The plots also suggest that the variability of regret under the optimal Bayes policy is much smaller than under Thompson sampling.
In Appendix E, we compare the optimal Bayes policy against the UCB policy (Lai and Robbins, 1985). The UCB policy is sensitive to tuning parameters, but even under the best tuning parameter (which is infeasible) its risk is 35-85% higher, the gap being greater for larger $n$.

7.4. **Minimax risk.** We compute the least favorable prior, $m_0^*$, by solving the two player game (2.8) numerically. The procedure is described in Appendix D. Recall that we need only solve for $m_0^*$ under $\sigma = 1$; the least favorable prior more generally involves multiplying the support points of $m_0^*$ by $\sigma$. As conjectured, $m_0^*$ has a two point support at $\bar{\mu} = -2.5$ and $\bar{\mu} = 1.7$ with $m_0^*(\bar{\mu}) = 0.415$. Figure 7.4, Panel A plots the frequentist risk profile of the corresponding minimax optimal policy. Consistent with the two point support, the risk profile has two peaks. The value of minimax risk is 0.373. For comparison, the risk of Thompson sampling under the least favorable prior is 0.551, about 48% higher. Panel B of Figure 7.4 plots the finite sample performance of the minimax policy (under the least favorable prior). The theoretical estimate of minimax risk is very close to the finite sample ones.

8. **Conclusion**

Bandit algorithms have been studied for a long time, but there are relatively few results describing optimal policy rules under the commonly used notions of Bayes and minimax risk. Working under diffusion asymptotics, we obtain lower bounds for these risk measures, and suggest ways to numerically compute the corresponding...
optimal policies. We also show that all bandit problems, whether parametric or non-parametric, are asymptotically equivalent to Gaussian bandits. Furthermore, it is sufficient (asymptotically) to restrict attention to just two state variables per arm. For minimax risk, the paper only proves a lower bound. We believe the bound is tight, but a formal proof will require proving a minimax theorem.

As with any asymptotic analysis, there are some shortcomings: e.g., replacing unknown variances with consistent estimates has no effect. One could achieve optimal risk by sampling all arms equally for \( \bar{\nu} := \frac{n}{\rho} \) periods, \( \rho \in (0, 1) \), obtaining estimates of \( \sigma \), and applying the optimal policies based on those estimates from \( \bar{\nu} \) onwards. But in practice, the choice of \( \rho \) will matter.

The techniques introduced here are applicable to other sequential experiments. Adusumilli (2022a) applies insights from this paper to derive the minimax optimal policy for best-arm identification with two arms, while Adusumilli (2022b) does the same for the problem of costly sampling.

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Appendix A. Proofs

Lemma 1. Let \( p(Y|h) \) denote the likelihood of the outcome \( Y \) given some parameter \( h \) with prior distribution \( m_0(h) \). The posterior distribution of \( h \) satisfies

\[
p_n(h|\xi_t) \propto \left\{ \prod_{i=1}^{\lfloor nq(t) \rfloor} p(Y_i|h) \right\} \cdot m_0(h).
\]

In particular, the posterior distribution is independent of the past values of actions.

Proof. Let \( \mathcal{F}_t \) be the sigma-algebra generated by the set of random variables, \( \xi_t \equiv \{\{A_j\}_{j=1}^{nt}, \{Y_i\}_{i=1}^{\lfloor nq(t) \rfloor}\} \); here, and in what follows, \( j = a \) refers to the \( a \)-th time period while \( i = a \) refers to the \( a \)-th pull of the arm. Denote by \( p_n(\cdot|\xi_t) \equiv p_n(\cdot|\mathcal{F}_t) \) the posterior distribution of some parameter \( h \) given \( \xi_t \) (or equivalently \( \mathcal{F}_t \)).

The claim is shown using induction. Clearly the claim is true for \( t = 1 \). For any \( t > 1 \), we can think of \( p_n(h|\xi_{t-1}) \) as the revised prior for \( \mu \). Suppose that \( A_t = 1 \). Then \( nq(t) = nq(t-1) + 1 \), and

\[
p_n(h|\xi_t) \propto p(Y_t, A_t = 1|\xi_t, h) \cdot p_n(h|\xi_{t-1})
\]

\[
\propto \pi(A_t = 1|\xi_{t-1}) \cdot p(Y_t|h) \cdot p_n(h|\xi_{t-1})
\]

\[
\propto p(Y_t|h) \cdot p_n(h|\xi_{t-1}) = \left\{ \prod_{i=1}^{\lfloor nq(t) \rfloor} p(Y_i|h) \right\} \cdot m_0(h).
\]

Alternatively, suppose \( A_t = 0 \). Then \( nq(t) = nq(t-1) \), and

\[
p_n(h|\xi_t) \propto p(A_t = 0|\xi_t, h) \cdot p_n(h|\xi_{t-1})
\]

\[
\propto \pi(A_t = 0|\xi_{t-1}) \cdot p_n(h|\xi_{t-1})
\]

\[
\propto p_n(h|\xi_{t-1}) = \left\{ \prod_{i=1}^{\lfloor nq(t) \rfloor} p(Y_i|h) \right\} \cdot m_0(h).
\]

Thus the induction step holds under both possibilities, and the claim follows. \( \square \)

A.1. Proof of Theorem 2. For the proof of this theorem, we make the time change \( \tau := 1 - t \). Let \( s := (x, q, \tau), \mathbb{I}_n \equiv \{\tau < 1/n\} \) and denote the domain of \( s \) by \( \mathcal{S} \). Also, let \( C^\infty(\mathcal{S}) \) denote the set of all infinitely differentiable functions \( \phi : \mathcal{S} \to \mathbb{R} \) such that \( \sup_{q \geq 0} |D^q \phi| \leq M \) for some \( M < \infty \) (these are also known as test functions).
We can represent the solution, $V^*_n(\cdot)$, to (3.2) as a solution to the approximation scheme

$$S_n(s, \phi(s), [\phi]) = 0 \text{ for } \tau > 0; \quad \phi(x, q, 0) = 0,$$

(A.2)

where for any $u \in \mathbb{R}$ and $\phi_2 : \mathcal{S} \to \mathbb{R},$

$$S_n(s, u, [\phi_2]) := -\min \left\{ \frac{\mu^+(s) - \pi \mu(s)}{n} \right\} + \mathbb{E} \left[ \mathbb{I}_n \cdot \phi_2 \left( x + \frac{A \pi Y_{nq+1}}{\sqrt{n}}, q + \frac{A \pi}{n} \tau - \frac{1}{n} \right) - u \bigg| s \right\}.$$

Here, $[\phi_2]$ refers to the fact that it is a functional argument. Define

$$F(D^2 \phi, D\phi, s) = \partial_{\tau} \phi - \mu^+(s) - \min \{ -\mu(s) + L[\phi](s), 0 \}.$$

Barles and Souganidis (1991) show that the solution, $V^*_n(\cdot)$, of (A.2) converges to the solution, $V^*(\cdot)$, of $F(D^2 \phi, D\phi, s) = 0$ with the boundary condition $\phi(x, q, 0) = 0$ if the scheme $S_n(\cdot)$ satisfies the properties of monotonicity, stability and consistency.

Monotonicity requires $S_n(s, u, [\phi_1]) \leq S_n(s, u, [\phi_2])$ for all $s \in \mathcal{S}$, $u \in \mathbb{R}$ and $\phi_1 \geq \phi_2$. This is clearly satisfied.

Stability requires (A.2) to have a unique solution, $V^*_n(\cdot)$, that is uniformly bounded. That a unique solution exists follows from backward induction. To obtain an upper bound, note that following a state $s$, the agent may choose to pull the arm in all subsequent periods. This results in a risk of $\tau (\mu^+(s) - \mu(s))$. Alternatively, the agent may choose not to pull the arm in all subsequent periods, in which case the resulting risk is $\tau \mu^+(s)$. Hence, by definition of $V^*_n(\cdot)$ as the risk under an optimal policy,

$$0 \leq V^*_n(s) \leq \tau \min \{ \mu^+(s) - \mu(s), \mu^+(s) \} \leq C \tau.$$

(A.3)

Finally, consistency requires that for all $\phi \in C^\infty(\mathcal{S})$, and $s \equiv (x, q, \tau) \in \mathcal{S}$ such that $\tau > 0$,

$$\limsup_{n \to \infty} n S_n(z, \phi(z) + \rho, [\phi + \rho]) \leq F(D^2 \phi(s), D\phi(s), s), \quad \text{and}$$

$$\liminf_{n \to \infty} n S_n(z, \phi(z) + \rho, [\phi + \rho]) \geq F(D^2 \phi(s), D\phi(s), s).$$

(A.4)

(A.5)
It suffices to restrict attention to \( \tau > 0 \) because (A.3) implies that for any \( s \) on the boundary, i.e., of the form \((x,q,0)\),

\[
\limsup_{n \to \infty} V_n^*(z) = 0 = \liminf_{n \to \infty} V_n^*(z).
\]

When the above holds, an analysis of the proof of Barles and Souganidis (1991) shows that we only need prove (A.4) and (A.5) for interior values of \( s \), i.e., when \( \tau > 0 \).

We now show (A.4). The argument for (A.5) is similar. Since any \( z \equiv (x',q',\tau') \) converging to \( s \equiv (x,q,\tau) \) with \( \tau > 0 \) will eventually satisfy \( \tau' > 1/n \), we can drop \( \mathbb{I}_n \) in the definition of \( S_n(\cdot) \) while taking the lim sup operation in (A.4). Now, for any \( s \in \mathcal{S} \), a third order Taylor expansion gives

\[
n \mathbb{E} \left[ \phi \left( x + \frac{\mathbb{I}(A_\pi = 1)Y_{nq+1} + \mathbb{I}(A_\pi = 1)}{n}, \frac{\mathbb{I}(A_\pi = 1)}{n}, \frac{\pi - 1}{n} \right) - \phi(s) \right] s
\]

\[
= \mathbb{E} \left[ \sqrt{n} \mathbb{I}(A_\pi = 1)Y_{nq+1} \right] \partial_x \phi + \frac{1}{2} \mathbb{E} \left[ \mathbb{I}(A_\pi = 1)Y_{nq+1}^2 \right] \partial_x^2 \phi
\]

\[
+ \mathbb{E} \left[ \mathbb{I}(A_\pi = 1) \right] \partial_y \phi - \partial_x \phi + \frac{R(s)}{\sqrt{n}}
\]

where \( R(s) \) is a continuous function of \( \mu(s), \sigma(s) \) and \( \mathbb{E} \left[ |Y_{nq+1}|^3 | s \right] \) that is bounded as long as these three functions are also bounded. Because \( A_\pi \sim \text{Bernoulli}(\pi) \) independent of \( s \) (given \( \pi \), we have \( \mathbb{E} \left[ \sqrt{n} \mathbb{I}(A_\pi = 1)Y_{nq+1} \right] = \pi \mu(s), \mathbb{E} \left[ \mathbb{I}(A_\pi = 1)Y_{nq+1}^2 \right] = \pi(\sigma^2 + n^{-1} \mathbb{E}[\mu^2|s]) \) and \( \mathbb{E} \left[ \mathbb{I}(A_\pi = 1) \right] = \pi \). Furthermore, recalling that \( Y|\mu \sim \mathcal{N}(\mu, \sigma^2) \), the properties of the Gaussian distribution imply

\[
\mathbb{E} \left[ |Y_{nq+1}|^3 | s \right] = \mathbb{E} \left[ \mathbb{E} \left[ |Y_{nq+1}|^3 | \mu \right] | s \right] \leq \mathbb{E} \left[ |\mu|^3 | s \right] < \infty
\]

under the stated assumptions. We thus obtain

\[
n S_n(z, \phi(z) + \rho, [\phi + \rho])
\]

\[
\leq - \min_{\pi \in [0,1]} \left\{ \left( \mu^+(z) - \pi \mu(z) \right) + \pi L[\phi](z) - \partial_x \phi(z) + \frac{R(z)}{\sqrt{n}} + \frac{\mathbb{E}[\mu^2|z]}{n} \right\}
\]

\[
\leq - \min_{\pi \in [0,1]} \left\{ \left( \mu^+(z) - \pi \mu(z) \right) + \pi L[\phi](z) - \partial_x \phi(z), 0 \right\} + \frac{|R(z)|}{\sqrt{n}} + \frac{M \mathbb{E}[\mu^2|z]}{n}
\]

\[
= \partial_x \phi(z) - \mu^+(z) - \min \left\{ -\mu(z) + L[\phi](z), 0 \right\} + \frac{|R(z)|}{\sqrt{n}} + \frac{M \mathbb{E}[\mu^2|z]}{n}.
\]
Because \( \limsup_{z \to s} \{|R(z)| + E[\mu^2 |z]\} < \infty \), \( \phi \in C^\infty(S) \) and \( \mu^+(\cdot), \mu(\cdot) \) are continuous functions,

\[
\limsup_{n \to \infty} \limsup_{\rho \to 0} \limsup_{z \to s} nS_n(z, \phi(z) + \rho, [\phi + \rho]) \\
\leq \limsup_{z \to s} \partial_t \phi(z) - \mu^+(z) - \min \{-\mu_k(z) + L[\phi(z)], 0\} \\
= F(D^2 \phi(s), D\phi(s), s).
\]

This completes the proof of consistency.

