INVERSION OF A CLASS OF SINGULAR INTEGRAL OPERATORS ON ENTIRE FUNCTIONS

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Abstract. Given constants $x, \nu \in \mathbb{C}$ and the space $\mathcal{H}_0$ of entire functions in $\mathbb{C}$ vanishing at 0, we consider the integro-differential operator

$$\mathcal{L} = \left( \frac{x \nu (1 - \nu)}{1 - x} \right) \delta \circ \mathfrak{M},$$

with $\delta = z \frac{d}{dz}$ and $\mathfrak{M} : \mathcal{H}_0 \to \mathcal{H}_0$ defined by

$$\mathfrak{M} f(z) = \int_{1}^{0} e^{-ztz} \left( 1 - (1 - x)t \right)^{\nu - 1} f \left( tz^{\nu} (1 - t) \right) \frac{dt}{t}, \quad z \in \mathbb{C},$$

for any $f \in \mathcal{H}_0$. Operator $\mathcal{L}$ originates from an inversion problem in Queuing Theory. Bringing the inversion of $\mathcal{L}$ back to that of $\mathfrak{M}$ translates into a singular Volterra integral equation, but with no explicit kernel.

In this paper, the inverse of operator $\mathcal{L}$ is derived through a new inversion formula recently obtained for infinite matrices with entries involving Hypergeometric polynomials. For $x \notin \mathbb{R}^+ \cup \{1\}$ and $\text{Re}(\nu) < 0$, we then show that the inverse $\mathcal{L}^{-1}$ of $\mathcal{L}$ on $\mathcal{H}_0$ has the integral representation

$$\mathcal{L}^{-1} g(z) = \frac{1 - x}{2\pi i x} e^z \int_{1}^{0} \frac{e^{-ztz} \left( 1 - (1 - x)t \right)^{\nu - 1}}{t(t - 1)} g \left( z(-t)^{\nu} (1 - t)^{1-\nu} \right) dt, \quad z \in \mathbb{C},$$

for any $g \in \mathcal{H}_0$, where the bounded integration contour in the complex plane starts at point 1 and encircles the point 0 in the positive sense. Other related integral representations of $\mathcal{L}^{-1}$ are also provided.

1. Introduction

The inversion of an integro-differential operator acting on entire functions in $\mathbb{C}$ is related to a new class of linear inversion formulas with coefficients involving Hypergeometric polynomials. After an overview of the state-of-the-art in the associated fields, we summarize our main contributions.

1.1. Motivation. Consider the following problem:

let constants $x \in [0, 1], \nu < 0$ and the function $\mathfrak{R}$ defined by

$$\mathfrak{R}(\zeta) = x \cdot (1 - \zeta)^{-\nu} \cdot (1 - (1 - x) \zeta)^{\nu - 1}, \quad \zeta \in [0, 1].$$

Let $\mathcal{H}_0$ be the linear space of entire functions in $\mathbb{C}$ vanishing at $z = 0$ and define the integro-differential operator $\mathcal{L} : \mathcal{H}_0 \to \mathcal{H}_0$ by

$$\mathcal{L} f(z) = \int_{0}^{1} \left[ (1 + z \mathfrak{R}(\zeta)) \cdot f'(\zeta) \mathfrak{R}(\zeta) \cdot z - c z \mathfrak{R}(\zeta) \cdot f(\zeta \mathfrak{R}(\zeta) \cdot z) \right] e^{-\mathfrak{R}(\zeta) \cdot z} d\zeta.$$
for all \( z \in \mathbb{C} \), where \( f' \) denotes the derivative of \( f \in \mathcal{H}_0 \) and with the constant \( c \) in the integrand equal to
\[ c = \frac{1 - \nu x}{1 - x}. \]

Given \( K \in \mathcal{H}_0 \), solve the equation
\[ (1.3) \quad \mathcal{L} E^*(z) = K(z), \quad z \in \mathbb{C}, \]
for the unknown \( E^* \in \mathcal{H}_0 \).

This inversion problem has been motivated by an integral equation arising from a problem of Queuing Theory [1], namely, the study of the sojourn time in a Processor-Sharing queue with batch customer arrivals.

The operator \( \mathcal{L} = \mathcal{L}_{x,\nu} \) depends on parameters \( x \) and \( \nu \). Solving equation (1.3) for such parameters is thus equivalent to prove that this operator from \( \mathcal{H}_0 \) to itself is onto. As detailed in this paper, the following Properties (I) and (II) for \( \mathcal{L} \) and the associated equation (1.3) can be successively outlined:

(I) Reduction to a Linear System: power series expansions
\[ (1.4) \quad E^*(z) = \sum_{\ell=1}^{+\infty} E_{\ell} \frac{z^\ell}{\ell!}, \quad K(z) = \sum_{b=1}^{+\infty} (-1)^b K_b \frac{z^b}{b!}, \quad z \in \mathbb{C}, \]
for a solution \( E^* \in \mathcal{H}_0 \) and the given \( K \in \mathcal{H}_0 \) reduce the resolution of (1.3) to that of the infinite lower-triangular linear system
\[ (1.5) \quad \forall b \in \mathbb{N}^*, \quad \sum_{\ell=1}^{b} (-1)^{\ell} \binom{b}{\ell} Q_{b,\ell} E_{\ell} = K_b, \]
with unknown \( E_{\ell} \), \( \ell \in \mathbb{N}^* \), and where the coefficient matrix \( Q = (Q_{b,\ell})_{b,\ell\in\mathbb{N}^*} \), on account of the specific function \( \mathfrak{R} \) introduced in (1.1), is given by
\[ (1.6) \quad Q_{b,\ell} = \frac{\Gamma(b)\Gamma(1-b\nu)}{\Gamma(b-b\nu)} \frac{x}{x-1} F(\ell-b, -b\nu; -b; x), \quad 1 \leq \ell \leq b. \]

In (1.6), \( \Gamma \) is the Euler Gamma function and \( F(\alpha, \beta; \gamma; \cdot) \) denotes the Gauss Hypergeometric function with complex parameters \( \alpha, \beta, \gamma \notin -\mathbb{N} \). Recall that \( F(\alpha, \beta; \gamma; \cdot) \) reduces to a polynomial with degree \(-\alpha\) (resp. \(-\beta\)) if \( \alpha \) (resp. \( \beta \)) equals a non positive integer; expression (1.6) for coefficient \( Q_{b,\ell} \) thus involves a Hypergeometric polynomial with degree \( b-\ell \) in both arguments \( x \) and \( \nu \).

