TWO NOTES ON THE O’HARA ENERGIES

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Abstract. The O’Hara energies, introduced by Jun O’Hara in 1991, were proposed to answer the question of what is a “good” figure in a given knot class. A property of the O’Hara energies is that the “better” the figure of a knot is, the less the energy value is. In this article, we discuss two topics on the O’Hara energies. First, we slightly generalize the O’Hara energies and consider a characterization of its finiteness. The finiteness of the O’Hara energies was considered by Blatt in 2012 who used the Sobolev-Slobodeckij space, and naturally we consider a generalization of this space. Another fundamental problem is to understand the minimizers of the O’Hara energies. This problem has been addressed in several papers, some of them based on numerical computations. In this direction, we discuss a discretization of the O’Hara energies and give some examples of numerical computations. Particular one of the O’Hara energies, called the Möbius energy thanks to its Möbius invariance, was considered by Kim-Kusner in 1993, and Scholtes in 2014 established convergence properties. We apply their argument in general since the argument does not rely on Möbius invariance.

1. Introduction. The family of O’Hara energies was introduced by O’Hara [10, 11] and is defined as

\[ E^{\alpha,p}(f) := \int\int_{(\mathbb{R}/L\mathbb{Z})^2} \left( \frac{1}{\| f(s_2) - f(s_1) \|^\alpha_{\mathbb{R}^d}} - \frac{1}{\mathcal{D}(f(s_1), f(s_2))^\alpha} \right)^p ds_2 ds_1, \]

where \( \alpha, p \in (0, \infty) \) are constants, \( f : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^d \) is a curve embedded in \( \mathbb{R}^d \) parametrized by arc-length with total length \( L \), and \( \mathcal{D}(f(s_1), f(s_2)) \) is the intrinsic distance between \( f(s_1) \) and \( f(s_2) \). The purpose of these energies is to give an answer to the question: “What is the most beautiful knot in a given knot class?” Therefore, the O’Hara energies were constructed so that the more a knot is well-balanced, the less the energy is. Also, when we deform a knot, it is desirable that the knot stays within its (ambient) isotopy class. Thus, these energies were also constructed so that the energy value diverges if the curve has self-intersection. A study of minimizers of the O’Hara energies under length constraint were carried out in [1, 4, 12]. In particular, round circles attain the minimum of these energies for \( \alpha \in (0, \infty) \) and \( p \in [1, \infty) \) with \( 0 < \alpha < 2 + 1/p \). Indeed, this result was shown by Adams et al. [1] for the more general energy

\[ E^F(f) := \int\int_{(\mathbb{R}/L\mathbb{Z})^2} F(\| f(s_2) - f(s_1) \|_{\mathbb{R}^d}, \mathcal{D}(f(s_1), f(s_2))) ds_2 ds_1, \]

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where $F = F(x,y)$ is increasing and convex in $x \in (0,y]$ for $y \in (0,\mathcal{L}/2)$. For example, $F(x,y) = (x^{-\alpha} - y^{-\alpha})^p$ satisfies this assumption when $p \in [1,\infty)$ and $0 < \alpha < 2 + 1/p$.

The purpose of this article is two-fold. Firstly, we study a slightly generalized energy

$$E^{\Phi,p}(f) := \int\int_{(\mathbb{R}/\mathbb{Z})^2} \left( \frac{1}{\Phi(f(s_2) - f(s_1))} - \frac{1}{\Phi(\mathcal{D}(f(s_1), f(s_2)))} \right)^p ds_2 ds_1$$

under suitable assumptions on $\Phi$, and we should see that such a generalization brings out certain properties of $E^{\alpha,p}$ in a clearer manner. It is known that the finiteness of $E^{\alpha,p}(f)$ implies bi-Lipschitz continuity and some regularity of $f$, see [2]. We generalize this fact to the case $E^{\Phi,p}$, and we clarify what properties of $\Phi$ give rise to these properties of $f$. In particular, we define a function space $W^{k+\Phi,p}$ which is a generalization of the Sobolev-Slobodeckij space, and discuss the relation between our new space and the domain of $E^{\Phi,p}$.

The second aim is to study energies for polygonal knots, that is, discretization of the original energy. More precisely, a discrete version of $E^{\alpha,p}$ is proposed together with some numerical results. Several discrete versions of the energy $E^{2,1}$, called the Möbius energy, have been introduced earlier; one is by Kim-Kusner [7], and another is by Simon [15]. Their convergence was shown by Scholtes [14] and Rawdon-Simon [13] respectively. Although $E^{2,1}$ is invariant under Möbius transformations (cf. [4]), the proof of the result of [14] did not use the Möbius invariance. Here, we extend the results by Kim-Kusner [7] and Scholtes [14] to the case $E^{\alpha,p}$, and improve the rate of convergence of $E^{2,1}$.

2. A generalization of the O’Hara energy. Although minimizers of $E^F$ were obtained in [1], other fundamental properties of $E^F$ have not been investigated in the existing literature. Here, we consider the problem of characterizing the finiteness of generalized energies. At the level of generality of $E^F$, this seems to be a very difficult problem so we restrict ourself to the case $E^{\alpha,p}$ defined above, where $p \in [1,\infty)$ is a constant, and $\Phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that $\Phi(0) = 0$. Note that finiteness of the O’Hara energies is discussed by Blatt in [2], where he showed that if $\alpha \in (0,\infty)$ and $p \in [1,\infty)$ satisfy $2 \leq \alpha p < 2p + 1$, then $E^{\alpha,p}(f) < \infty$ if and only if $f$ is bi-Lipschitz continuous and belongs to the Sobolev-Slobodeckij space

$$W^{1+\sigma,2p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d)$$

where $\sigma = (\alpha p - 1)/(2p)$. Hence, to establish a condition for finiteness of $E^{\Phi,p}$, it is natural to consider a generalization of the Sobolev-Slobodeckij space.

**Definition 2.1.** Let $\Omega$ be a non-empty subset of $\mathbb{R}$. For $p \in [1,\infty)$, $k \in \mathbb{N} \cup \{0\}$, and measurable function $\Psi : [0,\infty) \to [0,\infty)$, we define

$$W^{k+\Psi,p}(\Omega, \mathbb{R}^d) := \{ f \in W^{k,p}(\Omega, \mathbb{R}^d) \mid [f^{(k)}]_{\Psi,p} < \infty \},$$

where

$$[f^{(k)}]_{\Psi,p} := \left( \int_{\Omega \times \Omega} \frac{\| f^{(k)}(s_2) - f^{(k)}(s_1) \|_{\mathbb{R}^d}^p}{\Psi(|s_2 - s_1|)^p} \frac{1}{|s_2 - s_1|} ds_2 ds_1 \right)^{1/p}.$$
We equip the space $W^{k+\Psi,p}$ with the norm
\[ \|f\|_{W^{k+\Psi,p}} := \|f\|_{W^{k,p}} + \|f^{(k)}\|_{\Psi,p}, \]
in which case it becomes a Banach space. Moreover, the dual space of $W^{\Psi,p}(\Omega, \mathbb{R}^d)$ is characterized by the following proposition which is proven by using the argument of [9, pp. 38–42]. The author gives its proof below, since [9] is written in Japanese, and since he is unable to find suitable references in English.

