A network of bosons evolving among different modes while passing through beam splitters and phase shifters has been applied to demonstrate quantum computational advantage. While such networks have mostly been implemented in optical systems using photons, alternative realizations addressing major limitations in photonic systems such as photon loss have been explored recently. Quantized excitations of vibrational modes (phonons) of trapped ions are a promising candidate to realize such bosonic networks. Here, we demonstrate a minimal-loss programmable phononic network in which any phononic state can be deterministically prepared and detected. We realize networks with up to four collective vibrational modes, which can be extended to reveal quantum advantage. We benchmark the performance of the network for an exemplary tomography algorithm using arbitrary multi-mode states with fixed total phonon number. We obtain high reconstruction fidelities for both single- and two-phonon states. Our experiment demonstrates a clear pathway to scale up a phononic network for quantum information processing beyond the limitations of classical and photonic systems.

It is of interest to demonstrate the power of quantum computers that outperform their classical counterparts for certain problems. Bosonic systems spanning large Hilbert spaces enable promising and useful applications. For instance, a network composed of a number of bosons evolving among different modes while passing through beam splitters and phase shifters has been proposed and applied to demonstrate quantum advantage. Boson sampling devices can also be applied to solve quantum chemistry problems and enhance stochastic algorithms or quantum machine learning. Such networks have mostly been implemented by using optical systems with photons. However, technical bottlenecks exist in photon systems. In particular, photon loss and non-deterministic generation and inefficient detection of photonic states hinder their further scalability and the demonstration of quantum advantage. It thus becomes desirable to explore new experimental platforms.

In a trapped ion system, the quantized vibrational modes give rise to phonons that can be used as alternative bosons to build a bosonic network. The phonon states can be deterministically prepared and detected by coupling between the internal states of the ions and the vibrational modes. Recently, there have been various developments...
Vibrational ground-state cooling. In order to manipulate the internal and vibrational energy levels, providing global (blue) and local (red) control, vibrational modes have been demonstrated to generate entangled states such as N00N. The interferometer is composed of an ion trap and (3) interferometer and (3) output measurements. These procedures are realized by using individually addressed Raman laser beams that couple the vibrational modes. The frequencies of the modes are \{\nu_1, \nu_2, \nu_3, \nu_4\} = \{1.985, 2.057, 2.114, 2.153\} MHz. As shown in Fig. 1a, a phononic network consists of three main stages: (1) state preparation, (2) a programmable interferometer and (3) output measurements. These procedures are realized by using individually addressed Raman laser beams that manipulate the interaction between ion qubits and vibrational modes. The ion qubits are encoded in the hyperfine levels of \(^{171}\)Yb\(^{+}\) ions in the \(^{2S1/2}\) manifold denoted as \(|\downarrow\rangle \equiv |F = 0, m_F = 0\rangle\) and \(|\uparrow\rangle \equiv |F = 1, m_F = 0\rangle\) with an energy splitting of \(\omega_{\downarrow\uparrow} = 12.642812\) GHz. The ion qubit is initialized to \(|\downarrow\rangle\) and measured through qubit-state-dependent fluorescence, detected individually by using a multichannel photomultiplier tube (PMT) (Methods).

**Preparation and detection of phonon states**

Our phononic network consists of four collective vibrational modes in one of the radial directions, except for the centre-of-mass mode of a five-ion linear chain, as a result of the Coulomb interaction between the ions. The frequencies of the modes are \{\nu_1, \nu_2, \nu_3, \nu_4\} = \{2\pi \times [1.905, 1.985, 2.057, 2.114, 2.153]\} MHz. As shown in Fig. 1a, a phononic network consists of three main stages: (1) state preparation, (2) a programmable interferometer and (3) output measurements. These procedures are realized by using individually addressed Raman laser beams that manipulate the interaction between ion qubits and vibrational modes. The ion qubits are encoded in the hyperfine levels of \(^{171}\)Yb\(^{+}\) ions in the \(^{2S1/2}\) manifold denoted as \(|\downarrow\rangle \equiv |F = 0, m_F = 0\rangle\) and \(|\uparrow\rangle \equiv |F = 1, m_F = 0\rangle\) with an energy splitting of \(\omega_{\downarrow\uparrow} = 12.642812\) GHz. The ion qubit is initialized to \(|\downarrow\rangle\) and measured through qubit-state-dependent fluorescence, detected individually by using a multichannel photomultiplier tube (PMT) (Methods).

