Collisions at infinity in hyperbolic manifolds

BY D. B. MCREYNOLDS†
Purdue University
e-mail: dmcreyno@math.purdue.edu

ALAN W. REID‡
University of Texas at Austin
e-mail: areid@math.utexas.edu

AND MATTHEW STOVER§
University of Michigan
e-mail: stoverm@umich.edu

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Abstract

For a complete, finite volume real hyperbolic $n$-manifold $M$, we investigate the map between homology of the cusps of $M$ and the homology of $M$. Our main result provides a proof of a result required in a recent paper of Frigerio, Lafont and Sisto.

1. Introduction

Let $M$ be a cusped finite volume hyperbolic $n$-orbifold. Recall that the thick part of $M$ is the quotient $M_0 = X_M / \pi_1(M)$, where $X_M$ is the complement in $H^n$ of a maximal $\pi_1(M)$-invariant collection of horoballs (see for instance [11]). It is known that $M$ and $M_0$ are homotopy equivalent and $M_0$ is a compact orbifold with boundary components $E_1, \ldots, E_r$. Each $E_j$ is called a cusp cross-section of $M$. Since horoballs in $H^n$ inherit a natural Euclidean metric, each cusp cross-section is naturally a flat $(n - 1)$-orbifold. Changing the choice of horoballs preserves the flat structure up to similarity.

The aim of this paper is to provide a proof of a result required in Frigerio, Lafont, and Sisto [3] for their construction in every $n \geq 4$ of infinitely many $n$-dimensional graph manifolds that do not support a locally $\text{CAT}(0)$ metric. Specifically, the following is our principal result.

THEOREM 1.1. For every $n \geq 3$ and $n > k \geq 2$, there exist infinitely many commensurability classes of orientable non-compact finite volume hyperbolic $n$-manifolds $M$ containing a properly embedded totally geodesic hyperbolic $k$-submanifold $N$ with the following properties. Let $E = \{E_1, \ldots, E_r\}$ be the cusp cross-sections of $M$ and $F = \{F_1, \ldots, F_s\}$ the cusp cross-sections of $N$. Then:

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also noncompact with cusp cross-sections

Then proving Theorem 1

is known. However, our proof yields the more general result stated above. The main difficulty
in proving Theorem 1 is (2), which will follow from a separation result that requires some terminology.

Consider a totally geodesic hyperbolic k-manifold N immersed in M. Assume that N is
also noncompact with cusp cross-sections $F_1, \ldots, F_s$. Though they are not freely homotopic in N, it is possible that two distinct ends of N become freely homotopic inside M. When this occurs, we say that the two ends of N collide at infinity inside M. For certain N, it is a well-known consequence of separability properties of $\pi_1(N)$ in $\pi_1(M)$ (e.g., see [1] and [6]) that one can find a finite covering $M'$ of M into which N embeds. However, N may still have collisions at infinity in $M'$ and such collisions can lead to the continued failure of (2). Removing these collisions is the content of the next result; see §2 for the definition of virtual retraction.

**Theorem 1-2.** Suppose M is a cusped finite volume hyperbolic n-manifold and N is an immersed totally geodesic cusped hyperbolic k-manifold. If $\pi_1(M)$ virtually retracts onto $\pi_1(N)$, then there exists a finite covering $M'$ of M such that N embeds in $M'$ and has no collisions at infinity.

That there are infinitely many commensurability classes of manifolds for which Theorem 1-2 applies is discussed in the remark at the end of Section 2. In particular, any noncompact arithmetic hyperbolic n-manifold has the required property, and these manifolds determine infinitely many commensurability classes in every dimension $n > 2$. Given that there are infinitely many commensurability classes to which Theorem 1-2 applies, we now assume Theorem 1-2 and (1) and prove (2) and (3) of Theorem 1-1.

**Proof of Theorem 1-1 (2) and (3).** Let M and N satisfy the conditions of Theorem 1-2 and assume that M satisfies (1). We replace M with the covering $M'$ satisfying the conclusions of Theorem 1-2 and let $\mathcal{E}' = \{E'_j\}$ be the set of cusp ends of $M'$. Note that $M'$ also satisfies (1). Now, each $(k - 1)$-torus $F_j$ is realized as an embedded homologically essential submanifold of some $(n - 1)$-torus $E'_j$. In particular, (3) is immediate as the homology class $\sum [F_j]$ bounds the class $[N_0] = [N \cap M'_0]$ inside $M'$. Since $F_j$ is not freely homotopic to $F_k$ in $M'$ for any $j \neq k$, it follows that they cannot be freely homotopic in $\mathcal{E}'$ for any $j \neq k$. Hence the induced map on $(k - 1)$-homology is an injection, which gives (2). This completes the proof.

2. The proof of Theorem 1-2

The proof of Theorem 1-2 is an easy consequence of the virtual retract property of [7] (see also [2]) which has found significant applications in low-dimensional topology and geometric group theory of late (see [2], [7] and the references therein).

