Discrete Painlevé system associated with Unitary matrix model

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Abstract. We discuss the emergence of discrete Painlevé system in the context of irregular conformal block which is also a $SU(2) N_f = 2$ supersymmetric gauge theory. We develop a framework of orthogonal polynomials in unitary matrix model.

1. Introduction
In this paper, we consider the generic following type of the \( \beta \)-deformed “matrix model” depending on two integration contours \( C_L \) and \( C_R \):

\[
Z(N_L, N_R) = C \left( \prod_{i=1}^{N_L} \int_{C_L} dw_i \right) \left( \prod_{j=1}^{N_R} \int_{C_R} dw_{N_L+j} \right) \Delta^{2\beta}(w) \exp \left( \sqrt{\beta} \sum_{I=1}^{N_L+N_R} W(w_I) \right), \quad (1.1)
\]

where \( C \) is a normalization constant and \( \Delta(w) \) is the Vandermonde determinant:

\[
\Delta(w) = \prod_{1 \leq I < J \leq N_{L+N_R}} (w_I - w_J). \quad (1.2)
\]

Following [1], we introduce their generating function

\[
Z(N; \mu_L, \mu_R) = \sum_{N_L+N_R=N} \frac{\mu_L^{N_L} \mu_R^{N_R}}{N_L! N_R!} Z(N_L, N_R) \quad (1.3)
\]

where \( C = \mu_L C_L + \mu_R C_R \), i.e.,

\[
\int_C = \mu_L \int_{C_L} + \mu_R \int_{C_R} \quad (1.4)
\]

As we recall from [2] (see, [3, 4, 5, 6, 7, 8] for earlier references), in the case of \( \beta = 1 \) which we consider in the body of this paper, the \( N_f = 2 \) matrix model of the above form with \( \alpha_{1+2} \in \mathbb{Z} \) in fact reduces to the unitary matrix model with cosine + log potential in section 3.
2. Unitary matrix model

In this section, we briefly review the unitary matrix model, the method of orthogonal polynomials and the string equations to explain our notation.

The partition function of the unitary matrix model is defined by

\[ Z_U(N) := \frac{1}{\text{vol}(U(N))} \int [dU] \exp \left( \text{Tr} W_U(U) \right), \]  
(2.1)

where \( U \) is an \( N \times N \) unitary matrix and \( W_U(U) \) is a potential. We define a unitary Haar measure \([dU]\) from the metric

\[ ds^2 = \text{Tr}(dU^\dagger dU) = -\text{Tr}(U^{-1}dU)^2. \]  
(2.2)

With this normalization of the measure, the volume of the unitary group \( U(N) \) is given by

\[ \text{vol}(U(N)) = \int [dU] = (2\pi)^{N(N+1)/2} G_2(N+1), \]  
(2.3)

where \( G_2(z) \) is the Barnes function. Explicitly, \( G_2(N+1) \) is given by

\[ G_2(N+1) = N \prod_{j=1}^{N-1} j! = \prod_{k=1}^{N} k^{N-k}. \]  
(2.4)

If we diagonalize the unitary matrix \( U \) as

\[ U = V^{-1} U_D V, \quad U_D = \text{diag}(z_1, z_2, \cdots, z_N), \quad |z_i| = 1, \]  
(2.5)

we have

\[ Z_{U(N)} = \frac{1}{N!} \left( \prod_{i=1}^{N} \int \frac{dz_i}{2\pi i z_i} \right) \Delta(z) \Delta(z^{-1}) \exp \left( \sum_{i=1}^{N} W_U(z_i) \right), \]  
(2.6)

where

\[ \Delta(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j). \]  
(2.7)

Let

\[ d\mu(z) := \frac{dz}{2\pi i z} \exp \left( W_U(z) \right). \]  
(2.8)

Then

\[ Z_{U(N)} = \frac{1}{N!} \prod_{i=1}^{N} d\mu(z_i) \Delta(z) \Delta(z^{-1}). \]  
(2.9)

