Quasi-Bell states in a strongly coupled qubit-oscillator system and their delocalization in phase space

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Abstract

We study evolution of bipartite entangled quasi-Bell states in a strongly coupled qubit-oscillator system, and extend it to ultra-strong coupling regime. We assume the presence of a static bias that may be easily generated, say, for the flux qubits. Using the adiabatic approximation the reduced density matrix of the qubit is obtained for the strong coupling domain in closed forms involving linear combinations of Jacobi theta functions. The analytical results based on theta functions are found to be in good agreement with their series counterparts. The evaluation of entropy suggests that the entanglement and mixedness of the individual subsystems may be reduced to very low values for short intervals of time. The reduced density matrix of the oscillator is employed for analytically obtaining the phase space distributions such as the Husimi $Q$-function. For the ultra-strong coupling regime delocalization in the phase space of the oscillator is studied by using the Wehrl entropy, the Heisenberg uncertainty function, and the complexity of the quantum state. Following maximum delocalization a condensation of the $Q$-function in the phase space is evident for a wide class of states with increasing coupling strength.
I Introduction

One of the ubiquitous and most important models in quantum physics describes a two-level system (qubit) interacting with a simple harmonic oscillator. In quantum optics it describes the interaction of an atom with an electromagnetic field mode in a cavity. For a small detuning between the oscillator and the qubit frequencies, and also for a weak qubit-oscillator coupling, the dynamical behavior of the interacting system is accurately described by the exactly solvable Jaynes-Cummings model [1] that employs the rotating wave approximation. Recently, however, a variety of experimental situations pertaining to stronger coupling domain, where the rotating wave approximation no longer remains valid, have been investigated. Various experimental realizations such as a nanomechanical resonator capacitively coupled to a Cooper-pair box [2], a quantum semiconductor microcavity undergoing excitonic transitions [3], a flux-biased superconducting quantum circuit that uses large nonlinear inductance of a Josephson junction to achieve ultrastrong coupling with a coplanar waveguide resonator [4] have been achieved. One advantage of the superconducting qubits over the natural atoms is the additional flexibility available in selecting the control parameters. For instance, in the context of cavity electrodynamics usually unbiased qubits operating through their degeneracy points are encountered, whereas the external static bias of the superconducting qubits may be generated and controlled by the gate voltage applied to a Cooper-pair box or the magnetic flux linked with a Josephson junction [5].

Moving beyond the rotating wave approximation in these strongly interacting systems one needs to incorporate terms in the qubit-oscillator interaction Hamiltonian that do not preserve the total excitation number. In the parametric regime where the oscillator frequency dominates the qubit frequency the authors of Ref. [6, 7] advanced an adiabatic approximation scheme that utilizes the separation of the time scales of the qubit and the oscillator. The dynamical state of the fast-moving component, say the oscillator, is assumed to instantaneously adjust to the slow-changing state of the qubit. This facilitates decoupling of the full Hamiltonian into sectors related to each time scale, and in this setting approximate eigenstates of the system have been derived [6]. In addition, analytical expression of the time-dependent behavior of the two level system was obtained [6] in a relatively weak coupling limit. Studying a system of two qubits coupled to a single oscillator mode in the context of the Tavis-Cummings model the authors of Ref. [8] analytically evaluated the collapse and revival dynamics of the qubit reduced density matrix elements. Their procedure involves using Poisson sum formula [9] that leads to expressions of the said elements as infinite sums. Following the experimental developments for the superconducting qubits, a static bias term has been included in the qubit Hamiltonian and the corresponding energy spectrum has been investigated [7, 10].

In the context of coupled qubit-oscillator system the nonclassical quasi-Bell states are found to be of much interest. They exhibit entanglement of microscopic atomic states and the photonic coherent states that can be regarded as mesoscopic for reasonably large values of the coherent state amplitudes. When the amplitude of the coherent states are large enough they are often called Schrödinger cat states as they exhibit entanglement between a microscopic and a classical object. Experimental observation of these states involving circular Rydberg atoms interacting with a mesoscopic coherent field trapped in a high-Q...
microwave cavity has been realized [11]. In the instance of cavity quantum electrodynamics they have also been used [12] for generating even or odd coherent states as well as more generalized configurations of macroscopic field superposition states. Bell inequality tests involving these qubit-field entangled states have been proposed [13] recently. These states also play a crucial role in the non-destructive measurement [14] of the photon number in a field stored in a cavity. Moreover, it has been observed [7] that in the large coupling regime a state of the generic quasi-Bell type becomes the approximate ground state of the combined system. These states are likely to be fruitful for studying quantum phase transitions of qubit-oscillator systems. Therefore, we consider it is of importance to study the dynamical evolution of such a state in a strongly coupled qubit-oscillator system. In particular, the phase-space dynamics of the oscillator may shed light on the macroscopic quantum state.

Our objective in the present work is twofold. Within the adiabatic approximation scheme we analytically study the evolution of quasi-Bell states in a coupled qubit-oscillator system possessing a dominant oscillator frequency and an external static bias of the qubit. Our study includes both strong and ultra-strong coupling domains. Starting with a quasi-Bell bipartite initial state we obtain the time-evolution of the reduced density matrices of the qubit and the oscillator, respectively. The reduced density matrix of the qubit provides the von Neumann and the linear entropies of the system that measure the entanglement and the mixedness of the state. Proceeding further, the reduced density matrix of the oscillator that yields the Husimi $Q$-function [15] on the phase space is studied. In the strong coupling regime we retain up to the quadratic terms in the Laguerre polynomials in the phase factors and analytically express the reduced density matrix elements in closed forms via the Jacobi theta functions. To evaluate the analytical expressions we treat the bias perturbatively, and retain terms of second order in the expansion parameter. The summation procedure may, however, be extended to an arbitrary order. Employing the $Q$-distribution we also express the quadrature variables of the oscillator and the Heisenberg uncertainty thereof via linear combinations of Jacobi theta functions. We will later specify the parametric range of validity of our derivation.

In the ultra-strong coupling regime we employ the long-range time as well as the coupling strength dependence of the classical Wehrl entropy [16] and the complexity [17] of the $Q$-distribution that measure the delocalization of the mixed state of the oscillator reduced density matrix in the phase space. An analytical evaluation of the complexity of the coupled qubit-oscillator system is developed. These measures of delocalization are found to be qualitatively similar. The long-range time dependence of Wehrl entropy shows that the reduced density matrices in the ultra-strong coupling domain act as long-living metastable states undergoing Markovian fluctuations. Another observed feature is that in the ensemble of states there exist candidates which, following a peak dispersion in the phase space, exhibit diminishing delocalization with increasing coupling strength. Condensation of the $Q$-distribution in the phase space in the ultra-strong coupling regime is evident from the decrease of Wehrl entropy thereat. We set the general framework in Sec. II. In Secs. III and IV we focus on obtaining closed-form expressions of qubit density matrix elements, $Q$-distribution, and expectation values of the quadrature variables of the oscillator in the strong coupling regime. The phase space description of the system for the ultra-strong coupling strength is given in Sec. V. We conclude in Sec. VI.
II The reduced density matrices

We study a coupled qubit-oscillator system with the Hamiltonian that reads in natural units

\[ H = -\frac{\Delta}{2} \sigma_x - \frac{\epsilon}{2} \sigma_z + \omega a^\dagger a + \lambda \sigma_x (a^\dagger + a), \]  

(2.1)

where the harmonic oscillator with a frequency \( \omega \) is described by the raising and lowering operators \((a^\dagger, a)\), and the qubit characterized by an energy splitting \( \Delta \) as well as an external static bias \( \epsilon \) is expressed via the spin variables \((\sigma_x, \sigma_z)\). The qubit-oscillator coupling strength is denoted by \( \lambda \). The Fock states \(|n\rangle, n = 0, 1, \ldots; a |n\rangle = \sqrt{n} |n - 1\rangle, a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle \) provide the basis for the oscillator, and the eigenstates \(\sigma_z |\pm 1\rangle = \pm |\pm 1\rangle \) span the space of the qubit. The Hamiltonian (2.1) is not known to be exactly solvable. In the present work we follow the adiabatic approximation [6] that relies on the separation of the time scales governed by the high oscillator frequency and the (renormalized) low qubit frequency. The fast-moving oscillator then adiabatically adjusts to the slow changes of the state of the qubit.

