Form factors in quantum integrable models with $GL(3)$-invariant $R$-matrix

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Abstract

We study integrable models solvable by the nested algebraic Bethe ansatz and possessing $GL(3)$-invariant $R$-matrix. We obtain determinant representations for form factors of off-diagonal entries of the monodromy matrix. These representations can be used for the calculation of form factors and correlation functions of the XXX $SU(3)$-invariant Heisenberg chain.

1 Introduction

The calculation of correlation functions in quantum integrable models is a very important and complex problem. A form factor approach is one of the most effective methods for solving this problem. For this reason, the study of form factors of local operators has attracted the attention of many authors. There are different methods to address the problem of the calculation of form factors. In the integrable models of the quantum field theory there exists the ‘form factor bootstrap approach’ $^{1,2}$. It is based on a set of form factors axioms $^2$, which represent

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a set of difference equations that define specific analytic properties of form factors. These equations can be solved to provide the integral representations for form factors. The form factor bootstrap program is closely related to the approach based on the conformal field theory and its perturbation \[8–11\]. A purely algebraic method to calculate form factors in the infinite chain spin models was developed by the Kyoto group \[12–14\] using the representation theory of quantum affine algebras. This approach also yields integral formulas for the form factors of the local operators in such models. An alternative way to calculate form factors in the spin chain models was developed by the Lyon group after the inverse scattering problem was solved and local operators in the spin chain models were expressed in terms of the monodromy matrix elements \[15\]. In this framework one can obtain determinant formulas for the form factors of local spin operators. These determinant representations appeared to be very effective for the calculation of correlation functions \[16–18\].

In this article we try to address this problem and continue our study of form factors in \(GL(3)\)-invariant quantum integrable models solvable by the algebraic Bethe ansatz \[19–23\]. More precisely, we calculate matrix elements of the monodromy matrix entries \(T_{ij}(z)\) with \(|i-j|=1\) between on-shell Bethe vectors (that is, the eigenstates of the transfer matrix). Recently, in the work \[24\], we obtained determinant representations for form factors of the diagonal elements \(T_{ii}(z)\) \((i=1,2,3)\) of the monodromy matrix. Our method was based on the use of a twisted monodromy matrix \[25–27\]. However this approach fails if we deal with form factors of off-diagonal matrix elements. In this last case, one has to apply a more general method, which is based on the explicit calculation of the action of the monodromy matrix entries onto Bethe vectors. As we have shown in \[28\] this action gives a linear combination of Bethe vectors. Then the resulting scalar products can be evaluated in terms of sums over partitions of Bethe parameters \[29\].

The form factors of the monodromy matrix entries play a very important role. For a wide class of models, for which the inverse scattering problem can be solved \[15, 30\], such matrix elements can be easily associated with form factors of local operators \[15\]. In particular, if \(E_m^{\alpha,\beta}\), \(\alpha,\beta=1,2,3\), is an elementary unit \(\left((E^{\alpha,\beta})_{jk} = \delta_{j\alpha}\delta_{k\beta}\right)\) associated with the \(m\)-th site of the \(SU(3)\)-invariant XXX Heisenberg chain, then

\[
E_m^{\alpha,\beta} = (\text{tr} T(0))^{m-1}T_{\beta\alpha}(0)(\text{tr} T(0))^{-m}.
\] (1.1)

Since the action of the transfer matrix \(\text{tr} T(0)\) on on-shell Bethe vectors is trivial, we see that the form factors of \(E_m^{\alpha,\beta}\) are proportional to those of \(T_{\beta\alpha}\). Thus, if we have an explicit and compact representations for form factors of \(T_{ij}\), we can study the problem of two-point and multi-point correlation functions, expanding them into series with respect to the form factors.

We have mentioned already that the problem considered in this paper is closely related to the calculation of Bethe vectors scalar products. In these scalar products, one of the vectors is on-shell, while the other one is off-shell (that is, this vector generically is not an eigenstate of the transfer matrix). A determinant representation for such type of scalar product was obtained in \[31\] for \(GL(2)\)-based models. This representation allows one to obtain various determinant formulas for form factors. Unfortunately, so far an analog of this determinant formula is not known in the case of integrable models based on the \(GL(3)\) symmetry. In our previous publications \[24, 32\] we argued that such an analog hardly exists for the scalar products.
involving on-shell Bethe vector and a generic off-shell Bethe vector. However, calculating the form factors of the operators $T_{ij}$ we obtain scalar products involving very specific off-shell Bethe vectors. For such particular cases of scalar products we succeed to find representations in terms of a determinant, which is analogous to the determinant representation of [31].

The article is organized as follows. In section 2 we introduce the model under consideration and describe the notation used in the paper. We also give there explicit formulas for the scalar product of Bethe vectors obtained in [29] and explain the relationship between different form factors. In section 3 we formulate the main results of this paper. Section 4 is devoted to the derivation of the determinant representation for the form factor of the operator $T_{12}$. In section 5 we prove the results for form factors of other operators $T_{ij}$ with $|i-j|=1$. Appendix A contains the properties of the partition function of the six-vertex model with domain wall boundary conditions and several summation identities for it.

2 General background

2.1 Generalized \( GL(3) \)-invariant model

The models considered below are described by a \( GL(3) \)-invariant \( R \)-matrix acting in the tensor product of two auxiliary spaces \( V_1 \otimes V_2 \), where \( V_k \sim \mathbb{C}^3 \), \( k = 1, 2 \):

\[
R(x, y) = I + g(x, y)P, \quad g(x, y) = \frac{c}{x - y}.
\]

(2.1)

In the above definition, \( I \) is the identity matrix in \( V_1 \otimes V_2 \), \( P \) is the permutation matrix that exchanges \( V_1 \) and \( V_2 \), and \( c \) is a constant.

The monodromy matrix \( T(w) \) satisfies the algebra

\[
R_{12}(w_1, w_2)T_1(w_1)T_2(w_2) = T_2(w_2)T_1(w_1)R_{12}(w_1, w_2).
\]

(2.2)

Equation (2.2) holds in the tensor product \( V_1 \otimes V_2 \otimes \mathcal{H} \), where \( V_k \sim \mathbb{C}^3 \), \( k = 1, 2 \), are the auxiliary linear spaces, and \( \mathcal{H} \) is the Hilbert space of the Hamiltonian of the model under consideration. The matrices \( T_k(w) \) act non-trivially in \( V_k \otimes \mathcal{H} \).

The trace in the auxiliary space \( V \sim \mathbb{C}^3 \) of the monodromy matrix, \( \text{tr} T(w) \), is called the transfer matrix. It is a generating functional of integrals of motion of the model. The eigenvectors of the transfer matrix are called on-shell Bethe vectors (or simply on-shell vectors). They can be parameterized by sets of complex parameters satisfying Bethe equations (see section 2).

2.2 Notations

We use the same notations and conventions as in the paper [24]. Besides the function \( g(x, y) \) we also introduce a function \( f(x, y) \)

\[
f(x, y) = \frac{x - y + c}{x - y}.
\]

(2.3)

Two other auxiliary functions will be also used

\[
h(x, y) = \frac{f(x, y)}{g(x, y)} = \frac{x - y + c}{c}, \quad t(x, y) = \frac{g(x, y)}{h(x, y)} = \frac{c^2}{(x - y)(x - y + c)}.
\]

(2.4)
The following obvious properties of the functions introduced above are useful:

\[ g(x, y) = -g(y, x), \quad h(x - c, y) = g^{-1}(x, y), \quad f(x - c, y) = f^{-1}(y, x), \quad t(x - c, y) = t(y, x). \]

Before giving a description of the Bethe vectors we formulate a convention on the notations. We denote sets of variables by bar: \( \bar{w}, \bar{u}, \bar{v} \) etc. Individual elements of the sets are denoted by subscripts: \( w_j, u_k \) etc. Notation \( \bar{x} + c \) means that the constant \( c \) is added to all the elements of the set \( \bar{x} \). Subsets of variables are denoted by roman indices: \( \bar{u} \). For example, \( \bar{u} \) is the corresponding set. For example, \( \bar{u} \) is divided into two disjoint subsets \( \bar{u}_1 \) and \( \bar{u}_2 \). We assume that the elements in every subset of variables are ordered in such a way that the sequence of their subscripts is strictly increasing. We call this ordering natural order.

In order to avoid too cumbersome formulas we use shorthand notations for products of functions depending on one or two variables. Namely, if functions \( g, f, h, t \), as well as \( r_1 \) and \( r_3 \) (see (2.10)) depend on sets of variables, this means that one should take the product over the corresponding set. For example,

\[ r_1(\bar{u}) = \prod_{u_j \in \bar{u}} r_1(u_j); \quad g(z, \bar{w}) = \prod_{w_j \in \bar{w}} g(z, w_j); \quad f(\bar{u}_g, \bar{u}_i) = \prod_{u_j \in \bar{u}_g} \prod_{u_k \in \bar{u}_i} f(u_j, u_k). \] (2.6)

In the last equation of (2.6) the set \( \bar{u} \) is divided into two subsets \( \bar{u}_1 \) and \( \bar{u}_2 \), and the double product is taken with respect to all \( u_k \) belonging to \( \bar{u}_1 \) and all \( u_j \) belonging to \( \bar{u}_2 \). We emphasize once more that this convention is only valid in the case of functions, which are by definition dependent on one or two variables. It does not apply to functions that depend on sets of variables.

One of the central object in the study of scalar products of \( GL(3) \) invariant models is the partition function of the six-vertex model with domain wall boundary conditions (DWPF) \[25, 33\]. We denote it by \( K_n(\bar{x} \upharpoonright \bar{y}) \). It depends on two sets of variables \( \bar{x} \) and \( \bar{y} \); the subscript shows that \( \#\bar{x} = \#\bar{y} = n \). The function \( K_n \) has the following determinant representation \[33\]

\[ K_n(\bar{x} \upharpoonright \bar{y}) = \Delta'_n(\bar{x}) \Delta_n(\bar{y}) h(\bar{x}, \bar{y}) \det \begin{pmatrix} x_j & y_k \end{pmatrix}_n, \] (2.7)

where \( \Delta'_n(\bar{x}) \) and \( \Delta_n(\bar{y}) \) are

\[ \Delta'_n(\bar{x}) = \prod_{j < k} g(x_j, x_k), \quad \Delta_n(\bar{y}) = \prod_{j > k} g(y_j, y_k). \] (2.8)

It is easy to see that \( K_n \) is symmetric over \( \bar{x} \) and symmetric over \( \bar{y} \), however \( K_n(\bar{x} \upharpoonright \bar{y}) \neq K_n(\bar{y} \upharpoonright \bar{x}) \).

Below we consider \( K_n \) depending on combinations of sets of different variables, for example \( K_n(\bar{\xi} \upharpoonright \{\bar{\alpha}, \bar{\beta} + c\}) \). Due to the symmetry properties of DWPF \( K_n(\bar{\xi} \upharpoonright \{\bar{\alpha}, \bar{\beta} + c\}) = K_n(\bar{\xi} \upharpoonright \{\bar{\beta} + c, \bar{\alpha}\}) \).

### 2.3 Bethe vectors

Now we pass to the description of Bethe vectors. A generic Bethe vector is denoted by \( \mathbb{B}^{a,b}(\bar{u}; \bar{v}) \). It is parameterized by two sets of complex parameters \( \bar{u} = u_1, \ldots, u_a \) and \( \bar{v} = v_1, \ldots, v_b \) with \( a, b = 0, 1, \ldots \). Dual Bethe vectors are denoted by \( \mathbb{C}^{a,b}(\bar{u}; \bar{v}) \). They also depend on two sets of complex parameters \( \bar{u} = u_1, \ldots, u_a \) and \( \bar{v} = v_1, \ldots, v_b \). The state with \( \bar{u} = \bar{v} = \emptyset \) is called a
pseudovacuum vector $|0\rangle$. Similarly the dual state with $\bar{u} = \bar{v} = \emptyset$ is called a dual pseudovacuum vector $|0\rangle$. These vectors are annihilated by the operators $T_{ij}(w)$, where $i > j$ for $|0\rangle$ and $i < j$ for $|0\rangle$. At the same time both vectors are eigenvectors for the diagonal entries of the monodromy matrix

$$T_{ii}(w)|0\rangle = \lambda_i(w)|0\rangle, \quad (0|T_{ii}(w) = \lambda_i(w)|0\rangle, \quad (2.9)$$

where $\lambda_i(w)$ are some scalar functions. In the framework of the generalized model, $\lambda_i(w)$ remain free functional parameters. Actually, it is always possible to normalize the monodromy matrix $T(w) \to \lambda_2^{-1}(w)T(w)$ so as to deal only with the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \quad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}. \quad (2.10)$$

If the parameters $\bar{u}$ and $\bar{v}$ of a Bethe vector satisfy a special system of equations (Bethe equations), then it becomes an eigenvector of the transfer matrix (on-shell Bethe vector). The system of Bethe equations can be written in the following form:

$$r_1(\bar{u}_i) = \frac{f(\bar{u}_i, \bar{u}_\pi)}{f(\bar{u}_\pi, \bar{u}_i)} f(\bar{v}, \bar{u}_i),$$

$$r_3(\bar{v}_i) = \frac{f(\bar{v}_i, \bar{v}_\pi)}{f(\bar{v}_\pi, \bar{v}_i)} f(\bar{v}_i, \bar{u}). \quad (2.11)$$

These equations should hold for arbitrary partitions of the sets $\bar{u}$ and $\bar{v}$ into subsets $\{\bar{u}_i, \bar{u}_\pi\}$ and $\{\bar{v}_i, \bar{v}_\pi\}$ respectively. Obviously, it is enough to demand that the system (2.11) is valid for the particular case when the sets $\bar{u}_i$ and $\bar{v}_i$ consist of only one element. Then (2.11) coincides with the standard form of Bethe equations.

