PSEUDO B-FREDHOLM OPERATORS AND SPECTRAL THEORY

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Abstract. In this paper, we show that every pseudo B-Fredholm operator is a pseudo Fredholm operator. Afterwards, we prove that the pseudo B-Weyl spectrum is empty if and only if the pseudo B-Fredholm spectrum is empty. Also, we study a symmetric difference between some parts of the spectrum.

1. Introduction and Preliminaries

Throughout, $X$ denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on $X$, we denote by $T^*$, $R(T)$, $R^\infty(T) = \bigcap_{n \geq 0} R(T^n)$, $K(T)$, $H_0(T)$, $\rho(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma(T)$, respectively the adjoint, the range, the hyper-range, the analytic core, the quasinilpotent part, the resolvent set, the approximate point spectrum, the surjectivity spectrum and the spectrum of $T$.

Next, let $T \in \mathcal{B}(X)$, $T$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighbourhood $U \subseteq \mathbb{C}$ of $\lambda_0$, the only analytic function $f : U \to X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. $T$ is said to have the SVEP if $T$ has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, hence $T$ and $T^*$ have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi-Fredholm) if $\dim N(T) < \infty$ and $R(T)$ closed (resp, $\text{codim} R(T) < \infty$). $T$ is semi-Fredholm if it is a lower or upper. The index of a semi-Fredholm operator $T$ is defined by $\text{ind}(T) = \dim N(T) - \text{codim} R(T)$.

$T$ is a Fredholm operator if it is a lower and upper semi-Fredholm, and is called a Weyl operator if it is a Fredholm of index zero. The essential and Weyl spectrum of $T$ are closed and defined by :

$\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not a Fredholm operator} \}$

$\sigma_W(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not a Weyl operator} \}$.

Recall that $T \in \mathcal{B}(X)$ is said to be Kato operator or semi-regular, if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$. Denote by $\rho_K(T) : \rho_K(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is Kato} \}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C} \setminus \rho_K(T)$ the Kato spectrum of $T$. It is well known that $\rho_K(T)$ is an open subset of $\mathbb{C}$.

Let $T \in \mathcal{B}(X)$ such that $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$.

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$T$ is a B-Fredholm operator if $T_1$ is Fredholm and $T_2$ is nilpotent. The B-Fredholm spectrum defined by:

$$\sigma_{BF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not B-Fredholm} \}.$$  

This class of operators, introduced and studied by Berkani et al. in a series of papers which extends the class of semi-Fredholm operators. In the beginning this class was defined by: An operator $T \in \mathcal{B}(X)$, is said to be B-Fredholm, if for some integer $n \geq 0$ the range $R(T^n)$ is closed and $T_n$, the restriction of $T$ to $R(T^n)$ is a Fredholm operator. $T$ is said to be a B-Weyl operator if $T_n$ is a Fredholm operator of index zero which is also equivalent to the fact that $T_1$ is a Weyl operator and $T_2$ is nilpotent. The B-Weyl spectrum defined by

$$\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl} \}.$$ 

Note that, Berkani gave the equivalence onto this two definitions of B-Fredholm operator, see [4, Theorem 2.7]. It is easily seen that every nilpotent operator, as well as any idempotent bounded operator, is B-Fredholm.

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl [6], [24]. Precisely, $T$ is a pseudo B-Fredholm operator if $T_1$ is a Fredholm operator and $T_2$ is a quasi-nilpotent operators. The pseudo B-Fredholm spectrum defined by

$$\sigma_{pBF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not pseudo B-Fredholm} \}.$$  

An operator $T$ is a pseudo B-Weyl operator if $T_1$ is a Weyl operator and $T_2$ is a quasi-nilpotent operator. The pseudo B-Weyl spectrum defined by

$$\sigma_{pBW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not pseudo B-Weyl} \}.$$  

$\sigma_{pBW}(T)$ and $\sigma_{pBF}(T)$ is not necessarily non empty. For example, the quasi nilpotent operator has empty pseudo B-Weyl and pseudo B-Fredholm spectrum. Evidently $\sigma_{pBF}(T) \subset \sigma_{pBW}(T) \subset \sigma(T)$.

