The measurement of coherence of fields is an important concept with applications in many disciplines of physics. Radio astronomers measure coherence by correlating radio signals received by an interferometer array and characterize the morphology of radio emission received from the sky [1]. The signals received at each telescope are usually corrupted due to propagation effects and local imperfections in the array receiver elements. Finding interferometric invariants, quantities that are immune to this corruption, and so accurately reflect the true structure of the sky emission, is of significant value. This is the principal focus of this paper.

The subject of closure phases and closure amplitudes, two popular interferometric invariants in astronomy, has a long history [2, 3]. Concepts analogous to closure invariants (interferometric invariants defined on closed loops) occur in other areas like speckle interferometry [4], crystallography [5] and quantum mechanics [6, 7]. Our approach, as in the recent work [8], is inspired by the geometric phase [7, 9] of quantum physics and classical optics [10, 11]. A common thread that ties all these diverse problems together is gauge theory. We adapt the general framework to the problem of closure invariants in astronomy.

Closure invariants in co-polar correlations, in which signals of the same polarization are correlated between all pairs of telescopes, are well studied in both theory [1, 12] and practice [13, 14]. In a recent significant extension to polarimetric measurements, “closure traces” were introduced and explored in detail for a system of four array elements [15]. These closure traces were used in studying the magnetic properties near the event horizon of the supermassive black hole at the center of the galaxy M87 [16]. Generalizing and extending this work [15] on polarimetric interferometry and closure invariants, our paper applies the gauge theory framework to the group of nonsingular 2 × 2 matrices (known to mathematicians as GL(2, C)) and Lorentz groups. We go beyond earlier work by transparently determining the complete and independent set of invariants from an N-element interferometer array. We show that auto-correlation measurements, which are highly susceptible to systematic errors [4], are not necessary. If available, they are easily included in the scheme. This should be useful in future mapping of polarized sources, especially using long baseline interferometry, where calibration is challenging.

Notation.—We use indices $a, b = 0, \ldots, N - 1$ to label the array elements, and indices $p, q = 1, 2$ to label the

1 In radio astronomy, auto-correlations are problematic because of their susceptibility to systematic effects caused by emission unrelated to the object being imaged (e.g., atmospheric emission and radio frequency interference) and the gain errors affecting these strong unwanted signals.
two polarizations. $2 \times 2$ and $4 \times 4$ matrices are shown in uppercase boldface. 2-element vectors and 4-vectors are written using lowercase boldface in regular and italicized fonts, respectively. For 4-vectors and $4 \times 4$ matrices written in component form, we use relativity conventions, like the Einstein summation convention for repeated Greek indices, which range over 0, 1, 2, 3.

Polarimetric Interferometry.—Consider an interferometer array with $N$ elements. The $p$-th component of polarization (in some orthonormal basis) of the electric field incident on element, $a$, is denoted by $e^p_a$, represented in amplitude and phase by a complex 2-element column vector, $e_a$. The true correlation matrix between the array elements is obtained via pairwise cross-multiplication and averaging, $S_{ab} := \langle e_a e_b^\dagger \rangle$, where $\dagger$ denotes a conjugate-transpose operation, and $\langle \rangle$ denotes the average.

Due to the propagation medium and the non-ideal measurement process, the incoming amplitudes are corrupted by an element-based linear transformation – a general $2 \times 2$ complex matrix, $G_a$. The off-diagonal entries of $G_a$ represent leakage between the two polarized receiver channels at array element $a$. The measured amplitudes, $v_a$, are related to the true amplitudes, $e_a$, by $v_a = G_a e_a$.

The corrupted correlation matrix for a pair of elements, $(a, b)$, is constructed from the measurements as

$$C_{ab} = \langle v_a v_b^\dagger \rangle = G_a \langle e_a e_b^\dagger \rangle G_b^\dagger = G_a S_{ab} G_b^\dagger. \quad (1)$$

$G_a$, $C_{ab}$, and $S_{ab}$ are $2 \times 2$ matrices. Generically, their eigenvalues are non-zero and we will assume this to be true. These matrices are then invertible and so belong to the general linear group, GL$(2, \mathbb{C})$. Evidently, $C_{ab}^\dagger = C_{ba}$. The auto-correlations, $A_{aa} := C_{aa}$, are Hermitian and positive definite (strictly positive eigenvalues). We use a special symbol to distinguish them from general cross-correlations, $C_{ab}$.

