ASYMPTOTICS OF RESONANCES FOR A SCHRÖDINGER OPERATOR WITH MATRIX VALUES

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1. Introduction

We obtain the asymptotics for the counting function of resonances for a matrix valued Schrödinger operator in dimension one. This theorem is a generalization of the result of Zworski [?], using ideas from Froese [?], both of which only treat the scalar case.

There has been much work in proving upper and lower bounds for the number of resonances in various situations (for a review of this subject see [?]), but only [?,?] obtain results about asymptotic. Here we generalize these results to matrix potentials which appear in such physically important models as the Born Oppenheimer approximation, Dirac operator etc. Resonances are defined as poles of the continuation of the scattering matrix. In physics, the main interest is in resonances close to the real axis, since these can be measured in experiments. From a mathematical point of view, their asymptotics are important in order to establish trace formula.

Compared to [?], where the resolvent was used and the potential V was a scalar, the proof here uses the determinant of the scattering matrix to characterize the resonances. We use the Born approximation to approximate the scattering matrix. This approximation, based on WKB solution, is widely used in inverse scattering theory and is a good tool here as well.

For Schrödinger operators in dimensions greater than one, sharp upper bounds on the number of resonances are known [?], but only weak lower bounds exist [?].

The result for the scalar case is

**Theorem 1.1.** Apart from a set of density zero for the Lebesgue measure, all resonances of $-\Delta + V$ are contained in arbitrarily small sectors about the real axis. Let $n_\pm(r)$ denote the number of resonances of modulus less than $r$ contained in some sector about $\mathbb{R}^\pm$ then

$$n_\pm(r) = C(V)r + o(r).$$

$$C(V) = \frac{1}{\pi} \left( \sup_{x,y \in \text{supp} V} |x - y| \right).$$

\(\ast\).
The quantity $C(V)$ get no clear generalization in the matrix case.

We study the resonances of the operator $-\Delta + V$ in dimension one, where $V$ is a matrix with value $V(x) \in M_n(C)$ for $x \in \mathbb{R}$, real $\nabla = V$ and of compact support and bounded (so $V \in L^1$). Define

$$t_{i,j} = \inf_{x \in \text{supp} V_{ij}} x, \quad u_{i,j} = \sup_{x \in \text{supp} V_{ij}} x,$$

We obtain

**Theorem 1.2.** Apart from a set of density zero for Lebesgue measure, all resonances of $-\Delta + V$ are contained in arbitrarily small sectors about the real axis. Let $n_\pm(r)$ denote the number of resonances of modulus less than $r$ contained in some sector about $\mathbb{R}^\pm$ then there exist a constant depending only on $V$ such that

$$n_\pm(r) = C(V)r + o(r),$$

$$C(V) \leq \frac{1}{\pi} \left( \sum_i \sup_{\sigma \in S_n} \sum_{p, p'} t_{i,p} - u_{p,\sigma(i)} \right).$$

In some case the constant $C(V)$ could be determined,

(H1) : if $V$ is a triangular matrix then

$$C(V) = \frac{1}{\pi} \left( \sum_i u_{i,i} - t_{i,i} \right).$$

(H2) : if $V$ is a such that for all $i, j V_{i,j} \geq 0$,

$$\sup_{\sigma \in S_n} \sum_i \sup_p t_{i,p} - u_{p,\sigma(i)}$$

is attained for only one $\sigma_0 \in S_n$ and

$$\sup_{\sigma \in S_n} \sum_i \sup_{p \neq p'} t_{i,p} - u_{p',\sigma(i)} < \sum_i \sup_p t_{i,p} - u_{p,\sigma_0(i)}$$

and

$$\sup_p \sum_i t_{i,p} - u_{p,\sigma_0(i)}$$

is attained for only one $p$, denote by $p(i)$, then

$$C(V) = -\sup_{\sigma \in S_n} \sum_i \sup_p \sum_i t_{i,p} - u_{p,\sigma(i)}.$$

