BOCHNER-RIESZ MEANS FOR THE HERMITE AND SPECIAL HERMITE EXPANSIONS

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ABSTRACT. We consider the Bochner-Riesz means for the Hermite and special Hermite expansions and study their $L^p$ boundedness with the sharp summability index in a local setting. In two dimensions we establish the boundedness on the optimal range of $p$ and extend the previously known range in higher dimensions. Furthermore, we prove a new lower bound on the $L^p$ summability index for the Hermite Bochner-Riesz means in $\mathbb{R}^d$, $d \geq 2$. This invalidates the conventional conjecture which was expected to be true.

1. INTRODUCTION

Let $\mathcal{H}$ denote the Hermite operator

$$-\Delta + |x|^2 = -\sum_{i=1}^{d} \partial^2_i + x_i^2, \quad x = (x_1, \cdots, x_d), \quad d \geq 1$$

which is non-negative and selfadjoint with respect to the Lebesgue measure on $\mathbb{R}^d$. The spectrum of the operator $\mathcal{H}$ is given by the set $2\mathbb{N}_0 + d$. Here $\mathbb{N}_0$ denotes the set of nonnegative integers. For each $k \in \mathbb{N}_0$, the Hermite polynomial $H_k(t)$ on $\mathbb{R}$ is given by Rodrigues’ formula $H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} (e^{-t^2})$, and the $L^2$ normalized Hermite functions $h_k(t) := (2^{k}k!\sqrt{\pi})^{-1/2} H_k(t)e^{-t^2}/2, \quad k \in \mathbb{N}_0$ form an orthonormal basis of $L^2(\mathbb{R})$. In higher dimensions the $d$-dimensional Hermite functions are given by the tensor products of $h_k$:

$$\Phi_\alpha(x) = \prod_{i=1}^{d} h_{\alpha_i}(x_i), \quad \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d.$$ 

The Hermite operator and functions respectively represent the Hamiltonian and quantum states of the particle for the quantum harmonic oscillator. The functions $\Phi_\alpha$ can also be interpreted as basis functions for the bosonic Fock space via the Bargmann transform. For a detailed discussion regarding the matters, we refer the reader to [8]. The Hermite operator also appears in the representation theory of the Heisenberg group $\mathbb{H}^d$ (see for example [32]).

The set $\{\Phi_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ forms a complete orthonormal system in $L^2(\mathbb{R}^d)$ and the functions $\Phi_\alpha$ are eigenfunctions for the Hermite operator with eigenvalue $2|\alpha| + d$ where

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On the other hand, Thangavelu [29] showed that (1.1) fails to hold if \( \delta < \frac{1}{\lambda} \) does not converge to \( f \equiv 0 \) (1.2). However, it looks that the estimate (1.1) (or its weaker variants) with \( \delta = \frac{1}{\lambda} \) holds with a uniform constant (1.1) ≤ 2. This can be shown making use of the transplantation theorem due to Kenig-Stanton-Tomas [16], and Fefferman’s counterexample for \( L^p \) boundedness of the ball multiplier [7] (also see [30, Theorem 3.1.2]). Thus we are naturally led to consider the Bochner-Riesz mean:

\[
\Pi^H_\lambda f = \sum_{2|\alpha| + d = \lambda} (f, \Phi_\alpha) \Phi_\alpha.
\]

1.1. Bochner-Riesz means for the Hermite expansion. The Hermite expansion is convergent in \( L^2(\mathbb{R}^d) \) space, but when \( d \geq 2 \) the expansion \( \sum_{\lambda \leq N} \Pi^H_\lambda f \) does not converge to \( f \) as \( N \to \infty \) in \( L^p(\mathbb{R}^d) \) unless \( p = 2 \). This can be shown making use of the transplantation theorem due to Kenig-Stanton-Tomas [16] and Fefferman’s counterexample for \( L^p \) boundedness of the ball multiplier [7] (also see [30, Theorem 3.1.2]). Thus we are naturally led to consider the Bochner-Riesz mean:

\[
S^\delta_\lambda (H)f(x) := \left(1 - \frac{\|H\|_\lambda}{\lambda}\right)^\delta f(x) := \sum_{\lambda \in 2\mathbb{N}_0 + d} \left(1 - \frac{\|H\|_\lambda}{\lambda}\right)^\delta \Pi^H_\lambda f.
\]

The summability exponent \( \delta \) mitigates the influence of new summands \( \Pi^H_\lambda f \) which enter into the summation as \( \lambda \) increases. So, the operator \( S^\delta_\lambda (H) \) has more favorable behavior in perspective of \( L^p \) summability as \( \delta \) becomes larger. The classical Bochner-Riesz problem is to determine the optimal summability order \( \delta \) for which \( S^\delta_\lambda (H)f \) converges to \( f \) in \( L^p \) for a given \( p \in [1, \infty] \). When \( d = 2 \), the problem was settled by Carleson-Sjölin [4]. In higher dimensions progress has been made, however the problem is still left open. See [25, 28, 19] and also see [9, 33] for most recent results and references therein.

In this paper we are concerned with \( L^p \) convergence of the Hermite Bochner-Riesz means, that is to say, the problem of determining the optimal \( \delta \) for which \( S^\delta_\lambda (H)f \) converges to \( f \) in \( L^p \). By the uniform boundedness principle, this problem is equivalent to that of characterizing the optimal \( \delta \) for which the estimate

\[
\|S^\delta_\lambda (H)f\|_p \leq C
\]

holds with a uniform constant \( C \) where \( \|T\|_p := \sup_{\|f\|_p \leq 1} \|Tf\|_p \).

When \( d = 1 \), the problem is almost completely settled except some endpoint cases. Askey and Wainger [11] proved that (1.1) holds with \( \delta = 0 \) if and only if \( 4/3 < p < 4 \). When \( p \leq 4/3 \) or \( p \geq 4 \), combining this with the result due to Thangavelu [29], one can show that \( S^\delta_\lambda \) converges to \( f \) in \( L^p \) for \( \delta > \max\{\frac{2}{3}, \frac{1}{p}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, 0\} \). On the other hand, Thangavelu [29] showed that (1.1) fails to hold if \( \delta < \min\{\frac{2}{3}, \frac{1}{p}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, 0\} \). However, it looks that the estimate (1.1) (or its weaker variants) with \( \delta = \frac{2}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \) still remains open when \( p < 4/3 \) or \( p > 4 \).

In higher dimensions, unlike one dimension, only partial results are known. By the transplantation theorem due to Kenig, Stanton, and Tomas [19], the bound (1.1) implies that the classical Bochner-Riesz means \( S^\delta_\lambda (\Delta) \) is uniformly bounded on \( L^p(\mathbb{R}^d) \) (see Proposition [4.1 and its proof). Thus, by the well known necessary condition for \( L^p \) boundedness of \( S^\delta_\lambda (\Delta) \) (see for example [10, 7]) we have

\[
\delta > \delta(d, p) := \max\{d \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0\}, \quad p \neq 2
\]
if the uniform bound \([1.1]\) holds. This naturally leads to the following conjecture.

**Conjecture 1.1.** Let \(p \in [1, \infty[ \setminus \{2\}\). The uniform estimate \([1.1]\) holds if and only if \([1.2]\) holds.

Karadzhov \([15]\) verified Conjecture \([1.1]\) for \(\max(p, p') \geq 2d/(d - 2)\). His result was based on the optimal \(L^2 - L^p\) spectral projection estimate

\[
\| \Pi^H_{\lambda} \|_{2 \rightarrow p} \leq C \lambda^{\frac{d}{2} + \frac{d}{p} - \frac{1}{2}}, \quad 2d/(d - 2) \leq p \leq \infty.
\]

The estimate plays the role of \(L^2 - L^p\) restriction estimate for the sphere in Stein’s argument \([7]\) which deduces the sharp \(L^p\) bound on \(S^\lambda_\Delta(-\Delta)\) from the \(L^2\) restriction estimates. However, as shown by Koch and Tataru \([18]\), the range of \(p\) where the above estimate is valid can not be extended any further. We refer the reader to \([13, 15]\) and references therein for more about the Hermite spectral projection operator. This means the approach in \([15]\) relying on the \(L^2 - L^p\) spectral estimate is no longer viable when one tries to prove \(L^p\) boundedness for \(\delta > \delta_b\)

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**Local \(L^p\) estimate for \(S^\lambda_\Delta(\mathcal{H})\).** Meanwhile, the kernel of \(S^\lambda_\Delta(\mathcal{H})\) is expressed on a critical region as an Airy type integral and such phenomenon does not occur in the case of the classical Bochner-Riesz operator \(S^\lambda_\Delta(-\Delta)\). Taking this into account, Thangavelu \([31]\) speculated that Conjecture \([1.1]\) may fails \([1]\) when \(\max(p, p') \in (2d/(d - 2), 2)\). Instead of the global estimate \([1.1]\) he considered a local variant of \([1.1]\). To be specific, let us consider the estimate

\[
\| \chi_E S^\lambda_\Delta(\mathcal{H}) \chi_F \|_p \leq C
\]

with a constant \(C\) independent of \(\lambda\) where \(E, F\) are measurable subsets of \(\mathbb{R}^d\). It was shown by Thangavelu \([31]\) that \([13]\) holds with a compact set \(E\) and \(F = \mathbb{R}^d\) for \(\frac{2(d+1)}{d-1} \leq p \leq \infty\) when \([1.2]\) holds. The result is clearly sharp in that the estimates fail if \(\delta < \delta(d, p)\) because of the aforementioned transplantation \([16]\). In analogy to Karadzhov’s approach, a form of local \(L^2 - L^{2(d+1)/(d-1)}\) estimate for \(\Pi^H_{\lambda}\) was utilized. As was shown in \([13]\), the local spectral projection estimate does not extend for \(p < 2(d+1)/(d - 1)\), so we can not expect any progress using \(L^2 - L^p\) estimate for \(\Pi^H_{\lambda}\).

We shall show that Conjecture \([1.1]\) is generally not true when \(\max(p, p') \in (2d/(d - 2), 2)\). In fact, on a certain range of \(p\) we obtain a new lower bound on the summability index \(\delta\) (see Proposition \([1.1]\) for the uniform bound \([1.1]\)). This invalidates Conjecture \([1.1]\). Thus, in order to prove boundedness for \(\delta > \delta(d, p)\) one has to consider a weaker alternative as was done \([31]\). It would be interesting to determine whether \([1.1]\) holds up to the new lower bound but for the present the problem seems to be beyond reach. Instead, we first look into \(L^p\) convergence of \(S^\lambda_\Delta(\mathcal{H})\) in a local setting to make progress on the current state regarding \(L^p\) boundedness of the Hermite Bochner-Riesz means.

As far as the authors are aware, concerning on \(L^p\) boundedness of the Hermite Bochner-Riesz means no further progress has been made beyond Thangavelu’s result \([31]\) until now. In this paper, we extend the range of \(p\) for which \([1.3]\) holds

\(\text{1That is to say, the summability index for } (1.3) \text{ may be bigger than that for the classical Bochner-Riesz means } S^\lambda_\Delta(-\Delta) \text{ when } d \geq 2.\)
under a suitable condition on $E$ and $F$. Even if \((1.3)\) is a weaker variant of the global estimate, the local estimate is still strong enough to imply the sharp $L^p$ bound on $S^d_\lambda(-\Delta)$. More precisely, if the estimate \((1.3)\) holds with $E, F = B(0, \epsilon)$ for any $\epsilon > 0$, from the transplantation theorem \((\cite{10})\) we see that the Bochner-Riesz operator $(\lambda + \Delta)^d_+\,\text{is uniformly bounded on } L^p$.\]

To state our first result, we introduce some notations. Let us set

$$p_0(d) = \begin{cases} 2 \frac{3d+2}{3d-2} & \text{if } d \equiv 0 \, (\text{mod } 2), \\ 2 \frac{3d+1}{3d-3} & \text{if } d \equiv 1 \, (\text{mod } 2), \end{cases}$$

and

$$D(x, y) := 1 + ((x, y)^2) - |x|^2 - |y|^2, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$ 

The following is our first result.

**Theorem 1.2.** Suppose that $E, F \subset \mathbb{R}^d$ are compact sets such that $E \times F \subset \mathcal{D}(c_0) := \{(x, y) \in \mathbb{R}^d : |x|, |y| \leq 1 - c_0, D(x, y) > c_0^2\}$ for some $0 < c_0 < 1$. Then there is a constant $C$ independent of $\lambda$ such that

\begin{equation}
\|\chi_{E_\lambda} S^d_\lambda(\mathcal{H}) \chi_{F_\lambda}\|_p \leq C, 
\end{equation}

provided that $p > p_0(d)$ and $\delta > \delta(d, p)$ where $E_\lambda, F_\lambda$ denote the dilated set $\sqrt{\lambda}E, \sqrt{\lambda}F$, respectively.

When $d = 2$, Theorem 1.2 establishes the estimate \(1.4\) on the optimal range $p$, that is to say, \(1.4\) holds if and only if $\delta > \delta(d, p), \; p \neq 2$. From Theorem 1.2, we have the following $L^p$ convergence result.

**Corollary 1.3.** Suppose that $p > p_0(d)$ and $\delta > \delta(d, p)$. Then for any compact set $K \subset \mathbb{R}^d$ and compactly supported $f \in L^p(\mathbb{R}^d)$, we have

$$\lim_{\lambda \to \infty} \int_K |S^d_\lambda(\mathcal{H}) f(x) - f(x)|^p d\lambda = 0.$$ 

1.2. **Bochner-Riesz means for the special Hermite expansion.** Now we consider the twisted Laplacian which is closely related to the Hermite operator. The twisted Laplacian $\mathcal{L}$ on $\mathbb{C}^d \cong \mathbb{R}^{2d}$ which is defined by

$$\mathcal{L} = -\sum_{j=1}^d \left( \frac{\partial}{\partial x_j} - \frac{i}{2} y_j \right)^2 + \left( \frac{\partial}{\partial y_j} + \frac{1}{2} i x_j \right)^2, \quad x, y \in \mathbb{R}^d$$

has the same discrete spectrum $2\mathbb{N}_0 + d$ as $\mathcal{H}$. The associated eigenfunctions are the special Hermite functions $\Phi_{\alpha, \beta}$ which are given by the Fourier-Wigner transform of the Hermite functions. Indeed, for any multi-index $\alpha, \beta \in \mathbb{N}_0^d$,

$$\Phi_{\alpha, \beta}(z) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i(x, \xi)} \Phi_\alpha(\xi - \frac{1}{2} iy) \Phi_\beta(\xi + \frac{1}{2} iy) d\xi, \quad z = x + iy.$$ 

Then it follows that $\mathcal{L}\Phi_{\alpha, \beta} = (2|\beta| + d)\Phi_{\alpha, \beta}$, thus $\Phi_{\alpha, \beta}$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue $2|\beta| + d$ and the eigenspaces of $\mathcal{L}$ are infinite dimensional. Additionally, $\Phi_{\alpha, \beta}$ satisfies $(-\Delta_z + \frac{d}{2} |z|^2)\Phi_{\alpha, \beta} = (|\alpha| + |\beta| + d)\Phi_{\alpha, \beta}$, which means
Thangavelu verified the conjecture for \(2\) regarding the Bochner-Riesz means for the special Hermite expansion. Fefferman's counterexample \([7]\) this series fails to converge in \(\mathbb{R}^d\).

As seen before in the case of Hermite expansion, by the transplantation in \([16]\) we see the estimate \((1.6)\) implies \(L^p(\mathbb{R}^d)\) is equivalent to the uniform estimate \((1.7)\) when \(2d/p < \delta(2d,p)\). Later, Koch and Ricci \([17]\) proved that \((1.7)\) holds if and only if \(2d/p > \delta(2d,p)\). The problem has been studied by many authors. By the transplantation theorem in \([16]\) and 

In \([30]\], Thangavelu verified the conjecture for \(2N_0 + d\), by \(\Pi^\delta_L\) we denote the projection to the eigenspace of \(\mathcal{L}\) with the eigenvalue \(\lambda\), i.e.,

\[
(1.5) \quad \Pi^\delta_L f(z) := \sum_{\beta:2|\beta| + d = \lambda_{\alpha} \in \mathbb{N}_0} \langle f, \Phi_{\alpha,\beta}\rangle \Phi_{\alpha,\beta}(z), \quad f \in L^2(\mathbb{C}^d).
\]

Since \(\{\Phi_{\alpha,\beta}\}_{\alpha,\beta}\) is a orthonormal basis of \(L^2(\mathbb{C}^d)\), so one can expand \(f\) into the series of special Hermite functions. In fact, we have

\[
f(z) = \sum_{\lambda \in 2N_0 + \delta} \Pi^\delta_L f(z), \quad f \in L^2(\mathbb{C}^d).
\]

By the uniform boundedness principle, the \(L^p\) convergence of \(S^\delta_L(\mathcal{L})f\) for all \(f \in L^p\) is equivalent to the uniform estimate

\[
(1.6) \quad ||S^\delta_L(\mathcal{L})f||_p \leq C||f||_p.
\]

The problem has been studied by many authors. By the transplantation theorem in \([16]\) we see the estimate \((1.6)\) implies \(L^p\) boundedness of the classical Bochner-Riesz operator in \(\mathbb{R}^{2d}\). Thus \((1.6)\) holds only if \(\delta > \delta(2d,p)\), \(p \neq 2\). It seems to be plausible to conjecture that the uniform estimate \((1.6)\) holds if \(\delta > \delta(2d,p)\). In \([30]\), Thangavelu verified the conjecture for \(2d/p < \delta \leq \infty\). Later, the range was extended to \(2d(2d+1)/2d-1 < \delta \leq \infty\) by Ratnakumar, Rawat, and Thangavelu \([21]\). Further progress was made by Thangavelu \([31]\) who showed the local estimate \(||\chi_\varepsilon S^\delta_L(\mathcal{L})||_p \leq C\) holds provided that \(\delta > \delta(2d,p)\) and \(2d(2d+1)/2d-1 < \delta \leq \infty\). The corresponding global estimate was later established by Stempak and Zienkiewicz \([26]\), i.e., they showed that the estimate \((1.6)\) holds for \(2d(2d+1)/2d-1 < \delta \leq \infty\) and \(\delta > \delta(2d,p)\). The common key ingredient of the previous results is the \(L^2-L^p\) projection estimate of the form

\[
(1.7) \quad ||\Pi^\delta_L||_{2 \to p} \leq C\lambda^{d-1/2}\delta^{d/2}\|
\]

with \(C\) independent of \(\lambda\) which was combined with Stein’s argument \([7]\). The projection estimate \((1.7)\) was shown by Stempak and Zienkiewicz \([26]\) for \(2d(2d+1)/2d-1 < \delta \leq \infty\). Later, Koch and Ricci \([17]\) proved that \((1.7)\) holds if and only if \(2d(2d+1)/2d-1 < \delta \leq \infty\) (also, see \([14]\) for \(L^p\)-\(L^q\) estimates for \(\Pi^\delta_L\)). So, further improvement is no longer possible via the estimate \((1.7)\) when \(2d(2d+1)/2d-1 > p\).

Local \(L^p\) convergence of \(S^\delta_L(\mathcal{L})\). Currently, no result with the sharp summability exponent \(\delta(2d,p)\) is known when \(2d(2d+1)/2d-1 > p\). Following the approach in \([31]\), we consider a local variant of \((1.6)\) and prove new estimate with the sharp summability exponent outside the aforementioned range of \(p\). The following is our result regarding the Bochner-Riesz means for the special Hermite expansion.

\[\Phi_{\alpha,\beta}\] is an eigenfunction of the Hermite operator \(-\Delta_z + 1/4|z|^2\). This is the reason that \(\Phi_{\alpha,\beta}\) is called the special Hermite function.
Theorem 1.4. Let $0 < c_0 < 2$, $p > p_0(2d)$, and $\delta > \delta(2d, p)$. Suppose that $E, F \subset \mathbb{R}^{2d}$ are compact sets satisfying $|z - z'| \leq 2 - c_0$ for all $z \in E$, $z' \in F$. Then there exists a constant $C$ independent of $\lambda$ such that

$$\|\chi_{E_\lambda} S_{\lambda}^d(\mathcal{L}) \chi_{F_\lambda}\|_p \leq C,$$

where $E_\lambda, F_\lambda$ denote the dilated set $\sqrt{\lambda}E$, $\sqrt{\lambda}F$, respectively.

Corollary 1.5. Let $p, \delta$ be as in Theorem 1.4. Then for any compact set $K \subset \mathbb{R}^{2d}$ and compactly supported $f \in L^p(K)$ we have

$$\lim_{\lambda \to \infty} \int_B |S_{\lambda}^d(\mathcal{L}) f(x) - f(x)|^p dx = 0.$$

In the case of $d = 1$, from this result we have a complete characterization of $(p, \delta)$ for which the local convergence (1.9) holds. The assumption that $|z - z'| \leq 2 - c_0$ for all $z \in E$, $z' \in F$ is made for technical reason and we don’t know whether the condition is necessary for the estimate (1.8) with the sharp summability index. The assumption can be regarded as a counterpart of the assumption on $\mathcal{D}(x, y)$ in Theorem 1.2. If $|z - z'| \geq 2 + c_0$ for all $z \in E$, $z' \in F$, the kernel of the operator $\chi_{E_\lambda} S_{\lambda}^d(\mathcal{L}) \chi_{F_\lambda}$ rapidly decays (See (5.11)). Thus the assumption can be relaxed so that $||z - z'| - 2| \geq c_0$ for all $z \in E$, $z' \in F$. In order to prove the global bound we need to understand the behavior of the kernel of $S_{\lambda}^d(\mathcal{L})(z, z')$ when $|z - z'|$ is close to 2.

Our approach. The proofs of Theorem 1.4 and 1.2 follow a similar strategy which is inspired by the recent works by Jeong and the authors [13, 14] on the spectral projection operators $\Pi^N_\delta$ and $\Pi^C_\delta$. First, we obtain a explicit expression for the kernels of the operators $S_{\lambda}^N(\mathcal{H})$, $S_{\lambda}^d(\mathcal{L})$ using the Schrödinger propagators $e^{-it\mathcal{H}}$, $e^{-it\mathcal{L}}$ (see (2.1), (3.1)) of which kernel representation is well known. Secondly, combining the expression of kernel with the method of stationary phase we obtain asymptotic expansions of the kernels. This reduces the matter to obtaining the sharp estimate for the oscillatory integral operators which satisfy the Carleson-Sjölin condition [4, 11, 24]. In two dimensions this allows us to obtain the optimal results. However, in higher dimensions the Carleson-Sjölin condition alone is not enough, as was shown by Bourgain [3], to give the sharp bound for $p < 2(d+1)/(d-1)$ (or $p < 2(2d+1)/(2d-1)$). However, an additional ellipticity assumption on the second fundamental form of the phase allows to get the bound on improved range. This observation was first made by one of the author [20]. Thirdly, we show that the phases in the asymptotic expansion satisfy the ellipticity condition (see Lemma 3.10 and 2.13). We combine this with the results regarding the oscillatory integral operator [20, 9]. We refer the reader forward to Section 2.5 for more regarding the oscillatory integral operators.

