$G_2$–invariant 7D Euclidean super Yang–Mills theory as a higher–dimensional analogue of the 3D super–BF theory

D. Mülsch$^a$ and B. Geyer$^b$

$^a$ Wissenschaftszentrum Leipzig e.V., D–04103 Leipzig, Germany

$^b$ Universität Leipzig, Naturwissenschaftlich-Theoretisches Zentrum and Institut für Theoretische Physik, D–04109 Leipzig, Germany

Abstract

A formulation of $N_T = 1$, $D = 8$ Euclidean super Yang–Mills theory with generalized self–duality and reduced $Spin(7)$–invariance is given which avoids the peculiar extra constraints of Ref. [7]. Its reduction to seven dimensions leads to the $G_2$–invariant $N_T = 2$, $D = 7$ super Yang–Mills theory which may be regarded as a higher–dimensional analogue of the $N_T = 2$, $D = 3$ super–BF theory. When reducing further that $G_2$–invariant theory to three dimensions one gets the $N_T = 2$ super–BF theory coupled a spinorial hypermultiplet.

1 Introduction

Recent developments in string duality and compactifications of M–theory renewed the interest in those field theories in dimensions $D > 4$ whose low energy effective action turns out to be (essentially) that of dimensionally reduced supersymmetric gauge theories. Such theories may arise naturally on the world volumes of Euclidean D–branes wrapping manifolds of special holonomy. In particular, compactifications of $D = 11$ supergravity on compact Joyce seven– and eight–folds with $G_2$ and $Spin(7)$ holonomy, respectively, have attracted some attention [1].

Moreover, independent of that development, it has been shown [2] [3] [4] how the notion of topological quantum field theories [5] or, more specifically, cohomological gauge theories being related to supersymmetric gauge theories by twisting, can be extended to dimensions greater than four. Such higher–dimensional (untwisted) cohomological theories are obtained when Euclidean supersymmetric gauge theories are considered on manifolds with reduced holonomy group $H \subset SO(D)$. These theories, which have a rather intriguing structure, localize onto the moduli space of certain generalized, higher–dimensional self–duality equations [6].

Recently, Euclidean super Yang–Mills theory (SYM) with generalized self–duality was explicitly established both in eight and seven dimensions with the $SO(8)$ and $SO(7)$ rotation invariance being broken down to $Spin(7)$ and $G_2$, respectively [7]. In four dimensions the main ingredient connecting self–duality with simple or extended supersymmetry is the chirality of fermions [8]. In Ref. [7] it has been verified that in the case of generalized self–duality in $D > 4$ one needs, in addition to usual chirality, certain constraints on the supersymmetry parameters. Moreover, these extra constraints for the $G_2$–invariant theory can not be obtained from the $Spin(7)$–invariant theory in eight dimensions via simple dimensional reduction.

In this Letter we will re–analyse that problem, thereby relaxing the reality condition on fermions. We explicitly verify that when hermiticity in Euclidean space is abandoned — which bears no problem here since hermiticity is primarily needed to ensure unitarity in Minkowski space — chirality of fermions is a consistent and sufficient constraint being compatible with generalized self–duality, $Spin(7)$–invariance and octonionic algebra. Then, in fact, the $G_2$–invariant theory can be obtained by ordinary dimensional reduction. Moreover, that theory has a nice interpretation: It may be regarded as the seven–dimensional analogue of the $N_T = 2$,
$D = 3$ super–BF theory \[9\], just as the $Spin(7)$–invariant theory may be regarded as the eight–dimensional analogue of the $N_T = 1, D = 4$ Donaldson–Witten theory \[3\]. Namely, replacing in the $G_2$–invariant theory the octonionic through the quaternionic structure constants and considering all the fields as three–dimensional ones one gets exactly the $N_T = 2, D = 3$ super–BF theory (without matter). On the other hand, compactifying the $G_2$–invariant theory down to three dimensions gives the $N_T = 2, D = 3$ super–BF theory with matter.

2 $Spin(7)$–invariant, $N_T = 1, D = 8$ Euclidean SYM

Now, let us formulate the $Spin(7)$–invariant $N_T = 1, D = 8$ SYM without requiring the reality condition for the fermions. Thereby, we avoid the subtlety of Ref. \[7\] which is associated with the compatibility of dimensional reduction (from eight to seven dimensions) and generalized self–duality (in seven dimensions).

First, we introduce the $SO(8)$–invariant action of the Euclidean $N = 2, D = 8$ SYM by ordinary dimensional reduction of the Minkowskian $N = 1, D = 10$ SYM \[10\] and subsequent Wick rotation into the Euclidean space. Its field content consists of an anti–hermitean vector field $A_a \ (a = 1, \ldots, 8)$, 16–component chiral and anti–chiral Weyl spinors, $\lambda$ and $\bar{\lambda}$, respectively, and scalar fields $\phi$ and $\bar{\phi}$, all of them taking their values in the Lie algebra $Lie(G)$ of some compact gauge group $G$. As a result, one obtains

$$S^{(N=2)} = \int_E d^8x \text{tr} \left\{ \frac{1}{4} F^{ab}_{\alpha \beta} F_{\alpha \beta} + 2 \bar{\lambda} \Gamma^a D_a \lambda - 2 D^a \bar{\phi} D_a \phi + 2 \bar{\lambda} T C_8^{-1} [\phi, \lambda] - 2 \bar{\lambda} C_8 [\phi, \bar{\lambda} T] - 2 [\bar{\phi}, \phi]^2 \right\}, \quad (1)$$

where $F_{ab} = \partial_a A_b + [A_a, A_b]$ and $D_a = \partial_a + [A_a, \cdot \cdot \cdot]$ and $C_8$ is the charge conjugation matrix, $C_8^{-1} \Gamma_a C_8 = - \Gamma_a^T$. For the moment, we do not specify the $SO(8)$ matrices $\Gamma_a$ explicitly, \(\frac{1}{2} \{ \Gamma_a, \Gamma_b \} = \delta_{ab} I_{16}\).

