CAMPANA POINTS ON BIEQUIVARIANT
COMPACTIFICATIONS OF THE HEISENBERG
GROUP

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Abstract. We study Campana points on biequivariant compactifications of the Heisenberg group and confirm the log Manin conjecture introduced by Pieropan, Smeets, Tanimoto and Várilly-Alvarado.

1. Introduction

Manin’s conjecture concerns the distribution of rational points of bounded height on Fano varieties over number fields and has been extensively studied. It was initially proposed by Franke, Manin and Tschinkel [17], later on the modern and more appropriate formulations of Manin’s conjecture were made in [2, 5, 22, 23]. Let $X$ be a Fano projective variety defined over a number field $F$ and $L$ an ample line bundle on $X$. Let $H_L$ be a height function

$$H_L : X(F) \to \mathbb{R}_{>0}$$

where $\mathcal{L}$ is an adelically metrized line bundle associated to $L$. Manin’s conjecture states that there is a subset $U$ of $X(F)$ such that the counting function

$$N(U, \mathcal{L}, T) := \#\{x \in U \mid H_L(x) \leq T\}$$

satisfies the asymptotic formula

$$N(U, \mathcal{L}, T) \sim c T^{a(X,L)} (\log T)^{b(X,F,L)-1}$$

as $T \to \infty$, where $c$ is a positive constant, $a(X,L)$ and $b(X,F,L)$ are certain geometric invariants stated by [2]. Manin’s conjecture for homogeneous spaces has been studied for example in [3, 4, 12, 18–20, 26, 28, 31], it has also been extensively studied for various varieties, such as Del Pezzo surfaces and so on, we refer the readers to excellent surveys [7, 33] and the references therein. Besides the distribution of
rational points, the problem of counting integral points has also been considered by [6, 14–16, 29, 30].

Let

\[ G = \left\{ g = g(x, z, y) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \]

be the Heisenberg group. A smooth projective variety \( X \) is called a biequivariant compactification of \( G \) if \( G \) is a dense Zariski open subset of \( X \) and \( X \) carries both left and right \( G \)-actions. A trivial example of biequivariant compactification of \( G \) is the projective space \( \mathbb{P}^3 \). For a general construction one may take \( X \) to be the Zariski closure of an orbit of a projective representation of \( G \), for details see [27].

Campana orbifolds and Campana points were introduced by Campana [10, 11], the distribution of Campana points over a number field was initiated in for example [8, 9, 34]. Recently, M. Pieropan, A. Smeets, S. Tanimoto and A. Várilly-Alvarado [25] initiated a systematic quantitative study of Campana points. In their paper, Pieropan et al [25] studied the distribution of Campana points of bounded height on equivariant compactifications of vector groups and posed a log version of Manin’s conjecture. Pieropan and Schindler [24] recently developed the hyperbola method to study the distribution of Campana points on toric varieties over \( \mathbb{Q} \).

Let us briefly review the notion of Campana points and the log Manin conjecture studied in Pieropan et al [25]. We begin by recalling some basics of Campana orbifolds and Campana points.

1.1. Campana orbifolds. The notion of Campana orbifolds was introduced in Campana [10, 11] and Abramovich [1]. Pieropan et al [25] applied the Campana orbifolds of smooth type to compactifications of vector groups. In this paper we shall discuss the Campana orbifolds in the sense of [25], through the paper we let \( F \) be a number field.

Let \( X \) be a smooth variety and \( D_\varepsilon \) an effective Weil \( \mathbb{Q} \)-divisor on \( X \), a pair \((X, D_\varepsilon)\) is called a Campana orbifold over \( F \) if

\[ D_\varepsilon = \sum_{\alpha \in \mathcal{A}} \varepsilon_\alpha D_\alpha \]

where \( D_\alpha \)'s are prime divisors,

\[ \varepsilon_\alpha \in \mathfrak{W} := \left\{ 1 - \frac{1}{m} \middle| m \in \mathbb{Z}_{\geq 1} \right\} \cup \{1\} \]

for all \( \alpha \in \mathcal{A} \), and the support \( D_{\text{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha \) is a divisor with strict normal crossings.
One sees that any Campana orbifold \((X, D_\varepsilon)\) is a dlt (divisorial log terminal) pair, i.e., \(\varepsilon_\alpha \leq 1\). If moreover \(\varepsilon_\alpha \neq 1\) for all \(\alpha \in \mathcal{A}\), we say that \((X, D_\varepsilon)\) is klt (Kawamata log terminal). For the definitions of dlt and klt one can see [21].

Let \(\text{Val}(F)\) be the set of all places and \(\text{Val}(F)_\infty\) all the archimedean places of the field \(F\). For \(v \in \text{Val}(F)\), we denote the completion of \(F\) at \(v\) by \(F_v\). If \(v\) is a nonarchimedean place, we denote the corresponding ring of integers by \(\mathcal{O}_v\) with the maximal ideal \(m_v\) and the residue field \(k_v\). We denote the ring of adeles of \(F\) by \(A_F\) and fix a finite set \(S\) of places of \(F\) containing all archimedean places.

**Definition 1.1.** [25, §3] We say \((\mathcal{X}, \mathcal{D}_\varepsilon)\) is a **good integral model** of \((X, D_\varepsilon)\) away from \(S\) if \((\mathcal{X}, \mathcal{D}_\varepsilon)\) is a flat, regular and proper model over \(\mathcal{O}_{F,S}\) where \(\mathcal{D}_\varepsilon\) is the Zariski closure of \(D_\varepsilon\) in \(\mathcal{X}\).

**1.2. Campana points.** Let us fix a good integral model \((\mathcal{X}, \mathcal{D}_\varepsilon)\) for \((X, D_\varepsilon)\) over \(\mathcal{O}_{F,S}\) and let \(\mathcal{A}_\varepsilon = \{\alpha \in \mathcal{A} : \varepsilon_\alpha \neq 0\}\) and \(X^\circ = X \setminus (\bigcup_{\alpha \in \mathcal{A}_\varepsilon} D_\alpha)\), if \(P \in X^\circ(F)\) and \(v \notin S\) then there is an induced point \(\mathcal{P}_v \in \mathcal{X}(\mathcal{O}_v)\). Let \(\alpha \in \mathcal{A}\) be such that \(\mathcal{P}_v \notin \mathcal{D}_\alpha\), we denote the co-length of the ideal defined by the pullback of \(\mathcal{D}_\alpha\) via \(\mathcal{P}_v\) by \(n_v(\mathcal{D}_\alpha, P)\) and call it intersection multiplicity of \(P\) and \(\mathcal{D}_\alpha\) at \(v\).

Following [25], a point \(P \in X^\circ(F)\) is called a **Campana \(\mathcal{O}_{F,S}\)-point** on \((\mathcal{X}, \mathcal{D}_\varepsilon)\) if for all places \(v \notin S\), \(n_v(\mathcal{D}_\alpha, P) = 0\) whenever \(\alpha \in \mathcal{A}_\varepsilon\) satisfies \(\varepsilon_\alpha = 1\) and

\[
n_v(\mathcal{D}_\alpha, P) \geq \frac{1}{1 - \varepsilon_\alpha}.
\]

whenever \(\alpha \in \mathcal{A}_\varepsilon\) satisfies \(\varepsilon_\alpha < 1\) and \(n_v(\mathcal{D}_\alpha, P) > 0\). So when writing \(\varepsilon_\alpha = 1 - \frac{1}{m_\alpha}\), \(n_v(\mathcal{D}_\alpha, P) \geq m_\alpha\) whenever \(n_v(\mathcal{D}_\alpha, P) > 0\). We denote by \((\mathcal{X}, \mathcal{D}_\varepsilon)(\mathcal{O}_{F,S})\) the set of all Campana \(\mathcal{O}_{F,S}\)-points on \((\mathcal{X}, \mathcal{D}_\varepsilon)\).

**1.3. A log Manin conjecture.** Let \((X, D_\varepsilon)\) be a **Fano orbifold** over a number field \(F\), that is, \((X, D_\varepsilon)\) is a Campana orbifold with \(-(K_X + D_\varepsilon)\) being ample where \(K_X\) is the canonical divisor of \(X\). Assume that \((X, D_\varepsilon)\) is klt. Let

\[
H_\mathcal{L} : X(F) \to \mathbb{R}_{>0}
\]

be the height function associated with the adelically metrized big divisor \(\mathcal{L} = (L, \| \cdot \|)\) on \(X\). Let \(U \subset X(F)\) and \(T > 0\), we denote the counting function by

\[
N(U, \mathcal{L}, T) = \#\{ P \in U | H_\mathcal{L}(P) \leq T \}.
\]

**Conjecture 1.2** (Log Manin conjecture). Suppose that the divisor \(L\) is big and nef and that the set of klt Campana points \((\mathcal{X}, \mathcal{D}_\varepsilon)(\mathcal{O}_{F,S})\) is
not thin in the sense of [25, Definition 3.6], then there is an exceptional set \( Z \subset (\mathcal{X}, D_\varepsilon)(\mathcal{O}_{E, S}) \) such that as \( T \to \infty \)

\[
N((\mathcal{X}, D_\varepsilon)(\mathcal{O}_{E, S}) \setminus Z, \mathcal{L}, T) \sim c T^a (\log T)^{b-1}
\]

where \( a, b, c \) are constants described in [25].

Pieropan et al [25, Theorem 1.2] proved the conjecture above for equivariant compactifications of vector groups, they also discussed dlt cases [25, Theorem 1.4].

1.4. Main results in this paper. It is shown in [27] that Manin’s conjecture is true for biequivariant compactifications of the Heisenberg group. Following the spirit of Pieropan et al [25], we study Campana points on biequivariant compactifications of the Heisenberg group over \( \mathbb{Q} \) and confirm the above log Manin conjecture for this case.