The partition function (2.6) expressed in eigenvalue integrals may be generalized to the form of the two contour model (1.1). A natural choice of the two contours \( C_L \) and \( C_R \) is take them as circles of radius \( r_L \) and \( r_R \) respectively. Suppose, \( r_L < r_R \) and there is no singularity in the region \( r_L \leq |w| \leq r_R \). Then the contours can be smoothly deformed to circles of the same radius, i.e., to the same contour: \( C_L = C_R \). Then, for the two contour unitary matrix model, \( Z_{U(N)}(N_L, N_R) \) depends only on \( N = N_L + N_R \), and the generating function \( Z_{U(N)}(N, 0) \) is essentially \( Z_U(N, 0) \). Because \( Z_{U(N)} = (\mu_L + \mu_R)^N Z_{U(0)} \), we can set \( \mu_L + \mu_R = 1 \) without loss of generality. Hence \( Z_{U(N)} = Z_U(N) \).
2.1. Orthogonal polynomials

The unitary matrix model can be solved [9, 10, 11] by the method of orthogonal polynomials [12, 13]. Let us use the monic orthogonal polynomials [9, 10] (In [11], orthogonal polynomials of different type have been introduced to solve the unitary matrix model).

Let $p_n$ and $\tilde{p}_n$ ($n \geq 0$) be monic polynomials satisfying orthogonality conditions with respect to the measure (2.8)

$$\int d\mu(z)p_n(z)\tilde{p}_m(1/z) = h_n\delta_{n,m}, \quad (2.10)$$

where

$$p_n(z) = z^n + \sum_{k=0}^{n-1} A_k^{(n)} z^k, \quad \tilde{p}_n(1/z) = z^{-n} + \sum_{k=0}^{n-1} B_k^{(n)} z^{-k}. \quad (2.11)$$

Let us introduce the moments $\mu_n$ for the measure (2.8) by

$$\mu_n := \int d\mu(z)z^n, \quad (n \in \mathbb{Z}). \quad (2.12)$$

For later convenience, we define $K_k^{(n)}$ by

$$K_k^{(n)} := \det(\mu_{j-i+k})_{1 \leq i,j \leq n}, \quad (n \geq 0, k \in \mathbb{Z}). \quad (2.13)$$

From the definition, the orthogonal polynomials have the following properties:

$$\int d\mu(z) p_n(z) z^{-k} = 0, \quad \int d\mu(z) z^k \tilde{p}_n(1/z) = 0, \quad (k = 0, 1, \cdots, n-1), \quad (2.14)$$

Using these and the monic properties, the orthogonal polynomials are determined as

$$p_n(z) = \frac{1}{\tau_n} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{-n} & \mu_{-n+2} & \mu_{-n+3} & \cdots & \mu_1 \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}, \quad (2.15)$$

$$\tilde{p}_n(1/z) = \frac{1}{\tau_n} \begin{vmatrix} \mu_0 & \mu_{-1} & \mu_{-2} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \mu_{-1} & \cdots & \mu_{-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \cdots & \mu_{-1} \\ 1 & z^{-1} & z^{-2} & \cdots & z^{-n} \end{vmatrix}, \quad (2.16)$$

where

$$\tau_n := K_0^{(n)} = \det(\mu_{j-i})_{1 \leq i,j \leq n}, \quad (2.17)$$

(We set $\tau_0 = 1$). We can easily see that these polynomials obey (2.14).

The normalization constants $h_n$ defined by (2.10) are given by

$$h_n = \frac{\tau_{n+1}}{\tau_n} = \frac{K_0^{(n+1)}}{K_0^{(n)}}. \quad (2.18)$$
The constant terms of these polynomials will play important roles.

\[ A_n := p_n(0) = A_n^{(n)} = (-1)^n \frac{K_1^{(n)}}{K_0^{(n)}}, \quad B_n := \tilde{p}_n(0) = B_n^{(n)} = (-1)^n \frac{K_{-1}^{(n)}}{K_0^{(n)}}. \] (2.19)

Note that

\[ \frac{h_n}{h_{n-1}} = \frac{\tau_n + 1}{\tau_{n-1}}, \quad 1 - \frac{h_n}{h_{n-1}} = \frac{\tau_n - \tau_{n-1}}{\tau_n^2}. \] (2.20)

Using an identity

\[ \tau_n^2 - \tau_{n+1}\tau_{n-1} = \left( K_0^{(n)} \right)^2 - K_0^{(n+1)} K_0^{(n-1)} = K_1^{(n)} K_{-1}^{(n)}, \] (2.21)

we can show that

\[ 1 - \frac{h_n}{h_{n-1}} = \frac{K_1^{(n)} K_{-1}^{(n)}}{\left( K_0^{(n)} \right)^2} = A_n B_n. \] (2.22)

Thus we have the following relations:

\[ \frac{h_n}{h_{n-1}} = 1 - A_n B_n. \] (2.23)

