CALDERÓN-ZYGJMUND ESTIMATES FOR QUASILINEAR ELLIPTIC DOUBLE OBSTACLE PROBLEMS WITH VARIABLE EXPONENT AND LOGARITHMIC GROWTH

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Abstract. Quasilinear elliptic double obstacle problems with variable exponent and logarithmic growth are studied. We obtain a global Calderón-Zygmund estimate for such an irregular obstacle problem by proving that the gradient of the solution is as integrable as both the nonhomogeneous term and the gradient of the associated double obstacles under minimal regularity requirements on the elliptic operator over the boundary of the nonsmooth domain.

1. Introduction. We are concerned with regularity of solutions for nonlinear elliptic equations that are not of variational form. If there is a function $f = f(x, z)$ whose gradient is the nonlinearity $A(x, \xi)$ of an equation, $D\xi f(x, \xi) = A(x, \xi)$, then the equation is the Euler-Lagrange equation corresponding to the Lagrangian $f = f(x, z)$, and the solution of the equation is a minimizer of the associated energy functional. However, if there is not such a function, we do not simply apply variational methods directly to the problem. Here we consider a very general problem that is not necessarily of variational form.

In this paper we are under the frame of function spaces from a natural outgrowth of both an Orlicz space and a variable exponent space. More precisely we are dealing with a highly nonlinear elliptic equation in divergence form whose nonlinearity $A(x, \xi)$ indirectly relates to the associated energy functional $f = f(x, \xi)$ like

$$f(x, \xi) \approx A(x, \xi) \cdot \xi \approx |\xi|^p(x) \log(e + |\xi|) \quad (x \in \Omega, \ \xi \in \mathbb{R}^n).$$

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A main point of the present article is the obstacle problem. It is a field in the mathematical study of variational inequalities. The purpose is to identify the density of some quantity whose boundary is governed and which is constrained by a given obstacle. Its applications include fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control and financial mathematics. Recently it has been extensively studied. We refer to [4, 7, 8, 17, 26] for the elliptic case, while to [3, 5, 6, 12, 13, 27] for the parabolic case.

Returning to our problem (1), a Calderón-Zygmund type estimate is obtained in [24] for the corresponding single obstacle problem whose solution is constrained to be greater than a given single obstacle. As a natural follow-up of the study with the same spirit, we investigate the double obstacle problem whose solution is constrained to remain between given two obstacles, proving that the gradient of the solution is as integrable as the double obstacles as well as the nonhomogeneous term in divergence form.

There have been some notable results in the literature of the double obstacle problem. For the existence issue, we refer to [2, 19]. For the regularity estimates including Hölder regularity, we mention [11, 16, 20, 25]. We also cite [9] for the Calderón-Zygmund type estimates for nonlinear elliptic equations of $p$-Laplacian type with polynomial growth of $p$-Laplacian type. Comparing with the previous works mentioned above, the present work treats a very general elliptic equation with nonstandard growth of a kind of variable exponent coupled with a logarithmic type in order to establish an optimal Calderón-Zygmund theory by identifying a possibly minimal regularity requirement on the nonlinearity $A(x, \xi)$ as well as on the boundary $\partial \Omega$ of the bounded domain with the desired higher maximality estimate.

Our proof is based on the so-called large-M-inequality principle introduced in [1] alongside a geometric approach used for the boundary regularity in [10]. A point is to derive the desired comparison estimate in Lemma 3.2. To this end we first reduce the given double obstacle problem to a localized single one, and then freeze the variable exponent $p(\cdot)$ to compare the resulting single obstacle problem with the limiting problem having an Orlicz growth of $L^p \log L$ type.

The paper is organized as follows. In the next section we introduce our double obstacle problem to state the main results. Section 3 provides comparison estimates in double obstacle problems. In section 4 we give the proof of our main theorem.

2. Main theorem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open domain. Throughout the paper, we assume that $p : \Omega \rightarrow (1, \infty)$ is a continuous function with

$$1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty$$

for some constants $\gamma_1, \gamma_2$. Then with a generalized $N$-function

$$\varphi(x, s) = s^{p(x)} \log(e + s) \quad (x \in \Omega, s \geq 0),$$

we deal with the Musielak-Orlicz space $L^\varphi(\Omega)$ comprising of measurable functions $w : \Omega \rightarrow \mathbb{R}$ for which

$$\int_{\Omega} \varphi(x, |w|) \, dx < \infty$$

equipped with its Luxemburg norm

$$\|w\|_{L^\varphi(\Omega)} = \inf \left\{ \mu > 0 : \int_{\Omega} \varphi \left( x, \frac{|w|}{\mu} \right) \, dx \leq 1 \right\}.$$
The Musielak-Orlicz-Sobolev space $W^{1,\varphi}(\Omega)$ is the set of all measurable functions $w \in L^\varphi(\Omega)$ whose distributional gradient $Dw$ again belongs to $L^\varphi(\Omega,\mathbb{R}^n)$ with the norm

$$\|w\|_{W^{1,\varphi}(\Omega)} = \|w\|_{L^\varphi(\Omega)} + \|Dw\|_{L^\varphi(\Omega,\mathbb{R}^n)}.$$ 

We denote by $W_0^{1,\varphi}(\Omega)$ $W_0^{1,\varphi}(\Omega)$ to mean the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$. For more details, we refer to [22, 15].

