THE FRACTAL DIMENSIONS OF THE SPECTRUM OF
STURM HAMILTONIAN

QING-HUI LIU, YAN-HUI QU, ZHI-YING WEN

ABSTRACT. Let \( \alpha \in (0, 1) \) be irrational and \([0; a_1, a_2, \ldots]\) be the continued fraction expansion of \( \alpha \). Let \( H_{\alpha, V} \) be the Sturm Hamiltonian with frequency \( \alpha \) and coupling \( V \), \( \Sigma_{\alpha, V} \) be the spectrum of \( H_{\alpha, V} \). The fractal dimensions of the spectrum have been determined by Fan, Liu and Wen (Erg. Th. Dyn. Sys., 2011) when \( \{a_n\}_{n \geq 1} \) is bounded. The present paper will treat the most difficult case, i.e, \( \{a_n\}_{n \geq 1} \) is unbounded. We prove that for \( V \geq 24 \),

\[
\dim_H \Sigma_{\alpha, V} = s_*(V) \quad \text{and} \quad \overline{\dim}_B \Sigma_{\alpha, V} = s^*(V),
\]

where \( s_*(V) \) and \( s^*(V) \) are lower and upper pre-dimensions respectively. By this result, we determine the fractal dimensions of the spectra for all Sturm Hamiltonians.

We also show the following results: \( s_*(V) \) and \( s^*(V) \) are Lipschitz continuous on any bounded interval of \([24, \infty)\); the limits \( s_*(V) \ln V \) and \( s^*(V) \ln V \) exist as \( V \) tend to infinity, and the limits are constants only depending on \( \alpha \); \( s^*(V) = 1 \) if and only if \( \limsup_{n \to \infty} (a_1 \cdots a_n)^{1/n} = \infty \), which can be compared with the fact: \( s_*(V) = 1 \) if and only if \( \liminf_{n \to \infty} (a_1 \cdots a_n)^{1/n} = \infty \) (Liu and Wen, Potential anal. 2004).

Key words: Sturm Hamiltonian; fractal dimensions; Gibbs like measure; Cookie-Cutter-like.

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1. INTRODUCTION

The Sturm Hamiltonian is a discrete Schrödinger operator

\[
(H \psi)_n := \psi_{n-1} + \psi_{n+1} + v_n \psi_n
\]

on \( \ell^2(\mathbb{Z}) \), where the potential \( (v_n)_{n \in \mathbb{Z}} \) is given by

\[
v_n = V \chi_{[1-\alpha, 1)}(n \alpha + \phi \mod 1), \quad \forall n \in \mathbb{Z}, \tag{1}
\]

where \( \alpha \in (0, 1) \) is irrational, and is called frequency, \( V > 0 \) is called coupling, \( \phi \in [0, 1) \) is called phase. It is known that the spectrum of Sturm Hamiltonian is independent of \( \phi \), so we take \( \phi = 0 \) and denote the spectrum by \( \Sigma_{\alpha, V} \). We often simplify the notation \( \Sigma_{\alpha, V} \) to \( \Sigma_V \) or \( \Sigma \) when \( \alpha \) or \( V \) are
fixed. The present paper is devoted to determine the fractal dimensions of \( \Sigma_{\alpha,V} \) for all irrational \( \alpha \).

The most prominent model among the Sturm Hamiltonian is the Fibonacci Hamiltonian, which is given by taking \( \alpha \) to be the golden number \( \alpha_0 \) := \( \frac{\sqrt{5} - 1}{2} \). This model was introduced by physicists to model the quasicrystal system ([11, 16]). Sütő showed that the spectrum always has zero Lebesgue measure ([18]).

\[
L(\Sigma_{\alpha_0,V}) = 0, \quad \text{for all } V > 0.
\]

Then it is natural to ask what is the fractal dimension of the spectrum. Raymond first estimated the Hausdorff dimension ([17]), and he showed that \( \dim_H \Sigma_{\alpha_0,V} < 1 \) for \( V > 4 \). Jitomirskaya and Last ([10]) showed that for any \( V > 0 \), the spectral measure of the operator has positive Hausdorff dimension, as a consequence \( \dim_H \Sigma_{\alpha_0,V} > 0 \). By using dynamical method, Damanik et al. ([3]) showed that if \( V \geq 16 \) then

\[
\dim_B \Sigma_{\alpha_0,V} = \dim_H \Sigma_{\alpha_0,V}.
\]

They also got lower and upper bounds for the dimensions. Due to these bounds they further showed that

\[
\lim_{V \to \infty} \dim_H \Sigma_{\alpha_0,V} \ln V = \ln(1 + \sqrt{2}).
\]

We remark that more than a natural question, the fractal dimensions of the spectrum are also related to the rates of propagation of the fastest part of the wavepacket (see [3] for detail).

Write \( d(V) = \dim_H \Sigma_{\alpha_0,V} \). Cantat [2], Damanik and Gorodetski [4] showed that: \( d(V) \in (0, 1) \) is analytic on \((0, \infty)\). In [5], Damanik and Gorodetski further showed that \( \lim_{V \downarrow 0} d(V) = 1 \) and the speed is linear.

Now we go back to the general Sturm Hamiltonian case. We fix an irrational \( \alpha \in (0,1) \) with continued fraction expansion \([0; a_1, a_2, \ldots] \). Write

\[
K_\alpha^*(\alpha) = \lim sup \frac{1}{k} \sum_{i=1}^{k} a_i^{1/k} \quad \text{and} \quad K_\alpha(\alpha) = \lim inf \frac{1}{k} \sum_{i=1}^{k} a_i^{1/k}.
\]

Bellissard et al. ([1]) showed that \( \Sigma_{\alpha,V} \) is a Cantor set of Lebesgue measure zero. Damanik, Killip and Lenz ([6]) showed that, if \( \limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} a_i < \infty \), then \( \dim_H \Sigma_{\alpha,V} > 0 \), notice that the set of such \( \alpha \) has Lebesgue measure
0 in $(0,1)$. Basing on the analysis of Raymond [17] about the structure of spectrum, Liu and Wen [13] showed that for $V \geq 20$
\[
\left\{\begin{array}{ll}
\dim_H \Sigma_{\alpha,V} \in (0,1) & \text{if } K_*(\alpha) < \infty \\
\dim_H \Sigma_{\alpha,V} = 1 & \text{if } K_*(\alpha) = \infty.
\end{array}\right.
\] (5)

Raymond [17], Liu and Wen [13] showed that the spectrum $\Sigma_{\alpha,V}$ has a natural covering structure. This structure makes it possible to define the so called pre-dimensions $s_*(V)$ and $s^*(V)$ (see (19) for the definition). Liu, Peyriere and Wen [12] showed that $\dim_H \Sigma_{\alpha,V} \leq s_*(V)$, $\dim_B \Sigma_{\alpha,V} \geq s^*(V)$. (6)

Moreover, they show that, for $\alpha$ of bounded type, i.e. $\{a_k\}_{k \geq 1}$ bounded
\[
\lim_{V \to \infty} s_*(V) \ln V = -\ln f_*(\alpha), \quad \lim_{V \to \infty} s^*(V) \ln V = -\log f^*(\alpha).
\] (see (34) for the definition of $f_*(\alpha)$ and $f^*(\alpha)$). When $\alpha = \alpha_0$ they proved that
\[
f_*(\alpha_0) = f^*(\alpha_0) = (1 + \sqrt{2})^{-1}.
\]

Recently Fan, Liu and Wen [8] showed that for $\alpha$ of bounded type, the two inequalities in (6) are indeed equalities. Moreover if $\{a_k\}_{k \geq 1}$ is eventually periodic, then $s_*(V) = s^*(V)$. Thus for $\alpha$ of bounded type, they determined the fractal dimensions of the spectrum and generalized (2) and (3).

In this paper we will complete the picture for the fractal dimensions of the spectrum of Sturm Hamiltonian by treating the most difficult part: $\alpha$ is of unbounded type, i.e., $\{a_k\}_{k \geq 1}$ is unbounded. We state now the main results of the paper and some remarks.

**Theorem 1.1.** Let $V \geq 24$, and $\alpha \in (0,1)$ be irrational. Then
\[
\dim_H \Sigma_{\alpha,V} = s_*(V) \quad \text{and} \quad \overline{\dim}_B \Sigma_{\alpha,V} = s^*(V).
\] (7)

Moreover
\[
\left\{\begin{array}{ll}
\overline{\dim}_B \Sigma_{\alpha,V} \in (0,1) & \text{if } K^*(\alpha) < \infty \\
\overline{\dim}_B \Sigma_{\alpha,V} = 1 & \text{if } K^*(\alpha) = \infty.
\end{array}\right.
\] (8)

**Theorem 1.2.** Fix $\alpha \in (0,1)$ irrational. Let $f_*(\alpha)$ and $f^*(\alpha)$ be defined as in (34), then
\[
\lim_{V \to \infty} s_*(V) \ln V = -\ln f_*(\alpha), \quad \lim_{V \to \infty} s^*(V) \ln V = -\log f^*(\alpha).
\] (9)

**Theorem 1.3.** $s_*(V)$ and $s^*(V)$ are Lipschitz continuous on any bounded interval of $[24, \infty)$. 
Remark 1. 1) Formula (8) is the box dimension counterpart of (5), and the formulas (5) and (8) give the sufficient and necessary conditions such that Hausdorff dimensions and box dimension are strictly less than 1 and positive.

2) In general we can not expect $s_*(V) = s^*(V)$. The simplest example is as follows: take $\alpha = [0, a_1, a_2, \cdots]$ such that $K_*(\alpha) = 1$ and $K^*(\alpha) = \infty$.

Then by (5) and (7) we have $s_*(V) < 1$, by (8) and (7) we have $s^*(V) = 1$.

3) Formula (9) is a complete generalization of (3).

4) We know that in the Fibonacci case, the dimension function $d(V)$ is real analytic ([2, 4]). For the Sturm case, we can not expect such strong regularity. However by Theorem [13] both Hausdorff and Box dimension functions are still Lipschitz continuous, which will be obtained essentially from the formula (7).

We will compare the present work with some previous works [13, 12, 8] to explain the main difficulties we will meet and indicate some new ideas and techniques we will introduce.

The main idea in [13] is essentially introducing a natural covering structure by construct spectral generating bands, and estimate the length of spectral generating bands by computing one-order derivative of spectral generating polynomial. The key points in [8] consists of, on the one hand, generalizing the Cookie-Cutter-like structure introduced by Ma, Rao and Wen [15] and developing some related techniques for establishing the Gibbs like measure; and on the other hand, giving a more exact formula for the derivative of spectral generating polynomial, and estimating of the two-order derivative.

But if $\{a_k\}_{k \geq 1}$ is unbounded, these techniques and methods are not enough. To see this, we recall first the definition of Cookie-Cutter set. Taking $I = [0, 1]$, $I_0, I_1 \subset I$ be two disjoint subintervals of $I$, let $f : I_0 \cup I_1 \to I$ satisfies

(C-i) $f|_{I_0}, f|_{I_1}$ are 1–1 mappings onto $I$;

(C-ii) $C^{1+\gamma}$ Hölder $(\gamma > 0)$, i.e., $\exists c > 0$

$$|f'(x) - f'(y)| \leq c|x - y|^{\gamma}, \quad \forall x, y \in I_0 \cup I_1;$$

(C-iii) expansion, i.e. there exist $B > b > 1$, for any $x \in I_0 \cup I_1$,

$$1 < b \leq |f'(x)| \leq B < \infty.$$
We call \( f \) a Cookie-Cutter map. The hyperbolic attractor of \( f \) is defined as

\[
E := \{ x \in \mathbb{R} \mid \forall k \geq 0, f^k(x) \in [0,1] \}.
\] (10)

\( E \) is called the Cookie-Cutter set associated with the Cookie-Cutter map \( f \).

Let \( \phi_0 = (f|_{I_0})^{-1}, \phi_1 = (f|_{I_1})^{-1} \) and \( \Sigma = \{0,1\} \). For any \( k \geq 1, \sigma = i_1 i_2 \cdots i_k \in \Sigma^k \), define \( I_\sigma = \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_k}(I) \), then \( f^k(I_\sigma) = I \) and \( E = \bigcap_{k \geq 1} \bigcup_{\sigma \in \Sigma^k} I_\sigma \).

As in [7], the system satisfies the principle of bounded variation, i.e., there exists \( \xi \geq 1 \) such that, for any \( k \geq 1, \sigma \in \Sigma^k \), and any \( x,y \in I_\sigma \),

\[
\left| \frac{(f^k)'(x)}{(f^k)'(y)} \right| \leq \xi;
\]

and the system also satisfies the principle of bounded distortion, i.e. for any \( x \in I_\sigma \),

\[
\xi^{-1} \leq \left| (f^k)'(x) \right| |I_\sigma| \leq \xi.
\]

Notice that by the chain rule, we have

\[
(f^k)'(x) = f'((f^{k-1}(x)) f'((f^{k-2}(x)) \cdots f'(x).
\] (11)

By these two principles, we see that the length of the interval \( I_\sigma \) could be estimated by the derivative of \( f^k \) at any point of \( I_\sigma \).

Moreover, Ma, Rao and Wen [15] showed that the system also satisfies the principle of bounded covariation, i.e., for any \( m > k > 0, \sigma_1, \sigma_2 \in \Sigma^k \), and \( \tau \in \Sigma^{m-k} \),

\[
\frac{|I_{\sigma_1 \tau}|}{|I_{\sigma_1}|} \leq \xi^2 \frac{|I_{\sigma_2 \tau}|}{|I_{\sigma_2}|}.
\]

With these principles, one can prove the existence of the Gibbs measure, i.e., for any \( 0 < \beta < 1 \), there exists probability measure \( \mu_\beta \) such that, for any \( k > 0 \) and \( \sigma \in \Sigma^k \),

\[
\xi^{-2} \frac{|I_\sigma|^\beta}{\sum_{\tau \in \Sigma^k} |I_\tau|^\beta} \leq \mu_\beta(I_\sigma) \leq \xi^2 \frac{|I_\sigma|^\beta}{\sum_{\tau \in \Sigma^k} |I_\tau|^\beta}.
\]

These measures are crucial for analyzing fractal dimensions of the attractors, such as formulas for Hausdorff dimension, box dimension and continuous dependence of dimensions with respect to \( f \).

Now we turn to the Cookie-Cutter-like set introduced by Ma, Rao and Wen [15], which generalizes the classical Cookie-Cutter set:

\[
E = \{ x \in \mathbb{R} \mid \forall k \geq 0, f_k \circ f_{k-1} \circ \cdots \circ f_1(x) \in [0,1] \},
\] (12)

where for any \( k \geq 1, f_k \) satisfies
(U-i) \( \exists I_j^k \subset I = [0, 1], j = 1, 2, \ldots, m_k, \) mutually disjoint, such that \( f_k|_{I_j^k} \) are \( 1-1 \) mappings onto \( I; \)

(U-ii) \( C^{1+\gamma} \) Hölder (\( \gamma > 0 \)), i.e., \( \exists c_k > 0, \)
\[ |f_k'(x) - f_k'(y)| \leq c_k |x - y|^{\gamma}, \quad \forall x, y \in \bigcup_j I_j^k; \]

(U-iii) expansion, i.e. there exist \( B_k > b_k > 1, \) for any \( x \in \bigcup_j I_j^k, \)
\[ 1 < b_k \leq |f_k'(x)| \leq B_k < \infty. \]

Comparing with (10), we see that the \( k \)-th iteration of the same mapping \( f \) is replaced by composition of \( k \) different mappings in (12).

Under the conditions of uniformly Hölder and uniformly bounded expansion, i.e.,
\[
\sup c_k < \infty, \quad 1 < \inf b_k \leq \sup B_k < \infty, \quad (13)
\]
the principles of bounded variation, bounded distortion, bounded covariation and the existence of Gibbs like measure were proven in [15]. They gave formulas for the dimensions and showed the continuous dependence of dimensions with respect to \( \{f_k\}_{k \geq 1}. \)

In [8], to study the dimensional property of spectrum with bounded type, they apply the technique of Cookie-Cutter-like set in the following way. For every spectral generating band \( B \), there is a generating polynomial \( h_B \) such that \( h_B \) is monotone on \( B \) and \( h_B(B) = [-2, 2] \). They estimated the length of \( B \) by help of \( h_B \). Suppose \( (B_k)_{k=0}^n \) is a sequence of spectral generating bands of order from 0 to \( n \) with
\[ B_n \subset B_{n-1} \subset \cdots \subset B_0, \]
and suppose their corresponding generating polynomials are \( (h_i)_{i=0}^n \). Noting that \( h_0' = 1 \), and
\[ h_n' = \frac{h_n'}{h_{n-1}'} \frac{h_{n-1}'}{h_{n-2}'} \cdots \frac{h_1'}{h_0'}. \]
Comparing with (11), they analyze \( h_{k+1}'/h_k' \) in stead of analyzing \( f'(f^k(x)) \).
Analogous to condition (U-iii), they proved
\[ 4 < h_k'(x)/h_{k-1}'(x) < B_k \quad (14) \]
And instead of Hölder condition (U-ii), they proved (see also (53))
\[
\left| \frac{h_{k+1}'(x)}{h_k'(x)} - \frac{h_{k+1}'(y)}{h_k'(y)} \right| \leq t_k \left( |h_k'(x) - h_k'(y)| + \frac{2}{6} |h_{k-1}'(x) - h_{k-1}'(y)| \right)
+ \frac{1}{6 t_k} \left| \frac{h_k'(x)}{h_{k-1}'(x)} - \frac{h_k'(y)}{h_{k-1}'(y)} \right|. \quad (15)
\]
Notice that all parameters $B_k, t_k, d_k, e_k$ in (14) and (15) depend on $a_k$. If $\{a_k\}$ is bounded, $B_k, t_k, d_k, e_k$ are also bounded, thus they can apply techniques of [15] directly.

But if the sequence $\{a_n\}$ is unbounded, then $\sup_k t_k = \infty, \sup_k B_k = \infty$. Return to the Cookie-Cutter case, comparing with (13), this is equivalent to

$$\sup_k c_k = \infty, \quad \sup_k B_k = \infty,$$

i.e., neither uniformly Hölder nor uniformly bounded expansion. By carefully analyzing the relation between $c_k$ and $B_k$, we find that the conclusion of [15] still holds if we relax the condition (13) to require that for some constant $C > 0$,

$$c_k \leq C \cdot \inf_{x \in \bigcup_{j} I_j} |f_k'(x)|, \quad \forall k > 0.$$

By this way, we could overcome the difficulty $c_k$ and $B_k$ not bounded.

This technique can be accommodated to our case to show the principles of bounded variation, distortion and covariation for the spectrum, by making more accurate estimations for $t_k, d_k, e_k$ and $h_k(x) - h_k(y)$ in (15).

It is more tricky to construct a Gibbs like measure, since when $\{a_k\}_{k \geq 1}$ is unbounded, we can not distribute mass evenly on different types of spectral generating bands. However, with much effort, we can still construct a weak type Gibbs like measure which plays the same role as Gibbs like measure.