Note that

\[ \Delta(z) = \det(p_{j-1}(z_i))_{1 \leq i, j \leq N} = \sum_{\sigma \in S_N} (-1)^{\varepsilon(\sigma)} \prod_{k=1}^{N} p_{\sigma(k)-1}(z_k), \] (2.24)

\[ \Delta(z^{-1}) = \det(\tilde{p}_{j-1}(1/z_i))_{1 \leq i, j \leq N} = \sum_{\sigma \in S_N} (-1)^{\varepsilon(\sigma)} \prod_{k=1}^{N} \tilde{p}_{\sigma(k)-1}(1/z_k), \] (2.25)

Using these relations, the partition function (2.6) is evaluated as

\[ Z_{U(N)} = \frac{1}{N!} \int \prod_{i=1}^{N} d\mu(z_i) \Delta(z) \Delta(z^{-1}) = \prod_{k=0}^{N-1} h_k = \prod_{k=0}^{N-1} \frac{\tau_{k+1}}{\tau_k} = \tau_N. \] (2.26)

Also, it can be written as

\[ Z_{U(N)} = h_0^N \prod_{j=1}^{N-1} (1 - A_j B_j)^{N-j}. \] (2.27)

The orthogonal polynomials \( p_n(z) \) obey the following relations:

\[ z p_n(z) = p_{n+1}(z) + \sum_{k=0}^{n} C_k^{(n)} p_k(z), \] (2.28)

where

\[ C_k^{(n)} = (-1)^{n-k} \frac{K_1^{(n+1)} K_{-1}^{(k)}}{K_0^{(n)} K_0^{(k+1)}} = \frac{h_n}{h_k} A_{n+1} B_k, \quad (0 \leq k \leq n). \] (2.29)

If we use \( p_k \) only, all lower degree polynomials appear in the expansion of \( z p_n(z) \). Similarly, \( \tilde{p}_n \) behave as follows:

\[ z^{-1} \tilde{p}_n(1/z) = \tilde{p}_{n+1}(1/z) + \sum_{k=0}^{n} C_k^{(n)} \tilde{p}_k(1/z), \] (2.30)
In the gauge theory parameters, 
\[ g \]
Recall that

2.2. String equations

In particular, let us consider the following three cases of (2.35): (i) \((k, \ell, m) = (-1, n, n - 1)\), (ii) \((k, \ell, m) = (0, n, n)\) and (iii) \((k, \ell, m) = (1, n - 1, n)\). They lead to the “string equations”

\[
\int d\mu(z) W_U'(z) p_n(z) \tilde{p}_{n-1}(1/z) = n(h_n - h_{n-1}),
\]

\[
\int d\mu(z) z W_U'(z) p_n(z) \tilde{p}_n(1/z) = 0,
\]

\[
\int d\mu(z) z^2 W_U'(z) p_n(z) \tilde{p}_{n-1}(1/z) = -n(h_n - h_{n-1}).
\]

3. Unitary matrix model with logarithmic potential

Let us consider the unitary matrix model with the following potential

\[ W_U(z) = -\frac{1}{2 g_s} \left( z + \frac{1}{z} \right) + M \log z. \]

In the gauge theory parameters, \( g_s = g_s/\Lambda_2 \) and \( M = \alpha_{1+2} + N = (m_2 - m_1)/g_s \). We assume that \( M \) is an integer. Note that \( \Gamma(2 g_s) = \Lambda_2/(2 g_s) = q_0/2 \).

3.1. Moments and related quantities

The moments for this potential are given by

\[
\mu_n = \int \frac{dz}{2\pi iz} \exp \left( -\frac{1}{2 g_s} \left( z + \frac{1}{z} \right) \right) z^{M+n} = \left( -\frac{1}{2 g_s} \right)^{|M+n|} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{|M+n|!} \left( \frac{1}{2 g_s} \right)^{2k},
\]
where \( I_\nu(z) \) is the modified Bessel function of the first kind:

\[
I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left( \frac{z}{2} \right)^{2k}.
\]  

(3.3)

Note that

\[
K_k^{(n)} = \det(\mu_{j-i+k})_{1\leq i,j\leq n} = \det \left( (-1)^{M+j-i+k} I_{|M+j-i+k|}(1/2) \right)_{1\leq i,j\leq n}
\]  

(3.4)

where

\[
K_k^{(n)} := \det \left( I_{j-i+\nu}(1/2) \right)_{1\leq i,j\leq n}, \quad (\nu \in \mathbb{C}; n = 0, 1, 2, \cdots).
\]  

