Representations of certain Banach algebras

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Abstract. For a space $X$ denote by $C_b(X)$ the Banach algebra of all continuous bounded scalar-valued functions on $X$ and denote by $C_0(X)$ the set of all elements in $C_b(X)$ which vanish at infinity.

We prove that certain Banach subalgebras $H$ of $C_b(X)$ are isometrically isomorphic to $C_0(Y)$, for some unique (up to homeomorphism) locally compact Hausdorff space $Y$. The space $Y$ is explicitly constructed as a subspace of the Stone–Čech compactification of $X$. The known construction of $Y$ enables us to examine certain properties of either $H$ or $Y$ and derive results not expected to be deducible from the standard Gelfand theory.

1. Introduction

By a space we mean a topological space; completely regular spaces are Hausdorff. Throughout this article the underlying field of scalars (which is fixed throughout each discussion) is assumed to be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, unless specifically stated otherwise.

Let $X$ be a completely regular space. We denote by $C_b(X)$ the set of all continuous bounded scalar-valued functions on $X$. If $f \in C_b(X)$, the zero-set of $f$, denoted by $Z(f)$, is $f^{-1}(0)$, the cozero-set of $f$, denoted by $Coz(f)$, is $X \setminus Z(f)$, and the support of $f$, denoted by $\text{supp}(f)$, is $\text{cl}_X Coz(f)$. Let

$$Coz(X) = \{Coz(f) : f \in C_b(X)\}.$$ 

The elements of $Coz(X)$ are called cozero-sets of $X$. Denote by $C_0(X)$ the set of all $f \in C_b(X)$ which vanish at infinity (that is, $|f|^{-1}([\epsilon, \infty))$ is compact for each $\epsilon > 0$) and denote by $C_{00}(X)$ the set of all $f \in C_b(X)$ with compact support.

This work is a continuation (though it is self-contained) of our previous work [12] (and [15]; see also the follow-up preprint [16]) in which we have studied the Banach algebra of continuous bounded scalar-valued functions with separable support defined on a locally separable metrizable space $X$. Indeed, this article is an outgrowth of author’s unsuccessful (partly successful, however) attempt to give an alternative proof of the celebrated commutative Gelfand–Naimark Theorem. We show that certain Banach subalgebras $H$ of $C_b(X)$ are representable as $C_0(Y)$ for some unique locally compact Hausdorff space $Y$. We construct $Y$ explicitly as a subspace of the Stone–Čech compactification of $X$. The known construction of $Y$ enables us to examine certain properties of either $H$ or $Y$ and derive results not expected to be deducible from the standard Gelfand theory.



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We now briefly review certain facts from General Topology. Additional information may be found in [5] and [6].

1.1. The Stone–Čech compactification. Let $X$ be a completely regular space. By a compactification $\gamma X$ of $X$ we mean a compact Hausdorff space $\gamma X$ containing $X$ as a dense subspace. The Stone–Čech compactification $\beta X$ of $X$ is the compactification of $X$ which is characterized among all compactifications of $X$ by the following property: Every continuous $f : X \to K$, where $K$ is a compact Hausdorff space, is continuously extendable over $\beta X$; denote by $f_\beta$ this continuous extension of $f$. The Stone–Čech compactification of a completely regular space always exists. In what follows use will be made of the following properties of $\beta X$.

- $X$ is locally compact if and only if $X$ is open in $\beta X$.
- Any open-closed subspace of $X$ has open-closed closure in $\beta X$.
- If $X \subseteq T \subseteq \beta X$ then $\beta T = \beta X$.
- If $X$ is normal then $\beta T = \text{cl}_{\beta X} T$ for any closed subspace $T$ of $X$.

1.2. Locally-$P$ spaces. Let $P$ be a topological property. A space $X$ is called locally-$P$, if each $x \in X$ has an open neighborhood $U$ in $X$ whose closure $\text{cl}_X U$ has $P$.

1.3. Metrizable spaces and separability. The density of a space $X$, denoted by $d(X)$, is defined by

$$d(X) = \min \{|D| : D \text{ is dense in } X\} + \aleph_0.$$ 

In particular, a space $X$ is separable if and only if $d(X) = \aleph_0$. Note that in any metrizable space the three notions of separability, being Lindelöf, and second countability coincide; thus any subspace of a separable metrizable space is separable. By a theorem of Alexandroff, any locally separable metrizable space $X$ can be represented as a disjoint union

$$X = \bigcup_{i \in I} X_i,$$

where $I$ is an index set, and $X_i$ is a non-empty separable open-closed subspace of $X$ for each $i \in I$. (See Problem 4.4.F of [5].) Note that $d(X) = |I|$ if $I$ is infinite.

