Loop integrals in three outstanding gauges: Feynman, Light-cone and Coulomb

Alfredo T. Suzuki, Alexandre G. M. Schmidt
Instituto de Física Teórica - UNESP
R.Pamplona, 145 São Paulo - SP CEP 01410-900, Brazil
E-mail: suzuki@ift.unesp.br, schmidt@ift.unesp.br

We apply negative dimensional integration method (NDIM) to three outstanding gauges: Feynman, light-cone and Coulomb gauges. Our aim is to show that NDIM is a very suitable technique to deal with loop integrals, being them originated from any gauge choice. In Feynman gauge we perform scalar two-loop four-point massless integrals; in the light-cone gauge we calculate scalar two-loop integrals contributing for two-point functions without any kind of prescriptions, since NDIM can abandon such devices – this calculation is the first test of our prescriptionless method beyond one-loop order; finally, for the Coulomb gauge we consider a four propagator massless loop integral, in the split dimensional regularization context.

Key Words: Quantum Field Theory, Negative dimensional integration, Dimensional regularization, Non-covariant gauges: Light-cone and Coulomb.

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1. INTRODUCTION

Perturbative approach for Quantum Field Theory in any gauge deals with Feynman diagrams, which are expressed as $D$-dimensional integrals. The success of such approach can be understood from the comparison between the $a = \frac{1}{3}(g-2)$ measure for the electron,

\begin{align}
    a_{th} &= 1159652201.2(2.1)(27.1) \times 10^{-12} \\
    a_{exp} &= 1159652188.4(4.3) \times 10^{-12},
\end{align}

see for instance [1].
This is the best motivation for studying Quantum Field Theory, no physical theory can give such accuracy in any measurement. In other words, it is the very best we have.

In section 2 we discuss some 2-loop 4-point functions, namely, on-shell double boxes with 5 and 6 massless propagators; section 3 is devoted to non-covariant gauges: the light-cone and Coulomb ones. The integrals we study for the former have 7 propagators (2-loops) and the latter is 1-loop and have 4, however it is also complicated since we have to use split dimensional regularization(SDR). In the final section, 4, we present our concluding remarks.

2. FEYNMAN GAUGE: SCALAR TWO-LOOP FOUR-POINT MASSLESS INTEGRALS

Of course, covariant gauges are the most popular, in what we could call “gauge market” [2]. Several methods were and are still developed to evaluate complicated Feynman loop integrals, being them concerned with analytic or numerical results[1, 3, 4], all in the context of dimensional regularization[5].

Our work in concerned with the application of negative-dimensional integration method (NDIM). It is a technique which can be applied to any gauge, covariant or non-covariant alike. The results are always expressed as hypergeometric series which have definite regions of convergence allowing one to study the referred diagrams or process in specific kinematical regions of external momenta and/or masses.

On the other side, NDIM has a drawback: the amazing number of series – in the case where one is considering massless diagrams – which must be summed. When such sums are of gaussian type, it is quite easy to write a small computer program that can do the job algebraically. However, when the series are of superior order, \( p+1 F_p \), for \( p \geq 2 \), there are no known formulas which can reduce it to a product of gamma functions for any value of its parameters. Despite this technical problem, NDIM proved to be an excellent method[6, 7, 8, 9].

A question which is often arised is: what is more difficult to handle, graphs with more loops or graphs with more legs? In our point of view, i.e., in the context of NDIM, the greater the number of loops the heftier the calculations will be needed to solve it. We will consider in this section a diagram which has both (great number of legs and loops, four and two respectively), a scalar two-loop double-box integral where all the particles are massless and the external legs are on-shell.

