Characterization of anisotropic Liouville-type mixed spaces and its application

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Abstract

Within the framework of the method of hypersingular integrals we obtain a characterization of the anisotropic spaces $L^{\alpha}_{p, r}$. These spaces are defined to consist of functions $f(x) \in L^{r}$ for which

$$F^{-1} \left( \sum_{j=1}^{n} |\xi_j|^{\alpha_j} \right) Ff \in L^{p}, \quad \alpha_j > 0.$$ 

The mentioned characterization is applied to prove the denseness of the class $C^{\infty}_0$ in $L^{\alpha}_{p, r}$.

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1 Introduction

We consider the anisotropic Liouville-type mixed spaces, which are defined via Fourier transform as follows:

$$L^{\overline{\alpha}}_{p, r} \equiv L^{\overline{\alpha}}_{p, r}(\mathbb{R}^n) = \left\{ f : f \in L^{r}, F^{-1} \left( \sum_{j=1}^{n} |\xi_j|^{\alpha_j} \right) Ff \in L^{p} \right\}, \quad (1.1)$$

where $\alpha_i > 0$, $\overline{\alpha} = (p_1, \ldots, p_n)$, $\overline{r} = (r_1, \ldots, r_n)$, $1 \leq p_i, r_i < \infty \ (i = 1, \ldots, n)$, $L^{\overline{r}} \equiv L^{\overline{r}}(\mathbb{R}^n)$ being the well-known space equipped with the mixed norm

$$\|f\|_{\overline{\alpha}} = \left( \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^1} \ldots \left( \int_{\mathbb{R}^1} |f(x)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} \ldots dx_{n-1} \right)^{\frac{p_n}{p_n-1}} \right)^{\frac{1}{p_n}} \quad (1.2)$$

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We note that some basic properties of the spaces $L^p_\pi$ were established in [1, 2]. The Fourier transform $Ff$ in (1.1) is treated in the sense of distributions on a suitable space of test functions (see Section 2).

In the isotropic case when $\alpha_1 = \cdots = \alpha_n = \alpha$, $p_1 = \cdots = p_n = p$, and $r_1 = \cdots = r_n = r$ the spaces (1.1) coincide with the spaces $L_{p,r}^\alpha$ of functions $f \in L_r$ having their Riesz derivatives $\partial^\alpha f \in L_p$. Such spaces were first introduced and investigated by S. G. Samko in [13, 14] (see also the books [16] and [17]). We note that the spaces (1.1) were treated by P. I. Lizorkin in the case $p = r$ (see [8]–[10]). The spaces $L_{p,r}^\alpha$ were studied by A. A. Davtyan in [3, 4] in the case $1 < p < n/\alpha^*$, $1/\alpha^* = \frac{1}{n} \sum_{j=1}^{n} 1/\alpha_j$, $1 < r < \infty$, and by V. A. Nogin and G. P. Emgusheva (see [5, 6]) for $1 \leq p, r < \infty$.

Here we deal with the most general anisotropic case of vector-valued $\pi$, $\overline{\pi}$, and $\overline{\tau}$. We obtain a characterization of the spaces $L_{p,r}^\pi$ via anisotropic hypersingular integrals (HSI)

$$\lim_{\varepsilon \to 0} (L^\pi_{\varepsilon} f)(x),$$

introduced by P. I. Lizorkin (see [3]), where $T_{\varepsilon}^\pi f$ is the ”truncated” integral

$$\int_{\rho(t) > \varepsilon} (\Delta_{t}^{2\overline{\ell}} f)(x) \frac{dt}{\rho_{\alpha + \alpha^*}(t)}, \quad 2\overline{\ell} > \max_j \alpha_j.$$  \hspace{1cm} (1.4)

Here $(\Delta_{t} f)(x)$ is the centered finite difference of the function $f(x)$. The function $\rho(t)$, which is referred to as ”anisotropic distance”, is a positive solution of the equation $\sum_{i=1}^{n} x_i^2 \rho^{-2\lambda_i} = 1$, $\lambda_i = \alpha^*/\alpha_i$. Namely, we prove that

$$L_{p,r}^\pi = \{ f : f \in L_{\pi}, \ T_{\varepsilon}^\pi f \in L_{\overline{\pi}} \},$$

(1.5)

$\alpha_i > 0$, $1 < p_i < \infty$, $1 \leq r_i < \infty$, under some additional restrictions on $\pi$ in the case $\alpha^* \geq n$ (see (2.1)).

We also give an application of the mentioned characterization. Basing ourselves on equality (1.5), we prove that the class $C^\infty_0$ is dense in $L_{p,r}^\pi$, $\alpha_i > 0$, $1 < p_i < \infty$, $1 \leq r_i < \infty$ (under the mentioned restrictions on $\pi$ in the case $\alpha^* \geq n$). It should be noted that the denseness of the class $C^\infty_0$ in $L_{p,r}^\alpha$ was proved in [13] for $1 < p < n/\alpha$ and in [11] for $p \geq n/\alpha$. In the
anysotropic case the corresponding result was established in [12], where the authors assumed that \( r_i > p_i \) or \( r_i \leq p_i \) for every \( i, 1 \leq i \leq n \), if \( \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} \leq 1 \). One of the goals of this paper is to get rid of these unnatural restrictions.

We note that the proof of denseness given in Theorem 2.2 looks very non-trivial. We first prove the statement of this theorem in the case \( 0 < \max_{j} \alpha_j < 1 \). After that, we extend it to the remaining values of parameters with the aid of induction.

We also point out the papers [19] and [20], where the author dealt with some generalization of the spaces of \( L^{\alpha_{p,r}} \)-type to the case of a more general anysotropic distance (in comparison with that used in (1.4)).

The paper is organized as follows. In Section 2 we formulate our main results, see Theorems 2.1 and 2.2. Sections 3 and 4 can be regarded as a background to the proof of the mentioned theorems. They contain the necessary preliminaries and some auxiliary statements respectively. In Section 5 we prove Theorems 2.1 and 2.2. For the sake of convenience we gather some technical fragments of the proofs in Appendix. Some results presented in this paper were announced in [12].

## 2 The main results

We define the Liouville-type spaces \( L^{\alpha_{p,r}}_{p,r} \), \( 1 \leq r_i < \infty, 1 \leq p_i < \infty, \alpha_i > 0, i = 1, \ldots, n \), by equality (1.1), where the Fourier transform \( Ff \) is interpreted in the sense of \( \Psi' \)-distributions (see Subsection 3.1). We put

\[
\|f\|_{L^{\alpha_{p,r}}_{p,r}} = \|f\|_{\tau} + \left\| F^{-1} \left( \sum_{j=1}^{n} |\xi_j|^{|\alpha_j|} \right) Ff \right\|_{\tau}.
\]

The following theorem provides a characterization of the space \( L^{\alpha_{p,r}}_{p,r} \) by means of HSI (1.3).

