Combinatorics of $A_2$-crystals

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Abstract. We show that a connected regular $A_2$-crystal (the crystal graph of an irreducible representation of $sl_3$) can be produced from two half-grids by replicating them and gluing together in a certain way. Also some extensions and related aspects are discussed.

Keywords: Simply-laced algebra, Crystal, Gelfand-Tsetlin pattern

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1 Introduction

The notion of crystals introduced by Kashiwara$^4$ embraces a wide class of edge-colored digraphs (directed graphs). Stembridge$^8$ pointed out a list of graph-theoretic axioms characterizing the crystals with finite monochromatic paths that are related to representations of simply-laced Lie algebras (i.e., with a Cartan matrix whose off-diagonal entries are 0 or $-1$), called regular simply-laced crystals. Each of these axioms imposes a simple local condition on a 2-colored subgraph of the digraph. In particular, a digraph is a regular simple-laced crystal if and only if each (inclusion-wise) maximal 2-colored subgraph in it is a regular simple-laced crystal (for a general result of this type on any crystals of representations that have a unique maximal vertex, see $^3$). This shows an importance of a proper study of 2-colored crystals.

This paper is the first in our series of works devoted to a combinatorial study of crystals of representations, and to related topics. Here we consider regular 2-colored simply-laced crystals for the Cartan matrix $A_2 = \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right)$. When such a crystal is connected and all monochromatic paths in it are finite, we refer to it as an RC-graph (abbreviating “regular crystal graph”). Our main structural theorem says that an RC-graph can be produced, by use replicating and gluing together in a certain way, from two RC-graphs of a very special form, viewed as triangular halves of two-dimensional square grids. As a result, the combinatorial structure of these objects becomes rather transparent, giving rise to revealing additional properties of RC-graphs and their

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extensions. In particular, it follows that an RC-graph is the Hasse diagram of a finite lattice.

The paper is organized as follows. Section 2 gives basic definitions and exhibits some elementary properties of RC-graphs. Section 3 contains the formulation and proof of the above-mentioned structural theorem. Section 4 explains how to extend this result to 2-colored digraphs having infinite monochromatic paths. Section 5 describes natural embeddings of RC-graphs in the Abelian groups \( \mathbb{Z}^4 \) and \( \mathbb{Z}^3 \). In particular, it explicitly describes a relation to Gelfand-Tsetlin patterns. The concluding Section 6 considers the disjoint union of all RC-graphs, called the universal RC-graph, and associate to it a certain semigroup of the Abelian group \( \mathbb{Z}^5 \).

2 RC-graphs

Let \( K \) be a digraph with vertex set \( V \) and with edge set \( E \) partitioned into two subsets \( E_1, E_2 \). We say that an edge in \( E_i \) has color \( i \) and call it an edge with color \( i \), or, briefly, an \( i \)-edge. Unless explicitly stated otherwise, any digraph in question is assumed to be (weakly) connected, i.e., it is not representable as the disjoint union of two nonempty digraphs. An RC-graph is defined by imposing on \( K = (V, E_1, E_2) \) four axioms (A1)–(A4) described below. (In fact, we reformulate axioms (P1)–(P6),(P5’),(P6’) given in [8] for \( n \)-colored simply-laced crystals, to our case \( n = 2 \).)

The first axiom concerns the structure of monochromatic subgraphs \((V, E_i)\).

(A1) For \( i = 1, 2 \), each connected component of \((V, E_i)\) is a finite simple (directed) path, i.e., a sequence \((v_0, e_1, v_1, \ldots, e_k, v_k)\), where \( v_0, v_1, \ldots, v_k \) are distinct vertices and each \( e_i \) is an edge going from \( v_{i-1} \) to \( v_i \).

In particular, each vertex \( v \) has at most one outgoing 1-edge and at most one incoming 1-edge, and similarly for 2-edges. For convenience, we refer to a maximal monochromatic path in \( K \), with color \( i \) on the edges, as an \( i \)-line. The \( i \)-line passing through a given vertex \( v \) (possibly consisting of the only vertex \( v \)) is denoted by \( P_i(v) \), its part from the first vertex to \( v \) by \( P_i^{\text{in}}(v) \), and its part from \( v \) to the last vertex by \( P_i^{\text{out}}(v) \). The lengths of \( P_i^{\text{in}}(v) \) and of \( P_i^{\text{out}}(v) \) (i.e., the numbers of edges in these paths) are denoted by \( t_i(v) \) and \( h_i(v) \), respectively.

The second axiom tells us how these lengths can change when one traverses an edge of the other color.

(A2) Each \( i \)-line \( P \) (\( i = 1, 2 \)) contains a vertex \( r \) satisfying the following property: for any edge \((u, v)\) (from a vertex \( u \) to a vertex \( v \)) in \( P_i^{\text{in}}(r) \), one holds \( t_{3-i}(v) = t_{3-i}(u) - 1 \) and \( h_{3-i}(v) = h_{3-i}(u) \), and for any edge \((u', v')\) in \( P_i^{\text{out}}(r) \), one holds \( t_{3-i}(v') = t_{3-i}(u') \) and \( h_{3-i}(v') = h_{3-i}(u') + 1 \).
Clearly such a vertex \( r \) is unique; it is called the \textit{critical} vertex of the given line \( P \). Axiom (A2) is illustrated in Fig. 1.

![Figure 1](image)

*Figure 1: A example of changing the lengths \( t_1 \) and \( h_1 \) along line \( P \) with color 2. The thick dot indicates the critical vertex in \( P \).*

The digraphs defined by axioms (A1),(A2) constitute a subclass of the class of so-called (locally finite) \textit{normal} A2-crystals of Kashiwara. (In the definition for the latter, axiom (A2) is replaced by a relaxed axiom: for an \( i \)-edge \((u, v)\), either \( t_{3-i}(v) = t_{3-i}(u) - 1 \) and \( h_{3-i}(v) = h_{3-i}(u) \), or \( t_{3-i}(v) = t_{3-i}(u) \) and \( h_{3-i}(v) = h_{3-i}(u) + 1 \).) We will refer to a digraph satisfying (A1) and (A2) as an \textit{NC-graph} (abbreviating “normal crystal graph”).

The quadruple \((t_1(v), h_1(v), t_2(v), h_2(v))\) giving an important information about a vertex \( v \) is called the \textit{length-tuple} of \( v \) and denoted by \( \tau(v) \). We can also associate with a vertex \( v \) the pair of integers \( \sigma(v) := (h_1(v) - t_1(v), h_2(v) - t_2(v)) \). From axiom (A2) it follows that the difference \( \sigma(v) - \sigma(u) \) is equal to \((-2, 1)\) for all \( 1 \)-edges \((u, v)\). In its turn, for each \( 2 \)-edge \((u, v)\), such a difference is equal to \((1, -2)\). So, under the map \( \sigma : V \to \mathbb{R}^2 \), each \( 1 \)-edge (2-edge) \( e \) becomes a parallel translation of the same vector \((-2, 1)\) (resp. \((1, -2)\)). This implies the following property (indicated for \( n \)-colored crystals in \[8\]).

**Corollary 2.1** An NC-graph \( K \) is graded w.r.t. each color \( i \), in the sense that any two directed paths in \( K \) having the same first vertex and the same last vertex contain equal numbers of \( i \)-edges.

In particular, \( K \) is acyclic and has no parallel edges. Also one can observe that the images by \( \sigma \) of the monochromatic subgraphs \((V, E_i)\) of \( K \) have certain symmetries (assuming that \( \sigma \) is properly extended to the edges of \( K \)). If a \( 1 \)-line has length \( p \), then \( \sigma \) brings it to the directed straightline segment in \( \mathbb{R}^2 \) with the beginning point \((p, q)\) and the end point \((-p, p + q)\) for some \( q \). Therefore, the median points of the
images of all 1-lines lie on the “vertical” coordinate axis in \( \mathbb{R}^2 \). Similarly, the median points of the images of all 2-lines lie on the “horizontal” coordinate axis.

