FILTERED STOCHASTIC CALCULUS

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Abstract

By introducing a color filtration to the multiplicity space $G$, we extend the quantum Itô calculus on multiple symmetric Fock space $\Gamma(L^2(R^+, G))$ to the framework of filtered adapted biprocesses. In this new notion of adaptedness, “classical” time filtration makes the integrands similar to adapted processes, whereas “quantum” color filtration produces their deviations from adaptedness. An important feature of this calculus, which we call filtered stochastic calculus, is that it provides an explicit interpolation between the main types of calculi, regardless of the type of independence, including freeness, Boolean independence (more generally, $m$-freeness) as well as tensor independence. Moreover, it shows how boson calculus is “deformed” by other noncommutative notions of independence. The corresponding filtered Itô formula is derived. Existence and uniqueness of solutions of a class of stochastic differential equations are established and unitarity conditions are derived.

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1. Introduction

In this paper we develop a filtered version of the quantum Itô calculus on multiple symmetric Fock spaces. It is an extension of the Hudson-Parthasarathy calculus [H-P1] and its multivariate version developed by Mohari and Sinha [Mo-Si] (see also [P]). Apart from boson calculus, it includes many other calculi, in particular a new version of free calculus, which was originally developed by Kümmerer and Speicher [K-Sp] for the Cuntz algebra,
as well as new examples of $m$-free calculi for the $m$-free Brownian motions introduced in [F-L], where $m \in \mathbb{N}$ (for an inclusion of the calculus on the finite difference algebra [B] see [P-Si]).

In [L2] we introduced filtered random variables, from which other random variables can be obtained by addition or strong limits, regardless of the notion of independence. In particular, this includes the three main types in the axiomatic approach to independence ([Sp1],[S2]), corresponding to tensor, free [V] and Boolean products of states. The same is true for $m$-free random variables for all $1 \leq m \leq \infty$ obtained from the hierarchy of freeness construction [L1] (see also [F-L] for limit theorems and [F-L-S] for the GNS construction).

By studying the asymptotic joint distributions of their normalized sums in limit theorems [L2], we were led to filtered creation, annihilation, number and time processes (see (2.4)-(2.7)). They live in a multiple symmetric Fock space $\Gamma(\mathcal{H})$, where $\mathcal{H} = L^2(\mathbb{R}^+, \mathcal{G})$ and $\mathcal{G}$ is a separable Hilbert space with a countable fixed orthonormal basis $(e_n)_{n \in \mathbb{N}}$ and are obtained from the CCR processes by multiplying them by canonical projections

$$P^{(V)} : \Gamma(\mathcal{H}) \to \Gamma(L^2(\mathbb{R}^+, \mathcal{G}^{(V)})),$$

where $V \subseteq \mathbb{N}$ and we set $\mathcal{G}^{(0)} = \{0\}$. In other words, $P^{(V)}$ is the projection onto the subspace built from the vacuum vector $\Omega$ and those copies (or colors) of $L^2(\mathbb{R}^+)$ which are associated with the set $V$.

This leads to the filtered stochastic calculus developed in this paper, where we deal with integrals of type

$$I^n(t) = \int_0^t FdA^nG,$$

defined on the exponential domain, with the integrator $A^n = (A^n_t)_{t \geq 0}$ being one of the CCR basic integrators: $A^{(k)_*}$ (creation), $A^{(k)}$ (annihilation), $A^{(k)_o}$ (number), associated with color $k$, or $A^{(0)}$ (time). The integrands are biprocesses $F \otimes G = (F(t) \otimes G(t))_{t \geq 0}$ which are not adapted, namely

$$F(t) = \bar{F}(t) \otimes P^{(D)}, \quad G(t) = \bar{G}(t) \otimes P^{(E)}$$

for all $t \geq 0$, where $D,E \subseteq \mathbb{N}$, according to the past-future decomposition $\Gamma(\mathcal{H}) = \Gamma(\mathcal{H}_d) \otimes \Gamma(\mathcal{H}_t)$. In other words, the identity corresponding to “the future” is replaced by color projections with filters $D,E$ showing which colors are filtered through. We will say that $F \otimes G$ is $(D,E)-$ adapted, whereas linear combinations of such biprocesses, corresponding to different filters, will be called filtered adapted.

We arrive at the filtered Itô formula, which takes a particularly nice form. Namely, let $A^{n_1}$ and $A^{n_2}$ be CCR integrators associated with colors $k_1$ and $k_2$, respectively, and let $dA^{n_1}dA^{n_2} = d[A^{n_1}, A^{n_2}]$ be the result of the Itô multiplication of the CCR differentials. Then all nontrivial Itô corrections for the differentials

$$dI_1^n = G_1dA^{n_1}F_1, \quad dI_2^{n_2} = F_2dA^{n_2}G_2$$

where $G_1 \otimes F_1$ and $F_2 \otimes G_2$ are $(E_1, D_1)-$ and $(D_2, E_2)-$ adapted locally square integrable biprocesses, respectively, can be written as

$$dI_1^n dI_2^{n_2} = \mathbb{1}_{D_1 \cap D_2}(k_1)G_1d[A^{n_1}, A^{n_2}]F_1 F_2 G_2$$
where \( 1_A \) is the indicator function of the set \( A \). We proceed further and study existence, uniqueness and unitarity of solutions of stochastic differential equations.

It can be seen that the key role in this approach is played by the new notion of adaptedness, which exhibits an interplay between “classical” time filtration and “quantum” color filtration. The corresponding calculus is a non-trivial, but quite natural and general extension of boson calculus. It includes many examples of quantum stochastic calculi, gives new ones, like \( m \)-free calculi for all natural \( m \), and shows connections between them.

Let us recall that the construction of the hierarchy of freeness \([L1]\) showed how to approximate the free product of states using tensor independence. Filtered stochastic calculus preserves this hierarchy and shows that the \( m \)-free calculus exhibits the \( m \)-th “level of adaptedness”. More importantly, it allows us to compare the \( m \)-free calculi against other calculi, in particular, boson calculus, by measuring their “deviations from adaptedness”. In order to do that, it is enough to give the collections of filters associated with the calculi. This gives

\[
\text{1-free calculus } - \mathcal{P}^{(1)} = \{\emptyset, \{1\}\}
\]

\[
\text{m-free calculus } - \mathcal{P}^{(m)} = \{\emptyset, \{1\}, \ldots, \{1, \ldots, m\}\}
\]

\[
\text{free calculus } - \mathcal{P}^{(\infty)} = \{\emptyset, \{1\}, \ldots, \{1, \ldots, m\}, \ldots\}
\]

for the “minimal” formulation, i.e. when the integrands belong to the *-algebra generated by the corresponding fundamental processes (if we add the unit or study unitarity, we need to add the filter \( N \)).

These can be compared against the two extreme cases

\[
\text{\( \Omega \)-adapted calculus } - \mathcal{P}^{(0)} = \{\emptyset\}
\]

\[
\text{boson calculus } - \mathcal{P} = \{N\},
\]

i.e. the “least adapted” calculus studied in \([Vi]\) and \([Be]\) associated with the projection \( P_\Omega \) on the zero-particle space, and the boson calculus – the “most adapted” calculus associated with the projection \( P(N) = I \) on the whole space. The “least adapted” calculus from the hierarchy of \( m \)-free calculi is the Boolean (or 1-free) calculus. In turn, the \( m \)-free calculus corresponds to a mixture of \( m + 1 \) types of adaptedness, whereas the free calculus – to a mixture of infinitely many types of adaptedness. Of course, this picture holds on \( \Gamma(\mathcal{H}) \).

When we restrict ourselves to suitable proper subspaces of \( \Gamma(\mathcal{H}) \), for instance, to \( m \)-free Fock spaces, one can construct calculi which become adapted on those subspaces.

Let us mention here other unified approaches to quantum stochastic calculus. A representation-free stochastic calculus was presented by Accardi, Fagnola and Quaegebeur in \([Ac-Fa-Qu]\) and \([Fa]\). We would like to mention here that it seems possible to treat the filtered calculus in a similar manner by proving semimartingale inequalities for \((D, E)\)-adapted bi-processes. Non-causal approaches to stochastic calculus were developed by Lindsay \([Li]\) and Belavkin \([Bel]\). For other calculi, see \([Ba-St-Wi]\), \([Ap-H]\), \([H-P2]\), \([Bi-Sp]\), \([Me]\), \([At-Li]\), \([Ma]\), \([S1]\), \([Sp2]\). Our approach to calculus is closest to that of Parthasarathy and Sinha who realized \([P-Si]\) free fundamental processes as stochastic integrals of non-adapted processes in boson calculus. However, we can go much further and treat in a
unified manner the Itô formula as well as stochastic differential equations and unitary evolutions.

We would like to stress that our approach is not restricted to the stochastic calculus. On the contrary, it provides a unified treatment of such elements of noncommutative probability as (i) product states, (ii) limit theorems, (iii) quantum Itô calculus, (iv) stochastic differential equations, of which the first two were treated in [L2]. At the same time, it provides a very concrete mathematical framework and exhibits connections with many other models, some of which were mentioned above. Although we have taken the most traditional (Hudson-Parthasarathy) approach to calculus, the core of our approach seems more universal. The main idea boils down to introducing the second filtration in the underlying Hilbert space (the infinite tensor product of Hilbert spaces for the first quantization and multiple symmetric Fock space for the second quantization). It seems likely that in other models one can use the same idea, which could be of importance in our understanding how noncommutative notions of independence “deform” classical probability.

2. Definitions and notation

**Exponential domain**

Let $G$ be a separable Hilbert space with a countably infinite fixed orthonormal basis $(e_n)_{n \in \mathbb{N}}$. It is sometimes called the *multiplicity space*. By a *multiple symmetric Fock space* over $K$ we understand the symmetric Fock space over $\mathcal{H} = L^2(\mathbb{R}^+, G) \cong \mathcal{K} \otimes G$, namely

$$\Gamma(\mathcal{H}) = C\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^n$$

where $\mathcal{H}^n$ denotes the $n$-th symmetric tensor power of $\mathcal{H}$ and $\Omega$ is the vacuum vector, with the scalar product given by $\langle \Omega, \Omega \rangle = 1$, $\langle \Omega, u \rangle = 0$ and

$$\langle u_1 \circ \ldots \circ u_n, v_1 \circ \ldots \circ v_m \rangle = \delta_{n,m} \frac{1}{n!} \sum_{\sigma \in S_n} \langle u_1, v_{\sigma(1)} \rangle \ldots \langle u_n, v_{\sigma(n)} \rangle$$

where

$$u_1 \circ \ldots \circ u_n = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(n)}$$

and $S_n$ denotes the symmetric group of order $n$.

The exponential vectors are given by

$$\varepsilon(u) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} u^\otimes n$$

where $u^\otimes 0 = \Omega$, and $u \in \mathcal{H}$. Thus, in particular, $\varepsilon(0) = \Omega$. The linear space $\mathcal{E}$ spanned by exponential vectors is usually called the exponential domain. It is well-known that $\mathcal{E}$ is dense in the symmetric Fock space $\Gamma(\mathcal{H})$. The scalar product of two exponential vectors is given by

$$\langle \varepsilon(u), \varepsilon(v) \rangle = e^{\langle u, v \rangle}.$$
where \( u, v \in \mathcal{H} \).

We will use the functorial property of \( \Gamma(\mathcal{H}) \) for the time filtration of the Hilbert space \( \mathcal{H} \). Namely, for the direct sum decomposition

\[
\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_{[s,t]} \oplus \mathcal{H}_{[t,\infty)},
\]

where \( \mathcal{H}_s = L^2([0, s]) \otimes \mathcal{G}, \mathcal{H}_{[s,t]} = L^2([s, t]) \otimes \mathcal{G} \) and \( \mathcal{H}_{[t,\infty)} = L^2([t, \infty)) \otimes \mathcal{G} \), we have

\[
\Gamma = \Gamma_s \otimes \Gamma_{[s,t]} \otimes \Gamma_{[t,\infty)}
\]

where \( \Gamma_s = \Gamma(\mathcal{H}_s), \Gamma_{[s,t]} = \Gamma(\mathcal{H}_{[s,t]}), \Gamma_{[t,\infty)} = \Gamma(\mathcal{H}_{[t,\infty)}) \) for any \( 0 < s < t < \infty \).

Another direct sum decomposition of \( \mathcal{H} \) is associated with the discrete color filtration in \( \mathcal{H} \). More generally, for arbitrary \( V \in \mathcal{P}(\mathbb{N}) \), let \( \Pi^{(V)} : \mathcal{H} \to \mathcal{H}^{(V)} \) be the canonical projection onto

\[
\mathcal{H}^{(V)} = \bigoplus_{k \in V} \mathcal{K} \otimes e_k
\]

with \( \mathcal{H}^{(0)} = \{0\} \). We will say that \( \Pi^{(V)} \) is the projection onto the subspace spanned by vectors of colors which are in \( V \). If \( V = \{1, \ldots, r - 1\} \), then a short-hand notation will be used, namely

\[
\mathcal{H}^{(r)} := \mathcal{H}^{(\{1, \ldots, r-1\})}.
\]

The vector subspace of \( \mathcal{H} \) spanned by all vectors \( u \) of finite color support, i.e. \( u \in \mathcal{H}^{(r)} \) for some \( r \in \mathbb{N} \), will be denoted by \( \mathcal{H}_0 \). Similarly, if \( x \in \Gamma(\mathcal{H}^{(r)}) \) for some \( r \in \mathbb{N} \), we will also say that it is of finite color support. Finally, let \( P^{(V)} : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}^{(V)}) \) denote the second quantization of \( \Pi^{(V)} \), thus \( P^{(V)} \varepsilon(u) = \varepsilon(\Pi^{(V)} u) \).

The space \( \Gamma(\mathcal{H}) \) will be extended to

\[
\tilde{\Gamma}(\mathcal{H}) = \mathcal{H}_0 \otimes \Gamma(\mathcal{H}),
\]

where \( \mathcal{H}_0 \) is a separable Hilbert space called the initial space. The ampliations \( 1 \otimes A^0 \) of fundamental operators \( A^0 \) will also be denoted by \( A^0 \). Moreover, by \( P^{(V)} \) we will denote the ampliations \( 1 \otimes P^{(V)} \) and also its restrictions to \( \tilde{\Gamma}_s \), \( \tilde{\Gamma}_{[s,t]} \) or \( \tilde{\Gamma}_{[t,\infty)} \).

The exponential domain \( \mathcal{E}_0 \), i.e. the span of \( \varepsilon(u) \), where \( u \) is locally bounded as a function of time and is of finite color support, is then replaced by \( \tilde{\mathcal{E}}_0 = \text{span}\{\mathcal{M}_0\} \) where

\[
\mathcal{M}_0 = \{w \otimes \varepsilon(u) \equiv w\varepsilon(u) : w \in \mathcal{D}_0, u \in \mathcal{H}_0\},
\]

\( \mathcal{D}_0 \) is a dense subset of \( \mathcal{H}_0 \) and \( u \) is locally bounded as a function of time. Clearly, \( \mathcal{M}_0 \) is total and \( \tilde{\mathcal{E}}_0 \) is dense in \( \tilde{\Gamma}(\mathcal{H}) \). We will also write

\[
x^{(V)} = w\varepsilon(u^{(V)}).
\]

for \( x \in \mathcal{M}_0 \), where \( V \in \mathcal{P}(\mathbb{N}) \).

