Positive Solutions for a Class of Quasilinear Schrödinger Equations with Nonlocal Term

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Abstract

This paper is considered the existence of positive solutions for a class of generalized quasilinear Schrödinger equations with nonlocal term in $\mathbb{R}^N$ which have appeared from plasma physics, as well as high-power ultrashort laser in matter. We use a change of variables and obtain the existence of solutions via minimization argument.

Keywords

Quasilinear Schrödinger Equation, Minimization, Implicit Function Theorem

1. Introduction

In this paper, we consider investigating the existence of solutions for the following generalized quasilinear Schrödinger equation with nonlocal term

\[-div\left( g(u) \nabla u \right) + g(u) g'(u) |\nabla u|^2 + V(x) u = \lambda |x|^\mu \ast |u|^\nu |u|^{p-2} u + \beta |u|^{q-1} u, \tag{1.1}\]

where $N \geq 3$, $0 < \mu < N$, $\beta < \frac{N-2}{2N}$, $1 \leq q \leq \frac{N}{N-2}$, $\frac{2N-\mu}{N} \leq p < \frac{2N-\mu}{N-2}$, the function $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, $g$ is a $C^1$ function with $g'(t) \leq 0$ for all $t > 0$, $g(0) = 0$, and $g(t) \to a$ as $t \to \infty$.

When $g(u) = 1$, (1.1) boils down to the so called nonlinear Choquard or Choquard-Pekar equation

\[-\Delta u + V(x) u = \lambda |x|^\mu \ast |u|^\nu |u|^{p-2} u + \beta |u|^{q-1} u \tag{1.2}\]

Such like equation has several physical origins. The problem

\[-\Delta u + u = |x|^{-\mu} \ast |u|^\nu |u|^{p-2} u, \tag{1.3}\]
appeared at least as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [1]. In 1976, Choquard used (1.3) to describe an electron trapped in its own hole and in a certain approximation to Hartree-Fock theory of one component plasma [2]. In 1996, Penrose proposed (1.3) as a model of self-gravitating matter, in a program in which quantum state reduction is understood as a gravitational phenomenon [3]. In this context, equation of type (1.3) is usually called the nonlinear Schrödinger-Newton equation. The first investigations for the existence and symmetry of the solutions to (1.3) go back to the works of Lieb [2] and Lions [4]. In [2], by using symmetric decreasing rearrangement inequalities, Lieb proved that the ground state solution of Equation (1.3) is radial and unique up to translations. Lions [4] showed the existence of a sequence of radially symmetric solutions. Ma and Zhao [5] considered the generalized Choquard equation

$$-\Delta u + u = \left[|x|^{-\mu} * |u|^p\right]|u|^{p-2} u \quad (q \geq 2),$$

(1.4)

and proved that every positive solution of it is radially symmetric and monotone decreasing about some fixed point, under the assumption that a certain set of real numbers, defined in terms of $N$ and $q$, is nonempty. Under the same assumption, Cingolani, Clapp, and Secchi [6] gave some existence and multiplicity results in the electromagnetic case and established the regularity and some decay asymptotically at infinity of the ground states. In [7], Moroz and Van Schaftingen eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters and derived decay asymptotically at infinity for them as well. Moreover, they [8] also obtained a similar conclusion under the assumption of Berestycki-Lions type nonlinearity. We point out that the existence, multiplicity, and concentration of such like equation have been established by many authors. We refer the readers to [9] [10] for the existence of sign-changing solutions, [11] [12] for the existence and concentration behavior of the semiclassical solutions and [13] for the critical nonlocal part with respect to the Hardy-Littlewood-Sobolev inequality. For more details associated with the Choquard equation, please refer to [14] [15] [16] and the references in. Li, Teng, Zhang, Nie [17] investigate the existence of solutions for the following generalized quasilinear Schrödinger equation with nonlocal term

$$-\text{div}\left(g^2(u)\nabla u\right) + g(u)g'(u) + |\nabla u|^2 + V(x) u = \lambda \left[|x|^{-\mu} * |u|^p\right]|u|^{p-2} u, \quad x \in \mathbb{R}^N$$

(1.5)

and prove the existence of solution.

