ALMOST EVERYWHERE CONVERGENCE OF SPECTRAL SUMS FOR SELF-ADJOINT OPERATORS

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Abstract. Let $L$ be a non-negative self-adjoint operator acting on the space $L^2(X)$, where $X$ is a metric measure space. Let $L = \int_0^\infty \lambda dE_L(\lambda)$ be the spectral resolution of $L$ and $S_R(L)f = \int_0^R dE_L(\lambda)f$ denote the spherical partial sums in terms of the resolution of $L$. In this article we give a sufficient condition on $L$ such that

$$\lim_{R \to \infty} S_R(L)f(x) = f(x), \text{ a.e.}$$

for any $f$ such that $\log(2 + \Delta)f \in L^2(X)$.

These results are applicable to large classes of operators including Dirichlet operators on smooth bounded domains, the Hermite operator and Schrödinger operators with inverse square potentials.

1. Introduction

The almost-everywhere convergence of the spherical partial sums

$$S_Rf(x) = \int_{|\xi| \leq R} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

on $L^2(\mathbb{R}^n)$ is a well-known classical problem in Fourier analysis, where $\widehat{f}$ denotes the Fourier transform of $f$. In the one-dimensional case $n = 1$, a celebrated theorem of Carleson[4] states that for $f \in L^2(\mathbb{R})$,

$$\lim_{R \to \infty} S_Rf(x) = f(x) \text{ for almost every } x.$$ 

For $n \geq 2$, Carbery and Soria [3, Theorem 3] proved that $\lim_{R \to \infty} S_Rf(x) = f(x)$ almost everywhere for any $f$ such that $\log(2 + \Delta)f \in L^2(\mathbb{R}^n)$, where $\Delta = -\sum_{i=1}^n \partial^2/\partial x_i^2$ denotes the classical Laplace operator on $\mathbb{R}^n$.

In [14], Meaney, Müller and Prestini extended the result of Carbery and Soria to arbitrary right-invariant sub-Laplacian $L$ on a connected Lie group $\mathbb{G}$. Let $L = \int_0^\infty \lambda dE_L(\lambda)$ be the spectral resolution of $L$ and

$$S_R(L)f(x) = \int_0^R dE_L(\lambda)f(x)$$

denote the spherical partial sums in terms of the resolution of $L$. They showed that $S_R(L)f(x)$ converges a.e. to $f(x)$ as $R \to \infty$ when $\log(2 + L)f \in L^2(\mathbb{G})$. Their proof is based on the Rademacher-Menshov theorem ([8, 16]). It also employs an extension of a Plancherel theorem as in [11] and [6] to arbitrary connected Lie groups $\mathbb{G}$, which says that for any Borel measurable essentially bounded function $F$ on $[0, \infty)$ and for the spectral multiplier $F(L)f = K_{F \ast f}$ corresponds a unique distribution $K_F$, there exists a unique $\sigma$-finite positive Borel measure $\omega$ on $[0, \infty)$ such that the following
holds:

$$\|K_\ell\|_2^2 = \int_0^\infty |F(\lambda)|^2 d\omega(\lambda).$$

In this article we assume that \((X, d, \mu)\) is a separable metric measure space, that is \(\mu\) is a Borel measure with respect to the topology defined by the metric \(d\). Next let \(B(x, r) = \{y \in X : d(x, y) < r\}\) be the open ball with center \(x \in X\) and radius \(r > 0\). Given a subset \(E \subseteq X\), we denote by \(\chi_E\) the characteristic function of \(E\) and set \(P_E f(x) = \chi_E(x)f(x)\). We consider a non-negative self-adjoint operator \(L\) acting on \(L^2(X)\). Such an operator admits a spectral resolution \(E_L(\lambda)\) and we define the spherical partial sums for \(L\) by

