1. Introduction

1.1. The natural action of the group $GL_2$ on $\mathbb{C}^2$ induces a $GL_2$-action on $\text{Hilb}^n \mathbb{C}^2$, the Hilbert scheme of $n$ points in the plane. There is also a similar action of the group $SL_2$ on $X_c$, the Calogero-Moser space. The fixed points of the corresponding maximal torus $\mathbb{C}^* \times \mathbb{C}^*$, resp. $\mathbb{C}^*$, of diagonal matrices, are labeled by partitions. Let $y_\lambda \in \text{Hilb}^n \mathbb{C}^2$, resp. $x_\lambda \in X_c$, denote the point labeled by a partition $\lambda$. It turns out that such a point is fixed by the group $SL_2$ if and only if $\lambda = (m, m-1, \ldots, 2, 1) =: m$ is a staircase partition. In the Hilbert scheme case, this has been observed by Kumar and Thomsen [KT]. The case of the Calogero-Moser space can be deduced from the Hilbert scheme case using "hyper-Kähler rotation". A different, purely algebraic proof is given in section 3 below.

The theory of rational Cherednik algebras gives an $SL_2 \times S_n$-equivariant vector bundle $R$ of rank $n!$ on the Calogero-Moser space. Thus, $R|_{x_m}$, the fiber of $R$ over the $SL_2$-fixed point, acquires the structure of a $SL_2 \times S_n$-representation. We find the character formula of this representation in terms of Kostka-Macdonald polynomials. The vector bundle $R$ is an analogue of the Procesi bundle $P$, a $GL_2 \times S_n$-equivariant vector bundle of rank $n!$ on $\text{Hilb}^n \mathbb{C}^2$. Our formula agrees with the character of the representation of $GL_2 \times S_n$ in $P|_{y_m}$, the fiber of $P$ over the $GL_2$-fixed point, obtained by Haiman [H]. It is, in fact, possible to derive our character formula for $R|_{x_m}$ from the one for $P|_{y_m}$. However, the character formula for $P|_{y_m}$, as well as the construction of the Procesi bundle itself, involves the $n!$-theorem.

In section 2 we review some general results about $SL_2$-actions. In section 3, we apply these results to show that, for any $\lambda$, the $SL_2$-orbit of $x_\lambda$ is closed in $X_c$. The $GL_2$-orbit of $y_\lambda$ is not closed in $\text{Hilb}^n \mathbb{C}^2$, in general, and we describe the closure in §4.

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2. $sl_2$-actions

Let $T \subset SL_2$ be the maximal torus of diagonal matrices. The group $T$ acts on the Lie algebra $sl_2$ by conjugation. Let $(E, H, F)$ be the standard basis of $sl_2$.

Let $X$ be an algebraic variety equipped with a $T$-action and let $\text{Vect}(X)$ be the Lie algebra of algebraic vector fields on $X$. The $T$-action on $X$ induces a $T$-action on $\text{Vect}(X)$ by Lie algebra automorphisms. An algebraic variety $X$ equipped with a $T$-action and with...
a Lie algebra homomorphism $\mathfrak{sl}_2 \to \text{Vect}(X)$ that intertwines the $T$-actions on $\mathfrak{sl}_2$ and on $\text{Vect}(X)$, respectively, will be referred to as an $(\mathfrak{sl}_2, T)$-variety.

Given a group $G$ and a $G$-variety $X$, we write $X^G$ for the fixed point set of $G$. Given an $(\mathfrak{sl}_2, T)$-variety $X$ we write $X^{\mathfrak{sl}_2}$ for the closed subset, with reduced scheme structure, of $X$, defined as the zero locus of all vector fields contained in the image of the map $\mathfrak{sl}_2 \to \text{Vect}(X)$. Clearly, we have $X^{\mathfrak{sl}_2} \subset X^T$. Any variety with an $SL_2$-action has an obvious structure of an $(\mathfrak{sl}_2, T)$-variety. In such a case we have $X^{SL_2} = X^{\mathfrak{sl}_2}$.

**Theorem 2.1.** Let $X$ be smooth quasi-projective variety equipped with an $(\mathfrak{sl}_2, T)$-action. Then,

(i) If $x \in X^T$ is an isolated fixed point, then $x \in X^{\mathfrak{sl}_2}$ if and only if all the weights of $T$ on $T_xX$ are odd.

(ii) If the $(\mathfrak{sl}_2, T)$-action on $X$ comes from a nontrivial $SL_2$-action with dense orbit then the set $X^{SL_2}$ is finite.

**Proof.** (i) Let $x \in X^{\mathfrak{sl}_2}$ and let $m$ be the maximal ideal in the local ring $\mathcal{O}_{X,x}$ defining this point. Then $\mathfrak{sl}_2$ acts on $m/m^2$. Since $x$ is a isolated fixed point for the $T$-action, the degree zero weight space is 0 so all $\mathfrak{sl}_2$-modules appearing in $m/m^2$ must have odd weight spaces only.

