Symmetry aspects of fermions coupled to torsion and electromagnetic Fields

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March 27, 2022

Abstract

We study and explore the symmetry properties of fermions coupled to dynamical torsion and electromagnetic fields. The stability of the theory upon radiative corrections as well as the presence of anomalies are investigated.

PACS numbers: 04.60.-m, 11.10.Gh, 11.15.-q
1 Introduction

The study of gauge field theories with torsion has been the object of intensive investigations in recent years due its possible relation with string theory - the most promising framework for overcoming the problems of quantum gravity [1] - which predicts, along with the metric, a totally antisymmetric rank-3 tensor field usually associated with torsion. On the other hand, the direct interaction of torsion with fermionic matter fields has received considerable attention for a long time for the purpose of formulating general relativity as a gauge theory [2, 3, 4, 5]. In fact, torsion is the geometric object that relates the spin of matter with the geometry of the space-time [6].

As it is well known, in Einstein-Cartan gravity, torsion is determined by the spin distributions and cannot propagate outside matter; however, there have been discussed recently some theoretical and phenomenological implications of the hypothesis that torsion is a propagating field [4, 5]. In particular, the analysis of the consistency of an effective quantum field theory of spinors interacting with propagating torsion and electromagnetic fields has been considered in the works of Ref. [3]; the perturbative analysis was performed at one- and two-loop orders, indicating severe restrictions for the theory.

The aim of our work is to provide a simple algebraic understanding [7] of these results and an extension of them to all orders of perturbations theory. We shall show the stability of such a model under radiative corrections and investigate the possible presence of anomalies, by making use of the symmetry properties of the theory. We shall consider a situation where matter interacts with a gravitational field whose torsion fluctuations dominate over the metric excitations. In practice, this means that we adopt a space-time as a flat background on which the torsion degrees of freedom propagate and interacts with matter and gauge fields - it might corresponds to a physical situation where the torsion field is produced by a cosmological neutrino sea [8].

Our results indicate that a consistent effective quantum field theory of torsion - stable under radiative corrections and anomaly free - requires non-
minimal gravitational coupling between the torsion and the (massless) fermionic sector. Moreover, it contains neither interaction nor coupling terms between torsion and electromagnetic gauge fields, while keeping the initial gauge and discrete symmetries.

2 The Action

We consider a Riemann-Cartan space-time where the torsion tensor is defined by
\[ T_{\lambda}^{\mu\nu} = 2\Gamma_{[\mu\nu]}^\lambda - \Gamma_{\mu\nu}^\lambda \] and the affine connection \( \Gamma_{\mu\nu}^\lambda \) is expressed through metric and torsion as \( \Gamma_{\mu\nu}^\lambda = \{^\lambda_{\mu\nu}\} + K_{\mu\nu}^\lambda \). Here \( \{^\lambda_{\mu\nu}\} \) is the Christoffel symbol while \( K_{\mu\nu}^\lambda = \frac{1}{2} (T_{\mu\nu}^\lambda + T_{\nu\mu}^\lambda - T_{\nu\mu}^\lambda) \) is the contorsion tensor.

In this space-time, the action for a massless fermion minimally coupled to torsion and electromagnetic fields has the form:
\[
\Sigma_\psi = \frac{i}{2} \int d^4x \sqrt{-g} \left( \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right),
\] (2.1)

where the covariant derivatives of the spinor fields are given by:
\[
\nabla_\mu \psi = \partial_\mu \psi + ieA_\mu \psi + \frac{1}{8} B_{\mu}^{ab} [\gamma_a, \gamma_b] \psi,
\] (2.2)
\[
\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - ieA_\mu \bar{\psi} - \frac{1}{8} B_{\mu}^{ab} \bar{\psi} [\gamma_a, \gamma_b].
\]

Here, Latin indices refer to frame components. \( B_{\mu}^{ab} = \omega_{\mu}^{ab} + K_{\mu}^{ab} \) are the components of the spin-connection, which is the gauge field of the local Lorentz group. \( \omega_{\mu}^{ab} \) is the Riemannian part of the spin-connection:
\[
\omega_{\mu}^{ab} = e_{\mu \nu} \omega_{\nu}^{cab} = \frac{1}{2} e_{\mu \nu} \left( \Omega_{cab} + \Omega_{abc} - \Omega_{bac} \right),
\] (2.3)

where \( \Omega_{abc} = e_a^\mu e_b^\nu (\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) \) stands for the rotation coefficients (see [6] for details); \( e_{\mu}^a \) stands for the vierbeins.

