Faster Deterministic All Pairs Shortest Paths in CONGEST Model

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May 20, 2020

Abstract

We present a new deterministic algorithm for distributed weighted all pairs shortest paths (APSP) in both undirected and directed graphs. Our algorithm runs in $\tilde{O}(n^{4/3})$ rounds in the CONGEST models on graphs with arbitrary edge weights, and it improves on the previous $\tilde{O}(n^{3/2})$ bound of Agarwal et al. [2]. The main components of our new algorithm are a new faster technique for constructing blocker set deterministically and a new pipelined method for deterministically propagating distance values from source nodes to the blocker set nodes in the network. Both of these techniques have potential applications to other distributed algorithms.

Our new deterministic algorithm for computing blocker set adapts the NC approximate hypergraph set cover algorithm in [4] to the distributed construction of a blocker set. It follows the two-step process of first designing a randomized algorithm that uses only pairwise independence, and then derandomizes this algorithm using a sample space of linear size. This algorithm runs in almost the same number of rounds as the initial step in our APSP algorithm that computes $h$-hops shortest paths. This result significantly improves on the deterministic blocker set algorithms in [2, 1] by removing an additional $n \cdot |Q|$ term in the round bound, where $Q$ is the blocker set.

The other new component in our APSP algorithm is a deterministic pipelined approach to propagate distance values from source nodes to blocker nodes. We use a simple natural round-robin method for this step, and we show using a suitable progress measure that it achieve the $\tilde{O}(n^{4/3})$ bound on the number of rounds. It appears that the standard deterministic methods for efficiently broadcasting multiple values, and for sending or receiving messages using the routing schedule in an undirected APSP algorithm [12, 15] do not apply to this setting.

1 Introduction

We study the computation of all pairs shortest path (APSP) in the widely-used CONGEST model of distributed computing (see, e.g., [2, 8, 13, 16]). In the CONGEST model (described in Section 1.1), the input is a directed (or undirected) graph $G = (V, E)$ and the distributed computation occurs at the nodes in this graph. The output of the APSP problem is to compute at each node $v$, the shortest path distances from every source node to $v$ in the network. We assume an arbitrary

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1We will refer to a vertex as a source if we compute shortest paths from that vertex.
Table 1: Table comparing our new results with previous known results for exact weighted APSP problem.

| Author                        | Arbitrary/ Integer weights | Randomized/ Deterministic | Undirected/ (Directed & Undirected) | Round Complexity |
|-------------------------------|-----------------------------|---------------------------|-------------------------------------|------------------|
| Huang et al. [13]             | Integer                     | Randomized                | Directed & Undirected               | $\tilde{O}(n^{5/4})$ |
| Elkin [8]                     | Arbitrary                    | Randomized                | Undirected                          | $\tilde{O}(n^{5/3})$ |
| Agarwal et al. [2]            | Arbitrary                    | Deterministic             | Directed & Undirected               | $\tilde{O}(n^{3/2})$ |
| Agarwal & Ramachandran [1]    | Integer                     | Deterministic             | Directed & Undirected               | $\tilde{O}(n^{5/4} \cdot W^{1/4})$ (W = max edge wt) |
| Bernstein & Nanongkai [5]     | Arbitrary                    | Randomized                | Directed & Undirected               | $\tilde{O}(n^{5/4} \cdot \Delta^{1/3})$ (Δ = max SP distance) |
| This Paper                    | Arbitrary                    | Deterministic             | Directed & Undirected               | $\tilde{O}(n^{4/3})$ |

non-negative weight on each edge, and we let $|V| = n$. In this paper we consider the computation of exact (and not approximate) shortest paths.

**Overview of our contributions.** In this paper we present a $\tilde{O}(n^{4/3})$ round deterministic algorithm for the weighted APSP problem. Table 1 lists earlier results for this problem [8, 13, 2, 1, 5]. All of these results as well as our new result can handle zero weight edges, and these algorithms are qualitatively different from algorithms for unweighted APSP.

Our algorithm follows the general 3-phase strategy initiated by Ullman and Yannakakis [20] for parallel computation of path problems in directed graphs:

1. Compute $h$-hop shortest paths for each source for a suitable value of $h$. (An $h$-hop path is a path that contains at most $h$ edges.)

2. Find a small blocker set $Q$ that intersects all $h$-hop paths computed in Step 1. (With randomization, this step is very simple: a random sample of the vertices of size $O((n/h) \cdot \log n)$ satisfies this property w.h.p. in $n$.)

3. Compute shortest paths between all pairs of vertices within $Q$. Then, use this information and the $h$-hop trees from Step 1 in a suitable algorithm to compute the APSP output at each node in $V$.

**CONGEST** directed APSP algorithms that fall in this framework include the randomized algorithm in Huang et al. [13] that runs in $\tilde{O}(n^{5/4})$ rounds for polynomial integer edge-weights, the deterministic algorithm in Agarwal et al. [2] that runs in $\tilde{O}(n^{3/2})$ rounds for arbitrary edge-weights, and the deterministic algorithm in Agarwal and Ramachandran [1] that improves on [2] for moderate integer edge-weights.

Our new deterministic algorithm directly improves on [2]. The algorithm in [2] computes Step 1 in $O(n \cdot h)$ rounds by running the distributed Bellman-Ford algorithm for $h$ hops from each source. Our algorithm leaves Step 1 unchanged from [2], but it improves on both Step 2 and Step 3. Our improved methods for implementing Steps 2 and 3 are the main technical contributions of this paper. We list them below, followed by an informal description of each of them.

1. A new deterministic algorithm for computing backer set (Algorithms 2, 7). The ideas are derived from the Berger et al.’s NC algorithm for finding a small set cover in a hypergraph [4],
and the result is an algorithm that significantly improves on the blocker set algorithm in Agarwal et al. [2] by removing an additional $n \cdot |Q|$ term in the round bound, where $Q$ is the blocker set.

2. A deterministic pipelined algorithm for propagating distance values from source nodes to blocker nodes (Algorithm 9). This algorithm deterministically propagates $\tilde{O}(n^{5/3})$ distance values from $n$ sources to $\tilde{O}(n^{2/3})$ blocker nodes in $\tilde{O}(n^{4/3})$ rounds, when the congestion at any node is at most $\tilde{O}(n^{4/3})$. Prior to this work, no deterministic algorithm was known that can implement this step in less than $n^{5/3}$ rounds.

(1) Deterministic Algorithm for Computing a Blocker Set (Step 2). For Step 2, [2] gives a deterministic algorithm that greedily chooses vertices to add to $Q$ at the cost of $O(n)$ rounds per vertex added, for the cleanup cost for removing paths that are covered by this newly chosen vertex; this is after an initial start-up cost of $O(n \cdot h)$. This gives an overall cost of $O(nh + nq)$ for Step 2, where $q = |Q| = O((n/h) \cdot \log n)$. Our new contribution is to construct $Q$ in a sequence of $\text{polylog}(n)$ steps, where each step adds several vertices to $Q$. Our method incurs a cleanup cost of $O(|S| \cdot h)$ rounds per step after an initial start-up cost of $O(|S| \cdot h) rounds for an arbitrary source set $S$, thereby removing the dependence on $q$ from this bound. ($S = V$ gives the standard setting used in previous APSP algorithms.) We achieve this by framing the computation of a small blocker set as an approximate set cover problem on a related hypergraph. We then adapt the efficient NC algorithm in Berger et al. [4] for computing an approximate minimum set cover in a hypergraph to an $\tilde{O}(|S| \cdot h)$-round CONGEST algorithm. As in [4] this involves two main parts. We first give a randomized $\tilde{O}(|S| \cdot h)$-round algorithm that computes a blocker set of expected size $\tilde{O}(n/h)$ using only pairwise independent random variables. We then derandomize this algorithm, again with an $\tilde{O}(|S| \cdot h)$-round algorithm.

(2) Deterministic Pipelined Algorithm for Propagating Distance Values (Step 3). For Step 3, [2] gave a deterministic $O(n \cdot q)$-round algorithm, and [13] gave a randomized $\tilde{O}(n \cdot \sqrt{q} + n \cdot \sqrt{h})$-round algorithm. We replace the $n \cdot \sqrt{h}$ randomized algorithm used in [13] with a simple $n \cdot h$ round algorithm (similar to Step 1). The randomized $O(n \cdot \sqrt{q})$ method in [13] computes the reversed $q$-sink shortest paths problem that appears to use randomization in a crucial manner, by invoking the randomized scheduling result of Ghaffari [9], which allows multiple algorithms to run concurrently in $O(d + c \cdot \log n)$ rounds, where $d$ bounds the dilation of any of the concurrent algorithms and $c$ bounds the congestion on any edge when considering all algorithms. It is known that this result in [9] cannot be derandomized in a completely general setting. For Step 3, our contribution is to give a deterministic $\tilde{O}(n \cdot \sqrt{q})$-round algorithm for the reversed $q$-sink shortest paths problem. Our algorithm uses a simple round-robin pipelined approach. To obtain the desired round bound we rephrase the algorithm to work in frames which allows us to establish suitable progress in the pipelining to show that it terminates in $\tilde{O}(n \cdot \sqrt{q})$ rounds. We note that the standard known results on efficiently broadcasting multiple values, and on sending or receiving messages using the routing schedule in an undirected APSP algorithm [12, 19, 15] do not apply to this setting.

Finally, we obtain the $\tilde{O}(n^{4/3})$ bound on the number of rounds by balancing the $\tilde{O}(nh)$ bound for Steps 1 and 2 with the $\tilde{O}(n \cdot \sqrt{q})$ bound for the reversed $q$-sink shortest path problem, as stated in the following theorem.

**Theorem 1.1.** There is a deterministic distributed algorithm that computes APSP on an $n$-node graph with arbitrary nonnegative edge-weights, directed or undirected, in $\tilde{O}(n^{4/3})$ rounds.
Algorithm 1 Overall APSP Algorithm

Input: number of hops $h = n^{1/3}$

1: Compute $h$-CSSSP for set $V$ using the algorithm in [1].
2: Compute a blocker set $Q$ of size $\tilde{O}(n/h)$ for the $h$-CSSSP computed in Step 1 (described in Section 3).
3: For each $c \in Q$ in sequence: Compute $h$-in-SSSP rooted at $c$.
4: For each $c \in Q$ in sequence: Broadcast $ID(c)$ and the shortest path distance value $\delta_h(c, c')$ for each $c' \in Q$.
5: Local Step at node $x \in V$: For each $c \in Q$ compute the shortest path distance values $\delta(x, c)$ using the distance values received in Step 4.
6: Run Alg. 8 and 9 described in Sec. 4 to propagate each distance value $\delta(x, c)$ from source $x \in V$ to blocker node $c \in Q$.
7: For each $x \in V$ in sequence: Compute extended $h$-hop shortest paths starting from every $c \in Q$ using Bellman-Ford algorithm (described in Section 5).

Theorem 1.1 improves on prior results for deterministic APSP on weighted graphs in the CONGEST model. If randomization is allowed, the very recent result in [5] gives an $\tilde{O}(n)$-round randomized algorithm, which is close to the known lower bound of $\Omega(n)$ rounds [6], that holds even for unweighted APSP.

**Derandomizing Distributed Algorithms.** The method of conditional expectations has been used for derandomizing randomized distributed algorithms in [7, 10]. Censor-Hillel et al. [7] semi-formalized a template of combining bounded independence with the method of conditional expectation for derandomizing an algorithm for computing a maximal independent set in the distributed setting. For unweighted graphs, Ghaffari and Kuhn [10] use a special case of the hitting set problem in a bipartite graph along with a network decomposition technique to obtain deterministic distributed algorithms for constructing certain types of spanners, small dominating sets, etc.

Instead of using the method of conditional expectations, our blocker set algorithm in Section 3.2 first gives an efficient distributed randomized algorithm for the problem which uses only pairwise independence. It uses a linear-sized sample space for generating pairwise independent random variables and then an aggregation of suitable parameters of sample point values to derandomize our randomized blocker set algorithm.

**Roadmap.** In Section 2 we present our overall APSP algorithm. Section 3 sketches our blocker set algorithm and Section 4 gives our pipelined algorithm for the reversed $q$-sink shortest path problem. Further details on all of our results are in Appendix A.

### 1.1 Congest Model

In the CONGEST model, there are $n$ independent processors interconnected in a network by bounded-bandwidth links. We refer to these processors as nodes and the links as edges. This network is modeled by graph $G = (V, E)$ where $V$ refer to the set of processors and $E$ refer to the set of links between the processors. Here $|V| = n$ and $|E| = m$.

Each node is assigned a unique ID between 1 and $\text{poly}(n)$ and has infinite computational power. Each node has limited topological knowledge and only knows about its incident edges. For the
weighted APSP problem we consider, each edge has an arbitrary real weight. Also if the edges are directed, the corresponding communication channels are bidirectional and hence the communication network can be represented by the underlying undirected graph $U_G$ of $G$ (as in [13, 19, 11]).