If $\bar{u}$ and $\bar{v}$ satisfy the system (2.11), then

$$\text{tr} T(w) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \tau(w|\bar{u}, \bar{v}) \mathbb{B}^{a,b}(\bar{u}; \bar{v}),$$

$$C^{a,b}(\bar{u}; \bar{v}) \text{tr} T(w) = \tau(w|\bar{u}, \bar{v}) C^{a,b}(\bar{u}; \bar{v}), \quad (2.12)$$

where

$$\tau(w) \equiv \tau(w|\bar{u}, \bar{v}) = r_1(w)f(\bar{u}, w) + f(w, \bar{u})f(\bar{v}, w) + r_3(w)f(w, \bar{v}). \quad (2.13)$$

### 2.4 Scalar products and form factors

The scalar products of Bethe vectors are defined as

$$S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C|\bar{u}^B, \bar{v}^B) = C^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B). \quad (2.14)$$

We use here superscripts $B$ and $C$ in order to distinguish the sets of parameters entering these two vectors. In other words, unless explicitly specified, the variables $\{\bar{u}^B, \bar{v}^B\}$ in $\mathbb{B}^{a,b}$ and $\{\bar{u}^C, \bar{v}^C\}$ in $C^{a,b}$ are not supposed to be related.

Before giving an explicit formula for the scalar product we introduce the notion of highest coefficient $Z_{a,b}(\bar{t}; \bar{x}|\bar{s}; \bar{y})$. This function depends on four sets of variables with cardinalities $\#\bar{t} = \#\bar{x} = a$, $\#\bar{s} = \#\bar{y} = b$, and $a, b = 0, 1, \ldots$. There exist several explicit representations for

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For simplicity here and below we do not distinguish between vectors and dual vectors.
the highest coefficient in terms of DWPF [34, 35]. In this paper we use two of them. The first one reads
\[ Z_{a,b}(\vec{t}, \vec{x}|\vec{s}, \vec{y}) = (-1)^b \sum K_b(\bar{s} - c|\bar{w}_1)K_a(|\bar{w}_n\vec{t})K_b(\bar{y}|\bar{w}_1)f(\bar{w}_1, \bar{w}_n). \] (2.15)
Here \( \bar{w} = \{ \bar{s}, \bar{x} \} \). The sum is taken with respect to all partitions of the set \( \bar{w} \) into subsets \( \bar{w}_i \) and \( \bar{w}_n \) with \( \#\bar{w}_i = b \) and \( \#\bar{w}_n = a \).

The second representation has the following form:
\[ Z_{a,b}(\vec{t}, \vec{x}|\vec{s}, \vec{y}) = (-1)^b f(\bar{y}, \bar{x})f(\bar{s}, \bar{t}) \sum K_b(\tilde{\alpha}_n - c|\tilde{y} + c)K_a(\tilde{x}|\tilde{\alpha}_t)K_b(\tilde{\alpha}_n - c|\tilde{s})f(\tilde{\alpha}_t, \tilde{\alpha}_n). \] (2.16)
Here \( \tilde{\alpha} = \{ \tilde{y} + c, \tilde{t} \} \). The sum is taken with respect to all partitions of the set \( \tilde{\alpha} \) into subsets \( \tilde{\alpha}_t \) and \( \tilde{\alpha}_n \) with \( \#\tilde{\alpha}_t = a \) and \( \#\tilde{\alpha}_n = b \).

The scalar product (2.15) is a bilinear combination of the highest coefficients. It was calculated in the work [29].

\[ S_{a,b}(\vec{u}^C, \vec{v}^C|\bar{u}^B, \bar{v}^B) = \sum r_1(\bar{u}_1^B)r_1(\bar{u}_n^B)r_3(\bar{v}_1^B)r_3(\bar{v}_n^B)f(\vec{u}^C, \bar{u}_n^B)f(\vec{u}^B, \bar{u}_1^B)f(\vec{v}^C, \bar{v}_n^B)f(\vec{v}^B, \bar{v}_1^B) \]
\[ \times \frac{f(\vec{u}_1^C, \bar{u}_1^B)}{f(\vec{v}_1^C, \bar{v}_1^B)}Z_{a-k,n}(\vec{u}_n^C, \bar{u}_n^B|\bar{v}_1^C, \bar{v}_1^B)Z_{k,b-n}(\vec{u}_1^C, \bar{u}_1^B|\bar{v}_n^C, \bar{v}_n^B). \] (2.17)

Here the sum is taken over the partitions of the sets \( \vec{u}^C, \bar{u}^B, \bar{v}^C \), and \( \bar{v}^B \):
\[ \vec{u}^C \Rightarrow \{ \vec{u}_1^C, \vec{u}_n^C \}, \quad \bar{v}^C \Rightarrow \{ \bar{v}_1^C, \bar{v}_n^C \}, \quad \bar{v}^B \Rightarrow \{ \bar{v}_1^B, \bar{v}_n^B \}. \] (2.18)

The partitions are independent except that \( \#\bar{u}_1^B = \#\vec{u}_1^C = k \) with \( k = 0, \ldots, a \), and \( \#\bar{u}_n^B = \#\bar{v}_1 = n \) with \( n = 0, \ldots, b \).

In this formula the parameters \( \vec{u}^C, \bar{u}^B, \bar{v}^C \), and \( \bar{v}^B \) are arbitrary complex numbers, that is \( \mathbb{B}^{a,b}(\vec{u}^B; \bar{v}^B) \) and \( \mathbb{C}^{a,b}(\vec{u}^C; \bar{v}^C) \) are generic Bethe vectors. If one of these vectors, say \( \mathbb{C}^{a,b}(\vec{u}^C; \bar{v}^C) \), is on-shell, then the parameters \( \vec{u}^C \) and \( \bar{v}^C \) satisfy the Bethe equations. In this case one can express the products \( r_1(\bar{u}_1^B) \) and \( r_3(\bar{v}_1^B) \) in terms of the function \( f \) via (2.11).

Form factors of the monodromy matrix entries are defined as
\[ \mathcal{F}^{(i,j)}_{a,b}(z) \equiv \mathcal{F}^{(i,j)}_{a,b}(z|\vec{u}^C, \bar{v}^C; \vec{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\vec{u}^C; \bar{v}^C)T_{ij}(z)\mathbb{B}^{a,b}(\vec{u}^B; \bar{v}^B), \] (2.19)
where both \( \mathbb{C}^{a',b'}(\vec{u}^C; \bar{v}^C) \) and \( \mathbb{B}^{a,b}(\vec{u}^B; \bar{v}^B) \) are on-shell Bethe vectors, and
\[ a' = a + \delta_{i1} - \delta_{j1}, \]
\[ b' = b + \delta_{j3} - \delta_{i3}. \] (2.20)

The parameter \( z \) is an arbitrary complex number. Acting with the operator \( T_{ij}(z) \) on \( \mathbb{B}^{a,b}(\vec{u}^B; \bar{v}^B) \) via formulas obtained in [28], we reduce the form factor to a linear combination of scalar products, in which \( \mathbb{C}^{a',b'}(\vec{u}^C; \bar{v}^C) \) is on-shell vector.
2.5 Relations between form factors

Obviously, there exist nine form factors of $T_{ij}(z)$ in the models with $GL(3)$-invariant $R$-matrix. However, not all of them are independent. In particular, due to the invariance of the $R$-matrix under transposition with respect to both spaces, the mapping

$$
\psi : T_{ij}(u) \mapsto T_{ji}(u)
$$

defines an antimorphism of the algebra $(2.2)$. Acting on the Bethe vectors this antimorphism maps them into the dual ones and vice versa

$$
\psi(\mathbb{B}^{a,b}(\bar{u}; \bar{v})) = \mathbb{C}^{a,b}(\bar{u}; \bar{v}), \quad \psi(\mathbb{C}^{a,b}(\bar{u}; \bar{v})) = \mathbb{B}^{a,b}(\bar{u}; \bar{v}).
$$

(2.22)

Therefore we have

$$
\psi(\mathcal{F}^{(ij)}_{a,b}(z|\bar{u}^C; \bar{v}^C; \bar{u}^B; \bar{v}^B)) = \mathcal{F}^{(ji)}_{a',b'}(z|\bar{u}^B; \bar{v}^B; \bar{u}^C; \bar{v}^C),
$$

(2.23)

and hence, the form factor $\mathcal{F}^{(ij)}_{a,b}(z)$ can be obtained from $\mathcal{F}^{(ji)}_{a',b'}(z)$ via the replacements $\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}$ and $\{a, b\} \leftrightarrow \{a', b'\}$.

One more relationship between form factors arises due to the mapping $\varphi$:

$$
\varphi : T_{ij}(u) \mapsto T_{1-j,4-i}(-u),
$$

(2.24)

that defines an isomorphism of the algebra $(2.2)$, $(2.8)$. This isomorphism implies the following transform of Bethe vectors:

$$
\varphi(\mathbb{B}^{a,b}(\bar{u}; \bar{v})) = \mathbb{B}^{b,a}(-\bar{v}; -\bar{u}), \quad \varphi(\mathbb{C}^{a,b}(\bar{u}; \bar{v})) = \mathbb{C}^{b,a}(-\bar{v}; -\bar{u}).
$$

(2.25)

Since the mapping $\varphi$ connects the operators $T_{11}$ and $T_{33}$, it also leads to the replacement of functions $r_1 \leftrightarrow r_3$. Therefore, if $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$ and $\mathbb{C}^{a,b}(\bar{u}; \bar{v})$ are constructed in the representation $\mathcal{V}(r_1(u), r_3(u))$, when their images are in the representation $\mathcal{V}(r_3(-u), r^*_1(-u))$. Thus,

$$
\varphi(\mathcal{F}^{(ij)}_{a,b}(z|\bar{u}^C; \bar{v}^C; \bar{u}^B; \bar{v}^B)) = \mathcal{F}^{(1-j,4-i)}_{b,a}(-z| -\bar{u}^C; -\bar{v}^C; -\bar{v}^B; -\bar{u}^B)|_{r_1 \leftrightarrow r_3}.
$$

(2.26)

3 Main results

The main result considered in this paper is the form factor $\mathcal{F}^{(1,2)}_{a,b}(z)$:

$$
\mathcal{F}^{(1,2)}_{a,b}(z) = \mathcal{F}^{(1,2)}_{a,b}(z|\bar{u}^C; \bar{v}^C; \bar{u}^B; \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)T_{12}(z)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B),
$$

(3.1)

where both $\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors and $a' = a + 1, b' = b$.

In order to describe the determinant representation for this form factor we first of all introduce a set of variables $\bar{x} = \{x_1, \ldots, x_{a'+b}\}$ as the union of three sets

$$
\bar{x} = \{\bar{u}^B, z, \bar{v}^C\} = \{u^B_1, \ldots, u^B_{a'}, z, v^C_1, \ldots, v^C_b\},
$$

(3.2)
and define a scalar function $\mathcal{H}_{a',b}^{(1,2)}$ as

$$
\mathcal{H}_{a',b}^{(1,2)} = \frac{h(\bar{x}, \bar{u}^b)h(\bar{v}^c, \bar{x})}{h(\bar{u}^c, \bar{u}^b)} \Delta_{a'}(\bar{u}^C) \Delta_b(\bar{v}^B) \Delta_{a'+b}(\bar{x}).
$$

(3.3)

Then for general $a$ and $b$ we introduce an $a' \times (a' + b)$ matrix $\mathcal{L}^{(1,2)}(x_k, u_j^C)$ as

$$
\mathcal{L}^{(1,2)}(x_k, u_j^C) = (-1)^{b'-1} t(u_j^C, x_k) \frac{r_1(x_k)h(\bar{u}^C, x_k)}{f(\bar{u}^C, x_k)h(x_k, \bar{v}^B)} + t(x_k, u_j^C) \frac{h(x_k, \bar{v}^B)}{h(x_k, \bar{u}^B)}.
$$

(3.4)

and a $b \times (a' + b)$ matrix $\mathcal{M}^{(1,2)}(x_k, v_j^B)$ as

$$
\mathcal{M}^{(1,2)}(x_k, v_j^B) = (-1)^{b-1} t(x_k, v_j^B) \frac{r_3(x_k)h(\bar{v}^B, x_k)}{f(x_k, \bar{u}^B)h(\bar{v}^C, x_k)} + t(v_j^B, x_k) \frac{h(\bar{v}^B, x_k)}{h(\bar{v}^C, x_k)}.
$$

(3.5)

**Proposition 3.1.** The form factor $\mathcal{F}_{a,b}^{(1,2)}(z)$ admits the following determinant representation:

$$
\mathcal{F}_{a,b}^{(1,2)}(z) = \mathcal{H}_{a',b}^{(1,2)} \det_{a'+b} \mathcal{N}^{(1,2)},
$$

(3.6)

where

$$
\mathcal{N}_{j,k}^{(1,2)} = \mathcal{L}^{(1,2)}(x_k, u_j^C), \quad j = 1, \ldots, a',
$$

$$
\mathcal{N}_{a'+j,k}^{(1,2)} = \mathcal{M}^{(1,2)}(x_k, v_j^B), \quad j = 1, \ldots, b.
$$

(3.7)

The proof of this Proposition is given section 4.

