Eigenvalue and Eigenfunction for the
$PT$-symmetric Potential $V = -(ix)^N$

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Abstract

If replace the Hermiticity from conventional quantum mechanics with the physically transparent condition of parity-time reflection symmetry ($PT$-symmetry), the non-Hermitian Hamiltonian still guarantees that its entire energy spectrum is real if the Hamiltonian has unbroken $PT$-symmetry. If its $PT$-symmetry is broken, then two cases can happen - its entire energy spectrum is complex for the first case, or a finite number of real energy levels can still be obtained for the second case. This was “officially” discovered on a paper by Bender and Boettcher since 1998 when the energy spectrum from the $PT$-symmetric Hamiltonian $H = p^2 - (ix)^N$ with $x \in \mathbb{C}$ was examined within one pair of Stokes wedges.

To better understanding differential equation in complex plane, for this Hamiltonian we discuss the following three questions in this paper. First, since their paper used a Runge-Kutta method to integrate along a path at the center of the Stokes wedges to calculate eigenvalues $E$ with high accuracy, we wonder if the same eigenvalues can be obtained if integrate along some other paths in different shapes. Second, what the corresponding eigenfunctions look like? Should the eigenfunctions be independent from the shapes of path or not? Third, since for large $N$ the Hamiltonian contains many pairs of Stokes wedges symmetric with respect to the imaginary axis of $x$, thus multiple families of real energy spectrum can be obtained. What do they look like? Any relation among them?

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1 INTRODUCTION

1 Introduction

In quantum mechanics, the sign of position operator $\hat{x}$ and the momentum operator $\hat{p}$ can be changed by the parity reflection operator $P$ in the following way[2]:

$$P\hat{x}P = -\hat{x}, \quad P\hat{p}P = -\hat{p}, \quad PiP = i,$$  \hspace{1cm} (1)

where, however, the sign of the complex number $i$ is unchanged. If we apply the time reversal operator $T$ instead, then

$$T\hat{x}T = \hat{x}, \quad T\hat{p}T = -\hat{p}, \quad TiT = -i,$$  \hspace{1cm} (2)

where the sign of the momentum operator and of the complex number are changed. We say a Hamiltonian $\mathcal{H}$ is $PT$-symmetric if the combined operator $PT$ commutes with $\mathcal{H}$ such that

$$\left[ PT, \mathcal{H} \right] = \mathcal{H}(PT) - (PT)\mathcal{H} = 0.$$  \hspace{1cm} (3)

For example, the Hamiltonian $\mathcal{H} = p^2 - (ix)^N$ with $x \in \mathbb{C}$ and $N \in \mathbb{R}$ is $PT$-symmetric.

The discovery[4, 9] that an entirely real energy spectrum can be obtained from the non-Hermitian but $PT$-symmetric Hamiltonian was a surprise to scientific communities in 1998. Since then, the developments in $PT$-symmetric quantum theory rapidly grew - at least 50 experiments to observe $PT$-symmetric system were published during the last 10 years. Those experiments told us that it was possible to experimentally measure complex eigenvalue, and observe broken and unbroken $PT$-symmetry.

It's well-known[2] that within a specific pair of Stokes wedges the $PT$-symmetry of $\mathcal{H} = p^2 - (ix)^N$ is unbroken if $N \geq 2$, so that the entire energy spectrum is positive; and broken if $N < 2$, so that only a finite number of positive energy levels can be found. However, this conclusion is not satisfied for the curiosity of any enthusiastic student. As far as we know, most students from physical science are unfamiliar with the concept of Stokes wedge. Driven by curiosity, they would ask something similar to the following questions:

1. Since Bender and Boettcher[4] used a Runge-Kutta method to integrate along a path at the center of the Stokes wedges to calculate eigenvalues $E$ with high accuracy, we wonder if the same eigenvalues can be obtained if integrate along some other paths in different shapes. In other words, are those eigenvalues independent from the shape of path or not?

2. So far we have not seen any research about its eigenfunctions, and do not know why the research of eigenfunction should be ignored. In this paper, can we provide a detailed study to fill the gap?
3. Due to the existence of multiple pairs of Stokes wedges, the Hamiltonian \( H = p^2 - (ix)^N \) must contain multiple families of real energy spectrum if \( N \) is large enough. We are really curious to see what the spectra look like.

It’s difficult to answer these three questions by using rigorous mathematics which is beyond university-level and in fact we don’t know. To answer them pedagogically, we implement the strategy - “seeing is believing”. A lot of figures and tables are shown in this paper to help students visualize data, Stokes wedges and the relation among them. How we answer the questions is based on the observation of the data, rather than rigorous mathematics. We believe empirical observation and conjecture are the first and crucial step to deepen our understanding upon rather complicated concept.

Our paper is organized in the following way. In Sec.(2), the concept of Stokes wedges is introduced. In Sec.(3), we show two ingredients needed for the numerical calculation of the eigenvalue - algorithm and parametrization, then we provide answers for the first two questions by discussing the results for the specific values of \( N \). In Sec.(4), after introduced the WKB approximation, we plot and discuss the first four families of the spectrum.

2 Local asymptotic analysis for the potential \( V = -(ix)^N \)

Consider 1D Schrodinger equation in the complex plane with \( N \in \mathbb{R} \) and \( x \in \mathbb{C} \),

\[
-\psi''(x) - (ix)^N \psi(x) = E\psi(x), \quad (4)
\]

with boundary condition \( \psi(x) \to 0 \) and \( \psi'(x) \to 0 \) as \( |x| \to \infty \).

We guess that (4) has a solution with the form \( \psi(x) = e^{S(x)} \) where \( S(x) = ax^b \) with \( a \in \mathbb{C} \), \( a \neq 0 \) and \( b > 0 \). So we substitute our ansatz into (4), neglect[7] those terms whose modulus are orders of magnitude smaller than the rest when \( |x| \to \infty \), and finally obtain the following asymptotic relations as \( |x| \to \infty \),

\[
\psi(x) \sim C_1 |x|^{-\frac{N}{4}} \exp \left[ \pm \frac{i\frac{N}{2} + 1}{\frac{N}{2} + 1} x^{-\frac{N}{2} + 1} \right] \quad \text{when } N \geq 2; \quad (5)
\]

\[
\psi(x) \sim C_2 |x|^{-\frac{N}{4}} \exp \left[ \pm \frac{i\frac{N}{2} + 1}{\frac{N}{2} + 1} x^{-\frac{N}{2} + 1} + \frac{E \left( x^{-\frac{N}{2} + 1} \right)}{(-N + 2) i \frac{N}{2} + 1} \right] \quad \text{when } \frac{2}{3} \leq N < 2; \quad (6)
\]

\[
\psi(x) \sim C_3 |x|^{-\frac{N}{4}} \exp \left[ \pm \frac{i\frac{N}{2} + 1}{\frac{N}{2} + 1} x^{-\frac{N}{2} + 1} + \frac{E \left( x^{-\frac{N}{2} + 1} \right)}{(-N + 2) i \frac{N}{2} + 1} \pm \frac{E^2}{8 i \frac{N}{2} + 1} \left( 1 - \frac{3\frac{N}{2}}{2} \right) x^{-3\frac{N}{2} + 1} \right] \quad (7)
\]

when \( 0 < N < \frac{2}{3} \).

