Some boundary-value problems for anisotropic quarter plane

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Abstract. To solve the mixed boundary-value problems of the anisotropic elasticity for the anisotropic quarter plane, a method based on the use of the space of generalized functions $\mathcal{I}'(\mathbb{R}^2_+)$ with slow growth properties was developed. The two-dimensional integral Fourier transform was used to construct the system of fundamental solutions for the anisotropic quarter plane in this space and a system of eight boundary integral relations was obtained, which allows one to reduce the mixed boundary-value problems for the anisotropic quarter plane directly to systems of singular integral equations with fixed singularities. The exact solutions of these systems were found by using the integral Mellin transform. The asymptotic behavior of solutions was investigated at the vertex of the quarter plane.

1. Introduction

The plane problems of the theory of elasticity for isotropic and anisotropic wedges and the quarter plane have been considered by many authors. Here we should mention the papers [1, 2], where mixed, first and second basic problems for the isotropic wedge were solved using the Mellin integral transform. In [3], the mixed plane problem for the elastic quadrant was solved. A similar problem was also solved in [4] by reduction to systems of singular integral equations. In [5], the Mellin integral transform was used to solve the first fundamental problem for an anisotropic wedge. In [6], the first basic problem for a composite anisotropic wedge was considered and the stress behavior at the vertex of the wedge was investigated.

In this paper, the mixed boundary-value problems of anisotropic elasticity for the anisotropic quarter plane is solved by an approach proposed in [7–13] for solving the problems for piecewise-homogeneous anisotropic unbounded media. The approach is implemented by constructing the system of fundamental solutions and singular integral relations connecting the boundary values of stresses and displacements on the faces of the anisotropic quarter plane. The latter permits reducing the mixed boundary-value problems for the anisotropic quarter plane directly to systems of singular integral equations (SIE).

2. Statement of the problem and construction of the fundamental system of solutions

The stress-strain state of the anisotropic quarter plane $(x, y) \in \mathbb{R}^2_+ = L_+ \times L_+ = (0, +\infty)$ is described by the vector $v = \{v_p(x, y)\}_{p=1,5} = \{\sigma_x, \sigma_y, \tau_{xy}, u, v\}$, whose boundary values of
components on the faces of the quarter plane are denoted by
\[ \chi_k(x) = v_{k+1}(x, +0), \quad k = \frac{1}{4}, \quad x \in L_+, \]
\[ \mu_1(y) = v_1(+0, y), \quad \mu_k(y) = v_{k+1}(+0, y), \quad k = \frac{2}{4}, \quad y \in L_. \]

The components of the vector \( v \) satisfy the equilibrium equations and the generalized Hooke’s law, which in the absence of all forces can be written as
\[ M(\partial_1, \partial_2)v = 0, \quad (x, y) \in \mathbb{R}^2_+, \]
where
\[ M(\partial_1, \partial_2) = \begin{pmatrix} D & O_{2 \times 2} \\ A & D^T \end{pmatrix}, \quad D = \begin{pmatrix} \partial_1 & 0 & \partial_2 \\ 0 & \partial_2 & \partial_1 \end{pmatrix}, \quad A = \begin{pmatrix} a_{kj} \end{pmatrix}, \quad \partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \]
a\(_{ij}\) are elastic constants of the anisotropic quarter plane [14], and \( O_{n \times m} \) is the zero \( n \times m \) matrix.

In the statement of the problem for anisotropic elasticity, it is necessary to know two functions of representations (1) and (2) on each face of the quarter plane. The remaining functions are to be determined by solving the problem. Following [7–9], one must construct the system of fundamental solutions and establish integral relations between the functions (1) and (2) on the faces of the quarter plane.