1.4. Paracompact spaces and the Lindelöf property. Let $X$ a regular space. For any open covers $\mathcal{U}$ and $\mathcal{V}$ of $X$ we say that $\mathcal{U}$ is a refinement of $\mathcal{V}$ if every element of $\mathcal{U}$ is contained in an element of $\mathcal{V}$. An open cover $\mathcal{U}$ of $X$ is called locally finite if each point of $X$ has an open neighborhood in $X$ intersecting only a finite number of the elements of $\mathcal{U}$. The space $X$ is called paracompact if for every open cover $\mathcal{U}$ of $X$ there is an open cover of $X$ which refines $\mathcal{U}$. Every metrizable space is paracompact and every paracompact space is normal. The Lindelöf number of $X$, denoted by $\ell(X)$, is defined by

$$\ell(X) = \min \{n : \text{any open cover of } X \text{ has a subcover of cardinality } \leq n\} + \aleph_0.$$ 

In particular, $X$ is Lindelöf if and only if $\ell(X) = \aleph_0$. Any locally compact paracompact space $X$ can be represented as a disjoint union

$$X = \bigcup_{i \in I} X_i,$$

where $I$ is an index set, and $X_i$ is a non-empty Lindelöf open-closed subspace of $X$ for each $i \in I$. (See Theorem 5.1.27 of [5].) Note that $\ell(X) = |I|$ if $I$ is infinite.
2. The representation theorem

The subspace $\lambda_{H}X$ of $\beta X$ defined below plays a crucial role in our study.

**Definition 2.1.** Let $X$ be a completely regular space and let $H \subseteq C_{b}(X)$. Define

$$\lambda_{H}X = \bigcup \{ \text{int}_{\beta X} \text{cl}_{\beta X} \text{Coz}(h) : h \in H \}.$$

The above definition of $\lambda_{H}X$ is motivated by the definition of $\lambda_{P}X$ (here $P$ is a topological property) as given in [9] (also, in [10], [11], [13] and [14]). Note that $\lambda_{H}X$ is open in $\beta X$ and is thus locally compact.

Recall that if $X$ is a space and $D$ is a dense subspace of $X$, then $\text{cl}_{X}U = \text{cl}_{X}(U \cap D)$ for any open subspace $U$ of $X$.

**Lemma 2.2.** Let $X$ be a completely regular space and let $H \subseteq C_{b}(X)$ such that

($\ast$) For any $x \in X$ there is some $h \in H$ with $h(x) \neq 0$.

Then

$$X \subseteq \lambda_{H}X.$$

**Proof.** Let $x \in X$. By ($\ast$) there is some $h \in H$ with $h(x) \neq 0$. Since

$$\text{Coz}(h_{\beta}) \subseteq \text{cl}_{\beta X} \text{Coz}(h_{\beta}) = \text{cl}_{\beta X} (X \cap \text{Coz}(h_{\beta})) = \text{cl}_{\beta X} \text{Coz}(h)$$

we have

$$x \in \text{Coz}(h_{\beta}) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} \text{Coz}(h) \subseteq \lambda_{H}X.$$

A version of the classical Banach–Stone Theorem states that if $X$ and $Y$ are locally compact Hausdorff spaces, the Banach algebras (or even the rings; see [3]) $C_{0}(X)$ and $C_{0}(Y)$ are isometrically isomorphic if and only if the spaces $X$ and $Y$ are homeomorphic (see Theorem 7.1 of [4]); this will be used in the proof of the following theorem.

**Theorem 2.3.** Let $X$ be a completely regular space. Let $H$ be a Banach subalgebra of $C_{b}(X)$ such that

1. For any $x \in X$ there is some $h \in H$ with $h(x) \neq 0$.
2. For any $f \in C_{b}(X)$, if $\text{supp}(f) \subseteq \text{supp}(h)$ for some $h \in H$, then $f \in H$.

Then $H$ is isometrically isomorphic to $C_{0}(Y)$ for some unique locally compact Hausdorff space $Y$, namely $Y = \lambda_{H}X$. Furthermore, the following are equivalent:

(a) $H$ is unital.
(b) $H$ contains 1.
(c) $Y$ is compact.
(d) $Y = \beta X$.

**Proof.** For any $f \in C_{b}(X)$ denote

$$f_{H} = f_{\beta} \mid \lambda_{H}X.$$

By Lemma 2.2 we know that $f_{H}$ extends $f$. Note that $\text{supp}(|f|) = \text{supp}(f)$ for any $f \in C_{b}(X)$; thus by (2) we have $|f| \in H$ if $f \in H$.

We divide the proof into verification of several claims.

**Claim.** For any $f \in C_{b}(X)$ the following are equivalent:
(i) $f \in H$.
(ii) $f_H \in C_0(\lambda_H X)$.

Proof of the claim. (i) implies (ii). Note that $\text{Coz}(f_\beta) \subseteq \lambda_H X$, as
\[
\text{int}_\beta \text{cl}_X \text{Coz}(f) \subseteq \lambda_H X
\]
and
\[
\text{Coz}(f_\beta) \subseteq \text{int}_\beta \text{cl}_X \text{Coz}(f),
\]
since $\text{Coz}(f_\beta) \subseteq \text{cl}_X \text{Coz}(f_\beta) = \text{cl}_X (X \cap \text{Coz}(f_\beta)) = \text{cl}_X \text{Coz}(f)$.

If $\epsilon > 0$ then
\[
|f_H|^{-1}((\epsilon, \infty)) = |f_\beta|^{-1}((\epsilon, \infty))
\]
is closed in $\beta X$ and is therefore compact.

