The Froth of the Universe

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Abstract

We consider a model of the Universe based on the equation of state
\[ p = \frac{1}{3} \rho \left( \frac{c}{F} \right)^2, \]
where \( F \) is the scale factor. This model behaves as an inflationary Universe from the beginning and during its early stages, and behaves as dust matter during the stages of maximum expansion.

1 Introduction

Let us consider the Robertson-Walker line-element written using one of its appropriate coordinates:

\[
ds^2 = -c^2 dt^2 + F^2(t) N^2(r) \delta_{ij} dx^i dx^j, \quad N(r) = 1/(1 + kr^2/4), \quad r = \sqrt{\delta_{kl} x^k x^l}
\]

(1)

to which we shall refer as the standard coordinates of the Robertson-Walker metrics. We shall assume that the scale factor \( F \) is dimensionless, that \( t \) has dimensions of time \( T \) and that \( r \) has dimensions of space \( L \). The curvature constant \( k \) will have therefore dimensions \( L^{-2} \) and the speed of light \( c \) dimensions \( LT^{-1} \).

From both the geometrical and physical points of view the line-element

\[
d\tau^2 = dt^2 - \frac{F^2(t)}{c^2} N^2(r) \delta_{ij} dx^i dx^j,
\]

(2)
which can be written also in the following suggestive form:

\[ d\tau^2 = dt^2 - \frac{1}{c_{\text{eff}}^2} N^2(r) \delta_{ij} dx^i dx^j, \]  

where:

\[ c_{\text{eff}} = c/F(t) \]  

For any local physical process whose evolution is fast compared with the time evolution of the scale factor, and is also local in space or \( k = 0 \), the line-element \( \mathcal{L} \) is telling us that this process will evolve as if the Universe were Minkowski space-time with an effective speed of light given by \( c_{\text{eff}} \). If the time scale of the evolution of the system that is considered is not short, like for instance if the system is the Universe itself, the conclusion above does not follow from the form of the line-element \( \mathcal{L} \) but it is an heuristic idea \( ^1 \) that in our opinion deserves to be analyzed.

More precisely, let \( \rho(t) \) and \( p(t) \) be the density and the pressure of the fluid filling homogeneously and isotropically the Universe. We are going to assume that in the phase during which the Universe is dominated by radiation the equation of state is not \( p = 1/3 \rho c^2 \) as it is usually assumed, but:

\[ p = \frac{1}{3} \rho c^2 / F^2 = 1/3 \rho c_{\text{eff}}^2 \]  

Section 2 will set up the basic equations that can be derived from Einstein’s equations and the equation of state above for any interval of time where it can be relevant. In section 3 we shall justify this equation of state from the point of view of Relativistic kinetic theory by proposing an appropriate re-definition of the energy-momentum tensor of a gas of zero mass particles. In section 4 we shall set up the equivalence of the equation of state \( \mathcal{E} \) with a particular inflationary scalar field in the very early phase of the evolution of the Universe. In section 5 we shall consider the very simple model that follows from assuming this equation of state all the way during the life-span of the Universe.

\(^1\)This idea was already mentioned in \([1]\). It has recently been substantiated in different contexts. See for example \([2]\) and references therein
2 The field equations

For a Robertson-Walker line element, Einstein’s field equations with a perfect fluid energy-momentum source term are

\[ S_{\alpha\beta} - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta}, \quad T_{\alpha\beta} = \rho u_\alpha u_\beta + p \hat{g}_{\alpha\beta}, \quad \hat{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \]  

(6)

where \( u^\alpha \) is the unit time-like vector of the world-lines of the cosmological fluid. Its components in standard coordinates are:

\[ u^0 = 1, \quad u^i = 0 \]  

(7)

\( \rho \) is the mass density with dimensions \( ML^{-3} \) and \( p \) is the pressure with dimensions \( ML^{-1}T^{-2} \).

Eqs. (6) can be reduced to the following two equations, a dot meaning a derivative with respect to \( t \):

\[ \dot{\hat{F}}^2 = \frac{1}{3}(8\pi G \rho + c^2 \Lambda) \hat{F}^2 - c^2 k \]  

(8)

\[ 2 \hat{F} \ddot{\hat{F}} + \dot{\hat{F}}^2 + k c^2 = -\frac{8\pi G}{c^2} p \hat{F}^2 + c^2 \Lambda \hat{F}^2 \]  

(9)

These two equations imply the conservation equation:

\[ \dot{\rho} + 3(\rho + p/c^2) \hat{F}/\hat{F} = 0 \]  

(10)

which is also equivalent to:

\[ \frac{d}{d\hat{F}} (\rho \hat{F}^3) = -(3p/c^2)\hat{F}^2 \]  

(11)

We shall use the standard notation for the Hubble and the deceleration functions:

\[ H = \dot{\hat{F}}/\hat{F}, \quad q = -\dot{\hat{F}} \hat{F}^2/\hat{F}^2 \]  

(12)

and for a while we shall use units of length, time and mass such that:

\[ c = 8\pi G = 1, \quad H_0 = 1 \]  

(13)

where \( H_0 \) is the present value of \( H \).

