\textbf{L}^\infty \text{ ILL-POSEDNESS FOR A CLASS OF EQUATIONS ARISING IN HYDRODYNAMICS}

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\textsc{Abstract.} Many questions related to well-posedness/ill-posedness in critical spaces for hydrodynamic equations have been open for many years. In this article we give a new approach to studying norm inflation (in some critical spaces) for a wide class of equations arising in hydrodynamics. As an application, we prove strong ill-posedness of the \textit{d}-dimensional Euler equations in the class $C^1 \cap L^2(X)$ and also in $C^k \cap L^2(X)$ where $X$ can be the whole space, a smooth bounded domain, or the torus.

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1. \textsc{Introduction}

1.1. \textbf{The concept of well-posedness.} In 1903, Jacques Hadamard set forth a concept of well-posedness for mathematical problems of physical origin, particularly for partial differential equations. Hadamard suggested that for a PDE problem to be well-posed (whether it be an initial value problem, a boundary value problem, or both) it should enjoy the following properties:

(1) Existence, \\
(2) Uniqueness, \\
(3) Continuous dependence on the data.

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These conditions were obviously physically motivated because if the equation is to model a physical phenomenon then one would expect that solutions to the model exist, are unique, and that small perturbations should not result in wild responses by the system (at least for small times). Of course, well-posedness or the lack thereof depends upon where one is looking for a solution; thus a particular Cauchy problem might be well-posed in one solution class but ill-posed in another.

Based on the definition of well-posedness, one can think at least of three types of ill-posedness: nonexistence, nonuniqueness, and discontinuous dependence on the data. In this work we are interested in nonexistence and discontinuity with respect to the data. In our investigations nonexistence in a space $X$ will be deduced from the fact that a solution uniquely exists in a larger space with a norm in $X$ that becomes immediately equal to infinity. Hence, nonexistence can be thought of as the strongest kind of ill-posedness while discontinuity with respect to the data is the weakest kind of ill-posedness.

There are weaker kinds of discontinuity that were studied in the literature: The flow map is not $C^1$ or $C^2$ with respect to the data (see, for example, [17] and [28]), or the the flow map is not uniformly continuous with respect to the data in a bounded set [23]. Many other ill-posednessed questions have been studied in the case of dispersive equations and we refer the reader to [11, 24, 20, 11, 10].

**Definition 1.1.** We say that a Cauchy problem

$$f_t = N(f),$$

$$f(0) = f_0$$

is **mildly ill-posed** in a space $X$ if there exists a space $Y$ embedded in $X$ such that for all $\epsilon > 0$ there exists $f_0 \in Y \subset X$, $|f_0|_X \leq \epsilon$ and there exists a unique solution to the equation which is bounded in time in $Y$, $f \in L^\infty([0,T];Y)$ for some $T > 0$ with initial data $f_0$ such that: $|f(t)|_X \geq \epsilon$ for some universal $c$ (independent of $\epsilon$) and for some $0 < t < \epsilon$. If $c$ can be taken to be equal to $\frac{1}{\epsilon}$, we will say that the equation is **strongly ill-posed**.

Typically, the space $Y$ will be a space which is smoother than $X$ and for which local existence is already known. The initial data $f_0$ will be chosen such that $|f_0|_X \leq \epsilon$ while $|f_0|_Y$ may be large. Of course, both mild ill-posedness and strong ill-posedness only imply discontinuity with respect to the initial data. However, in many cases, strong ill-posedness can be used to prove nonexistence (the strongest form of ill-posedness in the sense of Hadamard)—see section 9.

1.2. **The Euler equations of incompressible flow.** One of the most elusive and difficult issues to deal with in studying the equations of incompressible hydrodynamics is their inherent nonlocality. This can be seen intuitively: every part of the fluid should, in some way, affect every other part of the fluid. Recall the Euler equations for inviscid and incompressible flow modeling an ideal (frictionless) liquid in the whole space:

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0 \text{ on } \mathbb{R}^n \times (0, \infty),$$

(1.1)
\[
\text{(1.2) } \ \ \ \ \ \text{div}(u) = 0 \ \text{ on } \mathbb{R}^n \times (0, \infty),
\]
\[
\text{(1.3) } \ \ \ \ \ u(x,t) \to 0 \ \text{ as } |x| \to \infty,
\]
\[
\text{(1.4) } \ \ \ \ \ u(x,0) = u_0 \ \text{ on } \mathbb{R}^n.
\]

In (1.1)-(1.4), \(u(x,t) \in \mathbb{R}^n\) is the velocity of the fluid at position \(x \in \mathbb{R}^n\) and at time \(t \in [0, \infty)\). Equation (1.3) says that the initial velocity profile of the fluid is given by \(u_0\). Equation (1.4) is an idealized condition which says that the significant movement in the fluid is localized in space. (1.2) dictates that the fluid be incompressible, which means that if one tracks the evolution of a particular portion of the fluid in time then the volume of that portion cannot change in time. Equation (1.1) is just Newton’s second law, the momentum equation, which says that the only force acting on the fluid is that of internal pressure.

One of the most challenging basic problems in the study of fluid equations, is the question of well-posedness for the Euler equations. In two dimensions, the global well-posedness question was settled in \(C^{k,\alpha}\) spaces with \(k \geq 1, 1 > \alpha > 0\) by Wolibner \[35\] and Hölder \[19\] in the 1930’s. Note that well-posedness in \(C^k\) was left open. In three dimensions, it is not known whether the Euler equations are globally well-posed in the class of smooth solutions. The only results which exist in this direction are local well-posedness results which go back to Lichtenstein \[25\] in the \(C^{k,\alpha}\) case and blow-up criteria such as the blow-up criteria of Beale, Kato, and Majda \[5\] and the geometric criteria of Constantin, Fefferman, and Majda \[12\] (see \[14\] and the references therein for various improvements on these blow-up criteria). The criteria of Beale-Kato-Majda states that the growth of the high Sobolev norms of the velocity field is controlled by the growth of the \(L^\infty\) norm of the vorticity. In particular, they prove:

\[
|u|_{H^s} \leq |u_0|_{H^s} \exp(C \int_0^t |\text{curl}(u)(\tau)|_{L^\infty} \, d\tau), \text{ for all } s > \frac{d}{2} + 1,
\]

which was an improvement on the classical energy estimate:

\[
|u|_{H^s} \leq |u_0|_{H^s} \exp(C \int_0^t |\nabla u(\tau)|_{L^\infty} \, d\tau).
\]

Therefore, higher \(H^s\) norms cannot blow up in finite time unless \(\int_0^t |\text{curl}(u)|_{L^\infty} \, d\tau\) becomes infinite in finite time. In the two-dimensional case, the main tool in the well-posedness proof was the fact that the vorticity, \(\omega := \text{curl}(u)\), is uniformly bounded in time. No such result exists in three dimensions. In this regard, it is instructive to mention the following interesting example.

1.3. A first example. To motivate some of our results, we start with a simple example:

**Proposition 1.2.** There exists \(u_0 \in W^{1,2}(\mathbb{T}^3)\) such that \(\omega_0 = \nabla \times u_0 \in L^2 \cap L^\infty\) so that the 3-D Euler system has a global (and unique) weak solution, \(u(t)\), for which

\[
|\omega(t)|_{L^p} = |u(t)|_{W^{1,p}} = +\infty,
\]

for all \(p > \frac{2}{1-e^{-t}}\).
As a consequence, the 3D Euler equations are strongly ill-posed (in the strongest sense of Hadamard) in the class of finite-energy velocity fields with bounded vorticity.

The example is quite simple: consider the so-called 2\frac{1}{2} dimensional solutions of the 3D Euler equations on $\mathbb{T}^3$. Namely, those solutions depending only upon $x$ and $y$ (which are of finite energy on $\mathbb{T}^3$). These solutions are given by a solution of the 2D Euler equations $u_h(t)$ and a freely advected scalar $u_3$. If $u_h$ is taken to be such that its 2-D vorticity is $\text{sgn}(x)\text{sgn}(y)$, then it is well known that the flow-map associated with $u_h$ is of regularity $C^{e^{-t}}$ and no better. Thus, $u_3(x, t) = u_{30}(\Phi_h(x, -t))$ and $u_{30}$ is then chosen such that $u_3$ does not belong to $C^\alpha$ for $\alpha > e^{-t}$ (for example, one can take $u_{30}(x, y) = x$ in a small neighborhood of the origin). Proposition 6.2 then follows by Sobolev embedding.

We remark that this example can be modified to give an example of a global smooth solution of the 3D Euler equations on $\mathbb{T}^3$ for which the vorticity grows exponentially in time:

**Proposition 1.3.** There exists $u_0 \in C^\infty(\mathbb{T}^3)$ such that the 3-D Euler system has a global strong solution $u(t)$ with initial data $u_0$ for which:

$$|\omega(t)|_{L^\infty} \geq e^t$$

for all $t > 0$.

To see that this proposition is true, simply take $u_h$ to be the stationary 2-D solution $(\sin(x)\cos(y), -\cos(x)\sin(y))$, which has a hyperbolic point at the origin. A consequence of this choice is that the 2-D flow map induced by $u_h$ has an exponential contraction along the $y$-axis at the origin. Once $u_3$ is chosen to be non-constant along the $y$-axis the exponential growth is attained. Using some recent ideas of Kiselev and Sverák [22] one can actually prove that the exponential growth for data with a hyperbolic point for the 2\frac{1}{2} dimensional solutions is, in a sense, generic. This gives some idea as to why one might consider questions of ill-posedness for weak solutions: behind ill-posedness for weak solutions there may be uncontrollable growth for strong solutions.