$T$ is a pseudo-Fredholm operator (or admit generalized Kato decomposition) if $T_1$ is Kato operator and $T_2$ is quasi-nilpotent. The pseudo-Fredholm spectrum defined by

$$\sigma_{GK}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not a pseudo-Fredholm} \}.$$ 

Denote by $\rho_{GK}(T) : \rho_{GK}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is pseudo-Fredholm} \}$. If we assume in the definition above that $T_2$ is nilpotent, $T$ is said to be quasi-Fredholm (or admit a Kato decomposition or Kato type). The quasi-Fredholm spectrum defined by:

$$\sigma_{QK}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is quasi-Fredholm} \}.$$  

The Operators which admit a generalized Kato decomposition was originally introduced by M.Mbekhta [17] in the Hilbert spaces as a generalization of quasi-Fredholm operators have been introduced by J.P.Labrousse [14] and the semi-Fredholm operators.

In [4], Berkani showed that a B-Fredholm operator is a quasi-Fredholm, see [4 Proposition 4]. This result lead to ask the following question: If every pseudo B-Fredholm operator is a pseudo-Fredholm operator?
We organize our paper in the following way: In the next section we prove that every pseudo B-Fredholm operator is a pseudo-Fredholm operator. Also we study the relationships between the class of pseudo B-Fredholm and other class of operator. In section 3, we shall study the component of pseudo B-Fredholm resolvent \( \rho_{pBF}(T) \), to obtain a classification of the components by using the constancy of the subspaces quasi-nilpotent part and analytic core, some applications are also given. Finally, in section 4, we show that the symmetric difference \( \sigma_{K}(T) \Delta \sigma_{pBF}(T) \) is at most countable.

2. The class of Pseudo B-Fredholm Operators

In the following theorem we prove that every pseudo B-Fredholm operator is pseudo Fredholm.

**Theorem 2.1.** Let \( T \in B(X) \). If \( T \) is pseudo B-Fredholm, then \( T \) is pseudo Fredholm.

**Proof.** Let \( T \in B(X) \). If \( T \) is pseudo B-Fredholm operator, then there exists a subsets \( M \) and \( N \) of \( X \) such that \( X = M \oplus N \) and \( T = T_1 \oplus T_2 \) with \( T_1 = T|_M \) is a Fredholm operator and \( T_2 = T|_N \) is a quasi-nilpotent. Since \( T_1 \) is Fredholm then \( T \) admits a Kato decomposition, hence there exists \( M', M'' \) closed subsets of \( M \) such that \( M = M' \oplus M'' \), \( T_1 = T_1' \oplus T_1'' \) with \( T_1' = T|_{M'} \) is a Kato operator and \( T_1'' = T|_{M''} \) is nilpotent. Then \( X = M' \oplus M'' \oplus N \), and \( T = S \oplus R \) where \( S = T_1' \) is a Kato operator and \( R = T_1'' \oplus T_2 \) is a quasi-nilpotent operator, hence \( T \) is a pseudo Fredholm operator. \( \square \)

The following example shows that the class of pseudo B-Fredholm operator is a proper subclass of pseudo Fredholm operator.

**Example 1.** Consider the example given by Müller in [19]

Let \( H \) be the Hilbert space with an orthonormal basis \( \{e_{i,j}\} \), where \( i \) and \( j \) are integers such that \( ij \leq 0 \). Define operator \( T \in B(H) \) by:

\[
T e_{i,j} = \begin{cases} 
0 & \text{if } i = 0, j > 0 \\
e_{i+1,j} & \text{Otherwise}
\end{cases}
\]

We have \( N(T) = \bigvee_{j>0} \{e_{0,j}\} \subset R^\infty(T) \) and \( R(T) \) is closed, then \( T \) is a Kato operator but \( T \) is not a Fredholm operator, since \( \dim N(T) = \infty \).