Since the main goal of our work is to study the specific effects of a propagating torsion and its coupling to fermionic matter and electromagnetic fields,
we restrict our analysis to a flat-metric background \( g_{\mu\nu} = \eta_{\mu\nu} \). Moreover, in our considerations, we shall be dealing only with the pseudo-vector component of torsion, \( S^\mu \), since this is the only mode of \( T^\mu_\nu \) predicted by string theory and that survives a minimal coupling to fermions. Thus, we set

\[
T_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda} S^\lambda. \tag{2.4}
\]

From (2.1) and (2.3), one gets the matter action in presence of torsion and the electromagnetic fields:

\[
\Sigma_\psi = \int d^4x \left( i \bar{\psi} \gamma^\mu \partial_\mu \psi - e A_\mu \bar{\psi} \gamma^\mu \psi + \alpha S_\mu \bar{\psi} \gamma^5 \gamma^\mu \psi \right), \tag{2.5}
\]

where \( \alpha \) is the coupling constant governing the interaction between fermions and torsion. In the particular case \( \alpha = \frac{3}{4} \), the coupling is minimal in that it comes out from the spin connection with torsion present in the covariant derivative of the fermion (2.2). Notice that the coupling constants \( e \) and \( \alpha \) are dimensionless. Moreover, it is important to point out that only the totally antisymmetric component of the torsion minimally couples to the matter field.

In the works of ref. [9], the authors propose and discuss the coupling between fermions and a background 4-vector to discuss the issue of CPT and Lorentz violation in gauge theories. Here, our \( S_\mu \) is a pseudo-vector, so that our interaction term preserves CPT.

The action is left invariant under the usual \( U(1) \) gauge transformations for the fermion and the electromagnetic fields:

\[
\psi \rightarrow e^{ieA(x)} \psi, \\
\bar{\psi} \rightarrow e^{-ieA(x)} \bar{\psi}, \\
A_\mu \rightarrow A_\mu - \partial_\mu \Lambda(x). \tag{2.6}
\]

Along with such a symmetry, the theory has also an additional local chiral invariance for which \( S_\mu \) behaves as the corresponding gauge field:

\[
\psi \rightarrow e^{i\alpha \theta(x) \gamma^5} \psi, \\
\bar{\psi} \rightarrow \bar{\psi}(x) e^{i\alpha \theta(x) \gamma^5}, \\
S_\mu \rightarrow S_\mu - \partial_\mu \theta(x). \tag{2.7}
\]
Here, $\Lambda$ and $\theta$ are respectively scalar and pseudo-scalar gauge functions. We underline that, in contrast to the gauge transformation (2.6), (2.7) does not leave invariant a bilinear mass term, $\bar{\psi}\psi$, for the fermion.

In the same way as in QED, the action remains invariant under charge conjugation, $C$, symmetry:

$$
\begin{align*}
\psi & \rightarrow \psi^c = C (\bar{\psi})^t, \quad C = -\gamma^2, \\
A^\mu & \rightarrow -A^\mu, \\
S^\mu & \rightarrow S^\mu.
\end{align*}
$$

(2.8)

Our point of view is that $S^\mu$ is even under charge conjugation symmetry since it might only mediate, along with the metric, the gravitational interaction between spinor particles.

Furthermore the theory has an additional rigid gauge invariance, namely:

$$
\begin{align*}
\psi & \rightarrow e^{i\omega} \psi, \\
\bar{\psi} & \rightarrow e^{-i\omega} \bar{\psi},
\end{align*}
$$

(2.9)

where $\omega$ is a constant parameter.

The kinetic action for the gauge sector, compatible with the local and discrete symmetries (2.6) - (2.8), is as follows:

$$
\Sigma_g = -\frac{1}{4} \int d^4x \left( F^{\mu\nu} F_{\mu\nu} + S^{\mu\nu} S_{\mu\nu} \right),
$$

(2.10)

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $S^{\mu\nu} = \partial^\mu S^\nu - \partial^\nu S^\mu$. Here, we do not consider the gauge invariant torsion-photon coupling terms $\varepsilon^{\mu\nu\kappa\lambda} F_{\mu\nu} S_{\kappa\lambda}$ and $F^{\mu\nu} S_{\mu\nu}$ since they both break charge conjugation symmetry (2.8). In fact, as we shall see later, this action is stable under radiative corrections; namely, neither coupling nor interaction terms between torsion and electromagnetic fields are required in the action in order to fulfill the stability condition.

