Inner structure of Gauss-Bonnet-Chern Theorem and the Morse theory

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We define a new 1-form $H^A$ based on the second fundamental tensor $H_{ab}$, the Gauss-Bonnet-Chern form can be novelty expressed with this 1-form. Using the $\phi$-mapping theory we find that the Gauss-Bonnet-Chern density can be expressed in terms of the $\delta$-function $\delta(\phi)$ and the relationship between the Gauss-Bonnet-Chern theorem and Hopf-Poincaré theorem is given straightforward. The topological current of the Gauss-Bonnet-Chern theorem and its topological structure are discussed in details. At last, the Morse theory formula of the Euler characteristic is generalized.

I. INTRODUCTION

The Gauss-Bonnet-Chern (GBC) theorem is one of the most significant results in topology, which relates the curvature tensor of a compact and oriented even-dimensional Riemannian manifold $M$ to an important topological invariant, the Euler characteristic $\chi(M)$ of $M$. A great advance in this field is the discovery of the relationship between supersymmetry and the index theorem, which includes the derivation of the GBC theorem via supersymmetry and the path integral techniques as presented by Alvarez-Gaumé et al. [6]. On the physics side, the optical Berry phase is a direct result of the Gauss-Bonnet theorem [7] and the black hole entropy emerges as the Euler class through dimensional continuation of the Gauss-Bonnet theorem [8]. Furthermore, it must be pointed out that the topological current of GBC theorem is important to study the topological defects in Ginzburg-Landau theory et al. [4, 5].

It is well-known that there were three theoretical ways to study the Euler characteristic $\chi(M)$ of a compact oriented even-dimensional Riemannian manifold $M$. The first is the integral of GBC density in terms of Chern form [9]; the second is the Hopf-Poincaré theorem [10] which shown that the $\chi(M)$ is the sum of the indices of the zeroes of the vector field on $M$; and the third is the so called Morse theory [11] which related to the Hessian Matrices. The main purpose in this paper is to present a new theoretical framework which directly gives the inner relationships between GBC theorem, Hopf-Poincaré theorem and the Morse theory. For this purpose we introduce a new 1-form $H^A$ based on the second fundamental tensor $H_{ab}$, and give a decomposed expression of the curvature 2-form $F^{AB}$. Using this expression we find an important formula of the GBC form, which is simply determined by the edge production of the 1-form $H^A$. It gives a natural way to obtain the $\delta$-function form of the GBC density with the $\phi$-mapping theory [12]. By making use of the above formulation the GBC theorem is proved to be identify with the Hopf-Poincaré theorem. It is noted that the Chern’s form and the gauge potential decomposition of SO(n) [13] are unnecessary in the proof. Furthermore, according to the topological structure of the GBC density, a generalization of Morse theory can be obtained directly when the vector field on the manifold is considered as the gradient field of a Morse function $f$.

In Section II we give the definition of the 1-form $H^A$ and the expression of the curvature 2-form $F^{AB}$ in the Riemannian manifold. In Section III, in terms of the $\phi$-mapping theory we get the $\delta$-function expression of the GBC density and study the topological structure of the GBC density, in which the Hopf index and Brouwer degree of $\phi$-mapping at the zero points of $\phi(\vec{x})$ play an important role. The topological current of GBC theorem is given and the general velocity of the $i$-th zero of the vector field $\phi(\vec{x})$ is studied in detail in Section IV. At last, we discuss the relationship between Euler characteristic $\chi(M)$ and indices of the critical points of function $f$ and get the generalized formula of $\chi(M)$ in the Morse theory in Section V.