$$S_R(L)f(x) = \int_0^R dE_L(\lambda)f(x).$$

The aim of this article is to investigate when it is possible to replace condition (1.1) in the Meaney-Müller-Prestini theorem by other suitable condition to study almost everywhere convergence of spherical partial sums in the general setting of abstract operators rather than in a specific setting of group invariant operators acting on Lie groups. To do it, we recall that in ([9, 3.1]), Duong, Ouhabaz and Sikora introduced the so-called Plancherel-type estimate to establish the sharp Hörmander-type spectral multiplier theorems for \(L\). We say that \(L\) satisfies the Plancherel-type estimate if there exists \(C > 0\) such that for all \(M > 0\), \(y \in X\) and all Borel functions \(F\) such that \(\text{supp } F \subseteq [0, M]$, 

$$\int_X |K_{F, \sqrt{\ell}}(x, y)|^2 d\mu(x) \leq \frac{C}{\mu(B(y, M^{-1}))}||F(M \cdot)||_{L^2},$$

where \(K_{F, \sqrt{\ell}}(x, y) : X \times X \to \mathbb{C}\) denotes the kernel of the integral operator \(F(\sqrt{\ell})\), and \(m\) is positive constant and \(m \geq 2\). For the standard Laplace operator \(\Delta\) on \(\mathbb{R}^n\), it is well-known ([5, Proposition 2.4]) that condition (1.2) is equivalent to the (1, 2) restriction estimate of Stein-Tomas, i.e.

$$||dE \chi(\lambda)||_{L^1 \rightarrow L^\infty} \leq C\lambda^{n-1}.$$  

Alternative form of the Plancherel-type estimate was introduced in [12, (4.3)] by Kunstmann and Uhl, and can be formulated in the following way:

$$\|F(\sqrt{\ell})P_{B(x,1/M)}\|_{L^2} \leq \|F(M \cdot)\|_{L^2}$$

for all \(M > 0\), \(x \in X\) and all bounded Borel functions \(F\) with \(\text{supp } F \subseteq [0, M]\). Note that

$$\|F(\sqrt{\ell})P_{B(x,1/M)}\|_{L^2} \leq \|F(\sqrt{\ell})P_{B(x,1/M)}\|_{L^1} \|P_{B(x,1/M)}\|_{L^1}$$

so, by Hölder’s inequality, estimate (1.2) implies (1.3) provided that \(X\) is a space of homogeneous type (see Section 2 below). For more information about (1.2) and (1.3), we refer to [5, 9, 12] and the references therein.

Motivated by the Plancherel-type estimates (1.2) and (1.3) above, we have the following result.

**Theorem 1.1.** Let \((X, d, \mu)\) be a metric measure space and \(L\) satisfies the Plancherel-type estimate: for all compact subset \(K\), there exist positive constants \(C_K\) and \(a\) such that for all \(M > 1\), all Borel functions \(F\) with \(\text{supp } F \subseteq [M/4, M]$

$$\|F(L)\chi_K\|_{L^2} \leq C_K M^a \|F(M \cdot)\|_{L^2}.$$

If \(\log(2 + L)f \in L^2(X)\), then

$$\lim_{R \to \infty} S_R(L)f(x) = f(x).$$
for almost every \( x \in X \). Moreover, for every compact subset \( K \) of \( X \) there exists a constant \( C_K > 0 \) such that
\[
\int_K \left| \sup_{R > 0} |S_R(L)f(x)|^2 \right| d\mu(x) \leq C_K \| \log(2 + L)f \|_2^2.
\]

It is not difficult to see that the Plancherel type estimate (1.4) implies that the set of point spectrum of \( L \) is empty in \((1/4, \infty)\). Indeed, one has, for \( 0 \leq \lambda < M \),
\[
\| \mathbb{I}_{(\lambda)}(L)\|_{L^2} \leq C_K \| \mathbb{I}_{(\lambda)}(M^+) \|_{L^2} = 0,
\]
and thus \( \mathbb{I}_{(\lambda)}(L) = 0 \). Since \( \sigma(L) \subseteq [0, \infty) \), it is clear that the point spectrum of \( L \) is empty in \((1/4, \infty)\). In particular, (1.4) does not hold for elliptic operators on compact manifolds or for the harmonic oscillator. In order to treat these cases, we will prove the following result.

