Klebanov-Witten flows in M-theory

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Abstract: We study renormalization group flows among three dimensional superconformal gauge theories which closely resemble the renowned Klebanov-Witten flow in four dimensions. In the large $N$ limit, each theory appearing in the flow is holographically dual to M-theory on AdS\textsubscript{4} times a toric Sasaki-Einstein seven-manifold. The theories are obtained through the so-called flavoring method, which adds some fundamental matter fields to the dimensionally reduced Klebanov-Witten theories. We reconfirm the matching between the gauge theories and the dual geometries by comparing the chiral ring structure. As a more refined test of the flows, we compute the three-sphere partition function of the gauge theories. The square of the free energy, inversely proportional to the volume of the seven-manifold, decreases by a universal ratio $16/27$ for all flows considered in this paper.
1 Introduction

The discovery of Klebanov-Witten (KW) flow [1] was one of the most important landmarks in the early development of AdS/CFT correspondence. The KW flow describes a renormalization group (RG) flow from an $\mathcal{N} = 2$ supersymmetric orbifold quiver gauge theory in the ultraviolet (UV) to an $\mathcal{N} = 1$ theory in the infrared (IR). The IR theory, often called the KW theory, was the first non-orbifold theory to be studied in the AdS/CFT context.

The KW flow triggered rapid progress in several important directions. First, the classification of supersymmetric gravity backgrounds in terms of special holonomy manifolds was carried out in [2, 3]. Second, since the KW theory was physically realized as D3 branes at the tip of the conifold, a toric Calabi-Yau threefold ($\text{CY}_3$) cone, systematic studies relating toric $\text{CY}_3$ cones to supersymmetric gauge theories was initiated in [3]. A large number of
subsequent works gradually converged to a unifying framework of the brane tiling model [4, 5]. Finally, the KW flow partly motivated the general study of RG flows in AdS/CFT. A notable early work includes the Pilch-Warner (PW) flow [6, 7] described by a kink solution that interpolates between the maximally supersymmetric AdS$^5 \times S^5$ background and another AdS background for an $\mathcal{N} = 1$ theory obtained by a mass deformation. Unlike the KW flow, the PW flow preserves the topology of the ‘internal’ manifold $Y$ in the AdS$\times Y$ background. For instance, for the original PW flow, the supergravity background for the IR theory is the same $S^5$ but with a non-standard metric and Ramond-Ramond flux.

From early days, many attempts were made to carry over these important findings of the AdS$^5$/CFT$_4$ correspondence to the AdS$^4$/CFT$_3$ setup. However, progress has been severely limited due to the difficulty associated with strongly coupled infrared dynamics of M2-brane world-volume theories. Rather recently, the discovery of $\mathcal{N} \geq 4$ supersymmetric Chern-Simons-matter theories [8–17] marked a major breakthrough in the development of AdS$_4$/CFT$_3$ and revived many previous attempts. In particular, new methods were developed to construct a variety of $\mathcal{N} = 2$ theories, including those corresponding to M2-branes probing toric Calabi-Yau fourfold (CY$_4$) cones [18–22].

As for the study of RG flows, there had been gravity analysis of PW-like flows from early on [23–29]. More recently, field theory models have been given for some flows [30], and the gravity analysis have also been improved [31, 32]. In contrast, the possibility of a KW-like flow remained an open question even after all the new developments.

The main goal of this paper is to give an affirmative answer to this question by proposing two concrete examples of KW type flows in M-theory in the AdS$_4$/CFT$_3$ setup. The UV and IR CFT’s appearing in the new flows of this paper are constructed by using the so-called flavoring method [21, 22], which adds some fundamental matter fields to the dimensionally reduced Klebanov-Witten theories. Geometrically, the flavoring lifts the toric diagrams for the KW flow to higher dimensional ones. Once the theories are constructed, we can reconfirm the correspondence between the gauge theory and the geometry by comparing the vacuum moduli space and the chiral ring structure.

Independently of the AdS/CFT correspondence, recently there has been major progress in understanding RG-flows in three dimensional supersymmetric gauge theories. It was shown in [33–35] that the three-sphere partition function $Z$ of $\mathcal{N} = 2$ supersymmetric theories can be exactly computed via localization techniques. The exact partition function provides a systematic and quantitative way to study the strongly coupled infrared dynamics of three-dimensional theories. In particular, Ref. [34] proposed that the free energy $F = -\log |Z|$ can define a measure of the number of degrees of freedom that decreases monotonically along RG flows. See [36, 37] for possible proof of this conjectured ‘F-theorem’. When the gauge theory is AdS/CFT-dual to a toric CY$_4$, the free energy is related to the volume of the base $Y$ of the cone by $F \propto 1/\sqrt{\text{Vol}(Y)}$ [30, 38–40].

To test our proposal for the new RG flows at the quantum level, we compute the three-sphere partition function of the gauge theories. By extremizing the free energy with respect to trial $R$-charges of the matter fields, we find the extremal values of the $R$-charges and the value of $F$ that perfectly matches the expectation from the geometry.
We also note that, for all examples of flows considered in this paper, the free energy decreases along the RG flow by a universal ratio,
\[
\frac{F_{\text{IR}}}{F_{\text{UV}}} = \sqrt{\frac{\text{Vol}(Y_{\text{UV}})}{\text{Vol}(Y_{\text{IR}})}} = \sqrt{\frac{16}{27}},
\]
consistent with the F-theorem. Using the localization method, we give a field theoretic proof of this universal ratio for a large class of theories that includes, but is not limited to, all theories explicitly studied in this paper.

