Deformation of the Fermi Surface and Anomalous Mass Renormalization by Critical Spin Fluctuations through Asymmetric Spin-Orbit Interaction

Yukinobu Fujimoto¹, Kazumasa Miyake², and Hiroyasu Matsuura³

¹Division of Materials Physics, Department of Materials Engineering Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan
²Toyota Physical and Chemical Research Institute, Nagakute, Aichi 480-1192, Japan
³Department of Physics, University of Tokyo, Bunkyo, Tokyo 113-0033, Japan

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It is shown that an asymmetric spin-orbit interaction is appreciably renormalized by the effect of critical spin fluctuations. As an explicit example, the Rashba-type interaction is predicted to be suppressed by anti-ferromagnetic critical fluctuations near the hot line (or point), leading to a deformation of the Fermi surface near the hot line (or point) around which the band splitting is minimized, while ferromagnetic critical fluctuations are shown to enhance the Rashba-type interaction everywhere on the Fermi surface, leading to an increase in the band splitting. It is also found that the many-body mass renormalization, which diverges towards the magnetic quantum critical point, is in the opposite sign for the two split bands owing to the Rashba-type interaction. These predictions are observable through the de Haas-van Alphen experiment in principle.

The role of asymmetric spin-orbit (ASSO) interaction in giving rise to intriguing physical phenomena of strongly correlated electron systems has attracted much attention in the past decade. In particular, it has been discussed in relation to the superconductivity of non-centrosymmetric systems without the inversion centers of crystals in the field of heavy electron systems.¹,² It is also an essential ingredient for the surface states of topological insulators, which have been extensively discussed for the past several years since their discovery by Kane and Mele.³

It is an old problem how many-body effects, owing to the on-site Coulomb interaction, affect the atomic spin-orbit (SO) interaction in orbitally degenerate d-electron systems.⁴,⁵ Recently, phenomena of the growth in the spin Hall effect have attracted much attention, and theoretical proposals have been published from the viewpoint of understanding that these phenomena are due to the effect of the growth of the atomic SO interaction.⁶,⁷

On the other hand, it has also been discussed how the ASSO interaction of Rashba type influences crucially the effect of critical spin fluctuations (CSFs) on the superconducting transition temperature near the antiferromagnetic quantum critical point.⁸,⁹ Furthermore, it has recently been discussed how the Kondo effect is influenced if the conduction electrons are subject to strong ASSO interaction. For example, they are concerned with behaviors of magnetic impurities on the surface of topological insulators¹⁰,¹¹ and the Kondo effect in the system where conduction electrons are subject to the
Rashba-type spin-orbit interaction.\textsuperscript{12,13)} However, it has not been clarified how CSFs affect the ASSO interaction. This effect possibly gives rise to a deformation of the Fermi surface near the magnetic quantum critical point. In this Letter, we report on a perturbation calculation of a correction to the ASSO interaction under the influence of CSF modes, revealing the fact that CSFs, ferromagnetic (F) or antiferromagnetic (AF), give rise to a singular renormalization of the ASSO interaction. This fact suggests that the ASSO interaction is renormalized strongly by many-body effects.

We discuss the renormalization of the ASSO interaction due to CSFs near the magnetic quantum critical point by a perturbation calculation. Such a renormalization is expected to be visibly large considering that CSFs considerably enhance nonmagnetic impurity scattering.\textsuperscript{14)} We demonstrate that the ASSO interaction is markedly influenced by CSFs, leading to the deformation of the Fermi surface (FS). We also note that the dynamical effect on the renormalization for ASSO interaction gives rise to a nontrivial mass renormalization of quasiparticles and the deformation of the FS.

To provide an explicit discussion, we adopt a single-band model action, on the basis of the idea of an itinerant-localized duality model with tetragonal symmetry but without an inversion center of the crystal as follows:\textsuperscript{15)}

\begin{equation}
A = A_f + A_s + A_{\text{int}} + A_{\text{so}},
\end{equation}

\begin{equation}
A_f = - \sum_{k,\sigma,n} f_{k\sigma}^\dagger (-i\epsilon_n)(G^{(0)}_{\sigma}(k,i\epsilon_n))^{-1} f_{k\sigma}(i\epsilon_n),
\end{equation}

\begin{equation}
A_s = - \sum_{q,m} S_{-q}(-i\nu_m) \cdot S_{q}(i\nu_m)[\chi_0(i\nu_m)^{-1} - J(q)],
\end{equation}

\begin{equation}
A_{\text{int}} = -A \sum_{k,q,\sigma,\sigma'} \sum_{n,m} f_{-k-q,\sigma}^\dagger (-i\epsilon_n - i\nu_m) \hat{\sigma}_{\sigma,\sigma'} f_{k\sigma'}(i\epsilon_n) \cdot S_{q}(i\nu_m),
\end{equation}

\begin{equation}
A_{\text{so}} = \alpha \sum_{k,\sigma,\sigma'} f_{k\sigma,\gamma}^\dagger(k) \cdot \hat{\sigma}_{\sigma,\sigma'} f_{k\sigma'},
\end{equation}

where $f_{k\sigma}$ ($f_{k\sigma}^\dagger$) is the annihilation (creation) operator for the quasiparticle, with the momentum $k$ and the “spin” $\sigma$, renormalized by local correlation, and $S(q)$ is the local spin density operator constructed from high-energy states of correlated electrons. Note that “spin” stands for pseudo-spin specifying a degeneracy of the Kramers doublet in general. Hereafter, we call it as simply spin. The Green function $G^{(0)}_{\sigma}(k,i\epsilon_n)$ of the quasiparticle is given by

\begin{equation}
G^{(0)}_{\sigma}(k,i\epsilon_n) \approx \frac{z}{i\epsilon_n - \xi_k},
\end{equation}

where $\xi_k$ is the dispersion of the quasiparticle measured from the Fermi energy, $\epsilon_n = (2n + 1)\pi T$ is the fermionic Matsubara frequency, and $z$ is the renormalization amplitude due to the effect of the local correlation. $\chi_0(i\nu_m)$ (with $\nu_m = 2m\pi T$ being the bosonic Matsubara frequency) and $J(q)$ in Eq. (3) represent the local dynamical spin susceptibility and the exchange interaction among localized parts of spin degrees of freedom, respectively. $A_{\text{int}}$ [Eq. (4)] represents the exchange interaction between quasi-
particles and local spins. Finally, $A_{so}$ [Eq. (5)] represents the ASSO interaction, $\alpha$ is the coupling constant of the ASSO interaction, and $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector spanned by the Pauli matrices. The vector $\mathbf{\gamma}(\mathbf{k}) = (\gamma_k, \gamma_k, \gamma_k)$ in Eq. (5) obeys $\mathbf{\gamma}(-\mathbf{k}) = -\mathbf{\gamma}(\mathbf{k})$, thereby breaking the inversion symmetry and causing the ASSO interaction. The main purpose of this study is to demonstrate how the ASSO coupling constant $\alpha$ is renormalized by CSFs. Hereafter, to provide explicit discussions, we assume that the ASSO interaction is the Rashba-type SO interaction given by $^{16}$

$$\mathbf{\gamma}(\mathbf{k}) = (\sin k_x a, -\sin k_x a, 0),$$

(7)

where $a$ is the lattice constant in the tetragonal plane. However, it remains valid also for another type of ASSO interaction$^{17,18}$ that the ASSO coupling constant $\alpha$ is considerably influenced by CSFs.

The lowest-order correction to the coupling constant $\alpha$ of the ASSO interaction due to CSFs is depicted by the Feynman diagrams shown in Fig. 1. The off-diagonal (in spin space) self-energy of quasiparticles due to this correction $\Sigma_{\uparrow\downarrow}(\mathbf{k}, i\epsilon_n)$ appearing in Fig. 1(a) is given by

$$\Sigma_{\uparrow\downarrow}(\mathbf{k}, i\epsilon_n) = \lambda^2 T \sum_{\mathbf{k}'} \sum_{x} \alpha[\mathbf{\gamma}(\mathbf{k}') \cdot \hat{\sigma}_{\uparrow\downarrow}] \chi_{\uparrow\downarrow}(\mathbf{k} - \mathbf{k}', i\epsilon_n - i\epsilon_{n'}) [G^{(0)}(\mathbf{k}', i\epsilon_{n'})]^2,$$

(8)

where $\chi_{\uparrow\downarrow}$ is a part of the spin-fluctuation propagator $\chi_s = (\chi_{\uparrow\uparrow} - \chi_{\uparrow\downarrow})/2$.

The propagator $\chi_{\uparrow\downarrow}$ is expressed in terms of $\chi_s$ and the charge-fluctuation propagator $\chi_c = 2(\chi_{\uparrow\uparrow} + \chi_{\uparrow\downarrow})$ as $\chi_{\uparrow\downarrow} = -\chi_s + (\chi_c/4)$, in general. Near the magnetic critical point, the propagator $\chi_{\uparrow\downarrow}$ is dominated by the spin-fluctuation propagator $\chi_s$ and is given by $^{14,19-21}$

$$\chi_{\uparrow\downarrow}(\mathbf{q}, i\omega_n) \approx \frac{\chi_{Q}^{(0)}}{\eta + A (\mathbf{q} - \mathbf{Q})^2 + C_q |\omega_n|},$$

(9)

where $\omega_n = 2m \pi T$ is the boson Matsubara frequency, $\eta$ is the distance from the magnetic critical point, $\mathbf{Q}$ is the magnetic ordering vector, and $\chi_{Q}^{(0)}$ is the noninteracting static spin susceptibility at the magnetic ordering vector $\mathbf{Q}$. Note that $\eta > 0$ represents the paramagnetic state and $\eta = 0$ corresponds to the magnetic critical point.$^{20,21}$

Substituting Eq. (9) into Eq. (8), we obtain

$$\Sigma_{\uparrow\downarrow}(\mathbf{k}, i\epsilon_n) = \sum_{\gamma=x,y} (\sigma_{\gamma})_{\uparrow\downarrow}(\mathbf{k}, i\epsilon_n),$$

(10)

with

$$\Gamma_{\gamma}(\mathbf{k}, i\epsilon_n) \equiv -\alpha \lambda^2 T \sum_{\mathbf{k}'} \sum_{\gamma'} \gamma_{\gamma'}(\mathbf{k}') \frac{\chi_{Q}^{(0)}}{\eta + A (\mathbf{k} - \mathbf{k}' - \mathbf{Q})^2 + C_{|\mathbf{k} - \mathbf{k}'|} |\epsilon_n - \epsilon_{n'}| (i\epsilon_{n'} - i\epsilon_k)^2} (\gamma = x, y),$$

(11)

where $\lambda \equiv \zeta \lambda$. The other off-diagonal self-energy $\Sigma_{\downarrow\uparrow}(\mathbf{k}, i\epsilon_n)$, shown in Fig. 1(b), is given by

$$\Sigma_{\downarrow\uparrow}(\mathbf{k}, i\epsilon_n) = \sum_{\gamma=x,y} (\sigma_{\gamma})_{\downarrow\uparrow}(\mathbf{k}, -i\epsilon_n),$$

(12)

where we have used a general relation

$$G_{\uparrow\downarrow}(\mathbf{k}, i\epsilon_n) = [G_{\downarrow\uparrow}(\mathbf{k}, -i\epsilon_n)]^*,$$

(13)
Fig. 1. Feynman diagrams for the lowest-order correction to the coupling constant $\alpha$ of the ASSO interaction due to CSFs. The wavy line represents the spin-fluctuation propagator $\chi_s$, the solid line the Green function $G$ of the quasiparticles, the filled circle the coupling constant $\lambda$ between spin fluctuations and quasiparticles, the upward arrow the up spin, the downward arrow the down spin, and the cross the ASSO interaction.

as shown in Sect. 1 of Ref. 22.

First, we discuss the effect of three-dimensional (3D) AF quantum critical fluctuations (QCFs) in which $C_q$ is $q$-independent, i.e., $C_q \equiv C$. We also discuss the case where the external wave vector $k$ is on the FS. To begin with, let us define $k^*$ as $k^* \equiv k - Q$ and investigate the case where $k^*$ is on the spherical FS; i.e., $k$ is on the hot line where $\xi_{k+Q} = -\xi_k$. Note that the results obtained in the present study are valid for any shape of FS as long as Eq. (9) is valid, although the calculations become far more tedious. Here we assume that the origin of the AF order is caused by a commensurate AF-exchange interaction through an incoherent process in the meaning of the itinerant-localized duality model so that the ordering vector $Q$ is not influenced by QCFs.\textsuperscript{15,23}

Then, the frequency-independent terms in the denominator of Eq. (9) are transformed into

$$\eta + A(k - k' - Q)^2 = \eta + A(k^* - k')^2 = \frac{A}{v_F^2}(\xi'^2 + 2b \xi' + a),$$

where $\xi'$ stands for $\xi_{k'}$, and $a$ and $b$ are defined by

$$a \equiv \frac{v_F^2}{A} \eta + 2k_F^2 v_F^2 (1 - \cos \theta),$$

$\eta$ and $A$ are defined as

$$\eta = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{2} - 1} \right) \right] = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{2} - 1} \right) \right] = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{2} - 1} \right) \right].$$

$A$ is given by

$$A = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{2} - 1} \right) \right] = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{2} - 1} \right) \right] = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2} - 1} \left( \frac{1}{\sqrt{2} - 1} \right) \right].$$
and
\[ b \equiv k_F v_F (1 - \cos \theta), \]  

(16)

where \( \cos \theta \equiv (\mathbf{k} \cdot \mathbf{k}' - (\mathbf{k}' \parallel \mathbf{q}' \mathbf{k})) \), and \( k_F \) and \( v_F \) are the Fermi wavenumber and velocity of quasiparticles on the FS, respectively. Thus, \( \Gamma_v (\mathbf{k}, i \varepsilon_n) \) [Eq. (11)] is expressed as
\[
\Gamma_v (\mathbf{k}, i \varepsilon_n) = -\alpha \lambda^2 \gamma_v (\mathbf{k} - \mathbf{Q}) T \sum_{n'} \int_{-1}^{1} \frac{d(\cos \theta)}{2} \int d\xi' \frac{v_F^2}{A} \frac{\lambda_Q^{(0)}}{\xi'^2 + 2b \xi' + a + C^*|\varepsilon_n - \varepsilon_{n'}| (\xi' - i\varepsilon_{n'})^2},
\]