**Theorem 1.3.** Let \( X \) be a smooth projective biequivariant compactification of the Heisenberg group \( G \) over \( \mathbb{Q} \) such that the boundary divisor \( D = X \setminus G \) is a strict normal crossings divisor on \( X \), let \( S \) be a finite set of places of \( \mathbb{Q} \) containing the archimedean place, assume that \( (X, D_\varepsilon) \) is klt and has a good integral model away from \( S \) and assume Assumption \( \theta \) see in \( \theta \). If \( aL + K_X + D_\varepsilon \) is rigid (i.e., its Iitaka dimension is 0), then Conjecture \( \theta \) holds for \( (\mathcal{X}, D_\varepsilon, \mathcal{L}) \) with exceptional set \( Z = (X \setminus G) \cap (\mathcal{X}, D_\varepsilon)(\mathbb{Z}_S) \).

**Theorem 1.4.** Let \( \mathcal{X}, D, \varepsilon, S \) be as above theorem and let \( L = -(K_X + D_\varepsilon) \). Assume that \( (X, D_\varepsilon) \) is dlt and has a good integral model away from \( S \) and assume Assumption \( \theta \) see in \( \theta \), set

\[
G(\mathbb{Q})_\varepsilon = G(\mathbb{Q}) \cap (\mathcal{X}, D_\varepsilon)(\mathbb{Z}_S).
\]

Then as \( T \to \infty \), there are constants \( a, b, c \) such that

\[
N(G(\mathbb{Q})_\varepsilon, \mathcal{L}, T) \sim \frac{c}{(b-1)!} T (\log T)^{b-1}.
\]

To prove Theorems 1.3 and 1.4, we use the height zeta function method. As we shall see later the height zeta function is defined to be

\[
Z_\varepsilon(s, g) = \sum_{\gamma \in G(\mathbb{Q})} \delta_\varepsilon(g) H(s, \gamma g)^{-1},
\]

where \( \delta_\varepsilon(g) \) is the indicator function. We consider the representation theory of the Heisenberg group in the adelic setting [27] and the spectral decomposition of a certain representation space:

\[
Z_\varepsilon(s, g) = Z_{0, \varepsilon}(s, g) + Z_{1, \varepsilon}(s, g) + Z_{2, \varepsilon}(s, g),
\]
where
\[
Z_{0,\varepsilon}(s, \text{id}) = \int_{G(A_Q)} H(s, g)^{-1}\delta_\varepsilon(g)dg,
\]
\[
Z_{1,\varepsilon}(s, g) = \sum_\eta \eta(g) \int_{G(A_Q)} H(s, g)^{-1}\eta(g)\delta_\varepsilon(g)dg.
\]
\[
Z_{2,\varepsilon}(s, g) = \sum_\psi \sum_\omega \omega^\psi(g) \int_{G(A_Q)} H(s, g)^{-1}\omega^\psi(g)\delta_\varepsilon(g)dg.
\]

For details of the above spectral decomposition see Lemma 3.3. We are going to obtain a meromorphic continuation of the function \(Z_\varepsilon(s, g)\) and then apply Tauberian theorems to derive our results. The way treating \(Z_{0,\varepsilon}(s, g)\) and \(Z_{1,\varepsilon}(s, g)\) is essentially analogous to that of [25], for \(Z_{2,\varepsilon}(s, g)\), we will use theta distribution and Schwartz-Bruhat function described in [27] as a tool to compute and estimate it.

In this paper for simplicity we only consider Heisenberg group over \(\mathbb{Q}\), we hope in the future to treat general unipotent groups over a general number field using the orbit method developed in Shalika-Tschinkel [28].

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2. Geometry of biequivariant compactifications of the Heisenberg group

In this section we recall some basic facts on the geometry of biequivariant compactifications of the Heisenberg group from Shalika and Tschinkel [27]. Hereafter for simplicity we suppose \(F = \mathbb{Q}\), let
\[
G = \left\{ g = g(x, z, y) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}
\]
be the Heisenberg group over \(\mathbb{Q}\). Let \(X\) be a smooth projective biequivariant compactification of \(G\) with boundary \(D = X\setminus G\) consisting of irreducible components \(D = \bigcup_{\alpha \in A} D_\alpha\), with strict normal crossings. We write the anticanonical divisor of \(X\) as \(-K_X = \sum_{\alpha \in A} \kappa_\alpha D_\alpha\) and denote the cone of effective divisors on \(X\) by \(\Lambda_{\text{eff}}(X)\).

Proposition 2.1 ([27], Proposition 1.5). With the notations above, then we have
(1) \[ \text{Pic}(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z}D_{\alpha}, \]

(2) \[ \Lambda_{\text{eff}}(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{R}_{\geq 0}D_{\alpha}, \]

(3) \[ \kappa_{\alpha} \geq 2 \text{ for all } \alpha \in \mathcal{A}. \]

**Corollary 2.2** ([27], Corollary 1.7). The divisor of every non-constant function \( f \in F[G] \) can be written as

\[
\text{div}(f) = E(f) - \sum_{\alpha \in \mathcal{A}} d_{\alpha}(f)D_{\alpha}
\]

where \( E(f) \) is the divisor of \( \{ f = 0 \} \) in \( G \) and \( d_{\alpha}(f) \geq 0 \) for all \( \alpha \). Moreover, there is at least one \( \alpha \in \mathcal{A} \) such that \( d_{\alpha}(f) > 0 \).

**Notation 2.3.** We introduce coordinates on \( \text{Pic}(X)_{\mathbb{C}} \) using the basis \((D_{\alpha})_{\alpha \in \mathcal{A}}\): a vector \( s = (s_{\alpha}) \) corresponds to \( \sum_{\alpha \in \mathcal{A}} s_{\alpha}D_{\alpha} \).

### 3. Height zeta functions and representation theory of the Heisenberg group

In this section we recall some basic properties of Height zeta functions and review the representation theory of the Heisenberg group in the adelic setting. Let \( G \) be the Heisenberg group over \( \mathbb{Q} \) and let \( X \) be a smooth projective bi equivariant compactification of \( G \) defined over \( \mathbb{Q} \). We mainly refer to Shalika and Tschinkel [27].

#### 3.1. Height functions.

Let us consider the decomposition of the boundary into irreducible components:

\[ D = X \setminus G = \bigcup_{\alpha \in \mathcal{A}} D_{\alpha}. \]

We fix a smooth adelic metrization \( (\| \cdot \|_v)_{v \in \text{Val}(\mathbb{Q})} \) for each line bundle \( \mathcal{O}(D_{\alpha}) \). Let \( f_{\alpha} \) be a section corresponding to \( D_{\alpha} \), for each place \( v \), the local height pairing is defined by

\[
H_v : G(\mathbb{Q}_v) \times \text{Pic}(X)_{\mathbb{C}} \to \mathbb{C}^\times, \quad \left( g_v, \sum_{\alpha \in \mathcal{A}} s_{\alpha}D_{\alpha} \right) \mapsto \prod_{\alpha \in \mathcal{A}} \| f_{\alpha}(g_v) \|_v^{-s_{\alpha}},
\]

and the global height pairing is

\[
H = \prod_{v \in \text{Val}(\mathbb{Q})} H_v : G(\mathbb{A}_\mathbb{Q}) \times \text{Pic}(X)_{\mathbb{C}} \to \mathbb{C}^\times.
\]

We have the following properties.
Lemma 3.1 ( [27], Proposition 2.3). With the notations above, the height pairing is linear in the Pic($X$) component:

$$H(g, s + s') = H(g, s)H(g, s')$$

for all $s, s' \in \text{Pic}(X)_C$, all $g \in G(A_\mathbb{Q})$ and there is a compact open subgroup

$$K = \prod_v K_v \subset G(A_{\text{fin}})$$

such that for all $v \in \text{Val}(\mathbb{Q})_{\text{fin}}$, one has $H_v(k_v g_v k'_v, s) = H_v(g_v, s)$ for all $s \in \text{Pic}(X)_C$, $k_v, k'_v \in K_v$ and $g_v \in G(Q_v)$.

Moreover, if

1. there is a smooth projective $\mathbb{Z}_v$-model $X$ for $X$ which comes equipped with an action of the $\mathbb{Z}_v$-group scheme $G_{\mathbb{Z}_v}$ extending the given action of $G$ on $X$,
2. the metric $\| \cdot \|_v$ is induced by the integral model $(X, D)$,
3. the unique linearisation on $O(D_\alpha)$ extends to $O(D_\alpha)$ for every $\alpha \in A$,

then we can take $K_v = G(\mathbb{Z}_v)$.

In particular we can take

$$K = \prod_{p \notin S'} G(Z_p) \cdot \prod_{p \in S'} G(p^{n_p} Z_p)$$

where $S'$ is a finite set of primes and $n_p$ are positive integers.

The height zeta function of $G$ associated to the height pairing is defined to be

$$Z_\varepsilon(s, g) = \sum_{\gamma \in G(\mathbb{Q})} \delta_\varepsilon(g) H(s, \gamma g)^{-1}. \tag{3.1}$$

The height zeta function $Z_\varepsilon(s, g)$ is holomorphic in $s$ and continuous in $g$ when $\Re(s) \gg 0$ by [27, Proposition 2.6].

3.2. Representation theory of the Heisenberg group. Here we recall the representation theory of the Heisenberg group $G$ from Shalika and Tschinkel [27, §3].

