1. Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^d$-valued random variables, and define the random walk $S_0 = 0$, $S_n = \sum_{j=1}^n X_j$, for $n \geq 1$. The special case with $P(X_j = e) = 1/(2d)$, for all $e \in \mathbb{Z}^d$ with $|e| = 1$, is known as the simple random walk in $\mathbb{Z}^d$ and will be denoted by $(\text{SRW})_{n \in \mathbb{N}_0}$.

Let $l(n, x) = \sum_{j=1}^n \delta_{x_j}$ be the local time of $(S_n)_{n \in \mathbb{N}_0}$ at the site $x \in \mathbb{Z}^d$, and define a positive integer $\alpha$ the $\alpha$-fold self-intersection local time

$$L_n = L_n(\alpha) = \sum_{x \in \mathbb{Z}^d} l(n, x)^\alpha = \sum_{i_1, \ldots, i_n = 0}^n \mathbb{1} \left( S_{i_1} = \cdots = S_{i_n} \right).$$

We will denote the corresponding quantities for simple random walk in $\mathbb{Z}^d$ by $L_n^{\text{SRW}}(\alpha, d)$ or simply $L_n^{\text{SRW}}(\alpha)$ when the dimension is clear from the context.

Let $R^+$ and $R^-$ be, respectively, the semigroup and the group generated by the support of $X$,

$$R^+ = \left\{ x \in \mathbb{Z}^d \mid P \left( S_n = x \right) > 0 \text{ for some } n \geq 0 \right\},$$

$$R^- = \left\{ x \in \mathbb{Z}^d \mid x = y - z \text{ for some } x, y \in R^+ \right\}.$$

Following Spitzer [1], we call the random variable $X$ and the random walk it generates genuinely $d$-dimensional if the group $R$ is $d$-dimensional.

The quantity $L_n(\alpha)$ has received considerable attention in the literature due to its relation to self-avoiding walks and random walks in random scenery. In particular let the random scenery $\{\xi_x, x \in \mathbb{Z}^d\}$ be a collection of i.i.d. random variables, independent of $(S_n)_{n \in \mathbb{N}_0}$, and define the process $Z_0 = 0, Z_n = \sum_{j=1}^n \xi_{S_j}$. Then $(Z_n)_{n \in \mathbb{N}_0}$ is commonly referred to as random walk in random scenery and was introduced in Kesten and Spitzer [2], where functional limit theorems were obtained for $Z_{[nt]}$ under appropriate normalization for the case $d = 1$. The case $d = 2$, with $X_i$ centered with nonsingular covariance matrix, was treated in [3] where it
was shown that $Z_{[n]}(\sqrt{n} \log n)$ converges weakly to Brownian motion. As is obvious from the identities $Z_n = \sum_{x \in \mathbb{Z}^d} I(n, x)$ and $\text{var}(Z_n) = \text{var}(\text{var}(Z_n))$, limit theorems for $(Z_{n})_{n}$ usually require asymptotic results for the local times of the random walk $(S_n)_{n}$.

Such asymptotic results are usually obtained from Fourier techniques applied to the characteristic function $f(t) = \mathbb{E}[\exp(it \cdot X)]$, under the additional assumption of a Taylor expansion of the form $f(t) = 1 - (\Sigma t, t) + o(|t|^2)$, where $\Sigma$ is a positive definite covariance matrix [3–7], which further requires that $E|X|^2 < \infty$ and $EX = 0$. Similar restrictions are also required for the application of local limit theorems such as in [8, 9].

In this paper, motivated by the results of Spitzer [1] for genuinely $d$-dimensional random walks and the approach of Becker and König [10], we will study the asymptotic behavior of $(S_n)_{n}$, which corresponds to genuinely $d$-dimensional recurrent random walks. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7) that the asymptotic results for the genuinely $d$-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [2]. The proofs are based on adapting the Tauberian approach developed in [13].

**Theorem 2.** Let $X, X_1, X_2, \ldots$ be independent, identically distributed, and genuinely $d$-dimensional with $d \leq 3$. If

$$\lim_{n \to \infty} \frac{\text{var}(L_n(\alpha))}{\text{var}(\text{SRW}(\alpha))} > 0,$$

then $E|X|^2 < \infty$ and $EX = 0$.

As it follows from Theorem 3 given below for $d = 2, 3$ and from Theorem 5.2.3 in Chen [12] for $d = 1$, if $EX = 0$ and $0 < E|X|^2 < \infty$, then $\lim_{n \to \infty} \frac{\text{var}(L_n(\alpha))}{\text{var}(\text{SRW}(\alpha))}$.

For any genuinely $d$-dimensional random walk with finite second moments and zero mean, the asymptotic behaviour of $\text{var}(L_n(\alpha))$ is similar to that of the $d$-dimensional simple symmetric random walk. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7) that the asymptotic results for the genuinely $d$-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [2]. The proofs are based on adapting the Tauberian approach developed in [13].

**Theorem 3.** Let $d = 1, 2, 3$, and suppose that for $t \in \Gamma := [-\pi, \pi]^d$ one has

$$f(t) = 1 - \gamma |t| + R(t), \quad \text{for } d = 1,$$

$$f(t) = 1 - (\Sigma t, t) + R(t), \quad \text{for } d = 2, 3,$$

where $\Sigma$ is a nonsingular covariance matrix and $R(t) = o(|t|)$.