A.2. **Proof of Theorem 4.** For this proof, we use \( |f| \) to represent the sup norm of \( f \). Let \( V^*_{\Delta t, n, k}(x, q) \) denote the Bayes risk in the fixed \( n \) setting at state \((x, q, t_{K-k})\) under \( \pi^*_\Delta(\cdot) \). Then \( V^*_{\Delta t, n, k}(x, q) = 0 \), and \( V^*_{\Delta t, n, k+1}(\cdot) \) satisfies

\[
V^*_{\Delta t, n, k+1}(x, q) = \tilde{\Gamma}_{\Delta t} \left[ V^*_{\Delta t, n, k} \right] (x, q), \; k = 0, \ldots, K - 1. \tag{A.6}
\]

Here,

\[
\tilde{\Gamma}_{\Delta t} [\phi](x, q) := \min \left\{ \tilde{S}_{\Delta t} [\phi] (x, q), \phi(x, q) + \Delta t \cdot \mu^+(x, q) \right\}
\]

and \( \tilde{S}_{\Delta t} [\phi] (x, q) \) denotes the solution at \((x, q, \Delta t)\) of the recursive equation

\[
f(x, q, \tau) = E \left[ \frac{\mu^+(x, q) - \mu(x, q)}{n} + f \left( x + \frac{Y}{\sqrt{n}}, q + \frac{1}{n}, \tau - \frac{1}{n} \right) \right]; \; \tau > 0
\]

\[
f(x, q, 0) = \phi(x, q). \tag{A.7}
\]

In other words, \( \tilde{S}_{\Delta t} [\phi] (x, q) \) is the discrete time counterpart of the operator \( S_{\Delta t} [\cdot] \) defined in Section 3.3.

For any \( k > 0 \), it can be seen from the recursive definitions of \( V^*_{\Delta t, n, k} \) and \( V^*_{\Delta t, k} \),

\[
|V^*_{\Delta t, n, k+1} - V^*_{\Delta t, k+1}| \leq |\tilde{S}_{\Delta t} [V^*_{\Delta t, k+1}] - S_{\Delta t} [V^*_{\Delta t, k+1}]| + |\tilde{\Gamma}_{\Delta t} [V^*_{\Delta t, n, k}] - \tilde{\Gamma}_{\Delta t} [V^*_{\Delta t, k}]|.
\]

Recall that \( \tilde{S}_{\Delta t} [\phi] \) denotes the solution to (A.7), while \( S_{\Delta t} [\phi] \) denotes the solution to (3.6), when the initial condition in both cases is \( \phi \). Hence, by the use of Barles and Jakobsen (2007, Theorem 3.1) as described in Appendix B, \(|\tilde{S}_{\Delta t} [V^*_{\Delta t, k+1}] - S_{\Delta t} [V^*_{\Delta t, k+1}]| \lesssim n^{-1/14}\). Additionally, it is straightforward to verify that \(|\tilde{\Gamma}_{\Delta t} [\phi_1] - \tilde{\Gamma}_{\Delta t} [\phi_2]| \leq |\phi_1 - \phi_2| \) for all \( \phi_1, \phi_2 \). Together, these results imply

\[
|V^*_{\Delta t, n, k+1} - V^*_{\Delta t, k+1}| \lesssim n^{-1/14} + |V^*_{\Delta t, n, k} - V^*_{\Delta t, k}| \lesssim kn^{-1/14},
\]

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where the last inequality follows by iterating on \( k \). Since \( k \) is finite for a fixed \( \Delta t \), we have thereby shown \( \lim_{n \to \infty} |V_{\Delta t, n, k+1}^* - V_{\Delta t,k+1}^*| = 0 \) for all \( k \). The claim follows by combining this result with Theorem 3.

A.3. **Proof of Theorem 5.** Recall that \( y_i = \{Y_k\}_{k=1}^i \) denotes the rewards after \( i \) pulls of the arms. Let \( \nu(y_n) := \prod_{i=1}^n \nu(Y_i) \), \( p_{n,\theta}(y_n) := \prod_{k=1}^n p_{\theta}(Y_k) \) and \( \bar{P}_n \) be the probability measure corresponding to the density \( d\bar{P}_n(y_n) := \int p_{n,\theta_0+h/\sqrt{n}}(y_n) \cdot m_0(h)dh \). The latter quantity is the marginal probability measure of \( y_n \), and we use \( \bar{E}_n[\cdot] \) to denote its corresponding expectation. Similarly, define \( \tilde{P}_n \) as the measure (not necessarily a probability) corresponding to the density \( d\tilde{P}_n(y_n) := \int \lambda_{n,h}(y_n) \cdot m_0(h)dh \), where

\[
\lambda_{n,h}(y_n) := \frac{d\Lambda_{n,h}(y_n)}{d\nu(y_n)} = \exp\left\{ \frac{1}{\sigma^2} hx_n - \frac{1}{2\sigma^2} h^2 \right\} p_{n,\theta_0}(y_n).
\]

Finally, \( \|\cdot\|_{TV} \) denotes the total variation metric between two measures.

The proof proceeds in the following steps:

**Step 1 (Sequential local asymptotic normality):** Lemma 2 in Appendix B shows that quadratic mean differentiability, (4.1), implies the SLAN property (4.2).

**Step 2 (Approximation of posterior and marginal measures):** Let \( A_n \) denote the event \( \{y_n : \sup_q |x_{nq}| \leq M\} \). For any measure \( P \), define \( P \cap A_n \) as the restriction of \( P \) to the set \( A_n \). By Lemma 6 in Appendix B, for any \( \epsilon > 0 \) there exists \( M < \infty \) such that

\[
\lim_{n \to \infty} \tilde{P}_n(A_n^c) \leq \epsilon, \quad \lim_{n \to \infty} \|P_n \cap A_n - \tilde{P}_n \cap A_n\|_{TV} = 0, \quad \text{and} \quad \lim_{n \to \infty} \sup_q \bar{E}_n \left[ \|A_n \|_{TV} \right] = 0.
\]

The measures \( \Lambda_{n,h}(\cdot), \tilde{P}_n(\cdot) \) are not probabilities as they need not integrate to 1. But Lemma 6 also shows the following: \( \Lambda_{n,h}(\cdot), \tilde{P}_n(\cdot) \) are \( \sigma \)-finite and contiguous with respect to \( P_{\theta_0} \), and letting \( Y_n \) denote the same space of \( y_n \),

\[
\lim_{n \to \infty} \tilde{P}_n(Y_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} \tilde{P}_n(A_n^c) \leq \epsilon_M. \quad (A.11)
\]

The first result in (A.11) implies that \( \tilde{P}_n \) is almost a probability measure.
Step 3 (Approximation of $V^*_{\pi,n}(0)$): By the law of iterated expectations

$$V^*_{\pi,n}(0) = \mathbb{E}_{\{y_n,h\}} \left[ \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right] = \tilde{\mathbb{E}}_{n} \left[ \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right],$$

where $\tilde{\mathbb{E}}_{n}[\cdot]$ is the expectation expectation with respect to the marginal probability measure $\tilde{P}_n$, and we explicitly write $q_j(\pi)$ as function of $\pi$ to make it clear that the value of $q$ at period $j$ depends on $y_n$ through the policy $\pi$. The second equality in the above expression uses the fact that $(y_{nq_j}, q_j, \xi_j, \pi_j)$ are all deterministic functions of $y_n$ when $\pi$ is deterministic.

Recall that $\mathbb{E}[\cdot | y_{nq}]$ is the expectation under $\tilde{P}_n(\cdot | y_{nq})$ as defined in the main text (see the discussion following (4.7)). Also, let $\tilde{\mathbb{E}}_{n}[\cdot]$ denote the expectation under the measure $\tilde{P}_n$ and define

$$\tilde{V}^*_{\pi,n}(0) = \tilde{\mathbb{E}}_{n} \left[ \frac{1}{n} \sum_{j=1}^{n} \tilde{\mathbb{E}} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right].$$

We show that

$$\lim_{n \to \infty} \sup_{\pi \in \Pi} \left| V^*_{\pi,n}(0) - \tilde{V}^*_{\pi,n}(0) \right| = 0 \quad (A.12)$$

by bounding each term in the following expansion:

$$V^*_{\pi,n}(0) - \tilde{V}^*_{\pi,n}(0) = \tilde{\mathbb{E}}_{n} \left[ \mathbb{I}_{A_n} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right] + \tilde{\mathbb{E}}_{n} \left[ \mathbb{I}_{A_n} \frac{1}{n} \sum_{j=1}^{n} \tilde{\mathbb{E}} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right]$$

$$+ \left( \tilde{\mathbb{E}}_{n} - \tilde{\mathbb{E}}_{n} \right) \left[ \mathbb{I}_{A_n} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right]$$

$$+ \tilde{\mathbb{E}}_{n} \left[ \mathbb{I}_{A_n} \frac{1}{n} \sum_{j=1}^{n} \left\{ \mathbb{E} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] - \tilde{\mathbb{E}} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right\} \right]. \quad (A.13)$$

Because of the compact support of the prior, the posteriors $p_n(\cdot | y_{nq}), \tilde{p}_n(\cdot | y_{nq})$ are also compactly supported on $|h| \leq \Gamma$ for all $q$. On this set $|R_n(h, \pi_j)| \leq b\Gamma$ for some $b < \infty$ by Assumption 1(iii). The first two quantities in (A.13) are therefore bounded by $b\Gamma \mathbb{P}_n(A_n)$ and $b\Gamma \tilde{\mathbb{P}}_n(A_n)$. By (A.8) and (A.11), these can be made arbitrarily small by choosing a suitably large $M$ in the definition of $A_n$. The third term in (A.13) is bounded by $b\Gamma \| \mathbb{P}_n \cap A_n - \tilde{\mathbb{P}}_n \cap A_n \|_{TV}$. By (A.9) it converges to 0 as $n \to \infty$. The expression within $\{ \}$ brackets in the fourth term of (A.13) is smaller
than \( b \Gamma \| p_n(\cdot|y_{nq_j(\pi)}) - \tilde{p}_n(\cdot|y_{nq_j(\pi)}) \|_{TV} \). Hence, by the linearity of expectations, the term overall is bounded by

\[
b \Gamma \sup_q \mathbb{E}_n \left[ I_{A_n} \left\| p_n(\cdot|y_q) - \tilde{p}_n(\cdot|y_q) \right\|_{TV} \right],
\]

which is \( o(1) \) because of (A.10). We have thus shown (A.12).

**Step 4 (A recursive formula for the welfare approximation):** The measure, \( \tilde{P}_n \), used in the outer expectation in the definition of \( \hat{V}_{\pi,n}^*(0) \) is not a probability. This can be rectified as follows: First, note that the density, \( \lambda_{n,h}(\cdot) \), of \( \Lambda_{n,h}(\cdot) \) can be written as

\[
\lambda_{n,h}(y_n) = \prod_{i=1}^n \left\{ \exp \left( \frac{h}{\sigma \sqrt{n}} \psi(Y_i) - \frac{h^2}{2\sigma^2 n} \right) p_{\theta_0}(Y_i) \right\} = \prod_{i=1}^n \tilde{p}_n(Y_i|h),
\]

where\(^5\)

\[
\tilde{p}_n(Y_i|h) := \exp \left( \frac{h}{\sigma \sqrt{n}} \psi(Y_i) - \frac{h^2}{2\sigma^2 n} \right) p_{\theta_0}(Y_i).
\]

Using (A.14), Lemma 7 shows that \( \tilde{P}_n \) can be disintegrated as

\[
d \tilde{P}_n(y_n) = \prod_{i=1}^n \left\{ \int \tilde{p}_n(Y_i|h) \tilde{p}_n(h|y_{i-1})dh \right\},
\]

with \( \tilde{p}_n(h|y_0) := m_0(h) \). Now define \( c_{n,i} := \int \{ \int \tilde{p}_n(Y_i|h) \psi(Y_i) \} \tilde{p}_n(h|y_{i-1})dh \), and let \( \tilde{P}_n \) denote the probability measure

\[
d \tilde{P}_n(Y_i|y_{i-1}) := \frac{1}{c_{n,i}} \int \tilde{p}_n(Y_i|h) \tilde{p}_n(h|y_{i-1})dh.
\]

Note that \( c_{n,i} \) is a random (because it depends on \( y_{i-1} \)) integration factor ensuring \( \tilde{P}_n(y_{i+1}|y_i) \), and therefore \( \tilde{P}_n \), is a probability. In Lemma 8, it is shown that there exists some non-random \( C < \infty \) such that

\[
\sup_i |c_{n,i} - 1| \leq Cn^{-c} \text{ for any } c < 3/2,
\]

and furthermore, \( \| \tilde{P}_n - \tilde{P}_n \|_{TV} \to 0 \) as \( n \to \infty \). Hence, letting

\[
\hat{V}_{\pi,n}^*(0) := \mathbb{E}_{\tilde{P}_n} \left[ \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ R_n(h, \pi_j) | y_{nq_j(\pi)} \right] \right],
\]

\(^5\)Despite the notation, \( \tilde{p}_n(Y_i|h) \) is not a probability density.
where $\mathbb{E}_{\tilde{P}_n}[\cdot]$ is the expectation with respect to $\tilde{P}_n$, one obtains the approximation

$$
\sup_{\pi \in \Pi} |\bar{V}_{\pi,n}^*(0) - \bar{V}_{\pi,n}^*(0)| \leq b \Gamma \|\bar{P}_n - \tilde{P}_n\|_{TV} \rightarrow 0.
$$

See the arguments following (A.13) for the definition of $b$.

Since $\bar{p}_n(h|x_{i-1}, q = i - 1)$ by (4.6) in the main text, it follows by the definition (A.16) that $\bar{P}_n(Y_i|x_{i-1}, i - 1)$, i.e., the conditional distribution, under $\tilde{P}_n$, of future rewards $Y_i$ given the past rewards depends only on the current $x$ and $q$.