For phononic networks, but building programmable scalable networks remains a challenge. In principle, the number of vibrational modes can be increased by simply adding more ions to the system. Such modes can be divided into two categories: local modes and collective modes. When ions are confined in separated trap potentials or the distances between the ions in a single trap potential are relatively large, the vibration of an ion is localized and almost independent of the motion of the other ions. This is the local phonon mode regime. With two local modes, coupling and hopping in the level of a single quanta and Hong–Ou–Mandel interference have been observed. However, the coupling between local modes is always present owing to the Coulomb interaction between the ions. The frequencies of the modes are \{\nu_1, \nu_2, \nu_3, \nu_4\} = \{2\pi \times [1.905, 1.985, 2.057, 2.114, 2.153]\} MHz. As shown in Fig. 1a, a phononic network consists of three main stages: (1) state preparation, (2) a programmable interferometer and (3) output measurements. These procedures are realized by using individually addressed Raman laser beams that manipulate the interaction between ion qubits and vibrational modes. The ion qubits are encoded in the hyperfine levels of \(^{171}\)Yb\(^{+}\) ions in the \(^{2S1/2}\) manifold denoted as \(|\downarrow\rangle \equiv |F = 0, m_F = 0\rangle\) and \(|\uparrow\rangle \equiv |F = 1, m_F = 0\rangle\) with an energy splitting of \(\omega_{\downarrow\uparrow} = 12.642812\) GHz. The ion qubit is initialized to \(|\downarrow\rangle\) and measured through qubit-state-dependent fluorescence, detected individually by using a multichannel photomultiplier tube (PMT) (Methods).
We deterministically prepare and detect phonon states of the vibrational modes of interest by using a properly chosen single ion (Methods). We choose the ion with the largest coupling strength for the manipulation of a mode. Figure 1c illustrates the preparation of a two-phonon state of mode 1 \((|n = 2⟩)\) by using ion 3, as an example. The zero-phonon state \((|1⟩_1\,|0⟩_2\rangle)\) in Fig. 1c is prepared by Doppler cooling (1 ms) and resolved side-band cooling (3 ms)\(^{27}\). Then, the initial state of the two-phonon state \((|1⟩_1\,|2⟩_2\rangle)\) is prepared by using a combination of carriers (single qubit rotations) and red side-band transitions (RSBs) applied on individually ion 3 (Fig. 1c). As shown in Fig. 1e, the detection of the phonon state of the fourth mode is realized by an adiabatic RSB transition\(^{29,41-44}\) followed by a qubit state detection sequence, with no fluorescence for only \(|n = 0⟩\); and fluorescence for any state \(|n > 0⟩\). To minimize the operation time, all the adiabatic RSBs are applied simultaneously on the chosen ions. The duration of the carrier π pulse is 3 μs, while that of the red side-bands is about 200 μs.

We realize a programmable interferometer that consists of beam splitters with arbitrary rotation angle \((θ_{bs})\) and phase \((φ_{bs})\) by using Raman transitions on a single ion (Methods). By taking advantage of the full connectivity between collective vibrational modes and ions, a beam splitter between arbitrary pairs of modes can be realized. Two Raman transitions that have the same frequency detuning \(δ_{bs}\) from the RSBs of two modes (labelled with \(m\) and \(n\)) are applied to the single ion (labelled with \(j\)) at the same time, with different Rabi frequencies \(Ω_{jm}\) and \(Ω_{jn}\) which satisfies \(η_{jm}\,\,Ω_{jm} = η_{jn}\,\,Ω_{jn} = δ_{bs}/2\). The evolution operator of the effective Hamiltonian takes the form

\[
U_{bs,j}^{(m,n)}(t) = \exp \left[ i θ_{bs} \left( η_{jm} a_m^\dagger e^{-iφ_{bs}} + a_m^\dagger a_n e^{iφ_{bs}} \right) \right]
\]

(1)

\[
θ_{bs} = \frac{η_{jm}\,\,Λ_{jm} + η_{jn}\,\,Λ_{jn}}{4\,\,δ_{bs}}.
\]

(2)
where $a_m^\dagger (a_m)$ is the creation (annihilation) operator of the $m$th mode. Here, $\phi_m = \phi_m - \phi$, is the phase difference between two Raman transitions, which can be considered as a phase shifter integrated into a beam splitter. With this laser-activated beam splitter, the mixing of different modes is no longer limited to the nearest neighbour and can be programmed arbitrarily to construct interferometers for different applications.

Figure 2 shows the performance of the beam splitters acting between various pairs of modes. In the beginning, a single-phonon state of each mode is prepared and detected by using the schemes shown in Figs. 1c,e. The average fidelities of the state preparation and detection for single- and two-phonon states are 96.7% and 95.6%, respectively (Methods). The typical durations of the state preparation and detection are about 300 and 200 \mu s, respectively. The imperfections mainly come from the intensity fluctuations of the Raman laser beams and off-resonant couplings to other modes, which could be improved by further technological developments (Methods).