**Theorem 1.2.** Let \((X, d, \mu)\) be a metric measure space and assume that the spectrum of \( L \) is purely discrete, i.e. the essential spectrum is empty. Let \( \lambda_1 < \lambda_2 < \cdots \lambda_k < \cdots \) be all the different eigenvalues of \( L \). Assume that there exist constants \( A, a > 0 \) such that for large enough natural number \( k \)
\[
k \leq A \lambda_k^a.
\]
If \( \log(2 + L)f \in L^2(X) \), then
\[
\lim_{R \to \infty} S_R(L)f(x) = f(x)
\]
for almost every \( x \in X \). Moreover, for every compact subset \( K \) of \( X \) there exists a constant \( C_K > 0 \) such that
\[
\int_K \left| \sup_{R > 0} |S_R(L)f(x)|^2 \right| d\mu(x) \leq C_K \| \log(2 + L)f \|_2^2.
\]

We would like to mention that in Theorem 1.1, when \( X \) is a space of homogeneous type, either (1.2) or (1.3) implies estimate (1.4), see Lemma 2.2 below. There are several examples of operators discussed in [5, 9, 12] which satisfy the Plancherel-type estimate (1.2) or (1.3). In particular, (1.2) holds for positive definite self-adjoint right invariant operators and quasi-homogeneous operators acting on a homogeneous group, see [9, Section 7.1]. However, it is not clear for us whether or not estimate (1.2) holds for the right-invariant sub-Laplacian \( L \) on a connected Lie group \( G \).

Note that in Theorem 1.2, if the number \( N(\lambda) \) of eigenvalues in \([0, \lambda]\), counted with the multiplicities of each eigenvalue, satisfies
\[
N(\lambda) \leq A \lambda^a,
\]
then for eigenvalue \( \lambda_k \),
\[
k \leq N(\lambda_k) \leq A \lambda_k^a.
\]
Estimate (1.8) can be derived from the Weyl formula for \( L \), see for examples, Sections 5.1 and 5.2 below. As pointed in [9, p. 470], in the case of group invariant operators on compact Lie groups the Plancherel-type estimates and the sharp Weyl formula are equivalent.

Our Theorems 1.1 and 1.2 are applicable to large classes of operators including Dirichlet operators on bounded domains, the Hermite operator and Schrödinger operators with the inverse square potentials. See Section 5 below for details.

## 2. Preliminary results

As mentioned in Introduction, the proofs of Theorems 1.1 and 1.2 are based on the following Rademacher-Menshov Theorem (see [1, 16]).
Theorem 2.1 (Rademacher-Menshov Theorem). Suppose that \((X, \mu)\) is a positive measure space. There is a positive constant \(c\) with the following property: For each orthogonal subset \(\{f_k : k \in \mathbb{N}\}\) in \(L^2(X)\) satisfying
\[
\sum_{k=0}^{\infty} (\log(2 + k))^2 \|f_k\|_2^2 < \infty,
\]
the maximal function
\[
F^*(x) := \sup_{N \in \mathbb{N}} \left| \sum_{k=0}^{N} f_k(x) \right|
\]
is in \(L^2(X)\), and
\[
\|F^*\|_2^2 \leq c \sum_{k=0}^{\infty} (\log(2 + k))^2 \|f_k\|_2^2.
\]

In particular, when \(2.1\) holds, then the series \(\sum_{n=1}^{\infty} f_k(x)\) converges almost everywhere on \(X\).

Proof. For the proof, we refer to Theorem XIII.10.21 from [16], Proposition 2.3.1, and Theorem 2.3.2 from [1, pp. 79-80].

