Computing a Minimum-Width Cubic and Hypercubic Shell*

Sang Won Bae†

16th April, 2019 01:21

Abstract

In this paper, we study the problem of computing a minimum-width axis-aligned cubic shell that encloses a given set of \( n \) points in a three-dimensional space. A cubic shell is a closed volume between two concentric and face-parallel cubes. Prior to this work, there was no known algorithm for this problem in the literature. We present the first nontrivial algorithm whose running time is \( O(n \log^2 n) \). Our approach easily extends to higher dimension, resulting in an \( O(n^{\lceil d/2 \rceil} \log^{d-1} n) \)-time algorithm for the hypercubic shell problem in \( d \geq 3 \) dimension.

Keywords: facility location, geometric optimization, exact algorithm, cubic shell, hypercubic shell, minimum width

1 Introduction

The minimum-width circular annulus problem asks to find an annulus of the minimum width, determined by two concentric circles, that encloses a given set \( P \) of \( n \) points in the plane. It has an application to the points-to-circle matching problem, the minimum-regret facility location, and the roundness problem. After early results on the circular annulus problem [12, 13], the currently best algorithm that computes a minimum-width circular annulus that encloses \( n \) input points takes \( O(n^{3/2} \log n) \) time [2,3].

Along with these applications and with natural theoretical interests, the minimum-width annulus problem and its variants have recently been attained a lot of attention by many researchers, resulting in various efficient algorithms. Abellanas et al. [1] considered minimum-width rectangular annuli that are axis-parallel, and presented two algorithms taking \( O(n) \) or \( O(n \log n) \) time: one minimizes the width over rectangular annuli with arbitrary aspect ratio and the other does over rectangular annuli with a prescribed aspect ratio, respectively. Gluchshenko et al. [10] presented an \( O(n \log n) \)-time algorithm that computes a minimum-width axis-parallel square annulus, and proved a matching lower bound, while the second algorithm by Abellanas et al. can do the same in the same time bound. If one considers rectangular or square annuli in arbitrary orientation, the problem gets more difficult. Mukherjee et al. [11] presented an \( O(n^2 \log n) \)-time algorithm that computes a minimum-width rectangular annulus in arbitrary orientation and arbitrary aspect ratio. The author [5] recently showed that a minimum-width square annulus in arbitrary orientation can be computed in \( O(n^3 \log n) \) time.

Despite of these recent progress and successful generalizations, little is known about the high dimensional variants of the annulus problem. For \( d \geq 3 \), the \( d \)-dimensional generalization of annuli is often referred to shells of a certain body of volume. Mukherjee et al. [11] showed

---

*This work was supported by Kyonggi University Research Grant 2018.
†Division of Computer Science and Engineering, Kyonggi University, Suwon, Korea. Email: swbae@kgu.ac.kr
that a minimum-width shell of $d$-dimensional axis-parallel boxes (or hyper-rectangles) can be computed in $O(dn)$ time. For the minimum-width spherical or hyperspherical shells, Chan [8] showed an $O(n^{(d/2) + 1})$-time exact algorithm, and some approximation algorithms are known [4]. However, to our best knowledge, there is no known result for the cubic or hypercubic shell problem in the literature. In fact, it is not difficult to apply Chan’s approach and algorithm [8] for the hyperspherical shells to the hypercubic shells, which implies $O(n^{[d/2] + 1})$-time algorithm exact algorithm that computes a minimum-width hypercubic shell enclosing $n$ points in $\mathbb{R}^d$. This in particular implies an $O(n^2)$-time algorithm that computes a minimum-width cubic shell for $d = 3$.

In this paper, we address the minimum-width hypercubic shell problem in three or higher dimensions. We first handle the three dimensional case, and present a new algorithm that computes a minimum-width axis-aligned cubic shell that encloses $n$ input points. Our algorithm is based on a new approach which is different from that of Chan [8], and takes $O(n \log^2 n)$ time in the worst case. Next, we show that our approach can be extended to higher dimensions $d > 3$, and present an algorithm that runs in $O(n^{[d/2]} \log^{d-1} n)$ expected time.

The rest of the paper is organized as follows: We start with introducing some preliminaries in Section 2. After providing basic observations on hypercubes in $\mathbb{R}^d$ for $d \geq 3$ in Section 2, we discuss the case of $d = 3$ dimension in Section 3 and present our algorithm that computes a minimum-width axis-aligned cubic shell enclosing $n$ input points in $\mathbb{R}^3$. We then extend our approach and algorithm to higher dimensions in Section 5. We finally conclude our paper with Section 6.

## 2 Preliminaries

In this section, we introduce some preliminaries for our discussions. We consider the $d$-dimensional space $\mathbb{R}^d$ for $d \geq 1$ with a standard coordinate system of $d$ axes, namely, the $x_1$-axis, $x_2$-axis, ..., and $x_d$-axis. For any point $p \in \mathbb{R}^d$, its coordinates will be referred to $x_1(p), x_2(p), \ldots, x_d(p)$ in this order, so $p = (x_1(p), x_2(p), \ldots, x_d(p))$. The $L_\infty$ norm of $p \in \mathbb{R}^d$, denoted here by $\|p\|$, is defined to be

$$\|p\| := \sum_{i=1}^{d} |x_i(p)|.$$

For any two points $p, q \in \mathbb{R}^d$, the $L_\infty$ distance between $p$ and $q$ is $\|p - q\|$. The $L_\infty$-ball centered at $p$ with radius $r \in \mathbb{R}$, denoted by $B(p, r)$, is the set of points $q \in \mathbb{R}^d$ such that $\|q - p\| \leq r$. A $d$-dimensional axis-aligned hypercube is a synonym to an $L_\infty$-ball in $\mathbb{R}^d$. In particular, an axis-aligned hypercube is called an interval if $d = 1$; an axis-aligned square if $d = 2$; and an axis-aligned cube if $d = 3$. The side length of a hypercube is twice its radius. Throughout this paper, we only discuss axis-aligned hypercubes, so we shall mean axis-aligned hypercubes without the adjective “axis-aligned.”

Two hypercubes are called concentric if they share a common center. A hypercubic shell $A$ in $\mathbb{R}^d$ is the closed volume between two concentric hypercubes, called the outer hypercube $B$ and the inner hypercube $B'$ of $A$, respectively, where the radius of $B$ is at least that of $B'$. Specifically, $A = B \setminus \text{int} B'$, where int $B'$ denotes the interior of $B'$. The width of a hypercubic shell is the difference between the radii of its inner and outer hypercubes. A hypercubic shell is also called a square annulus, in particular, for $d = 2$, and a cubic shell for $d = 3$. See Figure 1 for an illustration of a square annulus in $\mathbb{R}^2$ and a cubic shell in $\mathbb{R}^3$.