Define $\tilde{V}_{n}^*(0) = \inf_{\pi \in \Pi} \tilde{V}_{\pi,n}^*(0)$. Recall that for a given $\pi \in \{0, 1\}$, $\mathbb{E} \left[ R_n(h, \pi) | y_{nq}, \right] \equiv \mathbb{E} \left[ R_n(h, \pi) | x_{nq}, q \right]$ by (4.6). Furthermore, we have noted above that the conditional distribution of the future values of the rewards, $\tilde{P}_n(Y_{nq+1}|y_{nq})$, also depends only on $(x_{nq}, q)$. Based on this, standard backward induction/dynamic programming arguments imply $\tilde{V}_{\pi,n}^*(0)$ can be obtained as the solution at $(x, q, t) = (0, 0, 0)$ of the recursive problem

$$
\tilde{V}_{n}^*(x, q, t) = \min_{\pi \in \{0, 1\}} \{ \mathbb{E} \left[ R_n(h, \pi) | x, q \right] + \mathbb{E}_{\tilde{P}_n} \left[ \mathbb{I}_n \cdot \tilde{V}_{n}^* \left( x + \pi Y_{nq+1} / \sqrt{n}, q + \pi / n, t + 1 / n \right) \right] \} ;
$$

if $t < 1$,

$$
\tilde{V}_{n}^*(x, q, 1) = 0,
$$

(A.18)

where $\mathbb{E}_{\tilde{P}_n}[\cdot | s]$ denotes the expectation under $\tilde{P}_n(Y_{nq+1}|y_{nq}) \equiv \tilde{P}_n(Y_{nq+1}|x = x_{nq}, q)$ and $\mathbb{I}_n = \{ t \leq 1 - 1/n \}$.

The value $\pi^* \in \{0, 1\}$ that attains the minimum in (A.18) is a function only of $s$. We would have thus obtained the approximation, $\bar{V}_{n}^*(0)$, to minimal Bayes risk even if we restricted the policy class to $\Pi^F$. This proves the first claim of the theorem.

**Step 5 (Auxiliary results for showing PDE approximation of (A.18)):** We now state a couple of results that will be used to show convergence of (A.18) to a PDE.

The first result is that $\mathbb{E} \left[ R_n(h, \pi) | x, q \right]$ can be approximated by $\mu^+(s) - \pi \mu(s)$ uniformly over $(x, q)$. To this end, denote $\bar{R}(h, \pi) = \bar{\mu}_0 h (\mathbb{I}(\bar{\mu}_0 h > 0) - \pi)$. Assumption (1(iii)) implies $\sup_{|h| \leq \Gamma} |\mu_n(h) - \bar{\mu}_0 h / \sqrt{n}| \leq C / n$ for some $C < \infty$. Because the function $x \mathbb{I}(x > 0) - \pi x = \max\{x, 0\} - \pi x$ is Lipschitz continuous,

$$
\sup_{|h| \leq \Gamma; \pi \in \{0, 1\}} |R_n(h, \pi) - \bar{R}(h, \pi)| \leq 2C / \sqrt{n}.
$$

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Recalling the definitions of $\mu^+(s), \mu(s)$ from the main text, the above implies

$$\sup_{(x,q): \pi \in \{0,1\}} \left| \mathbb{E} \left[ R_n(h, \pi) \mid x, q \right] - \left( \mu^+(s) - \pi \mu(s) \right) \right| \leq 2C/\sqrt{n}. \quad (A.19)$$

The next result is given as Lemma 9 in Appendix B. It states that there exists $\xi_n \to 0$ independent of $s$ and $\pi \in \{0,1\}$ such that

$$\sqrt{n} \mathbb{E}_n \left[ \pi \psi(Y_{nq+1}) \mid s \right] = \pi \mu(s) + \xi_n; \quad (A.20)$$

$$\mathbb{E}_n \left[ \pi \psi^2(Y_{nq+1}) \mid s \right] = \pi \sigma^2 + \xi_n. \quad (A.21)$$

Furthermore,

$$\mathbb{E}_n \left[ |\psi(Y_{nq+1})|^3 \mid s \right] < \infty. \quad (A.22)$$

**Step 6 (PDE approximation of (A.18)):** The solution, $\tilde{V}^*_n(s)$, of (A.18) converges locally uniformly to $V^*_n(s)$, the viscosity solution to PDE(2.7). This follows by similar arguments as in the proof of Theorem 2:

Clearly the scheme defined in (A.18) is monotonic. Because $\sup_n |\tilde{R}(h, \pi)| \leq \bar{\mu}_0 \Gamma$, the solution to (A.18) is uniformly bounded, with $|\tilde{V}^*_n(s)| \leq \bar{\mu}_0 \Gamma$ independent of $s$ and $n$. Finally, consistency of the scheme follows by similar arguments as in the proof of Theorem 2, after making use of (A.19) and (A.20) - (A.22).

This completes the proof of the second claim of the theorem.

**Step 7 (Proof of the third claim):** Steps 1-4 imply $\lim_{n \to \infty} V^*_{n\Delta t,n,k}(0) - \tilde{V}^*_{\Delta t,n}(0) = 0$. In addition we can follow the arguments in Step 4 to express $\tilde{V}^*_{\Delta t,n}(0)$ in recursive form, in a manner similar to the definition of $V^*_{\Delta t,n,k}($) in the proof of Theorem 4; the only difference is that the operator $\tilde{S}_{\Delta t}[\phi](x,q)$ in that proof is now defined as the solution at $(x,q,\Delta t)$ of the recursive equation

$$f(x,q,\tau) = \frac{\mathbb{E}[R_n(h, \pi) \mid x, q]}{n} + \mathbb{E}_n \left[ f \left( x + \frac{Y}{\sqrt{n}}, q + \frac{1}{n}, \tau - \frac{1}{n} \right) \mid s \right]; \quad \tau > 0$$

$$f(x,q,0) = \phi(x,q).$$

Now, using the results from Step 5, an application of Barles and Jakobsen (2007, Theorem 3.1) as described in Appendix B gives $|\tilde{S}_{\Delta t} V^*_{\Delta t,k+1} - S_{\Delta t} V^*_{\Delta t,k+1}| \lesssim n^{-1/4}$. The remainder of the proof is analogous to that of Theorem 4.
A.4. Proof of Theorem 6. The proof consists of two steps. First, we show that $V^*_n(0)$ converges to the solution of a PDE with state variables $(\chi, q, t)$ where $\chi(t) := \chi_{nq}(t)$ with $\chi_{nq}$ defined in Section 5. Recall that the first component of $\chi$ is $x/\sigma$.

Next, we show that the PDE derived in the first step can be reduced to one involving just the state variables $s = (x, q, t)$.

The first step follows the proof of Theorem 5 with straightforward modifications. Indeed, the setup is equivalent to taking $\chi(Y_i)$ to be the vector valued score function in the parametric setting (see, Section 4.4). The upshot of these arguments is that $V^*_n(0)$ converges to $V^*(0)$, where $V^*(\cdot)$ solves the PDE

$$
\partial_t f(\chi, q, t) + \mu^+(x, q) + \min \{-\mu(x, q) + L[f](\chi, q, t), 0\} = 0 \text{ if } t < 1 \quad \text{(A.23)}
$$

$$
f(\chi, q, t) = 0 \text{ if } t = 1,
$$

with the infinitesimal generator (here $\triangle$ denotes the Laplace operator)

$$
\bar{L}[f](\chi, q, t) := \partial_q f + h(\chi, q)^\top D\chi f + \frac{1}{2} \triangle \chi f.
$$

The function $h(\chi, q)$ is defined in Section 5. Note that $\mu^+(\cdot), \mu(\cdot)$ depend only on $(x, q)$. This is because the expected rewards depend only on the first component, $h_0/\sigma$, of $h$ and its posterior distribution can be approximated by $\tilde{p}_n(h_0|x, q)$, defined in (5.6).

By the arguments leading to (5.6), the first component of $h(\chi, q)$ is $\sigma^{-1} \tilde{E}[h_0|\chi, q] = \sigma^{-1} \tilde{E}[h_0|x, q] = \sigma^{-1} \mu(x, q)$. Let $\chi^c, h^c(\chi, q)$ denote $\chi, h(\chi, q)$ without their first components $\chi_1 = x/\sigma$ and $h_1(\chi, q) = \sigma^{-1} \mu(x, q)$. Then, defining

$$
L[f](x, q, t) := \partial_q f + \mu(x, q) \partial_x f + \frac{1}{2} \sigma^2 \partial^2_x f,
$$

we see that $\bar{L}[f] = L[f] + h^c(\chi, q)^\top D\chi^c f + \frac{1}{2} \triangle \chi^c f$. Note that in defining $L[f](\cdot)$, we made use of the change of variables $\partial_{\chi_1} f = \sigma \partial_x f$ and $\partial^2_{\chi_1} f = \sigma^2 \partial^2_x f$. We claim that the solution of PDE (A.23) is the same as that of PDE (2.7), reproduced here:

$$
\partial_t f(x, q, t) + \mu^+(x, q) + \min \{-\mu(x, q) + L[f](x, q, t), 0\} = 0 \text{ if } t < 1 \quad \text{(A.24)}
$$

$$
f(x, q, t) = 0 \text{ if } t = 1.
$$
Intuitively, $\chi^c$ do not affect instantaneous payoffs $\mu^+(x,q) - \mu(x,q), \mu^+(x,q)$, nor do they affect the boundary condition, so these state variables are superfluous. The formal proof makes use of the theory of viscosity solutions: Under Assumption 2(v), Theorem 1 implies there exists a unique viscosity solution to (A.23), denoted by $V^*(x,q,t)$. Then, it is a consequence of the definition that $\bar{V}^*(x,q,t) = \sup_{\chi^c} V^*(\chi,q,t)$ is a viscosity sub-solution to (A.24). In a similar fashion, $\underline{V}^*(x,q,t) = \inf_{\chi^c} V^*(\chi,q,t)$ is a viscosity super-solution to (A.24). Under Assumption 2(v), a comparison principle (see, Crandall et al., 1992) holds for (A.24) implying any super-solution is larger than a sub-solution. So, $\bar{V}^*(x,q,t) = \underline{V}^*(x,q,t) = V^*(x,q,t)$, where $V^*(x,q,t)$ is the unique viscosity solution to (A.24). This suffices to prove the theorem.

---

6See Crandall et al. (1992) for the definition of viscosity sub- and super-solutions using test functions. To show $\bar{V}^*$ is a sub-solution one can argue as follows: First, $\bar{V}^*(x,q,t)$ is upper-semicontinuous because of the continuity of the solution $V^*(x,q,t)$ to PDE (A.23). Second, $\bar{V}^*$ satisfies the boundary condition in PDE (A.24) by construction. Third, let $\phi \in C^\infty(X,Q,T)$ denote a test function such that $\phi \geq \bar{V}^*$ everywhere. By the definition of $\bar{V}^*$ we also have $\phi(x,q,t) \geq V^*(\chi,q,t)$ everywhere. Since $V^*(\chi,q,t)$ is a solution to PDE (A.23), $\phi$ must satisfy the viscosity requirement for a sub-solution to PDE (A.23). But because $\phi$ is constant in $\chi^c$, this implies it also satisfies the viscosity requirement for a sub-solution to PDE (A.24). These three facts suffice to show $\bar{V}^*$ is a sub-solution.
Appendix B. Supplementary results

B.1. Rates of convergence to the PDE solution. The results of Barles and Jakobsen (2007, Theorem 3.1) provide a bound on the rate of convergence of $V_n^*$ to $V^*$. The technical requirements to obtain this are described in their Assumptions A2, S1-S3. Assumptions A2, S1-S2 are straightforward to verify using the regularity conditions given for Theorem 2 with the additional requirement $\sup_n |\mu^+(s)| < \infty$.

Assumption S3 of Barles and Jakobsen (2007) is a strengthening of the consistency requirement in (A.4) and (A.5). Suppose that the test function $\phi \in C^\infty(S)$ is such that $|\partial_0^\beta D_{(x,q)}^\beta \phi(x,q,t)| \leq K \varepsilon^{1-2\beta_0-\|\beta\|}$ for all $\beta_0 \in \mathbb{N}, \beta \in \mathbb{N} \times \mathbb{N}$. Then by a third order Taylor expansion as in the proof of Theorem 2 and some tedious but straightforward algebra,

$$|nS_n(z, \phi(z) + \rho, [\phi + \rho]) - F(D^2\phi(s), D\phi(s), s)| \leq E(n, \varepsilon) \equiv \bar{K} \frac{n^{1/2} \varepsilon^2}{n},$$

where $\bar{K}$ depends only on $K$, defined above, and the upper bounds on $\mu^+(\cdot), \mu(\cdot)$. The above suffices to verify the Assumption S3 of Barles and Jakobsen (2007); note that the definition of $S(\cdot)$ in that paper is equivalent to $nS_n(\cdot)$ here.

Under the above conditions, Barles and Jakobsen (2007, Theorem 3.1) implies

$$V^* - V_n^* \leq \sup_{\varepsilon} (\varepsilon + E(n, \varepsilon)) \leq n^{-1/6}$$ and

$$V_n^* - V^* \leq \sup_{\varepsilon} \left(\varepsilon^{1/3} + E(n, \varepsilon)\right) \leq n^{-1/14}.$$

The asymmetry of the rates is an artifact of the techniques of Barles and Jakobsen (2007). The rates are also far from optimal. The results of Barles and Jakobsen (2007), while being relatively easy to apply, do not exploit any regularity properties of the approximation scheme. There exist approximation schemes for PDE (2.7) that converge at $n^{-1/2}$ rates, see e.g., Krylov (2005), but it is unknown (to the author) if (3.2) is one of them.

B.2. Supporting lemmas for the proof of Theorem 5. We implicitly assume Assumption 1 for all the results in this section.

Lemma 2. Suppose $P_\theta$ is quadratic mean differentiable as in (4.1). Then $P_\theta$ satisfies the SLAN property as defined in (4.2).
Proof. The proof builds on Van der Vaart (2000, Theorem 7.2). Set \( p_n := dP_{\theta_0 + h/\sqrt{n}}/d\nu \), \( p_0 := dP_{\theta_0}/d\nu \) and \( W_{ni} := 2 \left[ \sqrt{p_n/\theta_0(Y_i)} - 1 \right] \). We use \( E[\cdot] \) to denote expectations with respect to \( P_{n,\theta_0} \). Quadratic mean differentiability implies \( E[\psi(Y_i)] = 0 \) and \( E[\psi^2(Y_i)] = 1/\sigma^2 \), see Van der Vaart (2000, Theorem 7.2).