II. A NEW 1-FORM $H^A$ IN RIEMANNIAN MANIFOLD

Let $M$ be a compact and oriented $(n(even))$-dimensional Riemannian manifold, which can be immersed in a Euclidean space $R^{m+n}(m+n) = n(n+1)/2$ with local coordinate $x^\mu$ which satisfy the parametric equation [12]

$$x^\mu = x^\mu(u^1, u^2, ..., u^n), \quad \mu = 1, 2, ..., m+n$$

where $u^a$ is the local coordinate of $M$. In the $(m+n)$-dimensional Euclidean space, the tangent vector $B^\mu_a$ on the $M$ is defined as

$$B^\mu_a = \frac{\partial x^\mu}{\partial u^a}, \quad a = 1, 2, ..., n$$
and the metric tensor \( g_{ab} \) in the \( M \) is determined by
\[
g_{ab} = B^\mu_a B^\mu_b.
\]
The unit normal vector \( N^\mu_A \) satisfy
\[
N^\mu_A B^\mu_a = 0, \quad \mathbf{A} = n + 1, n + 2, \ldots, (m + n),
\]
and the second fundamental tensor \( H_{ab,\mathbf{A}} \) is introduced as follow:
\[
H_{ab,\mathbf{A}} = N^\mu_A \nabla_a B^\mu_b, \quad \mu = 1, 2, \ldots, (m + n), \quad a, b = 1, 2, \ldots, n.
\]
Using the Gauss-Codazzi formula \[12\] the curvature tensor \( R_{ab,cd} \) is expressed with the second fundamental tensor
\[
R_{ab,cd} = H_{ac,\mathbf{A}} H_{bd,\mathbf{A}} - H_{ad,\mathbf{A}} H_{bc,\mathbf{A}}.
\]
Then the curvature tensor of the SO(n) principle bundle, i.e., the SO(n) gauge field tensor \( F_{AB}{}^{ab} = -R_{ab,cd} e^A e^B d \) can be written as
\[
F_{AB}{}^{ab} = (H_{ac,\mathbf{A}} H_{bd,\mathbf{A}} - H_{ad,\mathbf{A}} H_{bc,\mathbf{A}}) e^A e^B d,
\]
where \( e^A \) is the veilbein on \( M \), which is defined by
\[
g^{ab} = e^A e^B, \quad A = 1, 2, \ldots, n.
\]
It is well known that the metric \( g^{ab} \) is invariant under the SO(n) gauge transformation
\[
e^A = L_B^A(u) e^B, \quad L_B^A(u) \in SO(n).
\]
From the above formulation we see that if we define a 1-form \( H_A^A \)
\[
H_A^A = e^A H_{abA} d^b,
\]
the field strength 2-form \( F_{AB} \) can be expressed as
\[
F_{AB} = \frac{1}{2} F_{ab} d^a \wedge d^b = H_A^A \wedge H_B^B.
\]

### III. GAUSS-BONNET-CHERN DENSITY

The GBC form \( \Lambda \) is a \( n \)-form over \( M \), i.e.
\[
\Lambda = \frac{(-1)^{n/2}}{2^{n/2} \pi n/2 (n/2)!} e^{A_1 A_2 \ldots A_n} F^{A_1 A_2} \wedge F^{A_3 A_4} \wedge \ldots \wedge F^{A_{n-1} A_n}.
\]
The famous GBC theorem can thus be expressed as
\[
\chi(M) = \int_M \Lambda.
\]
With the 1-form \( H_A^A \) we can obtain a new elegant expression of the GBC form
\[
\Lambda = \frac{(-1)^{n/2}}{2^{n/2} \pi n/2 (n/2)!} e^{A_1 A_2 \ldots A_n} H_A^{A_1} \wedge H_A^{A_2} \wedge \ldots \wedge H_A^{A_{n/2}}.
\]
Let \( n^a (a = 1, 2, \ldots, n) \) be a unit tangent vector on \( M \)
\[
n^a n^a = 1,
\]
by which two new unit tangent vectors \( n^\mu \) and \( n^A \) can be defined as

\[
n^\mu = B^\mu_a n_a, \quad n^A = e^A_{a^1} n_{a^1}, \quad (8)
\]

and it is easy to verify that

\[
n^\mu N^\mu_A = 0, \quad n^A n^A = 1. \quad (9)
\]

From Eqs. (8) and (9) one can find

\[
H^A_A n^A = N^\mu_A dn^\mu. \quad (10)
\]

To study the topology of manifold \( M \) by making use of the unit vector field \( n^\mu \) in \( R^{(m+n)} \), it must be required that both \( n^\mu \) and \( dn^\mu \) should be intrinsic on \( M \), i.e.,

\[
N^\mu_A dn^\mu = 0. \quad (11)
\]

Thus one can find a relation

\[
H^A_A n^A = 0. \quad (12)
\]

