The Arbitrarily Varying Broadcast Channel with Degraded Message Sets with Causal Side Information at the Encoder

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Abstract

In this work, we study the arbitrarily varying broadcast channel (AVBC), when state information is available at the transmitter in a causal manner. We establish inner and outer bounds on both the random code capacity region and the deterministic code capacity region with degraded message sets. The capacity region is then determined for a class of channels satisfying a condition on the mutual informations between the strategy variables and the channel outputs. As an example, we consider the arbitrarily varying binary symmetric broadcast channel with correlated noises. We show cases where the condition holds, hence the capacity region is determined, and other cases where there is a gap between the bounds.

Index Terms

Arbitrarily varying channel, broadcast channel, degraded message sets, causal state information, Shannon strategies, side information, minimax theorem, deterministic code, random code, symmetrizability.

The arbitrarily varying channel (AVC) was first introduced by Blackwell et al. [5] to describe a communication channel with unknown statistics, that may change over time. It is often described as communication in the presence of an adversary, or a jammer, attempting to disrupt communication.

The arbitrarily varying broadcast channel (AVBC) without side information (SI) was first considered by Jahn [13], who derived an inner bound on the random code capacity region, namely the capacity region achieved by encoder and decoders with a random experiment, shared between the three parties. As indicated by Jahn, the arbitrarily varying broadcast channel inherits some of the properties of its single user counterpart. In particular, the random code capacity region is not necessarily achievable using deterministic codes [5]. Furthermore, Jahn showed that the deterministic code capacity region either coincides with the random code capacity region or else, it has an empty interior [13]. This phenomenon is an analogue of Ahlswede’s dichotomy property [2]. Then, in order to apply Jahn’s inner bound, one has to verify whether the capacity region has non-empty interior or not. As observed in [12], this can be resolved using the results of Ericson [10] and Csiszár and Narayan [8]. Specifically, a necessary and sufficient condition for the capacity region to have a non-empty interior is that both user marginal channels are non-symmetrizable.

Various models of interest involve SI available at the encoder. In [19], the arbitrarily varying degraded broadcast channel with non-causal SI is addressed, using Ahlswede’s Robustification and Elimination Techniques [1]. The single user AVC with causal SI is addressed in the book by Csiszár and Körner [7], while their approach is independent of Ahlswede’s work. A straightforward application of Ahlswede’s Robustification Technique (RT) would violate the causality requirement.

In this work, we study the AVBC with causal SI available at the encoder. We extend Ahlswede’s Robustification and Elimination Techniques [2, 1], originally used in the setting of non-causal SI. In particular, we derive a modified version of Ahlswede’s RT, suited to the setting of causal SI. In a recent paper by the authors [15], a similar proof technique is applied to the arbitrarily varying degraded broadcast channel with causal SI. Here, we generalize those results, and consider a general broadcast channel with degraded message sets with causal SI.

We establish inner and outer bounds on the random code and deterministic code capacity regions. Furthermore, we give conditions on the AVBC under which the bounds coincide, and the capacity region is determined. As an example, we consider the arbitrarily varying binary symmetric broadcast channel with correlated noises. We show that in some cases, the conditions hold and the capacity region is determined. Whereas, in other cases, there is a gap between the bounds.

I. Definitions and Previous Results

A. Notation

We use the following notation conventions throughout. Calligraphic letters \( \mathcal{X}, \mathcal{S}, \mathcal{Y}, \ldots \) are used for finite sets. Lowercase letters \( x, s, y, \ldots \) stand for constants and values of random variables, and uppercase letters \( X, S, Y, \ldots \) stand for random variables. The distribution of a random variable \( X \) is specified by a probability mass function (pmf) \( P_X(x) = p(x) \) over a finite set \( \mathcal{X} \). The set of all pmfs over \( \mathcal{X} \) is denoted by \( \mathcal{P}(\mathcal{X}) \). We use \( x^n = (x_1, x_2, \ldots, x_j) \) to denote a sequence of letters from \( \mathcal{X} \). A random sequence \( X^n \) and its distribution \( P_{X^n}(x^n) = p(x^n) \) are defined accordingly. For a pair of integers \( i \) and \( j, 1 \leq i \leq j \), we define the discrete interval \( [i : j] = \{i, i + 1, \ldots, j\} \).

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B. Channel Description

A state-dependent discrete memoryless broadcast channel \((\mathcal{X} \times \mathcal{S}, W_{Y_{1},Y_{2}|X,S}, \mathcal{Y}_{1}, \mathcal{Y}_{2})\) consists of a finite input alphabet \(\mathcal{X}\), two finite output alphabets \(\mathcal{Y}_{1}\) and \(\mathcal{Y}_{2}\), a finite state alphabet \(\mathcal{S}\), and a collection of conditional pmfs \(W_{Y_{1},Y_{2}|X,S}\). The channel is memoryless without feedback, and therefore \(W_{Y_{1}^{n},Y_{2}^{n}|X^{n},s^{n}}(y_{1}^{n},y_{2}^{n}|x^{n},s^{n}) = \prod_{i=1}^{n} W_{Y_{1},Y_{2}|X,S}(y_{1,i},y_{2,i}|x_{i},s_{i})\). The marginals \(W_{Y_{1}|X,S}\) and \(W_{Y_{2}|X,S}\) correspond to user 1 and user 2, respectively. Throughout, unless mentioned otherwise, it is assumed that the users have degraded message sets. That is, the encoder sends a private message which is intended for user 1, and a public message which is intended for both users. For state-dependent broadcast channels with causal SI, the channel input at time \(i \in [1 : n]\) may depend on the sequence of past and present states \(s^{i}\).

The arbitrarily varying broadcast channel (AVBC) is a discrete memoryless broadcast channel \(W_{Y_{1},Y_{2}|X,S}\) with a state sequence of unknown distribution, not necessarily independent nor stationary. That is, \(S^{n} \sim q(s^{n})\) with an unknown joint pmf \(q(s^{n})\) over \(\mathcal{S}^{n}\). In particular, \(q(s^{n})\) can give mass 1 to some state sequence \(s^{n}\). We denote the AVBC with causal SI by \(B = \{W_{Y_{1},Y_{2}|X,S}\}\).

To analyze the AVBC with degraded message sets with causal SI, we consider the compound broadcast channel. Different models of compound broadcast channels have been considered in the literature, as e.g. in [18] and [3]. Here, we define the compound broadcast channel as a discrete memoryless broadcast channel with a discrete memoryless state, where the state distribution \(q(s)\) is not known in exact, but rather belongs to a family of distributions \(Q\), with \(Q \subseteq \mathcal{P}(S)\). That is, \(S^{n} \sim \prod_{i=1}^{n} q(s_{i})\), with an unknown pmf \(q \in Q\) over \(S\). We denote the compound broadcast channel with causal SI by \(B^{Q}\).

The random parameter broadcast channel is a special case of a compound broadcast channel where the set \(Q\) consists of a single distribution, i.e. when the state sequence is memoryless and distributed according to a given state distribution \(q(s)\). Hence, we denote the random parameter broadcast channel with causal SI by \(B^{q}\).

In Figure 1, we set the basic notation for the broadcast channel families that we consider. The columns correspond to the channel families presented above, namely the random parameter broadcast channel, the compound broadcast channel and the AVBC. The rows indicate the role of SI, namely the case of no SI and causal SI. In the first row, and throughout, we use the subscript ‘0’ to indicate the case where SI is not available.

C. Coding with Degraded Message Sets

We introduce some preliminary definitions, starting with the definitions of a deterministic code and a random code for the AVBC \(B\) with degraded message sets with causal SI. Note that in general, the term ‘code’, unless mentioned otherwise, refers to a deterministic code.

**Definition 1** (A code, an achievable rate pair and capacity region). A \((2^{nR_{0}},2^{nR_{1}},n)\) code for the AVBC \(B\) with degraded message sets with causal SI consists of the following; two message sets \([1 : 2^{nR_{0}}]\) and \([1 : 2^{nR_{1}}]\), where it is assumed throughout that \(2^{nR_{0}}\) and \(2^{nR_{1}}\) are integers, a sequence of \(n\) encoding functions \(f_{1} : [1 : 2^{nR_{0}}] \times [1 : 2^{nR_{1}}] \times S^{i} \rightarrow \mathcal{X}, i \in [1 : n]\), and two decoding functions, \(g_{1} : \mathcal{Y}_{1}^{n} \rightarrow [1 : 2^{nR_{0}}] \times [1 : 2^{nR_{1}}]\) and \(g_{2} : \mathcal{Y}_{2}^{n} \rightarrow [1 : 2^{nR_{1}}]\).

At time \(i \in [1 : n]\), given a pair of messages \((m_{0},m_{1}) \in [1 : 2^{nR_{0}}] \times [1 : 2^{nR_{1}}]\) and a sequence \(s^{i}\), the encoder transmits \(x_{i} = f_{i}(m_{0},m_{1},s^{i})\). The codeword is then given by

\[
x^{n} = f^{n}(m_{0},m_{1},s^{n}) \triangleq (f_{1}(m_{0},m_{1},s_{1}),\ldots,f_{n}(m_{0},m_{1},s_{n})) .
\]

(1)

Decoder 1 receives the channel output \(y_{1}^{n}\), and finds an estimate for the message pair \((\hat{m}_{0},\hat{m}_{1}) = g_{1}(y_{1}^{n})\). Decoder 2 only estimates the common message with \(\hat{m}_{0} = g_{2}(y_{2}^{n})\). We denote the code by \(\mathcal{C} = (f^{n}(\cdot,\cdot,\cdot),g_{1}(\cdot),g_{2}(\cdot))\).

Define the conditional probability of error of \(\mathcal{C}\) given a state sequence \(s^{n} \in \mathcal{S}^{n}\) by

\[
P_{e}^{(n)}(\mathcal{C}) = \frac{1}{2^{n(R_{0}+R_{1})}} \sum_{m_{0}=1}^{2^{nR_{0}}} \sum_{m_{1}=1}^{2^{nR_{1}}} \sum_{s^{n}} W_{Y_{1}^{n},Y_{2}^{n}|X^{n},S^{n}}(y_{1}^{n},y_{2}^{n}|f^{n}(m_{0},m_{1},s^{n}),s^{n}) ,
\]

(2)

where \(D(m_{0},m_{1}) \triangleq \{(y_{1}^{n},y_{2}^{n}) \in \mathcal{Y}_{1}^{n} \times \mathcal{Y}_{2}^{n} : g_{1}(y_{1}^{n}) = (m_{0},m_{1}), g_{2}(y_{2}^{n}) = m_{0}\}\).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{SI} & \text{Random Parameter} & \text{Compound} & \text{AVBC} \\
\hline
\text{without SI} & - & B_{0}^{Q} & B_{0} \\
\text{causal SI} & B^{q} & B^{Q} & B \\
\hline
\end{array}
\]

Fig. 1. Notation of broadcast channel families. The columns correspond to the channel family, and the rows indicate the role of SI at the encoder.
Now, define the average probability of error of $\mathcal{C}$ for some distribution $q(s^n) \in \mathcal{P}(S^n)$,

$$P_e^{(n)}(q, \mathcal{C}) = \sum_{s^n \in S^n} q(s^n) \cdot P_e^{(n)}(q, \mathcal{C}).$$

We say that $\mathcal{C}$ is a $(2^{nR_0}, 2^{nR_1}, n, \varepsilon)$ code for the AVBC $\mathcal{B}$ if it further satisfies

$$P_e^{(n)}(q, \mathcal{C}) \leq \varepsilon, \quad \text{for all } q(s^n) \in \mathcal{P}(S^n).$$

We say that a rate pair $(R_0, R_1)$ is achievable if for every $\varepsilon > 0$ and sufficiently large $n$, there exists a $(2^{nR_0}, 2^{nR_1}, n, \varepsilon)$ code. The operational capacity region is defined as the closure of the set of achievable rate pairs and it is denoted by $\mathcal{C}(\mathcal{B})$. We use the term ‘capacity region’ referring to this operational meaning, and in some places we call it the deterministic code capacity region in order to emphasize that achievability is measured with respect to deterministic codes.

We proceed now to define the parallel quantities when using stochastic-encoder stochastic-decoders triplets with common randomness. The codes formed by these triplets are referred to as random codes.

**Definition 2** (Random code). A $(2^{nR_0}, 2^{nR_1}, n)$ random code for the AVBC $\mathcal{B}$ consists of a collection of $(2^{nR_0}, 2^{nR_1}, n)$ codes $\{\mathcal{C}_\gamma = (f_\gamma^n, g_1, g_2, \gamma)\}_{\gamma \in \Gamma}$, along with a probability distribution $\mu(\gamma)$ over the code collection $\Gamma$. We denote such a code by $\mathcal{C} = (\mu, \Gamma, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma})$.

Analogously to the deterministic case, a $(2^{nR_0}, 2^{nR_1}, n, \varepsilon)$ random code has the additional requirement

$$P_e^{(n)}(q, \mathcal{C}) \leq \varepsilon, \quad \text{for all } q(s^n) \in \mathcal{P}(S^n).$$

The capacity region achieved by random codes is denoted by $\mathcal{C}^*(\mathcal{B})$, and it is referred to as the random code capacity region.

Next, we write the definition of superposition coding [4] using Shannon strategies [16]. See also [17], and the discussion after Theorem 4 therein. Here, we refer to such codes as Shannon strategy codes.

**Definition 3** (Shannon strategy codes). A $(2^{nR_0}, 2^{nR_1}, n)$ Shannon strategy code for the AVBC $\mathcal{B}$ with degraded message sets with causal SI is a $(2^{nR_0}, 2^{nR_1}, n)$ code with an encoder that is composed of two strategy sequences

$$u_0^n : [1 : 2^{nR_0}] \rightarrow U_0^n,$$

$$u_1^n : [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \rightarrow U_1^n,$$

and an encoding function $\xi(u_0^n, u_1^n, s)$, where $\xi : U_0 \times U_1 \times S \rightarrow X$, as well as a pair of decoding functions $g_1 : Y^n \rightarrow [1 : 2^{nR_0}]$ and $g_2 : Y^n \rightarrow [1 : 2^{nR_1}]$. The codeword is then given by

$$x^n = \xi(u_0^n(m_0), u_1^n(m_0, m_1), s^n) \triangleq [\xi(u_0^n(m_0), u_1^n(m_0, m_1), s_i)]_{i=1}^n.$$

We denote the code by $\mathcal{C} = (u_0^n, u_1^n, \xi, g_1, g_2)$.

**D. In the Absence of Side Information – Inner Bound**

In this subsection, we briefly review known results for the case where the state is not known to the encoder or the decoder, i.e. SI is not available.

Consider a given AVBC with degraded message sets without SI, which we denote by $\mathcal{B}_0$. Let

$$R_{0, in}^* \triangleq \bigcup_{p(x,u)} \bigcap_{q(s)} \left\{ (R_0, R_1) : \begin{array}{ll} R_0 & \leq I_q(U; Y_2), \\
R_1 & \leq I_q(X; Y_1|U), \\
R_0 + R_1 & \leq I_q(X; S) \end{array} \right\}.$$  

In [13, Theorem 2], Jahn introduced an inner bound for the arbitrarily varying general broadcast channel. In our case, with degraded message sets, Jahn’s inner bound reduces to the following.

**Theorem 1** (Jahn’s Inner Bound [13]). Let $\mathcal{B}_0$ be an AVBC with degraded message sets without SI. Then, $R_{0, in}^*$ is an achievable rate region using random codes over $\mathcal{B}_0$, i.e.

$$\mathcal{C}^*(\mathcal{B}_0) \supseteq R_{0, in}^*.$$  

Now we move to the deterministic code capacity region.

**Theorem 2** (Ahlswede’s Dichotomy [13]). The capacity region of an AVBC $\mathcal{B}_0$ with degraded message sets without SI either coincides with the random code capacity region or else, its interior is empty. That is, $\mathcal{C}(\mathcal{B}_0) = \mathcal{C}^*(\mathcal{B}_0)$ or else, int($\mathcal{C}(\mathcal{B}_0)$) = $\emptyset$.

By Theorem 1 and Theorem 2, we have that $R_{0, in}^*$ is an achievable rate region, if the interior of the capacity region is non-empty. That is, $\mathcal{C}(\mathcal{B}_0) \supseteq R_{0, in}^*$, if int($\mathcal{C}(\mathcal{B}_0)$) $\neq \emptyset$.

**Theorem 3** (see [10, 8, 12]). For an AVBC $\mathcal{B}_0$ without SI, the interior of the capacity region is non-empty, i.e. int($\mathcal{C}(\mathcal{B}_0)$) $\neq \emptyset$, if and only if the marginals $W_{Y_1|X,S}$ and $W_{Y_2|X,S}$ are not symmetrizable.
II. MAIN RESULTS

We present our results on the compound broadcast channel and the AVBC with degraded message sets with causal SI.

A. The Compound Broadcast Channel with Causal SI

We now consider the case where the encoder has access to the state sequence in a causal manner, i.e. the encoder has $S^t$.

1) Inner Bound: First, we provide an achievable rate region for the compound broadcast channel with degraded message sets with causal SI. Consider a given compound broadcast channel $B^Q$ with causal SI. Let

$$R_{in}(B^Q) \triangleq \bigcup_{p(u_0, u_1), \xi(u_0, u_1, s) \in Q} \bigcap_{q(s) \in Q} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq I_q(U_0; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_0), \\ R_0 + R_1 \leq I_q(U_0, U_1; Y_1) \end{array} \right\}$$

subject to $X = \xi(U_0, U_1, S)$, where $U_0$ and $U_1$ are auxiliary random variables, independent of $S$, and the union is over the pmf $p(u_0, u_1)$ and the set of all functions $\xi : U_0 \times U_1 \times S \to \mathcal{X}$. This can also be expressed as

$$R_{in}(B^Q) = \bigcup_{p(u_0, u_1), \xi(u_0, u_1, s)} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq \inf_{q \in Q} I_q(U_0; Y_2), \\ R_1 \leq \inf_{q \in Q} I_q(U_1; Y_1|U_0), \\ R_0 + R_1 \leq \inf_{q \in Q} I_q(U_0, U_1; Y_1) \end{array} \right\}.$$ 

Lemma 4. Let $B^Q$ be a compound broadcast channel with degraded message sets with causal SI available at the encoder. Then, $R_{in}(B^Q)$ is an achievable rate region for $B^Q$, i.e.

$$\mathcal{C}(B^Q) \supseteq R_{in}(B^Q).$$

Specifically, if $(R_0, R_1) \in R_{in}(B^Q)$, then for some $a > 0$ and sufficiently large $n$, there exists a $(2^{nR_0}, 2^{nR_1}, n, e^{-an})$ Shannon strategy code over the compound broadcast channel $B^Q$ with degraded message sets with causal SI.