To facilitate our construction of the evolution of the quasi-Bell states, we, following [6], now give a short review of the diagonalization process of the Hamiltonian (2.1) in the adiabatic approximation scheme, where the oscillator quickly adjusts to the slow-changing instantaneous state of the qubit observable \(\sigma_z\) so that for the oscillator degree of freedom \(\sigma_z\) may be replaced with its eigenvalue: \(\pm 1\). The effective Hamiltonian of the harmonic oscillator now reads

\[ H_O = \omega a^\dagger a + \lambda (a^\dagger + a). \]  

(2.2)

The number states which are shifted symmetrically by the displacement operator diagonalize the above Hamiltonian:

\[ |n_\pm\rangle = D^\dagger (\pm \lambda/\omega) |n\rangle, \quad D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \]  

(2.3)

where the degenerate eigenenergies are as follows:

\[ E_n^\pm \equiv E_n = n\omega - \frac{\lambda^2}{\omega}. \]  

(2.4)

Identically displaced states maintain orthonormality: \(\langle m_\pm |n_\pm\rangle = \delta_{m,n}\), whereas the overlap between the Fock states displaced in opposite directions is given by

\[ \langle m_- |n_+\rangle = \begin{cases} (-1)^{m-n} \exp \left( -\frac{x}{2} \right) x^{\frac{m+n}{2}} \sqrt{\frac{m!}{n!}} L_n^{m-n}(x), & m \geq n \\ \exp \left( -\frac{x}{2} \right) x^{\frac{n-m}{2}} \sqrt{\frac{m!}{n!}} L_m^{n-m}(x), & m < n, \end{cases} \]  

(2.5)

where the associated Laguerre polynomial reads \(L_n^j(x) = \sum_{k=0}^{n} (-1)^k \binom{n+j}{n-k} \frac{x^k}{k!}, \) and the parameter \(x = (2\lambda/\omega)^2\) acts as the coefficient of the perturbative expansion.

After diagonalizing the high frequency oscillator component of the Hamiltonian one now turns attention to the low frequency qubit part. Tensoring the qubit states with the displaced oscillator basis \(|\pm 1, n_\pm\rangle |\forall n = 0, 1, \ldots\rangle\) the Hamiltonian may now be readily constructed. For a dominant oscillator frequency \(\Delta/\omega \ll 1\) one may neglect [6]...
the matrix elements that mix oscillator states with different eigenvalues of its number operator. This reduces the Hamiltonian to a block-diagonal form where each block mixes displaced oscillator states with identical number of photonic excitations. The Hamiltonian for the \( n \)-th block assumes the form

\[
H_n = \begin{pmatrix}
  n\omega - \frac{\lambda^2}{\omega} - \delta_n & \frac{\delta_n}{n\omega - \lambda^2/\omega + \delta_n} \\
  \frac{\delta_n}{n\omega - \lambda^2/\omega + \delta_n} & n\omega - \frac{\lambda^2}{\omega} + \delta_n
\end{pmatrix}, \quad \delta_n = -\frac{\Delta}{2}L_n(x), \quad \Delta = \Delta \exp \left(-\frac{x}{2}\right), \quad \delta = \frac{\epsilon}{2}
\]

that may be diagonalized in the basis of the eigenstates listed below

\[
|E^\pm_n\rangle = \sqrt{\frac{\chi_n + \delta}{2\chi_n}} |1, n_+\rangle \pm \frac{\delta_n}{|\delta_n|} \sqrt{\frac{\chi_n + \delta}{2\chi_n}} |1, n_-\rangle,
\]

where the corresponding eigenvalues read

\[
E^\pm_n = n\omega - \lambda^2/\omega \pm \chi_n, \quad \chi_n = \sqrt{\delta_n^2 + \delta^2/4}.
\]

Given the above construction of the eigenstates of the system we now study the time evolution of the qubit-oscillator entanglement in the presence of a bias (\( \epsilon \neq 0 \)). The quasi-Bell initial states of the coupled system read

\[
|\psi(0)\rangle^{(\pm)} = \frac{1}{\sqrt{2}} (|1, \alpha\rangle \pm |1, -\alpha\rangle), \quad |\alpha\rangle = D(\alpha)|0\rangle,
\]

where the coherent state \( \{|\alpha\rangle \forall \alpha = \Re(\alpha) + i\Im(\alpha) \in \mathbb{C}\} \) for the oscillator degree of freedom is realized by the action of the displacement operator on the vacuum. In the present work we, as an example of a typical nonclassical state, analytically study the properties such as time evolution of its entropy and the phase space distributions of the oscillator for strong qubit-oscillator coupling. The precise limits of the coupling strength and the bias parameter will be made explicit later. The evolution of the initial state \( (2.9) \) is given by

\[
|\psi(t)\rangle^{(\pm)} = \frac{1}{\sqrt{2}} \exp \left(-\frac{1}{2}g_\alpha^2 - i \left(\Im(\alpha) \frac{\lambda}{\omega} + E_n t\right)\right) \times \\
\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \hat{\alpha}^n \left[ \Xi^+_n \exp(-i\chi_n t)|E^+_n\rangle \mp (-1)^n \frac{\delta_n}{|\delta_n|} \Xi^-_n \exp(i\chi_n t)|E^-_n\rangle \right],
\]

where \( \hat{\alpha} = \alpha + \lambda/\omega, \ \ g_\alpha = |\hat{\alpha}|, \ \ \text{and} \ \ \Xi^\pm_n = \sqrt{\frac{\chi_n - \epsilon}{2\chi_n}} \pm (-1)^n \frac{\delta_n}{|\delta_n|} \sqrt{\frac{\chi_n + \epsilon}{2\chi_n}}. \) On the rhs of \( (2.10) \) and hereafter the upper and lower signs refer to respective quasi-Bell states given in \( (2.9) \). Employing the state \( (2.10) \) the evolution of the density matrix of the composite bipartite system may be readily constructed:

\[
\rho^{(\pm)}(t) = |\psi(t)\rangle^{(\pm)}\langle\psi(t)|.
\]