The plan of the article is as follow. The two first sections recall known results which we will use. The first estimates the number of zero of an analytic function already used in [?], and it allowed him to give asymptotic when $V$ is scalar but does not have compact support. The second is essentially the Paley Wiener Theorem. Section 4 concerns the approximation of the transmission matrix (Born Approximation). Section 5 is the definition of the transmission matrix. Section 6 just recall some simplifications due to the symmetry of the potential. The section 7 contain the proof using all the tools of the previous sections.
2. Zero of entire functions

Let denote by $S(k)$ the scattering matrix. The zeroes of $\det S(k)$ are the complex conjugates of the resonances of our problem so it suffice to count the number of zeroes of $\det S(k)$ in the upper half plane. The function $\det S(k)$ is analytic in the upper half plane so we going to use Theorem 2.1.

A function is said to be of exponential type in the upper half plane if

$$\limsup_{r \to \infty} \sup_{|z|=r, \Im z \geq 0} \frac{\ln|F(z)|}{r} < \infty.$$ 

We call is type the number $\limsup_{r \to \infty} \sup_{|z|=r, \Im z \geq 0} \frac{\ln|F(z)|}{r}$. Denote by $n(r, \theta_1, \theta_2)$ the number of zeroes of $F$ in the sector

$$\{z; |z| \leq r, \theta_1 \leq \Im z \leq \theta_2\}.$$

The following result appears in [?] p 243 and p 251.

**Theorem 2.1.** Let $F$ be an holomorphic function on the half plane $\{z, \arg z \geq 0\}$ of exponential type which satisfies

$$\int_{-\infty}^{\infty} \ln+ |F(x)| \frac{1}{1 + x^2} dx < \infty.$$ 

Then the function $\tau(\theta)$ defined by $\tau(\theta) = \limsup_{r \to \infty} \ln|F(re^{i\theta})|/r$ satisfies

$$\tau(\theta) = \tau |\sin \theta| \text{ for } \theta \in [0, \pi],$$

where $\tau$ is constant. For $\theta_1, \theta_2 \in [0, \pi]$, 

$$\lim_{r \to \infty} \frac{n(r, \theta_1, \theta_2)}{r} = \frac{1}{2\pi} \left[ \tau'(\theta_1) - \tau'(\theta_2) + \int_{\theta_1}^{\theta_2} \tau(\theta)d\theta \right] = 0.$$

For $\theta > 0$

$$\lim_{r \to \infty} \frac{n(r, 0, \theta)}{r} = \frac{\tau}{\pi}.$$

3. An variant of Paley-Wiener

We quote another lemma from [?], that we need, which is a variant of the Paley-Wiener Theorem.

**Lemma 3.1.** Assume $W \in L^\infty$, and $\text{supp}W \subset [-1, 1]$ but $\text{supp}W$ contain in no smaller interval. Let $f(k, x)$ be holomorphic in $k$ in the lower half plane, and satisfies uniformly in $x$

$$|f(k, x)| \leq C/|k|.$$ 

Then $\int_{-1}^{1} e^{\pm ikx}W(x)(1 + f(k, x))$ is of type at least 1 in the lower half plan.
4. APPROXIMATION USING WKB SOLUTIONS

To apply the theorem 2.1 to the function $\det S(k)$, we are going to give an approximation of the scattering matrix by computing the transmission matrix to compute $\tau$, and prove that $\det S(k)$ satisfies the hypotheses of Theorem 2.1. The following considerations will give asymptotics in $k$ of solutions of the equation

$$(1) \quad (-\Delta + V - k^2)u = 0,$$

or equivalently, for the solutions of the system,

$$(2) \quad \left( \begin{array}{cc} 1 & \frac{1}{i} \frac{d}{dx} \\ 0 & 1 \end{array} \right) v = \left( \begin{array}{cc} -k + \frac{V}{2k} & -\frac{V}{2k} \\ \frac{V}{2k} & k - \frac{V}{2k} \end{array} \right) v.$$

where $v_1 = -iku + \frac{d}{dx} u$ and $v_2 = iku + \frac{d}{dx} u$, which is a convenient reformulation for studying the behavior at infinite of the solution.

