A PARTITION IDENTITY CONNECTED TO THE SECOND RANK MOMENT

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Abstract. The purpose of this note is to offer some partition implications of a $q$-series that is connected to the second Atkin-Garvan moment. Inequalities and relationships among the number of divisors and partitions are provided as consequences.

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1. Introduction and main theorems

We use the familiar basic-hypergeometric notation [6] $(a)_n = (a; q)_n := \prod_{0 \leq k \leq n-1} (1 - aq^k)$ to display our $q$-series in compact form. It is now well-known in the literature [2, 3, 10] that

$$\sum_{n \geq 0} \left( \frac{1}{(q)_{\infty}} - \frac{1}{(q)_n} \right) = \frac{1}{(q)_{\infty}} \sum_{n \geq 1} \frac{q^n}{1 - q^n},$$

which is sometimes noted as a simple example of a “sum of tails” $q$-series. It also has been noted in [10] as equivalent to the result found in [12], that $d(n)$ is equal to the sum of smallest parts of partitions of $n$ into an odd number of distinct parts minus those with an even number of distinct parts. Here we put $\sigma_i(n) = \sum_{d|n} d^i$, $d(n) = \sigma_0(n)$, and $p(n)$ shall denote the number of unrestricted partitions of $n$.

Theorem 1.1. Let $p(m, n)$ be the number of partitions of $n$ with $m$ parts, and put $p_2(n) = \sum_m m^2 p(m, n)$. Let $N_2(n)$ be the second Atkin-Garvan moment [5]. Let $f(n, N)$ equal $p(n)$ minus the number of partitions of $n$ where parts are $\leq N$, and $g(n, N)$ the number of divisors $\leq N$ of $n$. Put $S(n) = \sum_{k,N \geq 1} f(k, N) g(n-k, N)$. Then,

$$\sum_{n \geq 1} S(n) q^n = \sum_{n \geq 1} \left( \frac{1}{(q)_{\infty}} - \frac{1}{(q)_n} \right) \sum_{i \geq 1} \frac{q^i}{1 - q^i} = \sum_{n \geq 1} (p_2(n) - np(n) - \frac{1}{2} N_2(n)) q^n.$$
We mention that the appearance of the finite sum \(\sum_{i\geq 1} q^i (1 - q^i)^{-1}\) suggests a connection with (1.1) through the identity of Van Hamme [7]. The non-negativity of \(S(n)\) implies our next result, which includes \(M_2(n)\) [5], the second crank moment function.

**Theorem 1.2.** For each non-negative integer \(n\), \(M_2(n) + N_2(n) \leq 2p_2(n)\).

In general it is known through Andrew’s [4] \(\text{spt}(n) = \frac{1}{2}(M_2(n) - N_2(n))\), that \(M_2(n) > N_2(n)\). Hence a natural consequence of this theorem is that \(N_2(n) \leq p_2(n)\), since \(2N_2(n) \leq M_2(n) + N_2(n)\).

To provide a further partition theorem connected to our Theorem 1.1 we will need the following definition.

**Definition 1.** We define \(P_{i,j}(n_1, n_2, n)\) to be a partition of \(n\) into parts \(\leq n_1 + n_2\) wherein each part \(\leq n_1 + n_2\) appears at least once as a part, and \(n_1\) appears at least \((1 + i)\) times and \(n_2\) appears at least \((1 + j)\) times.

The generating function is

\[
(1.2) \quad \frac{q}{1 - q_1 - q_2} \cdot \frac{q^{(1+i)n_1}}{1 - q^{n_1}} \cdot \frac{q^{(i+j)n_2}}{1 - q^{n_2}} \cdot \frac{q^{n_1+n_2}}{1 - q^{n_1+n_2}}.
\]

and so we see that \(\sum_{i,j,n_1,n_2,n\geq 1} (-1)^{n_1+n_2} P_{i,j}(n_1, n_2, n) q^n\) is our \(q\)-series on the left hand side of (2.5).

**Theorem 1.3.** Let \(D_o(n, N)\) (resp. \(D_e(n, N)\)) denote the number of partitions of \(n\) into an odd (resp. even) number of distinct parts, each \(\geq N + 1\). We have,

\[
\sum_{i,j,n_1,n_2\geq 1} (-1)^{n_1+n_2} P_{i,j}(n_1, n_2, n) = -\sum_{k,N\geq 1} (D_o(k, N) - D_e(k, N)) \eta(n - k, N).
\]

We mention this may be viewed as a two-dimensional analogue of the identity [6, 10]

\[
\sum_{n\geq 1} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q)_n (1 - q^n)} = \sum_{n\geq 0} ((q^{n+1})_\infty - 1),
\]

which, in our notation is also

\[
\sum_{i,n_1,n\geq 1} (-1)^{n_1} P_{i,0}(n_1, 0, n) q^n.
\]

