Orbital evolution of a particle around a black hole: II. Comparison of contributions of spin-orbit coupling and the self force

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We consider the evolution of the orbit of a spinning compact object in a quasi-circular, planar orbit around a Schwarzschild black hole in the extreme mass ratio limit. We compare the contributions to the orbital evolution of both spin-orbit coupling and the local self force. Making assumptions on the behavior of the forces, we suggest that the decay of the orbit is dominated by radiation reaction, and that the conservative effect is typically dominated by the spin force. We propose that a reasonable approximation for the gravitational waveform can be obtained by ignoring the local self force, for adjusted values of the parameters of the system. We argue that this approximation will only introduce small errors in the astronomical determination of these parameters.

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I. INTRODUCTION AND SUMMARY

One of the most promising sources of gravitational radiation that can be detected by the Laser Interferometer Space Antenna (LISA) is the capture of a compact object by a supermassive black hole. In the extreme mass ratio limit this problem can be solved using perturbation theory, in which the compact object (e.g., a stellar mass black hole or a neutron star) is treated as a test particle moving on the fixed background of the central, supermassive black hole.

A test particle, in the absence of self interaction and internal structure, moves along a geodesic of the spacetime background. When the particle’s mass is much smaller than the typical length scale of the curvature of the background spacetime, it is useful to use perturbation theory. The particle’s mass (or charge, if any) contributes to the spacetime curvature, such that the particle moves along a geodesic of a perturbed spacetime. (These perturbation fields can be interpreted as those of free gravitational waves, which are produced by the particle at retarded times. These perturbations are smooth on the particle’s worldline.) Alternatively, the particle can be construed as moving along an accelerated, nongeodesic trajectory of the unperturbed background spacetime: Let the mass of the companion compact object be \( \mu \), and the mass of the central black hole be \( M \), such that \( \mu \ll M \). The particle’s acceleration at order \( \mu \) is driven by forces at order \( \mu^2 \) imparted on the particle. To obtain the corrected orbit of the particle [to \( O(\mu) \)] one needs to include all the forces, at \( O(\mu^2) \), which act on the particle.

Two sources can contribute forces at \( O(\mu^2) \): i) the particle’s self force, and ii) the particle’s spin angular momentum: When the companion of a central black hole is a black hole of mass \( \mu \), the latter carries spin angular momentum \( j = s\mu^2 \), where \(-1 \leq s \leq 1 \). (Neutron stars also have maximal spin angular momentum which is quadratic in their mass.) The force imparted on the particle is at order \( \mu^2 \), which endows the particle acceleration at order \( \mu \). Numerous authors have considered the orbital evolution under either self force effects or spin effects. However, in order to obtain the particle’s orbit to order \( \mu \), one needs to include both the self interaction and the spin effects.

The relative importance of the two contributing effects can be evaluated using an order-of-magnitude estimate for \( r \gg M \). The component of the self-force responsible for the decay of the orbit (dissipation, “radiation reaction”) scales like \( (\mu/M)^2 (M/r)^3 \). The contribution of the particle’s spin angular momentum to the decay of the orbit can be found from the Papapetrou equation. Specifically, an order-of-magnitude estimate for the spin force can be found from the fact that it is linear in the particle’s spin, linear in the spacetime curvature, and quadratic in the particle’s four-velocity. (One needs to be careful here, because the various components of the four-velocity scale differently in \( M/r \). Specifically, \( u^i \sim 1 \), whereas \( u^2 \sim (M/r)^{1/2} \) — see below.) Taking the typical curvature to scale as \( M/r^3 \), and the typical velocity of a particle in a quasi-circular orbit to be given by Kepler’s law, one expects the contribution of the spin force to dissipation to scale as \( (\mu/M)^3 (M/r)^{13/2} \). Consequently, we expect the decay of the orbit to be dominated by the radiation reaction, with only small corrections (which we neglect) due to the particle’s spin angular momentum.

The situation is much different when one considers the conservative correction to the orbit at \( O(\mu) \). The component of the self force contributing to the conservative effect scales like \( (\mu/M)^2 (M/r)^6 \), whereas the component of the spin force contributing to the conservative effect scales like \( (\mu/M)^2 (M/r)^{7/2} \). (See below the justification for these scaling laws.) Both forces are at \( O(\mu^2) \), but because of the slower drop off with \( M/r \), naively one might expect the spin force under these conditions to be much more important than the self force. In a strict sense, however, a direct comparison of the spin force and the self force tells us only little about the relative importance of the two forces. The reason is that the self force is gauge-dependent. In particular, one can always choose a gauge in which the self force vanishes. Consequently, the scaling law used above is not unambiguous, at least as long as we do not specify the gauge in which it is written.
A useful way to compare the spin force and the self force then is to compare between gauge-independent quantities. The generated waveform or the number of cycles that the system spends in a logarithmic interval of frequency are two such quantities. In practice we evolve a quasi-circular, equatorial orbit around a Schwarzschild black hole using both forces, or either one, and compare these gauge-independent quantities. [An initially equatorial orbit will remain nearly equatorial because the total angular momentum is conserved, and the spin angular momentum is much smaller than the orbital angular momentum. Specifically, the spin angular momentum is at $O(\mu^2)$, such that its rate of change is at $O(\mu^3)$. Conservation of total angular momentum imply that changes in the orbital angular momentum are also at $O(\mu^2)$, whereas the orbital angular momentum itself is at $O(\mu)$. Consequently, changes in the direction of the spin vector imply only very small changes in the direction of the orbital angular momentum, or very small changes in the orbital plane.] Indeed, we find that our naive expectations are realized: the conservative effect is controlled by the spin force, which overwhelms the self force. More general orbits around a Kerr black hole will include also dissipation due to the spin force and spin-spin coupling. Here, we ignore such effects, and focus on the spin-orbit coupling. Our motivation is that for such a simplified orbit we can already demonstrate our main point: spin forces for astrophysical systems may be much more important than the self forces, as far as conservative effects on the orbital evolution are concerned.

