NON-ARCHIMEDEAN TRANSPORTATION PROBLEMS AND KANTOROVICH ULTRA-NORMS

MICHAEL MEGRELI\textregistered HVILI AND MENACHEM SHLOSSBERG

Abstract. We study a non-archimedean (NA) version of transportation problems and introduce naturally arising ultra-norms which we call Kantorovich ultra-norms. For every ultra-metric space and every discretely valued NA field (e.g., the field \( \mathbb{Q}_p \) of \( p \)-adic numbers) the naturally defined inf-max cost formula achieves its infimum. We also present NA versions of the Arens-Eells construction and of the integer value property. We introduce and study free NA locally convex spaces. In particular, we provide conditions under which these spaces are normable by Kantorovich ultra-norms and also conditions which yield NA versions of Tkachenko-Uspenskij theorem.

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1. Introduction

Kantorovich norm [6] plays a major role in various areas of mathematics, economics and computer science (see [3, 5, 11, 12, 13, 19, 22]). For instance, in Monge-Kantorovich transportation problem. The norms that determine the topology of the free real locally convex space are in fact Kantorovich norms (see [5, 11, 12, 19]). Uspenskij [21] provided a simplified formula for these norms.

In this paper we deal with discrete transportation problems. In Subsection 2.2 we present a slightly more flexible ("democratic") approach to the classical Kantorovich problem. Continuing in this direction, we study in Section 3 non-archimedean transportation problems. Since the word non-archimedean appears many times in this work we write shortly: NA.

In Section 4 we present an NA version of the Arens-Eells embedding (Theorem 4.3) and introduce the naturally arising Kantorovich ultra-norms defined on free vector spaces \( L_f(X) \) via inf-max formula. Probably one can encounter a variety of min-max optimization problems when dealing with the Kantorovich ultra-norms. It is worth noting that different

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algorithms for solving such problems are known (see [2] for example). Theorem 4.3 also shows that for an arbitrary NA valued field \( F \) and for any vector \( u \in L_G(X) \) the value of \( ||u|| \) can be approximated by computations on the support elements of \( u \). Note that the analogue property in the archimedean case does not hold in general. Indeed, it is no longer true when \( F = \mathbb{C} \), the field of complex numbers, in contrast to the case \( F = \mathbb{R} \) (see Flood [4, p. 90] and Remark 2.3.3 below).

In case that \( F \) is an NA field with discrete valuation the infimum is attained for any ultra-metric space \( X \) (Min-attaining Theorem 5.8). Note that discretely valued fields form a major subclass in the class of NA valued fields. This subclass is closed under taking arbitrary subfields, completions and finite extensions. It contains, in particular, the locally compact field \( \mathbb{Q}_p \) of \( p \)-adic numbers and the field \( \mathbb{C}\{\{T\}\} \) of formal Laurent series (which is not locally compact).

We generalize the so-called integer value property (well known in the case \( F = \mathbb{R} \) of reals) and provide an NA version to this generalization. We call both generalizations the \( G \)-value property (Theorems 5.2 and 5.9). NA local \( G_u \)-value property 5.5 asserts that in Formula 3.1 it is enough to take the elements \( c_{ij} \) from \( G_u \), where \( G_u \) is the additive subgroup of \( F \) generated by the coefficients \( \lambda_i \) of \( u \). As a consequence, the infimum is attained for every NA field \( F \) with characteristic \( \text{char}(F) > 0 \) (Corollary 5.10.1).

In Section 6 we introduce the free NA locally convex spaces for NA uniform spaces. We describe its topology in terms of Kantorovich ultra-seminorms (Theorem 6.2). We show that for an ultra-metric space \((X, d)\) and a trivially valued field \( F \) the free NA locally convex space \( L_F(X, U(d)) \) (of the uniformity \( U(d) \) of \( d \)) is normable by the Kantorovich ultranorm induced by \( d \) (Theorem 6.5). By Tkachenko-Uspenskij theorem (in the archimedean case \( F = \mathbb{R} \)) the free abelian topological group \( A(X) \) is a topological subgroup of \( L(X) \). Using Ostrowski’s classical theorem we prove that in case \( F \) is an NA valued field of zero characteristic, the uniform free NA abelian topological group \( A_{y_A}(X, U) \) is a topological subgroup of \( L_F(X, U) \) if and only if the restricted valuation on \( \mathbb{Q} \) is trivial (Theorem 6.13). For example, this happens for the Levi-Civita field (Example 6.14).

2. Kantorovich norm

For a nonempty set \( X \) and a field \( F \) denote by \( L_F(X) \) the free vector \( F \)-space on the set \( X \). We simply write \( L(X) \) in case \( F = \mathbb{R} \). Define \( X := X \cup \{0\} \) where \( 0 \notin X \) is the zero element of \( L_F(X) \). The zero element of the field \( F \) is denoted by \( 0_F \). Denote by \( L^0_F(X) \) the kernel of the linear functional

\[
L_F(X) \to F, \quad \sum_{i=1}^n \lambda_i x_i \mapsto \sum_{i=1}^n \lambda_i.
\]

**Notation 2.1.** Every non-zero vector \( u \in L_F(X) \) has a normal form as follows: \( u = \sum_{i=1}^n \lambda_i x_i \in L_F(X) \), where \( x_i \in X \), \( \lambda_i \in F \setminus \{0_F\} \) \( \forall 1 \leq i \leq n \) and \( x_i \neq x_j \) whenever \( i \neq j \). If \( u \in L^0_F(X) \) then define the support of \( u \) as \( \text{supp}(u) := \{x_1, x_2, x_3, \ldots, x_n\} \). Otherwise, let \( \text{supp}(u) := \{x_1, x_2, x_3, \ldots, x_n, x_{n+1}\} \) where \( x_{n+1} = 0 \). We denote by \( m := |\text{supp}(u)| \) the length of the support, so \( m \) is either \( n \) or \( n + 1 \). The support of \( 0 \) is \( \{0\} \). Below writing \( u = \sum_{i=1}^n \lambda_i x_i \in L_F(X) \) we mean that it is a normal form.

2.1. Classical transportation problem. Recall the following transportation problem from the historical work of Kantorovich [6]. Let \((X, d)\) be a metric space and denote by \( \mathbb{R}_{\geq 0} \) the set of non-negative reals. Suppose that a network of railways connects a number of production locations \( x_1, \ldots, x_n \in X \) with daily output of \( \lambda_1, \ldots, \lambda_n \) carriages
of a certain good, respectively, to a number of consumption locations \( y_1, \ldots, y_m \in X \) with daily demand of \( \mu_1, \ldots, \mu_m \) carriages. So, we have \( \sum_{i=1}^n \lambda_i = \sum_{j=1}^m \mu_j \), where \( \lambda_i, \mu_j \) are positive. We want to minimize our cost. Let \( c_{ij} \) denote the real number transferred from the point \( x_i \) to \( y_j \). The value we are seeking is

\[
\inf \left\{ \sum_{i=1}^n \sum_{j=1}^m c_{ij}d(x_i, y_j) : c_{ij} \in \mathbb{R}_{\geq 0}, \sum_{i=1}^n c_{ij} = \mu_j, \sum_{j=1}^m c_{ij} = \lambda_i \right\}. \tag{2.1}
\]

This infimum is known as the Kantorovich distance in \( L(X) \) between \( \sum \lambda_i x_i \) and \( \sum \mu_j y_j \). It coincides with \( ||u|| \) where \( u = \sum \lambda_i x_i - \sum \mu_j y_j \in L^0(X) \) and \( || \cdot || \) is the norm defined on \( L^0(X) \) as follows. For every \( v = \sum \lambda_i x_i \in L^0(X) \)

\[
||v|| = \inf \left\{ \sum_{i=1}^l |\rho_i|d(a_i, b_i) : v = \sum_{i=1}^l \rho_i(a_i - b_i), \rho_i \in \mathbb{R}, a_i, b_i \in X \right\}. \tag{2.2}
\]

This norm on \( L^0(X) \) is called a Kantorovich norm, \([11]\). If \((X,d)\) is a pseudometric space then 2.1 and 2.2 define the Kantorovich pseudometric and the Kantorovich seminorm respectively.

It is known that the infimum in Formula 2.1 is attained. The fact that it is attained at a matrix with entries belonging to \( G_v \), the subgroup generated by the coefficients of \( v \), seems to be new (see Theorem 5.2.2 and Remark 2.3.2 below).

Let \((X, \mathcal{U})\) be a uniform space. Denote by \( D \) the family of all uniformly continuous pseudometrics on \( X := X \cup \{0\} \). For each \( d \in D \) consider the corresponding Kantorovich seminorm \( s_d \) on \( L^0(X) \). Since \( L^0(X) \) is a subspace of co-dimension 1 in \( L(X) \) for every (semi)norm \( s_d \) we have a natural extension \( \bar{s}_d \), a (semi)norm on \( L(X) \). Then \( L(X) \) together with the family \( \{\bar{s}_d : d \in D\} \) determines a uniform structure \( \mathcal{U}_L \) on \( L(X) \) such that \((L(X), \mathcal{U}_L)\) is the free locally convex space over \((X, \mathcal{U})\). See Pestov \([12]\) for example and compare with Raikov \([11]\) in the case of pointed spaces.

2.2. "Democratic" reformulation. We want to highlight a point that will become important in the sequel. In the problem described above two disjoint sets \( A = \{x_1, \ldots, x_n\} \) and \( B = \{y_1, \ldots, y_m\} \) are considered. The distances between the elements in each set seem irrelevant. Indeed, every distance which appears in Formula 2.1 is between an element of \( A \) and an element of \( B \).

Now we consider a more flexible form of the transportation problem. Let \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) with \( \sum_{i=1}^n \lambda_i = 0 \). We have to transfer real numbers between the points \( x_1, \ldots, x_n \in X \) in the following way. The sum of numbers transferred from \( x_i \) minus the sum of numbers transferred to \( x_i \) is \( \lambda_i \). Let \( c_{ij} \) denote the real number transferred from the point \( x_i \) to \( x_j \). In particular, \( c_{ii} = 0 \). We want to minimize as possible our cost, that is, the value of \( \sum_{i=1}^n \sum_{j=1}^n |c_{ij}|d(x_i, x_j) \).

The Kantorovich norm serves both of the above approaches as the following lemma suggests.