Notation. Throughout the rest of the paper, we identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$.

- For given $A, B > 0$, we write $B \lesssim A$ if there is a constant $C > 0$ such that $B \leq CA$. Here, if $C$ has to be taken to be small enough, we use the notation $B \ll A$ to mean that $A$ is sufficiently larger than $B$. Furthermore, $A \sim B$ denotes that $A \lesssim B$ and $B \lesssim A$.
- $B_d(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$.
- For an operator $T$ we denote by $T(x,y)$ (or $T(z,z')$) the kernel of $T$. 
• $\partial_x := (\partial_{x_1}, \ldots, \partial_{x_d})^T$, $\partial_{x}^T := (\partial_{x_1}, \ldots, \partial_{x_d})$, so $\partial_x \partial_y = (\partial_x \partial_{y_1}, \ldots, \partial_x \partial_{y_d})_{1 \leq i,j \leq d}$. In particular, if $a(x) = (a_1(x), \ldots, a_d(x))$ is a $\mathbb{C}^d$-valued differentiable function on $\mathbb{R}^d$, then we have $\partial^T a(x) = (\partial_x a_i(x))_{1 \leq i,j \leq d}$.

• By $I_d$ we denote the $d \times d$ identity matrix. If the value of $d$ is clear from the context, we simply denote $I_d$ by $I$.

• For $1 \leq i \leq d$, $e_i$ denotes the $i$-th standard basis in $\mathbb{R}^d$.

• For $S \subset \mathbb{R}^d$ and a constant $a > 0$, we denote $aS = \{ax : x \in S\}$.

2. Bochner-Riesz means for the Hermite expansion:

Proof of Theorem 1.2

In this section we prove Theorem 1.2. We begin by obtaining an explicit expression of the kernel of the Bochner-Riesz means. For the purpose we make use of Mehler’s formula for the Schrödinger propagator.

2.1. Decomposition of $S^\delta_\lambda(\mathcal{H})$. We start by considering the Hermite-Schrödinger propagator $e^{-it\mathcal{H}}$ which is the solution to the Cauchy problem $(i\partial_t - \mathcal{H})u = 0$ and $u(0, x) = f(x)$. The propagator $e^{-it\mathcal{H}}$ can be expressed by the spectral projection operators $\Pi^\mathcal{H}_\lambda$:

$$e^{-it\mathcal{H}} = \sum_{\lambda \in 2\mathbb{N}_0 + d} e^{-it\lambda} \Pi^\mathcal{H}_\lambda.$$ 

On the other hand, by virtue of Mehler’s formula we have an explicit expression of the kernel of $e^{-it\mathcal{H}}$. It is well known that

$$e^{-it\mathcal{H}} f(x) = (2\pi i \sin 2t)^{-\frac{d}{4}} e^{\pi d/4} \int e^{\frac{1}{2}((|x|^2 + |y|^2) \cot 2t - 2(x, y) \csc 2t)} f(y) dy$$

for $f \in S(\mathbb{R}^d)$. For example, see [30] and also see [22] for a detailed discussion regarding derivation of (2.1).

To study $L^p$ boundedness of the Bochner-Riesz means we need to properly decompose the operator $S^\delta_\lambda(\mathcal{H})$. Let $\psi \in C^\infty_c([\frac{1}{4}, 1])$ be a smooth bump function such that $\sum_{j \in \mathbb{Z}} \psi(2^j t) = 1$ for all $t > 0$. Then, setting $\psi^\delta(t) := t^\delta \psi(t)$, we denote

$$\psi_j(t) = \psi^\delta(2^j t), \quad j \geq 1,$$

$$\psi_0(t) = t^\delta \sum_{j \geq 0} \psi(2^j t),$$

so that $t^\delta = \sum_{1 \leq 2^j \leq 4\lambda} 2^j \psi_j(t)$ for $0 \leq t \leq \lambda$. For any bounded continuous function $m$ on $\mathbb{R}$, we define an operator $m(\mathcal{H})$ by setting

$$m(\mathcal{H}) = \sum_{\lambda \in 2\mathbb{N}_0 + d} m(\lambda) \Pi^\mathcal{H}_\lambda.$$ 

Then, since $S^\delta_\lambda(\mathcal{H}) = \lambda^{-\delta} (\lambda - \mathcal{H})^\delta_+$ and since the above summation is taken over the set $2\mathbb{N}_0 + d$, we can write

$$S^\delta_\lambda(\mathcal{H}) = \lambda^{-\delta} (\lambda - \mathcal{H})^\delta_+ = \lambda^{-\delta} \sum_{1 \leq 2^j \leq 4\lambda} 2^j \psi_j(\lambda - \mathcal{H}).$$

Though $\psi_0$ is not smooth, we may assume $\psi_0$ is smooth replacing it with a suitable smooth function because $\mathcal{H}$ has the spectrum $2\mathbb{N}_0 + d$. Thus, the proof of Theorem...
Theorem 1.2 follows if we show

\[ \| \mathcal{X}_e \psi_j (\lambda - \mathcal{H}) \mathcal{X}_e \|_p \lesssim (\lambda 2^{-j})^\delta(d,p), \quad 1 \leq 2^j \leq 4\lambda \]

for \( p > p_0(d) \).

We now relate the operators \( \psi_j (\lambda - \mathcal{H}) \) to the propagator \( e^{it\mathcal{H}} \) via Fourier inversion. In fact, for \( \eta \in \mathcal{S}(\mathbb{R}) \) we have

\[ \tilde{\eta}(\lambda - \mathcal{H}) = \frac{1}{2\pi} \int \eta(t) e^{it(\lambda - \mathcal{H})} dt. \]

Combining this with (2.1) and changing variables \( t \to t/2 \), we get

\[ \tilde{\eta}(\lambda - \mathcal{H})(x,y) = C_d \int \eta(t/2)(\sin t)^{-\frac{d}{2}} e^{i\phi_\lambda(t,x,y)} dt, \]

where

\[ \phi_\lambda(t, x, y) := \frac{\lambda t}{2} + \frac{|x|^2 + |y|^2}{2} \cot t - \langle x, y \rangle \csc t. \]

Instead of dealing with \( \tilde{\eta}(\lambda - \mathcal{H}) \) it is more convenient to work with the rescaled operator. So, for \( \zeta \in \mathcal{S}(\mathbb{R}) \) we define an operator \( [\zeta]_\mathcal{H}^\lambda \) whose kernel is given by

\[ [\zeta]_\mathcal{H}^\lambda(x,y) := \tilde{\zeta}(2\cdot)(\lambda - \mathcal{H})(\sqrt{\lambda x}, \sqrt{\lambda y}) \]

\[ = C_d \int \zeta(t)(\sin t)^{-\frac{d}{2}} e^{i\lambda \mathcal{P}_\mathcal{H}(t,x,y)} dt, \]

where \( C_d \) is a constant depending on \( d \) and

\[ \mathcal{P}_\mathcal{H}(t,x,y) := \frac{t}{2} + \frac{(|x|^2 + |y|^2) \cos t}{2 \sin t} - \frac{\langle x, y \rangle}{\sin t} \]

By scaling, the estimate (2.3) is now equivalent to

\[ \| \mathcal{X}_e \frac{\psi_j (\cdot/2)}{\lambda} \mathcal{X}_e \|_p \lesssim \lambda^{-\frac{d}{2}} (\lambda 2^{-j})^\delta(d,p), \quad 1 \leq 2^j \leq 4\lambda. \]

We occasionally use the following lemma which is a simple consequence of the formula (2.5) for the kernel.

**Lemma 2.1.** Let \( \zeta \in C^\infty_c((0,\infty)) \). Then, for any measurable sets \( E, F \subset \mathbb{R}^d \),

\[ \| \mathcal{X}_E [\zeta]_\mathcal{H}^\lambda \mathcal{X}_F \|_p = \| \mathcal{X}_E [\zeta(-\cdot)]_\mathcal{H}^\lambda \mathcal{X}_F \|_p, \]

\[ \| \mathcal{X}_E [\zeta]_\mathcal{H}^\lambda \mathcal{X}_F \|_p = \| \mathcal{X}_E [\zeta(\cdot + \pi)]_\mathcal{H}^\lambda \mathcal{X}_F \|_p. \]

*Proof.* We observe that \( \mathcal{P}_\mathcal{H}(-t,x,y) = -\phi(t,x,y) \) and \( \mathcal{P}_\mathcal{H}(t + \pi, x,y) = \tilde{\pi} + \mathcal{P}_\mathcal{H}(t,x,-y) \). Using this, (2.10) and changing of variables, we have

\[ [\zeta(-\cdot)]_\mathcal{H}^\lambda(x,y) = C_1 [\zeta]_\mathcal{H}^\lambda(x,y), \]

\[ [\zeta(\cdot + \pi)]_\mathcal{H}^\lambda(x,y) = C_2 [\zeta]_\mathcal{H}^\lambda(x,-y), \]

where \( C_1, C_2 \in \mathbb{C} \) such that \( |C_1| = |C_2| = 1 \). Thus, (2.10), (2.11) give (2.8), (2.9), respectively. \( \square \)
In the expression (2.14) there are singularities at $t = k\pi, k \in \mathbb{Z}$, so we need to make further decomposition by breaking $\hat{\psi}_j(\cdot / 2)$. To do so, we set

$$\nu^l(t) = \sum_{\pm} \psi(\pm 2^l t) + \psi(2^l (\pi \pm t)), \quad l \geq 1,$$

(2.12)

$$\nu^0(t) = \chi_{(-\pi, \pi)}(t) - \sum_{l=1}^{\infty} \nu^l(t),$$

so that $\text{supp} \nu^l \subset (-\pi, \pi)$ and $\sum_{l=1}^{\infty} \nu^l(t - 2k\pi) = 1$ a.e. on $(-\pi, \pi)$. We also set

$$\nu_{j,k}^l(t) = \hat{\psi}_j(t/2) \nu^l(t - 2k\pi),$$

so we have $\sum_{k=\infty}^{\infty} \sum_{l=0}^{\infty} \nu^l(t - 2k\pi) = 1$ a.e. Thus it follows

\[
(2.13) \quad \hat{\psi}_j(\cdot / 2)_{\lambda}^H = \sum_{k=\infty}^{\infty} \sum_{l=0}^{\infty} [\nu_{j,k}^l]_{\lambda}^H.
\]

Now the proof of (2.7) (and Theorem 1.2) is essentially reduced to showing the following.

**Theorem 2.2.** Let $0 < \rho < \pi - 2^{-5}$ and $\lambda \in 2\mathbb{N}_0 + d$. Let $\eta_{\rho}$ be a smooth function such that $\text{supp} \eta_{\rho} \subset [2^{-2}\rho, \rho]$ and $|\eta_{\rho}^{(n)}(t)| \leq C_n \rho^{-n}, n \in \mathbb{N}_0$. Suppose that the sets $E, F$ satisfy the condition in Theorem 1.2, i.e., $E \times F \subset \mathcal{D}(c_0)$. Then, for $p > p_0(d)$ we have

\[
(2.14) \quad \| \chi_E [\eta_{\rho}]_X^H \chi_F \|_p \lesssim \begin{cases} 
\lambda^{-\frac{d}{2}} \rho, & \lambda \leq \lambda^{-1}, \\
\lambda^{-\frac{d}{2}} \lambda^{\delta(d,p)} \rho^{\delta(d,p)+1}, & \lambda > \lambda^{-1}.
\end{cases}
\]

Once we have Theorem 2.2, the proof of (2.7) is rather straightforward. Assuming Theorem 2.2 for the moment, we prove (2.7).

**Proof of (2.7).** Recalling (2.13) we need to obtain bounds on $\| \chi_E [\nu_{j,k}^l]_{\lambda}^H \chi_F \|_p$. By (2.10) we need only to consider, instead of $\nu_{j,k}^l$, the multipliers given by $\hat{\psi}_j(k\pi + t/2) \nu^l(t)$. Then by (2.9) and (2.8) the matter reduces to dealing with the multipliers given by

$$\nu_{j,k}^l := \hat{\psi}_j(k\pi + t/2) \psi(2^l t), \quad \nu_{j,k}^l := \hat{\psi}_j(k\pi \pm (t - \pi)/2) \psi(2^l t),$$

which are supported in $[2^{-2-l}, 2^{-l}]$. One can easily see that the estimates

$$\left| \frac{d^n}{dt^n} \nu_{j,k}^l(t) \right| \leq C 2^n \max(2^n, 2^{n_1}(1 + 2^{j-l})^{-N}(1 + 2^j |k|)^{-N}, n \in \{\pm, \pm\},$$

hold for any $N \in \mathbb{N}$ with a constant $C = C(N)$. Recalling $1 \leq 2^j \leq 4\lambda$, after normalizing the functions $\nu_{j,k}^l$, we apply Theorem 2.2 to $\| \chi_E [\nu_{j,k}^l]_{\lambda}^H \chi_F \|_p$. Thus, for any $N$ we get

\[
\| \chi_E [\nu_{j,k}^l]_{\lambda}^H \chi_F \|_p \lesssim \begin{cases}
\lambda^{-\frac{d}{2}} 2^{-l} (1 + 2^{j} |k|)^{-N}, & 4\lambda < 2^j, \\
\lambda^{-\frac{d}{2}} \lambda^{\delta(d,p)} 2^{-(l-j)N} (1 + 2^j |k|)^{-N}, & 2^j \leq 4\lambda, \\
\lambda^{-\frac{d}{2}} \lambda^{\delta(d,p)} 2^{-(1-j)N} (1 + 2^j |k|)^{-N}, & 2^j > 4\lambda.
\end{cases}
\]

Using (2.13) and taking summation over $l, k$ give (2.7). In fact, taking $N$ large enough, it is sufficient to consider the case $k = 0$. \hfill \square
2.2. **Further decomposition and reduction.** From now on we extensively utilize the expression \[(2.5)\]. To obtain estimates from the oscillatory integral we need to collect some properties of the phase function \(P_{H}\).

Using \[(2.6)\], a computation gives

\[
\partial_{t} P_{H}(t, x, y) = -\frac{\cos^{2} t - 2 \langle x, y \rangle \cos t + |x|^{2} + |y|^{2} - 1}{2 \sin^{2} t}.
\]

Since \(D(x, y) \geq c_{0}^{2}\) for \((x, y) \in \mathcal{D}(c_{0})\), we can factor \(\cos^{2} t - 2 \langle x, y \rangle \cos t + |x|^{2} + |y|^{2} - 1 = (\cos t - \langle x, y \rangle - D(x, y))(\cos t - \langle x, y \rangle + D(x, y))\). One can easily see \(-1 < \langle x, y \rangle \pm \sqrt{D(x, y)} < 1 \) for \((x, y) \in \mathcal{D}(c_{0})\) because \(|\langle x, y \rangle| < 1\) if \((x, y) \in \mathcal{D}(c_{0})\).

Thus we can define smooth functions \(S_{c}(x, y), S_{s}(x, y) \in (0, \pi)\) by setting

\[
\cos S_{c}(x, y) = \langle x, y \rangle + \sqrt{D(x, y)}, \quad (x, y) \in \mathcal{D}(c_{0}),
\]

\[
\cos S_{s}(x, y) = \langle x, y \rangle - \sqrt{D(x, y)}, \quad (x, y) \in \mathcal{D}(c_{0}).
\]

So, we may also write

\[
\partial_{t} P_{H}(t, x, y) = -\frac{(\cos t - \cos S_{c}(x, y))(\cos t - \cos S_{s}(x, y))}{2 \sin^{2} t}.
\]

Also, since \(\sin^{2} S_{c}(x, y) = (1 - \langle x, y \rangle - \sqrt{D(x, y)})(1 + \langle x, y \rangle + \sqrt{D(x, y)})\) and \((1 - \langle x, y \rangle)^{2} - D(x, y) = |x - y|^{2}\), we have

\[
\sin S_{c}(x, y) = |x - y| \left(1 - \frac{\langle x, y \rangle + \sqrt{D(x, y)}}{1 + \langle x, y \rangle + \sqrt{D(x, y)}}\right)^{\frac{1}{2}}.
\]

Note that \(1 \pm \langle x, y \rangle + \sqrt{D(x, y)} \geq c_{0}\) for \((x, y) \in \mathcal{D}(c_{0})\). Thus we have \(S_{c}(x, y) \sim |x - y|\). A similar computation shows \(S_{s}(x, y) \sim |x + y|\).

Since the integral in \[(2.5)\] has slower decay near the critical points, the kernel \([\eta_{p}]_{x}^{H}\) becomes singular on the sets \(\{x = y\}, \{x = -y\}\), as \(S_{c}(x, y), S_{s}(x, y) \to 0\) while \(S_{c}(x, y), S_{s}(x, y) \in \text{supp } \eta_{p}\). We note that

\[
|S_{c}(x, y) - S_{s}(x, y)| \geq 2c_{0}
\]

for any \((x, y) \in E \times F\). This follows by the mean value theorem and the fact \(\cos S_{c}(x, y) - \cos S_{s}(x, y) = 2\sqrt{D(x, y)} \geq 2c_{0}\).

As a first step of the proof of \[(2.14)\], we decompose \(E \times F\) into smaller balls to localize the values of \(S_{c}(x, y), S_{s}(x, y)\). We choose a constant \(c > 0\) small enough so that

\[
|\cos S_{c}(x_{1}, y_{1}) - \cos S_{c}(x_{2}, y_{2})| \leq c_{0}/10, \quad |\cos S_{s}(x_{1}, y_{1}) - \cos S_{s}(x_{2}, y_{2})| \leq c_{0}/10,
\]

for \((x_{1}, y_{1}), (x_{2}, y_{2}) \in \mathcal{D}(c_{0})\) if \(|(x_{1}, y_{1}) - (x_{2}, y_{2})| \leq 4c_{0}\). Since \(E\) and \(F\) are compact, so there are finite collections of balls \(\{B(x_{j}, c_{0})\}_{j=1}^{N}, \{B(y_{j}', 2c_{0})\}_{j=1}^{N'}\) which cover \(E, F\), respectively. Let \(\{\varphi_{j}\}_{1 \leq j \leq N}, \{\varphi_{j}'\}_{1 \leq j' \leq N}\) be collections of smooth functions such that \(\sum_{j=1}^{N} \varphi_{j} = 1\) on \(E\) and \(\text{supp } \varphi_{j} \subset B(x_{j}, 2c_{0})\) and \(\sum_{j'=1}^{N'} \varphi_{j}' = 1\) on \(F\) and \(\text{supp } \varphi_{j}' \subset B(y_{j}', 2c_{0})\). In order to prove \[(2.14)\] it suffices to prove the corresponding estimates for \(\varphi_{j}[\eta_{p}]_{x}^{H} \varphi_{j}'\) with the same bound.

If \(\rho \sim 1\), then the support of \(\eta_{p}\) may contain both critical points \(S_{c}(x, y)\) and \(S_{s}(x, y)\) for some \((x, y) \in \text{supp } \varphi_{j} \times \text{supp } \varphi_{j}'\). However, if we use the decomposition
in the above we can exclude the case $S_*(x, y) \in \text{supp } \eta_\rho$ breaking $\eta_\rho$ into smooth functions supported in small intervals. In fact, combining (2.21) with the symmetric properties (2.28), (2.29) and the separation condition (2.20), we may assume that

$$\text{dist}(S_*(x, y), \text{supp } \eta_\rho) \geq c_0/2$$

for all $(x, y) \in \text{supp } \varphi_j \times \text{supp } \varphi'_j$.

To see this, we fix $j, j'$ and decompose $\eta_\rho$ into $O(\rho/c_0)$ many cutoff functions $\eta_{\rho, k}$ which are supported in finitely overlapping intervals of length $c_0/2$. Then, from the construction (2.21) we have either $\text{dist}(S_*(x, y), \text{supp } \eta_{\rho, k}) \geq c_0/2$ for all $(x, y) \in B(x_j, 2c_0) \times B(y_j', 2c_0)$ or $\text{dist}(S_*(x, y), \text{supp } \eta_{\rho, k}) \leq c_0$ for all $(x, y) \in B(x_j, 2c_0) \times B(y_j', 2c_0)$. The former case is acceptable. For the latter case, by (2.20) it follows that $\text{dist}(S_*(x, y), \text{supp } \eta_{\rho, k}) \geq c_0/2$ for all $(x, y) \in B(x_j, 2c_0) \times B(y_j', 2c_0)$. We note that $S_*(x, y) + S_*(x, -y) = \pi$ for any $x, y$ since $\cos S_*(x, y) = -\cos S_*(x, -y)$. Thus $\text{dist}(\eta_{\rho, k, \pi}, S_*(x, y)) \geq c_0/2$ for all $(x, y) \in B(x_j, 2c_0) \times B(y_j', 2c_0)$. From (2.28) and (2.29) we note that

$$\|\chi B(x_j, 2c_0)[\eta_{\rho, j}]\chi B(y_j', 2c_0)\|_p = \|\chi B(x_j, 2c_0)[\eta_{\rho, k, \pi}]\chi(-B(y_j', 2c_0))\|_p,$$

where $\eta_{\rho, k, \pi}(t) := \eta_{\rho, k}(\pi - t)$. Clearly, $B(x_j, 2c_0) \times (-B(y_j', 2c_0)) \subset \mathcal{D}(c_0)$ and we have $\text{dist}(\eta_{\rho, k, \pi, S_*(x, y)}) \geq c_0/2$ for all $(x, y) \in B(x_j, 2c_0) \times (-B(y_j', 2c_0))$ as desired.

