THE REGULARIZATION OF SOLUTION FOR THE COUPLED NAVIER-STOKES AND MAXWELL EQUATIONS

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ABSTRACT. The purpose of this paper is to build the existence of time-spatial global regular solution to the coupled Navier-Stokes and Maxwell equations.

1. Introduction. It is customary to describe the dynamics of homogeneous, incompressible, conducting fluids under the influence of body forces and applied currents by the system

\[
\begin{align*}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \mu \mathbf{H} \times \nabla \mathbf{p} &= \mathbf{f}, \\
\text{div} \mathbf{v} &= 0, \\
\text{curl} \mathbf{E} &= -\mu \mathbf{H}_t, \\
\text{curl} \mathbf{H} &= \sigma (\mathbf{E} + \mu \mathbf{v} \times \mathbf{H}) + \mathbf{j}, \\
\text{div} \mathbf{H} &= 0,
\end{align*}
\]

see, e.g.,[4, 9]. Here \(\mathbf{v}\) denotes the velocity field, \(p\) the pressure, \(\mathbf{f}\) the known body force, \(\mathbf{j}\) the known applied current, and \(\mathbf{E}\) and \(\mathbf{H}\) the electric and magnetic fields, respectively. \(\mu, \sigma, \text{ and } \nu\) are the magnetic permeability, the electric conductivity and the kinematic viscosity. The density of the fluid is assumed to equal to one. In this paper, we consider the standard case where \(\mu, \sigma, \text{ and } \nu\) are all positive constants. The displacement current is proportional to \(\mathbf{E}_t\) and is assumed to be negligible.

According to [5] generalized solutions of the system with three different initial-boundary value problems are exist. For the unique solvability of those problems, some results that analogous to the results of A.Kiselev and O.Ladyzhenskaya [3] for the three-dimensional Navier-Stokes system were derived. Furthermore, in [2], it was shown that the problem is globally uniquely generalized solution to the coupled modified Navier-Stokes and Maxwell equations in three dimensional case of initial-boundary value problems.

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In order to investigate the regularity of solutions of the coupled modified Navier-Stokes and Maxwell equations, we concern the coupled Navier-Stokes and Maxwell equations with nonlinear dispersive long waves as the same as considering the mathematical models for the unidirectional propagation of long waves in nonlinear and dispersive systems

$$u_t + uu_x + u_{xxx} = 0.$$  

The term $u_{xxx}$ represents dispersive and dissipative effects when they are considered. The term $-u_{xxt}$ we chosen instead of $u_{xxx}$ not only obviates the difficulties in question but also makes the model more suitably posed for long waves systems; see, e.g., [1, 7]. Therefore in this paper we consider the following equation:

$$v_t + v \cdot \nabla v - \nu \Delta v - \Delta v_t + \mu H \times \text{curl} H + \nabla p = f.$$  

As discussing the long wave propagation equation by the following equation

$$u_t + uu_x - u_{xxt} = 0,$$

we first consider the coupled Navier-Stokes and Maxwell equations with nonlinear dispersive long waves

$$\begin{cases} 
    v_t + v \cdot \nabla v - \nu \Delta v - \Delta v_t + \mu H \times \text{curl} H + \nabla p = f, \\
    \text{div} v = 0, \\
    \text{curl} E = -\mu H_t, \\
    \text{curl} H = \sigma(E + \mu v \times H) + j, \\
    \text{div} H = 0.
\end{cases}$$

Next we consider the regularization of solution of the coupled Navier-Stokes and Maxwell equations over bounded spatial-time domain. Finally we show the existence of regular solution over global spatial-time space.

The regularization of the initial-boundary value problem in a bounded, three-dimensional domain with fixed perfectly conducting boundaries is discussed for this system with the smooth boundary condition by energy method and inductive method. In the rest of this section, we establish some notation that will be used throughout the paper and then provide a description of the problems we consider.

1.1. Notation. In this section, we introduce the notation that will be used throughout the paper. Vector-valued function will be denoted in bold-face, i.e., $u = (u_1, u_2, ..., u_l) \in R^l$ and furthermore, $u \cdot v = \sum_{k=1}^{l} u_k v_k$, we also denote by $u \cdot v = u_k v_k$ unless explicitly noted. Points in Euclidean space $R^l$ are denoted by $x = (x_1, ..., x_l)$ and $k$ times partial derivatives are denoted by $D^k u = u_{x^k}$, for example $D^3 u = u_{x^3} = u_{xxx}$. Notice that $u_{x^k} = \frac{\partial u}{\partial x^k}$, $u^k = \{\partial^\alpha u \mid |\alpha| = k\}$.

Let $\Omega \subset R^l$ denotes an open domain with boundary $\partial \Omega$. For $x \in \partial \Omega$, $n = n(x)$ denotes an outward unit normal of $\partial \Omega$ and $\tau = \tau(x)$ denotes unit tangential vectors of $\partial \Omega$.

The Lebesgue spaces are denoted by $L^p(\Omega)$ and have norms

$$\|\phi\|_{p,\Omega} = \left(\int_{\Omega} |\phi(x)|^p \, dx\right)^{\frac{1}{p}} \text{ for } p \in [1, \infty).$$

The inner product in $L^2(\Omega)$ is denoted by $(\phi, \varphi) = \int_{\Omega} \phi \varphi \, dx$. Sobolev spaces are denoted by $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ is defined to be the closure of $W^{k,p}(\Omega)$ with zero boundary.
The set of all infinitely differentiable functions with compact support with respect to \( \Omega \) is denoted by \( C^\infty_c(\Omega) \). We then introduce the set

\[
K^\infty(\Omega) = \{ v \in C^\infty_c \left| \text{div } v = \sum_{i=1}^l v_{i,i} = 0 \right. \}
\]

and the subspace of \( L^2(\Omega) \)

\[
K(\Omega) = \{ v \in L^2(\Omega) \mid \text{div } v = 0 \}
\]

where \( \text{div } v = 0 \) is understood in the sense of distributions, i.e.,

\[
\int_\Omega v \cdot \nabla \phi \, dx = 0 \quad \forall \phi \in C^\infty_c(\Omega).
\]

Then, \( K_0(\Omega) \) is defined to be the closure of \( K^\infty(\Omega) \) in the norms of \( L^2(\Omega) \) and we also define

\[
K^{k,p}(\Omega) = W^{k,p}(\Omega) \cap L^2(\Omega)
\]

and \( K_0^{k,p}(\Omega) \), the closure of \( K^\infty(\Omega) \) in the norms of \( W^{k,p}(\Omega) \) which are given by

\[
\| \phi \|_{W^{k,p}(\Omega)} = \sum_{|i| = 0}^k \left\| \frac{\partial^{|i|} \phi}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_\ell^{i_\ell}} \right\|_{p,\Omega},
\]

where \( i_1, i_2, \ldots, i_\ell \) are non-negative integers and \( |i| = \sum_{j=1}^\ell i_j \).

Finally, \( C, C_1, C_2 \) will denote positive constants whose value changes with context and some other notation in this paper will explain appropriately.

1.2. Formulation of problems. Set \( \Omega \subset \mathbb{R}^3 \) a bounded domain, \( v = (v_1, v_2, v_3) \), \( H = (H_1, H_2, H_3) \), and the system is supplemented by the initial data

\[
v|_{t=0} = v^0 \quad \text{and} \quad H|_{t=0} = H^0
\]

and the boundary conditions:

\[
v|_{S_T} = 0, H|_{S_T} = 0 \quad \text{and} \quad (\text{curl } H)|_{S_T} = 0,
\]

where \( S_T = \partial \Omega \times [0, T] \).