Let \( Q \) a quasinilpotent operator in \( H \) which is not nilpotent and no commute with \( T \), then \( S = T \oplus Q \) is a pseudo Fredholm operator but is not pseudo B-Fredholm operator, hence the class of pseudo B-Fredholm operator is a proper subclass of pseudo Fredholm operator.

**Remark 1.** In [24 Remark 2.5] and [7 Proposition 1.2], If \( T \) is a bilateral shift on \( l^2(\mathbb{N}) \), we have :

1. \( T \) is pseudo B-Weyl if and only if \( T \) is Weyl or \( T \) is quasi-nilpotent operator.
2. \( T \) is pseudo Fredholm if and only if \( T \) is semi-regular or \( T \) is quasi-nilpotent operator.

By the same argument we can prove :

1. \( T \) is pseudo B-Fredholm if and only if \( T \) is Fredholm or \( T \) is quasi-nilpotent operator.
(2) $T$ is generalized Drazin if and only if $T$ is invertible or $T$ is quasi-nilpotent operator.

Corollary 2.1. Let $T \in B(X)$. Then
\[ \sigma_{\text{GR}}(T) \subset \sigma_{pBF}(T) \subset \sigma_{pBW}(T) \]

Lemma 2.1. \([16]\) Let $T \in B(X)$ and let $G$ a connected component of $\rho_K(T)$. Then
\[ G \cap \rho(T) \neq \emptyset \implies G \subset \rho(T) \]

Lemma 2.2. \([7]\) Let $T \in B(X)$. 
\[ \rho_{\text{GR}}(T) \setminus \rho_K(T) \text{ is at most countable} \]

Since $\rho_{pBF}(T) \setminus \rho_K(T) \subset \rho_{\text{GR}}(T) \setminus \rho_K(T)$, we can easily obtain that:

Corollary 2.2. Let $T \in B(X)$. 
\[ \rho_{pBF}(T) \setminus \rho_K(T) \text{ is at most countable.} \]

Proposition 2.1. Let $T \in B(X)$. Then the following statements are equivalent:

1. $\sigma_{pBF}(T)$ is at most countable.
2. $\sigma_{pBW}(T)$ is at most countable.
3. $\sigma(T)$ is at most countable.

Proof. 1) $\implies$ 3) Suppose that $\sigma_{pBF}(T)$ is at most countable then $\rho_{pBF}(T)$ is connexe, by corollary \([2.2]\) $\rho_{pBF}(T) \setminus \rho_K(T)$ is at most countable. Hence $\rho_K(T) = \rho_{pBF}(T) \setminus (\rho_{pBF}(T) \setminus \rho_K(T))$ is connexe. By lemma \([2.1]\) $\sigma(T) = \sigma_K(T)$. Therefore $\sigma(T) = \sigma_{pBF}(T) \cup (\rho_{pBF}(T) \setminus \rho_K(T))$ is at most countable.

3) $\implies$ 1) Obvious.

2) $\implies$ 3) If $\sigma_{pBW}(T)$ is at most countable then $\rho_{pBW}(T)$ is connexe, since every pseudo B-Weyl operator is a pseudo B-Fredholm operator by corollary \([2.2]\) $\rho_{pBW}(T) \setminus \rho_K(T)$ is at most countable. Hence $\rho_K(T) = \rho_{pBW}(T) \setminus (\rho_{pBW}(T) \setminus \rho_K(T))$ is connexe. By lemma \([2.1]\) $\sigma(T) = \sigma_K(T)$. Therefore $\sigma(T) = \sigma_{pBW}(T) \cup (\rho_{pBW}(T) \setminus \rho_K(T))$ is at most countable.

3) $\implies$ 2) Obvious. \[\square\]

Corollary 2.3. Let $T \in B(X)$, if $\sigma_{GR}(T)$ is at most countable. Then:

1. $T$ is a spectral operator if and only if $T$ is similar to a paranormal operator.