The action \[1\] is invariant under the following supersymmetry transformations

$$\delta A_a = \bar{\zeta} \Gamma_a \lambda - \bar{\lambda} \Gamma_a \zeta,$$
$$\delta \phi = \bar{\zeta} C_8 \bar{\lambda} T,$$
$$\delta \bar{\phi} = \lambda T C_8^{-1} \zeta,$$
$$\delta \lambda = - \frac{1}{4} \Gamma^{ab} \bar{\zeta} T F_{ab} + \Gamma^a \bar{C}_8 \bar{\zeta} T D_a \bar{\phi} - \zeta [\phi, \bar{\phi}],$$
$$\delta \bar{\lambda} = \frac{1}{4} \bar{\zeta} T F_{ab} - \bar{\zeta} T C_8^{-1} \Gamma^a D_a \phi - \bar{\zeta} [\phi, \bar{\phi}], \quad (2)$$

Here, $\zeta$ and $\bar{\zeta}$ are constant chiral and anti–chiral Weyl spinors, respectively, with $\Gamma_9 \zeta = - \zeta$, where $\Gamma_9 := \Gamma_1 \cdots \Gamma_8$, and $\Gamma_{ab} = \frac{1}{2} [\Gamma_a, \Gamma_b]$ are the generators of $SO(8)$ rotations.

In order to get from \[1\] a cohomological action with an underlying $N_T = 1$ equivariantly nilpotent shift symmetry we break down the Euclidean rotation group $SO(8)$ to $Spin(7)$, i.e., we replace the $SO(8)$ matrices by the standard embedding $\Gamma_a = (\Gamma_A, \Gamma_8)$, $A = 1, 2, \ldots, 7$, of $Spin(7)$ into $SO(8)$. In this representation we have (see, e.g., \[6\])

$$\Gamma_A = \begin{pmatrix} 0 & -i (\gamma_A)_{\alpha \beta} \\ i (\gamma_A)_{\beta \alpha} & 0 \end{pmatrix}, \quad \Gamma_8 = \begin{pmatrix} 0 & \delta_{\alpha \beta} \\ \delta_{\beta \alpha} & 0 \end{pmatrix}, \quad \Gamma_9 = \begin{pmatrix} \delta_{\alpha \beta} & 0 \\ 0 & -\delta_{\alpha \beta} \end{pmatrix}, \quad (3)$$

and for the charge conjugation matrix we may choose $C_8 = \Gamma_9$. The 7 imaginary antisymmetric $Spin(7)$ matrices $(\gamma_A)_{\alpha \beta}$ ($\alpha = 1, \ldots, 8$) are defined by

$$(\gamma_A)_{AB} = i \delta_{AB}, \quad (\gamma_A)_{BC} = i \Psi_{ABC}, \quad 1 \leq A, B, C \leq 7,$$
Besides, we introduce the dual octonionic structure constants, Φ. Below, we also need the following basic relations. The non–vanishing components of this tensor are

\[ \Phi_{123} = \Psi_{246} = \Psi_{435} = \Psi_{367} = \Psi_{651} = \Psi_{572} = \Psi_{714} = 1. \]

Besides, we introduce the dual octonionic structure constants, \( \Phi_{ABCD} = -\frac{1}{6} \epsilon_{ABCD}EFG \Phi^{EFG} \), which, together with

\[ \Phi_{8ABC} = \Psi_{8ABC}, \quad \Phi_{8AB} = \Psi_{8AB}, \]

define the 8–dimensional Spin(7)–invariant self–dual Cayley tensor \( \Phi_{abcd} = \frac{1}{24} \epsilon_{abcdefgh} \Phi^{abcdefgh}. \)

The non–vanishing components of this tensor are

\[ \Phi_{1238} = \Phi_{2468} = \Phi_{3458} = \Phi_{3678} = \Phi_{6518} = \Phi_{5728} = \Phi_{1748} = -1, \]
\[ \Phi_{4567} = \Phi_{3751} = \Phi_{6172} = \Phi_{5214} = \Phi_{7423} = \Phi_{1346} = \Phi_{2635} = -1. \]

Below, we also need the following basic relations,

\[ \Psi^{ABC} \Psi_{CDE} = \delta^{[A}_{C} \delta^{B]}_{D} + \Psi^{AB}_{CD}, \]
\[ \Psi^{ABF} \Phi_{CDE} = \frac{1}{2} \Psi^{A}_{[CD} \delta^{B]}_{E]}, \]
\[ \Phi^{ABCG} \Phi_{DEFG} = \delta^{[A}_{D} \delta^{B} \delta^{C]}_{F} + \frac{1}{4} \delta^{[AB}_{D} \delta^{C]}_{E} F ] - \Psi^{ABC} \Psi_{DEF}. \]