1.4. Previous ill-posedness results for weak solutions. In recent years, the question of well-posedness at low-regularity has become of great interest. As we stated above, ill-posedness can mean one of three things: the initial data can start in $X$ and then leave $X$, or we may have non-uniqueness, or the solution map may be discontinuous—the first case being the strongest form of ill-posedness. There are still many questions which are unanswered in the well-posedness theory of weak solutions even in two spatial dimensions. Existence of weak solutions in the class $W^{1,p}, p > 1$ was established in two dimensions. Uniqueness has been proven only for weak solutions which are Lipschitz or "almost" Lipschitz (see the works of Yudovich [37], [36], and Vishik [33], [34] for example). It does not seem that continuous dependence on data has been established for weak solutions in any sort of generality.

Previous works in the direction of ill-posedness include results of DiPerna-Lions [15] (ill-posedness (non-existence) in $W^{1,p}, p < \infty$), Bardos-Titi [4] (non-existence in $C^\alpha, \alpha < 1$), and Misiolek-Yoneda [26] (non-existence in critical Besov-spaces),
to mention a few. All of the above cases were done by explicit examples. A very interesting preprint by Bourgain and Li \[6\] studies the ill-posedness of the Euler equations with velocity in \(H^{\frac{d}{2}+1}\), the difficulty being that \(H^{\frac{d}{2}+1}\) is a critical space sitting at the lower threshold of the classes of strong solutions where local well-posedness holds (because the Euler equations are locally well-posed in \(H^s\) with \(s > \frac{d}{2} + 1\)).

Finally, nonuniqueness of weak solutions was shown by Scheffer \[31\], Shnirelman \[32\], De Lellis and Szekelyhidi \[13\], Isett \[21\], and Buckmaster \[8\] for weak solutions of the Euler equations in various "very weak" spaces, the smallest of which is \(C^{\alpha}_{t,x}\), \(\alpha < \frac{1}{5}\). It is conjectured that non-uniqueness should hold up to \(C^{\frac{3}{2}}\).

1.5. Comparison with some recent results. This paper is devoted to establishing a general framework to study non-linear and non-local transport equations in critical spaces based on \(L^\infty\) (such as \(L^\infty\), \(C^1\), etc.). As an application, we prove non-existence of \(C^1\) solutions to the incompressible Euler equations coming from \(C^1\) data (this is done in Section 8). Recently, the same result was proven by Bourgain and Li \[7\] using completely different methods. Misiolek and Yoneda also proved non-continuity of the solution map on \(C^1\) \[27\].

1.6. Brief outline of the paper. In section 2 we will give our main technical linear result from which all the applications will follow. Section 3 will be the proof of the linear result. Section 4 we will prove ill-posedness for linear transport equations with singular integral forcing. In sections 5 and 6 we will prove ill-posedness for perturbations of the 2D Euler equations and the full 3D Euler equations in vorticity form in the class of flows with bounded vorticity. In section 7 we prove the strong ill-posedness of the Euler equations in the class of \(C^1\) velocity fields and that the solution map on \(C^{1,\alpha}\) does not have a bounded extension to \(C^1\). In section 8, we show non-existence of a \(C^1\) solution to the Euler equations from some \(C^1\) initial data.

2. The Building Block : A Linear Result

We consider linear equations for the following form:

\[
(2.1) \quad f_t + u \cdot \nabla f = R(f),
\]

\[
(2.2) \quad \text{div}(u) = 0
\]

\[
(2.3) \quad f(t = 0) = f_0.
\]

Here, \(R\) is a Calderón-Zygmund singular-integral operator, \(u\) is a Lipschitz function and \(b\) is a bounded function. A natural question to ask is:

\[
(2.4) \quad \text{If } f_0 \in L^\infty, \text{ is it true that } f(t) \in L^\infty \text{ for even a short time?}
\]
2.1. A one-dimensional example. It is instructive to consider the following one-dimensional example when the velocity field is not present ($u \equiv 0$):

$$f_t = H(f),$$

where $H$ is the Hilbert transform.

Simple calculations, using the fact that $H^2 = -1$ yield that the solution to this simple evolution equation can be written explicitly:

$$f(t) = \cos(t)f_0 + \sin(t)H(f_0).$$

There are two points which are important to take from this calculation:

1. If $t$ is small enough, $\exp(tH)$ is unbounded on $L^\infty$.
2. If $t$ is small enough $\exp(tH)$ is as singular in $L^\infty$ as $H$.

Since there exist functions $f_0$ which are bounded for which $H(f_0)$ is unbounded, the answer to question (1.3) is negative for the case $u \equiv 0$ and $R = H$.

2.2. Preliminaries and notation. Before stating our main result we will first introduce a little bit of necessary background material. Due to the criticality of our problem, we will need Besov spaces.

Throughout this paper we will use the convention that $C$ is an absolute constant which changes from line to line. By $L^p$ we mean the space of measurable functions $f$ on $\mathbb{R}^d$ so that $|f|$ is integrable. By $W^{1,p}$ we mean the Sobolev space on $\mathbb{R}^d$ of $L^p$ functions whose derivative also belongs to $L^p$. We will also define the Lipschitz class using the following norm:

$$|f|_{Lip} = |f|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$  

We will be making use of two kinds of commutators: a commutator of a singular integral operator and composition with a given function and a commutator of a singular integral operator and multiplication by a given function $[\cdot, \cdot]$ and $[\cdot, \cdot]_*$.

$$[R, \Phi] f := R(f \circ \Phi) - R(f) \circ \Phi,$$

and

$$[R, g]_* f := R(fg) - R(f)g.$$  

We recall here the Littlewood-Paley decomposition of a function. We define $C$ to be the ring of center 0, of small radius $1/2$ and great radius 2. There exist two nonnegative radial functions $\chi$ and $\phi$ belonging respectively to $C_0^\infty(B(0, 1))$ and to $C_0^\infty(C)$ so that

$$\chi(\xi) + \sum_{q \geq 0} \phi(2^{-q}\xi) = 1,$$

and

$$|p - q| \geq 2 \Rightarrow \text{Supp } \phi(2^{-q}) \cap \text{Supp } \phi(2^{-p}) = \emptyset.$$  

For instance, one can take $\chi \in C_0^\infty(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and take $\phi(\xi) = \chi(\xi/2) - \chi(\xi)$.

Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let $h$, $\tilde{h}$, $\Delta_q$, $S_q$ ($q \in \mathbb{Z}$) be defined as follows:

$$h = \mathcal{F}^{-1}\phi \quad \text{and} \quad \tilde{h} = \mathcal{F}^{-1}\chi,$$

$$\Delta_q u = \mathcal{F}^{-1}(\phi(2^{-q}\xi)\mathcal{F}u) = 2^{qd} \int h(2^q y)u(x-y)dy,$$

$$S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u) = 2^{qd} \int \tilde{h}(2^q y)u(x-y)dy.$$

We point out that $S_q u = \sum_{q' \leq q-1,q' \in \mathbb{Z}} \Delta_{q'} u$.

We define the inhomogeneous Besov spaces by

**Definition 2.1.** Let $s$ be a real number, $p$ and $r$ two real numbers greater than 1. Then we define the following norm

$$|u|_{B_{p,r}^s} = |S_0 u|_{L^p} + \left(2^{qs} |\Delta_q u|_{L^p}\right)_{q \in \mathbb{N}} \ell_r^p(\mathbb{N}).$$

**Definition 2.2.** Let $s$ be a real number, $p$ and $r$ two real numbers greater than 1. We denote by $B_{p,r}^s$ the space of tempered distributions $u$ such that $\|u\|_{B_{p,r}^s}$ is finite. We refer to [3] for the proof of the following results:

**Lemma 2.3.**

$$|\Delta_q u|_{L^b} \leq 2^{d\left(1 - \frac{1}{b}\right)q} |\Delta_q u|_{L^a} \quad \text{for} \quad b \geq a \geq 1.$$

The following corollary is straightforward:

**Corollary 2.4.** If $b \geq a \geq 1$, then, we have the following continuous embeddings

$$B_{a,r}^s \subset B_{b,r}^{s-d\left(\frac{1}{a} - \frac{1}{b}\right)}.$$

**Corollary 2.5.**

$$B_{p,1}^a \subset L^\infty$$

if $ap = d$ and the imbedding is continuous.

**Proof.** By Corollary 2.4,

$$B_{p,1}^a \subset B_{\infty,1}^0.$$ 

Now, the norm of a function on $B_{\infty,1}^0$ is given as follows:

$$|f|_{B_{\infty,1}^0} = \sum_j |\Delta_j f|_{L^\infty}$$

and

$$f = \sum_j \Delta_j f.$$

Thus, $|f| \leq |f|_{B_{\infty,1}^0}$ and we are done.
Finally, we recall that singular integral operators are bounded on Besov spaces.

**Lemma 2.6.** If $R$ is a Calderón-Zygmund singular integral operator with a kernel $K(x) = \frac{\Omega(x)}{|x|^d}$, with $\Omega$ smooth and mean-zero on the unit-sphere, then $R$ is bounded from $B^{a}_{\rho,q}$ to itself for all $a \geq 0, \rho \geq 1, \text{ and } q \geq 1$.

### 2.3. The main (linear) result.

**Theorem 2.7.** Let $u$ be a given divergence-free Lipschitz function, $u \in L^{\infty}_{\text{loc}} \text{Lip}$. Suppose that $R$ is a Calderón-Zygmund singular integral operator. Let $f_0$ be a Schwartz class function and let $f$ be the unique solution of (2.1)-(2.2) with initial data $f_0$.