To implement this approach, we pass to the space of generalized slowly growing functions \( \mathcal{F}'(\mathbb{R}^2_+) \). Using the boundary conditions (1), (2) and the connection formulas \([8, 9]\) between the ordinary derivatives \( \partial_k \) \((k = 1, 2)\) and generalized derivatives \( \tilde{\partial}_k \) \((k = 1, 2)\), instead of the matrix differential equation (3), we obtain the following matrix differential equation in the space of generalized functions \( \mathcal{F}'(\mathbb{R}^2_+) \):
\[ (M(\tilde{\partial}_1, \tilde{\partial}_2)v, \zeta) = (f, \zeta), \quad \tilde{v} = \{\tilde{v}_j\}^5, \quad \tilde{v}_j \in \mathcal{F}'(\mathbb{R}^2_+), \quad \zeta \in \mathcal{F}(\mathbb{R}^2_+), \]
where
\[ f = \{f_j\}^5, \quad f_1 = \chi_2(x)\delta(L_1) + \mu_1(y)\delta(L_2), \quad f_2 = \chi_1(x)\delta(L_1) + \mu_2(y)\delta(L_2), \]
\[ f_3 = \mu_3(y)\delta(L_2), \quad f_4 = \chi_4(y)\delta(L_1), \quad f_5 = \chi_3(x)\delta(L_1) + \mu_3(y)\delta(L_2), \]
\( \delta(L_k) \in \mathcal{F}'(L_k) \) is the generalized function concentrated on the semiaxis \( L_k, \ L_1 = \{y = 0, x \in [0, +\infty)\}, \) and \( L_2 = \{x = 0, y \in [0, +\infty)\} \).

Following [7–13], we solve equation (4) using the system of fundamental solutions by which we mean the system of vectors \( v^0_j = \{v^0_{kj}\}^5, \ j = \frac{1}{18}, \ v^0_{kj} \in \mathcal{F}'(\mathbb{R}^2_+) \). In the space \( \mathcal{F}'(\mathbb{R}^2_+) \), these vectors satisfy the matrix differential equation
\[ (M(\tilde{\partial}_1, \tilde{\partial}_2)v_j, \zeta) = (f^0_j, \zeta), \quad \zeta \in \mathcal{F}(\mathbb{R}^2_+) \quad (j = \frac{1}{18}), \]
where
\[ f^0 = \{f^0_j\}^5, \quad f^0_1 = \delta_{2j}\delta(x-x_0, y) + \delta_{5j}\delta(x, y-y_0), \quad f^0_2 = \delta_{1j}\delta(x-x_0, y) + \delta_{0j}\delta(x, y-y_0), \]
\[ f^0_3 = \delta_{7j}\delta(x, y-y_0), \quad f^0_4 = \delta_{kj}\delta(x-x_0, y), \quad f^0_5 = \delta_{3j}\delta(x-x_0, y) + \delta_{8j}\delta(x, y-y_0), \]
and \( \delta_{kj} \) is the Kronecker symbol.

The support of generalized functions \( \tilde{v}_j, v^0_{kj} \) is the domain \( \mathbb{R}^2_+ \), i.e., \( \text{supp}(\tilde{v}_j, v^0_{kj}) = \mathbb{R}^2_+ \). The latter allows one to apply the two-dimensional Fourier transform to solve the matrix differential
equation (5). As a result, we obtain explicit expressions for the vector components of the system of fundamental solutions

\[
v^{0}_{p,j}(x,y) = \frac{1}{\pi} \text{Im} \sum_{k,m=1}^{2} \frac{1}{q_k} \left[ M^{0}_{p,m}(z_k)\delta_{m,j} + \frac{M^{0}_{p,2+m}(z_k)\delta_{2+m,j}}{(x-x_0+z_ky)^2} \right] + \frac{M^{0}_{p,4+m}(z_k)\delta_{4+m,j}}{x+z_k(y-y_0)} + \frac{M^{0}_{p,6+m}(z_k)\delta_{6+m,j}}{(x+z_k(y-y_0))^2} \right], \quad p = 1,3,
\]

\[
v^{0}_{p,j}(x,y) = \frac{1}{\pi} \text{Im} \sum_{k,m=1}^{2} \frac{1}{q_k} \left[ M^{0}_{p,m}(z_k)\delta_{m,j} \left( \ln(x-x_0+z_ky) + \gamma - \frac{\pi i}{2} \right) + \frac{M^{0}_{p,2+m}(z_k)\delta_{2+m,j}}{x-x_0+z_ky} + \frac{M^{0}_{p,4+m}(z_k)\delta_{4+m,j}}{x+z_k(y-y_0)} + \frac{M^{0}_{p,6+m}(z_k)\delta_{6+m,j}}{(x+z_k(y-y_0))^2} \right], \quad p = 4, 5,
\]

\[
M^{0}(z) = \{M^{0}_{p,m}(z)\}_{p=1,5, m=0}^{3},
\]