With the representation of the SO(8) matrices \( \Gamma_{a} = (\Gamma_{A}, \Gamma_{8}) \) we get the generators \( \Gamma_{ab} = (\Gamma_{AB}, \Gamma_{A8}) \) of the SO(8) rotations as follows,

\[ \Gamma_{AB} = \left( \begin{array}{cc} (\gamma_{AB})_{a\beta} & 0 \\ 0 & (\gamma_{AB})_{\alpha\beta} \end{array} \right), \quad \Gamma_{A8} = \left( \begin{array}{cc} -i(\gamma_{A})_{a\beta} & 0 \\ 0 & i(\gamma_{A})_{\alpha\beta} \end{array} \right), \]

where \((\gamma_{AB})_{a\beta}\) are the antisymmetric Spin(7) generators,

\[ (\gamma_{AB})_{C8} = \Psi_{ABC}, \quad (\gamma_{AB})_{CD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} - \Phi_{ABCD}. \]

Now, let us construct the Spin(7)–invariant action. First, we write the Weyl spinors as

\[ \lambda = -\Gamma_{9}\lambda = \left( \begin{array}{c} 0 \\ \lambda_{\alpha} \end{array} \right), \quad \bar{\lambda} = \lambda \Gamma_{9} = (\bar{\lambda}_{\alpha}, 0), \]

and consider, in Euclidean space, \( \lambda_{\alpha} \) and \( \bar{\lambda}_{\alpha} \) as two independent 8–component spinors. Hence, \( \lambda \) and \( \bar{\lambda} \) are no longer subjected to the reality condition. More precisely, just as in Minkowski space, we take the adjoint spinor as \( \bar{\lambda} = \lambda \Gamma_{8} = (\bar{\lambda}_{\alpha}, 0) \) and, afterwards, we drop the reality condition between \( \lambda \) and \( \bar{\lambda} \). In addition, for clarity, we change notation as \( \bar{\lambda}_{\alpha} = \bar{\lambda}_{\alpha} \).

Next, we introduce a Spin(7)–octet of vector fields \( \psi_{a} \), which is obtained from the 8–spinor \( \bar{\lambda}_{\alpha} \) by identifying the spinor index \( \alpha \) with the vector index \( a \), i.e., \( \psi_{a} = \bar{\lambda}_{\alpha} \) \((a, \alpha = 1, \ldots, 8)\), a Spin(7)–septet of self–dual tensor fields, \( \chi_{ab} = \frac{1}{4} \Phi_{abcd} \chi^{cd} \), and a Spin(7)–singlet scalar field \( \eta \), which are obtained from the 8–spinor \( \lambda_{\alpha} = (\lambda_{A}, \lambda_{8}) \) according to \( \chi_{A8} = \lambda_{A}, \chi_{8} = \Psi_{A8} = (\lambda_{A}, \lambda_{8}) \) and \( \chi_{AB} = \Psi_{ACB} = \chi_{AB} = \Psi_{ACB} = \chi_{AB} \) and \( \eta = \lambda_{8} \), respectively.

Then, after substituting in the action for \( \Gamma_{a} \) the representation, replacing \( C_{8} \) through \( \Gamma_{9} \), and introducing the fields \( \eta, \psi_{a} \) and \( \chi_{ab} \), one gets the following Spin(7)–invariant action,

\[ S^{(N_{T} = 1)}_{\text{Spin}(7) \subset SO(8)} = \int_{E} dx^{8} \text{tr} \left\{ \frac{1}{4} F^{ab} D_{ab} - 2 D^{a} \phi D_{a} \phi - 2 \chi_{ab} D_{a} \psi_{b} + 2 \eta D^{a} \psi_{a} \right. \]
\[ + 2 \bar{\phi} \{ \psi_{a}, \psi_{a} \} + \frac{1}{4} \phi \{ \chi_{ab}, \chi_{ab} \} + 2 \phi \{ \eta, \eta \} - 2 [\bar{\phi}, \phi]^{2} \right\}. \]

\(^{3}\)By the help of \( \Phi_{abcd} \) one can define Spin(7) as that subgroup of SO(8) whose action on the 8–dimensional Euclidean space preserves the Cayley four–form \( \Phi = \frac{1}{24} \Phi_{abcd} dx^{a} \wedge dx^{b} \wedge dx^{c} \wedge dx^{d} \).
Furthermore, from (2), decomposing the (anti)chiral spinors $\zeta$ and $\bar{\zeta}$ in the same manner as $\lambda$ and $\bar{\lambda}$ and performing the same replacements as before, after a straightforward but lengthy calculation, one gets the following on-shell supersymmetry transformations:

\[
Q A_a = \psi_a, \quad Q \bar{\psi}_a = D_a \phi, \\
Q \phi = 0, \quad Q \bar{\phi} = \eta, \\
Q \eta = [\bar{\psi}, \psi], \quad Q \chi_{ab} = \frac{1}{2} \Theta_{abcd} F^{cd}, \\
Q a A_b = \delta_{ab} \eta + \chi_{ab}, \quad Q a \psi_b = F_{ab} - \frac{1}{4} \Theta_{abcd} F^{cd} + \delta_{ab}[\bar{\phi}, \phi], \\
Q a \phi = \psi_a, \quad Q a \bar{\phi} = 0, \\
Q a \eta = D_a \bar{\phi}, \quad Q a \chi_{cd} = \Theta_{abcd} D^b \bar{\phi} 
\]