2.1. Double box with 5 and 6 propagators

Let us consider the diagram of figure 1. Consider as the generating functional for our negative-dimensional integral the gaussian one, where all external legs are on-shell,

\[
G_b = \int d^D q \int d^D r \exp \left[ -\alpha q^2 - \beta(q-p)^2 - \gamma(q-p-p')^2 - \theta(q-r-p_1)^2 - \phi r^2 - \omega(q-r)^2 \right],
\]

\[= \left( \frac{\pi^2}{\Lambda} \right)^{D/2} \exp \left[ \frac{1}{\Lambda}(-\gamma \phi s - \beta \theta t) \right], \tag{2}
\]

\[= \left( \frac{\pi^2}{\Lambda} \right)^{D/2} \exp \left[ \frac{1}{\Lambda}(-\gamma \phi s - \beta \theta t) \right], \tag{3}
\]
where \((s, t)\) are the usual Mandelstam variables and we use \(s + t + u = 0\). Observe that in the particular case where \(\alpha = 0\) we recover the gaussian integral for the diagram of figure 2. We also define \(\Lambda = \alpha \theta + \alpha \phi + \alpha \omega + \beta \theta + \beta \phi + \beta \omega + \gamma \theta + \gamma \phi + \gamma \omega + \phi \omega + \theta \phi\).

The usual technique reveals that there are thirteen sums and seven equations. From the combinatorics one can solve such constraints in 1716 different ways. Of course several systems have no solution – not even in the homogeneous case – and from our previous works we know that some results are \(n\)-fold degenerated and others are related by analytic continuation. The result for the integral in question

\[
\text{BOX} = \int d^D q d^D r \frac{(q^2)^i(q-p)^j(q-p-p')^{2k}(q-r-p_1)^m(q-r)^n}{s^{X_1} t^{X_2} X_{11}! X_{12}! Y_1! Y_2! Y_3! Y_4! Y_5! Y_6! Y_7! Y_8! Y_9! Z_1! Z_2!} \delta,
\]

where “all” means \(\{X_1, X_2, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Z_1, Z_2\}\) and

must be understood and \(\text{delta}\) represents the system of constraints. The above expression can be expressed, in principle, as a seven-fold hypergeometric series, there are three possibilities,

\[
\mathcal{F}(..|z, z^{-1}, 1), \quad \mathcal{F}(..|z, 1), \quad \text{and} \quad \mathcal{F}(..|z^{-1}, 1),
\]

where \(z = -s/t\). Some series with unit argument, if they were gaussian can be summed up. However, a hypergeometric function is meaningful only if the series
which defines it was convergent. Since the first possibility cannot be convergent we disregard it.

Among the 624 total solutions of the system of constraints we look for the simplest solution, namely, the one in which we can sum the great number of series. It is not difficult to find it using computer facilities,

$$\mathcal{B}O\mathcal{X}^{AC}(i, j, k, l, m, n) = f_1(i, j, k, l, m, n) \, {}_3F_2\{1\}|z), \quad (7)$$

where the five parameters are quoted in the table, $$\sigma_b = i + j + k + l + m + n + D$$ and

$$f_1(i, j, k, l, m, n) = \pi^D (t^2)^{\sigma_b} (-j|\sigma_b)(-l|\sigma_b)(\sigma_b + D/2) - 2\sigma_b - D/2)$$

\[ \times \frac{(i + j + k + m + D| - m - D/2)}{(i + j + k + m + n + D/2)(i + j + k + m + n + D/2)} \times \frac{(l + m + n + D| - l - n - D/2)}{(l + m + n + D/2)(l + m + n + D/2)}, \quad (8) \]

besides this one, we can have Appel’s, Lauricella’s and even more complicated hypergeometric functions. Moreover,

$$\Gamma(x|y) = \frac{\Gamma(x + y)}{\Gamma(x)}. \quad (x|y) \equiv (x)_y = \frac{\Gamma(x + y)}{\Gamma(x)}.$$

If we remember that the final result should be the sum of linearly independent series[7, 10], we can rightfully ask if one is not missing two other $$_3F_2$$ functions. According to Luke and Slater[12], the differential equation for $$_pF_q$$ has $p$ linearly independent solutions, so we should write a sum of three terms. On the other hand, according to Nørlund[13], if the difference between an upper parameter and a lower one was an integer number, then some series do not exist — we used this theorem in [7]. So, eq.(7) is the final result for the referred integral in the region where $|z| < 1$.

The expression for the same graph outside this region can be obtained making the substitutions,

$$s \leftrightarrow t, \quad j \leftrightarrow k, \quad l \leftrightarrow n, \quad (9)$$

so we have other $$_3F_2$$ hypergeometric function as the result for $|z| > 1$.