**Theorem 2.1.** Let \( 1 < p_i < \infty, 1 \leq r_i < \infty, \alpha_i > 0, i = 1, \ldots, n \), and

\[
\sum_{j=1}^{n} \frac{1+k_j}{\alpha_j} \neq 1, \quad |k| = 0,1,\ldots, m-1, \quad m = [\gamma],
\]

\[
\gamma = \max_{j} \alpha_j \left( 1 - \sum_{j=1}^{n} \frac{1}{\alpha_j} \right).
\]
Then equality (1.5) holds. Moreover, the norms \( \|f\|_{L^\alpha p, r} \) and \( \|f\|_r + \|T^\alpha f\|_p \) are equivalent.

**Remark 2.1.** For the rest of the paper we assume the condition (2.1) to be fulfilled.

We apply the characterization (1.5) to prove the next theorem.

**Theorem 2.2.** Let \( 1 < p_i < \infty, 1 \leq r_i < \infty, \alpha_i > 0, i = 1, \ldots, n \). Then the space \( C^\infty_0 \) is dense in \( L^\alpha p, r \).

### 3 Preliminaries

#### 3.1 Notation:

\( S \) is the Schwartz space of rapidly decreasing smooth functions; \( C^\infty_0 \) is its subspace of finite functions; \( \Psi_0, \Psi \) are the Lizorkin spaces of functions in \( S \), vanishing together with all their derivatives at the origin and on the coordinate hyperplanes respectively; \( \Phi_0, \Phi \) are their Fourier duals; \( \Psi', \Phi' \) are the corresponding spaces of distributions; \( C^\infty_0(0, \infty) \) is the class of finite \( C^\infty \)-functions on the semi-axis \( (0, \infty) \) supported outside zero; \( M_{\overline{p}} \) is the class of \( p \)-multipliers; \( X \rightarrow Y \) is the continuous embedding of the normed space \( X \) into the normed space \( Y \); \( (\Delta^\ell_y f)(x) = \sum_{k=0}^\ell (-1)^k C^k_\ell f(x + (\ell/2 - k)y) \) is the centered finite difference of the function \( f(x) \) of order \( \ell \) with the step \( y \); \( (\bar{\Delta}^\ell_y f)(x) = \sum_{k=0}^\ell (-1)^k C^k_\ell f(x - ky) \) is the non-centered finite difference; \( (Ff)(y) = \widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{ix\cdot y}dx \) is the Fourier transform of the function \( f(x) \); \( (F^{-1}f)(x) = \widehat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y)e^{-ix\cdot y}dy \) is the inverse Fourier transform; \( \langle f, \omega \rangle = \int_{\mathbb{R}^n} \overline{f(x)}\omega(x)dx \). All constants in various estimates not necessarily the same at each occurrence are denoted by the same letter \( C \). The end of proof is denoted by ■.
3.2 Estimates for some integrals and finite differences

The following lemmas were proved in [13] in the isotropic situation (when \( \rho(x) = |x|, p_1 = \cdots = p_n, \) and \( s_1 = \cdots = s_n \)). The proofs of their analogues in the anisotropic case are much in lines with those given in [13] and we omit them.

**Lemma 3.1.** Let \( \gamma > 0, 1 \leq s_i < \infty, i = 1, \ldots, n, \)

\[
I(t) = \left( \int_{\mathbb{R}^n} dx_n \cdots \left( \int_{\mathbb{R}^1} \left( \prod_{k=0}^{\ell} [1 + \rho(x - kt)]^{-\gamma s_1} \right) dx_1 \right)^{s_2/s_1} \cdots \right)^{1/s_n},
\]

\( \ell = 1, 2, \ldots. \) Then

\[
I(t) \leq C(1 + \rho(t))^{\sum_{i=1}^{n} \lambda_i/s_i - (\ell+1)\gamma}, \quad \text{if} \quad \frac{1}{\ell + 1} \sum_{i=1}^{n} \frac{\lambda_i}{s_i} < \gamma < \sum_{i=1}^{n} \frac{\lambda_i}{s_i}, \tag{3.1}
\]

\[
I(t) \leq C(1 + \rho(t))^{-\gamma \ell}, \quad \text{if} \quad \gamma > \sum_{i=1}^{n} \frac{\lambda_i}{s_i}. \tag{3.2}
\]

**Lemma 3.2.** If \( a(x) \in C_0^\infty, \) then the following inequality holds:

\[
\left| (\tilde{\Delta}^l a)(x) \right| \leq C \left[ \frac{\rho^l(t)}{\prod_{k=0}^{l} (1 + \rho(x - kt))} \right]^\theta, \quad \ell = 1, 2, \ldots, \theta = \min_i \left( \frac{\alpha_i^*}{\alpha_i} \right).
\]

\( x, t \in \mathbb{R}^n. \)

**Lemma 3.3.** If \( a(x) \in C_0^\infty, \) then

\[
\left\| \tilde{\Delta}^l a \right\|_{\mathcal{F}} \leq C \left[ \frac{\rho(t)}{1 + \rho(t)} \right]^{\theta \ell}, \quad \ell = 1, 2, \ldots, \theta = \min_i \left( \frac{\alpha_i^*}{\alpha_i} \right). \tag{3.4}
\]
3.3 Some properties of the mixed spaces $L_{\pi}$

We need the following properties of functions in $L_{\pi}$.

**Theorem 3.1 (\[2\]).** Let $f(x) \in L_{\pi}(\mathbb{R}^n)$, $1 \leq p_i < \infty$, $i = 1, \ldots, n$, and let $a(t)$ be an averaging kernel, that is, $a(t) \in L_1$ and $\int a(t) dt = 1$. Then

$$\lim_{\delta \to 0} \|f - f_\delta\|_{\pi} = 0,$$

where

$$f_\delta(x) = \int_{\mathbb{R}^n} a(t)f(x - \delta t) dt.$$  \hspace{1cm} (3.5)

The following anysotropic interpolation inequality “for $m$ points” was proved in [18].

**Theorem 3.2.** Let $\pi^j = (p_1^j, \ldots, p_n^j)$, $1 \leq p_i^j \leq \infty$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $m \leq n + 1$, and $f \in \bigcap_{j=1}^m L_{\pi^j}$, $\theta_k \geq 0$, $\sum_{k=1}^m \theta_k = 1$. Then $f \in L_{\pi}$ for any $\pi$ such that $1/p_i = \sum_{j=1}^m \theta_j/p_i^j$, $i = 1, \ldots, n$, and

$$\|f\|_{\pi} \leq \|f\|_{\pi^1}^{\theta_1} \|f\|_{\pi^2}^{\theta_2} \ldots \|f\|_{\pi^m}^{\theta_m}. \hspace{1cm} (3.6)$$

3.4 Anysotropic potentials

Let $K_{\pi}(x)$ be a smooth function in $\mathbb{R}^n \setminus \{0\}$ such that $K_{\pi}(t^\lambda x) = t^{\alpha* - n} K_{\pi}(x)$, where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n_+$, and $t > 0$ (that is, the function $K_{\pi}(x)$ is $\lambda$-homogeneous of order $\alpha^* - n$).