For the system, along with the identity

\[
\text{curl}(v \times H) = H \cdot \nabla v - v \cdot \nabla H
\]

and the Maxwell equations, we can easily derived the relation

\[
v_t + v \cdot \nabla v - \nu \Delta v - \Delta v_t + \mu H \times \text{curl } H + \nabla p = f, \quad (1.1)
\]

\[
\text{div } v = 0, \quad (1.2)
\]

\[
\mu H_t + \frac{1}{\sigma} \text{curl } \text{curl } H + \mu (v \cdot \nabla H - H \cdot \nabla v) = \frac{1}{\sigma} \text{curl } j, \quad (1.3)
\]

\[
\text{div } H = 0. \quad (1.4)
\]

We consider the system (1.1)-(1.4) in \( Q_T = \Omega \times (0, T) \) with some fixed time \( 0 < T < \infty \).
1.3. Main results. Now we state our main result.

**Theorem 1.1.** Suppose that $\Omega$ is a bounded domain in $R^3$ with smooth boundary and let $Q_T = \Omega \times (0, T)$ with some fixed times $T < \infty$. $v^0 \in W^{k+1,2}(\Omega)$, $H^0 \in W^{k,2}(\Omega)$, $f, \text{curl} \tilde{f} \in L^2(0, T; W^{k-1,2}(\Omega))$, $\text{div} j = 0$, $k = 0, 1, 2, \ldots$, and suppose that $v_{x,k}$ and $H_{x,k}$ equal to 0 on $\partial \Omega$ in the trace sense. Then the system has generalized solution $v$, $H$ and the generalized solution has the properties

$$\max_{t \in [0, T]} \|v_{x,k}\|_{2, \Omega}, \max_{t \in [0, T]} \|H_{x,k}\|_{2, \Omega} < \infty$$

and

$$\|v_{x,k+1}\|_{2, Q_T}, \|H_{x,k+1}\|_{2, Q_T} < \infty.$$ 

**Theorem 1.2.** Let $\Omega = R^3$,

i) If $Q = R^3 \times (0, T)$, $v^0 \in D^{k,2}(R^3)$, $H^0 \in D^{k-2}(R^3)$, $f, \text{curl} f \in L^2(0, T; D^{k-1,2}(\Omega))$, $\text{div} j = 0$, $k = 0, 1, 2, \ldots$. Then the system has generalized solution $v$, $H$ with the properties

$$\max_{t \in [0, T]} \|v_{x,k}\|_{2, \Omega}, \max_{t \in [0, T]} \|H_{x,k}\|_{2, \Omega} < \infty$$

and

$$\|v_{x,k}\|_{2, Q}, \|H_{x,k}\|_{2, Q} < \infty.$$

ii) If $Q = R^3 \times (0, 0)$, $v^0 \in C^\infty_0(R^3)$, $H^0 \in C^\infty_0(R^3)$, $f, \text{curl} f \in C^\infty_0(\tilde{Q})$, $\text{div} j = 0$, $k = 0, 1, 2, \ldots$. Then the system has generalized solution $v$, $H$ with the properties

$$\max_{\tilde{Q}} |v_{x,k}|, \max_{\tilde{Q}} |H_{x,k}| < \infty$$

for every $k$.

2. Estimates. In this section, we divide into three parts: three order for the estimates at most, some additional estimates and higher order estimates under the appropriate conditions.

2.1. An energy relation and corresponding estimates. Instead of the equations (1.1), (1.3) and the identity

$$H \times \text{curl} H = -H \cdot \nabla H + \frac{1}{2} \nabla |H|^2,$$

we will use the integral identities

$$(v_t + v \cdot \nabla v, \eta) - (\nu \Delta v + (v \cdot \nabla) v, \eta) - \mu (H \cdot \nabla H, \eta) = (f, \eta), \quad (2.1)$$

$$(\mu H_t, \zeta) - \frac{1}{\sigma} (\Delta H, \zeta) + \mu (v \cdot \nabla H - H \cdot \nabla v, \zeta) = (\frac{1}{\sigma} \text{curl} j, \zeta) \quad (2.2)$$

for any $\eta, \zeta \in K_0(\Omega)$.

For the identity (2.1) with $\eta = v$ and (2.2) with $\zeta = H$, we get the energy relation

$$\frac{1}{2} \frac{d}{dt} \|v\|^2_{2, \Omega} + \mu \|H\|^2_{2, \Omega} + \|\nabla v\|^2_{2, \Omega} + \nu \|\nabla v\|^2_{2, \Omega} + \frac{1}{\sigma} \|\nabla H\|^2_{2, \Omega}$$

$$= (f, v) + \frac{1}{\sigma} (\text{curl} j, H). \quad (2.3)$$

In order to obtain the identity (2.3), we have used the identity (1.2), (1.4) and the boundary conditions. Furthermore, we have used the relation

$$(u \cdot \nabla v, w) = -(u \cdot \nabla w, v)$$
that holds for arbitrary elements \( u, v, w \) of \( W^{1,2}(\Omega) \) satisfying the conditions

\[
\text{div } v = 0, \quad \int_{\partial \Omega} \sum_{i=1}^{3} \sum_{k=1}^{3} u_k v_i w_i n_k dS = 0,
\]

and the equality

\[
(curl \, u, v) = (curl \, v, u)
\]

holds if \( u |_{\partial \Omega} = 0 \).

From (2.3), we get the inequality

\[
\frac{1}{2} \frac{d}{dt} (\|v\|_{L^2(\Omega)}^2 + \mu \|H\|_{L^2(\Omega)}^2 + \|v_x\|_{L^2(\Omega)}^2 + \nu \|v_{xx}\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2) + 1 \|\text{curl} \, j\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + \|v_x\|_{L^2(\Omega)}^2 + \|H\|_{L^2(\Omega)}^2 + \|v_{xx}\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2).
\]

The result of integrating (2.4) over \( t \) and using the Gronwall’s inequality, we can yield the relation

\[
\max_{t \in [0, T]} \|v(t)\|_{H^1(\Omega)}^2 + \max_{t \in [0, T]} \|H(t)\|_{H^1(\Omega)}^2 + \max_{t \in [0, T]} \|v_x(t)\|_{L^2(\Omega)}^2 + \|v_x\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2)
\]

\[
= \Phi(T; \|v^0\|_{H^1(\Omega)}^2, \|v^0\|_{H^1(\Omega)}^2, \|v^0\|_{H^1(\Omega)}^2, \|H^0\|_{H^1(\Omega)}^2, \|f\|_{L^2(\Omega)}^2, \|\text{curl} \, j\|_{L^2(\Omega)}^2).
\]

Here, \( \Phi \) is a continuous function.

In the identity (2.1) choosing \( \eta = -\Delta v \) and (2.2) choosing \( \zeta = -\Delta H \), we can gain the relation

\[
\frac{1}{2} \frac{d}{dt} (\|v_x\|_{L^2(\Omega)}^2 + \mu \|H_{xx}\|_{L^2(\Omega)}^2 + \|v_{xx}\|_{L^2(\Omega)}^2) + \frac{1}{\sigma} \|H_{xx}\|_{L^2(\Omega)}^2 + \nu \|v_{xx}\|_{L^2(\Omega)}^2
\]

\[
= \mu (H \cdot \nabla H, -\Delta v) - (v \cdot \nabla v, -\Delta v) - \mu (v \cdot \nabla v, H, -\Delta H) + \mu (H \cdot \nabla v, -\Delta H) + \frac{1}{\sigma} (\text{curl} \, j, -\Delta H) + (f, -\Delta v)
\]

\[
\leq \mu \int_{\Omega} |
\nabla H|^2 |\nabla v| \, dx + \mu \int_{\Omega} |H| |
\nabla v| |\Delta H| \, dx + \nu \|\Delta v\|_{L^2(\Omega)}^2 + \frac{1}{8\sigma} \|\Delta H\|_{L^2(\Omega)}^2 + C(\|H \cdot \nabla v\|_{L^2(\Omega)}^2 + \|v \cdot \nabla H\|_{L^2(\Omega)}^2 + \|H \cdot \nabla v\|_{L^2(\Omega)}^2 + \|v \cdot \nabla H\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2 + \|H_{xx}\|_{L^2(\Omega)}^2))
\]

\[
\leq A_1 + A_2 + A_3 + \nu \|\Delta v\|_{L^2(\Omega)}^2 + \frac{1}{8\sigma} \|\Delta H\|_{L^2(\Omega)}^2 + C(\|f\|_{L^2(\Omega)}^2 + \|\text{curl} \, j\|_{L^2(\Omega)}^2),
\]

where

\[
A_1 = \mu \int_{\Omega} |
\nabla H|^2 |\nabla v| \, dx, \quad A_2 = \mu \int_{\Omega} |H| |
\nabla v| |\Delta H| \, dx,
\]

\[
A_3 = C(\|v \cdot \nabla v\|_{L^2(\Omega)}^2 + \|v \cdot \nabla H\|_{L^2(\Omega)}^2 + \|H \cdot \nabla v\|_{L^2(\Omega)}^2).
\]