**Proposition 1.** Let $\Omega$ be a non-empty subset of $\mathbb{R}$, and let $\Psi : [0, \infty) \to [0, \infty)$ be a measurable function. For $p \in [1, \infty)$, let $q \in (1, \infty]$ satisfy $1/p + 1/q = 1$. Then, for all $T \in (W^{\Psi,p}(\Omega, \mathbb{R}^d))^\prime$, there exists $(\phi, \psi) \in L^q(\Omega, \mathbb{R}^d) \times L^q(\Omega \times \Omega, \mathbb{R}^d)$ such that
\[ \|T\|_{(W^{\Psi,p}(\Omega))^\prime} = \max\{\|\phi\|_{L^q(\Omega)}, \|\psi\|_{L^q(\Omega \times \Omega)}\} \]
and
\[ T(f) = \int_\Omega f(s) \cdot \phi(s) ds + \int_{\Omega \times \Omega} \left( \frac{f(s_2) - f(s_1)}{\Psi([s_2 - s_1])} \cdot \psi(s_1, s_2) \right) \frac{1}{|s_2 - s_1|^{1/p}} ds_2 ds_1 \]
for any $f \in W^{\Psi,p}(\Omega, \mathbb{R}^d)$. In particular, if $1 < p < \infty$, then $W^{\Psi,p}(\Omega, \mathbb{R}^d)$ is reflexive.

**Proof.** The map $\tau : W^{\Psi,p}(\Omega, \mathbb{R}^d) \to L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d)$ is defined by
\[ \tau(f) = \left( f, \frac{f(s_2) - f(s_1)}{\Psi([s_2 - s_1])} \frac{1}{|s_2 - s_1|^{1/p}} \right) \]
for $f \in W^{\Psi,p}(\Omega, \mathbb{R}^d)$. Then, it holds that $\tau$ is injective and isometric. Therefore, $\tau(W^{\Psi,p}(\Omega, \mathbb{R}^d))$ is equal to a closed subspace $\mathcal{W} \subset L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d)$. Hence, $T \circ \tau^{-1}$ is a linear functional on $\mathcal{W}$, and we have $\|T \circ \tau^{-1}\|_{\mathcal{W}'} = \|T\|_{(W^{\Psi,p}(\Omega, \mathbb{R}^d))^\prime}$. By the Hahn-Banach theorem, there exists a bounded linear functional $G \in (L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime$ such that
\[ (T \circ \tau^{-1})(f) = G(f) \]
for $f \in \mathcal{W}$.

The spaces $(L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime$ and $(L^p(\Omega, \mathbb{R}^d))^\prime \times (L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime$ are isometric and isomorphic. Indeed, the map $\tau_1 : (L^p(\Omega, \mathbb{R}^d))^\prime \times (L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime \to (L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime$ given by
\[ (\tau_1(\varphi, \psi))(u, v) = \varphi(u) + \psi(v) \]
for $(\varphi, \psi) \in (L^p(\Omega, \mathbb{R}^d))^\prime \times (L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime$ and $(u, v) \in L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d)$ is isometric and isomorphic. Note that
\[ \|(\varphi, \psi)\|_{(L^p(\Omega))^\prime \times (L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime} = \max\{\|\varphi\|_{(L^p(\Omega))^\prime}, \|\psi\|_{(L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime}\}. \]

Moreover, the map $\tau_2 : L^q(\Omega, \mathbb{R}^d) \to (L^q(\Omega, \mathbb{R}^d))^\prime$ defined by
\[ (\tau_2(\phi))(u) = \int_\Omega u(s) \cdot \phi(s) ds \]
for $\phi \in L^q(\Omega, \mathbb{R}^d)$ and $u \in L^p(\Omega, \mathbb{R}^d)$ is isometric and isomorphic. Therefore, we have $(L^p(\Omega, \mathbb{R}^d))^\prime \cong L^q(\Omega, \mathbb{R}^d)$.

Hence, there exists $(\phi, \psi) \in L^q(\Omega, \mathbb{R}^d) \times L^q(\Omega \times \Omega, \mathbb{R}^d)$ such that
\[ \|G\|_{(L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \Omega, \mathbb{R}^d))^\prime} = \max\{\|\phi\|_{L^q(\Omega, \mathbb{R}^d)}, \|\psi\|_{L^q(\Omega \times \Omega, \mathbb{R}^d)}\}, \]
\[ G((u, v)) = \int_\Omega u(s) \cdot \phi(s) ds + \int_{\Omega \times \Omega} v(s_1, s_2) \cdot \psi(s_1, s_2) ds_2 ds_1 \]
for \((u, v) \in L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \mathbb{R}^d)\). Since \(G \circ \tau = T\) and from the definition of \(\tau\), it follows that

\[ \|T\|_{(W^{s,p}(\Omega))'} = \max\{\|\phi\|_{L^s(\Omega)}, \|\psi\|_{L^s(\Omega \times \mathbb{R}^d)}\}, \]

\[ T(f) = \int_{\Omega} f(s) \cdot \phi(s) ds + \int_{\Omega \times \mathbb{R}^d} \left( \frac{f(s_2) - f(s_1)}{\|s_2 - s_1\|} \cdot \psi(s_1, s_2) \right) \frac{1}{|s_2 - s_1|^{1/p}} ds_2 ds_1 \]

for \(f \in W^{s,p}(\Omega, \mathbb{R}^d)\).

As mentioned previously, \(W^{s,p}(\Omega, \mathbb{R}^d)\) and the closed subspace \(W \subset L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \mathbb{R}^d)\) are isometric and isomorphic. Moreover, if \(1 < p < \infty\), \(W\) is reflexive because of the reflexivity of \(L^p(\Omega, \mathbb{R}^d) \times L^p(\Omega \times \mathbb{R}^d)\). Therefore, \(W^{s,p}(\Omega, \mathbb{R}^d)\) is reflexive when \(1 < p < \infty\). \(\square\)

In [2], it was shown that \(f\) is bi-Lipschitz continuous for all embedded regular curves \(f \in C^{0,1}(\mathbb{R}/\mathbb{LZ}, \mathbb{R}^d) \cap W^{1+\Psi,2p}(\mathbb{R}/\mathbb{LZ}, \mathbb{R}^d)\), which suggests that \(f\) does not bend sharply. It is natural to expect that all embedded regular curves belonging to the generalized Sobolev space are bi-Lipschitz; we confirm this expectation with the following theorem which we establish by modifying the argument of Blatt [2].

**Theorem 2.2** (The bi-Lipschitz continuity). Let an increasing function \(\Phi : [0, \infty) \rightarrow [0, \infty)\) satisfy \(\Phi(0) = 0\) and \(\Phi(x) = O(x^{2/p})\) as \(x \rightarrow +0\) for \(p \in [1, \infty)\). Set \(\Psi(x) := (x^{-1/p}\Phi(x))^{1/2}\). Assume that \(f\) belongs to \(C^{0,1}(\mathbb{R}/\mathbb{LZ}, \mathbb{R}^d) \cap W^{1+\Psi,2p}(\mathbb{R}/\mathbb{LZ}, \mathbb{R}^d)\) whose image is a closed embedded curve in \(\mathbb{R}^d\) parametrized by arc-length. Then, \(f\) is bi-Lipschitz continuous.

**Remark 1.** For \(\Phi(x) = x^\alpha\) with \(2 \leq \alpha p < 2p + 1\), the assertion in Theorem 2.2 corresponds to [2, Lemma 2.1].

**Proof of Theorem 2.2.** We only have to prove that there exists \(C > 0\) such that

\[ \|f(s_2) - f(s_1)\|_{\mathbb{R}^d} \geq C \varphi(f(s_1), f(s_2)) \]

for \(s_1, s_2 \in \mathbb{R}/\mathbb{LZ}\).