It is without loss of generality for this proof to take the domain of \( q \) to be \( \{0, 1/n, 2/n, \ldots, 1\} \). For any such \( q \),

\[
E \left[ \sum_{i=1}^{nq} W_{ni} \right] = 2nq \left( \int \sqrt{p_n \cdot p_0} d\nu - 1 \right) = -nq \int (\sqrt{p_n} - \sqrt{p_0})^2 d\nu.
\]

Now, (4.1) implies there exists \( \epsilon_n \to 0 \) such that

\[
\mathbf{|} n \int (\sqrt{p_n} - \sqrt{p_0})^2 d\nu - \frac{h^2}{4\sigma^2} \mathbf{|} \lesssim \epsilon_n h^2.
\]

Hence, for any given \( h \),

\[
\sup_q \mathbf{|} E \left[ \sum_{i=1}^{nq} W_{ni} \right] - qh^2/4\sigma^2 \mathbf{|} \to 0. \quad \text{(B.1)}
\]

Next, denote \( Z_{ni} = W_{ni} - h\psi(Y_i)/\sqrt{n} - E[W_{ni}] \) and \( S_{nq} = \sum_{i=1}^{nq} Z_{ni} \). Observe that \( E[Z_{ni}] = 0 \) since \( E[\psi(Y_i)] = 0 \). Furthermore, by (4.1),

\[
\text{Var}[\sqrt{n}Z_{ni}] = E \left[ \left( \sqrt{n}W_{ni} - h\psi(Y_i) \right)^2 \right] \lesssim \epsilon_n h^2 \to 0. \quad \text{(B.2)}
\]

Now, an application of Kolmogorov’s maximal inequality for partial sum processes gives

\[
P \left( \sup_q |S_{nq}| \geq \lambda \right) \leq \frac{1}{\lambda^2} \text{Var} \left[ \sum_{i=1}^{n} Z_{ni} \right] = \frac{1}{\lambda^2} \text{Var}[\sqrt{n}Z_{ni}].
\]

Combined with (B.1) and (B.2), the above implies

\[
\sum_{i=1}^{nq} W_{ni} = \frac{h}{\sqrt{n}} \sum_{i=1}^{nq} \psi(Y_i) - \frac{qh^2}{4\sigma^2} + o_{P_{n,\theta_0}}(1) \text{ uniformly over } q. \quad \text{(B.3)}
\]

We now employ a Taylor expansion of the logarithm \( \ln(1+x) = x - \frac{1}{2}x^2 + x^2R(2x) \) where \( R(x) \to 0 \) as \( x \to 0 \), to expand the log-likelihood as

\[
\ln \prod_{i=1}^{nq} \frac{p_n(X_i)}{p_0} = 2 \sum_{i=1}^{nq} \ln \left( 1 + \frac{1}{2} W_{ni} \right)
\]

\[
= \sum_{i=1}^{nq} W_{ni} - \frac{1}{4} \sum_{i=1}^{nq} W_{ni}^2 + \frac{1}{2} \sum_{i=1}^{nq} W_{ni}^2 R(W_{ni}). \quad \text{(B.4)}
\]
Because of (B.2), we can write $\sqrt{n}W_{ni} = h\psi(Y_i) + C_{ni}$ where $E[|C_{ni}|^2] \to 0$. Defining $A_{ni} := 2h\psi(Y_i)C_{ni} + C_{ni}^2$, some straightforward algebra then gives $nW_{ni}^2 = h^2\psi^2(Y_i) + A_{ni}$ with $E[|A_{ni}|] \to 0$. Now, by the uniform law of large numbers for partial sum processes, see e.g., Bass and Pyke (1984), $n^{-1}\sum_{i=1}^{nq} h^2\psi^2(Y_i)$ converges uniformly in $P_{n,0}$-probability to $qh^2/\sigma^2$. Furthermore, $E\left[\sup_q n^{-1}\sum_{i=1}^{nq} |A_{ni}|\right] \leq E\left[n^{-1}\sum_{i=1}^{nq} |A_{ni}|\right] = E[|A_{ni}|] \to 0$ and therefore $n^{-1}\sum_{i=1}^{nq} A_{ni}$ converges uniformly in $P_{n,0}$-probability to 0. These results yield

$$\sum_{i=1}^{nq} W_{ni}^2 = \frac{qh^2}{\sigma^2} + o_{P_{n,0}}(1) \text{ uniformly over } q.$$ 

Next, by the triangle inequality and Markov’s inequality

$$nP_{n,0}\left(|W_{ni}| > \varepsilon \sqrt{2}\right) \leq nP_{n,0}\left(h^2\psi^2(Y_i) > n\varepsilon^2\right) + nP_{n,0}\left(|A_{ni}| > n\varepsilon^2\right) \leq \varepsilon^{-2}h^2E\left[\psi^2(Y_i)\mathbb{1}\left\{\psi^2(Y_i) > n\varepsilon^2\right\}\right] + \varepsilon^{-2}E[|A_{ni}|] \to 0$$

for any given $h$. The above implies $\max_{1 \leq i \leq n}|W_{ni}| = o_{P_{n,0}}(1)$ and consequently, $\max_{1 \leq i \leq n}|R(W_{ni})| = o_{P_{n,0}}(1)$. The last term on the right hand side of (B.4) is bounded by $\max_{1 \leq i \leq n}|R(W_{ni})| \cdot \sum_{i=1}^{nq} W_{ni}^2$ and is therefore $o_{P_{n,0}}(1)$ by the above results. We thus conclude

$$\ln \prod_{i=1}^{nq} \frac{p_n(X_i)}{p_0} = \sum_{i=1}^{nq} W_{ni} - \frac{qh^2}{4\sigma^2} + o_{P_{n,0}}(1) \text{ uniformly over } q.$$ 

The claim follows by combining the above with (B.3). \hfill\Box

**Lemma 3.** For any $\varepsilon > 0$, there exist $M(\varepsilon), N(\varepsilon) < \infty$ such that $M \geq M(\varepsilon)$ and $n \geq N(\varepsilon)$ implies $P_n(A^q_n) < \varepsilon$. Furthermore, letting $A^q_n = \{y_n : \sup_q |x_{nq}| < M\}$, and $E_{n,0}[\cdot]$, the expectation under $P_{n,0}$,

$$\sup_q E_{n,0}\left[\mathbb{I}_{A^q_n} \left\| \frac{dP_{n,q,0+h/\sqrt{n}}}{dP_{n,0}}(y_{nq}) - \frac{d\Lambda_{n,q,h}}{dP_{n,q,0}}(y_{nq}) \right\| \right] = o(1) \forall \{h : |h| \leq \Gamma\}.$$ 

**Proof.** Set $A_{n,M} = \mathbb{1}\{\sup_q |x_{nq}| < M\}$, $P_{n,h} = P_{n,q,0+h/\sqrt{n}}$ and $P_{n,0} = P_{n,0}$. Note that $x_{nq}$ is a partial sum process with mean 0 under $P_{n,0}$. By Kolmogorov’s maximal inequality, $P_{n,0}\left(\sup_q |x_{nq}| \geq M\right) \leq M^{-1}\text{Var}[x_{n}] = M^{-1}\sigma^2$. Hence, $P_{n,0}(A_{n,M}^c) \to 0$ for any $M_n \to \infty$. But by (4.2) and standard arguments involving Le Cam’s first lemma, $P_{n,h}$ is contiguous to $P_{n,0}$ for all $h$. This implies $\bar{P}_n := \int P_{n,h}dm_0(h)$ is also contiguous to $P_{n,0}$ (this can be shown using the dominated convergence theorem; see
also, Le Cam and Yang, p.138). Consequently, \( \bar{P}_n(A_{n,M_n}) \to 0 \) for any \( M_n \to \infty \). The first claim is a straightforward consequence of this.

For the second claim, we follow Le Cam and Yang (2000, Proposition 6.2):

We first argue that \( P_{n,q_n,h} \) is contiguous to \( P_{n,q_n,0} \) for any deterministic \( q_n \to q \). We have

\[
\ln \frac{dP_{n,q_n,h}}{dP_{n,q_n,0}} = \frac{h x_{nq}}{2\sigma^2} + o_{P_n,0}(1)
\]

\[
\frac{d}{P_{n,0}} \to N \left( -\frac{qh^2}{2\sigma^2}, \frac{qh^2}{\sigma^2} \right),
\]

where the equality follows from (4.2), and the weak convergence limit follows from:

(i) weak convergence of \( x_{nq} \) under \( P_{n,0} \) to a Brownian motion process \( W(q) \), see e.g., Van Der Vaart and Wellner (1996, Chapter 2.12), and (ii) the extended continuous mapping theorem, see Van Der Vaart and Wellner (1996, Theorem 1.11.1).

Since \( E_{P_{n,0}}[f(y_{nq})] = E_{P_{n,q_n,0}}[f(y_{nq})] \) for any \( f(\cdot) \), we conclude from (B.5) and the definition of weak convergence that

\[
\ln \frac{dP_{n,q_n,h}}{dP_{n,q_n,0}} \overset{d}{\to} N \left( -\frac{qh^2}{2\sigma^2}, \frac{qh^2}{\sigma^2} \right).
\]

An application of Le Cam’s first lemma then implies \( P_{n,q_n,h} \) is contiguous to \( P_{n,q_n,0} \).

Now, let \( q_n \) denote the value of \( q \) attaining the supremum below (or reaching values arbitrarily close to the supremum if the supremum is not attainable) so that

\[
\sup_q E_{P_{n,0}} \left[ I_{A_n} \left| \frac{dP_{n,q_n,h}}{dP_{n,q_n,0}} - \frac{d\Lambda_{n,q_n,h}}{dP_{n,q_n,0}} \right| \right] = E_{P_{n,0}} \left[ I_{A_n} \left| \frac{dP_{n,q_n,h}}{dP_{n,q_n,0}} - \frac{d\Lambda_{n,q_n,h}}{dP_{n,q_n,0}} \right| \right].
\]

Without loss of generality, we may assume \( q_n \to q \); otherwise we can employ a subsequence argument since \( q_n \) lies in a bounded set. Define

\[
G_n(q) := I_{A_n} \left| \frac{dP_{n,q_n,h}}{dP_{n,q_n,0}} - \frac{d\Lambda_{n,q_n,h}}{dP_{n,q_n,0}} \right|.
\]

The claim follows if we show \( E_{P_{n,0}}[G_n(q_n)] \to 0 \). By Lemma 2 and the definition of \( \Lambda_{n,q_n,h}(\cdot) \),

\[
G_n(q) = I_{A_n} \cdot \exp \left\{ \frac{1}{\sigma^2} h x_{nq} - \frac{q}{2\sigma^2} h^2 \right\} \left( \exp \delta_{n,q} - 1 \right),
\]

where \( \sup_q |\delta_{n,q}| = o(1) \) under \( P_{n,0} \). Since \( I_{A_n} \cdot \exp \left\{ \frac{1}{\sigma^2} h x_{nq} - \frac{q}{2\sigma^2} h^2 \right\} \) is bounded for \(|h| \leq \Gamma\) by the definition of \( I_{A_n} \), this implies \( G_n(q_n) = o(1) \) under \( P_{n,0} \). Next, we argue \( G_n(q_n) \) is uniformly integrable. The first term \( I_{A_n} \cdot \frac{d\Lambda_{n,q_n,h}}{dP_{n,q_n,0}} \) in the

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definition of $G_n(q_n)$ is bounded, and therefore uniformly integrable, for $|h| \leq \Gamma$. We now prove uniform integrability of $dP_{nq,h}/dP_{nq,0}$, and thereby that of $I_{A_{nq}} \cdot dP_{nq,h}/dP_{nq,0}$. For any $b < \infty$,

$$
E_{n,0} \left[ dP_{nq,h} \left\{ \frac{dP_{nq,h}}{dP_{nq,0}} > b \right\} \right] = \int dP_{nq,h} \left\{ \frac{dP_{nq,h}}{dP_{nq,0}} > b \right\} dP_{nq,0} \\
\leq P_{nq,h} \left( \frac{dP_{nq,h}}{dP_{nq,0}} > b \right).
$$

But,

$$
P_{nq,0} \left( \frac{dP_{nq,h}}{dP_{nq,0}} > b \right) \leq b^{-1} \int \frac{dP_{nq,h}}{dP_{nq,0}} dP_{nq,0} \leq b^{-1},
$$

so the contiguity of $P_{nq,h}$ with respect to $P_{nq,0}$ implies we can choose $b$ and $\bar{n}$ large enough such that

$$
\limsup_{n \geq \bar{n}} P_{nq,h} \left( \frac{dP_{nq,h}}{dP_{nq,0}} > b \right) < \epsilon
$$

for any arbitrarily small $\epsilon$. These results demonstrate uniform integrability of $G_n(q_n)$ under $P_{n,0}$. Since convergence in probability implies convergence in expectation for uniformly integrable random variables, we have thus shown $E_{n,0} [G_n(q_n)] \to 0$, which concludes the proof. 