Finally, in applying mass distribution principle to get a good lower bound for Hausdorff dimension, we will meet the following difficulty: for a spectral generating band $B$ of order $k$ and type III, it contains $a_k$ spectral generating bands (denote as $B_l$ for $1 \leq l \leq a_k$) of order $k+1$ and type I with contraction ratios

$$|B_l|/|B| \sim a_k^{-1} \sin^2 \frac{l \pi}{a_k + 1}, \quad l = 1, 2, \cdots, a_k.$$

The contraction ratios vary from $a_k^{-1}$ to $a_k^{-3}$, so the weak Gibbs like measure fails to give desired estimation.

To overcome this difficulty, we introduce a truncation technique. For any small $\varepsilon > 0$, we delete the intervals $B_l$ with

$$0 < l/(a_k + 1) < \varepsilon \quad \text{or} \quad 1 - \varepsilon < l/(a_k + 1) < 1.$$

So the remaining intervals satisfy

$$|B_l|/|B| \geq \varepsilon^2 a_k^{-1}.$$
Denote the remaining set by $E$. Now we can apply weak Gibbs like measure supported on $E$ to get lower bound of Hausdorff dimension for $E$ (here we use idea from [9]), and then obtain the desired lower bound for $E$ as $\varepsilon$ tends to 0.

The plan of the paper is as follows. In Section 2, we will study the structure of the spectrum, especially we will give a coding for the spectrum. In Section 3, we state some results which we need to prove the main Theorems. Section 4 and Section 5 are devoted to the proofs of Theorem 1.1 and Theorem 1.2 respectively. The proof of Theorem 1.3 will be postponed to Section 8 since the proof of which need a techinque lemma. The rest sections are devoted to the proofs of the results stated in section 3.

2. The structure and coding of the spectrum

We describe the structure of the spectrum $\Sigma = \Sigma_{\alpha,V}$ for some fixed $\alpha$ and $V$. We will see that $\Sigma$ has a natural covering structure which can be associated with a natural coding.

Let $\alpha = [0; a_1, a_2, \ldots] \in (0, 1)$ be irrational, let $p_k/q_k (k > 0)$ be the $k$-th partial quotient of $\alpha$ given by:

\[
p_{-1} = 1, \quad p_0 = 0, \quad p_{k+1} = a_{k+1}p_k + p_{k-1}, \quad k \geq 0,
\]

\[
q_{-1} = 0, \quad q_0 = 1, \quad q_{k+1} = a_{k+1}q_k + q_{k-1}, \quad k \geq 0.
\]

Let $k \geq 1$ and $x \in \mathbb{R}$, the transfer matrix $M_k(x)$ over $q_k$ sites is defined by

\[
M_k(x) := \begin{bmatrix}
x - v_{q_k} & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x - v_{q_{k-1}} & -1 \\
1 & 0
\end{bmatrix} \cdots \begin{bmatrix}
x - v_1 & -1 \\
1 & 0
\end{bmatrix},
\]

where $v_n$ is defined in (1). By convention we take

\[
M_{-1}(x) = \begin{bmatrix}
1 & -V \\
0 & 1
\end{bmatrix}, \quad M_0(x) = \begin{bmatrix}
x & -1 \\
1 & 0
\end{bmatrix}.
\]

For $k \geq 0$, $p \geq -1$, let $t_{(k,p)}(x) = \text{tr} M_{k-1}(x) M_k^p(x)$ and

\[
\sigma_{(k,p)} = \{ x \in \mathbb{R} : |t_{(k,p)}(x)| \leq 2 \}
\]

where $\text{tr} M$ stands for the trace of the matrix $M$.

With these notations, we collect some known facts that will be used later, for more details, we refer to [1, 17, 18, 19].

(A) Renormalization relation. For any $k \geq 0$

\[
M_{k+1}(x) = M_{k-1}(x) (M_{k}(x))^{a_{k+1}},
\]
so, \( t_{(k+1,0)} = t_{(k-1,a_k)} \), \( t_{(k-1)} = t_{(k-1,a_k-1)} \).

(B) Structure of \( \sigma_{(k,p)}(k \geq 0, p \geq -1) \). For \( V > 0 \), \( \sigma_{(k,p)} \) is made of \( \deg t_{(k,p)} \) disjoint closed intervals.

(C) Trace relation. Defining \( \Lambda(x, y, z) = x^2 + y^2 + z^2 - xyz - 4 \), then

\[
\Lambda(t_{(k+1,0)}, t_{(k,p)}, t_{(k,p+1)}) = V^2.
\]

Thus for any \( k \in \mathbb{N}, p \geq 0 \) and \( V > 4 \),

\[
\sigma_{(k+1,0)} \cap \sigma_{(k,p)} \cap \sigma_{(k,p-1)} = \emptyset.
\] (16)

(D) Covering property. For any \( k \geq 0, p \geq -1 \),

\[
\sigma_{(k,p+1)} \subset \sigma_{(k+1,0)} \cup \sigma_{(k,p)},
\]

then

\[
(\sigma_{(k+2,0)} \cup \sigma_{(k+1,0)}) \subset (\sigma_{(k+1,0)} \cup \sigma_{(k,0)}).
\]

Moreover

\[
\Sigma = \bigcap_{k \geq 0} (\sigma_{(k+1,0)} \cup \sigma_{(k,0)}).
\]

The intervals of \( \sigma_{(k,p)} \) will be called the bands, when we discuss only one of these bands, it is often denoted as \( B_{(k,p)} \). Property (B) also implies \( t_{(k,p)}(x) \) is monotone on \( B_{(k,p)} \), and

\[
t_{(k,p)}(B_{(k,p)}) = [-2, 2],
\]
we call \( t_{(k,p)} \) the generating polynomial of \( B_{(k,p)} \).

\{\sigma_{(k+1,0)} \cup \sigma_{(k,0)} : k \geq 0\} form a covering of \( \Sigma \). However there are some repetitions between \( \sigma_{(k,0)} \cup \sigma_{(k-1,0)} \) and \( \sigma_{(k+1,0)} \cup \sigma_{(k,0)} \). It is possible to choose a coverings of \( \Sigma \) elaborately such that we can get rid of these repetitions, as we will describe in the follows:

**Definition 1.** (\[17\] \[13\]) For \( V > 4, k \geq 0 \), we define three types of bands as follows:

- \((k, I)\)-type band: a band of \( \sigma_{(k,1)} \) contained in a band of \( \sigma_{(k,0)} \);
- \((k, II)\)-type band: a band of \( \sigma_{(k+1,0)} \) contained in a band of \( \sigma_{(k,-1)} \);
- \((k, III)\)-type band: a band of \( \sigma_{(k+1,0)} \) contained in a band of \( \sigma_{(k,0)} \).

By the property (B), \[16\] and \[17\], all three kinds of types of bands are well defined, and we call these bands spectral generating bands of order \( k \) (the type I band is called the type I gap in \[17\]). Note that for order 0, there is only one \((0, I)\)-type band \( \sigma_{(0,1)} = [V - 2, V + 2] \) (the corresponding generating polynomial is \( t_{(0,1)} = x - V \)), and only one \((0, III)\) type band \( \sigma_{(1,0)} = [-2, 2] \) (the corresponding generating polynomial is \( t_{(1,0)} = x \)). They are contained
in $\sigma_{(0,0)} = (-\infty, +\infty)$ with corresponding generating polynomial $t_{(0,0)} \equiv 2$. For the convenience, we call $\sigma_{(0,0)}$ the spectral generating band of order $-1$.

For any $k \geq -1$, denote by $\mathcal{G}_k$ the set of all spectral generating bands of order $k$, then the intervals in $\mathcal{G}_k$ are disjoint. Moreover ([13, 8])

- any $(k, I)$-type band contains only one band in $\mathcal{G}_{k+1}$, which is of $(k+1, II)$-type.
- any $(k, II)$-type band contains $2a_{k+1} + 1$ bands in $\mathcal{G}_{k+1}$, $a_{k+1} + 1$ of which are of $(k+1, I)$-type and $a_{k+1}$ of which are of $(k+1, III)$-type.
- any $(k, III)$-type band contains $2a_{k+1} - 1$ bands in $\mathcal{G}_{k+1}$, $a_{k+1}$ of which are of $(k+1, I)$-type and $a_{k+1} - 1$ of which are of $(k+1, III)$-type.

Thus $\{\mathcal{G}_k\}_{k \geq 0}$ forms a natural covering([14, 12]) of the spectrum $\Sigma$. For any $k \geq 1$, let $s_k$ be the unique real number in $[0,1]$ satisfies

$$\sum_{B \in \mathcal{G}_k} |B|^{s_k} = 1,$$

and define the pre-dimensions of $\Sigma$ by

$$s^*(V) = \limsup_{k \to \infty} s_k, \quad s^*(V) = \liminf_{k \to \infty} s_k. \quad (19)$$

In the following we will give a coding for $\Sigma$. Let

$$\mathcal{E} := \{(I, II), (II, I), (II, III), (III, I), (III, II), (III, III)\}$$

be the admissible edges. To simplify the notation, we write

$$e_{12} = (I, II), e_{21} = (II, I), e_{23} = (II, III), e_{31} = (III, I), e_{33} = (III, III).$$

For each $n \in \mathbb{N}$ define

$$\tau_e(n) = \begin{cases} 1 & e = e_{12} \\ n+1 & e = e_{21} \\ n & e = e_{23} \\ n & e = e_{31} \\ n-1 & e = e_{33}. \end{cases}$$

Then define

$$\mathcal{E}_n = \{(e, \tau_e(n), l) : e \in \mathcal{E}, 1 \leq l \leq \tau_e(n)\}$$
\[ E_n^* = \{(e, \tau_e(n), l) \in E_n : e \neq e_{21}, e_{23}\}. \]

For any \( w = (e, \tau_e(n), l) \in E_n \), we use the notation \( e_w := e \).

For any \( n, n' \in \mathbb{N} \) and any \( (e, \tau_e(n), l) \in E_n \) and \( (e', \tau_e'(n'), l') \in E_{n'} \) we say \( (e, \tau_e(n), l)(e', \tau_e'(n'), l') \) is admissible if the end point of \( e \) is the initial point of \( e' \). We denote it by \( (e, \tau_e(n), l) \rightarrow (e', \tau_e'(n'), l') \).

Define

\[ \Omega = \{ \omega \in E_n^* \times \prod_{k=2}^{\infty} E_{a_k} : \omega = \omega_1 \omega_2 \cdots \text{ s.t. } \omega_k \to \omega_{k+1} \text{ for all } k \geq 1 \}. \]

Define \( \Omega_1 = E_{a_1}^* \) and for \( n \geq 2 \), define

\[ \Omega_n = \{ w \in E_{a_1}^* \times \prod_{k=2}^{n} E_{a_k} : w = w_1 \cdots w_n \text{ s.t. } w_k \to w_{k+1} \text{ for all } 1 \leq k < n \}. \]

Define finally \( \Omega_* = \bigcup_{n \geq 1} \Omega_n \).

Given any \( w \in \Omega_n \), 1 \( \leq k < n \), we write \( w = u * v \) or \( w = uv \), where \( u = w_1 \cdots w_k \), \( v = w_{k+1} \cdots w_n \).

Given any \( w \in \Omega_n \), define \( B_w \) inductively as follows: Let \( B_I = [V - 2, V + 2] \) be the unique \((0, I)\)-type band in \( \mathcal{G}_0 \) and let \( B_{III} = [-2, 2] \) be the unique \((0, III)\)-type band in \( \mathcal{G}_0 \).

Given \( w \in \Omega_1 \). If \( w = (e_{12}, 1, 1) \), then define \( B_w \) to be the unique \((1, II)\)-type band contained in \( B_I \). If \( w = (e_{31}, \tau_{e_{31}}(a_1), l) \), then define \( B_w \) to be the unique \( l \)-th \((1, I)\)-type band contained in \( B_{III} \). If \( w = (e_{33}, \tau_{e_{33}}(a_1), l) \), then define \( B_w \) to be the unique \( l \)-th \((1, III)\)-type band contained in \( B_{III} \), where we order the bands of the same type from left to right.

If \( B_w \) has been defined for any \( w \in \Omega_{n-1} \). Given \( w \in \Omega_n \) and write \( w = w' * (e, \tau_e(a_n), l) \), then \( w' \in \Omega_{n-1} \). If \( e = (T, T') \), define \( B_w \) to be the unique \( l \)-th \((n, T')\)-type band inside \( B_{w'} \).

With these notations we can rewrite (15) as

\[ \Sigma = \bigcap_{n \geq 0} \bigcup_{w \in \Omega_n} B_w. \]

Given \( w \in \Omega_k \), we say \( w \) has length \( k \) and denote by \( |w| = k \). If \( B_w \) is of \((k, T)\) type, sometimes we also say simply that \( B_w \) has type \( T \).

3. Variation, Covariation and Gibbs Like Measure

In this section, we will present three properties related to the spectrum, that is, bounded variation; bounded covariation and the existence of Gibbs like measures. These properties play essential roles in the proof of the main theorems of the paper. We fix \( V > 0 \) and \( \alpha \in (0, 1) \) irrational with continued
fraction expansion \([0; a_1, a_2, \cdots]\). Since now \((a_k)_{k \geq 1}\) can be unbounded, the proofs of these properties are much more difficult than \([8]\), and we put the proofs to the sections \(7, 8\) and \(9\).

We also collect several other basic properties which will be used in the proofs of the main theorems.

3.1. Bounded variation.

**Theorem 3.1 (Bounded variation).** Let \(V \geq 20\) and \(\alpha\) be irrational. Then there exists a constant \(\xi = \exp (180V) > 1\) such that for any spectral generating band \(B\) of \(\Sigma_{\alpha,V}\) with generating polynomial \(h\),

\[
\xi^{-1} \leq \frac{|h'(x_1)|}{|h'(x_2)|} \leq \xi, \quad \forall x_1, x_2 \in B.
\]

**Corollary 3.2 (Bounded distortion).** Let \(V \geq 20\) and \(\alpha\) be irrational. Then there exists a constant \(\xi = 4 \exp (180V) > 1\) such that for any spectral generating band \(B\) of \(\Sigma_{\alpha,V}\) with generating polynomial \(h\),

\[
\xi^{-1} \leq |h'(x)| \cdot |B| \leq \xi, \quad \forall x \in B.
\]

We will prove Theorem 3.1 and Corollary 3.2 in Section 7.

3.2. Bounded covariation.

**Theorem 3.3 (Bounded covariation).** Let \(V \geq 24\) and \(\alpha\) be irrational. Then there exist absolute constants \(C_1, C_2 > 1\) such that if \(w, \tilde{w}, wu, \tilde{w}u \in \Omega_*\), then, for \(\eta = C_1 \exp(2C_2V)\),

\[
\eta^{-1} \frac{|B_{wu}|}{|B_w|} \leq \frac{|B_{\tilde{w}u}|}{|B_{\tilde{w}}|} \leq \eta \frac{|B_{wu}|}{|B_w|}.
\]

**Corollary 3.4.** Let \(V \geq 24\) and \(\alpha = [0; a_1, a_2, \cdots]\) be irrational. Write \(N = \{n : n = a_i \text{ for some } i\}\). Then there exist absolute constants \(C_1, C_2 > 1\) and sequence \(\{\zeta_n : 0 < \zeta_n \leq 1, n \in N\}\) depending on \(\alpha, V\) and \(n\), such that for any \(k \in \mathbb{N}\) if \(a_{k+1} = n, w \in \Omega_k, u = (e_{12}, 1, 1)\) and \(wu \in \Omega_{k+1}\) then

\[
\eta^{-1} \zeta_n \leq \frac{|B_{wu}|}{|B_w|} \leq \eta \zeta_n
\]

with \(\eta = C_1 \exp(C_2V)\). Moreover \(\zeta_1\) can be taken as 1.

We will prove Theorem 3.3 and Corollary 3.4 in Section 8.
3.3. Existence of Gibbs like measures.

At first we introduce some notations used in this paper. For two positive sequences \( \{a_n\} \) and \( \{b_n\} \), \( a_n \sim b_n \) means that there exist constants \( 0 < d_1 \leq d_2 \) such that \( d_1 a_n \leq b_n \leq d_2 a_n \) for every \( n \in \mathbb{N} \). \( a_n \preceq b_n \) means that there exists a constant \( d > 0 \) such that \( a_n \leq db_n \) for every \( n \in \mathbb{N} \). \( a_n \simeq b_n \) can be defined similarly.

For any \( \beta > 0 \) define
\[
  b_{k,\beta} := \sum_{w \in \Omega_k} |B_w|^\beta = \sum_{B \in \mathcal{G}_k} |B|^\beta.
\]

**Theorem 3.5** (Existence of Gibbs like measures). For any \( 0 < \beta < 1 \), there exists a probability measure \( \mu_\beta \) supported on \( \Sigma \) such that

- If \( B_w \) has type \( (k, I) \), let \( u = (e_{12}, 1, 1) \), then
  \[
  \mu_\beta(B_w) \sim \left( \frac{\zeta_k}{a_{k+1}} \right)^{1-\beta} \frac{b_{k,\beta}}{b_{k+1,\beta}} \sum |B_{wu}|^\beta.
  \]
- If \( B_w \) has type \( (k, II) \), then
  \[
  \mu_\beta(B_w) \sim \frac{b_{k,\beta}}{b_{k,\beta}}.
  \]
- If \( B_w \) has type \( (k, III) \), then
  \[
  \mu_\beta(B_w) \sim \left\{ \begin{array}{ll}
  \frac{|B_w|^\beta}{b_{k,\beta}} & a_{k+1} > 1; \\
  \frac{\zeta_k}{a_k^{1-\beta}} \frac{|B_w|^\beta}{b_{k,\beta}} & a_{k+1} = 1.
  \end{array} \right.
  \]

We will prove a generalized version of this theorem, i.e. Theorem 9.4 in Section 9. Indeed the measure constructed in this theorem is a weak type Gibbs like measure, compared with that constructed in [15, 8]. However we still call it Gibbs like measure for convenience.

3.4. Other useful facts.

In this subsection we collect several other useful facts, which are essentially contained in [8].

**Lemma 3.6.** ([8]) Assume \( w \in \Omega_k, wu \in \Omega_{k+1} \) with \( u = (e, p, l) \). Let \( h_w, h_{wu} \) be the generating polynomials of \( B_w, B_{wu} \) respectively. Then for any \( x \in B_{wu} \), if \( e \neq e_{12} \),
\[
  \frac{V - 8}{3} (p + 1) \csc^2 \frac{l\pi}{p + 1} \leq \left| \frac{h'_{wu}(x)}{h'_w(x)} \right| \leq (V + 5) (p + 1) \csc^2 \frac{l\pi}{p + 1},
\]
if \( e = e_{12} \), then \( p = 1 \), we have
\[
\left( \frac{2(V - 8)}{3} \right)^{a_{k+1} - 1} \leq \left| \frac{h'_{w_0}(x)}{h'_w(x)} \right| \leq (2(V + 5))^{a_{k+1} - 1}.
\]

We remark that here \( p = \tau_e(a_{k+1}) \). This lemma is stated in another way in Proposition 6.3. See [8] Proposition 3.3 for a proof.