An operator of the form

$$(K_{\pi} \varphi)(x) = \int_{\mathbb{R}^n} K_{\pi}(x - y) \varphi(y) dy$$  \hspace{1cm} (3.7)$$
is called anisotropic potential (of order $\alpha^*$). This operator is well-defined on the whole space $L_{\pi}$, $1 \leq p_i \leq \infty$ ($i = 1, \ldots, n$), if $\sum_{i=1}^n \frac{1}{\alpha_i p_i} > 1$. The following theorem can be regarded as an anysotropic analogue of the well-known Sobolev theorem for the Riesz potential operator.
Theorem 3.3 ([10]). Let \( 1 < p_i \leq q_i \leq \infty \ (i = 1, \ldots, n - 1), 1 < p_n < q_n < \infty, \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} > 1 \), and
\[
\sum_{i=1}^{n} \frac{1}{\alpha_i q_i} = \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} - 1.
\] (3.8)

Then the operator \( K^\alpha \) is bounded from \( L^p \) into \( L^q \).

Remark 3.1. The statement of Theorem 3.3 can be generalized to the case of \( \lambda \)-homogeneous kernels of some order \( \gamma < 0 \) (not necessarily \( \gamma = \alpha^* - n \)). One can obtain such a generalization replacing (3.8) by the following condition
\[
\sum_{i=1}^{n} \frac{1}{\alpha_i q_i} = \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} - \frac{\gamma + n}{\alpha^*}.
\]

For \( \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} \leq 1 \) we treat the potential \( K^\alpha \varphi, \varphi \in L^p \), in the sense of \( \Phi_0^\prime \)-distributions:
\[
\langle K^\alpha \varphi, \omega \rangle = \langle \varphi, K^\alpha \omega \rangle, \quad \omega \in \Phi_0, \quad \varphi \in L^p.
\] (3.9)

Such a treatment is correct since \( K_\alpha^\varphi(x) \) is a convolute in \( \Phi_0 \), in accordance with the Gel'fand-Shilov theorem (see [7], P.155), because \( \hat{K_\alpha^\varphi}(\xi) \) is a multiplier in \( \Psi_0 \).

We consider some anysotropic potentials of special-type. Let \( Q^\alpha \) be the operator with the kernel \( Q^\alpha_\alpha(x) = F^{-1} \left( \frac{1}{\sqrt{2\pi}} S^\alpha_\alpha(\xi) \right) (x) \), where \( S^\alpha_\alpha(\xi) \) is a symbol of HSI (1.3). The kernel \( Q^\alpha_\alpha(x) \) admits the following representations (see [4] and [3] in the cases \( \alpha^* < n \) and \( \alpha^* \geq n \) respectively):
\[ Q_\alpha(x) = \int_0^\infty t^{n-\alpha^* - 1} \tilde{g}(t^\lambda x) dt, \quad \text{if} \quad \alpha^* < n, \] (3.10)

\[ Q_\alpha(x) = m \sum_{k=[m]} \frac{x^k}{k!} \int_0^1 t^{n-\alpha^* + \lambda k - 1} dt \int_0^1 (1-u)^{m-1} (D^k \tilde{g})(ut^\lambda x) du + \]
\[ + \int_1^\infty t^{n-\alpha^* - 1} \tilde{g}(t^\lambda x) dt, \quad m = \lceil \gamma \rceil + 1, \]

\[ \gamma = (\alpha^* - n) \max_j \frac{\alpha_j}{\alpha^*}, \quad \text{if} \quad \alpha^* \geq n, \]

where \( g(\xi) = \phi(\rho(\xi))/S_\alpha(\xi), \varphi \in C_0^\infty(0, \infty), \varphi(t) \geq 0, \) and \( \int_0^\infty \frac{\varphi(t)}{t} dt = 1. \)

The function \( Q(\alpha)(x) \) is \( \lambda \)-homogeneous of order \( \alpha^* - n \), if \( \alpha^* < n \). In the case \( \alpha^* \geq n \) it is \( \lambda \)-homogeneous in the \( \Phi'_0 \)-sense, that is,

\[ t^{\alpha^* - n} \langle Q(\alpha)(x/t^\lambda), \omega \rangle = \langle Q(\alpha), \omega(t^\lambda x) \rangle, \quad \omega \in \Phi_0. \]

We also consider the modified potential \( R(\alpha) \) with the kernel \( R(\alpha)(x) \), being \( \lambda \)-homogeneous (in the regular sense) for any \( \alpha^* > 0 \), which differs from \( Q(\alpha)(x) \) by a polynomial in the case \( \alpha^* \geq n \) (see [3]):

\[ R(\alpha)(x) = Q(\alpha)(x), \quad \text{if} \quad \alpha^* < n, \]
\[ R(\alpha)(x) = Q(\alpha)(x) + \sum_{|k| \leq m-1} \frac{x^k}{k!} \frac{(D^k \tilde{g})(0)}{\lambda k + n - \alpha^*}, \quad \text{if} \quad \alpha^* \geq n. \] (3.11)

We observe that

\[ R(\alpha) \varphi = Q(\alpha) \varphi, \quad \varphi \in L_\alpha, \] (3.12)

in the case \( \alpha^* \geq n \), when both potentials are treated in the \( \Phi'_0 \)-sense, in view of the fact that \( \Phi_0 \)-functions are orthogonal to all polynomials.

4 Some auxiliary statements
4.1 Characterization of the space $L_{\overline{p},\overline{r}}$ in terms of anyisotropic potentials

Let $Q^{\overline{\sigma}}(L_{\overline{\sigma}})$ be the space of anyisotropic potentials:

$$Q^{\overline{\sigma}}(L_{\overline{\sigma}}) = \{ f : f = Q^{\overline{\sigma}} \varphi, \varphi \in L_{\overline{\sigma}} \},$$

where $\overline{\sigma} = (\alpha_1, \ldots, \alpha_n)$, $\overline{\sigma} = (p_1, \ldots, p_n)$, $\alpha_i > 0$, $1 \leq p_i < \infty$; we set

$$\| f \|_{Q^{\overline{\sigma}}(L_{\overline{\sigma}})} = \| \varphi \|_{\overline{\sigma}}.$$

**Theorem 4.1.** Let $1 < p_i < \infty$, $1 \leq r_i < \infty$, $\alpha_i > 0$, $i = 1, \ldots, n$. Then

$$L_{\overline{\sigma},\overline{r}} = L_{\overline{\sigma}} \cap Q^{\overline{\sigma}}(L_{\overline{\sigma}}).$$

Moreover, the norms $\| f \|_{L_{\overline{\sigma},\overline{r}}}$ and $\| f \|_{\overline{\sigma}} + \| f \|_{Q^{\overline{\sigma}}(L_{\overline{\sigma}})}$ are equivalent.