Now, the proof of (2.14) reduces to showing, for $p > p_0(d)$,

$$\|\varphi_j[\eta_\rho]^{\mathcal{H}} \varphi'_j\|_p \leq \begin{cases} \lambda^{-\frac{4}{d} + \rho}, & \rho \leq \lambda^{-1}, \\ \lambda^{-\frac{4}{d} + \delta(d, p) + \rho} & \rho > \lambda^{-1} \end{cases}$$

for each $j, j'$. To show this we break the kernel dyadically away from the diagonal $\{x = y\}$:

$$\varphi_j[\eta_\rho]^{\mathcal{H}} \varphi'_j(x, y) = \sum_l \mathcal{T}_l(x, y) := \sum_l [\eta_\rho]^{\mathcal{H}}(x, y) \psi(2^l|x - y|) \varphi_j(x) \varphi'_j(y).$$

For each $l$ we further decompose $\varphi_j$, $\varphi'_j$ into smooth functions which are supported in finitely overlapping balls of radius $c2^{-l}$. Since $|x - y| \sim 2^{-l}$, by this additional decomposition it now suffices to consider the operator

$$\chi_l[\eta_\rho]^{\mathcal{H}} \chi'_l$$

while $\chi_l, \chi'_l$ satisfy the following:

(2.24) $\text{supp } \chi_l \times \text{supp } \chi'_l \subset \mathcal{D}(c_0/2),$

(2.25) $\text{supp } \chi_l, \text{supp } \chi'_l \subset B(x_l, 2c_2^{-l})$, for some $x_l \in \mathcal{D}(c_0),$

(2.26) $2^{-l-2} \leq \text{dist}(\text{supp } \chi_l, \text{supp } \chi'_l) \leq 2^{-l},$

(2.27) $|\partial^s \chi_l|, |\partial^s \chi'_l| \leq C_{\alpha} 2^{\alpha s}.$

The following is clear.

**Lemma 2.3.** Let $\chi_l, \chi'_l$ be smooth functions satisfying (2.24)–(2.27). Suppose that $\|\chi_l[\eta_\rho]^{\mathcal{H}} \chi'_l\|_p \leq B$ holds, then we have $\|\mathcal{T}_l\|_p \leq CB$ for a constant $C$. 
2.3. Estimates for $\chi[\eta_\rho]^N\chi'_i$. Therefore, we only need to obtain estimate for $\chi[\eta_\rho]^N\chi'_i$. To do this, we consider the kernel of $[\eta_\rho]^N$ which is given by the integral in \(2.29\). So, the decay property of the kernel of $[\eta_\rho]^N$ largely depends on whether the support of $\eta_\rho$ contains the critical point $S_c(x, y)$ or not. As be seen below, it is easy to handle $\chi[\eta_\rho]^N\chi'_i$ when $\text{dist}(S_c(x, y), \text{supp}(\eta_\rho)) \geq c > 0$.

From (2.29) and (2.26) we have
\[
S_c(x, y) \sim 2^{-t}, \quad (x, y) \in \text{supp} \chi_i \times \text{supp} \chi'_i.
\]
We also occasionally make use of the following elementary lemmas.

**Lemma 2.4.** Let $1 \leq p \leq \infty$ and let $E, F$ be measurable subsets of $\mathbb{R}^d$. Suppose that the kernel $T(x, y)$ of $T$ satisfies $|T(x, y)| \leq B$ for $(x, y) \in E \times F$. Then $\|\chi_E T \chi_F\|_p \leq B|E|^{\frac{1}{p}}|F|^{\frac{1}{p'}}$.

**Lemma 2.5** (\cite{13} Lemma 2.8). Let $\phi \in C^\infty(I)$ and $A$ be a smooth bump function on $\mathbb{R}$ supported in an interval $I$ of length $0 < \rho < 2^2$. Suppose that $|\phi'(t)| \geq L$, $|\phi^{(n)}(t)| \leq CLp^{1-n}$, and $|A^{(n)}(t)| \leq C\rho^{-n}$ for any $n \in \mathbb{N}_0$ and $t \in I$. Then, for $\lambda \geq 1$ and $N = 0, 1, 2, \ldots$,
\[
\left| \int A(t)e^{i\lambda \phi(t)} dt \right| \leq C_N \rho(1 + \lambda \rho L)^{-N}.
\]

Lemma 2.4 follows from interpolation between the trivial estimates $\|\chi_E T \chi_F\|_\infty \leq B|E|$ and $\|\chi_E T \chi_F\|_1 \leq B|E|$. Lemma 2.5 can be shown by routine integration by parts, also see \cite{13} Lemma 2.8 for a proof based on a scaling argument.

In order to show (2.22) we separately handle the cases $\rho \leq \lambda^{-1}$ and $\rho > \lambda^{-1}$. The former case is easier to show.

**Lemma 2.6.** Let $1 \leq p \leq \infty$. If $\rho \leq \lambda^{-1}$, then
\[
\|\chi[\eta_\rho]^N\chi'_i\|_p \lesssim \bigg\{ \begin{array}{ll}
\lambda^{-N-\frac{1}{2}+N\rho^{2(2N-d)}} & , \quad 2^{-t} \gg \lambda^{-\frac{1}{2}+\frac{1}{2}}
\rho^{2-2-d} & , \quad 2^{-t} \lesssim \lambda^{-\frac{1}{2}+\frac{1}{2}} \rho^{2-2-d}
\end{array}
\bigg\}
\]

**Proof.** The bound (2.29) follows from estimates for the kernel of $\chi[\eta_\rho]^N\chi'_i$, so we assume $(x, y) \in \text{supp} \chi_i \times \text{supp} \chi'_i$ and $t \in \text{supp} \eta_\rho$. We first deal with the bound (2.29) in the case $2^{-t} \gg \lambda^{-\frac{1}{2}+\frac{1}{2}} \rho^{2-2-d}$.

From (2.28) we have $1 - \cos S_c(x, y) \sim 2^{-2t}$. Thus it follows $|\cos t - \cos S_c(x, y)| \geq |1 - \cos S_c(x, y)| - |1 - \cos t| \geq 2^{-2t}$ because $2^{-t} \gg \rho$. Using this and (2.18), we get the lower bound
\[
|\partial_t \mathcal{P}_H(t, x, y)| \sim 2^{-2t} \rho^{-2}
\]
since $t \sim \rho$. We now obtain bounds on $\partial_t^n \mathcal{P}_H$, $n \geq 2$. From (2.18) we note
\[
\partial_t^n \mathcal{P}_H(t, x, y) = \frac{1}{2} \sum_{n_1+n_2+n_3=n-1} \partial_{t_1}^{n_1} \cos t - \cos S_c \partial_{t_2}^{n_2} \cos t - \cos S_* \partial_{t_3}^{n_3} \sin t)^{-2}.
\]

From now on, we occasionally denote $S_c(x, y)$, $S_*(x, y)$ by $S_c$, $S_*$, respectively, concealing $x, y$ for simplicity. The absolute value of the summand $\partial_{t_1}^{n_1} \cos t - \cos S_c \partial_{t_2}^{n_2} \cos t - \cos S_*$.
cos \( S_c \) \( \partial_x^n (\cos t - \cos S_c) \partial_x^n (\sin t)^{-2} \) is bounded by \( C \rho^{1-n} \) if \( n_3 \leq n-2 \), or \( C2^{-2l} \rho^{-n-1} \) if \( n_3 = n-1 \). Since \( 2^{-l} \gg \rho \), we thus obtain \( |\partial_t^n \mathcal{P}_H(t, x, y)| \lesssim 2^{-2l} \rho^{-n-1} \). Also, by a computation it is clear that
\[
(2.32) \quad |\partial_t^n (\eta_\rho(t) (\sin t)^{-\frac{2}{t}})| \lesssim \rho^{\frac{2}{t}-n}, \quad n \in \mathbb{N}_0,
\]
Putting these estimates together and using Lemma 2.3, we get
\[
|\langle \chi_l | \eta_\rho | \mathcal{H} \rangle (x, y) | \lesssim \lambda^{-N} \rho^{1-\frac{2}{t}+N} 2^{2Nl}, \quad N \in \mathbb{N}.
\]
Then by (2.32) and Lemma 2.4 we get the desired estimate (2.34) when \( 2^{-l} \gg \lambda^{-\frac{2}{t}} \rho^{\frac{2}{t}} \).

Now we turn to the case \( 2^{-l} \lesssim \lambda^{-\frac{2}{t}} \rho^{\frac{2}{t}} \). In this case, we shall make use of the following well known estimate (see [30], p. 70):
\[
(2.33) \quad |\Pi_\lambda^H (x, y)| \lesssim \lambda^{\frac{2}{t}-1}, \quad \lambda \in 2\mathbb{N}_0 + d,
\]
for any \( x, y \in \mathbb{R}^d \). From (2.24) we note that \( |\eta_{\lambda} | H (\lambda^{-1/2} x, \lambda^{-1/2} y) \) can be expressed as \( \sum_{\lambda' \in 2\mathbb{N}_0 + d} \int \eta_{\lambda} (2t) e^{\iota t (\lambda - \lambda')} dt \Pi_{\lambda'} (H) (x, y) \). Thus we have
\[
|\eta_{\lambda} | H (\lambda^{-1/2} x, \lambda^{-1/2} y) = 2^{-1} \sum_{\lambda' \in 2\mathbb{N}_0 + d} \bar{\eta}_{\lambda} (\lambda - \lambda')/2 \Pi_{\lambda'} (H) (x, y).
\]
Since \( |\bar{\eta}_{\lambda} (\tau)| \lesssim \rho (1 + \rho |\tau|)^{-N} \) for any \( N \in \mathbb{N} \), we get
\[
|\langle \eta_{\lambda} | H (\lambda^{-1/2} x, \lambda^{-1/2} y) | \leq \sum_{\lambda' \in 2\mathbb{N}_0 + d} \rho (1 + \rho |\lambda - \lambda'|)^{-N} (\lambda')^{\frac{2}{t}-1} \lesssim \rho^{1-\frac{2}{t}}
\]
for \( x, y \in \mathbb{R}^d \). Thus, \( |\langle \eta_{\lambda} | H (x, y) | \leq \rho^{1-\frac{2}{t}} \). Applying Lemma 2.4 with (2.29), we get the second estimate in (2.29).

Estimates for \( \chi_l | \eta_\rho | H \chi' \) when \( \rho > \lambda^{-1} \). We distinguish the three cases
\[
2^{-l} \ll \rho, \quad 2^{-l} \gg \rho, \quad 2^{-l} \sim \rho.
\]
Unlike the first and second cases, the support of \( \eta_\rho \) may contain \( S_c (x, y) \) in the third case. So handling the case is the main part of the proof (see Proposition 2.8). We first deal with the two easier cases.

Lemma 2.7. Let \( 1 \leq p \leq \infty \). If \( \rho > \lambda^{-1} \), then
\[
(2.34) \quad \| \chi_l | \eta_\rho | H \chi' \|_p \lesssim \begin{cases} \lambda^{-N} \rho^{1-\frac{2}{t}+N} 2^{-2l}, & 2^{-l} \ll \rho, \\ \lambda^{-N} \rho^{1-\frac{2}{t}+N} 2^{(2N-1)d}, & 2^{-l} \gg \rho. \end{cases}
\]

Proof. As before we show the bound (2.34) by obtaining estimates for the kernel of \( \chi_l | \eta_\rho | H \chi' \). Throughout the proof we assume \( (x, y) \in \text{supp} \chi_l \times \text{supp} \chi'_l \) and \( t \in \text{supp} \eta_\rho \).

We first deal with the case \( \rho \gg 2^{-l} \). By (2.29) and (2.19), we have \( 1 - \cos S_c (x, y) \sim 2^{-2l} \), so \( |\cos t - cos S_c (x, y)| \geq 1 - \cos t - |1 - \cos S_c (x, y)| \geq \rho^2 \) because \( \rho \gg 2^{-l} \).

Using this and (2.18), we get \( |\partial_t^n \mathcal{P}_H(t, x, y)| \gtrsim 1 \) and
\[
|\partial_t^n \mathcal{P}_H(t, x, y)| \lesssim \rho^{1-n}, \quad n \in \mathbb{N}.
\]
Since \( |\partial_t^n (\eta_\rho(t) (\sin t)^{-\frac{2}{t}})| \lesssim \rho^{\frac{2}{t}-n} \), by Lemma 2.5 we get \( |\langle \chi_l | \eta_\rho | H \chi' | (x, y) | \lesssim \lambda^{-N} \rho^{1-\frac{2}{t}-N} \), \( N \in \mathbb{N}_0 \). Applying Lemma 2.4 yields the first case estimate in (2.34).
The proof of the second case in (2.34) similar to that of the first case of (2.29). Note
that \( |\cos t - \cos S_c(x, y)| \gtrsim 2^{-2l} \) because \( \rho \ll 2^{-l} \). Thus, from (2.18) and
(2.31), we have \( |\partial_l \mathcal{P}_H(t, x, y)| \gtrsim 2^{-2l}\rho^{-2} \) and \( |\partial^d_l \mathcal{P}_H(t, x, y)| \lesssim 2^{-2l}\rho^{-1-n}, n \in \mathbb{N} \). The rest
of proof is identical, so we omit the detail.

The following is the key estimate which we need to prove Theorem 2.2.

**Proposition 2.8.** Let \( \rho > \lambda^{-1} \) and \( \rho \sim 2^{-l} \). Then we have
(2.35) \( \|\chi I[\eta_0]H_\lambda \chi'\|_p \lesssim \lambda^{-\frac{d}{2}+\delta(d,p)}\rho^{1+\delta(d,p)} \)
provided that \( p_0(d) < p \leq \infty \).

Assuming Proposition 2.8 we show the estimate (2.22). We combine Lemma 2.6
Lemma 2.7 and Proposition 2.8.

**Proof of (2.22).** Let us first consider the case \( \rho \leq \lambda^{-1} \). By (2.23) we see
(2.36) \( \|\varphi_j[\eta_0]H_\lambda \varphi'\|_p \leq \sum l \|T^l\|_p \).

Splitting the sum \( \sum l = \sum_{2^{-l} \lesssim \lambda^{-\frac{d}{2}}\rho \gtrsim 2^{-l}} + \sum_{2^{-l} \sim \lambda^{-\frac{d}{2}}\rho} \), we combine Lemma 2.4 with
a large \( N \) and Lemma 2.3. Thus, we see that \( \|\varphi_j[\eta_0]H_\lambda \varphi'\|_p \) is bounded by a constant times
\( \sum_{2^{-l} \lesssim \lambda^{-\frac{d}{2}}\rho \gtrsim 2^{-l}} \rho^{1-\frac{d}{2}} 2^{-dl} \) \( \sum_{2^{-l} \sim \lambda^{-\frac{d}{2}}\rho} \lambda^{-N} \rho^{1-\frac{d}{2}+N(2N-d)l} \).

Thus, we get \( \|\varphi_j[\eta_0]H_\lambda \varphi'\|_p \lesssim \lambda^{-\frac{d}{2}\rho} \) as desired.

We now consider the case \( \rho > \lambda^{-1} \). Similarly, using Lemma 2.7 with a large \( N \) and
Proposition 2.8 together with Lemma 2.3, we have
(2.38) \( \sum_{2^{-l} \lesssim \lambda^{-\frac{d}{2}}\rho \gtrsim 2^{-l}} \sum \sum l \|T^l\|_p \lesssim (\rho\lambda)^{-N} \rho^{1+\frac{d}{2}} + \lambda^{-\frac{d}{2}+\delta(d,p)}\rho^{1+\delta(d,p)} \)
for \( p_0(d) < p \leq \infty \). By (2.36) it follows \( \|\varphi_j[\eta_0]H_\lambda \varphi'\|_p \lesssim \lambda^{-\frac{d}{2}+\delta(d,p)}\rho^{1+\delta(d,p)} \) since
\( \rho > \lambda^{-1} \). Therefore we get (2.22). \( \square \)

To complete the proof of Theorem 2.2 it now remains to prove Proposition 2.8. The
rest of this section is devoted to the proof of Proposition 2.8.

**2.4. Asymptotic expansion of the kernel when \( 2^{-l} \sim \rho > \lambda^{-1} \).** As discussed
before, in this case the support of \( \eta_0 \) may contain the critical point \( S_c(x, y) \).

**Additional decomposition of \( \chi_I, \chi'_I \) and \( \eta_0 \).** If we further break the cutoff functions
\( \chi_I, \chi'_I \) and \( \eta_0 \) into finitely many smooth functions, then in addition to (2.24)–(2.27)
we may assume that
(2.37) \( \text{supp } \chi_I \subset B(x_0, \epsilon_0\rho), \quad \text{supp } \chi'_I \subset B(y_0, \epsilon_0\rho) , \quad \text{supp } \chi_I \subset B(x_0, \epsilon_0\rho) \),
(2.38) \( \text{supp } \eta_0 \subset S_c(x_0, y_0) + (-\epsilon_0\rho, \epsilon_0\rho), \quad \chi_I \in \text{supp } \chi_I \times \text{supp } \chi'_I \),
(2.39) \( \text{supp } \eta_0 \subset S_c(x_0, y_0) + (-2\epsilon_0\rho, 2\epsilon_0\rho) \),
for a small \( \epsilon_0 > 0 \) and \( (x_0, y_0) \in \mathcal{D}(\epsilon_0) \). Indeed, just breaking \( \chi_I, \chi'_I \) into cutoff
functions which are supported in finitely overlapping balls of radius \( \epsilon_0\rho \) with a
small $c > 0$, trivially we have (2.37) and (2.38) and this gives rise to only $O(\epsilon_0^{-2d})$
many such functions. However, the third condition is not completely trivial. This
is achieved by breaking $\chi[\eta_0]\chi_t'$ into major and minor parts. In fact, let $\tilde{\eta}_0$ be
a smooth bump function such that $supp \tilde{\eta}_0 \subset S_c(x_0, y_0)$, $\tilde{\eta}_0(t) = 1$
if $t \in S_c(x_0, y_0) + (-2\epsilon_0, 2\epsilon_0)$, and $|\eta_0^{-1}(n)| \leq C\rho^{-n}$ for any $n \in \mathbb{N}$. Then we split
\[\chi[\eta_0]^{\chi}_t = \chi[\eta_0]^{\chi}_t + \chi[\eta_0(1 - \tilde{\eta}_0)]^{\chi}_t.\]
The second term can be handled in the same manner as before. From (2.38) we
note that $|\cos t - \cos S_c(x, y)| \geq \rho^2$ on the support of $\eta_0(1 - \tilde{\eta}_0)$. Thus, using this,
(2.39) and (2.41), we have $|\partial_t P_H(t, x, y)| \geq 1$ and $|\partial^n P_H(t, x, y)| \leq \rho^{1-n}$, $n \in \mathbb{N}$.
Following the argument in the proofs of Lemma 2.6 and 2.7 we get $\|\chi[\eta_0(1 - \tilde{\eta}_0)]^{\chi}_t\| \leq (\lambda\rho)^{-N} \rho^{1+\frac{d}{2}}$ for $\rho > \lambda^{-1}$. Therefore we may disregard the contribution
from $\chi[\eta_0(1 - \tilde{\eta}_0)]^{\chi}_t$ and we may therefore assume (2.39).

We obtain an asymptotic development of the kernel $[\eta_0]^{\chi}_t$ via the method of stationary phase. We recall the formula (2.31) with $n = 2$. Then, using (2.40) and (2.17) we have
\[\partial_t^2 P_H(S_c(x, y), x, y) = \frac{\cos S_c(x, y) - \cos S_c(x, y)}{2 \sin S_c(x, y)} = \frac{\sqrt{D(x, y)}}{\sin S_c(x, y)}\]
Since $c_0 \leq \sqrt{D(x, y)} \leq 1$, from (2.20) and (2.28) we have
\[\partial_t^2 P_H(S_c(x, y), x, y) \sim \rho^{-1}, \quad (x, y) \in \text{supp } \chi_1 \times \text{supp } \chi'_1.\]
One can easily see
\[|\partial_\alpha^\beta P_H(t, x, y)| \leq \rho^{1-n-|\alpha|-|\beta|},\]
where $(t, x, y) \in \text{supp } \eta_0 \times \text{supp } \chi_1 \times \text{supp } \chi'_1$. To show the estimate (2.42), recalling
(2.6), we only need to consider the case $|\alpha| + |\beta| \leq 2$. When $|\alpha| + |\beta| = 0$, (2.42) can be shown using (2.31) similarly as before. The other cases can be handled in a more straightforward manner. For example, note that $\partial_\alpha P_H(t, x, y) = (\cos tx - y)/\sin t$.
Thus $\partial_\alpha P_H(t, x, y) = O(1)$ because $\cos tx - y = (\cos t - 1) x + x - y = O(\rho)$ and we also have $\partial_\alpha^\beta P_H(t, x, y) = O(\rho^{-\alpha})$. The remaining cases can be handled similarly.

We make change of variables so that the associated phase and amplitude functions
have uniformly bounded derivatives in $\rho$. Let us set
\[\sigma(t, x, y) = S_c(\rho x + x_0, \rho y + y_0) + \rho t,\]
\[\tilde{P}_h(t, x, y) = \rho^{-1} P_H(\sigma(t, x, y), \rho x + x_0, \rho y + y_0),\]
\[\tilde{\chi}_1(x) = \chi_1(\rho x + x_0), \quad \tilde{\chi}'_1(y) = \chi'_1(\rho y + y_0),\]
\[a(t, x, y) = \rho^\frac{3}{2} (\sin \sigma(t, x, y))^{-\frac{d}{2}} \eta_0(\sigma(t, x, y)) \tilde{\chi}_1(x) \tilde{\chi}'_1(y).\]
As to be seen later, all of these functions have uniformly bounded derivatives and the support of $a$ is contained in $(-2\epsilon_0, 2\epsilon_0) \times B(0, \epsilon_0) \times B(0, \epsilon_0)$. Let us set
\[I^H_\rho(x, y) = \rho^{1+\frac{d}{2}} \int a(t, x, y) e^{i\lambda x} e^{i\lambda y} \tilde{P}_h(t, x, y) dt.\]
By the change of variables $t \rightarrow \sigma(t, x, y)$, we have
\[(\chi[\eta_0]^{\chi}_t)(\rho x + x_0, \rho y + y_0) = C_d I^H_\rho(x, y),\]
where \( C_d \) is a constant depending on \( d \). We obtain an asymptotic expansion of the integral \( I^H_\rho(x, y) \) using the method of stationary phase.

To do so, we first note

\[
|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \sigma(t, x, y)| \leq C\rho
\]

for \((t, x, y) \in \text{supp } a \times \text{supp } \tilde{\chi}_t \times \text{supp } \tilde{\chi}'_t\). This can be shown using the following.

