SIMPLIFIED EXACT SICS

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Abstract:
In the standard basis exact expressions for the components of SIC vectors (belonging to a symmetric informationally complete POVM) are typically very complicated. We show that a simple transformation to a basis adapted to the symmetries of a fiducial SIC vector can result in a massive reduction in complexity. We rely on a conjectural number theoretic connection between SICs in dimension $d_j$ and SICs in dimension $d_{j+1} = d_j(d_j - 2)$. We focus on the sequence 5, 15, 195, ... We rewrite Zauner’s exact solution for the SIC in dimension 5 to make its simplicity manifest, and use our adapted basis to convert numerical solutions in dimensions 15 and 195 to exact solutions. Comparing to the known exact solutions in dimension 15 we find that the simplification achieved is dramatic. The proof that the exact vectors are indeed SIC fiducial vectors, also in dimension 195, is a long calculation guided by the standard ray class hypothesis about the algebraic number fields generated by the SICs. We conjecture that our result generalizes to every dimension in the particular sequence we consider.
1. Introduction

A SIC is a delicately placed collection of $d^2$ unit vectors in $\mathbb{C}^d$. The definition requires that the vectors give rise to a resolution of the identity, and that all scalar products between distinct vectors have the same absolute value. Among alternative names “maximal equiangular tight frame” is more descriptive, “SIC-POVM” less so. The question whether SICs exist in all finite dimensions is of foundational interest in quantum theory, and it arises as a kind of engineering problem in classical signal processing [1]. Twenty years ago Zauner conjectured that SICs exist in every dimension, that they form orbits under the finite Weyl–Heisenberg group, and that they enjoy certain symmetries derived from the unitary automorphism group of the latter [2]. Until recently progress in the area consisted in finding numerical and exact solutions, and studying their properties [3–7]. It then emerged that the components of the SIC vectors, relative to the standard basis singled out by the group, generate algebraic number fields of considerable pure mathematical interest [8, 9]. As a result the motivation for studying SICs has shifted to pure mathematics, and to the possibility of establishing a link between physics and an area of pure mathematics that so far has not interacted with physics.

Since it forms an orbit under a group, a SIC can be presented by listing the components of a single fiducial vector. The highest dimension in which a numerical solution is known is $2208 = 48 \cdot 46$ [10]. The record for an exact solution is dimension $323 = 19 \cdot 17$, with $124$ taking second place [11]. The catch is that the actual numbers look complicated. In high dimensions they look ghastly. In this paper we will perform a simple transformation from the standard basis to a basis that is adapted to the (known or conjectured) symmetries of the SIC fiducial vectors [12]. We then consider a number theoretically connected sequence of dimensions, and show that for the first three dimensions in this sequence (namely dimensions 5, 15, and 195) there exist SIC fiducials such that all their components relative to the adapted basis are of the form

$$
(q_1 + q_2 \sqrt{3})^2 (q_3 + iq_4)^{q_5} (q_6 + iq_7 \sqrt{3})^{q_8} e^{2\pi i q_9},
$$

where $q_1, \ldots, q_9$ are rational numbers. The full expressions for the components of fiducial 195d are given in Table 2 below supplemented by eqs. (10) to (13). This is a massive reduction of complexity, because the expressions for the components relative to the standard basis occupy a text file of size 3.3 MB. We conjecture that similarly simplified solutions occur in an infinite sequence of dimensions.

The reason for seeking a simpler way of writing SICs is not simply a matter of aesthetics, or the desire to save space. It is related to the surprising link between the SIC existence problem on the one hand, and major unsolved problems in algebraic number theory on the other [9]. To see what is involved, it is interesting to compare SICs to complete sets of mutually unbiased bases, another special configuration of vectors of interest in quantum information theory [13]. At first glance the standard expressions for the vectors in the latter are much simpler than the expressions for a SIC in the same dimension. However, as observed in an appendix of ref. [14], that is only because we are not comparing like with like. The SICs are expressed in terms of radicals. The mutually unbiased bases are expressed in terms of roots of unity, which in their turn are expressed using the exponential function. If one writes the roots of unity in terms of radicals then the expressions can become equally complicated. This suggests that the problem is to find functions which play the same role for the SIC numbers that the exponential function does for the roots of unity. If one could do that, and if one could find appropriate identities satisfied by these functions, then one might be able to solve the SIC existence problem and, incidentally, one may have made a significant step in the direction of solving Hilbert’s 12th problem. Recently Kopp has made
a specific proposal along these lines: namely that in prime dimensions equal to \(2\) modulo \(3\) ray class \(L\)-functions play such a role \cite{15}. The result we prove establishes a different kind of simplification, using Hilbert space geometry rather than number theory as a guide. We hope it will prove useful too.

The first link between SICs and algebraic number theory was the observation \cite{8} that (ratios between) the components of the SIC vectors in dimension \(d \geq 4\), when expressed in the basis singled out by the Weyl–Heisenberg group, lie in number fields that contain the real quadratic field \(\mathbb{Q}(\sqrt{D})\), where

\[
D = \text{the square free part of } (d + 1)(d - 3) .
\]  

Given an integer \(D \geq 2\) there is an infinite number of integers \(d\) leading to this value of \(D\). What is more, given \(D\) and \(d\) there exists a unique ray class field with base field \(\mathbb{Q}(\sqrt{D})\) and conductor \(d\). For a long time algebraic number theorists have been concerned with trying to obtain a better understanding of such ray class fields. Remarkably, in every dimension \(d\) where exact SICs have been found, the components of one of them generates the ray class field with conductor \(d\), or \(2d\) if \(d\) is even \cite{9}. The Ray Class Hypothesis states that this happens in all dimensions, so that the number fields containing these SICs is conjecturally known. To avoid any misunderstanding, the ray class fields typically have much higher degrees than their base fields. If there are several inequivalent SICs, the others lie in fields containing the ray class field \cite{10}.

We can now see how the dimension ladders arise. Make the substitution

\[
d \rightarrow d(d - 2) \quad \Rightarrow \quad (d + 1)(d - 3) \rightarrow (d - 1)^2(d + 1)(d - 3) .
\]  

The square free part \(D\), and hence the base field of the SIC, is unchanged. What is more, a ray class field is contained in another if its conductor divides that of the other. This gives rise to infinite sequences of dimensions \(d_j\), defined by the recursion relation \(d_{j+1} = d_j(d_j - 2)\), for which the ray class hypothesis implies number theoretic connections between SICs in different dimensions. These connections are reflected by geometrical properties of the SICs \cite{11, 14}, and raise the hope that closed form expressions for SICs can be found in infinite sequences of dimensions. However, here we climb only two rungs up the ladder that starts with \(d = 5\) and \(D = 3\).

A curious feature of our simplified solutions is that the components \cite{11} of the SIC fiducial vectors relative to the normalized adapted basis typically lie in quadratic extensions of the minimal number field. On closer inspection one finds that this feature can be regarded as an artefact of the normalization of the basis vectors, so that the ray class hypothesis is vindicated.

We have organized our paper as follows. The solutions are given in Section \(\text{2}\). Section \(\text{3}\) is a quick review of known facts about the relevant groups, and the adapted bases are computed in Section \(\text{4}\). Section \(\text{5}\) explains how, in favourable cases, the adapted bases can be used to convert numerical fiducial vectors to exact ones using only modest precision. In Section \(\text{6}\) we explain the lengthy calculations we use to prove that the candidate solutions from Section \(\text{2}\) are indeed SIC fiducials. These calculations depend heavily on the ray class hypothesis. In the concluding Section \(\text{7}\) we set our solutions in context.
2. Solutions

We present the matter back to front, that is we first give solutions for SIC fiducials, and afterwards define the bases in which they are expressed. The solution in dimension 5 is taken from Zauner’s thesis [2]. It is

\[ |\Psi_0\rangle = \sqrt{p_0}|e_0^{(5)}\rangle + \sqrt{p_1}(P_5)^{\frac{1}{4}}|e_1^{(5)}\rangle, \]

where

\[ p_0 = \frac{3 - \sqrt{3}}{4}, \quad p_1 = \frac{1 + \sqrt{3}}{4}, \]

\[ P_5 = -\frac{3}{5} + \frac{4i}{5}. \]

The vectors \(|e_0^{(5)}\rangle\) and \(|e_1^{(5)}\rangle\) are eigenvectors of a unitary operator connected to the SIC. They span what is known as the Zauner subspace, and will be easy to describe in words once we have presented the background (in Section 3). The form of the components of the vector is interesting. It is remarkable that the squares of their absolute values (\(p_0\) and \(p_1\)) lie in the base field \(Q(\sqrt{3})\). Moreover the phase factor is the fourth root of a phase factor formed from a Pythagorean triple of integers. That is, \(P_5\) is a phase factor because

\[ 3^2 + 4^2 = 5^2. \]

This relation was remarked upon in ancient Egypt.