It can be seen from Eq. (11) that the 1-form \( H^A_A \) is invariant in general covariant coordinates transformation on \( M \) and covariant in the \( SO(n) \) gauge transformation, i.e.,

\[
n^A = L^A_B (u) n^B, \quad H^A_A = L^A_B (u) H^B_B. \quad (13)
\]

Since the covariant 1-form \( H^A_A \) is perpendicular to \( n^A \), for each fixed index \( A \), the only covariant 1-form of \( H^A_A \) related to \( n^A \) should be

\[
H^A_A = k^A_A (u) Dn^A, \quad (13)
\]

where \( k^A_A (u) \) is a scalar function for each fixed index on \( M \) and

\[
Dn^A = dn^A - \omega^{AB} n^B, \quad (14)
\]

in which \( \omega^{AB} \) is the spin connection, i.e., the connection of the \( SO(n) \) principle bundle. Then from Eq. (13) the field strength \( F^{AB} \) can be given

\[
F^{AB} = k^2(u) Dn^A \wedge Dn^B, \quad (14)
\]

where \( k^2(u) = k^A_A (u) k^A_A (u) \).

With the Eqs. (13) and (14) the GBC form can be expressed as

\[
\Lambda = \left( \frac{(-1)^{n/2} k^u(u)}{2^n \pi^{n/2} (n/2)!} \right) \epsilon^{A_1 A_2 \ldots A_n} Dn^{A_1} \wedge Dn^{A_2} \wedge \ldots \wedge Dn^{A_n}. \quad (15)
\]

Here following Chern’s work [6] and let \( P \) be an arbitrary but fixed point of \( R^n \). In the neighborhood of \( P \) one can choose a family of veilbein \( \{e^{\mu_a} \} \) such that at \( P \)

\[
\omega^{AB} = 0, \quad (16)
\]

which gives

\[
\Lambda = \left( \frac{(-1)^{n/2} k^u(u)}{2^n \pi^{n/2} (n/2)!} \right) \epsilon^{A_1 A_2 \ldots A_n} dn^{A_1} \wedge dn^{A_2} \wedge \ldots \wedge dn^{A_n}. \quad (17)
\]

Since the integral region is the manifold \( M \), at the same time the integral kernel is only function defined in this manifold \( M \), the last result should be independent of the choice of Euclidean space’s dimension. With the help of \((n+1)\)-dimensional Euclidean space \( R^{n+1} \), we can give

\[
k = \left( \frac{n!}{(n-1)!} \right)^{1/n}, \quad (18)
\]
which is discussed in detail in the appendix. Then the above GBC form can be simply expressed as

$$\Lambda = \frac{1}{A(S^{n-1})(n-1)!} \varepsilon^{A_1 A_2 \ldots A_n} dn^{A_1} \wedge dn^{A_2} \wedge \ldots \wedge dn^{A_n},$$

(19)

which is just the result obtained verbosely by Chern in Ref. [3]. In the \( \phi \)-mapping theory [3, 7] the unit vector \( n^A \) should be further determined by the smooth vectors \( \phi^A \), i.e.

$$n^A = \frac{\phi^A}{||\phi||}, \quad ||\phi||^2 = \sqrt{\phi^A \phi^A}. \quad (20)$$

In fact \( n \) is identified as a section of the sphere bundle over \( M \) (or a partial section of the vector bundle over \( M \)). We see that the zeroes of \( \phi \) are just the singular points of \( n \). Since the global property of a manifold has close relation with zeroes of a smooth vector fields on it, the above expression of the unit vector \( n \) is a very powerful tool in the discussion of the global topology.

