Modeling an efficient Brownian heat engine

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We discuss the effect of subdividing the ratchet potential on the performance of a tiny Brownian heat engine that is modeled as a Brownian particle hopping in a viscous medium in a sawtooth potential (with or without load) assisted by alternately placed hot and cold heat baths along its path. We show that the velocity, the efficiency and the coefficient of performance of the refrigerator maximize when the sawtooth potential is subdivided into series of smaller connected barrier series. When the engine operates quasistatically, we analytically show that the efficiency of the engine can approach and study the effect of subdividing the ratchet potential on the performance of the engine. We show that the velocity, the efficiency and the coefficient of performance of the refrigerator tend to increase as the ratchet potential is subdivided into series of barriers.

Unlike macroscopic heat engines, the Carnot efficiency is unattainable for Brownian heat engines when the engines work quasistatically because of the irreversible heat flow via the kinetic energy [3, 9]. In this work, we obtain a simple analytic expression for the efficiency at quasistatic limit. The analytic result reveals that the efficiency of the engine never goes to the Carnot efficiency at quasistatic limit. Another important but unexplored issue is the influence of the heat flow via the kinetic energy on the coefficient of performance of the refrigerator. In the present work, we analytically show that the coefficient of performance of the refrigerator is always less than the Carnot refrigerator when the engine operates quasistatically.

The paper is organized as follows: In section II, we present the model. In section III, we study the dependence of the efficiency and the velocity on the model parameters in the absence of external force. We show that the velocity and the efficiency attain optimum values at a particular value of barrier subdivision $N$. At quasistatic limit, the efficiency never goes to Carnot efficiency for any $N$ as the heat transfer via the kinetic energy is irreversible. In section IV, we consider the model in the presence of external load. We find that the velocity, the efficiency and the coefficient of performance of the refrigeration...
II. THE MODEL

Consider a Brownian particle which walks in a viscous medium in a periodic sawtooth potential whose potential profile (see Fig. 1) is described by

\[ U(x) = \begin{cases} 
U_0\left[\frac{x}{L_0} + 1\right], & \text{if } -L_0 < x \leq 0; \\
U_0\left[\frac{x}{L_0} + 1\right], & \text{if } 0 < x \leq L_0; 
\end{cases} \tag{1} \]

where \( U_0 \) and \( L_0 \) denote the barrier height and the width of the sawtooth potential, respectively. The viscous medium is alternatively in contact with the hot \( T_h \) and the cold \( T_c \) reservoirs along the reaction coordinate as shown in Fig. 1. Using the same theoretical frame work [16], the sawtooth potential is subdivided into series of smaller connected barrier series. For example, Fig. 2 shows the left and the right sides of the sawtooth potential shown in Fig. 1 are subdivided into three small steps \( N = 3 \). For single barrier step between \( x_2 \) and \( x_4 \), for simplicity, we choose \( U_1 = 2U_2 \) and \( a = 2b \) where \( x_4 - x_2 = a + b \). In general, for \( N \) equally spaced intervals, from the top of the barrier to either side, \( L_0 \) and \( U_0 \) are given by \( L_0 = Na + (N-1)b \) and \( U_0 = NU_1 - (N-1)U_2 \). Such parameterization is physically reasonable as the barrier height, the barrier width and the area under the barrier remain approximately constant as \( N \) is varied [16, 10].

The Brownian particle attains a directional motion when it is exposed to the potential coupled with spatially variable temperature. For such a system, the general expression for the steady state current \( J \) for the Brownian particle in any periodic potential with or without load is reported in the work [14]. The closed form expression for the steady state current \( J \) (please refer Appendix A, ref. [14]) is given by

\[ J = \frac{-F}{G_1G_2 + HF}. \tag{2} \]

The drift velocity \( v \) of the particle is associated to the steady state current \( J \) and it is given by \( v = 2JL_0 \).