**Proof.** Let $f \in L_{\overline{\sigma},\overline{r}}$. We observe that

$$f = Q^{\overline{\sigma}} B \varphi,$$

where $\varphi = F^{-1} \left( \sum_{j=1}^{n} |\xi_j|^{\alpha_j} \right) F f \in L_{\overline{\sigma}}$, $B$ is the bounded operator in $L_{\overline{\sigma}}$, generated by the $\overline{\sigma}$-multiplier $b(\xi) = S_{\alpha}(\xi)/\sum_{j=1}^{n} |\xi_j|^{\alpha_j}$ (the relation $b(\xi) \in M_{\overline{\sigma}}$ is verified with the aid of Lizorkin theorem, see [11]). Indeed, for $\omega \in \Phi$ we have

$$\langle f, \omega \rangle = \frac{1}{(2\pi)^n} \langle \hat{f}, \hat{\omega} \rangle = \frac{1}{(2\pi)^n} \langle \hat{\varphi}, b(\xi) \hat{\omega}(\xi) \rangle = \langle B \varphi, Q^{\overline{\sigma}} \omega \rangle.$$

From here we derive equality (4.1), which yields $f \in Q^{\overline{\sigma}}(L_{\overline{\sigma}})$. Relatively, $L_{\overline{\sigma},\overline{r}} \subset L_{\overline{\sigma}} \cap Q^{\overline{\sigma}}(L_{\overline{\sigma}})$ and $\| f \|_{\overline{\sigma}} + \| f \|_{Q^{\overline{\sigma}}(L_{\overline{\sigma}})} \leq C \| f \|_{L_{\overline{\sigma},\overline{r}}}$. Let now $f \in L_{\overline{\sigma}}$ and $f = Q^{\overline{\sigma}} \varphi$, $\varphi \in L_{\overline{\sigma}}$. Then

$$F^{-1} \left( \sum_{j=1}^{n} |\xi_j|^{\alpha_j} \right) F f = B^{-1} \varphi \in L_{\overline{\sigma}}.$$ Therefore $f \in L_{\overline{\sigma},\overline{r}}$ and $\| f \|_{L_{\overline{\sigma},\overline{r}}} \leq C(\| f \|_{\overline{\sigma}} + \| f \|_{Q^{\overline{\sigma}}(L_{\overline{\sigma}})}).$ $\blacksquare$

**Corollary 4.1.** The spaces $L_{\overline{\sigma},\overline{r}}$ are complete.
4.2 On a continuous imbedding of the spaces $L^\overline{\alpha}_{p,r}$ with respect to $\overline{\alpha}$

**Theorem 4.2.** Let $1 \leq r_i < \infty$, $1 < p_i < \infty$, $\alpha_i > 0$, and $\beta_i > 0$ be such that $0 < \alpha^* - \beta^* < n$ and $\sum_{i=1}^{n} \frac{1}{\alpha_ip_i} > 1 - \frac{\beta^*}{\alpha^*}$. Then

$$L^\overline{\alpha}_{p,r} \rightarrow L^\overline{\alpha}_{\overline{\pi},\overline{r}},$$

$$\overline{\pi} = (s_1, \ldots, s_n)$$

being an arbitrary vector such that $s_i \geq p_i$, if $i = 1, \ldots, n-1$, $s_n > p_n$, and

$$\sum_{i=1}^{n} \frac{1}{\alpha_is_i} = \sum_{i=1}^{n} \frac{1}{\alpha_ip_i} - \left(1 - \frac{\beta^*}{\alpha^*}\right).$$

**Proof.** Let us consider the anysotropic potential $I^\overline{\alpha}_\varphi = k^\overline{\alpha} * \varphi$ with kernel $k^\overline{\alpha}(x) \overset{(\Phi_0)}{=} F^{-1}(\rho^{-\alpha^*}(\xi))(x)$, which is possessed of the same properties, as the kernel $Q^\overline{\alpha}(x)$ (see (3.10)).

Since both functions $\rho^{\alpha^*}(\xi)/S^\overline{\alpha}(\xi)$ and $\rho^{-\alpha^*}(\xi)S^\overline{\alpha}(\xi)$ belong to $M^\overline{\alpha}$ by the Lizorkin theorem (see [10]), we have

$$Q^\overline{\alpha}(L_{\overline{\rho}}) = I^\overline{\alpha}(L_{\overline{\rho}}) \overset{\text{def}}{=} \{f : f = I^\overline{\alpha}_\varphi, \varphi \in L_{\overline{\rho}}\}.\$$

Therefore

$$L^\overline{\alpha}_{\overline{p},\overline{r}} = L_{\overline{r}} \cap I^\overline{\alpha}(L_{\overline{\rho}})$$

by Theorem [11].

Let $f = I^\overline{\alpha}_\varphi$, $\varphi \in L_{\overline{\rho}}$, and $\omega \in \Phi_0$. We have

$$\langle f, \omega \rangle = \frac{1}{(2\pi)^n} \langle \hat{\varphi}, \rho^{-\alpha^*}(\xi)\hat{\omega}(\xi) \rangle = \langle \psi, I^\beta \omega \rangle,$$

where $\psi = F^{-1}(\rho^{\beta^* - \alpha^*}(\xi)) * \varphi$. Making use of Remark [3.1], we easily obtain the imbedding $Q^\overline{\alpha}(L_{\overline{\pi}}) \rightarrow I^\overline{\beta}(L_{\overline{\pi}})$, which yields (4.2) by (4.3).

5 Proof of the main results
5.1 Proof of Theorem 2.1

Let \( f \in L^\overline{\alpha}_p \), then \( f = Q^\overline{\alpha} \varphi \), \( \varphi \in L_\overline{\alpha} \), by Theorem 4.1. Relatively, \( f = R^\overline{\alpha} \varphi \).

We prove the following integral representation

\[
(T_{\varepsilon}^\overline{\alpha} R^\overline{\alpha} \varphi)(x) = \int_{\mathbb{R}^n} K(y) \varphi(x - \varepsilon^\lambda y) dy,
\]

where the kernel

\[
K(y) = \rho^{-n}(y) \int_{\rho(t) > 1/\rho(y)} \frac{(\Delta_t^{2t} \overline{\alpha} \varphi)(\frac{y}{\rho(y)})}{\rho^{n+\alpha}(t)} dt
\]

is averaging. We note that the representation (5.1) was proved in [6] for \( \varphi \in \Phi_0 \).