Let \(s_1, s_2, s_3 \in \mathbb{R}/\mathbb{LZ}\) with \(|s_2 - s_1| = 2r\) and \(\varphi(f(s_1), f(s_3)) = \varphi(f(s_3), f(s_2))\). Then, we have

\[ \|f(s_2) - f(s_1)\|_{\mathbb{R}^d} = \sup_{\|x\|_{\mathbb{R}^d} \leq 1} \int_{s_3 - r}^{s_3 + r} f'(s) \cdot x ds \]

\[ = 2r + \sup_{\|x\|_{\mathbb{R}^d} \leq 1} \int_{s_3 - r}^{s_3 + r} f'(s) \cdot (f'(s) - x) ds \]

\[ \geq 2r - \inf_{\|x\|_{\mathbb{R}^d} \leq 1} \int_{s_3 - r}^{s_3 + r} \|f'(s) - x\|_{\mathbb{R}^d} ds \]

\[ = \left(1 - \inf_{\|x\|_{\mathbb{R}^d} \leq 1} \frac{1}{2r} \int_{s_3 - r}^{s_3 + r} \|f'(s) - x\|_{\mathbb{R}^d} ds \right)|s_2 - s_1|. \]

Note that there exists \(M, \delta > 0\) such that if \(x < \delta\), then we have \(\Phi(x) \leq Mx^{2/p}\) because \(\Phi(x) = O(x^{1/p})\) as \(x \rightarrow +0\). By the assumption \(f \in W^{1+\Psi,2p}(\mathbb{R}/\mathbb{LZ}, \mathbb{R}^d)\), we have

\[ \|f\|^{2p}_{W^{1+\Psi,2p}} = \int_{\mathbb{R}/\mathbb{LZ}} \int_{-L/2}^{L/2} \|f'(s_1 + s_2) - f'(s_1)\|_{\mathbb{R}^d}^{2p} \Phi(|s_2|)^p ds_2 ds_1 < \infty. \]
Using Lebesgue’s dominated convergence theorem, we have
\[
\lim_{r \to +0} \int_{\mathbb{R}/\mathcal{L}Z} \int_{-r}^{r} \frac{\|f'(s_1 + s_2) - f'(s_1)\|^{2p}_{\mathbb{R}^d}}{\Phi(|s_2|)^p} ds_2 ds_1 = 0.
\]
Therefore, there exists \( \eta \in (0, \min\{\delta, 1, \mathcal{L}/2\} ) \) such that
\[
\sup_{s \in \mathbb{R}/\mathcal{L}Z} \int_{s-r}^{s+r} \frac{\|f'(s_1 + s_2) - f'(s_1)\|^{2p}_{\mathbb{R}^d}}{\Phi(|s_2|)^p} ds_2 ds_1 \leq \frac{1}{2^p \mathcal{M}^p}
\]
if \( r \leq \eta \). Putting
\[
x = \frac{1}{2r} \int_{s_3-r}^{s_3+r} f'(s_2) ds_2,
\]
we get
\[
\frac{1}{2r} \int_{s_3-r}^{s_3+r} \|f'(s_1) - x\|_{\mathbb{R}^d} ds_1
\]
\[
= \frac{1}{2r} \int_{s_3-r}^{s_3+r} \|f'(s_1) - \frac{1}{2r} \int_{s_3-r}^{s_3+r} f'(s_2) ds_2\|_{\mathbb{R}^d} ds_1
\]
\[
\leq \frac{1}{4r^2} \int_{s_3-r}^{s_3+r} \int_{s_3-r}^{s_3+r} \|f'(s_2) - f'(s_1)\|_{\mathbb{R}^d} ds_2 ds_1
\]
\[
\leq \left( \frac{\Phi(2r)^p}{4r^2} \int_{s_3-r}^{s_3+r} \int_{s_3-r}^{s_3+r} \|f'(s_2) - f'(s_1)\|^{2p}_{\mathbb{R}^d} ds_2 ds_1 \right)^{1/2p} \leq \frac{1}{2}.
\]
Therefore, it holds that
\[
\|f(s_2) - f(s_1)\|_{\mathbb{R}^d} \geq \frac{1}{2} |s_2 - s_1|.
\]

Next, we consider the case where \( \mathcal{D}(f(s_1), f(s_2)) \geq 2\eta \). Let
\[
I_\eta := \{(s_1, s_2) \in (\mathbb{R}/\mathcal{L}Z)^2 \mid \mathcal{D}(f(s_1), f(s_2)) \geq 2\eta \}.
\]
Then, we have
\[
C_\eta := \inf_{(s_1, s_2) \in I_\eta} \|f(s_2) - f(s_1)\|_{\mathbb{R}^d} > 0
\]
because \( f \) has no self-intersection. Therefore, we obtain
\[
\|f(s_2) - f(s_1)\|_{\mathbb{R}^d} \geq C_\eta \mathcal{D}(f(s_1), f(s_2)).
\]

Using the space \( W^{k+\Psi, 2p} \), we establish the following theorem concerning the finiteness of the energies \( \mathcal{E}^{\Phi, p} \).

**Theorem 2.3** (Finiteness of \( \mathcal{E}^{\Phi, p}(f) \)). Let \( p \in [1, \infty) \), and let \( f \in C^{0,1}(\mathbb{R}/\mathcal{L}Z, \mathbb{R}^d) \) be a function whose image is a closed curve parametrized by arc-length embedded in \( \mathbb{R}^d \) with total length \( \mathcal{L} \). Assume that the measurable function \( \Phi : [0, \infty) \to [0, \infty) \) satisfies the following.

(A0) \( \Phi(0) = 0, \Phi \in C^1 \), and \( \Phi'(x) > 0 \) for \( x > 0 \).

(A1) There exists \( K > 0 \) such that \( \lim_{x \to +0} G(x) = K \), where \( G(x) := \frac{x \Phi'(x)}{\Phi(x)} \).

(A2) There exists a measurable function \( \varphi : [0, \infty) \to [0, \infty) \) such that

(A2-1) \( \Phi(kx) \leq \varphi(k) \Phi(x) \) for \( k, x \geq 0 \).
and \( M(a) := \int_0^a \frac{\varphi(t)^p}{t} dt \) for \( a > 0 \) satisfies

(A2-2) \( M(\varepsilon) = o(\varepsilon) \) as \( \varepsilon \to +0 \),

(A2-3) \( M(a) < \infty \) for \( a > 0 \).

(A3) \( \int_0^{a} \frac{t^{2p}}{\Phi(t)^p} dt < \infty \) for \( a > 0 \).

Set \( \Psi(x) := (x^{-1/p} \Phi(x))^{1/2} \) for \( x > 0 \). Then, we have the following two properties.

1. If \( f \in W^{1,\Psi,2p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^d) \) and \( f \) is bi-Lipschitz continuous, then we have \( \mathcal{E}^{\Phi,p}(f) < \infty \).

2. If \( \mathcal{E}^{\Phi,p}(f) < \infty \), then \( f \) belongs to \( W^{1,\Psi,2p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^d) \).

Moreover, there exists \( C > 0 \) depending only on \( p, L \), and \( \Phi \) such that

\[
\|f\|_{W^{\Psi,2p}} \leq C(\mathcal{E}^{\Phi,p}(f) + \|f\|_{L^2}). \tag{1}
\]

Remark 2. Suppose we assume (A2-2)' \( \varphi(x) = O(x^{2/p}) \) as \( x \to 0 \)

instead of (A2-2) in Theorem 2.3. Then, we have \( M(\varepsilon) = o(\varepsilon) \) as \( \varepsilon \to +0 \), and using the argument of [12], we can prove that \( f \) is bi-Lipschitz continuous if \( \mathcal{E}^{\Phi,p}(f) < \infty \). Thus, it holds that \( \mathcal{E}^{\Phi,p}(f) < \infty \) if and only if \( f \in W^{1,\Psi,2p}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^d) \cap C^{0,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^d) \) and \( f \) is bi-Lipschitz continuous.