Conversely, assume that all non-zero weight spaces in $m/m^2$ have odd weight. We need to show that $\mathfrak{sl}_2$ acts in this case i.e. $\mathfrak{sl}_2(m) \subset m$. It is known that any $T$-orbit is contained in an affine $T$-stable Zariski open subset of $X$. Therefore, replacing $\mathcal{O}_{X,x}$ by some affine $T$-stable neighborhood, we may assume that $X$ is an affine $T$-variety with $\mathfrak{sl}_2$-action and isolated fixed point defined by $m < \mathbb{C}[X]$. Then $\mathbb{C}[X] = \mathbb{C}1 \oplus m$ as a $T$-module. In particular, every homogeneous element of non-zero degree belongs to $m$. If $z \in m$ is homogeneous of degree $\neq -2$ then $\deg E(z) = \deg z + 2 \neq 0$. Thus, $E(z) \in m$. On the other hand, if $\deg z = -2$ then our assumptions imply that $z \in m^2$ and hence $E(z) \in m$. A similar argument applies for $F$.

Part (ii) is a result of Bialynicki-Birula, [BB, Theorem 1].

Let $N(T)$ be the normalizer of $T$ in $SL_2$. The Borel of upper-triangular matrices in $SL_2$ is denoted $B$. Its opposite is $B^{-}$.

**Lemma 2.2.** Let $O$ be a one-dimensional homogeneous $SL_2$-space. Then $O \simeq SL_2/B$.

**Proof.** Let $K = \text{Stab}_{SL_2}(x)$ for some $x \in O$, a closed subgroup of $SL_2$. Let $t$ be the Lie algebra of $K$. Since $\dim t = 2$ it is a solvable subalgebra of $\mathfrak{sl}_2$. Therefore it is conjugate to $b$. Without loss of generality, $t = b$. This means that $K^0 = B \subset K \subset N_{SL_2}(B) = B$. □

**Lemma 2.3.** Let $O$ be an $SL_2$-orbit in an affine variety $X$. Assume that the stabilizer of $x \in O$ contains $T$. Then $O$ is closed in $X$ and $\text{Stab}_{SL_2}(x)$ is one of: $T, N(T)$ or $SL_2$.

**Proof.** Let $K = \text{Stab}_{SL_2}(x)$ and $t = \text{Lie } K$. Then $t \subset t$ implies that either $t = \mathfrak{sl}_2$ (and hence $O = \{x\}$), $t \simeq b$ or $t = t$. We assume that $t \neq \mathfrak{sl}_2$.

Let $Y := \overline{O} \setminus O$ and assume $Y$ is nonempty. Then $Y$ is affine and $\dim Y < \dim O = 2$, i.e. $\dim Y \leq 1$. Assume first $\dim Y = 1$. Since $\overline{O}$ is affine, it has a unique closed orbit $O'$. Therefore $Y$ also has a unique closed orbit $O'$ and all other orbits in $Y$ have dimension greater than $\dim O'$. In particular, $Y$ contains at most 1 orbit of dimension zero. Therefore, $Y$ must contain an orbit of dimension one. The stabilizer $K$ of such an orbit has dimension 2 and, without loss of generality, contains $T$. This implies that $\text{Lie } K$ is a Borel subalgebra of $\mathfrak{sl}_2$. Hence $K^0$ is a Borel subgroup of $SL_2$. Since $K \subset N_{SL_2}(K^0) = K^0$, it follows that $K = K^0$ is a Borel subgroup. This is impossible since $SL_2/K$ does not embedded into an affine variety. Thus, the only remaining possibility is that $\dim Y = 0$. Let $Z$ be the normalization of $\overline{O}$. Thus, we have an open dense embedding $O \hookrightarrow Z$ such that $\dim(Z \setminus O) = 0$. Since
both $\mathcal{O}$ and $Z$ are affine, this is impossible: it would imply that any regular function on $\mathcal{O}$ extends to $Z$, so the the restriction $\mathbb{C}[Z] \to \mathbb{C}[\mathcal{O}]$ is an isomorphism, hence $\mathcal{O} = Z$. □

**Lemma 2.4.** Let $X$ be a complete $SL_2$-variety and $\mathcal{O}$ an orbit such that the stabilizer of $x \in \mathcal{O}$ equals $T$, resp. $N(T)$.

1. There is a finite (surjective) equivariant morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{O}$, resp. $\mathbb{P}^2 \to \mathcal{O}$, which is the identity on $\mathcal{O}$.

2. This morphism is an isomorphism if and only if $\mathcal{O}$ is normal.

3. In all cases $\mathcal{O} \setminus \mathcal{O} \simeq \mathbb{P}^1$ and $\mathcal{O}^{SL_2} = \emptyset$.

**Proof.** We explain how the lemma can be deduced from the results of [M].

Matsushima’s Theorem implies that $\mathcal{O}$ is affine. Therefore, by [EGA Corollaire 21.12.7], the complement $Y = \overline{\mathcal{O}} \setminus \mathcal{O}$ has pure codimension one. By Theorem 2.1(ii), there are only finitely many zero-dimensional orbits in $Y$. Therefore Lemma 2.2 implies that each irreducible component $Y_i$ of $Y$ (being one-dimensional) must contain an orbit $\simeq SL_2/B$. Since this orbit is complete, it is closed in $Y_i$ i.e. $Y_i \simeq SL_2/B$. Moreover, this implies that $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and hence $\mathcal{O}^{SL_2} = Y^{SL_2} = \emptyset$.