Since the symbol

\[
S_{\overline{\alpha},\varepsilon}(\xi) = (-1)^\varepsilon 2^{2\varepsilon} \int_{\rho(t) > \varepsilon} \frac{\sin(\frac{\xi t}{2})}{\rho^{n+\alpha}(t)} dt
\]

of the "truncated" integral (1.4) is a multiplier in \( \Psi_0 \) (see [3]), we conclude that the space \( \Phi_0 \) is invariant with respect to the operator \( T_{\varepsilon}^\overline{\alpha} \). Therefore

\[
\langle T_{\varepsilon}^\overline{\alpha} f, \omega \rangle = \langle f, T_{\varepsilon}^\overline{\alpha} \omega \rangle = \langle \varphi, R^\overline{\alpha} T_{\varepsilon}^\overline{\alpha} \omega \rangle = \langle \varphi, \int_{\mathbb{R}^n} K(y) \omega(x - \varepsilon^\lambda y) dy \rangle =
\]

\[
= \langle \int_{\mathbb{R}^n} K(y) \varphi(x - \varepsilon^\lambda y) dy, \omega \rangle, \quad \omega \in \Phi_0.
\]

Thus (5.1) is valid in the \( \Phi'_0 \)-sense.

Since two locally integrable functions, which agree in the \( \Phi'_0 \)-sense, may differ from each other only by a polynomial, we obtain (5.1) for almost all \( x \in \mathbb{R}^n \) (because both sides of (5.1) belong to the weighted spaces \( L_{1,\gamma} = \{ f(x) : \int_{\mathbb{R}^n} | f(x) | (1 + | x |)^\gamma dx < \infty \} \) for some \( \gamma > -n \).

Letting \( \varepsilon \to 0 \) in (5.1), in view of Theorem 5.1 we have

\[
T^\overline{\alpha} f = \varphi \in L_{\overline{\alpha}}, \quad \|T^\overline{\alpha} f\|_{\overline{\alpha}} = \|f\|_{Q^\overline{\alpha}(L_{\overline{\alpha}})}.
\]
Let now \( f \in L^r, T^\alpha f \in L^r \). The statement of theorem will follow from the relation \( f = Q^\alpha \varphi \), where \( \varphi = T^\alpha f \). For \( \omega \in \Phi_0 \) we have

\[
\langle \varphi, Q^\alpha \omega \rangle = \langle \lim_{\varepsilon \to 0} (T^\alpha_{\varepsilon} f, R^\alpha \omega) = \lim_{\varepsilon \to 0} \langle T^\alpha_{\varepsilon} f, R^\alpha \omega \rangle = \lim_{\varepsilon \to 0} \langle f, \int_{\mathbb{R}^n} K(y) \omega(x - \varepsilon^\lambda y) dy \rangle = \langle f, \omega \rangle. \quad (5.3)
\]

The second of equalities of this chain follows from the fact that the convergence in \( L^r \) implies that in \( \Phi'_0 \), the last one is justified by the Hölder inequality. Thus we have proved that

\[
\langle f, \omega \rangle = \langle \varphi, Q^\alpha \omega \rangle. \quad (5.4)
\]

In the case \( \sum_{i=1}^n \frac{1}{\alpha_i p_i} > 1 \) we have

\[
\langle f, \omega \rangle = \langle Q^\alpha \varphi, \omega \rangle
\]

hence, \( f = Q^\alpha \varphi \) for almost all \( x \in \mathbb{R}^n \). In the case \( \sum_{i=1}^n \frac{1}{\alpha_i p_i} \leq 1 \) the relation (5.4) itself means that \( f = Q^\alpha \varphi \). Now the application of Theorem 4.1 yields \( f \in L^r \). \( \square \)

**Corollary 5.1.** Let \( f \in L^r, \alpha_i > 0, 1 < p_i < \infty, 1 \leq r_i < \infty, i = 1, \ldots, n \). Then the following integral representation holds:

\[
(\Delta^m f)(x) = \int_{\mathbb{R}^n} (\Delta^m R^\alpha)(x-y)(T^\alpha f)(y)dy, \quad m > \max_j \alpha_j. \quad (5.5)
\]

**Corollary 5.2.** Let \( f \in L^r, \alpha_i > 0, 1 < p_i < \infty, 1 \leq r_i < \infty \) (\( i = 1, \ldots, n \)). Then

\[
\|\Delta^m f\|_p \leq c \rho^\alpha(t) \|T^\alpha f\|_p, \quad m > \max_j \alpha_j. \quad (5.6)
\]

We give the proofs of Corollaries 5.1 and 5.2 in Appendix.
5.2 Proof of Theorem 2.2

We first approximate the function $f \in L_{\overline{p},\overline{r}}$ by smooth functions using the averages (3.5) with a "very nice" kernel. Namely, we assume that $a(t) \in C_0^\infty$.

Then $\|f - f_\delta\|_{L_{\overline{p},\overline{r}}} \to 0$, as $\delta \to 0$. Therefore it remains to approximate $f \in C_\infty \cap L_{\overline{p},\overline{r}}$ by $C_0^\infty$-functions.

For this goal, we consider the function $\mu(x) \in C_0^\infty$ supported in the ball $\rho(x) < 2$ such that $0 \leq \mu(x) \leq 1$, $\mu(x) = 1$ if $\rho(x) < 1$, and $\mu(x) = 0$ in the case $\rho(x) \geq 2$. Let $\mu_N(x) = \mu(N^{-\lambda}x)$. We show that the smooth truncations $f_N(x) = \mu_N(x)f(x) (\in C_0^\infty)$ approximate $f(x)$ in the norm of $L_{\overline{p},\overline{r}}$, as $N \to \infty$. Let $\nu(x) = 1 - \mu(x)$, $\nu_N(x) = \nu(x/N)$. We have to prove that

$$\|T^{\overline{p}}(\nu_N f)\|_{\overline{p}} \to 0, \quad as \quad N \to \infty, \quad f \in C_\infty \cap L_{\overline{p},\overline{r}}, \quad (5.7)$$

where

$$(T^{\overline{p}}f)(x) = \int_{\mathbb{R}^n} \frac{\Delta_t^\ell f(x)}{\rho_n + \alpha^*(t)} \, dt, \quad 2\ell \geq \max_j \alpha_j. \quad (5.8)$$

It should be noted that the integral (5.8) converges absolutely for the functions $f(x)$ bounded in $\mathbb{R}^n$ together with all their partial derivatives up to the order $[\alpha^*] + 1$.