Then given $t > 0$, there exists a constant $C > 0$ independent of $u$ and $f_0$ such that

$$|f(t)|_{L^{\infty}} \geq |tR(f_0) + f_0|_{L^{\infty}} - Ct^2(1 + |u|_{L^{\infty}} \text{Lip}) \exp(tC|u|_{L^{\infty}} \text{Lip}))|f_0|_{B^{\frac{1}{2}}_{2d,1}}. $$

**Remark:** Note that estimate (2.5) is only useful for $t$ small as the right-hand side of the estimate is negative for $t$ large.

The proof of Theorem 2.1 is based upon a non-trivial commutator estimate.

### 3. Proof of the Linear Theorem

The subject of this section is the proof of Theorem 2.7. The proof is based upon a non-trivial commutator estimate which we now present.

#### 3.1. The Commutator Estimate

**Proposition 3.1.** Let $\Phi$ be a bi-Lipschitz mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$ which is measure preserving. Define the following commutator operating on $L^2$:

$$[R, \Phi] \omega = R(\omega \circ \Phi) - R(\omega) \circ \Phi,$$

where $R$ is as in Lemma 2.6.

Let $0 < a < 1$ and $1 < \rho < \infty$. Then $[R, \Phi] : B^a_{\rho,1} \to B^a_{\rho,1}$. Furthermore, there exists a universal constant $c$ depending only upon $R$ such that if

$$M := \max\{|\Phi - \text{Id}|_{\text{Lip}}, |\Phi^{-1} - \text{Id}|_{\text{Lip}}\} \leq c,$$

where $\text{Id}$ is the identity matrix, then,

$$|[R, \Phi] \omega|_{B^a_{\rho,1}} \lesssim M|\omega|_{B^a_{\rho,1}}, \quad (3.1)$$

with constant only depending upon $|\Phi|_{\text{Lip}}$ and $|\Phi^{-1}|_{\text{Lip}}$ and the dimension $n$, and the operator $R$.

We would first like to relate this proposition to a well-known problem in harmonic analysis, which is the boundedness of the Calderón commutators:

Indeed, write

$$[R, \Phi] \omega \circ \Phi^{-1} = R(\omega \circ \Phi) \circ \Phi^{-1} - R(\omega) = K \ast (\omega \circ \Phi) \circ \Phi^{-1} - K \ast \omega.$$
Now, using the fact that $\Phi$ is measure preserving,
\[
[R, \Phi]_\omega \circ \Phi^{-1}(x) = \int_{\mathbb{R}^d} [K(\Phi^{-1}(x) - \Phi^{-1}(y)) - K(x-y)]\omega(y)dy.
\]

In the special case where $K = pv\frac{1}{x}$ in one dimension (the Hilbert transform), then we see that
\[
[R, \Phi]_\omega = \int_{\mathbb{R}} \frac{1}{|x-y|} \omega(y)dy
\]

\[
= \int_{\mathbb{R}} \frac{(I - \Phi^{-1})(x) - (I - \Phi^{-1})(y)}{x-y} \cdot \frac{x-y}{\Phi^{-1}(x) - \Phi^{-1}(y)} \cdot \frac{\omega(y)}{x-y}dy
\]

thus we see that in order to estimate this commutator, we would need to estimate operators of the form:
\[
T_A(\omega) = \int_{\mathbb{R}} F\left(\frac{A(x) - A(y)}{x-y}\right) \frac{\omega(y)}{x-y}dy,
\]

where $A$ is a Lipschitz function. Estimates of this type have been studied in $L^p$ spaces by many authors. We refer the reader to the recent book of Muscalu and Schlag [30] and the references therein. Fortunately, due to the large literature on these operators, we will be able to use some of the existing results to prove estimate (3.1). In particular, in [29], Murray proved the lemma in the $L^2$ case. Here, we show how the modern theory of Calderón-Zygmund operators is used to extend her result to the $L^p$ case. This is, admittedly, a simple exercise for the specialists.

We recall here a simple consequence of Murray’s theorem (Theorems 6.1 and 6.2 of [29]):

**Theorem 3.2.** *(Murray, 1985) Proposition 3.1 holds with $B^a_{p,1}$ replaced by $L^2$.***

In fact, Murray proved that the mapping $\Phi \to \Phi R \Phi^{-1}$ is analytic for $\Phi$ in a neighborhood of the identity.

We will use Theorem 3.2 to prove Proposition 3.1. The strategy is as follows:

1. Use Theorem 3.2 and pass from the commutator estimate on $L^2$ to the same estimate on $L^p$; here we will have to be careful not to lose the $|\Phi - Id|_{Lip}$ factor when passing from the $L^2$ to the $L^p$ estimate.

2. Use the $L^p$ estimate to prove a $W^{1,p}$ estimate. Here we will make use of the classical Coifman-Rochberg-Weiss commutator estimate, which will allow us to commute multiplication by a bounded function and a Riesz operator.

3. Conclude using the method of real interpolation.

**Proof.** *(Of Proposition 3.1)* The proof relies upon first observing that $[R, \Phi]$ is a linear operator and thus, using results from the theory of interpolation, it suffices to show that $[R, \phi]$ satisfies estimate (3.1) on $L^p$ and $W^{1,p}$. Namely, it suffices to prove the following two inequalities:
\[
|[R, \Phi]_\omega|_{L^p} \lesssim M|\omega|_{L^p}
\]
and
\[
\|[R, \Phi]\omega|_{W^{1,p}} \lesssim M|\omega|_{W^{1,p}},
\]
for all \(1 < p < \infty\).

For the \(L^p\) estimate, we use, as a black box, Theorem 3.2—the work will go into writing our commutator in the form of the operator in Theorem 3.2. A direct consequence of their results is the \(L^p\) estimate above. As for the \(W^{1,p}\) estimate, we show that this is a consequence of the \(L^p\) estimate.

3.1.1. The \(L^p\) estimate. To get the \(L^p\) estimate, we are first going to show that the commutator can be written as an operator with a so-called standard kernel [18]. Indeed, define \(T\) as follows:

\[
T(\omega) := [R, \Phi^{-1}]\omega \circ \Phi(x) = \int_{\mathbb{R}^d} J(x, y)\omega(y)dy,
\]

with \(J(x, y) := K(\Phi(x) - \Phi(y)) - K(x - y)\).

(note we have interchanged \(\Phi^{-1}\) and \(\Phi\) just for notational convenience; they play the same role throughout)

**Claim:**

\[
|J(x, y)| \lesssim \frac{M}{|x - y|^n},
\]

\[
|\nabla J(x, y)| \lesssim \frac{M}{|x - y|^{n+1}},
\]

where \(M = \max\{|\Phi - Id|_{Lip}, |\Phi^{-1} - Id|_{Lip}\}\) and with constants depending only on the dimension and on the Lipschitz norm of \(\Phi\) and its inverse.

**Proof of Claim:**

Recall that

\[
K(x) = \frac{\rho(x)}{|x|^n}
\]

where \(\rho\) is smooth on the unit sphere.

Therefore,

\[
J(x, y) = \frac{\rho(x - y)|\Phi(x) - \Phi(y)|^n - \rho(\Phi(x) - \Phi(y))|x - y|^n}{|x - y|^n|\Phi(x) - \Phi(y)|^n}
\]

Now call

\[
\Psi := \Phi - Id.
\]

Then,

\[
J(x, y) = \frac{\rho(x - y)|\Psi(x) - \Psi(y) - x - y|^n - \rho(\Phi(x) - \Phi(y))|x - y|^n}{|x - y|^n|\Phi(x) - \Phi(y)|^n}
\]

\[
= \frac{\rho(x - y)|\Psi(x) - \Psi(y) - x - y|^n - \rho(x - y)|x - y|^n + \rho(x - y)|x - y|^n - \rho(\Phi(x) - \Phi(y))|x - y|^n}{|x - y|^n|\Phi(x) - \Phi(y)|^n}
\]
So we get two terms:

\[ I := \frac{\rho(x - y)|\Psi(x) - \Psi(y) - x - y|^n - \rho(x - y)|x - y|^n}{|x - y|^n|\Phi(x) - \Phi(y)|^n} \]

and

\[ II := \frac{\left(\rho(x - y) - \rho(x - y + \Psi(x) - \Psi(y))\right)|x - y|^n}{|x - y|^n|\Phi(x) - \Phi(y)|^n} \]

The fact that \( I \) satisfies the claim is obvious because:

\[ |\Psi|_{Lip} \leq M, \quad \frac{|\Phi(x) - \Phi(y)|}{|x - y|} \lesssim 1, \quad \frac{|x - y|}{|\Phi(x) - \Phi(y)|} \lesssim 1. \]

\( II \) is similarly controlled because \( \rho \) is smooth on the unit sphere. This completes the proof of the claim.

Now that \( J \) satisfies the above estimates, and since, by Theorem 3.2, an \( L^2 \) commutator estimate holds:

\[ [R, \Phi]_{L^2 \to L^2} \leq M, \]

we are in a position to now pass to the \( L^p \) estimate.

Recall the following standard theorem [18]:

**Proposition 3.3.** Assume \( J(x, y) \) satisfies

\[ |J(x, y)| \lesssim \frac{A}{|x - y|^n} \]

and

\[ |\nabla J(x, y)| \leq \frac{A}{|x - y|^{n+1}}. \]

Let \( T \) be the singular integral operator associated with the kernel \( J \). Assume that

\[ |T|_{L^2 \to L^2} \leq B. \]

Then, \( T \) maps \( L^p \) to itself for \( 1 < p < \infty \).