The universal ratio 16/27 was first observed in [30] which considered the RG flows triggered by mass deformation of three dimensional CFT’s describing M2-branes probing CY$_3 \times \mathbb{C}$. Here CY$_3$’s are of a special class described algebraically by $xy = z^{n_1}w^{n_2}$. This work provided strong hint on the existence of various PW flows in M-theory whose explicit solutions were constructed recently in [32]. Furthermore, a general proof of the universal ratio was given in [31] based on a certain scaling behavior of the gravitational free energy for the PW solutions. In the present work, we consider KW flows from CY$_3 \times \mathbb{C}$ that are qualitatively different from the PW flows. The two geometries dual to the UV and IR fixed points of the KW flow are related by a complex deformation and, in contrast to the PW flow, have different topology. Many of CY$_3$’s in our discussion are beyond the examples considered in [30]. Our field theoretical proof of the universal ratio (1.1) can be regarded as AdS/CFT dual to the gravitational proof presented in [31]. See [41] for a discussion of similar universal ratio in four dimensions.

This paper is organized as follows. In section 2, we review the basics of toric geometry. To prepare for a later comparison to the dual gauge theories, we compute explicitly the Hilbert series, which contains the complete information on the chiral ring in the large $N$ limit and also gives the volume of the Sasaki-Einstein base manifold of the toric cone. In section 3, after reviewing the general method of constructing M2-brane CFT’s with fundamental matter fields via the flavoring process, we construct the UV and IR theories for the two basic examples of RG flows. To confirm the validity of the construction, we compute the chiral ring of the gauge theory and find agreement with the results from the geometry. In section 4, we generalize the main examples of section 3 to infinite families by orbifolding. In section 5, we subject the RG flows to a more stringent test by computing the three-sphere partition functions. The partition function of the UV theory and that of IR theory are related in a simple manner. Using this relation, we give a field theoretic explanation of the universal ratio 16/27. Section 6 contains a brief discussion on the so-called brane crystal model [42–44] which anticipated the KW type flows of this paper before the breakthrough with Chern-Simons-matter theories.

2 Geometry

The $\mathcal{N} = 2$ superconformal gauge theories we consider in this paper are the world-volume theories of a stack of $N$ M2-branes near the tip of some toric Calabi-Yau 4-fold (CY$_4$), $X = C(Y)$ whose base (unit radius section) $Y$ is by definition a toric Sasakian 7-manifold.
In what follows, we will often use the names for the base manifold $Y$, the CY$_4$ cone $X$ and the gauge theory dual to the geometry interchangeably.

2.1 GLSM and toric diagram

We follow the notation of [42–46] for toric geometry. The cone $X$ is constructed by the gauged linear sigma model (GLSM) [47] which take a quotient of $\mathbb{C}^d$ for some $d \geq 4$. Given some integer-valued charge matrix $Q_{\alpha}^I$ ($I = 1, \ldots, d; \alpha = 1, \ldots, d - 4$), the quotient is defined by

$$X = \left\{ \{\phi_I\} \in \mathbb{C}^d \left| \sum_{I=1}^{d} Q_{\alpha}^I |\phi_I|^2 = 0 \right. \right\} / \left( \phi_I \sim e^{Q_{\alpha}^I \theta^\alpha} \phi_I \right).$$

The toric diagram is a convex polyhedron composed of a set of lattice points $\{v_I\} \in \mathbb{Z}^4$ ($i = 1, \cdots, 4$) satisfying

$$\sum_{I=1}^{d} Q_{\alpha}^I v_I = 0.$$

The CY condition, $\sum_{I=1}^{d} Q_{\alpha}^I v_I = 0$, enforces the $v_I$ to lie on the same $\mathbb{Z}^3$ sublattice. It is customary to choose a basis to set $v_4^I = 1$ for all $I$ and specify other three coordinates of the vertices on the $\mathbb{Z}^3$ sublattice. Hence, the resulting toric diagram for the CY$_4$ is effectively three dimensional. Similarly, the toric diagram for a CY$_3$ is two dimensional. The toric diagram defines a solid cone $\Delta_X = \{ y_i \in \mathbb{R}^4; (v_I \cdot y) \geq 0 \}$ over which the CY$_4$ space $X$ is a $U(1)^4$ bundle.

The moduli space of Kähler metrics on $X$ is parameterized by the Reeb vector $b_i \in \mathbb{R}^4$, which also defines the base of the cone by $Y = X \cap \{ b \cdot y = 1/2 \}$. In the basis mentioned above, the CY condition fixes $b_i = 4$. The volume of $Y$ as an explicit function of $v_I$ and $b$ is known [45, 46]. To obtain the Ricci-flat metric, one minimizes the volume with respect to $b_i$ ($(i = 1, 2, 3)$) with the domain of $(b_i/4)$ being precisely the interior of the toric diagram.

The CY$_4$ cone inherits a $U(1)^4$ global symmetry, $F_i$ ($i = 1, 2, 3, 4$), from the GLSM construction. They all correspond to global symmetries of the dual gauge theory. One particular combination determined by the Reeb vector, $R = \frac{1}{2} b_i F_i$, is dual to the superconformal $U(1)$ $R$-symmetry of the gauge theory.

One of the most basic checks of AdS/CFT correspondence with eight supercharges is the comparison of chiral rings. In the gauge theory, the chiral ring is defined by the space of all gauge invariant monomials modded out by F-term conditions. In addition to the classical F-term conditions $d\mathcal{W} = 0$ implied by the superpotential $\mathcal{W}$ of the gauge theory, the three dimensional theories considered in this paper are governed by some quantum F-term conditions.

On the geometry side, the elements of chiral ring correspond to integer points in the cone, $\Delta_X \cap \mathbb{Z}^4 = \{ m_I \in \mathbb{R}^4; (v_I \cdot m) \geq 0 \}$ for all $I$. Their $R$-charge can be computed by a simple formula $R(m) = (b \cdot m)/2$. In terms of the GLSM fields, the chiral ring elements are gauge invariant monomials of $\phi_I$. It is possible to assign a value of $R$-charge, $R^I$, to each GLSM field $\phi_I$. To determine $R^I$, one can use the correspondence between the vertices $v_I$
of the toric diagram and supersymmetric cycles $\Sigma_I$ of the base manifold $Y$. Then, $R^I$ is proportional to the volume of $\Sigma_I$ and satisfies $\sum_I R^I = 2$. Alternatively, one can compare the monomials in $\phi_I$ against the formula $R(m) = (b \cdot m)/2$ and deduce $R^I$.