The notations associated with the continuous tensor product decompositions of \( \tilde{\Gamma}(\mathcal{H}) \) as well as \( \mathcal{M}_0 \) will also be standard. For instance, if \( x = w\varepsilon(u) \in \mathcal{M}_0 \), then we will write

\[
x = x_s \otimes x_{[s,t]} \otimes x_{[t]},
\]

where \( x_s = w\varepsilon(u_s) \), \( x_{[s,t]} = \varepsilon(u_{[s,t]}) \) and \( x_{[t]} = \varepsilon(u_{[t]}) \) for \( s < t \).
Filtered fundamental processes

The filtered creation, annihilation and number operators introduced in [L2] lead to the filtered fundamental processes. Namely, they are expressed in terms of the canonical ones on $\Gamma(H)$ by the following formulas:

\begin{align*}
A_t^{(k,V)*} &= A_t^{(k)*}P(V) = a^*(\chi_{[0,t]} \otimes e_k)P(V) \quad (2.2) \\
A_t^{(k,V)} &= P(V)A_t^{(k)} = P(V)a(\chi_{[0,t]} \otimes e_k) \quad (2.3) \\
A_t^{(k,V)o} &= A_t^{(k)o}P(V \cup \{k\}) = \lambda(I_{[0,t]} \otimes |e_k\rangle\langle e_k|)P(V \cup \{k\}) \quad (2.4) \\
A_t^{(0,V)} &= A_t^{(0)}P(V) = tP(V) \quad (2.5)
\end{align*}

where $k \in \mathbb{N}$, $V \in \mathcal{P}(\mathbb{N})$, $I_{[0,t]}$ denotes the operator of multiplication by the characteristic function $\chi_{[0,t]}$ on $L^2(\mathbb{R}^+)$ and $\lambda(T)$ is the differential second quantization of $T$. The families of processes given by (2.2-2.5) will be called filtered creation, annihilation, number and time processes, respectively.

Clearly, if $V = \mathbb{N}$, equations (2.2-2.5) give CCR creation, annihilation, number and time processes, respectively. By

$$\mathcal{T} = \{(k), (k)*, (k)o, (0)|k \in \mathbb{N}\}$$

we denote the set of indices associated with these processes. Here, we do not follow the notation of [Mo-Si] in order to have a clear connection with free processes. We will also find it convenient to use the “duals” of $A^n$, $\eta \in \mathcal{T}$. Thus $A^{(k)*} = A^{(k)}$, $A^{(k)+} = A^{(k)+}$, $A^{(k)+} = A^{(k)+}$, $A^{(0)+} = A^{(0)}$.

$m$-free fundamental processes

From the extended $m$-free fundamental operators defined in [L2] we obtain extended $m$-free fundamental processes

\begin{align*}
l_t^{(m)*} &= \sum_{k=1}^{m} A_t^{(k)*}P^{[k-1]}, \quad (2.6) \\
l_t^{(m)} &= \sum_{k=1}^{m} P^{[k-1]}A_t^{(k)} \quad (2.7) \\
l_t^{(m)o} &= \sum_{k=1}^{m} P^{[k]}A_t^{(k)o} \quad (2.8) \\
l_t^{(m)} &= P^{(m)}A_t^{(0)} \quad (2.9)
\end{align*}

i.e. the extended $m$-free creation, annihilation, number and time processes, respectively, where $P^{[k-1]} = P^{(k)} - P^{(k-1)}$, $P^{[0]} = P_{\Omega}$, and $m \in \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$.

Note that

$$P^{(k)} : \tilde{\Gamma}(\mathcal{H}) \to \tilde{\Gamma}(\mathcal{H}^{(k)})$$

$$P^{[k-1]} : \tilde{\Gamma}(\mathcal{H}) \to \tilde{\Gamma}(\mathcal{H}^{(k)}) \oplus \tilde{\Gamma}(\mathcal{H}^{(k-1)}),$$

6
are the orthogonal projections on the closed subspaces of $\tilde{\Gamma}(\mathcal{H})$ spanned by elementary tensors constructed from vectors of colors $0, \ldots, k-1$, and those with largest color $k-1$, respectively.

In the sequel we will make the identifications: $l_t \equiv l_t^{(\infty)}$, $l_t^* \equiv l_t^{(\infty)*}$, $l_t^0 \equiv l_t^{(\infty)0}$ and $l_t \equiv l_t^{(\infty)}$. We should mention that in this paper we also identify $l_t^{(m)}$, $l_t^{(m)*}$, $l_t^{(m)0}$, $l_t^{(m)}$, with their ampliations to $\tilde{\Gamma}(\mathcal{H}) = h_0 \otimes \Gamma$. The “duality” relations read: $l_t^{(m)} = l_t^{(m)*}$, $l_t^{(m)0} = l_t^{(m)}$, and $l_t^{(m)} = l_t^{(m)}$.

A shorthand notation for (2.6)-(2.9) will be used, namely

$$l_t^0 = \sum_{(\eta, V) \sim \alpha} A_{\alpha}^{(\eta, V)} \tag{2.10}$$

where $(\eta, V) \sim \alpha$ means that $A_{\alpha}^{(\eta, V)}$ appears on the RHS of (2.6)-(2.9) and $\alpha \in \mathcal{F}_m = \{(m), (m)*, (m)_0, (m)\}$.

The set of the indices of $m$-free fundamental processes, for $m \in \mathbb{N}^*$.

It was shown in [L2] that the extended $m$-free fundamental processes approximate extended free fundamental processes as $m \to \infty$, the latter being obtained for $m = \infty$. This holds on all of $\tilde{\Gamma}(\mathcal{H})$ in the case of creation, annihilation and time processes since they have unique bounded extensions to $\tilde{\Gamma}(\mathcal{H})$, or on a dense domain in the case of number processes, which are unbounded on $\tilde{\Gamma}(\mathcal{H})$. One should also note that if $m = \infty$, then the restrictions of formulas (2.6)-(2.9) to $\tilde{\Gamma}(\mathcal{H}^{(r)})$ are always finite sums.

Notation 

We will follow [Bi-Sp] and use the symbol $\#$ to write stochastic integrals and their matrix elements in such a way that $F$ and $G$ from an integrated elementary biprocess $F \otimes G$ are not separated by the integrators. Thus

$$dI = F \otimes G \#dM := FdMG$$

will be the differential w.r.t. $dM$ and, if $X = \sum_i F_i \otimes G_i$ is a biprocess integrable w.r.t. $dM$, we will write the integrals as

$$\int_0^t X \#dM := \sum_i \int_0^t F_i dMG_i.$$

When calculating matrix elements of stochastic integrals, we will also use. In turn, when using matrix elements, we will use

$$(x, F \otimes G y) \#\langle I, Q \rangle := \langle x, F Q G y \rangle$$

where $F \otimes G$ is a suitable stochastic biprocess, $Q$ is a projection and $x, y \in \tilde{\mathcal{E}}_0$, and extend it by linearity.

3. Filtered adapted biprocesses
By $\mathcal{L}(h, \mathcal{D})$, where $h$ is a separable Hilbert space and $\mathcal{D}$ is a dense subset of $h$, we denote the vector space of all linear operators $F$ on $h$ such that $\mathcal{D} \subset D(F) \cap D(F^*)$, where $F^*$ is the adjoint of $F$. In this paper we will use $\mathcal{L}(h_0, \mathcal{D}_0)$ and $\mathcal{L}(\Gamma, \mathcal{E}_0)$.

Let us first specify the notion of adaptedness. The dependence on $\mathcal{E}_0$ and $\mathcal{D}_0$ of this notion of adaptedness is suppressed in the notation.

**Definition 3.1.** Let $t \geq 0$ and let $D, E \in \mathcal{P}(\mathbb{N})$. We will say that an operator $A \otimes B$, where $A, B \in \mathcal{L}(h_0, \mathcal{D}_0) \otimes \mathcal{L}(\Gamma, \mathcal{E}_0)$, is $(t, D, E)$-adapted if

1. $\mathcal{E}_0 \subset D(AB)$ and $\mathcal{E}_0 \subset D(B^*A^*)$
2. $A = \tilde{A} \otimes P^{(D)}$ and $B = \tilde{B} \otimes P^{(E)}$ according to the decomposition $\tilde{\Gamma} = \tilde{\Gamma}_t \otimes \tilde{\Gamma}_s$,
3. $A$ and $B$ leave invariant the span of vectors of finite color support, i.e.

$$\forall r \in \mathbb{N} \exists p, q \in \mathbb{N} : \tilde{B} : (\mathcal{E}_0^{(r)}_t) \rightarrow \tilde{\Gamma}(\mathcal{H}^{(p)}_t) \quad \text{and} \quad \tilde{A} : \tilde{\Gamma}(\mathcal{H}^{(q)}_t) \cap D(\tilde{A}) \rightarrow \tilde{\Gamma}(\mathcal{H}^{(q)}_t).$$

**Definition 3.2.** Let $D, E \in \mathcal{P}(\mathbb{N})$. By an elementary $(D, E)$-adapted stochastic biprocess we will understand a family $(F(t) \otimes G(t))_{t \geq 0}$, where

1. $F(t) \otimes G(t)$ is $(t, D, E)$-adapted for all $t \geq 0$ and such that for each $r$ of Definition 3.1, the numbers $p, q$ can be chosen the same for all $t \geq 0$,
2. the map $t \rightarrow F(t)G(t)x$ is strongly measurable for all $x \in \tilde{\mathcal{E}}_0$.

We will often denote this biprocess by $F \otimes G$ understanding that $F = F(t)$ and $G = G(t)$.

**Definition 3.3.** An elementary $(D, E)$-adapted stochastic biprocess $F \otimes G$ will be called simple if there exists a partition of $\mathbb{R}^+$ given by $0 = t_0 < t_1 < \ldots < t_n < \ldots$, where $t_n \uparrow \infty$, such that

$$F(t) \otimes G(t) = \sum_{k=0}^{\infty} F(t_k) \otimes G(t_k) \chi_{[t_k, t_{k+1})}(t)$$

for any $t \in \mathbb{R}^+$. It will be called regular (or, continuous) if the map $t \rightarrow F(t)G(t)x$ is strongly continuous for all $x \in \tilde{\mathcal{E}}_0$.

The vector space spanned by elementary $(D, E)$-adapted biprocesses will be denoted by $\mathcal{A}(D, E)$. The vector subspaces of $\mathcal{A}(D, E)$ spanned by elementary simple $(D, E)$-adapted biprocesses, and elementary regular $(D, E)$-adapted biprocesses, will be denoted $\mathcal{S}(D, E)$ and $\mathcal{C}(D, E)$, respectively. Arbitrary elements of $\mathcal{A}(D, E)$, $\mathcal{S}(D, E)$, or $\mathcal{C}(D, E)$ will be called $(D, E)$-adapted, simple $(D, E)$-adapted, and regular $(D, E)$-adapted biprocesses, respectively.

**Definition 3.4.** Let $\mathcal{P}_0$ be a finite subset of $\mathcal{P}(\mathbb{N})$. A finite linear combination of the form

$$X = \sum_i F_i \otimes G_i$$

where $F_i \otimes G_i \in \mathcal{A}(D_i, E_i)$, $D_i, E_i \in \mathcal{P}_0$ for all $i$, will be called a $(\mathcal{P}_0, \mathcal{P}_0)$-adapted stochastic biprocess. The vector space spanned by $(\mathcal{P}_0, \mathcal{P}_0)$-adapted stochastic biprocesses will be denoted by $\mathcal{A}(\mathcal{P}_0, \mathcal{P}_0)$. Any element of the algebraic direct sum

$$\mathcal{A} = \bigoplus_{D, E \in \mathcal{P}(\mathbb{N})} \mathcal{A}(D, E)$$

is an $\mathcal{A}(\mathcal{P}_0, \mathcal{P}_0)$-adapted stochastic biprocess.
will be called a filtered adapted stochastic biprocess.

A family \( F = (F(t))_{t \geq 0} \equiv (F_t)_{t \geq 0} \) of operators from \( \mathcal{L}(h_0, D_0) \otimes \mathcal{L}(\Gamma, \mathcal{E}_0) \) will be called a \( V \)-adapted stochastic process, where \( V \in \mathcal{P}(\mathbb{N}) \) if \( F \otimes I \) and \( I \otimes F \) are \( (V, \mathbb{N}) \)– and \( (\mathbb{N}, V) \)–adapted stochastic biprocesses, respectively. The vector space spanned by \( V \)-adapted processes will be denoted by \( A(V) \). Note that if \( F \) is \( \mathbb{N} \)– adapted, then it is adapted in the usual ([H-P1]) sense. Note also that if \( F \otimes G \in A(D, E) \), then \( FG \in A(D \cap E) \). A process \( F \) will be called simple, or regular if the biprocesses \( F \otimes I, I \otimes F \) are simple, or regular, respectively. A stochastic process \( F \) will be called \( \mathcal{P}_0 \)-adapted if

\[
F(t) = \sum_{V \in \mathcal{P}_0} F_V(t)
\]

for all \( t \), where \( F_V \in A(V) \) for all \( V \in \mathcal{P}_0 \).

Filtered fundamental processes \( A^{(k,V)}, A^{(k,V)*}, A^{(k,V)_\circ} \) and \( A^{(0,V)} \) are natural examples of \( V \)-adapted regular stochastic processes for all \( k \in \mathbb{N}, V \in \mathcal{P}(\mathbb{N}) \). In view of (2.2)=(2.5), however, the integrals with filtered fundamental processes as integrators can be expressed as stochastic integrals with boson integrators. Namely, given a filtered adapted biprocess \( X = \sum_i F_i \otimes G_i, \eta \in \mathcal{T} \) and \( V \in \mathcal{P}(\mathbb{N}) \), there exists a filtered adapted biprocess \( X[\eta, V] \) such that

\[
\int_0^t X \# dA^{(\eta,V)} = \int_0^t X[\eta, V] \# dA^n
\]

on \( \tilde{E}_0 \), provided the integral on the RHS exists, where

\[
X[\eta, V] = \begin{cases} 
\sum_i F_i P^{(V)} \otimes G_i & \text{if} \ \eta = (k) \\
\sum_i F_i \otimes P^{(V)} G_i & \text{if} \ \eta = (k)^* \\
\sum_i F_i P^{(V \cup \{k\})} \otimes G_i & \text{if} \ \eta = (k)^\circ \\
\sum_i F_i P^{(V)} \otimes G_i & \text{if} \ \eta = (0)
\end{cases}
\]

(3.2)

Therefore, our study will concentrate on stochastic integrals w.r.t. the CCR processes. Note that in the case of number and time operators, the projections commute with \( A^n \), so one can also flip the projections to the other side of the tensor product.
4. Filtered fundamental lemmas

Let us begin with the definition of stochastic integrals of elementary simple biprocesses.

