Towards Efficient and Reliable Prediction-Based Control Using EDMD

Manuel Schaller\textsuperscript{1}, Karl Worthmann\textsuperscript{1} Friedrich Philipp\textsuperscript{1}, Sebastian Peitz\textsuperscript{2} and Felix Nüske\textsuperscript{2,3}

Abstract. While Koopman-based techniques like extended Dynamic Mode Decomposition (eDMD) are nowadays ubiquitous in the data-driven approximation of dynamical systems, quantitative error estimates were only recently established. To estimate the full approximation error, both sources of error resulting from a finite dictionary (projection error) and only finitely-many data points in the generation of the surrogate model (estimation error) have to be taken into account. We generalize the rigorous analysis of the approximation error to the control setting while simultaneously avoiding the curse of dimensionality by using a recently proposed bilinear approach. In particular, we establish uniform bounds on the approximation error of state-dependent quantities like constraints or a performance index enabling data-based optimal and model predictive control with guarantees.

Keywords. data-based, predicted control, eDMD, finite-data error bound, Koopman operator

1. Introduction

While optimal and predictive control based on models derived from first principles is nowadays well established, data-driven control design is becoming more and more popular. We present an approach via extended Dynamic Mode Decomposition (eDMD) using the Koopman framework to construct a data-driven surrogate model suitable to optimal and predictive control.

The Koopman framework provides the theoretical foundation for data-driven approximation techniques like extended Dynamic Model Decomposition (eDMD), see [15] Chapters 1 and 8: Using the Koopman operator semi-group \((K_t)_{t \geq 0}\) or—equivalently—the Koopman generator \(\mathcal{L}\), so-called observables \(\varphi\) (real- or complex-valued \(L^2\)-functions of the state, e.g., representing a state constraint) can be propagated forward in time via

\[
K_t \varphi = K_0 \varphi + \mathcal{L} \int_0^t K^s \varphi \, ds.
\]

The propagated observable \(K_t \varphi\) can be evaluated for some state \(x_0\) instead of first calculating the solution \(x(t; x_0)\) of the underlying Ordinary Differential Equation (ODE) and then evaluating the observable as depicted in Figure 1.

\textsuperscript{1}Technische Universität Ilmenau, Institute of Mathematics, Optimization-based Control group, Germany (e-mail: \{friedrich.philipp,manuel.schaller,karl.worthmann\}@tu-ilmenau.de).
\textsuperscript{2}Paderborn University, Department of Computer Science, Data Science for Engineering, Germany, (e-mail: sebastian.peitz@upb.de).
\textsuperscript{3}Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany, (e-mail: nueske@mpi-magdeburg.mpg.de).
In the analysis of the eDMD-based approximation $(\tilde{K}^t)_{t \geq 0}$ of the Koopman semi-group $(K^t)_{t \geq 0}$, two sources of error have to be taken into account: The projection and the estimation error. First, a dictionary is chosen, which consists of finitely-many observables $\varphi_1, \ldots, \varphi_N$ and, thus, spans a finite-dimensional subspace $V$. Since the eDMD-based surrogate model is constructed on $V$, a projection error occurs. Second, only a finite number of data points $x_1, \ldots, x_m$ is used to generate the surrogate model, which induces an additional estimation error on $V$. Whereas the convergence of the eDMD-based approximation to the Koopman semi-group in the infinite-data limit, i.e., for $N$ and $m$ tending to infinity, was shown in [13], error estimates for a finite dictionary and finite data explicitly depending on $N$ and $m$ were provided in [25] and [17] for identically-and-independently distributed (i.i.d.) data (points) for the approximation step. While in the former reference also the projection error is analysed, the latter covers the estimation error even for stochastic differential equations and ergodic sampling.

We consider dynamics governed by the nonlinear control-affine differential equation

$$\dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{n_c} g_i(x(t))u_i(t)$$

with initial condition $x(0) = x_0$ and locally Lipschitz-continuous vector fields $g_0, g_1, \ldots, g_{n_c} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Further, we impose the control constraints $u(t) \in U$ for some compact, convex, and nonempty set $U \subset \mathbb{R}^{n_c}$ and define the set of admissible control functions by

$$\mathcal{U}_T(x_0) \triangleq \left\{ u : [0, T] \rightarrow \mathbb{R}^{n_c} \mid \begin{array}{l} u \text{ measurable} \\ \exists ! x(\cdot; x_0, u) \\ u(t) \in U, t \in [0, T] \end{array} \right\},$$

where $x(t; x_0, u)$ denotes the unique solution at time $t \geq 0$. In the following, we assume that $\mathcal{U}_T(x_0)$ is nonempty whenever the set of initial values is suitably chosen.