(7)

and

\[
Q a b A_c = -\Theta_{abcd} \psi^d, \quad Q a b \psi_c = \Theta_{abcd} D^d \phi, \\
Q a b \phi = 0, \quad Q a b \bar{\phi} = \chi_{ab}, \\
Q a b \eta = -\frac{1}{4} \Theta_{abcd} F^{cd}, \quad Q a b \chi_{cd} = \frac{1}{4} \Theta_{abef} \Theta_{cdef} F^{ef} + \Theta_{abcd} [\bar{\psi}, \phi]. 
\]

(8)

Here, we have introduced the (unnormalized) projector

\[
\Theta_{abcd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} + \Phi_{abcd}, \quad \frac{1}{8} \Theta_{abef} \Theta_{cdef} = \Theta_{abcd},
\]

(10)

which projects any antisymmetric second rank tensor onto its self–dual part, according to the decomposition $28 = 7 \oplus 21$ of the adjoint representation of $SO(8) \sim SO(8) / Spin(7) \otimes Spin(7)$.

In order to verify that the transformations (7) – (9) really leave the action (6) invariant, one needs the following two identities,

\[
\frac{1}{2} (\Theta_{abef} \Theta_{cdef} - \Theta_{abfg} \Theta_{cdef}^g) = -\Theta_{efac} \delta_{bd} + \Theta_{efad} \delta_{bc} + \Theta_{efbd} \delta_{ac} - \Theta_{efbd} \delta_{ac} \\
\quad + \Theta_{abde} \delta_{cf} - \Theta_{abec} \delta_{df} - \Theta_{abde} \delta_{cf} + \Theta_{abef} \delta_{cd} \\
\quad - \Theta_{cdbe} \delta_{af} + \Theta_{cdbe} \delta_{af} + \Theta_{cdaf} \delta_{be} - \Theta_{cdaf} \delta_{be}, \\
\frac{1}{2} (\Theta_{abef} \Theta_{cdef}^g + \Theta_{abfg} \Theta_{cdef}^g) = \Theta_{abcd} \delta_{ef},
\]

(11)

which encode all of the algebraic properties of the structure constants $\Psi_{ABC}$ and $\Phi_{ABCD}$ being displayed in Eqs. (4) and (5). Moreover, by the help of (11), one can check that the transformations (7) – (9) satisfy the following superalgebra on–shell,

\[
\{Q, Q\} \doteq -2\delta_G(\phi), \quad \{Q, Q_a\} \doteq \partial_a + \delta_G(A_a), \quad \{Q_A, Q_b\} \doteq -2\delta_{ab} \delta_G(\bar{\phi}), \\
\{Q, Q_{cd}\} \doteq 0, \quad \{Q_a, Q_{cd}\} \doteq \Theta_{abcd} (\partial^b + \delta_G(A^b)), \quad \{Q_{ab}, Q_{cd}\} \doteq -2\Theta_{abcd} \delta_G(\phi),
\]

where $\delta_G(\varphi)$ denotes a gauge transformation with field–dependent parameter $\varphi = (A_a, \phi, \bar{\phi})$ being defined by $\delta_G(\varphi) = -D_a \varphi$ and $\delta_G(\varphi) X = [\varphi, X]$ for all the other fields. (The symbol $\doteq$ means that the corresponding relation is fulfilled only on–shell.)

Finally, by adding to (6) a topological term,

\[
S^{(N_T = 1)} = S^{(N_T = 1)}_{Spin(7) \subset SO(8)} + \int_E d^8 x \text{tr} \left\{ \frac{1}{8} \Theta^{abcd} F_{ab} F_{cd} \right\},
\]

for the cohomological action we are looking for, by virtue of (10), one immediately obtains

\[
S^{(N_T = 1)} = \int_E d^8 x \text{tr} \left\{ \frac{1}{8} \Theta^{abcd} F_{ab} F_{cd} - 2D^a \bar{\phi} D_a \phi - 2\chi^{ab} D_a \psi_b + 2\eta D^a \psi_a \\
\quad + 2\bar{\phi} \{\psi^a, \psi_a\} + \frac{1}{4} \bar{\phi} \{\chi^{ab}, \chi_{ab}\} + 2\phi \{\eta, \eta\} - 2[\bar{\phi}, \phi]^2 \right\}. 
\]

(12)
Except for some field rescalings, this is precisely the action given in Refs. [4]. Notice that, by virtue of \( \frac{1}{2} \Theta_{aef} \Theta_{cd} e^f = \Theta_{abcd} \), only the self–dual part \( F^+_{ab} = \frac{1}{8} \Theta_{abcd} F^{cd} \) enters into the first term of (12).

On–shell, upon using the equation of motion of \( \chi^{ab} \), the action (12) can be recast into the \( Q \)–exact form \( Q \Psi \), with the gauge fermion

\[
\Psi = \int_E d^8 x \text{tr} \left\{ \chi^{ab} (F_{ab} - \frac{1}{16} \Theta_{abcd} F^{cd}) - 2 \psi^a D_a \phi - 2 \langle [\eta, \bar{\phi}] \phi \rangle \right\}.
\]

Here, the first term enforces the localization onto the moduli space and the third term ensures that pure gauge degrees of freedom are projected out. The second and fourth term are typically for the Feynman type gauge; they could be omitted, leading to the Landau type gauge.

### 3 \( G_2 \)–invariant, \( N_T = 2, D = 7 \) SYM with global \( SU(2) \) symmetry

After having established the \( Spin(7) \)–invariant \( N_T = 1, D = 8 \) SYM — without introducing extra constraints — the \( G_2 \)–invariant \( N_T = 2, D = 7 \) SYM can be simply obtained by ordinary dimensional reduction.