Proof. See \([18]\) Theorem 2.4 and Corollary 2.5 \[\square\]

Let $T \in B(X)$. The operator range topology on $R(T)$ is the topology induced by the norm $\|\cdot\|_T$ defined by $\|y\|_T := \inf_{x \in X} \{\|x\| : y = Tx\}$.

For a detailed discussion of operator ranges and their topology we refer the reader to \([21]\).

$T$ is said to have uniform descent for $n \geq d$ if $R(T^n) = R(T) + N(T^d)$ for $n \geq d$. If in addition, $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then $T$ is said to have topological uniform descent (TUD for brevity) for $n \geq d$. The topological uniform descent spectrum:

\[ \sigma_{\text{ud}}(T) = \{ \lambda \in \mathbb{C}, T - \lambda \text{ does not have TUD} \} \]

Let $T \in B(X)$, the ascent of $T$ is defined by $a(T) = \min\{ p \in \mathbb{N} : N(T^p) = N(T^{p+1}) \}$, if such $p$ does not exist we let $a(T) = \infty$. Analogously the descent of $T$ is $d(T) = \min\{ q \in \mathbb{N} : R(T^q) = R(T^{q+1}) \}$, if such $q$ does not exist we let $d(T) = \infty$ \([15]\). It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we
Corollary 2.4. Let corollary 2.1, [7, Theorem 3.3] and [11, corollary 3.4], we have the following:

For a pseudo B-Fredholm operator, these properties do not necessarily hold. Indeed:

Let \( T \) be the Banach space of continuous functions on \([0,1]\), denoted by \( C([0,1])\), provided with the infinity norm. We define by \( V \), the Volterra operator, \( X \) by:

\[
V f(x) := \int_0^d f(x) \, dx
\]

\( V \) is injective and quasi-nilpotent. In addition, \( N^\infty(V) = \{0\}, K(V) = \{0\} \) and we have \( R^\infty(V) = \{ f \in C^\infty[0,1] : f^{(n)}(0) = 0, \, n \in \mathbb{N} \} \), thus \( R^\infty(V) \) is not closed. Hence:
(1) \( K(V) \neq R^\infty(V) \)
(2) \( H_0(V) \neq N^\infty(V) \)

Note that \( V \) is a compact operator, then \( R(V) \) is not closed.

**Theorem 2.2.** There exists a pseudo B-Fredholm operator \( T \) such that:

1. \( K(T) \neq R^\infty(T) \)
2. \( H_0(T) \neq N^\infty(T) \)
3. \( R(T) \) is not closed.

**Proposition 2.2.** Let \( T \in \mathcal{B}(X) \). Then the following statements are equivalent

1. \( \sigma_{pBF}(T) \) is empty
2. \( \sigma_{pBW}(T) \) is empty
3. \( \sigma_{GK}(T) \) is empty
4. \( \sigma(T) \) is finite

**Proof.** 3 \( \iff \) 4 see [7] Theorem 3.3. 

1. \( \iff \) 4 If \( \sigma_{pBF}(T) \) is empty then \( \sigma(T) = \rho_{pBW}(T) \setminus \rho_K(T) \). By corollary 2.2 \( \rho_{pBW}(T) \setminus \rho_K(T) \) is at most countable and this set is bounded, hence it is finite.

4. \( \implies \) 1 Suppose that \( \sigma(T) \) is finite then for all \( \lambda \in \sigma(T) \) is isolated, then \( X = H_0(T - \lambda_0) \oplus K(T - \lambda_0) \), [24] Theorem 4 \( (T - \lambda_0)_K(T - \lambda_0) \) is quasi-nilpotent and \( (T - \lambda_0)_K(T - \lambda_0) \) is surjective, hence \( (T - \lambda_0)_K(T - \lambda_0) \) is Fredholm. Indeed, \( \lambda_0 \) is an isolated point, then \( T \) has the SVEP at \( \lambda_0 \), hence \( (T - \lambda_0)_K(T - \lambda_0) \) has the SVEP at 0 and it is surjective by [11] corollary 2.24 \( (T - \lambda_0)_K(T - \lambda_0) \) has bijective

2. \( \iff \) 4) similar to 1) \( \iff \) 4).