The proof of Lemma 4 is given in Appendix A.

2) The Capacity Region: We determine the capacity region of the compound broadcast channel $B^Q$ with causal SI available at the encoder. In addition, we give a condition, for which the inner bound in Lemma 4 coincides with the capacity region. Let

$$R_{out}(B^Q) \triangleq \bigcap_{q(s) \in Q} \bigcup_{p(u_0, u_1, s) \in D} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq I_q(U_0; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_0), \\ R_0 + R_1 \leq I_q(U_0, U_1; Y_1) \end{array} \right\}.$$ 

Now, our condition is defined in terms of the following.

Definition 4. We say that a function $\xi : U_0 \times U_1 \times S \to \mathcal{X}$ and a set $D \subseteq \mathcal{P}(U_0 \times U_1)$ achieve both $R_{in}(B^Q)$ and $R_{out}(B^Q)$ if

$$R_{in}(B^Q) = \bigcup_{p(u_0, u_1) \in D} \bigcap_{q(s) \in Q} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq I_q(U_0; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_0), \\ R_0 + R_1 \leq I_q(U_0, U_1; Y_1) \end{array} \right\},$$

and

$$R_{out}(B^Q) = \bigcap_{q(s) \in Q} \bigcup_{p(u_0, u_1) \in D} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq I_q(U_0; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_0), \\ R_0 + R_1 \leq I_q(U_0, U_1; Y_1) \end{array} \right\},$$

subject to $X = \xi(U_0, U_1, S)$. That is, the unions in (12) and (15) can be restricted to the particular function $\xi(u_0, u_1, s)$ and set of strategy distributions $D$.

Observe that by Definition 4, given a function $\xi(u_0, u_1, s)$, if a set $D$ achieves both $R_{in}(B^Q)$ and $R_{out}(B^Q)$, then every set $D'$ with $D \subseteq D' \subseteq \mathcal{P}(U_0 \times U_1)$ achieves those regions, and in particular, $D' = \mathcal{P}(U_0 \times U_1)$. Nevertheless, the condition defined below requires a certain property that may hold for $D$, but not for $D'$.

Definition 5. Given a convex set $Q$ of state distributions, define Condition $\mathcal{Q}$ by the following: for some $\xi(u_0, u_1, s)$ and $D$ that achieve both $R_{in}(B^Q)$ and $R_{out}(B^Q)$, there exists $q^* \in Q$ which minimizes the mutual informations $I_q(U_0; Y_2)$, $I_q(U_1; Y_1|U_0)$, and $I_q(U_0, U_1; Y_1)$, for all $(u_0, u_1) \in D$, i.e.

$$q^* = \arg\min_{q \in Q} I_q(U_0; Y_2) = \arg\min_{q \in Q} I_q(U_1; Y_1|U_0) = \arg\min_{q \in Q} I_q(U_0, U_1; Y_1),$$

and $p(u_0, u_1) \in D$.  

$$\forall p(u_0, u_1) \in D.$$ 

(17)
Intuitively, when Condition \( T \) holds, there exists a single jamming strategy \( q^*(s) \) which is worst for both users simultaneously. That is, there is no tradeoff for the jammer. As the optimal jamming strategy is unique, this eliminates ambiguity for the users as well.

**Theorem 5.** Let \( B^Q \) be a compound broadcast channel with causal SI available at the encoder. Then,

1) the capacity region of \( B^Q \) follows

\[
C(B^Q) = R_{\text{out}}(B^Q), \quad \text{if } \text{int}(C(B^Q)) \neq \emptyset,
\]

and it is identical to the corresponding random code capacity region, i.e. \( C^*(B^Q) = C(B^Q) \) if \( \text{int}(C(B^Q)) \neq \emptyset \).

2) Suppose that \( Q \subseteq P(S) \) is a convex set of state distributions. If Condition \( T \) holds, the capacity region of \( B^Q \) is given by

\[
C(B^Q) = R_{\text{in}}(B^Q) = R_{\text{out}}(B^Q),
\]

and it is identical to the corresponding random code capacity region, i.e. \( C^*(B^Q) = C(B^Q) \).

The proof of Theorem 5 is given in Appendix B. Regarding part 1, we note that when \( \text{int}(C(B^Q)) = \emptyset \), then the inner bound \( R_{\text{in}}(B^Q) \) has an empty interior as well (see (13)). Thus, \( \text{int}(R_{\text{in}}(B^Q)) \neq \emptyset \) is also a sufficient condition for \( C(B^Q) = R_{\text{out}}(B^Q) \).

3) The Random Parameter Broadcast Channel with Causal SI: Consider the random parameter broadcast channel with causal SI. Recall that this is simply a special case of a compound broadcast channel, where the set of state distributions consists of a single member, i.e. \( Q = \{ q(s) \} \). Then, let

\[
C(B^Q) \triangleq \bigcup_{p(w_0,u_1), \xi(w_0,u_1,s)} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq I_q(U_0; Y_2), \\
R_1 \leq I_q(U_1; Y_1 | U_0), \\
R_0 + R_1 \leq I_q(U_0, U_1; Y_1)
\end{array} \right\},
\]

with

\[
|U_0| \leq |X||S| + 2, \quad |U_1| \leq |X||S|(|X||S| + 2).
\]

**Theorem 6.** The capacity region of the random parameter broadcast channel \( B^Q \) with degraded message sets with causal SI is given by

\[
C(B^Q) = C(B^Q).
\]

Theorem 6 is proved in Appendix C.

**B. The AVBC with Causal SI**

We give inner and outer bounds, on the random code capacity region and the deterministic code capacity region, for the AVBC \( B \) with degraded message sets with causal SI. We also provide conditions, for which the inner bound coincides with the outer bound.

1) **Random Code Inner and Outer Bounds:** Define

\[
R_{\text{in}}^*(B) \triangleq \left. R_{\text{in}}(B^Q) \right|_{Q=P(S)}, \quad R_{\text{out}}^*(B) \triangleq \left. R_{\text{out}}(B^Q) \right|_{Q=P(S)}.
\]

and

\[
T = T^Q \bigg|_{Q=P(S)}.
\]

**Theorem 7.** Let \( B \) be an AVBC with degraded message sets with causal SI available at the encoder. Then,

1) the random code capacity region of \( B \) is bounded by

\[
R_{\text{in}}^*(B) \subseteq C^*(B) \subseteq R_{\text{out}}^*(B).
\]

2) If Condition \( T \) holds, the random code capacity region of \( B \) is given by

\[
C^*(B) = R_{\text{in}}^*(B) = R_{\text{out}}^*(B).
\]

The proof of Theorem 7 is given in Appendix D.

Before we proceed to the deterministic code capacity region, we need one further result. The following lemma is a restatement of a result from [2], stating that a polynomial size of the code collection \( \{ \mathcal{C}_\gamma \} \) is sufficient. This result is a key observation in Ahlswede’s Elimination Technique (ET), presented in [2], and it is significant for the deterministic code analysis.

**Lemma 8.** Consider a given \( (2^nR_0, 2^nR_1, n, \varepsilon_n) \) random code \( \mathcal{C}^T = (\mu, \Gamma, \{ \mathcal{C}_\gamma \}_{\gamma \in \Gamma}) \) for the AVBC \( B \), where \( \lim_{n \to \infty} \varepsilon_n = 0 \). Then, for every \( 0 < \alpha < 1 \) and sufficiently large \( n \), there exists a \( (2^nR_0, 2^nR_1, n, \alpha) \) random code \( (\mu^*, \Gamma^*, \{ \mathcal{C}_\gamma \}_{\gamma \in \Gamma^*}) \) with the following properties:
1) The size of the code collection is bounded by $|\Gamma^*| \leq n^2$.
2) The code collection is a subset of the original code collection, i.e., $\Gamma^* \subseteq \Gamma$.
3) The distribution $\mu^*$ is uniform, i.e., $\mu^*(\gamma) = \frac{1}{\Gamma^*}$, for $\gamma \in \Gamma^*$.

The proof of Lemma 8 follows the same lines as in [2, Section 4] (see also [13, 19]). For completeness, we give the proof in Appendix E.

2) Deterministic Code Inner and Outer Bounds: The next theorem characterizes the deterministic code capacity region, which demonstrates a dichotomy property.

**Theorem 9.** The capacity region of an AVBC $B$ with degraded message sets with causal SI either coincides with the random code capacity region or else, it has an empty interior. That is, $C(B) = C^*(B)$ or else, int$(C(B)) = \emptyset$.

The proof of Theorem 9 is given in Appendix F. Let $U = (U_0, U_1)$, hence $U = U_0 \times U_1$. For every pair of functions $\xi : U \times S \rightarrow X$ and $\xi' : U_0 \times S \rightarrow X$, define the DMCs $V_{Y_1[U,S]}^\xi$ and $V_{Y_2[U_0,S]}^{\xi'}$ specified by

$$
V_{Y_1[U,S]}^\xi(y_1|u,s) = W_{Y_1|X,S}(y_1|x,s), \quad (27a)
$$

$$
V_{Y_2[U_0,S]}^{\xi'}(y_2|u_0,s) = W_{Y_2|X,S}(y_2|x',s), \quad (27b)
$$

respectively.

**Corollary 10.** The capacity region of $B$ is bounded by

$$
C(B) \supseteq R^*_in(B), \text{ if } \text{int}(C(B)) \neq \emptyset, \quad (28)
$$

$$
C(B) \subseteq R^*_out(B). \quad (29)
$$

Furthermore, if $V_{Y_1[U,S]}^\xi$ and $V_{Y_2[U_0,S]}^{\xi'}$ are non-symmetrizable for some $\xi : U \times S \rightarrow X$ and $\xi' : U_0 \times S \rightarrow X$, and Condition $\mathcal{F}$ holds, then $C(B) = R^*_in(B) = R^*_out(B)$.

The proof of Corollary 10 is given in Appendix G.

### III. Degraded Broadcast Channel with Causal SI

In this section, we consider the special case of an arbitrarily varying degraded broadcast channel (AVDBC) with causal SI, when user 1 and user 2 have private messages.

#### A. Definitions

We consider a degraded broadcast channel (DBC), which is a special case of the general broadcast channel described in the previous sections. Following the definitions by [17], a state-dependent broadcast channel $W_{Y_1,Y_2|X,S}$ is said to be physically degraded if it can be expressed as

$$
W_{Y_1,Y_2|X,S}(y_1,y_2|x,s) = W_{Y_1|X,S}(y_1|x,s) \cdot \beta(y_2|y_1), \quad (30)
$$

i.e. $(X,S)\leftrightarrow Y_1 \leftrightarrow Y_2$ form a Markov chain. User 1 is then referred to as the stronger user, whereas user 2 is referred to as the weaker user. More generally, a broadcast channel is said to be stochastically degraded if $W_{Y_2|X,S}(y_2|x,s) = \sum_{y_1 \in Y_1} W_{Y_1|X,S}(y_1|x,s) \cdot \beta(y_2|y_1)$ for some conditional distribution $\beta(y_2|y_1)$. We note that the definition of degradedness here is stricter than the definition in [13, Remark 11B5]. Our results apply to both the physically degraded and the stochastically degraded broadcast channels. Thus, for our purposes, there is no need to distinguish between the two, and we simply say that the broadcast channel is degraded. We use the notation $\mathcal{B}_D$ for an AVDBC with causal SI.

We consider the case where the users have private messages. A deterministic code and a random code for the AVDBC $\mathcal{B}_D$ with causal SI are then defined as follows.

**Definition 6** (A private-message code, an achievable rate pair and capacity region). A $(2^{nR_1},2^{nR_2},n)$ private-message code for the AVDBC $\mathcal{B}_D$ with causal SI consists of the following; two message sets $[1 : 2^{nR_1}]$ and $[1 : 2^{nR_2}]$, where it is assumed throughout that $2^{nR_1}$ and $2^{nR_2}$ are integers, a set of $n$ encoding functions $f_i : [1 : 2^{nR_1}] \times [1 : 2^{nR_2}] \times S_i \rightarrow X$, $i \in [1 : n]$, and two decoding functions, $g_1 : \mathcal{Y}_1^n \rightarrow [1 : 2^{nR_1}]$ and $g_2 : \mathcal{Y}_2^n \rightarrow [1 : 2^{nR_2}]$.

At time $i \in [1 : n]$, given a pair of messages $m_1 \in [1 : 2^{nR_1}]$ and $m_2 \in [1 : 2^{nR_2}]$ and a sequence $s^i$, the encoder transmits $x_i = f_i(m_1, m_2, s^i)$. The codeword is then given by

$$
x^n = f^n(m_1, m_2, s^n) \triangleq (f_1(m_1, m_2, s_1), f_2(m_1, m_2, s_2), \ldots, f_n(m_1, m_2, s^n)). \quad (31)
$$

Decoder $k$ receives the channel output $y^n_k$, for $k = 1, 2$, and finds an estimate for the $k^{th}$ message, $\hat{m}_k = g_k(y^n_k)$. Denote the code by $C = (f^n(\cdot, \cdot, \cdot), g_1(\cdot), g_2(\cdot))$. 

Define the conditional probability of error of $\mathcal{C}$ given a state sequence $s^n \in \mathcal{S}^n$ by

$$P_{e|s^n}(\mathcal{C}) = \frac{1}{2^{n(R_1 + R_2)}} \sum_{m_1=1}^{2^{nR_1}} \sum_{m_2=1}^{2^{nR_2}} \sum_{D(m_1,m_2)} W_{Y_1^n,Y_2^n|X^n,S^n}(y_1^n,y_2^n|f^n(m_1,m_2,s^n),s^n),$$

(32)

where

$$D(m_1,m_2) \triangleq \{ (y_1^n,y_2^n) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n : g_1(y_1^n) = m_1, g_2(y_2^n) = m_2 \}.$$  

(33)

We say that $\mathcal{C}$ is a $(2^{nR_1},2^{nR_2},n,\varepsilon)$ code for the AVDBC $\mathcal{B}$ if it further satisfies

$$P_{e|s^n}(\mathcal{C},\Gamma) = \sum_{q(s^n) : P_{e|s^n}(\mathcal{C}) \leq \varepsilon, \text{ for all } q(s^n) \in \mathcal{P}(S^n).} (34)$$

An achievable private-message rate pair $(R_1,R_2)$ and the capacity region $\mathbb{C}(B_D)$ are defined as usual.

We proceed now to define the parallel quantities when using stochastic-encoder stochastic-decoders triplets with common randomness.

**Definition 7 (Random code).** A $(2^{nR_1},2^{nR_2},n)$ private-message random code for the AVDBC $\mathcal{B}_D$ consists of a collection of $(2^{nR_1},2^{nR_2},n)$ codes $\{\mathcal{C}_\gamma = (f^n_\gamma,g_1,\gamma,g_2,\gamma)\}_{\gamma \in \Gamma}$, along with a probability distribution $\mu(\gamma)$ over the code collection $\Gamma$.

Analogously to the deterministic case, a $(2^{nR_1},2^{nR_2},n,\varepsilon)$ random code has the additional requirement

$$P_{e}(q,\mathcal{C},\Gamma) = \sum_{\gamma \in \Gamma} \mu(\gamma) P_{e}(q,\mathcal{C}_\gamma) \leq \varepsilon, \text{ for all } q(s^n) \in \mathcal{P}(S^n).$$

(35)

The private-message capacity region achieved by random codes is denoted by $\mathbb{C}^*(B_D)$, and it is referred to as the random code capacity region.

By standard arguments, a private-message rate pair $(R_1,R_2)$ is achievable for the AVDBC $\mathcal{B}_D$ if and only if $(R_0,R_1)$ is achievable with degraded message sets, with $R_0 = R_2$. This immediately implies the following results.

**B. Results**

The results in this section are a straightforward consequence of the results in Section II.

1) Random Code Inner and Outer Bounds: Define

$$R^*_n(B_D) \triangleq \bigcup_{p(u_1,u_2),\xi(u_1,u_2,s)} \bigcap_{q(s)} \{ (R_1,R_2) : R_2 \leq I(q(U_2;Y_2), R_1 \leq I(q(U_1;Y_1|U_2) \big},$$

(36)

and

$$R^*_n(B_D) \triangleq \bigcap_{q(s)} \bigcup_{p(u_0,u_1),\xi(u_0,u_1,s)} \{ (R_1,R_2) : R_2 \leq I(q(U_2,Y_2), R_1 \leq I(q(U_1,Y_1|U_2) \big \}.$$  

(37)

Now, we define a condition in terms of the following.

**Definition 8.** We say that a function $\xi : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{S} \rightarrow \mathcal{X}$ and a set $\mathcal{D}^* \subseteq \mathcal{P}(\mathcal{U}_1 \times \mathcal{U}_2)$ achieve both $R^*_n(B_D)$ and $R^*_n(B_D)$ if

$$R^*_n(B_D) = \bigcup_{p(u_0,u_1) \in \mathcal{D}^*} \bigcap_{q(s)} \{ (R_1,R_2) : R_2 \leq I(q(U_2;Y_2), R_1 \leq I(q(U_1;Y_1|U_2) \big \},$$

(38a)

and

$$R^*_n(B_D) = \bigcap_{q(s)} \bigcup_{p(u_0,u_1) \in \mathcal{D}^*} \{ (R_1,R_2) : R_2 \leq I(q(U_2;Y_2), R_1 \leq I(q(U_1;Y_1|U_2) \big \},$$

(38b)

subject to $X = \xi(U_1,U_2,S)$. That is, the unions in (36) and (37) can be restricted to the particular function $\xi(u_1,u_2,s)$ and set of strategy distributions $\mathcal{D}^*$.