where \( C_1, C_2 \) and \( C_3 \) are some constants. Note that the asymptotic relation (5) for \( N \geq 2 \) is independent from the eigenvalue \( E \). To satisfy the boundary condition, we expect that its dominant contribution in the leading-order behavior vanishes such that

\[
\exp [S_{1,2}(x)] \equiv \exp \left[ \pm \frac{i\frac{N}{2} + 1}{\frac{N}{2} + 1} x^{-\frac{N}{2} + 1} \right] \to 0 \quad \text{as } |x| \to \infty.
\]
exp \[S_{1,2}(x)\] approaches to zero in the fastest speed if the oscillatory behavior of the exponential is zero, in other words\[7\],

\[
\text{Im} \left[ S_1(x) - S_2(x) \right] = 0 \implies \text{Im} \left[ i^{N+1} x^{\frac{N}{2}+1} \right] = 0,
\]

which is our definition of “Stokes lines”. With \(x = re^{i\theta}\), (8) becomes

\[
\frac{\pi}{2} \left( \frac{N}{2} + 1 \right) + \theta \left( \frac{N}{2} + 1 \right) = \pm k\pi \quad \text{for } k = 0, 1, 2, 3, \cdots,
\]

which yields

\[
\begin{align*}
\theta_{\text{left}} &= -\pi + \frac{N-4k+2}{N+2} \frac{\pi}{2} \\
\theta_{\text{right}} &= -\frac{N-4k+2}{N+2} \frac{\pi}{2}
\end{align*}
\]

for \(k = 0, 1, 2, 3, \cdots\),

so that

\[
\theta_{\text{left}} = -\pi - \theta_{\text{right}}.
\]

We plot all those Stokes lines on Fig.(1). When \(k = 1\) we obtain

\[
\begin{align*}
\theta_{\text{left}} &= -\pi + \frac{N-2}{N+2} \frac{\pi}{2} \\
\theta_{\text{right}} &= -\frac{N-2}{N+2} \frac{\pi}{2}
\end{align*}
\]

These Stokes lines are plotted on Fig.(2).

The locations of “anti-Stokes lines” are defined as

\[
\text{Re} \left[ S_1(x) - S_2(x) \right] = 0,
\]

when the exponential is purely oscillatory. Solving (12) yields

\[
\theta_1 = \frac{\pi}{N+2} - \frac{\pi}{2} + \frac{2(j-1) \pi}{N+2} \quad \theta_2 = \frac{\pi}{N+2} - \frac{\pi}{2} + \frac{2j \pi}{N+2} \quad \text{for } j = 0, 1, 2, 3, \cdots,
\]

which define the width of the “Stokes sector” or “Stokes wedge”

\[
\Delta = |\theta_1 - \theta_2| = \frac{2\pi}{N+2}.
\]

The shape of a Stokes wedge is not really like a wedge or a slice of pie. They are asymptotic concepts. The angular opening \(\Delta\) from (13) of the wedge only refers to the opening for \(|x|\) at certain range of complex infinity.
Figure 1: All Stokes wedges for non-negative integer \( N \) and all corresponding turning points (yellow point with black edge). Although the locations of turning points are eigenvalue \( E \)-dependent, for the purpose of visualization here we set \( E \) as an appropriate constant.

Except the top and bottom wedges which contain the imaginary axis, Fig.(1) shows that the rest of all wedges form pairs symmetric with respect to the imaginary axis. Each pair are labeled with a color - orange, green, pink, yellow, red, etc. We may sometimes call those pairs as “\( PT \)-symmetric Stokes wedges”. Each pair contains a pair of turning points which are also symmetric with respect to the imaginary axis. The larger the \( N \) is, the more pairs of wedges and of turning points are. Any wedge whose anti-Stokes line coincides with the imaginary axis only shares one singular turning point with its pair, and that singular turning point must be located on the imaginary axis. On Fig.(1), when \( N = 0 \) and \( N = 1 \), we label the wedges as “hypothetical wedges”, because the locations of these wedges on the figure are actually not true for \( N < 2 \) according to (6) and (7), where these wedges are eigenvalue \( E \)-dependent.

Since different pair of wedges will pose different eigenvalue problem, to proceed, we now only focus on one pair of wedges by choosing the orange pair with the Stokes lines defined from (11) shown on Fig.(2) to calculate the eigenvalue.
3 Numerical approximation

3.1 Levenberg-Marquardt algorithm

To solve the eigenvalue problem (4), by Levenberg-Marquardt algorithm (LMA) we choose to minimize a square-function \( F(x_\infty, E) \) of the following complex modulus with respect to the energy \( E \),

\[
F(x_\infty, E) \equiv |f(x_\infty, E) - \psi(x_\infty, E)|^2
\]

where \( x_\infty \) is the right boundary point located within a right Stokes wedge and \( f(x_\infty, E) \) can be calculated by Gauss-Legendre integration method[10] (GLI), an implicit Runge-Kutta method. Due to the boundary condition \( \lim_{|x| \to \infty} \psi(x) = 0 \),

\[
F(x_\infty, E) = |f(x_\infty, E)|^2. \tag{14}
\]

LMA is an iterative procedure, where the previous estimate \( E \) is replaced by a new estimate, \( E + \delta E \), for each iterative step. We can approximate \( f(x_\infty, E + \delta E) \) by

\[
f(x_\infty, E + \delta E) \approx f(x_\infty, E) + J \delta E, \tag{15}
\]

where \( J \) is the gradient of \( f(x_\infty, E) \) with respect to \( E \),

\[
J = \frac{\partial f(x_\infty, E)}{\partial E}.
\]
In our case, \( f(x_\infty, E) \) is complex and so is \( E \). Let \( u, v, a \) and \( b \) be real such that

\[
f(x_\infty, E) = u(a, b) + iv(a, b) \quad E = a + ib,
\]

then we have the following Jacobian matrix \( J \)

\[
J = \begin{bmatrix}
\frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} \\
\frac{\partial v}{\partial a} & \frac{\partial v}{\partial b}
\end{bmatrix},
\]

and (15) in vector notation is

\[
f(x_\infty, E + \delta E) \approx f(x_\infty, E) + J \delta E,
\]

where

\[
f(x_\infty, E) = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \delta E = \begin{bmatrix} \delta a \\ \delta b \end{bmatrix}.
\]

By (14) and (17), we obtain

\[
F(x_\infty, E + \delta E) = [f(x_\infty, E) + J \delta E]^2 = [f(x_\infty, E) + J \delta E]^T [f(x_\infty, E) + J \delta E].
\]

To find the minimum, we set

\[
\frac{\partial F(x_\infty, E + \delta E)}{\partial (\delta E)} = 0.
\]

Hence,

\[
\delta E = -(J^T J)^{-1} J^T f(x_\infty, E).
\]

Due to Levenberg’s and Marquardt’s modification on the last equation, we have a damped factor \( \lambda \), which is a positive parameter, such that

\[
\delta E = - [J^T J + \lambda \text{diag}(J^T J)]^{-1} J^T f(x_\infty, E),
\]

where \( \text{diag}(J^T J) \) means a diagonal matrix with entries on the diagonal from the matrix \( J^T J \). If the function \( F(x_\infty, E_{\text{new}}) \leq F(x_\infty, E_{\text{old}}) \) after a single iterative step, we update \( E_{\text{old}} \) by

\[
E_{\text{new}} = E_{\text{old}} + \delta E = E_{\text{old}} - [J^T J + \lambda \text{diag}(J^T J)]^{-1} J^T f(x_\infty, E),
\]

and meanwhile decrease the value \( \lambda_{\text{old}} \) by a factor, for example, \( \lambda_{\text{new}} = \lambda_{\text{old}} / \sqrt{2} \). If after a single iterative step \( F(x_\infty, E_{\text{new}}) > F(x_\infty, E_{\text{old}}) \), this means our \( \lambda \) value is too small and we then increase \( \lambda_{\text{old}} \) by a factor, for example, \( \lambda_{\text{new}} = 10 \lambda_{\text{old}} \). And the eigenvalue will not be updated so that we still have

\[
E_{\text{new}} = E_{\text{old}}.
\]

How we adjust the value of \( \lambda \) becomes important to efficiently find the eigenvalue, yet so far there is no absolutely best way to optimize the value of \( \lambda \).
3.2 Parametrization, eigenvalue and eigenfunction

We set up the following initial condition at the numerical infinity \( x_0 \equiv \infty_{\text{left}} \) within the left Stokes wedge on Fig.(2):

\[
x_0 = r_0 \exp (i \theta_{\text{left}}) \quad \psi (x_0) = 0 \quad \frac{d\psi (x_0)}{dx} = 10^{-7},
\]

where \( \theta_{\text{left}} \) is defined by (11) and \( r_0 \) is the complex modulus of the numerical infinity \( x_0 \). In other words, \( r_0 \) is the distance between the origin and the point where the wave function and the derivative of the wave function almost vanish. By observation, we set \( r_0 = 4 \).