The diagonal coefficients \( Q_{b,b}, b \geq 1 \), are non-zero so that lower-triangular system (1.5) has a unique solution; equivalently, this proves the uniqueness of the solution \( E^* \in \mathcal{H}_0 \) to (1.3). To make this solution explicit in terms of parameters, write system (1.5) equivalently as
\[ (1.7) \quad \forall b \in \mathbb{N}^*, \quad \sum_{\ell=1}^{b} A_{b,\ell}(x,\nu) E_{\ell} = \tilde{K}_b, \]
with the reduced right-hand side \( (\tilde{K}_b) \) defined by
\[ \tilde{K}_b = \frac{\Gamma(b-b\nu)}{\Gamma(b)\Gamma(1-b\nu)} \frac{x-1}{x} \cdot K_b, \quad b \geq 1, \]
and with matrix $A(x, \nu) = (A_{b,\ell}(x, \nu))$ given by
\begin{equation}
A_{b,\ell}(x, \nu) = (-1)^\ell \binom{b}{\ell} F(\ell - b, -b\nu; -b; x), \quad 1 \leq \ell \leq b.
\end{equation}

As recently shown [2], the linear relation (1.7) to which initial system (1.5) has been recast can be explicitly inverted for any right-hand side ($K_b\in\mathbb{N}^*$, the inverse matrix $B(x, \nu) = A(x, \nu)^{-1}$ involving also Hypergeometric polynomials as well. This consequently solves system (1.5) explicitly, hence integral equation (1.3):

(II) Factorization: operator $\mathcal{L}$ can be factored as
\begin{equation}
\mathcal{L} = \frac{x\nu(1-\nu)}{1-x} \cdot \delta \circ \mathfrak{M}
\end{equation}

where $\delta = z \frac{d}{dz}$ and $\mathfrak{M}$ is the integral operator defined by
\begin{equation}
\mathfrak{M}f(z) = \int_0^1 e^{-zt-\nu(1-x)t} f(z t^{-\nu}(1-t)) \frac{dt}{t}, \quad z \in \mathbb{C},
\end{equation}

for all $f \in \mathfrak{H}_0$. Using factorization (1.9), the resolution of (1.3) is thus equivalent to solving equation
\begin{equation}
\mathfrak{M}E^* = K_1
\end{equation}

with right-hand side
\begin{equation}
K_1(z) = \frac{1-x}{\nu(1-\nu)x} \cdot \int_0^z K(\xi) \frac{d\xi}{\xi}, \quad z \in \mathbb{C},
\end{equation}

where $K_1 \in \mathfrak{H}_0$ as soon as $K \in \mathfrak{H}_0$. Integral equation (1.11) can be in turn recast into the Volterra equation
\begin{equation}
\int_{\hat{\tau}z}^{\hat{\tau}z} \Psi \left( z, \frac{\xi}{z} \right) E^*(\xi) d\xi = z \cdot K_1(z), \quad z \in \mathbb{C},
\end{equation}

for some constant $\hat{\tau}$ and a kernel $\Psi(z, \cdot)$. As $\Psi(z, \cdot)$ has an integrable singularity of order $\Psi(z, \tau) = O(\tau - \hat{\tau})^{-1/2}$ near point $\tau = \hat{\tau}$, (1.12) is therefore a singular Volterra integral equation of the first kind.

1.2. State-of-the-art. As equation (1.3) or (1.11) can be recast into the singular Volterra equation of the first kind (1.12), we here briefly review known results for this class of integral equations.

Given the constant $\alpha \in [0, 1]$, the standard case for such singular equations is that of the classical Abel’s equation
\begin{equation}
\int_0^z \frac{E(\xi)}{(z-\xi)^\alpha} d\xi = \kappa(z), \quad z \in [0, r],
\end{equation}
on a real interval $[0, r]$, for the unknown function $E$ and some given function $\kappa$ (see [3, Chap.7], [4, Chap.2], [5, Chap.1]. If $\kappa$ is absolutely continuous on $[0, r]$, then Abel’s equation has the unique solution $E \in L^1[0, r]$ given by
\begin{equation}
E(z) = \frac{\sin(\pi \alpha)}{\pi} \cdot \frac{d}{dz} \left[ \int_0^z \frac{\kappa(\xi)}{(z-\xi)^{1-\alpha}} d\xi \right] = \frac{\sin(\pi \alpha)}{\pi} \left[ \frac{\kappa(0)}{z^{1-\alpha}} + \int_0^z \frac{\kappa'(\xi)}{(z-\xi)^{1-\alpha}} d\xi \right], \quad z \in [0, r].
\end{equation}
This solution extends to a complex variable \( z \in \mathbb{C} \) pertaining to a neighborhood of point 0 where function \( \kappa \) is assumed to be analytic; the solution \( E \) is then analytic in a neighborhood of \( z = 0 \) if condition \( \kappa(0) = 0 \) holds, that is, if and only if \( \kappa \in \mathcal{H}_0 \).

More generally, let a compact subset \( \Omega \subset \mathbb{C} \) and a singular operator \( \mathfrak{J} : E \mapsto \mathfrak{J}E \) defined by

\[
\mathfrak{J}E(z) = \int_0^z N(z, \xi) E(\xi) \, d\xi, \quad z \in \Omega,
\]

where the kernel \( N \) verifies

\[
|N(z, \xi)| \leq \frac{M}{|z - \xi|^\alpha}, \quad z, \xi \in \Omega, \quad z \neq \xi,
\]

for some constant \( M > 0 \) and \( \alpha \in ]0, 1[ \). Operator \( \mathfrak{J} \) is known to be continuous (and also compact) on space \( \mathcal{C}^0[\Omega] \) [6, Theorem 2.29]. No general results are available, however, on the inverse of \( \mathfrak{J} \) on a subspace of \( \mathcal{C}^0[\Omega] \).

This standard framework may nevertheless suggest the existence of an integral representation of the kind (1.13) for the entire solution to either equation (1.3) (1.11) or (1.12). In this paper, we will show how such integral representations can be obtained for the solutions of these singular equations.

1.3. Paper contribution. The main contributions of this paper can be summarized as follows:

(A) we first prove the above mentioned Reduction Property I (Section 3.1) whereby integral equation (1.3) is reduced to linear system (1.5) with coefficients related to Hypergeometric polynomials;

(B) we next justify the Factorization Property II (Section 3.2) for the integro-differential operator \( \mathfrak{L} \). We further specify how equation (1.11) can be recast into a Volterra integral equation with singular kernel (Section 3.3);

(C) the previous results finally enable us to derive an integral representation of the inverse \( \mathfrak{L}^{-1} \) of operator \( \mathfrak{L} \) in space \( \mathcal{H}_0 \) in the form of the contour integral

\[
\mathfrak{L}^{-1} g(z) = \frac{1 - x}{2\pi i x} e^z \int_1^{(0+)} \frac{e^{-xtz}}{t(t - 1)} g \left( z \left( -t \right)^\nu (1 - t) \right) \, dt, \quad z \in \mathbb{C},
\]

for any \( g \in \mathcal{H}_0 \), where the finite contour in the complex plane starts at 1 and encircles 0 in the positive sense (Section 4). By using suitable variable change in the latter, other related integral representations of the inverse \( \mathfrak{L}^{-1} \) are also provided.

2. Preliminaries

2.1. Infinite matrices inversion. We first recall the results established in [2] for the inversion of some class of lower-triangular matrices. These results will be used below for the inversion of operator \( \mathfrak{L} \).