(17)

where \( C^* \equiv v_F^2 C/A \). Since the singular behavior in \( \Gamma_v \) appears through the contributions from the regions \( \xi' \approx 0 \) and \( \cos \theta \approx 1 \), in deriving Eq. (17), we have used the approximation \( \mathbf{k}' = \mathbf{k} = \mathbf{k} - \mathbf{Q} \); i.e., \( \gamma_v (\mathbf{k}') = \gamma_v (\mathbf{k} - \mathbf{Q}) \). Similarly, the density of states of quasiparticles, \( N(\xi') \), is approximated by that at the Fermi level, \( N_F \). Therefore, \( \Gamma_v (\mathbf{k}, i \varepsilon_n) \) is reduced to
\[
\Gamma_v (\mathbf{k}, i \varepsilon_n) = -J \gamma_v (\mathbf{k} - \mathbf{Q}) \int_{-1}^{1} \frac{d(\cos \theta)}{2} T \sum_{n'} \int d\xi' \frac{1}{\xi'^2 + 2b \xi' + a + C^*|\varepsilon_n - \varepsilon_{n'}| (\xi' - i\varepsilon_{n'})^2},
\]

(18)

where \( J \) is defined by
\[
J \equiv \alpha \lambda^2 N_F \chi_0^{(0)} \frac{v_F^2}{A},
\]

(19)

Performing the integration with respect to \( \xi' \) in Eq. (18) with the use of the contour integration, we obtain
\[
\Gamma_v (\mathbf{k}, i \varepsilon_n) = -J \gamma_v (\mathbf{k} - \mathbf{Q}) \int_{-1}^{1} \frac{d(\cos \theta)}{2} 
\pi T \sum_{n' \geq 0, n' \neq n} \left\{ \frac{1}{\sqrt{a_+ - b^2} \left[ b + i(\sqrt{a_+ - b^2} + \varepsilon_{n'})^2 \right]} + \frac{1}{\sqrt{a_- - b^2} \left[ b - i(\sqrt{a_- - b^2} + \varepsilon_{n'})^2 \right]} \right\},
\]

(20)

where \( a_\pm \equiv a + C^*|\varepsilon_n - \varepsilon_{n'}| \). Here we note that Eq. (20) is valid regardless of the sign of \( \varepsilon_n \), positive or negative, although we consider the case with \( \varepsilon_n > 0 \) hereafter. Equation (20) is rearranged and approximated as
\[
\Gamma_v (\mathbf{k}, i \varepsilon_n) = \tilde{\Gamma}_v (\mathbf{k}, i \varepsilon_n) + J \gamma_v (\mathbf{k} - \mathbf{Q}) \int_{-1}^{1} \frac{d(\cos \theta)}{2}
\pi T \sum_{n' \geq 0, n' \neq n} \left\{ \frac{1}{\sqrt{a_+} (\sqrt{a_+} + \varepsilon_{n'})^2} + \frac{1}{\sqrt{a_-} (\sqrt{a_-} + \varepsilon_{n'})^2} \right\}
+ 2ib \sum_{n' \geq 0, n' \neq n} \left\{ \frac{1}{\sqrt{a_+} (\sqrt{a_+} + \varepsilon_{n'})^3} - \frac{1}{\sqrt{a_-} (\sqrt{a_-} + \varepsilon_{n'})^2} \right\},
\]

(21)

where \( \tilde{\Gamma}_v (\mathbf{k}, i \varepsilon_n) \) is the contribution from \( n' = n > 0 \) in the first term in the brace of Eq. (20), which corresponds to the static term of fluctuation and should be treated separately, as discussed in Sect. 3 of Ref. 22. In deriving Eq. (21), we have neglected the terms of \( O(b^2) \) because the singular contribution to \( \Gamma_v \) arises from the regions \( \cos \theta \approx 1 \) and \( b^2 \propto (1 - \cos \theta)^2 \). Then, the retarded function \( \Gamma^R_v (\mathbf{k}, \varepsilon + i\delta) \)
at $T \to 0$ in the static limit ($\varepsilon \to 0$) is given by $\lim_{T \to 0} \Gamma_{\nu}(k, i\varepsilon T)$:

$$
\Gamma_{\nu}^{R}(k, i\delta) = \Gamma_{\nu}^{R}(k, i\delta) + J \gamma_{\nu}(k - Q) \int_{-1}^{1} \frac{d(\cos \theta)}{2} \int_{0}^{\infty} \frac{d\varepsilon'}{\sqrt{\varepsilon' + \varepsilon'^{2}}} \cdot \frac{1}{\sqrt{\varepsilon' + \varepsilon'^{2} + \varepsilon' + \varepsilon'^{2}}},
$$

(22)

where the term $\Gamma_{\nu}^{R}(k, i\delta)$ is given by the first term in the bracket of Eq. (23) in Sect. 3 of Ref. 22.

By changing the integration variable from $\varepsilon' \to y \equiv \sqrt{\varepsilon' + \varepsilon'^{2}} / \sqrt{\varepsilon}$, Eq. (22) is reduced to a more compact form as

$$
\Gamma_{\nu}^{R}(k, i\delta) = \alpha_{af}(\eta) \gamma_{\nu}(k - Q),
$$

(23)

where the excess contribution $\alpha_{af}$ to the SO coupling from the spin fluctuations is given by

$$
\alpha_{af}(\eta) = \frac{\Gamma_{\nu}^{R}(k, i\delta)}{\gamma_{\nu}(k - Q)} \cdot \frac{2J}{C} \int_{-1}^{1} \frac{d(\cos \theta)}{2} \int_{1}^{\infty} \frac{dy}{y} \frac{1}{\sqrt{y + \frac{\varepsilon}{\varepsilon'}(y^{2} - 1)}},
$$

(24)

With the use of the expression for $a$ [Eq. (15)], it is easier to perform first the integration with respect to $\cos \theta$ and then with respect to $y$. According to Sects. 2 and 3 of Ref. 22, the result for $\alpha_{af}$ up to $O(\sqrt{\eta})$ is given by

$$
\alpha_{af}(\eta) \approx \frac{J}{2k_{F}v_{F}} \left[ F\left(\frac{4m^{*}A}{C}\right) - \frac{2\sqrt{A}}{\sqrt{C}} \frac{\sqrt{A}k_{F}^{2} 2\pi}{k_{F}v_{F}} \sqrt{\eta} + O(\eta) \right],
$$

(25)

where $m^{*}$ is the effective mass of quasiparticles and $F(x)$ is defined by

$$
F(x) \equiv \log \frac{x}{2} - \frac{1}{\sqrt{1 + x^{2}}} \log \frac{x + 1 - \sqrt{1 + x^{2}}}{x + 1 + \sqrt{1 + x^{2}}}.
$$

(26)

The function $F(x)$ is an increasing function and positive definite for $x > 0$, and $F(1) \approx 0.55$. Therefore, $\Gamma_{\nu}^{R}(k, i\delta)$ has a cusp singularity at $\eta = 0$.

The $\varepsilon_{n}$ dependence of $\Gamma_{\nu}(k, i\varepsilon_{n})$ for $\varepsilon_{n} > 0$, up to $O(\varepsilon_{n})$, is given in Sect. 3 of Ref. 22 as

$$
\Gamma_{\nu}(k, i\varepsilon_{n}) = \Gamma_{\nu}^{R}(k, i\delta) + \gamma_{\nu}(k - Q) h(i\varepsilon_{n}) + O(\varepsilon_{n}^{3}),
$$

(27)

where $h$ is defined by

$$
h \equiv J \left[ -\frac{\sqrt{A}k_{F}^{2} 2\pi}{(k_{F}v_{F})^{4}} \frac{\sqrt{\eta}}{\sqrt{\eta} + 4(k_{F}v_{F})^{3}} \left( \log \frac{4\sqrt{A}k_{F}^{2} 2\pi}{\eta e} \right) \right].
$$

(28)

We note that $h$ is divergent because $\eta \propto T^{3/2}$ just at the 3D-AFQCP. In the case where the result of the first-order perturbation calculation is divergent, we need to consider the effect of higher-order corrections with respect to CSFs in general. This problem is beyond our scope and will be left as a future work.

Substituting Eq. (27) into Eqs. (10) and (12) and performing the analytic continuation $i\varepsilon_{n} \to \varepsilon + i\delta$, we obtain

$$
\Sigma_{\Pi_{1}}^{R}(k, \varepsilon + i\delta) = \gamma (k - Q) \cdot (\partial)^{\dagger}_{\Pi_{1}}[\alpha_{af}(\eta) + h \varepsilon],
$$

(29)
and
\[ \Sigma_{I}^{R}(\mathbf{k}, \varepsilon + i\delta) = \gamma (\mathbf{k} - \mathbf{Q}) \cdot (\hat{\sigma})_{1}^{\dagger} [\alpha_{af}(\eta) + h \varepsilon]. \]  
(30)

Then, the retarded Green function matrix \( \hat{G}^{R} \) for quasiparticles, renormalized by QCFs, is given by
\[ \hat{G}^{R}(\mathbf{k}, \varepsilon) = \left[ (\varepsilon - \xi_{k}) \hat{1} + \alpha \gamma (\mathbf{k}) \cdot \hat{\sigma} - \hat{\Sigma}^{R}(\mathbf{k}, \varepsilon) \right]^{-1}, \]  
(31)

with
\[ \hat{\Sigma}^{R}(\mathbf{k}, \varepsilon) = \Sigma_{I}^{R}(\mathbf{k}, \varepsilon) \hat{1} + \hat{\Sigma}_{I}^{R}(\mathbf{k}, \varepsilon). \]  
(32)

Here, \( \Sigma_{I}^{R}(\mathbf{k}, \varepsilon) \) is the self-energy of the quasiparticles due to the effect of spin fluctuations without considering the Rashba-type SO interaction.

According to Sect. 4 of Ref. 22, by diagonalizing the Green function matrix \( \hat{G}^{R} \), the dispersion of quasiparticles is given by
\[ \tilde{\xi}_{k}^{\pm} = \left[ \xi_{k} + \Sigma_{I}(\mathbf{k}, 0) \pm |\alpha \gamma (\mathbf{k}) + \alpha_{af}(\eta)\gamma (\mathbf{k} - \mathbf{Q})| \right] \tilde{\zeta}_{k}^{\pm}, \]  
(33)

where
\[ \left( \tilde{\zeta}_{k}^{\pm} \right)^{-1} = 1 - \frac{\partial \Sigma_{I}^{R}(\mathbf{k}, \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \frac{1}{h} \frac{\gamma (\mathbf{k}) \cdot \alpha_{af}(\eta)\gamma (\mathbf{k} - \mathbf{Q})}{|\alpha \gamma (\mathbf{k}) + \alpha_{af}(\eta)\gamma (\mathbf{k} - \mathbf{Q})|}. \]  
(34)

It can be seen from Eqs. (33) and (34) that the band (0) with the energy \( \xi_{k}^{0} = [\xi_{k} + \Sigma_{I}(\mathbf{k}, 0)] \tilde{\zeta}_{k} \), where \( \tilde{\zeta}_{k} \) is equal to \( \tilde{\zeta}_{k}^{\pm} \) with \( h = 0 \), is split into the two bands (±) with the energies \( \tilde{\xi}_{k}^{\pm} \), and the term of \( \alpha_{s}(\eta)\gamma (\mathbf{k} - \mathbf{Q}) \) is newly involved into the coupling constant of the Rashba-type SO interaction due to CSFs. The FS is deformed by an effect of \( \Sigma_{I}(\mathbf{k}, 0) \) in general. Therefore, precisely speaking, the calculation of \( \hat{\Sigma}_{I}^{R}(\mathbf{k}, \varepsilon) \) in Eq. (32) should have been performed with the use of quasiparticles with this deformed FS. However, we are interested in the effect of \( \hat{\Sigma}_{I}^{R}(\mathbf{k}, \varepsilon) \) on the band splitting due to the ASSO interaction, and we have taken an approximation that the spin-fluctuation propagator (9) is maintained intact. Note also that the divergences in \( h \) [Eq. (28)] can make either of \( \left( \tilde{\zeta}_{k}^{\pm} \right)^{-1} \) negative, which is unphysical. To remedy this fault, we need to consider the effect of higher-order corrections with respect to CSFs, as pointed out above.

The derivations above are restricted to the case where \( \mathbf{k} \) is on the hot line(s) on the FS without the ASSO interaction. To discuss a deformation of the FS due to the ASSO interaction renormalized by the AF-CSF, the effect of the distance of the position of \( \mathbf{k} \) from the hot line on the FS is parameterized by shifting \( \eta \) to \( \eta + Aq_{m}^{2} \) in Eq. (25),\(^{14} \) where \( q_{m} \) is the perpendicular distance between \( \mathbf{k} \) and the hot line on the FS without the ASSO interaction. Then, in the case where the ordering vector is commensurate, i.e., \( \mathbf{Q} = (\pi/a, \pi/a, \pi/c) \), so that \( \gamma_{s}(\mathbf{k} - \mathbf{Q}) = -\gamma_{s}(\mathbf{k}) \), the band energy \( \tilde{\xi}_{k}^{\pm} \) in Eq. (33) is approximated as
\[ \tilde{\xi}_{k}^{\pm} \approx \left[ \xi_{k} + \Sigma_{I}(\mathbf{k}, 0) \pm |\alpha - \alpha_{af}(\eta + Aq_{m}^{2})| \gamma (\mathbf{k}) \right] \tilde{\zeta}_{k}^{\pm}, \]  
(35)

where \( \tilde{\zeta}_{k}^{\pm} \) is given by modifying \( h \) in Eq. (34) with the replacement of \( \eta \) by \( \eta + Aq_{m}^{2} \) in Eq. (28). Therefore, according to Eq. (34), we expect that the effect of dynamical mass renormalization is opposite in
the two split bands: the effective mass in band (+) is enhanced while that in band (-) is suppressed.

