Cycles of Time in Classical Cosmology

A. E. Pavlov

Institute of Mechanics and Power Engineering,
Russian State Agrarian University – Moscow Timiryazev Agricultural Academy,
Timiryazevskaya str. 49, Moscow 127550, Russia
alexpavlov60@mail.ru

We present exact solutions to the Friedmann equation in standard ΛCDM cosmology in Weierstrass and Jacobi functions. The right hand side of the Friedmann equation describing various contributions of matter sources is considered in generic form. It is proved that the problem of integration of the Friedmann equation for simple EoS of medium is reduced to solving Abel integrals for algebraic functions.

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I. INTRODUCTION

The last few decades have seen a remarkable progress in cosmology [1]. Observations with more advanced instruments will resolve mysterious questions in the coming decades. The Friedmann equation is used to interpret the observational data. It connects the rate of expansion of the Universe with the energy density of matter and spatial curvature. For different states of matter, different scenarios of the evolution of the Universe are obtained [2, 4].

Cosmologists prefer the numerical integration of the Friedmann equation. Since the right hand side of the Friedmann equation has a polynomial form, it is of interest to use the theory of doubly periodic functions to find analytical solutions developed in papers of Jacobi, Abel, Weierstrass, Kovalevskaya. The theory has found effective application in problems of analytical mechanics [3] and celestial mechanics [6].

Kovalevskaya raised the problem of finding all integrable cases of rigid body rotations in class of meromorphic at all plane of complex time variable. This statement of the question presented essential character and was not demanded by any mechanical argumentations. This remarkable mathematical extension of the original mechanical problem. Moreover, this extension had purely mathematical character and was not demanded by any mechanical argumentations. This remarkable mathematical extension of studying the mechanical problem on the plane of complex time permitted applying the theory of analytical functions excellent elaborated in XIX-th century. Later on, successfully, the approach was taken into consideration by mathematicians in studying various problems in applied mechanics.

We get more information about real functions considering them on a complex plane [2]. The application of meromorphic functions of complex time in theoretical cosmology is of interest. In this work, by including the most significant contributions from different states of matter, we obtain analytical solutions of the Friedmann equation. The author hopes the analytical solutions will be useful in the interpretation of the modern cosmological data.

II. CONFORMAL FRIEDMANN EQUATION

In the standard ΛCDM cosmology the conformal Friedmann equation \(\dot{a}/a\) is represented by the first integral

\[
\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3} a^2 \rho - k. \tag{1}
\]

Here \(a(\eta)\) is the scale factor, the prime denotes the derivative with respect to the conformal time \(\eta\). \(G\) is the Newton constant, \(\rho\) is the matter sources density, \(k\) is the constant of space curvature. The quantity \(\mathcal{H} \equiv a'/a\) is the conformal Hubble parameter defining the expansion rate of the Universe. Its present value \(H_0\) calls the Hubble constant. The energy continuity equation in conformal time

\[
\rho' = -3(\rho + P) \left(\frac{a'}{a}\right) \tag{2}
\]

with an equation of state \(P = w\rho\), connecting the pressure \(P\) and density \(\rho\), yields the matter density sources dependence of the scale factor

\[
\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}. \tag{3}
\]

Here, \(\rho_0\) and \(a_0\) are the modern values of the corresponding characteristics. So, for simplest cases, one has

- interstellar dust: \(w = 0, P = 0 : \rho = \rho_0 (a_0/a)^3\);
- radiation: \(w = 1/3, P = \rho/3 : \rho = \rho_0 (a_0/a)^4\);
- De Sitter vacuum: \(w = -1, P = -\rho : \rho = \rho_0\).

Taking into consideration the vacuum density, non-relativistic and relativistic matter with the corresponding cosmological parameters \(\Omega_\Lambda, \Omega_M, \Omega_{\text{rad}}\), we get the sum of their contributions

\[
\rho = \frac{3H_0^2}{8\pi G} \left[\Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_{\text{rad}} \left(\frac{a_0}{a}\right)^4\right]. \tag{4}
\]

The cosmological parameters are constrained

\[
\Omega_\Lambda + \Omega_M + \Omega_{\text{rad}} + \Omega_{\text{curv}} = 1, \tag{5}
\]
where for curvature term is
\[
\Omega_{\text{curv}} \equiv -\frac{k}{a_0^2 H_0^2}. 
\]