Denote the standard basis of $\mathbb{R}^2$ by $\{e_j\}_{j \in \{1, \cdots, 2n\}}$, and set $v^{-}_j = e^{-i k x} e_j$ and $v^{+}_j = e^{i k x} e_{j+n}$ for $j \in \{1, \cdots, n\}$, these form a base of the solution of the system

$$(3) \quad \left( \begin{array}{cc} 1 & \frac{1}{i} \frac{d}{dx} \\ 0 & 1 \end{array} \right) v = \left( \begin{array}{cc} 0 & 0 \\ 0 & k \end{array} \right) v.$$

The function $v = \sum_{j=1}^{n} \beta^{-}_j v^{-}_j + \beta^{+}_j v^{+}_j$ is a solution of the system (2) if and only if

$$(4) \quad \left( \begin{array}{cc} 1 & \frac{1}{i} \frac{d}{dx} \\ 0 & 1 \end{array} \right) \left( \beta^{-} \right) = \left( \begin{array}{cc} \frac{V}{2k} \gamma^{-} & -e^{2ikx}V \gamma^{-} \\ \frac{-V}{2k} & -\frac{V}{2k} \end{array} \right) \left( \beta^{+} \right).$$

Writing

$$\gamma^{+}_j = \beta^{+}_j e^{2ikx}, \quad \gamma^{-}_j = \beta^{-}_j,$$

$$(5) \quad \left( \begin{array}{cc} 1 & \frac{1}{i} \frac{d}{dx} \\ 0 & 1 \end{array} \right) \left( \gamma^{-} \right) = \left( \begin{array}{cc} \frac{V}{2k} \gamma^{-} & \frac{-V}{2k} \\ \frac{-V}{2k} & \frac{-V}{2k} \end{array} \right) \left( \gamma^{+} \right).$$

We also write this in expanded form as

$$(6) \quad \frac{d}{dx} \gamma^{-}_j = \sum_k \frac{V_{jk}}{2k} \gamma^{-}_k + \sum_k -\frac{V_{jk}}{2k} \gamma^{+}_k,$$

$$\frac{d}{dx} \gamma^{+}_j - 2k \gamma^{+}_j = \sum_k \frac{V_{jk}}{2k} \gamma^{-}_k + \sum_k -\frac{V_{jk}}{2k} \gamma^{+}_k.$$

Let $x_0 \in \mathbb{R}$ such that $x_0 > \text{sup}\{x, x \in \text{supp} V\}$. This system (6) can be solved by iteration as follows. Let $\gamma^{-}_j = \sum_{n=0}^{\infty} \gamma^{-}_{j,n}$, $\gamma^{+}_j = \sum_{n=0}^{\infty} \gamma^{+}_{j,n}$, where we define an initial condition

$$(7) \quad \gamma^{+}_{j,0}(x) = 0, \quad \gamma^{-}_{j,0}(x) = c_j$$

or $\gamma^{+}_{j,0}(x) = d_j e^{2ikx}$, $\gamma^{-}_{j,0}(x_0) = 0$

and an iteration procedure

$$\gamma^{-}_{j,n+1}(x) = J^{+}_j (\gamma^{-}_n - \gamma^{+}_n)(x),$$

$$\gamma^{+}_{j,n+1}(x) = J^{-}_j (\gamma^{-}_n - \gamma^{+}_n)(x),$$
The series \( k \) is a matrix holomorphic in \( J^{-} \) and \( J^{+} \) are linear operator acting on vector

\[
J^{-}(u)(x) = i \int_{x_{0}}^{x} \sum_{k} \frac{V_{jk}(y)}{2k} u_{k}(y) dy = -i \int_{y_{0} \geq x} \sum_{k} \frac{V_{jk}(y)}{2k} u_{k}(y) dy
\]

\[
J^{+}(u)(x) = (8)i \int_{x_{0}}^{x} e^{2ik(x-y)} \sum_{k} \frac{V_{jk}(y)}{2k} u_{k}(y) dy = -i \int_{y_{0} \geq x} e^{2ik(x-y)} \sum_{k} \frac{V_{jk}(y)}{2k} u_{k}(y) dy.
\]