Group theoretically preferred adapted bases for the Zauner subspaces exist also in the higher dimensions considered next. They will be explicitly constructed in Section 4. For now, suffice it to say that they can be calculated by hand in all the cases we consider. Given these adapted bases we expand the SIC fiducial as

\[ |\Psi_0\rangle = \sum_{r=0}^{d_s-1} \sqrt{p_r}e^{i\nu_r}|e_r^{(d)}\rangle, \]

where \(d\) is the dimension and \(d_s\) is the dimension of the subspace in which the fiducials live. In the Tables we give the exact solutions for the squares \(p_r\) of the absolute values and for the phases \(e^{i\nu_r}\). The normalization of the vector is ensured by

\[ \sum_r p_r = 1. \]

An overall phase factor is fixed by the requirement that \(e^{i\nu_0} = 1\).

In dimension 15 = 5 · 3 there exist four unitarily and anti-unitarily inequivalent SICs. The ones we are interested in are labelled 15d and 15b [5]. 15d is the ray class SIC. It has higher symmetry and sits in a four dimensional subspace of the six dimensional Zauner subspace. In dimension 195 four SICs are available (see our Acknowledgements) in numerical form. The ones we are interested in are 195d, which is aligned to 15d in a sense that can be made precise, and 195b which is similarly aligned to 15b [14]. They both live in a 36 dimensional...
subspace of the Zauner subspace. 195d has higher symmetry and sits in a 19 dimensional subspace.

As we move up the dimension ladder new Pythagorean triple phase factors enter, including some of a new kind constructed using also $\sqrt{3}$ in the numerator. It remains true that all the phase factors occurring in the dimensions 15 and 195 are triple phase factors multiplied with roots of unity. Since the triples enjoy a group structure, so that those with composite numbers in their denominators can be expressed as products of ones with prime denominators, we have expressed everything in terms of

$$P_5 = -\frac{3}{5} + \frac{4i}{5} \quad Q_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$P_{13} = -\frac{5}{13} + \frac{12i}{13} \quad Q_{13} = -\frac{11}{13} + \frac{4i\sqrt{3}}{13}$$

$$P_{37} = -\frac{12}{37} + \frac{35i}{37} \quad Q_{37} = -\frac{13}{37} + \frac{20i\sqrt{3}}{37}$$

$$P_{241} = -\frac{120}{241} + \frac{209i}{241} \quad Q_{241} = -\frac{143}{241} + \frac{112i\sqrt{3}}{241}$$

The primes that occur in the denominators are necessarily equal to 1 mod 4.

The absolute values squared and the phases needed in eq. (8) are given in Table 1 for 15db and Table 2 for 195d. The candidate solution for 195b is placed in an Appendix as Table 4, because for 195b we have not carried out the entire proof that the candidate solution is indeed an exact solution. We observe that a Pythagorean triple with 37 in the denominator occurs as a phase factor only in fiducial 195d, while a triple with 241 in the denominator occurs only in 195b. The simplicity of the components when expanded in our adapted basis should be judged by means of a comparison to how fiducial 15d looks when expanded in the standard basis [5].

Table 1. Moduli squared and relative phases in two $d = 15$ SICs.

| 15d (Moduli)$^2$ | Phases | 15b (Moduli)$^2$ | Phases |
|------------------|--------|------------------|--------|
| $p_0 = \frac{\sqrt{3}}{3}$  | $e^{i\nu_0} = 1$ | $p_0 = \frac{\sqrt{3}}{3}$  | $e^{i\nu_0} = 1$ |
| $p_1 = \frac{2-\sqrt{3}}{4}$  | $e^{i\nu_1} = \omega_4^3 (P_5)^\frac{1}{4}$ | $p_1 = \frac{4-\sqrt{3}}{8}$  | $e^{i\nu_1} = \omega_4^3 \left( -\frac{Q_{13}P_{13}}{P_5} \right)\frac{1}{4}$ |
| $p_2 = \frac{\sqrt{3}}{7}$  | $e^{i\nu_2} = \omega_3^3 (iP_5)\frac{1}{3}$ | $p_2 = \frac{2\sqrt{3} - 2}{8}$  | $e^{i\nu_2} = \omega_3^3 \left( -\frac{Q_{13}^{P_{13}}}{\sqrt{5}} \right)\frac{1}{4}$ |
| $p_3 = \frac{4-\sqrt{3}}{8}$  | $e^{i\nu_3} = \omega_4^{12} \left( -\frac{P_{13}}{Q_{13}^{P_{13}}} \right)\frac{1}{17}$ | $p_3 = \frac{1}{5}$  | $e^{i\nu_3} = \omega_4^{12} \left( -\frac{1}{P_5} \right)\frac{1}{17}$ |
| $p_4 = \frac{6-3\sqrt{3}}{8}$  | $e^{i\nu_4} = -1$ | $p_4 = \frac{6-3\sqrt{3}}{8}$  | $e^{i\nu_4} = -1$ |
| $p_5 = \frac{\sqrt{3}}{8}$  | $e^{i\nu_5} = \left( \frac{1}{P_5} \right)\frac{1}{8}$ | $p_5 = \frac{\sqrt{3}}{8}$  | $e^{i\nu_5} = \left( \frac{1}{P_5} \right)\frac{1}{8}$ |
We must now introduce the Weyl–Heisenberg group and the Clifford group. The latter is the unitary automorphism group of the former. Our account is intended to fix notation and to highlight some special features that we will make use of. All the dimensions we will encounter are odd prime numbers or products of odd prime numbers with multiplicity one, and here we consider only such choices of the dimension \( d \). As it happens this leads to considerable simplifications, because arithmetic modulo \( d \) plays a large role, and in prime dimensions the set of integers modulo \( d \) is a field, not just a ring. We will be able to deal with square-free odd dimensions by using the direct product structure of the relevant groups. For more complete accounts we refer elsewhere \([4, 5, 12]\).

Keeping these limitations in mind, let us start. The Weyl–Heisenberg group acting on \( \mathbb{C}^d \) is generated by the unitary clock and shift operators \( Z \) and \( X \). In the basis where \( Z \) is diagonal they are represented by