Substituting Eq. (20) into Eq. (19), the GBC form can be given as [9, 11]

$$\Lambda = \delta(\tilde{\phi})D(\frac{\tilde{\phi}}{\tilde{u}})d^n u$$

(21)

where the Jacobian \( D(\frac{\tilde{\phi}}{\tilde{u}}) \) is defined as

$$\varepsilon^{A_1 A_2 \ldots A_n} D(\frac{\tilde{\phi}}{\tilde{u}}) = \varepsilon^{a_1 a_2 \ldots a_n} \partial_{a_1} \phi^1 \partial_{a_2} \phi^2 \ldots \partial_{a_n} \phi^n = \delta(\tilde{\phi})D(\frac{\tilde{\phi}}{\tilde{u}}), \quad (22)$$

which shows that only at the zero points of the vector field \( \tilde{\phi}(u) \), i.e.

$$\phi^1(u^1, u^2, \ldots, u^n) = 0,$$

$$\phi^2(u^1, u^2, \ldots, u^n) = 0,$$

$$\ldots$$

$$\phi^n(u^1, u^2, \ldots, u^n) = 0, \quad (23)$$

we can get the nontrivial GBC density. The expressions (21) and (22) are of great importance: they yield, in our case, the evident result of the Hopf theorem.

Suppose that the vector field \( \tilde{\phi}(u) \) possesses \( l \) isolated zeroes, according to the implicit function theorem [13, 14], when the Jacobian \( D(\tilde{\phi}/u) \neq 0 \), the solutions of Eq. (23) are generally expressed as

$$u_i^A = z_i^A, \quad A = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, l.$$

In terms of the \( \phi \)-mapping theory [3, 10] and the \( \delta \)-function theory [15], one can rigorously prove that the \( \delta(\tilde{\phi}(u)) \) can be expanded as

$$\delta(\tilde{\phi}) = \sum_{i=1}^l \frac{W_i \delta(\tilde{u} - \tilde{z}_i)}{D(\tilde{\phi}/u)|_{u=\tilde{z}_i}}, \quad W_i = \beta_i \eta_i,$$

where \( W_i \) is the winding number of the vector field, \( \beta_i = |W_i| \) is the Hopf index and \( \eta_i \) is the Brouwer degree of map \( x \rightarrow \phi \) [13]

$$\eta_i = \text{sgn} D(\frac{\tilde{\phi}}{\tilde{u}})|_{u=\tilde{z}_i} = \pm 1.$$

This leads to the following topological structure

$$\Lambda = \delta(\tilde{\phi})D(\frac{\tilde{\phi}}{\tilde{u}})d^n u = \beta_i \eta_i \delta(\tilde{u} - \tilde{z}_i)d^n u,$$

which means that GBC form is labeled by the Brouwer degree and Hopf index. Therefore, the Euler characteristic \( \chi(M) \) can be represented as

$$\chi(M) = \int_M \Lambda = \sum_{i=1}^l W_i. \quad (24)$$

Here, Eq. (24) states that the sum of indices of the zeroes of vector \( \tilde{\phi} \) is the Euler characteristic. Therefore, the topological structure of GBC form reveals the expected result of the Hopf-Poincaré theorem.
IV. THE GBC TOPOLOGICAL CURRENT

Let us consider the \((n+1)\)-dimensional manifold \(M \times \mathbb{R}\) with coordinate \(u^0 = t \in \mathbb{R}\) denoted as a time variable. The line element of \(M \times \mathbb{R}\) is
\[
ds^2 = \bar{g}_{\alpha\beta} du^\alpha du^\beta = (du^0)^2 - g_{ab} du^a du^b \quad (\alpha, \beta = 0, 1, \cdots, n; \ a, b = 1, 2, \cdots, n),
\]
which implies
\[
\bar{g} = det(\bar{g}_{\alpha\beta}) = -g, \quad \sqrt{-g} = \sqrt{g}.
\]

A generally covariant GBC topological current is defined as
\[
j^\alpha := \frac{(-1)^{n/2}}{2^{n} \pi^{n/2} (n/2)!} \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_n} F^{A_1 A_2 \cdots A_n}.
\]
It follows that
\[
j^\alpha := \frac{1}{A(s^n-1)/(n-1)!} \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_n} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_n} \partial_{\alpha_1} n^{A_1} \partial_{\alpha_2} n^{A_2} \cdots \partial_{\alpha_n} n^{A_n}, \quad (25)
\]
where \(j^0\) is just the GBC density \(1/\sqrt{|g|}\). Obviously the current \((25)\) is identically conserved,
\[
\nabla_\alpha j^\alpha = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} j^\alpha) = 0.
\]

