The recent experimental confirmation of a transition between meron and skyrmion topological spin textures in the chiral magnet Co$_2$Zn$_8$Mn$_3$ confirms that the skyrmion crystals discovered in 2009 are just the tip of the iceberg. Crystals of topological textures, including skyrmions, merons, vortices and monopoles can be stabilized by combining simple physical ingredients, such as lattice symmetry, frustration and spin anisotropy. The current challenge is to find the combinations of these ingredients that produce specific topological spin textures. Here we report a simple mechanism for the stabilization of meron, skyrmion, and vortex crystals in centrosymmetric tetragonal magnets. In particular, the meron/skyrmion crystals can form even in absence of magnetic field. The application of magnetic field leads to a variety of novel topological spin textures that survive in the long wavelength limit of the theory. When conduction electrons are coupled to the spins, these topological spin textures twist the electronic wave functions to induce Chern insulators and Weyl semimetals for specific band filling fractions.

Introduction. Inspired by the work of Herman Hemholtz, William Thompson proposed in 1867 that atoms could be vortices in ether. While later experiments put this proposal out of business, thinking of topological solitons as emerging building blocks or artificial atoms is very appealing. Indeed, more recent developments, that started around the 1960’s, have demonstrated that nature has plenty of room for finding updated versions of ether. The “ether of quantum magnets” is the vector field of magnetic moments, whose topological solitons can be regarded as emergent mesoscale atoms. Like real atoms, these solitons form crystal structures dictated by symmetry, anisotropy and competing interactions.

Periodic arrays of topological spin textures typically arise from the superposition of small-$\mathbf{Q}$ spirals propagating along symmetry related directions (multi-$\mathbf{Q}$ ordering), whose wave-length is dictated by the competition between ferromagnetic (FM) and antiferromagnetic (AFM) interactions. In chiral magnets, the FM interaction competes against the antisymmetric component of the effective exchange tensor, known as Dzyaloshinskii-Moriya (DM) interaction. However, the selection of a small-$\mathbf{Q}$ spiral ordering is not enough to stabilize topological spin textures because the superposition of multiple spirals requires additional energetic considerations. For instance, it is known that three-fold symmetric lattices are more favorable for the formation of skyrmion crystals (triple-$\mathbf{Q}$ ordering) induced by a magnetic field parallel to the $c$-axis, because the Ginzburg-Landau (GL) free energy can include terms of the form $(\mathbf{S}_0 \cdot \mathbf{Q}_1)(\mathbf{S}_0 \cdot \mathbf{Q}_2)\quad (\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 = 0$ because the three ordering wavevectors differ by $2\pi/3$ rotations). This simple consideration explains why the vast majority of magnetic skyrmion crystals have been found by applying magnetic field along a three-fold symmetry axis of different materials.

While less common, square skyrmion and meron crystals have been recently reported in chiral, polar, and centrosymmetric tetragonal magnets. The observation of skyrmion crystals in four-fold symmetric lattices forces us to think about alternative stabilization mechanisms. Here, we report a new guiding principle for the formation of meron crystals (MX), which are also skyrmion crystals (SkX), in four-fold (tetragonal) lattices even in absence of magnetic field. The merons form a square lattice and the magnetic unit cell includes four merons with a net skyrmion charge $n_{sk} = \pm 1$ (MX-I) or $n_{sk} = \pm 2$ (MX-II). The phase diagram also includes a field-induced vortex crystal (VX). Remarkably, this rich phase diagram is obtained from a very simple model for centrosymmetric magnets that only includes competing easy-axis and compass anisotropies.

The different phases of the phase diagram are obtained by minimizing the energy over all the possible spin configurations for a fixed magnetic unit cell, whose size is dictated by the ordering wavevector. We also provide approximated expansions of the relevant spin configurations that include the fundamental Fourier components $\mathbf{q} = Q_\nu$ and a few higher harmonics. Our analysis is complemented with the derivation of the GL theory that describes the long wavelength limit of the microscopic model, allowing us to demonstrate the universal character of the phase diagram. We also analyze the anomalous Hall response of itinerant electrons coupled to the local magnetic moments, and the possible realization a magnetic Weyl semimetal induced by the MX-II phase.

Model. We consider the classical Heisenberg model on the square lattice:

$$\mathcal{H} = J_1 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_3 \sum_{\langle \langle ij \rangle \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \Gamma \sum_i \left( S_i^x S_{i+x}^x + S_i^y S_{i+y}^y \right) - D \sum_i (S_i^z)^2 - H \sum_i S_i^z$$

where $\{J_1, J_2, J_3\}$ are the Heisenberg interactions up to the third nearest neighbor, $\Gamma$ is the compass anisotropy, $D$ is the single-ion anisotropy, and $H$ is the magnetic field along the $z$-axis. In this work, we consider the case of FM nearest neighbor Heisenberg exchange ($J_1 < 0$). The magnitude of the spins is fixed by the normalization condition $|S_i| = 1$. The compass anisotropy can either be generated by the spin-orbit coupling or by the dipolar interaction.
The characteristic length scale of the spin structure is determined by the competition between different symmetric exchange interactions\textsuperscript{18–23}. In absence of anisotropies and magnetic field, the ordering wavevector $\mathbf{Q}$ is obtained by minimizing the exchange interaction in momentum space:

$$J(\mathbf{q}) = J_1 \left( \cos q_x \cos q_y + 2 J_2 \cos q_x \cos q_y \right) + J_3 \left( \cos 2q_x + \cos 2q_y \right)$$  

(2)

In the long wavelength limit we have:

$$J(\mathbf{q}) \approx J_0 - \frac{I_1}{2} q^2 + \frac{I_2}{2} q^4 - I_3 q_x^2 q_y^2$$  

(3)

with $I_0 = 2(J_1 + J_2 + J_3)$ and

$$I_1 = J_1 + 2J_2 + 4J_3, \quad I_2 = \frac{J_1 + 2J_2 + 16J_3}{12}, \quad I_3 = I_2 - \frac{J_2}{2}$$  

(4)

Depending on the sign of the quartic anisotropy, $I_3$, the competing exchange interactions lead to spiral phases with ordering wavevectors $\mathbf{Q} = (\pm Q_0, 0)$ or $\mathbf{Q} = (0, \pm Q_0)$ with $Q_0 = \sqrt{I_1/I_2}$ for $I_3 < 0$, and $\mathbf{Q} = (\pm Q_0, \pm Q_0)/\sqrt{2}$ with $Q_0 = \sqrt{I_1/(2I_2 - I_3)}$ for $I_3 > 0$\textsuperscript{24}. In this situation, the ground state can either be a single-$\mathbf{Q}$ spiral or a multi-$\mathbf{Q}$ structure depending on the values of the spin anisotropies and the magnetic field. Once these extra terms are included in the Hamiltonian, the optimal value of $\mathbf{Q}$ can also change. We note, however, that this change has been found to be negligible for several cases of interest\textsuperscript{19,25}.