**Definition 4.1.** Let \( F \otimes G \in \mathcal{S}(D, E) \), where \( D, E \in \mathcal{P}(\mathbb{N}) \), be given by Definition 3.3. For any \( \eta \in \mathcal{T} \), define

\[
I^n(t) = \int_0^t F \otimes G \#dA^n = \sum_{k=1}^{n+1} F(t_{k-1}) \otimes G(t_{k-1}) \#(A^n_{t_k \wedge t} - A^n_{t_{k-1} \wedge t})
\]
on \( \tilde{\mathcal{E}}_0 \), where \( t_n \leq t < t_{n+1} \).

Note that this definition does not depend on the partition of \( \mathbb{R}^+ \) in the sense that one can take a refinement of the given partition to obtain the same result. Therefore, for convenience, we will fix \( t \) and from now on assume that \( t = t_n \).

It is convenient to introduce some notation for complex-valued measures which appear in boson multivariate calculus. Thus, for given \( u, v \in \mathcal{H}_0 \) (suppressed in the notation), let

\[
\mu^n([s, t]) = \begin{cases} 
  \int_s^t v^{(k)}(r)dr & \text{if } \eta = (k) \\
  \int_s^t \bar{u}^{(k)}(r)dr & \text{if } \eta = (k)^* \\
  \int_s^t \bar{u}^{(k)}(r)v^{(k)}(r)dr & \text{if } \eta = (k)^{\circ} \\
  t - s & \text{if } \eta = (0)
\end{cases}
\]  

be the measures associated with the annihilation, creation, number and time processes, respectively, of boson multivariate calculus. They are all absolutely continuous with respect to the Lebesgue measure.

**Lemma 4.2.** Let \( 0 \leq s \leq t \), \( x = w\varepsilon(u), y = z\varepsilon(v) \), where \( w, y \in \mathcal{D}_0 \), \( u, v \in \mathcal{H}_0 \) and let \( F \otimes G \in \mathcal{S}(D, E) \), where \( D, E \in \mathcal{P}(\mathbb{N}) \). Then

\[
\langle x, I^n(t)y \rangle = \int_0^t \langle x, F(s)G(s)y \rangle d\mu^n_{D,E}(s)
\]

where \( \mu^n_{D,E} = \mathbb{1}^n_{D,E} \mu^n \) and

\[
\mathbb{1}^n_{D,E} = \begin{cases} 
  \mathbb{1}_E(k) & \text{if } \eta = (k) \\
  \mathbb{1}_D(k) & \text{if } \eta = (k)^* \\
  \mathbb{1}_{D \cap E}(k) & \text{if } \eta = (k)^{\circ} \\
  1 & \text{if } \eta = (0)
\end{cases}
\]

for all \( \eta \in \mathcal{T} \). Here, \( \mathbb{1}_A \) denotes the indicator function of the set \( A \). Moreover, \( I^n_t \) is a \( D \cap E \)-adapted regular process.

**Proof.** Denote \( P = P^D \), \( Q = P^E \). Since \( F \otimes G \in \mathcal{S}(D, E) \), and \( A^n_t \) is \( \mathbb{N} \)-adapted, therefore, using the continuous tensor product decomposition of exponential vectors, we obtain

\[
\langle x, F(s) \otimes G(s) \#(A^n_t - A^n_s)y \rangle
\]
Proof. We only prove the case which completes the proof of the first part of the lemma. The second part is obvious. \(\Box\)

If we set \(D = E = V = N\), then \(\mathbb{1}_{D,E}^n \equiv 1\) for all \(\eta \in \mathcal{T}\) and we obtain the formulas of boson calculus for adapted processes (see [Mo-Si] or [P]). Thus, the new feature in this lemma is that apart from the time-dependent measures given by formula (4.1), which are the same as in boson calculus and can be associated with time, we also have the 0–1 color multipliers \(\mathbb{1}_{D,E}^n\) given by formula (4.2). We choose to incorporate them into the measures \(\mu_{D,E}^n\) (thus we can get trivial measures) for notational convenience.

Let us now evaluate matrix elements of the form \(\langle I_1^n (t) x, I_2^n (t) y \rangle\) for \(x = w\varepsilon(u), y = z\varepsilon(v), u, v \in \mathcal{H}_0, w, z \in \mathcal{D}_0\), where

\[
I_i^n (t) = \int_0^t F_i \otimes G_i \# dA^n_i,
\]

for elementary simple biprocesses \(F_i \otimes G_i \in S(D_i, E_i)\), where \(D_i, E_i \in \mathcal{P}(N)\) and \(\eta_i \in \mathcal{T}\), \(i = 1, 2\).

Let us also use a short-hand notation

\[
\Delta_{1,2} = \langle P_1 (A_{1,1}^n - A_{1,2}^n) Q_1 x, P_2 (A_{2,1}^n - A_{2,2}^n) Q_2 y \rangle
\]

where \(P_i = P^{(D_i)}, Q_i = P^{(E_i)}, i = 1, 2\) and \(0 \leq s < t < \infty\). We will also use the following convenient notation. Instead of sets \(E_1, D_1, D_2, E_2\) we will use integers \(-2, -1, 1, 2\), respectively. Moreover, we will identify \(\mathbb{1}_{[-2]} \equiv \mathbb{1}_{E_1}, \mathbb{1}_{[-1]} \equiv \mathbb{1}_{D_1}, \mathbb{1}_{[1]} \equiv \mathbb{1}_{D_2}, \mathbb{1}_{[2]} \equiv \mathbb{1}_{E_2}\) and extend this notation multiplicatively

\[
\mathbb{1}_{[n,m]} = \mathbb{1}_{[n]} \mathbb{1}_{[n+1]} \cdots \mathbb{1}_{[m]}
\]

for \(n, m \in \{-2, -1, 1, 2\}\). Thus, for instance \(\mathbb{1}_{D_1 \cap D_2} = \mathbb{1}_{[-1,1]}, \mathbb{1}_{D_1 \cap D_2 \cap E_2} = \mathbb{1}_{[-2,2]}\), etc.

**Proposition 4.3.** Let \(F_i \otimes G_i \in S(D_i, E_i)\), where \(D_i, E_i \in \mathcal{P}(N)\) and let \(\eta_i \in \mathcal{T}\), where \(i = 1, 2\). Then

\[
\Delta_{1,2} = [\kappa_{1,2} \mu_{E_1,E_2}^n ([s,t]) \mu_{E_1,E_2}^n ([s,t]) + \mu_{1,2}([s,t]) \langle P_1 Q_1 x, P_2 Q_2 y \rangle
\]

where \(\kappa_{1,2} \in \{0, 1\}\) and the non-trivial part of the table of measures \(\mu_{1,2}\) is given by

| \(\mu_{1,2}\) | \((k)\ast\) | \((k)\diamond\) |
|---|---|---|
| \((k)\ast\) | \(\mathbb{1}_{[-1,1]}(k) \mu^{(0)}\) | \(\mathbb{1}_{[-1,2]}(k) \mu^{(k)}\) |
| \((k)\diamond\) | \(\mathbb{1}_{[-2,1]}(k) \mu^{(k)}\) | \(\mathbb{1}_{[-2,2]}(k) \mu^{(k)}\) |

Proof. We only prove the case \(\eta_1 = (k_1)\ast, \eta_2 = (k_2)\ast\). Using the relation

\[
P^{(V)} A_{i}^{(k)} = \mathbb{1}_{V}(k) A_{i}^{(k)} P^{(V)}
\]
which holds for all $V \in \mathcal{P}(\mathbb{N})$, $k \in \mathbb{N}$ and $t \geq 0$, we obtain
\[
\Delta_{1,2} = \mathbb{I}_{D_1(k_1)} \mathbb{I}_{D_2(k_2)} ((A^{(k_1)*}_t - A^{(k_1)*}_s) P_1 x + (A^{(k_2)*}_t - A^{(k_2)*}_s) P_2 y)
\]
\[
= \sum_{j=1}^n [\llbracket \mathbb{E}_1 \cap D_2(k_2) \rrbracket_{\mathbb{E}_1 \cap D_1(k_1)}(k_1) \mu^{(k_1)}([s, t]) \mu^{(k_2)*}([s, t]) + \delta_{k_1, k_2} \mathbb{I}_{[-1, 1]}(k_1)(t - s)]
\times \langle P_1 Q_1 x, P_2 Q_2 y \rangle.
\]
The other cases are proved in a similar way. We do not give the explicit formulas for $\kappa_{1,2}$ since they are not relevant for the calculus (see Lemma 4.4).

**Lemma 4.4.** Let $x = w \varepsilon(u)$, $y = z \varepsilon(v)$, where $w, z \in \mathcal{D}_0$ and $u, v \in \mathcal{H}_0$. Under the assumptions of Proposition 4.3 we have
\[
\langle I^n_1(t)x, I^n_2(t)y \rangle = \int_0^t \langle F_1(s)G_1(s)x, F^n_2(s)y \rangle d\mu_1(s)
\]
\[
+ \int_0^t \langle I^n_1(s)x, F_2(s)G_2(s)y \rangle d\mu_2(s)
\]
\[
+ \int_0^t \langle F_1(s)G_1(s)x, F_2(s)G_2(s)y \rangle d\mu_{1,2}(s)
\]
where the measures $\mu_{1,2}$ are given by Proposition 4.3, and

\[
\mu_1 = \begin{cases}
\mathbb{I}_{[-2]}(k_1) \mu^{(k_1)*} & \text{if } \eta_1 = (k_1) \\
\mathbb{I}_{[-2]}(k_1) \mu^{(k_1)} & \text{if } \eta_1 = (k_1)^* \\
\mathbb{I}_{[-2, 2]}(k_1) \mu^{(k_1)} & \text{if } \eta_1 = (k_1) \circ \\
\mu^{(k_1)} & \text{if } \eta_1 = (0)
\end{cases}
\]
(4.3)

\[
\mu_2 = \begin{cases}
\mathbb{I}_{[2]}(k_2) \mu^{(k_2)} & \text{if } \eta_2 = (k_2) \\
\mathbb{I}_{[-2, 1]}(k_2) \mu^{(k_2)*} & \text{if } \eta_2 = (k_2)^* \\
\mathbb{I}_{[-2, 2]}(k_2) \mu^{(k_2)} & \text{if } \eta_2 = (k_2) \circ \\
\mu^{(k_2)} & \text{if } \eta_2 = (0)
\end{cases}
\]
(4.4)

**Proof.** The proof is based on the Hudson-Parthasarathy theory (for instance, see [HP1] or [P]). The main changes that come into play in our case are due to 0-1 color multipliers which produce more zeros in our formulas. We provide only the basic algebraic calculations.

Using standard decompositions of exponential vectors, we obtain
\[
x = x_{t_{j-1}} \otimes x_{[t_{j-1}, t_j]} \otimes x_{[t_j, t]}
\]
for each $j = 1, \ldots, n$, and an analogous formula for $y$, which gives the formula
\[
\langle I^n_1(t)x, I^n_2(t)y \rangle = S_1 + S_2 + S_3
\]
where
\[
S_1 = \sum_{j=1}^n \langle F_1(t_{j-1}) G_1(t_{j-1}) x_{t_{j-1}}, I^n_2(t_{j-1}) y_{t_{j-1}} \rangle
\]

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\begin{align*}
&\times \quad \langle P_1(A^n(t_j) - A^n(t_{j-1}))Q_1x_{[t_{j-1}, t_j]}, P_2Q_2y_{[t_{j-1}, t_j]} \rangle \\
&\times \quad \langle P_1Q_1x_{[t_{j-1}, t_j]}, P_2(A^n(t_j) - A^n(t_{j-1}))Q_2y_{[t_{j-1}, t_j]} \rangle \\
&\times \quad \langle P_1Q_1x_{[t_{j-1}, t_j]}, P_2Q_2y_{(t_{j-1})} \rangle.
\end{align*}

\begin{align*}
S_2 &= \sum_{j=1}^{n} \langle I^n(t_{j-1})x_{t_{j-1}}, F_2(t_{j-1})G_2(t_{j-1})y_{t_{j-1}} \rangle \\
&\times \quad \langle P_1Q_1x_{[t_{j-1}, t_j]}, P_2(A^n(t_j) - A^n(t_{j-1}))Q_2y_{[t_{j-1}, t_j]} \rangle \\
&\times \quad \langle P_1Q_1x_{[t_{j-1}, t_j]}, P_2Q_2y_{[t_{j-1}, t_j]} \rangle.
\end{align*}

\begin{align*}
S_3 &= \sum_{j=1}^{n} \langle F_1(t_{j-1})G_1(t_{j-1})x_{t_{j-1}}, F_2(t_{j-1})G_2(t_{j-1})y_{t_{j-1}} \rangle \\
&\times \quad \langle P_1(A^n(t_j) - A^n(t_{j-1}))Q_1x_{[t_{j-1}, t_j]}, \\
&\quad \quad P_2(A^n(t_j) - A^n(t_{j-1}))Q_2y_{[t_{j-1}, t_j]} \rangle \\
&\times \quad \langle P_1Q_1x_{[t_{j-1}, t_j]}, P_2Q_2y_{[t_{j-1}, t_j]} \rangle.
\end{align*}

In each of those sums, the middle factor produces a complex-valued measure, denoted by \( \mu_1, \mu_2 \) and \( \mu_{1,2} \), respectively. Note that Proposition 4.3 gives \( \mu_{1,2} \). In turn, \( \mu_1 \) and \( \mu_2 \) are defined by

\[
\mu_1([s, t]) = \langle P_1(A^n(t) - A^n(s))Q_1x_{[s, t]}, P_2Q_2y_{[s, t]} \rangle \\
\mu_2([s, t]) = \langle P_1Q_1x_{[s, t]}, P_2(A^n(t) - A^n(s))Q_2y_{[s, t]} \rangle.
\]

It can be seen that it is enough to apply Lemma 4.2 to get formulas (4.3)-(4.4). The remaining arguments are the same as in the usual case (see [P]). \( \square \)

\textbf{Remark.} It is worth pointing out that the 0-1 color multipliers which appear in all measures in both fundamental lemmas follow easy-to-remember rules:

\begin{itemize}
  \item [Rule 1.] all filters to the right of \( dA^{(k)} \) must contain \( k \),
  \item [Rule 2.] all filters to the left of \( dA^{(k)} \) must contain \( k \),
  \item [Rule 3.] all filters on both sides of \( dA^{(k)} \) must contain \( k \).
\end{itemize}

We can of course assume that the integrated biprocesses satisfy these rules and then we can skip the 0-1 color multipliers. However, when we take a linear combination of biprocesses with different types of adaptedness as integrands, it is important to keep track of what survives after integration and what is killed, which is the main source of non-commutativity leading, for instance, to \( m \)-free calculi.

