The Spectrum of \( \text{QED}_1^1 \) in the framework of the DLCQ method *

Stephan Elser  
Max-Planck-Institut für Kernphysik  
Postfach 10 39 80  
69029 Heidelberg

March 27, 2022

This work is based on a diploma thesis done in 1993/94 at the Max-Planck-Institute for Nuclear Physics, Heidelberg, under supervision of Prof. H.C. Pauli, MPIK Heidelberg.

1 Motivation

We concern ourselves with \( \text{QED}_1^1 \) in the Discretized Light-Cone Quantization formalism suggested by Pauli and Brodsky [Pau85] as presented in Eller, Pauli and Brodsky [Ell87] to investigate the influence of the fermion field boundary conditions on the spectrum and wavefunctions.

The DLCQ approach is chosen because its relativistic, non-perturbative treatment of gauge theories enables us to aim for hadronic physics, i.e. a solution of \( \text{QCD}_3^1 \) obtaining spectra and wavefunctions. The boundary condition question is motivated by the so-called “background field” or “zero mode problem” generally existing for photons and electrons. In the case of photons this problem, despite some recent investigations e.g. by McCartor [McC88] or Heinzl, Krusche and Werner [Hei92], is still unsolved. Insights quantizing fermion fields in the ’toy model’ of \( \text{QED}_1^1 \) can be generalized to the fundamental problem.

2 Method

Following Dirac [Dir49, Dir64] physical degrees of freedom are quantized on a equal light-cone time \( x^+ = (x^0 + x^3)/\sqrt{2} = 0 \) surface instead of on the usual \( t = 0 \) surface. To find the dynamical degrees of freedom an analysis of the constraints using e.g. the Dirac-Bergman algorithm [Dir64, Sim82] is used. The time evolution operator propagating in a generalized time direction \( x^+ \) is regarded as the Hamiltonian. This light-front scheme, somewhat improperly called light-cone quantisation (LCQ), offers advantages for field theories, mainly a simpler vacuum structure [Wei66]. For details, we refer to [Bro91].

Pauli and Brodsky [Pau85] realized the existence of a compactified formulation equivalent to lattice gauge theory, where space is compactified using periodic boundary conditions for physical observables on a finite interval \([-L, +L]\) in lightcone longitudinal dimension \( x^- = (x^0 - x^3)/\sqrt{2} \) and \([-L_\perp, +L_\perp]\) in the transverse dimensions \( x_\perp = (x^1, x^2) \) and a basis of plane waves in momentum space.

*invited talk given at the Hadron Structure ’94 conference, Kosice, Slovakia, 19/9 – 23/9/1994
problem can be formulated as a Schrödinger eigenvalue equation for the time evolution operator or equivalently for the Lorentz-invariant mass squared. Writing the theory in creation and annihilation operators representing particles with positive longitudinal momentum as elements of a denumerable Fock-Hilbert space results in a finite matrix equation

$$\langle i | : 2P^+P^- - (\vec{P}_\perp)^2 : | j \rangle \langle j | \Psi \rangle = m^2 \langle j \rangle \langle \Psi \rangle,$$

which can be diagonalized numerically. Approximations like restrictions of particle number, in the literature mostly called *Tamm-Dancoff truncation* [Tam45], or the construction of effective potentials are solidly based [Pau93]. Real physics of the system can be investigated at each step as the continuum limit of arbitrarily large volume both analytically or numerically.

Not touching on the still existing problems of DLCQ (“left movers”, “zero modes”, “renormalization of a Hamiltonian light-cone theory”), we concentrate on the question still under discussion whether the choice of boundary conditions of the fields can have impact on the results of the theory. For details of the calculation we refer to the thesis of Elser [Els94].

We therefore consider three cases:

- anti-periodic boundary conditions for fermions (i.e. no zero Fourier mode)
- periodic boundary conditions for fermions (zero Fourier mode neglected)
- periodic boundary conditions for fermions (zero Fourier mode included)

To depict our results, two coupling constants are chosen for convenience by

$$\lambda^2 = \frac{1}{1 + \frac{\pi m_f^2}{g^2}}$$

and

$$\rho = \frac{m_f}{g},$$

using the existence of an arbitrary mass scale. The ultra-violet cut-off connected to the periodicity length is the so-called harmonic resolution $K$.