By [M, Theorem 5.1], $\mathbb{P}^1 \times \mathbb{P}^1$ (in the case $\mathcal{O} \simeq SL_2/T$), resp. $\mathbb{P}^2$ (in the case $\mathcal{O} \simeq SL_2/N(T)$), is the unique normal completion of $\mathcal{O}$. In both these cases the complement is a single copy of $\mathbb{P}^1$. Since we have shown that the boundary $Y$ is a finite union of codimension one orbits, the lemma follows from the Observation of [M, Section 0]. □

### 3. Calogero-Moser spaces

Let $(W, \mathfrak{h})$ be a finite Coxeter group, with $S$ the set of all reflections in $W$ and $\mathfrak{c}: S \to \mathbb{C}$ a conjugate invariant function. For each $s \in S$, we fix eigenvectors $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ with eigenvalue $-1$. Associated to this data is the rational Cherednik algebra $H_c(W)$ at $t = 0$. It is the quotient of the skew group ring $T^*(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$ by the relations

$$[y, x] = -\sum_{s \in S} \mathfrak{c}(s) \frac{\alpha_s(y) x(\alpha_s^\vee)}{\alpha_s(\alpha_s^\vee)}, \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}$$

and $[x, x'] = [y, y'] = 0$ for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. We choose a $W$-invariant inner product $(-, -)$ on $\mathfrak{h}$. The form defines an $W$-isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$, $x \mapsto \hat{x}$.

#### 3.1. The center $Z(H_c(W))$ of $H_c(W)$ has a natural Poisson structure, making $H_c(W)$ into a Poisson module.

Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{h}^*$ and $y_1, \ldots, y_n$ dual basis. Then the elements

$$E = -\frac{1}{2} \sum_i x_i^2, \quad F = \frac{1}{2} \sum_i y_i^2, \quad H = \frac{1}{2} \sum_i x_i y_i + y_i x_i. \quad (3.1)$$

are central and form an $sl_2$-triple under the Poisson bracket. Their action on $H_c(W)$ is given by $[E, x] = [F, \hat{x}] = 0$, $[E, \hat{x}] = [F, x] = x$, $[F, \hat{x}] = \hat{x}$ and $[H, x] = x$, $[H, \hat{x}] = -\hat{x}$. Their action on $H_c(W)$ is locally finite. Therefore this action can be integrated to get a locally finite action of $SL_2(\mathbb{C})$ on $H_c(W)$ by algebra automorphisms. Explicitly, this action is given on generators by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = ax + cx, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \hat{x} = bx + dx, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot w = w, \quad \forall x \in \mathfrak{h}^*, w \in W.$$

The Calogero-Moser space $X_c(W)$ is an affine variety defined as $\text{Spec} Z(H_c(W))$. The action of $SL_2(\mathbb{C})$ restricts to $Z(H_c(W))$ and induces a Hamiltonian action on $X_c(W)$, such that its differential is the action of $sl_2$ given by the vector fields $\{E, -\}$, $\{F, -\}$ and $\{H, -\}$. 

3
There are only finitely many $T$-fixed points on $X_c(W)$. When the Calogero-Moser space is smooth, the $T$-fixed points are naturally labeled $x_\lambda$, with $\lambda \in \text{Irr}(W)$. These fixed points are uniquely specified by the fact that the simple head $L(\lambda)$ of the baby Verma module $\Delta(\lambda)$ is supported at $x_\lambda$; see [G] for details.

Consider the element $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $SL_2$. It normalizes $T$.

**Lemma 3.2.** Assume that $X_c(W)$ is smooth. Let $x_\lambda \in X_c(W)$ be the $T$-fixed point labeled by the representation $\lambda \in \text{Irr}(W)$. Then $w_0 \cdot x_\lambda$ is the fixed point labeled by $\lambda \otimes \text{sgn}$, where $\text{sgn}$ is the sign representation.

**Proof.** The automorphism of $H_c(W)$ defined by $w_0$ is the Fourier transform, of order 4. The fixed point $w_0 \cdot x$ is the support of $w_0 L(\lambda)$. Thus, it suffices to show that $w_0 L(\lambda) \cong L(\lambda \otimes \text{sgn})$. This is a standard result. \qed

**Definition 3.3.** A $(H_c, sl_2)$-module $M$ is both a left $H_c(W)$-module and left $sl_2$-module such that the morphism $H_c(W) \otimes M \to M$ is a morphism of $sl_2$-modules.

Every finite dimensional $(H_c(W), sl_2)$-module is set-theoretically supported at a $SL_2$-fixed point. However, not every finite dimensional $H_c(W)$-module set-theoretically supported at a $SL_2$-fixed point has a compatible $sl_2$-action.

Let $e$ denote the trivial idempotent in $CW$. Then $e$ is $SL_2$-invariant and hence $H_c(W)e$ is a $(H_c, sl_2)$-module. Thinking of $H_c(W)e$ as a finitely generated $Z(H_c(W))$-module, we get a $SL_2 \times W$-equivariant coherent sheaf $\mathcal{R}$ on $X_c(W)$. When the latter space is smooth, $\mathcal{R}$ is a vector bundle of rank $|W|$.

**3.2. Type A.** Let $H_c$ be the rational Cherednik algebra for the symmetric group $\mathfrak{S}_n$ at $t = 0$ and $c \neq 0$. In this case both the set of $T$-fixed points in the CM-space $X_c := X_c(\mathfrak{S}_n)$ and the set of (isomorphism classes of) simple irreducible representations of $\mathfrak{S}_n$ are labeled by partitions of $n$. We write $m_\lambda$ for the maximal ideal of the $T$-fixed point corresponding to a partition $\lambda$.

