RINGS OF INVARIANT MODULE TYPE AND AUTOMORPHISM-INARIANT MODULES

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Dedicated to T. Y. Lam on his 70th Birthday

ABSTRACT. A module is called automorphism-invariant if it is invariant under any automorphism of its injective hull. In [Algebras for which every indecomposable right module is invariant in its injective envelope, Pacific J. Math., vol. 31, no. 3 (1969), 655-658] Dickson and Fuller had shown that if $R$ is a finite-dimensional algebra over a field $F$ with more than two elements then an indecomposable automorphism-invariant right $R$-module must be quasi-injective. In this paper we show that this result fails to hold if $F$ is a field with two elements. Dickson and Fuller had further shown that if $R$ is a finite-dimensional algebra over a field $F$ with more than two elements, then $R$ is of right invariant module type if and only if every indecomposable right $R$-module is automorphism-invariant. We extend the result of Dickson and Fuller to any right artinian ring. A ring $R$ is said to be of right automorphism-invariant type (in short, RAI-type) if every finitely generated indecomposable right $R$-module is automorphism-invariant. In this paper we completely characterize an indecomposable right artinian ring of RAI-type.

1. Introduction

All our rings have identity element and modules are right unital. A right $R$-module $M$ is called an automorphism-invariant module if $M$ is invariant under any automorphism of its injective hull, i.e. for any automorphism $\sigma$ of $E(M)$, $\sigma(M) \subseteq M$ where $E(M)$ denotes the injective hull of $M$.

Indecomposable modules $M$ with the property that $M$ is invariant under any automorphism of its injective hull were first studied by Dickson and Fuller in [5] for the particular case of finite-dimensional algebras over fields $F$ with more than two elements. But for modules over arbitrary rings, study of such a property has been initiated recently by Lee and Zhou in [14]. The dual notion of these modules has been proposed by Singh and Srivastava in [17].

The obvious examples of the class of automorphism-invariant modules are quasi-injective modules and pseudo-injective modules. Recall that a module $M$ is said to be $N$-injective if for every submodule $N_1$ of the module $N$, all homomorphisms $N_1 \to M$ can be extended to homomorphisms $N \to M$. A right $R$-module $M$ is injective if $M$ is $N$-injective for every $N \in \text{Mod-}R$. A module $M$ is said to be

2000 Mathematics Subject Classification. 16U60, 16D50.

Key words and phrases. rings of invariant module type, automorphism-invariant modules, quasi-injective modules, pseudo-injective modules.
quasi-injective if $M$ is $M$-injective. A module $M$ is called pseudo-injective if every monomorphism from a submodule of $M$ to $M$ extends to an endomorphism of $M$.

Thus we have the following hierarchy;

injective $\implies$ quasi-injective $\implies$ pseudo-injective $\implies$ automorphism-invariant

It is well known that a quasi-injective module need not be injective. In [18] Teply gave construction of a pseudo-injective module which is not quasi-injective. We do not know yet an example of an automorphism-invariant module which is not pseudo-injective.

Dickson and Fuller [5] studied automorphism-invariant modules in case of finite-dimensional algebras over a field $F$ with more than two elements. They proved that if $R$ is a finite-dimensional algebra over a field $F$ with more than two elements then an indecomposable automorphism-invariant right $R$-module must be quasi-injective. We show that this result fails to hold if $F$ is a field with two elements. A ring $R$ is said to be of right invariant module type if every indecomposable right $R$-module is quasi-injective. Dickson and Fuller had further shown that if $R$ is a finite-dimensional algebra over a field $F$ with more than two elements, then $R$ is of right invariant module type if and only if every indecomposable right $R$-module is automorphism-invariant. We extend the result of Dickson and Fuller to any right artinian ring.

We call a ring $R$ to be of right automorphism-invariant type (in short, RAI-type), if every finitely generated indecomposable right $R$-module is automorphism-invariant. In this paper we study the structure of indecomposable right artinian rings of RAI-type.

Lee and Zhou in [14] asked whether every automorphism-invariant module is pseudo-injective. In this paper we show that the answer is in the affirmative for modules with finite Goldie dimension.

We also prove that a simple right noetherian ring $R$ is a right SI ring if and only if every cyclic singular right $R$-module is automorphism-invariant.

Before presenting the proofs of these results, let us recall some basic definitions and facts. A module $M$ is said to have finite Goldie (or uniform) dimension if it does not contain an infinite direct sum $\bigoplus_{n \in \mathbb{N}} M_n$ of non-zero submodules.

A module $M$ is said to be directly-finite if $M$ is not isomorphic to a proper summand of itself. Clearly, a module with finite Goldie dimension is directly-finite. A module $M$ is called a square if $M \cong X \oplus X$ for some module $X$; and a module is called square-free if it does not contain a non-zero square.

A module $M$ is said to have the internal cancellation property if whenever $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$. For details on internal cancellation property, the reader is referred to [13]. Now, if an injective module $M$ is directly-finite, then it has internal cancellation property (see [15] Theorem 1.29).

A module $M$ is said to be uniserial if any two submodules of $M$ are comparable with respect to inclusion. A ring $R$ is called a right uniserial ring if $R_R$ is a uniserial module. Any direct sum of uniserial modules is called a serial module. A ring $R$ is said to be a right serial ring if the module $R_R$ is serial. A ring $R$ is called a serial ring if $R$ is both left as well as right serial.
If $A$ is an essential submodule of $B$, then we denote it as $A \subseteq_e B$. For any module $M$, we define $Z(M) = \{ x \in M : \text{ann}_r(x) \subseteq_e R_R \}$. It can be easily checked that $Z(M)$ is a submodule of $M$. It is called the singular submodule of $M$. If $Z(M) = M$, then $M$ is called a singular module. If $Z(M) = 0$, then $M$ is called a non-singular module.

Consider the following three conditions on a module $M$:

- **C1**: Every submodule of $M$ is essential in a direct summand of $M$.
- **C2**: Every submodule of $M$ isomorphic to a direct summand of $M$ is itself a direct summand of $M$.
- **C3**: If $N_1$ and $N_2$ are direct summands of $M$ with $N_1 \cap N_2 = 0$ then $N_1 \oplus N_2$ is also a direct summand of $M$.