Substituting (11), (5), (6) into the right hand side of the conformal Friedmann equation (1), we obtain
\[
\frac{1}{x^2} \left( \frac{dx}{d\eta} \right)^2 = \left( \frac{H_0 a_0}{c} \right)^2 \left[ \Omega_{\text{tot}} \right] x + \Omega_{\text{curv}} \frac{\Omega_M}{x} + \Omega_{\text{rad}} \right]. 
\]

Here the variable \( x \) is given as a ratio of a scale \( a(\eta) \) to a modern value \( a_0 \):
\[
x \equiv \frac{a(\eta)}{a_0} = \frac{1}{1+z}, 
\]

\( z \) is a redshift of spectral lines,
\[
H_0 = \frac{h}{10^5 m/s/Mpc}, \quad h = 0.72 \pm 0.08
\]
is the Hubble constant.

From the above differential equation (7) the formula for getting the conformal time as a function of the scale is followed:
\[
\eta(x) = \frac{c}{H_0 a_0 \sqrt{\Omega_\Lambda}} \int_0^x \frac{dx}{\sqrt{x^4 + 6a_2 x^2 + 4a_3 x + a_4}} 
\equiv \frac{c}{H_0 a_0 \sqrt{\Omega_\Lambda}} I(x). 
\]

Here the following notations are introduced
\[
6a_2 \equiv \left( \frac{\Omega_{\text{curv}}}{\Omega_\Lambda} \right), \quad 4a_3 \equiv \left( \frac{\Omega_M}{\Omega_\Lambda} \right), \quad a_4 \equiv \left( \frac{\Omega_{\text{rad}}}{\Omega_\Lambda} \right).
\]

Notice, that it is possible to consider the subradical polynomial of generic kind
\[
P(x) = x^4 + 4A_1 x^3 + 6A_2 x^2 + 4A_3 x + A_4. 
\]

The second term \( 4A_1 x^3 \) corresponds to domain walls contribution \( \Phi \) characterized by
\[
\begin{align*}
\cdot & w = -2/3, \quad p = -(2/3)p: \quad p = \rho_0(a_0/a) \\
\end{align*}
\]
However, by shift the variable \( x \rightarrow x - A_1 \) the polynomial \( P(x) \) (11) is reduced to the standard form
\[
P_4(x) \equiv x^4 + 6a_2 x^2 + 4a_3 x + a_4 
\]
with the coefficients
\[
a_2 = A_2 - A_1^2, \quad a_3 = A_3 - 3A_1 A_2 + 2A_1^3, \\
a_4 = A_4 - 4A_1 A_3 + 6A_2 A_1^2 - 3A_1^4.
\]

The polynomial (12) is of the fourth order, to low its order let us implement the change of the variable \( x \rightarrow y \) by the rule (4):
\[
\sqrt{x^4 + 6a_2 x^2 + 4a_3 x + a_4} \equiv x^2 - 2y + a_2. 
\]

Raising to square both sides of the equation (13), one gets the polynomial expression
\[
4a_2 x^2 + 4a_2 x^2 y + 4a_3 x - 4y^2 + 2a_2 y + a_4 - a_2^2 = 0. 
\]
The differential of the equality (14) can be written in the form of a proportion
\[
\frac{dx}{x^2 - 2y + a_2} = -\frac{dy}{2a_2 x + 2xy + a_3}. 
\]

With regard to equality (14), we rewrite the proportion (15) in the form
\[
\frac{dx}{\sqrt{x^4 + 6a_2 x^2 + 4a_3 x + a_4}} = -\frac{dy}{2a_2 x + 2xy + a_3}, 
\]
Let us express the variable \( x \) from the (14)
\[
x(y) = -a_3 \pm \sqrt{4y^3 - 2g_2 y - g_3} 
\frac{2(y+a_2)}{2(y+a_2)}, 
\]

where the following notations were introduced:
\[
g_2 \equiv a_4 + 3a_2^2, \quad g_3 \equiv a_2 a_4 - a_2^3 - a_2^3.
\]
Let us express the variable \( y \) from the equality (14)
\[
y(x) = \frac{1}{2} \left( x^2 + a_2 \pm \sqrt{x^4 + 6a_2 x^2 + 4a_3 x + a_4} \right).
\]
The equality (17) can be presented in the form
\[
(2xy + A_2 x + A_3)^2 \equiv 4y^3 - 2g_2 y - g_3. 
\]
Then we implement the next substitution \( y \rightarrow u \):
\[
y = \varphi(u; g_2, g_3), 
\]
where \( \varphi(u; g_2, g_3) \) is the Weierstrass function (10). Then the differential equation (16) takes the exact form
\[
\frac{dy}{2a_2 x + 2xy + a_3} = \frac{dy}{\sqrt{4y^3 - 2g_2 y - g_3}} = du. 
\]
The integral (10) can be presented to the standard form after the substitution (18)
\[
I(x) \equiv \int_0^x \frac{dx}{\sqrt{P_4(x)}} = -\int_{(a_2 - \sqrt{a_2})/2}^{y(x)} \frac{dy}{\sqrt{4y^3 - 2g_2 y - g_3}} \\
- \int_{(a_2 - \sqrt{a_2})/2}^{+\infty} \frac{dy}{\sqrt{4y^3 - 2g_2 y - g_3}} + \int_{y(x)}^{+\infty} \frac{dy}{\sqrt{4y^3 - 2g_2 y - g_3}}.
\]