□

**Lemma 4.** $\lim_{n \to \infty} \left\| \bar{P}_n \cap A_n - \tilde{P}_n \cap A_n \right\|_{TV} = 0$.

**Proof.** Set $P_{n,h} := P_{n,\theta_0 + h/\sqrt{n}}$. By the properties of the total variation metric, contiguity of $\bar{P}_n$ with respect to $P_{n,0}$ and the absolute continuity of $\Lambda_{n,h}$ with respect to $P_{n,0}$,

$$
\lim_{n \to \infty} \left\| \bar{P}_n \cap A_n - \tilde{P}_n \cap A_n \right\|_{TV} = \frac{1}{2} \lim_{n \to \infty} \int \left\{ \int \mathbb{I}_{A_n} \left| \frac{dP_{n,h}}{dP_{n,0}}(y_n) - \frac{d\Lambda_{n,h}}{dP_{n,0}}(y_n) \right| dP_{n,0}(y_n) \right\} m_0(h)dh.
$$

In the last expression, denote the term within the \{ \} brackets by $f_n(h)$. By Lemma 3, $f_n(h) \to 0$ for each $h$. Additionally, $\mathbb{I}_{A_n} \cdot (d\Lambda_{n,h}/dP_{n,0})$ is bounded because of the definition of $A_n$ and the fact $|h| \leq \Gamma$, while

$$
\int \mathbb{I}_{A_n} \left| \frac{dP_{n,h}}{dP_{n,0}} \right| dP_{n,0} \leq \int \frac{dP_{n,h}}{dP_{n,0}} dP_{n,0} \leq 1.
$$

Hence, $f_n(h)$ is dominated by a (suitably large) constant for all $n$. The dominated convergence theorem then implies $\int f_n(h)m_0(h)dh \to 0$. This proves the claim. 

□
Lemma 5. \[ \sup_q \mathbb{E}_n \left[ \mathbb{I}_{A_n} \left\| p_n(\cdot | y_{nq}) - \tilde{p}_n(\cdot | y_{nq}) \right\|_{TV} \right] = o(1). \]

Proof. Set \( P_{n,h} = P_{n,\theta_0+h/\sqrt{n}}, p_{nq,h}(y_{nq}) = dP_{nq,h}(y_{nq})/d\nu \) and \( \lambda_{nq,h}(y_{nq}) = d\Lambda_{nq,h}(y_{nq})/d\nu. \) Let \( S_{nq} \) and \( \tilde{S}_{nq} \) denote joint measures over \( (y_{nq}, h) \), corresponding to \( dS_{nq}(y_{nq}, h) = p_{nq,h}(y_{nq}) \cdot m_0(h) \) and \( d\tilde{S}_{nq}(y_{nq}, h) = \lambda_{nq,h}(y_{nq}) \cdot m_0(h) \). In the main text, \( \tilde{p}_n(h|y_{nq}) \) was defined through the disintegration \( d\tilde{S}_{nq} = \tilde{p}_n(h|y_{nq}) \cdot \tilde{p}_n(y_{nq}), \) where \( \tilde{p}_n(y_{nq}) \) is the marginal. In a similar vein, we can disintegrate \( dS_{nq} = p_n(h|y_{nq}) \cdot \tilde{p}_n(y_{nq}). \)

Since \( p_n(h|y_{nq}), \tilde{p}_n(h|y_{nq}) \) are both conditional probabilities, we obtain \( p_{nq}(y_{nq}) = \int p_{nq,h}(y_{nq})m_0(h)dh \) and \( \tilde{p}_n(y_{nq}) = \int \lambda_{nq,h}(y_{nq})m_0(h)dh. \)

By the properties of the total variation metric,

\[ 2\mathbb{I}_{A_n} \cdot \left\| p_n(\cdot | y_{nq}) - \tilde{p}_n(\cdot | y_{nq}) \right\|_{TV} = \mathbb{I}_{A_n} \cdot \int \left\| p_n(h|y_{nq}) - \tilde{p}_n(h|y_{nq}) \right\| |dh. \]

By the disintegration formula, the right hand side can be written as

\[ \mathbb{I}_{A_n} \int \left| \frac{p_{nq,h}(y_{nq}) \cdot m_0(h)}{\tilde{p}_n(y_{nq})} - \frac{\lambda_{nq,h}(y_{nq}) \cdot m_0(h)}{\tilde{p}_n(y_{nq})} \right| |dh. \]

Hence,

\[ 2\mathbb{E}_n \left[ \mathbb{I}_{A_n} \left\| p_n(\cdot | y_{nq}) - \tilde{p}_n(\cdot | y_{nq}) \right\|_{TV} \right] \leq \mathbb{E}_n \left[ \mathbb{I}_{A_n} \left\| \frac{p_{nq,h}(y_{nq}) - \lambda_{nq,h}(y_{nq})}{\tilde{p}_n(y_{nq})} \right\| m_0(h) |dh \right] \]

\[ + \mathbb{E}_n \left[ \mathbb{I}_{A_n} \left\| \lambda_{nq,h}(y_{nq}) \left| \frac{1}{\tilde{p}_n(y_{nq})} - \frac{1}{\tilde{p}_n(y_{nq})} \right| m_0(h) |dh \right\] \]

\( : = B_{1n}(q) + B_{2n}(q) \)

We start by bounding \( \sup_q B_{1n}(q). \) Recall the definition of \( A_n \supseteq A_n \) from the statement of Lemma 3. By Fubini’s theorem and the definition of \( \tilde{p}_n(\cdot) \) as the density of \( \tilde{P}_{nq}, \)

\[ B_{1n}(q) \leq \int \left\{ \int A_n^q \left| p_{nq,h}(y_{nq}) - \lambda_{nq,h}(y_{nq}) \right| d\nu(y_{nq}) \right\} m_0(h) |dh \]

\[ = \int \left\{ \int A_n^q \left| \frac{dP_{nq,h}(y_{nq})}{dP_{nq,0}(y_{nq})} - \frac{d\Lambda_{nq,h}(y_{nq})}{dP_{nq,0}(y_{nq})} \right| dP_{nq,0}(y_{nq}) \right\} m_0(h) |dh. \]  

(B.6)

Hence,

\[ \sup_q B_{1n}(q) \leq \int \left\{ \sup_q \int A_n^q \left| \frac{dP_{nq,h}(y_{nq})}{dP_{nq,0}(y_{nq})} - \frac{d\Lambda_{nq,h}(y_{nq})}{dP_{nq,0}(y_{nq})} \right| dP_{nq,0}(y_{nq}) \right\} m_0(h) |dh. \]  

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In the above expression, denote the term within the \{ \} brackets by \( g_n(h) \). By Lemma 3, \( g_n(h) \to 0 \) for each \( h \). Furthermore, by similar arguments as in the proof of Lemma 4, \( g_n(h) \) is bounded by a constant for all \( n \) (it is easy to see that the bound derived there applies uniformly over all \( q \)). The dominated convergence theorem then gives \( \int g_n(h)m_0(h)dh \to 0 \), and therefore, \( \sup_q B_{1n}(q) = o(1) \).

We now turn to \( B_{2n}(q) \). The disintegration formula implies \( \lambda_{nq,h}(y_{nq}) \cdot m_0(h) = \tilde{p}_{nq}(y_{nq}) \cdot \tilde{p}_n(h|y_{nq}) \). So,

\[
B_{2n}(q) = \mathbb{E}_n \left[ \mathbb{I}_{A_n} \int \tilde{p}_n(h|y_{nq}) \left| \frac{\tilde{p}_{nq}(y_{nq}) - \tilde{p}_{nq}(y_{nq})}{\tilde{p}_{nq}(y_{nq})} \right| dh \right]
= \mathbb{E}_n \left[ \mathbb{I}_{A_n} \left| \frac{\tilde{p}_{nq}(y_{nq}) - \tilde{p}_{nq}(y_{nq})}{\tilde{p}_{nq}(y_{nq})} \right| \right]
\leq \int \mathbb{I}_{A_n^q} \left| \frac{\tilde{p}_{nq}(y_{nq}) - \tilde{p}_{nq}(y_{nq})}{\tilde{p}_{nq}(y_{nq})} \right| d\nu(y_{nq}). \tag{B.7}
\]

By the integral representation for \( \tilde{p}_{nq}(y_{nq}), \tilde{p}_{nq}(y_{nq}) \) the right hand side of \( (B.7) \) equals

\[
\int \mathbb{I}_{A_n^q} \left| \frac{\frac{d\Lambda_{nq,h}(y_{nq})}{dP_{nq,0}}(y_{nq}) - \frac{d\Lambda_{nq,h}(y_{nq})}{dP_{nq,0}}(y_{nq})}{\tilde{p}_{nq}(y_{nq})} \right| dP_{nq,0}(y_{nq})
\leq \int \left\{ \int \mathbb{I}_{A_n^q} \left| \frac{\frac{d\Lambda_{nq,h}(y_{nq})}{dP_{nq,0}}(y_{nq}) - \frac{d\Lambda_{nq,h}(y_{nq})}{dP_{nq,0}}(y_{nq})}{\tilde{p}_{nq}(y_{nq})} \right| dP_{nq,0}(y_{nq}) \right\} m_0(h)dh, \tag{B.8}
\]

where the second step makes use of Fubini’s theorem. The right hand side of \( (B.8) \) is the same as in \( (B.6) \). So, by the same arguments as before, \( \sup_q B_{2n}(q) = o(1) \). The claim can therefore be considered proved.

\[ \square \]

**Lemma 6.** Let \( \mathcal{Y}_n \) denote the domain of \( y_n \). Then, \( \lim_{n \to \infty} \sup_{|h| \leq r} \Lambda_{n,h}(\mathcal{Y}_n) = 1 \), and \( \Lambda_{n,h} \) is contiguous to \( P_{n,h_0} \). Furthermore, \( \lim_{n \to \infty} \tilde{P}_n(\mathcal{Y}_n) = 1 \), \( \tilde{P}_n \) is contiguous to \( P_{n,h_0} \) and for each \( \epsilon > 0 \) there exists \( M(\epsilon), N(\epsilon) < \infty \) such that \( \tilde{P}_n(A_n^\epsilon) < \epsilon \) for all \( M \geq M(\epsilon) \) and \( n \geq N(\epsilon) \).

**Proof.** Set \( P_{n,h} := P_{n,h_0+h/\sqrt{n}} \) and \( p_{n,h} = dP_{n,h}/d\nu \). Note that \( p_{n,0}(y_n) = \Pi_{i=1}^n p_0(Y_i) \), where \( p_0(\cdot) \) is the density function of \( P_{b_0}(Y) \). Then, by the definition of \( \Lambda_{n,h} \) and \( \lambda_{n,h}(\cdot) \), we can write \( \Lambda_{n,h}(\mathcal{Y}_n) \equiv \int \lambda_{n,h}(y_n)d\nu(y_n) \) as

\[
\Lambda_{n,h}(\mathcal{Y}_n) = (a_n(h))^n \quad \text{where} \quad a_n(h) := \int \exp \left\{ \frac{h}{\sigma^2 \sqrt{n}} \psi(Y_i) - \frac{h^2}{2\sigma^4 n} \right\} p_0(Y_i)d\nu(Y_i).
\]
Denote \( g_n(h, Y) = \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2 \sigma^2 n} \), \( \delta_n(h, Y) = \exp\{g_n(h, Y)\} - \{1 + g_n(h, Y) + g_n^2(h, Y)/2\} \) and \( \mathbb{E}_{p_0}[\cdot] \), the expectation corresponding to \( p_0(Y) \). Then,

\[
a_n(h) = \mathbb{E}_{p_0} \left[ \exp \left\{ \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2 \sigma^2 n} \right\} \right] \\
= \mathbb{E}_{p_0} \left[ 1 + g_n(h, Y) + \frac{1}{2} g_n^2(h, Y) \right] + \mathbb{E}_{p_0} [\delta_n(h, Y)] \\
:= Q_{n1}(h) + Q_{n2}(h).
\] (B.9)

Since \( \psi(\cdot) \) is the score function at \( \theta_0 \), \( \mathbb{E}_{p_0}[\psi(Y)] = 0 \) and \( \mathbb{E}_{p_0}[\psi^2(Y)] = \sigma^2 \). Using these results and the fact \( |h| \leq \Gamma \), straightforward algebra implies

\[
Q_{n1}(h) = 1 + b_n, \text{ where } b_n \leq \Gamma^4/4\sigma^4 n^2.
\]

We can expand \( Q_{n2} \) as follows:

\[
Q_{n2}(h) = \mathbb{E}_{p_0} \left[ \mathbb{I}_{\psi(Y) \leq K} \delta_n(h, Y) \right] + \mathbb{E}_{p_0} \left[ \mathbb{I}_{\psi(Y) > K} \delta_n(h, Y) \right].
\] (B.10)

Since \( |h| \leq \Gamma \) and \( e^x - (1 + x + x^2/2) = O(|x|^3) \), the first term in (B.10) is bounded by \( \sigma^{-6} K^3 \Gamma^2 n^{-3/2} \). Furthermore, for large enough \( n \), the second term in (B.10) is bounded by \( \mathbb{E}_{p_0} |\exp(\psi(Y))|/\exp(aK) \) for any \( a < 1 \). Hence, setting \( K = \ln n \) gives \( \sup_{|h| \leq \Gamma} Q_{n2}(h) = O(\ln^3 n/n^{3/2}) \). In view of the above,

\[
\sup_{|h| \leq \Gamma} |a_n(h) - 1| = O(n^{-c}) \text{ for any } c < 3/2.
\]

Thus, \( \sup_{|h| \leq \Gamma} |\Lambda_{n,h}(Y_n) - 1| = |\{1 + O(n^{-c})\}^n - 1| = O(n^{-(c-1)}) \). Since it is possible to choose any \( c < 3/2 \), this proves the first claim.

Under \( P_{n,0} \), the likelihood \( d\Lambda_{n,h}/dP_{n,0} \) converges weakly to some \( V \) satisfying \( \mathbb{E}_{P_{n,0}}[V] = 1 \) (the argument leading to this is standard, see, e.g., Van der Vaart, 2000, Example 6.5). Since \( \Lambda_{n,h}(Y_n) \to 1 \), an application of Le Cam’s first lemma implies \( \Lambda_{n,h} \) is contiguous with respect to \( P_{n,0} \).