A module $M$ is called a continuous module if it satisfies conditions C1 and C2. A module $M$ is called $\pi$-injective (or quasi-continuous) if it satisfies conditions C1 and C3. A module $M$ is called a CS module (or extending module) if it satisfies condition C1.

In general, we have the following implications.

$$\text{Injective} \implies \text{Quasi-injective} \implies \text{Continuous} \implies \pi\text{-injective} \implies \text{CS}$$

The socle of a module $M$ is denoted by $\text{Soc}(M)$. A right $R$-module $M$ is called semi-artinian if for every submodule $N \neq M$, $\text{Soc}(M/N) \neq 0$. A ring $R$ is called right semi-artinian if $R_R$ is semi-artinian. We denote by $J(R)$, the Jacobson radical of a ring $R$. For any term not defined here, the reader is referred to [9], [11], [12], and [15].

### 2. Basic Facts about Automorphism-invariant Modules

Lee and Zhou proved the following basic facts about automorphism-invariant modules [14].

- A module $M$ is automorphism-invariant if and only if every isomorphism between any two essential submodules of $M$ extends to an automorphism of $M$.
- A direct summand of an automorphism-invariant module is automorphism-invariant.
- If for two modules $M_1$ and $M_2$, $M_1 \oplus M_2$ is automorphism-invariant, then $M_1$ is $M_2$-injective and $M_2$ is $M_1$-injective.
- Every automorphism-invariant module satisfies the property C3.
- A CS automorphism-invariant module is quasi-injective.
3. Results

Dickson and Fuller in [5] considered a finite-dimensional algebra \( R \) over a field \( F \) with more than two elements and proved that if an indecomposable right \( R \)-module \( M \) is automorphism-invariant, then \( M \) is quasi-injective. They further obtained the following.

**Theorem 1.** (Dickson and Fuller, [5]) Let \( R \) be a finite-dimensional algebra over a field \( F \) with more than two elements. Then the following statements are equivalent:

(i) Each indecomposable right \( R \)-module is automorphism-invariant.
(ii) Each indecomposable right \( R \)-module is quasi-injective.
(iii) Each indecomposable right \( R \)-module has a square-free socle.

We will provide an example to show that if \( R \) is a finite-dimensional algebra over a field \( F \) with two elements, then an indecomposable automorphism-invariant right \( R \)-module need not be quasi-injective.

First, note that in an artinian serial ring \( R \), any indecomposable summand of \( R_R \) of maximum length is injective. Thus if \( T_n(D) \) is the upper triangular matrix ring over a division ring \( D \), then \( e_1 T_n(D) \) is injective and uniserial.

**Example.** Let \( R = \begin{bmatrix} F & F & F \\ F & 0 & F \\ 0 & 0 & F \end{bmatrix} \) where \( F \) is a field of order 2.

We know that \( R \) is a left serial ring. Note that \( e_{11} R \) is a local module, \( e_{12} F \cong e_{22} R \), \( e_{13} F \cong e_{33} R \) and \( e_{11} J(R) = e_{12} F \oplus e_{13} F \), a direct sum of two minimal right ideals. So the injective hull of \( e_{11} R \) is \( E(e_{11} R) = E_1 \oplus E_2 \), where \( E_1 = E(e_{12} F) \) and \( E_2 = E(e_{13} F) \).

Now set \( A = \text{ann}_r(e_{12} F) \). Then \( A = e_{12} F + e_{33} F \). Thus \( R = R/A \cong \begin{bmatrix} F & F \\ F & 0 \\ 0 & F \end{bmatrix} = S \). Denote the first row of \( S \) by \( S_1 \). It may be checked that \( S_1 \) is injective. As \( F \) has only two elements, \( S_1 \) has only two endomorphisms, zero and the identity. Take the pre-image \( L_1 \) of \( S_1 \) in \( R \). It is uniserial with composition length 2, and \( e_{12} F \) naturally embeds in \( L_1 \). There is no mapping of \( e_{12} F \) into \( L_1 \). It follows that \( L_1 \) is \( e_{11} R \)-injective and \( e_{12} F \)-injective. As \( e_{22} R \cong e_{12} F \), \( L_1 \) is \( e_{22} R \)-injective. There is no map from \( e_{33} R \) into \( L_1 \) so it is also \( e_{33} R \)-injective. Hence \( L_1 \) is injective. Thus \( E_1 = L_1 \) and its ring of endomorphisms has only two elements.

If \( B = \text{ann}_r(e_{13} F) \), then \( B = e_{12} F + e_{22} F \). Thus \( R/B \cong \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} \). The pre-image of \( S_1 \) in \( R/B \) is \( L_2 \), which is uniserial, and injective. We have \( E_2 \cong L_2 \) and its ring of endomorphism has only two elements.

Note that \( e_{11} R \) has all its composition factors non-isomorphic, both \( L_1 \) and \( L_2 \) have composition length 2 with \( \frac{L_1}{L_1 J(R)} \cong \frac{e_{11} R}{e_{11} J(R)} \), \( L_1 J(R) \cong e_{22} R \), \( \frac{L_2}{L_2 J(R)} \cong \frac{e_{12} R}{e_{11} J(R)} \), and \( L_2 J(R) \cong e_{33} R \). Thus \( L_1, L_2 \) have isomorphic tops but non-isomorphic socles.

Suppose there exists a non-zero mapping \( \sigma : L_1 \to L_2 \). Then \( \sigma(L_1) = L_2 J(R) \). Thus \( \frac{e_{11} R}{e_{11} J(R)} \cong e_{33} R \), which is a contradiction. Therefore, there is no non-zero map between \( L_1 \) and \( L_2 \).
Hence the only automorphism of \( L_1 \oplus L_2 \) is the identity. So \( e_{11}R \) is trivially automorphism-invariant but it is not uniform. Then clearly \( e_{11}R \) is not quasi-injective as an indecomposable quasi-injective module must be uniform.