Our objective is to construct quantities which are immune to the corruptions and actually reflect the true coherence properties of the source of radio emission, rather than local conditions (represented by $G_a$) at the interferometer elements. This problem is mathematically related to the “gauge” theories of fundamental interactions, such as electromagnetism (EM) or the nuclear force. We regard multiplication of the true signal by the local gains, $G_a$, as gauge transformations. We wish to eliminate these spurious effects introduced by $G_a$ and identify a maximal, independent set of gauge-invariant quantities (independent of $G_a$), which we will call “closure invariants”.

The gains are elements of GL$(2, \mathbb{C})$, which is not unitary, resulting in features not seen in the usual gauge theories of particle physics. The case of co-polar observations is described in detail in [1] [2], where the gains are just nonzero complex numbers, and the gauge group is GL$(1, \mathbb{C})$, which is Abelian — a commutative group. In a companion paper [Paper I. [17]], we have introduced these ideas in the simpler context of GL$(1, \mathbb{C})$, which serves as a stepping stone to the polarized theory. The current paper deals with full polarimetric measurements governed by the gauge group GL$(2, \mathbb{C})$. An important difference between the two cases is that matrices GL$(2, \mathbb{C})$ group do not, in general, commute. This introduces a level of complexity not seen in the co-polar (Abelian) case [17].

Counting Invariants.—How many independent closure invariants can we expect to find? The number of true cross-correlations, $S_{ab}$, in the measured cross-correlations, $C_{ab}$, is equal to the number of array element pairs, $N(N - 1)/2$, which are assumed to be non-redundant in this paper. Each $C_{ab}$ gives us one complex $2 \times 2$ matrix, with 8 real parameters. If auto-correlations are also measured, we would have to add $4n_A$ to this count, since $A_{aa}$ is Hermitian and has 4 real parameters, and $n_A$ is the number of true auto-correlations in the measurements. When the objects occupies a small fraction of the field of view, all elements would measure essentially the same auto-correlation (with an inherent assumption of spatial stationarity). Therefore, $S_{aa} = S_{ab}$ for all $a, b$, and we can set $n_A$ to zero or one without loss of generality. Thus, if $M$ is the space of all measured correlations, its dimension is the total number of real parameters in the measured correlations, $\dim(M) = 8N(N - 1)/2 + 4n_A$.

Each of the $N$ gain matrices, $G_a$, is described by $4$ complex numbers (or, 8 real numbers), resulting in $8N$ real parameters in total. The gains $G_a$ from the $N$ array elements corrupt the signals according to Eq. (1) and the $C_{ab}$ could lie anywhere on a surface of points containing $S_{ab}$ (see Fig. 1). Any function of the measurements which is constant on each surface is immune to corruption by the gains. These are the interferometric invariants we seek. The number of independent interferometric invariants is the dimension of $M$ minus the dimension of the surface.

The dimension of the surface can at most be the dimension of the gains $G_a$, which is $8N$. It could be smaller if there are gain variations which do not corrupt the measured correlations. We refer to these as Non-Corrupting Gains, NCGs for short. As a result, we would expect

$$N_T = 4N^2 - 12N + 4n_A + s \quad (2)$$

independent invariants, where $s$ is the number of parameters in NCGs (dimension of the stabilizer subgroup). One well-known example of an NCG is a constant phase shift applied to all the signals on all the interferometer elements, which would not affect $C_{ab}$. Such a uniform phase shift applied to all elements belongs to the U$(1)$ subgroup, which forms the entire stabilizer group. This is the only NCG ($s = 1$) in all cases except $N = 3, n_A = 0$ (see below). For $N = 4$ this yields 17 and 21 invariants,

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1 In gauge theory, these correspond to a subgroup (called the stabilizer), which acts trivially (i.e., does not affect the measured correlations).
The measured correlation, \( C_{ab} \), could lie anywhere on the gray two-dimensional surface because of gain distortions. For \( N > 3 \), \( n_A = 0 \), the correlation space and the surface have real dimensions \( 4N(N - 1) \) and \( 8N - 1 \), respectively. The figure shows these spaces as three- and two-dimensional, respectively, for illustrative purposes.

For \( n_A = 0 \) and \( n_A = 1 \), respectively. We note that our count differs from that of [15], where NCGs were not considered and the dimension of the surface was assumed to be \( 8N \) (from \( 4N \) complex gain parameters).