**Lemma 2.2** (Democratic reformulation). Let \( v = \sum_{i=1}^n \lambda_i x_i \in L^0(X) \). Then

\[
||v|| = \inf \left\{ \sum_{i=1}^n \sum_{j=1}^m |c_{ij}|d(x_i, x_j) : \sum_{j=1}^m c_{ij} = \sum_{j=1}^m c_{ji} = \lambda_i \forall i \right\}. \tag{2.3}
\]
Proof. Denote by $||v||'$ the expression on the right hand side of Formula 2.3. We want to show that $||v|| = ||v||'$. Let $(c_{ij}) \in \mathbb{R}^{n \times n}$ such that $\sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji} = \lambda_i$. The coefficient of $x_i$ in $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(x_i - x_j)$ is just $\sum_{j=1}^{n} c_{ij} - \sum_{j=1}^{n} c_{ji}$. It follows that

$$v = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(x_i - x_j).$$

So, by Formula 2.2

$$||v|| \leq ||v||'.$$

On the other hand, using reductions from [21], we show that if $v = \sum_{i=1}^{l} \rho_i(a_i - b_i)$ then there exists a decomposition $v = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(x_i - x_j)$ with

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}|d(x_i, x_j) \leq \sum_{i=1}^{l} |\rho_i|d(a_i, b_i).$$

To see this, first observe that we may assume that $\rho_i > 0 \ \forall i$. Consider the following reductions which do not increase the value of the corresponding sum:

1. Deleting any term of $v$ of the form $\rho_i(x - x)$.
2. If there exist two terms $\lambda(x_i - x_j)$ and $\mu(x_i - x_j)$ with $\lambda, \mu > 0$ we replace them with the single term $(\lambda + \mu)(x_i - x_j)$.
3. Assuming the decomposition contains the term $\lambda(x - z)$ where $z \notin \text{supp}(v)$, then we necessarily have also a term of the form $\mu(z - y)$ where $\lambda, \mu > 0$. We have three subcases to consider replacing in each case the terms $\lambda(x - z)$ and $\mu(z - y)$.
   a. If $\lambda = \mu$ then we can replace the pair of terms above with one term $\lambda(x - y)$ because
      $$\lambda d(x, y) \leq \lambda d(x, z) + \lambda d(z, y).$$
   b. If $\lambda < \mu$ then we can replace the terms with $\lambda(x - y)$ and $(\mu - \lambda)(z - y)$. The value of the sum does not increase since
      $$\lambda d(x, z) + \mu d(z, y) = \lambda d(x, z) + d(z, y) + (\mu - \lambda)d(z, y) \geq \lambda d(x, y) + (\mu - \lambda)d(z, y).$$
   c. If $\lambda > \mu$ we replace the terms with $(\lambda - \mu)(x - z)$ and $\mu(x - y)$. This time we have
      $$\lambda d(x, z) + \mu d(z, y) = (\lambda - \mu)d(x, z) + \mu d(x, z) + d(z, y) \geq (\lambda - \mu)d(x, z) + \mu d(x, y).$$

Using reduction (3) the number of terms in which the element $z$ appears decreases. Applying finitely many substitutions of this form and taking into account that the sum of the $z'$s coefficients in any decomposition of $v$ is equal to zero, we obtain a decomposition of $v$ with only two terms in which $z$ appears: $\lambda(x - z)$ and $\lambda(z - y)$. Now use reduction (3.a). Therefore, we can assume that the decomposition only contains terms with support elements. That is, terms of the form $\lambda(x_i - x_j)$ where $\lambda > 0$. At this point we use reduction (2) if it is necessary. We obtain that there exists a decomposition $v = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(x_i - x_j)$.
with
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| d(x_i, x_j) \leq \sum_{i=1}^{l} |\rho_i| d(a_i, b_i).
\]

It follows that \( ||v'|| \leq ||v|| \). We conclude that \( ||v|| = ||v'|| \). \( \square \)

**Remark 2.3.**

1. Every non-zero element \( v \in L^0(X) \) has the form \( v = \sum_{i=1}^{n} a_i x_i - \sum_{j=1}^{m} b_j y_j \) where \( \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j \) and \( \forall 1 \leq i \leq n \ \forall 1 \leq j \leq m \ a_i, b_j > 0 \). Using this fact one can move back from the democratic approach to the classical one as in Section 2.1.

2. The infimum in Lemma 2.2 is attained at a matrix with entries from \( G_v \) as it follows from Theorem 5.2.2.

3. Replacing \( \mathbb{R} \) with the field \( \mathbb{C} \) of complex numbers completely changes the situation. As it follows from Flood [4, p. 90] in the latter case of \( F = \mathbb{C} \) we cannot even guarantee that the infimum in the inf-sum Formula 2.2 can be approximated by computations on support elements of a vector \( u \in L^0_{\mathbb{C}}(X) \).

3. **Non-archimedean transportation problem**

In this section we discuss the main object of our work: a non-archimedean transportation problem (NATP). First we recall some definitions.

3.1. **Preliminaries.** A uniform space \( (X, U) \) is NA if it has a base \( B \) consisting of equivalence relations on \( X \). A metric space \( (X, d) \) is an ultra-metric space if \( d \) is an ultra-metric, i.e., it satisfies the strong triangle inequality
\[
d(x, z) \leq \max \{d(x, y), d(y, z)\}.
\]

Allowing the distance between distinct elements to be zero we obtain the definition of an ultra-pseudometric.

For every ultra-pseudometric \( d \) on \( X \) the open balls of radius \( \varepsilon > 0 \) form a clopen partition of \( X \). So, the uniformity induced by any ultra-pseudometric \( d \) on \( X \) is NA. A uniformity is NA if and only if it is generated by a system \( \{d_i\}_{i \in I} \) of ultra-pseudometrics.

Recall that a topological group is non-archimedean if it has a base at the identity consisting of open subgroups. For some properties of this class of topological groups see for example [8, 9]. We say that a topological ring (or field or vector space) is NA if its additive group is NA. Note that Lyudkovskii [7] studied NA free Banach spaces.

A valuation on a field \( F \) is a function \( | \cdot | : F \to [0, \infty) \) which satisfies the following \( (x, y \in F) \):

1. \( |x| \geq 0 \);
2. \( |x| = 0 \) if and only if \( x = 0_F \);
3. \( |x + y| \leq |x| + |y| \);
4. \( |xy| = |x||y| \).

Replacing condition (3) with \( |x + y| \leq \max \{|x|, |y|\} \) we obtain a non-archimedean valuation. In this case the metric \( d \) defined by \( d(x, y) = |x - y| \) is an ultra-metric. An (NA) valued field is a field \( F \) with a (resp., NA) valuation \( | \cdot | \). Every NA valued field is NA as a topological group because every open ball \( \{x \in F : |x| < r\} \) is a (clopen) additive subgroup.

Let \( (F, | \cdot |) \) be a valued field. A seminorm on an \( F \)-vector space \( V \) is a map \( || \cdot || : V \to [0, \infty) \) such that \( (x, y \in V, \alpha \in F) \):

1. \( ||0_V|| = 0 \);
(2) $||x + y|| \leq ||x|| + ||y||$;
(3) $||a x|| = |a||x||$.

If instead of condition (1) we have: $||x|| = 0$ if and only if $x = 0$, then $||\cdot||$ is called a norm. If the valuation on $\mathbb{F}$ is NA and condition (2) is replaced by $||x + y|| \leq \max\{||x||, ||y||\}$ then the norm (seminorm) $||\cdot||$ is an ultra-norm (respectively, ultra-seminorm).

Let $(\mathbb{F}, |\cdot|)$ be an NA valued field. The set $\{ |x| : |x| \neq 0 \}$ is a subgroup of the multiplicative group $\mathbb{R}_{>0}$ of all positive reals and is said to be the value group of the valuation $|\cdot|$. The value group is either discrete or dense in $\mathbb{R}_{>0}$. Accordingly the valuation is called discrete or dense. If the value group is the trivial subgroup $\{ 1 \}$ then the valuation is said to be trivial. For any non-trivial discrete valuation the value group is the infinite cyclic closed subgroup $\{ a^k : k \in \mathbb{Z} \}$ of $\mathbb{R}_{>0}$, where $a := \max\{|x| : |x| < 1\}$.

In most applications NA fields are discretely valued. The $p$-adic valuation on the field $\mathbb{Q}$ of rationals is a classical particular case (for every prime $p$). The completion is the field of $p$-adic numbers $\mathbb{Q}_p$, a locally compact NA valued field.

In most applications NA fields are discretely valued. The $p$-adic valuation on the field $\mathbb{Q}$ of rationals is a classical particular case (for every prime $p$). The completion is the field of $p$-adic numbers $\mathbb{Q}_p$, a locally compact NA valued field. The valuation of every locally compact NA valued field is discrete (see [15]). The natural valuation on the field of formal Laurent series (which is not locally compact) is discrete. Below we use several times the following well known theorem of Ostrowski (see for example [15, Theorem 1.2]) which shows that the $p$-adic valuation, up to the natural equivalence, is the only NA non-trivial valuation on $\mathbb{Q}$. In particular, any NA valuation on $\mathbb{Q}$ is discrete.

**Theorem 3.1.** (Ostrowski’s Theorem) Let $|\cdot|$ be a non-trivial NA valuation on the field $\mathbb{Q}$ of rationals. Then there exists a prime $p$ such that $|\cdot|$ is equivalent to the usual $p$-adic valuation $|\cdot|_p$ (namely, there exists $c > 0$ such that $|x| = |x|^c_p \forall x \in \mathbb{Q}$).

3.2. **Formulation of NATP.** We formulate here a non-archimedean transportation problem using a democratic approach (compare Section 2.2). Let $\mathbb{F}$ be an NA valued field, $(X, d)$ be an ultra-metric space and $x_i \in X$ for every $1 \leq i \leq n$. We have to transfer elements from the field between these points in the following way. The sum of elements transferred from $x_i$ minus the sum of elements transferred to $x_i$ is $\lambda_i$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ is a given $n$-tuple of elements with $\sum_{i=1}^n \lambda_i = 0$.

Let $c_{ij} \in \mathbb{F}$ denote the element transferred from $x_i$ to $x_j$. We want to minimize as much as possible our max-cost, that is, the value of

$$\max_{1 \leq i, j \leq n} |c_{ij}| d(x_i, x_j).$$

A natural question arises:

**Question 3.2.** Is the infimum

$$\inf\left\{ \max_{1 \leq i, j \leq n} |c_{ij}| d(x_i, x_j) : \forall i \sum_{j=1}^n c_{ij} - \sum_{j=1}^n c_{ji} = \lambda_i \right\}$$

attained?