(ii) implies (i). Let $n$ be a positive integer. Since $|f_H|^{-1}([1/n, \infty))$ is a compact subspace of $\lambda_H X$, we have
\[
\left| |f_H|^{-1}([1/n, \infty)) \right| \subseteq \bigcup_{i=1}^{k_n} \text{int}_\beta \text{cl}_X \text{Coz}(h_i)
\]
\[
\subseteq \bigcup_{i=1}^{k_n} \text{cl}_X \text{Coz}(h_i)
\]
\[
= \text{cl}_X \left( \bigcup_{i=1}^{k_n} \text{Coz}(h_i) \right)
\]
\[
= \text{cl}_X \left( \bigcup_{i=1}^{k_n} \text{Coz}(|h_i|) \right) = \text{cl}_X \text{Coz} \left( \sum_{i=1}^{k_n} |h_i| \right)
\]
for some $h_1, \ldots, h_{k_n} \in H$. Let
\[
g_n = |h_1| + \cdots + |h_{k_n}| \in H.
\]
We have
\[
|f|^{-1}([1/n, \infty)) = X \cap |f_H|^{-1}([1/n, \infty))
\]
\[
\subseteq X \cap \text{cl}_X \text{Coz}(g_n) = \text{cl}_X \text{Coz}(g_n) = \text{supp}(g_n).
\]
Let
\[
g = \sum_{n=1}^{\infty} 2^{-n} \frac{g_n}{\|g_n\|}.
\]
(We may assume that $g_n \not= 0$ for each positive integer $n$.) Then $g \in H$. Since
\[
\text{Coz}(f) = \bigcup_{n=1}^{\infty} |f|^{-1}([1/n, \infty)) \subseteq \bigcup_{n=1}^{\infty} \text{supp}(g_n) \subseteq \text{supp}(g)
\]
we have $\text{supp}(f) \subseteq \text{supp}(g)$, which by (2) implies that $f \in H$. This proves the claim.

Claim. Let
\[
\psi : H \to C_0(\lambda_H X)
\]
be defined by $\psi(h) = h_H$ for any $h \in H$. Then $\psi$ is an isometric isomorphism.
Proof of the claim. By the first claim the function ψ is well-defined. It is clear that ψ is a homomorphism and that ψ is injective. (Note that X ⊆ λH X by Lemma 2.2 and that any two scalar-valued continuous functions on λH X coincide, provided that they agree on X.) To show that ψ is surjective, let g ∈ C0(λH X). Then (g|X)H = g and thus g|X ∈ H by the first claim. Now ψ(g|X) = g. Finally, to show that ψ is an isometry, let h ∈ H. Then

\[ |h|H |(λH X) = |hh|H |(cl_λh X) ⊆ cl_λ |h|H |(X) = cl_λ |h|H |(X) ≤ [0, ||h||] \]

which yields ||hH|| ≤ ||h||. That ||h|| ≤ ||hH|| is clear, as hH extends h. This proves the claim.

The uniqueness part of the theorem follows from the Banach–Stone Theorem. (Note that λH X is a locally compact Hausdorff space.)

We now verify the final assertion of the theorem. Suppose that H is unital with the unit element u. For each x ∈ H let h_x ∈ H such that h_x(x) ≠ 0 (h_x exists by (1)). Then u(x)h_x(x) = h_x(x) which yields u(x) = 1. That is u = 1. Note that Coz(u) = X. Now, by the way λH X is defined we have λH X = βX. For the converse, note that if Y is compact then C0(Y) = Cb(Y). Thus H is unital, as it is isometrically isomorphic to C0(Y) and the latter is so. □

Remark 2.4. In Theorem 2.3 the existence of the locally compact space Y may also be deduced from the commutative Gelfand–Naimark Theorem in which Y would be the structure space or the maximal ideal space of H. (See [7].) The point in Theorem 2.3 is that we construct Y explicitly as a subspace of the Stone–Čech compactification of X. The known construction of Y will then enable us to derive certain of its properties. This will be illustrated in Section 3 where we consider special cases of the Banach subalgebra H.

Remark 2.5. Condition (2) in Theorem 2.3 can be replaced by the following rather more abstract condition.

(2)′ For any f ∈ Cb(X), if ann(h) ⊆ ann(f) for some h ∈ H, then f ∈ H.

Here ann(f) = \{g ∈ Cb(X) : fg = 0\}

for any f ∈ Cb(X).

To show that (2) (in Theorem 2.3) implies (2)′, let f ∈ Cb(X) such that ann(h) ⊆ ann(f) for some h ∈ H. Suppose to the contrary that supp(f) ⊈ supp(h). Let x ∈ supp(f) with x ∉ supp(h). There exists a continuous k : X → [0, 1] such that

k(x) = 1 and k|supp(h) = 0.

Now k−1((1/2, 1]) is an open neighborhood of x in X. Therefore

I = k−1((1/2, 1]) ∩ Coz(f) ≠ ∅.

If we let t ∈ I then k(t)f(t) ≠ 0. But k ∈ ann(h), as kh = 0 by the way we have defined k, and therefore k ∈ ann(f), which is a contradiction. Thus supp(f) ⊆ supp(h). By (2), this implies that f ∈ H.