The hot reservoir is the ultimate source of energy for the engine. When the engine works as a heat engine, the net flux of the particle is from hot to the cold heat baths. Hence when the particle moves from the hot to the cold heat baths, for any \( N \), the particle takes \( U_0 + v\gamma L_0 + fL_0 \) amount of energy from the hot reservoir to surmount the potential of magnitude \( U_0 \) and to overcome the viscous drag force \( v\gamma \) as well as the external force of amount \( f \). On the other hand, \( 1/2k_BT_h(T_h - T_c) \) amount of energy is transferred from the hot to the cold heat baths via kinetic energy \( \frac{1}{2}k_BT_h \) when the particle walks from the hot to the cold heat baths. Hence the Brownian particle takes an amount of heat from the hot reservoir. The heat flow to the cold reservoir is given by

\[ Q_h = (U_0 + fL_0 + \gamma vL_0) + \frac{1}{2}k_BT_h(T_h - T_c) \tag{3} \]

When the engine acts as refrigerator, the net flow of the particle is from the cold to the hot reservoirs. Note that, due to the particle recrossing between the hot and the cold reservoirs, heat is leaking from the hot to the cold reservoir of magnitude \( 1/2k_BT_h(T_h - T_c) \). This is in opposite direction to the heat being taken out of the cold reservoir. Hence, this quantity contributes as negative to \( Q_c \). Thus, the net heat flow out of the cold reservoir is given by \( Q_c = U_0 - L_0(f + \gamma v) + \frac{1}{2}k_BT_h(T_h - T_c) \).

Not all motors are designed to pull loads and alternative proposals for efficiency depend on the task each...
motor performs. Some motors may have to achieve high velocity against a frictional drag. This basically implies that the objective of the motor is to move a certain distance in a given time interval. For such motors \((f = 0)\), the useful work is the difference between \(Q_h\) and \(Q_c\): \(W = Q_h - Q_c = 2\gamma v L_0\). For motors designed to pull loads \(f \neq 0\), the useful work is given by \(W = 2f L_0\). The efficiency \(\eta\) and the coefficient of performance \((\text{COP})\) of the refrigerator \(P_{\text{ref}}\) of the engine is given by \(\eta = W/Q_h\) and \(P_{\text{ref}} = Q_c/W\).

The purpose of this theoretical work, for given parameter values of \(U_0\) and \(L_0\), to find the velocity, \(v\), the efficiency, \(\eta\), and the coefficient of performance of the refrigerator, \(\text{COP}\), for various values of barrier subdivision, \(N\). Next, the energetics of the Brownian heat engine will be explored as a function of model parameters both in the absence and in the presence of external force.

III. THE EFFICIENCY AND THE VELOCITY IN THE ABSENCE OF EXTERNAL FORCE

In the absence of the external force \(f = 0\), the analytically obtained steady state current for \(N = 1\) is given by

\[
J = \frac{1}{2\gamma(T_h + T_c)} \frac{U_0}{L_0} \left( \frac{1}{e^{\frac{U_0}{k_B T_h}} - 1} - \frac{1}{e^{\frac{U_0}{k_B T_c}} - 1} \right)
\]

where \(\gamma\) denotes the coefficient of friction of the Brownian particle. The magnitude of the coefficient of friction of the Brownian particle \(\gamma\) depends on temperature of the viscous medium. As approximation, it is considered to be a constant. In Appendixes A and B, the expressions for \(F, G_1, G_2,\) and \(H\) are given for \(N = 2\) and \(N = 3\), respectively. For \(N \geq 4\), the expressions for \(F, G_1, G_2,\) and \(H\) are lengthy and will not be presented in this work.

When one omits the heat exchange via kinetic energy (neglecting the term \(1/2k_B(T_h - T_c)\)) in (3) and (4)), the efficiency goes to Carnot efficiency \(\eta_c\) at quasistatic limit. The quasistatic limit of the engine is obtained when \(U_0\) goes to zero. For any \(N\), we explore the efficiency at quasistatic limit and find that \(\lim_{U_0 \to 0} \eta = (T_h - T_c)/T_h\), which is exactly equal to the efficiency of the Carnot engine.