In order to predict control systems by means of the Koopman framework, [19] as well as [12] proposed a method in which the state is first augmented by the control variable and then, a linear surrogate model depending of the extended state is generated by means of eDMD, see also [15, Chapter 1] for details. Other popular methods are given by, e.g., using a coordinate transformation into Koopman eigenfunctions [8] or a component-wise Taylor series expansion [14]. In this work, however, we will use the bilinear approach, exploiting the control-affine structure of (1) as suggested, e.g., in [25, 22, 18], see also [15, Section 4], for which estimation error estimates were derived.

Acknowledgments: F. Philipp was funded by the Carl Zeiss Foundation within the project DeepTurb—Deep Learning in und von Turbulenz. K. Worthmann gratefully acknowledges funding by the German Research Foundation (DFG; grant WO 2056/6-1, project number 406141926).
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The advantages of this approach are twofold. First, one can observe a superior performance when considering bilinear systems where the control is coupled to the state, which we briefly showcase in Example 1. Second, as the state dimension is not augmented, the data-requirements are less demanding. In particular, consequences of the curse of dimensionality is alleviated in the multi-input case.

The probabilistic bounds on the estimation error of the propagated state observable derived in [17] depend on the control function. However, for optimal and predictive control, it is essential to derive uniform estimates. Our first key contribution is, thus, to establish a bound on the approximation, which uniformly holds for all control functions on the prediction horizon in Section 3.

Our second key contribution is the additional estimation of the projection error using a dictionary consisting of only finitely many observables using techniques well-known for finite-element methods in Section 4, see [2, 20]. The derived bound decays with increasing size of the dictionary. In conclusion and to the best of the authors’ knowledge, this is the first rigorous finite-data error estimate for the eDMD-based prediction for nonlinear control systems taking into account both sources of errors, i.e., the projection and the estimation error.

The paper is organized as follows: In Section 2, we briefly recap eDMD and the bilinear surrogate model obtained for control-affine control systems. Section 3 is devoted to rigorous error bounds on the estimation error—uniform w.r.t. the control, while the projection error is considered in Section 4. Then, the application of the derived bounds in optimal and predictive control is discussed in Section 5 before conclusions are drawn in Section 6.

2. KOOPMAN GENERATOR AND EXTENDED DMD

In this section, we recap extended Dynamic Mode Decomposition (eDMD) as an established methodology to derive a data-based approximation for the Koopman operator \( \mathcal{K}_0 \) and its generator \( \mathcal{L} \) for control-affine systems [1].

2.1. eDMD for autonomous systems. Extended Dynamic Mode Decomposition is a data-based technique to approximately describe the dynamics of observable functions by means of the corresponding Koopman theory [11], see [3, 16, 21].

In this first part, we introduce the data-based finite-dimensional approximation of the Koopman generator and the corresponding Koopman operator for autonomous systems using eDMD, i.e., setting \( u \equiv \bar{u} \in U \), see, e.g., [24, 9] and defining \( x(t) = f(x(t)) \) by

\[
\begin{align*}
  f(x) &= g_0(x) + \sum_{i=1}^{n_c} g_i(x) \bar{u}. \\
  \text{We consider this dynamical system on a compact, convex set } X = \{ x \in \mathbb{R}^n \mid h_j(x) \leq 0 \text{ for all } j \in \{1, 2, \ldots, p\} \} \subseteq \mathbb{R}^n.
\end{align*}
\]

For initial value \( x_0 \in X \), the Koopman semigroup acting on bounded measurable functions \( \varphi \in L^2(X) \) is defined by \( (\mathcal{K}^t \varphi)(x_0) = \varphi(x(t; x_0)) \) on the maximal interval of existence of the solution \( x(t; x_0) \). The corresponding Koopman generator \( \mathcal{L} : D(\mathcal{L}) \subseteq L^2(X) \to L^2(X) \) is defined as

\[
\mathcal{L} \varphi := \lim_{t \to 0} \frac{(\mathcal{K}^t - \text{Id}) \varphi}{t}
\]