**Notation 3.4.** From now on, the staircase partition $(m, m-1, \ldots, 1)$ will be denoted $m$. Given a partition $\lambda$, the corresponding representation of the symmetric group will be denoted $\pi_\lambda$. The finite dimensional, irreducible $SL_2$-module with highest weight $m \geq 0$ will be denoted $V(m)$.

\[
\begin{array}{cccc}
7 & 5 & 3 & 1 \\
5 & 3 & 1 \\
3 & 1 \\
1 \\
\end{array}
\]

(3.5)

Let $x$ be a box of the partition $\lambda$. The **hook length** $h(x)$ of $x$ is the number boxes strictly to the right of $x$ plus the number strictly below plus one. In the above staircase partition the entry of the box is the corresponding hook length. The **hook polynomial** of $\lambda$ is defined to be

\[H_\lambda(q) = \prod_{x \in \lambda} (1 - q^{h(x)}) .\]

Let $(q)_n = \prod_{i=1}^{n} (1 - q^i)$ and denote by $n(\lambda)$ the partition statistic $\sum_{i \geq 1} (i - 1) \lambda_i$.

We write $\chi_T$ for the character of a finite dimensional $T$-representation.
Lemma 3.6. Let $x_\lambda$ be the $T$-fixed point of $X_c$ labeled by the partition $\lambda$. Then

$$\chi_T(T_{x_\lambda}X_c) = \sum_{x \in \lambda} q^{h(x)} + q^{-h(x)}.$$ 

Proof. It is known that the graded multiplicity of $\pi_\lambda$ in the coinvariant ring $\mathbb{C}[h]/\langle \mathbb{C}[h]^{T} \rangle$ is given by $(q)_n q^{n(h)} H_\lambda(q)^{-1}$, the so called “fake polynomial”. If we decompose $T_y X_c = (T_y X_c)^+ \oplus (T_y X_c)^-$ into its positive and negative weight parts, then Theorem 4.1 and Corollary 4.4 of [B2] imply that

$$\chi_T((T_y X_c)^+) = \sum_{x \in \lambda} q^{h(x)}, \quad \text{since} \quad \chi_T(C[(T_y X_c)^+]) = \frac{1}{H_\lambda(q)}.$$ 

The fact that $T$ preserves the symplectic form on $X_c$ implies that $\chi_T((T_y X_c)^-) = \sum_{x \in \lambda} q^{-h(x)}$. 

The following observation is elementary.

Lemma 3.7. Let $\lambda$ be a partition such that every hook length in $\lambda$ is odd. Then $\lambda$ is a staircase partition.

Lemma 3.7 together with Lemma 3.6 and Theorem 2.1 imply that $SL_2$-fixed points in $X_c$ are very rare. Namely,

Theorem 3.8. If $n = \frac{m(m+1)}{2}$, for some integer $m$, then $X^{sl_2}_c = \{x_m\}$. Otherwise, $X^{sl_2}_c = \emptyset$.

The lemma, together with Theorem 2.1 implies

Proposition 3.9. There exists a finite dimensional $(H_c, sl_2)$-module if and only if $n = \frac{m(m+1)}{2}$ for some $m$. In this case, any such module $M$ is set-theoretically supported at the fixed point $x_m$ labeled by the staircase partition.

Proof. If $M$ is a $(H_c, sl_2)$-module, then its set-theoretic support must be $SL_2$-stable. If $M$ is also finite dimensional, then this support is a finite collection of points. These points must be $SL_2$-fixed since the group is connected. The result follows from Theorem 3.8

Finally, we must show that there exists at least one $(H_c, sl_2)$-module supported at $x_m$. Let $m \triangleleft Z(H_c)$ be the maximal ideal of $x_m$. Then $\{sl_2, m\} \subset m$. Recall that the $H_c$-module $H_c e$ is an $(H_c, sl_2)$-module. Thus, $H_c e/mH_c e$ is a (simple) $(H_c, sl_2)$-module supported at $x_m$. 

Recall that there is a unique simple $H_c$-module $L(\lambda)$ supported at each of the $T$-fixed points $x_\lambda$. Notice that we have shown,

Corollary 3.10. The simple module $L(m) \simeq H_c e/mH_c e$ is a $(H_c, sl_2)$-module.

Equivalently, the above arguments show that $sl_2$ acts on the fiber $R_m$ of $R$ at $x_m$. The formula for the character of the tangent space of $X_c(\mathfrak{g}_n)$ at $x_m$ given by Lemma 3.6 shows that

$$T_{x_m}X_c \simeq V(m) \otimes V(m-1),$$

as $SL_2$-modules.

Next we describe the $SL_2$-orbits $O_\lambda := SL_2 \cdot x_\lambda$ of the $T$-fixed points $x_\lambda$. First, we note that Lemma 2.3 implies that

Lemma 3.12. The orbit $O_\lambda$ is closed and $\text{Stab}_{SL_2}(x_\lambda)$ is reductive.

