Instability Analysis of Cylindrical Stellar Object in Brans-Dicke Gravity

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Abstract

This paper investigates instability ranges of a cylindrically symmetric collapsing stellar object in Brans-Dicke theory of gravity. For this purpose, we use perturbation approach in the modified field equations as well as dynamical equations and construct a collapse equation. The collapse equation with adiabatic index ($\Gamma$) is used to explore the instability ranges of both isotropic as well as anisotropic fluid in Newtonian and post-Newtonian approximations. It turns out that the instability ranges depend on the dynamical variables of collapsing fluid. We conclude that the system always remains unstable for $0 < \Gamma < 1$ while $\Gamma > 1$ provides instability only for the special case.

Keywords: Brans-Dicke theory; Instability; Newtonian and post-Newtonian regimes.

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1 Introduction

Dark energy and gravitational collapse are the most fascinated and interesting phenomena of cosmology as well as gravitational physics. Number of astronomical observations such as Supernova type I, Sloan Digital Sky Survey, large scale-structure, Wilkinson Microwave Anisotropy Probe, galactic cluster emission of X-rays and weak lensing describe accelerated behavior of the expanding universe [1]. It is suggested that a mysterious type of energy known as dark energy is responsible for this accelerated expansion of the universe. This induces the problem of correct theory of gravity and thus number of modified theories of gravity are constructed using modified Einstein-Hilbert actions. Brans-Dicke (BD) theory is one of the most explored example among various modified theories that provides convenient evidences of various cosmic problems like inflation, early and late behavior of the universe, coincidence problem and cosmic acceleration [2]. This is a generalized form of general relativity (GR) which is constructed by the coupling of scalar field $\phi$ and tensor field $R$. It contains a constant coupling parameter $\omega_{BD}$ (tuneable parameter) which can be adjusted according to suitable observations. This theory is compatible with Mach’s principle, weak equivalence principle and Dirac’s large number hypothesis [3]. It is also consistence with solar system observations and experiments (weak field regimes test) for $|\omega| \geq 40,000$ [4].

Gravitational collapse is a process in which stable stellar objects turn into unstable ones under the effects of their own gravity. The formation and dynamics of large scale structures such as stars, celestial cluster and galaxies are investigated through this phenomenon. It is believed that different instability ranges for astronomical bodies lead to different structure formation of collapsing models. Chandrasekhar [5] was the first who explored stability ranges of a spherically symmetric isotropic fluid in GR. He used equation of state involving adiabatic index ($\Gamma$) and concluded that the fluid remains unstable for $\Gamma < \frac{4}{3}$. Later on, many researchers [6] investigated dynamical instability of different types of fluids (anisotropic fluid, adiabatic, non adiabatic as well as shearing viscous fluid) in spherical as well as cylindrical configurations and found that stability ranges depend on physical properties of the respective fluid.

It is believed that study of collapse phenomenon in modified theories may reveal modification hidden in the formation of astronomical structures [7]. In 1969, Nutku [8] explored instability ranges of spherically symmetric
isotropic fluid in BD theory and concluded that BD fluid remains unstable for $\Gamma > \frac{4}{3}$. Kwon et al. [9] discussed instability analysis of the Schwarzschild black hole in BD gravity. Sharif and Kauser [10] investigated stability ranges for spherical as well as cylindrical collapsing models in $f(R)$ theory and found that instability ranges depend upon characteristics of fluids and dark energy components. Sharif and Yousaf [11] studied the effects of electromagnetic field on instability ranges for various models of $f(R)$ gravity. Sharif and Rani [12] explored dynamical instability of spherically symmetric fluid in $f(T)$ theory and concluded that modified terms control instability ranges. In a recent paper [13], we have discussed collapse of spherically symmetric anisotropic BD fluid through instability analysis and found that $0 < \Gamma < 1$ always leads to instability while $\Gamma > 1$ provides instability only for one particular case.

In this paper, we investigate dynamical instability of cylindrically symmetric collapsing fluid in BD gravity. The paper is organized in the following format. The next section discusses BD equations, Darmois junction conditions and dynamical equations. In section 3, we use perturbation technique to construct hydrostatic equilibrium (collapse equation). Section 4 describes instability ranges at Newtonian and post-Newtonian limits for isotropic as well as an anisotropic fluid distributions. Finally, the last section summarizes the results.

2 Brans-Dicke Theory and Dynamical Equations

The BD theory has the following action [3]

$$S = \int d^4 x \sqrt{-g} \left[ \phi R - \frac{\omega_{BD}}{\phi} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) + L_m \right],$$

where $8\pi G_0 = c = 1$, $V(\phi)$ is the self-interacting potential and $L_m$ represents matter distribution. Varying Eq.(1) by $g_{\alpha\beta}$ and $\phi$, we obtain the following BD equations

$$G_{\alpha\beta} = \frac{1}{\phi} (T^m_{\alpha\beta} + T^\phi_{\alpha\beta}),$$

$$\Box \phi = \frac{T^m}{3 + 2\omega_{BD}} + \frac{1}{3 + 2\omega_{BD}} \left[ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right].$$
Here $G_{\alpha\beta}$ is the Einstein tensor, $T^m_{\alpha\beta}$ is the energy-momentum tensor for matter distribution with $T^m$ as its trace and $\Box$ represents d’Alembertian operator. The energy distribution due to scalar field is given by

$$T^\phi_{\alpha\beta} = \phi_{,\alpha} \phi_{,\beta} - g_{\alpha\beta} \Box \phi + \frac{\omega_{BD}}{\phi} [\phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\mu} \phi_{,\mu}] - \frac{V(\phi)}{2} g_{\alpha\beta}. \quad (4)$$

Equation (2) gives the BD field equations and (3) is a wave equation for the evolution of scalar field.

