Saturation of Turbulent Drag Reduction in Dilute Polymer Solutions

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Drag reduction by polymers in turbulent wall-bounded flows exhibits universal and non-universal aspects. The universal maximal mean velocity profile was explained in a recent theory. The saturation of this profile and the crossover back to the Newtonian plug are non-universal, depending on Reynolds number $Re$, concentration of polymer $c_p$ and the degree of polymerization $N_p$. We explain the mechanism of saturation stemming from the finiteness of extensibility of the polymers, predict its dependence on $c_p$ and $N$ in the limit of small $c_p$ and large $Re$, and present the excellent comparison of our predictions to experiments on drag reduction by DNA.

**Introduction.** The onset of turbulence in fluid flows is accompanied by a significant increase in the drag. This drag poses a real technological hindrance to the transport of fluids and to the navigation of ships. It is interesting that the addition of long chained polymers to wall-bounded turbulent flows can result in a significant reduction in the drag. The basic experimental knowledge of the phenomenon had been reviewed and systematized by Virk [2]: the increase in mean velocity profile as a function of the distance from the wall depends on the characteristics of the polymer and its concentration, but cannot exceed a universal asymptote known as the “Maximum Drag Reduction” (MDR) curve which is independent of the polymer’s concentration or its characteristics. When the concentration is not large enough, the mean velocity profile follows the MDR for a while and then crosses back to a Newtonian-like profile, known as the “Newtonian plug” cf. Fig. 1. Recently the nature of the MDR and the mechanism leading to its establishment were explained [4]. For Newtonian flows the momentum flux is dominated by the so-called Reynolds stress, leading to a logarithmic (von-Karman) dependence of the mean velocity on the distance from the wall [6]. With the addition of polymers, while momentum is produced at a fixed rate by the forcing, polymer stretching results in a suppression of the Reynolds stress (and thus of the momentum flux from the bulk to the wall). Accordingly the mean velocity in the channel must increase. It was shown that when the concentration of the polymers is large enough there exists a new logarithmic law for the mean velocity with a slope that fits existing numerical and experimental data. The law is universal, thus explaining the MDR asymptote. It turned out that the polymer stretching leads to an effective viscosity which increases with the distance from the wall. This effective viscosity attains a self-consistent profile, increasing linearly with the distance from the wall. With this profile the reduction in the momentum flux from the bulk to the wall overwhelms the increased dissipation that results from the increased viscosity. Thus the mean momentum increases in the bulk, and this is how drag reduction is realized. In [4] it was demonstrated by DNS that Navier-Stokes flows with viscosity profiles that vary linearly with the distance from the wall indeed show drag reduction in close correspondence with the phenomena seen in full viscoelastic simulations.

**Theory of the saturation of drag reduction.** The aim of this Letter is to provide a theory of the crossovers from the MDR to the Newtonian plug for small polymer concentration $c_p$ and large $Re$. We show that the mecha-
2: a viscous sublayer, a logarithmic polymeric sublayer (region 3 in the Fig. 2 with the slope greater then the Newtonian one:

\[ V^+(y^+) = \kappa_v^{-1} \log y^+ + B_v, \quad \text{(MDR profile),} \]

\((\kappa_v^{-1} \approx 11.7, B_v \approx -17, y^+ \equiv y^+/\nu)\), and a Newtonian plug (region 4). In the last region the velocity follows a log law with the Newtonian slope, but with some velocity increment \( \Delta V^+ \):

\[ V^+ = \kappa_N^{-1} \log y^+ + B_N + \Delta V^+. \]

Note that the three profiles Eqs. (2), (3), and (4) intercept at one point \( y^+=y^+_N \approx \kappa_N^{-1} \approx 11.7 \).

The increment \( \Delta V^+ \) which determines the amount of drag reduction is in turn determined by the crossover from the MDR to the Newtonian plug (see Fig. 2). We refer to this cross over point as \( y^+_N \). To measure the quality of drag reduction we introduce a dimensionless drag reduction parameter

\[ Q \equiv \frac{y^+_N}{y^+_N} - 1. \]

The velocity increment \( \Delta V^+ \) is related to this parameter as follows

\[ \Delta V^+ = (\kappa_v^{-1} - \kappa_N^{-1}) \log \left(\frac{y^+_N}{y^+_N}\right) = \alpha \log(1 + Q). \]

Here \( \alpha \equiv \kappa_v^{-1} - \kappa_N^{-1} \approx 9.4 \). The Newtonian flow is then a limiting case of the viscoelastic flow corresponding to \( Q = 0 \).