| \((\text{Moduli})^2\) | Phases |
|-----------------|--------|
| \(p_0 = \frac{12\sqrt{3} - 9}{182}\) | \(e^{i\nu_0} = 1\) |
| \(p_1 = \frac{19 - 8\sqrt{3}}{182}\) | \(e^{i\nu_1} = \omega_4 ( -Q_{13}P_5P_{13})^{\frac{1}{12}}\) |
| \(p_2 = \frac{12\sqrt{3} - 9}{364}\) | \(e^{i\nu_2} = \omega_3^2 (P_5)^{\frac{1}{12}}\) |
| \(p_3 = \frac{41 - 20\sqrt{3}}{364}\) | \(e^{i\nu_3} = \omega_6^2 (iP_3P_{37}Q_{37})^{\frac{1}{12}}\) |
| \(p_4 = \frac{55\sqrt{3} - 90}{546}\) | \(e^{i\nu_4} = \omega_6^3 ( -\frac{1}{P_5})^{\frac{1}{12}}\) |
| \(p_5 = \frac{20 - 5\sqrt{3}}{182}\) | \(e^{i\nu_5} = \omega_6^4 ( -P_5^{\frac{1}{12}})\) |
| \(p_6 = \frac{4\sqrt{3} - 3}{27}\) | \(e^{i\nu_6} = \omega_6^2 (P_5)^{\frac{1}{12}}\) |
| \(p_7 = \frac{19 - 8\sqrt{3}}{91}\) | \(e^{i\nu_7} = \omega_6^2 ( -\frac{P_5}{Q_{13}P_{13}})^{\frac{1}{12}}\) |
| \(p_8 = \frac{5 - 2\sqrt{3}}{14}\) | \(e^{i\nu_8} = \omega_6^2 ( -\frac{P_5Q_2Q_{13}}{P_{13}})^{\frac{1}{12}}\) |
| \(p_9 = \frac{2\sqrt{3} - 3}{14}\) | \(e^{i\nu_9} = \omega_6^2 (P_5Q_2Q_{13})^{\frac{1}{12}}\) |
| \(p_{10} = \frac{2\sqrt{3} - 3}{21}\) | \(e^{i\nu_{10}} = \omega_6^2 (P_5Q_2Q_{13})^{\frac{1}{12}}\) |
| \(p_{11} = \frac{\sqrt{3}}{21}\) | \(e^{i\nu_{11}} = \omega_6^2 (\frac{P_5}{Q_2Q_{13}})^{\frac{1}{12}}\) |
| \(p_{12} = \frac{2 - \sqrt{3}}{7}\) | \(e^{i\nu_{12}} = \omega_6^2 (\frac{P_5}{Q_2Q_{13}})^{\frac{1}{12}}\) |
| \(p_{13} = \frac{2 - \sqrt{3}}{14}\) | \(e^{i\nu_{13}} = ( -1)^{\frac{1}{12}}\) |
| \(p_{14} = \frac{\sqrt{3}}{14}\) | \(e^{i\nu_{14}} = \omega_6^2 ( -P_5^{\frac{1}{12}})\) |
| \(p_{15} = \frac{4\sqrt{3} - 3}{42}\) | \(e^{i\nu_{15}} = \omega_6^2 ( -P_5^{\frac{1}{12}})\) |
| \(p_{16} = \frac{7 - 4\sqrt{3}}{14}\) | \(e^{i\nu_{16}} = \omega_6^2 ( -\frac{P_5Q_2Q_{13}}{P_{13}})^{\frac{1}{12}}\) |
| \(p_{17} = \frac{4\sqrt{3} - 6}{21}\) | \(e^{i\nu_{17}} = \omega_6^2 ( -\frac{P_5Q_2Q_{13}}{P_{13}})^{\frac{1}{12}}\) |
| \(p_{18} = \frac{1}{12}\) | \(e^{i\nu_{18}} = \omega_6^2 ( -\frac{P_5Q_2Q_{13}}{P_{13}})^{\frac{1}{12}}\) |

3. Group theoretic background

Table 2. Moduli squared and relative phases in 195d.
This effectively defines the standard basis. Its basis vectors are labelled by integers modulo $d$. For any $d$ we define the primitive $d$th root of unity

$$\omega_d = e^{\frac{2\pi i}{d}}.$$ \hfill (15)

The group generators $X$ and $Z$ are of order $d$. The group itself is of order $d^3$, but up to phase factors there are only $d^2$ group elements. They are conveniently represented by the $d^2$ displacement operators

$$D_p = D_{i,j} = \tau^{ij}X^iZ^j,$$ \hfill \tau = -e^{\frac{2\pi i}{d}}, \quad \tau^2 = \omega_d.$$ \hfill (16)

Here $i, j$ are integers modulo $d$, and we think of $p$ as a two component vector with entries $i$ and $j$. An orbit of the group is obtained by choosing a fiducial unit vector $|\Psi_0\rangle$ and acting on it with all the displacement operators. The orbit is a SIC if and only if

$$|\langle \Psi_0|D_p|\Psi_0\rangle|^2 = \frac{1}{d+1}.$$ \hfill (17)

for all $p \neq 0$. This is a highly non-trivial set of real quartic polynomial equations for the components of the fiducial vector. The complex quantities $\langle \Psi_0|D_p|\Psi_0\rangle$ are known as SICs overlaps (or as radar ambiguity functions, for those who like to keep the engineering background in mind).

Zauner [2] and Grassl [17] drew attention to the group containing all unitary operators that take the Weyl–Heisenberg group into itself under conjugation. This includes the symplectic group defined by unimodular two-by-two matrices with entries that are integers modulo $d$. With the representation of the Weyl–Heisenberg group fixed as above, the unitary representation of the symplectic group for odd prime $d$ is

$$U_F = \sum_s \omega^{2^{-1}\beta^{-1}(\delta^2 - 2rs + \alpha s^2)}|s\rangle \langle s| \quad \beta \neq 0$$

$$U_F = \sum_s \omega^{2^{-1}\alpha s^2}|s\rangle \langle s| \quad \beta = 0.$$ \hfill (18)

Here $(\alpha|d) = \pm 1$ is the Legendre symbol; recall that we have assumed that 2 has the multiplicative inverse $2^{-1}$ modulo $d$. One finds that

$$U_F D_p U_F^{-1} = D_{Fp}.$$ \hfill (19)

A symplectic matrix is represented by a monomial unitary matrix if and only if $\beta = 0$, as seen in eq. (18). This happens because such matrices transforms the maximal abelian subgroup generated by the diagonal operator $Z = D_{0,1}$ into itself. To obtain a faithful representation the phase factors in eq. (18) are chosen to be

$$e^{i\theta} = -\frac{1}{\sqrt{2}}(-\beta|d) = \begin{cases} (-1)^k(-\beta|d) & \text{if } d = 4k + 1 \\ (-1)^{k+1}i(-\beta|d) & \text{if } d = 4k + 3. \end{cases}$$ \hfill (20)
The faithful representation is also known as the metaplectic representation. The choice of phase factors ensures—via a Gauss sum—that the representation is in terms of numbers belonging to the cyclotomic field \( \mathbb{Q}(\omega_d) \). Thus, taking dimension \( d = 5 \) as an example, it holds that

\[ \sqrt{5} = \omega_5 + \omega_5^4 - \omega_5^2 - \omega_5^3. \]  

(21)

It follows that \( \sqrt{5} \) belongs to the cyclotomic number field \( \mathbb{Q}(\omega_5) \).

Let us now consider composite dimensions where the factors are relatively prime. Then the Weyl–Heisenberg group in the composite dimension splits as a direct product of Weyl–Heisenberg groups in the factors. This is important to us because we are interested in dimensions of the form \( d(d-2) \) for some choice of \( d \). If \( d \) is odd then \( d \) and \( d-2 \) are relatively prime and the group structure dictates that we write

\[ \mathbb{C}^{d(d-2)} = \mathbb{C}^{d-2} \otimes \mathbb{C}^d. \]  

(22)

The symplectic group also splits as a direct product. Thus, using a subscript to imply the modulus of the arithmetic,

\[ \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)_{d(d-2)} \sim \left( \begin{array}{cc} \alpha & \kappa^{-1} \beta \\ \kappa \gamma & \delta \end{array} \right)_{d-2} \times \left( \begin{array}{cc} \alpha & \kappa^{-1} \beta \\ \kappa \gamma & \delta \end{array} \right)_d, \]  

(23)

where

\[ \kappa = \frac{d-1}{2}. \]  

(24)

(This is an integer because \( d \) is odd.) The details of this isomorphism consist in a judicious application of the Chinese remainder theorem. In this context they were first worked out by David Gross, and are described elsewhere [14].

There do exist anti-unitary operators obeying eq. (19). They represent two-by-two matrices of determinant \(-1\) modulo \( d \), and fill out what is called the extended Clifford group. Unitarily or anti-unitarily equivalent SICs belong to a single orbit of this group.