**Definition 9.** Define Condition $\mathcal{F}_D$ by the following: for some $\xi(u_1,u_2,s)$ and $\mathcal{D}^*$ that achieve both $R^*_n(B_D)$ and $R^*_n(B_D)$, there exists $q^* \in \mathcal{P}(S)$ which minimizes both $I_q(U_2;Y_2)$ and $I_q(U_1;Y_1|U_2)$, for all $p(u_1,u_2) \in \mathcal{D}^*$, i.e.

$$\mathcal{F}_D : \text{For some } q^* \in \mathcal{P}(S), \quad q^* = \arg \min_{q(s)} I_q(U_2;Y_2) = \arg \min_{q(s)} I_q(U_1;Y_1|U_2) \quad \forall p(u_1,u_2) \in \mathcal{D}^*.$$  

(39)

**Theorem 11.** Let $\mathcal{B}_D$ be an AVDBC with causal SI available at the encoder. Then,
1) the random code capacity region of $B_D$ is bounded by
\[ R^*_\text{in}(B_D) \subseteq C^*(B_D) \subseteq R^*_\text{out}(B_D). \] (39)

2) If Condition $\mathcal{T}_D$ holds, the random code capacity region of $B_D$ is given by
\[ C^*(B_D) = R^*_\text{in}(B_D) = R^*_\text{out}(B_D). \] (40)

Theorem 11 is a straightforward consequence of Theorem 7.

2) Deterministic Code Inner and Outer Bounds: The next theorem characterizes the deterministic code capacity region, which demonstrates a dichotomy property.

**Theorem 12.** The capacity region of an AVDBC $B_D$ with causal SI either coincides with the random code capacity region or else, it has an empty interior. That is,
\[ C(B_D) = C^*(B_D) \text{ or else, } \text{int}(C(B_D)) = \emptyset. \] (41)

Theorem 12 is a straightforward consequence of Theorem 9. Now, Theorem 11 and Theorem 12 yield the following corollary.

**Corollary 13.** The capacity region of $B_D$ is bounded by
\[ C(B_D) \supseteq R^*_\text{in}(B_D), \text{ if } \text{int}(C(B_D)) \neq \emptyset, \] (42)
\[ C(B_D) \subseteq R^*_\text{out}(B_D). \] (43)

Furthermore, if $V^{\xi'}_{Y_2|U_2,S}$ is non-symmetrizable for some $\xi' : U_2 \times S \to X$, and Condition $\mathcal{T}_D$ holds, then $C(B_D) = R^*_\text{in}(B_D) = R^*_\text{out}(B_D)$.

**IV. EXAMPLES**

To illustrate the results above, we give the following examples. In the first example, we consider an AVDBC and determine the private-message capacity region. Then, in the second example, we consider a non-degraded AVBC and determine the capacity region with degraded message sets.

**Example 1.** Consider an arbitrarily varying binary symmetric broadcast channel (BSBC),
\[ Y_1 = X + Z_S \mod 2, \]
\[ Y_2 = Y_1 + K \mod 2, \]
where $X, Y_1, Y_2, S, Z_S, K$ are binary, with values in $\{0, 1\}$. The additive noises are distributed according to

$$Z_s \sim \text{Bernoulli}(\theta_s), \text{ for } s \in \{0, 1\},$$

$$K \sim \text{Bernoulli}(\alpha),$$

with $\theta_0 \leq 1 - \theta_1 \leq \frac{1}{2}$ and $\alpha < \frac{1}{2}$, where $K$ is independent of $(S, Z_S)$. It is readily seen the channel is physically degraded. Then, consider the case where user 1 and user 2 have private messages.

We have the following results. Define the binary entropy function $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$, for $x \in [0, 1]$, with logarithm to base 2. The private-message capacity region of the arbitrarily varying BSBC $B_{D, 0}$ without SI is given by

$$C(B_{D, 0}) = \{(0, 0)\}.$$  

The private-message capacity region of the arbitrarily varying BSBC $B_D$ with causal SI is given by

$$C(B_D) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq 1 - h(\alpha * \beta * \theta_1), \\
R_1 \leq h(\beta * \theta_1) - h(\theta_1)
\end{array} \right\}.$$  

(45)

It will be seen in the achievability proof that the parameter $\beta$ is related to the distribution of $U_1$, and thus the RHS of (45) can be thought of as a union over Shannon strategies. The analysis is given in Appendix H.

It is shown in Appendix H that Condition $\mathcal{F}_D$ holds and $C(B_D) = R^*_D(B_D) = R^*_{SI}(B_D)$. Figure 2 provides a graphical interpretation. Consider a DBC $W_{Y_1, Y_2 | X, S}$ with random parameters with causal SI, governed by an i.i.d. state sequence, distributed according to $S \sim \text{Bernoulli}(q)$, for a given $0 \leq q \leq 1$, and let $C(B_D^q)$ denote the corresponding capacity region. Then, the analysis shows that Condition $\mathcal{F}_D$ implies that there exists $0 \leq q^* \leq 1$ such that $C(B_D) = C(B_D^{q^*})$, where $C(B_D^{q^*}) \subseteq C(B_D^q)$ for every $0 \leq q \leq 1$. Indeed, looking at Figure 2, it appears that the regions $C(B_D^q)$, for $0 \leq q \leq 1$, form a well ordered set, hence $C(B_D) = C(B_D^{q^*})$ with $q^* = 1$.

Next, we consider an example of an AVBC which is not degraded in the sense defined above.

**Example 2.** Consider a state-dependent binary symmetric broadcast channel (BSBC) with correlated noises,

$$Y_1 = X + Z_S \mod 2,$$

$$Y_2 = X + N_S \mod 2,$$

where $X, Y_1, Y_2, S, Z_S, N_S$ are binary, with values in $\{0, 1\}$. The additive noises are distributed according to

$$Z_s \sim \text{Bernoulli}(\theta_s), \text{ for } s \in \{0, 1\},$$

$$N_s \sim \text{Bernoulli}(\varepsilon_s), \text{ for } s \in \{0, 1\},$$

where $S, Z_0, Z_1, N_0, N_1$ are independent random variables, with $\theta_0 \leq \varepsilon_0 \leq \frac{1}{2}$ and $\frac{1}{2} \leq \varepsilon_1 \leq \theta_1$.

Intuitively, this suggests that $Y_2$ is a weaker channel. Nevertheless, observe that this channel is not degraded in the sense defined in Section III-A (see (30)). For a given state $S = s$, the broadcast channel $W_{Y_1, Y_2 | X, S}((\cdot, \cdot), s)$ is stochastically degraded. In particular, one can define the following random variables,

$$A_s \sim \text{Bernoulli}(\pi_s), \text{ where } \pi_s = \frac{\varepsilon_s - \theta_s}{1 - 2\theta_s},$$

$$\tilde{Y}_2 = Y_1 + A_S \mod 2.$$  

(47)

Then, $\tilde{Y}_2$ is distributed according to $\Pr(\tilde{Y}_2 = y_2 | X = x, S = s) = W_{Y_2 | X, S}(y_2 | x, s)$, and $X \leftrightarrow (Y_1, S) \leftrightarrow \tilde{Y}_2$ form a Markov chain. However, since $X$ and $A_S$ depend on the state, it is not necessarily true that $(X, S) \leftrightarrow Y_1 \leftrightarrow \tilde{Y}_2$ form a Markov chain, and the BSBC with correlated noises could be non-degraded.

We have the following results.

**Random Parameter BSBC with Correlated Noises**

First, we consider the random parameter BSBC $B^q$, with a memoryless state $S \sim \text{Bernoulli}(q)$, for a given $0 \leq q \leq 1$. Define the binary entropy function $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$, for $x \in [0, 1]$, with logarithm to base 2. We show that the capacity region of the random parameter BSBC $B^q$ with degraded message sets with causal SI is given by

$$C(B^q) = C(B^q) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq 1 - h(\beta * \delta_q^{(2)}), \\
R_1 \leq h(\beta * \delta_q^{(1)}) - h(\delta_q^{(1)})
\end{array} \right\},$$  

(48)

where

$$\delta_q^{(1)} = (1 - q)\theta_0 + q(1 - \theta_1),$$

$$\delta_q^{(2)} = (1 - q)\varepsilon_0 + q(1 - \varepsilon_1).$$  

(49)
provides a graphical interpretation. The analysis shows that Condition (48) holds and the capacity region of the AVBC is given by $C(B) = R_{\text{out}}(B) = C(B^q)$ for $q = 1$ (see (52)).

The proof is given in Appendix I-A. It can be seen in the achievability proof that the parameter $\beta$ is related to the distribution of $U_1$, and thus the RHS of (48) can be thought of as a union over Shannon strategies.

Arbitrarily Varying BSBC with Correlated Noises

We move to the arbitrarily varying BSBC with correlated noises. As shown in Appendix I-B, the capacity region of the arbitrarily varying BSBC $B_0$ with degraded message sets without SI is given by $C(B_0) = \{(0,0)\}$. For the setting where causal SI is available at the encoder, we consider two cases.

**Case 1:** Suppose that $\theta_0 \leq 1 - \theta_1 \leq \epsilon_0 \leq 1 - \varepsilon_1 \leq \frac{1}{2}$. That is, $S = 1$ is a noisier channel state than $S = 0$, for both users. The capacity region of the arbitrarily varying BSBC $B$ with degraded message sets with causal SI is given by

$$C(B) = C(B^q) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq 1 - h(\beta \ast \varepsilon_1) , \\ R_1 \leq h(\beta \ast \theta_1) - h(\theta_1) \end{array} \right\}.$$

(50)

It is shown in Appendix I-B that Condition $\mathcal{T}$ holds and $C(B) = R_{\text{in}}^*(B) = R_{\text{out}}^*(B)$. Figure 3 provides a graphical interpretation. The analysis shows that Condition $\mathcal{T}$ implies that there exists $0 \leq q^* \leq 1$ such that $C(B) = C(B^{q^*})$, where $C(B^{q^*}) \subseteq C(B^0)$ for every $0 \leq q \leq 1$. Indeed, looking at Figure 3, it appears that the regions $C(B^q)$, for $0 \leq q \leq 1$, form a well ordered set, hence $C(B) = C(B^{q^*})$ with $q^* = 1$.

**Case 2:** Suppose that $\theta_0 \leq 1 - \theta_1 \leq 1 - \varepsilon_1 \leq \epsilon_0 \leq \frac{1}{2}$. That is, $S = 1$ is a noisier channel state for user 1, whereas $S = 0$ is noisier for user 2. The capacity region of the arbitrarily varying BSBC $B$ with degraded message sets with causal SI is bounded by

$$C(B) \supseteq R_{\text{in}}^*(B) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq 1 - h(\beta \ast \varepsilon_0) , \\ R_1 \leq h(\beta \ast \theta_1) - h(\theta_1) \end{array} \right\},$$

(51)

and

$$C(B) \subseteq R_{\text{out}}^*(B) \subseteq C(B^{q=0}) \cap C(B^{q=1})$$

$$= \bigcup_{0 \leq \beta_0 \leq 1 \atop 0 \leq \beta_1 \leq 1} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq 1 - h(\beta_0 \ast \varepsilon_0) , \\ R_0 \leq 1 - h(\beta_1 \ast \varepsilon_1) , \\ R_1 \leq h(\beta_0 \ast \theta_0) - h(\theta_0) , \\ R_1 \leq h(\beta_1 \ast \theta_1) - h(\theta_1) \end{array} \right\}.$$

(52)
Fig. 4. The inner and outer bounds on the capacity region of the AVBC in Example 2, the arbitrarily varying binary symmetric broadcast channel with correlated noises, with parameters that correspond to case 2, namely, $\theta_0 = 0.12$, $\theta_1 = 0.85$, $\varepsilon_0 = 0.22$ and $\varepsilon_1 = 0.88$.

(a) The dashed and dotted lines depict the boundaries of $C(B^q=0)$ and $C(B^q=1)$, respectively. The colored lines depict $C(B^q)$ for a range of values of $0 < q < 1$.

(b) The area under the thick blue line is the inner bound $R^*_i(B)$, and the area under the thin line is the outer bound $R^*_o(B)$.

The analysis is given in Appendix I. Figure 4 provides a graphical interpretation. The dashed and dotted lines in Figure 4(a) depict the boundaries of $C(B^q=0)$ and $C(B^q=1)$, respectively. The colored lines depict $C(B^q)$ for a range of values of $0 < q < 1$. It appears that $R^*_o(B) = \cap_{0 \leq q \leq 1} C(B^q)$ reduces to the intersection of the regions $C(B^q=0)$ and $C(B^q=1)$. Figure 4(b) demonstrates the gap between the bounds in case 2.

APPENDIX A

PROOF OF LEMMA 4

We show that every rate pair $(R_0, R_1) \in R_{in}(B^Q)$ can be achieved using deterministic codes over the compound broadcast channel $B^Q$ with causal SI. We construct a code based on superposition coding with Shannon strategies, and decode using joint typicality with respect to a channel state type, which is “close” to some $q \in Q$.

We use the following notation. Basic method of types concepts are defined as in [7, Chapter 2]; including the definition of a type $P_{x^n}$ of a sequence $x^n$; a joint type $P_{x^n,y^n}$ and a conditional type $P_{x^n|y^n}$ of a pair of sequences $(x^n, y^n)$; and a $\delta$-typical
set $\mathcal{A}^i(P_{X,Y})$ with respect to a distribution $P_{X,Y}(x,y)$. Define a set $\hat{Q}_n$ of state types

$$\hat{Q}_n = \left\{ \hat{P}_s^n : s^n \in \mathcal{A}^i(q), \text{ for some } q \in Q \right\},$$

where

$$\delta_1 \triangleq \frac{\delta}{2 \cdot |S|},$$

where $\delta > 0$ is arbitrarily small. That is, $\hat{Q}_n$ is the set of types that are $\delta_1$-close to some state distribution $q(s)$ in $Q$. Note that for any fixed $\delta$ (or $\delta_1$), for a sufficiently large $n$, the set $\hat{Q}_n$ covers the set $Q$, and it is in fact a $\delta_1$-blowup of $Q$. Now, a code for the compound broadcast channel with causal SI is constructed as follows.

**Codebook Generation:** Fix the distribution $\hat{P}_{U_0,U_1}$ and the function $\xi(u_0,u_1,s)$. Generate $2^{nR_0}$ independent sequences at random,

$$u^n_0(m_0) \sim \prod_{i=1}^{n} P_{U_0}(u_0,i), \text{ for } m_0 \in [1 : 2^{nR_0}].$$

For every $m_0 \in [1 : 2^{nR_0}]$, generate $2^{nR_1}$ sequences at random,

$$u^n_1(m_0,m_1) \sim \prod_{i=1}^{n} P_{U_1|U_0}(u_1,i|u_0,(m_0)), \text{ for } m_1 \in [1 : 2^{nR_1}],$$

conditionally independent given $u^n_0(m_0)$.

**Encoding:** To send a pair of messages $(m_0,m_1) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}]$, transmit at time $i \in [1 : n]$,

$$x_i = \xi(u_0,i(m_0),u_1,i(m_0,m_1),s_i).$$

**Decoding:** Let

$$P^q_{U_0,U_1,Y_1,Y_2}(u_0,u_1,y_1,y_2) = \sum_{s \in S} q(s)P_{U_0,U_1}(u_0,u_1)W_{Y_1,Y_2|X,s}(y_1,y_2|\xi(u_0,u_1,s),s).$$

Observing $y^n_2$, decoder 2 finds a unique $\hat{m}_0 \in [1 : 2^{nR_0}]$ such that

$$(u^n_0(\hat{m}_0),y^n_2) \in \mathcal{A}^i(\hat{P}_{U_0}P^q_{Y_2|U_0}), \text{ for some } q \in \hat{Q}_n.$$  

If there is none, or more than one such $\hat{m}_0 \in [1 : 2^{nR_0}]$, then decoder 2 declares an error. Observing $y^n_1$, decoder 1 finds a unique pair of messages $(\hat{m}_0,\hat{m}_1) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}]$ such that

$$(u^n_0(\hat{m}_0),u^n_1(\hat{m}_0,\hat{m}_1),y^n_1) \in \mathcal{A}^i(\hat{P}_{U_0,U_1}P^q_{Y_1,Y_2|U_0}), \text{ for some } q \in \hat{Q}_n.$$  

If there is none, or more than such pair $(\hat{m}_0,\hat{m}_1)$, then decoder 1 declares an error. We note that using the set of types $\hat{Q}_n$ instead of the original set of state distributions $Q$ alleviates the analysis, since $Q$ is not necessarily finite nor countable.

**Analysis of Probability of Error:** Assume without loss of generality that the users sent the message pair $(M_0,M_1) = (1,1)$. Let $q(s) \in Q$ denote the actual state distribution chosen by the jammer. By the union of events bound,

$$P_c^{(n)}(q,\mathcal{E}) \leq \Pr(\hat{M}_0 \neq 1) + \Pr(\hat{M}_0,\hat{M}_1) \neq (1,1),$$

where the conditioning on $(M_0,M_1) = (1,1)$ is omitted for convenience of notation. The error event for decoder 2 is the union of the following events.

$$\mathcal{E}_{2,1} = \{(U^n_0(1),Y^n_2) \notin \mathcal{A}^i(\hat{P}_{U_0}P^q_{Y_2|U_0}) \text{ for all } q' \in \hat{Q}_n\},$$

$$\mathcal{E}_{2,2} = \{(U^n_0(m_0),Y^n) \notin \mathcal{A}^i(\hat{P}_{U_0}P^{q''}_{Y_1,Y_2|U_0}) \text{ for some } m_0 \neq 1, q' \in \hat{Q}_n\}.$$

Then, by the union of events bound,

$$\Pr(\hat{M}_2 \neq 1) \leq \Pr(\mathcal{E}_{2,1}) + \Pr(\mathcal{E}_{2,2}).$$

Considering the first term, we claim that the event $\mathcal{E}_{2,1}$ implies that $(U^n_0(1),Y^n_2) \notin \mathcal{A}^i(\hat{P}_{U_0}P^q_{Y_2|U_0})$ for all $q' \in Q$. Assume to the contrary that $\mathcal{E}_{2,1}$ holds, but there exists $q'' \in Q$ such that $(U^n_0(1),Y^n_2') \in \mathcal{A}^i(\hat{P}_{U_0}P^{q''}_{Y_2|U_0})$. Then, for a sufficiently large $n$, there exists a type $q'(s)$ such that $|q'(s) - q''(s)| \leq \delta_1$ for all $s \in S$. It can then be inferred that $q' \in \hat{Q}_n$ (see (53)), and

$$|P^{q''}_{Y_2|U_0}(y_2|u_0) - P^{q''}_{Y_2|U_0}(y_2|u_0)| \leq |S| \cdot \delta_1 = \frac{\delta}{2},$$

(65)
for all \( u_0 \in \mathcal{U}_0 \) and \( y_2 \in \mathcal{Y}_2 \) (see (54) and (58)). Hence, \((U_0^n(1), Y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}|U_0^n)\), which contradicts the first assumption. Thus,

\[
\Pr(\mathcal{E}_{2,1}) \leq \Pr \left( (U_0^n(1), Y_2^n) \notin \mathcal{A}^{\nu'2}(P_{U_0} P_{Y_2}^{\nu'}|U_0^n) \text{ for all } \nu'' \in \mathcal{Q} \right)
\leq \Pr \left( (U_0^n(1), Y_2^n) \notin \mathcal{A}^{\nu'2}(P_{U_0} P_{Y_2}^{\nu'}|U_0^n) \right).
\]

(66)

The last expression tends to zero exponentially as \( n \to \infty \) by the law of large numbers and Chernoff’s bound.