First of all we introduce some necessary notations. We denote by $Z$ the center of $G$ and by $G^{ab} = G/Z$ the abelianization of $G$. Let $U \subset G$ be the subgroup

$$U := \{ u \in G | u = (0, z, y) \},$$

and

$$W := \{ w \in G | w = (x, 0, 0) \}. $$
For the compact open subgroup
\[ K := \prod_{p \not\in S'} G(\mathbb{Z}_p) \cdot \prod_{p \in S'} G(\mathfrak{p}^{n_p} \mathbb{Z}_p), \]
we put
\[ n(K) = \prod_{p \in S'} p^{n_p}, \]
and denote
\[ K^{ab} := K \cap \mathcal{G}^{ab}, \]
\[ K_Z := K \cap \mathbb{Z}. \]

Let \( v \) be a place of \( \mathbb{Q} \), through the paper we define the local zeta function by
\[ \zeta_{\mathbb{Q}_v}(s) = \begin{cases} 
s^{-1} & \text{if } \mathbb{Q}_v = \mathbb{R} \text{ or } \mathbb{C}, \\
\frac{1}{1 - p^{-s}} & \text{if } v = p \text{ is nonarchimedean}. \end{cases} \]

**Notation 3.2.** Through the paper, for \( s = (s_1, \cdots, s_n) \in \mathbb{C}^n \) and \( c \in \mathbb{R} \), by \( \mathbb{R}(s) > c \) we mean that \( \mathbb{R}(s_i) > c \) for all \( i \in \{1, \cdots, n\} \). For \( c \in \mathbb{R} \) we denote the tube domain
\[ (3.2) \quad T > c = \{ s \in \text{Pic}(X)_\mathbb{C} : \mathbb{R}(s) > \kappa_\alpha - \varepsilon_\alpha + c, \alpha \in \mathcal{A} \}. \]

We denote the Haar measure on \( G(\mathbb{A}_Q) \) by \( dg = \prod_p dg_p \cdot dg_\infty \) where \( dg_\infty = dx_\infty dy_\infty dz_\infty \) and \( dg_p = dx_p dy_p dz_p \) with the normalizations
\[ \int_{\mathbb{Z}_p} dx_p = 1, \int_{\mathbb{Z}_p} dy_p = 1, \int_{\mathbb{Z}_p} dz_p = 1 \]
and \( dx_\infty, dy_\infty, dz_\infty \) are usual Lebesgue measures on \( \mathbb{R} \). We denote \( du_p = dy_p dz_p \) (resp. \( du_\infty, du \) for the Haar measure on \( U(\mathbb{Q}_p) \) (resp. \( U(\mathbb{R}), U(\mathbb{A}_Q) \)) and \( dk_p \) for the Haar measure on \( K_p \) with the normalization \( \int_{K_p} dk_p = 1 \).

Let \( \rho \) be the right regular representation of \( G(\mathbb{A}_Q) \) on the Hilbert space
\[ \mathcal{H} := L^2 \left( G(\mathbb{Q}) \backslash G(\mathbb{A}_Q) \right). \]
By Peter-Weyl theorem there is a decomposition
\[ \mathcal{H} = \bigoplus \mathcal{H}_\psi \]
where
\[ \mathcal{H}_\psi = \{ \varphi \in \mathcal{H} : \rho(z)(\varphi)(g) = \psi(z)\varphi(g) \} \]
and \( \psi \) is over the set of unitary characters of \( Z(\mathbb{A}_Q) \) which are trivial on \( Z(\mathbb{Q}) \). For nontrivial \( \psi \), the corresponding representation \( (\rho_\psi, \mathcal{H}_\psi) \) of \( G(\mathbb{A}_Q) \) is nontrivial, irreducible and unitary. When \( \psi \) is the trivial character, the corresponding representation \( \rho_0 \) decomposes as a direct sum of one dimensional representations \( \rho_\eta \)
\[ \mathcal{H}_0 = \bigoplus_\eta \mathcal{H}_\eta \]
where \(\eta\) runs over all unitary characters of the group \(G^{ab}(\mathbb{Q})\backslash G^{ab}(\mathbb{A}_\mathbb{Q})\).

One considers \(\eta\) as a function on \(G(\mathbb{A}_\mathbb{Q})\), trivial on the \(Z(\mathbb{A}_\mathbb{Q})\) cosets, namely, let \(\psi_1 = \prod_p \psi_{1,p} \cdot \psi_{1,\infty}\) be the Tate character, for \(a = (a_1, a_2) \in \mathbb{A}_\mathbb{Q} \oplus \mathbb{A}_\mathbb{Q}\), consider the corresponding linear form on \(G^{ab}(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_\mathbb{Q} \oplus \mathbb{A}_\mathbb{Q}\) given by
\[
g(x, z, y) \mapsto a_1 x + a_2 y
\]
and denote by \(\eta = \eta_a\) the corresponding adelic character
\[
\eta : g(x, z, y) \mapsto \psi_1(a_1 x + a_2 y)
\]
of \(G(\mathbb{A}_\mathbb{Q})\). For \(a \in \mathbb{A}_\mathbb{Q}\), we denote by \(\psi_a\) the adelic character of \(Z(\mathbb{A}_\mathbb{Q})\) given by
\[
z \mapsto \psi_1(az).
\]
Accordingly there is the spectral decomposition of \(\mathcal{H}\).

**Lemma 3.3** ([27], Proposition 3.3). There is a constant \(\delta > 0\) such that for all \(s \in T_{> \delta}\), one has
\[
(3.3) \quad Z_\varepsilon(s, g) = Z_{0, \varepsilon}(s, g) + Z_{1, \varepsilon}(s, g) + Z_{2, \varepsilon}(s, g),
\]
where
\[
(3.4) \quad Z_{0, \varepsilon}(s, \text{id}) = \int_{G(\mathbb{A}_\mathbb{Q})} H(s, g)^{-1} \delta_\varepsilon(g) dg,
\]
\[
(3.5) \quad Z_{1, \varepsilon}(s, g) = \sum_{\eta} \eta(g) \int_{G(\mathbb{A}_\mathbb{Q})} H(s, g)^{-1} \bar{\eta}(g) \delta_\varepsilon(g) dg,
\]
\[
(3.6) \quad Z_{2, \varepsilon}(s, g) = \sum_{\psi} \sum_{\omega^\psi} \omega^\psi(g) \int_{G(\mathbb{A}_\mathbb{Q})} H(s, g)^{-1} \bar{\omega^\psi}(g) \delta_\varepsilon(g) dg.
\]
Here \(\eta\) ranges over all nontrivial characters of \(G^{ab}(\mathbb{Q}) \cdot K^{ab} \backslash G^{ab}(\mathbb{A}_\mathbb{Q})\),
\(\psi\) ranges over all nontrivial characters of \(Z(\mathbb{Q}) \cdot K_Z \backslash Z(\mathbb{A}_\mathbb{Q})\),
and \(\omega^\psi\) ranges over a fixed orthonormal basis of \(\mathcal{H}_\psi^K\).

**Lemma 3.4.** The height function \(H(s, g)\) and \(\delta_\varepsilon\) are bi-\(K\)-invariant, that is,
\[
H(s, gk) = H(s, kg) = H(s, g),
\]
\[
\delta_\varepsilon(gk) = \delta_\varepsilon(kg) = \delta_\varepsilon(g),
\]
for \(g \in G\) and \(k \in K\).
Proof. This follows from [27] and [25, Lemma 6.2].

We are going to compute
\[ Z_\epsilon(s, g) = Z_{0,\epsilon}(s, g) + Z_{1,\epsilon}(s, g) + Z_{2,\epsilon}(s, g). \]

Recall that \( S \) is a finite set of places of \( \mathbb{Q} \) containing the archimedean place such that there is a good integral model \( (\mathcal{X}, D) \) for \( (X, D) \) over the ring of \( S \)-integers \( \mathbb{Z}_S \), accordingly \( (\mathcal{X}, D_\epsilon) \) is a good integral model for \( (X, D_\epsilon) \) over \( \mathbb{Z}_S \), we are going to count the Campana \( \mathbb{Z}_S \)-points on \( (\mathcal{X}, D_\epsilon) \).

We are concerned with the sets
\[ G(\mathbb{Q})_\epsilon = G(\mathbb{Q}) \cap (\mathcal{X}, D_\epsilon)(\mathbb{Z}_S), \]
and
\[ G(\mathbb{Q}_v)_\epsilon = G(\mathbb{Q}_v) \cap (\mathcal{X}, D_\epsilon)(\mathbb{Z}_v). \]

For \( v \notin S \), let \( \delta_{\epsilon,v} \) denote the local indicator function detecting whether or not a given point in \( G(\mathbb{Q}_v) \) belongs to \( G(\mathbb{Q}_v)_\epsilon \), if \( v \in S \), set \( \delta_{\epsilon,v} \equiv 1 \). The global indicator function is thus \( \delta_\epsilon = \prod_v \delta_{\epsilon,v} \).

4. Height integrals I

In this section, we study the height integral
\[ Z_{0,\epsilon}(s, g) = \int_{G(\mathbb{Q})_\epsilon} H(s, g)^{-1} \delta_\epsilon(g) dg. \]

Our analysis here is similar to that of Pieropan et al [25]. Let us first set up some notations. Write
\[ D \otimes_{\mathbb{Q}} \mathbb{Q}_v = \bigcup_{\beta \in \mathcal{A}_v} D_{v,\beta}, \]
\[ D_\alpha \otimes_{\mathbb{Q}} \mathbb{Q}_v = \bigcup_{\beta \in \mathcal{A}_v(\alpha)} D_{v,\beta}, \]
where \( D_{v,\beta} \)'s are irreducible components.

Let \( \beta \in \mathcal{A}_v \), we denote the field of definition for one of the geometric irreducible components of \( D_{v,\beta} \) by \( \mathbb{Q}_{v,\beta} \) and denote the extension degree \([\mathbb{Q}_{v,\beta} : \mathbb{Q}_v]\) by \( f_{v,\beta} \).