To majorize the terms in the right-hand side of (2.6), we will introduce the imbedding inequalities

\[
\|u\|_{m, \Omega} \leq C \|u\|_{r, \Omega} + C_1 \|u\|_{L^2(\Omega)}
\]

with \( m \leq \frac{3r}{2} \) for \( r \in [1, 3] \) and with constants \( C, C_1 \) that depend on \( m, r \). Particularly, \( C_1 = 0 \) if \( \|u\|_{\partial \Omega} = 0 \) or \( \int_{\Omega} u \, dx = 0 \). We will also apply the multiplicative inequalities

\[
\|u\|_{q, \Omega} \leq C \|u\|_{L^2(\Omega)}^\alpha \|u\|_{L^2(\Omega)}^{1-\alpha} + C_1 \|u\|_{L^2(\Omega)}
\]
with $\alpha = 3\left(\frac{1}{2} - \frac{1}{q}\right) \in [0, 1]$ and $q \in [2, 6]$ and with constants $C, C_1$ that depend on $q$. Particularly, if $u|_{\Omega} = 0$ or $\int_{\Omega} u \nabla \sigma = 0$, then $C_1 = 0$.

For the estimation of $A_1$, we apply the Hölder inequality, the multiplicative inequalities, the imbedding inequalities and the Young’s inequalities. In more detail,

$$A_1 \leq C \left\| v_x \right\|_{3,\Omega} \left\| H_x \right\|_{3,\Omega}^2 \leq C \left\| v_x \right\|_{3,\Omega} \left( \left\| H_{xx} \right\|_{2,\Omega} \left\| H_x \right\|_{2,\Omega} + \left\| H_x \right\|_{2,\Omega}^2 \right)$$

$$\leq \frac{1}{4\sigma} \left\| H_{xx} \right\|_{2,\Omega}^2 + C \left\| v_x \right\|_{3,\Omega} \left( \left\| v_x \right\|_{2,\Omega} + \left\| v_{xx} \right\|_{2,\Omega} \right) \left\| H_x \right\|_{2,\Omega}^2$$

$$\leq \frac{1}{4\sigma} \left\| H_{xx} \right\|_{2,\Omega}^2 + C \left\| H_x \right\|_{2,\Omega}^2 \left( \left\| v_x \right\|_{2,\Omega} + \left\| v_{xx} \right\|_{2,\Omega} \right) \left\| H_x \right\|_{2,\Omega}^2$$

$$\leq \frac{1}{4\sigma} \left\| H_{xx} \right\|_{2,\Omega} + \varepsilon \left\| v_x \right\|_{2,\Omega} + C(\varepsilon) \left\| v_x \right\|_{2,\Omega} \left\| H_x \right\|_{2,\Omega}^2 + C_1(\varepsilon) \left\| H_x \right\|_{2,\Omega}^4$$

$$+ C \left( \left\| v_x \right\|_{2,\Omega} + \left\| v_{xx} \right\|_{2,\Omega} \right) \left\| H_x \right\|_{2,\Omega}^2$$

$$\leq \frac{1}{4\sigma} \left\| H_{xx} \right\|_{2,\Omega} + \varepsilon \left\| v_x \right\|_{2,\Omega} + CF_1 \left\| H_x \right\|_{2,\Omega}^2,$$  \hspace{1cm} (2.7)

where $F_1 = (1 + \left\| v_x \right\|_{2,\Omega}^2) \left\| H_x \right\|_{2,\Omega}^2 + \left\| v_x \right\|_{2,\Omega} \left\| v_{xx} \right\|_{2,\Omega} + \left\| v_x \right\|_{2,\Omega}^2$.

It is easy to obtain the estimations of $A_2$ and $A_3$ analogous to $A_1$. In more detail,

$$A_2 \leq C \left( \left\| H \right\|_{3,\Omega}^2 \left\| v_x \right\|_{3,\Omega} \right) \leq C \left( \left\| H_x \right\|_{2,\Omega}^2 + \left\| H \right\|_{2,\Omega}^2 \right) \left( \left\| v_x \right\|_{2,\Omega} + \left\| v_{xx} \right\|_{2,\Omega} + \left\| v_x \right\|_{2,\Omega}^2 \right)$$

$$\leq \varepsilon_1 \left\| v_x \right\|_{2,\Omega} + C(\varepsilon_1) \left\| v_x \right\|_{2,\Omega} \left( \left\| H_x \right\|_{2,\Omega}^4 + \left\| H \right\|_{2,\Omega}^2 \right)$$

$$+ \varepsilon_1 \left\| v_x \right\|_{2,\Omega} + CF_2 \left\| H_x \right\|_{2,\Omega} + C \left\| v_x \right\|_{2,\Omega} \left( \left\| H \right\|_{2,\Omega}^4 + \left\| H \right\|_{2,\Omega}^2 \right),$$ \hspace{1cm} (2.8)

where $F_2 = \left\| v_x \right\|_{2,\Omega}^2 + \left\| v_x \right\|_{2,\Omega} \left\| H_x \right\|_{2,\Omega}^2$.

$$A_3 = C \left( \int_{\Omega} \left| v \cdot \nabla v \right|^2 \right) dx + \int_{\Omega} \left| v \cdot \nabla H \right|^2 dx + \int_{\Omega} \left| H \cdot \nabla v \right|^2 dx$$

$$\leq C \left( \left\| v \right\|_{6,\Omega}^2 \left\| v_x \right\|_{3,\Omega} \right) + \left\| H \right\|_{6,\Omega} \left\| H \right\|_{3,\Omega} \leq C \left( \left\| v \right\|_{2,\Omega} + \left\| H \right\|_{2,\Omega}^2 \right) \left( \left\| v_x \right\|_{2,\Omega} + \left\| v_{xx} \right\|_{2,\Omega} + \left\| v_x \right\|_{2,\Omega}^2 \right)$$

$$+ C \left( \left\| H \right\|_{2,\Omega} \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega} \right) \left\| H \right\|_{2,\Omega} \left\| H \right\|_{2,\Omega} \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega}^2 \right)$$

$$\leq \varepsilon_2 \left\| v_x \right\|_{2,\Omega} + C(\varepsilon_2) \left\| v_x \right\|_{2,\Omega} \left( \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega} \right)$$

$$+ C \left( \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega} \right) \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega}^2 + \left\| v \right\|_{2,\Omega}^2$$

$$+ \varepsilon_3 \left\| H \right\|_{2,\Omega}^{2,\Omega} + C \left( \left\| H \right\|_{2,\Omega}^4 + \left\| H \right\|_{2,\Omega}^2 \right) \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega}^2$$

$$+ C \left( \left\| H \right\|_{2,\Omega} \left\| v \right\|_{2,\Omega} + \left\| v \right\|_{2,\Omega} \right)$$

$$\leq \varepsilon_2 \left\| v_x \right\|_{2,\Omega} + \varepsilon_3 \left\| H \right\|_{2,\Omega}^4 + CF_3 \left\| v \right\|_{2,\Omega}^2$$

$$+ C \left( \left\| v \right\|_{2,\Omega} \left\| H \right\|_{2,\Omega}^4 + \left\| v \right\|_{2,\Omega} \right) + \left\| v \right\|_{2,\Omega}^2 + \left\| v \right\|_{2,\Omega}^2 \right) \left\| v \right\|_{2,\Omega}^2,$$ \hspace{1cm} (2.9)

where $F_3 = \left\| v \right\|_{2,\Omega} \left\| H \right\|_{2,\Omega}^4 + \left\| v \right\|_{2,\Omega} \left\| v \right\|_{2,\Omega}^2 + \left\| v \right\|_{2,\Omega}^4 + \left\| v \right\|_{2,\Omega}^2$.