The computation proceeds in rounds. In each round each processor can send a constant number of words along each outgoing edge, and it receives the messages sent to it in the previous round. The CONGEST model normally assumes that a word has $O(\log n)$ bits. Since we allow arbitrary edge-weights, here we assume that a constant number of node ids, edge-weights, and distance values can be sent along every edge in every round (similar assumptions are made in [5, 2, 8]). The model allows a node to send different message along different edges though we do not need this feature in our algorithm.

The performance of an algorithm in the CONGEST model is measured by its round complexity, which is the worst-case number of rounds of distributed communication. As noted earlier, for the APSP problem, each node in the network needs to compute its shortest path distance from every other node as well as the last edge on each such shortest path.

## 2 Overall APSP Algorithm

Algorithm 1 gives our overall APSP algorithm. In Step 1 we use the (simple) $O(n \cdot h)$-round algorithm in [1] to compute an $h$-hop Consistent SSSP Collection (or $h$-CSSSP for short) for the vertex set $V$, defined as follows (and described in detail in Section A.2). Here, $\delta(u, v)$ denotes the shortest path distance from $u$ to $v$ and $\delta_h(u, v)$ denotes the $h$-hop shortest path distance from $u$ to $v$.

**Definition 2.1 (CSSSP [1]).** Let $H$ be a collection of rooted trees of height $h$ for a set of sources $S \subseteq V$ in a graph $G = (V, E)$. Then $H$ is an $h$-hop CSSSP collection (or simply an $h$-CSSSP) if for every $u, v \in V$ the path from $u$ to $v$ is the same in each of the trees in $H$ (in which such a path exists), and is the $h$-hop shortest path from $u$ to $v$ in the $h$-hop tree $T_u$ rooted at $u$. Further, each $T_u$ contains every vertex $v$ that has a path with at most $h$ hops from $u$ in $G$ that has distance $\delta(u, v)$.

The advantage of using $h$-CSSSP instead of other types of $h$-hop shortest paths is that the trees in an $h$-CSSSP create a consistent collection of paths across all trees in the collection, i.e. a path from $u$ to $v$ is same in all trees in the CSSSP collection $C$ (in which such a path exists). We exploit this useful property of CSSSPs throughout this paper.

Step 2 computes a blocker set $Q$, which is defined as follows:

**Definition 2.2 (Blocker Set [14, 2]).** Let $H$ be a collection of rooted $h$-hop trees for a set of vertices $S \subseteq V$ in a graph $G = (V, E)$. A set $Q \subseteq V$ is a blocker set for $H$ if every root to leaf path of length $h$ in every tree in $H$ contains a node in $Q$. Each node in $Q$ is called a blocker node for $H$.

Our deterministic blocker set algorithm for Step 2 is completely different from the blocker set algorithms in [2, 1] with significant improvement in the round complexity. We describe this algorithm in Section 3. Our blocker set algorithm is based on the NC approximate Set Cover algorithm of Berger et al. [4] and runs in $\tilde{O}(|S| \cdot h)$ rounds, where $S$ is the set of vertices from which we want to compute the shortest paths. Previous deterministic blocker set algorithms in [2, 1] have an additional $\tilde{O}(n \cdot |Q|)$ term in the round complexity.
In Step 3 we compute, for each $c \in Q$, the $h$-hop in-SSSP rooted at $c$, which is the set of in-coming $h$-hop shortest paths ending at node $c$. We can compute these $h$-hop in-SSSPs in $O(h)$ rounds per source using Bellman-Ford algorithm [3]. In Step 4 every blocker node $c \in Q$ broadcasts its ID and the corresponding $h$-hop shortest path distance values $\delta_h(c, c')$ for every $c' \in Q$. Step 5 is a local computation step where every node $x$ computes its shortest path distances $\delta(x, c)$ to every $c \in Q$ using the shortest path distance values it computed and received in Steps 3 and 4 respectively.

In Step 6 every node $x$ wants to send each shortest path distance value $\delta(x, c)$ it computed in Step 5 to blocker node $c \in Q$. This is the reversed $q$-source shortest path problem, where $q = |Q|$, and is the other crucial step in our APSP algorithm. This step requires sending $\tilde{O}(n^{5/3})$ different distance values to $\tilde{O}(n^{2/3})$ different blocker nodes (using $|Q| = \tilde{O}(n^{2/3})$). A trivial solution is to broadcast all these messages in the network, resulting in a round complexity of $\tilde{O}(n^{5/3})$ rounds. However this is the only method known so far to implement this step deterministically. In Sec. 4 we give a pipelined algorithm for implementing this step more efficiently in $\tilde{O}(n^{4/3})$ rounds. After the execution of Step 6 every blocker node $c \in Q$ knows its shortest path distance from every node $x \in V$.

Finally, in Step 7 for every $x \in V$, we run Bellman-Ford algorithm for $h$ hops with distance values $\delta(x, c)$ used as the initialization values at every blocker node $c \in Q$. These constructed paths are also known as extended $h$-hop shortest paths [13]. After this step, each $t \in V$ knows the shortest path distance value $\delta(x, t)$ from every $x \in V$, which gives the desired APSP output. We describe Step 7 in Section 5. With these results in place we can now prove Theorem 1.1, whose statement we reproduce here for convenience.

**Theorem.** 1.1 There is a deterministic distributed algorithm that computes APSP on an $n$-node graph with arbitrary nonnegative edge-weights, directed or undirected, in $\tilde{O}(n^{4/3})$ rounds.

**Proof.** Fix a pair of nodes $x$ and $t$. If the shortest path from $x$ to $t$ has less than $h$ hops, then $\delta(x, t) = \delta_h(x, t)$ and the correctness is straightforward (see Lemma A.4).

Otherwise, we can divide the shortest path from $x$ to $t$ into subpaths $x$ to $c_1$, $c_1$ to $c_2$, ..., $c_l$ to $t$ where $c_i \in Q$ for $1 \leq i \leq l$ and each of these subpaths have hop-length at most $h$. Since $x$ knows $\delta_h(x, c_1)$ from Step 3 and $\delta_h(c_i, c_{i+1})$ distance values from Step 4, it can correctly compute $\delta(x, c_l)$ distance value in Step 5. And from Lemmas 4.1 and 4.4, $c_l$ knows the distance value $\delta(x, c_l)$ after Step 6. Since the shortest path from $c_l$ to $t$ has hop-length at most $h$, from Lemma 5.1 it will compute $\delta(x, t)$ in Step 7.

Step 1 runs in $O(nh) = O(n^{4/3})$ rounds [1] (Lemma A.4). In Section 3, we will give an $\tilde{O}(nh) = \tilde{O}(n^{4/3})$ rounds algorithm to compute a blocker set of size $q = \tilde{O}(n/h) = \tilde{O}(n^{2/3})$ (Step 2). Step 3 takes $O(|Q| \cdot h) = \tilde{O}(n)$ rounds using Bellman-Ford algorithm (Lemma A.4). Since $|Q|^2 = \tilde{O}(n^2/h^2) = \tilde{O}(n^{4/3})$, Step 4 takes $\tilde{O}(n^{4/3})$ rounds (see Lemma A.2). Step 5 is local computation and has no communication. From Lemmas 4.1 and 4.5, Step 6 takes $\tilde{O}(n^{4/3})$ rounds and Step 7 can be computed in $O(nh) = O(n^{4/3})$ rounds using Lemma 5.1. Hence the overall algorithm runs in $\tilde{O}(n^{4/3})$ rounds.

### 3 Computing a Blocker Set

In this section we describe our algorithm to compute a small blocker set. We frame this problem as that of finding a small set cover in an associated hypergraph. We then adapt the efficient NC
Given a hypergraph $H = (V, F)$, a subset of vertices $R$ is a set cover for $H$ if $R$ contains at least one vertex in every hyperedge in $F$. Computing a set cover of minimum size is NP-hard. Berger et al. [4] gave an efficient NC algorithm to compute an $O(\log n)$ approximation to the minimum set cover.

We now briefly describe the set cover algorithm of Berger et al. [4]. The algorithm runs in phases, which are further subdivided into subphases, in order to construct a suitable blocker set $Q$. In phase $i$ only vertices with degree between $(1 + \epsilon)^{-1}$ and $(1 + \epsilon)^i$ are considered for selection (let $V_i$ be this set of vertices), and subphase $j$ consists of only those hyperedges that have at least one vertex in $V_i$.

In each subphase $j$, the algorithm performs a series of selection steps such that all the hyperedges considered in subphase $j$ are covered by the vertices added to $Q$. This process is repeated for all phases and their subphases and the set cover is then constructed by taking the union of all the vertices selected by these selection steps across all phases.

We map the problem of computing a minimum blocker set for an $h$-CSSSP collection $C$ in a graph $G = (V, E)$ to the minimum set cover problem in a hypergraph $H = (V, F)$ as follows. The vertex set $V$ remains the vertex set of $G$ and each edge in $F$ consists of the vertices in a root-to-leaf path in a tree in $C$. This hypergraph has $n$ vertices and at most $n \cdot |S|$ edges, where $S$ is the number of sources (i.e., trees) in $C$. Each edge in $F$ has exactly $h$ vertices (since we do not need to cover

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We use pairwise independence extensively in our analysis of the randomized blocker set algorithm, specifically lemmas A.16, A.17 where we use pairwise independence to get bounds for the terms of the form $E[X_v \cdot X_{v'}]$. This analysis needs pairwise independence as does the derandomization algorithm in Section 3.2.
paths that have less than \(h\) hops). We now use this mapping to rephrase the algorithm in [4] in our setting, and we derive an \(O(|S| \cdot h)\)-round randomized algorithm to compute a blocker set of expected size within a \(O(\log n)\) factor of the optimal size, using only pairwise independent random variables. Since we know there exists a blocker set of size \(O((n/h) \cdot \log n)\) (which is constructed in [2, 1]) the size of the blocker set constructed by this randomized algorithm is \(\tilde{O}(n/h)\).

Our randomized blocker set method is in Algorithm 2. Table 2 presents the notation we use for this section. In Step 1 for each node \(v\) we compute \(score(v)\), the number of \(h\)-hop shortest paths in CSSSP collection \(C\) that contain node \(v\). This can be done in \(O(|S| \cdot h)\) rounds for all nodes \(v \in V\) using Algorithm 3 in [2]. Our algorithm proceeds in stages from \(i = \log_{1+\epsilon} n^2\) down to 2 (Steps 2-16), where \(\epsilon\) is a small positive constant \(\leq 1/12\), such that at the start of stage \(i\), all nodes in \(V\) have score value at most \((1 + \epsilon)^i\) and in stage \(i\) we focus on \(V_i\), the set of nodes \(v\) with score value greater than \((1 + \epsilon)^{i-1}\). (This ensures that the nodes that are added to the blocker set have their score values near the maximum score value). Let \(P_i\) be the set of paths in \(C\) that contain a vertex in \(V_i\) and let \(P_i^v\) be the set of paths in \(P_i\) with \(v\) as the leaf node. These sets are readily computed in \(O(|S| \cdot h)\)-rounds (Sec. 3.1.1).

| \(C\) | \(h\)-CSSSP collection |
| \(S\) | set of source nodes in \(C\) |
| \(h\) | number of hops in a path |
| \(n\) | number of nodes |
| \(\epsilon, \delta\) | positive constants \(\leq 1/12\) |
| \(Q\) | blocker set (being constructed) |

\[score(v)\] number of root-to-leaf paths in \(C\) that contain \(v\) (local var. at \(v\))

\[V_i\] set of nodes \(v\) with \(score(v) \geq (1 + \epsilon)^{i-1}\)

\[P_i\] set of paths in \(C\) with at least one node in \(V_i\)

\[P_{ij}\] set of paths in \(P_i\) with at least \((1 + \epsilon)^{i-1}\) nodes in \(V_i\)

\[P_{ij}^v\] set of paths in \(P_{ij}\) with \(v\) as the leaf node

\[score^{ij}(v)\] number of paths in \(P_{ij}\) that contain \(v\) (local var. at \(v\))

Similar to [4], in order to ensure that the average number of paths covered by the newly chosen blocker nodes is near the maximum score value, we further divide our algorithm for stage \(i\) into a sequence of \(\log_{1+\epsilon} n^{1/3}\) phases, where in each phase \(j\) we focus on the paths in \(P_i\) with at least \((1 + \epsilon)^{j-1}\) nodes in \(V_i\). We call this set of paths \(P_{ij}\) and let \(P_{ij}^v\) be the set of paths in \(P_{ij}\) with \(v\) as the leaf node. We maintain that at the start of phase \(j\), every path in \(P_i\) has at most \((1 + \epsilon)^j\) nodes in \(V_i\). We now describe our algorithm for phase \(j\) (Steps 5-16). The algorithm for phase \(j\) consists of a series of selection steps (Steps 6-16) (similar to [4]) which are performed until there are no more paths in \(P_{ij}\).