For any subset \( B \subseteq \mathcal{A}_v \), define
\[ D_{v,B} := \bigcap_{\beta \in B} D_{v,\beta}, \quad D_{v,B}^\circ := D_{v,B} \setminus \bigcup_{B \subseteq B' \subset \mathcal{A}_v} \left( \bigcap_{\beta \in B'} D_{v,\beta} \right) \]
where we assume that \( D_{v,\varnothing} = X_{\mathbb{Q}_v} \) and \( D_{v,\varnothing}^\circ = \mathcal{G}_{\mathbb{Q}_v} \). For \( v \notin S \), we denote by \( \mathcal{D}_{v,B} \) the Zariski closure of \( D_{v,B} \) in \( \mathcal{X} \otimes_{\mathbb{Z}_S} \mathbb{Z}_v \). We define \( \mathcal{D}_{v,B}^\circ \) as above.
4.1. **Places not in** \( S \). Assume that \( p \not\in S \), we shall consider the integral

\[
\int_{G(\mathbb{A}_S^p)} H(s, g)^{-1} \delta_\varepsilon(g) dg = \prod_{p \notin S} \int_{G(\mathbb{Q}_p)} H_p(s, g_p)^{-1} \delta_{\varepsilon, p}(g_p) dg_p,
\]

we denote

\[
Z_{0,\varepsilon, p}(s, g_p) := \int_{G(\mathbb{Q}_p)} H_p(s, g_p)^{-1} \delta_{\varepsilon, p}(g_p) dg_p.
\]

4.1.1. **Places of good reduction.** We assume the integral model \( X \) has good reduction at \( p \) and the metrics at \( p \) are induced by the integral model.

Set \( \kappa = (\kappa_\alpha)_{\alpha \in A} \) and let

\[
d\tau = \frac{dg_p}{\|\omega\|_p}
\]

denote the Tamagawa measure where

\[
\|\omega\|_p = H_p(\kappa, g_p)
\]
is a gauge form on \( G \). We see that

\[
Z_{0,\varepsilon, p}(s, g_p) = \int_{G(\mathbb{Q}_p)} H_p(s, g_p)^{-1} H_p(\kappa, g_p) \delta_{\varepsilon, p}(g_p) \frac{dg_p}{\|\omega\|_p}
\]

\[
= \int_{G(\mathbb{Q}_p)} H_p(s - \kappa, g_p)^{-1} \delta_{\varepsilon, p}(g_p) d\tau.
\]

Consider the reduction map

\[
\rho_p : G(\mathbb{Q}_p) \rightarrow X(\mathbb{F}_p),
\]

let \( y \in X(\mathbb{F}_p) \) such that \( \rho_p^{-1}(y) \subset G(\mathbb{Q}_p) \), we have

\[
Z_{0,\varepsilon, p}(s, g_p) = \sum_{B \subset A_p} \sum_{y \in D_{p,B}^0(\mathbb{F}_p)} \int_{\rho_p^{-1}(y)} H_p(s - \kappa, g_p)^{-1} \delta_{\varepsilon, p}(g_p) d\tau.
\]

**Proposition 4.1.** We have

\[
Z_{0,\varepsilon, p}(s, g_p) = \sum_{B \subset A_p} \frac{\# D_{p,B}^0(\mathbb{F}_p)}{p^r \# B} \prod_{\beta \in B} \left( 1 - \frac{1}{p} \right) \frac{p^{-m_{\alpha(\beta)}(\kappa_{\alpha(\beta)} - \kappa_{\alpha(\beta)} + 1)}}{1 - p^{-(s_{\alpha(\beta)} - \kappa_{\alpha(\beta)} + 1)}}
\]

where we interpret the term \( p^{-m_{\alpha(\beta)}(\kappa_{\alpha(\beta)} - \kappa_{\alpha(\beta)} + 1)} \) if \( \varepsilon_{\alpha(\beta)} = 1 \).

**Proof.** The proof is analogous to that of [25, Theorem 7.1]. \( \square \)
4.1.2. Places of bad reduction. Again let \( p \not\in S \), we assume the integral model \( \mathcal{X} \) has bad reduction at \( p \) or the metrics at \( p \) are not induced by the integral model.

**Proposition 4.2.** The function \( Z_{0,\varepsilon,p}(s, g_p) \) is holomorphic in \( s \) when \( \Re(s_\alpha) > \kappa_\alpha - 1 \) for \( \alpha \in A \) with \( \varepsilon_\alpha < 1 \).

**Proof.** This follows from the arguments in [15, §3.2.5]. \( \square \)

4.2. Places in \( S \). Let \( v \in S \), then \( \delta_{\varepsilon,v} \equiv 1 \) by definition. We have

**Proposition 4.3.**

1. The function
   \[
   Z_{0,\varepsilon,v}(s, g) := \int_{G(Q_v)} H_v(s, g_v)^{-1} \delta_{\varepsilon,v}(g_v) \, dg_v
   \]
   is holomorphic when \( \Re(s_\alpha) > \kappa_\alpha - 1 \) for all \( \alpha \in A \).

2. Suppose \( L = \sum_{\alpha \in A} \lambda_\alpha D_\alpha \) is a big divisor on \( X \) and let
   \[
   a := a((X, D_{\text{red}}), L), \quad b := b(Q_v, (X, D_{\text{red}}), L)
   \]
   be as defined in [25, §4]. Then there is a constant \( \delta > 0 \) such that the function
   \[
   s \mapsto \zeta_{Q_v}(s - a)^{-b} Z_{0,\varepsilon,v}(sL, g)
   \]
   admits a holomorphic continuation to the domain \( \Re(s) > a - \delta \) and the function \( s \mapsto Z_{0,\varepsilon,v}(sL, g) \) has a pole at \( s = a \) of order \( b \).

**Proof.**

1. The assertion follows from [27, Lemma 4.1].
2. This follows from [13, Proposition 4.3]. \( \square \)

4.3. Euler products. Let \( \alpha \in A \), we denote the field of definition for one of the geometric irreducible components of \( D_\alpha \) by \( F_\alpha \), that is, \( F_\alpha \) is the algebraic closure of \( F \) in the function field of \( D_\alpha \).

**Proposition 4.4.** Let \( p \not\in S \), recall the decomposition
   \[
   D_\alpha \otimes \mathbb{Q}_p = \bigcup_{\beta \in A_{p}(\alpha)} D_{p,\beta}
   \]
   of \( D_\alpha \otimes \mathbb{Q}_p \) into irreducible components.

1. The function
   \[
   s \mapsto \prod_{\alpha \in A} \prod_{\beta \in A_{p}(\alpha)} \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1} Z_{0,\varepsilon,p}(s, g_p)
   \]
   is holomorphic on \( T_{> \delta} \) for \( \delta > 0 \) sufficiently small, if \( \varepsilon_\alpha = 1 \), we take \( \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1} = 1 \).
(2) Let $\delta > 0$ be sufficiently small, then there is a $\delta' > 0$ such that
\[
\prod_{\alpha \in A} \prod_{\beta \in A_p(\alpha)} \zeta_{Q_p,\beta}(m_{\alpha}(s_{\alpha} - \kappa_{\alpha} + 1))^{-1} Z_{0,\varepsilon,p}(s, g_p) = 1 + O(p^{-1+\delta'})
\]
for $s \in T_{>-\delta}$.

**Proof.** The proof is analogous to that of [25, Proposition 7.4]. 

**Corollary 4.5.** The following function
\[
s \mapsto \left( \prod_{\alpha \in A} \zeta_{F_{\alpha}}(m_{\alpha}(s_{\alpha} - \kappa_{\alpha} + 1))^{-1} \prod_{p \notin S} Z_{0,\varepsilon,p}(s, g_p) \right)
\]
is holomorphic on the domain $T_{>-\delta'}$.

**Proof.** This follows from the propositions 4.2, 4.4 and the fact that
\[
F_{\alpha} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{\beta \in A_p(\alpha)} \mathbb{Q}_{p,\beta}
\]
for all $\alpha \in A$.

### 5. Height integrals II

In this section, we study the height integral
\[
\int_{G(\mathbb{A}_\mathbb{Q})} H(s, g)^{-1} \eta(g) \delta_{\varepsilon}(g) dg = \prod_v \int_{G(\mathbb{Q}_v)} H_v(s, g_v)^{-1} \eta(g_v) \delta_{\varepsilon,v}(g_v) dg_v.
\]

The analysis in this section is parallel to that of [25, §8]. For each nonzero $a = (a_1, a_2) \in \mathbb{Q}^2$, we denote the linear functional
\[
(x, y) \mapsto a_1 x + a_2 y
\]
by $f_a$. Recall from the Corollary 2.2 that
\[
\text{div}(f_a) = E(f_a) - \sum_{\alpha \in A} d_{\alpha}(f_a) D_{\alpha}
\]
with $d_{\alpha}(f_a) \geq 0$ and at least one $\alpha$ such that $d_{\alpha}(f_a) > 0$. Define
\[
\mathcal{A}^0(a) := \{\alpha \in A | d_{\alpha}(f_a) = 0\},
\]
\[
\mathcal{A}^{\geq 1}(a) := \{\alpha \in A | d_{\alpha}(f_a) \geq 1\},
\]
and for any place $v \in \text{Val}(\mathbb{Q})$ we define
\[
H_v(a) = \max\{|a_1|_v, |a_2|_v\}
\]
and for any nonarchimedean place $p$ define
\[
j_p(a) = \min\{v(a_1), v(a_2)\}
so that \( H_p(a) = p^{-j_p(a)} \). Denote
\[
H_{\text{fin}}(a) = \prod_p H_p(a),
\]
then
\[
(5.1) \quad H_\infty(a) \gg H_{\text{fin}}(a)^{-1}.
\]

5.1. Places not in \( S \). Let \( p \not\in S \), we shall consider the integral
\[
(5.2) \quad Z_{1,\varepsilon,p}(s,\eta) := \int_{G(Q_p)} H_p(s,g_p)^{-1}\eta(g_p)\delta_{\varepsilon,p}(g_p)dg_p.
\]

5.1.1. Places of good reduction. We assume the integral model \( X \) has good reduction at \( p \) and the metrics at \( p \) are induced by the integral model. There are two cases, \( j_p(a) = 0 \) or \( j_p(a) \neq 0 \). We first assume \( j_p(a) = 0 \).