Below in Theorem 5.8 we prove that the answer to Question 3.2 is positive for every NA discretely valued field $\mathbb{F}$ (e.g., $\mathbb{Q}_p$) and any ultra-(pseudo)metric space $(X, d)$. In fact we will show in Theorem 6.2 that, in general, Formula 3.1 can be studied via special ultra-(semi)norms $|\cdot|_d$ on $L(X)$. We call them Kantorovich ultra-(semi)norms (Definition 4.2) because their role is similar to the role of Kantorovich seminorms in the classical version of the transportation problem (with the field $\mathbb{R}$).
4. Kantorovich ultra-norms

Let \((X, d)\) be an ultra-pseudometric space. Consider the set \(\overline{X} := X \cup \{0\}\), where \(0 \notin X\). In the sequel we repeatedly use the following simple lemma.

**Lemma 4.1.** For every ultra-pseudometric \(d\) on \(X\) there exists an ultra-pseudometric (denoted also by \(d\)) which extends \(d\) on \(\overline{X} := X \cup \{0\}\), such that \(0\) is an isolated point in \((\overline{X}, d)\).

**Proof.** Fix \(x_0 \in X\) and extend the definition of \(d\) from \(X\) to \(\overline{X}\) by letting \(d(x, 0) = \max\{d(x, x_0), 1\}\). For more details see [9]. □

**Definition 4.2.** Let \((X, d)\) be an ultra-(pseudo)metric space and \(F\) be an NA valued field. Let us say that an ultra-(semi)norm \(p\) on \(L^0(X)\) is \(d\)-compatible if the pseudometric induced on \(X\) by \(p\) is \(d\). We say that \(p\) is a Kantorovich ultra-seminorm for \(d\) if \(p\) is the maximal \(d\)-compatible ultra-(semi)norm on \(L^0(X)\).

The maximal property of the Kantorovich norm in the classical non-discrete transportation problem was proved in [11]. This justifies Definition 4.2.

The Kantorovich ultra-norm \(\| \cdot \|\) in Theorem 4.3 serves the NA transportation problem described in Section 3.2. To see this observe that one of the reformulations of this ultra-norm (see Claim 3 below) coincides with the infimum in Formula 3.1 above.

The classical analogue of the following Theorem 4.3 is the Arens-Eells embedding, [23] the usual verification of which is based on the dual space. In our case the approach is different. If \(d\) is a metric on \(X\) then the Kantorovich seminorm defined on \(L^0(X)\) is, in fact, a norm. This fact relies on the classical Hahn-Banach theorem (see [23, Corollary 2.2.3]) which does not always hold for general NA Banach spaces, [17, 10]. The proof that the ultra-seminorm in the following theorem is an ultra-norm uses only the fact that the valuation of \(F\) is NA.

**Theorem 4.3.** (Non-archimedean Arens-Eells embedding)

Let \((X, d)\) be an ultra-pseudometric space and \(F\) be an NA valued field.

1. There exists a Kantorovich ultra-seminorm \(\| \cdot \| := \| \cdot \|_d\) on \(L^0_F(X)\) for \(d\). Furthermore, if \(d\) is an ultra-metric then \(\| \cdot \|_d\) is an ultra-norm.

2. \(\|u\|\) can be computed on the support of \(u\) for every \(u \in L^0_F(X)\). That is,

\[
\|u\| = \inf \left\{ \max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i) : u = \sum_{i=1}^{n} \lambda_i (x_i - y_i), \ x_i, y_i \in \text{supp}(u), \ \lambda_i \in F \right\}.
\]

3. Moreover, if \(u = \sum_{i=1}^{n} \lambda_i x_i\) (normal form) then

\[
\|u\| = \inf \left\{ \max_{1 \leq i, j \leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in F, \ \forall 1 \leq i \leq n, \ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\},
\]

where \(c_{ii} = 0\).

**Proof.** For \(u \in L^0_F(X)\) define

\[
\|u\| := \inf \left\{ \max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i) : u = \sum_{i=1}^{n} \lambda_i (x_i - y_i), \ x_i, y_i \in \overline{X}, \ \lambda_i \in F \right\}.
\]

**Claim 1:** \(\| \cdot \|\) is an ultra-seminorm on \(L^0_F(X)\).
Proof. Clearly, $||u|| \geq 0$ for every $u \in L_\mathcal{F}(X)$. Since $0 = 0 - 0$ we also have $||0|| \leq d(0,0) = 0$ and thus $||0|| = 0$. The equality $||\lambda u|| = ||\lambda|| ||u||$ follows from the fact that for every $\lambda \neq 0$ if $u = \sum_{i=1}^{n} \lambda_i(x_i - y_i)$ then $\lambda u = \sum_{i=1}^{n} \lambda \lambda_i(x_i - y_i)$ and if $\lambda u = \sum_{i=1}^{n} \lambda_i(x_i - y_i)$ then $u = \sum_{i=1}^{n} \lambda^{-1} \lambda_i(x_i - y_i)$. Of course, we use also axiom (4) in the definition of valuation.

Finally, observe that

$$||u + v|| \leq \max\{||u||, ||v||\} \quad \forall u, v \in L_\mathcal{F}(X).$$

Assuming the contrary, there exist decompositions

$$u = \sum_{i=1}^{n} \lambda_i(x_i - y_i), \quad v = \sum_{i=n+1}^{m} \lambda_i(x_i - y_i)$$

such that

$$||u + v|| > c := \max\{\max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i), \quad \max_{n+1 \leq i \leq m} |\lambda_i| d(x_i, y_i)\}.$$

This contradicts the definition of $||u + v||$ since $u + v = \sum_{i=1}^{m} \lambda_i(x_i - y_i)$ with

$$||u + v|| > \max\{\max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i), \quad \max_{n+1 \leq i \leq m} |\lambda_i| d(x_i, y_i)\} = \max_{1 \leq i \leq m} |\lambda_i| d(x_i, y_i).$$

\[\square\]

Claim 2: For every $u \in L_\mathcal{F}(X)$ the value of $||u||$ can be computed on the support of $u$. That is,

$$||u|| = \inf\left\{\max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i) : u = \sum_{i=1}^{n} \lambda_i(x_i - y_i), \quad x_i, y_i \in \text{supp}(u), \quad \lambda_i \in \mathcal{F}\right\}.$$

Proof. If $u = 0$ then we clearly have $u = 0 - 0$ and $||u|| = 0 = d(0,0)$.

Now assume that $u$ is a non-zero element of $L_\mathcal{F}(X)$. Consider the following elementary steps which do not increase the value of $\max_{1 \leq i \leq n} |\lambda_i| d(x_i, y_i)$:

1. Deleting any term of $u$ of the form $0_\mathcal{F}(x - y)$ or $\lambda_i(x - x)$.
2. Replacing the term $\lambda_i(x_i - y_i)$ with $-\lambda_i(y_i - x_i)$.
3. Assume there exist $1 \leq i_0 \leq n$ and $z \notin \text{supp}(u)$ such that $z = x_{i_0}$ or $z = y_{i_0}$. Using steps (1) – (2) we may assume without loss of generality that the terms $\lambda(x - z)$ and $\mu(z - y)$ appear in the decomposition of $u = \sum_{i=1}^{n} \lambda_i(x_i - y_i)$ with $|\lambda| \leq |\mu|$. Then we replace them with $\lambda(x - y)$ and $(\mu - \lambda)(z - y)$.

Using reduction (3) the number of terms in which the element $z$ appears decreases. The value of the corresponding maximum $\max_{1 \leq j \leq n} |\mu_j| d(x_j, y_j)$ does not increase under such a substitution, because

$$\max\{||\lambda|| d(x, y), |\mu - \lambda|| d(z, y)\} \leq \max\{|\lambda|| d(x, z), |\mu|| d(z, y)\}.$$

Indeed, using the strong triangle inequality and the fact that $|\lambda| \leq |\mu|$ we obtain

$$|\lambda|| d(x, y) \leq \max\{||\lambda|| d(x, z), |\lambda|| d(z, y)\} \leq \max\{||\lambda|| d(x, z), |\mu|| d(z, y)\}.$$

Also, assuming that $|\mu - \lambda|| d(z, y) > |\mu|| d(z, y)$ we obtain that $|\mu - \lambda| > \max\{|\lambda|, |\mu|\}$, which contradicts the strong triangle inequality. Thus, $|\mu - \lambda|| d(z, y) \leq |\mu|| d(z, y)$.

Applying finitely many substitutions of this form and taking into account that the sum of $z$’s coefficients in any decomposition of $u$ is equal to zero, we obtain a decomposition of $u$ with only two terms in which $z$ appears: $\lambda(x - z)$ and $\lambda(z - y)$. These terms
can be replaced by the single term \( \lambda(x - y) \) since \( \lambda(x - z) + \lambda(z - y) = \lambda(x - y) \) and \( |\lambda|d(x, y) \leq \max\{|\lambda|d(x, z), |\lambda|d(z, y)\} \). Since the term \( \lambda(x - y) \) and all other terms in the new decomposition do not contain the element \( z \) the proof is completed. \[ \square \]

**Claim 3:** For \( u = \sum_{i=1}^{n} \lambda_i x_i \in L_F(X) \) let \( m = |\text{supp}(u)| \) (so by Notation 2.1 we have \( m = n \), or \( m = n + 1 \)). Then,

\[
||u|| = \inf \left\{ \max_{1 \leq i,j \leq m} |c_{ij}|d(x_i, x_j) : c_{ij} \in \mathbb{F}, \forall 1 \leq i \leq n \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\},
\]

where \( c_{ii} = 0 \).

**Proof.** By Notation 2.1, \( \sum_{i=1}^{n} \lambda_i x_i \) is a normal form of \( u \). It follows that a matrix \((c_{ij}) \in \mathbb{F}^{m \times m}\) satisfies the equations

\[
\sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \forall 1 \leq i \leq n
\]

if and only if \( u = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} (x_i - x_j) \). Indeed, on the one hand the coefficient of \( x_i \) in the left expression is \( \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} \) for all \( 1 \leq i \leq n \). On the other hand, \( \lambda_i \) is the coefficient of \( x_i \) in \( u \). Note that by our convention if \( m = n + 1 \) then \( x_{n+1} = \mathbf{0} \). Since \( d(x_i, x_i) = 0 \) and \( c_{ii} - c_{ii} = 0 \) we may assume without loss of generality that \( c_{ii} = 0 \).

By Claim 2, \( ||u|| \) can be computed on the support of \( u \). If we have two terms of the form \( \lambda(x_i - x_j), \mu(x_i - x_j) \) we can replace them with the single term \( (\lambda + \mu)(x_i - x_j) \) since \( |\lambda + \mu| \leq \max\{|\lambda|, |\mu|\} \). Thus, we may consider only decompositions of the form \( u = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} (x_i - x_j) \).