To have the fastest convergence to the eigenvalue, we’re tempted to use GLI to integrate along the two Stokes lines given by (11). However, they are connected at the origin where is non-differentiable for \( N \neq 2 \). Since this causes non-smoothness (See Fig.(3)) on the eigenfunction at the origin, one way to have smooth-looking eigenfunction is to integrate along some new paths, which should satisfy the following four conditions:

1. The potential \(- (ix)^N\) has a numerical cut on the positive-imaginary axis. So the new path must not cross it; otherwise we must have a different eigenvalue problem.

2. The new path must go from one complex infinity within one Stokes wedge and back to the other complex infinity in the other Stokes wedge. These two complex infinities are symmetric with respect to the imaginary axis of \( x \).

3. The new path is smooth everywhere and can be parametrized by a differentiable function.

4. Since GLI converges fastest if integrate along the two Stokes lines, it would be more efficient if the path or the differentiable function has two asymptotic lines coincident with the locations of the two Stokes lines.
Figure 3: The eigenfunction of the ground state for $N = 4$ along the Stokes lines (non-differentiable at origin) with $r_0 = 4$. The vertical-blue dotlines represent two numerical infinities $\pm \text{Re}(x_0) = \pm r_0 \cos 30^\circ$. Along the Stokes lines, the numerical result for the eigenvalue $E$ does not change even though the shape of the eigenfunction is not smooth, in comparison with the smooth eigenfunctions associated with the hyperbolic paths.

![Figure 3](image1)

Figure 4: The eigenfunctions of the ground state for $N = 4$ along the differentiable (hyperbolic) paths with three different values of $\alpha$ defined by the hyperbolic equation $Y_- = -a \sqrt{1 + \frac{x^2}{b^2}}$. All three paths have the same eigenvalue. The vertical-blue dotlines represent two numerical infinities $\pm \text{Re}(x_0) = \pm r_0 \cos 30^\circ$ with $r_0 = 4$.

![Figure 4](image2)
To satisfy all four conditions and since Fig.(2) shows that the two Stokes lines move below the real axis when $N > 2$, the best differentiable function used for the parametrization when $N > 2$ must be hyperbola shown on the miniplot on Fig.(4). In this hyperbolic parametrization, we treat $\text{Re}(x)$ as the parameter so that

$$
x = \text{Re}(x) + i \text{Im}(x) = \text{Re}(x) - i \sqrt{a^2 + [\text{Re}(x)]^2 (\tan \theta)^2},$$

(22)

where the angle $\theta$ between one of the asymptotic lines and the horizontal axis is $\theta = \arctan \left( \frac{a}{b} \right)$. Also, $\theta = \theta_{\text{right}}$ from (11). Fig.(4) shows that the shapes of the corresponding eigenfunctions are smooth and different-looking since the values of $a$ defined by the hyperbolic equation $Y_\pm = -a \sqrt{1 + \frac{X^2}{b^2}}$ is different. For the upcoming work, we choose $a = 0.2$ since this hyperbola is quite close to the location of the two Stokes lines and meanwhile keeps the shape of eigenfunction smooth.

For $0 < N < 2$, Fig.(2) shows that the two Stokes lines move above the real axis. Does the function satisfy the four conditions exist? Yes. As $X \to \pm \infty$ the following function with $k > 0$, real parameters $c$ and $t$

$$
f(X) = \frac{X^2 - c}{\sqrt{kX^2 + t}}$

(23)

has two asymptotic lines:

$$
Y = \frac{1}{\sqrt{k}} X \quad \text{as} \quad X \to +\infty, \quad Y = -\frac{1}{\sqrt{k}} X \quad \text{as} \quad X \to -\infty.
$$

The angle $\theta$ between the asymptotic line associated with $X \to +\infty$ and the horizontal axis satisfies

$$
k = \left( \frac{1}{\tan \theta} \right)^2.
$$

Fig.(5) shows a good news that the function satisfies all four conditions.

**Figure 5:** The differentiable path for $N < 2$ by choosing $c = \frac{1}{10}$ and $t = 8$.  

\[ Y = \frac{(X^2 - \frac{1}{10})}{\sqrt{kX^2 + 8}} \text{ with } k = 39.86 \]
With these parametrizations (22), (23) and along the real axis of $x$, we are able to use LMA and GLI to find all eigenvalues and eigenfunctions associated with all pairs of wedges defined by Fig.(1). The final result of eigenvalues is shown on Fig.(29), and we will explain it later.

### 3.2.1 When $N = 2$

The concept of Stokes wedge implies that the same eigenvalue is obtained if we integrate along different paths, as along as the conditions 1 and 2 are satisfied. However, can we justify this by numerical analysis? How about their eigenfunctions? Are they independent from the shape of path or not?

**Figure 6: Four distinct contour paths we follow for $N = 2$.**

The first step to answer these questions is to parametrize various paths and calculate the corresponding eigenvalues to see if they agree each other or not. We start with the case when $N = 2$ as a harmonic oscillator, and use LMA and GLI to search for the eigenvalue by integrating along the four different paths shown on Fig.(6), where two symmetric points $A$ and $B$ are located on the real axis and treated as two numerical infinities. These four different paths start from the same boundary point $A$ and end to the same boundary point $B$. One of the four paths is along the real axis, two complex paths (sym. path and non-sym. path) are defined by (24), and we add another complex path (sin. path) defined by a sinusoidal function.

It's a bit of a challenge to parametrize the non-sym. path on Fig.(6). We start with the following parametric curve with real parameter $p$ shown on Fig.(7)
which crosses itself once and is symmetric with respect to the \( Y \)-axis. So this curve can be used to define the sym. path on Fig.(6). To get the non-symmetric version, we can just rotate the curve by an given angle.