If we define \(n+1\) Jacobians as
\[
\epsilon^{A_1 \cdots A_n} D^n(\phi/u) = \epsilon_{\alpha_1 \cdots \alpha_n} \partial_{\alpha_1} \phi^{A_1} \cdots \partial_{\alpha_n} \phi^{A_n}, \quad (26)
\]
in which \(D^n(\phi/u) = D(\phi/u)\) is the usual \(n\)-dimensional Jacobian. Making use of the Laplacian relation, we obtain the \(\delta\)-function-like current
\[
j^\alpha = \frac{1}{\sqrt{g}} \delta(\phi) D^n(\phi/u).
\]

Suppose \(\phi^A(u)\) possesses \(l\) isolated zeroes on \(M\) and let the \(i\)-th zero be \(\bar{u} = \bar{z}_i\) i.e.
\[
\phi^A(\bar{z}_i, t) = 0, \quad i = 1, 2, \cdots, l, \quad A = 1, 2, \cdots, n,
\]
according to the implicit function theorem whose solution can be expressed as \((11)\)
\[
\bar{z}_i = \bar{z}_i(t), \quad (28)
\]
which is the trajectory of the \(i\)-th zero. One can prove that the general velocity of the \(i\)-th zero \([3, 4]\).
\[
V^\alpha := \frac{d\bar{z}_i^\alpha}{dt} = \left. \frac{D^n(\phi/u)}{D(\phi/u)} \right|_{\bar{u} = \bar{z}_i}. \quad V^0 = 1, \quad (30)
\]
Then the topological current \(j^\alpha\) can be written as the current density form of a system including \(l\) classical point particles with topological charge \(W_i = \beta_i \eta_i\) moving in the \((n+1)\)-dimensional space-time
\[
j^\alpha = \frac{1}{\sqrt{g}} \sum_{i=1}^{l} W_i \delta(\bar{u} - \bar{z}_i(t)) \frac{d\bar{z}_i^\alpha}{dt}, \quad j^0 = \frac{1}{\sqrt{g}} \sum_{i=1}^{l} W_i \delta(\bar{u} - \bar{z}_i(t)). \quad (31)
\]
The total charge of the system is
\[
W := \int_M j^0 \sqrt{|g|} du = \sum_{i=1}^{l} \beta_i \eta_i = \sum_{i=1}^{l} W_i, \quad (32)
\]
which is none other than the topological invariant $\chi(M)$, i.e. $\chi(M) = W$. From (22) and (27), we get a concise expression for the topological current,

$$j^\alpha = \rho \frac{D^\alpha(\phi/u)}{\sqrt{g} D(\phi/u)} = \frac{1}{\sqrt{g}} \rho V^\alpha,$$

which takes the same form as the current density in hydrodynamics. From (32) the topological charge can be expressed as

$$W = \int_M \rho d^n u = \int_M \delta(\vec{\phi}) D(\phi/u) d^n u = \deg n \int_{n(M)} \delta(\vec{\phi}) d^n \phi,$$

where $\deg n$ is the degree of the mapping $n$ [16]. It indicates $W = \deg n$. Expressions (31)-(34) show that the topological structure of the GBC topological current is characterized by the Brouwer degrees and Hopf indices. The structure of the GBC topological current and its formulation is of great use in studying the topological defects in the Ginzburg-Landau theory [4, 5] and topological field theories, especially in low-dimensional cases [10].

In our theory the point-like particles with topological charges $W_i = \beta_i \eta_i (i = 1, 2, \cdots, l)$ are called GBC topological particles. These particles are just located at the zeros of $\vec{\phi}(u)$, i.e. the singularities of the unit vector $\vec{n}(u)$. The charges of them are topologically quantized.

V. FROM GBC THEOREM TO MORSE THEORY

In this section we will study the relation between Euler characteristic $\chi(M)$ and indices of the critical points in Morse theory via the topological structure. We will show that the formula of $\chi(M)$ in Morse theory is only a corollary of the GBC theorem.