With the beam splitters of equation (1) after initial state preparation, the phonon states are coherently evolved between two modes. The beam splitter is realized through the ion with the largest mode-coupling strengths $\eta_m$ for the related pairs of modes (Methods). For example, ion 2 is used for the beam splitters between mode 1 and mode 2. In Fig. 2, each data point is obtained by averaging over 300 repetitions. We fit the data using exponentially decaying sinusoids and obtained time constants over 10 ms, which is more than ten times longer than the duration of the 50:50 beam splitter. The average population fidelity of the 50:50 beam splitters is 95.6 \pm 1.72% including the errors owing to the preparation procedures and measurements. The fidelity of the 50:50 beam splitter itself is 99.1%, which is estimated by fitting the fidelity decay of multiple beam splitters. The fidelity of the beam splitter can be further improved by suppressing heating and decoherence (Methods).

**Tomography with a programmable phonon network**

The performance of our phononic network is verified by demonstrating the boson sampling tomography protocol\textsuperscript{46}, which allows for the reconstruction of an arbitrary input state, with a definite total number of phonons in multiple modes, by measuring the outcomes of the interferometric configurations. The number of configurations can be reduced to one when we include additional vacuum modes\textsuperscript{45}. Having access to the full tomography from the sampling data, we can easily verify and quantify the performance of our phononic network, in contrast to other sampling algorithms.

In our realization, two vibrational modes are used for input states while the other two modes serve as ancillary vacuum modes. We choose modes 1 and 2 as input and modes 3 and 4 as ancillary modes. For a given input and interferometric configuration, the output probability of a basis state $|\nu\rangle$ of four modes takes the form

$$p_{\nu'} = \langle \nu'|U_{\text{IFO}} U_{\text{FO}}|\nu\rangle^*.$$  

**Fig. 3 | Tomography experiment with single phonon.** a. The four-mode interferometric configuration for tomography of two-mode input states. Two ancillary vacuum states are required for the reconstruction with a single configuration. Here, $\phi$ and $\phi'$ denote respectively the optimal choice of rotation angle and phase for each beam splitter for tomography. b,c. The output population of states $|1\rangle$, $|0\rangle$, and $|2\rangle$, and $|1\rangle$, $|0\rangle$, and $|3\rangle$ (b) and $|1\rangle$, $|0\rangle$, and $|4\rangle$ (c), depending on the phase of the beam splitter $U_{bs}^{2,4}$. The experimental outputs of the phonon states (points) are compared with theoretically predicted values without any fitting parameters (dashed lines). The red line denotes the experimentally chosen value for the interferometer. It is shifted from the ideal value (n) owing to the ac Stark shifts induced by previous beam splitters, which are in agreement with calculations. d. The experiment outputs (blue) and ideal values (orange) for the output population of phonon states after the interferometric configurations with the programmed parameters in a, e. The density matrix reconstructed from the experimental results. f. The density matrix of the ideal input state. All data points and bars were obtained from the mean of 2,000 samples, and the error bars represent the 95% CI.
where $\rho' = \rho \otimes |0,0\rangle \langle 0,0|$ denotes the density matrix of a four-mode state including two input modes and two ancillary vacuum modes, and $U_{fo}$ is the unitary operation for the interferometric configuration. The reconstruction of the input state can be realized with a singe $U_{fo}$ by measuring the probabilities $p_{\nu}$ for all possible output states.

Figure 3a shows the interferometric configuration with optimal rotation angles and phases of four sequentially applied beam splitters for efficient and reliable reconstruction of the density matrix (Methods). For a single-phonon case, we choose the input state $|\psi\rangle = (|1\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2)/\sqrt{2}$ as an example. We verify the phase coherence of our interferometer by scanning the phase $\phi^{(1,2)}_{bs}$ of the last beam splitter $U_{bs}^{(1,2)}$. As shown in Fig. 3b, the changes of the final state populations are in agreement with the theoretical predictions without any fitting parameters, which clearly shows the programming capability for the parameters of the beam splitters (Methods). Figure 3c shows no influence of the $U_{bs}^{(1,2)}$ on unrelated modes 3 and 4, which implies negligible crosstalk of the beam splitters. As shown in Fig. 3d, the final output populations for the interferometric settings of Fig. 3a are in agreement with the ideal values, which are used for the reconstruction of the density matrix of the input state. Only four output states are detected because the input is a single-phonon state. Figure 3e shows the density matrix reconstructed from the experimental results with a fidelity of $94.5 \pm 1.95\%$ and a purity of $0.893$, in comparison with the ideal case shown in Fig. 3f. The fidelity includes operations for state preparation, state manipulation and measurements, thus demonstrating the accuracy of our highly controllable platform.

The tomography of the two-phonon state is also demonstrated in Fig. 4. The two-phonon experiments include pure quantum interference, that is, Hong–Ou–Mandel interference as shown in Fig. 4a, which demonstrates the bosonic nature of the phononic network.\(^{(1,10)}\) We prepare an input state of $|\psi\rangle = (|1\rangle_1 |1\rangle_2 + |1\rangle_1 |2\rangle_2 + |2\rangle_1 |0\rangle_2)/\sqrt{3}$. With the same interferometric configuration as in Fig. 3a, the output probabilities of the phononic network are shown in Fig. 4b, which are used for the reconstruction of the density matrix of the input state. The number of output states increases to ten for an input state with larger phonon number. We note that the detections do not resolve phonon numbers. We assume that the phonon numbers are conserved for the measurements of ten output states (Methods). The density matrix reconstructed from the experimental results is shown in Fig. 4c, with a fidelity of $93.4 \pm 3.15\%$ and a purity of $0.920$ in comparison with the ideal one shown in Fig. 4d. We do not observe any noticeable reduction of the fidelity in the two-phonon experiment, which demonstrates the high-quality performance of our platform with two phonons.

