Strong openness of multiplier ideal sheaves and optimal $L^2$ extension

In memory of Professor LU QiKeng (1927–2015)

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Abstract In this paper, we reveal that our solution of Demailly’s strong openness conjecture implies a matrix version of the conjecture; our solutions of two conjectures of Demailly-Kollár and Jonsson-Mustaţă implies the truth of twisted versions of the strong openness conjecture; our optimal $L^2$ extension implies Berndtsson’s positivity of vector bundles associated to holomorphic fibrations over a unit disc.

Keywords strong openness conjecture, plurisubharmonic function, multiplier ideal sheaf, optimal $L^2$ extension theorem

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1 Introduction

The multiplier ideal sheaves giving an invariant of plurisubharmonic singularities play an important role in several complex variables and complex geometry. Various important and fundamental properties about the multiplier ideal sheaves have been established, e.g., coherence, integrally closedness, Nadel vanishing theorem, the restriction formula and subadditivity property, the strong openness property (i.e., solution of Demailly’s strong openness conjecture), and other important properties. There have been found many interesting applications of these properties.

The Ohsawa-Takegoshi $L^2$ extension and its optimal version was used in the proofs of some above properties and their applications, e.g., the restriction formula and subadditivity property, the strong openness property, and some other properties and applications.

In this paper, we give some further consequences of our recent works about the strong openness of the multiplier ideal sheaves and optimal $L^2$ extension.

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1.1 A matrix version of Demailly’s strong openness conjecture

Let $X$ be a complex manifold with dimension $n$ and $\varphi$ be a plurisubharmonic function (psh for short) on $X$ (see [31, 48, 49, 56]). The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is defined to be the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-2\varphi}$ is locally integrable (see [11, 12, 38, 50, 52, 53, 58]). The basic properties of multiplier ideal sheaves include: $\mathcal{I}(\varphi)$ is a coherent analytic and integrally closed sheaf and satisfies the Nadel vanishing theorem.

Let $\mathcal{I}_+(\varphi) := \bigcup_{\epsilon > 0} \mathcal{I}((1 + \epsilon)\varphi)$.

In [22], Guan and Zhou proved Demailly’s strong openness conjecture posed in [11, 12] (see also [27, 30, 34, 36]) (related effectiveness result see [25]).

Theorem 1.1 (See [22]). $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ holds.

This means that the multiplier ideal sheaf has strong openness property. As immediate applications, several questions are solved in [24], such as, a problem about the existence of an analytic weight in a multiplier ideal sheaf, a conjecture about a more general vanishing theorem than Nadel’s posed by Demailly (see also [8]), a conjecture posed by Demailly et al. [14] and Lazarsfeld [33], and a conjecture posed by Boucksom et al. [7], etc.

Recently, Guan and Zhou [26] characterized the multiplier ideal sheaves with psh weights of Lelong number one by using the strong openness property of the multiplier ideal sheaf (see Theorem 1.1 (dimension two case was solved in [6, 18])). When Lelong number is smaller than one, it was proved by Skoda [55] that the multiplier ideal sheaves are trivial (see also [12, 13]).

When $\mathcal{I}(\varphi)$ is trivial, Demailly’s strong openness conjecture degenerates to the openness conjecture posed in [16] which was proved by Berndtsson [5] (dimension two case was proved by Favre and Jonsson [18]).

After the strong openness conjecture was proved, Ohsawa was invited by the second author to the Institute of Mathematics in Chinese Academy of Sciences in January 2014 and gave three lectures. During his lectures, Ohsawa asked the following matrix version of the strong openness conjecture:

Question. Let $(F_{i,j})_{1 \leq i \leq k, 1 \leq j \leq l}$ be the matrix such that $F_{i,j}$ are holomorphic functions on $X$ and

$$\int_{\Delta^n} \det((F_{i,j}) \times (F_{i,j})^t) e^{-\varphi} d\lambda_n < +\infty.$$  

Does there exist a number $p > 1$, such that

$$\int_{\Delta^n} \det((F_{i,j}) \times (F_{i,j})^t) e^{-p\varphi} d\lambda_n < +\infty,$$

where $r \in (0, 1)$?