\(^2\)There are many good books on cosmology. Two of them are \[3\] and \[4\].
As it was said in the introductory section we are going to discuss a model of the universe for which the equation of state \(\frac{d}{dF} \ln \rho = -(F^{-2} + 3)F^{-1}\) holds. The equation \(\frac{d}{dF} \ln \rho = -(F^{-2} + 3)F^{-1}\) can be written as:

\[
\frac{d}{dF} \ln \rho = -(F^{-2} + 3)F^{-1} \tag{14}
\]

which integrated yields the \(F\) dependence of \(\rho\):

\[
\rho = AF^{-3} \exp\left[\frac{1}{2F^2}\right] \tag{15}
\]

where \(A\) is a constant of integration. The \(t\) dependence of \(F\) can be inferred from the result of integrating Eq. \(\frac{d}{dF} \ln \rho = -(F^{-2} + 3)F^{-1}\):

\[
t(F_2) - t(F_1) = \int_{F_1}^{F_2} \frac{dF}{\left[1/3(\rho + \Lambda)F^2 - k\right]^{1/2}} \tag{16}
\]

Eq. \(\frac{d}{dF} \ln \rho = -(F^{-2} + 3)F^{-1}\) can be re-written:

\[
k = F^2(H^2 - \Lambda/3)(\Omega - 1) \tag{17}
\]

where as usual we have defined \(\Omega\) as the quotient of \(\rho\) and the critical density \(\rho_c\):

\[
\Omega = \rho/\rho_c, \quad \rho_c = 3H^2 - \Lambda \tag{18}
\]

Eq. \(\frac{d}{dF} \ln \rho = -(F^{-2} + 3)F^{-1}\) becomes:

\[-2F^2qH^2 + F^2H^2\Omega - 1/3F^2\Lambda\Omega - 2/3\Lambda F^2 + H^2\Omega - 1/3\Lambda\Omega = 0 \tag{19}\]

Solving the system of Eqs. \(18\) and \(19\) for \(\Omega\) and \(\Lambda\) we get:

\[
\Omega = \frac{2F^2\rho}{6F^2H^2(q + 1) - \rho(F^2 + 1)} \tag{20}
\]

and:

\[
\Lambda = -\frac{6F^2qH^2 - \rho(F^2 + 1)}{2F^2} \tag{21}
\]

The two equations \(17\) and \(21\) would yield directly the values of the two free parameters of the model derived from the the density and the Hubble and deceleration functions at some given time.
3 Relativistic kinetic theory

Let \( f(x^\alpha, k_\beta) \) be the distribution function of a gas of particles with mass \( m = 0 \). The energy momentum tensor of such gas has always been defined as

\[
T_{\alpha\beta}(x^\mu) = \int_{C^+_x} f(x^\mu, k_\nu) k_\alpha k_\beta \sqrt{-g} \frac{dk^1 \wedge dk^2 \wedge dk^3}{-k_0}
\]

where \( C^+_x \) is the future-poynting light-cone with vertex at the event \( x \).

A kinetic theory model of a domain of space-time filled with radiation is then governed by a system of equations which are:

- Einstein’s equations \footnote{Relativistic kinetic theory for a gas of massless particles filling the Universe at some stage has been thoroughly studied in \cite{5}. A gas of massive particles may behave also as a gas of massless particles at some stages of the evolution of the Universe; see the preceding reference and \cite{6}. See also \cite{7} for a discussion of the concept of equilibrium in a cosmological context.}

- A kinetic equation for the distribution function \( f(x^\alpha, k_\beta) \).