### TABLE 1
**Parameters of hypergeometric functions $$_3F_2$$ representing box integrals**

| Parameters | $$_3F_2\{1\}|z)$$ | $$_3F_2\{2\}|z)$$ |
|------------|----------------------|----------------------|
| a          | $-k$                 | $-k$                 |
| b          | $-n$                 | $-n$                 |
| c          | $-\sigma_b$          | $-\sigma_b$          |
| e          | $1 + j - \sigma_b$   | $1 + j - \sigma_b'$  |
| f          | $1 + l - \sigma_b$   | $1 + l - \sigma_b'$  |
Another solution for the Feynman integral can be written as a triple hypergeometric series,

\[ B_{\text{OX3}} = \frac{D^{1/2} \Gamma(\sigma_{\nu}-j) f_{3}}{\Gamma(\sigma_{\nu}-j)} \sum_{l=0}^{\infty} \frac{(-l|Y_{1247})(m + D/2|Y_{147})(-l|Y_{1})}{Y_{1247}!Y_{147}!Y_{1}!(1 + \sigma_{\mu} - j|Y_{1247})} \times \frac{(i + l + m + n + D|Y_{147})(i + j + n + D|Y_{1247})(l + m + n + D|Y_{147})}{(1 - j + l)|Y_{147}|(i + j + m + n + D|Y_{147})(l + m + n + D|Y_{147})}, \]

\[ + (j \leftrightarrow l), \]

where

\[ f_{3} = (-l|j)(l + m + n + D|i)(\sigma_{\mu} + D/2) - j - n - D/2)(-n|i + 2n + D/2) \times (i + j + k + m + D|\sigma_{\mu} - j - \sigma_{\nu})(-m|2m + D/2), \]

\[ (11) \]

obviously the above series converges if \(|z| < 1\), besides other possible condition on \(z\). So there is an overlapping between the regions of convergence of \(B_{\text{OX3}}\) and \(B_{\text{OX4}}\), so there exists an analytic continuation formula which relates both. As far as we know textbooks do not show formulas relating triple hypergeometric series with simple ones.

We have also 4-fold series,

\[ B_{\text{OX4}} = \frac{D^{1/2} \Gamma(\sigma_{\nu}-l) f_{4}}{\Gamma(\sigma_{\nu}-l)} \sum_{l=0}^{\infty} \frac{(-l|Y_{1247})(m + D/2|Y_{147})(-l|Y_{1})}{Y_{1247}!Y_{147}!Y_{1}!(1 + \sigma_{\mu} - l|Y_{1247})} \times \frac{(i + l + m + n + D|Y_{147})(i + j + n + D|Y_{1247})(l + m + n + D|Y_{147})}{(1 - j + l)|Y_{147}|(i + j + n + D|Y_{1247})(l + m + n + D|Y_{147})}, \]

\[ + (j \leftrightarrow l), \]

where

\[ f_{4} = (-l|j)(l + m + n + D|i + j - l)(\sigma_{\mu} + D/2) - k - l - D/2)(-n|2m + D/2) \times (k|l + D - \sigma_{\nu})(-n|l + 2n + D/2). \]

\[ (13) \]

Observe that the two previous results are singular when \(j - l = \) integer, since we have \(\Gamma(j - l)\) or \(\Gamma(l - j)\) in the numerator. However, such singularity cancels if one consider propagators exponents in the analytic regularization context, i.e., introduce \([7, 11]\) for instance \(j = -1 + \delta\), then expand the whole expression around \(\delta = 0\). Proceeding in this way the pole in \(\delta\) cancels.

We have above reduction formulas which transform a hypergeometric function defined by triple and 4-fold series in a simpler function defined by a unique sum. These formulas are not in the textbooks on the subject. It is an original result.