Lemma 3.2, Theorem 3.8 and Lemma 3.12 imply that

Proposition 3.13. Let $\lambda$ be a partition of $n$. Then, one has the following 3 alternatives:
therefore peeling away the hooks gives a natural factorization. The largest hook in \( m \) the character of the right hand side of equation (3.15). The character of exponents

\[ \text{Proof.} \]

Where the final term Proposition 3.14. There is an isomorphism of \( L \)

deckkeke as a \( 2 \)-module, \( L \)

deckkeke is isomorphic to \( L \) as an \( SL_2 \)-module equals the character of the right hand side of equation \( 3.15 \). The character of \( L \) is given in \( B1 \) Lemma 3.3. However, we must shift the grading on \( L \) from the one given in loc. cit. so that the isomorphism \( H_c e / m H_c e \rightarrow L \) is graded i.e. we require that the one-dimensional space \( e L \) lies in degree zero. Then,

\[ \chi_T(L(m)) = q^{-n(m)} \frac{H_m(q)}{(1-q)^n} \dim \pi_m. \]

Note that \( n(m) = \frac{1}{6}(m-1)m(m+1) \). For the staircase partition, the character of \( L \) has a natural factorization. The largest hook in \( m \) is \( (m, 1^{m-1}) \) and \( m = (m, 1^{m-1}) + [m-2] \), therefore peeling away the hooks gives \( q^{-n(m)} \frac{H_m(q)}{(1-q)^n} = q^{-m+2} \) and

\[ \frac{H_m(q)}{(1-q)^{2m-1} H_{[m-2]}(q)} = \frac{1}{(1-q)^{2m-1}} \left( \prod_{i=1}^{m-1} \frac{1}{1-q^{2^{i-1}} - 1} \right)^2. \]

Thus,

\[ \frac{H_m(q) q^{-m+2}}{(1-q)^{2m-1} H_{[m-2]}(q)} = \frac{1}{1-q} \prod_{i=1}^{m-1} \frac{1}{1-q^{2^{i-1}}} \cdot \frac{1}{1-q}. \]

This is precisely the character of \( U_m \). \[ \square \]

One would like to refine this character by taking into account the action of \( W \) too. We decompose \( L(m) \) as a \( W \times SL_2 \)-module,

\[ L(m) = \bigoplus_{\lambda \vdash m} \pi_{\lambda} \otimes V_{\lambda}. \]  (3.16)

Then the exponents of \( \lambda \) are defined to be the positive integers \( 0 \leq e_1 \leq e_2 \leq \cdots \) such that \( V_{\lambda} = \bigoplus_i V(e_i) \). The fact that \( L(m) \) is the regular representation as a \( W \)-module implies that

\[ \dim \pi_{\lambda} = \sum_i (e_i + 1) = \dim V_{\lambda}. \]
Example 3.17. For \( m = 3 \), we have \( n = 6 \) and

\[
\begin{array}{c|c}
\lambda & e_1, e_2, \ldots \\
(6) & 0 \\
(5, 1) & 1, 2 \\
(4, 2) & 1, 2, 3 \\
(4, 1, 1) & 0, 1, 2, 3 \\
(3, 3) & 0, 3 \\
(3, 2, 1) & 0, 1^2, 2^2, 4 \\
(3, 1, 1, 1) & 0, 1, 2, 3 \\
(2, 2, 2) & 0, 3 \\
(2, 2, 1, 1) & 1, 2, 3 \\
(2, 1, 1, 1, 1) & 1, 2 \\
(1, 1, 1, 1, 1) & 0 \\
\end{array}
\]

Lemma 3.18. The exponents of \( \lambda \) equal the exponents of \( \lambda' \).

Proof. There is an algebra isomorphism \( \text{sgn} : H_e \rightarrow H_{-c} \) defined by \( \text{sgn}(x) = x \), \( \text{sgn}(y) = y \) and \( \text{sgn}(w) = (-1)^{\ell(w)} w \), where \( x \in h^*, y \in h \), \( w \in \mathfrak{S}_n \) and \( \ell \) is the length function. It is clear from (3.1) that \( \text{sgn} \) is \( SL_2 \)-equivariant. Moreover \( \text{sgn} L(\lambda) \simeq L(\lambda') \). In particular, \( \text{sgn} L(m) \simeq L(m) \). This isomorphism maps \( V_\lambda \) to \( V_{\lambda'} \) since \( \text{sgn} \pi_\lambda \simeq \pi_\lambda \otimes \text{sgn} \simeq \pi_{\lambda'} \). \( \square \)

Using the deeper combinatorics of Macdonald polynomials, we prove

Proposition 3.19. \( \chi_T(V_\lambda) = \bar{K}_{\lambda, m}(q, q^{-1}) \).

Proof. Let \( s_\lambda \) denote the Schur polynomial associated to the partition \( \lambda \) so that \( s_\lambda \left[ \frac{Z}{1-q} \right] \) is a particular plethystic substitution of \( s_\lambda \); we refer the reader to [H] for details.

The module \( L(m) \) is a graded quotient of the Verma module \( \Delta(m) = H_e(W) \otimes_{\mathbb{C}[h^*] \times W} \pi_m \). The graded \( W \)-character of \( \Delta(m) \) is given by \( s_m \left[ \frac{Z}{1-q} \right] \). As shown in [G], the graded multiplicity of \( L(m) \) in \( \Delta(m) \) is given by

\[
(q)_n^{-1} q^{-n(m)} f_m(q) = H_m(q)^{-1} = \prod_{i=1}^{m} (1 - q^{2i-1})^{-(m-i)}
\]

Therefore, the graded \( W \)-character, shifted by \( q^{-n(m)} \) so that \( eL(m) \) is in degree zero, of \( L(m) \) equals \( q^{-n(m)} H_m(q) s_m \left[ \frac{Z}{1-q} \right] \). This implies that

\[
\chi_T(V_\lambda) = \left< s_\mu, q^{-n(m)} \prod_{i=1}^{m} (1 - q^{2i-1})^{m-i} s_m \left[ \frac{Z}{1-q} \right] \right>.
\] (3.20)

The fact that the right hand side of (3.20) equals \( \bar{K}_{\lambda, m}(q, q^{-1}) \) follows from the property of transformed Macdonald polynomials, [H] Proposition 3.5.10]. \( \square \)

Remark 3.21. A similar analysis can be done for other Coxeter groups \( W \). For instance, when \( W \) is a Weyl group of type \( B \) and \( c \) generic, it is easily seen that \( X_c(W)^{sl_2} = \emptyset \).