**Conclusion and outlook**

Our programmable number-conserving phononic network of collective transverse vibrational modes can be scaled up to reach quantum advantage with large numbers of modes $M$ and phonons $N$. In our experiment, the average fidelity of preparing and detecting a single phonon in a mode is $96.70 \pm 1.31\%$. When the number of phonons increases to $N$, the probability of detecting all the phonons scales as $0.967^N$, which surpasses the success probability of phononic systems, which scale as $0.3^N$ for the best performance at the moment.\(^{(12)}\) In our realization, the imperfection of each 50:50 beam splitter is around 1% and a single phonon may pass through the number of $M - 1$ beam splitters. In principle, we can simultaneously perform many beam splitters to directly create an arbitrary interferometer with the capability of full connectivity of our phononic network, which further suppresses imperfections.\(^{(17)}\)

We estimate that about 100 vibrational modes can be utilized for the phononic network in a single trap (Supplementary Information). The number of phonons can be easily increased thanks to the deterministic preparation of phononic states. The duration of preparing phonon states for each mode scales with $N$, the number of ions, due to the reduced Lamb–Dicke parameters. The number of phonons in each mode can be simultaneously prepared using ions properly assigned to the modes. More than single phonons at each mode can be deterministically prepared with a time cost scaling as $\sum(1/\sqrt{n_m})$, where $n_m$ is the number of phonons in mode $m$. Since the number of possible states in the bosonic network grows as $(N + M - 1)!/N!M!$, the increase of the total phonon number $N$ for a given number of modes $M$ can be considered as a complementary implementation for a bosonic network to demonstrate quantum advantage. When the number of phonons is larger than the number of modes, it may be necessary to implement number-resolving detection of phonons to obtain the full statistics of the phonon distributions. Schemes capable of phonon number
resolving detection have been realized for both a single mode and two modes\textsuperscript{13,14,15}. This can be extended to multiple modes by using fast detection\textsuperscript{46} or mapping to multiple levels of ions\textsuperscript{47}.

With the help of multi-mode phonon number resolving detection, the phononic network could be extended to more complex phonon problems such as Gaussian boson sampling. Various Gaussian states, including coherent and squeezed states, have been implemented in trapped ion systems. Moreover, the combination with the qubit degrees of freedom in a trapped ion system can enable the realization of hybrid quantum computing with both discrete and continuous variables\textsuperscript{48}. This may further enhance the capability of the phononic system, which could introduce nonlinearities into the vibrational modes\textsuperscript{49} for applications to continuous-variable quantum computations\textsuperscript{2,40} and quantum chemistry\textsuperscript{11–14}.

Online content
Any methods, additional references, Nature Portfolio reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41567-023-01952-5.