The different rows of Fig. 1 include the $T = 0$ phase diagrams of $\mathcal{H}$ for three different sets of exchange interactions, \{J\_1, J\_2, J\_3\} that produce relatively small values of $\mathbf{Q}$ and quartic anisotropy: $I_3 = 0.02|J_1|$, $I_3 = 0$ and $I_3 \approx -0.037|J_1|$. The different columns correspond to four different values of the compass anisotropy $\Gamma$. Since $I_3$ and $\Gamma$ are the main sources of tetragonal anisotropy, the first column of Fig. 1 (Fig. 1a,e,i) describes cases with weak lattice anisotropy. Thus, it is not surprising that the phase diagrams are similar to the ones obtained in the continuum limit of the isotropic theory\textsuperscript{26}. The ordering wavevectors $\{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$ differ by $120^\circ$, implying that the skyrmion crystal SkX($\Delta$) has hexagonal symmetry in spite of the underlying tetragonal atomic lattice. The conical spiral (CS) is just a canted cycloidal spiral induced by the effective easy-plane anisotropy produced by the applied magnetic field. In contrast, the easy-axis anisotropy term $D > 0$ favors a proper-screw or vertical spiral (VS). The additional two phases, $2\mathbf{Q}$-conical spiral [2\mathbf{Q}-CS($\Delta$)] and the vertical spiral with in-plane modulation [VS($\Delta$)] have also been reported for centrosymmetric magnets with hexagonal symmetry\textsuperscript{18–22,25}.

The phase diagram is qualitatively different for $\Gamma/|J_1| \geq 0.1$ because the compass anisotropy term is strong enough to enforce the tetragonal anisotropy on the spin textures. The propagation wavevectors, $\mathbf{Q}_1 = (\mathbf{Q}, 0)$ and $\mathbf{Q}_2 = (0, Q)$ of the conical and the vertical spirals are now pinned along the principal $x$ and $y$ directions of the tetragonal lattice\textsuperscript{27}. However, the most remarkable effect of the $\Gamma$-term is the stabilization of double-$\mathbf{Q}$
orderings that are superpositions of two proper screw spirals with propagation wavevectors $Q_1$ and $Q_2$. In particular, the MX-II, MX-I and ViX phases, shown in Fig. 2a–c, have the same intensity in the spin structure factor at both wavevectors; while the vertical spiral with in-plane modulation (VS') has different intensities (Fig. 2d), implying that it is not invariant under a 90 degree rotation.

The magnetic unit cell of the MX-II state includes four merons (Fig. 2a), whose skyrmion charge adds up to $|n_{sk}| = 2$ (Fig. 2a, Supplementary Fig. S2a), meaning that the MX-II phase is simultaneously a double-SkX state with net scalar spin chirality $\chi_{ijk} = S_i \cdot (S_j \times S_k)$. Note that the scalar chiralities of the merons have the same sign because a change of sign in the vector spin chirality $\hat{z} \cdot (S_j \times S_k)$ (vorticity) is always accompanied by a sign change of $S_i^z$. The MX-I phase is induced by magnetic field, but it can also exist at zero magnetic field (Fig. 1). Its magnetic unit cell includes four merons with a total skyrmion charge $|n_{sk}| = 1$ (Fig. 2b, Supplementary Fig. S2b), implying that this phase is also a SkX. In this case, one of the merons has opposite scalar chirality relative to the other three because of a sign change in $S_i^z$ near its core. It is interesting to note that the MX-I state is topologically equivalent to the spin configurations that have been observed in chiral crystals (Fig. 1e of ref. 2) and centrosymmetric magnets (Fig. 1d of ref. 15). The finite skyrmion charges of the MX-I and MX-II phases make them qualitatively different from the MXs with $n_{sk} = 0$ that have been reported in previous works28,29.

The VtX state can in principle exist all the way from zero magnetic field up to the saturation (Fig. 1). Similarly to the MX-I and MX-II states, this texture includes four vortices in each magnetic unit cell. The main difference is that their topological charges cancel with each other: $n_{sk} = 0$ (Fig. 2c, Supplementary Fig. S2c). The origin of this cancellation is easy to understand at high fields, where $S_i^z$ does not change sign and the net scalar chirality becomes proportional to the net vector chirality. Hexagonal versions of this VtX phase have also been reported for frustrated quantum magnets below the saturation field30,31.

The in-plane spin components are almost identical for the MX-I, MX-II, and ViX states (Fig. 2a–c), making them equally good candidates for the double-$Q$ magnetic textures revealed by Lorentz transmission electron microscopy (TEM)2,15. We must then rely on other measurements to discriminate between the three possibilities. As shown in Fig. 2a–c, the $zz$ compo-
The resulting GL energy functional reads

$$\mathcal{H}_{GL} = \int d\mathbf{r} \left[ -\frac{1}{2} \left( \partial_\mathbf{r} S_r \right)^2 + \frac{\tilde{\Gamma}}{2} \left( \partial_\mathbf{r} S_r \right)^2 \right] + \frac{1}{2} \left( \partial_\mathbf{r}^2 S_r \right)^2 - \tilde{\Delta} \left( \partial_\mathbf{r} S_r \right)^2 - \tilde{H} S_r^2 \right] - \frac{I_3}{I_2} \int d\mathbf{r} \left( \partial_{\mathbf{r}^2} S_r \right) \cdot \left( \partial_{\mathbf{r}^2} S_r \right) \right]$$

(9)