Because \( m_0(\cdot) \) is supported on \( |h| \leq \Gamma \), \( \int \Lambda_{n,h}(Y_n) - 1 \leq \int |\Lambda_{n,h}(Y_n) - 1|m_0(h)dh = O(n^{-(c-1)}) \). Thus, \( \lim_{n \to \infty} \tilde{P}_n(Y_n) = 1 \). Contiguity of \( \tilde{P}_n \) with respect to \( P_{n,0} \) follows from the contiguity of \( \Lambda_{n,h} \) with respect to \( P_{n,0} \). The final claim, that \( \tilde{P}_n(A_n^c) < \epsilon \), follows by similar arguments as in the proof of Lemma 3. \( \square \)

**Lemma 7.** The measure, \( \tilde{P}_n \), can be disintegrated as in equation (A.15).
Proof. Let $\lambda_{n,h}(\cdot)$, $\tilde{S}_{nq}$ be defined as in the proof of Lemma 5. Equation (A.14) implies

$$\lambda_{n,h}(y_n) \cdot m_0(h) = \lambda_{n-1,h}(y_{n-1}) \cdot m_0(h) \cdot \tilde{p}(Y_n|h). \quad (B.11)$$

Let $\tilde{S}_{n-1}$ denote the probability measure corresponding to the density $d\tilde{S}_{n-1} = \lambda_{n-1,h}(y_{n-1}) \cdot m_0(h)$. As argued in the proof of Lemma 5, one can disintegrate this as $d\tilde{S}_{n-1} = p_n(h|y_{n-1}) \cdot \tilde{p}_{n-1}(y_{n-1})$, where $p_n(h|y_{n-1})$ is a conditional probability density and $\tilde{p}_{n-1}(y_{n-1}) = \int \lambda_{n-1,h}(y_{n-1})m_0(h)dh$. Thus,

$$\lambda_{n-1,h}(y_{n-1}) \cdot m_0(h) = p_n(h|y_{n-1}) \cdot \tilde{p}_{n-1}(y_{n-1}).$$

Combining the above with (B.11) gives

$$\lambda_{n,h}(y_n) \cdot m_0(h) = p_n(h|y_{n-1}) \cdot \tilde{p}_{n-1}(y_{n-1}) \cdot \tilde{p}(Y_n|h).$$

Taking the integral with respect $h$ on both sides, and making use of the definition of $\tilde{p}_n(\cdot)$,

$$\tilde{p}_n(y_n) = \tilde{p}_{n-1}(y_{n-1}) \cdot \int \tilde{p}(Y_n|h)p_n(h|y_{n-1})dh. \quad (B.12)$$

There is nothing special about the choice of $n$ here, so iterating the above expression gives the desired result:

$$\tilde{p}_n(y_n) = \prod_{i=1}^n \int \tilde{p}(Y_i|h)p_n(h|y_{i-1})dh.$$  

□

Lemma 8. Let $c_{n,i}$ and $\bar{P}_n$ denote the quantities defined in Step 4 of the proof of Theorem 5. There exists some non-random $C < \infty$ such that $\sup_i |c_{n,i} - 1| \leq Cn^{-c}$ for any $c < 3/2$. Furthermore, $\lim_{n \to \infty} \|\bar{P}_n - \tilde{P}_n\|_{TV} = 0$.

Proof. Denote

$$a_n(h) := \int \tilde{p}_n(Y_i|h)d\nu(Y_i) = \int \exp \left\{ \frac{h}{\sigma^2 n} \psi(Y_i) - \frac{h^2}{2\sigma^2 n} \right\} p_0(Y_i)d\nu(Y_i).$$

It is shown in the proof of Lemma 6 that $\sup_{|h| \leq \Gamma} |a_n(h) - 1| = O(n^{-c})$ for any $c < 3/2$. Since $c_{n,i} = \int a_n(h)\tilde{p}_n(h|y_{i-1})dh$, and $\tilde{p}_n(h|y_{i-1})$ is a probability density, this proves the first claim.
For the second claim, denote \( \tilde{p}_n(Y_i | y_{i-1}) := \int \tilde{p}_n(Y_i | h) \tilde{p}_n(h | y_{i-1}) dh \). We also write \( c_{n,i}(y_{i-1}) \) for \( c_{n,i} \) to make it explicit that this quantity depends on \( y_{i-1} \). The properties of the total variation metric, along with (A.15) and (A.16) imply

\[
\| \tilde{p}_n - \tilde{P}_n \|_{TV} = \frac{1}{2} \int \left| \frac{d\tilde{p}_n}{d\nu} - \frac{d\tilde{P}_n}{d\nu} \right| d\nu \leq \frac{1}{2} \sup_{y_n} \left| \prod_{i=1}^n \frac{1}{c_{n,i}(y_{i-1})} - 1 \right| \cdot \int \prod_{i=1}^n \tilde{p}_n(Y_i | y_{i-1}) d\nu(y_n).
\]

Recall from (A.15) that \( \prod_{i=1}^n \tilde{p}_n(Y_i | y_{i-1}) \) is the density (wrt \( \nu \)) of \( \tilde{P}_n \), so the integral in the above expression equals \( \int d\tilde{P}_n = \tilde{P}_n(Y) \to 1 \) by Lemma 6. Furthermore, using the first claim of this lemma, it is straightforward to show

\[
\sup_{y_n} \left| \prod_{i=1}^n \frac{1}{c_{n,i}(y_{i-1})} - 1 \right| = O(n^{-(c-1)}).
\]

Thus, \( \| \tilde{p}_n - \tilde{P}_n \|_{TV} = O(n^{-(c-1)}) \) and the claim follows. \( \Box \)

**Lemma 9.** For the probability measure \( \tilde{P}_n \) defined in Step 4 of the proof of Theorem 5, there exists a deterministic sequence \( \xi_n \to 0 \) independent of \( s \) and \( \pi \in \{0, 1\} \) such that equations (A.20) - (A.22) hold.

**Proof.** Start with (A.20). We have

\[
\mathbb{E}_{\tilde{P}_n} \left[ \psi(Y_{nq+1}) | s \right] = c_{n,nq+1}^{-1} \int \left\{ \int \psi(Y_{nq+1}) \tilde{p}_n(Y_{nq+1} | h) d\nu(Y_{nq+1}) \right\} \tilde{p}(h | x, q) dh
\]

\[
= c_{n,nq+1}^{-1} \int \mathbb{E}_{\tilde{P}_0} \left[ \psi(Y) \exp \left( \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2\sigma^2 n} \right) \right] \tilde{p}(h | x, q) dh
\]

\[
= \int \mathbb{E}_{\tilde{P}_0} \left[ \psi(Y) \exp \left( \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2\sigma^2 n} \right) \right] \tilde{p}(h | x, q) dh + O(n^{-c}),
\]

where the second equality follows by the definition of \( \tilde{p}(Y_i | h) \), and the third equality follows by (A.17). Define \( g_n(h, Y) = \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2\sigma^2 n} \) and \( \delta_n(h, Y) = \exp \{ g_n(h, Y) \} - \{1 + g_n(h, Y)\} \).

Then,

\[
\mathbb{E}_{\tilde{P}_0} \left[ \psi(Y) \exp \left( \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2\sigma^2 n} \right) \right]
\]

\[
= \mathbb{E}_{\tilde{P}_0} \left[ \psi(Y) \left\{ 1 + \frac{h}{\sigma^2 \sqrt{n}} \psi(Y) - \frac{h^2}{2\sigma^2 n} \right\} \right] + \mathbb{E}_{\tilde{P}_0} \left[ \psi(Y) \delta_n(h, Y) \right].
\]
Assumption 1(i) implies, see e.g., Van der Vaart (2000, Theorem 7.2), \( \mathbb{E}_{p_{n_0}}[\psi(Y)] = 0 \) and \( \mathbb{E}_{p_{n_0}}[\psi^2(Y)] = \sigma^2 \). Hence, the first term in the above expression equals \( h/\sqrt{n} \).

For the second term,

\[
\mathbb{E}_{p_{n_0}}[\psi(Y)\delta_n(h,Y)] = \mathbb{E}_{p_{n_0}}\left[ I_{|\psi(Y)| \leq K \psi(Y)} \psi(Y)\delta_n(h,Y) \right] + \mathbb{E}_{p_{n_0}}\left[ I_{|\psi(Y)| > K \psi(Y)} \psi(Y)\delta_n(h,Y) \right].
\]

(B.13)

Since \( |h| \leq \Gamma \) and \( e^x - (1 + x) = o(x^2) \), the first term in in (B.13) is bounded by \( \sigma^{-4} K^3 \Gamma^2 n^{-1} \). The second term in (B.13) is bounded by \( \mathbb{E}_{p_{n_0}}[|\psi(Y)|]/\exp(aK) \) for any \( a < 1 \). Hence, setting \( K = \ln n \) gives \( \sup_{|h| \leq \Gamma} |\mathbb{E}_{p_{n_0}}[\psi(Y)\delta_n(h,Y)]| = O(\ln n/n) \). In view of the above,

\[
\sqrt{n}\mathbb{E}_{\tilde{P}_n}\left[ |\psi(Y_n) - \psi(Y)|^2 \right] = \int h\tilde{p}(h|x,q)dh + O(\ln n/\sqrt{n}) = \mu(s) + \xi_n,
\]

where \( \xi_n \approx \ln n/\sqrt{n} \). This proves (A.20). The proofs of (A.21) and (A.22) are similar. \( \square \)

**B.3. Proof of Corollary 2.** We only sketch the argument: the proof follows the same outline as that of Theorem 5. The main difference is the need to allow for randomized policies. Under randomized policies, the variables \( \xi_j \) are deterministic functions of \( (y_n, u_n) \) where \( u_n := (u_1, \ldots, u_n) \) is a sequence of exogenous uniform random variables modeling the randomization in each period \( j \).

The actions are given by \( A_j = \{ \pi(y_{nq_j}) > u_j \} \). Let \( \mathcal{U}_n \) denote the joint probability measure of \( u_n \).

We make the following adjustments to the proof of Theorem 5:

\[ V_{\pi,n}^*(0) = \mathbb{E}_n\left[ \frac{1}{n} \sum_{j=1}^n \mathbb{E}\left[ R_n(h, \pi_j) | y_{nq_j(x)} \right] \right], \]

as in Step 3, but \( \mathbb{E}_n[\cdot] \) should now read as the expectation under the joint density \( \tilde{P}_n \times \mathcal{U}_n \). Similarly, \( \tilde{E}_n[\cdot] \) and \( \tilde{E}_{\tilde{P}_n}[\cdot] \) should read as the joint expectations under \( \tilde{P}_n \times \mathcal{U}_n \) and \( \tilde{P}_n \times \mathcal{U}_n \) respectively. Because \( \tilde{P}_n, \mathcal{U}_n \) are independent, and so are \( \tilde{P}_n, \mathcal{U}_n \), we have \( \left\| (\tilde{P}_n \times \mathcal{U}_n) \cap A_n - (\tilde{P}_n \times \mathcal{U}_n) \cap A_n \right\|_{TV} \leq \left\| \tilde{P}_n \cap A_n - \tilde{P}_n \cap A_n \right\|_{TV} \rightarrow 0. \) Similarly, \( \left\| \tilde{P}_n \times \mathcal{U}_n - \tilde{P}_n \times \mathcal{U}_n \right\|_{TV} \leq \left\| \tilde{P}_n - \tilde{P}_n \right\|_{TV} \rightarrow 0. \) On the other hand, the posterior distribution of \( h \) does not depend on \( u_n \), so the posterior densities and their approximations remain unchanged. With these changes, we can employ the same arguments as in Steps 1-4. The upshot of these arguments is a recursive expression for \( V_{\pi,n}^*(x,q,t) \) that is analogous to (3.4); the only difference from the proof of Theorem 5 is that while (A.18) there gives an expression for
inf_{\pi \in \Pi} \tilde{V}^*_{\pi,n}(s)$, we now write down a recursive expression for $\tilde{V}^*_{\pi,n}(s)$ itself. Given the recursive expression, we can apply the PDE approximation arguments in Steps 5 and 6 to show that $\tilde{V}^*_{\pi,n}(s)$ converges locally uniformly to $V^*_\pi(s)$.

**Appendix C. Theory for extensions of the bandit problem**

**C.1. Multi-armed bandits.**

*Existence of the solution to PDE (6.2).* If $\mu^{\max}(\cdot)$ and $\mu_k(\cdot)$ are Hölder continuous for all $k$, there exists a unique viscosity solution to PDE (6.2). This is a straightforward consequence of Barles and Jakobsen (2007, Theorem A.1).

*Convergence to the PDE.* In addition to Hölder continuity of $\mu^{\max}(\cdot)$ and $\mu_k(\cdot)$, we also require $\sup_s \varpi(s) < \infty$, where $\varpi(s) = \mu^{\max}(s) - \max_k \mu_k(s)$, and $\mathbb{E}[|\mu_k|^3 | s] < \infty$ for all $k$. An analogue of Theorem 2 can then be shown with a straightforward modification to the proof.

*Piece-wise constant policies.* The results on piece-wise constant policies continue to apply under the additional requirement of $\sup_s \mu^{\max}(s) < \infty$.

*Parametric and non-parametric distributions.* As described in Section 6, we set $\mathbb{P}_\theta := \mathbb{P}_{\theta}^{(1)} \times \cdots \times \mathbb{P}_{\theta}^{(K)}$ to be the probability distribution over the vector of outcomes, and it is without loss of generality to assume the distributions across arms are independent of each other. For each arm $k$, we have the SLAN property

$$\sum_{i=1}^{\lfloor nq_k(t) \rfloor} \ln \frac{dp_{\theta_0+h/\sqrt{n}}^{(k)}}{dp_{\theta_0}^{(k)}} = \frac{h}{\sigma_k^2} x^{(k)} - \frac{q_k}{2\sigma_k^2} h^2 + o_{p_{\theta_0}^{(k)}}(1),$$

uniformly over $q$, (C.1) where $p_{\theta_0}^{(k)}$ is the density of $\mathbb{P}_{\theta_0}^{(k)}$ with respect to some dominating measure $\nu := \nu^{(1)} \times \cdots \times \nu^{(K)}$, $x^{(k)} := \sigma_k^2 n^{-1/2} \sum_{i=1}^{\lfloor nq_k(t) \rfloor} \psi_k(Y_{ik})$, $\psi_k(\cdot)$ is the score function corresponding to $\mathbb{P}_{\theta_0}^{(k)}$ and $\sigma_k^2$ is the corresponding information matrix, i.e., $\sigma_k^2 = \left(\mathbb{E}_{\mathbb{P}_{\theta_0}^{(k)}}[\psi_k^2]\right)^{-1}$.

