On the Preservation of Commutation and Anticommutation Relations of 

\( n \)-Level Quantum Systems

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Abstract—The goal of this paper is to provide conditions under which a quantum stochastic differential equation (QSDE) preserves the commutation and anticommutation relations of the \( SU(n) \) algebra, and thus describes the evolution of an open \( n \)-level quantum system. One of the challenges in the approach lies in the handling of the so-called anomaly coefficients of \( SU(n) \). Then, it is shown that the physical realizability conditions recently developed by the authors for open \( n \)-level quantum systems also imply preservation of commutation and anticommutation relations.

I. INTRODUCTION

The property of physical realizability of open quantum systems has attracted considerable interest in recent years with its main motivation being its role in quantum coherent control [1], [2], [11], [15]. In simple words, this property systematically characterizes the quantum nature of a control system from a state space point of view, which is relevant in many engineering areas, and is closely related to algebraic properties of the QSDE coefficients. The case of open quantum linear systems is well understood [8]. More recently, the physical realizability property was introduced for \( n \)-level quantum systems [4]. This was the product of a nontrivial extension of the two-level case presented in [3]. From the state space point of view, physical realizability has to be complemented by the preservation of commutation relations since the latter is an exhibitor of the quantum behavior of the system. In this regard, the authors provided conditions for the preservation of commutation relations in the case of QSDE’s evolving in \( SU(2) \) [5]. However, in order to extend the formalism to the general \( n \)-level case it is necessary to deal with the so-called anomaly coefficients of \( SU(n) \), which forms the completely symmetric tensor \( d_{ijk} \) that appears out of the anticommutation relation of the generators of \( SU(n) \). These generators form a complete orthonormal basis spanning the set of \( n \)-dimensional complex matrices and are known as the generalized Gell-Man matrices [10], [12], [13]. The commutation relations of these generators also generate the completely antisymmetric tensor \( f_{ijk} \). In the case of QSDEs describing two-level quantum systems evolving in \( SU(2) \), it suffices to consider preserving the commutation relations of \( SU(2) \) since the anomaly tensor \( d \) is zero for all indices. On the other hand, the situation is nontrivial for \( n \geq 3 \) since one now requires extra machinery on the tensor \( d \) that in some cases depend on the value of \( n \). In other words, the QSDE for an \( n \)-level quantum system must preserve both commutation and anticommutation relations for all times.

The approach followed in this manuscript is that of open quantum systems defined in terms of a triple \((S, L, H)\), where \( S \) is an operator, known as the scattering matrix, describing the interaction of the environment fields among themselves, \( L \) is a vector of coupling operators expressing the interaction of the environment fields with the system variables and \( H \) is a Hamiltonian operator [6], [9]. This description implicitly takes in consideration the quantum nature of the evolution equations. For example, the \((S, L, H)\) description will preserve, in time, the canonical commutation relations of the system under consideration. The physical realizability conditions in [4] do not, in general, assure the preservation of commutation and anticommutation relations. It is thus the main goal of this paper to provide conditions for the preservation of commutation and anticommutation relations for a state space model evolving in \( SU(n) \) independently of the physical realizability conditions. Then, as a second result it is shown that physical realizability does imply the preservation of commutation and anticommutation relations.

The paper is organized as follows. Section II summarizes the necessary tools obtained from the algebra of \( SU(n) \). In Section III the basic preliminaries of open \( n \)-level quantum systems as well as the property of physical realizability for such systems are introduced. This is followed by Section IV in which conditions for the preservation of commutation and anticommutation relations are provided. In addition, it is also shown that those conditions are implied by the physical realizability conditions. Finally, Section V gives the conclusions.