### 2.1.1. Double box with 5 propagators

The graph of figure 2, is a special case of the previous one. In the gaussian integral \(\alpha\) must be zero, so in the final result we must merely take \(i = 0\),

\[ B_{\text{OX4AC}}(0, j, k, l, m, n) = f(0, j, k, l, m, n) \frac{\Gamma(\sigma_{\mu})}{\Gamma(\sigma_{\mu} - j)} \times \frac{(i + l + m + n + D|Y_{147})(i + j + n + D|Y_{1247})(l + m + n + D|Y_{147})}{(1 - j + l)|Y_{147}|(i + j + n + D|Y_{1247})(l + m + n + D|Y_{147})} + (j \leftrightarrow l), \]

\[ (14) \]
where the parameters are listed in the table and we define $\sigma'_b = j + k + l + m + n + D$ and

$$f(0, j, k, l, m, n) = \pi^D (t^2)^{\sigma'_b}(-j|\sigma'_b)(-l|\sigma'_b)(\sigma'_b + D/2) - 2\sigma'_b - D/2)$$

$$\times(-m|l + m + n + D/2)(j + k + m + D) - m - D/2)$$

$$\times(l + m + n + D| - l - n - D/2),$$

we can proceed with the same substitutions (9) to obtain the result outside the region $|z| < 1$.

Finally, when all the exponents are equal to minus one, the $\binom{3}{F}^2_2$ collapses to a $\binom{2}{F}^1_1$ which can be written as an elementary function.

The results $BOX_3$ and $BOX_4$, in the same special case ($i = 0$), are hypergeometric series representations for the integral in question. Since they are different and depend on the same variable we must sum them in order to get a triple series representation for $BOX_3(0, j, k, l, m, n)$.

3. NON-COVARIANT GAUGES: LIGHT-CONE AND COULOMB

Recently there have been many works on non-covariant gauges, namely, light-cone[14, 15], Coulomb[16] and radial and axial gauges[17]. Despite they are not so popular as covariant ones, they have some important features which can help our study on certain physical problems.

Light-cone gauge, as far as we know, is the only one where certain supersymmetric theories can be shown to be UV finite and possess a local Nicolai map[18]. Moreover, ghosts decouple from physical particles and one is left with a reduced number of diagrams. On the other hand, the price to pay seemed to be so high, since the gauge boson propagator did generate spurious poles in physical amplitudes. This problem was overcame when andelstam and Leibbrandt[19] introduced causal
prescriptions to treat such poles (there are also other causal prescription which can be implemented, proposed by Pimentel and Suzuki[20], known as causal Cauchy principal value prescription.) However, the famous ML-prescription necessarily forces one to use partial fractioning tricks and integration over components, which turn the calculations rather involved[21].

Negative-dimensional approach can avoid at all the use of prescriptions and provide physically acceptable results, i.e., causality preserving results. The calculation we will present is the very first test beyond 1-loop order without invoking ML-prescription, as we called in [8] NDIM is a prescriptionless method. Still, integration over components and partial fractioning tricks can be completely abandoned as well as parametric integrals. The important point to note[8] is that the dual light-like 4-vector $n^\mu$ is necessary in order to span the needed 4-dimensional space[18, 22]. This is the very reason why our calculation for one-degree of covariance violation failed[6].

The second non-covariant gauge we deal with in this paper is the Coulomb gauge. Potential between quarks and studies on confinement are easily performed in this gauge[15, 16]. Besides, the ghost propagator has no pole in this gauge! As light-cone gauge, Coulomb also have problems with gauge boson propagator. In the former, loop integrals generated aditional poles; in the latter, such integrals are not even defined[23] since they have the form,

$$\int \frac{dq_4 d^3q}{q^2},$$

(16)
such objects are the so-called energy-integrals. Doust and Taylor[23] presented a solution for this issue in a form of a interpolating gauge (between Feynman and Coulomb). Leibbrandt and co-workers[24] presented also a solution, a procedure they called split dimensional regularization, (SDR), which introduces two regulating parameters, one for the energy component and another for the 3-momentum one. So, the measure becomes,

$$d^Dq = dq_4 d^{D-1}q^{\text{sd}}$$
$$d^Dq = dq_4 d^Dq,$$

(17)
in Euclidean space.

NDIM can also deal with Coulomb gauge loop integrals, but it needs to make use of SDR. In this work we propose to apply NDIM to scalar integrals with four massless propagators. Our results are given in terms of hypergeometric series involving external momenta, exponents of propagators and regulating parameters $\omega$ and $\rho$.