Moving to the second term in the RHS of (64), we use the classic method of type considerations to bound \( \Pr(\mathcal{E}_{2,2}) \). By the union of events bound and the fact that the number of type classes in \( S^n \) is bounded by \((n + 1)|S|\), we have that

\[
\Pr(\mathcal{E}_{2,2}) 
\leq (n + 1)|S| \cdot \sup_{\nu' \in \mathcal{Q}_n} \Pr \left( (U_0^n(m_0), Y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}) \text{ for some } m_0 \neq 1 \right).
\]

(67)

For every \( m_0 \neq 1 \),

\[
\Pr \left( (U_0^n(m_0), Y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}|U_0^n) \right) = \sum_{u_0^n \in \mathcal{U}_0^n} P_{U_0^n}(u_0^n) \cdot \Pr \left( (u_0^n, Y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}|U_0^n) \right)
= \sum_{u_0^n \in \mathcal{U}_0^n} P_{U_0^n}(u_0^n) \cdot \sum_{y_2^n : (u_0^n, y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}|U_0^n)} P_{Y_2}^{\nu'}(y_2^n),
\]

(68)

where the last equality holds since \( U_0^n(m_0) \) is independent of \( Y_2^n \) for every \( m_0 \neq 1 \). Let \((u_0^n, y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}|U_0^n)\). Then, \( y_2^n \in \mathcal{A}^\delta(P_{Y_2}^{\nu'}) \) with \( \delta_2 \triangleq |U_0^n| \cdot \delta \). By Lemmas 2.6 and 2.7 in [7],

\[
P_{Y_2}^{\nu'}(y_2^n) = 2^{-nH(P_{Y_2}^\nu + D(P_{Y_2}^\nu||P_{Y_2}^{\nu'}))} \leq 2^{-nH(P_{Y_2}^\nu)} \leq 2^{-n(H(\nu')(Y_2) - \varepsilon_1(\delta))},
\]

(69)

where \( \varepsilon_1(\delta) \to 0 \) as \( \delta \to 0 \). Therefore, by (67)–(69),

\[
\Pr(\mathcal{E}_{2,2}) 
\leq (n + 1)|S| 
\cdot \sup_{\nu' \in \mathcal{Q}_n} \left[ 2^{-nR_0} \cdot \sum_{u_0^n \in \mathcal{U}_0^n} P_{U_0^n}(u_0^n) \cdot \left| \{y_2^n : (u_0^n, y_2^n) \in \mathcal{A}^\delta(P_{U_0} P_{Y_2}^{\nu'}|U_0^n)\} \right| \cdot 2^{-nH(\nu'(Y_2) - \varepsilon_1(\delta))} \right]
\leq (n + 1)|S| \cdot \sup_{\nu' \in \mathcal{Q}_n} 2^{-n[I_{\nu'}(U_0; Y_2) - R_0 - \varepsilon_2(\delta)]},
\]

(70)

with \( \varepsilon_2(\delta) \to 0 \) as \( \delta \to 0 \), where the last inequality is due to [7, Lemma 2.13]. The RHS of (70) tends to zero exponentially as \( n \to \infty \), provided that \( R_0 < \inf_{\nu' \in \mathcal{Q}} I_{\nu'}(U_0; Y_2) - \varepsilon_2(\delta) \).

Now, consider the error event of decoder 1. For every \((m_0, m_1) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}]\), define the event

\[
\mathcal{E}_{1,1}(m_0, m_1) = \{(U_0^n(m_0), U_1^n(m_0, m_1), Y_2^n) \in \mathcal{A}^\delta(P_{U_0, U_1} P_{Y_2}^{\nu'}|U_0, U_1, Y_2) \}, \text{ for some } \nu' \in \mathcal{Q}_n\}.
\]

(71)

Then, the error event is bounded by

\[
\left\{ (\hat{M}_0, \hat{M}_1) \neq (1, 1) \right\} \subseteq \bigcup_{m_1 \neq 1} \mathcal{E}_{1,1}(m_1, 1) \cup \bigcup_{m_1 \in [1 : 2^{nR_1}], m_0 \neq 1} \mathcal{E}_{1,1}(m_0, m_1).
\]

(72)

Thus, by the union of events bound,

\[
\Pr \left( (\hat{M}_0, \hat{M}_1) \neq (1, 1) \right) 
\leq \Pr(\mathcal{E}_{1,1}(1, 1)^c) + \sum_{m_1 \in [1 : 2^{nR_1}], m_0 \neq 1} \Pr(\mathcal{E}_{1,1}(m_0, m_1)) + \sum_{m_1 \neq 1} \Pr(\mathcal{E}_{1,1}(m_1, 1))
\]

\[
\leq 2^{-\theta n} + 2^{-n \inf_{\nu' \in \mathcal{Q}} I_{\nu'}(U_0, U_1; Y_2) - R_0 - R_1 - \varepsilon_3(\delta)} + \sum_{m_1 \neq 1} \Pr(\mathcal{E}_{1,1}(m_1, 1)),
\]

(73)

where the last inequality follows from the law of large numbers and type class considerations used before, with \( \varepsilon_3(\delta) \to 0 \) as \( \delta \to 0 \). The middle term in the RHS of (73) exponentially tends to zero as \( n \to \infty \) provided that \( R_0 + R_1 <
\[
\inf_{q' \in Q} I_{q'}(U_0, U_1; Y_1) - \varepsilon_3(\delta). \]
It remains for us to bound the last sum. Using similar type class considerations, we have that for every \( q' \in \hat{Q}_n \) and \( m_1 \neq 1 \),
\[
\Pr \left( \left( U^n_0(1), U^n_1(m_1, 1), Y^n_1 \right) \in A^\delta(P_{U_0, U_1} P_{Y_1|U_0, U_1}) \right) = \sum_{(u^n_0, u^n_1, y^n_1) \in A^\delta(P_{U_0, U_1} P_{Y_1|U_0, U_1})} P_{U_0^n} (u^n_0) \cdot P_{U_1^n|U_0^n} (u^n_1|u^n_0) \cdot P_{Y_1^n|U_0^n} (y^n_1|u^n_0)
\leq 2^n (H_{q'}(U_0, U_1, Y_1) + \varepsilon_4(\delta)) 
= 2^{-n} (H_{q'}(U_0, U_1, Y_1) - \varepsilon_4(\delta)),
\]
where \( \varepsilon_4(\delta) \to 0 \) as \( \delta \to 0 \). Therefore, the sum term in the RHS of (73) is bounded by
\[
\sum_{m_1 \neq 1} \Pr (E_{1,1}(m_1, 1)) = \sum_{m_1 \neq 1} \Pr \left( \left( U^n_0(1), U^n_1(m_1, 1), Y^n_1 \right) \in A^\delta(P_{U_0, U_1} P_{Y_1|U_0, U_1}) \right) \leq (n + 1)|S| \cdot 2^{-n} \left( \inf_{q' \in Q} I_{q'}(U_1; Y_1|U_0) - R_1 - \varepsilon_5(\delta) \right),
\]
where the last line follows from (74), and \( \varepsilon_5(\delta) \to 0 \) as \( \delta \to 0 \). The last expression tends to zero exponentially as \( n \to \infty \) and \( \delta \to 0 \) provided that \( R_1 < \inf_{q' \in Q} I_{q'}(U_1; Y_1|U_0) - \varepsilon_5(\delta) \).

The probability of error, averaged over the class of the codebooks, exponentially decays to zero as \( n \to \infty \). Therefore, there must exist a \((2^{nR_0}, 2^{nR_1}, n, e^{-an})\) deterministic code, for a sufficiently large \( n \).

\[\square\]

**APPENDIX B**

**PROOF OF THEOREM 5**

**PART 1**

At the first part of the theorem it is assumed that the interior of the capacity region is non-empty, i.e. \( \text{int}(\mathcal{C}(\mathcal{B}^Q)) \neq \emptyset \).

**Achievability proof.** We show that every rate pair \((R_0, R_1) \in \mathcal{R}_{\text{out}}(\mathcal{B}^Q)\) can be achieved using a code based on Shannon strategies with the addition of a codeword suffix. At time \( i = n + 1 \), having completed the transmission of the messages, the type of the state sequence \( s^n \) is known to the encoder. Following the assumption that the interior of the capacity region is non-empty, the type of \( s^n \) can be reliably communicated to both receivers as a suffix, while the blocklength is increased by \( \nu > 0 \) additional channel uses, where \( \nu \) is small compared to \( n \). The receivers first estimate the type of \( s^n \), and then use joint typicality with respect to the estimated type. The details are provided below.

Following the assumption that \( \text{int}(\mathcal{C}(\mathcal{B}^Q)) \neq \emptyset \), we have that for every \( \varepsilon_1 > 0 \) and sufficiently large blocklength \( \nu \), there exists a \((2^{nR_0}, 2^{nR_1}, \nu, \varepsilon_1)\) code \( \mathcal{C} = (f^\nu, g_1, g_2) \) for the transmission of a type \( \hat{P}_{s^n} \) at positive rates \( \tilde{R}_0 > 0 \) and \( \tilde{R}_1 > 0 \). Since the total number of types is polynomial in \( n \) (see [7]), the type \( \hat{P}_{s^n} \) can be transmitted at a negligible rate, with a blocklength that grows a lot slower than \( n \), i.e.
\[
\nu = o(n).
\]

We now construct a code \( \mathcal{C} \) over the compound broadcast channel with causal SI, such that the blocklength is \( n + o(n) \), and the rate \( \tilde{R}_{s^n} \) approaches \( R \) as \( n \to \infty \).

**Codebook Generation:** Fix the distribution \( P_{U_0, U_1} \) and the function \( \xi(u_0, u_1, s) \). Generate \( 2^{nR_0} \) independent sequences \( u^n_0(m_0), m_0 \in [1 : 2^{nR_0}] \), at random, each according to \( P_{U_0}|u_0=0,1|^n \). For every \( m_0 \in [1 : 2^{nR_0}] \), generate \( 2^{nR_1} \) sequences at random,
\[
u_1^0(m_0, m_1) \sim \prod_{i=1}^n P_{U_1|u_0=0,1,i|^n}(u_1|u_0=0,1,m_0), \text{ for } m_1 \in [1 : 2^{nR_1}],
\]
conditionally independent given \( u^n_0(m_0) \). Reveal the codebook of the message pair \( (m_0, m_1) \) and the codebook of the type \( \hat{P}_{s^n} \) to the encoder and the decoders.

**Encoding:** To send a message pair \( (m_0, m_1) \in [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \), transmit at time \( i \in [1 : n] \),
\[
x_i = \xi(u_{0,i}(m_0), u_{1,i}(m_0, m_1), s_i).
\]

At time \( i \in [n + 1 : n + \nu] \), knowing the sequence of previous states \( s^n \), transmit
\[
x_i = \tilde{f}_i(\hat{P}_{s^n}, s_{n+1}, \ldots, s_{n+i}),
\]
where $\tilde{P}_s$ is the type of the sequence $(s_1, \ldots, s_n)$. That is, the encoded type $\tilde{P}_s$ is transmitted as a suffix of the codeword. We note that the type of the sequence $(s_{n+1}, \ldots, s_{n+i})$ is not necessarily $\tilde{P}_s$, and it is irrelevant for that matter, since the assumption that $\int(C(B^0)) \neq \emptyset$ implies that there exists a $(2^{R_0}, 2^{R_1}, \nu, \varepsilon_1)$ code $\tilde{C} = (\tilde{f}, \tilde{g}_1, \tilde{g}_2)$ for the transmission of $\tilde{P}_s$, with $R_0 > 0$ and $R_1 > 0$.

**Decoding:** Let

$$P^q_{U_0, U_1, Y_1, Y_2}(u_0, u_1, y_1, y_2) = \sum_{s \in S} q(s) P_{U_0, U_1}(u_0, u_1) W_{Y_1, Y_2|X,S}(y_1, y_2|\xi(u_0, u_1, s), s).$$

(80)

Decoder 2 receives the output sequence $y_{2}^{n+\nu}$. As a pre-decoding step, the receiver decodes the last $\nu$ output symbols, and finds an estimate of the type of the state sequence, $\hat{q}_2 = \tilde{g}_2(y_{2,n+1}, \ldots, y_{2,n+\nu})$. Then, given the output sequence $y_{2}^{n}$, decoder 2 finds a unique $\tilde{m}_0 \in [1 : 2^{R_0}]$ such that

$$(u_0^n(\tilde{m}_0), y_2^n) \in A^\delta(P_{U_0}P_{Y_2|U_0}).$$

(81)

If there is none, or more than one such $\tilde{m}_0 \in [1 : 2^{R_0}]$, then decoder 2 declares an error.

Similarly, decoder 1 receives $y_{1}^{n+\nu}$ and begins with decoding the type of the state sequence, $\hat{q}_1 = \tilde{g}_1(y_{1,n+1}, \ldots, y_{1,n+\nu})$. Then, decoder 1 finds a unique pair of messages $(\tilde{m}_0, \tilde{m}_1) \in [1 : 2^{R_0}] \times [1 : 2^{R_1}]$ such that

$$(u_0^n(\tilde{m}_0), u_1^n(\tilde{m}_0, \tilde{m}_1), y_1^n) \in A^\delta(P_{U_0, U_1, Y_1|U_0}).$$

(82)

If there is none, or more than one such pair $(\tilde{m}_0, \tilde{m}_1) \in [1 : 2^{R_0}] \times [1 : 2^{R_1}]$, then decoder 1 declares an error.

**Analysis of Probability of Error:** By symmetry, we may assume without loss of generality that the users sent $(M_0, M_1) = (1, 1)$. Let $q(s) \in Q$ denote the actual state distribution chosen by the jammer, and let $q(s^n) = \prod_{i=1}^{n} q(s_i)$. Then, by the union of events bound, the probability of error is bounded by

$$P_e^{(n)}(q, \tilde{C}) \leq \Pr \left( \tilde{M}_0 \neq 1 \right) + \Pr \left( (\tilde{M}_0, \tilde{M}_1) \neq (1, 1) \right),$$

(83)

where the conditioning on $(M_0, M_1) = (1, 1)$ is omitted for convenience of notation.

Define the events

$$\mathcal{E}_{1.0} = \{ \hat{q}_1 \neq \tilde{P}_s \}$$

(84)

$$\mathcal{E}_{1.1}(m_0, m_1, q') = \{ (U_0^n(m_0), U_1^n(m_0, m_1), Y_1^n) \in A^\delta(P_{U_0, U_1, Y_1|U_0}) \}$$

(85)

and

$$\mathcal{E}_{2.0} = \{ \hat{q}_2 \neq \tilde{P}_s \}$$

(86)

$$\mathcal{E}_{2.1}(m_0, q') = \{ (U_0^n(m_0), Y_2^n) \in A^\delta(P_{U_0, Y_2|U_0}) \},$$

(87)

for every $m_0 \in [1 : 2^{R_0}]$, $m_1 \in [1 : 2^{R_1}]$, and $q' \in \mathcal{P}(S)$. The error event of decoder 2 is bounded by

$$\{ \tilde{M}_2 \neq 1 \} \subseteq \mathcal{E}_{2.0} \cup \mathcal{E}_{2.1}(1, \hat{q}_2) \cup \bigcup_{m_0 \neq 1} \mathcal{E}_{2.1}(m_0, \hat{q}_2) \cup \left( \bigcup_{m_0 \neq 1} \mathcal{E}_e^{c}_{2.0} \cap \mathcal{E}_{2.1}(m_0, \hat{q}_2) \right).$$

By the union of events bound,

$$\Pr \left( \tilde{M}_2 \neq 1 \right) \leq \Pr (\mathcal{E}_{2.0}) + \Pr (\mathcal{E}_e^{c}_{2.0} \cap \mathcal{E}_{2.1}(1, \hat{q}_2)) + \Pr \left( \bigcup_{m_0 \neq 1} \mathcal{E}_e^{c}_{2.0} \cap \mathcal{E}_{2.1}(m_0, \hat{q}_2) \right).$$

(88)

Since the code $\tilde{C}$ for the transmission of the type is a $(2^{R_0}, 2^{R_1}, \nu, \varepsilon_1)$ code, where $\varepsilon_1 > 0$ is arbitrarily small, we have that the probability of erroneous decoding of the type is bounded by

$$\Pr (\mathcal{E}_{1.0} \cup \mathcal{E}_{2.0}) \leq \varepsilon_1.$$  

(89)
Thus, the first term in the RHS of (88) is bounded by $\varepsilon_1$. Then, we manipulate the last two terms as follows.

$$
\Pr \left( \tilde{M}_2 \neq 1 \right) \leq \sum_{s^n \in \mathcal{A}^{\hat{s}_2}(q)} q(s^n) \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{q}_2)^c \mid S^n = s^n \right) + \sum_{s^n \notin \mathcal{A}^{\hat{s}_2}(q)} q(s^n) \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{q}_2)^c \mid S^n = s^n \right)
$$

where

$$
\delta_2 \triangleq \frac{1}{2|\mathcal{S}|} \cdot \delta.
$$

Next we show that the first and the third sums in (90) tend to zero as $n \to \infty$.

Consider a given $s^n \in \mathcal{A}^{\hat{s}_2}(q)$. For notational convenience, denote

$$
q'' = \hat{P}_s^n.
$$

Then, by the definition of the $\delta$-typical set, we have that $|q''(s) - q(s)| \leq \delta_2$ for all $s \in \mathcal{S}$. It follows that

$$
|P_{U_0}(u_0)P_{Y_2|U_0}(y_2|u_0) - P_{U_0}(u_0)P_{Y_2|U_0}(y_2|u_0)| \leq \delta_2 \cdot \sum_{s,u_1} P_{U_1|U_0}(u_1|u_0)W_{Y_2|X,S}(y_2|x,u_1,s) \leq \delta_2 \cdot |\mathcal{S}| = \frac{\delta}{2},
$$

for all $u_0 \in U_0$ and $y_2 \in Y_2$, where the last equality follows from (91).