(This solve then the relation

\[
\frac{d}{dx} \gamma_{j,n+1}^{+} = \sum_{k} \frac{V_{jk}}{2k} \gamma_{k,n}^{+} + \sum_{k} \frac{V_{jk}}{2k} \gamma_{k,n}^{+},
\]

\[
\frac{d}{dx} \gamma_{j,n+1}^{-} - 2k \gamma_{j,n+1}^{+} = \sum_{k} \frac{V_{jk}}{2k} \gamma_{k,n}^{-} + \sum_{k} \frac{V_{jk}}{2k} \gamma_{k,n}^{+}
\]

We remark that the \( \gamma_{j,n+1}(x) \) are constant in \( x \) for \( x \geq x_{0} \). The equation could be see also as

\[
\gamma_{j,n+1}^{-}(x) - \gamma_{j,n+1}^{+}(x) = (J^{-} - J^{+})(\gamma_{n}^{-} - \gamma_{n}^{+})(x),
\]

\[
\gamma_{j,n+1}^{+}(x) = J^{+}(\gamma_{n}^{-} - \gamma_{n}^{+})(x).
\]

So we get the expression

\[
\gamma_{j,n}^{-}(x) - \gamma_{j,n}^{+}(x) = [(J^{-} - J^{+})^{n}](\gamma_{0}^{-} - \gamma_{0}^{+})(x), \text{ for } n \geq 0
\]

\[
\gamma_{j,n}^{-}(x) = J^{+}(J^{-} - J^{+})^{n-1}(\gamma_{0}^{-} - \gamma_{0}^{+})(x) \text{ for } n \geq 1,
\]

\[
\gamma_{j,n}^{-}(x) = J^{-}(J^{-} - J^{+})^{n-1}(\gamma_{0}^{-} - \gamma_{0}^{+})(x) \text{ for } n \geq 1.
\]

**Theorem 4.1.** Let \( \text{Im} k \leq 0 \), \( |k| \) large enough, the formal series \( \sum_{n} \gamma_{j,n}^{\pm} \) are absolutely convergent. We have for each \( n \in \mathbb{N} \) and \( N \in \mathbb{N} \)

\[
\sum_{n=0}^{\infty} \gamma_{j,n}^{\pm} - \sum_{n=0}^{N} \gamma_{j,n}^{\pm} = O(|k|^{-N-1}),
\]

The series \( \sum_{n=N}^{\infty} (J^{-} - J^{+})^{n} \) are convergent and theirs kernels, denote \( k_{N}(x,y) \), satisfy for \( N \geq 1 \)

\[
k_{N}(x,y) = B_{N}(k,x,y)V(y) \text{ with } B_{N}(k,x,y)
\]

is a matrix holomorphic in \( k \) that satisfy

\[
|B_{N}(k,x,y)| \leq \frac{\|V\|_{L_{1}}^{N-1}}{(2|k|)^{N}}
\]

uniformly in \( x \).

Proof:

Let \( \text{Im} k \leq 0 \). We get \( |e^{2ik(x-y)}| \leq 1 \) for all \( y > x \).