Following [7, Chapter 3)], a space of homogeneous type \((X, d, \mu)\) is a set \(X\) together with a metric \(d\) and a nonnegative measure \(\mu\) on \(X\) such that \(\mu(B(x, r)) < \infty\) for all \(x \in X\) and all \(r > 0\), and there exists a constant \(C > 0\) such that
\[
V(x, 2r) \leq CV(x, r) \quad \forall \ r > 0, \ x \in X,
\]
where \(V(x, r) = \mu(B(x, r))\). If this is the case, there exist \(C, n\) such that for all \(\lambda \geq 1\) and \(x \in X\)
\[
V(x, \lambda r) \leq C\lambda^n V(x, r).
\]
for some \(c, n > 0\) uniformly for all \(\lambda \geq 1\) and \(x \in X\). The parameter \(n\) is a measure of the dimension of the space. There also exist \(c\) and \(N, 0 \leq N \leq n\) so that
\[
V(y, r) \leq c \left(1 + \frac{d(x, y)}{r}\right)^N V(x, r)
\]
uniformly for all \(x, y \in X\) and \(r > 0\). Indeed, the property \(2.5\) with \(N = n\) is a direct consequence of triangle inequality of the metric \(d\) and the strong homogeneity property. In the cases of Euclidean spaces \(\mathbb{R}^n\) and Lie groups of polynomial growth, \(N\) can be chosen to be 0.

As mentioned in Introduction, estimate \((1.2)\) implies \((1.3)\) when \(X\) is a space of homogeneous type. Now we discuss the relationship between two Plancherel-type estimates \((1.3)\) and \((1.4)\). We have the following result.

Lemma 2.2. Let \(X\) be a space of homogeneous type. Suppose that the operator \(L\) satisfies the condition \((1.3)\), then estimate \((1.4)\) holds.

Proof. Let \(M > 1\) and let \(F\) be a Borel functions such that \(\text{supp} \ F \subseteq [M/4, M]\). Since \(K\) is compact, we have a ball \(B = B(x_K, r_K)\) with \(r_K \geq 1/M\) such that \(K \subseteq B(x_K, r_K)\). Then we take a function \(G(\lambda) = F(\lambda^m)\) such that \(G(\sqrt[1/m]{L}) = F(L)\) and so \(\text{supp} \ G \subseteq [0, M^{1/m}]\).

For every \(1/M > 0\), we choose a sequence \((x_i)_{i=1}^{N_M} \in B(x_K, r_K)\) for some \(N_M < \infty\) such that \(d(x_i, x_j) > 1/2M\) for \(i \neq j\) and \(\inf_{x \in X} d(x, x_i) \leq 1/2M\). Such a sequence exists because \(X\) is separable. Set \(B(x_K, r_K) \subseteq \bigcup_{i \in \mathbb{N}_M} B(x_i, 1/M)\). Note that for every \(1 \leq i, j \leq N_M\)
\[
V(x_i, 1/M) \leq C \left(1 + \frac{d(x_i, x_j)}{1/M}\right)^{N} V(x_j, 1/M) \leq C(r_K M)^n V(x_i, 1/M).
\]
Without loss of generality, we assume that $x_1 = x_K$. Then we have

\[
N_M(r_K M)^{-n} V(x_1, 1/M) \leq C \sum_{i \in N_M} V(x_i, 1/M) \leq CV(x_K, r_K)
\]

\[
\leq C \frac{V(x_K, r_K)}{V(x_K, 1/M)} V(x_K, 1/M)
\]

\[
\leq C(r_K M)^n V(x_1, 1/M),
\]

so $N_M \leq C r_K^2 M^{2n}$. Therefore,

\[
\|F(L)\chi_K\|_{L^2} \leq \|G(\sqrt{L})\chi_{B(x_K, r_K)}\|_{L^2} \leq \sum_{i \in N_M} \|G(\sqrt{L})\chi_{B(x_i, 1/R)}\|_{L^2}
\]

\[
\leq C \sum_i \|G(M^{1/m_j})\|_{L^2}
\]

\[
\leq C r_K^2 M^{2n} \|F(M\lambda^n)\|_{L^2}
\]

\[
\leq C r_K^2 M^{2n} \|F(M^n)\|_{L^2}.
\]