**Lemma 2.9.** If \((x, y) \in \text{supp } \chi_t \times \text{supp } \chi'_t\), then

\[
|\partial_x^\alpha \partial_y^\beta S_c(x, y)| \lesssim \rho^{1-|\alpha|-|\beta|}.
\]

**Proof.** From \ref{eq:2.19} we see \ref{eq:2.45} is trivially true if \(|\alpha| + |\beta| = 0\). We may assume \(|\alpha| + |\beta| \geq 1\) and by symmetry we may also assume \(|\alpha| \geq 1\). Using \ref{eq:2.44}, \ref{eq:2.32}, and \ref{eq:2.45}, one can easily see}

\[
|\partial_x^\alpha \partial_y^\beta S_c(x, y)| \lesssim \rho^{1-|\alpha|-|\beta|}.
\]

where \( E \) is a linear combination of the terms

\[
\partial_x^\alpha \partial_y^\beta (\sin S_c(x, y))^{-1} \partial_x^\gamma \cos S_c(x, y) + E
\]

with \(|\gamma| \geq 2\) and \(|\alpha| = |\alpha| + |\beta| - |\gamma|\). On the other hand, for any multi-indices \( \alpha', \beta' \), using \ref{eq:2.19}, we get \(|\partial_x^\alpha \partial_y^\beta \cos S_c(x, y)| \leq C_{\alpha', \beta'}\) because \( D(x, y) \geq c_0/4 \) and from \ref{eq:2.19} it similarly follows that \(|\partial_x^\alpha \partial_y^\beta (\sin S_c(x, y))^{-1} = O(|x-y|^{-|\alpha|-|\beta'|-1})\). Thus, we see \(|E| \lesssim |x-y|^{-|\alpha|-|\beta|}\) and \(|\partial_x^\alpha \partial_y^\beta (\sin S_c(x, y))^{-1} \partial_x^\gamma \cos S_c(x, y)| \lesssim |x-y|^{-|\alpha|-|\beta|}\). Hence we get \ref{eq:3.31}. \(\square\)

Using \ref{eq:2.44}, \ref{eq:2.32}, and \ref{eq:2.45}, one can easily see

\[
|\partial_x^\alpha \partial_y^\beta a(t, x, y)| \leq C_{n, \alpha, \beta}
\]

for \((t, x, y) \in \text{supp } a \times \text{supp } \tilde{\chi}_t \times \text{supp } \tilde{\chi}'_t\). Similarly, combining \ref{eq:2.45} with \ref{eq:2.45}, we also have

\[
|\partial_x^\alpha \partial_y^\beta \tilde{P}_h(t, x, y)| \leq C_{n, \alpha, \beta}
\]

for \((t, x, y) \in \text{supp } a \times \text{supp } \tilde{\chi}_t \times \text{supp } \tilde{\chi}'_t\). From \ref{eq:2.41} it follows that

\[
\partial_t \tilde{P}_h(t, x, y) \sim 1
\]

for \((t, x, y) \in \text{supp } a \times \text{supp } \tilde{\chi}_t \times \text{supp } \tilde{\chi}'_t\). Since \( \partial_t \tilde{P}_h(0, x, y) = 0 \), the map \( t \to \tilde{P}_h(t, x, y) \) has a nondegenerate critical point at \( t = 0 \). Thus we may now apply the method of stationary phase. In fact, applying \[12\] Theorem 7.7.5] together with \ref{eq:2.46}, \ref{eq:2.47}, and \ref{eq:2.48}, we obtain the following.

**Lemma 2.10.** For \( N \in \mathbb{N} \), we have

\[
I^H_\rho(x, y) = \rho^{-\frac{d}{2}} \sum_{n=0}^{N-1} (\lambda \rho)^{-\frac{1}{2}-n} A_n(x, y) e^{i\lambda \rho \tilde{P}_h(0, x, y)} + E_N(x, y),
\]

where \( \text{supp } A_n \subset B(0, \varepsilon_0) \times B(0, \varepsilon_0) \) and

\[
|\partial_x^\alpha \partial_y^\beta A_n(x, y)| \leq C_{\alpha, \beta}, \quad |E_N(x, y)| \leq C_N \rho^{2-d} (\lambda \rho)^{-N}
\]

with \( C_{\alpha, \beta} \) and \( C_N \) independent of \( \lambda, \rho \).
In the expansion (2.49), the error term $E_N(x,y)$ is negligible if we take $N$ large enough. Indeed, from Lemma 2.4 we have $\| \chi(1) E_N \chi \|_p \lesssim \rho^{-\frac{d}{2} + \delta} (\lambda \rho)^{-N} \rho^d$. With a large $N$ it is clear that $\rho^{\frac{d}{2} + \delta}(\lambda \rho)^{-1} \rho^d \lesssim \lambda^{-\frac{d}{2} + \delta(d,p)} \rho^{1 + \delta(d,p)}$. Thus, to obtain the estimate (2.35) we are led to consider the operators with the oscillatory kernels $A_n(x,y) e^{i \lambda \rho \mathcal{P}_h(0,x,y)}$. Boundness of properties of such an operator are determined by the phase function $\mathcal{P}_h(0,x,y)$. So, we need to take a close look at it.

Let us define

(2.50) $\Phi_H(x,y) = \mathcal{P}_H(S_c(x,y), x,y).$

From (2.0), using (2.37) and (2.17), we see

$$\Phi_H(x,y) = \frac{1}{2} \left( S_c(x,y) + \frac{(|x|^2 + |y|^2) \cos S_c(x,y) - 2\langle x,y \rangle}{\sin S_c(x,y)} \right)$$

$$= \frac{1}{2} \left( S_c(x,y) - \cos S_c(x,y) \sin S_c(x,y) \right).$$

Combining this with (2.19) one can easily see $\Phi_H(x,y) - 2^{-1} |x-y| = O(||x-y||^2)$ as $|x-y| \to 0$. Thus, this suggest the phase function $\Phi_H(x,y)$ can be viewed as a small perturbation of the function $|x-y|$ when $(x,y)$ is contained in a small ball with center 0. This is natural in view of the transplantation theorem due to Kenig, Stanton, and Tomas [16] because the phase function $|x-y|$ arises as a counterpart of $\Phi_H(x,y)$ for the classical Bochner-Riesz operator $(1 + \Delta)^\lambda$. 

2.5. Carleson-Sjölin type operator. Let $A \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^{d-1})$ and $\phi \in C^\infty(\text{supp } A)$. Let $T_\lambda[\phi, A]$ denote the operator defined by

$$T_\lambda[\phi, A]f(x) = \int_{\mathbb{R}^{d-1}} e^{i \lambda \phi(x,\xi)} A(x,\xi) f(\xi) d\xi.$$

We assume that the mixed Hessian $\partial_\xi \partial_x^T \phi$ has the maximal rank, i.e.,

(C1) $\text{rank}(\partial_\xi \partial_x^T \phi(x,\xi)) = d - 1$, \hspace{0.5cm} $(x, \xi) \in \text{supp } A$

This means that the image of $\xi \to \partial_x \phi(x_0, \xi)$ is a smooth immersed surface in $\mathbb{R}^d$. The condition guarantees that, for any $(x_0, \xi_0) \in \text{supp } A$, there is a unique vector $\nu(x_0, y_0) \in \mathbb{S}^{d-1}$ modulo $\pm$ such that

$$\partial_\xi \langle \partial_x \phi(x_0, \xi), \nu(x_0, \xi_0) \rangle |_{\xi = \xi_0} = 0.$$

We further assume that the parameterized surface $\xi \to \partial_x \phi(x_0, \xi)$ has nonvanishing Gaussian curvature. Equivalently, the second fundamental form of the surface parameterized by $\xi \to \partial_x \phi(x_0, \xi)$ is not singular, i.e.,

(C2) $\text{rank}(\partial_\xi \partial_x^T \phi(x_0, \xi), \nu(x_0, \xi_0)) |_{\xi = \xi_0} = d - 1$, \hspace{0.5cm} $(x_0, \xi_0) \in \text{supp } A$.

The conditions (C1) and (C2) together are called the Carleson-Sjölin condition, and if an oscillatory integral operator $T_\lambda[\phi, A]$ satisfies both (C1) and (C2), we say $T_\lambda[\phi, A]$ is a Carleson-Sjölin type operator. Concerning the Carleson-Sjölin type operators, the estimate of the form

(2.51) $\|T_\lambda[\phi, A]f\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{-\frac{d}{2}} \|f\|_{L^p(\mathbb{R}^{d-1})}$.
Lemma 2.12. Let \( (2.55) \) holds for \( \frac{1}{q} < \frac{d-1}{2d} \) and \( \frac{1}{q} \leq \frac{d-1}{(d+1)p} \) if \( T_{\lambda}[\phi, A] \) is a Carleson-Sjölin type operator. This was verified by Hörmander when \( d = 2 \). He also showed that this range of \( p, q \) is optimal for \( (2.51) \).

In higher dimensions, Stein [24] obtained \( (2.51) \) for \( \frac{2(d+1)}{d-1} \leq q \leq \infty \) and \( \frac{1}{q} \leq \frac{d-1}{(d+1)p} \) but Bourgain [3] essentially disproved the Hörmander’s conjecture by constructing a phase function \( \phi \) which satisfies the Carleson-Sjölin condition but the estimate \( (2.51) \) does fail for any \( q > \frac{2(d+1)}{d} \) if \( d \) is odd. Nevertheless, one of the authors [20] proved that the range of \( p, q \) for \( (2.51) \) can be improved to \( q > \frac{2(d+2)}{d} \) under the additional condition:

\[
\text{For } (x_0, \xi_0) \in \text{supp } A, \text{ all nonzero eigenvalues of the matrix (C3) } \partial_x \partial_y \langle \langle \partial_x \phi(x_0, \xi), \nu(x_0, \xi_0) \rangle \rangle \xi = \xi_0 \text{ have the same sign.}
\]

It was also shown that the range is optimal when \( d = 3 \). Recently, Guth, Hickman and Iliopoulou [9] obtained the sharp result for \( (2.51) \) with \( p = q \) under the assumptions \( (C1)-(C3) \) for \( d \geq 4 \). The main ingredients of their result are multilinear estimates due to Bennett, Carbery and Tao [2] and the method of polynomial partitioning. For our purpose, we summarize the previously known results when \( p = q \) as follows ([11] [20] [9]).

**Theorem 2.11.** Suppose that \( d \geq 2 \) and \( \phi \) satisfies the conditions \( (C1)-(C3) \). Then we have \( (2.51) \) whenever \( p = q > p_0(d) \).

Furthermore, the range of \( p \) is sharp up to the endpoint in that there exist counterexamples of \( T_{\lambda}[\phi, A] \) whose phase function \( \phi \) satisfies the conditions \( C1)-(C3) \) such that \( (2.51) \) with \( p = q \) fails whenever \( p < p_0(d) \) (see [9]).

2.6. The phase function \( \Phi_{\mathcal{H}} \). In this section we investigate the phase function \( \Phi_{\mathcal{H}} \) and its curvature condition. We begin with considering the vectors \( \mathbf{a}(x, y), \mathbf{b}(x, y) \) which are given by

\[
(2.52) \quad \mathbf{a}(x, y) := \cos S_c(x, y)x - y,
\]

\[
(2.53) \quad \mathbf{b}(x, y) := x - \cos S_c(x, y)y.
\]

The following show how the vectors \( \mathbf{a}(x, y), \mathbf{b}(x, y) \), and \( x, y \) are related.

**Lemma 2.12.** Let \( \mathcal{D}(x, y) > 0 \). Then, we have the following:

\[
(2.54) \quad |\mathbf{a}(x, y)|^2 = (1 - |x|^2) \sin^2 S_c(x, y),
\]

\[
(2.55) \quad |\mathbf{b}(x, y)|^2 = (1 - |y|^2) \sin^2 S_c(x, y),
\]

\[
(2.56) \quad \langle \mathbf{a}(x, y), \mathbf{b}(x, y) \rangle = \sqrt{\mathcal{D}(x, y)} \sin^2 S_c(x, y).
\]

In particular, \( (2.54) \) shows that the map \( y \rightarrow \partial_y \Phi_{\mathcal{H}}(x, y) \) can not have the maximal rank \( d \). To see this, note that

\[
\partial_y \Phi_{\mathcal{H}}(x, y) = \partial_y \mathcal{P}_{\mathcal{H}}(S_c(x, y), x, y) = \frac{\cos S_c(x, y)x - y}{\sin S_c(x, y)}.
\]

\[\text{The range of } p, q \text{ is the best possible one. This follows from the well known necessity condition for the restriction estimate to the surfaces with nonzero curvature since the estimate } (2.51) \text{ implies the adjoint restriction estimate to such surfaces.}\]
For the first equality we use \( \partial_t \mathcal{P}_H(S_c(x,y), x,y) = 0 \). Thus, by (2.54) we have \(|\partial_x \Phi_H(x,y)| = \sqrt{1 - |x|^2} \). So, the image of \( y \rightarrow \partial_x \Phi_H(x,y) \) is contained in the sphere of radius \( \sqrt{1 - |x|^2} \).

**Proof of Lemma 2.12** From (2.15) we note that 
\[
(\partial_x \Phi_H(x,y)) = \left( \frac{\cos S_c x - y}{|x|^2} \cos^2 S_c - 2 \langle x, y \rangle \cos S_c + |y|^2 = (1 - |x|^2) \sin^2 S_c.
\]
This gives (2.54). Since \( \mathbf{b}(x,y) = -\mathbf{a}(y,x) \), (2.55) follows from (2.54). It remains to show (2.56). Let us set \( \mathbf{Q}(t, x,y) = \langle x, y \rangle \cos^2 t - (\langle x \rangle^2 + |y|^2) \cos t + \langle x, y \rangle \). Then we note
\[
\langle \mathbf{a}(x,y), \mathbf{b}(x,y) \rangle = -\mathbf{Q}(S_c, x,y).
\]
We also note that \( \partial_t^2 \mathcal{P}_H(t,x,y) = -\frac{\partial(\mathbf{t}, x,y)}{\sin^2 t} \). Combining this with (2.40) we get
\[
(2.57) \quad \mathbf{Q}(S_c(x,y), x,y) = -\sqrt{\mathcal{D}} \sin^2 S_c
\]
and hence (2.56).

Let us set
\[
(2.58) \quad M(x,y) = \partial_x \partial_t \langle \partial_x \Phi_H(x,z), \frac{\mathbf{a}(x, y)}{|\mathbf{a}(x, y)|} \rangle \bigg\vert_{z=y}.
\]
What follows is crucial in showing that the phase \( \Phi_H \) satisfies C1)–C3) after rescaling and freezing a suitable coordinate.

**Lemma 2.13.** Let \( (x,y) \in \text{supp} \chi \times \text{supp} \chi' \). Then, (i) the matrix \( \partial_y \partial^T_x \Phi_H(x,y) \) has rank \( d - 1 \) and
\[
(2.59) \quad \partial_y \partial^T_x \Phi_H(x,y) \mathbf{a}(x,y) = 0.
\]
Additionally, (ii) if \( (x,y) \in \text{supp} \chi \times \text{supp} \chi' \) satisfies
\[
(2.60) \quad \frac{\mathbf{b}(x,y)}{|\mathbf{b}(x,y)|} = e_d,
\]
then the submatrix \( \mathbf{M}(x,y) := \{ M(x,y)_{i,j} \}_{1 \leq i,j \leq d-1} \) of \( M(x,y) \) has negative eigenvalues \( \lambda_1, \ldots, \lambda_{d-1} \) such that
\[
-\lambda_i \sim |x-y|^{-2}, \quad 1 \leq i \leq d - 1.
\]

From now on, to simplify the notation, we denote by \( \mathbf{a}, \mathbf{b} \) the vectors \( \mathbf{a}(x,y), \mathbf{b}(x,y) \), respectively, and we also drop the variables \( x, y \) from \( \mathcal{D}(x,y) \) as long as no ambiguity arises. From (2.60) a computation shows
\[
(2.61) \quad \partial_y \partial^T_x \Phi_H(x,y) = \partial_y \partial^T_x \mathcal{P}_H(S_c, x,y) + \partial_y \partial_t \mathcal{P}_H(S_c, x,y) \partial^T_x S_c
\]
\[
+ \partial_y S_c \partial_x^T \partial_t \mathcal{P}_H(S_c, x,y) + (\partial_y S_c \partial_x^T S_c) \partial^T_x \mathcal{P}_H(S_c, x,y).
\]
Here we use $\partial_t \mathcal{P}_\mathcal{H}(S_c, x, y) = 0$. Using \eqref{eqn:2.10} it is easy to show

$$
\partial_y \partial_x ^\text{T} \mathcal{P}_\mathcal{H}(S_c, x, y) = -\frac{\mathbf{I}}{\sin S_c},
$$

$$
\partial^2_x \partial_t \mathcal{P}_\mathcal{H}(S_c, x, y) = -\frac{\mathbf{b}^\top}{\sin^2 S_c},
$$

$$
\partial_y \partial_t \mathcal{P}_\mathcal{H}(S_c, x, y) = \frac{\mathbf{a}}{\sin^2 S_c}.
$$

Since $\partial_x ^\text{T} S_c = -\partial_x (\cos S_c)/\sin S_c$ and $\partial_y S_c = -\partial_y (\cos S_c)/\sin S_c$, using \eqref{eqn:2.10} we also have

$$
\partial_x ^\text{T} S_c = -M \mathcal{P}_\mathcal{H}(S_c, x, y), \quad \partial_y S_c = -\frac{\mathbf{a}}{\sin S_c}. \tag{2.62}
$$

Thus, putting these identities and \eqref{eqn:2.40} into \eqref{eqn:2.61}, we have

$$
\partial_y \partial_x ^\text{T} \mathcal{P}_\mathcal{H}(x, y) = \frac{\mathbf{b}^\top}{\sin S_c \sqrt{D}}, \quad \partial_y S_c = -\frac{\mathbf{a}}{\sin S_c \sqrt{D}}. \tag{2.63}
$$

The second equality follows from \eqref{eqn:2.56}. This allows us to obtain an explicit expression for the matrix $\mathbf{M}(x, y)$.

**Lemma 2.14.** Let $(x, y) \in \mathcal{D}(c_0)$ and $\omega(x, y) = \sqrt{(1 - |x|^2)D(x, y)/\sin^2 S_c(x, y)}$. Then we have

$$
\mathbf{M}(x, y) = \frac{\mathbf{a}^\top \mathbf{b} - \mathbf{b}^\top}{\omega(x, y)} \mathbf{a}^\top \mathbf{b} \left( \mathbf{b}^\top - \cos S_c \mathbf{a} \mathbf{a}^\top - \mathbf{a}^\top \mathbf{b} \mathbf{I} \right). \tag{2.64}
$$

**Proof.** From \eqref{eqn:2.62} and \eqref{eqn:2.63}, using \eqref{eqn:2.51}, we see

$$
\mathbf{M}(x, y) = \frac{-1}{\sqrt{1 - |x|^2 \sin S_c(x, y)}} \partial_x ^\text{T} \left( \frac{\mathcal{G}(x, y, z)}{\sin^3 S_c(x, z) \sqrt{D(x, z)}} \right) \bigg|_{z=y},
$$

where

$$
\mathcal{G}(x, y, z) = (\mathbf{a}(x, z), \mathbf{b}(x, z)) \mathbf{a}(x, y) - (\mathbf{a}(x, y), \mathbf{b}(x, z)) \mathbf{a}(x, z).
$$

Since $\mathcal{G}(x, y, y) = 0$, it follows that

$$
\mathbf{M}(x, y) = -\frac{\partial_y \mathcal{G}(x, y, z) \big|_{z=y}}{\omega(x, y)}. \tag{2.65}
$$

Via a straightforward calculation we have

$$
\partial_x ^\text{T}((\mathbf{a}(x, z), \mathbf{b}(x, z)) \mathbf{a}(x, y)) = (\mathbf{a}(x, y) \mathbf{a}(x, z))^\top \partial_x ^\text{T} \mathbf{b}(x, z) + (\mathbf{a}(x, y) \mathbf{b}(x, z))^\top \partial_x ^\text{T} \mathbf{a}(x, z),
\partial_x ^\text{T}((\mathbf{a}(x, y), \mathbf{b}(x, z)) \mathbf{a}(x, z)) = (\mathbf{a}(x, y) \mathbf{b}(x, z))^\top \partial_x ^\text{T} \mathbf{a}(x, z) + (\mathbf{a}(x, y) \mathbf{a}(x, z))^\top \partial_x ^\text{T} \mathbf{b}(x, z).
$$

Thus we have

$$
\partial_x ^\text{T} \mathcal{G}(x, y, z) \big|_{z=y} = (\mathbf{b} - \mathbf{a})^\top \mathbf{I} \partial_x ^\text{T} \mathbf{a}.
$$

Differentiating both sides of the equations \eqref{eqn:2.52} and \eqref{eqn:2.10}, one can easily see

$$
\partial_y \mathbf{a}(x, y) = D^{-2} x \mathbf{a}^\top - \mathbf{I}. \tag{2.66}
$$

Combining this with an identity $x = \frac{\mathbf{b} - \cos S_c \mathbf{a}^\top}{\sin^2 S_c \sqrt{D}}$ gives

$$
\partial_y \mathbf{a}(x, y) = \frac{(\mathbf{b} - \cos S_c \mathbf{a}^\top) \mathbf{a}^\top - \mathbf{b} \mathbf{I}}{\sin^2 S_c \sqrt{D}} = \frac{\mathbf{b} - \cos S_c \mathbf{a}^\top - \mathbf{b} \mathbf{I}}{\mathbf{a}^\top \mathbf{b}}.
$$

For the last inequality we use \eqref{eqn:2.56}. Hence, combining the above identities with \eqref{eqn:2.65} we obtain \eqref{eqn:2.64}. \qed
We are now ready to prove Lemma 2.13.

**Proof of Lemma 2.13.** We begin by noting from Lemma 2.12 that $|a|, |b|, |a^\top b| \gtrsim \rho$ since $(x, y) \in \text{supp} \chi_i \times \text{supp} \chi_i' \subset \mathcal{D}(c_0/2)$. Using (2.63) we have

$$\partial_y \partial^2_x \Phi_R v = \begin{cases} 0, & v = a, \\ \frac{\langle b, v \rangle a - (a, b) v}{\sin^3 S \sqrt{D}}, & v \perp \text{span}\{a\}. \end{cases}$$

This proves (i) in Lemma 2.13 because $(b, v)a - (a, b)v \neq 0$ by (2.56).

We now verify the next assertion (ii). Let us set

$$G(x, y) = (a^\top b I - ab^\top)(ba^\top - \cos S_a a^\top - a^\top b I).$$

Then we note that $b^\top G(x, y) = 0$ and $G(x, y)b = (a^\top b I - ab^\top)(-\cos S_a a) = 0$. Since $b(x, y) = |b(x, y)|e_d$, it follows that

$$G(x, y) = \begin{pmatrix} \tilde{G}(x, y) & 0 \\ 0 & 0 \end{pmatrix}$$

where $\tilde{G}(x, y)$ is a $d - 1 \times d - 1$ submatrix of $G(x, y)$. Also, we observe that

$$G(x, y)v = \begin{cases} -|a|^2 |b|^2 v, & \text{if } v = a - \frac{a^\top b}{|b|^2} b, \\ -a^\top (a^\top b) v, & \text{if } v \perp \text{span}\{a, b\}. \end{cases}$$

A computation gives $G(x, y)a = -|a|^2 |b|^2 (a - \frac{a^\top b}{|b|^2} b)$. Thus, the first follows since $G(x, y)b = 0$ and the second is easy to see. Thus, $\tilde{G}(x, y)$ has two eigenvalues $|a|^2 |b|^2$, $a^\top b$ of multiplicity $1, d - 2$, respectively. Therefore, from (2.65) we see that $\tilde{M}(x, y)$ has eigenvalues $\lambda_1, \lambda_2 = \cdots = \lambda_{d-1}$, where

$$-\lambda_1 = \frac{|a|^2 |b|^2}{a^\top b \sqrt{(1 - |x|^2)D \sin^4 S_c}}, \quad -\lambda_2 = \frac{a^\top b}{\sqrt{(1 - |x|^2)D \sin^4 S_c}}.$$
after replacing $Rx_0, Ry_0$ with $x_0, y_0$.