The first observation is that the ordering wavevectors $Q_1$ and $Q_2$ of the double-$Q$ states remain parallel to the principal $x$ and $y$ directions even for $I_3 > 0$. The reason is that the $q$-dependent part of the $\Gamma$-term favors proper screw spirals, $S_r = (\cos(Q_1 \cdot r), \sin(Q_1 \cdot r))$ and $S_r = (\cos(Q_2 \cdot r), 0, \sin(Q_2 \cdot r))$, that propagate along these directions. Indeed, these principal directions become energetically favorable for $I_3/I_2 < \tilde{\Gamma}/(1 - \tilde{\Gamma}/8)$. This observation allows us to understand the stabilization mechanism of the double-$Q$ MX-II phase that is approximately described by the equation

$$m_r^x = \pm a_1 \sin(Q_2 \cdot r)$$

(10a)

$$m_r^y = \pm a_1 \sin(Q_1 \cdot r)$$

(10b)

$$m_r^z = a_0 + a_2 \cos(Q_1 \cdot r) \cos(Q_2 \cdot r)$$

(10c)

where $S_r \equiv m_r/|m_r|$. As it is clear from the phase diagram, the competing zero field phases are the conical (cycloidal) and vertical (proper screw) spiral orderings. A simple evaluation of the energies of these competing phases, which does not include the small higher harmonic components, produces a rough estimate of the stability interval of the MX-II phase at zero field (Supplementary Sec. IV):

$$-\frac{1}{2} \frac{5}{32} \left( 2 - \Gamma \right)^2 - \frac{I_3}{4 I_2} \leq \Delta \leq -\frac{1}{12} + \frac{I_3}{6 I_2}$$

(11)

Note that a positive quartic anisotropy $0 < I_3/I_2 < \tilde{\Gamma}/(1 - \tilde{\Gamma}/8)$ lowers the exchange energy cost of the MX-II phase (the...
first harmonics of this phase are \( \pm \mathbf{Q}_1 \pm \mathbf{Q}_2 \). According to equations (6) and (8), the \( g \)-independent contribution of the \( \Gamma \)-term can turn the single-ion term into an effective in-plane anisotropy \( D < 0 \). This explains the suppression of the VS phase in favor of the MX-II phase, which has stronger in-plane (XY) spin components. The MX-II phase is also more stable than the CS because of the \( g \)-dependent contribution of the \( \Gamma \) term increases the energy cost of the CS phase by an amount proportional to \( \Gamma Q^2 \) (Supplementary Sec. IV).

For positive \( I_3 \), the compass anisotropy \( \Gamma \) must be strong enough to guarantee that the ordering wavevectors remain parallel to the principal \( x \) and \( y \) directions. These conditions can be naturally fulfilled by tetragonal metallic systems, such as 4 \( f \)-electron materials, with dominant RKKY interactions and a small Fermi surface. The combination of a small Fermi surface and strong spin-orbit coupling can naturally lead to 0 < \( I_3 \) < \( 1/2 \).

**Anomalous Hall effect in 2D.** The net scalar chirality of the double-\( Q \) states MX-I and MX-II creates an effective U(1) gauge flux \( ^{32} \) for itinerant electrons that are coupled via exchange to the magnetic moments \( S_{\alpha} \). The resulting non-zero Berry curvature of the (reconstructed) electronic bands leads to anomalous Hall effect (AHE). Unlike typical realizations of field-induced SkX, the MX-I and MX-II states can produce a spontaneous topological Hall effect at zero field.

This simple phenomenon can be illustrated by coupling the local moments \( S_{\alpha} \) to the spins of itinerant electrons \( ^{16} \):

\[
\mathcal{H}_\text{K} = -i \sum_{(ij)} \sum_{\sigma} \left( c_{i\sigma}^\dagger c_{j\sigma} + h.c. \right) + J \sum_i \sum_{\alpha \beta} c_{i\alpha}^\dagger \sigma_{\alpha\beta} c_{i\beta} \cdot S_i
\]

where the first term is the nearest neighbor hopping, and the second term represents the exchange interaction \( J \) between the itinerant electrons and the local moments. By assuming that the effective spin-spin interaction mediated by the conduction electrons is much smaller than the relevant exchange energy scales in equation (1), we can still use the magnetic phase diagram obtained for \( \mathcal{H} \) (Fig. 1). As discussed below, the topological properties of the itinerant electrons are mainly dictated by symmetry, so the choice of nearest neighbor hopping in equation (12) is a matter of convenience.

For long period \( (Q \ll 1) \) incommensurate states, the (folded) first Brillouin zone (BZ) includes a large number of reconstructed electronic bands. For simplicity, here we focus on the four lowest energy bands \( ^{33} \). Figure 4a shows the reconstructed electronic bands for the MX-II state, which turn out to be doubly degenerate at \( H = 0 \). This degeneracy is protected by the combined symmetries \( O_1 = IC_2 \) and \( O_2 = \left\{ C_{2x}, \overline{M}_y \right\} \) (Supplementary Sec. V). A finite magnetic field along the \( z \) axis breaks \( O_2 \) and partially lifts the degeneracy except for the folded BZ boundaries, which are protected by the non-symmorphic symmetries \( O_3 = \left\{ \mathcal{T}, \overline{M}_y \right\} \) and \( O_4 = \left\{ \mathcal{T}, \overline{M}_y \right\} \) (Supplementary Sec. V). The remaining degeneracies can be fully lifted by applying an in-plane magnetic field along the [110] direction (Fig. 4a) that distorts the MX-II state.

The MX-II state produces anomalous Hall conductivity even in absence of magnetic field, implying that the Hall effect is purely of topological origin. As shown in Fig. 4b, a value as large as \( |\sigma_{xy}| = 2e^2/h \) can be achieved by completely filling the four lowest bands. Figures 4c and 4d show the Berry curvatures of the two lowest bands depicted in Fig. 4a. The direct field-induced gap between these two bands reaches the minimum value at the M point of the folded BZ (see inset of Fig. 4a) producing a sharp increase of the Berry curvature (Fig. 4c,d) and a sharp peak of \( \sigma_{xy} \) around band filling \( \nu = 2 \) (Fig. 4b) \( ^{35} \). The resulting Chern number of the lower (higher) energy band is \( C = 0 \) (–1). Although the Berry curvature of the lowest band is large and positive at the M point, it is negative and small in the rest of the BZ (Fig. 4c), leading to a cancellation of the Chern number. Consequently, the Hall conductivity becomes quantized at \( \sigma_{xy} = -e^2/h \) for filling fraction \( \nu = 2 \). Similarly, \( \sigma_{xy} = -2e^2/h \) for filling fraction \( \nu = 4 \) because the total Chern number of the four lowest bands is \( C = -2 \) (see Fig. 4b). Note that the signs of \( C \) and \( \sigma_{xy} \) are controlled by the sign of scalar chirality of the MX-II state, which can be flipped without energy cost.

