FACTORIAL ALGEBRAIC GROUP ACTIONS
AND CATEGORICAL QUOTIENTS

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Abstract. Given an action of an affine algebraic group with only trivial characters on a factorial variety, we ask for categorical quotients. We characterize existence in the category of algebraic varieties. Moreover, allowing constructible sets as quotients, we obtain a more general existence result, which, for example, settles the case of a finitely generated algebra of invariants. As an application, we provide a combinatorial GIT-type construction of categorical quotients for actions on, e.g., complete, varieties with finitely generated Cox ring via lifting to the universal torsor.

1. Introduction

We consider the action of an affine algebraic group $G$ on a normal variety $X$, all defined over an algebraically closed field $K$, and investigate existence of a categorical quotient, i.e., a $G$-invariant morphism $\pi: X \to Y$ such that for every other $G$-invariant morphism $\varphi: X \to Z$, there exists a unique morphism $\psi: Y \to Z$ with $\varphi = \psi \circ \pi$. Similarly to the reductive case, the algebra of invariants plays a central role, compare also the work on unipotent group actions [7], [11], [12] and, recently [6]. In contrast to the reductive case, even for affine $X$, the algebra of invariants need not be finitely generated. However, considering finitely generated normal subalgebras $A \subseteq \Gamma(X, O)^G$, which are large in the sense that they have $K(X)^G$ as their field of fractions, provides at least candidates $\pi': X \to Y'$ with $Y' := \text{Spec } A$ for a quotient.

An obvious obstruction to being a categorical quotient is non-surjectivity; this even happens if $X$ is affine and the algebra of invariants is finitely generated, i.e., we may take $A = \Gamma(X, O)^G$. In general, the image $Y = \pi'(X)$ is a constructible set. This motivates an excursion to the category of constructible spaces, i.e., spaces locally isomorphic to constructible subsets of affine varieties, see Section 2 for details and [3] for a related concept. We ask whether the map $\pi: X \to Y$ sending $x \in X$ to $\pi'(x) \in Y$ is a categorical quotient in the category of constructible spaces, i.e., every $G$-invariant morphism $X \to Z$ to a constructible space $Z$ factors uniquely through $\pi: X \to Y$; note that a positive answer allows in particular to associate a unique quotient to the action. Here comes our first result.

Theorem 1.1. Suppose that any $G$-invariant closed hypersurface $D \subseteq X$ is the zero set of a $G$-invariant function on $X$. Then the following statements are equivalent.

(i) The morphism $\pi: X \to Y$ is a categorical quotient in the category of constructible spaces for the $G$-action on $X$.

(ii) The pullback $\pi^* : \Gamma(Y', O) \to \Gamma(X, O)^G$ is an isomorphism.

(iii) There is an open subset $Y'' \subseteq Y'$ with $Y \subseteq Y''$ and $Y'' \setminus Y$ is of codimension at least two in $Y''$.

If one of these statements holds, then $\pi: X \to Y$ is even a strong categorical quotient, i.e., $\pi : \pi^{-1}(V) \to V$ is a categorical quotient for every open $V \subseteq Y$.

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The assumptions of Theorem 1.1 are what we mean by a “factorial algebraic group action”. For example, they are satisfied if $G$ has trivial character group $\Xi(G)$, e.g., is semisimple or unipotent, and $X$ has finite divisor class group, e.g., is a vector space. If the algebra of invariants is finitely generated, then it obviously fulfills the second condition of Theorem 1.1 and thus we obtain the following.

**Corollary 1.2.** Let $G$ act on $X$ as in Theorem 1.1 and suppose that $\Gamma(X,\mathcal{O})^G$ is finitely generated. Then $\pi: X \to Y$, $x \mapsto \pi'(x)$, where $\pi': X \to \text{Spec } \Gamma(X,\mathcal{O})^G$ is the canonical map and $Y = \pi'(X)$, is a strong categorical quotient in the category of constructible spaces for the action of $G$ on $X$.

We come back to the problem of existence of quotients in the category of varieties. An important observation is that $\Gamma(X,\mathcal{O})^G$ admits a finitely generated separating subalgebra in the sense of Derksen and Kemper [5], i.e., a subalgebra that separates any pair of points, which can be separated by invariant functions, see Proposition 3.1. Combining this with our first result, we obtain the following characterization of existence of categorical quotients.

**Theorem 1.3.** Suppose that any $G$-invariant closed hypersurface $D \subseteq X$ is the zero set of a $G$-invariant function on $X$. Then the following statements are equivalent.

(i) There exists a categorical quotient $\pi: X \to Y$ in the category of varieties for the action of $G$ on $X$.

(ii) There is a finitely generated normal subalgebra $A \subseteq \Gamma(X,\mathcal{O})^G$ with quotient field $\mathbb{K}(X)^G$ such that the canonical map $X \to \text{Spec } A$ has an open image.