For $d = 1$ and $o(|t|^2)$, for $d = 2, 3$ as $t \to 0$. Then

$$\text{var}(L_n(\alpha)) \sim \begin{cases} \left(\frac{\pi^2 + 6}{12} \frac{\alpha^2}{(\gamma^2)^{3/2}} \right) \log(n) n^{2\alpha-4}, & \text{for } d = 1, \\ \frac{\alpha^2}{2(2\pi \sqrt{\alpha})^2} \log(n) \alpha^4, & \text{for } d = 2, \\ (k_1 + k_2) n \log(n), & \text{for } d = 3, \alpha = 2, \end{cases}$$

where

$$\kappa = \int_0^\infty \int_0^\infty dr ds \left(1 + r\right) \left(1 + s\right) \sqrt{(1 + r + s)^2 - 4rs}^{-1}$$

$$\frac{n^2}{6},$$

and $k_1$ and $k_2$ are defined in (58) and (63), respectively.

Moreover, if $L'(n,\alpha)$ is the self-intersection local time of another random walk, independent of $(S_n)_{n}$, whose characteristic function also satisfies (6), then $\text{var}(L'_n(\alpha)) = \text{var}(L_n(\alpha))(1 + o(1))$.

### 2. Proofs

#### 2.1. General Bounds

We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.
Proposition 4 (general upper bound). Assume that $X_1, X_2, \ldots$ are independent $\mathbb{Z}^d$-valued random variables and let $S_{u,v} := X_u + \cdots + X_{u+v}$. Suppose further that for all $n \in \mathbb{N}$ and integers $a, b, v \geq 0$, with $a + u \leq b$ and any $x \in \mathbb{Z}^d$, one has
\[
P\left(S_{a,u} + S_{b,v} = x\right) \leq \phi(u + v),
\]
\[
P\left(S_{a,u} = 0\right) - P\left(S_{a,u} + S_{b,v} = 0\right) \leq \psi(u, v),
\]
where $\phi(u)$ is nonincreasing and $\psi(u, v)$ is nonincreasing in $u$ and is nondecreasing and subadditive in $v$ in the sense that $\psi(u, v + w) \leq A_v[\psi(u, v) + \psi(u, w)]$, for some constant $A_v$ independent of $u, v, w$. Then, for some constant $K = cA_v(1 + A_v)^{\alpha - 2}$ depending only on $\alpha$
\[
\var\left(L_n(\alpha)\right) \leq Kn \left(\sum_{i=0}^{n-1} \phi(i)\right)^{2\alpha - 4} \cdot \sum_{i,j=0}^{n-1} \left[\phi(j_v) \phi(k_v) + \phi(j) \psi(i + k, j)\right].
\]
Proof of Proposition 4. We first write out the variance as a sum
\[
\var\left(L_n(\alpha)\right) = (\alpha!)^2 \cdot \sum_{k_1 \leq \cdots \leq k_n} \sum_{l_1 \leq \cdots \leq l_n} \sup_{\mathbf{w} \in \mathbb{Z}^d} \left[\mathbb{P}\left[S_{k_1} = \cdots = S_{k_n}, S_{l_1} = \cdots = S_{l_n}\right]\right]
\]
\[
- \mathbb{P}\left[S_{k_1} = \cdots = S_{k_n}\right] \mathbb{P}\left[S_{l_1} = \cdots = S_{l_n}\right].
\]
An important role is played by the manner in which the two sequences are interlaced, since, for example, if $k_1 \leq l_1$ or $l_\alpha \leq k_1$, the term vanishes by the Markov property.

We will treat the sum over indices with $k_1 \leq l_1$. The sum over the remaining index set with $k_1 > l_1$ can be treated in a similar fashion and will contribute a constant factor. Therefore, we assume that $k_1 \leq l_1$ and we arrange the two sequences in an ordered sequence of combined length $2\alpha$ which we denote as $(p_1, \ldots, p_{2\alpha})$; we also define $(\epsilon_1, \ldots, \epsilon_{2\alpha})$ where $\epsilon_i = 0$ if $p_i$ came from $k := \{k_1, \ldots, k_\alpha\}$, and $\epsilon_i = 1$ if $p_i$ came from $l := \{l_1, \ldots, l_\alpha\}$. Finally we define two new sequences $m_0, m_1, \ldots, m_{2\alpha - 1}$, and $\delta_1, \ldots, \delta_{2\alpha - 1}$, where $m_0 = p_1$, $m_1 = p_{h+1} - p_1$, and $\delta_i = \epsilon_{i+1} - \epsilon_i$, for $i = 1, \ldots, 2\alpha - 1$. Notice that since we assume that $k_1 \leq l_1$, we have $p_1 = k_1$ and $\epsilon_1 = 0$. Let $\delta := \sum_{i=1}^{2\alpha - 1} |\delta_i|$ denote the interlacement index.

The terms with $v = 1$ vanish, while the terms with $v = 2$ will be considered separately.