But the kernel of \( J^{-} - J^{+} \) is

\[
k(x,y) = -i1_{y \geq x}(1 - e^{2ik(x-y)}) \frac{V(y)}{2k}
\]

which satisfy \( |k(x,y)| \leq \frac{|V(y)|}{2k} \) uniformly in \( x \). So \( ||J^{-} - J^{+}||_{\infty} \leq \frac{1}{|2k|} \int_{-\infty}^{\infty} \sum_{p,m} |V_{p,m}(y)| dy \).
The kernel of $J^-$ is

$$k^-(x, y) = -i1_{y \geq x} \frac{V(y)}{2k}$$

which satisfy $|k^-(x, y)| \leq \frac{|V(y)|}{2k}$ uniformly in $x$. So $\|J^-\|_\infty \leq \frac{1}{|2k|} \int_{-\infty}^{\infty} \sum_{p,m} |V_{p,m}|(y)dy$.

The kernel of $J^+$ is

$$k^+(x, y) = -i1_{y \geq x} e^{2ik(x-y)} \frac{V(y)}{2k}$$

which satisfy $|k^+(x, y)| \leq \frac{|V(y)|}{2k}$ uniformly in $x$. So $\|J^+\|_\infty \leq \frac{1}{|2k|} \int_{-\infty}^{\infty} \sum_{p,m} |V_{p,m}|(y)dy$.

Form which the results (9) and (10) holds.

**Remark 4.2.** The equation (5) is equivalent to

$$\left( \begin{array}{c} d \frac{d}{dx} \\ 0 \end{array} \right) \left( \begin{array}{c} \gamma^- - \gamma^+ \\ \gamma^+ \\ \gamma^- \\
\gamma^+ \\
\end{array} \right) = \left( \begin{array}{cc} 0 & -2k \\ \frac{V}{2k} & 0 \end{array} \right) \left( \begin{array}{c} \gamma^- - \gamma^+ \\ \gamma^+ \\ \gamma^- \\
\gamma^+ \\
\end{array} \right).$$

We note, that the preceding equations for $\gamma_{j,n}^\pm$ are similar to the ones obtained by an exact WKB construction for scalar Schrödinger equations, see for example the work of C. Gerard and A. Grigis [?] or T. Ramond [?].

5. THE TRANSMISSION MATRIX

Let $\text{Im}k \leq 0$, $k \neq 0$, and $c = t(c_1, \cdots, c_n)$. Let $v$ a solution of the equation (2) that behave like

$$v(x) = \sum_{j=1}^{n} \gamma_j^-(x, c) e^{-ikx} e_j + \gamma_j^+(x, c) e^{-2ikx} e_j e_{j+n},$$

with $\gamma_j^-(\infty, c) = c_j$ and $\gamma_j^+(\infty, c) = 0$ and $\gamma_j^\pm(x, c)$ are solution of (6) (here $x_0 = \infty$).

let $d = t(d_1, \cdots, d_n)$. Let $w$ be solution of the equation (2) that behave like

$$w(x) = \sum_{j=1}^{n} \gamma_j^-(x, de^{2iky}) e^{-ikx} e_j + \gamma_j^+(x, de^{2iky}) e^{-2ikx} e_j e_{j+n},$$

with $\gamma_j^-(\infty, de^{2iky}) = 0$ and $\gamma_j^+(\infty, de^{2iky}) = de^{2iky}$.

The transmission matrix of the system (2) with is also the one of the system (1) and (4) is by definition the matrix of the linear map associating to $t(c,d)$ the vector

$$\lim_{x \rightarrow -\infty} \gamma_j^-(x, c) + \lim_{x \rightarrow -\infty} \gamma_j^-(x, de^{2iky})$$

$$\lim_{x \rightarrow -\infty} \gamma_j^+(x, c) e^{-ikx} + \lim_{x \rightarrow -\infty} \gamma_j^+(x, de^{2iky}) e^{-2ikx}.$$