The marginalized density matrices for the individual degrees of freedom may be obtained by partial tracing of the composite density matrix \( (2.11) \). For instance tracing of the
oscillator degree of freedom yields the reduced density matrix of the qubit. Its explicit structure obtained via (2.11, 2.10) assumes the form

\[
\rho_Q^{(\pm)}(t) \equiv \text{Tr}_O \rho^{(\pm)}(t) = \begin{pmatrix}
\frac{1}{2} \mp \zeta & \pm \zeta^{(\pm)} \\
\pm (\zeta^{(\pm)})^* & \frac{1}{2} \pm \zeta
\end{pmatrix},
\]

where the components read

\[
\zeta = \hat{\epsilon} \exp\left(-g_n^2 \sum_{n=0}^{\infty} (-1)^n \frac{\frac{2n}{n!} \sin^2 \chi_n t}{\lambda n^2}\right),
\]

\[
\zeta^{(\pm)} = \frac{1}{2} \exp(-g_n^2) \sum_{n=0}^{\infty} (-1)^m \frac{\left(\hat{\alpha}^n \hat{\alpha}^* \right)^m}{\sqrt{n! m!}} \mathcal{C}_n^\pm(t) \mathcal{C}_m^\mp(t) \times
\]

\[
\times \exp\left(-i(n - m)\omega t\right) \langle m_- | n_+ \rangle,
\]

and the coefficients are given by

\[
\mathcal{C}_n^\pm(t) = \mathfrak{A}_n^\pm \exp(i \chi_n t) + \mathfrak{B}_n^\pm \exp(-i \chi_n t),
\]

\[
\mathfrak{A}_n^\pm = \frac{\lambda_n + \hat{\epsilon} \pm (-1)^n \delta_n}{2 \lambda_n}, \quad \mathfrak{B}_n^\pm = \frac{\lambda_n - \hat{\epsilon} \pm (-1)^n \delta_n}{2 \lambda_n}.
\]

The reduced density matrix (2.12) of the qubit produces its pair of eigenvalues:

\[
\frac{1}{2} + \omega^{(\pm)}, \quad \frac{1}{2} - \omega^{(\pm)}; \quad \omega^{(\pm)} = \sqrt{\zeta^2 + |\zeta^{(\pm)}|^2}.
\]

The von Neumann entropy of the qubit given by \( S(\rho_Q) = -\text{Tr}(\rho_Q \log \rho_Q) \) now reads

\[
S_Q^{(\pm)} = -\left(\frac{1}{2} + \omega^{(\pm)}\right) \log\left(\frac{1}{2} + \omega^{(\pm)}\right) - \left(\frac{1}{2} - \omega^{(\pm)}\right) \log\left(\frac{1}{2} - \omega^{(\pm)}\right).
\]

Similarly the trace over the qubit basis produces the reduced density matrix of the oscillator:

\[
\rho_O^{(\pm)}(t) \equiv \text{Tr}_Q |\psi(t)\rangle^{(\pm)} \langle \psi(t)|
\]

that may be expressed, via (2.10), in terms of the displaced number states as follows:

\[
\rho_O^{(\pm)}(t) = \frac{1}{2} \exp(-g_n^2) \sum_{n,m=0}^{\infty} \frac{\left(\hat{\alpha}^n \hat{\alpha}^* \right)^m}{\sqrt{n! m!}} \mathcal{C}_n^\pm(t) \mathcal{C}_m^\mp(t) |n_+\rangle \langle m_+| + (-1)^{n+m} \mathcal{C}_n^\mp(t) \mathcal{C}_m^\pm(t) |n_-\rangle \langle m_-| \exp\left(-i(n - m)\omega t\right).
\]

We note that apart from employing the adiabatic approximation no other simplification has been made in the derivation leading to the reduced density matrix elements (2.13, 2.14, 2.20).

To obtain a measure of entanglement and mixedness of the bipartite system the von Neumann entropy of the reduced density matrix of the qubit may be considered. It is
well-known that the entropies of each of these subsystems corresponding to a pure state of the composite system are equal. This may be explicitly checked in the present case by employing the expansion

$$\log \rho = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} (\rho - 1)^{\ell} \quad \Rightarrow \quad S(\rho) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} \text{Tr}[\rho(\rho - 1)^{\ell}]. \quad (2.21)$$

The equality of the entropies of two subsystems \(S_O(\rho_O) = S_Q(\rho_Q)\) follows from the equality of traces of arbitrary integral powers of the reduced density matrices (2.12, 2.20). It may be verified via an order by order computation that the reduced density matrix of the oscillator endowed with the infinite dimensional Fock space and that of the qubit with its two dimensional Hilbert space satisfy the relation

$$\text{Tr}[\rho^{(\pm)}(t)^{\ell}] = \text{Tr}[\rho^{(\pm)}(t)^{\ell}] = \left(\frac{1}{2} + \omega^{(\pm)}\right)^{\ell} + \left(\frac{1}{2} - \omega^{(\pm)}\right)^{\ell}. \quad (2.22)$$

This demonstrates that the present model satisfies the criterion of equality of entropy of the subsystems. Moreover, above equality immediately allows, via (2.12), computation of the linear entropy \(S^{(\pm)} = 1 - \text{Tr}[\rho^{(\pm)}(t)^{2}] = 1 - \text{Tr}[\rho^{(\pm)}(t)\rho^{(\pm)}(t)] = \frac{1}{2} - 2(\zeta^2 + |\xi^{(\pm)}|^2). \quad (2.23)\)

We mention a byproduct of the above computation. During the revival of the qubit density matrix elements the entropy of the system is reduced from its near-maximal value. Consequently, the reduced density matrices of the subsystems represent almost pure states of the subsystems at certain times and at certain values of coupling strength. We notice two such instances in Fig. 1 where we plot von Neumann and linear entropies with varying coupling strength while fixing the times. Towards making an order of magnitude estimate we assume that in the vicinity of the minimum value of entropy, the reduced density matrix of, say, the oscillator remains close to the projection operator for a state \(|\Psi_y\rangle\) with a residual contribution arising from the orthogonal states \(|\Psi_\ell\rangle\):

$$\rho_O|_{\min. \ ent.} \sim (1-x)|\Psi_y\rangle\langle\Psi_y| + \frac{x}{N} \sum_{\ell=1}^{N} |\Psi_\ell\rangle\langle\Psi_\ell|, \quad (2.24)$$

which yields leading contribution to the entropy as \(S \sim -x(\log \frac{x}{N} - 1)\). In (2.24) we, for the purpose of estimation, introduced an arbitrary cut-off \(N\) in the expansion of the oscillator state. The minimum values of entropy observed in Figs. 1 (a) and (b) correspond to the coefficients \(x \lesssim 0.0092\) and \(x \lesssim 0.0247\), respectively. These minimum entropy states are, up to a good approximation, pure states of the subsystems. This property may be important for relatively large values of \(|\alpha|\) that characterize mesoscopic states of the oscillator.

### III The qubit density matrix

We now focus our attention towards closed-form analytical evaluation of the qubit density matrix in the parametric regime where the oscillator frequency sets the dominant scale
and the following hierarchy is maintained: $\epsilon < \Delta < \omega$. In the perturbative expansion given below we retain terms $O(\epsilon^2)$ in the bias parameter. Furthermore, as we investigate the first few collapses and revivals of the density matrix elements, we, for the purpose of simplicity, neglect $\epsilon$ dependent terms in phases as they will be important only in the very long time limit. The higher order terms in the bias parameter $\epsilon$ may, however, be systematically taken into account in the present perturbative scheme. Previous attempts in analytical evaluation of the elements of the density matrices include Refs. [6, 8]. The summation technique adopted in [6] retained terms of the order $O(x) \sim O((\lambda/\omega)^2)$ and was valid in the weak coupling limit $\lambda \ll \omega$. For analytically evaluating the density matrix elements in the context of Tavis-Cummings model at higher values of the qubit-oscillator coupling strength the authors of Ref. [8] developed a method based on Poisson sum formula. They have retained contributions of the order $O(x^2)$ that amounts to considering up to the quadratic terms in the expansion of the Laguerre polynomials: $L_n(x) = 1 - nx + \frac{1}{2!} \binom{n}{2} x^2 + O(x^3)$.