**Figure 7:** Parametric curve crosses itself.

The eigenvalues we found by following all four paths defined by Fig.(6) are shown on the following Tab.(1), where the corresponding residues are obtained by calculating the complex modulus of the numerical eigenfunctions at the right boundary point \( B \). Ideally, these values should be zero, so we set our tolerance of residue to be \( 10^{-13} \). Through observation, the real parts of eigenvalues are almost the same for all four paths, whereas the imaginary parts are so small that they can be ignored. When the energy level increases, both real and imaginary parts of eigenvalues \( E \) start to shift from the analytic results, which are \( E_i \in \{1, 3, 5, 7, \ldots \} \). The higher energy level, the larger the shift. This is typical, since we fix the locations of boundary points \( A \) and \( B \), and define them as numerical infinities. When energy level becomes higher, the pattern of corresponding eigenfunctions becomes more complicated - they wiggle more before vanishing at the infinity so that the length of non-vanishing parts becomes longer. Ideally, to minimize the shift, we need to separate \( A \) and \( B \) even farther to accommodate longer eigenfunction for higher energy level. For those, who are meticulous, you may observe from Tab.(1) that following the real path has relatively smaller imaginary parts of eigenvalues than following complex paths. The reason is the same as what we have just said. The non-vanishing parts of corresponding eigenfunctions by following the complex paths are longer than by following the real path, because the shapes of these complex paths are more complicated than of the real path. In example from the next Sec.3.2.2, we will numerically demonstrate that within the same pair of Stokes wedges and independent from the shape of path, imaginary parts of eigenvalues become smaller by separating \( A \) and \( B \) farther.
Table 1: Eigenvalues $E$ from four distinct paths for $N = 2$.

| Path Type | $\text{Re}(E)$ | $\text{Im}(E)$ | Residue |
|-----------|-----------------|-----------------|---------|
| Sym. path | 1.000000000000000 | -0.1E-17 | 0.6E-14 |
| Non-sym. path | 1.000000000000000 | -0.9E-16 | 0.6E-15 |
| Sin. path | 1.000000000000000 | -0.1E-19 | 0.6E-13 |
| Real path | 1.000000000000000 | -0.6E-24 | 0.4E-13 |
| Sym. path | 3.000000000000000 | -0.3E-17 | 0.7E-14 |
| Non-sym. path | 3.000000000000000 | -0.1E-14 | 0.6E-13 |
| Sin. path | 3.000000000000000 | -0.3E-16 | 0.2E-13 |
| Real path | 3.000000000000000 | -0.9E-25 | 0.4E-16 |
| Sym. path | 5.000000000000000 | 0.2E-15 | 0.2E-13 |
| Non-sym. path | 5.000000000000000 | -0.5E-14 | 0.7E-15 |
| Sin. path | 5.000000000000000 | -0.1E-14 | 0.2E-13 |
| Real Path | 5.000000000000000 | -0.3E-21 | 0.4E-14 |
| Sym. path | 7.000000000000000 | 0.7E-14 | 0.8E-14 |
| Non-sym. path | 7.000000000000000 | -0.2E-13 | 0.9E-14 |
| Sin. path | 7.000000000000000 | -0.3E-13 | 0.5E-13 |
| Real Path | 7.000000000000000 | -0.1E-21 | 0.5E-16 |
| Sym. path | 9.000000000000000 | 0.1E-12 | 0.9E-13 |
| Non-sym. path | 9.000000000000000 | -0.5E-13 | 0.6E-15 |
| Sin. path | 9.000000000000000 | -0.4E-13 | 0.3E-14 |
| Real path | 9.000000000000000 | -0.5E-18 | 0.1E-13 |
| Sym. path | 11.000000000000000 | 0.2E-11 | 0.6E-13 |
| Non-sym. path | 11.000000000000000 | -0.9E-14 | 0.1E-13 |
| Sin. path | 11.000000000000000 | -0.8E-11 | 0.2E-13 |
| Real path | 11.000000000000000 | -0.2E-18 | 0.4E-15 |
| Sym. path | 13.000000000000000 | 0.2E-10 | 0.6E-13 |
| Non-sym. path | 13.000000000000000 | 0.1E-11 | 0.1E-13 |
| Sin. path | 13.000000000000000 | -0.1E-09 | 0.3E-13 |
| Real path | 13.000000000000000 | -0.1E-18 | 0.1E-16 |
**Figure 8:** $\text{Re}(x)$ versus the eigenfunction of the ground level along the real and sinusoidal path for $N = 2$ (Before using (25) to normalize).

**Figure 9:** $\text{Re}(x)$ versus the eigenfunction of the ground level along the real and sinusoidal path for $N = 2$ (after using (25) to normalize).
Now we plot the eigenfunctions associated with different paths. Fig.(10) shows two mini-plots. One mini-plot is to show the shape of paths, while the other shows the parameter \( p \) defined by (24) versus the corresponding eigenfunction \( \psi(p) \) by following the sym. path. For \( N = 2 \), since all imaginary parts of eigenfunctions associated with the real path are zero, that is why we only see a vertical segment (the grey dash line) contributed by \( \text{Re} [\psi(x)] \) along the real path on Fig.(11), Fig.(14) and Fig.(17).

It seems that the different path has different eigenfunction even though the eigenvalues are the same. This is an illusion! All corresponding eigenfunctions are independent from the shape of paths as well. To demonstrate this, we draw the vertical-red dot lines on all those figures to indicate where two different paths intersect each other. At all those intersection points, the corresponding two eigenfunctions cross each other as well. We call these behaviors as “crossing events”. Suppose that two different paths intersect at one point \( x_s \), then a single crossing event is that the eigenfunction \( \psi(x_s) \) from the one path and \( \psi(x_s) \) from the other path are crossed at \( x_s \). For example, on Fig.(8), \( \text{Re} [\psi(x)] \) from the real path and \( \text{Re} [\psi(x)] \) from the sinusoidal path are crossed at 9 locations, and meanwhile \( \text{Im} [\psi(x)] \) from the real path and \( \text{Im} [\psi(x)] \) from the sinusoidal path are crossed at another 9 locations. These two sets of 9 locations can be connected pair by pair by 9 vertical-red dot lines, whose horizontal coordinates are the real coordinates of the 9 intersection points between the two paths. On Fig.(11), however, the horizontal coordinates are changed to be the imaginary coordinates of the corresponding intersection points. By observation on all figures from Fig.(10) to Fig.(18), we conclude that the number of crossing events is equal to the number of vertical-red dot lines, and further conclude that the eigenfunctions are independent from the shape of paths so long as those paths all start from the same boundary point \( A \) and end at the same boundary point \( B \). In the later case (See Fig.(19)), we will numerically demonstrate that the eigenfunction is not independent from path if that path starts and ends on different boundary point.

The amplitude of eigenfunction depends on the shape of path. If a path contains some points whose distances are far away from the origin, then the amplitude of the corresponding eigenfunction must be large. That is why the amplitude along the real axis is the smallest; whereas the amplitude along the sym. path (on Fig.(10) and Fig.(11)) is the largest.

How about normalization? We’re tempted to use standard normalization from conventional quantum mechanics. In numerical approximation, the normalized wavefunction \( \phi_n(x) \) would be

\[
\phi_n(x) = \frac{\psi_n(x)}{\sqrt{\int_c \psi_n^*(x) \psi_n(x) \, dx}} \approx \frac{\psi_n(p)}{\sqrt{\int_{p_1}^{p_2} \psi_n^*(p) \psi_n(p) \, dp}} \approx \frac{\psi_n(p)}{\sqrt{\sum_p \psi_n^*(p) \cdot \psi_n(p) \cdot dp}}.
\]

(25)

where \( p \) is the real parameter which parametrizes the path. By this way, we find that \( \phi_n(x) \) satisfies

\[
\int_c \phi_n^*(x) \phi_n(x) \, dx \approx \int_{p_1}^{p_2} \phi_n^*(p) \phi_n(p) \, dp \approx \sum_p \phi_n^*(p) \cdot \phi_n(p) \cdot dp = 1.
\]