The generating function of the chiral ring is usually called the Hilbert series $[46, 48]$. 

$$H_X(t) = \sum_{\{m\}} \prod_{i=1}^4 t_i^{m_i} (\{m_i\} \in \Delta_X \cap \mathbb{Z}^4) . \quad (2.3)$$

As explained in $[46, 48]$, instead of actually counting the chiral ring elements, one can compute the Hilbert series by a simple localization formula involving a triangulation of the toric diagram. In this paper, we will use the results of $[46, 48]$ without reviewing the derivation of the localization formula.

An important application of Hilbert series is the computation of the volume of the base manifold $Y$. The normalized volume of $Y$, defined by

$$V_Y = \frac{\text{Vol}(Y)}{\text{Vol}(S^4)} = \frac{\text{Vol}(Y)}{\left(\pi^4/3\right)} , \quad (2.4)$$

can be obtained from the Hilbert series as follows $[46, 48]$,

$$V_Y(b) = \lim_{\varepsilon \to 0} \left[ \varepsilon^4 H_X(t_i = e^{-\varepsilon b_i}) \right]_{b^i = 4} . \quad (2.5)$$

By minimizing the volume with respect to $b^{i=1,2,3}$, we find the critical value of the Reeb vector $b^\ast$. The critical Reeb vector assigns $R$-charge $(b_i^\ast/2)$ to each $t_i$.

The new KW flows in this paper is related to the original KW flow by dimensional reduction and flavoring process. Geometrically, toric diagrams for the new flows are related to those of the original KW flow by projection of toric diagrams. We will always choose the basis of the $\mathbb{Z}^3$ sublattice such that the ‘vertical’ projection of the toric diagrams for CY$_4$ reproduces those of CY$_3$ appearing in the original KW flow.

2.2 Hilbert series and volume

In this subsection, we compute the Hilbert series and volume of the four toric CY$_4$’s dual to the two pairs of gauge theories appearing in the basic examples of the RG flows.

![Figure 1](image-url) Toric diagrams for the flow from $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}$ to $D_3$. 

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The toric diagram for the CY4’s corresponding to the first flow is shown in Figure 1. Note that the ‘vertical’ projection to the \(z = 0\) plane reproduces the toric diagrams for \((\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}\) and \(C(T^{1,1})\) relevant to the original KW flow.

Using the methods of [46, 48], we find the Hilbert series for \(\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}\),

\[
H_{\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}}(t) = \frac{1 + t_4/t_1}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4/t_1^2t_2t_3)},
\]

from which we can compute the normalized volume as a function of the Reeb vector,

\[
V_{\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}}(b) = \frac{2}{b_1b_2b_3(8 - 2b_1 - b_2 - b_3)}.
\]

It is easy to obtain the Sasakian Reeb vector which minimizes the volume,

\[
b_* = (1, 2, 2, 4), \quad V_{\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}}(b_*) = \frac{1}{4}.
\]

Note that upon making the substitution,

\[
t_1 = s_4, \quad t_2 = s_2^2, \quad t_3 = s_2^2, \quad t_4 = s_1s_2s_3s_4,
\]

we recover the familiar orbifold form (average over mirror images) of the Hilbert series,

\[
H_{\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}} = \frac{1 + s_1s_2s_3}{(1 - s_1^2)(1 - s_2^2)(1 - s_3^2)(1 - s_4)}.
\]

As indicated in (2.8) and (2.9), each \(s_i\) carries \(R\)-charge \(1/2\).

\[D_3\]

The Hilbert series for \(D_3\) is given by

\[
H_{D_3}(t) = \frac{1 - t_4}{(1 - t_1)(1 - t_4/t_1)(1 - t_2)(1 - t_3)(1 - t_4/t_2t_3)},
\]

from which we can compute the normalized volume as a function of the Reeb vector,

\[
V_{D_3}(b) = \frac{4}{b_1(4 - b_1)b_2b_3(4 - b_2 - b_3)}.
\]

The volume-minimizing Sasakian Reeb vector respects the symmetry of the toric diagram,

\[
b_* = (2, 4/3, 4/3, 4), \quad V_{D_3}(b_*) = \frac{27}{64}.
\]

To make the symmetries more manifest, we make the following change of variable,

\[
t_1 = t^3s_1, \quad t_2 = t^2s_2, \quad t_3 = t^2s_3, \quad t_4 = t^6,
\]

such that [20]

\[
H_{D_3} = \frac{1 - t^6}{(1 - t^3s_1)(1 - t^3/s_1)(1 - t^2s_2)(1 - t^2s_3)(1 - t^2/s_2s_3)}.
\]

The variable \(t\) carries \(R\)-charge \(1/3\). The \(s_i\) are fugacities for flavor symmetries.
The volume-minimizing Sasakian Reeb vector respects the symmetry of the toric diagram, so that again that the ‘vertical’ projection to the $z = 0$ plane reproduces the toric diagrams relevant to the original KW flow.

Using the methods of [46, 48], we find the Hilbert series for $dP_3 \times \mathbb{C}$,

$$H_{dP_3 \times \mathbb{C}}(t) = \frac{(1 - t_4/t_1)f_{dP_3 \times \mathbb{C}}(t)/(t_1^4t_2t_3)}{(1 - t_1)(1 - t_2)(1 - t_2t_3)(1 - t_4/t_1t_3)/(1 - t_4t_1/t_3)/1^2/t_2t_3)(1 - 1^2/t_2t_3)},$$

$$f_{dP_3 \times \mathbb{C}}(t) = t_1^4t_2t_3 + 2t_1^3t_2t_3t_4 + 2t_1t_2t_3t_4 + t_2t_3^2,$$

$$- t_1^3t_2^2t_3t_4 - t_1^3t_2^2t_3^2 - t_1^3t_2^2 - t_1^2t_2t_3^2 - t_1^3t_2 - t_1t_3^2;$$

from which we can compute the normalized volume as a function of the Reeb vector,

$$V_{dP_3 \times \mathbb{C}}(b) = \frac{2(4 - b_1)(32 - 16b_1 + 2b_1^2 + 8b_2 - 2b_1b_2 - b_3^2 + 4b_3 - b_1b_3 - b_2b_3 - b_3^2)}{b_1b_2(b_2 + b_3)(4 - b_1 + b_3)(4 - b_1 - b_3)(8 - 2b_1 - b_2)(8 - b_1 - b_2 - b_3)}.$$  

