Multifractal Analysis of Turbulence by Statistics based on Non-Extensive Tsallis’ or Extensive Rényi’s Entropy

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An analytical expression of probability density function (PDF) of velocity fluctuation is derived with the help of the statistics based on generalized entropy (the Tsallis entropy or the Rényi entropy). It is revealed that the derived PDF explains the detailed structure of experimentally observed PDF as well as the scaling exponents of velocity structure function. Every parameters appeared in the analysis, including the index proper to the Tsallis entropy or the Rényi entropy, are determined, self-consistently, by making use of observed value of intermittency exponent. The experiments conducted by Lewis and Swinney (1999) are analyzed.

Our analysis is based on the fact that, for high Reynolds number Re ≫ 1, the Navier-Stokes equation for incompressible fluid is invariant under the scale transformation 3: \( r' \rightarrow \lambda^{\alpha/3} r, \quad u' \rightarrow \lambda F, \quad x' \rightarrow \lambda^{-1-\alpha/3} t \) and \( (p/\rho)_0 \rightarrow \lambda^{2\alpha/3} (p/\rho)_0 \). Here, the exponent \( \alpha \) is an arbitrary real quantity which specifies the degrees of singularity in the velocity gradient \( |\partial u(x)/\partial x| = \lim_{\ell_n \to 0} |u(x + \ell_n) - u(x)}/\ell_n = \lim_{\ell_n \to 0} \delta u_n/\ell_n \). This can be seen with the relation \( \delta u_n/\delta u_0 = (\ell_n/\ell_0)^{\alpha/3} \), which leads to the singularity in the velocity gradient \( \delta \) for \( \alpha < 3 \), since \( \delta u_n/\delta u_0^{\alpha/3} = \text{const.} \).

The distribution \( P(n)(\alpha)\, d\alpha \propto P^{(1)}(\alpha)^n\, d\alpha \) of singularities at the nth step in the cascade with

\[
P^{(1)}(\alpha) \propto \left[ 1 - (\alpha - \alpha_0)^2/(\Delta \alpha)^2 \right]^{1/(1-q)},
\]

\( (\Delta \alpha)^2 = 2X/[(1 - q)\ln 2] \) was derived 4, by taking an extreme of the Tsallis entropy \( S_q[P^{(1)}(\alpha)] = (1 - q)^{-1} \int d\alpha P^{(1)}(\alpha)^q - 1 \) which is non-extensive, or the Rényi entropy \( S_q[P^{(1)}(\alpha)] = (1 - q)^{-1} \int d\alpha P^{(1)}(\alpha)^q \) which has the extensive character, with appropriate constraints, i.e., the normalization of distribution function, \( \int d\alpha P^{(1)}(\alpha) = \text{const.} \), and the q-variance being kept constant as a known quantity, \( \sigma_q^2 = \langle (\alpha - \alpha_0)^2 \rangle_q = (\int d\alpha P^{(1)}(\alpha)^q (\alpha - \alpha_0)^2)/\int d\alpha P^{(1)}(\alpha)^q \). In deriving \( P(n)(\alpha) \), we assumed that each step in the cascade is statistically independent. Note that the values of \( \alpha \) are restricted within the range \( [\alpha_{\text{min}}, \alpha_{\text{max}}] \), where \( \alpha_{\text{max}} - \alpha_0 = \alpha_0 - \alpha_{\text{min}} = \Delta \alpha \).

With the help of the relation \( P(n)(\alpha) \propto \delta_n^{-1} f(\alpha) \), we can extract the multi-fractal spectrum \( f(\alpha) \) in the form 4, 12:

\[
f(\alpha) = 1 + (1 - q)^{-1} \log_2 \left[ 1 - (\alpha - \alpha_0)^2/(\Delta \alpha)^2 \right],
\]

which reveals how dense each singularity, labeled by \( \alpha \), fills physical space.