I. Proof of (5.7) in the case $\sum_{i=1}^n \frac{1}{\alpha_i p_i} > 1$.

We first suppose that $0 < \max \alpha_j < 1$ and take $\ell = 1$ in (5.8). Then

$$|T^{\overline{p}}(\nu_N f)(x)| \leq |\nu_N(x)(T^{\overline{p}}f)(x)| + 2 |(B_N f)(x)|, \quad (5.9)$$

where

$$(B_N f)(x) = \int_{\mathbb{R}^n} \frac{\nu_N(x - t) - \nu_N(x)}{\rho_n + \alpha^*(t)} f(x - t) \, dt.$$ 

In order to obtain (5.7), it suffices to verify that

$$\|B_N f\|_{\overline{p}} \to 0, \quad N \to \infty \quad (5.10)$$

(since $\|\nu_N(x)f(x)\|_{\overline{p}} \to 0$, as $N \to \infty$, for every $f(x) \in L_{\overline{p}}$).
We justify (5.10) for $f \in L_q$, where $\mathbf{q}$ is an arbitrary vector such that $\mathbf{q} \succ \mathbf{p}$ and equality (3.8) is fulfilled. This implies (5.10) in view of the imbedding $L_{\mathbf{q},\mathbf{r}} \subset L_{\mathbf{q}}$, which is valid by virtue of Theorems 4.1 and 3.3.

Owing to the uniform estimate

$$\|B_N f\|_{\mathbf{q}} \leq C \|f\|_{\mathbf{q}}$$

(5.11)

with the constant $C$ not depending on $N$ (see Appendix), in accordance with the Banach–Steinhaus theorem, it remains to verify (5.10) for $f \in C_0^\infty$ (since the class $C_0^\infty$ is dense in $L_{\mathbf{q}}$).

Assuming that $f(x) \in C_0^\infty$ is supported in the ball $\rho(x) < a$, we have

$$\|B_N f\|_{\mathbf{q}} \leq \frac{C}{N^{\theta}} + C N^{\alpha^*} \sum_{i=1}^n \frac{1}{\alpha_i p_i} \alpha^* \int_{\rho(t) > \frac{a N}{\rho^*}} \frac{dt}{\rho^{n+\alpha^*-\theta} t} \times$$

$$\times \left( \int_{\mathbb{R}^1} dx \cdots \left( \int_{\mathbb{R}^1} \frac{|f(N^\lambda \cdot x)|^{p_1} dx_1}{(1 + \rho(x))^{p_1 \theta}(1 + \rho(x) - t)^{p_1 \theta}} \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{1}{p_n}},$$

where $\theta = \alpha^*/\max_j \alpha_j$, $C_1$ being such that $\rho(x - t) \geq C_1 \rho(t) - \rho(x)$. Since $\rho(x - t) \geq C_1 \rho(t) - \frac{a}{N} > 0$, if $\rho(x) < \frac{a}{N}$ and $\rho(t) > \frac{2a}{C_1 N}$, we have

$$\|B_N f\|_{\mathbf{q}} \leq \frac{C}{N^{\theta}} + \frac{C}{N^{\alpha^*}} \int_{\mathbb{R}^1} \frac{dt}{\rho^{n+\alpha^*-\theta} t} \left( 1 + C_1 \rho(t) - \frac{a}{N_0} \right)^{\theta} \to 0,$$

as $N \to \infty$ ($N > N_0 > a$). Thus we have obtained the desired result in the case $0 < \max \alpha_j < 1$.

In order to prove (5.7) in the case $\max_j \alpha_j \geq 1$, we use the induction argument. It should be noted that such an idea is due to S.G. Samko (see [13]) in the isotropic case. Its realization in the most general anisotropic case of vector-valued $\mathbf{p}$, $\mathbf{r}$, and $\mathbf{q}$, treated here, is much more difficult. The following lemma, which enables one us to apply the induction argument, plays a crucial role in the proof of Theorem 2.2.

**Lemma 5.1.** Let $m = 1, 2, \ldots$ and $\sum_{i=1}^n \frac{1}{\alpha_i p_i} > 1$. If the class $C_0^\infty$ is dense in $L_{\mathbf{q},\mathbf{r}}$ for $0 < \max \alpha_j < m$, then it is also dense in $L_{\mathbf{q},\mathbf{r}}$ for $m \leq \max \alpha_j < m + 1$. 

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Proof. As above, we have to check (5.7) for $m \leq \max_j \alpha_j < m + 1$.

Owing to the evident formulas $(\Delta^2 \nu_N f)(x) = (\tilde{\Delta}^2 \nu_N f)(x + \ell t)$,

$$(\tilde{\Delta}^2 \nu_N f)(x) = \sum_{k=0}^{2\ell} C_{2\ell}^k \left( \tilde{\Delta}^k \nu_N \right)(x) \left( \tilde{\Delta}^{2\ell-k} f \right)(x - kt),$$

we have

$$T^\alpha(\nu_N f) = B_N f + \sum_{k=1}^{2\ell} C_{2\ell}^k B_{N,k} f,$$

where

$$(B_N f)(x) = \int \frac{\nu_N(x + \ell t) \left( \tilde{\Delta}^{2\ell} f \right)(x + \ell t)}{\rho_n + \alpha^*(t)} dt;$$

$$(B_{N,k} f)(x) = \int \frac{\left( \tilde{\Delta}^k \nu_N \right)(x + \ell t) \left( \tilde{\Delta}^{2\ell-k} f \right)(x + (\ell - k)t)}{\rho_n + \alpha^*(t)} dt,$$

$k = 1, 2, \ldots, 2\ell$.

Let us show that

$$\|B_{N,k} f\|_p \to 0, \quad as \quad N \to \infty, \quad k = 1, 2, \ldots, 2\ell, \quad (5.12)$$

and

$$\|B_N f\|_p \to 0, \quad as \quad N \to \infty. \quad (5.13)$$

In the case $k = 2\ell$ the relation (5.12) is proved for $f \in L_{\pi} \supset L_{\pi,\alpha}^\tau$ in just the same way, as (5.7) for $0 < \max_j \alpha_j < 1$.