The main purpose of this paper is to solve the minimum-width hypercubic shell problem, in which we are given a set $P$ of $n$ points in $\mathbb{R}^d$ for $d \geq 1$ and want to find a hypercubic shell of
minimum width that encloses $P$. The problem is also called the minimum-width square annulus problem for $d = 2$ and the minimum-width cubic shell problem for $d = 3$.

As introduced above, the minimum-width square annulus problem can be solved in $O(n \log n)$ time in the worst case, and its matching lower bound is also known \cite{10}. The case of $d = 1$ would be less interesting, while it is worth mentioning for completeness. For $d = 1$, the problem is to compute two intervals of equal length that contain $n$ given numbers $P \subset \mathbb{R}$, and it can be easily done in $O(n)$ time.

**Theorem 1** The minimum-width hypercubic problem can be solved in $\Theta(n)$ time for $d = 1$ and $\Theta(n \log n)$ time for $d = 2$, both in the worst case.

In the following, we consider the problem for $d = 3$ and higher. For the purpose, we need a basic geometry of cubes and hypercubes enclosing the given set $P$ of points. Throughout the paper, we shall say that a facet of a hypercube or a box contains a point $p \in \mathbb{R}^d$ if the facet or any face of less dimension incident to it contains the point $p$.

**Lemma 1** Let $P \subset \mathbb{R}^d$ for $d \geq 2$ be a set of points, and $B$ be a hypercube that encloses $P$. Then, $B$ is a smallest enclosing hypercube for $P$ if and only if there are two parallel facets of $B$ such that each of them contains a point of $P$.

**Proof.** If there is no pair of parallel facets of $B$, each of which contains a point of $P$, then $B$ is certainly not of the smallest size. Conversely, suppose that there are two parallel facets of $B$ containing a point of $P$ on each. Let $p, p' \in P$ be these two points on the parallel facets of $B$. Then the radius of $B$ is determined by $p$ and $p'$, $\|p - p'\|/2$. On the other hand, any hypercube $B'$ enclosing $P$ should includes these two points $p$ and $p'$, so the radius of such hypercube $B'$ cannot be smaller than $\|p - p'\|/2$. Hence, $B$ is a smallest enclosing hypercube for $P$.

---

### 3 Basic Observations on Hypercubic Shells

In this section, we observe some general properties of hypercubic shells enclosing a set of points.

Let $d \geq 3$ be an integer, being a constant, and $P$ be a set of $n$ points in $\mathbb{R}^d$.

Let $R$ be the smallest axis-aligned box, or hyperrectangle, that encloses $P$, that is,

$$R = \left[ \min_{p \in P} x_1(p), \max_{p \in P} x_1(p) \right] \times \left[ \min_{p \in P} x_2(p), \max_{p \in P} x_2(p) \right] \times \cdots \times \left[ \min_{p \in P} x_d(p), \max_{p \in P} x_d(p) \right].$$
In particular, if sides of \( R \) are parallel to the \( x_d \)-axis. Consider the hyperplane \( \Pi_0 \) orthogonal to the \( x_d \)-axis that halves \( R \). Again, we assume that \( \Pi_0 \) contains the origin \( o = (0, 0, \ldots, 0) \), i.e., \( \Pi_0 \) coincides the \( x_1x_2 \cdots x_{d-1} \)-hyperplane, which can be easily achieved by a translation of \( P \) along the \( x_d \)-axis. See Figure 2(a) for an illustration for \( d = 3 \).

We then consider any smallest axis-aligned hypercube \( B \) that encloses \( P \). Let \( C \) be the set of centers of all such smallest hypercubes that enclose \( P \).

**Lemma 2** We have \( C \subset \Pi_0 \) and \( C \) forms a \((d-1)\)-dimensional box in \( \Pi_0 \). Therefore, a hypercube \( B \) is a smallest hypercube enclosing \( P \) if and only if \( B = B(c, h/2) \) for some \( c \in C \).

**Proof.** Let \( B \) be any smallest axis-aligned hypercube \( B \) that encloses \( P \). Since the side length of \( B \) is \( h \), its center should lie on \( \Pi_0 \). Hence, the set \( C \) of centers of all smallest hypercubes that enclose \( P \) is a subset of \( \Pi_0 \). Furthermore, note that a hypercube \( B \) encloses \( P \) if and only if \( B \) encloses the smallest enclosing box \( R \) for \( P \). This implies that \( C \) forms a \((d-1)\)-dimensional box in \( \Pi_0 \), which may be degenerate to a box of lower dimension.

In particular, if \( d = 3 \), then \( \Pi_0 \) is the \( x_1x_2 \)-plane, and \( C \) forms a rectangle in \( \Pi_0 \). See Figure 2 for an illustration.

If we fix a center \( c \in \mathbb{R}^d \), then the minimum-width cubic shell \( A^*(c) \) enclosing \( P \) is uniquely determined as follows: Since the outer cube \( B \) of \( A^*(c) \) should enclose all points of \( P \), we have \( B = B(c, r) \) with \( r = \max_{p \in P} \| p - c \| \); while the interior of the inner cube \( B' \) of \( A(c) \) should avoid \( P \), we have \( B' = B(c, r') \) with \( r' = \min_{p \in P} \| p - c \| \).

For \( d = 2 \), Gluchshenko et al. [10] proved that there always exists a minimum-width square annulus enclosing \( P \) such that its center lies in \( C \). Here, we generalize this observation into higher dimensions.

**Lemma 3** There exists a minimum-width hypercubic shell enclosing \( P \) centered at some \( c \in C \).

**Proof.** Consider any minimum-width hypercubic shell \( A = A^*(c^*) \) enclosing \( P \) for \( c^* \in \mathbb{R}^d \). That is, the center \( c^* \) minimizes the width of \( A^*(c) \) over all \( c \in \mathbb{R}^d \). Let \( B = B(c^*, r) \) and \( B' = B(c^*, r') \) be its outer and inner hypercubes. Note that \( r = \max_{p \in P} \| p - c^* \| \) and \( r' = \min_{p \in P} \| p - c^* \| \). If \( c^* \in C \), then we are done.

Suppose that \( c^* \notin C \). Then, by Lemma 2 \( B \) is not a smallest enclosing hypercube for \( P \). Thus by Lemma 1 there is no pair of parallel facets of \( B \) both of which contain a point of \( P \). On the other hand, for each pair of parallel facets of \( B \), at least one should contain a point of
Figure 3: An illustration to the proof of Lemma 3. (a) A minimum-width square annulus $A = A^*(c^*)$ with $c^* \notin C$. Only $d = 2$ facets (edges) of its outer square contain a point of $P$. (b) A new square annulus $A(c')$ whose width is the same as that of $A^*(c^*)$ such that $d + 1 = 3$ facets of its outer square $B(c')$ contains a point of $P$.