We now provide an alternate method of summing the series in (2.13, 2.14) by employing the Jacobi theta functions. This method has an additional benefit in that it expresses the matrix elements in closed form. The series expansions of the Jacobi theta functions relevant in our case read [9]:

$$
\vartheta_3(q, z) = \sum_{n=-\infty}^{\infty} q^n \exp(2inz), \quad \vartheta_4(q, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^n \exp(2inz).
$$

(3.1)

These interrelated functions depend on two complex arguments and the series sums in (3.1) are convergent for the choice $\Re(q) < 1$. In our summation scheme we retain contributions $O(x^2)$ in the density matrix elements and consider up to quadratic terms in the Laguerre polynomials. To proceed, we, following [8], approximate at a large value of $g_\alpha^2$ the Poisson distribution in (2.13, 2.14) with the corresponding Gaussian limit:

$$
P(n) \equiv \exp(-g_\alpha^2) \frac{g_\alpha^{2n}}{n!} \to \frac{1}{\sqrt{2\pi g_\alpha^2}} \exp \left( -\frac{(n - g_\alpha^2)^2}{2g_\alpha^2} \right).
$$

(3.2)

For a large value of $g_\alpha^2$ the lower limit of the series in (2.13, 2.14) may be extended to $(-\infty)$. Due to the sharp decay of the Gaussian function for the parametric regime considered.

Figure 1: The dependence of von Neumann entropy (red solid) and linear entropy (green dashed) with varying coupling strength for $\Delta = 0.15\omega, \epsilon = 0.3\omega$: case (a) $\alpha = 0.5, t = 201$, (b) $\alpha = 1, t = 1916.4$. 
here the error encountered is quite small. This process is similar to the evaluation [8] of Fourier components via the Gaussian integration technique in the Poisson-sum method, where the lower limit of integration is extended to \((-\infty)\). We note that a truncated series at the lower limit in the sum [3.1] remains convergent in the domain \(\Re(q) < 1\), and may be viewed as a close analog of the theta function. Since the difference between these alternate cases in our parametric regime is quite small we retain the standard definition [3.1]. The resultant closed-form expressions for the elements of the density matrix [2.12] now assume the form:

\[
\zeta = -\frac{\epsilon}{2\Delta} \left[ \exp\left( -2\theta_0^2 \right) \left( 1 - f(1 + x) + \frac{3}{4} f^2 \right) - \frac{\exp\left( -\frac{\theta_0^2}{2} \right)}{\sqrt{2\pi \theta_0^2}} \Re(\Theta(\tau)) \right], \quad (3.3)
\]

\[
\xi^{(\pm)} = \frac{1}{2} \left( 1 + f + \frac{f^2}{4} \right) \exp\left( -2\theta_0^2 - \frac{x}{2} \right) + \frac{\epsilon}{\Delta} \zeta + \frac{\exp(-\left( x + \frac{\theta_0^2}{2} \right)/2)}{\sqrt{2\pi \theta_0^2}} \left( x^{\frac{3}{2}} R_1 \times \right.

\times R \left( \varphi_{1,0} \vartheta_4(q, z_1) + \frac{f}{2} \varphi_{3,1} \vartheta_4(q, z_3) \right) + \frac{x^2}{2} R_2 R \left( \left( \frac{1}{2} + x \right) \varphi_{2,2} \vartheta_4(q, z_2) - \frac{5f}{6} \varphi_{4,3} \vartheta_4(q, z_4) - \left( \frac{1}{2} + x \right) \varphi_1 \right)

+ \frac{x^2}{6} R_3 R (\varphi_{3,3} \vartheta_4(q, z_3)) + \frac{x^2}{24} R_4 R (\varphi_{4,4} \vartheta_4(q, z_4)) - i \frac{\epsilon}{2\Delta} \left[ \left( \frac{1}{2} + x \right) \varphi_{1,0} \vartheta_3(q, z_1) + \frac{f}{2} \varphi_{3,1} \vartheta_3(q, z_3) - \left( \frac{1}{2} + x \right) \varphi_1 \right)

+ \frac{x^2}{6} R_2 R \left( \left( \frac{1}{2} + x \right) \varphi_{2,2} \vartheta_3(q, z_2) - \varphi_1 \vartheta_3(q, z_2) \right) + \frac{x^2}{24} R_3 R (\varphi_{3,3} \vartheta_3(q, z_3)) - i \frac{\epsilon}{\Delta} \left[ \left( \frac{1}{2} + x \right) \varphi_{1,0} \vartheta_3(q, z_1) + \frac{f}{2} \varphi_{3,1} \vartheta_3(q, z_3) - \left( \frac{1}{2} + x \right) \varphi_1 \right)

+ \frac{x^2}{6} R_2 R \left( \left( \frac{1}{2} + x \right) \varphi_{2,2} \vartheta_3(q, z_2) - \varphi_1 \vartheta_3(q, z_2) \right) + \frac{x^2}{24} R_3 R (\varphi_{3,3} \vartheta_3(q, z_3)) + \frac{x^2}{6} R_3 R (\varphi_{4,4} \vartheta_3(q, z_4)) + \frac{x^2}{24} R_4 R (\varphi_{4,4} \vartheta_3(q, z_4))

\left. \left. + \frac{x^2}{6} R_2 R (\varphi_{3,3} \vartheta_3(q, z_3)) \right) \right] \quad (3.4)
\]
Figure 2: The evaluation of the diagonal component $\zeta$ of the density matrix for the values $\lambda = 0.15\omega, \Delta = 0.15\omega, \epsilon = 0.01\omega$ and $\alpha = 2$ using the series (2.13) (red solid), and theta function (3.3) (green dashed).

Figure 3: The evaluation of the (a) real and (b) imaginary parts of the off-diagonal component $\xi^{(+)}$ of the density matrix for the values $\lambda = 0.15\omega, \Delta = 0.15\omega, \epsilon = 0.01\omega$ and $\alpha = 2$ using the series (2.14) (red solid), and theta function (3.4) (green dashed).