**Proposition 5.1.** We have

1. there exists \( \delta > 0 \), independent of \( a \) such that the function
   \[
   s \mapsto \prod_{\alpha \in A^0(a)} \prod_{\beta \in A_p(a)} \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1}Z_{1,\varepsilon,p}(s,\eta)
   \]
   is holomorphic on \( T_{>-\delta} \), if \( \varepsilon_\alpha = 1 \), we take \( \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1} = 1 \).

2. there exists \( \delta' > 0 \), independent of \( a \) such that
   \[
   \prod_{\alpha \in A^0(a)} \prod_{\beta \in A_p(a)} \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1}Z_{1,\varepsilon,p}(s,\eta) = 1 + O(p^{-1+\delta'}),
   \]
   for \( s \in T_{>-\delta} \).

**Proof.** The proof is analogous to that of [25, Proposition 8.1]. \( \square \)

Next we assume \( j_p(a) \neq 0 \).

**Proposition 5.2.** Let \( p \) be a nonarchimedean place such that \( p \not\in S \) and \( j_p(a) \neq 0 \). Then there is a \( \delta > 0 \), independent of \( a \), such that the following function
\[
\prod_{\alpha \in A^0(a)} \prod_{\beta \in A_p(a)} \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1}Z_{1,\varepsilon,p}(s,\eta)
\]
is holomorphic on \( T_{>-\delta} \) and moreover there is a \( \delta' > 0 \), independent of \( a \) such that we have
\[
\prod_{\alpha \in A^0(a)} \prod_{\beta \in A_p(a)} \zeta_{Q_p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha + 1))^{-1}Z_{1,\varepsilon,p}(s,\eta) \ll (1 + H_p(a)^{-1})^{\delta'}.
\]

**Proof.** The proof is analogous to that of [25, Proposition 8.2]. \( \square \)
5.1.2. Places of bad reduction. Again let $p \notin S$, we assume the integral model $\mathcal{X}$ has bad reduction at $p$ or the metrics at $p$ are not induced by the integral model.

Proposition 5.3. The height function $Z_{1,\varepsilon,p}(s,\eta)$ is holomorphic in $s$ when $\Re(s_\alpha) > \kappa_\alpha - 1$ for $\alpha \in \mathcal{A}(a)$ with $\varepsilon_\alpha < 1$. And for $\delta > 0$ there are $\delta' > 0$ and $C_\delta > 0$ such that

$$|Z_{1,\varepsilon,p}(s,\eta)| < C_\delta (1 + H_\infty(a))^{\delta'}$$

when $\Re(s_\alpha) > \kappa_\alpha - 1 + \delta$ for $\alpha \in \mathcal{A}(a)$ with $\varepsilon_\alpha < 1$.

Proof. This follows from [15, Corollary 3.3.7].

5.2. Places in $S$. For the case $v \in S$, $\delta_{\varepsilon,v} \equiv 1$ by definition.

Proposition 5.4. Let $v \in S$, then we have

(1) The function

$$s \mapsto Z_{1,\varepsilon,v}(s,\eta)$$

is holomorphic in the domain $\Re(s_\alpha) > \kappa_\alpha - 1$ for all $\alpha \in \mathcal{A}$ and there exist $n > 0, m_n > 0$ such that

$$|\prod_{v \in S} Z_{1,\varepsilon,v}(s,\eta)| \ll \frac{(1 + |s|)^{mn}}{(1 + H_\infty(a))^n}$$

in the above domain.

(2) If $L = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha D_\alpha$ is a big divisor on $X$, let

$$a := a((X, D_{\text{red}}), L), \quad b_v := b(Q_v, (X, D_{\text{red}}), L, f_a)$$

be as defined in [25, §4]. Then there is a constant $\delta > 0$ such that the function

$$s \mapsto \zeta_{Q_v}(s - a)^{-b_v} Z_{1,\varepsilon,v}(sL,\eta)$$

admits a holomorphic continuation to the domain $\Re(s) > a - \delta$. Moreover

$$|\prod_{v \in S} \zeta_{Q_v}(s - a)^{-b_v} Z_{1,\varepsilon,v}(sL,\eta)| \ll_n \frac{(1 + |s|)^{mn}}{(1 + H_\infty(a))^n}$$

in the above domain.

Proof. (1) The assertion on holomorphy follows from [27, Lemma 4.1] and the estimate [5.3] follows from [25, Proposition 8.4].

(2) The proof of the assertion on holomorphic continuation is essentially analogous to that of [15, Proposition 3.4.4]. The estimate [5.4] follows from [25, Proposition 8.4].
5.3. **Euler products.** Finally we consider the Euler product

\[ Z_{1,\varepsilon}(s, g) = \sum_{\eta} \eta(g) \prod_{v \in \text{Val}(Q)} Z_{1,\varepsilon,v}(s, \eta). \]

We set

\[ Z_{1,\varepsilon}(s, \eta) = \prod_{v \in \text{Val}(Q)} Z_{1,\varepsilon,v}(s, \eta), \]

and for every \( \alpha \in \mathcal{A} \) set

\[ \zeta_{F_\alpha,S}(s) = \prod_{v \not\in S} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{Q_{v,\beta}}(s). \]

**Proposition 5.5.** Assume that \((X, D_\varepsilon)\) is klt, then there is a \( \delta > 0 \), independent of \( a \), such that the function

\[ s \mapsto \left( \prod_{\alpha \in \mathcal{A}^0(a)} \zeta_{F_\alpha,S}(m_\alpha(s_\alpha - \kappa_\alpha + 1)) \right)^{-1} Z_{1,\varepsilon}(s, \eta) \]

is holomorphic on \( T_{> -\delta} \).

Moreover, for \( n > 0 \), there exists \( m_n > 0 \) such that

\[ \left| \left( \prod_{\alpha \in \mathcal{A}^0(a)} \zeta_{F_\alpha,S}(m_\alpha(s_\alpha - \kappa_\alpha + 1)) \right)^{-1} Z_{1,\varepsilon}(s, \eta) \right| \ll \frac{(1 + |s|)^{m_n}}{(1 + H_{\infty}(a))^n}. \]

**Proof.** This follows from Propositions 5.1, 5.2, 5.3 and 5.4, together with the estimate (5.1). \( \square \)

6. **Height integrals III**

In this section, we study the height integral \( Z_{2,\varepsilon}(s, g) \).

Let \( Y \subset X \) be the induced compactification of \( U \subset G \) and denote by \( (Y, D_Y) \) a good integral model for \((Y, D^Y)\).

**Assumption 6.1** ([27], Assumption 4.7). We assume that the boundary \( D_Y := Y \setminus U \) is a strict normal crossing divisor whose components are intersections of the boundary components of \( X \) with \( Y \):

\[ Y \setminus U = \bigcup_{\alpha \in \mathcal{A}^Y} D_\alpha = \bigcup_{\alpha \in \mathcal{A}} (D_\alpha \cap Y) \]

where \( \mathcal{A}^Y \subseteq \mathcal{A} \).

We have for the anticanonical divisor of \( Y \)

\[ -K_Y = \sum_{\alpha \in \mathcal{A}^Y} \kappa_\alpha^Y D_\alpha \]

with \( \kappa_\alpha^Y \leq \kappa_\alpha \) for all \( \alpha \).
For nonzero $a \in \mathbb{Q}$ denote by $f_a$ the linear form on $G$

$$z \mapsto a \cdot z.$$  

The linear form $f_a$ defines an adelic character $\psi = \psi_a$ of $\mathrm{U}(A_{\mathbb{Q}})/\mathrm{U}(\mathbb{Q})$ by

$$\psi_a(g(0, z, y)) = \psi_1(a z)$$

where $\psi_1$ is the Tate character. Write the divisor of $f_a$ as

$$\text{div}(f_a) = E(f_a) - \sum_{\alpha \in \mathcal{A}^Y} d_\alpha(f_a) D_\alpha^Y$$

where $E(f_a)$ is the hyperplane along which $f_a$ vanishing in $Y$. Define

$$\mathcal{A}^0 := \{ \alpha \in \mathcal{A}^Y | d_\alpha(f_a) = 0 \},$$

$$\mathcal{A}^{\geq 1} := \{ \alpha \in \mathcal{A}^Y | d_\alpha(f_a) \geq 1 \},$$

note that the set $\mathcal{A}^0$ is a proper subset of $\mathcal{A}$ by [27]. For $\psi = \psi_a$ with $a \in \frac{1}{\nu(n(K))} \mathbb{Z}$, we define $S_\psi$ by

$$(6.1) \quad S_\psi := \{ p : p | n(K)a \} \cup \{ S \}.$$  

For any place $v \in \mathrm{Val}(\mathbb{Q})$ we define

$$H_v(a) = |a|_v,$$

and for any nonarchimedean place $p$ define

$$j_p(a) = v(a)$$

so that $H_p(a) = p^{-j_p(a)}$. Denote

$$H_{\text{fin}}(a) = \prod_p H_p(a),$$

then

$$(6.2) \quad H_\infty(a) \gg H_{\text{fin}}(a)^{-1}.$$  

We are going to analyse the integral

$$(6.3) \quad \int_{G(A_{\mathbb{Q}})} H(s, g)^{-1} \overline{\omega}(g) \delta_\varepsilon(g) dg.$$  

Let $p \not\in S_\psi$, recall [27, Definition 3.14] that the normalized spherical function $f_p$ on $G(\mathbb{Q}_p)$ is defined to be

$$f_p(g_p) = \langle \pi_{\psi_p}(g_p)e_p, e_p \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2(\mathbb{Q}_p)$, $\pi_{\psi_p}$ is the representation of $G(\mathbb{Q}_p)$ induced by $\psi_p$ and $e_p \in V_p^{K_p}$ is a vector of a representation space for $G(\mathbb{Q}_p)$.
There are decompositions

\[ G(A) = G(A_{S_{\psi}}^1) \cdot G(A_{S_{\psi}}) = \prod_{p \notin S_{\psi}} G(Q_p) \cdot \prod_{p \in S_{\psi}} G(Q_p) \]

and

\[ g = g^S \cdot g_S = \prod_{p \notin S_{\psi}} g_p \cdot \prod_{p \in S_{\psi}} g_p \]

where \( \prod_{p \notin S_{\psi}} \) means the restricted product. From Lemma 3.15 and Corollary 3.16 of [27] we have

\[
\int_{G(A_Q)} H(s, g) \frac{-1}{\omega_{\psi}(g)} \delta_\varepsilon(g) dg = \prod_{p \notin S_{\psi}} \int_{G(Q_p)} H_p(s, g_p)^{-1} f_p(g_p) \delta_{\varepsilon, p}(g_p) dg_p \cdot \int_{G(A_{S_{\psi}})} H(s, g_{S_{\psi}})^{-1} \omega_{S_{\psi}}(g_{S_{\psi}}) \delta_\varepsilon(g_{S_{\psi}}) dg_{S_{\psi}},
\]

where \( \omega_{S_{\psi}} \) is the restriction of \( \omega \) to \( G(A_{S_{\psi}}) \).