The transmission matrix is analytic in $k$. Let denote the transmission matrix

$$T(k) = \left( \begin{array}{cc} \tau_{11}(k) & \tau_{12}(k) \\ \tau_{21}(k) & \tau_{22}(k) \end{array} \right),$$
where $\tau_{ij}$ are $n \times n$ matrix given by the expression

$$
\tau_{11}(k)(c) = c + \sum_{n=1}^{\infty} \lim_{x \to -\infty} J^-(J^- - J^+)^{n-1}(c)
$$

$$
\tau_{12}(k)(d) = -\sum_{n=1}^{\infty} \lim_{x \to -\infty} J^-(J^- - J^+)^{n-1}(de^{2iky})
$$

$$
\tau_{21}(k)(c) = \sum_{n=1}^{\infty} \lim_{x \to -\infty} e^{-2ikx} J^+(J^- - J^+)^{n-1}(c)
$$

$$
\tau_{22}(k)(d) = d - \sum_{n=1}^{\infty} \lim_{x \to -\infty} e^{-2ikx} J^+(J^- - J^+)^{n-1}(de^{2iky})
$$

For $k \in \mathbb{R}$, and $d \in \mathbb{R}^n$, \( \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} w \end{pmatrix} \) is a solution of (2) (we use here $V = V$) that behave like

$$
\begin{pmatrix} w(x) \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} \gamma_j^-(x, de^{2iky})e^{2ikx}e^{-ikx}e_j + \gamma_j^+(x, de^{2iky})e^{2ikx}e^{-ikx}e_j, & \text{with } \gamma_j^- (\infty, de^{2iky}) = 0 \text{ and } \gamma_j^+ (\infty, de^{2iky})e^{2ikx} = d. \end{pmatrix}
$$

This proves that

$$
\begin{pmatrix} \gamma_j^+(x, de^{2iky})e^{-2ikx} = \gamma_j^-(x, d) \end{pmatrix}
$$

i.e. $\tau_{22}(k) = \tau_{11}(k)$ for $k \in \mathbb{R}$, so $\tau_{22}(k) = \overline{\tau_{11}(k)}$ for $k \in \mathbb{C}$. We get also using the same symmetry $\tau_{12}(k) = \tau_{21}(k)$.

6. Symmetry of the Problem

The scattering matrix is defined form the transmission matrix by the relation

$$
S = \begin{pmatrix}
\tau_{11} - \tau_{12} \tau_{22}^{-1} \tau_{21} & \tau_{12} \tau_{22}^{-1} \\
-\tau_{22}^{-1} \tau_{21} & \tau_{22}^{-1}
\end{pmatrix}.
$$

We remark that

$$
S = \begin{pmatrix}
\tau_{12} & 0 \\
0 & \tau_{22}^{-1}
\end{pmatrix} \begin{pmatrix}
\tau_{12}^{-1} & \tau_{22}^{-1} \\
0 & \tau_{11}^{-1}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
-\tau_{21} & I
\end{pmatrix}.
$$

So

$$
\det S(k) = \det \tau_{11} \times \det \tau_{22}^{-1} = \frac{\det \tau_{11}(k)}{\det \tau_{11}^{-1}(k)}.
$$

In particular

$$
|\det S(k)| = 1 \text{ for } k \in \mathbb{R}.
$$

The zero of $\det S(k)$ are the complex conjugate of the pole of $\det S$. The pole of $\det S$ are the resonances of our problem so we just want to count the number of zero of $\det S(k)$ in the upper half plan. The function $\det S(k)$ is holomorphic in the upper half plan so we going to use the theorem 2.1. We
need only to show that \( \det S \) is a function of exponential type in the full plan, the other hypothesis is satisfy using (13). By (12) the type of \( \det S \) in a direction is given by the type of \( \det \tau_{11}(k) \) minus the type of \( \det \tau_{22}(k) \) in the same direction.