and hence, for \( \varphi \in L^2(X) \), \( z(t) = \mathcal{K}^t \varphi \in L^2(X) \) describes the Cauchy problem \( \dot{z} = \mathcal{L} z \), \( z(0) = \varphi \).
For a dictionary of observables $\psi_1, \ldots, \psi_N \in D(\mathcal{L})$, we consider the finite-dimensional subspace

$$V := \text{span}\{\{\psi_j\}_{j=1}^N\} \subset D(\mathcal{L}).$$

The orthogonal projection onto $V$ and the Galerkin projection of the Koopman generator are denoted by $P_V$ and $\mathcal{L}_V := P_V \mathcal{L}|_V$, respectively. Along the lines of [10], we have the representation $\mathcal{L}_V = C^{-1}A$ with $C, A \in \mathbb{R}^{N \times N}$ given by

$$C_{i,j} = \langle \psi_i, \psi_j \rangle_{L^2(\mathcal{X})} \quad \text{and} \quad A_{i,j} = \langle \psi_i, \mathcal{L}\psi_j \rangle_{L^2(\mathcal{X})}.$$  

Consider data points $x_0, \ldots, x_{m-1} \in \mathcal{X}$ and the matrices

$$\Psi(X) := \begin{pmatrix} \psi_1(x_0) \\ \vdots \\ \psi_N(x_0) \end{pmatrix}, \quad \mathcal{L}\Psi(X) := \begin{pmatrix} \langle \mathcal{L}\psi_1(x_0) \rangle \\ \vdots \\ \langle \mathcal{L}\psi_N(x_0) \rangle \end{pmatrix}.$$

Then, defining matrices $\tilde{C}_m, \tilde{A}_m \in \mathbb{R}^{N \times N}$ by

$$\tilde{C}_m = \frac{1}{m}\Psi(X)\Psi(X)^T \quad \text{and} \quad \tilde{A}_m = \frac{1}{m}\Psi(X)\mathcal{L}\Psi(X)^T,$$

an empirical, i.e., purely data-based estimator for the Galerkin projection $\mathcal{L}_V$ is given by $\tilde{L}_m = \tilde{C}_m^{-1}\tilde{A}_m$.

2.2. Bilinear surrogate control system. We briefly sketch the main steps of the bilinear surrogate modeling approach as presented in [23] [22] [18], for which a finite-data estimation error estimate was given in [17]. Considering a control $u \in L^\infty([0, T], \mathbb{R}^{n_c})$, it turns out that by control affinity of the system, also the Koopman generators are control affine.

We set

$$\mathcal{L}^u(t) = \mathcal{L}^0 + \sum_{i=1}^{n_c} u_i(t) \left( \mathcal{L}^{e_i} - \mathcal{L}^0 \right),$$

where $\mathcal{L}^{e_i}, i = 0, \ldots, n_c$, is the Koopman generator for the autonomous system with constant control $\bar{u} \in \{e_0 := 0, e_1, \ldots, e_{n_c}\}$. Then, we can describe the time evolution of an observable function $\varphi \in L^2(\mathcal{X})$ via the bilinear system

$$\dot{z}(t) = \mathcal{L}^u(t)z(t), \quad z(0) = \varphi,$$

where we omitted the control argument in $z(t) = z(t; u)$ for the sake of brevity. This observable function can then be evaluated for an initial state $x_0$ via $z(t; u)(x_0)$, cp. Figure [1]. Analogously to Subsection 2.1, we denote the projection of (1) onto a finite dictionary $V$ by

$$\mathcal{L}^u_V(t) := \mathcal{L}^0_V + \sum_{i=1}^{n_c} u_i(t) \left( \mathcal{L}^{e_i}_V - \mathcal{L}^0_V \right).$$

Hence, the propagation of observable functions $\varphi \in L^2(\mathcal{X})$ projected onto the dictionary $V$ is given by

$$\dot{z}_V(t) = \mathcal{L}^u_V(t)z_V(t), \quad z_V(0) = P_V\varphi.$$

The corresponding approximation by means of eDMD using $m$ data points is defined analogously via

$$\tilde{L}_m(t) := \tilde{L}_m^0 + \sum_{i=1}^{n_c} u_i(t) \left( \tilde{L}_m^{e_i} - \tilde{L}_m^0 \right),$$
where for $i = 1, \ldots, n_c$, $\hat{L}^e_i$ are eDMD-based approximations of $L^e_i$ as described in Subsection 2.1. The corresponding data-based surrogate prediction dynamics read

$$\dot{\hat{z}}_m(t) = \hat{L}^u_m(t)\hat{z}_m(t), \quad \hat{z}_m(0) = P_V\varphi.$$ (9)

Let us highlight that, contrary to the popular DMD with control (DMDc) approach [19, 12], which yields linear surrogate models of the form $A_x + Bu$, numerical simulation studies indicate that bilinear surrogate models are better suited if control and state are coupled, see Example 1. Another key feature of the bilinear approach is that the state-space dimension is not augmented by the number of inputs, which counteracts the curse of dimensionality in comparison to DMDc.