\section{5. An Extension of the Stochastic Integral}

For given \( D, E \in \mathcal{P}(\mathbb{N}) \) and \( \eta \in \mathcal{T} \), we will now take elementary simple biprocesses \( X \in \mathcal{S}(D, E) \) as integrands and \( A^n \) as integrators to approximate locally square integrable \((D, E)\)-adapted biprocesses in a topology specified below. If \( X = \sum_i F_i \otimes G_i \in \mathcal{A}(D, E) \) and the associated \( D \cap E \)-adapted process \( B = (B(t))_{t \geq 0} \) is given by \( B(t) = \sum_i F_i(t)G_i(t) \) for all \( t \geq 0 \), then we will write \( B \models X \).
Lemma 5.1. Let $X \in \mathcal{S}(D, E)$, $x = w\varepsilon(u)$, where $D, E \in \mathcal{P}(\mathbb{N})$, $w \in \mathcal{D}_0$, $u \in \mathcal{H}_0$, and let

$$I^n(t)x = \int_0^t X \#dA^n x$$

where $\eta \in \mathcal{T}$. Then

$$\|I^n(t)x\|^2 \leq C^n_{D,E}(t) \int_0^t \|B(s)x\|^2 \xi^n_{D,E}(ds)$$

where $B \models X$ and

$$\xi^n_{D, E} = |\sigma^n_{D, E}| + \nu^n_{D, E}$$

with $C^n_{D, E}(t) = e^{\sigma^n_{D, E}([0, t])}$,

$$\nu^n_{D, E} = \begin{cases} 
\mathbb{I}_D(k)\mu^{(0)} & \text{if } \eta = (k)^* \\
\mathbb{I}_{D \cap E}(k)\mu^{(k)} & \text{if } \eta = (k)^) \\
0 & \text{otherwise}
\end{cases}$$

$$\sigma^n_{D, E} = \begin{cases} 
\mathbb{I}_E(k)\mu^{(k)} & \text{if } \eta = (k) \\
\mathbb{I}_{D \cap E}(k)\mu^{(k)^*} & \text{if } \eta = (k)^* \\
\mathbb{I}_{D \cap E}(k)\mu^{(k)^)} & \text{if } \eta = (k)^) \\
\mu^{(0)} & \text{if } \eta = (0)
\end{cases}$$

and $|\mu|$ denoting the variation of $\mu$.

Proof. This proof is based on Lemma 4.4 and is similar to that in the adapted case (see [P]). \hfill \Box

Definition 5.2. For fixed $\eta \in \mathcal{T}$, define a family of seminorms $\|\|_{x,t,\eta}$ on $\mathcal{A}(D, E)$, where $t \geq 0$, $x = w\varepsilon(u) \in \mathcal{M}_0$, by

$$\|X\|_{x,t,\eta}^2 = \int_0^t \|B(s)x\|^2 \xi^n(ds)$$

(5.1)

where $X \in \mathcal{A}(D, E)$ $t \geq 0$, $x \in \mathcal{M}_0$ and $\xi^n = \xi^n_{\mathcal{N}, \mathcal{N}, \mathbb{N}}$. Denote by $L^2_{\text{loc}}(D, E, dA^n)$ the linear vector space of all $(D, E)$–adapted biprocesses $X$ such that $\|X\|_{x,t,\eta} < \infty$ for all $x \in \mathcal{M}_0$ and $t \geq 0$. We will say that $X$ is locally square integrable with respect to $dA^n$ (in our notation the dependence of this notion on the domain $\tilde{E}_0$ and $\mathbb{R}^+$ is suppressed).

Theorem 5.3. The stochastic integral with respect to the fundamental process $A^n$ can be extended by continuity from $\mathcal{S}(D, E)$ to $L^2_{\text{loc}}(D, E, dA^n)$ for any $\eta \in \mathcal{T}$ and $D, E \in \mathcal{P}(\mathbb{N})$.

Proof. To show that $\mathcal{S}(D, E)$ is dense in $L^2_{\text{loc}}(D, E, dA^n)$ for any $D, E \in \mathcal{P}(\mathbb{N})$ and $\eta \in \mathcal{T}$, it is enough to slightly modify the ideas of [Ac-Fa-Qu] and [H-P1], where we refer the reader for details. \hfill \Box

For the approximating sequence of elementary simple biprocesses $H^{(n)} \otimes K^{(n)}$ we set

$$I^n(t)x = s - \lim_{n \to \infty} I^n(t)x = s - \lim_{n \to \infty} \int_0^t H^{(n)} \otimes K^{(n)} \#dA^n$$
on $\tilde{E}_0$. In view of Lemma 5.1, the sequence $(I^n_{\eta}(t)x)_{n\in\mathbb{N}}$ is Cauchy for each $t \geq 0$ and $x \in \mathcal{M}_0$, hence convergent. Moreover, the convergence is uniform for $t$ in finite intervals and the limit does not depend on the choice of approximating simple biprocesses. In this way we obtain a $D \cap E$-adapted process $I^n$.

**Theorem 5.4.** Lemmas 4.2, 4.4 and 5.1 remain true for locally square integrable integrands.

**Proof.** By Lemma 5.1, the convergence of $I^n_{\eta}$ to $I$ on $[0, T]$ is uniform for fixed $T$. Therefore we conclude that Lemmas 4.2 and 4.4 as well as the norm estimate of Lemma 5.1 hold for all biprocesses which are locally square integrable w.r.t. appropriate fundamental processes in the sense of Definition 5.2. $\square$

6. The filtered Itô formula

In the multiplicative version of the filtered Itô formula we will use the differentials

\begin{align*}
\text{d}I^n_{\eta_1} &= G_1 \otimes F_1 \# \text{d}A^n_{\eta_1} = G_1 \text{d}A^n_{\eta_1} F_1 \\
\text{d}I^n_{\eta_2} &= F_2 \otimes G_2 \# \text{d}A^n_{\eta_2} = F_2 \text{d}A^n_{\eta_2} G_2
\end{align*}

instead of integrals $I^n_{\eta_1}, I^n_{\eta_2}$, respectively, where $\eta_i \in \mathcal{T}, i = 1, 2$ (we switch the order of $F_1$ and $G_1$ in the first differential in order to make a direct connection with Lemma 4.4).

Recall that the boson Itô table in the multivariate case is of the form

\[
\begin{array}{c|cc}
\text{d}A^n_{\eta_1} \text{d}A^n_{\eta_2} & \text{d}A^{(k)\circ} & \text{d}A^{(k)\diamond} \\
\text{d}A^{(k)} & \text{d}A^{(0)} & \text{d}A^{(k)} \\
\text{d}A^{(k)\diamond} & \text{d}A^{(k)*} & \text{d}A^{(k)\circ}
\end{array}
\]

where we adopted the convention that only the non-trivial part of the Itô table is given, thus the usual $\delta_{k_1,k_2}$ does not appear here. By $[A^n_{\eta_1}, A^{n^2}_{\eta_2}]$ we will denote the process obtained in the multiplication.

A pair of biprocesses, $(X, X^\dagger)$, where $X \in \mathcal{A}(D, E)$ and $X^\dagger \in \mathcal{A}(E, D)$, will be called an adjoint pair if

\[
\langle x, B(s)y \rangle = \langle B^\dagger(s)x, y \rangle
\]

for all $x, y \in \mathcal{M}_0$, where $B \models X$ and $B^\dagger \models X^\dagger$ (cf. [P]). A pair of $V$-adapted processes, $(F, F^\dagger)$ is an adjoint pair if $(F \otimes 1, 1 \otimes F^\dagger)$ is an adjoint pair of biprocesses. For instance, $F^\dagger = A^{(k,V)}, A^{(k,V)*}, A^{(k,V)\circ}$ or $A^{(0,V)}$ according to whether $F = A^{(k,V)*}, A^{(k,V)}, A^{(k,V)\circ}$ or $A^{(0,V)}$. Then $(F, F^\dagger)$ is an adjoint pair of $V$-adapted processes in $\Gamma$. It follows easily from Lemma 4.2 that if $X \in L^2_{\text{loc}}(D, E, dA^n), X^\dagger \in L^2_{\text{loc}}(E, D, dA^{n^\dagger})$ and $(X, X^\dagger)$ is an adjoint pair, then

\[
I_1(t) = \int_0^t X \# dA^n, \quad I_2(t) = \int_0^t X^\dagger \# dA^{n^\dagger}.
\]

is an adjoint pair of $D \cap E$-adapted processes.
Theorem 6.1. (Filtered Itô Formula) Let $G_1 \otimes F_1 \in L^2_{\text{loc}}(E_1, D_1, dA^{n_1})$, $F_1^* \otimes G_1^* \in L^2_{\text{loc}}(D_1, E_1, dA^{n_1})$, $F_2 \otimes G_2 \in L^2_{\text{loc}}(D_2, E_2, dA^{n_2})$, where $D_1, D_2, E_1, E_2 \in \mathcal{P}(\mathbb{N})$ and $\eta_1, \eta_2 \in \mathcal{T}$. Suppose that $I_1^{n_1} I_2^{n_2}$ is a $D_1 \cap D_2 \cap E_1 \cap E_2$-adapted process and that $I_1^{n_1} F_2 \otimes G_2$, $G_1 \otimes F_1 I_2^{n_2}$, $G_1 F_1 F_2 \otimes G_2$ and $G_1 \otimes F_1 F_2 G_2$ are locally square integrable with respect to $dA^{n_1}$, $dA^{n_2}$, $d[A^{n_1}, A^{n_2}]$, and $d[A^{n_1}, A^{n_2}]$, respectively. Then

$$d(I_1^{n_1} I_2^{n_2}) = dI_1^{n_1} + dI_2^{n_2}$$

where

$$I_1^{n_1} I_2^{n_2} = I_1^{n_1} F_2 \otimes G_2 #dA^{n_2}$$

and the Itô correction can be written in two equivalent ways:

$$dI_1^{n_1} = G_1 \otimes \rho_{\eta_1, \eta_2}(F_1 F_2) G_2 #dA^{n_2}$$

where $\rho_{\eta_1, \eta_2}(F_1 F_2) = \mathbb{1}_{[-1,1]}(k_1) F_1 F_2 \equiv \mathbb{1}_{D_1 \cap D_2}(k_1) F_1 F_2$ for those values of $\eta_1, \eta_2$ which belong to the non-trivial part of the Itô table, and for the other ones, it is zero.

Proof. Without loss of generality assume that $I_1^{n_i}(0) = 0$, $i = 1, 2$. For all $t \in \mathbb{R}^+$, and $x = w \varepsilon(u), y = z \varepsilon(v)$, where $w, y \in D_0$ and $u, v \in H_0$, we have

$$\langle x, I_1^{n_1}(t) I_2^{n_2}(t) y \rangle = \langle I_1^{n_1}(t) x, I_2^{n_2}(t) y \rangle$$

where

$$I_1^{n_1} = \int_0^t F_1^* \otimes G_1^* #dA^{n_1}$$

since $(I_1^{n_1}, I_1^{n_1})$ is an adjoint pair and the product $I_1^{n_1} I_2^{n_2}$ has $\mathcal{E}_0$ in its domain. By Lemma 4.4, this gives

$$\langle x, I_1^{n_1}(t) I_2^{n_2}(t) y \rangle = \int_0^t \langle I_1^{n_1}(s) x, F_2(s) G_2(s) y \rangle d\mu_2(s)$$

$$+ \int_0^t \langle F_1^*(s) G_1^*(s) x, I_2^{n_2}(s) y \rangle d\mu_1(s)$$

$$+ \int_0^t \langle F_1^*(s) G_1^*(s) x, F_2(s) G_2(s) y \rangle d\mu_{1,2}(s)$$

$$= \int_0^t \langle x, I_1^{n_1}(s) F_2(s) G_2(s) y \rangle d\mu_2(s)$$

$$+ \int_0^t \langle x, G_1(s) F_1(s) I_2^{n_2}(s) y \rangle d\mu_1(s)$$

$$+ \int_0^t \langle x, G_1(s) F_1(s) F_2(s) G_2(s) y \rangle d\mu_{1,2}(s)$$

where the measures $\mu_1, \mu_2$ and $\mu_{1,2}$ are determined appropriately depending on the fundamental processes $A^{n_1}$ and $A^{n_2}$ and are given by Lemma 4.4. Let us look closer at the first two integrals. Note that

$$F_2 := I_1^{n_1} F_2 \in \mathcal{A}(D_2), \ F_1' := F_1 I_2^{n_2} \in \mathcal{A}(D_1')$$

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where $D'_2 = D_1 \cap D_2 \cap E_1$ and $D'_1 = D_1 \cap D_2 \cap E_2$. Moreover,

$$F'_2 \otimes G_2 \in L^2_{loc}(D'_2, E_2, dA'^{n})$$

by assumption. Therefore, we just need to check if the measures $\mu_1$, $\mu_2$ are also obtained when these biprocesses are integrated w.r.t. $A'^{n}$ and $A'^{n_1}$, respectively. This is verified by using Lemma 4.2. For instance, let

$$\eta = \eta_{k_2}$$

Comparing this with $\mu_2$, we conclude that $\mu_2 = \mu_{D'_2, E_2}^n$, which enables us to write the first integral as

$$\langle x, \int_0^t F'_2 \otimes G_2 \# dA'^{n}y \rangle$$

where

$$\mu_{D'_2, E_2}^n = \mu_{D_1 \cap D_2 \cap E_1, E_2}^n = \begin{cases} 
\mathbb{1}_{[2]}(k_2) \mu^{(k_2)}_\eta & \text{if } \eta_2 = (k_2) \\
\mathbb{1}_{[-2, 1]}(k_2) \mu^{(k_2)*}_\eta & \text{if } \eta_2 = (k_2)* \\
\mathbb{1}_{[-2, 2]}(k_2) \mu^{(k_2)\circ}_\eta & \text{if } \eta_2 = (k_2) \circ \\
\mu^{(0)} & \text{if } \eta_2 = (0)
\end{cases}$$

Comparing this with $\mu_2$, we conclude that $\mu_2 = \mu_{D'_2, E_2}^n$, which enables us to write the first integral as

$$\langle x, \int_0^t I_1^n F_2 \otimes G_2 \# dA'^{n}y \rangle$$

which, in differential notation, corresponds to

$$I_1^n dI_2^n = I_1^n F_2 \otimes G_2 \# dA'^{n} = I_1^n F_2 dA'^{n} G_2.$$ 

A similar reasoning gives $\mu_1 = \mu_{E_1, D'_1}^n$, which enables us to write the differential of the second integral as

$$dI_1^n I_2^n = G_1 \otimes F_1 I_2^n \# dA'^{n} = G_1 dA'^{n} F_1 I_2^n.$$ 

Let us finally evaluate the Itô correction. This boils down to straightforward examination of 4 non-trivial cases. For instance, let $\eta_1 = (k_1)$ and $\eta_2 = (k_2) \circ$. Then

$$G_1 \otimes F_1 F_2 G_2 \in L^2_{loc}(E_1, D_1 \cap D_2 \cap E_2, dA^{(k_1)})$$

by assumption since $[A^{(k_1)}, A^{(k_2)\circ}] = \delta_{k_1, k_2} A^{(k_1)}$. We have

$$\mu_{1, 2} = \delta_{k_1, k_2} \mathbb{1}_{[-1, 2]}(k_1) \mu^{(k_1)}_\eta = \delta_{k_1, k_2} \mathbb{1}_{[-1, 1]}(k_1) \mu^{(k_1)}_\eta$$

This gives

$$dI_1^{(k_1)} dI_2^{(k_2)\circ} = \delta_{k_1, k_2} \mathbb{1}_{[-1, 1]}(k_1) G_1 \otimes F_1 F_2 G_2 \# dA^{(k_1)}.$$ 

The second formula for the Itô correction as well as other cases are proved in a similar way. 