4. The Hilbert Scheme of Points in the Plane

The group \( SL_2 \) also acts naturally on the Hilbert scheme \( \text{Hilb}^n \mathbb{C}^2 \) of \( n \) points in the plane. This is the restriction of a \( GL_2 \)-action, induced by the natural action of \( GL_2 \) on \( \mathbb{C}^2 \).
4.1. The $T$-fixed points $y_{\lambda}$ in $\Hilb^n \mathbb{C}^2$ are also labeled by partitions $\lambda$ of $n$. If $I$ is the $T$-fixed, codimension $n$ ideal labeled by $\lambda$, then it is uniquely defined by the fact that the corresponding quotient $\mathbb{C}[x, y]/I_\lambda$ has basis given by $x^iy^j$ with

$$(i, j) \in Y_\lambda := \{(i, j) \in \mathbb{Z}^2 \mid 0 \leq j \leq \ell(\lambda) - 1, 0 \leq i \leq \lambda_j - 1\},$$

the Young tableau of $\lambda$. The orbit $GL_2 \cdot y_{\lambda}$ is denoted $O_\lambda$. Identify $\mathbb{C}^x$ with the scalar matrices in $GL_2$. Then $(\Hilb^n \mathbb{C}^2)^{GL_2}$ is the moduli space of homogeneous ideals of codimension $n$ in $\mathbb{C}[x, y]$, as studied in [1]. It is a smooth, projective $GL_2$-stable subvariety of $\Hilb^n \mathbb{C}^2$, containing the points $y_{\lambda}$. Notice that the $GL_2$-orbits and $SL_2$-orbits in $(\Hilb^n \mathbb{C}^2)^{GL_2}$ agree since the action factors through $PGL_2$.

**Lemma 4.1.** If $n = \frac{m(m+1)}{2}$, for some integer $m$, then $(\Hilb^n \mathbb{C}^2)^{GL_2} = \{y_m\}$. Otherwise, $(\Hilb^n \mathbb{C}^2)^{GL_2} = \emptyset$.

**Proof.** This follows from [KT] Lemma 12. Alternatively, notice that if $y_{\lambda}$ is fixed by $GL_2$, then $\mathbb{C}[x, y]/I_\lambda$ is an $GL_2$-module. Since each graded piece of $\mathbb{C}[x, y]$ is an irreducible $GL_2$-module, this implies that there is some $m$ such that $I_\lambda = \mathbb{C}[x, y]_{\geq m}$ and hence $\lambda = m$. \qed

We say that a partition $\lambda$ is steep if $\lambda_1 > \cdots > \lambda_t > 0$.

**Proposition 4.2.** Let $\lambda \neq m$ be a partition of $n$ and set $K = \Stab_{SL_2}(y_{\lambda})$.

1. If $\lambda$ is steep then $K = B$, and if $\lambda^t$ is steep then $K = B_-$. In both cases, $O_\lambda \simeq \mathbb{P}^1$.
2. If neither $\lambda$ or $\lambda^t$ is steep, then $K = T$ if $\lambda \neq \lambda^t$ and $K = N(T)$ if $\lambda = \lambda^t$. In both cases the complement to $O_\lambda$ in $\overline{O_\lambda}$ equals $\mathbb{P}^1$.
3. The orbit $O_\lambda$ is closed if and only if $\lambda$ or $\lambda^t$ is steep.

**Proof.** If $\lambda$ is steep then [KT] Lemma 12 shows that $B \subset K$. If $\dim K > \dim B$, then $\dim K = 3$ i.e. $K = SL_2$ and $\lambda = m$ (notice that $m$ is the only partition such that both $\lambda$ and $\lambda^t$ are steep). Therefore $\dim B = \dim K$ and hence $K^0 = B$. But then $N_{SL_2}(B) = B$ implies that $K = B$. Since $y_\lambda = w_0 \cdot y_{\lambda}$, if $\lambda^t$ is steep then $K = w_0 B w_0^{-1} = B_-$. This proves part (1).

Assume now that neither $\lambda$ nor $\lambda^t$ are steep. Let $\text{Lie} K = \mathfrak{t}$. Since $\mathfrak{t} \subset \mathfrak{t}$, but $\mathfrak{t} \neq \mathfrak{b, sl}_2$, we have $\mathfrak{t} = \mathfrak{t}$ and hence $K = T$ or $N(T)$. Then part (2) follows from Lemma 2.4. Notice that Lemma 2.4 is applicable here even though $\Hilb^n \mathbb{C}^2$ is not complete; this is because $O_\lambda$ is contained in the punctual Hilbert scheme $\Hilb^n_0 \mathbb{C}^2 \subset \Hilb^n \mathbb{C}^2$ of all ideals supported at $0 \in \mathbb{C}^2$. This $SL_2$-stable subvariety is complete.