3.1. The Light-Cone Gauge

So far, we have tested our NDIM for integrals pertaining to one-loop class. Now we apply such technology to some massless two-loop integrals. Let us consider an integral studied by Leibbrandt and Nyeo[21], since they did not present the full result for it,

$$C_3 = \int d^Dq d^Dk \frac{k^2}{q^2(q-k)^2(k-p)^2(k \cdot n)(q \cdot n)},$$

(18)
where in their calculation ML-prescription must be understood. On the other hand, in the NDIM context the key point is to introduce the dual vector $n^*$ in order to span the needed space\[8, 18, 22\]. If we do not consider it, our result will violate causality, giving the Cauchy principal value of the integral in question, as we conclude in \[6\].

NDIM can consider lots of integrals in a single calculation. Our aim to perform,

$$\mathcal{N} = \int d^D q \, d^D k_1 \, (q^2)^i (q^2 - k_1)^{2k_1} (k_1 - p)^{2l} (k_1 \cdot n)^m (q \cdot n)^s (k_1 \cdot n^*)^r, \quad (19)$$

we will carry out this integral and then present results for special cases, including Leibbrandt and Nyeo’s $C_3$, where $i = -1, r = 0$ and the other exponents equal to minus one. Observe that the integral must be considered as function of external momentum, exponents of propagators and dimension,

$$\mathcal{N} = \mathcal{N}(i, j, k, l, m, r, s; P, D), \quad (20)$$

where $P$ represents $(p^2, p^+, p^-, \frac{1}{2}(n \cdot n^*))$, and we adopt the usual notation of light-cone gauge\[2\].

Our starting point is the generating function for our negative-dimensional integrals,

$$G_N = \int d^D q \, d^D k \, \exp \left[ -\alpha k^2 - \beta q^2 - \gamma (q - k)^2 - \theta (k - p)^2 - \phi (k \cdot n) - \omega (q \cdot n) - \eta (k \cdot n^*) \right], \quad (21)$$

then after a little bit of algebra we integrate it,

$$G_N = \left( \frac{\pi^2}{\lambda} \right)^{D/2} \exp \left\{ \frac{1}{\lambda} \left[ -g_1 p^2 - g_2 (p \cdot n) - g_3 (p \cdot n^*) + g_4 \left( \frac{1}{2} n \cdot n^* \right) \right] \right\}, \quad (22)$$

where

$$g_1 = (\alpha \beta + \alpha \gamma + \beta \gamma) \theta, \quad g_2 = (\beta \phi + \gamma \omega + \gamma \phi) \theta, \quad g_3 = (\beta + \gamma) \eta \theta, \quad g_4 = n \frac{g_2}{\theta},$$

and $\lambda = \alpha \beta + \alpha \gamma + \beta \gamma + \beta \theta + \gamma \theta$.

Taylor expanding the exponentials one obtain,

$$\mathcal{N} = (-\pi^D)^i j k l m r s \Gamma (1 - \sigma_n - D/2) \sum_{\delta} \frac{\delta}{X_1! \ldots X_5! Y_1! Y_2! Y_3!} \times \left( \frac{p^2}{Z_1! \ldots Z_5!} \right)^{Y_{123}} \left( \frac{n \cdot n^*}{2} \right)^{Y_{123}}, \quad (23)$$

where $\delta$ represents the system of constraints $(8 \times 16)$ for the negative-dimensional integral. In the end of the day we have 12870 possible solutions for such system! Most of them, 9142, have no solution while 3728 present solutions which can be written as hypergeometric series. Of course several of these will provide the same series representation, these solutions we call degenerate.
We present a result for the referred integral as a double hypergeometric series,

\[
N = \pi^D f_n P_n \sum_{Z_4} \frac{(\sigma_n + D/2|Z_4)(i + j + k + m + s + D|Z_4)(D/2 + k|Z_4)}{Z_4!Z_5!(1 + i + j + k + \sigma_n + D|Z_4)(j + k + s + D|Z_4)}
\]

\[
\times \left(\frac{2n \cdot n^*}{p^2 p^*}\right)^{Z_4},
\]

where

\[
f_n = (-m|s)(-i - j - k - D/2|\sigma_n - D/2)\frac{(j + k + s + D|i + s + r)}{(1 + r|j - k - m + s + D/2)}
\]

\[
\times \left((-l|k + l + D/2)(-k - j - D/2)\frac{(-m|j + m + s + D/2)}{(-j - i - k - m - r - s - D)}\right).
\]

