Conformal mappings of nearly quasi-Einstein manifolds

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CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

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Abstract. In this paper, we consider a conformal mapping between two nearly quasi-Einstein manifolds \( V_n \) and \( V_n' \). We find some properties of this transformation from \( V_n \) to \( V_n' \) and some theorems are proved.

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1. INTRODUCTION

A non-flat \( n \)-dimensional Riemannian or a semi-Riemannian manifold \((M, g)\), \((n > 2)\) is said to be an Einstein manifold if the condition

\[
S(X, Y) = \frac{r}{n} g(X, Y)
\]

holds on \( M \), where \( S \) and \( r \) denote the Ricci tensor and the scalar curvature of \((M, g)\), respectively. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat \( n \)-dimensional Riemannian manifold \((M, g)\), \((n > 2)\) is defined to be a quasi-Einstein manifold if its Ricci tensor \( S \) of type \((0, 2)\) is not identically zero and satisfies the following condition

\[
S(X, Y) = a g(X, Y) + b A(X) A(Y)
\]

where \( a, b \in \mathbb{R} \) and \( A \) is a non-zero 1-form such that \( g(X, U) = A(X) \) for all vector fields \( X \) on \( M \), [5]. Then \( A \) is called the associated 1-form and \( U \) is called the generator of the manifold. Also M. C. Chaki and R. K. Maity [2] studied the quasi-Einstein manifolds by considering \( a \) and \( b \) as scalars such that \( b \neq 0 \) and \( U \) as a unit vector field.

A non-flat \( n \)-dimensional Riemannian manifold \((M, g)\), \((n > 2)\) is called a nearly quasi-Einstein manifold if its Ricci tensor \( S \) of type \((0,2)\) is not identically zero and...
satisfies the following condition

\[ S(X, Y) = a g(X, Y) + b E(X, Y) \]  

(1.3)

where \( a \) and \( b \) are non-zero scalars and \( E \) is a non-zero symmetric tensor of type \((0,2)\), [4]. Then \( E \) is called the associated tensor and \( a \) and \( b \) are called the associated scalars of \( M \). An \( n \)-dimensional nearly quasi-Einstein manifold is denoted by \( N(QE)_n \). An example of \( N(QE)_4 \) has been given in [4].

Putting \( X = Y = e_i \) in (1.3), we get

\[ r = na + b \tilde{E}. \]  

(1.4)

Here \( r \) is the scalar curvature of \( N(QE)_n \) and \( \tilde{E} = E(e_i, e_i) \) where \( \{e_i\}, i = 1, 2, \ldots, n \) is an orthonormal basis of the tangent space at each point of the manifold.

In this paper, we investigate a conformal mapping between two nearly quasi-Einstein manifolds.

2. CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

In this section, we suppose that \( V_n \) and \( \tilde{V}_n \), \( (n > 2) \) are two nearly quasi-Einstein manifolds with metrics \( g \) and \( \tilde{g} \), respectively.

Definition 1. A conformal mapping is a diffeomorphism of \( V_n \) onto \( \tilde{V}_n \) such that

\[ \tilde{g} = e^{2\sigma} g \]  

(2.1)

where \( \sigma \) is a function on \( V_n \). If \( \sigma \) is constant, then it is called a homothetic mapping. In local coordinates, (2.1) is written as

\[ \tilde{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \quad \tilde{g}^{ij} = e^{-2\sigma} g^{ij}. \]  

(2.2)

Besides those equations, we have the Christoffel symbols, the components of the curvature tensor, the Ricci tensor, and the scalar curvature, respectively

\[ \hat{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \]  

(2.3)

\[ \hat{R}^h_{ijk} = R^h_{ijk} + s^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + s^h g^{ko} (\sigma_{ok} g_{ij} - \sigma_{oj} g_{ik}), \]  

(2.4)

\[ \hat{S}_{ij} = S_{ij} + (n - 2) \sigma_{ij} + (\Delta_2 \sigma + (n - 2) \Delta_1 \sigma) g_{ij}, \]  

(2.5)

\[ \tilde{r} = e^{-2\sigma} (r + 2(n - 1) \Delta_2 \sigma + (n - 1)(n - 2) \Delta_1 \sigma), \]  