II. THE ALGEBRA OF \( SU(n) \)

In what follows the necessary tools regarding the algebra of the special unitary group \( SU(n) \) will be provided; see [10], [12], [13] for more details. Basically, this group is formed by all \( n \times n \) complex matrices that are Hermitian and have zero trace. Consider the set of elementary vectors spanning \( \mathbb{C}^n \), namely, \( \{e_1, \cdots, e_n\} \). Define \( P_{kl} \in \mathbb{C}^{n \times n} \) as
Define \( u_{jk} = P_{j,k} + P_{k,j}, \) \( v_{jk} = i ( P_{j,k} - P_{k,j} ), \) 
\[
w_l = -\sqrt{\frac{2}{l(l+1)}} \left( \sum_{s=1}^{k} P_{s,s} - k P_{l+1,l+1} \right)
\]
for \( 1 \leq j < k \leq n, 1 \leq l \leq n - 1. \) Note that the identity matrix \( I \) must be included in the basis in order to form a complete set. The identity, \((n^2 - n)/2\) symmetric matrices \( u_{jk}, \) the \((n^2 - n)/2\) antisymmetric matrices \( v_{jk} \) and the \( n - 1 \) mutually commutative matrices \( w_l \) together form the generators of \( SU(n). \) These generators are known as the generalized Gell-Mann matrices. Without any particular order, the generators are relabeled \( \{ I, \lambda_1, \ldots, \lambda_s \}, \) where \( s = n^2 - 1. \) Here the orthonormality condition these matrices satisfy is \( \text{Tr}(\lambda_i \lambda_j) = 2 \delta_{ij}, \) where \( \delta_{ij} \) denotes the Kronecker delta. Their commutation and anticommutation relations are
\[
[\lambda_i, \lambda_j] = 2i \sum_{k=1}^{s} f_{ijk} k, \\
\{\lambda_i, \lambda_j\} = \frac{4}{n} \delta_{ij} + 2 \sum_{k=1}^{s} d_{ijk} k.
\]
Thus, the product \( \lambda_i \lambda_j \) can be easily computed as
\[
\lambda_i \lambda_j = \frac{1}{2} ( [\lambda_i, \lambda_j] + \{\lambda_i, \lambda_j\} )
\]
\[
= \frac{2}{n} \delta_{ij} + \sum_{k=1}^{s} (4 f_{ijk} + d_{ijk} k).
\]
where the real completely antisymmetric tensor \( f_{ijk} \) and the real completely symmetric tensor \( d_{ijk} \) are called the structure constants of \( SU(n). \) The tensors \( f_{ijk} \) and \( d_{ijk} \) satisfy
\[
f_{ilm} f_{mj k} + f_{ilm} f_{jm k} + f_{kml} f_{ij m} = 0, \\
f_{ilm} d_{mj k} + f_{ilm} d_{jm k} + f_{kml} d_{ij m} = 0,
\]
\[
\sum_{k=1}^{s} f_{ikk} f_{mj k} = \frac{2}{n} \delta_{ij} \delta_{lm} - \delta_{ij} \delta_{ml},
\]
\[
+ \sum_{k=1}^{s} (d_{imk} f_{ijk} - d_{ijk} d_{lmk}).
\]
\[
\sum_{m,k} f_{imk} f_{mj k} = n \delta_{ij}.
\]
Define \( F_i, D_i \in \mathbb{R}^{s \times s}, i \in \{ 1, \ldots, s \}, \) such that their \((j,k)\) component is \( (F_i)_{jk} = f_{ijk} \) and \( (D_i)_{jk} = d_{ijk} \), respectively. In particular, the set \( \{ -i F_1, \ldots, -i F_s \} \) is the adjoint representation of \( SU(n). \) In [10], [12], identities \[24, 23\] were employed to obtain the following useful relationships
\[
[F_i, F_j] = -\sum_{k=1}^{s} f_{ijk} F_k \\
[F_i, D_j] = -\sum_{k=1}^{s} f_{ijk} D_k \\
F_i D_j + F_j D_i = \sum_{k=1}^{s} d_{ijk} F_k
\]
\[
D_i F_j + D_j F_i = \sum_{k=1}^{s} d_{ijk} F_k
\]
\[
(D_i D_j - F_j F_i)_{ml} = \sum_{k=1}^{s} d_{ijk} (D_k)_{ml}
\]
\[
+ \frac{2}{n} \left( \delta_{ij} \delta_{ml} - \delta_{im} \delta_{jl} \right).
\]
\[
\text{Definition 1: Let } \beta \in \mathbb{C}^s. \text{ The linear mappings } \Theta^- : \mathbb{C}^s \rightarrow \mathbb{C}^{s \times s} \text{ are defined as}
\]
\[
\Theta^-(\beta) = \left( F_1^T \beta, \cdots, F_s^T \beta \right) = \left( \begin{array}{c} \beta^T F_1^T \\ \vdots \\ \beta^T F_s^T \end{array} \right),
\]
\[
\Theta^+(\beta) = \left( D_1^T \beta, \cdots, D_s^T \beta \right) = \left( \begin{array}{c} \beta^T D_1^T \\ \vdots \\ \beta^T D_s^T \end{array} \right).
\]
Observe that the nature of the \( f \) and \( d \)-tensors make \( \Theta^- (\beta) \) and \( \Theta^+ (\beta) \) be antisymmetric and symmetric, respectively. When \( \beta \) is an \( s \)-dimensional row vector then it will be understood hereafter that \( \Theta^- (\beta) = \Theta^- (\beta^T) \) and \( \Theta^+ (\beta) = \Theta^+ (\beta^T). \) Consider now the stacking operator \( \text{vec} : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{mn} \) whose action on a matrix creates a column vector by stacking its columns below one another. With the help of \( \text{vec} \), the matrices \( \Theta^- (\beta) \) and \( \Theta^+ (\beta) \) can be reorganized so that
\[
\text{vec} (\Theta^- (\beta)) = \left( \begin{array}{c} \Theta_1^- (\beta) \\ \vdots \\ \Theta_s^- (\beta) \end{array} \right) = F \beta,
\]
and
\[
\text{vec} (\Theta^+ (\beta)) = \left( \begin{array}{c} \Theta_1^+ (\beta) \\ \vdots \\ \Theta_s^+ (\beta) \end{array} \right) = D \beta,
\]
where \( \beta \in \mathbb{C}^s, \Theta_1^- (\beta) = F_1^T \beta, \) \( F = (F_1, \ldots, F_s)^T, \) \( \Theta_1^+ (\beta) = D_1 \beta \) and \( D = (D_1, \ldots, D_s)^T. \) From \[24, 23\], \( F \) satisfies
\[
F^T F = n I.
\]
The properties of \( \Theta^- \) and \( \Theta^+ \) are summarized in the next lemma.
\[
\text{Lemma 1: Let } \beta, \gamma \in \mathbb{C}^s. \text{ The mappings } \Theta^- \text{ and } \Theta^+ \text{ satisfy}
\]
\[
\Theta^- (\beta) \gamma = -\Theta^- (\gamma) \beta,
\]
\[
\Theta^+ (\beta) \gamma = \Theta^+ (\gamma) \beta,
\]
\[
\Theta^- (\beta) \beta = 0,
\]
\[
\Theta^- (\Theta^- (\beta) \gamma) = [\Theta^- (\beta), \Theta^- (\gamma)],
\]
\[
\Theta^- (\Theta^+ (\beta) \gamma) = \Theta^- (\beta) \Theta^+ (\gamma) + \Theta^- (\gamma) \Theta^+ (\beta),
\]
\[
\Theta^+ (\Theta^- (\beta) \gamma) = \Theta^+ (\beta) \Theta^- (\gamma) + \Theta^+ (\gamma) \Theta^- (\beta),
\]
\[
\Theta^+ (\Theta^+ (\beta) \gamma) = \Theta^+ (\beta) \Theta^+ (\gamma) - \Theta^- (\gamma) \Theta^- (\beta)
\]
\[
- \frac{2}{n} (\beta^T \gamma I - \gamma \beta^T).
\]
\[
\text{Proof: The proofs of identities \[5a, 5f\] where provided by the authors in [4]. Only \[5g\] requires a proof. The left-hand-}