Thus, this ring \( R \) is an example of a finite-dimensional algebra over a field \( F \) with two elements such that there exists an indecomposable right \( R \)-module which is automorphism-invariant but not quasi-injective. □

Next, we proceed to extend the result of Dickson and Fuller [5] to any right artinian ring. But, first we obtain a useful result on decomposition property of automorphism-invariant modules.

We will show that under certain conditions a decomposition of injective hull \( E(M) \) of an automorphism-invariant module \( M \) induces a natural decomposition of \( M \).

We will denote the identity automorphism on any module \( M \) by \( I_M \).

Lemma 2. Let \( M \) be an automorphism-invariant right module over any ring \( R \). If \( E(M) = E_1 \oplus E_2 \) and \( \pi_1 : E(M) \to E_1 \) is an associated projection, then \( M_1 = \pi_1(M) \) is also automorphism-invariant.

Proof. Let \( E(M) = E_1 \oplus E_2 \) and \( M_1 = \pi_1(M) \), where \( \pi_1 : E(M) \to E_1 \) is a projection with \( E_2 \) as its kernel. Let \( \sigma \) be an automorphism of \( E_1 \) and \( x_1 \in M_1 \).

For some \( x \in M \), and \( x_2 \in E_2 \), we have \( x = x_1 + x_2 \). Now \( \sigma = \sigma_1 \oplus I_{E_2} \) is an automorphism of \( E \). Thus \( \sigma(x) = \sigma_1(x_1) + x_2 \in M \), which gives \( \sigma_1(x_1) \in M_1 \).

Hence \( M_1 \) is automorphism-invariant. □

Lemma 3. Let \( M \) be an automorphism-invariant right module over any ring \( R \). Let \( E(M) = E_1 \oplus E_2 \) such that there exists an automorphism \( \sigma_1 \) of \( E_1 \) such that \( I_{E_1} - \sigma_1 \) is also an automorphism of \( E_1 \). Then

\[
M = (M \cap E_1) \oplus (M \cap E_2).
\]

Proof. Set \( E = E(M) \). Set \( I_E = I_{E_1} \oplus I_{E_2} \), and \( \sigma = \sigma_1 \oplus I_{E_2} \). Clearly, both \( I_E \) and \( \sigma \) are automorphisms of \( E \). Since \( M \) is assumed to be an automorphism-invariant module, \( M \) is invariant under automorphisms \( I_E \) and \( \sigma \). Consequently, \( M \) is invariant under \( I_E - \sigma \) too. Note that \( (I_E - \sigma)(M) = (I_{E_1} - \sigma_1)(M) \subseteq M \). Let \( \pi_1 : E \to E_1 \) and \( \pi_2 : E \to E_2 \) be the canonical projections. Set \( M_1 = \pi_1(M) \) and \( M_2 = \pi_2(M) \). Now \( M \cap E_1 \subseteq M_1 \) and \( M \cap E_2 \subseteq M_2 \).

Let \( 0 \neq u_1 \in E_1 \). For some \( r \in R \), \( 0 \neq u_1r \in M \) and \( u_1r \in M_1 \). Thus \( M_1 \subseteq E_1 \). By Lemma 2, \( M_1 \) is automorphism-invariant. Therefore, \( M_1 = (I_{E_1} - \sigma)(M_1) \). Let \( x_1 \in M_1 \). Then, we have for some \( x \in M, x = x_1 + x_2, x_2 \in E_2 \).

Now, as \( I_{E_1} - \sigma_1 \) is an automorphism on \( E_1 \), there exists an element \( y_1 \in E_1 \) such that \( (I_{E_1} - \sigma_1)(y_1) = x_1 \), which gives \( y_1 \in (I_{E_1} - \sigma_1)(M_1) = M_1 \). This yields an element \( y \in M \) such that \( y = y_1 + y_2 \) for some \( y_2 \in E_2 \). We get \((I_E - \sigma)(y) = (I_{E_1} - \sigma_1)(y_1) = x_1 \). Thus \( x \in (I_E - \sigma)(M) \). As \((I_E - \sigma)(M) \subseteq M \), we get \( x_1 \in M \). Hence \( M_1 \subseteq M \).

Now, let \( u_2 \in M_2 \) be an arbitrary element. For some \( u_1 \in M_1 \), we have \( u = u_1 + u_2 \in M \). But we have shown in the previous paragraph that \( M_1 \subseteq M \), so \( u_1 \in M \). Therefore \( u_2 = u - u_1 \in M \). Hence \( M_2 \subseteq M \). This gives \( M_1 \oplus M_2 \subseteq M \) and hence \( M = M_1 \oplus M_2 \). Thus \( M = (M \cap E_1) \oplus (M \cap E_2) \). □
A quasi-injective module is obviously automorphism-invariant. In the next result we give a condition under which an automorphism-invariant module must be quasi-injective.

**Theorem 4.** Let $M$ be a right module over any ring $R$ such that every summand $E_1$ of $E(M)$ admits an automorphism $\sigma_1$ such that $I_{E_1} - \sigma_1$ is also an automorphism of $E_1$, then $M$ is automorphism-invariant if and only if $M$ is quasi-injective.

**Proof.** Let $M$ be automorphism-invariant. Set $E = E(M)$. Suppose every summand $E_1$ of $E$ admits an automorphism $\sigma_1$ such that $I_{E_1} - \sigma_1$ is also an automorphism of $E_1$.

Let $\sigma \in \text{End}(E)$ be an arbitrary element. Since $\text{End}(E)$ is a clean ring [1], $\sigma = \alpha + \beta$ where $\alpha$ is an idempotent and $\beta$ is an automorphism.

Let $E_1 = \alpha E$, and $E_2 = (1 - \alpha)E$. Then $E = E_1 \oplus E_2$. By Lemma 3 we have $M = M_1 \oplus M_2$ where $M_1 = M \cap E_1, M_2 = M \cap E_2$.

Then clearly $\alpha(M) \subseteq M$. Since $M$ is automorphism-invariant, $\beta(M) \subseteq M$. Thus $\sigma(M) \subseteq M$. Hence $M$ is quasi-injective.