Terms with $v \geq 3$. We first consider the sum $I_n$ over the terms with $v \geq 3$ for which we drop the negative part and obtain the bound
\[
I_n \leq (n + 1)
\]
\[
\cdot \sum_{m_1, m_2, \ldots, m_{2\alpha - 1}} \sum_{x \in \mathbb{Z}^d} \delta_{2\alpha - 1} \sup_{\mathbf{w} \in \mathbb{Z}^d} \left[\mathbb{P}\left(S_{m_1, m_2, \ldots, m_{2\alpha - 1}} = \delta_{2\alpha - 1}\right)\right].
\]
Letting $(\bar{S}_n)_{n \in \mathbb{N}_0}$ denote an independent copy of the random walk $(S_n)_{n \in \mathbb{N}_0}$ and assuming without loss of generality that $j_1 \leq \cdots \leq j_{2\alpha}$, we have that for any $\delta \in \{-1, +1\}^r$
\[
\sum_{y \in \mathbb{Z}^d} \sup_{\mathbf{w} \in \mathbb{Z}^d} \mathbb{P}\left(S_{w, j_1} = \delta_j y\right) \leq \left(\sum_{y \in \mathbb{Z}^d} \sup_{\mathbf{w} \in \mathbb{Z}^d} \mathbb{P}\left(S_{w, j_1} = y\right)\right) \cdot \sup_{\mathbf{w} \in \mathbb{Z}^d} \mathbb{P}\left(S_{w, j_1} - \delta_j \bar{S}_{w, j_1} = 0\right) \leq \left(\prod_{t=2}^{2\alpha - 1} \phi(j_t)\right) \cdot \phi(j_1 + j_{2\alpha}) \leq \prod_{t=2}^{2\alpha - 1} \phi(j_t).
\[ \Delta_{n,v} = \sum_{0 \leq s \leq v} \prod_{t=2}^{v} \phi(j_t \vee j_{t+1}) \leq \sum_{j=0}^{n} \phi(j_{r=0}) \sum_{0 \leq s \leq v} \prod_{t=2}^{v} \phi(j_t \vee j_{t+1}) \]
\[ = G_{v} \Delta_{n,v-1}, \]
and iterating this procedure, for \( v \geq 3 \), we have that \( \Delta_{n,v} \leq \Delta_{n,3} G_{n,3}^{-3} \). Combining the two bounds and summing over \( v = 3, \ldots, 2\alpha - 1 \), we have that
\[ I_n \leq \sum_{v=3}^{2\alpha - 1} c(\alpha) n^2 \alpha - 1 - v \Delta_{n,v} \leq c(\alpha) n^2 \alpha - 1 - v \Delta_{n,3} \]
\[ = c(\alpha) n G_{n,\alpha - 1} \Delta_{n,3}, \]
where \( c(\alpha) \) is a constant depending only on \( \alpha \).

**Terms with \( v = 2 \).** Next we consider the sum \( J_n \) over the terms with \( v = 2 \), which occurs when, for some \( j \), the indices \( l_1, \ldots, l_\alpha \) all lie in \([k_0, k+1]\). Then it is easy to see that this sum \( J_n \) is bounded above by
\[ J_n \leq C \max_{\alpha - 1} \sum_{m_\alpha - 1 = 0}^{n} \prod_{t=1}^{\alpha - 1} \phi(m_t) \psi(m_0 + m_\alpha, m_1) \]
\[ \cdot \cdots + m_{\alpha - 1}) \leq C n G_{n,\alpha - 1}^{-2} A \psi(1 + A \psi)^{\alpha - 2} \]
\[ \cdot \left( \sum_{m_{\alpha - 1} = 0}^{n} \prod_{t=2}^{\alpha - 1} \phi(m_t) \right) \sum_{m_0, m_\alpha} \phi(m_1) \psi(m_0 + m_\alpha) \]
\[ m_1) \leq C \alpha A \psi(1 + A \psi)^{\alpha - 2} n G_{n,\alpha - 1}^{2\alpha - 4} \sum_{i,j,k=0}^{n} \phi(i) \psi(i + k, m_1) \]

The following corollary provides explicit bounds in the cases that are usually considered in the literature.

**Corollary 5.** Assume that the conditions of Proposition 4 are satisfied with \( \phi(m) = T(m) \) and \( \psi(m, k) = T(m - k) \) for \( m, k \in \mathbb{Z}^+ \). Then,
\[ \var(L_n(\alpha)) \begin{cases} n^2 \log(n)^{2\alpha - 4}, & \text{if } r = 1, \\ n^{4-2r}, & \text{if } 1 < r < \frac{3}{2}, \\ n \log(n), & \text{if } r = \frac{3}{2}, \\ n, & \text{if } r > \frac{3}{2}. \end{cases} \]

It is straightforward to see that Corollary 5 includes random walks with mean zero and finite second moment; for example, \( d = 2 \) corresponds to \( r = 1 \) and \( d = 3 \) to \( r = 3/2 \). Therefore several relevant results in [3, 7–13] are obtained as a special case of Corollary 5 and extended to the case of independent but not necessarily identically distributed variables, for example, by applying the local limit theorem, as conducted in [8].

Also when the random walk increment \( X \) is in the domain of attraction of the one-dimensional symmetric Cauchy law [13, 14] or in the case of planar random walk with second moments [3, 7–9, 11], it is well known that the conditions of Proposition 4 are satisfied with \( \phi(m) = T(m) \) and \( \psi(m, k) = T(m - k) \).

However, we can do better for symmetric variables and show that condition (A) implies (B), which together with the comparison technique motivates the following results. For a real number \( x \), we write \( [x] \) for the integer part of \( x \).

**Proposition 6** (bounds via comparison with characteristic function of symmetric random variables). Let \( X_1, X_2, \ldots \), be independent \( \mathbb{Z}^d \)-valued random variables and let \( f_i(t) := \mathbb{E} \exp(i t X_i) \). Assume that there exist a measurable function \( f : \Gamma \to [0, 1] \) and a positive nonincreasing sequence \( (\phi(m))_{m \in \mathbb{N}_0} \), such that
\[ |1 - f_i(t)| \leq T f(t), \]
\[ |f_i(\pm t)| \leq f(t), \]
\[ \int_{\Gamma} f(t)^m \, dt \leq \phi(m), \]