In this paper, our main ideas come from [18] and the assumption of $g$ from [19]. Our purpose is to search for the existence of nontrivial solutions of (1.1) by implicit function theorem. For convenience, we introduce several notations: $C$ denotes a positive (possibly different) constant, $L^p\left(\mathbb{R}^N\right)$ denotes the usual Lebesgue space with norms

$$\|u\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad C_0^\infty\left(\mathbb{R}^N\right) \quad \text{be the collec-}$$
tion of smooth functions with compact support. Next, we introduce the energy functional of Equation (1.1)
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ g^2(u) |\nabla u|^2 + V(x) u^2 \right] \, dx
- \frac{\lambda}{2p} \int_{\mathbb{R}^N} \left[ |u|^p \star |u|^p \right] |u|^{p-1} \, dx - \frac{\beta}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} \, dx
\]
(1.6)
however, \( J \) is not well defined in \( H^1(\mathbb{R}^N) \) because of the term \( \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 \, dx \).
To overcome this difficulty, we make a change of variable constructed by Shen and Wang in [20]: \( v = G(u) = \int_0^u g(s) \, ds \), then,
\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla v|^2 + V(x) G^{-1}(v) \right] \, dx
- \frac{\lambda}{2p} \int_{\mathbb{R}^N} \left[ |v| \star |G^{-1}(v)| \right] |G^{-1}(v)| |\nabla v|^2 \, dx + \frac{\beta}{q+1} \int_{\mathbb{R}^N} |G^{-1}(v)|^{q+1} \, dx
= \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla v|^2 + V(x) G^{-1}(v) \right] \, dx
- \frac{\lambda}{2p} \int_{\mathbb{R}^N} \left( \frac{|G^{-1}(v(y))|^{p+1}}{|x-y|^\mu} \right) \, dx dy - \frac{\beta}{q+1} \int_{\mathbb{R}^N} |G^{-1}(v)|^{q+1} \, dx
\]
(1.7)
We say that \( u \) is a weak solution of (1.1), if
\[
\langle J'(u), \phi \rangle = \int_{\mathbb{R}^N} \left( g^2(u) \nabla u \nabla \phi + \frac{g(u) g'(u)}{2} |u|^2 \phi + V(x) u \phi \right) \, dx
- \frac{\lambda}{p} \int_{\mathbb{R}^N} \left[ |x| \star |u|^p \right] |u|^{p-2} u \phi - \frac{\beta}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} u \phi \, dx
\]
(1.8)
for all \( \phi \in C^0_c(\mathbb{R}^N) \). Let \( \phi = \left( \frac{1}{g(u)} \right) \), by [20], we know that the above formula is equivalent to
\[
\langle J'(v), \psi \rangle = \int_{\mathbb{R}^N} \left( \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \right) \psi \, dx
- \frac{\lambda}{p} \int_{\mathbb{R}^N} \left[ |x| \star |G^{-1}(v)|^{p+1} \right] g(g^{-1}(v)) |G^{-1}(v)|^{p-2} G^{-1}(v) \psi \, dx
- \frac{\beta}{q+1} \int_{\mathbb{R}^N} \frac{|G^{-1}(v)|^{q+1} G^{-1}(v)}{g(G^{-1}(v))} \psi \, dx
\]
(1.9)
for all \( \psi \in C^0_c(\mathbb{R}^N) \). Therefore, in order to find the solution of (1.1), it suffices to study the solution of following equation:
\[
-\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{\lambda}{p} \left[ |x| \star |G^{-1}(v)|^{p+1} \right] \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))}
- \frac{\beta}{q+1} \left( \frac{|G^{-1}(v)|^{q+1} G^{-1}(v)}{g(G^{-1}(v))} \right) = 0
\]
(1.10)
\( J \) is defined on the space
we can define the norm on $H^1_0(\mathbb{R}^N)$ by
\[
\|v\|^2_{H^1_0(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla v|^2 + V(x)v^2 \, dx
\]
then, $H^1_0(\mathbb{R}^N)$ is a Banach space. In the following, we always assume $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $\inf_{\mathbb{R}^N} V(x) \geq 1$. Let us consider the following assumptions of potential function $V(x)$:
\begin{enumerate}
\item[(V1)] $\lim_{x \to \infty} V(x) = +\infty$;
\item[(V2)] $V(x)$ is radially symmetric.
\end{enumerate}
Next, we will introduce the properties of some functions.

**Lemma 1.1.** [19] The function $g(t), G^{-1}(t), G(t)$ enjoys the following properties.
\begin{enumerate}
\item[(g1)] the function $G(t)$ and $G^{-1}(t)$ are strictly increasing and odd;
\item[(g2)] $|t| \leq |G(t)| \leq |t|/a$ for all $t \in \mathbb{R}$;
\item[(g3)] $G^{-1}(t)/t$ is nondecreasing for all $t \in \mathbb{R}$ and $\lim_{t \to 0^+} G^{-1}(t)/t = 1$, \( \lim_{t \to -\infty} G^{-1}(t)/t = 1/a \);
\item[(g4)] $t^2 \leq (t/g(t))G(t) \leq t^2/a$ for all $t \in \mathbb{R}$.
\end{enumerate}
Next, we set forth some preliminary results.