The Pochhammer symbols and

\[
P_n = (p^2)^{\sigma_n + i + j + k + D}(p^+)^{l + m + s - \sigma_n}(p^-)^{l + r - \sigma_n}\frac{(n \cdot n^*)}{2} \sigma_n = l.
\]

Now we can consider the special case \((i = 1, j = k = l = m = s = -1, r = 0)\), studied in [21],

\[
N_{SC} = \pi^D \frac{\Gamma(5 - 2D)\Gamma(D - 1)\Gamma(D - 1)\Gamma(2 - D/2)\Gamma(2 - D/2)}{\Gamma(1 - D/2)\Gamma(D - 3)}(p^2)^{2D - 5}
\]

\[
\times (p^+)^{1 - D}(p^-)^{3 - D} \left(\frac{n \cdot n^*}{2}\right)^{D - 3}\sum_{Z_4, Z_5} \frac{(3D/2 - 4|Z_4)}{Z_4!Z_5!(2D - 4|Z_4)}
\]

\[
\times \left(\frac{(D/2 - 1|Z_4)(D/2 - 2|Z_5)(D - 1|Z_4)}{(D - 2|Z_4)}\right)^{Z_4} \left(\frac{p^2 n \cdot n^*}{p^* p^-}\right)^{Z_4}.
\]

observe that it exhibits a double pole, as stated by Leibbrandt and Nyeo[21].

### 3.2. The Coulomb Gauge

We will present the full calculation of an integral which has four propagators,

\[
J(i, j, k, m) = \int d^D q \left(q^2\right)^i(q - p)^2 q^{2k}(q + p)^{2m},
\]

in order to regulate the possible divergences originated by the energy component, SDR must be understood, namely,

\[
d^D q = d^D q d^2 q.
\]

where \(D = \rho + \omega\).

The generating functional for our negative-dimensional integrals is the gaussian-like integral,

\[
G_c = \int d^D q \exp \left[-\alpha q^2 - \beta(q + p)^2 - \gamma q^2 - \theta(q + p)^2\right],
\]
which can be easily integrated,

\[ G_c = \frac{\pi^{D/2}}{\lambda_1^{\rho/2} \lambda_2^{\omega/2}} \exp \left(-\frac{\alpha \beta}{\lambda_1} p_1^2\right) \exp \left[-\frac{(\alpha + \gamma)(\beta + \theta)}{\lambda_2} p^2\right]. \]  

(31)

There are results given by double, triple, 4-fold and 5-fold hypergeometric series in the variable \( p^2/p_1^2 \) or its inverse.

We will present two of such hypergeometric series representations, the first one is a 4-fold series,

\[ J_4(i, j, k, m) = C_4(i, j, k, m) \sum_{|X_1| = 0}^\infty \frac{(-i|X_{1234})(j + m + D/2|X_{34})}{X_1!X_2!X_3!X_4!} \frac{(p^2/p_1^2)^{X_{1234}}}{X_{1234}} \]
\[ \times \frac{(-1)^{X_3}(1 + j + m + \rho/2|X_3 - X_{12})(-m|X_2)}{(1 + j + k + m + D/2|X_3)} \]
\[ \times \frac{(-j - \rho/2|X_1 - X_3)(k + \omega/2|X_{124})}{(1 - i - \rho/2|X_{124})} \]
\[ +(i \leftrightarrow j, k \leftrightarrow m), \]  

(32)

where

\[ C_4(i, j, k, m) = \pi^{D/2}(p_1^2)^i(p_2^2)^j(p_3^2)^k(p_4^2)^m(-i| - \rho/2)(-j - m - \rho/2| - k - \omega/2) \]
\[ \times (-k|2k + \omega/2)(j + k + m + D/2 + \omega/2| - k - \omega/2), \]  

(33)

where \( \sigma_c = i + j + k + m + D/2 \) and the second a 5-fold hypergeometric series,

\[ J_5(i, j, k, m) = C_5(i, j, k, m) \sum_{|X_1| = 0}^\infty \frac{(-i - j - \rho/2|2X_1 + X_{2345})}{X_1!X_2!X_3!X_4!X_5!} \frac{(p^2/p_1^2)^{2X_1 + X_{2345}}}{X_{1234}} \]
\[ \times \frac{(-1)^{X_3}(m + \omega/2|X_{1345})}{(1 + k + m + \omega/2|X_{145})} \]
\[ \times \frac{(k + \omega/2|X_{1245})}{(1 - j - \rho/2|X_{135})(1 - i - \rho/2|X_{124})}, \]  