for all integers \( i, m \geq 0, \) all \( t \in \Gamma \), and some positive constant \( T \). Then there exists another positive constant \( K = c(\alpha, d, T) \) such that
\[ \var(L_n(\alpha)) \]
\[ \leq \max_{\Gamma} \left( \sum_{i=0}^{n-1} \phi \left( \left[ \frac{i}{2} \right] \right) \right) \sum_{j=0}^{n} i \phi \left( \left[ \frac{j}{2} \right] \right) \sum_{k=j}^{2n} \phi \left( \left[ \frac{k}{2} \right] \right). \]
Proof of Proposition 6. Using the notation of Proposition 4, for positive integers $a, u, b$, and $v$, with $a + u \leq b$, $e_j = \pm 1$, and any $x \in \mathbb{Z}^d$

\[
P (S_{a,u} + e \cdot S_{b,v} = x) \leq \frac{1}{(2\pi)^d} \int f (t)^{a+u} \prod_{j=0}^{b} |f_j (\varepsilon_j t)| \, dt
\]  
\[\leq \frac{1}{(2\pi)^d} \int f (t)^{u+v} \, dt \leq \frac{1}{(2\pi)^d} \phi (u+v).
\]

To find $\psi (u, v)$, notice that since $f (t) \geq 0$, $\phi (m) \geq \int f (t)^m \left[ 1 - f (t)^m \right] \, dt$  
\[= \sum_{j=0}^{m-1} \int f (t)^{m+j} \left( 1 - f (t) \right) \, dt \geq m \int f (t)^2 (1 - f (t)) \, dt = m Q (2m)
\]
whence $Q (m) \leq 2 \phi ([m/2]) / m$. Therefore,

\[
\left| \mathbb{P} (S_{a,u} = 0) - \mathbb{P} (S_{a,u} + S_{b,v} = 0) \right| 
= \left| \frac{1}{(2\pi)^d} \int \prod_{j=0}^{a+u} f_j (t) \left( 1 - f_{b+v} (t) \right) \, dt \right| 
\leq CT \left| \int f (t)^{a+u} \left[ 1 - f (t) \right] \, dt \right| \leq \frac{CT \phi ([u/2])}{u}.
\]

A telescoping argument implies that

\[
\left| \mathbb{P} (S_{a,u} = 0) - \mathbb{P} (S_{a,u} + S_{b,v} = 0) \right| \leq CT \phi \left( \frac{u}{2} \right) \frac{v}{u}.
\]

On the other hand for $u \leq v$ we can obtain a tighter bound through

\[
\mathbb{P} (S_{a,u} = 0) - \mathbb{P} (S_{a,u} + S_{b,v} = 0) \leq \mathbb{P} (S_{n} = 0) 
\leq \phi (u).
\]

Combining the two bounds above it follows that (B) is satisfied with $\psi (u, v) = \phi ([u/2]) \min (u, v) / u$. Thus all conditions of Proposition 4 are satisfied and the result follows. \hfill \Box

The following corollary allows for the case where $\phi (m)$ is regularly varying.

Corollary 7. Assume that the conditions of Proposition 6 are satisfied with $\phi (m) = h (m)m^{-r}, r \geq 1$, where $h (\cdot)$ is slowly varying at $\infty$. Then,

\[
\var (\mathbb{L}_n (\alpha)) \leq K \Delta_n (\alpha, \phi)
\]

\[
\text{for } r = 1,
\]

\[
\leq c d T^{2r-2}
\]

\[
\text{for } 1 < r < \frac{3}{2},
\]

\[
\left( \begin{array}{l}
n^2 \sum_{k=1}^{n} h (k) \frac{1}{k}, \\
\text{for } r = \frac{3}{2},
\end{array} \right.
\]

\[
\left( \begin{array}{l}
n, \\
\text{for } r > \frac{3}{2},
\end{array} \right.
\]

Several results in [3, 7–13] are obtained as a special case of Corollary 7 and can be extended to dependent variables, for example, a random walk driven by a hidden Markov chain. In addition, following [2], we can construct a one-dimensional symmetric random walk with characteristic function $f (t) = 1 - c |t|^{1/2} + o (|t|^{1/2})$, where $r = 2/d$ for $d = 2, 3$ and $r = 1/2$ for $d \geq 4$, whose asymptotic behaviour is similar to that of genuinely $d$-dimensional random walk.

The following example of genuinely $2$-dimensional recurrent walk with infinite variance was motivated by Spitzer [1, pp. 87].

Example 8. Let $X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^2$-valued random variables, such that $\mathbb{P} (X_1 = k) = c / (k^2 \log (k)^g)$, for $k \geq 4$ and $g \in [0, 1)$. Let $\mathbb{S}_{n \in \mathbb{N}}$ be the corresponding random walk in $\mathbb{Z}^2$. Then we have

\[
\var (\mathbb{L}_n (\alpha)) \leq c n^2 \max \left\{ \left| \log n \right|^g, \log n \right\}^{2a-4} \left( \log n \right)^{2(1-g)},
\]

for $n \geq 10$. Under these assumptions we have that $\mathbb{P} (S_n = 0) \leq c / n \log n \log n^{1-g}$, which is in the critical range, where the random walk is recurrent, without second moment. To see why, we note that by a lengthy but straightforward calculation it can be shown that the characteristic function of $X$ satisfies (19) with

\[
\phi (n) = \frac{c}{n \log (e \vee n)^{1-g}},
\]

\[
f (t) = \exp \left[ -A |t|^2 h \left( |t|^2 \right) \right],
\]

where $h (r) = \left[ 1 + \log \left( \frac{1}{r} \right) \right]^{1-g}$.