**Lemma 3.7.** Assume \( V \geq 20 \). Write \( t_1 = (V - 8)/3 \) and \( t_2 = 2(V + 5) \). Then for any \( w = w_1 \cdots w_n \in \Omega_n \) with \( w_i = (e_i, \tau_{e_i}(a_i), l_i) \) we have
\[
\prod_{e_i = e_{12}}^{1} \frac{1}{t_2^{{a_i} - 1}} \cdot \prod_{e_i \neq e_{12}}^{1} \frac{1}{t_2^{{a_i} - 1}} \leq \left| B_w \right| \leq 4 \prod_{e_i = e_{12}}^{1} \frac{1}{t_1^{{a_i} - 1}} \cdot \prod_{e_i \neq e_{12}}^{1} \frac{1}{t_1^{{a_i} - 1}}.
\](20)

Especially we have
\[
\left| B_w \right| \leq 4^{1 - n/2}.
\](21)

We will prove this lemma in the end of Section 6.

### 4. Dimension Formulas

This section is devoted to the proof of (7) in Theorem 1.1. At first we will show the box dimension formula, which is easier. Then we will propose a truncation procedure to derive the Hausdorff dimension formula. The formula (8) will be proven in Section 5.

#### 4.1. \( s^*(V) = \overline{\dim}_B \Sigma \)

By (9), we only need to show that \( \overline{\dim}_B \Sigma \leq s^*(V) \). We recall an equivalent definition of the upper box dimension (see for example [20]). Let \( A \subset \mathbb{R} \) be a Cantor set. Let \( a = \inf E \) and \( b = \sup E \). The complement of \( A \), i.e. \([a, b] \setminus A\), consists of countable many open intervals \( \{G_i\}_{i \geq 1} \), which is called the gaps of \( A \). For \( i \geq 1 \), let \( t_i = |G_i| \), then
\[
\overline{\dim}_B A = \inf \left\{ \beta \mid \sum_{i=1}^{\infty} t_i^{\beta} < \infty \right\}.
\]

We now consider all the gaps of the spectrum \( \Sigma \). We call a gap of order \( k \) if it is covered by some band in \( \mathcal{G}_k \) but is not covered by any band in \( \mathcal{G}_{k+1} \). Let \( \mathcal{P}_k \) be the collection of all gaps of order \( k \), then
\[
\overline{\dim}_B \Sigma = \inf \left\{ \beta \mid \sum_{k \geq 0} \sum_{J \in \mathcal{P}_k} |J|^{\beta} < \infty \right\}.
\]

Now let us prove \( \overline{\dim}_B \Sigma \leq s^*(V) \).
If \( s^*(V) = 1 \), the result is trivial. So in the following we assume \( s^*(V) < 1 \). Fix \( s \) such that \( s^*(V) < s < 1 \).

Let \( B \) be a generating band of order \( k \). Suppose it contain \( n \) generating bands of order \( k+1 \), then it contains \( n-1 \) gaps of order \( k \), which we denote by \( J_1, \cdots, J_{n-1} \). It is seen that \( |J_1| + \cdots + |J_{n-1}| \leq |B| \). By concavity of the function \( x^s \) we get

\[
\sum_{i=1}^{n-1} |J_i|^s = (n-1) \sum_{i=1}^{n-1} \frac{|J_i|^s}{n-1} \leq (n-1) \frac{\sum_{i=1}^{n-1} |J_i|}{n-1} \leq (n-1)^{1-s}|B|^s.
\]

For any generating band of order \( k \), it contain at most \( 2a_{k+1}+1 \) generating bands of order \( k+1 \), so we have

\[
\sum_{J \in \mathcal{P}_k} |J|^s \leq (2a_{k+1})^{1-s} \sum_{B \in \mathcal{G}_k} |B|^s = (2a_{k+1})^{1-s} b_{k,s}.
\]

By (71) and Lemma 9.1 (taking \( \varepsilon \) to be 0), we have \( b_{k,s}/b_{k+1,s} \sim a_{k+1}^{s-1} \), so there exists \( C > 0 \) such that

\[
\sum_{J \in \mathcal{P}_k} |J|^s \leq C b_{k+1,s}.
\]

Let \( \varepsilon = s - s^*(V) \). Since \( s^*(V) < s \), there exists \( N > 0 \) such that for any \( k > N, s > s_k + \varepsilon/2 \) and for any \( B \in \mathcal{G}_k \) we have \( |B| < 1 \). By (21) we have

\[
b_{k,s} \leq \sum_{B \in \mathcal{G}_k} |B|^{s_k+\varepsilon/2} \leq 4^{1-\varepsilon k/4} \sum_{B \in \mathcal{G}_k} |B|^{s_k} = 4^{1-\varepsilon k/4}.
\]

Hence we have

\[
\sum_{k 
\geq 0} \sum_{J \in \mathcal{P}_k} |J|^s = \tilde{C} + \sum_{k=N}^{\infty} \sum_{J \in \mathcal{P}_k} |J|^s \leq \tilde{C} + C \sum_{k=N+1}^{\infty} b_{k,s} < \infty.
\]

So we get \( \dim_B \Sigma \leq s \). Since \( s > s^*(V) \) is arbitrary, we conclude that \( \dim_B \Sigma \leq s^*(V) \).

4.2. \( s^*(V) = \dim_H \Sigma \).

Recall that \( \alpha \) has expansion \([0; a_1, a_2, \cdots]\). If \( a_k \) is very large, as discussed in the introduction, the length of the bands of order \( k \) contained in the same band of order \( k-1 \) can differ from each other very much, which makes the estimation very difficult. To overcome this difficulty, we propose the following truncation procedure.

Fix \( 0 \leq \varepsilon < 1/12 \). Define

\[
\mathcal{E}_n(\varepsilon) = \{(e_{12}, 1, 1) \cup \bigcup_{e \neq e_{12}} \{(e, \tau_e(n), l) : (\tau_e(n)+1)e < l < (\tau_e(n)+1)(1-\varepsilon)\}.
\]

(22)
It is seen that if \( n \leq 10 \), then \( E_n(\varepsilon) = \varepsilon_n \). Define

\[
\Omega_n(\varepsilon) = \Omega_n \cap \left( \varepsilon_{a_1}^* \times \prod_{k=2}^{n} \varepsilon_{a_k}(\varepsilon) \right) \quad (n \geq 2), \quad \Omega_\ast(\varepsilon) = \bigcup_{n \geq 2} \Omega_n(\varepsilon) \tag{23}
\]

and

\[
E_\varepsilon := \bigcap_{n \geq 1} \bigcup_{w \in \Omega_n(\varepsilon)} B_w. \tag{24}
\]

It is obvious that \( E_\varepsilon \subset \Sigma \). For this set we can also define the associated pre-dimensions \( s_\ast(\varepsilon) \).

**Proposition 4.1.** \( s_\ast(\varepsilon) \to s_\ast \) when \( \varepsilon \to 0 \).

**Proof.** We begin with the comparison of \( s_n \) and \( s_n(\varepsilon) \). By the definitions

\[
\sum_{w \in \Omega_n} |B_w|^{s_n} = 1; \quad \sum_{w \in \Omega_n(\varepsilon)} |B_w|^{s_n(\varepsilon)} = 1.
\]

It is seen that \( s_n(\varepsilon) \leq s_n \) for \( n \in \mathbb{N} \).

**Claim:** There exists a constant \( C > 0 \) such that, for any small \( \varepsilon \),

\[
\sum_{w \in \Omega_n(\varepsilon)} |B_w|^{s_n} \geq (1 - C\varepsilon)^n. \tag{25}
\]

We first show that the claim implies the result. In fact if the claim holds, then

\[
(1 - C\varepsilon)^n \leq \sum_{w \in \Omega_n(\varepsilon)} |B_w|^{s_n} \\
= \sum_{w \in \Omega_n(\varepsilon)} |B_w|^{s_n(\varepsilon)}|B_w|^{s_n - s_n(\varepsilon)} \\
\leq 4 \cdot 4^{-(s_n - s_n(\varepsilon))n/2} \sum_{w \in \Omega_n(\varepsilon)} |B_w|^{s_n(\varepsilon)} \\
= 4 \cdot 4^{-(s_n - s_n(\varepsilon))n/2},
\]

where the second inequality is due to (21). Consequently

\[
0 \leq s_n - s_n(\varepsilon) \leq \frac{2}{n} - \frac{\ln(1 - C\varepsilon)}{\ln 2}.
\]

From this we can conclude that \( s_\ast(\varepsilon) \to s_\ast \) when \( \varepsilon \to 0 \).

Now we go back to the proof of the claim. For this purpose we introduce the following intermediate symbolic spaces. For \( 1 \leq j \leq n - 1 \) define

\[
\Omega_n^{(j)}(\varepsilon) := \Omega_n \cap \left( \varepsilon_{a_1}^* \times \prod_{l=2}^{j} \varepsilon_{a_l} \times \prod_{l=j+1}^{n} \varepsilon_{a_l}(\varepsilon) \right).
\]
Thus \( \Omega^{(1)}_n(\varepsilon) = \Omega_n(\varepsilon) \). We also write \( \Omega^{(n)}_n(\varepsilon) := \Omega_n \) to unify the notation.

To prove (25), we only need to show that, for \( j = 1, \cdots, n-1 \),
\[
\sum_{w \in \Omega^{(j)}_n(\varepsilon)} |B_w|^{s_n} \geq (1 - C\varepsilon) \sum_{w \in \Omega^{(j+1)}_n(\varepsilon)} |B_w|^{s_n}. \tag{26}
\]

By the definition if \( a_{j+1} \leq 10 \), then \( \varepsilon_{a_{j+1}}(\varepsilon) = \varepsilon_{a_{j+1}} \) and consequently \( \Omega^{(j)}_n(\varepsilon) = \Omega^{(j+1)}_n(\varepsilon) \), thus (26) is trivial.

Now assume \( a_{j+1} > 10 \). We can write
\[
\sum_{w \in \Omega^{(j+1)}_n(\varepsilon)} |B_w|^{s_n} = \sum_{e \in \mathcal{E}} Z_e \text{ and } \sum_{w \in \Omega^{(j)}_n(\varepsilon)} |B_w|^{s_n} = \sum_{e \in \mathcal{E}} Z_e(\varepsilon),
\]
where for \( e = e_{12} \)
\[
Z_{e_{12}} = Z_{e_{12}}(\varepsilon) = \sum_{w \in \Omega^{(j+1)}_n(\varepsilon), w_{j+1} = (e_{12}, 1, 1)} |B_w|^{s_n}
\]
and for \( e \neq e_{12} \)
\[
Z_e = \sum_{l=1}^{\tau_e(a_{j+1})} \sum_{w \in \Omega^{(j+1)}_n(\varepsilon), w_{j+1} = (e, \tau_e(a_{j+1}), l)} |B_w|^{s_n},
\]
\[
Z_e(\varepsilon) = \sum_{l=1}^{\tau_e(a_{j+1})} \sum_{w \in \Omega^{(j+1)}_n(\varepsilon), w_{j+1} = (e, \tau_e(a_{j+1}), l)} |B_w|^{s_n}
\]

Note that for \( e \neq e_{12} \), \( Z_e(\varepsilon) \) is different from \( Z_e \) only in the range of the index \( l \). To show (26) it is enough to show that
\[
Z_e(\varepsilon) \geq (1 - C\varepsilon) Z_e, \quad e \in \mathcal{E}.
\]

The case \( e = e_{12} \) is trivial. Now we consider \( e = e_{21} \). In this case we have \( \tau_e(a_{j+1}) = a_{j+1} + 1 \). Write \( \theta_t = (e, \tau_e(a_{j+1}), l) \). Then we can write \( Z_e(\varepsilon) \) as
\[
Z_e(\varepsilon) = \sum_{l=[a_{j+1}], u \in \Omega^{(j+1)}_n(\varepsilon)} |B_w|^{s_n}
\]

Fix \( u \in \Omega_j \), write
\[
\gamma_l := \sum_{u \in \Omega^{(j+1)}_n(\varepsilon)} |B_{u\theta_l}|^{s_n}.
\]

Let \( l_0 = [a_{j+1}/2] \). By Theorem 3.3 we have
\[
\gamma_l = |B_{u\theta_l}|^{s_n} \sum_{u \in \Omega^{(j+1)}_n(\varepsilon)} |B_{u\theta_l}|^{s_n} \sim |B_{u\theta_l}|^{s_n} \gamma_{l_0}.
\]
By Corollary 3.2 and Lemma 3.6,

\[ \frac{|B_{u\theta_l}|}{|B_{u\theta_l_0}|} = \frac{|B_{u\theta_l}|/|B_u|}{|B_{u\theta_l_0}|/|B_u|} \sim \sin^2 \frac{l\pi}{a_{j+1}+1}. \]

Then there exists \( c > 1 \) independent to \( l, j \) such that

\[ c^{-1}\gamma_{l_0}\sin^{2s_n} \frac{l\pi}{a_{j+1}+1} \leq \gamma_l \leq c\gamma_{l_0}\sin^{2s_n} \frac{l\pi}{a_{j+1}+1}. \]

So we have

\[ \sum_{l=1}^{a_{j+1}+1} \gamma_l \geq c^{-1}\gamma_{l_0} \sum_{l=1}^{a_{j+1}+1} \sin^{2s_n} \frac{l\pi}{a_{j+1}+1} \]

\[ \geq c^{-1}\gamma_{l_0} \sum_{l=l_0/2}^{3l_0/2} \sin^{2s_n} \frac{l\pi}{a_{j+1}+1} \]

\[ \geq c^{-1}\gamma_{l_0} l_0 \sin^{2s_n} \pi/4. \]

Let \( A = \{1 \leq l \leq a_{j+1}+1 \mid l < [a_{j+1}\varepsilon] \text{ or } l > [a_{j+1}(1-\varepsilon)]\} \), we have

\[ \sum_{l \in A} \gamma_l \leq c\gamma_{l_0} \sum_{l \in A} \sin^{2s_n} \frac{l\pi}{a_{j+1}+1} \leq c\gamma_{l_0} \cdot 2a_{j+1}\varepsilon \cdot \sin^{2s_n} \pi/4. \]

This implies for all \( u \in \Omega_j, \)

\[ \sum_{l=[a_{j+1}(1-\varepsilon)]}^{[a_{j+1}(1-\varepsilon)]} \gamma_l \geq (1-4c^2\varepsilon) \sum_{l=1}^{a_{j+1}+1} \gamma_l, \]

so we get \( Z_{\varepsilon}(\varepsilon) \geq (1-4c^2\varepsilon)Z_{\varepsilon} \).

For other \( \varepsilon \neq \varepsilon_{12} \), the proof is the same. Thus the proof is completed. \( \square \)

**Proposition 4.2.** \( \dim_H E_\varepsilon = s_*(\varepsilon) \).

**Proof.** It is known that \( \dim_H E_\varepsilon \leq s_*(\varepsilon) \), so it only need to show \( \dim_H E_\varepsilon \geq s_*(\varepsilon) \). It is trivial if \( s_*(\varepsilon) = 0 \), we thus assume \( s_*(\varepsilon) > 0 \). Fix any \( 0 < \beta < s_*(\varepsilon) \) and define

\[ b_{k,\beta}(\varepsilon) := \sum_{\omega \in \Omega_k(\varepsilon)} |B_\omega|^\beta. \quad (27) \]

Then there exists \( K \in \mathbb{N} \) such that \( b_{k,\beta}(\varepsilon) \geq 1 \) for any \( k \geq K \).

By (71) and Lemma 9.1 we have

\[ \frac{b_{k-1,\beta}(\varepsilon)}{b_{k,\beta}(\varepsilon)} \sim a_{k-1}^{\beta-1}. \quad (28) \]

By Theorem 9.4 we can construct a Gibbs like measure \( \mu_{\beta,\varepsilon} \) supported on \( E_\varepsilon \). Define \( \delta_0 := \min\{|B_w| : w \in \Omega_{K+1}(\varepsilon)\} \).
Claim: for any open interval $U \subset \mathbb{R}$ with $|U| \leq \delta_0/2$ we have
$$\mu_{\beta,\varepsilon}(U) \lesssim |U|^\beta.$$

Take any open interval $U \subset \mathbb{R}$ with $|U| \leq \delta_0/2$, define
$$\Xi := \{w \in \Omega_\varepsilon : B_w \cap U \neq \emptyset, |B_w| \leq |U| < |B_w^-|\},$$
where $w^-$ is gotten by deleting the last symbol of $w$. At first we claim that $|w| \geq K + 1$ for any $w \in \Xi$. In fact if otherwise, there exists $\tilde{w} = wu \in \Omega_{K+1}(\varepsilon)$. Then
$$\delta_0 \leq |B_{\tilde{w}}^-| \leq |B_w| \leq |U| \leq \delta_0/2,$$
which is a contradiction.