$P$ by our definition of $A = A^*(c^*)$. Summarizing, there are exactly $d$ facets of $B$ containing a point of $P$ and no two of them are parallel. Hence, there is a unique vertex $q$ of $B$ that is incident to these $d$ facets. See Figure 3(a) for an illustration of the case of $d = 2$.

We now try to slide the center $c'$ of the shell $A$ towards $q$. For each $c$ on the line segment between $c^*$ and $q$, we define a new hypercubic shell $A(c)$ such that its outer hypercube is $B(c) = B(c, r - \delta)$ and its inner hypercube is $B'(c) = B(c, r' - \delta)$, where $\delta = ||c - c^*||$. For any such $c$ with $\delta < r'$, observe that $q$ is still a vertex of $B(c)$, $B'(c) \subseteq B'(c^*)$ avoids the points in $P$ from its interior, and the width of $A(c)$ is exactly $r - r'$, being the same as that of $A = A^*(c^*)$. As $c$ continuously moves from $c^*$ towards $q$, $B(c)$ encloses $P$ and thus $A(c)$ also encloses $P$ until another facet of $B(c)$ hits the $(d + 1)$-st point $p' \in P$ at $c = c'$. Hence, $A(c')$ is also another minimum-width hypercubic shell enclosing $P$. See Figure 3(b) for an illustration.

Finally, we show that $c' \in C$. At $c = c'$, observe that $B(c')$ has $d + 1$ facets containing a point of $P$, so two of the $d + 1$ facets should be parallel. Since $B(c')$ encloses $P$, we conclude that $B'(c')$ is the smallest enclosing hypercube for $P$ by Lemma 1. Hence, we have $c' \in C$. \hfill \square

This implies that we can now solve the problem by searching a center in $C \subseteq \Pi_0$. For each $p \in P$ and $c \in \Pi_0$, define $f_p(c) := ||c - p||$ be the $L_\infty$ distance from $c$ to $p$. Consider any minimum-width hypercubic shell $A^*(c)$ centered at $c \in C$. Then, the radius of its outer hypercube is always fixed as $h/2$. Hence, our problem of computing a minimum-width hypercubic shell enclosing $P$ is a bit simplified to the problem of maximizing the radius of inner hypercube:

$$\text{maximize } \min_{p \in P} f_p(c) \text{ over } c \in C.$$ 

That is, we want to find a highest point in the lower envelope of the functions $f_p$. We define $\Phi(c) := \min_{p \in P} f_p(c)$ to be the lower envelope of the functions $f_p$.

Looking into the function $f_p$, it is defined on the $(d - 1)$-dimensional space $\Pi_0$ and

$$f_p(c) = ||p - c|| = \max_{i=1,...,d} |x_i(p) - x_i(c)| = \max\{\max_{i=1,...,d-1} |x_i(p) - x_i(c)|, |x_d(p)|\},$$

since $c \in \Pi_0$ and so $x_d(c) = 0$. Observe that the first term $\max_{i=1,...,d-1} |x_i(p) - x_i(c)|$ is the $L_\infty$ distance in a $(d - 1)$-dimensional subspace, while the second term $|x_d(p)|$ is a constant.
Thus, the graph \( \{(c, z) \in \Pi_0 \times \mathbb{R} \mid z = f_p(c), c \in \Pi_0\} \) of \( f_p \) is an \( L_\infty \)-cone cut by the hyperplane \( \{z = |x_d(p)|\} \) parallel to \( \Pi_0 \). See Figure 4 for an illustration. From this graphical intuition, one can easily derive the following properties of function \( f_p \).

**Lemma 4** Let \( p \in P \). Then, the following hold.

(1) \( f_p \) is convex.

(2) \( f_p \) is piecewise linear with \( 2^{d-1} + 1 \) patches, unless \( x_d(p) = 0 \). One of the patches forms a \((d - 1)\)-dimensional hypercube in \( \Pi_0 \times \mathbb{R} \), being parallel to \( \Pi_0 \). We call it the plateau of \( f_p \).

(3) Any point on the plateau of \( f_p \) is a lowest point in the graph of \( f_p \). That is, the global minimum of \( f_p \) is attained at \( c \in \Pi_0 \) if and only if \((c, f_p(c))\) is a point on the plateau of \( f_p \), or equivalently, \( f_p(c) = |x_d(p)| \).

**Proof.** From the fact that \( f_p(c) = \max\{\max_{i=1, \ldots, d-1} |x_i(p) - x_i(c)|, |x_d(p)|\} \), it is obvious that \( f_p \) is convex. The \( L_\infty \) distance function \( c \mapsto \max_{i=1, \ldots, d-1} |x_i(p) - x_i(c)| \) in \((d - 1)\)-dimensional space is convex and piecewise linear with exactly \( 2^{d-1} \) patches. If \( |x_d(p)| \neq 0 \), then the function \( f_p(c) \) adds one more patch to it, which is parallel to \( \Pi_0 \). Thus, properties (1) and (2) are true. This patch, called the plateau of \( f_p \), forms the minimum of convex function \( f_p \), so property (3) holds.

From the above observations on the functions \( f_p \), we can discuss local maxima of their lower envelope \( \Phi \).

**Lemma 5** Let \( c^* \in C \) be a local maximum of \( \Phi \) on subdomain \( C \subset \Pi_0 \). Then, either

(i) \( \Phi(c^*) = f_p(c^*) = |x_d(p)| \) for some \( p \in P \), or

(ii) for some \( 0 \leq d' \leq d - 1 \), \( c^* \) lies in a face of \( C \) of dimension \( d' \) and there are \( d' + 1 \) distinct points \( p_1, \ldots, p_{d'+1} \in P \) such that \( \Phi(c^*) = f_{p_1}(c^*) = \cdots = f_{p_{d'+1}}(c^*) \).

**Proof.** We make use of a general theorem on local maxima of the lower envelope of convex functions, which was proved by Bae et al. \[6\], stated as follows:

(*) Let \( d' \) be any positive integer. Let \( F \) be a finite family of real-valued convex functions defined on a convex subset \( C' \subseteq \mathbb{R}^{d'} \) and \( g(c) := \min_{f \in F} f(c) \) be their pointwise minimum. Suppose that \( g \) attains a local maximum at \( c^* \in C' \) and there are exactly \( m \leq d' \) functions \( f_1, \ldots, f_m \in F \) such that \( f_i(c^*) = g(c^*) \) for each \( i = 1, \ldots, m \). Then, there exists a \((d' + 1 - m)\)-flat \( \varphi \subseteq \mathbb{R}^{d'} \) through \( c^* \) such that \( g \) is constant on \( \varphi \cap U \) for some neighborhood \( U \subseteq \mathbb{R}^{d'} \) of \( c^* \) with \( U \subset C' \).