As in the context for \( Z_{0,\varepsilon}(s, g) \) and \( Z_{1,\varepsilon}(s, g) \), we write

\[
D^Y \otimes_Q Q_v = \bigcup_{\beta \in \mathcal{A}_v^Y} D_{Y,\beta}^Y,
\]

\[
D^Y_{\alpha} \otimes_Q Q_v = \bigcup_{\beta \in \mathcal{A}_v^Y(\alpha)} D^Y_{Y,\beta}
\]

where \( D_{Y,\beta}^Y \)'s are irreducible components.

Let \( \beta \in \mathcal{A}_v^Y \), we denote the field of definition for one of the geometric irreducible components of \( D_{Y,\beta}^Y \) by \( Q_{Y,\beta}^v \) and denote the extension degree \( [Q_{Y,\beta}^v : Q_v] \) by \( f_{Y,\beta} \).

For any subset \( B \subseteq \mathcal{A}_v^Y \), we define

\[
D_{Y,B}^v := \bigcap_{\beta \in B} D_{Y,\beta}^Y, \quad D_{\alpha,B}^{\leq Y} := D_{Y,B}^v \setminus \bigcup_{B \subseteq B' \subseteq \mathcal{A}_v^Y} \left( \bigcap_{\beta \in B'} D_{Y,\beta}^Y \right)
\]

where we assume that \( D_{Y,\varnothing}^v = X_{Q_v} \) and \( D_{\alpha,\varnothing}^{\leq Y} = G_{Q_v} \). For \( v \notin S \), we denote by \( D_{v,B}^Y \) the Zariski closure of \( D_{Y,B}^v \) in \( Y \otimes_{Z_v} Z_v \). We define \( D_{\alpha,B}^{\leq Y} \) as above.

### 6.1 Places not in \( S_{\psi} \).

Let \( p \notin S_{\psi} \). By [27, Lemma 3.17], we have

\[
Z_{2,\varepsilon,p}(s, \omega^p) := \int_{G(Q_p)} H_p(s, g_p)^{-1} f_p(g_p) \delta_{\varepsilon,p}(g_p) dg_p = \int_{U(Q_p)} H_p(s, u_p)^{-1} \omega_p(u_p) \delta_{\varepsilon,p}(u_p) du_p.
\]
6.1.1. **Places of good reduction.** We assume the integral model $X$ has good reduction at $p$ and the metrics at $p$ are induced by the integral model. There are two cases, $j_p(a) = 0$ or $j_p(a) \neq 0$. We first assume $j_p(a) = 0$.

**Proposition 6.2.** We have

1. There exists $\delta > 0$, independent of $a$ such that the function
   $$s \mapsto \prod_{\alpha \in A^0} \prod_{\beta \in A_p^Y(\alpha)} \zeta_{Q,p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))^{-1}Z_{2,\varepsilon,p}(s,\omega^\psi)$$

   is holomorphic on $T_{\varepsilon - \delta}$, if $\varepsilon_\alpha = 1$, we take $\zeta_{Q,p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))^{-1} = 1$.

2. There exists $\delta' > 0$, independent of $a$ such that
   $$\prod_{\alpha \in A^0} \prod_{\beta \in A_p^Y(\alpha)} \zeta_{Q,p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))^{-1}Z_{2,\varepsilon,p}(s,\omega^\psi) = 1 + O(p^{-(1+\delta')})$$

   for $s \in T_{\varepsilon - \delta}$.

**Proof.** The proof is analogous to that of [25, Proposition 8.1]. \hfill \Box

Next we assume $j_p(a) \neq 0$.

**Proposition 6.3.** Let $p$ be a nonarchimedean place such that $p \notin S_\psi$ and $j_p(a) \neq 0$. Then there is a constant $\delta > 0$, independent of $a$, such that the following function

$$s \mapsto \prod_{\alpha \in A^0} \prod_{\beta \in A_p^Y(\alpha)} \zeta_{Q,p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))^{-1}Z_{2,\varepsilon,p}(s,\omega^\psi)$$

is holomorphic on $T_{\varepsilon - \delta}$ and moreover there exists $\delta' > 0$, independent of $a$ such that

$$|\prod_{\alpha \in A^0} \prod_{\beta \in A_p^Y(\alpha)} \zeta_{Q,p,\beta}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))^{-1}Z_{2,\varepsilon,p}(s,\omega^\psi)| \ll (1 + H_\infty(a)^{-1})^{\delta'}.$$

**Proof.** The proof is analogous to that of [25, Proposition 8.2]. \hfill \Box

6.1.2. **Places of bad reduction.** Again let $p \notin S_\psi$, we assume the integral model $X$ has bad reduction at $p$ or the metrics at $p$ are not induced by the integral model.

**Proposition 6.4.** The height function $Z_{2,\varepsilon,p}(s,\omega^\psi)$ is holomorphic in $s$ when $\Re(s_\alpha) > \kappa_\alpha - 1$ for $\alpha \in A^0$ with $\varepsilon_\alpha < 1$. And for $\delta > 0$ there are constants $\delta' > 0$ and $C_\delta > 0$ such that

$$|Z_{2,\varepsilon,p}(s,\omega^\psi)| < C_\delta (1 + H_\infty(a))^{\delta'}$$

when $\Re(s_\alpha) > \kappa_\alpha - 1 + \delta$ for $\alpha \in A^0$ with $\varepsilon_\alpha < 1$. 
6.2. Places in \( S_\psi \). Let \( v \in S_\psi \). We first compute \( \omega_{S_\psi}(g_{S_\psi}) \), let \( \mathcal{S}(\mathbb{A}_Q) \subset L^2(\mathbb{A}_Q) \) denote the space of Schwartz-Bruhat functions, recall from [27] that for a function \( \varphi \in \mathcal{S}(\mathbb{A}_Q) \), the theta distribution is defined as

\[
\Theta(\varphi) := \sum_{\gamma \in Q} \varphi(\gamma).
\]

The theta distribution gives a map

\[
j_\psi : \mathcal{S}(\mathbb{A}_Q) \rightarrow L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}_Q))
\]

\[
j_\psi(\varphi)(g) = \Theta(\pi_\psi(g) \varphi)
\]

where \( \pi_\psi \) is the representation of \( G(\mathbb{Q}) \) induced by \( \psi \). For details of the representations see [27, §3.5]. Let \( K_{S_\psi} := \prod_{p \not\in S_\psi} K_p \), from the proof of [27, Lemma 3.15] we have

\[
\omega_{S_\psi}(g_{S_\psi}) = \int_{K_{S_\psi}} j_\psi(\varphi)(g_{S_\psi} k_{S_\psi}) dk_{S_\psi},
\]

therefore

\[
\omega_{S_\psi}(g_{S_\psi}) = \int_{K_{S_\psi}} \Theta(\pi_\psi(g_{S_\psi} k_{S_\psi}) \varphi) dk_{S_\psi} = \int_{K_{S_\psi}} \sum_{\gamma \in Q} \left( \pi_\psi(g_{S_\psi} k_{S_\psi}) \varphi(\gamma) \right) dk_{S_\psi}.
\]

Let

\[
g(x_v, z_v, y_v) \in g_{S_\psi},
\]

\[
g(x_{k_v}, z_{k_v}, y_{k_v}) \in k_{S_\psi},
\]

then

\[
g(x_v, z_v, y_v) \cdot g(x_{k_v}, z_{k_v}, y_{k_v}) = g(x_v + x_{k_v}, z_v + z_{k_v} + x_v y_{k_v}, y_v + y_{k_v}).
\]

Thus by (6.7) and [27, (3.6)] we have

\[
\omega_{S_\psi}(g_{S_\psi}) = \sum_{\gamma \in Q} \prod_{v \in S_\psi} \int_{K_{S_\psi}} \psi \left( (y_v + y_{k_v}) \gamma + z_v + z_{k_v} + x_v y_{k_v} \right) \varphi(x_v + x_{k_v} + \gamma) dx_{k_v} dy_{k_v} dz_{k_v}.
\]

\[
= \sum_{\gamma \in Q} \left( \prod_{v \in S_\psi} \psi(y_v \gamma + z_v) \varphi(x_v + \gamma) \cdot \prod_{v \in S_\psi} \int_{K_{S_\psi}} \psi(y_{k_v} \gamma + z_{k_v}) \varphi(x_{k_v} + \gamma) dk_{S_\psi} \right)
\]

\[
= \sum_{\gamma \in \mathbb{Z}_{S_\psi}} \prod_{v \in S_\psi} \psi(y_v \gamma + z_v) \varphi(x_v + \gamma) \cdot 1_{\mathbb{Z}_{S_\psi}}(\gamma),
\]

where \( 1_{\mathbb{Z}_{S_\psi}} \) is the characteristic function of \( \mathbb{Z}_{S_\psi} \).
Proposition 6.5. (1) The function
\[ s \mapsto Z_{2,\varepsilon, S_\psi}(s, \omega^\psi) := \int_{G(AS_\psi)} H(s, gS_\psi)^{-1} \omegaS_\psi(gS_\psi) \delta_\varepsilon(gS_\psi) dg_{S_\psi} \]
is holomorphic in the domain \( \Re(s_\alpha) > \kappa_\alpha - 1 \) for all \( \alpha \in A \) and
\[ | \prod_{v \in S_\psi} Z_{2,\varepsilon, S_\psi}(s, \omega^\psi) | \ll \frac{(1 + | a |)^{n'}}{(1 + | \lambda |)^n} \]
in the above domain where \( n > 0, n' \) is some fixed positive number and \( \lambda = \lambda(\omega^\psi_S) \) is the eigenvalue of \( \omega^\psi_S \).