Using the the inequalities (2.7), (2.8) and (2.9) with sufficiently small $\varepsilon, \varepsilon_1, \varepsilon_2$ and $\varepsilon_3$, we can conclude from (2.6) that

$$\frac{1}{2} \frac{d}{dt} \left( \left\| v_x \right\|_{2,\Omega}^2 + \left\| H \right\|_{2,\Omega}^2 + \left\| v_{xx} \right\|_{2,\Omega}^2 \right) + \frac{1}{2\sigma} \left\| H_{xx} \right\|_{2,\Omega}^2 + \frac{\nu}{2} \left\| v_{xx} \right\|_{2,\Omega}^2$$

$$\leq C \left( \left\| F \right\|_{2,\Omega}^2 + \left\| \nabla \sigma \right\|_{2,\Omega}^2 \right) + C \left( F_1 + F_2 + F_3 \right) \left\| H \right\|_{2,\Omega}^2.$$
\[ + C \|v_x\|^2_{2,\Omega} (\|v_x\|^4_{2,\Omega} + \|v\|^2_{2,\Omega} + \|H\|^4_{2,\Omega} + \|v_x\|^2_{2,\Omega} + \|v\|^2_{2,\Omega} + \|H\|^2_{2,\Omega}). \quad (2.10) \]

The result of integrating (2.10) over \( t \), we apply the Gronwall’s inequality to yield the relation
\[
\max_{t \in [0,T]} \|v_x\|^2_{2,\Omega} + \max_{t \in [0,T]} \|H_x\|^2_{2,\Omega} + \max_{t \in [0,T]} \|v_{xx}\|^2_{2,\Omega} + \|H_{xx}\|^2_{2,\Omega} + \|v_{xx}\|^2_{2,\Omega} \leq \Phi_1(T, \|v_x\|^2_{2,\Omega} \|H_x\|^2_{2,\Omega})
\]
for a continuous function \( \Phi_1 \) that depends on the arguments of \( \Phi \) from (2.5).

### 2.2. Some additional estimates.
In (2.1), let \( \eta = v_t \) (this is allowable since \( \text{div} \ v_t = 0 \) and \( v_t \big|_{\partial \Omega} = 0 \)), see, e.g., [5] and we have the relation
\[
(v_t, v_t) - \langle \Delta v_t, v_t \rangle = (f, v_t) - (v \cdot \nabla v, v_t) + (\nu \Delta v, v_t) + \mu (\nabla v, \text{curl} v_t, v_t).
\]
From this we can get the relation
\[
\frac{1}{2} \|v_t\|^2_{2,\Omega} + \|\nabla v_t\|^2_{2,\Omega} \leq C \|v\| \|\nabla v\|^2_{2,\Omega} + \|H\| \|\nabla H\|^2_{2,\Omega} + \|\Delta v\|^2_{2,\Omega} + 2 \|f\|^2_{\Omega}
\]
\[
\leq C \left( \|H\|^2_{6,\Omega} \|H_x\|^2_{4,\Omega} + \|v\|^4_{4,\Omega} + \|v_x\|^4_{4,\Omega} + \|v_{xx}\|^2_{2,\Omega} + 2 \|f\|^2_{\Omega} \right)
\]
\[
\leq C \left( \|H\|^2_{2,\Omega} + \|H_x\|^2_{2,\Omega} \|H_{xx}\|^2_{2,\Omega} \right)
\]
\[
+ C_2 (\|v\|^4_{2,\Omega} + \|v_x\|^2_{2,\Omega} + \|v_{xx}\|^2_{2,\Omega}) + 2 \|f\|^2_{2,\Omega}. \quad (2.11)
\]
According to previous estimates and integrating over \( t \) in (2.11), we obtain the inequality
\[
\|v_t\|_{2,\Omega} + \|\nabla v_t\|_{2,\Omega} \leq D,
\]
where \( D \) depends on the function \( \Phi \) and \( \Phi_1 \).

Similarly let \( \eta = -\Delta v_t \) in (2.1) and it is easy to gain the relation
\[
\frac{1}{2} \|\Delta v_t\|^2_{2,\Omega} + \frac{\nu}{2} \frac{d}{dt} \|\Delta v\|^2_{2,\Omega}
\]
\[
\leq C \left( \|v\| \|\nabla v\|^2_{2,\Omega} + \|H\| \|\nabla H\|^2_{2,\Omega} + \|v_t\|^2_{2,\Omega} \right) + 2 \|f\|^2_{\Omega}. \quad (2.12)
\]
From the result of (2.11) and integrating over \( t \) in (2.12), we can obtain the inequality
\[
\|\Delta v_t\|_{2,\Omega} \leq D_1
\]
with \( D_1 \) under our control.

It is also easy to get estimates for \( H_t, \nabla p \) analogous to estimates in [2].

### 2.3. Higher order estimates.
In this section, in order to get higher order estimates, we have to get the three order relation and corresponding estimates first.

Let \( \eta = \Delta^2 v \) in (2.1) and \( \zeta = \Delta^2 H \) in (2.2), then we can obtain the relation
\[
\frac{1}{2} \frac{d}{dt} (\|v_{xx}\|^2_{2,\Omega} + \mu \|H_{xx}\|^2_{2,\Omega} + \|v_{xxx}\|^2_{2,\Omega} + \nu \|v_{xxx}\|^2_{2,\Omega} + \frac{1}{\sigma} \|H_{xxx}\|^2_{2,\Omega} \Omega)
\]
\[
= (f, \Delta^2 v) + \frac{1}{\sigma} (\text{curl} j, \Delta^2 H) + (\mu \nabla H, \Delta^2 v) - (v \cdot \nabla v, \Delta^2 v)
\]
\[
- \mu (v \cdot \nabla H - H \cdot \nabla v, \Delta^2 H)
\]
\[
\leq C \left( \|v \cdot \nabla v\|^2_{2,\Omega} + \|\nabla (H \cdot \nabla H)\|^2_{2,\Omega} + \|v \cdot (\nabla H)\|^2_{2,\Omega} + \|\nabla (H \cdot \nabla v)\|^2_{2,\Omega} \right)
\]
\[
+ \frac{\nu}{2} \|v_{xxx}\|^2_{2,\Omega} + \frac{1}{2\sigma} \|H_{xxx}\|^2_{2,\Omega} + C (\|f\|_{2,\Omega}^2 + \|\text{curl} j\|_{2,\Omega}^2). \quad (2.13)
\]
For the estimation of \( \|\nabla(\mathbf{v} \cdot \nabla\mathbf{v})\|^2_{2,\Omega} \) in (2.13), we apply the Hölder inequalities, the multiplicative inequalities, the Young’s inequalities. In more detail,

\[
\|\nabla(\mathbf{v} \cdot \nabla\mathbf{v})\|^2_{2,\Omega} \leq C \int_{\Omega} (|\nabla\mathbf{v}|^2 + |\mathbf{v}| |\Delta\mathbf{v}|)^2 dx \leq C (\|\mathbf{v}\|_{4,\Omega}^4 + \|\mathbf{v}\|_{6,\Omega}^2 \|\mathbf{v}_{xx}\|_{3,\Omega}^2)
\]

\[
\leq C (\|\mathbf{v}\|_{2,\Omega}^2 + \|\mathbf{v}_x\|_{2,\Omega}^2) (\|\mathbf{v}_{xx}\|_{2,\Omega} + \|\mathbf{v}_{x,x}\|_{2,\Omega}^2) + C_1 \|\mathbf{v}_x\|_{4,\Omega}^4
\]