It is a remarkable fact that every known Weyl–Heisenberg SIC contains an eigenvector of a symplectic unitary of order three [5,7]. However, in composite dimensions it is not always true that every symplectic unitary of order three gives rise to SICs. Using the characteristic equation for two-by-two matrices one can show that a symplectic matrix \( F \) whose trace equals \(-1\) must obey \( F^3 = 1 \). When the symplectic group is defined over a field the converse holds, that is if \( F^3 = 1 \) then either \( \text{Tr} F = -1 \) or else \( F \) is the unit matrix. Hence symplectic matrices of order three are easy to recognize in the cases we consider. If \( d > 3 \) is an odd prime there is a unique conjugacy class of order three matrices, consisting of all matrices with trace \(-1\), and it contains the representative

\[ F_Z = \left( \begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array} \right). \]  

(25)

Matrices in this conjugacy class are known as Zauner matrices, and the eigenspaces of the corresponding unitaries as Zauner subspaces. The case \( d = 3 \) is special because \(-1 = 2 \) modulo 3, and when the trace equals 2 there are always three distinct conjugacy classes. As representatives we may choose \( 1 \), \( F_Z \), and \( F_Z^2 \), or more conveniently
The point is that, according to eq. (18), these matrices are represented by monomial unitary matrices. The ambiguity in dimension three propagates through the direct product when the Chinese remainder theorem is applied, so that when \( d = 3 \cdot (3k + 1) \) or \( d = 3 \cdot (3k + 2) \) there exist more than one conjugacy class of order three symplectic matrices with trace \(-1\). Moreover, when the dimension is divisible by 3 there exist order three Clifford unitaries not conjugate to any symplectic unitary. But it seems that SICs invariant under order three Clifford unitaries not conjugate to \( U_{FZ} \) or its square exist only if \( d = 3 \cdot (3k + 1) \). In this paper the only composite dimensions we consider are of the form \( d = 3 \cdot (3k + 2) \). The available evidence suggests that every SIC in these dimensions is equivalent to one invariant under the unitary corresponding to \( FZ \) [5].

Sometimes a symmetry of order larger than three is observed among the SICs. Then the unitary symmetry group is a subgroup of the centralizer of an order three element. For Zauner matrices belonging to the same conjugacy class as the one given in eq. (25) the centralizer is abelian. Again suppose \( d \) is an odd prime. Then the centralizer within the symplectic group has order \( d + 1 \) if \( d = 2 \) mod 3 and order \( d - 1 \) if \( d = 1 \) mod 3 [12]. Thus it is large enough, or almost large enough, to define a preferred basis in \( \mathbb{C}^d \).

Published lists of exact and numerical solutions include only one SIC from each orbit under the extended Clifford group, and by convention the matrix \( FZ \) is chosen as a representative of its conjugacy class [5,7]. However, when \( d = 1 \) mod 3 it is possible to choose the representative of the conjugacy class so that the entire centralizer is given by diagonal two-by-two matrices, hence by monomial unitary matrices. This leads to considerable simplifications that we will make use of.

Readers familiar with the representation theory of \( SL(2, \mathbb{R}) \) will recognize several ingredients of our discussion, such as the occurrence of parabolic conjugacy classes when the trace \( t = 2 \), and diagonalizable matrices when the trace \( t \) obeys \( t^2 - 4 > 0 \). In the present case the inequality translates to the requirement that \( t^2 - 4 \) be a quadratic residue. The story would have been more complicated had we not restricted ourselves to dimensions that are products of odd prime numbers of multiplicity one. We refer to the literature for this [4,5]. For an in-depth study of order three elements of the Clifford group in the general case, see Bos and Waldron [19].

4. The adapted basis

In Section 2 we left the adapted basis undefined. This must now be remedied. It is an easy task because we view the higher dimensional Hilbert spaces as tensor products,

\[
\mathbb{C}^{15} = \mathbb{C}^3 \otimes \mathbb{C}^5, \quad \mathbb{C}^{195} = \mathbb{C}^{13} \otimes \mathbb{C}^{15} = \mathbb{C}^{13} \otimes \mathbb{C}^3 \otimes \mathbb{C}^5,
\]

and so on as we walk up the dimension ladder. The bases consist of product vectors, so it is enough to consider the factor spaces separately. In dimensions 13 (a prime equal to 1 mod 3) and 3 the conjugacy class of Zauner matrices has a member represented by a monomial unitary matrix, so in these two dimensions the calculations are trivial. The calculations needed in dimension 5 are somewhat lengthy but still straightforward.

Following Zauner, what we do to find a SIC in dimension 5 is to choose any order three symplectic matrix. Then we calculate the eigenvectors of the corresponding unitary. Thus
\[ F_Z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}_5 \implies U_Z = \frac{-1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \omega_5^3 & \omega_3^4 & 1 & \omega_5 \omega_3^2 \\ \omega_5^2 & \omega_3^3 & \omega_5^4 & 1 \\ \omega_3^2 & 1 & \omega_5^3 & \omega_5^4 \\ \omega_3 & \omega_5^3 & \omega_5 & \omega_5^2 \end{pmatrix}. \] (28)

The Zauner matrix \( F_Z \) clearly commutes with

\[ F_P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \] (29)

The latter is represented by a monomial unitary of order two, known as the parity operator.

The order six operator \( U_{ZP} \) represents \( F_Z F_P \), a generator of the centralizer of \( F_P \), and it has a non-degenerate spectrum. Hence it defines a preferred basis in \( \mathbb{C}^5 \), adapted to the SIC because—according to a refined form of Zauner’s conjecture—there must be a SIC fiducial that is an eigenvector of \( U_Z \).

After a routine calculation one finds eigenvectors that, when expressed as vectors relative to the standard basis, are

\[ |f_0^{(5)}\rangle = \begin{pmatrix} 4\omega_5 - 2\omega_3^2 + 2\omega_3(\omega_5^3 - \omega_3^3) \\ \omega_5^3 + \omega_3^4 + \omega_3(1 + 2\omega_5^2 + 2\omega_3^2) \\ 4\omega_5^2 + 3\omega_3^2 + 3\omega_3(\omega_3^3 - 1) \\ \omega_3^3 + \omega_3^4 + \omega_3(1 + 2\omega_3^2 + 2\omega_5^2) \end{pmatrix} \]

\[ |f_1^{(5)}\rangle = \begin{pmatrix} \langle 5 \rangle \\ \omega_5 - \omega_3^3 + \sqrt{5}\omega_3 \\ 1 - \omega_5^3 \\ -1 + \omega_5^3 \end{pmatrix} \]

\[ |f_2^{(5)}\rangle = \begin{pmatrix} 4\omega_5 - 2\omega_3^4 + 2\omega_3(\omega_5^3 - \omega_3^3) \\ 4 + 3\omega_3^2 + 3\omega_3(\omega_3^3 - \omega_5^3) \\ 1 + \omega_5 + \omega_5^2(2\omega_3^2 + 2\omega_3^2 + \omega_5^2) \\ 1 + \omega_5 - \omega_3(2\omega_3^2 + 2\omega_3^2 + \omega_5^2) \\ 4 + 3\omega_3^2 + 3\omega_3(\omega_3^3 - \omega_5^3) \end{pmatrix} \]

\[ |f_3^{(5)}\rangle = \begin{pmatrix} \omega_5^3 - \omega_5^2 \\ \omega_5^3 - \omega_3^3 + \sqrt{5}\omega_3^2 \omega_5^2 \\ \omega_3^3 - \omega_5^3 - \sqrt{5}\omega_3^2 \omega_5^2 \\ \omega_3^3 - \omega_3^2 \end{pmatrix} \]

\[ |f_4^{(5)}\rangle = \frac{\omega_3^2 - \omega_3}{15} \begin{pmatrix} 4 + 3\omega_5 + 2\omega_3^2 + 2\omega_3^3 \\ -1 - 2\omega_5 - 3\omega_3^2 - \omega_3^3 \\ 4 + 3\omega_5 + 2\omega_3^2 + \omega_3^3 \\ -1 - 2\omega_5 - 3\omega_3^2 + \omega_3^3 \end{pmatrix}. \] (32)

The vector \( |f_4^{(5)}\rangle \) is a unit vector, but the others are not. Actually we constructed the second pair from the first by applying the anti-unitary transformation representing the matrix \( F = 21 \), so the only independent normalization factors are

\[ n_0 = \langle f_0|f_0\rangle = \langle f_2|f_2\rangle \neq 1, \quad n_1 = \langle f_1|f_1\rangle = \langle f_3|f_3\rangle \neq 1. \] (33)

The cyclotomic numbers \( n_0 \) and \( n_1 \) are easily calculated. To normalize the vectors we need their square roots \( \sqrt{n_0} \) and \( \sqrt{n_1} \). These numbers do not belong to the cyclotomic field \( \mathbb{Q}(\omega_5, \omega_3) \). Nevertheless we introduce the normalized basis
Labelling the normalized eigenvectors with the eigenvalues of $U_Z$ and $U_P$ will be useful later. We refer to the intermediate basis \{|$f_r^{(5)}$\}\textsubscript{r=0} as the cyclotomic basis because its components lie in the cyclotomic field $\mathbb{Q}(\omega_5, \omega_3) = \mathbb{Q}(\omega_{15})$.

