Towards Optimal Range Medians

Beat Gfeller
ETH Zurich, Switzerland
gfeller@inf.ethz.ch

Peter Sanders*
Universität Karlsruhe, Germany
sanders@ira.uka.de

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Abstract
We consider the following problem: given an unsorted array of \( n \) elements, and a sequence of intervals in the array, compute the median in each of the subarrays defined by the intervals. We describe a simple algorithm which uses \( O(n) \) space and needs \( O(n \log k + k \log n) \) time to answer the first \( k \) queries. This improves previous algorithms by a logarithmic factor and matches a lower bound for \( k = O(n) \). Since the algorithm decomposes the range of element values rather than the array, it has natural generalizations to higher dimensional problems – it reduces a range median query to a logarithmic number of range counting queries.

1 Introduction and Related Work

The classical problem of finding the median is to find the element of rank \( \lceil n/2 \rceil \) in an unsorted array of \( n \) elements. Clearly, the median can be found in \( O(n \log n) \) time by sorting the elements. However, a classical algorithm finds the median in \( O(n) \) time [BFP+72], which is asymptotically optimal.

More recently, the following generalization, called the Range Median Problem (RMP), has been considered [KMS05, HPM08]:

**Input:** An unsorted array \( A \) with \( n \) elements, each having a value. Furthermore, a sequence of \( k \) queries \( Q_1, \ldots, Q_k \), each defined as an interval \( Q_i = [l_i, r_i] \). In general, the sequence is given in an online fashion, we want an answer after every query, and \( k \) is not known in advance.

**Output:** A sequence \( x_1, \ldots, x_k \) of values, where \( x_i \) is the median of the elements in \( A[l_i, r_i] \). Here, \( A[l, r] \) denotes the set of all elements whose index in \( A \) is at least \( l \) and at most \( r \).

This RMP naturally fits into a larger group of problems, in which an unsorted array is given, and in a query one wants to compute a certain function of all the elements in a given interval. Instead of the median, natural candidates for such a function are:

- Sum: this problem can be trivially solved in \( O(n) \) preprocessing time and \( O(1) \) query time by computing prefix sums.

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*An element has rank \( i \) if it is the \( i \)-th element in some sorted order. Actually, any specified rank might be of interest. We restrict ourselves to the median to simplify notation but a generalization to arbitrary ranks will be straightforward for all our results.
• Semigroup operator: this problem is significantly more difficult than the sum. However, there exists a very efficient solution: for any constant $c$, a preprocessing in $O(nc)$ time and space allows to answer queries in $O(\alpha_k(n))$ time, where $\alpha_k$ is a certain functional inverse of the Ackerman function [Yao82]. A matching lower bound was given in [Yao85].

• Maximum, Minimum: This is a special case of a semigroup operator, for which the problem can be solved slightly more efficiently: $O(n)$ preprocessing time and space is sufficient to allow $O(1)$ time queries (see e.g. [BFC04]).

In addition to being a natural extension of the median problem, the RMP has some applications in practice, namely obtaining a “typical” element in a given time series out of a given time interval [HPM08].

Natural special cases of the RMP are an offline variant, where all queries are given in a batch and a variant where we want to do all preprocessing up front and are then interested in good worst case bounds for answering a single query.

The authors of [HPM08] give a solution of the RMP which requires $O(n \log k + k \log n \log k)$ time and $O(n \log k)$ space. In addition, they give a lower bound of $\Omega(n \log k)$ time for comparison-based algorithms. They basically use a one-dimensional range tree over the input array, where each inner node corresponds to a subarray defined by an interval. Each such subarray is sorted, and stored with the node. A range median query then corresponds to selecting the median from $O(\log k)$ sorted subarrays (whose union is the queried subarray) of total length $O(n)$, which requires $O(\log n \log k)$ time. The main difficulty of their approach is to show that the subarrays need not be fully sorted, but only presorted in a particular way, which reduces the construction time of the tree from $O(n \log n)$ to $O(n \log k)$.