Gauge Theory on a Graph.—Given the measured correlations, we wish to find a complete and independent set of invariants. In a standard gauge theory, this is done by fixing a base point and considering parallel transport (holonomy) along closed loops which start and end at the base point. Such parallel transport measures curvature and is called a Wilson loop. Wilson loops generate all the gauge invariant quantities in a gauge theory and are the key to constructing closure invariants. We fix a base vertex at one of the array elements labeled 0. We regard each array element, \( a \), as a vertex in a graph with \( N \) vertices. Each element pair, \((a, b)\), is a directed edge or a “link” in the graph, carrying the variable, \( C_{ab} \), a \( 2 \times 2 \) complex matrix from the \( GL(2, \mathbb{C}) \) group. This link variable is called a “connection” and defines parallel transport from \( a \) to \( b \). The gains, \( G_a \), are local (i.e., element-based) gauge transformations by the \( GL(2, \mathbb{C}) \) group.

In standard gauge theories, such as \( SU(2) \) for example, the gauge group describing \( G_a \) is unitary and Hermitean conjugation in Eq. (1) is the same as matrix inversion. Since our gauge group is not unitary, we have to proceed differently. So, we define a hat operator on invertible matrices, \( \hat{P} = (P^\dagger)^{-1} \) (its relevant properties are listed in appendix I). This convenient notation lets us adapt methods employed by [15]. We first define covariants, \( C_{\Gamma_0} \), as products of an even number of \( C_{ab} \) matrices around a closed loop, \( \Gamma_0 \), pinned at vertex 0, with the even terms ‘hatted’. For example, for \( \Gamma_0 = 0abcd0 \), \( C_{\Gamma_0} = C_{0a} \hat{C}_{ab} C_{bc} \hat{C}_{cd} C_{da} \). Under gauge transformations, covariants transform as

\[
C_{\Gamma} \rightarrow G_0 CG_0^{-1}.
\]

Note that by defining covariants pinned at vertex 0, we have considerably reduced the number of gains contributing to corruptions. All that remains is a single gain \( G_0 \) at the base vertex 0. Closure invariants can be found by taking traces, \( Z = \text{tr}(C) \), of covariants as in [15] or indeed in gauge theories. These invariants will be unchanged under any gauge transformation with \( G_a \). However, the need for an even number of links to construct covariants would seem to force us towards rectangles. The problem now is that it is far from clear which set will give us a complete and independent set of invariants.

Our solution to this problem is to define and use quantities called ‘advariants’, which we introduced in [17]. They are constructed by multiplying an odd number of correlations, \( C_{ab} \), around a closed loop, \( \Gamma_0 \), pinned at vertex 0 with even terms hatted. For example, for \( \Gamma_0 = 0abcd0 \), \( A_{\Gamma_0} = C_{0a} \hat{C}_{ab} C_{bc} \hat{C}_{cd} C_{da} \). Under gauge transformations, advariants transform as

\[
A_{\Gamma} \rightarrow G_0 ACG_0^{-1},
\]

One special case of an advariant is the auto-correlation, \( A_0 := A_{00} \), which obeys Eq. (1). The next non-trivial example is a 3-vertex advariant defined on an elementary triangle \( \Delta_{(0ab)} = (0, a, b) \), \( A_{ab} := C_{0a} \hat{C}_{ab} C_{b0} \). Note that \( A_{ab} \) is also an advariant. Advariants are the building blocks for covariants. In fact, any covariant can be obtained by multiplying an even number of such elementary advariants with even terms hatted as described earlier. For example, a 4-vertex covariant on a closed loop, \( \Gamma_0 = 0abcd0 \), can be written as \( C_{\Gamma_0} = A_{ab} A_{bc} \). Note that all these covariants transform in accordance with Eq. (3) for a closed loop pinned at vertex 0. The traces of these covariants are the closure invariants [15]. The key advance in this paper is that we use elementary triangular advariants to arrive at a complete and independent set
of closure invariants, using general arguments based on Lorentz and scaling symmetries. An advariant, $\mathcal{A}$, can be expanded in terms of the identity matrix, $\sigma_0 := I$, and Pauli matrices, $\sigma_m, m = 1, 2, 3$ as
\[ \mathcal{A} = z^\mu \sigma_\mu, \]
where, $z^\mu$ are complex coefficients. The $z^\mu$ can be thought of as similar to complex Stokes parameters $\{I, Q, U, V\}$. In the case of the autocorrelation advariant, which is Hermitian, these are real and are the ordinary Stokes parameters of polarization optics [18].