The sequence $\phi (m)$ is identified via Fourier inversion, polar coordinates, and a Laplace argument,

\[
\int f (t)^u \, dt \leq c \int_0^\infty \exp \left( -nr \left( 1 + \log \left( \frac{1}{r} \right) \right)^{1-g} \right) \]

\[\leq \frac{c}{n \log (e \vee n)^{1-g}} = \phi (n).
\]

2.2. Bounds for Identically Distributed Variables

Proposition 9 (general upper bound for i.i.d.). Let $X, X_1, X_2, \ldots$ be independent, identically distributed,
\(Z^d\)-valued random variables. Suppose that for any \(x \in Z^d\) and all positive integers \(a, u, b,\) and \(v,\) with \(a + u \leq b,\) it holds that
\[
\mathbb{P}(S_{a,u} + S_{b,v} = x) \leq \phi(a + u),
\]
where \(\phi(m)_{m \in \mathbb{N}_0}\) is a nonincreasing sequence. Then for some constant \(K = c(\alpha)\) we have that
\[
\text{var}(L_n(\alpha)) \leq Kn \sum_{j=0}^{n-1} \sum_{i \leq j \leq 2n-1} \phi(i) \sum_{k=j}^{i} \phi\left(\frac{k}{\alpha}\right).
\]  

Proof of Proposition 9. By inspecting the proof of Proposition 6, we notice that we only need to bound the term \(I_n\). Consider typical ordering
\[
0 \leq i_1 \leq \cdots \leq i_k \leq \cdots \leq i_{k+1} \leq \cdots \leq i_n \leq n,
\]
and let us change variables to \((m_0, \ldots, m_{2n})\) such that \(m_0 + \cdots + m_{2n} = n\). Then the contribution to \(I_n\) is given by
\[
\sum_{m_0+\cdots+m_{2n}=n} \prod_{j \leq k \leq \alpha} \mathbb{P}(S_{m_j} = 0) \cdot \left[\mathbb{P}(S_{m_0+\cdots+m_{k-\alpha}} = 0) - \mathbb{P}(S_{m_0+\cdots+m_{k+\alpha}} = 0)\right].
\]

We keep \(m_j\) fixed for \(j \neq \alpha, k + \alpha\) and we sum over \(m = m_k + m_{k+\alpha}\) from 0 to some \(M = m(n, (m_j)_{j \leq k \leq \alpha})\). Then for given \(m_{k+1}, \ldots, m_{k+\alpha-1}\), the term in the sum is
\[
\sum_{m=0}^{M} (m+1) \left[\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)\right],
\]
where \(q = m_{k+1} + \cdots + m_{k+\alpha-1}\). Then since \(M \leq n - q\), it is an easy exercise to show that this sum is bounded above by
\[
\sum_{m=0}^{M} (m+1) \left[\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)\right] \leq \sum_{m=0}^{q-1} (m+1) \mathbb{P}(S_m = 0) + q \delta(n-q \geq q)
\]
\[
\leq \sum_{m=0}^{n-q} \mathbb{P}(S_m = 0) \leq \sum_{m=0}^{n-q} (m+1) \mathbb{P}(S_m = 0)
\]
\[
+ \alpha m^* \sum_{m=m^*}^{n} \mathbb{P}(S_m = 0),
\]
where \(m^* = \max(m_{k+1}, \ldots, m_{k+\alpha-1})\). The result follows by summing over all indices apart from \(m^*\) and changing the order of summation. \(\square\)

2.3. Proofs of Main Results

Proof of Theorem 1. We apply a comparison argument found to be useful in many areas (e.g., Montgomery-Smith and Pruss [15], and Lefèvre and Utev [16]). More specifically we bound the quantity \(\text{var}(L_n)\) by the corresponding quantity for the symmetrised random walk.

Following Spitzer’s argument we notice that with \(f(t) = \exp(i \cdot X)\)
\[
\mathbb{P}(S_{a,u} + eS_{b,v} = x) \leq c \int \mathbb{E}[|f(t)|^\alpha |f(-t)|^\alpha] dt
\]
\[
= c \int \left[|f(t)|^2|f(-t)|^2\right]^\gamma dt.
\]

Since \(|f(t)|^2|f(-t)|^2\) is the characteristic function of a symmetric random variable in \(Z^d\), for some positive \(\lambda\), we have \(1 - |f(t)|^2 \geq \lambda |t|^2\), and, hence,
\[
\mathbb{P}(S_{a,u} + eS_{b,v} = x) \leq c \exp\left[-\frac{\lambda (u+v)}{2} |t|^2\right] dt
\]
\[
\leq c (u+v)^{-d/2}.
\]

The result follows from Proposition 9 applied with \(\phi(m) = m^{-d/2}\). \(\square\)

The proof of Theorem 2 will be based on the following lemma.

Lemma 10. Assume \(X, X_1, X_2, \ldots\) are independent, identically distributed, genuinely \(d\)-dimensional random variables such that \(E[X]^2 = \infty\). Then there exists a monotone, slowly varying sequence \((h_n)_{n \in \mathbb{N}_0}\), such that \(h_n \to 0\) as \(n \to \infty\) and
\[
\sup_{x \in \mathbb{R}^d} \mathbb{P}(S_n = x) \leq c_d \int \mathbb{E}[|X|^\delta] dt \leq h_n n^{-d/2}.
\]

Proof of Lemma 10. Without loss of generality we assume that \(X\) is symmetric. Let \(\sigma_{X} := \mathbb{E}[|X|^\delta\mathbb{I}(|X| \leq L)]\). Following Spitzer, since \(X\) is genuinely \(d\)-dimensional, we may assume that there exist positive constants \(c, W\), such that for any unit vector \(|e| = 1\) we have that \(\sigma_{X} \geq c (1 - f(t)) \geq |t|^2\) for all \(t \in \Gamma\). Let \(\lambda_d\) be the \(d\)-dimensional Lesbegue measure on \(\mathbb{R}^d\) and \(\mu_d\) the Lebesgue-Haar measure on \(\mathbb{S}^{d-1} := \{e \in \mathbb{R}^d : |e| = 1\}\). Notice that since \(E[X]^2 = \infty\), for any \(K\), we have \(\mu_d[e : \sigma_{e \perp} < K] = 0\).