References
1. Aaronson, S. & Arkhipov, A. The computational complexity of linear optics. In Proc. 43rd Annual ACM Symposium on Theory of Computing. (eds. STOC 11 Conference Committee) 333–342 (ACM, 2011).
2. Spring, J. B. et al. Boson sampling on a photonic chip. Science \textbf{339}, 798–801 (2013).
3. Broome, M. A. et al. Photonic boson sampling in a tunable circuit. Science \textbf{339}, 794–798 (2013).
4. Tillmann, M. et al. Experimental boson sampling. Nat. Photon. \textbf{7}, 540–544 (2013).
5. Crespi, A. et al. Integrated multimode interferometers with arbitrary designs for photonic boson sampling. Nat. Photon. \textbf{7}, 545–549 (2013).
6. Carolan, J. et al. Universal linear optics. Science \textbf{349}, 711–716 (2015).
7. Wang, H. et al. High-efficiency multiphoton boson sampling. Nat. Photon. \textbf{11}, 361–365 (2017).
8. Wang, H. et al. Boson sampling with 20 input photons and a 60-mode interferometer in a 10\textsuperscript{th}-dimensional Hilbert space. Phys. Rev. Lett. \textbf{123}, 250503 (2019).
9. Zhong, H.-S. et al. Quantum computational advantage using photons. Science \textbf{370}, 1460–1463 (2020).
10. Arrazola, J. M. et al. Quantum circuits with many photons on a programmable nanophotonic chip. Nature \textbf{591}, 54–60 (2021).
11. Huh, J., Guerreucci, G. G., Peropadre, B., McClean, J. R. & Aspuru-Guzik, A. Boson sampling for molecular vibronic spectra. Nat. Photon. \textbf{9}, 615–620 (2015).
12. Sawaya, N. P. D. & Huh, J. Quantum algorithm for calculating molecular vibronic spectra. J. Phys. Chem. Lett. \textbf{10}, 3586–3591 (2019).
13. Shen, Y. et al. Quantum optical emulation of molecular vibronic spectroscopy using a trapped-ion device. Chem. Sci. \textbf{9}, 836–840 (2018).
14. Banchi, L., Fingerhuth, M., Babej, T., Ing, C. & Arrazola, J. M. Molecular docking with Gaussian boson sampling. Sci. Adv. \textbf{6}, eaax1950 (2020).
15. Arrazola, J. M. & Bromley, T. R. Using Gaussian boson sampling to find dense subgraphs. Phys. Rev. Lett. \textbf{121}, 30503 (2018).
16. Arrazola, J. M., Bromley, T. R. & Rebentrost, P. Quantum approximate optimization with Gaussian boson sampling. Phys. Rev. A \textbf{98}, 12322 (2018).
42. Marshall, K. & James, D. F. V. Linear mode-mixing of phonons with trapped ions. Appl. Phys. B 123, 1–8 (2017).
43. An, S. et al. Experimental test of the quantum Jarzynski equality with a trapped-ion system. Nat. Phys. 11, 193–199 (2015).
44. Um, M. et al. Phonon arithmetic in a trapped ion system. Nat. Commun. 7, 11410 (2016).
45. Lu, D. et al. Reconstruction of the Jaynes–Cummings field state of ionic motion in a harmonic trap. Phys. Rev. A 95, 43813 (2017).
46. Banchi, L., Kolthammer, W. S. & Kim, M. S. Multiphoton tomography with linear optics and photon counting. Phys. Rev. Lett. 121, 250402 (2018).
47. Lu, Y. et al. Global entangling gates on arbitrary ion qubits. Nature 572, 363–367 (2019).
48. Noek, R. et al. High speed, high fidelity detection of an atomic hyperfine qubit. Opt. Lett. 38, 4735–4738 (2013).
49. Ding, S., Maslennikov, G., Hablützel, R. & Matsukevich, D. Cross-Kerr nonlinearity for phonon counting. Phys. Rev. Lett. 119, 193602 (2017).

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

© The Author(s), under exclusive licence to Springer Nature Limited 2023
Adiabatic phonon state preparation and detection
We use adiabatic side-band transitions to compensate for the difference in Rabi frequencies between phonon number states used to prepare and detect phonon states. The typical duration for the adiabatic side-band transition is 200 μs, three times longer than the π time of ground-state side-band transition. We verify the performance of the adiabatic side-band transition by preparing and detecting the target states (one-phonon and two-phonon states) on each mode. The measurement results are shown in Extended Data Table 1.

Scheme of beam splitters
An example of a detailed beam splitter scheme is illustrated in Extended Data Fig. 2 for mode 1 and mode 2 with ion 2. We simultaneously apply a pair of off-resonant side-band transitions with a detuning of both Δ0 and Δm, on ion 2. One of the Raman lasers has a frequency of f2, and the other has two frequencies off f2, and f2, where f2 denotes the frequency applied to ion i that couples to mode m (Extended Data Fig. 2a). Extended Data Fig. 2b shows the energy diagram of this transition. We connect in |f2, 1⟩1 |f2⟩0 to |f2⟩1 |f2⟩0, using off-resonant red side-band transitions of mode 1 and mode 2, respectively. In practice, we choose f2, 2f = Δbs and ω0 = ωm, where Δbs denotes the off-resonant detuning, ω0 is the energy gap of the ion qubit and ωm is the frequency of mode m. Ignoring all of the off-resonant terms, the effective Hamiltonian of this system is written as

\[ H_{bs}^{(2,2)} = -\frac{\eta}{\Delta_0} a_2^\dagger a_2^\dagger a_2^\dagger a_2 + H_{ac}. \]

where

\[ H_{ac}^{(2,2)} = \frac{\Omega_{bs}^2}{4} \left( \frac{\nu_{m} - \nu_{M}}{\Delta_{m}} \right) a_2^\dagger a_2, \]

\[ + \frac{\Omega_{bs}^2}{4} \left( \frac{\nu_{m} - \nu_{M}}{\Delta_{m}} \right) a_2^\dagger a_2^\dagger a_2^\dagger a_2 + \Delta_{bs} a_2^\dagger a_2. \]

Equation (4) describes a spin-dependent beam splitter between two vibrational modes with an effective Rabi frequency of η2, 1f2, 1f2, 1f2, 2f and (4, Δbs) or η2, m and Δm, as the Lamb–Dicke parameter and the coupling strength between mode m and ion 2, respectively. Equation (5) is the ac Stark shift term. We can compensate for the ac Stark shift effect by adding a corresponding frequency shift on |f2⟩1 as f2, 1 = f2, 1 + Δbs (ω0 − ωm). The magnitude of the ac Stark shift is around several hundred hertz, and we note that compensation of the ac Stark shift is important for the proper operation of beam splitters. The experimentally measured ac Stark shifts are in agreement with the theoretical values. We note that we can simultaneously implement two or more beam splitters by applying properly selected multiple frequencies on Raman II with careful compensation of ac Stark shifts. We can use different ions for different pairs of modes, for example, ion 1 for modes 1 and 2 and ion 2 for modes 3 and 4, etc. to reduce the crosstalk when simultaneous beam splitters are applied.