Next, we verify that (2)′ implies (2). Let f ∈ Cb(X) such that supp(f) ⊆ supp(h) for some h ∈ H. Suppose to the contrary that ann(h) ⊈ ann(f). Let g ∈ ann(h) with g ∉ ann(f). Then gf ≠ 0. Let t ∈ X such that g(t)f(t) ≠ 0. Then
$r = |g(t)| > 0$. Note that $t \in \text{supp}(f)$, as $f(t) \neq 0$, and thus $t \in \text{supp}(h)$. Now $|g|^{-1}((r/2, \infty))$ is an open neighborhood of $t$ in $X$. Therefore

$$E = |g|^{-1}\left(\left(\frac{r}{2}, \infty\right)\right) \cap \text{Coz}(h) \neq \emptyset.$$ 

But if $x \in E$ then $g(x)h(x) \neq 0$, which is a contradiction, as $gh = 0$ by the choice of $g$. Thus $\text{ann}(h) \subseteq \text{ann}(f)$. By (2)', this implies that $f \in H$.

**Remark 2.6.** In Theorem 2.3 observe that the space $Y$ is compact if and only if $X = \text{supp}(h)$ for some $h \in H$. To show this, suppose that $\lambda_H X$ is compact. Then

$$\lambda_H X = \bigcup_{i=1}^{k_n} \text{int}_{\beta X} \text{cl}_{\beta X} \text{Coz}(h_i) \subseteq \bigcup_{i=1}^{k_n} \text{cl}_{\beta X} \text{Coz}(h_i)$$

for some $h_1, \ldots, h_{k_n} \in H$. Note that $X \subseteq \lambda_H X$ by Lemma 2.2. Now, if we intersect both sides of (2.1) with $X$ we obtain

$$X = X \cap \lambda_H X \subseteq \bigcup_{i=1}^{k_n} (X \cap \text{cl}_{\beta X} \text{Coz}(h_i)) = \bigcup_{i=1}^{k_n} \text{cl}_X \text{Coz}(h_i) = \bigcup_{i=1}^{k_n} \text{cl}_X \text{Coz}(|h_i|)$$

and thus

$$X = \bigcup_{i=1}^{k_n} \text{cl}_X \text{Coz}(|h_i|) = \text{cl}_X \left(\bigcup_{i=1}^{k_n} \text{Coz}(|h_i|)\right) = \text{cl}_X \text{Coz}\left(\sum_{i=1}^{k_n} |h_i|\right).$$

That is

$$X = \text{cl}_X \text{Coz}(h) = \text{supp}(h),$$

where

$$h = |h_1| + \cdots + |h_{k_n}| \in H.$$

The converse is trivial, as if $X = \text{supp}(h)$ for some $h \in H$, then $\lambda_H X = \beta X$ by the definition of $\lambda_H X$.

**Remark 2.7.** In Theorem 2.3 it is natural to ask what would happen if we start with $H = C_0(X)$ from the outset, with $X$ a locally compact Hausdorff space. Given that $x \in X$, there exists an open neighborhood $U_x$ of $x$ in $X$ with compact closure $\text{cl}_X U_x$. Since $X$ is completely regular, there is a continuous $h_x : X \to \mathbb{R}$ such that

$$h_x(x) = 1 \quad \text{and} \quad h_x|(X \setminus U_x) = 0.$$ 

Then $\text{supp}(h_x)$ is compact, as $\text{supp}(h_x) \subseteq \text{cl}_X U_x$ (and $\text{cl}_X U_x$ is compact). Thus in particular $h_x \in C_0(X)$. In other words, if $H = C_0(X)$, then condition (1) in Theorem 2.3 holds, though, as we will see now, condition (2) therein may fail.

Let $X = \mathbb{R}$. Define $h : X \to \mathbb{R}$ by

$$h(x) = \frac{1}{1 + x^2}$$

for any $x \in X$. Then obviously $h \in C_0(X)$. Now, if we let $f = 1$ then $\text{supp}(f) \subseteq \text{supp}(h)$ while $f \notin C_0(X)$.

More generally, we show that $C_0(X)$ fails to satisfy condition (2) in Theorem 2.3 for any non-compact locally compact $\sigma$-compact Hausdorff space $X$. Let $X$ be a non-compact locally compact $\sigma$-compact Hausdorff space. Then, there exists a sequence $X_1, X_2, \ldots$ of compact subspaces of $X$ such that

$$X = \bigcup_{n=1}^{\infty} X_n,$$
and $X_n \subseteq \text{int}_X X_{n+1}$ for each positive integer $n$. (See Exercise 3.8.C of [5].) By complete regularity of $X$, for each positive integer $n$ there exists a continuous $h_n : X \to [0,1]$ such that

$$h_n|X_n = 1 \quad \text{and} \quad h_n|(X\setminus \text{int}_X X_{n+1}) = 0.$$ 

Let

$$h = \sum_{n=1}^{\infty} \frac{h_n}{2^n}.$$ 

Then $h : X \to [0,1]$ is continuous. Fix some positive integer $n$. Let $x \in X \setminus X_{n+2}$. Then $x \notin X_i$ if $1 < i \leq n+2$ and thus $h_{i-1}(x) = 0$ (by the definition of $h_{i-1}$). Therefore

$$h(x) = \sum_{i=n+2}^{\infty} \frac{h_i(x)}{2^i} \leq \sum_{i=n+2}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} < \frac{1}{2^n}.$$ 

That is

$$E = h^{-1}\left([\frac{1}{2^n}, \infty)\right) \subseteq X_{n+2}.$$ 

Since $X_{n+2}$ is compact, so is its closed subspace $E$. This shows that $h \in C_0(X)$. Note that supp($h$) = $X$ by (2.2) (and the definitions of $h$ and $h_n$’s). Now, as argued in the case when $X = \mathbb{R}$, it follows that $C_0(X)$ does not satisfy condition (2) in Theorem 2.3.