The basis vectors may look complicated, but they are the result of a routine calculation and are easily described in words. Following Zauner \cite{Zauner} we have chosen their overall phase factors by first ensuring that their first non-vanishing components are either real or purely imaginary, and afterwards we have multiplied them with suitable powers of $\omega_5$ with a view to simplify the calculations in Section 5.

The logic behind the adapted bases in dimensions 15 and 195 is similar, but because of eq. (27) the lengthy calculations are already behind us. The point is that the dimensions of the Hilbert spaces we will be factoring in, $\mathbb{C}^3$ and $\mathbb{C}^{13}$, admit monomial Zauner unitaries. We are free to choose the representative of the conjugacy class of the symplectic symmetries in the higher dimension to exploit this fact.

In dimension 15 we therefore use the Zauner matrix

$$F_{Z_{15}} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}_3 \times \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}_5 \sim \begin{pmatrix} -5 & 3 \\ -2 & 4 \end{pmatrix}_{15}. \tag{35}$$

This choice ensures that the matrix in the dimension 3 factor is represented by a monomial unitary matrix. According to by now standard conventions, Scott and Grassl \cite{ScottGrassl} use the Zauner matrix (25) in their tables of solutions. The transformation between the two representatives is

$$\begin{pmatrix} -7 & -1 \\ -7 & 1 \end{pmatrix}_{15} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}_{15} \begin{pmatrix} 1 & 1 \\ 7 & -7 \end{pmatrix}_{15} = \begin{pmatrix} -5 & 3 \\ -2 & 4 \end{pmatrix}_{15}. \tag{36}$$

We will transform numerical fiducials from the Scott and Grassl form to the form we prefer. We use the metaplectic representation of the symplectic group in the prime dimensional factors, and find that the SIC fiducials lie in the six dimensional Zauner subspace corresponding to eigenvalue $\omega_3^2$.

We use the Zauner and parity unitaries to define an adapted basis in the $\mathbb{C}^3$ factor. Again we will have two bases, one whose elements are in the cyclotomic field $\mathbb{Q}(\omega_3)$ and one which is made from unit vectors only. The cyclotomic basis is

$$|f_0^{(3)}\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad |f_1^{(3)}\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad |f_2^{(3)}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{37}$$

Accidentally, in this case the cyclotomic entries of the vectors are rational integers. The normalized basis is

$$|e_0^{(5)}\rangle = \frac{1}{\sqrt{n_0}}|f_0^{(5)}\rangle = |\omega_3, +\rangle_5, \quad |e_1^{(5)}\rangle = \frac{1}{\sqrt{n_1}}|f_1^{(5)}\rangle = |\omega_3, -\rangle_5, \tag{34}$$

$$|e_2^{(5)}\rangle = \frac{1}{\sqrt{n_0}}|f_2^{(5)}\rangle = |\omega_2^2, +\rangle_5, \quad |e_3^{(5)}\rangle = \frac{1}{\sqrt{n_1}}|f_3^{(5)}\rangle = |\omega_2^2, -\rangle_5,$$

$$|e_4^{(5)}\rangle = |f_4^{(5)}\rangle = |1, +\rangle_5.$$
\begin{equation}
|e_0^{(3)}\rangle = \frac{1}{\sqrt{2}}|f_0^{(3)}\rangle = |\omega_3, -\rangle_3, \quad |e_1^{(3)}\rangle = \frac{1}{\sqrt{2}}|f_1^{(3)}\rangle = |\omega_3, +\rangle_3,
\end{equation}
\begin{equation}
|e_2^{(3)}\rangle = |f_2^{(3)}\rangle = |1, -\rangle_3.
\end{equation}

For the case of \(\mathbb{C}^{15}\) the adapted normalized basis in the Zauner subspace (eigenvalue \(\omega_3^2\)) is the product basis

\begin{align*}
|e_0^{(15)}\rangle &= |\omega_3, -\rangle_3 \otimes |\omega_3, +\rangle_5 = |e_0^{(3)}\rangle |e_0^{(5)}\rangle, \\
|e_1^{(15)}\rangle &= |\omega_3, -\rangle_3 \otimes |\omega_3, -\rangle_5 = |e_0^{(3)}\rangle |e_1^{(5)}\rangle, \\
|e_2^{(15)}\rangle &= |1, -\rangle_3 \otimes |\omega_3^2, +\rangle_5 = |e_2^{(3)}\rangle |e_2^{(5)}\rangle, \\
|e_3^{(15)}\rangle &= |1, -\rangle_3 \otimes |\omega_3^2, -\rangle_5 = |e_2^{(3)}\rangle |e_3^{(5)}\rangle \tag{39}
\end{align*}

The labelling of the basis vectors may seem odd. We have divided them into two groups because fiducial 15d has an additional symmetry, implying that it lies in a four dimensional subspace of the Zauner subspace. The first group of basis vectors spans that subspace within the subspace. Within the two groups of vectors lexicographical ordering is being used, in a way that is manifested after the second equality sign.

We now factor in \(\mathbb{C}^{13}\) in order to reach dimension 195. Fiducial 195d has a symmetry group of order 12 [14]. We choose the representative of the conjugacy class to ensure that it is represented by monomial unitary matrices in two of the factors, and obtain two commuting generators of the symmetry group in the form

\begin{align*}
F_Z &= \begin{pmatrix} 55 & 156 \\ 169 & 139 \end{pmatrix}_{195} \sim \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}_{13} \times \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}_3 \times \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}_5 \tag{40} \\
F_S &= \begin{pmatrix} 161 & 0 \\ 0 & 86 \end{pmatrix}_{195} \sim \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}_{13} \times \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_3 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_5 \tag{41}
\end{align*}

The first matrix here is a Zauner matrix, while the order four matrix \(F_S\) is represented by a unitary operator \(U_S\). Fiducial 195d sits in an eigenspace of \(U_S\) having eigenvalue 1. Fiducial 195b has a lower symmetry, and is invariant under \(U_S^2\) only.

An adapted basis in the \(\mathbb{C}^{13}\) factor is easily constructed. The centralizer of its Zauner matrix is of order 12, and is represented by monomial matrices. From eq. [13] we see that, in the unitary representation we use, the vector \(|0\rangle\) is special so that there is a natural split

\begin{equation}
\mathbb{C}^{13} = \mathbb{C}^1 \oplus \mathbb{C}^{12} \tag{42}
\end{equation}

Thus the centralizer is generated by a unitary matrix of the form

\begin{equation}
U_{13} = -\begin{pmatrix} 1 & 0 \\ 0 & \Pi \end{pmatrix}, \tag{43}
\end{equation}
where $\Pi$ is a 12 by 12 permutation matrix with a non-degenerate spectrum consisting of 12th roots of unity. The eigenvectors are easily written down, and normalized, using only numbers from the cyclotomic field $\mathbb{Q}(\omega_{12})$. For definiteness we choose their phases so that their first non-zero entry is real. In this factor the normalized and cyclotomic bases coincide,

$$|e_r^{(13)}\rangle = |f_r^{(13)}\rangle .$$

Incidentally this feature would continue to hold if we were to take another step up the dimensional ladder, reaching dimension $\mathbb{C}^{193} \otimes \mathbb{C}^{195}$.