**Remark 1.** The order of the elements in the set $\bar{x}$ is not essential, because the prefactor $\Delta_{a'+b}(\bar{x})$ and $\det_{a'+b} \mathcal{N}^{(1,2)}$ are antisymmetric under permutations of any two elements of $\bar{x}$. We used the ordering as in (3.2), because it is more convenient for the derivation of the determinant representation (3.6).

**Remark 2.** It is straightforward to check that due to (2.13) the entries of the matrix $\mathcal{N}^{(1,2)}$ are proportional to the Jacobians of the transfer matrix eigenvalues

$$
\mathcal{L}^{(1,2)}(x_k, u_j^C) = \frac{c}{f(x_k, \bar{u}^B)f(\bar{v}^C, x_k)} \frac{g(x_k, \bar{u}^B)}{g(x_k, \bar{v}^C)} \frac{\partial \tau(x_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C},
$$

(3.8)

$$
\mathcal{M}^{(1,2)}(x_k, v_j^B) = \frac{-c}{f(x_k, \bar{u}^B)f(\bar{v}^C, x_k)} \frac{g(\bar{v}^C, x_k)}{g(\bar{u}^B, x_k)} \frac{\partial \tau(x_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B}.
$$

(3.9)

In this sense the representation (3.6) is an analog of the determinant representations for form factors in the $GL(2)$-based models [13]. In particular, at $b = 0$ the equation (3.10) reproduces the result of [13].

Determinant representations for other form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i-j| = 1$ can be derived from (3.6) by the mappings (2.23), (2.26). First, we give the explicit formulas for the form factor of the operator $T_{23}$

$$
\mathcal{F}_{a,b}^{(2,3)}(z) = \mathcal{F}_{a,b}^{(2,3)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathcal{C}^{a'B'}(\bar{u}^C, \bar{v}^C)T_{23}(z)\mathcal{B}^{a,b}(\bar{u}^B, \bar{v}^B),
$$

(3.10)
and a function
\[ H_{a,b}^{(2,3)} = \frac{h(\bar{y}, \bar{u}^C) h(\bar{u}^B, \bar{y})}{h(\bar{u}^B, \bar{u}^C)} \Delta_a' (\bar{u}^B) \Delta_b' (\bar{v}^C) \Delta_{a+b'} (\bar{y}). \]

Then for general \(a\) and \(b\) we define an \(a \times (a + b')\) matrix \(L^{(2,3)}(y_k, u_j^B)\) as
\[ L^{(2,3)}(y_k, u_j^B) = (-1)^{a-1} t(u_j^B, y_k) \frac{r_1(y_k) h(\bar{u}^B, y_k) h(\bar{v}^C, y_k)}{f(\bar{u}^B, y_k) h(\bar{u}^B, \bar{u}^C)} + t(y_k, u_j^B) \frac{h(y_k, \bar{u}^B) h(\bar{u}^B, \bar{v}^C)}{h(\bar{v}^C, y_k)}, \]
and a \((a + b') \times (a + b')\) matrix \(M^{(2,3)}(y_k, v_j^C)\) as
\[ M^{(2,3)}(y_k, v_j^C) = (-1)^{b'} t(y_k, v_j^C) \frac{r_3(y_k) h(\bar{v}^C, y_k)}{f(\bar{v}^C, y_k) h(\bar{v}^C, \bar{u}^B)} + t(v_j^C, y_k) \frac{h(y_k, \bar{v}^C) h(\bar{v}^C, \bar{u}^B)}{h(\bar{v}^C, y_k)}. \]

**Proposition 3.2.** The form factor \(F_{a,b}^{(2,3)}(z)\) admits the following determinant representation:
\[ F_{a,b}^{(2,3)}(z) = H_{a,b}^{(2,3)} \det N^{(2,3)} \]
where
\[ N^{(2,3)}_{j,k} = L^{(2,3)}(y_k, u_j^B), \quad j = 1, \ldots, a, \]
\[ N^{(2,3)}_{a+j,k} = M^{(2,3)}(y_k, v_j^C), \quad j = 1, \ldots, b'. \]

Similarly to the case considered in Proposition 3.1 the order of the elements in the set \(\bar{y}\) is not essential, and the entries of the matrix \(N^{(2,3)}\) can be expressed in terms of the Jacobians of the transfer matrix eigenvalues.

**Proposition 3.3.** The form factor \(F_{a,b}^{(2,3)}(z)\) admits the following determinant representation:
\[ F_{a,b}^{(2,3)}(z) = H_{a,b}^{(1,2)} \det N^{(1,2)} \]
where \(H_{a,b}^{(1,2)}\) and \(N^{(1,2)}\) are given by (3.3) and (3.6) respectively.

The form factor \(F_{a,b}^{(2,1)}(z)\) admits the following determinant representation:
\[ F_{a,b}^{(2,1)}(z) = H_{a,b}^{(2,3)} \det N^{(2,3)} \]
where \(H_{a,b}^{(2,3)}\) and \(N^{(2,3)}\) are given by (3.12) and (3.15) respectively.

The proofs of Proposition 3.2 and Proposition 3.3 are given section 5.

Remark. We would like to stress that although the representations (3.17) and (3.18) formally coincide with (3.6) and (3.15), the values of \(a'\) and \(b'\) in these formulas are different. Indeed, one has \(a' = a + 1\) and \(b' = b\) in (3.6), while \(a' = a\) and \(b' = b + 1\) in (3.17). Similarly \(a' = a\) and \(b' = b + 1\) in (3.15), while \(a' = a - 1\) and \(b' = b\) in (3.18). Therefore, in particular, the matrices \(N^{(1,2)}\) and \(N^{(2,3)}\) in (3.6) and (3.15) have a size \((a + b + 1) \times (a + b + 1)\), while in the equations (3.17) and (3.18) the same matrices have a size \((a + b) \times (a + b)\).
4 Derivation of the determinant representation

In this section we prove the determinant representation (3.3) for the form factor of the operator $T_{12}(z)$. We use the same technique as in the work [32].

First of all we need a formula for the action of $T_{12}$ on the Bethe vectors [28].

\[
T_{12}(z)B^{a,b}(u; v) = - \sum f(\xi_i, \xi_i)K_1(\xi_i|z + c)B^{a+1,b}(\eta; \xi_i). \tag{4.1}
\]

Here \(\{v, z\} = \xi\) and \(\{\bar{u}, \bar{z}\} = \eta\). The sum is taken over partitions \(\xi = \{\xi_i, \xi_{ii}\}\) with \(\#\xi_i = 1\). Multiplying (4.1) from the left by \(C^{a+1,b}(\bar{u}^C; \bar{v}^C)\) we reduce the form factor \(F_{a,b}^{(1,2)}(z)\) to a linear combination of scalar products

\[
F_{a,b}^{(1,2)}(z) = - \sum f(\xi_i, \xi_i)K_1(\xi_i|z + c)S_{a+1,b}(\bar{u}^C, \bar{v}^C | \eta, \xi_{ii}). \tag{4.2}
\]

Now we can substitute here the expression (2.17) for the scalar product, replacing there the set \(\bar{u}^B\) by \(\bar{\eta}\) and the set \(\bar{v}^B\) by \(\bar{\xi}_{ii}\). Using the Bethe equations for the set \(\bar{u}^C\)

\[
r_1(\bar{u}_n^C)f(\bar{u}_i^C, \bar{u}_n^C) = f(\bar{u}_n^C, \bar{u}_i^C)f(\bar{v}_i^C, \bar{u}_n^C), \tag{4.3}
\]

we obtain

\[
F_{a,b}^{(1,2)}(z) = - \sum f(\xi_i, \xi_i)K_1(\xi_i|z + c)r_1(\bar{\eta})r_3(\bar{\xi}_i)Z_{a+1-k,n}(\bar{u}_n^C; \bar{\eta}_n^C | \bar{v}_i^C, \bar{\xi}_i)Z_{k,b-n}(\bar{\eta}_i; \bar{u}_i^C | \bar{v}_n^C). \tag{4.4}
\]

The sum is taken with respect to partitions:

\[
\bar{u}^C \Rightarrow \{\bar{u}_n^C, \bar{u}_i^C\}, \quad \bar{v}^C \Rightarrow \{\bar{v}_i^C, \bar{v}_n^C\}, \\
\bar{\eta} \Rightarrow \{\bar{\eta}_n, \bar{\eta}_i\}, \quad \bar{\xi} \Rightarrow \{\bar{\xi}_i, \bar{\xi}_n\}, \tag{4.5}
\]

where \(\#\bar{\eta} = \#\bar{\xi} = k\) with \(k = 0, \ldots, a + 1\); \(\#\bar{\xi}_i = 1\); and \(\#\bar{\xi}_i = \#\bar{\xi}^C = n\) with \(n = 0, \ldots, b\).

Substituting here (2.15) for \(Z_{a+1-k,n}\) and (2.16) for \(Z_{k,b-n}\) we find

\[
F_{a,b}^{(1,2)}(z) = \sum (-1)^{k+1}f(\xi_i, \xi_i)K_{b-n}(\bar{a}_n - c|\bar{\xi}_n)f(\bar{\xi}_i, \bar{\xi}_n)f(\bar{v}_i, \bar{v}_n)f(\bar{v}_n, \bar{v}_i)\]

\[
\times \left[ K_1(\xi_i |z + c)K_{b-n}(\bar{a}_n - c|\bar{\xi}_n)f(\bar{\xi}_i, \bar{\xi}_n) \right] \cdot \left[ K_{k}(\bar{u}_n^C | \bar{a}_1)K_{a-k+1}(\bar{\bar{\eta}}_n, \bar{\bar{\xi}}^C_n) f(\bar{\bar{\xi}}^C_n, \bar{\bar{u}}^C_n) \right] \\
\times K_{k}(\bar{v}_i^C - c|\bar{\bar{\eta}}_i)K_{n}(\bar{\bar{\xi}}^C_n | \bar{\bar{\eta}}_i)K_{b-n}(\bar{\bar{a}}_n - c|\bar{\bar{v}}^C_n + c). \tag{4.6}
\]

Here \(\bar{\bar{\bar{\eta}}}_i = \bar{\bar{\bar{\eta}}}_n, \bar{\bar{\bar{v}}}_i^C\) and \(\bar{\bar{\bar{\alpha}}}_1 = \bar{\bar{\bar{\alpha}}}_n = \bar{\bar{\bar{\alpha}}}_i^C\). The sum is taken with respect to the partitions (4.5) and two additional partitions: \(\bar{\bar{\bar{\eta}}} = \{\bar{\bar{\bar{\eta}}}_i, \bar{\bar{\bar{\eta}}}_n\}\) and \(\bar{\bar{\bar{\alpha}}} = \{\bar{\bar{\bar{\alpha}}}_1, \bar{\bar{\bar{\alpha}}}_n\}\) with \(\#\bar{\bar{\bar{\eta}}}_i = n\) and \(\#\bar{\bar{\bar{\alpha}}}_1 = k\).

**Remark.** Note that the restrictions on the cardinalities of subsets are explicitly specified by the subscripts of DWPF. For example, the DWPF \(K_{k}(\bar{u}_n^C | \bar{a}_1)\) is defined only if \(\#\bar{u}_n^C = \#\bar{a}_1 = k\). Therefore below we do not specify the cardinalities of subsets in separate comments.
Now we can apply (A.4) to the terms in the square brackets in the second line of (4.6). The sum with respect to the partitions $\bar{u}^c \to \{\bar{u}^c, \bar{u}^c_n\}$ gives

$$\sum K_k(\bar{u}^c | \bar{a}_1) K_{a-k+1}(\bar{w}_n | \bar{u}^c_n) f(\bar{u}^c, \bar{w}^c_n) = (-1)^k f(\bar{u}^c, \bar{a}_1) K_{a+1}(\{\bar{a}_1 - c, \bar{w}_n\} | \bar{u}^c).$$  

(4.7)

Similarly, setting $\{\bar{\xi}_i, \bar{\xi}_n\} = \bar{\xi}_m$ we calculate the sum with respect to the partitions $\bar{\xi}_m \to \{\bar{\xi}_1, \bar{\xi}_n\}$:

$$\sum K_1(\bar{\xi}_1 | z + c) K_{b-n}(\bar{a}_n - c | \bar{\xi}_n) f(\bar{\xi}_n, \bar{\xi}_1) = -f^{-1}(z, \bar{\xi}_n) K_{b-n+1}(\{z, \bar{a}_n - c\} | \bar{\xi}_m),$$

(4.8)

where we have used (2.5). Then (4.6) turns into

$$f^{(1,2)}_{a,b}(z) = \sum (-1)^{k+k} r_1(\bar{\eta}) r_3(\bar{\xi}_i) r_3(\bar{v}^B)^c f(\bar{\eta}^c_n, \bar{\eta}) f(\bar{v}^c_n, \bar{v}^c) f(\bar{\xi}_i, \bar{\xi}_m) f(\bar{w}, \bar{w}_n) f(\bar{a}_i, \bar{a}_n)$$

$$\times f(\bar{u}^c, \bar{a}_1) K_{a+1}(\{\bar{a}_1 - c, \bar{w}_n\} | \bar{u}^c) K_n(\bar{v}^c - c | \bar{w}_1)$$

$$\times K_{b-n+1}(\{z, \bar{a}_n - c\} | \bar{\xi}_m) K_{b-n}(\bar{a}_n - c | \bar{v}^B + c).$$

(4.9)

Now one should distinguish between two cases: $z \in \bar{\xi}_m$ or $z \in \bar{\xi}_i$. In the first case the contribution to the form factor does not depend on $r_3(z)$, while in the second case it is proportional to $r_3(z)$. Thus, we can write

$$f^{(1,2)}_{a,b}(z) = r_3(z) \Omega_2 + \Omega_1.$$  

(4.10)

We will calculate $\Omega_1$ and $\Omega_2$ separately.