This completes the proof of Lemma 2.2. \(\square\)

**Remark 2.3.** Note that in our Theorems 1.1 and 1.2, we assume that $(X, d, \mu)$ is a separable metric measure space, and we do not need the assumption that $X$ is a space of homogeneous type.

### 3. Proof of Theorem 1.2

To show Theorem 1.2, we note that the spectrum of $L$ is purely discrete and the eigenvalues satisfy condition (1.6). In this case,

\[
S_R(L)f(x) = \sum_{k=0}^{[R]} \sum_{\ell, \lambda_k} \langle f, \phi_{\ell, k} \rangle \phi_{\ell, k}(x) = \sum_{k=0}^{[R]} \mathcal{P}_k f(x)
\]

where $\{\phi_{\ell, k}(x)\}$ are the eigenfunctions corresponding to the eigenvalue $\lambda_k$ and $[R]$ denotes the largest integer number such that $\lambda_{[R]} \leq R$. From condition (1.6), we see that there exists constant $C > 0$ such that

\[
\log(2 + k) \leq C \log(2 + \lambda_k).
\]

Taking $f_k = \mathcal{P}_k(L)f$ in (2.1), we have

\[
\sum_{k=0}^{\infty} (\log(2 + k))^2 \|\mathcal{P}_k(L)f\|_2^2 = \sum_{k=0}^{\infty} (\log(2 + k))^2 \sum_{\ell, \lambda_k} \langle f, \phi_{\ell, k} \rangle^2
\]

\[
\leq C \sum_{k=0}^{\infty} (\log(2 + \lambda_k))^2 \sum_{\ell, \lambda_k} \langle f, \phi_{\ell, k} \rangle^2
\]

\[
= C \sum_{k=0}^{\infty} \sum_{\ell, \lambda_k} \langle f, \log(2 + \lambda_k) \phi_{\ell, k} \rangle^2
\]

\[
= C \sum_{k=0}^{\infty} \sum_{\ell, \lambda_k} \langle f, \log(2 + L) \phi_{\ell, k} \rangle^2
\]
Following an argument as in Theorem 1.2, we take \( f_k = P_k f \) in (2.1) to get
\[
\sum_{k=1}^{\infty} (\log(2 + k))^2 \|P_k f\|_2^2 = \sum_{k=1}^{\infty} (\log(2 + k))^2 \int_{(\lambda_k-1, \lambda_k]} d\langle E_L(\lambda) f, f \rangle \\
\leq 4 \sum_{k=1}^{\infty} (\log(2 + k - 1))^2 \int_{(\lambda_k-1, \lambda_k]} d\langle E_L(\lambda) f, f \rangle \\
\leq C \sum_{k=1}^{\infty} \int_{(\lambda_k-1, \lambda_k]} (\log(2 + \lambda_{k-1}))^2 d\langle E_L(\lambda) f, f \rangle \\
\leq C \sum_{k=1}^{\infty} \int_{(\lambda_k-1, \lambda_k]} (\log(2 + \lambda))^2 d\langle E_L(\lambda) f, f \rangle \\
\leq C \|\log(2 + L) f\|_2^2.
\]