We split 4D geometry into interior and exterior regions by considering a timelike 3D hypersurface $\Sigma^{(e)}$ as an external boundary of the respective cylindrical body. The line element of interior spacetime is represented by

$$ds^2_2 = A^2(t, r) dt^2 - B^2(t, r) dr^2 - C^2(t, r) d\phi^2 - dz^2. \quad (5)$$

In stationary or static region, a scalar field becomes constant and all stationary black holes in BD gravity are identical with GR solutions [14]. Therefore, for exterior region to $\Sigma^{(e)}$, we take line element of the form

$$ds^2_+ = -\frac{2M}{r} d\nu^2 + 2 dr d\nu - r^2(d\phi^2 + \gamma^2 dz^2), \quad (6)$$

where $M$, $\nu$ and $\gamma$ describe the total gravitating mass, retarded time, and arbitrary constant, respectively [15]. The interior region is filled with anisotropic matter distribution represented by

$$T^m_{\alpha\beta} = (\rho + p_r) u_\alpha u_\beta - p_r g_{\alpha\beta} + (p_z - p_r) S_\alpha S_\beta + (p_\phi - p_r) K_\alpha K_\beta, \quad (7)$$

where $\rho$, $p_r$, $p_\phi$ and $p_z$ indicate energy density and principal pressure stresses, respectively. The four velocity $u_\alpha$, unit four-vectors $S_\alpha$ and $K_\alpha$ are calculated as $u_\alpha = A \delta^0_\alpha$, $S_\alpha = \delta^3_\alpha$, $K_\alpha = C \delta^2_\alpha$ satisfying $u^\alpha u_\alpha = 1$, $S^\alpha S_\alpha = K^\alpha K_\alpha = -1$, $S^\alpha u_\alpha = K^\alpha u_\alpha = S^\alpha K_\alpha = 0$. For the interior region, the BD equations are given in Appendix A.

Junction conditions provide smooth connection between interior and exterior regions over $\Sigma^{(e)}$. We consider Darmois junction conditions to discuss connection between two regions and for this purpose we take C-energy (mass function) [14] given by

$$\tilde{E}(t, r) = m(t, r) = \frac{1}{8} (1 - t^{-2} \nabla^\beta \tilde{r} \nabla_\beta \tilde{r}). \quad (8)$$
Here $\tilde{E}(t, r)$ is the gravitational energy per unit specific length of the cylinder, $\tilde{r}$ represents the areal radius, $\mu$ shows the circumference radius and $l$ indicates specific length. These are given as follows

$$\tilde{r} = \mu l, \quad \mu^2 = \xi_{(1)}^{(1)\beta} \xi_{(1)}^{\beta}, \quad l^2 = \xi_{(2)}^{(2)\beta} \xi_{(2)}^{\beta},$$

where $\xi_{(1)} = \frac{\partial}{\partial \theta}$ and $\xi_{(2)} = \frac{\partial}{\partial z}$ are the respective Killing vectors. For the interior spacetime, Eq.(8) takes the form

$$m(t, r) = \frac{l}{8} \left( 1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right), \quad (9)$$

where dot and prime show derivatives with respect to $t$ and $r$, respectively. Since in BD gravity, scalar field and metric tensor are indicated as gravitational variables, therefore $\phi = \phi_{\Sigma(e)} = constant$ at the hypersurface $\Sigma(e)$. The continuity of first and second fundamental forms (Darmois conditions) yield the following relations

$$r = r_{\Sigma(e)} = constant, \quad m(t, r) - M \frac{\Sigma(e)}{8}, \quad l \frac{\Sigma(e)}{4} = 4C,$$

$$\frac{p_r}{\phi} \frac{\Sigma(e)}{B^2} - \frac{T_{01}^\phi}{AB} = - \frac{V(\phi)}{2\phi}. \quad (10)$$

Dynamical equations obtained from the contracted Bianchi identities describe the conservation of total energy of the system given by

$$\left( \frac{T_{m}^{\alpha\beta}}{\phi} + \frac{T_{\phi}^{\alpha\beta}}{\phi} \right) ;_{\alpha} u_{\beta} = 0, \quad \left( \frac{T_{m}^{\alpha\beta}}{\phi} + \frac{T_{\phi}^{\alpha\beta}}{\phi} \right) ;_{\alpha} \chi_{\beta} = 0, \quad (11)$$

where $\chi_{\beta} = -B \delta_{\beta}^{1}$ (unit four-vector) which provides

$$\left[ \frac{\dot{p}_r}{A} - \frac{p_{r}}{\phi^2 A} + (\rho + p_r) \frac{\dot{B}}{AB} + (\rho + p_{\phi}) \frac{\dot{C}}{AC} \right] + K_1 = 0, \quad (12)$$

$$\left[ \frac{p'_r}{B} + \frac{\phi' p_r}{\phi^2 B} + (\rho + p_r) \frac{A'}{AB} + (p_r - p_{\phi}) \frac{C'}{BC} \right] + K_2 = 0, \quad (13)$$

$K_1$ and $K_2$ are mentioned in Appendix A.
3 Perturbation Scheme

In this section, we use perturbation approach to construct collapse equation. We assume that initially the system is in static equilibrium (metric as well as material parts have radial dependence only) and after that all the dynamical variables along with metric functions are perturbed and time dependence appears. The scalar field, scalar potential and metric tensors have the same time dependence, while the density and pressure bear the same time dependence as follows

\[ A(t, r) = A_0(r) + \epsilon T(t)a(r), \]  
\[ B(t, r) = B_0(r) + \epsilon T(t)b(r), \]  
\[ C(t, r) = C_0(r) + \epsilon T(t)c(r), \]  
\[ \phi(r, t) = \phi_0(r) + \epsilon T(t)\Phi(r), \]  
\[ p_r(t, r) = p_{r0}(r) + \epsilon \overline{p}_r(t, r), \]  
\[ p_\phi(t, r) = p_{\phi0}(r) + \epsilon \overline{p}_\phi(t, r), \]  
\[ \rho(t, r) = \rho_0(r) + \epsilon \overline{\rho}(t, r), \]  
\[ V(\phi) = V_0(r) + \epsilon T(t)\overline{V}(r), \]

where \( 0 < \epsilon \ll 1 \) and the static distribution is expressed by zero subscript. For static and perturbed configurations of the field as well as dynamical equations, we take \( C_0 = r \). The static configuration of BD formalism, perturbed form of BD equations and junction condition \([10]\) are given in Appendix A.