**Remark 1.** NC-graphs have a rather loose structure, in contrast to RC-graphs. In particular, the local finiteness (in the sense that all monochromatic paths are finite, as required in axiom (A1)) and even the finiteness of the list of different length-tuples of vertices does not guarantee that the set of vertices is finite. (We shall see later, in Remark 3 in Section 3, that a similar behavior is possible even if axiom (A3) is added.) Indeed, for an arbitrary NC-graph \( K \) having an undirected cycle, one can construct an NC-graph with the same list of length-tuples, which is viewed as an infinite tree (the “free covering” over \( K \)). Also one can combine NC-graphs as follows. Suppose a vertex \( v \) of an NC-graph \( K_1 \) and a vertex \( v' \) of an NC-graph \( K_2 \) have equal length-tuples (\( K_2 \) may be taken as a copy of \( K_1 \)). Choose an edge in \( K_1 \) incident with \( v \) and the corresponding edge in \( K_2 \) incident with \( v' \); let for definiteness these edges be incoming 1-edges \((u, v)\) and \((u', v')\). Then the digraph obtained by replacing these edges by \((u, v')\) and \((u', v)\) is also an NC-graph (provided it is connected).

To formulate two remaining axioms defining RC-graphs, we need some definitions and notation. (These axioms give combinatorial analogs of the Serre relations.)

In an NC-graph \( K \), the edges with color \( i \) (\( i = 1, 2 \)) are naturally associated with operator \( F_i \) acting on the corresponding subset of vertices. So, for a 1-edge (2-edge) \((u, v)\), we write \( v = F_1(u) \) and \( u = F_1^{-1}(v) \) (resp. \( v = F_2(u) \) and \( u = F_2^{-1}(v) \)). Using this notation, one can express any vertex via another one (since \( K \) is connected). For example, the expression \( F_2^{-1}F_3F_2F_1(v) \) determines the vertex \( w \) obtained from a vertex \( v \) by traversing 1-edge \((v, u')\), followed by traversing 2-edges \((v', u)\) and \((u, u')\), followed by traversing 1-edge \((w, u')\) in backward direction. Emphasize that every time we use an expression with \( F_1 \) or \( F_2 \) in what follows, this automatically says that all involved edges do exist in \( K \).

For each edge \( e = (u, v) \) with color \( i \), we assign label \( \ell(e) := 0 \) if \( h_{3-i}(u) = h_{3-i}(v) \), and label \( \ell(e) := 1 \) otherwise. Axiom (A2) shows that the labels are monotonically nondecreasing along any \( i \)-line \( P \). In terms of labels, the critical vertex in \( P \) is just the vertex where the incoming \( i \)-edge, if exists, is labeled 0 and the outgoing \( i \)-edge, if exists, is labeled 1.

In further illustrations we will draw 1-edges by horizontal arrows directed to the right, and 2-edges by vertical arrows directed up.

The first additional axiom describes situations when the operators \( F_1 \) and \( F_2 \) commute.

\( \text{(A3) (a)} \) If a vertex \( u \) has two outgoing edges \((u, v), (u, v')\) and if \( \ell(u, v) = 0 \), then \( \ell(u, v') = 1 \) and \( F_2F_1(u) = F_1F_2(u) \). Symmetrically: \( \text{(b)} \) if a vertex \( v \) has two incoming edges \((u, v), (u', v)\) and if \( \ell(u, v) = 1 \), then \( \ell(u', v) = 0 \) and \( F_2^{-1}F_1^{-1}(v) = F_1^{-1}F_2^{-1}(v) \)
Let us say that vertices \( \tilde{u}, \tilde{v}, \tilde{u}', \tilde{v}' \) form a square if, up to renaming them, \( \tilde{v} = F_1(\tilde{u}), \tilde{u}' = F_2(\tilde{u}) \) and \( \tilde{v}' = F_1(\tilde{u}') = F_2(\tilde{v}) \). The opposite 1-edges \((\tilde{u}, \tilde{v})\) and \((\tilde{u}', \tilde{v}')\) for this square have equal labels, because of the obvious relations \( t_2(\tilde{u}') = t_2(\tilde{u}) + 1 \) and \( t_2(\tilde{v}') = t_2(\tilde{v}) + 1 \), and similarly for the opposite 2-edges \((\tilde{u}, \tilde{v})\) and \((\tilde{v}, \tilde{v}')\). Therefore, (a) in (A3) implies \( \ell(v, w) = 1 \) and \( \ell(v', w) = 0 \), where \( w := F_2F_1(u) \), and (b) implies \( \ell(w, u) = 0 \) and \( \ell(w, u') = 1 \), where \( w := F_2^{-1}F_1^{-1}(v) \). The picture illustrates the cases when edge \((u, v)\) has color 1.

\[
\begin{array}{cc}
\begin{array}{c}
v' \\
u \end{array} & \Rightarrow & \begin{array}{c}
0 \\
1 \\
1 \\
0 \\
v \end{array} \\
\begin{array}{c}
u' \\
u \end{array} & \Rightarrow & \begin{array}{c}
1 \\
0 \\
1 \\
0 \\
v \end{array}
\end{array}
\]

From (A3) it follows that

\[(1) \quad \text{if } v \text{ is the critical vertex in an } i\text{-line, then } v \text{ is simultaneously the critical vertex in the } (3 - i)\text{-line passing through } v.\]

Indeed, let for definiteness \( i = 1 \) and assume that \( v \) has outgoing 2-edge \((v, w)\). Suppose this edge is labeled 0. Then \( t_1(v) = t_1(w) \) implies that \( v \) has incoming 1-edge \((u, v)\). It is labeled 0 (since \( v \) is the critical vertex in \( P_1(v) \)). This means that \( h_2(u) = h_2(v) > 0 \), and therefore, \( u \) has outgoing 2-edge \((u, u')\). Axiom (A3) implies \( w = F_1(u') \) and \( \ell(u, u') = 1 \). But the latter contradicts the fact that \((v, w)\) is labeled 0. Thus, \( \ell(v, w) = 1 \). Arguing similarly and using (b) in (A3), one shows that if \( v \) has incoming 2-edge, then this edge is labeled 0.

Therefore, we can speak of critical vertices without indicating line colors. The final axiom indicates situations of “remote commuting” \( F_1 \) and \( F_2 \).

**(A4)** (i) If a vertex \( u \) has two outgoing edges both labeled 1, then \( F_1F_2F_1(u) = F_2F_2F_1(u) \). Symmetrically: (ii) if \( v \) has two incoming edges both labeled 0, then \( F_1^{-1}(F_2^{-1})^2F_1^{-1}(v) = F_2^{-1}(F_1^{-1})^2F_2^{-1}(v) \).

Note that in case (i), we have \( F_2F_1(u) \neq F_1F_2(u) \) (otherwise the vertices \( u, v := F_1(u), v' := F_2(u) \) and \( w := F_2(v) \) would form a square; then both edges \((v, w)\) and \((v', w)\) have label 1, contrary to (A3)(b)). Similarly, \( F_2^{-1}F_1^{-1}(v) \neq F_1^{-1}F_2^{-1}(v) \) in case (ii). The picture below illustrates axiom (A4). Here also the labels for all involved edges are indicated and the critical vertices are surrounded by circles. (These labels and critical vertices are determined uniquely, which is not difficult to show by use of (A3) and (A4). These facts will be seen from the analysis in the next section as well.)
Clearly the digraph obtained by reversing the orientation of all edges of $K$, while preserving their colors, again satisfies axioms (A1)–(A4) (thereby the label of each edge changes). The resulting RC-graph is called dual to $K$ and is denoted by $K^*$.

3 Structural theorem

In this section we present a theorem that clarifies the combinatorial structure of RC-graphs defined by axioms (A1)–(A4). According to this theorem, each RC-graph can be produced from two elementary RC-graphs by use of a certain operation of replicating and gluing together. First of all we introduce this operation in a general form.

Consider arbitrary graphs or digraphs $G = (V, E)$ and $H = (V', E')$. Let $S$ be a distinguished subset of vertices of $G$, and $T$ a distinguished subset of vertices of $H$. Take $|T|$ disjoint copies of $G$, denoted as $G_t$ ($t \in T$), and $|S|$ disjoint copies of $H$, denoted as $H_s$ ($s \in S$). We glue these copies together in the following way: for each $s \in S$ and each $t \in T$, the vertex $s$ in $G_t$ is identified with the vertex $t$ in $H_s$. The resulting graph consisting of $|V||T| + |V'||S| - |S||T|$ vertices and $|E||T| + |E'||S|$ edges is denoted by $(G, S) \bowtie (H, T)$.