where the parameters and functions abbreviated above read as follows: $f = x_{g_{\alpha}}^2, \tau = \tilde{\Delta} t, q(\tau) = \exp\left(-\frac{1}{2g_{\alpha}^2} - i\frac{\pi x}{4}\right), q = \exp\left(-\frac{1}{2g_{\alpha}^2}\right), z_j \mid_{0,1,2,3,4} = \frac{1}{2} (x + (1 - j) \frac{x}{4}) \tau - i), \varphi_j \mid_{1,2,3,4} = -\frac{1}{2} (j\frac{x}{4} \tau + i), \varphi_{3,4} = \exp\left(-i\tau \left(1 - j\frac{x}{4} + \ell\frac{x^3}{4}\right)\right), \varphi_{\tilde{z}} = \exp \left(i\tau x (1 - j\frac{x}{4})\right), R_n = \Re(\tilde{\alpha}^n e^{-i\omega t}), I_n = \Im(\tilde{\alpha}^n e^{-i\omega t}).$ The linear combination of theta functions used above is given by $\Theta(x) = \varphi_{\gamma_0,\gamma_1}(q, z_0) - f(1 + x)\varphi_{2,1}(q, z_2) + \frac{x^2}{4} f^2 \varphi_{4,2,\gamma_1}(q, z_4).$ This completes our closed-form evaluations of the elements of the reduced density matrix of the qubit via the Jacobi theta functions. For moderately large coupling ($\lambda/\omega \approx 0.20$) this summation technique has good agreement with the numerical evaluations based on (2.13, 2.14). Concerning our choice of bias parameter we note that as we have included terms up to $O(\epsilon^2)$, we need to maintain the order of the dimensionless ratio $\epsilon/\tilde{\Delta} \sim 0.1$. As $g_{\alpha}$ appears in the coefficients of the perturbative terms as well as in the Gaussian factors, we choose it in the optimum range $g_{\alpha} \sim 1.5 - 2$. These restrictions describe the parametric regime in which our analytical evaluations based on theta functions hold. In Figs. 2-3 we compare the closed-form expressions of the density matrix elements (3.3, 3.4) with the corresponding sums given in (2.13, 2.14). For a significant number of revivals of the density matrix elements two sets agree to a good extent.

The time evolution of the entanglement of the initially chosen quasi-Bell state (2.9) is of interest. For the bipartite system considered here this may be measured by the von Neumann entropy of the reduced density matrix of a subsystem. Employing (2.18)
and our evaluations (3.3, 3.4) we may find a theta-function dependent expression for the time evolution of the said entropy in the parametric domain discussed in the preceding paragraph. This may be compared with expression of the entropy based on the sums (2.13, 2.14) as a check on the validity of our closed-form evaluation scheme. As these expressions are voluminous, we only reproduce here in Fig. 4 the corresponding diagrams for the time evolution of entropy. For first few revivals in the qubit density matrix elements a good agreement between the analytical closed-form expression and the series sum is obtained. As physically expected the entropy dips from its maximal value during successive revivals in the quantum oscillations of the density matrix elements (2.13, 2.14). The large decreases in the entropy indicate that during its course of evolution the components of the system repeatedly come close to pure states of the individual subsystems, as discussed at the end of Sec. II. The entanglement between the qubit and oscillator for these short time periods characterized by low entropy is reduced to very small values. The extent of the dip in entropy increases with increase of the bias parameter.

IV The Q-function of the oscillator density matrix

The Husimi Q-function [15] is a distribution on the complex phase space that provides a description of the time evolution of the quantum state of the oscillator degree of freedom. It assumes non-negative values and is normalized to unity on the phase space. For our choice of quasi-Bell initial states (2.9) it is defined as the expectation value of the reduced density matrix of the oscillator (2.20) in an arbitrary coherent state:

\[
Q^{(\pm)}(\beta) = \frac{1}{\pi} \langle \beta | \rho^{(\pm)}_O (t) | \beta \rangle,
\]  

which for the state under consideration assumes the explicit positive definite form

\[
Q^{(\pm)}(\beta) = \frac{1}{2\pi} \exp(-\delta^2) \left( \exp(-\delta^2) |X^{\pm}|^2 + \exp(-\delta^2) |Y^{\pm}|^2 \right). 
\]  

The Fourier sums in (4.2) are given by

\[
X^{\pm} = \sum_{n=0}^{\infty} \frac{(\hat{\alpha}^{\dagger} \beta^*)^n}{n!} \mathcal{C}^{\pm}_n (t) \exp(-in\omega t), \quad Y^{\pm} = \sum_{n=0}^{\infty} (-1)^n \frac{(\hat{\alpha}^{\dagger} \beta^*)^n}{n!} \mathcal{C}^{\pm}_n (t)^* \exp(-in\omega t). 
\]
In (4.2) we have abbreviated: \( \hat{\beta} = \beta + \lambda/\omega, \tilde{\beta} = |\hat{\beta}| \) and \( \tilde{\beta} = \beta - \lambda/\omega, \tilde{\beta} = |\tilde{\beta}| \).

The Husimi distribution (4.2) of the reduced density matrix (2.20) does not have any zero on the phase space except at asymptotically large radial distances. It satisfies the normalization condition: \( \int Q(\pm)(\beta) d^2\beta = 1 \). The property of positivity and normalization to unity facilitate interpretation of \( Q(\pm)(\beta) \) as a probability distribution function on the phase space, which, in turn, allows determination of expectation values of operators.

Since, for this purpose we need to determine the \( Q(\pm)(\beta) \) function on the whole complex phase space, the analytical approximation scheme based on theta functions described earlier is not suitable. Instead we may adopt the procedure developed in [6] where the Laguerre functions are approximated by their linear part. Under this approximation valid for the regime \( \lambda/\omega \lesssim 0.1 \) the Fourier sums (4.3) read:

\[
X^{(\pm)} = \exp(\Phi^-) \cos \left( \Phi^- + \frac{\tau}{2} \right) \pm i \exp(-\Phi^-) \sin \left( \Phi^- + \frac{\tau}{2} \right) \\
-\frac{\epsilon}{\Delta} \exp(\Phi^-) \left[ \sin \left( \Phi^- + \frac{\tau}{2} \right) + x \Phi^- \sin \left( \Phi^- - \frac{(1-x)\tau}{2} \right) \right] \\
+ \frac{\epsilon^2}{2\Delta^2} \exp(-\Phi^-) \left[ \sin \left( \Phi^- + \frac{\tau}{2} \right) - 2x\Phi^- \sin \left( \Phi^- + \frac{(1-x)\tau}{2} \right) \right],
\]

\[
Y^{(\pm)} = \exp(-\Psi^-) \cos \left( \Psi^- + \frac{\tau}{2} \right) \mp i \exp(\Psi^-) \sin \left( \Psi^- - \frac{\tau}{2} \right) \\
-\frac{\epsilon}{\Delta} \exp(-\Psi^-) \left[ \sin \left( \Psi^- + \frac{\tau}{2} \right) - x \Psi^- \sin \left( \Psi^- + \frac{(1-x)\tau}{2} \right) \right] \\
\pm \frac{\epsilon^2}{2\Delta^2} \exp(\Psi^-) \left[ \sin \left( \Psi^- - \frac{\tau}{2} \right) + 2x\Psi^- \sin \left( \Psi^- - \frac{(1-x)\tau}{2} \right) \right]. \tag{4.4}
\]

In (4.3) we have used the notations \( \Phi^- = \hat{\alpha}\hat{\beta}^* \exp(-i\omega t), \Psi^- = \hat{\alpha}\tilde{\beta}^* \exp(-i\omega t), \Phi^+_t = \Phi_t \cos \frac{\tau}{2}, \Psi^+_t = \Psi_t \cos \frac{\tau}{2}, \Phi^+_t = \Phi_t \sin \frac{\tau}{2} \) and \( \Psi^+_t = \Psi_t \sin \frac{\tau}{2} \). In the said parametric regime where the linear approximation holds we now compare in Fig. 5 the Fourier sum (4.3) with the corresponding analytical evaluation (4.4) of the Q-function. The time evolution of the Q-function is faithfully reproduced by the closed-form expression (4.4).