A bounded operator \( T \in \mathcal{B}(X) \) is said to be a Riesz operator if \( T - \lambda I \) is a Fredholm operator for every \( \lambda \in \mathbb{C} \setminus \{0\} \).

**Corollary 2.5.** Let \( T \in \mathcal{B}(X) \) a Riesz operator, then the following statements are equivalent

1. \( \sigma_{pBF}(T) \) is empty
2. \( \sigma_{pBW}(T) \) is empty
3. \( \sigma_{GK}(T) \) is empty
4. \( \sigma(T) \) is finite,
5. \( K(T) \) is closed,
6. \( K(T^*) \) is closed,
7. \( K(T) \) is finite-dimensional,
8. \( K(T - \lambda) \) is closed for all \( \lambda \in \mathbb{C} \),
9. \( \text{codim} H_0(T) < \infty \),
10. \( \text{codim} H_0(T^*) < \infty \),
11. \( T = Q + F, \) with \( Q, F \in \mathcal{B}(X), \) \( QF = FQ = 0, \) \( \sigma(Q) = \{0\} \) and \( F \) is a finite rank operator.

**Proof.** Direct consequence of Proposition 2.2 and [8] Theorem 2.3 and [20] Corollary 9.

In the following, we will prove that if \( T \) is with finite descent, then \( T \) is pseudo B-Fredholm if and only if \( T \) is a B-Fredholm operator.

**Proposition 2.3.** Let \( T \in \mathcal{B}(X) \) with finite descent. Then \( T \) is a pseudo B-Fredholm if and only if \( T \) is a B-Fredholm.
Proof. Obviously if \( T \) is B-Fredholm then \( T \) is pseudo B-Fredholm. If \( T \) is a pseudo B-Fredholm then \( T = T_1 \oplus T_2 \) with \( T_1 \) is Fredholm operator and \( T_2 \) is quasinilpotent. Since \( T \) has finite descent then \( T_1 \) and \( T_2 \) have finite descent, we have \( T_2 \) is quasinilpotent with finite descent implies that is a nilpotent operator. Thus \( T \) is a B-Fredholm operator.

In the following, we show that an operator with dense range is pseudo Fredholm if and only if it is a semi regular.

**Proposition 2.4.** Let \( T \in \mathcal{B}(X) \). If \( T \) is with dense range, then :
\[
T \text{ is a pseudo Fredholm if and only if } T \text{ is semi regular.}
\]

Proof. Every semi regular operator is a pseudo Fredholm. Conversely, if \( T \) admits a GKD, there exists a pair of \( T \)-invariant closed subspaces \((M, N)\) such that \( X = M \oplus N \), the restriction \( T_M \) is semi-regular, and \( T_N \) is quasinilpotent. \( T \) has dense range give \( \overline{R(T)} = X \Rightarrow N(T^*) = \{0\} \) then \( T^* \) have the SVEP at 0. According to [1, Theorem 3.15], we have \( K(T) = M \). Since \( M \) is closed and \( T_M \) is semi-regular (then \( M = X \)), therefore \( T \) is semi-regular.

**Corollary 2.6.** Let \( T \in \mathcal{B}(X) \), with dense range. Then :
\[
T \text{ is a pseudo B-Fredholm } \Rightarrow T \text{ is semi regular.}
\]
In particular, \( T \) is a generalized Drasin invertible \( \Rightarrow T \) is semi regular.