### 4.1 The first particular case

Here we consider the case $z \in \bar{\xi}_m$. The corresponding contribution $\Omega_1$ to the form factor does not depend on $r_3(z)$. Therefore without loss of generality below we will set $r_3(z) = 0.$

We can set $\bar{\xi}_i = \bar{v}^B_i$ and $\bar{\xi}_m = \{z, \bar{v}^B_n\}$. Then the product $f^{-1}(z, \bar{\xi}_m)$ vanishes, however this zero is compensated by the pole of $K_{b-n+1}(\{z, \bar{a}_n - c\} | \bar{\xi}_m)$ (see (A.1)):

$$f^{-1}(z, \bar{\xi}_m) K_{b-n+1}(\{z, \bar{a}_n - c\} | \bar{\xi}_m) = f^{-1}(z, \bar{a}_n) K_{b-n}(\bar{a}_n - c | \bar{v}^B + c).$$

(4.11)

Substituting this into (4.10) and using Bethe equations for $r_3(\bar{\xi}_i) = r_3(\bar{v}^B)$ we obtain after simple algebra

$$\Omega_1 = \sum (-1)^{b+k} r_1(\bar{\eta}) r_3(\bar{v}^B_i) f(\bar{u}^c, \bar{a}_1) f(\bar{v}^c_n, \bar{v}^B) f(\bar{v}^c_i) f(\bar{w}_1, \bar{w}_n) f(\bar{a}_1, \bar{a}_n) K_n(\bar{v}^c - c | \bar{w}_1)$$

$$\times K_{b-n}(\bar{a}_n - c | \bar{v}^B + c) K_{a+1}(\{\bar{a}_1 - c, \bar{w}_n\} | \bar{u}^c) \left[ K_n(\bar{v}^B_i | \bar{w}_1) K_{b-n}(\bar{a}_n - c | \bar{v}^B + c) f(\bar{v}^B_n, \bar{v}^B_i) \right].$$

(4.12)

The sum with respect to the partitions $\bar{v}^B \to \{\bar{v}^B_i, \bar{v}^B_n\}$ (see the terms in the square brackets in (4.12)) can be calculated via (A.4)

$$\sum K_n(\bar{v}^B_i | \bar{w}_1) K_{b-n}(\bar{a}_n - c | \bar{v}^B_n) f(\bar{v}^B_n, \bar{v}^B_i) = (-1)^n f(\bar{v}^B, \bar{w}_1) K_b(\{\bar{w}_1 - c, \bar{a}_n - c\} | \bar{v}^B).$$

(4.13)
Thus, we obtain
\[
\Omega_1 = \sum (-1)^{b+k+n} \frac{\Omega_1 \Omega_3(v_C)}{f(z, \alpha_\eta)} f(u_C, \alpha_1) f(v_B, \omega_1) f(\eta_\alpha, \eta_1) f(\bar{v}_\alpha, \bar{v}_1) f(\bar{w}_1, \omega_\alpha) f(\alpha_1, \alpha_\eta)
\]
\[
\times K_n(\bar{v}_\alpha - c|\bar{w}_1) K_{b-n}(\alpha_\eta - c|\bar{v}_\alpha + c) K_b(\{\bar{v}_\alpha - c, \bar{w}_1 - c\}|\bar{v}_\alpha) K_{a+1}(\{\alpha_1 - c, \bar{w}_1\}|\bar{u}_C).
\]  \hspace{1cm} (4.14)

Now it is necessary to specify the partitions of the sets \(\bar{\alpha}\) and \(\bar{\omega}\). We set
\[
\bar{\alpha}_1 = \{\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C + c\}, \quad \bar{\eta}_1 = \{\bar{\eta}_{\bar{i}_1}, \bar{\eta}_{\bar{i}_2}\},
\]
\[
\bar{\alpha}_n = \{\bar{\eta}_n, \bar{v}_n^C + c\}, \quad \bar{\eta}_n = \{\bar{\eta}_{\bar{i}_n}, \bar{\eta}_{\bar{i}_n}\},
\]
\[
\bar{\omega}_1 = \{\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C\}, \quad \bar{v}_1^C = \{\bar{v}_1^C, \bar{v}_1^C\},
\]
\[
\bar{\omega}_n = \{\bar{\eta}_{\bar{i}_n}, \bar{v}_n^C\}, \quad \bar{v}_n^C = \{\bar{v}_n^C, \bar{v}_n^C\}.
\]  \hspace{1cm} (4.15)

We denote the cardinalities of these subsets as \(#\bar{\eta}_j = k_j\) and \(#\bar{v}_j^C = n_j\), where \(j = i, ii, iii, iv\). Evidently \(\sum_{j=1}^{iv} k_j = a + 1\) and \(\sum_{j=1}^{iv} n_j = b\). It is also easy to see that
\[
n_i + n_{ii} = b - n, \quad k_i + k_{ii} = k,
\]
\[
n_{ii} + n_{iv} = n, \quad k_{ii} + k_{iv} = a + 1 - k,
\]
\[
n_i = k_i, \quad n_{iv} = k_{iv}.
\]  \hspace{1cm} (4.16)

In terms of the subsets introduced above the equation (4.14) takes the form
\[
\Omega_1 = \sum (-1)^{b+k+n} \frac{r_1(\bar{\eta}_1)r_3(\bar{v}_1^C)r_3(\bar{v}_1^C)}{f(z, \alpha_\eta)} f(u_C, \bar{v}_1^C) f(\eta_\alpha, \bar{v}_1^C) f(\bar{v}_1^C, \bar{v}_1^C) f(\bar{w}_1, \omega_\alpha) f(\alpha_1, \alpha_\eta)
\]
\[
\times K_n(\{\bar{v}_1^C - c, \bar{v}_1^C\} | \{\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C\}) K_{b-n}(\{\bar{\eta}_{\bar{i}_1} - c, \bar{v}_1^C\} | \{\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C\} | \bar{v}_1^C) K_b(\{\bar{\eta}_{\bar{i}_1} - c, \bar{\eta}_{\bar{i}_1}, \bar{v}_1^C\} | \bar{v}_1^C)
\]
\[
\times K_{a+1}(\{\bar{\eta}_{\bar{i}_1} - c, \bar{\eta}_{\bar{i}_1}, \bar{v}_1^C\} | \bar{v}_1^C).
\]  \hspace{1cm} (4.17)

Now several simplifications are possible. First of all the DWPF \(K_n\) and \(K_{b-n}\) can be transformed via \(\{A.2\}, \{A.3\}\)
\[
K_n(\{\bar{v}_1^C - c, \bar{v}_1^C\} | \{\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C\}) = (-1)^n K_n, (\bar{\eta}_{\bar{i}_1} | \bar{v}_1^C) f^{-1}(\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C)
\]  \hspace{1cm} (4.18)
\[
K_{b-n}(\{\bar{\eta}_{\bar{i}_1} - c, \bar{v}_1^C\} | \{\bar{v}_1^C + c, \bar{v}_1^C + c\}) = (-1)^{b-n} K_{b-n}(\bar{v}_1^C + c | \bar{\eta}_{\bar{i}_1} f^{-1}(\bar{v}_1^C + c, \bar{\eta}_{\bar{i}_1}.
\]  \hspace{1cm} (4.19)

Then one should express \(r_1(\bar{\eta}_1)\) and \(r_3(\bar{v}_1^C)\) in terms of the Bethe equations. Observe that \(z \notin \bar{\eta}_1\), due to the factor \(f^{-1}(z, \bar{\eta}_1)\). Therefore the subset \(\bar{\eta}_1\) consists of the elements \(u_j^B\) only, and one do can use the Bethe equations for \(r_1(\bar{\eta}_1)\). Therefore
\[
r_1(\bar{\eta}_1) = \frac{f(\bar{\eta}_1, \bar{\eta}_{\bar{i}_1}) f(\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C) f(z, \bar{\eta}_1)}{f(\bar{\eta}_1, \bar{\eta}_{\bar{i}_1}) f(\bar{\eta}_{\bar{i}_1}, \bar{\eta}_1) f(\bar{\eta}_{\bar{i}_1}, \bar{v}_1^C) f(z, \bar{\eta}_1)},
\]  \hspace{1cm} (4.20)
and
\[ r_3(v_i^C) = \frac{f(v_i^C, \bar{v}_i^C)f(v_{ii}^C, \bar{v}_i^C)f(v_{iii}^C, \bar{v}_i^C)}{f(v_i^C, \bar{v}_i^C)f(v_{ii}^C, \bar{v}_{ii}^C)f(v_{iii}^C, \bar{v}_{iii}^C)} f(v_i^C, \bar{v}_i^C). \]  
(4.21)

These expressions should be substituted into (4.17).

**Remark.** Formally one can also use the Bethe equations for the product \( r_3(\bar{v}_{ii}^C) \). However it is more convenient to keep this product as it is.

Finally, we introduce new subsets
\[ \tilde{\eta}_i = \{ \tilde{\eta}_i, \tilde{\eta}_{ii} \}, \quad \tilde{v}_i^C = \{ \tilde{v}_i^C, \tilde{v}_{ii}^C \}, \]
\[ \tilde{\eta}_n = \{ \tilde{\eta}_n, \tilde{\eta}_{nn} \}, \quad \tilde{v}_n^C = \{ \tilde{v}_n^C, \tilde{v}_{nn}^C \}, \]  
(4.22)

and we denote \( n_i = \# \tilde{\eta}_i = \# \tilde{v}_i^C \). We draw the readers attention that these new subsets have nothing to do with the subsets used in (4.14). We use, however, the same notation, as we deal with the sum over partitions, and therefore it does not matter how we denote separate terms of this sum.

Then the equation (4.17) can be written in the following form:
\[ \Omega_1 = \sum (-1)^b f(\bar{v}_n^C, \bar{\eta}_n) f(\bar{v}_n^C, \bar{\eta}_n) G_{n_i}(\bar{\eta}_i | \bar{v}_i^C) L_{a+1}(\{ \bar{\eta}_n, \bar{v}_n^C \}| \bar{u}^C) M_b(\{ \bar{\eta}_n, \bar{v}_n^C \}| \bar{u}^a), \]  
(4.23)

where we have introduced three new functions: \( G_{n_i}, L_{a+1}, \) and \( M_b \). Originally all of them are defined as sums over partitions. The function \( G_{n_i} \) is given by
\[ G_{n_i}(\bar{\eta}_i | \bar{v}_i^C) = \sum_{ \{ \bar{\eta}_{ii} \}, \{ \bar{v}_{ii}^C \} } \frac{f(\bar{\eta}_i, \bar{\eta}_{ii})f(\bar{v}_{ii}^C, \bar{v}_i^C)}{f(\bar{\eta}_i, z)} K_{n_{\eta_i}}(\bar{\eta}_{ii} | \bar{v}_{ii}^C) K_{n_{\eta_i}}(\bar{v}_i^C + c | \bar{\eta}_i), \]  
(4.24)

where the sum is taken over partitions \( \tilde{\eta}_i \Rightarrow \{ \tilde{\eta}_i, \tilde{\eta}_{ii} \} \) and \( \tilde{v}_i^C \Rightarrow \{ \tilde{v}_i^C, \tilde{v}_{ii}^C \} \).

The function \( L_{a+1}(\{ \bar{\eta}_n, \bar{v}_n^C \}| \bar{u}^C) \) reads
\[ L_{a+1}(\{ \bar{\eta}_n, \bar{v}_n^C \}| \bar{u}^C) = \sum (-1)^b f(\bar{u}^C, \bar{\eta}_n) f(\bar{u}^C, \bar{\eta}_n) f(\bar{v}_n^C, \bar{\eta}_n) K_{a+1}(\{ \bar{\eta}_n - c, \bar{\eta}_{nn} \}, \{ \bar{v}_n^C, \bar{v}_i^C \}, | \bar{u}^C), \]  
(4.25)

where the sum is taken over partitions \( \tilde{\eta}_n \Rightarrow \{ \tilde{\eta}_n, \tilde{\eta}_{nn} \} \).

Finally, the function \( M_b(\{ \bar{\eta}_n, \bar{v}_n^C \}| \bar{u}^a) \) is given by
\[ M_b(\{ \bar{\eta}_n, \bar{v}_n^C \}| \bar{u}^a) = \sum (-1)^b r_3(\bar{v}_n^C) f(\bar{u}^a, \bar{\eta}_n) f(\bar{u}^a, \bar{\eta}_n) \times f(\bar{v}_n^C, \bar{\eta}_n) f(\bar{v}_n^C, \bar{\eta}_n) K_b(\{ \bar{\eta}_n - c, \bar{\eta}_{nn} - c, \bar{\eta}_{nn} \}| \bar{u}^a), \]  
(4.26)

where the sum is taken over partitions \( \tilde{v}_n^C \Rightarrow \{ \tilde{v}_n^C, \tilde{v}_{nn}^C \} \). It is straightforward to check that substituting the definitions (4.24)–(4.26) into (4.23) we reproduce (4.17).