In our case the role of $G$ and $H$ play 2-colored digraphs $K(a, 0)$ and $K(0, b)$ depending on parameters $a, b \in \mathbb{Z}_+$, each of which being a certain triangular part of the Cartesian product of two paths. More precisely, the vertices of $K(a, 0)$ correspond to the pairs $(i, j)$ for $i, j \in \mathbb{Z}$ with $0 \leq i \leq j \leq a$, and the vertices of $K(0, b)$ correspond to the pairs $(i, j)$ for $0 \leq j \leq i \leq b$. The edges with color 1 in these graphs correspond to all possible pairs of the form $((i, j), (i + 1, j))$, and the edges with color 2 to the pairs of the form $((i, j), (i, j + 1))$. It is easy to check that $K(a, 0)$ satisfies axioms (A1)–(A4) and that the diagonal $\{(i, i) : i = 0, \ldots, a\}$ is exactly the set of critical vertices in it. Similarly, $K(0, b)$ is an RC-graph in which the set of critical vertices coincides with the diagonal $\{(i, i) : i = 0, \ldots, b\}$. These diagonals are just considered as the distinguished subsets $S$ and $T$ in these digraphs.

We refer to the digraph formed by applying the operation $\bowtie$ in this case as the diagonal-product of $K(a, 0)$ and $K(0, b)$, and for brevity, denote it by $K(a, 0) \bowtie K(0, b)$. This digraph is 2-colored, where the edge colors are inherited from $K(a, 0)$ and $K(0, b)$ in a natural way. The case $a = 2$ and $b = 1$ is shown in the picture; here the critical vertices are marked with circles. The trivial (degenerate)
RC-graph \(K(0,0)\) consists of a unique vertex; clearly \(K(a,0) \cong K(0,0) = K(a,0)\) and \(K(0,0) \cong K(0,b) = K(0,b)\) for any \(a,b\).

![Figure 2: (a) \(K(2,0)\), (b) \(K(0,1)\), (c) \(K(2,0) \cong K(0,1)\).](image)

It will be convenient for us to refer to a subgraph of an RC-graph \(K\) isomorphic to \(K(a,0)\) (respecting colors and labels of the edges), including \(K(a,0)\) itself, as a \textit{left sail} of size \(a\). Symmetrically, a subgraph of \(K\) isomorphic to \(K(0,b)\) is referred to as a \textit{right sail} of size \(b\). In a left or right sail we specify, besides the diagonal, the 1-side (the largest 1-line) and the 2-side (the largest 2-line).

It is a relatively easy exercise to verify validity of axioms (A1)–(A4) for \(K(a,0) \cong K(0,b)\) with any \(a,b \in \mathbb{Z}_+\), i.e. such a digraph is always an RC-graph. Our main theorem asserts that the converse also takes place.

**Theorem 3.1** Every RC-graph \(K\) is representable as \(K(a,0) \cong K(0,b)\) for some \(a,b \in \mathbb{Z}_+\). In particular, \(K\) is finite.

**Proof of the theorem.** The proof falls into several claims.

**Claim 1.** (i) For any edge \((u,v)\) with color \(i\) and label 0, there exists edge \((w,u)\) with color \((3-i)\) and this edge has label 1. Symmetrically: (ii) for any \(i\)-edge \((u,v)\) labeled 1, there exists \((3-i)\)-edge \((v,w)\) and this edge has label 0.

**Proof.** (i) For an \(i\)-edge \((u,v)\) labeled 0, one has \(t_{3-i}(u) > t_{3-i}(v)\) (by axiom (A2)). Therefore, \(u\) has incoming \((3-i)\)-edge \((w,u)\). Suppose \(\ell(w,u) = 0\). Then \(h_i(w) = h_i(u) > 1\). So \(w\) has outgoing \(i\)-edge \((w,w')\). By axiom (A3) applied to the pair \((w,u),(w,w')\), the vertices \(w,w',u,v\) form a square, and \(\ell(w,w') = 1\). But the edge \((u,v)\) opposite to \((w,w')\) for this square is labeled 0. This contradiction shows that \((w,u)\) must be labeled 1.

Part (ii) in this claim follows from part (i) applied to the dual RC-graph \(K^*\). ■

**Claim 2.** (i) Let \((u,v)\) be an \(i\)-edge labeled 0 and let \(h_{3-i}(v) > 0\). Then there exist \((3-i)\)-edges \((u,u'),(v,v')\) labeled 1 and \(i\)-edges \((u',v'),(v',v'')\) labeled 0. Symmetrically:
(ii) if \((u, v)\) is an \(i\)-edge labeled 1 and if \(t_{3-i}(u) > 0\), then there exist \((3-i)\)-edges \((u', u), (v', v)\) labeled 0 and \(i\)-edges \((u'', u'), (u', v')\) labeled 1.

**Proof.** (i) For an \(i\)-edge \((u, v)\) labeled 0, one has \(h_{3-i}(u) = h_{3-i}(v)\). Therefore, \(u\) has outgoing \((3-i)\)-edge \((u, u')\). By axiom (A3) applied to edges \((u, v), (u, u')\), the vertices \(u, v, u'\) and 
\(v' := F_2(v)\) form a square, and \(\ell(u, u') = 1\). Then \(\ell(u', v') = \ell(u, v) = 0\) and \(\ell(v, v') = \ell(u, u') = 1\). The existence of \(i\)- \((v', v'')\) labeled 0 follows from part (ii) in Claim 1 applied to the edge \((v, v')\).

The second part of the claim follows from the first one applied to \(K^*\). ■

The picture below illustrates Claims 1 and 2 for the cases when \((u, v)\) is a 1-edge labeled 0 or a 2-edge labeled 1.

For a path \((v_0, e_0, v_1, \ldots, e_k, v_k)\), we may use the abbreviate notation \(v_0v_1\ldots v_k\).

**Claim 3.** Let \(v\) be a critical vertex in \(K\) and let \(L\) be a left sail of maximum size that contains \(v\). Then \(L\) has size \(d := t_1(v) + h_2(v)\) and contains the paths \(P_{1}^\text{in}(w)\) and \(P_{2}^\text{out}(w)\) for all vertices \(w\) in \(L\) (which is equivalent to saying that \(h_2(w') = 0\) for each vertex \(w'\) on the 1-side of \(L\), and \(t_1(w'') = 0\) for each vertex \(w''\) on the 2-side of \(L\)).

\((L\) exists since the vertex \(v\) itself forms the trivial sail \(K(0, 0)\).)

**Proof.** The claim is obvious if \(d = 0\). Let \(d > 0\). If \(v\) has incoming 1-edge \((u, v)\), then \(\ell(u, v) = 0\) (since \(v\) is critical). By Claim 1, \(v\) belongs to the left sail of size 1 formed by the edge \((u, v)\) and the 2-edge incoming \(u\). Similarly, if \(v\) has outgoing 2-edge, then it belongs to a sail of size 1.

Thus, one may assume that the maximum-size left sail \(L\) has size \(k \geq 1\). Consider the 1-side \(P = v_0v_1\ldots v_k\) of \(L\). All 1-edges of \(L\) are labeled 0, therefore, \(h_2(v_0) = h_2(v_1) = \ldots = h_2(v_k)\). Suppose \(h_2(v_i) > 0\). Applying Claim 2 to the edges of \(P\), one can conclude that there exists a path \(u_0u_1\ldots u_{k+1}\) whose edges have color 1 and label 0 and whose vertices are connected with the vertices of \(P\) by the 2-edges \((v_i, u_i)\) labeled 1, \(i = 0, \ldots, k\). But this implies that the sail \(L\) is not maximum. Hence \(h_2(v_i) = 0\) \(v_i\) \(P\). Considering the 2-side of \(L\) and arguing in a similar fashion, we obtain \(t_1(w'') = 0\) for all vertices on this side, and the claim follows. ■
Note that, in Claim 3, the vertex $v$ lies on the diagonal of the sail $L$ (since the edges in $P_1^{\text{out}}(v)$ are labeled 1). One can see that $v$ determines $L$ uniquely, and therefore, we may call $L$ the maximal left sail containing $v$. Also each vertex $q$ in the diagonal of $L$ is critical. Indeed, suppose this is not so for some $q$ and consider the 1-line $P$ passing through $q$ and the critical vertex $r$ in $P$. Then $q$ occurs in $P$ earlier than $r$, i.e., $q \in P_1^{\text{out}}(r)$ and $q \neq r$. Applying Claim 3 to the maximum-size left sail $L'$ containing $r$, one can conclude that $L'$ includes $L$, contrary to the maximality of $L$.

