Mesh Denoising and Inpainting using the Total Variation of the Normal∗

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In this paper we present a novel approach to solve surface mesh denoising and inpainting problems. The purpose is not only to remove noise while preserving important features such as sharp edges, but also to fill in missing parts of the geometry. A discrete variant of the total variation of the unit normal vector field serves as a regularizing functional to achieve this goal. In order to solve the resulting problem, we present a novel variant of the split Bregman (ADMM) iteration. Numerical examples are included demonstrating the performance of the method with some complex 3D geometries.

Keywords. mesh denoising, mesh inpainting, total variation of the normal vector, split Bregman iteration

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1 Introduction

Meshes are widely employed in computer graphics and computer vision, where they are utilized to model arbitrary shapes and real geometries. Meshes can be produced by 3D scanners and can be

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efficiently processed numerically with appropriate software. Unfortunately, the process of geometry
collection by scanning leads to unavoidable errors in the form of noise or missing parts. The process
of removing such noise while preserving relevant features is known as mesh denoising. When missing
parts of the geometry must be filled in we speak of mesh inpainting.

The aforementioned problems have been of interest in the community of image processing since late
in the 1980s; see for instance Caselles, Chambolle, Novaga, 2015. Also for mesh denoising, many
algorithms exist, and we refer the reader, e.g., to Botsch et al., 2007 for a survey.

Before giving a brief overview of the different types of methods for mesh denoising, we would like
to clarify that surface fairing (or smoothing) and mesh denoising should be clearly distinguished.
The goal of the latter is to remove spurious information from the geometry while preserving sharp
features in it, while the goal of the former is to smooth the geometry represented by the mesh with
less emphasis on the recovery of edges.

Denoising methods based on diffusion can be classified either as isotropic or anisotropic. They can
also often be traced back to perimeter minimization and curvature flows. On the one hand, isotropic
methods are applied for mesh fairing and their main characteristic is that they do not take into account
the geometric features. As examples of such methods we mention Field, 1988; Taubin, 1995; Desbrun
et al., 1999 and Vollmer, Mencl, Müller, 1999. These methods generally have the drawback that they
may suffer from surface shrinkage and tend to blur geometric features. On the other hand, anisotropic
diffusion methods, as detailed in Section 2.1, can give good recovery results but meshes with sharp
edges are still a challenge to them. To overcome this drawback, several approaches have been developed
in the literature, as detailed in the following section.

2 Related Work for Mesh Denoising

In this section we provide a brief overview of different types of mesh denoising approaches. Since
mesh inpainting problems are similar, we do not explicitly discuss specific algorithms for them.

2.1 Anisotropic Diffusion

One group of methods for mesh denoising is based on anisotropic diffusion. As their characteristic,
these methods take into account feature directions while filtering the normal vector. References Bajaj,
Xu, 2003; Clarenz, Diewald, Rumpf, 2000; Tasdizen et al., 2002 and Hildebrandt, Polthier, 2004 fall
into this category. Generally speaking, these methods can preserve genuine features such as edges,
but they may be numerically unstable during the diffusion process Zhao et al., 2018.
2.2 Bilateral Filtering

Bilateral filtering methods are one-stage iterative approaches that use the normal and tangential directions to determine a filter at every vertex of the mesh. They then compute a displacement correction for the vertex and update its position. This estimation, however, may be inaccurate due to the presence of noise in the mesh, which may lead to edges not being well preserved. In this category, we mention Fleishman, Drori, Cohen-Or, 2003; Jones, Durand, Desbrun, 2003 where the authors extend the bilateral filter method in imaging processing from Tomasi, Manduchi, 1998 to denoise 3D meshes.

2.3 Normal Filtering and Vertex Update

The class of normal filtering and vertex update methods are characterized as two-stage methods whose idea is first to filter the facet normals and subsequently update the vertex positions according to the filtered normals. Nevertheless, they might blur small-scale features since most of them filter facet normals by averaging neighboring (facet) normals. Particular methods of this class differ w.r.t. the treatment of the facet normal vector since the vertex update usually is quite straightforward. Among others, we mention Ohtake, Belyaev, Seidel, 2002; Sun et al., 2007; Zheng et al., 2011; Zhang et al., 2015; Wang, Fu, et al., 2015 and more recently, Wang, Liu, Tong, 2016; Yadav, Reitebuch, Polthier, 2018 and Centin, Signoroni, 2018.