**Claim 4:** For \( u = \sum_{i=1}^{n} \lambda_i x_i \in L_F(X) \) let \( m = |\text{supp}(u)| \). Then,

\[
||u|| \geq r \cdot l_0
\]

where \( r = \max\{|\lambda_i| : 1 \leq i \leq n\} \) and \( l_0 = \min\{d(x_i, x_j) : 1 \leq i \neq j \leq m\} \).

**Proof.** Assuming the contrary let \( ||u|| < r \cdot l_0 \). By Claim 3 there exists a matrix \((c_{ij}) \in \mathbb{F}^{m \times m}\) such that \( \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \forall 1 \leq i \leq n \) and in addition \( r \cdot l_0 > \max_{1 \leq i,j \leq m} |c_{ij}|d(x_i, x_j) \).

Taking into account the definition of \( l_0 \) we get \( r > |c_{ij}| \forall i,j \). By the definition of \( r \) there exists \( 1 \leq i_0 \leq n \) such that \( r = |\lambda_{i_0}| \). Thus, \( |\lambda_{i_0}| > |c_{ij}| \forall i,j \). In particular,

\[
|\lambda_{i_0}| > \max\{\max_{1 \leq j \leq m} |c_{i_0j}|, \max_{1 \leq j \leq m} |c_{j_0i}|\}.
\]

Applying the strong triangle inequality to the equation \( \sum_{j=1}^{m} c_{i_0j} - \sum_{j=1}^{m} c_{j_0i} = \lambda_{i_0} \) we obtain the contradiction

\[
|\lambda_{i_0}| \leq \max\{\max_{1 \leq j \leq m} |c_{i_0j}|, \max_{1 \leq j \leq m} |c_{j_0i}|\}.
\]

\[ \square \]
Claim 5: \( \iota: (X, d) \rightarrow (L_{\overline{F}}(X), || \cdot ||), \iota(x) = \{ x \} \) is an isometric embedding, i.e.
\[ ||x - y|| = d(x, y) \quad \forall \, x, y \in X. \]

Proof. If \( x = y \) the assertion is trivial so we may assume that \( u = x - y \neq 0 \). By Claim 2 the value \( ||x - y|| \) can be computed on the support \( \{ x, y \} \). Using also some reductions we mentioned above it suffices to consider only the trivial decomposition \( u = x - y \). It follows that \( ||x - y|| = d(x, y) \).

Claim 6: \( || \cdot ||_d \) is an ultra-norm on \( L_{\overline{F}}(X) \) if (and only if) \( d \) is an ultra-metric.

Proof. By Claim 1 \( || \cdot || \) is an ultra-seminorm on \( L_{\overline{F}}(X) \). By Claim 4 for \( u \neq 0 \) we have \( ||u|| \geq r \cdot l_0 \). Clearly, \( r > 0 \). Since \( d \) is a metric we have \( l_0 > 0 \) and thus the ultra-seminorm \( || \cdot || \) is an ultra-norm.

Claim 7: (Maximality property) Let \( \sigma \) be an ultra-seminorm on \( L_{\overline{F}}(X) \) such that
\[ \sigma(x - y) \leq d(x, y) \quad \forall\, x, y \in X. \]
Then \( || \cdot || \geq \sigma \).

Proof. Let \( u \) be a non-zero element of \( L_{\overline{F}}(X) \) and \( \sigma \) is an ultra-norm which satisfies Equation 4.1. Then for every decomposition \( u = \sum_{i=1}^{n} \lambda_i(x_i - y_i) \) we obtain
\[ \sigma(u) = \sigma\left(\sum_{i=1}^{n} \lambda_i(x_i - y_i)\right) \leq \max_{1 \leq i \leq n} |\lambda_i|\sigma(x_i - y_i) \leq \max_{1 \leq i \leq n} |\lambda_i|d(x_i, y_i). \]
It follows from the definition of the ultra-seminorm \( || \cdot || \) that \( \sigma(u) \leq ||u||. \)

Summing up all previous claims we complete the proof of Theorem 4.3.

Remark 4.4.

(1) Theorem 5.5 shows that in Theorem 4.3.3 we can assume, in addition, that the coefficients \( c_{ij} \) belong to the additive subgroup \( G_u \) of \( F \) generated by the normal coefficients \( \lambda_i \) of \( u \).

(2) For \( u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\overline{F}}(X) \) let \( m = |\text{supp}(u)| \). Claim 3 and additional computations lead to the following formula:
\[ ||u|| = \inf \left\{ \max_{1 \leq i < j \leq m} |c_{ij}|d(x_i, x_j) : \forall i \geq j \, c_{ij} = 0, \, \forall 1 \leq i \leq n \sum_{j=i+1}^{m} c_{ij} - \sum_{j=1}^{i-1} c_{ji} = \lambda_i \right\}. \]

5. Generalized integer value property

5.1. \( G \)-value property for subgroups \( G \subset \mathbb{R} \). First recall the integer value property for the case \( F = \mathbb{R} \). Let \( d \) be an ultra-(pseudo)metric on \( X \) and \( || \cdot || \) is its Kantorovich ultra-(semi)norm. For an element of \( L_{\overline{F}}(X) \) with integer coefficients the inf-sup cost Formula 2.1 achieves its infimum at an integer matrix \( (c_{ij}) \). See, for example, Sakarovitz [16, p. 179], and also Uspenskij [21]. Our arguments below were inspired by Uspenskij [21].

Replacing the set of integers \( \mathbb{Z} \) with any other additive subgroup \( G \) of \( \mathbb{R} \) we obtain a natural generalization. We call it the \( G \)-value property. It means that whenever we have an element of \( L_{\overline{F}}(X) \) with coefficients from \( G \) the minimum in the formula is obtained at a matrix with elements from \( G \). In the sequel we prove the \( G \)-value property also for the NA case assuming that the field \( F \) is discretely valued. Before proving this property for the real case we need the following lemma.
Lemma 5.1. Let $u = \sum_{i=1}^{n} \lambda_i x_i \in L^0(X)$ such that $\forall 1 \leq i \leq n \lambda_i \in G$ where $G$ is an additive subgroup of $\mathbb{R}$. Then for every decomposition $u = \sum_{j=1}^{m} \mu_j(a_j - b_j)$ with $\mu_j \geq 0 \ \forall j$ there exists a decomposition $u = \sum_{k=1}^{l} \rho_k(s_k - t_k)$ with $\rho_k \in G \ \forall 1 \leq k \leq l$ such that $\rho_k \geq 0$ and

$$\sum_{k=1}^{l} \rho_k d(s_k, t_k) \leq \sum_{j=1}^{m} \mu_j d(a_j, b_j).$$

Proof. By deleting any term of $u$ of the form $\mu_j(x - x)$ we may assume that $a_j \neq b_j \ \forall j$. If $\mu_j \in G \ \forall 1 \leq j \leq m$ there is nothing to prove. So, without loss of generality, we may assume that $\mu_1 \notin G$. Consider the set of indices

$$A := \{j \neq 1 : a_j = a_1 \lor b_j = a_1\}.$$

We show that $A \setminus G$ is not empty. If $a_1 \in \text{supp}(u)$ then there exists $1 \leq i \leq n$ such that

$$\mu_1 + \sum_{j \in A} (-1)^{k_j} \mu_j = \lambda_i$$

where $k_j = 1$ if $a_j = a_1$ and $k_j = -1$ if $b_j = a_1$. If $a_1 \notin \text{supp}(u)$ then

$$\mu_1 + \sum_{j \in A} (-1)^{k_j} \mu_j = 0.$$

We know that $G$ is an additive subgroup of $\mathbb{R}$, $\mu_1 \notin G$ and $\{0, \lambda_1\} \subseteq G$. So we conclude that $A \setminus G$ is not empty. Thus, without loss of generality, there exists $j \neq 1$ such that $b_j = a_1$, $\mu_j \notin G$. Consider the following cases.

1. $\mu_j \leq \mu_1$.

   Then we replace the terms $\mu_1(a_1 - b_1)$ and $\mu_j(a_j - a_1)$ with $\mu_j(a_j - b_1)$ and $(\mu_1 - \mu_j)(a_1 - b_1)$.

   Using the triangle inequality we obtain

   $$(\mu_1 - \mu_j)d(a_1, b_1) + \mu_j d(a_j, b_1) \leq (\mu_1 - \mu_j)d(a_1, b_1) + \mu_j d(a_1, b_1) + d(a_j, a_1) =$$
   $$= \mu_1 d(a_1, b_1) + \mu_j d(a_j, a_1).$$

2. $\mu_j > \mu_1$.

   Replace the terms $\mu_1(a_1 - b_1)$ and $\mu_j(a_j - a_1)$ with $\mu_1(a_j - b_1)$ and $(\mu_j - \mu_1)(a_j - a_1)$.

   Using again the triangle inequality we obtain

   $$(\mu_j - \mu_1)d(a_j, a_1) + \mu_1 d(a_j, b_1) \leq (\mu_j - \mu_1)d(a_j, a_1) + \mu_1 d(a_1, b_1) + d(a_j, a_1) =$$
   $$= \mu_1 d(a_1, b_1) + \mu_j d(a_j, a_1).$$

In both cases we decrease the number of terms in which the element $a_1$ appears with scalar coefficient outside of $G$. Applying finitely many substitutions of this form to terms in which the element $a_1$ appears and in which the coefficients are not taken from $G$, we obtain a decomposition in which all coefficients of $a_1$ (if there are any) are from $G$. Repeating this algorithm, if necessary, for other elements we obtain a decomposition of the form

$$u = \sum_{k=1}^{l} \rho_k(s_k - t_k)$$
with \( \rho_k \in G \) \( \forall 1 \leq k \leq l \) such that \( \rho_k \geq 0 \) and

\[
\sum_{k=1}^{l} \rho_k d(s_k, t_k) \leq \sum_{j=1}^{m} \mu_j d(a_j, b_j).
\]

The following theorem shows in particular that the infimum in Lemma 5.1 is attained.

**Theorem 5.2. (G-value property)** Let \( G \) be an additive subgroup of \( \mathbb{R} \). Let \( u = \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{m} \mu_j y_j \in L^0(X) \) with \( \lambda_i, \mu_j \in G \cap \mathbb{R}_{\geq 0} \) \( \forall 1 \leq i \leq n, \ 1 \leq j \leq m. \)

1. There exists a matrix \((c_{ij}) \in G^{n \times m}\) such that

\[
||u|| = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} d(x_i, y_j)
\]

where \( \sum_{i=1}^{n} c_{ij} = \mu_j \), \( \sum_{j=1}^{m} c_{ij} = \lambda_i \).