Let $f$ be an arbitrary function on $M$. A critical point of $f$ is a point $p \in M$ at which $df$ vanishes

$$df|_p = \partial_a f du^a|_p = 0,$$

and at such point the Hessian $H_f(p)$ is well defined quadratic form on $T_p M$, the tangent space to $M$ at $p$. In local coordinates $\{ u^a \}$ centered at $p$, the matrix of $H_f(p)$ relative to the base $\partial_a$ at $p$ is then given by

$$\{ H_f(p) \}_{ab} = \frac{\partial^2 f}{\partial u^a \partial u^b},$$

which is called the Hessian matrix. If there exist $l$ critical points on $f$, we denote them by $p_i (i = 1, 2, \cdots, l).$ The index of $p_i$ is the number of negative eigenvalues of $det H_f(p_i)$ and it will be denoted by $\lambda_i(f)$.

Now let the smooth vector field $\vec{\phi}$ be a gradient field [8] of the function $f$ on $M$ as

$$\vec{\phi}^A = e^{Aa} \partial_a f,$$

which means that the critical points of $f$ are just the zero point of $\vec{\phi}$. From Eq. (24), we get

$$\partial_b \vec{\phi}^A|_p = e^{Aa} \partial_a \partial_b f|_p.$$

In terms of the above equation and the formula

$$\varepsilon_{A_1 A_2 \cdots A_n} e^{A_1 a_1} e^{A_2 a_2} \cdots e^{A_n a_n} = \varepsilon^{a_1 a_2 \cdots a_n} \frac{1}{\sqrt{g}},$$

one can find

$$D(\vec{\phi}|_u)_p = \frac{1}{g} det H_f(p).$$

Therefore the GBC form and Euler characteristic $\chi(M)$ can be represented in terms of the Hopf indices $\beta_i$ and the Hessian $H_f(p_i)$

$$\vec{n}^* d\Omega = \sum_{i=1}^l \beta_i (\vec{n} - \vec{n}_i) \frac{det H_f(p_i)}{|det H_f(p_i)|} \sqrt{g} d^n u,$$
\[ \chi(M) = \sum_{i=1}^{l} \beta_i \frac{\det H_f(p_i)}{|\det H_f(p_i)|}. \] (38)

In the Morse theory \[8, 17\] it is well-known that Morse function \( f \) has only the non-degenerate critical points \( p \) at which \( f \) satisfies

\[ \det H_p f \neq 0. \]

At the neighborhood of any critical point \( p_i \), Morse function \( f \) can take following form

\[ f = f(p_i) - (u^1)^2 - ... - (u^{\lambda_i})^2 + ... + (u^n)^2, \] (39)

where \( \lambda_i = 0, 1, ..., N \). Substituting (39) into (38), one can get the generalized expression of \( \chi(M) \) in the Morse theory

\[ \chi(M) = \sum_{i=1}^{l} \beta_i (-1)^{\lambda_i}, \] (40)

when \( f \) is taken as a Morse function. In the case of \( \beta_i = 1 \), one can get the common Morse theory formula of \( \chi(M) \)

\[ \chi(M) = \sum_{i=1}^{l} (-1)^{\lambda_i}. \] (41)

Since the meaning of Hopf index \( \beta_i \) is that when the point \( \vec{u} \) covers the neighborhood of the zero \( z_i \) on \( U_i \) once, the vector field \( \vec{\phi} \) covers the corresponding region \( \beta_i \) times, we can think that the Hopf index corresponding to some physical degeneracy. The formula (41) is only the special case of non-degeneracy.

VI. ACKNOWLEDGMENTS

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VII. APPENDIX: THE COEFFICIENT IN EQ. (18)

Here we give a proof of the Eq. (18). From Eqs. (13) and (16) one can obtain

\[ H^A = k(u)du^A. \] (42)
In terms of the Eqs. (1), (8) and (10) one can get
\[ dN^\mu = -B_\alpha^\mu e^{A_\alpha} H^A. \]

From Eq. (50) we can find the relationship
\[ dN^\mu = -k(u) B_\alpha^\mu e^{A_\alpha} dn^A, \]  
(43)

where \( n^A \) are the unit tangent vectors which satisfy
\[ n^A n^A = 1. \]  
(44)

It is well-known that the area element of \( S^n \) is
\[ ds = \frac{1}{n!} \varepsilon_{\mu_1 \mu_2 \ldots \mu_n} N^{\lambda} dN^{\mu_1} \wedge dN^{\mu_2} \wedge \ldots \wedge dN^{\mu_n}. \]