The $d = 195$ SIC fiducials we are interested in are invariant under

$$U_S^2 = U_P \otimes \textbf{1}_{15} ,$$

where $U_P$ is the parity operator in dimension 13 [7,14]. This means that only sixth roots of unity occur as eigenvalues in the relevant subspace of $\mathbb{C}^{13}$. It is convenient to label the eigenvectors with the eigenvalues of the Zauner unitary and of the order four operator $U_S$. The eigenvector in the $C^1$ summand in eq. (42) is

$$|e_0^{(13)}\rangle = |1, -, a\rangle .$$

To span the relevant subspace of $\mathbb{C}^{195}$ we will need the additional eigenvectors

$$|e_1^{(13)}\rangle = |1, -, b\rangle , \quad |e_2^{(13)}\rangle = |\omega_2^3, +\rangle , \quad |e_3^{(13)}\rangle = |\omega_3, -\rangle ,
|e_4^{(13)}\rangle = |1, +, \rangle , \quad |e_5^{(13)}\rangle = |\omega_2^3, -, \rangle , \quad |e_6^{(13)}\rangle = |\omega_3^2, +\rangle .$$

We are now ready to write down the adapted basis for the eigenspace in $\mathbb{C}^{195}$ that holds our SIC fiducials. Its dimension is 36, but fiducial 195d has a further symmetry and sits in a smaller subspace of dimension 19 [14]. We order the product basis vectors spanning the latter lexicographically, and arrive at

$$|e_0\rangle = |1, -, a\rangle|\omega_3, -\rangle|\omega_3, +\rangle \quad |e_1\rangle = |1, -, a\rangle|\omega_3, +\rangle|\omega_3, -\rangle \quad |e_2\rangle = |1, -, a\rangle|1, -, \rangle|\omega_3^2, +\rangle \quad |e_3\rangle = |1, -, a\rangle|1, -, \rangle|\omega_3^2, -\rangle \quad |e_4\rangle = |1, +, \rangle|\omega_3, -\rangle|\omega_3, +\rangle \quad |e_5\rangle = |1, +, \rangle|\omega_3, +\rangle|\omega_3, -\rangle \quad |e_6\rangle = |1, +, \rangle|1, -, \rangle|\omega_3^2, +\rangle \quad |e_7\rangle = |1, +, \rangle|1, -, \rangle|\omega_3^2, -\rangle \quad |e_8\rangle = |\omega_3^2, +\rangle|\omega_3, +\rangle|\omega_3^2, +\rangle \quad |e_9\rangle = |\omega_3^2, +\rangle|\omega_3, +\rangle|\omega_3^2, -\rangle \quad |e_{10}\rangle = |\omega_3, -\rangle|\omega_3, -\rangle|1, +\rangle \quad |e_{11}\rangle = |\omega_3, -\rangle|1, +\rangle|\omega_3, +\rangle \quad |e_{12}\rangle = |\omega_3, -\rangle|1, +\rangle|\omega_3, -\rangle \quad |e_{13}\rangle = |1, +\rangle|\omega_3, +\rangle|\omega_3, +\rangle \quad |e_{14}\rangle = |1, +\rangle|\omega_3, +\rangle|\omega_3, -\rangle \quad |e_{15}\rangle = |\omega_3^2, -\rangle|\omega_3, -\rangle|\omega_3^2, +\rangle \quad |e_{16}\rangle = |\omega_3^2, +\rangle|\omega_3, -\rangle|\omega_3^2, -\rangle \quad |e_{17}\rangle = |\omega_3^2, -\rangle|1, +\rangle|1, +\rangle \quad |e_{18}\rangle = |\omega_3, +\rangle|\omega_3, +\rangle|1, +\rangle .$$

Fiducial 195d can be expanded in terms of these. An additional set of 17 basis vectors is needed to obtain a basis in which fiducial 195b can be expanded. Again we order the product vectors lexicographically within this set, as explained below eqs. (39).
This concludes the construction of the adapted basis. To sum up: Due to complications in the dimension 5 and 3 factors it comes in two versions, a normalized basis containing only unit vectors and a cyclotomic basis constructed purely from roots of unity. The calculations needed to construct these bases are admittedly somewhat lengthy, but they are entirely straightforward except that a choice of phase factor has to be made for each vector. The choices we made were partly taken from Zauner [2], and partly determined by trial and error in the course of the calculations described in the next section.

5. Converting numerical SICs to exact SICs

Although in a way this is the key section it is a brief one. This is so because, with the groundwork laid, the procedure by means of which candidate exact solutions are obtained is very simple. It works as well as it does because we carefully selected the cases we looked at—and also because the phase factors in the adapted basis vectors were carefully chosen.

We start with a numerical SIC with 150 digits precision in the standard basis (kindly supplied by Andrew Scott). We apply unitary transformations and Chinese remaindering to it, so that it takes the form in eq. (5) with respect to the adapted basis defined in Section 4. In this basis we calculate the squares \( p_r \) of the absolute values and suitable powers of the relative phases \( e^{i\nu_r} \). Then we apply the ‘RootApproximant’ command in Mathematica to these decimal numbers. This command implements an integer relation algorithm which turns out to be surprisingly effective. For the absolute values squared the answer comes out directly. For suitable powers of the phases Mathematica typically returns a fourth order polynomial equation which we solve. The results are as stated in Table 1 for fiducials 15d and 15b, Table 2 for fiducial 195d, and Table 4 for fiducial 195b.

To test the result we lower the precision in steps of 5 digits, in order to find out the minimum precision needed. Worst case results are given in Table 3. Finally the SIC condition is tested using 1000 digits precision.

In dimension 15 there exists a total of four inequivalent fiducials, belonging to different orbits of the extended Clifford group. Table 3 includes some information about how our procedure fares when we apply it to the fiducials 15a and 15c, which lie in a larger number field than 15b (and 15d) [16]. We did establish that for 15a and 15c the absolute values squared of the components are not expressible in terms of numbers from the base field \( \mathbb{Q}(\sqrt{3}) \).

**Table 3.** Minimum precision needed for the calculation. For 15ac we did not pursue the only partially successful calculation to higher than 300 digits.

| Fiducial | 15d | 15b | 15ac | 195d | 195b |
|----------|-----|-----|------|------|------|
| Moduli   | 10  | 10  | >300 | 20   | 20   |
| Phases   | 45  | 30  | ?    | 65   | 90   |

6. Checking the results

It remains to carry out an exact check of the SIC condition. In trying to do so one quickly realizes that the adapted basis is not well suited to describe the action of the Weyl–Heisenberg
displacement operators. Moreover, while the form taken by the components of the fiducials with respect to the orthonormal basis (see Tables 1, 2, and 3) is interesting, it also poses a problem. The absolute values squared lie in the base field \( \mathbb{Q}(\sqrt{3}) \), but one can check that the absolute values themselves, the \( \sqrt{p_r} \), do not belong to the minimal field predicted by the ray class hypothesis. Neither do the phases \( e^{i \nu_r} \), nor the actual components \( \sqrt{p_r} e^{i \nu_r} \). This causes a significant problem for an exact calculation to check the SIC conditions (17), because when the degree of the number field goes up so does the time needed for the computer program Magma to do the exact calculations. A related problem is posed by the normalizing factors \( \sqrt{n_0}, \sqrt{n_1}, \sqrt{2} \), used in going from the cyclotomic to the ON basis. One can check that they do not belong to the minimal field.

To proceed we need some information about the relevant ray class fields. They are ramified at both infinite places [9]. In this paper we are concerned with three such ray class field over \( \mathbb{Q}(\sqrt{3}) \), with conductors 5, 15, and 195. The degrees of these fields, that is their dimensions when considered as linear spaces over the rational numbers, raises quickly. For \( d = 5 \) the degree is 32, for \( d = 15 \) the degree is 96, and for \( d = 195 \) the degree is 6912. In practice this means that calculations using Magma are quick for dimension 15, and time consuming for dimension 195. One helpful fact is that the sixthieth root of unity \( \omega_{60} \) belongs to the ray class field with conductor 5, and hence to all three of them. Hence we can use fifteenth and twelfth roots of unity with the comfortable knowledge that they belong to the minimal number fields admitted by the ray class hypothesis.