**Magnetic Weyl semimetal in 3D.** Weyl semimetals are
realized in crystals with broken time-reversal or inversion symmetry\textsuperscript{36}. In particular, it was proposed that topological spin textures and Weyl points could affect each other in the same material\textsuperscript{37}. As we demonstrate below, Weyl points can be systematically generated by coupling the conduction electrons to vertically stacked layers of MX-II states. The $O_{h=1,2,3,4}$ symmetries of 2D MX-II state can be generalized to 3D MX-II state as $O_{h}\overline{M}_z$ (Supplementary Sec. V). In the $H = 0$ case, all the bands are doubly degenerate in the whole BZ, and the degeneracy is protected by the combined symmetries $O_1\overline{M}_z$ and $O_2\overline{M}_z$ (Supplementary Sec. V). Weyl points can be generated by partially lifting this double degeneracy, which can be done in different ways. For instance, the distortion of the spin texture induced by an in-plane field $H_y$ along the [010] direction breaks the $O_1\overline{M}_z$, $O_2\overline{M}_z$, and $O_1\overline{M}_2$ symmetries, leaving a nodal surface at the BZ side $k = (k_x, \pi/L, k_z)$ protected by $O_2\overline{M}_z$, where $L$ is the lattice constant of the magnetic superlattice. Further inclusion of the spin-orbit coupling (SOC) in the interlayer hopping $t_z\sum_{i,j}\sum_{\alpha\beta} \left(c_{i+x,\alpha}^\dagger \sigma_{\alpha\beta}^x c_{j+\beta} + h.c. \right)$ gaps out the nodal surfaces except for two nodal lines at $k = (k_x, \pi/L, \pm \pi/2)$, where the distance between two nearest neighbor MX-II layers is set as unity (Fig. 5a). The SOC-induced conversion of the nodal surface into two nodal lines can be easily understood by noticing that the Fourier transform of the SOC term vanishes at $k_z = \pm \pi/2$ (see Supplementary Sec. V).

Finally, a further inclusion of SOC in the intralayer hopping, $t_x\sum_{i,j}\sum_{\alpha\beta} \left(c_{i+x,\alpha}^\dagger \sigma_{\alpha\beta}^y c_{j+\beta} + h.c. \right)$, lifts the degeneracy of the nodal lines and generates pairs of Weyl points with opposite chirality in the folded first BZ (Fig. 5b,c). The chirality of a Weyl point is characterized by the topological charge that counts the total Berry curvature flux emitted or absorbed by the Weyl point\textsuperscript{38}. In our system, each pair of vertically aligned Weyl points carry opposite topological charges $C = \pm 1$ (Fig. 5b). Then 2D energy bands for fixed $k_z$ values between the two Weyl points (highlighted in Fig. 5b) carry a nonzero Chern number. Thus, the contribution of each pair of Weyl points to the quantum anomalous Hall effect ($\sigma_{xy}$) increases with their separation, which can be tuned by changing the ratio $t_x/t_z$\textsuperscript{39}.

**Summary and discussion.** Our work introduces a new guiding principle for the stabilization of novel topological spin textures in centrosymmetric tetragonal magnets. The magnetic unit cell of these crystals comprises four merons, whose net skyrmion charge can be $\pm 2$ (MX-II), $\pm 1$ (MX-I) and 0 (ViX). The three phases are double-$Q$ orderings obtained from the superposition of two proper screw (vertical) spirals that propagate along perpendicular directions with a wavenumber $Q$ dictated by competing exchange interactions (magnetic frustration). The key observation is that the above-mentioned double-$Q$ vertical spiral orderings result from the competition between easy-axis and compass anisotropies. The combination of these two terms can produce a relatively weak effective easy-plane anisotropy $D < 0$ and a rather strong $q$-dependent component of the compass anisotropy $\Gamma$, that penalizes the conical phase. An effective easy-plane single-ion anisotropy favors the double-$Q$ vertical spiral orderings relative to the single-$Q$ vertical spiral phase because, on average, the MXs and the ViX have a stronger in-plane component of the spins than the VS.

It is important to note that these double-$Q$ phases could be further stabilized by effective four-spin interactions that are naturally generated in systems where the interaction between local moments is mediated by conduction electrons\textsuperscript{32,40,41}. In this case, the compass anisotropy alone (without the need of a bare single-ion anisotropy) should be enough to stabilize the MX-I, MX-II and ViX phases. Since these conditions can be naturally fulfilled by $f$-electron mangets, our results indicate that tetragonal $f$-electron materials with relatively weak single-ion anisotropy are natural candidates to host these phases. This hypothesis is supported by the recent experimental results in the centrosymmetric tetragonal material GdRu$_2$Si$_2$\textsuperscript{15}.

Finally, the effective U(1) gauge flux\textsuperscript{32} generated by the MXs produces anomalous Hall effect in metallic systems even at zero field. Moreover, a symmetry analysis reveals that the reconstructed electronic bands can naturally include Weyl

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**Figure 5.** Nodal lines and Weyl points generated by coupling vertically stacked MX-II states to itinerant electrons. The MX-II states used in this calculation were obtained with $J_1 = -1, J_2 = 0.35, J_3 = 0.15, \Gamma = 0.3$, and $D = 0.16$ in the $L = 8$ variational space (Methods). a, Nodal lines at $k = (k_x, \pi/L, \pm \pi/2)$ are generated by distorting the MX-II state with an in-plane field $H_y$ along the [010] direction, and a finite SOC in the interlayer hopping $t_z$. b, Weyl points marked by red and blue spots with topological charges $\pm 1$, respectively, are generated by further adding intralayer hopping $t_x$ with finite SOC. c, Electronic dispersion of the four lowest bands along $A \rightarrow M \rightarrow A$ direction, where the inset shows a zoomed-in view near the Weyl points. The parameters used in (c) are $J = t, H_y = 0.012, t_z = 0.1t$, and $t_x = 0.05t$. 

