A supersymmetric Sawada-Kotera equation

Kai Tian and Q. P. Liu*

Department of Mathematics,
China University of Mining and Technology,
Beijing 100083, P.R. China

Abstract

A new supersymmetric equation is proposed for the Sawada-Kotera equation. The integrability of this equation is shown by the existence of Lax representation and infinite conserved quantities and a recursion operator.

PACS: 02.30.Ik; 05.45.Yv

Key Words: integrability; Lax representation; recursion operator; supersymmetry

1 Introduction

The following fifth-order evolution equation

\[ u_t + u_{xxxxx} + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x = 0 \] (1)

is a well-known system in soliton theory. It was proposed by Sawada and Kotera, also by Caudrey, Dodd and Gibbon independently, more than thirty years ago [1][2], so it is referred as Sawada-Kotera (SK) equation or Caudrey-Dodd-Gibbon-Sawada-Kotera equation in literature. Now there are a large number of papers about it and thus its various properties are established. For example, its Bäcklund transformation and Lax representation were given in [3][4], its bi-Hamiltonian structure was worked out by Fuchssteiner and Oevel [5], and a Darboux transformation was derived for this system [6][7], to mention just a few (see also [8][9]).

Soliton equations or integrable systems have supersymmetric analogues. Indeed, many equations such as KdV, KP, and NLS equations were embedded into their supersymmetric counterparts and it turns out that these supersymmetric systems have also remarkable properties. Thus, it is interesting to work out supersymmetric extensions for a given integrable equation.

*Email: qpl@cumtb.edu.cn  Tel: 86 10 62339015
The aim of the Note is to propose a supersymmetric extension for the SK equation. In this regard, we notice that Carstea [10], based on Hirota bilinear approach, presented the following equation

\[ \phi_t + \phi_{xxxx} + \left[ 10(D\phi)\phi_{xx} + 5(D\phi)_{xx}\phi + 15(D\phi)^2\phi \right]_x = 0 \]

where \( \phi = \phi(x,t,\theta) \) is a fermionic super variable depending on usual temporal variable \( t \) and super spatial variables \( x \) and \( \theta \). \( D = \partial_\theta + \theta \partial_x \) is the super derivative. Rewriting the equation in components, it is easy to see that this system does reduce to the SK equation when the fermionic variable is absent. However, apart from the fact that the system can be put into a Hirota’s bilinear form, not much is known for its integrability. We will give an alternative supersymmetric extension for the SK equation and will show the evidence for the integrability of our system.

The paper is organized as follows. In section two, by considering a Lax operator and its factorization, we construct the supersymmetric SK (sSK) equation. In section three, we will show that our sSK equation has an interesting property, namely, it does not have the usual bosonic conserved quantities since those, resulted from the super residues of a fractional power for Lax operator, are trivial. Evermore, there are infinite fermionic conserved quantities. In the section four, we construct a recursion operator for our sSK equation. Last section contains a brief summary of our new findings and presents some interesting open problems.

## 2 Supersymmetric Sawada-Kotera Equation

The main purpose of this section is to construct a supersymmetric analogy for the SK equation. To this end, we will work with the algebra of super-pseudo differential operators on a \((1 \mid 1)\) superspace with coordinates \((x, \theta)\). We start with the following general Lax operator

\[ L = \partial_x^3 + \Psi \partial_x D + U \partial_x + \Phi D + V. \] (2)

By the standard fractional power method [12], we have an integrable hierarchy of equations given by

\[ \frac{\partial L}{\partial t_n} = [(L^\#)_+, L] \] (3)

where we are using the standard notations: \([A, B] = AB - (-1)^{|A||B|}BA\) is the supercommutator and the subscript \( _+ \) means taking the projection to the differential part for a given super-pseudodifferential operator. It is remarked that the system (3) is a kind of even order generalized SKdV hierarchies considered in [11].