**2. Preliminary Results**

To begin with, we prove some functions are continuous, more detailed see [21].

**Lemma 2.1** If $\|v_n - v\|_{H^1_0(\mathbb{R}^N)} \to 0$, then
\[
\int_{\mathbb{R}^N} V(x)G^{-1}(v_n)^2 - V(x)G^{-1}(v)^2 \, dx \to 0.
\]

**Proof:** By sobolev imbedding inequality, Lemma 1.1 and definition of $g$, we have
\[
\int_{\mathbb{R}^N} V(x)G^{-1}(v_n)^2 - V(x)G^{-1}(v)^2 \, dx \\
= 2\int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v + \theta(v_n - v))}{G(G^{-1}(v + \theta(v_n - v)))}(v_n - v) \, dx \\
\leq \frac{2}{a^2} \int_{\mathbb{R}^N} V(x)||v + \theta(v_n - v)||v_n - v||v_n - v|| \, dx \\
\leq \frac{2}{a^2} \int_{\mathbb{R}^N} V(x)||v_n - v||^2 \, dx \int_{\mathbb{R}^N} V(x)||v + \theta(v_n - v)||^2 \, dx \\
\leq \frac{2}{a^2} \int_{\mathbb{R}^N} V(x)||v_n - v||^2 \, dx \int_{\mathbb{R}^N} V(x)||v_n - v||^2 \, dx \\
\leq \frac{2}{a^2} \int_{\mathbb{R}^N} V(x)||v_n - v||^2 \, dx \int_{\mathbb{R}^N} V(x)||v_n - v||^2 \, dx \\
\leq \frac{2}{a^2} \|v_n - v\|_{H^1_0(\mathbb{R}^N)} \left( C + \|v_n - v\|_{H^1_0(\mathbb{R}^N)} \right) \to 0, n \to +\infty
\]
where $\theta \in (0, 1)$.

**Lemma 2.2.** The map $v \to G^{-1}(v)$ from $H^1_0(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ is continuous for $2 \leq r < 2^*$.

**Proof:** By the definition of $g$, we have
\[ \int_{\mathbb{R}^n} \left| G^{-1}(v_n) - G^{-1}(v) \right|^p \, dx \leq \int_{\mathbb{R}^n} \left| G^{-1}(v_n) - G^{-1}(v) \right|^p \, dx \leq \frac{1}{a^p} \int_{\mathbb{R}^n} \left( |v_n| + |v| \right)^p \, dx. \]

Assume \( v_n \to v \) in \( H^1_r (\mathbb{R}^N) \), moreover, the imbedding from \( H^1_r (\mathbb{R}^N) \) into \( L^r (\mathbb{R}^N) \) is compact where \( r \in [2, 2^*] \), from Lemma 3.4 [22], we get the result. ■

Next, we introduce some minimization with corresponding energy functional and define

\[ m_b = \inf_{u \in M_b} E(u) \]

where

\[ M_b = \left\{ u \in H^1 (\mathbb{R}^N) : \| u \|_{L^{p+1}} = b \right\}, \quad a > 0 \]

and

\[ E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ g^2(u) \left| \nabla u \right|^2 + V(x)u^2 \right] \, dx - \frac{\beta}{q+1} \int_{\mathbb{R}^N} \left| u \right|^{q+1} \, dx. \]

We also define

\[ \omega_b = \inf_{v \in \mathbb{R}^N} F(v) \]

where

\[ W_b = \left\{ v \in H^1_r (\mathbb{R}^N) : \left\| G^{-1}(v) \right\|_{L^{p+1}} = b \right\}, \quad b > 0 \]

and

\[ F(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ \left| \nabla v \right|^2 + V(x)G^{-1}(v)^2 \right] \, dx - \frac{\beta}{q+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{q+1} \, dx. \]

Therefore, we have following fact.

