A Note on Ward’s Chiral Model

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Abstract. A one parameter generalization of Ward’s chiral model in 2+1 dimensions is given. Like the original model the present one is integrable and possesses a positive-definite and conserved energy and y-momentum. The details of the scattering depend on the value of the parameter of the generalisation.

In this note we take a fresh look at Ward’s integrable chiral model in (2+1) dimensions\cite{1}, which is defined by its time evolution equation

\[(\eta^{\mu\nu} + \varepsilon^{\mu\nu\alpha}V_\alpha)\partial_\mu(\partial_\nu \Psi \Psi^\dagger) = 0.\] (1)

\(\Psi\) is a map from \(\mathbb{R}^{2+1}\) to \(SU(2)\) and can be thought of as a \(2 \times 2\) unitary matrix valued function of coordinates \(x^\mu = (t, x, y)\) and \(\dagger\) denotes the hermitian conjugation. Greek indices range over the values 0, 1, 2, \(\partial_\mu\) denotes partial differentiation with respect to \(x^\mu\), \(\varepsilon^{\mu\nu\alpha}\) is the totally skew tensor with \(\varepsilon^{012} = 1\), and \(V_\alpha\) is a constant unit vector, \(i e V_\alpha V^\alpha = 1\). Indices are raised and lowered using the Minkowski metric \(\eta^{\mu\nu} = \text{diag}(−1, 1, 1)\).

This model was derived by Ward, by dimensional reduction, from the self-dual Yang-Mills equation in (2+2) dimensions. The dimensional reduction first gave him a Yang-Mills Higgs system in (2+1) dimensions governed by the equation:

\[D_\mu \Phi = \frac{1}{2}\varepsilon_{\mu\alpha\beta}F^{\alpha\beta},\] (2)

where \(\Phi\), the Higgs field, is a function on \(\mathbb{R}^{2+1}\) with values in the Lie algebra \(su(2)\). Its covariant derivative is \(D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]\), the SU(2) gauge potential is \(A_\mu\) and the corresponding gauge field is \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\). He then rewrote (2)

\(^1\)To Appear in *Physics Letter A*
as (1) and having made a choice of $V_\alpha$ as $V_\alpha = (0, 1, 0)$ discussed many properties of his model. Ward’s model resembles the (2+1) dimensional reduction of the self-dual equation introduced by Manakov and Zakharov [2]. In their case $V_\alpha = (i, 0, 0)$, and the behaviour of the solutions of both models is similar. However, in the Manakov-Zakharov model, an energy functional seems not to exist. Therefore, in this paper we restrict our attention to Ward’s model and its generalization.

The choice of the vector $V_\alpha$ is very important. First of all, note that the standard $SU(2)$ chiral equation in (2+1) dimensions has $V_\alpha = (0, 0, 0)$ and, although is Lorentz covariant, it does not seem to be integrable. By contrast, the existence of the vector $V_\alpha$ in (1) breaks explicitly the Lorentz covariance of the model by picking out a particular direction in space-time. Actually, Ward’s choice of $V_\alpha$ was motivated by the energy conservation. In [1] he took

$$T^{\mu\nu} = (-\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}) \text{tr}(J^{-1}J_{\alpha}J^{-1}J_{\beta}), \quad (3)$$

and then showed that its divergence due to (1) is,

$$\partial_\mu T^{\mu\nu} = -\frac{1}{3} V_\nu \varepsilon^{\alpha\beta\gamma} \text{tr}(J^{-1}J_{\alpha}J^{-1}J_{\beta}J^{-1}J_{\gamma}). \quad (4)$$

Here $\Psi(\lambda = 0, t, x, y) = J^{-1}(t, x, y)$ and $J_\mu = \partial_\mu J$. So $T^{\mu0}$ is conserved, if and only if, $V_0 = 0$: which is true in this case. Note that Ward’s model has the same energy functional as the standard $SU(2)$ chiral equation, since the additional term in (1) is analogous to a background magnetic field, and so does not affect the energy.