When we derive properties of energies, it is not important that \( \Phi \) is a monomial or a power-function. Indeed, \( \Phi(x) = x^\alpha, x^\alpha \log(x+1), 1 - e^{-x^\alpha} + x^{2\alpha}/2 \) satisfy the assumptions of Theorems 2.2 and 2.3 for suitable \( \alpha \). The following table shows the allowable range of the exponent \( \alpha \) corresponding to the assumptions in Theorems 2.2 and 2.3. The row “Remark 2” shows the range of \( \alpha \) corresponding to (A0), (A1), (A2-1), (A2-2)', (A2-3), and (A3).

| | \( \Phi(x) = x^\alpha \) | \( \Phi(x) = x^\alpha \log(x+1) \) | \( \Phi(x) = 1 - e^{-x^\alpha} + x^{2\alpha}/2 \) |
|---|---|---|---|
| Theorem 2.2 | \([2/p, \infty)\) | \([2/p - 1, \infty)\) | \([1/p, \infty)\) |
| Theorem 2.3 | \((1/p, 2 + 1/p)\) | \((1/p, 1/p + 1)\) | \((1/p, 2 + 1/p)\) |
| Remark 2 | \([2/p, 2 + 1/p)\) | \([2/p, 1/p + 1)\) | \([2/p, 2 + 1/p)\) |

\( p > 1 \)

Table 1. Examples of \( \Phi \) (Ranges of \( \alpha \))

**Notation.** For \( s_1, s_2 \in \mathbb{R}/L\mathbb{Z} \) and \( v : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^d \), we write \( \Delta_{s_{2}}; v := v(s_2) - v(s_1) \).

The proof of Theorem 2.3 is based on an argument by Blatt [2]. Before proving Theorem 2.3, we establish the following lemma which is used in the proof of inequality (1). Let

\[
\mathcal{E}^{\Phi,p}(g) := \int_{\mathbb{R}/L\mathbb{Z}} \int_{-L/2}^{L/2} \left( \int_{0}^{1} \int_{0}^{1} \left\| \Delta_{s_2}^{\alpha} g \right\|_{\mathbb{R}^d} ds_3 ds_4 \right)^p \Phi(|s_2|)^p ds_2 ds_1
\]

for \( g : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^d \).
Lemma 2.4. There exists $C = C(p, L, \Phi) > 0$ such that

$$\| \Phi \|_{L^{2p}}^2 \leq C \left( \tilde{\Phi}(g) + \| g \|_{L^{2p}}^2 \right)$$

holds for each function $g : \mathbb{R}/L \mathbb{Z} \to \mathbb{R}^d$ that is continuous almost everywhere.

Proof. First, we consider the case where $g \in C^\infty(\mathbb{R}/L \mathbb{Z}, \mathbb{R}^d)$. For $\varepsilon \in (0, 1)$, we decompose

$$\| g \|_{L^{2p}}^2 = J_1^\varepsilon (g) + J_2^\varepsilon (g),$$

where

$$J_1^\varepsilon (g) := \int_{\mathbb{R}/L \mathbb{Z}} \int_{|s_2| \geq \varepsilon L/2} \frac{\| \Delta g \|_{L^2}^2}{\Phi(|s_2|)} ds_2 ds_1,$$

$$J_2^\varepsilon (g) := \int_{\mathbb{R}/L \mathbb{Z}} \int_{|s_2| \leq \varepsilon L/2} \frac{\| \Delta g \|_{L^2}^2}{\Phi(|s_2|)} ds_2 ds_1.$$

Now, we have

$$J_1^\varepsilon (g) \leq \frac{2^p L}{\Phi(\varepsilon L)} \| g \|_{L^{2p}}^2$$

because $\Phi$ is an increasing function. Moreover, because

$$\int_0^\varepsilon \int_{1-\varepsilon}^1 ds_4 ds_3 = \varepsilon^2,$$

it holds that

$$\varepsilon^{2p} J_2^\varepsilon (g) \leq \int_{\mathbb{R}/L \mathbb{Z}} \int_{|s_2| \leq \varepsilon L/2} \left( \int_0^\varepsilon \int_{1-\varepsilon}^1 \frac{\| \Delta g \|_{L^2}^2}{\Phi(|s_2|)} ds_4 ds_3 \right)^p ds_2 ds_1 \leq 3^{p-1} (K_1^\varepsilon (g) + K_2^\varepsilon (g) + K_3^\varepsilon (g)),$$

where

$$K_1^\varepsilon (g) := \int_{\mathbb{R}/L \mathbb{Z}} \int_{|s_2| \geq \varepsilon L/2} \frac{\| \Delta g \|_{L^2}^2}{\Phi(|s_2|)} ds_4 ds_3 ds_1,$$

$$K_2^\varepsilon (g) := \int_{\mathbb{R}/L \mathbb{Z}} \int_{|s_2| \leq \varepsilon L/2} \frac{\| \Delta g \|_{L^2}^2}{\Phi(|s_2|)} ds_4 ds_3 ds_1,$$

$$K_3^\varepsilon (g) := \int_{\mathbb{R}/L \mathbb{Z}} \int_{|s_2| \leq \varepsilon L/2} \frac{\| \Delta g \|_{L^2}^2}{\Phi(|s_2|)} ds_4 ds_3 ds_1.$$
By the definition of $\mathcal{E}^{\Phi, p}(g)$, we have $K_3^2(g) \leq \mathcal{E}^{\Phi, p}(g)$. Moreover, using Jensen’s inequality, Fubini’s Theorem, and (A2-1), we have

$$K_3^2(g) = \varepsilon^p \int_{\mathbb{R}/LZ} \int_{|s_2| \leq \varepsilon L/2} \left( \int_0^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \right)^p ds_2 ds_1$$

$$\leq \varepsilon^{2p-1} \int_{\mathbb{R}/LZ} \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p ds_2 ds_3 ds_1$$

$$= \varepsilon^{2p-1} \int_{\mathbb{R}/LZ} \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \Phi(|s_3|)^p ds_2 ds_3 ds_1$$

$$\leq \varepsilon^{2p-1} \int_{\mathbb{R}/LZ} \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \Phi(|s_3|)^p \bar{s} ds_2 ds_3 ds_1$$

$$\leq M(\varepsilon) \varepsilon^{2p-1-1} J_1^2(g)$$

by the change of variable $\bar{s} = s_3 s_2$. Also, we have

$$\int_{1-\varepsilon}^1 \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 ds_4 = \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2-s_3 s_2} \|_{\mathbb{R}/LZ}^2 ds_4$$

by the change of variable $\bar{s} = 1 - s_4$. Therefore, we obtain

$$K_3^2(g) = \varepsilon^p \int_{\mathbb{R}/LZ} \int_{-\varepsilon L/2}^{\varepsilon L/2} \left( \int_{1-\varepsilon}^1 \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \right)^p ds_2 ds_1$$

$$\leq \varepsilon^p \int_{\mathbb{R}/LZ} \int_{-\varepsilon L/2}^{\varepsilon L/2} \left( \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \right)^p ds_2 ds_1$$

$$= \varepsilon^p \int_{-\varepsilon L/2}^{\varepsilon L/2} \left( \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \right)^p ds_2 ds_1$$

$$\leq \varepsilon^p \int_{-\varepsilon L/2}^{\varepsilon L/2} \left( \int_{0}^\varepsilon \| \Delta_s^{s_1+s_2} \|_{\mathbb{R}/LZ}^2 \Phi(|s_2|)^p \right)^p ds_2 ds_1$$

$$= K_3^2(g)$$

by Fubini’s theorem and the change of variables $\bar{s} = s_1 + s_2$ and $\bar{s} = -s_2$. Hence, we get

$$J_2^2(g) \leq \frac{3p-1}{\varepsilon^{2p}} \mathcal{E}^{\Phi, p}(g) + 3p-1 \cdot \frac{2M(\varepsilon)}{\varepsilon} J_1^2(g).$$