\[
M^{0}_{14}(z) = M^{0}_{16}(z) = M^{0}_{27}(z) = zl_4(z), \quad M^{0}_{12}(z) = M^{0}_{15}(z) = M^{0}_{17}(z) = l_3(z), \quad M^{0}_{23}(z) = M^{0}_{28}(z) = M^{0}_{31}(z) = z, \quad M^{0}_{30}(z) = M^{0}_{32}(z) = M^{0}_{33}(z) = M^{0}_{35}(z) = l_1(z), \quad M^{0}_{41}(z) = M^{0}_{42}(z) = M^{0}_{50}(z) = z, \quad M^{0}_{40}(z) = M^{0}_{46}(z) = zl_2(z), \quad M^{0}_{44}(z) = M^{0}_{45}(z) = -l_2(z), \quad M^{0}_{46}(z) = M^{0}_{45}(z) = -l_0(z), \quad M^{0}_{42}(z) = M^{0}_{46}(z) = M^{0}_{50}(z) = zl_3(z), \quad M^{0}_{44}(z) = M^{0}_{45}(z) = -l_3(z), \quad M^{0}_{46}(z) = M^{0}_{45}(z) = -l_0(z),
\]

where \(z_k\) are roots of the characteristic equation [14] and \(A_{ij}\) is the cofactor of the matrix \(A = [a_{kl}]^{3}\).

The presence of explicit expressions (6) for the vector components \(v^{0}_{j} = \{v^{0}_{k,j}\}_{j=1,5,}^{5}\), \(j = 1,5\), permits expressing the stresses and displacements in the space \(\mathfrak{S}'(R^2_+)\) in terms of the boundary values (1), (2) as the convolution

\[
\tilde{v}_{k}(x,y) = \frac{4}{\pi} \sum_{j=1}^{4} \chi_{j}(x) * v^{0}_{k,j}(x,y) + \mu_{j}(y) * v^{0}_{k,4+j}(x,y), \quad k = 1,5.
\]

Passing in the limit to the faces of the quarter plane and using the boundary values of displacements and stresses (1) and (2), we obtain the integral relations on the boundary:

\[
\sum_{j=1}^{2} \left\{ A^{1}_{pj} \chi_{j}(x) + \frac{B^{1}_{pj}}{\pi} \int_{0}^{+\infty} \chi_{j}(x_0) dx_0 \right\} x_0 - x + A^{1}_{p,2+j} \chi_{2+j}(x) + \frac{B^{1}_{p,2+j}}{\pi} \int_{0}^{+\infty} \chi_{2+j}(x_0) dx_0 \right\} x_0 - x
\]
the following system of SIE with fixed singularities:

\[\begin{align*}
\sum_{j=1}^{2} \left\{ A_{pj}^{2} \mu_{j}(y) + \frac{B_{pj}^{2}}{\pi} \int_{0}^{\infty} \frac{\mu_{j}(y_{0}) \, dy_{0}}{y_{0} - y} \right\} &= 0, \quad p = \frac{1}{4}, \quad x \in L_{1}, \\
+ \text{Im} \sum_{k=1}^{2} \left[ C_{pj}^{1}(z_{k}) \int_{0}^{\infty} \chi_{j}(x_{0}) \frac{dx_{0}}{x_{0} - z_{k}y_{0}} + C_{pj}^{2}(z_{k}) \int_{0}^{\infty} \chi_{2+j}(x_{0}) \frac{dx_{0}}{x_{0} - z_{k}y_{0}} \right] &= 0, \quad p = \frac{1}{4}, \quad y \in L_{2},
\end{align*}\]

\[\begin{align*}
A_{pj}^{1} &= 1 + \text{Re} \sum_{k=1}^{2} M_{p+1,j}^{0}(z_{k}), \quad p, j = 1, 2, \quad p, j = 3, 4, \\
B_{pj}^{1} &= \text{Im} \sum_{k=1}^{2} M_{p+1,j}^{0}(z_{k}), \quad p = \frac{1}{4}, \quad j = 1, 2, \quad p, j = 3, 4,
\end{align*}\]

\[\begin{align*}
C_{pj}^{1}(z) &= z^{-1} M_{p+1,j}^{0}(z), \quad p, j = 1, 2, \quad j = 3, 4, \\
C_{pj}^{1}(z) &= -z^{-1} M_{p+1,j}^{0}(z), \quad p, j = 3, 4,
\end{align*}\]