**Example 1.** We briefly present an example with a Duffing oscillator, cf. [17, Section 4.2.1] for more details, using the bilinear surrogate model approach to showcase its superior performance compared to DMDc if state and control are coupled. Consider the dynamics

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ -\delta x_2 - \alpha x_1 - 2\beta x_1 u \end{pmatrix}, \quad x(0) = x_0,$$ (10)

with $\alpha = -1$, $\beta = 1$, $\delta = 0$, $x_0 = (1, 1)^\top$. Figure 2 shows the prediction accuracy for $m = 100$ and the dictionary $\{\psi_j\}_{j=1}^N$ consisting of monomials with maximal degree five. We observe an excellent agreement for the bilinear surrogate model. In particular, the relative error is below $0.1\%$ for almost three seconds, whereas eDMDc yields a large error of approximately $10\%$ from the start and becomes unstable within the first second.

![Figure 2](image-url)

**Figure 2.** Comparison of (10), the bilinear and the DMDc approach for a random control input $u(t) \in [-1, 1]$.

### 3. Uniform finite-data bounds on the estimation error

In this section, we present an error bound that is uniform in the chosen control, which is constrained to values within a compact set $U$, refining the error estimate of [17]. We require the following standard assumption to derive the data-based surrogate model and probabilistic bounds on the estimation error depending on the number of data points.

**Assumption 2.** Assume that the data, for each autonomous system with constant control $u \equiv e_i$ for $i \in \{0, \ldots, n_c\}$, is sampled i.i.d. from the Lebesgue measure and contained in the compact subset $X$. 


Combining error bounds for each individual autonomous system \((u \equiv e_i, i = 0, \ldots, n_c)\) and using the control-affine structure of (11) allows to derive the following error bound, extending our previous work by incorporating control constraints and providing a uniform estimation error bound independently of the chosen control function.

**Proposition 3.** Suppose that Assumption 2 holds and let \(\mathbb{U} \subset \mathbb{R}^{n_c}\) be compact, and nonempty. Then, for given error bound \(\varepsilon > 0\), confidence level \(1 - \delta \in (0, 1)\) and \(t \geq 0\), the probabilistic error bound

\[
P\left(\|\mathcal{L}_t^u - \hat{\mathcal{L}}_t^m\|_F \leq \varepsilon\right) \geq 1 - \delta.
\]

holds for all measurable control functions \(u \in L^\infty(0, \infty; \mathbb{R}^{n_c})\) satisfying \(u(t) \in \mathbb{U}\) for almost all \(t \geq 0\) if the number of data points satisfies \(m \geq m = \mathcal{O}\left(\frac{N^2}{\varepsilon^2}\right)\).

**Proof.** As \(\mathbb{U}\) is compact, we set

\[
C = \frac{\varepsilon}{(n_c + 1)(1 + \max_{u \in \mathbb{U}} \|u\|_1)}
\]

with \(\|u\|_1 := \sum_{i=1}^{n_c} |u_i|\) and denote by \(A^{(k)}\), \(k = 0, \ldots, n_c\), the matrix defined by

\[
(A^{(k)})_{i,j} = \langle \psi_i, \mathcal{L}^{(k)} \psi_j \rangle_{L^2(\mathbb{X})}
\]

with \(e_0 := 0\). Further, set \(\tilde{\delta} = \delta \left(\frac{n}{(n_c + 1)}\right)^2\) and for \(k = 0, \ldots, n_c\)

\[
\tilde{\delta}_k = \min \left\{1, \frac{1}{\|A^{(k)}\| \|C^{-1}\|} \right\} \cdot \frac{\|A^{(k)}\| \varepsilon_0}{2\|A^{(k)}\| \|C^{-1}\| + \varepsilon_0}.
\]

