ALMOST POSITIVE CURVATURE
ON THE GROMOLL-MEYER 7-SPHERE

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(Communicated by Wolfgang Ziller)

Abstract. D. Gromoll and W. Meyer have represented a certain exotic 7-
sphere $M$ as a biquotient of the compact Lie group $Sp(2)$. Thus any invariant
normal homogeneous metric on $Sp(2)$ induces a metric of nonnegative sectional
curvature on $M$. We show that the simplest such metrics (except the bi-
invariant one) induce metrics which have in fact strictly positive curvature
outside a subset of $M$ with measure zero.

There are only very few compact manifolds known which allow metrics of strictly
positive sectional curvature. But recently it has been shown ([PW], [Wk]) that
much more spaces satisfy a condition which seems to be only slightly weaker: A
Riemannian manifold $M$ is said to have almost positive curvature if it has positive
curvature on an open subset $M_0 \subset M$ such that $M \setminus M_0$ is a set of measure zero.

D. Gromoll and W. Meyer [GM] constructed a metric of nonnegative sectional
curvature on the exotic 7-sphere $M = G/U$ where $G = Sp(2)$ and
$\quad U = \{((q_1),\,(q_2)) ; \, q \in Sp(1)\} \subset G \times G.$

In fact, a subgroup $U \subset G \times G$ acts on $G$ by left and right multiplication: $(u_1, u_2).g := u_1 g u_2^{-1}$. If this action is free, the orbit space $G/U$ is a smooth manifold, called a biquotient. Any normally homogeneous metric on $G$ has nonnegative curvature, and if this metric is also $U$-invariant, it induces a metric on the orbit space which has also nonnegative curvature by O'Neill's formulas for Riemannian submersions. For the bi-invariant metric and many other normal homogeneous metrics on $Sp(2)$, the curvature on $M = Sp(2)/U$ is even strictly positive near the point $U.e$ where $e \in Sp(2)$ is the identity, but this cannot hold on the whole manifold ([E1]). How large is the subset $M_0 \subset M$ where the curvature is strictly positive? It is known ([W]) that for the bi-invariant metric $M \setminus M_0$ contains an open subset, so this metric does not have almost positive curvature in the above sense. However the property does hold for the simplest normally homogeneous metrics on $Sp(2)$ which are not bi-invariant. Using arguments taken from [E1] we will show that $M \setminus M_0$ is essentially a hypersurface. F. Wilhelm [W] has shown almost positivity for another set of metrics on $M$, but his computations are much more involved.

Let $K = Sp(1) \times Sp(1) \subset Sp(2) = G$. Then $G$ is equivariantly diffeomorphic
to the homogeneous space $(G \times K)/K$ where $K$ sits diagonally in $G \times K$. A bi-
invariant metric on $G \times K$ thus induces a normally homogeneous metric on $G$. Note
that $G/K = \mathbb{H}P^1 = S^4$ is a symmetric space. Such metrics are described in detail in [E2]. They are induced by certain $Ad(K)$-invariant inner products on the Lie algebra $\mathfrak{g}$ and have nonnegative curvature (by O'Neill's formula). Moreover, the 2-planes with curvature zero are those spanned by two orthogonal vectors $X, Y \in \mathfrak{g}$ with

\[(1) \quad [X, Y] = [X_t, Y_t] = [X_p, Y_p] = 0\]

where $X_t$ and $X_p$ are the components of $X$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Since $G/K$ is a rank-one symmetric space, there are no vanishing commutators in $\mathfrak{p}$; thus we may assume that $Y$ has no $\mathfrak{p}$-component, i.e.

$Y = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{k}$ where $y, z$ are imaginary quaternions. Let $X_p = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$ for some nonzero $x \in \mathbb{H}$. Then $[X_p, Y] = 0$ iff $zx = xy$ or

\[(2) \quad z = xyx^{-1}.\]

The infinitesimal action of the Lie algebra $\mathfrak{u}$ of $U$ on $G$ is given as follows: For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$ we have $V_g := g^{-1}(u.g) = \{v_g; \; v \in \mathbb{R}^3\}$ where $\mathbb{R}^3 \subset \mathbb{H}$ denotes the set of imaginary quaternions (the Lie algebra of $Sp(1)$) and where

\[(3) \quad v_g = Ad(g^*) \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} av_a - v & \bar{a}v \bar{b} \\ bva & bvb - v \end{pmatrix}.
\]

In order to have zero curvature at the point $U.g \in G/U$ we need to find perpendicular $X, Y \perp V_g$ satisfying (1), thus spanning a horizontal zero curvature plane at $g$, and in fact this condition is also sufficient (cf. [E1], p. 31, and [GM]).