Now we describe how we select nodes to add to blocker set \(Q\). Let \(\delta\) be some fixed positive constant less than or equal to \(1/12\). In Step 9 we check if there exists a node \(v\) which covers at least \(\delta^3/(1 + \epsilon)^i\) fraction of paths in \(P_{ij}\) and if so, we add this node to the blocker set in Step 10. In case of multiple such nodes, we pick the one with the maximum \(score^{ij}\) value and break ties using node IDs. Otherwise in Step 12, we randomly pick every node with probability \(\delta/(1 + \epsilon)^i\), pairwise independently, and form a set \(A\). In Step 14 we check if \(A\) is a good set, otherwise we try again and form a new set \(A\) in Step 12. As in [4] we define the notion of a good set as given below and we will later show that \(A\) is a good set with probability at least \(1/8\).

**Definition 3.1.** A set of nodes \(A \subseteq V_i\) is a good set if \(A\) covers at least \(|A| \cdot (1 + \epsilon)^i \cdot (1 - 3\delta - \epsilon)\) paths in \(P_i\) and at least a \(\delta/2\) fraction of paths from \(P_{ij}\).
Algorithm 3  \textsc{Compute-}P_i:  Algorithm for computing paths in $P_i$ for source $x$ at node $v$

\begin{algorithm}
\KwInput{$V_i$; $h$: number of hops; $T_x$: tree for source $x$}
1. \textbf{(Round 0):} if $v \in V_i$ then set \textsf{flag} $\leftarrow$ \text{true} else \textsf{flag} $\leftarrow$ \text{false}
2. \textbf{Round} $h \geq r > 0$:
3. \textbf{Send:} if $r = h_x(v) + 1$ then send \langle \textsf{flag} \rangle to all children
4. \textbf{receive} [lines 5-8]:
5. if $r = h_x(v)$ then
6. let $M$ be the incoming message to $v$
7. let the sender be $w$ and let $M = \langle \textsf{flag}_w \rangle$
8. if $w$ is a parent of $v$ in $T_x$ then \textsf{flag} $\leftarrow$ \textsf{flag} $\lor$ \textsf{flag}_w
9. \textbf{Local Step} at $v$: if $v$ is a leaf node and \textsf{flag} = \text{true} then the path from $x$ to $v$ is in $P_i$.
\end{algorithm}

Before the next selection step, we remove the paths covered by these newly chosen nodes from the collection $C$ along with recomputing the score values and sets $V_i$ and $P_i$ (Steps 15-16).

### 3.1.1 Helper Algorithms for Randomized Blocker Set Algorithm

Here we describe the helper algorithms for Algorithm 2.

**Algorithms for Computing $V_i$ and $P_i$.** Here we describe our algorithm for computing Steps 3 and 4 of Algorithm 2, which computes the set $V_i$ and identifies which paths belong to $P_i$ respectively. Since every node with score value greater than or equal to $(1 + \epsilon)^{i-1}$ belongs to $V_i$, computing $V_i$ is quite trivial. And to determine if a path $p$ belong to $P_i$, we only need to check if one of the nodes in $p$ is in $V_i$.

Our algorithm for computing $V_i$ works as follows: Every node $v$ checks if its score value is greater than or equal to $(1 + \epsilon)^{i-1}$ and if so, it broadcast its ID to every other node. The set $V_i$ is then constructed by including the IDs of all such nodes. Since there are at most $n$ messages involved in the broadcast step, this algorithm takes $O(n)$ rounds. This leads to the following lemma.

**Lemma 3.2.** Given the score($v$) values for every $v \in V$, the set $V_i$ can be constructed in $O(n)$ rounds.

We now describe our algorithm for computing $P_i$. Fix a source node $x \in V$. In Round 0 $x$ initializes \textsf{flag} to \text{true} if it belongs to $V_i$, otherwise set it to \text{false} (Step 1). It then sends this \textsf{flag} to its children in next round (Step 3). In round $r \geq 1$, a node $v$ that is $r$ hops away from $x$ receives the \textsf{flag} from its parent (Steps 5-8) and $v$ updates the \textsf{flag} value in Step 8 (set it to true if $v \in V_i$) and send it to its children in $x$’s tree in round $r + 1$ (Step 3).

**Lemma 3.3.** Using \textsc{Compute-}P_i (Algorithm 3), $P_i$ can be computed in $O(h)$ rounds per source node.

**Proof.** Fix a path $p$ from source $x$ to leaf node $v$. After $h$ rounds, $v$ will know if any node in $p$ belongs to $V_i$ (using the \textsf{flag} value it received in Steps 5-8).

The algorithm takes $h$ rounds per source $x$ and thus $P_i$ can be computed in $O(|S| \cdot h)$ rounds in total (since we need to run the algorithm for every source $x$).
Algorithm 4 COMPUTE-$P_{ij}$: Algorithm for computing paths in $P_{ij}$ for source $x$ at node $v$

Input: $V_i$; $h$: number of hops; $T_x$: tree for source $x$
1: (Round 0): if $v \in V_i$ set $\beta \leftarrow 1$ else $\beta \leftarrow 0$
2: Round $h \geq r > 0$:
3: Send: if $r = h_x(v) + 1$ then send $\langle \beta \rangle$ to all children
4: receive [lines 5-8]:
5: if $r = h_x(v)$ then
6: let $\mathcal{M}$ be the incoming message to $v$
7: let the sender be $w$ and let $M = \langle \beta_w \rangle$ and
8: if $w$ is a parent of $v$ in $T_x$ then $\beta \leftarrow \beta + \beta_w$
9: Local Step at $v$: if $v$ is a leaf node and $\beta \geq (1 + \epsilon)^{j-1}$ then the path from $x$ to $v$ is in $P_{ij}$.

Algorithm 5 COMPUTE-$|P_{ij}|$

Input: $P_{ij}^v$: paths in $P_{ij}$ with $v$ as the leaf node
1: Local Step at $v \in V_i$: set $\alpha_{P_{ij}^v} \leftarrow |P_{ij}^v|$
2: For each $v \in V$: Broadcast $ID(v)$ and the value $\alpha_{P_{ij}^v}$.
3: Local Step at $v \in V$: $|P_{ij}| \leftarrow \sum_{v' \in V} \alpha_{P_{ij}^v}$

Algorithm for Computing $P_{ij}$. Here we describe our algorithm for computing Step 7(a) of Algorithm 2, which identifies the paths in $P_i$ that also belong to $P_{ij}$. Since every path in $P_{ij}$ has at least $(1 + \epsilon)^{j-1}$ nodes from $V_i$, for each path $p$ in $P_{ij}$ we need to determine the number of nodes in $p$ that belong to $V_i$. We do this by counting the number of nodes that are in $V_i$, starting from root to leaf node.

Our algorithm works as follows: Fix a source node $x \in V$. In Round 0 $x$ initializes $\beta$ value to 1 if it belongs to $V_i$, otherwise set it to 0 (Step 1). It then sends this $\beta$ value to its children in next round (Step 3). In round $r \geq 1$, a node $v$ that is $r$ hops away from $x$ receives the $\beta$ value from its parent (Steps 5-8) and $v$ updates the $\beta$ value in Step 8 (increment it by 1 if $v \in V_i$) and send it to its children in $x$’s tree in round $r + 1$ (Step 3).

Lemma 3.4. Using COMPUTE-$P_{ij}$ (Algorithm 4), $P_{ij}$ can be computed in $O(h)$ rounds per source node.

Proof. Fix a path $p$ from source $x$ to leaf node $v$. After $h$ rounds, $v$ will know the number of nodes that belong to $V_i$ (using the $\beta$ values it received in Steps 5-8).

The algorithm takes $h$ rounds per source $x$ and thus $P_{ij}$ can be computed in $O(|S| \cdot h)$ rounds in total (since we need to run the algorithm for every source $x$).

Algorithm for Computing $|P_{ij}|$. Algorithm 5 describes our algorithm for computing Step 7(b) of Algorithm 2, which computes the value of $|P_{ij}|$. Let $P_{ij}^v$ represents the set of paths $p$ in $P_{ij}$ with $v$ as the leaf node. Every node $v$ knows the set $P_{ij}^v$ after running the algorithm described in the previous section. Our algorithm works as follows: Every node $v$ first compute $|P_{ij}^v|$ (Step 1) and then broadcast this value in Step 2. Every node $v$ then compute $|P_{ij}|$ by summing up the values received in Step 2 (Step 3).

Lemma 3.5. COMPUTE-$|P_{ij}|$ (Algorithm 5) computes $|P_{ij}|$ in $O(n)$ rounds.
Algorithm 6 REMOVE-SUBTREES: Algorithm for Removing Subtrees rooted at \( z \in Z \) for source \( x \) at node \( v \)

Input: \( S \): set of sources; \( \mathcal{C} \): \( h \)-CSSSP collection for set \( S \)

1: (Round 0:) If \( v \in Z \) then send \( \langle \text{ID}(v) \rangle \) to all children in \( T_x \) and set \( \text{parent}_x(v) \) to NIL.
2: (Round \( r > 0 \)) If \( v \) received a message \( M \) in round \( r - 1 \) then set \( \text{parent}_x(v) \) to NIL and send \( M \) to all children in \( T_x \).

Proof. Steps 1 and 3 are local steps and involves no communication. Step 2 involves a broadcast of \( n \) messages and takes \( O(n) \) rounds using Lemma A.2.

Remove Subtrees rooted at \( z \in Z \). In this Section we describe a deterministic algorithm for implementing Step 15 of Algorithm 2, which removes subtrees rooted at nodes \( z \in Z \) from the trees in the given \( h \)-CSSSP collection \( \mathcal{C} \). This algorithm (Algorithm 6) is quite simple and works as follows: Fix a source \( x \) and let its corresponding tree in \( \mathcal{C} \) be \( T_x \). Every node \( z \in Z \) in \( T_x \) send its ID to all its children in \( T_x \) (Step 1). Every node \( v \) on receiving a message from its parent in \( T_x \), forwards it to all its children and set the parent pointer in \( T_x \) to NIL (Step 2).

Lemma 3.6. Given a source \( x \in S \) and tree \( T_x \in \mathcal{C} \), then REMOVE-SUBTREES (Algorithm 6) removes all subtrees rooted at \( z \in Z \) in \( T_x \).

Proof. Every \( z \in Z \) in \( T_x \) removes its parent pointer in \( T_x \) in Step 1. Any node \( v \in V \) that lies in the subtree rooted at a \( z \in Z \) in \( T_x \) would have received a message with \( \text{ID}(z) \) from its parent by \( h \) rounds (since height of \( T_x \) is at most \( h \)) and hence would have set its parent pointer to NIL in Step 2.

Lemma 3.7. REMOVE-SUBTREES (Algorithm 6) requires at most \( h \) rounds per source node \( x \in S \).

Proof. Since the height of \( T_x \) is at most \( h \), any node \( v \in V \) which lies in the subtree rooted at a \( z \in Z \) will receive the message from \( z \) by \( h \) rounds. This establishes the lemma.

3.1.2 Correctness of Algorithm 2

Similar to [4] we get the following Lemmas 3.8-3.10 which give us a bound on the number of selection steps and a bound on the size of \( Q \).

Lemma 3.8. The set \( A \) constructed in Step 12 is a good set with probability at least \( 1/8 \).

Lemma 3.9. The while loop in Steps 6-16 runs for at most \( O\left( \frac{\log^4 n}{n^{3/2}} \right) \) iterations in total.

We provide the full proof of Lemmas 3.8 and 3.9 in Sec A.4.

Lemma 3.10. The blocker set \( Q \) constructed by Algorithm 2 has size \( O(n \log n/h) \).

Proof. As shown in [14, 2] the size of the blocker set computed by an optimal greedy algorithm is \( \Theta\left( \frac{n \ln p}{h} \right) \), where \( p \) is the number of paths that need to be covered. We will now argue that the blocker set constructed by Algorithm 2 is at most a factor of \( \frac{1}{(1-3\delta - \epsilon)} \) larger than the greedy solution.
Algorithm 7 Deterministic Algorithm for picking good set $A$

Input: $h$: number of hops; $S$: set of source nodes; $C$: $h$-CSSSP collection; $X^{(\mu)}$: $\mu$-th vector in sample space
1: For each $x \in S$ in sequence: Collect at each $v$ the ids of the nodes on the path ending at leaf node $v$ in $T_x$ (using Alg. 4 in [2]). (This computes both $P_i^v$ and $P_{ij}^v$)
2: Compute BFS in-tree $T$ rooted at leader $l$.
3: Compute $\sigma^{(\mu)}_{P_i,u}$ and $\sigma^{(\mu)}_{P_{ij},u}$ terms locally at each $v \in V$, for each sample point $\mu$, and then using the pipelined algorithm in Sec. A.5, send these values to the leader $l$.
4: Local Step at $l$: For each $1 \leq \mu \leq n$, compute $\nu^{(\mu)}_{P_i}$ and $\nu^{(\mu)}_{P_{ij}}$. Let $\mu'$ be such that $X^{(\mu')}$ corresponds to a good set $A$.
5: Node $l$ broadcast $X^{(\mu')}$ values. (This corresponds to good set $A$)

thus showing that the constructed blocker set $Q$ has size at most $O \left( \frac{n \log p \cdot \frac{1}{(1-3\delta-\epsilon)}}{n} \right) = O \left( \frac{n \log n}{h} \right)$ since $p \leq n^2$ and $0 < \delta, \epsilon \leq \frac{1}{17}$.