**Lemma 2.3.** \( m_b = \omega_b \) for every \( a > 0 \).

**Proof:** For any \( v \in W_b \), let \( u = G^{-1}(v) \), from the definition of \( g \), we get

\[ \int_{\mathbb{R}^N} \left| \nabla u \right|^2 \, dx = \int_{\mathbb{R}^N} \frac{\left| \nabla v \right|^2}{g^{-2}} \, dx \leq \frac{1}{a^2} \int_{\mathbb{R}^N} \left| \nabla v \right|^2 \, dx < +\infty, \]

\[ \int_{\mathbb{R}^N} u^2 \, dx \leq \int_{\mathbb{R}^N} V(x)G^{-1}(v)^2 \, dx < +\infty, \]

so \( u \in M_b \). It follows that \( F(v) = E(G^{-1}(v)) = E(u) \geq m_b \), hence \( \omega_b \geq m_b \), moreover, for any \( v \in M_b \), let \( v = G(u) \), then \( u = G^{-1}(v) \). We assume \( E(u) < +\infty \), since \( u \in H^1_r (\mathbb{R}^N) \), \( 2 < q+1 < 2^* \), then \( u \in L^{q+1} (\mathbb{R}^N) \). We have

\[ \frac{1}{2} \int_{\mathbb{R}^N} \left[ g^2(u) \left| \nabla u \right|^2 + V(x)u^2 \right] \, dx = E(u) + \frac{\beta}{q+1} \int_{\mathbb{R}^N} \left| u \right|^{q+1} \, dx < +\infty. \]

Then \( \int_{\mathbb{R}^N} V(x)G^{-1}(v)^2 \, dx = \int_{\mathbb{R}^N} V(x)u^2 \, dx < +\infty \). It shows that \( v \in W_b \), which implies that \( E(u) = E(G^{-1}(v)) = F(v) \geq \omega_b \), hence \( m_b \geq \omega_b \), this completes the proof. ■

**Lemma 2.4.** 1) \( F(v) \) is well defined and continuous for \( 2 \leq r \leq 2^* \).

2) \( F(v) \) is Gateaux-differentiable. For \( v \in H^1_r (\mathbb{R}^N) \), the \( G \)-derivative \( F'(v) \) is a continuous function, and \( F'(v) \) is continuous in \( v \) in the strongly-weak
topology, that is, if \( v_n \to v \) strongly in \( H^1_\vartheta (\mathbb{R}^N) \), then \( F'(v_n) \to F'(v) \) weakly.

**Proof:** (1) For any \( v \in H^1_\vartheta (\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{q+1} \, dx \leq C \int_{\mathbb{R}^N} |v|^{q+1} \, dx < +\infty ,
\]
then
\[
F(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V(x) \left| G^{-1}(v) \right|^2 \, dx - \frac{\beta}{q+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{q+1} \, dx
\]
\[
\leq \frac{1}{2} a^2 \int_{\mathbb{R}^N} |\nabla v|^2 + V(x) v^2 \, dx + \frac{\beta}{q+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{q+1} \, dx
\]
\[
\leq \frac{1}{2} a^2 \int_{\mathbb{R}^N} |\nabla v|^2 + V(x) v^2 \, dx + \frac{\beta}{(q+1) a} \int_{\mathbb{R}^N} |v|^{q+1} \, dx < +\infty .
\]

with the proof of continuity, note that \( J \) consists of three terms. By Lemma 1.1, we need to check the superlinear term only.

\[
\left| \frac{1}{q+1} \int_{\mathbb{R}^N} \left| G^{-1}(v_n) \right|^{q+1} \, dx - \frac{1}{q+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{q+1} \, dx \right|
\]
\[
= \left| \int_{0}^{1} \int_{\mathbb{R}^N} \frac{\left| G^{-1}(v+t(v_n-v)) \right|^{q+1} G^{-1}(v+t(v_n-v))}{g \left( G^{-1}(v+t(v_n-v)) \right)} (v_n-v) \, dx \right|
\]
\[
\leq C \int_{0}^{1} \int_{\mathbb{R}^N} \left| v+t(v_n-v) \right|^q |v_n-v| \, dx \, dt
\]
\[
\leq \left\| v+t(v_n-v) \right\|_{L^q(\mathbb{R}^N)} \left\| v_n-v \right\|_{L^q(\mathbb{R}^N)}
\]
\[
\leq C \left\| v_n-v \right\|_{H^1_\vartheta (\mathbb{R}^N)}
\]

where \( 1 \leq q \leq \frac{N}{N-2} \).