The crossover point \( y^+_N \) is non-universal, depending on \( \Re \), the number of polymers per unit volume \( c_p \), the chemical nature of the polymer, etc. According to the theory of drag reduction \( 8 \), the total viscosity of the fluid \( \nu_{tot}(y^+) = \nu_0 + \nu_p(y^+) \) [where \( \nu_p(y^+) \) is the polymeric contribution to the viscosity] is linear in \( y^+ \) in the MDR region:

\[ \nu_{tot}(y^+) = \nu_0 y^+/y^+_N, \quad y^+_N < y^+ < y^+_v. \]

When the concentration of polymers is small and \( \Re \) is large enough, the crossover to the Newtonian plug at \( y^+_v \) occurs when the polymer stretching can no longer provide the necessary increase of the total fluid viscosity. In other words, in that limit the crossover is due to the finite extensibility of the polymer molecules. Obviously, the polymeric viscosity can not be greater than \( \nu_p\max \) which is the viscosity of the fully stretched polymers. Thus the total viscosity is limited by \( \nu_0 + \nu_p\max \). Equating \( \nu_0 + \nu_p\max \) and \( \nu_{tot}(y^+_v) \) gives us the crossover position

\[ y^+_v = \frac{y^+_N (\nu_0 + \nu_p\max) / \nu_0. \]

It follows from Eq. (\( \ref{eq:q} \)) that the drag reduction parameter is determined very simply by

\[ Q = \nu_p\max / \nu_0, \quad c_p \text{ small, } \Re \text{ large.} \]
At this point we need to relate the maximum polymeric viscosity \( \nu_p \) to the polymer properties. To this aim we estimate the energy dissipation due to a single, fully stretched, polymer molecule. In a reference frame co-moving with the polymer’s center of mass the fluid velocity can be estimated as \( u \approx r \sqrt{u} \) (the polymer’s center of the mass moves with the fluid velocity due to negligible inertia of the molecule). The friction force exerted on the \( i \)-th monomer is estimated using Stokes law,

\[
F_i \approx \rho_0 \nu_0 a_i \delta u_i = \rho_0 \nu_0 a_i r_i \nabla u ,
\]

where \( a \) is an effective hydrodynamic radius of one monomer (depending on the chemical composition), and \( r_i \) is the distance of the \( i \)-th monomer from the center of the mass. In a fully stretched state \( r_i \approx a i \) (the monomers are aligned along a line). The energy dissipation rate (per unit volume) is equal to the work performed by the external flow

\[
- \frac{dE}{dt} \approx c_p \sum_{i=1}^{N_p} F_i \delta u_i \approx \rho_0 \nu_0 a^3 c_p N_p^3 (\nabla u)^2
\]

\[
\equiv \rho_0 \nu_{p, \text{max}} (\nabla u)^2 .
\]

We thus can estimate \( \nu_{p, \text{max}} \):

\[
\nu_{p, \text{max}} = \nu_0 a^3 c_p N_p^3.
\]

Finally, the drag reduction parameter \( Q \) is given by

\[
Q = a^3 c_p N_p^3 c_p \text{ small, } \Re \text{ large} .
\]

This is the central theoretical results of this Letter, relating the concentration \( c_p \) and degree of polymerization \( N_p \) to the increment in mean velocity \( \Delta V^+ \) via Eq. (11).

**Testing and explaining a DNA experiment** [11].

A particularly interesting experiment suitable for testing our prediction was described in [11]. Here turbulence was produced in a rotating disk apparatus, with \( \lambda \)-DNA molecules used to reduce the drag. The Reynolds number was relatively high (the results below pertain to \( \Re \approx 1.2 \times 10^6 \) and the initial concentrations of DNA relatively low (results employed below pertain to 2.70 and 1.35 wppm). During the experiment DNA degrades; fortunately the degradation is very predictable: double stranded molecules with 48 502 bp in size degrade to double stranded molecules with 23 100 bp. Thus invariably the length \( N_p \) reduces by a factor of approximately 2, and the concentration \( c_p \) increases by a factor of 2. The experiment followed the drag reduction efficacy measured in terms of the percentage drag reduction defined by

\[
\% \text{DR} = \frac{T_N - T_\nu}{T_N} \times 100 ,
\]

where \( T_N \) and \( T_\nu \) are the torques needed to maintain the disk to rotate at a particular Reynolds number \( \Re \) without and with polymers, respectively. The main experimental results which are of interest to us are summarized in Fig. 3. We see from the experiment that both initially (with un-degraded DNA) and finally (with degraded DNA) the \( \% \text{DR} \) is proportional to \( c_p \). Upon degrading, which amounts to decreasing the length \( N_p \) by a factor of approximately 2 and, simultaneously increasing \( c_p \) by factor of 2, \( \% \text{DR} \) decreases by a factor 4.