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**Table 1.** Summary of the effective easy-plane single-ion anisotropy $O_{h}\overline{M}_z$ (Supplementary Sec. V).
points for 3D versions of the MX phases, that lead to a magnetic Weyl semimetal for certain filling fractions.

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\section*{Methods}

\subsection*{Variational methods.}

The variational methods used in this Article follow closely ref.\textsuperscript{25} except for some fine details.

We start by assuming that the ground state magnetic supercell contains $L \times L$ spins spanned by the basis \{\(La_1, La_2\)\}, where $a_1$ and $a_2$ are the basis of the square lattice. In other words, the ground state ordering wavenumber $Q$ is fixed by $Q = |\vec{b}|/L$, where $l$ is another integer satisfying $l < L$. We first try several integer values of $L$ which is given by $L \approx 2\pi/Q_0, \quad 2\pi/Q_0 \pm 1, \quad 2\pi/Q_0 \pm 2...$ (13) where $Q_0$ is the wavenumber that minimizes equation (2). Clearly, if the true ground state is incommensurate to the choices of $L$ (typically ranges from 4 to 10 for the parameters considered in this work), a systematic error is introduced in the energy calculation. To resolve this issue, we will relax this variational constraint later.

The spin states are described by $2 \times L \times L$ variational parameters \{\(\theta_\nu, \phi_\nu\)\} that parametrize the spin space with $|S_\nu| = 1$. The total energy density of equation (1) is then numerically minimized in this variational space\textsuperscript{42}. Finally, for any given parameter set \(\{J_1, J_2, J_3, \Gamma, D, H\}\), we pick the lowest energy solution among the different choices of $L$.

Such a variational calculation is unbiased, in the sense that we are not making any assumption of the spin structure, except for the size of the underlying magnetic unit cell. However, the phase boundaries will not be very accurate since the ordering wavenumber $Q$ is generally incommensurate. To further improve the phase boundaries, we will perform another calculation where $Q$ is included as a variational parameter.

To prepare for the refined variational calculation, we first perform a Fourier analysis of the phases discovered by the “unbiased” variational method and parametrize them by a few dominant Fourier components (Supplementary Sec. III). With such parametrization, we can relax the magnetic supercell assumption, and include the ordering wavenumber as a variational parameter. For each point in the phase diagrams, we compare the optimized energies of different ansatze and pick the lowest one.

The energy minimization of different ansatze are performed on a finite $512 \times 512$ square lattice for the microscopic model\textsuperscript{32}. For the GL theory, we obtain the $\Gamma = 0$ phase diagrams by discretizing the real space continuum and replacing derivatives by finite difference. Then, we optimize different variational ansatze and find the lowest energy solution. The linear dimension of the real space continuum is set to 200, and we typically discretize it by a $200 \times 200$ mesh. Note that the phase boundary between VS (\(\Delta\)) and VS states (Fig. 3a) requires a mesh as fine as 1600 $\times$ 1600 for a good convergence.

\section*{Data availability}

The data that support the findings of this study are available from the corresponding authors upon reasonable request.

\section*{Code availability}

The codes that support the findings of this study are available from the corresponding authors upon reasonable request.

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\section*{Author contributions}

Z.W. and S.Z.L. performed the numerical simulations. Z.W. and C.D.B. designed the variational analysis and derived the effective GL theory and the guiding principles for the stabilization of the MX phases. Y.S. computed the anomalous Hall response. Y.S. and S.Z.L. performed the symmetry analysis. All authors contributed to the writing and discussion.
The authors declare no competing interests.

---Supplementary information---

1. ORDERING WAVEVECTORS OF THE HEISENBERG EXCHANGES

The ordering wavevectors of the square lattice Heisenberg model (no anisotropy or magnetic field) can be determined by minimizing the exchange interaction $J(q)$ in momentum space:

$$J(q) = J_1 \left( \cos q_x + \cos q_y \right) + 2J_2 \cos q_x \cos q_y + J_3 \left( \cos 2q_x + \cos 2q_y \right).$$  \hfill (S1)

The results for $J_1 < 0$ (FM) are presented in Fig. S1. Note that this analysis is no longer valid in the presence of magnetic field (Zeeman) and spin anisotropies because lowest energy state is not necessarily a single-$\mathbf{Q}$ spiral ordering. Consequently, $\mathbf{Q}$ must be treated as an additional variational parameter to determine the lowest energy state (Methods).

Depending on the sign of $2J_3 - J_2$, the incommensurate ordering wavevectors are located on different principle axes (Fig. S1):

- For $2J_3 < J_2$, $\mathbf{Q} = (\pm Q_0, 0)$ or $(0, \pm Q_0)$, where $Q_0 = \arccos \left( -\frac{J_1 + 2J_3}{4J_2} \right)$;
- For $2J_3 > J_2$, $\mathbf{Q} = (\pm Q_0, \pm Q_0)/\sqrt{2}$, where $Q_0 = \sqrt{2} \arccos \left( -\frac{J_1}{2J_3 + J_2} \right)$.

In Fig. 1 in the main text, we take three points in the region with $\mathbf{Q} = (\pm Q_0, 0)/(0, \pm Q_0)$, all of which gives roughly $Q_0 \approx \frac{2\pi}{a}$ (see Fig. S1).

In the long wavelength limit, we can expand equation (S1):

$$J(q) \approx I_0 - \frac{I_1}{2} q_x^2 + \frac{I_2}{2} q_y^2 - I_3 q_x^2 q_y^2,$$

with $I_0 = 2(J_1 + J_2 + J_3)$ and

$$I_1 = J_1 + 2J_2 + 4J_3, \quad I_2 = \frac{J_1 + 2J_2 + 16J_3}{12}, \quad I_3 = I_2 - \frac{J_2}{2}. \hfill (S3)$$

In this limit, the direction of the ordering wavevector $\mathbf{Q}$ is determined by the sign of the quartic anisotropy $I_3$ (see dashed line in Fig. S1):

- For $I_3 < 0$, $\mathbf{Q} = (\pm Q_0, 0)$ or $(0, \pm Q_0)$, where $Q_0 = \sqrt{I_1/I_2}$;
- For $I_3 > 0$, $\mathbf{Q} = (\pm Q_0, \pm Q_0)/\sqrt{2}$, where $Q_0 = \sqrt{I_1/(2I_2 - I_3)}$.