2.4 $L_0$ Minimization

Another popular class of methods for mesh denoising is based on the so-called $L_0$ minimization, mixing both vertex and normal regularization; see He, Schaefer, 2013 and Zhao et al., 2018. The $L_0$ term introduces sparsity into a discrete gradient operator describing the variation of the surface. One of the weak points of this class of methods is that the non-convexity of the model leads to a high demand in computational resources.

2.5 Vertex Classification and Denoising

Vertex classification and denoising methods can also be understood as two-stage methods since they classify the vertices of the mesh first, and then apply a denoising method per class or cluster of vertices. As examples of such methods we highlight Wei, Yu, et al., 2015; Lu, Deng, Chen, 2016; Wang, Zhang, Yu, 2012; Zhu et al., 2013; Wei, Liang, et al., 2017. Although the idea of separating vertices in homogeneous classes seems to be promising, the presence of noise can make the vertex classification difficult or unreliable; see Lu, Deng, Chen, 2016. As a consequence, this class of methods depends heavily on the level of noise that the mesh has.
2.6 **Total Variation (TV) for Mesh Denoising**

We now provide a brief overview over the utility of the total variation semi-norm for mesh denoising. Since the seminal paper Rudin, Osher, Fatemi, 1992, where the so-called ROF model was introduced for image denoising, TV-regularization terms have been heavily utilized in image processing; see Chan et al., 2006; Chambolle et al., 2010 for a good state-of-the-art on TV and its applications. It is well-known that TV-regularizers are especially good at preserving high frequency features in image denoising. Despite this advantage, the total variation based solutions may suffer from the so-called staircasing effect. Moreover, the non-differentiability of the TV-seminorm requires special algorithmic treatment; see for instance the survey Caselles, Chambolle, Novaga, 2015 and the references therein.

To the best of our knowledge, Tasdizen et al., 2002 was the first paper to point out the possibility of applying the idea of total variation for mesh optimization as future research. In Elsey, Esedoglu, 2009 the authors proposed an analogue of the ROF model based on the Gaussian curvature. The work closest to ours is Wu et al., 2015 where the authors present a variational model for mesh denoising featuring the same TV-regularization term we propose, plus a fairing term. However, they then approximate the TV-seminorm for algorithmic purposes and propose a Euclidean ADMM (alternating direction method of multipliers) to solve the problem.

2.7 **Our Contribution and Organization of the Paper**

In this paper we present a total variation (TV) approach for mesh reconstruction problems involving mesh denoising and inpainting. While our definition of the TV of the normal agrees with the one in Wu et al., 2015, our algorithmic treatment of the ensuing variational problems is quite different. We employ a differential geometric framework Lellmann et al., 2013 which avoids approximations of the TV term. In fact, a Riemannian version of the ADMM (also known as split Bregman iteration Goldstein, Osher, 2009) appropriate for this setting was recently proposed in Bergmann et al., 2020; see also Bergmann et al., 2019.

Our contribution here is two-fold. First, we significantly simplify the Riemannian split Bregman iteration compared to Bergmann et al., 2020, and second, we apply it not only to mesh denoising, but also to mesh inpainting problems. The rest of the paper is organized as follows. In Section 3 we review the notion of total variation of the normal vector field. Section 4 is devoted to the derivation of the novel split Bregman iteration and its implementation for mesh denoising problems. In Section 5 we discuss an extension to mesh inpainting. To conclude the paper, in Section 6 we provide a brief summary and identify future research directions.