(2.6)

where \( S_{ij} = R_{ij}^{o\alpha}, r = S_{\alpha\beta} g^{o\beta}, \sigma_i = \frac{\partial \sigma}{\partial x^i}, \sigma^h = \sigma_0 g^{o\alpha} \) and \( \sigma_{ij} = \nabla_j \nabla_i \sigma - \nabla_i \sigma \nabla_j \sigma \). \( \Delta_1 \sigma \) and \( \Delta_2 \sigma \) are the first and the second Beltrami’s symbols which are determined by

\[ \Delta_1 \sigma = g^{o\beta} \nabla_\alpha \sigma \nabla_\beta \sigma, \quad \Delta_2 \sigma = g^{o\beta} \nabla_\beta \nabla_\alpha \sigma \]  

(2.7)

(2.8)
where $\nabla$ is the covariant derivative according to the Riemannian connection in $V_n$. We denote the objects of space conformally corresponding to $V_n$ by a bar, i.e., $\bar{V}_n$. If $\bar{V}_n$ is a $N(QE)_n$, then we have, from (1.3), (2.2), and (2.5),

$$\bar{b}E_{ij} = bE_{ij} + (n - 2)\sigma_{ij} + (\Delta_2 \sigma + (n - 2)\Delta_1 \sigma + a - \bar{a}e^{2\sigma})g_{ij}. \quad (2.9)$$

**Definition 2.** A vector field $\xi$ in a Riemannian manifold $M$ is called torse-forming if it satisfies the condition $\nabla_\xi \xi = \rho X + \phi(X)\xi$ where $X \in TM$, $\phi(X)$ is a linear form and $\rho$ is a function, [12]. In the local transcription, this reads

$$\nabla_i \xi^h = \rho \delta_i^h + \xi^h \phi_i \quad (2.10)$$

where $\xi^h$ and $\phi_i$ are the components of $\xi$ and $\phi$, and $\delta_i^h$ is the Kronecker symbol. A torse-forming vector field $\xi$ is called recurrent if $\rho = 0$; concircular if the form $\phi_i$ is a gradient covector, i.e., there is a function $\psi(x)$ such that $\phi = d\psi(x)$; convergent, if it is concircular and $\rho = \text{const} \cdot \exp(\psi)$.

Therefore, recurrent vector fields are characterized by the following equation

$$\nabla_X \xi = \phi(X)\xi. \quad (2.11)$$

Also, from the Definition 2., for a concircular vector field $\xi$, we get

$$(\nabla_Y \xi)X = \rho g(X, Y) \quad (2.12)$$

for all $X, Y \in TM$. A Riemannian space with a concircular vector field is called equidistant, [10,11].

Conformal mappings of Riemannian spaces (or semi-Riemannian spaces) have been studied by many authors, [1,3,6,9]. In this section, we investigate the conformal mappings of nearly quasi-Einstein manifolds preserving the associated tensor $E$.

**Theorem 1.** If $V_n$ admits a conformal mapping preserving the associated tensor $E$ and the associated scalar $b$, then $V_n$ is an equidistant manifold.

**Proof.** Suppose that $V_n$ admits a conformal mapping preserving the associated tensor $E$ and the associated scalar $b$. Using (2.9), we obtain

$$(n - 2)\sigma_{ij} + (\beta + a - \bar{a}e^{2\sigma})g_{ij} = 0 \quad (2.13)$$

where $\beta = \Delta_2 \sigma + (n - 2)\Delta_1 \sigma$. In this case, we get

$$\sigma_{ij} = \alpha g_{ij} \quad (2.14)$$

where $\alpha = \frac{1}{n-2}(\bar{a}e^{2\sigma} - a - \beta)$ is a function. Putting $\xi = -\exp(-\sigma)$ and using (2.7), (2.12) and (2.14), we get that $V_n$ is an equidistant manifold. Hence, the proof is complete. \qed
Theorem 2. An equidistant manifold $V_n$ admits a conformal mapping preserving the associated tensor $E$ if the associated scalars $\tilde{a}$ and $\tilde{b}$ satisfy both of the conditions

$$\tilde{b} = b,$$

$$\tilde{a} = e^{-2\sigma} (a + \gamma),$$

where $\gamma = \left(\frac{n-1}{n}\right)[2 \Delta_2 \sigma + (n-2)\Delta_1 \sigma]$.