While stated for general matrices \( A(x, \nu; \alpha, \beta, \gamma) \) depending on \( x, \nu \) and three other complex parameters \( \alpha, \beta \) and \( \gamma \) [2, Theorem 2.3], we here only use the inversion property particularized to the matrix \( A(x, \nu) \) introduced in (1.8) and corresponding to the sub-case \( \alpha = \beta = \gamma = 0 \). In such a case, the inversion formula for lower-triangular matrices involving Hypergeometric polynomials \( F(m, \cdot; \cdot; x) \), \( m \in -\mathbb{N} \), can be stated as follows.
Theorem 2.1. ([2 Sect. 2.2]) Let \( x, \nu \in \mathbb{C} \) and define the lower-triangular matrices \( A(x, \nu) \) and \( B(x, \nu) \) by

\[
\begin{align*}
A_{n,k}(x, \nu) &= (-1)^k \binom{n}{k} F(k-n, -n\nu; -n; x), \\
B_{n,k}(x, \nu) &= (-1)^k \binom{n}{k} F(k-n, k\nu; k; x)
\end{align*}
\]

for \( 1 \leq k \leq n \). The inversion formula

\[
T_n = \sum_{k=1}^{n} A_{n,k}(x, \nu) S_k \iff S_n = \sum_{k=1}^{n} B_{n,k}(x, \nu) T_k, \quad n \in \mathbb{N}^*,
\]

holds for any pair of complex sequences \((S_n)_{n \in \mathbb{N}^*}\) and \((T_n)_{n \in \mathbb{N}^*}\).

As a direct consequence of Theorem 2.1, a remarkable functional identity can be derived for the exponential generating functions of sequences related by the inversion formula.

Corollary 2.1. ([2 Sect. 3.2]) Given sequences \( S \) and \( T \) related by the inversion formulae

\[
S = B(x, \nu) \cdot T \iff T = A(x, \nu) \cdot S,
\]

the exponential generating function \( \mathcal{G}_S(z) \) of the sequence \( S \) can be expressed by

\[
\mathcal{G}_S(z) = \exp(z) \cdot \sum_{k \geq 1} (-1)^k T_k \frac{z^k}{k!} \Phi(k\nu; k; -xz), \quad z \in \mathbb{C},
\]

where \( \Phi(\alpha; \beta; \cdot) \) denotes the Confluent Hypergeometric function with parameters \( \alpha, \beta \notin -\mathbb{N} \).

2.2. Parameters range. Operator \( \mathfrak{L} \) has been initially introduced for real parameters \( x \in [0,1] \) and \( \nu < 0 \). Following the results recalled in Section 2.1 and stated for arbitrary complex parameters, we hereafter extend the definition (1.2) of \( \mathfrak{L} \) to complex values, namely

\- \( x \in \mathbb{C} \setminus (\mathbb{R}^- \cup \{1\}) \) (so that \( 1/(1-x) \) is finite and does not belong to the integration interval \([0,1]\))
\- and \( \nu \in \mathbb{C} \) such that \( \text{Re}(\nu) < 0 \).

Within these assumptions, it is easily verified that \( \mathfrak{L}(\mathcal{H}_0) \subset \mathcal{H}_0 \) where \( \mathcal{H}_0 \) is again the linear space of entire functions in \( \mathbb{C} \) vanishing at 0.

Remark 2.1. Operator \( \mathfrak{L} \), well-defined for \( \text{Re}(\nu) < 0 \), may not exist for other values of \( \nu \). In fact, consider the function \( f \in \mathcal{H}_0 \) defined by \( f(z) = ze^{(1-x)z} \), \( z \in \mathbb{C} \). By definition (1.2), it is easily verified that, for \( \text{Re}(\nu) < 0 \), its image \( \mathfrak{L}f \in \mathcal{H}_0 \) is given by

\[
\mathfrak{L}f(z) = -\frac{z}{1-x} \Phi\left(1 - \frac{1}{\nu}, 1 - \frac{1}{\nu}; -z\right), \quad z \in \mathbb{C},
\]

\( \Phi(\cdot; \cdot; \cdot) \) denoting the Kummer Confluent Hypergeometric function. For \( \text{Re}(\nu) > 0 \), however, its image is given by

\[
\mathfrak{L}f(z) = -\frac{1-\nu}{\nu(1-x)} z^{1/2} \Gamma\left(1 - \frac{1}{\nu}; z\right), \quad z \neq 0
\]
(where $\Gamma(\cdot; \cdot)$ is the incomplete Gamma function), so that $\mathcal{L}f \notin \mathcal{H}_0$ in this case.

3. Properties of operator $\mathcal{L}$

3.1. Reduction to a linear system. We have claimed in 1.1 (I) that the integrodifferential equation (1.3) reduces to the infinite system (1.5). We justify this assertion by showing how the coefficients of system (1.5) can be expressed in terms of Hypergeometric polynomials.

Proposition 3.1. The Reduction Property (I) holds, that is, equation (1.3) reduces to system (1.5) with matrix $Q = (Q_{b,\ell})_{1 \leq \ell \leq b}$ related to Hypergeometric polynomials as given in (1.6).

Proof. To derive system (1.5), we expand both sides of (1.3) into power series of variable $z$ and identify like powers on each side. The series expansion (1.4) of $E^*(Z)$ in powers of $Z$ first provides

\begin{equation}
(1 + Z)E^*(\zeta Z) - c Z \frac{dE^*}{dz}(\zeta Z) = \sum_{b \geq 1} \Lambda_b(\zeta) \frac{Z^b}{b!}
\end{equation}

where we set $\Lambda_b(\zeta) = \zeta^b E_b + b \zeta^{b-1} E_{b-1} - b c \zeta^{b-1} E_b$ for all $\zeta$ and with the constant $c = (1 - \nu x)/(1 - x)$; applying equality (3.1) to the argument $Z = \mathcal{R}(\zeta) \cdot z$, the integrand of $\mathcal{L}E^*(z)$ in (1.2) can then be expanded into a power series of $z$ as

\begin{equation}
\mathcal{L}E^*(z) = \int_0^1 \left[ \sum_{b \geq 1} \Lambda_b(\zeta) \frac{\zeta^b \mathcal{R}(\zeta)^b z^b}{b!} \right] e^{-\mathcal{R}(\zeta)z} \, d\zeta.
\end{equation}

Now, expanding the exponential $e^{-\mathcal{R}(\zeta)z}$ of the integrand in (3.2) into a power series of $z$ gives the expansion

\begin{equation}
\mathcal{L}E^*(z) = \sum_{b \geq 0} (-1)^b \frac{z^b}{b!} \sum_{\ell = 0}^{b} (-1)^\ell \binom{b}{\ell} \int_0^1 \zeta^\ell \Lambda_\ell(\zeta) \mathcal{R}(\zeta)^b \, d\zeta
\end{equation}