(34)

where

\[ C_5(i, j, k, m) = \pi^{D/2}(p_1^2)^{i+j+p/2}(p_2^2)^{k+m+\omega/2}(-i|i + j + \rho/2)(-j|i + j + \rho/2) \]
\[ \times (-k|k + m + \omega/2)(-m|k + m + \omega/2)(i + j + \rho| - \sigma_c - \rho/2) \]
\[ \times (k + m + \omega| - \sigma_c - \omega/2), \]  

(35)

observe that the above result is also symmetric in \((i \leftrightarrow j, k \leftrightarrow m)\), which means in the loop integral, \( q^\mu \rightarrow q^\mu + p^\mu \).

Another important point to observe is that the final result must a sum of linearly independent hypergeometric series\([6, 7]\). The above 5-fold series, \( J_5 \), appears only one time whereas \( J_4 \) is degenerate since several systems give its two hypergeometric functions. This must be considered if one wants to apply NDIM to more complicated diagrams which can in principle generate hypergeometric series representations even more involved.
Moreover, the above expressions, \( J_4 \) and \( J_5 \), are related by direct analytic continuation, since both are convergent for \(|p^2/p_4^2| < 1\). When one is considering simple hypergeometric function, several formulas are known; on the other hand, for rather complicated hypergeometric series, as we obtained — four and five-fold series —, there are very few of such formulas. NDIM can fill this gap, since it is the only method which provides hypergeometric series representations for Feynman loop integrals, in different kinematical regions, and related by analytic continuation direct or indirect alike.

The above hypergeometric series (only the series, not the factors!), \( J_4 \) and \( J_5 \), can be written as generalized hypergeometric functions \([12]\) of four and five variables,

\[
\mathcal{F}_{6:0;0;0;0:0;0}^{2:0;0;0;0} \left[ \begin{array}{c}
(i : 1, 1, 1, 1), (j + m + D/2 : 0, 0, 1, 1)(1 + j + m + \rho/2 : -1, -1, 1, 0) \\
(1 + j + k + m + D/2 : 0, 0, 1, 1) \\
(1 - i - \rho/2 : 1, 1, 0, 1, 0) \\
(x, x, -x, x, x)
\end{array} \right],
\]

and

\[
\mathcal{F}_{3:0;0;0;0;0:0}^{3:0;0;0;0;0} \left[ \begin{array}{c}
(i - j - \rho/2 : 2, 1, 1, 1, 1), (m + \omega/2 : 1, 0, 1, 1, 1) \\
(1 + k + m + \omega/2 : 1, 0, 0, 1, 1), (1 - j - \rho/2 : 1, 0, 1, 0, 1) \\
(1 - i - \rho/2 : 1, 1, 0, 1, 0) \\
x^2, x, x, x, -x
\end{array} \right],
\]

where \( x = p^2/p_4^2 \).

4. CONCLUSION

The technique of Feynman parametrization can of course be used to perform loop integrals in different gauges but it is very difficult to perform the parametric integrals for arbitrary exponents of propagators. Not so with NDIM, carry loop integrals out with particular exponents is as easy as dealing with arbitrary ones – besides, one can come across with singularities which depend on them and not on dimension \( D \) – this fact is very important when we are studying light-cone gauge Feynman integrals, because one could have to handle products like \((q^+)^a ((q - p)^+)^b\), being \( a \) and \( b \) negative. NDIM can calculate all of them simultaneously, but if one chooses partial fractioning tricks then he/she will be forced to carry out each integral separately. Besides usual covariant integrals and the trickier light-cone gauge ones, NDIM was probed in the Coulomb gauge, where a procedure – introduced by Leibbrandt and co-workers – called split dimensional regularization is needed in order to render the energy integrals well-defined.

In this paper, we studied Feynman loop integrals pertaining to three outstanding gauges: the usual, and more popular, covariant Feynman gauge and two of the trickiest non-covariant gauges, the light-cone and the Coulomb ones. Our results are given in terms of hypergeometric functions and in the dimensional regularization context.

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