The perturbed distribution of first Bianchi identity takes the form

\[ \dot{\bar{\rho}} + \left[ \frac{\Phi \rho_0}{\phi_0} + \frac{b(\rho_0 + p_{r0})}{B_0} + \frac{c(\rho_0 + p_{\phi0})}{r} + A_0 \phi_0 K_1 \right] \dot{T} = 0, \]  

(22)

which, after integration, gives

\[ \dot{\bar{\rho}} = - \left[ \frac{(\rho_0 + p_{r0})b}{B_0} + \frac{(\rho_0 + p_{\phi0})c}{r} + \frac{\Phi \rho_0}{\phi_0} + A_0 \phi_0 K_1 \right] T. \]  

(23)

The perturbed form of Eq.(13) provides

\[ \ddot{\bar{p}}_r + (\dot{\bar{p}} + \overline{\rho}_r) \frac{A'_0}{A_0} + (\overline{\rho}_r - \overline{\rho}_\phi) \frac{1}{r} + \frac{\overline{p}_r \phi'_0}{\phi_0 B_0} + \bar{K}_2 \phi_0 B_0 = 0, \]  

(24)

where \( \bar{K}_1 \) and \( \bar{K}_2 \) are given in appendix A.
Equation (60) along with junction conditions can be expressed as

\[ u \ddot{T} + v T \Sigma^{(e)} = 0, \tag{25} \]

where

\[ u^{(e)} = \frac{\Phi}{\phi_0 A_0^2} - \frac{2c}{r A_0^2}, \quad v^{(e)} = \frac{\Phi}{\phi_0} \left[ \frac{\omega_{BD}}{\phi_0} - \frac{1}{B_0 r} \right]. \]

The general solution of Eq. (25) is given by

\[ T(t) = c_1 \exp(\gamma^{(e)} t) + c_2 \exp(\lambda^{(e)} t), \tag{26} \]

where \( \gamma^{(e)} = +\sqrt{\frac{v}{u}} \), \( \lambda^{(e)} = -\sqrt{\frac{v}{u}} \), and \( c_1, c_2 \) indicate arbitrary constants. Equation (26) shows static and non-static distributions leading to stable as well as unstable phases of gravitating system. For instability analysis, we assume only static part \((t = -\infty, T(-\infty) = 0)\), i.e., when the instability phase begins, the system possesses complete hydrostatic equilibrium. Using these assumptions in Eq. (26), we have \( c_2 = 0 \) whereas \( c_1 = -1 \) is chosen arbitrarily. The corresponding result is described by

\[ T(t) = -\exp(\gamma^{(e)} t). \tag{27} \]

For a real static instability regime, we assume only positive values of \( \frac{v}{u} \).

4 Instability Analysis

For the investigation of instability ranges, we use an equation of state involving adiabatic index \( \Gamma \) \[17\] given by

\[ \bar{p}_j = \Gamma \frac{p_j \rho_0 + \bar{p} \rho_0}{\rho_0 + p_j \rho_0}. \tag{28} \]

The adiabatic index evaluates variation of principal stresses (pressures) with respect to density and represents rigidity of the gravitating fluid. We consider \( \Gamma \) to be constant throughout the stability analysis of the fluid. Equations (23) and (28) lead to

\[ \bar{p}_r = -\Gamma \left[ b \frac{p_r \rho_0}{B_0} + \frac{c \rho_0 + p \rho_0}{r \rho_0 + p r_0} + \frac{p r_0 \Phi \rho_0}{\rho_0 + p r_0 \phi_0} + \frac{p r_0 A_0 \phi_0}{\rho_0 + p r_0 \bar{K}_1} \right] T. \tag{29} \]
Using Eqs. (56), (23), (29) and (30) in (24), we construct a hydrostatic equation given by

\[
\bar{p}_\phi = -\Gamma \left[ \frac{b}{B_0} \frac{\rho_0 + p_\phi}{p_\phi} + \frac{c p_\phi}{r} + \frac{p_\phi \Phi \rho_0}{(\rho_0 + p_\phi) \phi_0} + \frac{p_\phi A_0 \phi_0}{\rho_0 + p_\phi} \bar{K}_1 \right] T\cdot (30)
\]

This represents the general form of collapse equation which describes the instability of hydrostatic equilibrium of a gravitating fluid in BD gravity.

### 4.1 Isotropic Fluid

Here, we analyze instability ranges of isotropic fluid in Newtonian and post-Newtonian limits. In isotropic fluid, all principal stresses are equal \((p_r = p_\phi = p_\theta)\). Using this condition in Eq. (31) we obtain the corresponding collapse equation

\[
\Gamma \left[ p_{ro} \left[ \frac{b T}{B_0} + \frac{(\rho_0 + p_\phi) c T}{(\rho_0 + p_{ro})} + \frac{1}{\rho_0 + p_{ro}} A_0 \phi_0 \bar{K}_1 T \right] \right] - \Gamma \left[ p_{ro} \left[ \frac{b T}{B_0} \right] \right] - \Gamma \left[ p_{ro} \left[ \frac{b T}{B_0} + \frac{(\rho_0 + p_\phi) c T}{(\rho_0 + p_{ro})} + \frac{1}{\rho_0 + p_{ro}} A_0 \phi_0 \bar{K}_1 T \right] \frac{\phi_0}{\phi_0 B_0} \right] + \phi_0 B_0 \bar{K}_2 = 0.
\]

(31)

This equation represents the general form of collapse equation which describes the instability of hydrostatic equilibrium of a gravitating fluid in BD gravity.
Newtonian Limit

The Newtonian limit in BD theory leads to the following
\[ \rho_0 \gg p_r, \quad \rho_0 \gg p_{\phi 0}, \quad B_0 = 1, \quad A_0 = 1 - \frac{m_0}{r c^2}, \]
\[ \phi_0 = \text{constant}, \quad V_0 = \ddot{V} = 0. \] (33)

Using these limits along with (27) and \( t \rightarrow -\infty \), the collapse equation with at most \( O(c^{-2}) \) turns out to be
\[ \Gamma \left[ (p_r Z_N)_{,1} - \frac{m_0}{r^2 c^2} p_r Z_N \right] - \rho_0 Z_N \frac{m_0}{r^2 c^2} + K_3 = 0, \] (34)
where
\[ Z_N = \left( b + \frac{c}{r} \right), \quad K_3 = -\left[ \frac{d' m_0}{r^2 c^2} \frac{\Phi}{\phi_0} (1 - \frac{m_0}{r c^2}) - 2 p_{r0} \frac{\Phi}{\phi_0} \right]. \]

Equation (34) describes the hydrostatic equilibrium condition of a cylindrically symmetric isotropic fluid distribution. The system becomes unstable (collapses) if
\[ \Gamma \left[ (p_r Z_N)_{,1} - \frac{m_0}{r^2 c^2} p_r Z_N \right] - \rho_0 Z_N \frac{m_0}{r^2 c^2} + K_3 < 0, \]
which gives
\[ \Gamma < \frac{\rho_0 Z_N \frac{m_0}{r^2 c^2} - K_3}{(p_r Z_N)_{,r} - \frac{m_0}{r^2 c^2} p_{r0} Z_N}. \] (35)