\[
\leq \varepsilon_1 \|\mathbf{H}_{xx}\|_{2,\Omega}^2 + C(\varepsilon_1) \|\mathbf{H}_{xx}\|_{2,\Omega}^4 (\|\mathbf{H}\|_{2,\Omega}^4 + \|\mathbf{H}_x\|_{2,\Omega}^4) + C_1 \|\mathbf{H}_{xx}\|_{2,\Omega}^4 (\|\mathbf{H}\|_{2,\Omega}^4 + \|\mathbf{H}_x\|_{2,\Omega}^4);
\]

\[
(2.15)
\]

\[
\|\nabla(\mathbf{v} \cdot \nabla\mathbf{v})\|^2_{2,\Omega} \leq C (\|\mathbf{v}_x\|_{2,\Omega}^2 + \|\mathbf{H}\|_{2,\Omega}^4 \|\mathbf{H}_{xx}\|_{2,\Omega}^2)
\]

\[
\leq C (\|\mathbf{v}_x\|_{4,\Omega}^2 (\|\mathbf{H}_x\|_{4,\Omega}^2 + \|\mathbf{H}\|_{6,\Omega}^2 \|\mathbf{v}_{xx}\|_{3,\Omega}^2)
\]

\[
\leq \varepsilon_2 \|\mathbf{H}_{xx}\|_{2,\Omega}^2 + C(\varepsilon_2) \|\mathbf{H}_{xx}\|_{2,\Omega}^4 (\|\mathbf{v}_x\|_{2,\Omega}^4 + \|\mathbf{H}_x\|_{2,\Omega}^4) + C_1 \|\mathbf{H}_{xx}\|_{2,\Omega}^4 (\|\mathbf{v}_x\|_{2,\Omega}^2 + \|\mathbf{H}_x\|_{2,\Omega}^4)
\]

\[+ C_2 \|\mathbf{H}_{xx}\|_{2,\Omega}^2 (\|\mathbf{v}_x\|_{2,\Omega}^2 + \|\mathbf{H}_x\|_{2,\Omega}^4)
\]

(2.16)

Using the inequalities (2.14)-(2.17) with sufficiently small \( \varepsilon, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), we can conclude from (2.13) that

\[
\frac{1}{2} \frac{d}{dt} \left( \|\mathbf{v}_{xx}\|_{2,\Omega}^2 + \|\mathbf{H}_{xx}\|_{2,\Omega}^2 + \|\mathbf{v}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2 \right)
\]

\[
\leq C(\|\nabla\mathbf{f}\|_{2,\Omega}^2 + \|\nabla\text{curl}\mathbf{j}\|_{2,\Omega}^2) + C_1 G(\|\mathbf{H}_{xx}\|_{2,\Omega}^2) + C_2 G_1,
\]

(2.18)

where \( G = \|\mathbf{H}\|_{4,\Omega}^4 + \|\mathbf{H}_x\|_{2,\Omega}^4 + \|\mathbf{H}_{xx}\|_{2,\Omega}^2 + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{v}_x\|_{4,\Omega}^4
\]

\[+ \|\mathbf{v}_x\|_{2,\Omega}^2 (\|\mathbf{H}\|_{2,\Omega}^4 + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{v}_x\|_{2,\Omega}^4)
\]

\[+ \|\mathbf{v}_x\|_{2,\Omega}^2 (\|\mathbf{H}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{v}_x\|_{2,\Omega}^2 + \|\mathbf{v}_{x,x}\|_{2,\Omega}^2) + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2 \|\mathbf{v}_x\|_{2,\Omega}^2
\]

The result of integrating (2.18) over \( t \), we apply Gronwall’s inequality to obtain the estimate

\[
\max_{t \in [0,T]} \|\mathbf{v}_{xx}\|_{2,\Omega}^2 + \max_{t \in [0,T]} \|\mathbf{H}_{xx}\|_{2,\Omega}^2 + \max_{t \in [0,T]} \|\mathbf{v}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2
\]

\[+ \|\mathbf{v}_x\|_{2,\Omega}^2 (\|\mathbf{H}\|_{2,\Omega}^4 + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{v}_x\|_{2,\Omega}^4)
\]

\[+ \|\mathbf{v}_x\|_{2,\Omega}^2 (\|\mathbf{H}_{x,x}\|_{2,\Omega}^2 + \|\mathbf{v}_x\|_{2,\Omega}^2 + \|\mathbf{v}_{x,x}\|_{2,\Omega}^2) + \|\mathbf{H}_{x,x}\|_{2,\Omega}^2 \|\mathbf{v}_x\|_{2,\Omega}^2
\]
for continuous $\Phi_2$ that depends on the result of $\Phi$ and $\Phi_1$.

Now we suppose that when $\eta = (-\Delta)^k v$ in (2.1) and $\zeta = (-\Delta)^k H$ in (2.2), $k = 2, 3, 4, ...$, the relation

$$\max_{t \in [0,T]} \|v_{x+k}\|_{2,\Omega}^2 + \max_{t \in [0,T]} \|H_{x+k}\|_{2,\Omega}^2 + \mu \|v_{x+k+1}\|_{2,\Omega}^2 + \mu \|H_{x+k+1}\|_{2,\Omega}^2 \leq \Phi_k(T, \|v^0_{x+k}\|_{2,\Omega}^2, \|H^0_{x+k}\|_{2,\Omega}^2),$$

where $\Phi_k$ depends on $\Phi$, $\Phi_1$, $\Phi_2$, ..., $\Phi_{k-1}$, is established.

So when $\eta = (-\Delta)^{k+1} v$ in (2.1) and $\zeta = (-\Delta)^{k+1} H$ in (2.2), we can obtain the relation

$$(v_t + v \cdot \nabla v, (-\Delta)^{k+1} v) - \nu \Delta v + \Delta v_t, (-\Delta)^{k+1} v) - \mu (H \cdot \nabla H, (-\Delta)^{k+1} v)$$

$$= (f, (-\Delta)^{k+1} v)$$

and

$$(\mu H_t, (-\Delta)^{k+1} H) - \frac{1}{\sigma} (\Delta H, (-\Delta)^{k+1} H) + \mu (v \cdot \nabla H - H \cdot \nabla v, (-\Delta)^{k+1} v)$$

$$= \left(\frac{1}{\sigma} \text{curl} j, (-\Delta)^{k+1} H\right),$$

that is,

$$\frac{1}{2} \frac{d}{dt} \left(\|v_{x+k}\|_{2,\Omega}^2 + \mu \|H_{x+k}\|_{2,\Omega}^2 + \|v_{x+k+1}\|_{2,\Omega}^2 + \|v_{x+k+2}\|_{2,\Omega}^2\right) + \nu \|v_{x+k+2}\|_{2,\Omega}^2 + \frac{1}{\sigma} \|H_{x+k+2}\|_{2,\Omega}^2$$

$$= (f, (-\Delta)^{k+1} v) + \left(\frac{1}{\sigma} \text{curl} j, (-\Delta)^{k+1} H\right) - (v \cdot \nabla v, (-\Delta)^{k+1} v)$$

$$+ \mu (H \cdot \nabla H, (-\Delta)^{k+1} v) - \mu (v \cdot \nabla H - H \cdot \nabla v, (-\Delta)^{k+1} v)$$

$$\leq \frac{\nu}{2} \|v_{x+k+2}\|_{2,\Omega}^2 + \frac{1}{\sigma} \|H_{x+k+2}\|_{2,\Omega}^2 + C (\|\nabla f\|_{2,\Omega}^2 + \|\nabla \text{curl} j\|_{2,\Omega}^2) + C_1 B$$

where $B = \|\nabla^k (v \cdot \nabla v)|_{2,\Omega}^2 + \|\nabla^k (H \cdot \nabla H)|_{2,\Omega}^2 + \|\nabla^k (v \cdot \nabla H)|_{2,\Omega}^2 + \|\nabla^k (H \cdot \nabla v)|_{2,\Omega}^2$.