2. The matrix \((c_{ij})\) in (1) can be taken from \( G_u^{n \times m}\) where \( G_u \) is the additive subgroup of \( \mathbb{R} \) generated by the coefficients \( \lambda_i, \mu_j \) of \( u \).

**Proof.** (1): By Formula 2.1

\[
||u|| = \inf \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} d(x_i, y_j) : c_{ij} \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} c_{ij} = \mu_j, \sum_{j=1}^{m} c_{ij} = \lambda_i \right\}.
\]

By the Heine-Borel Theorem the set

\[
C := \left\{ (c_{ij}) \in (\mathbb{R}_{\geq 0})^{n \times m} : \sum_{i=1}^{n} c_{ij} = \mu_j, \sum_{j=1}^{m} c_{ij} = \lambda_i \right\}
\]

is compact in \((\mathbb{R}_{\geq 0})^{n \times m}\). It follows that there exists a matrix \((c_{ij}) \in (\mathbb{R}_{\geq 0})^{n \times m}\) such that \( \sum_{i=1}^{n} c_{ij} = \mu_j, \sum_{j=1}^{m} c_{ij} = \lambda_i \) and

\[
||u|| = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} d(x_i, y_j).
\]

Use Lemma 5.1 to conclude that the matrix \((c_{ij})\) can be taken from \( G_u^{n \times m}\).

(2): Follows immediately from (1). \( \square \)

Theorem 5.2 remains true replacing \( \mathbb{R} \) with an arbitrary subfield \( F \subseteq \mathbb{R} \).

5.2. G-value property in the non-archimedean case. In this subsection let \( F \) be an NA valued field and \((X, d)\) be an ultra-(pseudo)metric space.

**Lemma 5.3.** Let \( G \) be an additive subgroup of an NA valued field \( F \). Let \( u = \sum_{i=1}^{n} \lambda_i x_i \in L_F(X) \) with \( \lambda_i \in G \ \forall i \). Then the (semi)norm \( ||u|| \) can be computed using only the coefficients from \( G \). That is, in the formula of Theorem 4.3.2 we get

\[
||u|| := \inf \left\{ \max_{1 \leq k \leq l} |\rho_k| d(s_k, t_k) : u = \sum_{k=1}^{l} \rho_k (s_k - t_k), \ s_k, t_k \in X, \ \rho_k \in G \right\}.
\]
Proof. It is equivalent to show that for every decomposition \( u = \sum_{j=1}^{m} \mu_j(a_j - b_j) \) there exists a decomposition \( u = \sum_{k=1}^{l} \rho_k(s_k - t_k) \) with \( \rho_k \in G \forall 1 \leq k \leq l \) such that
\[
\max_{1 \leq k \leq l} |\rho_k|d(s_k, t_k) \leq \max_{1 \leq j \leq m} |\mu_j|d(a_j, b_j).
\]

By deleting any term of \( u \) of the form \( \mu_j(x - x) \) we may assume that \( a_j \neq b_j \forall j \). If \( \mu_j \in G \forall 1 \leq j \leq m \) there is nothing to prove. So, without loss of generality, we may assume that \( \mu_1 \notin G \) and \( a_1 \neq 0 \). Consider the set of indices
\[
A := \{ j \neq 1 : a_j = a_1 \land b_j = a_1 \}. 
\]
We show that \( A \setminus G \) is not empty. If \( a_1 \in \text{supp}(u) \) then there exists \( 1 \leq i \leq n \) such that
\[
\mu_1 + \sum_{j \in A} (-1)^{k_j} \mu_j = \lambda_i
\]
where \( k_j = 1 \) if \( a_j = a_1 \) and \( k_j = -1 \) if \( b_j = a_1 \). If \( a_1 \notin \text{supp}(u) \) then
\[
\mu_1 + \sum_{j \in A} (-1)^{k_j} \mu_j = 0_F.
\]
Since \( G \) is an additive subgroup of \( F \), \( \mu_1 \notin G \) and \( \{0_F, \lambda_i\} \subseteq G \) we conclude that \( A \setminus G \) is not empty. Thus, without loss of generality, there exists \( j \neq 1 \) such that \( b_j = a_1 \), \( \mu_j \notin G \) and \( |\mu_j| \leq |\mu_1| \). Replace the terms \( \mu_1(a_1 - b_1) \) and \( \mu_j(a_j - a_1) \) with \( \mu_j(a_j - b_1) \) and \( (\mu_1 - \mu_j)(a_1 - b_1) \). This way we decrease the number of terms in which the element \( a_1 \) appears with scalar coefficient not from \( G \). Since \( |\mu_j| \leq |\mu_1| \) it follows from the strong triangle inequality for the valuation \( | \cdot | \) that
\[
|\mu_1 - \mu_j|d(a_1, b_1) \leq \max\{|\mu_1|d(a_1, b_1), |\mu_j|d(a_1, b_1)| = |\mu_1|d(a_1, b_1) \leq \max\{|\mu_j|d(a_1, a_1), |\mu_1|d(a_1, b_1)|.
\]
From the strong triangle inequality for \( d \) we obtain
\[
|\mu_j|d(a_j, b_1) \leq \max\{|\mu_j|d(a_j, a_1), |\mu_j|d(a_1, b_1)| \leq \max\{|\mu_j|d(a_j, a_1), |\mu_1|d(a_1, b_1)|.
\]
Therefore,
\[
\max\{|\mu_j|d(a_j, b_1), |\mu_1 - \mu_j|d(a_1, b_1)| \leq \max\{|\mu_j|d(a_j, a_1), |\mu_1|d(a_1, b_1)|.
\]
Applying finitely many substitutions of this form to terms in which the element \( a_1 \) appears and in which the coefficients are not taken from \( G \), we obtain a decomposition in which all coefficients of \( a_1 \) (if there are any) are from \( G \). Repeating this algorithm if necessary also for other elements we obtain a decomposition of the form
\[
u = \sum_{k=1}^{l} \rho_k(s_k - t_k)
\]
with \( \rho_k \in G \forall 1 \leq k \leq l \) such that
\[
\max_{1 \leq k \leq l} |\rho_k|d(s_k, t_k) \leq \max_{1 \leq j \leq m} |\mu_j|d(a_j, b_j).
\]
\[\square\]

Notation 5.4. For every \( u = \sum_{i=1}^{n} \lambda_i x_i \in L_F(X) \) (normal form) denote by \( G_u \) the additive subgroup of \( F \) generated by the coefficients \( \lambda_i \) of \( u \).
Theorem 5.5. (NA local $G_u$-value property) For every $u = \sum_{i=1}^{n} \lambda_i x_i \in L_\mathbb{F}(X)$ we have

\[
||u|| = \inf \left\{ \max_{1 \leq i,j \leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in G_u, \ \forall 1 \leq i \leq n \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}.
\]

Proof. Combine Lemma 5.3 with Claims 2, 3 of Theorem 4.3 taking into account the following property. Consider a decomposition $u = \sum_{k=1}^{l} \rho_k(s_k - t_k)$ with $\rho_k \in G \ \forall 1 \leq k \leq l$. Since $G$ is an additive subgroup of $\mathbb{F}$ each reduction appearing in the proof of Claim 2 yields a decomposition of the same form. That is, the coefficients in the resulting decomposition are from $G$. \hfill \Box

Proposition 5.6. For every $u = \sum_{i=1}^{n} \lambda_i x_i \in L_\mathbb{F}(X)$ we have

\[
r \cdot l_0 \leq ||u|| \leq r \cdot l_1
\]

where $r = \max\{|\lambda_i| : 1 \leq i \leq n\}$, $l_1 = \max\{d(x_i, x_j) : 1 \leq i, j \leq m\}$, $l_0 = \min\{d(x_i, x_j) : 1 \leq i \neq j \leq m\}$ and $m = |\text{supp}(u)|$.

Proof. Claim 4 of Theorem 4.3 provides a lower bound $r \cdot l_0 \leq ||u||$.

By Theorem 5.5

\[
||u|| = \inf \left\{ \max_{1 \leq i,j \leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in G_u, \forall 1 \leq i \leq n \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\},
\]

while $|c_{ij}| \leq r$ by Equation 5.1. So, we have

\[
||u|| \leq \max_{1 \leq i,j \leq m} |c_{ij}| d(x_i, x_j) \leq r \cdot l_1.
\]

\hfill \Box

Corollary 5.7. Let $u = \sum_{i=1}^{n} \lambda_i x_i \in L_\mathbb{F}(X)$. Suppose that $l = d(x_i, x_j)$ for every $x_i \neq x_j \in \text{supp}(u)$. Then $||u|| = r \cdot l$ where $r = \max\{|\lambda_i| : 1 \leq i \leq n\}$.

Theorem 5.8. (Min-attaining Theorem) Let $u = \sum_{i=1}^{n} \lambda_i x_i \in L_\mathbb{F}(X)$. Suppose that the NA valuation of $\mathbb{F}$ restricted on $G_u \setminus \{0\}$ is discrete and closed in $\mathbb{R}_{>0}$. Then

\[
||u|| = \min \left\{ \max_{1 \leq i,j \leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in G_u, \forall 1 \leq i \leq n \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}.
\]

Proof. We have to show that the infimum in Theorem 5.5 is attained. Assuming the contrary and taking into account Formula 5.1, there exists a sequence of matrices $\{(c_{ij}^k) : k \in \mathbb{N}\} \subseteq G_u^{m \times m}$ with the following properties:

1. $\forall i,j,k \ |c_{ij}^k| \leq r$;
2. $\forall k \in \mathbb{N} \ \forall 1 \leq i \leq n \sum_{j=1}^{m} c_{ij}^k - \sum_{j=1}^{m} c_{ji}^k = \lambda_i$;
3. $\max_{1 \leq i,j \leq m} |c_{ij}^k| d(x_i, x_j) > \max_{1 \leq i,j \leq m} |c_{ij}^{k+1}| d(x_i, x_j) > ||u||$. 