Let $\mathcal{L}$ be the set of all maximal left sails (each containing a critical vertex). By above reasonings, the members of $\mathcal{L}$ are pairwise disjoint and they cover all critical vertices, all 1-edges labeled 0 and all 2-edges labeled 1.

Since any left sail of $K$ turns into a right sail of $K^*$, we have similar properties for the set $\mathcal{R}$ of all maximal right sails of $K$: the diagonal of each member of $\mathcal{R}$ consists of critical vertices, the members of $\mathcal{R}$ are pairwise disjoint and they cover all critical vertices, all 1-edges labeled 1 and all 2-edges labeled 0.

For a sail $Q$, the vertices in the diagonal $D$ of $Q$ are ordered in a natural way, where the minimal (maximal) element is the vertex with zero indegree (resp. zero outdegree) in $Q$. According to this ordering, the elements of $D$ are numbered by $0, 1, \ldots, |D|$. So if $u, v$ are vertices in $D$ with numbers $i, i + 1$, respectively, then $v = F_1F_2(u)$ in case of left sail, and $v = F_2F_1(u)$ in case of right sail.

Choose a maximal left sail $L \in \mathcal{L}$. Let it have size $a$ and diagonal $D = (v_0, v_1, \ldots, v_a)$, where $v_i$ is the vertex with number $i$ in $D$. For $i = 0, \ldots, a$, denote by $R_i$ the maximal right sail containing $v_i$. It has size $h_1(v_i) + t_2(v_i)$. Since $v_{i+1} = F_1F_2(v_i)$ and since the 2-edge $(v_i, F_2(v_i))$ of $L$ is labeled 1, we have $h_1(v_{i+1}) = h_1(F_2(v_i)) - 1 = h_1(v_i)$. In a similar way, the fact that the 1-edge $(F_2(v_i), v_{i+1})$ of $L$ is labeled 0 implies $t_2(v_{i+1}) = t_2(v_i)$. Therefore, the values $h_1(v_i)$ ($i = 0, \ldots, a$) are equal to one and the same number $p$, and the values $t_2(v_i)$ are equal to the same number $q$. This gives the following important property:

(2) the maximal right sails $R_0, \ldots, R_a$ have the same size $b := p + q$, and for $i = 0, \ldots, a$, the vertex $v_i$ has number $q$ in the diagonal of $R_i$ (and therefore, this number does not depend on $i$).

Then the sails $R_0, \ldots, R_a$ are different, or, equivalently, $L$ and $R_i$ have a unique vertex in common, namely, $v_i$ (since $R_i = R_j$ for $i \neq j$ would imply $t_2(v_i) \neq t_2(v_j)$.)

Considering an arbitrary $R \in \mathcal{R}$ and arguing similarly, we have:

(3) the maximal left sails $L' \in \mathcal{L}$ intersecting $R$ have the same size, and in the diagonals of these sails $L'$, the vertices common with $R$ have equal numbers.

Since $K$ is connected, one can conclude from (2) and (3) that all members of $\mathcal{L}$ have size $a$ and all members of $\mathcal{R}$ have size $b$. 

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Now let \( u_0, \ldots, u_b \) be the vertices in the diagonal of the maximal right sail \( R_0 \) (where the vertices are indexed according to the ordering in the diagonal), and for \( j = 0, \ldots, b \), define \( L_j \) to be the maximal left sail containing \( u_j \). One may assume that \( L_0 \) is just the left sail \( L \) chosen above, i.e., \( u_0 = v_0 \).

So far we have applied only axioms (A1)–(A3). The final claim essentially uses axiom (A4).

**Claim 4.** For each \( i = 0, \ldots, a \) and each \( j = 0, \ldots, b \), the sails \( R_i \) and \( L_j \) are intersecting. Moreover, their (unique) common vertex \( w \) has number \( i \) in the diagonal of \( L_j \) and has number \( j \) in the diagonal of \( R_i \).

**Proof.** We use induction on \( i + j \). In fact, we have seen above that the claim is valid for \( j = 0 \) and any \( i \), as well as for \( i = 0 \) and any \( j \). So let \( 0 < i \leq a \) and \( 0 < j \leq b \). By induction there exist a common vertex \( u \) for \( L_{i-1} \) and \( R_{i-1} \), a common vertex \( v \) for \( L_j \) and \( R_j \), and a common vertex \( v' \) for \( R_{i-1} \) and \( L_j \). Furthermore, \( u \) and \( v \) have numbers \( i - 1 \) and \( i \) (respectively) in the diagonal of \( L_{i-1} \), and therefore, \( v = F_2F_1(u) \).

In their turn, \( u \) and \( v' \) have numbers \( j - 1 \) and \( j \) (respectively) in the diagonal of \( R_{i-1} \), and therefore, \( v' = F_2F_1(u) \). Also \( v \) has number \( j - 1 < b \) in the diagonal of \( R_i \), while \( v' \) has number \( i - 1 < a \) in the diagonal of \( L_j \). Hence the diagonal \( D \) of \( R_i \) contains vertex \( w \) next to \( v \) (i.e., \( w = F_2F_1(v) \) and \( w \) has number \( j \) in \( D \)), and the diagonal \( D' \) of \( L_j \) contains vertex \( w' \) next to \( v' \) (i.e., \( w' = F_1F_2(v') \) and \( w' \) has number \( i \) in \( D' \)).

Since \( (u, F_2(u)) \) is a 2-edge of a left sail and \( (u, F_1(u)) \) is a 1-edge of a right sail, both edges are labeled 1. Applying axiom (A4)(i) to them, we obtain \( w = w' \), and the result follows. ■

Thus, the union \( K' \) of sails \( L_0, \ldots, L_b, R_0, \ldots, R_a \) is isomorphic to \( K(a, 0) \Join K(0, b) \). Also each of these sails meets any other member of \( \mathcal{L} \cup \mathcal{R} \) only within the set of critical vertices in the former (by Claim 3), and these critical vertices belong to \( K' \). Therefore, the connectedness of \( K \) implies \( K = K' \). This completes the proof of the theorem. ■ ■

We denote the RC-graph isomorphic to \( K(a, 0) \Join K(0, b) \) by \( K(a, b) \).

**Corollary 3.2** \( K(a, b) \) contains \( (a + 1)(b + 1) \) critical vertices and has exactly one vertex \( s \) with zero indegree (the minimal vertex, or the source of the RC-graph) and exactly one vertex \( t \) with zero outdegree (the maximal vertex, or the sink). This \( s \) is the common vertex of the sails \( L_0 \) and \( R_0 \) (defined in the above proof), while \( t \) is the common vertex of the sails \( L_b \) and \( R_a \), and one holds \( h_1(s) = t_2(t) = b \) and \( h_2(s) = t_1(t) = a \). In particular, an RC-graph with source \( s \) is determined by the parameters \( h_1(s), h_2(s) \). Also \( K(a, b) \) has equal numbers of edges of each color, namely, \( (a + 1)(b + 1)(a + b)/2 \).
Remark 2. In the proof of Theorem 3.1 we never used part (ii) of axiom (A4). This means that this part is redundant. In fact, one can show directly that (A4)(ii) is implied by (A4)(i) and (A1)–(A3). (However, part (ii) of axiom (A4) becomes essential for infinite RC-graphs considered in the next section.)

Remark 3. The connected 2-colored digraphs satisfying axioms (A1)–(A3), which may be named weakened RC-graphs, or WC-graphs, form an interesting class WC lying between the classes of NC-graphs and RC-graphs. This class can be completely characterized, relying on the fact (seen from the above proof) that each member of it is the union of a set of pairwise disjoint left sails of the same size a and a set of pairwise disjoint right sails of the same size b. More precisely, each \( K \in \mathcal{WC} \) can be encoded by parameters \( a, b \in \mathbb{Z}_+ \), a graph \( \Gamma = (V(\Gamma), E(\Gamma)) \) and a map \( \omega : V(\Gamma) \to \mathbb{Z}_+ \) such that:

\[ (*) \quad \begin{align*}
(a) \ & \Gamma \text{ is a connected finite or infinite bipartite graph with vertex parts } V_1, V_2; \\
(b) \ & \text{each vertex in } V_1 \text{ has degree } a + 1 \text{ and each vertex in } V_2 \text{ has degree } b + 1; \\
(c) \ & \omega(v) \text{ does not exceed } b \text{ for each vertex in } V_1, \text{ and } a \text{ for each vertex in } V_2; \\
(d) \ & \text{for each } v \in V(\Gamma), \text{ the vertices } w \text{ adjacent to } v \text{ have different values } \omega(w).
\end{align*} \]

The vertices in \( V_1 \) are associated with left sails of size \( a \), and the vertices in \( V_2 \) with right sails of size \( b \). The edges of \( \Gamma \) indicate how these sails are glued together, namely: if \( u \in V_1 \) and \( v \in V_2 \) are connected by an edge, then the vertex with number \( \omega(v) \) in the diagonal of the sail corresponding to \( u \) is identified with the vertex with number \( \omega(u) \) in the diagonal of the sail corresponding to \( v \). The resulting \( K \) is finite if and only if \( \Gamma \) is finite. If \( \Gamma \) is a complete bipartite graph, we just obtain the RC-graph \( K(a, b) \). A somewhat different way to characterize \( \mathcal{WC} \) is as follows.