In the following, we will consider the particular \( t_5 \) flow. Our interest here is to find a minimal supersymmetric extension for the SK equation, so we have to do reductions for the general Lax operator (2). To this end, we impose

\[ L + L^* = 0 \]
where * means taking formal adjoint. Then we find

\[ \Psi = 0, \quad V = \frac{1}{2}(U_x - (D\Phi)) \]

that is

\[ L = \partial_x^3 + U\partial_x + \Phi D + \frac{1}{2}(U_x - (D\Phi)) \]

a Lax operator with two field variables. In this case, we take

\[ B = 9(\xi_5^4)_+, \quad \xi = \frac{1}{2} (U_x - \Phi) \]

for convenience. Then, the flow of equations, resulted from

\[ \frac{\partial L}{\partial t} = [B, L] \]

reads as

\[ \begin{align*}
U_t + U_{xxxx} + 5 \left( UU_{xx} + \frac{3}{4} U_x^2 + \frac{1}{3} U^3 + \Phi_x (DU) + \frac{1}{2} \Phi (DU_x) + \frac{1}{2} \Phi \Phi_x - \frac{3}{4} (D\Phi)^2 \right)_x &= 0 \quad (4a) \\
\Phi_t + \Phi_{xxxx} + 5 \left( U\Phi_{xx} + \frac{1}{2} U_x \Phi + \frac{1}{2} U \Phi_x + U^2 \Phi + \frac{1}{2} \Phi (DU_x) - \frac{1}{2} (D\Phi)\Phi_x \right)_x &= 0 \quad (4b)
\end{align*} \]

where we identify \( t_5 \) with \( t \) for simplicity.

Remarks:

1. It is interesting to note that the above system has an obvious reduction. Indeed, setting \( \Phi = 0 \), we will have the standard Kaup-Kupershimdt (KK) equation. Therefore, we may consider it as a supersymmetric extension of the KK equation.

2. The coupled system (4a,4b) admits the following simple Hamiltonian structure

\[ \left( \begin{array}{c} U_t \\ \Phi_t \end{array} \right) = \left( \begin{array}{cc} 0 & \partial_x \\ \partial_x & 0 \end{array} \right) \delta \mathcal{H} \]

where the Hamiltonian is given by

\[ \mathcal{H} = \int \left[ \frac{5}{4} \Phi (D\Phi)^2 - (DU_x) (DU_{xx}) - \frac{5}{3} \Phi U^3 - \frac{5}{4}(DU_x) (DU) \Phi \\
+ \frac{5}{4} (DU) U_x (D\Phi) + 5(DU) U (D\Phi_x) + \frac{5}{2} (DU) \Phi_x \Phi \right] \ dx d\theta. \]
At this point, it is not clear how this system (4a, 4b) is related to the SK equation. To find a supersymmetric SK equation from it, we now consider the factorization of the Lax operator in the following way

\[ L = \partial_x^3 + U \partial_x + \Phi \partial + \frac{1}{2} (U_x - (D \Phi)) \]
\[ = (D^3 + W \partial + \Upsilon)(D^3 - D W + \Upsilon), \] (5)

which gives us a Miura-type transformation

\[ U = -2W_x - W^2 + (D \Upsilon), \]
\[ \Phi = -\Upsilon_x - 2\Upsilon W, \]