Observe that by the strong triangle inequality for every $c \in G_u$ we have

\[
||c|| \leq r := \max\{|\lambda_i| : 1 \leq i \leq n\}.
\]
Passing to a subsequence, if necessary, we can also assume that there exists a pair of indices \((i_0, j_0)\) such that
\[
\forall k \in \mathbb{N} \quad \max_{1 \leq i,j \leq m} |c_{ij}^k|d(x_i, x_j) = |c_{i_0j_0}^k|d(x_{i_0}, x_{j_0}).
\]
It follows that
\[
\forall k \in \mathbb{N} \quad r \geq |c_{i_0j_0}^k| > |c_{i_0j_0}^{k+1}| > \frac{|||u|||}{d(x_{i_0}, x_{j_0})} > 0.
\]
Since the valuation \(| \cdot |\) restricted on \(G_u \setminus \{0\}\) is discrete and closed in \(\mathbb{R}_{>0}\) the set
\[
A = \{|a| : a \in G_u, \ r \geq |a| > \frac{|||u|||}{d(x_{i_0}, x_{j_0})} > 0\}
\]
is finite. This contradicts the fact that the set \(\{|c_{i_0j_0}^k| : k \in \mathbb{N}\}\) is infinite being a strictly decreasing sequence. \(\square\)

**Theorem 5.9** (NA G-value property). Let \((\mathbb{F}, | \cdot |)\) be an NA discretely valued field. Let \(G\) be an additive subgroup of \(\mathbb{F}\) and \(u = \sum_{i=1}^{n} \lambda_i x_i \in L_{\mathbb{F}}(X)\) with \(\lambda_i \in \mathbb{G} \forall i\). Then
\[
||u|| = \min \left\{ \max_{1 \leq i,j \leq m} |c_{ij}|d(x_i, x_j) : c_{ij} \in G, \ \forall 1 \leq i \leq n \ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\}.
\]

**Proof.** Apply Theorem 5.8. \(\square\)

By \(\text{char}(\mathbb{F})\) we denote the characteristic of the field \(\mathbb{F}\). Recall that if \(\text{char}(\mathbb{F}) = 0\) then the field \(\mathbb{Q}\) of rationals is naturally embedded in \(\mathbb{F}\). If \(\text{char}(\mathbb{F}) > 0\) then every finitely generated additive subgroup of \(\mathbb{F}\) is finite.

**Corollary 5.10.** Let \(u = \sum_{i=1}^{n} \lambda_i x_i\). In each of the following cases the infimum computing \(||u||\) in Theorem 5.5 is attained in a matrix with entries from \(G_u\):

1. \(\text{char}(\mathbb{F}) > 0\).
2. \(\text{char}(\mathbb{F}) = 0\) and \(\lambda_i \in cl(\mathbb{Q})\) (where \(cl(\mathbb{Q})\) is the closure of \(\mathbb{Q}\) in \(\mathbb{F}\)).
3. \(\mathbb{F}\) is \(\mathbb{Q}_p\) the field of \(p\)-adic numbers.

**Proof.** (1) If \(\text{char}(\mathbb{F}) > 0\) then any finitely generated additive subgroup of \(\mathbb{F}\) is finite. In particular, \(G_u\) is finite. Now apply Theorem 5.5 (or Theorem 5.8).

(2) Use Ostrowski’s Theorem 3.1 which implies that the restriction of the valuation on \(\mathbb{Q} \subseteq \mathbb{F}\) is discrete. Then the valuation on the closure \(cl(\mathbb{Q})\) is also discrete (having even the same value group). So we can apply Min-attaining Theorem 5.8.

(3) Apply Theorem 5.9 (or use assertion (2)). \(\square\)

**Remark 5.11.** Corollary 5.10.1 can be applied to \(\mathbb{F} := \mathbb{Z}_2\) the discrete field of two elements. Note that in this case \((L_{\mathbb{F}}(X), || \cdot ||)\), as a topological group, coincides with \(B_{\mathbb{N}_A}\) the uniform free NA Boolean group over \((X, d)\). Indeed, this follows from the fact that \(B_{\mathbb{N}_A}\) is metrizable by a Graev type ultra-norm (see [9]).

6. **Free NA locally convex space**

For the free locally convex spaces (classical fields) on uniform spaces we refer to Raikov [11]. Here we consider its NA analogue. Let \(\mathbb{F}\) be an NA valued field Recall [17, 10] that a Hausdorff NA \(\mathbb{F}\)-vector space \(V\) is said to be *locally convex* if its topology can be generated by a family of ultra-seminorms.

Assigning to every NA locally convex \(\mathbb{F}\)-space \(V\) its uniform space \((V, \mathcal{U})\) defines a forgetful functor from the category \(\mathbb{F}\text{LCS}_{\mathbb{N}_A}\) of all Hausdorff NA locally convex spaces to the category of all NA Hausdorff uniform spaces \(\text{Unif}_{\mathbb{N}_A}\).
**Definition 6.1.** Let $\mathbb{F}$ be an NA valued field and $(X, U) \in \text{Unif}_{\mathbb{F}}$ be an NA uniform space. By a free NA locally convex $\mathbb{F}$-space of $(X, U)$ we mean a pair $(L_{\mathbb{F}}(X, U), i)$ (or, simply, $L_{\mathbb{F}}(X, U)$ and $L_{\mathbb{F}}(X)$ when $i$ and $U$ are understood), where $L_{\mathbb{F}}(X, U)$ is a locally convex $\mathbb{F}$-space and $i : X \to L_{\mathbb{F}}(X, U)$ is a uniform map satisfying the following universal property. For every uniformly continuous map $\varphi : (X, U) \to V$ into a locally convex $\mathbb{F}$-space $V$ there exists a unique continuous linear homomorphism $\Phi : L_{\mathbb{F}}(X, U) \to V$ for which the following diagram commutes:

\[
\begin{array}{ccc}
(X, U) & \xrightarrow{i} & L_{\mathbb{F}}(X, U) \\
\downarrow \varphi & & \downarrow \Phi \\
& & V
\end{array}
\]

A categorical reformulation of this definition is that $i : X \to L_{\mathbb{F}}(X, U)$ is a universal arrow from $(X, U)$ to the forgetful functor $\mathbb{F}\text{LCS}_{\mathbb{F}} \to \text{Unif}_{\mathbb{F}}$. The uniformity $\overline{U}$ in the following theorem is obtained from the uniformity $U$ by adding to $X$ the element $0$ as an isolated point. In particular, if $U$ is metrizable and $d$ is the corresponding ultra-metric one can extend $d$ from $X$ to $\overline{X}$ such that $d$ induces the uniformity $\overline{U}$ (apply Lemma 4.1).

**Theorem 6.2.** For every Hausdorff NA uniform space $(X, U)$ the uniform NA free locally convex $\mathbb{F}$-space exists. Its structure can be defined as follows. Let $D$ be the set of all uniformly continuous ultra-pseudometrics on $\overline{X} := X \cup \{0\}$. For every $d \in D$ we have the corresponding Kantorovich ultra-seminorm $||| \cdot |||_d$ on $L_{\mathbb{F}}(X)$. Then $L_{\mathbb{F}}(X)$ endowed with the family $\Gamma := \{||| \cdot |||_d : d \in D\}$ of Kantorovich ultra-seminorms defines the desired uniform NA free locally convex $\mathbb{F}$-space which we denote by $L_{\mathbb{F}}(X, U)$. The corresponding arrow $i : (X, U) \to L_{\mathbb{F}}(X, U)$ is a uniform embedding.

**Proof.** First of all, observe that $L_{\mathbb{F}}(X, U)$ is Hausdorff. Indeed, this follows by analyzing Claims 4 and 6 of Theorem 4.3 (or, Proposition 5.6).

Next we have the following commutative diagram

\[
\begin{array}{ccc}
(X, U) & \xrightarrow{i} & L_{\mathbb{F}}(X, U) \\
\downarrow \varphi & & \downarrow \Phi \\
& & V
\end{array}
\]

Now we only have to show that $\Phi$ is continuous. By our assumption $V$ has a family $\Gamma_V$ of ultra-seminorms which generate its topology. Every $\rho \in \Gamma_V$ induces an ultra-seminorm $\sigma_{\rho}$ on $L_{\mathbb{F}}(X)$ and an ultra-pseudometric $d_{\rho}$ on $\overline{X}$ defined by

\[\sigma_{\rho}(u) := \rho(\Phi(u)), \quad d_{\rho}(x, y) := \rho(\varphi(x) - \varphi(y)),\]

respectively. Consider the corresponding Kantorovich ultra-seminorm $||| \cdot |||_{d_{\rho}}$ on $L_{\mathbb{F}}(X)$. Then $\sigma_{\rho}(x - y) = d_{\rho}(x, y)$ for every $x, y \in \overline{X}$. By the maximality property (Definition 4.2 and Theorem 4.3) we obtain that $||| \cdot |||_{d_{\rho}} \geq \sigma_{\rho}$. This guarantees that $\rho(\Phi(u)) \leq ||u||_{d_{\rho}}$ for every $u \in L_{\mathbb{F}}(X)$, which implies the continuity of $\Phi$.

Finally note that by Lemma 4.1 and Theorem 4.3 the family $\Gamma$ of Kantorovich ultraseminorms generates the original uniform structure $U$ on $X = i(X) \subset L_{\mathbb{F}}(X)$. Hence $i$ is a uniform embedding. 

**Proposition 6.3.** Let $\mathbb{F}$ be an NA valued field and $K$ is its subfield. Then for every Hausdorff NA uniform space $(X, U)$ the natural algebraic inclusion $j : L_K(X) \to L_{\mathbb{F}}(X)$ of $K$-vector spaces is a topological embedding.
Proof. Let $d$ be a uniformly continuous ultra-pseudometric on $X$. Denote by $||\cdot||^K$ and $||\cdot||^\mathbb{F}$ the corresponding Kantorovich ultra-seminorms of $d$ in $L_K(X)$ and $L_\mathbb{F}(X)$ respectively. Let $u = \sum_{i=1}^{n} \lambda_i x_i \in L_K(X) \subset L_\mathbb{F}(X)$. Then clearly $G_u$ is an additive subgroup of $K$. Therefore by Theorem 5.5 we have $||u||^K = ||u||^\mathbb{F}$. Now Theorem 6.2 guarantees that $j : L_K(X) \to L_\mathbb{F}(X)$ is a topological embedding. □

As in the classical case of the fields $\mathbb{R}$ or $\mathbb{C}$ (see [14]) we have the following property for the NA case.

Proposition 6.4. The universal arrow $i : (X, \mathcal{U}) \to L_\mathbb{F}(X, \mathcal{U})$ is a closed embedding for any NA valued field $\mathbb{F}$.

Proof. We have to show that $X = i(X)$ is closed in $L_\mathbb{F}(X)$. Let $v \in L_\mathbb{F}(X)$ be a vector such that $v \notin X$. It is enough to find a locally convex space $V$ and a continuous linear morphism $\Phi : L_\mathbb{F}(X) \to V$ such that $\Phi(v) \notin cl(\Phi(X))$. For $v = \lambda x$ with $\lambda \neq 1$ and $x \in X$ consider the continuous functional

$$
\Phi : L_\mathbb{F}(X) \to \mathbb{F}, \quad \sum_{k=1}^{m} \lambda_k x_k \mapsto \sum_{k=1}^{m} \lambda_k.
$$

Then $\Phi(v) = \lambda \notin cl(\Phi(X)) = \{1\}$. The same $\Phi$ works for the case of $v = 0$.