Note that, by using the Cauchy-Binet formula,

$$\det((F_{i,j}) \times (F_{i,j})^t) = \sum_{|J|=k} \det((F_{i,j})_{1 \leq i \leq k, j \in J} \times (F_{i,j})^t_{1 \leq i \leq k, j \in J})$$

$$= \sum_{|J|=k} \det((F_{i,j})_{1 \leq i \leq k, j \in J}) \times \det((F_{i,j})^t_{1 \leq i \leq k, j \in J})$$

$$= \sum_{|J|=k} \det((F_{i,j})_{1 \leq i \leq k, j \in J}) \times \det((F_{i,j})^t_{1 \leq i \leq k, j \in J}),$$

(1.1)

where $J \subset \{1, \ldots, l\}$. Then the above question degenerates to the strong openness conjecture, which can be answered by our solution of the strong openness conjecture.

Proposition 1.2. The matrix version of the strong openness conjecture holds.
1.2 Twisted versions of Demailly’s strong openness conjecture

Let $I$ be an ideal of $\mathcal{O}_{\Delta^n, o}$, which is generated by $\{f_j\}_{j=1, \ldots, l}$. Denote
$$\log |I| := \log \max_{1 \leq j \leq l} |f_j|.$$ 

The jumping number is defined to be $c^I_o(\varphi) = \sup\{c \geq 0 : |I|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } o\}$ (see [29]). One can check that $\nu(\varphi, o) = 0 \Leftrightarrow c_o(\varphi) = +\infty \Leftrightarrow c^I_o(\varphi) = +\infty$, where $I \neq \{0\}$, and $c_o(\varphi)$ is the complex singularity exponent, which was also called the log canonical threshold in algebraic geometry (see [32, 47]).

It is known that $L_+(\varphi) = \mathcal{I}(\varphi)$ for any psh $\varphi$ is equivalent to the statement that $|I|^2 \exp(-2c^I_o(\varphi)\varphi)$ is not locally integrable near $o$ for any $I$ and $\varphi$ satisfying $c^I_o(\varphi) < +\infty$. In [23, 25], Guan and Zhou proved two conjectures posed by Demailly and Kollár [16] and Jonsson and Mustată [28], respectively by using the following result.

**Theorem 1.3** (See [23, 25]). If $c^I_o(\varphi) < +\infty$, then
$$\frac{1}{r^2} \int |I|^2 \chi_{\{c^I_o(\varphi)\varphi < \log r\}}$$
and
$$\frac{1}{r^2} \int \chi_{\{c^I_o(\varphi)\varphi < \log |I| < \log r\}}$$
have positive lower bounds independent of $r \in (0, 1)$.

Dimension two case of the above theorem was proved by Jonsson and Mustata [28] ($I$ is trivial, see [17, 18]). The proof of the general case is based on our solution of Demailly’s strong openness conjecture [22] and our solution of the $L^2$ extension problem with optimal estimates [19–21].

In the present paper, we deduce that Theorem 1.3 implies the the following twisted version of Demailly’s strong openness conjecture.

**Theorem 1.4.** Let $a(t)$ be a positive measurable function on $(-\infty, +\infty)$ such that $a(t)e^t$ is strictly increasing and continuous near $+\infty$. Then the following three statements are equivalent:

(A) $a(t)$ is not integrable near $+\infty$;

(B) $a(-2c^I_o(\varphi)\varphi) \exp(-2c^I_o(\varphi)\varphi + 2\log |I|)$ is not integrable near $o$ for any $\varphi$ and $|I|$ satisfying $c^I_o(\varphi) < +\infty$;

(C) $a(-2c^I_o(\varphi)\varphi + 2\log |I|) \exp(-2c^I_o(\varphi)\varphi + 2\log |I|)$ is not integrable near $o$ for any $\varphi$ and $|I|$ satisfying $c^I_o(\varphi) < +\infty$.

When one takes $a \equiv 1$, both $B$ and $C$ degenerate to strong openness of multiplier ideal sheaves.

When one takes $a(t) = \frac{1}{t^\alpha}$ and $\alpha > 0$, it is clear that $B$ holds if and only if $\alpha \in (0, 1]$, i.e.,
$$\frac{1}{|I|^2} \exp(-2c^I_o(\varphi)\varphi)$$
is not locally integrable if and only if $\alpha \in (0, 1]$ (the case when $I = \mathcal{O}$ was expected in [9]).