Fix a small positive \(x\) such that \(\sqrt{c/x} \geq 2W\), and for any \(e > 0\) let \(K = K(e) = e^{-d/2}\). Then there exists \(L = L(e) > 0\) small enough so that \(\mu_d[e : \sigma_{e \perp} < K] \leq c e^{d/2}\). We partition \(S^{d-1}\) in two sets
\[
A_{LK} = \{e \in S_d : \sigma_{e \perp} \geq K\},
\]
\[
\overline{A}_{LK} = \{e \in S_d : \sigma_{e \perp} < K\},
\]
so that, for any direction \(e \in \overline{A}_{LK}\),
\[
\{z \in \mathbb{R} : 1 - f(z e) \leq x\} \subseteq \{z : cz^2 \leq x\}
\]
\[
\subseteq \{z : |z| \leq \sqrt{\frac{x}{c}}\}.
\]
Hence, using $d$-dimensional spherical coordinates,

$$
\lambda_d \left\{ (z, \epsilon) \in \mathbb{R} \times \mathcal{A}_{L,K} : 1 - f(\epsilon z) \leq x \right\} \leq \mu_d \left( \mathcal{A}_{L,K} \right) \left( \frac{x}{c} \right)^{d/2} \left( \frac{1}{d} \right) \leq e^{d/2} \left( \frac{x}{c} \right)^{d/2} \left( \frac{1}{d} \right).
$$

(41)

On the other hand, for any $t$,

$$
1 - f(t) = 2 \sum_{k \in \mathbb{Z}} \sin \left( \frac{|t - k|}{2} \right) P(X = k)
$$

$$
\geq \left( \frac{1}{4} \right) E \left[ (t \cdot X)^2 \right] I \left( |t \cdot X| \leq \frac{1}{2} \right)
$$

$$
= \left( \frac{|t|^2}{4} \right) \sigma_{1/2, 1/2} e.
$$

(42)

Now, assume that $\sqrt{|c/x|} \geq 2L$. Then for any direction $\epsilon \in \mathcal{A}_{L,K}$, by choice of $x$ and since $\sigma_{1/2}$ is increasing in $L$, for $cz^2 \leq 1 - f(\epsilon z) \leq x$ or $|z| \leq \sqrt{x/c}$, it must be the case that

$$
x \geq 1 - f(\epsilon z) \geq \left( \frac{z^2}{4} \right) \sigma_{1/2, 1/2} \geq \left( \frac{z^2}{4} \right) \sigma_{e, L}.
$$

(43)

implying that, on the set $\mathcal{A}_{L,K}$, it must be that $|z| \leq 2\sqrt{x/K}$. Changing to $d$-dimensional polar coordinates, we find that

$$
\lambda_d \left\{ (z, \epsilon) \in \mathbb{R} \times \mathcal{A}_{L,K} : 1 - f(\epsilon z) \leq x \right\}
$$

$$
\leq \int_{\mathcal{A}_{L,K}} \int_{|r| \leq \sqrt{x/K}} r^{d-1} dr de \leq C e^{d/2} x^{d/2}. \tag{44}
$$

Overall, for $x \leq c/4L^2$, $\lambda_d \left\{ t : 1 - f(t) \leq x \right\} \leq c_d (xe)^{d/2}$, and hence $|t| \in \Gamma : 1 - f(t) \leq x$ has Lebesgue measure $o(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of the random variable log$(1/f(t))$ defined on the probability space $\Gamma$ with normalised Lebesgue measure. Then $F$ is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, we have that $F(x) = o(x^{d/2})$ as $x \downarrow 0$. Therefore, for some positive sequence $(\epsilon_n)_{n \in \mathbb{N}_0}$ with $\epsilon_n \to 0$, we have that

$$
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(t)^n \, dt = \int_0^\infty e^{-nx} dF(x)
$$

$$
= n \int_0^\infty e^{-nx} F(x) \, dx \leq n^{-d/2} \epsilon_n. \tag{45}
$$

It remains to show that there exists a positive, monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}_0}$, such that $\epsilon_n \leq h(n) \to 0$ as $n \to \infty$. Let $\delta_n = \sup_{j \geq n} \delta_j$ and $\delta_0 = 0$ and for $n \geq 1$ define $a_n$ recursively by $a_0 = \min(2a_{n-1}, 1/\delta_n)$, for $2^{-1} < n \leq 2^x$, so that $a_n \to 0$ is monotone, $a_n \leq 2a_{n-1}$ implying that $a_n \leq 4a_{n-1}$, and $1/a_n \geq \delta_n \geq \epsilon_n$. Finally, take $h_n = 1/\max(a_0, \log a_n)$.

**Proof of Theorem 2.** Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying sequence $h_n \to 0$ as $n \to \infty$ such that $\int \mathbb{E}\exp(t \cdot X) \, dt \leq h_n n^{-d/2}$. Applying Corollary 7 with $r = 1$ and $r = 3/2$ we, respectively, find that

$$
\text{var}(L_n(\alpha)) \leq \begin{cases} 
Kn \left( \sum_{k=1}^n \frac{h'(k)}{k} \right)^{2a-4} = o(n^2 (\log n)^{2a-4}), & \text{for } d = 2, \tag{46}
\end{cases}
$$

$$
Kn \left( \sum_{k=1}^n \frac{h'(k)}{k} \right)^{2a-4} = o(n(\log n)^{2a-4}), & \text{for } d = 3.
$$

Finally assume that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}[X] = \mu \neq 0$. Then $P(S_n = 0) = P(S_n = -n\mu)$ whence it follows that $P(S_n = 0) = o(n^{-d/2})$ (see, e.g., [17, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the $J_n$ term, while with slight modification the bound for the $I_n$ term also follows.