This equation is the Liouville equation if one assumes either i) that the radiation is so diluted that collisions can be neglected or ii) that there is equilibrium balance. This equation then reads:

\[
Df \equiv k^\alpha \partial_\alpha f - \Gamma^\alpha_{\lambda\mu} u^\lambda u^\mu \frac{\partial f}{\partial u^\alpha} = 0
\]

where \( \Gamma^\alpha_{\lambda\mu} \) are the Christoffel symbols of the line-element \footnote{In a cosmological Universe filled with black-body radiation these equations reduce to two groups of equations which using a system of standard coordinates are:}

- Einstein’s equations \footnote{\cite{8}, \cite{9} or/and \cite{10}}

where now \( \rho \) and \( p \) are given by:

\[
\rho = 4\pi \int_0^\infty f(t, \nu) \nu^3 d\nu
\]

\[
p = 4\pi \int_0^\infty f(t, \nu) \nu^2 d\nu
\]
\[ f = \frac{2h}{\exp((h\nu/k_B T(t)) - 1)} \]  

(26)

\( k_B \) being Boltzman’s constant and \( T(t) \) the local temperature function:

\[ T(t) = T_0 F(t) \]  

(27)

\( T_0 \) being the temperature at some particular time.

- The Liouville equation that in this particular situation reads:

\[ Df = \nu \partial_t f - H\nu^2 \frac{\partial f}{\partial \nu} = 0 \]  

(28)

Since \( k^\alpha \) is a null vector the trace of the energy-momentum tensor is zero, \( g^{\alpha\beta} T_{\alpha\beta} = 0 \), and this is equivalent to the equation of state \( p = \frac{1}{3} \rho \) and not to the equation of state \( \frac{4}{3} \).

Let us remind that the world-lines of the cosmological fluid in a Robertson-Walker universe are the world-lines of a conformal Killing vector. That is to say that there exists a function \( \xi(x^\alpha) \) such that the vector field \( \xi^\alpha = \xi u^\alpha \) satisfies the following equations:

\[ \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 2\Psi g_{\alpha\beta} \quad \Psi = 1/4 \nabla_\sigma \xi^\sigma \]  

(29)

Consider the new definition of the energy-momentum tensor:

\[ T_{\alpha\beta}(x^\sigma) = \int_{C^+} f(x^\mu, k_\nu) \bar{k}_\alpha \bar{k}_\beta \sqrt{-g} \frac{dk^1 \wedge dk^2 \wedge dk^3}{-k_0} \]  

(30)

where \( \bar{k}_\mu \) is

\[ \bar{k}_\mu = \xi^{-1}(k_\mu + (1 - \xi^{-1})(k^\sigma u_\sigma)u_\mu) \]  

(31)

Consider now the following metric:

\[ \bar{g}^{\alpha\beta} = g^{\alpha\beta} + (1 - \xi^{-2}) u^\alpha u^\beta \]  

(32)

We now have:

\[ \bar{g}^{\alpha\beta} T_{\alpha\beta} = 0 \]  

(33)
and taking into account the fact that in standard coordinates we have $\xi = F$ from Eq. 33 it follows that the new expressions for $\rho$:

$$\rho(t) = aT^4(t), \quad a = \frac{8\pi^5 k_B^4}{15\hbar^3}$$

(34)

and $p$ satisfy the equation of state 5.

The fact that a small modification of the standard formalism of Relativistic kinetic theory for a gas of zero mass particles allows to derive Eq. 5 is, we believe, a sufficient justification for this equation of state. Nevertheless one should be aware of an important conceptual innovation. Contrary to what happened with 22, the modified energy-momentum tensor 30 will not satisfy automatically the conservation equation 10 as a consequence of the Liouville equation 23. It will satisfy this conservation equation only as a consequence of Einstein’s equations 8 and 9 and this means: i) that the Liouville equation can not be implemented and ii) that the local temperature function $T(t)$ in Planck’s distribution function 26 has to be derived by equating the expression of the density above and that that we derived before in section 2, i.e. Eq. 15:

$$A F^{-3} \exp[1/(2F^2)] = aT^4$$

(35)

This temperature function corresponds to a distribution function that satisfies a kinetic equation that can be written in many ways; a possibility being:

$$Df(t, \nu) = 1/8H(F^2 - 1)F^{-2}\nu h^{-1} f(f + 2h) \ln(1 + 2hf^{-1})$$

(36)

Therefore while we can still speak about a black-body radiation, the equilibrium described by the distribution function 26 is now local and not global. This is a satisfactory situation for an expanding Universe.

4 Inflationary equivalent model

It is well known that it is not necessary to impose that the source of a Robertson-Walker space-time is a perfect fluid because this follows directly from Einstein’s equations and the assumptions that lead to the line-element 1. This is only necessary if one wants to specify local conditions, like for example choosing a particular equation of state. But by the same token we can always assume that the source of a Robertson-Walker universe is a scalar field for which the energy-momentum tensor is:
\[ T_{\alpha\beta} = \partial_\alpha \Phi \partial_\beta \Phi - g_{\alpha\beta} [\partial_\mu \partial^\mu \Phi - V(\Phi)] \] (37)

where \( V(\Phi) \) is a potential function depending on the model being considered.