\(^1\)A \( d''\)-flat is an affine subspace of dimension \( d'' \).
Informally speaking, the above theorem tells us that if the number of functions that simultane-
ously appear on the lower envelope $g$ at a local maximum $c^*$ is not enough, then $g$ is constant
near $c^*$. See [6] for its proof and discussion.

We apply the above theorem (*) to our situation. Let $F$ be a face of $C$ of dimension $d'$ for
$0 \leq d' \leq d - 1$. Note that, in particular, if $d' = d - 1$, then $F$ is the interior of $C$. Assume that
$c^* \in F$ is a local maximum of $\Phi$ on $C$. If $d' = 0$, then $F$ is a vertex of $C$ and there must be at
least one $p \in P$ such that $\Phi(c^*) = f_p(c^*)$, so we are done. Thus, in the following, we assume
$1 \leq d' \leq d - 1$.

Now, assume that there are $m$ distinct points $p_1, \ldots, p_m \in P$ such that $\Phi(c^*) = f_{p_1}(c^*) =
\cdots = f_{p_m}(c^*)$. If $m \geq d' + 1$, then this is case (ii) and we are done. Suppose that $m \leq d'$.
Consider the restriction $f_p|_F$ of functions $f_p : \Pi_0 \to \mathbb{R}$ to $F$, for each $p \in P$, also the restriction
$\Phi|_F$ of $\Phi$ to $F$. Let $\mathcal{F} := \{f_p|_F \mid p \in P\}$. Note that $\Phi|_F(c) = \min_{f \in \mathcal{F}} f(c)$ and $f_{p_i}|_F(c^*) =
\Phi|_F(c^*)$ for each $i \in \{1, \ldots, m\}$. Since $c^*$ is a local maximum of $\Phi$, it is also a local maximum
of $\Phi|_F$ in $F$. Hence, we can apply the theorem (*), concluding that $\Phi|_F$ is constant near $c^*$.
This implies that every $f_{p_i}|_F$ must be constant near $c^*$ since $\Phi|_F(c^*) = f_{p_i}|_F(c^*)$. From the
properties of $f_p$ observed in Lemma 4, this is possible only if $f_{p_i}(c^*) = |x_d(p_i)|$. So, this is case
(i) of the lemma.

4 Algorithm for the Minimum-Width Cubic Shell

Let $P$ be a set of $n$ points in $\mathbb{R}^3$. In this section, we present an $O(n \log^2 n)$ time algorithm that
computes a minimum-width cubic shell enclosing $P$.

The function $f_p$ is piecewise linear of constant complexity, defined on domain $\Pi_0$, which is a
two-dimensional subspace. Thus, one can apply an available machinery that computes the
lower envelope of the piecewise linear functions. It was successful for the case of $d = 2$; it is
just computing the lower envelope of line segments in the plane, and can be done in $O(n \log n)$
worst-case time using a known algorithm, as shown by Ghoshen et al. [10]. However, for
d = 3, it takes $O(n^2 \alpha(n))$ time [9] to compute the envelope $\Phi$, and this is too much for us.

We suggest another approach which does not explicitly compute the whole envelope $\Phi$. Here,
we consider the case of $d = 3$. Thus, $\Pi_0$ is the $x_1x_2$-plane and $C$ is a rectangle in $\Pi_0$.

For the purpose, we define for each $p \in P$ and $c \in \Pi_0$

$$
\overline{f}_p(c) := \max\{|x_1(p) - x_1(c)|, |x_2(p) - x_2(c)|\}.
$$

Note that $f_p(c) = \max\{|x_1(p) - x_1(c)|, |x_2(p) - x_2(c)|, |x_3(p)|\}$. Thus, $\overline{f}_p(c)$ is the $L_\infty$
distance between $c \in \Pi_0$ and the orthogonal projection of $p$ onto $\Pi_0$, so basically $L_\infty$
distance between two points in a plane. Let $w^*$ be the width of a minimum-width cubic shell enclosing $P$. As
discussed above, we have

$$
w^* = h/2 - \max_{c \in C} \Phi(c),
$$

where $h$ is the longest side length of the smallest enclosing box $R$ for $P$ as defined above.
Let $r^* = \max_{c \in C} \Phi(c)$ and $c^* \in C$ be such that $\Phi(c^*) = r^*$. Since $\overline{f}_p(c) = f_p(c)$ unless
$f_p(c) = |x_3(p)|$, Lemma 5 implies the following.

Lemma 6 One of the following cases (i) and (ii) holds:

(i) $r^* = |x_3(p)|$ for some $p \in P$.
(ii) $c^*$ is either

(a) a point in the interior of $C$ such that $r^* = \overline{f}_{p_1}(c^*) = \overline{f}_{p_2}(c^*) = \overline{f}_{p_3}(c^*)$ for some
$p_1, p_2, p_3 \in P$,
(b) a point on an edge of $C$ such that $r^* = \mathcal{F}_{p_1}(c^*) = \mathcal{F}_{p_2}(c^*)$ for some $p_1, p_2 \in P$, or
(c) a vertex of $C$.

**Proof.** Recall that $c^* \in C$ maximizes $\Phi$ over $C$ and $r^* = \Phi(c^*)$, so $c^*$ is a local maximum of $\Phi$ in $C$. Hence, we can apply Lemma 5 for $c^*$.

Suppose that we are not in case (i) of the lemma, in which we have $r^* = |x_3(p)|$ for some $p \in P$. In other words, we suppose that $r^* \neq |x_3(p)|$ for all $p \in P$. Note that this implies that $f_p(c^*) = \mathcal{F}_p(c^*)$ for all $p \in P$, as discussed above. This also excludes case (i) of Lemma 5, so this should be case (ii) of Lemma 5. Specifically, it holds that for $0 \leq d' \leq 2$, $c^*$ lies in a $d'$-face of $C$ and there are $d' + 1$ distinct points $p_1, \ldots, p_{d'+1} \in P$ such that

$$r^* = \Phi(c^*) = f_{p_1}(c^*) = \cdots = f_{p_{d'+1}}(c^*).$$

There are three cases according to the dimension $d'$ of the face in which $c^*$ lies.

(a) If $c^*$ lies in a 2-face of $C$, or the interior of $C$, then there are three points $p_1, p_2, p_3 \in P$ such that $r^* = f_{p_1}(c^*) = f_{p_2}(c^*) = f_{p_3}(c^*)$. Since $f_p(c^*) = \mathcal{F}_p(c^*)$ for all $p \in P$, we have $r^* = \mathcal{F}_{p_1}(c^*) = \mathcal{F}_{p_2}(c^*) = \mathcal{F}_{p_3}(c^*)$.