In particular, we find that neglecting the local, conservative self force in the construction of templates may introduce only a small error in the astronomical determination of the parameters of the system. Specifically, we can match a template made without the local self force almost exactly to a waveform which does include the local, conservative self force, but with a slightly different value for the spin of the companion.

In a more general system, the component of the spin force which induces the conservative effect also contributes to dissipation. Consider a Kerr black hole in Boyer-Lindquist coordinates. The rates of change of the particle’s energy, $\hat{E}$, and angular momentum (in the direction of the spin axis), $\hat{L}_z$, can be found from the fluxes of energy and angular momentum to infinity and through the central black hole’s event horizon. The rate of change of Carter’s constant, $\hat{Q}$, however, cannot be found using balance arguments and global conservations laws, because $\hat{Q}$ is non-additive. To find $\hat{Q}$ one needs to find first the local forces which act on the particle. Specifically, $\hat{Q} = G_E(\text{COM}, g) \hat{E} + G_{L_z}(\text{COM}, g) \hat{L}_z - 2 \Sigma \frac{u^r}{u^t} f_r$, where $G_E, G_{L_z}$ are certain (known) functions of the constants of motion (COM) in the absence of dissipation, and the metric $g$, and $\Sigma = r^2 + a^2 \cos^2 \theta$, $a$ being the spin parameter of the black hole. To have a full description of dissipation then, one needs the total radial force, $f_r$, which acts on the particle. Similarly to our argument above, we expect the contribution to $\hat{Q}$ due to the self force to be small compared with the contribution of the spin force.

As generically the companion is expected to spin (and even spin fast), we propose that a reasonable approximation for the orbital evolution of a spinning particle can be found without finding first the local self force. Specifically, we propose that a practical way to obtain the orbital evolution (and the waveforms) to high accuracy could be the following: First, find $\hat{E}, \hat{L}_z$ using the fluxes to infinity and down the event horizon of the central black hole. Next, find $\hat{Q}$ using $\hat{E}, \hat{L}_z$ and the spin contributions to $f_r$. Undoubtedly, that will introduce an error in the determination of $\hat{Q}$. However, because of the smallness of the radial component of the self force with respect to the radial component of the spin force, we believe that this may be a useful approximation for the total $\hat{Q}$. To find the orbital evolution one needs also to include conservative effects. Again, we propose to neglect the contribution of the self force to the conservative effects, and approximate the full conservative effect by the conservative effect due to the spin force.

In the simple problem we consider here the motion is quasi-circular and equatorial around a Schwarzschild black hole, such that the spin angular momentum of the companion is aligned (or anti-aligned) with the orbital angular momentum. It is not hard to let the central black hole be spinning — with the spin axis pointing along the same direction as the companion’s spin angular momentum and the orbital angular momentum — and include also spin-spin coupling effects. We are hoping to return to that problem in the future. We note, that when the spins are not aligned as in our assumptions, the precession of the particle’s spin will induce a changing quadrupole moment, such that the spin force will cause a strong dissipative effect. The implications of such a spin dissipative effect are beyond the scope of the present paper.

Our goal in this paper is more to point out that the accurate determination of the local self force may perhaps be less crucial for determination of the orbital evolution of extreme mass-ratio binaries than previously thought, than to give a definite prediction. First, some companions may have only little spin angular momentum, in which case one would no longer be justified in neglecting the local self force. Second, and most importantly, our analysis in what follows is incomplete, in the sense that some of the necessary pieces of information for carrying out the comparison of the spin and self force effects are as yet unknown. We consequently make a number of assumptions, that allow us to obtain what we believe to be at least a reasonable order-of-magnitude estimate for the effect of interest. We emphasize that our analysis is based on a number of assumptions, which are as yet unproven. We believe that our main conclusion — namely, that spin-orbit coupling may overwhelm the conservative self force effect — is insensitive to the accuracy of these
assumptions and is robust. At three places our assumptions are speculative (see below for details): First, we assume that the terms linear in second-order forces in the expression for the second-order radial velocity are small compared with terms quadratic in first-order forces. Second, we assume that finite-mass effects in the luminosity in gravitational waves, which are small in the weak field regime, are small also in the strong field regime. Third, we assume that the radial component of the gravitational self force is proportional to its scalar field counterpart, with a specific proportionality constant. Of these three assumptions, the last one is the easiest to test. When this is done, the actual gravitational self force can be used to replace the assumed expression \( \frac{d\mu}{d\tau} \). While these assumptions appear to be quite natural to make, they are by no means guaranteed to be justified under all circumstances. However, even if any of them turns out to be incorrect, the implications on our main point in this paper are not expected to be strong. We therefore believe that our results are at least reasonable order-of-magnitude estimates, and that our main point is relevant for realistic extreme mass-ratio binaries.

The organization of this paper is as follows. In Section II we describe the equations of motion and derive the equations for a quasi-circular, equatorial orbit with aligned spins, using the Papapetrou equations and linearizing in the spin covector. Appendix A includes more details on the definitions of the spin force and the local self force. In section III we derive the perturbative solutions for the equations of motion following the method of Ref. [5] (paper I), but with slightly different definitions and notation, which will make the generalization to a Kerr black hole simple. Then, we discuss the domain of validity of the equation. In Section IV we describe the modeling of the self force. First, we discuss the fitting of the numerically-derived luminosity in gravitational waves to a smooth function using a match to two different asymptotic expansions, and discuss the associated errors, and then we discuss our conjecture for the as-yet undetermined (radial component of the) self force. Finally, in Section VI we compute the orbit and the waveforms, and discuss the relative importance of the spin force and the self force.

II. EQUATIONS OF MOTION

A. Equations of motion to order \( \mu^2 \)

The total force which acts on the particle at order \( \mu^2 \) is just the sum of the well known spin force and the self force. The deviation of the orbit from the geodesic because of the spin force is at order \( \mu \). Its effect on the self force is therefore at order \( \mu^3 \), and hence negligible, as we are interested here in the forces only at order \( \mu^2 \). Similarly, the deviation of the orbit from the geodesic because of the self force is also at order \( \mu \), such that its effect on the spin force is again negligible. It is clear then, that the total force, at order \( \mu^2 \), which acts on the particle and pushes it off the geodesic is just the sum of the self force and the spin force. Specifically, the equations of motion for a particle of mass \( \mu \), whose center of mass travels along the worldline \( z^\mu \) with four-velocity \( u^\mu = dz^\mu/d\tau \) are given by

\[ \mu \frac{Du^\alpha}{d\tau} = f^\alpha = f_{SF}^\alpha + f_{spin}^\alpha. \]  

The expressions we use for \( f_{SF}^\alpha \) and \( f_{spin}^\alpha \) are given in Appendix A.