In order to discuss the coefficient we introduce the following \( n \)-form
\[ I = \frac{2}{A(S^n)n!} \varepsilon_{\mu_1 \mu_2 \ldots \mu_n} N^{\lambda} dN^{\mu_1} \wedge dN^{\mu_2} \wedge \ldots \wedge dN^{\mu_n}, \]
in terms of Eq. (51) which can be expressed as
\[ I = \frac{2}{A(S^n)n!} k^n(u) \varepsilon_{A_1 A_2 \ldots A_n} dn^{A_1} \wedge dn^{A_2} \wedge \ldots \wedge dn^{A_n}. \]  
(45)

We split \( S^n \) into hemispheres \( S^\pm \), i.e. \( S^n = S^+ + S^- \), and \( \partial S^+ = S^{n-1} \), we have
\[ \int_{S^+} I = \frac{2}{A(S^n)} \int_{S^+} ds \]
\[ = \frac{2}{A(S^n)} \frac{1}{2} A(S^n) \]
\[ = 1. \]  
(46)

For the unit vector \( n^A \), from Eq. (44) we have
\[ n^A dn^A = 0, \]
which can be considered as the linear homogeneous system of equations about \( n^A \). In case that \( n^A \) has the nontrivial solution, i.e. \( n^A \) has no singularity in the Riemannian manifold \( M \), there is
\[ det(\partial_a n^A) = 0, \]
or
\[ \varepsilon^{a_1 a_2 \ldots a_n} \varepsilon_{A_1 A_2 \ldots A_n} \partial_{a_1} n^{A_1} \partial_{a_2} n^{A_2} \ldots \partial_{a_n} n^{A_n} = 0. \]

Thus we have \( I = 0 \), i.e. the coefficient \( k \) in the Eq. (50) can be treated arbitrarily in the non-singularities, so we only need to consider the \( k \) in the singularity. Suppose that there are \( l \) isolated singularities in the \( M \), the \( i \)th singularity is
\[ u = z_i, \quad i = 1, 2, \ldots, l. \]

We know that there is at least a singularity when we enclose the sphere using a open face, and set this singularity is \( z_i \). With Eq. (53) and Stokes’ theorem we have
\[ \int_{S^+} I = \frac{2}{A(S^n)} \int_{S^+} \frac{1}{n!} k^n(u) \varepsilon_{A_1 A_2 \ldots A_n} dn^{A_1} \wedge dn^{A_2} \wedge \ldots \wedge dn^{A_n} \]
\[ = \frac{2}{A(S^n)} \frac{k^n(z_i)}{n!} \int_{S^{n-1}} \varepsilon_{A_1 A_2 \ldots A_n} n^{A_1} dn^{A_2} \wedge dn^{A_3} \wedge \ldots \wedge dn^{A_n} \]
\[ = \frac{2}{A(S^n)} \frac{k^n(z_i)}{n!} (n-1)! A(S^{n-1}). \]  
(47)
Considering the Eq. (52) one can find
\[ k^n(z_i) = \frac{A(S^n)n!}{2A(S^{n-1})(n-1)!} = \frac{n!!}{(n-1)!!} \]  
which shows the relationship of the Gauss mappings between the \( n \)-dimension and the \((n+1)\)-dimension. Obviously the Eq. (53) is valid in other singularities, in the other words, we have the result (54) for all singularities. Then we obtain the relation
\[ H^A = \left(\frac{n!!}{(n-1)!!}\right)^{1/n} dn^A, \]
this is just the Eq. (13).