**Definition 6.2.** A $(D, E)$-adapted stochastic biprocess $X = (X(t))_{t \geq 0}$ will be called **locally-bounded** if

$$(1) \; X(t) \in (\mathcal{B}(h_0) \otimes \mathcal{B}(\Gamma)) \otimes (\mathcal{B}(h_0) \otimes \mathcal{B}(\Gamma)) \text{ for all } t \geq 0$$

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(2) the map \( t \to B(t)x \) is strongly measurable for all \( x \in \tilde{\Gamma}(\mathcal{H}) \), where \( B = X \)
(3) \( \sup_{0 \leq s \leq t} \|B(s)\| < \infty \) for all \( t > 0 \).

The space of locally bounded \((D, E)\)-adapted biprocesses will be denoted by \( \mathcal{B}_{\text{loc}}(D, E) \).

**Remark.** One can give a shorter formulation of Theorem 6.1 if one makes stronger assumptions. Namely, if we assume that \( G_1 \otimes F_1 \in \mathcal{B}_{\text{loc}}(E_1, D_1) \) and \( F_2 \otimes G_2 \in \mathcal{B}_{\text{loc}}(D_2, E_2) \), then the assumptions of Theorem 6.1 are satisfied.

Let us now give conditions under which an infinite sum of stochastic integrals associated with the same pair of filters \((D, E)\) but different \( \eta \in \mathcal{T} \) is well-defined on the domain \( \bar{\Xi}_0 \). Namely, we want to define

\[
I(t) = \sum_{\eta \in \mathcal{T}} \int_{0}^{t} X_{\eta} \, dA_{\eta}
\]

on \( \bar{\Xi}_0 \) for all \( t \in \mathbb{R}^+ \), where \( X_{\eta} \) is \((D, E)\)-adapted and locally square integrable with respect to \( dA_{\eta} \) for each \( \eta \in \mathcal{T} \) and the sum is possibly infinite. The case \( D = E = \mathbb{N} \) was studied in [Mo-Si] (see also [P]).

An approximating sequence of integrals will be given by

\[
I^n(t) = \sum_{\eta \in \mathcal{T}(n)} \int_{0}^{t} X_{\eta} \, dA_{\eta}
\]

where

\[
\mathcal{T}(n) = \{(k), (l)*, (r)*, (0) : 1 \leq k, l, r \leq n\}
\]

denotes the set of indices associated with fundamental processes of colors less than or equal to \( n \). Now, let

\[
N(u) = \max\{k : u^{(k)} \text{ is a non-zero function in } L^2(\mathbb{R}^+)\}
\]

with

\[
\mathcal{T}(n, u) = \{(k), (l)*, (r)*, (0) : 1 \leq k, r \leq n \land N(u), 1 \leq l \leq n\}
\]

for \( u \in \mathcal{H}_0 \), and we set \( \mathcal{T}(u) = \mathcal{T}(\infty, u) \). Note that in the case of creation process there is no constraint on the color support of \( u \) – the reason is that the Itô correction term corresponding to the creation–creation pair always appears in the filtered Itô formula irrespective of \( u \).

**Theorem 6.3.** Suppose \( X_{\eta} \in L^2_{\text{loc}}(D, E, dA_{\eta}) \) for each \( \eta \in \mathcal{T} \) and that

\[
\sum_{\eta \in \mathcal{T}} \int_{0}^{t} \|B_{\eta}(s)x\|^2 \, d\nu_u(s) < \infty.
\]

(6.1) for all \( t \geq 0 \) and \( x \in \mathcal{M}_0 \), where the real-valued measures \( \nu_u, u \in \mathcal{H}_0 \), are given by

\[
\nu_u(t) = \int_{0}^{t} \left( \sum_{k \in \mathbb{N}} |u^{(k)}(s)|^2 + 1 \right) ds
\]
and $B^n = X^n$. Then there exists a regular $D \cap E$-adapted process $I$ such that

$$\lim_{n \to \infty} \sup_{0 \leq s \leq t} \|I(n)(s)x - I(t)x\| = 0$$

$$\|I(t)x\|^2 \leq 2e^{\nu_\omega(t)} \sum_{\eta \in T(u)} \int_0^t \|B^n(s)x\|^2 d\nu_\omega(s)$$

for all $x \in \mathcal{M}_0$ and $t \geq 0$.

**Proof.** The proof is similar to that in [P]. \hfill \Box

### 7. The $m$-free calculi

In order to include $m$-free calculi for $1 \leq m \leq \infty$, we need to add biprocesses associated with different pairs of filters $(D, E)$ and integrate them w.r.t. fundamental processes. In this section we use filtered stochastic calculus to recover $m$-free calculi.

**Definition 7.1.** For all $m \in \mathbb{N}^*$ and $\alpha \in \mathcal{F}_m$ we define on $\tilde{E}_0$ the integrals w.r.t. extended $m$-free fundamental processes by the linear extension of

$$\int_0^t X#dl^\alpha := \sum_{(\eta,V) \sim \alpha} \int_0^t X[\eta,V]#dA^\eta$$

where $X = F \otimes G$ is a $(D, E)$-adapted stochastic biprocess for which the integrands on the RHS are locally square integrable biprocesses and $X[\eta,V]$ is given by (3.2). We will say that $X$ is locally square integrable w.r.t. $dl^\alpha$. The space spanned by such biprocesses will be denoted by $L^2_{\text{loc}}(D, E, dl^\alpha)$. Note that if $X$ is locally bounded, then $X \in L^2_{\text{loc}}(D, E, dl^\alpha)$ for all $\alpha \in \mathcal{F}_m$. It can be shown that the integrals w.r.t. extended free fundamental processes always reduce to finite sums on $\tilde{E}_0$ and thus are well-defined.

**Proposition 7.2.** Let $x = w\varepsilon(u)$, $y = z\varepsilon(v)$, where $w, z \in D_0$, $u, v \in H_0$, $\alpha \in \mathcal{F}_m$ and $m \in \mathbb{N}^*$, and assume that $F \otimes G \in L^2_{\text{loc}}(D, E, dl^\alpha)$, where $D, E \in \mathcal{P}(\mathbb{N})$. Then

$$\langle x, \int_0^t F \otimes G#dl^\alpha y \rangle = \int_0^t (x, F(s) \otimes G(s)y)#d\tilde{\nu}^\alpha(s)$$

where

$$\tilde{\nu}^\alpha = \begin{cases} 
\sum_{k \in E(m)} \langle I, P^{[k-1]} \rangle \mu^{(k)} & \text{if } \alpha = (m) \\
\sum_{k \in D(m)} \langle I, P^{[k-1]} \rangle \mu^{(k)*} & \text{if } \alpha = (m)^* \\
\sum_{k \in D(m)} \langle I, P^{[k]} \rangle \mu^{(k)} & \text{if } \alpha = (m)^\circ \\
\langle I, P^{(0)} \rangle \mu^{(0)} & \text{if } \alpha = (m) \end{cases}$$

and $D(m) = D \cap \{1, \ldots, m\}$.

**Proof.** It is a straightforward consequence of (2.6)-(2.9) and Lemma 4.2. \hfill \Box

Before we state the general version of the $m$-free Itô formula, let us first establish its easy case, the $m$-free Itô table.
Proposition 7.3. Let $\alpha_1, \alpha_2 \in F_m$, where $m \in \mathbb{N}^\ast$. Then $l^{\alpha_1}l^{\alpha_2}$ satisfies the relation

$$d(l^{\alpha_1}l^{\alpha_2}) = l^{\alpha_1}dl^{\alpha_2} + dl^{\alpha_1}l^{\alpha_2} + dl^{\alpha_1}dl^{\alpha_2}$$

where the Itô correction is given by the following multiplication table:

| $dl^{\alpha_1}dl^{\alpha_2}$ | $dl^{(m)*}$ | $dl^{(m)\circ}$ |
|-----------------------------|--------------|------------------|
| $dl^{(m)}$                  | $dl^{(m)*}$  | $dl^{(m)\circ}$ |
| $dl^{(m)\circ}$            | $dl^{(m)*}$  | $dl^{(m)\circ}$ |

with the convention that only the non-trivial part of the table is given.

Proof. The proof is based on the Itô table for boson calculus (see Section 7). Assume first that $m$ is finite. We will consider one case, leaving the other ones to the reader. Using bi-linearity, we obtain

$$d(l^{(m)}l^{(m)*}) = \sum_{k,l=1}^{m} P^{[k-1]} d(A^{(k)} A^{(l)*}) P^{[l-1]}$$

$$+ \sum_{k,l=1}^{m} \left\{ P^{[k-1]} A^{(k)} dA^{(l)*} P^{[l-1]} + P^{[k-1]} dA^{(k)} A^{(l)*} P^{[l-1]} \right\}$$

$$= l^{(m)} dl^{(m)*} + dl^{(m)*} l^{(m)*} + dA^{(0)} \sum_{k=1}^{m} P^{[k-1]}$$

$$= l^{(m)} dl^{(m)*} + dl^{(m)*} l^{(m)*} + dl^{(m)}.$$ 

where we used the fact that the time differential $dA^{(0)} \equiv dt$ commutes with the projections $P^{[k-1]}$ for all $k \in \mathbb{N}$. This gives $dl^{(m)} dl^{(m)*} = dl^{(m)}$. The strong limit as $m \to \infty$ on $\tilde{E}_0$ of this relation gives $ddl^{*} = dl^{*} \equiv dt$. The other cases are analogous. \(\square\)

The processes obtained from the above Itô table will be denoted $[l^{\alpha_1}, l^{\alpha_2}]$, as in Section 6 for filtered fundamental processes, i.e.

$$dl^{\alpha_1}dl^{\alpha_2} = d[l^{\alpha_1}, l^{\alpha_2}].$$

Another notation will also be needed in the Itô formula. Namely, for a given densely defined $V$-adapted linear operator $H$ on $\tilde{\Gamma}(\mathcal{H})$, let

$$\mathbb{P}_0(H) = \sum_{k \in V} P^{[k-1]} HP^{[k-1]} \quad (7.1)$$

$$\mathbb{P}_1(H) = \sum_{k \in V} P^{[k]} HP^{[k]} \quad (7.2)$$

The two operators $\mathbb{P}_0$ and $\mathbb{P}_1$ play a role of “partial traces”. Of course, the only difference between $\mathbb{P}_0$ and $\mathbb{P}_1$ is that the first one includes the vacuum in its trace if $k = 1 \in V$, whereas $\mathbb{P}_1$ doesn’t.
The differentials which enter the \( m \)-free Itô formula are given by

\[
\begin{align*}
    dJ_1^{\alpha_1} &= G_1 \otimes F_1 \# dl^{\alpha_1} = G_1 dl^{\alpha_1}, \\
    dJ_2^{\alpha_2} &= F_2 \otimes G_2 \# dl^{\alpha_2} = F_2 dl^{\alpha_2}G_2
\end{align*}
\]

where \( \alpha_1, \alpha_2 \in \mathcal{F}_m \). They correspond to integrals \( J_1^{\alpha_1}, J_2^{\alpha_2} \), respectively, for which we assume, without loss of generality, that \( J_1^{\alpha_1}(0) = J_2^{\alpha_2}(0) = 0 \).

**Theorem 7.4.** Let \( G_1 \otimes F_1 \in L_{loc}^2(E_1, D_1, dl^{\alpha_1}), \) \( F_1^* \otimes G_1^* \in L_{loc}^2(D_1, E_1, d[l^{\alpha_1}]), \) \( F_2 \otimes G_2 \in L_{loc}^2(D_2, E_2, dl^{\alpha_2}) \), where \( D_1, D_2, E_1, E_2 \in \mathcal{P}(\mathbb{N}) \) and \( \alpha_1, \alpha_2 \in \mathcal{F}_m, m \in \mathbb{N}^* \). Suppose that \( J_1^{\alpha_1}, J_2^{\alpha_2} \) is a filtered adapted process and that \( J^{\alpha_1} F_2 \otimes G_2, G_1 \otimes F_1 J_2^{\alpha_2}, G_1 F_1 F_2 \otimes G_2 \) and \( G_1 \otimes F_1 F_2 G_2 \) are locally square integrable with respect to \( dl^{\alpha_2}, dl^{\alpha_1}, d[l^{\alpha_1}, l^{\alpha_2}] \) and \( d[l^{\alpha_1}, l^{\alpha_2}] \), respectively. Then

\[
    d(J_1^{\alpha_1} J_2^{\alpha_2}) = dJ_1^{\alpha_1} J_2^{\alpha_2} + dJ_1^{\alpha_1} J_2^{\alpha_2} + dJ_1^{\alpha_1} dJ_2^{\alpha_2}
\]

where

\[
\begin{align*}
    dJ_1^{\alpha_1} J_2^{\alpha_2} &= G_1 \otimes F_1 J_2^{\alpha_2} \# dl^{\alpha_1}, \\
    J_1^{\alpha_1} dJ_2^{\alpha_2} &= J_1^{\alpha_1} F_2 \otimes G_2 \# dl^{\alpha_2},
\end{align*}
\]

and the Itô correction can be written in two equivalent ways:

\[
\begin{align*}
    dJ_1^{\alpha_1} dJ_2^{\alpha_2} &= G_1 \otimes P_{\alpha_1, \alpha_2}(F_1 F_2) G_2 \# d[l^{\alpha_1}, l^{\alpha_2}] \\
    &= G_1 P_{\alpha_1, \alpha_2}(F_1 F_2) \otimes G_2 \# d[l^{\alpha_1}, l^{\alpha_2}]
\end{align*}
\]

where

\[
\begin{array}{c|cc}
    P_{\alpha_1, \alpha_2} & (m)^\ast & (m)\circ \\
    \hline
    (m) & \mathbb{P}_0 & \mathbb{P}_1 \\
    (m)\circ & \mathbb{P}_1 & \mathbb{P}_1
\end{array}
\]

**Proof.** We will restrict our attention to the Itô correction. Let \( m \) be finite and take \( \alpha_1 = (m) \) and \( \alpha_2 = (m)^\ast \). Using bi-linearity and Theorem 6.1, we obtain

\[
\begin{align*}
    dJ_1^{(m)} dJ_2^{(m)^\ast} &= G_1 \otimes F_1 F_2 \otimes G_2 \# \langle \sum_{k=1}^{m} P^{[k-1]} dA^{(k)} \sum_{l=1}^{m} dA^{(l)^\ast} P^{[l-1]} \rangle \\
    &= \sum_{k,l=1}^{m} G_1 P^{[k-1]} \otimes F_1 F_2 \otimes P^{[l-1]} G_2 \# \langle dA^{(k)} dA^{(l)^\ast} \rangle \\
    &= \sum_{k \in D_1(m) \cap D_2(m)} G_1 P^{[k-1]} \otimes F_1 F_2 P^{[k-1]} G_2 \# dA^{(0)} \\
    &= \sum_{k \in D_1 \cap D_2} G_1 \otimes P^{[k-1]} F_1 F_2 P^{[k-1]} G_2 \# dl^{(m)} \\
    &= G_1 \otimes \mathbb{P}_0(F_1 F_2) G_2 \# dl^{(m)^\ast}.
\end{align*}
\]

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Using the second way of writing the Itô correction as indicated in Theorem 6.1 we obtain a similar expression in which the differential \(dl^{(m)}\) is placed right before \(G_2\).