Proof: (θ^+(βγ))_{ij} = \sum_{k=1}^{s} d_{ijk} \beta^T D_k γ
= \beta^T \left( \sum_{k=1}^{s} d_{ijk} D_k \right) γ
= \beta^T (D_k D_j - F_j F_k) γ
= \frac{1}{n} \left( \delta_{ij} \sum_{m,l=1}^{s} \beta_m δ_{ml} γ_l + \sum_{m,l=1}^{s} \beta_m δ_{im} δ_{jl} γ_l \right)
= \frac{1}{n} \left( \beta^T D_k D_j - \beta^T F_j F_k γ_l - \frac{2}{n} (\beta^T γ_δ) \delta_{ij} - \beta_δ γ_δ \right)
= \frac{1}{n} \left( (θ^+(β)θ^+(γ) - θ^-(γ)θ^+(β))_{ij} - \frac{2}{n} (\beta^T γ 1 - \beta^T γ 1) \right).

Some additional identities regarding matrices F and D with respect to the Kronecker product are given next. They will become useful when proving the main results of the paper in Section IV. Define the tensor permutation matrix \( I_⊗ = \{I_{ji}\}_{i,j=1} \) such that it satisfies \( I_⊗ (A ⊗ B) I_⊗ = (B ⊗ A) \) for any \( A, B \in C^{s×s} \).

Lemma 2: Let \( F = (F_1, F_2, \cdots, F_s)^T \) and \( F = (D_1, D_2, \cdots, D_s)^T \), and A, B ∈ \( \mathbb{R}^{s×s} \). Then

i. \( F = -I_⊗ F \),

ii. \( D = I_⊗ D \),

iii. \( F^T (A ⊗ B) F = F^T (B ⊗ A) F \),

iv. \( D^T (A ⊗ B) D = D^T (B ⊗ A) D \).

Proof: Identity \[2\] is proved by multiplying directly \( I_⊗ \) and F. Therefore,

\[ I_⊗ F = \begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix} \]

\[ I_⊗ D = \begin{pmatrix} D_1 \\ \vdots \\ D_s \end{pmatrix} \]

Observe that \( (F_i)_δ = (f_{i,j})_{j=1} \) since \( f_{i,j} \) is an antisymmetric tensor. Using this fact in the previous expression gives

\[ 1_⊗ F = - \begin{pmatrix} F_1^T \\ \vdots \\ F_s^T \end{pmatrix} = -F \]

An analogous procedure is used to show identity \[11\], but in this case the permutation of indices in \( d_{ijk} \) does not produce a negative sign due to the fact that \( d_{ijk} \) is a completely symmetric tensor. For \[11\], one will employ \[1\] as follows

\[ F^T (A ⊗ B) F = -(1_⊗ F)^T (A ⊗ B)(-1_⊗ F) \]

\[ = F^T (1_⊗ (A ⊗ B) 1_⊗) F \]

\[ = F^T (B ⊗ A) F. \]

Finally, identity \[11\] is proved similarly but using \[1\] instead.

### III. Physical Realizability of Open n-Level Quantum Systems

Consider the separable Hilbert space \( \mathcal{H} \), \( \mathcal{F}(\mathcal{S}) \) the set of operators in \( \mathcal{S} \), and \( \mathcal{F}(\mathcal{S})^{n×m} \) the set of \( n \times m \) dimensional arrays of operators in \( \mathcal{F}(\mathcal{S}) \). The commutator of \( x \) and \( y \) in \( \mathcal{F}(\mathcal{S}) \) is \( [x, y] = xy - yx \). Let \( x, y \in \mathcal{F}(\mathcal{S})^{n_1} \) and \( y \in \mathcal{F}(\mathcal{S})^{n_2} \), then \( [x, y^T] ≜ xy^T - (yx)^T \) is an antisymmetric tensor. The adjoint of \( x \) is denoted by \( x^\dagger = (x^#)^T \) with

\[ x^\dagger = \begin{pmatrix} x_1^\dagger \\ \vdots \\ x_n^\dagger \end{pmatrix} \]

and \( ^\dagger \) denotes the adjoint or conjugate transpose. For the case of complex vectors and matrices, \( ^\dagger \) denotes the complex conjugate while \( ^\dagger \) denotes the transpose.

The Heisenberg evolution of \( x \in \mathcal{F}(\mathcal{S}) \) interacting with \( n_w \) quadrature Boson quantum fields \( W_1 \) and \( W_2 \) in the parametrization \( (S, L, H) \) is given by

\[ dx = \mathcal{L}(x) dt + \frac{1}{2} \left( [x, L^T] - [x, L^T] \right) dW_1 \]

\[ - \frac{i}{2} \left( [x, L^T] + [x, L^T] \right) dW_2 \]

where \( \mathcal{L}(X) \) is the Lindblad operator defined as

\[ \mathcal{L}(x) = -i[x, H] + \frac{1}{2} \left( (L^T \{x, L^T\}^T + [L^#, x^T] L^T \right) \]

The output field in its quadrature form is

\[ \begin{pmatrix} dY_1 \\ dY_2 \end{pmatrix} = \begin{pmatrix} L + L^# \\ i(L^# - L) \end{pmatrix} dt + \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}. \]
Here the operator $S$ is assumed to be the identity operator ($S = I$), and the Itô table ([7]) for the fields $W_1$ and $W_2$ is

$$\begin{pmatrix}
\frac{dW_1}{dt} & dW_2
\end{pmatrix} = \begin{pmatrix}
I_{n_w} & iI_{n_w} \\
-iI_{n_w} & I_{n_w}
\end{pmatrix} dt.
\tag{8}
$$