B. Specializing to equatorial motion with an aligned spin

1. General equatorial motion

The line element, in the usual Schwarzschild coordinates, is given by

\[ ds^2 = -F(r)\,dt^2 + \frac{dr^2}{F(r)} + r^2\,d\Omega^2 \]  

where \( F(r) = 1 - 2M/r \), and \( d\Omega^2 = d\theta^2 + \sin^2 \theta \,d\varphi^2 \). (Notice, that the form of the Papapetrou equations of motion depends on the choice of signature.)

In general, the spin covector precesses along the motion, and the orbit is not planar. In particular, with arbitrary alignment of the spin covector, an otherwise equatorial motion will no longer be equatorial. To facilitate the analysis, let us specialize to a particular solution of the Papapetrou equations, in which the entire motion is on the equatorial plane. Under the requirement that the entire motion is equatorial, Eq. (A9) for the parallel transport of the spin covector becomes very simple. Specifically, Eq. (A9) becomes

\[ \frac{dS^\alpha}{d\tau} = -\frac{1}{r}u^\alpha S^\sigma \delta^\sigma_\alpha \]  

which implies that \( S^\theta = s\mu^2/r \), and \( S^t = S^r = S^\varphi = 0 \), \( s \) being a constant. Notice that the magnitude of the spin vector then is \( S = s\mu^2 \), such that \( s \) satisfies \(-1 < s < 1 \) for black holes. Equation (A8) then becomes

\[ \mu \frac{Du^t}{d\tau} = 3 \frac{M}{r^3} \frac{v^2}{F(r)}u^\varphi u^\sigma S^\theta \]  

\[ \mu \frac{Du^r}{d\tau} = 3 \frac{M}{r} F(r)u^t u^\varphi S^\theta \]  

\[ \mu \frac{Du^\varphi}{d\tau} = 0 \]
We next solve the equations of motion perturbatively, for the case of quasi-circular, equatorial orbit. We use the normalization condition for $u^a$, namely $u^a u_a = -1$, to eliminate $u^i$ from the equations of motion. We next use the $t$ component of the equations of motion (EOM) to eliminate $\dot{u}^t$. We can simplify the EOM to first-order (nonlinear) ODEs by taking $\dot{r} = V(r)$, $\dot{x} = V x(r)$, and $x$ denotes any quantity. We find the EOM to be

$$VV' - 3M V^2 \frac{\Delta}{\Delta r} + (M - \omega^2 r^3) \frac{\Delta}{r^4} = \frac{1}{\mu u^2} \left( \Delta^2 f_r + \Delta r V f_t \right)$$

(9)

$$V \omega' + 2 \omega (r - 3M) \frac{V}{\Delta r} = (r - 2M) f_\phi + r^3 \omega f_t$$

(10)

where $\omega$ is the angular velocity, $\Delta = r^2 - 2Mr$, and $1/u^2 = 1 - 2M/r - rV^2/\Delta - r^2 \omega^2$. We next expand Eqs. (9)–(10) in powers of $\epsilon = \mu/M$: $\omega = \omega(0) + \omega(1) + \omega(2)$, $V = V(1) + V(2)$, and $a_i = a_i^{(1)} + a_i^{(2)}$. Here, $x_{(i)}$ denotes the term in $x$ which is at $O(\epsilon^i)$, and $a_i$ being the self acceleration. We then expand the self force as $f_\phi^{SF} = f_\phi^{(1)} + f_\phi^{(2)}$, where $f_\phi^{(i)} = \mu a_i^{(j)}$. We then expand Eqs. (9)–(10), and solve perturbatively order by order.

The zeroth order term of Eq. (9) recovers Kepler’s law. Specifically, it yields

$$\omega(0) = \frac{M}{r^2}.$$  

(11)

The first order corrections to Kepler’s law are obtained from the terms at $O(\epsilon)$ of Eq. (9). Specifically, we find that

$$\omega^{(1)} = \frac{r f_r^{(1)} r^3 \omega_{(0)}^2 - (r - 2M) f_t}{2 \mu r^3}.$$  

(12)

We next define

$$\omega^2 := r^2 \Delta \omega_{(0)} + 2 \omega^2 (r - 3M) \omega_{(0)}.$$  

(13)

The first order term of the radial velocity is obtained from the terms at $O(\epsilon)$ of Eq. (10):

$$V^{(1)} = - \frac{1}{\mu \omega^2} \left[ r^3 \omega_{(0)}^2 - (r - 2M) \right] \times \left[ (r - 2M) f_\phi^{(1)} + r^3 \omega_{(0)} f_t^{(1)} \right].$$  

(14)

The second order correction to the radial velocity is found from the terms at $O(\epsilon^2)$ of Eq. (10):

$$V^{(2)} = - \frac{1}{\mu \omega^2} \left[ r^3 \omega_{(1)} f_t^{(1)} \left[ 3r^3 \omega_{(0)}^2 - (r - 2M) \right] \right].$$  

(15)
As we shall see below, $\omega^{(2)}$ does not contribute at the order to which we are solving the EOM. The explicit perturbative solution of the EOM is listed in Appendix B.

We remark that we express all quantities here as functions of $r$. For other purposes, e.g., for analysis of detected signals, it is frequently more convenient to express quantities as functions of $\omega$. It is easy to translate our expressions to functions of $\omega$ by noting that $d\omega = \omega' dr$.