Fidelity and duration of beam splitters
The fidelity and duration of the beam splitter are mainly determined by the choices of both Δ0 and Δm, since major errors come from the off-resonant coupling to other side-band transitions and carriers. Extended Data Fig. 3 shows the numerical simulation with an average mode spacing of δνm = 0.05 MHz. The fidelity and duration of the beam splitter increase with Rf = Δ0/√(Sln(\Omega_{bs}^2/\Delta_{bs}) and Rf = Δ0/Δbs, which means a long duration of the beam splitter can suppress infidelities from the off-resonant coupling. In the experiment, we set Rf, 1 from 1.5 to 3 and Rf, 2 from 3 to 7. We also use a pulse shaping method by modulating Ω with a sinusoidal function at the beginning and the end of the pulse.

Extended Data Fig. 4 shows the numerical simulation of the beam splitters with systematic errors, which is performed by solving the Lindblad master equation with Lindblad operators L heating = a m a m / δνm, L mdc = ∑ P m k l exp (−δj / δj) (|m⟩⟨k| + |k⟩⟨m|), and L dep = √(K m a m a m / δνm). Here, a m and k m denote the heating and decoherence rates of mode m and k, and δj is the decoherence rate of ion i. The subscripts mdc and sdc denote motional and spin decoherence, respectively. We perform the simulation with an average mode spacing of 50 kHz and a 50:50 beam splitter duration of 250 μs. According to the fitting curves, the error is proportional to the heating rate and inversely proportional to the coherence time in the interval where the coherence time is notably longer than that of the beam splitter. Green stars in Extended Data Fig. 4a–c denote the measured system heating rate (Supplementary Information) and coherence times.

Fidelity measurements of the beam splitters
We verify the performance of the beam splitters by comparing the probability distributions shown in Fig. 2 with the ideal probability distribution. We estimate the population fidelity of the beam splitter by using the formula F(m,n) = \sum k √(p(m,k) p(n,k)), where k denotes different output states, p(m,k) is the ideal output probability of the |k⟩ and F(m,n) is the measured output probability of state |k⟩ in the experiment as shown in Fig. 2. We estimate the fidelity of the beam splitter by using a linear fitting model F(m,n) = F ini + c(m,n) L, where F ini denotes the initial population fidelity, which is mainly limited by imperfect state preparation and detection, and c(m,n) is a fitting parameter for the decay of fidelities from errors caused by off-resonant coupling, heating and decoherence. The measured probability fidelities and the fitting results are shown in Extended Data Fig. 5. Here, the average value of F ini for four different beam splitters is 94.6% and the average fidelity at the time of the beam splitter F(m,n) = F ini + c(m,n) L is 95.8%, where F ini denotes the duration of the 50:50 beam splitter between mode m and n. This result reveals that the error induced by a 50:50 beam splitter itself is less than 1%.

Compensation of phase shift induced by ac Stark shift
The laser components for a beam splitter between two modes cause ac Stark shifts for all the other modes. The amount of the phase shift for the kth mode introduced by the ac Stark shifts from a beam splitter between mode m and n is written as

\[ \int_{T_s}^{T_f} \frac{\Omega_{bs}^2}{2} \left( \frac{\nu_{m} - \nu_{M}}{\Delta_{bs}} \right)^2 dt, \]

where T and T are the starting and ending times of the beam splitter, respectively, and δj = (νm − νj) / (ω0 − νj) is the detuning of the laser component to the RSB of the kth mode. The Rabi frequency is time dependent because of the pulse shaping. The beam splitter performance is shown in Fig. 2, and the related experimental parameters.
and fidelities are listed in Extended Data Table 2. In the interferometric configuration, the phase shift of each beam splitter is calculated by including the effects of ac Stark shifts from all the previous beam splitters, which is consistent with the experimental shift.