(2.17)

The volume-minimizing Sasakian Reeb vector respects the symmetry of the toric diagram,

$$b_\ast = (1, 3, 0, 4), \quad V_{dP_3 \times \mathbb{C}}(b_\ast) = \frac{2}{9}. \quad (2.18)$$

To see the symmetries of $dP_3 \times \mathbb{C}$ more clearly, we make the following change of variable,

$$t_1 = s_4, \quad t_2 = ts_1, \quad t_3 = s_2, \quad t_4 = ts_4,$$

(2.19)

so that

$$H_{dP_3} = \frac{1}{1 - s_4}H_{dP_3}(t, s_1, s_2),$$

$$H_{dP_3}(t, s_1, s_2) = \frac{(1 - t)(1 + 2t - t^2 \sum_{i=1}^3 (s_i + 1/s_i) + 2t^3 + t^4)}{\prod_{i=1}^3(1 - ts_i)(1 - t/s_i)} \bigg|_{s_1s_2s_3=1}.$$  

(2.20)

As indicated in (2.18) and (2.19), the variables $(t, s_1, s_2, s_4)$ carry $R$-charges $(3/2, 0, 0, 1/2)$. 

Figure 2. Toric diagrams for the flow from $dP_3 \times \mathbb{C}$ to $Q^{1,1,1}$. 

$dP_3 \times \mathbb{C}$

The toric diagram for the CY’s associated to the second flow is shown in Figure 2. Note again that the ‘vertical’ projection to the $z = 0$ plane reproduces the toric diagrams relevant to the original KW flow.
The Hilbert series of $Q^{1,1,1}$ in the basis of Figure 2(b) is rather lengthy and not instructive as it does not show the symmetries of $Q^{1,1,1}$ manifestly. To recover the symmetries, we make the following change of variable,

$$t_1 = t(s_1s_2/s_3), \quad t_2 = t(s_1/s_2s_3), \quad t_3 = s_3^2, \quad t_4 = t^2.$$  \hspace{1cm} (2.21)

In this new basis, the Hilbert take the simple form [20],

$$H_{Q^{1,1,1}} = \sum_{n=1}^{\infty} \chi_n(s_1)\chi_n(s_2)\chi_n(s_3)t^{n-1},$$  \hspace{1cm} (2.22)

where

$$\chi_n(s) = \frac{s^n - s^{-n}}{s - s^{-1}},$$  \hspace{1cm} (2.23)

is the character for the $n$-dimensional representation of $SU(2)$. Thus we identify $s_i$ as the fugacities for the $SU(2)^3$ flavor symmetry of $Q^{1,1,1}$. The variable $t$ carries $R$-charge 1/3. The partial symmetries of the toric diagram in Figure 2(b) uniquely determine the Sasakian Reeb vector, from which we find the minimal volume by using (2.5),

$$b_\star = (2, 2, 0, 4), \quad V_{Q^{1,1,1}}(b_\star) = \frac{3}{8}.$$  \hspace{1cm} (2.24)

### 2.3 Chiral ring

To prepare for comparison with gauge theories and later generalizations, we take a closer look at how the chiral ring can be constructed from the GLSM (2.1) and how its information is encoded in the Hilbert series.

$\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}$

For the toric diagram of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}$ shown in Figure 1(a), let $\phi_{a,b}$ denote the GLSM variable assigned to the vertex at $(0,a,b)$. We ignore the vertex at $(1,0,0)$ which accounts for the $\mathbb{C}$ factor and contributes to the chiral ring in a trivial way. In other words, we are effectively dealing with a toric diagram for CY$_3$ with six GLSM fields. From the explicit value of the GLSM charges,

| $\phi_{00}$ | $\phi_{10}$ | $\phi_{20}$ | $\phi_{01}$ | $\phi_{02}$ | $\phi_{11}$ |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 1 | 0 | 0 |
| $v$ | 0 | 0 | 0 | 1 | 2 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

$$Q = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix},$$  \hspace{1cm} (2.25)

we find that there are four elementary gauge invariant monomials subject to one constraint

$$z_1 = \phi_{00}^2\phi_{10}\phi_{01}, \quad z_2 = \phi_{02}^2\phi_{01}\phi_{11}, \quad z_3 = \phi_{20}^2\phi_{10}\phi_{11}, \quad w = \phi_{00}\phi_{10}\phi_{20}\phi_{01}\phi_{02}\phi_{11} \implies z_1z_2z_3 = w^2.$$  \hspace{1cm} (2.26)
Thus, the chiral ring is the polynomial ring of \((z_1, z_2, z_3, w)\) modded out by \(z_1z_2z_3 = w^2\). This fact is reflected in the Hilbert series. For instance, the expansion,

\[
H_{\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)} = \frac{1 + s_1s_2s_3}{(1 - s_1^2)(1 - s_2^2)(1 - s_3^2)} = 1 + \frac{s_1^2 + s_2^2 + s_3^2 + s_1s_2s_3 + \cdots}{z_1z_2z_3 = w^2} + \cdots,
\]

(2.27)

shows that there is precisely one monomial with \(R\)-charge 3 that is fully invariant under the permutations of \(z_1, z_2, z_3\).