In the cases $k = 1, 2, \ldots, 2\ell - 1$ we prove the stronger relation:

$$\|B_{N,k} f\|_p \to 0, \quad as \quad N \to \infty, \quad f \in L_{\pi,\alpha}^\tau, \quad (5.14)$$

the vectors $\gamma^k = (\gamma^k_1, \ldots, \gamma^k_n)$ and $\tau^k = (\tau^k_1, \ldots, \tau^k_n)$ being such that $\sum_{i=1}^n \frac{1}{\alpha_i \tau^k_i} = \sum_{i=1}^n \frac{1}{\alpha_i p_i} - \left(1 - \frac{(\gamma^k)^*_s}{\alpha^*_s}\right)$, $\tau^k_i > p_i$, and

$$0 < \max_i \gamma^k_i < m. \quad (5.15)$$
(Later we will put some additional restrictions on $\gamma^k_i$). Then (5.14) yields (5.12) by (4.2) for $f \in L^p_{\tau^k}$ and $\max_i \gamma^k_i < m \leq \max_i \alpha_i$. We first obtain the uniform estimate

$$\|B_{N,k}f\|_p \leq C \|f\|_{L^p_{\tau^k}}, \quad f \in L^p_{\tau^k}.$$  \hfill (5.16)

Making use of (3.3) and the Hölder inequality, we obtain

$$\|B_{N,k}f\|_p \leq \frac{C}{N^{k\theta}} \int_{\mathbb{R}^n} \left\|\tilde{\Delta}^{2l-k} f\right\|_p dt \times$$

$$\times \left(\int_{\mathbb{R}^1} dx_1 \cdots \left(\int_{\mathbb{R}^1} dx_1 \prod_{i=0}^k \left(1 + \frac{\rho(x + (l-i)t)}{N} \theta \xi^k_i\right)\right) \cdots \right)^{1/\xi^k_n},$$

where $\xi^k_i = \frac{p_i \gamma^k_i}{\tau^k_i - p_i}$. Applying Lemma 3.2 under the additional condition $2l - k > \max_j \gamma^k_j$, we have

$$\|\Delta^{2l-k} f\|_{\tau^k} \leq c \rho^{(\gamma^k)^*}(t) \|T^{\gamma^k} f\|_{\tau^k}, \quad f \in L^p_{\tau^k},$$

where

$$\sum_{i=1}^n \frac{1 + j_i}{\gamma^k_i} \neq 1, \quad |j| = 0, 1, \ldots, \left[\max_i \gamma^k_i \left(1 - \sum_{i=1}^n \frac{1}{\gamma^k_i}\right)\right] - 1. \hfill \text{(5.17)}$$

Besides this, we assume that

$$\gamma^k_i < \frac{(2l - k)\alpha_i}{\max_j \alpha_j}, \quad i = 1, \ldots, n, \hfill \text{(5.18)}$$

then

$$\|B_{N,k}f\|_p \leq C \|T^{\gamma^k} f\|_{\tau^k} \int_{\mathbb{R}^n} \frac{dt}{\rho^{n + \alpha^* - k\theta - (\gamma^k)^*}} \times$$

$$\times \left(\int_{\mathbb{R}^1} dx_1 \cdots \left(\int_{\mathbb{R}^1} dx_1 \prod_{i=0}^k \left(1 + \rho(x + (l-i)t)^{\theta \xi^k_i}\right)\right) \cdots \right)^{1/\xi^k_n}.$$
In order to apply Lemma 3.1, we also put the following restrictions on $\gamma_k^k$:

$$(\gamma^k)^* > \alpha^* - \theta k. \quad (5.19)$$

Then Lemma 3.1 yields

$$\|B_{N,k}f\|_p \leq C \|T_{\gamma^k}^k f\|_{\gamma^k}. \quad (5.20)$$

Taking into account (5.15), (5.18), and (5.19), we arrive at the following inequalities for the coordinates $\gamma^k_i$:

$$\max \left( 0, \alpha_i - \frac{\alpha_i k}{\max_j \alpha_j} \right) < \gamma^k_i < \min \left( \frac{m \alpha_i}{\max_j \alpha_j}, \frac{(2\ell - k)\alpha_i}{\max_j \alpha_j} \right). \quad (5.21)$$

Since $\max_j \alpha_j < 2\ell$ and $m \leq \max_j \alpha_j < m + 1$, this interval is not empty.

Thus we have proved (5.16). In accordance with the Banach–Steinhaus theorem, it suffices to verify (5.14) on a dense set in $L_{\gamma^k,\gamma^k}$. Since $0 < \max_i \gamma^k_i < m$, the class $C^\infty_0$ is dense in $L_{\gamma^k,\gamma^k}$ according to the assumption of induction. For $f(x) \in C^\infty_0$, in view of (3.3) we have

$$\|B_{N,k}f\|_p \leq C \int_{\mathbb{R}^n} \frac{dt}{\rho^{n+\alpha^*-2\ell \theta}(t)(1 + N \rho(t))^{(2\ell - k)\theta}} \times$$

$$\times \left( \int_{\mathbb{R}^1} \frac{dx_1}{(1 + \rho(x - it))^{\theta \xi_{1,k}^k}} \prod_{i=0}^k (1 + \rho(x - it))^{\theta \xi_{i,k}^k} \right)^{1/\xi_{n,k}^k}.$$

Applying the Hölder inequality to the inner integral, we get

$$\|B_{N,k}f\|_p \leq \frac{c}{N^{(\gamma^k)^* - (2\ell - k)\theta + d}} \int_{\mathbb{R}^n} \frac{dt}{\rho^{n+\alpha^*-2\ell \theta+d}(t)(1 + \rho(t))^{(k+1)\theta - \alpha^* + (\gamma^k)^*}}, \quad (5.22)$$

where $0 \leq d \leq (2\ell - k)\theta$. The integral on the right-hand side converges, if we take $d \in ((2\ell - k)\theta - (\gamma^k)^*, \min\{(2\ell - k)\theta, 2\ell \theta - \alpha^*\})$ (this interval is not empty in view of (5.19)). Then the right-hand side of (5.22) tends to zero, as $N \to \infty$. 

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Let us prove (5.13). We have
\[(B_N f)(x) = \nu_N(x) (T^\alpha f)(x) + (M_N f)(x),\]
where
\[(M_N f)(x) = \int_{\mathbb{R}^n} \frac{\nu_N(x + \ell t) - \nu_N(x)}{\rho^{n+\alpha^*}(t)} \left(\tilde{\Delta}_t^\alpha f\right)(x + \ell t) \, dt. \tag{5.23}\]
The validity of (5.13) follows from the relation
\[\|M_N f\|_p \to 0, \quad \text{as } N \to \infty, \quad f \in L_{\overline{\gamma}^1}.\]
which is verified in just the same way, as (5.12).

II. The case \[\sum_{i=1}^{n} \frac{1}{\alpha_i p_i} \leq 1.\]

In the case \(0 < \max_j \alpha_j < 1\), as above, we have (5.9).