Concerning the preprocessing variant of the RMP, [KMS05] give a data structure and answers queries in $O(\log n)$ time, which uses $O(n \log^2 n / \log \log n)$ space. They do not analyze the required preprocessing time, but it is clearly at least as large as the required space in words. Moreover, they give a structure which uses only $O(n)$ space, but query time $O(n^\epsilon)$ for arbitrary $\epsilon > 0$. Another data structure given in [Pet08] can answer queries in $O(1)$ time and uses $O(n^2 \log^{(p)} n / \log n)$ space, where $p$ is an arbitrary integer, and $\log^{(p)} n$ is the $p$ times iterated logarithm of $n$.

Our results.

First, in Section 2 we give an algorithm for the pointer-machine model which solves the RMP in $O(n \log k + k \log n)$ time and $O(n \log k)$ space. This improves the running time of $O(n \log k + k \log n \log k)$ reported in [HPM08] for $k \in \omega(n / \log n)$. Our algorithm is also considerably simpler. The idea is to reduce a range median query to a logarithmic number of related range counting queries by recursively partitioning the values in array $A$ in a tree in a fashion similar to Quicksort. The final time bound is achieved using the technique of Fractional Cascading. In Section 2.1 we explain why our algorithm is optimal for $k \in O(n)$ and at most $\Omega(\log n)$ from optimal for $k \in \omega(n)$.

\(^2\)Note that the data structures in [KMS05, Pet08] work only for a specific quantile (e.g. the median), which must be the same for all queries.
Section \[3\] achieves linear space in the RAM model using techniques from succinct data structures – the range counting problems are reduced to rank computations in bit arrays. To achieve the desired bound, we compress the recursive subproblems in such a way that the bit arrays remain dense at all times.

The latter algorithm can be easily modified to obtain a data structure using \( O(n) \) space, can be constructed in \( O(n \log n) \) time, and allows to answer arbitrary range median queries (or an arbitrary rank, which may be specified online together with the query interval) in \( O(\log n) \) time. Note that the previously best linear-space data structure required \( O(n^{\epsilon}) \) query time \([KMS05]\).

After a few remarks on generalizations for higher dimensional inputs in Section \[4\], we discuss dynamic variants of the RMP problem in Section \[5\] before Section \[6\] concludes with a summary and some open problems.

## 2 A Pointer Machine Algorithm

Our algorithm is based on the following key observation (see also Figure 1): Suppose that we partition the elements in array \( A \) of length \( n \) into two smaller arrays: \( A.low \) which contains all elements with the \( n/2 \) smallest values in \( A \), and \( A.high \) which contains all elements with the \( n/2 \) largest values. The elements in \( A.low \) and \( A.high \) are sorted by their index in \( A \), and each element \( e \) in \( A.low \) and \( A.high \) is associated with its index \( e.i \) in the original input array, and its value \( e.v \). Now, if we want to find the element of rank \( p \) in the subarray \( A[L, R] \), we can do the following: We count the number \( m \) of elements in \( A.low \)

\[3\]To simplify notation we ignore some trivial rounding issues and also sometimes assume that all elements have unique values. This is without loss of generality because we could artificially expand the size \( A \) to the next power of two and because we can use the position of an element in \( A \) to break ties in element comparisons.
which are contained in $A[L, R]$. To obtain $m$, we can do a binary search for both $L$ and $R$ in $A\.low$ (using the $e\.i$ fields). If $p \leq m$, then the element of rank $p$ in $A[L, R]$ is the element of rank $p$ in $A\.low[L, R]$. Otherwise, the element of rank $p$ is the element of rank $p - m$ in $A\.high[L, R]$.

Hence, using the partition of $A$ into $A\.low$ and $A\.high$, we can reduce the problem of finding an element of a given rank in array $A[L, R]$ to the same problem, but on a smaller array (either $A\.low[L, R]$ or $A\.high[L, R]$). Our algorithm applies this reduction recursively.