This shows that the adiabatic index depends on dynamical properties such as density, pressure, scalar field. To preserve difference between configurations of pressure gradient and gravitational forces, we assume \( \Gamma > 0 \). Thus the celestial objects remain unstable until Eq. (35) is satisfied which leads to
\[ \frac{\rho_0 Z_N \frac{m_0}{r^2 c^2} - K_3}{(p_r Z_N)_{,r} - \frac{m_0}{r^2 c^2} p_{r0} Z_N} > \Gamma > 0. \] (36)

The fraction
\[ \frac{\rho_0 Z_N \frac{m_0}{r^2 c^2} - K_3}{(p_r Z_N)_{,r} - \frac{m_0}{r^2 c^2} p_{r0} Z_N} > 0 \]
leads to the following possibilities:
1. $\rho_0 Z_N \frac{m_0}{r^2c^2} - K_3 = [(p_r0 Z_N)_r - \frac{m_0}{r^2c^2} p_r0 Z_N]$;

2. $\rho_0 Z_N \frac{m_0}{r^2c^2} - K_3 < [(p_r0 Z_N)_r - \frac{m_0}{r^2c^2} p_r0 Z_N]$;

3. $\rho_0 Z_N \frac{m_0}{r^2c^2} - K_3 > [(p_r0 Z_N)_r - \frac{m_0}{r^2c^2} p_r0 Z_N]$.

The first case along with (36) shows that the isotropic system becomes unstable for $0 < \Gamma < 1$. The corresponding expression leads to first order differential equation

$$p'_r0 + p_r0 \frac{Z'_N}{Z_N} = \left( (\rho_0 - 1) \frac{m_0}{r^2c^2} - Z_N^{-1} K_3 \right),$$

which yields

$$p_r0 = Z_N^{-1} \int_{r_0}^r Z_N \left( (\rho_0 - 1) \frac{m_0}{r^2c^2} - Z_N^{-1} K_3 \right) dr'.$$

This is a constraint equation for a collapsing cylindrical isotropic fluid with $0 < \Gamma < 1$. The second case also gives the same range of adiabatic index as in the first case. Thus, the corresponding expression provides the following constraint equation

$$p_r0 < Z_N^{-1} \int_{r_0}^r Z_N \left( (\rho_0 - 1) \frac{m_0}{r^2c^2} - Z_N^{-1} K_3 \right) dr'.$$  

(38)

In the third case, the denominator of is less than its numerator and hence in Eq. (36), $\Gamma$ can be taken greater than 1. The corresponding instability constraint is given by

$$p_r0 > Z_N^{-1} \int_{r_0}^r Z_N \left( (\rho_0 - 1) \frac{m_0}{r^2c^2} - Z_N^{-1} K_3 \right) dr',$$

for which $\Gamma > 1$ and isotropic cylindrical system becomes unstable. It is obvious that if the system is unstable for $\Gamma > 1$, then it will also be unstable for $0 < \Gamma < 1$.

**Post-Newtonian Limit**

The post-Newtonian (pN) regimes are found upto order $c^{-4}$ by taking

$$A_0 = 1 - \frac{m_0}{r^2c^2} + \frac{m_0^2}{r^4c^4}, \quad B_0 = 1 + \frac{\alpha m_0}{r^2c^2}, \quad \phi_0 = constant, \quad V_0 = \dot{V} = 0, (40)$$
where

\[ \alpha = \frac{1 + \omega_{BD}}{2 + \omega_{BD}}. \]

Using pN limits along with \( t \to -\infty \) and Eq. (27) in (31), we obtain

\[
\Gamma \left[ [p_{ro} X_{pN}] - [p_{ro} X_{pN}] \left( \frac{m_0}{r^2 c^2} - 2 \frac{m_0^2}{r^3 c^4} \right) \right] - [(\rho_0 + p_{ro}) X_{pN}]
\times \left( \frac{m_0}{r^2 c^2} - 2 \frac{m_0^2}{r^3 c^4} \right) - K_5 = 0,
\]

(41)

where

\[ X_{pN} = \left[ (b(1 - \frac{\alpha m_0}{r c^2}) + \frac{c}{r}) + \frac{K_4}{\rho_0 + p_{ro}} \right]. \]

Here \( K_4 \) and \( K_5 \) are given in Appendix A. The system becomes unstable whenever

\[
\Gamma < \frac{[(\rho_0 + p_{ro}) X_{pN}] \left( \frac{m_0}{r^2 c^2} - 2 \frac{m_0^2}{r^3 c^4} \right) - K_5}{[[p_{ro} X_{pN}] - [p_{ro} X_{pN}] \left( \frac{m_0}{r^2 c^2} - 2 \frac{m_0^2}{r^3 c^4} \right)]}
\]

(42)

with constraint

\[
\frac{[(\rho_0 + p_{ro}) X_{pN}] \left( \frac{m_0}{r^2 c^2} - 2 \frac{m_0^2}{r^3 c^4} \right) - K_5}{[[p_{ro} X_{pN}] - [p_{ro} X_{pN}] \left( \frac{m_0}{r^2 c^2} - 2 \frac{m_0^2}{r^3 c^4} \right)]} > 0.
\]

Similar to the Newtonian case, the system collapses for \( 0 < \Gamma < 1 \) with the following constraints

1. \( p_{ro} = X_{pN}^{-1} \exp \left[ 2r \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) + \int_{r_0}^r Y_{pN} dr' \int_{r_0}^r X_{pN} \exp \left( -2r \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) - \int_{r_0}^{r'} Y_{pN} dr' \right) \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) \right] \]
\[ \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) + X_{pN}^{-1} \left[ \frac{c}{\rho_0} (1 + 2 \frac{m_0}{r c^2} (1 - \alpha)) + \frac{m_0^2}{r^3 c^4} (1 + 4 \alpha) \right] - X_{pN}^{-1} \frac{4m_0}{r c^2} \gamma_{\Sigma(e)} \]

2. \( p_{ro} < X_{pN}^{-1} \exp \left[ 2r \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) + \int_{r_0}^r Y_{pN} dr' \int_{r_0}^r X_{pN} \exp \left( -2r \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) - \int_{r_0}^{r'} Y_{pN} dr' \right) \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) \right] \]
\[ \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) + X_{pN}^{-1} \left[ \frac{c}{\rho_0} (1 + 2 \frac{m_0}{r c^2} (1 - \alpha)) + \frac{m_0^2}{r^3 c^4} (1 + 4 \alpha) \right] - X_{pN}^{-1} \frac{4m_0}{r c^2} \gamma_{\Sigma(e)} \]