### B. Approximating the solution and its validity regime

Notice that each term inside the curly brackets in Eq. (15) is at $O(\mu^3)$. These terms come in two kinds: first, there are terms which are quadratic in $f^{(1)}_{\mu}$ (or their gradients), and second, there are terms which are linear in $f^{(2)}_{\mu}$. The latter, of course are as yet unknown. Their derivation requires second-order perturbation theory, and the extension of regularization techniques to that order. One should therefore view Eq. (15) as a formal expression. However, we propose that the expression for $V^{(2)}$ is dominated by the terms quadratic in $f^{(1)}_{\mu}$, such that neglecting the terms involving $f^{(2)}_{\mu}$ introduces only a small error: Consider the time and the extension of regularization techniques to that order. One should therefore view Eq. (15) as a formal expression. However, we propose that the expression for $V^{(2)}$ is dominated by the terms quadratic in $f^{(1)}_{\mu}$, such that neglecting the terms involving $f^{(2)}_{\mu}$ introduces only a small error: Consider the time and the four-force. Expand the four-acceleration as $a = a^{(1)} + a^{(2)} + ...$ where $a^{(n)}$ is at order $\varepsilon^n$. Then obviously $a^{(2)} \propto a^{(1)}$, with some proportionality constant (with dimensions of $1/M$) which we expect to be neither very large nor very small. (Recall that we solve perturbatively, such that $a^{(1)} \sim \mu/M^2$. In this case $a^{(2)} \ll a^{(1)}$.) The force $f^{(2)}_{\mu} = \mu a^{(2)}_{\mu}$ can be written as $f^{(2)}_{\mu} \propto \mu a^{(1)}_{\mu}$. Substituting $a^{(1)}_{\mu} = f^{(1)}_{\mu}/\mu$ then find that $f^{(2)}_{\mu} \propto \mu^{-1} f^{(1)}_{\mu}$. As $f^{(2)}_{\mu}$ has the same dimensions as $f^{(1)}_{\mu}$ (namely, it is dimensionless in a normalized basis), we expect that $f^{(2)}_{\mu} = \alpha_{\mu} M \mu^{-1} f^{(1)}_{\mu}$, (no summation over repeated indices implied) where $\alpha_{\mu}$ is an (unknown) dimensionless function of $r/M$. Next consider the $\varphi$ component. We expect $f^{(2)}_{\varphi} = -f^{(1)}_{\varphi}/\omega$. Substituting Kepler’s law for $\omega$, and repeating our arguments above, we find that $f^{(2)}_{\varphi} = \alpha_{\varphi} (M/r)^{3/2} \mu^{-1} f^{(1)}_{\varphi}$, where again $\alpha_{\varphi}$ is a dimensionless function of $r/M$.

Consider now the terms in Eq. (15). The coefficients of the terms proportional to $f^{(2)}_{\mu}$ are typically much smaller than the coefficients of the terms quadratic in $f^{(1)}_{\mu}$. For example, compare the term involving $f^{(2)}_{t}$ and the term involving $f^{(1)}_{t}$ in Eq. (15). The ratio of these two terms is

$$R := \frac{B_t f^{(2)}_{t} / f^{(1)}_{t}}{A_{tr} f^{(2)}_{t} / f^{(1)}_{tr}} \sim \frac{\alpha_t (M/\mu) B_t f^{(2)}_{t}}{A_{tr} f^{(1)}_{tr}} \sim \frac{\alpha_t (M/\mu) B_t f^{(1)}_{t}}{A_{tr} f^{(1)}_{tr}}.$$ (16)

Next, for quasi-circular orbits at $r \gg M$, $f^{(1)}_{t} / f^{(1)}_{r} \sim (M/r)^{1/2}$, such that

$$R \sim \frac{\alpha_t (M/\mu) B_t (M/r)^{1/2}}{A_{tr} \left(\frac{M}{r}\right)^{5/2}}.$$ (17)

Here, $B_t = 2(r - 3M)/\mu(r - 6M)$ and $A_{tr} = 4r^3(r - 3M)^2/\mu^2 M(r - 6M)^2$ are the coefficients of the terms proportional to $f^{(2)}_{t}$ and $f^{(1)}_{t}$, respectively, in $V^{(2)}$. Demanding that $R \ll 1$ introduces then restrictions on the function $\alpha_t$. Specifically, for $r \gg M$ our assumption is justified if $\alpha_t$ increases as a function of $r/M$ slower than $(r/M)^{3/2}$. (Even if $\alpha_t$ violates this condition, our assumption can still be valid in a neighborhood of a point $r$ at which it is valid. If the condition is satisfied, our assumptions are valid globally.) Presently, as was already discussed above, we have no knowledge of the functions $\alpha_{\mu}$. We proceed by assuming that $\alpha_{\mu}$ are such that indeed the terms proportional to $f^{(2)}_{\mu}$ are small compared with the terms quadratic in $f^{(1)}_{\mu}$ in Eq. (15). Similarly, we can introduce analogous constraints on $\alpha_{\varphi}$ and $\alpha_{t}$ such that any term linear in $f^{(2)}_{\mu}$ in Eq. (15) would indeed be much smaller than any term quadratic in $f^{(1)}_{\mu}$. Although our discussion above is limited to $r \gg M$, we assume the applicability of our conclusions for all $r > 6M$. We therefore neglect all the terms in Eq. (15) which are linear in $f^{(2)}_{\mu}$.

We emphasize that even if our assumption that all the terms in Eq. (15) which are linear in $f^{(2)}_{\mu}$ are negligible compared with terms which are quadratic in $f^{(1)}_{\mu}$ turns out to be incorrect, our main conclusion in this paper is still likely to be relevant for at least a portion of the orbit (see below).