The blocker set $Q$ constructed by Algorithm 2 has 2 types of nodes: (1) node $c$ added in Step 10, (2) set of nodes $A$ added in Step 12. Since the while loop in Steps 6-16 runs for at most $O \left( \frac{\log^3 n}{\delta^2 \epsilon^2} \right)$ iterations (by Lemma 3.9), hence there are at most $O \left( \frac{\log^3 n}{\delta^2 \epsilon^2} \right)$ nodes of type 1. Since $\frac{\log^3 n}{\delta^2 \epsilon^2} = o(\frac{n}{h})$, hence we only need to bound the number of nodes added in Steps 12-14.

Since $A$ is a good set, by Lemma 3.8 the number of paths covered by $A$ is at least $|A| \cdot (1+\epsilon)^i \cdot (1-3\delta-\epsilon)$, where $(1+\epsilon)^i$ is the maximum possible score value across all nodes in $V$ (in the current iteration). Since maximum possible score value is $(1+\epsilon)^i$, any greedy solution must add at least $|A| \cdot (1-3\delta-\epsilon)$ nodes in the blocker set to cover these paths. Hence the choice of $A$ is at most a factor of $\frac{1}{(1-3\delta-\epsilon)}$ larger than the greedy solution. This establishes the lemma. \qed

Lemma 3.11. Algorithm 2 computes the blocker set $Q$ in $\tilde{O}(|S| \cdot h/(\epsilon^2 \delta^3))$ rounds, in expectation.

Proof. Step 1 runs in $O(|S| \cdot h)$ rounds [2]. The for loop in Steps 2-16 runs for $\log_{1+\epsilon} n^2 = O \left( \log n / \epsilon \right)$ iterations. Each iteration takes $O \left( \frac{|S| \cdot h}{(\epsilon \delta^3)} \right)$ rounds in expectation: Step 4 is readily seen to run in $O(|S| \cdot h)$ rounds (Lemma 3.3). The inner for loop in Steps 5-16 runs for $\log_{1+\epsilon} h = O \left( \log n / \epsilon \right)$ iterations, with each iteration taking $O \left( \frac{|S| \cdot h}{\delta^2} \right)$ rounds in expectation (Lemma A.12). \qed

3.2 Deterministic Blocker Set Algorithm

The only place where randomization is used in Algorithm 2 is in Steps 12-14, where a good set $A$ (see Definition 3.1) is chosen. Fortunately, the $X_v$’s are pairwise-independent random variables, where $X_v = 1$ if $v \in A$ and 0 otherwise. We use an $O(n)$ size sample space [18, 17] for generating pairwise independent random variables and then find a good sample point (i.e., a good set $A$) in this $O(n)$-sized sample space in $O(|S| \cdot h + n)$ rounds. This $O(n)$ size sample space can be generated locally at each node by going over all 0-1 strings of length about $\log n$ using the techniques in [18, 17]. We provide more details about the construction of this sample space in Appendix A.3.

Algorithm 7, our derandomized algorithm, works as follows. Each vertex $v$ generates the sample set locally. Let $P_i^v$ and $P_{ij}^v$ denote the set of paths in $P_i$ and $P_{ij}$, respectively, that have $v$ as the leaf node. Initially every node $v$ determines these sets $P_i^v$ and $P_{ij}^v$, which can be done in $O(|S| \cdot h)$
sets. Then for each sample point \( \mu \), every node \( v \) locally computes the number of paths in sets \( P^v_i \) and \( P^v_{ij} \) covered by \( \mu \). Each vertex sends its computed values for all sample points to a leader node \( l \in V \) which then computes the total number of paths covered in both \( P_i \) and \( P_{ij} \) for every sample point \( \mu \) and picks one that satisfies the good set criterion (Definition 3.1). Such a set is guaranteed to exist from Lemma 3.8.

To compute sets \( P^v_i \) and \( P^v_{ij} \), we collect the ids of the vertices in each \( P_i \) at the leaf node of \( P_i \), for each tree \( T_x \) in turn, using the ANCESTORS algorithm in [2] (Step 1, Alg. 7). We then create an incoming BFS tree rooted at \( l \) (Step 2, Alg. 7). We assume that the \( X \) values are enumerated in order and every node knows this enumeration. Let \( X^{(\mu)} \) refer to the \( \mu \)-th vector in this enumeration and let \( \sigma_{P^v_i}^{(\mu)} \) and \( \sigma_{P^v_{ij}}^{(\mu)} \) refer to the number of paths covered by \( X^{(\mu)} \) in sets \( P^v_i \) and \( P^v_{ij} \) respectively. Similarly let \( \nu_{P^v_i}^{(\mu)} \) and \( \nu_{P^v_{ij}}^{(\mu)} \) refer to the total number of paths covered by \( X^{(\mu)} \) in sets \( P_i \) and \( P_{ij} \) respectively. In Step 3 (Alg. 7), the leader \( l \) receive sums of the \( \nu_{P^v_i,u} \) and \( \nu_{P^v_{ij},u} \) values for all sample points from the nodes \( u \) using the algorithm in Sec. A.5. The leader then is able to compute the number of paths covered in both \( P_i \) and \( P_{ij} \) for each \( \mu \) and then picks one that satisfies the good set criterion (Step 4, Alg. 7). It then broadcasts the corresponding \( X \) vector to every node in the network (Step 5, Alg. 7). Algorithm 7 gives the pseudocode for this algorithm.

**Lemma 3.12.** The leader node \( l \) can identify a good sample point \( X \in \{0,1\}^{|V|} \), and thus a good set \( A \) in \( \tilde{O}(|S| \cdot h + n) \) rounds.

Details of the steps in Alg. 7 and proof of Lemma 3.12 are in Sec. A.5. Let Algorithm 2' be the blocker set algorithm obtained after replacing Steps 12-14 in Algorithm 2 with the deterministic algorithm for generating a good set \( A \) (Algorithm 7). Lemma 3.12, together with Lemma 3.11, gives us the following Corollary.

**Corollary 3.13.** Algorithm 2' computes the blocker set \( Q \) deterministically in \( \tilde{O}(|S| \cdot h/\epsilon^2 3^3) \) rounds.

### 4 A \( \tilde{O}(n^{4/3}) \) Rounds Algorithm for Step 6 of Algorithm 1

In Step 6 of Algorithm 1, the goal is to send the distance values \( \delta(x,c) \) (which are already computed at node \( x \)) from source node \( x \in V \) to the corresponding blocker node \( c \). Since there are \( n \) sources and \( |Q| = \tilde{O}(n^{2/3}) \) blocker nodes, this step can be implemented in \( \tilde{O}(n^{5/3}) \) rounds using all-to-all broadcast (Lemma A.2). One could conjecture that the techniques in [12, 15] could be used to send these \( \tilde{O}(n^{5/3}) \) messages from the source nodes in \( V \) to the blocker nodes by constructing trees rooted at each \( c \). However, it is not clear how these methods can distribute the \( \tilde{O}(n^{5/3}) \) different source-destination messages in \( o(n^{5/3}) \) rounds.

We now describe a method to implement this step more efficiently in \( \tilde{O}(n^{4/3}) \) rounds deterministically. A randomized \( \tilde{O}(n^{4/3}) \)-round algorithm for this problem is given in Huang et al. [13]. Our algorithm uses the concept of bottleneck nodes from that result but is otherwise quite different.

Our algorithm is divided into two cases: (i) when \( \text{hops}(x,c) > n^{2/3} \) and, (ii) when \( \text{hops}(x,c) \leq n^{2/3} \) (\( \text{hops}(x,c) \) denotes the number of edges on the shortest path from \( x \) to \( c \)).
Algorithm 8 Compute $\delta(x,c)$ at $c$: when $\text{hops}(x,c) > n^{2/3}$

| Step | Description |
|------|-------------|
| 1:   | Compute $n^{2/3}$-in-CSSSP for source set $Q$ using the algorithm in [1]. |
| 2:   | Compute a blocker set $Q'$ of size $\tilde{O}(n/n^{2/3}) = \tilde{O}(n^{1/3})$ for the $n^{2/3}$-CSSSP computed in Step 1 using the blocker set algorithm described in Section 3. |
| 3:   | For each $c' \in Q'$ in sequence: Compute in-SSSP and out-SSSP rooted at $c'$ using Bellman-Ford algorithm. |
| 4:   | For each $x \in V$ in sequence: Broadcast $ID(x)$ and the shortest path distance values $\delta(x,c')$ for each $c' \in Q'$. |
| 5:   | Local Step at node $c \in Q$: For each $x \in V$ compute the shortest path distance value $\delta(x,c)$ using the $\delta(x,c')$ distance values received in Step 4 and the $\delta(c',c)$ distance values computed in Step 3. |

4.1 Case (i): $\text{hops}(x,c) > n^{2/3}$

Algorithm 8 describes our algorithm for this case. We first construct an $n^{2/3}$-in-CSSSP collection (i.e., CSSSP in-trees) using the blocker set $Q$ as the source set (Step 1, Alg. 8). In Step 2 (Alg. 8) we construct a blocker set $Q'$ of size $\tilde{O}(n^{1/3})$ for this CSSSP collection using deterministic Algorithm 2’ in Sections 3 and 3.2. Then for each $c' \in Q'$ we construct the incoming and outgoing shortest path tree rooted at $c'$ (Step 3, Alg. 8). In Step 4 (Alg. 8), every source $x \in V$ broadcasts the distance value $\delta(x,c')$ for each $c' \in Q'$. The lemma below shows that each $c \in Q$ can determine the $\delta(x,c)$ values for all $x$ for which $\text{hops}(x,c) > n^{2/3}$, and the algorithm runs in $\tilde{O}(n^{4/3})$ rounds.

Lemma 4.1. Let $V'$ be the set of nodes $x$ such that there is a shortest path from $x$ to a blocker node $c \in Q$ with hop-length greater than $n^{2/3}$. Using Algorithm 8 each blocker node $c$ can correctly compute $\delta(x,c)$ for all such $x \in V'$ in $\tilde{O}(n^{4/3})$ rounds.

Proof. Since $\text{hops}(x,c) > n^{2/3}$, there exists a blocker node $c' \in Q'$ (constructed in Step 2) such that the shortest path from $x$ to $c$ passes through $c'$. Thus $c$ can compute the distance value $\delta(x,c)$ by adding $\delta(x,c')$ (received in Step 4) and $\delta(c',c)$ (computed in Step 3) values in Step 5.

Step 1 takes $\tilde{O}(n^{2/3} \cdot |Q|) = \tilde{O}(n^{4/3})$ rounds using Bellman-Ford algorithm. Step 2 requires $\tilde{O}(n^{2/3} \cdot n^{2/3}) = \tilde{O}(n^{4/3})$ rounds by Corollary 3.13. Since $|Q'| = \tilde{O}(n/n^{2/3}) = \tilde{O}(n^{1/3})$, Step 3 takes $\tilde{O}(n \cdot n^{1/3}) = \tilde{O}(n^{4/3})$ rounds using Bellman-Ford algorithm and so does Step 4 using Lemma A.2. Step 5 is a local step and has no communication.

4.2 Case (ii): $\text{hops}(x,c) \leq n^{2/3}$

This case deals with sending the distance values from source nodes $x \in V$ to the blocker nodes $c$ when the shortest path between $x$ and $c$ has hop-length at most $n^{2/3}$. Recall that using an all-to-all broadcast or the techniques in [12, 15] for sending these $\tilde{O}(n^{5/3})$ messages appears to require at least $\tilde{O}(n^{5/3})$ rounds.

Let $C^Q$ be the $n^{2/3}$-in-CSSSP collection for source set $Q$. A set $B \subset V$ is a set of bottleneck nodes if removing the nodes in $B$, along with their descendants in the trees in the collection $C^Q$, reduces the congestion to at most $\tilde{O}(n^{4/3})$, i.e. every node would need to send at most $\tilde{O}(n^{4/3})$ messages if all nodes $x$ transmitted their $\delta(x,c)$ values along the pruned CSSSP trees in the collection $C^Q$. 

\[14\]
Algorithm 9 Compute $\delta(x, c)$ at $c$ when $\text{hops}(x, c) \leq n^{2/3}$

Input: $Q$: blocker set; $|Q| \leq n^{2/3} \log n$; $C^Q$: $n^{2/3}$-in-CSSSP collection for set $Q$

1: Compute a set of bottleneck nodes $B$ of size $\tilde{O}(n^{1/3})$ using Algorithm 13 (Sec. A.6.1).
2: For each $b \in B$ in sequence: Compute both in-SSSP and out-SSSP tree rooted at $b$ using Bellman-Ford algorithm.
3: For each $x \in V$ in sequence: Broadcast $ID(x)$ and the shortest path distance values $\delta(x, b)$ for each $b \in B$.
4: Local Step at node $c \in Q$: For each $x \in V$ compute $\delta^{(B)}(x, c) = \min_{b \in B} \{\delta(x, b) + \delta(b, c)\}$ using the $\delta(x, b)$ distance values received in Step 3 and $\delta(b, c)$ distance values computed in Step 2.
5: Remove subtrees rooted at $b \in B$ from the collection $C^Q$ using Algorithm 6.
6: Reset round counter to 0.
7: Assume the nodes in $Q$ are ordered in a (cyclic) sequence $O$.
8: Round $0 < r \leq (n^{4/3} \log n + n^{4/3}) \cdot ((1/3) \cdot \log n / \log \log n - 1)$.
9: Round-robin sends: At each node $v$, forward an unsent message for the next blocker node $c$ in $O$ to its parent in $c$’s tree.