(26)

for all wave functions from different energy level and different path. Hence, it is possible to “normalize” all wave functions we have encountered so far! However, except of the case by following the real path when \( N = 2 \), the condition of orthogonality \( \int_c \phi_m^*(x) \phi_n(x) \, dx = 0 \)

\[1\] However, we are not able to use \( \text{Re}(x) \) to “normalize” \( \psi(x) \) if \( \text{Re}(x) \) is not used to parametrize the contour path (For example, see (24)).
may not hold to be true in all other paths. For example, following the sinusoidal path on Fig.(9), we find that, numerically,

\[
\int_c \phi_0^*(x) \phi_1(x) \, dx \approx 0 \quad \int_c \phi_3^*(x) \phi_3(x) \, dx \approx 0 \quad \int_c \phi_5^*(x) \phi_5(x) \, dx \approx 0,
\]

but

\[
\int_c \phi_0^*(x) \phi_2(x) \, dx = -0.92145 + 1.56493i \neq 0.
\]

The “official” way of normalization introduced by Bender[2] is to use the recipe, which at first requires to find the \(PT\)-normalized eigenfunction through

\[
\phi_n(x) = \exp \left( i \frac{\theta_n}{2} \right) \psi_n(x), \quad (27)
\]

which satisfies \(\phi_n^*(-x) = \phi_n(x)\). Then we can verify

\[
\int_c \phi_n(x) \phi_n(x) \, dx = (-1)^n. \quad (28)
\]

After that, we can use \(CPT\)-normalization defined by

\[
\langle \phi_m(x), \phi_n(x) \rangle_{CPT} = \int_c \int_c' \hat{C}(x,y) \phi_m(y) \, dy \phi_n(x) \, dx = \delta_{mn}, \quad (29)
\]

which in some case may require to find the charge operator \(\hat{C}\) first. For the potential \(- (ix)^N\), the most difficult part is to find the phase angle \(\theta_n\) from (27). In this paper, we made no attempt to find \(\theta_n\), and therefore no attempt to normalize any eigenfunction we have encountered.

Since (25) is only a numerical approximation of the normalization from conventional quantum mechanics, Fig.(9) indicates that due to numerical error or using the conventional/wrong method to normalize, the two eigenfunctions no longer cross each other precisely at those intersection points between the two paths. This means that the crossing events no longer happen. We believe that such conundrum will still exist even if we undertake the procedure of the \(PT\)-normalization initiated at (27), because eventually (27) and (29) require us to use numerical approximation again. This adds another reason why we don’t normalize eigenfunctions.
**Figure 10:** $\text{Re}(x)$ or $p$ versus the eigenfunction of the ground level along three paths (real, sin. and sym. path) for $N = 2$.

**Figure 11:** $\text{Im}(x)$ versus the eigenfunction of the ground level along three paths (real, sin. and sym. path) for $N = 2$. 
Figure 12: \( \text{Re}(x) \) or \( p \) versus the eigenfunction of the ground level along three paths (real, sin. and non-sym. path) for \( N = 2 \).

Figure 13: \( \text{Re}(x) \) versus the eigenfunction of the ground level along three paths (real, sin. and non-sym. path) for \( N = 2 \) (after magnifying).
Figure 14: $\text{Im}(x)$ versus the eigenfunction of the ground level along three paths (real, sin. and non-sym. path) for $N = 2$.

For higher energy state, we plot the 4th level on the following figures where we observe that the number of crossing events is equal to the number of intersection points between two different paths as well.
**Figure 15:** \( \text{Re} \{ x \} \) or \( p \) versus the eigenfunction of the 4th level along three paths (real, sin. and non-sym. path) for \( N = 2 \).

**Figure 16:** \( \text{Re} \{ x \} \) versus the eigenfunction of the 4th level along three paths (real, sin. and non-sym. path) for \( N = 2 \) (after magnifying).
Figure 17: $\text{Im}(x)$ versus the eigenfunction of the 4th level along three paths (real, sin. and non-sym. path) for $N = 2$.

Figure 18: $\text{Im}(x)$ versus the eigenfunction of the 4th level along three paths (real, sin. and non-sym. path) for $N = 2$ (after magnifying).
3.2.2 When $N = 3$ and $N = 2.9$

First, we draw three pairs of boundary points on Fig.(19). All these pairs $AB$, $CD$, and $C'D'$ are symmetric with respect to the imaginary axis of $x$. If label the origin as $O$, then $OA = OB$ and $OC = OD = OC' = OD'$. For further test, we set six different paths for the case of $N = 3$. Poly. path $AB$ and poly. path $CD$ are parametrized by two different polynomials, and both of paths are non-symmetric with respect to the imaginary axis, and start on the left Stokes line and end on the right Stokes line. In comparison, we add another four different paths on Fig.(19), one of which is our old friend the sinusoidal path $CD$, the other one from $C$ to $D$ crosses the positive-imaginary axis, and the rest two are straight lines. One straight line path (real path $C'D'$) is along the real axis from $C'$ to $D'$. The other straight line path (line path $CD'$) is slant, non-symmetric and connects $C$ to $D'$.

**Figure 19:** Six distinct contour paths we follow for $N = 3$. 
Table 2: Eigenvalues of the 0th and 1st level from six distinct paths for $N = 3$.

|          | Re ($E$) | Im ($E$) | Residue |
|----------|----------|----------|---------|
| Ploy. AB | 1.156267071989019 | 0.8E-14 | 0.5E-15 |
| Poly. CD | 1.156267071988113  | 0.2E-23 | 0.3E-13 |
| Sin. CD  | 1.156267071988113  | -0.1E-18| 0.1E-13 |
| Real path| 1.156267071988114  | -0.8E-17| 0.4E-14 |
| Line CD' | Unknown       | Unknown | Unknown |
| Cross cut CD | 1.156267071988113 | 0.1E-22 | 0.1E-13 |
| Ploy. AB | 4.109228752783768  | 0.2E-12 | 0.5E-15 |
| Ploy. CD | 4.109228752809652  | 0.2E-21 | 0.8E-14 |
| Sin. CD  | 4.109228752809652  | 0.1E-17 | 0.9E-14 |
| Real path| 4.109228752809730   | -0.1E-15| 0.1E-14 |
| Line CD' | Unknown       | Unknown | Unknown |
| Cross cut CD | 4.109228752809652 | -0.1E-22| 0.2E-14 |

As shown on Tab.(2), when we separate $AB$ even farther to $CD$, all imaginary parts of eigenvalues $E$ along the poly. path become smaller. These demonstrate the claim we made in the previous example.

Since no real eigenvalue associated with the straight line path $CD'$ is found, we then conclude that the two infinities $\infty_{\text{left}}$ and $\infty_{\text{right}}$ or two boundary points have to be symmetric with respect to the imaginary axis to have real eigenvalue.

Figure 20: $\text{Re}(x)$ versus the eigenfunction of the ground level along two paths (poly. CD and sin. CD) for $N = 3$. The number of crossing events is equal to the number of intersection points between the two paths.
**Figure 21:** \( \text{Re}(x) \) versus the eigenfunction of the ground level along two paths (poly. AB and sin. CD) for \( N = 3 \). No crossing event happens.

**Figure 22:** \( \text{Re}(x) \) versus the eigenfunction of the ground level along two paths (sin. CD and real C'D') for \( N = 3 \). No crossing event happens.
On Fig.(20), Fig.(21) and Fig.(22), our purpose is not to plot the entire eigenfunctions, but only to show whether the crossing events occur or not. On Fig.(20), we observe that the number of crossing events is equal to the number of intersection points between the two paths. This is not only true for integer $N$ but also for fractional $N$ (e.g. $N = 2.9$). However, if any two paths start and end at different boundary points within a pair of Stokes wedges, then the crossing event will not happen - for example, on Fig.(21) one path goes from $C$ to $D$ while the other goes from $A$ to $B$. Since $A$ and $B$ are closer to the origin, the amplitude of the wave function is smaller so that no crossing event happens. Another example is shown on Fig.(22), where one path goes from $C$ to $D$ while the other goes from $C'$ to $D'$, and no crossing event happens even though $OC = OD = OC' = OD'$.