The solution we find for the EOM is a perturbative solution about a circular orbit. So long as the orbit does not get too far from circularity we therefore expect our solution to be valid. However, when the particle arrives at the innermost stable circular orbit (ISCO) at $r = 6M$ the orbit changes from an adiabatic, quasi-circular orbit to a plunge. (We do not consider here the change in the ISCO itself under radiation reaction and spin-orbit effects.) When that happens the radial velocity can no longer be considered as small, and the perturbative approach to the solution (i.e., expanding the solution about
a circular orbit) breaks down. To find where our perturbative approach loses its validity let us compare the magnitude of \( V^{(2)} \) with the magnitude of \( V^{(1)} \). We consider the perturbative approach as valid only if \( V^{(2)} \) is smaller than \( V^{(1)} \). Specifically, let us introduce the condition that \( |V^{(2)}/V^{(1)}| \lesssim \gamma \) for the perturbative approach to be valid, where \( \gamma \) is a positive constant smaller than unity. (In practice, we arbitrarily fix \( \gamma = 0.2 \).) For a typical term in \( V^{(2)} \) we take the term proportional to \( f(t)^{(1)} f(t)^{(1)} \) in Eq. (B3), and for a typical term in \( V^{(1)} \) we take the term proportional to \( f(t)^{(1)} \) in Eq. (B2). Their ratio is

\[
\left| \frac{V^{(2)}}{V^{(1)}} \right| \approx \frac{2}{\mu M} \frac{r^2(r - 3M)}{r - 6M} f(t)^{(1)} \lesssim \gamma. \tag{18}
\]

As \( f(t)^{(1)} \) is bounded, clearly this inequality is violated at some value of \( r > 6M \). However, as \( f(t)^{(1)} \) is at \( O(\mu^2) \), the domain of validity of the perturbative approach extends to smaller values of \( r \) the smaller \( \mu \).

\section{IV. COMPUTATION OF THE ORBIT AND THE WAVEFORM}

In the previous section we found \( V(r) \) to order \( \mu^2 \) and \( \omega(r) \) to order \( \mu \) for quasi-circular equatorial orbits. Next, we find the orbit by computing

\[
t(r) = \int_{r_{\text{initial}}}^{r} \frac{d\bar{r}}{V(\bar{r})} \quad \text{and} \quad \varphi(r) = \int_{r_{\text{initial}}}^{r} \frac{\omega(\bar{r})d\bar{r}}{V(\bar{r})},
\tag{19}
\]

where the integrands are evaluated using the perturbative solution for \( V(r) \) and \( \omega(r) \). In practice, we solve these integrals using a fourth-order Runge-Kutta integrator. This yields the triad \( t, t(r), \varphi(r) \), which we invert to \( t, r(t), \varphi(t) \), which is just the required orbit.

Next, we compute the number of cycles \( N_{\text{cyc}} \) spent in a logarithmic interval of frequency \( f \), \( dN_{\text{cyc}}/df \), by noting that \( dN_{\text{cyc}}/df = \omega^2/(\pi \omega) \). Re-expressing the last equation as \( dN_{\text{cyc}}/df = \omega^2/(\pi \omega V) \) and using the perturbative solution for \( V(r) \) and for \( \omega(r) \), we find that

\[
\frac{dN_{\text{cyc}}}{df} = \frac{2}{3\pi} \left( \frac{M}{r} \right)^{\frac{1}{2}} \left( \frac{1}{V^{(1)}} \right) \left\{ 1 + \left( \frac{r}{M} \right)^{\frac{3}{2}} r^3 \right\} \\
\times \left[ 2\omega^{(1)} \frac{\omega'}{r^2} + 2\omega^{(1)} \frac{\omega'}{r^3} \right] - \frac{V^{(2)}}{V^{(1)}} + O(\mu^2) \]
\tag{20}

Finally, the total change in the number of cycles can be found by integrating

\[
\Delta N_{\text{cyc}} := \int_{r_{\text{initial}}}^{r} \frac{\omega}{\omega} \frac{dN_{\text{cyc}}}{df} d\bar{r}. \tag{21}
\]

\section{V. MODELING THE SELF FORCE}

\subsection{A. Fitting the luminosity in gravitational waves to a smooth function}

The local gravitational self force has not been computed yet for a point particle in motion around a Schwarzschild black hole, not even for quasi-circular, equatorial orbits. (An exception is a radial plunge, for which it has been calculated \[12\].) However, the radiation reaction for quasi-circular, equatorial orbits in Schwarzschild (and also for other types of orbits, including in Kerr) was calculated using balance arguments, and the results for the rate of change of the constants of motion can be translated into the local self force. As the local self force is not available to us, we shall proceed by using results obtained from the non-local approach. That approach can give us the dissipative part of the self force, but not the conservative part, which is not encoded in the gravitational-wave luminosity. We will therefore estimate the conservative piece of the self force by analogy the the case of scalar field self interaction. Although we do not expect our estimate to be accurate, nevertheless hope that it can still provide us with a crude order-of-magnitude estimate for the actual conservative piece of the self force.

In practice, we take the results for the luminosity of gravitational waves (i.e., the flux to infinity) for a test mass in circular orbit around a Schwarzschild black hole \[11\]. These results are given in the form of a table, which gives \( dE/dt(r) \). For the purpose of integrating a slowly-evolving orbit, we need a smooth function. We can obtain such a function by fitting the data in the table to a smooth function. The most natural functional form to use is that of the post-Newtonian (PN) expansion. While the PN expansion converges very rapidly at large distances, it does so only very poorly close to the ISCO, where the motion is very relativistic. On the other hand, a general polynomial fit converges very rapidly at small distances, but poorly at large distances. We find that the two asymptotic expansions match at an overlap region, such that a matched asymptotic expansion method can be very useful.

In practice, we use the 3.5PN expansion for a test particle at large distances. The flux of energy to infinity is given by \[13, 14\]

\[
\frac{dE^{3.5\text{PN}}}{dt} = \frac{32}{5} \mu^2 M^3 r^5 \left\{ -1 - \frac{1247}{336} v^2 + 4\pi v^3 - \frac{44711}{9072} v^4 - \frac{8191}{672} \pi v^5 - \frac{6643739519}{69854400} - \frac{1712}{105} \gamma_e + \frac{16}{3} \pi^2 - \frac{856}{105} \ln(16\pi^2) \right\} v^6 - \frac{16285}{504} \pi v^7, \tag{22}
\]

where \( v = (M \omega)^{1/3} \) is the orbital velocity, and where \( \gamma_e \) is Euler’s constant.