and the modified system corresponding to this factorization is given by

\[ W_t + W_{xxxx} + 5W_{xxx}W_x - 5W_{xx}W^2_x - 5W^2_{xx} + 10W_x(D \Upsilon_x) \]
\[ - 20W_{xx}W_x W - 5W_{xx}W(D \Upsilon) - 5W_x^3 - 5W^2(D \Upsilon) + 5W_x(D \Upsilon xx) + 5W_x(D \Upsilon)^2 \]
\[ + 5W_x W - 5W_x(D \Upsilon_x) + 10W(D \Upsilon_x)(D \Upsilon) - 10\Upsilon_x \Upsilon W_x + 5(DW_{xx})\Upsilon_x \]
\[ + 5(DW_x)\Upsilon_{xx} + 5(DW_x)\Upsilon_W + 10(DW_x)\Upsilon W^2 - 5(DW)\Upsilon_x W_x + 10(DW)\Upsilon_x W^2 \]
\[ + 10(DW)\Upsilon(D \Upsilon_x) - 5(DW)\Upsilon W_{xx} + 30(DW)\Upsilon W_x W = 0, \]
\[ \Upsilon_t + \Upsilon_{xxxx} + 5\Upsilon_{xxx}(D \Upsilon) - 5\Upsilon_{xx}W^2 + 5\Upsilon_{xx}(D \Upsilon_x) + 5\Upsilon_{xx}W_{xx} - 25\Upsilon_{xx}W_x W \]
\[ + 5\Upsilon_{xx}W(D \Upsilon) + 5\Upsilon_x(D \Upsilon)^2 + 5\Upsilon_x W_{xxx} - 25\Upsilon_x W_{xx} W - 25\Upsilon_x W^2 + 5\Upsilon_x W_x (D \Upsilon) \]
\[ + 10\Upsilon_x W_x W^2 + 5\Upsilon_x W^4 - 10\Upsilon_x W^2(D \Upsilon) + 5\Upsilon_x W(D \Upsilon_x) - 10\Upsilon W_{xx} W - 20\Upsilon W_{xx} W_x \]
\[ + 10\Upsilon W_{xx} W^2 + 30\Upsilon W_x W^2 + 20\Upsilon W_x W^3 - 30\Upsilon W_x W(D \Upsilon) - 10\Upsilon W^2(D \Upsilon_x) \]
\[ - 5(DW_x)\Upsilon_x \Upsilon - 5(DW)\Upsilon_{xx} \Upsilon - 10(DW)\Upsilon_x \Upsilon W = 0. \]

Although this modification does indeed have a complicated form, the remarkable fact is that it allows a simple reduction. What we need to do is simply putting W to zero, namely

\[ W = 0, \quad \Upsilon = \phi. \]

In this case, we have

\[ \phi_t + \phi_{xxxx} + 5\phi_{xxx}(D \phi) + 5\phi_{xx}(D \phi_x) + 5\phi_x(D \phi)^2 = 0 \] (6)

this equation is our supersymmetric SK equation. To see the connection with the original SK equation (1), we let \( \phi = \theta u(x, t) + \xi(x, t) \) and write the equation (6) out in components

\[ u_t + u_{xxxx} + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x - 5\xi_{xxx}\xi_x = 0, \] (7a)
\[ \xi_t + \xi_{xxxx} + 5u\xi_{xxx} + 5u_x\xi_{xx} + 5u^2\xi_x = 0. \] (7b)

It is now obvious that the system reduces to the SK equation when \( \xi = 0 \). Therefore, our system (3) does qualify as a supersymmetric SK equation.
Our system (6) is integrable in the sense that it has a Lax representation. In fact, the factorization (5) implies that the reduced Lax operator has the following appealing form

\[ L = (D^3 + \phi)(D^3 + \phi) \]  

or

\[ L = \partial_x^3 + (D\phi)\partial_x - \phi_x D + (D\phi_x) \]

3 Conserved Quantities

In general, an integrable system has infinite number of conserved quantities. Since the sSK equation has a simple Lax operator (8), it is natural to take advantage of the fractional power method of Gel'fand and Dickey [12] to find conserved quantities. In the present situation, we have to work with the super residue of a pseudodifferential operator.

The obvious choice in this case is to consider the operators \( L^\frac{4}{3} \) and their super residues. Then, we have the

**Proposition 1**

\[ \text{sres}L^\frac{4}{3} \in \text{Im}D \].

where \( \text{sres} \) means taking the super residue of a super pseudodifferential operator.

**Proof:** As observed already in [13], there exists a unique odd operator \( \Lambda = D + O(1) \), whose coefficients are all differential polynomials of \( \phi \), such that

\[ (D^3 + \phi) = \Lambda^3, \]

thus, the Lax operator (8) is written as

\[ L = (D^3 + \phi)(D^3 + \phi) = \Lambda^6. \]

From it we have

\[ \text{sres}L^\frac{4}{3} = \text{sres}\Lambda^{2n} = \frac{1}{2} \text{sres}\{\Lambda^{2n-1}\Lambda + \Lambda\Lambda^{2n-1}\} = \frac{1}{2} \text{sres}[\Lambda^{2k-1}, \Lambda] \in \text{Im}D. \]

This completes the proof.