Now we may suppose that $v = \sum_{i=1}^{n} \lambda_i x_i$ with non-zero coefficients $\lambda_i$ and supp$\,(v)$ contains at least two elements from $X$. That is, supp$\,(u) = \{x_1, x_2, x_3, \ldots, x_n\}$, where $x_1, x_2 \in X$ and $n \geq 2$. Define $V$ as the 2-dimensional NA normed $\mathbb{F}$-space $\mathbb{F}^2$ (with the max ultra-norm). Since the uniform space $(X, \mathcal{U})$ is NA and Hausdorff one may divide it into three clopen disjoint subsets

$$
X = X_1 \cup X_2 \cup X_3
$$
such that

$$
x_1 \in X_1, x_2 \in X_2, x_k \in X_3 \quad \forall \, 3 \leq k \leq n.
$$

Now define

$$
\varphi : X \to V = \mathbb{F}^2, \quad \varphi(x) = \begin{cases} 
(1, 0) & \text{for } x \in X_1 \\
(0, 1) & \text{for } x \in X_2 \\
(0, 0) & \text{for } x \in X_3.
\end{cases}
$$

This map is uniformly continuous and $\mathbb{F}^2$ is a locally convex NA $\mathbb{F}$-space. Hence, by the universality property, there exists the continuous extension $\Phi : L_\mathbb{F}(X) \to V$. Now observe that

$$
\Phi(v) = (\lambda_1, \lambda_2) \notin cl(\Phi(X)) = \{(1, 0), (0, 1), (0, 0)\}.
$$

□

6.1. Normability and metrizability.

Theorem 6.5. Let $\mathbb{F}$ be an NA valued field with a trivial valuation, $(X, d)$ be an ultrametric space and $\mathcal{U}(d)$ is the uniformity of $d$. Then the free NA locally convex space $L_\mathbb{F}(X, \mathcal{U}(d))$ is normable by the Kantorovich ultra-norm $||\cdot||_d$.

Proof. As in Lemma 4.1 consider the extension of $d$ on $X$. Next by Theorem 4.3 we have the corresponding Kantorovich ultra-norm $||\cdot||$. It suffices to show that if $\varphi : (X, d) \to V$ is a uniformly continuous map to a locally convex space $V$ then the linear extension $\Phi : (L_\mathbb{F}(X), ||\cdot||) \to V$ is continuous. Being a locally convex space the topology of $V$ is defined by a collection of ultra-seminorms $\{\rho_i\}_{i \in I}$. Clearly, $\varphi : (X, d) \to V$ is uniformly continuous. Fix $\varepsilon > 0$ and $i_0 \in I$. It follows that there exists $\delta > 0$ such that $\rho_{i_0}(\varphi(x) -$
By (3) and the strong triangle inequality we have \(|\varphi(y)| < \varepsilon \forall x, y \in X\) with \(d(x, y) < \delta\). Now assume that \(u \in L_F(X)\) with \(||u|| < \delta\). We prove the continuity of \(\Phi\) by showing that \(\rho_0(\Phi(u)) < \varepsilon\). By the definition of the ultra-norm \(||\cdot||\) there exists a decomposition \(u = \sum_{i=1}^{n} \lambda_i(x_i - y_i)\) such that \(\max_{1 \leq i \leq n} |\lambda_i|d(x_i, y_i) < \delta\). Since the valuation \(|\cdot|\) is trivial we obtain that \(\max_{1 \leq i \leq n} d(x_i, y_i) < \delta\). It follows that

\[
\rho_0(\Phi(u)) = \rho_i(\Phi(\sum_{i=1}^{n} \lambda_i(x_i - y_i))) = \rho_i(\sum_{i=1}^{n} \lambda_i(\varphi(x_i) - \varphi(y_i))) \\
\leq \max_{1 \leq i \leq n} \lambda_i |\rho_0(\varphi(x_i) - \varphi(y_i))| = \max_{1 \leq i \leq n} \rho_0(\varphi(x_i) - \varphi(y_i)) < \varepsilon.
\]

It is known that if a Tychonoff space \(X\) is non-discrete, then \(A(X)\) is not metrizable (see [1, Theorem 7.1.20]). This result inspired us to obtain the following.

**Proposition 6.6.** Let \((X, \mathcal{U})\) be a non-discrete NA uniform space. Let \(F\) be a complete NA valued field with a non-trivial valuation. Then \(L_F(X, \mathcal{U})\) is not metrizable.

**Proof.** Assuming the contrary, there exists a decreasing sequence \(\{U_n\}_{n \in \mathbb{N}}\) which forms a local base at \(0 \in L_F(X, \mathcal{U})\). Since the valuation \(|\cdot|\) is non-trivial there exists \(\lambda \in F\) with \(|\lambda| > 1\). In view of Theorem 6.2 \((X, \mathcal{U})\) is a uniform subspace of \(L_F(X, \mathcal{U})\). By the continuity of the scalar multiplication it follows that there exists a sequence of entourages \(\varepsilon_n \in \mathcal{U}\) such that \(\lambda^n(x - y) \in U_n \forall x, y \in \varepsilon_n\). Since \(\mathcal{U}\) is non-discrete and Hausdorff we can find a sequence \((x_n, y_n)\) \(\in \varepsilon_n\) such that \(x_n \neq y_n \ \forall n \in \mathbb{N}\) and \(\forall i < n \ x_n \notin \{x_i, y_i\}\). Clearly, the sequence \(u_n = \lambda^n(x_n - y_n) \in U_n\) converges to \(0\). Let us show that this leads to a contradiction. By induction on \(n\), we define a sequence \(\{f_n : n \in \mathbb{N}\}\) of uniformly continuous functions on \((X, \mathcal{U})\) with values in \(F\) such that for every \(n \geq 1\):

1. \(|f_n(x)| \leq |\lambda|^{-n} \forall x \in X\);
2. \(f_n(x_k) = f_n(y_k) = f_n(y_n) = 0_F \ \forall k < n\);
3. \(f_n(x_n) = \lambda^{-n} - \sum_{k=1}^{n-1} (f_k(x_n) - f_k(y_n))\) if \(|\sum_{k=1}^{n-1} (f_k(x_n) - f_k(y_n))| \leq |\lambda|^{-n}\) and \(f_n(x_n) = \lambda^{-n}\) otherwise.

By (3) and the strong triangle inequality we have \(|f_n(x_n) + \sum_{k=1}^{n-1} (f_k(x_n) - f_k(y_n))| \geq |\lambda|^{-n}\).

By (1) for every \(x \in X\) the sequence of partial sums \(\left\{\sum_{k=1}^{n} f_k(x)\right\}_{n \in \mathbb{N}}\) is Cauchy. Since the field \(F\) is complete it follows that the function \(f = \sum_{n=1}^{\infty} f_n\) is well defined. From (1) it follows that \(f\) is uniformly continuous. Therefore, it admits an extension to a linear continuous map \(\tilde{f} : L_F(X, \mathcal{U}) \to F\). For every \(n \in \mathbb{N}\) we have

\[
|\tilde{f}(u_n)| = |\sum_{k=1}^{\infty} (f_k(x_n) - f_k(y_n))| = \\
= |\lambda|^{n} \cdot |\sum_{k=1}^{\infty} (f_k(x_n) - f_k(y_n))| = \\
\geq |\lambda|^{n} \cdot |\lambda|^{-n} = 1.
\]

It follows that the sequence \(\{\tilde{f}(u_n)\}\) does not converge to \(0\) contradicting the continuity of \(\tilde{f}\).

We know that the uniform free NA abelian topological group \(A_{N_A}\) (Definition 6.7) is metrizable for every metrizable NA uniform space \((X, \mathcal{U})\) (see [9] and also Remark 6.9).
6.2. Free abelian NA groups and NA Tkachenko-Uspenskij theorem. Recall the following definition from [9].

**Definition 6.7.** Let $(X,\mathcal{U})$ be an NA uniform space. The uniform free NA abelian topological group of $(X,\mathcal{U})$ is denoted by $A_{NA}$ and defined as follows: $A_{NA}$ is an NA abelian topological group for which there exists a universal uniform map $i : X \to A_{NA}$ satisfying the following universal property. For every uniformly continuous map $\varphi : (X,\mathcal{U}) \to G$ into an abelian NA topological group $G$ there exists a unique continuous homomorphism $\Phi : A_{NA} \to G$ for which the following diagram commutes:

$$
\begin{array}{ccc}
(X,\mathcal{U}) & \xrightarrow{i} & A_{NA} \\
\downarrow{\varphi} & & \downarrow{\Phi} \\
G & & G
\end{array}
$$

Let $(X,\mathcal{U})$ be an NA uniform space and $Eq(\mathcal{U})$ be the set of all equivalence relations from $\mathcal{U}$.

**Theorem 6.8.** [9, Theorem 4.14] Let $(X,\mathcal{U})$ be NA and let $\mathcal{B} \subseteq Eq(\mathcal{U})$ be a base of $\mathcal{U}$. For every $\varepsilon \in \mathcal{B}$ denote by $< \varepsilon >$ the subgroup of $A(X)$ algebraically generated by the set

$$
\{ x - y \in A(X) : (x,y) \in \varepsilon \},
$$

then $\{ < \varepsilon > \}_{\varepsilon \in \mathcal{B}}$ is a local base at the zero element of $A(X)$.

**Remark 6.9.** It is easy to see from the above description that if $(X,d)$ is an ultrametric space then $A_{NA}$ is metrizable. The following theorem provides a specific metrization which can be viewed as a Graev type ultra-norm.

**Lemma 6.10.** Let $(X,d)$ be an ultrametric space treated as an ultrametric subspace of $(\overline{X},d)$ as in Lemma 4.1. Then $A_{NA}$ is metrizable by the Graev type ultra-norm defined as follows. For $u \in A(X)$ let

$$
||u|| := \inf \left\{ \max_{1 \leq i \leq n} d(x_i, y_i) : u = \sum_{i=1}^{n} (x_i - y_i), \, x_i, y_i \in \overline{X} \right\}.
$$

**Proof.** Observe that for $\varepsilon < 1$ for the $\varepsilon$-ball we have $B_d(0, \varepsilon) = < \varepsilon >$. \hfill $\square$

**Remark 6.11.** Suppose that $(X,\mathcal{U})$ is an NA uniform space generated by a collection of ultra-seminorms $\{d_i\}_{i \in I}$. Then using the idea of Lemma 6.10 one can show that the topology of $A_{NA}$ is generated by the set of the corresponding Graev type ultra-norms $\{|| \cdot ||_{d_i}\}_{i \in I}$. So we have an analogy with Theorem 6.2. At the same time we have one key difference. In the description of $A_{NA}$ it is enough to consider any set of ultra-pseudometrics $\{d_i\}_{i \in I}$ which generate the uniformity $\mathcal{U}$ on $X$.