Under gauge transformations of $\mathcal{A}$, the action of the gauge group, $G_0 \in \text{GL}(2, \mathbb{C})$, on advariants can be split into two parts. First, the subgroup, $G_0 = \Lambda I$, acts on $z^\mu$ by scaling,
\[ z^\mu \mapsto |\lambda|^2 z^\mu. \]
Second, the subgroup $\text{SL}(2, \mathbb{C})$ of matrices with unit determinant preserves $\det(\mathcal{A}) = I^2 - Q^2 - U^2 - V^2 = z^\mu \eta_{\mu\nu} z^\nu$, which shows that it acts on $z^\mu$ by a Lorentz transformation,
\[ z^\mu \mapsto \Lambda^\mu_\nu z^\nu, \]
where, $\Lambda \equiv \Lambda^\mu_\nu$ is a real $4 \times 4$ matrix representing a Lorentz transformation. Note that the real and imaginary parts of $z$ separately transform as 4-vectors. Closure invariants must remain unchanged under transformations by both these subgroups.

**Complete and independent set.**—The triangular advariants (pinned at vertex 0) can be used to build covariants for every closed loop with even links in the graph. Each triangular advariant, $\mathcal{A}_{ab}$, can be expanded as $\mathcal{A}_{ab} = z^\mu_{ab} \sigma_\mu$ as in Eq. (5), where, $z^\mu_{ab}$ are complex 4-vectors.

We first consider only the Lorentz transformations in Eq. (7). Each triangle, $\Delta_{(ab)}$, gives us a complex 4-vector, $z_{ab}^\mu$, which can be decomposed into two real 4-vectors. The number of triangles with one vertex at 0 is $N_\Delta = \binom{N-2}{2}$. In addition, for $n_\Lambda = 1$, there will be one more Hermitian advariant, $\mathcal{A}_0$, which gives us one more real 4-vector. Thus, we have a set of $M = 2N_\Delta + n_\Lambda$ real, independent, 4-vectors, $y_m, m = 1, 2, \ldots, M$. Invariants of the Lorentz subgroup (Lorentz invariants), can only be functions of Minkowskian inner products among these vectors.

We construct an independent set of closure invariants (for $N > 3$) as follows: choose any four real 4-vectors to be the basis set, say $h_k = y_k, k = 1, 2, 3, 4$. Between these four independent basis vectors, there are 10 Minkowskian inner products, $\tilde{\mathcal{I}}_{kl} = h_k \cdot h_l$, with $k, l = 1, 2, 3, 4$ and $k < l$. Further, the remaining $(M - 4)$ 4-vectors give us $4(M - 4)$ Minkowski inner products with the four basis vectors, $\tilde{\mathcal{I}}_{kn} = h_k \cdot y_n$ for $n = 5, \ldots, M$. Together, we have $10 + 4(M - 4) = 4N^2 - 12N + 2 + 4n_\Lambda$ real Minkowski inner products. There is no need to take any more inner products since a 4-vector is completely specified by its components in a basis. The $M$ triangular advariants are independent and all covariants can be constructed from these triangular advariants. This ensures that these Minkowskian inner products are both complete and independent.

The inner products (Lorentz invariants) are still not the invariants we seek since they acquire a scale factor from the element-based gains [Eq. (9)]. We can obtain true invariants by forming ratios of the Lorentz invariants $\{\tilde{I}_{kl}, \tilde{I}_{kn}\}$, for example, dividing the set by any one of them, say $\tilde{I}_{11}$. Thus, the number of closure invariants will be $4N^2 - 12N + 1 + 4n_\Lambda$, that is, one less than the number of independent inner products. Comparing with Eq. (2), we find that $s = 1$ (one NCG corresponding to a uniform phase across all elements) for $N \geq 4$.

The algorithmic summary for finding the closure invariants from the measured visibilities is: 1) Choose a base element 0 and construct the $N_\Delta = \binom{N-2}{2}$ complex triangular advariants. 2) Use these to construct $2N_\Delta$ real 4-vectors. 3) If $n_\Lambda = 1$, include one more real 4-vector to get $M = 2N_\Delta + n_\Lambda$ 4-vectors. 4) Pick four of these 4-vectors as a basis and compute the 10 Minkowski dot products between them. 5) Compute the Minkowski dot products between the remaining $(M - 4)$ 4-vectors and the basis vectors. 6) Divide the remaining $4(M - 4)$ Minkowski dot products by any one of them to find all the closure invariants. Finding a complete and independent set of closure invariants in polarized interferometry is the main new result of this paper.