By making use of an observed value of the intermittency exponent \( \mu \) as an input, the quantities \( \alpha_0, X \) and the index \( q \) can be determined, self-consistently, with the help of the three independent equations, i.e., the energy conservation: \( \langle \epsilon_n \rangle = \epsilon \), the definition of the intermittency exponent \( \mu : \langle \epsilon_n^2 \rangle = \epsilon^2 \delta_n^{-\mu} \), and the scaling relation 4, 12: \( 1/(1-q) = 1/\alpha_- - 1/\alpha_+ \) with \( \alpha_\pm \) satisfying \( f(\alpha_\pm) = 0 \), where the average \( \langle \cdots \rangle \) is taken by
where the singular part $\Pi_S^{(n)}(x_n)$, stemmed from multifractal distribution of singularities in velocity gradient that may be related to the fluctuations caused by turbulent viscosity, is determined by $\Pi_S^{(n)}(x_n)dx_n = P^{(n)}(\alpha)d\alpha$ with the transformation of the variables between $|x_n|$ and $\alpha$: $|x_n| = \delta_0^{\alpha/3}$, leading to

$$\Pi_S^{(n)}(x_n)dx_n = 3G^{(n)}(|x_n|)/\left(2^{\gamma^{(n)}}\ln \delta_n\right)dx_n$$ (5)

with

$$G^{(n)}(x) = x^{-1} \left[1 - (\alpha(x) - \alpha_0)^2/(\Delta \alpha)^2\right]^{n/(1-q)},$$ (6)

$$\alpha(x) = 3\ln x/\ln \delta_n.$$  Note that the singular part of the probability density function has a dependence on $\ln \delta_u$, and that the values of $|x_n|$ are restricted within the range $[\delta_0^{\max/3}, \delta_0^{\min/3}]$. The first term, $\Pi_S^{(n)}(x_n)$ assumed to come from thermal or dissipative fluctuation caused by the kinematic viscosity, will be considered later in this paper after estimating the moments of the velocity fluctuation.

FIG. 3. The scaling exponent $\gamma_n$ of the velocity structure function. The experimental results obtained by Lewis and Swinney are shown by circles (Re = 540 000) and by triangles (Re = 69 000). The present theoretical result (I) is drawn by solid line for the intermittency exponent $\mu = 0.28$ taken from the experimental data. Dotted line represents K41, whereas dashed line She-Leveque. The prediction of the log-normal model is given by short-dashed line with the best fit but inconsistent value $\mu = 0.27$ of the intermittency exponent following Lewis and Swinney.

The $m$th moments of the velocity fluctuation, defined by $\langle|x_n|^m\rangle = \int_{-\infty}^{\infty} dx_n |x_n|^m \Pi^{(n)}(x_n)$, are given by

$$\langle|x_n|^m\rangle = 2\zeta_m + (1 - 2\zeta_0) a_m \delta_n^{\gamma_m}$$ (7)

where $\zeta_{\delta q} = \left\{2/(C_q^{1/2}(1 + C_q^{1/2}))\right\}^{1/2}$ with $C_q = 1 + 2q^2(1 - q)X \ln 2$, and

$$2\zeta_m = \int_{-\infty}^{\infty} dx_n |x_n|^m \Pi_N^{(n)}(x_n).$$ (8)

Adopting the value $\mu = 0.28$ observed in the experiment [10] with the Taylor-scale Reynolds number $R_\lambda = 262$ (Re = 540 000) or $R_\lambda = 80$ (Re = 69 000), and solving the above three equations self-consistently, we have $q = 0.471$, $\alpha_0 = 1.162$, $X = 0.334$. Then, we obtain $\alpha_+ - \alpha_0 = \alpha_+ - \alpha_- = 0.748$, $\Delta \alpha = 1.350$ [10].