The interest in this paper is in systems evolving with respect to the special unitary group $SU(n)$. Consider the Hilbert space for these systems to be $H = \mathbb{C}^n$. It is standard to associate a vector $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{C}^s$ with the vector of operators $\hat{\beta} \in \mathfrak{su}(n)$ by simply considering the identity operator in each component of the vector, i.e., $\beta = (\beta_1 I, \ldots, \beta_s I)$. In a similar manner, any complex matrix is associated to a matrix of operators by considering the identity operator $I$ in each component. The identity operator $I$ is usually suppressed. Therefore, abusing of the notation slightly, the mappings $\Theta^-$ and $\Theta^+$ are allowed to act on a vector of operators. In this manner, it is valid to multiply complex matrices and operator matrices. This convention allows to express the commutation and anticommutation relations of the system variables $x$ as

$$
[x, x^T] = 2i\Theta^-(x),
\tag{9a}
$$

$$
\{x, x^T\} = \frac{4}{n} I + 2\Theta^+(x).
\tag{9b}
$$

The vector of system variables for (6) evolving in $SU(n)$ is then

$$
x = \begin{pmatrix}
x_1 \\
\vdots \\
x_s
\end{pmatrix} \triangleq \begin{pmatrix}
\hat{\lambda}_1 \\
\vdots \\
\hat{\lambda}_s
\end{pmatrix},
$$

where $\hat{\lambda}_1, \ldots, \hat{\lambda}_s$ are self-adjoint and spanned by the generalized Gell-Mann matrices. They are usually called spin operators. The initial value of the system variables can be set to $x(0) = (\lambda_1, \ldots, \lambda_s)$ with $\lambda_1, \ldots, \lambda_s$ being the generators of $SU(n)$ introduced in Section III. Due to the product relation (3) any polynomial of spin operators can be written as a linear combination of generalized Gell-Mann matrices. Therefore, assuming linearity captures a large class of Hamiltonian and coupling operators. This allows to assume, without losing generality, that the Hamiltonian $H$ is $H = \alpha x$ with $\alpha \in \mathbb{R}^2$, and the multiplicative coupling operator is of the form $L = \Lambda x$ with $\Lambda \in \mathbb{C}^{n_w \times n_s}$.

In general, the evolution of $x$ in quadratic form falls into a class of bilinear QSDEs expressed as

$$
dx = A_0 dt + A x dt + (B_{11} x, \ldots, B_{1n_w} x, B_{21} x, \ldots, B_{2n_w} x) \left(\frac{dW_1}{dt}, \frac{dW_2}{dt}\right),
\tag{10}
$$

$$
\left(\frac{dY_1}{dt}, \frac{dY_2}{dt}\right) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x dt + \left(\frac{dW_1}{dt}, \frac{dW_2}{dt}\right),
\tag{11}
$$

where $A_0 \in \mathbb{R}^s, A, B_{1k} \triangleq B_{1k} + \bar{B}_{2k}, B_{2k} \triangleq i(B_{2k} - \bar{B}_{1k}) \in \mathbb{R}^{s \times s}$ and $C_1, C_2 \in \mathbb{R}^{n_w \times s}, k = 1, \ldots, n_w$. It is assumed that all matrices in (10) and (11) are real due to the fact that the class of quantum systems considered in this paper are in quadratic form. The next results in physical realizability were given in [4] and are employed later in Section IV.

Definition 2: [4] A system described by equations (10) and (11) is said to be physically realizable if there exist $H$ and $L$ such that (10) can be written as (6). The explicit form of matrices $A_0, A, B_{1k}, B_{2k}, C_1$ and $C_2$ in terms of the Hamiltonian and coupling operator is given next.

Theorem 1: [4] Let $H = \alpha x$, with $\alpha^T \in \mathbb{R}^s$, and $L = \Lambda x$, with $\Lambda \in \mathbb{C}^{n_w \times s}$. Then

$$
A_0 = \frac{4i}{n} \sum_{k=1}^{n_w} \Theta^-(\Lambda_k^\#) \Lambda_k^T,
\tag{12a}
$$

$$
A = -2\Theta^- (\alpha) + \sum_{k=1}^{n_w} (R_k - iQ_k),
\tag{12b}
$$

$$
B_{1k} = \Theta^-(i(\Lambda_k^\# - \Lambda_k)),
\tag{12c}
$$

$$
B_{2k} = -\Theta^-(\Lambda_k + \Lambda_k^\#),
\tag{12d}
$$

$$
C_1 = \Lambda + \Lambda^\#,
\tag{12e}
$$

$$
C_2 = i(\Lambda^\# - \Lambda),
\tag{12f}
$$

where

$$
R_k \triangleq \Theta^- (\Lambda_k) \Theta^- (\Lambda_k^\#) + \Theta^- (\Lambda_k^\#) \Theta^- (\Lambda_k),
$$

$$
Q_k \triangleq \Theta^- (\Lambda_k) \Theta^+ (\Lambda_k^\#) - \Theta^- (\Lambda_k^\#) \Theta^+ (\Lambda_k).
$$

The next theorem present conditions for physical realizability in terms of $(A_0, A, B_i, C)$ in (10) and (11).