For (2) we consider the second and the third terms of the functional \( J \), we see for \( \phi \in H^1_\vartheta (\mathbb{R}^N) \), using Hölder inequality, we get
\[
\left| \frac{1}{2} \int_{\mathbb{R}^N} V(x) \left( \frac{G^{-1}(v+t\phi)^2 - G^{-1}(v)^2}{g \left( G^{-1}(v) \right)} \right) \, dx - \int_{\mathbb{R}^N} \frac{V(x) G^{-1}(v)}{g \left( G^{-1}(v) \right)} \phi \, dx \right|
\]
\[
= \left| \int_{0}^{1} \int_{\mathbb{R}^N} V(x) \left( \frac{G^{-1}(v+t\phi)}{g \left( G^{-1}(v+t\phi) \right)} - \frac{G^{-1}(v)}{g \left( G^{-1}(v) \right)} \right) \phi \, dx \right|
\]
\[
\leq \int_{0}^{1} \int_{\mathbb{R}^N} V(x) \left| \frac{G^{-1}(v+t\phi)}{g \left( G^{-1}(v+t\phi) \right)} - \frac{G^{-1}(v)}{g \left( G^{-1}(v) \right)} \phi \right| \, dx \int_{0}^{1} \int_{\mathbb{R}^N} V(x) \phi^2 \, dx \right|^{\frac{1}{2}}
\]

Using the definition of \( g \) and Lemma 1.1, we know
\[
\left| \frac{G^{-1}(v+t\phi)}{g \left( G^{-1}(v+t\phi) \right)} - \frac{G^{-1}(v)}{g \left( G^{-1}(v) \right)} \right|^2 \leq \left| g \left( G^{-1}(v+t\phi) \right) + G^{-1}(v) \right|^2
\]
\[
\leq C \left( \left| G^{-1}(v+t\phi) \right|^2 + \left| G^{-1}(v) \right|^2 \right)
\]
\[
\leq C \left( \left| v+t\phi \right|^2 + \left| v \right|^2 \right)
\]
\[
\leq C \left( \left| v \right|^2 + \left| \phi \right|^2 \right).
\]
By the dominated convergence theorem
\[
\left| \int_{\mathbb{R}^N} \nabla \phi \cdot \nabla V(x) + \int_{\mathbb{R}^N} \frac{G^{-1}(v) G^{-1}(v)}{g(G^{-1}(v))} \phi \right| \to 0 \quad (t \to 0)
\]

For the third term, we have
\[
= \left| \int_{\mathbb{R}^N} \left( G^{-1}(v) G^{-1}(v) - G^{-1}(v) G^{-1}(v) \right) \phi \right| \to 0
\]

Similarly to above, by the dominated convergence theorem
\[
\lim_{t \to 0} \left| \int_{\mathbb{R}^N} \left( G^{-1}(v) G^{-1}(v) - G^{-1}(v) G^{-1}(v) \right) \phi \right| = 0
\]

The Gateaux derivative $J'(v)$ has the form
\[
\langle J'(v), \phi \rangle = \int_{\mathbb{R}^N} \nabla V(x) \phi dx + \int_{\mathbb{R}^N} \frac{G^{-1}(v) G^{-1}(v)}{g(G^{-1}(v))} \phi dx - \int_{\mathbb{R}^N} \frac{G^{-1}(v) G^{-1}(v)}{g(G^{-1}(v))} \phi dx
\]
\[
\leq C \| \phi \|_{L^2(\mathbb{R}^N)} + C \| G^{-1}(v) \|_{L^2(\mathbb{R}^N)}
\]

from Sobolev imbedding theorem, we get $J'(v)$ is a continuous linear functional on $H^1_{v} (\mathbb{R}^N)$.

Finally, the continuity with strong-weak topology is easy to check, as $v_n \to v$ in $H^1_{v} (\mathbb{R}^N)$, for any $\phi \in H^1_{v} (\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n)}{\left| G^{-1}(v_n) \right|} - V(x) \frac{G^{-1}(v)}{\left| G^{-1}(v) \right|} dx \to 0
\]

Remark 2.5. Lemma 2.4 does not show that $F(v)$ is $C^\infty$, so we cannot use the Lagrange multiplier theorem. But we can get our conclusion we want exactly by a similar argument for the Lagrange multiplier theorem. Next, we state our main conclusion. The idea of our proof is based on the work in [18] [22] [23].

3. Main Conclusion

Theorem 3.1. Let $N \geq 3, 1 \leq q < \frac{N}{N-2}, \frac{2N-\mu}{N} \leq p < \frac{2N-\mu}{N-2}, \beta \in \mathbb{R}$ and $q < p$. Assume $(V_1)$ or $(V_2)$ holds. Then for every $b > 0$, there exists $\lambda(b) \in \mathbb{R}$ such that Equation (1.1) with $\lambda = \lambda(b)$ has a positive weak solution $u \in M_b$.

Remark 3.2. From the assumption of $V$, we know $H^1_{v} (\mathbb{R}^N)$ embedding into $L^p (\mathbb{R}^N)$ is compact. In the process of the proof of theorem 3.1, it is important for us to construct auxiliary function, then by implicit function theorem to prove it and lemma 3.4 [22] play a great role in this paper. Moreover, when $\frac{N-2}{N} < q \leq 2$.
is a open question for Equation (1.1), someone could do it if they are interested.