The converse is obvious. $\square$

As a consequence of this theorem, we may now deduce the following which extends the result of Dickson and Fuller [5] to any algebra (not necessarily finite-dimensional) over a field $F$ with more than two elements.

**Corollary 5.** Let $R$ be any algebra over a field $F$ with more than two elements. Then the following are equivalent:

(i) Each indecomposable right $R$-module is automorphism-invariant.

(ii) Each indecomposable right $R$-module is quasi-injective, that is, $R$ is of right invariant module type.

**Proof.** Clearly, for any right $R$-module $E$, the multiplication by an element $u \in F$ where $u \neq 0$ and $u \neq 1$ gives an automorphism $\sigma$ of $E$ such that $I_E - \sigma$ is also an automorphism of $E$. Hence the result follows from the above theorem. $\square$

**Corollary 6.** ([14]) Let $R$ be a ring in which 2 is invertible. Then any automorphism-invariant module over $R$ is quasi-injective.

**Proof.** Let $M$ be an automorphism-invariant right $R$-module. Let $E = E(M)$. Let $E_1$ be any summand of $E$. We have automorphism $\sigma_1 : E_1 \rightarrow E_1$, given by $\sigma_1(x) = 2x, x \in E_1$. Clearly, $I_{E_1} - \sigma_1 = -I_{E_1}$ is also an automorphism of $E_1$. By Theorem 4 $M$ is quasi-injective. $\square$

In the next lemma we give another useful result on decomposition of automorphism-invariant modules.

**Lemma 7.** Let $M$ be an automorphism-invariant right module over any ring $R$. If $E(M) = E_1 \oplus E_2 \oplus E_3$, where $E_1 \cong E_2$, then

$$M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3).$$

**Proof.** Set $E(M) = E$. Let $E = E_1 \oplus E_2 \oplus E_3$. Let $\sigma : E_1 \rightarrow E_2$ be an isomorphism and let $\pi_1 : E \rightarrow E_1, \pi_2 : E \rightarrow E_2, \pi_3 : E \rightarrow E_3$ be the canonical projections. Then $M \cap E_1 \subseteq \pi_1(M), M \cap E_2 \subseteq \pi_2(M)$ and $M \cap E_3 \subseteq \pi_3(M)$. $\square$
Let $\eta = \sigma^{-1}$. Consider the map $\lambda_1 : E \to E$ given by $\lambda_1(x_1, x_2, x_3) = (x_1, \sigma(x_1) + x_2, x_3)$. Clearly, $\lambda_1$ is an automorphism of $E$. Since $M$ is automorphism-invariant, $M$ is invariant under $\lambda_1$ and $I_E$. Consequently, $M$ is invariant under $\lambda_1 - I_E$. Thus $(\lambda_1 - I_E)(M) \subseteq M$. Next, we consider the map $\lambda_2 : E \to E$ given by $\lambda_2(x_1, x_2, x_3) = (x_1 + \eta(x_2), x_2, x_3)$. This map $\lambda_2$ is also an automorphism of $E$. Thus, as explained above, $M$ is invariant under $\lambda_2 - I_E$ too, that is $(\lambda_2 - I_E)(M) \subseteq M$.

Let $x = (x_1, x_2, x_3) \in M$. Then $(\lambda_1 - I_E)(x) = (0, \sigma(x_1), 0) \in M$. Similarly, we have $(\lambda_2 - I_E)(x) = (\eta(x_2), 0, 0) \in M$. This gives $(\lambda_1 - I_E)(\sigma(x_2), 0, 0) = (0, \sigma\eta(x_2), 0) = (0, x_2, 0) \in M$. Thus $\pi_2(M) \subseteq M$. Similarly, $(\lambda_2 - I_E)(0, \sigma(x_1), 0) = (\eta\sigma(x_1), 0, 0) = (x_1, 0, 0) \in M$. Thus $\pi_1(M) \subseteq M$. This yields that $(0, 0, x_3) \in M$, that is, $\pi_3(M) \subseteq M$. This shows that $\pi_1(M) \oplus \pi_2(M) \oplus \pi_3(M) \subseteq M$ and therefore, $M = \pi_1(M) \oplus \pi_2(M) \oplus \pi_3(M)$. Hence $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$. □

As a consequence of the above decomposition, we have the following for socle of an indecomposable automorphism-invariant module.

**Corollary 8.** If $M$ is an indecomposable automorphism-invariant right module over any ring $R$, then $\text{Soc}(M)$ is square-free.

**Proof.** Let $M$ be an indecomposable automorphism-invariant module. Suppose $M$ has two isomorphic simple submodules $S_1$ and $S_2$. Then $E(M) = E_1 \oplus E_2 \oplus E_3$, where $E_1 = E(S_1), E_2 = E(S_2)$ and $E_1 \cong E_2$. By Lemma 7, $M$ decomposes as $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$, a contradiction to our assumption that $M$ is indecomposable. Hence $\text{Soc}(M)$ is square-free. □

Next, we have the following for any indecomposable semi-artinian automorphism-invariant module.

**Corollary 9.** Let $R$ be any ring and let $M$ be any indecomposable semi-artinian automorphism-invariant right $R$-module. Then one of the following statements holds:

(i) $M$ is uniform and quasi-injective.

(ii) Any simple submodule $S$ of $M$ has identity as its only automorphism.

**Proof.** Let $M$ be an indecomposable semi-artinian automorphism-invariant right $R$-module. Since $M$ is semi-artinian, $\text{Soc}(M) \neq 0$. By Corollary 8, we know that $\text{Soc}(M)$ is square-free. Suppose $S$ is a simple submodule of $M$. Now $D = \text{End}(S)$ is a division ring.