The heat flow via the kinetic energy significantly affects the efficiency of the Brownian heat engines. Next, considering the heat flow via the potential and the kinetic energies, we explore the thermodynamic properties of the engine. Let us introduce dimensionless parameters before exploring the dependence of the velocity and the efficiency on different values of barrier subdivision \(N\). We introduce scaled parameters: scaled length \(\ell = 1\), scaled barrier height \(u = U_0/k_B T_c\), scaled current \(j = J/J_0\) where \(J_0 = k_B T_c/\gamma L_0^2\), scaled velocity \(v = 2j/\ell\) and scaled temperature \(\tau = \frac{k_B T}{k_B T_h} - 1\). Here \(k_B\) denotes Boltzmann’s constant. For simplicity, it is considered to be unity. We also introduce dimensionless parameters \(\alpha_i = \eta_i/\eta_1\) \((i = 2, 3, \ldots N)\) where \(\eta_i\) and \(\eta_1\) are the efficiencies when \(N = i\) and \(N = 1\), respectively.

The dependence of the steady state current or equivalently the drift velocity on the potential \(u\) for parameter values of \(\tau = 2\) and \(\ell = 1\). In the limit \(u \to 0\), \(\eta_{\text{rev}} \to 2/3\) which is equal to the Carnot efficiency for the given parameter values. In the limit \(u \to 0\), \(\eta_{\text{rev}} \to 0\). When \(u \to \infty\), \(\eta \to 0\).
which qualitatively agrees with this work. The analytical
determination of efficiency \( \eta \) is plotted as a function of \( u \) in Fig. 4 for the case \( N = 1 \). When one considers the heat flow via the potential energy, in the
limit \( u \to 0 \), \( \eta_{irr} \to 2/3 \) which is equal to the Carnot
efficiency for parameter values of \( \tau = 2 \) and \( \ell = 1 \). On the
other when we consider the heat flow both via the po-
tential and the kinetic energies, \( \eta_{irr} \to 0 \) when \( u \to 0 \)
and \( u \to \infty \). This exhibits that quasistatic process may
not be the best working condition for the Brownian heat
ingines which agrees with the claim of Hondou and Seki-
moto. In addition, \( \eta_{irr} \) attains a maximum value at
finite value of \( u \). The same figure depicts that \( \eta_{irr} < \eta_{rev} \)
and \( \eta_{irr,rev} \to 0 \) when \( u \to \infty \).

The plot of \( \alpha \) as a function of \( u \) is displayed in Fig. 5. The enhancement in the efficiency is high when \( \tau \) is small.
For \( u < 29 \), \( \alpha_4 > \alpha_2, \alpha_3...\alpha_{10} \). This implies \( N = N_{op} = 4 \) is the optimal
value of barrier subdivision. On the other hand, when \( u > 29 \), \( N = N_{op} = 6 \) is the optimal value of barrier subdivision.

limit \( u \to \infty \), \( j \to 0 \) as the particle encounters a diffi-
culty of jumping the high potential barrier of the ratchet
potential, see Fig. 3. The velocity \( v \) attains maximum
value at a particular value of \( u \). Note that the engine
operates with maximum power at this particular value of \( u \). The potential \( u \), at which the velocity of the particle
is maximum, shifts to wards the right as the number of
barrier subdivisions increase as it can be readily seen in
Fig. 3. Note that in the system we consider, the left and
the right sides of the sawtooth potential are coupled
with the hot and the cold baths, respectively. For such
a system, positive velocity exhibits that the net flux of
the particle is from the hot to the cold reservoirs and
the engine operates only as a heat engine. Figure 3 shows
that the velocity is positive for any \( N \).

For high potential barriers, the Brownian particle en-
counters a difficulty of jumping the sawtooth potential
when the background temperature is weak. Subdivid-
ing the barrier along the reaction coordinates enables
the Brownian particle to cross each small barrier with
small thermal kicks and ultimately the particle crosses
the high potential barrier within short period of time
than the time taken by the particle when it crosses the
smooth potential barrier. Hence subdividing the saw-
tooth potential enhances the drift velocity \( v \) as depicted
in Fig. 3. The possibility of enhancing the escape rate
of a Brownian particle over sawtooth potential under spe-
cific conditions was envisaged. The analytical
finding revealed that the escape rate of the Brownian
particle is enhanced for subdivided reaction coordinate
which qualitatively agrees with this work.