Equation (1) with Ward’s choice for the unit vector $V_\alpha$ has many properties of an integrable system. It arises as a consistency condition for a pair of linear equations, it passes the Painlevé test [3] for integrability and has an inverse scattering transformation [4]. It admits multisoliton solutions [1, 5, 6, 7] (more properly, lumps, since they are algebraically decaying) and possesses an infinite set of conserved quantities [11]. In this note we return to (1) and look at other choices of the vector $V_\alpha$. In fact, we will show that most observations of Ward can be extended to the case of $V_\alpha = (\lambda, 1, \lambda)$ where $\lambda$ is arbitrary. In this case, $\Psi$ in (1) becomes a function of $\lambda$; while for $\lambda = 0$ (1) reduces to Ward’s model.

This method is similar in nature to the dressing method introduced by Zakharov and Shabat [8] which has been a powerful tool for obtaining new integrable nonlinear
equations as well as characterizing large classes of solutions of these equations. This method is applicable to both equations in (1+1) dimensions (cf. [8]-[9]), as well as to equations in (2+1) dimensions (cf. [9]-[10]).

System (1) is integrable in the sense that can be written as the compatibility condition for the following linear system:

\[(\zeta \partial_x - \partial_t - \partial_y) Z = -AZ,\]
\[(\zeta \partial_t - \zeta \partial_y - \partial_x) Z = -BZ.\]  \hspace{1cm} (5)

Here \(\zeta\) is a complex parameter, \(A\) and \(B\) are 2 \(\times\) 2 anti-Hermitian trace-free matrices, depending only on \((t, x, y)\) but not on \(\zeta\), and \(Z(\zeta, t, x, y)\) is a 2 \(\times\) 2 matrix satisfying \(\det Z = 1\), and the reality condition

\[Z(\bar{\zeta}, t, x, y)^\dagger = Z(\zeta, t, x, y)^{-1}.\]  \hspace{1cm} (6)

The system (5) is overdetermined and in order for a solution \(Z\) to exist, \(A\) and \(B\) have to satisfy the integrability conditions,

\[\partial_x B = \partial_t A - \partial_y A, \quad \partial_x A - \partial_t B - \partial_y B + [A, B] = 0.\]  \hspace{1cm} (7)

If we put \(\Psi(\lambda, t, x, y) = Z(\zeta = \lambda, t, x, y)\), we find by comparing (5) and (7) that

\[A = -\lambda \Psi_x \Psi^\dagger + \Psi_t \Psi^\dagger + \Psi_y \Psi^\dagger,\]
\[B = -\lambda \Psi_t \Psi^\dagger + \lambda \Psi_y \Psi^\dagger + \Psi_x \Psi^\dagger.\]  \hspace{1cm} (8)

Therefore, the integrability condition for (5) implies that there exist a field \(\Psi\) that satisfies the equation of motion (1); and is unitary (due to the reality condition).

Let us return now to the Yang-Mills Higgs system (2) from which Ward’s model was derived. Note that for this system there exist a gauge in which the fields have the term

\[A_t \equiv A_y = \frac{1}{2} A, \quad A_x \equiv -\Phi = \frac{1}{2} B.\]  \hspace{1cm} (9)

where \(A\) and \(B\) given by (8). Then, it is easily checked that (2) is equivalent to (1). This relation does not depend on the value of \(\lambda\) and so it suggests that our discussion of a more general \(V_\alpha\) is not misguided.

One advantage of the \(\Psi\)-description is that, as shown by Ward (when \(\lambda = 0\)), the system has a conserved, positive-definite energy given by \(T^{00}\) (3). However, Ward’s choice
holds only for \( \lambda = 0 \). When \( \lambda \neq 0 \) we have to consider a different expression. To do this we consider

\[
\Lambda_{\rho \nu} = \begin{pmatrix}
(1 + \lambda^2)^{1/2} & \lambda(1 + \lambda^2)^{-1/2} & \lambda^2(1 + \lambda^2)^{-1/2} \\
\lambda & 1 & \lambda \\
0 & -\lambda(1 + \lambda^2)^{-1/2} & (1 + \lambda^2)^{-1/2}
\end{pmatrix}. 
\]

(10)

This tensor represents a Lorentz transformation which takes \( V_\alpha = (\lambda, 1, \lambda) \) to \( V_\alpha = (0, 1, 0) \).