3 **Total Variation of the Normal Vector**

For the rest of the paper, suppose that $\Gamma$ is a triangulated surface in $\mathbb{R}^3$ consisting of flat triangles $T$, edges $E$ and vertices $V$. Throughout, we denote vector-valued quantities by bold-face symbols. The authors in Wu et al., 2015 proposed the total absolute edge-lengthed supplementary angle of the dihedral
angle (TESA)  
\[
\sum_E \theta_E |E|_2
\]  
(3.1)
for surface denoising. Here $|E|_2$ represents the length of the edge $E$ and $\theta_E$ is the exterior dihedral angle between two neighboring triangles, i.e., the angle between the respective outward pointing normal vectors. In Bergmann et al., 2020 we proposed the total variation of the normal
\[
\sum_E |\log_{n_E^\pm} n_E^\pm|_g |E|_2,
\]  
(3.2)
which can be shown to agree with (3.1). Here the ‘+’ and ‘−’ refer to the two sides of an edge $E$ and $n_E^+, n_E^−$ are the outer unit normal vectors of the respective triangles on either side of $E$. The normal vectors belong to the unit sphere $S^2$ in $\mathbb{R}^3$. On this manifold, the logarithmic map $\log_{n_E^\pm} n_E^\pm$ returns the tangent vector pointing from $n_E^\pm$ to $n_E^\mp$ and its length $|\log_{n_E^\pm} n_E^\pm|_g$ equals the geodesic distance (angle) between $n_E^\pm$ and $n_E^\mp$. Since the angle is always non-negative, the functional (3.2) is non-differentiable whenever at least two neighboring normals $n_E^\pm$ agree. The Riemannian metric $|\cdot|_g$ we use is the pull-back of the Euclidean metric in $\mathbb{R}^3$ to the tangent space $T_{n_E^\pm} S^2$ of the unit sphere $S^2$ at $n_E^\pm$.

In this paper, we propose another reformulation of (3.2) which gives rise to a simpler algorithm. It is based on the observation that $\log_{n_E^\pm} n_E^\pm$ is parallel to the so-called co-normal vector $\mu_E^\pm$. The latter is the unit vector in the same plane as the triangle $T_E^\pm$, perpendicular to $E$ and outward pointing from $T_E^\pm$; see Fig. 3.1. Therefore, we obtain

\[
|\log_{n_E^\pm} n_E^\pm|_g = |(\log_{n_E^\pm} n_E^\pm) \cdot \mu_E^\pm| = \arccos (n_E^\pm \cdot n_E^\mp)
\]  
(3.3)

Figure 3.1: Illustration of the geodesic distance (angle) between normals $n_E^\pm$ and $n_E^\mp$ and the logarithmic map $\log_{n_E^\pm} n_E^\pm$ (shown in black) of two triangles $T_E^\pm$, $T_E^\mp$ (shown in blue) which share the edge $E$. The triangles’ co-normals $\mu_E^\pm$ are shown in orange. Notice that $\log_{n_E^\pm} n_E^\mp$ is parallel to $\mu_E^\mp$.

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and it is easy to see that
\[
(\log n_E^+ \cdot \mu_E^+ = \text{sign} \left( \mu_E^+ \cdot n_E^+ \right) \arccos \left( n_E^+ \cdot n_E^- \right)
\]
holds. Compared to the common definition of the discrete total variation semi-norm in imaging, which involves the absolute value of the difference of neighboring function values, the arccos in (3.3) appears to be highly non-linear. However, it agrees with the geodesic distance and is thus the natural extension of the absolute value of the difference for \(S^2\)-valued data. In addition, (3.4) can be viewed as the signed distance between neighboring normal vectors.