The analytically determined efficiency \( \eta \) is plotted as

\[
\eta = \frac{\int_0^\infty \left[ j(T_h, T_c) - j(T_c, T_h) \right] dt}{\int_0^\infty \left[ j(T_h, T_c) + j(T_c, T_h) \right] dt}
\]

where \( j(T_h, T_c) \) is the net flux of the particle under finite
temperature difference between the hot and the cold baths.

In the limit \( \tau \to 0 \) (since \( T_h = T_c \), the system is in
thermal equilibrium), the steady state current \( j \) vanishes:
\( j \to 0 \). The drift velocity \( v \) intensifies as \( \tau \) and \( N \) increase.

FIG. 5: The plot of \( \alpha \) as a function of \( u \) for parameter values
of \( \tau = 1.5 \) and \( \ell = 1 \). \( \alpha \) is increasing function of \( u \). For \( u < 29 \), \( \alpha_4 > \alpha_2, \alpha_3...\alpha_{10} \). This implies \( N = N_{op} = 4 \) is the optimal
value of barrier subdivision. On other hand, when \( u > 29 \), \( N = N_{op} = 6 \) is the optimal value of barrier subdivision.

FIG. 6: The drift velocity \( v \) versus \( \tau \) for values of \( u=24 \)
and \( \ell=1 \). In the limit \( \tau \to 0 \) (since \( T_h = T_c \), the system is
in thermal equilibrium), the steady state current \( j \) vanishes:
\( j \to 0 \). The drift velocity \( v \) intensifies as \( \tau \) and \( N \) increase.
with \( N \). The plot of \( \alpha \) as a function of \( \tau \) is displayed in Fig. 7. Significant enhancement of the efficiency is observed when \( \tau \) is small.

IV. THE EFFICIENCY, THE VELOCITY AND THE PERFORMANCE OF THE REFRIGERATOR IN THE PRESENCE OF EXTERNAL FORCE

In the presence of external force, the net flow of the particle depends on the magnitude of the external force. For large load, current reversal may occur and this indicates that the engine operates not only as a heat engine but also as a refrigerator. In the presence of constant external force \( f \), similar to the previous section, the closed form expression for steady state is given by \( J = -F/(G_1 G_2 + H F) \). For \( N = 1 \), the expressions for \( F, G_1, G_2, \) and \( H \) are given by

\[
F = e^{a-b} - 1, \\
G_1 = \frac{L_0}{a T_h} (1 - e^{-a}) + \frac{L_0}{b T_c} e^{-a} (e^b - 1), \\
G_2 = \frac{\gamma L_0}{a} (e^a - 1) + \frac{\gamma L_0}{b} e^a (1 - e^{-b}).
\]

(6)

On the other hand, \( H = A + B + C \), where

\[
A = \frac{\gamma}{T_h} \left( \frac{L_0}{a} \right)^2 (a + e^{-a} - 1), \\
B = \frac{\gamma L_0 L_0}{a b T_c} (1 - e^{-a}) (e^b - 1), \\
C = \frac{\gamma}{T_c} \left( \frac{L_0}{b} \right)^2 (e^b - 1 - b).
\]

(7)

Here \( a = (U_0 + f L_0)/T_h \) and \( b = (U_0 - f L_0)/T_c \). The expressions for \( N \geq 2 \) are lengthy and will not be presented in this work. The drift velocity \( v \) is related to the steady state current and it is given by \( v = 2 J L_0 \). Introducing additional rescaled parameter: \( \lambda = f L_1/T_c \), we study how the velocity, the efficiency and the coefficient of performance of the refrigerator behave as \( N \) varies.