(2) If \( L = \sum_{\alpha \in A} \lambda_\alpha D_\alpha \) is a big divisor on \( X \), let
\[ a' := a'(X, D_{\text{red}}), b_v := b(Q_v, (X, D_{\text{red}}), L, f_a) \]
be as defined in [25, §4]. Then there is a constant \( \delta > 0 \) such that the function
\[ s \mapsto \prod_{v \in S_\psi} \zeta_{Q_v}(s - a') - b_v Z_{2,\varepsilon, S_\psi}(sL, \omega^\psi) \]
admits a holomorphic continuation to the domain \( \Re(s) > a' - \delta \) and
\[ | \prod_{v \in S_\psi} \zeta_{Q_v}(s - a') - b_v Z_{2,\varepsilon, S_\psi}(sL, \omega^\psi) | \ll_n \frac{(1 + | a |)^{n'}}{(1 + | \lambda |)^n} \]

Proof. (1) The assertion on holomorphy follows from [27, Lemma 4.1] and the estimate (6.9) follows from [27, Proposition 4.12].

(2) By (6.8) we have
\[ Z_{2,\varepsilon, S_\psi}(sL, \omega^\psi) = \sum_{\gamma \in S_\psi} \int_{G(AS_\psi)} H(s, gS_\psi)^{-1} \overline{\psi}(y_v \gamma + z_v) \varphi(x_v + \gamma) \mathbf{1}_{Z_{2,\varepsilon, \psi}}(\gamma) \delta_\varepsilon(gS_\psi) dg_{S_\psi}. \]
The assertion on holomorphic continuation follows from the fact that \( \varphi \) is a Schwartz-Bruhat function and the estimate (6.10) follows from [27, Proposition 4.12] and [27, Lemma 4.14].

\[ \square \]

6.3. Euler products. Finally we consider the Euler product
\[ Z_{2,\varepsilon}(s, g) = \sum_\psi \sum_\omega \omega^\psi(g) \left( \prod_{v \in S_\psi} Z_{2,\varepsilon, \psi}(s, \omega^\psi) \cdot Z_{2,\varepsilon, S_\psi}(s, \omega^\psi) \right). \]
For every $\alpha \in \mathcal{A}^Y$ we set

$$\zeta_{F^Y_\alpha, S_\psi}(s) = \prod_{v \notin S_\psi} \prod_{\beta \in \mathcal{A}_v^Y(\alpha)} \zeta_{Q_\psi, S_\psi}(s),$$

and we denote

$$Z_{2,\varepsilon}(s, \omega^\psi) = \prod_{p \notin S_\psi} Z_{2,\varepsilon, p}(s, \omega^\psi) \cdot Z_{2,\varepsilon, S_\psi}(s, \omega^\psi).$$

**Proposition 6.6.** Assume that $(X, D_\varepsilon)$ is klt, then there is a $\delta > 0$, independent of $a$, such that the following function

$$s \mapsto \left(\prod_{\alpha \in A^0} \zeta_{F^Y_\alpha, S_\psi}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))\right)^{-1} Z_{2,\varepsilon}(s, \omega^\psi)$$

is holomorphic on the domain $T > -\delta$.

Moreover, for $n > 0$ we have

$$\left|\left(\prod_{\alpha \in A^0} \zeta_{F^Y_\alpha, S_\psi}(m_\alpha(s_\alpha - \kappa_\alpha^Y + 1))\right)^{-1} Z_{2,\varepsilon}(s, \omega^\psi)\right| \ll \frac{(1 + |a|)^{n'}}{(1 + |\lambda|)^n}$$

where $n'$ is some fixed positive number and $\lambda = \lambda(\omega^\psi_{S_\psi})$ is the eigenvalue of $\omega^\psi_{S_\psi}$ described in [27].

**Proof.** This follows from Propositions 6.2, 6.3, 6.4 and 6.5, together with the estimate (6.2). \qed

7. Proof of main results

Let $X$ be a smooth projective biequivariant compactification of the Heisenberg group $G$ over $F = \mathbb{Q}$ with boundary $D$ a strict normal crossings divisor on $X$, with irreducible components $(D_\alpha)_{\alpha \in \mathcal{A}}$. Let $S$ be a finite set of places containing the infinite place such that there is a good integral model $(\mathcal{X}, \mathcal{D})$ of $(X, D)$ over $\mathbb{Z}_S$ as in the Introduction. We fix $\varepsilon_\alpha \in \mathfrak{M}$ for each $\alpha \in \mathcal{A}$ and let $D_\varepsilon = \sum_{\alpha \in \mathcal{A}} \varepsilon_\alpha D_\alpha$ and $\mathcal{D}_\varepsilon = \sum_{\alpha \in \mathcal{A}} \varepsilon_\alpha \mathcal{D}_\alpha$. We denote by $\overline{X}$ the base change of $X$ to $\overline{F}$ and we write $X_v$ for the base change of $X$ to $F_v$.

Let $L$ be a big line bundle $L$ on $X$ equipping with a smooth adelic metrization induced by $\mathcal{X}$ for all places $v \notin S$. We are concerned with the asymptotic behavior of the counting function

$$\mathcal{N}(G(\mathbb{Q})_\varepsilon, L, T)$$

where

$$G(\mathbb{Q})_\varepsilon = G(\mathbb{Q}) \cap (\mathcal{X}, \mathcal{D}_\varepsilon)(\mathbb{Z}_S).$$
7.1. **Proof of Theorem 1.3.** In this subsection we assume that the pair \((X, D)\) is klt.

**Proposition 7.1.** The following function

\[
s \mapsto \left( \prod_{\alpha \in A} \zeta_{F_{\alpha}}(m_{\alpha}(s_{\alpha} - \kappa_{\alpha} + 1)) \right)^{-1} Z_\varepsilon(s, g)
\]

is holomorphic on \(T > 0\).

**Proof.** It follows from §4 that \(Z_{0,\varepsilon}(s, g)\) converges absolutely, we now show the convergence of \(Z_{1,\varepsilon}(s, g)\) and \(Z_{2,\varepsilon}(s, g)\).

\[
Z_{1,\varepsilon}(s, g) = \sum_\eta \eta(g) \cdot Z_{1,\varepsilon}(s, \eta).
\]

It follows from Proposition 5.5 that \(Z_{1,\varepsilon}(s, \eta)\) is defined for \(s \in T > 0\) and

\[
|Z_{1,\varepsilon}(s, \eta)| \ll_n \frac{1}{(1 + H_\infty(a))^n},
\]

as the series

\[
\sum_\eta \frac{1}{(1 + H_\infty(a))^n}
\]

converges for sufficiently large \(n\), the convergence of \(Z_{1,\varepsilon}(s, g)\) follows.

Next we show the convergence of \(Z_{2,\varepsilon}(s, g)\).

\[
Z_{2,\varepsilon}(s, g) = \sum_\psi \sum_\omega \omega_\psi(g) \cdot Z_{2,\varepsilon}(s, \omega_\psi).
\]

It follows from Proposition 6.6 that \(Z_{2,\varepsilon}(s, \omega_\psi)\) is defined for \(s \in T > 0\) and

\[
|Z_{2,\varepsilon}(s, \omega_\psi)| \ll \frac{1}{(1 + |\lambda|)^n} \cdot (1 + |a|)^{n'},
\]

as in the proof of [27, Theorem 4.15], it then suffices to prove the convergence of the series

\[
\sum_\psi \sum_\omega |\lambda|^{-n+m'} |a|^n
\]

where \(m', n'\) are certain positive numbers. The remaining proof is the same as that in [27, Theorem 4.15].

We conclude that the spectral decomposition

\[
Z_\varepsilon(s, g) = Z_{0,\varepsilon}(s, g) + Z_{1,\varepsilon}(s, g) + Z_{2,\varepsilon}(s, g)
\]

holds for \(\Re(s) \gg 0\). Now the proposition follows from Proposition 4.3, Corollary 4.5, Proposition 5.5, and Proposition 6.6. \(\square\)
We now discuss the case where \( s = s_L \), write \( L = \sum_{\alpha \in A} \lambda_\alpha D_\alpha \) where \( \lambda_\alpha > 0 \) for all \( \alpha \in A \), then \( s_\alpha = s\lambda_\alpha \). It follows from Proposition \[\text{7.1}\] that the rightmost pole along \( \Re(s) \) of \( Z_\varepsilon(sL,g) \) is
\[
a = a((X, D_\varepsilon), L) = \max_{\alpha \in A} \left\{ \frac{\kappa_\alpha - \varepsilon_\alpha}{\lambda_\alpha} \right\}.
\]

We set
\[
A_\varepsilon(L) = \left\{ \alpha \in A : \frac{\kappa_\alpha - \varepsilon_\alpha}{\lambda_\alpha} = a \right\},
\]
and
\[
b = b(\mathbb{Q}, (X, D_\varepsilon), L) := \#A_\varepsilon(L).
\]
Assume that the divisor \( aL + K_X + D_\varepsilon \) is rigid, which we mean that its Iitaka dimension is 0. By the spectral decomposition
\[
Z_\varepsilon(sL, g) = Z_{0,\varepsilon}(sL, g) + Z_{1,\varepsilon}(sL, g) + Z_{2,\varepsilon}(sL, g),
\]
we shall investigate the poles of \( Z_\varepsilon(sL, g) \) by investigating individually \( Z_{0,\varepsilon}(sL, g) \), \( Z_{1,\varepsilon}(sL, g) \) and \( Z_{2,\varepsilon}(sL, g) \).