Then, choose a number of data points \(m \in \mathbb{N}\) such that

\[
m \geq \max_{k=0,\ldots,n_c} \frac{N^2}{\tilde{\delta}_k^2} \max \left\{\|\Sigma_{A^{(k)}}\|^2, \|\Sigma_C\|^2\right\}
\]

where \(\Sigma_{A^{(k)}}\) and \(\Sigma_C\) are variance matrices defined via

\[
(S_{A^{(k)}})_{i,j} = \int_{\mathbb{X}} \psi_i^2(x) (g_k(x) \cdot \nabla \psi_j(x))^2 \, dx - \left(\int_{\mathbb{X}} \psi_i(x) g_k(x) \cdot \nabla \psi_j(x) \, dx\right)^2,
\]

\[
(S_C)_{i,j} = \int_{\mathbb{X}} \psi_i^2(x) \psi_j^2(x) \, dx - \left(\int_{\mathbb{X}} \psi_i(x) \psi_j(x) \, dx\right)^2.
\]

for \(i, j = 1, \ldots, N\). Using the minimal amount of data \(m\) specified in (11), we obtain probabilistic error estimates for the individual generators \(\hat{\mathcal{L}}_m\) via [17; Theorem 12]:

\[
P\left(\|\mathcal{L}_t^u - \hat{\mathcal{L}}_t^m\| \leq C\right) \geq 1 - \frac{\delta}{n_c + 1}
\]

for all \(i = 0, \ldots, n_c\). Having obtained an error bound on the individual generators, this estimate can straightforwardly be used to obtain the estimate for

\[
\mathcal{L}_t^u - \hat{\mathcal{L}}_t^m = \mathcal{L}_t^u - \hat{\mathcal{L}}_t^0 + \sum_{i=1}^{n_c} u_i(t) (\mathcal{L}_t^{e_i} - \hat{\mathcal{L}}_t^{e_i} - (\mathcal{L}_t^0 - \hat{\mathcal{L}}_t^0))
\]

analogously to [17; Proof of Theorem 17].

Having an estimation error estimate on the projected non-autonomous generator at hand, a bound on resulting trajectories of observables can be derived using Gronwall’s inequality.
Corollary 4. Let Assumption 2 hold. Let $T, \varepsilon > 0$ and $\delta \in (0, 1)$ and $z^0 \in \mathbb{V}$. Then there is a number of data points $m = O\left(\frac{N}{\varepsilon^2}\right)$ such that for any $m \geq m$, the solutions $z, \tilde{z}$ of

$$
\dot{z}(t) = L^\varphi(t)z \quad z(0) = z^0
$$

and

$$
\dot{\tilde{z}}(t) = \tilde{L}^\varphi(t)\tilde{z} \quad \tilde{z}(0) = z^0
$$

satisfy

$$
\min_{t \in [0, T]} \mathbb{P}\left(\|z(t)(x_0) - \tilde{z}(t)(x_0)\| \leq \varepsilon\right) \geq 1 - \delta
$$

for all $x_0 \in \mathbb{X}$ and measurable control functions $u \in L^\infty(0, T; \mathbb{R}^n_c)$ satisfying $u(t) \in \mathbb{U}$ such that the state response along the dynamics is contained in $\mathbb{X}$.

Proof. The proof follows by straightforward modifications of [17, Proof of Corollary 18] using the uniform data requirements of Proposition 3. □

Note that our approach to approximate the generator only requires the state to be contained in $\mathbb{X}$ up to any arbitrary small time $t > 0$ to be able to define the generator as in (3). Then, in order to obtain error estimates for arbitrary long time horizons when going to a control setting, we have to ensure that the state trajectories remain in the set $\mathbb{X}$ by means of our chosen control function. Besides a controlled forward-invariance of the set $\mathbb{X}$, this can be ensured by choosing an initial condition contained in a suitable sub-level set of the optimal value function of a respective optimal control problem, see, e.g., [1] or [5] for an illustrative application of such a technique in showing recursive stability of Model Predictive Control (MPC) without stabilizing terminal constraints for discrete- and continuous-time systems, respectively.

4. Uniform finite-data bounds on the projection error

In this section, we present the second main result of this paper. If the dictionary $\mathbb{V}$ forms a Koopman invariant subspace, Corollary 4 directly yields an estimate for the observables, as the original system and the projected system coincide.