Since the theory possesses two local symmetries, one has to perform a gauge fixing for both, i.e.:

$$
\Sigma_{gf} = -\frac{1}{2} \int d^4x \left[ \frac{1}{\Lambda} (\partial A)^2 + \frac{1}{\beta} (\partial S)^2 \right],
$$

(2.11)

In view of that, let us start from the following action for fermions interacting with electromagnetic and torsion fields in a flat background:

$$
\Sigma = \Sigma_g + \Sigma_{gf} + \Sigma_\psi.
$$

(2.12)
The (continuous) symmetries properties of this model are translated to a set of the Ward identities, namely:

\[ W_A \Sigma = \frac{1}{\lambda} \partial^2 \partial A, \quad W_S \Sigma = \frac{1}{\beta} \partial^2 \partial S \quad \text{and} \quad W_{\text{rig}} \Sigma = 0 \quad (2.13) \]

where

\[ W_A = \partial_\mu \delta \frac{\delta}{\delta A_\mu} - ie \left( \bar{\Psi} \frac{\delta}{\delta \bar{\Psi}} - \frac{\delta}{\delta \Psi} \bar{\Psi} \right), \quad (2.14) \]

\[ W_S = \partial_\mu \delta \frac{\delta}{\delta S_\mu} + i\alpha \left( \bar{\Psi} \gamma^5 \frac{\delta}{\delta \bar{\Psi}} + \frac{\delta}{\delta \Psi} \gamma^5 \Psi \right), \quad (2.15) \]

\[ W_{\text{rig}} = \omega \left( \bar{\Psi} \delta \frac{\delta}{\delta \bar{\Psi}} - \frac{\delta}{\delta \Psi} \bar{\Psi} \right). \quad (2.16) \]

We point out that, in the same way as the gauge fixing terms, bilinear mass terms for the abelian gauge fields are allowed since they provide only a linear breaking of the identities (2.14) and (2.15) - whose linear structure is preserved by radiative corrections. However, the same conclusion does not hold for the fermionic sector: a fermionic mass term is not allowed since it provides a non-linear breaking of the Ward identity \( W_S \Sigma = \frac{1}{\beta} \partial^2 \partial S \).

### 3 Stability

Once we have established some symmetry properties of the theory, let us proceed further by studying the stability of the action under radiative corrections. Let us consider

\[ \Sigma \to \Sigma + \varepsilon \bar{\Sigma}, \quad (3.17) \]

where \( \bar{\Sigma} \) is the most general integrated local polynomial of dimension bounded by four, due to power-counting renormalizability; it has the same quantum numbers as \( \Sigma \). Moreover, it is invariant under local and rigid gauge transformations, namely:

\[ W_{A(S)} \bar{\Sigma} = 0 \quad \text{and} \quad W_{\text{rig}} \bar{\Sigma} = 0. \quad (3.18) \]
The most general counterterm, $\Sigma$, respecting the discrete $\mathcal{C}$-symmetry along with the conditions (3.18) is given by:

$$\bar{\Sigma} = -\frac{Z_A}{2}F_{\mu\nu}F^{\mu\nu} - \frac{Z_S}{2}S_{\mu\nu}S^{\mu\nu} + 2Z\psi \left( \bar{\psi}\gamma_\mu \psi A^\mu + \bar{\psi}\gamma^5\gamma_\mu \psi S^\mu + \bar{\psi}\gamma_\mu \partial^\mu \psi \right),$$  

(3.19)

where the $Z$’s are arbitrary coefficients. This corresponds to a redefinition of the fields and coupling constants of the theory, namely:

$$\Sigma + \varepsilon \bar{\Sigma} = \Sigma (\phi_0; e_0, \alpha_0, \lambda_0, \beta_0),$$  

(3.20)

with

$$\begin{align*}
\phi_0 &= (1 + \varepsilon Z_A) \phi, & e_0 &= (1 - \varepsilon Z_A) e, & \alpha_0 &= (1 - \varepsilon Z_S) \alpha, \\
\lambda_0 &= (1 + 2\varepsilon Z_A), & \beta_0 &= (1 + 2\varepsilon Z_S),
\end{align*}$$  

(3.21)

where $\phi$ stands for $A$, $S$ or $\psi$.