**Proof of Theorem 3.1:** Step 1: By the assumptions of \((V_1)\) or \((V_2)\), \(b_\omega\) is achieved at some \(0 \leq \nu_0 \leq W_b\) with \(\nu_0 \neq 0\). Let \(\{v_n\} \in W_b\) be a minimizing sequence for \(b_\omega\). Set \(u_n = G^{-1}(v_n)\). Then \(\{u_n\} \in M_b\) is a minimizing sequence for \(m_b\). We can assume \(u_n \geq 0\). It shows that \(E(u_n) \rightarrow m_b\) so there exists \(C > 0\) such that

\[
C \geq E(u_n) \geq \frac{1}{2} \int_{\mathbb{R}^N} \left[ g^2(u_n) |\nabla u_n|^2 + V(x) u_n^2 \right] dx - \frac{\beta}{q+1} \int_{\mathbb{R}^N} |u_n|^{q+1} dx.
\]

By Hölder inequality,

\[
\int_{\mathbb{R}^N} |u_n|^{q+1} dx \leq \left( \int_{\mathbb{R}^N} |u_n|^2 dx \right)^{\frac{q+1}{2}} \left( \int_{\mathbb{R}^N} |u_n|^{p+1} dx \right)^{\frac{1}{p+1}}
\]

\[
\leq \frac{\lambda (q+1)}{2} \int_{\mathbb{R}^N} |u_n|^2 dx - \frac{(1-\lambda)(q+1)}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1} dx.
\]

where \(\lambda = \frac{2(p-q)}{(q+1)(p-1)}\). Then

\[
C \geq E(u_n) \geq \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla u_n|^2 + V(x) u_n^2 \right] dx - \frac{\beta}{q+1} \left( \frac{\lambda (q+1)}{2} \int_{\mathbb{R}^N} V(x) |u_n|^2 dx - \frac{(1-\lambda)(q+1)}{p+1} a^{p+1} \right)
\]

\[
\geq \left( \frac{1}{2} - \frac{\beta (p-q)}{(q+1)(p-1)} \right) \left( \int_{\mathbb{R}^N} \left[ |\nabla u_n|^2 + V(x) u_n^2 \right] dx - \frac{\beta (q-1)}{(q+1)(p-1)} a^{p+1} \right)
\]

Because of \(\beta < \frac{N-2}{2N}, \frac{1}{2} - \frac{\beta (p-q)}{(q+1)(p-1)} > 0\) it implies that \(u_n(x)\) is bounded in \(H^1_r(\mathbb{R}^N)\). By the compact embedding result from \(H^1_r(\mathbb{R}^N)\) into \(L^r(\mathbb{R}^N)\) for \(2 \leq r < 2^*\). We may assume that \(u_n \rightarrow u_b\) in \(H^1_r(\mathbb{R}^N)\), \(u_n \rightarrow u_b\) in \(L^r(\mathbb{R}^N)\) for \(2 \leq r < 2^*\) and \(u_n(x) \rightarrow u_b(x)\) a.e \(x \in \mathbb{R}^N\). Hence \(u_b \in M_b\), since \(u_n \geq 0, u_b \geq 0\) and \(u_b \neq 0\).

Using the same argument as the process of the proof of Lemma 2.1 in [20] and noting that \(u_n \rightarrow u_b\) in \(L^{p+1}(\mathbb{R}^N)\). We have

\[
m_b = \lim_{n \rightarrow \infty} E(u_n)
\]

\[
\geq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} \left[ g^2(u_n) |\nabla u_n|^2 + V(x) u_n^2 \right] dx - \frac{\beta}{q+1} \int_{\mathbb{R}^N} |u_n|^{q+1} dx \right\}
\]

Hence \(m_b\) is achieved at \(u_b\) and.
and the property of $g$ implies $v_b \geq 0$ and $v_b \neq 0$.