Figure 8 presents the plot of the velocity \( v \) versus rescaled load \( \lambda \) for \( N = 1, 2 \) and 3. As shown in the figure for \( \lambda < 4 \), the load is not strong enough to reverse the direction of the net flux of the particle, i.e., the net flow of the particle is from the hot to the cold reservoirs and hence the model works as a heat engine. On other hand for \( \lambda > 4 \), the current becomes negative and the model acts as a refrigerator. The particle velocity is zero at \( \lambda = 4 \). In general, for any number of barrier subdivisions \( N \) the velocity \( v = 0 \), when the load is

\[
f_0 = \frac{\tau U_0}{(\tau + 2) L_0}.
\]

(8)

When one omits the heat exchange via kinetic energy (neglecting the term \( 1/2 k_B (T_h - T_c) \) in (3) and (4)), for any \( N \), in the quasistatic limit \( v^+ \to 0 \), the efficiency is equal to Carnot efficiency:

\[
\lim_{v^+ \to 0} \eta_C = \frac{T_h - T_c}{T_h}
\]

(9)

and in the quasistatic limit \( v^- \to 0 \), the coefficient of performance of the refrigerator is equal to Carnot refrigerator:

\[
\lim_{v^- \to 0} \eta_{ref} = \frac{T_c}{T_h - T_c}.
\]

(10)

We further investigate the thermodynamic property of the engine by including the heat exchange via the kinetic and the potential energies. At a quasistatic limit, for any \( N \), the efficiency takes a simple form:

\[
\lim_{v^+ \to 0} \eta_{irr} = \frac{T_h - T_c}{T_h} \delta = \eta C \delta
\]

(11)
The refrigerator converges to $T$ played in Fig. 9. The figure shows that in the limit energy is considerable when exploring how the analytical expressions (9) and (11), one can explore how $\eta_{irr}$ and $\eta_C$ behave as a function of $\tau$ as displayed in Fig. 9. The figure shows that in the limit $T_h \rightarrow T_c$, $\eta_{irr} \rightarrow \eta_C$ while $\eta_{irr} \ll \eta_C$ when $T_c \ll T_h$. This exhibits that the heat transfer via the kinetic energy is considerable when $\tau$ steps up.

In the limit $v^{-} \rightarrow 0$, the coefficient of performance of the refrigerator converges to

$$\lim_{v^{-} \rightarrow 0} P_{ref} = \frac{T_c}{T_h - T_c} \Delta. \quad (13)$$

where

$$\Delta = \frac{U_0 T_h - 0.25 \frac{\tau^2}{T_c} + 0.25 T_c}{U_0}. \quad (14)$$

When $T_h > T_c$ and within the region where the model works as a refrigerator, $\Delta < 1$. This reveals that the coefficient of performance of the refrigerator $P_{ref}$ is always less than the Carnot refrigerator at a quasistatic limit. The quasistatic behavior of the engine can be explored by exploiting the analytic expressions (10) and (13) as shown in Fig. 10. The figure depicts that $P_{ref}^{*} < P_{ref}^{C}$ and, in the limit $T_h \rightarrow T_c$, $P_{ref}^{*} \rightarrow P_{ref}^{C}$.

We next explore how $v$ and $P_{ref}$ behave as a function of $\tau$ for $N = 1$, 2 and 3. The dependence of the velocity $v$ on the rescaled temperature $\tau$ is demonstrated in Fig. 11 for $N = 1$, 2 and 3. As shown in the Fig. 11, when $\tau < 0.5$, the load is strong enough to reverse the net particle flow and the velocity is negative. On other hand, when $\tau = 0.5$, the temperature renormalizes the effect of the load and $v = 0$. For $\tau > 0.5$, the temperature gains strength to overcome the load and hence the current is positive in this region. Within the region where the model works as a heat engine, $v$ strengthens with $N$ and $\tau$. In the region where the model works as a refrigerator, $|v|$ intensifies as $N$ increases as shown in Fig. 11. This is because subdividing the sawtooth potential into small barriers enables the Brownian particle...
FIG. 12: The coefficient of performance of the refrigerator \( P_{\text{ref}} \) versus \( \tau \) (for reversible case) for fixed \( u = 10, \ell = 1 \) and \( \lambda = 2.0 \). \( P_{\text{ref}} \) is a decreasing function of \( \tau \) and it rises up with \( N \). In the limit \( \tau \to 0.5 \), \( P_{\text{ref}} \to 2 \) which is equal to the Carnot refrigerator for the given parameter values.

to cross these small barriers at small thermal kicks. On the other hand, when \( \tau \) is small, the background thermal kick is weak for the particle to cross the smooth sawtooth potential barrier.