As a concluding remark, observe that if $C_0(X)$ satisfies the assumption of Theorem 2.3 where $X$ is a locally compact Hausdorff space, then $C_0(X)$ is isometrically isomorphic to $C_0(\lambda H X)$ by the theorem. Now (a version of) the Banach–Stone Theorem implies that $X$ and $\lambda H X$ are homeomorphic spaces.

3. Examples

In this section we give examples of spaces $X$ and Banach subalgebras $H$ of $C_b(X)$ for which Theorem 2.3 is applicable.

Recall that a topological property $\mathcal{P}$ is called hereditary with respect to closed subspaces, if each closed subspace of a space with $\mathcal{P}$ has $\mathcal{P}$. We will always assume that a topological property is non-empty, that is, for a given topological property $\mathcal{P}$ there always exists a space with $\mathcal{P}$. This in particular implies that $\emptyset$ has $\mathcal{P}$ for any topological property $\mathcal{P}$ which is hereditary with respect to closed subspaces.

**Theorem 3.1.** Let $\mathcal{Q}$ be a topological property hereditary with respect to closed subspaces. Let $\mathcal{P}$ be a topological property such that

1. $\mathcal{P}$ is hereditary with respect to closed subspaces of spaces with $\mathcal{Q}$.
2. Any space with $\mathcal{Q}$ containing a dense subspace with $\mathcal{P}$ has $\mathcal{P}$.
3. Any space which is a countable union of its closed subspaces each with $\mathcal{P}$ has $\mathcal{P}$.

Let $X$ be a completely regular locally-$\mathcal{P}$ space with $\mathcal{Q}$. Let

$$H = \{ f \in C_b(X) : \text{supp}(f) \text{ has } \mathcal{P} \}.$$ 

Then $H$ is a Banach algebra isometrically isomorphic to $C_0(Y)$ for some unique locally compact Hausdorff space $Y$, namely $Y = \lambda H X$. Furthermore, the following are equivalent:

(a) $X$ has $\mathcal{P}$.
(b) $H$ is unital.
(c) $H$ contains $1$.
(d) $Y$ is compact.
(e) $Y = \beta X$.

**Proof.** We verify that $H$ satisfies the assumption of Theorem 2.3. First, we need to show that $H$ is a subalgebra of $C_b(X)$. Note that $0 \in H$, as $\emptyset$ has $\mathcal{P}$ by (1). Let $f, g \in H$. Then $\text{supp}(f) \cup \text{supp}(g)$ has $\mathcal{Q}$, as it is closed in $X$, and has $\mathcal{P}$, as it is the union of two of its closed subspaces each with $\mathcal{P}$; see (3). Since

$$\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g)$$

it then follows that $\text{supp}(f + g)$ has $\mathcal{P}$, by (1). Analogously, $fg \in H$ and $cf \in H$ for any scalar $c$. (Note that $\text{supp}(fg) \subseteq \text{supp}(f)$.)

Next, we show that $H$ is closed in $C_b(X)$. Let $h_1, h_2, \ldots$ be a sequence in $H$ converging to some $f \in C_b(X)$. Note that $E = \text{cl}_X \left( \bigcup_{n=1}^{\infty} \text{supp}(h_n) \right)$ has $\mathcal{Q}$, as it is closed in $X$, and thus by (2) has $\mathcal{P}$, as it contains $\bigcup_{n=1}^{\infty} \text{supp}(h_n)$ as a dense subspace, and by (3) the latter has $\mathcal{P}$, as it is a countable union of its closed subspaces each with $\mathcal{P}$. Note that $\text{Coz}(f) \subseteq \bigcup_{n=1}^{\infty} \text{Coz}(h_n)$.

Therefore $\text{supp}(f)$ has $\mathcal{P}$ by (1), as it is closed in $E$. That is $f \in H$. This shows that $H$ is a Banach subalgebra of $C_b(X)$.

Next, we verify that $H$ satisfies conditions (1) and (2) of Theorem 2.3.

To show that $H$ satisfies condition (1) of Theorem 2.3, let $x \in X$. Let $U$ be an open neighborhood of $x$ in $X$ such that $\text{cl}_X U$ has $\mathcal{P}$. Note that $\text{cl}_X U$ has also $\mathcal{Q}$, as it is closed in $X$. Let $f : X \to [0, 1]$ be continuous with $f(x) = 1$ and $f|(X \setminus U) = 0$.

Then $\text{supp}(f)$ has $\mathcal{P}$ by (1), as it is closed in $\text{cl}_X U$. Therefore $f \in H$.

That $H$ satisfies condition (2) in Theorem 2.3 is clear, as if $f \in C_b(X)$ with $\text{supp}(f) \subseteq \text{supp}(h)$ for some $h \in H$, then $\text{supp}(h)$ has $\mathcal{P}$ (and also $\mathcal{Q}$, as it is closed in $X$) and thus, by (1) again, so does its closed subspace $\text{supp}(f)$. Therefore $f \in H$.