(after noting that $\Lambda_0(\zeta) = 0$ since $E_0 = 0$ by definition). On account of expansion (3.3) with the above definition (3.1) of $\Lambda_\ell(\zeta)$, together with the expansion (1.4) for $K(z)$, the identification of like powers of these expansions readily yields the relation

\begin{equation}
\sum_{\ell = 1}^{b} (-1)^\ell \binom{b}{\ell} B_{b,\ell} E^*_\ell + \sum_{\ell = 1}^{b} (-1)^\ell \binom{b}{\ell} \ell M_{b,\ell-1} E_{\ell-1} = K_b, \quad b \geq 1,
\end{equation}

with $B_{b,\ell} = M_{b,\ell} - \ell c M_{b,\ell-1}$, where $M_{b,\ell}$ denotes the definite integral

\begin{equation}
M_{b,\ell} = \int_0^1 \zeta^\ell \mathcal{R}(\zeta)^b \, d\zeta, \quad 1 \leq \ell \leq b.
\end{equation}

By first changing the index in the second sum in the left-hand side of (3.4) and then using identity $\binom{b}{\ell+1} = (b - \ell) \cdot \binom{b}{\ell}/(\ell + 1)$, (3.4) reduces to (1.5) with coefficients

\begin{equation}
Q_{b,\ell} = (\ell + 1 - b) M_{b,\ell} - \ell c M_{b,\ell-1}, \quad 1 \leq \ell \leq b.
\end{equation}

The calculation of integral $M_{b,\ell}$ in (3.5) in terms of Hypergeometric functions and its reduction to Hypergeometric polynomials is detailed in Appendix 5.1; this eventually provides expression (1.6) for the coefficients of matrix $Q = (Q_{b,\ell})$.
We can now deduce the unique solution to system (1.5).

**Corollary 3.1.** Let \( \nu \in \mathbb{C} \) with Re\((\nu) < 0 \). Given the sequence \((K_b)_{b \geq 1}\), the unique solution \((E_b)_{b \geq 1}\) to system (1.5) is given by

\[
E_b = \frac{1}{x} \sum_{\ell=1}^{b} (-1)^{\ell} \binom{b}{\ell} F(\ell - b, \ell \nu; \ell; x) \frac{\Gamma(\ell - \ell \nu)}{\Gamma(\ell) \Gamma(1 - \ell \nu)} K_\ell
\]

for all \( b \geq 1 \).

**Proof.** By expression (1.6) for the coefficients of lower-triangular matrix \( Q \), equation (1.5) equivalently reads

\[
\sum_{\ell=1}^{b} (-1)^{\ell} \binom{b}{\ell} F(\ell - b, -b \nu; -b; x) \cdot E_\ell = \tilde{K}_b, \quad 1 \leq \ell \leq b,
\]

when setting

\[
\tilde{K}_b = \frac{\Gamma(b - b \nu)}{\Gamma(b) \Gamma(1 - b \nu)} \frac{x - 1}{x} \cdot K_b, \quad b \geq 1.
\]

The application of inversion Theorem 2.1 to lower-triangular system (3.8) readily provides the solution sequence \((E_\ell)_{\ell \in \mathbb{N}}\) in terms of the sequence \((\tilde{K}_b)_{b \in \mathbb{N}}\); using then transformation (3.9), the final solution (3.7) for the sequence \((E_\ell)_{\ell \in \mathbb{N}}\) follows. \(\square\)

### 3.2. Factorization of \( \mathcal{L} \)

We now prove the factorization property 1.1 (II) for integro-differential operator \( \mathcal{L} \).

**Proposition 3.2.** The Factorization Property (II) holds, that is, the linear operator \( \mathcal{L} \) on space \( \mathcal{H}_0 \) can be factored as in (1.9) in terms of operators \( \delta = z \frac{d}{dz} \) and \( \mathfrak{M} \).

**Proof.** Calculating the exponential generating function of the sequence \((-1)^b K_b\), \( b \geq 1 \), from relation (1.5) with help of (1.6) for the coefficients of matrix \( Q \) gives

\[
\mathcal{L} E^*(z) = K(z) = \sum_{b \geq 1} (-1)^b K_b \cdot z^b = 
\]

\[
\sum_{b \geq 1} - \frac{\Gamma(b) \Gamma(1 - b \nu)}{\Gamma(b - b \nu)} \frac{x}{1 - x} \frac{(-z)^b}{b!} \sum_{\ell=1}^{b} (-1)^{\ell} \binom{b}{\ell} F(\ell - b, -b \nu; -b; x) E_\ell,
\]

for all \( z \in \mathbb{C} \), that is,

\[
\mathcal{L} E^*(z) = \sum_{\ell \geq 1} \frac{(-1)^{\ell}}{\ell!} E_\ell \times 
\]

\[
\sum_{b \geq \ell} - \frac{\Gamma(b) \Gamma(1 - b \nu)}{\Gamma(b - b \nu)} \frac{x}{1 - x} \frac{(-z)^b}{(b - \ell)!} F(\ell - b, -b \nu; -b; x)
\]

(after changing the summation order on indexes \( b \) and \( \ell \)). Applying the general identity (5.14) to parameters \( m = b - \ell \geq 0 \), \( \beta = -b \nu \) and \( \gamma = \ell - b \nu + 1 \) to express
polynomial $F(\ell-b,-b\nu,-b;x)$ in terms of polynomial $F(\ell-b,-b\nu,\ell-b\nu+1;1-x)$, we further obtain

$$
- \frac{\Gamma(b)\Gamma(1-b\nu)}{\Gamma(b-b\nu)} F(\ell-b,-b\nu,-b;x) = \\
- \frac{(1-\nu)\Gamma(\ell+1)\Gamma(1-b\nu)}{\Gamma(\ell-b\nu+1)} F(\ell-b,-b\nu,\ell-b\nu+1;1-x);
$$

(3.11)

using the integral representation recalled in Appendix 5.1 - Equ.(5.1) for the factor $F(\ell-b,-b\nu,\ell-b\nu+1;1-x)$ in the right-hand side of (3.11) eventually yields

$$
- \frac{\Gamma(b)\Gamma(1-b\nu)}{\Gamma(b-b\nu)} F(\ell-b,-b\nu,-b;x) = b\nu(1-\nu) \int_0^1 t^{-b\nu-1}(1-t)^\ell(1-(1-x)t)^{b-\ell} \, dt.
$$

Now, replacing the latter into the right-hand side of (3.10) provides

$$
\mathcal{L}E^*(z) = \frac{x\nu(1-\nu)}{1-x} \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} E_\ell \cdot S_\ell(z)
$$

where

$$
S_\ell(z) = \sum_{b \geq \ell} \frac{b}{(b-\ell)!} \cdot (-z)^b \int_0^1 t^{-b\nu-1}(1-t)^\ell(1-(1-x)t)^{b-\ell} \, dt.
$$

With the index change $b' = b - \ell$, the sum $S_\ell(z)$ for given $\ell$ equivalently reads

$$
S_\ell(z) = \sum_{b' \geq 0} \frac{b' + \ell}{b'!} \cdot (-z)^{b'+\ell} \int_0^1 t^{-(b'+\ell)\nu}(1-t)^\ell(1-(1-x)t)^{b'} \, \frac{dt}{t}
$$

$$
= -\frac{z}{1-x} \frac{d}{dz} \left[ (-z)^\ell \int_0^1 t^{-\nu}(1-t)^\ell \, \frac{dt}{t} \times \exp\left( -z \, t^{-\nu}(1-(1-x)t) \right) \right].
$$

(3.13)