Notice that by the natural covering property, any two generating bands are either disjoint or one of them is included in another, thus we conclude that $\#\{w^- : w \in \Xi\} \leq 2$. Thus to show the claim we only need to show that
$$\mu_{\beta,\varepsilon}(U \cap B_w^-) \lesssim |U|^\beta, \quad (\forall w \in \Xi).$$

If $B_{w^-}$ is of type $(k,I)$, then by (81),
$$\mu_{\beta,\varepsilon}(U \cap B_{w^-}) \leq \mu_{\beta,\varepsilon}(B_{w^-}) \sim \frac{|B_w|^\beta}{b_{k+1,\beta}(\varepsilon)} \leq |U|^\beta.$$

Now we assume $B_{w^-}$ is a band of type $(k,II)$ or $(k,III)$. In this case, $w$ has the form $w^-(e,p,l)$ with $e \neq e_{12}$ and $p = \tau_\varepsilon(a_{k+1})$. By bounded variation and Lemma 3.6 if $a_{k+1} \leq 10$, then
$$|B_w| \sim \frac{1}{a_{k+1}}.$$

If $a_{k+1} > 10$, then we have $\varepsilon \leq l/a_{k+1} \leq 1-\varepsilon$. Consequently, also by bounded variation and Lemma 3.6 there exists constants $C > 1$ not depending on $\varepsilon$ such that
$$\frac{\varepsilon^2}{Ca_{k+1}} \leq \frac{\sin^2 \varepsilon \pi}{Ca_{k+1}} \leq \frac{|B_w|}{|B_{w^-}|} \leq \frac{C}{a_{k+1}}.$$

So in both cases we have
$$\frac{|B_w|}{|B_{w^-}|} \sim \frac{1}{a_{k+1}}. \quad (29)$$

Let
$$\Upsilon := \{u : |u| = k + 1; B_u \subset B_{w^-}; B_u \cap U \neq \emptyset\}.$$

Then by (29), (82) and (83)
$$\mu_{\beta,\varepsilon}(U \cap B_{w^-}) = \sum_{u \in \Upsilon} \mu_{\beta,\varepsilon}(B_u) \leq \#\Upsilon \cdot \max_{u \in \Upsilon} \mu_{\beta,\varepsilon}(B_u) \mu_{\beta,\varepsilon}(U \cap B_u).$$
\[ \mu_{\beta, \varepsilon}(U \cap B_{\omega^-}) \leq \mu_{\beta, \varepsilon}(B_{\omega^-}) \leq |B_{\omega^-}|^{1 - \beta} b_{k+1, \beta}(\varepsilon) =: d_2. \]

So, we have

\[ \mu_{\beta, \varepsilon}(U \cap B_{\omega^-}) \leq \min\{d_1, d_2\} \leq d_1^{\beta} d_2^{1 - \beta} \]

where the last inequality is due to (28) and \( b_{k, \beta}(\varepsilon) \geq 1. \)

Then by the mass distribution principle we conclude that \( \dim_H E_\varepsilon \geq \beta. \)

Since \( \beta < s_*(\varepsilon) \) is arbitrary, we get \( \dim_H E_\varepsilon \geq s_*(\varepsilon). \)

**Proof of \( \dim_H \Sigma = s_*(V) \).** By (6), we only need to prove

\[ \dim_H \Sigma \geq s_*(V). \]

Since \( E_\varepsilon \subset \Sigma \), by Proposition 4.2 we have

\[ \dim_H \Sigma \geq \dim_H E_\varepsilon = s_*(\varepsilon). \]

By Proposition 4.1 we get \( \dim_H \Sigma \geq s_*(V). \)

**5. Proof of (8) and Theorem 1.2**

The main result of this section is Proposition 5.3. Formula (8) and Theorem 1.2 are direct consequences of this proposition.

Let us do some preparation. Recall that we have defined \( K_*(\alpha) \) and \( K^*(\alpha) \) in (1). To simplify the notation, we write \( K_* = K_*(\alpha) \) and \( K^* = K^*(\alpha) \). It is obvious that \( 1 \leq K_* \leq K^*. \)

Throughout this section, for a matrix \( A \in M(3, \mathbb{R}) \) we define

\[ \|A\| := \max\{|a_{ij}| : 1 \leq i, j \leq 3\}. \]

For any \( n \geq 1, 0 \leq x \leq 1 \) define

\[ R_n(x) := \begin{pmatrix} 0 & x^{(a_n-1)} & 0 \\ (a_n + 1)x & 0 & a_nx \\ a_nx & 0 & (a_n - 1)x \end{pmatrix}, \]

\[ S_n(x) := R_1(x) \cdots R_n(x). \]
For $x \in [0,1]$ we define
\[ \psi(x) := \liminf_{n \to \infty} \|S_n(x)\|^{1/n}, \quad \phi(x) := \limsup_{n \to \infty} \|S_n(x)\|^{1/n}. \]

Then $\psi(x) \leq \phi(x)$ for $x \in [0,1]$. It is direct to get the following inequality, assume $0 < x < y \leq 1$, then for any $k > 0$,
\[ R_k(x)R_{k+1}(x) \leq \frac{x}{y} \cdot R_k(y)R_{k+1}(y). \tag{31} \]

Lemma 5.1. (1) If $K_\ast < \infty$, then $\psi : [0,1] \to \mathbb{R}^+$ is strictly increasing and
\[ \left( \frac{K_\ast}{2} \lor \sqrt{2} \right) x \leq \psi(x) \leq K_\ast \sqrt{2x}, \quad \forall x \in [0,1]. \]

If $K_\ast = \infty$, then $\psi(0) = 0$ and $\psi(x) = \infty$ for any $x \in (0,1]$.

(2) If $K_\ast < \infty$, then $\phi : [0,1] \to \mathbb{R}^+$ is strictly increasing and
\[ \left( \frac{K_\ast}{2} \lor \sqrt{2} \right) x \leq \phi(x) \leq K_\ast \sqrt{2x}, \quad \forall x \in [0,1]. \]

If $K_\ast = \infty$, then $\phi(0) = 0$ and $\phi(x) = \infty$ for any $x \in (0,1]$.

Proof. We only prove (1), since the proof of (2) is analogous. It is easy to check that $\psi(0) = 0$. Define
\[ J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{32} \]

At first assume $K_\ast < \infty$. It is ready to check the following inequality
\[ R_n(x)R_{n+1}(x) \leq 2xa_n a_{n+1}J. \]

Consequently we have
\[ \|R_n(x)R_{n+1}(x)\| \leq 2xa_n a_{n+1}, \]
which implies $\psi(x) \leq K_\ast \sqrt{2x}$.

Claim: There exists a path $i_0 i_1 \cdots i_n \in \{1,2,3\}^{n+1}$ such that for $k = 1, \cdots, n$,
\[ (R_k(x))_{i_{k-1}i_k} \geq (1 \lor \frac{a_k}{2})x, \]
and for $k = 1, \cdots, n - 2$,
\[ (R_k(x))_{i_{k-1}i_k} (R_{k+1}(x))_{i_ki_{k+1}} (R_{k+2}(x))_{i_{k+1}i_{k+2}} \geq 2x^3. \]

Take a path $i_0 i_1 \cdots i_n \in \{1,2,3\}^{n+1}$ as follows:
- take $i_0 = 2$;
- for $0 < j \leq n$, if $i_{j-1} = 1$, then take $i_j = 2$;
• for $0 < j \leq n$, if $i_{j-1} = 2$ or 3 and $a_{j+1} > 2$, then take $i_j = 3$;
• for $0 < j \leq n$, if $i_{j-1} = 2$ or 3 and $a_{j+1} \leq 2$, then take $i_j = 1$.

Fix $1 \leq k \leq n$ let us discuss the following cases:

**Case 1:** $i_{k-1}i_k = 12$. By the way we choose the path, we have $a_k \leq 2$. Thus
\[
(R_k(x))_{i_{k-1}i_k} = (R_k(x))_{12} = x^{a_k-1} \geq (1 \vee \frac{a_k}{2})x.
\]

**Case 2:** $i_{k-1}i_k = 21$. Thus
\[
(R_k(x))_{i_{k-1}i_k} = (R_k(x))_{21} = (a_k + 1)x \geq (2 \vee a_k)x.
\]

**Case 3:** $i_{k-1}i_k = 23$. Thus
\[
(R_k(x))_{i_{k-1}i_k} = (R_k(x))_{23} = a_kx \geq x.
\]

**Case 4:** $i_{k-1}i_k = 31$ or 33. By the way we choose the path, we have $a_k > 2$. Thus
\[
(R_k(x))_{i_{k-1}i_k} = \begin{cases} a_kx & i_k = 1 \\ (a_k - 1)x & i_k = 3 \end{cases} \geq (2 \vee \frac{a_k}{2})x.
\]

Since all the possible sequences of $i_{k-1}i_ki_{k+1}i_{k+2}$ are
\[
\{1212; 1231; 1233; 2121; 2123; 2312; 2331; 2333; 3121; 3123; 3312; 3331; 3333\},
\]
by the conclusions of four cases above we get the result. ▷

Thus by the claim above, on the one hand we have
\[
\|R_1(x) \cdots R_n(x)\| \geq (R_1(x))_{i_0i_1} \cdots (R_n(x))_{i_{n-1}i_n} \geq 2^{[n/3]}x^n,
\]
which implies $\psi(x) \geq \sqrt[3]{2}x$. On the other hand we get
\[
\|R_1(x) \cdots R_n(x)\| \geq (R_1(x))_{i_0i_1} \cdots (R_n(x))_{i_{n-1}i_n} \geq a_1 \cdots a_n 2^{-n}x^n, \quad (33)
\]
which implies $\psi(x) \geq K_\star x/2$.

Now we are going to show that $\psi$ is strictly increasing. By the definition of $\psi$ and (31) we get
\[
\psi(x) \leq \frac{x}{y} \psi(y).
\]
Since $\psi(x) \geq \sqrt[3]{2}x$ we have $\psi(x) > 0$ when $x > 0$, thus we conclude that $\psi$ is strictly increasing.

If $K_\star = \infty$, by (33) we have $\psi(x) = \infty$ for any $x \in (0, 1]$. □
Due to the strictly increasing property of $\psi$ and $\phi$ we can define
\[
\begin{align*}
\left\{ 
\begin{array}{l}
f_*(\alpha) := \inf\{ x > 0 : \psi(x) \geq 1 \} = \sup\{ x \geq 0 : \psi(x) \leq 1 \}; \\
f^*(\alpha) := \inf\{ x > 0 : \phi(x) \geq 1 \} = \sup\{ x \geq 0 : \phi(x) \leq 1 \}.
\end{array}
\right.
\end{align*}
\] (34)

**Corollary 5.2.** If $K_* < \infty$, then
\[
f_*(\alpha) \in \left[ \frac{1}{2K_*^2}, \frac{2}{K_* \sqrt{2}} \right] \subset (0, 1);
\]
if $K_* = \infty$, then $f_*(\alpha) = 0$.

If $K^* < \infty$, then
\[
f^*(\alpha) \in \left[ \frac{1}{2K^{*2}}, \frac{2}{K^* \sqrt{2}} \right] \subset (0, 1);
\]
if $K^* = \infty$, then $f^*(\alpha) = 0$.

Now we state the main result of this section:

**Proposition 5.3.** Let $V \geq 24$ and $\alpha = [0; a_1, a_2, \ldots]$ be irrational.

(i) If $K_* < \infty$, then $0 < f_*(\alpha) < 1$ and
\[
\begin{align*}
-\ln f_*(\alpha) &\leq \frac{-\ln f_*(\alpha)}{6 \ln 4K_*^2 + \ln 2(V + 5)} \leq S_*(V) \leq \frac{-\ln f_*(\alpha)}{\ln(V - 8)/3}. \\
\end{align*}
\] (35)

If $K_* = \infty$, then $f_*(\alpha) = 0$ and $s_*(V) = 1$.

(ii) If $K^* < \infty$, then $0 < f^*(\alpha) < 1$ and
\[
\begin{align*}
-\ln f^*(\alpha) &\leq \frac{-\ln f^*(\alpha)}{6 \ln 2K* + \ln 2(V + 5)} \leq \frac{-\ln f^*(\alpha)}{\ln(V - 8)/3} \wedge \frac{\ln K* + \ln \sqrt{2}}{\ln K^* + \ln(V - 8)/3}. \\
\end{align*}
\] (36)

If $K^* = \infty$, then $f^*(\alpha) = 0$ and $s^*(V) = 1$.

**Proof.** Let $t_1 = (V - 8)/3$, $t_2 = 2(V + 5)$. Write
\[
\bar{a}_k = a_k + 1, \quad a_k = a_k - 1, \quad \delta_k = (a_1 \cdots a_k)^{1/k}.
\]

For any $0 \leq \gamma \leq 1$ define
\[
\begin{align*}
Q_k(\gamma) := \begin{pmatrix}
0 & t_1^{-\gamma a_k} & 0 \\
\bar{a}_k(t_1a_k)^{-\gamma} & 0 & a_k(t_1a_k)^{-\gamma} \\
a_k(t_1a_k)^{-\gamma} & 0 & \bar{a}_k(t_1a_k)^{-\gamma}
\end{pmatrix}, \\
\bar{Q}_k(\gamma) := \begin{pmatrix}
0 & t_2^{-\gamma a_k} & 0 \\
\bar{a}_k(t_2a_k^3)^{-\gamma} & 0 & a_k(t_2a_k^3)^{-\gamma} \\
a_k(t_2a_k^3)^{-\gamma} & 0 & \bar{a}_k(t_2a_k^3)^{-\gamma}
\end{pmatrix}.
\end{align*}
\]
By (20) we have
\[ b_{k,\gamma} \leq 4^\gamma(1, 0, 1)Q_1(\gamma) \cdots Q_k(\gamma)(1, 1, 1)^t, \]
\[ b_{k,\gamma} \geq (1, 0, 1)\overline{Q}_1(\gamma) \cdots \overline{Q}_k(\gamma)(1, 1, 1)^t, \]  
(37)
where \((1, 1, 1)^t\) is a column vector. We have, by definition of norm and (37),
\[ b_{k,\gamma} \leq 6 \cdot 4^\gamma \|Q_1(\gamma) \cdots Q_k(\gamma)\|. \]  
(38)
By (30), for any \(k \geq 1\), \(Q_k(\gamma) \leq R_k(t_1^{-\gamma})\), then by (37),
\[ b_{k,\gamma} \leq 4^\gamma(1, 0, 1)S_k(t_1^{-\gamma})(1, 1, 1)^t \leq 6 \cdot 4^\gamma \|S_k(t_1^{-\gamma})\|. \]  
(39)
Since for any \(k \geq 1\), \(\overline{Q}_k(\gamma) \geq a_k^{-3}R_k(t_2^{-\gamma})\), by (37),
\[ b_{k,\gamma} \geq \delta_k^{-3k\gamma}(1, 0, 1)S_k(t_2^{-\gamma})(1, 1, 1)^t. \]
Since there exists \(c > 1\) such that \(c^{-1}J < S_5(t_2^{-\gamma}) < cJ\) (see (32) for definition of \(J\)) and by definition of norm,
\[ b_{k,\gamma} \geq c^{-1}\delta_k^{-3k\gamma}\|R_0(t_2^{-\gamma})R_\gamma(t_2^{-\gamma}) \cdots R_k(t_2^{-\gamma})\| \]
\[ \geq c^{-1}\delta_k^{-3k\gamma}\|S_k(t_2^{-\gamma})\|/\|S_5(t_2^{-\gamma})\| \]
\[ \geq c^{-2}\delta_k^{-3k\gamma}\|S_k(t_2^{-\gamma})\|. \]
This implies, on one hand, by Claim in Lemma 5.1,
\[ b_{k,\gamma} \geq c^{-2}\delta_k^{-3k\gamma}a_1 \cdots a_k t_2^{-k\gamma} = c^{-2}\left((\delta_k^3 t_2)^{-\gamma}\delta_k/2\right)^k; \]  
(40)
on the other hand, by (31),
\[ b_{k,\gamma} \geq c^{-2}\|S_k((\delta_k^6 t_2)^{-\gamma})\|. \]  
(41)
We discuss first upper bound of pre-dimensions.

Assume \(K_\ast < \infty\). Then \(f_\ast(\alpha) > 0\) by Corollary 5.2. Take \(\gamma > 0\) such that \(t_1^{-\gamma} < f_\ast(\alpha)\), i.e. \(\gamma > -\ln f_\ast(\alpha)/\ln t_1\). Then by the definition of \(f_\ast(\alpha)\) and the fact that \(\psi\) is strictly increasing on \([0, 1]\) we conclude that \(\psi(t_1^{-\gamma}) < 1\). Thus for any \(\lambda \in (\psi(t_1^{-\gamma}), 1)\) and any \(n\) big enough, there exists \(k_n \geq n\) such that
\[ \|S_{k_n}(t_1^{-\gamma})\| < \lambda^{k_n}. \]
Thus \(b_{k_n,\gamma} < 1\) when \(n\) is big enough by (39). Consequently \(\gamma > s_{k_n}\) for \(n\) big. Thus we conclude that \(s_\ast(V) \leq \gamma\). Since \(\gamma > -\ln f_\ast(\alpha)/\ln t_1\) is arbitrary, we get
\[ s_\ast(V) \leq \frac{-\ln f_\ast(\alpha)}{\ln t_1} = \frac{-\ln f_\ast(\alpha)}{\ln(V-8)/3}, \]
which is the second inequality in (35).
Assume $K^* < \infty$. By essentially repeating the above proof (indeed it is simpler), we get

$$s^*(V) \leq \frac{-\ln f^*(\alpha)}{\ln(V - 8)/3},$$

which is one of the second inequality in (36).

On the other hand, by definition of the norm, it is direct to check that

$$||Q_k(\gamma)Q_{k+1}(\gamma)|| \leq 2(a_k a_{k+1})^{1-\gamma} t_1^{-\gamma}.$$

(Note that if $a_k = 1$, then $t_1^{-\gamma} a_k = 1$, we cannot get $||Q_k(\gamma)|| \leq 2 a_k^{1-\gamma} t_1^{-\gamma}$ in this case.) So, by (38), for some $c > 0$,

$$b_{k,\gamma} \leq 6c \cdot 4^\gamma (2t_1^{-\gamma})^{k/2} (a_1 \cdots a_k)^{1-\gamma} \leq 24c \left( \frac{\sqrt{2} \delta_k}{(\delta k t_1)^{\gamma}} \right)^k.$$

This implies

$$s^*(V) \leq \ln K^* + \ln \sqrt{2 \ln K^* + \ln t_1},$$

and we get the second inequality in (36).

Next we discuss the low bound of pre-dimensions.

Assume $K_* < \infty$. We will show the first inequality in (35). By Corollary 5.2 we have $0 < f_*(\alpha) < 1$. Fix $g \in (f_*(\alpha), 1)$.

Claim 1. If $\gamma < \frac{-\ln g}{6\ln 4K_*^2 + \ln t_2}$, then $\lim_{k \to \infty} b_{k,\gamma} = \infty$.

Notice that by Corollary 5.2 we have $g > f_*(\alpha) \geq \frac{1}{2K_*^2}$.

If $\delta_k \geq 4K_*^2$, then

$$\gamma < \frac{\ln 2K_*^2}{6\ln 4K_*^2 + \ln t_2} < \frac{\ln 4K_*^2 - \ln 2}{3\ln 4K_*^2 + \ln t_2} \leq \frac{\ln \delta_k - \ln 2}{3\ln \delta_k + \ln t_2}.$$

Write

$$c := \frac{\ln 4K_*^2 - \ln 2}{3\ln 4K_*^2 + \ln t_2} - \gamma > 0,$$

then by direct computation we get

$$(\delta_k^{3/2} t_2)^{-\gamma} \delta_k^{1/2} \geq (\delta_k^2 t_2)^c \geq ((4K_*^2)^3 t_2)^c =: \mu > 1.$$

Consequently by (40) we have

$$b_{k,\gamma} \geq \mu^k.$$

If $\delta_k \leq 4K_*^2$, then by direct computation we get

$$g < (\delta_k^2 t_2)^{-\gamma}.$$
Consequently by (41) we get

\[ b_{k, \gamma} \gtrsim \| S_k(g) \| \]

Since \( g > f_\ast(\alpha) \), there exists \( \tilde{\mu} > 1 \) such that for big \( k \) we have \( \| S_k(g) \| \geq \tilde{\mu}^k \).

Thus we conclude that in either case we have

\[ b_{k, \gamma} \gtrsim \mu^k \wedge \tilde{\mu}^k. \]

By the claim we have \( \gamma \leq s_k \) for \( k \) big. Thus \( \gamma \leq s_\ast(V) \).