## 3 Schwinger model

QED in one space and one time dimension with massless fermions, i.e. $m_f = 0$, the Schwinger model [Sch62], is known from an analytic solution to allow only uncharged states. This is sometimes called “quark” trapping [Sch63]. The calculated spectrum is that of free bosons, the ground state with mass $m_B$ corresponding to a quark-antiquark pair, i.e. a meson. Above that we find a continuum of two, three, . . . meson states.

QED$_{1+1}$ proper, also called the massive Schwinger model, shows a number of stable particles calculated by Coleman which at vanishing coupling diverges as expected. In the strong coupling limit the result is three stable particles for the case considered [Col73].

Basic to the expansion around the massless case is the proof that this limit is allowed and leads to the soluble massless Schwinger model [Kog73]. Although this still being under discussion [Par93], the mass squared of the ground state was calculated perturbatively to be linear with the bare fermion mass $m_f$ [Kog75, Ban76]. Low-lying state masses and this behaviour were numerically confirmed by lattice gauge theory [Kog75, Ban76, Car76, Cre80].
Figure 1: **Comparison to lattice results.** In a) results of a DLCQ calculation with antiperiodic boundary conditions (full lines) are compared to lattice gauge (crosses) and DLCQ with extrapolation scheme (stars) results. The parameter $\lambda$ is used to depict all mass and coupling possibilities. In b) an DLCQ calculation with antiperiodic boundary conditions is compared to one with periodic boundary conditions neglecting fermion zero modes in c) for low fermion masses corr. to small $\rho$. 


4 Results

All results are shown at a fixed harmonic resolution of $K = 16$ unless extrapolated data is mentioned. Here a quadratic fit routine was used to go beyond this limit. Comparison to lattice gauge theory. The main impression in fig. (1a) considering the lowest states in the spectra is the good agreement for the whole range of possible couplings and masses to the lattice results of Crewther and Hamer [Cre80]. Especially the extrapolated data (stars) is excellent.

In figs. (1b+c) this is investigated in more detail. Focussing on the small mass limit, only the antiperiodic case of fig. (2a) is able to regain the Schwinger ($m_f = 0$) limit, whereas the periodic case neglecting zero modes is 50% off, showing the importance of the zero mode. It also can be seen that both do not obtain the linear behaviour found in lattice gauge calculations.

Complete mass spectrum. The mass eigenvalues in fig. (2) are normalized to the ground state mass. Decomposition of scattering states with invariant masses below two ground state masses and bound stable states is then obvious. States just below threshold are identified as weakly bound dipositronium molecules [Ell87, Els94]. A qualitative consideration shows good agreement to Coleman’s prediction of stable states in the cases with no neglected zero modes in figs. (2a+c).

Tamm-Dancoff truncation. For space reasons we do not depict results, but simply state that a good quality of this approximation in the extreme case of a two-particle cut-off can be found for small coupling constants, where all stable states can be thus obtained. But already at a coupling of $g = 0.9 m_f$ ($\lambda = 0.45$) this is no longer true. Exactly the interesting weakly bound states are four particle states and therefore unobtainable in this extreme approximation.

Small mass limit. Recent investigation based on work by K. Hornbostel [Hor90] showed that the problems in the small mass limit can be regarded as numerical artefacts from the limited harmonic resolution. A extrapolation scheme is able to improve results by a factor of about 20 using the same computer facilities. This is depicted in fig. (3) suggesting that the linear behaviour of the mass squared can be obtained to a very high precision.

5 Extensions

In this work we did for space reasons not include some rather preliminary investigations into ground and exited states wave functions and structure functions calculated in the full Fock space. This is different to other approaches, where only the two-particle sector was investigated, and results in detailed information on the internal structure of bound many-particle states [Ell87]. Following a suggestion by L.C.L. Hollenberg [Hol94] we also calculated finite temperature quantities like the energy density and the specific heat from our mass spectra results assuming the partition function of a micro-canonical ensemble [Els94b].