At small distances we fit the luminosity to a polyno-
As we already discussed in the preceding subsection, the error in the determination of the luminosity in gravitational waves for a test mass is no worse that $10^{-3}$. The other sources of errors which affect our determination of the $t$-component of the self force are finite mass effects (i.e., the effect of a finite $\mu/M$), and the flux of energy through the black hole’s event horizon.

The total rate of change of energy of the particle is the sum of the luminosity in gravitational waves (i.e., the flux of energy to infinity) and the flux in energy through the event horizon of the black hole. Specifically,

$$\frac{dE}{dt} = \frac{dE}{dt}^\infty + \frac{dE}{dt}^{EH}. \quad (25)$$

As $(dE/ dt)^{EH} \approx \alpha v^8 (dE/ dt)^\infty$, with $\alpha$ being a constant of order unity \cite{15}, the flux of energy through the event horizon where it is the greatest (when the particle is at $r = 6M$) is smaller than the flux to infinity by a factor of $8 \times 10^{-4}$. Consequently, neglecting the absorption by the black hole introduces an error in the determination of the $t$-component of the self force which is compatible with our tolerance.

The other source for errors in the determination of the $t$-component of the self force are finite-mass effects. The finite-mass effects are easy to estimate for large distances, where they are fully known at the 3.5PN level. (Recall however the ambiguity in determination of the finite-mass effect of the 3PN terms.) The error introduced by neglecting the finite-mass effects in the 3.5PN expansion is a few $\times \mu/M$. The error will be compatible with our tolerance if we take the mass ratio to be $\mu/M \lesssim$ a few $\times 10^{-4}$. We shall therefore restrict the systems we study here to compatible mass ratios. We cannot estimate finite-mass effects close to the ISCO, as the luminosities available to us were obtained from a linearized analysis. We are encouraged, however, that at all PN orders (up to 3.5PN) the errors introduced by neglecting finite-mass effects are comparable. If that behavior persists at all PN orders, we are guaranteed that our approximation is valid also near the ISCO. Although we do not know whether this is indeed the case, we hope that our approximation remains qualitatively valid also near the ISCO, and that it does not introduce errors significantly bigger than our tolerance.

C. Determination of the self force

The $t$-component of the self force is easily determined by

$$f_t^{SF} = \frac{dt}{dt} \frac{dE}{dt}. \quad (26)$$

In Eq. (20), $dE/ dt$ is the flux of energy to infinity. The rate of change of the particle’s energy then is

\begin{align*}
\frac{dE}{dt} &= \frac{32 \mu^2 M^3}{5} \sum_{k=0}^n C_k r^k, \quad (23)
\end{align*}

with coefficients $C_k$ given in Appendix C.

Figure 2 shows the temporal component of the self force and the relative errors in its determination in terms of both methods. As expected, the relative error in the 3.5PN expansion is very small at large distances, but grows rapidly at small distances. Similarly, the power-law approximation yields highly accurate results at small distances, but is inaccurate at large distances. There is an overlap region, for $12 \lesssim r/M \lesssim 24$, where the two expansions have comparable accuracy. Specifically, the 3.5PN expansion yields at $r = 12M$ a relative error of $1 - (dE/ dt)^{3.5PN}/(dE/ dt) \approx 4 \times 10^{-4}$, and the power-law expansion yields at $r = 24M$ a relative error of $1 - (dE/ dt)^{PL}/(dE/ dt) \approx 1.1 \times 10^{-3}$. Allowing our accuracy in the determination of the luminosity to be 1 part in $10^3$, we evaluate the self force at large distances ($r/M > 24$) using the 3.5PN expansion, and at small distances ($r/M < 12$) using the power-law expansion. For the region $12 \leq r/M \leq 24$ we use a sliding average of the two methods, i.e., we simply take

$$\frac{dE}{dt}^{mix} = \frac{dE}{dt}^{PL, 24M - r} + \frac{dE}{dt}^{3.5PN, r - 12M}. \quad (24)$$

This guarantees that our fit to the luminosity, and to the temporal component of the self force, will nowhere be worse than our tolerance.
Notice, that we only need to determine \( f_t^{(1)} \), such that in Eq. \( 20 \) the factor \( dt/\sigma \) can be evaluated for geodesic motion, i.e., \( dt/\sigma = 1/(1 - 3M/r)^{1/2} \). The \( \varphi \)-component of the self force can be determined from its proportionality to the \( t \)-component.

The \( r \)-component of the self force cannot be found by using non-local methods. Instead, it needs to be evaluated using a fully local calculation. As the results of such a calculation are as yet unavailable to us, we shall instead use a crude order-of-magnitude estimate for \( f_r^{\mathrm{SF}} \).

Because of the smallness of the effects of this component, and because it is a posteriori found to be much smaller than the magnitude of the \( r \)-component of the spin force, we do not expect our crude estimate to be problematic. Note, that once \( f_r^{\mathrm{SF}} \) is calculated, it can easily be used to replace our estimate here. Specifically, we estimate the radial component of the self force by using the known results for the radial self force on a scalar charge \( q \) in circular orbit around a Schwarzschild black hole \([16,17]\). In order to do that estimate, recall that for \( r \gg M \), grav\( f_r^{\mathrm{SF}} \)/scalar \( f_r^{\mathrm{SF}} \approx 96 (\mu/q)^2 (M/r) \) \([8,10]\). We next assume that an analogous scaling is satisfied also by \( f_r^{(2)} \) (which is entirely conjectural, although not implausible). Consequently, because \( f_r^{\mathrm{SF}} u^\alpha = 0 \), it is reasonable that the same proportionality is satisfied also by the \( r \)-components, \( f_r^{(1)} \). For strict circular orbits the radial velocity vanishes, such that no restrictions on \( f_r^{(1)} \) can be applied, but for quasi-circular orbits the orthogonality of the force and the orbit imply that it is not implausible that such a proportionality is satisfied. In view of this argument, we assume

\[
\text{grav} f_r^{\mathrm{SF}} \approx \frac{\beta \mu^2}{q^2} \frac{M}{r} \times \text{scalar} f_r^{\mathrm{SF}}.
\]  

(27)

Although we expect \( \beta \approx 96/5 \), in practice we parametrize our results with \( \beta \). Although our arguments may apply to the far field region only, we extrapolate the scaling relation \((27)\) for all values of \( r \). This choice, in the absence of numerical data, appears to be reasonable, at least as an approximation for the actual radial component of the self force.