Claim 2. If \( \gamma < \frac{-\ln g}{6 \ln 2K^\ast + \ln t_2} \), then \( \limsup_{k \to \infty} b_{k, \gamma} = \infty \).

For any \( k \geq N \), we have

\[ \gamma < \frac{-\ln g}{6 \ln 2K^\ast + \ln t_2} \leq \frac{-\ln g}{6 \ln \delta_k + \ln t_2}. \]

Consequently \( (\delta_k t_2)^{-\gamma} > g \). By (31) and (41),

\[ b_{k, \gamma} > c^{-2} \left( \frac{(\delta_k t_2)^{-\gamma}}{g} \right)^{k/2} \| S_k(g) \|. \]

This prove the claim by definition of \( f_\ast(\alpha) \) and \( g > f_\ast(\alpha) \).

By the choices of \( g \) and \( \gamma \) we conclude that

\[ s_\ast(V) \geq \frac{-\ln f_\ast(\alpha)}{6 \ln 2K^\ast + \ln t_2}. \]

which is the first inequality in (36).

Finally, we consider the cases of \( K_\ast = \infty \) and \( K^\ast = \infty \). Let us define

\[ \hat{Q}_k(\gamma) := \begin{pmatrix} 0 & t_2^{-\gamma} & 0 \\
 a_k/4(t_2a_k/4)^{-\gamma} & 0 & a_k/4(t_2a_k/4)^{-\gamma} \\
 a_k/4(t_2a_k/4)^{-\gamma} & 0 & a_k/4(t_2a_k/4)^{-\gamma} \end{pmatrix}. \]

Fix \( \varepsilon_0 = 1/4 \) and take any \( w \in \Omega_\varepsilon(\varepsilon_0) \). Similar with the proof of Lemma 3.7 we can show that

\[ |B_w| \geq \prod_{\varepsilon_i = \varepsilon_{12}} \frac{1}{t_2^{\varepsilon_i - 1}} \prod_{\varepsilon_i \neq \varepsilon_{12}} \frac{4}{t_2a_i}. \]
Then by a direct computation we get

$$b_{k, \gamma}(\varepsilon_0) \geq (1, 0, 1)\hat{Q}_1(\gamma) \cdots \hat{Q}_k(\gamma)(1, 1, 1)^k.$$  

(See (27) for the definition of $b_{k, \gamma}(\varepsilon)$.)

Analogous to the selection procedure in the proof of the claim in Lemma 5.1, we can get

$$b_{k, \gamma} \geq b_{k, \gamma}(\varepsilon_0) \gtrsim k \prod_{i=1}^{a_i} a_i \delta_k - \ln \delta_k + \ln t/4.$$  

By taking $\gamma = s_k$ and using the fact that $b_{k, s_k} = 1$ we get

$$s_k \geq \frac{\ln \delta_k - \ln 8}{\ln \delta_k + \ln t/4}.$$  

From this inequality it is seen that if $K^* = \infty$, then $s^*(V) \equiv 1$. If $K^* = \infty$, then $s^*(V) \equiv 1$. This finish the proof of Proposition 5.3. □

6. Generating Polynomial and Ladders

In this section we give some preparations for the proofs of bounded variation, bounded covariation and Gibbs like measure.

Consider the equation

$$\Lambda(x, y, z) = x^2 + y^2 + z^2 - xyz = V^2.$$  

We can solve $z$ as

$$z_{\pm}(x, y, V) = \frac{xy}{2} \pm \frac{1}{2} \sqrt{4V^2 + (4 - x^2)(4 - y^2)}.$$  

For two branches $z = z_+$ or $z = z_-$, let

$$z_1(x, y, V) := \frac{\partial z(x, y, V)}{\partial x}, \quad z_2(x, y, V) := \frac{\partial z(x, y, V)}{\partial y}, \quad z_3(x, y, V) := \frac{\partial z(x, y, V)}{\partial V}.$$  

We also define $z_{ij}(x, y, V)$ by the obvious way. For any $|x| \leq 2$, $|y| \leq 2$ and $V > 4$, by a simple computation we get

$$V - 2 \leq |z_{\pm}(x, y, V)| \leq V + 2,$$

$$|z_j(x, y, V)| \leq 1, \quad j = 1, 2, 3$$

$$|z_{ij}(x, y, V)| \leq 1, \quad ij = 11, 12, 13, 21, 22, 23.$$  

We will estimate the derivatives of generating polynomials by using Chebyshev polynomial $S_p(x)$, which is defined by

$$S_0(x) \equiv 0, \quad S_1(x) \equiv 1,$$

$$S_{p+1}(x) = xS_p(x) - S_{p-1}(x), \quad p \geq 1.$$
6.1. Ladders and modified ladders.

In [8], the authors introduce the notion of ladder and modified ladder which is very useful for estimating the derivatives of the generating polynomials. Now we recall the definitions and state some related results which will be used later.

Given \( w \in \Omega_n \), write \( w = w_1 \cdots w_n \) and \( w|_m = w_1 \cdots w_m \) for \( m = 1, \cdots, n \). Write \( B_m = B_{w|_m} \). Then for any \( k \leq n \)

\[
B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_k
\]

is a sequence of spectral generating bands from order \( n \) to \( k \). We call the sequence \( (B_i)_{i=k}^n \) an initial ladder, and the bands \( B_i(k \leq i \leq n) \) are called initial rungs. Now we are going to modify the initial ladder by the following way: for any \( i \) (\( k \leq i \leq n-1 \)),

- if \( B_i \) is of \((i,I)\)-type with \( a_{i+1} = 1 \): delete the rung \( B_{i+1} \) (in this case \( B_{i+1} \) must be \((i+1,II)\)-type, then \( t_{(i+2,0)} = t_{(i,1)} \) and \( t_{(i+1,-1)} = t_{(i,0)} \) implies \( B_{i+1} = B_i \));
- if \( B_i \) is of \((i,I)\)-type with \( a_{i+1} = 2 \): change nothing;
- if \( B_i \) is of \((i,I)\)-type with \( a_{i+1} > 2 \): add rungs \( (B_{(i,p)})_{p=2}^{a_{i+1}-1} \) between \( B_i \) and \( B_{i+1} \):

\[
B_{i+1} = B_{(i,a_{i+1})} \subset B_{(i,a_{i+1}-1)} \subset \cdots \subset B_{(i,2)} \subset B_{(i,1)} = B_i;
\]

where \( B_{(i,p)} \) is the unique band in \( \sigma_{(i,p)} \) which is included in \( B_i \).
- if \( B_i \) is of \((i,II)\) or \((i,III)\)-type: change nothing.

By this way we get a unique modified ladder which we relabel as

\[
B_n = \hat{B}_m \subset \cdots \subset \hat{B}_1 \subset \hat{B}_0 = B_k.
\]

We call \( (\hat{B}_i)_{i=0}^m \) the modified ladder, and we denote the corresponding generating polynomials by \( (\hat{h}_i)_{i=0}^m \). Note that any two consecutive initial rungs can not be of type \( I \) simultaneously, so we have

\[
\frac{(n-k)}{2} \leq m \leq a_{k+1} + a_{k+2} + \cdots + a_n.
\] (43)

Given an initial ladder \( (B_i)_{i=k}^n \). Let \( (\hat{B}_i)_{i=0}^m \) be the related modified ladder. For \( i = 0, \cdots, m-1 \) define

\[
(p_i, l_i) = \begin{cases} 
(\tau_e(a_j), l), & \text{if } (\hat{B}_i, \hat{B}_{i+1}) = (B_{j-1}, B_j) \\
(1, 1), & \text{otherwise}
\end{cases}
\] (44)
We call \((p_i)_{i=0}^{m-1}\) and \((l_i)_{i=0}^{m-1}\) the type sequence and index sequence of the modified ladder.

The following key formula is proved in [8]:

\[
\hat{h}_{i+1}(x) = z_\pm(\hat{h}_i(x), \hat{h}_{i-1}(x), V)S_{p_{i+1}}(\hat{h}_i(x)) - \hat{h}_{i-1}(x)S_{p_i}(\hat{h}_i(x)). \tag{45}
\]

For convenience we denote \(z_\pm(\hat{h}_i(x), \hat{h}_{i-1}(x), V)\) by \(z_\pm(x)\). By taking derivative on both side of (45), we get

\[
\frac{\hat{h}'_{i+1}(x)}{\hat{h}'_i(x)} = S'_{p_{i+1}}(\hat{h}_i(x))z_\pm(x) - S'_{p_i}(\hat{h}_i(x))\hat{h}_{i-1}(x) + \frac{S_{p_{i+1}}(\hat{h}_i(x))z_\pm'(x)}{\hat{h}'_i(x)} \frac{\hat{h}'_{i-1}(x)}{\hat{h}'_i(x)} - \frac{S_{p_i}(\hat{h}_i(x))\hat{h}'_{i-1}(x)}{\hat{h}'_i(x)}, \tag{46}
\]

where

\[
\frac{z_\pm'(x)}{\hat{h}'_i(x)} = z_1(\hat{h}_i(x), \hat{h}_{i-1}(x), V) + z_2(\hat{h}_i(x), \hat{h}_{i-1}(x), V) \frac{\hat{h}'_{i-1}(x)}{\hat{h}'_i(x)}. \tag{47}
\]

We will use this relation later.

Let \(p \geq 1, 1 \leq l \leq p\). Define

\[
I_{p,l} := \left\{ 2\cos \frac{l + c}{p + 1} : |c| \leq \frac{1}{10} \text{ and } |S_{p+1}(2\cos \frac{l + c}{p + 1})| \leq \frac{1}{4} \right\}.
\]

The following property is proved in [8].

**Proposition 6.1.** Assume \(V \geq 20\). Let \((\tilde{B}_i)_{i=0}^m\) be a modified ladder, \((\hat{h}_i)_{i=0}^m\) the corresponding generating polynomials, and \((p_i)_{i=0}^{m-1}\), \((l_i)_{i=0}^{m-1}\) the type sequence and index sequence respectively. Then for any \(0 \leq i < m\)

\[
\hat{h}_i(\tilde{B}_{i+1}) \subset I_{p_i,l_i}.
\]

We collect some useful estimations of Chebyshev polynomials on the interval \(I_{p,l}\), which is essentially the Proposition 7 of [13].

**Proposition 6.2 ([13]).** Fix \(p \geq 1, 1 \leq l \leq p\). For any \(t \in I_{p,l}\),

\[
|S_{p+1}(t)| \leq \frac{1}{4}, \quad |S_p(t)| \leq \frac{5}{4}, \quad \frac{p+1}{4} \csc^2 \frac{lt}{p+1} \leq |S'_{p+1}(t)| \leq (p+1) \csc^2 \frac{lt}{p+1}, \quad |S'_p(t)| \leq 2|S'_{p+1}(t)|, \quad |S''_{p+1}(t)| \leq 4p^2 \csc^3 \frac{lt}{p+1}, \quad |S''_p(t)| \leq 4p^2 \csc^3 \frac{lt}{p+1}.
\]

The following result shown in [8] will also be useful later.
Proposition 6.3. Assume $V ≥ 20$. Let $(\hat{B}_i)_{i=0}^m$ be a modified ladder, $(\hat{h}_i)_{i=0}^m$, $(p_i)_{i=0}^{m-1}$ and $(l_i)_{i=0}^{m-1}$ be the corresponding generating polynomials, type sequence and index sequence. For any $0 ≤ i < m$, $x ∈ \hat{B}_{i+1}$, we have,

$$\frac{V - 8}{3}(p_i + 1)\csc^2 \frac{l_i \pi}{p_i + 1} ≤ \left| \frac{\hat{h}'_{i+1}(x)}{\hat{h}'_i(x)} \right| ≤ \frac{(V + 5)(p_i + 1)\csc^2 \frac{l_i \pi}{p_i + 1}}{(V - 8)(p_i + 1)}.$$ 

Proof of Lemma 3.7. Given $w ∈ \Omega_n$. Consider the initial ladder $(B_i)_{i=0}^n$ with $B_0$ the unique band in $\mathcal{G}_0$ containing $B_w$ and $B_n = B_w$. Let $(\hat{B}_i)_{i=0}^m$ be the related modified ladder and $(\hat{h}_i)_{i=0}^m$, $(p_i)_{i=0}^{m-1}$ and $(l_i)_{i=0}^{m-1}$ be the corresponding generating polynomials, type sequence and index sequence. Since $\hat{h}_m(\hat{B}_m) = [-2, 2]$, there exists $x_0 ∈ \hat{B}_m$ such that $|\hat{h}_m'(x_0)||\hat{B}_m| = 4$. Notice also that $|\hat{h}_0| = 1$(see the explanation after Definition 1), then by Proposition 6.3, the definition of modified ladder and (41)

$$|B_w| = |\hat{B}_m| = 4\frac{|\hat{h}_0(x_0)|}{|\hat{h}_m'(x_0)|} ≤ 4 \prod_{i=0}^{m-1} \frac{3\sin^2 \frac{l_i \pi}{p_i + 1}}{(V - 8)(p_i + 1)}$$

$$≤ 4 \prod_{e_j = e_{12}} \frac{1}{(2t_1)^{a_j - 1}} \cdot \prod_{e_j \neq e_{12}} \frac{1}{(\tau_{e_j}(a_j) + 1)t_1}$$

$$≤ 4 \prod_{e_j = e_{12}} \frac{1}{t_1^{a_j - 1}} \cdot \prod_{e_j \neq e_{12}} \frac{1}{a_j t_1}.$$ 

Similarly by using the facts that $\sin x ≥ 2x/\pi$ for $x ∈ [0, \pi/2]$ and $\tau(e(n) + 1 ≤ 3n$ we have

$$|B_w| ≥ 4 \prod_{i=0}^{m-1} \frac{\sin^2 \frac{l_i \pi}{p_i + 1}}{(V + 5)(p_i + 1)}$$

$$≥ 4 \prod_{e_j = e_{12}} \frac{1}{(2(V + 5))^{a_j - 1}} \cdot \prod_{e_j \neq e_{12}} \frac{4}{(\tau_{e_j}(a_j) + 1)^3(V + 5)}$$

$$≥ \prod_{e_j = e_{12}} \frac{1}{t_2^{a_j - 1}} \cdot \prod_{e_j \neq e_{12}} \frac{1}{a_j^3 t_2}.$$ 

Notice that $a_j ≥ 1$ and $t_1 ≥ 4$ since $V ≥ 20$. Also notice that any two consecutive initial rungs can not be type $I$ simultaneously, we get

$$|B_w| ≤ 4 \prod_{e_j \neq e_{12}} \frac{1}{a_j t_1} ≤ 4 \cdot 4^{-n/2},$$

which implies the result.

\[\square\]
The following properties is fundamental for the proof of bounded variation, bounded covariation and continuity of pre-dimensions.

**Proposition 7.1.** Assume \( V, \tilde{V} \geq 20 \). Let \((\hat{B}_i^m)_{i=0}^m \) and \((\tilde{B}_i^m)_{i=0}^m \) be modified ladders of \( \Sigma_{\alpha,V} \) and \( \Sigma_{\alpha,\tilde{V}} \) respectively with the same type sequence \((p_i)_{i=0}^{m-1}\) and index sequence \((l_i)_{i=0}^{m-1}\). Let \((\hat{h}_i)_{i=0}^m \) and \((\tilde{h}_i)_{i=0}^m \) be their corresponding generating polynomials. Let \( x_1 \in \hat{B}_m, \ x_2 \in \tilde{B}_m \). Write \( q_i = p_i + 1 \) and \( \theta_i = l_i\pi/q_i \).

1. If there exist constants \( c_1, c_2 > 1 \) and \( \lambda > 1 \) such that
\[
|\hat{h}_i(x_1) - \tilde{h}_i(x_2)| \leq q_i^{-1}\sin^2 \theta_i \left( c_1\lambda^{-i} + c_2|V - \tilde{V}| \right), \quad \forall \ 0 \leq i < m, \quad (48)
\]
Then there exists constants \( \xi, M \geq 1 \) such that
\[
\frac{|\hat{h}'_m(x_1)/\hat{h}_0(x_1)|}{|\hat{h}'_m(x_2)/\hat{h}_0(x_2)|} \leq \xi \exp(Mm|V - \tilde{V}|). \quad (49)
\]
More explicitly we can take
\[
\xi = \exp \left( (V + \tilde{V} + 130 + (4V + 53)c_1)(3 + \frac{\lambda^2}{(\lambda-1)^2}) \right), \quad M = (8V + 56)c_2 + 6. \quad (50)
\]

2. If there exist constants \( c_1, \lambda > 1 \) such that
\[
|\hat{h}_i(x_1) - \tilde{h}_i(x_2)| \leq c_1 q_i^{-1}\sin^2 \theta_i \lambda^{-m+i}, \quad \forall \ 0 < i < m. \quad (51)
\]
Then there exists a constant \( \xi \geq 1 \) such that
\[
\frac{|\hat{h}'_m(x_1)/\hat{h}_0(x_1)|}{|\tilde{h}'_m(x_2)/\tilde{h}_0(x_2)|} \leq \xi \exp(6m|V - \tilde{V}|). \quad (52)
\]
Moreover \( \xi \) take the same form as in (50).