For \( k = 0 \), it is clear that \((\log(2 + 0))^2 \|f_0\|_2^2 \leq C\|f\|_2^2 \leq C\|\log(2 + L) f\|_2^2\). By the Rademacher-Menshov Theorem 2.1,
\[
(4.1) \quad \left\| \sup_{N \in \mathbb{N}} \sum_{k=0}^{N} P_k f(x) \right\|_2^2 \leq C \sum_{k=0}^{\infty} (\log(2 + k))^2 \|P_k f\|_2^2 \leq C \|\log(2 + L) f\|_2^2.
\]

Let us estimate the term II. To do it, it follows from the fact that \( \ell^2 \subseteq \ell^\infty \) and the dual space of \( L^2(K, L^1[\lambda_k, \lambda_{k+1}]) \) is \( L^2(K, L^\infty[\lambda_k, \lambda_{k+1}]) \) (see [2]) that
\[
\left\| \sup_{k \in \mathbb{N}} \left( \sup_{\lambda_k \leq r < \lambda_{k+1}} \langle E_L(\lambda_k, r) f \rangle \right) \right\|_{L^2(K)}^2 \leq C \sum_{k \in \mathbb{N}} \left\| \sup_{\lambda_k \leq r < \lambda_{k+1}} \langle E_L(\lambda_k, r) f \rangle \right\|_{L^2(K)}^2
\]
Integration by parts gives us

\[
\int_{[\lambda_k, \lambda_{k+1}]} E_L(\lambda_k, r) f(x) g(r, x) dr
\]

where the equality \((4.3)\) makes sense in \(L^2(X)\). To go on, we make a partition of the interval \((\lambda_k, \lambda_{k+1}]\): \(\lambda_k = \lambda_{k,0} < \lambda_{k,1} < \ldots < \lambda_{k,j} = \lambda_{k+1}\). From the Plancherel type estimate \((1.2)\), we see that

\[
\|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_{L^2(K)} \leq \|X_K E_L(\lambda_{k,j-1}, \lambda_{k,j})\|_{L^2} \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_2
\]

\[
\leq C_K \lambda_{k,j}^{\alpha} \|X_K(\lambda_{k,j-1}, \lambda_{k,j})\|_2 \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_2
\]

\[
\leq C_K \lambda_{k,j}^{\alpha - \frac{1}{2}} (\lambda_{k,j} - \lambda_{k,j-1})^{\frac{1}{2}} \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_2,
\]

where \(C_K\) is a constant depending on \(K\) only, but does not depend on \(j\) and \(k\). This, in combination with the properties of Riemann-Stieltjes integral and the Fatou Lemma, yields that

\[
\left| \int_{[\lambda_k, \lambda_{k+1}]} \left( \int_{\lambda_k}^{\lambda_{k+1}} g(s, x) ds \right) dE_L(\lambda_k, r) f(x) dx \right|
\]

\[
\leq \lim \sum_{j=1}^{J} \int_{[\lambda_k, \lambda_{k+1}]} \left| \int_{\lambda_k}^{\lambda_{k+1}} g(s, x) ds \right| E_L(\lambda_{k,j-1}, \lambda_{k,j}) f(x) dx
\]

\[
\leq \lim \sum_{j=1}^{J} \| \int_{\lambda_k}^{\lambda_{k+1}} g(s, x) ds \|_{L^2(K)} \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_{L^2(K)}
\]

\[
= \lim \sum_{j=1}^{J} \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_{L^2(K)}
\]

\[
\leq C_K \sum_{j=1}^{J} \lambda_{k,j}^{\alpha - \frac{1}{2}} (\lambda_{k,j} - \lambda_{k,j-1})^{\frac{1}{2}} \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_2
\]

\[
\leq C_K \left( \sum_{j=1}^{J} \lambda_{k,j}^{2\alpha - 1} (\lambda_{k,j} - \lambda_{k,j-1}) \right)^{1/2} \left( \sum_{j=1}^{J} \|E_L(\lambda_{k,j-1}, \lambda_{k,j}) f\|_2^2 \right)^{1/2}
\]

\[
(4.4)
\]