**Algorithm overview.** The basic idea is therefore to subdivide the $n$ elements in the array into two parts of (almost) equal size by computing the median of their values and using it to split the list into a list of the $n/2$ elements with smaller values and a list of the $n/2$ elements with larger values. The two parts are recursively subdivided further, but only when required by a query. To answer a range median query, we determine in which of the two parts the element of the desired rank lies (initially, this rank corresponds to the median, but this may change during the search). Once this is known, the search continues recursively in the appropriate part until a trivial problem of constant size is encountered.

We will show that the total work involved in splitting the subarrays is $O(n \log k)$ and that the search required for any query can be completed in $O(\log n)$ time using Fractional Cascading [CG86]. Hence, the total running time is $O(n \log k + k \log n)$.

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**Algorithm 1:** Query($A, L, R, p$)

1. Input: range select data structure $A$ of elements, query range $[L, R]$, desired rank $p$
2. if $|A| = 1$ then return $A[1]$
3. if $A\.low$ is undefined then
4.  Compute median $x$ value of the pairs in $A$
5.  $A\.low := \{ e \in A : e\.v \leq x \}$
6.  $A\.high := \{ e \in A : e\.v > x \}$
7.  \{ $\langle e \in A : Q \rangle$ is an array containing all elements $e$ of $A$ satisfying the given condition $Q$, ordered as in $A$ \}
8.  \{ Find($A, q$) returns $\max\{ j : A[j].i \leq q \}$ (with Find($A, 0$) = 0) \}
9.  $l := $ Find($A\.low, L - 1$) ; \hspace{1cm} // # of low elements left of $L$
10. $r := $ Find($A\.low, R$) ; \hspace{1cm} // # of low elements up to $R$
11. $m := r - l$ ; \hspace{1cm} // # of low elements between $L$ and $R$
12. if $p \leq m$ then return Query($A\.low, L, R, p$)
13. else return Query($A\.high, L, R, p - m$)

**Detailed description and analysis.** Algorithm 1 gives pseudo-code for the query, which performs preprocessing (i.e., splitting the array into two smaller arrays) only where needed. Note that we have to keep three things separate here: values that are relevant for median computation and partitioning the input, positions in the input sequence that are relevant for finding the elements within the range $[L, R]$, and positions in the subdivided arrays that are important for counting elements.
Let us first analyze the time required for processing a query not counting the ‘preprocessing’ time within lines 4–6: The query descends $\log_2 n$ levels of recursion. On each level, $\text{Find}$-operations for $L$ and $R$ are performed on the lower half of the current subproblem. If we used binary search, we would get a total execution time of up to $\sum_{i=1}^{\log_2 n} O(\log \frac{n}{2^i}) = \Theta(\log^2 n)$. However, the fact that in all these searches, we search for the same key ($L$ or $R$) allows us to use a standard technique called Fractional Cascading [CG86] that reduces the search time to a constant, once the position in the first search is known. Indeed, we only need a rather basic variant of Fractional Cascading, which applies when each successor list is a sublist of the previous one [dBvKOS00]. Here, it suffices to augment an element $e$ of a list with a pointer to the position of some element $e'$ in each subsequent list (we have two successors – $A.\text{low}$ and $A.\text{high}$). In our case, we need to point to the largest element in the successor that is no larger than $e$. We get a total search time of $O(\log n)$.