**Proof.** To simplify the notation we write
\[
\Gamma_i := \left| \frac{\hat{h}'_{i+1}(x_1) - \tilde{h}'_{i+1}(x_2)}{\hat{h}'_i(x_1) - \tilde{h}'_i(x_2)} \right| \quad \text{and} \quad \Delta_i := |\hat{h}_i(x_1) - \tilde{h}_i(x_2)|.
\]
We will prove that for any \( 0 < i < m \),
\[
\Gamma_i \leq (4V + 23)q_i^2 \csc^3 \theta_i \cdot \Delta_i + 5q_i \csc^2 \theta_i \cdot \Delta_{i-1} + (6q_i^2)^{-1} \sin^4 \theta_{i-1} \cdot \Gamma_{i-1} + 3|V - \tilde{V}| q_i \csc^2 \theta_i. \quad (53)
\]
We show first that (53) implies (49) and (52). Notice that by Proposition 6.3
\( \Gamma_0 \leq q_0 \csc^2 \theta_0 (V + \bar{V} + 10). \) (54)
(49) and (52) can be shown through the following two claims.
Let \( \bar{M} = V + \bar{V} + 130 + (4V + 53)c_1, \) \( M_1 = (4V + 28)c_2 + 3 \) and \( M = 2M_1. \)

**Claim 1:** If (48) holds, then for \( 0 < i < m \)
\[ \Gamma_i \leq 2q_i \csc^2 \theta_i \left( \bar{M}(i + 1)(6^{-i} + \lambda^{-i}) + M_1|V - \bar{V}| \sum_{l=0}^{i-1} 6^{-l} \right). \]

Assume first \( 1 < \lambda \leq 6, \) we will show by induction
\[ \Gamma_i \leq q_i \csc^2 \theta_i \left( \bar{M}(i + 1)(6^{-i} + \lambda^{-i}) + M_1|V - \bar{V}| \sum_{l=0}^{i-1} 6^{-l} \right). \]

It is trivially \( \Delta_0 \leq 4. \) By using (53) and (48) for \( i = 1, \) together with (54), we get
\[ \Gamma_1 \leq (4V + 23)q_1^2 \csc^3 \theta_1 \cdot \Delta_1 + 5q_1 \csc^2 \theta_1 \cdot \Delta_0 \\
+ (6q_0^2)^{-1} \sin^4 \theta_0 \cdot \Gamma_0 + 3|V - \bar{V}|q_1 \csc^2 \theta_1 \\
\leq (4V + 23)q_1 \csc \theta_1 \left( c_1 \lambda^{-1} + c_2 |V - \bar{V}| \right) + 20q_1 \csc^2 \theta_1 \\
+ (V + \bar{V} + 10)/6 + 3|V - \bar{V}|q_1 \csc^2 \theta_1 \\
\leq q_1 \csc^2 \theta_1 \left( 2\bar{M}(6^{-1} + \lambda^{-1}) + M_1|V - \bar{V}| \right). \]

Assume \( i > 1 \) and the statement holds for \( i - 1. \) By (53), (48) and 1 \( < \lambda \leq 6, \) we get
\[ \Gamma_i \leq (4V + 23)q_i \csc \theta_i \left( c_1 \lambda^{-i} + c_2 |V - \bar{V}| \right) \\
+ 5q_i \csc^2 \theta_i \left( c_1 \lambda^{-i+1} + c_2 |V - \bar{V}| \right) \\
+ \frac{1}{6} \left( \bar{M}i(6^{-i+1} + \lambda^{-i+1}) + M_1|V - \bar{V}| \sum_{l=0}^{i-2} 6^{-l} \right) + 3|V - \bar{V}|q_i \csc^2 \theta_i \\
\leq q_i \csc^2 \theta_i \left( \bar{M}(i + 1)(6^{-i} + \lambda^{-i}) + M_1|V - \bar{V}| \sum_{l=0}^{i-1} 6^{-l} \right). \]

Thus the result holds for \( i. \)

Now if \( \lambda > 6, \) then (48) fulfills for \( \lambda_0 = 6. \) Thus by what have been proven we get
\[ \Gamma_i \leq q_i \csc^2 \theta_i \left( \bar{M}(i + 1)(6^{-i} + \lambda_0^{-i}) + M_1|V - \bar{V}| \sum_{l=0}^{i-1} 6^{-l} \right) \]
\[ \leq 2q_i \csc^2 \theta_i \left( \tilde{M}(i + 1)(6^{-i} + \lambda^{-i}) + M_1|V - \tilde{V}| \sum_{l=0}^{i-1} 6^{-l} \right). \]

Claim 2: If (51) holds, then for \(0 < i < m\)

\[ \Gamma_i \leq q_i \csc^2 \theta_i \left( \tilde{M}(6^{-i} + \lambda^{-i-m}) + 3|V - \tilde{V}| \sum_{l=0}^{i-1} 6^{-l} \right). \]

We show it by induction. By (53) and (51) for \(i = 1\), together with (54),

\[ \Gamma_1 \leq (4V + 23)q_1^2 \csc^3 \theta_1 \cdot \Delta_1 + 5q_1 \csc^2 \theta_1 \cdot \Delta_0 + (6q_0^2)^{-1} \sin^4 \theta_0 \cdot \Gamma_0 + 3|V - \tilde{V}||q_1 \csc^2 \theta_1 \]

\[ \leq q_1 \csc^2 \theta_1 \left( \tilde{M}(6^{-1} + \lambda^{-1-m}) + 3|V - \tilde{V}| \right). \]

Assume \(i > 1\) and the statement holds for \(i - 1\). By (53) and induction, if the condition (51) holds, we can get

\[ \Gamma_i \leq (4V + 23)c_1 q_i \csc \theta_i \lambda^{i-m} + 5c_1 q_i \csc^2 \theta_i \lambda^{i-1-m} + \tilde{M} \frac{1}{6} (6^{i-1} + \lambda^{i-1-m}) \]

\[ + 3|V - \tilde{V}| \sum_{l=1}^{i-1} 6^{-l} + 3|V - \tilde{V}||q_i \csc^2 \theta_i \]

\[ \leq q_i \csc^2 \theta_i \left( \tilde{M}(6^{-i} + \lambda^{-i-m}) + 3|V - \tilde{V}| \sum_{l=0}^{i-1} 6^{-l} \right), \]

so the conclusion holds.

The following inequality is basic for us: for any \(x, y > 0\), we have

\[ |\ln y - \ln x| \leq (x \land y)^{-1}|y - x|. \]  

(55)

As what have been done for Cookie-cutter set, we have

\[ \left| \ln \frac{\hat{h}_i'(x_1)}{\hat{h}_i'(x_2)} - \ln \frac{\hat{h}_i'(x_1)}{\hat{h}_i'(x_2)} \right| \]

\[ \leq \sum_{i=0}^{m-1} \left| \ln \frac{\hat{h}_{i+1}'(x_1)}{\hat{h}_i'(x_1)} - \ln \frac{\hat{h}_{i+1}'(x_2)}{\hat{h}_i'(x_2)} \right| \]

\[ \leq \sum_{i=0}^{m-1} \left| \ln \frac{\hat{h}_{i+1}'(x_1)}{\hat{h}_i'(x_1)} - \ln \frac{\hat{h}_{i+1}'(x_2)}{\hat{h}_i'(x_2)} \right| \]

\[ \leq \sum_{i=0}^{m-1} \frac{3}{V \land V - 8} q_i \Gamma_i \]  

(56)
\[
\sum_{i=0}^{m-1} \frac{\sin^2 \theta_i}{q_i} \Gamma_i \leq \frac{1}{4} \sum_{i=0}^{m-1} \frac{\sin^2 \theta_i}{q_i} \Gamma_i
\]

\[
\leq \tilde{M} (1 + 36/25 + \lambda^2/(\lambda - 1)^2) + \begin{cases} 
Mm|V - \tilde{V}|, & \text{by claim 1} \\
6m|V - \tilde{V}|, & \text{by claim 2},
\end{cases}
\]

where (56) is due to (55) and Proposition 6.3. (57) is due to $V, \tilde{V} > 20$ and (58) is due to (54) and the two claims above. Consequently (49) and (52) follow.

Now fix $0 < i < m$, we are going to prove (53).

For convenience, we denote $z_\pm (\hat{h}_i(x_1), \hat{h}_{i-1}(x_1), V)$ as $z_\pm (x_1)$, and denote $z_\pm (\hat{h}_i(x_2), \hat{h}_{i-1}(x_2), \tilde{V})$ as $z_\pm (x_2)$. Both $(\hat{h}_{i+1}(x_1), \hat{h}_i(x_1), \hat{h}_{i-1}(x_1))$, and $(\hat{h}_{i+1}(x_2), \hat{h}_i(x_2), \hat{h}_{i-1}(x_2))$ satisfy (45) with the same $p_i$, so by using (46), the quantity

\[
\frac{\hat{h}'_{i+1}(x_1)}{\hat{h}'(x_1)} - \frac{\hat{h}'_{i+1}(x_2)}{\hat{h}'(x_2)}
\]

is equal to

\[
[S_{p_i+1}(\hat{h}_i(x_1)) - S_{p_i+1}(\hat{h}_i(x_2))]z_\pm (x_1) + [z_\pm (x_1) - z_\pm (x_2)]S_{p_i+1}(\hat{h}_i(x_2))
\]

\[
- [S_{p_i}(\hat{h}_i(x_1)) - S_{p_i}(\hat{h}_i(x_2))] \hat{h}_{i-1}(x_1) - [\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)]S_{p_i}'(\hat{h}_i(x_2))
\]

\[
+ [S_{p_i+1}(\hat{h}_i(x_1)) - S_{p_i+1}(\hat{h}_i(x_2))] \frac{z'(x_1)}{z'(x_1)} + [\frac{z'(x_1)}{z'(x_1)} - \frac{z'(x_2)}{z'(x_2)}]S_{p_i+1}(\hat{h}_i(x_2))
\]

\[
- [S_{p_i}(\hat{h}_i(x_1)) - S_{p_i}(\hat{h}_i(x_2))] \frac{\hat{h}'_{i-1}(x_1)}{\hat{h}'(x_1)} - [\frac{\hat{h}'_{i-1}(x_1)}{\hat{h}'(x_1)} - \frac{\hat{h}'_{i-1}(x_2)}{\hat{h}'(x_2)}]S_{p_i}(\hat{h}_i(x_2)).
\]

There are eight terms in (59), we will estimate them one by one.

By proposition 6.1, $\hat{h}_i(x_1), \hat{h}_i(x_2) \in I_{p_i, i_1}$, thus by Proposition 6.2

\[
\begin{align*}
|S_{p_i+1}(\hat{h}_i(x_1)) - S_{p_i+1}(\hat{h}_i(x_2))| & \leq q_i \csc^2 \theta_i \Delta_i \\
|S'_{p_i+1}(\hat{h}_i(x_1)) - S'_{p_i+1}(\hat{h}_i(x_2))| & \leq 4q_i^2 \csc^3 \theta_i \Delta_i \\
|S_{p_i}(\hat{h}_i(x_1)) - S_{p_i}(\hat{h}_i(x_2))| & \leq 2q_i \csc^2 \theta_i \Delta_i \\
|S'_{p_i}(\hat{h}_i(x_1)) - S'_{p_i}(\hat{h}_i(x_2))| & \leq 4q_i^2 \csc^3 \theta_i \Delta_i.
\end{align*}
\]
By \((12)\) and \((17)\), we have

\[
|z_\pm(x_1) - z_\pm(x_2)| \leq \Delta_i + \Delta_{i-1} + |V - \tilde{V}|
\]

\[
\left| \frac{z'_\pm(x_1)}{\hat{h}'_i(x_1)} - \frac{z'_\pm(x_2)}{\hat{h}'_i(x_2)} \right| \leq \left| z_1(x_1) - z_2(x_2) \right| + \left| \frac{\hat{h}_{i-1}(x_1)}{\hat{h}'_i(x_1)} \right| \left| \hat{h}_{i-1}(x_1) \right|
\]

and

\[
\left| z_1(\hat{h}_i(x_1), \hat{h}_{i-1}(x_1), V) - z_1(\hat{h}_i(x_2), \hat{h}_{i-1}(x_2), \tilde{V}) \right| \leq \Delta_i + \Delta_{i-1} + |V - \tilde{V}|
\]

\[
\left| z_2(\hat{h}_i(x_1), \hat{h}_{i-1}(x_1), V) - z_2(\hat{h}_i(x_2), \hat{h}_{i-1}(x_2), \tilde{V}) \right| \leq \Delta_i + \Delta_{i-1} + |V - \tilde{V}|.
\]

By a direct computation and Proposition 6.3,

\[
\left| \frac{\hat{h}'_{i-1}(x_1)}{\hat{h}'_i(x_1)} - \frac{\hat{h}'_{i-1}(x_2)}{\hat{h}'_i(x_2)} \right| = \left| \frac{\hat{h}'_{i-1}(x_1) \hat{h}'_{i-1}(x_2)}{\hat{h}'_i(x_1) \hat{h}'_i(x_2)} \right| \Gamma_{i-1} \leq \frac{9 \sin^4 \theta_{i-1}}{(V - 8)(V - 8)q_i^2 \Delta_i} \Gamma_{i-1}.
\]

Now we estimate the eight terms in (59) one by one. By (42) and (60), the first term is bounded by

\[4(V + 2)q_i^2 \csc^3 \theta_i \Delta_i.\]

By (61) and Proposition 6.2, the second term is bounded by

\[q_i \csc^2 \theta_i (\Delta_i + \Delta_{i-1} + |V - \tilde{V}|).\]

By (60) and \(|\hat{h}_{i-1}(x_1)| \leq 2\), the third term is bounded by

\[8q_i^2 \csc^3 \theta_i \Delta_i.\]

By Proposition 6.2 the 4th term is bounded by

\[2q_i \csc^2 \theta_i \Delta_{i-1}.\]

By (60) and (61), the 5th term is bounded by

\[2q_i \csc^2 \theta_i \Delta_i.\]
By (12) and Proposition 5.2, \(|z_2(x_2)| \leq 1\) and \(|S_{p_1+1}(\hat{h}_i(x_2))| \leq 1/4\), then by \(\hat{h}'_{i-1}(x_1) / \hat{h}'_i(x_1) | \leq 1/4\), (61), (62) and (63), the 6th term is bounded by

\[
2 \left( \Delta_i + \Delta_{i-1} + |V - \tilde{V}| \right) + \frac{9\sin^4 \theta_{i-1}}{(V - 8)(V - 8)q_{i-1}^2} \Gamma_{i-1}.
\]

By (60) and \(|\hat{h}'_{i-1}(x_1)| / |\hat{h}'_i(x_1)| \leq 1/4\), the 7th term is bounded by

\[
2q_i \csc \theta_i \Delta_i.
\]

By Proposition 6.2 and (63), the 8th term is bounded by

\[
\frac{45\sin^4 \theta_{i-1}}{4(V - 8)(V - 8)q_{i-1}^2} \Gamma_{i-1}.
\]

Take sum on the eight bounds, we get (53). This proves the proposition. □

**Proof of Theorem 3.1** Let

\[
B_n \subset B_{n-1} \subset \cdots \subset B_0
\]

be a sequence of spectral generating bands (with orders from \(n\) to 0), which form an initial ladder. Let \((\hat{B}_i)_{i=0}^m\) be the corresponding modified ladder, \((\hat{h}_i)_{i=0}^m\) the corresponding generating polynomials. Note that \(\hat{B}_0 = B_0\) and \(\hat{h}_0 \equiv 1\). To apply Proposition 7.1, we only need to verify (51).

Let \(\lambda := (V - 8)/3\). For any \(0 \leq i < m\) and \(x, y \in \hat{B}_{i+1}\), since \(\hat{h}_i\) is monotone on \(\hat{B}_i\), we have

\[
|\hat{h}_i(x) - \hat{h}_i(y)| = \left| \int_x^y \frac{\hat{h}'_i(t)}{\hat{h}'_{i+1}(t)} \hat{h}'_{i+1}(t) dt \right|
\]

\[
\leq |\lambda| q_{i-1} \sin^2 \theta_i \left| \int_x^y \hat{h}'_{i+1}(t) dt \right|
\]

\[
= |\lambda| q_{i-1} \sin^2 \theta_i |\hat{h}_{i+1}(x) - \hat{h}_{i+1}(y)|,
\]

where the inequality is due to Proposition 6.3. Since \(\hat{h}_m(\hat{B}_m) = [-2, 2]\), for any \(x_1, x_2 \in \hat{B}_m\), \(|\hat{h}_m(x_1) - \hat{h}_m(x_2)| \leq 4\), hence we have

\[
|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \leq 4\lambda^{i-m} \prod_{l=i}^{m-1} q_{l-1} \sin^2 \theta_l \leq 4q_{i-1} \sin^2 \theta_i \lambda^{i-m}, \quad \forall 0 \leq i < m.
\]

Now by Proposition 7.1 the result follows for some constant \(\xi\) which only depends on \(V\). More explicitly notice that \(\lambda > 4\), then by (50) we can take

\[
\xi = \exp (180V).
\]

□
Proof of Corollary 3.2 Write $B = [a, b]$. We know that $h$ is monotone on $B$ and $h(B) = [-2, 2]$. By mean value theorem, there exists $x_0 \in B$ such that
\[ 4 = |h(a) - h(b)| = |h'(x_0)||B|. \]
By Proposition 3.1 the result follows. □

8. Bounded covariation

Proposition 8.1. Assume $V, \tilde{\nu} \geq 24$. Suppose that $(\hat{B}_i)_{i=0}^m$ and $(\hat{\tilde{B}}_i)_{i=0}^m$ are modified ladders of $\Sigma_{\alpha, V}$ and $\Sigma_{\alpha, \tilde{\nu}}$ respectively with the same type sequence $(p_i)_{i=0}^{m-1}$, and the same index sequence $(l_i)_{i=0}^{m-1}$. Let $(\hat{h}_i)_{i=0}^m$ and $(\hat{\tilde{h}}_i)_{i=0}^m$ be their corresponding generating polynomials. Then we have

(i) There exists positive constant $c$ depending only on $V$ with $c \leq 18/37$ such that for any $0 < i < m$ and any $x_1 \in \hat{B}_{i+1}$, $x_2 \in \hat{\tilde{B}}_{i+1},$
\[ \Delta_i \leq q_i^{-1} \sin^2 \theta_i \left( c\Delta_{i+1} + c\Delta_{i-1} + |V - \tilde{\nu}| \right). \] (64)
where $\Delta_i = |\hat{h}_i(x_1) - \hat{\tilde{h}}_i(x_2)|$, $q_i = p_i + 1$ and $\theta_i = l_i \pi/q_i.$

(ii) Let $\lambda = \frac{1 + \sqrt{1 - 4x_1^2}}{2x_1}$. Then there exist absolute constants $c_1, c_2 > 1$
such that for any $x_1 \in \hat{B}_m$, there exists $x_2 \in \hat{\tilde{B}}_m$ such that,
\[ \Delta_i \leq q_i^{-1} \sin^2 \theta_i \left( c_1 \lambda^{-i} + c_2 |V - \tilde{\nu}| \right), \quad 0 < i < m. \] (65)

(iii) There exists absolute constants $C_1, C_2, C_3 > 1$ such that
\[ \eta^{-1} \frac{|\hat{B}_m|}{|\hat{B}_0|} \leq \frac{|\hat{\tilde{B}}_m|}{|\hat{\tilde{B}}_0|} \leq \eta \frac{|\hat{\tilde{B}}_m|}{|\hat{B}_0|}, \]
where $\eta = C_1 \exp \left( C_2(V + \tilde{\nu}) + C_3m|V - \tilde{\nu}| \right).$

Proof. (i) Take $0 < i < m$ and $x_1 \in \hat{B}_{i+1}$, $x_2 \in \hat{\tilde{B}}_{i+1}.$