**Interferometric configurations for tomography**

The unitary rotation matrix of the interferometric configurations is denoted as $U(g)$, where the number $g$ denotes different settings. Then a superoperator$^{46}$ can be constructed as $\mathcal{L}_{\nu g} = \langle \nu | U(g)^\dagger \beta \rho | U(g) | \nu \rangle$, where the numbers $\alpha$ and $\beta$ denote the elements of the Fock space for the input states and $\nu$ for the output states. Then, we get the best choice for tomography when $\det(\mathcal{L}^\dagger \mathcal{L})$ has the largest value. Based on the output probabilities, the reconstruction of the input density matrix is realized by

$$\rho_{\text{best}} = (\mathcal{L}^\dagger \mathcal{L})^{-1} \mathcal{L}^\dagger [p]$$

(ref. $^{46}$), where $p$ denotes the output probabilities. Owing to errors in the experiment, a maximum-likelihood method is used to predict the possible reconstructed density matrix. We assume a positive density matrix $\rho'$ with $\text{Tr} [\rho_{\text{best}}] = 1$. By minimizing the 2-norm $|\rho_{\text{best}} - \rho'|$, we get the possible density matrix for our reconstruction scheme.

For a system with an input state having two modes and one phonon, we need three interferometric configurations$^{46}$, where each consists of one beam splitter. In the experiment, we choose the rotating phases of the three beam splitters as $\{0, 2/3\pi, 4/3\pi\}$ with an optimized rotating angle of $0.304\pi$. Extended Data Table 3 shows the reconstructed matrix of various one-phonon states for two transverse modes of three ions.

The number of interferometric configurations can be reduced to one with two additional vacuum modes. Extended Data Table 4 shows an interferometric configuration with four different beam splitters. This configuration can be used to reconstruct any phonon-state between two modes$^{46}$. The reconstruction results are shown in Extended Data Table 5.

**Data availability**

All relevant data are available from the corresponding authors upon request.

**References**

50. Shen, C. & Duan, L. M. Correcting detection errors in quantum state engineering through data processing. *N. J. Phys.* **14**, 53053 (2012).
Extended Data Fig. 1 | Imaging system for the fluorescence detection of individual ion-qubit states. The fluorescence of five ions is collected by a high NA lens and imaged to 32-channel PMT. To reduce crosstalk, we image each ion into alternative channels of PMT and put a slit at the second focused plane of the imaging system to suppress horizontal and vertical components, respectively.
Extended Data Fig. 2: Raman schematic diagram of beam-splitter.

a. Frequency arrangement of two Raman lasers from perpendicular directions. Raman I is a global laser with one frequency component, and Raman II is an individually addressed laser with two components focused on one ion. b. Energy diagram of beam-splitter. $\Delta_{BS}$ is a frequency detuning between Raman I and Raman II, effectively introducing two off-resonant RSB on a single ion. $\omega_0$ is the frequency of ion-qubit. Here two energy levels are connected by the Raman transition, $|↓\rangle_i |1\rangle_1 |0\rangle_2$ and $|↓\rangle_i |0\rangle_1 |1\rangle_2$. 
Extended Data Fig. 3 | Raman schematic diagram of beam-splitter. a, Fidelity of a 50:50 beam-splitter with $R_1 = \Delta_{bs}/\sqrt{J_m \Omega_{jm} J_n \Omega_{jn}}$ and $R_2 = \eta_j/\Delta_{bs}$ where $\eta_j$ and $\eta_n$ are Lamb-Dicke parameters of two modes and $\Omega_{jm}$ is the Rabi frequency of frequency component $f_{jm} - f_0$. b, Prediction of operation time for a 50:50 beam-splitter. Here the simulation based on an average mode spacing of 50 kHz.
Extended Data Fig. 4 | Numerical simulation for systematic errors of beam-splitters. We simulate the performance of 50:50 beam-splitters with similar parameters used in the experiment. We achieve population errors induced by (a) heating, (b) motional decoherence, and (c) spin decoherence by subtracting inevitable errors caused by off-resonant couplings shown in Extended Data Fig. 3a. The blue points represent simulation results, with the orange fitting curves representing the error trend. The green stars represent the measurement results of the corresponding values.
Extended Data Fig. 5 | Fidelity measurements of beam-splitters. We calculate the population fidelity of beam-splitters between modes (a) 1 and 2, (b) 2 and 4, (c) 1 and 3, (d) 3 and 4, using data shown in Fig. 2. The blue circles represent the calculated fidelity, and the blue dashed lines represent the linear fitting results. The intersection of the blue and red lines represents the fidelities of the 50:50 beam splitters. All the data points occur in the figures were obtained from the mean value of 100 samples, and the error bars represent 95% confidence intervals.
### Extended Data Table 1 | Detection and preparation fidelities of each mode

| Mode | One-phonon State | Two-phonon State |
|------|------------------|------------------|
| 1    | 96.52±1.81%      | 96.12±1.23%      |
| 2    | 97.46±1.19%      | 95.03±1.89%      |
| 3    | 96.66±1.05%      | 96.03±1.64%      |
| 4    | 96.16±1.20%      | 95.25±1.49%      |
| **Average** | **96.70±1.34%** | **95.61±1.58%** |