Now, after the substitution (20), the integral is expressed via the inverse Weierstrass functions
\[
I(x) = -\int_{\varphi^{-1}[(a_2 - \sqrt{a_2})/2]}^{+\infty} du + \int_{\varphi^{-1}[y(x)]}^{+\infty} du \\
\varphi^{-1}[(a_2 - \sqrt{a_2})/2] + \varphi^{-1}[y(x)]. 
\]
Finally, we yield the exact solution: the conformal time as a function of the scale \( \eta = \eta(x) \):
\[
\frac{H_0a_0\sqrt{\Delta}}{c} \eta(x) = -\varphi^{-1} [(a_2 - \sqrt{a_4})/2] + \varphi^{-1} [y(x)].
\] (23)
The expression (23) can be converted where we denoted
\[
\varphi[u] = y(x),
\]
where we denoted
\[
u = \frac{H_0a_0\sqrt{\Delta}}{c} \eta(x) + \varphi^{-1}[y(0)].
\]

III. PERIODS OF DOUBLY PERIODIC FUNCTIONS

Let us focus our attention on the cubic polynomial in Weierstrass form
\[
P_3(y) \equiv 4y^3 - g_2y - g_3, \tag{24}
\]
where \( g_2, g_3 \) are its invariants. Depending on the sign of the discriminant of the polynomial, we have three cases, which we consider in order.

- The discriminant
\[
\Delta = g_3^3 - 27g_3^2 \quad \tag{25}
\]
is positive: \( \Delta > 0 \). Then the roots of the polynomial (24) are real and also \( e_1 > e_2 > e_3 \):
\[
e_2 = s_1 + s_2,
\]
where
\[
s_1 \equiv \frac{1}{2} \left[ g_3 + \frac{1}{3\sqrt{3}} \sqrt{-\Delta} \right]^{1/3},
\]
\[
s_2 \equiv \frac{1}{2} \left[ g_3 - \frac{1}{3\sqrt{3}} \sqrt{-\Delta} \right]^{1/3};
\]
and the rest two are:
\[
e_1 = \frac{1}{2}(s_1 + s_2) + \frac{\sqrt{3}}{2}(s_1 - s_2); \quad e_3 = \frac{1}{2}(s_1 + s_2) - \frac{\sqrt{3}}{2}(s_1 - s_2).
\]
The \( \varphi \)-function has one real semi-period \( \omega_1 \)
\[
\omega_1 = \int_{e_1}^{+\infty} \frac{dy}{\sqrt{4(y-e_1)(y-e_2)(y-e_3)}}
\]
and one is imaginary \( \omega_3 \)
\[
\omega_3 = i \int_{-\infty}^{e_3} \frac{dy}{\sqrt{4(e_1-y)(e_2-y)(e_3-y)}}
\]
The Weierstrass function is two-parameter, and for practical use it is more convenient to work with the one-parameter elliptic Jacobi function. If we change the variable
\[
\varphi(u) = e_3 + \frac{e_1 - e_3}{\sin^2(u\sqrt{e_1 - e_3})}
\]
with the modulus of the elliptic sine
\[
k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}},
\]
we yield the solution (23) in Jacobi form
\[
\sin^2(u\sqrt{e_1 - e_3}) = \frac{e_1 - e_3}{y(x) - e_3}. \quad (26)
\]
The basic periods of the elliptic sine are \( 4K \) and \( 2K' \), where
\[
K \equiv \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad K' \equiv \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}},
\]
and \( k^2 + k'^2 = 1 \).
- The discriminant (24) is negative: \( \Delta < 0 \). Therefore, one root \( e_2 \) of the polynomial (24) is real:
\[
e_2 = s_1 + s_2,
\]
and the rest two are complex conjugated:
\[
e_1 = m + m; \quad e_3 = m - m.
\]
Present the polynomial (24) as a product
\[
4y^3 - g_2y - g_3 = 4(y - e_2)[(y - m)^2 + n^2].
\]
The semi-periods of the Weierstrass \( \varphi \)-function:
\[
\omega_2 = \int_{e_2}^{+\infty} \frac{dy}{\sqrt{4(y-e_2)[(y-m)^2 + n^2]}},
\]
\[
\omega'_2 = i \int_{-\infty}^{e_2} \frac{dy}{\sqrt{4(e_2-y)[(y-m)^2 + n^2]}},
\]
If we make the substitution
\[
\varphi(u) = e_2 + H \frac{1 + \text{cn}(2u\sqrt{H})}{1 - \text{cn}(2u\sqrt{H})} \quad \tag{27}
\]
with \( H = \sqrt{9m^2 + n^2} \) we present the solution (23) in Jacobi form
\[
\text{cn}(2u\sqrt{H}) = \frac{y(x) - (e_2 + H)}{y(x) - (e_2 - H)},
\]
where the modulus is equal to
\[
k = \sqrt{\frac{1}{2} - \frac{3e_2}{4H}}.
\]
Basic periods for the elliptic cosine are $4K$ and $2K'$.