(iii) For every normal separating subalgebra $A \subseteq \Gamma(X,\mathcal{O})^G$ with quotient field $\mathbb{K}(X)^G$, the canonical map $X \to \text{Spec } A$ has an open image.

Moreover, if one of the statements holds, then the categorical quotient $\pi: X \to Y$ is even a strong categorical quotient.

In the case of a finitely generated ring of invariants, we obtain as an immediate consequence the following characterization for existence of a categorical quotient.

**Corollary 1.4.** Let $G$ act on $X$ as in Theorem 1.1 and suppose that $\Gamma(X,\mathcal{O})^G$ is finitely generated. Then the following statements are equivalent.

(i) The $G$-action on $X$ has a categorical quotient in the category of varieties.

(ii) The canonical morphism $\pi: X \to \text{Spec } \Gamma(X,\mathcal{O})^G$ has an open image.

Moreover, if one of these statements holds, then $\pi: X \to \pi(X)$ is a categorical quotient, and it is even a strong one.

The results presented so far are proven in Sections 2 and 3. In Section 4, we discuss examples. An application is given in Section 5. There, we consider the action of an affine algebraic group $G$ with trivial character group $\Xi(G)$ on an, e.g., complete, variety $X$ and assume that the Cox ring $\mathcal{R}(X)$ as well as the subring $\mathcal{R}(X)^G$ are finitely generated. Using factoriality of the Cox ring and Corollary 1.2, we obtain $G$-invariant open subsets $U \subseteq X$ with a strong categorical quotient by constructing geometric quotients for a certain torus action on the factorial affine variety $\text{Spec } \mathcal{R}(X)^G$. Among the resulting sets $U \subseteq X$, there are many sets of finitely generated semistable points as introduced by Doran and Kirwan in [6]. They fit into a combinatorial picture given by the GIT-fan of a torus action on $\text{Spec } \mathcal{R}(X)^G$.

2. Constructible quotients

In this section, we prove Theorem 1.1. We begin with presenting the basic concepts concerning constructible spaces.

By a space with functions we mean a topological space $X$ together with a sheaf $\mathcal{O}_X$ of $\mathbb{K}$-valued functions. A morphism of spaces $X$ and $Y$ with functions is a continuous map $\varphi: X \to Y$ such that for every open subset $V \subseteq Y$ and every
$g \in \mathcal{O}_Y(V)$, we have $g \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$. If $Y \subseteq X$ is a subset of a space $X$ with functions, then $Y$ is in a natural manner a subspace with functions: firstly, it inherits the subspace topology from $X$ and, secondly, it inherits the sheaf $\mathcal{O}_Y$ of functions that are locally represented as restrictions of functions of $\mathcal{O}_X$. A subset $Y \subseteq X$ is called constructible if it is a union of finitely many locally closed subsets. By a constructible subspace $Y \subseteq X$, we mean a constructible subset $Y \subseteq X$ together with the subspace structure. We are ready to introduce the category of constructible spaces.

- A quasiaffine constructible space is a space with functions isomorphic to a constructible subspace of an affine $\mathbb{K}$-variety.
- A constructible space is a space with functions admitting a finite cover by open quasiaffine constructible subspaces.
- A morphism of constructible spaces is a morphism of the underlying spaces with functions.

Note that the prevarieties form a full subcategory of the category of constructible spaces. Moreover, every constructible subset of a constructible space inherits the structure of a constructible space. We will need the following basic observation.

**Lemma 2.1.** Let $X'$ be a normal affine variety and $X \subseteq X'$ a dense constructible subspace. If every closed hypersurface $D \subseteq X'$ meets $X$, then the restriction $\Gamma(X', \mathcal{O}_{X'}) \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.

**Proof.** Locally every $f \in \Gamma(X, \mathcal{O}_X)$ extends to $X'$. Since $X \subseteq X'$ is dense, the local extensions can be glued together and thus $f$ extends to an open neighbourhood $X'' \subseteq X'$ of $X$. Normality then gives the claim.

 Similarly one obtains that, given two constructible subspaces $X \subseteq X'$ and $Y \subseteq Y'$ of varieties $X'$ and $Y'$, every morphism $X \to Y$ extends to a morphism $U' \to Y'$ with an open neighbourhood $U' \subseteq X'$ of $X$. This shows in particular that the category of dc-subsets defined by A. Białynicki-Birula [2] is a full subcategory of the category of constructible spaces.

**Proof of Theorem 1.1.** In order to obtain “(i)$\Rightarrow$(ii)”, apply the universal property of the categorical quotient to $G$-invariant functions.