Moreover,

\[ |T|_{L^p \to L^p} \leq C(n) \max\{p, (p - 1)^{-1}\}(A + B) \]

Thus, using the claim, Proposition 3.3, as well as Theorem 3.2, we conclude:

\[ [R, \Phi]_{L^p \to L^p} \leq C(|\Phi|_{Lip}, |\Phi^{-1}|_{Lip}, n) \max\{p, (p - 1)^{-1}\} M, \]

where

\[ M = \max\{|\Phi - Id|_{Lip}, |\Phi^{-1} - Id|_{Lip}\}. \]
3.1.2. The $W^{1,p}$ estimate. Consider the commutator applied to some $\omega \in W^{1,p}$, $[R, \Phi] \omega$. In order to prove the desired estimate, we need to estimate $\partial_x ([R, \Phi] \omega)$ in $L^p$.

$$\partial_x ([R, \Phi] \omega) = \partial_x R(\omega \circ \Phi) - \partial_x (R(\omega) \circ \Phi).$$

Note that $\partial_x$ and $R$ commute.

Now we compute:

$$\partial_x R(\omega \circ \Phi) - \partial_x (R(\omega) \circ \Phi) = R(\nabla \omega \circ \Phi \cdot \partial_x \Phi) - R(\nabla \omega) \circ \Phi \cdot \partial_x \Phi$$

$$= R(\nabla \omega \circ \Phi \cdot \partial_x \Phi) - R(\nabla \omega \circ \Phi) \cdot \partial_x \Phi + R(\nabla \omega \circ \Phi) \cdot \partial_x \Phi - R(\nabla \omega) \circ \Phi \cdot \partial_x \Phi$$

$$= R(\nabla \omega \circ \Phi \cdot \partial_x \Phi) - R(\nabla \omega \circ \Phi) \cdot \partial_x \Phi + [R, \Phi] \nabla \omega \cdot \partial_x \Phi$$

where

$$[R, A]_s B = R(A \cdot B) - R(A) \cdot B.$$

Thus,

$$\partial_x ([R, \Phi] \omega) = [R, \partial_x \Phi]_s \nabla \omega \circ \Phi + [R, \Phi] \nabla \omega \cdot \partial_x \Phi.$$

Finally,

$$\partial_x ([R, \Phi] \omega) = [R, \partial_x (\Phi - I)]_s \nabla \omega \circ \Phi + [R, \Phi] \nabla \omega \cdot \partial_x \Phi.$$

Now, the first two terms can be estimated using commutator estimates for the application of a Calderón-Zygmund operator and pointwise multiplication. Note, further, that we can subtract the identity mapping from $\Phi$ in the first term without changing anything because the identity commutes with multiplication (as opposed to the second commutator which is a commutator with composition).

To estimate the first term, we use the Coifman-Rochberg-Weiss commutator estimate \cite{IS} and to estimate the second we use the $L^p$ estimate from above.

Thus,

$$|\partial_x ([R, \Phi] \omega)|_{L^p} \lesssim |\nabla \omega|_{L^p} (|\Phi - I|_{Lip} + |\Phi|_{Lip} |I - \Phi|_{Lip}).$$

This concludes the $W^{1,p}$ estimate. Now that we have the $L^p$ and the $W^{1,p}$ estimate, we can use the method of real interpolation to conclude the corresponding $B^2_{p,1}$ estimate for all $0 < a < 1$. This concludes the proof of Proposition 3.1. $\square$
3.2. **Estimates on the flow.** Given a Lipschitz velocity field $u$ we may solve the following ordinary differential equation, to find the flow induced by $u$:

$$
\dot{\Phi}(x, t) = u(\Phi(t, x), t)
$$

$$\Phi(x, 0) = x.
$$

Because $u$ is divergence free, $\Phi$ is measure preserving. Now, we may write

$$
\Phi(x, t) = x + \int_0^t u(\Phi(x, \tau), \tau) d\tau.
$$

Thus,

$$\Phi(\cdot, t) - I = \int_0^t u(\Phi(\cdot, \tau), \tau) d\tau.
$$

Consequently,

$$|\Phi - I|_{\text{Lip}} \leq t|u|_{\text{Lip}}|\Phi|_{\text{Lip}}
$$

and similarly for $\Phi^{-1}(\cdot, t)$.

Furthermore, by Gronwall’s lemma,

$$|\Phi|_{\text{Lip}} \leq \exp(t|u|_{\text{Lip}}).
$$

In particular,

$$(3.3) \quad |\Phi - I|_{\text{Lip}} \leq t|u|_{\text{Lip}} \exp(t|u|_{\text{Lip}}).
$$

Here, we have abused notation and wrote $|u|_{\text{Lip}}$ for $|u|_{L^\infty_{\text{Lip}}}$.

3.3. **Local well-posedness in the critical besov space.** Because the velocity field is Lipschitz, we will have that the transport equation is well-posed in all the Besov spaces $B^{a}_{\rho,1}$, $0 < a < 1$, $\rho < \infty$. In particular, the transport equation is locally well-posed in the critical Besov spaces $B^{d}_{p,1}$ for all $d < p < \infty$ (note that these spaces imbed in $L^\infty$).

In particular, we have the following proposition.

**Proposition 3.4.** Let $u \in L^\infty([0,1], \text{Lip})$ and let $R$ be a Calderón-Zygmund singular integral operator. Then (2.1)-(2.3) is well-posed in $B^{a}_{\rho,1}$ for every $a \in [0,1]$ and every $\rho \in [1,\infty]$ in the sense that if $f_0 \in B^{a}_{\rho,1}$ then there exists a unique solution $f \in C([0,1], B^{a}_{\rho,1})$ which solves (2.1)-(2.2) with initial condition $f_0$.

Moreover,

$$(3.4) \quad |f(t)|_{B^{a}_{\rho,1}} \leq |f_0|_{B^{a}_{\rho,1}} \exp(tC|\nabla u|_{L^\infty_{\text{Lip}}})
$$

for all $t \in [0,1]$. 
3.4. Write equation along the flow. We have

\[ \frac{\partial f}{\partial t} + u \cdot \nabla f = R(f). \]

Then consider the flow map \( \Phi \) as in subsection 3.2.

Then \( f \) satisfies the following equation,

\[ (f \circ \Phi)_t = R(f) \circ \Phi. \]

In particular,

\[ (f \circ \Phi)_t = R(f \circ \Phi) + [R, \Phi] f. \]

Thus, by Duhamel’s principle,

\[ f \circ \Phi = \exp(Rt)f_0 + \int_0^t \exp(R(t-s))[R, \Phi]f(\tau) d\tau. \]

In particular,

\[ \|f\|_{L^\infty} \geq \|\exp(Rt)f_0\|_{L^\infty} - C \int_0^t \|\exp(R(t-s))[R, \Phi]f(\tau)\|_{B^{\frac{1}{2},1}_{2d,1}} d\tau, \]

where we have used that \( B^{\frac{1}{2},1}_{2d,1} \hookrightarrow L^\infty \).

Note now that \( R \) is bounded on \( B^{\frac{1}{2},1}_{2d,1} \) and \( t \in [0, 1] \). Thus,

\[ \|f\|_{L^\infty} \geq \|\exp(Rt)f_0\|_{L^\infty} - C \int_0^t \|[R, \Phi]f(\tau)\|_{B^{\frac{1}{2},1}_{2d,1}} \]

and thus

\[ \|\exp(Rt)f_0\|_{L^\infty} \geq tR(f_0) + f_0 \|_{L^\infty} - t^2C \|f\|_{B^{\frac{1}{2},1}_{2d,1}} \]

In particular,

\[ \|f\|_{L^\infty} \geq \|tR(f_0) + f_0\|_{L^\infty} - t^2C \|f\|_{B^{\frac{1}{2},1}_{2d,1}} - C \int_0^t \|[R, \Phi]f(\tau)\|_{B^{\frac{1}{2},1}_{2d,1}} d\tau. \]

We now use the commutator estimate (3.1) as well as estimate (3.2). For \( t \) small enough, \( \Phi \) and its inverse will be arbitrarily close to the identity. In particular,

\[ \|f\|_{L^\infty} \geq \|tR(f_0) + f_0\|_{L^\infty} - t^2C \|f\|_{B^{\frac{1}{2},1}_{2d,1}} - Ct^2 \|u\|_{L^\infty \text{Lip}} \exp(tC \|u\|_{L^\infty \text{Lip}}) \sup_{\tau \in [0,1]} \|f(\tau)\|_{B^{\frac{1}{2},1}_{2d,1}}. \]

Now using estimate (3.3) we see that

\[ \|f\|_{L^\infty} \geq \|tR(f_0) + f_0\|_{L^\infty} - t^2C \|f_0\|_{B^{\frac{1}{2},1}_{2d,1}} - Ct^2 \|u\|_{L^\infty \text{Lip}} \exp(tC \|u\|_{L^\infty \text{Lip}}) \|f_0\|_{B^{\frac{1}{2},1}_{2d,1}}. \]

This concludes the proof of Theorem 2.1.

\[ \square \]
4. General Application of the Linear Estimate

In this section we give a very mild condition on $R$ which ensures that the linear system (2.1)-(2.3) is strongly ill-posed in $L^\infty(\mathbb{R}^d)$ in the sense of Definition 1.1.

Assumption 1. There exists a sequence of functions $g^N \in B^{a}_\rho$ such that the following holds:

\[(4.1) \quad |g_N|_{L^\infty} \leq 1,\]
\[(4.2) \quad |R(g_N)|_{L^\infty} \geq cN,\]
\[(4.3) \quad |R(g_N)|_{B^{a}_\rho} \leq CN,\]

where $c$ and $C$ are constants independent of $N$, $0 < a < 1$, $1 < \rho < \infty$, and $a\rho = d$ (d is the dimension).