• The discriminant \( \Delta \) is equal to zero: \( \Delta = 0 \). Therefore, all roots of the polynomial \( \Delta \) are real. If invariants \( g_2 \) and \( g_3 \) are not zero, two roots are equal. Let \( e_1 = e_2 \), then \( k^2 \equiv (e_2 - e_3)/(e_1 - e_3) = 1 \). Weierstrass function \( \wp(u) \) is expressed via hyperbolic one.

Let \( e_2 = e_3 \), then \( k^2 \equiv (e_2 - e_3)/(e_1 - e_3) = 0 \). Weierstrass function \( \wp(u) \) is expressed via trigonometric sine:

\[
\wp(u) = -\frac{3g_3}{2g_2} + \frac{g_3}{2g_2} \sin^2 \left( \frac{\sqrt{g_2}}{2} \right).
\]

If invariants \( g_2 \) and \( g_3 \) are zero, all three roots are equal. Then, one yields

\[
u = \int_{\gamma} \frac{dy}{\sqrt{y^3 - a_2y^2 + a_4}} = \frac{1}{\sqrt{a_2}}
\]

and the substitution is rational

\[
y = \wp(u) = \frac{1}{\nu^2}.
\]

### IV. FRIEDMANN EQUATION IN COORDINATE TIME

In the standard cosmology the Friedmann equation in coordinate time \( \ddot{a}/a \)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2},
\]

where the dot denotes differentiation with respect to coordinate time \( t \). By analogy with the quadrature in conformal time \( \sigma \) we get the quadrature in coordinate time

\[
t = \frac{c}{H_0 \sqrt{\Omega_\Lambda}} \int_{0}^{x} \frac{x dx}{\sqrt{x^2 + 6a_2x^2 + 4a_3 x + a_4}} = \frac{c}{H_0 \sqrt{\Omega_\Lambda}} \int_{0}^{x} \frac{x dx}{\sqrt{P_2(x)}} \equiv \frac{c}{H_0 \sqrt{\Omega_\Lambda}} J(x).
\]

After substitution \( x \to y \) \( \beta \) one gets for the integral the expression

\[
J(x) = \int_{\wp^{-1}(y)}^{\wp^{-1}(y)} \frac{x(y) dy}{\sqrt{P_3(y)}},
\]

with the polynomial of the third order \( \Delta \). For the variable \( x(y) \) in \( \beta \) we have the substitution \( \beta \) in terms of Weierstrass function \( \wp \):

\[
x[y(u)] = -\frac{a_3 \pm \sqrt{4\wp'^2(u) - g_2\wp(u) - g_3}}{2\wp(u) + a_2} = -\frac{a_3 + \wp'(u)}{2\wp(u) + a_2}.
\]

After substitution \( \beta \) into \( \beta \), the integral \( \beta \) takes the form

\[
J(x) = -\frac{a_3}{2} \int_{\wp^{-1}(y)}^{\wp^{-1}(y)} \frac{\left( \wp'(u) + a_2 \right)^2 \wp'(u) du}{\wp(u) + a_2}
\]

\[
+ \frac{1}{2} \int_{\wp^{-1}(y)}^{\wp^{-1}(y)} \frac{\wp'(u) du}{\wp'(u) + a_2} \equiv J_1 + J_2.
\]