Here we give a proof of the Eq. (18). In order to discussing the coefficient we introduce the following \( n \)-form
\[ I = \frac{2}{A(S^n)n!} k^n(u) \varepsilon_{A_1 A_2 \ldots A_n} dn^{A_1} \wedge \ldots \wedge \ldots \wedge dn^{A_n}. \]  
For the unit vector \( n^A \), from Eq. (4) we have
\[ n^A dn^A = 0, \]
which can be considered as the linear homogeneous system of equations about \( n^A \). In case that \( n^A \) has the nontrivial solution, i.e. \( n^A \) has no singularity in the Riemannian manifold \( M \), there is
\[ det(\partial_a n^A) = 0, \]
or
\[ \varepsilon^{a_1 a_2 \ldots a_n} \varepsilon_{A_1 A_2 \ldots A_n} \partial_a n^{A_1} \partial_{a_2} n^{A_2} \ldots \partial_{a_n} n^{A_n} = 0. \]
Thus we have \( I = 0 \), i.e. the coefficient \( k \) can be treated arbitrarily in the non-singularities, so we only need to consider the \( k \) in the singularity for the \( n \)-form \( I \). Considering the property of manifold, each closed neighborhood \( M_i \) of singularity can always be immersed in a \((n+1)\)-dimensional Euclidean space \( R^{n+1} \). In this Euclidean space \( R^{n+1} \) there exist a vector \( N^\mu \)
\[ N^\mu = \frac{1}{n!} \frac{1}{\sqrt{g}} \varepsilon^{\mu_1 \mu_2 \ldots \mu_n} \varepsilon_{a_1 a_2 \ldots a_n} B^{\mu_1}_{a_1} B^{\mu_2}_{a_2} \ldots B^{\mu_n}_{a_n}, \quad \mu = 1, 2, \ldots, n+1 \]
which is normal to \( M_i \), i.e. \( N^\mu B_\mu = 0 \). Similarly we have
\[ H^A = -B_\mu e^{A_\mu} dN^\mu. \]
From Eqs. (13) one can obtain
\[ H^A = kdn^A. \]  
then,
\[ dn^A = -k^{-1} dN^\mu B^\mu e^{A_\mu}. \]  
In terms of Eq. (51) the \( n \)-form \( I \) can be expressed as
\[ I = \frac{2}{A(S^n)n!} \varepsilon^{\lambda \mu_1 \mu_2 \ldots \mu_n} \lambda N^\mu_1 dN^{\mu_1} \wedge dN^{\mu_2} \wedge \ldots \wedge dN^{\mu_n}. \]
Let \( S^n = S^+ + S^- \), and \( \partial S^+ = S^{n-1} \), we have
\[ \int_{S^+} I = \int_{S^-} \int_{S^+} ds \]  
\[ = \frac{2}{A(S^n)n!} \int_{S^n} ds \]  
\[ = \frac{1}{2} A(S^n) \]  
\[ = 1. \]  
\[ \therefore \]
where $ds$ is the area element of $S^n$

$$ds = \frac{1}{n!} \varepsilon_{\lambda\mu_1\mu_2...\mu_n} N^\lambda dN^{\mu_1} \wedge dN^{\mu_2} \wedge ... \wedge dN^{\mu_n}.$$ 

Suppose that there are $l$ isolated singularities in the $M$, the $i$th singularity is

$$u = z_i, \quad i = 1, 2, ..., l.$$ 

We know that there is at least a singularity when we enclose the sphere using an open face, and set this singularity is $z_i$. With Eq. (49) and Stokes’ theorem we have

$$\int_{S^+} I = \frac{2}{A(S^n)} \int_{S^+} \frac{1}{n!} k^n \varepsilon_{A_1 A_2...A_n} dn^{A_1} \wedge dn^{A_2} \wedge ... \wedge dn^{A_n}$$

$$= \frac{2}{A(S^n)} \frac{k^n}{n!} \int_{S^{n-1}} \varepsilon_{A_1 A_2...A_n} n^{A_1} dn^{A_2} \wedge dn^{A_3} \wedge ... \wedge dn^{A_n}$$

$$= \frac{2}{A(S^n)} \frac{k^n}{n!} (n-1)! A(S^{n-1}). \quad (53)$$

Considering the Eq. (52) one can find

$$k^n = \frac{A(S^n)n!}{2A(S^{n-1})(n-1)!} = \frac{n!!}{(n-1)!!}, \quad (54)$$

which shows the relationship of the Gauss mappings between the $n$-dimension and the $(n + 1)$-dimension. Obviously the Eq. (53) is valid in other singularities, in the other words, we have the result (54) for all singularities. Then for the $n$-form $\Lambda$ we obtain the relation

$$H^n = \left( \frac{n!!}{(n-1)!!} \right)^{1/n} dn^n,$$

this is just the Eq. (18).