\(D_3\)

For the toric diagram of \(D_3\) shown in Figure 1(b), let \(\phi_{a,b}\) denote the GLSM variable assigned to the vertex at \((0, a, b)\) and similarly use \(\tilde{\phi}_{a,b}\) for the vertex at \((1, a, b)\). There are 5 elementary gauge invariant monomials subject to one constraint,

\[
z_1 = \phi_{00}\tilde{\phi}_{00}, \quad z_2 = \phi_{01}\tilde{\phi}_{01}, \quad z_3 = \phi_{10}\tilde{\phi}_{10},
\]

\[
w = \phi_{00}\tilde{\phi}_{10}\tilde{\phi}_{01}, \quad \tilde{w} = \tilde{\phi}_{00}\phi_{10}\phi_{01} \implies z_1z_2z_3 = w\tilde{w}.
\]

(2.28)

So, the chiral ring is the polynomial ring of \((z_1, z_2, z_3, w, \tilde{w})\) modded out by \(z_1z_2z_3 = w\tilde{w}\). This fact is reflected in the Hilbert series. For instance, the expansion,

\[
H_{\mathbb{D}_3} = \frac{1 - t^6}{(1 - t^3s_1)(1 - t^3/s_1)(1 - t^2s_2)(1 - t^2/s_2s_3)} = 1 + \frac{(s_2 + s_3 + 1/s_2s_3)t^2 + (s_1 + 1/s_1)t^3 + \cdots}{z_1z_2z_3 = w\tilde{w}} + \cdots,
\]

(2.29)

shows that there is precisely one monomial at \(O(t^6)\) that is fully invariant under all flavor symmetries.

We note that the two geometries \(\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}\) and \(D_3\) are related by a complex deformation; compare the algebraic descriptions (2.26) and (2.28) for the two geometries

\[
\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C} : \quad z_1z_2z_3 = w^2, \quad v \text{ free}
\]

\[
D_3 : \quad z_1z_2z_3 = w\tilde{w}.
\]

(2.30)

We can regard the two geometries as limiting cases of a single family,

\[
z_1z_2z_3 = w(w + \epsilon v),
\]

(2.31)

such that \(\epsilon \to 0\) leads to \(\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}\) and \(\epsilon = 1\) with \(\tilde{w} = w + v\) gives \(D_3\).

dP_3 \times \mathbb{C}

For the toric diagram of \(dP_3 \times \mathbb{C}\) shown in Figure 2(a), let \(\phi_{a,b}\) denote the GLSM variable assigned to the vertex at \((0, a, b)\) and ignore the trivial one at \((1, 0, 0)\). There are 7
At the Hilbert series correctly captures the structure of the chiral ring. Setting the fugacities to 1 for simplicity, we observe that
\begin{equation}
H_{dP_3} = \frac{1 + 4t + t^2}{(1 - t)^3} = 1 + 7t + 19t^2 + \cdots.
\end{equation}

The 7 terms at \( \mathcal{O}(t) \) represent the elementary gauge invariant variables \( (s_{1,2}, t_{1,2}, u_{1,2}; w) \). At \( \mathcal{O}(t^2) \), there are \( \frac{7 \cdot 8}{2} = 28 \) quadratic monomials in total, but the chiral ring relation (2.33) removes 9 of them, so we are left with 19 terms.

For the toric diagram of \( Q^{1,1,1} \) shown in Figure 2(b), let \( \tilde{\phi}_{a,b} \) denote the GLSM variable assigned to the vertex at \( (0, a, b) \) and similarly use \( \tilde{\phi}_{a,b} \) for the vertex at \( (1, a, b) \). There are 8 elementary gauge invariant monomials,
\begin{eqnarray}
s_1 = \phi_{01} \tilde{\phi}_{00} \tilde{\phi}_{10}, & t_1 = \phi_{00} \tilde{\phi}_{00} \tilde{\phi}_{11} , & u_1 = \phi_{10} \tilde{\phi}_{10} \tilde{\phi}_{1,-1} , & w_1 = \phi_{00} \phi_{10} \phi_{01}, \\
s_2 = \phi_{00} \tilde{\phi}_{10} \tilde{\phi}_{1,-1} , & t_2 = \phi_{10} \tilde{\phi}_{01} \tilde{\phi}_{10} , & u_2 = \phi_{00} \phi_{01} \phi_{00}, & w_2 = \tilde{\phi}_{00} \tilde{\phi}_{10} \tilde{\phi}_{1,-1},
\end{eqnarray}
subject to 9 constraints,
\begin{eqnarray}
w_1 w_2 = s_1 s_2 = t_1 t_2 = u_1 u_2, \\
w_1 s_1 = t_2 u_2, & w_1 t_1 = u_2 s_2, & w_1 u_1 = s_2 t_2, \\
w_2 s_2 = t_1 u_1, & w_2 t_2 = u_1 s_1, & w_2 u_2 = s_1 t_1.
\end{eqnarray}

The Hilbert series correctly captures the structure of the chiral ring. Setting the fugacities to 1 for simplicity, we observe that
\begin{equation}
H_{Q^{1,1,1}} = \sum_{n=1}^{\infty} n^3 t^{n-1} = 1 + 8t + 27t^2 + \cdots.
\end{equation}

The 8 terms at \( \mathcal{O}(t) \) represent the elementary gauge invariant variables \( (s_{1,2}, t_{1,2}, u_{1,2}, w_{1,2}) \). At \( \mathcal{O}(t^2) \), there are \( \frac{8 \cdot 9}{2} = 36 \) quadratic monomials in total, but the chiral ring relation (2.36) removes 9 of them, so we are left with 27 terms.

Again, the algebraic descriptions of the two geometries (2.32) and (2.35) are related by a complex deformation in a way similar to (2.31).
3 Gauge theory

3.1 M2-brane CFT with flavors

To construct the gauge theories dual to the toric geometry we described in the previous section, we will follow the recent work [22] (see also [21]) which proposed a systematic method to read off the $\mathcal{N} = 2$ superconformal field theories on M2-branes probing a large class of toric Calabi-Yau fourfolds CY$_4$. A given toric CY$_4$ can be described as a CY$_3$ fibration over a real line $\mathbb{R} = \{ \sigma \}$ with the RR two-form field strength $F_{RR}$ turned on, where CY$_3$ can be obtained from a certain Kähler quotient of the CY$_4$, i.e., $\text{CY}_3 = \text{CY}_4//U(1)_M$. The Kähler moduli of CY$_3$ changes linearly as $\sigma$ varies. In terms of the 3d toric diagram for CY$_4$ with a suitable choice of $SL(3,\mathbb{Z})$ basis, such a Kähler quotient can be understood as a vertical projection down to the 2d toric diagram for CY$_3$ on the $z = 0$ plane. Performing the Kaluza-Klein (KK) reduction along the circle $U(1)_M$, the M2-branes at the tip of the CY$_4$ singularity can be reduced to D2-branes probing the CY$_3$ with the two-form flux turned on. One can construct a low-energy quiver gauge theory living on the D2-branes using standard methods such as the brane-tiling model [4, 5]. One key feature in the type IIA background is that the two-form flux $F_{RR}$ can induce the Chern-Simons coupling to the low-energy quiver gauge theory.