Theorem 2: [4] System (10) with output equation (11) is physically realizable if and only if

$$
i. A_0 = \frac{1}{n} \sum_{k=1}^{n_w} (iB_{1k} + B_{2k}) ((C_1)^k + i(C_2)^k)^T,
\tag{12a}
$$

$$
ii. B_{1k} = \Theta^- ((C_2)^k),
\tag{12b}
$$

$$
iii. B_{2k} = \Theta^- ((C_1)^k),
\tag{12c}
$$

$$
iv. A + A^T + \sum_{i,k=1}^{2,n_w} B_{ik} B_{ik}^T = \frac{n}{2} \Theta^+(A_0),
\tag{12d}
$$

where $(C_i)^k$ indicates the $k$-th row of $C_i$. In which case, the coupling matrix can be identified to be

$$
\Lambda = \frac{1}{2} (C_1 + iC_2),
$$

and $\alpha$, defining the system Hamiltonian, is

$$
\alpha = \frac{1}{4n} \vec{\left( A^T - A + \frac{1}{2} \sum_{k=1}^{n_w} ([B_{2k}, \Theta^+ ((C_2)^k)] - [B_{1k}, \Theta^- ((C_1)^k)])\right)}^T F.
\tag{13}
$$

IV. PRESERVATION OF COMMUTATION AND ANTI-COMMUTATION RELATIONS

In this section, the main results of the paper are provided. It is shown first under what conditions a state space model, described by the QSDE (10), preserves the commutation and anticommutation relations of $SU(n)$. The employed procedure for this task is independent of the physical realizability conditions given in Theorem III. Also, recall from Section III that $G \in \mathbb{C}^{s \times s}$ can be always regarded as $G \in \mathfrak{su}(n)$ since the identity operator in $\mathfrak{su}(n)$ can be considered attached to each component of $G$. The following Lemma plays a key role in the forthcoming theorems.

Lemma 3: Let $G \in \mathfrak{su}(n)$ be antisymmetric and $x$ be a vector comprised by elements of a complete linearly
independent set of operators spanning $\mathfrak{g}(\mathfrak{s})$. If $G$ satisfies
\begin{equation}
G\Theta^-(x) + \Theta^-(x)G^T - \Theta^-(Gx) = 0,
\end{equation}
then there exist $g \in \mathfrak{g}(\mathfrak{s})$ such that
\begin{equation}
G = \Theta^-(g),
\end{equation}
where the unique $g$ is given by
\begin{equation}
g \overset{\Delta}{=} -\frac{1}{n} \left( \frac{\text{Tr}(F_1G)}{\text{Tr}(F_2G)} \right).
\end{equation}
Conversely, if (15) holds, then (14) is true for any $x \in \mathfrak{g}(\mathfrak{s})^\ast$.

**Proof:** Assume (14) holds. By applying vec to (14) and recalling that $F = (F_1, \cdots, F_s)^T$, it follows that
\[\text{vec} \left( G\Theta^-(x) + \Theta^-(x)G^T - \Theta^-(Gx) \right) = (I \otimes G)Fx + (G \otimes I)Fx - FGx.\]
Since by assumption the components of $x$ are linearly independent, it can be concluded that
\begin{equation}
(I \otimes G)F + (G \otimes I)F - FG = 0.
\end{equation}
The application of identity (12) in Lemma 2 and (4) after multiplying on the left by $F^T$ gives
\[2 \sum_{k=1}^s F_kGF_k + nG = 0.\]
The equation for the $(i, j)$ component of $G$ is then
\begin{equation}
2 \sum_{k, r, l=1}^s f_{kjr}f_{kil}G_{lr} - nG_{ij} = 0.
\end{equation}
From (24) and noting that $\text{Tr}(AB) = \sum_{r, t=0}^s A_{rt}B_{tr}$ for any $A, B \in \mathbb{R}^{s \times s}$, (13) is rewritten as
\[0 = 2 \sum_{k, r, l=1}^s f_{kjr}f_{kil}G_{lr} - nG_{ij}
= -2 \sum_{r, l=1}^s G_{tr} \sum_{k=1}^s \left( f_{jkr}f_{ilk} + f_{lrk}f_{ijk} \right) - nG_{ij}
= -2 \sum_{k, r, l=1}^s f_{jkr} \sum_{k=1}^s \left( f_{ilk} + f_{lrk}G_{ir} \right) - nG_{ij}
= -2 \sum_{k, r, l=0}^s f_{jkr} \sum_{k=1}^s \left( f_{ilk} + f_{lrk}G_{ir} \right)
= -2 \sum_{k=1}^s f_{jkr} \text{Tr}(F_kG) - nG_{ij}.
\end{equation}
Also (13) implies that
\[2 \sum_{k, r, l=1}^s f_{kjr}f_{kil}G_{lr} = 2 \sum_{k, r, l=1}^s f_{kjr}(-f_{kir})(-G_{rl}) = nG_{ij}.
\]
Substituting this on (19) gives
\[0 = -nG_{ij} - 2 \sum_{k=1}^s f_{ijk} \text{Tr}(F_kG) - nG_{ij}.
\]
which means that $G_{ij} = -\frac{1}{n} \sum_{k=1}^s f_{ijk} \text{Tr}(F_kG)$. Then,
\begin{align*}
G_{ij} &= -\frac{1}{n} \sum_{k=1}^s f_{ijk} \text{Tr}(F_kG) \\
&= \frac{1}{n} \left( f_{j1}, \cdots, f_{js} \right) \begin{pmatrix}
\text{Tr}(F_1G) \\
\vdots \\
\text{Tr}(F_sG)
\end{pmatrix} \\
&= -\frac{1}{n} \left( (F_j)_{i1}, \cdots, (F_j)_{is} \right) \begin{pmatrix}
\text{Tr}(F_1G) \\
\vdots \\
\text{Tr}(F_sG)
\end{pmatrix}.
\end{align*}
Therefore,
\[G_j = -\frac{1}{n} F_j^T \begin{pmatrix}
\text{Tr}(F_1G) \\
\vdots \\
\text{Tr}(F_sG)
\end{pmatrix}.
\]
By defining $g$ as in (19), one obtains that
\[G = (F_1^T g, \cdots, F_s^T g) = \Theta^-(g).
\]
Now, assuming that there exist a $g$ such that $G = \Theta^-(g)$, it then follows directly from (13d) that the left-hand side of (14) satisfies
\[\Theta^-(g)\Theta^-(x) - \Theta^-(x)\Theta^-(g) = 0.
\]
for an arbitrary $x \in \mathfrak{g}(\mathfrak{s})^\ast$, which concludes the proof. 