Step 2: Set $h_p = \frac{1}{2p} \int_{\mathbb{R}^{2N}} \frac{|G^{-1}(v(x))|^p - |G^{-1}(v(y))|^p}{|x-y|^p} \, dx \, dy$ for

$$2N - \mu \frac{N}{N-2} < p < 2N - \mu \frac{N}{N-2}.$$ Then $h_p(v) \in C^1 \left( H^1_v \left( \mathbb{R}^N \right), \mathbb{R} \right)$. Actually, for any $\varphi \in H^1_v \left( \mathbb{R}^N \right)$, by Sobolev inequality and Hölder inequality, we get

$$\left| \int_{\mathbb{R}^N} \left| x'' \ast |G^{-1}(v)|^p \right| \left| G^{-1}(v(x)) \right|^{-2} G^{-1}(v(x)) \right| \varphi \, dx \leq C \left\| \int_{\mathbb{R}^N} \left| G^{-1}(v(y)) \right|^p \left| G^{-1}(v(x)) \right|^{p-1} \varphi \, dy \right\|_{L^p(\mathbb{R}^N)} \leq C \left\| G^{-1}(v) \right\|_{L^p(\mathbb{R}^N)} \left\| \varphi \right\|_{L^r(\mathbb{R}^N)} \leq C \left\| \varphi \right\|_{L^r(\mathbb{R}^N)}$$

where $r = \frac{2N}{2N - \mu}$ and $2 \leq pr < 2'$, so $h_p(v) \in \left( H^1_v \left( \mathbb{R}^N \right) \right)^*$. Let $v_n \to v$ in $H^1_v \left( \mathbb{R}^N \right)$. Up to a subsequence, we can assume $v_n \to v$ a.e. in $\mathbb{R}^N$ and $v_n \to v$ in $L^r \left( \mathbb{R}^N \right)$ for $2 \leq r < 2'$. Hence

$$\left| \int_{\mathbb{R}^N} \left| x'' \ast |G^{-1}(v_n)|^p - |G^{-1}(v)|^p \right| \left| G^{-1}(v_n) \right|^{p-2} G^{-1}(v_n) \right| \varphi \, dx \leq C \left\| G^{-1}(v_n) \right\|_{L^p(\mathbb{R}^N)} \left\| \varphi \right\|_{L^r(\mathbb{R}^N)} + C \left\| G^{-1}(v_n) \right\|_{L^p(\mathbb{R}^N)} \left\| \varphi \right\|_{L^r(\mathbb{R}^N)} \left( 3.1 \right)$$

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Since $v_n \to v$ in $H^1_0(\mathbb{R}^N)$, we get $\|v_n\|_{d(\mathbb{R}^N)} \leq C$, $2 \leq p^* < 2^*$ and by definition $g$, then
\[
\left\| G^{-1}(v_n) \right\|_{L^p(\mathbb{R}^N)} \left\| G^{-1}(v) \right\|_{L^p(\mathbb{R}^N)} \leq C \left( \|v_n\|_{d(\mathbb{R}^N)} + \|v\|_{d(\mathbb{R}^N)} \right),
\]
then by Lemma 3.4 [22] and assumption, we get
\[
\left\| G^{-1}(v_n) \right\|_{L^p(\mathbb{R}^N)} \left\| G^{-1}(v) \right\|_{L^p(\mathbb{R}^N)} \|G^{-1}(v_n)\| \|G^{-1}(v)\| \phi \left\| \right\|_{L^p(\mathbb{R}^N)} \to 0, \quad n \to +\infty.
\]
Similarly,
\[
\left\| G^{-1}(v) \right\|_{L^p(\mathbb{R}^N)} \left\| \frac{G^{-1}(v_n) - G^{-1}(v)}{v} \right\|_{L^p(\mathbb{R}^N)} \phi \left\| \right\|_{L^p(\mathbb{R}^N)} \to 0, \quad n \to +\infty.
\]
Hence, follows that $h_p(v) \in C^1 \left( H^1_0(\mathbb{R}^N), \mathbb{R} \right)$ for $\frac{2N - \mu}{N} < p < \frac{2N - \mu}{N - 2}$.