We will consider in this paper some examples where the M-theory circle become degenerate in the KK reduction. In particular, we are interested in the case where $U(1)_M$ action has fixed loci of codimension two which are non-compact. In terms of the 3d toric diagram, the degeneration happens when two adjacent external toric vertices are projected down to the same external point in the 2d toric diagram. In the type IIA background, it leads to D6-branes wrapping a toric divisor, a non-compact four-cycle, corresponding to the external point in the 2d toric diagram. Let $X_\alpha$ denote a bi-fundamental matter field in the quiver gauge theory associated to the toric divisor. Then, one can argue that D6-branes introduce to the quiver gauge theory $n_I$ pairs of flavors, coupled to $X_I$ via the superpotential

$$W_{\text{flavor}} = \sum_{a=1}^{n_\alpha} p^a_\alpha X_\alpha q^a_\alpha.$$  \hspace{1cm} (3.1)

We refer to the above process as flavoring the gauge theory.

It is possible to reverse the flavoring process and construct the CY$_4$ background in M-theory from a flavored gauge theory by analyzing the vacuum moduli space. It often turns out that (diagonal) monopole operators $T^{(n)}$ play a key role in identifying the quantum moduli space of the quiver gauge theory. The monopole operator $T^{(n)}$ carries the same flux $n$ for all diagonal $U(1)$ gauge groups in the quiver together with electric charges $(n k_{I1}, \ldots, n k_{I_G})$ where $k_I$ denotes the Chern-Simons level for each $U(1)$ gauge group.

As shown in [22], for the flavored quiver gauge theories, the monopole operators becomes charged under both gauged and global $U(1)$ symmetry groups via quantum correc-
tions
\[ \delta Q[T^{(n)}] = -\frac{|n|}{2} \sum_f Q_f, \quad (3.2) \]
where \( Q_f \) denote the \( U(1) \) charge for matter fermions. Based on (3.1) and (3.2), one can show that the monopole operator \( T^{(n)} \) carries flavor \( U(1) \) charges
\[ Q[T^{(n)}] = |n| \sum_\alpha n_\alpha Q[X_\alpha], \quad (3.3) \]
a \( U(1)_R \) charge
\[ R[T^{(n)}] = |n| \sum_\alpha n_\alpha R[X_\alpha], \quad (3.4) \]
and gauge charges \( g_a \) \((a = 1, 2, \ldots, G)\)
\[ g_a[T^{(n)}] = n k_a + |n| \sum_\alpha n_\alpha g_a[X_\alpha]. \quad (3.5) \]
Those quantum-mechanically generate charges strongly implies that the following holomorphic quantum relation, known as ‘quantum \( F \)-relation’, should hold
\[ T^{(n)} T^{(-n)} = \left( \prod_\alpha X_\alpha^{h_\alpha} \right)^{|n|}. \quad (3.6) \]
As a consequence, the simplest monopole operators \( T \equiv T^{(1)} \) and \( \tilde{T} \equiv T^{(-1)} \) parametrizes one extra dimension beyond those for CY\(_3\), needed for the CY\(_4\) background in M-theory.

3.2 Klebanov-Witten flow

The UV theory of the original KW flow has the superpotential,
\[ \mathcal{W}_{\text{UV}} = \tr \left[ \Phi(A_1 B_2 - A_2 B_1) - \tilde{\Phi}(B_2 A_1 - B_1 A_2) \right]. \quad (3.7) \]
The flow is triggered by the addition of the relevant operator,
\[ \mathcal{W}_{\text{def}} = \frac{m}{2} \tr(\Phi^2 - \Phi^2), \quad (3.8) \]
which is the field theoretic counterpart of the complex deformation of geometry discussed in section 2.3. Integrating out \( \Phi, \tilde{\Phi} \), we obtain the superpotential of the IR theory,
\[ \mathcal{W}_{\text{IR}} = \frac{1}{m} \tr(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1). \quad (3.9) \]
The key idea in the construction of the gauge theories in this paper is that, for both the UV theory and the IR theory of the KW flow, we can apply dimensional reduction to three dimensions and addition of fundamental chiral multiplets according to the rules of
`flavoring’ explained earlier. Our main claim is that there exists an RG flow between the resulting UV and IR theories in three dimensions.

For the flow from $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}$ to $D_3$, we put ‘flavors’ on $A_1$ and $B_1$ by adding the superpotential term, $W_{\text{flavor}} = pA_1q + \tilde{p}B_1\tilde{q}$, as depicted in Figure 3. Then, we begin with the UV theory with

$$W_{\text{UV}} = \text{tr} \left[ \Phi(A_1B_2 - A_2B_1) - \tilde{\Phi}(B_2A_1 - B_1A_2) \right] + pA_1q + \tilde{p}B_1\tilde{q}, \quad (3.10)$$

and trigger the KW flow to end up with the IR theory with

$$W_{\text{IR}} = \frac{1}{m} \text{tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1) + pA_1q + \tilde{p}B_1\tilde{q}, \quad (3.11)$$

Both for the UV and IR theories, the quantum F-term relation $T\tilde{T} = A_1B_1$ plays a crucial role.