Suppose $|D| > 2$. Then there exists a $\sigma \in D$ such that $\sigma \neq 0$ and $\sigma \neq I_S$. Then $I_S - \sigma$ is an automorphism of $S$. Let $E = E(M)$ and $E_1 = E(S) \subseteq E$. Then $E = E_1 \oplus E_2$ for some submodule $E_2$ of $E$. Let $\sigma_1 \in \text{End}(E_1)$ be an extension of $\sigma$. Then $\sigma_1$ is an automorphism of $E_1$ and $(I_{E_1} - \sigma_1)(S) = (I_S - \sigma)(S) \neq 0$. Hence $I_{E_1} - \sigma_1$ is an automorphism of $E_1$. Thus, by Lemma 8, $M = (M \cap E_1) \oplus (M \cap E_2)$. As $M$ is indecomposable, we must have $M = M \cap E_1$. Therefore, $M$ is uniform. Then $\text{End}(E(M))$ is a local ring. Therefore for any $\alpha \in \text{End}(E(M))$, $\alpha$ is an automorphism or $I - \alpha$ is an automorphism. In any case $\alpha(M) \subseteq M$. Therefore $M$ is quasi-injective.

Now, if $M$ is not uniform then $|D| = 2$, that is $D = \text{End}(S) \cong \mathbb{F_2}$. In this case, the only automorphism of $S$ is the identity automorphism. □
Remark 10. Recall that an algebra \( A \) is said to be of finite module type if \( A \) has only a finite number of non-isomorphic indecomposable right modules. In regard to Corollary \[\text{5}\] we would like to mention here that Curtis and Jans proved that if \( A \) is an algebra over an algebraically closed field \( F \) any field \( \mathbb{F} \), then \( A \) is of finite module type (see \[\text{31}\]). This was extended by Dickson and Fuller who proved that if \( A \) is an algebra over any field \( \mathbb{F} \) such that \( A \) is of right invariant module type then \( A \) has finite module type \[\text{5}\].

We call a ring \( R \) to be of right automorphism-invariant type (in short, RAI-type), if every finitely generated indecomposable right \( R \)-module is automorphism-invariant. We would like to understand the structure of right artinian rings of RAI-type.

Lemma 11. Let \( R \) be a right artinian ring of RAI-type. Let \( e \in R \) be an indecomposable idempotent such that \( eR \) is not uniform. Let \( A \) be a right ideal contained in \( \text{Soc}(eR) \). Then \( \text{Soc}(eR) = A \oplus A' \) where \( A' \) has no simple submodule isomorphic to a simple submodule of \( A \) and \( eR/A' \) is quasi-projective.

Proof. As \( \text{Soc}(eR) \) is square-free, \( \text{Soc}(eR) = A \oplus A' \) where \( A' \) has no simple submodule isomorphic to a simple submodule of \( A \). If for some \( eR \in eRe, eReA' \not\subseteq A' \), then for some minimal right ideal \( S \subset A' \), \( eReS \not\subseteq A' \). This gives that \( S \) is isomorphic to a simple submodule contained in \( A \), a contradiction. Hence \( eR/A' \) is quasi-projective.

\[\square\]

Lemma 12. Let \( R \) be a right artinian ring of RAI-type. Then any uniserial right \( R \)-module is quasi-projective.

Proof. Let \( A \) be a uniserial right \( R \)-module with composition length \( l(A) = n \geq 2 \). We will prove the result by induction. Suppose first that \( n = 2 \). In this case, we can take \( J(R)^2 = 0 \). For some indecomposable idempotent \( e \in R \), we have \( A \cong eR/B \) for some \( B \subseteq \text{Soc}(eR) \). By Lemma \[\text{11}\] \( A \) is quasi-projective.

Now consider \( n > 2 \) and assume that the result holds for \( n - 1 \). Let \( 0 \neq \sigma : A \rightarrow A/C \) be a homomorphism where \( C \neq 0 \). Suppose \( \sigma \) cannot be lifted to a homomorphism \( \eta : A \rightarrow A \). Let \( F = \text{Soc}(A) \). Then \( F \subseteq \text{Ker}(\sigma) \). We get a mapping \( \bar{\sigma} : \frac{A}{F} \rightarrow \frac{A}{C} \). By the induction hypothesis, there exists a homomorphism \( \bar{\eta} : \frac{A}{F} \rightarrow \frac{A}{F} \) such that \( \bar{\sigma} = \pi \bar{\eta} \), where \( \pi : \frac{A}{F} \rightarrow \frac{A}{C} \) is a natural homomorphism.

Let \( M = A \times A \), and \( N = \{(a, b) \in M : (a + F) = b + F \} \). Then \( N \) is a submodule of \( M \). Now there exist elements \( x \in A \) and indecomposable idempotent \( e \in R \) such that \( A = xR \) and \( xe = x \). Fix an element \( y \in A \) such that \( \bar{\eta}(x + F) = y + F \) and \( ye = y \). Set \( z = (x, y) \). Then \( z \in N \) and \( N_1 = zR \) is local. Let \( \pi_1, \pi_2 \) be the associated projections of \( M \) onto the first and second components of \( M \), respectively. Then \( \pi_1(N_1) = A \).

Now, we claim that \( N_1 \) is uniserial. If \( N_1 \) is not uniform, then \( \text{Soc}(N_1) = \text{Soc}(M) \). Therefore \( \text{Soc}(N_1) \) is not square-free, which is a contradiction by Lemma \[\text{8}\]. Thus \( N_1 \) is uniform. It follows that \( N_1 \) embeds in \( A \) under \( \pi_1 \) or \( \pi_2 \). Hence \( N_1 \) is uniserial. As \( \pi_1(N_1) = A \), and \( l(N_1) \leq l(A) \), it follows that \( \pi_1|_{N_1} \) is an isomorphism. Thus given any \( x \in A \), there exists a unique \( y \in A \) such that \( (x, y) \in N_1 \). We get a homomorphism \( \lambda : A \rightarrow A \) such that \( \lambda(x) = y \) if and only if \( (x, y) \in N_1 \). Clearly \( \lambda \) lifts \( \bar{\eta} \) and hence it also lifts \( \sigma \). This proves that \( A \) is quasi-projective.

\[\square\]
Lemma 13. Let \( R \) be a right artinian ring of RAI-type. Let \( A_R \) be any uniserial module. Then the rings of endomorphisms of different composition factors of \( A \) are isomorphic.