It may be reasonable to assume that the probability $P^{(n)}(x_n)dx_n$ to find the scaled velocity fluctuation $|x_n| = \delta u_n/\delta u_0$ in the range $x_n \sim x_n + dx_n$ can be divided into two parts:

$$P^{(n)}(x_n)dx_n = P_N^{(n)}(x_n)dx_n + P_S^{(n)}(|x_n|)dx_n$$ (4)
Here, we used the normalization $\langle \xi \rangle = 1$. The quantity
\[
\zeta_m = \alpha_0 m/3 - 2Xm^2/\left[9 \left(1 + C_m^{1/2}\right)\right] - \left[1 - \log_2 \left(1 + C_m^{1/2}\right)\right] / (1 - q),
\]
(9)
is the so-called scaling exponents of velocity structure function, whose expression was derived first by the present authors [9-12]. In Fig. 3, we compare the present result (9) with the experimentally measured scaling exponents [17] at Re = 540 000 (circles) and Re = 69 000 (triangles). We use the observed value of the intermittency exponent, i.e., $\mu = 2 - \zeta_6 = 0.28$, for theoretical analysis. There are also represented, as references, the predictions of K41 (dotted line) [2] of She-Leveque (dashed line) [17] and of the log-normal (short-dashed line) [18, 20]. Note that our formula (9) can also explain the scaling exponent for higher moments [9-12].

Let us introduce new variable
\[
\xi_n = \delta u_n / \sqrt{\langle \delta u_n^2 \rangle} = x_n / \sqrt{\langle x_n^2 \rangle} = \bar{\xi}_n \delta_n^{\alpha/3 - \zeta_2/2}
\]
(10)
scaled by the variance of velocity fluctuation where $\xi_n = [2\gamma_2^{(n)} \delta_n^{-\zeta_2} + (1 - 2\gamma_0^{(n)})]^{-1/2}$, and the probability density function (PDF) $\hat{H}^{(n)}(\xi_n)$ in terms of this variable through the relation
\[
\hat{H}^{(n)}(\xi_n) d\xi_n = \hat{H}^{(n)}(x_n) dx_n.
\]
(11)
Making use of the relation between $\xi_n$ and $\alpha$, the PDF responsible for the distribution of singularities $\hat{H}^{(n)}_S(\xi_n)$ can be rewritten in terms of $\alpha$ in the form
\[
\hat{H}^{(n)}_S(\xi_n) = \hat{H}^{(n)}_S \delta \xi_n^{2 - \alpha/3 + 1 - f(\alpha)}
\]
(12)
with $\tilde{H}^{(n)}_S = 3(1 - 2\gamma_0^{(n)})/(2\xi_n \sqrt{2\pi} X \ln \delta_n)$. In the following, we will derive the symmetric part of the PDF of velocity difference. The origin of the asymmetry in the PDF (i.e., skewness) may be attributed to dissipative evolution of eddies or to experimental setup and situations. The consideration of the latter will be reported elsewhere.

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**FIG. 4.** The $n$-dependence of the PDF $\hat{H}^{(n)}(\xi_n)$ given by the present analysis with $q = 0.471$ ($\mu = 0.280$) for integer values $n$ from 5 to 20, from left to right at the axis $\xi$.

**FIG. 5.** Experimentally measured PDF of the velocity fluctuations by Lewis and Swinney for $R_\lambda = 262$ (Re = 540 000) are compared with the present theoretical results $\hat{H}^{(n)}(\xi_n)$. Open triangles are the experimental data points on the left hand side of the PDF taken from Lewis and Swinney, whereas open squares are those on the right hand side. Closed circles are the symmetrized points obtained by taking averages of the left and the right hand experimental data. For the experimental data, the distances $r/\eta = \ell_n/\eta$ are, from top to bottom: 11.6, 23.1, 46.2, 92.5 208, 399, 830 and 1440. Solid lines represent the curves given by the present theory with $q = 0.471$. For the theoretical curves, the number of steps in the cascade $n$ are, from top to bottom: 14, 13, 11, 10, 9.0, 8.0, 7.5, 7.0. For better visibility, each PDF is shifted by -1 unit along the vertical axis.