(b) If $c^*$ lies in a 1-face, or an edge, of $C$, then there are two points $p_1, p_2 \in P$ such that $r^* = f_{p_1}(c^*) = f_{p_2}(c^*)$. Since $f_p(c^*) = \mathcal{F}_p(c^*)$ for all $p \in P$, we have $r^* = \mathcal{F}_{p_1}(c^*) = \mathcal{F}_{p_2}(c^*)$.

(c) If $c^*$ lies in a 0-face, then $c^*$ is a vertex of $C$.

Hence, the lemma follows.

Our algorithm computes $r^* = \max_{c \in C} \Phi(c)$ and a corresponding center $c^*$ such that $r^* = \Phi(c^*)$ by separately handling two cases (i) and (ii) of Lemma 5. For case (i), let $r^*_1$ be the largest value in $\{|x_3(p)| \mid p \in P\}$ such that there exists a cubic shell of width $r^*_1$ and center $c^*_1 \in C$ that encloses $P$. If the solution $r^*$ falls in case (i), then it should hold that $r^* = r^*_1$.

For case (ii), any point $c \in C$ is called a candidate center if it satisfies the condition of case (ii); more precisely, if $c$ is either

(a) a point in the interior of $C$ such that $\Phi(c) = \mathcal{F}_{p_1}(c) = \mathcal{F}_{p_2}(c) = \mathcal{F}_{p_3}(c)$ for some $p_1, p_2, p_3 \in P$,
(b) a point on an edge of $C$ such that $\Phi(c) = \mathcal{F}_{p_1}(c) = \mathcal{F}_{p_2}(c)$ for some $p_1, p_2 \in P$, or
(c) a vertex of $C$.

Let $Q$ be the set of all candidate centers, and let $r^*_2 := \max_{c \in Q} \Phi(c)$ and $c^*_2 \in Q$ be such that $r^*_2 = \Phi(c^*_2)$. If the solution $r^*$ and $c^*$ does not fall in case (i), then we will have $r^* = r^*_2$.

Our algorithm thus computes $r^*_1$ and $r^*_2$ and then $r^* = \max\{r^*_1, r^*_2\}$ by Lemma 5. Hence, we are done by reporting $r^* = \max\{r^*_1, r^*_2\}$ and its corresponding center and cubic shell. Note that the width of the minimum-width shell is $h/2 - r^*$.

In the following, we describe how to handle each case and compute $r^*_1$ and $r^*_2$.

### 4.1 Case (i)

Note that there are only $n$ candidate values $\{|x_3(p)| \mid p \in P\}$ for $r^*_1$. Here, we consider the following decision problem:

**given a real $w \geq 0$, is there a cubic shell $A$ enclosing $P$ with width at most $w$ and center in $C$?**

This is equivalent to deciding if the sublevel set $U(w)$ of $\Phi$ covers $C$, where

$$U(w) := \{c \in \Pi_0 \mid \Phi(c) < w\}$$
that is, whether or not $C \subseteq U(w)$.

For a given real number $w$ and any $c \in \Pi_0$, $\Phi(c) < w$ if and only if there exists a point $p \in P$ such that $f_p(c) < w$. Since $f_p(c) = \|c - p\|$, the above condition is again equivalent to $c \in \text{int} B(p, w)$ for some $p \in P$ or $c \in \bigcup_{p \in P} \text{int} B(p, w)$, where $\text{int} B(p, w)$ denotes the interior of $B(p, w)$. Hence, $U(w)$ is indeed the intersection of the union $\bigcup_{p \in P} \text{int} B(p, w)$ of $n$ cubes by $\Pi_0$.

Let $B_0(p, r) := B(p, r) \cap \Pi_0$ be the intersection of the $L_\infty$ ball $B(p, r)$ by $\Pi_0$. We then have

$$U(w) = \bigcup_{p \in P} \text{int} B_0(p, w).$$

Note that $B_0(p, w)$ is either empty if $|x_3(p)| \geq w$, or a square of radius $w$.

After specifying $B_0(p, w)$ for each $p \in P$, we can explicitly compute the union $U(w)$ of squares of equal radius $w$. It is well known that the complexity of the union of $n$ squares is $O(n)$ \cite{1}, and one can compute it in $O(n \log n)$ time by a standard plane-sweep algorithm. We then intersect $U(W)$ by $C$. If there is a point $c \in C$ such that $c \notin U(w)$, then we have $\Phi(c) \geq w$ and thus the cubic shell $A^*(c)$ centered at $c$ has width at most $w$, so we report that there exists a cubic shell of width $w$ enclosing $P$. Otherwise, if $C \subseteq U(w)$, then there is no such shell.

Thus, we conclude the following.

**Lemma 7** Given a set $P$ of $n$ points in $\mathbb{R}^3$ and a real $w \geq 0$, we can decide if there exists a cubic shell enclosing $P$ of width $w$ in $O(n \log n)$ time in the worst case. If exists, such a cubic shell can be output in the same time bound.

After sorting $\{|x_3(p)| \mid p \in P\}$ in $O(n \log n)$ time, we can find the biggest value $r_1^*$ for which the above decision algorithm returns “yes” in $O(n \log^2 n)$ time by a binary search. Such a point $c_1^* \in C$ that $r_1^* = \Phi(c_1^*)$ can also be found in the same time bound.

**4.2 Case (ii)**

Next, we describe how to compute $r_1^*$ and $c_1^*$. As defined above, $Q \subseteq C$ is the set of all candidate centers. Again, recall that $\overline{T}_P(c)$ is equivalent to the $L_\infty$ distance in the plane $\Pi_0$ between the projection of $p$ onto $\Pi_0$ and a point $c \in \Pi_0$. This means that each candidate center $c$ is, unless it is a vertex of $C$, a point on the locus of equidistant points from two or more points in on the plane $\Pi_0$ under the $L_\infty$ distance. This naturally suggests an application of the $L_\infty$ Voronoi diagram in the plane $\Pi_0$.

For each $p \in P$, let $\overline{p}$ be the orthogonal projection of $p$ onto the plane $\Pi_0$, and $\overline{P} := \{\overline{p} \mid p \in P\}$. Let $\overline{VD}(\overline{P})$ be the $L_\infty$ Voronoi diagram for points $\overline{P}$ on $\Pi_0$, that is, the decomposition of $\Pi_0$ into vertices, edges, and cells, each of which is the set of points having a common set of nearest points in $\overline{P}$ under the $L_\infty$ distance. It is well known that $\overline{VD}(\overline{P})$ consists of $O(n)$ vertices, edges, and faces, and can be computed in $O(n \log n)$ time \cite{1}. In particular, we have the following:

(a) A point $c \in \Pi_0$ is a vertex of $\overline{VD}(\overline{P})$ if and only if we have three nearest points $\overline{p}_1, \overline{p}_2, \overline{p}_3 \in \overline{P}$ so that $\overline{T}_{\overline{p}_1}(c) = \overline{T}_{\overline{p}_2}(c) = \overline{T}_{\overline{p}_3}(c)$.