1.3 The optimal $L^2$ extension and applications

Now let us recall some notation in [41]. Let $M$ be a complex $n$-dimensional manifold, and $S$ be a closed complex subvariety of $M$. Let $dV_M$ be a continuous volume form on $M$. We consider a class of upper-semi-continuous function $\Psi$ from $M$ to the interval $[-\infty, A)$, where $A \in (-\infty, +\infty)$, such that

(1) $\Psi^{-1}(-\infty) \supset S$, and $\Psi^{-1}(-\infty)$ is a closed subset of $M$;

(2) if $S$ is $l$-dimensional around a point $x \in S_{\text{reg}}$ ($S_{\text{reg}}$ is the regular part of $S$), there exists a local coordinate $(z_1, \ldots, z_n)$ on a neighborhood $U$ of $x$ such that $z_1 = \cdots = z_n = 0$ on $S \cap U$ and
$$\sup_{U \setminus S} \left| \Psi(z) - (n - l) \log \sum_{j=1}^n |z_j|^2 \right| < \infty.$$ 

The set of such polar functions $\Psi$ will be denoted by $\#_A(S)$. 
For each $\Psi \in \#_A(S)$, one can associate a positive measure $dV_M[\Psi]$ on $S_{\text{reg}}$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_t} f d\mu \geq \limsup_{t \to \infty} \frac{2(n - l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{I}_{\{-1 - t < \Psi < -t\}} dV_M$$

for any nonnegative continuous function $f$ with $\text{supp} f \subset M$, where $\mathbb{I}_{\{-1 - t < \Psi < -t\}}$ is the characteristic function of the set $\{-1 - t < \Psi < -t\}$. Here denote by $S_t$ the $l$-dimensional component of $S_{\text{reg}}$, denote by $\sigma_m$ the volume of the unit sphere in $\mathbb{R}^{m+1}$.

Let $M$ be a complex manifold with a continuous volume form $dV_M$, and $S$ be a closed complex subvariety of $M$. We call a pair $(M, S)$ an almost Stein pair if $M$ and $S$ satisfy the following conditions:

There exists a closed subset $X \subset M$ such that

(a) $X$ is locally negligible with respect to $L^2$ holomorphic functions, i.e., for any local coordinate neighborhood $U \subset M$ and for any $L^2$ holomorphic function $f$ on $U \setminus X$, there exists an $L^2$ holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f}|_{U \setminus X} = f$ with the same $L^2$ norm.

(b) $M \setminus X$ is a Stein manifold which intersects with every component of $S$, such that $S_{\text{sing}} \subset X$.

Given $\delta > 0$, let $c_A(t)$ be a positive function on $(-A, +\infty)$ ($A \in (-\infty, +\infty)$), which is in $C^\infty((-A, +\infty))$ and satisfies both $\int_{-\infty}^A c_A(t)e^{\delta t} dt < \infty$ and

$$\left(\frac{1}{\delta} c_A(-A)e^A + \int_{-A}^t c_A(t_1)e^{-\delta t_1} dt_1\right)^2 > c_A(t)e^{-\delta t} \left(\int_{-A}^t \frac{1}{\delta} c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-\delta t_1} dt_1\right) dt_2 + \frac{1}{\delta^2} c_A(-A)e^A,$$

for any $t \in (-A, +\infty)$.

An easy example of such functions $c_A(t)$ is when $c_A(t)e^{-t}$ is decreasing with respect to $t$. We have given several interesting examples in [21] to solve several open questions. We remark that the solutions of the questions are reduced to the choice of such functions $c_A(t)$.

In [21], Guan and Zhou established the optimal $L^2$ extension theorem as follows.