The problems we observed can now be remedied. The first step is to rewrite the fiducials in terms of the cyclotomic rather than the orthonormal bases. Thus

\[
|\Psi_0\rangle = \sum_{r=0}^{d-1} \sqrt{p_r} e^{i \nu_r} |e^{(d)}_r\rangle = \Omega \sum_{r=0}^{d-1} z_r |f^{(d)}_r\rangle ,
\]

where \( \Omega \) is an overall factor adjusted so that

\[
z_0 = 1 .
\]

The square of \( \Omega \) belongs to the cyclotomic field \( \mathbb{Q}(\omega_{60}) \), which means that we do not have to consider this overall factor further. According to the Ray Class Hypothesis, if our expressions for fiducials 5a, 15d, and 195d, are correct, then the numbers \( z_r \) have to belong to the ray class field with discriminant 3 and conductor equal to the dimension \( d \). For fiducials 15b and 195b the number field containing the \( z_r \) must contain the relevant ray class field, and is in fact expected to be a degree 2 extension thereof.

At first sight the numbers \( z_r \) are less appealing than the numbers \( p_r \) and the Pythagorean triple phases. They include the non-cyclotomic normalizing factors \( \sqrt{n_0}, \sqrt{n_1}, \sqrt{2} \) from eqs. (34) and (38). As an example, when we trace things through for 195d we find that

\[
z_1 = \sqrt{\frac{P_1 P_0}{P_1 P_1}} \omega_4 \left( -Q_2 P_2 P_1 \right)^{\frac{1}{4}}.
\]

The minimal polynomial of this particular component has degree 32. Going through all the components we find that fiducials 15d, 15b, 195d, and 195b, all have some components of degree 96, but not higher than that. (For fiducial 5a the single non-trivial component has degree 32. Actually, in dimension 5 a direct check of the SIC conditions is not very hard to do.)

We explain the procedure for how to prove that the candidate solutions are, indeed, exact solutions using fiducial 195d (see Table 2) as an example. Fiducials 15d and 15b are easily
handled in the same way. For 195b we have only completed the first step, that is we have verified that all its components lie in the expected number field.

The components \( z_r \) lie in the ray class field if their minimal polynomials factor completely over the latter. The degree of the ray class field with conductor 195 is

\[
6912 = 2^8 \cdot 3^3 .
\]

An explicit set of generators of this number field, in terms of which arbitrary numbers in the field can be expanded, is

\[
a = \sqrt{3} \quad i = \sqrt{-1} \quad r_1 = \sqrt{5} \quad r_2 = \sqrt{13}
\]

\[
b_1 = \sqrt{10 + 2r_1} \quad b_2 = \sqrt{6r_2 + 26} \quad b_3 = \sqrt{3 - 4a}
\]

\[
b_4 = \sqrt{300 - 140a + (60 - 28a)r_1 + (47a - 84 + (3a - 12)r_1)b_1}
\]

\[
c_1 : x^3 - 39x + 65 \quad c_2 : x^3 - 507x - 1014a + 2535 \quad c_3 : x^3 - 15x - 20.
\]

The degree of the field containing 195b is twice as large, and it needs the extra generator

\[
e_1 = \sqrt{-2a} .
\]

To speed up the calculations that Magma does when it factors the minimal polynomials, we first ask it to do the factorization over the degree \( 2^8 \) and degree \( 3^3 \) subfields. These subfields are then embedded in the full field and the greatest common divisors of the factors is calculated. We end up with explicit expressions for the components. As an example, the component \( z_1 \) in 195d now reads

\[
z_1 = \left( \left( \frac{(11a - 34)i}{3120} + \frac{(a - 9)r_1}{1040} + \frac{(11a - 8)i}{624} + \frac{(3a - 1)}{624} \right) b_1 + \right.
\]

\[
+ \left( \frac{(19a - 80)i}{1560} + \frac{(17a - 10)r_1}{1560} + \frac{(a - 2)i}{24} + \frac{a}{24} \right) b_3 b_4 .
\]

Because it is expressed as a linear combination of products of the generators, it follows that \( z_1 \) belongs to the ray class field, and Magma can do calculations with it. The same is true for all the components \( z_r \), although the actual expressions are often much lengthier.

When the factoring is completed all the numbers inside the SIC fiducial can be expressed as linear combinations of products of generators of the field. Magma is then asked to calculate all the SIC overlaps. Because an order 12 symmetry is built into the fiducial it is not necessary to calculate all the 195·195 overlaps. A subset of 1604 overlaps suffices. Due to the high degree of the field this is still a considerable calculation, as is shown by the fact that the final Magma file containing the 1604 overlaps takes up 75 MB. All of the overlaps have absolute value squared equal to \( 1/196 \), which completes the proof that Table 2 indeed gives a SIC fiducial.

One curious feature of the fiducials 15b and 195b is worth mentioning, namely that if they are split into two, one part lying in the subspace containing fiducials 15d respectively
195d, and one part lying in the orthogonal complement, then their first parts are found to lie in the ray class field. In other words, the generator \( e_1 \) enters only in the components in the orthogonal complements.

7. Summary and discussion

We have been following a three step procedure to find exact expression for SIC fiducials:

1. Express a numerical SIC in an adapted basis related to the standard basis in a definite way.

2. Use Mathematica’s ‘RootApproximant’ command to convert the numerical solution to an exact one.

3. Check that the solution lies in the expected number field, and using this check the SIC condition.

We comment on these steps in turn. Our choice of adapted basis makes use of the Chinese remaindering isomorphism. Then it is enough to find bases in the prime dimensional factors, and in prime dimensions the basis singled out by the centralizer of the Zauner matrix suggests itself. It is adapted to the symmetries of the SIC \([12]\). The calculations in this step are tedious but straightforward, except that a judicious choice of phase factors is made in order to simplify the next step.

Remarkably, the second step of the procedure requires only very modest precision. In previous work numerical SICs were converted to exact ones using the basis independent overlap phases. This has the advantage that it works in general, but the disadvantage that the actual conversion has to be carried out in a much more sophisticated way using a precision sometimes running to several thousand digits \([16]\). We needed only 90 digits in the worst case, and could manage with less at the expense of more work.

The exact expression obtained in the second step turns out to involve numbers that do not lie in the minimal field. This is easily repaired by rewriting the expression in terms of the cyclotomic version of the adapted basis. It is then found that the ray class hypothesis holds. Moreover the maximal degree of the redefined components, considered as algebraic numbers, does not increase as the dimension increases from 15 to 195. Nevertheless, because of the high degree of the field it is a computationally heavy task to check the SIC condition in dimension 195. One is left wondering if there is a better way to do it, using the more compact expression actually obtained in the second step, or a suitably adapted basis for the number field, or both.

Our procedure has provided strikingly simple exact expressions for the SIC fiducials 5a, 15d, 15b, and 195d. We also have a candidate exact expression for 195b, with the expected number theoretic properties. In the normalized adapted basis the absolute values squared of all the components lie in the quadratic base field of the SIC, and all the phase factors are obtained from Pythagorean triples and similar triples constructed using the base field.

Are there other SICs for which a basis adapted to the symmetries of a SIC vector will force the absolute values squared of the components into their basefields?

According to the calculations presented in Section 5 for fiducials 15a and 15c the answer is “no”. Their number fields are in fact larger than that of 15b \([16]\), so this is perhaps not a surprise. We also investigated the ten SICs in dimension 53 \([7]\), which share the same basefield with the ones considered here. Adapting the basis to the centralizer of the Zauner
unitary in dimension 53 we found that the answer for all of them is \textquotedblleft no\textquotedblright. An exact solution in dimension 53 has been found by other means [10].