In the third case, \( \Gamma > 1 \) leads to unstable configuration with the following constraint

\[ p_{ro} > X_{pN}^{-1} \exp \left( 2r \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) + \int_{r_0}^r Y_{pN} dr' \int_{r_0}^r X_{pN} \exp \left( -2r \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) - \int_{r_0}^{r'} Y_{pN} dr' \right) \left( \frac{m_0}{r^2 c^2} - \frac{2m_0^2}{r^3 c^4} \right) \right] \]

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\[ (\rho_0 \frac{m_0}{r^2c^2} - \frac{2m_0^2}{r^3c^4}) + X_{pN}^{-1} \left( \frac{\alpha'}{\phi_0} \left( 1 + \frac{2m_0}{rc^2} (1 - \alpha) \right) + \frac{m_0^2}{r^2c^4} (1 + 4\alpha) \right) \]

\[-X_{pN}^{-1} \Sigma(0) \left( \frac{m_0}{r^2c^2} - \frac{4m_0^2}{r^3c^4} + \frac{2\alpha m_0}{r^4c^4} \right), \]

where \( Y_{pN} = X_{pN}^{-1} \left[ \frac{\alpha'}{\phi_0} (1 + \frac{2m_0}{rc^2} (1 - \alpha)) + \frac{m_0^2}{r^2c^4} (1 + 4\alpha) \right] \). In this case, \( 0 < \Gamma < 1 \) is also an instability range.

### 4.2 Anisotropic Fluid

Here, we have \( p_r \neq p_\phi \neq p_z \) and hydrostatic equilibrium is described by Eq.(31).

**Newtonian limit**

Using Eq.(33) in (31), we obtain

\[ \Gamma \left[ [p_{r0}Z_N]' + \frac{(p_{r0} - p_{\phi0}) Z_N - p_{r0}Z_N \frac{m_0}{r^2c^2}}{r} \right] - \frac{2p_{r0}}{\phi_0} - \frac{(p_{r0} - p_{\phi0})}{r} \left[ \frac{\rho}{r} \right]' \]

\[ = \rho_0 (Z_N + K_6) - K_7, \tag{43} \]

where \( K_6 \) and \( K_7 \) are mentioned in Appendix A. Thus, anisotropic fluid collapses in Newtonian approximation for

\[ 0 < \Gamma < \frac{2p_{r0} \Phi}{\phi_0} + \frac{(p_{r0} - p_{\phi0})}{r} \left[ \frac{\rho}{r} \right]' + \rho_0 (Z_N + K_6) + K_7 \]

\[ \left[ [p_{r0}Z_N]' + \frac{(p_{r0} - p_{\phi0}) Z_N - p_{r0}Z_N \frac{m_0}{r^2c^2}}{r} \right]. \tag{44} \]

Similar to isotropic case, the above equation implies that for

\[ p_{r0} \leq r^{-1} Z_N^{-1} \exp \left( \frac{m_0}{r^2c^2} \int_{r_0}^r \phi + \frac{1}{2} [\phi]' dr' \right) \left[ \int_{r_0}^r r Z_N \exp \left( -\frac{m_0}{r^2c^2} \int_{r_0}^r \phi + \frac{1}{2} [\phi]' dr' \right) \frac{P_{\phi0}}{r} \right. \]

\[ + \rho_0 (1 + K_6) + K_7] dr', \]

\( \Gamma \) lies in \((0,1)\) and the system collapses. If

\[ p_{r0} > r^{-1} Z_N^{-1} \exp \left( \frac{m_0}{r^2c^2} \int_{r_0}^r \phi + \frac{1}{2} [\phi]' dr' \right) \left[ \int_{r_0}^r r Z_N \exp \left( -\frac{m_0}{r^2c^2} \int_{r_0}^r \phi + \frac{1}{2} [\phi]' dr' \right) \frac{P_{\phi0}}{r} \right. \]

\[ + \rho_0 (1 + K_6) + K_7] dr', \]

de the system becomes unstable for \( \Gamma > 1 \).
Post-Newtonian Limit

The collapse equation of anisotropic cylindrical fluid in pN regime is
\[
\Gamma \left[ \dot{p}_{r0} U_{pN} - \frac{\rho_{\phi0}}{r} V_{pN} \right] - p_{r0} U_{pN} \left( \frac{m_0}{r^2 c^2} - \frac{m_0}{r^3 c^4} \right) = (p_{r0} + \rho_0) \left[ U_{pN} \right] - K_9, (45)
\]
where
\[
U_{pN} = \left[ b(1 - \alpha \frac{m_0}{r c^2}) + \frac{(p_{\phi0} + \rho_0) c}{(p_{r0} + \rho_0) r} + \frac{1}{(p_{r0} + \rho_0) K_8} \right],
\]
\[
V_{pN} = \left[ b(1 - \alpha \frac{m_0}{r c^2}) \frac{(p_{r0} + \rho_0) c}{(p_{\phi0} + \rho_0) r} + \frac{1}{(p_{\phi0} + \rho_0) K_8} \right].
\]
The values of $K_8$ and $K_9$ are given in Appendix A. The system becomes to unstable for
\[
\Gamma < \frac{\frac{\rho_{\phi0}}{r} U_{pN} \left( \frac{m_0}{r^2 c^2} - \frac{m_0}{r^3 c^4} \right) - (p_{r0} + \rho_0) \left[ U_{pN} \right] - K_9}{\left[ \dot{p}_{r0} U_{pN} \right] + \frac{\rho_{\phi0}}{r} U_{pN} - \frac{\rho_{\phi0}}{r} V_{pN}},
\]
which provides instability ranges $0 < \Gamma < 1$ for
- $p_{r0} r U_{pN} \left( \frac{m_0}{r^2 c^2} - \frac{m_0}{r^3 c^4} \right) - (p_{r0} + \rho_0) \left[ U_{pN} \right] - K_9 = \left[ \dot{p}_{r0} U_{pN} \right] + \frac{\rho_{\phi0}}{r} U_{pN} - \frac{\rho_{\phi0}}{r} V_{pN}$;
- $p_{r0} r U_{pN} \left( \frac{m_0}{r^2 c^2} - \frac{m_0}{r^3 c^4} \right) - (p_{r0} + \rho_0) \left[ U_{pN} \right] - K_9 < \left[ \dot{p}_{r0} U_{pN} \right] + \frac{\rho_{\phi0}}{r} U_{pN} - \frac{\rho_{\phi0}}{r} V_{pN}$,
and becomes unstable for $\Gamma > 1$ if
- $p_{r0} r U_{pN} \left( \frac{m_0}{r^2 c^2} - \frac{m_0}{r^3 c^4} \right) - (p_{r0} + \rho_0) \left[ U_{pN} \right] - K_9 > \left[ \dot{p}_{r0} U_{pN} \right] + \frac{\rho_{\phi0}}{r} U_{pN} - \frac{\rho_{\phi0}}{r} V_{pN}$.