Let $m_0$ denote the prior over $h$. Also, denote $y_{nk}^{(k)} := (Y_{1k}, \ldots, Y_{Kq_k})$ as the vector of stacked outcomes for each arm. Then, in the fixed $n$ setting, the posterior distribution of $h$ is (compare the equation below with (4.4))

$$p_n(h|\mathcal{F}_t) = p_n(h|y_{nk}^{(1)}, \ldots, y_{nk}^{(K)}) \propto \left\{ \prod_{k=1}^{K} \prod_{i=1}^{\lfloor nq_k(t) \rfloor} p_{\theta_0+h/\sqrt{n}}^{(k)}(Y_{ik}) \right\} \cdot m_0(h).$$

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As in Section 4, we aim to approximate the bracketed term above with a product of tilted measures implied by (C.1). The approximate posterior is given by

\[ \tilde{p}_n(h|s) \propto \prod_{k=1}^{K} \tilde{p}_{q_k}(x_k|h) \cdot m_0(h); \text{ where } \tilde{p}_{q_k}(-|h) \equiv N(q_k h, q_k \sigma^2_k). \]

The above suggests that we can obtain an analogous result to Theorem 5. Formally, the argument employs the same steps as in the proof of Theorem 5, with the key difference being that the relevant likelihood is \( \prod_{k=1}^{K} \prod_{i=1}^{|nq(t)|} p_{\theta_0+h/\sqrt{n}}(Y_k) \) as compared to \( \prod_{i=1}^{|nq(t)|} p_{\theta_0+h/\sqrt{n}}(Y_k) \) before. The assumption of independent distributions across arms is convenient here, and helps simplify the proofs. The requirements for the proof are also similar to Assumption 1: the only difference is that Assumptions 1(i)-(iii) need to apply to each arm \( k \) and Assumption 1(v) uses \( \mu^{\max}(\cdot) \) in place of \( \mu^+(\cdot) \) and \( \varpi(s) \).

The modifications to Section 5 are analogous.

C.2. Best arm identification. The assumptions required are the same as that for multi-armed bandits.

Existence of the solution to PDE (6.4). Under the stated assumptions on the boundary condition \( \varpi(s) \), i.e., Hölder continuity and \( \sup_s \varpi(s) < \infty \), along with Hölder continuity of \( \mu^{\max}(s) \), there exists a unique viscosity solution to PDE (6.4). This again follows from Barles and Jakobsen (2007, Theorem A.1).

Convergence to the PDE. Recall that the relevant state variables are \( s = (x_1, q_1, \ldots, x_K, q_K) \).

In analogy with (3.2), the discrete analogue to PDE (6.4) is given by the recursive equation

\[
V^*_n(x_1, q_1, \ldots, x_k, q_k, t) = \mathbb{I}_n \cdot \varpi(s) + \cdots \\
\cdots + \min_{\pi_1, \ldots, \pi_K \in [0,1]} \mathbb{E} \left[ \mathbb{I}_n \cdot V^*_n \left( \ldots, x_k + \frac{\pi_k Y_{nq_k+1, k} \sqrt{n}}{n}, q_k + \frac{\pi_k}{n}, \ldots, t + \frac{1}{n} \right) \bigg| s \right] \]  (C.2)

where \( \mathbb{I}_n := \mathbb{I}\{t \geq 1/n\} \). The solution, \( V^*_n(\cdot) \), of the above converges locally uniformly to the viscosity solution, \( V^*(\cdot) \), of PDE (6.4). We can show this by modifying

\( \text{For instance, it implies that the joint probability } P^{(1)}_{nq_{a_1},h} \times \cdots \times P^{(K)}_{nq_{a_K},h} \text{ is contiguous to } P^{(1)}_{nq_{a_1},0} \times \cdots \times P^{(K)}_{nq_{K},0} \text{ for any } (q_{a_1}, \ldots, q_{a_K}) \to (q_1, \ldots, q_K) \text{ as long as } P^{(k)}_{nq_{a_k},h} \text{ is contiguous with respect to } P^{(k)}_{nq_{a_k},0} \text{ for each } k. \) This enables us to prove the counterpart of Lemma 3, which is the key element to the proof of Theorem 5.

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the proof of Theorem 2 to account for the non-zero boundary condition. As in that
proof, after a change of variables \( \tau = 1 - t \), we can characterize \( V_n^*(\cdot) \) as the solution
to \( S_n(s, \phi(s), [\phi]) = 0 \), where for any \( u \in \mathbb{R} \) and \( \phi : S \to \mathbb{R} \), and \( \mathbb{I}_n := \{ \tau > 1/n \}, \)
\[
S_n(s, u, [\phi]) := -\mathbb{I}_n \cdot \frac{(\varpi(s) - u)}{n} - \ldots
\]
\[
\ldots - \mathbb{I}_n \cdot \min_{\pi \in [0,1]} \left\{ \mathbb{E} \left[ \phi \left( \ldots, x_k + \frac{\pi_k Y_{nq_k+1,k}}{\sqrt{n}}, q_k + \frac{\pi_k}{n}, \ldots, \tau - \frac{1}{n} \right) - u \mid s \right] \right\}.
\]
Define \( F(D^2\phi, D\phi, s) = \partial_s \phi - \min_k L_k[\phi](s) \).

We need to verify monotonicity, stability and consistency of \( S_n(\cdot) \). Monotonicity of
\( S_n(s, u, [\phi]) \) is clearly satisfied. Stability is also straightforward under the assumption
\( \sup u \varpi(s) < \infty \). The consistency requirement is more subtle. For interior values,
i.e., when \( s := (x, q, \tau) \) is such that \( \tau > 0 \), the usual conditions (A.4) and (A.5) are required to hold with the definitions of \( S_n(\cdot), F(\cdot) \) above. These can be shown using
the same Taylor expansion arguments as in the proof of Theorem 2. For boundary
values, \( s \in \{ (x, q, 0) : x \in \mathcal{X}, q \in [0,1] \} := \partial \mathcal{S} \), the consistency requirements are
(see, Barles and Souganidis, 1991)
\[
\limsup_{n \to \infty} \limsup_{\rho \to 0} \frac{n S_n(z, \phi(z) + \rho, [\phi + \rho])}{\rho} \leq \max \left\{ F(D^2\phi(s), D\phi(s), s), \phi(s) - \varpi(s) \right\}, \quad (C.3)
\]
\[
\liminf_{n \to \infty} \liminf_{\rho \to 0} \frac{n S_n(z, \phi(z) + \rho, [\phi + \rho])}{\rho} \geq \min \left\{ F(D^2\phi(s), D\phi(s), s), \phi(s) - \varpi(s) \right\}. \quad (C.4)
\]
We can show (C.3) as follows (the proof of (C.4) is similar): By the definition of
\( S_n(\cdot) \), for every sequence \( (n \to \infty, \rho \to 0, z \to s \in \partial \mathcal{S}) \), there exists a sub-sequence
such that either \( n S_n(z, \phi(z) + \rho, [\phi + \rho]) = \phi + \rho - \varpi(z) \) or
\[
n S_n(z, \phi(z)+\rho, [\phi+\rho]) = -\min_{\pi \in [0,1]} \left\{ \mathbb{E} \left[ \phi \left( \ldots, x_k + \frac{\pi_k Y_{nq_k+1,k}}{\sqrt{n}}, q_k + \frac{\pi_k}{n}, \ldots, \tau - \frac{1}{n} \right) - u \mid s \right] \right\}.
\]
In the first instance, \( n S_n(z, \phi(z)+\rho, [\phi+\rho]) \to \phi(s) - \varpi(s) \) by the continuity of \( \varpi(\cdot) \),
while the second instance gives rise to the same expression for \( S_n(\cdot) \) as being in the
interior, so that \( n S_n(z, \phi(z)+\rho, [\phi+\rho]) \to F(D^2\phi(s), D\phi(s), s) \) by similar arguments
as in the proof of Theorem 2. Thus, in all cases, the limit along subsequences is
smaller than the right hand side of (C.3).
**Piecewise-constant policies.** The results on piece-wise constant policies continue to apply since Barles and Jakobsen (2007, Theorem 3.1) holds under any continuous boundary condition.

**Parametric and non-parametric distributions.** The analogues of Theorems 5 and 6 follow by the reasoning as that employed for multi-armed bandits. In fact, the arguments are simplified since the loss function now only involves the value of regret at $t = 1$.

### C.3. Discounting

The assumptions required are the same as in Theorems 2-6 in the main text.

**Existence of the solution to PDE (6.5).** A unique viscosity solution for PDE (6.5) exists under the same assumptions as Theorem 1, see Barles and Jakobsen (2007, p. 29).

**Convergence to PDE.** Under Gaussian rewards, the discrete analogue to PDE (6.5) is

$$V^*_n(x, q) = \min_{\pi \in [0, 1]} \mathbb{E} \left[ \frac{\mu^+(s) - \pi \mu(s)}{n} + e^{-\beta/n} V^*_n \left( x + \frac{A_n Y_{nq+1}}{\sqrt{n}}, q + \frac{A_n}{n} \right) \right] . \quad (C.5)$$

A straightforward modification of the proof of Theorem 2 then shows $V^*_n(\cdot)$ converges locally uniformly to $V^*(\cdot)$, the viscosity solution of PDE (6.5). There is no analogue to piece-wise constant policies in the discounted setting.

**Parametric and non-parametric distributions.** The proofs of Theorems 5 and 6 are slightly complicated by the fact $q$ is now unbounded. While the SLAN property (4.2) applies even if $q > 1$, it does require $q < \infty$. We can circumvent this issue by exploiting the fact that the infinite horizon problem is equivalent to a finite horizon problem with a very large time limit. In other words, we prove the relevant results for the PDE

$$\partial_t V^* - \beta V^* + \mu^+(s) + \min \{ -\mu(s) + L[V^*](s), 0 \} = 0 \text{ if } t < 1,$$

$$V^*(s) = 0 \text{ if } t = T, \quad (C.6)$$

with the boundary condition set at $t = T$, and then let $T \to \infty$.  

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Let \( V^*(0), V^*(0; T) \) denote the viscosity solutions to PDEs (6.5) and (C.6) - the latter when the boundary condition is set at \( t = T \) - evaluated at \( s_0 \). Following (4.10), the ex-ante expected Bayes under a policy \( \pi \) in the fixed \( n \) setting with discounting is

\[
V_{\pi,n}(0) = \mathbb{E}_{(y_n,h)} \left[ \frac{1}{n} \sum_{j=1}^{\infty} e^{-\beta j/n} R(h, \pi_j) \right].
\] (C.7)

Analogously, if we terminate the experiment at a suitably large \( T \), we have

\[
V_{\pi,n}(0; T) = \mathbb{E}_{(y_n,h)} \left[ \frac{1}{n} \sum_{j=1}^{nT} e^{-\beta j/n} R(h, \pi_j) \right].
\]

Under Assumption 1, \( R(h, \pi) \leq C < \infty \) (due to the compactness of the prior \( m_0 \)), so \( \sup_{\pi \in \Pi} |V_{\pi,n}(0) - V_{\pi,n}(0; T)| \leq e^{-\beta T} \). Now, a straightforward modification of the proof of Theorem 5 implies \( \lim_{n \to \infty} \inf_{\pi \in \Pi} V_{\pi,n}(0; T) = V^*_n(0; T) \), where \( V^*(0; T) \) is the viscosity solution to PDE (C.6) evaluated at \( s_0 \). Finally, it can be shown, e.g., by approximating the PDEs with dynamic programming problems as in Theorem 2, that \( |V^*(0; T) - V^*(0)| \leq e^{-\beta T} \). Since we can choose \( T \) as large as we want, it follows \( \lim_{n \to \infty} \inf_{\pi \in \Pi} V_{\pi,n}(0) = V^*_n(0) \). The proof of Theorem 6 can be modified in a similar manner.

**Appendix D. Details on computation**

PDE (2.7) can be solved efficiently using ‘upwind’ finite-difference methods. This involves discretizing both the spatial (i.e. \( \mathcal{X} \) and \( \mathcal{Q} \)) and time domains. Let \( i,j \) index the grid points for \( x,q \) respectively, with the grid lengths being \( \Delta x, \Delta q \). For this section, we switch the direction of time (so \( t = 1 \) earlier is now \( t = 0 \)) and discretize it as \( 0, \Delta t, \ldots, m \Delta t, \ldots, 1 \).

We approximate the second derivative \( \partial_x^2 V \) using

\[
\partial_x^2 V \approx \frac{V_{i+1,j}^m + V_{i-1,j}^m - 2V_{i,j}^m}{(\Delta x)^2}.
\]

As for the first order derivatives, we approximate by either \( \frac{V_{i+1,j}^m - V_{i,j}^m}{\Delta x} \) or \( \frac{V_{i,j}^m - V_{i-1,j}^m}{\Delta x} \) depending on whether the associated drift, i.e., the coefficient multiplying \( \partial_x V \) is positive or negative. This is known as up-winding and is crucial for ensuring the resulting approximation procedure is ‘monotone’ (see Appendix A, and also Achdou...
et al. (2017) for a discussion of monotonicity). In our setting, this implies
\[
\partial_x V \approx \frac{V_{i+1,j}^m - V_{i,j}^m}{\Delta x} \mathbb{I}(\mu(s) \geq 0) + \frac{V_{i,j}^m - V_{i-1,j}^m}{\Delta x} \mathbb{I}(\mu(s) < 0)
\]
:= \left( \frac{V_{i+1,j}^m - V_{i,j}^m}{\Delta x} \right)_+,
\]
while \(\partial_q V\), which is associated with the coefficient 1, is approximated as
\[
\partial_q V \approx \frac{V_{i,j}^{m+1} - V_{i,j}^m}{\Delta q}.
\]
Finally, let \(\mu_{i,j}^+, \mu_{i,j}\) denoted the values of \(\mu^+(\cdot), \mu(\cdot)\) evaluated at the grid points \(i, j\).