VI. CALCULATED ORBITS AND WAVEFORMS

In order to compare between different evolutions we use \( dN_{\mathrm{cyc}}/d\ln f \), the number of cycles \( N_{\mathrm{cyc}} \) that the system spends in a logarithmic interval of frequency \( f \), as given in Eq. \( 20 \). The total number of cycles that the system undergoes between two values of \( r \), \( \Delta N_{\mathrm{cyc}} \) (this, of course, can be translated to the number of cycles between two values of the frequency \( f \)), is found by using Eq. \( 21 \). First, we discuss the contribution to \( \Delta N_{\mathrm{cyc}} \) from the spin-orbit coupling. Figure \( 4 \) shows the difference between \( \Delta N_{\mathrm{cyc}} \) which was obtained for \( s = 1 \) and a number of values of \( s \), for a system with \( \mu = 5 \times 10^{-5}M \) that starts decaying at \( r = 10M \). The maximum effect is obtained when \( s = -1 \), where it is just above 1 cycle at \( r = 6M \). This effect is at \( O(\mu^0) \), i.e., it is independent of the mass ratio \( \beta \). That is, ignoring the spin force would result in a maximal error of about a full cycle, which could reduce the correlation integral of the signal with a theoretical template considerably. (Notice that the total number of cycles that the system undergoes is \( N_{\mathrm{cyc}} \approx 2.6 \times 10^3 \), and that \( N_{\mathrm{cyc}} \propto \mu^{-1} \).)

The waveforms, which we present using the “restricted waveform” approximation \([11]\), are displayed in Fig. \( 6 \) (Our calculation of the \( \Delta N_{\mathrm{cyc}} \) is independent of the assumption of the “restricted waveform” approximation.) Figure \( 6 \) indeed shows that the maximal ambiguity in phase due to ignoring the spin-orbit coupling is about two wavelengths.

In order to examine how important that self force may be we fix \( s = 1 \), and compute \( \delta(\Delta N_{\mathrm{cyc}}) \), the difference in the total number of cycles for the case in which we include the self force in the calculations of the orbital evolution, and the case in which we do not. The results, for \( \beta = 1 \), are presented in Fig. \( 6 \). We find that the difference in the total number of cycles is \( 2 \times 10^{-3} \beta \) at \( r = 6M \). Even for \( \beta \approx 20 \) the difference in the total number of cycles is only \( 4 \times 10^{-2} \), which is very small indeed. This effect is at \( O(\mu^0) \) \([5]\). Figure \( 6 \) displays a fraction of the last oscillation in the waveforms with and without the inclusion of the self force (for \( s = 1 \)) before \( r = 6M \).

Most importantly, we can make the waveform without the inclusion of the self force coincide almost exactly with the waveform with its inclusion, if we modify the value of \( s \) by a small number which is comparable to the difference in the number of cycles for the cases with and without
FIG. 4: The waveform $-h$ as a function of $t/M$. The data are shown for $\mu = 5 \times 10^{-4}M$. We show the last few oscillations before $r = 6M$. Plotted are the waveforms for two values of $s$: $s = 1$ (solid curve) and $s = -1$ (dashed curve). The two curves were in phase at $t = 0$ ($r = 10M$).

the inclusion of the self force. This situation is shown in Fig. 7, which displays two waveforms: one waveform is the same as the waveform without including the self force effect in Fig. 6. The other waveform is a waveform with the self force effect, but with a corrected value of the spin $s$. The two waveforms overlap almost exactly over the entire wave train, and in particular during the last fraction of the orbit before $r = 6M$.

Therefore, we propose that neglecting the self force will only introduce a small error in the determination of the spin rate of the companion, at the order of $\delta[\Delta N_{\text{cyc}}]$.

FIG. 6: The waveforms for $\mu = 5 \times 10^{-4}M$ and $s = 1$. Solid curve: including the effects of the self force. Dashed curve: neglecting the effect of the self force. Here, we used $\beta = 1$, and the two waveforms were in phase at $r = 10$. Shown is a fraction of the last oscillation before $r = 6M$. We set $t = 0$ at $r = 6M$.

FIG. 5: $\delta[\Delta N_{\text{cyc}}]$ — the difference between $\Delta N_{\text{cyc}}$ with the inclusion of the self force, and $\Delta N_{\text{cyc}}$ without its inclusion — as a function of $r/M$. The data shown are for $\mu = 5 \times 10^{-4}M$, $s = 1$, and the system starts decaying at $r = 10M$.

FIG. 7: The waveforms for $\mu = 5 \times 10^{-4}M$. Solid curve: excluding the effects of the self force for $s = 1$. Dashed curve: including the effect of the self force for $s = 0.997$. Here, we used $\beta = 1$, and the two waveforms were in phase at $r = 10$. Shown is a fraction of the last oscillation before $r = 6M$. The two waveforms are indistinguishable on this scale.

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APPENDIX A: EXPRESSIONS FOR \( f_{\alpha}^{S} \) AND \( f_{\alpha}^{\text{spin}} \)

In this Appendix we write explicitly the expressions for \( f_{\alpha}^{S} \) and for \( f_{\alpha}^{\text{spin}} \) which appear in Eq. (11). Specifically,

\[
f_{\alpha}^{S} = \mu^{2} k^{\alpha \beta \gamma \delta} \lim_{\tau \to 0} \int_{-\infty}^{\tau} \nabla_{\delta} G_{\beta \gamma} \left[ z^{\mu}(\tau); z^{\mu}(\tau') \right] \times u^{\beta}(\tau') u^{\gamma}(\tau') \, d\tau' \tag{A1}
\]

where

\[
k^{\alpha \beta \gamma \delta} = \frac{1}{2} g^{\alpha \beta} u^{\gamma} u^{\delta} - \frac{1}{2} g^{\alpha \gamma} u^{\beta} u^{\delta} + \frac{1}{4} g^{\beta \gamma} g^{\alpha \delta} \tag{A2}
\]

and \( G_{\beta \gamma} \left[ z^{\mu}(\tau); z^{\mu}(\tau') \right] \) is the two-point retarded Green’s function.