**Remark:** The triviality of \( L^\frac{4}{3} \) implies that the Lax operator could not generate any Hamiltonian structures for the equation (6).

To find nontrivial conserved quantities, we now turn to \( L^\frac{4}{3} \) rather than \( L^\frac{8}{3} \). It is easy to prove that

\[ \frac{\partial}{\partial t} L^\frac{4}{3} = [9(L^\frac{4}{3})_+, L^\frac{4}{3}] \]

thus

\[ \frac{\partial}{\partial t} L^\frac{8}{3} = [9(L^\frac{8}{3})_+, L^\frac{8}{3}]. \]
Consequently, the super residue of $L$ is conserved.

By direct calculation, we obtain the first two nontrivial conserved quantities

\[
\int \text{res} L \tilde{\phi} dx d\theta = -\frac{1}{9} \int [2(D\phi_{xx}) + (D\phi)^2 - 6\phi_x\phi] dx d\theta
\]

\[
\int \text{res} L \hat{\phi} dx d\theta = -\frac{1}{81} \int [6(D\phi_{xxx}) + 18(D\phi_{xx})(D\phi) + 9(D\phi_x)^2 \\
+ 4(D\phi)^3 - 18\phi_{xxx}\phi + 6\phi_{xx}\phi_x - 36\phi_x(\phi\phi)] dx d\theta
\]

Remarks:

1. What is remarkable is that the conserved quantities found in this way, unlike the supersymmetric KdV case [15][14], are local.

2. All those conserved quantities are fermionic. To our knowledge, this is the first supersymmetric integrable system whose only conserved quantities are fermionic.

4 Recursion operator

An integrable system often appears as a particular flow of hierarchy equations and an important ingredient in this aspect is the existence of recursion operators. In this section, we deduce the recursion operator for the sSK equation (6) following the method proposed in [16]. We first notice that the sSK hierarchy can be written as

\[
\frac{\partial}{\partial t_n} L = [(L, L)]
\]

where $L$ is given by (8). It is easy to see that the flow equations are nontrivial only if $n$ is an integer satisfying

\[
n \neq 0 \mod 3 \quad \text{and} \quad n = 1 \mod 2.
\]

Therefore the next flow which is achieved by applying recursion operator to (9) should be

\[
\frac{\partial}{\partial t_{n+6}} L = [(L, L)]
\]

But

\[
[(L, L)] = [(L^2(L, L) + L^2(L, L)], L]
\]

\[
= L^2 [(L, L)] + [(L^2(L, L)], L]
\]

\[
= L^2 [(L, L)] + [R_n, L]
\]

\[
= L^2 \frac{\partial}{\partial t_n} L + [R_n, L]
\]
where
\[ R_n = \left( L^2 \left( \frac{\partial}{\partial t} \right)^n \right)_+ \]
is a differential operator of \( O(\partial^n \phi D) \), that is,
\[ R_n = (\alpha \partial_x^5 + \beta \partial_x^4 + \gamma \partial_x^3 + \delta \partial_x^2 + \xi \partial_x + \eta)D \]
\[ + a \partial_x^5 + b \partial_x^4 + c \partial_x^3 + d \partial_x^2 + e \partial_x + f. \]

Therefore,
\[ \frac{\partial}{\partial t_{n+6}} L = L^2 \frac{\partial}{\partial t_n} L + [R_n, L]. \] (12)