The appearance of the square of the generating function for \(d(n)\) creates a bit more of a challenge to work with than some similar identities related to \(N_2(n)\), such
as the smallest part identity [4]. We were, however, able to apply an interesting
formula due to B. Kim [9, eq.(1.6)], \( n \geq 2 \),
\[
\sum_{k}^{{n-1}} d(k)d(n-k) = \sigma_1(n) - \sigma_0(n) + 2b(2,n),
\]
where \( b(m,n) \) denotes the number of partitions of \( n \) into \( m \) different parts. Therefore, we have from Theorem 1.1 and (1.3) that
\[
S(n) = \sum_{k} p(n-k)(b(2,k) - d(k)) - \frac{1}{2}N_2(n),
\]
where we apply the known [6] identity of Euler \( np(n) = \sum \sigma_1(k)p(n-k) \). Hence
\( S(n) \equiv - \sum p(n-k)d(k) - \frac{1}{2}N_2(n) \pmod{2} \). Another simple inequality may be
obtained from (1.4) with the observation that \( p(m,n) \geq b(m,n) \).

2. Proof of Theorems

To prove our theorems we will use some familiar tools that have appeared in previous
studies [1, 6, 8, 11].

Proof of Theorem [1.1] From [1], we have that for a 2-fold Bailey pair \((\alpha_{n_1,n_2}, \beta_{n_1,n_2})\)
relative to \(a_i \), \( i = 1, 2 \),
\[
\beta_{n_1,n_2} = \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} \frac{\alpha_{r_1,r_2}}{(a_1q;q)_{n_1+r_1}(q;q)_{n_1-r_1}(a_2q;q)_{n_2+r_2}(q;q)_{n_2-r_2}}.
\]
The needed general formula is given by
\[
\sum_{n_1 \geq 0} \sum_{n_2 \geq 0} (x)_{n_1}(y)_{n_1}(z)_{n_2}(w)_{n_2} (a_1q/xy)^{n_1}(a_2q/zw)^{n_2} \beta_{n_1,n_2}
\]
\[
= \frac{(a_1q/x)_{\infty}(a_1q/y)_{\infty}(a_2q/z)_{\infty}(a_2q/w)_{\infty}}{(a_1q)_{\infty}(a_1q/xy)_{\infty}(a_2q)_{\infty}(a_2q/zw)_{\infty}} \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} (x)_{n_1}(y)_{n_1}(z)_{n_2}(w)_{n_2} (a_1q/xy)^{n_1}(a_2q/zw)^{n_2} \alpha_{n_1,n_2}.
\]
Using the Joshi and Vyas [8] 2-fold Bailey pair with relative to \(a_j = 1 \), \( j = 1, 2 \),
where \( \alpha_{0,0} = 1 \),
\[
\alpha_{n_1,n_2} = \begin{cases} (-1)^n q^{n(n-1)/2}(1 + q^n), & \text{if } n_1 = n_2 = n, \\ 0, & \text{otherwise}, \end{cases}
\]
and
(2.4) \[ \beta_{n_1, n_2} = \frac{q^{n_1 n_2}}{(q)_{n_1} (q)_{n_2} (q)_{n_1 + n_2}}. \]

with (2.2) and differentiating (2.2) with respect to \( x \), setting \( x = 1, y \rightarrow \infty \), differentiating with respect to \( z \), setting \( z = 1, w \rightarrow \infty \),

(2.5) \[ \sum_{n_1, n_2 \geq 1} (-1)^{n_1 + n_2} q^{\left(\frac{n_1 + n_2 + 1}{2}\right)} \frac{(q)_{n_1 + n_2} (1 - q^{n_1}) (1 - q^{n_2})}{(q)_{n_1 + n_2}} = \left( \sum_{n \geq 1} \frac{q^n}{1 - q^n} \right)^2 + \sum_{n \geq 1} (-1)^n (1 + q^n)q^{n(3n+1)/2} (1 - q^n)^2. \]

By Fine’s identity [6, pg.13, eq.(12.2), \( a = 1, t = q^n \)], we have

(2.6) \[ \frac{1 - q^n}{(q)_m} \sum_{n \geq 0} \frac{(q)_n}{(bq)_n} q^{nm} = \sum_{n \geq 0} \frac{(b)_n (-1)^n q^{n(n+1)/2 + nm}}{(bq)_n (q)_{n+m}}. \]