The structure of the \( Q(\pm)(\beta) \) indicates that interference pattern that develops between the oscillator frequency and other interaction-dependent modes causes splitting of the probability peaks and spreading of the distribution in the phase space. This signifies transfer of energy between the oscillator and the qubit. The splitting and rejoined splitting of the peaks the Q-distribution correspond to the collapse and the revival of the qubit reduced density matrix elements (2.13, 2.14). In general, newer modes arise and the interference effect increases with increasing coupling constant of the qubit-oscillator interaction. This causes, in a time-averaged sense, a wider distribution of the \( Q(\pm)(\beta) \) function on the phase space with increasing coupling strength. We will discuss these issues in Sec. VI.

One of the utilities of the Q-distribution is that it provides a convenient evaluation of the expectation values of the operators expressed in their antinormal ordered form. The first and second moments of the quadrature operators defined as

\[
E = \frac{1}{\sqrt{2}i}(a - a^\dagger), \quad B = \frac{1}{\sqrt{2}}(a + a^\dagger) \tag{4.5}
\]
may be expressed \[15\] via the following phase space integrals over the complex plane:

\[
\langle E \rangle^{(\pm)} = \text{Tr}(E(a, a^\dagger)\rho_\ell(t)) = \frac{1}{\sqrt{2i}} \int (\beta - \beta^*)Q^{(\pm)}(\beta)d^2\beta,
\]

\[
\langle B \rangle^{(\pm)} = \text{Tr}(B(a, a^\dagger)\rho_\ell(t)) = \frac{1}{\sqrt{2}} \int (\beta + \beta^*)Q^{(\pm)}(\beta)d^2\beta,
\]

\[
\langle E^2 \rangle^{(\pm)} = \text{Tr}(E^2(a, a^\dagger)\rho_\ell(t)) = -\frac{1}{2} \int ((\beta - \beta^*)^2 + 1)Q^{(\pm)}(\beta)d^2\beta,
\]

\[
\langle B^2 \rangle^{(\pm)} = \text{Tr}(B^2(a, a^\dagger)\rho_\ell(t)) = \frac{1}{2} \int ((\beta + \beta^*)^2 - 1)Q^{(\pm)}(\beta)d^2\beta. \tag{4.6}
\]

Employing the Fourier sum \[14\] and using the following notations for the coefficients

\[
G_{n,\ell}^{\pm}(t) = \hat{\alpha}^\ell \left( c_{n+\ell}^{\pm}(t)c_{n}^{\mp}(t)^* + c_{n+\ell}^{\mp}(t)c_{n}^{\pm}(t)^* \right) \exp(-i\omega t),
\]

\[
G_{n,\ell}^{\pm}(t) = \hat{\alpha}^\ell \left( c_{n+\ell}^{\pm}(t)c_{n}^{\mp}(t)^* - c_{n+\ell}^{\mp}(t)c_{n}^{\pm}(t)^* \right) \exp(-i\omega t). \tag{4.7}
\]
the first two moments of the quadrature variables are enlisted using mode sums as

\[
\langle E \rangle^{(\pm)} = \frac{1}{\sqrt{2}} \exp(-\frac{g^2}{2}) \sum_{n=0}^{\infty} \frac{g^{2n}}{n!} \mathfrak{R} \left( G_{n,1}^{\pm}(t) \right),
\]

(4.8)

\[
\langle B \rangle^{(\pm)} = \frac{1}{\sqrt{2}} \exp(-\frac{g^2}{2}) \sum_{n=0}^{\infty} \frac{g^{2n}}{n!} \mathfrak{R} \left( G_{n,1}^{\pm}(t) \right) \pm \sqrt{2} x \zeta,
\]

(4.9)

\[
\langle E^2 \rangle^{(\pm)} = g^2 + \frac{1}{2} \left(1 - \exp(-\frac{g^2}{2}) \sum_{n=0}^{\infty} \frac{g^{2n}}{n!} \mathfrak{R} \left( G_{n,1}^{\pm}(t) \right)\right),
\]

(4.10)

\[
\langle B^2 \rangle^{(\pm)} = g^2 + \frac{1}{2} \left(1 + x + \exp(-\frac{g^2}{2}) \sum_{n=0}^{\infty} \frac{g^{2n}}{n!} \mathfrak{R} \left( G_{n,2}^{\pm}(t) - 2\sqrt{x} G_{n,1}^{\pm}(t) \right)\right).
\]

(4.11)

The time evolution of Heisenberg uncertainty

\[
\mathcal{H} = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \langle B^2 \rangle - \langle B \rangle^2
\]

(4.12)

for various values of coupling strength may be directly obtained from the above moments. We will employ the uncertainty function \( \mathcal{H} \) for studying delocalization in phase space in Sec. V.

Using the procedure developed in Sec. III, we now provide the first two moments of the quadrature variables (1.8) as linear combinations of the Jacobi theta functions. To this end we retain up to the quadratic terms in the expansion of the Laguerre polynomials. The first moments of the quadrature variables are non-vanishing only in the presence of a bias term \( (\epsilon \neq 0) \) in the qubit Hamiltonian (2.1):

\[
\langle E \rangle^{(\pm)} = -\frac{\epsilon}{\Delta} \exp(-\frac{g^2}{2}) \left( \mathcal{R}_1 \mathfrak{R}(T) \mp \mathcal{I}_1 \mathfrak{R}(H) \right),
\]

(4.13)

\[
\langle B \rangle^{(\pm)} = \frac{\epsilon}{\Delta} \exp(-\frac{g^2}{2}) \left( \mathcal{I}_1 \mathfrak{R}(T) \pm \mathcal{R}_1 \mathfrak{R}(H) \right) \pm \sqrt{2} x \zeta,
\]

(4.14)

where the linear combinations of \( \vartheta_3 \) and \( \vartheta_4 \) functions read, respectively, as

\[
T = \left(2 + x + x^2\right)\sqrt{\varphi_0} + \left(2 + \frac{7x}{2}\right) f\sqrt{\varphi_2} + \frac{3}{2} x f \varphi_4 \vartheta_3(q, \bar{\vartheta}_1)
\]

\[
+ x(1 + x)\varphi_{1,0} \vartheta_3(q, z_1) + \frac{3}{2} x f \varphi_{3,1} \vartheta_3(q, z_3),
\]

\[
H = \left(x(1 + x)\sqrt{\varphi_0} - \frac{3}{2} x f \sqrt{\varphi_2}\right) \vartheta_4(q, \bar{\vartheta}_1)
\]

\[
- x(1 + x)\varphi_{1,0} \vartheta_4(q, z_1) + \frac{3}{2} x f \varphi_{3,1} \vartheta_4(q, z_3).
\]

(4.15)

Similarly the second moments of the quadrature variables may also be expressed via Jacobi theta functions as follows:

\[
\langle E^2 \rangle^{(\pm)} = \frac{1}{2} + g^2 + \frac{\exp(-\frac{g^2}{2})}{\sqrt{2\pi} g^2} \left( \mathcal{R}_2 \mathfrak{R}(T) \pm \mathcal{I}_2 \mathfrak{R}(H) \right),
\]

(4.16)

\[
\langle B^2 \rangle^{(\pm)} = \frac{1}{2} \left(1 + x\right) + g^2 + \frac{\exp(-\frac{g^2}{2})}{\sqrt{2\pi} g^2} \times
\]

\[
\times \left( \mathcal{R}_2 \mathfrak{R}(T) \pm \mathcal{I}_2 \mathfrak{R}(H) \right) - 2\sqrt{x} \left( \mathcal{R}_1 \mathfrak{R}(T) \mp \mathcal{I}_1 \mathfrak{R}(H) \right),
\]