Consider the first sum in the RHS of (90). Given a state sequence $s^n \in \mathcal{A}^{\hat{s}_2}(q)$, we have that

$$
\Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{q}_2)^c \mid S^n = s^n \right) = \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{P}_s^n)^c \mid S^n = s^n \right)
$$

where the first equality follows from (86), and the second equality follows from (92). Then,

$$
\Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{q}_2)^c \mid S^n = s^n \right) \leq \Pr \left( \mathcal{E}_{2,1}(1, q'')^c \mid S^n = s^n \right) = \Pr \left( \mathcal{E}_{2,0}^c \mid \mathcal{E}_{2,1}(1, q'')^c, S^n = s^n \right) \cdot \Pr \left( \mathcal{E}_{2,1}(1, q'')^c \mid S^n = s^n \right),
$$

Now, suppose that $(U_0^n(1), Y_2^n) \in \mathcal{A}^{\delta/2}(P_{U_0}P_{Y_2|U_0}^q)$, where $q$ is the actual state distribution. By (93), in this case we have that $(U_0^n(1), Y_2^n) \in \mathcal{A}^{\delta}(P_{U_0}P_{Y_2|U_0})$. Hence, (95) implies that

$$
\Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{q}_2)^c \mid S^n = s^n \right) \leq \Pr \left( \left( U_0^n(1), Y_2^n \right) \notin \mathcal{A}^{\delta/2}(P_{U_0}P_{Y_2|U_0}) \mid S^n = s^n \right).
$$

The first sum in the RHS of (90) is then bounded as follows.

$$
\sum_{s^n \in \mathcal{A}^{\hat{s}_2}(q)} q(s^n) \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \hat{q}_2)^c \mid S^n = s^n \right) \leq \sum_{s^n \in \mathcal{A}^{\hat{s}_2}(q)} q(s^n) \Pr \left( \left( U_0^n(1), Y_2^n \right) \notin \mathcal{A}^{\delta/2}(P_{U_0}P_{Y_2|U_0}) \mid S^n = s^n \right)
$$

for a sufficiently large $n$, where the last inequality follows from the law of large numbers.
We bound the third sum in the RHS of (90) using similar arguments. If \((U_0^n(m_0), Y_2^n) \in \mathcal{A}(P_{U_0} P_{Y_2^n | U_0}), \) then \((U_0^n(m_0), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_0} P_{Y_2^n | U_0}), \) due to (93). Thus, for every \(s^n \in \mathcal{A}^{\delta_2}(q), \)

\[
\Pr \left( \bigcup_{m_0 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_0, \hat{q}_2) \mid S^n = s^n \right) \leq \sum_{m_0 \neq 1} \Pr(\mathcal{E}_{2,1}(m_0, q''') \mid S^n = s^n) \\
= \sum_{m_0 \neq 1} \Pr \left( (U_0^n(m_0), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_0} P_{Y_2^n | U_0}) \mid S^n = s^n \right) \\
\leq \sum_{m_0 \neq 1} \Pr \left( (U_0^n(m_0), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_0} P_{Y_2^n | U_0}) \mid S^n = s^n \right).
\]  

(98)

This, in turn, implies that the third sum in the RHS of (90) is bounded by

\[
\sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q(s^n) \Pr \left( \bigcup_{m_0 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_0, \hat{q}) \mid S^n = s^n \right) \\
\leq \sum_{s^n \in \mathcal{S}} \sum_{m_0 \neq 1} q(s^n) \cdot \Pr \left( (U_0^n(m_0), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_0} P_{Y_2^n | U_0}) \mid S^n = s^n \right) \\
eq \sum_{m_0 \neq 1} \Pr \left( (U_0^n(m_0), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_0} P_{Y_2^n | U_0}) \right) \\
\leq 2^{-n[I_q(U_0; Y_2) - R_0 - \varepsilon_2(\delta)]},
\]

(99)

with \(\varepsilon_2(\delta) \to 0\) as \(\delta \to 0.\) The last inequality follows from standard type class considerations. The RHS of (99) tends to zero as \(n \to \infty,\) provided that \(R_0 < I_q(U_0; Y_2) - \varepsilon_2(\delta).\) Then, it follows from the law of large numbers that the second and fourth sums in the RHS of (90) tend to zero as \(n \to \infty.\) Thus, by (97) and (99), we have that the probability of error of decoder 2, \(\Pr(\hat{M}_2 \neq 1),\) tends to zero as \(n \to \infty.\)

Now, consider the error event of decoder 1,

\[
\{(M_0, \hat{M}_1) \neq (1, 1)\} \subseteq \mathcal{E}_{1,0} \cup \mathcal{E}_{1,1}(1, 1, \hat{q}_1)^c \cup \bigcup_{m_0 \neq 1, m_1 \in [1:2^n R_1]} \mathcal{E}_{1,1}(m_0, m_1, \hat{q}_1) \cup \bigcup_{m_1 \neq 1} \mathcal{E}_{1,1}(1, m_1, \hat{q}_1).
\]  

(100)

Thus, by the union of events bound,

\[
\Pr \left( (M_0, \hat{M}_1) \neq (1, 1) \right) \leq \Pr(\mathcal{E}_{1,0}) + \Pr(\mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(1, 1, \hat{q}_1)^c) + \Pr \left( \bigcup_{m_0 \neq 1, m_1 \in [1:2^n R_1]} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(m_0, m_1, \hat{q}_1) \right) \\
+ \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(1, m_1, \hat{q}_1) \right).
\]

(101)

By (89), the first term is bounded by \(\varepsilon_1,\) and as done above, we write

\[
\Pr \left( (M_0, \hat{M}_1) \neq (1, 1) \right) \leq \sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q(s^n) \Pr \left( \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(1, 1, \hat{P}_{s^n})^c \mid S^n = s^n \right) \\
+ \sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q(s^n) \Pr \left( \bigcup_{m_1 \in [1:2^n R_1]} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(m_1, 1, \hat{P}_{s^n}) \mid S^n = s^n \right) \\
+ \sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q(s^n) \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(1, m_1, \hat{P}_{s^n}) \mid S^n = s^n \right) \\
+ 3 \cdot \Pr \left( S^n \notin \mathcal{A}^{\delta_2}(q) \right) + \varepsilon_1,
\]

(102)

where \(\delta_2\) is given by (91). By the law of large numbers, the probability \(\Pr \left( S^n \notin \mathcal{A}^{\delta_2}(q) \right)\) tends to zero as \(n \to \infty.\) As for the sums, we use similar arguments to those used above.
We have that for a given \( s^n \in A^{\Delta_2}(q) \),

\[
|P_{U_0,U_1}(u_0,u_1)P_{Y_1|U_0,U_1}(y_1|u_0,u_1) - P_{U_0,U_1}(u_0,u_1)P_{Y_1|U_0,U_1}(y_1|u_0,u_1)| \\
\leq \delta_2 \cdot \sum_{s \in S} W_{Y_1|X,S}(y_1) \mathbb{E}(u_0,u_1,s) \leq |S| \cdot \delta_2 = \frac{\delta}{2},
\]

(103)

with \( q'' = \hat{P}_s \), where the last equality follows from (91).

The first sum in the RHS of (102) is bounded by

\[
\sum_{s^n \in A^{\Delta_2}(q)} q(s^n) \Pr \left( \bigcap_{m_1 \in [1:2nR_1]_{m \neq 1}} \mathcal{E}_{1,0}^{m_1} | S^n = s^n \right) \\
\leq \sum_{s^n \in S^n} q(s^n) \Pr \left( (U_0^n(1),U_1^n(1,1),Y_1^n) \notin A^{\Delta_2}/(P_{U_0,U_1}P_{Y_1|U_0,U_1}) | S^n = s^n \right) \\
= \Pr \left( (U_0^n(1),U_1^n(1,1),Y_1^n) \notin A^{\Delta_2}/(P_{U_0,U_1}P_{Y_1|U_0,U_1}) \right) \leq \varepsilon_2.
\]

(104)

The last inequality follows from the law of large numbers, for a sufficiently large \( n \).

The second sum in the RHS of (102) is bounded by

\[
\sum_{s^n \in A^{\Delta_2}(q)} q(s^n) \Pr \left( \bigcup_{m_1 \in [1:2nR_1]_{m \neq 1}} \mathcal{E}_{1,0}^{m_1} | S^n = s^n \right) \\
\leq 2^{-n(I_q(U_0,U_1;U_1) - R_0 - R_1 - \varepsilon_3(\delta))},
\]

(105)

with \( \varepsilon_3(\delta) \to 0 \) as \( n \to \infty \) and \( \delta \to 0 \). This is obtained following the same analysis as for decoder 2. Then, the second sum tends to zero provided that \( R_0 + R_1 < I_q(U_0,U_1;Y_1) - \varepsilon_3(\delta) \).

The third sum in the RHS of (102) is bounded by

\[
\sum_{s^n \in A^{\Delta_2}(q)} q(s^n) \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{1,0}^{m_1} | S^n = s^n \right) \\
\leq \sum_{s^n \in A^{\Delta_2}(q)} \sum_{m_1 \neq 1} q(s^n) \Pr \left( \mathcal{E}_{1,1}(m_1,1,\hat{P}_s) | S^n = s^n \right).
\]

(106)

For every \( s^n \in A^{\Delta_2}(q) \), it follows from (103) that the event \( \mathcal{E}_{1,1}(m_1,1,\hat{P}_s) \) implies that \( (U_0^n(1),U_1^n(m_1,1),Y_1^n) \in A^{\Delta_2}/(P_{U_0,U_1}P_{Y_1|U_0,U_1}) \). Thus, the sum is bounded by

\[
\sum_{s^n \in A^{\Delta_2}(q)} q(s^n) \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{1,0}^{m_1} | S^n = s^n \right) \leq 2^{-n(I_q(U_0;U_1) - R_1 - \delta_3)},
\]

(107)

where \( \delta_3 \to 0 \) as \( \delta \to 0 \). Then, the RHS of (107) tends to zero as \( n \to \infty \) provided that \( R_1 < I_q(U_1;Y_1|U_0) - \delta_3 \).

We conclude that the RHS of both (90) and (102) tend to zero as \( n \to \infty \). Thus, the overall probability of error, averaged over the class of the codebooks, decays to zero as \( n \to \infty \). Therefore, there must exist a \( (2^{nR_0},2^{nR_1},n,\varepsilon) \) deterministic code, for a sufficiently large \( n \).

\[ \square \]

Converse proof. First, we claim that it can be assumed that \( U_0 \oplus U_1 \oplus X \) form a Markov chain. Define the following region,

\[
R_{M,\text{out}}(B^Q) = \bigcap_{q(s) \in \mathbb{Q}_p(u_0,u_1)} \bigcup_{\xi(u_0,u_1)} \left\{ (R_0,R_1) : \begin{array}{c} R_0 \leq I_q(U_0;Y_2), \\
R_1 \leq I_q(U_1;Y_1|U_0) \\
R_0 + R_1 \leq I_q(U_0,U_1;Y_1) \end{array} \right\},
\]

(108)

subject to \( X = \xi(U_1,S) \). Clearly, \( R_{M,\text{out}}(B^Q) \subseteq R_{\text{out}}(B^Q) \), since \( R_{M,\text{out}}(B^Q) \) is obtained by restriction of the function \( \xi(u_0,u_1,s) \) in the union on the RHS of (15). Moreover, we have that \( R_{M,\text{out}}(B^Q) \supseteq R_{\text{out}}(B^Q) \), since, given some \( U_0, U_1 \) and \( \xi(u_0,u_1,s) \), we can define a new strategy variable \( \hat{U}_1 = (U_1,S) \), and then \( X \) is a deterministic function of \( (U_1,S) \).

As \( R_{\text{out}}(B^Q) = R_{M,\text{out}}(B^Q) \), it can now be assumed that \( U_0 \oplus U_1 \oplus X \) form a Markov chain, hence \( I_q(U_0,U_1;Y_1) = I_q(U_1;Y_1) \). Then, by similar arguments to those used in [14] (see also [7, Chapter 16]), we have that

\[
R_{\text{out}}(B^Q) = \bigcap_{q(s) \in \mathbb{Q}_p(u_0,u_1)} \bigcup_{\xi(u_0,u_1)} \left\{ (R_0,R_1) : \begin{array}{c} R_0 \leq I_q(U_0;Y_2), \\
R_0 + R_1 \leq I_q(U_1;Y_1|U_0) + I_q(U_0;Y_2) \\
R_0 + R_1 \leq I_q(U_1;Y_1) \end{array} \right\}.
\]

(109)

We show that for every sequence of \( (2^{nR_0},2^{nR_1},n,\theta_n) \) codes, with \( \lim_{n \to \infty} \theta_n = 0 \), we have that \((R_0,R_1)\) belongs to the set above.
Define the following random variables,
\[ U_{0,i} \triangleq (M_0, Y_1^{i-1}, Y_{2,i+1}^n), \quad U_{1,i} \triangleq (M_0, M_1, S^{i-1}). \] (110)

It follows that \( X_i \) is a deterministic function of \((U_{0,i}, S_i)\), and since the state sequence is memoryless, we have that \( S_i \) is independent of \((U_{0,i}, U_{1,i})\). Next, by Fano’s inequality,
\[ nR_0 \leq I_q(M_0; Y_2^n) + n\varepsilon_n, \] (111)
\[ n(R_0 + R_1) \leq I_q(M_0, M_1; Y_2^n) + n\varepsilon_n, \] (112)
\[ n(R_0 + R_1) \leq I_q(M_1; Y_1^n|M_0) + I_q(M_0; Y_2^n) + n\varepsilon_n, \] (113)
where \( \varepsilon_n \to 0 \) as \( n \to \infty \). Applying the chain rule, we have that (111) is bounded by
\[ I_q(M_0; Y_2^n) = \sum_{i=1}^{n} I_q(M_0; Y_{2,i}|Y_{2,i+1}^n) \leq \sum_{i=1}^{n} I_q(U_{0,i}; Y_{2,i}), \] (114)
and (112) is bounded by
\[ I_q(M_0, M_1; Y_1^n) = \sum_{i=1}^{n} I_q(M_0, M_1; Y_{1,i}|Y_{1,i+1}^{i-1}) \leq \sum_{i=1}^{n} I_q(U_{0,i}; Y_{1,i}) = \sum_{i=1}^{n} I_q(U_{1,i}; Y_{1,i}), \] (115)
where the last equality holds since \( U_{0,i} \关停 U_{1,i} \关停 Y_{1,i} \) form a Markov chain. As for (113), we have that
\[
I_q(M_1; Y_1^n|M_0) + I_q(M_0, Y_2^n) = \sum_{i=1}^{n} I_q(M_1; Y_{1,i}|M_0, Y_1^{i-1}) + \sum_{i=1}^{n} I_q(M_0; Y_{2,i}|Y_{2,i+1}^n) \\
\leq \sum_{i=1}^{n} I_q(M_1, Y_{2,i+1}; Y_{1,i}|M_0, Y_1^{i-1}) + \sum_{i=1}^{n} I_q(M_0, Y_{2,i+1}; Y_{2,i}) \\
= \sum_{i=1}^{n} I_q(M_1; Y_{1,i}|M_0, Y_1^{i-1}, Y_{2,i+1}^n) + \sum_{i=1}^{n} I_q(Y_{2,i+1}; Y_{1,i}|M_0, Y_1^{i-1}) \\
+ \sum_{i=1}^{n} I_q(M_0, Y_{1,i}^{i-1}, Y_{2,i+1}^n; Y_{2,i}) - \sum_{i=1}^{n} I_q(Y_1^{i-1}; Y_{2,i}|M_0, Y_{2,i+1}^n). \] (116)

Then, the second and fourth sums cancel out, by the Csiszár sum identity [9, Section 2.3]. Hence,
\[ I_q(M_1; Y_1^n|M_0) + I_q(M_0, Y_2^n) \leq \sum_{i=1}^{n} I_q(U_{1,i}; Y_{1,i}|U_{0,i}) + \sum_{i=1}^{n} I_q(U_{0,i}; Y_{2,i}). \] (117)

Thus, by (111)–(113) and (115)–(117), we have that
\[ R_0 \leq \frac{1}{n} \sum_{i=1}^{n} I_q(U_{0,i}; Y_{2,i}) + \varepsilon_n, \] (118)
\[ R_0 + R_1 \leq \frac{1}{n} \sum_{i=1}^{n} I_q(U_{1,i}; Y_{1,i}) + \varepsilon_n, \] (119)
\[ R_0 + R_1 \leq \frac{1}{n} \sum_{i=1}^{n} I_q(U_{1,i}; Y_{1,i}|U_{0,i}) + \sum_{i=1}^{n} I_q(U_{0,i}; Y_{2,i}) + \varepsilon_n. \] (120)

Introducing a time-sharing random variable \( K \), uniformly distributed over \([1 : n]\) and independent of \((S^n, U_0^n, U_1^n)\), we have that
\[ R_0 \leq I_q(U_{0,K}; Y_{2,K}|K) + \varepsilon_n, \] (121)
\[ R_0 + R_1 \leq I_q(U_{1,K}; Y_{1,K}|K) + \varepsilon_n, \] (122)
\[ R_0 + R_1 \leq I_q(U_{1,K}; Y_{1,K}|U_{0,K}, K) + I_q(U_{0,K}; Y_{2,K}|K) + \varepsilon_n. \] (123)

Define \( U_0 \triangleq (U_{0,K}, K) \) and \( U_1 \triangleq (U_{1,K}, K) \). Hence, \( P_{Y_{1,K}, Y_{2,K}|U_0, U_1} = P_{Y_{1,2}|U_0, U_1} \). Then, by (109) and (121)–(123), it follows that \((R_0, R_1) \in R_{out}(B^Q)\).
PART 2

We show that when the set of state distributions $Q$ is convex, and Condition $\mathcal{S}^Q$ holds, the capacity region of the compound broadcast channel $B^Q$ with causal SI is given by $C(B^Q) = C^*(B^Q) = R_{in}(B^Q) = R_{out}(B^Q)$ (and this holds regardless of whether the interior of the capacity region is empty or not).