For \( a, b \in \mathbb{Z}_+ \), let \( \Gamma_{a,b} \) denote the Cartesian product \( P_a \times P_b \) of directed paths with length \( a \) and \( b \), respectively. A covering over \( \Gamma_{a,b} \) is a nonempty (finite or infinite) connected digraph \( G \) along with a homomorphism \( \gamma : G \to \Gamma_{a,b} \) under which the 1-neighborhood of each vertex \( v \) of \( G \) (i.e., the subgraph induced by the edges incident with \( v \)) is isomorphically mapped to the 1-neighborhood of \( \gamma(v) \). For such a \( (G, \gamma) \), the preimage in \( G \) of each path \( (P_a, \cdot) \) of \( \Gamma_{a,b} \) is a collection of pairwise disjoint paths \( Q \) of length \( a \), and we can replace each of these \( Q \) by a copy of the left sail \( L \) of size \( a \) in a natural way (the vertices of \( Q \) are identified with the elements of the diagonal of \( L \) in the natural order). The preimages of each path \( (\cdot, P_b) \) are replaced by copies of the right sail \( R \) of size \( b \) in a similar fashion. One can check that the resulting digraph \( K \), with a due assignment of edge colors, satisfy (A1)–(A3), and conversely, each member of \( \mathcal{WC} \) can be obtained by this construction.

Proposition 3.3 There is a bijection between the set \( \mathcal{WC} \) of 2-colored digraphs satisfying axioms (A1)–(A3) and the set of coverings over grids \( \Gamma_{a,b} \) for all \( a, b \in \mathbb{Z}_+ \).

Adding axiom (A4) removes all nontrivial coverings (i.e., those different from the grids themselves).
indegree and zero outdegree vertices, respectively) implies that the quantity of sources (sinks) in a covering $G$ is equal to the quantity of preimages in $G$ of a vertex of $\Gamma_{a,b}$. This gives the following important property.

**Corollary 3.4** Under validity of (A1)–(A3), axiom (A4) is equivalent to the requirement that the digraph has only one source or only one sink. In other words, the RC-graphs are precisely the WC-graphs with one source (one sink).

**Remark 4.** The fact that an RC-graph $K = K(a, b)$ is graded w.r.t. each color (cf. Corollary 2.1) implies that $K$ is the Hasse diagram of a poset $(V, \preceq)$ on the vertex set (it is generated by the relations $u < v$ for edges $(u, v)$). Considering the sail structure of $K$ as above, it is not difficult to obtain the following sharper property. Here $r_{ij}$ stands for the common critical vertex of sails $R_i$ and $L_j$, and for a vertex $v$, we denote by $p(v)$ ($q(v)$) the minimal (resp. maximal) critical vertex greater (resp. smaller) than or equal to $v$ in a maximal sail containing $v$.

**Proposition 3.5** The poset $(V, \preceq)$ is a lattice, that is, any two vertices $u, v$ have a unique minimal upper bound $u \lor v$ and a unique maximal lower bound $u \land v$. More precisely:

(i) if $u, v$ are critical vertices $r_{ij}, r_{i'j'}$, then $u \lor v = r_{\max\{i,i\'}, \max\{j,j\'}}$ and $u \land v = r_{\min\{i,i\'}, \min\{j,j\'}}$;

(ii) if $u, v$ occur in the same maximal sail, then both $u \lor v$ and $u \land v$ belong to this sail (they are computed in a straightforward way; in particular, $p(u \lor v) = p(u) \lor p(v)$ and $q(u \lor v) = q(u) \lor q(v)$);

(iii) for vertices $u, v$ occurring in different maximal sails: (a) if $p(u) \preceq p(v)$, then $v$ and $w := u \lor v$ belong to the same maximal sail (sails) $Q$ and one holds $p(w) = p(v)$ and $q(w) = p(u) \lor q(v)$ (the latter vertex belongs to $Q$ as well); and (b) if $p(u)$ and $p(v)$ are incomparable, then $u \lor v = p(u) \lor p(v)$.

(For vertices $u, v$ in different maximal sails, computing $u \land v$ is symmetric to (iii).) Note that this lattice is not distributive already for $a, b = 1$.

### 4 Infinite RC-graphs

The notion of RC-graphs can be extended, with a due care, to (connected) infinite 2-colored digraphs $K = (V, E_1, E_2)$. By Theorem 3.1, the finiteness of each monochromatic path leads to the finiteness of an RC-graph, so we now should allow infinite monochromatic paths and accordingly modify axiom (A1). There are three types of infinite paths. A **fully infinite path** is a sequence of the form $\ldots, v_i, e_i, v_{i+1}, e_{i+1}, \ldots$, where the index set $I$ of vertices $v_i$ ranges $\mathbb{Z}$ (and, as before, $e_i$ is the edge from $v_i$,
to $v_{i+1})$. If $I = \mathbb{Z}_+$ (resp. $I = \mathbb{Z}_-$), we deal with a *semiinfinite path* in forward (resp. backward) direction.

There are two methods to define an RC-graph so as to involve both finite or infinite cases. The first method is based on a generalization of Theorem 3.1 and uses the construction $(G, S) \bowtie (H, T)$ defined in Section 3 and applicable to arbitrary finite or infinite graphs $G, H$ and distinguished subsets $S, T$.

In our case the role of $G$ plays a left sail, which can be either finite (defined as before) or infinite of any of three possible sorts. The sail denoted by $L_\infty$ ("infinite up and to the left") has vertex set \{(i, j) : i, j ∈ \mathbb{Z}, i ≤ j\}, the sail $L^\infty$ ("infinite up") has the vertices (i, j) for 0 ≤ i ≤ j, and the sail $L_{\infty}$ ("infinite to the left") has the vertices (i, j) for i ≤ j ≤ 0. Analogously, $H$ is a finite of infinite right sail, and the latter can be of three sorts: $R_\infty$ ("infinite down and to the right") with the vertices (i, j) for j ≤ i, $R^\infty$ ("infinite to the right") with the vertices (i, j) for 0 ≤ j ≤ i, and $R_{\infty}$ ("infinite down") with the vertices (i, j) for j ≤ i ≤ 0. In all cases the 1-edges correspond to the pairs ((i, j), (i + 1, j)), and 2-edges to the pairs ((i, j), (i, j + 1)). As before, in a left sail all 1-edges are labeled 0, and all 2-edges are labeled 1, while in a right sail the labels are interchanged. The distinguished subsets $S, T$ are the corresponding "diagonals" consisting of the vertices (i, i) (being critical). Infinite sails are illustrated in the picture.

Combining any of the four sorts of left sails (one of which is finite and the other three are infinite) with any of the four sorts of right sails, we obtain 16 types of RC-graphs $L \bowtie R$, of which one is finite, while the other 15 contain a fully infinite or semiinfinite monochromatic paths. For example, $L_\infty \bowtie R_\infty$ has fully infinite 1-lines and finite 2-lines, and $K(a, 0) \bowtie R_{\infty}$ has semiinfinite in forward direction 1-lines and semiinfinite in backward direction 2-lines. The largest RC-graph $L_\infty \bowtie R_{\infty}$ contains the other ones as induced subgraphs.