We are now in a position to introduce a nonlinear elliptic problem with double obstacles under consideration in this paper. With two functions $\psi_1$, $\psi_2 \in W^{1,\varphi}(\Omega)$ such that $\psi_1 \leq \psi_2$ a.e. in $\Omega$ and $\psi_1 \leq 0 \leq \psi_2$ on $\partial\Omega$, a convex admissible set is

$$\mathcal{A}_0(\Omega) = \left\{ \phi \in W_0^{1,\varphi}(\Omega) : \psi_1 \leq \phi \leq \psi_2 \text{ a.e. in } \Omega \right\}. \quad (4)$$

If $u \in \mathcal{A}_0(\Omega)$ satisfies the following variational inequality

$$\int_{\Omega} A(x, Du) \cdot (u - \phi) \, dx \leq \int_{\Omega} H(x, F) \cdot (u - \phi) \, dx \quad (\phi \in \mathcal{A}_0(\Omega)), \quad (5)$$

then $u$ is called to be a weak solution to (5). Here the nonlinearity $A = A(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in $x$, differentiable in $\xi \neq 0$, and satisfies

$$\begin{align*}
&|A(x, \xi)| + |D_\xi A(x, \xi)||\xi| \leq L|\xi|^{p(x)-1} \log(e + |\xi|), \\
&D_\xi A(x, \xi)\eta \cdot \eta \geq \nu|\xi|^{p(x)-2} \log(e + |\xi|)|\eta|^2
\end{align*} \quad (6)$$

for almost all $x \in \mathbb{R}^n$, every $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{R}^n$, and some constants $0 < \nu \leq L < \infty$. The nonhomogeneous term $H = H(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|H(x, \xi)| \leq L|\xi|^{p(x)-1} \log(e + |\xi|), \quad (7)$$

where $F : \Omega \to \mathbb{R}^n$ is the associated vector-valued function with $|F| \in L^\varphi(\Omega)$.

According to the classical theory for the double obstacle problem with a monotone and coercive nonlinearity, there exists a unique weak solution to (5), as follows from Theorem 3.2 in [25].

Throughout this paper, we write $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$ for $y \in \mathbb{R}^n$ and $r > 0$, $B_r = B_r(0)$ and

$$\Omega_r(y) = B_r(y) \cap \Omega \quad \text{and} \quad B^+_r(y) = B_r(y) \cap \{x_n > 0\}.$$ 

We also denote by

$$w_U = \int_U w(x) \, dx = \frac{1}{|U|} \int_U w(x) \, dx,$$

to mean the integral average of a locally integrable function $w$ over a bounded set $U \subset \mathbb{R}^n$.

To obtain the required Calderón-Zygmund type estimate for the double obstacle problem (4)-(7), it is required to impose some extra regular assumptions on $A, p(\cdot)$ and $\partial\Omega$, which we now present as follows.

**Assumption 2.1.** Let $0 < \delta < \frac{1}{8}$ and $R > 0$. We say that $(A, \Omega, p(\cdot))$ is $(\delta, R)$-vanishing if the followings hold:

(i) \[ \sup_{0 < r \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_r(y)} N(A, B_r(y))(x) \, dx \leq \delta, \]
where

\[ N(A, B_r(y))(x) := \sup_{\xi \in \mathbb{R}^{n-1}(0)} \left| \frac{A(x, \xi)}{|x|^{|p(x)-1| \log(e + |\xi|)}} - \int_{B_r(y)} \frac{A(x, \xi)}{|x|^{|p(x)-1| \log(e + |\xi|)}} \, dx \right|. \]

(ii) For any \( y \in \partial \Omega \) and for every number \( r \in (0, R] \), there exists a coordinate system depending on \( y \) and \( r \) such that in this new coordinate system, \( y \) is the origin and

\[ B_r \cap \{ x_n > \delta r \} \subset B_r \cap \Omega \subset B_r \cap \{ x_n > -\delta r \}. \]

(iii) \( p(\cdot) \) admits a nondecreasing function \( \omega : [0, \infty) \to [0, \infty) \) with

\[ \sup_{0 < r \leq R} \omega(r) \log \left( \frac{1}{r} \right) \leq \delta \]

as a modulus of continuity.

Remark 1. Note that the boundary of a domain satisfying (ii), especially called a \((\delta, R)\)-Reifenberg flat domain, is so irregular that it can go beyond the Lipschitz category but it still satisfies measure density condition like

\[ \sup_{0 < r \leq R} \frac{|B_r(y)|}{|B_r(y) \cap \Omega|} \leq \left( \frac{2}{1 - \delta} \right)^n \leq \left( \frac{16}{7} \right)^n \]

under which Sobolev embedding theorem and Sobolev extension theorem hold, see [18, 21, 28].

The main result of this paper is the following.