Note that for $d = 1$ the situation is much simpler since then $\text{var}(L_n^{SRW}(\alpha)) = C \mathbb{E}[L_n^{SRW}](\alpha, d)^2$ and if $\mathbb{E}|X|^2 = \infty$ or $\mathbb{E}[X] \neq 0, \mathbb{E}[L_n^{SRW}(\alpha, d)] = o(n^{1+o(1)/2})$.

**Proof of Theorem 3.** We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (10) and define the sequences $P_i$ and $\delta_i$ for $i = 1, \ldots, 2a - 1$, and the quantity $v(\delta) = \sum_{i=1}^{2a-1} |\delta_i|$. Recall that $v(\delta)$ measures the interlacement of the two sequences $k_1, \ldots, k_n$ and $l_1, \ldots, l_n$. For example, $v(\delta) = 1$ occurs when either $k_1 \leq l_1$ or $l_1 \leq k_1$, in which case the contribution vanishes by the Markov property. On the other hand $v(\delta) = 2$ when, for example, $l_1, \ldots, l_1 \in [k_1, k_1]$, for some $i$. Finally $v(\delta) = 3$ occurs when, for example,

$$
k_1 \leq \cdots \leq k_\alpha \leq l_\alpha \leq \cdots \leq k_{\alpha+1} \leq \cdots \leq k_{\alpha} \leq l_{\alpha+1} \leq \cdots \leq l_n \leq n. \tag{47}
$$

From the proof of Proposition 4, and using the bound $P(S_n = 0) \leq c/n$, the terms of the sum are bounded above by $n^2 \log(n)^{2a-1-v(\delta)}$, and thus the leading term appears when either $v(\delta) = 2, 3$, with other terms giving strictly lower order. We will therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $v = 3$, the two terms in the difference individually give the correct order and will be treated by the classical Tauberian theory. However for $v = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata’s Tauberian theorem since the monotonicity restriction would require roughly that $X'_i$ is symmetric. Thus the complex Tauberian approach, as developed in [13], is required to justify the answer.

**Case 1 ($v(\delta) = 3$).** Assume that part of the sequence $1 = [l_1, \ldots, l_\alpha]$ lies between $k_\alpha$ and $k_{\alpha+1}$ and the rest between $k_\alpha$ and $k_{\alpha+1}$. Then using the change of variables
\[
i_1 = m_0, \\
i_2 = m_0 + m_1, \\
\vdots \\
i_r = m_0 + \cdots + m_{r-1} \\
j_1 = m_0 + \cdots + m_r, \\
j_2 = m_0 + \cdots + m_{r+1}, \\
\vdots \\
j_s = m_0 + \cdots + m_{r+s-1}, \\
i_{r+1} = m_0 + \cdots + m_{r+s}, \\
i_{r+2} = m_0 + \cdots + m_{r+s+1}, \\
\vdots \\
i_\alpha = m_0 + \cdots + m_{\alpha+s-1}, \\
j_\alpha = m_{\alpha+1}, \\
n = m_0 + \cdots + m_{2\alpha},
\]

we rewrite the positive term in (10) as

\[
\begin{align*}
\sum_{n \geq 0} \lambda^n a(n) &= c_j \left(1 - \lambda\right) a(\lambda) \\
&\cdot \sum_{x \in \mathbb{Z}, k, k_2 \geq 0} \lambda^{k_1+k_2+k_3} \mathbb{P}(S_{k_1} = x) \mathbb{P}(S_{k_2} = -x) \\
&\cdot \int_{\mathbb{R}} \frac{dt \, ds}{(1 - \lambda f(t)) (1 - \lambda f(s)) (1 - \lambda f(t + s))} \\
&\sim c_j \left(1 - \lambda\right) a(\lambda) \frac{1}{(2\pi)^2 \gamma^2} - 1 - \lambda \\
&\cdot c_j \left(1 - \lambda\right) a(\lambda).
\end{align*}
\]

Next we consider the negative term in (10)

\[
\begin{align*}
b(n) &= \sum_{m_0 \cdots m_{2\alpha-1}, r, s, \alpha = 0} \mathbb{P} \left[ S_{m_1} = \cdots = S_{m_{r-1}} = S_{m_r} + \cdots + S_{m_{r+s}} = 0 \right] \\
&+ S_{m_{r+s}} = S_{m_{r+s+1}} = \cdots = S_{m_{2\alpha-1}} = 0 \right] \mathbb{P} \left[ S_{m_1} = \cdots = \right. \\
&= S_{m_{r+s}} + \cdots + S_{m_{2\alpha-1}} = 0 
\end{align*}
\]