Differentiating with respect to \( b \) and setting \( b = 1 \), dividing by \( 1 - q^m \) and inverting the desired series we have that the left side of (2.5) is

\[
\sum_{n, m \geq 1} \frac{(-1)^{m-1}q^{nm+m(m+1)/2}}{(q)_m} \sum_{i \geq 1} \frac{q^i}{1 - q^i} = - \sum_{n \geq 1} ((q^{n+1})_\infty - 1) \sum_{i \geq 1} \frac{q^i}{1 - q^i}.
\]

Inserting this into (2.5) and dividing both sides by \( (q)_\infty \) gives our main identity once we note that

\[ \sum_{n \geq 1} p_2(n) q^n = \frac{1}{(q)_\infty} \left( \sum_{n \geq 1} \frac{q^n}{1 - q^n} \right)^2 + \sum_{n \geq 1} np(n) q^n. \]

Now interpreting our \( q \)-series may be done as follows.

\[ \frac{1}{(q)_\infty} - \frac{1}{(q)_N} \]

is the generating function for \( f(n, N) \), \( p(n) \) minus the number of partitions of \( n \) where parts are \( \leq N \). The sum \( \sum_{1 \leq i \leq N} q^i (1 - q^i)^{-1} \) is the generating function for \( g(n, N) \), the number of divisors \( \leq N \) of \( n \). Hence we may write

\[ \left( \frac{1}{(q)_\infty} - \frac{1}{(q)_N} \right) \sum_{1 \leq i \leq N} q^i (1 - q^i)^{-1} = \sum_{n \geq 1} \left( \sum_{k} f(k, N) g(n - k, N) \right) q^n, \]

and therefore \( \sum_{k, N \geq 1} f(k, N) g(n - k, N) \) is the coefficient of \( q^n \) on the left side of Theorem 1. \( \square \)

**Proof of Theorem 1.2** Since it is clear from the generating function that \( S(n) \geq 0 \), we have \( p_2(n) - np(n) - \frac{1}{2} N_2(n) \geq 0 \), and with \( 2np(n) = M_2(n) \) [5] we see the result follows. \( \square \)
Proof of Theorem 1.3. From Definition 1, we paraphrase the identity
\[
\sum_{n_1, n_2 \geq 1} \frac{(-1)^{n_1+n_2} q^{\left(\frac{n_1+n_2+1}{2}\right)}}{(q)_{n_1+n_2}(1-q^{n_1})(1-q^{n_2})} = -\sum_{n \geq 1} \left((q^{n+1})_\infty - 1\right) \sum_{i \geq 1} \frac{q^i}{1-q^i},
\]
and apply Definition 1.

3. Concluding remark

Here we were able to obtain an interesting sum of tails identity without the use of the methods in [2, 3]. It would appear that one should be able to extend the ideas found in [3], particularly [3, Theorem 4.1, t = 0], to obtain our Theorem 1.1. It would also be of interest to see if S(n) has similar divisibility properties as Andrews spt(n) function [4]. Lastly, a natural question would be if Theorem 1.2 holds for kth moments as well.

References

[1] G. E. Andrews, Umbral Calculus, Bailey Chains, and Pentagonal Number Theorems, J. Comb. Theory, Ser. A 91 (1-2): 464–475 (2000)
[2] G. E. Andrews, J. Jimenez-Urroz, K. Ono, q-series identities and values of certain L-functions, Duke Math. J. 108 (2001), no. 3, 395–419.
[3] G. E. Andrews, and P. Freitas, Extension of Abel’s lemma with q-series implications, Ramanujan J. 10 (2005), 137–152.
[4] G. E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008), 133–142.
[5] A.O.L. Atkin and F.G. Garvan, Relations between the ranks and cranks of partitions, (Rankin memorial issues) Ramanujan J. 7 (2003), no. 1-3, 343–366.
[6] N. J. Fine, Basic Hypergeometric Series and Applications, Math. Surveys 27, AMS Providence, 1988.
[7] L. Van Hamme, Advanced problem 6407, Amer. Math. Monthly 40 (1982), 703–704.
[8] C.M. Joshi and Y. Vyas, Bailey Type Transforms and Applications, http://arxiv.org/abs/math/0701192
[9] B. Kim, On the number of partitions of n into k different parts, J. Number Theory, Volume 132, Issue 6, 1306–1313, 2012.
[10] A. E. Patkowski, Divisors, partitions and some new q-series identities, Colloq. Math. 117 No. 2 (2009), 289–294.
[11] A. E. Patkowski, An interesting q-series related to the 4-th symmetrized rank function, Discrete Mathematics, Volume 341, Issue 11, 2018, pages 2965–2968.
[12] Z. B. Wang, R. Fokkink, and W. Fokkink, A relation between partitions and the number of divisors, Amer. Math. Monthly 102 (1995), 345–347.
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