**Definition 1.** Let G be a group and $H < G$ be a subgroup. Then G virtually retracts onto H if there exists a finite index subgroup $G' < G$ with $H < G'$ and a homomorphism
\[ \rho: G^\prime \to H \] such that \( \rho|_H = \text{id}_H \). In addition we say that \( G^\prime \) retracts onto \( H \), and \( \rho \) is called the retraction homomorphism.

With this definition we note the following lemma.

**Lemma 2.1.** Let \( G \) be a group and \( H < G \) a subgroup such that \( G \) retracts onto \( H \). Then two subsets \( S_1, S_2 \) of \( H \) are conjugate in \( G \) if and only if they are conjugate in \( H \).

**Proof.** One direction is trivial. Suppose that there exists \( g \in G \) such that \( S_1 = gS_2g^{-1} \). Then

\[ S_1 = \rho(S_1) = \rho(gS_2g^{-1}) = \rho(g)S_2\rho(g)^{-1}, \]

so \( S_1 \) and \( S_2 \) are conjugate in \( H \).

**Proof of Theorem 1.2.** Let \( M = \mathbb{H}^n/\Gamma \) be a cusped finite volume hyperbolic \( n \)-manifold, \( N = \mathbb{H}^k/\Lambda \) be a noncompact finite volume totally geodesic hyperbolic \( k \)-manifold immersed in \( M \) such that that \( \Gamma \) virtually retracts onto \( \Lambda \). Let \( F_1, \ldots, F_r \) be the cusp cross-sections of \( N \) and \( \Delta_1, \ldots, \Delta_r < \Lambda \) representatives for the associated \( \Lambda \)-conjugacy classes of peripheral subgroups, i.e., \( \Delta_j = \pi_1(F_j) \).

Two ends \( F_{j_1} \) and \( F_{j_2} \) of \( N \) collide at infinity in \( M \) if and only if any two representatives \( \Delta_{j_1} \) and \( \Delta_{j_2} \) for the associated \( \Lambda \)-conjugacy classes of peripheral subgroups are conjugate in \( \Gamma \) but not in \( \Lambda \). Let \( \Gamma_N \) denote the finite index subgroup of \( \Gamma \) that retracts onto \( \Lambda \), and \( \rho: \Gamma_N \to \Lambda \) the retracting homomorphism. By Lemma 2.1, \( \Delta_{j_1} \) and \( \Delta_{j_2} \) are not conjugate in \( \Gamma_N \) for any \( j_1 \neq j_2 \). Thus \( N \) has no collisions at infinity inside \( M' = \mathbb{H}^n/\Gamma_N \).

Moreover, since \( \Lambda \) is a retract of \( \Gamma_N \), it follows that \( \Lambda \) is separable in \( \Gamma_N \) (see [4, lemma 9.2]). Now a well-known result of Scott [12] shows that we can pass to a further covering \( M'' \) of \( M' \) such that the immersion of \( N \) into \( M' \) lifts to an embedding in \( M'' \). This proves the theorem.

**Remark.**

(i) Examples where the virtual retract property holds are abundant. From [2], if \( M = \mathbb{H}^n/\Gamma \) is any non-compact finite volume hyperbolic \( n \)-manifold, which is arithmetic or arises from the construction of Gromov–Piatetskii-Shapiro, then \( \Gamma \) has the required virtual retract property. Briefly, the arithmetic case follows from [2, theorem 1.4] and the discussion at the very end of [2, section 9], and for the examples from the Gromov–Piatetskii-Shapiro construction it follows from [2, theorem 9.1] and the same discussion at the very end of [2, section 9].

(ii) We have in fact shown something stronger, namely that two essential loops in a cusp cross-section \( F_j \) of \( N \) are homotopic inside \( M' \) if and only if they are freely homotopic in \( N \). Therefore, the kernel of the induced map from \( H_*(\mathbb{F}; \mathbb{Q}) \) to \( H_*(\mathbb{M'}; \mathbb{Q}) \) is precisely equal to the kernel of the homomorphism from \( H_*(\mathbb{F}; \mathbb{Q}) \) to \( H_*(\mathbb{N}; \mathbb{Q}) \).

(iii) Lemma 2.1 also implies that \( N \) cannot have positive-dimensional essential self-intersections inside \( M' \). In particular, if \( n < 2k \), then \( N \) automatically embeds in \( M' \).

**3. Covers with torus ends**

The following will complete the proof of Theorem 1.1.
THEOREM 3.1. Let $M$ be a complete finite volume cusped hyperbolic $n$-manifold. Then $M$ has a finite covering $M'$ such that $M'$ has at least two ends and each cusp cross-section is a flat $(n-1)$-torus.