(4.17)
where the respective linear sums of $\vartheta_3$ and $\vartheta_4$ functions read:

$$
T = \left( 1 - \frac{\epsilon^2 x^2}{\Delta^2} \right) \tilde{\varphi}_3 \vartheta_3(q, z_2) + \frac{\epsilon^2 x^2}{\Delta^2} \varphi_{2,1} \vartheta_3(q, z_2),
$$

$$
H = \left( \frac{1}{2} - \frac{\epsilon^2}{\Delta^2} \left( 1 + 2x + \frac{11x^2}{2} \right) \right) \tilde{\varphi}_3 + \frac{\epsilon^2}{\Delta^2} f(1 + 4x) \varphi_{2,1} \vartheta_3(q, z_2)
- \frac{5\epsilon^2}{4\Delta^2} f^2 \tilde{\varphi}_3 \vartheta_4(q, z_2) - \frac{\epsilon^2}{\Delta^2} f(1 + \frac{11x}{4}) \varphi_{2,1} \vartheta_4(q, z_2)
+ \frac{5\epsilon^2}{2\Delta^2} f \varphi_{4,3} \vartheta_4(q, z_4),
$$

$$
T = \left( 1 - \frac{\epsilon^2}{\Delta} \right) \varphi_{1,0} \vartheta_3(q, z_1) - \frac{2\epsilon^2}{\Delta^2} f \varphi_{3,1} \vartheta_3(q, z_3)
+ \frac{\epsilon^2}{\Delta^2} \left( (1 + x) \sqrt{\tilde{\varphi}_0} + 2f \sqrt{\tilde{\varphi}_2} \right) \vartheta_3(q, z_1)
$$

$$
H = \left( 1 - \frac{\epsilon^2}{2\Delta} \right) \varphi_{1,0} \vartheta_4(q, z_1) + \frac{\epsilon^2}{\Delta^2} f \varphi_{3,1} \vartheta_4(q, z_3)
- \frac{\epsilon^2}{2\Delta} f \sqrt{\tilde{\varphi}_0} \vartheta_4(q, z_1).
$$

This completes our analytical evaluations of the expectation values of antinormally ordered operators in closed forms involving linear combinations of theta functions. The procedure is valid for the parametric regime described in Sec. III. In Fig. 6 we compare the values of the second moment $\langle E^2 \rangle^{(+)}$ given by the Fourier sum (4.10) and its theta function-based analytical evaluation (4.16). We observe a good agreement between two sets of values.

**V Complexity and its variation with coupling strength**

Our objective in this section is to study the phase space behavior of the system in the ultra-strong coupling domain: $\lambda \sim \omega$. Here we do not adhere to the parametric restrictions
mentioned following (3.4). For instance, our subsequent analysis may be applied to values $|\alpha| \gg 1$ that correspond to macroscopic quantum states. In principle, the evolution (4.2) of the phase space density $Q^{(\pm)}(\beta)$ contains all the relevant informations. An useful object that characterizes the physical property of the state is the Wehrl entropy [16] defined as

$$S_Q^{(\pm)} = -\int Q^{(\pm)}(\beta) \log Q^{(\pm)}(\beta) d^2\beta,$$

(5.1)

which is a semiclassical evaluation of the delocalization in the phase space. It has been interpreted [20] as the number of widely separated coherent states acting as a measure of a given phase space $Q$-distribution of the oscillator. The rate of growth of Wehrl entropy and its connection with Liapunov exponents have been studied [21] as limiting cases to quantum behavior. It has recently been used for studying quantum phase transitions in several models [22, 23]. In our model we, using the time evolution (4.2) and the definition (5.1), first numerically study (Fig. 7) the long-range time dependence of Wehrl entropy for various values of coupling strength. The physical picture may be stated as follows. With increasing qubit-oscillator coupling in the ultra-strong regime newer high-frequency quantum fluctuation modes are rapidly generated. Interference among these modes causes, in general, quick spread of the $Q$-distribution on the phase space resulting in fast initial production of Wehrl entropy:

$$\frac{dS_Q}{dt} |_{t \ll \lambda^{-1}} \gg 0,$$

where the typical time scale of interaction is given by $t \sim \lambda^{-1}$ and a coarse-graining i.e. local averaging over short time interval is implied. Due to randomization of the interferences among a large number of frequency modes at time $t \gg \lambda^{-1}$ we, after a statistical averaging over short time span, obtain a quasi-stationary behavior $S_Q^{-1} \frac{dS_Q}{dt} |_{t \gg \lambda^{-1}} \approx 0$. High frequency quantum fluctuations are superimposed on an almost stationary value. In the ultra-strong coupling domain (Figs. 7(c) and (d)) quantum fluctuations approach towards a random white noise. For the purpose of demonstrating the quasi-stationary property of long-time behavior of the Wehrl entropy in this coupling regime we use a smoothing filter [24] that eliminates the high frequency part of the fluctuations. The contrast between the long-time evolution in the strong coupling (Fig. 7(a)) and the ultra-strong coupling (Figs. 7(c) and (d)) regimes may be stated as follows. In the former case the system undergoes an almost periodic dynamical evolution, while in the latter case recurrences do not occur within a long span of time showing a chaotic property. The coarse-grained structure of Wehrl entropy for ultra-strong qubit-oscillator coupling therefore suggests existence of long-living metastable states of the system. Another significant fact in Fig. 7 is that in a time-averaged sense the Wehrl entropy gradually increases with increasing coupling strength pointing towards a wider spread of the $Q$-distribution in phase space. This is caused by the said randomization of the interferences of the more and more frequency modes that become active due to increasing interaction strength. We also note that the fluctuations in Wehrl entropy $|\Delta S_Q|/S_Q$ without local time-averaging is relatively large.

The above inference regarding the long-time evolution of the state may be further supported by studying the Heisenberg uncertainty function (4.12). As the reduced density matrix of the oscillator at an arbitrary time corresponds to a mixed state, the uncertainty function (4.12) gets contributions from pure quantum states as well as from their classical statistical mixing. It, therefore, provides another measure of the delocalization of the
$Q$-distribution in the phase space. Long-time evolution of the uncertainty function $\mathcal{H}$ given in Fig. 8 is qualitatively similar to that of the Wehrl entropy. In the ultra-strong coupling regime the existence of long-living metastable states with superimposed random Markovian fluctuations resembling white noise is observed in Fig. 8 (b), where a local smoothing filter [24] has been applied.

![Graphs](a) (b) (c) (d)

Figure 7: The evolution of the Wehrl entropy for a long time for $\Delta = 0.15\omega, \epsilon = 0.03\omega$ and $\alpha = 3$ at various $\lambda$ (a)0.1$\omega$, (b)0.13$\omega$, (c)0.7$\omega$, (d)0.9$\omega$. For the cases (c) and (d) a smoothing filter (indicated by the green curve) has been used to eliminate the high frequency part of the fluctuations.