Now we turn to the preprocessing code in lines 4–6 of Algorithm 1. Let $s(i)$ denote the level of recursion at which query $i$ encountered an undefined array $A.\text{low}$ for the first time. Then the preprocessing time performed during query $i$ is $O(n/2^{s(i)})$ if a linear time algorithm is used for median selection [BFP72] (note that we have a linear recursion with geometrically decreasing execution times). This preprocessing time also includes the cost of finding the pointers for Fractional Cascading while splitting the list in lines 4–6. Since the preprocessing time during query $i$ decreases with $s(i)$, the total preprocessing time is maximized if small levels $s(i)$ appear as often as possible. However, level $j$ can appear no more than $2^j$ times in the sequence $s(1), s(2), \ldots, s(k)$. Hence, we get an upper bound for the preprocessing time when the smallest $\lfloor \log k \rfloor$ levels are used as often as possible (‘filled’) and the remaining levels are $\lceil \log k \rceil$. The preprocessing time at every used level is $O(n)$ giving a total time of $O(n \log k)$. The same bound applies to the space consumption since we never allocate memory that is not used later. We summarize the main result of this section in a theorem:

**Theorem 1.** The online range median problem (RMP) on an array with $n$ elements and $k$ range queries can be solved in time $O(n \log k + k \log n)$ and space $O(n \log k)$.

Another variant of the above algorithm invests $O(n \log n)$ time and space into complete preprocessing up front. Subsequently, any range median query can be answered in $O(\log n)$ time. This improves the preprocessing space of the corresponding result in [KMS05] by a factor $\log n / \log \log n$ and the preprocessing time by at least this factor.

### 2.1 Lower Bounds

We shortly discuss how far our algorithm is from optimality. In [HPM08], a comparison-based lower bound of $\Omega(n \log k)$ is shown for the range median problem 4. As our algorithm shows, this bound is (asymptotically) tight if

4Indeed, for $j > 0$ the maximal number is $2^{j-1}$ since the other half of the available subintervals have already been covered by the preprocessing happening in the layer above.

5The authors derive a lower bound of $l := \frac{n^2}{\log(n/k-1)^{n/k}}$, where $n$ is a multiple of $k < n$.

Unfortunately, the analysis of the asymptotics of $l$ given in [HPM08] is erroneous; however, a corrected analysis shows that the claimed $\Omega(n \log k)$ bound holds.
$k \in O(n)$. For larger $k$, the above lower bound is no longer valid, as the construction requires $k < n$. Yet, a lower bound of $\Omega(n \log n)$ is immediate for $k \geq n$, considering only the first $n - 1$ queries. Furthermore, $\Omega(k)$ is a trivial lower bound. Note that in the analysis of our algorithm, the number of levels of the recursion is actually bounded by $O(\min\{\log k, \log n\})$, and thus for any $k \geq n$ our algorithm has running time $O(n \log n + k \log n)$, which is up to $\Omega(\log n)$ from optimal for any $k$.

In a very restricted model (sometimes called “Pointer Machine”), where a memory location can be reached only by following pointers, and not by direct addressing, our algorithm is indeed optimal also for $k \geq n$: it takes $\Omega(\log n)$ time to even access an arbitrary element of the input. Since every element of the input is the answer to at least one range query (e.g. the query whose range contains only this element), the bound follows. An interesting question is whether a lower bound $\Omega(k \log n)$ could be shown in stronger models. However, note that any comparison-based lower bound (as the one in [HPM08]) cannot be higher than $\Omega(n \log n)$: With $O(n \log n)$ comparisons, an algorithm can determine the permutation of the array elements, which suffices to answer any query without further element comparisons. Therefore, one would need to consider stronger models (e.g. the “cell-probe” model), in which proving lower bounds is significantly more difficult.

3 A Linear Space RAM Implementation

**Algorithm 2: Query($A, L, R, p$)**

1. **Input:** range select data structure $A$, query range $[L, R]$ and desired rank $p$
2. **if** $|A| = 1$ **then return** $A[1]$
3. **if** $A\.low$ **is undefined** **then**
   4. Compute median $x$ value of the values in $A$
   5. $A\.lowbits := \text{BitVector}(|A|, \{i \in 1..|A| : A[i] \leq x\})$
   6. $A\.low := \{A[i] : i \in 1..|A|, A[i] \leq x\}$
   7. $A\.high := \{A[i] : i \in 1..|A|, A[i] > x\}$
   8. deallocate the value array of $A$ itself
9. $l := A\.lowbits\.rank(L - 1)$
10. $r := A\.lowbits\.rank(R)$
11. $m := r - l$
12. **if** $p \leq m$ **then** return Query($A\.low$, $l + 1$, $r$, $p$)
13. **else** return Query($A\.high$, $L - l$, $R - r$, $p - m$)