Figure S1. Phase diagram showing the positions of the ordering wavevectors for the square lattice Heisenberg model with $J_1 < 0$ (FM) at $T = 0$. The symbols correspond to the values of $\{J_1, J_2, J_3\}$ used in Fig. 1 in the main text. The dashed line corresponds to $I_3 = 0$. 
II. SKYRMION CHARGE OF MX-I, MX-II, AND VTX STATES

The skyrmion charge is defined by

\[ n_{sk} = \frac{1}{4\pi} \int \mathbf{d}r \mathbf{S}_r \cdot (\partial_x \mathbf{S}_r \times \partial_y \mathbf{S}_r), \]  

which corresponds to the area spanned by the unit vector field \( \mathbf{S}_r \) (in units of the area \( 4\pi \) of the unit sphere) when the two dimensional real space \( \mathbb{R}^2 \cup \{\infty\} \) is mapped to \( S^2 \) via stereographic projection. A single skyrmion “wraps” the sphere only once and yields \( n_{sk} = \pm 1 \) (the sign is defined by the sign of scalar chirality). A meron corresponds to a half covering, which yields \( n_{sk} = \pm 1/2 \), while the skyrmion charge of a vortex is generally an arbitrary number different from 1 or 1/2.

Figure S2 shows the vector-contour plots of the MX-II, MX-I, and Vtx states. It is clear that the MX-II state contains four merons (or equivalently, four overlapping merons) in each magnetic unit cell (Fig. S2b). Denote the skyrmion charge of the largest vortex (\( S_{sk}^0 \) has the same energy for the model equation (1) considered in the main text, which can be obtained by simply changing the sign of \( S_{sk}^0 \) at every site. The MX-I state contains four vortices (or equivalently, four overlapping merons) in each magnetic unit cell (Fig. S2b). Denote the skyrmion charge of the largest vortex (\( S_{sk}^0 < 0 \) in the center) as \( \alpha \) (\( 1/2 < \alpha < 1 \)), then the two vortices centered at the elliptical contours have skyrmion charge roughly \( 1-\alpha \) each, and the last vortex has skyrmion charge \( -(1-\alpha) \). Thus, the net skyrmion charge of MX-I is \( n_{sk} = \alpha + 2(1-\alpha) - (1-\alpha) = 1 \). Same as the MX-II case, the \( n_{sk} = \pm 1 \) solutions have the same energy in MX-I. The Vtx state has four vortices in the magnetic unit cell, whose skyrmion charges cancel out with each other, giving \( n_{sk} = 0 \) (Fig. S2c).

![Figure S2](image)

**Figure S2.** Vector-contour plots of the MX-II, MX-I, and Vtx states. The arrows represent the \( xy \) components of the spins, and the background colors represent the \( z \) component of the spins. The contours are taken at linear incremental level with \( \Delta S^z = 0.2 \).

III. FOURIER ANALYSIS OF DIFFERENT STATES

The states that appear in this paper can be parametrized by a few dominant Fourier components. We first focus on the single-\( \mathbf{Q} \) and double-\( \mathbf{Q} \) states, which involve at most two perpendicular wavevectors

\[ \mathbf{Q}_1 = (Q, 0), \quad \mathbf{Q}_2 = (0, Q), \]  

and \( 0 < Q < \pi \).

The normalized spin configurations \( \mathbf{S}_r \equiv \mathbf{m}_r/|\mathbf{m}_r| \) of the VS and VS' states can be parametrized by

\[ m^x_{r-x_0} = a_1 \sin (\mathbf{Q}_2 \cdot r) + a_2 \sin (2\mathbf{Q}_2 \cdot r) + a_3 \sin (3\mathbf{Q}_2 \cdot r), \]  

\[ m^y_{r-x_0} = a_4 \sin (\mathbf{Q}_1 \cdot r), \]  

\[ m^z_{r-x_0} = a_0 + a_5 \cos (\mathbf{Q}_2 \cdot r) + a_6 \cos (2\mathbf{Q}_2 \cdot r) + a_7 \cos (3\mathbf{Q}_2 \cdot r). \]

The difference between the VS and VS' states is that \( a_4 = 0 \) in the VS while it is nonzero in the VS'.

The normalized spin configurations \( \mathbf{S}_r \equiv \mathbf{m}_r/|\mathbf{m}_r| \) of the CS state can be parametrized by

\[ m^x_{r-x_0} = a_1 \cos (\mathbf{Q}_1 \cdot r) + a_2 \cos (3\mathbf{Q}_1 \cdot r), \]  

\[ m^y_{r-x_0} = a_3 \sin (\mathbf{Q}_1 \cdot r) + a_4 \sin (3\mathbf{Q}_1 \cdot r), \]  

\[ m^z_{r-x_0} = a_0 + a_5 \cos (2\mathbf{Q}_1 \cdot r). \]
The normalized spin configurations \(S_r \equiv m_r / |m_r|\) of the MX-I, MX-II, and VtX states can be universally parametrized by

\[
\begin{align*}
    m^x_{r-r_0} &= \pm \sin(\mathbf{Q}_2 \cdot \mathbf{r}) \left[ a_1 + a_2 \cos(\mathbf{Q}_1 \cdot \mathbf{r}) + a_3 \cos(2\mathbf{Q}_1 \cdot \mathbf{r}) + a_4 \sin(2\mathbf{Q}_1 \cdot \mathbf{r}) + a_9 (1 + 2 \cos(\mathbf{Q}_2 \cdot \mathbf{r})) \right], \\
    m^y_{r-r_0} &= \pm \sin(\mathbf{Q}_1 \cdot \mathbf{r}) \left[ a_1 + a_2 \cos(\mathbf{Q}_2 \cdot \mathbf{r}) + a_3 \cos(\mathbf{Q}_1 \cdot \mathbf{r}) + a_4 \cos(2\mathbf{Q}_2 \cdot \mathbf{r}) + a_9 (1 + 2 \cos(\mathbf{Q}_1 \cdot \mathbf{r})) \right], \\
    m^z_{r-r_0} &= a_0 + a_5 \cos(\mathbf{Q}_1 \cdot \mathbf{r}) + a_6 \cos(2\mathbf{Q}_2 \cdot \mathbf{r}) + a_8 \cos(\mathbf{Q}_1 \cdot \mathbf{r}) + a_7 \cos(\mathbf{Q}_1 \cdot \mathbf{r}) + a_8 \cos(2\mathbf{Q}_2 \cdot \mathbf{r}) + a_7 \cos(\mathbf{Q}_1 \cdot \mathbf{r}),
\end{align*}
\]

(S8a)

(S8b)

(S8c)

where the \(\pm\) choice controls the sign of the skyrmion charge of each meron and vortex.