In order to be considered a quantum system, the system variables of (10) must preserve the product rule (1) for all times. This amounts to preserve the commutation and anticommutation relations of $SU(n)$. Thus, the conditions that (10) must satisfy are
\begin{align}
d[x, x^T] &- 2i\Theta^-(dx) = 0, \\
d\{x, x^T\} &- 2\Theta^+(dx) = 0.
\end{align}
Note by the linearity of $\Theta^-$ and $\Theta^+$ that
\[\Theta^-(dx) = \Theta^-(A_0)dt + \Theta^-(Ax)dt + \Theta^-(B_1x)dtW + \Theta^-(B_2x)dtW_2,
\]
\[\Theta^+(dx) = \Theta^+(A_0)dt + \Theta^+(Ax)dt + \Theta^+(B_1x)dtW + \Theta^+(B_2x)dtW_2.
\]
A condition for system (10) to satisfy (20a) and (20b) is given in the next theorem.

**Theorem 3:** Let $x(0) \in \mathfrak{g}(\mathfrak{s})^\ast$ be comprised by the generators of $SU(n)$. Then, system (10) implies
\[x(t), x(t)] = 2i\Theta^-(x(t)),
\]
\[x(t), x(t)] = \frac{4}{n} I + 2\Theta^+(x(t))
\]
for all $t \geq 0$, if and only if
\begin{align}
B_i &= \Theta^-(b_i) \\
\sum_{k=1}^n B_{1k}B_{2k}^T - B_{2k}B_{1k}^T &= \frac{n}{2} \Theta^-(A_0) \\
A &= \Theta^-(a) - \frac{1}{2} \sum_{k=1}^n \left( B_{1k}B_{1k}^T + B_{2k}B_{2k}^T \right) + \frac{1}{2} \sum_{k=1}^n \left( B_{2k} \Theta^+(b_{1k}) - B_{1k} \Theta^+(b_{2k}) \right).
\end{align}
where $b_{ik}$ and $a$ are $s$-dimensional vectors as in (16) for $i = 1, \ldots, s$ and $k = 1, \ldots, n_a$.

**Proof:** Without loss of generality, one can consider interacting with only one quadrature field. That is,

$$dx = A_0 \, dt + A_x \, dt + B_1 \, dw_1 + B_2 \, dw_2.$$

Using the Itô table (8), it follows that

$$d(x x^T) = (dx) x^T + x (dx)^T + (dx)(dx)^T = (A_0 x^T + x A_0^T) \, dt + (A x x^T + x x^T A^T) \, dt + (B_1 x x^T + x B_1^T) \, dw_1 + (B_2 x x^T + x B_2^T) \, dw_2 + B_1 x x^T B_1^T \, dw_1 \, dw_1 + B_2 x x^T B_2^T \, dw_1 \, dw_2 + B_2 x x^T B_2^T \, dw_2 \, dw_1 + B_2 x x^T B_2^T \, dw_2 \, dw_2.$$

Similarly,

$$d([x, x^T]) = (x x^T)^T = (A_0 x^T + x A_0^T) \, dt + (A x x^T + x x^T A^T) \, dt + (B_1 x x^T + x B_1^T) \, dw_1 + (B_2 x x^T + x B_2^T) \, dw_2 + B_1 x x^T B_1^T \, dw_1 \, dw_1 + B_2 x x^T B_2^T \, dw_1 \, dw_2 + B_2 x x^T B_2^T \, dw_2 \, dw_1 + B_2 x x^T B_2^T \, dw_2 \, dw_2.$$

Computing $d([x, x^T])$ then gives

$$d([x, x^T]) = d(x x^T) - d(x x^T)^T = 2i \left( \frac{2}{n} B_1 B_2^T - \frac{2}{n} B_2 B_1^T \right) \, dt + 2i \left( A x^T - x A^T \right) \, dt + 2i \left( B_1 x^T B_1^T + B_2 x^T B_2^T \right) \, dt + 2i \left( B_1 x^T B_1^T - B_2 x^T B_2^T \right) \, dw_1 + 2i \left( B_2 x^T B_2^T - B_1 x^T B_1^T \right) \, dw_2. \quad (22)$$

Replacing (25) into (20a) amounts to

$$2i \left( B_1 B_2^T - B_2 B_1^T - \Theta^-(A_0) + A \Theta^-(x) + \Theta^-(x) A^T \right) \, dt + 2i \left( B_1 \Theta^-(x) B_1^T + B_2 \Theta^-(x) B_2^T - \Theta^-(A x) \right) \, dt + 2i \left( B_1 \Theta^-(x) B_1^T + B_2 \Theta^-(x) B_2^T \right) \, dw_1 + 2i \left( B_1 \Theta^-(x) - B_2 \Theta^-(x) B_2^T - \Theta^-(B_2 x) \right) \, dw_1 = 0. \quad (23)$$