Heuristically, Assumption 1 says that there exists an $L^\infty$ function $g$ such that $R(g)$ has a logarithmic singularity. Assumption 1 can be shown to hold for many singular integral operators such as the Hilbert transform, the Riesz transforms (and compositions of Riesz transforms), and others.

Now we can state the (linear) ill-posedness theorem.

**Theorem 4.1.** If $R$ satisfies Assumption 1 and if $u \in L^\infty \text{Lip}$ then (2.1)-(2.3) is strongly ill-posed.

**Proof.** The proof is a direct consequence of Theorem 2.1. By Assumption 1, there exists $g_N$ satisfying (4.1)-(4.3). Then solve (2.1)-(2.3) in $B^{a}_\rho$ with initial data $\epsilon g_N$ and call the solution $f_N(t)$. Theorem 2.1 now applies so that:

\[|f_N(t)|_{L^\infty} \geq \epsilon \left( t |R(g_N)|_{L^\infty} - |g_N|_{L^\infty} - C t^2 \left( 1 + |u|_{L^\infty \text{Lip}} \exp(tC|u|_{L^\infty \text{Lip}}) \right) |g_N|_{B^{a}_{\rho,1}} \right).\]

Using (4.1)-(4.3), we see:

\[|f_N(t)|_{L^\infty} \geq \epsilon t cN - \epsilon - \epsilon C t^2 (1 + |u|_{L^\infty \text{Lip}} \exp(tC|u|_{L^\infty \text{Lip}})) N.\]

Now take $t = \frac{\epsilon}{1 + |u|_{L^\infty \text{Lip}}} \frac{\epsilon cN}{cN - \epsilon}$ for some small constant $c$, then

\[|f_N(t)|_{L^\infty} \geq \frac{\epsilon cN}{1 + |u|_{L^\infty \text{Lip}}} N.\]

Now take $N = \frac{1 + |u|_{L^\infty \text{Lip}}}{cN - \epsilon}$. Then, $|f_N(t)|_{L^\infty} \geq \frac{1}{\epsilon}$ even though $|f_N(0)|_{L^\infty} \leq \epsilon$. This concludes the proof.

In the coming sections, the proof of Theorem 4.2 will be used to show mild ill-posedness for some non-linear equations.
5. Perturbations of the 2D Euler equations

As stated in the introduction, a very interesting open problem in mathematical fluid dynamics is to prove global well-posedness for the following type of equation:

\begin{align}
(5.1) \quad &u_t + (u \cdot \nabla)u + \nabla p = Au \\
(5.2) \quad &\div(u) = 0,
\end{align}

where $A$ is some constant matrix. It is possible to prove global well-posedness in only one case: when $Au = \lambda u + \gamma u^\perp$, for constants $\lambda$ and $\gamma$ (of course $A$ can be taken to depend on $x$ in a similar fashion). Indeed, in this case $\curl(Au) = \lambda \omega$ and thus $\omega$ will satisfy a maximum principle which will lead to global well-posedness using the standard technique.

When $A$ is not of the above form, we will use Theorem 2.1 to prove a mild ill-posedness result for (5.1)-(5.2) (in other words, to show that $\omega$ does not satisfy a maximum principle). The reason that we will not be able to prove the strong ill-posedness is that the non-linear term starts to play a prominent role once the vorticity grows. As a special case, we consider the following system:

\begin{align}
(5.3) \quad &u_t + (u \cdot \nabla)u + \nabla p = \begin{pmatrix} -u_1 \\ 0 \end{pmatrix}, \\
(5.4) \quad &\div(u) = 0.
\end{align}

Notice that the right side of this equation is a drag term—it causes the energy of the system to decrease. A simple computation shows that $|\omega|_{L^2}$ should also be decreasing. Therefore, on the level of kinetic energy, we should expect this system to behave 'better' than 2-D Euler. It turns out that this drag term ruins the conventional well-posedness proof for 2-D Euler. Upon passing to the equation for the vorticity we get:

\begin{align}
(5.5) \quad &\omega_t + u \cdot \nabla \omega = -u_{1y} \\
(5.6) \quad &u = \nabla^\perp(-\Delta)^{-1}\omega
\end{align}

In particular, using (5.6),

\[-u_{1y} = R_2^2 \omega,\]

where $R_2$ is the Riesz transform with symbol $-\frac{i\xi}{|\xi|}$.

Using our linear theorem, we will prove the following non-linear ill-posedness result.

**Theorem 5.1.** (5.5)-(5.6) is mildly ill-posed in $L^\infty$. In other words, there exists a sequence of functions $\omega_0$ belonging to $H^s$ for every $s > 0$ and a universal constants $C_1$, independent of $\epsilon$, with the following properties:

1. $|\omega_0|_{L^\infty} \leq \epsilon$
2. $|\omega_0|_{B^1_{4,1}} \leq C_1$
Let $\omega^\epsilon(t)$ be the (local) solution of (2) in $L^\infty([0,C_2];B^\frac{1}{2}_{4,1})$ with $\omega^\epsilon(0) = \omega^\epsilon_0$. Then there exists some $t \in (0,\epsilon]$ so that

$$|\omega(t)^\epsilon|_{L^\infty} \geq C_3.$$  

We remark that we may take $t < \delta$ independent of $\epsilon$.
We conjecture the following more striking result:

**Conjecture:** (5.5)-(5.6) is strongly ill-posed in $L^\infty$.

We have a few obvious corollaries.

**Corollary 5.2.** The zero solution of (5.5)-(5.6) is (non-linearly) unstable with respect to $L^\infty$ perturbations.

**Corollary 5.3.** The map $J_t$ taking an initial data $\omega_0$ to the solution at time $\omega(t)$ is discontinuous in $L^\infty$.

To prove Theorem 5.1 we first need to prove that $R^2_2$ satisfies Assumption 1.

We remark here that our result holds for more general equations of the following type

$$\omega_t + u \cdot \nabla \omega = R \omega$$

$$u = (-\Delta)^{-\alpha} \nabla \perp \omega,$$

where $\alpha \geq 1$ and $R$ is a linear operator mapping $B^0_{\infty,1}$ to itself and for which there exists some $\omega_0 \in L^\infty$ for which $R \omega_0$ has a logarithmic singularity. Therefore any Calderon-Zygmund operator which is unbounded on $L^\infty$ would work. It is easy to show that for $\alpha > 1$, the system is strongly ill-posed in $L^\infty$.

**5.1. The Proof of Theorem 5.1.** The proof of theorem 5.1 is based upon the linear Theorem 2.1. Indeed, suppose that $R := R^2_2$ satisfies Assumption 1. Note that, following a result of Vishik [34], one can prove local well-posedness of (5.3)-(5.5) in all spaces $B^a_{p,1}$ with $a \rho = 1$ (in fact this is a consequence of proposition 3.3).

Indeed, the following is standard:

**Lemma 5.4.** Let $R$ be a Calderón-Zygmund operator. Consider the following equation in the plane:

(5.7)  

$$\omega_t + u \cdot \nabla \omega = R \omega$$

(5.8)  

$$u = (-\Delta)^{-1} \nabla \perp \omega.$$  

Then, this system is locally well-posed in the Besov space $B^a_{p,1}$ and the following estimate holds for all $t$ with $Ct|\omega_0|_{B^a_{p,1}} < \frac{1}{2}$:

(5.9)  

$$|\omega(t)|_{B^a_{p,1}} \leq C \frac{|\omega_0|_{B^a_{p,1}}}{1 - Ct|\omega_0|_{B^a_{p,1}}}.$$
By Lemma 5.5 below, there exists \( g_N \) satisfying:

\[
|g_N|_{B^\frac{1}{2},1} \leq CN, \\
|g_N|_{L^\infty} \leq 1, \\
|Rg_N|_{L^\infty} \geq cN.
\]

Now solve the (5.3)-(5.5) with initial data \( \epsilon g_N \). Note that in what follows, \( \epsilon t N \) will always be smaller than some fixed constant \( c \). This will ensure that we have existence on a uniform time interval.

Call the solution \( f_N(t) \) and its corresponding velocity field \( u_N(t) \). Following the proof of Theorem 4.2 above, we see that using the linear estimate (2.7),

\[
|f_N(t)|_{L^\infty} \geq \epsilon(t|R(g_N)|_{L^\infty} - |g_N|_{L^\infty} - Ct^2(1 + |u_N|_{L^\infty \text{Lip}} \exp(tC|u_N|_{L^\infty \text{Lip}}))|f_N|_{L^\infty B^\frac{1}{2d},1}).
\]

We then see, due to the local well-posedness of (5.3)-(5.5), that

\[
|f_N(t)|_{L^\infty} \geq \epsilon t cN - \epsilon - \epsilon Ct^2(1 + |u_N|_{L^\infty \text{Lip}} \exp(tC|u_N|_{L^\infty \text{Lip}}))N.
\]

But note that \( |\nabla u_N|_{L^\infty} \lesssim |\omega|_{B^\frac{1}{2},1} \lesssim \epsilon N. \)

Therefore,

\[
|f_N(t)|_{L^\infty} \geq \epsilon t cN - \epsilon - \epsilon Ct^2(1 + \epsilon N \exp(tC\epsilon N))N.
\]

Take \( \epsilon N t \) small (but fixed independent of \( \epsilon \)) and we see that

\[
|f_N(t)| \geq \epsilon \epsilon N t - C \epsilon^2 N^2 t^2.
\]

Upon taking \( \epsilon N t \) smaller yet (on the order of \( \frac{\epsilon}{C^2} \)) we see that

\[
|f_N(t)| \geq \epsilon \epsilon N t.
\]

We are done once we note that \( |f(0)|_{L^\infty} \leq \epsilon \) and \( |f(t)| \geq c \) where \( c \) is an absolute constant.

\[\square\]

Remark: The reason that we are unable to prove the strong ill-posedness for (5.7)-(5.8) is that once the vorticity becomes large, the commutator estimate we have becomes uncontrollable. If there were a way to control the non-linear term by something less than \( \Phi \) in the Lipschitz class (say, if one were able to do with only a \( C^\alpha \) bound on \( \Phi \)), then the strong ill-posedness would be within reach. This would be a challenge.
5.2. Proof that \( R_2^2 \) satisfies Assumption 1.