This notion is defined in Huang et al. [13], where they present a randomized algorithm using the randomized scheduling algorithm in Ghaffari [9] to identify such a set of bottleneck nodes. Here we deterministically identify a set of bottleneck nodes $B$ where $|B| = \tilde{O}(n^{1/3})$ (Step 1, Alg. 9) using a pipelined strategy (Sec. A.6.1). Clearly, after we remove these bottleneck nodes, any remaining node needs to send at most $\tilde{O}(n^{4/3})$ messages.

After we identify the set of bottleneck nodes $B$ we run Bellman-Ford algorithm [3] for each $b \in B$ to compute both the incoming and outgoing shortest path tree rooted at $b$ (Step 2, Alg. 9). We then broadcast the $\delta(x, b)$ distance values from every source $x \in V$ to the corresponding $b \in B$ (Step 3, Alg. 9). Thus for all vertices $x \in V$ such that the shortest path from $x$ to a blocker node $c$ passes through some other $b \in B$, the blocker node $c$ can compute the shortest path distance value, $\delta(x, c)$ by adding $\delta(x, b)$ and $\delta(b, c)$ distance values (Step 4, Alg. 9).

It remains to send the distance value $\delta(x, c)$ to blocker node $c$ if $x$ is not part of a subtree of any bottleneck node $b$ in $c$’s shortest path tree. Since the maximum congestion at any node is at most $\tilde{O}(n^{4/3})$ after removing bottleneck nodes in $B$, we are able to perform this computation deterministically. In Steps 8-9 (Alg. 9), we use a simple round-robin strategy to propagate these distance values from each source $x \in V$ to all blocker nodes $c$ in the network. We show in Section 4.3, using the notion of frames, that this simple strategy achieves the desired $\tilde{O}(n^{4/3})$-round bound.

**Lemma 4.2.** If the shortest path from $x \in V$ to a blocker node $c \in Q$ has hop-length at most $n^{2/3}$ and there exists a bottleneck node $b \in B$ on this path, then after executing Steps 1-4 of Algorithm 9 blocker node $c$ knows the distance value $\delta(x, c)$ for all such $x \in V$.

**Proof.** This is immediate from Step 4 (Alg. 9) where $c$ will compute $\delta(x, c)$ by adding the distance values $\delta(x, b)$ (received in Step 3, Alg. 9) and $\delta(b, c)$ value (computed at $c$ in Step 2, Alg. 9).

**Lemma 4.3.** If a source node $x \in V$ lies in a blocker node $c$’s tree in the CSSSP collection $C^Q$ after the execution of Step 5 of Algorithm 9, then $c$ would have received $\delta(x, c)$ value by $(n^{4/3} \log n + n^{4/3}) \cdot ((1/3) \cdot \log n / \log \log n - 1)$ rounds of Step 9 of Algorithm 9.
Lemma 4.3 is established below in Section 4.3. Lemmas 4.2 and 4.3 establish the following lemma.

**Lemma 4.4.** If the shortest path from $x \in V$ to a blocker node $c \in Q$ has hop-length at most $n^{2/3}$, then after running Algorithm 9 blocker node $c$ knows the distance value $\delta(x, c)$ for all such $x \in V$.

**Lemma 4.5.** Algorithm 9 runs for $\tilde{O}(n^{4/3})$ rounds in total.

**Proof.** Step 1 takes $\tilde{O}(n^{4/3})$ rounds (Lemma A.17). Since $|B| = \tilde{O}(n^{1/3})$, Step 2 takes $\tilde{O}(n \cdot n^{1/3}) = \tilde{O}(n^{4/3})$ rounds using Bellman-Ford algorithm and so does Step 3 using Lemma A.2. Step 4 is a local step and involves no communication. Step 5 takes $\tilde{O}(n^{2/3} |Q|) = \tilde{O}(n^{4/3})$ rounds (Lemma 3.7). Step 9 runs for $\tilde{O}(n^{4/3})$ rounds, thus establishing the lemma.

### 4.3 Correctness of Step 9 of Algorithm 9

In this section we will establish that the simple round-robin approach used in Steps 8-9 of Algorithm 9 is sufficient to propagate distance values $\delta(x, c)$ from source nodes $x \in V$ to blocker nodes $c \in Q$ in $\tilde{O}(n^{4/3})$ rounds, when the congestion at any node\textsuperscript{4} is at most $n \sqrt{|Q|}$. While this looks plausible, the issue to resolve is whether a node could be left idling when there are more messages it needs to pass on from its descendants to its parents in some of the trees. This could happen because each node forwards at most one message per round and these descendants might have forwarded messages for other blocker nodes. The round robin scheme appears to only guarantee that a message for a chosen blocker node will be sent from a node to its parent at least once every $|Q|$ rounds.

We now present and analyze a more structured version of Steps 9-10 to establish the bound. In this Algorithm 6 we divide Step 9 (Alg. 9) into $(1/3) \cdot (\log n / \log \log n) - 1$ different stages, with each stage running for at most $n^{4/3} \log n + n^{4/3}$ rounds (we assume $|Q| \leq n^{2/3} \log n$). Our key observation (in Lemma 4.8) is that at the start of Stage $i$, every node $v$ only needs to send the distance values for at most $n^{2/3} / \log^{4/3+1/2} n$ different blocker nodes (note that $i$ is not a constant), thus more messages can be sent by $v$ to each blocker node in later stages.

Let $Q_{v,i}$ be the set of blocker nodes for which node $v$ has messages to send at start of stage $i$. We introduce the notion of a *frame*, where each frame has a single round available for each blocker node in $Q_{v,i}$. Stage $i$ is divided into $n^{2/3} \log^{i+1} n + n^{2/3}$ frames (we will show that each frame consists of $n^{2/3} / \log^{i+1/2} n$ rounds). In each frame, node $v$ sends out an unsent message for each $c \in Q_{v,i}$ to its parent in $c$’s tree (Step 4, Alg. 10).

\textsuperscript{4}Congestion at a node refers to the maximum number of messages sent by a node during the execution of an algorithm.
Lemma 4.6. For all blocker nodes \( c \in Q_{v,i} \), node \( v \) would have sent \( \alpha \) messages to its parent in \( c \)'s tree by \( \alpha + n^{2/3} - h_c(v) \) frames of Stage \( i \), where \( h_c(v) = \text{hops}(v,c) \), provided at least \( \alpha \) messages are routed through \( v \) in Step 4 of Algorithm 10.

Proof. Fix a blocker node \( c \). Let \( i' \) be the smallest \( i \) for which the above statement does not hold and let \( v \) be a node with maximum \( h_c(v) \) value for which this statement is violated in Stage \( i' \). Node \( v \) is not a leaf node since \( \alpha \) is 0 or 1 for a leaf and a leaf would have sent its distance value to its parent in the first frame of Stage-0.

So \( v \) must be an internal node. Since the statement does not hold for \( v \) for the first time for \( \alpha \), it implies that \( v \) has already sent \( \alpha - 1 \) messages (including its own distance value \( \delta(v,c) \)) by \( (\alpha - 1) + n^{2/3} - h_c(v) \) frames and now does not have any message to send to its parent in \( c \)'s tree in the next frame. However since the statement holds for all of \( v \)'s children, \( v \) should have received at least \( \alpha - 1 \) messages from its children by \( (\alpha - 1) + n^{2/3} - (h_c(v) + 1) \)-th frame, resulting in a contradiction. \( \square \)

Since \( h_c(v) \leq n^{2/3} \), Lemma 4.6 leads to the following Corollary.

Corollary 4.7. After the completion of Stage \( i \), every node \( v \) would have sent all or at least \( n^{2/3} \log^{i+1} n \) different distance values for all blocker nodes \( c \in Q_{v,i} \).

Lemma 4.8. The set \( Q_{v,i} \) has size at most \([n^{2/3}/\log^{i+1/2} n]\).

Proof. By Corollary 4.7 after the completion of Stage \( i - 1 \), every node \( v \) would have sent all or at least \( n^{2/3} \log^{i+1} n \) different distance values for all blocker nodes in \( Q_{v,i-1} \). Thus the set \( Q_{v,i} \) will consist of only those nodes from \( Q \) for which \( v \) needs to send at least \( n^{2/3} \log^{i+1} n \) different distance values. Since congestion at any node \( v \) is at most \( n \sqrt{|Q|} = n^{4/3} \log^{1/2} n \) (using Lemma A.15), the size of \( Q_{v,i} \) is at most \( n^{4/3} \log^{1/2} n/n^{2/3} \log^{i+1} n = n^{2/3} / \log^{i+1/2} n \). This establishes the lemma. \( \square \)

Proof of Lemma 4.3. Since \( |Q_{v,i}| \leq n^{2/3} / \log^{i+1/2} n \) (by Lemma 4.8), Stage \( i \) runs for \( n^{2/3} / \log^{i+1/2} n \cdot (n^{2/3} \log^{i+1} n + n^{2/3}) \leq n^{4/3} \log^{1/2} n + n^{4/3} \) rounds. Lemma 4.3 is immediately established from Corollary 4.7 and the fact that there are \((1/3) \cdot \log n / \log \log n - 1\) stages. \( \square \)

5 Overview of \( h \)-hop Shortest Path Extension Algorithm

We now describe an algorithm for computing \( h \)-hop extensions (Step 7 of Algorithm 1) based on the Bellman-Ford algorithm [3]. This algorithm is also used as a step in the randomized APSP algorithm of Huang et al. [13]. Here every blocker node \( c \in Q \) knows its shortest path distance value from every source node \( x \in V \) and the goal is to extend the shortest path from \( x \) to \( c \) by additional \( h \) hops.

This algorithm works as follows: Fix a source \( x \in V \). Every blocker node \( c \in Q \) initializes the shortest path distance from \( x \) to \( \delta(x,c) \) (this value is already known to every \( c \)). We then run Bellman-Ford algorithm at every node \( v \in V \) for source node \( x \) for \( h \) rounds using these initialized values. We repeat this for every \( x \in V \).

After this algorithm terminates, every sink node \( t \in V \) knows the shortest path distance from every \( x \in V \). Since we run Bellman-Ford for \( h \) rounds per source node, for each \( x \in V \), this whole algorithm takes \( O(nh) \) rounds in total. This leads to the following lemma.
Lemma 5.1. The $h$-hop shortest path extensions can be computed in $O(nh)$ rounds for every source $x \in V$ using Bellman-Ford algorithm.

6 Conclusion

We have presented a new deterministic distributed algorithm for computing exact weighted APSP in $\tilde{O}(n^{4/3})$ rounds in both directed and undirected graphs with arbitrary edge weights. This algorithm improves on the $\tilde{O}(n^{3/2})$ round APSP algorithm of [2]. At the heart of our algorithm is an efficient distributed algorithm for sending the distance values from source nodes to the blocker nodes and an improved deterministic algorithm for computing the blocker set using pairwise independence and derandomization. We believe that both these techniques may be of independent interest for obtaining results for other distributed graph problems.

The main open question left by our work is whether we can get a deterministic algorithm that can match the current $\tilde{O}(n)$ randomized bound for computing weighted APSP [5].

Acknowledgement.

We thank Valerie King for suggesting using the techniques in Berger et al. [4] for the blocker set construction.

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A Appendix

A.1 Broadcast Primitives

In this paper we use the following two broadcast primitives quite extensively. These primitives are widely known and we restate them here only for completeness. See [2] for more details.

Lemma A.1 ([2]). A node $v$ can broadcast $k$ local values to all other nodes reachable from it deterministically in $O(n+k)$ rounds.

Lemma A.2 ([2]). All $v \in V$ can broadcast a local value to every other node they can reach in $O(n)$ rounds deterministically.
A.2 Consistent $h$-hop SSSP ($h$-CSSSP)

The notion of $h$-hop Consistent SSSP (CSSSP) was introduced recently in [1]. The goal of this new notion was to create a consistent collection of paths across all trees in the collection, i.e. a path from $u$ to $v$ is same in all trees $T$ in the CSSSP collection $C$ (in which such a path exists).