Figure 12 depicts the plot of \( P_{\text{ref}} \) versus \( \tau \). The coefficient of performance of the refrigerator \( P_{\text{ref}} \) is a decreasing function of \( \tau \) and it attends a maximum value when \( N \) steps up. When the rescaled temperature \( \tau \) increases, \( P_{\text{ref}} \) declines towards the Carnot refrigerator.

V. SUMMARY AND CONCLUSION

In this work, we consider a model of Brownian heat engine. The dependence of the velocity, the efficiency and the coefficient of performance of the refrigerator is investigated for different number of barrier subdivisions, \( N \). We show that the velocity and the efficiency attain optimum values at a particular value of barrier subdivision \( N \). In the presence of external load we find that the velocity, the efficiency and the coefficient of performance of the refrigerator attain maximum values when the sawtooth potential is subdivided into series of smaller connected barrier series.

Considering the heat exchange via the potential and the kinetic energies, we show that Carnot efficiency is unachievable for Brownian heat engines when the engines work quasistatically. Quasistatic consideration for the Brownian heat engines also reveals that the coefficient of performance of the refrigerator is always less than the Carnot refrigerator.

In this work, considering an exactly solvable model, we explore the energetics of a Brownian heat engine not only at quasistatic limit but also at any finite time. This theoretical work suggests that the performance of the heat engine can be improved by subdividing the sawtooth potential into series of small barrier steps systematically by considering physically reasonable parameterization.

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APPENDIX A

In this Appendix we will give the expressions for \( F \), \( G_1 \), \( G_2 \) and \( H \) which define the value of the steady state current, \( J^L \), for zero external load case when \( N = 2 \).

\[
F = e^{\frac{U_0}{\ell T_T^h} + \frac{U_0}{\ell T_H^h}} - 1, \quad \text{(A1)}
\]

\[
G_1 = \frac{3L_0}{5U_0} e^{-\frac{U_0}{\ell T_T^h}} \left[ -2 - 2e^{\frac{2U_0}{\ell T_T^h}} + e^{\frac{U_0}{\ell T_T^h}} - 2e^{\frac{U_0}{\ell T_H^h}} + 2e^{\frac{2U_0}{\ell T_H^h}} \right] + \frac{3L_0}{5U_0}, \quad \text{(A2)}
\]

\[
G_2 = \frac{L_0 \gamma}{5U_0} \left( -3e^{-\frac{U_0}{\ell T_H^h} - \frac{U_0}{\ell T_T^h}} T_C - 3T_H - 6e^{\frac{U_0}{\ell T_T^h}} T_H + 6e^{\frac{2U_0}{\ell T_H^h}} - \frac{3L_0}{5U_0} \right) e^{\frac{U_0}{\ell T_T^h}} \left[ -6T_C + 6e^{\frac{U_0}{\ell T_T^h}} T_C + 3e^{\frac{2U_0}{\ell T_H^h}} (T_C + T_H) \right], \quad \text{(A3)}
\]

\[
H^L = T_1(T_2 + T_3 + T_4 + T_5 + T_6), \quad \text{(A4)}
\]

\[
T_1 = \frac{L_0 \gamma}{25T_T^h U_0} e^{-\frac{1}{2} \left( \frac{U_0}{\ell T_T^h} + \frac{U_0}{\ell T_H^h} \right)}, \quad \text{(A5)}
\]

\[
T_2 = 36e^{\frac{U_0}{\ell T_H^h} T_H} + 18e^{\frac{U_0}{\ell T_T^h} T_T^2} + 18e^{\frac{2U_0}{\ell T_H^h} T_H^2} - 18e^{\frac{U_0}{\ell T_T^h} T_T^2} - 9e^{\frac{U_0}{\ell T_H^h} T_H^2}, \quad \text{(A6)}
\]