It is remarkable that all the sums with respect to partitions in (4.24)–(4.26) can be explicitly computed. The function \( G_{n_i}(\bar{\eta}_i | \bar{v}_i^C) \) can be calculated via (A.13)
\[ G_{n_i}(\bar{\eta}_i | \bar{v}_i^C) = (-1)^{n_i} t(\bar{v}_i^C, \bar{\eta}_i) h(\bar{v}_i^C, \bar{v}_i^C) h(\bar{\eta}_i, \bar{\eta}_i) h(\bar{v}_i^C, \bar{z}) h(\bar{\eta}_i, \bar{z}). \]  
(4.27)
The functions $L_{a+1}({\bar{\eta}_a, \bar{v}^C})$ and $M_b({\bar{\eta}_b, \bar{v}^B})$ can be calculated via lemma A.2. Let us set in (A.5)

$$
\bar{\gamma} = \{\bar{\eta}_a, \bar{v}^C\}, \quad C_1(\gamma) = -\frac{r_1(\gamma)}{f(\bar{v}^C, \gamma)}, \quad C_2(\gamma) = 1.
$$

(4.28)

Observe that $C_1(v^C_k) = 0$ due to the factor $f^{-1}(\bar{v}^C, \gamma)$. Therefore, dividing in (A.6) the set $\bar{\gamma}$ into two subsets $\{\bar{\gamma}_1, \bar{\gamma}_a\}$ one should consider only the partitions for which $\bar{v}^C_i \subset \bar{\gamma}_a$. It means that actually we deal with the partitions of the subset $\bar{\eta}_a$ only. Namely, we can set $\bar{\gamma}_1 = \bar{\eta}_a$ and $\bar{\gamma}_a = \{\bar{v}^C, \bar{\eta}_{\bar{u}}\}$. Then the sum in (A.5) coincides with the sum (4.26) and we obtain

$$
L_{a+1}(\bar{\gamma} | \bar{u}^C) = \Delta_{a+1}(\bar{u}^C) \Delta_{a+1}(\bar{\eta}_a) \det_{a+1} \left[ L^{(1,2)}(\gamma_k, u^C_j) h(\gamma_k, \bar{u}^B) \right], \quad \bar{\gamma} = \{\bar{\eta}_a, \bar{v}^C\},
$$

(4.29)

where the matrix $L^{(1,2)}$ is given by (3.4).

Similarly for the calculation (4.26) one should set in the sum (A.5)

$$
\bar{\gamma} = \{\bar{\eta}_a, \bar{v}^C\}, \quad C_1(\gamma) = 1, \quad C_2(\gamma) = -\frac{r_3(\gamma)}{f(\gamma, \bar{u}^B)}.
$$

(4.30)

Then $C_2(\eta_k) = 0$ either due to the product $f^{-1}(\gamma, \bar{u}^B)$ or due to the condition $r_3(z) = 0$ (that we freely imposed in this subsection). Therefore we can set $\bar{\gamma}_1 = \{\bar{v}^C, \bar{\eta}_a\}$ and $\bar{\gamma}_a = \bar{v}^B$ in (A.5). Then the sum (A.5) turns into (4.26) and we obtain

$$
M_b(\bar{\gamma} | \bar{v}^B) = (-1)^b \Delta_b(\bar{v}^B) \Delta_b(\bar{\gamma}) \det_b \left[ M^{(1,2)}(\gamma_k, v^B_j) h(\bar{v}^C, \gamma_k) \right], \quad \bar{\gamma} = \{\bar{\eta}_a, \bar{v}^C\},
$$

(4.31)

where the matrix $M^{(1,2)}$ is given by (3.5).

Introducing

$$
\hat{L}_{a+1}(\bar{\gamma} | \bar{u}^C) = \frac{L_{a+1}(\bar{\gamma} | \bar{u}^C)}{h(\gamma, \bar{u}^B)}, \quad \hat{M}_b(\bar{\gamma} | \bar{v}^B) = \frac{M_b(\bar{\gamma} | \bar{u}^B)}{h(\bar{v}^C, \gamma)}.
$$

(4.32)

we obtain after simple algebra

$$
\Omega_1 = \frac{f(\bar{v}^C, \bar{\eta}) h(\bar{v}^B, \bar{v}^C) h(\bar{\eta}, \bar{\eta})}{h(\bar{\eta}, z)} \sum (-1)^{n_1} g(\bar{\eta}, \bar{v}^B) g(\bar{v}^C, \bar{\eta}_a) g(\bar{v}^C, \bar{\eta}) \Delta_{a+1}(\{\bar{\eta}_a, \bar{v}^C\} | \bar{u}^C) \hat{M}_b(\{\bar{\eta}_a, \bar{v}^C\} | \bar{v}^B).
$$

(4.33)

Define a set $\bar{x}$ as in (4.32)

$$
\bar{x} = \{\bar{\eta}, \bar{v}^C\} = \{u^B_1, \ldots, u^B_a, \bar{z}, v^C_1, \ldots, v^C_b\}.
$$

(4.34)

For arbitrary partition $\bar{x} \Rightarrow \{\bar{x}_1, \bar{x}_a\}$ with $\# \bar{x}_1 = a + 1$ and $\# \bar{x}_a = b$ we have

$$
1 = \frac{\Delta_{a+b+1}(\bar{x})}{\Delta_{a+b+1}(\bar{x})} = (-1)^{P_{a+b+1}} \Delta_{a+b+1}(\bar{x}) \Delta_b(\bar{x}_a) g(\bar{\eta}, \bar{x}_1) \Delta_{a+1}(\bar{\eta}) \Delta_b(\bar{v}^C) g(\bar{v}^C, \bar{\eta}),
$$

(4.35)

where $P_{a+b+1}$ is the parity of the permutation mapping the sequence $\{\bar{x}_1, \bar{x}_a\}$ into the ordered sequence $x_1, \ldots, x_{a+b+1}$. Setting $\bar{x}_1 = \{\bar{\eta}_a, \bar{v}^C\}$ and $\bar{x}_a = \{\bar{\eta}, \bar{v}^C\}$ we obtain after elementary algebra

$$
1 = (-1)^{P_{a+b+1}} \frac{\Delta_{a+1}(\bar{x}_1) \Delta_b(\bar{x}_a) g(\bar{\eta}, \bar{\eta}_a) g(\bar{v}^C, \bar{v}^C)}{\Delta_{a+1}(\bar{\eta}) \Delta_b(\bar{v}^C) g(\bar{v}^C, \bar{\eta}) g(\bar{v}^C, \bar{\eta}_a)}
$$

(4.36)
Thus, the equation \((4.33)\) can be written in the form

\[
\Omega_1 = f(\bar{v}^C, \bar{\eta})h(\bar{v}^C, v^C)h(\bar{u}^B, \bar{v}^B)h(z, \bar{u}^B)\Delta_{a+1}^1(u^B_1)\Delta_{a+1}(\bar{\eta})\Delta_b(\bar{v}^C) \\
\times \sum (-1)^{P_{a+1}} \det [\mathcal{L}^{(1,2)}(x_k^1, u_j^C)] \det \mu^{(1,2)}(x_k^1, v_j^B). \tag{4.37}
\]

Here \(x_k^1\) (resp. \(x_k^B\)) is the \(k\)-th element of the subset \(x_i\) (resp. \(x_n\)). It is easy to see that the prefactor in the last line of \((4.37)\) coincides with the function \(\mathcal{H}^{(1,2)}_{a+1, b}\) (see \((4.36)\)). The sum \((4.37)\) is nothing but the expansion of the determinant of the \((a + b + 1) \times (a + b + 1)\) matrix \(\mathcal{N}^{(1,2)}\) with respect to the first \((a + 1)\) rows. Thus, we finally obtain

\[
\Omega_1 = \mathcal{H}^{(1,2)}_{a+1, b} \det \mu^{(1,2)} \bigg|_{r_3(z) = 0}. \tag{4.38}
\]

### 4.2 The second particular case

Now we turn back to the equation \((4.9)\) and consider the case \(z \in \xi_i\), that is we compute the term \(\Omega_2\) in \((4.11)\). The general idea of the calculation is the same as in the case of \(\Omega_1\), however there are several subtleties.

Since \(\bar{\eta} = \{z, \bar{u}^B\}\), the product \(f^{-1}(\xi_i, \bar{\eta})\) vanishes. The only possible way to obtain non-zero contribution to \(\Omega_2\) is to compensate this zero by the pole of \(K_n(\xi_i | \bar{w}_i)\). The last one occurs if and only if \(z \in \bar{w}_i\), which implies \(z \in \bar{\eta}_n\). Thus, we can set

\[
\xi_i = \{z, \bar{u}^B\}, \qquad \bar{\xi}_n = \bar{v}^B_n, \\
\bar{w}_i = \{z, \bar{w}_0\}, \qquad \bar{w}_n = \bar{w}_n, \\
\bar{\eta}_n = \bar{u}^B_n, \qquad \bar{\eta}_n = \{z, \bar{u}_n^B\}. \tag{4.39}
\]

Substituting this into \((4.9)\) we obtain

\[
\Omega_2 = \sum (-1)^{b+k} r_1(\bar{u}_n^B) r_3(v^C_1) f(u^C, \bar{\alpha}_1) f(z, v^C_1) f(\bar{u}_n^B, \bar{v}_1^B) f(\bar{v}_n^C, v^C_1) f(\bar{w}_0, w^B_n) f(\bar{\alpha}_1, \bar{\alpha}_n) \\
\times K_n(\bar{v}^C_1 - c | \{\bar{w}_0, z\}) K_{b-n}(\bar{\alpha}_n - c | \bar{v}^C_n + c) K_{a+1}(\{\bar{\alpha}_1 - c, \bar{w}_n\} | \bar{u}^C) \\
\times K_{b-n+1}(\{z, \bar{\alpha}_n - c\} | \bar{v}^B_n) K_{n-1}(\bar{v}^B_n | \bar{w}_0) f(\bar{v}^B_n, \bar{v}^B_1), \tag{4.40}
\]

where we have also used the Bethe equations for \(r_3(\bar{v}^B_n)\):

\[
r_3(\bar{v}^B_1) = \frac{f(\bar{v}^B_1, \bar{v}^B_0)}{f(\bar{v}^B_1, \bar{u}^B_n)} f(\bar{v}^B_1, \bar{u}^B_n). \tag{4.41}
\]

Applying \((4.4)\) to the terms in the last line of \((4.40)\) we take the sum over partitions \(\bar{v}^B \Rightarrow \{\bar{v}^B_1, \bar{v}^B_0\}\):

\[
\sum K_{b-n+1}(z, \bar{\alpha}_n - c | \bar{v}^B_n) K_{n-1}(\bar{v}^B_n | \bar{w}_0) f(\bar{v}^B_n, \bar{v}^B_1) = (-1)^{n-1} f(\bar{v}^B_n, \bar{w}_0) K_b(\{\bar{w}_0 - c, \bar{\alpha}_n - c, z\} | \bar{v}^B_n). \tag{4.42}
\]
Thus, we arrive at
\[
\Omega_2 = \sum (-1)^{b+k+n+1} r_1(\bar{u}_i^B) r_2(\bar{v}_i^C) f(z, \bar{v}_i^C) f(\bar{u}_i^B, \bar{u}_i^B) f(\bar{v}_i^C, \bar{v}_i^C) f(\bar{w}_0, \bar{w}_n)
\times f(\bar{\alpha}_1, \bar{\alpha}_n) K_n(\bar{v}_i^C - c \vert \{\bar{w}_0, z\}) K_{b-n}(\bar{\alpha}_n - c \vert \bar{v}_n^C + c) K_b(\{\bar{w}_0 - c, \bar{\alpha}_n - c, z\} \vert \bar{v}_n^B) K_{n+1}(\{\bar{\alpha}_1 - c, \bar{w}_n\} \vert \bar{u}_n^C).
\]
(4.43)

Now we should specify the subsets similarly to (4.15)
\[
\bar{\alpha}_i = \{\bar{u}_i^B, \bar{v}_i^C + c\}, \quad \bar{u}_i^B = \{\bar{u}_i^B, \bar{u}_i^B\}, \quad \bar{u}_n^B = \{\bar{u}_n^B, \bar{u}_n^B\}, \quad \bar{v}_i^C = \{\bar{v}_i^C, \bar{v}_i^C\}, \quad \bar{v}_n^C = \{\bar{v}_n^C, \bar{v}_n^C\}.
\]
(4.44)

We again denote the cardinalities of the subsets above as \#\bar{u}_j^B = k_j and \#\bar{v}_j^C = n_j. Now \sum_{j=1}^{iv} k_j = a, \sum_{j=1}^{iv} n_j = b and
\[
\begin{align*}
n_i + n_{ii} &= b - n, & k_i + k_{ii} &= k, \\
n_{iii} + n_{iv} &= n, & k_{iii} + k_{iv} &= a - k, \\
n_i &= k_i, & n_{iv} &= k_{iv} + 1.
\end{align*}
\]
(4.45)