The difference between an $h$-hop SSSP for source $x$ and the tree for source $x$ in the $h$-CSSSP collection $C$ is that the former contains path from $x$ to every $t \in V$ for which there exists a path with at most $h$ hops. However this is not guaranteed in the latter case as the CSSSP only guarantees that if the shortest path from $x$ to $t$ has at most $h$ hops, then this path will be present in the corresponding tree for source $x$ in the CSSSP collection. This is the major difference between these two notions. We now re-state the definition of CSSSP from [1] here:

**Definition A.3 (CSSSP [1]).** Let $H$ be a collection of rooted trees of height $h$ in a graph $G = (V,E)$. Then $H$ is an $h$-CSSSP collection (or simply an $h$-CSSSP) if for every $u, v \in V$ the path from $u$ to $v$ is the same in each of the trees in $H$ (in which such a path exists), and is the $h$-hop shortest path from $u$ to $v$ in the $h$-hop tree $T_u$ rooted at $u$. Further, each $T_u$ contains every vertex $v$ that has a path with at most $h$ hops from $u$ in $G$ that has distance $\delta(u,v)$.

[1] describes a very simple algorithm for constructing $h$-CSSSP collection: First compute $2h$-hop SSSPs for every source $x$. To compute CSSSP, just retain the initial $h$ hops of each of these $2h$-hop SSSPs. We can construct these $h$-hop SSSPs using Bellman-Ford algorithm, leading to the following lemma. (See [1] for further details.)

**Lemma A.4 ([1]).** $h$-CSSSPs for source set $S$ can be computed in $O(|S| \cdot h)$ rounds using the Bellman-Ford algorithm.

The CSSSP collection have the following two important properties which we use throughout in this paper. We call a tree $T$ rooted at a vertex $c$ an out-tree if all the edges incident to $c$ are outgoing edges from $c$ and we call $T$ an in-tree if all the edges incident to $c$ are incoming edges.

**Lemma A.5 ([1]).** Let $C$ be an $h$-CSSSP collection. Let $c$ be a node in $G$ and let $T$ be the union of the edges in the collection of subtrees rooted at $c$ in the trees in $C$. Then $T$ forms an out-tree rooted at $c$.

**Lemma A.6 ([1, 2]).** Let $C$ be an $h$-CSSSP collection. Let $c$ be a node in $G$ and let $T$ be the union of the edges on the tree-path from the root of each tree in $C$ to $c$ (for the trees that contain $c$). Then $T$ forms an in-tree rooted at $c$.

A.3 $O(n)$ Sample Space Construction

In this section we describe Luby’s approach [17] to speed up exhaustive search for finding a good sample point (or good set $A$, Def. 3.1) by replacing the sample space $\{0,1\}^n$ of size $2^n$ with another sample space of size $O(n)$. Let $l$ be such that $2n < 2^l \leq 4n$ and let this new sample space be $\{0,1\}^l$. For each $i \in \{1, n\}$, we consider $i$ as a binary string of length $l$, where the last bit $i_l$ is 1. We define $X_i(v) = \oplus_{k=1}^{l} (i_k \cdot z_k)$ for each $i \in \{1,n\}$ and $z \in \{0,1\}^l$, where $\oplus$ is addition modulo 2. For a random string $w$ of length $l$, Luby [17] showed that the variables $X_1(w), \ldots, X_n(w)$ are pairwise independent and are identically distributed uniformly in $\{0,1\}$. This claim allows us to find a good sample point (or good set) in this sample space instead of performing an exhaustive search.
A.4 Correctness Proofs for Algorithm 2

In this Section we provide proofs for our randomized algorithm for computing blocker set for a given $h$-CSSSP collection $C$. Note that the proof of Lemmas A.9-A.11, 3.9 and 3.10 is based on the analysis in [4] and we adapt them here in our setting. Table 3 presents the notation we use in our analysis in this section.

**Lemma A.7.** The set $Q$ constructed in Algorithm 2 is a blocker set for the CSSSP collection $C$.

**Proof.** To show that $Q$ is a blocker set, we need to show that the computed blocker set $Q$ indeed covers all paths in the CSSSP collection $C$. The while loop in Steps 6-16 runs as long as there is a path in $P_i$ with at least $(1 + \epsilon)j^{j-1}$ nodes in $V_i$ and since this loop terminated for $i = 1$ and $j = 1$, it implies that there is no path in $C$ which is not covered by some node in $Q$. \(\square\)

**Lemma A.8.** If the check in Step 9 fails, then $|V_i| > \frac{(1+\epsilon)^j}{\delta^3}$.

**Proof.** Since no node in $V_i$ covers a $\frac{\delta^3}{(1+\epsilon)}$ fraction of paths from $P_{ij}$, hence the total $\text{score}_{ij}$ values (defined in Step 8) for all nodes in $V_i$ has value at most $|V_i| \cdot \frac{\delta^3}{(1+\epsilon)} \cdot |P_{ij}|$. And since every path in $P_{ij}$ has at least $(1 + \epsilon)^{j-1}$ nodes in $V_i$,

$$|V_i| \cdot \frac{\delta^3}{(1+\epsilon)} \cdot |P_{ij}| > |P_{ij}| \cdot (1 + \epsilon)^{j-1}$$

This establishes that $|V_i| > \frac{(1+\epsilon)^j}{\delta^3}$. \(\square\)
Lemma A.9. The set $A$ constructed in Step 12 of Algorithm 2 has size at most $(\delta + 2\delta^2) \cdot \frac{|V_i|}{(1+\epsilon)^i}$ and at least $(\delta - 2\delta^2) \cdot \frac{|V_i|}{(1+\epsilon)^i}$ with probability at least $3/4$.

Proof. Consider random variable $X_v$ where $X_v = 1$ if $v$ is present in $A$, otherwise $X_v = 0$. Thus $\sum_{v \in V_i} X_v$ denotes the size of $A$. We now calculate its expectation and variance.

\[
E[\sum_{v \in V_i} X_v] = |V_i| \cdot \frac{\delta}{(1+\epsilon)^i} \tag{1}
\]

\[
Var[\sum_{v \in V_i} X_v] = |V_i| \cdot Var[X_v] \leq |V_i| \cdot E[X_v^2] = |V_i| \cdot \frac{\delta^2}{(1+\epsilon)^i} \tag{2}
\]

We now use Chebyshev’s inequality to get an upper bound on the size of $A$. Using Chebyshev’s inequality the following holds with probability at least $3/4$:

\[
||A| - E[|A|]| \leq 2 \sqrt{Var[|A|]}
\]

\[
\leq 2 \sqrt{|V_i| \cdot \frac{\delta}{(1+\epsilon)^i}}
\]

\[
\leq 2 \cdot |V_i| \cdot \frac{\delta^2}{(1+\epsilon)^i} \quad \text{(by Lemma A.8)}
\]

\[
|A| \leq |V_i| \cdot \frac{\delta}{(1+\epsilon)^i} + 2 \cdot |V_i| \cdot \frac{\delta^2}{(1+\epsilon)^i}
\]

Using the above analysis we can also show that $|A| \geq (\delta - 2\delta^2) \cdot \frac{|V_i|}{(1+\epsilon)^i}$ with probability at least $3/4$. \qed

Lemma A.10. The set $A$ constructed in Step 12 of Algorithm 2 covers at least $|A| \cdot (1+\epsilon)^i \cdot (1 - 3\delta - \epsilon)$ paths in $P_i$ with probability at least $1/2$.

Proof. Consider random variable $X_v$ as defined in the proof of Lemma A.9. A path $p$ is covered by $A$ if $v \in A$ for some $v \in V_i \cap p$. To get a lower bound on the number of paths covered by $A$, we use the term $\sum_{v \in V_i \cap p} X_v - \sum_{v,v' \in V_i \cap p} X_v \cdot X_{v'}$ to denote if a path $p$ is covered by $A$ or not. Note that this term has value at most 1 which is attained when either 1 or 2 nodes from $p$ are picked in $A$ and otherwise the value is non-positive. Thus the term $\sum_{p \in P_i} \left( \sum_{v \in V_i \cap p} X_v - \sum_{v,v' \in V_i \cap p} X_v \cdot X_{v'} \right)$ gives a lower bound on the number of paths covered by $A$ in $P_i$. Let this term be $Y$. Now we show that value of $Y$ is $\geq |A| \cdot (1+\epsilon)^i \cdot (1 - 3\delta - \epsilon)$ with probability at least $1/2$.

We first split $Y$ into $Y_1$ and $Y_2$ where $Y_1 = \sum_{p \in P_i} \sum_{v \in V_i \cap p} X_v$ and $Y_2 = \sum_{p \in P_i} \sum_{v,v' \in V_i \cap p} X_v \cdot X_{v'}$. We first get a lower bound on the term $Y_1$.

\[
Y_1 = \sum_{p \in P_i} \sum_{v \in V_i \cap p} X_v
\]
\[
\sum_{v \in V_i} \sum_{p \in P_i, v \in p} X_v \\
\geq (1 + \epsilon)^{i-1} \cdot \sum_{v \in V_i} X_v \quad \text{(since every node in } V_i \text{ lies in } i \text{-paths in } P_i) \\
= (1 + \epsilon)^{i-1} \cdot |A|
\]

We now need to get an upper bound on the term \(Y_2\). We first compute an upper bound on \(E[Y_2]\) and then use Markov inequality to get an upper bound on \(Y_2\). (Let \(n_{V_i,p}\) denote the number of nodes from \(V_i\) in \(p\). Clearly \(n_{V_i,p} \leq (1 + \epsilon)^j\))

\[
E[Y_2] = \sum_{p \in P_i} \sum_{v, v' \in V_i \cap p} E[X_v \cdot X_{v'}] \\
= \sum_{p \in P_i} \left( \frac{n_{V_i,p}}{2} \right) \left( \frac{\delta}{(1 + \epsilon)^j} \right)^2 \\
\leq (1 + \epsilon)^j \cdot \sum_{p \in P_i} \frac{n_{V_i,p}}{2} \left( \frac{\delta}{(1 + \epsilon)^j} \right)^2 \quad \text{(since } n_{V_i,p} \leq (1 + \epsilon)^j) \\
\leq (1 + \epsilon)^j \cdot \frac{|V_i|}{2} \cdot \max_{v \in V_i} \text{score}(v) \cdot \left( \frac{\delta}{(1 + \epsilon)^j} \right)^2 \\
\leq \frac{|V_i|}{2} \cdot (1 + \epsilon)^{i-j} \cdot \delta^2
\]

Now using Markov inequality we get the following upper bound on \(Y_2\) with probability at least 3/4:

\[
Y_2 \leq 4E[Y_2] \leq 2\delta^2 \cdot (1 + \epsilon)^{i-j} \cdot |V_i|
\]

Since \(|A| \geq (\delta - 2\delta^2) \cdot \frac{|V_i|}{(1 + \epsilon)^j}\) with probability at least 3/4 by Lemma A.9, \(Y_2 \leq 2\delta^2 \cdot (1 + \epsilon)^j \cdot \frac{|A|}{(\delta - 2\delta^2)}\) with probability at least 1/2.

Combining the bounds for \(Y_1\) and \(Y_2\) we get the following lower bound on \(Y\) with probability at least 1/2:

\[
Y = Y_1 - Y_2 \\
\geq (1 + \epsilon)^{i-1} \cdot |A| - 2\delta^2 \cdot (1 + \epsilon)^j \cdot \frac{|A|}{(\delta - 2\delta^2)} \\
= (1 + \epsilon)^j \cdot |A| \cdot \left( \frac{1}{1 + \epsilon} - \frac{2\delta}{1 - 2\delta} \right) \\
= (1 + \epsilon)^j \cdot |A| \cdot \left( 1 - \frac{\epsilon}{1 + \epsilon} - \frac{3\delta}{3/2 - 3\delta} \right)
\]
\[ \geq (1 + \epsilon)^j \cdot |A| \cdot (1 - \epsilon - 3\delta) \]

This establishes the lemma.

**Lemma A.11.** The set \( A \) constructed in Step 12 of Algorithm 2 covers at least a \( \delta/2 \) fraction of paths in \( P_{ij} \) with probability at least \( 5/8 \).

**Proof.** Similar to the proof of Lemma A.10 we can lower bound the number of paths covered by set \( A \) in \( P_{ij} \) by the term
\[
\sum_{p \in P_{ij}} \left[ \sum_{v \in V_i \cap p} X_v - \sum_{v, v' \in V_i \cap p} X_v \cdot X_{v'} \right].
\]
Let this term be \( Y' \), with first term \( Y_3 \) and the second term \( Y_4 \). We need to show that \( Y' \geq \frac{\delta}{2} \cdot |P_{ij}| \) with probability at least \( 5/8 \).