We continue with the parametrization of the triple-\(\mathbf{Q}\) states, which are known to exist in the long wavelength limit from the GL theory \(^{20}\). Note that these states are not favored by the spatial anisotropy of the square lattice.

The ordering wavevectors \(\{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}\) are parametrized by a single parameter in the continuum limit because \(|\mathbf{Q}_1| = |\mathbf{Q}_2| = |\mathbf{Q}_3|\) and their directions differ by \(\pm 120^\circ\) rotations. The following parametrization

\[
\mathbf{Q}_1 = (Q_a \cos \alpha, -Q_a \sin \alpha), \quad \mathbf{Q}_2 = (0, Q_b), \quad \mathbf{Q}_3 = -(Q_a \cos \alpha, Q_a \sin \alpha),
\]

(S9)

includes the three ordering wavevectors in the continuum limit of the theory (very weak lattice anisotropy) as well as the corrections by the square lattice anisotropy. The optimal values from the minimization are found to be \(Q_a \approx Q_b \approx \pi / b\) on a finite square lattice (the approximated relations become exact in the thermodynamic limit). Then, the parametrization of the

\[\text{the triple-}\mathbf{Q}\text{ states follow the ones in Refs. 19, 20, and 25.}\]

IV. STABILITY ESTIMATE OF THE MX-II PHASE

To understand the stabilization mechanism of the MX-II phase at zero field, it is helpful to obtain rough estimates of the energies of the CS, MX-II and VS states. Here, we work in the long wavelength limit with the rescaled GL functional.

The CS state at zero field can be approximated by

\[
S_r = (\cos(\mathbf{Q}_1 \cdot \mathbf{r}), \sin(\mathbf{Q}_1 \cdot \mathbf{r}), 0),
\]

(S13)

whose energy density is

\[
\epsilon_{CS} = \frac{Q^2}{2} + \frac{Q^4}{2} + \frac{\hat{\Gamma}}{4} Q^2 = \frac{(2 - \hat{\Gamma})^2}{32},
\]

(S14)

where the optimal \(Q = \sqrt{2 - \hat{\Gamma}} / 2\).
The VS state at zero field can be approximated by

\[ S_r = (\sin (Q_2 \cdot r), 0, \cos (Q_2 \cdot r)) , \]  

(S15)

whose energy density is

\[ e_{VS} = -\frac{Q_2^2}{2} + \frac{Q_1^4}{2} - \frac{D}{2} = -\frac{1}{8} + \frac{D}{2} , \]  

(S16)

where the optimal \( Q = \frac{1}{\sqrt{2}} \).

The MX-II state at zero field is approximated by

\[ m_x^r = \pm a_1 \sin (Q_2 \cdot r) , \]  

(S17a)

\[ m_y^r = \pm a_1 \sin (Q_1 \cdot r) , \]  

(S17b)

\[ m_z^r = a_2 \cos (Q_1 \cdot r) \cos (Q_2 \cdot r) , \]  

(S17c)

where \( S_r = m_r / |m_r| \).

For \( a_1 = a_2 \), we obtain

\[ S_x^q = \pm \sqrt{\frac{4}{5\pi}} \frac{\delta(q + Q_2) - \delta(q - Q_2)}{i} , \]  

(S18a)

\[ S_y^q = \pm \sqrt{\frac{4}{5\pi}} \frac{\delta(q + Q_1) - \delta(q - Q_1)}{i} , \]  

(S18b)

\[ S_z^q = \sqrt{\frac{4}{5\pi}} \frac{\delta(q + Q_1 + Q_2) + \delta(q - Q_1 - Q_2) + \delta(q + Q_1 - Q_2) + \delta(q - Q_1 + Q_2)}{2} , \]  

(S18c)

after applying the sum rule in momentum space (note that we are neglecting higher harmonics). The energy density of this state is

\[ e_{MX-II} = -\frac{3}{5} Q_2^2 + \frac{2}{5} \left( 2 - \frac{I_3}{2I_2} \right) Q_1^4 - \frac{1}{5} D = -\frac{1}{10} - \frac{I_3}{20I_2} - \frac{1}{5} D , \]  

(S19)

where we have fixed \( Q = \frac{1}{\sqrt{2}} \).

A comparison of the energies leads to an estimate of the stability condition for the MX-II state at zero field

\[ -\frac{1}{2} + \frac{5}{32} (2 - \Gamma)^2 - \frac{I_3}{4I_2} \leq D \leq -\frac{1}{12} + \frac{I_3}{6I_2} . \]  

(S20)

V. SYMMETRY ANALYSIS OF BAND DEGENERACY

In this section, we analyze the symmetries of the MX-II state which put restrictions on the electronic band structures of equation (12) in the main text.

The zero-field MX-II state has four important symmetries:

1. \( O_1 = IC_{2z} \);

2. \( O_2 = \{ C_{2x} M_y | \frac{1}{2}, 0, 0 \} \);

3. \( O_3 = \{ T M_x | \frac{1}{2}, 0, 0 \} \);

4. \( O_4 = \{ T \bar{M}_x | 0, \frac{1}{2}, 0 \} \).