In standard coordinates the equivalence between a perfect fluid source and a scalar field \( \Phi(t) \) depending on the time only is set up by writing:

\[
\rho = \frac{1}{2} \dot{\Phi}^2 + V(\Phi), \quad p = \frac{1}{2} \dot{\Phi}^2 - V(\Phi)
\] (38)

equations that could be considered as the parametric representation of an equation of state.

Let us assume that the model presented in the preceding sections is a good description of the Universe during an early period of time for which the scale factor \( F \) is very small. It is an elementary exercise to obtain the potential function \( V(\Phi) \) that would make this model look as a particular inflationary model. From (38) and the above equations we get:

\[
\ddot{\Phi} + 3H \dot{\Phi} + V' = 0, \quad V' = \frac{\partial V}{\partial \Phi}
\] (39)

Deriving the first Eq. (38) with respect to time and using the equation above we obtain:

\[
\dot{\Phi}^2 = -\frac{\dot{\rho}}{3H}
\] (40)

which substituted in the first Eq. (38) yields:

\[
V = \rho + \frac{\dot{\rho}}{6H}
\] (41)

Now we consider our particular solution characterized by a density function \( \rho \). Deriving this function with respect to \( t \) we obtain:

\[
\dot{\rho} = -\frac{\rho}{F^3}(3F^2 + 1)\dot{F}^2
\] (42)

and substituting into (10) and (11) we get:

\[
\dot{\Phi}^2 = \frac{\rho}{3F^2}(3F^2 + 1)
\] (43)

\[
V = \frac{\rho}{6F^2}(3F^2 - 1)
\] (44)
Let us consider the scalar field as a function $\Phi(F)$ of the scale factor. Deriving this function with respect to $t$ and using Eqs. 13 and 8 we obtain:

$$\frac{d\Phi}{dF} = \frac{1}{F} \sqrt{\frac{3F^2 + 1}{F^2 + (\Lambda F^2 - 3k)/\rho}}$$ (45)

If in the interval under consideration $F$ is small enough then the r-h-t can be approximated to $1/F^2$ and therefore we obtain:

$$\Phi(F_2) - \Phi(F_1) = \mp\left(\frac{1}{F_2} - \frac{1}{F_1}\right)$$ (46)

and therefore, up to a sign factor and an arbitrary additive constant, we get:

$$\Phi = \frac{1}{F}$$ (47)

in the largest interval for which the assumptions made are valid.

Using the expression above into 44 and keeping only the largest term we obtain the inflationary potential $V_4$ that would be equivalent to the equation of state considered in 5:

$$V(\Phi) = -\frac{A}{6}\Phi^5 \exp[(1/2)\Phi^2]$$ (48)

5 A pure radiation very simple model

Either from 5 but still better from 15 one can see that for large values of $F(t)$ a model based on the equation of state 5 will be equivalent to a dust model, i.e. a matter model with zero pressure. In fact under this condition the density behaves as:

$$\rho(F) = AF^{-3},$$ (49)

It is therefore tempting to examine the properties of a Universe that would satisfy the equation of state 5 all the way during its life-span. Such a model contains only two parameters: the curvature constant $k$ and the cosmological constant $\Lambda$. We shall determine them using the available theory presented before and the present observable values of the Hubble constant $H_0$, the deceleration parameter $q_0$ and the present density.

4Concerning the general ideas about inflation see for instance 8 or 10
The point of view that we presented in the introductory section raises the following important question: What is the constant $c$ appearing in the Robertson-Walker metric but also in Einstein’s constant, $8\pi G/c^2$, and in the energy-momentum tensor. It is called the speed of light in vacuum but we are also saying here that since measuring the speed of light is a local process the result of our measure is actually $c_{\text{eff}}$, not $c$. It turns out then that for fixed units of length and time $c$ can take any value as long as we choose the present value of $F$ to be: $F_0 = c/c_{\text{eff}}$. In particular we mankind, leaving for a very short period of the history of the Universe, can choose to avoid the use of two different quantities $c$ and $c_{\text{eff}}$ and decide to answer our question by saying that $c$ is the measured value of the speed of light in vacuum with the proviso that we take the present value of $F$ to be:

$$F_0 = 1$$

But remember this is not a choice of units condition as that that we made in \[\text{13}\]. This is a conceptual decision.