To illustrate this behaviour, let \(\alpha \in (-\pi, \pi)\) be the angle between the normal vectors of two neighbouring triangles \(T^+\) and \(T^-\), such that \(\alpha = 0\) refers to the case where the two triangles are co-planar, \(\alpha < 0\) represents the concave situation and \(\alpha > 0\) the convex one. Without loss of generality, the two normal vectors can be parametrized by
\[
n^+ = (\sin \alpha, \cos \alpha, 0)^\top \quad \text{and} \quad n^- = (0, 1, 0)^\top,
\]
which yields
\[
\mu^+ = (-\cos \alpha, \sin \alpha, 0)^\top.
\]
Then (3.3) is simplified to
\[
\arccos (n^+ \cdot n^-) = \arccos (\cos \alpha) = |\alpha|
\]
and (3.4) becomes
\[
\text{sign} (\mu^+ \cdot n^-) \arccos (n^+ \cdot n^-) = \text{sign} (\sin \alpha) |\alpha| = \alpha.
\]
In the following section, we employ functional (3.2) as a prior in different shape optimization problems and solve these with the split Bregman method. To this end, we have to differentiate the expression
\[
\text{sign} (\mu^+ \cdot n^-) \arccos (n^+ \cdot n^-)
\]
with respect to the shape, i.e., with respect to the vertex positions of the triangles involved. Notice that \(n^+, n^-\) and \(\mu^+\) all depend smoothly on the vertex positions. Therefore, in order to demonstrate the differentiability of (3.5), it is sufficient to differentiate with respect to \(n^+, n^-\) and \(\mu^+\), and apply the chain rule afterwards. Due to the jump of sign, we exclude the case \(n^+ = n^-\) for now. Then the derivative of \(\text{sign} (\mu^+ \cdot n^-)\) w.r.t. all three quantities is zero and its dependency on the shape can be ignored when differentiating (3.5).

The derivative of \(\arccos (n^+ \cdot n^-)\) with respect to \(n^+ \in S^2\) reads
\[
\frac{\partial \arccos \left( \frac{n^+}{|n^+|} \cdot n^- \right)}{\partial n^+} = \frac{-(|n^+|_g n^- - (n^+ \cdot n^-) \frac{n^+}{|n^+|_g})}{\sqrt{1 - (n^+ \cdot n^-)^2|n^+_g|^2}} = \frac{-(n^- - (n^+ \cdot n^-) n^+)}{\sqrt{1 - (n^+ \cdot n^-)^2}}.
\]
A simple calculation shows that this resulting vector is normalized and the numerator is parallel to \(\mu^+\). Proceeding similarly for the derivative with respect to \(n^-\), we can summarize our findings as
\[
\frac{\partial \text{sign} (\mu^+ \cdot n^-) \arccos \left( \frac{n^+}{|n^+|_g} \cdot n^- \right)}{\partial n^+} = -\mu^+,
\]
\[
\frac{\partial \text{sign} (\mu^+ \cdot n^-) \arccos \left( \frac{n^-}{|n^-|_g} \cdot n^+ \right)}{\partial n^-} = -\mu^-.
\]
From here we also infer that the assumption \(n^+ \neq n^-\) is no longer necessary, since both expressions in (3.7) are continuous even across \(n^+ = n^-\).

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4 Mesh Denoising Problem

In this section we consider an analogue of the ROF model for mesh denoising. The vertex positions $x_v \in \mathbb{R}^3$ serve as optimization variables, and they implicitly determine the triangles’ normal and co-normal vectors. Using the reformulation (3.3), we consider the following variational model:

$$\text{Minimize } \frac{1}{2} \sum_V |x_v - x_v^{\text{data}}|^2 + \beta \sum_E |(\log_{n_E^+} n_E^-) \cdot \mu_E^+| |E|^2.$$  \hspace{1cm} (4.1)

Here $x_v^{\text{data}} \in \mathbb{R}^3$ are the given, noisy vertex positions which serve as data in the fidelity term in (4.1).

To solve this problem, we apply a variant of the split Bregman method Goldstein, Osher, 2009. To this end, we introduce the additional variable $d_E = (\log_{n_E^+} n_E^-) \cdot \mu_E^+$ and let $b_E$ the associated (scaled) Lagrange multiplier.