If this is not the case, one further has to analyze the error resulting from projection onto the dictionary $\mathbb{V}$. To this end, we choose a dictionary that consists of our constraint functions $h_i, \ i = 1, \ldots, p$, as introduced in Section 1 and is further enriched by finite elements, i.e.,

$$
\mathbb{V} = \text{span}\{\{\psi^i\}_{i=1}^l\}
$$

where $\psi_i, i = 1, \ldots, l$, are the usual linear hat functions defined on a regular uniform triangulation of the compact set $\mathbb{X}$ with meshsize $\Delta x > 0$ (e.g., the incircle diameter of each cell) and nodes $x_j, j = 1, \ldots, l$, i.e., we have $\psi_i(x_j) = \delta_{ij}$, where the latter is the Kronecker symbol. As usual in finite-element theory, we assume that $\mathbb{X}$ has a locally Lipschitz boundary. Note that, if the constraint functions are linearly independent, the size of the dictionary is proportional to $\frac{1}{\Delta x^d}$ and $p$. For details on finite elements, we refer to, e.g., [2] [20].

Next, we provide a bound on the full approximation error extending [25, Proposition 5.1] to non-autonomous and control systems.

Theorem 5. Suppose that Assumption 2 holds and consider an observable $\varphi \in C^2(\mathbb{X}, \mathbb{R})$, a probabilistic tolerance $\varepsilon > 0$ and a confidence level $1 - \delta \in (0, 1)$ be given. Then, there
is a mesh size $\Delta x = O(\varepsilon)$ and a required amount of data $m = O\left(\frac{1}{\varepsilon^2} \left(\frac{1}{\varepsilon} + p\right)^2\right)$, such that, for $\tilde{z}_m(0)$, $P\{\varphi(x(t; x_0, u)) - \tilde{z}_m(t; x_0, u)\} \leq \varepsilon \geq 1 - \delta$

is guaranteed for the data-based prediction using the bilinear surrogate dynamics generated with $m \geq m$ data points — independently of the initial value $x_0 \in X$ and the control $u \in U_T(x_0)$ if $x(t; x_0, u) \in X$ holds on $[0, T]$.

Proof. First, we have $\varphi(x(t_0; x_0, u)) = z(t; u)_0$, where $z$ solves $z(t) = z(t; u)$, initial datum $z(0) = \varphi$, that is, using $L^0 \varphi = g_0 \cdot \nabla \varphi$, $L^\varepsilon \varphi = (g_0 + g_i) \cdot \nabla \varphi$ and abbreviating $z(t) = z(t; u), \digamma(t) = L^\varepsilon z(t) = \left(\begin{array}{c} L^0 + \sum_{i=0}^{n_e} u_i(t) \left(L^\varepsilon - L^0\right) \end{array}\right) z(t) = \left(\begin{array}{c} g_0 + \sum_{i=0}^{n_e} u_i(t) (g_i - g_0) \end{array}\right) \cdot \nabla z(t)$.

This can be viewed as a linear transport equation

$$\frac{d}{dt} z(t, x) = a(t, x) \cdot \nabla z(t, x), \quad z(0, \cdot) = \varphi(\cdot),$$

along the time- and space-dependent vector field

$$a(t, x) := g_0(x) + \sum_{i=0}^{n_e} u_i(t) (g_i(x) - g_0(x)),$$

which by compactness of the control constraint set is uniformly bounded in $u$, that is, there are $a, \overline{a} \in \mathbb{R}$ such that for all $u \in L^\infty(0, T; U)$, a.e. $t \in (0, T)$ and $x \in X$, we have that

$$a \leq a(t, x) \leq \overline{a}.$$

In variational form, this evolution equation reads, for all $v \in L^2(X)$ and $t \in (0, T)$,

$$\frac{d}{dt} \langle z(t, \cdot), w \rangle_{L^2(X)} = \langle a(t, \cdot) \cdot \nabla z(t, \cdot), w \rangle_{L^2(X)}$$

$$\langle z(0, \cdot), w \rangle_{L^2(X)} = \langle \varphi, w \rangle_{L^2(X)}$$

By $H_v$, we denote the solution of the projected system

$$\hat{z}_V(t) = \hat{L}_v z(t)$$

with $\hat{L}_v$ as defined in (12), or in variational form, for all test functions $w_V \in V$ and $t \in (0, T)$,

$$\frac{d}{dt} \langle z_V(t, \cdot), w_V \rangle_{L^2(X)} = \langle a(t, \cdot) \cdot \nabla z_V(t, \cdot), w_V \rangle_{L^2(X)}$$

$$\langle z_V(0, \cdot), w_V \rangle_{L^2(X)} = \langle \varphi, w_V \rangle_{L^2(X)}.$$