The singular behavior given by Eq. (25) fades away at positions on the FS far from the hot lines. As a result, the suppression of the band splitting is confined to the region near the hot line, as shown in Fig. 2(b). Thus, it means that the deformation of the FS is caused by the band splitting under the influence of critical AF spin fluctuations. In particular, the suppression of the band splitting seems to be followed by an AF-gap opening around the hot line(s), which is reasonable from the physical viewpoint. In the “AF ordered state”, the ordering vector \( Q \) itself should be determined self-consistently by taking into account the effect of the enhancement of the ASSO interaction \( \alpha \). Therefore, it is possible that a resultant ordering vector \( Q \) can be incommensurate even though the critical mode is commensurate in the paramagnetic state. This problem will be discussed elsewhere.

Fig. 2. Fermi surfaces split by the Rashba-SO interaction for (a) the system without the effect of CSFs, (b) the system with critical AF spin fluctuations, and (c) the system with critical F spin fluctuations. The circle indicated by the dotted line represents the Fermi surface of the band (0), the large circle indicated by the solid line with the (±) represents the Fermi surface of the band (±), the long leftward arrow represents the AF-order vector \( Q \), the short arrow represents the direction of the spin, and the small open circle indicated by the solid line represents the hot spot.

There exists a singular contribution to the renormalization amplitude \( \tilde{z}_k \) near the criticality \( \eta = 0 \) through a singularity included in \( h \) [Eq. (28)]. Another nontrivial aspect suggested by Eqs. (33) and (34) is that the mass renormalization effects [through the factor \( (\tilde{z}_k)^{-1} \)] for the two split bands are in the opposite sign, as explicitly shown in Eq. (34). This effect can be observed through the de Haas-van Alphen (dHvA) experiment in principle.

Note that the aspects discussed above are expected to survive even if the ordering vector \( Q \) is not commensurate as long as the deviation from the commensurate one is not very large.

Next, let us discuss the effect of 3D F-QCP in which \( C_q \) in Eq. (9) is proportional to \( q^{-1} \): \( C_q = \sqrt{2} k_F \tilde{C} / q \), with \( q = |k - k'| \approx \sqrt{2} k_F \sqrt{1 - \cos \theta} \) and \( \tilde{C} \) being constant. Then, after calculations similar
to deriving Eq. (24), $\Gamma^R_\nu(k, i\delta)$ is expressed as

$$\Gamma^R_\nu(k, i\delta) = \alpha_\nu(\eta)\gamma_\nu(k),$$

(36)

where $\alpha_\nu$ is given by

$$\alpha_\nu(\eta) \equiv \frac{\Gamma^R_\nu(k, i\delta)}{\gamma_\nu(k)} + 2J \int_{-1}^{1} \frac{d\cos \theta}{2} \frac{\sqrt{1 - \cos \theta}}{C^* \sqrt{\eta} \int_{1}^{\infty} dy \frac{1}{y + \sqrt{4(1 - \cos \theta)/(y^2 - 1)}}},$$

(37)

where $C^* \equiv v^2_F / C$. As shown in Sects. 5 and 6 of Ref. 22, by calculations similar to those deriving Eq. (25) from Eq. (24), the $\Gamma^R_\nu(k, i\delta)$ is, up to a logarithmic accuracy in $\eta$, reduced to

$$\alpha_\nu(\eta) \approx \frac{J}{4k_F^2 v_F^2} \left( F \left( \frac{4 \sqrt{2m^*} A}{C} \right) + \frac{2 \sqrt{2m^*} A}{16k_F^2} \log \frac{e \eta}{16Ak_F^2} \log \frac{e^2 \eta}{16Ak_F^2} \right) + \frac{2 \sqrt{Ak_F^2 \pi T}}{k_F v_F \sqrt{\eta}} \right),$$

(38)

where the functions $G(x)$ and $H(x)$ are defined by

$$G(x) \equiv \frac{x^2}{8(1 + x^2)} \left[ 1 + \log \frac{x}{2} - F(x) \right],$$

(39)

and

$$H(x) \equiv \frac{x^2}{2} \int_{1}^{\infty} dy \frac{y^2 - 1}{[y + x(y^2 - 1)]^2}.$$

(40)

Both the functions $G(x)$ and $H(x)$ are positive definite.

The coefficient $\alpha_\nu(\eta)$ increases sharply as the critical point is approached, i.e., $\eta \to 0$, so that it exhibits a sharp cusp as a function of $\eta$. As a result, the band energy $\tilde{\xi}_k^\pm$ in Eq. (33) is approximated as

$$\tilde{\xi}_k^\pm \approx \left[ \xi_k^0 + \Sigma^R_\nu(k, 0) \pm |\alpha_\nu(\eta)| \gamma_\nu(k) \right] \tilde{z}_k^\pm,$$

(41)

where $\tilde{z}_k^\pm$ is defined by the same expression as Eq. (34) with the same expression as $h$ [Eq. (28)], as discussed in Sect. 6 of Ref. 22:

$$h \equiv J \left[ -\frac{\sqrt{Ak_F^2}}{(k_F v_F)^3} \frac{\pi T}{\sqrt{\eta}} + \frac{1}{4(k_F v_F)^3} \left( \log \frac{4Ak_F^2}{\eta e} \right) \right].$$

(42)

This means that the coupling constant of the Rashba-SO interaction is enhanced due to the critical F spin fluctuations, resulting in an increase in the splitting of the band (0) into the bands ($\pm$), which we can easily see if we compare Fig. 2(c) with Fig. 2(a), in contrast to the AF critical fluctuations where the band splitting is suppressed. This marked difference arises from the fact that $\gamma_\nu(k - Q) = \gamma_\nu(Q)$ for $Q = 0$ and $\gamma_\nu(k - Q) = -\gamma_\nu(Q)$ for $Q = (\pi/a, \pi/a, \pi/c)$, because of Eq. (7). As in the case of 3D-AFQCP, we expect that the effect of dynamical mass renormalization is opposite in two split bands: the effective mass in band (+) is suppressed while that in band (-) is enhanced, which is in contrast with the case of 3D AF-QCP.

We note that the singular behaviors in Eqs. (25) and (38) become more prominent in two dimen-
that the deformation of the FS in directions on the FS near the AF-QCP while it is uniformly increased over the whole FS near the F-QCP. As a result, the ASSO has been shown to be decreased near the hot diagonal (in spin space) self-energy of quasiparticles, which gives rise to a considerable renormalization of the ASSO interaction. Higher-order effects in CSFs, and a self-consistent treatment of the ASSO interaction and CSFs, deserve further investigations. This type of effect on the ASSO interaction arising from enhanced spin fluctuations may be possible in other physical situations, which are also interesting subjects to be explored in future studies.

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Supplemental Materials: Deformation of the Fermi Surface and Anomalous Mass Renormalization by Critical Spin Fluctuations through Asymmetric Spin-Orbit Interaction

Yukinobu Fujimoto¹, Kazumasa Miyake², and Hiroyasu Matsuura³

¹Division of Materials Physics, Department of Materials Engineering Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan
²Toyota Physical and Chemical Research Institute, Nagakute, Aichi 480-1192, Japan
³Department of Physics, University of Tokyo, Bunkyo, Tokyo 113-0033, Japan

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1. Derivation of Eqs. (12) and (13) in the text

Off-diagonal components of fermionic Matsubara Green function matrix in spin space, \( \hat{G}(k, \tau) \), are defined by

\[
G_{\uparrow\downarrow}(k, \tau) \equiv -\langle a_{k\uparrow}(\tau) a_{k\downarrow}^\dagger(0) \rangle
\]

and

\[
G_{\downarrow\uparrow}(k, \tau) \equiv -\langle a_{k\downarrow}(\tau) a_{k\uparrow}^\dagger(0) \rangle,
\]

for \( 0 < \tau < \beta \). Then, it is easy to show that the following relation holds:

\[
G_{\downarrow\uparrow}(k, \tau) = [G_{\uparrow\downarrow}(k, \tau)]^*. \tag{1}
\]

The Fourier transform of the off-diagonal components, \( G_{\downarrow\uparrow}(k, i\epsilon_n) \) and \( G_{\uparrow\downarrow}(k, i\epsilon_n) \), are defined by

\[
G_{\downarrow\uparrow}(k, i\epsilon_n) \equiv \int_0^\beta d\tau e^{i\epsilon_n \tau} G_{\downarrow\uparrow}(k, \tau), \tag{2}
\]

and

\[
G_{\uparrow\downarrow}(k, i\epsilon_n) \equiv \int_0^\beta d\tau e^{i\epsilon_n \tau} G_{\uparrow\downarrow}(k, \tau). \tag{3}
\]

With the use of Eq. (1), it is easy to show that the following relation holds:

\[
G_{\downarrow\uparrow}(k, i\epsilon_n) = [G_{\uparrow\downarrow}(k, -i\epsilon_n)]^*. \tag{4}
\]

This is nothing but the relation Eq. (13) in the text.

Since the inverse matrix of \( \hat{G}(k, i\epsilon_n)^{-1} \) satisfies the same relation as Eq. (1), the off-diagonal components of the matrix self-energy, \( \hat{\Sigma}(k, i\epsilon_n) \), satisfies the same relation as Eq. (4). Then, considering the relation \( (\sigma_{\uparrow\downarrow})_{\uparrow\downarrow} = (\sigma_{\downarrow\uparrow})_{\downarrow\uparrow}^* \), Eq. (12) in the text is immediately derived.

2. Derivation of Eq. (25) in the text

Let us introduce \( S_{3dAF}(\eta) \) representing double integration part in Eq. (24) in the text. Substituting the expression (15) in the text into the expression of \( S_{3dAF}(\eta) \), \( S_{3dAF}(\eta) \) is explicitly written as

\[
S_{3dAF}(\eta) = \int_{-1}^{1} dx \left( \frac{1}{2\sqrt{\eta + K(1-x)}} \right) \int_{1}^{\infty} dy \frac{1}{\left[ \frac{1}{e} \sqrt{\eta + K(1-x)} (y^2 - 1) + y \right]^2}. \tag{5}
\]
where \( x = \cos \theta, \ \tilde{\eta} \equiv v_F^2 \eta/A \) and \( K \equiv 2k_F^2 v_F^2 \). Changing the integration variable from \( x \) to \( u \equiv \sqrt{\tilde{\eta} + K(1 - x)} \), Eq. (5) is reduced to

\[
S_{3\text{dAF}}(\eta) = \frac{1}{K} \int_{\sqrt{\tilde{\eta}}}^{\tilde{\eta} + 2K} du \int_1^{\infty} dy \frac{1}{\left[ 2 + (y^2 - 1) + y \right]^2}.
\]

(6)

Then, the integration with respect to \( u \) is easily performed:

\[
S_{3\text{dAF}}(\eta) = -\frac{C^*}{K} \int_1^{\infty} dy \frac{1}{y - 1} \left[ \frac{B_0}{B_0(y^2 - 1) + y} - \frac{B_1}{B_1(y^2 - 1) + y} \right].
\]

(7)

where \( B_0 \) and \( B_1 \) are defined as follows:

\[
B_0 = \frac{\sqrt{\tilde{\eta}}}{C^*}, \\
B_1 = \frac{\sqrt{\tilde{\eta} + 2K}}{C^*}.
\]

(8)

Then, the integrand in Eq. (7) is transformed into

\[
S_{3\text{dAF}}(\eta) = -\frac{C^*}{K} \int_1^{\infty} dy \left[ \frac{B_0}{y - 1} \frac{1}{B_0(y^2 - 1) + y} - \frac{B_1}{y - 1} \frac{1}{B_1(y^2 - 1) + y} \right].
\]

(10)

There holds the following definite integral:

\[
\int_1^{\infty} \frac{dy}{y - 1} \frac{1}{B(y^2 - 1) + y} = \frac{1}{2} \log B - \frac{1}{\sqrt{1 + 4B^2}} \log \frac{2B + 1 - \sqrt{1 + 4B^2}}{2B + 1 + \sqrt{1 + 4B^2}}.
\]

(11)

With the use of definition (26) in the text, this definite integral is expressed as

\[
\int_1^{\infty} \frac{dy}{y - 1} \frac{1}{B(y^2 - 1) + y} = \frac{1}{2} F(2B).
\]

(12)

Therefore, Eq. (10) is expressed as

\[
S_{3\text{dAF}}(\eta) = -\frac{C^*}{2K} \left[ F(2B_0) - F(2B_1) \right].
\]

(13)

In the limit of \( \eta \to 0, B_0 \to 0 \) and \( B_1 \to \sqrt{2K}/C^* \equiv B^* \). Then,

\[
\lim_{\eta \to 0} S_{3\text{dAF}}(\eta) = -\frac{C^*}{2K} \left[ F(0) - F(2B^*) \right].
\]

(14)

It is easily shown that \( F(0) = 0 \) and \( B^* = 2m^*/A \) since \( K = 2k_F^2 v_F^2 \) and \( C^* \equiv v_F^2 C/A \). Therefore,

\[
\lim_{\eta \to 0} S_{3\text{dAF}}(\eta) = \frac{C^*}{2K} F \left( \frac{4m^*}{C} \right).
\]

(15)

This gives the first term in the bracket of Eq. (25) in the text. The second term arises from \( F(2B_0) \) but \( F(2B_1) \) gives only the term of \( O(\eta) \) except for the first term of \( O(\eta^0) \). Namely,

\[
F(2B_0) = \log B_0 - \frac{1}{\sqrt{1 + 4B_0^2}} \log \frac{2B_0 + 1 - \sqrt{1 + 4B_0^2}}{2B_0 + 1 + \sqrt{1 + 4B_0^2}} \approx -\log \left( \frac{1 - B_0}{1 + B_0} \right) \approx 2B_0.
\]

(16)
Then, substituting the definition (9) into Eq. (13), we obtain

\[ S_{3dAF}(\eta) - S_{3dAF}(0) = -\frac{C^n}{2K} F(2B_0) \approx -\frac{C^n}{2K} 2B_0 \]

\[ = -\frac{\sqrt{\eta}}{K} \approx -\frac{\nu F}{K \sqrt{A}} \sqrt{\eta} + O(\eta), \]  

(17)

and the second term in the bracket of Eq. (25) in the text.