Part (3) follows directly from parts (1) and (2). \qed

**Question 4.3.** For which $\lambda$ is $\overline{O_\lambda}$ normal?

Associate to a partition $\lambda$ diagonals $d_k := \{(i, j) \in Y_\lambda \mid i + j = k\}$, where $k = 0, 1, \ldots$. For instance, if $\lambda = (4, 3, 3, 1, 1)$, then the diagonals $(d_0, d_1, \ldots)$ are $(1, 2, 3, 4, 2)$. Now construct a new partition $U(\lambda)$ from $\lambda$ by setting $U(\lambda)_i = \{|d_k| \geq i\}$. It is again a partition of $|\lambda|$. Pictorially, we if we visualize the Young tableau $Y_\lambda$ in the English style, as in (3.5), then on the $k$th diagonally (where there are $d_k$ boxes), we have simply moved all boxes as far to the top-right as possible. E.g. $U(4, 3, 3, 1, 1) = (5, 4, 2, 1)$. If instead we move all boxes on the $k$th diagonally as far to the bottom left as possible, we get $U(\lambda)^t$.

**Lemma 4.4.** Let $\lambda$ be a partition.

1. The partition $U(\lambda)$ is steep and $U(\lambda) = \lambda$ if and only if $\lambda$ is steep.
2. $U(\lambda) = m$ if and only if $\lambda = m$. 

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Proof. It is clear from the construction that \( U(\lambda) \) is steep; if \( \lambda_{i-1} = \lambda_i \) for some \( i \) then one can move the box at the end of \( i \)th row further up and to the right on the diagonal that it belongs to. Similarly, if \( \lambda \) is steep, then \( \lambda_{i-1} > \lambda_i \) for all \( i \) such that \( \lambda_i \neq 0 \) implies that there is always a box “above and to the right” of a given box i.e. if \((i, j) \in Y_\lambda \) and \( i \neq 0 \) then \((i - 1, j + 1) \in Y_\lambda \) (this can be viewed as an alternative definition of steep).

Part (2) is also immediate from the construction. \( \square \)

**Proposition 4.5.** Let \( \lambda \) be a partition such that neither \( \lambda \) nor \( \lambda^t \) is steep then \( \overline{O_\lambda} = O_\lambda \cup O_{U(\lambda)} \).

*Proof.* Grade \( \mathbb{C}[x, y] \) by putting \( x \) and \( y \) in degree one. Then every \( I \in O_\lambda \) is graded, \( I = \bigoplus_{k \geq 0} I_k \) and \( \dim I_k \) is independent of \( I \). Since \( \dim(I_{\lambda})_k = k + 1 - d_k \), we deduce that \( \dim I_k = k + 1 - d_k \) for all \( I \in O_\lambda \). By Proposition 4.2 and Lemma 2.4, we know that \( \overline{O_\lambda} = O_\lambda \cup O' \). Thus, there exists a steep partition \( \mu \neq m \) such that \( O' = O_\mu \).

The Hilbert-Mumford criterion implies that there exists some \( I \in O_\lambda \) such that \( J = \lim_{t \to 0} t \cdot I \) is a \( T \)-fixed point in \( O_\mu \). Thus, either \( J = I_\mu \) or \( J = I_\mu^t \). Without loss of generality, \( J = I_\mu \). This implies that \( \dim(I_\mu)_k = k + 1 - d_k \). Since \( \mu \) is steep, \( (I_\mu)_k \) is a \( T \)-submodule of \( \mathbb{C}[x, y]_{k} \), cf. Proposition 4.2(1). Therefore, \( \{x^k, x^{k-1}y, \ldots, x^{k+1-d_k}y^{d_k-1}\} \) is a basis of \( (\mathbb{C}[x, y]/I_\mu)_k \). \( \{x, y\} \subseteq \mathbb{C} \) and such that the map \( \lambda : \mathbb{C}[x, y]/I_\mu \) is equivariant, with \( \lambda \) acting trivially on \( \mathbb{C}[x, y]/I_\mu \). Thus, \( \lambda \) is uniquely defined by this property. Hence \( \lambda = U(\lambda) \). \( \square \)

**Remark 4.6.** For any (homogeneous) ideal \( I \in (\text{Hilb}^n \mathbb{C}^2)^{\mathbb{C}} \), \( I \) is fixed by \( B \) if and only if each \( I_k \) is a \( B \)-submodule of \( \mathbb{C}[x, y]_k \). But the \( B \)-submodules of \( \mathbb{C}[x, y]_k \) are the same as the \( U \)-submodules of \( \mathbb{C}[x, y]_k \). This implies that \( I \) is \( B \)-fixed if and only if it is \( U \)-fixed.

It is known, see eg [GS, Theorem 5.6], that the Hilbert scheme fits into a flat family \( p : \mathcal{X} \to \mathbb{A}^1 \) such that \( p^{-1}(0) \simeq \text{Hilb}^n \mathbb{C}^2 \) and \( p^{-1}(c) \simeq \mathcal{X}_c \) for \( c \neq 0 \). Moreover, \( SL_2 \) acts on \( \mathcal{X} \) such that the map \( p \) is equivariant, with \( SL_2 \) acting trivially on \( \mathcal{X} \). The identification of the fibers is also equivariant. The set-theoretic fixed point set \( \mathcal{X}^{T} \) decomposes

\[
\mathcal{X}^{T} = \bigcup_{\lambda \vdash n} \mathbb{A}_\lambda,
\]

into a union of connected components \( \mathbb{A}_\lambda \) where \( \mathbb{A}_\lambda \simeq \mathbb{A}^1 \) with \( p^{-1}(c) \cap \mathbb{A}_\lambda = \{\lambda\} \) for \( \lambda \neq 0 \) and \( p^{-1}(0) \cap \mathbb{A}_\lambda = \{y_\lambda\} \). The only thing that is not immediate here is that the parameterization of the fixed points in \( \mathcal{X}_c \) match those of \( \text{Hilb}^n \mathbb{C}^2 \). But this can be seen from Lemma 3.6 [H] Lemma 5.4.5 and the fact that a partition is uniquely defined by its hook polynomial.