From \((4.4)\) and \((4.3)\), we have that

\[
\left| \int_{[\lambda_k, \lambda_{k+1}]} E_L(\lambda_k, r) f(x) g(r, x) dr dx \right|
\]

\[
= \left| \int_{[\lambda_k, \lambda_{k+1}]} E_L(\lambda_k, \lambda_{k+1}) f(x) \left( \int_{\lambda_k}^{\lambda_{k+1}} g(s, x) ds \right) dx - \int_{[\lambda_k, \lambda_{k+1}]} \left( \int_{\lambda_k}^{\lambda_{k+1}} g(s, x) ds \right) dE_L(\lambda_k, r) f(x) dx \right|
\]

\[
\leq \|E_L(\lambda_k, \lambda_{k+1}) f\|_2 + C_K \left( \sum_{j=1}^{J} \lambda_{k,j}^{2\alpha - 1} (\lambda_{k,j} - \lambda_k) \right)^{1/2} \|E_L(\lambda_k, \lambda_{k+1}) f\|_2.
\]
Recall that \( \lambda_k = k^{1/(2a)} \) and it implies that \( \lambda_{k+1}^{2a-1}(\lambda_{k+1} - \lambda_k) \leq C \) with \( C \) independent of \( k \). Then by (4.2), we see that
\[
\text{LHS of (4.2)} \leq C_K \sum_{k \in \mathbb{N}} \left( |E_k(\lambda_k, \lambda_{k+1})| + \sum_{k \in \mathbb{N}} \left( |E_k(\lambda_k, \lambda_{k+1})||f||^2 \right) \right) 
\leq C_K \sum_{k \in \mathbb{N}} |E_k(\lambda_k, \lambda_{k+1})| + \sum_{k \in \mathbb{N}} \left( \sum_{k \in \mathbb{N}} \left( |E_k(\lambda_k, \lambda_{k+1})||f||^2 \right) \right) 
\leq C_K ||f||^2 \leq C_K \log(2 + L)||f||^2,
\]
which, together with (4.1), completes the proof of (1.5) in Theorem 1.1. \( \square \)

5. Applications

5.1. Dirichlet operators on smooth bounded domains. Let \( \Omega \) be a connected bounded open subset of \( \mathbb{R}^n \) with \( C^\infty \) boundary and \( L = P(x, D) \) be a second order differential operator of the form
\[
P(x, D) = -\sum_{j,k} \frac{\partial}{\partial x_j} g^{jk}(x) \frac{\partial}{\partial x_k},
\]
where \( (g^{jk}) \in C^\infty(\Omega) \) is real and positive definite in \( \bar{\Omega} \). Define the operator with Dirichlet boundary conditions. In [10, Section 17.5], it is proved that the number \( N(\lambda) \) of eigenvalues \( \leq \lambda \) of \( P(x, D) \) satisfies
\[
N(\lambda) = O(\lambda^{n/2}),
\]
and so condition (1.6) holds. From Theorem 1.2, we have the following proposition.

**Proposition 5.1.** Let \( \Omega \) be a connected bounded open subset of \( \mathbb{R}^n \) with \( C^\infty \) boundary and \( P(x, D) \) be a second order differential operator as above. Assume that \( \log(2 + P)f \in L^2(\Omega) \). Then
\[
\lim_{R \to \infty} S_R(P)f(x) = f(x)
\]
for almost every \( x \in \Omega \).

5.2. Schrödinger operators with growth potentials. Assume that the potential \( V : \mathbb{R}^n \to \mathbb{R} \) is smooth and satisfies the growth conditions:
\[
|\partial^\alpha V(x)| \leq C_\alpha (1 + |x|)^k \text{ for each multiindex } \alpha
\]
and
\[
V(x) \geq c(1 + |x|)^k \text{ for } |x| \geq R,
\]
where \( k, c, C_\alpha, R > 0 \) are appropriate constants.