By direct calculations and (6),

\[
\begin{align*}
\sum_{n \geq 0} \lambda^n b(n) &= \left( \frac{1}{\pi \gamma} \log \left( \frac{1}{1 - \lambda} \right) \right)^{-\alpha - s + r - 2} (1 - \lambda)^{-2} \\
&\cdot \sum_{m_r \cdots m_{2\alpha-1}, r, s, \alpha = 0} \mathbb{P}(S_{m_1} = 0) \\
&\cdot \mathbb{P}(S_{m_r} + \cdots + S_{m_{r+s}} = 0) \\
&\cdot \mathbb{P}(S_{m_{r+s}} + \cdots + S_{m_{2\alpha-1}} = 0),
\end{align*}
\]

and using Fourier inversion and (6) the internal sum behaves as

\[
\begin{align*}
(2\pi)^{\alpha - s + r} \left[ \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (1 - \lambda \phi(x))^{-1} (1 - \lambda \phi(x) \phi(y))^{-1} (1 - \lambda \phi(y))^{-1} \\
\cdot \left[ \prod_{j=r+s+1}^{\alpha+s-1} \prod_{k=r+s+1}^{\alpha+s-1} (1 - \lambda \phi(x) \phi(t_j))^{-1} (1 - \lambda \phi(y) \phi(t_k))^{-1} \right] \\
\cdot \log \left( \frac{1}{1 - \lambda} \right)^{\alpha - s + r - 2} \pi^2 \right. \\
\cdot \log (1 - \lambda)^{\alpha - s + r - 2} \pi^2 / 6.
\end{align*}
\]
Then, we have $\sum \lambda^n b(n) \sim (\pi^2/6(\pi)^2)\alpha(\lambda)$, whence the Tauberian theorem implies that $a(n) - b(n) \sim n^2 \log(n)^{3\alpha-4}/24\pi^{3\alpha-4}n^{2\alpha-2}$. Most importantly we see that the lengths and locations of the chains, $r$ and $s$, do not affect the asymptotic behaviour. Noting that if $1 \leq r, s \leq \alpha - 1$, we can partition $2\alpha = r + s + (\alpha - r) + (\alpha - s)$ in $(\alpha - 1)^2$ ways, and thus overall the main contribution of terms with $v = 3$ is

$$
\left[ \frac{(\pi^2(\alpha - 1)^2}{12\pi^{3\alpha-4}n^{2\alpha-2}} \right] n^2 \log(n)^{3\alpha-4}. \quad (55)
$$

Case 2 ($v(\delta) = 2$). The typical term $c(n)$ was introduced in (33) in the proof of Proposition 9. Now we let $\lambda \in C$, with $|\lambda| < 1$. By lengthy but direct calculations we can derive an expression of the form

$$
\sum \lambda^n c(n) = \frac{\alpha - 1}{(\pi\gamma)^2} a(\lambda) + o(a(\lambda)), \quad \lambda \rightarrow 1. \quad (56)
$$

The approach developed in [13] can then be used to bound the error terms and show that $c(n) \sim [(\alpha - 1)/2(\pi\gamma)^2] n^2 \log(n)^{3\alpha-4}$.

Finally taking into account the fact that $l_1, \ldots, l_\alpha$ can be in any of the $\alpha - 1$ intervals $[k_i, k_{i+1}]$, for $i = 1, \ldots, \alpha - 1$, the result follows the overall contribution of terms with $v(\delta) = 2$

$$
(\alpha - 1)^2 \frac{2}{(\pi\gamma)^2} n^2 \log(n)^{3\alpha-4}. \quad (57)
$$

The case for $d = 2$ is very similar, so we move on to the case $d = 3$.

Case 3 ($d = 3$ and $\alpha = 2$). Using the same notation as before, we have three terms to consider $a(n), b(n)$, and $c(n)$. We first consider $c(n)$. Letting $K := \epsilon/\sqrt{1 - \lambda}$ and using the usual power series construction and spherical coordinates

$$
\sum \lambda^n c(n) = (1 - \lambda)^2 (2\pi)^{-6}
$$

and thus overall we must have that

$$
I_1(\lambda) \sim |\Sigma|^{-1} \frac{\pi}{2} \log K
$$

$$
\cdot \int_{\theta_1, \theta_2, \phi_1, \phi_2 = 0}^{2\pi} \sin(\theta_1) \sin(\theta_2) d\phi_1 d\phi_2 d\theta_1 d\theta_2,
$$

The other integral is slightly easier

$$
I_2(\lambda) \sim |\Sigma|^{-1} \frac{\pi}{2} \log K
$$

$$
\cdot \int_{\theta_1, \theta_2, \phi_1, \phi_2 = 0}^{2\pi} \sin(\theta_1) \sin(\theta_2) d\phi_1 d\phi_2 d\theta_1 d\theta_2,
$$

and thus overall we must have

$$
(I_1 - I_2)(\lambda) \sim \frac{1}{2} (2\pi)^{-6} |\Sigma|^{-1} (1 - \lambda)^2 \log \left( \frac{1}{1 - \lambda} \right)
$$

$$
\cdot \int_{\theta_1, \theta_2, \phi_1, \phi_2 = 0}^{2\pi} \frac{\arcsin(A)}{\sqrt{1 - A^2}} \frac{\pi}{2} \sin(\theta_1)
$$
\[
\sin(\theta_2) d\phi_1 d\phi_2 d\theta_1 d\theta_2 = \kappa_2 (1 - \lambda)^2 \log \left( \frac{1}{1 - \lambda} \right),
\]
whence it follows that \( \text{var}(L_n(2)) \sim (\kappa_1 + \kappa_2) n \log n. \)

To prove the last claim let \( S'_n = X'_1 + \cdots + X'_n \) be another random walk, independent of \( S_n \), such that its characteristic function \( f'(t) = \mathbb{E}[\exp(itX'_1)] \) also satisfies the expansion (6). Then using [13, Lemma 3.1], one can adapt the proof of [13, Theorem 2.1] to show that \( L'_n(\alpha) = L_n(\alpha) + o(L_n(\alpha)). \)  

\[\square\]

**Competing Interests**

The authors declare that they have no competing interests.

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