To majorize the terms in the right-hand side in the side of (2.17), we just calculate four terms in $B$. In more detail,

$$\|\nabla^k (v \cdot \nabla v)|_{2,\Omega}^2 = \int \Omega |\nabla^k (v \cdot \nabla v)|^2 dx \leq C \int \Omega \sum_{0 \leq i, j, k \leq k} |v_{x+i}|^2 |\nabla v_{x+j}|^2 dx$$

$$\leq C \int \Omega |v|^2 |\nabla v_{x+k}|^2 dx + B_1 \leq C \|v\|^2_{6,\Omega} \|v_{x+k+1}\|_{3,\Omega}^2 + B_1$$

$$\leq C (\|v\|_{2,\Omega}^2 + \|v_{x+k+2}\|_{2,\Omega}^2) (\|v_{x+k+2}\|_{2,\Omega}^2 + \|v_{x+k+1}\|_{2,\Omega}^2) + B_1$$

$$\leq \varepsilon \|v_{x+k+2}\|_{2,\Omega}^2 + C(\varepsilon) \|v_{x+k+1}\|_{2,\Omega}^2 (\|v\|_{2,\Omega}^2 + \|v_{x+k+2}\|_{2,\Omega}^2)$$

$$+ C \|v_{x+k+1}\|_{2,\Omega}^2 (\|v\|_{2,\Omega}^2 + \|v_{x+k+2}\|_{2,\Omega}^2) + B_1$$

where $B_1 = C \int \Omega \sum_{0 \leq i, j, k \leq k} |v_{x+i}|^2 |\nabla v_{x+j}|^2 dx$;
\[
\begin{align*}
&\leq C \|H\|_{6,\Omega}^2 \|H_{xk+1}\|_{3,\Omega}^2 + C \|H\|_{6,\Omega}^2 \|H_{xk}\|_{3,\Omega}^2 + B_2 \\
&\leq C(\|H\|_{2,\Omega}^2 + \|v\|_{2,\Omega}^2)(\|H_{xk+1}\|_{2,\Omega}^2 + \|H_{xk+1}\|_{2,\Omega}^2) \\
&\quad + C_1(\|H\|_{2,\Omega}^2 + \|H_{xx}\|_{2,\Omega}^2)(\|H_{xk}\|_{2,\Omega}^2 + \|H_{xk+1}\|_{2,\Omega}^2) + B_2 \\
&\leq \varepsilon_1 \|H_{xk+1}\|_{2,\Omega}^2 + C(\varepsilon_1) \|H_{xk+1}\|_{2,\Omega}^2 (\|H\|_{2,\Omega}^4 + \|H_{xk}\|_{2,\Omega}^4) \\
&\quad + C \|H_{xk+1}\|_{2,\Omega}^2 (\|H\|_{2,\Omega}^2 + \|H_{xk}\|_{2,\Omega}^2) \\
&\quad + C_1(\|H\|_{2,\Omega}^2 + \|H_{xx}\|_{2,\Omega}^2)(\|H_{xk}\|_{2,\Omega}^2 + \|H_{xk+1}\|_{2,\Omega}^2) + B_2 \\
&\leq \varepsilon_1 \|H_{xk+1}\|_{2,\Omega}^2 + C \|H_{xk+1}\|_{2,\Omega}^2 (\|H\|_{2,\Omega}^4 + \|H_{xk}\|_{2,\Omega}^4 + \|H_{xx}\|_{2,\Omega}^2) \\
&\quad + \|H_{xx}\|_{2,\Omega}^2 + C_1 \|H_{xk}\|_{2,\Omega}^2 (\|H_{xk}\|_{2,\Omega}^2 + \|H_{xx}\|_{2,\Omega}^2) + B_2,
\end{align*}
\]
where \(B_2 = C \int_{\Omega} \sum_{0 \leq i,j < k, i+j = k} |v_{x_i}|^2 \|\nabla v_{x_j}\|^2 dx;\)

\[
\|\nabla^k (v \cdot \nabla H)\|_{2,\Omega}^2 = \int_{\Omega} |\nabla^k (v \cdot \nabla H)|^2 dx \leq C \int_{\Omega} \sum_{0 \leq i,j \leq k, i+j = k} |v_{x_i}|^2 \|\nabla v_{x_j}\|^2 dx
\]

\[
\leq C \int_{\Omega} |v_{x_i}|^2 \|\nabla H_{xk}\|^2 + |v_{x_j}|^2 \|\nabla H_{xk-1}\|^2 dx + B_3
\]

\[
\leq C(\|v_{x_i}\|_{2,\Omega}^2 + \|v_{x_j}\|_{2,\Omega}^2)(\|H_{xk+1}\|_{2,\Omega}^2 + \|H_{xk+1}\|_{2,\Omega}^2) \\
+ C(\|v_{x_i}\|_{2,\Omega}^4 + \|v_{x_j}\|_{2,\Omega}^4)(\|H_{xk+1}\|_{2,\Omega}^4 + \|H_{xk+1}\|_{2,\Omega}^4) + B_3
\]

\[
\leq \varepsilon_2 \|H_{xk+1}\|_{2,\Omega}^2 + C(\varepsilon_2) \|H_{xk+1}\|_{2,\Omega}^2 (\|H\|_{2,\Omega}^4 + \|H_{xk}\|_{2,\Omega}^4) \\
+ C \|H_{xk+1}\|_{2,\Omega}^2 (\|v_{x_i}\|_{2,\Omega}^2 + \|v_{x_j}\|_{2,\Omega}^2) \\
+ C_1 \|H_{xk+1}\|_{2,\Omega}^2 (\|v_{x_i}\|_{2,\Omega}^2 + \|v_{x_j}\|_{2,\Omega}^2) + B_3
\]

\[
\leq \varepsilon_2 \|H_{xk+1}\|_{2,\Omega}^2 + C \|H_{xk+1}\|_{2,\Omega}^2 (\|v\|_{2,\Omega}^4 + \|v_{x_i}\|_{2,\Omega}^4 + \|v_{x_j}\|_{2,\Omega}^4 + \|v_{x\Omega}^2
\]

\[
\|\nabla^k (H \cdot \nabla v)\|_{2,\Omega}^2 = \int_{\Omega} |\nabla^k (H \cdot \nabla v)|^2 dx \leq C \int_{\Omega} \sum_{0 \leq i,j \leq k, i+j = k} |H_{x_i}|^2 \|\nabla v_{x_j}\|^2 dx
\]

\[
\leq C \int_{\Omega} |v_{x_i}|^2 \|\nabla H_{xk}\|^2 + |H_{xj}|^2 \|\nabla v\|^2 dx + B_4
\]

\[
\leq C(\|H_{x_i}\|_{2,\Omega}^2 \|v_{x_i}\|_{3,\Omega}^2 + \|\nabla v\|_{2,\Omega}^2)(\|H_{xj}\|_{2,\Omega}^2 + \|v_{x_j}\|_{2,\Omega}^2) + B_4
\]

\[
\leq C(\|H\|_{2,\Omega}^2 + \|H_{xk}\|_{2,\Omega}^2)(\|v_{xk+1}\|_{2,\Omega}^2 + \|v_{xk+1}\|_{2,\Omega}^2) \\
+ C_1(\|H_{xk+1}\|_{2,\Omega}^2 + \|H_{xk}\|_{2,\Omega}^2)(\|v_{xk}\|_{2,\Omega}^2 + \|v_{xk}\|_{2,\Omega}^2) + B_4
\]

\[
\leq \varepsilon_3 \|v_{xk+1}\|_{2,\Omega}^2 + C(\varepsilon_3) \|v_{xk+1}\|_{2,\Omega}^2 (\|H\|_{2,\Omega}^4 + \|H_{xk}\|_{2,\Omega}^4) + C_1 \|v_{xk+1}\|_{2,\Omega}^2 (\|H\|_{2,\Omega}^4
\]

\[
+ \|H_{xk}\|_{2,\Omega}^2 + C_1(\|H_{xk+1}\|_{2,\Omega}^2 + \|H_{xk}\|_{2,\Omega}^2)(\|v_{xk}\|_{2,\Omega}^2 + \|v_{xk}\|_{2,\Omega}^2) + B_4,
\]
where \( B_4 = C \int_{\Omega} \sum_{0 \leq i, j \leq k \atop x+1, y=k} |H_{x+i} x^2 + \nabla v_i x^2|^2 \, dx \).