By (2.43) and (2.49) the matter is reduced to obtaining estimates for the operator

$$\mathcal{T} f(x) = \int A(x, y) e^{i\lambda \rho \tilde{F}_h(0, x, y)} f(y) dy,$$

where $A \in C_\infty^o (B(0, \epsilon_0) \times B(0, \epsilon_0))$. Since $\|\chi_1 [\eta_\rho] H \chi'_1 \|_p = \rho^d \| (\chi_1 [\eta_\rho] H \chi'_1)(\rho \cdot x_0, \rho \cdot y) \|_p$, taking a large enough $N$ in (2.49) and using Lemma 2.10, the estimate (2.66) follows if we show

$$\|\mathcal{T} f\|_p \leq C(\lambda \rho)^{-\frac{d}{p}} \|f\|_p$$

for $p_0(d) < p \leq \infty$. We note that

$$(2.68) \quad \tilde{F}_h(0, x, y) = \rho^{-1} \Phi_h(\rho x + x_0, \rho y + y_0).$$

Let us write $y = (\xi, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and set

$$\phi_{y_d}(x, \xi) = \tilde{F}_h(0, x, \xi, y_d).$$

Then by (2.68) and (2.59) it follows that

$$\partial_\xi \left( \partial_x \phi_{y_d}(x, \xi), \frac{a_p(x, \xi, y_d)}{a_p(x, \xi, y_d)} \right) \bigg|_{\xi = \xi} = 0,$$

for $(x, \xi, y_d) \in B(0, \epsilon_0) \times B(0, \epsilon_0)$ where $a_p(x, y) = a(\rho x + x_0, \rho y + y_0)$. From (2.58) we also see

$$\partial_\xi \partial_\xi' \left( \partial_x \phi_{y_d}(x, \xi), \frac{a(x_0, y_0)}{a(x_0, y_0)} \right) \bigg|_{\xi = \xi} = \rho^2 M(x_0, y_0).$$

Recalling (2.69) by Lemma 2.13 we see the matrix has negative eigenvalues $-\lambda_1, \ldots, -\lambda_{d-1}$ while $\lambda_i \sim 1$, $i = 1, \ldots, d - 1$. We now note that

$$|\partial_x \partial_y \tilde{F}_h(0, x, y)| \lesssim 1, \quad |\partial_x \partial_y \tilde{F}_h(0, x, y)| \lesssim 1.$$

The former follows from (2.47) and Lemma 2.9. The latter can be shown similarly. Therefore, taking small enough $\epsilon_0 > 0$, by continuity we see that the matrix

$$\partial_\xi \partial_\xi' \left( \partial_x \phi_{y_d}(x, \xi), \frac{a_p(x, \xi, y_d)}{a_p(x, \xi, y_d)} \right) \bigg|_{\xi = \xi}$$

has negative eigenvalues $-\lambda_1, \ldots, -\lambda_{d-1}$ with $\lambda_i \sim 1$, $i = 1, \ldots, d - 1$ for $(x, \xi, y_d) \in B(0, \epsilon_0) \times B(0, \epsilon_0)$. Therefore the phase function $\phi_{y_d}$ satisfies the elliptic Carleson-Sjölin condition, i.e., (C1)–(C3). We now set

$$T_{y_d} g(x) = \int A(x, \xi, y_d) e^{i\lambda \rho \tilde{F}_h(0, x, \xi, y_d)} g(\xi) d\xi,$$

Thus we can use Theorem 2.11 to obtain

$$\|T_{y_d} g\|_p \leq C(\lambda \rho)^{-\frac{d}{p}} \|g\|_p.$$

Since $\mathcal{T} f = \int T_{y_d} f(\cdot, y_d) dy_d$, by Minkowski’s inequality followed by Hölder’s inequality gives (2.67). This completes the proof. □
3. *BOCHNER-RIESZ MEANS FOR THE SPECIAL HERMITE EXPANSION:*

**Proof of Theorem 1.4**

In this section we consider the Bochner-Riesz means for the special Hermite expansion. Basically, we follow the same strategy for the Hermite expansion. We begin with noting that the expression (1.5) can be simplified by making use of the twisted convolution which is defined as follows:

\[ f \times g(z) = \int_{\mathbb{R}^{2d}} f(z-w)g(w)e^{i(z,\mathbf{S}z')}dw, \]

where \( \mathbf{S} \) is a skew-symmetric \( 2d \times 2d \) matrix given by

\[ \mathbf{S} = \begin{pmatrix} 0 & -\mathbf{I}_d \\ \mathbf{I}_d & 0 \end{pmatrix}. \]

By the argument using the Weyl transform, it can be shown that \( \Pi^d_k f = f \times \varsigma_k \), where \( k = \frac{n-d}{2} \) and \( \varsigma_k(z) = (2\pi)^{-d}L^{d-1}_{\frac{1}{2}|z|^2}e^{-\frac{|z|^2}{2}} \), and \( L^\alpha_k \) is the k-th Laguerre polynomial of type \( \alpha \). Thus the Schrödinger propagator \( e^{-itL}f \) is given by

\[ e^{-itL}f = \sum_{k=0}^{\infty} e^{-it(2k+d)}f \times \varphi_k. \]

If \( f \in \mathcal{S}(\mathbb{R}^{2d}) \), by Lemma 3.1 below we see that the sum in the right-hand side is uniformly and absolutely convergent. Using the kernel formula for the heat operator \( e^{-tL} \) by replacing \( t \) with \( it \) (see [23, p.37]) we have

\[ e^{-itL}f(z) = \frac{1}{(-4\pi i \sin t)^d} \int_{\mathbb{R}^{2d}} e^{i\frac{t}{4}|z-z'|^2\cot t + \frac{1}{4}(z,\mathbf{S}z')}f(z')dz', \quad f \in \mathcal{S}(\mathbb{R}^{2d}). \]

Here and henceforth, we regard the variables \( z, z' \) as real variables, i.e., \( z, z' \in \mathbb{R}^{2d} \).

**Lemma 3.1.** Let \( f \in \mathcal{S}(\mathbb{R}^{2d}) \) and \( N \in \mathbb{N} \). Then, there exists a constant \( C = C(N, f, d) \) that satisfies the estimate \( \|f \times \varphi_k\|_\infty \leq C|k|^{-N} \) for any \( k \in \mathbb{N}_0 \).

**Proof.** We recall the known identity \( \varphi_k(z) = (2\pi)^{-d/2}\sum_{|\alpha| = k} \Phi_{\alpha,\alpha}(z) \) (see [30, p. 30]). From this, we have

\[ \|\varphi_k\|_2 = (2\pi)^{-d/2}(\|\alpha\| = k)^{\frac{d}{2}} \sim \lambda^{\frac{d-1}{2}}, \]

where \( \lambda = 2k+d \). Clearly we have \( f \times \mathcal{L}^N \varphi_k = \mathcal{L}^N(f \times \varphi_k) = \lambda^N f \times \varphi_k \) and, on the other hand, by routine integration by parts we also have \( f \times \mathcal{L}g = (Lf) \times g \) for a second order differential operator \( L \) whose coefficients are \( O(|z|^2) \). Thus we obtain \( f \times \varphi_k = \lambda^{-N} (f \times \mathcal{L}^N \varphi_k) = \lambda^{-N} (\mathcal{L}^N f \times \varphi_k) \). By Hölder’s inequality and (3.2) we get \( |f \times \varphi_k(z)| \leq \lambda^{-N}\|\mathcal{L}^N f\|_2\|\varphi_k\|_2 \leq C_N \lambda^{-N+\frac{d}{2}} \) for any \( N \) since \( f \in \mathcal{S}(\mathbb{C}^d) \). \( \square \)

Using the same notation as in Section 2 we decompose (c.f. (2.2))

\[ S^\delta(\mathcal{L}) = \left(1 - \frac{\mathcal{L}}{\lambda}\right)^\delta = \lambda^{-\delta} \sum_{1 \leq 2^j \leq 4\lambda} 2^{j\delta}\psi_j(\lambda - \mathcal{L}). \]

As before, Theorem 1.4 follows if we show

\[ \|\chi_{E_\lambda} \psi_j(\lambda - \mathcal{L})\chi_{F_\lambda}\|_p \lesssim (\lambda 2^{-j})^{\delta(2d,p)} \]

\[ \text{In fact, one can make it rigorous via analytic continuation using the operator } e^{-zL} \text{ with } \Re z > 0. \]
for \( p > p_0(d) \). As before, for any bounded continuous function \( m \) on \( \mathbb{R} \), by \( m(\mathcal{L}) \) we denote the operator defined by \( m(\mathcal{L}) = \sum_{\lambda \in 2\mathbb{N}_0 + d} m(\lambda) \mathcal{L}_\lambda^d \).

For \( \eta \in \mathcal{S}(\mathbb{R}) \) we define the scaled operator \([\eta]\mathcal{L}^d \) of which kernel is given by

\[
[\eta]\mathcal{L}^d(z, z') = \int \eta(t)(\sin t)^{-\frac{d}{2}} e^{i\lambda \mathcal{L}(t, z, z')} dt, 
\]

where

\[
\mathcal{L}(t, z, z') := t + \frac{|z - z'|^2 \cos t}{4 \sin t} + \langle z, S z' \rangle. 
\]

The explicit kernel form \((3.5)\) gives the following periodic and symmetric property of \([\eta]\mathcal{L}^d \).

**Lemma 3.2.** Let \( \eta \in C^\infty_c((0, \infty)) \). Let \( \mathbf{L} \) denote a \( 2d \times 2d \) rotation matrix

\[
\mathbf{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_d & -1_d \\ 1_d & 1_d \end{pmatrix}. 
\]

Then we have

\[
\|\chi_E[\eta]^{\mathcal{L}^d} \chi_F\|_p = \|\chi_{\mathbf{L}(E)}[\eta(- \cdot)]^{\mathcal{L}^d} \chi_{\mathbf{L}(F)}\|_p, 
\]

\[
(3.7) \quad \|\chi_E[\eta]^k \chi_F\|_p = \|\chi_E[\eta(\cdot + k\pi)]^k \chi_F\|_p, \quad k \in \mathbb{Z}. 
\]

**Proof of Lemma 3.2** The first identity \((3.7)\) follows from

\[
[\eta(- \cdot)]^k(z, z') = c[\eta]^{\mathcal{L}^d}(\mathbf{L}z, \mathbf{L}z'), \quad z, z' \in \mathbb{R}^{2d},
\]

where \( c \) is a constant such that \(|c| = 1 \). This can be shown changing variables \( t \rightarrow -t \) and using the fact that \( S = -\mathbf{L}^t S \mathbf{L} \). For \((3.8)\), the change of variable \( t \rightarrow t + k\pi \) gives \([\eta(\cdot + k\pi)]^k(z, z') = c[\eta]^{\mathcal{L}^d}(z, z')\) with \(|c| = 1 \). Hence we get \((3.8)\). \( \square \)

### 3.1. Reductions.

Similarly as before, we observe that

\[
\tilde{\psi}_j(\lambda - \mathcal{L})(x, y) = [\tilde{\psi}_j]^{\mathcal{L}}(\sqrt{x}, \sqrt{y}).
\]

Thus, by scaling and \((3.1)\) we see the estimate \((3.4)\) is equivalent to

\[
\|\chi_E[\tilde{\psi}_j]^{\mathcal{L}} \chi_F\|_p \lesssim \lambda^{-d}(\lambda^2)^{d(2d, p)}. 
\]

To prove the estimate \((3.9)\), we proceed in the similar manner as in the Hermite case. We decompose \([\tilde{\psi}_j]^{\mathcal{L}} \chi \) using the cutoff functions \( \psi_j^{(l, k)} \) in Section 2 (see \((2.12)\)).

Thus we have \([\tilde{\psi}_j]^{\mathcal{L}} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} [\psi_j^{(l, k)}]^{\mathcal{L}} \). If we combine this with the symmetric and periodic properties in Lemma 3.2, the proof of \((3.9)\) is basically reduced to showing the following. (See Proof of \((2.7)\).)

**Theorem 3.3.** Let \( 0 < \rho < \pi - \epsilon_0 \) for some \( \epsilon_0 > 0 \). Let \( \eta_\rho \) be a smooth function such that \( \text{supp} \eta_\rho \subset [2^{-2\rho}, \rho] \) and \( |\eta_\rho^{(n)}(t)| \leq C_n \rho^{-n} \) for \( t > 0 \), \( n \in \mathbb{N}_0 \). Then for any \( \lambda \in 2\mathbb{N}_0 + d \) and the sets \( E, F \) satisfying the condition in Theorem 1.4, we have

\[
\|\chi_E[\eta_\rho]^{\mathcal{L}} \chi_F\|_p \lesssim \begin{cases} 
\lambda^{-d}\rho, & \rho \leq \lambda^{-1}, \\
\lambda^{-d}\rho^{d(2d, p)}\rho^{d(2d, p)+1}, & \rho \geq \lambda^{-1}
\end{cases}
\]

whenever \( p > p_0(2d) \).
In order to show Theorem 3.3 we make a simple observation. From the assumption on $E, F$ in Theorem 1.4 there exists $z_0 \in \mathbb{R}^{2d}$ such that $E, F \subset B(z_0, 4)$. Using (3.5) it is easy to see that

$$\left[ \eta_0 \right]_{\chi}^F (z + z_0, z' + z_0) = \left[ \eta_0 \right]_{\chi}^F (z, z') e^{i \frac{1}{2} \chi ((z_0, Sz_0) + (z, Sz_0))}.$$ 

Clearly the oscillating term $e^{i \frac{1}{2} \chi ((z_0, Sz_0) + (z, Sz_0))}$ does not affect the $L^p$ norm of $\chi_E \left[ \eta_0 \right]_{\chi}^F \chi_F$, so we may assume that $E, F \subset B(0, 4)$ after making a change of variable $(z, z') \to (z + z_0, z' + z_0)$.

We now recall (3.5) and (3.6). To obtain estimates for the kernel of $\chi_E \left[ \eta_0 \right]_{\chi}^F \chi_F$ we consider the phase function $\mathcal{P}_c(t, z, z')$. A computation gives

$$\partial_t \mathcal{P}_c(t, z, z') = 1 - \frac{|z - z'|^2}{4 \sin^2 t}, \quad t \in (0, \pi).$$

Since $|z - z'| \leq 2 - c_0$, $\mathcal{P}_c(t, z, z')$ has two critical points $\mathcal{S}_c(z, z')$ and $\mathcal{S}_c^* (z, z')$ such that $\mathcal{S}_c(z, z') \in (0, \pi/2 - 2c_0)$ and $\mathcal{S}_c^* (z, z') \in (\pi/2 + 2c_0, \pi)$ for a small $c_0 > 0$ and

$$\sin \mathcal{S}_c(z, z') = \sin \mathcal{S}_c^*(z, z') = \frac{|z - z'|}{2}.$$ 

So we have $\pi - \mathcal{S}_c^*(z, z') = \mathcal{S}_c(z, z')$.

As in the Hermite case which we have handled, we may further assume that

$$\text{dist}(\mathcal{S}_c^*(z, z'), \text{supp } \eta_0) \geq c_0, \quad (z, z') \in E \times F.$$ 

To justify this, let us define $\eta_1, \eta_2, \eta_3 \in C_c^\infty([0, \pi])$ such that $\text{supp } \eta_1 \subset [0, \pi/2 - c_0)$, $\text{supp } \eta_2 \subset (\pi/2 - 3c_0, \pi/2 + 3c_0/2)$, $\text{supp } \eta_3 \subset (\pi/2 + c_0, \pi]$ and $\eta_1 + \eta_2 + \eta_3 = 1$ on $[0, \pi]$. Then we split $\left[ \eta_0 \right]_{\chi}^F$ into the sum

$$\left[ \eta_0 \right]_{\chi}^F + \left[ \eta_0 \eta_1 \right]_{\chi}^F + \left[ \eta_0 \eta_2 \right]_{\chi}^F + \left[ \eta_0 \eta_3 \right]_{\chi}^F.$$ 

Since $\text{supp } (\eta_0 \eta_1) \subset [0, \pi/2 - c_0)$, so the desired assumption is valid in this case. Also, from (3.11) it follows that $|\partial_t \mathcal{P}_c(t, z, z')| \geq c_0^3$ on $\text{supp } (\eta_0 \eta_2)$. So, applying Lemma 2.5 yields that $|\langle \chi_E \eta_0 \eta_3 \chi_F\rangle (z, z')| \lesssim \lambda^{-N}$ for any $(z, z') \in E \times F$. Hence we may disregard the contribution from $\left[ \eta_0 \eta_2 \right]_{\chi}^F$. For the remaining term $\left[ \eta_0 \eta_3 \right]_{\chi}^F$, by utilizing (3.7), (3.8) we note

$$\|\chi_E \eta_0 \eta_3 \chi_F\|_p = \|\chi_L (E) \left[ \eta_0 \eta_3 \right] (\pi - \cdot)\|_p^F \chi_L (F)\|_p$$ 

where $L$ is the rotation matrix in Lemma 3.3. Obvioulsy $L(E), L(F)$ satisfy the assumption of Theorem 3.3 and $\text{supp } ((\eta_0 \eta_3)(\pi - \cdot)) \subset [0, \pi/2 - c_0)$. Hence it is sufficient to show the estimate (3.10) under the additional assumption (3.13).

In order to show (3.10), by additional decomposition and the standard argument as in Section 2 it is sufficient to show

$$\|\varphi \left[ \eta_0 \right]_{\chi}^F \varphi'\|_p \lesssim \begin{cases} \lambda^{-d} \rho, & \rho \leq \lambda^{-1}, \\ \lambda^{-d + s(2d, p) \rho^{d(2d, p) + 1}}, & \rho > \lambda^{-1} \end{cases},$$ 

where $\varphi, \varphi'$ are smooth functions such that $\text{supp } \varphi \times \text{supp } \varphi' \subset \{(z, z') : |z|, |z'| < 4, |z - z'| < 2 - c_0\}$. For the purpose we decompose the integral kernel in dyadic manner:

$$\langle \varphi \left[ \eta_0 \right]_{\chi}^F \varphi' \rangle (z, z') = \sum_l |T^l (z, z') := \sum_i |\eta_0 |_{\chi}^F (z, z') \psi (2^l |z - z'|) \varphi (\cdot) \varphi' (\cdot).$$
The matter now is reduced to showing the estimate for $\mathcal{I}_1$. For the purpose it is sufficient to consider the operator which is given by the kernel
\[
(\chi_l[\eta_\rho]z\chi_l')(z, z') = [\eta_\rho]z\chi_l(z)\chi_l(z'),
\]
where $\chi_l, \chi_l'$ satisfy (2.25) and supp $\chi_l \times \text{supp} \chi_l' \subset \{(z, z') : |z|, |z'| < 4, |z - z'| < 2 - c_0\}$. Thus, by (3.12) we have
\[
(\chi_l[\eta_\rho]z\chi_l')(z, z') \sim 2^{-l}, \quad (z, z') \in \text{supp} \chi_l \times \text{supp} \chi_l'.
\]

**Lemma 3.4.** Let $1 \leq p \leq \infty$. If $\rho \leq \lambda^{-1}$, then for any $N \geq 0$ we have
\[
\|\chi_l[\eta_\rho]z\chi_l'\|_p \lesssim \begin{cases} \lambda^{-N}\rho^{1-d-N}2^{2N-2d}|l|^{1+d-N}2^{N-3}, & 2^{-l} \gg \lambda^{-\frac{1}{2}}\rho^{\frac{1}{2}}, \\ \rho^{1-d-2^{N-3}}, & 2^{-l} \lesssim \lambda^{-\frac{1}{2}}\rho^{\frac{1}{2}}. \end{cases}
\]

**Proof.** We assume $(t, z, z') \in \text{supp} \eta_\rho \times \text{supp} \chi_l \times \text{supp} \chi_l'$. We first consider the case $2^{-l} \gg \lambda^{-\frac{1}{2}}\rho^{\frac{1}{2}}$. Since $2^{-l} \gg \rho$, by (3.16) it follows that $0 < \sin t \ll |z - z'|$. From (3.11), we have
\[
|\partial_t \mathcal{P}_l(t, z, z')| \sim \frac{|z - z'|^2}{4 \sin^2 t} \sim 2^{-2l} \rho^{-2}.
\]
Also a simple calculation shows
\[
|\partial_t^\nu \mathcal{P}_l(t, z, z')| \lesssim \frac{|z - z'|^2}{(\sin t)^{n+1}} \lesssim 2^{-2l} \rho^{-n-1}, \quad n \geq 2,
\]
\[
|\partial_t^\nu (\eta_\rho(t)(\sin t)^{-d})| \lesssim \rho^{-d-n}, \quad n \geq 1.
\]
Hence Lemma 2.5 gives $|\langle \chi_l[\eta_\rho]z\chi_l' \rangle(z, z')| \lesssim_N \lambda^{-N}\rho^{1-d-N}2^{N/2}$, Using Lemma 2.4 we get $\|\chi_l[\eta_\rho]z\chi_l'\|_p \lesssim_N \lambda^{-N}\rho^{1-d-N}2^{N-3d}|l|$ for any $1 \leq p \leq \infty$.

We now handle the case $2^{-l} \lesssim \lambda^{-\frac{1}{2}}\rho^{\frac{1}{2}}$. We first recall
\[
\|\Pi^\perp_z\|_{1 \to \infty} \lesssim \lambda^{d-1}, \quad \lambda \in 2\mathbb{N}_0 + d.
\]
We refer the reader to (30, Section 2.6) for the proof of (32). Then, by following the same argument in the proof of Lemma 2.6 we have
\[
|\eta_\rho(z, z')| \lesssim \sum_{\nu \leq 2\mathbb{N}_0 + d} \rho(1 + \rho|\lambda - \nu|)^{-N}\rho^{-d-1} \lesssim \rho^{1-d}, \quad z, z' \in \mathbb{R}^d.
\]
This and Lemma 2.5 give the desired estimate. 

**Estimate for $\chi_l[\eta_\rho]z\chi_l'$ when $\rho > \lambda^{-1}$**. Now we deal with the case $\rho > \lambda^{-1}$. We separately consider three sub-cases $2^{-l} \ll \rho$, $2^{-l} \gg \rho$, and $2^{-l} \sim \rho$. The first two cases are easy to handle.