For convenience, we denote $z_\pm(\hat{h}_i(x_1), \hat{h}_{i-1}(x_1), V)$ as $z_\pm(x_1)$, and also denote $z_\pm(\hat{\tilde{h}}_i(x_2), \hat{\tilde{h}}_{i-1}(x_2), V)$ as $z_\pm(x_2)$. Both $(\hat{h}_{i+1}(x_1), \hat{h}_i(x_1), \hat{h}_{i-1}(x_1))$, and $(\hat{\tilde{h}}_{i+1}(x_2), \hat{\tilde{h}}_i(x_2), \hat{\tilde{h}}_{i-1}(x_2))$ satisfy (15) with the same $p_i$. So, we have
\[
\hat{h}_{i+1}(x_1) - \hat{h}_{i+1}(x_2) = z_\pm(x_1) [S_{p_{i+1}}(\hat{h}_i(x_1)) - S_{p_{i+1}}(\hat{\tilde{h}}_i(x_2))] + [z_\pm(x_1) - z_\pm(x_2)] S_{p_{i+1}}(\hat{\tilde{h}}_i(x_2)) - \hat{h}_{i-1}(x_1) [S_{p_i}(\hat{h}_i(x_1)) - S_{p_i}(\hat{\tilde{h}}_i(x_2))] - [\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)] S_{p_i}(\hat{\tilde{h}}_i(x_2)).
\]
By Proposition 6.1, \( \hat{h}_i(x_1), \hat{h}_i(x_2) \in I_{p_i,l_i} \), then by Proposition 6.2,
\[
\left| S_{p_i+1}(\hat{h}_i(x_1)) - S_{p_i+1}(\hat{h}_i(x_2)) \right| \geq \frac{q_i}{3} \cdot \csc \theta_i \cdot \Delta_i.
\]
By Proposition 6.2 again,
\[
\left| S_{p_i}(\hat{h}_i(x_1)) - S_{p_i}(\hat{h}_i(x_2)) \right| \leq 2q_i \csc \theta_i \Delta_i.
\]
So by Proposition 6.2, (61) and the above three formulas, we have
\[
\Delta_{i+1} \geq (\frac{V - 2}{3} - 4)q_i \csc \theta_i \Delta_i - \frac{1}{4} \Delta_i - \frac{3}{2} \Delta_{i-1} - \frac{1}{4} |V - \tilde{V}|.
\]
Let \( c := 18/(4V - 59) \), then \( 0 < c \leq 18/37 \) since \( V \geq 24 \). We get
\[
\Delta_i \leq q_i \sin^2 \theta_i \left( c \Delta_{i+1} + c \Delta_{i-1} + |V - \tilde{V}| \right).
\]

(ii) Take any \( x_1 \in \hat{B}_m \). By \( \hat{h}_m(\hat{B}_m) = [-2, 2], \hat{\tilde{h}}_m(\hat{\tilde{B}}_m) = [-2, 2] \), there exists \( x_2 \in \hat{\tilde{B}}_m \) such that
\[
\hat{h}_m(x_1) = \hat{h}_m(x_2).
\]
Thus \( \Delta_m = 0 \). Take any integer \( i \in \{0, \cdots, m - 1\} \). By \( \hat{h}_i(\hat{B}_m) \subset [-2, 2], \hat{\tilde{h}}_i(\hat{\tilde{B}}_m) \subset [-2, 2] \), we get
\[
\Delta_i = |\hat{h}_i(x_1) - \hat{\tilde{h}}_i(x_2)| \leq 4.
\]

(64) implies that for \( 0 < i < m \)
\[
\Delta_i \leq c(\Delta_{i+1} + \Delta_{i-1}) + |V - \tilde{V}|.
\]
Notice that \( \lambda = \frac{1+\sqrt{1-4c^2}}{2c} \geq 37/36 \) is the larger root of \( x^2 - x/c + 1 = 0 \). Write \( c' = \lambda + \lambda^{-1} \), hence (66) can be rewritten as
\[
\lambda \Delta_i - \Delta_{i+1} \leq \lambda^{-1}(\lambda \Delta_{i-1} - \Delta_i) + c'|V - \tilde{V}|.
\]

Claim: For \( 0 < i < m \) we have
\[
\Delta_i \leq 8 \lambda^{-i} \sum_{k=0}^{\infty} \lambda^{-2k} + c'|V - \tilde{V}| \sum_{k=1}^{\infty} k \lambda^{-k}.
\]
\( \Leftrightarrow \) We show it by induction.

At first by using (67), for \( 0 < i < m \) we can get
\[
\lambda \Delta_i - \Delta_{i+1} \leq \lambda^{-i}(\lambda \Delta_0 - \Delta_1) + c'|V - \tilde{V}| \sum_{k=0}^{i-1} \lambda^{-k}.
\]
Take \( i = m - 1 \) and notice that \( \Delta_i \leq 4, \Delta_m = 0 \) we get

\[
\Delta_{m-1} \leq 8\lambda^{1-m} + c'|V - \tilde{V}| \sum_{j=1}^{m-1} \lambda^{-j}.
\]

Assume the result holds for \( i + 1 \). Then

\[
\Delta_i \leq \lambda^{-i} \left( \Delta_{i+1} + \lambda^{-i}(\lambda\Delta_0 - \Delta_1) + c'|V - \tilde{V}| \sum_{k=0}^{i-1} \lambda^{-k} \right)
\]

\[
\leq 8\lambda^{-i} \sum_{k=0}^{\infty} \lambda^{-2k} + c'|V - \tilde{V}| \sum_{k=1}^{\infty} k\lambda^{-k}.
\]

Thus for \( 0 < i < m \) we have

\[
\Delta_i \leq M_1\lambda^{-i} + c'M_2|V - \tilde{V}|
\]

with \( M_1 = 8\lambda^2/(\lambda^2 - 1) \) and \( M_2 = \sum_{k=1}^{\infty} k\lambda^{-k} = \lambda/(\lambda - 1)^2 \). Since \( \Delta_0 \leq 4 \) and \( \Delta_m = 0 \), the inequality also holds for \( i = 0, m \).

Consequently for \( 0 < i < m \), by (64) we get

\[
\Delta_i \leq q_i^{-1} \sin^2 \theta_i \left( c\Delta_{i+1} + c\Delta_{i-1} + |V - \tilde{V}| \right)
\]

\[
\leq q_i^{-1} \sin^2 \theta_i \left( M_1 c(\lambda^{-(i+1)} + \lambda^{-i-1}) + (1 + 2cc'M_2)|V - \tilde{V}| \right)
\]

\[
\leq q_i^{-1} \sin^2 \theta_i \left( M_1 \lambda^{-i} + (1 + c'M_2)|V - \tilde{V}| \right).
\]

Recall that \( \lambda \geq 37/36 =: \lambda_0 > 1 \), thus

\[
M_1 \leq \frac{8\lambda^2}{\lambda^2 - 1} =: c_1 \quad \text{and} \quad 1 + c'M_2 \leq 1 + \frac{\lambda^2_0 + 1}{(\lambda_0 - 1)^2} =: c_2.
\]

Consequently (65) holds with two absolute constants \( c_1 \) and \( c_2 \).

(iii) Proposition 7.1 and (65) imply that there exist absolute constants \( C_1', C_2', C_3' > 1 \) such that, for any \( \tilde{x} \in \hat{B}_m \), there exists \( \hat{y} \in \hat{B}_m \) such that

\[
\xi_1^{-1} \leq \left| \frac{\hat{h}_m'(\tilde{x})/\hat{h}_0'(\tilde{x})}{\hat{h}_m'(\hat{y})/\hat{h}_0'(\hat{y})} \right| < \xi_1,
\]

where \( \xi_1 = C_1' \exp \left( C_2'(V + \tilde{V}) + C_3'Vm|V - \tilde{V}| \right) \).

By the definition of generating polynomial, there exist \( x_1 \in \hat{B}_m, x_2 \in \hat{B}_0 \) such that

\[
|\hat{B}_m| |\hat{h}_m'(x_1)| = 4, \ |\hat{B}_0| |\hat{h}_0'(x_2)| = 4.
\]
By Theorem 3.1 and 3.2, we have
\[ \left| \hat{B}_m \right| = \left| B_m \right| \left| h_m'(x_1) \right| \left| \hat{h}_m'(\hat{x}) \right| \leq 16 \exp(720V) \left| \frac{\hat{h}_0'(\hat{x})}{h_m'(\hat{x})} \right|. \]

By the same discussion, we have
\[ \frac{\left| \hat{B}_m \right|}{\left| B_0 \right|} \geq \frac{1}{16} \exp(-720\hat{V}) \left| \frac{\hat{h}_0'(\hat{y})}{\hat{h}_m'(\hat{y})} \right|. \]

Then by (68), we have
\[ \frac{\left| \hat{B}_m \right|}{\left| B_0 \right|} \leq \frac{\eta}{\left| B_0 \right|} \left| \hat{B}_m \right| \]
with \( \eta = C_1 \exp \left( C_2(V + \hat{V}) + C_3Vm|V - \hat{V}| \right) \), where \( C_1, C_2, C_3 \) are still absolute constants.

The opposite direction of the inequality can be got by the same way. \( \square \)

**Proof of Theorem 3.3.** This is a direct consequence of Proposition 8.1 (iii).
\( \square \)

**Proof of Corollary 3.4.** For each \( n \in \mathcal{N} \), fix some \( w^{(n)} \in \Omega_{i_n} \) such that \( B_{w^{(n)}} \) is of type I and \( a_{i_n+1} = n \). Then define
\[ \zeta_n := \left| \frac{B_{w^{(n)}u}}{B_{w^{(n)}}} \right|. \] (69)

By applying Theorem 3.3, we get the result. If moreover \( a_{i_n+1} = 1 \), then we know that \( B_{w^{(n)}u} = B_{w^{(n)}} \), thus \( \zeta_1 = 1 \).
\( \square \)

Now we can give the proof of Theorem 1.3.

**Proposition 8.2.** For \( V, \hat{V} \geq 24 \), there exists an absolute constant \( C > 0 \) such that
\[ |s_+(V) - s_+(\hat{V})| \leq CV|V - \hat{V}|, \]
\[ |s^-(V) - s^-(\hat{V})| \leq CV|V - \hat{V}|. \]

**Proof.** For any \( w \in \Omega_n \), let \( B_w \) and \( \tilde{B}_w \) be the related bands of \( \Sigma_{\alpha,V} \) and \( \Sigma_{\alpha,\hat{V}} \) respectively. Let \( (B_i)_{i=0}^n \) and \( (\tilde{B}_i)_{i=0}^n \) be the ladders of \( \Sigma_{\alpha,V} \) and \( \Sigma_{\alpha,\hat{V}} \) respectively with \( B_n = B_w \) and \( \tilde{B}_n = \tilde{B}_w \). Let \( (\hat{B}_i)_{i=0}^{m_w} \) and \( (\hat{\tilde{B}}_i)_{i=0}^{m_w} \) be the related modified ladders. Then by (43) we have \( m_w \geq n/2 \). By (21),
\[ |B_w|, |\tilde{B}_w| \leq 4^{1-m_w}. \]

By Proposition 8.1 (iii), there exists absolute constants \( C_1, C_2, C_3 > 1 \) such that
\[ \eta^{-1} \leq \frac{|B_w|}{|\tilde{B}_w|} \leq \eta, \]
where $\eta = C_1 \exp \left( C_2 (V + \tilde{V}) + C_3 V m_\omega |V - \tilde{V}| \right)$.

Write $s_n = s_n(V)$ and $\tilde{s}_n = s_n(\tilde{V})$. Let $d := \limsup_{n \to \infty} |s_n - \tilde{s}_n|$, then $d \leq 1$ and it is easy to show that

$$|s^*(V) - s^*(\tilde{V})|, |s^* (V) - s^*(\tilde{V})| \leq d.$$ 

If $d = 0$ the result holds trivially. So in the following we assume $d > 0$.

Then there exist infinitely many $n$ such that $s_n \geq \tilde{s}_n + d/2$ or $\tilde{s}_n \geq s_n + d/2$.

At first we assume that there are infinitely many $n$ such that $s_n \geq \tilde{s}_n + d/2$.

For those $n$ big enough we have

$$1 = \sum_{|w|=n} |B_w|^s_n \leq \sum_{|w|=n} |B_w|^\tilde{s}_n + d/2$$

$$\leq \sum_{|w|=n} \eta^{\tilde{s}_n + d/2} |\tilde{B}_n| \tilde{s}_n + d/2 \leq \sum_{|w|=n} \eta^{\tilde{s}_n + d/2} 4^{(1-m_w)d/2} |\tilde{B}_n| \tilde{s}_n$$

$$\leq C(V, \tilde{V}) \sum_{|w|=n} \exp\left[-m_w \left( d \ln 2 - C_3 V (\tilde{s}_n + d/2)|V - \tilde{V}| \right)\right]|\tilde{B}_n| \tilde{s}_n.$$

We claim that $d \ln 2 \leq 2C_3 V |V - \tilde{V}|$. In fact if otherwise, notice that $\tilde{s}_n, d \leq 1$ and $m_\omega \geq n/2$, we should get

$$1 \leq C(V, \tilde{V}) \exp[-C_3 n |V - \tilde{V}|/4] \sum_{|w|=n} |\tilde{B}_n| \tilde{s}_n = C(V, \tilde{V}) \exp[-C_3 n |V - \tilde{V}|/4],$$

which leads to contradiction for large $n$. So we have

$$d \leq \frac{2C_3 V |V - \tilde{V}|}{\ln 2}.$$ 

For the case that there are infinitely many $n$ such that $\tilde{s}_n \geq s_n + d/2$, the argument is the same. \sqcup \quad \Box$

**Proof of Theorem 1.3** It is a direct consequence of Proposition 8.2. \sqcup \quad \Box

9. **Gibbs like measure**

Throughout this section we take $V \geq 24$, $0 \leq \varepsilon < 1/12$ and consider the set $E_\varepsilon$ defined in (24). We will construct a Gibbs like measure on $E_\varepsilon$.

For any $m \geq k$, $T = I$, $II$, or $III$, define

$$\Omega_m^{(k,T)}(\varepsilon) = \{ w \in \Omega_m(\varepsilon) : e_{w_k} = (\ast, T) \},$$

where $\Omega_m(\varepsilon)$ is defined in (23). For any $0 < \beta < 1$ define

$$b_m^{(k,T)}(\varepsilon) = \sum_{w \in \Omega_m^{(k,T)}(\varepsilon)} |B_w|^\beta.$$
Fix $0 < \beta < 1$. At first we discuss the relationship between $b_{k-1,\beta}(\varepsilon)$ and $b_{k,\beta}(\varepsilon)$ (see (27) for definition). As a preparation we define the following sequence

$$A_{0,\beta}(\varepsilon) := 0, \quad A_{n,\beta}(\varepsilon) := \sum_{(n+1)\varepsilon < j < (n+1)(1-\varepsilon)} \frac{1}{(n+1)^{\beta}} \sin^{2\beta} \frac{j\pi}{n+1} \quad (n \geq 1).$$

**Lemma 9.1.** $A_{n,\beta} := A_{n,\beta}(0) \sim A_{n,\beta}(\varepsilon) \sim n^{1-\beta}$. And $A_{n,\beta} \sim A_{n+1,\beta}$ for $n \geq 1$.

**Proof.** Since

$$A_{n,\beta}(\varepsilon) = (n+1)^{1-\beta} \sum_{(n+1)\varepsilon < j < (n+1)(1-\varepsilon)} \frac{1}{n+1} \sin^{2\beta} \frac{j\pi}{n+1} \sim \frac{(n+1)^{1-\beta}}{\pi} \int_{\varepsilon \pi}^{(1-\varepsilon)\pi} \sin^{2\beta} x dx.$$

Since $\varepsilon < 1/12$, the result follows. \(\square\)

**Remark 2.** Here the constants related to “$\sim$” only depend on $\beta$.

**Proposition 9.2.** For any $k \geq 1$, we have

$$\frac{b_{k,\beta}(\varepsilon)}{b_{k,\beta}(\varepsilon)} \sim 1; \quad \frac{b_{k,\beta}(\varepsilon)}{b_{k,\beta}(\varepsilon)} \sim \frac{\zeta_{a_k}}{A_{a_k,\beta}}, \quad \frac{b_{k,\beta}(\varepsilon)}{b_{k,\beta}(\varepsilon)} \sim \begin{cases} 1 & a_k > 1, \\ \frac{\zeta_{a_{k-1}}}{\zeta_{a_{k-1},\beta}} & a_k = 1 \end{cases},$$

(70)

(where $\xi_n$ is defined in (69)) and

$$\frac{b_{k,\beta}(\varepsilon)}{b_{k-1,\beta}(\varepsilon)} \sim A_{a_k,\beta}.$$ (71)

Moreover the constants related to “$\sim$” only depend on $V$ and $\beta$.

**Proof.** By the definition and Lemma 9.1, $A_{n,\beta} \sim A_{n+1,\beta}$ for $n \geq 1$. By Proposition 6.3 we have

$$b_{k,\beta}(\varepsilon) = \sum_{w \in \Omega_{k,\beta}(\varepsilon)} |B_w|^{\beta} = \sum_{w \in \Omega_{k-1,II}(\varepsilon)} |B_w|^{\beta} \sum_{j=[\varepsilon(a_k+2)]}^{[(1-\varepsilon)(a_k+2)]} |B_w(\varepsilon 21, a_k+1, j)|^{\beta}$$

$$+ \sum_{w \in \Omega_{k-1,II}(\varepsilon)} \sum_{j=[\varepsilon(a_k+1)]}^{[(1-\varepsilon)(a_k+1)]} |B_w(\varepsilon 31, a_k, j)|^{\beta}$$

$$\sim \sum_{w \in \Omega_{k-1,II}(\varepsilon)} |B_w|^{\beta} \sum_{j=[\varepsilon(a_k+2)]}^{[(1-\varepsilon)(a_k+2)]} (a_k + 2)^{-\beta} \sin^{2\beta} \frac{j\pi}{a_k+2}$$

$$+ \sum_{w \in \Omega_{k-1,II}(\varepsilon)} |B_w|^{\beta} \sum_{j=[\varepsilon(a_k+1)]}^{[(1-\varepsilon)(a_k+1)]} (a_k + 1)^{-\beta} \sin^{2\beta} \frac{j\pi}{a_k+1}$$

$$= A_{a_k,\beta} \cdot b_{k-1,\beta}(\varepsilon) + A_{a_k,\beta} \cdot b_{k-1,II}(\varepsilon)$$

$$\sim A_{a_k,\beta} \left( b_{k-1,\beta}(\varepsilon) + b_{k-1,II}(\varepsilon) \right).$$ (72)
Similarly we have
\begin{equation}
\beta_{k,\beta}^{(k,III)}(\varepsilon) \sim A_{\alpha_k,\beta} \cdot \beta_{k-1,\beta}^{(k-1,II)}(\varepsilon) + A_{\alpha_k-1,\beta} \cdot \beta_{k-1,\beta}^{(k-1,III)}(\varepsilon).
\end{equation}

We can see
\begin{equation}
\begin{aligned}
\beta_{k,\beta}^{(k,I)}(\varepsilon) &\sim \beta_{k,\beta}^{(k,III)}(\varepsilon), \quad \text{if } a_k > 1, \\
\beta_{k,\beta}^{(k,I)}(\varepsilon) &\geq \beta_{k,\beta}^{(k,III)}(\varepsilon), \quad \text{if } a_k = 1.
\end{aligned}
\end{equation}

On the other hand by Corollary 3.4 we get
\begin{equation}
\begin{aligned}
\beta_{k,\beta}^{(k,II)}(\varepsilon) &= \sum_{w \in \Omega_{k-1}} |B_w|^\beta = \sum_{w \in \Omega_{k-1}} |B_w^{((k-1,II),1,1)}|^\beta \\
&\sim \zeta_{\alpha_k} \sum_{w \in \Omega_{k-1}} |B_w|^\beta = \zeta_{\alpha_k} \cdot \beta_{k-1,\beta}^{(k,II)}(\varepsilon).
\end{aligned}
\end{equation}

We remark that for the three relations above, the constants related to "\sim" only depend on $\varepsilon$ and $\beta$.