We verify “(ii)$\Rightarrow$(iii)”. Consider $C := Y' \setminus Y$, let $C_1, \ldots, C_r \subseteq C$ denote the irreducible components, which are closed in $Y'$, and set $Y'' := Y' \setminus (C_1 \cup \ldots \cup C_r)$. By Lemma 2.1 we have

$$\Gamma(Y'', \mathcal{O}) = \Gamma(Y, \mathcal{O}).$$

We show that $Y'' \setminus Y$ is small. Otherwise, let $D_1, \ldots, D_s \subseteq Y''$ be the (nonempty) collection of prime divisors such that $D_i \setminus Y$ is dense in $D_i$. Choose non-zero functions $f, g \in \Gamma(Y', \mathcal{O})$ with

$$D_i \subseteq V(Y'', f), \quad D_i \not\subseteq V(Y'', g), \quad V(Y, f) \subseteq V(Y, g).$$

Then, for any $m \in \mathbb{Z}_{\geq 0}$, the function $g^m f^{-1}$ is not regular on $Y''$ and hence not on $Y$. On the other hand, for $m$ big enough, we have $m \, \text{div}(\pi^*(g)) > \text{div}(\pi^* f)$ and thus $\pi^*(g^m f^{-1})$ belongs to $\Gamma(X, \mathcal{O})^G$. This contradicts (ii).

We check “(iii)$\Rightarrow$(ii)”. Clearly, $\pi^*: \Gamma(Y, \mathcal{O}) \to \Gamma(X, \mathcal{O})^G$ is injective. To see surjectivity, let $f \in \Gamma(X, \mathcal{O})^G$ be given. Then we have $f = \pi^* g$ with a rational function $g \in \mathbb{K}(Y'')$. But condition (iii) ensures that $g$ has no poles and thus, we have $g \in \Gamma(Y, \mathcal{O})^G$.

We show that (ii) and (iii) imply (i). Let $\varphi: X \to Z$ be a $G$-invariant morphism. Cover $Z$ by open subspaces $Z_1, \ldots, Z_r$ such that we have open embeddings $Z_i \subseteq Z'_i$ with affine varieties $Z'_i$. Then $X$ is covered by the open subsets $X_i := \varphi^{-1}(Z_i)$, and we have

$$X \setminus X_i = D_i \cup B_i,$$
Assume that for every finitely generated subalgebra $B_i \subseteq X$ is of codimension at least two. Choose $G$-invariant functions $f_i \in \Gamma(X, \mathcal{O})$ having precisely $D_i$ as their set of zeroes. These $f_i$ descend to $Y$, and, by Lemma 2.1 extend to $Y''$. Set $Y_i'' := Y_i''$. Then we have

$$\Gamma(X_i, \mathcal{O})^G = \Gamma(X, \mathcal{O})_{f_i}^G = \pi^* \Gamma(Y, \mathcal{O})_{f_i} = (\pi')^* \Gamma(Y_i'', \mathcal{O}),$$

where, for the last equality, we again use Lemma 2.1. As a consequence, we obtain a commutative diagram

$$\begin{array}{ccc}
X_i & \xrightarrow{\varphi} & Z_i' \\
\downarrow{\pi_i} & & \downarrow{\psi_i'} \\
Y_i'' & \xrightarrow{\psi_i} & Y_i''
\end{array}$$

Consider $Y_i := \pi_i'(X_i) \subseteq Y_i''$. Then we have $Y_i = \pi_i(X_i)$. Moreover, because of $\psi_i'(Y_i) = \varphi(X_i) \subseteq Z_i$, we obtain morphisms $\psi_i : Y_i \rightarrow Z_i, y \mapsto \psi_i'(y)$ of constructible spaces. By construction, these morphisms glue together to the desired factorization $\psi : Y \rightarrow Z$.

In order to see that the categorical quotient $\pi : X \rightarrow Y$ is even strong, first note that for every principal open subset $Y_f$ the restriction $\pi : X_{\pi, f} \rightarrow Y_f$ is a categorical quotient, because it satisfies the second condition of the theorem. Then the desired property is obtained by gluing.

\[\Box\]

**Remark 2.2.** Let $G$ act on $X$ as in Theorem 1.1. If there is a categorical quotient $\pi : X \rightarrow Y$ with a quasiaffine constructible space $Y$, then this quotient is obtained by the procedure of Theorem 1.1. Indeed, by the universal property of a categorical quotient, the pullback $\pi^* : \Gamma(Y, \mathcal{O}) \rightarrow \Gamma(X, \mathcal{O})^G$ is an isomorphism. Now choose an embedding $Y \subseteq Y'$ into an affine variety $Y'$. Then $A := \pi^* \Gamma(Y', \mathcal{O})$ is as wanted.

Note that, given a subalgebra $A$ of the algebra of invariants as in Theorem 1.1, the equivalent conditions of 1.1 need not be fulfilled, see [12, Section 4].