The case $N = 3$ is special. For $n_\Lambda = 1$, we get 5 invariants as expected (from $4N^2 - 12N + 1 + 4n_\Lambda$). For $n_\Lambda = 0$, we may likewise expect to get 1 invariant. Instead, we get 2 invariants: the three inner products of the two 4-vectors associated with $\mathcal{A}_{12}$ give us two invariant ratios. The “extra” invariant appears because there is an extra one parameter family of NCGs. These are Lorentz transformations in the plane orthogonal to the two 4-vectors. We identified a uniform phase shift on all elements as an example of an NCG. Are there others? The work of this paper and comparison with Eq. (2) clearly rules this out in all cases but one ($N = 3, n_\Lambda = 0$). While this special case is discussed in appendix B, a simple proof of this general statement can be given: any Lorentz transformation that preserves all the visibilities also preserves the advariants and the associated 4-vectors. The only Lorentz transformation that preserves three (or more) independent vectors is the identity. The only case where there are fewer than three independent 4-vectors occurs when $N = 3, n_\Lambda = 0$, as was noted earlier as an example of an NCG. For $N > 3$, there are no NCGs in the Lorentz subgroup. There is just one in the scaling subgroup, a constant phase factor $\eta$.

**Comparison with earlier work.**—We now return to the formulation of invariants in terms of closure traces introduced in [13], and relate it to the framework of Lorentz invariants constructed from 4-vectors developed in this
paper. An elementary calculation shows that
\begin{equation}
\mathcal{A} = \frac{z^{0\ast} \sigma_0 - z^{1\ast} \sigma_1 - z^{2\ast} \sigma_2 - z^{3\ast} \sigma_3}{(z \cdot z)^{\ast}}.
\end{equation}
Given two advariants, \( \mathcal{A} \) and \( \mathcal{A}' \), we form a covariant, \( \mathcal{A}\mathcal{A}' \). Taking the trace and using Eq. \( \text{(8)} \), we find an invariant,
\begin{equation}
\frac{1}{2} \text{tr}(\mathcal{A}\mathcal{A}') = \frac{z \cdot z^\ast}{(z' \cdot z')^\ast},
\end{equation}
which is a ratio of Minkowskian inner products. This equation provides the link between the closure trace formulation \([15]\) and the Minkowskian one presented here. Appendix \([11]\) discusses an interesting point that the use of closely spaced elements not resolving the object, introduced in \([15]\) as a diagnostic, can be effectively extended to provide auto-correlation information \((n_A = 1)\).

We have performed numerical tests to check the analytical theory described in this paper. The expected number of independent invariants was confirmed by computing the rank of the Jacobian of the numerical partial derivatives of the invariants with respect to the components of the correlation matrices. We used multiple realizations of the correlations, and examined the rank of the Jacobian of the numerical partial derivatives, ranked by signal to noise, via singular value decomposition. Given that direct geometric interpretation of invariants is still being explored even in the case of co-polar measurements \([8]\), there is clearly much work to be done on the 4-vector approach. Finally, we anticipate that the general concept of gauge invariance we have introduced may have wider applicability, in other fields where general linear transformations of multi-channel data by unknown or ill-constrained factors have corrupted the correlations.

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I. THE “HAT” OPERATOR

Some of the basic properties of the hat operator are:

- The order of the inverse and conjugate-transpose operations is irrelevant, \( \hat{P} = (P^{-1})^\dagger = (P^\dagger)^{-1} \)

- If \( Q = P_1 \ldots P_j \ldots P_n \) is a product of invertible matrices, \( \hat{P}_j \), then \( \hat{Q} = \hat{P}_1 \ldots \hat{P}_j \ldots \hat{P}_n \) retains the same order of multiplication of the ‘hatted’ matrices, \( \hat{P}_j \).