(b) A point $c \in \Pi_0$ lies on an edge of $\overline{VD}(\overline{P})$ if and only if we have exactly two nearest points $\overline{p}_1, \overline{p}_2 \in \overline{P}$ so that $\overline{T}_{\overline{p}_1}(c) = \overline{T}_{\overline{p}_2}(c)$.

This gives us a necessary condition of candidate centers.

**Lemma 8** Let $c \in Q$ be a candidate center. Then, $c$ is either a vertex of $\overline{VD}(\overline{P})$, an intersection of an edge of $\overline{VD}(\overline{P})$ and an edge of $C$, or a vertex of $C$. 

9
Proof. By definition, any candidate center \( c \) should satisfy either (a) there are three points \( p_1, p_2, p_3 \in P \) such that \( \mathcal{I}(p_1(c)) = \mathcal{I}(p_2(c)) = \mathcal{I}(p_3(c)) \), (b) \( c \) is a point on an edge of \( C \) and there are two points \( p_1, p_2 \in P \) such that \( \mathcal{I}(p_1(c)) = \mathcal{I}(p_2(c)) \), or (c) \( c \) is a vertex of \( C \). From the property of the Voronoi diagram \( \mathcal{V}(\mathcal{P}) \) discussed above, in case (a), \( c \) is a vertex of \( \mathcal{V}(\mathcal{P}) \); in case (b), \( c \) is also a point on an edge of \( \mathcal{V}(\mathcal{P}) \). Hence, the lemma is proved.

Now, we are ready to describe our algorithm computing \( r^*_2 \): We first compute \( \mathcal{V}(\mathcal{P}) \) and then compute the intersection points between edges of \( \mathcal{V}(\mathcal{P}) \) and edges of \( C \). Initially, we let \( Q \) include all vertices of \( \mathcal{V}(\mathcal{P}) \) and an edge of \( \mathcal{V}(\mathcal{P}) \) and an edge of \( C \), and all vertices of \( C \). By Lemma \( \ref*{lemma:case} \) \( Q \) contains all candidate centers. For each \( c \in Q \), test if \( f_p(c) = \mathcal{I}_p(c) \) for every nearest point \( p \) from \( c \) among \( \mathcal{P} \). This test can be done in \( O(1) \) time since \( \mathcal{V}(\mathcal{P}) \) stores nearest points for each vertex, edge, and cell. If the test is passed, \( c \) is a candidate center by definition and so we keep \( c \) in \( Q \); otherwise, we discard \( c \) and remove \( c \) from \( Q \). Now, \( Q \) consists of all candidate centers. Note that if \( c \) is a candidate center, it holds that \( \Phi(c) = f_p(c) = \mathcal{I}_p(c) \) for each nearest point \( p \in \mathcal{P} \). Then we pick a candidate center \( c^*_2 \in Q \) such that \( \Phi(c^*_2) = \max_{c \in Q} \Phi(c) = r^*_2 \). All the effort to compute \( r^*_2 \) and \( c^*_2 \) is bounded by \( O(n \log n) \) time.

Summarizing, we handle two cases (i) and (ii) of Lemma \( \ref*{lemma:radius} \) separately, computing \( r^*_1 \) and \( r^*_2 \), and choose the bigger one as \( r^* \). Then, a minimum-width cubic shell enclosing \( P \) is obtained from the corresponding center and the radii \( h/2 \) and \( r^* \) of its outer and inner cubes.

**Theorem 2** Let \( P \) be a set of \( n \) points in \( \mathbb{R}^3 \). A minimum-width cubic shell enclosing \( P \) can be computed in \( O(n \log^2 n) \) time in the worst case.

### 5 Minimum-Width Hypercubic Shell

Our approach for cubic shells in \( \mathbb{R}^3 \) easily extends to hypercubic shells in \( \mathbb{R}^d \) for \( d > 3 \). In this section, let \( d > 3 \) be a constant.

As done for \( d = 3 \), we define

\[
\mathcal{I}_p(c) := \max_{i=1,\ldots,d-1} |x_i(p) - x_i(c)|.
\]

Note that \( f_p(c) = \max\{\mathcal{I}_p(c), |x_d(p)|\} \) and \( f_p(c) = \mathcal{I}_p(c) \) unless \( f_p(c) = |x_d(p)| \). Thus, \( \mathcal{I}_p(c) \) is the \( L_\infty \) distance between \( c \in \Pi_0 \) and the orthogonal projection of \( p \) onto \( \Pi_0 \), so the \( L_\infty \) distance between two points in the \((d-1)\)-dimensional space. Let \( w^*, r^* \) be defined as above. So, we have an analogue of Lemma \( \ref*{lemma:radius} \).

**Lemma 9**

(i) \( r^* = |x_d(p)| \) for some \( p \in P \), or

(ii) \( c^* \) is a point in a face of \( C \) of dimension \( d' \) with \( 0 \leq d' \leq d - 1 \) and \( r^* = \mathcal{I}_{p_1}(c^*) = \cdots = \mathcal{I}_{p_{d'+1}}(c^*) \) for \( d' + 1 \) distinct points \( p_1, \ldots, p_{d'+1} \in P \).

**Proof.** The proof is almost identical to that of Lemma \( \ref*{lemma:radius} \). Since \( c^* \) is a local maximum of \( \Phi \) in \( C \), we can apply Lemma \( \ref*{lemma:local_max} \) for \( c^* \).

Suppose that we are not in case (i) of the lemma, in which we have \( r^* = |x_d(p)| \) for some \( p \in P \). In other words, we suppose that \( r^* \neq |x_d(p)| \) for all \( p \in P \). Note that this implies that \( f_p(c^*) = \mathcal{I}_p(c^*) \) for all \( p \in P \), as discussed above. This also excludes case (i) of Lemma \( \ref*{lemma:radius} \) so this should be case (ii) of Lemma \( \ref*{lemma:radius} \). Specifically, it holds that for \( 0 \leq d' \leq d - 1 \), \( c^* \) lies in a \( d' \)-face of \( C \) and there are \( d' + 1 \) distinct points \( p_1, \ldots, p_{d'+1} \in P \) such that

\[
r^* = \Phi(c^*) = f_{p_1}(c^*) = \cdots = f_{p_{d'+1}}(c^*).
\]
Since we have $f_p(c^*) = \mathcal{F}_p(c^*)$ for all $p \in P$, this implies that $r^* = \mathcal{F}_{p_1}(c^*) = \cdots = \mathcal{F}_{p_{d+1}}(c^*)$, as claimed.