Our starting point for a more space efficient implementation of Algorithm 2 is the observation that we do not actually need all the information available in the arrays stored at the interior nodes of our data structure. All we need is support for the operation $\text{Find}(x)$ that counts the number of elements $e$ in $A\.low$ that have index $e.i \leq x$. This information can already be obtained from a bit-vector where a 1-bit indicates whether an element of the original array is in $A\.low$. For this bit-vector, the operation corresponding to $\text{Find}$ is called $\text{rank}$. In the RAM model, there are data structures that need space
$n + o(n)$ bits, can be constructed in linear time and support \textit{rank} in constant time (e.g., [Cla88, OS06]). Unfortunately, this idea alone is not enough since we would need to store $2^j$ bit arrays consisting of $n$ positions each on level $j$. Summed over all levels, this would still need $\Omega(n \log^2 n)$ bits of space even if optimally compressed data structures were used. This problem is solved using an additional idea: for a node of our data structure with value array $A$, we do not store a bit array with $n$ positions but only with $|A|$ positions, i.e., bits represent positions in $A$ rather than in the original input array. This way, we have $n$ positions on every level leading to a total space consumption of $O(n \log n)$ bits. For this idea to work, we need to be able to transform the query range in the recursive call in such a way that \textit{rank} operations in the contracted bit arrays are meaningful. Fortunately, this is easy because the rank information we compute also defines the query range in the contracted arrays. Algorithm 2 gives pseudocode specifying the details. Note that the algorithm is largely analogous to Algorithm 1. In some sense, the algorithm becomes simpler because the distinction between query positions and array positions for counting disappears (If we still want to report the positions of the median values in the input, we can store this information at the leaves of the data structure using linear space). Using an analysis analogous to the analysis of Algorithm 1 we obtain the following theorem:

\textbf{Theorem 2.} The online range median problem (RMP) on an array with $n$ elements and $k$ range queries can be solved in time $O(n \log k + k \log n)$ and space $O(n)$ words on the RAM model.

By doing all the preprocessing up front, we obtain an algorithm with preprocessing time $O(n \log n)$ using $O(n)$ space and query time $O(\log n)$. This improves the space consumption compared to [KMS05] by a factor $\log^2 n / \log \log n$.

4 Higher Dimensions

Since our algorithm decomposes the values rather than the positions of elements, it can be naturally generalized to higher dimensional point sets. We obtain an algorithm that needs $O(n \log k)$ preprocessing time plus the time for supporting range counting queries on each level. The amortized query time is the time for $O(\log n)$ range counting queries. Note that query ranges can be specified in any way we wish: (hyper)-rectangles, circles, etc., without affecting the way we handle values. For example, using the data structure for 2D range counting from [JMS04] we obtain a data structure for the 2D rectangular range median problem that needs $O(n \log n \log k)$ preprocessing time, $O(n \log k / \log \log n)$ space, and $O(\log^2 n / \log \log n)$ query time. This not only applies to 2D arrays consisting of $n$ input points put to arbitrary two-dimensional point sets with $n$ elements.

Unfortunately, further improvements, e.g. to logarithmic query time seem difficult. Although the query range is the same at all levels of recursion, Fractional Cascading becomes less effective when the result of a rectangular range

\footnote{Indeed, since we only need the rank operation, there are very simple and efficient implementations: store a table with ranks for indices that are a multiple of $w = \Theta(\log n)$. General ranks are then the sum of the next smaller table entry and the number of 1-bits in the bit array between this rounded position and the query position. Some processors have a POPCNT instruction for this purpose. Otherwise we can use lookup tables.}
counting query is defined by more than a constant number of positions within
the data structure because we would have to follow many forwarding pointers.
Also, the array contraction trick that allowed us to use dense bit arrays in Sec-
tion 3 does not work anymore because an array with half the number of bits
need not contain any empty rows or columns.