**D.0.1. Explicit schemes.** The explicit finite difference scheme for estimating \(V^*(\cdot)\) is obtained as the solution to
\[
V_{i,j}^{m+1} = V_{i,j}^m + \mu_{i,j}^+ + \min \left\{ 0, -\mu_{i,j} + \frac{V_{i,j}^m - V_{i+1,j}^m}{\Delta q} \right\} + \mu_{i,j} \left( \frac{V_{i+1,j}^m - V_{i,j}^m}{\Delta x} \right)_+ + \frac{1}{2} \sigma^2 \frac{V_{i,j}^{m+1} - V_{i,j}^m}{(\Delta x)^2} = 0,
\]
\[
V_{i,j}^0 = 0 \quad \forall \quad i, j.
\]
The above can be computed straightforwardly by starting from the boundary condition and progressively computing the values of \(V_{i,j}^m\) at each time step \(m\). However, to ensure convergence to the PDE solution, explicit schemes require the CFL condition to hold: \(\Delta t \leq 0.5 \min \{ (\Delta x)^2, (\Delta q)^2 \} \). In practice, this implies very small time steps, which raises the scope for numerical instabilities.

**D.0.2. Implicit schemes.** Implicit finite difference schemes do not require any condition on the time-step and are therefore much more stable numerically. The implicit upwind scheme solves
\[
V_{i,j}^{m+1} = V_{i,j}^m + \mu_{i,j}^+ + \min \left\{ 0, -\mu_{i,j} + \frac{V_{i,j}^{m+1} - V_{i,j}^m}{\Delta q} \right\} + \mu_{i,j} \left( \frac{V_{i+1,j}^{m+1} - V_{i,j}^m}{\Delta x} \right)_+ + \frac{1}{2} \sigma^2 \frac{V_{i,j}^{m+1} - V_{i,j}^m}{(\Delta x)^2} = 0, \quad (D.1)
\]
\[
V_{i,j}^0 = 0 \quad \forall \quad i, j.
\]
Algorithm 1 Implicit FD using Howard algorithm

Initialize $V_{i,j}^0 = 0$

For $m = 0, \ldots, M - 1$:

Write (D.1) as $\min_{a \in \{0,1\}} \{A_{m+1}^a V_{m+1} - V_m + X^a\}$ where $V_m := \text{vec}(V_{i,j}^m : i, j)$

Randomly initialize $a_0 = \text{vec}(a_{i,j}^m \in \{0,1\} : i, j)$

Repeat:

$w = (A_{m+1}^{a_0})^{-1} (V_m - X^{a_0})$

$a_0 = \arg\min_{a} \{A_{m+1}^a w - V_m + X^a\}$

Until: Convergence criteria for $w$ are reached

$V_{m+1} = w$

The implicit scheme solves a nonlinear equation at each time step. This can be solved efficiently using policy iteration, also known in this setting as the Howard algorithm, see Algorithm 1.

D.0.3. Convergence of numerical solutions. Let $\Delta := \max\{\Delta x, \Delta q, \Delta t\}$, and define $V_{\Delta}^\ast(\cdot)$ as the solution to either the upwind or downwind schemes (even though the schemes only define $V_{\Delta}^\ast(\cdot)$ on a grid, the discussion below is valid under any interpolation method). The following theorem guarantees convergence of $V_{\Delta}^\ast(\cdot)$ to $V^\ast(\cdot)$ when $\Delta \to 0$:

**Theorem 7.** Suppose that $\mu^+(\cdot), \mu(\cdot)$ are uniformly continuous, and that the CFL condition $\Delta t \leq 0.5 \min \{(\Delta x)^2, (\Delta q)^2\}$ holds for the explicit scheme. Then, $V_{\Delta}^\ast(\cdot)$ converges uniformly to $V^\ast(\cdot)$ as $\Delta \to 0$.

The proof is a straightforward application of the results of Barles and Souganidis (1991). See Barles and Jakobsen (2007) for results on the rates of convergence as a function of $\Delta$.

D.0.4. Piece-wise constant policy approximations. The optimal piece-wise constant policy can obtained by recursively computing (3.5). This involves solving the linear PDE (3.6) - which can be done efficiently using the implicit scheme described above, or through Monte-Carlo using the Feynman-Kac formula.
The procedure is particularly well-suited for multi-armed bandits. Let \( j = 1, \ldots, J \) index the various arms. We then have the following recursion for the value function \( V_{\Delta t,k}^*(x_1, q_1, \ldots, x_J, q_J) \), under a piece-wise constant policy with grid points \( T_{\Delta t} := \{ t_1, \ldots, t_K \} \):

\[
V_{\Delta t,k+1}^*(x_1, q_1, \ldots, x_J, q_J) = \min_{j=1,\ldots,J} \left\{ S_{\Delta t}^{(j)} \left[ V_{\Delta t,k}^* \right] (x_1, q_1, \ldots, x_J, q_J) \right\}, \quad k = 0, \ldots, K-1,
\]

\[
V_{\Delta t,0}(x_1, q_1, \ldots, x_J, q_J) = 0, \tag{D.2}
\]

where for each \( j \), the operator \( S_{\Delta t}^{(j)}[\phi](x_1, q_1, \ldots, x_J, q_J) \) denotes the solution at \((x_1, q_1, \ldots, x_J, q_J, \Delta t)\) of the linear second order PDE

\[
-\partial_t f(s) + \mu_{\text{max}}(s) - \mu_j(s) + L_j[f](s) = 0, \quad \text{if } t > 0; \quad f = \phi, \quad \text{if } t = 0. \tag{D.3}
\]

Recall from Section 6.1 that the operator \( L_j[f] \) only involves derivatives with respect to \((x_j, q_j)\). Thus computing (D.2) involves solving \( J \) different linear PDEs of the form (D.3), with derivatives involving only two dimensions. The computational cost of solving such PDEs is fairly low.

**D.0.5. Hybrid schemes.** Consider a piece-wise policy approximation where the linear PDEs are solved by iterating only once under an implicit FD scheme. In other words, we take the time step in the implicit FD schemes solving the \( K \) linear PDEs to be the same as the grid size \( \Delta t \). Here, \( \Delta t \) is chosen to be a small value with the idea \( \Delta t \to 0 \), similar to the choice in Section D.0.2. This results in a hybrid scheme that is faster than the standard implicit scheme as it does not require policy iteration. At the same time, it is more numerically stable than the explicit scheme as it does not require the CFL condition. To describe the resulting algorithm in the one armed bandit setting, define \( \tilde{V}_{i,j}^{m+1,1} \) in terms of \( V_{i,j}^m \) as the solution to

\[
\tilde{V}_{i,j}^{m+1,1} = V_{i,j}^m + \mu_{i,j}^+ - \mu_{i,j} + \frac{\tilde{V}_{i,j+1}^{m+1,1} - \tilde{V}_{i,j}^{m+1,1}}{\Delta q} + \mu_{i,j} \left( \frac{\tilde{V}_{i+1,j}^{m+1,1} - \tilde{V}_{i,j}^{m+1,1}}{\Delta x} \right) + \frac{1}{2} \sigma^2 \frac{\tilde{V}_{i+1,j}^{m+1,1} + \tilde{V}_{i-1,j}^{m+1,1} - 2 \tilde{V}_{i,j}^{m+1,1}}{(\Delta x)^2} = 0, \tag{D.4}
\]

and also set

\[
\tilde{V}_{i,j}^{m+1,0} := V_{i,j}^m + \mu_{i,j}^+.
\]

The hybrid FD scheme is described in Algorithm 2.
**Algorithm 2** Hybrid FD

Initialize $V_{i,j}^0 = 0$

For $m = 0, \ldots, M - 1$:

Write (D.4) as $A\tilde{V}_{m+1}^1 - V_m + X = 0$ where $\tilde{V}_m^{(1)} = \text{vec}(\tilde{V}_{m;i,j}^{m+1})$

$\tilde{V}_{m+1}^1 = A^{-1}(V_m - X)$

$\tilde{V}_{m+1}^0 = V_m + \mu^+$ where $\mu^+ = \text{vec}(\mu_{i,j}^+; i, j)$

$V_{m+1} = \min\{\tilde{V}_{m+1}^1, \tilde{V}_{m+1}^0\}$ where the minimum is computed element-wise

D.0.6. **Solving the discounted bandit problem.** Under discounting, the value function is independent of time, so we only need discretize the spatial domains. Let $i, j$ index the grid points for $x, q$ respectively with grid lengths $\Delta x, \Delta q$. The implicit upwind scheme for PDE (6.5) solves

$$\beta V_{i,j} = \mu_{i,j}^+ + \min \left\{ 0, -\mu_{i,j} + \frac{V_{i,j+1} - V_{i,j}}{\Delta q} \right. $$

$$+ \mu_{i,j} \left( \frac{V_{i+1,j} - V_{i,j+1}}{\Delta x} \right) + \frac{1}{2} \sigma^2 \left( \frac{V_{i+1,j} + V_{i-1,j} - 2V_{i,j}}{\Delta x} \right) \left\} = 0. \quad (D.5)$$

As there is no time-stepping, equation (D.5) has to be solved for $V_{i,j}$ only once. Thus, the discounted bandit problem can be computed much faster than the one with a fixed number of periods.

D.0.7. **Implementation details for the results in Section 7.** For the numerical results in Section 7, we used the hybrid finite-difference scheme with $\Delta x = 1/1000$, $\Delta q = 1/500$ and $\Delta t = 1/1000$. Since $x$ is unbounded, for the purposes of computation we set its upper and lower bounds to $-2.5\sigma$ and $2.5\sigma$. The accuracy of the scheme was evaluated using Monte-Carlo (see Section 7.3 in the main text).

D.0.8. **Computation of least favorable prior in Section 7.4.** We solve the two player game (2.8) numerically. The basic idea is to search over priors with two-point support by varying $\mu$, $\bar{\mu}$ and $p := m_0^*(\bar{\mu})$. Given any prior, we compute the corresponding optimal Bayes policy by solving PDE (2.7) as described previously. We then compute the frequentist risk profile of this Bayes policy using Monte-Carlo (we used 20000 Monte-Carlo replications with $n = 5000$ over a grid of $\mu$ values with interval-width 0.1). The risk profile typically involves two peaks at (say) $\mu^l$ and
\( \mu^r \). Let \( R^l \) and \( R^r \) denote the corresponding frequentist risks at these values. The parameters of the prior are then updated as

\[
\begin{align*}
\mu &\leftarrow \mu + \alpha_1 \frac{\mu^l - \mu}{\mu^l}, \\
\bar{\mu} &\leftarrow \bar{\mu} + \alpha_2 \frac{\mu^r - \bar{\mu}}{\mu^r}, \\
p &\leftarrow p + \alpha_3 \frac{R^r - R^l}{\min\{R^l, R^r\}}.
\end{align*}
\]

Here \((\alpha_1, \alpha_2, \alpha_3)\) are learning rates that determine the speed of updates. We took these to be 0.1, and the updates were rounded to the nearest 0.5 decimal in the case of \( \mu \) and \( \bar{\mu} \) and to the nearest 0.005 decimal in the case of \( p \). The accuracy is limited by the computational time required to solve PDE (2.7). We started off with \((\mu = -2.5, \bar{\mu} = 2.5, p = 0.5)\). The final iterations required some manual tweaking.

**Appendix E. Comparison with UCB**

The UCB algorithm (Lai and Robbins, 1985) is another popular bandit algorithm. In the context of one-armed bandits, UCB policy rule is given by

\[
\pi_{\text{UCB}}(s) = \mathbb{I}\left\{ \hat{\mu}(t) + \sqrt{\frac{2\delta \ln n}{q(t)}} \geq 0 \right\},
\]

where \( \hat{\mu}(t) = x(t)/q(t) \) is the estimate of \( \mu \) at time \( t \), and \( \delta \) is a tuning parameter.

We compare the Bayes risk of UCB to the minimal Bayes risk under the same parameter values as in Section 7.3. The performance of UCB is quite sensitive to the choice of \( \delta \). The asymptotically optimal choice (Lattimore and Szepesvári, 2020, Chapter 8), which replaces \( \delta \ln n \) in the definition of \( \pi_{\text{UCB}}(\cdot) \) with \( \ln (1 + t(\ln t)^2) \), performs poorly in the current setting. We instead chose \( \delta \) by performing a grid search; the optimal choice turns out to be \( \delta^* = 7.8 \). Under \( \delta^* \), which is infeasible, UCB outperforms Thompson sampling. Even with this optimal choice, however, Figure E.1 shows that the risk of UCB is much higher than that of the optimal Bayes policy. For \( n = 200 \), the risk of UCB is around 35% higher, while for \( n = 2500 \), the risk is 85% higher. It is seen that the risk of UCB increases with \( n \), perhaps because we do not let \( \delta \) vary with \( n \). As noted in Section 3.2, Corollary 1 does not apply to the UCB policy, so it is unclear whether the Bayes risk of UCB converges to some bounded value as \( n \to \infty \).
A: UCB

B: Optimal Bayes policy

Note: The parameter values are $\mu_0 = 0$, $\nu = 50$ and $\sigma = 5$. Black lines within the bars denote the Bayes risk in finite samples. The bars provide the interquartile range for the regret.

**Figure E.1.** Comparison with UCB