Here \( O_2, O_3, \) and \( O_4 \) are nonsymmetric symmetries with a glide reflection operation. \( \bar{M}_y \) is the space reflection operator under which \( y \rightarrow -y \), while \( M_y \) is the reflection operator acting on both space coordinate and spin as \( y \rightarrow -y, S^x \rightarrow -S^x, \) and \( S^z \rightarrow -S^z \). \( I \) is the inversion operator and \( T \) is the time-reversal operator. \( C_{2x} \) and \( C_{2z} \) are two-fold rotation operators with the rotation axis along the \( x \) and \( z \) directions, respectively. \( \{ \frac{1}{2}, 0, 0 \} \) represents a translation along the \( x \) direction by half a lattice constant of the magnetic superlattice.
As shown in the main text, the MX-II state is stable even in the absence of magnetic field. The four symmetries are present at zero field, resulting in double degeneracy of the electronic bands over the full Brillouin zone (BZ) (see Fig. S3a). To prove this point, consider the symmetries $O_1$ and $O_2$. Because $O_2^2 = -1$ and $[O_1, H(k)] = 0$, we can label the Bloch eigenfunctions of the Hamiltonian given in equation (12) by the eigenvalues $\pm i$ of the $O_1$:

$$H(k)|\psi_\pm(k)\rangle = E_\pm(k)|\psi_\pm(k)\rangle,$$

(S21)

and

$$O_1|\psi_\pm(k)\rangle = \pm i|\psi_\pm(k)\rangle.$$  

(S22)

Meanwhile, $[O_2, H(k)] = 0$ gives

$$H(k)O_2|\psi_\pm(k)\rangle = E_\pm(k)O_2|\psi_\pm(k)\rangle.$$  

(S23)

We then have two sets of eigenstates $|\psi_\pm(k)\rangle$ and $O_2|\psi_\pm(k)\rangle$ that have the same eigenvalues $E_\pm(k)$. Furthermore, since $O_1$ and $O_2$ anticommute, $(O_1, O_2) = 0$, we have

$$O_1O_2|\psi_\pm(k)\rangle = -O_2O_1|\psi_\pm(k)\rangle = \mp iO_2|\psi_\pm(k)\rangle.$$  

(S24)

Therefore, $O_2|\psi_\pm(k)\rangle$ is also the eigenstate of $O_1$ with eigenvalue $\mp i$ that is opposite to that of $|\psi_\pm(k)\rangle$. Namely, $|\psi_\pm(k)\rangle$ and $O_2|\psi_\pm(k)\rangle$ are orthogonal states, implying that the energy bands are doubly degenerate. This combined symmetry can be broken by a net magnetization induced by an external magnetic field.

For the MX-II state that is obtained in the presence of a finite magnetic field along the $z$ axis, the combined symmetry of $O_1$ and $O_2$ is broken. Consequently, the double degeneracy is no longer present in the whole BZ. However, it is still present along the boundary of the folded first BZ, which is protected by the symmetries $O_3$ and $O_4$.

The double degeneracy along the BZ boundary $k = (\pi/L, k_y)$ is protected by $O_3$, where $L$ is the lattice constant of the magnetic superlattice. Because $O_3^2 = e^{-i k_y L}$, $O_3$ becomes an antiunitary operator at $k_y = \pi/L$ where $O_3^2 = -1$. Moreover $k = (\pi/L, k_y)$ is invariant under the action of $O_3$ that changes $(k_x, k_y) \rightarrow (-k_x, k_y)$. Therefore, there is a Kramers degeneracy of energy bands along the BZ boundary $k = (\pi/L, k_y)$ (see Fig. S3b). This symmetry can be broken by a net magnetization along $y$ direction.

For the same reason, the double degeneracy along the BZ boundary $k = (k_x, \pi/L)$ is protected by $O_4$ (see Fig. S3c). This symmetry can be broken by a net magnetization along $x$ direction.

We can further generalize the analysis to three-dimensional (3D) systems, where we couple electrons to vertically stacked MX-II states. Similarly to the 2D case, there are four relevant symmetries at zero field: $\{O_1\bar{M}_z, O_2\bar{M}_z, O_3\bar{M}_z, O_4\bar{M}_z\}$, where $\bar{M}_z$ is the space reflection under which $z \rightarrow -z$.

These symmetries protect the band degeneracy at different parts of the 3D BZ:

1. $O_1\bar{M}_z$ and $O_2\bar{M}_z$ together protect the double degeneracy of all energy bands over the whole BZ. The combined symmetries can be broken by a net magnetization or magnetic field in any direction (Fig. S4a).

2. $O_3\bar{M}_z$ protects the double degeneracy of energy bands over the BZ surface at $k = (\pi/L, k_y, k_z)$. This symmetry can be broken by a net magnetization or magnetic field along $y$ direction (Fig. S4b).
Figure S4. **Double degeneracy of the electronic bands coupled to vertically stacked MX-II states.** Blue shades mark the double degeneracy protected by the corresponding symmetries.

3. $O_3\overline{M}_z$ protects the double degeneracy of energy bands over the BZ surface at $\mathbf{k} = (k_x, \pi/L, k_z)$. This symmetry can be broken by a net magnetization or magnetic field along $x$ direction (Fig. S4c).

Now we consider a transverse field in the $y$ direction that induces a net magnetization in the same direction and breaks the $O_1\overline{M}_z$, $O_2\overline{M}_z$, and $O_3\overline{M}_z$ symmetries. According to the analysis above, the remaining $O_4\overline{M}_z$ symmetry protects the double degeneracy of the energy bands only at the surface of the BZ: $\mathbf{k} = (k_x, \pi/L, k_z)$ (Fig. S4c). Further inclusion of the spin-orbit coupling in the interlayer hopping

$$t_z \sum_i \sum_{\alpha\beta} (c^\dagger_{i+\hat{k},\alpha} \sigma^x_{\alpha\beta} c_{i,\beta} + h.c.) = \sum_{k_z} 2t_z \cos(k_z) \sum_{\alpha\beta} \sum_{\gamma} c^\dagger_{k_z,\gamma,\alpha} \sigma^x_{\alpha\beta} c_{k_z,\gamma,\beta}, \quad (S25)$$

breaks the $O_4\overline{M}_z$ symmetry ($\gamma$ is the index that labels different sublattices of the magnet supercell). However, equation (S25) vanishes at $k_z = \pm\pi/2$ resulting in two nodal lines at $\mathbf{k} = (k_x, \pi/L, \pm\pi/2)$ (Fig. 5a).

To further reduce the nodal lines to Weyl points, we also include spin-orbit coupling in the intralayer hopping,

$$t_x \sum_i \sum_{\alpha\beta} (c^\dagger_{i+\hat{x},\alpha} \sigma^x_{\alpha\beta} c_{i,\beta} + h.c.), \quad (S26)$$

which leads to the Weyl points shown in Fig. 5b in the main text.