Replacing expression (3.13) into the left-hand side of (3.12), the linearity of operator $\delta = z \frac{d}{dz}$ and the permutation of the summation on index $\ell$ with the integration with respect to variable $t \in [0,1]$ enable us to obtain

$$
\mathcal{L}E^*(z) = \frac{x\nu(1-\nu)}{1-x} \cdot \delta \left[ \int_0^1 \frac{dt}{t} \, e^{-z \, t^{-\nu}(1-(1-x)t)} \sum_{\ell \geq 1} \frac{E_\ell}{\ell!} z^\ell \, t^{-\nu}(1-t)^\ell \right],
$$

that is,

$$
\mathcal{L}E^*(z) = \frac{x\nu(1-\nu)}{1-x} \cdot \delta \left[ \int_0^1 \frac{dt}{t} \, e^{-z \, t^{-\nu}(1-(1-x)t)} \times E^* \left( z \, t^{-\nu}(1-t) \right) \right]
$$

$$
= \frac{x\nu(1-\nu)}{1-x} \cdot (\delta \circ \mathfrak{M}) E^*(z), \quad z \in \mathbb{C},
$$

for any function $E^* \in \mathcal{H}_0$, as claimed in (1.9), with the corresponding definition of integral operator $\mathfrak{M}$ on space $\mathcal{H}_0$. □
3.3. The Volterra equation. As outlined in the Introduction, the factorization (1.9) of operator $\mathcal{L}$ allows one to write equation (1.3) equivalently as

$$\mathfrak{M} E^* = K_1$$

where $K_1 \in \mathcal{H}_0$ relates to the initial function $K$ as in (1.11). As the real function $\tau : t \in [0, 1] \mapsto t^{-\nu}(1 - t)$ has a unique maximum at point $\hat{t} = \nu/(\nu - 1)$ for $\nu < 0$, we can introduce the variable changes $t \in [0, \hat{t}] \mapsto \tau_-(t) = t^{-\nu}(1 - t)$ and $t \in [\hat{t}, 1] \mapsto \tau_+(t) = t^{-\nu}(1 - t)$ on segments $[0, \hat{t}]$ and $[\hat{t}, 1]$, respectively; we further denote by

$$\theta_- = \tau_-^{-1}, \quad \theta_+ = \tau_+^{-1}$$

the respective inverse mappings of $\tau_-$ and $\tau_+$, both defined on segment $[0, \hat{\tau}]$ where $\hat{\tau} = \tau(\hat{t}) = (\hat{t})^{-\nu}(1 - \hat{t})$ (see illustration on Fig. 1). The variable changes $\tau_-$ and $\tau_+$ then allow us to write equation (3.14) as a singular Volterra equation.

$$\int_0^{\hat{\tau}} \left[ \Psi_-(z, \xi) - \Psi_+(z, \xi) \right] E^*(\xi) d\xi = z \cdot K_1(z), \quad z \in \mathbb{C},$$

where we set

$$\Psi_\pm(z, \tau) = \frac{e^{-z\theta_\pm(\tau)\tau^{-\nu}(1 - (1 - \tau)\theta_\pm(\tau))}}{\theta_\pm(\tau)^{-\nu}(-\nu + (\nu - 1)\theta_\pm(\tau))}, \quad 0 \leq \tau \leq \hat{\tau},$$

with $\theta_\pm$ introduced in (3.15), and where $K_1 \in \mathcal{H}_0$ is defined by (1.11).

We refer to Appendix 5.2 for the proof of Corollary 3.2. It is noted there that the kernel $\tau \mapsto \Psi_-(z, \tau) - \Psi_+(z, \tau)$ of Volterra equation (3.16) is singular with an integrable singularity at the boundary $\tau = \hat{\tau}$ of order $O((\hat{\tau} - \tau)^{-1/2})$. 

**Figure 1. Graph of function $\tau : t \in [0, 1] \mapsto t^{-\nu}(1 - t)$ (here $\nu = -2$ for illustration).**

**Corollary 3.2.** Given constants $x$ and $\nu$ as above, the equivalent equation (3.14) can be recast into the singular Volterra integral equation

$$\int_0^{\hat{\tau}} \left[ \Psi_-(z, \xi) - \Psi_+(z, \xi) \right] E^*(\xi) d\xi = z \cdot K_1(z), \quad z \in \mathbb{C},$$

where we set

$$\Psi_\pm(z, \tau) = \frac{e^{-z\theta_\pm(\tau)\tau^{-\nu}(1 - (1 - \tau)\theta_\pm(\tau))}}{\theta_\pm(\tau)^{-\nu}(-\nu + (\nu - 1)\theta_\pm(\tau))}, \quad 0 \leq \tau \leq \hat{\tau},$$

with $\theta_\pm$ introduced in (3.15), and where $K_1 \in \mathcal{H}_0$ is defined by (1.11).
Although giving a reformulation to initial equation (1.3), equation (3.16) remains difficult to solve as its kernel depends on inverse functions \( \theta_- \) and \( \theta_+ \) which cannot be made explicit simply.

4. Inversion of operator \( \mathcal{L} \)

We now provide integral representations for the inverse of operator \( \mathcal{L} \) on \( \mathcal{H}_0 \) or, equivalently, integral representations for the solution of integral equation (1.3) addressed in the Introduction.

**Theorem 4.1.** Let \( \nu \in \mathbb{C} \) with \( \text{Re}(\nu) < 0 \). Then

a) the operator \( \mathcal{L} : \mathcal{H}_0 \to \mathcal{H}_0 \) is a bijection;

b) given \( K \in \mathcal{H}_0 \), the unique solution \( E^* = \mathcal{L}^{-1}K \in \mathcal{H}_0 \) to the integral equation (1.3) has the integral representation

\[
E^*(z) = \mathcal{L}^{-1}K(z) = \frac{1 - x}{2i\pi x} e^z \int_1^{(0+)} \frac{e^{-zt}z (t - 1)^{1 - \nu}}{t\left(t - 1\right)} K(z(-t)^\nu(1 - t)^{1 - \nu}) \, dt, \quad z \in \mathbb{C},
\]

where the contour in integral (4.4) in variable \( t \) is a loop starting and ending at point \( t = 1 \), and encircling the origin \( t = 0 \) once in the positive sense (see Fig. 2, red solid line).