Theorem 2.2. Let \( u \in A_0(\Omega) \) be the weak solution to the variational inequality (5) under the basic structural assumptions (6) and (7). Suppose that

\[ \varphi(x, |F|), \varphi(x, |D\psi_1|), \varphi(x, |D\psi_2|) \in L^q(\Omega) \quad \text{for} \quad q \in (1, +\infty). \]

Then there exists a small constant \( \delta_0 = \delta_0(n, \gamma_1, \gamma_2, \nu, L, q) > 0 \) such that for every \( \delta \in (0, \delta_0] \), if \((A, \Omega, p(\cdot))\) is \((\delta, R)\)-vanishing for some \( 0 < \delta < 1 \) as in Assumption 2.1, then \( \varphi(x, |Du|) \in L^q(\Omega) \) with the estimate

\[ \int_{\Omega} \varphi(x, |Du|)^q \, dx \leq c M^{n(q-1)} \int_{\Omega} \Psi^q \, dx, \]

where

\[ \Psi(x) = \varphi(x, |F|) + \varphi(x, |D\psi_1|) + \varphi(x, |D\psi_2|) + 1, \]

\[ M = \int_{\Omega} \Psi \, dx + 1 \]

and \( c \) is a positive constant depending only on \( n, \gamma_1, \gamma_2, \nu, L, q, \Omega \) and \( \omega(\cdot) \).

Remark 2. Comparing with previous regularity results as in [24], there are two improvements in the present article. One is from a single obstacle to double obstacles. The other is from a local estimate to a global one. We also point out that it is so general enough to cover a large of class of non-variational problems.

For the sake of simplicity of notation, we denote \( c \) to mean a universal constant that can be computed in terms of known quantities such as \( \nu, L, n, \gamma_1, \gamma_2, q, \Omega \) and \( \omega(\cdot) \). It varies from line to line in the context.
3. Estimates in double obstacle problems. This section provides comparison estimates in double obstacle problems which will be used later.

Recalling (2) and (3), we first introduce some basic properties of the $N$-function $\varphi$. Its complementary function is defined as

$$
\varphi^*(x, t) = \sup_{s > 0} \{ st - \varphi(x, s) \} = \int_0^t (\varphi')^{-1}(x, s) ds,
$$

where $(\varphi')^{-1}(x, s) = \sup\{ s \in [0, \infty) : \varphi'(x, s) \leq s \}$. Note that there exist $s_0 > 1$ and $s_1 > 2$ depending on $\gamma_1$ and $\gamma_2$ such that $\varphi(x, \theta_1 s) \leq 2s_0 \theta_1^{1+\log_2 s_1} \varphi(x, s)$ and $\varphi(x, \theta_2 s) \leq s_1 \theta_2^{\log_2 s_1} \varphi(x, s)$ for any $0 < \theta_1 \leq 1 < \theta_2 < \infty, s \geq 0$ and $x \in \Omega$. In particular, we observe that

$$
\varphi(x, s + t) \leq s_1 2^{\log_2 s_1 - 1} (\varphi(x, s) + \varphi(x, t)) \quad (s, t \geq 0, \ x \in \Omega), \tag{12}
$$

and that for any $\kappa > 0$ there exists a constant $c(\kappa) = c(\gamma_1, \gamma_2, \kappa) > 0$ satisfying

$$
s \cdot t \leq \kappa \varphi(x, s) + c(\kappa) \varphi^*(x, t) \quad (s, t \geq 0, \ x \in \Omega), \tag{13}
$$

which is Young’s inequality in the generalized Orlicz space. We also remark that for $s > 0$ and $x \in \Omega$, there is a constant $c = c(\gamma_1, \gamma_2) > 1$ such that

$$
\frac{1}{c} \varphi(x, s) \leq \varphi^* \left(x, \frac{\varphi(x, s)}{s}\right) \leq c \varphi(x, s). \tag{14}
$$

In light of (6), we find that

$$
A(x, \xi) \cdot \xi \geq c \varphi(x, |\xi|) \quad \text{for all } x, \xi \in \mathbb{R}^n, \tag{15}
$$

for some constant $c = c(n, \nu, \gamma_1, \gamma_2) > 0$, and that for any $\kappa > 0$ there exists $c = c(n, \gamma_1, \gamma_2, \nu, \kappa) > 0$ such that

$$
\varphi(x, |\xi_1 - \xi_2|) \leq \kappa \left( \varphi(x, |\xi_1|) + \varphi(x, |\xi_2|) \right) + c \left( A(x, \xi_1) - A(x, \xi_2) \right)(\xi_1 - \xi_2) \quad (x, \xi_1, \xi_2 \in \mathbb{R}^n). \tag{16}
$$

We now state and prove the following energy estimate for the double obstacle problem (4)-(7).

**Lemma 3.1.** Let $u \in A_0(\Omega)$ be the weak solution to (5) with (6) and (7). Then there holds

$$
\int_{\Omega} \varphi(x, |Du|) \, dx \leq c \int_{\Omega} \left( \varphi(x, |F|) + \varphi(x, |D\psi_1|) + \varphi(x, |D\psi_2|) \right) \, dx \tag{17}
$$

for some constant $c = c(\gamma_1, \gamma_2, \nu, L) > 0$.