**Theorem 1.5** (See [21]). Let $(M, S)$ be an almost Stein pair, $h$ be a smooth metric on a holomorphic vector bundle $E$ on $M$ with rank $r$. Let $\Psi \in \#_A(S) \cap C^\infty(M \setminus S)$, which satisfies

1. $he^{-\Psi}$ is semi-positive in the sense of Nakano on $M \setminus (S \cup X)$ ($X$ is as in the definition of condition (a, b));

2. there exists a continuous function $a(t)$ on $(-A, +\infty)$, such that $0 < a(t) \leq s(t)$ and $a(-\Psi)\sqrt{-1\partial\bar{\partial}\Psi} + \sqrt{-1\partial\bar{\partial}\Psi}$ is semi-positive in the sense of Nakano on $M \setminus (S \cup X)$, where

$$s(t) = \frac{\int_{-A}^t \frac{1}{\delta} c_A(-A)e^A + \int_{-A}^{t_2} c_A(t_1)e^{-\delta t_1} dt_1} {\frac{1}{\delta} c_A(-A)e^A + \int_{-A}^t c_A(t_1)e^{-\delta t_1} dt_1} + \frac{1}{\delta^2} c_A(-A)e^A.$$

Then for any holomorphic section $f$ of $K_M \otimes E|_S$ on $S$ satisfying

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2_h dV_M[\Psi] < \infty,$$

there exists a holomorphic section $F$ of $K_M \otimes E$ on $M$ satisfying $F = f$ on $S$ and

$$\int_M c_A(-\Psi)|F|^2_h dV_M \leq C \left(\frac{1}{\delta} c_A(-A)e^A + \int_{-A}^\infty c_A(t)e^{-\delta t} dt\right) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2_h dV_M[\Psi],$$

where $c_A(t)$ satisfies $c_A(-A)e^A := \lim_{t \to -A^+} c_A(t)e^{-t} < \infty$ and $c_A(-A)e^A \neq 0$, $C = 1$ (which is optimal).

The optimal $L^2$ extension theorem (see [21, Theorem 1.5]) gives unified optimal estimate versions of various well-known $L^2$ extension theorems in [1, 4, 10, 15, 35, 37, 39, 40] and [43, 44, 51, 52, 54, 61], etc. Some interesting relations between the optimal $L^2$ extension and some questions are found, so that the
questions are solved in [21] by using optimal $L^2$ extension (see Theorem 1.5), such as Suita’s conjecture (see [45,57]), L-conjecture (see [59]), some background of finite Riemann surfaces could be referred to [46], extended Suita’s conjecture (see [59]), and an open question posed by Ohsawa [42], etc. A survey should be implied by the main theorem in [21].

Let $c_{\infty}(t)$ be a positive function in $C^\infty(-\infty, +\infty)$ satisfying $\int_{-\infty}^{\infty} e_{\infty}(t)e^{-t}dt < \infty$ and

$$\left(\int_{-\infty}^{t} e_{\infty}(t_1)e^{-t_1}dt_1\right)^2 > c_{\infty}(t)e^{-t} \int_{-\infty}^{t} \int_{-\infty}^{t_2} e_{\infty}(t_1)e^{-t_1}dt_1dt_2,$$

for any $t \in (-\infty, +\infty)$. This class of functions will be denoted by $C_{\infty}$.

**Remark 1.6** (See [21]). Assume that $\frac{d}{dt} c_{\infty}(t)e^{-t} > 0$ for $t \in (-\infty, a)$, $\frac{d}{dt} c_{\infty}(t)e^{-t} \leq 0$ for $t \in [a, +\infty)$, and $\frac{d^2}{dt^2} \log(c_{\infty}(t)e^{-t}) < 0$ for $t \in (-\infty, a)$, where $a > -\infty$. Then (1.5) holds.

Especially, when we take $X$ to be a pseudo-convex domain in $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$ and $\Psi = \log |z_n|$, Theorem 1.5 degenerates to the following.

**Theorem 1.7** (See [21]). Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain, $\varphi$ a plurisubharmonic function on $D$ and $H = \{z_n = 0\}$. Then for any holomorphic function $f$ on $H$ satisfying

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty,$$

there exists a holomorphic function $F$ on $D$ satisfying $F = f$ on $H$ and

$$\int_D c_{\infty}(-2\log |z_n|)|F|^2 e^{-\varphi} dV_D \leq C \pi \int_{-\infty}^{\infty} c_{\infty}(t)e^{-t}dt \int_H |f|^2 e^{-\varphi} dV_H,$$

where $C = 1$ (which is optimal).