**Proof.**

a) Given \( K \in \mathcal{H}_0 \), equation (1.3) for \( E^* \in \mathcal{H}_0 \) is equivalent to system (1.5) for the coefficients \( (E_\ell)_{\ell\geq1} \) of the exponential series expansion of \( E^* \). For \( \text{Re}(\nu) < 0 \), Corollary 3.1 entails these coefficients are uniquely determined by expression (3.7). The linear operator \( \mathcal{L} : \mathcal{H}_0 \to \mathcal{H}_0 \) is consequently one-to-one and onto, and has an inverse \( \mathcal{L}^{-1} \) on \( \mathcal{H}_0 \).

b) An integral representation for the inverse operator \( \mathcal{L}^{-1} \) is now derived as follows. Setting \( S = E \) and \( T = \tilde{K} \) in (2.3), with the sequence \( \tilde{K} = (\tilde{K}_b)_{b\geq1} \)
defined as in (3.9), we obtain

\[ G_b^\nu(z) = e^z \cdot \sum_{b \geq 1} (-1)^b \frac{\tilde{K}_b}{b!} \Phi(b\nu; b; -xz) \]

(4.2)

\[
= \frac{x-1}{x} e^z \cdot \sum_{b \geq 1} (-1)^b \frac{\Gamma(b-b\nu)}{\Gamma(b)\Gamma(1-b\nu)} \cdot K_b \frac{z^b}{b!} \Phi(b\nu; b; -xz)
\]

after using (3.9) to express (4.3) Φ(α; β) in terms of K_b, b ≥ 1. Invoke then the integral representation

\[ \Phi(\alpha; \beta; Z) = -\frac{1}{2\pi i} \frac{\Gamma(1-\alpha)\Gamma(\beta)}{\Gamma(\beta-\alpha)} \int_{1}^{(0+)} e^{zt}(-t)^{\alpha-1}(1-t)^{\beta-\alpha-1} \, dt \]

(4.3)

of the Confluent Hypergeometric function Φ(α; β; ·) for Re(β-α) > 0 [7, Sect.6.11.1, (3)], where the integration contour is specified as in Fig.2 red solid line. On account of (4.3) applied to α = b\nu and β = b ∈ \mathbb{N}^* with Re(\nu) < 1, expression (4.2) now reads

\[ G_b^\nu(z) = \frac{1-x}{2i\pi x} e^z \int_{1}^{(0+)} \frac{e^{-xt} \, dt}{t(t-1)} \sum_{b \geq 1} (-1)^b \frac{K_b}{b!} (z(-t)^{\nu}(1-t)^{1-\nu})^b \]

(4.4)

\[
= \frac{1-x}{2i\pi x} e^z \int_{1}^{(0+)} \frac{e^{-xt} \, dt}{t(t-1)} K(z(-t)^{\nu}(1-t)^{1-\nu}) \, dt
\]

for all z ∈ \mathbb{C}. As E^*(z) = G_b^\nu(z) by definition, expression (4.4) readily yields the final representation (4.1), as claimed.

□

As mentioned in the latter proof, the representation (4.1) of the inverse \( \mathcal{L}^{-1} \) is actually valid for Re(\nu) < 1, although the operator \( \mathcal{L} \) is defined on space \( \mathcal{H}_0 \) for Re(\nu) < 0 only.

Now, using suitable variable changes in formula (4.1), alternative integral representations for \( \mathcal{L}^{-1} \) with Re(\nu) < 0 can be asserted as follows.

**Corollary 4.1.** Let \( \nu \in \mathbb{C} \) with Re(\nu) < 0.

Given \( K \in \mathcal{H}_0 \), the unique solution \( E^* = \mathcal{L}^{-1}K \in \mathcal{H}_0 \) to integral equation (4.3) has the equivalent integral representations

\[ E^*(z) = \mathcal{L}^{-1}K(z) \]

(4.5)

\[
= \frac{1-x}{2i\pi x} e^{(1-x)z} \int_{0}^{(1+)} \frac{e^{zt} \, dt}{t(t-1)} K(z t^{1-\nu}(t-1)^{\nu}) \, dt, \quad z \in \mathbb{C},
\]

where the contour in (4.5) in variable t is a loop starting and ending at point \( t = 0 \), and encircling point \( t = 1 \) once in the positive sense (see Fig.3 blue dotted line), and

\[ E^*(z) = \mathcal{L}^{-1}K(z) \]

(4.6)

\[
= \frac{1-x}{2i\pi x} e^{(1-x)z} \int_{c_0-i\infty}^{c_0+i\infty} \frac{e^{zr}}{1-r} K \left( \frac{z(1-r)^{\nu}}{r} \right) \, dr, \quad z \in \mathbb{C},
\]

where the contour in (4.6) is the vertical line \( \text{Re}(r) = c_0 \), for any real abscissa \( 0 < c_0 < 1 \) (see Fig.3 red dotted line).
Proof. \( \bullet \) By the variable change \( t \mapsto 1 - t \), formula (4.1) readily entails (4.5) which is defined for \( \Re(\nu) < 0 \).

\( \bullet \) Let \( 0 < c_0 < 1 \). As a contour in integral (4.1), choose the circle centered at point \( 1 - 1/(2c_0) \) on the real axis and with radius \( 1/(2c_0) \); this circle passes through point 1 and encircles the origin (see Fig.3). It is easily verified that the homographic transformation \( t \mapsto r' = t/(1 - t) \) maps this circle 1-to-1 and onto the vertical line \( \Re(r') = c_0 - 1 \). Applying the latter variable change \( t \mapsto r' \) to (4.1) with

\[ (-t)\nu (1 - t)^{1-\nu} = \frac{(1 - r')\nu}{1 + r'}, \quad \, dt = \frac{dr'}{(1 + r')^2} \]

then readily gives

\[ \mathcal{L}^{-1}K(z) = \frac{1 - x}{2\pi i x} \int_{\Re(r')=c_0-1} e^{\frac{z}{1+r'}} K \left( \frac{z}{1+r'} \right) \frac{dr'}{r'} \]

for all \( z \in \mathbb{C} \). The mapping \( r' \mapsto r = r' + 1 \) then eventually transforms the latter integral to the expected representation (4.6) with integration contour the vertical line \( \Re(r) = c_0, 0 < c_0 < 1 \).

By the factorization (1.9), we readily deduce that the inverse of operator \( \mathcal{M} \) is given by

\[ \mathcal{M}^{-1}g(z) = \frac{x\nu(1-\nu)}{1-x} \cdot \mathcal{L}^{-1}(zg'(z)), \quad z \in \mathbb{C}, \]

for all \( g \in \mathcal{H}_0 \), with inverse \( \mathcal{L}^{-1} \) provided by either integral representation (4.1), (4.5) or (4.6). The involvement of the derivative \( g' \) for the inverse \( \mathcal{M}^{-1}g \) in (4.7) reminds us of formula (1.13) in the particular case of the Abel’s equation.

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Lemma 5.1. We have
for real parameters $Q$ terms of Hypergeometric polynomials only. We first calculate coefficients $Q_{b,\ell}(s)$, $1 \leq \ell \leq b$, in terms of the general Gauss Hypergeometric function $F$. Recall that $F = F(\alpha, \beta; \gamma; \cdot)$ has the integral representation [8, Chap.15, Sect.15.6.1]

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-zt)^{\alpha}} \, dt, \quad |z| < 1,$$

for real parameters $\alpha$, $\beta$, $\gamma$ where $\gamma > \beta > 0$.

Lemma 5.1. We have

$$Q_{b,\ell} = \frac{\Gamma(\ell)\Gamma(1-b\nu)}{\Gamma(\ell + 1 - b\nu)} \left( x^{1+b} \right) \left[ \nu (b-\ell) \times \right.$$  

$$F(b(1-\nu), \ell; \ell + 1 - b\nu; 1-x) \right]$$

for $1 \leq \ell \leq b$.