**Theorem.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$ with $a, b \neq 0$. There exists a zero curvature plane at $U.g \in G/U$ iff

\[(*) \quad \det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0.\]

**Proof.** Let $X, Y \perp V_g$ with (1), spanning a zero curvature plane. Our first claim is that $X_t$ and $Y_t$ are linearly dependent. In fact, since $[X_t, Y_t] = 0$, we may assume $X_t = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$ and $Y_t = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$ for $x, y \in \mathbb{R}^3$. Thus $\langle v_g, X \rangle = \langle \bar{a}v - v, x \rangle = \langle v, ax\bar{a} - x \rangle$ and likewise $\langle v_g, Y \rangle = \langle v, by\bar{b} - y \rangle$. This vanishes for all $v \in \mathbb{R}^3$ iff $ax\bar{a} = x$ and $by\bar{b} = y$. If both $x, y$ are nonzero, we have $|a|^2 = |b|^2 = 1$ which is impossible since $|a|^2 + |b|^2 = 1$ (recall that $g$ is unitary).

Thus we may assume $X_t = 0$ and hence by (2)

\[(4) \quad X = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y & 0 \\ 0 & xyx^{-1} \end{pmatrix}.
\]

Now

\[\langle v_g, X \rangle = 2\langle bva, x \rangle = 2\langle v, bxa \rangle,
\]

and this vanishes if $bx\bar{a}$ is perpendicular to $\mathbb{R}^3 \subset \mathbb{H}$, hence a real number. Thus if $a \neq 0$, we get

\[(5) \quad bx = ta \]

for some nonzero $t \in \mathbb{R}$. Moreover, $\langle v_g, Y \rangle = \langle v, a\bar{y}a - y + bxxy^{-1}\bar{b} - xyx^{-1} \rangle$ vanishes for all $v \in \mathbb{R}^3$ iff

\[(6) \quad a\bar{y}a - y + bxxy^{-1}\bar{b} - xyx^{-1} = 0.
\]
By (5) we have \( bxyx^{-1}b = |b|^2 bxy(bx)^{-1} = |b|^2 aya^{-1} \) if also \( b \neq 0 \). Hence
\[
(7) \quad ayb + bxyx^{-1}b = |a|^2 aya^{-1} + |b|^2 aya^{-1} = aya^{-1} = Ad(a)y.
\]
Further (5) implies \( Ad(x) = Ad(b^{-1}a) \). Therefore \( \langle v_y, Y \rangle = 0 \) iff
\[
(8) \quad Ad(a)y - Ad(b^{-1})Ad(a)y - y = 0.
\]
Thus \( Ad(a)y \neq 0 \) is in the kernel of \( I - Ad(b^{-1}) - Ad(a^{-1}) \) which implies that the determinant of that matrix vanishes.

Vice versa, if \( \det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0 \), we find a nonzero \( y \in \mathbb{R}^3 \) such that \( Ad(a)y \) is in the kernel of this matrix. Now putting \( x = b^{-1}a \) and defining \( X, Y \) by (4), we obtain a horizontal zero curvature plane at \( g \).

Remarks. 1. We can determine the horizontal zero curvature planes also in the cases \( a = 0 \) or \( b = 0 \), using (6). E.g. if \( b = 0 \), then (6) becomes \( aya^{-1} - y - xyy^{-1} = 0 \) which is solvable precisely for those \( a \) such that \( Ad(a) \) turns some vector \( y \in \mathbb{R}^3 \) by the angle \( \pi/3 \); then \( |Ad(a)y - y| = |y| \), and we find some \( x \in \mathbb{H} \) with \( Ad(x)y = Ad(a)y - y \). Thus a horizontal zero curvature plane at such \( g \) exists if and only if the (minimal) rotation angle of \( Ad(a) \) is \( \geq \pi/3 \).

2. Note that equation (*) for \( g \) in the Theorem is invariant under the action of \( U \) and thus determines a hypersurface (possibly with singularities) in \( G/U \). In fact, if \( u = ((q_1), (q_2)) \in U \), then
\[
uug = \begin{pmatrix} qaq^{-1} & qbq^{-1} \\ cq^{-1} & dq^{-1} \end{pmatrix}.
\]
Thus \( a \) and \( b \) become conjugated by \( q \) which does not change the determinant equation.

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