- \( \hat{C}_{ab}C_{ba} = I \) (the identity matrix)

II. INVARIANTS WITH \( N = 3 \)

As mentioned in the main text, the dimension of the stabilizer for any \( N \geq 4 \) is \( \dim(\mathcal{H}) = 1 \). The stabilizer subgroup, \( \mathcal{H} \), consists of a constant phase shift applied to the signals from all the interferometer elements, which would change the signals. However, the stabilizer \( G_a \) does not conserve correlations in standard astronomical interferometry applications depend only on the relative and not absolute phases. This means that the stabilizer contains at least the subgroup of multiplication of the ‘hatted’ matrices, \( \hat{P}_j \).
There is just one triangle, \(\Delta_{(012)}\), which gives us only one complex advariant (and so, only one complex 4-vector, \(z^{0}_1\)), which gives us just two real 4-vectors, \(y^0_1\) and \(y^2_0\). We can form three Lorentz invariants from Minkowski dot products, \(\tilde{T}_{k\ell} = y_k \cdot y_\ell = y_k^\mu \eta_{\mu\nu} y_\ell^\nu\), with \(k, \ell = 1, 2\) and \(\ell \geq k\), which will give us two closure invariants, \(\tilde{T}_{11}/\tilde{T}_{11}\), after eliminating the unknown scale factor. This is one more than that given by the formula \(4N^2 - 12N + 4n_A + 1\) that works for \(N > 3\).

This increase in closure invariants occurs because the stabilizer now has \(\dim(\mathcal{H}) = 2\). To show this, we use the fact that there is an extra one-parameter subgroup of Lorentz transformations in the plane orthogonal to the two 4-vectors, \(y_1\) and \(y_2\), which acts trivially on them (i.e., leave them unaffected).

Corresponding to these Lorentz transformations, we have a one-parameter group of gauge transformations at the base 0, \(G_0(s) = e^{sK}\), which leaves \(A_{12}\) unchanged

\[ G_0(s)A_{12}G_0(s)^\dagger = A_{12}. \tag{10} \]

Note that the Lorentz transformations, \(G_0(s)\), belong to the subgroup SL(2, \(\mathbb{C}\)) and have determinant unity, and so do not affect the scale of the vectors. The question now is whether we can find a one-parameter subgroup of gauge transformations, acting at all elements, \(\{G_0(s), G_1(s), G_2(s)\}\), such that none of the measured correlations is affected:

\[
\begin{align*}
C_{01} &= G_0(s)C_{01}G_0(s)^\dagger \quad (11) \\
C_{02} &= G_0(s)C_{02}G_0(s)^\dagger \quad (12) \\
C_{12} &= G_1(s)C_{12}G_1(s)^\dagger \quad (13)
\end{align*}
\]

The answer is yes: we solve Eq. (11) for \(G_1(s)\) and Eq. (12) for \(G_2(s)\) to express them in terms of \(G_0(s)\) and the measured correlations. This yields \(G_1(s) = C_{01}^\dagger G_0(s)C_{01}\), and \(G_2(s) = C_{02}^\dagger G_0(s)C_{02}\). It then follows from Eq. (10) that \(C_{12}\) is also unchanged by the one-parameter group of gauge transformations, \(\{G_0(s), G_1(s), G_2(s)\}\). Thus, this subgroup fixes all the measured correlations \(C_{ab}\), and so belongs to the stabilizer subgroup.

This increase in the dimension of the stabilizer increases the number of invariants by one. The two invariants can also be exhibited by taking the trace of the covariant, \(A_{12}\bar{A}_{12}\), and its inverse. This is formed from the six-edged loop in which the triangle, \(\Delta_1 = \Delta_{(012)}\), is traversed twice. While the traces of this matrix, and its inverse, are both complex, they are conjugates of each other. So, we get two real invariants from this combination. This is also clear from the representation of the trace in terms of inner products of 4-vectors,

\[ \tilde{\mathcal{T}}_{\Delta_1, \Delta_1} = \frac{1}{2} \text{tr}[A_{12}\bar{A}_{12}] = \frac{z_{12} \cdot z_{12}^*}{(z_{12} \cdot z_{12})^*}. \tag{14} \]

This exceptional behavior disappears if an auto-correlation, \(A_0\), is measured. This yields one more real 4-vector, \(y^0_0\), so that we now have three real 4-vectors \(\{y^0_0, y^1_1, y^2_2\}\), thereby removing the residual Lorentz gauge freedom (present earlier in the plane orthogonal to the plane containing the first two real 4-vectors, \(\{y^0_0, y^1_1\}\)). The number of Lorentz invariants, \(\tilde{T}_{k\ell}\) with \(k, \ell = 0, 1, 2\) and \(\ell \geq k\), is six. After eliminating the unknown scaling factor by dividing by one of them, the number of closure invariants is \(N_T = 5\). In matrix language, \(G_0(s)\) was chosen to leave the advariant \(A_{12}\) unchanged. So if it left \(A_0\) in the case \(N = 3\), or indeed, for \(N > 3\), any other advariant unchanged, it could only be by accident, since these are independent measurements. In the most general case, \(G_0(s)\) will no longer be a stabilizer when \(n_A = 1\) and \(N = 3\), or when \(N > 3\).