**Remark 5.** There are five more infinite RC-graphs (defined up to swapping the edge colors). They have a simple structure and do not contain critical vertices at all. The vertices of these RC-graphs are the pairs (i, j), where either (i) $i, j ∈ \mathbb{Z}$, or (ii) $i ∈ \mathbb{Z}$ and $j ∈ \mathbb{Z}_+$, or (iii) $i ∈ \mathbb{Z}$ and $j ∈ \mathbb{Z}_-$, or (iv) $i ∈ \mathbb{Z}_-$ and $j ∈ \mathbb{Z}_+$, or (v) $i ∈ \mathbb{Z}_+$ and $j ∈ \mathbb{Z}_-$. Formally, in cases (ii),(iv), the 1-edges ((i, j), (i + 1, j)) are labeled 1, and 2-edges ((i, j), (i, j + 1)) are labeled 0, while in cases (iii),(v), the labels are interchanged.
The second, alternative, method of unifying the definition of RC-graphs to include infinite cases consists in modifying axioms (A2) and (A3) (while preserving (A4)). We replace them by a single axiom that postulates properties exposed in Claims 1 and 2 from the proof of Theorem 3.1. As before, each edge $e$ is endowed with label $\ell(e) \in \{0,1\}$, the labels are monotonically nondecreasing along each monochromatic path, and $t_i(v), h_i(v)$ denote the lengths of corresponding paths (which may be infinite). The new axiom is stated as follows:

\[ (A') \quad K \text{ is graded w.r.t. each color (cf. Corollary 2.1). Also: (a) for each } i \text{-edge } (u, v) \text{ labeled } 0, \text{ there exists } (3-i) \text{-edge } (w, u) \text{ labeled } 1; \text{ moreover, } u \text{ has outgoing } (3-i) \text{-edge } (u, u') \text{ if and only if } v \text{ has outgoing } (3-i) \text{-edge } (v, v'), \text{ and in this case both } (u, u'), (v, v') \text{ are labeled } 1 \text{ and there exist } i \text{-edges } (u', v'), (v', u'') \text{ labeled } 0. \text{ Symmetrically: (b) for each } i \text{-edge } (u, v) \text{ labeled } 1, \text{ there exists } (3-i) \text{-edge } (v, w) \text{ labeled } 0; \text{ moreover, } u \text{ has incoming } (3-i) \text{-edge } (u', u) \text{ if and only if } v \text{ has incoming } (3-i) \text{-edge } (v', v) \text{, and in this case both } (u', u), (v', v) \text{ are labeled } 0 \text{ and there exist } i \text{-edges } (u', v'), (u'', u') \text{ labeled } 1. \]

In reality both definitions are equivalent. A verification that a generalized RC-graph constructed by the first method satisfies (A') (and (A4)) is relatively simple. The converse assertion, that a digraph satisfying (A'),(A4) is one of those described in the first method, can be proved by following the method of proof of Theorem 3.1 with necessary extensions and refinements; we omit details here. (Unlike the finite case (cf. Remark 2), part (ii) of axiom (A2) becomes essential for the general case.) In particular, one shows that if some (infinite) monochromatic line has no critical vertex, then the RC-graph is one of those indicated in Remark 5.

**Remark 6.** The construction of diagonal-product can be used for extending the notion of RC-graphs to more abstract structures. More precisely, let $I, J$ be two fully ordered sets. (For example, we can take as $I, J$ intervals in $\mathbb{R}$ or $\mathbb{Q}$. In essence, so far we have dealt with intervals in $\mathbb{Z}$.) We define the left sail over $I$ in a natural way, to be the set $L := \{(x, y) \in I^2 : x \leq y\}$, and define the right sail over $J$ to be $R := \{(x, y) \in J^2 : x \geq y\}$. The distinguished subsets $D, D'$, or the diagonals, in $L, R$, respectively, consist of the identical pairs $(x, x)$. Then we can form the corresponding “diagonal-product” $K := L \bowtie R$. Fixing the second coordinate $y$ (resp. the first coordinate $x$) in the left sail $L$ or in the right sail $R$ gives a line of color 1 (resp. 2) in this sail. When $I, J$ are intervals in $\mathbb{R}$ (in which case $K$ may be named a continuous crystal), one can introduce a reasonable metric on $K$, which determines its intrinsic topological structure, as follows. The distance between points $(x, y)$ and $(x', y')$ in each sail of $K$ is assigned to be the $\ell_1$-distance $|x - x'| + |y - y'|$, and the distance within the critical set $W := D \times D'$ is also assigned to be the corresponding distance of $\ell_1$-type. This induces a metric $d$ on the entire $K$: for different sails $Q, Q'$ of $K$ and points $u \in Q$ and $v \in Q'$, $d(u, v)$ is equal to $\inf\{d(u, u') + d(u', v') + d(v', v) : u' \in Q \cap W, v' \in Q' \cap W\}$. Note that the resulting metric space need not be compact even...
5 Polyhedral aspects and a relation to Gelfand-Tsetlin patterns

In this section we return to a (finite) RC-graph $K = K(a,b)$, with node set $V$, and discuss some natural embeddings of $K$ and other properties, using definitions, notation and results from Section 3. (The results can be extended to infinite RC-graphs as well.)

Recall that $K$ has set $\mathcal{L} = \{L_0, \ldots, L_b\}$ of maximal left sails (with size $a$) and set $\mathcal{R} = \{R_0, \ldots, R_a\}$ of maximal right sails (with size $b$). These sails and the vertices in their diagonals are numbered as in the proof of Theorem 3.1. Under these numerations, sails $L_j$ and $R_i$ intersect at the critical vertex that has number $i$ in the diagonal $D_j$ of $L_j$ and number $j$ in the diagonal $D'_i$ of $R_i$. We denote the vertex with number 0 in $D_j$ (the source of $L_j$) by $s_j$, and denote the vertex with number 0 in $D'_i$ (the source of $R_i$) by $s'_i$. Each vertex $v$ of $L_j$ is determined by (local) coordinates $(p,q)$, where $p$ (resp. $q$) is the number of 1-edges (resp. 2-edges) in a path from $s_j$ to $v$, i.e., $v = F^p_j F^q_j(s_j)$. Analogous coordinates are assigned in $R_i$ with respect to $s'_i$. The vertex $s_0 = s'_0$ is the source of the whole $K$, denoted by $s_K$.

1. One way to embed $K$ in an Abelian group relies on the observation that the vertices $v \in V$ have different length-tuples $\tau(v) = (t_1(v), h_1(v), t_2(v), h_2(v))$. Moreover, the vertices differ from each other even if three parameters involved in $\tau$ are considered, e.g., $t_1(v), h_1(v), t_2(v)$. This is seen from the following lemma.

**Lemma 5.1** For $v \in V$, define $\epsilon := a - t_1(v) - h_2(v)$ and $\delta := b - t_2(v) - h_1(v)$. (i) If $v$ occurs in a left sail $L_j$, then $-2\delta = \epsilon \geq 0$, $v$ has coordinates $(t_1(v), a-h_2(v))$ in $L_j$, and $j$ is equal to $t_2(v) - \epsilon = b - h_1(v) + \epsilon$. (ii) If $v$ occurs in a right sail $R_i$, then $-2\epsilon = \delta \geq 0$, $v$ has coordinates $(b - h_1(v), t_2(v))$ in $R_i$, and $i$ is equal to $t_1(v) - \delta = a - h_2(v) + \delta$. (iii) The vertex $v$ is critical if and only if $\epsilon = \delta = 0$.

**Proof.** The assertions are obvious when $v$ is critical. If $v$ lies in a left sail $L_j$, then the assertions in (i) can be obtained by comparing $\tau(v)$ with the length-tuples of the critical vertices in the lines $P_1(v)$ and $P_2(v)$ and by using the fact that both critical vertices have number $j$ in the diagonals of the corresponding maximal right sails. If $v$ lies in a right sail $R_i$, the proof is analogous. ■

Note that the edges $(u,v)$ with the same color and the same label have the same difference $\tau(v) - \tau(u)$ (e.g., for color 1 and label 0, the difference is $(1,-1,-1,0)$). So $\tau$ induces an embedding of $K$ in the corresponding subgroup of $\mathbb{Z}^4$ shifted by the vector $-\tau(s_K) = (0, -b, 0, -a)$.
2. Next we are interested in embeddings with the property that the edges of $K$ correspond to parallel translations of unit base vectors. For $l = 0, 1$ and a path $P$ in $K$, we denote the number of 1-edges (2-edges) of $P$ with label $l$ by $\alpha_l(P)$ (resp. by $\beta_l(P)$). The next lemma strengthens Corollary 2.1, showing that $K$ is graded w.r.t. each combination of color and label.