3. **Quotients in the category of varieties**

Here, we prove Theorem 3.1. A first observation is existence of separating subalgebras; note that for affine $G$-varieties, an elementary proof is given in [5, Theorem 3.15].

**Proposition 3.1.** Let $G$ be any affine algebraic group and $X$ any $G$-variety. Then there exists a finitely generated separating subalgebra $A \subseteq \Gamma(X, \mathcal{O})^G$. Moreover, if $X$ is irreducible, then we may assume that $A$ has $\mathbb{K}(X)^G$ as its field of fractions, and if $X$ is normal, then we may assume in addition that $A$ is normal.

**Proof.** Assume that for every finitely generated subalgebra $B \subseteq \Gamma(X, \mathcal{O})^G$ there exist $x_1, x_2 \in X$ such that $F(x_1) = F(x_2)$ for all $F \in B$, but $f(x_1) \neq f(x_2)$ for some $f \in \Gamma(X, \mathcal{O})^G$. Then we may construct an infinite strictly increasing sequence of finitely generated subalgebras

$$B_1 \subset B_2 \subset B_2 \subset \ldots$$

in $\Gamma(X, \mathcal{O})^G$ such that for any $i \geq 1$ there exist $x_{1i}, x_{2i} \in X$ with $F(x_{1i}) = F(x_{2i})$ for all $F \in B_i$, but $f(x_{1i}) \neq f(x_{2i})$ for some $f \in B_{i+1}$. This sequence of subalgebras gives us the affine varieties $Y_i := \text{Spec} B_i$ and the morphisms $\psi_i : X \rightarrow Y_i$ and $\varphi_i : Y_{i+1} \rightarrow Y_i$ defined by the inclusions $B_i \subseteq \Gamma(X, \mathcal{O})^G$ and $B_i \subseteq B_{i+1}$.

The images $V_i := \psi_i(X) \subseteq Y_i$ and the maps $\varphi_i : V_{i+1} \rightarrow V_i$ form a dominated inverse system of dc-subsets, see [4, Section 3]. By [4, Theorem 0.1], there exists $m \geq 1$ such that the maps $\varphi_i : V_{i+1} \rightarrow V_i$ are bijective for any $i \geq m$. This implies that the fibers of the morphisms $\psi_i$ and $\psi_{i+1}$ coincide for any $i \geq m$, a contradiction. \[\Box\]
The basic property of a separating subalgebra $A \subseteq \Gamma(X, O)^G$ we will use is that it realizes the categorical closure of the equivalence relation given by the $G$-action on $X$ in the following sense.

**Proposition 3.2.** Assumptions as in Theorem 1.3. If $A \subseteq \Gamma(X, O)^G$ is a finitely generated separating subalgebra and $U \subseteq X$ a $G$-invariant open subset, then every $G$-invariant morphism $\varphi: U \to Z$ to a prevariety $Z$ is constant along the fibers of the map $\pi': X \to \text{Spec } A$.

*Proof.* Consider $x_1, x_2 \in U$ with $\varphi(x_1) \neq \varphi(x_2)$. Let $Z_1 \subseteq Z$ be an open affine neighbourhood of $\varphi(x_1)$. Set $U_1 := \varphi^{-1}(Z_1)$, and write $U \setminus U_1 = D_1 \cup B_1$, where $D_1 \subseteq U$ is of pure codimension one, and $B_1 \subseteq U$ is of codimension at least two. Then there is a function $f \in \Gamma(U_1, O)^G$ with $f(x_1) \neq f(x_2)$. If $x_2 \in U_1$ holds, then choose a function $f_2 \in \Gamma(U_1, O)^G$ having inside $U$ precisely $D_1$ as its set of zeroes. If $x_2 \not\in U_1$ holds, then there is a function $f \in \Gamma(U_1, O)^G$ with $f(x_1) \neq f(x_2)$. Since $\Gamma(U_1, O)^G = \Gamma(U, O)^G_{f_1}$, we find a function $f' \in \Gamma(U, O)^G$ with $f'(x_1) \neq f'(x_2)$.

*Proof of Theorem 1.3.* The implication “(iii)$\Rightarrow$(ii)” is obvious. Moreover, “(ii)$\Rightarrow$(i)” and the supplement are clear by Theorem 1.1. To verify “(i)$\Rightarrow$(iii)”, let $\pi: X \to Y$ be a categorical quotient. Given any normal separating subalgebra $A \subseteq \Gamma(X, O)^G$, the universal property yields a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi'} & \text{Spec } A \\
\downarrow{\pi} & & \\
Y & \xrightarrow{\psi} & 
\end{array}
\]

By assumption, the morphism $\psi: Y \to \text{Spec } A$ is birational. Moreover, using surjectivity of the categorical quotient $\pi: X \to Y$ and Proposition 3.2, we see that it is injective. Consequently, since Spec $A$ is normal, Zariski’s Main Theorem yields that $\psi: Y \to \text{Spec } A$ is an open embedding. Using once more surjectivity $\pi: X \to Y$, we conclude that $\pi'(X) = \psi(Y)$ is open in Spec $A$.