First, we introduce two \( SU(2) \)–doublets of scalar and vector fields, \( \eta^\alpha \) and \( \psi_A^{\alpha} (\alpha = 1, 2) \), and a \( SU(2) \)–triplet of scalar fields \( G^{\alpha \beta} \),

\[
\eta^\alpha = \left( \begin{array}{c} \psi_8 \\ \eta \end{array} \right), \quad \psi_A^{\alpha} = \left( \begin{array}{c} \psi_A \\ \chi_{AS} = -\frac{1}{2} \Psi_{ABC} \chi^{BC} \end{array} \right), \quad G^{\alpha \beta} = \left( \begin{array}{c} \phi \\ \frac{1}{2} A_8 \\ \chi \end{array} \right),
\]

where the spinor fields \( \eta^\alpha \) and \( \psi_A^{\alpha} \) are singlets and septets of the group \( G_2 \), respectively.\(^4\)

The internal group index \( \alpha \) is raised and lowered as follows: \( \varphi^\alpha = \epsilon^{\alpha \beta} \varphi_\beta \) and \( \varphi_\alpha = \varphi_\beta \epsilon^{\beta \gamma} \), with \( \epsilon^{\alpha \gamma \beta} = \delta^{\alpha \beta} \), where \( \epsilon^{\alpha \beta} \) is the antisymmetric invariant tensor of the group \( SU(2) \).

Then, by dimensional reduction, from (12) one arrives at the following \( G_2 \)–invariant cohomological action with an underlying \( N_T = 2 \) equivariantly nilpotent shift symmetry \( Q^\alpha \) and global symmetry group \( SU(2) \),

\[
S^{(N_T=2)} = \int_E d^7 x \text{tr} \left\{ \frac{1}{8} \Theta_{ABCD} F_{AB} F_{CD} - D_A G_{\alpha \beta} D_A G^{\alpha \beta} - \Psi_{ABC} \psi_{A\alpha} D_B \psi_{C\alpha} - 2 \eta_\alpha D_A \psi^A_{\alpha} - 2 G_{\alpha \beta} \left\{ \psi_{A\alpha}, \psi^A_{\beta} \right\} + 2 G_{\alpha \beta} \left\{ \eta^\alpha, \eta^\beta \right\} + [G_{\alpha \beta}, G_\gamma] [G^{\alpha \beta}, G^{\gamma \delta}] \right\}.
\]

Here, analogous to (10), we have introduced the (unnormalized) projector (c.f., Eq. (5))

\[
\Theta_{ABCD} = \Psi_{ABE} \Psi_{CDE}, \quad \frac{1}{6} \Theta_{ABEF} \Theta_{CDE}^{EF} = \Theta_{ABCD},
\]

which projects any antisymmetric second rank tensor onto its self–dual part \( T = 7 \oplus 14 \) of \( Spin(7) \sim Spin(7)/G_2 \otimes G_2 \).

Next, we put \( Q, Q_a = (Q_A, Q_8) \) and \( Q_{ab} = (Q_{AB}, Q_{AS} = -\frac{1}{6} \Psi_{ABC} Q^{BC}) \) into the following \( SU(2) \)–doublets of scalar and vector supercharges,

\[
Q^{\alpha} = \left( \begin{array}{c} Q \\ -Q_8 \end{array} \right), \quad Q_A^{\alpha} = \left( \begin{array}{c} -Q_{AS} \\ Q_A \end{array} \right).
\]

\(^4\)By the help of \( \Psi_{ABC} \) one can define \( G_2 \) as the subgroup of \( SO(7) \) whose action on the 7–dimensional Euclidean space preserves the associative 3–form \( \Psi = \frac{1}{4} \Psi_{ABC} dx^A \wedge dx^B \wedge dx^C \). Alternatively, it can be also characterized as the maximal common subgroup of \( SO(7) \) and \( Spin(7) \), i.e., \( G_2 = SO(7) \cap Spin(7) \). [11]
Once more, performing the same dimensional reduction, from (7) – (9) one obtains the following on–shell supersymmetry transformations:

\[ Q^\alpha_A = \psi^\alpha_A, \]
\[ Q^\alpha\psi^\beta_B = D_A G^{\alpha\beta} - \frac{1}{4} \epsilon^{\alpha\beta} \Psi_{ABC} F^{BC}, \]
\[ Q^\alpha\eta^\beta_B = -\epsilon_{\gamma\delta}[G^{\alpha\gamma}, G^{\beta\delta}], \]
\[ Q^\alpha G^{\beta\gamma} = \frac{1}{2} \epsilon^{\alpha(\beta\gamma)}, \]

(14)

and

\[
\begin{align*}
Q_A^\alpha A_B &= \delta_{AB} \eta^\alpha - \Psi_{ABC} \psi_C^\alpha, \\
Q_A^\alpha \psi_B^\beta &= -\epsilon^{\alpha\beta} F_{AB} + \frac{1}{2} \epsilon^{\alpha\beta} \Theta_{ABCD} F^{CD} + \Psi_{ABC} D^C G^{\alpha\beta} + \delta_{AB} \epsilon_{\gamma\delta}[G^{\alpha\gamma}, G^{\beta\delta}], \\
Q_A^\alpha \eta^\beta &= D_A G^{\alpha\beta} - \frac{1}{4} \epsilon^{\alpha\beta} \Psi_{ABC} F^{BC}, \\
Q_A^\alpha G^{\beta\gamma} &= -\frac{1}{2} \epsilon^{\alpha(\beta\gamma)}. \\
\end{align*}
\]