Next we may determine the coefficients in \( R_n \). Using (12), we obtain
\[ a = \frac{1}{3}(D^{-1} \phi_n), \quad b = 2(D \phi_n), \]
\[ c = \frac{44}{9}(D \phi_{n,x}) + \frac{5}{3}(D \phi)(D^{-1} \phi_n) + \frac{4}{9}(\partial^{-1}_x \phi \phi_n), \]
\[ d = \frac{55}{9}(D \phi_{n,xx}) + \frac{19}{9}(D \phi)(D \phi_n) + \frac{5}{9} \phi \phi_n + \frac{10}{9}(D \phi_x)(D^{-1} \phi_n) \]
\[ e = \frac{1}{27} \left\{ 106(D \phi_{n,xxx}) + 74(D \phi)(D \phi_{n,x}) - 14 \phi \phi_{n,x} + 79(D \phi_x)(D \phi_n) \right. \]
\[ + 27 \phi \phi_{n,x} + [23(D \phi_{xxx}) + 4(D \phi)^2](D^{-1} \phi_n) + 16(D \phi)(\partial^{-1}_x \phi \phi_n) \]
\[ + 2D^{-1}[(\phi_{xxx} + \phi_x(D \phi))(D^{-1} \phi_n) - 3(D \phi)(D^{-1} \phi \phi_n) \]
\[ - 2 \phi_x(\partial^{-1}_x \phi \phi_n) + 2D^{-1}(\phi_{xxx} \phi_n + 2 \phi_x(D \phi \phi_n)) \right\}, \]
\[ f = \frac{1}{27} \left\{ 28(D \phi_{n,xxxx}) + 32(D \phi)(D \phi_{n,xx}) - 20 \phi \phi_{n,xx} + 54(D \phi_x)(D \phi_n) \right. \]
\[ + 16 \phi \phi_{n,xx} + [30(D \phi_{xxx}) + 4(D \phi)^2](D \phi_n) + [8 \phi_{xxx} + 4 \phi_x(D \phi)] \phi_n \]
\[ + [10(D \phi_{xxx}) + 10(D \phi_x)(D \phi)](D^{-1} \phi_n) - 8 \phi_x(D^{-1} \phi \phi_n) \]
\[ + 12(D \phi_x)(\partial^{-1}_x \phi \phi_n) \right\} \]
\[ \alpha = 0, \quad \beta = -\frac{1}{3} \phi_n, \quad \gamma = \frac{5}{3} \phi_{n,x}, \]
\[ \delta = -\frac{1}{9} \left\{ 29 \phi_{n,xx} + 5 \phi(D \phi) + 5 \phi_x(D^{-1} \phi_n) - 2(D^{-1} \phi \phi_n) \right\}, \]
\[ \xi = -\frac{1}{9} \left\{ 26 \phi_{n,xxx} + 16 \phi_x(D \phi_n) + 3 \phi_n(D \phi_x) + 14 \phi_{n,xx}(D \phi) + 5 \phi_{xx}(D^{-1} \phi_n) \right\}, \]
\[ \eta = -\frac{1}{27} \left\{ 28 \phi_{n,xxxx} + 32(D \phi)(D \phi_{n,xx}) + 28 \phi_x(D \phi_{n,xx}) + 26(D \phi_x)(D \phi_n) \right. \]
\[ + 28 \phi_{xxx}(D \phi_n) + [2(D \phi_{xxx}) + 4(D \phi)^2] \phi_n + [10 \phi_{xxx} + 10 \phi_x(D \phi)](D^{-1} \phi_n) \]
\[ - 2(D \phi)(D^{-1} \phi_x \phi_n) + 12 \phi_x(\partial^{-1}_x \phi \phi_n) - 2 \phi_x(D \phi \phi_n) \right\} \]
Finally, we have the recursion operator