On Tab.(2), it is a little surprise to see that the path (cross cut $CD$) yields the same eigenvalues as those paths without crossing the cut. The crossing events also happen in this case, where the two paths (cross cut $CD$ and sin. $CD$) are involved. However, it is not “safe” to cross the cut if $N$ is not an integer. For example, in case when $N = 2.9$, the locations of Stokes lines and anti-Stokes lines on Fig.(23) are slightly changed, so that we shift the boundary points $A$, $B$, $C$, $D$ accordingly and calculate eigenvalues again. Tab.(3) shows that the eigenvalue associated with the path (cross cut $CD$) is drastically changed even if $N$ is changed only by 0.1. We only find one real and negative eigenvalue. The rest eigenvalues may be complex. Tab.(2) and Tab.(3) imply that the path which crosses the cut on the positive-imaginary axis must give the same eigenvalue as those paths without crossing it, only if $N$ is an integer.

**Figure 23:** Six distinct contour paths we follow for $N = 2.9$. 

\[\text{RE} \quad \text{IM} \]

\[\begin{align*}
A & \quad B \\
C & \quad D \\
C' & \quad D'
\end{align*}\]

**Sin. path CD**

**Poly. path AB**

**Poly. path CD**

**Line path CD'**

**Real path C'D'**

**Cross cut CD**

**Anti-Stokes**

**Stokes**
Table 3: Eigenvalues of the 0th and 1st level from six distinct paths for $N = 2.9$.

| Path            | Re ($E$)               | Im ($E$)       | Residue |
|-----------------|------------------------|----------------|---------|
| Poly. AB        | 1.131396959784506      | 0.5E-13        | 0.8E-14 |
| Poly. CD        | 1.131396959777214      | 0.8E-22        | 0.9E-14 |
| Sin. CD         | 1.131396959777214      | -0.8E-20       | 0.4E-13 |
| Real path       | 1.131396959777217      | -0.9E-16       | 0.5E-14 |
| Line CD'        | Unknown                | Unknown        | Unknown |
| Cross cut CD    | -0.1948727126451554    | -0.4E-22       | 0.2E-13 |
| Poly. AB        | 3.958636971974068      | 0.1E-11        | 0.1E-14 |
| Poly. CD        | 3.958636972135053      | 0.1E-19        | 0.3E-13 |
| Sin. CD         | 3.958636972135053      | 0.1E-18        | 0.1E-13 |
| Real path       | 3.958636972135127      | 0.3E-15        | 0.2E-13 |
| Line CD'        | Unknown                | Unknown        | Unknown |
| Cross cut CD    | Unknown                | Unknown        | Unknown |

3.2.3 Summary

1. For the potential $-(ix)^N$ with $N > 1$, one necessary condition to have real-positive eigenvalue is that the two boundary points for any path must be symmetric with respect to the imaginary axis of $x$.

2. If none of paths crosses the cut, and suppose that one path has boundary points $A$ and $B$ symmetric with respect to the imaginary axis, whereas the other path has boundary points $C$ and $D$ symmetric with respect to the imaginary axis, and $A$, $B$, $C$, $D$ all lie within the same pair of Stokes wedges, then eigenvalues for these two paths must be the same even if $A \neq C$ and $B \neq D$. However, their eigenfunctions may be different.

3. Suppose that two paths have the same boundary points $A$ and $B$ symmetric with respect to the imaginary axis, one path crosses the cut on the positive-imaginary axis and the other does not, and $A$, $B$ lie within a pair of Stokes wedges, then their eigenvalues and eigenfunctions must be all independent from the shape of path if $N$ is an integer; and dependent if $N$ is an non-integer.

4. Suppose that two paths have the same boundary points $A$ and $B$ symmetric with respect to the imaginary axis, none of the paths crosses the cut on the positive-imaginary axis, and $A$, $B$ lie within a pair of Stokes wedges, then their eigenvalues and eigenfunctions must be all independent from the shape of path.
4 Multiple families of real energy spectrum

4.1 Comparison between two pairs of $PT$-symmetric wedges when $N = 5$

*Figure 24:* Two distinct paths we follow for $N = 5$.

![Diagram showing two distinct paths for $N = 5$.](image)

**Table 4:** Eigenvalues $E$ from the two distinct paths for $N = 5$.

|                | Re ($E$)       | Im ($E$)       | Residue | Ratio of Re ($E$) |
|----------------|----------------|----------------|---------|-------------------|
| Hyper. path    | 1.908264578170778 | -0.1E-29      | 0.2E-15 | 1.638318217184208 |
| Real path      | 1.164770407943415 | -0.8E-21      | 0.4E-14 |                   |
| Hyper. path    | 8.587220836207222 | -0.1E-29      | 0.7E-15 | 1.967838030619607 |
| Real path      | 4.363784367712109 | -0.1E-19      | 0.4E-13 |                   |
| Hyper. path    | 17.71080901173115 | -0.4E-28      | 0.4E-14 | 1.977719568513977 |
| Real Path      | 8.955166998240672 | -0.2E-18      | 0.9E-13 |                   |
| Hyper. path    | 28.59510331173597 | 0.4E-27       | 0.5E-13 | 1.983325673682043 |
| Real path      | 14.41775483027413 | -0.1E-17      | 0.2E-13 |                   |
| Hyper. path    | 40.91889089052085 | -0.9E-26      | 0.1E-13 | 1.985376898653392 |
| Real path      | 20.61013751004891 | -0.1E-16      | 0.4E-14 |                   |

A typical question is what if we follow a path whose boundary points are outside of the chosen (orange) wedges. Can we find any real eigenvalue? The answer is yes. As shown
on Tab.(4), we have one family of real spectrum by following the hyperbolic path within
the orange wedges and another family of real spectrum by the real path within the green
wedges on Fig.(24), and the ratios between the two families approach to a constant as the
energy level increases.

4.2 The leading-order WKB approximation

The constant ratio from Tab.(4) can be predicted by conventional WKB approximation[11].
For the first step to do this, it’s tempted to find turning points by equating $E$ and the
potential from (4), so that

$$E = -(ix)^N \implies x = E^{1/N} e^{i\left(\frac{3-2N}{N} + \frac{2j}{N}\right)\pi} \quad \text{for } j = 0, \pm 1, \pm 2 \cdots. \quad (30)$$

However, this expression may overestimate the total number of turning points on the prin-
cipal branches. For example, (30) suggests 7 turning points for $N = 3.5$, however, only 4
turning points on the principal branches.