For the final assertion of the theorem, note that (b), (c), (d) and (e) are equivalent by Theorem 2.3. That (a) implies (c) is trivial. To show that (d) implies (a), note that if $Y$ is compact then $X$ has $\mathcal{P}$, as $X = \text{supp}(h)$ for some $h \in H$ (see Remark 2.6). □

**Remark 3.2.** Observe that in the proof of Theorem 3.1 we actually proved that $H$ is a closed subalgebra of $C_b(X)$.

**Remark 3.3.** The set

$$C_{\mathcal{P}}(X) = \{ f \in C_b(X) : \text{supp}(f) \text{ has } \mathcal{P} \}$$

where $X$ is a space and $\mathcal{P}$ is a topological property, has been also considered in [1] and [17]. (See also [2].) In [1] (Theorem 2.2) conditions are given which are necessary and sufficient for $C_{\mathcal{P}}(X)$ to be a Banach space. The approach in [17] is quite algebraic.
The next two theorems are to provide examples of topological properties $\mathcal{P}$ and $\mathcal{Q}$ satisfying the assumption of Theorem 3.1. Specifically, in Theorem 3.4 we let $\mathcal{P}$ and $\mathcal{Q}$ be, respectively, separability and metrizability, and in Theorem 3.5 we let $\mathcal{P}$ and $\mathcal{Q}$ be, respectively, the Lindelöf property and paracompactness. Among other things, we will show that in these cases (with the notation of Theorem 3.1) $Y$ is countably compact, and is non-normal if $X$ is non-$\mathcal{P}$.

Theorem 3.4 is known (see [12]); we include it here (along with its proof) partly to demonstrate an application of Theorem 3.1, and partly because of the existing duality between Theorem 3.4 and Theorem 3.5. Besides, the proof in Theorem 3.4 illustrates how our methods may be used to extract results not generally expected to be deducible from the standard Gelfand Theory. Let us recall the following.

A space $X$ is called countably compact if every countable open cover of $X$ has a finite subcover, or equivalently, if every countable infinite subspace of $X$ has a limit point in $X$. It is well known that if $X$ is a locally compact Hausdorff space then $C_0(X) = C_{00}(X)$ if and only if every $\sigma$-compact subspace of $X$ is contained in a compact subspace of $X$ (see Problem 7G.2 of [6]); in particular, $C_0(X) = C_{00}(X)$ implies that $X$ is countably compact.

Let $D$ be an uncountable discrete space. Denote by $D^\lambda$ the subspace of $\beta D$ consisting of elements in the closure in $\beta D$ of countable subsets of $D$. In [18], the author proves the existence of a continuous (2-valued) function $f : D^\lambda \setminus D \to [0,1]$ which is not continuously extendible to $\beta D \setminus D$. This, in particular, proves that $D^\lambda$ is not normal. (To see this, suppose the contrary. Note that $D^\lambda \setminus D$ is closed in $D^\lambda$, as $D$ is locally compact and then open in $\beta D$. By the Tietze–Urysohn Extension Theorem, $f$ is extendible to a continuous bounded function over $D^\lambda$, and therefore over $\beta D^\lambda = \beta D$; note that $D \subseteq D^\lambda$. But this is not possible.)

A theorem of Tarski guarantees for an infinite set $I$ the existence of a collection $I$ of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of $I$ such that the intersection of any two distinct elements of $I$ is finite. (See [8].) Note that the collection of all subsets of cardinality at most $m$ in a set of cardinality $n \geq m$ is of cardinality at most $n^m$.

Observe that if $X$ is a space and $D \subseteq X$, then

$$U \cap \text{cl}_X D = \text{cl}_X (U \cap D)$$

for any open-closed subspace $U$ of $X$.

**Theorem 3.4.** Let $X$ be a locally separable metrizable space. Then

$$C_s(X) = \{ f \in C_b(X) : \text{supp}(f) \text{ is separable} \}$$

is a Banach algebra isometrically isomorphic to $C_0(Y)$ for some unique locally compact Hausdorff space $Y$, namely $Y = \lambda_{C_s(X)}X$. Furthermore,

1. $Y$ is countably compact.
2. $Y$ is non-normal, if $X$ is non-separable.
3. $C_0(Y) = C_{00}(Y)$.
4. $\dim C_s(X) = d(X)^{\aleph_0}$.
5. The following are equivalent:
   (a) $X$ is separable.
   (b) $C_s(X)$ is unital.
   (c) $C_s(X)$ contains $1$.
   (d) $Y$ is compact.
(e) $Y = \beta X$.

Proof. Let $\mathcal{P}$ be separability and $\mathcal{Q}$ be metrizability. Then the pair $\mathcal{P}$ and $\mathcal{Q}$ satisfies the assumption of Theorem 3.1 (Observe that any subspace of a separable metrizable space is separable.) As remarked in Part 1.3 the space $X$ may be represented as a disjoint union

$$X = \bigcup_{i \in I} X_i,$$

where $I$ is an index set and $X_i$ is a non-empty separable open-closed subspace of $X$ for each $i \in I$. To simplify the notation, for any $J \subseteq I$ denote

$$H_J = \bigcup_{i \in J} X_i.$$

Observe that $H_J$ is open-closed in $X$, thus it has open-closed closure in $\beta X$. Also, as we will see now,

$$\lambda_{C_s(X)}X = \bigcup \{\text{cl}_{\beta X}H_J : J \subseteq I \text{ is countable} \}.$$  