Another indication that logarithmic query time in two dimensions might be
difficult to achieve is that there has been intensive work on the more specialized
median-filtering problem in image processing where we ask for all range medians
with query ranges that are squares of size $2r+1 \times 2r+1$ in an image with $n$ pixels.
The best previous algorithms known here need time $\Theta(n \log^2 r)$ [GW93] unless
the range of values is very small [PH07, CWE07]. Our result above improves
this by a factor $\log \log r$ (by applying the general algorithm to input pieces of
size $3r \times 3r$) but this seems to be of theoretical interest only.

5 Dynamic Range Medians

In this section, we consider a dynamic variant of the RMP, where we have a
linked list instead of an array, and elements can be deleted or inserted arbitrarily.
In this setting, we still want to answer median queries, whose range is given by
two pointers to the first and the last element in the query range.

In the following, we sketch a solution which allows inserts and deletes in
$O(\log^2 n)$ amortized time each, and range median queries in $O(\log^2 n)$ worst
case time. The basic idea is to use a BB($\alpha$) tree [NR72] as a primary structure,
in which all elements are ordered by their value. With each inner node, we
associate a secondary structure, which contains all the elements of the node’s
subtree, ordered by their position in the input list. More precisely, we store
these elements in a balanced binary search tree, where nodes are augmented
with a field indicating the size of their subtree, see e.g. [Ron01]. This data
structure permits to answer a range query by a simple adaptation of Algorithm 1:
starting at the root, we determine the number of elements within the query range
which are in the left subtree, and depending on the result continue the search
for the median in the left or in the right subtree. The required counting in
each secondary structure takes $O(\log n)$ time, and we need to perform at most
$O(\log n)$ such searches for any query. When an element is inserted or deleted,
we follow the search path in the BB($\alpha$) tree according to its value, and update
all the $O(\log n)$ secondary structures of the visited nodes. The main difficulty
arises when a rotation in the BB($\alpha$) tree is required: in this case, the secondary
structures are rebuilt from scratch, which costs $O(p \log p)$ time if the subtree
which is rotated contains $p$ nodes. However, as shown in [Meh84, WL85], such
rotations are required so rarely that the amortized time of such an event is only
$O(\log p \log n) = O(\log^2 n)$.

We note that using this dynamic data structure for the one-dimensional
RMP, we can implement a two-dimensional median filter, by scanning over the
image, maintaining all the pixels in a strip of width $r$. In this way, we obtain a
running time of $O(\log^2 r)$ per pixel, which matches the state-of-the-art solution
for this problem [GW93]. This indicates that obtaining a solution with $O(\log n)$
time for all operations could be difficult.
6 Conclusion

We have presented improved upper bounds for the range median problem. The query time of our solution is asymptotically optimal for \( k \in O(n) \), or when all preprocessing has to be done up front. For larger values of \( k \), our solution is at most a factor \( \log n \) from optimal. In a very restricted model where no arrays are allowed, our solution is optimal for all \( k \). Moreover, in the RAM model, our data structure requires only \( O(n) \) space, which is clearly optimal. Making the data structure dynamic adds a factor \( \log n \) to the query time. Using \( O(n^2) \) space, it is trivial to precompute all medians of a given array so that the query time becomes constant. However, it is open whether the term \( O(k \log n) \) in the query time could be reduced towards \( O(k) \) in the RAM model when \( k = o(n^2) \).

Given the simplicity of our data structure, a practical implementation would be easily possible. To avoid the large constants involved when computing medians for recursively splitting the array, one could use a pivot chosen uniformly at random. This should work well in expectation.

It would be interesting to find faster solutions for the dynamic RMP or the two-dimensional (static) RMP: Either would lead to a faster median filter for images, which is a basic tool in image processing.

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