Due to part 1, we have that

$$C^*(B^Q) \subseteq R_{out}(B^Q).$$

By Lemma 4,

$$C(B^Q) \supseteq R_{in}(B^Q).$$

Thus,

$$R_{in}(B^Q) \subseteq C(B^Q) \subseteq C^*(B^Q) \subseteq R_{out}(B^Q).$$

To conclude the proof, we show that Condition $\mathcal{S}^Q$ implies that $R_{in}(B^Q) \supseteq R_{out}(B^Q)$, hence the inner and outer bounds coincide. By Definition 4, if a function $\xi(u_0, u_1, s)$ and a set $D$ achieve $R_{in}(B^Q)$ and $R_{out}(B^Q)$, then

$$R_{in}(B^Q) = \bigcup_{p(u_0, u_1) \in D} \left\{ (R_0, R_1) : R_0 \leq \min_{q \in Q} I_q(U_0; Y_2), R_1 \leq \min_{q \in Q} I_q(U_1; Y_1|U_0), R_0 + R_1 \leq \min_{q \in Q} I_q(U_0, U_1; Y_1) \right\},$$

and

$$R_{out}(B^Q) = \bigcap_{q(s) \in Q} \bigcup_{p(u_0, u_1) \in D} \left\{ (R_0, R_1) : R_0 \leq I_q(U_0; Y_2), R_1 \leq I_q(U_1; Y_1|U_0), R_0 + R_1 \leq I_q(U_0, U_1; Y_1) \right\}.$$

Hence, when Condition $\mathcal{S}^Q$ holds, we have by Definition 5 that for some $\xi(u_0, u_1, s)$, $D \subseteq \mathcal{P}(U_0 \times U_1)$, and $q^* \in Q$,

$$R_{in}(B^Q) = \bigcup_{p(u_0, u_1) \in D} \left\{ (R_0, R_1) : R_0 \leq I_{q^*}(U_0; Y_2), R_1 \leq I_{q^*}(U_1; Y_1|U_0), R_0 + R_1 \leq I_{q^*}(U_0, U_1; Y_1) \right\},$$

where the last line follows from (127b).

\[\square\]

APPENDIX C

PROOF OF THEOREM 6

At first, ignore the cardinality bounds in (21). Then, it immediately follows from Theorem 5 that $C(B^s) = C(B^t)$, by taking the set $Q$ that consists of a single state distribution $q(s)$.

To prove the bounds on the alphabet sizes of the strategy variables $U_0$ and $U_1$, we apply the standard Carathéodory techniques (see e.g. [7, Lemma 15.4]). Let

$$L_0 \triangleq (|\mathcal{X}| - 1)|S| + 3 \leq |\mathcal{X}||S| + 2,$$

where the inequality holds since $|S| \geq 1$. Without loss of generality, assume that $\mathcal{X} = [1 : |\mathcal{X}|]$ and $S = [1 : |S|]$. Then, define the following $L_0$ functionals,

$$\varphi_j(P_{U_1,X|S}) = \sum_{u_1 \in U_1} P_{U_1,X|S}(u_1, i|j) = P_{X|S}(i|j), \quad i = 1, \ldots, |\mathcal{X}| - 1, j = 1, \ldots, |S|,$$

$$\psi_1(P_{U_1,X|S}) = - \sum_{s, u_1, x, y_1} q(s)P_{U_1,X|S}(u_1, x|s)W_{Y_1|X,S}(y_1|x, s) \log \left[ \sum_{s', u_1', x'} q(s')P_{U_1,X|S}(u_1', x'|s')W_{Y_1|X,S}(y_1|x', s') \right],$$

$$\psi_2(P_{U_1,X|S}) = - \sum_{s, u_1, x, y_2} q(s)P_{U_1,X|S}(u_1, x|s)W_{Y_2|X,S}(y_2|x, s) \log \left[ \sum_{s', u_1', x'} q(s')P_{U_1,X|S}(u_1', x'|s')W_{Y_2|X,S}(y_2|x', s') \right],$$

$$\psi_3(P_{U_1,X|S}) = - \sum_{u_1, x, s} q(s)P_{U_1,X|S}(u_1, x|s) \log \left[ \sum_{x', s'} q(s')P_{U_1,X|S}(u_1, x'|s') \right]$$

$$- \sum_{u_1, x, s} q(s)P_{U_1,X|S}(u_1, x|s)W_{Y_1|X,S}(y_1|x, s) \log \left[ \sum_{u_1', x''', s'''} q(s''')P_{U_1,X|S}(u_1', x'''|s''')W_{Y_1|X,S}(y_1|x'', s''') \right].$$
Then, observe that
\[ \sum_{u_0 \in U_0} p(u_0) \varphi_{i,j}(P_{U_1,X|S,U_0}(\cdot, \cdot, \cdot), u_0)) = P_{X|S}(i|j), \]  
(134)
\[ \sum_{u_0 \in U_0} p(u_0) \psi_1(P_{U_1,X|S,U_0}(\cdot, \cdot, \cdot), u_0)) = H(Y_1|U_0), \]  
(135)
\[ \sum_{u_0 \in U_0} p(u_0) \psi_1(P_{U_1,X|S,U_0}(\cdot, \cdot, \cdot), u_0)) = H(Y_2|U_0), \]  
(136)
\[ \sum_{u_0 \in U_0} p(u_0) \psi_1(P_{U_1,X|S,U_0}(\cdot, \cdot, \cdot), u_0)) = I(U_1; Y_1|U_0). \]  
(137)

By [7, Lemma 15.4], the alphabet size of \( U_0 \) can then be restricted to \( |U_0| \leq L_0 \), while preserving \( P_{X,S,Y_1,Y_2}; I(U_0; Y_2) = H(Y_2) - H(Y_2|U_0); I(U_0; Y_1|U_0) \); and \( I(U_0, U_1; Y_2) = I(U_0; Y_1|U_0) + H(Y_1) - H(Y_1|U_0) \).

Fixing the alphabet of \( U_0 \), we now apply similar arguments to the cardinality of \( \mathcal{U}_1 \). Then, less than \( |X||S|L_0 - 1 \) functionals are required for the joint distribution \( P_{U_0,X|S} \) and an additional functional to preserve \( H(Y_1|U_1, U_0) \). Hence, by [7, Lemma 15.4], the alphabet size of \( U_0 \) can then be restricted to \( |U_1| \leq |X||S|L_0 \leq |X||S|(|X||S| + 2) \) (see (129)).

**APPENDIX D**

**PROOF OF THEOREM 7**

**A. Part 1**

First, we explain the general idea. We devise a causal version of Ahlswede’s Robustification Technique (RT) [1, 19]. Namely, we use codes for the compound broadcast channel to construct a random code for the AVBC using randomized permutations. However, in our case, the causal nature of the problem imposes a difficulty, and the application of the RT is not straightforward.

In [1, 19], the state information is noncausal and a random code is defined via permutations of the codeword symbols. This cannot be done here, because the SI is provided to the encoder in a causal manner. We resolve this difficulty using Shannon strategy codes for the compound broadcast channel to construct a random code for the AVBC, applying permutations to the strategy sequence \((u_1^n, u_0^n)\), which is an integral part of the Shannon strategy code, and is independent of the channel state. The details are given below.

1) **Inner Bound:** We show that the region defined in (23) can be achieved by random codes over the AVBC \( \mathcal{B} \) with causal SI, i.e. \( \mathbb{C}(\mathcal{B}) \supseteq \mathcal{R}^n_\alpha(\mathcal{B}) \). We start with Ahlswede’s RT [1], stated below. Let \( h : S^n \rightarrow [0,1] \) be a given function. If, for some fixed \( \alpha_n \in (0,1) \), and for all \( q(s^n) = \prod_{i=1}^n q(s_i) \), with \( q \in \mathcal{P}(S) \),

\[ \sum_{s^n \in S^n} q(s^n) h(s^n) \leq \alpha_n, \]  
(138)

then,

\[ \frac{1}{n!} \sum_{\pi \in \Pi} h(\pi s^n) \leq \beta_n, \]  
(139)

where \( \Pi_n \) is the set of all \( n \)-tuple permutations \( \pi : S^n \rightarrow S^n \), and \( \beta_n = (n+1)^{|S|} \cdot \alpha_n \).

According to Lemma 4, for every \((R_0, R_1) \in \mathcal{R}^n_\alpha(\mathcal{B}) \), there exists a \((2^n R_0, 2^n R_1, n, e^{-2n} \alpha_n)\) Shannon strategy code for the compound broadcast channel \( \mathcal{B}^n(\mathcal{S}) \) with causal SI, for some \( \theta > 0 \) and sufficiently large \( n \). Given such a Shannon strategy code \( \mathcal{C} = (u_0^n(m_0), u_1^n(m_0, m_1), \xi(u_0, u_1, s), g_1(y^n_1), g_2(y^n_2)) \), we have that (138) is satisfied with \( h(s^n) = P^{(n)}_{e|\pi s^n}(\mathcal{C}) \) and \( \alpha_n = e^{-2n} \). As a result, Ahlswede’s RT tells us that

\[ \frac{1}{n!} \sum_{\pi \in \Pi} P^{(n)}_{e|\pi s^n}(\mathcal{C}) \leq (n+1)^{|S|} e^{-2n} \leq e^{-\theta n}, \]  
(140)

for a sufficiently large \( n \), such that \((n+1)^{|S|} \leq e^{\theta n} \).

On the other hand, for every \( \pi \in \Pi_n \),

\[ P^{(n)}_{e|\pi s^n}(\mathcal{C}) \overset{(a)}{=} \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1} \sum_{(\pi y^n_1, \pi y^n_2) \not\in \mathcal{D}(m_0, m_1)} W_{Y^n_1, Y^n_2|X^n, S^n}(\pi y^n_1, \pi y^n_2 | \xi^n(u_0^n(m_0), u_1^n(m_0, m_1), \pi s^n), \pi s^n) \]  
(134)
\[ \overset{(b)}{=} \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1} \sum_{(\pi y^n_1, \pi y^n_2) \not\in \mathcal{D}(m_0, m_1)} W_{Y^n_1, Y^n_2|X^n, S^n}(y^n_1, y^n_2 | \pi^{-1} \xi^n(u_0^n(m_0), u_1^n(m_0, m_1), \pi s^n), s^n), \pi s^n) \]  
(135)
\[ \overset{(c)}{=} \frac{1}{2^{n(R_0+R_1)}} \sum_{m_0, m_1} \sum_{(\pi y^n_1, \pi y^n_2) \not\in \mathcal{D}(m_0, m_1)} W_{Y^n_1, Y^n_2|X^n, S^n}(y^n_1, y^n_2 | \xi^n(\pi^{-1} u_0^n(m_0), \pi^{-1} u_1^n(m_0, m_1), s^n), s^n) \]  
(136)
where (a) is obtained by plugging $\pi s^n$ and $x^n = \xi^n(\cdot, \cdot)$ in (2) and then changing the order of summation over $(y^n_1, y^n_2)$; (b) holds because the broadcast channel is memoryless; and (c) follows from that fact that for a Shannon strategy code, $x_i = \xi(u_{0,i}, u_{1,i}, s_i)$, $i \in [1:n]$, by Definition 3. The last expression suggests the use of permutations applied to the encoding strategy sequence and the channel output sequences.

Then, consider the $(2^n R_0, 2^n R_1, n)$ random code $\mathcal{C}^\Pi$, specified by

$$f^\Pi_n(m_0, m_1, s^n) = \xi^n(\pi^{-1}u^n_1(m_0, m_1), \pi^{-1}u^n_0(m_0), s^n),$$

and

$$g_{1,\pi}(y^n_1) = g_1(\pi y^n_1), \quad g_{2,\pi}(y^n_2) = g(\pi y^n_2),$$

for $\pi \in \Pi_n$, with a uniform distribution $\mu(\pi) = \frac{1}{|\Pi_n|} = \frac{1}{n!}$. Such permutations can be implemented without knowing $s^n$, hence this coding scheme does not violate the causality requirement.

From (141), we see that

$$P^{(n)}_{\epsilon|s^n}(\mathcal{C}^\Pi) = \sum_{\pi \in \Pi_n} \mu(\pi) P^{(n)}_{\epsilon|\pi s^n}(\mathcal{C}),$$

for all $s^n \in \mathcal{S}^n$, and therefore, together with (140), we have that the probability of error of the random code $\mathcal{C}^\Pi$ is bounded by

$$P^{(n)}_{\epsilon}(q, \mathcal{C}^\Pi) \leq e^{-\theta n},$$

for every $q(s^n) \in \mathcal{P}(\mathcal{S}^n)$. That is, $\mathcal{C}^\Pi$ is a $(2^n R_0, 2^n R_1, n, e^{-\theta n})$ random code for the AVBC $B$ with causal SI at the encoder. This completes the proof of the inner bound.

2) Outer Bound: We show that the capacity region of the AVBC $B$ with causal SI is bounded by $\mathcal{C}^*(B) \subseteq R_{out}^*(B)$ (see (23)). The random code capacity region of the AVBC is included within the random code capacity region of the compound broadcast channel, namely

$$\mathcal{C}^*(B) \subseteq \mathcal{C}^*(B^{P(S)}).$$

By Theorem 5 we have that $\mathcal{C}^*(B^{\Omega}) \subseteq R_{out}^*(B^{\Omega})$. Thus, with $Q = \mathcal{P}(S)$,

$$\mathcal{C}^*(B^{P(S)}) \subseteq R_{out}^*(B).$$

It follows from (145) and (146) that $\mathcal{C}^*(B) \subseteq R_{out}^*(B)$. Since the random code capacity region always includes the deterministic code capacity region, we have that $\mathcal{C}(B) \subseteq R_{out}^*(B)$ as well.

Part 2

The second equality, $R_{in}^*(B) = R_{out}^*(B)$, follows from part 2 of Theorem 5, taking $Q = \mathcal{P}(S)$. By part 1, $R_{in}^*(B) \subseteq \mathcal{C}^*(B) \subseteq R_{out}^*(B)$, hence the proof follows.

APPENDIX E

PROOF OF LEMMA 8

The proof follows the lines of [2, Section 4]. Let $k > 0$ be an integer, chosen later, and define the random variables

$$L_1, L_2, \ldots, L_k \text{ i.i.d. } \sim \mu(\ell).$$

Fix $s^n$, and define the random variables

$$\Omega_j(s^n) = P^{(n)}_{\epsilon|s^n}(\mathcal{C}_{L_j}), \quad j \in [1:k],$$

which is the conditional probability of error of the code $\mathcal{C}_{L_j}$ given the state sequence $s^n$.

Since $\mathcal{C}^\Pi$ is a $(2^n R_1, 2^n R_2, n, \varepsilon_n)$ code, we have that $\sum_{\gamma} \mu(\gamma) \sum_{s^n} q(s^n) P^{(n)}_{\epsilon|s^n}(\mathcal{C}_\gamma) \leq \varepsilon_n$, for all $q(s^n)$. In particular, for a kernel, we have that

$$\mathbb{E} \Omega_j(s^n) = \sum_{\gamma} \mu(\gamma) \cdot P^{(n)}_{\epsilon|s^n}(\mathcal{C}_\gamma) \leq \varepsilon_n,$$

for all $j \in [1:k]$. 

Now take \( n \) to be large enough so that \( \varepsilon_n < \alpha \). Keeping \( s^n \) fixed, we have that the random variables \( \Omega_j(s^n) \) are i.i.d., due to (147). Next the technique known as Bernstein’s trick\(^2\) is applied.

\[
\Pr \left( \sum_{j=1}^{k} \Omega_j(s^n) \geq k\alpha \right) \leq \mathbb{E} \left\{ \exp \left[ \beta \left( \sum_{j=1}^{k} \Omega_j(s^n) - k\alpha \right) \right] \right\}
\]

(150)

\[
= e^{-\beta k \alpha} \cdot \mathbb{E} \left\{ \prod_{j=1}^{k} e^{\beta \Omega_j(s^n)} \right\}
\]

(151)

\[
\leq e^{-\beta k \alpha} \cdot \prod_{j=1}^{k} \mathbb{E} \left\{ e^{\beta \Omega_j(s^n)} \right\}
\]

(152)

\[
\leq e^{-\beta k \alpha} \cdot \prod_{j=1}^{k} \left( 1 + e^{\beta} \cdot \Omega_j(s^n) \right)
\]

(153)

\[
\leq e^{-\beta k \alpha} \cdot \left( 1 + e^{\beta} \varepsilon_n \right)^k
\]

(154)

where \((a)\) is an application of Chernoff’s inequality; \((b)\) follows from the fact that \( \Omega_j(s^n) \) are independent; \((c)\) holds since \( e^{\beta x} \leq 1 + e^{\beta} x \), for \( \beta > 0 \) and \( 0 \leq x \leq 1 \); \((d)\) follows from (149). We take \( n \) to be large enough for \( 1 + e^{\beta} \varepsilon_n \leq e^{\alpha} \) to hold. Thus, choosing \( \beta = 2 \), we have that

\[
\Pr \left( \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha \right) \leq e^{-\alpha k},
\]

for all \( s^n \in \mathcal{S}^n \). Now, by the union of events bound, we have that

\[
\Pr \left( \max_{s^n} \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha \right) = \Pr \left( \exists s^n : \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha \right)
\]

(156)

\[
\leq \sum_{s^n \in \mathcal{S}^n} \Pr \left( \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha \right)
\]

(157)

\[
\leq |\mathcal{S}|^n \cdot e^{-\alpha k}.
\]

(158)

Since \( |\mathcal{S}|^n \) grows only exponentially in \( n \), choosing \( k = n^2 \) results in a super exponential decay.