Then we can define a new energy-momentum tensor by

\[
\Theta^{\rho \mu} = \Lambda_{\rho \nu} ( -\eta^{\mu \alpha} \eta^{\nu \beta} + \frac{1}{2} \eta^{\mu \nu} \eta^{\alpha \beta}) \text{tr}(\Psi_\alpha \Psi_\beta \Psi_\gamma \Psi_\delta),
\]

(11)

which, if we set \( \Psi(\lambda) = J^{-1} \), is just a Lorentz transform of \( \Xi \). Clearly \( \Theta^{\rho \mu} \) is non-symmetric in \( \mu \) and \( \rho \). If we impose (I), the divergence of \( \Theta^{\rho \mu} \) is

\[
\partial_\mu \Theta^{\rho \mu} = \frac{1}{3} R^\rho \varepsilon^{\alpha \beta \gamma} \text{tr}(\Psi_\alpha \Psi_\beta \Psi_\gamma \Psi_\delta),
\]

(12)

with \( R^\rho \equiv \Lambda^\rho \nu V^\nu = (0, 1, 0) \). So the conservation of \( \Theta^{\rho \mu} \) mirrors the conservation of \( T^{\mu \nu} \) in the Ward case, and so does the energy-momentum vector \( P^\mu = \Theta^{\mu 0} \).

Consequently, we have \( \partial_\mu \Theta^{i \mu} = 0 \), for \( i = 0, 2 \) and so the energy and \( y \)-momentum, which are the integrals of the densities

\[
P^0 = \left( \frac{1}{2} + \frac{\lambda^2}{2} \right) \text{tr}(\Psi_t \Psi_t^\dagger + \Psi_x \Psi_x^\dagger + \Psi_y \Psi_y^\dagger) - \lambda \text{tr}(\Psi_x \Psi_t^\dagger) - \lambda^2 \text{tr}(\Psi_y \Psi_t^\dagger),
\]

\[
P^2 = \lambda \text{tr}(\Psi_x \Psi_t^\dagger) - \text{tr}(\Psi_y \Psi_t^\dagger),
\]

(13)

are well-defined and independent of \( t \). By contrast, the \( x \)-momentum density is not conserved, since \( R^1 = 1 \). In order to ensure the finiteness of the energy, we require \( \Psi \) to be smooth and be given by

\[
\Psi = \Psi_0 + \Psi_1(\theta) O(r^{-1}) + O(r^{-2}),
\]

at spatial infinity; where \( \Psi_0 \) denotes a constant \( SU(2) \) matrix, \( \Psi_1 \) is independent of \( t \) and of \( x + iy = re^{i\theta} \).

To obtain explicit solutions of (I) Ward [II] used \( \Psi \) of the Lax pair formulation of his model, and then using the standard methods of Riemann problem with zeros derived families of solutions which correspond to localized lumps of energy. This \( \Psi(\lambda) \) is a solution of
describing extended structures which interact with each other trivially or nontrivially and we shall study the effects of the parameter $\lambda$ on this interaction.

Let us take, as an example, a solution of (ii) representing two lumps which undergo a $90^\circ$ scattering. In this case the solution $\Psi(\lambda)$ takes the form of a product

$$\Psi = \left(1 + \frac{2i}{\lambda - i} \frac{q^\dagger \otimes q}{|q|^2}\right)\left(1 + \frac{2i}{\lambda - i} \frac{p^\dagger \otimes p}{|p|^2}\right),$$

(15)

where $q$ and $p$ are two-dimensional vectors

$$q = (1 + |f|^2)(1, f) - 2i(tf' + h)(\bar{f}, -1),$$
$$p = (1, f).$$

(16)

So we have a family of solutions depending on the value of the real parameter $\lambda$ and on two arbitrary meromorphic rational functions $f$ and $h$ of $z = x + iy$. Let us look in more detail at the case of $f(z) = z$ and $h(z) = z^2$.

Using the same arguments as in [5] we see that $\Psi$ departs from its asymptotic value when $(tf' + h) = 0$ and at this time we can approximately identify two separate lumps. For $t$ negative, they are on the $x$-axis at $x = \pm \sqrt{-t}$, while for $t$ positive, they are on the $y$-axis at $y = \pm \sqrt{t}$. So the evolution can be described as being given by two lumps accelerating towards each other, scattering at right angles, and then decelerating as they separate. In fact, their acceleration is such as if the force of their attraction were proportional to the inverse cube of the distance between them. Similar behaviour has recently also, been observed in some other integrable (2+1) dimensional models (cf. [12]). This suggests that our results are more generic in their nature.