(15)

By making use of (5) one easily verifies that the above transformations satisfy the following superalgebra on–shell,

\[
\left\{ Q^\alpha, Q^\beta \right\} = -2 \delta_G (G^{\alpha\beta}), \quad \left\{ Q^\alpha, Q_A^\beta \right\} = \epsilon^{\alpha\beta}(\partial A + \delta_G(A_A)), \\
\left\{ Q_A^\alpha, Q_B^\beta \right\} = -2 \delta_{AB} \delta_G (G^{\alpha\beta}) + \epsilon^{\alpha\beta} \Psi_{ABC}(\partial C + \delta_G(A_C)).
\]

Furthermore, on–shell, upon using the equation of motion for \( \psi^\alpha_A \), the action (13) can be rewritten as

\[ S^{(N_T=2)} = \frac{1}{2} \epsilon_{\alpha\beta} Q^\alpha Q^\beta \Omega \]

with the gauge boson

\[ \Omega = S_{CS} + \int_E d^7 x \text{tr} \left\{ \psi^A_A \psi^\alpha_A - \eta^\alpha \eta^\alpha \right\}, \]

(16)

where \( S_{CS} \) is the 7–dimensional Chern–Simons action,

\[ S_{CS} = - \int_E d^7 x \text{tr} \left\{ \Psi_{ABC}(A_A \partial_B A_C + \frac{2}{3} A_A A_B A_C) \right\}. \]

(17)

Let us notice that, with regard to the particular structure of \( \Omega \), in Ref. [4] the question has been raised whether a 7–dimensional analogue of the Schwarz–type topological Chern–Simons theory [14] exists which can be obtained directly from the action [17]. However, it was pointed out that, in contrast to the 3–dimensional case, the quantization of such an action remains an open question because the Gauss law in the \( A_T = 0 \) gauge is not sufficient to consistently solve the theory. Here instead, according to our construction, we observe that the action [17] appears quite natural as the relevant part of the gauge fermion [16] in the \( G_2 \)–invariant Witten–type cohomological Yang–Mills theory.

4 Dimensional reduction to three dimensions

Independently, one may ask whether the \( G_2 \)–invariant action [13] may be regarded as higher–dimensional analogue of some topologically twisted action (in \( D \leq 4 \)). In this section we show that the \( N_T = 2, D = 3 \) super–BF theory [9], with a spinorial hypermultiplet coupled to it, gets unified in the \( G_2 \)–invariant action of the \( N_T = 2, D = 7 \) SYM.

In fact, identifying the octonionic structure constants \( \Psi_{ABC} \) for \( 1 \leq A, B, C \leq 3 \) with the totally antisymmetric Levi–Civita tensor \( \epsilon_{ijk} \) (\( i = 1, 2, 3 \)) and considering all the fields as 3–dimensional ones, we recover from that part of [13] precisely the on–shell formulation of the
$N_T = 2$, $D = 3$ super–BF model with global symmetry group $SU(2)$,\(^5\)

$$S_{\text{BF}}^{(N_T=2)} = \int d^3x \text{tr} \left\{ \frac{1}{4} F_{ij} F_{ij} - D_i G_{\alpha\beta} D_j G^{\alpha\beta} - \epsilon^{ijk} \psi_{i\alpha} D_j \psi_k^{\alpha} - 2\eta_{\alpha} D^i \psi_i^{\alpha} + 2G_{\alpha\beta} \{\psi_i^{\alpha}, \psi_j^{\beta}\} + 2G_{\alpha\beta} \{\eta^\alpha, \eta^\beta\} + [G_{\alpha\beta}, G_{\gamma\delta}] [G^{\alpha\beta}, G^{\gamma\delta}] \right\}. \quad (18)$$

In addition, let us (formally) replace the structure constants $\Psi_{iAB}$ and $\delta_{AB}$ for $4 \leq A, B \leq 7$ by $i(\sigma_i)_{a^b} \alpha^\beta$ and $-\epsilon_{ab} \alpha^c$, respectively, $\sigma_i = (\sigma_i)_a^b$ being the Pauli matrices, and let us put $A_A$ and $\psi_A^a$ for $4 \leq A \leq 7$ into the $SU(2)$–doublets $M_a^\alpha$ and $\alpha^\beta_{a^b}$ ($a = 1, 2$), respectively. Then, after dimensional reduction to three dimensions, from that remaining part of (13) one gets the following action with global symmetry group $SU(2) \otimes SU(2)$,\(^6\)

$$S^{(N_T=2)} = S_{\text{BF}}^{(N_T=2)} + \int d^3x \text{tr} \left\{ i\lambda_{\alpha\beta}(\sigma_i)^{ab} D_i \lambda_{\beta} - \frac{i}{2} D^i M_{\alpha\beta} D_i M^{\alpha\beta} - \epsilon_{\alpha\beta} B_i \right\}, \quad (19)$$

where the relations $(\sigma_i)_a^c (\sigma_j)_c^b = \delta_{ij} \delta^b \delta^b + i\epsilon_{ijk} \sigma_k^b$ and $(\sigma_i)^b (\sigma_i)_c^d = \epsilon_{abc} b^d - \delta^d \delta^b c^d$. The action (19) describes the $N_T = 2$, $D = 3$ super–BF model with matter $(M_a^\alpha, \alpha^\beta_{a^b})$ in the adjoint representation. It may be regarded as the dimensionally reduced non–Abelian version of the Seiberg–Witten (monopole) theory.