Step 3: For any $b \geq 0$, there exists $\lambda(b) \in \mathbb{R}$ such that $0 < u_b = G^{-1}(v_b) \in M_b$ is a weak solution of Equation (1.1) with $\lambda = \lambda(b)$. In fact, by Lemma 2.4,
\[
\langle F(v), \phi \rangle = \int_{\mathbb{R}^N} \nabla v \nabla \phi + \int_{\mathbb{R}^N} \frac{V(x)G^{-1}(v)}{g(G^{-1}(v))} \phi dx - \beta \int_{\mathbb{R}^N} \left| \frac{G^{-1}(v)^{p-1}G^{-1}(v)}{g(G^{-1}(v))} \right| \phi dx
\]
and $F(v) \in \left( H^1_0(\mathbb{R}^N) \right)^*$. Since $h_p(v) \in C \left( H^1_0(\mathbb{R}^N), \mathbb{R} \right)$ and $v_b \in W_b$, the implicit function theorem implies that for all $v \in N(h_p(v_b))$ (the null space of $h_p(v_b)$), there exist a $C^1$-map $g : [0,1] \to W_b$ such that $f(0) = v_a$ and $g'(0) = v$. Now, we prove $\langle F(v_a), v \rangle = 0$ for all $v \in N(h_p(v_b))$. Indeed, for every $t > 0$, $f(t) = v_a + \alpha(t) \in W_b$, where $\alpha(t) \to 0$ as $t \to 0$. Let
\[
\Phi(r) = F(v_a + r(v + \alpha(t))
\]
By Lemma 2.4
\[
\Phi'(r) = \lim_{\delta \to 0} \frac{\Phi(r + \delta) - \Phi(r)}{\delta} = \lim_{\delta \to 0} \frac{F'(v_a + (r + \delta)(v + \alpha(t)) - F'(v_a + r(v + \alpha(t)))}{\delta} = \langle F'(v_a + r(v + \alpha(t))), v + \alpha(t) \rangle.
\]
Hence there exist an $\theta \in (0,1)$ such that
\[
F'(v_a + tv + \alpha(t)) - F(v_a) = \langle F'(v_a + \theta(tv + \alpha(t))), tv + \alpha(t) \rangle
\]
\[
= t \left( F'(v_a + \theta(v + \alpha(t))), v \right) + t \left( F'(v_a + \theta(v + \alpha(t))), \alpha(t) \right) / t.
\]
Take limit $t \to 0$, by Lemma 2.4, one has $F'(v_a + \theta(tv + \alpha(t))) \rightarrow F'(v_a)$ weakly. It follows that $\langle F'(v_a + \theta(tv + \alpha(t))), v \rangle \rightarrow \langle F'(v_a), v \rangle$ and
\[ \left\{ F'(v_b + \theta (tv + o(t))) \right\} \text{ is bounded. Since } \frac{o(t)}{t} \to 0 \text{ as } t \to 0, \text{ we have} \]
\[ \left\{ F'(v_b + \theta (tv + o(t))), \frac{o(t)}{t} \right\} \to 0. \]

Since \( F(v_b) = \omega_b \), one has
\[ 0 \leq F(v_b + tv + o(t)) - F(v_b) = t \left( F'(v_b + \theta (tv + o(t))), v \right) + t \left( F'(v_b + \theta (tv + o(t))), \frac{o(t)}{t} \right). \]

Hence
\[ 0 \leq \left( F'(v_b + \theta (tv + o(t))), v \right) + \left( F'(v_b + \theta (tv + o(t))), \frac{o(t)}{t} \right). \]

Take limit \( t \to 0 \), we get \( \left( F'(v_b), v \right) \geq 0 \). By arbitrariness of \( v \), one has \( \left( F'(v_b), -v \right) \geq 0 \). It follows that \( \left( F'(v_b), v \right) = 0 \), for every \( v \in N(h'_p(v_b)) \). Set \( v' \in H^1_\omega(\mathbb{R}^N) \) be such that \( \left( h'_p(v_b), v' \right) = 1 \), for every \( \varphi \in H^1_\omega(\mathbb{R}^N) \), let
\[ \psi = \varphi - \left( h'_p(v_b), \varphi \right) v'. \]

Then \( \psi \in N(h'_p(v_b)) \). It means \( \left( F'(v_b), \psi \right) = 0 \), i.e.
\[ \left( F'(v_b), \varphi \right) = \left( F'(v_b), v' \right) \left( h'_p(v_b), \varphi \right). \]

Put \( \lambda = \lambda(b) = \left( F'(v_b), v' \right) \), we have
\[ \left( F'(v_b), \varphi \right) = \lambda \left( h'_p(v_b), \varphi \right). \]

namely,
\[
\int_{\mathbb{R}^N} \nabla v_b \nabla \varphi dx + \int_{\mathbb{R}^N} \frac{V(x) G^{-1}(v_b)}{g(G^{-1}(v_b))} \varphi dx
\]
\[ = \lambda \int_{\mathbb{R}^N} \left[ \left| x \right|^\mu \ast \left| G^{-1}(v_b) \right|^p \right] \frac{G^{-1}(v_b)}{g(G^{-1}(v_b))} \varphi dx
\]
\[ + \beta \int_{\mathbb{R}^N} \frac{G^{-1}(v_b)}{g(G^{-1}(v_b))} \varphi dx. \]

It implies that \( u_b = G^{-1}(v_b) \in M_b \) is a weak solution of Equation (1.1). Moreover, the maximum principle implies \( u_b > 0 \).

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**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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