Summarizing the unknowns $x_v$, $d_E$ and $b_E$ into vectors, the associated augmented Lagrangian reads

$$\mathcal{L}(x, d, b) := \frac{1}{2} \sum_V |x_v - x_v^{\text{data}}|^2 + \beta \sum_E |d_E| |E|^2 + \frac{\lambda}{2} \sum_E \left[ d_E - (\log_{n_E^+} n_E^-) \cdot \mu_E^+ - b_E \right]^2 |E|^2.$$  \hspace{1cm} (4.2)

Split Bregman iterations are characterized by the fact that $\mathcal{L}$ is minimized iteratively independently with respect to $x$ and $d$, respectively, and subsequently an update of the Lagrange multiplier $b$ is performed. We now discuss these subproblems.

The $x$-problem, i.e., the minimization of (4.2) w.r.t. all vertex positions $x_v$, can be thought of as a discrete shape optimization problem. Notice that $|E|^2$, $\log_{n_E^+} n_E^-$ and $\mu_E^+$ depend in a nonlinear but smooth way on $x$ as long as all triangles maintain positive area, which we ensure. In our implementation, we utilize the algorithmic differentiation capabilities of FEniCS Alnæs et al., 2015; Logg, Mardal, Wells, 2012 in order to obtain the total derivative of $\mathcal{L}(x, d, b)$ w.r.t. the vector $x$ of vertex positions. Rather than to minimize (4.2) exactly, we only perform a few shape gradient steps.

The $d$-problem is non-smooth, but it completely decouples, and the problem on each edge is well known to have a closed-form solution expressed in terms of the shrinkage operator, namely

$$d_E = \max \{ |v_E| - \beta \lambda^{-1} \cdot \text{sign}(v_E), 0 \} \cdot \text{sign}(v_E).$$ \hspace{1cm} (4.3)

where $v_E = (\log_{n_E^+} n_E^-) \cdot \mu_E^+ + b_E$. Finally, the update for the Lagrange multiplier is simply given by

$$b_E \leftarrow b_E + (\log_{n_E^+} n_E^-) \cdot \mu_E^+ - d_E.$$  \hspace{1cm} (4.4)

For completeness, our split Bregman method for (4.1) is summarized in Algorithm 1.

Let us briefly discuss the differences to the Riemannian split Bregman method for (4.1) recently proposed in Bergmann et al., 2020. In the latter, the alternative splitting $\tilde{d}_E = \log_{n_E^+} n_E^- \in T_n S^2$ was used, which required the Lagrange multiplier estimate $\tilde{b}_E$ to also belong to $T_n S^2$. Although $\tilde{d}_E$ can still be expressed explicitly in terms of a (vectorial) shrinkage operation, the multiplier estimate needs to be parallely transported once per iteration into the new tangent space $T_n S^2$ after the geometry update (which entails an update of the normal vectors). We refer the reader to Bergmann et al., 2020,
Algorithm 3.1 for details. This parallel transport, although computable by an explicit formula, is the most costly step. By contrast, our new Algorithm 1 is simpler and it involves only a scalar shrinkage for $d_E$ and, most importantly, does not require the parallel transport step for the Lagrange multiplier estimate.

Fig. 4.1 shows a denoising result obtained using the model (4.1) and Algorithm 1 after 200 outer iterations, while performing one gradient step with step length 0.01 per outer iteration. The data $x_{\text{data}}$ is from the well-known fandisk benchmark problem, where noise was added to each vertex. The noise at a vertex is added in normal direction using a Gaussian distribution with standard deviation $\sigma = 0.3$ times the average length of all edges. Our implementation was done in FEniCS (version 2018.2.devo).

![Figure 4.1: Mesh denoising using the split Bregman iteration on (4.1) with $\beta = 0.01$, $\lambda = 0.1$. Original geometry (left), noisy geometry (middle) and reconstruction (right).](image)

The same procedure is performed on the Stanford bunny mesh, a less regular object. Numerical results for two different values of $\beta$ are presented in Fig. 4.2.