Let $c_{\infty}(t) = e^{-ae^{-t}}$ in Theorem 1.7. By Remark 1.6, it suffices to find $a \in \mathbb{R}$ such that

(A) $\frac{d}{dt} (c_{\infty}(t)e^{-t}) > 0$ for $(-\infty, a)$;

(B) $\frac{d}{dt} (c_{\infty}(t)e^{-t}) \leq 0$ for $[a, +\infty)$;

(C) $\frac{d^2}{dt^2} \log(c_{\infty}(t)e^{-t}) < 0$ for $(-\infty, +\infty)$.

Denote $G(t) := e^{-ae^{-t}+t} = e^{-\alpha(e^{-t}+t)}$, where $\alpha > 0$. Then

$$G'(t) = -(ae^{-t}+t)'G(t) = -(1-ae^{-t})G(t) = (ae^{-t} - 1)G(t).$$

Take $a = \log \alpha$, then (A) and (B) holds. Note that

$$\log(c_{\infty}(t)e^{-t}) = -(ae^{-t}+t), \quad \frac{d^2}{dt^2}(-(ae^{-t}+t)) = -ae^{-t} < 0$$

for $t \in (-\infty, +\infty)$, then (C) holds. Then we obtain the following corollary of Theorem 1.7.

**Corollary 1.8.** Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain, and let $\varphi$ be a plurisubharmonic function on $D$ and $H = \{z_n = 0\}$. Then for any holomorphic function $f$ on $H$ satisfying

$$\int_H |f|^2 e^{-\varphi} dV_H < \infty,$$

there exists a holomorphic function $F$ on $D$ satisfying $F = f$ on $H$ and

$$\int_D e^{-\alpha |z_n|^2} |F|^2 e^{-\varphi} dV_D \leq \pi \int_H |f|^2 e^{-\varphi} dV_H.$$

The above result was listed as an open problem by Ohsawa in Winter School of Sanya School in Complex Analysis and Geometry in 2016. In his recent paper “On the extension of $L^2$ holomorphic functions VIII—a remark on a theorem of Guan and Zhou”, Ohsawa recognized that Corollary 1.8 could be implied by the main theorem in [21].
Let $p$ be subjective family holomorphic map, from compact manifold $M$ to $\Delta$ (with coordinate $z$) with surjective differential, and all the fibers $M_z = p^{-1}(z)$ are assumed to be compact. Let $(L, h_L)$ be an Hermitian line bundle on $M$. One can define a holomorphic vector bundle $E$ over $\Delta$ with $E_z = H^0(M_z, K_M |_{M_z} \otimes L |_{M_z})$, and the $H^0(\Delta, E) = H^0(M, K_M \otimes L)$. Let $u$ be a holomorphic section of $E$, and let $\|u\| := \|u\|_z = \int_{M_z} \frac{1}{2\pi h_L} \overline{\partial} \partial \overline{\partial} |u|^2$ be as in [4], which deduces an Hermitian metric on $E$.

In [4], Berndtsson established the following positivity of vector bundles associated to holomorphic fibrations.

**Theorem 1.9** (See [4]). If the total space $M$ is Kähler and $L$ is (semi)positive over $M$, then $(E, \| \cdot \|)$ is (semi)positive in the sense of Nakano.

In [21], Guan and Zhou found that the optimal $L^2$ extension theorem (see Theorem 1.5) implies Berndtsson’s theorem on log-plurisubharmonicity of Bergman kernel [2–4]. We remark here that only the optimal estimate could do so.

In the present paper, we reveal the following.

**Proposition 1.10.** Theorem 1.5 implies Theorem 1.9.

## 2 Preparations

The following lemma will be used to prove Theorem 1.4.

**Lemma 2.1.** For any two measurable spaces $(X_i, \mu_i)$ and two measurable functions $g_i$ on $X_i$, respectively $(i \in \{1, 2\})$, if $\mu_1\{g_1 \geq r^{-1}\} \geq \mu_2\{g_2 \geq r^{-1}\}$ for any $r \in (0, r_0]$, then $\int_{g_1 \geq r_1^{-1}} g_1 d\mu_1 \geq \int_{g_2 \geq r_2^{-1}} g_2 d\mu_2$.