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In order to determine the normal part $\tilde{H}^{(n)}_N(\xi_n)$ of the PDF, we divide $\hat{H}^{(n)}(\xi_n)$ into two parts [21]: $\hat{H}^{(n)}(\xi_n) = \tilde{H}^{(n)}_S(\xi_n)$ for $|\xi_n| \leq \xi_n^*$, and $\hat{H}^{(n)}(\xi_n) = \hat{H}^{(n)}_S(\xi_n)$ for $\xi_n^* \leq |\xi_n| \leq \xi_n\delta_n^{\alpha/3 - \zeta_2/2}$. The point $\xi_n^*$ is defined by $\xi_n^* = \xi_n\delta_n^{\alpha/3 - \zeta_2/2}$ giving $\tilde{H}^{(n)}_S(|\xi_n^*|) = \tilde{H}^{(n)}_S(\xi_n^*)$, where $\alpha^*$ is the solution of $\zeta_2/2 - \alpha/3 + 1 - f(\alpha) = 0$. The PDF function $\tilde{H}^{(n)}_S(\xi_n)$ is constituted both by thermal fluctuation and by the multifractal distribution of singularities, and is assumed to be given by Gaussian function [21]. On the other hand, the contribution to $\hat{H}^{(n)}_S(\xi_n)$ is assumed
to come only from the multifractal distribution of singularities, i.e., \( \tilde{\Pi}(n) = \tilde{\Pi}(\xi_n) \) \[^{[21]}\]. Connecting \( \tilde{\Pi}(n) = \tilde{\Pi}(\xi_n) \) at \( \xi_n \) having the same value and the same derivative there, we have
\[
\tilde{\Pi}(\xi_n) = \tilde{\Pi}(e^{-[1+3/2'(\alpha^*)][\xi_n/\xi_1^*]^2 - 1/2}.) \quad \text{(13)}
\]

The \( n \)-dependence of \( \tilde{\Pi}(\xi_n) \) is shown in Fig. 3 with the self-consistently determined parameters \( \alpha_0, X \) and \( q \) \[^{[21]}\]. We notice that there are two points which look like independent of the number \( n \) of steps in the cascade at \( \xi_n \approx 0.8 \) and at \( \xi_n \approx 2.3 \). These points are also observed in the experimental data \[^{[16]}\] at about the same values of \( \xi_n \).

The comparison of experimentally measured PDF’s of the velocity fluctuation \[^{[14]}\] and those obtained by the present analysis is given in Fig. 5. In order to extract the symmetrical part from experimental data, we took the power \( w \) in \( \text{Re} = (\ell_0/\eta)^w \) turns out to be \( w = 1.13 \) in the present intermittent turbulence, which is \( 4/3 \) in the absence of intermittency, and that the number \( n_\gamma \) corresponding to the Kolmogorov scale is estimated as \( n_\gamma = 17.2 \) for the experiments under consideration.

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\[ \text{FIG. 6. Dependence of } n \text{ on } r/\eta, \text{ extracted from Fig. 3, is given by circles, which can be fitted by solid line (14). The dashed line represents } \text{put for reference, which is derived in the absence of intermittency.} \]

\[
n = -1.019 \times \log_2 r/\eta + 0.901 \times \log_2 \text{Re} \quad \text{(14)}
\]

with \( \text{Re} = 540,000 \). The dashed line represents \[^{[14]}\] valid in the absence of intermittency. From \[^{[14]}\), we see that the power \( w \) in \( \text{Re} = (\ell_0/\eta)^w \) turns out to be \( w = 1.13 \) in the present intermittent turbulence, which is \( 4/3 \) in the absence of intermittency, and that the number \( n_\gamma \) corresponding to the Kolmogorov scale is estimated as \( n_\gamma = 17.2 \) for the experiments under consideration.

The success of the present theory in the analysis of the experiments \[^{[16]}\] may indicate the robustness of singularities even for the case of no inertial range. The same experiments are investigated in \[^{[22]}\] by a rather different analysis from the present one. Comparison of these two approaches is given in \[^{[21]}\]. The present theory works quite well also for another systematic numerical experiments conducted by Gotoh \[^{[24]}\]. It will be reported elsewhere.

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