3. Derivation of Eqs. (27) and (28) in the text

Here we derive Eqs. (27) and (28) in the text for 3D-AFQCP. Neglecting the \( b^2 \) terms in Eq. (20) in the text, \( \Gamma_v(k, i\epsilon_n) \) is expressed as

\[
\Gamma_v(k, i\epsilon_n) = J\gamma_v(k - Q) \pi T \int_{-1}^1 \frac{d(\cos \theta)}{2} \left\{ \frac{1}{\sqrt{\alpha_+} (\sqrt{\alpha_+} + \epsilon_n')^2} + \frac{1}{\sqrt{\alpha_-} (\sqrt{\alpha_-} + \epsilon_n')^2} + 2ib \left[ \frac{1}{\sqrt{\alpha_+} (\sqrt{\alpha_+} + \epsilon_n')^3} - \frac{1}{\sqrt{\alpha_-} (\sqrt{\alpha_-} + \epsilon_n')^3} \right] \right\},
\]  

(18)

where \( \alpha_k \equiv a + C'| \pm \epsilon_n - \epsilon_n'| \) with \( a \) given by Eq. (15) in the text as \( a = (\nu F^2/\eta A) + 2k_F^2\nu_F^2 (1 - \cos \theta) \), and \( b \equiv k_F\nu_F (1 - \cos \theta) \). Eq. (16) in the text. Here, we have neglected terms of \( O(b^3) \) because the singular contribution to \( \Gamma_v(k, i\epsilon_n) \) arises from the region \( \cos \theta \approx 1 \). We also note that the expression (18) is valid irrespective of the sign of \( \epsilon_n \), positive or negative.

Since we discuss the retarded function of low-energy quasiparticles, we need a linear dependence of \( \Gamma_v(k, i\epsilon_n) \) for \( \epsilon_n > 0 \). In the limit \( T \to 0 \), the summation over \( \epsilon_n' \) is approximated by integration with respect to \( \epsilon' \) as in Eq. (21) in the text, except for the term \( \epsilon_n' = \epsilon_n \) which corresponds to taking static limit \( (\omega_m = 0) \) of spin-fluctuation propagator given by Eq. (9) in the text. The latter term is usually neglected since it usually gives no contribution in the limit \( T \to 0 \). However, it turns out that the most singular contribution to the \( \epsilon_n \) dependence arises as shown below.

Let us define the term arising from \( \epsilon_n' = \epsilon_n \) in the first and the third terms in the brace of Eq. (18) (or Eq. (18) in the text) as \( \tilde{\Gamma}_v(k, i\epsilon_n) \). Then,

\[
\tilde{\Gamma}_v(k, i\epsilon_n) = J\gamma_v(k - Q) \pi T \int_{-1}^1 \frac{d(\cos \theta)}{2} \left[ \frac{1}{\sqrt{\alpha} (\sqrt{\alpha} + \epsilon_n)^2} + 2ib \frac{1}{\sqrt{\alpha} (\sqrt{\alpha} + \epsilon_n)^3} \right].
\]  

(19)

The expression in the bracket is expanded in \( \epsilon_n \) up to linear order as follows:

\[
\left[ \frac{1}{\alpha^{1/2}} - \frac{2}{2a^2} \epsilon_n + \frac{2b}{a^2} \epsilon_n - \frac{2ib}{a^{3/2}} \epsilon_n + O(\epsilon_n^2) \right].
\]  

(20)

With the use of \( a = \tilde{\eta} + K(1 - x) \) and \( b = \sqrt{K}/2(1 - x) \), with \( x \equiv \cos \theta \), \( \tilde{\eta} \equiv (\nu_F^2/\eta A) \), and \( K \equiv 2k_F^2\nu_F^2 \), Eq. (19) is reduced to

\[
\tilde{\Gamma}_v(k, i\epsilon_n) = J\gamma_v(k - Q) \pi T \int_{-1}^1 \frac{dx}{2} \left\{ \frac{1}{[\tilde{\eta} + K(1 - x)]^{3/2}} - \frac{2\epsilon_n}{[\tilde{\eta} + K(1 - x)]^2} + 2i \sqrt{\frac{K}{2}} \frac{(1 - x)\epsilon_n}{[\tilde{\eta} + K(1 - x)]^{1/2}} - 6i \sqrt{\frac{K}{2}} \frac{(1 - x)\epsilon_n}{[\tilde{\eta} + K(1 - x)]^{3/2}} \right\}.
\]  

(21)
The integrations with respect to $x$ in Eq. (21) are easily performed to give

$$\tilde{\Gamma}_\nu(k, i\varepsilon_n) \simeq J\gamma_{\nu}(k - Q) \frac{\pi T}{K} \left( \frac{1}{\sqrt{\eta}} - \frac{\varepsilon_n}{\eta} + \frac{i}{\sqrt{2K}} \log \left( \frac{2K}{\eta} \right) - \frac{4i}{\sqrt{2K}} \frac{\varepsilon_n}{\eta} \right),$$  

(22)

where we have retained the most singular term in each contribution in the curly bracket. Substituting $\tilde{\eta} \equiv v_F^2 \eta/A$ and $K \equiv 2k_F^2 v_F^2$ into Eq. (22), we obtain

$$\tilde{\Gamma}_\nu(k, i\varepsilon_n) \simeq J\gamma_{\nu}(k - Q) \left( \frac{\sqrt{AK^2}}{2(k_Fv_F)^3 \sqrt{\eta}} - \frac{\varepsilon_n}{2} \frac{AK^2}{(k_Fv_F)^4 \eta} \frac{\pi T}{\eta} + \frac{i\pi T}{4} \log \left( \frac{4AK^2}{\eta} - i\varepsilon_n \frac{\sqrt{AK^2}}{(k_Fv_F)^4 \sqrt{\eta}} \right) \right).$$  

(23)

Since $\eta \propto T^{3/2}$ just at the 3d-AFQCP, \({}^1\) the first and the fourth terms in Eq. (23) vanish as $T^{1/4}$ while the second term diverges as $T^{-1/2}$. Although the first term in Eq. (23) vanishes as $T^{1/4}$ in the limit $T \to 0$ at the criticality, this $T$ dependence gives a sharper cusp than the second term of Eq. (25) in the text because $\sqrt{\eta} \propto T^{3/4}$ there.

The singular behavior as Eq. (23), which arises from the term $\varepsilon_{n'} = \varepsilon_n$ in the first and the third terms in the brace of Eq. (18), does not appear for the self-energy $\Sigma_{\nu}(k, i\varepsilon_n)$ in Eq. (32) in the text. This is because the denominators $[b \pm i(\sqrt{a} - \sqrt{b}^2 + \varepsilon_{n'})^2]$ in Eq. (20) in the text should be replaced by $[b \pm i(\sqrt{a} - \sqrt{b}^2 + \varepsilon_{n'})]$ for $\Sigma_{\nu}(k, i\varepsilon_n)$ so that $(\sqrt{a} + \varepsilon_n)^2$ and $(\sqrt{a} + \varepsilon_n)^3$ are replaced by $(\sqrt{a} + \varepsilon_n)^1$ and $(\sqrt{a} + \varepsilon_n)^2$, respectively, in the denominator of Eq. (19), so that the contribution vanishes in the limit $T \to 0$.

The summation in Eq. (18) over $\varepsilon_{n'}$ (except for the term $\varepsilon_{n'} = \varepsilon_n$ in the first and the third terms in the brace) can be approximated, in the region $T \to 0$, by integration with respect to $\varepsilon'$ as follows:

$$\Gamma_{\nu}(k, i\varepsilon_n) - \tilde{\Gamma}_{\nu}(k, i\varepsilon_n) = J\gamma_{\nu}(k - Q) \int_{-\infty}^{\varepsilon_n - \pi T} \frac{d(cos \theta)}{2} \left[ L(\varepsilon_n, \cos \theta) + 2ib M(\varepsilon_n, \cos \theta) \right],$$  

(24)

where $L(\varepsilon_n, \cos \theta)$ and $M(\varepsilon_n, \cos \theta)$ are given by

$$L(\varepsilon_n, \cos \theta) = \int_{0}^{\varepsilon_n - \pi T} \frac{d\varepsilon'}{2} \frac{1}{\sqrt{a + C^\prime(\varepsilon_n - \varepsilon')(\sqrt{a + C^\prime(\varepsilon_n - \varepsilon') + \varepsilon'}^2)}},$$

$$+ \int_{\varepsilon_n + \pi T}^{\infty} \frac{d\varepsilon'}{2} \frac{1}{\sqrt{a + C^\prime(\varepsilon_n' - \varepsilon_n)(\sqrt{a + C^\prime(\varepsilon_n' - \varepsilon_n) + \varepsilon'}^2))},$$

$$+ \int_{0}^{\infty} \frac{d\varepsilon'}{2} \frac{1}{\sqrt{a + C^\prime(\varepsilon_n' + \varepsilon_n)(\sqrt{a + C^\prime(\varepsilon_n' + \varepsilon_n) + \varepsilon'}^2)}},$$

(25)

and

$$M(\varepsilon_n, \cos \theta) = \int_{0}^{\varepsilon_n - \pi T} \frac{d\varepsilon'}{2} \frac{1}{\sqrt{a + C^\prime(\varepsilon_n - \varepsilon')(\sqrt{a + C^\prime(\varepsilon_n - \varepsilon') + \varepsilon'}^3)}},$$

$$+ \int_{\varepsilon_n + \pi T}^{\infty} \frac{d\varepsilon'}{2} \frac{1}{\sqrt{a + C^\prime(\varepsilon_n' - \varepsilon_n)(\sqrt{a + C^\prime(\varepsilon_n' - \varepsilon_n) + \varepsilon'}^3))},$$

$$- \int_{0}^{\infty} \frac{d\varepsilon'}{2} \frac{1}{\sqrt{a + C^\prime(\varepsilon_n' + \varepsilon_n)(\sqrt{a + C^\prime(\varepsilon_n' + \varepsilon_n) + \varepsilon'}^3))}.$$  

(26)

In deriving Eqs. (25) and (26) from Eq. (18), we have used the relation between a definite integral...
with respect to $\varepsilon'$ and a summation over $n'$ in the region $T \to 0$ such as

$$
\pi T \sum_{0 \leq n' < n} F(\varepsilon'_n; \varepsilon_n, T) + \pi T \sum_{n' > n} F(\varepsilon'_n; \varepsilon_n, T)
\approx \left[ \frac{1}{2} \int_0^{\varepsilon_n - \pi T} d\varepsilon' F(\varepsilon'; \varepsilon_n, T) + \frac{1}{2} \int_{\varepsilon_n + \pi T}^{\infty} d\varepsilon' F(\varepsilon'; \varepsilon_n, T) \right] + O\left( \frac{C^*}{a^{3/2} \pi T \varepsilon_n} \right),
$$

(27)

and

$$
\pi T \sum_{n' \geq 0} F(\varepsilon'_n; \varepsilon_n, T) \approx \frac{1}{2} \int_0^{\infty} d\varepsilon' F(\varepsilon'; \varepsilon_n, T) \left[ 1 + O\left( \frac{C^*}{a^{3/2} \pi T \varepsilon_n} \right) \right],
$$

(28)

where $F(\varepsilon'_n; \varepsilon_n, T)$ stands for the functional forms in the brace of Eq. (18). The explicit $T$ dependence represents that of $\eta$ included in the parameter $a = (v_F^2 \eta/A) + 2k_F^2 v_F^2 (1 - \cos \theta)$. In deriving the approximate relation in Eqs. (27) and (28), we have used the trapezoidal rule and the estimation of its relative accuracy.

By changing the integration variable from $\varepsilon'$ to $\varepsilon'' \equiv \varepsilon_n - \varepsilon'$ in the first term in Eq. (25), from $\varepsilon'$ to $\varepsilon'' = \varepsilon' - \varepsilon_n$ in the second term in Eq. (25), and from $\varepsilon'$ to $\varepsilon'' = \varepsilon_n + \varepsilon'$ in the third term in Eq. (25), $L(\varepsilon_n, \cos \theta)$ is reduced to

$$
L(\varepsilon_n, \cos \theta) = \int_{\pi T}^{\varepsilon_n} \frac{d\varepsilon''}{2} \frac{1}{\sqrt{a + C^* \varepsilon''} (\sqrt{a + C^* \varepsilon'' + \varepsilon_n - \varepsilon''})^2} + \int_{\pi T}^{\infty} \frac{d\varepsilon''}{2} \frac{1}{\sqrt{a + C^* \varepsilon''} (\sqrt{a + C^* \varepsilon'' + \varepsilon' + \varepsilon_n})^2} + \int_{\pi T}^{\infty} \frac{d\varepsilon''}{2} \frac{1}{\sqrt{a + C^* \varepsilon''} (\sqrt{a + C^* \varepsilon'' + \varepsilon'' - \varepsilon_n})^2} - \int_{\pi T}^{\varepsilon_n} \frac{d\varepsilon''}{2} \frac{1}{\sqrt{a + C^* \varepsilon''} (\sqrt{a + C^* \varepsilon'' + \varepsilon'' - \varepsilon_n})^2}.
$$

(29)

It is easy to see that the sum of the second and the third terms of Eq. (29) are even functions in $\varepsilon_n$. The summation of the first and the fourth terms in Eq. (29) is calculated in the following way.