Let define a Wronskian for our problem in some situation. Let \( P \) be a projector on a space of dimension \( p \) and \( \mathcal{P} \) the projector defined by
\[
\mathcal{P} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} P u_1 \\ P u_2 \end{pmatrix}.
\]
Denote by \( P \) the matrix of \( \mathcal{P} \). \( W_P(u,v)(x) = \langle \mathcal{P} u(x), \mathcal{P} v(x) \rangle \) if \( u \) and \( v \) are solution of the system (2), and if \( \mathcal{P} V = \mathcal{P} V P \) then the Wronskian is constant in \( x \). This imply that if \( \mathcal{P} V = \mathcal{P} V P \) the transmission matrix satisfy the relation
\[
\mathcal{T}(k) \begin{pmatrix} 0 \\ -t P P \end{pmatrix} \mathcal{T}(k) = \begin{pmatrix} 0 \\ -t P P \end{pmatrix}.
\]
So we obtained \( |\det \mathcal{P} T(k)\mathcal{P}| = 1 \).

7. EXPONENTIAL TYPE OF THE SCATTERING MATRIX

Using the result (9) for \( \Im k < 0 \) and \( |k| \) big enough, we obtain
\[
\tau_{11}(k)(c) = c + O\left(\frac{1}{k}\right),
\]
so \( \det \tau_{11}(k) \) is of type 0 in the lower half plan. We have
\[
\tau_{22}(k)(d) = d + \frac{i}{2k} \int V(z) ddz + \frac{1}{(2k)^2} \int e^{-2ikz} V(z) 1_{y > z} V(y) de^{2iky} dydz - \frac{1}{(2k)^2} \int V(z) 1_{y > z} V(y) ddydz + \frac{1}{(2k)^2} \int e^{-2ikz} V(z) B_2(z, y) V(y) de^{2iky} dydz.
\]
The type of
\[
d + \frac{i}{2k} \int V(z) ddz - \frac{1}{(2k)^2} \int \int V(z) 1_{y > z} V(y) ddydz
\]
is zero in the lower half plan. The type of
\[
\int \int e^{-2ikz} \sum_p V_{i,p}(z) 1_{y > z} V_{p,j}(y) de^{2iky} dydz
\]
is in the lower half plan at most \( \sup_p -u_{i,p} + t_{p,j} \) in the lower half plan.
The type of
\[
\frac{1}{(2k)^2} \int \int e^{-2ikz} \sum_{p,p'} V_{i,p}(z) B_2(p,p')(z, y) V_{p',j}(y) de^{2iky} dydz
\]
is at most \( \sup_{p,p'} -u_{i,p} + t_{p',j} \) in the lower half plan. Taking the determinant this give that \( \det \tau_{22}(k) \) is of exponential type in the lower half plan with type at most
\[
\sup_{\sigma \in S_n} \sum_i \sup_{p,p'} -u_{i,p} + t_{p',\sigma(i)}.
\]
We also get the relation \( \overline{\det S(k)} = \frac{1}{\det S(k)} \) form the expression (12). So \( \det S \) is of exponential type in \( \{ \text{Im} k > 0 \} \), in the real axis also by (13), as a conclusion \( \det S \) is of exponential type in all the plan. So we have proof the first part of Theorem 1.2.

Now let study the type of \( \det S \) in the direction \(-i\mathbb{R}^+\), this type is the constant \( \pi C(V) \) of Theorem 1.2.

Proof of theorem 1.2 if \( V \) is triangular. Let write \( V(x) = D(x) + N(x) \) then any product of the form \( V(x_1)V(x_2)...V(x_n) \) is equal to \( D(x_1)D(x_2)...D(x_n) \) plus a nilpotent matrix \( N(x_1, \cdots, x_n) \). In particular this apply to the kernel of product of operators \( J^+ J^- \) and \( J^- J^+ \). Comparing the transition matrix for \( V \) and \( D \) this give that they exist nilpotent matrix \( N_{11}, N_{12}, N_{21}, N_{22} \) such that

\[
\tau_{11}(V)(k)(c) = \tau_{11}(D)(k)(c) + N_{11}(k)
\]
\[
\tau_{22}(V)(k)(c) = \tau_{22}(D)(k)(c) + N_{22}(k)
\]
\[
\tau_{12}(V)(k)(c) = \tau_{12}(D)(k)(c) + N_{12}(k)
\]
\[
\tau_{21}(V)(k)(c) = \tau_{21}(D)(k)(c) + N_{21}(k)
\]