### III. Auto-Correlations as the Coincidence Limit of Cross-Correlations

As we have emphasised in the text, auto-correlations tend to be severely affected by additional systematic effects (which cross-correlations eliminate) and are therefore regarded as less reliable. However, when the angular size of the object is small compared to the angular resolution determined by the element spacing, we can find a way around this problem through the use of cross-correlation following the discussion in Paper I [17].

This requires an extra element \(0'\) in close proximity with the base element 0. This can be provided by adding a new element, \(0'\), or by splitting a dense array of phased elements, as in the case of the Atacama Large Millimeter/submillimeter Array (ALMA) in the EHT/VLBI observations of M87 and Centaurus A [19, 20], into two independent portions that are geographically coincident and phasing them separately to act as independent elements 0 and 0'. This provides us with an extra short-spacing pair, \((0, 0')\), with separation, \(D_{00'}\), that is too short to resolve the object \((r_{00'} < \lambda/D_{00'})\), where, \(r_{00'}\) is the angular extent of the object, and \(\lambda\) is the wavelength of observation), but distant enough that the gains, \(G_0\) and \(G_{0'}\), are independent. Note that because \(S_{00'} \approx S_{00}\), no new true cross-correlations are provided by this additional element beyond that provided by element 0, excepting \(S_{00'}\), where, \(S_{00'} \approx S_{00}\). Using the extra short-spacing cross-correlation measured, \(C_{00'}\), we can construct an advariant that closely approximates that from an auto-correlation as described above.

Let us construct the advariant based on a “thin” triangle \(\Delta_{(00'a)}\), where \(a \notin \{0, 0'\}\): \(C_{00'}C_{00'a}C_{00'a} = G_0S_{00}S_{00'}S_{00}G_0^\dagger\). Under our assumption of closeness of elements 0 and 0’, this can be written as \(G_0S_{00}S_{00}G_0^\dagger = G_0S_{00}S_{00}G_0^\dagger\) since \(S_{00}S_{00} = I\). This is effectively the same advariant that we would obtain from a direct measurement of auto-correlation at element 0. This means the earlier discussion with \(n_A = 1\) is applicable and we gain four invariants. The new adariant
defines a 4-vector and the four components of this 4-vector in an already established frame give us four more real invariants. The use of extra short baselines was proposed in [15] as an error diagnostic. We note here that they can also be used to incorporate auto-correlations into the construction of invariants.

IV. CONNECTION TO CLOSURE TRACES: COMPLETENESS AND INDEPENDENCE

In the main paper, we chose to express the closure invariants as Minkowskian inner products. This has the advantage that their independence and completeness is manifest. However, for comparison with [15], we express the same invariants in the language of closure traces. Let us single out two triangles (labelled $\Delta_1 \equiv \Delta_{(012)}$ and $\Delta_2 \equiv \Delta_{(023)}$), and their reversed forms $\nabla_1 \equiv \Delta_{(021)}$ and $\nabla_2 \equiv \Delta_{(032)}$) and construct the complex closure traces

$$T_{\Delta_1 \Delta_2} = \frac{1}{2} \text{tr}[A_{12} \hat{A}_{23}] = \frac{z_{12} \cdot z_{23}^*}{(z_{23} \cdot z_{23})^*},$$
$$T_{\nabla_1 \Delta_2} = \frac{1}{2} \text{tr}[A_{12}^\dagger \hat{A}_{23}] = \frac{z_{12} \cdot z_{23}^*}{(z_{23} \cdot z_{23})^*},$$
$$T_{\Delta_2 \Delta_1} = \frac{1}{2} \text{tr}[A_{23} \hat{A}_{12}] = \frac{z_{23} \cdot z_{12}^*}{(z_{12} \cdot z_{12})^*},$$
$$T_{\nabla_2 \Delta_1} = \frac{1}{2} \text{tr}[A_{23}^\dagger \hat{A}_{12}] = \frac{z_{23} \cdot z_{12}^*}{(z_{12} \cdot z_{12})^*},$$
$$T_{\Delta_2 \nabla_1} = \frac{1}{2} \text{tr}[A_{12} \hat{A}_{23}] = \frac{z_{12} \cdot z_{23}^*}{(z_{23} \cdot z_{23})^*},$$
$$T_{\nabla_2 \nabla_1} = \frac{1}{2} \text{tr}[A_{23} \hat{A}_{12}] = \frac{z_{23} \cdot z_{12}^*}{(z_{12} \cdot z_{12})^*}. \quad (15)$$

The other pairwise advariant combinations not written here can be shown to be dependent on the above.

