Hipergeometric solutions to some nonhomogeneous equations of fractional order

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Abstract. In this paper a study is performed to the solution of the linear non homogeneous fractional order alpha differential equation equal to $I_0(x)$, where $I_0(x)$ is the modified Bessel function of order zero, the initial condition is $f(0)=0$ and $0<\alpha<1$. Caputo definition for the fractional derivatives is considered. Fractional derivatives have become important in physical and chemical phenomena as visco-elasticity and visco-plasticity, anomalous diffusion and electric circuits. In particular in this work the values of $\alpha=1/2$, $1/4$ and $3/4$ are explicitly considered. In these cases Laplace transform is applied, and later the inverse Laplace transform leads to the solutions of the differential equation, which become hypergeometric functions.

1. Introduction

Differential equations of fractional order has become important in Physics, Chemistry and several areas of Engineering as anomalous diffusion [1,2]. Gamma and Beta functions, and in general special functions playing an important role in Fractional Calculus [3,4]. Bessel functions are a kind of special functions, which play an important role in problems with cylindrical symmetry, and in this paper an investigation as carried out in the connection of these functions with Fractional Calculus. Here simple non-homogeneous fractional differential equation are considered and it is shown how its solution leads to hypergeometric functions. The procedure here used is through the Lapace transform, which is often used in this kind of calculations. In this work an explanation of the differential equation and the theoretical treatment is done as well as its solution for different values of the fractional differential equations. The last section is devoted to the Conclusion.

2. Planning of the problem and its solution

Here the differential equation

$$D^\alpha G(x) = \frac{d^{\alpha} G(x)}{dx^{\alpha}} = I_0(x)$$

is considered with the condition $G_0 = 0$. The letter $\alpha$ is a fractional number between zero and one. Four values of $\alpha$ will be considered $1/4$, $1/2$ and $3/4$, however other values of $\alpha$ could be considered in other works. The Bessel function $I_0(x)$ is taken as the non-homogeneous part of the differential equation, however most general treatment could be performed in future papers.
The procedure to be followed is to apply Laplace transform to both sides of the equation. The second step will be to separate the function $F(x)$, followed by the application of the inverse Laplace transform.

Caputo definition of fractional derivatives will be considered in this paper. Let us considerer first the case of $\alpha = 1/4$, thus the differential equation will be

$$D^{1/4} G(x) = I_0(x)$$

(2)

$$LD^{1/4} G_{1/4}(x) = s^{1/4} \tilde{G}_{1/4}(s) - s^{3/4} G_{1/4}(0) = L I_0(x) = \frac{1}{\sqrt{s^2 - 1}}$$

(3)

where $\tilde{G}(s)$ is the Laplace transform of $G(x)$, that is,

$$\tilde{F}(s) = L F(x) = \int_0^\infty e^{-sx} F(x) \, dx$$

(4)

With the condition that $G(0) = 0$, the Laplace transform $\tilde{G}(s)$ becomes

$$\tilde{G}(s) = \frac{1}{s^{1/4}\sqrt{s^2 - 1}}$$

(5)

The inverse transform of $\tilde{G}(s)$ is now

$$G(x) = L^{-1}\tilde{G}_{1/4}(s) = L^{-1}\left[ \frac{1}{s^{1/4}\sqrt{s^2 - 1}} \right] = \frac{2\Gamma(3/4)x^{1/4}}{\pi} 2F_1 \left( \frac{1}{2}; \frac{5}{8}; \frac{9}{8}; \frac{x^2}{4} \right)$$

(6)

where $2F_1 \left( \frac{1}{2}; \frac{5}{8}; \frac{9}{8}; \frac{x^2}{4} \right)$ is an hypergeometric function, $F(a,a;b;c;z)$

In similar way, the case of $\alpha = 1/2$ and $\alpha = 3/4$ can be also treated, giving

$$G_{1/2}(x) = L^{-1}\left[ \frac{1}{s^{1/2}\sqrt{s^2 - 1}} \right] = \frac{2}{\pi} x^{1/2} 2F_1 \left( \frac{1}{2}; \frac{3}{4}; \frac{5}{4}; \frac{x^2}{4} \right)$$

(7)

and

$$G_{3/4}(x) = L^{-1}\left[ \frac{1}{s^{3/4}\sqrt{s^2 - 1}} \right] = \frac{4}{3} x^{3/4} 2F_1 \left( \frac{1}{2}; \frac{7}{8}; \frac{11}{8}; \frac{x^2}{4} \right)$$