Let us introduce $\Delta X(\varepsilon_n)$ representing the summation of the first and the fourth terms of Eq. (29). Then, it is reduced to

$$
\Delta X(\varepsilon_n) = \int_{\pi T}^{\varepsilon_n} \frac{d\varepsilon''}{2} \frac{1}{\sqrt{a + C^* \varepsilon''} (\sqrt{a + C^* \varepsilon'' + \varepsilon_n - \varepsilon''})^2} - \frac{1}{\sqrt{a + C^* \varepsilon'' - \varepsilon'' + \varepsilon_n})^2}.
$$

(30)

By changing the integration variable from $\varepsilon''$ to $y \equiv \varepsilon''/\varepsilon_n$, we obtain

$$
\Delta X(\varepsilon_n) = -4 \int_{\pi T / \varepsilon_n}^{\varepsilon_n} \frac{dy}{|\sqrt{a + C^* \varepsilon_n y - \varepsilon_n^2 (1 - y)^2}|}.
$$

(31)

(32)

By changing the integration variable from $\varepsilon''$ to $y \equiv \varepsilon''/\varepsilon_n$, we obtain

$$
\Delta X(\varepsilon_n) = -4 \int_{\pi T / \varepsilon_n}^{\varepsilon_n} \frac{dy}{|\sqrt{a + C^* \varepsilon_n y - \varepsilon_n^2 (1 - y)^2}|}.
$$

Here, we note that $\pi T / \varepsilon_n$, the lower limit of integration in Eq. (32), must satisfy $\pi T / \varepsilon_n \leq 1$ because we are discussing the analytic continuation from the upper half plane of $i \varepsilon_n$. Then, up to the order of $O(\varepsilon_n^2)$, the denominator of Eq. (32) is approximated by $a^2$. Therefore, $\Delta X(\varepsilon_n)$ is given by

$$
\Delta X(\varepsilon_n) = - \frac{\varepsilon_n^2}{a^2} \left[ 1 - \frac{2 \pi T}{\varepsilon_n} + \frac{(\pi T)^2}{\varepsilon_n^2} \right].
$$
where the term \((\pi T)^2/a^2\) has been discarded because it gives only a negligible contribution to Eq. (18) in the asymptotic region \(T \sim 0\) as far as terms up to \(O(\varepsilon_n)\) are concerned. Therefore, up to \(O(\varepsilon_n)\), \(L(\varepsilon_n, \cos \theta)\) is given by

\[
L(\varepsilon_n, \cos \theta) \simeq \int_{\pi T}^{\pi T} \frac{d\varepsilon''}{\sqrt{\alpha + C^\star \varepsilon'' + \varepsilon'' + \varepsilon''}} + \pi T \frac{2}{a^2} \varepsilon_n + O(\varepsilon_n^2),
\]

where the first term is the contribution from the second and the third terms of Eq. (29).

Similarly, the expression Eq. (26) is reduced to

\[
M(\varepsilon_n, \cos \theta) = \int_{\pi T}^{\pi T} \frac{d\varepsilon''}{\sqrt{\alpha + C^\star \varepsilon'' + \varepsilon'' + \varepsilon''}} + \pi T \frac{2}{a^2} \varepsilon_n + O(\varepsilon_n^2),
\]

The summation of the first and the fourth terms in Eq. (35) is calculated in parallel to the derivation of Eq. (33). Let us introduce \(\Delta Y(\varepsilon_n)\) representing the summation of the first and the fourth terms of Eq. (35). Then, it is reduced to

\[
\Delta Y(\varepsilon_n) = \int_{\pi T}^{\pi T} \frac{d\varepsilon''}{2} \left[ \frac{1}{\sqrt{\alpha + C^\star \varepsilon''}} \left( \frac{1}{\varepsilon'' + \varepsilon_n - \varepsilon''} + \frac{1}{\varepsilon'' + \varepsilon_n + \varepsilon''} \right) \right]
\]

By changing the integration variable from \(\varepsilon''\) to \(y \equiv \varepsilon''/\varepsilon_n\), we obtain

\[
\Delta Y(\varepsilon_n) = \varepsilon_n \int_{\pi T/\varepsilon_n}^{1} \frac{dy}{\alpha + C^\star \varepsilon_n y - \varepsilon_n^2 (1 - y)^2}. \tag{37}
\]

Then, up to the order of \(O(\varepsilon_n)\), the numerator and the denominator of Eq. (37) are approximated by \(a\) and \(a^3\), respectively. Therefore, \(\Delta Y(\varepsilon_n)\) is given by

\[
\Delta Y(\varepsilon_n) = \frac{\varepsilon_n}{a^3} \frac{\pi T}{\varepsilon_n} \tag{38}
\]

Therefore, by expanding the second and the third terms of Eq. (35) up to \(O(\varepsilon_n)\), \(M(\varepsilon_n, \cos \theta)\) is given by

\[
M(\varepsilon_n, \cos \theta) \simeq -\frac{\pi T}{a^2} + \frac{\varepsilon_n}{a^2} - \int_{\pi T}^{\pi T} \frac{d\varepsilon''}{\sqrt{\alpha + C^\star \varepsilon'' + \varepsilon'' + \varepsilon''}} + O(\varepsilon_n^2). \tag{39}
\]
By changing the integration variable from $\varepsilon''$ to $p \equiv \varepsilon''/a$, the third term on the r.h.s. of Eq. (39) is reduced to

$$
- \frac{1}{a^{3/2}} \int_{\pi T/a}^{\infty} dp \frac{3\varepsilon_n}{\sqrt{1 + C'p (\sqrt{1 + C'p} + \sqrt{a}p)^2}}.
$$

(40)

Singularity arising from this term is much smaller than that from the second term on the r.h.s. of Eq. (39) because the singularity stems from the vanishing behavior in $a \equiv (v_F^2 \eta/A) + 2k_F^2v_F^2(1 - \cos \theta)$ near $\cos \theta \approx 1$ and $\eta \approx 0$. Therefore, neglecting the third term on the r.h.s. of Eq. (39), we obtain

$$
M(\varepsilon_n, \cos \theta) \approx -\frac{\pi T}{a^2} + \frac{\varepsilon_n}{\varepsilon_n^2} + O(\varepsilon_n^2).
$$

(41)

Collecting the results of Eqs. (24), (34), and (41), we obtain the $\Gamma_v(k, i\varepsilon_n)$ up to $O(\varepsilon_n)$ as follows:

$$
\Gamma_v(k, i\varepsilon_n) \approx \tilde{\Gamma}_v(k, i\varepsilon_n) + J\gamma_v(k - Q) \int_{-1}^{1} \frac{d(\cos \theta)}{2} \left[ \int_{\pi T}^{\infty} \frac{d\varepsilon''}{2} \frac{1}{\sqrt{1 + C'\varepsilon''(\sqrt{1 + C'\varepsilon''} + \varepsilon'')^2}} + \pi T \frac{2}{a^2} \varepsilon_n + 2ib \left( -\frac{\pi T}{a^2} + \frac{\varepsilon_n}{\varepsilon_n^2} \right) + O(\varepsilon_n^2) \right],
$$

(42)

where $a \equiv (v_F^2 \eta/A) + 2k_F^2v_F^2(1 - \cos \theta)$ and $b \equiv k_Fv_F(1 - \cos \theta)$. Then, the linear terms of $\Gamma_v(k, i\varepsilon_n)$ in $\varepsilon_n$ is obtained as follows:

$$
\Gamma_v(k, i\varepsilon_n) - \Gamma_v(k, i\varepsilon_n) \big|_{\varepsilon_n=0} \approx \tilde{\Gamma}_v(k, i\varepsilon_n) - \tilde{\Gamma}_v(k, i\varepsilon_n) \big|_{\varepsilon_n=0} + J\gamma_v(k - Q) \int_{-1}^{1} \frac{d(\cos \theta)}{2} \left[ \pi T \frac{2}{a^2} \varepsilon_n + 2b \left( \frac{i\varepsilon_n}{\varepsilon_n^2} \right) \right].
$$

(43)

We note that, in the limit $T \to 0$, the first term in the square bracket of Eq. (42) is reduced to the second term of Eq. (22) in the text. Integrations with respect to $\cos \theta$ on the r.h.s. of Eq. (42) are easily performed:

$$
\int_{-1}^{1} \frac{d(\cos \theta)}{2} \frac{1}{a^2} \frac{1}{\sqrt{1 + C'\varepsilon''(\sqrt{1 + C'\varepsilon''} + \varepsilon'')^2}} = \frac{Ak_F^2}{4(k_F^2v_F^2)\eta} \left( \frac{1}{\eta} - \frac{1}{\eta + 4Ak_F^2} \right) \approx \frac{4Ak_F^2}{4(k_F^2v_F^2)\eta} \frac{1}{\eta},
$$

(44)

and

$$
\int_{-1}^{1} \frac{d(\cos \theta)}{2} \frac{b}{a^2} \frac{1}{8(k_F^2v_F^2)} \left( \log \frac{4Ak_F^2 + \eta}{\eta} - \frac{4Ak_F^2}{4Ak_F^2 + \eta} \right) \approx \frac{1}{8(k_F^2v_F^2)} \log \frac{4Ak_F^2}{\eta^2}.
$$

(45)

where we have retained the most singular terms and within the logarithmic accuracy to obtain the last approximate expressions.

Finally, substituting Eqs. (23), (44), and (45) into Eq. (43), we obtain

$$
\Gamma_v(k, i\varepsilon_n) \approx \Gamma_v(k, i\varepsilon_n) \big|_{\varepsilon_n=0} + J\gamma_v(k - Q) \left[ \frac{\sqrt{Ak_F^2}}{(k_F^2v_F^2)\sqrt{\eta}} + \frac{1}{4(k_F^2v_F^2)} \left( \log \frac{4Ak_F^2}{\eta} \right) \right](i\varepsilon_n) + O(\varepsilon_n^2).
$$

(46)

Here we note that the term obtained from the first term in the square bracket of Eq. (43) cancels out the second term in the bracket of Eq. (23), so that the term proportional to $\varepsilon_n$ disappears but only that proportional to $i\varepsilon_n$ remains. This result verifies the fact that the first line in the brace of Eq. (18), i.e., its real part, is an even function in $\varepsilon_n$. Equation (46) gives Eqs. (27) and (28) in the text. We also note
that the term $-2ib\pi T/a^2$ in Eq. (42) cancels out the third term in Eq. (20) or Eq. (23).

4. Derivation of Eqs. (33) and (34) in the text

With the use of Eqs. (10) and (12) in the text, an explicit form of Eq. (31) in the text is given by

$$
\left[ \hat{G}^R (k, \epsilon + i\delta) \right]^{-1} = \begin{pmatrix}
\epsilon - \xi_k - \Sigma^R_1 (k, \epsilon) & -a\gamma (k) \cdot \hat{\sigma}_{\uparrow \downarrow} - \Sigma^R_{\uparrow \downarrow} (k, \epsilon) \\
-a\gamma (k) \cdot \hat{\sigma}_{\downarrow \uparrow} - \Sigma^R_{\downarrow \uparrow} (k, \epsilon) & \epsilon - \xi_k - \Sigma^R_0 (k, \epsilon)
\end{pmatrix},
$$

(47)

where $\Sigma^R_{\uparrow \downarrow}$ and $\Sigma^R_{\downarrow \uparrow}$ are defined by Eqs. (29) and (30) in the text, respectively. For concise presentation, let us introduce the following notation:

$$
\tilde{\Gamma}_v (k, \epsilon) = a\gamma_v (k) + \gamma_v (k - Q) [\alpha_{af} (\eta) + (h + ig)e].
$$

(48)

Although $g = 0$ as discussed in Sect. 3 (e.g., as derived from Eq. (46)), we here give a general expression for the dispersion of quasiparticles assuming $g \neq 0$ in general. Then, Eq. (47) is written in a concise form as follows:

$$
\left[ \hat{G}^R (k, \epsilon + i\delta) \right]^{-1} = \begin{pmatrix}
\epsilon - \xi_k - \Sigma^R_1 (k, \epsilon) & -\sum_{v=x,y} \tilde{\Gamma}_v (k, \epsilon) \sigma_{v}^{\uparrow \downarrow} \\
-\sum_{v=x,y} \tilde{\Gamma}_v (k, \epsilon) \sigma_{v}^{\downarrow \uparrow} & \epsilon - \xi_k - \Sigma^R_0 (k, \epsilon)
\end{pmatrix}.
$$

(49)

Dispersion of quasiparticles is determined by the condition that $|\hat{G}^R (k, \epsilon + i\delta)|^{-1} = 0$:

$$
\left[ \epsilon - \xi_k - \Sigma^R_1 (k, \epsilon) \right]^2 = \sum_{v=x,y} \tilde{\Gamma}_v (k, \epsilon) \sigma_{v}^{\uparrow \downarrow} \times \sum_{v=x,y} \tilde{\Gamma}_v (k, \epsilon) \sigma_{v}^{\downarrow \uparrow}
$$

(50)

The r.h.s. of Eq. (50) is transformed into

$$
\sum_{v=x,y} \tilde{\Gamma}_v \sigma_{v}^{\uparrow \downarrow} \times \sum_{v=x,y} \tilde{\Gamma}_v \sigma_{v}^{\downarrow \uparrow} = (|\tilde{\Gamma}_x|^2 + |\tilde{\Gamma}_y|^2) + i \left( \tilde{\Gamma}_x \times \tilde{\Gamma}_y \right)_z,
$$

(51)

where we have abbreviated $\tilde{\Gamma}_v (k, \epsilon)$ to $\tilde{\Gamma}_v$ for conciseness. With the use of Eq. (48), we calculate the two terms on the r.h.s. of Eq. (51):

$$
|\tilde{\Gamma}_x|^2 + |\tilde{\Gamma}_y|^2 = [a\gamma (k) + \alpha_{af} \gamma (k - Q)]^2 + 2\hbar e \gamma (k - Q) \cdot [a\gamma (k) + \alpha_{af} \gamma (k - Q)] + O(\epsilon^2),
$$

(52)

and

$$
i \left( \tilde{\Gamma}_x \times \tilde{\Gamma}_y \right)_z = 2a\hbar e \gamma (k) \times \gamma (k - Q)_z.
$$

(53)

Then, the condition Eq. (50), determining the dispersion of quasiparticles, is reduced to

$$
\epsilon - \xi_k - \Sigma^R_1 (k, 0) - \frac{\partial \Sigma^R_1 (k, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} \epsilon = \pm \left[ (a\gamma (k) + \alpha_{af} \gamma (k - Q))^2 + 2a\hbar e \gamma (k) \times \gamma (k - Q)_z \right]^{1/2} + 2\hbar e \gamma (k - Q) \cdot [a\gamma (k) + \alpha_{af} \gamma (k - Q)]^{1/2}.
$$

(54)

Therefore, the dispersion of the quasiparticles is given by

$$
\epsilon \left[ 1 - \frac{\partial \Sigma^R_1 (k, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} \right] = \pm \frac{\hbar [a\gamma (k) + \alpha_{af} \gamma (k - Q)] \cdot \gamma (k - Q)}{|a\gamma (k) + \alpha_{af} \gamma (k - Q)|}.
$$

(54)
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\[ \Delta k + \sum_{i=1}^{N} (k, 0) \pm |\alpha \gamma (k) + \alpha_{df} \gamma (k - Q)| \pm \frac{\alpha (ge)}{|\alpha \gamma (k) + \alpha_{df} \gamma (k - Q)|}. \]  (55)

This gives the dispersion relation presented as Eqs. (33) and (34) in the text if \( g = 0 \).