We first give a lower bound on \( Y_3 \). To get the lower bound, we first compute a lower bound on \( E[Y_3] \) and an upper bound on \( Var[Y_3] \) and then use Chebyshev’s inequality. (Let \( n_{v,P_{ij}} \) represent the number of paths in \( P_{ij} \) that contain node \( v \). Since no node covers at least \( \frac{\delta^3}{(1+\epsilon)} \) fraction of paths in \( P_{ij} \), \( n_{v,P_{ij}} < \frac{\delta^3}{(1+\epsilon)} \).)

\[
E[Y_3] = E[ \sum_{p \in P_{ij}} \sum_{v \in V_i \cap p} X_v ]
\]
\[
\geq E[ \sum_{p \in P_{ij}} (1 + \epsilon)^{j-1} \cdot X_v ] \quad \text{(since every path in } P_{ij} \text{ has}
\]
\[
\geq (1 + \epsilon)^{j-1} \text{ nodes from } V_i)
\]
\[
= (1 + \epsilon)^{j-1} \cdot |P_{ij}| \cdot \frac{\delta}{(1 + \epsilon)^j}
\]
\[
= |P_{ij}| \cdot \frac{\delta}{1 + \epsilon}
\]

\[
Var[Y_3] = Var[ \sum_{p \in P_{ij}} \sum_{v \in V_i \cap p} X_v ]
\]
\[
= Var[ \sum_{v \in V_i} \sum_{p \{P_{ij}, v \in p\}} X_v ]
\]
\[
= Var[ \sum_{v \in V_i} n_{v,P_{ij}} X_v ]
\]
\[
= \sum_{v \in V_i} n_{v,P_{ij}}^2 \cdot Var[X_v]
\]
\[
\leq \frac{\delta}{(1 + \epsilon)^j} \cdot \frac{\delta^3}{(1 + \epsilon)} \cdot |P_{ij}| \cdot \sum_{v \in V_i} n_{v,P_{ij}} \quad \text{(since}
\]
\[
n_{v,P_{ij}} < \frac{\delta^3}{(1 + \epsilon)} \cdot |P_{ij}| \)
\]
\[
\leq \frac{\delta^4}{(1 + \epsilon)^{j+1}} \cdot |P_{ij}| \cdot |P_{ij}| \cdot (1 + \epsilon)^j \quad \text{(since every path in}
\]
\[
P_{ij} \text{ has } (1 + \epsilon)^j \text{ nodes from } V_i)
\]

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\[ \leq \delta^4 \cdot |P_{ij}|^2 \]

We now use Chebyshev’s inequality to get a lower bound on the value of \( Y_3 \). Using Chebyshev’s inequality the following holds with probability at least \( 7/8 \):

\[
|Y_3 - E[Y_3]| \leq 2 \sqrt{\text{Var}[Y_3]} \\
Y_3 \geq E[Y_3] - 2\sqrt{\delta^2 \cdot |P_{ij}|} \\
\geq |P_{ij}| \cdot \frac{\delta}{(1 + \epsilon)} - 2\sqrt{2}\delta^2 \cdot |P_{ij}| 
\]

We now need to get an upper bound on the term \( Y_4 \). We first compute an upper bound on \( E[Y_4] \) and then use Markov inequality to get an upper bound on \( Y_4 \).

\[
E[Y_4] = \sum_{p \in P_{ij}} \sum_{v, v' \in V_i \cap p} E[X_v \cdot X_{v'}] \\
\leq |P_{ij}| \cdot \frac{(1 + \epsilon)^{2j}}{2} \cdot \left( \frac{\delta}{(1 + \epsilon)^j} \right)^2 \quad \text{(since there are \( (1 + \epsilon)^j \) nodes from} \ V_i \ \text{in any path in} \ P_{ij}) \\
= |P_{ij}| \cdot \frac{\delta^2}{2} 
\]

Now using Markov inequality we get the following upper bound on \( Y_4 \) with probability at least \( 3/4 \):

\[
Y_4 \leq 4E[Y_4] \leq 2\delta^2 \cdot |P_{ij}| 
\]

Combining the bounds for \( Y_3 \) and \( Y_4 \) we get the following lower bound on \( Y' \) with probability at least \( 5/8 \):

\[
Y' = Y_3 - Y_4 \\
\geq |P_{ij}| \cdot \frac{\delta}{(1 + \epsilon)} - 2\sqrt{2}\delta^2 \cdot |P_{ij}| - 2\delta^2 \cdot |P_{ij}| \\
\geq |P_{ij}| \cdot \delta \cdot (1 - \epsilon - 5\delta) \\
\geq |P_{ij}| \cdot \frac{\delta}{2} \quad \text{(since} \ \epsilon, \ \delta \ \leq 1/12) 
\]

This establishes the lemma. \( \square \)

\textbf{Lemma. 3.8} The set \( A \) constructed in Step 12 of Algorithm 2 is a good set with probability at least \( \frac{1}{8} \), i.e. \( A \) covers at least \( |A| \cdot (1 + \epsilon)^i \cdot (1 - 3\delta - \epsilon) \) paths in \( P_i \), including at least a \( \delta/2 \) fraction of paths in \( P_{ij} \).

\textit{Proof.} This is immediate from Lemma A.10 and A.11. \( \square \)
Lemma 3.9 The number of selection steps is $O \left( \frac{\log^3 n}{\delta^3 \cdot \epsilon^2} \right)$, i.e. the while loop in Steps 6-16 runs for at most $O \left( \frac{\log^3 n}{\delta^3 \cdot \epsilon^2} \right)$ iterations in total.

Proof. The while loop runs until $P_{ij}$ is non-empty, i.e. there exists a path in $P_i$ with at least $(1+\epsilon)^{j-1}$ nodes in $V_i$. In each iteration, the algorithm either covers at least $\frac{\delta^3}{(1+\epsilon)}$ fraction of paths in $P_{ij}$ (if node $c$ is added to blocker set $Q$ in Step 10) or at least $\frac{\delta}{2}$ fraction of paths from $P_{ij}$ (if set $A$ is added to $Q$ in Step 14). Since there are at most $n^2$ paths and each iteration of the while loop covers at least $\frac{\delta^3}{(1+\epsilon)}$ fraction of $P_{ij}$, there are at most $O \left( \frac{\log n^2}{\log \frac{1}{1-\frac{\delta^3}{(1+\epsilon)}}} \right) = O \left( \frac{(1+\epsilon) \log n}{\delta^3} \right) = O \left( \frac{\log n}{\delta^3} \right)$ iterations. Since both the inner and outer for loop runs for $O(\log_{1+\epsilon} n) = O \left( \frac{\log n}{\epsilon} \right)$ iterations, this establishes the lemma.

Lemma A.12. Each iteration of the inner for loop (Steps 5-16) in Algorithm 2 takes $\tilde{O} \left( \frac{|S| \cdot h}{\delta^3} \right)$ rounds in expectation.

Proof. We first show that each iteration of the while loop in Steps 6-16 takes $O(|S| \cdot h)$ rounds in expectation. Step 7 takes $O(|S| \cdot h)$ rounds by Lemmas 3.4 and 3.4 and so does Step 8 [2] and by Lemma A.2. The check in Step 9 involves no communication and so does Step 10, since every node knows the score values for every other node and also the value of $|P_{ij}|$, i.e. the number of paths that belong to $P_{ij}$. Steps 12 and 14 are also local steps and does not involve any communication. Step 13 involves broadcasting at most $n$ messages and hence takes $O(n)$ rounds using Lemma A.2. Since by Lemma 3.8 the set $A$ constructed in Step 12 is good with probability at least $1/8$, Steps 12-14 are executed $O(1)$ times in expectation. Step 16 takes $O(|S| \cdot h)$ rounds [2] and using Lemma 3.3. Since the while loop runs for at most $O \left( \frac{\log n}{\delta^3} \right)$ iterations (by Lemma 3.9), this establishes the lemma.

A.5 Helper Algorithms for Deterministic Blocker Set Algorithm: Distributed Computation of Terms $\nu_{P_i}$ and $\nu_{P_{ij}}$

In this Section we describe a simple pipelined algorithm to compute $\nu_{P_i}$ and $\nu_{P_{ij}}$ terms at leader node $l$. Both algorithms are similar to an algorithm in [2] (for computing ‘initial scores’). Recall that $\sigma^{(\mu)}_{P_i,v}$ refers to the number of paths in $P_i^v$ covered by the sample point $X^{(\mu)}$ and $\sigma^{(\mu)}_{P_{ij},v}$ refers to the total number of paths in $P_{ij}^v$ covered by the sample point $X^{(\mu)}$. Let $\nu^{(\mu)}_{P_i,v}$ refers to the sum total of the $\sigma^{(\mu)}_{P_i,v,w}$ values of all descendant nodes $w$ of $v$ and similarly let $\nu^{(\mu)}_{P_{ij},v}$ refers to the sum total of the $\sigma^{(\mu)}_{P_{ij},v,w}$ values of all descendant nodes $w$ of $v$. Also recall from Section 3.2 that $\nu_{P_i}$ and $\nu_{P_{ij}}$ refers to the total number of paths covered by $X^{(\mu)}$ in sets $P_{ij}$ and $P_{ij}$ respectively. Table 4 presents the notations that we use in this Section.
Table 4: List of Notations Used in the Analysis of the Deterministic Algorithm

| Notation | Description |
|----------|-------------|
| $A$ | set constructed in Step 12 of Randomized Blocker Set Algorithm (Alg. 2) |
| $X_v$ | 1 if $v$ is present in $A$, otherwise 0 |
| $X$ | vector composed of $X_v$'s |
| $X^{(\mu)}$ | $\mu$-th vector in the enumeration of $X$ in the sample space |
| $S$ | set of sources in $C$ |
| $h$ | number of hops in a path |
| $n$ | number of nodes |
| $V_i$ | set of nodes $v$ with $\text{score}(v) \geq (1 + \epsilon)^{i-1}$ |
| $P_i$ | set of paths in $C$ with at least one node in $V_i$ |
| $P_{ij}$ | set of paths in $P_i$ with at least $(1 + \epsilon)^{j-1}$ nodes in $V_i$ |
| $P_{iu}$ | set of paths in $P_i$ with leaf node $u$ |
| $P_{iju}$ | set of paths in $P_{ij}$ with leaf node $u$ |
| $\sigma_{P_i, u}$ | $\sum_{p \in P_{iu}} \sum_{v \in V_i \cap p} X_v$ |
| $\sigma_{P_{ij}, u}$ | $\sum_{p \in P_{iju}} \sum_{v \in V_{ij} \cap p} X_v$ |
| $\nu_{P_i, u}$ | sum total of $\sigma_{P_i, w}$ values for all descendant nodes $w$ of $v$ |
| $\nu_{P_{ij}, u}$ | sum total of $\sigma_{P_{ij}, w}$ values for all descendant nodes $w$ of $v$ |
| $\nu_{P_i}^{(\mu)}$ | value of $\nu_{P_i}$ with $X^{(\mu)}$ as the input |
| $\nu_{P_{ij}}^{(\mu)}$ | value of $\nu_{P_{ij}}$ with $X^{(\mu)}$ as the input |
| $\nu_{P_i, u}^{(\mu)}$ | value of $\nu_{P_i, u}$ with $X^{(\mu)}$ as the input |
| $\nu_{P_{ij}, u}^{(\mu)}$ | value of $\nu_{P_{ij}, u}$ with $X^{(\mu)}$ as the input |
| $C$ | $h$-hop CSSSP collection |
| $Tx$ | $h$-hop shortest path tree rooted at $x$ in collection $C$ |
| $Q$ | blocker set (being constructed) |
| $l$ | leader node |
Algorithm 11 Compute sum of $\nu_{Pi}$ values at leader node $l$

Input: $h$: number of hops; $S$: set of sources; $C$: $h$-CSSP collection; $X^{(\mu)}$: $\mu$-th vector in sample space; $T$: BFS in-tree rooted at leader $l$

1: **Local Step at $v \in V$:** Let $\mathcal{P}$ be the set of paths in $P_i$ with $v$ as the leaf node. For each $1 \leq \mu \leq n$, set $\nu^{(\mu)}_{P_{i,v}} = \sum_{p \in \mathcal{P}, \forall z \in p} X^{(\mu)}_z$

2: **In round $r > 0$ (for all nodes $v \in V - \{l\}$):**

3: **send:** if $r = n - h(v) + \mu - 1$ then send $\langle \nu^{(\mu)}_{P_{i,v}} \rangle$ to parent($v$) in $T$

4: **receive [lines 5-9]:**

5: if $r = n - h(v) + \mu - 2$ then

6: let $\mathcal{I}$ be the set of incoming messages to $v$

7: for each $M \in \mathcal{I}$ do

8: let the sender be $w$ and let $M = \langle \nu^{(\mu)}_{P_{i,w}} \rangle$ and

9: if $w$ is a child of $v$ in $T$ then $\nu^{(\mu)}_{P_{i,v}} \leftarrow \nu^{(\mu)}_{P_{i,v}} + \nu^{(\mu)}_{P_{i,w}}$

10: **Local Step at leader $l$:** Compute the total sum $\nu^{(\mu)}_{P_i}$ for each sample point $\mu$, by summing up the received $\nu^{(\mu)}_{P_{i,w}}$ values from all its children $w$.