**Lemma 3.5.** Let $1 \leq p \leq \infty$. If $\rho > \lambda^{-1}$, then
\[
\|\chi_l[\eta_\rho]z\chi_l'\|_p \lesssim \begin{cases} \lambda^{-N}\rho^{1-d-N}2^{-d-1}, & 2^{-l} \ll \rho, \\ \lambda^{-N}\rho^{1-d-N}2^{2N-d}|l|, & 2^{-l} \gg \rho. \end{cases}
\]
Proposition 3.6. Let argument as before. □

Thus, the estimates \(3.19\), \(3.20\) remains valid. So, from Lemma 2.5 it follows that \(|\chi_l\eta_p|^l\chi_l(z, z')| \lesssim_N \lambda^{-N} p^{1-d-N}\) for any \(N \in \mathbb{N}\). This estimate combined with Lemma 2.4 yields \(\|\chi_l|\eta_p|^l\chi_l\|_p \lesssim \lambda^{-N} p^{1-d-N} 2^{-2d} \) for \(2 \leq p \leq \infty\).

Now we consider the case \(2^{-l} \gg \rho\). In this case we have \(0 < \rho \ll \mathcal{S}_r(z, z') < \pi/2\) because of \(3.16\), so as in the previous case the derivatives of \(\mathcal{P}_l\) satisfy \(3.13\) and \(3.19\). Also the amplitude function \(\eta_p(t)(\sin t)^{-d}\) satisfies the bound \(3.20\). With these estimates, we obtain the second case estimate in \(3.22\) following the same argument as before.

What follows is the key estimate which we need for the proof of Theorem 3.3.

**Proposition 3.6.** Let \(\rho > \lambda^{-1}\) and \(\rho \sim 2^{-l}\). Then we have
\[
\|\chi_l|\eta_p|^l\chi_l\|_p \lesssim \lambda^{-d+\delta(2d,p)} p^{1+\delta(2d,p)}
\]
provided that \(p_0(2d) \leq p \leq \infty\).

Now, putting Lemma 3.4, Lemma 3.5 and Proposition 3.6 together, we can prove Theorem 3.3 by establishing \(3.13\). Let us consider the case \(\rho > \lambda^{-1}\) first. By \(3.19\) we have
\[
\|\varphi|\eta_p|^l\varphi\|_p \leq \sum_l \|\mathcal{I}^l\|_p.
\]

Using Lemma 3.5 and Proposition 3.6 by a standard argument (for example, Lemma 2.3) we have
\[
\|\mathcal{I}^l\|_p \lesssim \begin{cases} 
\lambda^{-N} p^{1-\frac{1}{2}d-N} 2^{-dl}, & 2^{-l} \ll \rho, \\
\lambda^{-d+\delta(2d,p)} p^{1+\delta(2d,p)}, & 2^{-l} \sim \rho, \\
\lambda^{-N} p^{1-\frac{1}{2}d-N} 2^{N-d}l, & 2^{-l} \gg \rho.
\end{cases}
\]

Taking summation over \(l\) as in the Proof of \(2.22\), we get the desired bound \(3.14\). The remaining case \(\rho > \lambda^{-1}\) can be handled similarly using Lemma 3.4, so we omit the detail.

### 3.2. Asymptotic expansion of the kernel when \(2^{-l} \sim \rho\).

Unlike the previous cases, the support of \(\eta_p\) may contain the critical point \(\mathcal{S}_c(z, z')\).

**Additional decomposition of \(\chi_l, \chi_l'\) and \(\eta_p\).** Let \(\epsilon_0 > 0\) be a small number. If we further break \(\chi_l, \chi_l'\) and \(\eta_p\) into finitely many the cutoff functions, then in addition to \(2.26\) we may assume that
\[
\text{supp } \chi_l \times \text{supp } \chi_l' \subset \{ (z, z') : |z|, |z'| < 2, |z - z'| < 2 - 2^{-1}\epsilon_0\},
\]
\[
\text{supp } \chi_l \subset B(z_0, \epsilon_0\rho), \quad \text{supp } \chi_l' \subset B(z_0, \epsilon_0\rho),
\]
\[
\mathcal{S}_c(z, z') \in \mathcal{S}_c(z_0, z_0') + (-\epsilon_0\rho, \epsilon_0\rho), \quad (z, z') \in \text{supp } \chi_l \times \text{supp } \chi_l',
\]
\[
\text{supp } \eta_p \subset \mathcal{S}_c(z_0, z_0') + (-2\epsilon_0\rho, 2\epsilon_0\rho).
\]
for some \( z_0 \) and \( z'_0 \). The conditions (3.24) – (3.26) can be achieved splitting \( \chi_i, \chi'_i \) into cutoff functions which are supported in balls of radius \( c\epsilon_0 \rho \) with a small \( c > 0 \). For the last (3.27) we break

\[
\chi_i(\eta_\rho) |^ \xi \chi'_i = \chi_i(\eta_\rho \tilde{\eta}_\rho) |^ \xi \chi'_i + \chi_i(1 - \tilde{\eta}_\rho) |^ \xi \chi'_i
\]

where \( \tilde{\eta}_\rho \) is a smooth bump function such that \( \text{supp} \eta_\rho \subset S_c(z_0, z'_0) + (-3c\epsilon_0 \rho, 3c\epsilon_0 \rho) \), \( \tilde{\eta}_\rho(t) = 1 \) if \( t \in S_c(z_0, z'_0) + (-2c\epsilon_0 \rho, 2c\epsilon_0 \rho) \), and \( |\eta_\rho(\rho)| \leq C \rho^{-n} \) for any \( n \in \mathbb{N}_0 \). By the same argument as before we get

\[
|\chi_i(\eta_\rho(1 - \tilde{\eta}_\rho)) |^ \xi \chi'_i |_p \lesssim (\lambda \rho)^{-N} \rho^{1+d}
\]

for \( \rho > \lambda^{-1} \). Therefore, the contribution from \( \chi_i(\eta_\rho(1 - \tilde{\eta}_\rho)) |^ \xi \chi'_i \) is negligible. Thus, it is sufficient to deal with \( \chi_i(\eta_\rho(1 - \tilde{\eta}_\rho)) |^ \xi \chi'_i \), so we may assume (3.27). To show (3.28), we note that \( |\partial_t P_H(t, x, y)| \gtrsim 1 \) and \( |\partial_\beta P_H(t, x, y)| \lesssim \rho^{1-n} \), \( n \in \mathbb{N} \) if \( t \in \text{supp}(\eta_\rho(1 - \tilde{\eta}_\rho)) \) and \( (z, z') \in \text{supp} \chi_i \times \text{supp} \chi'_i \). Then, using Lemma 2.5 and Lemma 2.4 we obtain the estimate (3.28).

Since \( \text{supp}(z, z') \sim \rho \) for \( (z, z') \in \text{supp} \chi_i \times \text{supp} \chi'_i \), we have

\[
\partial_t^2 P_C(\text{supp}(z, z'), z, z') = \frac{|z - z'|^2 \cos \theta_c(z, z')}{2 \sin^3 \theta_c(z, z')} \sim \rho^{-1}
\]

for \( (z, z') \in \text{supp} \chi_i \times \text{supp} \chi'_i \). Recalling (3.30) it is easy to see

\[
\partial_\alpha^2 \partial_\beta^2 \partial_\gamma^2 \partial_\delta P_C(t, y, z) = O(\rho^{1-n-|\alpha|-|\delta|}), \quad n \geq 1
\]

for \( (t, z, z') \in \text{supp} \eta_\rho \times \text{supp} \chi_i \times \text{supp} \chi'_i \). Since sin \( \theta_c(z, z') = \frac{|z - z'|}{2} \) and \( \partial_\alpha \text{supp} \theta_c(z, z') = \partial_\alpha \cos \theta_c(z, z') \), we see that \( \partial_\alpha \text{supp} \theta_c(z, z') = (4 - |z - z'|^2)^{-1/2} \partial_\alpha (|z - z'|) \). Thus a routine computation gives

\[
\partial_\alpha^2 \partial_\beta^2 \partial_\gamma^2 \partial_\delta \theta_c(z, z') = O(|z - z'|^{-|\alpha|-|\delta|}) = O(\rho^{1-|\alpha|-|\delta|})
\]

for \( (z, z') \in \text{supp} \chi_i \times \text{supp} \chi'_i \) since \( \rho \sim 2^{-l} \).

The derivatives of the phase \( P_c \) and the amplitude \( \eta_\rho \) are not uniformly bounded in \( \rho \). Following the same approach in the previous section, we can get around this making change of variables. Let us set

\[
\phi(t, z, z') = S_c(\rho z + z_0, \rho z' + z'_0) + \rho t,
\]

\[
\phi(t, z, z') = \rho^{-1} P_C(\phi(t, z, z'), \rho z + z_0, \rho z' + z'_0),
\]

\[
\chi_i(z) = \chi_i(\rho z + z_0), \quad \chi'_i(z') = \chi'_i(\rho z' + z'_0),
\]

\[
a(t, z, z') = \rho^d (\sin \phi(t, z, z') - d \eta_\rho(\phi(t, z, z'))) \chi_i(z) \chi'_i(z').
\]

Then we also set

\[
I_\rho^C(x, y) = \rho^{1-d} \int a(t, z, z') e^{i\lambda t} \tilde{P}_C(t, z, z') dt.
\]

By the change of variables \( t \rightarrow \phi(t, z, z') \), we write

\[
(\chi_i(\eta_\rho) |^ \xi \chi'_i)(\rho z + z_0, \rho z' + z'_0) = C_d I_\rho^C(x, y)
\]

where \( C_d \) is a constant depending on \( d \). Then, the support of \( a \) is contained in the interval \((-2c\epsilon_0, 2c\epsilon_0) \times B(0, \epsilon_0) \times B(0, \epsilon_0)\). From (3.31) we first note

\[
|\partial_\alpha^2 \partial_\beta^2 \partial_\gamma^2 \phi(t, z, z')| \leq C_{n, \alpha, \beta, \rho}
\]
for \((t, z, z') \in \text{supp} \, a \times \text{supp} \, \tilde{\chi}_I \times \text{supp} \, \tilde{\chi}'_I\). Using this and \((3.30)\), one can easily see
\[
|\partial_t^\alpha \partial_z^\beta \partial_{\tilde{t}}^\gamma a(t, z, z')| \leq C_{n, \alpha, \beta, \gamma},
\]
(3.35)
\[
|\partial_t^\alpha \partial_z^\beta \partial_{\tilde{t}}^\gamma \tilde{P}_c(t, z, z')| \leq C_{n, \alpha, \beta, \gamma}
\]
(3.36)
for \((t, z, z') \in \text{supp} \, a \times \text{supp} \, \tilde{\chi}_I \times \text{supp} \, \tilde{\chi}'_I\). From \((3.29)\) we have
\[
\partial_t^2 \tilde{P}_c(t, z, z') \sim 1,
\]
(3.37)
and \(\partial_t \tilde{P}_c(0, z, z') = 0\). As before, we now apply the method of stationary phase ([12] Theorem 7.7.5) to the integral \(I^t_c(x, y)\), and we obtain

**Lemma 3.7.** Let \(N \in \mathbb{N}\). Then, for \((z, z') \in \text{supp} \, \chi_I \times \text{supp} \, \chi'_I\), we have
\[
I^t_c(x, y) = \rho^{1-d} \sum_{n=0}^{N-1} (\lambda \rho)^{-\frac{n}{2}} A_n(z, z') e^{i \lambda \rho \tilde{P}_c(0, z, z')} + E_N(z, z'),
\]
(3.38)
where \(A_n, E_N \in C_\infty(B(0, \epsilon_0))\) satisfy \(|\partial_t^\alpha \partial_z^\beta A_n(z, z')| \leq C_{\alpha, \beta}\) and \(\sup_{z, z'} |E_N(x, y)| \leq C_N \rho^{\frac{d-2}{2}} (\lambda \rho)^{-N}\) with \(C_{\alpha, \beta}\) and \(C_N\) independent of \(\lambda, \rho\).

Let us set
\[
\Phi_c(z, z') := \tilde{P}_c(\mathcal{S}_c(z, z'), z, z').
\]
(3.39)
In the next section we investigate the curvature condition of the phase function \(\Phi_c\).

### 3.3. The phase function \(\Phi_c(z, z')\).

For \((z, z') \in \text{supp} \, \tilde{\chi}_I \times \text{supp} \, \tilde{\chi}'_I\), we set
\[
v(z, z') := z - z',
\]
and
\[
R(z, z') := \cos \mathcal{S}_c(z, z') \mathbf{1}_{2d} - \sin \mathcal{S}_c(z, z') S.
\]
It should be noted that \(R(z, z') R(z, z')^\top = \mathbf{1}_{2d}\), so \(R(z, z')\) is a rotation matrix. We occasionally denote \(v = v(z, z')\), \(\mathcal{S}_c = \mathcal{S}_c(z, z')\), and \(R = R(z, z')\) for simplicity.

Since \(\partial_r \tilde{P}_c(\mathcal{S}_c, z, z') = 0\), from \((3.39)\) we have \(\partial_r \Phi_c(z, z') = \partial_r \tilde{P}_c(\mathcal{S}_c, z, z')\). Thus, using \((3.30)\), we get \(\partial_z \Phi_c(z, z') = \frac{(z - z') \cos \mathcal{S}_c}{2 \sin \mathcal{S}_c} + \frac{S z'}{2}\). Now, by \((3.12)\) we have
\[
\partial_z \Phi_c(z, z') = \frac{R(z, z')(z - z')}{|z - z'|} + \frac{S z'}{2}.
\]
Thus, we see \(|\partial_z \Phi_c(z, z') - S z'/2| = 1\) for any \(z'\). Hence the map \(z' \rightarrow \partial_z \Phi_c(z, z')\) has its rank at most \(2d - 1\). As we shall see later, \(\text{rank}(\partial_z \tilde{P}_c(\mathcal{S}_c, z, z')) = 2d - 1\) for any \((z, z') \in \text{supp} \chi_I \times \text{supp} \chi'_I\).

We compute the mixed Hessian \(\partial_z \partial_{\tilde{t}}^2 \Phi_c\). Using the chain rule, we have
\[
\partial_z \partial_{\tilde{t}}^2 \Phi_c = \partial_z \partial_{\tilde{t}}^2 \tilde{P}_c(\mathcal{S}_c, z, z') + \partial_z \partial_t \tilde{P}_c(\mathcal{S}_c, z, z') \cdot \partial_t^2 \mathcal{S}_c
\]
(3.40)
\[
+ \partial_z \mathcal{S}_c \cdot \partial_t^2 \partial_t \tilde{P}_c(\mathcal{S}_c, z, z') + (\partial_z \mathcal{S}_c \cdot \partial_t^2 \mathcal{S}_c) \partial_t^2 \tilde{P}_c(\mathcal{S}_c, z, z').
\]
Here we use $\partial_z \partial_z^\top \mathcal{P}_c(z, z') = 0$. From (3.40), one can easily see
\[
\partial_z \partial_z^\top \mathcal{P}_c(\mathcal{S}_c, z, z') = -\frac{\cos \mathcal{S}_c}{2 \sin \mathcal{S}_c} \mathbf{I} - \frac{1}{2} \mathbf{S},
\]
\[
\partial_z^\top \partial_z^\top \mathcal{P}_c(\mathcal{S}_c, z, z') = -\frac{\mathbf{v}^\top}{2 \sin^2 \mathcal{S}_c},
\]
\[
\partial_{z'} \partial_{z'}^\top \mathcal{P}_c(\mathcal{S}_c, z, z') = \frac{\mathbf{v}}{2 \sin^2 \mathcal{S}_c}.
\]

Since $\sin \mathcal{S}_c = |z - z'|/2$, by (3.29) we also have
\[
\partial_z^2 \mathcal{P}_c(\mathcal{S}_c, z, z') = \frac{4 \cos \mathcal{S}_c}{|z - z'|},
\]
Now we note that $\partial_z^2 \mathcal{S}_c = \partial_z^2 (\sin \mathcal{S}_c)/\cos \mathcal{S}_c$, $\partial_{z'} \mathcal{S}_c = \partial_{z'} (\sin \mathcal{S}_c)/\cos \mathcal{S}_c$. Thus, differentiating $\sin \mathcal{S}_c = |z - z'|/2$, we obtain
\[
\partial_z^2 \mathcal{S}_c = \frac{\mathbf{v}^\top}{2 \cos \mathcal{S}_c |z - z'|},
\]
\[
\partial_{z'} \mathcal{S}_c = -\frac{\mathbf{v}}{2 \cos \mathcal{S}_c |z - z'|}.
\]

Combining the identities together with (3.40), we obtain
\[
\partial_z \partial_z^\top \Phi_L = \frac{1}{|z - z'|^3 \cos \mathcal{S}_c} (\mathbf{v} \mathbf{v}^\top - \cos^2 \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{I} - \sin \mathcal{S}_c \cos \mathcal{S}_c \mathbf{v}^\top \mathbf{S}).
\]

Since $\mathbf{S}^2 = -\mathbf{I}$ and $\mathbf{R}(z, z')^\top = \cos \mathcal{S}_c \mathbf{I} + \sin \mathcal{S}_c \mathbf{S}$, it is easy to see the matrix $\mathbf{v} \mathbf{v}^\top - \cos^2 \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{I} - \sin \mathcal{S}_c \cos \mathcal{S}_c \mathbf{v}^\top \mathbf{S}$ can be factored as follows:
\[
(\cos \mathcal{S}_c \mathbf{v} \mathbf{v}^\top - \sin \mathcal{S}_c \mathbf{v}^\top \mathbf{S} - \cos \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{I}) \mathbf{R}(z, z')^\top.
\]

Now we observe
\[
(\cos \mathcal{S}_c \mathbf{v} \mathbf{v}^\top - \cos \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{I} - \sin \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{S}) \mathbf{v} = \begin{cases} 0, & v = \mathbf{v}, \\ -\mathbf{v}^\top \mathbf{v} \mathbf{R}(z, z') \mathbf{v}, & v = \mathbf{S} \mathbf{v}, \\ -\cos \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{v}, & v \in (\text{span}\{\mathbf{v}, \mathbf{S} \mathbf{v}\})^\perp. \end{cases}
\]

For the second case we use the fact that $\mathbf{S}$ is skew symmetric, i.e., $\mathbf{v}^\top \mathbf{S} \mathbf{v} = 0$. Note that the vectors $\mathbf{v}^\top \mathbf{v} \mathbf{R}(z, z') \mathbf{v}$ and $\cos \mathcal{S}_c \mathbf{v}^\top \mathbf{v} \mathbf{v}$, $\mathbf{v} \neq 0$ are nonzero. From this and (3.42) we see that the mixed Hessian $\partial_z \partial_z^\top \Phi_L$ has rank $2d - 1$ and $\partial_z \partial_z^\top \Phi_L \mathbf{R}(z, z') \mathbf{v} = 0$ for any $\mathbf{v}$. So, we obtain the following.

**Lemma 3.8.** For $(z, z') \in \text{supp} \chi_1 \times \text{supp} \chi_1'$ the matrix $\partial_z \partial_z^\top \Phi_L(z, z')$ has rank $2d - 1$ and its null space is generated by the vector
\[
\nu(z, z') = \frac{\mathbf{R}(z, z') \mathbf{v}(z, z')}{|\mathbf{v}(z, z')|}.
\]

The vector $\nu(z, z')$ is the unique (modulo $\pm$) unit vector such that $\partial_z \partial_z^\top \Phi_L \nu(z, z') = 0$. We now consider the matrix $\partial_{z'} \partial_{z'}^\top (\partial_z \Phi_L(z, w), \nu(z, z'))|_{w=z}$ for which we can obtain an explicit expression.

**Lemma 3.9.** For $(z, z') \in \text{supp} \chi_1 \times \text{supp} \chi_1'$ we have
\[
\partial_{z'} \partial_{z'}^\top (\partial_z \Phi_L(z, w), \nu(z, z'))|_{w=z} = \frac{-1}{\cos^2 \mathcal{S}_c |z - z'|^4} \mathbf{M}(z, z'),
\]
where the matrix $M(z, z')$ is given by

$$M(z, z') = vv^T - 2\cos^2c_vv^T + 2\cos^2c_vv^TS - Svv^T.$$  

Proof of Lemma 3.9. For simplicity we set $v_w(z) = v(z, w), \ \ G_w(z) = G_c(z, w).$

Using (3.42) and $R(z, z') = \cos G_cI - \sin G_cS,$ after some computation we have

$$\partial_w \partial_w^T \Phi(z, w) \cdot \nu(z, z') = (\cos G_w|z - w|^2|z - z'|)^{-1}k(z, z', w),$$

where $k(z, z', w) \in \mathbb{R}^{2d}$ is given by

$$k(z, z', w) = \cos G_c v_w^Tv\sin G_c v_wv^T v - \sin G_c v_wv^T v$$

Clearly $k(z, z', z') = 0$ for any $z'.$ Thus, we have

$$\partial_w \partial_w^T (\partial_w^T \Phi(z, w) \cdot \nu(z, z'))|_{w=z'} = \frac{1}{\cos G_c|z - z'|} \partial_w^T k(z, z', z').$$

Now, (3.43) follows if we express the matrix $\partial_w^T k(z, z', z')$ in the desired form, i.e.,

$$\partial_w^T k(z, z', z') = -\frac{1}{\cos G_c} M(z, z').$$

Via a routine computation we write

$$\partial_w^T k(z, z', w) = \cos G_c \partial_w^T (v_wv_w^Tv) - \sin G_c \partial_w^T (v_wv_w^Tv)$$

$$- \cos S_w \cos S_w - \sin S_w \cos S_w - \cos S_w \sin S_w \cos S_w$$

$$+ v_w^Tv_w \cos S_w \cos S_w - \cos S_w \sin S_w \cos S_w$$

Here we can discard the 5, 6, 7-th terms on the right hand side since they vanish if we put $w = z'.$ Also, we have

$$\partial_w^T (v_wv_w^Tv) = -(v_wv^T + v_w^Tv),$$

$$\partial_w^T (v_wv_w^Tv) = (v_wv^T S - v_w^TvSv I).$$

Using these identities, (3.41), and $4\sin^2 G_c = v^Tv$ (see (3.12)), we now get

$$\partial_w^T k(z, z', z') = -\cos G_c (v^Tv + v^Tv I) - \sin G_c (v^Tv S - v^TvSv I) + 2\cos G_c v^Tv$$

$$- (\cos G_c)^{-1} \sin^2 G_c v^Tv + \sin G_c Svv^T.$$  

Since $v^TvSv = 0,$ we therefore obtain (3.44). \qed

Lemma 3.10. Let $(z, z') \in \operatorname{supp} \chi_l \times \operatorname{supp} \chi_l'$ with $2^{-l} \sim \rho.$ Define the matrix $B(z, z')$ by setting

$$B(z, z') = 1_{2d-2} \frac{Sv(z, z')}{v(z, z')}, \ \ v(z, z')$$

where $\{v_i\}_{i=1}^{2d-2}$ denotes an orthonormal basis of $(\operatorname{span} \{v(z, z'), Sv(z, z')\})^\perp.$ Then the $2d \times 2d$ matrix

$$B^T(z, z')R(z, z')M(z, z')R(z, z')^TB(z, z')$$
is a diagonal matrix such that (2d, 2d)-th entry is zero and the other diagonal entries \(\lambda_1, \ldots, \lambda_{2d-1}\) satisfy \(\lambda_i \sim \rho^2\), \(1 \leq i \leq 2d - 1\).