The flow geometry is rather complicated: with a rotating disk the linear velocity depends on the radius, and the local \( \Re \) is a function of the radius. The drag reduction occurs however in a relatively small near-wall region, where the flow can be considered as a flow near the flat plate. Thus, we consider an equivalent channel flow – with the same \( \Re \) and a half width \( L \) of the order of height/radius of the cylinder. In this plane geometry the torques in (15) should be replaced by the pressure gradients \( p'_N, V' \):

\[
\% \text{DR} = \frac{p'_N - p'_\nu}{p'_N} \times 100 .
\]

In order to relate \( \% \text{DR} \) with the drag reduction parameter \( Q \), we re-write Eq. (5) in natural units

\[
V(y) = \sqrt{\nu L} \left[ \kappa_N^{-1} \log \left( \frac{y\sqrt{\nu L/V_0}}{\nu} + B_N + \Delta V^+ \right) \right] .
\]

With constant \( \Re \) the centerline velocity \( V_0 = V(L) \) is kept fixed:

\[
\Re \equiv \frac{V_0 L}{\nu_0} = \Re_\tau \left[ \kappa_N^{-1} \log \Re_\tau + B_N + \Delta V^+ \right] .
\]
This equation implicitly determines the pressure gradient and therefore the $\% DR$ as a function of $Q$ and $Re$. The set of Eqs. (7) and (18) is readily solved numerically, and the solution for three different values of $Re$ is shown in Fig. 4. The middle curve corresponds to $Re = 1.2 \times 10^6$, which coincides with the experimental conditions [11]. One sees, however, that the dependence of $\% DR$ on the $Re$ is rather weak.

One important consequence of the solutions shown in Fig. 4 is that for small $Q$ (actually for $Q \leq 0.5$ or $\% DR \leq 20$), $\% DR$ is approximately a linear function of $Q$. The experiments [11] lie entirely within this linear regime, in which we can linearize Eq. (18) and find an approximate solution for the $\% DR$:

$$\% DR = \frac{2\alpha Q}{\kappa_n^{-1} \log(\epsilon Re^{-2}) + B_n} \times 100 \ .$$

(19)

Here $Re^{-2}$ is the friction Reynolds number for the Newtonian flow, i.e. the solution of Eq. (18) for $\Delta V^+ = 0$.

It is interesting to note, that while the $\% DR$ depends on the Reynolds number, the ratio of different $\% DR$'s does not [to $O(Q)$]:

$$\frac{\% DR^{(1)}}{\% DR^{(2)}} = \frac{Q^{(1)}}{Q^{(2)}} = \frac{\nu_p^{(1)}_{\max}}{\nu_p^{(2)}_{\max}} \ .$$

(20)

This result, together with Eq. (18), rationalizes completely the experimental finding of [11] summarized in Fig. 3. During the DNA degradation, the concentration of polymers increases by a factor of 2, while the number of monomers $N_p$ decreases by the same factor. This means that $\% DR$ should decrease by a factor of 4, as is indeed the case.

**Summary.** We have presented a theory of the crossover from the universal MDR mean velocity profiles to the Newtonian plug. We have connected the theory to measured percentages of drag reduction, and tested the theory against experiments in which $\lambda$-DNA is used as the drag-reducing agent. The experimental results pertain to high $Re$ and small $c_p$, where we can assert that the crossover results from exhausting the stretching of the polymers such that the maximal available viscosity is achieved. In the linear regime that pertains to this experiment the degradation has a maximal effect on the quality of drag reduction $Q$, leading to the precise factor of 4 in the results shown in Fig. 4. Larger values of the concentration of DNA will exceed the linear regime as is predicted by Eq. (18) then the degradation is expected to have a smaller influence on the drag reduction efficacy. It is worthwhile to test the predictions of this theory also in the nonlinear regime.

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