The spin force is given (at the pole-dipole approximation, i.e., taking into consideration only the mass monopole and the spin dipole, neglecting higher multipoles, such as the tidal coupling, which is a mass quadrupole effect) by the Papapetrou equations

\[
\frac{Dp^{\alpha}}{d\tau} = -\frac{1}{2} S^{\mu \nu} u^{\mu} R_{\nu \sigma \mu} \tag{A3}
\]

and

\[
\frac{D S^{\alpha \beta}}{d\tau} = 2 u_{\mu} u^{[\alpha} D S^{\beta] \rho}_{\rho} \tag{A4}
\]

where \( p^{\mu} = \mu u^{\alpha} - u_{\beta} \frac{D S^{\alpha \beta}}{d\tau} \) is the particle’s four-momentum, \( S^{\alpha \beta} \) is the skew-symmetric spin tensor of the particle, which is given by

\[
S^{\alpha \beta} = 2 \int_{\Sigma} (x^{[\alpha} - z^{[\alpha}) T^{\beta]} d\Sigma_{\gamma}, \tag{A5}
\]

where \( \Sigma \) is an arbitrary spacelike hypersurface. Notice that the RHS of Eq. (A4) is just \( 2 \mu \alpha \beta \beta \delta \).

As the mass of the particle \( \mu \ll M \), and as the spin of the particle is consequently also small, we shall approximate the Papapetrou equations by considering the spin force to leading order, that is to order \( \mu^{2} \). We next introduce the Mathisson-Pirani spin supplementary condition (which physically identifies the particle’s center of mass) \( S_{\alpha} = 0 \) and the Pauli-Lubanski spin covector \( S_{\alpha} \), defined by

\[
S_{\alpha} = \frac{1}{2} \epsilon_{\beta \mu \nu} u^{\beta} S^{\mu \nu} \tag{A6}
\]

(\( \epsilon_{\alpha \beta \gamma \delta} u_{\alpha} S_{\beta} \delta \)) is the inverse of Eq. (A6) is given by

\[
S^{\alpha \beta} = \epsilon^{\alpha \beta \gamma \delta} u_{\gamma} S_{\delta} \tag{A7}
\]

The Papapetrou equations are then given by

\[
\mu \frac{Du^{\alpha}}{d\tau} = \frac{1}{2} \epsilon^{\lambda \mu \rho \sigma} R_{\lambda \mu}^{\nu \sigma} u_{\nu} u_{\rho} S_{\rho} + O(S^{2}) \tag{A8}
\]

and

\[
\frac{DS_{\alpha}}{d\tau} = u_{\alpha} S_{\rho} \frac{Du^{\rho}}{d\tau} = 0 + O(S^{2}) \tag{A9}
\]

which means that to leading order in the spin, the Pauli-Lubanski spin covector is parallel transported along \( u^{\alpha} \). (The spin covector is always Fermi-Walker transported.) In these equations we neglect all terms at order \( S^{2} \) or higher. From Eq. (A8) we identify the spin force as

\[
f_{\alpha}^{\text{spin}} = \frac{1}{2} \epsilon^{\lambda \mu \rho \sigma} R_{\lambda \mu}^{\nu \sigma} u_{\nu} u_{\rho} S_{\rho} \tag{A10}
\]

APPENDIX B: EXPONENTIAL EXPRESSIONS FOR SCHWARZSCHILD

We list in this Appendix the explicit expressions for the solution of the perturbative equations of motion. We remark that the following explicit solution appears different from the solutions of Ref. 8 because of the different perturbative expansion of the angular velocity that is used there.

\[
\omega(1) = -\frac{r - 3M}{2\mu(MR)^{1/2}} f_{r}^{(1)} \tag{B1}
\]

\[
\omega'(1) = -\frac{1}{4\mu(MR)^{1/2}} \left[ \frac{r + 3M}{r} f_{r}^{(1)} + 2(r - 3M) f_{r}^{(1)'} \right] \tag{B2}
\]

\[
V_{(1)} = \frac{2r}{\mu M} \left( M \right) ^{1/2} \left( 1 - 2M/r \right) f_{r}^{(1)} + M f_{r}^{(1)} \tag{B3}
\]

\[
V_{(2)} = \frac{M}{\mu^{2} M^{2}(r - 6M)^{2}} \left[ 2 \left( \frac{M}{r} \right) ^{1/2} f_{r}^{(1)} f_{r}^{(1)'} (r - 2M)^{2} \times (r - 3M) + \left( \frac{M}{r} \right) ^{1/2} f_{r}^{(1)} f_{r}^{(1)'} (5r - 6M)(r - 2M) \right.

\times \left. 2M f_{r}^{(1)} f_{r}^{(1)'} (r - 2M)(r - 3M) + 4M f_{r}^{(1)} f_{r}^{(1)'} (r - 2M) + 2M^{2} f_{r}^{(2)} (r - 6M) \right.

\times \left. 2M \left( \frac{M}{r} \right) ^{1/2} f_{r}^{(2)} (r - 2M)(r - 6M) \right] \tag{B4}
\]
APPENDIX C: THE EXPANSION COEFFICIENTS $C_k$

We fit the luminosity in gravitational waves (data taken from Ref. [11]) to a function of the form (23) with $n = 6$. The number of terms we sum over was determined by the accuracy of the approximation. The coefficients were determined to equal (not all the figures shown are significant)

\begin{align*}
C_0 &= 3.604707682 \\
C_1 &= -0.9253608775 \\
C_2 &= 0.130073368 \\
C_3 &= -9.25673735 \times 10^{-3} \\
C_4 &= 3.455531402 \times 10^{-4} \\
C_5 &= -6.363613595 \times 10^{-6} \\
C_6 &= 4.521868774 \times 10^{-8}.
\end{align*}