**Lemma 5.5.** Let \( R := R_2^2 \) For each \( N \), there exists a function \( f^N \) belonging to \( H^s \) for all \( s > 0 \) which satisfies the following conditions:

1. \( |f_N|_{B_{1,1}^{1/2}} \leq C_1 N \),
2. \( |f_N|_{L^\infty} \leq 1 \),
3. \( |Rf_N|_{L^\infty} \geq C_2 N \),

for universal constants \( C_1, C_2 \).

**Proof of the Lemma:**

Define \( f_N \) on the Fourier side by \( \hat{f}_N = \chi_{[-2N,2N]^2} \hat{f} \). Where \( f \) is the odd-odd extension of the characteristic function of \([0,1]^2\). Note that this is a regularization of the stationary solution of the Euler equations used in the work of Bahouri and Chemin \([2]\). Then clearly \( f_N \) belongs to \( H^s \) for all \( s \).

First note that

\[
\hat{f}(\xi_1, \xi_2) = 4 \frac{\sin(\xi_1) \sin(\xi_2)}{\xi_1 \xi_2}.
\]

**Proof that \( f_N \) satisfies condition (1).**

Note that \( f \) belongs to \( B_{1,\infty}^{1/2} \) this is because \( |D|^{1/2} f \) is a smooth function multiplied by \( \frac{1}{|\xi|^{1/2}} \) for \( \xi \) large. Showing that \( f \) belongs to \( B_{1,\infty}^{1/2} \) is then an exercise (see for example Proposition 2.21 of \([3]\)).

Then

\[
|\chi_{B_2^N} f|_{B_{1,1}^{1/2}} \leq \sum_{j=-1}^{CN} |\Delta_j f|_{B_{1,\infty}^{1/2}}.
\]

This implies condition (1).

**Proof that \( f_N \) satisfies condition (2).**

By the Fourier inversion formula we have that

\[
|f_N|_{L^\infty} \leq \sup_{x_1, x_2} \int_{-2N}^{2N} \int_{-2N}^{2N} \frac{\sin(x_1) \sin(x_2)}{\xi_1 \xi_2} \cos(x_1) \cos(y_2) d\xi_1 d\xi_2.
\]

To show condition (2) it suffices to show that the following quantity is bounded:

\[
\sup_x \int_{-2N}^{2N} \frac{\sin(x)}{\xi} \cos(x) d\xi = \sup_x \int_{-2N}^{2N} \frac{\sin(x + \xi) - \sin(x - \xi)}{2\xi} d\xi,
\]

\[
= \sup_x \int_{-2N}^{2N(x+1)} \frac{\sin(x)}{2\xi} d\xi + \int_{-2N}^{-2N(x+1)} \frac{\sin(x)}{2\xi} d\xi.
\]
which is bounded by a universal constant since the following quantity is known to be bounded:

\[ \sup_{a,b} \left| \int_a^b \frac{\sin(\xi)}{\xi} \, d\xi \right| < C. \]

**Proof that \( f_N \) satisfies condition (3)**

Using the Fourier inversion formula we see that

\[ Rf(x, y) = \int_{[-2^N, 2^N]^2} \sin(\xi_1) \sin(\xi_2) \sin(x \xi_1) \sin(y \xi_2) d\xi_1 d\xi_2. \]

\[ Rf(1, 1) = \int_{[-2^N, 2^N]^2} \frac{\sin^2(\xi_1) \sin^2(\xi_2)}{\xi_1^2 + \xi_2^2} d\xi_1 d\xi_2. \]

and condition (3) follows. \( \square \)

6. The 3D Euler equations

Consider the 3D vorticity equation:

(6.1) \[ \omega_t + u \cdot \nabla \omega = \nabla u \omega. \]

It is not clear at first that the 3-D Euler equations can be cast in the framework of the linear problem (2.1)-(2.3). As above, through the Biot-Savart law, one can view \( \nabla u \) as \( R(\omega) \) where \( R \) is now a matrix of singular integral operators.

So the 3D Euler equations can be seen as:

(6.2) \[ \omega_t + u \cdot \nabla \omega = R(\omega) \omega. \]

The quadratic nature of \( R(\omega) \omega \) is such that we cannot directly apply the analysis of (2.1)-(2.3). However, one can consider perturbing \( \omega \) by a shear flow in order to pull a linear \( R(\omega) \) out of the right hand side. Indeed, let \( \omega = \tilde{\omega} + e_3 \).

Then we see that

\[ \tilde{\omega}_t + u \cdot \nabla \tilde{\omega} = R(\tilde{\omega}) \tilde{\omega} + R(e_3) \tilde{\omega} + R(\tilde{\omega}) e_3. \]

Note that we may regard \( R(\tilde{\omega}) \tilde{\omega} \) as a quadratic term so that it would be of order \( \epsilon^2 \) if we follow the proof of Theorem 5.1. Following along the lines of Theorem 5.1 we can deduce the following theorem:

**Theorem 6.1.** There exists a sequence of functions \( \omega^\epsilon_0 \) belonging to \( H^s \) for every \( s > 0 \) and universal constants \( C_i \), independent of \( \epsilon \), with the following properties:

1) \( |\omega^\epsilon_0 - e_3|_{L^\infty} \leq \epsilon \)
2) \( |\omega^\epsilon_0 - e_3|_{B^s_{4,1}} \leq C_1 \)

Let \( \omega^\epsilon(t) \) be the (local) solution of the 3D Euler equations in \( L^\infty([0,C_2]; B^s_{4,1}) \) with \( \omega^\epsilon(0) = \omega^\epsilon_0 \). Then there exists some \( t \in (0, \epsilon] \) so that

\[ |\omega(t)^\epsilon - e_3|_{L^\infty} \geq C_3. \]
Remark 1: The proof of Theorem 6.1 follows from the same ideas we used in the proof of Theorem 5.1.

Remark 2: One might be concerned by the fact that $e_3$ is not of finite energy in the whole space; however, the result is very easily localized by considering $e_3$ multiplied by a smooth cutoff.

7. The Euler equations with $C^1$ data

As another bi-product of Proposition 3.1, we have that the incompressible Euler equations are strongly ill-posed for $u \in C^1 \cap L^2$. Indeed, consider the 3D Euler equations in velocity form:

\begin{align}
(7.1) & \quad u_t + (u \cdot \nabla)u + \nabla p = 0, \\
(7.2) & \quad \text{div}(u) = 0.
\end{align}

Notice that the equation for the gradient of $u$ is:

\begin{align}
(7.3) & \quad \nabla u_t + (u \cdot \nabla)\nabla u + D^2p + Q(\nabla u, \nabla u) = 0,
\end{align}

with, $Q$ a bilinear form.

The pressure is recovered from $u$ by the following equation:

\begin{align}
(7.4) & \quad \Delta p = \text{div}(u \cdot \nabla u) = \sum_{l \neq k} u_{l,k}u_{k,l},
\end{align}

with $u_{j,i} = \partial_{x_i} u_j$.

Then notice that $D^2p = R_i R_j (\sum_{l \neq k} u_{l,k}u_{k,l})$. Therefore, (7.3) becomes:

\begin{align}
\nabla u_t + (u \cdot \nabla)\nabla u + R_i R_j (\sum_{l \neq k} u_{l,k}u_{k,l}) + Q(\nabla u, \nabla u) = 0.
\end{align}

We will write this as:

\begin{align}
(7.5) & \quad \nabla u_t + (u \cdot \nabla)\nabla u + R(B(\nabla u, \nabla u)) + Q(\nabla u, \nabla u) = 0,
\end{align}

where $R := (R_i R_j)_{i,j}$ is a matrix of singular integral operators. Further, $B(\nabla u, \nabla u) := \sum_{l \neq k} u_{l,k}u_{k,l}$.

We have the following theorem:

**Theorem 7.1.** For every $\epsilon > 0, \delta > 0$ small enough there exists $u_0 \in \mathbb{S}(\mathbb{R}^d)$, the Schwartz class, with

$$|u_0|_{C^1 \cap L^2} \leq \epsilon,$$

such that if we denote by $u(t)$, the solution of the incompressible Euler equations in $\mathbb{R}^d$ with initial data $u_0$, then

$$\sup_{0 < t < \delta} |u(t)|_{C^1 \cap L^2} \geq \frac{1}{\epsilon}.$$

In section 8 we will prove a stronger result:
Theorem 7.2. For every $\epsilon > 0, \delta > 0$ small enough there exists $u_0 \in C^1 \cap L^2(\mathbb{R}^d)$, with

$$|u_0|_{C^1 \cap L^2} \leq \epsilon,$$

such that if we denote by $u(t)$, the solution of the incompressible Euler equations in $\mathbb{R}^d$ with initial data $u_0$,

$$\sup_{0 < t < \delta} |u(t)|_{C^1 \cap L^2} = +\infty.$$

Some remarks are in order.