Consider the code \( \mathcal{C}^{\Gamma*} = (\mu^*, \Gamma^* = [1 : k], \{ \mathcal{G}_{L_j} \}_{j=1}^{k} ) \) formed by a random collection of codes, with \( \mu^* (j) = \frac{1}{k} \). It follows that the conditional probability of error given \( s^n \), which is given by

\[
P_{e|s^n}(\mathcal{C}^{\Gamma*}) = \frac{1}{k} \sum_{j=1}^{k} P_{e|s^n}(\mathcal{G}_{L_j})
\]

(159)

exceeds \( \alpha \) with a super exponentially small probability \( \sim e^{-\alpha n^2} \), for all \( s^n \in \mathcal{S}^n \). Thus, there exists a random code \( \mathcal{C}^{\Gamma*} = (\mu^*, \Gamma^*, \{ \mathcal{G}_{L_j} \}_{j=1}^{k} ) \) for the AVBC \( \mathcal{B} \), such that

\[
P_{e|s^n}(\mathcal{C}^{\Gamma*}) = \sum_{s^n \in \mathcal{S}^n} q(s^n) P_{e|s^n}(\mathcal{C}^{\Gamma*}) \leq \alpha, \quad \text{for all } q(s^n) \in \mathcal{P}(\mathcal{S}^n).
\]

(160)

\[\square\]

APPENDIX F

PROOF OF THEOREM 9

Achievability proof. To show achievability, we follow the lines of [2], with the required adjustments. We use the random code constructed in the proof of Theorem 7 to construct a deterministic code.

Let \((R_0, R_1) \in \mathcal{C}(\mathcal{B})\), and consider the case where \( \text{int}(\mathcal{C}(\mathcal{B})) \neq \emptyset \). Namely,

\[
\mathcal{C}(\mathcal{W}_1) > 0, \quad \text{and } \mathcal{C}(\mathcal{W}_2) > 0,
\]

(161)

where \( \mathcal{W}_1 = \{ W_{Y_1|X,S} \} \) and \( \mathcal{W}_2 = \{ W_{Y_2|X,S} \} \) denote the marginal AVCs with causal SI of user 1 and user 2, respectively. By Lemma 8, for every \( \varepsilon_1 > 0 \) and sufficiently large \( n \), there exists a \((2^n R_0, 2^n R_1, n, \varepsilon_1)\) random code \( \mathcal{C}^{\Gamma} = (\mu(\gamma) = \frac{1}{n}, \Gamma = [1 : k], \{ \mathcal{G}_{\gamma} \}_{\gamma \in \Gamma}) \), where \( \mathcal{G}_\gamma = (f^n_\gamma, g_1, g_2, \gamma) \), for \( \gamma \in \Gamma \), and \( k = |\Gamma| \leq n^2 \). Following (161), we have that for every
\[ \varepsilon_2 > 0 \] and sufficiently large \( \nu \), the code index \( \gamma \in [1 : k] \) can be sent over \( B \) using a \((2^\nu \tilde{R}_0, 2^\nu \tilde{R}_1, \nu, \varepsilon_2)\) deterministic code \( \mathcal{C}_i = (f^\nu, \tilde{g}_i, \tilde{g}_0) \), where \( \tilde{R}_0 > 0, \tilde{R}_1 > 0 \). Since \( k \) is at most polynomial, the encoder can reliably convey \( \gamma \) to the receiver with a negligible blocklength, i.e. \( \nu = o(n) \).

Now, consider a code formed by the concatenation of \( \mathcal{C}_i \) as a prefix to a corresponding code in the code collection \( \{ \mathcal{C}_i \}_{i \in \Gamma} \). That is, the encoder sends both the index \( \gamma \) and the message pair \((m_0, m_1)\) to the receivers, such that the index \( \gamma \) is transmitted first by \( f^\nu(\gamma, s^\nu) \), and then the message pair \((m_0, m_1)\) is transmitted by the codeword \( x^n = f^n(\gamma_\nu, m_0, \nu_{s+1}, \ldots, \nu_{n+1}) \).

Subsequently, decoding is performed in two stages as well; decoder 1 estimates the index at first, with \( \hat{\gamma}_1 = \tilde{g}_1(y_1, \ldots, y_1, \nu) \), and decoder 2 estimates the index with \( \hat{\gamma}_2 = \tilde{g}_0(y_{2,1}, \ldots, y_{2,\nu}) \), and the message \( m_0 \) is then estimated by \( \tilde{m}_0 = g_2(\nu_{2,\nu+1}, \ldots, \nu_{2,\nu+n}) \).

By the union of events bound, the probability of error is then bounded by \( \varepsilon = \varepsilon_1 + \varepsilon_2 \), for every joint distribution in \( \mathcal{P}^{\nu+n}(S^{\nu+n}) \). That is, the concatenated code is a \((2^{\nu+n})\tilde{R}_1, (2^{\nu+n})\tilde{R}_2, \nu + n, \varepsilon)\) code over the AVBC \( B \) with causal SI, where \( \nu = o(n) \). Hence, the blocklength is \( n + o(n) \), and the rates \( \tilde{R}_0, n = \nu \nu + n, \tilde{R}_1, n = \nu \nu \cdot \tilde{R}_1 \) approach \( \tilde{R}_0 \) and \( \tilde{R}_1 \), respectively, as \( n \to \infty \).

Converse proof. In general, the deterministic code capacity region is included within the random code capacity region. Namely, \( \mathcal{C}(B) \subseteq \mathcal{C}^*(B) \).

**Appendix G**

**Proof of Corollary 10**

First, consider the inner and outer bounds in (28) and (29). The bounds are obtained as a direct consequence of part 1 of Theorem 7 and Theorem 9. Note that the outer bound (29) holds regardless of any condition, since the deterministic code capacity region is always included within the random code capacity region, i.e. \( \mathcal{C}(B) \subseteq \mathcal{C}^*(B) \subseteq \mathcal{R}_{out}^*(B) \).

Now, suppose that the marginals \( V_{Y_1,U,S}^\xi \) and \( V_{Y_2,U,S}^\xi \) are non-symmetrizable for some \( \xi : \mathcal{U} \times \mathcal{S} \to \mathcal{X} \) and Condition \( \mathcal{T} \) holds. Then, based on [8, 7], both marginal (single-user) AVCs have positive capacity, i.e. \( \mathcal{C}(W_1) > 0 \) and \( \mathcal{C}(W_2) > 0 \). Namely, \( \text{int}(\mathcal{C}(B)) \neq \emptyset \). Hence, by Theorem 9, the deterministic code capacity region coincides with the random code capacity region, i.e. \( \mathcal{C}(B) = \mathcal{C}^*(B) \). Then, the proof follows from part 2 of Theorem 7.

**Appendix H**

**Analysis of Example 1**

We begin with the case of an arbitrarily varying BSBC \( B_{D,0} \) without SI. We claim that the single user marginal AVC \( W_{1,0} \) without SI, corresponding to the stronger user, has zero capacity. Denote \( q \triangleq q(1) = 1 - q(0) \). Then, observe that the additive noise is distributed according to \( Z_S \sim \text{Bernoulli}(\varepsilon_0) \), with \( \eta_\theta \triangleq (1 - q) \cdot \theta_0 + q \cdot \theta_1 \), for \( 0 \leq q \leq 1 \). Based on [5], \( \mathcal{C}(W_{1,0}) \leq \mathcal{C}^*(W_{1,0}) \leq \min_{\eta_\theta \leq q \leq 1} [1 - h(\eta_\theta)] \). Since \( \theta_0 < \frac{1}{2} \leq \theta_1 \), there exists \( 0 \leq q \leq 1 \) such that \( \eta_0 \leq \frac{1}{2} \), thus \( \mathcal{C}(W_{1,0}) = 0 \). The capacity region of the AVDBC \( B_{D,0} \) without SI is then given by \( \mathcal{C}(B_{D,0}) = \{(0, 0)\} \).

Now, consider the arbitrarily varying BSBC \( B_{D} \) with causal SI. By Theorem 11, the random code capacity region is bounded by \( \mathcal{R}_{out}^*(B_{D}) \subseteq \mathcal{C}^*(B_{D}) \subseteq \mathcal{R}_{out}^*(B_{D}) \). We show that the bounds coincide, and are thus tight. Let \( B_{D}^q \) denote the random parameter DBC \( W_{Y_1,Y_2,X,S} \) with causal SI, governed by an i.i.d. state sequence, distributed according to \( S \sim \text{Bernoulli}(q) \). By [17], the corresponding capacity region is given by

\[
\mathcal{C}(B_{D}) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_1, R_2) : R_2 \leq 1 - h(\alpha \ast \beta + \delta_q), \quad R_1 \leq h(\beta + \delta_q), \right\},
\]

where

\[
\delta_q \triangleq (1 - q) \cdot \theta_0 + q \cdot (1 - \theta_1),
\]

for \( 0 \leq q \leq 1 \). For every given \( 0 \leq q' \leq 1 \), we have that \( \mathcal{R}_{out}^*(B_{D}) \cap [0 \leq q \leq 1] \mathcal{C}(B_{D}) \subseteq \mathcal{C}(B_{D}) \). Thus, taking \( q' = 1 \), we have that

\[
\mathcal{R}_{out}^*(B_{D}) \subseteq \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_1, R_2) : R_2 \leq 1 - h(\alpha \ast \beta \ast \theta_1), \quad R_1 \leq h(\beta \ast \theta_1) \right\},
\]

where we have used the identity \( h(\alpha \ast (1 - \delta)) = h(\alpha \ast \delta) \).

Now, to show that the region above is achievable, we examine the inner bound,

\[
\mathcal{R}_{in}^*(B_{D}) = \bigcup_{p(x_1, x_2), \xi(x_1, x_2, s)} \left\{ (R_1, R_2) : R_2 \leq \min_{0 \leq q \leq 1} I_{q}(U_2; Y_2), \quad R_1 \leq \min_{0 \leq q \leq 1} I_{q}(U_1; Y_1 | U_2) \right\}.
\]

Consider the following choice of \( p(u_1, u_2) \) and \( \xi(u_1, u_2, s) \). Let \( U_1 \) and \( U_2 \) be independent random variables,

\[
U_1 \sim \text{Bernoulli}(\beta), \quad \text{and} \quad U_2 \sim \text{Bernoulli} \left( \frac{1}{2} \right),
\]

for \( 0 \leq \beta \leq \frac{1}{2} \), and let

\[
\xi(u_1, u_2, s) = u_1 + u_2 + s \mod 2.
\]

Then,

\[
H_q(Y_1|U_1, U_2) = H_q(S + Z_s) = h(\delta_q),
\]

\[
H_q(Y_1|U_2) = H_q(U_1 + S + Z_s) = h(\beta \ast \delta_q),
\]

\[
H_q(Y_2|U_2) = H_q(U_1 + S + Z_s + V) = h(\alpha \ast \beta \ast \delta_q),
\]

\[
H_q(Y_2) = 1,
\]

where addition is modulo 2, and \( \delta_q \) is given by \((162b)\). Thus,

\[
I_q(U_2; Y_2) = 1 - h(\alpha \ast \beta \ast \delta_q),
\]

\[
I_q(U_1; Y_1|U_2) = h(\beta \ast \delta_q) - h(\delta_q),
\]

hence

\[
\mathcal{R}^*_\text{inn}(B_D) \supseteq \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq \min_{0 \leq q \leq 1} 1 - h(\alpha \ast \beta \ast \delta_q),
R_1 \leq \min_{0 \leq q \leq 1} h(\beta \ast \delta_q) - h(\delta_q)
\end{array} \right\}.
\]

Note that \( \theta_0 \leq \delta_q \leq 1 - \theta_1 \leq \frac{1}{2} \). For \( 0 \leq \delta \leq \frac{1}{2} \), the functions \( g_1(\delta) = 1 - h(\alpha \ast \beta \ast \delta) \) and \( g_2(\delta) = h(\beta \ast \delta) - h(\delta) \) are monotonic decreasing functions of \( \delta \), hence the minima in \((169)\) are both achieved with \( q = 1 \). It follows that

\[
\mathcal{C}^*(B_D) = \mathcal{R}^*_\text{inn}(B_D) = \mathcal{R}^*_\text{out}(B_D) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq 1 - h(\alpha \ast \beta \ast \theta_1),
R_1 \leq h(\beta \ast \theta_1) - h(\theta_1)
\end{array} \right\}.
\]

It can also be verified that Condition \( \mathcal{F}_D \) holds (see Definition 9), in agreement with part 2 of Theorem 11. First, we specify a function \( \xi(u_1, u_2, s) \) and a distributions set \( \mathcal{D}^* \) that achieve \( \mathcal{R}^*_\text{inn}(B_D) \) and \( \mathcal{R}^*_\text{out}(B_D) \) (see Definition 38). Let \( \xi(u_1, u_2, s) \) be as in \((166)\), and let \( \mathcal{D}^* \) be the set of distributions \( p(u_1, u_2) \) such that \( U_1 \) and \( U_2 \) are independent random variables, distributed according to \((165)\). By the derivation above, the requirement \((38a)\) is satisfied. Now, by the derivation in \([17, \text{Section IV}]\), we have that

\[
\mathcal{C}(\mathcal{B}_D^*) = \bigcup_{p(u_1, u_2) \in \mathcal{D}^*} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq I_q(U_2; Y_2),
R_1 \leq I_q(U_1; Y_1|U_2)
\end{array} \right\}.
\]

Then, the requirement \((38b)\) is satisfied as well, hence \( \xi(u_1, u_2, s) \) and \( \mathcal{D}^* \) achieve \( \mathcal{R}^*_\text{inn}(B_D) \) and \( \mathcal{R}^*_\text{out}(B_D) \). It follows that Condition \( \mathcal{F}_D \) holds, as \( q^* = 1 \) satisfies the desired property with \( \xi(u_1, u_2, s) \) and \( \mathcal{D}^* \) as described above.

We move to the deterministic code capacity region of the arbitrarily varying BSBC \( B_D \) with causal SI. If \( \theta_1 = \frac{1}{2} \), the capacity region is given by \( \mathcal{C}(\mathcal{B}_D) = \mathcal{C}^*(\mathcal{B}_D) = \{(0, 0)\} \), by \((170)\). Otherwise, \( \theta_0 < \frac{1}{2} < \theta_1 \), and we now show that the condition in Corollary 10 is met. Suppose that \( V_{Y_2|U_2,s}^\xi \) is symmetrizable for all \( \xi' : U_2 \times S \to \mathcal{X} \). That is, for every \( \xi'(u_2, s) \), there exists \( \lambda_{u_2} = J(1|u_2) \) such that

\[
(1 - \lambda_{u_2})W_{Y_2|X,S}(y_2|\xi'(u_2, 0), 0) + \lambda_{u_2}W_{Y_2|X,S}(y_2|\xi'(u_2, 1), 1) = (1 - \lambda_{u_2})W_{Y_2|X,S}(y_2|\xi'(u_2, 0), 0) + \lambda_{u_2}W_{Y_2|X,S}(y_2|\xi'(u_2, 1), 1)
\]

for all \( u_a, u_b \in U_2, y_2 \in \{0, 1\} \). If this is the case, then for \( \xi'(u_2, s) = u_2 + s \mod 2 \), taking \( u_a = 0, u_b = 1, y_2 = 1 \), we have that

\[
(1 - \lambda_1) \cdot (\alpha \ast \theta_0) + \lambda_1 \cdot (1 - \alpha \ast \theta_1) = (1 - \lambda_0) \cdot (\alpha \ast \theta_0) + \lambda_0 \cdot (\alpha \ast \theta_1).
\]

This is a contradiction. Since \( f(\theta) = \alpha \ast \theta \) is a monotonic increasing function of \( \theta \), and since \( 1 - f(\theta) = f(1 - \theta) \), we have that the value of the LHS of \((173)\) is in \([0, \frac{1}{2}]\), while the value of the RHS of \((173)\) is in \([\frac{1}{2}, 1]\). Thus, there exists \( \xi' : U_2 \times S \to \mathcal{X} \) such that \( V_{Y_2|X,S}^\xi \) is non-symmetrizable for \( \theta_0 < \frac{1}{2} < \theta_1 \). As Condition \( \mathcal{F}_D \) holds, we have that \( \mathcal{C}(\mathcal{B}_D) = \mathcal{R}^*_\text{inn}(B_D) = \mathcal{R}^*_\text{out}(B_D) \), due to Corollary 13. Hence, by \((170)\), we have that the capacity region of the arbitrarily varying BSBC \( B_D \) with causal SI is given by \((45)\). \( \square \)
A. Random Parameter BSBC with Correlated Noises

Consider the random parameter BSBC $B^q$ with causal SI. By Theorem 6, the capacity region of $B^q$ with degraded message sets with causal SI is given by $C(B^q) = C(B^q)$ (see (20)). Then, to show achievability, consider the following choice of $p(u_0, u_1)$ and $\xi(u_0, u_1, s)$. Let $U_0$ and $U_1$ be independent random variables, $U_0 \sim \text{Bernoulli} \left(\frac{1}{2}\right)$, and $U_1 \sim \text{Bernoulli}(\beta)$, for $0 \leq \beta \leq \frac{1}{2}$, and let

$$\xi(u_0, u_1, s) = u_0 + u_1 + s \mod 2.$$  \hfill (175)

Then,

$$H_q(Y_1|U_0, U_1) = H_q(S + Z_s) = h(\delta^{(1)}_q),$$
$$H_q(Y_1|U_0) = H_q(U_1 + S + Z_s) = h(\beta \ast \delta^{(1)}_q),$$
$$H_q(Y_2|U_0) = H_q(U_1 + S + N_s) = h(\beta \ast \delta^{(2)}_q),$$
$$H_q(Y_2) = 1,$$

where addition is modulo 2, and $\delta^{(1)}_q, \delta^{(2)}_q$ are given by (49). Thus,

$$I_q(U_0; Y_2) = 1 - h(\beta \ast \delta^{(2)}_q),$$
$$I_q(U_1; Y_1|U_0) = h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q).$$  \hfill (177)

The last inequality on the sum rate in (20) is redundant, as shown below. Since $\theta_0 \leq \epsilon_0 \leq \frac{1}{2}$ and $\frac{1}{2} \leq \theta_1 \leq \epsilon_1$, we have that $\delta^{(1)}_q \leq \delta^{(2)}_q \leq \frac{1}{2}$. Hence,

$$I_q(U_0; Y_2) = 1 - h(\beta \ast \delta^{(2)}_q) \leq 1 - h(\beta \ast \delta^{(1)}_q) = I_q(U_0; Y_1),$$

which implies that $I_q(U_0; Y_2) + I_q(U_1; Y_1|U_0) \leq I_q(U_0, U_1; Y_1)$. This completes the proof of the direct part.