Proof. To calculate the integral $M_{b,\ell}$ introduced in (3.5), use the definition of $\Re(t)$, to write

$$M_{b,\ell} = x^b \int_0^1 t^\ell (1-t)^{-b\nu} (1-(1-x)t)^{b(\nu-1)} \, dt;$$

using representation (5.1) for parameters $\alpha = -b(1-\nu)$, $\beta = \ell + 1$, $\gamma = 2 + \ell - b\nu$, this integral reduces to

$$M_{b,\ell} = \frac{\Gamma(\ell + 1)\Gamma(1-b\nu)}{\Gamma(2 + \ell - b\nu)} x^b F(b(1-\nu), \ell + 1; 2 + \ell - b\nu; 1-x);$$

after (5.3) and the expression (3.6) of coefficient $Q_{b,\ell}$, we then derive

$$Q_{b,\ell} = \frac{\Gamma(\ell)\Gamma(1-b\nu)}{\Gamma(\ell + 1 - b\nu)} x^b \times \left[ \frac{\ell}{\ell + 1 - b\nu} (\ell + 1 - b) \cdot F(b(1-\nu), \ell + 1; \ell + 2 - b\nu; 1-x) \right.$$  

$$\left. - \ell c \cdot F(b(1-\nu), \ell; \ell + 1 - b\nu; 1-x) \right].$$

To simplify further the latter expression, first invoke the identity

$$\beta F(\alpha, \beta + 1; \gamma + 1; z) = \gamma F(\alpha, \beta; \gamma; z) - (\gamma - \beta) F(\alpha, \beta; \gamma + 1; z)$$

easily derived from representation (5.1) for $F(\alpha, \beta + 1; \gamma + 1; z)$, after splitting the factor $t^\beta$ of the integrand into $t^\beta = t^{\beta-1} - t^{\beta-1}(1-t)$. Applying (5.5) to $\alpha = b(1-\nu)$, $\beta = \ell$ and $\gamma = \ell + 1 - b\nu$ then enables one to express the term
\[ F(b(1 - \nu), \ell + 1; \ell + 2 - b\nu; 1 - x) \]

in the r.h.s. of (5.4) as a combination of \( F(b(1 - \nu), \ell; \ell + 1 - b\nu; 1 - x) \) and \( F(b(1 - \nu), \ell; \ell + 2 - b\nu; 1 - x) \) hence, after simple algebra,

\[
Q_{b,\ell} = \frac{\Gamma(\ell)\Gamma(1 - b\nu)}{\Gamma(\ell + 1 - b\nu)} \cdot \left[ ((\ell + 1 - b - \ell c) \cdot F(b(1 - \nu), \ell; \ell + 1 - b\nu; 1 - x) \right.
\]

\[ - \frac{(\ell + 1 - b)(1 - b\nu)}{\ell + 1 - b\nu} \cdot F(b(1 - \nu), \ell; \ell + 2 - b\nu; 1 - x) \right]. \tag{5.6} \]

Furthermore, the contiguity identity \[8, \text{Sect.15.5.18}\]

\[ \gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)]z \left[ F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z F(\alpha, \beta; \gamma + 1; z) \right] = \gamma(\gamma - 1)(1 - z) F(\alpha, \beta; \gamma - 1; z) \]

applied to \( \alpha = b(1 - \nu), \beta = \ell \) and \( \gamma = \ell + 1 - b\nu \) allows us to write the last term \( F(b(1 - \nu), \ell; \ell + 2 - b\nu; 1 - x) \) in the bracket of the r.h.s. of (5.6) as a combination of \( F(b(1 - \nu), \ell; \ell - b\nu; 1 - x) \) and \( F(b(1 - \nu), \ell; \ell + 1 - b\nu; 1 - x) \), that is,

\[ F(b(1 - \nu), \ell; \ell + 2 - b\nu; 1 - x) = \frac{\ell + 1 - b\nu}{(\ell + 1 - b)(1 - b\nu)(1 - x)} \times \]

\[
\left[ (\ell - b\nu)x \cdot F(b(1 - \nu), \ell; \ell - b\nu; 1 - x) - [\ell - b\nu - (\ell + 1 - b\nu)(1 - x)] \cdot F(b(1 - \nu), \ell; \ell + 1 - b\nu; 1 - x) \right]. \]

inserting the latter relation into the right-hand side of (5.6) then yields

\[
Q_{b,\ell} = \frac{\Gamma(\ell)\Gamma(1 - b\nu)}{\Gamma(\ell + 1 - b\nu)} \cdot \left[ T_{b,\ell} \cdot F(b(1 - \nu), \ell; \ell + 1 - b\nu; 1 - x) - \right.
\]

\[ \frac{1}{1 - x}(\ell - b\nu)x \cdot F(b(1 - \nu), \ell; \ell - b\nu; 1 - x) \right] \tag{5.8} \]

where

\[ T_{b,\ell} = b\nu - \ell c + \frac{(\ell - b\nu)x}{1 - x} = \frac{(\ell - b\nu)x}{1 - x} \]

after the definition \( c = (1 - \nu x)/(1 - x) \) of constant \( c \). Inserting this value of \( T_{b,\ell} \) in the right-hand side of (5.8) readily provides expression (5.2) for \( Q_{b,\ell} \).

We finally show how coefficient \( Q_{b,\ell} \) can be written in terms of a Hypergeometric polynomial only. Applying the general identity \[9, \text{Chap.9, Sect.9.131.1}\]

\[ F(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma; z), \quad |z| < 1, \]

to each term \( F(b(1 - \nu), \ell; \ell + 1 - b\nu; 1 - x) \) and \( F(b(1 - \nu), \ell; \ell - b\nu; 1 - x) \) in (5.2), we obtain

\[
Q_{b,\ell} = -\frac{\Gamma(\ell)\Gamma(1 - b\nu)}{\Gamma(\ell + 1 - b\nu)} \cdot \frac{x^2}{b(1 - x)}(\ell - b\nu) \times R_{b,\ell} \tag{5.10}, \quad b \geq \ell \geq 1, \]

where we set

\[ R_{b,\ell} = \frac{b\nu(\ell - \ell)}{\ell - b\nu} \cdot F(\ell - b + 1, -b\nu + 1; \ell - b\nu + 1; 1 - x) + \]

\[ b x^{-1} \cdot F(\ell - b, -b\nu; \ell - b\nu; 1 - x). \]
From the identity [8, Chap.15, Sect.15.5.1]

\[
(5.11) \quad \frac{d}{dz} F(\alpha, \beta; \gamma; z) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; z), \quad |z| < 1,
\]

applied to parameters \(\alpha = \ell - b\), \(\beta = -b\nu\) and \(\gamma = \ell - b\nu\), the factor \(R_{b,\ell}\) above then equals the derivative

\[
R_{b,\ell} = x^b \frac{d}{dz} (1 - z)^{-b} F(\ell - b, -b\nu; \ell - b\nu; z)|_{z=1-x} = x^b \frac{d}{dz} F(b - b\nu, \ell; \ell - b\nu; z)|_{z=1-x}
\]

\[
= x^b \frac{(b - b\nu)\ell}{\ell - b\nu} F(b - b\nu + 1, \ell + 1; \ell - b\nu + 1; 1 - x)
\]

hence

\[
(5.12) \quad R_{b,\ell} = \frac{(b - b\nu)\ell}{\ell - b\nu} x^{-1} F(\ell - b, -b\nu; \ell - b\nu + 1; 1 - x)
\]

where we have successively applied identity (5.9), (5.11) and (5.9) again to derive the second, third and fourth equality, respectively. Using (5.12), expression (5.10) for \(Q_{b,\ell}\) then reads

\[
(5.13) \quad Q_{b,\ell} = -\frac{\Gamma(\ell)\Gamma(1 - b\nu)}{\Gamma(\ell - b\nu)} \frac{x}{1 - x} \frac{(1 - \nu)}{\ell - b\nu} S_{b,\ell}(\nu; 1 - x), \quad b \geq \ell \geq 1,
\]

where we set

\[
S_{b,\ell}(\nu; 1 - x) = F(\ell - b, -b\nu; \ell - b\nu + 1; 1 - x).
\]