Now, we can take $\varepsilon$ sufficiently small satisfying

$$3p-1 \cdot \frac{2M(\varepsilon)}{\varepsilon} < 1$$

by (A2-2). Then, we get

$$J_2^2(g) \leq C(p, \varepsilon, \varphi) \mathcal{E}^{\Phi, p}(g)$$

because $J_2^2(g) < \infty$ by $g \in C^\infty(\mathbb{R}/LZ, \mathbb{R})^4$ and because of (A3), where $C(p, \varepsilon, \varphi)$ is a positive constant. Therefore, we obtain

$$\| g \|_{\psi, 2p} \leq \frac{2^{2p} \Phi(\varepsilon L/2)}{\Phi(\varepsilon L/2)} g \|_{L_2^p}^\varepsilon + C(p, \varepsilon, \varphi) \mathcal{E}^{\Phi, p}(g).$$
Next, we assume that $g$ is continuous almost everywhere and that $\tilde{\mathcal{E}}^{\Phi,p}(g)$ is finite. Let $\rho \in C^\infty_0(\mathbb{R})$ with $\text{supp } \rho \subset [-L/2, L/2]$ and
\[
\int_{-L/2}^{L/2} \rho(x)dx = 1,
\]
and define $\rho_\lambda(x) := \lambda^{-1} \rho(x/\lambda)$ for $x \in \mathbb{R}$. Set
\[
g_\lambda(s) := \int_{-L/2}^{L/2} \rho(x)g(s-x)dx.
\]
Then, we have
\[
[g_\lambda]^{2p}_{\Psi,2p} \leq \frac{2^{2p}L}{\Phi(\varepsilon L/2)}\|g_\lambda\|^{2p}_{L^{2p}} + C(p, \varepsilon, \varphi)\tilde{\mathcal{E}}^{\Phi,p}(g_\lambda)
\]
because $g_\lambda \in C^\infty$. From properties of the mollifier, it holds that
\[
\|g_\lambda\|_{L^{2p}} \leq \|g\|_{L^{2p}}.
\]
Next, we show $\tilde{\mathcal{E}}^{\Phi,p}(g_\lambda) \leq \tilde{\mathcal{E}}^{\Phi,p}(g)$. By Hölder’s inequality, we have
\[
\|\Delta^{s_1+x+s_2}g_\lambda\|_{L^d}^2 = \left\|\int_{-L/2}^{L/2} \rho_\lambda(x)\Delta^{s_1+x+s_2}gdx \right\|_{L^d}^2 \\
\leq \left(\int_{-L/2}^{L/2} \rho_\lambda(x)\|\Delta^{s_1+x+s_2}g\|_{L^d}dx \right)^2 \\
\leq \int_{-L/2}^{L/2} \rho_\lambda(x)\|\Delta^{s_1+x+s_2}g\|_{L^d}^2 dx.
\]
Also, it holds that
\[
\left(\int_0^1 \int_0^1 \int_{-L/2}^{L/2} \rho_\lambda(x)\|\Delta^{s_1+x+s_2}g\|_{L^d}^2 dxdsdxds_3 \right)^p \\
= \left(\int_{-L/2}^{L/2} \rho_\lambda(x)\int_0^1 \int_0^1 \|\Delta^{s_1+x+s_2}g\|_{L^d}^2 dsds_3dx \right)^p \\
\leq \left(\int_{-L/2}^{L/2} \rho_\lambda(x)dx \right)^{p-1} \left\{\int_{-L/2}^{L/2} \rho_\lambda(x) \left(\int_0^1 \int_0^1 \|\Delta^{s_1+x+s_2}g\|_{L^d}^2 dsds_3 \right)^p dx \right\} \\
\leq \int_{-L/2}^{L/2} \rho_\lambda(x) \left(\int_0^1 \int_0^1 \|\Delta^{s_1+x+s_2}g\|_{L^d}^2 dsds_3 \right)^p dx.
\]
by Fubini’s theorem and Hölder’s inequality. Therefore, we get

\[
\mathcal{E}^{\Phi,p}(g) = \int_{\mathbb{R}/L^2} \int_{-L/2}^{L/2} \left( \int_0^1 \int_0^1 \frac{\| \Delta_{s_1+2s_2} g \|_{L^2}^2 ds_2 ds_1}{\Phi(|s_2|)^p} \right)^p ds_2 ds_1 \\
\leq \int_{\mathbb{R}/L^2} \int_{-L/2}^{L/2} \left( \int_0^1 \int_0^1 \frac{\| \Delta_{s_1+2s_2} g \|_{L^2}^2 ds_2 ds_3}{\Phi(|s_2|)^p} \right)^p ds_2 ds_1 \\
\leq \int_{\mathbb{R}/L^2} \int_{-L/2}^{L/2} \frac{\rho_\lambda(x)}{\Phi(|s_2|)^p} \left( \int_0^1 \int_0^1 \frac{\| \Delta_{s_1+2s_2} g \|_{L^2}^2 ds_2 ds_3}{\Phi(|s_2|)^p} \right)^p ds_2 ds_1 \\
= \left( \int_{-L/2}^{L/2} \rho_\lambda(x) dx \right) \left( \int_{-L/2}^{L/2} \int_{\mathbb{R}/L^2} \frac{\| \Delta_{s_1+2s_2} g \|_{L^2}^2 ds_3}{\Phi(|s_2|)^p} ds_2 ds_1 \right) \\
= \tilde{\mathcal{E}}^{\Phi,p}(g)
\]

by the change of variable \( \tilde{s}_1 = s_1 + x \).

Hence, we obtain

\[
\| g_\lambda \|_{W^{p,p}} \leq 2^{2p-1} \left\{ \left( 1 + \frac{2^{2p} L}{\Phi(\varepsilon/2)} \right) \| g \|_{L^{2p}}^{2p} + C(p, \varepsilon, \varphi) \mathcal{E}^{\Phi,p}(g) \right\},
\]

and we can see \( \{ g_\lambda \}_{\lambda > 0} \) is a \( W^{p,p} \)-bounded set. By reflexivity of \( W^{p,p} \), there exists a subsequence \( (g_{\lambda_j})_{j=0}^\infty \) such that

\[
g_{\lambda_j} \rightharpoonup g
\]

as \( j \to \infty \). Therefore, we obtain

\[
\| g \|_{W^{p,p}} \leq \liminf_{j \to \infty} \| g_{\lambda_j} \|_{W^{p,p}}^{2p} \\
\leq 2^{2p-1} \left\{ \left( 1 + \frac{2^{2p} L}{\Phi(\varepsilon/2)} \right) \| g \|_{L^{2p}}^{2p} + C(p, \varepsilon, \varphi) \tilde{\mathcal{E}}^{\Phi,p}(g) \right\}
\]

because the norm on a reflexive Banach space is lower semi-continuous with respect to the weak topology.

\[
\text{Proof of Theorem 2.3. We use the notation}
\]

\[
g_a(t) := \frac{1}{\Phi(t)} - \frac{1}{\Phi(a)}
\]

for \( 0 < t \leq a \). For \( \varepsilon \in (0, L/2) \), let

\[
\mathcal{E}^{\Phi,p}(f) := \int_{\mathbb{R}/L^2} \int_{-L/2}^{L/2} (g_{s_2}(\| \Delta_{s_1+2s_2} f \|_{L^2}))^p ds_2 ds_1.
\]