**Proof.** Consider the functions $g_{i, m} := r_0^{-1} \sum_{n=1}^{\infty} \chi_{(g_i \geq r_n^{-1})} + \frac{1}{m} \sum_{n=1}^{\infty} \chi_{(g_i \geq r_n^{-1} + \frac{1}{m})}$, $m \in \mathbb{N}^+$. One can check that $g_{i, m}$ is increasing with respect to $m$, and converges to $g_i$ on $\{g_i \geq r_0^{-1}\}$ when $m$ goes to $+\infty$. As $\mu_1\{g_1 \geq r^{-1}\} \geq \mu_2\{g_2 \geq r^{-1}\}$ for any $r \in (0, r_0]$, then $\int_{g_1 \geq r_1^{-1}} g_{1, m} d\mu_1 \geq \int_{g_2 \geq r_2^{-1}} g_{2, m} d\mu_2$ for any $m \in \mathbb{N}^+$. By Levi’s theorem, it follows that $\int_{g_i \geq r_1^{-1}} g_i d\mu_i = \lim_{m \to +\infty} \int_{g_i \geq r_1^{-1}} g_{i, m} d\mu_i$, which deduces the present lemma. \hfill $\square$

Let $c_A(t) = 1, A = 0, p : M \to \Delta$ be a projective family over unit disc $\Delta$ with coordinate $z$, and $L$ be a smooth semipositive line bundle on $M$, and $\Psi = 2r \log \frac{|z|}{|\delta|}$ (similar method in [21] implies the compact Kähler family case). Then the combination of Theorem 1.5 and the following lemma implies (the semi-positive part of) Proposition 1.10.

**Lemma 2.2.** Let $(V, h)$ be a trivial Hermitian vector bundle rank $m$ over the unit disc $\Delta$ with coordinate $z$. Assume that for any $u_0 \in \mathbb{C}^m$ and $r > 0$, there exists holomorphic section $u$ of $V |_{\Delta_r}$, such that $u(0) = u_0$ and $\frac{1}{4\pi^2} \int_{\Delta_r} |u|^2 < |u_0|^2$. Then $(V, h)$ is Nakano semi-positive at $0 \in \Delta \subset \mathbb{C}$.

**Proof.** It is known that there exists a holomorphic frame $(e_j)_{1 \leq j \leq m}$ of $V$ on a neighbourhood of $0 \in \Delta \subset \mathbb{C}$, such that
\[
\langle e_j, e_k \rangle = \delta_{j, k} - c_{j, k} |z|^2 + O(|z|^3),
\]
(2.1)

where $c_{j, k}$ is the Chern curvature tensor at 0 (see [13, Chapter V, Proposition 12.10]).

We prove the present lemma by contradiction: if not, there exists a holomorphic frame $(e_j)_{1 \leq j \leq m}$ on a neighbourhood of $0 \in \Delta \subset \mathbb{C}$, such that $c_{1, 1} < 0$ $c_{j, k} = 0$ for any $j \neq k$ in (2.1) (by the unitary transformation of $(e_j)_{1 \leq j \leq m}$).

Let $u_0$ be $e_1(0)$. Then there exists a section $u$ on $TV_{\Delta_r}$, such that
\[
u = \sum_{1 \leq j \leq m} e_j f_j \tag{2.2}
\]
and
\[
\frac{1}{4\pi^2} \int_{\Delta_r} (|f|^2 - \sum_{1 \leq j, k \leq m} (e_j f_j f_k f_k^* z^2 + O(|z|^3)|f|)^2) \leq |u_0|^2, \tag{2.3}
\]
where \(f_j\) are holomorphic functions on \(\Delta_r\), and \(f_j(0) = \delta_{1,j}, |f|^2 = \sum_{1 \leq j \leq m} |f_j|^2\), and \(O(|z|^3)\) is independent of \(r\). One can choose \(r\) small enough such that

1. \(|c_{j,j}|^2 \leq \frac{1}{6m}\), which implies \(|c_{j,j}f_j f_j||z|^2 \leq \frac{1}{6m}(|f_j|^2)\), where \(2 \leq j, k \leq m\);
2. \(|O(|z|^3)| \leq -\frac{1}{\delta}c_{1,1}|z|^2\) and \(|O(|z|^3)| \leq \frac{1}{\delta}\) on \(\Delta_r\), which implies \(|O(|z|^3)||f|^2 \leq \frac{1}{\delta}c_{1,1}|z|^2|f_1|^2\) + \(\frac{1}{\delta} \sum_{2 \leq j \leq m} |f_j|^2\) on \(\Delta_r\).