It would appear that we have twelve real invariants from these six complex closure traces. However, they have further dependencies satisfying the following three real relations:

$$\frac{|T_{\nabla_1 \Delta_2} T_{\nabla_2 \Delta_1}|}{T_{\Delta_1 \Delta_2} T_{\nabla_2 \Delta_1}} = 1, \quad (16)$$
$$\frac{T_{\Delta_1 \Delta_2} T_{\nabla_2 \Delta_1} T_{\Delta_2 \Delta_2}}{T_{\nabla_1 \Delta_2} T_{\Delta_2 \Delta_1} T_{\Delta_2 \Delta_2}} = 1, \quad (17)$$
and
$$\frac{T_{\nabla_1 \Delta_2} T_{\Delta_2 \nabla_1} T_{\Delta_2 \Delta_1}}{T_{\Delta_1 \Delta_2} T_{\Delta_2 \nabla_1} T_{\Delta_2 \Delta_1}} = 1. \quad (18)$$

This reduces the number of real invariants to 9, made from the advariants $A_{12}$ and $A_{23}$ alone. In addition, each of the remaining triangles, $\Delta_k$ with $k \geq 3$, contributes the following four complex invariants constructed by taking the traces of the products of each $A_{\Delta_k}$ with $A_{\Delta_1}$ and $A_{\Delta_2}$, and their Hermitian adjoints,

$$T_{\Delta_1 \Delta_k} = \frac{1}{2} \text{tr}[A_{\Delta_1} \hat{A}_{\Delta_k}], \quad T_{\Delta_2 \Delta_k} = \frac{1}{2} \text{tr}[A_{\Delta_2} \hat{A}_{\Delta_k}],$$
$$T_{\nabla_1 \Delta_k} = \frac{1}{2} \text{tr}[A_{\nabla_1} \hat{A}_{\Delta_k}], \quad T_{\nabla_2 \Delta_k} = \frac{1}{2} \text{tr}[A_{\nabla_2} \hat{A}_{\Delta_k}]. \quad (19)$$

As earlier, we can express these in terms of ratios of Lorentz-invariant Minkowskian inner products, and in this case it is easy to see that the four complex invariants are indeed independent. This gives us eight real invariants per additional triangle, $\Delta_k$, for $3 \leq k \leq (N - 1)(N - 2)/2$, in full agreement with the counting based purely on Lorentz invariants. Also, note that the argument in the main text based on Lorentz invariants is considerably simpler than the one described above.

V. HOLOMONY AND THE CHOICE OF BASE VERTEX

In gauge theory, it is usual to talk about the holonomy group at a point, $P$, which gives us the parallel transport along loops $\Gamma$ pinned at $P$. One can multiply two loops by traversing them in succession and the holonomy of the product of the loops is the product of the holonomy of the individual loops. In unitary lattice gauge theories (and more generally gauge theory on a graph), there are elementary plaquettes and all loops can be broken into these elementary plaquettes, which are both independent and complete. In our problem, the gauge group was not unitary, and the even holonomies were quadrilateral and therefore not independent. We got around this difficulty by using triangular advariants which are both independent and complete. The advariants are not themselves holonomies, but can be used to construct all holonomies.

The holonomy group depends on the base vertex, but only changes by conjugation. So, the choice of the base vertex is not important from the mathematical point of view, and we have arbitrarily singled out one array element (indexed 0) as the base vertex. In practice, the choice of the base vertex may be guided by other considerations like the quality of data received at different elements. A final remark about nomenclature: the word “holonomy” which is used in the mathematical literature is known to physicists as a “Wilson loop”. In the geometric phase literature, it is sometimes referred to as an “anholonomy”, following Michael Berry’s just observation [9, 21] that we are really describing a failure of integrability and therefore a lack of “wholeness”. This usage is consistent with “holonomic and anholonomic constraints” of classical mechanics.