**Proof.** We take $\phi = \psi_1^+ - \psi_2^- \in A_0(\Omega)$ as a test function in (5) to obtain that

$$
\int_{\Omega} A(x, Du) \cdot D(u - \psi_1^+ + \psi_2^-) \, dx \leq \int_{\Omega} H(x, F) \cdot D(u - \psi_1^+ + \psi_2^-) \, dx,
$$

where...
where $\psi_1^+ = \max\{\psi_1, 0\}$ and $\psi_2^- = \max\{-\psi_2, 0\}$. Then by (6), (7), (13)-(15), we have for any $\kappa > 0$

$$
\int_\Omega \varphi(x, |Du|) \, dx 
\leq c \int_\Omega A(x, Du) \cdot Du \, dx 
\leq c \int_\Omega A(x, Du) \cdot D(\psi_1^+ - \psi_2^-) \, dx + c \int_\Omega H(x, F) \cdot D(u - \psi_1^+ + \psi_2^-) \, dx 
\leq c \int_\Omega \frac{\varphi(x, |Du|)}{|Du|} (|D\psi_1| + |D\psi_1|) \, dx + c \int_\Omega \frac{\varphi(x, |F|)}{|F|} (|Du| + |D\psi_1| + |D\psi_1|) \, dx 
\leq c \kappa \int_\Omega \varphi(x, |Du|) \, dx + c \int_\Omega \left( \varphi(x, |F|) + \varphi(x, |D\psi_1|) + \varphi(x, |D\psi_2|) \right) \, dx.
$$

Then we choose sufficiently so small $\kappa > 0$ in order to get the desired result (17).

We now turn to a localization for the desired comparison estimates. With $0 < R < 1$ given, let $0 < 8\delta < R$ and assume that $\Omega_{4r} = \Omega_{4r}(0)$ satisfies

Either \( B_{4r} \subset \Omega \) or \( B_{4r}^+ \subset \Omega_{4r} \subset B_{4r} \cap \{ x \in \mathbb{R}^n : x_n > -8\delta r \} \),

and

$$
\omega(4r) \log \left( \frac{1}{4r} \right) \leq \delta
$$

where $0 < \delta < \frac{1}{8}$ is to be determined later. With the weak solution $u \in \mathcal{A}$ of (5) satisfying

$$
\int_{\Omega_{4r}} \varphi(x, |Du|) \, dx \leq \lambda \quad \text{and} \quad \int_{\Omega_{4r}} \Psi \, dx \leq \delta \lambda
$$

for the $\Psi$ given as in (10) and some $\lambda > 1$ to be selected, let $v \in \mathcal{A}_u(\Omega_{4r})$ be the weak solution to the following single obstacle problem

$$
\int_{\Omega_{4r}} A(x, Dv) \cdot D(v - \phi) \, dx \leq \int_{\Omega_{4r}} A(x, D\psi_2) \cdot D(v - \phi) \, dx \quad \text{for all } \phi \in \mathcal{A}_u(\Omega_{4r}),
$$

where

$$
\mathcal{A}_u(\Omega_{4r}) = \left\{ \phi \in u + W_0^{1,\infty}(\Omega_{4r}) : \phi \geq \psi_1 \text{ a.e. in } \Omega_{4r} \right\}.
$$

Then we have the following comparison estimate.

**Lemma 3.2.** For any $0 < \epsilon < 1$, there exists a constant $\delta = \delta(n, \epsilon, \gamma_1, \gamma_2, \nu, L) > 0$ such that if (18) and (21) hold, then we have

$$
\int_{\Omega_{4r}} \varphi(x, |Dv|) \, dx \leq c \lambda \quad \text{and} \quad \int_{\Omega_{4r}} \varphi(x, |Du - Dv|) \, dx \leq c \lambda
$$

for some constant $c = c(n, \gamma_1, \gamma_2, \nu, L) > 0$.