(3.5)

For an integer \( k \), it holds that \( I_{-k}(z) = I_k(z) \). Therefore, for \( j - i + M + k \in \mathbb{Z} \), we have \( I_{j-i+M+k}(1/2) = I_{j-i+M+k}(1/2) \). Also, we have \( K_k^{(n)} = K_k^{(n)} \) (\( k \in \mathbb{Z} \)). For later convenience, we have defined \( K_\nu^{(n)} \) (3.5) as a determinant of \( I_{j-i+\nu}(1/2) \) such that the index \( M \) in \( K_M^{(n)} \) can be analytically continued from an integer to any complex number.

Note that

\[
\tau_n = K_0^{(n)} = (-1)^n M K_M^{(n)}.
\]  

(3.6)

The normalization constants of the orthogonal polynomials are given by

\[
h_n = \frac{K_0^{(n+1)}}{K_0^{(n)}} = (-1)^M \frac{K_M^{(n+1)}}{K_M^{(n)}}.
\]  

(3.7)

In particular, \( h_0 = (-1)^M I_M(1/2) \).

The constant term of the orthogonal polynomials are written in terms of \( K_k^{(n)} \) as follows:

\[
A_n = p_n(0) = (-1)^n \frac{K_0^{(n)}}{K_0^{(n)}} = \frac{K_M^{(n+1)}}{K_M^{(n)}}.
\]  

(3.8)

\[
B_n = \tilde{p}_n(0) = (-1)^n \frac{K_0^{(n)}}{K_0^{(n)}} = \frac{K_M^{(n)}}{K_M^{(n)}}.
\]  

(3.9)

The partition function (2.27) can be written in terms of these objects:

\[
Z_{U(N)} = (-1)^MN K_M^{(N)} = \prod_{k=0}^{N-1} h_k = h_0^N \prod_{j=1}^{N-1} \left( 1 - A_j B_j \right)^{N-j}.
\]  

(3.10)

This partition function is essentially the tau function of the Painlevé III equation.

### 3.2. String equations

Let us write the string equations (2.36), (2.37) and (2.38) explicitly for the case of the potential (3.1). Since

\[
W'_U(z) = -\frac{1}{2^2} \left( 1 - \frac{1}{z^2} \right) + \frac{M}{z},
\]  

(3.11)
we have
\[\int d\mu(z) W_U'(z)p_n(z)\tilde{p}_{n-1}(1/z) = \frac{1}{2gs}(\tilde{C}_n^{(n)} + \tilde{C}_{n-1}^{(n-1)})h_n + Mh_n, \quad (3.12)\]
\[\int d\mu(z) z W_U'(z)p_n(z)\tilde{p}_n(1/z) = -\frac{1}{2gs}\left(C_n^{(n)} - \tilde{C}_n^{(n)}\right)h_n + Mh_n, \quad (3.13)\]
\[\int d\mu(z) z^2 W_U'(z)p_{n-1}(z)\tilde{p}_{n}(1/z) = -\frac{1}{2gs}\left(C_{n-1}^{(n-1)} + C_n^{(n-1)}\right)h_n + Mh_n. \quad (3.14)\]
Here we have used (2.28) and (2.30).

Then the string equations (2.36), (2.37), (2.38) for this potential become
\[\frac{1}{2gs}(\tilde{C}_n^{(n)} + \tilde{C}_{n-1}^{(n-1)}) + M = n\left(1 - \frac{h_n-1}{h_n}\right), \quad (3.15)\]
\[\frac{1}{2gs}\left(C_n^{(n)} - \tilde{C}_n^{(n)}\right) + M = 0, \quad (3.16)\]
\[\frac{1}{2gs}\left(C_{n-1}^{(n-1)} + C_n^{(n-1)}\right) + M = -n\left(1 - \frac{h_n-1}{h_n}\right), \quad (3.17)\]
Using
\[\frac{h_n}{h_{n-1}} = 1 - A_nB_n, \quad C_n^{(n)} = -A_{n+1}B_n, \quad \tilde{C}_n^{(n)} = -A_nB_{n+1}, \quad (3.18)\]
the string equations lead to the following recursion relations for \(A_n\) and \(B_n\):
\[A_{n+1} = -A_{n-1} + \frac{2ngs A_n}{1-A_nB_n}, \quad B_{n+1} = -B_{n-1} + \frac{2ngs B_n}{1-A_nB_n}, \quad (3.19)\]
\[A_nB_{n+1} - A_{n+1}B_n = 2Mgs. \quad (3.20)\]
With the initial conditions \(A_0 = B_0 = 1,\) and
\[A_1 = \frac{I_{M+1}(1/gs)}{I_M(1/gs)}, \quad B_1 = \frac{I_{M-1}(1/gs)}{I_M(1/gs)}, \quad (3.21)\]
the remaining constants \(A_n\) and \(B_n\) are characterized by the recursion relations (3.17), (3.18). We remark that one of recursion relations (3.17) can be obtained by combining the other of (3.17) with (3.18).