Similarly, for the flow from $dP_3 \times \mathbb{C}$ to $Q^{1,1,1}$, we put ‘flavors’ on $A_1$ and $A_2$ by adding the term, $W_{\text{flavor}} = p_1A_1q_1 + p_2A_2q_2$ as depicted in Figure 4. The resulting UV theory has the superpotential,

$$W_{\text{UV}} = \text{tr} \left[ \Phi(A_1B_2 - A_2B_1) - \tilde{\Phi}(B_2A_1 - B_1A_2) \right] + p_1A_1q_1 + p_2A_2q_2, \quad (3.12)$$

and the IR theory has

$$W_{\text{IR}} = \frac{1}{m} \text{tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1) + p_1A_1q_1 + p_2A_2q_2. \quad (3.13)$$
The quantum F-term relation reads $T\tilde{T} = A_1 A_2$.

### 3.3 Chiral ring and R-charge

It is straightforward to show that the chiral ring spectrum of the four theories above matches those from the corresponding geometries.

$$\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}$$

The gauge invariant generators of the chiral ring are

$$z_1 = T, \quad z_2 = \tilde{T}, \quad z_3 = A_2 B_2, \quad w = A_1 B_2 = A_2 B_1, \quad v = \Phi = \tilde{\Phi}. \quad (3.14)$$

The quantum F-term relation $T\tilde{T} = A_1 A_2$ implies that $z_1 z_2 z_3 = w^2$, which captures the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold geometry. The remaining variable $v$ parametrizes the $\mathbb{C}$ factor.

It is instructive to compare the chiral ring between the gauge theory and the geometry. Recall from section 2.3 the GLSM description of the chiral ring,

$$z_1 = \tilde{\phi}_0 \phi_{01} \phi_{01}, \quad z_2 = \tilde{\phi}_2 \phi_{01} \phi_{11}, \quad z_3 = \phi^2_2 \tilde{\phi}_{10} \phi_{11}, \quad w = \phi_0 \phi_{10} \phi_{02} \phi_{01} \phi_{02} \phi_{11}. \quad (3.15)$$

Ignoring the trivial $\mathbb{C}$ factor and comparing (3.14) and (3.15), we find the correspondence between the matter fields in gauge theory and the GLSM variables,

$$A_1 = \phi_{00} \phi_{01} \phi_{02}, \quad A_2 = \phi_{20}, \quad B_1 = \phi_{00} \phi_{01} \phi_{02} \phi_{10} \phi_{11}, \quad B_2 = \phi_{10} \phi_{11} \phi_{20}. \quad (3.16)$$

Using the $R$-charge of GLSM variables computed earlier and the correspondence (3.16), we find the $R$-charge of the matter fields,

$$R \quad \begin{array}{cccccc}
\Phi & \tilde{\Phi} & A_1 & A_2 & B_1 & B_2 \\
1/2 & 1/2 & 1 & 1/2 & 1 & 1/2 \\
\end{array} \quad p, q, \tilde{p}, \tilde{q} \quad 1/2. \quad (3.17)$$

We will show in section 5 that the quantum computation of the three-sphere partition function reproduces exactly the same $R$-charge spectrum.

### $D_3$

The gauge invariant generators of the chiral ring are

$$z_1 = T, \quad z_2 = \tilde{T}, \quad z_3 = A_2 B_2, \quad w = A_1 B_2, \quad \tilde{w} = A_2 B_1. \quad (3.18)$$

They satisfy the F-term relation $z_1 z_2 z_3 = w \tilde{w}$, which describes the $D_3$ geometry. Recall the GLSM description of the chiral ring,

$$z_1 = \phi_0 \tilde{\phi}_{00}, \quad z_2 = \phi_{10} \tilde{\phi}_{01}, \quad z_3 = \phi_{10} \tilde{\phi}_{10}, \quad w = \phi_0 \phi_{10} \phi_{01}, \quad \tilde{w} = \phi_{00} \tilde{\phi}_{10} \tilde{\phi}_{01}. \quad (3.19)$$

Comparing (3.18) and (3.19), we find

$$A_1 = \phi_{00} \phi_{01}, \quad A_2 = \tilde{\phi}_{10}, \quad B_1 = \tilde{\phi}_{00} \tilde{\phi}_{01}, \quad B_2 = \phi_{10}. \quad (3.20)$$

which implies the following $R$-charge assignments for the matter fields,

$$R \quad \begin{array}{cccccc}
\Phi & \tilde{\Phi} & A_1 & A_2 & B_1 & B_2 \\
1 & 1 & 2/3 & 1/3 & 2/3 & 1/3 \\
p, q, \tilde{p}, \tilde{q} & 2/3 & 2/3 \end{array}.$$

(3.21)
The gauge invariant generators of the chiral ring are

\[ s_1 = TB_1, \quad s_2 = \tilde{T} B_2, \quad t_1 = \tilde{T} B_1, \quad t_2 = TB_2, \]
\[ u_1 = A_2 B_2, \quad u_2 = A_1 B_1, \quad w = A_1 B_2 = A_2 B_1, \quad v = \Phi = \tilde{\Phi}. \quad (3.22) \]

The quantum F-term relation, \( T \tilde{T} = A_1 A_2 \), implies the following chiral ring relations,

\[ w_2 = s_1 s_2 = t_1 t_2 = u_1 u_2, \]
\[ w s_1 = t_2 u_2, \quad w t_1 = u_2 s_2, \quad w u_1 = s_2 t_2, \]
\[ w s_2 = t_1 u_1, \quad w t_2 = u_1 s_1, \quad w u_2 = s_1 t_1. \quad (3.23) \]

Recall the GLSM description of the chiral ring,

\[ s_1 = \phi_{00} \phi_{10} \phi_{20} (\phi_{01} \phi_{11})^2, \quad s_2 = \phi_{00} \phi_{10} \phi_{20} (\phi_{1-1} \phi_{2,-1})^2, \]
\[ t_1 = \phi_{00} \phi_{10} \phi_{20} (\phi_{01} \phi_{11})^2, \quad t_2 = \phi_{00} \phi_{10} \phi_{20} (\phi_{1-1} \phi_{2,-1})^2, \]
\[ u_1 = \phi_{10} \phi_{11} \phi_{1,-1} (\phi_{20} \phi_{2,-1})^2, \quad u_2 = \phi_{10} \phi_{11} \phi_{1,-1} (\phi_{00} \phi_{01})^2, \]
\[ w = \phi_{00} \phi_{10} \phi_{20} \phi_{01} \phi_{11} \phi_{1,-1} \phi_{2,-1}. \quad (3.24) \]