Proof. Let \( A \) be a uniserial right \( R \)-module with \( l(A) = 2 \). Let \( C = \text{ann}_e(A) \) and \( \overline{R} = R/C \). As \( A_R \) is quasi-projective, \( A \) is a projective \( \overline{R} \)-module. Thus there exists an indecomposable idempotent \( e \in R \) such that \( A \cong \overline{R}e \). As \( \overline{R}e \) embeds in a finite direct sum of copies of \( A \), there exists an indecomposable idempotent \( f \in R \) such that \( \text{Soc}(A) \cong \frac{\overline{R}f}{fJ(R)} \). Let \( \overline{R} = \frac{\overline{R}}{fJ(R)} \). We get an embedding \( \sigma : \frac{\overline{R}e}{fJ(R)e} \cong \frac{\overline{R}e}{fJ(R)} \) defined as \( \sigma(ere + eJ(R)e) = f\varepsilon f + fJ(R)f \) whenever \( \overline{e} \varepsilon f \overline{f} = \overline{e} \varepsilon f \overline{f} \); \( ere \in eRe, f\varepsilon f \in fRf \). Let \( z = f\varepsilon f \in fRf \). We get an \( \overline{R} \)-homomorphism \( \eta : \overline{J}(R) \to \overline{J}(R) \) such that \( \eta(\overline{e} \varepsilon f \overline{f}) = \overline{e} \varepsilon f \overline{f} \). As \( \overline{R} \) is quasi-injective, there exists an \( \overline{R} \)-homomorphism \( \lambda : \overline{R} \to \overline{R} \) extending \( \eta \). Now \( \lambda(r) = \overline{r}f \) for some \( r \in R \). Then \( \overline{e} \varepsilon f \overline{f} = \lambda(\overline{e} \varepsilon f \overline{f}) = \eta(\overline{e} \varepsilon f \overline{f}) = \overline{e} \varepsilon f \overline{f} \), which gives that \( \sigma \) is onto. Hence \( \frac{\overline{R}e}{fJ(R)e} \cong \frac{\overline{R}f}{fJ(R)} \). Thus the result holds whenever \( l(A) = 2 \). If \( l(A) = n > 2 \), the result follows by induction on \( n \). \( \square \)

Lemma 14. Let \( R \) be a right artinian ring of RAI-type. Then we have the following.

(i) Let \( D \) be a division ring and \( x \in R \). Let \( xR \) be a local module such that for any simple submodule \( S \) of \( \text{Soc}(xR) \), \( D = \text{End}(S) \). Then \( \text{End}(xR/xJ(R)) \cong D \).

(ii) Let \( xR \) be a local module and \( D = \text{End}(xR/xJ(R)) \) where \( x \in R \). Then \( \text{End}(S) \cong D \) for every composition factor \( S \) of \( xR \).

(iii) Let \( xR, yR \) be two local modules where \( x, y \in R \). If \( \text{End}(xR/xJ(R)) \not\cong \text{End}(yR/yJ(R)) \), then \( \text{Hom}(xR, yR) = 0 \).

Proof. (i) There exists an \( n \geq 1 \) such that \( xJ(R)^n = 0 \) but \( xJ(R)^n-1 \neq 0 \). If \( n = 1 \), then \( xR \) is simple, so the result holds. We apply induction on \( n \). Suppose \( n > 1 \) and assume that the result holds for \( n - 1 \). Now \( xJ(R)(J(R))^{n-1} = 0 \), but \( xJ(R)(J(R))^{n-2} \neq 0 \). Therefore there exists an element \( y \in xJ(R) \) such that \( yR \) is local and \( yJ(R)^{n-1} = 0 \) but \( yJ(R)^{n-2} \neq 0 \). By the induction hypothesis, \( \text{End}(yR/yJ(R)) \cong D \). In fact, for any simple submodule \( S' \) of \( xJ(R)/xJ(R)^2 \), \( \text{End}(S') \cong D \). Consider the local module \( M = xR/xJ(R)^2 \). Let \( S' \) be a simple submodule of \( M \). Then \( \text{Soc}(M) = S' \oplus B \) for some \( B \subset \text{Soc}(M) \). Then \( \text{End}(S') \cong D \). As \( A = M/B \) is uniserial, \( \text{Soc}(A) \cong S' \) and \( A/AJ(R) \cong xR/xJ(R) \). By Lemma 13, \( \text{End}(xR/xJ(R)) \cong D \).

(ii) Let \( S \) be a simple submodule of \( \text{Soc}(xR) \) and \( B \) be a complement of \( S \) in \( xR \). Then \( xR = xR/B \) is uniform and \( \text{Soc}(xR) \cong S \). By (i), \( \text{End}(S) \cong \text{End}(\overline{R}/J(R)) \cong \text{End}(xR/xJ(R)) \cong D \). Hence \( \text{End}(S) \cong D \) for any simple submodule \( S \) of \( xR \). Let \( S_1 \) be any composition factor of \( xR \). Then there exists a local submodule \( yR \) of \( xR \) such that \( S_1 \cong yR/yJ(R) \). By (i), \( \text{End}(S_1) \cong \text{End}(S) \cong D \), where \( S \) is a simple submodule of \( yR \).

(iii) It is immediate from (ii). \( \square \)
Now, we are ready to give the structure of indecomposable right artinian rings of RAI-type.

**Theorem 15.** Let $R$ be an indecomposable right artinian ring of RAI-type. Then the following hold.

(i) There exists a division ring $D$ such that $\text{End}(S) \cong D$ for any simple right $R$-module $S$. In particular, $R/J(R)$ is a direct sum of matrix rings over $D$.

(ii) If $D \not\cong \mathbb{Z}/2\mathbb{Z}$, then every finitely generated indecomposable right $R$-module is quasi-injective. In this case, $R$ is right serial.