Let \(p_j < r_j\) for \(j = m_1^1, \ldots, m_{k_1}^1\) and \(p_j \geq r_j\) for \(j = m_1^2, \ldots, m_{k_2}^2\), and let \(\overline{\gamma}^1\) be the vector with the following coordinates: \(r_j^1 = r_j\) for \(j = m_1^1, \ldots, m_{k_1}^1\) and \(r_j^1 = p_j\) for \(j = m_1^2, \ldots, m_{k_2}^2\). We observe that \(f \in L_{\overline{\gamma}^1}\) by virtue of (3.6). Similarly to (5.10) we prove that
\[\|B_N f\|_{\overline{\gamma}^1} \to 0, \quad \text{as } N \to \infty, \quad f \in L_{\overline{\gamma}^1}. \tag{5.24}\]

Making use of the Minkowsky and Hölder inequalities, we arrive at the estimate
\[\|B_N f\|_{\overline{\gamma}^1} \leq N \sum_{j=m_1^1}^{m_{k_1}^1} \frac{1}{r_j - p_j} \|f\|_{\overline{\gamma}^1} \int_{\mathbb{R}^n} \frac{\|\nu(x - t) - \nu(x)\|_{\overline{\gamma}^1}}{\rho^{n+\alpha^*}(t)} \, dt,\]
where \(s_j = \frac{p_j r_j}{r_j - p_j}\) for \(j = m_1^1, \ldots, m_{k_1}^1\) and \(s_j = \infty\) for \(j = m_1^2, \ldots, m_{k_2}^2\).

Since \(\alpha^* - \alpha^* \sum_{j=m_1^1}^{m_{k_1}^1} \frac{1}{\alpha_j p_j} + \alpha^* \sum_{j=m_1^1}^{m_{k_1}^1} \frac{1}{\alpha_j r_j} > 0\) and the integral on the right-hand side converges, we obtain (5.24).

In order to extend the result on denseness of \(C_0^\infty\) in \(L_{\overline{\gamma}^1}\) to the case \(\max_j \alpha_j \geq 1\), we use the following analogue of Lemma 5.1.
Lemma 5.2. Let $m = 1, 2, \ldots$, $1 < p_i < \infty$, $1 \leq r_i < \infty$, $\alpha_i > 0$, and 
\[ \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} \leq 1. \]
If $C_0^\infty$ is dense in $L_{\overline{p}, \overline{r}}$ for $0 < \max_j \alpha_j < m$, then it is dense in $L_{\overline{p}, \overline{r}}$ for $m \leq \max_j \alpha_j < m + 1$.

The proof of this lemma is much in lines with that of Lemma 5.1. The only difference is as follows. To prove (5.12), we have to use the assumption of induction only in the case \[ \max_j \alpha_j - \sum_{j=m_i}^{m_i} \frac{1}{\alpha_j p_j} + \sum_{j=m_i}^{m_i} \frac{1}{\alpha_j r_j} \leq 0 \]
and $k = 1, \ldots, [\max_j \alpha_j]$. As in the proof of Lemma 5.1, we base ourselves on the imbedding (4.2). Namely, we choose $\beta_i$ ($i = 1, \ldots, n$) such that $\alpha^* - \beta^* < \min \{n, \theta\}$, $0 < \max_j \beta_j < m$, and 
\[ \frac{1}{\alpha_i p_i} - \frac{1}{\alpha_i r_i} \neq 1, \]
where the components of $\zeta$ ($\zeta > p_i$, $i = 1, \ldots, n$) satisfy the equality 
\[ \sum_{i=1}^{n} \frac{1}{\alpha_i p_i} = \sum_{i=1}^{n} \frac{1}{\alpha_i r_i} - \left(1 - \frac{\beta^*}{\alpha^*}\right). \]
As regards the rest values of $k$, the relation (5.12) is proved by the direct estimation of the norm $\|B_{N,K} f\|_{\overline{p}}$. We leave the corresponding details to the reader.

6 Appendix

6.1 Proof of equality (5.5).

Let $f \in L_{\tilde{p}, \tilde{r}}$. We denote $\varphi_\varepsilon = T_{\varepsilon}^0 f$, then 
\[ \int_{\mathbb{R}^n} (\Delta_m^l \mathcal{R}_\alpha)(x - y) \varphi_\varepsilon(y) dy = \int_{\mathbb{R}^n} (\Delta_m^l f)(x - \varepsilon y) \mathcal{K}(y) dy, \]
where $\mathcal{K}(y)$ is the kernel (5.2). Let us pass to the limit, as $\varepsilon \to 0$, in (6.1). Making use of the estimate 
\[ (\Delta_m^l \mathcal{R}_\alpha)(y) \leq c \rho \frac{\alpha^* - n - \varepsilon}{\rho \alpha^*} \sum_{j=1}^{\max_j \alpha_j} (y), \rho(y) > A, \eta_t = t/\rho^\lambda(t) \]
(see [4]), we arrive at the relation $(\Delta_m^l \mathcal{R}_\alpha)(y) \in L_1$. Therefore the left-hand side of (6.1) converges in $L_{\tilde{p}}$, as $\varepsilon \to 0$, to $T\overline{p} f$, while the right-hand side
converges in $L_\rho$ to $\Delta^m_{\rho}f$, since $f(x) \in L_\rho$ and $\mathcal{K}(y) \in L_1$. This implies (5.3).

6.2 Proof of inequality (5.6).

With the aid of (5.5) we have

$$\|\Delta^m_{\rho}f\|_\rho \leq \rho^\alpha(t)\|T^{\alpha}\rho f\|_\rho \int_{\mathbb{R}^n} |(\Delta^m_{\eta(t)}\mathcal{R}_\alpha)(y)| dy,$$

where $\eta_t = t/\rho^\lambda(t)$. Since the integral on the right-hand side is finite, we obtain (5.6).

6.3 Proof of (5.11)

The application of (3.3) yields

$$|(B_N f)(x)| \leq \frac{C}{N^\theta} \int_{\mathbb{R}^n} \frac{\rho^\theta(t)|f(x-t)| dt}{(1 + \rho(x))^\theta(1 + \rho(x-t))^\theta \rho^{n+\alpha}(t)},$$

where $\theta = \frac{\alpha^*}{\max_j \alpha_j}$. Applying the Minkowsky and Hölder inequalities, we obtain

$$\|B_N f\|_\rho \leq C \|f\|_q \int_{\mathbb{R}^n} \frac{dt}{\rho^{n+\alpha-\theta}(t)} \times$$

$$\times \left( \int_{\mathbb{R}^1} \ldots \left( \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^1} \frac{dx_1}{(1 + \rho(x))^{\theta \beta_1}(1 + \rho(x-t))^{\theta \beta_1}} \right)^{\frac{\beta_2}{\beta_1}} \right)^{\frac{\beta_3}{\beta_2}} \ldots dx_n \right)^{\frac{1}{\beta_n}},$$

where $\beta_i = \frac{p_i q_i}{q_i - p_i}$, $i = 1, \ldots, n$. Since $\sum_{i=1}^n \frac{1}{\alpha_i \beta_i} < \frac{\theta}{\alpha^*}$, we arrive at (5.11) by virtue of (3.2).

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