As done for $d = 3$, our algorithm computes $r^* = \max_{c \in C} \Phi(c)$ and a corresponding center $c^* \in C$ such that $r^* = \Phi(c^*)$ by separately handling two cases (i) and (ii). Each case is also handled similarly: we define $r^*_1$ and $r^*_2$ analogously. In particular, a point $c \in C$ is called a candidate center if $c$ lies in a $d$-face of $C$ for $0 \leq d' \leq d - 1$ and there are $d' + 1$ distinct points $p_1, \ldots, p_{d+1} \in P$ such that $\Phi(c) = \mathcal{F}_{p_1}(c) = \cdots = \mathcal{F}_{p_{d+1}}(c)$.

For our algorithm for $d > 3$, an essential tool is again the $L_\infty$ Voronoi diagram $\text{VD}(\mathcal{P})$ in $d - 1$ dimensional space $\Pi_0$. The diagram $\text{VD}(\mathcal{P})$ decomposes $\Pi_0$ into faces of dimension $d' \in \{0, \ldots, d - 1\}$ such that each $d'$-face $F$ of $\text{VD}(\mathcal{P})$ is the maximal set of points $c \in \Pi_0$ having a common set $N(F)$ of $d - d'$ nearest points in $\mathcal{P}$. Fortunately, Boissonat et al. [7] proved the following:

Lemma 10 (Boissonat et al. [7]) The $L_\infty$ Voronoi diagram of $n$ points in $d - 1$ dimension has complexity $O(n^{d/2})$ and can be computed in $O(n^{d/2} \log^{d-2} n)$ expected time.

We again handle each case separately.

5.1 Case (i)

We again consider the decision problem, and solve it by testing $C \subseteq U(w)$. The only difference is that $U(w)$ is now the union of $(d - 1)$-dimensional hypercubes $\mathcal{B}_0(p, w)$ for $p \in P$ in $\Pi_0$.

It is known by Boissonat et al. [7] that the union of $n$ hypercubes of equal radius in $d - 1$ dimension has complexity $O(n^{(d-1)/2})$ for $d \geq 3$. We can compute the union $U(w)$ of hypercubes in $\Pi_0$ by using the $(d - 1)$-dimensional $L_\infty$ Voronoi diagram.

Lemma 11 Let $S$ be a set of $m$ hypercubes of equal radius in $d - 1$ dimensional space. Then, their union can be computed in $O(m^{d/2} \log^{d-2} m)$ expected time.

Proof. Let $w$ be the radius of hypercubes in $S$, and $P'$ be the set of centers of hypercubes in $S$. Let $U$ be their union $\bigcup_{B \in S} B$. We compute $U$ using the $L_\infty$ Voronoi diagram $\text{VD}(P')$. Note that, for each point $c \in U$, it holds that $\min_{p \in P'} ||c - p|| < w$, since each $B \in S$ is $\mathcal{B}(p, w)$ for some $p \in P'$.

We first compute the $L_\infty$ Voronoi diagram $\text{VD}(P')$. This takes $O(m^{d/2} \log^{d-2} m)$ expected time by Lemma 10. Then, for each face $F$ of $\text{VD}(P')$, we compute the set $U(F)$ of points $c \in F$ such that $||c - p|| < w$ for every $p \in N(F)$. Note that the set $N(F)$ of common nearest points for face $F$ consists of $d - d'$ points, if $F$ is a $d'$-face, and $U(F)$ is just the intersection

$$U(F) = F \cap \bigcap_{p \in N(F)} \mathcal{B}_0(p, w)$$

of $d - d'$ hypercubes and the face $F$. Since the complexity of $\text{VD}(P')$ is $O(m^{d/2})$, this iteration is done in time $O(m^{d/2})$. Since the faces of $\text{VD}(P')$ form a (disjoint) decomposition of the space, we have $U = \bigcup_F U(F)$. Thus, we can compute the union $U$ in the claimed time.

After specifying $\mathcal{B}_0(p, w)$ for each $p \in P$, we collect at most $n$ hypercubes of $d - 1$ dimension and compute their union $U(w)$ by the algorithm of Lemma 11. Then, we intersect $U(w)$ by $C$. Since the complexity of $U(w)$ is bounded by $O(n^{(d-1)/2})$, this can be also done in the same time bound. If there is a point $c \in C$ such that $c \notin U(w)$, then we have $\Phi(c) \geq w$ and thus the hypercubic shell $A^*(c)$ centered at $c$ has width at most $w$, so we report that there exists a hypercubic shell of width $w$ enclosing $P$. Otherwise, if $C \subseteq U(w)$, then there is no such shell.

Thus, we conclude the following.
Lemma 12 Let \( d \geq 4 \) be a constant. Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \) and a real \( w \geq 0 \), we can decide if there exists a hypercubic shell enclosing \( P \) of width \( w \) in \( O(n^{d/2} \log^{d-2} n) \) expected time. If exists, such a hypercubic shell can be output in the same time bound.

After sorting \( \{ |x_d(p)| \mid p \in P \} \) in \( O(n \log n) \) time, we can find the biggest value \( r^*_1 \) for which the above decision algorithm returns “yes” in \( O(n^{d/2} \log^{d-1} n) \) time by a binary search. Such a point \( c^*_1 \in C \) that \( r^*_1 = \Phi(c^*_1) \) can also be found in the same time bound.

5.2 Case (ii)

In order to compute \( r^*_2 \) and \( c^*_2 \) for \( d > 3 \), we show an analogous lemma of Lemma \[\]

**Lemma 13** Let \( c \in Q \) be a candidate center. Then, \( c \) is an intersection of a \( d' \)-face of \( C \) and a \( (d - d' - 1) \)-face of \( \text{VD}(\overline{P}) \) for some \( 0 \leq d' \leq d - 1 \).

**Proof.** By definition, any candidate center \( c \) is a point on a \( d' \)-face of \( C \) for \( 0 \leq d' \leq d - 1 \) such that there are \( d' + 1 \) distinct points \( p_1, \ldots, p_{d'+1} \in P \) such that \( \Phi(p_j) = f_{p_j}(c) = \cdots = f_{p_{d'+1}}(c) \). Since \( \Phi(c) \leq \min_{p \in P} f_p(c) \), those \( d' + 1 \) points are all nearest points in \( P \) from \( c \). From the property of the Voronoi diagram \( \text{VD}(\overline{P}) \), this implies that \( c \) lies in a face of \( \text{VD}(\overline{P}) \) of dimension \( d - (d'+1) = d - d' - 1 \). Hence, \( c \) is an intersection point of a \( d' \)-face of \( C \) and a \( (d - d' - 1) \)-face of \( \text{VD}(\overline{P}) \) for some \( 0 \leq d' \leq d - 1 \).