**Lemma 5.2** Let $P$ be a path in $K$ beginning at $s_K$ and ending at $v \in V$. (i) If $v$ occurs in a left sail $L_j$, then $\alpha_1(P) = \beta_0(P) = j$, and $v$ has coordinates $(\alpha_0(P), \beta_1(P))$ in $L_j$. (ii) If $v$ occurs in a right sail $R_i$, then $\alpha_0(P) = \beta_1(P) = i$, and $v$ has coordinates $(\alpha_1(P), \beta_0(P))$ in $R_i$.

**Proof.** Use induction on the length $|P|$ of $P$. The assertion is trivial when $|P| = 0$, so let $|P| > 0$. Suppose $v$ lies in $L_j$. If the vertex $u$ of $P$ preceding $v$ also lies in $L_j$, the assertion for $P$ easily follows by induction from that for the part of $P$ from $s_K$ to $u$.

Now let $u \notin L_j$. Then the vertex $v$ is critical and the edge $(u, v)$ is contained in some right sail $R_i$. Consider the last critical vertex $w$ of $P$ different from $v$ (it exists as the beginning vertex $s_K$ of $P$ is critical and $s_K \neq v$). Clearly $w$ belongs to $R_i$; let it have number $j'$ in the diagonal of $R_i$ (whereas $v$ has number $j$, and $j > j'$). For the part $P'$ of $P$ from $s_K$ to $w$, one has $\alpha_0(P') = \alpha_0(P)$, $\beta_1(P') = \beta_1(P)$, $\alpha_1(P') + j - j' = \alpha_1(P)$ and $\beta_0(P') + j - j' = \beta_0(P)$. By induction $\alpha_1(P') = \beta_0(P') = j'$ and $\alpha_0(P') = \beta_1(P') = i$ (since $w$ is also contained in the left sail $L_j$ and has number $i$ in its diagonal, whence $w$ has coordinates $(i, i)$ in $L_j$). This gives the desired result for $P$, taking into account that $v$ has the same coordinates $(i, i)$ in $L_j$.

When $v$ lies in a right sail $R_i$, we argue in a similar way. ■

Thus, for a path $P$ from the source $s_K$ to a vertex $v$, the numbers $\alpha_l(P), \beta_l(P)$ ($l = 0, 1$) depend only on $v$, and we can define $\alpha_l(v) := \alpha_l(P)$ and $\beta_l(v) := \beta_l(P)$. Also Lemma 5.2 shows that the quadruples $\overline{P}(v) := (\alpha_0(v), \alpha_1(v), \beta_0(v), \beta_1(v))$ are different for all vertices $v$, i.e., the map $\overline{P} : V \rightarrow \mathbb{Z}^4$ is injective. Under this map, traversing an edge of $K$ corresponds to adding a unit base vector associated with the color and label of the edge.

Since the local coordinates $(p, q)$ satisfy the relation $0 \leq p \leq q \leq a$ for the sails in $\mathcal{L}$, and $0 \leq q \leq p \leq b$ for the sails in $\mathcal{R}$, Lemma 5.2 implies the following.

**Corollary 5.3** For each vertex $v$, one has $0 \leq \alpha_0(v) \leq \beta_1(v) \leq a$ and $0 \leq \beta_0(v) \leq \alpha_1(v) \leq b$; moreover, at least one of $\alpha_0(v) \leq \beta_1(v)$ and $\beta_0(v) \leq \alpha_1(v)$ turns into equality (and both equalities here characterize the critical vertices). Conversely, if integers $p, p', q, q'$ satisfy $0 \leq p \leq q \leq a$ and $0 \leq q' \leq p' \leq b$ and if at least one of $p = q$ and $p' = q'$ holds, then there is a vertex $v$ with $\overline{P}(v) = (p, p', q, q')$. 

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Remark 7. Using Lemma 5.2 one can characterize the lattice \((V, \preceq)\) of \(K\) (cf. Proposition 3.5) via the vertex parameters \(\alpha_i, \beta_j\). More precisely, vertices \(u, v\) satisfy \(u \preceq v\) if and only if at least one of the following holds:

(a) \(\beta_1(u) \leq \alpha_0(v)\) and \(\alpha_1(u) \leq \beta_0(v)\);

(b) all \(\alpha_0(u), \alpha_0(v), \beta_1(u), \beta_1(v)\) are equal, \(\alpha_1(u) \leq \alpha_1(v)\), and \(\beta_0(u) \leq \beta_0(v)\).

(c) all \(\alpha_1(u), \alpha_1(v), \beta_0(u), \beta_0(v)\) are equal, \(\alpha_0(u) \leq \alpha_0(v)\), and \(\beta_1(u) \leq \beta_1(v)\).

(One can describe the lattice operations \(\lor, \land\) in terms of \(\alpha_i, \beta_j\); we leave this to the reader as an exercise.)

3. Corollary 5.3 enables us to transform the map \(\overline{\rho}\) defined in part 2 into an injective map \(\rho : V \to \mathbb{Z}^3\), by combining \(\beta_0\) and \(\beta_1\) into one coordinate. More precisely, define \(\beta := \beta_0 + \beta_1\) and \(\rho := (\alpha_0, \alpha_1, \beta)\). The fact that \(\rho\) is injective follows from the possibility of (uniquely) restoring \(\beta_0(v), \beta_1(v)\) if we know \(\rho(v)\), namely:

\[
\beta_0(v) = \alpha_1(v) \text{ if } \beta(v) \geq \alpha_0(v) + \alpha_1(v), \text{ and } \beta_1(v) = \alpha_0(v) \text{ otherwise}; \text{ equivalently: } \beta_0(v) = \min\{\alpha_1(v), \beta(v) - \alpha_0(v)\} \text{ and } \beta_1(v) = \max\{\alpha_0(v), \beta(v) - \alpha_1(v)\}.
\]

In a similar way, one can combine \(\alpha_0\) and \(\alpha_1\), by setting \(\alpha := \alpha_0 + \alpha_1\) and \(\rho' := (\alpha, \beta_0, \beta_1)\). Then the injectivity of \(\rho'\) is provided by:

\[
\alpha_0(v) = \beta_1(v) \text{ if } \alpha(v) \geq \beta_0(v) + \beta_1(v), \text{ and } \alpha_1(v) = \beta_0(v) \text{ otherwise}; \text{ equivalently: } \alpha_0(v) = \min\{\beta_1(v), \alpha(v) - \beta_0(v)\} \text{ and } \alpha_1(v) = \max\{\beta_0(v), \alpha(v) - \beta_1(v)\}.
\]

Consider the map \(\rho\) and identify \(V\) with the set \(\rho(V)\) of points in the space \(\mathbb{R}^3\) with coordinates \((\alpha_0, \alpha_1, \beta)\). Let \(\mathcal{P} = \mathcal{P}(a,b)\) denote the convex hull of \(V\). Using Corollary 5.3 it is not difficult to obtain the following description and properties of the polytope \(\mathcal{P}\).

**Proposition 5.4** \(P\) is formed by the vectors \((\alpha_0, \alpha_1, \beta) \in \mathbb{R}^3\) satisfying

\[
\begin{align*}
(6) \quad (i) & \quad 0 \leq \alpha_0 \leq a; \\
(ii) & \quad 0 \leq \alpha_1 \leq b; \\
(iii) & \quad \alpha_0 \leq \beta \leq \alpha_1 + a.
\end{align*}
\]

The polytope \(\mathcal{P}\) is represented as the Minkowski sum of the convex hulls of sails \(L_0\) and \(R_0\) (considered as sets of points) and the set of integer points in \(\mathcal{P}\) is exactly \(V\). The vertices of \(\mathcal{P}\) are \((0, 0, 0), (0, 0, a), (0, b, 0), (a, 0, a), (a, b, a), (0, b, a+b), (a, b, a+b)\) (some of which coincide when \(a = 0\) or \(b = 0\)).
So, in the nondegenerate case $a, b > 0$, $P$ has 6 facets and 7 vertices. All critical vertices of $K$ are contained in the cutting plane $\beta = \alpha_0 + \alpha_1$ (cf. Corollary 5.3). It intersects $P$ by the parallelogram $\Pi$ whose vertices are $(0, 0, 0), (a, 0, a), (a, b, a + b)$ and a point lying on the edge of $P$ connecting $(0, b, 0)$ and $(0, b, a + b)$. This “critical section” $\Pi$ subdivides $P$ into two triangular prisms being, respectively, the convex hulls of the sails in $L$ and of the sails in $R$. The polytope $P$ is illustrated in Fig. 3.