Let now \(\alpha_1 = (m)\circ\) and \(\alpha_2 = (m)\ast\). Then

\[
dJ_1^{(m)\circ}dJ_2^{(m)\ast} = G_1 \otimes F_1 F_2 \otimes G_2 \# \langle \sum_{k=1}^m dA^{(k)\circ} P^{[k]} , \sum_{l=1}^m dA^{(l)\ast} P^{[l-1]} \rangle
\]

\[
= \sum_{k,l=1}^m G_1 \otimes P^{[k]} F_1 F_2 \otimes P^{[k-1]} G_2 \# \langle dA^{(k)\circ} , dA^{(l)\ast} \rangle
\]

\[
= \sum_{k \in D_1(m) \cap D_2(m)} G_1 P^{[k]} F_1 F_2 \otimes P^{[k-1]} G_2 \# dA^{(k)\ast}
\]

\[
= \sum_{k \in D_1(m) \cap D_2(m)} G_1 P^{[k]} F_1 F_2 P^{[k]} \otimes G_2 \# dA^{(k)\ast} P^{[k-1]}
\]

where we used the fact that for any \(x \in \tilde{\Gamma}(\mathcal{H})\), we have \(x' = P^{[k-1]} x \in \tilde{\Gamma}(\mathcal{H}^{(k)})\) and thus

\[
x'' = dA^{(k)\ast} x' \in \tilde{\Gamma}(\mathcal{H}^{(k+1)}) \oplus \tilde{\Gamma}(\mathcal{H}^{(k)})
\]

since \(dA^{(k)\ast}\) adds color \(k\) to the given vector and thus \(P^{[k]} x'' = x''\). Therefore, we obtain

\[
dJ_1^{(m)\circ}dJ_2^{(m)\ast} = \sum_{k \in D_1 \cap D_2} G_1 P^{[k]} F_1 F_2 P^{[k]} \otimes G_2 \# (\sum_{r=1}^m dA^{(r)\ast} P^{[r-1]})
\]

\[
= G_1 P_1 (F_1 F_2) \otimes G_2 \# dl^{(m)\ast}
\]

which finishes the proof if the differential \(dl^{(m)\ast}\) is to be written right before \(G_2\). The second formula for this product as well as the two remaining cases are proved in a similar manner. If \(m = \infty\), one uses Definition 3.1.(assumption 3) and the fact that \(m\)-free processes converge strongly to free processes. 

\[\square\]

8. Stochastic differential equations

Let \(\mathcal{P}_0\) be a finite collection of subsets of the power set \(\mathcal{P}(\mathbb{N})\) which is closed under intersections. We will establish existence and uniqueness of solutions of systems of stochastic differential equations of the type

\[
dI_V = \sum_{C \cap D \cap E = V} \sum_{\eta \in T} X_{C,D}^{\eta} I_E \# dA^{\eta}
\]

\[
I_V(0) = I_V^{(0)}
\]

on \(\tilde{\mathcal{E}}_0\), where \(V \in \mathcal{P}_0\), \(X_{C,D}^{\eta}\) are suitable \((C,D)\)-adapted locally bounded biprocesses and

\[
I_V^{(0)} = I_V^{(0)} \otimes P(V) \in \mathcal{B}(h_0) \otimes \mathcal{B}(\Gamma)
\]

where \(I_V^{(0)}\) are bounded operators on \(h_0\). By \(\sum_{C \cap D \cap E = V}\) we understand the summation over all \(C,D,E \in \mathcal{P}_0\) such that \(C \cap D \cap E = V\). Thus, we suppress in the notation the fact that \(C,D,E \in \mathcal{P}_0\). This convention will also be adopted in expressions of similar
type in the sequel. Of course, the above system of stochastic differential equations should be interpreted as the system of stochastic integral equations

$$I_V(t) = I_V^{(0)} + \sum_{C \cap D \cap E = V} \sum_{\eta \in \mathcal{T}} \int_0^t X^n_{C,D}I^n_E \#dA^n$$

on $\tilde{\mathcal{E}}_0$, where $V \in \mathcal{P}_0$ and $t \geq 0$. Note that on both sides of (8.1) and (8.4) we have processes of the same type of adaptedness. The reason for this is that in a variety of interesting cases, processes of different types of adaptedness are linearly independent (see Lemma 8.3).

We will need the elementary estimate given below.

**Proposition 8.1.** Let $B(t), L(t) \in \mathcal{B}(h)$ for all $t \geq 0$, where $h$ is a separable Hilbert space and let $\mu$ be a numerically-valued measure on $\mathbb{R}^+$ of bounded variation. Suppose that the mapping $t \to B(t)y$ is strongly measurable for all $y \in h$ and locally bounded and that the mapping $t \to L(t)x$ is strongly measurable, where $x \in h$. Then

$$\int_0^t \|B(s)L(s)x\|^2 d\mu(s) \leq \sup_{0 \leq r \leq t} \|B(r)\|^2 \int_0^t \|L(s)x\|^2 d\mu(s)$$

for all $t \geq 0$.

**Proof.** Obvious. $\square$

**Theorem 8.2.** Let $X^n_{C,D} \in \mathcal{B}_{\text{loc}}(C,D)$ for all $\eta \in \mathcal{T}$ and $C,D \in \mathcal{P}_0$ and let $I_V^{(0)}$ be given by formula (8.3) for all $V \in \mathcal{P}_0$. If

$$\sum_{\eta \in \mathcal{T}} \sup_{0 \leq s \leq t} \|B^n_{C,D}(s)\|^2 < \infty$$

for all $t \geq 0$ and $C,D \in \mathcal{P}_0$, where $B^n_{C,D} = X^n_{C,D}$, then there exists a unique family of $V$-adapted regular processes $(I_V)_{V \in \mathcal{P}_0}$ satisfying equation (8.4).

**Proof.** The iterative scheme is established in the usual way. Thus, let $I_V^{(0)}$ be the zero-th order approximation of $I_V$, $V \in \mathcal{P}_0$, and let

$$I^{(n)}_V(t) = I_V^{(0)} + \sum_{C \cap D \cap E = V} \sum_{\eta \in \mathcal{T}} \int_0^t X^n_{C,D}I^{(n-1)}_E \#dA^n$$

for all $n \in \mathbb{N}$. We will show that the iterative scheme is well-defined, each $I^{(n)}_V$ is a regular $V$-adapted process and that the following estimate holds:

$$\| (I^{(n)}_V(t) - I_V^{(n-1)}(t))x \|^2 \leq \sum_{\eta \in \mathcal{T}} \sup_{0 \leq t \leq T} \|B^n_{C,D}(t)\|^2 \frac{(\nu(t))^n}{n!}$$

for each $n \in \mathbb{N}$, $t \in [0,T]$ and $x \in \mathcal{M}_0$, where $|\mathcal{P}_0|$ denotes the cardinality of $\mathcal{P}_0$,

$$l_0 = \max_{E \in \mathcal{P}_0} \|I^{(0)}_E\|^2$$

$$k_T = \max_{C,D \in \mathcal{P}_0} k_T(C,D)$$

$$k_T(C,D) = \sum_{\eta \in \mathcal{T}} \sup_{0 \leq t \leq T} \|B^n_{C,D}(t)\|^2$$

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and \( \nu_u(t) = \nu_u([0, t]) \).

First of all, \( X_{C, D}^n I_E^{(0)} \) is locally square integrable w.r.t. \( dA^n \), \( \eta \in \mathcal{T} \), for all \( C, D, E \in \mathcal{P}_0 \) since \( X_{C, D}^n \in \mathcal{B}_{loc}(D, E) \) and \( I_E^{(0)} \) is a bounded operator (Proposition 8.1 is used to show that the seminorms given by (5.1) are finite). Therefore, \( I_V^{(1)} \) will be well-defined if the estimate (8.7) holds, where 1 using Proposition 8.1 and condition (8.5), we obtain

\[
\| (I_V^{(1)}(t) - I_V^{(0)}(t)) x \|^2 
\]

\[
\leq |\mathcal{P}_0|^6 \max_{C \cap D \cap E = V} \left\{ \sum_{\eta \in \mathcal{T}} \int_0^t X_{C, D}^n I_E^{(0)} \# dA^n x \|^2 \right\} \]

\[
\leq 2|\mathcal{P}_0|^6 e^{\nu_u(T)} \max_{C \cap D \cap E = V} \left\{ \sum_{\eta \in \mathcal{T}} \sup_{0 \leq s \leq t} \| B_{C, D}^n(s) \|^2 \| I_E^{(0)} \|^2 \right\} \| x \|^2 \nu_u(t) 
\]

\[
\leq 2|\mathcal{P}_0|^6 e^{\nu_u(T)} k_T l_0 \| x \|^2 \nu_u(t) 
\]

using the estimate (5.1) and Proposition 8.1, which proves the estimate for \( n = 1 \). Moreover, note that \( I_V^{(1)} \) is a regular \( \mathcal{V} \)-adapted process (by Theorem 6.3).

Now, suppose that \( I_V^{(k)}, V \in \mathcal{P}_0 \), are well-defined regular \( \mathcal{V} \)-adapted processes for which the estimate (8.7) holds, where \( 1 \leq k \leq n - 1 \). Then the integrands \( X_{C, D}^n I_E^{(n-1)} \) are locally square integrable w.r.t. \( dA^n \) since \( X_{C, D}^n \in \mathcal{B}_{loc}(D, E) \) and \( I_E^{(n-1)} \) is regular. Now, using Proposition 8.1 and condition (8.7), we obtain

\[
\sum_{\eta \in \mathcal{T}} \int_0^t \| B_{C, D}^n(s) I_E^{(n-1)}(s) x \|^2 d\nu_u(s) 
\]

\[
\leq \sum_{\eta \in \mathcal{T}} \{ \sup_{0 \leq r \leq t} \| B_{C, D}^n(r) \|^2 \} \{ \max_{0 \leq s \leq t} \| I_E^{(n-1)}(s) x \|^2 \} \nu_u(t) < \infty 
\]

for all \( C, D, E \in \mathcal{P}_0, t \geq 0, \nu \in \mathcal{D}_0 \) and \( u \in \mathcal{H}_0 \). This implies that \( I_V^{(n)} \) is a well-defined, regular \( \mathcal{V} \)-adapted process since each term in the sum on the RHS of equation (8.7) is a regular \( \mathcal{V} \)-adapted process. Moreover, using Proposition 8.1 and then the inductive assumption, we arrive at

\[
\| (I_V^{(n)}(t) - I_V^{(n-1)}(t)) x \|^2 
\]

\[
\leq |\mathcal{P}_0|^6 \max_{C \cap D \cap E = V} \left\{ \sum_{\eta \in \mathcal{T}} \int_0^t X_{C, D}^n I_E^{(n-1)} - I_E^{(n-2)} \# dA^n x \|^2 \right\} 
\]

\[
\leq 2|\mathcal{P}_0|^6 e^{\nu_u(T)} \max_{C \cap D \cap E = V} \left\{ \sum_{\eta \in \mathcal{T}} \sup_{0 \leq s \leq t} \| B_{C, D}^n(s) \|^2 \right\} \times \int_0^t \| (I_E^{(n-1)}(s) - I_E^{(n-2)}(s)) x \|^2 d\nu_u(s) 
\]

\[
\leq 2^n |\mathcal{P}_0|^6 e^{\nu_u(T)} k_T l_0 \| x \|^2 \left( \frac{\nu_u(t)}{n!} \right)^n 
\]
This shows that the estimate (8.7) holds for all natural \( n \). Therefore, the strong limit 
\( I_V(t)x = s - \lim_{n \to \infty} I_V^{(n)} \) exists for all \( V \in \mathcal{P}_0, t \geq 0 \) and \( x \in \mathcal{M}_0 \) and defines a regular \( V \)-adapted process which satisfies equation (8.4).

Suppose there exist two solutions \((I_VV) \in \mathcal{P}_0, (I'V) \in \mathcal{P}_0\) of equation (8.4), where \( I_V, I'_V \) are regular \( V \)-adapted processes for all \( V \in \mathcal{P}_0 \). Then, setting \( I_V - I'_V = Z_V \) for all \( V \in \mathcal{P}_0 \), we have

\[
Z_V(t)x = \sum_{C \cap D \cap E = V} \sum_{\eta \in T} \int_0^t X_{C,D}^\eta Z_E dA^\eta x
\]

for all \( x \in \mathcal{M}_0, V \in \mathcal{P}_0 \) and \( 0 \leq t \leq T \). This gives (by Proposition 8.1)

\[
\max_V \|Z_V(t)x\|^2 \leq |\mathcal{P}_0|^6 \max_{C \cap D \cap E = V} \sum_{\eta \in T} \int_0^t \|B_{C,D}^\eta(s)Z_E(s)x\|^2 d\nu(s)
\]

and therefore, by Gronwall’s inequality (see [P]), we obtain

\[
\max_V \|Z_V(t)x\|^2 \leq 0
\]

for \( 0 \leq t \leq T \), which gives \( Z_V(t) = 0 \) on \( \tilde{E}_0 \) for \( 0 \leq t \leq T \) and \( V \in \mathcal{P}_0 \). Since \( T \) is arbitrary, this implies that \( Z_V(t) = 0 \) for all \( t \geq 0 \). \( \square \)

Using Theorem 8.2, which deals with systems of equations, we can establish existence and uniqueness of the solution of the stochastic differential equation

\[
dI = \sum_{\eta \in T} X^\eta I^\eta dA^\eta \quad \text{(8.8)}
\]

\[
I(0) = I^{(0)} \quad \text{(8.9)}
\]

in the class of \( \mathcal{P}_0 \)-adapted processes, where \( X^\eta \) is a suitable \((\mathcal{P}_0, \mathcal{P}_0)\)-adapted biprocess for all \( \eta \in T \). We just need to impose a condition on \( \mathcal{P}_0 \) which will ensure that processes with different types of adaptedness are linearly independent.