Every constructible subspace $X \subseteq X'$ of a quasiaffine variety has an open kernel, i.e., a unique maximal subset, which is open in the closure of $X$ in $X'$. This kernel does not depend on the embedding $X \subseteq X'$. Thus, given an arbitrary constructible space $X$, we can define the *set of varietic points* $X^{\text{var}} \subseteq X$ as the union of the open kernels of its quasiaffine open subspaces. Note that $X^{\text{var}} \subseteq X$ is the unique maximal open subspace of $X$, which is a prevariety.

**Corollary 3.3.** Assumptions as in Theorem 1.3. Then there is a unique maximal invariant open subset $U \subseteq X$ that admits a categorical quotient $\pi: U \to V$ in the category of varieties.

*Proof.* Let $A \subseteq \Gamma(X, O)$ be a finitely generated normal separating subalgebra. Then, by Proposition 3.2, this is a separating subalgebra for any invariant open subset of $X$. Now, set $Y' := \text{Spec } A$, let $\pi': X \to Y'$ be the canonical morphism and set $Y := \pi'(X)$. Then Theorem 1.3 tells us that $U := \pi'^{-1}(V)$ for $V := Y^{\text{var}}$ has a categorical quotient in the category of varieties. If another $G$-invariant open set $W \subseteq X$ admits a categorical quotient in the category of varieties, then Theorem 1.3 yields that $\pi'(W)$ is open in $Y'$ and hence $\pi'(W) \subseteq V$ holds. This implies $W \subseteq U$. \qed
4. Examples

Our first example is an action of the additive group $\mathbb{K}$ on a four-dimensional vector space having a finitely generated algebra of invariants but no categorical quotient in the category of varieties.

**Example 4.1.** See [9, Section 4.3] and [8, Example 6.4.10]. We regard $X := \mathbb{K}^4$ as the space of $(2 \times 2)$-matrices and consider the action of the additive group $G = \mathbb{K}$ given by

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + \lambda c & b + \lambda d \\ c & d \end{pmatrix}.$$ 

This action fulfills the assumptions of Theorem 1.1. The algebra of invariants is generated by $c, d$ and $ad - bc$. The corresponding morphism $\pi' : \mathbb{K}^4 \to \mathbb{K}^3$ has the non-open image

$$Y = V \cup \{(0, 0, 0)\}, \quad V := \mathbb{K}^* \times \mathbb{K} \times \mathbb{K} \cup \mathbb{K} \times \mathbb{K}^* \times \mathbb{K}.$$ 

According to Corollary 1.2, there is no categorical quotient in the category of constructible spaces. Moreover the set $U \subseteq X$ of Corollary 1.3 is $\pi^{-1}(V)$.

By a result of Sumihiro, every free torus action on a variety admits a geometric quotient with a possibly non-separated orbit space. The following example shows that this is not true for actions of the additive group $\mathbb{K}$, even if they admit a categorical quotient in the category of constructible spaces.

**Example 4.2.** See [11, Section 5]. Consider the (non-linear) action of the additive group $G = \mathbb{C}$ on $X = \mathbb{C}^4$ defined by

$$\lambda \cdot (x_1, x_2, x_3, x_4) := (x_1, x_2 + \lambda x_1, x_3 + \lambda x_2 + \frac{1}{2} \lambda^2 x_1, x_4 + \lambda(x_2^2 - 2x_1x_3 - 1)).$$

Then this action is free, and, according to [11, Lemma 10], the algebra of invariants is generated by

$$f_1 := x_1, \quad f_2 := x_1x_4 - x_2(x_2^2 - 2x_1x_3 - 1),$$

$$f_3 := x_2 - 2x_1x_3, \quad f_4 := \frac{1}{f_1}(f_3^2 - f_2(1 - f_2^2)).$$

The variety $Y' = \text{Spec}(\mathbb{C}, \mathcal{O})^G = V(\mathbb{C}^4; f_1, f_1 - f_3^2 + f_2(1 - f_2^2))$ is smooth, and the image of the canonical morphism $\pi' : \mathbb{C}^4 \to Y'$ is

$$Y = Y' \setminus \{f_1 = 0, f_2 = 1, f_3 = 0, f_4 \neq 0\}.$$ 

Thus, Theorem 1.1 says that $\pi : X \to Y, x \mapsto \pi'(x)$ is a categorical quotient in the category of constructible spaces. Since $\pi$ does not separate the orbits of the points $(0, 1, 0, 0)$ and $(0, -1, 0, 0)$, a geometric quotient cannot exist, even if we allow a non-separated orbit space.