Figure 3: The polytope $P(a, b)$. The critical section $\Pi$ is indicated by dots.

A similar description can be obtained for the convex hull $P'$ of the RC-graph $K$ when $K$ is embedded by use of $\rho'$ in the space $\mathbb{R}^3$ with coordinates $(\alpha, \beta_0, \beta_1)$. Comparing (4) and (5), one can determine the canonical bijection $\omega : P \rightarrow P'$ (preserving the vertices of $K$). This $\omega$ is piecewise-linear and maps a point $(\alpha_0, \alpha_1, \beta)$ to $(\alpha, \beta_0, \beta_1)$ such that

$$
(7) \quad \alpha = \alpha_0 + \alpha_1, \quad \beta_0 = \min\{\alpha_1, \beta - \alpha_0\}, \quad \beta_1 = \max\{\alpha_0, \beta - \alpha_1\}.
$$

4. Next we consider the shifted polytope $\tilde{P} := \{(0, a, 0)\} + P$. For a point $(\alpha_0, \alpha_1, \beta)$ in $P$, let $(z, x, y)$ be the corresponding point in $\tilde{P}$, i.e., $z = \alpha_0$, $x = \alpha_1 + a$, $y = \beta$. Following Proposition 5.4, $\tilde{P}$ is described by the linear inequalities

$$
(8) \quad 0 \leq z \leq a, \quad a \leq x \leq a + b, \quad z \leq y \leq x.
$$

A triple $(x, y, z)$ (where we change the order of entries) satisfying (8) is nothing else than a real Gelfand-Tsetlin array (or, briefly, GT-array) with border $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 := a + b, \lambda_2 := a, \lambda_3 := 0$. (The integer triples among these correspond to the semi-standard Young tableaux with shape $(\lambda_1, \lambda_2, \lambda_3)$; for a definition and survey see, e.g., [6]. More about combinatorial and polyhedral aspects of GT-arrays can be
In our case we deal with the simplest sort of such arrays, namely, with triangular arrays of size 2; they are usually associated with the diagram

\[
\begin{array}{ccc}
y \\
x & z \\
\lambda_1 & \lambda_2 & \lambda_3
\end{array}
\]

Thus, (6) and (8) explicitly indicate a one-to-one correspondence between the set \(V\) of vertices of the RC-graph \(K(a, b)\) and the set \(\mathcal{M}\) of integer GT-arrays \((x, y, z)\) with border \((a + b, a, 0)\). According to Kashiwara [5], there is the structure of a crystal graph on the set of GT-arrays. (See also [7, 2].) For the 2-colored crystal graph on GT-arrays with border \((a + b, a, 0)\), a 1-edge (2-edge) describes a feasible transformation in the lower row \((x, z)\) (resp. in the upper row \((y)\)) of an array in \(\mathcal{M}\). More precisely, for \(M = (x, y, z) \in \mathcal{M}\):

\[
(a) \text{ if } y < x, \text{ then } M \text{ is connected by 2-edge } (M, M') \text{ with the array } M' := (x, y + 1, z); \\
(b) \text{ if } z < a, y \text{ and } y - z > x - a, \text{ then there is 1-edge from } M \text{ to } (x, y, z + 1); \\
(c) \text{ if } (b) \text{ is not applicable and if } x < a + b, \text{ then there is 1-edge from } M \text{ to } (x + 1, y, z).
\]

One can check that the edges on \(\mathcal{M}\) defined in this way correspond to the edges of \(K(a, b)\), and therefore, the 2-colored crystal graph on GT-arrays with border \((a + b, a, 0)\) is isomorphic to \(K(a, b)\). (Thus, we have a proof, alternative to [8], that the set of (locally finite) \(A_2\)-regular crystals is isomorphic to the set of RC-graphs.)

Finally, recall that the map \(\rho'\) defined in part 3 gives another embedding of \(K\) to \(\mathbb{R}^3\) (with coordinates \((\alpha, \beta_0, \beta_1)\)). A point \((\alpha, \beta_0, \beta_1)\) in the polytope \(\mathcal{P}'\) corresponds to the point \((y', z', x')\) in the shifted polytope \(\tilde{\mathcal{P}}' := \{(0, 0, b)\} + \mathcal{P}'\), where \(y' = \alpha, z' = \beta_0, x' = \beta_1 + b\). The corresponding analog of Proposition 5.4 for \(\mathcal{P}'\) implies that \(\tilde{\mathcal{P}}'\) is described as

\[
0 \leq z' \leq b, \quad b \leq x' \leq a + b, \quad z' \leq y' \leq x',
\]

giving the set of (real) GT-arrays \((x', y', z')\) with border \((a + b, b, 0)\). The bijection \(\omega: \mathcal{P} \to \mathcal{P}'\) determines a bijection \(\tilde{\omega}\) of \(\mathcal{P}\) to \(\tilde{\mathcal{P}}'\). Using (7), one can obtain an explicit expression for \((x', y', z') = \tilde{\omega}(x, y, z)\):

\[
(11) \quad x' = b + \max\{z, y - x + a\}, \quad y' = x + z - a, \quad z' = \min\{x - a, y - z\}.
\]

6 The universal RC-graph

By the universal RC-graph we mean the disjoint union \(\mathbf{UC}\) of RC-graphs \(K(a, b)\) for all \(a, b \in \mathbb{Z}_+\). The characterization of RC-graphs \(K(a, b)\) given in Section 3 and
additional results from Section 5 enable us to construct a reasonable embedding for UC.

In this construction, each vertex of UC is encoded by a tuple \( \phi = (X, x, c, D, U) \in \mathbb{Z}^5 \) satisfying
\[
0 \leq x, c \leq X, \quad 0 \leq D, U.
\]
Moreover, there is a one-to-one correspondence between the vertices and tuples. Under this correspondence, the vertex set \( V \) of UC turns into a semi-group (“cone”) \( C \) in the Abelian group \( \mathbb{Z}^5 \).

To explain the correspondence, consider a vertex \( v \) in an RC-graph \( K(a, b) \), the 1-line \( P_1(v) \) passing through \( v \), and the critical vertex \( r \) in this line. We assign
\[
X := |P_1(v)|, \quad x := t_1(v), \quad c := t_1(r), \quad D := t_2(r), \quad U := h_2(r).
\]
Clearly (12) holds for these values.

Conversely, consider \( \phi = (X, x, c, D, U) \in \mathbb{Z}^5 \) satisfying (12). We associate with \( \phi \) the RC-graph \( K(a, b) \), where \( a := c + U \) and \( b := X - c + D \). Then the numbers \( X, c \) determine a (unique) critical vertex \( r \) in \( K(a, b) \), namely, \( r \) is the common vertex of the maximal left sail \( L_{X-c} \) and the maximal right sail \( R_c \) (using the numeration of maximal sails as in Section 5). The required vertex \( v \) belongs to the 1-line passing through \( c \); it is defined as having the local coordinates \((x, c)\) in the left sail \( L_{X-c} \) if \( x \leq c \), and coordinates \((x - c + D, D)\) in the right sail \( R_c \) if \( x \geq c \).

One can explicitly express how the partial operators \( F_1 \) and \( F_2 \) (corresponding to the 1-edges and 2-edges of UC) act on elements of the cone \( C \). Indeed, given \( \phi = (X, x, c, D, U) \in \mathbb{Z}^5 \) satisfying (12), we observe that the trivial RC-graph \( K(0, 0) \) is just the origin \( 0 \) of the cone \( C \). The RC-graph \( K(1, 0) \) consists of three points \( P = (0, 0, 0, 0, 1) \), \( Q = (1, 0, 1, 0, 0) \), \( R = (1, 1, 1, 0, 0) \) connected by the 1-edge \((Q, R)\) and the 2-edge \((P, Q)\). The RC-graph \( K(0, 1) \) consists of three points \( S = (1, 0, 0, 0, 0) \), \( T = (1, 1, 0, 0, 0) \), \( W = (0, 0, 0, 1, 0) \) connected by the 1-edge \((S, T)\) and the 2-edge \((T, W)\). One can check that any nontrivial RC-graph \( K(a, b) \) is obtained by taking the Minkowsky sum of \( a \) copies of \( K(1, 0) \) and \( b \) copies of \( K(0, 1) \). The cone \( C \) has six “extreme rays”, namely, those generated by \( P, Q, R, S, T, W \). The generators \( P \) and \( W \) are “free”, while \( Q, R, S, T \) obey the relation \( Q + T = R + S \).

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