5 Concluding Remarks

In this paper, we have investigated instability ranges of anisotropic cylindrically symmetric collapsing body in BD theory. We have used contracted Bianchi identities to obtain two dynamical equations of collapsing system. By applying perturbation technique on BD as well as dynamical equations, we separate the unperturbed (static) and perturbed (non-static) distributions of all dynamical relations. We have developed hydrostatic equation (collapse equation) through perturbed configuration of second dynamical equation.

The equation of state involving adiabatic index controls the ranges of instability for a collapsing system. We have used collapse equation along with
equation of state to investigate the instability ranges of both isotropic as well as anisotropic BD fluid at Newtonian and post-Newtonian limits. It is concluded that in both approximations the adiabatic index depending upon dynamical properties (energy density, pressure, scalar field terms and some constraints) controls the instability ranges. We have constructed constraints on static radial matter pressure under the effects of scalar field. It is found that the system always remains unstable for $0 < \Gamma < 1$, while $\Gamma > 1$ is the instability range for the special case. We would like to mention here that the instability ranges for spherical as well as cylindrical distributions in GR depend upon $\Gamma < \frac{4}{3}$ and $\Gamma < 1$. In $f(R)$ and $f(T)$ theories, physical variables such as density, pressure and respective modified dark terms provide the instability ranges. The instability range of spherically symmetric isotropic BD fluid is $\Gamma > \frac{4}{3}$ while anisotropic spherical BD fluid always remains unstable for $0 < \Gamma < 1$ and $\Gamma > 1$ leads to collapse only for the special case.

Appendix A

The non-zero components of BD equations for interior region are

$$G_{00} = \frac{1}{\phi} (T_{00}^m + T_{00}^s) = \frac{1}{\phi} \left( \rho A^2 + \frac{\omega_{BD}}{\phi^2} \left( \dot{\phi}^2 + \frac{A^2 \phi'^2}{B^2} \right) \right)$$

$$- \frac{\dot{\phi}}{\phi} \left( 2 \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \phi' \frac{A^2}{\phi B^2} \left( \frac{B'}{B} + \frac{C'}{C} \right) + \phi'' \frac{A^2}{B^2} - \frac{A^2 V(\phi)}{2\phi}, \quad (46)$$

$$G_{01} = \frac{1}{\phi} (T_{01}^m + T_{01}^s) = \frac{\omega_{BD}}{\phi^2} (\phi \phi') + \frac{1}{\phi} \left( \phi' - \frac{A' \phi}{A} - \frac{B \phi'}{B} \right), \quad (47)$$

$$G_{11} = \frac{1}{\phi} (T_{11}^m + T_{11}^s) = \frac{1}{\phi} \left( p_r B^2 + \frac{\omega_{BD}}{\phi^2} \left( \phi'^2 + \frac{B^2 \phi'^2}{A^2} \right) \right) + \frac{\dot{B} \phi}{A^2 \phi}$$

$$+ \frac{B^2 \phi'}{A^2} \left( \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) - \phi' \frac{A'}{A} - \frac{C'}{C} + \frac{B^2 V(\phi)}{2\phi}, \quad (48)$$

$$G_{22} = \frac{1}{\phi} (T_{22}^m + T_{22}^s) = \frac{1}{\phi} \left( p_{\perp} C^2 + \frac{\omega_{BD}}{\phi^2} \left( \frac{C^2 \phi'^2}{A^2} - \frac{C^2 \phi'^2}{B^2} \right) \right) + \frac{\dot{C} \phi}{A^2 \phi}$$

$$+ \frac{C^2 \phi'}{A^2 \phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{C^2 \phi'}{B^2} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{C^2 \phi''}{B^2 \phi} + \frac{C^2 V(\phi)}{2\phi}, \quad (49)$$
and Eq.(3) becomes

\[ G_{33} = \frac{1}{\phi} (T_{33}^m + T_{33}^S) = p_z + \frac{\omega_{BD}}{2\phi^2 B^2} \left( \frac{\phi^2}{A^2} - \frac{\phi'^2}{B^2} \right) + \frac{\phi}{A^2 \phi} \]
\[ + \frac{\phi}{A^2 \phi} \left[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right] - \frac{\phi'}{B^2 \phi} \left[ A' + B' + \frac{C'}{C} \right] - \frac{\phi''}{B^2 \phi} + \frac{V(\phi)}{2\phi} \] 

(50)

and Eq.(3) becomes

\[ \dot{\phi} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{A^2 B} - \frac{\dot{C}}{A^2 B} \right) + \frac{\phi}{A^2} + \phi' \left( \frac{A'}{AB^2} - \frac{B'}{B^3} - \frac{C'}{CB^2} \right) - \frac{\phi''}{B^2} \]
\[ = \frac{1}{2\omega_{BD} + 3} \left( (\rho + 3p_r + p_\phi + p_z) + \left( \frac{dV}{d\phi} - 2V \right) \right). \] 

(51)

The scalar terms \( K_1 \) and \( K_2 \) of Eqs.\([12]\) and \([13]\) are

\[ K_1 = \left( T_{00}^\phi \right)_t A^{-1} - \left( T_{01}^\phi \right)_x A^{-1} B^{-2} + \left( \rho A^{-1} + T_{00}^\phi A^{-2} \right) \phi^{-2} \dot{\phi} \]
\[ + T_{01}^\phi A^{-1} B^{-2} \phi^2 \phi' - 2T_{01}^\phi B^3 A'B' - T_{01}^\phi A^{-2} B^{-2} A', \]
\[ K_2 = T_{11}^\phi B^{-1} \phi' \phi^{-2} + \left( \rho + T_{00}^\phi \right) A^{-2} B^{-1} \phi^{-2} \dot{\phi} - \left( T_{01}^\phi A^{-2} B^{-2} \right)_t B \]
\[ - \left( T_{11} B^{-2} \right)_r B. \]