From \[\text{17}\] we have:

$$k = (1 - \Lambda/3)(\Omega_0 - 1)$$

and from \[\text{20}\] and \[\text{21}\] we get:

$$\Omega_0 = \frac{2\rho_0}{6(q_0 + 1) - 2\rho_0}$$

$$\Lambda = -3q_0 - \rho_0$$

where $q_0$ and $\rho_0$ are the present observed values.

We have now to make our mind about the present density of the Universe. To keep the model as simple as possible we offer two possibilities:

i) To imagine that the gas of zero mass particles that we have considered is in fact a faithful physical equivalent of the present mixture of radiation and massive particles as if, so to speak, the latter were still diluted in the form of radiation. From this point of view it would be safe to consider the present density, as a free parameter.

\[\text{5}\] It should be called otherwise because it is actually a property of space-time in the sense that it belongs to all causal interactions.
ii) To assume, as it has been repeatedly proposed, and as observations have always consistently suggested, that the mass distribution in the Universe is highly hierarchical. From this point of view there is no objection in considering that the density of the mass distribution could become much smaller than the density of the background black-body radiation at some scale of distances as if, so to speak, everything, from elementary particles to clusters of clusters of clusters of galaxies, were nothing more than the Froth of the Universe.

In both cases the present density and the present temperature would be given by the Stefan-Boltzman law:

\[ \rho_0 = aT_0^4 \]  

but the nicer aspect of the second choice is that for this case \( T_0 \) is a known quantity: the present temperature of the background black-body radiation. We give below a few numerical results corresponding to this case.

Using the following values: \( q_0 = .1, T_0 = 2.7 \) K, and assuming that \( H_0^{-1} \) is 13 billion years (\( H_0 = 75 \text{ km/s/Mpc} \)) we obtain using normalized units, i.e. those satisfying the conditions \[ \rho_0 = 1.3 \times 10^{-4}, \quad \Omega_0 = 3.8 \times 10^{-5} \]  

\[ \Lambda = -3, \quad k = -1.1 \]  

Reverting to the MKS system of units these values correspond to the following ones:

\[ \rho_0 = 4.5 \times 10^{-31} \text{ kg/m}^3 \]  

\[ \Lambda = -2. \times 10^{-51} \text{ m}^{-2}, \quad k = -7.3 \times 10^{-51} \text{ m}^{-2} \]

Eq. [16] with \( F_1 = 0 \) and \( F_2 = 1 \) gives the age of the Universe for this model:

\[ \text{Age} = .77(H_0^{-1}), \quad \text{or Age} = 9.83 \times 10^9 \text{ yr} \]

The maximum value \( F_{max} \) of \( F \) is given by the root of the Eq. \( \dot{F}^2 = 0 \). Using the data above we have \( F_{max} = 3.32 \) and therefore Eq. [16] with \( F_1 = 0 \) and \( F_2 = F_{max} \) gives half the life-span of the Universe, or:

\[ ^6\text{See for example [9] and references therein} \]
Life-span = 9.4, or Life-span = 12.2 \times 10^{10} \text{ yr} \quad (60)

Notice that although $F_{\text{max}}$ is not a large number it is still justified to claim that the energy density behaves as in 49 in the interval, say, $F = F_{\text{max}} \pm 1$.

6 Concluding remarks

This paper is based on three ideas. Namely:

- that the effective speed of light for any local process, in the cosmological sense, is given by 4

- that since the equation of state 5 leads to an inflationary behavior during the early stages of the Universe it might deserve consideration, for the same reasons that one has for other inflationary models,

- and that since this same equation of state is formally equivalent during the latest stages of the evolution, to assuming that the Universe is filled with dust matter, it is tempting to consider a model for which this equation of state would be valid from the beginning to the end.

We think that we have shown that these three ideas are connected. But nothing forbids to consider them separately. We could for instance keep the first one and forget the second and third. From this point of view the single interest of the idea would be to predict a variation of the effective speed of light given by:

$$\dot{c}_{\text{eff}} = -c_{\text{eff}}H_0 \quad (61)$$

but it is doubtful that a local uncorrupted test of this prediction, of cosmological origin, could be performed.

We could also keep the second idea and consider the Eq. 5 valid only during an early fraction of the history of the Universe to be connected with other models at some other stages. And so on. Any cosmological model can be improved making it more complicated. The model of section 5 is just a model incorporating the three ideas above in the simplest possible way. It has to be considered as a template ready to be modified to fit other particular points of view.
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