Still we conjecture that the answer to our question is \textquotedblleft yes\textquotedblright for every dimension on the particular number theoretic ladder we have been climbing. One reason for optimism is the fact that the dimensions continue to factor into primes equal to one modulo three, with all multiplicities equal to one, for the first eight rungs of the ladder. (We do not understand why, we simply let Mathematica do the factorization for us.) Hence adapted bases suitable for these rungs of the ladder are easily constructed, but numerical solutions for the SICs are lacking. Thus we cannot perform the first step in our procedure. The suspicion is that, if we could, the second step would be easy. The third step would need modification. Already at the third rung of the ladder the conductor is \(d = 195 \cdot 193\) and the degree of the ray class field is a staggering \(2^{19} \cdot 3^5\). Verifying the conjecture would therefore require considerable insight. We take Kopp\'s recent work [15] as an indication that there may be a passable road ahead.

Concerning SICs over other base fields, the SIC fiducials 7b, 7a, and 35j can be presented in an adapted basis so that all absolute values squared of the components lie in the base field \(\mathbb{Q}(\sqrt{2})\) [20]. They belong to a dimension ladder starting at \(d = 7\). It seems that SICs on the ladder starting at \(d = 9\) will not have the same feature. For ladders starting at an even dimension the adapted basis needs modification, because we cannot rely on the Chinese remaindering isomorphism in the same way. A solution of this technical difficulty has been announced [14], but not published as far as we know.

An interesting point is that we had to go slightly beyond the minimal number fields in order to reach the simplest form of the solutions. We do not understand why, but we feel that we have seen something similar before. The electromagnetic field can be expressed using the minimal number of two degrees of freedom per spatial point, but extending the description to include two additional degrees of freedom permits the elegant and useful description in terms of the vector potential. For SICs we have shown that elegance can be gained by going beyond the minimal number fields. This proved useful too, in the sense that we obtained previously unknown solutions. We do not know where this observation tends.

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### Appendix A. 195b

**Table 4.** Moduli squared and relative phases in 195b. This solution is placed here to emphasize that an exact check of the SIC condition has not been carried out for it.

| (Moduli)$^2$ | Phases |
|--------------|---------|
| $p_0 = 4\sqrt{2} \frac{3}{91}$ | $p_{19} = \frac{33 - 18\sqrt{2}}{182}$ | $e^{i\nu_{10}} = 1$ | $e^{i\nu_{19}} = -(-i)^{\frac{3}{2}}$ |
| $p_1 = 2\sqrt{2} \frac{\sqrt{3}}{7}$ | $p_{20} = 4\sqrt{2} \frac{3}{182}$ | $e^{i\nu_1} = \omega_4^3 \left(P_5 P_{13} \frac{Q_{13}}{Q_{13}}\right)^{\frac{1}{4}}$ | $e^{i\nu_{20}} = \omega_4^2 (-P_5)^{\frac{3}{4}}$ |
| $p_2 = 82\sqrt{2} \frac{3}{364}$ | $p_{21} = 12 - 3\sqrt{2} \frac{3}{364}$ | $e^{i\nu_2} = \omega_12 \left(P_{24} P_{13} Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{21}} = (-i)^{3/4}$ |
| $p_3 = 54\sqrt{2} \frac{3}{364}$ | $p_{22} = 6\sqrt{5} \frac{3}{364}$ | $e^{i\nu_3} = \omega_12 \left((-P_5)^{\frac{3}{4}}\right)^{\frac{1}{4}}$ | $e^{i\nu_{22}} = \omega_3^2 (-P_5)^{\frac{3}{4}}$ |
| $p_4 = 4\sqrt{2} \frac{3}{91}$ | $p_{23} = 5\sqrt{2} \frac{3}{91}$ | $e^{i\nu_4} = -1$ | $e^{i\nu_{23}} = \omega_4^4 \left(-\frac{P_5 P_{13} Q_{13}}{Q_{13}}\right)^{\frac{1}{4}}$ |
| $p_5 = 11 - 6\sqrt{2} \frac{7}{28}$ | $p_{24} = 7 - 4\sqrt{2} \frac{7}{14}$ | $e^{i\nu_5} = \omega_4^2 \left(P_5 Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{24}} = \omega_4^3 \left(-\frac{P_5 Q_{13}}{Q_{13}}\right)^{\frac{1}{4}}$ |
| $p_6 = -15 \sqrt{2} \frac{174}{1460}$ | $p_{25} = 2\sqrt{2} \frac{3}{91}$ | $e^{i\nu_6} = \omega_12 \left(P_{12} P_{13} Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{25}} = \omega_12 \left(P_{24} P_{13} Q_{13}\right)^{\frac{1}{4}}$ |
| $p_7 = -\sqrt{2} \frac{182}{182}$ | $p_{26} = 0$ | $e^{i\nu_7} = \omega_12 \left(-P_5\right)^{\frac{3}{4}}$ | $e^{i\nu_{26}} = (i)^{\frac{3}{4}}$ |
| $p_8 = -\sqrt{2} \frac{14}{14}$ | $p_{27} = \frac{9}{42}$ | $e^{i\nu_8} = \omega_12 \left(P_5 P_{13} Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{27}} = (i)^{\frac{3}{4}}$ |
| $p_9 = 2\sqrt{2} \frac{3}{14}$ | $p_{28} = 10 - 5\sqrt{2} \frac{3}{28}$ | $e^{i\nu_9} = \omega_12 \left(-P_5 P_{13} Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{28}} = \omega_3^2 (-P_5)^{\frac{3}{4}}$ |
| $p_{10} = \sqrt{2} \frac{7}{21}$ | $p_{29} = 2\sqrt{2} \frac{7}{14}$ | $e^{i\nu_{10}} = \omega_12 \left(-P_{24} P_{13} Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{29}} = \omega_3^3 (i P_5)^{\frac{3}{4}}$ |
| $p_{11} = \sqrt{2} \frac{42}{22}$ | $p_{30} = 2\sqrt{2} \frac{7}{14}$ | $e^{i\nu_{14}} = \omega_12 \left(P_5 Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{30}} = \omega_12 \left(-P_5\right)^{\frac{3}{4}}$ |
| $p_{12} = -\sqrt{2} \frac{14}{14}$ | $p_{31} = 6 - 3\sqrt{2} \frac{3}{14}$ | $e^{i\nu_{12}} = \omega_12 \left(P_{12} P_{13} Q_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{31}} = \omega_7 \left(-P_{24} Q_{13} P_{13}\right)^{\frac{1}{4}}$ |
| $p_{13} = \frac{1}{28}$ | $p_{32} = 2\sqrt{2} \frac{3}{14}$ | $e^{i\nu_{13}} = -i$ | $e^{i\nu_{32}} = \omega_12 \left(P_5 Q_{13} P_{13}\right)^{\frac{1}{4}}$ |
| $p_{14} = -\sqrt{2} \frac{28}{21}$ | $p_{33} = 4\sqrt{2} \frac{3}{6}$ | $e^{i\nu_{14}} = \omega_4 \left(P_5\right)^{\frac{3}{4}}$ | $e^{i\nu_{33}} = \omega_4^3 \left(P_5 Q_{13} P_{13}\right)^{\frac{1}{4}}$ |
| $p_{15} = 2\sqrt{2} \frac{3}{42}$ | $p_{34} = 16 - 3\sqrt{2} \frac{3}{42}$ | $e^{i\nu_{15}} = \omega_112 \left(P_{12} Q_{13} P_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{34}} = \left(-\frac{Q_1^3}{P_{13} Q_{13}}\right)^{\frac{1}{4}}$ |
| $p_{16} = -2\sqrt{2} \frac{7}{14}$ | $p_{35} = \frac{1}{14}$ | $e^{i\nu_{16}} = \omega_12 \left(-P_5 Q_{13} P_{13}\right)^{\frac{1}{4}}$ | $e^{i\nu_{35}} = \omega_12 \left(P_5 Q_{13} P_{13}\right)^{\frac{1}{4}}$ |
| $p_{17} = 2\sqrt{2} \frac{3}{42}$ | | $e^{i\nu_{17}} = \omega_12 \left(-P_{24} P_{13} Q_{13}\right)^{\frac{1}{4}}$ | |
| $p_{18} = 2\sqrt{2} \frac{3}{42}$ | | $e^{i\nu_{18}} = \omega_112 \left(-P_{24} Q_{13} P_{13}\right)^{\frac{1}{4}}$ | |
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