Combining (1) and (2), one can obtain that

\[
|f|^2 - \sum_{1 \leq j,k \leq n} (c_{j,k}f_j f_k)|z|^2 + O(|z|^3)|f|^2 \\
\geq |f|^2 - c_{1,1}|f_1|^2|z|^2 - \sum_{2 \leq j,k \leq m} |c_{j,k}f_j f_k||z|^2 - |O(|z|^3)||f|^2 \\
\geq |f|^2 - c_{1,1}|f_1|^2|z|^2 - \frac{1}{6m}(|f_j|^2) - \left( - \frac{1}{6}c_{1,1}|z|^2|f_1|^2 + \frac{1}{6} \sum_{2 \leq j \leq m} |f_j|^2 \right) \\
\geq |f_1|^2 - \frac{c_{1,1}}{2}|f_1|^2|z|^2.
\]

As \(c_{1,1} < 0\), it follows that \(\frac{1}{\pi r^2} \int_{\Delta_r} |f_1|^2|z|^2 > |u_0|^2\). Combining (2.3) and (2.4), we obtain \(\frac{1}{\pi r^2} \int_{\Delta_r} |u|^2 > |u_0|^2\), which contradicts \(\frac{1}{\pi r^2} \int_{\Delta_r} |u|^2 < |u_0|^2\). Then we obtain the present lemma.

Lemma 2.2 implies the following.

Lemma 2.3. Let \((V, h)\) be an Hermitian vector bundle of rank \(m\) on the unit disc \(\Delta\) with coordinate \(z\). Assume that there exists \(\varepsilon > 0\) such that \((V, he^{\varepsilon |z|^2})\) is Nakano semipositive at 0 \(\in \Delta \subset C\). Then \((V, h)\) is strictly Nakano positive at 0 \(\in \Delta \subset C\).

Let \(c_A(t) = 1, A = 0, p : M \to \Delta\) be a projective family over unit disc \(\Delta\) with coordinate \(z\), and \(L\) be a smooth positive line bundle on \(M\). Then the combination of Theorem 1.5 and Lemma 2.3 implies the compact Kähler family case. Then the proof of Theorem 1.4 is divided into three steps. It suffices to consider the case that \(a(t)e^t\) is continuous.

Step 1. We prove \(B \Rightarrow A\) and \(C \Rightarrow A\).

Consider \(I = 1\) and \(\varphi = \log |z_1|\) on the unit polydisc \(\Delta^n \subset C\), note that \(c_0(\log |z_1|) = 1\) and

\[
\int_{\Delta^n_0} a(-2 \log |z_1|) \frac{1}{|z_1|^2} = (\pi r_0^2)^{-n-1} \int_{\Delta^n_0} a(-2 \log |z_1|) \frac{1}{|z_1|^2} \\
= (\pi r_0^2)^{-n-1} 2\pi \int_{[0, r_0]} a(-2 \log r) r^{-2} dr \\
= (\pi r_0^2)^{-n-1} \pi \int_{[0, r_0]} a(t) dt,
\]

then we obtain \(B \Rightarrow A\) and \(C \Rightarrow A\).

Step 2. We prove \(A \Rightarrow B\).

Let \(X_1\) be a small neighborhood \(U \subset X, 0, X_2 = (0, 1]\). Let \(\mu_1(\cdot) = \mathcal{L}|I|^2 d\lambda\), where \(\lambda\) is the Lebesgue measure on \(U \subset \mathbb{C}^n\). Let \(\mu_2\) be the Lebesgue measure on \(X_2\). Let \(Y_r := \{\exp(-2c_0'(|\varphi|)\varphi) \geq r^{-1}\}\). Then Theorem 1.3 shows that there exists a positive constant \(C\) such that \(\mu_1(Y_r) \leq Cr\) holds for any \(r \in (0, 1]\).