**Proof.** (i) Let $e \in R$ be an indecomposable idempotent and $D = eRe/eJ(R)e$. By above lemma, every composition factor $S$ of $eR$ satisfies $\text{End}(S) \cong D$. Now $R_R = \oplus_{i=1}^n e_iR$ where $e_i$ are orthogonal indecomposable idempotents with $e_1 = e$. Let $A$ be the direct sum of those $e_jR$ for which $e_jR/e_jJ(R)e_j \cong D$. Consider any $e_k$ for which $e_kR/e_kJ(R)e_k \not\cong D$. It follows from Lemma 14(iii) that $Ae_kR = 0 = e_kRA$. Consequently, $A = e_kR$ and we get that $R = A \oplus B$ for some ideal $B$. As $R$ is indecomposable, we get $R = A$. This proves (i).

(ii) Suppose $D \not\cong \mathbb{Z}/2\mathbb{Z}$. It follows from Corollary 9 that every indecomposable right $R$-module is uniform and quasi-injective. In particular, if $e \in R$ is an indecomposable idempotent, then any homomorphic image of $eR$ is uniform, which gives that $eR$ is uniserial. Hence $R$ is right serial.

**Theorem 16.** Let $R$ be a right artinian ring such that $J(R)^2 = 0$. If every finitely generated indecomposable right $R$-module is local, then $R$ satisfies the following conditions.

(a) Every uniform right $R$-module is either simple or is injective with composition length 2.

(b) $R$ is a left serial ring.

(c) For any indecomposable idempotent $e \in R$ either $eJ(R)$ is homogeneous or $l(eJ(R)) \leq 2$.

Conversely, if $R$ satisfies (a), (b), (c) and $l(eJ(R)) \leq 2$, then every finitely generated indecomposable right $R$-module is local.

**Example.** Let $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$ where $F = \frac{Z}{2Z}$.

Then $R$ is a left serial ring. We have already seen that $e_{11}R$ is an indecomposable module which is automorphism-invariant but not quasi-injective. It follows from Theorem 16 that every finitely generated indecomposable right $R$-module is local. Thus the only indecomposable modules which are not simple are the homomorphic images of $e_{11}R$, which are $e_{11}R$, $\frac{e_{11}R}{e_{11}J(R)e_{11}}$, and $\frac{e_{11}R}{e_{11}J(R)e_{11}}$. These are all automorphism-invariant. It follows from Theorem 16 that any finitely generated indecomposable right $R$-module is local. Thus this ring $R$ is an example of a ring where every finitely generated indecomposable right $R$-module is automorphism-invariant. 

$\Box$
Example. Let $F = \mathbb{Z}/2\mathbb{Z}$ and $R = \begin{bmatrix} F & F & F & F \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}$.

This ring $R$ is left serial and $J(R)^2 = 0$. Now $e_{11}J(R) = e_{12}F \oplus e_{13}F \oplus e_{14}F$, a direct sum of non-isomorphic minimal right ideals. It follows from condition (c) in Theorem 16 that there exists a finitely generated indecomposable right $R$-module that is not local. We have $E_1 = E(e_{12}F)$, $E_2 = E(e_{13}F)$, $E_3 = E(e_{14}F)$, each of them has composition length 2. Now $e_{11}R$ has two homomorphic images $A_1 = \frac{e_{11}R}{e_{13}F}$ and $A_2 = \frac{e_{11}R}{e_{14}F}$ such that $\text{Soc}(A_1) \cong e_{12}F \oplus e_{13}F$ and $\text{Soc}(A_2) \cong e_{13}F \oplus e_{14}F$. So we get $B_1 \subseteq E_1 \oplus E_2 \subseteq E_1 \oplus E_2 \oplus E_3$ such that $A_1 \cong B_1$. Similarly, we have $A_2 \cong B_2 \subseteq E_2 \oplus E_3$. Let $E = E_1 \oplus E_2 \oplus E_3$. Its only automorphism is $I_E$. Thus any essential submodule of $E$ is automorphism-invariant. Now $B = B_1 + B_2 \subset E$, so $B$ is automorphism-invariant and $B$ is not local. We prove that $B$ is indecomposable. We have $B_1 \cap B_2 = e_{13}F$. Notice that any submodule of $E_1 \oplus E_2$ that is indecomposable and not uniserial is $B_1$. Suppose a simple submodule $S$ of $B$ is a summand of $B$. But $S \subset B_1$ or $S \subset B_2$, therefore $B_1$ or $B_2$ decomposes, which is a contradiction. As $l(B) = 5$, $B$ has a summand $C_1$ with $l(C_1) = 2$. Then $C_1$ being uniserial, it equals one of $E_i$.

Case 1. $C_1 = E_1$. Then $B = C_1 \oplus C_2$, where $\text{Soc}(C_2) \cong B_2$. As $C_2$ has no uniserial submodule of length 2, the projection of $B_1$ in $C_2$ equals $\text{Soc}(C_2)$, we get $B_1$ is semi-simple, which is a contradiction. Similarly other cases follow. Hence $B$ is indecomposable. $\square$

Now, we proceed to answer the question of Lee and Zhou [14] whether every automorphism-invariant module is pseudo-injective in the affirmative for modules with finite Goldie dimension.

Theorem 17. If $M$ is an automorphism-invariant module with finite Goldie dimension, then $M$ is pseudo-injective.

Proof. Let $N$ be a submodule of $M$. Let $\sigma : N \to M$ be a monomorphism. Then $\sigma$ can be extended to a monomorphism $\sigma' : E(N) \to E(M)$. Now, we may write $E(M) = \sigma'(E(N)) \oplus P = E(N) \oplus Q$ for some submodules $P$ and $Q$ of $E(M)$. Note that $\sigma'(E(N)) \cong E(N)$. Since $M$ has finite Goldie dimension, $E(M)$ has finite Goldie dimension. Thus $E(M)$ is a directly-finite injective module, and hence $E(M)$ satisfies internal cancellation property. Therefore, $P \cong Q$. Thus, there exists an isomorphism $\varphi : Q \to P$. Now consider the mapping $\lambda : E(M) \to E(M)$ defined as $\lambda(x + y) = \sigma'(x) + \varphi(y)$ where $x \in E(N)$ and $y \in Q$. Clearly, $\lambda$ is an automorphism of $E(M)$. Since $M$ is assumed to be automorphism-invariant, we have $\lambda(M) \subseteq M$. Thus $\lambda|_M : M \to M$ extends $\sigma$. This shows that $M$ is pseudo-injective. $\square$

It is well known that if $R$ is a ring such that each cyclic right $R$-module is injective then $R$ is semisimple artinian. For more details on rings characterized by properties of their cyclic modules, the reader is referred to [9]. We would like to understand the structure of rings for which each cyclic module is automorphism-invariant. In [14] it is shown that if every 2-generated right module over a ring $R$ is automorphism-invariant, then $R$ is semisimple artinian.
A ring $R$ is called a right SI ring if every singular right $R$-module is injective [6]. In [8], Huynh, Jain, and López-Permouth proved that a simple ring $R$ is a right SI ring if and only if every cyclic singular right $R$-module is $\pi$-injective. Their proof can be adapted to show that a simple right noetherian ring $R$ is a right SI ring if and only if every cyclic singular right $R$-module is automorphism-invariant.