The evolution of our lumps can be verified by looking in more detail at their energy density, which is,

$$P^0 = 16 \frac{1 + 10r^2 + 5r^4 + 4t^2(1 + 2r^2) - 8t(x^2 - y^2)}{[1 + 2r^2 + 5r^4 + 4t^2 + 8t(x^2 - y^2)]^2}$$
$$+ 64 \lambda \frac{(1 + r^2)[x(1 - t) - \lambda y(t+1)]}{(1 + \lambda^2)[1 + 2r^2 + 5r^4 + 4t^2 + 8t(x^2 - y^2)]^2}. \tag{17}$$

We see that for large (positive) $t$, $P^0$ has maxima at two points on the $y$-axis, namely $y \approx \pm \sqrt{t}$. Moreover, the height of the corresponding lumps is proportional to $1/t$, which means that the $y$-axis asymmetry vanishes at $t \to \infty$. Plots of the energy density are given in Figure 1 for some values of $\lambda$. 

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Obviously, the parameter $\lambda$ not only deforms the size of the corresponding lumps but also changes their velocity. This follows from the $y$-momentum density (see Figure 2) which is given by

$$P^2 = 64 \frac{(1 + r^2)(\lambda x(1 - t) + y(1 + t))}{(1 + \lambda^2)[1 + 2r^2 + 5r^4 + 4t^2 + 8t(x^2 - y^2)]^2}.$$  

(18)

Here we note the antisymmetry of $P^2$ under the interchange $x \rightarrow -x$, $y \rightarrow -y$.

Looking at Figure 1 we note that for $\lambda = 2$ the lumps are deformed in the $y$-direction. This is not very visible for $t < 0$, becomes quite clear at $t = 0$ and results, in a situation in which the lump moving in the positive $y$-direction has larger total energy than the other one. This is consistent with the behaviour of the $y$-momentum distribution. This behaviour is generic in nature and, in general, will hold for other lump field configurations.

In conclusion - we have presented a one parameter generalisation of Ward’s chiral model. The models possess a conserved total energy and a conserved $y$-component of the total momentum. For all values of the parameter the models are integrable and have solutions with “solitonic-like” properties. The scattering of these lumps is similar in nature but differs in detail; depending on the value of this parameter the lumps can be deformed at all stages of their evolution.

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**References**

[1] R. S. Ward, J. Math. Phys. 29, 386 (1988).

[2] S. V. Manakov and V. E. Zakharov, Lett. Math. Phys. 5, 247 (1981).

[3] R. S. Ward, Nonlinearity 1, 671 (1988).

[4] J. Villarroel, Stud. Appl. Math 83, 211 (1990).

[5] R. S. Ward, Phys. Lett. A 208, 203 (1995).

[6] T. Ioannidou, J. Math. Phys. 37, 3422 (1996).

[7] T. Ioannidou and W. J. Zakrzewski, Solutions of the Modified Chiral Model in (2+1) Dimension, in preparation (1997).
Figure Captions.

Figure 1: The energy density $P^0$ (17) at increasing times for (a) $\lambda = 0$ and (b) $\lambda = 2$.

Figure 2: The $y$-momentum $P^2$ (18) at various times for (a) $\lambda = 0$ (b) $\lambda = 2$ and (c) $\lambda = 10$. 

[8] V. E. Zakharov and P. B. Shabat, Funct. Anal. Appl. 8, 226 (1974).

[9] V. E. Zakharov and P. B. Shabat, Funct. Anal. Appl. 13, 166 (1979).

[10] V. E. Zakharov and S. V. Manakov, Funct. Anal. Appl. 19, No. 2, 11 (1985).

[11] T. Ioannidou and R. S. Ward, Phys. Lett. A 208, 209 (1995).

[12] M. Ablowitz and J. Villarroel, Phys. Rev. Lett. 78, 570 (1997).

M. Manas and P. M. Santini, Phys. Lett. A 227, 325 (1997).