3. Classification Of The Components Of Pseudo B-Fredholm Resolvent

**Lemma 3.1.** Let \( T \in \mathcal{B}(X) \) a pseudo B-Fredholm, then there exists \( \varepsilon > 0 \) such that for all \( |\lambda| < \varepsilon \), we have:
\[
\begin{align*}
(1) \quad & K(T - \lambda) + H_0(T - \lambda) = K(T) + H_0(T). \\
(2) \quad & K(T - \lambda) \cap \overline{H_0(T - \lambda)} = K(T) \cap \overline{H_0(T)}. 
\end{align*}
\]

Proof. By Theorem 2.1, \( T \) is a pseudo Fredholm operator, hence we conclude by [7, Theorem 4.2] the result.

The pseudo B-Fredholm resolvent set is defined as \( \rho_{pBF}(T) = \mathbb{C} \setminus \sigma_{pBF}(T) \).

**Corollary 3.1.** Let \( T \in \mathcal{B}(X) \) a pseudo B-Fredholm operator, then the mappings
\[
\lambda \mapsto K(T - \lambda) + H_0(T - \lambda), \quad \lambda \mapsto K(T - \lambda) \cap \overline{H_0(T - \lambda)}
\]
are constant on the components of \( \rho_{pBF}(T) \).

**Lemma 3.2.** Let \( T \) a pseudo B-Fredholm operator. Then the following statements are equivalent:
\[
\begin{align*}
(1) & \text{ } T \text{ has the SVEP at } 0, \\
(2) & \text{ } \sigma_{sp}(T) \text{ does not a cluster at } 0.
\end{align*}
\]

Proof. Without loss of generality, we can assume that \( \lambda_0 = 0 \).

1) \( \Rightarrow \) 2) Suppose that \( T \) is a pseudo B-Fredholm operator, then there exists two closed \( T \)-invariant subspaces \( X_1, X_2 \subset X \) such that \( X = X_1 \oplus X_2 \), \( T_{X_1} \) is Fredholm, \( T_{X_2} \) is quasinilpotent and \( T = T_{X_1} \oplus T_{X_2} \). Since \( T_{X_1} \) is Fredholm, then \( T_{X_1} \) is of Kato type by [2, Theorem 2.2] there exists a constant \( \varepsilon > 0 \) such that for all \( \lambda \in D^*(0, \varepsilon) \), \( \lambda I - T \) is bounded below. Since \( T_{X_2} \) is quasinilpotent, \( \lambda I - T \) is
bounded below for all $\lambda \neq 0$. Hence $\lambda I - T$ is bounded below for all $\lambda \in D^*(0, \varepsilon)$. Therefore $\sigma_{ap}(T)$ does not cluster at $\lambda_0$. 

By duality we have:

**Lemma 3.3.** Let $T$ a pseudo B-Fredholm operator. Then the following statements are equivalent:

1) $T^*$ has the SVEP at 0,
2) $\sigma_{su}(T)$ does not a cluster at 0.

**Theorem 3.1.** Let $T \in B(X)$ and $\Omega$ a component of $\rho_{BF}(T)$. Then the following alternative holds:

1) $T$ has the SVEP for every point of $\Omega$. In this case, $\sigma_{ap}(T)$ does not have limit points in $\Omega$, every point of $\Omega$ is not an eigenvalue of $T$ except a subset of $\Omega$ which consists of at most countably many isolated points.
2) $T$ has the SVEP at no point of $\Omega$. In this case, every point of $\Omega$ is an eigenvalue of $T$.

**Proof.** 1) Assume that $T$ has the SVEP at $\lambda_0 \in \Omega$. By [1, Theorem 3.14] we have $K(T - \lambda_0) \cap H_0(T - \lambda_0) = K(T - \lambda_0) \cap H_0(T - \lambda_0) = \{0\}$. According to corollary 3.1 we have $K(T - \lambda_0) \cap H_0(T - \lambda_0) = \{0\} = K(T - \lambda) \cap H_0(T - \lambda) = \{0\}$ for all $\lambda \in \Omega$. Hence $K(T - \lambda) + H_0(T - \lambda) = \{0\}$ and therefore $T$ has the SVEP at every $\lambda \in \Omega$. By Lemma 3.2 $\sigma_{ap}(T)$ does not cluster at any $\lambda \in \Omega$. Consequently every point of $\Omega$ is not an eigenvalue of $T$ except a subset of $\Omega$ which consists of at most countably many isolated points.