\[ \mathcal{R} = \partial_x^6 + 6(D\phi)\partial_x^4 + 9(D\phi_x)\partial_x^3 + 6\phi_{xxx}\partial_x^2 D + \{5(D\phi_{xx}) + 9(D\phi)^2\}\partial_x^2 \\
+ \{9\phi_{xxxx} + 12\phi_x(D\phi)\}\partial_x D + \{D\phi_{xxxx} + 9(D\phi_x)(D\Phi)\}\partial_x \\
+ \{5\phi_{xxxx} + 12\phi_{xx}(D\phi) + 6\phi_x(D\phi_x)\}D + \{4(D\phi_{xx})(D\phi) + 4(D\phi)^3 - 3\phi_{xx}\phi_x\} \\
+ \{\phi_{xxxxx} + 5\phi_{xxx}(D\phi) + 5\phi_{xx}(D\phi_x) + 2\phi_x(D\phi_{xx}) + 6\phi_x(D\phi)^2\}D^{-1} \\
- \{2(D\phi_{xx}) + 2(D\phi)^2\}D^{-1}\phi_x - 4\phi_x(D\phi)\partial_x^{-1}\phi_x - 2(D\phi)D^{-1}[\phi_{xxx} + 2\phi_x(D\phi)] \\
- 2\phi_xD^{-1}\{\phi_{xxx} + \phi_x(D\phi)D^{-1} - 3(D\phi)D^{-1}\phi_x - 2\phi_x\partial_x^{-1}\phi_x + 2D^{-1}[\phi_{xxx} + 2\phi_x(D\phi)]\} \]

**Remark:** When calculating the coefficients of \( R_n \), one should solve a system of differential equations. Due to nonlocality (those underlined terms), there is certain ambiguity and to avoid it, we used the \( t_7 \)-flow

\[ \phi_{t_7} = \phi_{xxxxxxx} + 7\phi_{xxxxx}(D\phi) + 14\phi_{xxx}(D\phi_x) + 14\phi_{xx}(D\phi_{xx}) \\
+ 14\phi_{xxx}(D\phi)^2 + 7\phi_{xx}(D\phi_{xxx}) + 28\phi_{xx}(D\phi_x)(D\phi) \\
+ 14\phi_x(D\phi_{xx})(D\phi) + 7\phi_x(D\phi_x)^2 + \frac{28}{3}\phi_x(D\phi)^3. \]

5 Conclusion

Summarizing, we find a supersymmetric SK equation which has Lax representation. We also obtain infinite conserved quantities and a recursion operator for this new proposed system. These imply that the system is integrable. It is interesting to establish other properties for it, such as Bäcklund transformation, Hirota bilinear form, etc..

**Acknowledgements** The calculations were done with the assistance of SUSY2 package of Popowicz [17]. We would like to thank him for helpful discussion about his package. The comments of anonymous referee has been very useful. The work is supported in part by National Natural Science Foundation of China under the grant numbers 10671206 and 10731080.

**References**

[1] K. Sawada and T. Kotera, *Prog. Theor. Phys.* **51** (1974) 1355.

[2] P. J. Caudrey, R. K. Dodd and J. D. Gibbon, *Proc. R. Soc. London A* **351** (1976) 407.

[3] J. Satsuma and D. J. Kaup, *J. Phys. Soc. Japan* **43** (1977) 692.

[4] R. K. Dodd and J. D. Gibbon, *Proc. R. Soc. London A* **358** (1977) 287.

[5] B. Fuchssteiner and W. Oevel, *J. Math. Phys.* **23** (1982) 358.

[6] R. N. Aiyer, B. Fuchssteiner and W. Oevel, *J. Phys. A* **19** (1986) 3755.
[7] D. Levi and O. Ragnisco, *Inverse Problems* **4** (1988) 815.

[8] A. P. Fordy and J. Gibbons, *Phys. Lett. A* **75** (1980) 325.

[9] R. Hirota, *J. Phys. Soc. Japan* **58** (1989) 2285.

[10] A. S. Carstea, *Nonlinearity* **13** (2000) 1645.

[11] J. M. Figueroa-O’Farrill, E. Ramos and J. Mas, *Rev. Math. Phys.* **3** (1991) 479.

[12] L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, 2nd Edition, (World Scientific, Singapore(2003)).

[13] Yu I Manin and A. O. Radul, *Commun. Math. Phys.* **98** (1985) 65.

[14] P. H. M. Kersten, *Phys. Letts, A* **134** (1988) 25.

[15] P. Labelle and P. Mathieu, *J. Math. Phys.* **32** (1991) 923.

[16] M. Gürses, A. Karasu and V. Sokolov, *J. Math. Phys.* **40** (1999) 6473.

[17] Z. Popowicz, *Compt. Phys. Commun.* **100** (1997) 277.