Then, we have \( \mathcal{E}^{\Phi,p}(f) = \lim_{\varepsilon \to 0^+} \mathcal{E}^{\Phi,p}_\varepsilon(f) \). By the mean value theorem, for \( s_1 \in \mathbb{R}/L^2, \varepsilon \leq |s_2| \leq L/2 \), there exists \( \theta \in (s_1, s_2) \in (\| \Delta_{s_1+2s_2} f \|_{L^2}, |s_2|) \) such that

\[
g_{s_2}(\| \Delta_{s_1+2s_2} f \|_{L^2}) = |g'_{s_2}(\theta)|(|s_2| - \| \Delta_{s_1+2s_2} f \|_{L^2}).
\]
where we change variables $\tilde{s}_j = s_1 + s_2$ for $j = 3, 4$. It follows from (3) and Jensen’s inequality that

$$
(|s_2|^2 - \|\Delta_{s_1}^1 + s_2 f\|_{\mathbb{R}^d})^p \leq \frac{1}{2^p} |s_2|^{2p} \left( \int_0^1 \int_0^1 \|\Delta_{s_1}^1 + s_2 f\|_{\mathbb{R}^d}^2 \right)^p
$$

Using (2), we have

$${\mathcal{E}}^{\Phi, p}(f) = \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \varepsilon \mathcal{L}/2} (g_{|s_2|}(\|\Delta_{s_1}^1 + s_2 f\|_{\mathbb{R}^d}))^p d s_2 d s_1
$$

By the bi-Lipschitz continuity of $f$ and (A2-1), we have

$$
|g_{|s_2|}(\theta)|_{|s_2|} = \frac{\Phi'(\theta)}{\Phi(\theta)^2} |s_2| = G(\theta) \frac{1}{\Phi(\theta)} |s_2| \frac{1}{\theta}
$$

$$
\leq C_b G(\theta) \frac{1}{\Phi(C_b^{-1} |s_2|)} \leq C_b G(\theta) \frac{\phi(C_b |s_4 - s_3|)}{\Phi(|s_4 - s_3| |s_2|)}
$$

for $s_1 \in \mathbb{R}/\mathcal{L}$, $\varepsilon \leq |s_2| \leq \varepsilon \mathcal{L}/2$, $s_3$, $s_4 \in [0, 1]$, where $C_b > 0$ is the bi-Lipschitz constant of $f$. Moreover, By (A0), (A1), and $0 \leq \theta \leq \mathcal{L}/2$, there exists $G > 0$ such that

$$
G(\theta) \leq G.
$$
Hence, it holds from Fubini’s theorem and (A2-1) that

\[
(*) \leq \frac{C_p}{2^p} \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \int_0^1 \int_0^1 G(\theta)^p \varphi(C_b |s_4 - s_3|)^p \times \left\| \Delta_{s_1 + s_3 s_2}^{s_1} f' \right\|_{\mathbb{R}/\phi}^{2p} ds_4 ds_3 ds_2 ds_1 \\
\times \Phi(|s_4 - s_3| |s_2|) \right| \right|^p ds_4 ds_3 ds_2 ds_1 \\
\leq \frac{C_p C_b}{2^p} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \int_{\mathbb{R}/\mathcal{L}} \left\| \Delta_{s_1 + s_3 s_2}^{s_1} f' \right\|_{\mathbb{R}/\phi}^{2p} \Phi(|s_4 - s_3| |s_2|) \right|^p ds_4 ds_3 ds_2 ds_1 \\
\times \frac{1}{\mathcal{L}^2} \int_{|s_4 - s_3| \varepsilon \leq |t_2| \leq |s_4 - s_3| \mathcal{L}/2} \int_{\mathbb{R}/\mathcal{L}} \frac{1}{\mathbb{R}/\phi} ds_1 dt_2 ds_3 ds_4 \\
\leq \frac{C_p C_b}{2^p} M(C_b) \left| f' \right|_{\Psi, 2p}^{2p} < \infty,
\]

where $s_1, s_2$ are transformed into $t_1 = s_1 + s_3 s_2, t_2 = (s_4 - s_3)s_2$ in the third equality. Thus we obtain $E_x^{\Phi, p}(f) < \infty$ for all $\varepsilon > 0$, i.e., $E^{\Phi, p}(f) < \infty$.

Next, we assume $E^{\Phi, p}(f) < \infty$. Since $\theta \leq |s_2|$, we have

\[
|g'_{s_2}(\theta)| |s_2| = G(\theta) \frac{1}{\Phi(\theta)} \frac{|s_2|}{\theta} \geq G(\theta) \frac{1}{\Phi(|s_2|)}
\]

using (A0). Moreover, by (A0), (A1), and $0 \leq \theta \leq \mathcal{L}/2$, there exists $\tilde{G} > 0$ such that

\[
G(\theta) \geq \tilde{G}.
\]

Therefore, it follows from (3) that

\[
\infty > E^{\Phi, p}(f) \geq E_x^{\Phi, p}(f) = \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \left( g_{|s_2|}(\|\Delta_{s_1 + s_3}^{s_1} f\|_{\mathbb{R}^4}) \right)^p ds_2 ds_1 \\
= \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \frac{|g'_{|s_2|}(\theta)| |s_2| - \|\Delta_{s_1 + s_3}^{s_1} f\|_{\mathbb{R}^4})^p ds_2 ds_1 \\
\geq \frac{1}{2^p} \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \frac{|g'_{|s_2|}(\theta)| |s_2| - \|\Delta_{s_1 + s_3}^{s_1} f\|_{\mathbb{R}^4})^p |s_2| ds_2 ds_1 \\
= \frac{1}{4^p} \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \frac{|g'_{|s_2|}(\theta)| |s_2| \left( \int_0^1 \int_0^1 \|\Delta_{s_1 + s_3}^{s_1} f\|_{\mathbb{R}^4} ds_3 ds_1 \right)^p ds_2 ds_1 \\
\geq \frac{C_p}{4^p} \int_{\mathbb{R}/\mathcal{L}} \int_{\varepsilon \leq |s_2| \leq \mathcal{L}/2} \left( \int_0^1 \int_0^1 \|\Delta_{s_1 + s_3}^{s_1} f\|_{\mathbb{R}^4} ds_3 ds_1 \right)^p ds_2 ds_1
\]

for all $\varepsilon \in (0, \mathcal{L}/2)$. Hence, we have

\[
E^{\Phi, p}(f) \geq \tilde{G}^{p/4} E^{\Phi, p}(f'),
\]

and we obtain inequality (1) because it holds that

\[
\|f'\|_{W^{\Phi, p}} \leq C_g \left( \|f'\|_{L^{2p}} + \tilde{E}^{\Phi, p}(f') \right) \leq C_g \left( \|f'\|_{L^{2p}} + E^{\Phi, p}(f) \right)
\]
by Lemma 2.4, where \( C_g \) is a positive constant depending only \( p, \mathcal{L} \), and \( \Phi \) and which may not be the same in each case. Therefore, we have \( \mathbf{f} \in W^{1+\Phi,2\Phi}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \). \( \square \)

3. A discretization of the O’Hara energies. Although minimizers of the O’Hara energies were studied, it is difficult to calculate the O’Hara energies directly, and as a result, it is not easy to evaluate well-balancedness. In [7], Kim and Kusner considered a discretization of the Möbius energy and numerically calculated values of Möbius energy of torus knots. Scholtes [14] discussed convergence of Kim-Kusner’s discretization, but he did not use the Möbius invariance. Therefore, we expect that we can consider convergence of a discretization of not only the Möbius energy but also of the other O’Hara energies such as \( \mathcal{E}^{\alpha,p} \) or \( \mathcal{E}^{\Phi,p} \). Actually, a discretization of \( \mathcal{E}^{\alpha,p} \) along with its \( \Gamma \)-convergence has been discussed in [6]. In this section, we mention the result of [6] and give some examples of numerical calculations in this discretization.