5. Derivation of Eq. (38) in the text

Let us introduce \( S_{3df}(\eta) \) representing double integration part in Eq. (38) in the text. Substituting the expression (15) in the text into the expression of \( S_{3df}(\eta) \), \( S_{3df}(\eta) \) is explicitly written as

\[ S_{3df}(\eta) = \int_{-1}^{1} \frac{dx}{2} \frac{\sqrt{1-x}}{C^* \sqrt{\eta + K(1-x)}} \int_{1}^{\infty} dy \frac{1}{\sqrt{\eta + K(1-x) (y^2 - 1) + y}}. \]  (56)

where \( x = \cos \theta, \eta \equiv v_T^2 \eta / A \) and \( K \equiv 2k_F^2 v_T^2 \). Changing the integration variable from \( x \) to \( s \equiv \sqrt{1-x} \), Eq. (56) is reduced to

\[ S_{3df}(\eta) = \int_{0}^{\sqrt{2}} ds \frac{s^2}{C^* \sqrt{\eta + K s^2}} \int_{1}^{\infty} dy \frac{1}{\sqrt{\eta + K s^2 (y^2 - 1) + y}}. \]  (57)

First, we calculate \( S_{3df}(0) \) which is given by

\[ S_{3df}(0) = \int_{0}^{\sqrt{2}} ds \frac{s}{C^* \sqrt{K}} \int_{1}^{\infty} dy \frac{1}{\sqrt{K s^2 (y^2 - 1) + y}}. \]  (58)

Changing the integration variable from \( s \) to \( z \equiv s^2 \), Eq. (58) is reduced to

\[ S_{3df}(0) = \frac{1}{2C^* \sqrt{K}} \int_{0}^{2} dz \int_{1}^{\infty} dy \frac{1}{\sqrt{K z} (y^2 - 1) + y}. \]  (59)

Then the integration with respect to \( z \) is easily performed:

\[ S_{3df}(0) = -\frac{1}{2K} \int_{1}^{\infty} dy \frac{1}{y^2 - 1} \left[ \frac{1}{2\sqrt{K} (y^2 - 1) + y} - \frac{1}{y} \right] \]

\[ = \frac{1}{2K} \int_{1}^{\infty} dy \frac{2\sqrt{K}}{y [2\sqrt{K} (y^2 - 1) + y]}. \]  (60)

Then, with the use of Eq. (12), we obtain

\[ S_{3df}(0) = \frac{1}{4K} F \left( \frac{4\sqrt{K}}{C^*} \right) = \frac{1}{4K} F \left( \frac{4\sqrt{2} m^* A}{C} \right), \]  (61)

where we have used definitions, \( C^* \equiv C v_T^2 / A \) and \( K \equiv 2k_F^2 v_T^2 \).

Next, let us discuss the lowest-order correction in \( \eta \), which will turn out to be of the order of \( O(\eta \log \eta) \). To this end, we calculate \( S_{3df}(\eta) - S_{3df}(0) \): With the use of the expression Eq. (57), we
obtain

\[ S_{3MF}(\eta) - S_{3MF}(0) = \frac{1}{C^*} \left\{ \int_0^{\sqrt{2}} ds \frac{s^2}{\sqrt{\eta + Ks^2}} \int_1^{\infty} dy \frac{1}{s \sqrt{\eta + Ks^2} (y^2 - 1) + y^2} \right\} \]

\[ - \frac{s}{\sqrt{K}} \int_1^{\infty} dy \frac{1}{s \sqrt{\eta + Ks^2} (y^2 - 1) + y^2} \]

\[ + \int_0^{\sqrt{2}} ds \frac{s}{C^* \sqrt{K}} \int_1^{\infty} dy \left\{ \frac{1}{s \sqrt{\eta + Ks^2} (y^2 - 1) + y^2} - \frac{1}{s \sqrt{\eta + Ks^2} (y^2 - 1) + y^2} \right\} \]  \hspace{1cm} (62)

It is easy to see that the second term of Eq. (62) can be expanded into the Taylor series so that this term gives only \(O(\eta)\) at most and can be safely neglected up to the logarithmic accuracy. Let us define the integration with respect \(y\) in the first term of Eq. (62) as \(L(s)\): i.e.,

\[ L(s) = \int_1^{\infty} dy \frac{1}{s \sqrt{\eta + Ks^2} (y^2 - 1) + y^2}. \]  \hspace{1cm} (63)

Then, the first term of Eq. (62), denoted by \(X(\eta)\), is expressed as

\[ X(\eta) = \frac{1}{C^*} \int_0^{\sqrt{2}} ds \left[ \frac{s^2}{\sqrt{\eta + Ks^2}} - \frac{s}{\sqrt{K}} \right] L(s). \]  \hspace{1cm} (64)

Changing the integration variable from \(s\) to \(z \equiv s^2\), Eq. (64) is transformed through partial integration with respect to \(z\) as follows:

\[ X(\eta) = \frac{1}{2C^* \sqrt{K}} \int_0^{\sqrt{2}} dz \left( \sqrt{\frac{z}{z + \frac{\eta}{K}}} - 1 \right) L(\sqrt{z}) \]

\[ = \frac{1}{2C^* \sqrt{K}} \left\{ \sqrt{2 \left( 2 + \frac{\eta}{K} \right)} - 2 + \frac{1}{2} \log \left[ \sqrt{2 + \frac{\eta}{K} + \sqrt{2 \left( 2 + \frac{\eta}{K} \right)}} \right] \right\} L(\sqrt{z}) \]

\[ - \frac{\eta}{2K} \int_0^{\sqrt{2}} dz \left[ \sqrt{z \left( z + \frac{\eta}{K} \right)} - z + \frac{1}{2} \log \left[ \frac{\sqrt{z + \frac{\eta}{K} + \sqrt{z \left( z + \frac{\eta}{K} \right)}}}{\sqrt{z + \frac{\eta}{K} + \sqrt{2 \left( 2 + \frac{\eta}{K} \right)}}} \right] \frac{dL(\sqrt{z})}{dz} \} \]  \hspace{1cm} (65)

In the limit of \(\eta \to 0\), the first term in the brace of Eq. (65), denoted by \(2 \sqrt{K} X^{(1)}(\eta)\), is given by

\[ X^{(1)}(\eta) = \frac{1}{4K^{3/2}} \log \frac{\eta}{8K} \times \lim_{\eta \to 0} L(\sqrt{z}), \]  \hspace{1cm} (66)
where, according to Eq. \((63)\), \(\lim_{\eta \to 0} L(\sqrt{\eta})\) is given by
\[
\lim_{\eta \to 0} L(\sqrt{\eta}) = \int_{1}^{\infty} dy \frac{1}{\left[\frac{2\sqrt{\eta}}{C} (y^2 - 1) + y\right]^2}.
\] (67)

After elementary integrations, an explicit form of Eq. (67) is
\[
\lim_{\eta \to 0} L(\sqrt{\eta}) = \frac{2\tilde{B}^*}{1 + 4\tilde{B}^*} \left[1 + \frac{1}{2\tilde{B}^*} + \log \tilde{B}^* - F(2\tilde{B}^*)\right] = \frac{4}{B^*} G(2\tilde{B}^*),
\] (68)
where
\[
\tilde{B}^* = \frac{2\sqrt{K}}{C^*}.
\] (69)

The last equality in Eq. (68) is obtained using the function \(G(x)\) defined by Eq. (40) in the text.

The second term in the brace of Eq. (65), denoted by \(2\sqrt{KX^{(2)}(\eta)}\), is estimated as follows: Changing the integration variable from \(z\) to \(w \equiv Kz/\eta\), \(X^{(2)}(\eta)\) is expressed as
\[
X^{(2)}(\eta) = -\frac{\tilde{\eta}}{2K^{3/2}} \int_{0}^{2K/\eta} dw \left[\sqrt{w(w+1)} - w + \frac{1}{2} \log \left|\frac{\sqrt{w+1} - \sqrt{w}}{\sqrt{w+1} + \sqrt{w}}\right|\right] \frac{dL(\sqrt{\eta w/K})}{dw}.
\] (70)

Since the upper limit of the integration with respect to \(w\) in Eq. (70) diverges in the limit \(\eta \to 0\), there is a possibility that a singularity in \(\eta\) arises. Therefore, the behavior of \(dL(\sqrt{\eta w/K})/dw\) at \(w = 2K/\tilde{\eta} \to \infty\) should be estimated. With the use of expression for \(L(s)\), Eq. (63), we obtain
\[
\frac{dL(\sqrt{\eta w/K})}{dw} \approx -\frac{\tilde{\eta}}{C^* \sqrt{K}} \int_{1}^{\infty} dy \frac{y^2 - 1}{\left[\frac{\sqrt{\eta w + 1}}{C^* \sqrt{K}} (y^2 - 1) + y\right]^3} \left(\frac{\sqrt{w+1} - \sqrt{w}}{\sqrt{w+1} + \sqrt{w}}\right).
\] (71)

Near the upper limit of \(w = 2K/\tilde{\eta}\) in the limit of \(\eta \to 0\), \(dL(\sqrt{\eta w/K})/dw\) is estimated as
\[
\frac{dL(\sqrt{\eta w/K})}{dw} \approx -\frac{2\tilde{\eta}}{C^* \sqrt{K}} \frac{2}{B^2} H(\tilde{B}^*).
\] (72)

With the use of definition (41) in the text, Eq. (72) is expressed as
\[
\frac{dL(\sqrt{\eta w/K})}{dw} \approx -\frac{2\tilde{\eta}}{C^* \sqrt{K}} \frac{2}{B^2} \tilde{B}^* H(\tilde{B}^*).
\] (73)

In the limit of \(w \to \infty\), the following asymptotic formulas hold:
\[
\log \left|\frac{\sqrt{w+1} - \sqrt{w}}{\sqrt{w+1} + \sqrt{w}}\right| \approx -\log |4w|,
\] (74)
and
\[
\sqrt{w(w+1)} - w \approx \frac{1}{2},
\] (75)
Therefore, \(X^{(2)}(\eta)\), Eq. (70), is given by
\[
X^{(2)}(\eta) \approx \frac{\tilde{\eta}}{2K^{3/2}} \frac{2\tilde{\eta}}{C^* \sqrt{K}} \frac{2}{B^2} H(\tilde{B}^*) \int_{0}^{2K/\eta} dw \left(-\frac{1}{2} - \frac{1}{2} \log |4w|\right)
\approx \frac{1}{4} \frac{\tilde{\eta}}{K^{3/2}} \log \frac{e^2}{8K} \times \frac{4}{B^*} H(\tilde{B}^*).
\] (76)
In deriving the last approximate result in Eq. (76), we have discarded the term of the order of $O(\eta^2 \log \eta)$, and used the definition of $\tilde{B}^*$, Eq. (69).

Collecting the above results, Eqs. (66), (68), and (76), near $\eta \sim 0$, $X(\eta)$, Eq. (65), is given by

$$X(\eta) \approx \frac{1}{4} \frac{\tilde{\eta}}{K^{3/2} C^{*} \tilde{B}^{*}} \left[ G(2\tilde{B}^{*}) \log \frac{e^{\tilde{\eta}}}{8K} + \tilde{H}^{*} \log \frac{e^{2\tilde{\eta}}}{8K} \right].$$

(77)

With the use of definitions of $\tilde{B}^*$, Eq. (69), $C^* \equiv \tilde{C} \nu^2 F_2 / A$, and $\tilde{\eta} \equiv v_0^2 \nu / A$, Eq. (77) is transformed into a compact form as

$$X(\eta) \approx \frac{1}{4K} \frac{\eta}{A k_F^2} \left[ G \left( \frac{4 \sqrt{2} m^* A}{C} \right) \log \frac{e \eta}{16A k_F^2} + H \left( \frac{2 \sqrt{2} m^* A}{C} \right) \log \frac{e^2 \eta}{16A k_F^2} \right].$$

(78)

Then, $S_{3dF}(\eta) - S_{3dF}(0)$, Eq. (68), is given by

$$S_{3dF}(\eta) - S_{3dF}(0) \approx \frac{1}{4K} \frac{\eta}{A k_F^2} \left[ G \left( \frac{4 \sqrt{2} m^* A}{C} \right) \log \frac{e \eta}{16A k_F^2} + H \left( \frac{2 \sqrt{2} m^* A}{C} \right) \log \frac{e^2 \eta}{16A k_F^2} \right].$$

(79)

Finally, adding Eq. (61) and Eq. (79), the second term of Eq. (37) in the text is reduced to the first and second terms in the brace of Eq. (38) in the text.

6. Derivation of Eq. (42) in the text

Here we derive Eq. (42) in the text for the case of 3D-FQCP. We start with Eq. (18), the same equation as that in the case of 3D-AFQCP except that $C^*$, in the parameter $\tilde{a}_s \equiv a + C^* | \pm \epsilon_n - \epsilon_n^* |$, is not constant but $C^* \equiv \tilde{C}^{*} / \sqrt{1 - \cos \theta}$ with $\tilde{C}^*$ being a constant. Then, we obtain the same expression for $\Gamma_\nu(k, i\epsilon_n)$ as Eq. (19). However, since Eq. (19) does not include the coefficient $C^*$, the expression Eq. (21) for $\Gamma_\nu(k, i\epsilon_n)$ can be also applied to the present case of 3D-FQCP:

$$\Gamma_\nu(k, i\epsilon_n) = J Y_\nu(k) \pi T \int_{-1}^{1} \frac{dx}{2} \left\{ \frac{1}{[\tilde{\eta} + K(1 - x}]^{3/2} \right\} - \frac{2\epsilon_n}{[\tilde{\eta} + K(1 - x)]^2}$$

$$+ 2i \sqrt{\frac{K}{2}} \left( \frac{1 - x}{[\tilde{\eta} + K(1 - x)]^2} \right) - 6i \sqrt{\frac{K}{2}} \left( \frac{\epsilon_n}{[\tilde{\eta} + K(1 - x)]^5/2} \right).$$

(80)

Then, the expression Eq. (23) for $\tilde{\Gamma}_\nu(k, i\epsilon_n)$ is also valid in the case of 3d-FQCP:

$$\tilde{\Gamma}_\nu(k, i\epsilon_n) \approx J Y_\nu(k) \left\{ \sqrt{\frac{A k_F^2}{2(2k_F^2)^3}} \sqrt{\frac{\pi T}{\eta}} - \frac{A k_F^2}{2(2k_F^2)^3} \frac{\pi T}{\eta} + \frac{i \pi T}{4(k_F^2)^2} \log \frac{4A k_F^2}{\eta} \right\}.$$ 

(81)

Since $\eta \propto T^{4/3}$ just at the 3D-FQCP, the first and fourth terms in Eq. (81) vanishes as $T^{1/3}$ while the second term diverges as $T^{-1/3}$. Although the first term in Eq. (81) vanishes as $T^{1/3}$ in the limit $T \to 0$ at the criticality, this $T$ dependence gives a sharper cusp than the second term of Eq. (39) in the text because $\eta \log \eta \propto T^{4/3} \log T$ there.