Instead of the expression from (30), let’s write the turning points $\{x_i\}$ from a principal
branch in terms of $x_\pm$ and

$$x_- = E^{1/N} e^{i\beta}, \quad (31)$$
$$x_+ = E^{1/N} e^{i\gamma}, \quad (32)$$

where $\beta \neq \gamma$ due to different values from $j$. By the leading-order WKB approximation on
the complex plane,

$$\int_{x_-}^{x_+} dx \sqrt{E - V(x)} = \left(n + \frac{1}{2}\right)\pi, \quad (33)$$

we finally obtain[4, 11]

$$E = \left[\frac{2 \left(n + \frac{1}{2}\right) \sqrt{\pi} \Gamma \left(\frac{3}{2} + \frac{1}{N}\right)}{(e^{i\gamma} - e^{i\beta}) \Gamma \left(1 + \frac{1}{N}\right)}\right]^{\frac{2}{N+2}}, \quad (34)$$

where $E$ can be real if

$$\text{Im} \left(e^{i\gamma} - e^{i\beta}\right) = 0, \quad (35)$$

which implies that

$$\beta = \pi - \gamma. \quad (36)$$

Therefore, the turning points $x_-$ and $x_+$ must be symmetrical with respect to the imaginary
axis to have real eigenvalue $E$. By (36), we obtain the leading-order approximation for the
eigenvalue $E_n$

$$E_n \sim \left[\frac{(n + \frac{1}{2}) \sqrt{\pi} \Gamma \left(\frac{3}{2} + \frac{1}{N}\right)}{\cos \gamma \Gamma \left(1 + \frac{1}{N}\right)}\right]^{\frac{2}{N+2}} \quad \text{for } n \to \infty. \quad (37)$$

Suppose that we have two families of real spectra associated with two pairs of Stokes
wedges, then these two families must be also associated with two pairs of turning points.
Assume that one pair of turning points is associated with $\gamma_1$ and the other pair with $\gamma_2$, then by (37) we have[11]

$$\frac{E_n (\gamma_2)}{E_n (\gamma_1)} \sim \left[\frac{\cos (\gamma_1)}{\cos (\gamma_2)}\right]^{\frac{2}{N+2}} \quad \text{for } n \to \infty. \quad (38)$$
The values of real spectra from the two families must approach to the constant ratio according to (38). For example, when \( N = 5 \),

\[
\left[ \cos \left( \gamma_1 \right) \right]^{\frac{2N}{N+2}} = \left[ \frac{\cos \left( \frac{1}{10} \pi \right)}{\cos \left( -\frac{3}{10} \pi \right)} \right]^{10} = 1.988629015490531 .
\] (39)

Our numerical results from Tab.(4) agrees with this WKB approximation as the energy level \( n \) increases.

So the conclusion[11] is that there exists more than one family of real spectra if \( N \) is large enough, and, as energy level increases, one family of real spectrum over the other family maybe approaches to a constant ratio, which sometimes can be predicted by the WKB approximation. For integer \( N \), how many families of real spectra there are depends on how many pairs of symmetric turning points there are or how many pairs of symmetric but non-contacting wedges there are. We will discuss more in Sec.(4) about the families of real spectra.

### 4.3 Energy spectra from the first four families versus \( N \)

*Figure 25: Energy spectrum of the 1st family from the pair of the orange wedges (The grey curves are the WKB approximation).*

Whether the Stokes lines on the pair of the orange wedges (see Fig.(2)) move above or below the real axis depends on whether \( N < 2 \) or \( N > 2 \). Based on this fact, in our code we set when \( N \leq 1.6 \), we integrate along the parametric path defined by (23); when \( 1.6 < N < 3.0 \), we follow the real axis; when \( N \geq 3.0 \), the hyperbolic path by (22) is followed. Using
LMA and GLI with the boundary condition applied within the orange wedges, the relation between $E$ and $N$ is obtained and shown on Fig.(25). The grey curves are obtained by the WKB approximation from (37). The other colored curves are numerical results, which are displayed in blue for $N \geq 2$, in orange for $N < 2$, and in green whenever $N$ is near to a location of degeneracy. On Fig.(25) the magnified region where the degeneracy occurs clearly shows the green data points, two of which are very close to the actual value of the degenerated eigenvalue. We summarize all these green points on the Tab.(5).

The leading-order WKB method is a very good approximation, since on the most part of Fig.(25) those grey curves are covered by the blue and orange curves so that we barely see them. However, whenever $N$ is approaching a location of degeneracy, the WKB approximation is no longer reliable. The WKB also fails when $N$ approaches to 1, where only the ground state has real eigenvalue. When $N = 1$ exactly, we did not find any real eigenvalue, including the ground level. We will talk more about the WKB approximation later.

**Table 5:** Locations of the green data points (two nearest to the degenerated eigenvalues).

|   | $N$  | Re($E$) |
|---|------|---------|
| Level 0 | 1.42210 | 3.798097503566341 |
|       | 1.42210 | 3.769947569720313 |
| Level 1 | 1.57145 | 6.931062951894809 |
|       | 1.57145 | 6.909904226441585 |
| Level 2 | 1.64860 | 10.19710564838468 |
|       | 1.64860 | 10.16647647154904 |
| Level 3 | 1.69810 | 13.56278552311201 |
|       | 1.69810 | 13.50221738682984 |
| Level 4 | 1.73330 | 16.98347623074032 |
|       | 1.73330 | 16.91803839426446 |
| Level 5 | 1.76000 | 20.46649448240978 |
|       | 1.76000 | 20.37974449784742 |

By the same way, on Fig.(26) we obtain the 2nd family of eigenvalues from the green wedges defined by Fig.(1). The most interesting discovery is that when $N$ is around 4 and $E$ is about 20, the eigenvalue curve starts to go in vertical direction with horizontal oscillation, whose amplitude decreases as $E$ increases. When this curve oscillates to the left so that $N < 4$, we use red color to plot the curve; when oscillates to the right so that $N > 4$, we use green color. When $E$ is around 20, the curve is in red color and inbetween $3.97 < N < 4$; when $E$ is above 90, the curve is confined within $3.9999999999 < N < 4$, which is on the top of the figure the red part of the curve, whose oscillation is too small and can be almost ignored. We guess that as $E$ further increases, the oscillation eventually breaks the limit so that no device is able to detect such tiny oscillation, which generates an illusion that the energy spectrum at $N = 4$ is continuous for high level (similar to the classical regime), rather than quantized. However, theoretically, when $N$ is exactly equal to 4, no matter how high the energy $E$ is, the eigenvalues are still quantized and discrete points whose locations distinguish the red and green part of the curve.

Since this eigenvalue curve has infinite number of degeneracies so we name it as a “Curve with Infinite Number of Degeneracies” or “CIND”. Within the pair of the green wedges, how many CINDs are there? Here is our conjecture without solid proof. Numerical
result shows that when $E$ is above 180, the eigenvalue curve for $N = 5$ becomes another CIND. However, we did not find such trend for the eigenvalue curve when $N = 6$, possibly because the entire region where $N \geq 6$ has unbroken $PT$-symmetry. The broken $PT$-symmetry happen within the three regions where $5 < N < 6, 4 < N < 5$ and $3 < N < 4$. We observe that because of the existence of CINDs, $PT$-symmetry is never broken when $N > 3$ and $N$ is an integer. So we conclude that CIND may only exist when $N$ is an integer, and there are two CINDs associated with the pair of the green wedges.

**Figure 26**: Energy spectrum of the 2nd family from the pair of the green wedges (The grey curves are the WKB approximation).

Before going further, we introduce the concept of open mouth. Most eigenvalue curves have a standard shape similar to Fig.(27), where we call the empty region between two adjacent and connected levels as an “open mouth”. The head of an open mouth is the location of the degeneracy. As $N$ increases, only two cases are observed: one is that the open mouth tilts upward for increasing $N$, and the other one is shown on Fig.(27) in which the open mouth initially tilts downward but eventually tilts upward if $N$ is large enough. Particularly, for the green wedges $N$ has to be around or larger than 6 for that part of the open mouth tilts upward. If the head of the open mouth is located far less than $N = 6$, then the tendency to initially tilt downward is more pronounced. This is especially obvious for those low-lying states as shown on Fig.(27) where the head of the open mouth is located at the region where $N < 4$ and far less than 6.