Using the new subsets we obtain an analog of (4.17)
\[
\Omega_2 = \sum (-1)^{b+k+n+1} r_1(\bar{u}_i^B) r_2(\bar{v}_i^C) f(z, \bar{v}_i^C) f(\bar{u}_i^B, \bar{u}_i^B) f(\bar{v}_i^C, \bar{v}_i^C) f(\bar{w}_0, \bar{w}_n)
\times f(\bar{\alpha}_1, \bar{\alpha}_n) K_n(\bar{v}_i^C - c \vert \{\bar{w}_0, z\}) K_{b-n}(\bar{\alpha}_n - c \vert \bar{v}_n^C + c) K_b(\{\bar{w}_0 - c, \bar{\alpha}_n - c, z\} \vert \bar{v}_n^B) K_{n+1}(\{\bar{\alpha}_1 - c, \bar{w}_n\} \vert \bar{u}_n^C).
\]
(4.46)

Now one should make the same transforms as before. Namely, we should simplify \(K_n\) and \(K_{b-n}\) via (A.2), (A.3); express \(r_1(\bar{u}_i^B)\) and \(r_3(\bar{v}_i^C)\) in terms of Bethe equations; introduce new subsets
\[
\bar{u}_i^B = \{\bar{u}_i^B, \bar{u}_i^B\}, \quad \bar{u}_n^B = \{\bar{u}_n^B, \bar{u}_n^B\}, \quad \bar{v}_i^C = \{\bar{v}_i^C, \bar{v}_i^C\}, \quad \bar{v}_n^C = \{\bar{v}_n^C, \bar{v}_n^C\}.
\]
(4.47)
Pay attention that now \( n_t = \#\bar{u}_t^B + 1 = \#\bar{v}_t^C \). We also introduce \( z' = z + c \). Then the equation (4.46) can be written in the following form:

\[
\Omega_2 = f^{-1}(\bar{v}^B, z') \sum (-1)^{b+1} \frac{f(\bar{v}_t^C, \bar{u}_t^B)}{f(\bar{v}_t^C, \bar{u}_t^B)} f(\bar{u}_t^B, \bar{u}_t^B) f(\bar{v}_t^C, \bar{v}_t^C) \\
\times \bar{\mathcal{G}}_{n_t}(\bar{u}_t^B, \bar{v}_t^C) \bar{\mathcal{L}}_{a+1}((\{\bar{u}_t^B, \bar{v}_t^C\}|\bar{u}^C) \bar{\mathcal{M}}_b((\{\bar{u}_t^B, \bar{v}_t^C, z'\}|\bar{v}^B)).
\] (4.48)

Here

\[
\bar{\mathcal{G}}_{n_t}(\bar{u}_t^B|\bar{v}_t^C) = \sum f(\bar{u}_t^B, \bar{u}_t^B) f(\bar{v}_t^C, \bar{v}_t^C) K_{n_t}(\{z, \bar{u}_t^B\})|\bar{v}_t^C| K_{n_t}(\bar{v}_t^C + c|\bar{u}_t^B),
\] (4.49)

where the sum is taken over partitions \( \bar{u}_t^B \Rightarrow \{\bar{u}_t^B, \bar{u}_t^B\} \) and \( \bar{v}_t^C \Rightarrow \{\bar{v}_t^C, \bar{v}_t^C\} \).

The function \( \bar{\mathcal{L}}_{a+1}(\{\bar{u}_t^B, \bar{v}_t^C\}|\bar{u}^C) \) is

\[
\bar{\mathcal{L}}_{a+1}(\{\bar{u}_t^B, \bar{v}_t^C\}|\bar{u}^C) = \sum (-1)^{b_k} r_1(\bar{u}_t^B) \frac{f(\bar{u}^C, \bar{u}_t^B) f(\bar{v}_t^C, \bar{u}_t^B) f(\bar{v}_t^C, \bar{v}_t^C) K_{a+1}(\{\bar{u}_t^B - c, \bar{u}_t^B, \bar{v}_t^C\}|\bar{u}^C),
\] (4.50)

where the sum is taken over partitions \( \bar{u}_t^B \Rightarrow \{\bar{u}_t^B, \bar{u}_t^B\} \).

The function \( \bar{\mathcal{M}}_b((\{\bar{u}_t^B, \bar{v}_t^C, z'\}|\bar{v}^B) \) is

\[
\bar{\mathcal{M}}_b((\{\bar{u}_t^B, \bar{v}_t^C, z'\}|\bar{v}^B) = \sum (-1)^{n_k} r_3(\bar{v}_t^C) \frac{f(\bar{u}^B, \bar{v}_t^C) f(\bar{v}_t^C, \bar{v}_t^C) f(\bar{v}_t^C, \bar{v}_t^C) K_b((\bar{u}_t^B - c, \bar{v}_t^C - c, z' - c, \bar{v}_t^C) |\bar{v}^B),
\] (4.51)

where the sum is taken over partitions \( \bar{v}_t^C \Rightarrow \{\bar{v}_t^C, \bar{v}_t^C\} \).

The function \( \bar{\mathcal{G}}_{n_t}(\bar{u}_t^B|\bar{v}_t^C) \) can be calculated via (A.10)

\[
\bar{\mathcal{G}}_{n_t}(\bar{u}_t^B|\bar{v}_t^C) = (-1)^{n_1} \frac{I(\bar{v}_t^C, \bar{u}_t^B) h(\bar{v}_t^C, \bar{v}_t^C) h(\bar{u}_t^B, \bar{u}_t^B)}{h(\bar{v}_t^C, z') g(z', \bar{u}_t^B)}.
\] (4.52)

The calculation of \( \bar{\mathcal{L}}_{a+1}(\{\bar{u}_t^B, \bar{v}_t^C\}|\bar{u}^C) \) is the same as the one of \( \mathcal{L}_{a+1}(\{\bar{u}_t^B, \bar{v}_t^C\}|\bar{u}^C) \) (one should only replace everywhere \( \bar{\eta}_t \) by \( \bar{u}_t^B \)).

The calculation of \( \bar{\mathcal{M}}_b((\{\bar{u}_t^B, \bar{v}_t^C, z'\}|\bar{v}^B) \) also is almost the same as before. The difference is that now it depends on additional parameter \( z' \). However this difference does not make sense, if we set by definition \( r_3(z') = 0 \). We can always do it, because the form factor anyway does not depend on \( r_3(z') \).

Thus, we find

\[
\bar{\mathcal{L}}_{a+1}(\bar{v}^C|\bar{u}^B) = \Delta'_{a+1}(\bar{v}^C) \bar{\Delta}_{a+1}(\bar{v}^C) \det_{a+1} \left[ \mathcal{L}^{(1,2)}(\gamma_k, \bar{u}_t^B) h(\gamma_k, \bar{u}_t^B) \right], \quad \bar{\gamma} = \{\bar{u}_t^B, \bar{v}_t^C\},
\] (4.53)

and

\[
\bar{\mathcal{M}}_b(\bar{v}^B|\bar{v}^C) = (-1)^b \Delta_b(\bar{v}^B) \Delta_b(\bar{v}^C) \det_b \left[ \mathcal{M}^{(1,2)}(\gamma_k, \bar{v}_t^B) h(\gamma_k, \bar{v}_t^C) \right], \quad \bar{\gamma} = \{\bar{u}_t^B, \bar{v}_t^C, \bar{v}^B\}.
\] (4.54)

Formally, the obtained representations coincide with (1.29), (4.31). However the sets \( \bar{\gamma} \) are different. In (4.53) the set \( \bar{\gamma} \) does not contain \( z \), while in (1.29) it could contain \( z \). Respectively, in (4.54) the set \( \bar{\gamma} \) contains \( z' \), while in (4.31) it was \( z \)-independent.
Introducing $\tilde{L}_{a+1}$ and $\tilde{M}_b$ by
\[
\tilde{L}_{a+1}(\gamma|\tilde{u}^c) = \frac{\tilde{L}_{a+1}(\gamma|u^c)}{h(\gamma, u^c)}, \quad \tilde{M}_b(\gamma|\tilde{u}^b) = (-1)\frac{\tilde{M}_b(\gamma|u^b)}{h(u^c, \gamma)},
\]
and substituting (4.52)–(4.54) into (4.48) we after simple algebra arrive at the analog of (4.33)

\[
\Omega_2 = \frac{f(\tilde{v}^c, \tilde{u}^b) h(\tilde{v}^c, \tilde{v}^c) h(\tilde{u}^b, \tilde{u}^b)}{f(\tilde{v}^b, z')} \sum (-1)^{n_1+1} g(\tilde{u}^b, \tilde{u}^b) g(\tilde{v}^c, \tilde{v}^c) \frac{\tilde{L}_{a+1}(\{\tilde{u}^b, \tilde{v}^c\}|\tilde{u}^b) \tilde{M}_b(\{\tilde{u}^b, \tilde{v}^c, z'\}|\tilde{u}^b)}{g(\tilde{v}^c, z') g(z', \tilde{u}^b)}. \tag{4.56}
\]

Similarly to (4.34) we introduce a set $\tilde{x}'$ as
\[
\tilde{x}' = \{\tilde{u}^b, z', \tilde{v}^c\} = \{u_1^b, \ldots, u_n^b, z', v_1^c, \ldots, v_b^c\}. \tag{4.57}
\]

Consider partitions $\tilde{x}' = \{\tilde{x}'_1, \tilde{x}'_2\}$ with $\#\tilde{x}'_1 = a$ and $\#\tilde{x}'_2 = b + 1$. One can set $\tilde{x}'_1 = \{\tilde{u}_a^b, \tilde{v}_c^c\}$ and $\tilde{x}'_2 = \{\tilde{u}_a^b, \tilde{z}', \tilde{v}_c^c\}$. Then the analog of (4.57) has the following form:
\[
1 = (-1)^{F_{a,n}} \frac{\Delta_{a+1}(\tilde{x}'_1) \Delta_{b}(\tilde{z}')} {\Delta_{a}(\tilde{u}^b) \Delta_{b}(\tilde{v}^c)} g(\tilde{u}_a^b, \tilde{u}_a^b) g(\tilde{v}_c^c, \tilde{v}_c^c) \frac{(-1)^{n_1}}{g(\tilde{v}_c^c, z') g(z', \tilde{u}^b)}. \tag{4.58}
\]

It is important that unlike the previous case we have $\#\tilde{v}_c^c = n_1$ and $\#\tilde{u}_a^b = n_1 - 1$. Therefore, in particular,
\[
g(\tilde{u}_a^b, \tilde{v}_c^c) = g(\tilde{v}_c^c, \tilde{u}_a^b). \tag{4.59}
\]

Thus, the equation (4.56) can be written in the form
\[
\Omega_2 = \frac{-P_{a+1,b}^{(1,2)}}{f(\tilde{v}^b, z') f(\tilde{z}, \tilde{u}^b) f(\tilde{v}^c, z)} \det \tilde{N}^{(1,2)}_{a+1,b}, \quad \tilde{x}' = \{u_1^b, \ldots, u_n^b, z', v_1^c, \ldots, v_b^c\}, \tag{4.61}
\]

where
\[
\tilde{N}^{(1,2)}_{j,k} = L^{(1,2)}(z_j, u_j), \quad j = 1, \ldots, a + 1, \quad k \neq a + 1,
\]
\[
\tilde{N}^{(1,2)}_{a+1,j,k} = M^{(1,2)}(z_j, v_j), \quad j = 1, \ldots, b, \quad k \neq a + 1,
\]
\[
\tilde{N}^{(1,2)}_{j,a+1} = 0, \quad j = 1, \ldots, a + 1,
\]
\[
\tilde{N}^{(1,2)}_{a+1+j,a+1} = M^{(1,2)}(z_j, v_j) \bigg|_{r_3(z')=0} = 0, \quad j = 1, \ldots, b.
\]
We see that all the columns of the obtained matrix coincide with the ones of the matrix in (4.38), except the \((a+1)\)-th column (associated with the parameter \(z'\)), where non-zero matrix elements are
\[
\mathcal{M}^{(1,2)}(z', v^j_j)|_{r_3(z')=0} = t(v^b_j, z') \frac{h(\bar{v}^b_j, z')}{h(\bar{v}^c_j, z')} = t(z, v^b_j) \frac{g(\bar{v}^c, z)}{g(\bar{v}^b_j, z)}. \tag{4.63}
\]

It is easy to see that
\[
\frac{-\mathcal{M}^{(1,2)}(z', v^j_j)}{f(\bar{v}^b_j, z') f(z, \bar{u}^b_j) f(\bar{v}^b_j, z)} \bigg|_{r_3(z')=0} = \lim_{r_3(z) \to \infty} \frac{1}{r_3(z)} \mathcal{M}^{(1,2)}(z, v^b_j). \tag{4.64}
\]

Thus we obtain for all the elements of the \((a+1)\)-th column
\[
\frac{-\tilde{\mathcal{N}}^{(1,2)}_{j,a+1}}{f(\bar{v}^b_j, z') f(z, \bar{u}^b_j) f(\bar{v}^b_j, z)} = \lim_{r_3(z) \to \infty} \frac{1}{r_3(z)} \mathcal{N}^{(1,2)}_{j,a+1}, \quad j = 1, \ldots, a + b + 1. \tag{4.65}
\]

Hence, we arrive at
\[
\Omega_2 = \lim_{r_3(z) \to \infty} \frac{\mathcal{N}^{(1,2)}_{a+1,b}}{a+b} \det \mathcal{N}^{(1,2)} \tag{4.66}
\]

It remains to combine (4.38) and (4.66). This can be easily done, because for any linear function of \(\phi(\zeta) = A\zeta + B\) one has
\[
\phi(\zeta) |_{\zeta = 0} + \zeta \lim_{\zeta \to \infty} \frac{1}{\zeta} \phi(\zeta) = \phi(\zeta). \tag{4.67}
\]