From [14, Proposition 27.3], one can also equate the integrands in (22) to zero. Also, recall that $x(0)$ is represented by the complete orthonormal set constituted by the generalized Gell-Mann matrices. This implies that any linear combination $\sum_{k=1}^{\infty} a_i x_i(0) \neq 0$ unless $a_i = 0$ for all $i$ and $a_i \in C$. In addition, no linear combination of generalized Gell-Mann matrices generates the identity. So, given that $x(0) \neq 0$, any equation involving the system variables of the form $A x = b$ ($A \in C^{s \times s}$) implies $A$ and $b$ must be identically 0. Thus, the equations to be satisfied for preservation of commutation relations are

$$B_1 \Theta^-(x) + \Theta^-(x) B_1^T - \Theta^-(B_1 x) = 0 \quad (24a)$$
$$B_2 \Theta^-(x) + \Theta^-(x) B_2^T - \Theta^-(B_2 x) = 0 \quad (24b)$$
$$\frac{2}{n} B_1 B_1^T - \frac{2}{n} B_2 B_2^T - \Theta^-(A_0) = 0 \quad (24c)$$
$$A \Theta^-(x) + \Theta^-(x) A^T + B_1 \Theta^+(x) B_1^T + B_2 \Theta^+(x) B_2^T + B_1 \Theta^+(x) B_1^T - B_2 \Theta^+(x) B_2^T - \Theta^-(A x) = 0. \quad (24d)$$

In the same vein for the anticommutator, one has that $d(\{x, x^T\})$ is given by

$$d([x, x^T]) = d(x x^T) + d(x x^T)^T = \frac{4}{n} (A + A^T + B_1 B_1^T + B_2 B_2^T) \, dt + 2 \left( A_0 x^T + x A_0^T \right) \, dt + 2 \left( A \Theta^T(x) + \Theta^+(x) A^T \right) \, dt + 2 \left( B_1 \Theta^+(x) B_1^T + B_2 \Theta^+(x) B_2^T \right) \, dt + 2 \left( B_1 \Theta^-(x) B_1^T - B_2 \Theta^-(x) B_2^T \right) \, dt + 2 \left( \frac{2}{n} B_1 + \frac{2}{n} B_1^T + B_1 \Theta^+(x) + \Theta^+(x) B_1^T \right) \, dw_1 + 2 \left( \frac{2}{n} B_2 + \frac{2}{n} B_2^T + B_2 \Theta^+(x) + \Theta^+(x) B_2^T \right) \, dw_2. \quad (25)$$

Using the same argument employed for the commutation relation of $x$, the integrands in (25) are equal to zero. Thus, for preservation of anticommutation relations one has that

$$B_1 \Theta^+(x) + \Theta^+(x) B_1^T - \Theta^+(B_1 x) = 0 \quad (26a)$$
$$B_2 \Theta^+(x) + \Theta^+(x) B_2^T - \Theta^+(B_2 x) = 0 \quad (26b)$$
$$A + A^T + B_1 B_1^T + B_2 B_2^T - \frac{n}{2} \Theta^+(A_0) = 0 \quad (26c)$$
$$B_1 + B_1^T = B_2 + B_2^T = 0 \quad (26d)$$
$$A_0 x^T + x A_0^T + A \Theta^+(x) + \Theta^+(x) A^T - \Theta^+(A x) + B_1 \Theta^+(x) B_1^T + B_2 \Theta^+(x) B_2^T + B_1 \Theta^+(x) B_1^T - B_2 \Theta^+(x) B_2^T - \Theta^-(A x) = 0. \quad (26e)$$

Assume that (24a)-(24d) and (26a)-(26d) hold. Condition (21a) appears explicitly in (24b). From (24a), (24b) and (26d), condition (21a) is obtained by direct application of Lemma 3 i.e., $B_i = \Theta^-(b_i)$ holds, where $b_i$ is given in (16) for $i \in \{1, 2\}$. In order to show (21a), $A_0$ need to be expressed in a more convenient form. This is achieved by using (5d) and (24b). That is,

$$\Theta^-(A_0) = \frac{2}{n} \left( (\Theta^-(b_2) \Theta^-(b_1) - \Theta^-(b_2) \Theta^-(b_1)) \right) = \frac{2}{n} \Theta^- (b_2) b_1.$$

Due to the linearity of $\Theta^-$, $A_0$ is then uniquely determined by $A_0 = \frac{2}{n} \Theta^-(b_2) b_1$. Computing $\Theta^+(A_0)$ in terms of $b_1$ and $b_2$ using (5c) gives

$$\Theta^+(A_0) = \frac{2}{n} \Theta^+(\Theta^-(b_2) b_1).$$
\[
\frac{1}{n} \left( \Theta^+(b_2) \Theta^-(b_1) - \Theta^-(b_1) \Theta^+(b_2) \right)
+ \Theta^-(b_2) \Theta^+(b_1) - \Theta^+(b_1) \Theta^-(b_2) \right). \tag{27}
\]

Using (26c) and (27), one obtains the following expression
\[
\Theta^-(b_1) \Theta^+(b_1) + \Theta^+(b_1) \Theta^-(b_1) = A - \frac{1}{2} \left( \Theta^-(b_2) \Theta^+(b_1) - \Theta^+(b_1) \Theta^-(b_2) \right)
+ \left( A - \frac{1}{2} \left( \Theta^-(b_2) \Theta^+(b_1) - \Theta^+(b_1) \Theta^-(b_2) \right) \right)^T. \tag{28}
\]

Now let \( \hat{P} = \frac{1}{2} \left( \Theta^-(b_2) \Theta^+(b_1) - \Theta^+(b_1) \Theta^-(b_2) \right) \). One can trivially decompose \( \hat{P} \) in its symmetric and antisymmetric parts. That is,
\[
\hat{P} = \frac{1}{2} \left( \hat{P} + \hat{P}^T \right) + \frac{1}{2} \left( \hat{P} - \hat{P}^T \right).
\]