**Proof.** By taking $\phi = u \in \mathcal{A}_u(\Omega_{4r})$ in (22) and applying (21), we find

$$
\int_{\Omega_{4r}} \varphi(x, |Dv|) \, dx \leq c \int_{\Omega_{4r}} \varphi(x, |Du|) \, dx + c \int_{\Omega_{4r}} \varphi(x, |D\psi_2|) \, dx \leq c \lambda.
$$
On the other hand, we take \( \phi = \min \{v, \psi_2\} = v - (v - \psi_2)^+ \in A_u(\Omega_{4r}) \) in (22) to get
\[
\int_{\Omega_{4r}} \left( A(x, Dv) - A(x, D\psi_2) \right) \cdot D(v - \psi_2)^+ \, dx \leq 0.
\]
This estimate and (16) imply that for any \( \kappa_1 > 0 \),
\[
\int_{\Omega_{4r}} \varphi(x, |D(v - \psi_2)^+|) \, dx \leq \kappa_1 \int_{\Omega_{4r}} \left( \varphi(x, |Dv|) + \varphi(x, |D\psi_2|) \right) \, dx.
\]
Thus it follows from (21) and (23) that
\[
\int_{\Omega_{4r}} \varphi(x, |D(v - \psi_2)^+|) \, dx \leq c\kappa_1 \lambda.
\]
Letting \( \kappa_1 \to 0 \), we get \( v \leq \psi_2 \) a.e. in \( \Omega_{4r} \). Consequently, we have
\[
v - u \in W^{1,p}_0(\Omega_{4r}) \quad \text{and} \quad \psi_1 \leq v \leq \psi_2 \quad \text{a.e. in} \; \Omega_{4r}.
\]
We next extend \( v \) to \( \Omega \) by \( u \) to see from that \( v \in A_0(\Omega) \). Then we substitute \( \phi = v \) into (5) and \( \phi = u \) into (22), respectively, we obtain
\[
\int_{\Omega_{4r}} \left( A(x, Dv) - A(x, Du) \right) \cdot D(v-u) \, dx \leq \int_{\Omega_{4r}} \left( A(x, D\psi_2) - H(x, F) \right) \cdot D(v-u) \, dx.
\]
In light of (16), (21) and (23), we discover that for any \( \kappa_2 > 0 \),
\[
\int_{\Omega_{4r}} \varphi(x, |Dv - Du|) \, dx \\
\leq \kappa_2 \int_{\Omega_{4r}} \left( \varphi(x, |Dv|) + \varphi(x, |Du|) \right) \, dx \\
+ c(\kappa_2) \int_{\Omega_{4r}} \left( A(x, Dv) - A(x, Du) \right) \cdot D(v-u) \, dx \\
\leq c\kappa_2 \lambda + c(\kappa_2) \int_{\Omega_{4r}} \left( A(x, D\psi_2) - H(x, F) \right) \cdot D(v-u) \, dx.
\]
(24)
On the other hand, it follows from (13), (14) and (21) that for any \( \kappa_3 > 0 \),
\[
\int_{\Omega_{4r}} \left( A(x, D\psi_2) - H(x, F) \right) \cdot D(v-u) \, dx \\
\leq c\kappa_3 \int_{\Omega_{4r}} \varphi(x, |Dv - Du|) \, dx + c(\kappa_3) \int_{\Omega_{4r}} \varphi(x, |D\psi_2|) \, dx + c(\kappa_3) \int_{\Omega_{4r}} \varphi(x, |F|) \, dx \\
\leq c\kappa_3 \int_{\Omega_{4r}} \varphi(x, |Dv - Du|) \, dx + c(\kappa_3)\delta \lambda.
\]
(25)
Combining (24) and (25) and then choosing \( \delta, \kappa_2, \kappa_3 > 0 \) sufficiently small, we finally finish the proof.

Once we have derived the comparison estimate in the single obstacle problem (22), the remaining comparison estimates with the reference problems follow from the existing regularity results, see [23, 24], which we now state.

**Lemma 3.3.** For any \( 0 < \epsilon < 1 \) and \( 0 < R < 1 \), there exist small constants \( \delta = \delta(n, \gamma_1, \gamma_2, \nu, L, \epsilon) > 0 \) and \( R_0 = R_0(n, \gamma_1, \gamma_2, \nu, L, \omega(\cdot)) \) with \( 0 < 2R_0 \leq R \) such
that if (18), (19), (20) and (21) hold, then there exist \( h \in W^{1,\varphi}(\Omega_{4r}) \cap W^{1,\varphi}_{x_M}(\Omega_{2r}) \) and \( w \in W^{1,\varphi}_{x_M}(\Omega_{2r}) \) such that

\[
\int_{\Omega_{4r}} \varphi(x, |Dv - Dh|) \, dx \leq \varepsilon \lambda,
\]

\[
\int_{\Omega_{2r}} \varphi_{x_M}(|Dh|) \, dx \leq \varepsilon \lambda \text{ and } \|\varphi_{x_M}(|Dh|)\|_{L^\infty(\Omega_r)} \leq c_1 \lambda
\]

for some positive constant \( c_1 = c_1(n, \gamma_1, \gamma_2, \nu, L) \), whenever \( 0 < 4Mr \leq R_0 \). Here \( M \) is given in (11), \( \rho(x, M) = \sup_{\Omega_{4r}} \rho(x) \) and \( \varphi_{x_M}(\cdot) = \varphi(x, \cdot) \).

4. **Proof of Theorem 2.2.** We start our proof with the following technical lemma.

**Lemma 4.1.** [14] Let \( \phi : [R_1, 2R_1] \to [0, \infty) \) be a function such that

\[
\phi(r_1) \leq \frac{1}{2} \phi(r_2) + B_0 (r_2 - r_1)^{-\beta} + A_0 \quad \text{for } R_1 \leq r_1 < r_2 \leq 2R_1,
\]

where \( B_0, A_0 \geq 0 \) and \( \beta > 0 \). Then

\[
\phi(R_1) \leq c(\beta) \left( B_0 R_1^{-\beta} + A_0 \right).
\]

**Proof of Theorem 2.2.** We now assume that \( (A, \Omega, p(\cdot)) \) is \( (\delta, R) \)-vanishing as Assumption 2.1, where \( 0 < \delta < 1 \) is given while \( \delta \in (0, \frac{1}{8}) \) is to be specified later. We also recall \( \Psi \) and \( M \) from (10) and (11), respectively.