For the term \( Z_{1,\varepsilon}(sL, g) \), Proposition \[\text{5.3}\] shows that \( Z_{1,\varepsilon}(sL, g) \) has a pole of the highest order equal to that of \( Z_{0,\varepsilon}(sL, g) \) if and only if
\[
A^0(a) \supset A_\varepsilon(L),
\]
which means that \( d_\alpha(f_a) = 0 \) whenever \( (\kappa_\alpha - \varepsilon_\alpha)/\lambda_\alpha = a \). Since
\[
E(f_a) \sim \sum_{\alpha \in A} d_\alpha(f_a)D_\alpha, \quad aL + K_X + D_\varepsilon = \sum_{\alpha \in A} (a\lambda_\alpha - \kappa_\alpha + \varepsilon_\alpha)D_\alpha,
\]
this means that \( E(f_a) \) is equivalent to a boundary divisor whose support is contained in that of the divisor \( aL + K_X + D_\varepsilon \), this is impossible because \( aL + K_X + D_\varepsilon \) is rigid. Similarly, Proposition \[\text{6.6}\] shows that the term \( Z_{2,\varepsilon}(sL, g) \) does not contribute to the main term of \( Z_\varepsilon(sL, g) \).

On the other hand, it follows from Corollary \[\text{4.5}\] that \( Z_{0,\varepsilon}(sL, g) \) has a pole at \( s = a \) of order \( b \) if we could show that the corresponding residue \( c \) is nonzero, i.e.,
\[
c := \lim_{s \to a} (s - a)^b Z_{0,\varepsilon}(sL, g) \neq 0.
\]
Recall that
\[
Z_{0,\varepsilon}(sL, g) = \int_{G(A_\varepsilon)} H(sL, g)^{-1} \delta_\varepsilon(g) \, dg = \int_{G(A_\varepsilon)} H(sL + K_X, g)^{-1} \, d\tau
\]
where \( \tau \) is the Tamagawa measure on \( G \). Let
\[
X^\circ = X \setminus \left( \cup_{\alpha \in A_\varepsilon(L)} D_\alpha \right),
\]
and let $\tau_{X^o}$ denote the Tamagawa measure on $X^o$ using certain virtual Artin $L$-function, for details see [25]. We also define
\[
\tau_{X^o,D_\varepsilon} = H(D_\varepsilon,g)\tau_{X^o}.
\]

**Lemma 7.2.** Let the notations be as above, we have
\[
c = \prod_{\alpha \in \mathcal{A}(L)} \frac{1}{m_\alpha \lambda_\alpha} \int_{X^o(\mathbb{A}_\varepsilon)} H(aL + K_X + D_\varepsilon,g)^{-1} d\tau_{X^o,D_\varepsilon} > 0.
\]

**Proof.** The proof is essentially analogous to that of [25, Lemma 9.3]. □

Applying a Tauberian theorem [32, II.7, Theorem 15] we have

**Theorem 7.3.** Let $X, L, D, a, b, c$ and $\varepsilon$ be as above. Assume that $(X, D_\varepsilon)$ is klt, if $aL + K_X + D_\varepsilon$ is rigid, then as $T \to \infty$,
\[
N(G(\mathbb{Q})_\varepsilon, L, T) \sim \frac{c}{a(b-1)!} T^a (\log T)^{b-1}.
\]

### 7.2. Proof of Theorem 1.4

Let notations be as former subsection but in this subsection we assume that the pair $(X, D_\varepsilon)$ is only dlt. We set
\[
\mathcal{A}^{klt} = \{ \alpha \in \mathcal{A} | \varepsilon_\alpha \neq 1 \},
\]
\[
\mathcal{A}^{nklt} = \{ \alpha \in \mathcal{A} | \varepsilon_\alpha = 1 \},
\]

**Proposition 7.4.** Let $L = -(K_X + D_\varepsilon)$, then the function
\[
s \mapsto \left( \prod_{\alpha \in \mathcal{A}^{klt}} \zeta_{F_\alpha}(1 + m_\alpha(\kappa_\alpha - \varepsilon_\alpha)(s-1)) \right)^{-1} \left( \prod_{v \in S} \zeta_{Q_v}(s-1)^{-b(Q_v,(X,D_\text{red}),L)} \right) Z_\varepsilon(sL,g)
\]
is holomorphic on $\Re(s) > 1$ where $b(Q_v,(X,D_\text{red}),L)$ is the $b$-invariant defined in [25, §4].

**Proof.** The proof is analogous to that of Proposition 7.1. □

The proposition above implies that $Z_\varepsilon(sL,g)$ has a pole at $s = 1$. We define
\[
b'(Q, S, (X,D_\varepsilon), L) = \#\mathcal{A}^{klt} + \sum_{v \in S} b(Q_v,(X,D_\text{red}),L).
\]

Let $D_\text{red} = \sum_{\alpha \in \mathcal{A}} D_\alpha$ and $v$ a place of $\mathbb{Q}$, fix an embedding $\overline{\mathbb{Q}} \subset \overline{Q_v}$ so that $\Gamma_v := \text{Gal}(\overline{Q_v}/Q_v)$ acts on $\overline{X} = X \otimes_{\mathbb{Q}} \overline{Q}$ and $\overline{D_\text{red}} = D_\text{red} \otimes_{\mathbb{Q}} \overline{Q}$. We denote by $\mathcal{A}$ the indexing set of $\overline{D_\text{red}}$ and by $\mathcal{A}_v$ the set of orbits of $\mathcal{A}$ under the action of $\Gamma_v$. 
By the definition of \( b(Q_v, (X, D_{\text{red}}), L) \) in [25, §4] we have

\[
\nu'(Q, S, (X, D_{\varepsilon}), L) = \#A^{\text{klt}} + \sum_{v \in S} \max_{B \subseteq A^\text{klt}} \left\{ \#B; \bigcap_{\beta \in B} D_{v, \beta, \text{red}}(Q_v) \neq \emptyset \right\}.
\]

**Lemma 7.5.** The height zeta function \( Z_{0, \varepsilon}(sL, g) \) has a pole at \( s = 1 \) of order \( b' \).

**Proof.** This follows from Proposition 4.3 and Corollary 4.5 \( \square \)

**Lemma 7.6.** The order of the pole at \( s = 1 \) of \( Z_{1, \varepsilon}(sL, g) \) is \( < b' \).

**Proof.** We denote the order of the pole at \( s = 1 \) of \( Z_{1, \varepsilon}(sL, g) \) by \( b_1 \), it is clear that \( b_1 \leq b' \). As the proof of [15, Lemma 3.5.4], we prove our result by contradiction. Assume that \( b_1 = b' \), then comparing the formulas of \( b_1 \) and \( b' \) we have

1. \( d_\alpha(f_a) = 0 \) for all \( \alpha \in \mathcal{A}^{\text{klt}} \),
2. for any \( v \in S \) there is a subset \( B \subseteq \mathcal{A}_v^{\text{klt}} \) of maximal cardinality with \( \bigcap_{\beta \in B} D_{v, \beta, \text{red}}(Q_v) \neq \emptyset \) such that \( d_\alpha(f_a) = 0 \) for all \( \alpha \in B \).

Fix a \( g(b_1, 0, b_2) \in G(Q) \) such that \( f_a(g(b_1, 0, b_2)) = 1 \). Let us fix a \( v \in S \) and let \( B \subseteq \mathcal{A}_v^{\text{klt}} \) satisfy condition (2). By definition the function \( f_a \) is defined and nonzero at general points of \( \bigcap_{\beta \in B} D_{v, \beta, \text{red}} \) and there is a general point \( g(x_0, z_0, y_0) \in \bigcap_{\beta \in B} D_{v, \beta, \text{red}}(Q_v) \) for a suitable \( B \).

Take \( g' = \lim_{t \to \infty} g(tb_1, 0, tb_2)g(x_0, z_0, y_0) \), then \( g' \in \bigcap_{\beta \in B} D_{v, \beta, \text{red}}(Q_v) \). The function \( t \mapsto f_a(g(tb_1, 0, tb_2)g(x_0, z_0, y_0)) \) is defined on \( \mathbb{P}^1 \) and \( f_a(g(tb_1, 0, tb_2)g(x_0, z_0, y_0)) \to \infty \) as \( t \to \infty \). Therefore \( g' \in D_\alpha \) for some \( \alpha \) such that \( d_\alpha(f_a) > 0 \), by condition (1), \( \alpha \in \mathcal{A}_{\text{klt}}^{\text{h}} \) and thus \( B' = B \cup \{ \alpha \} \subseteq \mathcal{A}_v^{\text{klt}} \) is such that \( \bigcap_{\beta \in B'} D_{v, \beta, \text{red}}(Q_v) \neq \emptyset \), contradicting the maximal cardinality condition (2). \( \square \)

**Lemma 7.7.** The order of the pole at \( s = 1 \) of \( Z_{2, \varepsilon}(sL, g) \) is \( < b' \).

**Proof.** The argument is analogous to that of Lemma 7.6 \( \square \)

**Theorem 7.8.** Let \( \mathcal{X}, \mathcal{D}, \varepsilon \) and \( b' \) be as above and let \( L = -(K_X + D_\varepsilon) \). Assume that \( (X, D_\varepsilon) \) is dlt, then as \( T \to \infty \), there is a constant \( c > 0 \) depending on \( S, (\mathcal{X}, \mathcal{D}_\varepsilon) \) and \( \mathcal{L} \) such that

\[
N(G(Q)_\varepsilon, \mathcal{L}, T) \sim \frac{c}{(b' - 1)!} T(\log T)^{b'-1}.
\]

**Proof.** This follows from Lemmas 7.5, 7.6, 7.7 and a Tauberian theorem [32, II.7, Theorem 15]. \( \square \)
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