The second integral is equal to

\[
J_2 = \frac{1}{2} \ln \left( \frac{y(x) + a_2}{a_2 - \sqrt{a_4}/2 + a_2} \right).
\]

If in the first integral \( a_2 \) is not equal to the semi-period of the function \( \wp(u) \), denoting

\[
a_2 \equiv -\wp(v),
\]

we have for the underintegral

\[
\frac{1}{\wp(u) - \wp(v)} = -\frac{1}{\wp'(v)} \left( \wp(u + v) - \wp(u - v) - 2\wp(v) \right).
\]

Here we introduced the Weierstrass function

\[
\wp'(u) = -\wp(u),
\]

Integration yields

\[
J_1 = -\frac{a_3}{2} \int_{\wp^{-1}(y)}^{\wp^{-1}(y)} \frac{\left( \wp'(u) + a_2 \right)^2 \wp'(u) du}{\wp(u) + a_2}
\]

\[
= -\frac{a_3}{2\wp'(v)} \ln \left( \frac{\wp'\left[ \wp^{-1}\left( y(x) - v \right) \right]}{\wp'\left( \wp^{-1}\left( y(x) + v \right) \right)} \right)
\]

\[
+ \frac{a_3}{2\wp'(v)} \ln \left( \frac{\wp'\left( \wp^{-1}\left( a_2 - \sqrt{a_4}/2 - v \right) \right)}{\wp'\left( \wp^{-1}\left( a_2 - \sqrt{a_4}/2 + v \right) \right)} \right)
\]

\[
- \frac{a_3 \wp'(v)}{\wp'(v)} \left[ \wp^{-1}\left( y(x) \right) - \wp^{-1}\left( a_2 - \sqrt{a_4}/2 \right) \right].
\]

Here we introduced once more Weierstrass function

\[
(\ln \sigma(u))' = \frac{\sigma'(u)}{\sigma(u)} = \zeta(u).
\]

The obtained solutions of the Friedmann equation in coordinate time look rather complicated. Therefore, it is more convenient to work with the Friedmann conformal equation.

### V. DISCUSSION

In theoretical mechanics, the solution of a mathematical pendulum problem with length of thread \( l \) is expressed in terms of an elliptic sine. Two periods of the doubly periodic Jacobi function reveal
dynamical sense. The period of oscillations is equal to

\[ T = 4\sqrt{\frac{l}{g}}K, \]

where \( g \) is the acceleration of gravity. \( K \) is the complete elliptic integral of the first kind

\[ K = \int_{0}^{1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}, \]

with modulus \( k^2 = h/(2l) \), and \( h \) is an initial height of the point. Paul Appel noticed [11] that the imaginary period is equal to

\[ T' = 4\sqrt{\frac{l}{g}}K' \]

where \( K' \) is the additional elliptic integral with

\[ k'^2 = 1 - k^2 = 1 - \frac{h}{2l} = \frac{(2l - h)}{2l}. \]

It corresponds to the pendulum whose initial height is equal to \((2l - h)\).

According to Roger Penrose ideas of conformal cyclic cosmology the universe undergoes cycles during its evolution [12]. The history of the Universe is considered without an inflation stage. One con is a continuation of an other one. There was con antecedent to the Big Bang. So, it should be another con after the Big Bang. This picture is prolonged to both directions: in past and towards future. The conformal cyclic cosmology predicts the presence of families of concentric low-variance circular rings in the cosmic microwave background picture [13].

In quantum cosmology, the wave function of the universe is studied in problems of its quantum origin [14]. Quantum tunnel transitions with a change in the signature of spacetime are described in the imaginary time formalism [15]. The description of the classically forbidden state using imaginary time means the complexification of the conformal super-space [16]. This complexification makes the conformal time variable purely imaginary and transforms the Wheeler–DeWitt equation from hyperbolic to elliptic. This is analogous to the transition from the hyperbolic to the elliptic Klein–Gordon equation for Wick rotation.

It is possible to consider exotic states of medium with EoS \( P = w\rho \) for various fractional values of \( w \) with integer \( n = 3(1 + w) \) [9]. The problem of integration of the Friedmann equation is reduced to Abel integrals for algebraic functions [5]. The Abel integrals for algebraic functions are named hyperelliptic. In this work there was found the class of solutions of the Friedmann equation. It was proven that the solutions belong to the Weierstrass doubly-periodic functions. It can take an addition information for understanding the evolution of the Universe.

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