Let \(g_1 = a(-2c_0'(|\varphi|)\varphi) \exp(-2c_0'(|\varphi|)\varphi)\) and \(g_2(x) = a(-\log x + \log C) Cx^{-1}\). As \(a(t)e^t\) is increasing near \(+\infty\), then it follows that \(g_1 \geq a(-\log r)r^{-1}\) on \(Y_r\), which gives

\[
\mu_1(\{g_1 \geq a(-\log r)r^{-1}\}) \geq \mu_1(Y_r) \geq Cr.
\]
Lemma 2.1, we obtain
\[ \mu_2(\{g_2 \geq a(-\log r)r^{-1}\}) = \mu_2(\{0 < x \leq Cr\}) = Cr, \] (3.3)
for small enough \( r > 0 \).

(3.2) and (3.3) give that
\[ \mu_1(\{g_1 \geq a(-\log r)r^{-1}\}) \geq \mu_2(\{g_2 \geq a(-\log r)r^{-1}\}) \]
for any \( r \) small enough. Using the continuity of \( a(-\log r)r^{-1} \) and that \( a(-\log r)r^{-1} \) converges to \( +\infty \) (\( r \to 0 + 0 \)), we obtain that \( \mu_1(\{g_1 \geq r^{-1}\}) \geq \mu_2(\{g_2 \geq r^{-1}\}) \) for any \( r > 0 \) small enough. By Lemma 2.1, we obtain \( A \Rightarrow B \).

Step 3. We prove \( A \Rightarrow C \).

Let \( X_1 \) be a small neighborhood near \( o \in \mathbb{C}^n \), and \( X_2 = (0, 1] \), and \( \mu_1 \) and \( \mu_2 \) be the Lebesgue measure on \( X_1 \) and \( X_2 \), respectively. Let \( Y_r := \{\exp(-2c_1(\varphi)\varphi + 2\log|I|) \geq r^{-1}\} \). Theorem 1.3 shows that there exists a positive constant \( C \) such that \( \mu_1(Y_r) \geq Cr \) holds for any \( r \in (0, 1] \).

Let
\[ g_1 = a(-2c_1(\varphi)\varphi + 2\log|I|) \exp(-2c_1(\varphi)\varphi + 2\log|I|) \]
and
\[ g_2(x) = a(-\log x + \log C)Cx^{-1}. \]

As \( a(t)e^t \) is increasing near \( +\infty \), then it follows that \( g_1 \geq a(-\log r)r^{-1} \) on \( Y_r \), which gives
\[ \mu_1(\{g_1 \geq a(-\log r)r^{-1}\}) \geq \mu_1(Y_r) \geq Cr. \] (3.4)

As \( a(t)e^t \) converges to \( +\infty \) (\( t \to +\infty \)), then \( a(-\log r)r^{-1} \) converges to \( +\infty \) (\( r \to 0 + 0 \)). As \( a(t)e^t \) is strictly increasing near \( +\infty \) and \( a(t)e^t \) is converges to \( +\infty \) (\( t \to +\infty \)), then \( \{g_2 \geq a(-\log r)r^{-1}\} = \{-\log x + \log C \geq -\log r\} = \{0 < x \leq Cr\} \) for small enough \( r > 0 \), which implies
\[ \mu_2(\{g_2 \geq a(-\log r)r^{-1}\}) = \mu_2(\{0 < x \leq Cr\}) = Cr, \] (3.5)
for small enough \( r > 0 \).

(3.4) and (3.5) give that
\[ \mu_1(\{g_1 \geq a(-\log r)r^{-1}\}) \geq \mu_2(\{g_2 \geq a(-\log r)r^{-1}\}) \]
for any \( r > 0 \) small enough. Using the continuity of \( a(-\log r)r^{-1} \) and that \( a(-\log r)r^{-1} \) converges to \( +\infty \) (\( r \to 0 + 0 \)), we obtain that \( \mu_1(\{g_1 \geq r^{-1}\}) \geq \mu_2(\{g_2 \geq r^{-1}\}) \) for any \( r > 0 \) small enough. By Lemma 2.1, we obtain \( A \Rightarrow C \).

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