**Proof of Lemma 3.10.** The matrix \(M = M(z, z')\) in Lemma 3.9 can be written as

\[
M = \cos^2 \Theta_v (v^\top v I + Svv^\top S - vv^\top) - R^T Svv^\top S R.
\]

Then, multiplying \(R = R(z, z')\) and \(R^\top = R(z, z')^\top\) to \(M(z, z')\), we have

\[
RMR^\top = \cos^2 \Theta_v (v^\top v I + \cos^2 \Theta_v R(Svv^\top S - vv^\top) R^\top - Svv^\top S).
\]

Using \(R = \cos \Theta I_{2d} - \sin \Theta S\), one can easily see \(R(Svv^\top S - vv^\top) R^\top = Svv^\top S - vv^\top\). Thus we have

\[
RMR^\top = \cos^2 \Theta_v (v^\top v I - vv^\top) - \sin^2 \Theta_v Svv^\top S.
\]

Then it is easy to verify

\[
RMR^\top v = \begin{cases} v^\top v \cos^2 \Theta_v v, & v \in (\text{span}\{v, Sv\})^\perp, \\ v^\top v v, & v = Sv, \\ 0, & v = v. \end{cases}
\]

Therefore, \(RMR^\top\) has the eigenvalues \(v^\top v \cos^2 \Theta_v (z, z')\), \(v^\top v, 0\) with multiplicity \(2d - 2\), \(2\), \(1\), respectively. It is clear that \(v^\top v, v^\top v \cos^2 \Theta_v (z, z') \sim |z - z'|^2 \sim \rho^2\) because \((z, z') \in \text{supp} \chi_l \times \text{supp} \chi'_l\). Now the elementary (diagonalization) argument gives (3.45).

### 3.4. Proof of Proposition 3.6

We now prove (3.23). Let us set

\[
\mathcal{T}_E f(z) = \int A(z, z') e^{i\lambda p \tilde{p}_E(0, z, z') f(z')dz'},
\]

where \(A \in C_c^\infty(B(0, c_0) \times B(0, c_0))\). Then we note

\[
\|\chi_l[\eta_0]|_{\chi_l'}\|_p = \rho^{2d} \|\chi_l[\eta_0]|_{\chi_l'}\rho \cdot + z_0, \rho \cdot + z_0'\|_p.
\]

If we use (3.33) and (3.38) with a large \(N\), for the proof of (3.23) it is sufficient to show

\[
\|\mathcal{T}_E f\|_p \leq C(\lambda \rho)^{-\frac{2d}{2}} \|f\|_p
\]

for \(p_0(2d) < p \leq \infty\). By (3.30) and (3.31) it follows that

\[
|\partial^\alpha z^\beta \tilde{p}_E(0, z, z')| \leq C_{\alpha, \beta}.
\]

Recalling \(\tilde{p}_E(0, z, z') = \rho^{-1} \Phi_E(\rho z + z_0, \rho z' + z_0')\) (see (3.39) and (3.32)), we put

\[
\tilde{\Phi}(z, z') = \rho^{-1} \Phi_E(\rho R_0 z + z_0, \rho R_0 z' + z_0'),
\]

where

\[
R_0 = R(z_0, z_0')^\top B(z_0, z_0).
\]

We define

\[
\mathcal{T}_E f(z) = \int A(R_0 z, R_0 z') e^{i\lambda p \tilde{\Phi}(z, z') f(z') dz'}.
\]

Since \(\mathcal{P}_E(0, z, z') = \rho^{-1} \Phi_E(\rho z + z_0, \rho z' + z_0')\), (3.40) is equivalent to

\[
\|\mathcal{T}_E f\|_p \leq C(\lambda \rho)^{-\frac{2d}{2}} \|f\|_p.
\]
Fixing $z_{2d} \in (-\epsilon_0, \epsilon_0)$, let us set

$$
\widetilde{T}_{E_{z_{2d}}}^g(z) = \int A(z, \zeta) e^{i\lambda \rho \tilde{\Phi}(z, \zeta, z_{2d})} g(\zeta) d\zeta, \quad \zeta \in \mathbb{R}^{2d-1},
$$

where $A \in C_c^\infty(B(0, \epsilon_0) \times B(0, \epsilon_0))$. By the same argument as before, the estimate (3.49) follows if we show

$$
\|\widetilde{T}_{E_{z_{2d}}}^g\|_p \leq C(\lambda \rho)^{-\frac{2d}{\alpha}} \|g\|_p, \quad z_{2d} \in (-\epsilon_0, \epsilon_0).
$$

To show the estimate it is sufficient to show that the function

$$
\Phi_{z_{2d}}(z, \zeta) := \tilde{\Phi}(z, \zeta, z_{2d}), \quad (z, \zeta) \in \mathbb{R}^d \times \mathbb{R}^{d-1}
$$

satisfies the Carleson-Sjölin condition with ellipticity for $z_{2d} \in (-\epsilon_0, \epsilon_0)$ on the support of $\tilde{A}$ since the desired estimate (3.49) follows from Theorem 2.11.

Now we note that

$$
\partial_z \partial_{z'}^2 \Phi(z, z') = \rho R_0^T \partial_z \partial_{z'}^2 \Phi_E(\rho R_0 z + z_0, \rho R_0 z' + z_0').
$$

By Lemma 3.8 we have

$$
\partial_z \partial_{z'}^3 \Phi(z, z') N(z, z') = 0,
$$

where $N(z, z') = R_0^T \nu(\rho R_0 z + z_0, \rho R_0 z' + z_0')$. Thus, it is clear that

$$
\partial_\zeta \langle \partial_z \partial_{z'}^2 \Phi \rangle (z, \zeta, N(z, z')) = 0.
$$

Since $\langle \partial_z \Phi(z, z'), N(z, z') \rangle = \partial_z^2 \Phi_E(\rho R_0 z + z_0, \rho R_0 z' + z_0') \nu(\rho R_0 z + z_0, \rho R_0 z' + z_0')$, we have

$$
\partial_z \partial_{z'}^2 \langle \partial_z \Phi(0, z'), N(0, 0) \rangle |_{z'=0} = \rho^2 R_0^T \partial_w \partial_{w'} \langle \partial_z \Phi_E(z_0, w), \nu(z_0, z_0') \rangle |_{w=z_0} R_0.
$$

Thus, using (3.43), we get

$$
\partial_z \partial_{z'}^2 \langle \partial_z \Phi(0, z'), N(0, 0) \rangle |_{z'=0} = -\frac{\rho^2}{\cos^2 \mathcal{S}_c(z_0, z_0') |z_0 - z_0'|^4} R_0^T M(z_0, z_0') R_0.
$$

From Lemma 3.10 $R_0^T M(z_0, z_0') R_0$ is a diagonal matrix with its nonzero diagonal entries $\sim \rho^2$. Thus, $N = \partial_z \partial_{z'}^2 \langle \partial_z \Phi(0, z'), N(0, 0) \rangle |_{z'=0}$ is a diagonal matrix such that $-N_{i,i} \sim 1$ for $1 \leq i \leq 2d-1$ and $N_{2d, 2d} = 0$. Here we use $\cos^2 \mathcal{S}_c(z_0, z_0') \gtrsim \epsilon_0$ and $|z_0 - z_0'| \sim \rho$. Therefore, the matrix $\partial_\zeta \partial_\zeta^3 \langle \partial_z \Phi_E(z, \zeta), N(0, 0) \rangle$ is a nonsingular diagonal matrix of which diagonal entries are all negative and their absolute values are comparable to 1. It is clear that $|\partial_z \partial_{z'}^2 N(z, z')| \leq C_{\alpha, \beta}$. By this and (3.47) we note that $\Phi(z, z')$ is a smooth function with bounded derivatives. Thus, now taking $\epsilon_0$ small enough, we see that the matrix

$$
\partial_\zeta \partial_\zeta^3 \langle \partial_z \partial_{z'}^2 \Phi(z, \zeta), N(z, z') \rangle, \quad z_{2d} \in (-\epsilon_0, \epsilon_0)
$$

has $2d - 1$ negative eigenvalues of which absolute values are comparable to 1 for all $(z, \zeta) \in B(0, \epsilon_0) \times B(0, \epsilon_0)$. This and (3.50) verify the conditions (C1) – (C3). Therefore, Theorem 2.11 gives the desired estimate (3.49). \[\square\]
4. LOWER BOUND ON THE SUMMABILITY INDEX OF $S^d_2(H)$

In this section we obtain a new lower bound on the summability index $\delta$ for uniform boundedness of $S^d_2(H)$ on $L^p$.

**Proposition 4.1.** Let $2 < p \leq \infty$. The uniform bound $\|S^d_2(H)\|_p \leq C$ holds only if $\delta > \delta(d,p)$ and

$$
\delta \geq \gamma(d,p) := -\frac{1}{3p} + \frac{d}{3} \left( \frac{1}{2} - \frac{1}{p} \right).
$$

In particular, when $d = 1$ the uniform bound holds only if $\delta \geq 0$ for $2 \leq p \leq 4$ and $\delta \geq -\frac{1}{3p} + \frac{1}{3} \left( \frac{1}{2} - \frac{1}{p} \right)$ for $p \geq 4$. This coincides with Thangavelu’s result [29, Theorem 2.1]. In higher dimensions, i.e., $d \geq 2$, it is necessary for (1.1) that $\delta > \delta(d,p)$ if $\frac{2(d+1)}{d} \leq p \leq \infty$; $\delta \geq \gamma(d,p) > \delta(d,p)$ if $\frac{2(d+1)}{d} \leq p \leq \frac{2(2d-1)}{2d-3}$; $\delta > 0$ if $2 < p \leq \frac{2(2d-1)}{2d-3}$. (See Figure 1.)

The second lower bound $\delta \geq \gamma(d,p)$ in Proposition 4.1 is a consequence of the following two lemmas.

**Lemma 4.2.** Let $1 \leq p, q \leq \infty$ and $\lambda \gg 1$. Then, for each $\lambda \gg 1$ there exists $f_\lambda \in \mathcal{S}(\mathbb{R}^d)$ such that

$$
\|\Pi_\lambda^H f_\lambda\|_q \geq C \lambda^{-\frac{d}{3} + \frac{1}{2} (1 - \frac{1}{d})} \|f_\lambda\|_p
$$

with $C$ independent of $\lambda$.

**Lemma 4.3.** Let $1 \leq p \leq \infty$, $\lambda \gg 1$, and let $E, F$ be measurable subsets of $\mathbb{R}^d$. Then, we have

$$
\|\chi_E \Pi_\lambda^H \chi_F\|_p \leq C \lambda^\delta \sup_{t > 0} \|\chi_E S^d_2(H) \chi_F\|_p.
$$

**Proof of Proposition 4.1.** Suppose that $\|S^d_2(H)\|_p \leq C$ holds for some $p \in (2, \infty]$ and $\delta$ and $E$ is a compact subset of $\mathbb{R}^d$ which contains the origin as a point of density. Since $\|\chi_E S^d_2(H) \chi_E\|_p \leq \|S^d_2(H)\|_p$, $\|\chi_E S^d_2(H) \chi_E\|_p \leq C$ with a constant $C$ independent of $\lambda$. Since the principal part of $H$ is $-\Delta$, the transplantation theorem of Kenig, Stanton, and Tomas [16, Theorem 3] gives $\|(1 - \Delta)^d\|_p \leq C$. It is well known that $\|(1 - \Delta)^d\|_p \leq C$ only if $\delta > \max\{0, -\frac{1}{2} + d(\frac{1}{2} - \frac{1}{p})\}$ when $p \neq 2$ (see for example [10, 7]).

Thus, it suffices to show that $\delta \geq \gamma(d,p)$. From Lemma 4.2 it follows that $\lambda^{\gamma(d,p)} \leq C \|\Pi_\lambda^H\|_p$. On the other hand, by Lemma 4.3 we have $\|\Pi_\lambda^H\|_p \leq C \lambda^\delta$. Thus, we obtain $\lambda^{\gamma(d,p)} \leq C \lambda^\delta$ for a constant $C$ independent of $\lambda$. Letting $\lambda$ tend to infinity, it follows that $\delta \geq \gamma(d,p)$ as desired.

Let $\Gamma$ denote the Gamma function and denote $t_+ = \max(t, 0)$ and $t_- = -\min(t, 0)$ for $t \in \mathbb{R}$. We consider the distributions

$$
\chi^\nu_{\pm} = \frac{x^\nu_{\pm}}{\Gamma(\nu + 1)}, \quad \nu \in \mathbb{C}
$$
which are given by analytic continuation of the functions \( \frac{x_+^\nu}{\Gamma(\nu + 1)} \), \( \text{Re}(\nu) > -1 \). See [12, Section 3.2] for more about the distribution. In order to prove Lemma 4.3 we use the identity

\[
F(\mathcal{H}) = \int F^{(\nu)}(t) \frac{(t - \mathcal{H})_{+}^{\nu-1}}{\Gamma(\nu)} dt = \int F^{(\nu)}(t)t^{\nu-1} \frac{S_t^{\nu-1}(\mathcal{H})}{\Gamma(\nu)} dt,
\]

which holds for any \( \nu \in \mathbb{C} \) and \( F \in C_c^\infty([0,\infty)) \). Here \( F^{(\nu)} \) denotes the Weyl fractional derivative of \( F \) of order \( \nu \) which is defined by \( F^{(\nu)} = F \ast \chi_{-\nu-1} \) for \( F \) supported in \([0,\infty)\). Note that \( t \) in the integrand can be assumed to be positive because \((t - \mathcal{H})_{+}^{\delta} = 0 \) if \( t \leq 0 \). The identity (4.2) can be shown using the convolution property of the distribution \( \chi_-^\nu \), see [5, pp. 3235-3236] for the detail.

Proof of Lemma 4.3. Now let \( \eta \) be a smooth bump function such that \( \eta(t) = 1 \) if \(|t| \leq \frac{1}{2} \) and \( \eta(t) = 0 \) if \(|t| \geq 1 \) and \( \lambda \in 2\mathbb{N}_0 + d \). Since \( \lambda \in 2\mathbb{N}_0 + d \), taking \( \nu = \delta + 1 \) and \( F = \eta(\cdot - \lambda) \) in (4.2), we get

\[
\Pi^{\mathcal{H}}_{\lambda} = \eta(H - \lambda) = \frac{1}{\Gamma(\delta + 1)} \int (\eta(\cdot - \lambda))^{(\delta+1)}(t)t^{\delta} S_t^{\delta}(\mathcal{H}) dt.
\]

A simple calculation shows \( |(\eta(\cdot - \lambda))^{(\delta+1)}(t)| \leq C(1 + |t - \lambda|)^{-(\delta+1)} \chi_{t \leq \lambda + 10} \) for a constant \( C \). Thus, combining this with the above identity, we have

\[
\|\chi_E \Pi^{\mathcal{H}}_{\lambda} \chi_F\|_p \leq C \int |(\eta(\cdot - \lambda))^{(\delta+1)}(t)t^{\delta} S_t^{\delta}(\mathcal{H})\chi_F\|_p dt
\leq \lambda^\delta \sup_{t>0} \|\chi_E S_t^{\delta}(\mathcal{H})\chi_F\|_p.
\]

We now turn to the proof of Lemma 4.3 which is similar to that of [13, Proposition 7.10]. Let \( Q \) be the cube which is given by

\[
Q = \{ x \in \mathbb{R}^d : \lambda^\frac{1}{2}/200 \leq |x_1| \leq \lambda^\frac{1}{2}/100, \ |x_i| \leq 10^2\lambda^\frac{1}{2}, \ 2 \leq i \leq d \},
\]
and set
\[ x_* = (\lambda^{\frac{2}{3}} - 10^2 \lambda^{-\frac{2}{3}})e_1. \]

We first claim that there is a point \( x_0 \in B(x_*, 10^2 \lambda^{-\frac{2}{3}}) \) such that
\[ \int_{Q} |\Pi^H(x_0, y)|^2 \, dy \gtrsim \lambda^{\frac{d-2}{3}}. \]  

Assuming this for the moment, we consider
\[ f_\lambda(x) = \chi_Q(x) \Pi^H_\lambda(x_0, x). \]

Then, we shall show the following two estimates:
\[ \|\Pi^H_\lambda f_\lambda\|_{L^2(B(x_*, c_\lambda \cdot x_*)]} \gtrsim \lambda^{\frac{d-2}{3} - \frac{4}{3}}, \]
\[ \|f_\lambda\|_p \lesssim \lambda^{-\frac{d}{3} + \frac{4}{3}} \]

with the implicit constants independent of \( \lambda \) and the constant \( c > 0 \) to be chosen sufficiently small. Combining (4.3) and (4.5) immediately yields the desired property (4.1).

Thus, it remains to show our claim (4.3) and the inequalities (4.4) and (4.5). Compared with the typical construction for the Laplacian, the proofs of them are somewhat involved.

4.1. Proof of (4.3). To show (4.3) we use the following which can be found in [18, Lemma 5.1].

Lemma 4.4. Let \( \mu = \sqrt{2k + 1} \) and define
\[ s_\mu^-(t) = \int_0^t \sqrt{\mu^2 - \mu^2} \, d\tau \quad \text{and} \quad s_\mu^+(t) = \int_0^t \sqrt{\tau^2 - \mu^2} \, d\tau. \]

Then the following hold:
\[ h_{2k}(t) = \begin{cases} a_{2k} (\mu^2 - t^2)^{-\frac{2}{3}} (\cos s_\mu^-(t) + \mathcal{E}), & |t| < \mu - \mu^{-\frac{1}{6}}, \\ O(\mu^{-\frac{1}{6}}), & \mu - \mu^{-\frac{1}{6}} < |t| < \mu + \mu^{-\frac{1}{6}}, \\ a_{2k} e^{-s_\mu^-(|t|)} (t^2 - \mu^2)^{-\frac{2}{3}} (1 + \mathcal{E}), & \mu + \mu^{-\frac{1}{6}} < |t|, \end{cases} \]
\[ h_{2k+1}(t) = \begin{cases} a_{2k+1} (\mu^2 - t^2)^{-\frac{2}{3}} (\sin s_\mu^-(t) + \mathcal{E}), & |t| < \mu - \mu^{-\frac{1}{6}}, \\ O(\mu^{-\frac{1}{6}}), & \mu - \mu^{-\frac{1}{6}} < |t| < \mu + \mu^{-\frac{1}{6}}, \\ a_{2k+1} e^{-s_\mu^-(|t|)} (t^2 - \mu^2)^{-\frac{2}{3}} (1 + \mathcal{E}), & \mu + \mu^{-\frac{1}{6}} < |t|, \end{cases} \]

where \(|a_\mu| \sim 1\) and \( \mathcal{E} = O(|t^2 - \mu^2|^{-\frac{2}{3}} |t| - \mu^{-1}) \).