Proof of Theorem 3.1 Let $M = \mathbb{H}^n/\Gamma$ be a cusped hyperbolic $n$-manifold of finite volume. Let $\Delta_1, \ldots, \Delta_r$ be representatives for the conjugacy classes of peripheral subgroups of $\Gamma$. For each $\Delta_j$ the Bieberbach Theorem [11, section 7.4] gives a short exact sequence

$$1 \rightarrow \mathbb{Z}^{n-1} \rightarrow \Delta_j \rightarrow \Theta_j \rightarrow 1$$

where $\Theta_j$ is finite. Then $E_j$ is a flat $(n-1)$-torus if and only if $\Theta_j$ is the trivial group. Note in the case when $M$ is a surface, the statement is trivial and thus we will assume $n > 2$.

Let $\gamma_j, \ldots, \gamma_{j,r_j}$ be lifts of the distinct nontrivial elements of $\Theta_j$ to $\Delta_j$. Since $n \geq 3$, it is a well-known consequence of Weil Local Rigidity (see [10, theorem 7.67]) that we can conjugate $\Gamma$ in $\text{PO}_0(n,1)$ inside $\text{GL}_n(\mathbb{C})$ so that it has entries in some number field $k$. Since $\Gamma$ is finitely generated, we can further assume that it has entries in some finitely generated subring $R \subset k$. Then $R/p$ is finite for every prime ideal $p \subset R$.

This determines a homomorphism from $\Gamma$ to $\text{GL}_n(R/p)$. For every $\gamma_{j,k}$, the image of $\gamma_{j,k}$ in the finite group $\text{GL}_n(R/p)$ is nontrivial for almost every prime ideal $p$ of $R$. Indeed, any off-diagonal element of $\Theta_j$ is congruent to zero modulo $p$ for only finitely many $p$ and there are only finitely many $p$ so that a diagonal element is congruent to 1 modulo $p$. Since there are finitely many $\gamma_{j,k}$, this determines a finite list of prime ideals $\mathcal{P} = \{p_1, \ldots, p_s\}$ such that $\gamma_{j,k}$ has nontrivial image in $\text{GL}_n(R/p)$ for any $p \notin \mathcal{P}$ and every $j, k$. If $\Gamma(p)$ is the kernel of this homomorphism, then $\Gamma(p)$ contains no conjugate of any of the $\gamma_{j,k}$.

The peripheral subgroups of $\Gamma(p)$ are all the form $\Gamma(p) \cap \gamma\Delta_j\gamma^{-1}$ for some $\gamma \in \Gamma$. Since no conjugate of any $\gamma_{j,k}$ is contained in $\Gamma(p)$, we see that $\Gamma(p) \cap \gamma\Delta_j\gamma^{-1}$ is contained in the kernel of the above homomorphism $\gamma\Delta_j\gamma^{-1} \rightarrow \Theta_j$. It follows that every cusp cross-section of $\mathbb{H}^n/\Gamma(p)$ is a flat torus. This proves the second part of the theorem.

To complete the proof of Theorem 3.1, it suffices to show that if $M$ is a noncompact hyperbolic $n$-manifold with $k$ ends, then $M$ has a finite sheeted covering $M'$ with strictly more than $k$ ends. We recall the following elementary fact from covering space theory. Let $\rho : \Gamma \rightarrow Q$ be a homomorphism of $\Gamma$ onto a finite group $Q$ and $\Gamma_{\rho}$ be the kernel of $\rho$. If $\Delta_j$ is a peripheral subgroup of $\Gamma$, then the number of ends of $\mathbb{H}^n/\Gamma_{\rho}$ covering the associated end of $\mathbb{H}^n/\Gamma$ equals the index $[Q : \rho(\Delta_j)]$ of $\rho(\Delta_j)$ in $Q$. Therefore, it suffices to find a finite quotient $Q$ of $\Gamma$ and a peripheral subgroup $\Delta_j$ of $\Gamma$ that $\rho(\Delta_j)$ is a proper subgroup of $Q$.

In our setting, the proof is elementary. From above, we can pass to a finite sheeted covering of $M$, for which all the cusp cross-sections are tori, i.e., all peripheral subgroups are abelian. It follows that for $\rho|_{\Delta_j}$ to be onto, $\rho(\Gamma)$ must be abelian. However, it is well-known that the above reduction quotients $\Gamma/\Gamma(p)$ are central extensions of non-abelian finite simple groups for all but finitely many prime ideals $p$ [8, Chapter 6]. The theorem follows.

Remark. Constructing examples with a small number of ends is much more difficult. For example, there are no known one-cusped hyperbolic $n$-orbifolds for $n > 11$. Furthermore, it is shown in [13] that for every $d$, there is a constant $c_d$ such that $d$-cusped arithmetic hyperbolic $n$-orbifolds do not exist for $n > c_d$. For example, in the case $d = 1$, there are no 1-cusped arithmetic hyperbolic $n$-orbifolds for any $n \geq 30$. Very recently, [5] announced the first construction of a one-cusped hyperbolic $n$-manifold.
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