The static distribution of BD field equations is

\[ \frac{\rho_0}{\phi_0} + \frac{\omega_{BD} \phi_0'^2}{2B_0^2 \phi_0} + \frac{B_0' \phi_0'}{B_0^2 \phi_0} + \frac{2\phi_0'}{B_0^2 r \phi_0} + \frac{\phi_0''}{B_0^2 \phi_0} - \frac{V_0}{2\phi_0} = \frac{1}{B_0^2 r} \frac{B_0'}{A_0}, \]

(52)

\[ \frac{p_r_0}{\phi_0} + \frac{\omega_{BD} \phi_0'^2}{2B_0^2 \phi_0} - \frac{A_0' \phi_0'}{A_0 B_0^2 \phi_0} - \frac{2\phi_0'}{B_0^2 r \phi_0} + \frac{V_0}{2\phi_0} = \frac{1}{B_0^2 r} \frac{A_0'}{A_0}, \]

(53)

\[ \frac{p_\phi_0}{\phi_0} - \frac{\omega_{BD} \phi_0'^2}{2B_0^2 \phi_0} - \frac{B_0' \phi_0'}{A_0 B_0^2 \phi_0} - \frac{A_0' \phi_0'}{A_0 B_0^2 \phi_0} - \frac{\phi_0''}{B_0^2 \phi_0} + \frac{V_0}{2\phi_0} \]
\[ = \frac{1}{B_0^2} \left[ \frac{A_0'}{A_0} + \frac{A_0' B_0'}{B_0} \right]. \]

(54)

\[ \frac{p_z}{\phi_0} - \frac{\phi_0' A_0'}{B_0^2 A_0 \phi_0} - \frac{\phi_0' B_0'}{B_0^2 r \phi_0} - \frac{\phi_0'}{B_0^2 \phi_0} - \frac{\phi_0''}{B_0^2} - \frac{\omega_{BD} \phi_0'^2}{2B_0^2} + \frac{V_0}{2\phi_0} \]
\[ = \left( \frac{A_0'}{r} + \frac{A_0''}{r} \right) \frac{1}{A_0 B_0^2} \frac{B_0'}{B_3} \left( \frac{1}{r} + \frac{A_0'}{A_0} \right). \]

(55)
The unperturbed wave equation is
\[
\frac{\phi''_{0} A'_{0}}{A_{0}} - \frac{\phi''_{0} B'_{0}}{B_{0}^{2}} + \frac{\phi''_{0}}{r B_{0}} = \frac{-1}{2 \omega_{BD} + 3} \left[ (\rho_{0} + 3 p_{r0} + p_{\phi0} + p_{z0}) + (\phi_{0} V_{0} - 2 V_{0}) \right].
\]

The static part of Eq. (10) is
\[
\phi_{0} = \frac{A_{0}'}{A_{0} B_{0} \phi_{0}} + \frac{1}{B_{0} \phi_{0}} (p_{r0} - p_{\phi0}) - K_{2}' = 0,
\]
where
\[
K_{2}' = T_{11_{unp}}^{\phi} \frac{\phi''_{0}}{\phi_{0} B_{0}^{2}} - \frac{p'_{r0}}{B_{0} \phi_{0}} - (p_{r0} + \rho_{0}) \frac{A'_{0}}{A_{0} B_{0} \phi_{0}} - (p_{r0} + p_{\phi0}) \frac{A'_{0}}{B_{0} r \phi_{0}} + 2 T_{11_{unp}}^{\phi} \frac{B'_{0}}{B_{0}^{2}} - \frac{(T_{11_{unp}}^{\phi})'}{B_{0}}.
\]

The static distribution of Eq. (12) is identically satisfied in static background while (13) turns out to be
\[
\frac{p'_{r0}}{B_{0} \phi_{0}} + \frac{\phi''_{0} p_{r0}}{\phi_{0} B_{0}^{2}} + (\rho_{0} + p_{r0}) \frac{A'_{0}}{A_{0} B_{0} \phi_{0}} + \frac{1}{B_{0} \phi_{0} r} (p_{r0} - p_{\phi0}) - K_{2}' = 0,
\]
where
\[
K_{2}' = T_{11_{unp}}^{\phi} \frac{\phi''_{0}}{\phi_{0} B_{0}^{2}} - \frac{p'_{r0}}{B_{0} \phi_{0}} - (p_{r0} + \rho_{0}) \frac{A'_{0}}{A_{0} B_{0} \phi_{0}} - (p_{r0} + p_{\phi0}) \frac{A'_{0}}{B_{0} r \phi_{0}} + 2 T_{11_{unp}}^{\phi} \frac{B'_{0}}{B_{0}^{2}} - \frac{(T_{11_{unp}}^{\phi})'}{B_{0}}.
\]

The perturb form of BD field equations are
\[
\frac{-2 T}{B_{0}^{2}} \left[ \left( \frac{c}{r} \right)'' - \frac{1}{r} \left( \frac{b}{B_{0}} \right)' \right] = \frac{-\rho}{\phi_{0}} - T \rho_{0} \Phi
\]
\[
+ \frac{T \omega_{BD} \phi_{0}^{2} b}{\phi_{0} B_{0}^{3}} - \frac{T \omega_{BD} \phi_{0}^{2} B_{0}^{2}}{\phi_{0} B_{0}^{3}} - \frac{T \omega_{BD} \phi_{0}^{2} B_{0}^{2}}{\phi_{0} B_{0}^{3}} + \frac{T \phi_{0}' \phi_{0}}{\phi_{0} B_{0}^{2}} \left( \frac{c}{r} \right)' + \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}} \left( \frac{\phi_{0} b}{B_{0}} \right)'
\]
\[
- \frac{2 T b \phi_{0}}{\phi_{0} B_{0}^{2}} - \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}} + \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}} - \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}} - \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}} - \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}}
\]
\[
- \frac{TV_{0} \Phi}{2 \phi_{0}^{2}} - \frac{TV}{2 \phi_{0}^{2}},
\]
\[
\frac{-c'}{c} + \frac{A_{0}'}{A_{0}} + \frac{b}{c B_{0}} = \frac{\omega_{BD} \dot{T} \Psi \phi_{0}'}{\phi_{0}^{2}} - \frac{\omega_{BD} \dot{T} \Psi \phi_{0}'}{\phi_{0}^{2}} + \frac{A_{0}'; \dot{T} c}{r A_{0}} - \frac{A_{0}'; \dot{T} c}{r A_{0}} - \frac{T p_{r0} \Phi}{\phi_{0}^{2}}
\]
\[
- \frac{2 \ddot{T} c}{r A_{0}^{2}} + \frac{T}{B_{0}^{2}} \left[ \left( \frac{a}{A_{0}} \right)' + \left( \frac{a}{A_{0}} \right)' \right] = \frac{2 b A_{0}'}{r A_{0} B_{0}^{2} - \frac{\phi_{0}^{2} b}{\phi_{0} B_{0}^{2}} + \phi' - \frac{\phi_{0}^{2} b}{\phi_{0} B_{0}^{2}} - \frac{T \phi_{0}'}{\phi_{0} B_{0}^{2}} \left( \frac{a}{A_{0}} \right)' + \frac{c'}{r} \right]
\]
\[
+ \frac{2 T b \phi_{0}}{\phi_{0} B_{0}^{2}} \left[ \frac{A_{0}'}{A_{0} - \frac{1}{r} + \frac{V_{0}}{2 B_{0}}} \right] - \frac{T}{B_{0}^{2}} \left[ \frac{T A_{0}'}{B_{0}^{2} A_{0} - \frac{T B_{0}'}{B_{0}^{2}}} \right] \left( \frac{\phi_{0}'}{\phi_{0}} \right)'.
\]
and perturbed wave equation is given by