From now on, we write \( \sigma = (\alpha p - 1)/(2p) \) for \( \alpha \in (0, \infty) \) and \( p \in [1, \infty) \) with \( 2 \leq \alpha p < 2p + 1 \). Note that the Sobolev-Slobodekij space \( W^{1+p,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \) naturally arises in connection with boundedness of the O’Hara energies; see [2]. Also, we call an \( n \)-gon a polygon with \( n \) edges. For a given regular curve \( \mathbf{f} \), we say that a polygon \( \mathbf{p} \) is inscribed in \( \mathbf{f} \) if \( \mathbf{p} \) satisfies

(i) the number of vertices is finite,

(ii) the set of vertices is \( \{\mathbf{f}(b_1), \ldots, \mathbf{f}(b_n)\} \) with \( b_1 < \cdots < b_n (b_1 + \mathcal{L}) \),

(iii) the \( k \)-th edge is the segment jointing \( \mathbf{f}(b_k) \) and \( \mathbf{f}(b_{k+1}) \), where we interpret \( b_{n+1} = b_1 \).

For \( \alpha, p \in (0, \infty) \), our discretization of the O’Hara energies is defined by

\[
\mathcal{E}^{\alpha,p}_n(\mathbf{p}_n) := \sum_{i,j=1, i \neq j}^n \left( \frac{1}{\|\mathbf{p}_n(a_j) - \mathbf{p}_n(a_i)\|_{\mathbb{R}^d}} - \frac{1}{\partial\mathbf{p}_n(a_i) \cdot \partial \mathbf{p}_n(a_j)} \right)^p \times \|\mathbf{p}_n(a_{i+1}) - \mathbf{p}_n(a_i)\|_{\mathbb{R}^d} \|\mathbf{p}_n(a_{j+1}) - \mathbf{p}_n(a_j)\|_{\mathbb{R}^d},
\]

where \( \mathbf{p}_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}^d \) is an \( n \)-gon parametrized by arc-length whose total length is \( \mathcal{L}_n \), and \( a_j \) is the value of arc-length parameter corresponding to vertex of \( \mathbf{p}_n \) and we assume \( 0 \leq a_1 < \cdots < a_n < \mathcal{L}_n (\text{mod} \mathcal{L}_n) \).

The following theorem obtained in [6] gives us convergence of our discretization as \( n \to \infty \) and the rate of convergence.

**Theorem 3.1** (Approximation of the O’Hara energies by inscribed polygons, [6]). Assume that \( \alpha \in (0, \infty) \) and \( p \in [1, \infty) \) satisfy \( 2 \leq \alpha p < 2p + 1 \). Let \( \mathbf{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \) be a function whose image is a closed curve parametrized by arc-length embedded in \( \mathbb{R}^d \), where \( \mathcal{L} \) is the length of \( \mathbf{f} \). Let \( c, \bar{c} > 0 \), and set \( K = \|\mathbf{f}'\|_{L^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)} \).

In addition, for \( n \in \mathbb{N} \), let \( \{b_k\}_{k=1}^n \) be a partition of \( \mathbb{R}/\mathcal{L}\mathbb{Z} \) satisfying

\[
\frac{c\mathcal{L}}{n} \leq \|\mathbf{f}(b_{k+1}) - \mathbf{f}(b_k)\|_{\mathbb{R}^d} \leq \frac{\bar{c}\mathcal{L}}{n},
\]

and let \( \mathbf{p}_n \) be the inscribed polygon in \( \mathbf{f} \) with vertices \( \mathbf{f}(b_1), \ldots, \mathbf{f}(b_n) \). Then, if the number \( n \) of points of the division is sufficiently large, there exists \( C > 0 \) such that

\[
|\mathcal{E}^{\alpha,p}(\mathbf{f}) - \mathcal{E}^{\alpha,p}_n(\mathbf{p}_n)| \leq C \frac{1}{n^{2p-\alpha p+1}}.
\]
Moreover, if \( f \in W^{1+\sigma,2p}(\mathbb{R}/\mathbb{LZ},\mathbb{R}^d) \), we have
\[
\lim_{n \to \infty} \mathcal{E}_{\alpha,p}^n(p_n) = \mathcal{E}_{\alpha,p}(f).
\]

Moreover, we obtained the \( \Gamma \)-convergence of \( \mathcal{E}_{\alpha,p}^n \) to \( \mathcal{E}_{\alpha,p} \) as \( n \to \infty \) in [6].

**Theorem 3.2** (\( \Gamma \)-convergence of \( \mathcal{E}_{\alpha,p}^n \), [6]). Let \( \alpha \in (0, \infty) \) and \( p \in [1, \infty) \) with \( 2 \leq \alpha p < 2p + 1 \). Then, the sequence \( \mathcal{E}_{\alpha,p}^n|_X \) converges to \( \mathcal{E}_{\alpha,p}|_X \) in the sense of \( \Gamma \)-convergence on the metric space \( X \) given by
\[
X := \left( \{C(K) \cap C^1(\mathbb{R}/\mathbb{LZ},\mathbb{R}^d)\} \cup \bigcup_{n \in \mathbb{N}} \mathcal{P}_n(K), d_X \right).
\]
Here, \( K \) is a tame knot class, \( C(K) \) is the set of simply closed curves of length 1 belonging to \( K \), \( \mathcal{P}_n(K) \) is the set of equilateral \( n \)-gons with total length 1 belonging to \( K \), and the metric \( d_X \) is such that, for some constants \( C_1, C_2 > 0 \), we have
\[
C_1\|f - g\|_{L^1} \leq d_X(f,g) \leq C_2\|f - g\|_{W^{1,\infty}}
\]
for \( f, g \in X \).

\( \Gamma \)-convergence implies convergence of the infima of \( \mathcal{E}_{\alpha,p}^n \) to that of \( \mathcal{E}_{\alpha,p} \) which is attained by a round circle. Furthermore, it also implies a somewhat weak statement on the convergence, for example, some information of the upper Kuratowski limit; see [3, Section 4]. Thus, it is natural to consider minimizers of \( \mathcal{E}_{\alpha,p}^n \). In [6], we have completely characterized minimizers of a generalized discrete functionals \( \mathcal{E}_F \) of the following form:

\[
\mathcal{E}_F^n(p_n) := \sum_{i,j=1}^n F(\|p_n(a_j) - p_n(a_i)\|_{\mathbb{R}^d}, \mathcal{D}(p_n(a_i), p_n(a_j)))
\]
\[
\times \|p_n(a_{i+1}) - p_n(a_i)\|_{\mathbb{R}^d}\|p_n(a_{j+1}) - p_n(a_j)\|_{\mathbb{R}^d},
\]
where \( p_n \) is an \( n \)-gon with total length 1, and \( F \) is a real-valued function on \( \Omega := \{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq y\} \).

**Theorem 3.3** (Minimizers of \( \mathcal{E}_F^n \), [6]). Assume that \( F : \Omega \to \mathbb{R} \) is such that if we set \( f_y(u) = F(\sqrt{u},y) \) for \( u \in (0,y^2) \) and \( y \in (0,1/2) \), then \( f_y \) is decreasing and convex. Moreover, for \( 0 < a < b \), set \( [a] := \min\{a,b-a\} \). Then, if \( p_n \) is an equilateral polygon, we have
\[
\mathcal{E}_F^n(p_n) \geq \frac{1}{n} \sum_{k=1}^{n-1} F \left( \frac{\sin([k]n\pi/n)}{n \sin(\pi/n)}, \mathcal{D}(p_n(a_k), p_n(a_0)) \right)
\]
and the minimizers of \( \mathcal{E}_F^n \) are regular \( n \)-gons.