Proof. Suppose that $V_n$ is an equidistant manifold. Then, there exists a concircular vector field $\xi$ satisfying the condition (2.12), that is, we have

$$\nabla_j \xi_i = \rho g_{ij} \quad (2.15)$$

where $\xi_i = \nabla_i \xi$. Putting $\sigma = -\ln(-\xi(x))$ and using the condition (2.5), we obtain

$$\tilde{S}_{ij} = S_{ij} + \gamma g_{ij} \quad (2.16)$$

where $\gamma = \left(\frac{n-1}{n}\right)[2 \Delta_2 \sigma + (n-2)\Delta_1 \sigma]$. Considering (1.3) in (2.16) and using (2.2), we get

$$\tilde{a} e^{2\sigma} g_{ij} + \tilde{b} \tilde{E}_{ij} = (a + \gamma) g_{ij} + b E_{ij}. \quad (2.17)$$

Taking $\tilde{a} = e^{-2\sigma} (a+\gamma)$ and $\tilde{b} = b$, (2.17) implies that $\tilde{E}_{ij} = E_{ij}$. This completes the proof. $\square$

The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have, [8]

$$\nabla_j \sigma_i + \frac{1}{2} (n-2) \sigma^l \sigma_l = 0. \quad (2.18)$$

Theorem 3. Let $V_n$ be a conformal mapping with preservation of the associated tensor $E$ and the associated scalar $b$. A necessary and sufficient condition for this conformal mapping to be conharmonic is that the associated scalar $\tilde{a}$ be transformed by $\tilde{a} = e^{-2\sigma} a$.

Proof. Suppose that $V_n$ admits a conformal mapping preserving the associated tensor $E$ and the associated scalar $b$. Using (2.7), (2.8) and (2.9), we obtain

$$(n-2) \nabla_j \nabla_i \sigma - (n-2) \sigma_l \sigma_j + [\nabla_h \sigma^h + (n-2)\sigma^h \sigma_h + a - \tilde{a} e^{2\sigma}] g_{ij} = 0. \quad (2.19)$$

Multiplying (2.19) by $g^{ij}$, we get

$$\nabla_h \sigma^h + \frac{1}{2} (n-2) \sigma^h \sigma_h + \frac{n}{2(n-1)} (a - \tilde{a} e^{2\sigma}) = 0. \quad (2.20)$$

If this mapping is conharmonic, using (2.18) in (2.20), we obtain $\tilde{a} = e^{-2\sigma} a$. The converse is also true. This completes the proof. $\square$
Definition 3. A $\varphi$($\text{Ric}$)-vector field is a vector field on an $n$-dimensional Riemannian manifold $(M, g)$ and Levi-Civita connection $\nabla$, which satisfies the condition
\[
\nabla \varphi = \mu \text{Ric}
\]  
(2.21)
where $\mu$ is a constant and Ric is the Ricci tensor, [7]. When $(M, g)$ is an Einstein space, the vector field $\varphi$ is concircular. Moreover, when $\mu = 0$, the vector field $\varphi$ is covariantly constant. In local coordinates, (2.21) can be written as
\[
\nabla_j \varphi_i = \mu S_{ij}
\]  
(2.22)
where $S_{ij}$ denote the components of the Ricci tensor and $\varphi_i = \varphi^\alpha g_{i\alpha}$.

Suppose that $V_n$ admits a $\sigma$($\text{Ric}$)-vector field. Then, we have
\[
\nabla_j \sigma_i = \mu S_{ij}
\]  
(2.23)
where $\mu$ is a constant. Now, we can state the following theorem.

Theorem 4. Let us consider the conformal mapping (2.1) of a nearly quasi-Einstein manifold $V_n$ with constant associated scalars being also conharmonic with the $\sigma$($\text{Ric}$)-vector field. A necessary and sufficient condition for the length of $\sigma$ to be constant is that the trace of the associated tensor $E$ of $V_n$ be constant.