Remarks:
(1) The growth in the $C^1$ case will come from the singular integral which arises in the pressure term. However, we will have to be careful because the pressure term is not linear in $u$, but bilinear.
(2) The construction in Theorem 7.1 is completely local. Therefore, the result holds on a bounded domain as well as on the torus.
(3) With the exception of choosing the right initial data, the proof of theorem 7.1 is quite soft–so it likely can be used in several other contexts.
(4) After the completion of this work we came to know that Misiolak and Yoneda [27] have proven ill-posedness for the Euler equations in $C^1$ in the sense that the solution map could not be $C^1$. Their result is not one about norm inflation but about discontinuity of the flow map. Their method relies upon a clever adaptation of the work of Bourgain and Li [6]; it does not seem that there is any apparent relation between our work and theirs.

7.1. A toy model. To understand the effect of the pressure term, $R(B(\nabla u, \nabla u))$, we may consider the following toy model:

$$f_t = R(f^2).$$

We want to see that this model is ill-posed on $L^\infty$. In the case of $f_t = R(f)$ we are able to solve this equation on the Fourier-side by a series expansion in order to deduce that

$$|f|_{L^\infty} \geq |f_0 + tR(f_0)|_{L^\infty} - t^2 C|f_0|^{\frac{1}{2}}. $$

However, in the case where we have $f_t = R(f^2)$, it is not clear how to solve the equation using any sort of similar expansion. Luckily, we are still able to manage.

Proposition 7.3. Let $B$ be a quadratic form acting on matrices. Consider the following matrix PDE:

\begin{align}
(7.6) \quad & f_t = R(B(f, f)), \\
(7.7) \quad & f(0, x) = f_0(x)
\end{align}

where $R$ is a Calderón-Zygmund singular integral operator. Then, (7.6)-(7.7) is locally well-posed on $B^a_{p,1}$ for all $a p \geq d$. Moreover, for $t$ small, smooth solutions satisfy the following bounds:
\( f(t) \) \( L^\infty \) ≥ \( f_0 + tR(B(f_0, f_0)) \) \( L^\infty \) − \( t^2C(\sup_{0 \leq \tau \leq t} |f(\tau)| L^\infty) f_0 \) \( B_{2d,1}^{\frac{1}{2}} \)

**Remark:** Bound (7.8) only holds so long as \( f \) exists. However, note that if the initial data \( f_0 \) belongs to \( B_{p,1}^a \) with \( a \rho \geq 1 \), then finite-time blow up in (7.6)-(7.7) can only happen if \( |f| L^\infty \) blows up. This will be important in what follows.

**Proof.** The local well-posedness is standard. Indeed, all that is needed is that \( R \) is bounded on \( B_{p,1}^a \) and that these spaces are algebras containing \( L^\infty \). Indeed, recall the following inequality:

\[ |fg|_{B_{p,1}^a} \leq |f|_{B_{p,1}^a} |g|_{L^\infty} + |f|_{L^\infty} |g|_{B_{p,1}^a}. \]

Now, write:

\[ f_t = R(B(f_0, f_0)) + (R(B(f, f)) - R(B(f_0, f_0))), \]

Now, note that \( |f_t|_{L_{t,x}^\infty} \leq C|B(f, f)|_{B_{2d,1}^\frac{1}{2}} \leq C(|f|_{L_{t,x}^\infty} f_0|_{B_{2d,1}^\frac{1}{2}} \), by local well-posedness.

Consequently,

\[ |B(f, f) - B(f_0, f_0)|_{B_{2d,1}^\frac{1}{2}} \leq C(|f|_{L_{t,x}^\infty} |f - f_0|_{B_{2d,1}^\frac{1}{2}}) \leq tC(|f|_{L_{t,x}^\infty} f_0|_{B_{2d,1}^\frac{1}{2}}). \]

Hence, so long as the solution \( f(t) \) exists,

(7.10) \[ |f(t)|_{L^\infty} \geq |f_0 + tR(B(f_0, f_0))|_{L^\infty} - t^2C(\sup_{0 \leq \tau \leq t} |f(\tau)| L^\infty) f_0 \] \( B_{2d,1}^{\frac{1}{2}} \).

**Corollary 7.4.** Let \( B \) be a quadratic form acting on matrices. Consider the following matrix PDE:

(7.11) \[ f_t = R(B(f, f)) + g, \]

(7.12) \[ f(0, x) = f_0(x) \]

where \( R \) is a Calderón-Zygmund singular integral operator and \( g \) is a given function belonging to \( B_{p,1}^a \), with \( a \rho \geq 1 \). Then, (7.11)-(7.12) is locally well-posed on \( B_{p,1}^a \) for all \( a \rho \geq d \). Moreover, for \( t \) small, smooth solutions satisfy the following bounds:

(7.13) \[ |f(t)|_{L^\infty} \geq |f_0 + tR(B(f_0, f_0))|_{L^\infty} - t|g|_{L^\infty} - t^2C(\sup_{0 \leq \tau \leq t} |f(\tau)| L^\infty) f_0 \] \( B_{2d,1}^{\frac{1}{2}} \).

We are now in a position to prove Theorem 7.1.

**Proof of Theorem 7.1.**
Call \( f := \nabla u \) and recall that \( \text{div}(u) = 0 \). Towards a contradiction, suppose that for all \( f \) with \( |f_0| < \epsilon \), \( \sup_{0 < t < \delta} |f(t)|_{L^\infty} \leq M \), for some given \( \epsilon, \delta, M \). Note if the assertion is true, we can solve the \( d \)-dimensional Euler equations on \([0, \delta]\) for any initial data with \( |f_0|_{L^\infty} < \epsilon \).

Now, \( f \) satisfies the equation:

\[
f_t + (u \cdot \nabla) f + Q(f, f) + R(B(f, f)) = 0.
\]

Now write this equation along the characteristics of \( u \). We first solve

\[
\dot{\Phi} = u(\Phi) \quad \Phi(0) = \text{Id}.
\]

Then we get:

\[
(f \circ \Phi)_t + Q(f \circ \Phi, f \circ \Phi) + R(B(f \circ \Phi, f \circ \Phi)) + [R, \Phi]B(f, f) = 0.
\]

Now, by Corollary 8.2, we have:

\[
|f \circ \Phi|_{L^\infty} \geq |f_0 + tR(B(f_0, f_0))|_{L^\infty} - t|g|_{L^\infty} - t^2C(\sup_{0 \leq \tau \leq t} |f(\tau)|_{L^\infty})|f_0|_{\frac{1}{2}, 1} \subseteq B_{\frac{1}{2d}, 1} \quad ,
\]

where

\[
g := Q(f \circ \Phi, f \circ \Phi) + [R, \Phi]B(f, f).
\]

Here, we have implicitly used the result of Vishik [33] that the Euler equations are locally well-posed on \( B_{\frac{1}{2d}, 1} \).

Now we need to estimate \( g \) using the commutator estimate (3.1). Since, \( |f(t)| = |\nabla u(t)| \leq M \) on \([0, \delta]\), we can choose \( t \) very small so that the conditions of Proposition 3.1 are satisfied (namely, that \( \Phi \) be sufficiently close to the identity). Hence, we will have that

\[
|g|_{L^\infty} \leq C|f|^2_{L^\infty} + tC(|\nabla u|_{L^\infty})B(f, f)|_{\frac{1}{2d}, 1} \subseteq C|f|^2_{L^\infty} + tC(|\nabla u|_{L^\infty})|f_0|_{\frac{1}{2d}, 1} \subseteq |f|_{L^\infty}.
\]

Consequently, we have:

\[
|f|_{L^\infty} \geq |f_0 + tR(B(f_0, f_0))|_{L^\infty} - t(C|f|^2_{L^\infty} + tC(|\nabla u|_{L^\infty})|f_0|_{\frac{1}{2d}, 1} \subseteq )
\]

Now, by assumption, \( |f|_{L^\infty} < M \). Hence,

\[
|f|_{L^\infty} \geq |f_0 + tR(B(f_0, f_0))|_{L^\infty} - tC(M) - t^2C(M)|f_0|_{\frac{1}{2d}, 1} \subseteq .
\]

**Lemma 7.5.** There exists a sequence of divergence-free functions \( g_N \in B^{a+1}_{\rho, 1} \) such that the following holds:

\[
|\nabla g_N|_{L^\infty} \leq 1,
\]

\[
|R(B(\nabla g_N, \nabla g_N))|_{L^\infty} \geq cN,
\]

\[
(7.14)
\]

\[
(7.15)
\]
\[ |g_N|_{B_{2d,1}^{2d}} \leq CN, \]

where \( c \) and \( C \) are constants independent of \( N \).

Assuming this lemma is true, take \( u_0 = \epsilon g^N \), where \( N \) is fixed for the moment. Then,

\[ |f|_{L^\infty} \geq ctN^2 - \epsilon - tC(M) - t^2C(M)N. \]

Recall that we need \( t < \frac{\epsilon}{cM} \) in order to apply Proposition 3.1 (because we need \( \Phi \) to be close enough to the identity). Now choose \( N \) large enough, \( t \) small enough and then \( |f|_{L^\infty} > M \), which is a contradiction. Consequently, for every \( \epsilon, \delta, M > 0 \), there exists \( \nabla u_0 \in \text{Lip} \) so that \( |u_0|_{\text{Lip}} \leq \epsilon \) and

\[ \sup_{0 \leq t \leq \delta} |\nabla u(t)| \geq M. \]

\[ \square \]

7.2. Proof of Lemma 7.5. We are interested in showing that for some \( i, j \) and for some divergence free \( u \), with \( \nabla u \in L^\infty \), \( D^2p = R_iR_j\det(\nabla u) \) has a logarithmic singularity. Once that is shown, Lemma 8.2 will follow by a regularization argument.