The following lemma due to Huynh et al [7, Lemma 3.1] is crucial for proving our result. This lemma is, in fact, a generalization of a result of J. T. Stafford given in [2, Theorem 14.1].

**Lemma 18.** ([7]) Let $R$ be a simple right Goldie ring which is not artinian and $M$ a torsion right $R$-module. If $M = aR + bR$, where $bR$ is of finite composition length and $f$ is a non-zero element of $R$ then $M = (a + bxf)R$ for some $x \in R$.

We know that for a prime right Goldie ring $R$, singular right $R$-modules are the same as torsion right $R$-modules. Now, we are ready to prove the following.

**Theorem 19.** Let $R$ be a simple right noetherian ring. Then $R$ is a right SI ring if and only if every cyclic singular right $R$-module is automorphism-invariant.

**Proof.** Let $R$ be a simple right noetherian ring such that every cyclic singular right $R$-module is automorphism-invariant. We wish to show that $R$ is a right SI ring. If $\text{Soc}(R_R) \neq 0$, then as $R$ is a simple ring, $R = \text{Soc}(R_R)$ and hence $R$ is simple artinian.

Now, assume $\text{Soc}(R_R) = 0$. Let $M$ be any artinian right $R$-module. Since any module is homomorphic image of a free module, we may write $M \cong F/K$ where $F$ is a free right $R$-module. We first claim that $K$ is an essential submodule of $F$. Assume to the contrary that $K$ is not essential in $F$. Let $T$ be a complement of $K$ in $F$. Note that $\overline{M} \cong \frac{F}{K} \supseteq \frac{M}{K} \cong T$. Since $M$ is an artinian module, $\text{Soc}(M) \neq 0$ and consequently $\text{Soc}(T) \neq 0$. This yields that $\text{Soc}(F) \neq 0$, a contradiction to the assumption that $\text{Soc}(R_R) = 0$. Therefore, $K$ is an essential submodule of $F$ and hence $M$ is a singular module. Let $C$ be a cyclic submodule of $M$. We have $\text{Soc}(C) \neq 0$. As $R$ is right noetherian and $C$ is a cyclic right $R$-module, $C$ is noetherian. Thus we have $\text{Soc}(C) = \bigoplus_{i=1}^t S_i$ where each $S_i$ is a simple right $R$-module. By the above lemma, it follows that $C \oplus S_1$ is cyclic. By induction, it may be shown that $C \oplus \text{Soc}(C)$ is cyclic. Now as $C \oplus \text{Soc}(C)$ is a cyclic singular right $R$-module, by assumption $C \oplus \text{Soc}(C)$ is automorphism-invariant. Hence $\text{Soc}(C)$ is $C$-injective. Therefore, $\text{Soc}(C)$ splits in $C$ and hence $C = \text{Soc}(C) \subset M$. Thus $M$ is semisimple. This shows that any artinian right $R$-module $M$ is semisimple.

Now, we prove that every singular module over $R$ is semisimple, or equivalently, for each essential right ideal $E$ of $R$, $R/E$ is semisimple. By the above claim, it suffices to show that $R/E$ is artinian. Set $N = R/E$. If $N$ is not artinian, then we get $0 \subset V_1 \subset N$ with $V_1$ not artinian. Now $N$ is torsion, so is $V_1$. Therefore, $Q = N \oplus V_1$ is torsion and hence cyclic by Lemma 18. Thus we can write $xR = N \oplus V_1$ for some $x \in R$. By the assumption, $xR$ is automorphism-invariant. Hence $V_1$ is $N$-injective. So $N = N_1 \oplus V_1$. Repeat the process with $V_1$, so $V_1 = N_2 \oplus V_2$, where $N_2 \neq 0$ and $V_2$ is not artinian. Continuing this process, we get an infinite direct sum of $N_i$ in $N$, which is a contradiction. Thus we conclude that any singular right $R$-module is artinian and consequently semisimple.
Thus $R$ is a right nonsingular ring such that every singular right $R$-module is semisimple. Hence, by [6], $R$ is a right SI ring.

The converse is obvious. \qed

4. Questions

Question 1: Does every automorphism-invariant module satisfy the property C2?

Lee and Zhou [14] have shown that every automorphism-invariant module satisfies the property C3.

Question 2: What is example of an automorphism-invariant module which is not pseudo-injective?

In Theorem 17 above, we have shown that such a module cannot have finite Goldie dimension.

A module $M$ is called skew-injective if for every submodule $N$ of $M$, any endomorphism of $N$ extends to an endomorphism of $M$. In [9] it is asked whether every skew-injective module with essential socle is quasi-injective. We ask the following

Question 3: Is every automorphism-invariant module with essential socle a quasi-injective module?

Call a ring $R$ to be a right a-ring if each right ideal of $R$ is automorphism-invariant.

Question 4: Describe the structure of a right a-ring.

Call a ring $R$ to be a right $\Sigma$-a-ring if each right ideal of $R$ is a finite direct sum of automorphism-invariant right ideals.

Question 5: Describe the structure of a right $\Sigma$-a-ring.

Question 6: Let $R$ be a simple ring such that $R_R$ is automorphism-invariant. Must $R$ be a right self-injective ring?

In fact, this question is open even when $R_R$ is pseudo-injective (see [3]).
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