By iterating (72), (73) and (75), we get
\begin{equation}
\begin{aligned}
\beta_{k,\beta}^{(k,I)}(\varepsilon) &\sim A_{\alpha_k,\beta} \cdot \beta_{k-1,\beta}^{(k-1,II)}(\varepsilon) + A_{\alpha_k,\beta} \cdot \beta_{k-1,\beta}^{(k-1,III)}(\varepsilon) \\
&\sim A_{\alpha_k,\beta} \cdot \beta_{k-1,\beta}^{(k-2,II)}(\varepsilon) \cdot \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon) \\
&\quad + A_{\alpha_k-1,\beta} \cdot \beta_{k-1,\beta}^{(k-2,II)}(\varepsilon) + A_{\alpha_k-1,\beta} \cdot \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\beta_{k,\beta}^{(k,III)}(\varepsilon) &\sim A_{\alpha_k,\beta} \cdot \beta_{k-1,\beta}^{(k-1,II)}(\varepsilon) + A_{\alpha_k-1,\beta} \cdot \beta_{k-1,\beta}^{(k-1,III)}(\varepsilon) \\
&\sim A_{\alpha_k,\beta} \cdot \beta_{k-1,\beta}^{(k-2,II)}(\varepsilon) \cdot \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon) \\
&\quad + A_{\alpha_k-1,\beta} \cdot \beta_{k-1,\beta}^{(k-2,II)}(\varepsilon) + A_{\alpha_k-1,\beta} \cdot \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\beta_{k,\beta}^{(k,II)}(\varepsilon) &\sim \zeta_{\alpha_k} \cdot \beta_{k-1,\beta}^{(k-1,II)}(\varepsilon) \\
&\sim \zeta_{\alpha_k} A_{\alpha_k-1,\beta} \left( \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon) + \beta_{k-2,\beta}^{(k-2,III)}(\varepsilon) \right).
\end{aligned}
\end{equation}

We now show that
\begin{equation}
\frac{\beta_{k,\beta}^{(k,II)}(\varepsilon)}{\beta_{k,\beta}^{(k,II)}(\varepsilon)} \leq \frac{\beta_{k-1,\beta}^{(k-1,II)}(\varepsilon)}{\beta_{k-1,\beta}^{(k-1,II)}(\varepsilon)}.
\end{equation}

If $a_k > 1$, by (74) we have $\beta_{k-1,\beta}^{(k-1,II)}(\varepsilon) \sim \beta_{k-1,\beta}^{(k-1,III)}(\varepsilon)$. Then by (75) we have
\begin{equation}
\frac{\beta_{k,\beta}^{(k,II)}(\varepsilon)}{\beta_{k,\beta}^{(k,II)}(\varepsilon)} \sim \frac{\beta_{k-1,\beta}^{(k-1,II)}(\varepsilon)}{\beta_{k-1,\beta}^{(k-1,II)}(\varepsilon)} \leq \frac{\beta_{k-1,\beta}^{(k,II)}(\varepsilon)}{\beta_{k-1,\beta}^{(k,II)}(\varepsilon)} = \frac{\zeta_{\alpha_k}}{A_{\alpha_k,\beta}}.
\end{equation}

If $a_k = 1$, recalling that $\zeta_1 = 1$ and $A_{1,\beta} = 2^{-\beta} \sim 1$, then by (76) we get
\begin{equation}
\begin{cases}
\beta_{k,\beta}^{(k,II)}(\varepsilon) \sim A_{\alpha_k,\beta} \left( \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon) + \beta_{k-2,\beta}^{(k-2,III)}(\varepsilon) \right), \\
\beta_{k,\beta}^{(k,II)}(\varepsilon) \sim \zeta_{\alpha_k} \left( \beta_{k-2,\beta}^{(k-2,II)}(\varepsilon) + \beta_{k-2,\beta}^{(k-2,III)}(\varepsilon) \right).
\end{cases}
\end{equation}

By (74) we get $\beta_{k-2,\beta}^{(k-2,II)}(\varepsilon) \sim \beta_{k-2,\beta}^{(k-2,III)}(\varepsilon)$, thus (77) holds.
We show further that
\[
\frac{b_{k,\beta}^{(k,II)}(\varepsilon)}{b_{k,\beta}^{(k,I)}(\varepsilon)} \sim \zeta_{ak}^\beta / A_{ak,\beta}.
\] (78)

In fact, by \(\zeta_{ak} \leq 1\), Lemma 9.1 and (77), we see \(b_{k,\beta}^{(k,II)}(\varepsilon) \leq b_{k,\beta}^{(k,I)}(\varepsilon)\) for any \(k > 0\), and also by (74), we have

\[
b_{k,\beta}^{(k,I)}(\varepsilon) \sim A_{ak,\beta} \left( b_{k-1,\beta}^{(k-1,II)}(\varepsilon) + b_{k-1,\beta}^{(k-1,III)}(\varepsilon) \right)
\]
\[
\leq A_{ak,\beta} b_{k-1,\beta}^{(k-1,II)}(\varepsilon)
\]
\[
\sim A_{ak,\beta} b_{k-1,\beta}^{(k-1,II)}(\varepsilon).
\]

Together with (75), we get the other direction of (78).

By (74) and (78), we have

\[
b_{k,\beta}(\varepsilon) = b_{k,\beta}^{(k,I)}(\varepsilon) + b_{k,\beta}^{(k,II)}(\varepsilon) + b_{k,\beta}^{(k,III)}(\varepsilon) \sim b_{k,\beta}^{(k,I)}(\varepsilon),
\]
(79)

which implies the first and the second formula of (70), also the third formula in case of \(a_k > 1\). If \(a_k = 1\), by (73) and (78),

\[
b_{k,\beta}^{(k,III)}(\varepsilon) \sim b_{k-1,\beta}^{(k-1,II)}(\varepsilon) \sim \frac{\zeta_{ak-1}^\beta}{A_{ak-1,\beta}} b_{k-1,\beta}^{(k-1,II)}(\varepsilon)
\]
\[
b_{k,\beta}(\varepsilon) \sim b_{k,\beta}^{(k,I)}(\varepsilon) \sim b_{k,\beta}^{(k,II)}(\varepsilon)
\]

Thus the third formula of (70) hold in the case of \(a_k = 1\).

Combine (72) and (79). If \(a_{k-1} > 1\), (74) and (78) implies (71). If \(a_{k-1} = 1\), by (78), we see \(b_{k-1,\beta}^{(k-1,II)}(\varepsilon)/b_{k-1,\beta}^{(k-1,II)}(\varepsilon) \sim 1\), and then we still have (71).

By Remark 2 and the remark given after the three relations, all the constants related to “\(\sim, \leq, \geq\)” only depend on \(V\) and \(\beta\). □

**Proposition 9.3.** For any \(m \geq k + 3\), we have

\[
\begin{align*}
\frac{b_{m,\beta}^{(k,I)}(\varepsilon)}{b_{m,\beta}(\varepsilon)} & \sim \frac{\zeta_{ak+1}^\beta}{A_{ak+1,\beta}}, & a_k > 1, a_{k+1} > 1; \\
\frac{b_{m,\beta}(\varepsilon)}{b_{m,\beta}(\varepsilon)} & \sim \frac{1}{A_{ak,\beta}}, & a_k > 1, a_{k+1} = 1; \\
\frac{b_{m,\beta}^{(k,II)}(\varepsilon)}{b_{m,\beta}(\varepsilon)} & \sim \begin{cases} \\
\frac{1}{A_{ak+2,\beta}} & a_k > 1, a_{k+1} = 1; \\
\frac{\zeta_{ak+1}^\beta}{A_{ak+1,\beta}} & a_k = 1, a_{k+1} > 1; \\
\frac{\zeta_{ak-1}^\beta}{A_{ak-1,\beta}} & a_k = 1, a_{k+1} = 1.
\end{cases}
\end{align*}
\] (80)
Proof. Take any \( \sigma_2 \in \Omega_{k+1}^{(k, I)}(\varepsilon) \), then \( B_{\sigma_2} \) is a band of type \((k + 1, II)\). Take \( \sigma_1, \sigma_3 \in \Omega_{k+1}^{(k, II)}(\varepsilon) \) such that \( B_{\sigma_1} \) is a band of type \((k + 1, I)\) and \( B_{\sigma_3} \) is a band of type \((k + 1, III)\). For any \( p \leq m \) and any \( T = I, II, III \) define

\[
\Omega_{p,m}^{(T)}(\varepsilon) = \{ w_p \cdots w_m \in \prod_{j=p}^m E_{a_j}(\varepsilon) \text{ admissible : } e_{w_p} = (T, \ast) \},
\]

where \( E_{a_j}(\varepsilon) \) is defined in (2.2).

Define

\[
\begin{align*}
&c_{k+1}^{(I)} = \sum_{\tau \in \Omega_{k+2,m}^{(I)}(\varepsilon)} |B_{\sigma_{1+\tau}}|^\beta |B_{\sigma_1}|^\beta \\
&c_{k+1}^{(II)} = \sum_{\tau \in \Omega_{k+2,m}^{(II)}(\varepsilon)} |B_{\sigma_{2+\tau}}|^\beta |B_{\sigma_2}|^\beta \\
&c_{k+1}^{(III)} = \sum_{\tau \in \Omega_{k+2,m}^{(III)}(\varepsilon)} |B_{\sigma_{3+\tau}}|^\beta |B_{\sigma_3}|^\beta.
\end{align*}
\]

We can also define \( c_{k+2}^{(I)}, c_{k+2}^{(II)}, c_{k+2}^{(III)} \) in an analogous way. Analogous to the arguments of (72), (73) and (75), we have

\[
\begin{align*}
&c_{k+1}^{(I)} \sim c_{k+2}^{(II)} \zeta_{k+2}^\beta \\
&c_{k+1}^{(II)} \sim A_{k+2,1,\beta} c_{k+2}^{(I)} + A_{k+2,1,\beta} c_{k+2}^{(III)} \\
&c_{k+1}^{(III)} \sim A_{k+2,1,\beta} c_{k+2}^{(I)} + A_{k+2,1,\beta} c_{k+2}^{(III)}.
\end{align*}
\]

And consequently

\[
\begin{align*}
&\frac{c_{k+1}^{(I)}}{c_{k+1}^{(II)}} \sim c_{k+2}^{(II)} \zeta_{k+2}^\beta \\
&\frac{c_{k+1}^{(II)}}{c_{k+1}^{(III)}} \sim \begin{cases} 
1 & a_{k+2} > 1 \\
\frac{c_{k+2}^{(I)}}{c_{k+2}^{(II)}} & a_{k+2} = 1
\end{cases}
\end{align*}
\]

Write

\[
\begin{align*}
\rho &:= (e_{12}, 1, 1), \quad \theta_j := (e_{21}, a_{k+1} + 1, j) \quad \text{and} \quad \phi_j := (e_{23}, a_{k+1}, j).
\end{align*}
\]

By Theorem 3.3 and Corollary 3.4 we have

\[
\begin{align*}
b_m^{(k, I)}(\varepsilon) &= \sum_{w \in \Omega_{m}^{(k, I)}(\varepsilon)} |B_w|^\beta \\
&= \sum_{\sigma \in \Omega_{m}^{(k, I)}(\varepsilon)} |B_\sigma|^\beta \frac{|B_{\sigma_{1+\tau}}|^\beta}{|B_{\sigma_1}|^\beta} \sum_{\tau \in \Omega_{k+1,II}^{(k+1, II)}(\varepsilon)} |B_{\sigma_{2+\tau}}|^\beta |B_{\sigma_2}|^\beta \\
&\sim \sum_{\sigma \in \Omega_{m}^{(k, I)}(\varepsilon)} |B_\sigma|^\beta \frac{|B_{\sigma_{2+\tau}}|^\beta}{|B_{\sigma_2}|^\beta} \sum_{\tau \in \Omega_{k+2,m}^{(II)}(\varepsilon)} |B_{\sigma_{3+\tau}}|^\beta |B_{\sigma_3}|^\beta \\
&\sim b_{k,\beta}^{(I)}(\varepsilon) \zeta_{k+1}^{\beta} c_{k+1}^{(II)} \\
b_m^{(k, II)}(\varepsilon) &= \sum_{\sigma \in \Omega_{m}^{(k, II)}(\varepsilon)} |B_\sigma|^\beta \\
b_m^{(k, III)}(\varepsilon) &= \sum_{\sigma \in \Omega_{m}^{(k, III)}(\varepsilon)} |B_\sigma|^\beta.
\end{align*}
\]
By a simple computation the result follows. □

We will prove the following theorem, which is Theorem 3.5 when ε = 0.
Theorem 9.4 (Existence of Gibbs like measures). For any \(0 < \beta < 1\), \(0 \leq \varepsilon < 1/12\), there exists a probability measure \(\mu_{\beta,\varepsilon}\) supported on \(E_\varepsilon\) such that

If \(w \in \Omega^{(k, I)}_k(\varepsilon)\), let \(u = (e_{12}, 1, 1)\), then

\[
\mu_{\beta,\varepsilon}(B_w) \sim \frac{\zeta^\beta_{k+1} |B_w|^\beta}{a^\beta_{k+1} b^\beta_{k,\beta}(\varepsilon)} \sim \frac{|B_{w, u}|^\beta}{b^\beta_{k+1,\beta}(\varepsilon)}. \tag{81}
\]

If \(w \in \Omega^{(k, II)}_k(\varepsilon)\), then

\[
\mu_{\beta,\varepsilon}(B_w) \sim \frac{|B_w|^\beta}{b^\beta_{k,\beta}(\varepsilon)}. \tag{82}
\]

If \(w \in \Omega^{(k, III)}_k(\varepsilon)\), then

\[
\mu_{\beta,\varepsilon}(B_w) \sim \begin{cases} 
\frac{|B_w|^\beta}{b^\beta_{k,\beta}(\varepsilon)} & a_{k+1} > 1; \\
\frac{\zeta^\beta_{k+2} |B_w|^\beta}{b^\beta_{k+2,\beta}(\varepsilon)} & a_{k+1} = 1. 
\end{cases} \tag{83}
\]

Proof. For any \(0 < \beta < 1\) and \(m > 0\), we define a probability \(\mu_{\beta,\varepsilon,m}\) on \(\mathbb{R}\) such that for any \(w \in \Omega_m(\varepsilon)\),

\[
\mu_{\beta,\varepsilon,m}(B_w) = \frac{|B_w|^\beta}{b^\beta_{m,\beta}(\varepsilon)},
\]

where \(\mu_{\beta,\varepsilon,m}\) is uniformly distributed on each band \(B_w\) for any \(w \in \Omega_m(\varepsilon)\).

Fix any \(k \geq 1\). For \(T \in \{I, II, III\}\) and \(w \in \Omega^{(k, T)}_k(\varepsilon)\), we will prove that \(\mu_{\beta,\varepsilon,m}(B_w)\) satisfy \((S1), (S2)\) or \((S3)\) respectively for \(m \geq k + 3\). Then by taking any weak limit of \(\{\mu_{\beta,\varepsilon,m}\}_{m>0}\), we prove the theorem.

For any \(\sigma \in \Omega^{(k, T)}_k(\varepsilon)\), by bounded covariation we have

\[
\mu_{\beta,\varepsilon,m}(B_w) = \frac{|B_w|^\beta}{b^\beta_{m,\beta}(\varepsilon)} \sum_{\tau \in \Omega^{(k, T)}_{k+1, m}(\varepsilon)} |B_{w \ast \tau}|^\beta
\]

\[
= \frac{|B_w|^\beta}{b^\beta_{m,\beta}(\varepsilon)} \sum_{\tau \in \Omega^{(k, T)}_{k+1, m}(\varepsilon)} \frac{|B_{w \ast \tau}|^\beta}{|B_w|^\beta}
\]

\[
\sim \frac{|B_w|^\beta}{b^\beta_{m,\beta}(\varepsilon)} \sum_{\tau \in \Omega^{(k, T)}_{k+1, m}(\varepsilon)} \frac{|B_{w \ast \tau}|^\beta}{|B_w|^\beta}.
\]

Hence

\[
|B_\sigma|^\beta \mu_{\beta,\varepsilon,m}(B_w) \sim \frac{|B_w|^\beta}{b^\beta_{m,\beta}(\varepsilon)} \sum_{\tau \in \Omega^{(k, T)}_{k+1, m}(\varepsilon)} |B_{\sigma \ast \tau}|^\beta.
\]

Take sum on both sides for all \(\sigma \in \Omega^{(k, T)}(\varepsilon)\), we get

\[
\sum_{\sigma \in \Omega^{(k, T)}(\varepsilon)} |B_\sigma|^\beta \mu_{\beta,\varepsilon,m}(B_w) \sim \frac{|B_w|^\beta}{b^\beta_{m,\beta}(\varepsilon)} \sum_{\sigma \in \Omega^{(k, T)}(\varepsilon)} |B_{\sigma \ast \tau}|^\beta.
\]
which implies that

\[ \mu_{\beta, \varepsilon, m}(B_w) \sim \frac{|B_w|^{\beta}}{b_{k, \beta}(\varepsilon)} \frac{b^{(k,T)}_{m, \beta}(\varepsilon)}{b_{m, \beta}(\varepsilon)}. \]

Combining with (70) and (80), if \( w \) has type \( I \) then

\[ \mu_{\beta, \varepsilon, m}(B_w) \sim \frac{\zeta^{\beta}_{ak+1}}{A_{ak+1, \beta}} \frac{|B_w|^{\beta}}{b_{k, \beta}(\varepsilon)}. \]

If \( w \) has type \( II \), then

\[ \mu_{\beta, \varepsilon, m}(B_w) \sim \frac{|B_w|^{\beta}}{b_{k, \beta}(\varepsilon)}. \]

If \( w \) has type \( III \), then

\[ \mu_{\beta, \varepsilon, m}(B_w) \sim \begin{cases} \frac{|B_w|^{\beta}}{b_{k, \beta}(\varepsilon)} & a_{k+1} > 1; \\ \frac{\zeta^{\beta}_{ak+2}}{A_{ak+2, \beta}} & a_{k+1} = 1. \end{cases} \]

Thus we get (82), (83) and the first relation of (81).

To get the second relation of (81), we proceed as follows. Write \( u = (\varepsilon_{12}, 1, 1) \). If \( w \in \Omega^{(k, I)}(\varepsilon) \), then \( w \ast u \in \Omega^{(k+1, II)}_{k+1}(\varepsilon) \) and \( \mu(B_{w \ast u}) = \mu(B_w) \).

Now the result follows by applying (82).

\[ \square \]

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