\[
\begin{align*}
- \frac{\ddot{T}b\phi'_0}{\phi_0 B_0^2} + \frac{\ddot{T} \Phi}{\phi_0^2} + \frac{TV_0 \Phi}{\phi_0^2} + \frac{T \ddot{V}}{\phi_0} + b \ddot{T} & = \frac{1}{2\omega_{BD}^2} \times \left[ \ddot{\phi}_r + \frac{3}{2} \ddot{\phi}_\phi + \ddot{\phi}_z + T \Phi V_0 - 2 \dddot{V} \right].
\end{align*}
\]
The perturbed terms \( \bar{K}_1 \) and \( \bar{K}_2 \) in Eqs. (22) and (24) are described as

\[
\bar{K}_1 = \hat{T} \left[ \rho_0 \Phi A_0 \phi_0 + T^\phi_{00(p)} A_0^{-1} + \left( T^\phi_{01(p)} \right)' A^{-1} B^{-2} - T^\phi_{01(p)} A_0' A^{-2} B^{-2} \right. \\
- \left. T^\phi_{01(p)} B_0' A^{-1} B^{-3} \right].
\]

\[
\bar{K}_2 = -T^\phi_{11(p)} \frac{\phi_0}{\phi_0^2 B_0^2} \phi_0 + \frac{2T^\phi}{\phi_0} + T^\phi' - \frac{2T b \phi_0}{\phi_0^2 B_0^2} \frac{T^\phi_{11 unp}}{B_0^2} - \frac{\bar{p}_r}{\phi_0 B_0} \\
+ \frac{p_v T}{B_0} \left[ \frac{b}{B_0} + \frac{\Phi}{\phi_0} \right] + \frac{\bar{\rho}_r - \bar{\rho}_\phi}{r B_0 \phi_0} + \left[ \frac{b \Phi}{B_0^2 r \phi_0} + \frac{1}{r B_0 \phi_0^2} (c')' \right] \\
\times (p_r - p_\phi) - \frac{(T^\phi_{10})_t}{A_0 B_0^2} + 4b B_0 T^\phi_{11 unp} - 2T^2_{11 unp} B_0^2 + T^\phi_{11 unp} B_0^2 \\
- \frac{2T^\phi_{11 unp} B_0'}{B_0^2} - \frac{(T^\phi_{11})'}{B_0^2} + \left[ \frac{b}{B_0} + \frac{\Phi}{\phi_0} + \frac{1}{B_0 \phi_0} \right]' \left[ \frac{a}{A_0} \right]' \frac{(p_r + \rho_0) A_0'}{A_0 B_0 \phi_0} \\
+ (\bar{\rho} + \bar{p}_r) \frac{A_0'}{A_0 B_0 \phi_0} + \frac{T_{11 unp} B_0'}{B_0^2}.
\]

Here \( T^\phi_{\mu\nu(unp)} \) and \( T^\phi_{\mu\nu(p)} \) indicate unperturbed as well as perturbed distributions of BD energy part, respectively.

The values of \( K_4 \) and \( K_5 \) in Eq. (44) are given by

\[
K_4 = \rho_0 \Phi \phi_0 + \Phi \frac{m_0}{r^3 c^2} + \Phi' \left( \frac{m_0}{r^3 c^2} + (3 + 2\alpha) \frac{m_0^2}{r^3 c^4} \right) + \Phi \left[ 1 + \frac{2m_0}{rc^2} \right] + \frac{m_0^2}{r^2 c^2} - \frac{2\alpha m_0}{rc^2} + (4\alpha + 1) \frac{m_0^2}{r^2 c^4},
\]

\[
K_5 = -\rho_0 \Phi \left( 1 - \frac{2\alpha}{r^2 c^2} \right) + (\rho_0 + p_{\phi 0}) \left[ \frac{a'}{\phi_0} \left( 1 + \frac{2m_0}{rc^2} (1 - \alpha) \right) + \frac{m_0}{r^2 c^4} (1 + 4\alpha) \right] \\
- \frac{\gamma_{3(c)} m_0}{r^2 c^2} - \frac{4m_0^2}{r^4 c^4} + \frac{2\alpha m_0^2}{r^4 c^4}).
\]

The scalar field terms of Eq. (43) are

\[
K_6 = \gamma_{3(c)} \left[ \rho_0 \Phi + (1 + \frac{\Phi'}{\phi_0}) + m_0 (\Phi \phi_0) \right] + \frac{\phi_0}{r} (1 - \frac{m_0}{rc^2}),
\]

\[
K_7 = \gamma_{3(c)} \left[ \frac{\Phi m_0^2 \phi_0}{r^2 c^2} - \frac{\rho_0 \phi_0}{c^2} \right].
\]

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The values of $K_8$ and $K_9$ in Eq. (45) are described as

$$
K_8 = \gamma_\Sigma(\epsilon) \rho_0 \Phi + \left[ \frac{\epsilon}{r} \right] (1 + \frac{m_0^2}{r^2 c^4} - \frac{2\alpha m_0}{rc^4}) + (1 + \frac{m_0^2}{r^2 c^4}) \frac{\Phi'}{\phi_0} + \Phi_0 \frac{m_0^2}{r^2 c^4} 
$$

$$
K_9 = 1 + \frac{\phi_0}{r} \left( 1 - \frac{m_0}{rc} \right) + \phi_0 \frac{m_0}{r^3 c^4} + \gamma_\Sigma(\epsilon) \frac{\phi_0 m_0^2}{r^2 c^4}.
$$

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