We can obtain the estimates of \( B_1, B_2, B_3, B_4 \) from the the previous arguments and hypothesis. In more detail,

\[
B_1 \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|v_{x+i} x^2 \Omega \|v_{x+i+1} x^2 \Omega \| \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|v_{x+i} x^2 \Omega \|^4 + \|v_{x+i+1} x^2 \Omega \|^4
\]

\[
\leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|v_{x+i} x^2 \Omega \|^4 + \|v_{x+i+1} x^2 \Omega \|^4 + \|v_{x+i+2} x^2 \Omega \|^4 + \|v_{x+i+1} x^2 \Omega \|^4 = B'_1;
\]

\[
B_2 \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|H_{x+i} x^2 \Omega \|H_{x+i+1} x^2 \Omega \| \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|H_{x+i} x^2 \Omega \|^4 + \|H_{x+i+1} x^2 \Omega \|^4
\]

\[
\leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|H_{x+i+1} x^2 \Omega \|^4 + \|H_{x+i} x^2 \Omega \|^4 + \|H_{x+i+1} x^2 \Omega \|^4 + \|H_{x+i+1} x^2 \Omega \|^4 = B'_2;
\]

\[
B_3 \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|v_{x+i} x^2 \Omega \|H_{x+i+1} x^2 \Omega \|^2 \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|v_{x+i} x^2 \Omega \|^4 + \|H_{x+i+1} x^2 \Omega \|^4
\]

\[
\leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|v_{x+i+1} x^2 \Omega \|^4 + \|v_{x+i+2} x^2 \Omega \|^4 + \|H_{x+i+1} x^2 \Omega \|^4 = B'_3;
\]

\[
B_4 \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|H_{x+i} x^2 \Omega \|v_{x+i+1} x^2 \Omega \|^2 \leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|H_{x+i} x^2 \Omega \|^4 + \|v_{x+i+1} x^2 \Omega \|^4
\]

\[
\leq C \sum_{0 \leq i, j \leq k, i+1, j+1} \|H_{x+i+1} x^2 \Omega \|^4 + \|H_{x+i} x^2 \Omega \|^4 + \|v_{x+i+2} x^2 \Omega \|^4 + \|v_{x+i+1} x^2 \Omega \|^4 = B'_4.
\]

From the previous hypothesis and arguments, we know that if \( j+2 \) or \( i+1 \leq k+1 \), then \( \|v_{x+i+2} x^2 \Omega \|^4 \) or \( \|v_{x+i+1} x^2 \Omega \|^4 \) is uniform bound for arbitrary \( t \in [0, T] \) and if \( j+2 \) or \( i+1 \leq k \), then \( \|H_{x+i+2} x^2 \Omega \|^4 \) or \( \|H_{x+i+1} x^2 \Omega \|^4 \) is uniform bound for any \( t \in [0, T] \).

From (2.19) and the estimates of \( B \), as well as sufficiently small \( \varepsilon, \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), we gain the relation

\[
\frac{1}{2} \frac{d}{dt} (\|v_{x+i+1} x^2 \Omega \|^2 + \|H_{x+i+1} x^2 \Omega \|^2 + \|v_{x+i+2} x^2 \Omega \|^2) + \|v_{x+i+1} x^2 \Omega \|^2 + \|H_{x+i+2} x^2 \Omega \|^2
\]

\[
\leq C (\|\nabla v \|^2 x^2 \Omega + \|\nabla v \|^2 x^2 \Omega + \|\nabla v \|^2 x^2 \Omega + \|v \|^2 x^2 \Omega
\]

\[
+ \|v \|^2 x^2 \Omega + \|v \|^2 x^2 \Omega + \|v \|^2 x^2 \Omega)
\]

\[
P_1 = \|v_{x+i+1} x^2 \Omega \|^2 + \|v_{x+i+2} x^2 \Omega \|^2 + \|H_{x+i+1} x^2 \Omega \|^2 + \|H_{x+i+2} x^2 \Omega \|^2 + \|v_{x+i+2} x^2 \Omega \|^2 + \|v_{x+i+1} x^2 \Omega \|^2
\]

\[
+ \|v \|^2 x^2 \Omega + \|v \|^2 x^2 \Omega + \|v \|^2 x^2 \Omega + \|H_{x+i+1} x^2 \Omega \|^2 + \|H_{x+i+2} x^2 \Omega \|^2 + \|v_{x+i+2} x^2 \Omega \|^2 + \|v_{x+i+1} x^2 \Omega \|^2
\]

\[
+ B'_1 + B'_2 + B'_3 + B'_4.
\]
The result of integrating (2.20) over $t$, we apply Gronwall’s inequality to obtain the estimate
\[
\max_{t \in [0,T]} \|v_{x,k+1}\|_{L^2}^2 + \max_{t \in [0,T]} \|H_{x,k+1}\|_{L^2}^2 + \max_{t \in [0,T]} \|v_{x,k+2}\|_{L^2}^2 + \|H_{x,k+2}\|_{L^2}^2
\]
\[
+ \|v_{x,k+2}\|_{L^2}^2 \leq \Phi_{k+1}(T, \|v_0\|_{L^2}, \|H_0\|_{L^2}).
\]

Here, continuous function $\Phi_{k+1}$ depends on the previous result or hypothesis of $\Phi$, $\Phi_1$, $\Phi_2$, ..., $\Phi_{k-1}$, $\Phi_k$.

3. Result on bounded domain. In this section, we prove Theorem 1.1 by the a priori estimates in Section 2.

3.1. Existence and uniqueness. We use Galerkin approximations methods to prove the existence of solution for the system in Theorem 1.1.

We choose basis functions $\{\omega_k(x)\}_{k=1}^\infty$ and $\{\varphi_k(x)\}_{k=1}^\infty$ which are smooth, and both are orthogonal bases of $W_0^{1,2}(\Omega)$ and orthonormal bases of $L^2(\Omega)$. Let
\[
v_m(x,t) = \sum_{k=1}^m a_m^k(t)\omega_k(x) \text{ and } H_m(x,t) = \sum_{k=1}^m b_m^k(t)\varphi_k(x), \ m = 1, 2, \ldots,
\]
where the coefficients $a_m^k$ and $b_m^k$ are determined from the system
\[
\begin{aligned}
\frac{d}{dt}v_m(t), \omega_k) + (v_m(t) \cdot \nabla v_m(t) - \nu \Delta v_m(t), \omega_k) - (\frac{d}{dt}\Delta v_m(t), \omega_k) \\
\mu(H_m(t) \cdot \nabla H_m(t), \omega_k) = (f(t), \omega_k), \ k = 1, 2, 3, \ldots, m
\end{aligned}
\]
and
\[
\begin{aligned}
\mu\frac{d}{dt}H_m(t), \varphi_k) - \frac{1}{\sigma}(\Delta H_m(t), \varphi_k) + \mu(v_m(t) \cdot \nabla H_m(t) - H_m(t) \cdot \nabla v_m(t), \varphi_k) \\
= \frac{1}{\sigma}(\text{curl} H_m(t), \varphi_k) \ k = 1, 2, 3, \ldots, m
\end{aligned}
\]

with the initial value
\[
v_m |_{t=0} = \sum_{k=1}^m (v_0, \omega_k)\omega_k \text{ and } H_m |_{t=0} = \sum_{k=1}^m (H_0, \varphi_k)\varphi_k.
\]

Equations (3.1) and (3.2) are ordinary differential equations about time $t \in [0,T]$ for the coefficients $a_m^k$ and $b_m^k$, k=1, 2, 3, ..., m. So the system has a unique solution on $[0,T]$ satisfying (3.3).