3. Solution of the mixed boundary-value problem and analysis of the obtaining results

Integral relations (8) allow one to reduce the boundary-value problems for the elastic anisotropic quarter plane directly to systems of SIE. In particular, let the conditions of complete cohesion be realized on one face and the stresses be known on the other:

\[\chi_{k}(x) = 0, \quad x \in L_{1}, \quad k = 3, 4; \quad \mu_{k}(y) = P_{k}(y), \quad y \in L_{2}, \quad k = 1, 2.\]  \hfill (9)

Having realized the first two conditions in (9) with the help of the third and fourth integral relations in (8) and the last two conditions in (9) and using the first and second integral relations, relatively to the unknown boundary values of the stresses \(\chi_{j}(x), \quad x \in L_{1} \quad (j = 1, 2)\) and the unknown boundary values of the displacement derivatives \(\mu'_{j}(y), \quad y \in L_{2} \quad (j = 3, 4)\), we obtain the following system of SIE with fixed singularities:

\[\begin{align*}
\sum_{j=1}^{2} A_{mj}^{1} \chi_{j}(x) + \frac{B_{mj}^{1}}{\pi} \int_{0}^{\infty} \frac{\chi_{j}(x_{0}) \, dx_{0}}{x_{0} - x} + \text{Im} \sum_{k=1}^{2} C_{m+2,j}^{1} \int_{0}^{\infty} \frac{\mu'_{2+j}(y_{0}) \, dy_{0}}{x - z_{k}y_{0}} &= F_{m}(x), \quad m = 1, 2, \\
\sum_{j=3}^{4} A_{mj}^{2} \mu'_{j}(y) + \frac{B_{mj}^{2}}{\pi} \int_{0}^{\infty} \frac{\mu'_{j}(y_{0}) \, dy_{0}}{y_{0} - y} + \text{Im} \sum_{k=1}^{2} C_{m+2,j}^{2} \int_{0}^{\infty} \frac{\chi_{j}(x_{0}) \, dx_{0}}{x_{0} - z_{k}y_{0}} &= F_{m}(y), \quad m = 3, 4.
\end{align*}\]
The solution of system (10) was obtained by the Mellin transform
\[
\tilde{\chi}_j(t) = \begin{cases} 
\chi_j(t), & j = 1, 2, \\
\mu'_j(t), & j = 3, 4,
\end{cases} 
\]
(11)
and
\[
\tilde{\chi}_j(t) = \frac{1}{2\pi i} \sum_{l=1}^{4} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\sin(\pi s)M_{jl}(s)F_l(s)}{\text{det}(M(s))} t^{-s} ds, \quad 0 < \sigma < 1,
\]
(12)
where
\[M(s) = \{M_{jl}(s)\}^4, \quad M_{jl}(s) = \text{Im} \sum_{k=1}^{2} C_{jl}^k(z_k)(-z_k)^{s-1}, \quad j = 1, 2, \ l = 3, 4;
\]
\[M_{jl}(s) = \text{Im}(\omega_{jl}e^{i\pi s}), \quad j, l = 1, 2, j, l = 3, 4, \quad M_{jl}(s) = \text{Im} \sum_{k=1}^{2} C_{jl}^1(z_k)(-z_k)^{-s}, \quad j = 3, 4, \ l = 1, 2,
\]
with the Mellin transform from the right-hand sides of system (10).

To calculate the integrals (12) and to determine the solution asymptotics at the vertex of the quarter plane, it is necessary to know the location of the roots of the transcendental equation
\[
\text{det}(M(s)) = 0,
\]
(13)
in the half plane \(\text{Re} \ s < 1\). Let \(\{s_k\}_k^{\infty}\) be the roots of equation (13) satisfying the conditions
\[
1 > \text{Re} \ s_1 > \text{Re} \ s_2 > \text{Re} \ s_3 > \ldots > \text{Re} \ s_k > \ldots,
\]
then
\[
\tilde{\chi}_j(t) \simeq k_j t^{-\text{Re} \ s_1}, \quad t \to 0.
\]
(14)

The behavior of the roots of transcendental equation (13) was analyzed numerically. In particular, it was established that, for known anisotropic materials equation (13), has at least one root whose real part belongs to the interval \((0; 1)\). Therefore, the mixed boundary-value problem (9) has a unique solution in the class of functions with integrable singularity at zero. In addition, the asymptotic behavior of the roots at infinity is established. Namely, for each anisotropic material, one can find a number \(s_\ast\) such that all real roots of transcendental equation (13) belong to the interval \((-s_\ast; 1)\) and their number \(n_\ast\) is bounded. All other roots are complex and are arranged along two straight lines symmetric with respect to the real axis.