As $\varphi \in C^2(X, \mathbb{R})$, the projection error, i.e., the difference of $z$ and $z_V$ can be bounded using finite element convergence results, cf. [20] Section 14.3]. In our case of linear finite elements, an application of the estimate [20] Eq. (14.3.16)] reads

$$\langle z(t, x) - z_V(t, x) \rangle \leq c \Delta x$$

where $c = c(\varphi, X)$ for some $c \geq 0$ and all $t \in (0, T)$ and a.e. $x \in X$. Thus, we conclude, for a.e. $x_0 \in X$,

$$|\varphi(x(t; x_0, u)) - \tilde{z}_m(t; x_0)| = |z(t, x_0) - \tilde{z}_m(t; x_0)| \leq |z(t, x_0) - z_V(t, x_0)| + |z_V(t, x_0) - \tilde{z}_m(t, x_0)|$$

The first term can be bounded by $\frac{\varepsilon}{2}$ by choosing a mesh width $\Delta x = O(\varepsilon)$ using (12). The second term can be estimated by $\frac{\varepsilon}{2}$ with confidence level $\delta$ by means of Corollary [1] with
\( m = \mathcal{O} \left( \frac{N^2}{\varepsilon^2} \right) \) data requirements. As the dictionary size is given by \( N = \mathcal{O} \left( \frac{1}{\Delta x^d} + p \right) = \mathcal{O} \left( \frac{1}{\varepsilon^2} + p \right) \), the result follows.

**Remark 6.** Theorem 5 yields data requirements \( m = \mathcal{O}(\varepsilon^{-(2d+2)}) \) to approximate the generator on a \( d \)-dimensional domain and, thus, suffers from the curse of dimensionality, see also [25] for a comparison of eDMD for system identification in comparison to other methods. We stress the advantages of the our approach in terms of data efficacy in view of the exponential scaling w.r.t. the state dimension since the state dimension is only incremented by one (instead of the input dimension \( n_c \)).

5. **Optimal and Model Predictive Control**

In this section, we demonstrate the effectiveness of the derived uniform approximation error bound in optimal and predictive control. To this end, we begin with the Optimal Control Problem

\[
\text{(OCP)} \quad \text{Minimize}_{u \in \mathcal{U}_T(x_0)} \int_0^T \ell(x(t; x_0, u), u(t)) \, dt
\]

subject to the initial condition \( x(0) = x_0 \), the control affine system dynamics (1), and the state constraints

\[
h_j(x(t; x_0, u)) \leq 0 \quad \forall j \in \{1, 2, \ldots, p\}
\]

where the set \( \mathcal{U}_T(x_0) \subset L^\infty([0, T], \mathbb{R}^{n_c}) \) of admissible control functions is given by (2).

The key challenge is to properly predict the performance index of (OCP) and satisfaction of the state constraints (13). Both quantities are evaluated along the state dynamics (1), i.e., the stage cost \( \ell \) and the constraint functions \( h_j, j = 1, 2, \ldots, p \). Instead of propagating the state dynamics and then evaluating these observables, we employ the Koopman framework as outlined in Section 2 to obtain, for all \( t \in [0, T] \), the equality

\[
(K^t_u \varphi)(x_0) = \varphi(x(t; x_0, u)).
\]

Using the observables \( \varphi = h_j \) for all \( j \in \{1, 2, \ldots, p\} \) enables us to ensure constraint satisfaction. We emphasize that the stage cost \( \ell \) is often separable, i.e.,

\[
\ell(x, u) = \ell_1(x) + \ell_2(u).
\]

In this case, \( \varphi = \ell_1 \) may be directly used as an observable while \( \ell_2 \) is at our disposal anyway. In the following, separability is assumed to streamline the presentation. Otherwise, one can consider the coordinate functions as observables, i.e., \( \varphi = x_i \) for all \( i \in \{1, \ldots, n\} \), to evaluate \( \ell \).

Since the Koopman operator \( K^t_u \) is, in general, not known analytically, we resort to eDMD as outlined in Section 2 to derive a data-based finite-dimensional approximation \( \tilde{K}^t_u \). Theorem 5 allows to rigorously ensure constraint satisfaction and a bound \( \varepsilon > 0 \) on the approximation error w.r.t. the stage cost—also termed Lagrange term in optimal control—with confidence level \( 1 - \delta \) if a sufficiently large amount of data \( m = \mathcal{O} \left( \frac{1}{\varepsilon^2} \left( \frac{1}{\varepsilon^2} \right)^2 \right) \) and a small enough finite element mesh size \( \Delta x = \mathcal{O}(\varepsilon) \) is used.