Recall that the modified Bessel function satisfies the following recursion relation:
\[I_{\nu-1}(z) - I_{\nu+1}(z) = (2\nu/z)I_{\nu}(z). \quad (3.22)\]
By examining (3.18) for \(n = 0,\) we can see that the range of the parameter \(M\) in the initial conditions (3.19) can be extended from the integers to any complex numbers. Furthermore,
\[A_n(M) = \frac{K_{M+1}^{(n)}}{K_{M}^{(n)}}, \quad B_n(M) = \frac{K_{M-1}^{(n)}}{K_{M}^{(n)}}, \quad (M \in \mathbb{C}) \quad (3.23)\]
indeed solve the string equations (3.17) and (3.18). Here \(K_n^{(n)}\) is defined by (3.5).
Note that the partition function (3.10) depends on \( A_j \) and \( B_j \) only through their product \( A_j B_j \). Let \( A_n = R_n D_n \) and \( B_n = R_n / D_n \). Then the partition function (3.10) becomes

\[
Z_U(N) = h_0^N \prod_{j=1}^{N-1} (1 - R_j^2)^{N-j}.
\]

(3.22)

The equation (3.18) turns into

\[
R_n R_{n+1} \left( \frac{D_n}{D_{n+1}} - \frac{D_{n+1}}{D_n} \right) = 2 M g_s.
\]

(3.23)

This leads to

\[
\begin{align*}
\frac{D_n}{D_{n+1}} &= \frac{M g_s + \sqrt{R_n^2 R_{n+1}^2 + M^2 g_s^2}}{R_n R_{n+1}}, \\
\frac{D_{n+1}}{D_n} &= -\frac{M g_s + \sqrt{R_n^2 R_{n+1}^2 + M^2 g_s^2}}{R_n R_{n+1}}.
\end{align*}
\]

(3.24)

(3.25)

By substituting these relations into the remaining relations (3.17), we find

\[
(1 - R_n^2) \left( \sqrt{R_n^2 R_{n+1}^2 + M^2 g_s^2} + \sqrt{R_n^2 R_{n-1}^2 + M^2 g_s^2} \right) = 2 n g_s R_n^2.
\]

(3.26)

This is equivalent to

\[
0 = \eta_n^2 \left[ \xi_n^2 (1 - \xi_n)^2 - \eta_n^2 \xi_n^2 + \zeta^2 (1 - \xi_n)^2 \right] + \frac{1}{2} \eta_n^2 \xi_n (1 - \xi_n)^2 (\xi_{n+1} - 2 \xi_n + \xi_{n-1}) - \frac{1}{16} (1 - \xi_n)^4 (\xi_{n+1} - \xi_{n-1})^2,
\]

(3.27)

where \( \xi_n \equiv R_n^2, \eta_n \equiv n g_s, \zeta \equiv M g_s \).

When \( M = 0 \) (i.e., with no logarithmic potential), (3.26) reduces to the string equation considered in [9]

\[
(1 - R_n^2) R_n (R_{n+1} + R_{n-1}) = 2 n g_s R_n^2.
\]

(3.28)

Let us introduce variables \( x_n \) and \( y_n \) by

\[
x_n := \frac{A_n}{A_{n+1}}, \quad y_n := \frac{B_{n+1}}{B_n}, \quad (n = 0, 1, 2, \ldots ).
\]

(3.29)

They respectively obey the alternate discrete Painlevé II equation [14, 15] with different values of the parameter \( \bar{\mu} \). With the initial conditions \( A_0 = 1 \) and \( B_0 = 1, A_n \) and \( B_n \) can be expressed by these variables:

\[
A_n = \prod_{k=0}^{n-1} x_k, \quad B_n = \prod_{k=0}^{n-1} y_k.
\]

(3.30)

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