Since the form factor \(\mathcal{F}^{(1,2)}_{a,b}(z)\) is a linear function of \(r_3(z)\), we obtain
\[
\mathcal{F}^{(1,2)}_{a,b}(z) = r_3(z) \Omega_2 + \Omega_1 = \mathcal{H}^{(1,2)}_{a+1,b} \det \mathcal{N}^{(1,2)}. \tag{4.68}
\]

5 Other form factors

Consider again the form factor of the operator \(T_{12}\)
\[
\mathcal{F}^{(1,2)}_{\tilde{a},\tilde{b}}(\tilde{z}|\tilde{v}^c, \tilde{v}^c; \tilde{u}^b, \tilde{u}^b) = C^{a+1,b} (\tilde{u}^c; \tilde{v}^c) T_{12}(\tilde{z}) \mathcal{E}^{\tilde{a},\tilde{b}}(\tilde{u}^b; \tilde{v}^b). \tag{5.1}
\]

Applying the mapping \(\varphi \) \(\mathcal{F}^{(2,3)}_{\tilde{a},\tilde{b}}(\tilde{z}|\tilde{u}^c, \tilde{v}^c; \tilde{u}^b, \tilde{v}^b) = \mathcal{F}^{(2,3)}_{\tilde{a},\tilde{b}}(-\bar{z}| \bar{u}^c, -\bar{v}^c; \bar{u}^b, -\bar{v}^b) \bigg|_{r_1=r_3}. \tag{5.2}
\]

Thus, in order to obtain the determinant representation for the form factor \(\mathcal{F}^{(2,3)}_{\tilde{a},\tilde{b}}(\tilde{z})\) one should take the resulting formulas for \(\mathcal{F}^{(1,2)}_{\tilde{a},\tilde{b}}(\tilde{z})\), set there
\[
\begin{align*}
\tilde{a} &= b, \quad \tilde{b} = a, \quad \tilde{z} = -z; \\
\tilde{u}^c &= -\bar{v}^c, \quad \tilde{v}^c = -\bar{u}^c; \\
\tilde{u}^b &= -\bar{v}^b, \quad \tilde{v}^b = -\bar{u}^b,
\end{align*} \tag{5.3}
\]
and replace the function $r_1$ by $r_3$ and vice versa. One can say that the mapping $\varphi$ actually acts on the determinant representation (3.6) via the replacements described above.

Consider how $\varphi$ acts on the prefactor $\mathcal{H}_{a+1,b}^{(2,1)} = \mathcal{H}_{a+1,b}^{(1,2)}(\tilde{x}; \bar{u}^C, \bar{v}^B)$, where $\tilde{x} = \{\bar{u}^a, \bar{z}, \bar{v}^C\}$. We have

$$\mathcal{H}_{a+1,b}^{(1,2)} = \frac{h(\tilde{x}, \bar{u}^B)h(\bar{v}^C, \tilde{x})}{h(\tilde{v}^C, \bar{u}^B)} \Delta_{a+1}(\tilde{v}^C)\Delta_{a+b+1}(\tilde{x}).$$

Thus, the action of the mapping $\varphi$ onto the matrix $\mathcal{N}^{(1,2)}$ gives the matrix $\mathcal{N}^{(2,3)}$ up to the permutation of the first $b + 1$ rows with the last $a$ rows. Hence,

$$\varphi(\det \mathcal{N}^{(1,2)}) = (-1)^{a(b+1)} \det \mathcal{N}^{(2,3)}.$$  \hspace{1cm} (5.9)

Combining (5.7) and (5.9), we arrive at (3.13), and thus, we prove Proposition 3.2.

The form factors $\mathcal{F}_{a,b}^{(j,j+1)}(z)$ with $j = 1, 2$ can be obtained from $\mathcal{F}_{a,b}^{(j,j+1)}(z)$ by the mapping $\psi$ (2.21). On the other hand one can easily check that making the replacements $\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}$ and $\{a,b\} \leftrightarrow \{a',b'\}$ in the determinant representation (3.6) for $\mathcal{F}_{a,b}^{(1,2)}(z)$ we arrive at (3.15) for $\mathcal{F}_{a,b}^{(2,3)}(z)$. Since the mapping $\psi$ yields the same replacements of the parameters, we conclude that applying $\psi$ to $\mathcal{F}_{a,b}^{(1,2)}(z)$ and $\mathcal{F}_{a,b}^{(2,3)}(z)$ we obtain the determinant representations for $\mathcal{F}_{a,b}^{(3,2)}(z)$ and $\mathcal{F}_{a,b}^{(2,1)}(z)$ respectively. In this way we prove Proposition 3.3.

**Conclusion**

In this paper we considered the form factors of the monodromy matrix entries in the models with $GL(3)$-invariant $R$-matrix. We obtained determinant representations for the form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ of the operators $T_{ij}(z)$ with $|i-j|=1$. In our previous publication [24] we have already calculated the form factors of the diagonal entries $T_{ii}(z)$. Thus, the only unknown remains the form factor $\mathcal{F}_{a,b}^{(1,3)}(z)$ of the operator $T_{13}(z)$ (the form factor $\mathcal{F}_{a,b}^{(3,1)}(z)$ can be obtained from
The question of whether or not there exists a determinant representation for this form factor remains open up to now.

One of possible ways to solve this problem is quite similar to the method used in this paper. The action of the operator $T_{13}(z)$ on the Bethe vectors is very simple

$$T_{13}(z)\mathbb{B}^{a,b}(\{\bar{u}, \bar{v}, z\}; \{\bar{u}, z\}) = \mathbb{B}^{a+1,b+1}\{\{\bar{u}, z\}; \{\bar{v}, z\}\}. \quad (5.10)$$

Thus, the form factor $F_{a,b}^{(1,3)}(z)$ is equal to the scalar product of the vectors $\mathbb{C}^{a+1,b+1}(\bar{u}^{C}, \bar{v}^{C})$ and $\mathbb{B}^{a+1,b+1}\{\{\bar{u}^{B}, z\}; \{\bar{v}^{B}, z\}\}$, and we can use the equation (2.17) for the general scalar product of Bethe vectors. However, further summation with respect to partitions of the Bethe parameters meets certain technical difficulties. In particular, one needs to have new generalizations of the summation lemma A.6.

Another possible way to solve the problem is to use the multiple integral representation for scalar products of the Bethe vectors obtained recently in [36]. This representation might be useful for the study of the form factor $F_{a,b}^{(1,3)}(z)$, if we understand how it can reproduce the results already obtained.

Concluding this paper we would like to say few words about possible applications of the results obtained. Models with higher rank symmetries play an important role in condensed matter physics. They appear for instance in two-component Bose (or Fermi) gas and in the study of models of cold atoms (for e.g. ferromagnetism or phase separation). One can also mention 2-band Hubbard models (mostly in the half-filled regime), in the context of strongly correlated electronic systems. In that case, the symmetry increases when spin and orbital degrees of freedom are supposed to play a symmetrical role leading to an $SU(4)$ or even an $SO(8)$ symmetry (see e.g. [37, 38]). All these studies require to look for integrable models with $SU(N)$ symmetry, the first step being the $SU(3)$ case. Compact determinant representations for form factors of the monodromy matrix entries give a possibility to study correlation functions of such models. We have mentioned already in the Introduction that these representations allow one to calculate the correlation functions of integrable spin chains via their form factor expansion. Furthermore, the explicit representations for the form factors also play an important role in the models, for which the solution of the inverse scattering problem is not known (see e.g. [39]). In this context it is worth mentioning the work [40], where the form factors in the model of two-component Bose gas were studied.

Apart from condensed matter physics, let us also mention super-Yang-Mills theories. Integrability has proved to be a very efficient tool for the calculation of scattering amplitudes in these models. The calculation of these amplitudes can be related to scalar products of Bethe vectors. In particular, in the $SU(3)$ subsector of the theory, one just needs the $SU(3)$ Bethe vectors. Hence, the knowledge of the form factors is also essential in this context.

Finally, in view of the potential applications, there is reason to wonder whether the results obtained in the present paper could be generalized to the models based on $GL(N)$ group with $N > 3$. However, the structure of the obtained determinant representations does not provide obvious clues about their possible generalization to the case $N > 3$. We would like to be very cautious with any ‘obvious’ predictions in this field. It is sufficient to recall some conjectures formulated previously on the basis of the results obtained in $GL(2)$-based models. Indeed, in the case $N = 2$ the analogs of the form factors considered in the present paper are proportional
to the Jacobian of the transfer matrix eigenvalue on one of the vectors. The natural hypothesis was that this structure is preserved in the case \( N > 2 \). We see, however, that already for \( N = 3 \) the determinant representations have a more complicated structure. In particular they contain the Jacobians of the transfer matrix eigenvalues on both vectors. It is very possible that in the case \( N > 3 \), the determinant representations for form factors (if they exist) have more sophisticated structure, that is difficult to see in the case of \( N = 3 \). Therefore we believe that the systematic study of the problem of generalization is the only way to solve it. In this context let us quote the work \[41\] where some preliminary results for \( GL(N) \)-based models were obtained.

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**A Properties of DWPF**

It follows from the representation \[2.7\] that the DWPF \( K_n(x|y) \) has simple poles at \( x_j = y_k \). The behavior of \( K_n \) near these poles can be expressed in terms of \( K_{n-1} \):

\[
\lim_{z' \to z} f^{-1}(z', z) K_{n+1}(\{x, z\}|\{y, z\}) = f(z, y)f(x, z)K_n(x|y). \tag{A.1}
\]

One can also easily check that the DWPF possesses the properties:

\[
K_{n+1}(\{x, z-c\}|\{y, z\}) = K_{n+1}(\{x, z\}|\{y, z+c\}) = -K_n(x|y). \tag{A.2}
\]

\[
K_n(x-c|y) = K_n(x|y+c) = (-1)^n f^{-1}(y, x)K_n(y|x). \tag{A.3}
\]

The DWPF \( K_n \) satisfies several summation identities.

**Lemma A.1.** Let \( \xi, \alpha \) and \( \beta \) be sets of complex variables with \#\( \alpha = m_1 \), \#\( \beta = m_2 \), and \#\( \xi = m_1 + m_2 \). Then

\[
\sum K_{m_1}(\xi_1|\alpha)K_{m_2}(\beta|\xi_1)f(\xi_2, \xi_1) = (-1)^{m_1}f(\bar{\xi}, \bar{\alpha})K_{m_1+m_2}(\{\bar{\alpha}, \bar{\beta}\}|\bar{\xi}). \tag{A.4}
\]

The sum is taken with respect to all partitions of the set \( \bar{\xi} \) into subsets \( \bar{\xi}_1 \) and \( \bar{\xi}_2 \) with \#\( \bar{\xi}_1 = m_1 \) and \#\( \bar{\xi}_2 = m_2 \).

**Lemma A.2.** Let \( \gamma \) and \( \tilde{\xi} \) be two sets of generic complex numbers with \#\( \gamma = \#\tilde{\xi} = m \). Let also \( C_1(\gamma) \) and \( C_2(\gamma) \) be two arbitrary functions of a complex variable \( \gamma \). Then

\[
\sum K_m(\{\gamma_1-c, \gamma_2\}|\tilde{\xi})f(\tilde{\xi}, \gamma_1)f(\gamma_2, \tilde{\xi})C_1(\gamma_1)C_2(\gamma_2)
= \Delta'_m(\tilde{\xi})\Delta_m(\gamma)\det_m(\{C_2(\gamma_k)\}f(\gamma_2, \tilde{\xi})h(\gamma_k, \tilde{\xi}) + (-1)^m C_1(\gamma_k)h(\gamma, \tilde{\xi})). \tag{A.5}
\]

Here we use the shorthand notation \[2.6\] for the products of the functions \( C_1 \) and \( C_2 \).
Lemma A.3. Let $\bar{\alpha}$ and $\bar{\beta}$ be two sets of generic complex numbers with $\#\bar{\alpha} = \#\bar{\beta} = m$, and $z$ is an arbitrary complex. Then

$$\sum_{\alpha = \{\bar{\alpha}_1, \bar{\alpha}_\text{r}\}} \sum_{\beta = \{\bar{\beta}_1, \bar{\beta}_\text{r}\}} f(\bar{\beta}_1, z) f(\bar{\beta}_\text{r}, \bar{\beta}_1) f(\bar{\alpha}_1, \bar{\alpha}_\text{r}) K_{m_1}(\bar{\beta}_1|\bar{\alpha}_1) K_{m_\text{r}}(\bar{\alpha}_\text{r} + c|\bar{\beta}_\text{r})$$

$$= (-1)^m t(\bar{\alpha}, \bar{\beta}) h(\bar{\alpha}, \bar{\beta}) h(\bar{\alpha}, \bar{z}) g(\bar{\beta}, \bar{z}), \quad (A.6)$$

where the sum is taken over all possible partitions of the sets $\bar{\alpha}$ and $\bar{\beta}$ with $\#\bar{\alpha} = \#\bar{\beta} = m$, $m_1 = 0, \ldots, m$, and $\#\bar{\alpha}_\text{r} = \#\bar{\beta}_\text{r} = m = m - m_1$.

This lemma is a generalization of the lemma 6.3 of the work [32]. In particular, the statement of the latter can be obtained from (A.6) in the limit $z \to \infty$.