Here some remarks are in order. Notice that the bilinear fermionic mass term, $\bar{\psi}\psi$, cannot show up in the counterterm while local chiral symmetry is required. In the same way, the term $\varepsilon^{\mu\nu\kappa\lambda} A_\mu S_\nu F_{\kappa\lambda}$ is not allowed by the requirement $W_A \bar{\Sigma} = 0$; it is a (electromagnetic) gauge invariant one only if torsion behaves as a longitudinal gauge field, namely, $S_\mu = \partial_\mu \phi$ [4].

### 4 Anomaly

Now, we shall investigate if the symmetry properties of the model can be extended to the quantum level for the vertex function

$$\Gamma = \Sigma + O(h).$$  

(4.22)

Firstly, we notice that the stable action is also invariant under the parity, $\mathcal{P}$, transformation:

$$\begin{align*}
x^\mu &\rightarrow x_\mu, \\
A^\mu &\rightarrow A_\mu, \\
S^\mu &\rightarrow -S_\mu, \\
\psi &\rightarrow \gamma^0 \psi,
\end{align*}$$  

(4.23)
where we emphasize, as usual, the pseudo-vector character of $S^\mu$, while $A^\mu$ behaves as a vector under $P$. When investigating the possible quantum breaking of the continuous symmetries we shall make use the fact that the discrete symmetries, including parity, must be preserved at the quantum level.

The Ward operators (2.14) and (2.15) transform in the following way under parity and charge conjugation symmetries, respectively:

$$W_{A(S)} \rightarrow + (-) W_{A(S)},$$
$$W_{A(S)} \rightarrow - (+) W_{A(S)}.$$  \hspace{1cm} (4.24)

Then, according to the quantum action principle, the Slavnov-Taylor identities (2.13) get the following quantum breaking:

$$W_{A} \Gamma = \frac{1}{\lambda} \partial^2 \partial A + \Delta_A \quad \text{and} \quad W_{S} \Gamma = \frac{1}{\beta} \partial^2 \partial S + \Delta_S,$$  \hspace{1cm} (4.25)

for $\Delta_{A(S)}$ a local polynomial of dimension four; in view of (4.24) it is even (odd) under parity and odd (even) under charge conjugation symmetries. Having these conditions in mind, we find out that the symmetry (2.6) is not anomalous: it is straightforward to show, resolving the corresponding Wess-Zumino consistency conditions \[10\], that $\Delta_A$ can be written as a $W_A$-variation of a local polynomial, $\hat{\Delta}$, odd under parity and even under charge conjugation symmetries, namely:

$$\Delta_A = W_A \hat{\Delta}.$$  \hspace{1cm} (4.26)

Rewriting the vertex function as $\bar{\Gamma} = \Gamma - h \hat{\Delta}$, the Ward identity, $W_A \bar{\Gamma} = \frac{1}{\lambda} \partial^2 \partial A$, is recovered. On the other hand, $\Delta_S$ cannot be totally written as a $W_S$-variation; its functional form reads as:

$$\Delta_S = a e^{\mu \nu \kappa \lambda} S_{\mu \nu} S_{\kappa \lambda} + W_{S} (\ldots),$$  \hspace{1cm} (4.27)

then we have a breakdown of the local chiral symmetry by quantum effects - the so called ABBJ anomaly \[11\] present on spaces with torsion \[12\]. However, it is well known that its coefficient, $a$, vanishes whenever one consider a model involving, for instance, two spinor fields interacting with torsion, with opposite coupling constants. Thus, in this case, the gauge transformation (2.7) for the spinor field should be replaced by $\psi_i \rightarrow e^{i \alpha \theta(x) \gamma_5} \psi_i \ (i = 1, 2)$,
with $\alpha_1 + \alpha_2 = 0$ - which means that one of the $\alpha_i$ could not be equal to $\frac{3}{4}$ - in agreement with the conjecture that a renormalizable model of fermions coupled to torsion requires non-minimal couplings \[13\].

Acknowledgments

Thanks are due to J.A. Helayël-Neto, O. Piguet and S.P. Sorella for helpful comments and clarifying discussions. The Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq, is acknowledged for its financial support.

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