A.5.1 Computing $\nu_{Pi}$

Consider computing the $\nu^{(\mu)}_{Pi}$ terms for each sample point $\mu$, at leader node $l$ (Algorithm 11) ($\nu_{Pi,j}$ can be computed similarly). First every node $v$ initializes its $\nu^{(\mu)}_{P_{i,v}}$ value, for each sample point $\mu$, in Step 1. Recall that we assume that all $X$ values are enumerated in order and every node knows this enumeration. In round $n - 1 - h + \mu$, the node $u$ at height $h$ sends its corresponding $\nu_{P_{i,u}}$ value for $X^{(\mu)}$ (Step 3) along with the total value of $\nu_{P_{i}}$ it received from its children for $X^{(\mu)}$ (Steps 5-9). Leader node $l$ then computes the total sum $\nu^{(\mu)}_{P_i}$ for each sample point $\mu$, by summing up the received $\nu^{(\mu)}_{P_{i,w}}$ values from all its children $w$ in Step 10. In Lemma A.13 we show that leader $l$ correctly computes $\nu_{P_i}$ values for all $X^{(\mu)}$’s in $O(n)$ rounds.

Lemma A.13. Algorithm 11 correctly computes the $\nu^{(\mu)}_{P_i}$ values at leader node $l$ for all $\mu$ in $O(n)$ rounds.

Proof. In Step 1, every node $v$ correctly initialize their contribution to the overall $\nu_{P_{i,v}}$ term for each $\mu$ locally. Since the height of tree $T$ is at most $n - 1$, it is readily seen that a node $v$ that is at depth $h(v)$ in $T$ will receive the $\text{count}^{(\mu)}_{P_i}$ values from its children in round $n - h(v) + \mu - 2$ (Steps 5-9) and thus will have the correct $\nu^{(\mu)}_{P_i}$ value to send in round $n - h(v) + \mu - 1$ in Step 3. Since $\mu = O(n)$, Steps 3-9 runs in $O(n)$ rounds. Step 10 is a local step and thus does not involve any communication. This establishes the lemma.

A.5.2 Computing $\nu_{Pi,j}$

Here we describe our algorithm for computing $\nu^{(\mu)}_{Pi,j}$ terms for each sample point $\mu$, at leader node $l$ (Algorithm 12). Every node $v$ first initializes its $\nu^{(\mu)}_{Pi,j}$ value in Step 1. Recall that we assume that all $X$ values are enumerated in order and every node knows this enumeration. In round $n - 1 - h + \mu$,
Algorithm 12 Compute sum of $\nu_{P_{ij}}$ values at leader node $l$

**Input:** $h$: number of hops; $S$: set of sources; $C$: $h$-CSSP collection; $X^{(\mu)}$: $\mu$-th vector in sample space; $T$: BFS in-tree rooted at leader $l$

1: **Local Step at $v \in V:** Let $P$ be the set of paths in $P_{ij}$ with $v$ as the leaf node. **For each** $1 \leq \mu \leq n$, set $\nu_{P_{ij},v}^{(\mu)} = \sum_{p \in P} \sum_{z \in p} X_{z}^{(\mu)}$

2: **In round** $r > 0$:

3: **send:** if $r = n - h(v) + \mu - 1$ then send $(\nu_{P_{ij},v}^{(\mu)})$ to parent($v$) in $T$

4: **receive [lines 5-9]:**

5: if $r = n - h(v) + \mu - 2$ then

6: let $I$ be the set of incoming messages to $v$

7: **for each** $M \in I$ do

8: let the sender be $w$ and let $M = (\nu_{P_{ij},w}^{(\mu)})$ and

9: if $w$ is a child of $v$ in $T$ then $\nu_{P_{ij},v}^{(\mu)} = \nu_{P_{ij},v}^{(\mu)} + \nu_{P_{ij},w}^{(\mu)}$

10: **Local Step at leader $l:** Compute the total sum $\nu_{P_{ij}}^{(\mu)}$ for each sample point $\mu$, by summing up the received $\nu_{P_{ij},w}^{(\mu)}$ values from all its children $w$.

the node $u$ at height $h$ sends its corresponding $\nu_{P_{ij},u}$ value for $X^{(\mu)}$ (Step 3) along with the total value of $\nu_{P_{ij}}$ it received from its children for $X^{(\mu)}$ (Steps 5-9). Leader node $l$ then computes the total sum $\nu_{P_{ij}}^{(\mu)}$ for each sample point $\mu$, by summing up the received $\nu_{P_{ij},w}^{(\mu)}$ values from all its children $w$ in Step 10. In Lemma A.14 we show that leader $l$ correctly computes $\nu_{P_{ij}}^{(\mu)}$ values for all $X^{(\mu)}$’s in $O(n)$ rounds.

**Lemma A.14.** Algorithm 12 correctly computes the $\nu_{P_{ij}}^{(\mu)}$ values at leader node $l$ for all $\mu$ in $O(n)$ rounds.

**Proof.** In Step 1, every node $v$ correctly initialize their contribution to the overall $\nu_{P_{ij}}$ term for each $\mu$ locally. Since the height of tree $T$ is at most $n - 1$, it is readily seen that a node $v$ that is at depth $h(v)$ in $T$ will receive the $\nu_{P_{ij}}^{(\mu)}$ values from its children in round $n - h(v) + \mu - 2$ (Steps 5-9) and thus will have the correct $\nu_{P_{ij}}^{(\mu)}$ value to send in round $n - h(v) + \mu - 1$ in Step 3. Since $\mu = O(n)$, Steps 3-9 runs for at most $2n$ rounds. Step 10 is a local step and thus does not involve any communication. This establishes the lemma.

**Proof of Lemma 3.12.** Every node $v$ correctly computes all the ancestor nodes in each tree $T_x$ in Step 1 using Algorithm 4 in [2] that takes $O(|S| \cdot h)$ rounds [2]. Step 2 computes the incoming BFS tree rooted at leader node $l$ in $O(n)$ rounds. Step 3 takes $O(n)$ rounds by Lemmas A.13 and A.14. Step 4 is a local step and involves no communication. Step 5 involves an all-to-all broadcast of at most $n$ messages and thus takes $O(n)$ rounds using Lemma A.2.
Algorithm 13 Compute-Bottleneck: Compute Bottleneck Nodes Set $B$

Input: $Q$: blocker set; $C^Q$: CSSSP collection for blocker set $Q$
Output: $B$: set of bottleneck nodes

1: For each $c \in Q$ in sequence: Compute $\text{count}_{v,c}$ values at every node $v \in V$ using Algorithm 14 (Section A.6.2).
2: Local Step at $v \in V$: Compute $\text{total\_count}_v \leftarrow \sum_{c \in Q} \text{count}_{v,c}$
3: while there is a node $v$ with $\text{total\_count}_v > n\sqrt{|Q|}$ do
4: For each $v \in V$: Broadcast $ID(v)$ and $\text{total\_count}_v$ value.
5: Add node $b$ to $B$ such that $b$ has maximum $\text{total\_count}_v$ value (break ties using IDs).
6: Update $\text{total\_count}_v$ values for the descendants and ancestors of $b$ across all trees in the collection $C^Q$ using results in $[2, 1]$.

A.6 Helper Algorithms for Algorithm 9

A.6.1 Computing Bottleneck Nodes

Here we describe our deterministic algorithm for computing Step 1 of Algorithm 9, which identifies a set $B$ of bottleneck nodes such that removing this set of nodes reduces the congestion in the network from $O(n \cdot |Q|)$ to $O(n \cdot \sqrt{|Q|})$. However when randomization is allowed, there is a $O(n \cdot \sqrt{|Q|})$ randomized algorithm of Huang et al. [13] that computes this set w.h.p. in $n$. Our deterministic algorithm is however very different from the randomized algorithm given in [13] and it uses ideas from our blocker set algorithm in [2].

We now give an overview of the randomized algorithm of [13] that computes this set of bottleneck nodes. For a source $x$ and its incoming shortest path tree $T_x$, every node in $T_x$ calculates the number of outgoing messages for source $x$. This is done by waiting for messages from all children nodes, followed by sending a message to its parent in $T_x$. This takes $O(n)$ rounds and can be run across multiple nodes in $Q$ as congestion is at most $O(|Q|)$. Thus using the randomized algorithm of Ghaffari [9], this algorithm can be run across all nodes in $Q$ concurrently in $\tilde{O}(n + |Q|) = \tilde{O}(n)$ rounds. After computing these values, a node $b$ with maximum count is selected to the set $B$ and is then removed from the network. The algorithm repeats this for $O(\sqrt{|Q|})$ times, thus eliminating all nodes that needed to send at least $n\sqrt{|Q|}$ messages (since removal of every such node eliminates $O(n\sqrt{|Q|})$ nodes across all trees and there are at most $n \cdot |Q|$ nodes).

Our deterministic algorithm for computing bottleneck nodes (Algorithm 13) works as follows: In Step 1, the algorithm computes the $\text{count}_{v,c}$ values (number of messages $v$ needs to send to its parent in $c$’s tree) using Algorithm 14 described in Section A.6.2. Every node $v$ calculates the total number of messages it needs to send by summing up the values computed in Step 1 (Step 2) and then broadcast this value in Step 4. The node with maximum value is added to the bottleneck node set $B$ (Step 5) and the values of its ancestors and descendants are updated using the algorithms in [1]. In Lemma A.17 we establish that the whole algorithm runs in $O(n\sqrt{|Q|} + h \cdot |Q|)$ rounds deterministically.

Lemma A.15. After Compute-Bottleneck (Algorithm 13) terminates, $\text{total\_count}_v \leq n\sqrt{|Q|}$ for all nodes $v$.

Proof. This is immediate since the while loop in Steps 3-6 terminates only when there is no node $v$ with $\text{total\_count}_v > n\sqrt{|Q|}$. \qed
Algorithm 14 Compute-Count: Algorithm for computing $count_{v,c}$ values for source $c$ at node $v$

Input: $h$: number of hops, $T_c$: tree for source $c$
1: (Round 0): if $v \in T_c$ then set $count_{v,c} \leftarrow 1$ else $count_{v,c} \leftarrow 0$
2: Round $h + 1 \geq r > 0$:
3: Send: if $r = h - h_c(v) + 1$ then send $\langle count_{v,c} \rangle$ to $v$’s parent
4: receive [lines 5-9]:
5: if $r = h - h_c(v)$ then
6: let $I$ be the set of incoming message to $v$
7: for $M \in I$ do
8: let the sender be $w$ and let $M = \langle count_{w,c} \rangle$
9: if $w$ is a child of $v$ in $T_c$ then $count_{v,c} \leftarrow count_{v,c} + count_{w,c}$

Lemma A.16. The set of bottleneck nodes, $B$, constructed by Compute-Bottleneck (Algorithm 13) has size at most $\sqrt{|Q|}$.

Proof. Since every node $b$ added to set $B$ has total $\_count_b > n\sqrt{|Q|}$, removing such $b$ is going to remove at least $n\sqrt{|Q|}$ nodes across all trees in $Q$ in Step 6. And since there are at most $n \cdot |Q|$ nodes across all trees, set $B$ has size at most $\sqrt{|Q|}$. \hfill \Box

Lemma A.17. Compute-Bottleneck (Algorithm 13) runs for $O(n\sqrt{|Q|} + h \cdot |Q|)$ rounds.

Proof. Step 1 takes $O(h \cdot |Q|)$ rounds using Lemma A.18. Step 2 is a local computation step and involves no communication. Step 4 involves a broadcast of at most $n$ messages and hence takes $O(n)$ rounds using Lemma A.2. Step 5 again do not involve any communication. Step 6 takes $O(n)$ rounds [2, 1]. Since $B$ has size at most $\sqrt{|Q|}$ (by Lemma A.16), the while loop runs for at most $\sqrt{|Q|}$ iterations, thus establishing the lemma. \hfill \Box

A.6.2 Computing $count_{v,c}$ Values

Here we describe our algorithm for computing Step 1 of Algorithm 13, which computes $count_{v,c}$ values in a given $h$-CSSSP collection $C$ for source set $S$. Our algorithm (Algorithm 14) is quite simple and works as follows: Fix a source $c \in S$ and let $T_c$ be the tree corresponding to source $c$ in $C$. The goal is to compute the number of messages each node $v \in T_c$ needs to send to its parent. In Step 1 every node $v \in T_c$ initializes its $count_{v,c}$ value to 1. Every node $v$ that is $h_c(v)$ hops away from $c$ receives the $count$ values from all its children by round $h - h_c(v)$ (Steps 5-9) and it then send it to its parent in round $h - h_c(v) + 1$ (Step 3) after updating it (Step 9).

Lemma A.18. Compute-Count (Algorithm 14) correctly computes $count_{v,c}$ for every $v \in T_c$ in $h + 1$ rounds per source node $c$.

Proof. Every leaf node $v$ can initialize their $count_{v,c}$ values to 1 in Step 1. For every other internal node $v$, $v$ correctly computes $count_{v,c}$ value after receiving the $count$ values from all its children by round $h - h_c(v)$ (Steps 5-9) and then send the correct $count_{v,c}$ value to its parent in round $h - h_c(v) + 1$ in Step 3.

Since $h_c(v) \geq 0$, this algorithm requires at most $h + 1$ rounds. \hfill \Box