Our algorithm thus computes \( \text{VD}(\overline{P}) \) and intersects it with \( C \). Initially, we let \( Q \) be the set of all intersection points between a \( d' \)-face of \( C \) and a \( (d - d' - 1) \)-face of \( \text{VD}(\overline{P}) \) for \( 0 \leq d' \leq d - 1 \). Since \( \text{VD}(\overline{P}) \) consists of \( O(n^{d/2}) \) faces (Lemma \[\]), \( Q \) consists of at most \( O(n^{d/2}) \) points.

By Lemma \[\] it is guaranteed that \( Q \) contains all candidate centers. We then test each \( c \in Q \) if \( f_p(c) = \overline{f}_p(c) \) for all nearest points \( p \in P \) from \( c \). This can be done in \( O(d) = O(1) \) time by storing the face \( F \) of \( \text{VD}(\overline{P}) \) that contains \( c \) and the set \( N(F) \) of its nearest points. If the test is passed, \( c \) is a candidate center; otherwise, we discard \( c \) and remove \( c \) from \( Q \). Now, \( Q \) consists of only candidate centers. Note that if \( c \) is a candidate center and \( F \) is the face of \( \text{VD}(\overline{P}) \) such that \( c \in F \), it holds that \( \Phi(c) = f_p(c) = \overline{f}_p(c) \) for each \( p \in N(F) \). Thus, we can find \( c_2 \) and \( r^*_2 \) simply taking the maximum \( r^*_2 = \max_{c \in Q} \phi(c) = \phi(c_2) \). The time consumed in this process is bounded by \( O(n^{d/2} \log^{d-2} n) \) expected time for computing \( \text{VD}(\overline{P}) \) by Lemma \[\].

Finally, we conclude the following.

**Theorem 3** Let \( d \geq 4 \) be a constant integer and \( P \) be a set of \( n \) points in \( \mathbb{R}^d \). Then, a minimum-width hypercubic shell enclosing \( P \) can be computed in \( O(n^{d/2} \log^{d-1} n) \) expected time.

6 Concluding Remarks

We addressed the minimum-width cubic and hypercubic shell problem in high dimension, generalizing the square annulus problem. Our algorithm runs in \( O(n \log^2 n) \) worst-case time for the cubic shell and \( O(n^{d/2} \log^{d-1} n) \) expected time for the hypercubic shell in \( \mathbb{R}^d \) for \( d \geq 4 \). It would be worth mentioning that the currently best time bound \( O(n^{d/2} \log^{d-1} n) \) holds for any \( d \geq 2 \). Theorems \[\] and \[\] together with the result in \[\], are summarized into the following corollary.

**Corollary 1** Let \( d \geq 2 \) be any constant integer, and \( P \) be a set of \( n \) points in \( \mathbb{R}^d \). Then, a minimum-width hypercubic shell enclosing \( P \) can be computed in \( O(n^{d/2} \log^{d-1} n) \) time.

There are several open questions. In particular for \( d = 3 \), our algorithm runs in \( O(n \log^2 n) \) time. Is it possible to reduce the time bound to \( O(n \log n) \)? As Gluckshenko et al. \[\] proved a lower bound of \( \Omega(n \log n) \) for \( d = 2 \), the same lower bound applies to the case of \( d \geq 3 \).
A bottleneck of our algorithm for $d = 3$ is the decision algorithm that takes $O(n \log n)$ time and the binary search using it. One could try to apply the parametric search technique, while it seems nontrivial to devise a proper parallel algorithm.

Another interesting question would be about the lower envelope of functions $f_p$. What is the correct complexity of the lower envelope $\Phi$ of functions $f_p$? We tried to obtain a nontrivial upper bound, i.e., a subquadratic bound for $d = 3$, on the complexity of $\Phi$, but failed. Note that the corresponding minimization diagram on $\Pi_0$ coincides the intersection of a $d$-dimensional $L_\infty$ Voronoi diagram by an axis-aligned hyperplane $\Pi_0$.

References

[1] M. Abellanas, F. Hurtado, C. Icking, L. Ma, B. Palop, and P. Ramos. Best fitting rectangles. In Proc. Euro. Workshop Comput. Geom. (EuroCG 2003), 2003.

[2] P. Agarwal and M. Sharir. Efficient randomized algorithms for some geometric optimization problems. Discrete Comput. Geom., 16:317–337, 1996.

[3] P. Agarwal, M. Sharir, and S. Toledo. Applications of parametric searching in geometric optimization. J. Algo., 17:292–318, 1994.

[4] P. K. Agarwal, B. Aronov, S. Har-Peled, and M. Sharir. Approximation algorithms for minimum-width annuli and shells. Discrete Comput. Geom., 24(4):687–705, 2000.

[5] S. W. Bae. Computing a minimum-width square annulus in arbitrary orientation. Theoret. Comput. Sci., 718:2–13, 2018.

[6] S. W. Bae, M. Korman, and Y. Okamoto. The geodesic diameter of polygonal domains. Discrete Comput. Geom., 50(2):306–329, 2013.

[7] J.-D. Boissonnat, M. Sharir, B. Tagansky, and M. Yvinec. Voronoi diagrams in higher dimensions under certain polyhedral distance functions. Discrete Comput. Geom., 19(4):485–519, 1998.

[8] T. Chan. Approximating the diameter, width, smallest enclosing cylinder, and minimum-width annulus. Int. J. Comput. Geom. Appl., 12:67–85, 2002.

[9] H. Edelsbrunner, L. J. Guibas, and M. Sharir. The upper envelope of piecewise linear functions: Algorithms and applications. Discrete Comput. Geom., 4(4):311–336, 1989.

[10] O. N. Gluchshenko, H. W. Hamacher, and A. Tamir. An optimal $O(n \log n)$ algorithm for finding an enclosing planar rectilinear annulus of minimum width. Operations Research Lett., 37(3):168–170, 2009.

[11] J. Mukherjee, P. Mahapatra, A. Karmakar, and S. Das. Minimum-width rectangular annulus. Theoretical Comput. Sci., 508:74–80, 2013.

[12] U. Roy and X. Zhang. Establishment of a pair of concentric circles with the minimum radial separation for assessing roundness error. Computer-Aided Design, 24(3):161–168, 1992.

[13] A. Wainstein. A non-monotonous placement problem in the plane. In Software Systems for Solving Optimal Planning Problems, Abstract: 9th All-Union Symp. USSR, Symp., pages 70–71, 1986.