So far, we saw examples of unipotent group actions having no categorical quotients in the category of varieties. Here comes a semisimple group action on a smooth quasi-affine variety.

**Example 4.3.** Let $V$ be the space of $(2 \times 3)$-matrices with the $\text{SL}(2)$-action by left multiplication. The algebra of invariants is generated by $(2 \times 2)$-minors $\Delta_{12}, \Delta_{23}, \Delta_{13}$, and the canonical morphism

$$\pi' : V \to \mathbb{K}^3, \quad M \mapsto (\Delta_{12}(M), \Delta_{23}(M), \Delta_{13}(M))$$

is surjective. Consider the open invariant subset $X \subset V$ consisting of matrices with non-zero first column. It has the same algebra of invariants as $V$. However, by
Corollary 1.4, it has no categorical quotient, because the image $Y = \pi'(X) \subset \mathbb{K}^3$ is not open: it is given by

$$(\mathbb{K}^3 \setminus V(\mathbb{K}^3; \Delta_{12}, \Delta_{13})) \cup \{(0,0,0)\}.$$ 

We now provide a class of examples, showing that the conditions of Theorem 1.3 may be fulfilled even without finite generation of the ring of invariants.

**Example 4.4.** Let $F$ be a connected simply connected semisimple algebraic group and $G \subseteq F$ a closed subgroup with $X(G) = 0$, and let $G$ act on $F$ by multiplication from the right. Then, in general, $\Gamma(F, \mathcal{O})^G$ is not finitely generated. Choose any finitely generated normal subalgebra $A \subseteq \Gamma(F, \mathcal{O})^G$ having $\mathbb{K}(F)^G$ as its field of fractions and being invariant with respect to the $F$-action by multiplication from the left. Then the morphism $\pi': F \to Y' := \text{Spec} A$ is $F$-equivariant and its image coincides with an open $F$-orbit.

The next example shows that without the assumption of a “factorial action”, even a surjective morphism $\pi': X \to \text{Spec} \Gamma(X, \mathcal{O})^G$ need not be a categorical quotient.

**Example 4.5.** Consider the action of the additive group $G = \mathbb{K}$ on the smooth quasifine variety

$$X = V(\mathbb{K}^4; x_1x_4 - x_2x_3) \setminus \{(0,0,0,0)\}$$

given by

$$\lambda \cdot (x_1, x_2, x_3, x_4) := (x_1, x_2, x_3 + \lambda x_1, x_4 + \lambda x_2).$$

The algebra of invariants is generated by $x_1$ and $x_2$, and the canonical morphism $\pi': X \to \text{Spec} \Gamma(X, \mathcal{O})^G$ is surjective. However, the following $G$-invariant morphism does not factor through $\pi'$:

$$X \to \mathbb{P}_1, \quad x \mapsto [x_1, x_2] = [x_3, x_4].$$

Finally, we give an example without quotient in the category of varieties, where we don’t know, if it has a quotient in the category of constructible spaces:

**Example 4.6.** Fix a number $m \in \mathbb{Z}_{\geq 2}$ and consider the action of the additive group $G = \mathbb{C}$ on $X = \mathbb{C}^7$ given by

$$\lambda \cdot (x, y, z, s, t, u, v) := (x, y, z, s + \lambda x^{m+1}, t + \lambda y^{m+1}, u + \lambda z^{m+1}, v + \lambda x^my^mz^m).$$

As observed in [1], the algebra of invariants is Roberts’ algebra [10]; in particular, it is not finitely generated. By [10 Lemma 2], any non-constant term of a $G$-invariant polynomial contains at least one of the variables $x$, $y$ and $z$. Let

$$f_1 = x, \ f_2 = y, \ f_3 = z, \ f_4, \ldots, \ f_n \in \Gamma(X, \mathcal{O})^G$$

generate a normal separating subalgebra and suppose that none of the $f_i$ has a constant term. Consider the morphism $\pi': \mathbb{C}^7 \to \mathbb{C}^n$ given by

$$(x, y, z, s, t, u, v) \mapsto (x, y, z, f_4(x, y, z, s, t, u, v), \ldots, f_n(x, y, z, s, t, u, v)).$$

We claim that the image $Y = \pi'((\mathbb{C}^7))$ is not open in its closure. Otherwise, it were a 6-dimensional variety. But if we restrict the projection

$$r: \mathbb{C}^n \to \mathbb{C}^3, \quad (x, y, z, \ldots) \mapsto (x, y, z)$$

to $Y$, then the preimage $r^{-1}(0,0,0)$ intersected with $Y$ is just one point; a contradiction to semicontinuity of the fiber dimension. Thus, by Theorem 1.3 there is no categorical quotient in the category of varieties.
5. A Combinatorial GIT-type construction

Let $G$ be an affine algebraic group with trivial character group $\chi(G)$. We consider an action of $G$ on a $\mathbb{Q}$-factorial variety $X$ with $\Gamma(X,\mathcal{O}_X^*) = K^*$ and free finitely generated divisor class group $\mathrm{Cl}(X)$. Our aim is to present a construction of open $G$-invariant subsets $U \subseteq X$ that admit a strong categorical quotient $U \to Y$. Passing, if necessary, to the action of the simply connected covering group, we may assume that $G$ itself is simply connected.