Why does the vertically straight line around $N = 6$ differentiate the behavior of the open mouth? When $N < 6$, Fig.(1) shows that the pair of the two green wedges moves above the real axis so that the $PT$-symmetry of the wedges is broken for any non-integer $N$, and this movement may cause the open mouth to tilt downward. When $N > 6$, Fig.(1) shows
that the green wedges moves below the real axis so that the $PT$-symmetry is unbroken for any real $N$, and consequently the open mouth may tilt upward. In short, the location of the $PT$-symmetric wedges may determine which direction the open mouth tilts in. This also implies that the open mouths of eigenvalue curves from the pink wedges (See Fig.(28)) will eventually tilt upward as $N > 10$.

**Figure 27:** A single eigenvalue curve from the pair of the green wedges.

![Figure 27](image1)

**Figure 28:** Energy spectrum of the 3rd family from the pair of the pink wedges (The grey curves are the WKB approximation).

![Figure 28](image2)
On Fig.(28) we plot the 3rd family of eigenvalues for the pair of the pink wedges defined by Fig.(1). There are four CINDs, where CINDs for \( N = 8 \) and \( N = 9 \) happen in much higher level. The region where \( N \geq 10 \) has unbroken \( PT \)-symmetry, whereas broken \( PT \)-symmetry happens within the region \( N < 10 \) except for integer \( N \). As the previous case of the 2nd family, the amplitude of horizontal oscillation for CIND decreases as \( E \) increases. Taking the CIND for \( N = 6 \) as an example, the part of the CIND where \( N < 6 \) is plotted in black color and \( N > 6 \) in pink color. Within the region around \( E = 15 \) and \( N = 6 \), the black part of the CIND is confined within \( 5.87 < N < 6 \); whereas for the region where \( E > 90 \), the CIND (in black color again) is within \( 5.9999998 < N < 6 \).

On Fig.(28), we magnify the most interesting region, where two levels fail to connect and form a degeneracy around \( E = 80 \) because they are too close to the integer \( N = 7 \), where \( PT \)-symmetry is never broken. Consequently, the curve from the lower level merges with the curve from the even lower level; whereas the curve from the upper level form a CIND for \( N = 7 \). This kind of behavior is somewhat similar to the cohesion of liquid, where similar or identical particles tend to cling to one another if the distance between them is small enough.

**Figure 29:** Four energy spectra of the first four families from four pairs of \( PT \)-symmetric (orange, green, pink, yellow) wedges.

By including real eigenvalues from the four \( PT \)-symmetric wedges (the orange, green, pink and yellow wedges), we plot the first four families of energy spectra altogether shown on Fig.(29). An interesting feature is immediately spotted. All open mouths from the 1st family from the orange wedges tilt upward, while the open mouths from the 3rd family from the pink wedges tilt downward, and the 4th family tilts downward even more. This is because that all heads of open mouths from the 1st family including the low-lying states
are quite near to the vertically straight line $N = 2$ where differentiates the regions with broken and unbroken $PT$-symmetry for the 1st family, while heads of open mouths from the 4th family are quite far away from $N = 14$ which differentiates the regions with broken and unbroken $PT$-symmetry for the 4th family.

Now let’s talk about another interesting feature. At exactly $N = 4$, the eigenvalues from the 1st and 2nd family are equal, which means that rather than two families, there is only one family of eigenvalues at $N = 4$. Similarly, at exactly $N = 6$, the eigenvalues from the 1st and 3rd family are equal, which means that rather than three families, there are only two families at $N = 6$. The most interesting part is at exactly $N = 8$ that not only the eigenvalues from the 1st and 4th family are equal, but the 2nd and 3rd family are also equal, so that rather than four families, there are only two families at $N = 8$. Do we expect this feature? Yes, because of symmetry. When $N = 4$, Fig.(1) shows that orange and green wedges are symmetric with respect to the real axis. When $N = 6$, the orange and pink wedges are symmetric with respect to the real axis while the pair of green wedges lies right on the real axis. When $N = 8$, not only the orange and yellow wedges but also the green and pink wedges are symmetric with respect to the real axis. This symmetry reduces the number of families of eigenvalues at even $N$ except when $N = 2$. The moment when $N$ is an odd integer, a new $PT$-symmetric wedges are born from the positive-imaginary axis and consequently, a new family of spectrum is born.

4.4 A comment on the WKB approximation

By observing on previous figures, the leading-order WKB method from (37) is a pretty good approximation in those regions with unbroken $PT$-symmetry, including those integers $N$ associated with CINDs. However, the WKB approximation may fail wherever the $PT$-symmetry is broken. The reason why it fails can be subtler than the reason provided by the paper[4], where it says that when $N < 2$, the path along which the integral $\int_{x_0}^{x_1} dx \sqrt{E - V(x)}$ is real is in the upper-half $x$ plane so that it crosses the cut on the positive-imaginary axis and thus is not a continuous path joining the turning points. This reason is only true if the pair of orange wedges from Fig.(1) is chosen. Let’s see why.

For the following 1D Schrodinger equation,

$$-\frac{d^2\psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x),$$

the leading-order WKB approximation gives two asymptotic solutions as $|x| \to \infty$

$$\psi_\pm(x) \sim \frac{1}{Q(x)^{1/4}} \exp \left( \pm i \int_{x_0}^{x} [Q(t)]^{1/2} dt \right) \quad \text{where} \quad Q(x) = E - V(x) = E + (ix)^N,$$  \hspace{1cm} (40)

where $x_0$ is a turning point, and $x$ is a complex variable. To be consistent with our previous definition, we define Stokes line as

$$\text{Re} \left\{ \int_{x_0}^{x} [Q(t)]^{1/2} dt \right\} = 0,$$  \hspace{1cm} (41)
and anti-Stokes line as

\[
\text{Im} \left\{ \int_{x_0}^x \left[ Q(t) \right]^{1/2} \, dt \right\} = 0,
\]

Fig.(30) is called Stokes diagram which shows the first-order approximation of the Stokes and anti-Stokes lines when \( N = 5 \). This diagram includes more detailed Stokes structure than on Fig.(24), where only the Stokes structure at \( |x| \to \infty \) is shown. (See [13, p.75] to know how to generate Stokes diagram.)

**Figure 30**: Stokes diagram for \( N = 5 \) and \( E = 1 \), where the yellow points are the turning points, the green dot-lines are Stokes lines and the orange dot-lines are anti-Stokes lines.

Fig.(26) shows that the WKB method from (37) is still pretty good approximation to the eigenvalue within the green wedges defined by Fig.(24) when \( N = 5 \). But Fig.(30) shows that the two turning points \( x_2 \) and \( x_4 \) within the green wedges are joined by an anti-Stokes line segment, which crosses the positive-imaginary axis and by its definition implies that the integral \( \int_{x_2}^{x_4} dx \sqrt{E - V(x)} \) is real. So the argument provided by the paper[4] only works for non-integer \( N \). For any integer \( N \) associated with a CIND, the \( PT \)-symmetry is not really broken for the given wedges.

### 5 Conclusion

By using numerical and WKB approximation, we have answered the three questions we posed in the abstract of this paper. Although these answers lack of mathematical rigor and
consequently can not guarantee to be 100% correct, we present students and researchers with wealthy information, user-friendly interface and keen insight which are very useful under the background of contemporary physics to understand the concept of Stokes wedge and eigenvalue problem in complex plane. We believe that a good understanding means not only understanding through rigorous mathematical proof but also understanding through empirical evidence, visualization and approximation. Those empirical observations are essential for students to develop mathematical intuition and finally come up with their own proofs.

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