To reduce further \(S_{b,\ell}(\nu; 1 - x)\), invoke the identity [8, Chap.15, Sect.15.8.7]

\[
(5.14) \quad F(-m, \beta, \gamma; 1 - x) = \frac{\Gamma(\gamma)\Gamma(\gamma - \beta + m)}{\Gamma(\gamma - \beta)\Gamma(\gamma + m)} F(-m, \beta + 1 - m - \gamma; x), \quad x \in \mathbb{C},
\]

for any non negative integer \(m\) and complex numbers \(\beta, \gamma\) such that \(\text{Re}(\gamma) > \text{Re}(\beta)\); applying (5.14) to factor \(S_{b,\ell}(\nu; 1 - x)\) in (5.13) then readily gives the final expression (1.10) for all indexes \(b \geq \ell \geq 1\). This concludes the proof of Proposition 3.1

5.2. Proof of Corollary 3.2. From the definition (1.10) of integral operator \(\mathcal{M}\), split the integral

\[
\mathcal{M} E^*(z) = \int_0^1 e^{-z^\tau(1 - (1 - z)t)} E^* \left( z^\tau(1 - t) \right) \frac{dt}{t}
\]

over adjacent segments \([0, \tilde{t}]\) and \([\tilde{t}, 1]\), respectively; applying the variable change \(\tau = t^{-\nu}(1 - t)\) on each of these two intervals with \(\tau = \tau_-(t) \Leftrightarrow t = \theta_-(\tau) \in [0, \tilde{t}]\) and \(\tau = \tau_+(t) \Leftrightarrow t = \theta_+(\tau) \in [\tilde{t}, 1]\) by the definition (3.15) of mappings \(\theta_-\) and \(\theta_+\), we then successively obtain

\[
\mathcal{M} E^*(z) = \int_0^{\tilde{t}} e^{-z\theta_-(\tau)^{-\nu}(1 - (1 - z)\theta_-(\tau))} E^* (z \tau) \frac{-d\tau}{\theta_-(\tau)^{-\nu}((1 - \nu)\theta_-(\tau))} + \int_{\tilde{t}}^1 e^{-z\theta_+(\tau)^{-\nu}(1 - (1 - z)\theta_+(\tau))} E^* (z \tau) \frac{-d\tau}{\theta_+(\tau)^{-\nu}((1 - \nu)\theta_+(\tau))}
\]
with \( \hat{\tau} = \tau_-(\hat{t}) = \tau_+(\hat{t}) \) and the differential \( \frac{dt}{t} = -d\tau/[t^{-\nu}(\nu + (1 - \nu)t)] \); this readily reduces to a single integral over segment \([0, \hat{\tau}]\), that is,
\[
\mathcal{M}E^*(z) = \int_0^{\hat{\tau}} [\Psi_-(z, \tau) - \Psi_+(z, \tau)] E^*(z, \tau) \, d\tau
\]
with \( \Psi_-(z, \tau) \) and \( \Psi_+(z, \tau) \) given as in the Corollary. The final variable change \( \xi = z \cdot \tau \) yields the right-hand side of (3.16) and the corresponding integral equation.

- We finally verify that the r.h.s. of (3.16) is well-defined for any \( E^* \in \mathcal{H}_0 \). The denominator \( t^{-\nu}(\nu + (1 - \nu)t) \) of \( \Psi_-(z, \tau) \) with \( t = \theta_-(\tau) \) (resp. of \( \Psi_+(z, \tau) \) with \( t = \theta_+(\tau) \)) vanishes at either \( \tau = 0 \) or \( \tau = \hat{\tau} \) (resp. at \( \tau = \hat{\tau} \)). As to the possible singularity at \( \tau = 0 \) for \( \Psi_-(z, \tau) \), we have \( \tau \sim t^{-\nu} \) for small \( t = \theta_-(\tau) \) so that
\[
\frac{1}{t^{-\nu}(\nu + (1 - \nu)t)} \sim -\frac{t^\nu}{\nu} \sim -\frac{1}{\nu \tau}, \quad \tau \to 0;
\]
the product \( E^*(z, \tau) \cdot \Psi_-(z, \tau) \) is thus integrable near \( \tau = 0 \) for any \( E^* \in \mathcal{H}_0 \). Furthermore, a second order Taylor expansion of \( \tau = \tau(t) \) near point \( t = \hat{t} \) yields \( \tau = \hat{\tau} + \tau''(\hat{t})(t - \hat{t})^2/2 + o(t - \hat{t})^2 \) with \( \tau'(\hat{t}) = 0 \) by definition and \( \tau''(\hat{t}) < 0 \); as a result,
\[
t - \hat{t} \sim \pm \sqrt{-\frac{2(\tau - \hat{\tau})}{\tau''(\hat{t})}}, \quad \tau \to \hat{\tau}.
\]
The denominator \( t^{-\nu}(\nu + (1 - \nu)t) \) of either \( \Psi_-(z, \tau) \) or \( \Psi_+(z, \tau) \) is consequently asymptotic to
\[
t^{-\nu}(\nu + (1 - \nu)t) \sim (\hat{t})^{-\nu}(\nu + (1 - \nu)(t - \hat{t}) \sim \pm(\hat{t})^{-\nu}(\nu + 1)\sqrt{\frac{2(\tau - \hat{\tau})}{-\tau''(\hat{t})}}
\]
when \( \tau \to \hat{\tau} \); the singularity of \( \Psi_-(z, \tau) \) (resp. \( \Psi_+(z, \tau) \)) at point \( \tau = \hat{\tau} \) is consequently of order
\[
\Psi_-(z, \tau) = O\left(\frac{1}{\sqrt{\hat{\tau} - \tau}}\right), \quad \Psi_+(z, \tau) = O\left(\frac{1}{\sqrt{\hat{\tau} - \tau}}\right)
\]
and the kernel \( \Psi(z, \tau) = \Psi_-(z, \tau) - \Psi_+(z, \tau) \) is thus integrable at \( \tau = \hat{\tau} \). This ensures that the singular integral (3.16) is well-defined for any \( E^* \in \mathcal{H}_0 \) ■

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