The idea is to lift the $G$-action to a universal torsor over $X$ and then reduce the problem to the case of a torus action on an affine variety by means of the results obtained so far. More precisely, the procedure is the following. Choose any subgroup $K \subseteq \mathrm{WDiv}(X)$ of the group of Weil divisors projecting isomorphically onto the divisor class group $\mathrm{Cl}(X)$ and define a sheaf of $K$-graded $\mathcal{O}_X$-algebras by

$$\mathcal{R} := \bigoplus_{D \in K} \mathcal{O}_X(D).$$

Then the $K$-grading of $\mathcal{R}$ defines an action of the torus $H := \text{Spec} \mathbb{K}[K]$ on the relative spectrum $\hat{X} := \text{Spec}_X \mathcal{R}$ and the canonical morphism $p: \hat{X} \to X$ is a geometric quotient for this action; for smooth $X$, this construction is classically called a universal torsor. Using $G$-linearization of the homogeneous components of $\mathcal{R}$, we may lift the $G$-action to $\hat{X}$ such that it commutes with the $H$-action and $p: \hat{X} \to X$ becomes $G$-equivariant, see [2, Section 4].

The variety $\hat{X}$ is quasiaffine and the Cox ring $\mathcal{R}(X) = \Gamma(\hat{X},\mathcal{O})$ is factorial, see [3]. In particular, the $G$-action on $\hat{X}$ satisfies the assumptions of Theorems 1.1 and 1.3. Suppose that the Cox ring $\mathcal{R}(X)$ and the algebra of invariants $\mathcal{R}(X)^G$ are finitely generated. This gives us factorial affine varieties

$$\overline{X} := \text{Spec} \mathcal{R}(X), \quad \overline{Y} := \text{Spec} \mathcal{R}(X)^G,$$

see [9, Theorem 3.17]. The variety $\hat{X}$ is an $(G \times H)$-invariant open subset of $\overline{X}$ and, by Corollary 1.2, there is a strong categorical quotient $\kappa: \hat{X} \to \overline{Y}$ with a constructible subset $\overline{Y} \subseteq \overline{V}$ such that $\overline{Y} \setminus \overline{V}$ is of codimension at least two. Moreover, since $\mathcal{R}(X)^G$ is $K$-graded, the $H$-action on $\overline{X}$ descends to an $H$-action on $\overline{Y}$ leaving $\overline{Y}$ invariant.

**Construction 5.1.** Let $\hat{V}' \subseteq \overline{Y}$ be an $H$-invariant open subset with $\kappa^{-1}(\hat{V}') \subseteq \hat{X}$ admitting a good quotient $q': \hat{V}' \to V'$ for the action of $H$. Set $\hat{V} := \overline{Y} \cap \hat{V}'$ and suppose we have ($*$): for each $v \in V := q(\hat{V})$, the closed $H$-orbit of $q^{-1}(v)$ lies in $\hat{V}$. Then $U := p(\hat{U})$, where $\hat{U} := \kappa^{-1}(\hat{V})$, is open in $X$, admits a strong categorical quotient $r: U \to V$ for the action of $G$ in the category of constructible spaces and $U$ is covered by $r$-saturated affine open subsets. For convenience, we summarize the data in a commutative diagram:

![Diagram](https://via.placeholder.com/150)
Lemma 5.2. Let a reductive group $H$ act on a normal variety $\hat{V}'$ with good quotient $q': \hat{V}' \to V'$ and let $\hat{V} \subseteq \hat{V}'$ be an $H$-invariant constructible subset. If $\hat{V}' \setminus \hat{V}$ is of codimension at least two in $\hat{V}'$ and for every $v \in V := q'(\hat{V})$ the closed $H$-orbit of $q'^{-1}(v)$ lies in $\hat{V}$, then $q: \hat{V} \to V$, $x \mapsto q'(x)$ is a strong categorical quotient for the action of $H$ on $\hat{V}$ in the category of constructible spaces.