**Algorithm 1** Split Bregman method for (4.1)

**Require:** noisy vertex positions $X_V$

**Ensure:** approx. solution of (4.1) with vertex positions $x_V$

1: Set $b^{(0)} := 0$ and $d^{(0)} := 0$
2: Set $k := 0$
3: while not converged do
4: Perform one or several gradient steps for $x \mapsto \mathcal{L}(x, d^{(k)}, b^{(k)})$ with initial guess $x^{(k)}$, to obtain $x^{(k+1)}$ and updated values for the edge lengths $|E|$, normal vectors $n_E$ and co-normal vectors $\mu_E$
5: Set $d^{(k+1)} := \arg\min \mathcal{L}(x^{(k+1)}, d^{(k)}, b^{(k)})$, see (4.3)
6: Set $b_E^{(k+1)} := b_E^{(k)} + (\log n_E^\star) \cdot \mu_E^\star - d_1^{(k+1)}$ for all edges $E$
7: Set $k := k + 1$
8: end while
Figure 4.2: Mesh denoising using the split Bregman iteration on (4.1). Original geometry (top left), noisy geometry (top right), reconstruction with $\beta = 0.003$, $\lambda = 0.01$ (bottom left) and with $\beta = 0.01$, $\lambda = 0.01$ (bottom right).
5 Mesh Inpainting Problem

This section is devoted to mesh inpainting problems. These problems differ from (4.1) in that there is no fidelity term. Instead, the exact positions of a number of vertices are given and not subject to noise, while the positions of the remaining vertices are unknown and there is no reference value known for them. In this setting, the augmented Lagrangian (4.2) is replaced by

\[
\mathcal{L}(x, d, b) := \beta \sum_{E} |d_E| \|E\|_2 + \frac{\lambda}{2} \sum_{E} \left[ d_E - \left( \log n_E^+ n_E^- \right) \cdot \mu_E^+ - b_E \right]_2^2 \|E\|_2.
\] (5.1)

In contrast to (4.1), the minimization w.r.t. \(x\) is carried out only for the vertices inside the inpainting region while the positions of the remaining vertices are fixed. Note that we need to construct an initial mesh on the inpainting subdomain before running Algorithm 1.

As a first test we consider a unit cube mesh with 10 × 10 × 2 triangles on each side. We select a subdomain on which we simulate the loss of data. We do so by solving a surface area minimization problem on this subdomain, which then solves as the initial guess for the subsequent split Bregman method based on (5.1). We optionally also remesh the subdomain in order to remove any information from the original, intact geometry. Remeshing was performed using the open source software Gmsh (version 3.0.6). The inpainting results obtained using FEniCS, once starting from the original and once from the newly generated mesh, are shown in Fig. 5.1.

Our algorithm yields different inpainting results, depending on the connectivity of the starting mesh. When using the original connectivity (no remeshing), the original cube was fully recovered. With remeshing, a result was found with one of the corners chopped off; see the bottom right of Fig. 5.1. In fact, the cube with the missing corner has a smaller value of the total variation of the normal than the original one. However, given the connectivity of the mesh, both are local minima to their respective problems.

To show the performance of our algorithm, another inpainting problem is solved on the more complex geometry *fandisk*, with remeshing of the inpainting subdomain. The corresponding results are shown in Fig. 5.2.

6 Conclusion

In this paper we introduced a new formulation of the total variation of the normal functional equivalent to previous expressions. The new formulation leads to a simplified variant of the previously proposed Riemannian split Bregman algorithm. Specifically, the relatively costly step of parallel transport of the Lagrange multiplier estimate can be avoided. We demonstrate the novel algorithm using mesh denoising and mesh inpainting problems.
Figure 5.1: Mesh inpainting using the split Bregman iteration for mesh inpainting, based on (5.1) with $\beta = 0.01$, $\lambda = 0.1$. Original mesh (top left) and visualization of the missing parts by painting the inside of the cube black (top right), starting mesh with original mesh connectivity (middle left) and corresponding reconstruction (middle right), previous starting mesh after remeshing with Gmsh (bottom left) and corresponding reconstruction (bottom right).
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Figure 5.2: Mesh inpainting using the split Bregman iteration for mesh inpainting, based on (5.1) with $\beta = 0.01$, $\lambda = 0.1$, after an initial remeshing was done on the affected area. Visualization of
the missing parts by painting the inside black (left), starting mesh obtained by solving a minimal surface problem (middle), and reconstruction (right).

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