Define \( P = \frac{1}{2} \left( \hat{P} - \hat{P}^T \right) \). Note from (28) that
\[
P + P^T = \Theta^-(b_1) \Theta^+(b_1) + \Theta^+(b_1) \Theta^-(b_1).
\]

Hence, \( P \) can be written as
\[
P = A - \frac{1}{2} \left( \Theta^-(b_2) \Theta^+(b_1) + \Theta^+(b_1) \Theta^-(b_2) \right)
- \frac{1}{2} \left( \Theta^-(b_2) \Theta^+(b_1) - \Theta^+(b_1) \Theta^-(b_2) \right).
\]

It is only left to show that \( P \) can be written in terms of \( \Theta \). The formula for \( P \) allows to calculate \( \Theta^{-}(P x) \) as
\[
\Theta^{-}(P x) = \Theta^{-}(A x) - \frac{1}{2} \Theta^{-}(\Theta(b_1) \Theta^{-}(b_1) x + \Theta^{-}(b_2) \Theta^{-}(b_2) x)
+ \Theta^{-}(b_2) \Theta^{-}(b_1) x - \Theta^{-}(b_1) \Theta^{-}(b_2) x).
\]

Hence,
\[
\Theta^{-}(P x) = \Theta^{-}(A x)
- \frac{1}{2} \left( \Theta^{-}(\Theta(b_1) \Theta^{-}(b_1) x + \Theta^{-}(b_2) \Theta^{-}(b_2) x) \right)
- \frac{1}{2} \left( \Theta^{-}(\Theta(b_2) \Theta^{-}(b_1) x - \Theta^{-}(b_1) \Theta^{-}(b_2) x) \right).
\]

It is only left to show that \( P \) can be written in terms of \( \Theta \). The formula for \( P \) allows to calculate \( \Theta^{-}(P x) \) as
\[
\Theta^{-}(P x) = \Theta^{-}(A x) - \frac{1}{2} \Theta^{-}(\Theta(b_1) \Theta^{-}(b_1) x + \Theta^{-}(b_2) \Theta^{-}(b_2) x)
+ \Theta^{-}(b_2) \Theta^{-}(b_1) x - \Theta^{-}(b_1) \Theta^{-}(b_2) x).
\]

Hence,
\[
\Theta^{-}(P x) = \Theta^{-}(A x)
- \frac{1}{2} \left( \Theta^{-}(\Theta(b_1) \Theta^{-}(b_1) x + \Theta^{-}(b_2) \Theta^{-}(b_2) x) \right)
- \frac{1}{2} \left( \Theta^{-}(\Theta(b_2) \Theta^{-}(b_1) x - \Theta^{-}(b_1) \Theta^{-}(b_2) x) \right).
\]

Now, from the symmetry of \( \Theta^+ \) and (26d), one observes that
\[
\Theta^+((b_1) x) = \Theta^+(\Theta^-((b_1) x))
+ \frac{1}{2} \left( \Theta^-(b_1) \Theta^+(b_1) x - \Theta^+(b_1) \Theta^-(b_1) x \right)
+ \frac{1}{2} \left( \Theta^+(b_1) \Theta^-(b_1) x - \Theta^-(b_1) \Theta^+(b_1) x \right).
\]

Therefore,
\[
\Theta^+(A x) = \Theta^+(\Theta^-((a)x))
+ \frac{1}{2} \left( \Theta^-(b_1) \Theta^+(b_1) x - \Theta^+(b_1) \Theta^-(b_1) x \right)
+ \frac{1}{2} \left( \Theta^+(b_1) \Theta^-(b_1) x - \Theta^-(b_1) \Theta^+(b_1) x \right).
\]

Conversely, assume that (21a) and (21c) are true. By Lemma 3 equations (26a), (26c) and (26d) are automatically satisfied. Equation (26d) it is easily obtained by computing \( A + A^T \) using (21c). Furthermore, equations (26a) and (26b) are a direct result of (23). The validity of (24) is shown by computing \( A \Theta^{-}(x) + \Theta^{-}(x) A^T - \Theta^{-}(A x) \). This calculation is analogous to the computation of (29), but in this case it is already known that \( P = \Theta^{-}(a) \). Thus, (24) is true given that (21a) hold. To show that (26c) holds, one first computes \( \Theta^+(A x) \) using (21c) as follows
\[
\Theta^+(A x) = \Theta^+((b_1) x)
+ \frac{1}{2} \left( \Theta^+(b_1) \Theta^+((b_1) x) - \Theta^+(b_1) \Theta^-((b_1) x) \right)
+ \frac{1}{2} \left( \Theta^-((b_1) x) - \Theta^-((b_1) x) \right).
\]

Therefore,
\[
\Theta^+(A x) = \Theta^+((b_1) x)
+ \frac{1}{2} \left( \Theta^+(b_1) \Theta^+((b_1) x) - \Theta^+(b_1) \Theta^-((b_1) x) \right)
+ \frac{1}{2} \left( \Theta^-((b_1) x) - \Theta^-((b_1) x) \right).
\]
Since \(2\) has from gives is physically realizable then adding (12b) and its transpose (12b) holds. Finally, since the considered system \(\Theta\) is preserved when it satisfies the physical realizability conditions since they also ensure preservation of commutation and anticommutation relations of \(SU(n)\). In addition, it is known that the physical realizability conditions can be employed to obtain the \((S, L, H)\) parameterization of that quantum system.

V. CONCLUSIONS

Conditions for preserving commutations and anticommutation relations have been provided for QSDE’s of the form (10). These results used explicitly the algebra of \(SU(n)\), and in particular algebraic expressions involving the anomaly tensor \(d\) had to be employed. Moreover, it was shown that the physical realizability implies the preservation of those relations, and therefore under physical realizability conditions the system given by (10) and (11) describes an open \(n\)-level quantum system.

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