Proof. We consider that the conformal mapping (2.1) of a nearly quasi-Einstein manifold $V_n$ admitting a $\sigma$($\text{Ric}$)-vector field is also conharmonic. In this case, comparing (2.18) and (2.23), we get
\[
r = \frac{(2-n)}{2\mu} \sigma^j \sigma_j
\]  
(2.24)
where $r$ is the scalar curvature of $V_n$. If $V_n$ is of the constant associated scalars, from (1.4) and (2.24), we find
\[
E = \frac{1}{b} \left( \frac{(2-n)}{2\mu} \sigma^j \sigma_j - na \right).
\]
If the length of $\sigma$ is constant, then $\sigma^j \sigma_j = c$, where $c$ is a constant. Thus, we can see that $E$ is constant. The converse is also true. Hence, the proof is complete. \(\Box\)

3. An Example of a Nearly Quasi-Einstein Manifold

In this section, we consider a Riemannian metric $g$ on $\mathbb{R}^4$ by the formula
\[
ds^2 = g_{ij} dx^i dx^j = (x^4)^2 [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2
\]  
(3.1)
where $i, j = 1, 2, 3, 4$ and $x^1, x^2, x^3, x^4$ are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor,
the Ricci tensor and the scalar curvature are

\[
\begin{align*}
\Gamma^1_{14} &= \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^3}, \\
\Gamma^4_{11} &= \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{1/3}, \\
R_{1441} &= R_{2442} = R_{3443} = -\frac{2}{9}(x^4)^{2/3}, \\
R_{1221} &= R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{2/3}, \\
S_{11} &= S_{22} = S_{33} = \frac{2}{3}(x^4)^{2/3}, \\
S_{44} &= -\frac{2}{3}(x^4)^2, \\
r &= \frac{4}{3}(x^4)^{2/3}.
\end{align*}
\]  
(3.2)

Therefore \( \mathbb{R}^4 \) with the considered metric is a Riemannian manifold \((M_4, g)\) of non-vanishing scalar curvature. Let us now consider the associated scalars \(a, b\), and the associated tensor \(E\) as follows:

\[
a = -\frac{2}{3(x^4)^2}, \\
b = -\frac{1}{9x^4} \tag{3.3}
\]

and

\[
E_{ij}(x) = \begin{cases} 
-12(x^4)^{1/3} & \text{for } i = j = 1, 2, 3 \\
0 & \text{for } i = j = 4 \text{ and } i \neq j 
\end{cases} \tag{3.4}
\]

at any point \(x \in M\). To verify the relation (1.3), it is sufficient to check the relations \(S_{ii} = ag_{ii} + bE_{ii}, i = 1, 2, 3, 4\) since for the other cases, (1.3) holds trivially. From (3.2), (3.3), and (3.4), we obtain

\[
R.H.S \ of \ S_{11} = ag_{11} + bE_{11} = \frac{2}{3(x^4)^{2/3}} = S_{11}.
\]

Similarly, \(S_{22}, S_{33}, \text{ and } S_{44}\) are also satisfied. Hence, \((M_4, g)\) endowed with the metric (3.1) is a \(N(QE)_4\) with the conditions (3.3) and (3.4).

Let \((M_4, g)\) endowed with the metric (3.1) be a conformal mapping with preservation of the associated tensor \(E\) and the associated scalar \(b\). Also, we choose \(\sigma\) and \(\tilde{a}\) as follows:

\[
\sigma = \ln(x^1x^2x^3), \quad \tilde{a} = -\frac{2}{3(x^1x^2x^3x^4)^2} \tag{3.5}
\]

where \(x^1, x^2, x^3 > 0\). Now, we show that these choices satisfy Theorem 3.

From (3.5), we get \(\nabla_i \sigma = \frac{\partial \sigma}{\partial x^i} = x_i = \frac{1}{x^i} \text{ for } i = 1, 2, 3 \text{ and } x_4 = 0\). Moreover, the only non-vanishing covariant derivatives of \(\sigma_i (i = 1, 2, 3, 4)\) are

\[
\begin{align*}
\nabla_1 \sigma_4 &= \nabla_4 \sigma_1 = -\frac{2}{3x^1x^4}, \\
\nabla_2 \sigma_4 &= \nabla_4 \sigma_2 = -\frac{2}{3x^2x^4}, \\
\nabla_3 \sigma_4 &= \nabla_4 \sigma_3 = -\frac{2}{3x^3x^4}.
\end{align*}
\]  
(3.6)

(3.7)

(3.8)
and
\[
\nabla_1 \sigma_1 = -\frac{1}{(x^1)^2}, \quad \nabla_2 \sigma_2 = -\frac{1}{(x^2)^2}, \quad \nabla_3 \sigma_3 = -\frac{1}{(x^3)^2}. \tag{3.9}
\]

Using (3.6)–(3.9), we find
\[
g^{11} \nabla_1 \sigma_1 + g^{11} \sigma_1 \sigma_1 = 0 \tag{3.10}
\]
and similarly the other cases hold. Therefore, the condition (2.18) is satisfied.