The full set of on–shell supersymmetry transformations which leave (19) invariant reads,

$$Q^\alpha A_i = \psi_i^\alpha, \quad Q^\alpha \psi_i^{\beta} = D_i G^{\alpha\beta} - \epsilon^{\alpha\beta} B_i,$$

$$Q^\alpha \eta^\beta = -\epsilon_{\gamma\delta} [G^{\alpha\gamma}, G^{\beta\delta}], \quad Q^\alpha G^{\beta\gamma} = \frac{1}{2} \epsilon^{\alpha(\beta \gamma)},$$

$$Q^\alpha M_a^\beta = \lambda_a^\beta, \quad Q^\alpha \lambda_{a^\beta}^{\gamma} = -[G^{\alpha\beta}, M_a^\gamma] - \epsilon^{\alpha\beta} C^{a^\gamma},$$

$$Q_i^\alpha A_j = \delta_{ij} \eta^\alpha - \epsilon_{ijk} \psi^{k^\alpha}, \quad Q_i^\alpha \psi_j^{\beta} = -\epsilon_{\alpha\beta} F_{ij} + \delta_{ij} \delta_{\gamma\delta} [G^{\alpha\gamma}, G^{\beta\delta}] + \epsilon_{ijk} (D^k G^{\alpha\beta} + \epsilon^{\alpha\beta} B^k),$$

$$Q_i^\alpha \eta^\beta = D_i G^{\alpha\beta} - \epsilon^{\alpha\beta} B_i, \quad Q_i^\alpha G^{\beta\gamma} = -\frac{1}{2} \epsilon^{\alpha(\beta \gamma)},$$

$$Q_i^\alpha M_a^\gamma = -i(\sigma_i)_{ab} \lambda_{ab}^{\alpha^\beta}, \quad Q_i^\alpha \lambda_{a^\beta}^{\gamma} = -\epsilon^{\alpha\beta} D_i M_a^\gamma - i(\sigma_i)_{ab} ([G^{\alpha\beta}, M^b\gamma] - \epsilon^{\alpha\beta} C^{b\gamma}),$$

and

$$Q_a^{\alpha\beta} A_i = i(\sigma_i)_{ab} \lambda_{ab}^{\alpha^\beta},$$

$$Q_a^{\alpha\beta} \psi_i^{\gamma} = \epsilon^{\alpha\gamma} D_i M_a^\beta + i(\sigma_i)_{ab} ([G^{\alpha\gamma}, M^b\beta] - \epsilon^{\alpha\gamma} C^{b\beta}),$$

$$Q_a^{\alpha\beta} \eta^\gamma = -[G^{\alpha\gamma}, M_a^\beta] - \epsilon^{\alpha\gamma} C_a^\beta,$$

$$Q_a^{\alpha\beta} G^{\gamma\delta} = -\frac{1}{2} \epsilon^{\alpha(\gamma \delta)},$$

$$Q_a^{\alpha\beta} M_b^\gamma = -\epsilon_{ab} \epsilon^{\gamma\delta} \eta^\alpha - i(\sigma_i)_{ab} \epsilon^{\gamma\delta} \psi_i^{\alpha},$$

$$Q_a^{\alpha\beta} \lambda_{b^\gamma}^{\delta} = -\epsilon^{\alpha\beta} [M_a^\beta, M_b^\delta] - \epsilon_{ab} [G^{\alpha\beta}, G^{\gamma\delta}] + i(\sigma_i)_{ab} \epsilon^{\gamma\delta} (D_i G^{\alpha\gamma} + \epsilon^{\alpha\gamma} B_i),$$

with the abbreviations $B_i = \frac{1}{4} (\epsilon_{ijk} F^{jk} + i\epsilon_{\alpha\beta}(\sigma_i)_{ab}[M_{\alpha^a}, M^b\beta])$ and $C_a^\alpha = \frac{1}{2} i(\sigma_i)_{ab} D_i M^{b\alpha}$.

\(^5\)After introducing in (13) an auxiliary field $B_i$, via its equation of motion, $B_i = \frac{1}{4} \epsilon_{ijk} F^{jk}$, one recognizes the usual off–shell formulation of the $N_T = 2$ super–BF theory (without matter).

\(^6\)In order to recast the (complex) matrices $i(\sigma_i)_{ab} \epsilon^{\alpha\beta}$ into the (real) octonionic structure constants $\Psi_{iAB}$ one has to perform an appropriate redefinition of $M_a^\alpha$ and $\alpha^\beta_{a^b}$.
These transformations obey the following superalgebra on–shell,

\[
\begin{align*}
\{Q^\alpha, Q^\beta\} &\equiv -2\delta_G(G^{\alpha\beta}), & \{Q^\alpha, Q^\beta_i\} &\equiv \epsilon^{\alpha\beta}(\partial_i + \delta_G(A_i)), \\
\{Q^i_\alpha, Q^j_\beta\} &\equiv -2\delta_{ij}\delta_G(G^{\alpha\beta}) + \epsilon^{\alpha\beta}\epsilon_{ijk}(\partial_k + \delta_G(A_k)), \\
\{Q^{\alpha\gamma}, Q_\beta^{\beta\gamma}\} &\equiv \epsilon^{\alpha\beta\gamma}(M^{\alpha\beta\gamma}), & \{Q^i_\alpha, Q_\beta^{\beta\gamma}\} &\equiv i\epsilon^{\alpha\beta}(\sigma_i)_{ab}\delta_G(M^{\beta\gamma}), \\
\{Q_a^{\alpha\gamma}, Q_b^{\beta\delta}\} &\equiv 2\epsilon_{ab}\epsilon^{\gamma\delta}\delta_G(G^{\alpha\beta}) + i\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}(\sigma_i)_{ab}(\partial_i + \delta_G(A_i)).
\end{align*}
\]