2) Suppose that $T$ has the SVEP at not point of $\Omega$. From [1, Theorem 2.22], we have $N(T - \lambda) \neq \{0\}$, for all $\lambda \in \Omega$, hence every point of $\Omega$ is an eigenvalue of $T$. 

**Theorem 3.2.** Let $T \in B(X)$ and $\Omega$ a component of $\rho_{BF}(T)$. Then the following alternative holds:

1) $T^*$ has the SVEP for every point of $\Omega$. In this case, $\sigma_{su}(T)$ does not have limit points in $\Omega$, every point of $\Omega$ is not a deficiency value of $T$ except a subset of $\Omega$ which consists of at most countably many isolated points.
2) $T^*$ has the SVEP at no point of $\Omega$. In this case, every point of $\Omega$ is a deficiency value of $T$.

**Proof.** 1) Assume that $T^*$ has the SVEP at $\lambda_0 \in \Omega$, by [1, Theorem 3.15] we have $K(T - \lambda_0) + H_0(T - \lambda_0) = X$. According to corollary 3.1 we have $K(T - \lambda_0) + H_0(T - \lambda_0) = K(T - \lambda) + H_0(T - \lambda) = X$ for all $\lambda \in \Omega$. Hence $K(T - \lambda) + H_0(T - \lambda) = X$ and therefore $T$ has the SVEP at every $\lambda \in \Omega$. By lemma 3.2 $\sigma_{su}(T)$ does not cluster at any $\lambda \in \Omega$. Consequently every point of $\Omega$ is not a deficiency value of $T$ except a subset of $\Omega$ which consists of at most countably many isolated points.

2) Suppose that $T^*$ has the SVEP at no point of $\Omega$. Assume that there exists a $\lambda_0 \in \Omega$ such that $T - \lambda$ is surjective, then $T^* - \lambda_0$ is injective this implies that $T^*$ has the SVEP at $\lambda_0$. Contraduction and hence every point of $\Omega$ is a deficiency value of $T$. 

□
Remark 2. We have $\sigma_{pBF}(T) \subset \sigma_{gD}(T)$, this inclusion is proper. Indeed: Consider the operator $T$ defined in $l^2(\mathbb{N})$ by

$$T(x_1, x_2, ...) = (0, x_1, x_2, ...), \quad T^*(x_1, x_2, ...) = (x_2, x_3, ...).$$

Let $B = T \oplus T^*$. Then $\sigma_{gD}(T) = \{ \lambda \in \mathbb{C}; |\lambda| \leq 1 \}$ and we have $0 \notin \sigma_{pBF}(T)$. This shows that the inclusion $\sigma_{pBF}(T) \subset \sigma_{gD}(T)$ is proper.

Next we obtain a condition on an operator such that its pseudo B-Fredholm spectrum coincide with the generalized Drazin spectrum.

Theorem 3.3. Suppose that $T \in B(X)$ and $\rho_{pBF}(T)$ has only one component. Then

$$\sigma_{pBF}(T) = \sigma_{gD}(T)$$

Proof. $\rho_{pBF}(T)$ has only one component, then $\rho_{pBF}(T)$ is the unique component. Since $T$ has the SVEP on $\rho(T) \subset \rho_{pBF}(T)$. By Theorem 3.1, $T$ has the SVEP on $\rho_{pBF}(T)$. Similar $T^*$ also has the SVEP on $\rho_{pBF}(T)$ by Theorem 3.2. This since $\rho(T^*) = \rho(T) \subset \rho_{pBF}(T))$. From Lemma 3.2 and Lemma 3.3 $\sigma(T)$ does not cluster at any $\lambda \in \rho_{pBF}(T)$. Therefor $\rho_{pBF}(T) \subset isos(T) \cup \rho(T) = \rho_{gD}(T)$, hence $\rho_{pBF}(T) = \rho_{gD}(T)$.