(8)

Futhermore it is known that

$$\int_0^x I_0(x) \, dx = x 2F_1 \left( \frac{1}{2}; 1; \frac{3}{2}; \frac{x^2}{4} \right)$$

(9)

In order to find some connections between the parameters of the tree hypergeometric functions, it is convenient to write the parameters as follows

$$\frac{3}{4} = \frac{6}{8} \quad ; \quad \frac{5}{4} = \frac{6}{8} \quad ; \quad 1 = \frac{8}{8} \quad ; \quad \frac{3}{2} = \frac{12}{8}$$

(10)

Now the characteristic parameters becomes $5/8; 6/8; 7/8$ and $8/8$, for $\alpha = 1/4$; $1/2=2/4$; $3/4$ and $1=4/4$, respectively. In this way, all the a’s parameters have the same denominator, and the numerators are obtained adding one to each previous numerator.

In the case of parameter b, this parameter can be also written as $9/8; 5/4=10/8; 11/8$ and $3/2=12/8$ for $\alpha = 1/4$; $1/2=2/4$; $3/4$ and $1=4/4$, respectively. Here the rule is similar to case of the parameter ”a”.

The parameter c is equal for the four cases of $\alpha$ here considered, similarly the independent
variable for all, the case will be $x^2/4$. The factor with the hypergeometric is always $x^\alpha$.
The rule for $\alpha=1/3$, $2/3$ and $3/3=1$ is a little similar to the cases of $1/4$; $2/4$; $3/4$ and $4/4$, since
now it is obtained

$$G_{1/\beta}(x) = \mathcal{L}^{-1} \left[ \frac{1}{s^{1/\beta} \sqrt{s^2 - 1}} \right] = \frac{3}{2} \Gamma(2/3) x^{1/3} \, _2F_1 \left( \frac{1}{2}, \frac{2}{3}; \frac{7}{6}; \frac{x^2}{4} \right)$$

and

$$G_{2/\beta}(x) = \mathcal{L}^{-1} \left[ \frac{1}{s^{2/\beta} \sqrt{s^2 - 1}} \right] = \frac{3}{2 \Gamma(2/3)} x^{2/3} \, _2F_1 \left( \frac{1}{2}, \frac{5}{6}; \frac{4}{3}; \frac{x^2}{4} \right)$$

Now the parameter $a$ and $b$ can be written as $a:2/3=4/6$; $5/6$ and $1=6/6$. The parameter $b$
will be $7/6$; $4/3=8/6$ and $3/2=9/6$, which correspond to $\alpha=1/3$; $2/3$ and $1=3/3$. The factor
to go with the hypergeometric function, the independent variable and the third parameter $c$
are obtained in similar to the cases of $\alpha=1/4$, $1/2$, $3/4$ and $1$.
Considering the two sets of values here analized it seems that the denominator of the parameters
$a$ and $b$, which are 6 and 8, can obtained as twice the denominator of $\alpha$.

3. Conclusion
Here it is shown how to determine the solution of some simple linear fractional differential
equation with some special functions as non homogeneous terms. The case of $I_0(x)$ is treated
in detail, however it seems that the procedure could be applied to other cases.
Here the fractional derivative of order $\alpha$ is considered to be between zero and one. The solutions
here obtained are always a combination of a fractional power of the independent variable and on
hypergeometric function. The fractional power was always the fractional derivative in the cases
here treated. Several values have been shown to determine the parameters of the hypergeometric
function.

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References
[1] H. M. Srivastava, R. K. Saxena, "Operators of fractional integration and their applications", Appl. Math.
Comput. 118, 1-52 (2001).
[2] Baojin Wang, Zhiyuan Liu, Shengo Eben Li and Scott-Jason Moure, "State-of-Charge Estimation for Lithium-
Ion Batteries Based on a Nonlinear Fractional Model", IEEE Trans. Control. Syst. Technol., 25, 3-11(2017).
[3] M. Ali Ozaslan, Emine Ozergin, "Some generating relations for extended hypergeometric functions via
generalized fractional derivative operator", Math. Comput. Modelling, 52, 1825-1833(2010).
[4] Igor Podlubny, "An introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of
their Solution and some of their Applications", (Academic Press, San Diego, Boston, New York, London,
Sydney, Tokyo, Toronto, 1999).