We multiply by the coefficients $a_m^k$ and $b_m^k$ (k=1, 2, 3, ..., m) two sides of the identity (3.1) and (3.2) and then add them up. Namely, we can derive
\[
\begin{aligned}
(\frac{d}{dt}v_m(t), v_m(t)) + (v_m(t) \cdot \nabla v_m(t) - \nu \Delta v_m(t), v_m(t)) - (\frac{d}{dt}\Delta v_m(t), v_m(t)) \\
- \mu(H_m(t) \cdot \nabla H_m(t), v_m(t)) = (f(t), v_m(t))
\end{aligned}
\]
and
\[
\begin{aligned}
\mu(\frac{d}{dt}H_m(t), H_m(t)) - \frac{1}{\sigma}(\Delta H_m(t), H_m(t)) + \mu(v_m(t) \cdot \nabla H_m(t) \\
- H_m(t) \cdot \nabla v_m(t), H_m(t)) = \frac{1}{\sigma}(\text{curl} H_m(t), H_m(t)).
\end{aligned}
\]

We have for $v_m$ and $H_m$ the same estimates for the norms in the section 2 for $v$ and $H$. Moreover, for $\eta = v, \eta = v_t, \eta = \Delta v$ or $\eta = \Delta v_t$ and $\zeta = H, \zeta = H_t$ or $\zeta = \Delta H_t$. 

\(\Delta H\), it is analogous relation for \(v\) and \(H\). On the basis of the previous estimate, we can get the following estimate

\[
\max_{t \in [0, T]} \|v_m(t)\|_{2, \Omega}^2 = \max_{t \in [0, T]} \sum_{k=1}^m \|a_m(t)\|_{L^2}^2 \leq C,
\]

\[
\max_{t \in [0, T]} \|H_m(t)\|_{2, \Omega}^2 = \max_{t \in [0, T]} \sum_{k=1}^m \|a_m(t)\|_{L^2}^2 \leq C
\]

and

\[
\max_{t \in [0, T]} \|\nabla v_m(t)\|_{2, \Omega}, \max_{t \in [0, T]} \|\Delta v_m(t)\|_{2, \Omega}, \max_{t \in [0, T]} \|\nabla H_m(t)\|_{2, \Omega}, \max_{t \in [0, T]} \|\Delta H_m(t)\|_{2, \Omega}, \max_{t \in [0, T]} \|\partial_t H_m(t)\|_{2, \Omega} \leq C,
\]

where the constants \(C\) depend only on the domain \(\Omega\) and the norms

\[
\|v^0\|_{2, \Omega}, \|v^0\|_{2, \Omega}, \|v^{0*}\|_{2, \Omega}, \|H^0\|_{2, \Omega}, \|H^0\|_{2, \Omega}, \|f\|_{2, Q_T}, \|\text{curl} j\|_{2, Q_T}.
\]

Now we can pass to the limit as \(m \to \infty\) and obtain the \(v\) and \(H\) as limit of \(v_m\) and \(H_m\).

For the uniqueness, the procedure is similar [2]. Having in hand \(v\) and \(H\), we can get the relation for the pressure \(p\) from the system (1.1) (see, e.g., [2]).

3.2. Regularization. For the initial boundary value, we can gain the Theorem 1.1 by the previous higher order estimates, hypothesis and argument in section 2.3, the procedure of proof is omit.

4. Existence of global solution on time and space. In this section, we give the outline of the proof of Theorem 1.2.

**Proof.** Taking \(\psi_l \in C(R^3), (l \geq 0),\)

\[
\psi_l(x) = \begin{cases} 
1, & |x| \leq 2^{l-1} + \frac{1}{2^l}, \\
0, & |x| \geq 2^{l+1} - \frac{1}{2^l}.
\end{cases} \tag{4.1}
\]

\[
\psi_l(x) = \begin{cases} 
1, & |x| \leq 2^{l-1} + \frac{1}{2^l}, \\
\text{linear}, & 2^{l-1} + \frac{1}{2^l} \leq |x| \leq 2^{l+1} - \frac{1}{2^l}, l = 1, 2, 3, \ldots, \\
0, & |x| \geq 2^{l+1} - \frac{1}{2^l}.
\end{cases} \tag{4.2}
\]

Let \(\phi_l(x) = \psi_l \ast \eta_{\frac{1}{2^l}}, l = 0, 1, 2, \ldots,\) then \(\phi_l \in C^\infty(R^3),\)

\[
\phi_l(x) = \begin{cases} 
1, & |x| \leq 2^{l-1}, \\
0, & |x| \geq 2^{l+1}.
\end{cases} \tag{4.3}
\]

and one may prove that there exists a \(C_l\) such that

\[
|\partial^\alpha \phi_l(x)| \leq C_l, \quad 2^{l-1} \leq |x| \leq 2^{l+1} \tag{4.4}
\]

for each \(|\alpha| \leq l\).

Let \(\Omega_t = B_{2^{l+1}}(0),\) then \(\Omega_t\) is a bounded domain in \(R^3\) with smooth boundary and let \(Q_T = \Omega_t \times (0, T)\) with some fixed times \(T < \infty\). If \(v^0 \in D^{k+1,2}(\Omega),\)

\(H^0 \in D^{k,2}(\Omega),\) then \(v^0 \in W^{k+1,2}(\Omega_t),\) \(\phi_l v^0 \in W^{k,2}(\Omega_t).\)

Since \(f, \text{curl} j \in L^2(0, T; D^{k-1,2}(\Omega)), \) \(\text{div} j = 0, (k = 0, 1, 2, \ldots),\) then \(f, \text{curl} j \in L^2(0, T; W^{k-1,2}(\Omega_t)), \) \(\text{div} j = 0, (k = 0, 1, 2, \ldots).\)
For every fixed $k$, let $l > k$. According to Theorem 1.1 the system has generalized solution $\mathbf{v}_l^0$, $\mathbf{H}_l^0$ and the generalized solution has the properties $\mathbf{H}_{l,k}^0$ equal to 0 on $\partial \Omega_k$ in the trace sense,
\[
\max_{t \in [0,T]} \| \mathbf{v}_{k+1}^t \|_{2, \Omega_k}, \max_{t \in [0,T]} \| \mathbf{H}_{k+1}^t \|_{2, \Omega_k} < C,
\]
\[
\| \mathbf{v}_{k+1}^t \|_{2, Q_T^l}, \| \mathbf{H}_{k+1}^t \|_{2, Q_T^l} < C,
\]
\[
\max_{t \in [0,T]} \| \mathbf{v}_m^l(t) \|_{2, \Omega_k}^2 \leq C,
\]
\[
\max_{t \in [0,T]} \| \mathbf{H}_m^l(t) \|_{2, \Omega_k}^2 \leq C,
\]
\[
\max_{t \in [0,T]} \| \nabla \mathbf{v}_m^l(t) \|_{2, Q_T^l} \| \Delta \mathbf{v}_m^l(t) \|_{2, Q_T^l} \| \Delta \mathbf{H}_m^l(t) \|_{2, Q_T^l} \| \frac{d}{dt} \Delta \mathbf{H}_m^l(t) \|_{2, Q_T^l} \| \frac{d}{dt} \mathbf{H}_m^l(t) \|_{2, Q_T^l} \| \mathbf{H}_m^l(t) \|_{2, Q_T^l} \leq C,
\]
where the constants $C$ depend only on the domain $\Omega_k$ and the norms
\[
\| \phi_0 \mathbf{v}^0 \|_{2, \Omega_k}, \| \phi_0 \mathbf{H}^0 \|_{2, \Omega_k}, \| \phi_0 \mathbf{v}^0 \|_{2, \Omega_k}, \| \phi_0 \mathbf{H}_m^l(t) \|_{2, Q_T^l}, \| \phi_0 \mathbf{H}_m^l(t) \|_{2, Q_T^l}.
\]
Notice that
\[
\| \phi_0 \mathbf{v}_m^l(t) - \mathbf{v}_m^0 \|_{2, \Omega_k}, \| \phi_0 \mathbf{H}_m^l(t) - \mathbf{H}_m^0 \|_{2, \Omega_k}, \| \phi_0 \mathbf{v}_m^l(t) - \mathbf{v}_m^0 \|_{2, \Omega_k} \rightarrow 0, \text{ as } l \rightarrow \infty.
\]

By using the induction method, we can prove the theorem 1.2.

\[ \square \]

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