4. Symmetric difference for pseudo B-Fredholm spectrum

Let in the following we give symmetric difference between $\sigma_{pBF}(T)$ and other parts of the spectrum. Denoted by $\rho_{fK}(T) = \{ \lambda \in \mathbb{C}; K(T - \lambda) \text{ is not closed} \}, \sigma_{fK}(T) = \mathbb{C} \setminus \rho_{fK}(T)$ and $\rho_{cr}(T) = \{ \lambda \in \mathbb{C}; R(T - \lambda) \text{ is closed} \}, \sigma_{cr}(T) = \mathbb{C} \setminus \rho_{cr}(T)$ the Goldberg spectrum. Most of the classes of operators, for example, in Fredholm theory, require that the operators have closed ranges. Thus, it is natural to consider the closed-range spectrum or Goldberg spectrum of an operator.

Proposition 4.1. If $\lambda \in \sigma_*(T)$ is non-isolated point then $\lambda \in \sigma_{pBF}(T)$, where $* \in \{fK, cr\}$.

Proof. Let $\lambda \in \sigma_*(T)$ an isolated point. Suppose that $T - \lambda$ is a pseudo B-Fredholm, by Lemma 2.4 there exists a constant $\varepsilon > 0$ such that for all $\lambda \in D^*(\lambda, \varepsilon)$, $\lambda - \lambda$ is semi regular. Then $R(T - \mu)$ and $K(T - \mu)$ are closed for all $\mu \in D^*(\lambda, \varepsilon)$, then $\lambda$ is an isolated point of $\sigma_*(T)$, contradiction.

Corollary 4.1. $\sigma_*(T) \setminus \sigma_{pBF}(T)$ is at most countable, where $* \in \{fK, cr\}$.

Proposition 4.2. Let $T \in B(X)$ such that $\sigma_{cr}(T) = \sigma(T)$ and every $\lambda$ is non-isolated in $\sigma(T)$. Then

$$\sigma(T) = \sigma_{cr}(T) = \sigma_{pBF}(T) = \sigma_{pBW}(T) = \sigma_*(T) = \sigma_{K}(T) = \sigma_{ap}(T)$$

Proof. Since every $\lambda \in \sigma(T) = \sigma_{cr}(T)$ is non-isolated then by Proposition 4.1 we have $\sigma(T) = \sigma_{cr}(T) \subseteq \sigma_{pBF}(T) \subseteq \sigma_{pBW}(T) \subseteq \sigma_*(T) \subseteq \sigma(T)$ and since $\sigma(T) = \sigma_{cr}(T) \subseteq \sigma_{K}(T) \subseteq \sigma_{ap}(T) \subseteq \sigma(T)$, we deduce the statement of the theorem.

Proposition 4.3. The symmetric difference $\sigma_{K}(T) \Delta \sigma_{pBF}(T)$ is at most countable.

Proof. By corollary 2.2 $\sigma_{K}(T) \setminus \sigma_{pBF}(T)$ is at most countable. We have $\sigma_*(T) \setminus \sigma_{K}(T)$ consists of at most countably many isolated points (see [1] Theorem 1.65) and $\sigma_{pBF}(T) \setminus \sigma_{K}(T) \subseteq \sigma_{cr}(T) \setminus \sigma_{K}(T)$, hence $\sigma_{pBF}(T) \setminus \sigma_{K}(T)$ is at most countable.
Since

\[ \sigma_K(T) \Delta \sigma_{pBF}(T) = (\sigma_K(T) \setminus \sigma_{pBF}(T)) \bigcup (\sigma_{pBF}(T) \setminus \sigma_K(T)) \]

Therefore \( \sigma_K(T) \Delta \sigma_{pBF}(T) \) is at most countable. \[\square\]

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