As for the converse, we need to show that if,

$$R_1 > h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q),$$

for some $0 \leq \beta \leq \frac{1}{2}$, then it must follows that $R_0 \leq 1 - h(\beta \ast \delta^{(2)}_q)$. Indeed, by (20) and (179),

$$H_q(Y_1|U_0) > h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q) + H_q(Y_1|U_0, U_1)$$
$$= h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q) + \min_{u_0, u_1} H_q(\xi(u_0, u_1, S) + Z_s)$$
$$= h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q) + \min \left( H_q(Z_s), H_q(S + Z_s) \right)$$
$$= h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q) + \max \left( h((1 - q)\theta_0 + q\theta_1), h(\delta^{(1)}_q) \right)$$
$$= h(\beta \ast \delta^{(1)}_q) - h(\delta^{(1)}_q) + h(\delta^{(1)}_q)$$
$$= h(\beta \ast \delta^{(1)}_q).$$  \hfill (180)

Then, since $\delta^{(1)}_q \leq \delta^{(2)}_q \leq \frac{1}{2}$, there exists a random variable $L \sim \text{Bernoulli}(\lambda_q)$, with

$$\delta^{(q)}_q = \delta^{(1)}_q \ast \lambda_q,$$

for some $0 \leq \lambda_q \leq \frac{1}{2}$, such that $Y_2 = Y_1 + L \mod 2$ is distributed according to $Pr \left( Y_2 = y_2 \mid U_0 = u_0, U_1 = u_1 \right) = \sum_{s \in S} q(s) W_{Y_2|X,S}(y_2|\xi(u_0, u_1, s), s)$. Thus,

$$H_q(Y_2|U_0) = H_q(Y_2|U_0) \geq h \left( [h^{-1}(H_q(Y_1|U_0))] - \lambda_q \right) \geq h(\beta \ast \delta^{(1)}_q \ast \lambda_q) \geq h(\beta \ast \delta^{(2)}_q),$$

where $\beta$ is due to Mrs. Gerber's Lemma [20], and (b)-(c) follow from (180) and (181), respectively.

B. Arbitrarily Varying BSBC with Correlated Noises

1) Without SI: We begin with the case of an arbitrarily varying BSBC $B_0$ without SI. We claim that the single user marginal AVCs $W_{1,0}$ and $W_{2,0}$ without SI, corresponding to user 1 and user 2, respectively, have zero capacity. Denote $q \equiv q(1) = 1 - q(0)$. Then, observe that the additive noises are distributed according to $Z_s \sim \text{Bernoulli}(\eta^{(1)}_q)$ and $N_s \sim \text{Bernoulli}(\eta^{(2)}_q)$, with

$$\eta^{(1)}_q \equiv (1 - q)\theta_0 + q\theta_1$$
and
$$\eta^{(2)}_q \equiv (1 - q)\phi_0 + q\phi_1,$$

for $0 \leq q \leq 1$. Based on [5], $C(W_{1,0}) \leq C(W_{1,0}) = \min_{0 \leq q \leq 1} [1 - h(\eta^{(1)}_q)]$. Since $\theta_0 < \frac{1}{2} \leq \theta_1$, there exists $0 \leq q_1 < 1$ such that $\eta_q^{(1)} = \frac{1}{2}$, thus $C(W_{1,0}) = 0$. Similarly, $\epsilon_0 < \frac{1}{2} \leq \epsilon_1$ implies that $\eta_q^{(2)} = -\frac{1}{2}$, for some $0 \leq q_2 \leq 1$, thus $C(W_{2,0}) = 0$ as well. The capacity region of the AVBC $B_0$ without SI is then given by $C(B_0) = \{(0, 0)\}$. 

\hfill \square
2) Causal SI – Case 1: Consider the arbitrarily varying BSBC $B$ with causal SI, with $\theta_0 \leq 1 - \theta_1 \leq \varepsilon_0 \leq 1 - \varepsilon_1 \leq \frac{1}{2}$. By Theorem 7, the random code capacity region is bounded by $R_{\text{in}}(B) \subseteq C(B) \subseteq R_{\text{out}}(B)$. We show that the bounds coincide, and are thus tight. By (15), (20) and (23), we have that $R_{\text{out}}(B) = \bigcap_{0 \leq q \leq 1} C(B^q) \subseteq C(B^\varepsilon)$, for every given $0 \leq q' \leq 1$. Thus, taking $q' = 1$, we have by (48) that

$$R_{\text{out}}(B) \subseteq \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_1, R_2) : \begin{array}{l} R_1 \leq 1 - h(\beta \ast \varepsilon_1) \\ R_2 \leq h(\beta \ast \theta_1) - h(\theta_1) \end{array} \right\},$$

(183)

where we have used the identity $h(\alpha \ast (1 - \delta)) = h(\alpha \ast \delta)$.

Now, to show that the region above is achievable, we examine the inner bound,

$$R_{\text{in}}(B) = \bigcup_{p(u_0, u_1; \xi(u_0, u_1, s))} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq \min_{0 \leq q \leq 1} I_q(U_0; Y_2) \\ R_1 \leq \min_{0 \leq q \leq 1} I_q(U_1; Y_1 | U_0) \\ R_0 + R_1 \leq \min_{0 \leq q \leq 1} I_q(U_0, U_1; Y_1) \end{array} \right\}.$$  

(184)

Consider the following choice of $p(u_0, u_1)$ and $\xi(u_0, u_1, s)$. Let $U_0$ and $U_1$ be independent random variables,

$$U_2 \sim \text{Bernoulli} \left( \frac{1}{2} \right), \text{ and } U_1 \sim \text{Bernoulli}(\beta),$$

(185)

for $0 \leq \beta \leq \frac{1}{2}$, and let

$$\xi(u_0, u_1, s) = u_0 + u_1 + s \mod 2.$$  

(186)

Then, as in Subsection I-A above, this yields the following inner bound,

$$R_{\text{in}}(B) \supseteq \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq \min_{0 \leq q \leq 1} 1 - h(\beta \ast \varepsilon_1^{(q)}) \\ R_1 \leq \min_{0 \leq q \leq 1} h(\beta \ast \theta_1) - h(\theta_1) \end{array} \right\}.$$  

(187)

Note that $\theta_0 \leq \delta_0^{(1)} \leq 1 - \theta_1 \leq \frac{1}{2}$ and $\varepsilon_0 \leq \delta_0^{(2)} \leq 1 - \varepsilon_1 \leq \frac{1}{2}$. For $0 \leq \delta \leq \frac{1}{2}$, the functions $g_1(\delta) = 1 - h(\beta \ast \delta)$ and $g_2(\delta) = h(\beta \ast \delta) - h(\delta)$ are monotonic decreasing functions of $\delta$, hence the minima in (187) are both achieved with $q = 1$. It follows that

$$C^\ast(B) = R_{\text{in}}^\ast(B) = R_{\text{out}}^\ast(B) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq 1 - h(\beta \ast \varepsilon_1) \\ R_1 \leq h(\beta \ast \theta_1) - h(\theta_1) \end{array} \right\}.$$  

(188)

It can also be verified that Condition $\mathcal{F}$ holds (see Definition 5 and (24)), in agreement with part 2 of Theorem 7. First, we specify a function $\xi(u_0, u_1, s)$ and a distribution set $D^\ast$ that achieve $R_{\text{in}}^\ast(B)$ and $R_{\text{out}}^\ast(B)$ (see Definition 4). Let $\xi(u_0, u_1, s)$ be as in (186), and let $D^\ast$ be the set of distributions $p(u_0, u_1)$ such that $U_0$ and $U_1$ are independent random variables, distributed according to (185). By the derivation above, the first requirement in Definition 5 is satisfied with $Q = P(S)$, and by our derivation in Subsection I-A,

$$C(B^\varepsilon) = \bigcup_{p(u_0, u_1) \in D^\ast} \left\{ (R_0, R_1) : \begin{array}{l} R_0 \leq I_q(U_0; Y_2) \\ R_1 \leq I_q(U_1; Y_1 | U_0) \\ R_0 + R_1 \leq I_q(U_0, U_1; Y_1) \end{array} \right\}.$$  

(189)

Then, the second requirement is satisfied as well, hence $\xi(u_0, u_1, s)$ and $D^\ast$ achieve $R_{\text{in}}^\ast(B)$ and $R_{\text{out}}^\ast(B)$. It follows that Condition $\mathcal{F}$ holds, as $q^* = 1$ satisfies the desired property with $\xi(u_0, u_1, s)$ and $D^\ast$ as described above.

We move to the deterministic code capacity region of the arbitrarily varying BSBC $B$ with causal SI. Consider the following cases. First, if $\theta_1 = \frac{1}{2}$, then $\varepsilon_1 = \frac{1}{2}$ as well, and the capacity region is given by $C(B) = C^\ast(B) = \{(0, 0)\}$, by (188). Otherwise, $\theta_0 < \frac{1}{2} < \theta_1$. Then, for the case where $\varepsilon_1 = \frac{1}{2}$, we show that the random code capacity region, $C^\ast(B) = \{(R_0, R_1) : R_0 = 0, R_1 \leq C^\ast(W_1) = 1 - h(\theta_1)\}$ can be achieved by deterministic codes as well. Based on [8, 7], it suffices to show that there exists a function $\xi : U \times S \rightarrow \mathcal{X}$ such that $V_{Y_1 | U, S}^\xi$ is non-symmetrizable.

Indeed, assume to the contrary that $\theta_0 < \frac{1}{2} < \theta_1$, yet $V_{Y_1 | U, S}^\xi$ is symmetrizable for all $\xi : U \times S \rightarrow \mathcal{X}$. That is, for every $\xi(u, s)$, there exists $\sigma_u = J(1 | u)$ such that

$$(1 - \sigma_u) W_{Y_1 | X, S}(y_1 | \xi(u_0, 0), 0) + \sigma_u W_{Y_1 | X, S}(y_1 | \xi(u_1, 1), 1) = (1 - \sigma_u) W_{Y_1 | X, S}(y_1 | \xi(u_0, 0), 0) + \sigma_u W_{Y_1 | X, S}(y_1 | \xi(u_1, 1), 1)$$

(190)

for all $u_a, u_b \in U$, $y_1 \in \{0, 1\}$. If this is the case, then for $\xi(u, s) = u + s \mod 2$, taking $u_a = 0$, $u_b = 1$, $y_1 = 1$, we have that

$$(1 - \sigma_1) \theta_0 + \sigma_1 (1 - \theta_1) = (1 - \sigma_0)(1 - \theta_0) + \sigma_0 \theta_1.$$  

(191)
This is a contradiction, since the value of the LHS of (191) is in \([0, \frac{1}{2}]\), while the value of the RHS of (191) is in \((\frac{1}{2}, 1]\). Hence, \(V_{Y_1|U,S}^\xi\) is non-symmetrizable, and \(\mathbb{C}(B) = \mathbb{C}'(B)\).

The last case to consider is when \(\theta_0 \leq \varepsilon_0 < \frac{1}{2} < \varepsilon_1 \leq \theta_1\). We now claim that the condition in Corollary 10 is met. Indeed, the contradiction in (191) implies that \(V_{Y_1|U_0,U_1,S}^\xi\) is non-symmetrizable with \(\xi(u_0, u_1, s) = u_0 + u_1 + s \mod 2\), given that \(\theta_0 < \frac{1}{2} < \theta_1\). Similarly, \(V_{Y_2|U_0,S}^\xi\) is non-symmetrizable with \(\xi'(u_0, s) = u_0 + s \mod 2\), given that \(\varepsilon_0 < \frac{1}{2} < \varepsilon_1\). As Condition 3 holds, we have that \(\mathbb{C}(B) = R^\ast_{in}(B) = R^\ast_{out}(B)\), due to Corollary 10. Hence, by (188), we have that the capacity region of the arbitrarily varying BSBC with correlated noises is given by (50).

3) Causal SI – Case 2: Consider the arbitrarily varying BSBC \(B\) with causal SI, with \(\theta_0 \leq 1 - \theta_1 \leq 1 - \varepsilon_1 \leq \varepsilon_0 \leq \frac{1}{2}\). By Theorem 7, the random code capacity region is bounded by \(R^\ast_{in}(B) \subseteq \mathbb{C}'(B) \subseteq R^\ast_{out}(B)\). Next, we show that the deterministic code capacity region is bounded by (51) and (52).

**Inner Bound.** Denote

\[
A_{in} \triangleq \bigcup_{0 \leq \beta \leq 1} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq 1 - h(\beta + \varepsilon_0), \\
R_1 \leq h(\beta \ast \theta_1) - h(\theta_1)
\end{array} \right\}.
\]  

(192)

We show that \(R^\ast_{in}(B) \subseteq A_{in}\) and \(R^\ast_{in}(B) \supseteq A_{in}\), hence \(R^\ast_{in}(B) = A_{in}\). As in the proof for case 1 above, consider the set of strategy distributions \(\mathcal{D}^\ast\) and function \(\xi(u_0, u_1, s)\) as specified by (185) and (186). Then, this results in the following inner bound,

\[
R^\ast_{in}(B) \supseteq \bigcup_{p \in \mathcal{D}^\ast} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq \min_{0 \leq q \leq 1} I_q(U_0; Y_2) , \\
R_1 \leq \min_{0 \leq q \leq 1} I_q(U_1; Y_1|U_0) \\
R_0 + R_1 \leq \min_{0 \leq q \leq 1} I_q(U_0, U_1; Y_1)
\end{array} \right\}.
\]  

(193)

where the last equality holds since in case 2, we assume that \(\theta_0 \leq 1 - \theta_1 \leq \frac{1}{2}\) and \(1 - \varepsilon_1 \leq \varepsilon_0 \leq \frac{1}{2}\).

Now, we upper bound \(R^\ast_{in}(B)\) by

\[
R^\ast_{in}(B) \subseteq \bigcup_{p(u_0, x)} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq \min_{s \in S} I_q(U_0; Y_2|S = s) , \\
R_1 \leq \min_{s \in S} I_q(X; Y_1|U_0, S = s) \\
R_0 + R_1 \leq \min_{s \in S} I_q(X; Y_1|S = s)
\end{array} \right\}.
\]  

(194)

We have replaced \(U_1\) with \(X\) since \((U_0, U_1) \rightarrow X \rightarrow Y_1\) form a Markov chain. Now, since \(X \rightarrow Y_1, S \rightarrow Y_2\) form a Markov chain, the third inequality in (194) is not necessary. Furthermore \(W_{Y_1|X,S}(y_1|x, 1)\) is degraded with respect to \(W_{Y_1|X,S}(y_1|x, 0)\), whereas \(W_{Y_2|X,S}(y_2|x, 0)\) is degraded with respect to \(W_{Y_2|X,S}(y_2|x, 1)\). Thus,

\[
R^\ast_{in}(B) \subseteq \bigcup_{p(u_0, x)} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq I_q(U_0; Y_2|S = 0) , \\
R_1 \leq I_q(X; Y_1|U_0, S = 1) \\
R_0 + R_1 \leq I_q(X; Y_1|U_0, S = 1)
\end{array} \right\}.
\]  

(195)

Observe that the RHS of (195) is the capacity region of a BSBC without a state, specified by \(Y_1 = X + Z_1 \mod 2, Y_2 = X + N_0 \mod 2\) [4, 11]. This upper bound can thus be expressed as in the RHS of (192) (see e.g. [6, Example 15.6.5]). Hence, \(R^\ast_{in}(B) = A_{in}\), which proves the equality in (51).

To show that the inner bound is achievable by deterministic codes, i.e. \(\mathbb{C}(B) \supseteq R^\ast_{in}(B)\), we consider the following cases. First, if \(\theta_1 = \frac{1}{2}\), then \(\varepsilon_0 = \frac{1}{2}\) as well, and \(R^\ast_{in}(B) = \{0, 0\}\), by (192). Otherwise, \(\theta_0 < \frac{1}{2} < \theta_1\). In particular, for \(\varepsilon_0 = \frac{1}{2}\), we have that \(R^\ast_{in}(B) = \{(R_0, R_1) : R_0 = 0, R_1 \leq 1 - h(\theta_1)\}\). Then, as shown in the proof of case 1, there exists a function \(\xi : \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{X}\) such that \(V_{Y_1|U,S}^\xi\) is non-symmetrizable. Thus, based on [8, 7], the deterministic code capacity of user 1 marginal AVC is given by \(\mathbb{C}(W_{Y_1}) = 1 - h(\theta_1)\), which implies that \(R^\ast_{in}(B)\) is achievable for \(\varepsilon_0 = \frac{1}{2}\).

It remains to consider the case where \(\theta_0 \leq \varepsilon_0 < \frac{1}{2} < \varepsilon_1 \leq \theta_1\). By Corollary 10, in order to show that \(\mathbb{C}(B) \supseteq R^\ast_{in}(B)\), it suffices to prove that the capacity region has non-empty interior. Following the same steps as in the proof of case 1 above, we have that the channels \(V_{Y_1|U,S}^\xi\) and \(V_{Y_2|U_0,S}^\xi\) are non-symmetrizable for \(\xi(u, s) = u + s \mod 2\) and \(\xi'(u_0, s) = u_0 + s \mod 2\) (see (27)). Thus, based on [8, 7], the deterministic code capacity of the marginal AVCs of user 1 and user 2 are positive, which implies that \(\text{int}(\mathbb{C}(B)) \neq \emptyset\), hence \(\mathbb{C}(B) \supseteq R^\ast_{in}(B)\). \(\square\)
Outer Bound. Since the deterministic code capacity region is included within the random code capacity region, it follows that \( C(B) \subseteq R^*_\text{out}(B) \). Based on (15), (20) and (23), we have that \( R^*_\text{out}(B) = \bigcap_{0 \leq q \leq 1} C(B^q) \). Thus,

\[
R^*_\text{out}(B) \subseteq C(B^{q=0}) \cap C(B^{q=1})
\]

\[
= \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq 1 - h(\beta \ast \varepsilon_0), \\
R_1 \leq h(\beta \ast \theta_0) - h(\theta_0)
\end{array} \right\} \cap \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq 1 - h(\beta \ast \varepsilon_1), \\
R_1 \leq h(\beta \ast \theta_1) - h(\theta_1)
\end{array} \right\}
\]

\[
= \bigcup_{0 \leq \beta_0 \leq \frac{1}{2}, 0 \leq \beta_1 \leq \frac{1}{2}} \left\{ (R_0, R_1) : \begin{array}{l}
R_0 \leq 1 - h(\beta_0 \ast \varepsilon_0), \\
R_1 \leq 1 - h(\beta_1 \ast \varepsilon_1), \\
R_0 \leq h(\beta_0 \ast \theta_0) - h(\theta_0), \\
R_1 \leq h(\beta_1 \ast \theta_0) - h(\theta_0)
\end{array} \right\}
\]

(196)

\[
\square
\]

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