We first note that
\[ \sup_{x \in B(x_*, 10^2 \lambda^{-\frac{1}{3}})} \int_{Q} \Pi_\lambda(\mathcal{H})(x, y)^2 \, dy = \|\chi_{B(x_*, 10^2 \lambda^{-\frac{1}{3}})} \Pi_\lambda(\mathcal{H}) \chi_Q\|_2^2 \to \infty. \]

Thus (4.3) follows once we find a function \( g \) such that
\[ \|\chi_{B(x_*, 10^2 \lambda^{-\frac{1}{3}})} \Pi_\lambda(\mathcal{H}) \chi_Q g\|_2 \gtrsim \lambda^{\frac{d-2}{3}} \|g\|_2. \]

Let \( k := \frac{\lambda^{\frac{2}{3}} - 10^2 \lambda^{-\frac{2}{3}}}{2} \in \mathbb{N}_0 \). Then, we define the set of indices \( J \) by
\[ J = \left\{ \alpha \in \mathbb{N}_0^d : |\alpha| = k, \lambda^{\frac{2}{3}}/d \leq \alpha_i \leq 2\lambda^{\frac{2}{3}}/d, 2 \leq i \leq d \right\}. \]
We first handle \( \sum_{i=1}^{\lambda^2 - 20\lambda^{-1/6}} \). We proceed to prove that for any \( \lambda \approx \lambda^2/d \) and \( \Phi_\alpha(x) = \prod_{i=1}^{d} h_{\alpha_i}(x_i) \), by using the estimates in the above we get

\[
\sum_{\alpha \in J} \int_{\lambda^2 - 10\lambda^{-1/6}} \int_{D} \Phi_\alpha(x_1, x')^2 dx' dx_1 \sim \lambda^{-\frac{d-2}{2}}.
\]

Thus there exists a point \( \bar{x}_* \in [\lambda^2 - 20\lambda^{-1/6}, \lambda^2/10\lambda^{-1/6}] \times D \subset B(x_*, 10^2\lambda^{-\frac{d}{2}}) \) such that \( \sum_{\alpha \in J} \Phi_\alpha^2(\bar{x}_*) \sim \lambda^{-\frac{d-2}{2}} \). Now we set

\[
g(x) = \sum_{\alpha \in J} \Phi_\alpha(\bar{x}_*) \Phi_\alpha(x).
\]

We proceed to prove that \( g \) satisfies (4.6). By orthogonality, we have \( \|g\|_2 = (\sum_{\alpha \in J} \Phi_\alpha(\bar{x}_*)^2)^{\frac{1}{2}} \leq \lambda^{-\frac{d-2}{2}} \), so (4.6) follows if we show

\[
\|\chi_{B(x_*, 10^2\lambda^{-1/6})} \Pi_\lambda(\mathcal{H}) \chi_Q g\|_\infty \sim \lambda^{-\frac{d-2}{2}}.
\]

We write

\[
\Pi_\lambda(\mathcal{H}) \chi_Q g(x) = \sum_{\alpha: |\alpha| = k} \sum_{\beta \in J} \Phi_\alpha(x) \Phi_\beta(\bar{x}_*) \int_Q \Phi_\alpha(y) \Phi_\beta(y) dy,
\]

where

\[
\Pi(x) = \sum_{\alpha \neq \beta} \Phi_\alpha(x) \Phi_\beta(\bar{x}_*) \int_Q \Phi_\alpha(y) \Phi_\beta(y) dy.
\]

We first handle \( \Pi \). By the choice of \( J \), \( \lambda - 2\lambda^2 \leq \alpha_1 \leq \lambda \) and \( \lambda^2/d \leq \alpha_i \leq 2\lambda^2/d \) for \( i = 2, \ldots, d \). Thus, using Lemma 4.3 we see that

\[
\int_{\lambda^2/200 \leq |t| \leq \lambda^2/100} h_{\alpha_1}(t)^2 dt \sim 1, \quad \int_{|t| \leq \lambda^2/10} h_{\alpha_i}(t)^2 dt \sim 1, \quad 2 \leq i \leq d
\]

for any \( \alpha \in J \). Thus, we have \( \int_Q \Phi_\alpha(x)^2 dx \geq 1 \) for \( \alpha \in J \), so \( \Pi(x) \) satisfies the estimate

\[
\Pi(x) \geq \lambda^{-\frac{d-2}{2}}
\]

because \( \sum_{\alpha \in J} \Phi_\alpha^2(x_*) \sim \lambda^{-\frac{d-2}{2}} \). For \( \Pi(x) \), we make use of the following formula

\[
\int_{-\tau}^{\tau} h_{\alpha}(t) h_\nu(t) dt = \frac{1 + (-1)^{\nu+\nu}}{\sqrt{2(u-v)}} (\sqrt{u+1} h_{\nu+1}(l) - \sqrt{v+1} h_{\nu}(l)), \quad u \neq v.
\]

See [13] pp.24–25 for its proof. By Lemma 4.3 it follows that \( |h_{\alpha_1}(10^2\lambda^2)| \lesssim e^{-c\lambda^2} \) for some \( c > 0 \) when \( v \leq 2\lambda^2 d^{-1} \). Also, if \( \alpha \neq \beta \), there exists at least one \( i \)'s such
that $2 \leq i \leq d$ and $\alpha_i \neq \beta_i$ because $|\alpha| = |\beta| = k$. Therefore, we have

$$\left| \int_Q \Phi_\alpha(y)\Phi_\beta(y)dy \right| \leq \prod_{i=2}^{d} \int_{-10^i \lambda^\frac{1}{2}}^{10^i \lambda^\frac{1}{2}} h_{\alpha_i}(t)h_{\beta_i}(t)dt \lesssim \lambda^N e^{-c \lambda^\frac{1}{2}}$$

for some constants $c > 0$ and $N$. From Lemma 4.4, we see $\Pi(x_*) = O(\lambda^N e^{-c \lambda^\frac{1}{2}})$. Combining this and the lower bound for $I(x_*)$ together with the estimate for $\|g\|_2$, we obtain (4.7) taking $\lambda$ large enough. $\square$

4.2. Proof of (4.4). To show (4.4), we make use of the following Lemma, which we prove later.

**Lemma 4.5.** Let $\lambda \in 2N_0 + d$ and $\mu \in [\lambda^{-\frac{1}{2}}, \frac{1}{4}]$. Suppose that $h \in S(\mathbb{R}^d)$ is a eigenfunction of $\mathcal{H}$ with the eigenvalue $\lambda$, i.e., $\mathcal{H}h(x) = \lambda h(x)$. If $\lambda^\frac{1}{2}(1 - 2\mu) \leq |y_0| \leq \lambda^\frac{1}{2}(1 - \mu)$, then for any $\alpha \in \mathbb{N}_0^d$ we have

$$|\partial^\alpha_y h(y_0)| \leq C(\lambda \mu)^{\frac{1}{2}} \|h\|_{L^\infty(B(y_0, 2(\lambda \mu)^{\frac{1}{2}}))}$$

with $C$ independent of $\lambda$, $\mu$ and $h$.

We start by recalling the estimate

$$\|\chi_{A_{\lambda,0}}\Pi^H_\lambda \chi_{A_{\lambda,0}}\|_{1 \to \infty} \lesssim \lambda^{\frac{d-2}{4}},$$

which can be found in ([13, p.79], [18, p.375]). Here $A_{\lambda,0} := \{x \in \mathbb{R}^d : |x| - \lambda^{\frac{1}{2}} \leq 10^3 \lambda^{-\frac{1}{2}}\}$. This is equivalent to the estimate $|\Pi^H_\lambda(x,y)| \lesssim \lambda^{\frac{d-2}{4}}$ for any $x, y \in A_{\lambda,0}$.

Now we note that

$$\Pi^H_\lambda f_\lambda(x) = \int_Q \Pi^H_\lambda(x_0,y)\Pi^H_\lambda(x,y)dy.$$ 

Using the Cauchy-Schwartz inequality and orthogonality between the Hermite functions $\Phi_\alpha$, we get

$$|\Pi^H_\lambda f_\lambda(x)| \leq \left( \int \Pi^H_\lambda(x_0,y)^2dy \int \Pi^H_\lambda(x,y)^2dy \right)^{\frac{1}{2}} \leq \left( \Pi^H_\lambda(x_0,x_0)\Pi^H_\lambda(x,x) \right)^{\frac{1}{2}}.$$ 

By (4.9), we see that $|\Pi^H_\lambda f_\lambda(x)| \lesssim \lambda^{\frac{d-2}{4}}$ for every $x \in A_{\lambda,0}$. Since $\Pi^H_\lambda f_\lambda \in S(\mathbb{R}^d)$ and trivially is an eigenfunction of $\mathcal{H}$ with the eigenvalue $\lambda$, by Lemma 4.5 we see

$$|\nabla(\Pi^H_\lambda f_\lambda)(x)| \lesssim C \lambda^{\frac{d-1}{4}}$$

if $||x| - \lambda^{\frac{1}{2}}| \leq 10^2 \lambda^{-\frac{1}{2}}$. On the other hand, from (4.3) and (4.10) we have $\Pi^H_\lambda f_\lambda(x_0) \gtrsim \lambda^{\frac{d-2}{4}}$. Thus, it follows that

$$\Pi^H_\lambda f(x) \gtrsim \lambda^{\frac{d-2}{4}}$$

if $x \in B(x_0, c\lambda^{-\frac{1}{2}})$ with a sufficiently small $c > 0$. We therefore get (4.4). $\square$

We now turn to the proof of Lemma 4.5.
Proof of Lemma 4.5. To show (4.8) we may assume \( \|h\|_{L^\infty(B(y_0,2\lambda\mu^{-\frac{1}{2}}))} = 1 \). Let \( \varphi \) be a smooth cutoff function such that \( \text{supp} \varphi \subset B(0,2) \) and \( \varphi \equiv 1 \) on \( B(0,1) \). Let us set \( \phi_{y_0}(y) = \varphi(\sqrt{\lambda \mu}(y-y_0)) \). Then by inversion we write

\[
\partial_x^\alpha h(y_0) = \partial_x^\alpha(\varphi_{y_0}h)(y_0) = (2\pi)^{-d} \int (i\xi)^\alpha \varphi_{y_0}(y)e^{i(y-y_0)\cdot\xi}h(y)dyd\xi.
\]

Setting \( \phi_K(\xi) = \varphi(\xi/K\sqrt{\lambda \mu}) \) with a large positive constant \( K \), we split the integral

\[
\partial_x^\alpha h(y_0) = I + II,
\]

where

\[
I := (2\pi)^{-d} \int (i\xi)^\alpha \phi_K(\xi)\phi_{y_0}(y)e^{i(y-y_0)\cdot\xi}h(y)dyd\xi,
\]

\[
II := (2\pi)^{-d} \int (i\xi)^\alpha (1 - \phi_K(\xi))\phi_{y_0}(y)e^{i(y-y_0)\cdot\xi}h(y)dyd\xi.
\]

For \( I \) we have

\[
|I| \lesssim \int |\xi|^{\alpha} |\varphi_K(\xi)\phi_{y_0}(y)h(y)|dyd\xi \leq C(\lambda \mu)^{\frac{\alpha}{2}}.
\]

Since \( (|\xi|^2 + \Delta_y)h(y) = |\xi|^2 + |y|^2 - \lambda \), we may write

\[
II = C \int (i\xi)^\alpha (1 - \varphi_K(\xi))\phi_{y_0}(y)e^{i(y-y_0)\cdot\xi}|\xi|^2 + \Delta_y h(y)dyd\xi.
\]

By integration by parts, this is equal to

\[
C \int \xi^\alpha (1 - \varphi_K(\xi))A_1(y,\xi)e^{i(y-y_0)\cdot\xi}h(y)dyd\xi,
\]

where

\[
A_1(y,\xi) := \Delta_y \left( \frac{\phi_{y_0}(y)}{\xi^2 + |y|^2 - \lambda} \right) - 2i\nabla_y \left( \frac{\phi_{y_0}(y)}{\xi^2 + |y|^2 - \lambda} \right) \cdot \xi.
\]

We now note that \( \xi^2 + |y|^2 - \lambda > \frac{K_0}{4}(\lambda \mu)^{\frac{1}{2}} + \frac{\xi^2}{2} \) for \( (y,\xi) \in \text{supp}(\phi_{y_0}) \times \text{supp}(\phi_K) \) with a sufficiently large \( K \) and \( |\partial_x^\alpha \phi_{y_0}(y)| \lesssim (\lambda \mu)^{\frac{\alpha}{2}} \) for any \( \alpha \in \mathbb{N}_0^d \). Thus, we have

\[
|A_1(y,\xi)| \lesssim (\lambda \mu)^{\frac{\alpha}{2}} |\xi|^{-1}. \]

Repeating this by \( N \) times, we have

\[
II = C_N \int \xi^\alpha (1 - \varphi_K(\xi))A_N(y,\xi)e^{i(y-y_0)\cdot\xi}h(y)dyd\xi,
\]

where \( A_N \) is a smooth function such that \( \text{supp} A_N \subset \text{supp}(\phi_{y_0} \otimes \varphi_K) \) and \( |A_N(y,\xi)| \lesssim |\xi|^{-N}(\lambda \mu)^{\frac{\alpha}{2}} \). Since \( \|h\|_{L^\infty(B(y_0,2\lambda\mu^{-\frac{1}{2}}))} = 1 \), we have

\[
|II| \lesssim (\lambda \mu)^{\frac{\alpha}{2}} \int_{K\sqrt{\lambda \mu} \leq |\xi|} |\xi|^{\alpha-N} \int_{|y-y_0| \leq 2/\sqrt{\lambda \mu}} dyd\xi \lesssim (\lambda \mu)^{\frac{\alpha}{2}}.
\]

Therefore, we obtain (4.8). \( \square \)
Let us set \( \tilde{x}_0 = \lambda^{-1/2}x_0 \). Changing variables, we note that the estimate (4.5) is equivalent to
\[
\| \chi_{Q} \Pi_{\lambda}^H (\lambda^{\frac{1}{2}} \tilde{x}, \lambda^{\frac{1}{2}} \cdot) \|_p \lesssim \lambda^{-\frac{1}{2} - \frac{d+1}{p}},
\]
where
\[
\tilde{Q} = \{ x \in \mathbb{R}^d : 1/200 \leq |x_1| \leq 1/100, |x_i| \leq 10^2 \lambda^{-\frac{1}{2}}, 2 \leq i \leq d \}.
\]
We recall the representation formula
\[
\Pi_{\lambda}^H = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\lambda - \lambda_{i})t} dt,
\]
which can be easily shown by utilizing the Hermite expansion and the fact that \( \lambda \in 2N_0 + d \). For example, see [13, p.11] for the detail. Using the cutoff functions \( \nu^l \) (see (2.12)) and scaling, we can write
\[
\Pi_{\lambda}^H (\lambda^{\frac{1}{2}} \tilde{x}_0, \lambda^{\frac{1}{2}} \cdot) = \sum_{l=0}^{\infty} [\nu^l]_{\lambda}^H (\tilde{x}_0, y).
\]
Thus, the desired estimate (4.11) follows if we show
\[
\| [\nu^l]_{\lambda}^H (\tilde{x}_0, \cdot) \|_{L^p(\tilde{Q})} \lesssim \lambda^{-\frac{1}{2} - \frac{d+1}{p}},
\]
\[
\| [\nu^l]_{\lambda}^H (\tilde{x}_0, \cdot) \|_{L^p(\tilde{Q})} \lesssim \lambda^{-\frac{d+1}{2} - N/2} - \frac{(d+1)}{2} - N, \quad l \geq 1.
\]
We recall (2.12). Then, using the properties (2.10) and (2.11), we see that the above estimates follow if we obtain
\[
\| [\nu^0]_{\lambda}^H (\tilde{x}_0, \cdot) \|_{L^p(\tilde{Q})} \lesssim \lambda^{-\frac{1}{2} - \frac{d+1}{p}},
\]
\[
\| [\nu^l]_{\lambda}^H (\tilde{x}_0, \cdot) \|_{L^p(\tilde{Q})} \lesssim \lambda^{-\frac{d+1}{2} - N/2} - \frac{(d+1)}{2} - N, \quad l \geq 1.
\]
Since \( \tilde{x}_0 \in \mathcal{B}((1 - 10^2 \lambda^{-2}) \epsilon, 10^2 \lambda^{-2}), \) for \( y \in \tilde{Q} \) we have
\[
|D(\tilde{x}_0, y)| = (1 - \tilde{x}_0^2)(1 - |y|^2) + |\tilde{x}_0|^2|y|^2 - (\tilde{x}_0, y)^2 \leq 10^3 \lambda^{-\frac{5}{2}}.
\]
We also have \(|(\tilde{x}_0, y)| \leq 10^{-2} \) for \( y \in \tilde{Q} \). Thus, from (2.15) we have
\[
\partial_t \mathcal{P}_H (t, \tilde{x}_0, y) = \frac{(\cos t - (\tilde{x}_0, y)^2 - D(\tilde{x}_0, y)}{2 \sin^2 t} \geq 2^l
\]
if \( t \in \text{supp} \psi(2^{l \cdot}), \) \( l \geq 1 \) since \( \lambda \) is assumed to be large. Recalling (2.3) we have
\[
[\psi(2^{l \cdot})]_{\lambda}^H (\tilde{x}_0, y) = C_2 \int \psi(2^{l t})(\sin t)^{-\frac{d}{2}} e^{i(\lambda - \lambda_{l \cdot})t} dt.
\]
Since \( |\partial^n(\psi(2^{l t})(\sin t)^{-\frac{d}{2}})| \lesssim 2^{l(n + \frac{d}{2})} \) for any \( n \geq 1 \), applying Lemma 2.5 to \( [\psi^l]_{\lambda}^H (\tilde{x}_0, y), \) we obtain
\[
[\psi^l]_{\lambda}^H (\tilde{x}_0, y) \lesssim \lambda^{-N} 2^{l\left(\frac{d+1}{2} - N\right)}, \quad N \in \mathbb{N}.
\]
Therefore, combining this with \( |\tilde{Q}| \sim \lambda^{-\frac{d+1}{2}}, \) we get (4.13).
Let \( t_* \) be a number in \([0, \pi]\) such that
\[

\cos t_* = (\tilde{x}_0, y).
\]
The point \( t_* \) can be contained in the support of \( \nu^0 \chi_{(0, \pi)} \). To show the estimate (4.12), we make additional decomposition away from \( t_* \). Let us set
\[
\psi(t) = \psi(2^l(t - t_*) + \psi(2^l(t - t_*)),
\]
and also set
\[ \psi_\lambda(t) := \sum_{2^{-1} \leq \lambda^{1/3}} \psi_l(t), \]
where \( C \) is a sufficiently large constant. We make a further decomposition on
\[ [\nu^0 \chi_{(0, \pi)}]_\lambda^H(\tilde{x}_0, y) \]
as follows. Then we may write
\[ [\nu^0 \chi_{(0, \pi)}]_\lambda^H(\tilde{x}_0, y) = [\psi_\lambda^\nu \chi_{(0, \pi)}]_\lambda^H(\tilde{x}_0, y) + \sum_{2^{-1} > \lambda^{1/3}} [\psi_l^\nu \chi_{(0, \pi)}]_\lambda^H(\tilde{x}_0, y). \]
Since \( \sin t \sim 1 \) on \( t \in \text{supp}(\nu^0 \chi_{(0, \pi)}) \subset [2^{-2}, \pi - 2^{-2}] \),
\[ ||\psi_l^\nu \chi_{(0, \pi)}|_\lambda^H(\tilde{x}_0, y)|| \lesssim \int |\psi_\lambda(t)| dt \lesssim \lambda^{-\frac{1}{2}}. \]
Now we note that \((\cos t - \cos t_\ast)^2 \sim 2^{-2l} \) on the support of \( \psi_l^\nu \chi_{(0, \pi)} \). Since
\[ \partial_t \mathcal{P}_H(t, \tilde{x}_0, y) = (\cos t - \cos t_\ast)^2 - D(\tilde{x}_0, y), \]
by (4.14) we have
\[ \partial_t \mathcal{P}_H(t, \tilde{x}_0, y) \gtrsim 2^{-2l} \]
for \( t \in \text{supp}(\psi_l^\nu \chi_{(0, \pi)}) \). So, Lemma 2.6 gives
\[ ||\psi_l^\nu \chi_{(0, \pi)}|_\lambda^H(\tilde{x}_0, y)|| \lesssim \lambda^{-N_2((-1+3)N)}, \]
for any \( N \in \mathbb{N} \). Thus,
\[ \sum_{2^{-1} > \lambda^{1/3}} ||\psi_l^\nu \chi_{(0, \pi)}|_\lambda^H(\tilde{x}_0, y)|| \lesssim \lambda^{-1/3}. \]
Combining these estimates, we now get
\[ ||\nu^0 \chi_{(0, \pi)}|_\lambda^H(\tilde{x}_0, y)|| \lesssim \lambda^{-\frac{1}{2}}. \]
Therefore, we get (4.12) because \(|\tilde{Q}| \sim \lambda^{-(d+1)/2} \). This completes the proof of (4.5).

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\section*{References}
[1] R. Askey, S. Wainger, Mean convergence of expansions in Laguerre and Hermite Series, Amer. J. Math. 87 (1965), 695–708.
[2] J. Bennett, A. Carbery, T. Tao, On the multilinear restriction and Kakeya conjectures, Acta Math. 196 (2006), no. 2, 261–302.
[3] J. Bourgain, Lp-estimates for oscillatory integrals in several variables, Geom. Funct. Anal. 1 (1991), no. 4, 321–374.
[4] J. Bourgain, Harmonic analysis in phase space, Ann. Math. Stud., Princeton Univ. Press (1989).
[5] L. Carleson, P. Sjölin, Oscillatory integrals and a multiplier problem for the disc, Studia Math. 44 (1972), 287–299.
[6] L. Carleson, P. Sjölin, On the mean inversion of Fourier and Hankel transform, Proc. Nat. Acad. Sci. U.S. A. 40 (1954), 996–999.
[7] L. Hörmander, Oscillatory integrals and multipliers on $L^p$, Ark. Mat. 11 (1973), no. 1–2, 1–11.
[8] G. B. Folland, Harmonic analysis in phase space, Ann. Math. Stud., Princeton Univ. Press (1989).
[9] L. Guth, J. Hickman, and M. Iliopoulou, Sharp estimates for oscillatory integral operators via polynomial partitioning, Acta Math 223 (2019), no. 2, 251–376.
[10] C. Herz, On the mean inversion of Fourier and Hankel transform, Proc. Nat. Acad. Sci. U.S. A. 40 (1954), 996–999.
[11] L. Hormander, Oscillatory integrals and multipliers on $L^p$, Ark. Mat. 11 (1973), no. 1–2, 1–11.
[12] The analysis of linear partial differential operators I. Distribution Theory and Fourier Analysis, Second edition, Springer-Verlag, Berlin, 1983.
[13] E. Jeong, S. Lee, J. Ryu, Estimates for the Hermite spectral projection, arXiv:2006.11762
[14] ________, Sharp $L^p$-$L^q$ estimate for the spectral projection associated with the Twisted Laplacian, arXiv:2008.09410
[15] G. B. Karadzhov, Riesz summability of multiple Hermite series in $L^p$ spaces, C. R. Acad. Bulgare Sci. 47 (1994), 5–8.
[16] C.E. Kenig, R. J. Stanton, and P. A. Tomas, Divergence of eigenfunction expansions, J. Funct. Anal. 46 (1982), 28-44.
[17] H. Koch, F. Ricci, Spectral projections for the twisted Laplacian, Studia Math. 180 (2007), no. 2, 103-110.
[18] H. Koch, D. Tataru, $L^p$ eigenfunction bounds for the Hermite operator, Duke Math. J. 128 (2005), 369–392.
[19] S. Lee, Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators, Duke Math. J. 122 (2004), no. 1, 205–232.
[20] S. Lee, Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces, J. Funct. Anal. 241 (2006), no. 1, 56–98.
[21] P. K. Ratnakumar, R. Rawat, and S. Thangavelu, A restriction theorem for the Heisenberg motion group, Studia Math. 126 (1997), 1-12.
[22] P. Sjögren, J. L. Torrea, On the boundary convergence of solutions to the Hermite-Schrödinger equation, Colloq. Math. 118 (2010), 161–174.
[23] C. D. Sogge, Fourier integrals in classical analysis, Cambridge tracts in math. 105, Cambridge Univ. Press, Cambridge, 1993.
[24] E. Stein, Oscillatory integrals in Fourier analysis, Beijing lectures in harmonic analysis (Beijing, 1984), pp. 307–355, Ann. of Math. Stud. 112, Princeton Univ. Press, Princeton, NJ, 1986.
[25] K. Stempak, J. Zienkiewicz, Twisted convolution and Riesz means, J. Anal. Math., 76, 93–107, 1998.
[26] G. Szego, Orthogonal polynomials, Amer. Math. Soc. Colloq. Pub. 23, Providence, R.I. (1967).
[27] T. Tao, A. Vargas, L. Vega, A bilinear approach to the restriction and Kakeya conjectures, J. Amer. Math. Soc. 11 (1998), 967–1000.
[28] S. Thangavelu, Summability of Hermite expansions I, Trans. Amer. Math. Soc. 314 (1989), no.1, 119–142.
[29] S. Thangavelu, Lectures on Hermite and Laguerre expansions, Math. notes 42, Princeton University Press, Princeton, N.J., 1993.
[30] ________, Hermite and special Hermite expansions revisited, Duke Math. J. 94 (1998), 257–278.
[31] ________, Harmonic analysis on the Heisenberg group, Progress in Math. Vol. 159, Birkhäuser, Boston, 1998.
[32] Shukun Wu, On the Bochner-Riesz operator in $\mathbb{R}^3$, arXiv:2008.13043.