To show (3.1), let $f \in C_s(X)$. Then $\text{supp}(f)$ is separable and is then Lindelöf. Therefore $\text{supp}(f) \subseteq H_J$ for some countable $J \subseteq I$. Thus

$$\text{cl}_{\beta X}\text{Coz}(f) \subseteq \text{cl}_{\beta X}H_J.$$ 

Now, let $J \subseteq I$ be countable. Let

$$g = \chi_{H_J}.$$ 

Then $g$ is continuous, as $H_J$ is open-closed in $X$, and $\text{supp}(g) = H_J$ is separable. Since $\text{cl}_{\beta X}H_J$ is open in $\beta X$ we have

$$\text{cl}_{\beta X}H_J = \text{int}_{\beta X} \text{cl}_{\beta X}H_J = \text{int}_{\beta X} \text{cl}_{\beta X}\text{Coz}(g) \subseteq \lambda_{C_s(X)}X.$$ 

Note that (5) follows from Theorem 3.1 and that (3) implies (1); as if (3) holds, then every countable infinite subspace of $Y$ (being $\sigma$-compact) is contained in a compact subspace of $Y$, and therefore has a limit point in $Y$. (See the remarks preceding the statement of the theorem.)

(3). We need to show that every $\sigma$-compact subspace of $\lambda_{C_s(X)}X$ is contained in a compact subspace of $\lambda_{C_s(X)}X$. Let $A$ be a $\sigma$-compact subspace of $\lambda_{C_s(X)}X$. Then

$$A = \bigcup_{n=1}^{\infty} A_n,$$

where $A_n$ is compact for each positive integer $n$. By compactness and the representation given in (3.1) we have

$$A_n \subseteq \text{cl}_{\beta X}H_{J_1} \cup \cdots \cup \text{cl}_{\beta X}H_{J_{k_n}}$$

for some countable $J_1, \ldots, J_{k_n} \subseteq I$. Let

$$J = \bigcup_{n=1}^{\infty} (J_{k_1} \cup \cdots \cup J_{k_n}).$$

Then $J$ is countable and $A \subseteq \text{cl}_{\beta X}H_J$.

(2). Let $x_i \in X_i$ for each $i \in I$. Then

$$D = \{x_i : i \in I\}.$$
is a closed discrete subspace of $X$, and since $X$ is non-separable, it is uncountable. Suppose to the contrary that $\lambda_{C_s(X)}X$ is normal. Then

$$\lambda_{C_s(X)}X \cap \cl_{\beta X}D = \bigcup \{\cl_{\beta X}H_J \cap \cl_{\beta X}D : J \subseteq I \text{ is countable}\}$$

is normal, as it is closed in $\lambda_{C_s(X)}X$. Now, let $J \subseteq I$ be countable. Since $\cl_{\beta X}H_J$ is open-closed in $\beta X$ (using the observation preceding the statement of the theorem) we have

$$\cl_{\beta X}H_J \cap \cl_{\beta X}D = \cl_{\beta X}H_J \cap D = \cl_{\beta X}(H_J \cap D) = \cl_{\beta X}\{x_i : i \in J\}.$$ 

But $\cl_{\beta X}D = \beta D$, as $D$ is closed in (the normal space) $X$. Therefore

$$\cl_{\beta X}\{x_i : i \in J\} = \cl_{\beta X}\{x_i : i \in J\} \cap \cl_{\beta X}D = \cl_{\beta D}\{x_i : i \in J\}.$$ 

Thus

$$\lambda_{C_s(X)}X \cap \cl_{\beta X}D = \bigcup \{\cl_{\beta D}E : E \subseteq D \text{ is countable}\} = D\lambda,$$

contradicting the fact that $D\lambda$ is not normal.

(4). Since $X$ is non-separable, $I$ is infinite and $d(X) = |I|$. The Tarski Theorem implies the existence of a collection $\mathcal{F}$ of cardinality $|I|^{|I|}$ consisting of countable infinite subsets of $I$, such that the intersection of any two distinct elements of $\mathcal{F}$ is finite. Let $f_J = \chi_{H_J}$ for any $J \in \mathcal{F}$. No element in

$$\mathcal{F} = \{f_J : J \in \mathcal{F}\}$$

is a linear combination of other elements (as each element of $\mathcal{F}$ is infinite and each pair of distinct elements of $\mathcal{F}$ has finite intersection). Observe that $\mathcal{F}$ is of cardinality $|\mathcal{F}|$. Thus

$$\dim C_s(X) \geq |\mathcal{F}| = |I|^{|I|} = d(X)^{\aleph_0}.$$ 

On the other hand, if $f \in C_s(X)$, then supp$(f)$ is Lindelöf (as it is separable) and thus supp$(f) \subseteq H_J$, where $J \subseteq I$ is countable; therefore, we may assume that $f \in C_b(H_J)$. Conversely, if $J \subseteq I$ is countable, then each element of $C_b(H_J)$ can be extended trivially to an element of $C_s(X)$ (by defining it to be identically 0 elsewhere). Thus $C_s(X)$ may be viewed as the union of all $C_b(H_J)$, where $J$ runs over all countable subsets of $I$. Note that if $J \subseteq I$ is countable, then $H_J$ is separable; thus any element of $C_b(H_J)$ is determined by its value on a countable set. This implies that for each countable $J \subseteq I$, the set $C_b(H_J)$ is of cardinality at most $\mathfrak{c}^{\aleph_0} = 2^{\aleph_0}$. There are at most $|I|^{|I|}$ countable $J \subseteq I$. Therefore

$$\dim C_s(X) \leq |C_s(X)| \leq \bigcup \{C_b(H_J) : J \subseteq I \text{ is countable}\} \leq 2^{\aleph_0} \cdot |I|^{|I|} = |I|^{|I|} = d(X)^{\aleph_0}.$$ 

\[\square\]