Table 1 shows the first ten roots of equation (5) for different angles between the coordinate axes of the plane and the main symmetry axes of anisotropic materials [14]: \(m1\) – glass-fiber orthogonally reinforced plastic, \(m2\) – glass-fiber plastic STET.

A numerical analysis has shown that, for the material \(m1\) (\(\varphi = 0\)), transcendental equation (5) has four real roots: \(n_\ast = 4\) and \(s_\ast = 1.34\). All roots in \(|s| > s_\ast = 1.34\) are arranged along the straight lines \(z = \text{Re} \ s \pm (0.034856 - 0.565527 \text{Re} \ s)i\) with step \(h = 1.5137\). For the material \(m2\) (\(\varphi = 0\)), the equation also has four real roots \(n_\ast = 4\) and \(s_\ast = 1.22\). All roots in \(|s| > s_\ast = 1.22\) are arranged along the straight lines \(z = \text{Re} \ s \pm (0.076057 - 0.413467 \text{Re} \ s)i\) with step \(h = 1.7067\). For the same materials but with \(\varphi = \pi/4\), the localization domain of the real roots is wider, so for the material \(m1\): \(s_\ast = 37.5\), and for the material \(m2\): \(s_\ast = 26.5\). Figure 1 shows the root location for materials \(m1\) and \(m2\) \(\varphi = 0\).

**Conclusions**

Thus, the received singular integral relations (8) allow the boundary mixed problems for the anisotropic quarter plane with any number of points of the boundary conditions’ change to reduce directly to the SIE systems with fixed singularities. The kernels of these systems have a simple structure and make it possible to reveal not only the main solutions’ asymptotics, but all up to any given order. This makes it possible to apply effective numerical methods to solving these SIE systems.
Figure 1. Location of roots of equation (13) for materials $m_1$ (a) and $m_2$ (b).

| Material $m_1$ $(\varphi = 0)$ | Material $m_1$ $(\varphi = \pi/4)$ | Material $m_2$ $(\varphi = 0)$ | Material $m_2$ $(\varphi = \pi/4)$ |
|-----------------------------------|-----------------------------------|---------------------------------|-----------------------------------|
| $s_1$ $0.96398 \pm 0.10714i$      | $0.68343$                         | $0.96782 \pm 0.07313i$         | $0.6501 \pm 0.43579i$            |
| $s_2$ $0.09662 \pm 0.43387i$      | $0.47475$                         | $0.10265 \pm 0.49705i$         | $-0.40598 \pm 0.49585i$          |
| $s_3$ $-0.00336$                  | $0.07335$                         | $-0.00653$                      | $-0.89475$                       |
| $s_4$ $-0.03593$                  | $0.04851$                         | $-0.02948$                      | $-0.98229$                       |
| $s_5$ $-0.66602 \pm 0.62959i$    | $-0.46505 \pm 0.14226i$          | $-0.7047 \pm 0.63859i$         | $-1.41454 \pm 0.47715i$          |
| $s_6$ $-1.00312$                  | $-1.48349 \pm 0.11995i$          | $-1.00558$                      | $-2.47249 \pm 0.43114i$          |
| $s_7$ $-1.32116$                  | $-1.90011$                        | $-1.21599$                      | $-2.89316$                       |
| $s_8$ $-1.7166 \pm 1.00599i$     | $-1.9648$                         | $-1.80739 \pm 0.83147i$        | $-2.91223$                       |
| $s_9$ $-3.35246 \pm 1.93375i$   | $-2.28077$                        | $-3.72018 \pm 1.60498i$        | $-3.46338 \pm 0.43883i$          |
| $s_{10}$ $-4.87467 \pm 2.78832i$| $-2.55499$                        | $-5.44624 \pm 2.32672i$        | $-4.39582 \pm 0.37442i$          |
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