Then the \( SL_2 \)-varieties \( SL_2 \cdot \mathbb{A}_\lambda \) are connected. Assume that neither \( \lambda \) nor \( \lambda^t \) is steep. Then there are equivariant trivializations

\[
SL_2 \cdot \mathbb{A}_\lambda \simeq SL_2/N(T) \times \mathbb{A}^1 \quad \text{or} \quad SL_2 \cdot \mathbb{A}_\lambda \simeq SL_2/T \times \mathbb{A}^1,
\]

depending on whether \( \lambda = \lambda^t \) or not.

Let \( \tilde{sl}_2 \to sl_2 \) be Grothendieck’s simultaneous resolution and write \( \tau = \) for the composition

\( \tilde{sl}_2 \to sl_2 \to \tilde{sl}_2/SL_2 \simeq \mathbb{A}^1 \), where the second map is \( a \mapsto \frac{1}{2} \text{Tr} a \).

**Conjecture 4.7.** Let \( \lambda \neq m \) be a steep partition. There exists a \( SL_2 \)-equivariant embedding \( sl_2 \hookrightarrow \mathcal{X} \) sending the \( B \)-fixed point \([1 : 0] \in \mathbb{P}^1 \subset \tilde{sl}_2 \) to \( y_\lambda \) and such that the following
diagram commutes

\[
\begin{array}{ccc}
\mathfrak{sl}_2 & \longrightarrow & \mathfrak{X} \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \mathbb{A}^1
\end{array}
\]

4.2. The Procesi bundle. The Procesi bundle \( P \) on \( \text{Hilb}^n \mathbb{C}^2 \) is a \( GL_2 \times \mathfrak{S}_n \)-equivariant vector bundle of rank \( n! \). See [H] and references therein, for details. The fiber \( P_m \) of a \( GL_2 \times \mathfrak{S}_n \)-module, decomposing as

\[
P_m = \bigoplus_{\mu \vdash n} V_\mu \otimes \pi_\mu.
\]

As \( GL_2 \)-modules, we have a decomposition \( V_\mu = \bigoplus_i V(m_i, n_i) \) into a direct sum of irreducible \( GL_2 \)-modules \( V(m_i, n_i) \) with highest weight \( (m_i, n_i) \); here \( m_i, n_i \in \mathbb{Z} \), with \( m_i \geq n_i \).

We call \( (m_1, n_1), (m_2, n_2), \ldots \) the graded exponents of \( \mu \). Let \( H \) denote the 2-torus of diagonal matrices in \( GL_2 \). The character of \( V_\mu \) is given by the cocharge Kostka-Macdonald polynomial,

\[
\chi_H(V_\lambda) = \tilde{K}_{\lambda, m}(q, t).
\]

Notice that this implies \( \tilde{K}_{\lambda, m}(q, t) = \tilde{K}_{\lambda, m}(t, q) \). This can also be deduced directly from the definition of Macdonald polynomials e.g. [H Proposition 3.5.10]. Similarly, equation (4.8), together with standard properties [H Proposition 3.5.12] of Macdonald polynomials imply that

\[
V_\lambda \simeq V_\lambda^* \otimes \det^\otimes n(m).
\]

Thus, if the exponents of \( \lambda \) are \( (m_1, n_1), \ldots \) then the exponents of \( \lambda' \) are

\[
(n(m) - n_1, n(m) - n_1, \ldots)
\]

**Question 4.9.** Is there an explicit formula for the graded exponents of \( \lambda \)?

Next we explain how Lemma 3.18 and Proposition 3.19 can be deduced from the statements of section 4.2 provided one uses Haiman’s \( n! \) Theorem.

Let \( u \) be a formal variable and \( H_{uc} \) the flat \( \mathbb{C}[u] \)-algebra such that \( H_{uc}/(u) \simeq H_0 \) and \( H_{uc}/(u - 1) \simeq H_c \). By [GS Theorem 5.5], the space \( \mathfrak{X} \) can be identified with a moduli space of \( \lambda \)-stable \( H_{uc} \)-modules \( L \) such that \( L|_{\mathfrak{S}_n} \simeq \mathfrak{C}_{\mathfrak{S}_n} \). Here \( \lambda \) is a generic stability parameter; see loc. cit. for definitions. As such, \( \mathfrak{X} \) comes equipped with a canonical bundle \( \mathfrak{P} \) such that each fiber is a \( H_{uc} \)-module. The action of \( SL_2 \) on \( \mathfrak{X} \) lifts to \( \mathfrak{P} \).

**Theorem 4.10.** For \( c \neq 0 \), \( \mathfrak{P}|_{p^{-1}(c)} \simeq \mathcal{R} \) and \( \mathfrak{P}|_{p^{-1}(0)} \simeq P \).

**Proof.** The first claim follows from [EG Section 3] and the second is a consequence of Haiman’s proof of the \( n! \)-conjecture; see the proof of [GS Theorem 5.3] and references therein. \( \square \)

**Corollary 4.11.** As \( \mathfrak{S}_n \times SL_2 \)-modules, \( R_m \simeq P_m \) and hence \( \chi_T(V_\lambda) = \chi_H(V_\lambda)|_{t = q^{-1}} \).

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