**Step 1.** Let \( x_0 \in \Omega \) and \( \rho \leq r_1 \leq r_2 \leq 2\rho \) with \( 0 < M\rho \leq R_0 \), where \( R_0 \) appears in Lemma 3.3. We write

\[
\lambda_0 = \left( \frac{9600 \rho}{7(r_2 - r_1)} \right)^n \left( \int_{\Omega_{2\rho}(x_0)} \varphi(x, |Du|) \, dx + \frac{1}{\delta} \int_{\Omega_{2\rho}(x_0)} \Psi \, dx \right),
\]

and

\[
E(\lambda, \Omega_{r_i}) = \{ x \in \Omega_{r_i}(x_0) : \varphi(x, |Du|) > \lambda \} \quad (\lambda \geq \lambda_0, \ i = 1, 2).
\]

For each fixed \( y \in E(\lambda, \Omega_{r_i}) \) and every \( r \in (0, r_2 - r_1] \), we define

\[
\tau(r) = \int_{\Omega_{r}(y)} \varphi(x, |Du|) \, dx + \frac{1}{\delta} \int_{\Omega_{\rho}(y)} \Psi \, dx.
\]

Simply enlarging the domain of integration and using (8), we find that for any \( r \leq r_2 - r_1 \),

\[
\tau(r) \leq \left( \frac{9600 \rho}{7(r_2 - r_1)} \right)^n \left( \int_{\Omega_{2\rho}(x_0)} \varphi(x, |Du|) \, dx + \frac{1}{\delta} \int_{\Omega_{2\rho}(x_0)} \Psi \, dx \right) = \lambda_0 \leq \lambda.
\]

On the other hand, by Lebesgue’s differentiation theorem, we see from (27) and (28) that for sufficiently small \( r > 0 \), \( \tau(r) > \lambda \). Therefore, we can pick a maximal radius \( r_y \in (0, \frac{r_2 - r_1}{300}] \) such that

\[
\tau(r_y) = \lambda \text{ and } \tau(r) < \lambda \quad (r_y < r \leq r_2 - r_1).
\]

We now consider the collection \( B_3 \) of all such subsets \( \Omega_{r_y}(y) \). By Vitali covering lemma, we extract a sub-collection \( \{\Omega_{r_i}(y_i)\}_{i \in \mathbb{N}} \in B_3 \) with \( r_i = r_y \), such that five times enlarged balls \( \Omega_{5r_i}(y_i) \) cover almost all \( E(\lambda, \Omega_{r_i}) \) and \( \{\Omega_{r_i}(y_i)\}_{i \in \mathbb{N}} \) is pairwise disjoint. In other words,

\[
\Omega_{r_i}(y_i) \cap \Omega_{r_j}(y_j) = \varnothing, \quad \text{whenever } i \neq j, \quad \text{and } E(\lambda, \Omega_{r_i}) \subset \bigcup_{i \in \mathbb{N}} \Omega_{5r_i}(y_i) \cup \text{null set}.
\]

(30)
In addition, by (29) we observe

$$|\Omega_{r_i}(y_i)| \leq \frac{2}{\lambda} \int_{\Omega_{r_i}(y_i) \cap (\varphi(x, |Du|) > \frac{1}{\lambda} \Psi(x))} \varphi(x, |Du|) \, dx + \frac{2}{\lambda} \int_{\Omega_{r_i}(y_i) \cap (\Psi > \frac{1}{\lambda})} 1 \Psi(x) \, dx. \quad (31)$$

**Step 2.** We first assume that $B_{20r_i}(y_i) \not\subset \Omega$. Then we have a boundary point $y_i \in B_{20r_i}(y_i) \cap \partial \Omega$. From Assumption 2.1 (ii), there exists a coordinate system, depending only on $y_i$ and $r_i$, such that in this new coordinate system

$$\begin{align*}
\begin{cases}
  z_i = y_i, & y_i + 272 \delta r_i (0, \cdots, 0, 1) \text{ is the origin,} \\
  B_{236r_i}^+(0) \subset \Omega_{236r_i}(0) \subset B_{236r_i}(0) \cap \{z \in \mathbb{R}^n : z_n > -544 \delta r_i\}. 
\end{cases}
\end{align*} \quad (32)$$

Note that $|z_i| \leq 20r_i + 272 \delta r_i < 54r_i$ to discover that

$$\Omega_{5r_i}(z_i) \subset \Omega_{59r_i}(0) \subset \Omega_{236r_i}(0) \subset \Omega_{290r_i}(z_i). \quad (33)$$

Then it follows from (29) and (32) that

$$\begin{align*}
\int_{\Omega_{236r_i}(0)} \varphi(z, |Du|) \, dz + \frac{1}{\delta} \int_{\Omega_{236r_i}(0)} \Psi(z) \, dz \\
\leq \frac{|B_{290r_i}(z_i)|}{|B_{236r_i}(0)|} \left( \int_{\Omega_{290r_i}(z_i)} \varphi(z, |Du|) \, dz + \frac{1}{\delta} \int_{\Omega_{290r_i}(z_i)} \Psi(z) \, dz \right) \\
\leq 2^{n+1} \lambda. \quad (34)
\end{align*}$$

In light of (32) and (34), we are under the hypothesis of Lemma 3.2 and Lemma 3.3. Therefore, we have functions $v_i, h_i, w_i$ for each $i \in \mathbb{N}$ and a constant $c_1 \geq 1$ such that

$$\begin{align*}
\int_{\Omega_{236r_i}(0)} \varphi(x, |Du - Dv_i|) \, dx \leq \epsilon 2^{n+1} \lambda, & \quad \int_{\Omega_{236r_i}(0)} \varphi(x, |Dv_i - Dh_i|) \, dx \leq \epsilon 2^{n+1} \lambda, \\
\int_{\Omega_{118r_i}(0)} \varphi_{x, h_i}(|Dh_i - Dw_i|) \, dx \leq \epsilon 2^{n+1} \lambda \quad \text{and} \quad \|\varphi_{x, h_i}(|Dw|)\|_{L^\infty(\Omega_{59r_i}(0))} \leq c_1 2^{n+1} \lambda 
\end{align*} \quad (35)$$

for any $\epsilon > 0$, where $p(x, M, i) = \sup_{\Omega_{236r_i}(0)} p(x)$.