We consider the Schrödinger operator \( -\Delta + V(x) \), where the potential \( V \) satisfies the above conditions (5.1) and (5.2). An example is the Hermite operator \( L = -\Delta + |x|^2 \). Then we have the following Weyl Law result:
\[
N(\lambda) \leq C \left| \{ (\xi, x) : |\xi|^2 + V(x) \leq \lambda \} \right|,
\]
where \( | \cdot | \) denotes the measure of the set in \( \mathbb{R}^{2n} \). See for example [15, Section 6.4]. Thus we have \( N(\lambda) \leq C\lambda^{n/2+n/k} \), and condition (1.6) holds. From Theorem 1.2, we have the following proposition.

**Proposition 5.2.** Let \( L \) be the Schrödinger operator \( -\Delta + V(x) \) where \( V(x) \) satisfies the above growth condition. Assume that \( \log(2 + L)f \in L^2(\mathbb{R}^n) \). Then
\[
\lim_{R \to \infty} S_R(L)f(x) = f(x)
\]
for almost every \( x \in \mathbb{R}^n \).
5.3. **Schrödinger operators with inverse-square potential.** Now we consider the inverse square potentials, that is \( V(x) = \frac{1}{|x|^2} \). Fix \( n \geq 3 \) and assume that \(- (n - 2)^2 / 4 < c \). Define by quadratic form method \( L = -\Delta + V \) on \( L^2(\mathbb{R}^n, dx) \). The classical Hardy inequality

\[
- \Delta \geq \frac{(n - 2)^2}{4} |x|^{-2},
\]

shows that for all \( c > -(n - 2)^2 / 4 \), the self-adjoint operator \( L \) is non-negative. Set \( p_c^* = n / \sigma \), \( \sigma = \max \{ (n - 2) / 2 - \sqrt{(n - 2)^2 / 4 + c}, 0 \} \). If \( c \geq 0 \) then the semigroup \( \exp(-tL) \) is pointwise bounded by the Gaussian semigroup and hence act on all \( L^p \) spaces with \( 1 \leq p \leq \infty \). If \( c < 0 \), then \( \exp(-tL) \) acts as a uniformly bounded semigroup on \( L^p(\mathbb{R}^n) \) for \( p \in (p_c^*, p_c^*) \) and the range \( (p_c^*, p_c^*) \) is optimal (see for example [13]).

It follows from [5, Theorem III.5] that \( L \) satisfies the Plancherel-type estimate (1.3). From Lemma 2.2 and Theorem 1.1, we obtain

**Proposition 5.3.** Suppose that \( n \geq 3 \) and \( -(n - 2)^2 / 4 < c \). Let \( L = -\Delta + c|x|^{-2} \) be defined as above. Assume that \( \log(2 + L)f \in L^2(\mathbb{R}^n) \). Then

\[
\lim_{R \to \infty} S_R(L)f(x) = f(x)
\]

for almost every \( x \in \mathbb{R}^n \).

5.4. **Scattering operators.** Assume now that \( n = 3 \) and \( V \) is a real-valued measurable function such that

\[
\int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x - y|^2} \, dx \, dy < (4\pi)^2 \quad \text{and} \quad \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} \, dy < 4\pi.
\]

Suppose that \( L = -\Delta + V \) on \( \mathbb{R}^3 \) with a real-valued \( V \) which satisfies (5.4).

From [5, Proposition III.6], we know that \( L \) satisfies the Plancherel-type estimate (1.2) and (1.3). From Lemma 2.2 and Theorem 1.1, we have the following proposition.

**Proposition 5.4.** Let \( L = -\Delta + V(x) \) be defined on \( \mathbb{R}^3 \) as above. Assume that \( \log(2 + L)f \in L^2(\mathbb{R}^3) \). Then

\[
\lim_{R \to \infty} S_R(L)f(x) = f(x)
\]

for almost every \( x \in \mathbb{R}^3 \).

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