**Theorem 3.5.** Let $X$ be a locally Lindelöf paracompact space. Then

$$C_s(X) = \{f \in C_b(X) : \text{supp}(f) \text{ is Lindelöf}\}$$

is a Banach algebra isometrically isomorphic to $C_0(Y)$ for some unique locally compact Hausdorff space $Y$, namely $Y = \lambda_{C_s(X)}X$. Furthermore, the following are equivalent:

(a) $X$ is Lindelöf.
(b) $C_s(X)$ is unital.
(c) $C_s(X)$ contains 1.
(d) \( Y \) is compact.
(e) \( Y = \beta X \).

If \( X \) is moreover locally compact then

1. \( Y \) is countably compact.
2. \( Y \) is non-normal, if \( X \) is non-Lindelöf.
3. \( C_0(Y) = C_00(Y) \).
4. \( \dim C(X) = \ell(X)^{\aleph_0} \).

**Proof.** Let \( \mathcal{P} \) be the Lindelöf property and \( \mathcal{Q} \) be paracompactness. We need to know that the pair \( \mathcal{P} \) and \( \mathcal{Q} \) satisfies the assumption of Theorem 3.1. It is well known that \( \mathcal{Q} \) is hereditary with respect to closed subspaces. (See Corollary 5.1.29 of [5].) It is obvious that \( \mathcal{P} \) and \( \mathcal{Q} \) satisfy conditions (1) and (3) of Theorem 3.1. It is also known that any paracompact space with a dense Lindelöf subspace is Lindelöf (see Theorem 5.25 of [5]), that is, \( \mathcal{P} \) and \( \mathcal{Q} \) satisfy conditions (2) of Theorem 3.1. The first part of the theorem together with the equivalence of conditions (a)–(c) now follows from Theorem 3.1. The proofs for (1)–(4) are analogous to the ones we have given for the corresponding parts of Theorem 3.4. (One needs to assume a representation for \( X \) as given in Part 1.4.) \( \square \)

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**References**

1. S.K. Acharyya and S.K. Ghosh, Functions in \( C(X) \) with support lying on a class of subsets of \( X \). Topology Proc. 35 (2010), 127-148.
2. S. Afrooz and M. Namdari, \( C_\infty(X) \) and related ideals. Real Anal. Exchange 36 (2010), 45–54.
3. A.R. Aliabad, F. Azarpanah and M. Namdari, Rings of continuous functions vanishing at infinity. Comment. Math. Univ. Carolin. 45 (2004), 519-533.
4. E. Behrends, M-structure and the Banach–Stone Theorem. Springer, Berlin, 1979.
5. R. Engelking, General Topology. Second edition. Heldermann Verlag, Berlin, 1989.
6. L. Gillman and M. Jerison, Rings of Continuous Functions. Springer–Verlag, New York–Heidelberg, 1976.
7. E. Hewitt and K.A. Ross, Abstract Harmonic Analysis I. Springer–Verlag, New York, 1979.
8. R.E. Hodel, Jr., Cardinal functions I, in: K. Kunen and J.E. Vaughan (Eds.), Handbook of Set-theoretic Topology, Elsevier, Amsterdam, 1984, pp. 1–61.
9. M.R. Koushesh, Compactification-like extensions. Dissertationes Math. (Rozprawy Mat.) 476 (2011), 88 pp.
10. M.R. Koushesh, The partially ordered set of one-point extensions. Topology Appl. 158 (2011), 509–532.
11. M.R. Koushesh, A pseudocompactification. Topology Appl. 158 (2011), 2191–2197.
12. M.R. Koushesh, The Banach algebra of continuous bounded functions with separable support. Studia Math. 210 (2012), 227–237.
13. M.R. Koushesh, One-point extensions and local topological properties. Bull. Aust. Math. Soc. (5 pp.), to appear, arXiv:1210.8074 [math.GN]
14. M.R. Koushesh, Topological extensions with compact remainder. J. Math. Soc. Japan (47 pp.), to appear.
15. M.R. Koushesh, Representations theorems for normed algebras. J. Aust. Math. Soc. (22 pp.), to appear, arXiv:1204.6660 [math.FA]
16. M.R. Koushesh, Continuous mappings with null support. (40 pp.), in preparation. arXiv:1302.2235 [math.FA]
17. A. Taherifar, Some generalizations and unifications of $C_0(X)$, $C_0^*(X)$ and $C_0(X)$. (13 pp.) arXiv:1210.6521 [math.GN]

18. N.M. Warren, Properties of Stone-Čech compactifications of discrete spaces. Proc. Amer. Math. Soc. 33 (1972), 599-606.

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