If we set \( F(x,y) := (x^{-\alpha} - y^{-\alpha})^p \), then \( \mathcal{E}_F^n \) corresponds to \( \mathcal{E}_{\alpha,p}^n \). Then, we get the following corollary.

**Corollary 1.** Let \( \alpha \in (0, \infty) \) and \( p \in [1, \infty) \). Then, minimizers of \( \mathcal{E}_{\alpha,p}^n \) in the set of equilateral \( n \)-gons are regular polygons. In particular, a regular polygon with \( n \) edges is the only minimizer up to congruence transformations and similarity transformations.
Remark 3. There do not exist minimizers of $\mathcal{E}_n^{\alpha,p}$ in the set of all $n$-gons which are not necessarily equilateral. Let us consider an $(n-1)$-gon as a degenerate $n$-gon. Here “degenerate” means that two vertices of the $n$-gon coincide. Note that it does not mean the degeneracy of parametrization. Then, we have

$$0 \leq \inf_{n\text{-gon}} \mathcal{E}_n^{\alpha,p} \leq \inf_{(n-1)\text{-gon}} \mathcal{E}_{n-1}^{\alpha,p} \leq \cdots \leq \inf_{3\text{-gon}} \mathcal{E}_3^{\alpha,p},$$

and because $\mathcal{E}_3^{\alpha,p}(p_3) = 0$ for all $3$-gons $p_3$, we obtain

$$\inf_{n\text{-gon}} \mathcal{E}_n^{\alpha,p} = 0.$$

Next, we show some examples of numerical experiments. Let $g_n$ be a regular $n$-gon. By the property of $\Gamma$-convergence and Corollary 1, we have

$$\inf \mathcal{E}_n^{\alpha,p} = \lim_{n \to \infty} \mathcal{E}_n^{\alpha,p}(g_n),$$

where $2 \leq \alpha p < 2p + 1$, and the infimum in the left-hand side is taken over the space of all embedded curves in $\mathbb{R}^d$. Therefore, considering [1], we can calculate the O’Hara energy of a round circle numerically by increasing the number of vertices $n$ in $\mathcal{E}_n^{\alpha,p}(g_n)$. Moreover, we calculate energies $L(g_n)^{\alpha-2}\mathcal{E}_n^{\alpha,1}(g_n)$, where $L(g_n)^{\alpha-2}$ is the total length of $g_n$, because the factor $L(g_n)^{\alpha-2}$ makes these energies scale invariant. Note that in [5], the values of the O’Hara energy $\mathcal{E}_n^{\alpha,1}$ ($2 \leq \alpha < 3$) of a round circle $f_0$ are obtained and expressed by

$$\mathcal{E}_n^{\alpha,1}(f_0) = \frac{1}{(\alpha-1)L(f_0)^{\alpha-2}} \left\{ \frac{(\alpha-2)\pi^{\alpha-1/2}\Gamma((3-\alpha)/2)}{\Gamma((4-\alpha)/2)} + 2^\alpha \right\}.$$

Here, we compare $L(g_n)^{\alpha-2}\mathcal{E}_n^{\alpha,1}(g_n)$ with $L(f_0)^{\alpha-2}\mathcal{E}_n^{\alpha,1}(f_0)$, and we tabulate the result of numerical calculation when $\alpha = 2, 2.1, 2.3, 2.5, 2.7, 2.9$ in Table 2. It follows from Theorem 3.1 that the convergence becomes slow, when $\alpha$ approaches to 3. We can see this fact from Table 2. Moreover, we investigate the behavior of

$$e_\alpha(n) := n^{3-\alpha} \left| L(f_0)^{\alpha-2}\mathcal{E}_n^{\alpha,1}(f_0) - L(g_n)^{\alpha-2}\mathcal{E}_n^{\alpha,1}(g_n) \right|$$

when the number of vertices $n$ increases, where $2 \leq \alpha < 3$. We expect that $e_\alpha(n)$ converges to a constant if the order of convergence in Theorem 3.1 is optimal, and we can see that this conjecture seems to be true in Figure 1.

Now, we show some interesting examples of $\mathcal{E}_n^{\alpha,p}(g_n)$ when the number of vertices $n$ is not so large. As we can see in Figure 2, $L(g_{2k})^{58}\mathcal{E}_{2k}^{2,30}(g_{2k})$ for $k \in \mathbb{N}$ takes the maximum value at $k = 4$, and the larger the value that $p$ takes, the larger the maximum value is. Therefore, we show a figure of $\mathcal{E}_n^{2,30}(g_n)$ for $n \geq 100$ in Figure 3. Note that $L(g_{2k+1})^{58}\mathcal{E}_{2k+1}^{2,30}(g_{2k+1})$ for $k \geq 2$ is monotonically increasing. However, $L(g_{2\ell})^{58}\mathcal{E}_{2\ell}^{2,30}(g_{2\ell})$ for $\ell \geq 2$ takes the maximum at $\ell = 10$ ($n = 20$) and is decreasing to the value of $L(f_0)^{58}\mathcal{E}_{20}^{2,30}(f_0)$ when $\ell \geq 10$. The cause of this phenomena think is as follows: when $n$ is much less than 20, the discrete energy is a summation which consists of a small number of terms with large value. On the other hand, when $n$ is much larger than 20, the discrete energy is a summation which consists of a large number of terms with small value. If $n$ is around 20, then the number of terms and the size of each term might make the energy large. This phenomena will be remarkable when $p$ becomes large. To the author, the reason seems to be as follows: when $p$ is large, the difference of the size of the terms becomes bigger. Moreover, we observe from Figure 3 that the energy with even $n$ is larger than that with odd $n$. The energy density becomes large when the difference between
the intrinsic distance and the extrinsic distance is large. The difference maximizes when two points are antipodal, which is a situation that occurs only when \( n \) is even.

4. Conclusions and future work. In Section 2, we considered the generalized O’Hara energy \( E^{\Phi,p} \) and characterized the finiteness of these energies by using the generalized Sobolev-Slobodeckij space \( W^{1+\Phi,2p} \). However, several problems concerning \( E^{\Phi,p} \) remain open, e.g., conditions such that \( E^{\Phi,p} \) is the knot energy (with regard to the definition, see [11]), and the existence of minimizers of \( E^{\Phi,p} \) in a given knot type. In Section 3, we discussed a discretization defined in [6] of not only the Möbius energy but also the O’Hara energy and numerically calculated the energy values of a round circle. Discretization of the general energies \( E^{\Phi,p} \) should also be considered. However, we have so far been unable to find suitable conditions on \( \Phi \) to carry out a discretization of \( E^{\Phi,p} \); we hope to address this in future work. Several researchers have considered numerical calculations of \( E^{\alpha,1} \) of not only circles but also various knots, for example, Kusner-Sullivan [8]. However, numerical calculation of \( E^{\alpha,p} \) \((p > 1)\), except round circles, is yet to be carried out; this will be addressed in forthcoming work of the author.

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TWO NOTES ON THE O’HARA ENERGIES

\[ \alpha = 2 \]
\[ \alpha = 2.1 \]
\[ \alpha = 2.3 \]
\[ \alpha = 2.5 \]
\[ \alpha = 2.7 \]
\[ \alpha = 2.9 \]

Figure 1. Graphs of \( e_\alpha(n) \) (The vertical and horizontal axes show values of \( e_\alpha(n) \) and numbers of vertices \( n = 2^k \) \((k = 2, 3, \ldots, 20)\), respectively)

Figure 2. Values of \( E_{2^k, 30} (g_{2^k}) \)

Figure 3. Values of \( E_{2^k, 30} (g_n) \) when \( n \leq 100 \) (Round points and diamond points show values when \( n \) is even and odd, respectively)

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