In conclusion, Proposition 7 allows us to approximately solve the problem (OCP) using the derived eDMD-based, bilinear surrogate model with guarantees w.r.t. constraint satisfaction and performance.
Proposition 7 (State constraint and stage cost). Let Assumption 2 hold. Further, suppose that $\ell_1, h_i \in C^2(\mathbb{X}, \mathbb{R})$, $i \in \{1, 2, \ldots, p\}$, hold. Then, for every probabilistic tolerance $\varepsilon > 0$ and every confidence level $1 - \delta \in (0, 1)$, every initial value $x_0 \in \mathbb{X}$ such that $x(t; x_0, u) \in \mathbb{X}$ for the solution of \( \mathcal{P} \) contained in $\mathbb{X}$,

1) the probabilistic performance bound

$$\mathbb{P} \left( \| \ell(x(t; x_0, u), u(t)) - \hat{\ell}_m(t; x_0, u) \| \leq \varepsilon \right) \geq 1 - \delta$$

for $\hat{\ell}_m(t; x_0, u) = \hat{\ell}_{1,m}(t; x_0, u) + \ell_2(u(t))$ hold and

2) the probabilistic state-constraint satisfaction, i.e.,

$$\mathbb{P}(h_i(x(t; x_0; u)) \leq 0) \geq 1 - \delta,$$

is satisfied provided that $\hat{h}_{i,m}(t; x_0, u) \leq -\varepsilon$ holds,

where $\hat{\ell}_{1,m}$, $\hat{h}_{i,m}$, $i \in \{1, 2, \ldots, p\}$, are predicted along the bilinear surrogate dynamics \( \mathcal{P} \) with initial states $\hat{\ell}_{1,m}(0; x_0, u) = P_{\mathcal{P}} \ell_1$, $\hat{h}_{i,m}(0; x_0, u) = P_{\mathcal{P}} h_i$ if the following data requirements are satisfied, i.e.,

- $m \geq \tilde{m}(\varepsilon, \delta)$ as defined in Proposition 3 for the estimation error, and
- $\Delta x \leq c(h, \ell_1, \mathbb{X})$ as defined in Condition 12 for the projection error.

Proof. For the first claim w.r.t. the stage cost $\ell$, we invoke the assumed separability to compute

$$\ell(x(t; x_0, u), u(t)) - \hat{\ell}_m(t; x_0, u) = \ell_1(x(t; x_0, u)) - \hat{\ell}_{1,m}(t; x_0, u).$$

The statement then follows by setting $\varphi = \ell_1$ in Theorem 5. The proof of the second claim 2) follows by setting $\varphi = h_i$, $i = 1, \ldots, p$ in Theorem 5. \( \square \) \( \square \)

In view of this result bounding the stage cost error and yielding chance constraint satisfaction, we briefly provide an outlook with respect to predictive control.

Towards Model Predictive Control: OCPs also play a predominant role in optimization-based control techniques like Model Predictive Control (MPC), where Problem (OCP) on an infinite-time horizon, i.e., $T = \infty$, is approximately solved by solving (OCP) at successive time instants $i \delta$, $i \in \mathbb{N}_0$, on the prediction horizon $[i \delta, i \delta + T]$ subject to the current state as initial value, see, e.g., the monographs and \( \square \) w.r.t. MPC for continuous-time systems. Having obtained rigorous error estimates in view of optimal control, this paves the way of analyzing data-driven MPC schemes as proposed in \( \square \) and \( \square \) w.r.t. recursive feasibility or stability.

6. CONCLUSION AND OUTLOOK

Motivated by data-based surrogate modeling for optimal control problems with state constraints, we derived quantitative error estimates for eDMD-approximations of control systems. In this context, we provided a novel uniform bound on the estimation error in terms of the control input and a projection error estimate. Further, using these results, we derived a stage cost error bound and chance constraint satisfaction for the original, possibly unknown system, if a tightened constraint for the data-based surrogate model holds.

In future work, we further elaborate the presented results towards optimal control to derive suboptimality estimates depending on both data and dictionary size. Moreover, a sensitivity analysis of the OCP could reveal robustness of optimal solutions w.r.t. approximation errors, that can be further exploited by numerical techniques, cf. \( \square \). Moreover, the presented results enable the rigorous analysis of data-driven MPC schemes.
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