Proof. Let $\varphi: \hat{V} \to Z$ be any $H$-invariant morphism to a constructible space. By assumption, we have $\varphi = \psi \circ q$ with a set-theoretical map $\psi: V \to Z$. In order to see that this map is a morphism note that firstly $V$ carries the quotient topology with respect to $q: \hat{V} \to V$, because $V'$ carries the quotient topology with respect to $q': \hat{V}' \to V'$, and secondly, that due to the fact that $\hat{V}' \setminus \hat{V}$ is of codimension at least two, the canonical morphism $O_V \to q_*\mathcal{O}_V^H$ is an isomorphism. Clearly, the arguments work as well locally with respect to $V$, and thus we have even a strong categorical quotient. □

Proof of Construction 5.1. Since $q: \hat{V}' \to V'$ is a good quotient, the set $\hat{V}'$ is covered by $q$-saturated affine open subsets, and these are of the form $\prod_{g_i}$. Thus, $\hat{U} := \kappa^{-1}(\hat{V}')$ is covered by the $q \circ \kappa$-saturated open subsets $\overline{X}_{f_i}$, where $f_i := \kappa^*(g_i)$. Since $\hat{U}$ is $(G \times H)$-invariant, its image $U = p(\hat{U})$ is open and $G$-invariant. Moreover, $U$ is covered by the $G$-invariant affine open subsets $U_i := p(\overline{X}_{f_i})$. Since $p: \hat{U} \to U$ is a categorical quotient, we have an induced morphism $r: U \to V$. By Lemma 5.2 and Theorem 1.1 this is a strong categorical quotient. Moreover, by construction, the sets $U_i$ give the desired $r$-saturated affine covering. □

Now, in addition to the assumptions made so far, let $X$ be projective. For every Weil divisor $D \in K$, we may define the associated set of semistable points $X^{ss}(D)$ as the union of all the affine sets $X_f$, where $n > 0$ and $f \in \mathcal{R}(X)^H_{n,D}$. Then, for any ample divisor $D \in K$, we have

$$X^{ss}(D) = p(\kappa^{-1}(\overline{Y}^{ss}(D))), \quad \overline{Y}^{ss}(D) := \bigcup_{f \in \mathcal{R}(X)^H_{n,D}} \overline{Y}_f.$$ 

Note that due to our finiteness assumptions on the Cox ring $\mathcal{R}(X)$ and the ring $\mathcal{R}(X)^H_{n,D}$ of invariants, the set $X^{ss}(D)$ coincides with the set of finitely generated semistable points introduced in [6, Definition 4.2.6]. Applying Construction 5.1 shows existence of a categorical quotient.

Corollary 5.3. Let $D \in K$ and suppose that $\hat{V}' = \overline{Y}^{ss}(D)$ satisfies Condition (a) of [5, 7] e.g., all points of $\hat{V}'$ are stable. Then there is a strong categorical quotient $X^{ss}(D) \to V$ for the $G$-action, where $V = q(\overline{Y} \cap \hat{V}')$ and $q: \hat{V}' \to \hat{V}'//H$ is the good quotient.

Now one may apply the combinatorial description of GIT-equivalence for torus actions on factorial affine varieties, see [2 Section 3], to the action of $H$ on $\overline{Y}$, and thus compute the variation of the Doran-Kirwan GIT-quotients. We demonstrate this by means of the following example.

Example 5.4. Compare also [6, Example 4.1.10]. Consider the action of the additive group $G = \mathbb{K}$ on $X = \mathbb{P}_1 \times \mathbb{P}_1$ given by

$$\lambda \cdot ([a, c], [c, d]) := ([a + \lambda c, c], [b + \lambda d, d]).$$

We have an obvious lifting of the action to the torsor $\hat{X}$: The extension to $X = \mathbb{K}^2 \times \mathbb{K}^2$ was discussed in Example 4.1. We have $\overline{Y} = \mathbb{K}^3$ and the quotient map is

$$\pi': \overline{X} \to \overline{Y}, \quad ((a, e), (b, d)) \mapsto (c, d, ad - bc).$$
The image is $\overline{Y} = K^* \times K \times K \cup K \times K \times K^* \cup \{(0,0,0)\}$. Now, the torus $H$ is $K^* \times K^*$ and it acts on $X$ via
\[(t_1, t_2) \cdot ((a, c), (b, d)) = ((t_1 a, t_1 c), (t_2 b, t_2 d)).\]
The induced $H$-action on $\overline{Y}$ is given by $t \cdot (u, v, w) = (t_1 u, t_2 v, t_1 t_2 w)$. Its GIT-fan in $X(H) = \mathbb{Z}^2$ looks like

The two full-dimensional chambers correspond via Construction 5.1 to the two sets $U_1 := \mathbb{P}^1 \times K$ and $U_2 := K \times \mathbb{P}^1$ of semistable points. Both of them have a strong categorical quotient $U_i \to \mathbb{P}^1$ in the category of varieties.

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