Take a harmonic polynomial, \( Q \), which is homogeneous of degree 4. In the two-dimensional case, we can take

\[ Q(x, y) := x^4 + y^4 - 6x^2y^2, \]

\[ \Delta Q = 0. \]

Define

\[ G(x, y) := Q(x, y)\log(x^2 + y^2). \]

Notice that

\[ \partial_i \partial_j \Delta G \in L^\infty(B_1(0)), i, j \in \{1, 2\}. \]

(7.17)

Notice, on the other hand, that

\[ \partial_{xxyy}G = -24\log(x^2 + y^2) + H(x, y), \]

with \( H \in L^\infty(B_1(0)) \). In particular, \( \partial_{xxyy}G \) has a logarithmic singularity at the origin—and the same can be said about \( \partial_{xxxx}G \) and \( \partial_{yyyy}G \).

Define \( \tilde{u} = \nabla^\perp \Delta G \). Then, by 8.12, \( \nabla \tilde{u} \in L^\infty(B_1(0)) \). Moreover, by definition,

\[ R_iR_j\nabla \tilde{u} = \nabla \nabla^\perp \partial_{ij}G. \]

Thus, for example, \( R_1R_2\nabla \tilde{u}_{1x} = \partial_{xxyy}G \) has a logarithmic singularity in \( B_1(0) \).

Unfortunately, we are interested in showing that \( R_iR_j\det(\nabla u) \) has a logarithmic singularity for some \( i, j \), not \( R_iR_j\nabla u \). To rectify this, we choose

\[ u = \delta \nabla^\perp \Delta(\chi G) + \eta \nabla^\perp(y \chi), \]

where \( \eta, \delta \) are small parameters which will be determined and \( \chi \) is a smooth cut-off function with:

\[ \chi = 1 \text{ on } B_1(0), \]
\[ \chi = 0 \text{ on } B_2(0)^c. \]

Note that \( u \) is divergence free and

\[ u \equiv \delta \nabla^\perp \Delta G + \eta(y, 0) \text{ on } B_1(0). \]

Therefore, on \( B_1(0) \),

\[ \nabla u = \delta \begin{bmatrix} -\partial_{xy}\Delta G & -\partial_{yy}\Delta G \\ \partial_{xx}\Delta G & \partial_{xy}\Delta G \end{bmatrix} + \eta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]

In particular,

\[ \det(\nabla u) = \eta \delta \partial_{xx} \Delta G + \delta^2 J(x, y), \]

where \( J \) is a bounded on \( B_1(0) \).

Now consider \( R_2 R_2 \det(\nabla u) : \)

\[ R_2 R_2 \det(\nabla u) = \eta \delta \partial_{xyy} G + \delta^2 R_2 R_2 J. \]

Now, by (7.18), we have

\[ R_2 R_2 \det(\nabla u) = \eta \delta (-24 \log(x^2 + y^2) + H(x, y)) + \delta^2 R_2 R_2 J, \]

with \( H \) and \( J \) bounded. Now, recall that \( R_2 R_2 \) maps \( L^\infty \) to BMO and that any BMO function can have at most a logarithmic singularity.

Thus,

\[ |R_2 R_2 \det(\nabla u)| \geq 24 \eta \delta \log(x^2 + y^2) - C \delta^2 \log(x^2 + y^2) - |H(x, y)|. \]

Choose \( \delta << \eta \) and we see that, near \((0, 0)\)

\[ |R_2 R_2 \det(\nabla u)| \geq 12 \delta^2 \log(x^2 + y^2). \]

Taking \( \delta \leq C \) small enough, we see that \( |\nabla u| \leq 1 \) but \( |R_2 R_2 \det(\nabla u)| \geq c \log(x^2 + y^2) \), for some small \( c \).

One may regularize the constructed velocity field by replacing \( \log(x^2 + y^2) \) by \( \log(x^2 + y^2 + \frac{1}{2N^2}} \) or by convolving \( u \) with an approximation of the identity.

8. **Strong ill-posedness in \( C^1 \): the \( L^p \) approach**

It is possible to prove the ill-posedness of the Euler equations in \( C^1 \) in a more direct fashion. In particular, one can show the existence of initial data \( u_0 \in C^1 \) so that the unique solution from initial data \( u_0 \) leaves \( C^1 \) immediately.

**Theorem 8.1.** Consider the 2D Euler equations in the plane, \( \mathbb{R}^2 \), the torus, \( \mathbb{T}^2 \), or a \( C^2 \) domain \( \Omega \). Then there exists \( u_0 \in C^1 \cap L^2 \) with \( \text{curl}(\omega_0) \in L^1 \cap L^\infty \) so that the unique solution \( u(t) \) of the Euler equations with initial data \( u_0 \) does not belong to \( C^1 \) for \( t < c \) for some \( c > 0 \).

**Proof.** Using the initial data constructed above in section 7, we see that there exists \( u_0 \) so that

\[ u_0 \in C^1 \]

but

\[ |D^2 p_0|_{L^p} = |B(\nabla u_0, \nabla u_0)|_{L^p} \geq cp, \forall p > 1. \]
Furthermore, as was noted in Proposition 3.3,
\[ \|\left[ R, \Phi \right]\|_{L^p \to L^p} \leq c_p |\Phi - I|_{\text{Lip}} \]
and \( c_p \approx p \) for \( p \) large.

Therefore, if we assume that the solution \( u(t) \) remains Lipschitz for positive time (say that \( |\nabla u| \leq M \) for \( t < c \)) then we see that \( \nabla u \) will satisfy the following estimate in \( L^p \).

\[ |\nabla u|_{L^p} \geq c p t - C(M) p t^2 \]

for all \( p \) and all \( t > 0 \).

This obviously leads to a contradiction for small \( t \) since \( |\nabla u(t)|_{L^p} > cp \) for \( t < c \) for some small \( c \) while \( \nabla u \) remains bounded. Thus the solution must leave \( C^1 \).

Note that our initial data can be taken to be as localized as we want so we can deal with the whole space, periodic boundary conditions, and the bounded domain case in one shot.

\[ \Box \]

9. The \( C^k \) Case

**Theorem 9.1.** The Euler equations are strongly ill-posed in \( C^k \) spaces for \( k \geq 1 \). In other words, for every \( \epsilon > 0 \) there exists initial data \( u_0 \in C^k \) such that the unique solution, \( u(t) \), of the Euler equations with initial data \( u_0 \) leaves \( C^k \) immediately.

We note that very recently Bourgain and Li have proven the same result as above [7]. We clarify here that strong ill-posedness in \( C^k \) can be proven quite easily only using commutator estimates without having to rely upon very intricate constructions.

**Proof.** We just sketch the proof since it is basically the same as the \( C^1 \) case. Note that it suffices to consider the two dimensional Euler equations (in the whole space case in higher dimensions a similar argument can be made simply by modifying the initial data slightly). Now consider the equation for \( D^k u := \partial_x^k u \) which means \( k \) spatial derivatives of \( u \) with respect to the first variable.

\[ \nabla D^{k-1} u \] satisfies the following equation:

\[ \partial_t \nabla D^{k-1} u + u \cdot \nabla D^k u + \sum_{j,l} Q(D^j u, D^l u) + D^{k-1} D^2 p = 0. \]

We are going to take data in \( C^k \). Then, locally in time, there will be a \( C^{k-\epsilon} \) solution by the result of Lichtenstein [25]. Assume that this solution remains in \( C^k \) for \( t \in [0, 1] \).

Now recall that

\[ \Delta p = \text{det}(\nabla u) \]

so that

\[ (D^2 p)_{ij} = (R_i R_j \text{det}(\nabla u))_{ij}. \]
Now, following the proof of Theorem 8.1, it suffices to construct $u_0 \in C^k$ such that $\|D^{k+1}p_0\|_{L^p} \geq cp$ as $p \to \infty$. Notice that $D^{k+1}p_0$ will consist of many terms all of which belong to $C^{2-\varepsilon}$ except for the terms where all of the derivatives hit one column of $\nabla u$ so that we only have to focus on these terms (because the $C^{2-\varepsilon}$ terms will be well-controlled) Now we can choose $P$ to be the $k + 3$ degree homogeneous polynomial which is just the $k^{th}$ integral with respect to $x$ of the $Q$ constructed in section 7. Then the argument is the same as in section 7 and we are done.

\hfill \Box

10. Conclusion

In section 3 we prove a linear ill-posedness result for general transport equations with Lipschitz velocity fields. As a consequence we proved strong ill-posedness for a particular linear equation. We saw in the previous sections that proving an $L^\infty$ mild ill-posedness result for non-linear equations is possible when three conditions are satisfied: first, that the velocity field be related to the advected quantity by a degree-zero operator (which is the case for the vorticity equation for example). Second, that the equation be locally well-posed in the critical Besov space which imbeds in $L^\infty$. Finally, that the non-local operator on the right-hand side satisfy Assumption 1. This method is quite robust. We use the method then to prove strong ill-posedness of the Euler equations in $C^k$ for integer $k$. In a forthcoming work it will be shown how this method can be used to show mild ill-posedness for other systems such as the 2D Boussinesq system, the surface quasi-geostrophic equations, and some viscoelastic systems.

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