We prepared single-phonon states and two-phonon states in each mode and used adiabatic sideband transitions to perform projection measurements of the phonon states. We use the bright state population of the chosen ion after adiabatic sideband transitions as the population fidelity of state preparation and detection.
## Extended Data Table 2 | Parameters and fidelities of 50:50 beam-splitters in four-mode setup

| Ion | Modes | $\Delta_{bs}$ (kHz) | $\eta_{m_1} \Omega_1$ (kHz) | $\eta_{m_2} \Omega_2$ (kHz) | Duration ($\mu$s) | Fidelity (With SPD Errors) |
|-----|-------|---------------------|-----------------------------|-----------------------------|------------------|---------------------------|
| 3   | 1&3   | -10                 | -6.3                        | -4.4                        | 286.6           | 95.89±1.26%               |
| 4   | 2&4   | -10                 | -6.3                        | 3.1                         | 468.1           | 94.85±2.70%               |
| 5   | 3&4   | -10                 | 4.4                         | 3.1                         | 453.7           | 95.15±1.19%               |
| 2   | 1&2   | -10                 | 6.3                         | 6.3                         | 254.8           | 96.46±1.26%               |

Here the beam-splitter fidelity is measured by the population overlap of the experimental state and the ideal state, where errors are mainly from the imperfections of state preparation and detection (SPD).
Extended Data Table 3 | Reconstruction results for two modes input states without using ancillary modes

| Input State                      | Ideal Density Matrix | Reconstructed Matrix | Fidelity       |
|---------------------------------|----------------------|----------------------|----------------|
| $|1\rangle_1|0\rangle_2$             | \[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.965 & 0.035 + 0.007i \\
0.035 - 0.007i & 0.001
\end{pmatrix}
\] | 96.52±1.35% |
| $|0\rangle_1|1\rangle_2$             | \[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0.003 + 0.016i \\
0.003 - 0.016i & 0.987
\end{pmatrix}
\] | 98.67±1.39% |
| $\frac{1}{\sqrt{2}}(|1\rangle_1|0\rangle_2 + |0\rangle_1|1\rangle_2)$ | \[
\begin{pmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.481 & 0.490 + 0.011i \\
0.490 - 0.011i & 0.5
\end{pmatrix}
\] | 98.09±0.66% |
| $\frac{1}{\sqrt{2}}(|1\rangle_1|0\rangle_2 + i|0\rangle_1|1\rangle_2)$ | \[
\begin{pmatrix}
0.5 & 0.5i \\
0.5i & 0.5
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.468 & 0.005 - 0.485i \\
0.005 + 0.485i & 0.503
\end{pmatrix}
\] | 97.12±0.78% |
| $\frac{1}{\sqrt{2}}(|1\rangle_1|0\rangle_2 - |0\rangle_1|1\rangle_2)$ | \[
\begin{pmatrix}
0.5 & 0.5i \\
-0.5i & 0.5
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.451 & -0.029 + 0.488i \\
-0.029 - 0.488i & 0.530
\end{pmatrix}
\] | 97.92±0.72% |
| $\frac{1}{\sqrt{2}}(|1\rangle_1|0\rangle_2 - |0\rangle_1|1\rangle_2)$ | \[
\begin{pmatrix}
0.5 & -0.5 \\
-0.5 & 0.5
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.455 & -0.488 + 0.003i \\
-0.488 - 0.003i & 0.523
\end{pmatrix}
\] | 97.65±0.80% |
Extended Data Table 4 | Interferometric configuration for a two-mode input state with two ancillary modes

| Modes of beam-splitter | \{\theta, \phi\} |
|------------------------|------------------|
| 1&3                    | \{0.696\pi, 0\pi\} |
| 2&4                    | \{0.304\pi, 0\pi\} |
| 3&4                    | \{0.5\pi, 0.5\pi\} |
| 1&2                    | \{0.5\pi, 1.0\pi\} |

Here only one configuration is used. The order keeps the same as in the real experiment.
Extended Data Table 5 | Reconstruction results for two modes input states using two ancillary modes

| Input State | Ideal Density Matrix | Reconstructed Matrix | Fidelity |
|-------------|----------------------|----------------------|----------|
| $\frac{1}{\sqrt{2}} (|1_1|0_2 + |0_1|1_2)$ | $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ | $\begin{pmatrix} 0.469 & 0.472 - 0.005i \\ 0.472 + 0.005i & 0.475 \end{pmatrix}$ | 94.49±1.95% |
| $\frac{1}{\sqrt{3}} (|1_1|1_2 + i|0_1|2_2 + i|2_1|0_2)$ | $\begin{pmatrix} 0.333 & 0.333i & 0.333i \\ -0.333i & 0.333 & 0.333 \\ -0.333i & 0.333 & 0.333 \end{pmatrix}$ | $\begin{pmatrix} 0.218 & 0.063 + 0.297i & 0.005 + 0.263i \\ 0.063 - 0.297i & 0.423 & 0.360 + 0.068i \\ 0.005 - 0.263i & 0.360 - 0.068i & 0.317 \end{pmatrix}$ | 93.36±3.15% |