Towards understanding the approach to ergodicity in the phase space the authors of Ref. [17] introduced another measure of delocalization. As it is difficult to analytically determine the Wehrl entropy due to the presence of the logarithmic function in it, they proposed the inverse of the second moment of the Husimi distribution as a measure of complexity of quantum states. This measure is defined [17] as

$$\mathcal{W}_2(Q) = (\mathcal{M}_2(Q))^{-1}, \quad \mathcal{M}_2(Q(\pm)) = \int (Q(\pm)(\beta))^2 d^2\beta. \quad (5.2)$$

The complexity of the quantum state $\mathcal{W}_2(Q)$ represents the effective phase space occupied by the Husimi distribution of the oscillator density matrix. The expansion [17] allows us to evaluate the second moment of the Husimi distribution as follows:

$$\mathcal{M}_2(Q(\pm)) = \frac{1}{8\pi} \exp(-2\tilde{g}_2^2) \left[ \sum_{N=0}^{\infty} \frac{\tilde{g}_2^2 N}{2^N N!} \mathcal{D}_N(t) + 2 \exp \left(-\frac{X}{2}\right) \sum_{N,M=0}^{\infty} \tilde{\alpha}^N (\tilde{\alpha}^*)^M \times \right.$$  

$$\times \exp \left(-i(N-M)\omega t\right) \sum_{\mu=0}^{\min(N,M)} \frac{(\lambda^2)^{N-M-2\mu}}{2^\mu \mu!(N-\mu)!(M-\mu)!} \times$$

$$\left. \times \tilde{\mathcal{F}}_{N,\mu}(t) \tilde{\mathcal{F}}_{M,\mu}^*(t) \right]. \quad (5.3)$$
Figure 8: The evolution of the uncertainty product ($\mathcal{H}$) for a long time for $\Delta = 0.15\omega$, $\epsilon = 0.03\omega$ and $\alpha = 3$ at values of $\lambda$ (a)0.1$\omega$ (b)0.9$\omega$. For the case (b) a smoothing filter (indicated by the green curve) has been used to eliminate the high frequency part of the fluctuations.

where the mode sums on the oscillator states are given below:

$$\mathcal{D}_N(t) = \left| \sum_{n=0}^{N} \binom{N}{n} \mathcal{C}_n^+(t) \mathcal{C}_{N-n}^-(t) \right|^2 + \left| \sum_{n=0}^{N} \binom{N}{n} \mathcal{C}_n^-(t) \mathcal{C}_{N-n}^+(t) \right|^2,$$

$$\mathfrak{F}_{N,\mu}(t) = \sum_{n=0}^{N} \Lambda_{N,n,\mu} \mathcal{C}_n^+(t) \mathcal{C}_{N-n}^-(t)^*, \quad \Lambda_{N,n,\mu} = \sum_{k} (-1)^k \binom{\mu}{k} \binom{N - \mu}{N - n - k}. \quad (5.4)$$

The above expression constitute our analytical evaluation of the complexity measure $\mathcal{W}_2(Q)$.

Lastly, in Fig. 9 we study the variation of different measures of complexity with respect to the qubit-oscillator coupling $\lambda$ at a fixed time. For the following discussion we view the time evolution \textsuperscript{(4.2)} of the $Q$-distribution as a parametrization of an ensemble of quantum mixed states of the oscillator, and one may study therein the dependence of the phase space complexity measures such as Wehrl entropy and $\mathcal{W}_2(Q)$ on the coupling strength $\lambda$. We observe that both these measures show qualitatively identical pattern of behavior. Disregarding small local fluctuations we find that for a class of states the Wehrl entropy (and also $\mathcal{W}_2(Q)$) first increases, and after reaching a maximum value it decreases with increasing coupling strength. This property is not in contradiction with our observation in Fig. 7 that locally time-averaged value of Wehrl entropy increases with increasing coupling strength. It is realized because of the relatively large value of the fluctuations in Wehrl entropy $|\Delta S_Q|/S_Q$ observed in Fig. 7. If we assume that the phase space distribution is equivalent to $W(\lambda)$ distinguishable widely separated nonoverlapping coherent state components, then the corresponding Wehrl entropy is given by [20]

$$S_Q(\lambda) = 1 + \log \pi + \log W(\lambda) \quad \Rightarrow \quad \frac{W(\lambda)}{W(\lambda_0)} = \exp(S_Q(\lambda) - S_Q(\lambda_0)), \quad (5.5)$$

where $\lambda_0$ is the coupling strength at the maximum delocalized state realized at the peak value of the Wehrl entropy. In (5.5) we have suppressed the time that parametrizes the state. The quantity $W(\lambda)$ has a role reminiscent of the statistical weight factor for a microcanonical ensemble. Let $m_<$ and $m_>$ denote the locally-averaged slopes of the Wehrl
Figure 9: The behavior of the Wehrl entropy and the complexity with varying coupling strength for $\Delta = 0.15 \omega, \epsilon = 0.1 \omega$ and $\alpha = 3$ at various time (a) $t = 76.5$ (b) $t = 540$ and (c) $t = 900$ in first and second rows, respectively. The averaged slopes $(m_\leq, m_\geq)$ of the Wehrl entropy below (above) its maximum value at the coupling strength $\lambda_0$ are marked by green (blue) dashed lines.

entropy in Fig. 9 below ($\lambda < \lambda_0$) and above ($\lambda > \lambda_0$) its maximum value, respectively. Then the corresponding effective nonoverlapping coherent state components in the said coupling ranges show exponential fall from the statistical weight of the maximum delocalized state at different rates:

$$W_\leq(\lambda) = \exp(-m_\leq |\lambda_0 - \lambda|), \quad W_\geq(\lambda) = \exp(-m_\geq |\lambda_0 - \lambda|).$$

In the coupling regime ($\lambda > \lambda_0$) this denotes a condensation in the phase space resembling an effective chemical potential term. A comparison between the Figs. 9(a), (b), (c) reveals that for a higher degree of delocalization of the state in the phase space as measured by the Wehrl entropy the onset of condensation occurs at a corresponding higher value of the coupling strength.

VI Conclusion

Using adiabatic approximation we investigated a qubit-oscillator bipartite system in the presence of a static bias term in the Hamiltonian for strong and ultra-strong coupling regimes. Starting with an entangled quasi-Bell state we obtained the evolution of the reduced density matrices of the qubit and the oscillator. Identifying the series with the Jacobi theta functions in the strong coupling domain we evaluated the qubit density matrix elements in closed form as linear combinations of theta functions. The reduced
density matrix of the qubit yields the entropy of the system. The time evolution of the entropy suggests that the individual subsystems repeatedly arrive at almost pure states reducing the entanglement and the mixedness to very low values for short time intervals. On the other hand the reduced density matrix of the oscillator provides a way to study the phase space density functions of the oscillator degree of freedom. In particular, we evaluate the $Q$-function and utilize this for obtaining closed-form expressions of the first two moments of the quadrature variables as linear combinations of theta functions. Our closed-form evaluations of various physical quantities for strong coupling domain and weak bias are compared with, and found to be good approximations of, their series values.

For an ultra-high qubit-oscillator coupling strength we study the Wehrl entropy, the Heisenberg uncertainty function, and the complexity $W_2(Q)$ that measure the delocalization of the oscillator in the phase space. A series expression of the complexity $W_2(Q)$ is obtained. The long-term time evolutions of the Wehrl entropy and the Heisenberg uncertainty function suggest existence of long-living metastable states with superimposed Markovian fluctuations that resemble white noise. Fixing the time we study the coupling strength dependence of the Wehrl entropy and complexity. For a class of states we observe that subsequent to the maximum delocalization of one of these ensemble of states a condensation of the effective occupied area in the phase space $Q$-distribution sets in with increasing qubit-oscillator coupling.

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