In the case of $B_{20r_i}(y_i) \subset \Omega$, we can also adopt Lemma 3.2 and 3.3 to find that

$$\begin{align*}
\int_{B_{20r_i}(y_i)} \varphi(x, |Du - Dv_i|) \, dx \leq \epsilon \lambda, & \quad \int_{B_{20r_i}(y_i)} \varphi(x, |Dv_i - Dh_i|) \, dx \leq \epsilon \lambda, \\
\int_{B_{10r_i}(y_i)} \varphi_{x, h_i}(|Dh_i - Dw_i|) \, dx \leq \epsilon \lambda \quad \text{and} \quad \|\varphi_{x, h_i}(|Dw|)\|_{L^\infty(\Omega_{59r_i}(y_i))} \leq c_2 \lambda 
\end{align*} \quad (37)$$

for some constant $c_2 \geq c_1 2^{n+1}$, where $p(x, M, i) = \sup_{B_{20r_i}(y_i)} p(x)$ and $c_1$ is given in (36).

Now setting $\alpha = 2 \left(s_1 \log s_1 - 1\right)^3 (c_2 + \log(e + 1))$ for $s_1$ and $c_2$ as in (12) and (38), respectively, we use (12), (33), (36), (38) and the fact that $\lambda \geq 1$ to find that
for all \( \lambda \) function:

**Step 3.** In this step, we prove our main result (9). Let us consider the truncated function:

\[
\varphi(x, |Du|) \leq \varphi(x, |Du|) + |Du| - |Du_k| + |Du_k - |Du| | + |Du - |Du_k| |
\]

\[
\leq (s_1^2 \log_2 s_1 - 1)^3 \left( \varphi(x, |Du|) + \varphi(x, |Du| - |Du_k|) + \varphi_{x_{\Omega}}(|Du_k - |Du| |)
\right)
\]

\[
+ \varphi_{x_{\Omega}}(|Du_k|) + \log(e + 1)
\]

\[
\leq (s_1^2 \log_2 s_1 - 1)^3 \left( \varphi(x, |Du|) + \varphi(x, |Du| - |Du_k|) + \varphi_{x_{\Omega}}(|Du_k - |Du| |)
\right)
\]

\[
+ (e_2 + \log(e + 1)) \lambda
\]

\[
\leq (s_1^2 \log_2 s_1 - 1)^3 \left( \varphi(x, |Du|) + \varphi(x, |Du| - |Du_k|) + \varphi_{x_{\Omega}}(|Du_k - |Du| |)
\right)
\]

\[
+ \frac{1}{2} \varphi(x, |Du|).
\]

That is,

\[
\varphi(x, |Du|) \leq 2 (s_1^2 \log_2 s_1 - 1)^3 \left( \varphi(x, |Du|) + \varphi(x, |Du| - |Du_k|) + \varphi_{x_{\Omega}}(|Du_k - |Du| |)
\right)
\]

on \( \{x \in \Omega_{\lambda} : \varphi(x, |Du|) > \lambda \} \). According to (35)-(38), this inequality yields that

\[
\int_{\Omega_{\lambda}} \varphi(x, |Du|) \, dx \leq c \lambda |\Omega_{\lambda}|,
\]

where we have used (8) and (33) when \( B_{2\rho_{\lambda}}(y) \subset \Omega \). On the other hand, it follows from (30) that

\[
\int_{\Omega_{\lambda}} \varphi(x, |Du|) \, dx \leq \sum_{i=1}^{\infty} \int_{\Omega_{\lambda} \cap \{\varphi(x, |Du|) > \lambda \}} \varphi(x, |Du|) \, dx.
\]

Then from (31) and (39), we finally obtain

\[
\int_{\Omega_{\lambda}} \varphi(x, |Du|) \, dx \leq c \int_{\Omega_{\lambda} \cap \{\varphi(x, |Du|) > \lambda \}} \varphi(x, |Du|) \, dx
\]

\[
+ \frac{e_\lambda}{\delta} \int_{\Omega_{\lambda} \cap \{\varphi(x, |Du|) > \lambda \}} \psi \, dx
\]

(40)

for all \( \lambda \geq \alpha \lambda_0 \) and for any \( \epsilon > 0 \).