6 Conclusions

Mainly, the method of DLCQ and the results of Eller, Pauli and Brodsky are confirmed. We obtained relativistic, non-pertubative mass spectra for a gauge field theory with dynamic fermions and (not shown) wave functions, structure functions and thermal quantities. Tamm-Dancoff particle number cut-offs are possible if treated with some care.

The basic lesson is that one Fourier mode (namely the zero mode) can be decisive for even the ground state mass. The choice of boundary conditions is seen to be not
important, if and only if all degrees of freedom are treated properly. This is still a problem for light-cone quantization in general.

References

[Ban76] T. Banks, L. Susskind, J. Kogut, Phys. Rev. D13 (1976) 1043
[Ber77] H. Bergknoff, Nucl. Phys. B122 (1977) 215
[Bro91] S.J. Brodsky, G. McCartor, H.C. Pauli, S.S. Pinsky, Particle World 3 (1993) 109
[Car76] A. Carroll, J. Kogut, D.K. Sinclair, L. Susskind, Phys. Rev. D13 (1976) 2270
[Col75] S. Coleman, R. Jackiw, L. Susskind, Ann. Phys. (N.Y.) 93 (1975) 267; S. Coleman, Ann. Phys. (N.Y.) 101 (1976) 239
[Cre80] D.P. Crewther, C.J. Hamer, Nucl. Phys. B170 (1980) 353
[Dir49] P.A.M. Dirac, Rev. Mod. Phys. 21 (1949) 392
[Dir64] P.A.M. Dirac, Lectures on Quantum Mechanics, Yeshiva University, New York 1964
[Ell87] Th. Eller, H.C. Pauli, S.J. Brodsky, Phys. Rev. D35 (1987) 1493
[Els94a] S. Elser, H.C. Pauli, to be published in CPC, 1994
[Els94b] S. Elser, work in progress
[Els94] S. Elser, The spectrum of QED_{1+1} in the framework of the DLCQ method, diploma thesis, Universität Heidelberg 1994 (available in english)
[Hei92] T. Heinzl, S. Krusche, E. Werner, Phys. Let. B275 (1992) 410
[Hol94] L.C.L. Hollenberg, private communication 1994
[Hor90] K. Hornbostel, S.J. Brodsky, H.C. Pauli, Phys. Rev. D41 (1990) 3814; K. Hornbostel, Ph.D. thesis, Stanford University, 1989 (SLAC-Report-333)
[Kal93] A.C. Kalloniatis, H.C. Pauli, Z. Phys. C 60 (1993) 255
[Kog75] J. Kogut, L. Susskind, Phys. Rev. D11 (1975) 3594
[McC88] G. McCartor, Z. Phys. C41 (1988) 271
[Mus90] D. Mustaki, Phys. Rev. D42 (1990) 1184
[Par93] M.B. Paranjape, Phys. Rev. D48 (1993) 4946
[Pau85] H.C. Pauli, S.J. Brodsky, Phys. Rev. D32 (1985) 1993 and 2001
[Pau93] H.C. Pauli, lecture at the workshop on “Quantum Field Theoretical Aspects of High Energy Physics”, Kyffhäuser 1993; to be published in the proceedings
[Sch62] J. Schwinger, Phys. Rev. 128 (1962) 2425
[Sch63] J. Schwinger in *Theoretical Physics, Trieste Lectures 1962* (IAEA, Vienna 1963 p89)

[Sun82] K. Sundermeyer, *Constrained Dynamics* (Lecture Notes in Physics, Vol. 169, Springer-Verlag, Heidelberg 1982)

[Tam45] I. Tamm, J. Phys. (USSR) 9 (1945) 449; S.M. Dancoff, Phys. Rev. 78 (1950) 382

[Wei66] S. Weinberg, Phys. Rev 150 (1966) 1313
Figure 2: Complete mass spectrum. All mass eigenvalues normalized on the physical ground state for a) antiperiodic, b) periodic boundary conditions neglecting zero modes and c) periodic boundary conditions incorporating a fermion zero mode at a resolution of $K = 16$ against the parameter $\lambda$.

Figure 3: Small mass limit. Results of a DLCQ calculation with antiperiodic boundary conditions (full lines) are compared to lattice gauge results (crosses) and DLCQ with extrapolation scheme (stars). The parameter $\rho$ is used to focus on extremely small fermion masses.