**Lemma 8.3.** Let \( \mathcal{P}_0 = \{V_1, \ldots, V_n\} \) be a finite subset of \( \mathcal{P}(N) \) such that

\[
V_i \setminus (V_1 \cup \ldots \cup V_{i-1}) \neq \emptyset \quad \text{(8.10)}
\]

for \( i = 2, \ldots, n \), and let \( Y_V \in \mathcal{A}(V) \), where \( V \in \mathcal{P}_0 \). If

\[
\sum_{V \in \mathcal{P}_0} Y_V(t)x = 0
\]

for all \( t \geq 0 \) and \( x \in \mathcal{M}_0 \), then \( Y_V(t)x = 0 \) for all \( t \geq 0 \), \( x \in \mathcal{M}_0 \) and \( V \in \mathcal{P}_0 \).

**Proof.** Let \( x = w \in \mathcal{A}(u) \). In view of the condition (8.10), \( W := V_n \) contains elements which
are not in \(V_1, \ldots, V_{n-1}\). Let us choose \(u\) in such a way that \(u_t^{(W)} \neq 0\) and \(u_t^{(V)} = 0\) for all \(V \in \{V_1, \ldots, V_{n-1}\}\). We have

\[
\sum_{V \in \mathcal{P}_0} Y_V(t) x = Y_W(t) w \varepsilon(u_t) \otimes \varepsilon(u_t^{(W)}) + \sum_{V \neq W} Y_V(t) w \varepsilon(u_t) = 0
\]

for all such \(u, t \geq 0\) and \(w \in h_0\). However, by assumption, we also must have

\[
\sum_{V} Y_V(t) x = \tilde{Y}_W(t) w \varepsilon(u_t) \otimes \varepsilon(u_t^{(W)}) = 0
\]

for all such \(u, t \geq 0\) and \(w \in h_0\). These two facts imply that

\[
\tilde{Y}_W(t) w \varepsilon(u_t) \otimes \varepsilon_0(u_t^{(W)}) = 0
\]

where

\[
\varepsilon_0(z) = \varepsilon(z) - \Omega
\]

for \(z \in \mathcal{H}_0\). Since \(\varepsilon_0(u_t^{(W)}) \neq 0\), we must have

\[
\tilde{Y}_W(t) w \varepsilon(u_t) = 0.
\]

Since \(u_t\) and \(w\) were arbitrary, it is now enough to use \(W\)-adaptedness to get

\[
Y_W(t) w \varepsilon(u) = \tilde{Y}_W(t) w \varepsilon(u_t) \otimes \varepsilon(u_t^{(W)}) = 0
\]

for all \(t \geq 0, w \in h_0\) and \(u \in \mathcal{H}_0\). Hence \(Y_W = 0\) on \(\tilde{E}_0\). We can continue this way for other sets in \(\mathcal{P}_0\) to show that \(Y_V = 0\) for all \(V \in \mathcal{P}_0\). \(\square\)

In view of Lemma 8.3, the problem of existence and uniqueness of solutions of (8.8)-(8.9) in the class of \(\mathcal{P}_0\)-adapted processes is equivalent to that of existence and uniqueness of solutions of the system of equations (8.1)-(8.2). The most natural examples of stochastic differential equations which “mix different types of adaptedness” appear in \(m\)-free calculi and will be presented below. Before we do that, let us establish another result which will be needed in Section 9.

Namely, it is desirable to establish existence and uniqueness of solutions for a more general class of equations than those given by (8.8)-(8.9). In particular, we would like to get uniqueness of solutions of the equation

\[
dI = \sum_{\eta \in \mathcal{T}} \{X^nIM^n \# dA^n + N^nIY^n \# dA^n\} \quad (8.11)
\]

\[
I(0) = I^{(0)} \quad (8.12)
\]

where

\[
X^n = \sum_{D, E \in \mathcal{P}_0} F^n_D \otimes G^n_E, \quad Y^n = \sum_{D, E \in \mathcal{P}_0} H^n_D \otimes K^n_E
\]

and

\[
M^n = \sum_{D \in \mathcal{P}_0} M^n_D, \quad N^n = \sum_{D \in \mathcal{P}_0} N^n_D
\]
under suitable assumptions on $X^n, Y^n, M^n, N^n$. For our purposes, it will be enough to establish conditions under which the solution of equations (8.11)-(8.12) is unique, if it exists. Note that they are not the most general (one can extend this result in the spirit of [Ac-Fa-Qu]) and that they also guarantee existence of a solution.

We will say that a $V$-adapted process $F_V$ is a $P(V)$-ampliation if

$$F_V(t) = \tilde{F}_V(t) \otimes P(V)$$

for all $t \geq 0$, according to the decomposition $\tilde{\Gamma} = h_0 \otimes \Gamma$.

**Theorem 8.4.** If $F^n_V, H^n_V, G^n_V, H^n_V, K^n_V, M^n_V, N^n_V$ are $P(V)$-ampliations and

$$\tilde{F}_V^n, \tilde{G}_V^n, \tilde{H}_V^n, \tilde{K}_V^n, \tilde{M}_V^n, \tilde{N}_V^n \in \mathcal{B}_{loc}(h_0)$$

for all $V \in \mathcal{P}_0$ and $\eta \in T$, then the solution of equations (8.11)-(8.12) is unique if it exists.

**Proof.** Denote by $Z$ the difference of two $\mathcal{P}_0$-adapted solutions of the above equation. Let

$$m(t) = \max_{V \in \mathcal{P}_0} \max_{W \in \mathcal{P}_0} \sup_{\|w\| \leq 1} \|Z_V(t)w\|_{u(W)}$$

for all $t \geq 0$. Then, as in the uniqueness proof of Theorem 8.2, for $0 \leq t \leq T$ and given $u \in \mathcal{H}_0$, there exists a non-negative constant $c(T, u)$ and a measure $\tau$ which is absolutely continuous w.r.t. the Lebesgue measure, such that

$$m(t) \leq c(T, u) \int_0^t m(s) d\tau_s$$

for all $0 \leq t \leq T$. By Gronwall’s inequality, we obtain $m(t) = 0$ on $[0, T]$ and this gives uniqueness of solutions. This completes the proof of the theorem. \qed

$m$-free stochastic differential equations

Assume that $m$ is finite and consider stochastic differential equations of the form

$$dI = F_1 \otimes G_1 I \# dl^{(m)} + F_2 \otimes G_2 I \# dl^{(m)*} + F_3 \otimes G_3 I \# dl^{(m)} + F_4 \otimes G_4 I \# dl^{(m)}$$

$$I(0) = I^{(0)}$$

where

$$I^{(0)} = \sum_{V \in \mathcal{P}_0^{(m)}} I_V^{(0)}$$

with $F_i \otimes G_i \in \mathcal{B}_{loc}(\mathcal{P}_0, \mathcal{P}_0)$ for $1 \leq i \leq 4$, where $\mathcal{P}_0^{(m)}$ is given by Example 2 in Section 9 and $\mathcal{B}_{loc}(\mathcal{P}_0, \mathcal{P}_0)$ denotes the space of locally bounded $(\mathcal{P}_0, \mathcal{P}_0)$-adapted biprocesses.
Theorem 8.6. Suppose that $F_i \otimes G_i \in \mathcal{B}_{\text{loc}}(\mathcal{P}_0, \mathcal{P}_0)$, $i = 1, \ldots, 4$ and that there exists $p \in \mathbb{N}$ such that the ranges of $F_i(t)$ are contained in $\overline{\Gamma(H^{(p)})}$ for all $t \geq 0$ and $i = 1, \ldots, 4$. Then

$$I(t)x =: s - \lim_{m \to \infty} I_{(m)}(t)x$$

exists for all $t \geq 0$ and $x \in \mathcal{M}_0$, and satisfies equations (8.13)-(8.14) for $m = \infty$. If $I_{(m)}(t)$ is an isometry for all $t \geq 0$, then $I(t)$ is an isometry for all $t \geq 0$.

Proof. From the assumption on the ranges of $F_i(t)$ and the iteration of solutions as in the proof of Theorem 8.2 we obtain $I_{(m)}^{(n)}(s)x \in \overline{\Gamma(H^{(p)})}$ and thus $I_{(m)}(s)x \in \overline{\Gamma(H^{(p)})}$ for all $s \geq 0, x \in \mathcal{M}_0$ and $m \in \mathbb{N}$. Therefore, by Definitions 3.1 and 3.2., there exists $q \in \mathbb{N}$ such that

$$G_\sigma(s)I_{(m)}(s)x, G_\sigma(s)I_{(n)}(s)x \in \overline{\Gamma(H^{(q)})}$$

for all $\sigma = 1, 2, 3, 4, s \geq 0$ and $n, m \in \mathbb{N}$. From this and relations (2.6)-(2.9) we infer that there exists $m \in \mathbb{N}$ such that for all $n \geq m$ we have

$$\int_0^t F_\sigma \otimes G_\sigma I_{(n)}x \#dl^{(n)} = \int_0^t F_\sigma \otimes G_\sigma I_{(n)}x \#dl^{(m)}$$

for all $x \in \mathcal{M}_0$, where $\sigma = 1, 2, 3, 4$ correspond to annihilation, creation, number and time processes, respectively. This leads to

$$(I_{(m)}(t) - I_{(n)}(t))x = \sum_{\sigma = 1}^4 \int_0^t F_\sigma \otimes G_\sigma (I_{(m)} - I_{(n)})x \#dl^{(m)}.$$
9. Unitary evolutions

In this section we establish necessary and also sufficient conditions under which the stochastic differential equation

\[ dU = \sum_{\eta \in T_0} X^\eta U#dA^\eta \]  
\[ U(0) = 1 \]

has a unique unitary solution \( U = (U(t))_{t \geq 0} \), i.e. \( U(t) \) is unitary for all \( t \geq 0 \), where \( \mathcal{P}_0 \) is a finite subset of the power set \( \mathcal{P}(N) \), \( T_0 \) is a finite subset of \( T \) and \( X^\eta \) are suitable \( (\mathcal{P}_0, \mathcal{P}_0) \)-adapted biprocesses for all \( \eta \in T_0 \).

Throughout this section we assume that \( D_0 = h_0 \) and we use the notation

\[ B^\eta(t) = \sum_{(D,E) \in \mathcal{P}_0} 1^\eta_{D,E} B^\eta_{D,E}(t) \]

where \( 1^\eta_{D,E} \) is given by (4.2) and \( B^\eta_{D,E} \models X^\eta_{D,E} \) for \( \eta \in T_0 \) and \( D, E \in \mathcal{P}_0 \). We begin with our second linear independence lemma.

**Lemma 9.1.** Let \( X^\eta \in C(\mathcal{P}_0, \mathcal{P}_0) \), where \( \eta \in T_0 \), and \( T_0, \mathcal{P}_0 \) are finite subsets of \( T \) and \( \mathcal{P}(N) \), respectively, and \( \mathcal{P}_0 \) is closed under intersections and satisfies the condition (8.10). If

(i) the map \( u \to B^\eta_{D,E}(t)w \varepsilon(u) \) is strongly continuous for each \( t \geq 0 \), \( w \in h_0 \), \( u \in \mathcal{H}_0 \), \( \eta \in T_0 \), and \( D, E \in \mathcal{P}_0 \),

(ii) \( \sum_{\eta \in T_0} \int_0^t X^\eta#dA^\eta x = 0 \) for all \( t \geq 0 \), \( x \in \mathcal{M}_0 \),

then

\[ B^\eta(t)x = 0 \]

for all \( \eta \in T_0 \), \( x \in \mathcal{M}_0 \), and \( t \geq 0 \).

**Proof.** The proof is a straightforward modification of that for adapted processes [Par] and therefore will be omitted. \( \square \)

Let us now address the question of unitarity of the solution of (9.1)-(9.2). In other words, we are looking for necessary and sufficient conditions under which \( U(t)U^*(t) = U^*(t)U(t) = 1 \) for all \( t \geq 0 \). If \( \mathcal{P}_0 \) is closed under intersections and the condition (8.10) is satisfied, then, for the solution to be unitary it is necessary that \( N \in \mathcal{P}_0 \) since we then must have

\[ \sum_{C,D \in \mathcal{P}_0} U^*_C(t)U_D(t) = \sum_{V \in \mathcal{P}_0} \sum_{C \cap D = V} U^*_C(t)U_D(t) = 1 \]
\[ \sum_{C,D \in \mathcal{P}_0} U_C(t)U^*_D(t) = \sum_{V \in \mathcal{P}_0} \sum_{C \cap D = V} U_C(t)U^*_D(t) = 1 \]

for all \( t \geq 0 \). This, by Lemma 8.3, implies that the sum over \( V \) must include \( V = N \) and that

\[ U^*_N(t)U_N(t) = U_N(t)U^*_N(t) = 1 \]
for all $t \geq 0$, whereas
\[
\sum_{C \cap D = V} U_C^*(t)U_D(t) = \sum_{C \cap D = V} U_C(t)U_D^*(t) = 0
\]
for all $t \geq 0$ and any $V \in \mathcal{P}_0$ such that $V \neq \mathbf{N}$. This means that in order to study unitarity on $\Gamma(\mathcal{H})$ one needs to include adapted biprocesses in the filtered adapted biprocesses. For that reason we will assume that $\mathcal{P}_0$ contains $\mathbf{N}$. However, in order to establish unitarity conditions, stronger conditions on $\mathcal{P}_0$ are needed. This motivates the following definition.

**Definition 9.2.** We will say that a collection $\mathcal{P}_0 \subset \mathcal{P}(\mathbf{N})$ is admissible if
\[
\mathcal{P}_0 = \{V_i, 1 \leq i \leq p\}, \quad 1 \leq p \leq \infty
\]
where $V_i$ is a proper subset of $V_{i+1}$ for all $1 \leq i \leq p$ and $V_p = \mathbf{N}$.

**Example 1.** Let $\mathcal{P}_0 = \{\mathbf{N}\}$, i.e. $\mathcal{P}_0$ consists of one filter which corresponds to adapted biprocesses, i.e. boson calculus (either with a finite or infinite number of degrees of freedom). Then $\mathcal{P}_0$ is admissible.

**Example 2.** Let $\mathcal{P}_0^{(m)} = \{V(k), \mathbf{N} : 1 \leq k \leq m + 1\}$, where $m \in \mathbf{N}$, with $V(k) = \{1, \ldots, k - 1\}$. Then $\mathcal{P}_0^{(m)}$ is finite and admissible for each $m \in \mathbf{N}$. These collections of filters appear in $m$-free calculi for finite $m$. In turn $\mathcal{P}_0^{(\infty)} = \{V(k), \mathbf{N} : 1 \leq k < \infty\}$ is an admissible collection of filters corresponding to free calculus.