Proof. Let us denote by $\Lambda_{m}^{(l)}(\bar{\alpha}|\bar{\beta})$ and $\Lambda_{m}^{(r)}(\bar{\alpha}|\bar{\beta})$ the l.h.s. and the r.h.s. of (A.6) respectively. Obviously, they are symmetric functions of $\bar{\alpha}$ and symmetric functions of $\bar{\beta}$. Consider them as functions of $\alpha_1$, while all other variables are fixed. Clearly, the functions $\Lambda_{m}^{(l)}$ and $\Lambda_{m}^{(r)}$ are rational functions of $\alpha_1$, and they both vanish if $\alpha_1 \to \infty$. They have poles at $\alpha_1 = \beta_k$ and $\alpha_1 + c = \beta_k$, $k = 1, \ldots, m$. Some terms in the l.h.s. of (A.6) may also have poles at $\alpha_1 = \alpha_k$, $k = 2, \ldots, m$, but it is not difficult to show that these singularities cancel each other. Indeed, the sum over partitions $\bar{\alpha} \to \{\bar{\alpha}_1, \bar{\alpha}_\text{r}\}$ can be explicitly calculated via (A.4).

$$\sum_{\alpha = \{\bar{\alpha}_1, \bar{\alpha}_\text{r}\}} f(\bar{\alpha}_1, \bar{\alpha}_\text{r}) K_{m_1}(\bar{\beta}_1|\bar{\alpha}_1) K_{m_\text{r}}(\bar{\alpha}_\text{r} + c|\bar{\beta}_\text{r}) = (-1)^m f(\bar{\alpha}, \bar{\beta}_\text{r} - c) K_{m}(\{\bar{\beta}_\text{r} - 2c, \bar{\beta}_1\} | \bar{\alpha}), \quad (A.7)$$

and we see that the r.h.s. of (A.7) is well defined at $\alpha_1 = \alpha_k$. Finally, it is easy to check that

$$\Lambda_{m}^{(l)}(\alpha_1|\beta_1) = \Lambda_{m}^{(r)}(\alpha_1|\beta_1) = -t(\alpha_1, \beta_1) h(\alpha_1, z) g(\beta_1, z). \quad (A.8)$$

The listed properties allow us to prove (A.6) via induction over $m$. We assume that it holds for $\#\bar{\alpha} = \#\bar{\beta} = m - 1$ and consider the case $\#\bar{\alpha} = \#\bar{\beta} = m$. Let us calculate the residues of $\Lambda_{m}^{(l)}(\bar{\alpha}|\bar{\beta})$ and $\Lambda_{m}^{(r)}(\bar{\alpha}|\bar{\beta})$ at $\alpha_1 = \beta_k$ and $\alpha_1 + c = \beta_k$. Obviously, due to the symmetry of $\Lambda_{m}^{(l)}$ and $\Lambda_{m}^{(r)}$ over $\bar{\beta}$ it is enough to consider $\beta_k = \beta_1$.

It is straightforward to establish the following recursions:

$$\left. \Lambda_{m}^{(r)}(\bar{\alpha}|\bar{\beta}) \right|_{\alpha_1 \to \beta_1} = -g(\alpha_1, \beta_1) f(\alpha_1, z) f(\bar{\beta}_1, \beta_1) f(\alpha_1, \bar{\alpha}_1) \cdot \Lambda_{m-1}^{(r)}(\bar{\alpha}_1|\bar{\beta}_1), \quad (A.9)$$

$$\left. \Lambda_{m}^{(r)}(\bar{\alpha}|\bar{\beta}) \right|_{\alpha_1 + c \to \beta_1} = h^{-1}(\alpha_1, \beta_1) f(\beta_1, \bar{\beta}_1) f(\bar{\alpha}_1, \alpha_1) \cdot \Lambda_{m-1}^{(r)}(\bar{\alpha}_1|\bar{\beta}_1), \quad (A.10)$$

where $\bar{\alpha}_1 = \bar{\alpha} \setminus \alpha_1$ and $\bar{\beta}_1 = \bar{\beta} \setminus \beta_1$. Let us check that $\Lambda_{m}^{(l)}(\bar{\alpha}|\bar{\beta})$ has the same recursion properties.

Consider, for example, the pole at $\alpha_1 = \beta_1$. This pole appears if and only if $\alpha_1 \in \bar{\alpha}_1$ and $\beta_1 \in \bar{\beta}_1$. Let $\bar{\alpha}_r = \bar{\alpha}_1 \setminus \alpha_1$ and $\bar{\beta}_r = \bar{\beta}_1 \setminus \beta_1$. Then using the recursion properties of the DWPF
we obtain
\[ \Lambda_m^{(l)}(\bar{\alpha}|\bar{\beta})\bigg|_{\alpha_1 \to \beta_1} = \left( \sum' f(\alpha_1, z) f(\bar{\beta}_1', z) f(\bar{\beta}_2, \bar{\beta}_1') f(\bar{\alpha}_1', \bar{\beta}_1) f(\alpha_1, \bar{\alpha}_2) \times g(\beta_1, \alpha_1) f(\alpha_1, \bar{\alpha}_1') f(\bar{\beta}_1', \beta_1) K_{m-1}(\bar{\beta}_1') |\bar{\alpha}_1') K_{m_{\bar{\beta}}}(\bar{\alpha}_2 + c|\bar{\beta}_2) \right) \tag{A.11} \]
where \( \sum' \) means that the sum is taken over partitions of the sets \( \bar{\alpha}_1 \) and \( \beta_1 \). Obviously
\[ f(\alpha_1, \bar{\alpha}_2) f(\alpha_1, \bar{\alpha}_1') = f(\alpha_1, \bar{\alpha}_1), \quad f(\bar{\beta}_2, \beta_1) f(\bar{\beta}_1', \beta_1) = f(\bar{\beta}_1, \beta_1), \tag{A.12} \]
and thus, these factors can be moved out of the sum over partitions. We obtain
\[ \Lambda_m^{(l)}(\bar{\alpha}|\bar{\beta})\bigg|_{\alpha_1 \to \beta_1} = \left( \sum' f(\bar{\beta}_1, z) f(\bar{\beta}_2, \bar{\beta}_1') f(\bar{\alpha}_1', \bar{\beta}_1) f(\alpha_1, \bar{\alpha}_1) \times g(\beta_1, \alpha_1) f(\alpha_1, \bar{\alpha}_1') f(\bar{\beta}_1', \beta_1) K_{m-1}(\bar{\beta}_1') |\bar{\alpha}_1') K_{m_{\bar{\beta}}}(\bar{\alpha}_2 + c|\bar{\beta}_2) \right) \]  
\[ = -g(\alpha_1, \beta_1) f(\alpha_1, \bar{\alpha}_1) f(\bar{\beta}_1', \beta_1) f(\alpha_1, z) \cdot \Lambda_m^{(l)}(\bar{\alpha}_1|\bar{\beta}_1). \tag{A.13} \]
Let now \( \alpha_1 + c = \beta_1 \). Then the pole appears if and only if \( \alpha_1 \in \bar{\alpha}_2 \) and \( \beta_1 \in \bar{\beta}_2 \). Let \( \bar{\alpha}_2' = \bar{\alpha}_2 \setminus \alpha_1 \) and \( \bar{\beta}_2' = \bar{\beta}_2 \setminus \beta_1 \). Then we obtain
\[ \Lambda_m^{(l)}(\bar{\alpha}|\bar{\beta})\bigg|_{\alpha_1 + c \to \beta_1} = \sum' f(\bar{\beta}_1, z) f(\bar{\beta}_2, \bar{\beta}_1') f(\bar{\alpha}_1', \bar{\beta}_1) f(\alpha_1, \bar{\alpha}_1) \times g(\alpha_1 + c, \beta_1) f(\bar{\alpha}_2', \alpha_1) f(\beta_1, \bar{\beta}_2') K_{m_{\bar{\beta}}}(\bar{\beta}_1 |\bar{\alpha}_1) K_{m_{\bar{\beta}}}(\bar{\alpha}_2' + c|\bar{\beta}_2') \]  
\[ = h^{-1}(\alpha_1, \beta_1) f(\beta_1, \bar{\beta}_1) f(\alpha_1, \bar{\alpha}_1) \cdot \Lambda_m^{(l)}(\bar{\alpha}_1|\bar{\beta}_1). \tag{A.14} \]
Thus, due to the induction assumption the residues of the functions \( \Lambda_m^{(l)}(\bar{\alpha}|\bar{\beta}) \) and \( \Lambda_m^{(r)}(\bar{\alpha}|\bar{\beta}) \) in the poles at \( \alpha_1 = \beta_1 \) and \( \alpha_1 + c = \beta_1 \) coincide. Hence, the difference \( \Lambda_m^{(l)}(\bar{\alpha}|\bar{\beta}) - \Lambda_m^{(r)}(\bar{\alpha}|\bar{\beta}) \) is a holomorphic function of \( \alpha_1 \) in the whole complex plane. Since this function vanishes at \( \alpha_1 \to \infty \), we conclude that \( \Lambda_m^{(l)}(\bar{\alpha}|\bar{\beta}) = \Lambda_m^{(r)}(\bar{\alpha}|\bar{\beta}) \).

**Corollary A.1.** At the conditions of lemma A.6
\[ \sum_{\bar{\alpha} = \{\bar{\alpha}_1, \bar{\alpha}_2\}} f^{-1}(\bar{\beta}_1, z) f(\bar{\beta}_2, \bar{\beta}_1) f(\bar{\alpha}_1, \alpha_2) K_{m_1}(\bar{\beta}_1 |\bar{\alpha}_1) K_{m_2}(\bar{\alpha}_2 + c|\bar{\beta}_2) \]  
\[ = (-1)^m t(\bar{\alpha}, \bar{\beta}) h(\bar{\alpha}, \bar{\beta}) h(\beta, \bar{\beta}) \frac{h(\bar{\alpha}, z)}{h(\beta, z)}. \tag{A.15} \]

**Proof.** Dividing both sides of (A.6) by \( f(\bar{\beta}, z) \) we immediately arrive at (A.15).

**Corollary A.2.** Let \( \bar{\alpha}, \bar{\beta}, \) and \( z \) be generic complex numbers with \( \#\bar{\alpha} = m \) and \( \#\bar{\beta} = m - 1 \). Then
\[ \sum_{\bar{\alpha} = \{\bar{\alpha}_1, \bar{\alpha}_2\}} f(\bar{\beta}_1, \bar{\beta}_1) f(\bar{\alpha}_1, \alpha_2) K_{m_1}(\{z, \bar{\beta}_1\} |\bar{\alpha}_1) K_{m_2}(\bar{\alpha}_2 + c|\bar{\beta}_2) \]  
\[ = (-1)^m t(\bar{\alpha}, \bar{\beta}) h(\bar{\alpha}, \bar{\beta}) h(\beta, \bar{\beta}) g(\bar{\alpha}, z) h(z, \bar{\beta}). \tag{A.16} \]
where the sum is taken over all possible partitions of the sets \( \bar{\alpha} \) and \( \bar{\beta} \) with \( \#\bar{\alpha} = \#\bar{\beta} + 1 = m_1 \), \( m_1 = 1, \ldots, m \), and \( \#\bar{\alpha}_I = \#\bar{\beta}_I + 1 = m \), \( m_1 = 1, \ldots, m \), and \( \#\bar{\alpha}_I I = \#\bar{\beta}_I I = m - m_1 \).

**Proof.** Consider the residue of (A.6) at \( \beta_1 = z \). We have in the r.h.s.

\[ \Lambda_m^{(l)} (\bar{\alpha} | \bar{\beta}) \mid_{\beta_1 \to z} = g(\beta_1, z) f(\bar{\beta}_1, z) \cdot (-1)^{m_1 t(\bar{\alpha}, \bar{\beta}_1)} h(\bar{\alpha}, \bar{\beta}_1) g(\bar{\alpha}, z) h(z, \bar{\beta}_1). \quad (A.17) \]

In the l.h.s. the pole occurs if and only if \( \beta_1 \in \bar{\beta}_I \). Setting \( \bar{\beta}_I' = \bar{\beta}_I \setminus \beta_1 \), we obtain

\[ \Lambda_m^{(l)} (\bar{\alpha} | \bar{\beta}) \mid_{\beta_1 \to z} = \sum' g(\beta_1, z) f(\bar{\beta}_1', z) f(\bar{\beta}_I', z) f(\bar{\beta}_I I, z) f(\bar{\alpha}_I I, z) \times K_m(\{z, \bar{\beta}_1', |\bar{\alpha}_I I\} | K_m \bar{\alpha}_I I + c | \bar{\beta}_I I). \quad (A.18) \]

where \( \sum' \) means that the sum is taken over partitions of the sets \( \bar{\alpha} \) and \( \bar{\beta}_1 \). Using

\[ f(\bar{\beta}_1', z) f(\bar{\beta}_I', z) = f(\bar{\beta}_1, z), \quad (A.19) \]

we obtain

\[ \Lambda_m^{(l)} (\bar{\alpha} | \bar{\beta}) \mid_{\beta_1 \to z} = g(\beta_1, z) f(\bar{\beta}_1, z) \sum' f(\bar{\beta}_I', \bar{\beta}_I') f(\bar{\alpha}_I I, \bar{\beta}_I') K_m(\{z, \bar{\beta}_I' | \bar{\alpha}_I I\}) K_m \bar{\alpha}_I I + c | \bar{\beta}_I I). \quad (A.20) \]

Comparing (A.20) and (A.17) we arrive at the statement of Corollary A.2.

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