**Step 3.** In this step, we prove our main result (9). Let us consider the truncated function:

\[
\varphi(x, |Du|)_k = \min \{\varphi(x, |Du|), k\} \quad \text{for} \quad x \in \Omega \quad \text{and} \quad k \in \mathbb{N}.
\]

We first observe for \( k \geq \lambda \) that \( \varphi(x, |Du|)_k \geq \lambda \) is preserved if and only if \( \varphi(x, |Du|) \geq \lambda \) is satisfied. We then employ Fubini’s theorem in order to get

\[
\int_{\Omega_{\lambda}} \varphi(x, |Du|)_k \varphi(x, |Du|) \, dx
\]

\[
\leq (q - 1) \int_0^{\alpha \lambda_0} \lambda^{q-2} d\lambda \int_{\Omega_{x}(x_0)} \varphi(x, |Du|) \, dx
\]

\[
+ (q - 1) \int_{\alpha \lambda_0}^{k} \lambda^{q-2} \int_{\Omega_{x}(x_0)} \varphi(x, |Du|) \, dx \, d\lambda
\]
for $k > \alpha \lambda$. By (40), this yields
\[
\int_{\Omega^1(x_0)} \varphi(x, |Du|)[\varphi(x, |Du|)_k]^{q-1} \, dx
\]
\[
\leq (\alpha \lambda)^{q-1} \int_{\Omega^2(x_0)} \varphi(x, |Du|) \, dx + \frac{c}{\delta^q} \int_{\Omega^2(x_0)} \Psi \, dx.
\]

Meanwhile, by a change of variable and Fubini’s theorem, we see that
\[
\int_0^k \lambda^{q-2} \int_{E(\frac{x_0}{\lambda}, \Omega^2_r)} \varphi(x, |Du|) \, dx \, d\lambda \leq c \int_{\Omega^2(x_0)} \varphi(x, |Du|)[\varphi(x, |Du|)_k]^{q-1} \, dx
\]
and
\[
\int_0^k \lambda^{q-2} \int_{\Omega^2(x_0) \cap \{\Psi > \frac{x_0}{\lambda^2}\}} \Psi \, dx \, d\lambda = \frac{c}{\delta^q} \int_{\Omega^2(x_0)} \Psi^q \, dx.
\]
Inserting this into (41), we yield
\[
\int_{\Omega^1(x_0)} \varphi(x, |Du|)[\varphi(x, |Du|)_k]^{q-1} \, dx
\]
\[
\leq (\alpha \lambda)^{q-1} \int_{\Omega^2(x_0)} \varphi(x, |Du|) \, dx + \frac{c}{\delta^q} \int_{\Omega^2(x_0)} \Psi^q \, dx.
\]
Selecting $\epsilon > 0$ small enough and recalling (26), we deduce
\[
\int_{\Omega^1(x_0)} \varphi(x, |Du|)[\varphi(x, |Du|)_k]^{q-1} \, dx
\]
\[
\leq \frac{1}{2} \int_{\Omega^2(x_0)} \varphi(x, |Du|)[\varphi(x, |Du|)_k]^{q-1} \, dx + \frac{c}{\delta^q} \int_{\Omega^2(x_0)} \Psi^q \, dx
\]
\[
+ \frac{c \rho^{n(q-1)}}{(r_2 - r_1)^{n(q-1)}} |\Omega^2| \left\{ \int_{\Omega^2(x_0)} \varphi(x, |Du|) \, dx + \frac{1}{\delta} \int_{\Omega^2(x_0)} \Psi \, dx \right\}^q.
\]

Here we remark that $\delta$ is determined as Lemma 3.2 and 3.3. We now apply Lemma 4.1 to find
\[
\int_{\Omega^1(x_0)} \varphi(x, |Du|)[\varphi(x, |Du|)_k]^{q-1} \, dx \leq c \left\{ \int_{\Omega^2(x_0)} \varphi(x, |Du|) \, dx \right\}^q + c \int_{\Omega^2(x_0)} \Psi^q \, dx.
\]

Passing $k \to \infty$ and employing Fatou’s lemma, we eventually conclude
\[
\int_{\Omega^1(x_0)} \varphi(x, |Du|)^q \, dx \leq c \left\{ \int_{\Omega^2(x_0)} \varphi(x, |Du|) \, dx \right\}^q + c \int_{\Omega^2(x_0)} \Psi^q \, dx.
\]
Recalling (8) and Lemma 3.1, we conclude
\[
\int_{\Omega^1(x_0)} \varphi(x, |Du|)^q \, dx \leq c \left( \frac{|\Omega|}{\rho^n} \right)^{q-1} \int_{\Omega} \Psi^q \, dx,
\]
whenever \( x_0 \in \Omega \) and \( 0 < M \rho \leq R_0 \).

In what follows, we turn to a standard finite covering argument to reach our global estimate. Since \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), there exist \( N \in \mathbb{N} \) and \( x_j \in \Omega \) for \( j = 1, 2, \ldots, N \) such that

\[
\Omega \subset \bigcup_{j=1}^{N} \Omega_{R_0}(x_j).
\]

Applying (42), we finally obtain

\[
\int_{\Omega} \varphi(x, |Du|^q) \, dx \leq \sum_{j=1}^{N} \int_{\Omega_{R_0}(x_j)} \varphi(x, |Du|^q) \, dx \leq c M^{n(q-1)} \int_{\Omega} \Psi^q \, dx,
\]

where \( c > 0 \) depends on \( n, \gamma_1, \gamma_2, \nu, L, q, \Omega \) and \( \omega(\cdot) \). This completes the proof. \( \square \)

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