So

\[
\det \tau_{11}(V)(k) = \det \tau_{11}(D)(k)(c)
\]

and

\[
\det \tau_{11}(V)(k) = \det \tau_{11}(D)(k)(c).
\]

So the type for the matrix \( V \) is the same as the type for a diagonal matrix \( D \). Then using the result of [?" or using the wronskian we get

\[
\tau_{22}(D)_{ii} \tau_{22}(D)_{ii} - \tau_{12}(D)_{ii} \tau_{21}(D)_{ii} = 1.
\]

The type of \( \tau_{22} \) in the lower half plan is the same as the type of \( \tau_{12}(\tau_{21})_{ii} \). Using the expression of \( \tau_{12}, \tau_{21} \) and using Lemma 3.1 we find that the type of \( \tau_{22}(D)_{ii} \) is \(-u_{i,i} + t_{i,i}\). So

\[
C(V) = C(D) = \frac{1}{\pi} \left( \sum_i u_{i,i} - t_{i,i} \right).
\]

Proof of theorem 1.2 if \( V \) satisfy (H2). We have

\[
(\tau_{22}(k))_{ij} = \delta_{ij} + \frac{1}{(2k)^2} \int V_{ij}(z)dz - \frac{1}{(2k)^2} \int \sum_p V_{i,p}(z)1_{y>z}V_{p,j}(y)dydz
\]
\[
+ \frac{1}{(2k)^2} \int e^{-2ikz} \sum_i V_{i,p}(z)1_{y>z}(1 + B_{2,p,p}(k, z, y))V_{p,j}(y)e^{2iky}dydz
\]
\[
+ \frac{1}{(2k)^2} \int e^{-2ikz} \sum_p \sum_{p \neq p'} V_{i,p}(z)B_{2,p,p'}(k, z, y)V_{p',j}(y)e^{2iky}dydz.
\]

The type of

\[
\frac{1}{(2k)^2} \int e^{-2ikz} V_{i,p(i)}(z)1_{y>z}(1 + B_{2,p,p}(k, z, y))V_{p(i),\sigma_0(i)}(y)e^{2iky}dydz
\]
is at least $t_{i,p(i)} - u_{p(i),\sigma_0(i)}$ in the direction $-i\mathbb{R}^+$. The proof is contain in the inequality, let $k = -ik'$ with $k' > 0$,

\begin{equation}
\begin{split}
\int_z \int_y e^{-2k'z}V_{i,p}(z)1_{y > z}(1 + B_{2,p,p}(k, z, y))V_{p,j}(y)e^{2iky}dydz \\
\geq \int_{z \leq t_{i,p} + \epsilon} \int_{y \geq u_{p(i),\sigma_0(i)} - \epsilon} e^{-2k'(t_{i,p} + \epsilon)}V_{i,p}(z)\frac{1}{2}V_{p,j}(y)e^{2k'(u_{p(i),\sigma_0(i)} - \epsilon)}dydz \\
\geq e^{-2k'(t_{i,p} + \epsilon)}e^{2k'(u_{p(i),\sigma_0(i)} - \epsilon)}C(\epsilon)
\end{split}
\end{equation}

with $C(\epsilon) > 0$ and independent of $k$. The type of

$$\frac{1}{(2k)^2} \int \int e^{-2ikz} \sum_{p \neq p'} V_{i,p}(z)B_{2,p,p'}(k, z, y)V_{p',j}(y)de^{2iky}dydz$$

is at most $\sup_{p \neq p'} t_{i,p} - u_{p',j}$ in the lower half plan. Using the hypotheses (H2) this give that $\tau_{22}(k)$ is of exponential type in the lower half plan with type

$$\sum_i \sup_p t_{i,p} - u_{p,\sigma_0(i)}.$$