The summation over $\epsilon_n^*$ (except for the term $\epsilon_n = \epsilon_n$) in Eq. (18) can be approximated, in the region $T \to 0$, by integration with respect to $\epsilon'$ as discussed in Sect. 3. Since the final results for this contribution in the case of 3D-AFQCP, i.e., Eqs. (43) ~ (45), also do not include the parameter $C^*$, they are also applied to the case of 3D-FQCP.
Then, $\Gamma_\nu(k, i\epsilon_n)$ is given by the same expression as Eq. (46):

$$
\Gamma_\nu(k, i\epsilon_n) \approx \Gamma_\nu(k, i\epsilon_n)|_{\epsilon_n=0} + J\gamma_\nu(k - Q) \left[ -\frac{\sqrt{AK_\nu^2}}{(kF_{\nu T})^2} \pi \sqrt{\eta} + \frac{1}{4(kF_{\nu T})^3} \left( \log \frac{4AK_\nu^2}{\eta e} \right) (i\epsilon_n) + O(\epsilon_n^2) \right]. \quad (82)
$$

This gives Eq. (42) in the text.

As in the case of 3D-AFQCP, the term $-2ib\pi T/a^2$ in Eq. (42) cancels out the third term in Eqs. (80) and (81).

7. Derivation of Eq. (43) in the text

Near the 2D-AFQCP, instead of Eq. (24) in the text, the correction to ASSO coupling $\alpha_{af}(\eta)$ is given by

$$
\alpha_{af}(\eta) = \frac{\tilde{\Gamma}_\nu(k, i\delta)}{\gamma_\nu(k - Q)} + \frac{2J}{C^*} \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_1^{\infty} \frac{dy}{y^2 \left[ \eta + \sqrt{\eta} + K(1 - \cos \varphi) \right]^2}, \quad (83)
$$

where $\cos \varphi \equiv (k \cdot k')/(|k||k'|)$, and

$$
a^* \equiv \tilde{\eta} + K(1 - \cos \varphi). \quad (84)
$$

Let us introduce $\tilde{\alpha}_{af}(\eta)$ expressing the second term on the r.h.s. of Eq. (83). Since the leading singularity for $\tilde{\alpha}_{af}(\eta)$ in $\eta$ arises from the integration with respect to $\varphi$ near $\cos \varphi \sim 1$ of the term $1/\sqrt{a^*}$, in the denominator of integration with respect to $y$ can be approximated by putting $a^* \rightarrow 0$:

$$
\tilde{\alpha}_{af}(\eta) \approx \frac{2J}{C^* \sqrt{\eta}} \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_1^{\infty} \frac{dy}{y^2 \left( \eta + \sqrt{\eta} + K(1 - \cos \varphi) \right)^2}. \quad (85)
$$

The integration with respect to $\varphi$ in Eq. (85) is given by $2F(\pi/2, 1/\sqrt{1 + (\tilde{\eta}/K)}) \pi \sqrt{2 + (\tilde{\eta}/K)}$, where $F(\pi/2, x) = K(x)$ is the complete elliptic function of first kind. The approximate form of $K(x)$ near $x \approx 1$ is given by

$$
K(x) \approx \frac{1}{2} \log \frac{16}{1 - x^2}. \quad (86)
$$

Therefore, in the limit of $\eta \rightarrow 0$, we obtain

$$
\tilde{\alpha}_{af}(\eta) \approx \frac{\sqrt{2J}}{\pi C^* \sqrt{\eta}} \log \frac{32K}{\eta}. \quad (87)
$$

Substituting definitions, $C^* \equiv v_F^2 C/A$, $K \equiv 2k_F^2 v_F^2$, and $\tilde{\eta} \equiv v_F^2 \eta/A$, into Eq. (87), we obtain

$$
\tilde{\alpha}_{af}(\eta) \approx \frac{J}{2k_F^2 v_F^2} \frac{2m^* A}{\pi v_F^2 C} \log \frac{64Ak_F^2}{\eta}. \quad (88)
$$

Finally, by shifting $\eta$ to $\eta + Aq_m^2$ to take into account the degree of distance from the hot point, we obtain the first term in the parenthesis on the r.h.s. of Eq. (43) in the text.
8. Derivation of Eq. (44) in the text

Near the 2D-FQCP, instead of Eq. (37) in the text, the correction to ASSO coupling $\alpha_\ell(\eta)$ is given by

$$\alpha_\ell(\eta) \equiv \frac{\bar{\Gamma}_\ell(k, i\delta)}{\gamma_\ell(k)} + 2J \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{\sqrt{1 - \cos \varphi}}{\tilde{C}^* \sqrt{a}} \int_1^\infty dy \frac{1}{y + \sqrt{a/(1 - \cos \varphi)} (y^2 - 1)^2},$$

where $\tilde{C}^* \equiv v_0^2 \tilde{C}/A$. Substituting Eq. (88) into the integrands of the second term on r.h.s. of Eq. (89), $S_{2\text{df}}(\eta)$ is expressed as

$$S_{2\text{df}}(\eta) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{\sqrt{1 - \cos \varphi}}{\tilde{C}^* \sqrt{\eta + K(1 - \cos \varphi)}} \int_1^\infty dy \frac{1}{y + \sqrt{1 - \cos \varphi} \sqrt{\eta + K(1 - \cos \varphi)} (y^2 - 1)^2}. \tag{90}$$

First, we calculate $S_{2\text{df}}(0)$ which is considerably simplified:

$$S_{2\text{df}}(0) = \frac{1}{\tilde{C}^* \sqrt{K}} \int_1^\infty dy \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{y + \sqrt{\frac{K}{\tilde{C}^*}} (y^2 - 1)^{3/2}}, \tag{91}$$

where we have interchanged the order of the integrations with respect to $\varphi$ and $y$. The integration with respect to $\varphi$ is easily performed to give

$$S_{2\text{df}}(0) = \frac{1}{\tilde{C}^* \sqrt{K}} \int_1^\infty dy \frac{1}{y^{3/2}} \frac{y + \sqrt{\frac{K}{\tilde{C}^*}} (y^2 - 1)}{y + \frac{\sqrt{K}}{\tilde{C}^*} (y^2 - 1)^{3/2}}. \tag{92}$$

With the use of a function $I(x)$, defined by Eq. (45) in the text, $S_{2\text{df}}(0)$ is expressed as

$$S_{2\text{df}}(0) = \frac{1}{K} I\left(\frac{\sqrt{K}}{\tilde{C}^*}\right). \tag{93}$$

Next, let us discuss the lowest-order correction in $\eta$, which will turn out to be of the order of $O(\sqrt{\eta})$. To this end, first, we change the integration variable in Eq. (90) from $\varphi$ to $x \equiv \cos(\varphi/2)$:

$$S_{2\text{df}}(\eta) = \frac{\sqrt{2}}{\pi \tilde{C}^*} \int_{-1}^{1} dx \frac{1}{\sqrt{\eta + 2K(1 - x^2)}} \int_1^\infty dy \frac{1}{y + \sqrt{\frac{K}{\tilde{C}^*}} \sqrt{1 - x^2} \sqrt{\eta + 2K(1 - x^2)} (y^2 - 1)^2}. \tag{94}$$

Let us define the integration with respect to $y$ in Eq. (94) as $M(x, \tilde{\eta})$: i.e.,

$$M(x, \tilde{\eta}) = \int_1^\infty dy \frac{1}{y + \sqrt{\frac{K}{\tilde{C}^*}} \sqrt{1 - x^2} \sqrt{\eta + 2K(1 - x^2)} (y^2 - 1)^2}. \tag{95}$$

Then,

$$S_{2\text{df}}(\eta) - S_{2\text{df}}(0) = \frac{2}{\pi \tilde{C}^* \sqrt{K}} \int_{0}^{1} dx \left[ \frac{1}{\sqrt{1 + (\tilde{\eta}/2K)}} x^2 M(x, \tilde{\eta}) - \frac{1}{\sqrt{1 - x^2}} M(x, 0) \right], \tag{96}$$

where we have used the fact that $M(x, \tilde{\eta})$ is an even function in $x$, and changed the interval of the integration with respect to $x$ from $-1 \leq x \leq 1$ to $0 \leq x \leq 1$, multiplying the result by a factor $2$. The
expression (96) is rearranged as
\[ S_{2\Delta F}(\eta) - S_{2\Delta F}(0) = \frac{2}{\pi C^* \sqrt{K}} \int_0^1 dx \left\{ \frac{1}{\sqrt{1 + (\eta/2K)x^2}} - \frac{1}{\sqrt{1 - x^2}} \right\} M(x, \tilde{\eta}) \]
\[ + \int_0^1 dx \frac{1}{\sqrt{1 - x^2}} [M(x, \tilde{\eta}) - M(x, 0)] \right\}. \tag{97} \]

Let us introduce \(Y^{(1)}(\eta)\) representing the first term in the brace of Eq. (97). By making partial integration with respect to \(x\), \(Y^{(1)}(\eta)\) is expressed as
\[ Y^{(1)}(\eta) = \left[ \sin^{-1} \frac{1}{\sqrt{1 + (\tilde{\eta}/2K)x^2}} - \sin^{-1} \frac{1}{x} \right] M(1, \tilde{\eta}) \]
\[ - \int_0^1 dx \left[ \sin^{-1} \frac{x}{\sqrt{1 + (\tilde{\eta}/2K)x^2}} - \sin^{-1} x \right] \frac{\partial M(x, \tilde{\eta})}{\partial x}. \tag{98} \]

In the limit of \(\eta \to 0\), Eq. (98) is reduced to
\[ Y^{(1)}(\eta) \approx -\sqrt{\frac{\tilde{\eta}}{2K}} M(1, 0) - \int_0^1 dx \left[ -\frac{1}{2} \frac{x}{\sqrt{1 - x^2}} \frac{\ddot{\eta}}{2K} + O(\ddot{\eta}^2) \right] \frac{\partial M(x, 0)}{\partial x}. \tag{99} \]

By straightforward calculations, \(M(1, 0)\) and \(\partial M(x, 0)/\partial x\) are reduced to
\[ M(1, 0) = \int_1^\infty dy \frac{1}{y^2} = 1, \tag{100} \]

and
\[ \frac{\partial M(x, 0)}{\partial x} = \frac{8 \sqrt{K}}{C^* x} \int_1^\infty dy \frac{y^2 - 1}{y + \frac{2\sqrt{K}}{C^*}(1 - x^2)(y^2 - 1)^3}. \tag{101} \]

Namely, up to the leading order in \(\eta\), \(Y^{(1)}(\eta)\) is given by
\[ Y^{(1)}(\eta) \approx -\sqrt{\frac{\ddot{\eta}}{2K}} + O(\ddot{\eta}). \tag{102} \]

It is easy to see that the second term in the brace of Eq. (97) gives only the term of the order of \(O(\ddot{\eta})\) at most. Therefore, \(S_{2\Delta F}(\eta) - S_{2\Delta F}(0)\), Eq. (97), is given by
\[ S_{2\Delta F}(\eta) - S_{2\Delta F}(0) \approx -\frac{2}{\pi \sqrt{2C^*K}} \sqrt{\ddot{\eta}} + O(\ddot{\eta}) \]
\[ = -\frac{1}{K \pi C_{V_F}} \sqrt{\ddot{\eta}} + O(\ddot{\eta}), \tag{103} \]

where we have used the definitions, \(C^* = C_{V_F}/A\) and \(\tilde{\eta} = \nu_v^2 \eta/A\), in deriving the last equality.

Finally, adding Eq. (103) and Eq. (103) and using the definition \(K \equiv 2k_F^2 \nu_v^2\), \(\alpha_{t}(\eta)\), Eq. (89), is given by
\[ \alpha_{t}(\eta) = \frac{\tilde{\Gamma}_v(k, i\tilde{\eta})}{\gamma_v(k)} + \frac{J}{2k_F^2 \nu_v^2} \left[ 2f\left(\frac{\sqrt{K}}{C^*}\right) - \frac{2\sqrt{2A}}{\pi C_{V_F}} \sqrt{\ddot{\eta}} + O(\ddot{\eta}) \right]. \tag{104} \]

The second term on the r.h.s. of Eq. (104) is nothing but the first and second terms in the bracket on the r.h.s. of Eq. (44) in the text because \(\sqrt{K}/C^* = \sqrt{2m^*A}/\tilde{C}\).
9. Derivation of Eq. (46) in the text for 2D-AFQCP

Here we derive Eq. (46) in the text for 2D-AFQCP. Instead of Eq. (18), \( \Gamma_v(\mathbf{k}, i\epsilon_n) \) in 2D is given by

\[
\Gamma_v(\mathbf{k}, i\epsilon_n) = J \mathcal{Y}_v(\mathbf{k} - \mathbf{Q}) \int_{0}^{2\pi} \frac{d\phi}{2\pi} \pi T \sum_{n \geq 0} \left\{ \frac{1}{\sqrt{a_+} (\sqrt{a_+} + \epsilon_n^*)} + \frac{1}{\sqrt{a_-} (\sqrt{a_-} + \epsilon_n^*)} + 2ib \left[ \frac{1}{\sqrt{a_+} (\sqrt{a_+} + \epsilon_n^*)^3} - \frac{1}{\sqrt{a_-} (\sqrt{a_-} + \epsilon_n^*)^3} \right] \right\},
\]

where \( \tilde{a}_n \equiv a + C|\epsilon_n - \epsilon_n'| \) with \( a \) given by Eq. (15) in the text as \( a = (v_F^2 \eta/A) + 2k_F^2 v_F^2 (1 - \cos \varphi) \), and \( b \equiv k_F v_F (1 - \cos \varphi) \).