By Tkachenko-Uspenskij theorem [20, 21] the free abelian topological group $A(X)$ is a topological subgroup of $L(X)$ (here $\mathbb{F} = \mathbb{R}$). This can be derived (as in [21]) using the usual integer value property and descriptions of Graev's extension. Consider an NA valued field $\mathbb{F}$ of characteristic zero. It is clear that, algebraically, $A(X)$ is a natural subgroup of $L_{\mathbb{F}}(X)$ since $\mathbb{Q}$ is embedded into $\mathbb{F}$ as a subfield. So, it is a natural question for which NA valued fields $\mathbb{F}$ we have an analogue of Tkachenko-Uspenskij theorem. Theorem 6.13 shows that this is true if and only if the valuation of $\mathbb{F}$ is trivial on $\mathbb{Q}$. First we give a particular example.

**Example 6.12.** Tkachenko-Uspenskij theorem is not true for the field $\mathbb{F} = \mathbb{Q}_p$ of $p$-adic numbers (with its standard valuation). Clearly, $\lim p^n = 0_\mathbb{F}$ in $\mathbb{F}$. Now, let $x, y \in X$ be a pair of distinct points in an ultrametric space $X$. By the continuity of the operations
$u_n := p^n(x - y)$ converges to zero in the free locally convex space $L_{\mathbb{F}}(X)$. At the same time it is not true in the free NA abelian group $A_{\mathbb{N},\mathbb{A}}(X)$ as it follows from the internal description of the topology of $A_{\mathbb{N},\mathbb{A}}(X)$ (see Theorem 6.8 or [9]).

**Theorem 6.13.** Let $\mathbb{F}$ be an NA valued field and $(X, U)$ be an NA uniform space. Suppose also that $\text{char}(\mathbb{F}) = 0$ and consider $A(X)$ as an algebraic subgroup of $L(X)$. Then the following conditions are equivalent:

1. $A_{\mathbb{N},\mathbb{A}}$ is a topological subgroup of $L_{\mathbb{F}}(X, U)$.
2. The valuation of $\mathbb{F}$ is trivial on $\mathbb{Q}$.

**Proof.** (1) $\Rightarrow$ (2): If the valuation on $\mathbb{Q}$ is not trivial then by Ostrowski’s Theorem 3.1 this restricted valuation is equivalent to the usual $p$-adic valuation. Now the proof is reduced to the concrete case of Example 6.12.

(2) $\Rightarrow$ (1): By Proposition 6.3 we know that $L_{\mathbb{Q}}(X, U)$ is a topological subgroup of $L_{\mathbb{F}}(X, U)$. So it suffices to show that $A_{\mathbb{N},\mathbb{A}}$ is a topological subgroup of $L_{\mathbb{Q}}(X, U)$. Let $\{d_i\}_{i \in I}$ be a family of ultra-pseudometrics generating the uniformity $U$. For every $i$ extend $d_i$ to $X$ as in Lemma 4.1. Then, consider the Kantorovich ultra-seminorm (Theorem 4.3) $\| \cdot \|_{d_i}$ on $L_{\mathbb{Q}}(X)$. Since the restricted valuation $\cdot$ on $\mathbb{Q}$ is trivial the topology of $L_{\mathbb{Q}}(X, U)$ is generated by the family $\{\| \cdot \|_{d_i}\}_{i \in I}$. It suffices to prove the following claim.

**Claim:** Let $(X, d)$ be an ultra-pseudometric space, $\| \cdot \|^L$ be the corresponding Kantorovich ultra-seminorm on $L_{\mathbb{Q}}(X)$ and $\| \cdot \|^A$ be the corresponding Graev type ultra-seminorm on $A_{\mathbb{N},\mathbb{A}}$ (from Lemma 6.10). Then $\| u \|^L = \| u \|^A$ for every $u \in A(X)$.

**Proof.** Since $Z$ is an additive subgroup of $\mathbb{Q}$ it follows by Lemma 5.3 that

$$\| u \|^L = \inf \left\{ \max_{1 \leq i \leq n} \lambda_i d(x_i, y_i) : u = \sum_{i=1}^n \lambda_i(x_i - y_i), x_i, y_i \in X, \lambda_i \in \mathbb{Z} \right\} =$$

$$= \inf \left\{ \max_{1 \leq i \leq n} d(x_i, y_i) : u = \sum_{i=1}^n \lambda_i(x_i - y_i), x_i, y_i \in X, \lambda_i \in \mathbb{Z} \right\} =$$

$$= \inf \left\{ \max_{1 \leq i \leq n} d(x_i, y_i) : u = \sum_{i=1}^n (x_i - y_i), x_i, y_i \in X \right\} = \| u \|^A.$$

Example 6.14. Theorem 6.13 can be applied to the Levi-Civita field $\mathcal{R}$ (see [18] for example). Recall that the elements of $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support. That is, for every rational number $q$ the set $A_q := \{ a < q | f(a) \neq 0 \}$ is finite. $\mathbb{R}$ is (algebraically) isomorphic to a subfield of $\mathcal{R}$. Indeed, the map $a \mapsto f_a$ from $\mathbb{R}$ to $\mathcal{R}$, where $f_a(0) = a$ and $f_a(x) = 0 \forall x \neq 0$ is a field embedding. Recall that $\mathcal{R}$ admits a natural NA valuation defined as follows. Due to the left-finiteness of the support for every non-zero element $x \in \mathcal{R}$ let $|x| = e^{-\min \text{supp}(x)}$, where $\text{supp}(x)$ denotes the support of $x$. It is easy to see that this valuation is dense. At the same time the restricted valuation on $\mathbb{Q}$ is trivial. We conclude by Theorem 6.13 that $A_{\mathbb{N},\mathbb{A}}$ is a topological subgroup of $L_{\mathcal{R}}(X, U)$ for every NA uniform space $(X, U)$.

7. Some problems

(1) One of the most attracting directions is to study concrete applications of the non-archimedean transportation problem.
(2) It would be interesting to look for additional properties of the free NA locally convex $F$-space.

(3) It is unclear if Theorems 5.8 and 5.9 remain true in general for any NA valued field $F$, that is if the valuation is dense (by Corollary 5.10.1 we can suppose in addition that $\text{char}(F) = 0$). For instance, in the particular case of the Levi-Civita field $F = \mathcal{R}$.

References

[1] A. Arhangel’skii and M. Tkachenko, *Topological groups and related structures*, v. 1 of Atlantis Studies in Math. Series Editor: J. van Mill. Atlantis Press, World Scientific, Amsterdam-Paris, 2008.

[2] R. Burkard, M. DellAmico and S. Martello, *Assignment Problems*, SIAM, Philadelphia, 2009.

[3] Y. Deng and W. Du, *The Kantorovich metric in computer science: A brief survey*, Electronic Notes in Theoretical Computer Science, **253** (2009), no. 3, 73-82.

[4] J. Flood, Free topological vector spaces, Dissertationes Math. CCXXI (1984), PWN, Warszawa.

[5] S. Gao and V. Pestov, *On a universality property of some abelian Polish groups*, Fund. Math. **179** (2003), no. 1, 1-15.

[6] L.V. Kantorovich, *On the transfer of masses*, Dokl. Akad. Nauk USSR **37** (1942), no. 7-8, 227-229 (in Russian).

[7] S.V. Lyudkovskii, *Non-Archimedean free Banach space*, Fundam. Prikl. Mat., **1:4** (1995), 979-987.

[8] M. Megrelishvili and M. Shlossberg, *Notes on non-archimedean topological groups*, Top. Appl., **159** (2012), 2497-2505.

[9] M. Megrelishvili and M. Shlossberg, *Free non-archimedean topological groups*, Comment. Math. Univ. Carolin. **54:2** (2013), 273-312.

[10] C. Perez-Garcia and W.H. Schikhof, *Locally Convex Spaces over Non-Archimedean Valued Fields*, Cambridge University Press, New York, 2010.

[11] J. Melleray, F.V. Petrov and A.M. Vershik, *Linearly rigid metric spaces and the embedding problem*, Fund. Math. **99** (2008), no. 2, 177-194.

[12] V. Pestov, *Topological groups: where to from here?* Topology Proc. **24** (1999), 421-502.

[13] S.T. Rachev and L. Rüschendorf, *Mass Transportation Problems*, Vol. I: Theory. Vol. II: Applications, Springer, New York-Berlin-Heidelberg (1998).

[14] D.A. Raikov, *Free locally convex spaces for uniform spaces*. (Russian) Mat. Sb. (N.S.) **63** (105) 1964, 582-590.

[15] A.C.M. van Rooij, *Non-Archimedean Functional Analysis*, Monographs and Textbooks in Pure and Applied Math. **51**, Marcel Dekker, Inc., New York, 1978.

[16] M. Sakarovitch, *Linear Programming*, Springer texts in electrical engineering, Springer-Verlag, New York, 1983.

[17] P. Schneider, *Nonarchimedean Functional Analysis*, Berlin, Springer, 2002.

[18] K. Shamseddine, *On the topological structure of the Levi-Civita field*, J. Math. Anal. Appl., **368** (2010), 281-292.

[19] O.V. Sipacheva, *The topology of free topological groups*, J. Math. Sci. (N.Y.) **131** (2005), no. 4, 5765-5838.

[20] M.G. Tkachenko, *On completeness of free abelian topological groups*, Soviet Math. Dokl. **27** (1983), 341-345.

[21] V.V. Uspenskij, *Free topological groups of metrizable spaces*, Math. USSR Izvestiya, **37** (1991), 657-680.

[22] A.M. Vershik, *The Kantorovich metric: the initial history and little-known applications*, J. Math. Sci. (N.Y.) **133** (2006), no. 4, 1410-1417.

[23] N. Weaver, *Lipschitz Algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999.

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

E-mail address: megereli@math.biu.ac.il

URL: http://www.math.biu.ac.il/~megereli

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

E-mail address: menyash@yahoo.com

URL: http://www.shlossberg.com