On–shell, upon using the equations of motion for \(\psi_i^\alpha\) and \(\lambda_a^{\alpha\beta}\), the action (19) can be recast into the form

\[
S(N_f=2) = \frac{1}{2} \epsilon_{\alpha\beta} Q^\alpha Q^\beta \Omega
\]

with the gauge boson

\[
\Omega = S_{CS} + \int_E d^3x \text{tr}\left\{i(\sigma^i)^{ab} M_{aa} D_i M_b^\alpha + \lambda^a_{\alpha\beta} \lambda_a^{\alpha\beta} + \psi^i_\alpha \psi_i^\alpha - \eta_\alpha \eta^\alpha\right\},
\]

where \(S_{CS}\) is the 3–dimensional Chern–Simons action (13).

Summarizing, we have shown that, in the Euclidean space, when relaxing the reality constraint on fermions, then generalized self–duality, simple supersymmetry, \(Spin(7)\) invariance and octonionic algebra are compatible with each other and with chirality in eight dimensions, just as self–duality and supersymmetry are compatible with usual chirality in four dimensions. Additionally, we have observed that the fields of the gauge and spinorial hypermultiplet of the \(N_f = 2, D = 3\) super BF–theory with matter gets unified in the fields of the gauge multiplet of the \(G_2\)–invariant \(N_f = 2, D = 7\) super Yang–Mills theory.

References

[1] G. Papadopoulos and P.K. Townsend, *Phys. Lett.* B 357 (1995) 300; B.S. Acharya, *On realising N = 1 super Yang–Mills in M–theory*, hep-th/0011089; M. Atiyah, J. Maldacena and C. Vafa, *J. Math. Phys.* 42 (2001) 3209; M. Atiyah and E. Witten, *Adv. Theor. Math. Phys.* 6 (2003) 1; E. Witten, *Anomaly cancellation on G_2 manifolds*, hep-th/0108165; S. Gukov and J. Sparks, *Nucl. Phys.* B 625 (2002) 3

[2] L. Baulieu, H. Kanno and I.M. Singer, *Commun. Math. Phys.* 194 (1998) 149; B.S. Acharya, M. O’Loughlin and B. Spence, *Nucl. Phys.* B 503 (1997) 657

[3] L. Baulieu, A. Losev and N. Nekrasov, *Nucl. Phys.* B 522 (1998) 82

[4] L. Baulieu and P. West, *Phys. Lett.* B 436 (1998) 97; M. Blau and G. Thompson, *Phys. Lett.* B 415 (1997) 242; B.S. Acharya, J.M. Figueroa-O’Farrill, M. O’Loughlin and B. Spence, *Nucl. Phys.* B 514 (1998) 583; J.M. Figueroa-O’Farrill, A. Imaanpur and J. McCarthy, *Phys. Lett.* B 419 (1998) 165

[5] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Reports* 209 (1991), 129; S. Cordes, G. Moore and S. Ramgoolam, *Lectures on 2D Yang–Mills Theory, Equivariant Cohomology and Topological Field Theories*, in Les Houches Session LXII, hep-th/9411210

[6] E. Corrigan, C. Devchand, D. Fairlie and J. Nuyts, *Nucl. Phys.* B 214 (1983) 452

[7] H. Nishino and S. Rajpoot, *Octonions, G_2 Symmetry, Generalized Self–Duality and Supersymmetries in Dimensions D \leq 8*, hep-th/0210132
[8] W. Siegel, *Phys. Rev.* **D 46** (1992) 3235, **D 47** (1993) 2504; A. Parkes, *Phys. Lett.* **B 286** (1992) 265; S. Ketov, S.J. Gates, Jr. and H. Nishino, *Phys. Lett.* **B 307** (1993) 323, **B 307** (1993) 331, **B 297** (1992) 99, *Nucl. Phys.* **B 393** (1993) 149; H. Nishino, *Int. Jour. Mod. Phys.* **A 9** (1994) 3077

[9] E. Witten, *Nucl. Phys.* **B 323** (1989) 113; D. Birmingham, M. Blau and G. Thompson, *Int. J. Mod. Phys.* **A 5** (1990) 4721; M. Blau and G. Thompson, *Commun. Math. Phys.* **152** (1993) 41

[10] L. Brink, J.H. Schwarz and J. Scherk, *Nucl. Phys.* **B 121** (1977) 77

[11] M. Günadarin and F. Gürsey, *J. Math. Phys.* **14** (1973) 1651

[12] B. de Wit and H. Nicolai, *Nucl. Phys.* **B 231** (1984) 506

[13] R. Dündarer, F. Gürsey and C.H. Tze, *J. Math. Phys.* **25** (1984) 1496

[14] A. S. Schwarz, *Lett. Math. Phys.* **2** (1978) 247