Moreover, from (3.3) and (3.5), we obtain
\[
\hat{a} e^{2\sigma} = -\frac{2}{3(x^1 x^2 x^3 x^4)^2} \times e^{2 \ln(x^1 x^2 x^3)} = -\frac{2}{3(x^4)^2} = a. \tag{3.11}
\]

From (3.10) and (3.11), we see that the equation (2.20) is satisfied. Hence, Theorem 3 holds for \((M_4, g)\) endowed with the metric (3.1) and the conditions (3.3) and (3.5).

Now, we also show that \((M_4, g)\) endowed with the metric (3.1) is not a quasi-Einstein manifold. If possible, we have \(S_{ij} = ag_{ij} + b A_i A_j\), where \(i, j = 1, 2, 3, 4\). For \(i = j\), we get from this relation
\[
S_{ii} = ag_{ii} + b A_i A_i, \tag{3.12}
\]
for all \(i = 1, 2, 3, 4\). Since \(S_{ii} \neq 0\) and \(g_{ii} \neq 0\), we can choose \(a \neq 0, b \neq 0\) and \(A_i \neq 0\) for all \(i = 1, 2, 3, 4\) such that (3.12) holds. However, for these values of \(a, b\) and \(A_i\) and for \(i \neq j\), the equation \(S_{ij} = ag_{ij} + b A_i A_j\) cannot be satisfied because for \(i \neq j\), \(S_{ij} = g_{ij} = 0\) but \(A_i \neq 0\).

Therefore, \((M_4, g)\) is not a quasi-Einstein manifold. Thus, a \(N(QE)_n\) is not necessarily a quasi-Einstein manifold.

References

[1] H. W. Brinkmann, “Einstein spaces which are mapped conformally on each other,” Math. Ann., vol. 94, no. 1, pp. 119–145, 1925.
[2] M. C. Chaki and R. K. Maity, “On quasi-Einstein manifolds,” Publ. Math. Debrecen, vol. 57, pp. 297–306, 2000.
[3] O. Chepurna, V. Kiosak, and J. Mikeš, “Conformal mappings of Riemannian spaces which preserve the Einstein tensor,” Aplimat - Journal of Applied Mathematics, vol. 3, no. 1, pp. 253–258, 2010.
[4] U. C. De and A. K. Gazi, “On nearly quasi-Einstein manifolds,” Novi Sad J. Math., vol. 38, no. 2, pp. 115–121, 2008.
[5] R. Deszcz, M. Głogowska, M. Hotlos, and Z. Senturk, “On certain quasi-Einstein semisymmetric hypersurfaces,” Annales Univ. Sci. Budapest., vol. 41, pp. 151–164, 1998.
[6] L. P. Eisenhart, Riemannian geometry. Princeton: Princeton Univ. Press, 1926.
[7] I. Hinterleitner and V. A. Kiosak, “\(\varphi(Ric)\)-vector fields in Riemannian spaces,” Archivum Mathematicum, vol. 44, no. 5, pp. 385–390, 2008.
[8] Y. Ishii, “On conharmonic transformations,” Tensor N. S., vol. 7, pp. 73–80, 1957.
[9] J. Mikeš, M. L. Gavrilchenko, and E. I. Gladysheva, “Conformal mappings onto Einstein spaces,” Mosc. Univ. Math. Bull., vol. 49, no. 3, pp. 10–14, 1994.
[10] N. S. Sinyukov, *Geodesic mappings of Riemannian spaces*. Moscow: Nauka, 1979.
[11] K. Yano, “Concircular geometry, I-IV,” *Proc. Imp. Acad.*, vol. 16, pp. 195–200, 354–360, 442–448, 505–511, 1940.
[12] K. Yano, “On the torse-forming directions in Riemannian spaces,” *Proc. Imp. Acad.*, vol. 20, no. 6, pp. 340–345, 1944.

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