\[
T_3 = -36e^{\frac{U_0}{\ell T_H^h} T_H} T_T + \frac{U_0}{\ell T_T^h} T_T^2 + 45e^{\frac{U_0}{\ell T_H^h} T_H} T_T^2 - 36e^{\frac{2U_0}{\ell T_H^h} T_H^2} T_T^2 - 2T_H + 72e^{\frac{U_0}{\ell T_H^h} T_H^2} T_T^2, \quad \text{(A7)}
\]

\[
T_4 = 54e^{\frac{U_0}{\ell T_H^h} T_H} T_T^2 + 36e^{\frac{U_0}{\ell T_T^h} T_H^2} T_T^2 + 18e^{\frac{2U_0}{\ell T_H^h} T_H^2} T_T^2 +
\]
In this Appendix we will give the expressions for $F$, $G_1$, $G_2$ and $H$ which define the value of the steady state current, $J$, for zero external load case when $N = 3$.

\begin{align}
F &= e^{\frac{\gamma_0}{c}\tau_+ + \frac{\gamma_0}{c}} - 1, \quad \text{(B1)} \\
G_1 &= \frac{L_0 e^{\frac{\gamma_0}{c}}}{2U_0}(-2 - 2e^{\frac{\gamma_0}{c}} + 2e^{\frac{\gamma_0}{c}} + e^{\frac{\gamma_0}{c}} - 2e^{\frac{\gamma_0}{4\tau_+ c}} + e^{\frac{\gamma_0}{4\tau_+ c}}), \quad \text{(B2)} \\
G_2 &= \frac{-L_0 e^{\frac{\gamma_0}{c}}}{2U_0}(e^{\frac{U_0}{c}\tau_+ + \frac{1}{\tau_+}})T_C + 2e^{\frac{\gamma_0}{c}} - 2e^{\frac{\gamma_0}{4\tau_+ c}} + 2e^{\frac{\gamma_0}{4\tau_+ c}} - 2e^{\frac{\gamma_0}{4\tau_+ c}} + 2e^{\frac{\gamma_0}{4\tau_+ c}} + e^{\frac{\gamma_0}{4\tau_+ c}}), \quad \text{(B3)} \\
H &= T_1 + T_2(T_3 + T_4), \quad \text{(B4)} \\
T_1 &= \frac{L_0^2 T_C}{4U_0^2}(-5 + 8e^{\frac{\gamma_0}{c}} - 12e^{\frac{\gamma_0}{4\tau_+ c}} + 4e^{\frac{\gamma_0}{4\tau_+ c}} + 4e^{\frac{\gamma_0}{4\tau_+ c}} + e^{\frac{\gamma_0}{4\tau_+ c}}), \quad \text{(B5)} \\
T_2 &= \frac{L_0^2 T_h}{4U_0^2}(e^{\frac{\gamma_0}{c}} - 1)e^{\frac{\gamma_0}{c}}, \quad \text{(B6)} \\
T_3 &= (-2 - 2e^{\frac{\gamma_0}{c}} + 2e^{\frac{\gamma_0}{4\tau_+ c}} + e^{\frac{\gamma_0}{4\tau_+ c}} + e^{\frac{0.5U_0}{c}\tau_+ + \frac{1}{\tau_+}}), \quad \text{(B7)} \\
T_4 &= 2e^{0.25U_0(\frac{\gamma_0}{c}\tau_+ + \frac{1}{\tau_+})} + 2e^{0.25U_0(\frac{\gamma_0}{c}\tau_+ + \frac{1}{\tau_+})} - 6e^{\frac{\gamma_0}{c}} - 6e^{\frac{\gamma_0}{4\tau_+ c}} + 8e^{\frac{\gamma_0}{4\tau_+ c}}, \quad \text{(B8)} \\
T_5 &= -4e^{\frac{U_0(T_h + T_C)}{\gamma_0 + \frac{\gamma_0}{c}}} - 2e^{\frac{U_0(T_h + 2T_C)}{\gamma_0 + \frac{\gamma_0}{c}}} + 4e^{\frac{U_0(T_h + 2T_C)}{\gamma_0 + \frac{\gamma_0}{c}}}. \quad \text{(B9)}
\end{align}