Corresponding to Eq. (19), the contribution from \( \epsilon_n' = \epsilon_n \) in the summation over \( \epsilon_n' \) in the first and the third terms in the brace of Eq. (105) is given by

\[
\Gamma_v(\mathbf{k}, i\epsilon_n) = J \mathcal{Y}_v(\mathbf{k} - \mathbf{Q}) \pi T \int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{\sqrt{a} (\sqrt{a} + \epsilon_n)} + 2ib \frac{1}{\sqrt{a} (\sqrt{a} + \epsilon_n)} \]

Expanding the expression in the bracket up to linear order in \( \epsilon_n \) as in Eq. (20), Eq. (106) is reduced to

\[
\Gamma_v(\mathbf{k}, i\epsilon_n) = J \mathcal{Y}_v(\mathbf{k} - \mathbf{Q}) \pi T \int_{0}^{2\pi} \frac{d\phi}{2\pi} \left\{ \frac{1}{[\tilde{\eta} + K(1 - \cos \varphi)]^{3/2}} - \frac{2\epsilon_n}{[\tilde{\eta} + K(1 - \cos \varphi)]^2} + 2i K \sqrt{\frac{1 - \cos \varphi}{2 [\tilde{\eta} + K(1 - \cos \varphi)]^2}} - 6i \sqrt{\frac{K}{2 [\tilde{\eta} + K(1 - \cos \varphi)]^{3/2}}} \right\}.
\]

The integration with respect to \( \varphi \) of the first term in Eq. (107) is performed as follows:

\[
\int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{[\tilde{\eta} + K(1 - \cos \varphi)]^{3/2}} = (-2) \frac{\partial}{\partial \tilde{\eta}} \int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{\sqrt{\tilde{\eta} + K(1 - \cos \varphi)}}
\]

\[
\approx -\frac{2}{\sqrt{K}} \frac{\partial}{\partial \tilde{\eta}} \int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{\sqrt{1 + (\tilde{\eta}/K) - \cos \varphi}}.
\]

The integration with respect to \( \varphi \) in the last line of Eq. (108) is the same as that appeared in Eq. (85) and is given by \( 2F\left(\pi/2, 1, \sqrt{1 + (\tilde{\eta}/2K)}\right)/\pi \sqrt{2 + (\tilde{\eta}/K)} \), where \( F(\pi/2, x) = K(x) \) is the complete elliptic function of first kind and its approximate form near \( x \approx 1 \) is given by Eq. (86). Therefore, in the limit \( \tilde{\eta} \to 0 \),

\[
\int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{\sqrt{1 + (\tilde{\eta}/K) - \cos \varphi}} \approx \frac{1}{\sqrt{2}} \frac{\log 32}{(\tilde{\eta}/K)}.
\]

Substituting this asymptotic form into the last line of Eq. (108), we obtain

\[
\int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{[\tilde{\eta} + K(1 - \cos \varphi)]^{3/2}} \approx \frac{1}{\sqrt{2}} \frac{1}{\pi} \frac{32}{(\tilde{\eta}/K)}.
\]
The integration with respect to $\varphi$ of the third term in Eq. (107) is performed as follows:

\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{(1 - \cos \varphi)}{\bar{\eta} + K(1 - \cos \varphi)^{3/2}} \approx \frac{2}{3} \frac{\partial}{\partial K} \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{\bar{\eta} + K(1 - \cos \varphi)^{3/2}}.
\]

(111)

Substituting the relation (110) into Eq. (111), we obtain

\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{(1 - \cos \varphi)}{\bar{\eta} + K(1 - \cos \varphi)^{3/2}} \approx \frac{\sqrt{2}}{3\pi \bar{\eta}}.
\]

(112)

The integration with respect to $\varphi$ of the last term in Eq. (107) is performed as follows:

\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{\bar{\eta} + K(1 - \cos \varphi)^2} = \frac{K}{(2K\bar{\eta})^{3/2}}.
\]

(113)

Then, to the leading order in $1/\bar{\eta} \gg 1$, we obtain

\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{\bar{\eta} + K(1 - \cos \varphi)^2} \approx \frac{K}{(2K\bar{\eta})^{3/2}}.
\]

(114)

The integration with respect to $\varphi$ of the third term in the bracket of Eq. (107) is performed as follows:

\[
\int_0^{2\pi} \frac{d\varphi}{2\pi a^2} b = \sqrt{\frac{K}{2}} \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{(1 - \cos \varphi)}{\bar{\eta} + K(1 - \cos \varphi)^2} \times \frac{1}{\bar{\eta} + K \bar{\eta}^2 - K^2}.
\]

(115)

Substituting Eqs. (110), (112), (114), and (115) into Eq. (107), we obtain

\[
\Gamma_v(k, i\varepsilon_n) \approx J_{\gamma_v}(k - Q) \frac{T}{\sqrt{K}} \left( \sqrt{\frac{2}{\bar{\eta}}} \frac{1}{\bar{\eta}} + \frac{\pi}{\bar{\eta}} \frac{\varepsilon_n}{\sqrt{2 \bar{\eta}^{3/2}}} + i \frac{\pi}{2 \sqrt{K}} \frac{1}{\sqrt{\bar{\eta}}} - \frac{2i}{\sqrt{K}} \frac{\varepsilon_n}{\bar{\eta}} \right).
\]

(116)

Substituting $\bar{\eta} \equiv v_f^2 \eta / A$ and $K \equiv 2k_F v_F^2$ into this expression, we obtain

\[
\Gamma_v(k, i\varepsilon_n) \approx J_{\gamma_v}(k - Q) \left[ \frac{AK_F^2}{(k_Fv_F)^3} \frac{T}{\sqrt{\eta}} - \varepsilon_n \frac{(AK_F^2)^{3/2}}{2(k_Fv_F)^4} \frac{\pi T}{\sqrt{\eta}} + i \frac{\sqrt{AK_F^2}}{4(k_Fv_F)^3} \frac{\pi T}{\sqrt{\eta}} - i \varepsilon_n \frac{AK_F^2}{(k_Fv_F)^4} \frac{T}{\sqrt{\eta}} \right].
\]

(117)

Since $\eta \propto T/(-\log T)$ in the case of 2D-AFQCP,$^{23}$ the first and the fourth terms in Eq. (117) diverge, in the limit $T \rightarrow 0$, as $(- \log T)$, while the second term diverges as $(- \log T)^{3/2} / T^{1/2}$. We remark here that the first term in Eq. (117) should be included in Eq. (43) in the text because it gives the same divergence in the $T$ dependence as the first term in Eq. (43) in the text at least on the hot line (i.e., $q_m = 0$) in the limit $T \rightarrow 0$.

The summation over $\varepsilon_n$ ($\neq \varepsilon_n$) in Eq. (105) is performed in accordance with the 3D case, and,
instead of Eq. (42), we obtain
\[
\Gamma_v(k, i\varepsilon_n) \approx \tilde{\Gamma}_v(k, i\varepsilon_n) + J\gamma_v(k - Q) \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{a + C^*e''(\sqrt{a + C^*e' + e'')}}} \right. \\
\left. + \pi T \frac{2\varepsilon_n}{a^2} + 2ib \left( -\frac{\pi T}{a^2} + \frac{\varepsilon_n}{\pi T} \right) + O(\varepsilon_n^2) \right] \tag{118}
\]
where \( a \equiv \bar{n} + K(1 - \cos \varphi) \) and \( b \equiv \sqrt{K}/2(1 - \cos \varphi) \). Then, instead of Eq. (43), the linear terms of \( \Gamma_v(k, i\varepsilon_n) \) in \( \varepsilon_n \) is obtained as follows:
\[
\Gamma_v(k, i\varepsilon_n) - \Gamma_v(k, i\varepsilon_n) \bigg|_{\varepsilon_n = 0} \approx \tilde{\Gamma}_v(k, i\varepsilon_n) - \tilde{\Gamma}_v(k, i\varepsilon_n) \bigg|_{\varepsilon_n = 0} \\
+ J\gamma_v(k - Q) \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ \frac{\pi T}{a^2} \varepsilon_n + \frac{2b}{a^2}(i\varepsilon_n) \right]. \tag{119}
\]
Finally, substituting Eqs. (114), (117), and (115) into Eq. (119), we obtain
\[
\Gamma_v(k, i\varepsilon_n) \approx \Gamma_v(k, i\varepsilon_n) \bigg|_{\varepsilon_n = 0} + J\gamma_v(k - Q) \left[ -\frac{Ak^2}{(k_Fv)^3} \frac{\pi T}{\eta} + \frac{\sqrt{Ak^2}}{4(k_Fv)^3} \frac{1}{\sqrt{\eta}} \right] (i\varepsilon_n) + O(\varepsilon_n^2). \tag{120}
\]
Here, we note that the term obtained from the first term in the square bracket of Eq. (119) cancels out the second term in the bracket of Eq. (117), so that the term proportional to \( \varepsilon_n \) disappears but only that proportional to \( i\varepsilon_n \) remains. As in the case of 3D-AFQCP, this result verifies the fact that the first line in the brace of Eq. (118), i.e., its real part, is an even function in \( \varepsilon_n \). Equation (120) gives Eq. (46) in the text.

As in the case of 3d-AFQCP, the term \(-2ib\pi T/a^2\) in Eq. (118) cancels out the third term in Eq. (116) or (117).

10. Derivation of (46) in the text for 2D-FQCP

Here, we derive Eq. (46) in the text for the case of 2D-FQCP. We start with Eq. (105), the same equation as that in the case of 2D-AFQCP except that \( C^* \), in the parameter \( \tilde{a}_s \equiv a + C^*|\varepsilon_n - \varepsilon_n'| \), is not constant but \( C^* \equiv \bar{C}^* / \sqrt{1 - \cos \varphi} \) with \( \bar{C}^* \) being a constant. Then, we obtain the same expression for \( \tilde{\Gamma}_v(k, i\varepsilon_n) \) as Eq. (106). However, since Eq. (106) does not include \( C^* \), the expression Eq. (107) for \( \tilde{\Gamma}_v(k, i\varepsilon_n) \) can be applied to the present case of 2D-FQCP:
\[
\tilde{\Gamma}_v(k, i\varepsilon_n) = J\gamma_v(k) \pi T \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ \frac{1}{\sqrt{\frac{1}{\bar{n} + K(1 - \cos \varphi)}^{3/2}}} - \frac{2\varepsilon_n}{\sqrt{\frac{1}{\bar{n} + K(1 - \cos \varphi)^2}}} \right] \\
+ 2i \left( \frac{\sqrt{K}}{2} \frac{1 - \cos \varphi}{\bar{n} + K(1 - \cos \varphi)} - 6i \frac{\varepsilon_n}{\sqrt{\frac{1}{\bar{n} + K(1 - \cos \varphi)^3}}} \right) \tag{121}
\]
Then, the expression Eq. (117) for \( \tilde{\Gamma}_v(k, i\varepsilon_n) \) is also valid in the case of 2D-FQCP:
\[
\tilde{\Gamma}_v(k, i\varepsilon_n) \approx J\gamma_v(k - Q) \left[ -\frac{Ak^2}{(k_Fv)^3} \frac{\pi T}{\eta} - \frac{\sqrt{Ak^2}}{4(k_Fv)^3} \frac{\pi T}{\sqrt{\eta}} - i\varepsilon_n \frac{Ak^2}{(k_Fv)^3} \frac{T}{\eta} \right]. \tag{122}
\]
Since \( \eta \propto (-T \log T) \) in the case of 2D-FQCP,\(^3\) the first and the fourth terms in Eq. (117) vanish, in
the limit $T \to 0$, as $1/(-\log T)$, while the second term diverges as $1/[\sqrt{T}(-\log T)^{3/2}]$. On the other hand, the first term in Eq. (122) vanishes as $1/(-\log T)$ in the limit $T \to 0$, which should be included in Eq. (44) in the text because it gives a sharper cusp in $T$ dependence in the limit $T \to 0$ than the second term of Eq. (44) in the text.

The summation over $\varepsilon_n'(\neq \varepsilon_n)$ in Eq. (105) is performed in the same way as the case of 2D-AFQCP, and we obtain the same relation as Eq. (119):

$$
\Gamma_\nu(k, i\varepsilon_n) - \Gamma_\nu(k, i\varepsilon_n)|_{\varepsilon_n=0} \approx \tilde{\Gamma}_\nu(k, i\varepsilon_n) - \tilde{\Gamma}_\nu(k, i\varepsilon_n)|_{\varepsilon_n=0} + J_\gamma(k - Q) \int_0^{2\pi} d\varphi \pi T \left[ \frac{2}{a^2} \varepsilon_n + 2 \frac{b}{a^2} (i\varepsilon_n) \right].
$$

(123)

where $a \equiv \tilde{\eta} + K(1 - \cos \varphi)$ and $b \equiv \sqrt{K/2}(1 - \cos \varphi)$. Then, $\Gamma_\nu(k, i\varepsilon_n)$ is given by the same expression as Eq. (120):

$$
\Gamma_\nu(k, i\varepsilon_n) \approx \Gamma_\nu(k, i\varepsilon_n)|_{\varepsilon_n=0} + J_\gamma(k - Q) \left[ -\frac{Ak_F^2}{(k_F\nu)^2} \frac{\pi T}{\tilde{\eta}} + \frac{\sqrt{Ak_F^2}}{4(k_F\nu)^2} \frac{1}{\sqrt{\tilde{\eta}}} \right] (i\varepsilon_n) + O(\varepsilon_n^2).
$$

(124)

This gives Eq. (46) in the text.

As in the case of 2d-AFQCP, the term $-2ib\pi T/a^2$ in Eq. (118) cancels out the third term in Eq. (122).
References

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