**Lemma 9.3.** Let $\mathcal{P}_0 \subset \mathcal{P}(\mathbf{N})$ be admissible and finite and suppose that $X^n \in \mathcal{B}_{\text{loc}}(\mathcal{P}_0, \mathcal{P}_0) \cap \mathcal{C}(\mathcal{P}_0, \mathcal{P}_0)$ and are non-zero if and only if $\eta \in \mathcal{T}_0$, where $\mathcal{T}_0$ is a finite subset of $\mathcal{T}$ and that the continuity condition (i) of Lemma 9.1 is satisfied. Then, for the unique solution of equations (9.1)-(9.2) to be an isometry it is necessary that
\[
(B^{n\dagger}(t))^* + B^n(t) + \sum_{[\eta_1, \eta_2] = \eta} (B^{n_{\eta_1\eta_2}}(t))^*B^{n_{\eta_2}}(t) = 0
\]
for all $t \geq 0$ and $\eta \in \mathcal{T}$, where
\[
\sum_{[\eta_1, \eta_2] = \eta} := \sum_{[A^{\eta_1\eta_2} = A^n]} \quad \text{(9.4)}
\]

**Proof.** The proof is based on the filtered Itô formula. We will use elementary biprocesses, i.e. $X^n_{D,E} = F^n_D \otimes G^n_E$ for all $\eta \in \mathcal{T}_0$ and $D, E \in \mathcal{P}_0$. In all summations it is implicitly assumed that $\eta \in \mathcal{T}_0$ and $D, E \in \mathcal{P}_0$ and only additional conditions on the summation indices are shown. We have
\[
dU = \sum_{\eta} \sum_{D,E} F^n_D \otimes G^n_E U \#dA^n
\]
\[
U(0) = 1
\]
and since the number of terms in the sum is finite, it is clear that \( U^* = (U^*(t))_{t \geq 0} \) satisfies
\[
d U^*(t) = \sum_{\eta, D,E} U^*(G^\eta_E)^* \otimes (F^\eta_D)^* \# dA^\eta_t
\]
\[
U^*(0) = 1
\]

Applying the filtered Itô formula to the isometry condition \( U^*(t)U(t) = 1 \), we obtain (skipping \( t \) to save some space and choosing one way of writing the Itô correction of Theorem 6.1)
\[
0 = d U^*U + U^*dU + dU^*dU
\]
\[
= \sum_{\eta, D,E} U^*(G^\eta_E)^* \otimes (F^\eta_D)^* U \# dA^\eta_t + \sum_{\eta, D,E} U^* F^\eta_D \otimes G^\eta_E U \# dA^\eta_u
\]
\[
+ \sum_{\eta_1, \eta_2} \sum_{D_1,E_1,D_2,E_2} U^*(G^\eta_1_{E_1})^* \otimes \rho_{\eta_1,\eta_2} \left[ (F^\eta_1_{D_1})^* F^\eta_2_{D_2} \right] G^\eta_2_{E_2} U \# d[ A^{\eta_1}, A^{\eta_2} ]
\]

Note that \( U \) and \( U^* \) are \( \mathcal{P}_0 \)-adapted, i.e. in general, they contain mixed types of adaptedness. Using equation (9.3), we obtain
\[
\sum_{(W,Z) \in \mathcal{P}_0} 1^\eta_{W,Z} U^*_W \{(B^\eta)^* + B^n + \sum_{[\eta_1,\eta_2]=\eta} (B^{\eta_1}_{n_1})^* B^{n_2}_{n_2}\} U_Z = 0 \tag{9.5}
\]
for each \( \eta \in \mathcal{T}_0 \).

Since \( \mathcal{P}_0 \) is increasing, it is easy to check that we have
\[
\sum_{V \in \mathcal{P}_0} U_V(t) \sum_{W \in \mathcal{P}_0} U^*_W(t) = U_{N}(t)U^*_{N}(t) = 1
\]
and therefore, by multiplying equation (9.3) by \( U \) or \( \sum_{V \in \mathcal{P}_0} U_V(t) \) from the left and by \( U^* \) or \( \sum_{k \in V} U^*_k(t) \) from the right (that depends on \( \eta \)), we arrive at
\[
(B^{\eta_1}(t))^* + B^n(t) + \sum_{[\eta_1,\eta_2]=\eta} (B^{\eta_1}_{n_1}(t))^* B^{n_2}_{n_2}(t) = 0
\]
which ends the proof. \( \square \)

**Lemma 9.4.** Under the assumptions of Lemma 9.3, for the unique solution of equations (9.1)-(9.2) to be a co-isometry it is necessary that
\[
(B^{\eta_1}(t))^* + B^n(t) + \sum_{[\eta_1,\eta_2]=\eta} B^{n_1}_{n_1}(t)(B^{n_2}_{n_2}(t))^* = 0
\]
for all \( t \geq 0 \) and \( \eta \in \mathcal{T}_0 \).

**Proof.** The proof is similar to that of Lemma 9.3 and is based on differentiating the co-isometry condition \( U(t)U(t)^* = 1 \) and then using the filtered Itô formula. \( \square \)
THEOREM 9.5. Under the assumptions of Lemma 9.3, for the unique solution of equations (9.1)-(9.2) to be unitary, it is necessary that for all $t \geq 0$

(i) $B^{(k)}(t) + 1$ is unitary $\forall k \in \mathbb{N}$

(ii) $(B^{(k)}(t))^* + B^{(k)}(t) + (B^{(k)}(t))^*B^{(k)}(t) = 0$

(iii) $B^{(0)}(t) + (B^{(0)}(t))^*+ \sum_{k \geq 1}(B^{(k)}(t))^*B^{(k)}(t) = 0$

Proof. It is a straightforward consequence of Lemmas 9.3-9.4.

Remark. Although the unitarity conditions of Theorem 9.5 have the same form as in boson calculus, it is important that $B^n$ are, in general, $\mathcal{P}_0$-adapted and not $\mathbf{N}$-adapted processes. Note also that, in general, certain summands in (9.4) are equal to zero. This tells us which components of $(\mathcal{P}_0, \mathcal{P}_0)$-adapted biprocesses may give a non-zero contribution to the differential equation (only these enter the unitarity conditions).

THEOREM 9.6. Suppose $\mathcal{P}_0$ is admissible and finite and $X^{n}_{D,E} = F^n_{D} \otimes G^n_{E}$, with $F^n_{D}(t) = F^{n}_{D}(t) \otimes P^{(D)}$, $G^n_{E}(t) = G^{n}_{E}(t) \otimes P^{(E)}$ according to the decomposition $\Gamma = \eta_0 \otimes \Gamma$, where $F^n_{D}, G^n_{E} \in \mathcal{B}_{loc}(h_0)$ for all $n \in \mathcal{T}_0$ and $D, E \in \mathcal{P}_0$. Then the conditions (i)-(iii) of Theorem 9.5 are sufficient for the unique solution of equations (9.1)-(9.2) to be unitary.

Proof. Let

$$U(t) = 1 + \sum_{\eta} \int_{0}^{t} X^{\eta}U \# dA^{\eta}$$

for all $t \geq 0$. We will first show that if conditions (i)-(iii) of Theorem 9.5 are satisfied, then $U$ is an operator-valued isometric process. Denote $x = w \in (u), z = y \in (v)$, where $w, y \in h_0$ and $u, v \in H_0$ and let

$$I_{Z}(t) = \sum_{\eta} \sum_{D,E \in V = Z} \int_{0}^{t} F^{\eta}_{D} \otimes G^{\eta}_{E}U_{V} \# dA^{\eta}$$

for $Z \in \mathcal{P}_0$ and $t \geq 0$. We have

$$\langle U(t)x, U(t)y \rangle - \langle x, y \rangle = \sum_{\eta} \sum_{W} \int_{0}^{t} \langle U_{W}x, (B^{n})^{*}y \rangle d\mu_{W,n}$$

$$+ \sum_{\eta} \sum_{W} \int_{0}^{t} \langle x, B^{n}U_{W}y \rangle d\mu_{W,n,W}$$

$$+ \sum_{\eta_1 \in W_1} \sum_{Z} \int_{0}^{t} \langle I_{Z}x, (B^{n})^{*}I_{Z}y \rangle d\mu_{W_1,n,Z}$$

$$+ \sum_{\eta_2 \in W_2} \sum_{Z} \int_{0}^{t} \langle I_{Z}x, B^{n}U_{W_2}y \rangle d\mu_{W_2,n,Z}$$

$$+ \sum_{\eta, [n_1,n_2]} \sum_{W_1,W_2} \int_{0}^{t} \langle U_{W_1}x, (B^{n_1})^{*}B^{n_2}U_{W_2}y \rangle d\mu_{W_1,n_1,W_2,n_2}$$

where we used the filtered Itô formula. Now, using

$$U_{Z} = I_{Z}, \ Z \neq \mathbf{N}$$

$$U_{\mathbf{N}} = 1 + I_{\mathbf{N}},$$

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then replacing $Z$ in the 3rd and 4th terms by $W_2$ and $W_1$, respectively, and taking into account all cancellations, we arrive at

$$\sum_{\eta} \sum_{W_1,W_2} \int_0^t \langle U_{W_1} x, (B^\eta)^* U_{W_2} y \rangle d\mu_{W_1,W_2}^\eta$$

$$+ \sum_{\eta} \sum_{W_1,W_2} \int_0^t \langle U_{W_1} x, B^\eta U_{W_2} y \rangle d\mu_{W_1,W_2}^\eta$$

$$+ \sum_{\eta} \sum_{[\eta_1, \eta_2]=\eta} \sum_{W_1,W_2} \int_0^t \langle U_{W_1} x, (B^{\eta_1})^* (B^{\eta_2})^* U_{W_2} y \rangle d\mu_{W_1,W_2}^\eta = 0$$

in view of the isometry conditions of Lemma 9.3. Therefore, $U$ is an operator-valued isometric process.

Proceeding in a similar way, we can show that if, say

$$\mathcal{P}_0 = \{V_1, \ldots, V_n\}, \text{ where } V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{N}$$

then

$$\sum_{i=k}^n U_{V_i}(t)$$

is also an isometry for all $1 \leq k \leq n$ and $t \geq 0$. Therefore, each $U_{V_i}(t)$, $V_i \in \mathcal{P}_0$, is an operator-valued process.

Let us show that $U(t)$ is a co-isometry for all $t \geq 0$. The adjoint process $U^*(t)$ obeys the equation

$$dU^*(t) = \sum_{\eta} \sum_{D,E} U^*(t)(G^\eta_{E}(t))^* \otimes (F^\eta_{D}(t))^* \#dA^\eta_t$$

$$U^*(0) = 1.$$

Proceeding as in the isometry case, we obtain

$$\langle U^*(t)x, U^*(t)y \rangle - \langle x, y \rangle = \sum_{\eta} \sum_{W_1,W_2} \int_0^t \langle x, U_{W_1} U_{W_2}^*(B^\eta)^* y \rangle d\mu_{W_1,W_2}^\eta$$

$$+ \sum_{\eta} \sum_{W_1,W_2} \int_0^t \langle x, B^\eta U_{W_1} U_{W_2}^* y \rangle d\mu_{N,W_1,W_2}^\eta$$

$$+ \sum_{\eta} \sum_{[\eta_1, \eta_2]=\eta} \sum_{W_1,W_2} \int_0^t \langle x, B^{\eta_1} U_{W_1} U_{W_2}^* (B^{\eta_2})^* \rangle d\mu_{W_1,W_2}^\eta.$$

This equation is equivalent to the stochastic differential equation

$$d(UU^*) = \sum_{\eta} \sum_{D,E} \{ UU^*(G^\eta_{E})^* \otimes (F^\eta_{D})^* \#dA^\eta + F^\eta_{D} \otimes G^\eta_{E} UU^* \#dA^\eta \}$$

$$+ \sum_{\eta} \sum_{[\eta_1, \eta_2]=\eta} \sum_{D_1, E_1, D_2, E_2} F^\eta_{D_1} \otimes G^\eta_{E_1} UU^* (G^\eta_{E_2})^* (F^\eta_{D_2})^* \#dA^\eta.$$

Now, $U(t)U^*(t) = 1$ is a solution of this equation if the co-isometry conditions of Lemma 9.4 hold. This solution is unique in view of Theorem 8.4, which completes the proof. □
Unitarity conditions for boson calculus

Note that the unitarity conditions of Theorem 9.5 have the same form as those for boson calculus on multiple symmetric Fock space [Mo-Si] (cf. [H-P1], see also [P]). To recover them, set \( P_0 = \{ \mathbb{N} \} \), \( B^{(k)} = L_k \), \( B^{(k)} = S_k - 1 \) and \( R = H \).

Unitarity conditions for \( m \)-free calculi

Let us show that Theorem 9.5 covers unitarity conditions for \( m \)-free calculi. Let \( m \in \mathbb{N} \) and let \( P_0 = \mathcal{P}_0^m \) (see Example 2 in this section).

For a given \( P_0 \)-adapted process \( F = \sum_{V \in P_0} F_V \), let

\[
[F]_k = \sum_{V \in P_0} F_V
\]

where \( k \in \mathbb{N} \). Using this notation, we can write

\[
B^{(k)}(t) = F_1(t)P^{[k-1]}[G_1(t)]_k
\]

\[
B^{(k)}(t) = [F_2(t)]_kP^{[k-1]}G_2(t)
\]

\[
B^{(k)}(t) = [F_3(t)]_kP^{[k+1]}[G_3(t)]_k
\]

\[
B^{(0)}(t) = F_4(t)P^mG_4(t).
\]

It is important to notice that \( [P[k]F]_k = P^{[k+1]}[F]_k \) which gives the third equation above. The other ones just express relations between two notations.

In particular, when \( F_3(t) = G_3(t) = 0 \) and \( F_2(t) = G_1(t) = 1 \) for all \( t \geq 0 \), the above conditions can be written in an equivalent form

\[
F_4(t)P^mG_4(t) + G_4^*(t)P^mF_4^*(t) + G_2^*(t)G_2(t) = 0 \quad (9.6)
\]

\[
F_1(t)P^m + G_2^*(t)P^m = 0 \quad (9.7)
\]

which are the \( m \)-truncated versions of the unitarity conditions in [K-Sp].

From the considerations of Section 8 it follows that if \( F_i, G_i, i = 1, \ldots, 4 \), are linear combinations of \( P^V \)-ampliations for \( V \in \mathcal{P}_0 \) which satisfy the assumptions of Theorem 9.6, then the strong limit of unitary solutions \( s-\lim_{m \to \infty} U_m \) exists and is unitary. Therefore, it is sufficient that the \( m \)-truncated unitarity conditions hold for all \( m \in \mathbb{N} \) which is equivalent to

\[
F_4(t)G_4(t) + G_4^*(t)F_4^*(t) + G_2^*(t)G_2(t) = 0
\]

\[
F_1(t) + G_2^*(t) = 0
\]

i.e. the unitarity conditions for the free calculus.

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