Comparing (3.22) and (3.24), we find

\[ A_1 = \phi_{00} \phi_{01}, \quad A_2 = \phi_{20} \phi_{2,-1}, \quad B_1 = \phi_{00} \phi_{01} \phi_{1,1} \phi_{10} \phi_{11}, \quad B_2 = \phi_{1,1} \phi_{10} \phi_{11} \phi_{20} \phi_{2,-1}, \quad (3.25) \]

which implies the following \( R \)-charge assignments of the matter fields,

\[ \begin{array}{c|cccccccc}
\Phi & \tilde{\Phi} & A_1 & A_2 & B_1 & B_2 & p & q & \tilde{p} & \tilde{q} \\
R & 1/2 & 1/2 & 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 \\
\end{array} \quad (3.26) \]

\( Q^{1,1,1} \)

The gauge invariant generators of the chiral ring are

\[ s_1 = TB_1, \quad s_2 = \tilde{T} B_2, \quad t_1 = \tilde{T} B_1, \quad t_2 = TB_2, \]
\[ u_1 = A_2 B_2, \quad u_2 = A_1 B_1, \quad w_1 = A_1 B_2, \quad w_2 = A_2 B_1. \quad (3.27) \]

The chiral ring relations read

\[ w_1 w_2 = s_1 s_2 = t_1 t_2 = u_1 u_2, \]
\[ w_1 s_1 = t_2 u_2, \quad w_1 t_1 = u_2 s_2, \quad w_1 u_1 = s_2 t_2, \]
\[ w_2 s_2 = t_1 u_1, \quad w_2 t_2 = u_1 s_1, \quad w_2 u_2 = s_1 t_1. \quad (3.28) \]

Recall the GLSM description of the chiral ring,

\[ s_1 = \phi_{01} \tilde{\phi}_{00} \tilde{\phi}_{10}, \quad t_1 = \phi_{00} \tilde{\phi}_{00} \tilde{\phi}_{1,-1}, \quad u_1 = \phi_{10} \tilde{\phi}_{10} \tilde{\phi}_{1,-1}, \quad w_1 = \phi_{00} \tilde{\phi}_{10} \tilde{\phi}_{01}, \]
\[ s_2 = \phi_{00} \tilde{\phi}_{10} \tilde{\phi}_{1,-1}, \quad t_2 = \phi_{10} \tilde{\phi}_{01} \tilde{\phi}_{10}, \quad u_2 = \phi_{00} \tilde{\phi}_{01} \tilde{\phi}_{00}, \quad w_2 = \tilde{\phi}_{00} \tilde{\phi}_{10} \tilde{\phi}_{1,-1}. \quad (3.29) \]
Comparing (3.27) and (3.29), we find
\[ A_1 = \phi_{00}\phi_{01}, \quad A_2 = \tilde{\phi}_{10}\tilde{\phi}_{1,1-1}, \quad B_1 = \phi_{00}, \quad B_2 = \phi_{10}, \] (3.30)
which implies the following \( R \)-charge assignments of the matter fields,
\[
\begin{array}{cccccc}
\Phi & \tilde{\Phi} & A_1 & A_2 & B_1 & B_2 \\
R & 1 & 1 & 2/3 & 2/3 & 1/3 & 2/3 & p, q, \tilde{p}, \tilde{q}
\end{array}
\] (3.31)

4 Generalization to Orbifolds

4.1 Orbifolding the 4d flows

\( \mathbb{C}^2/\mathbb{Z}_{2n} \times \mathbb{C} \rightarrow T^{1,1}/\mathbb{Z}_n \)

The original KW flow admits a simple generalization to an infinite family of orbifolds. On the geometry side, the orbifolded flows relate \( \mathbb{C}^2/\mathbb{Z}_{2n} \times \mathbb{C} \) to \( T^{1,1}/\mathbb{Z}_n \). The toric diagrams for the two orbifolds are depicted in Figure 5.

On the gauge theory side, the orbifold action results in ‘necklace’ quiver theories whose quiver diagrams are shown in Figure 6.

The UV theory of the orbifolded KW flow has the superpotential,
\[
W_{\text{UV}} = \sum_{I=1}^{2n} \text{tr} \left[ \Phi_I (C_I \bar{C}_I - \bar{C}_{I-1} C_{I-1}) \right].
\] (4.1)
The flow is triggered by the addition of the relevant operator,

$$\mathcal{W}_{\text{def}} = \frac{m}{2} \sum_{I=1}^{2n} (-1)^I \tr(\Phi^2_I).$$

Integrating out the adjoint fields, we obtain the superpotential of the IR theory,

$$\mathcal{W}_{\text{IR}} = \frac{1}{m} \sum_{I=1}^{2n} (-1)^I \tr \left( C_I \tilde{C}_I C_{I-1} \right).$$

To compare with the notations of the original \((n=1)\) KW flow discussed in section 3.2, we identify the fields as follows

$$\Phi, \tilde{\Phi}, A_1, A_2, B_1, B_2 \rightarrow (\Phi_1, \Phi_2, C_1, \tilde{C}_2, C_2, \tilde{C}_1).$$

To see the matching between the quiver theory and the orbifold geometry, we examine the chiral ring. In the UV theory, enumerating all elementary gauge invariant operators from the (abelian) field theory and modding out by the F-term conditions, we find the generators of the chiral ring,

$$x = \prod_{I=1}^{2n} C_I, \quad y = \prod_{I=1}^{2n} \tilde{C}_I, \quad z = C_1 \tilde{C}_1 = \ldots = C_{2n} \tilde{C}_{2n}, \quad v = \Phi_1 = \ldots = \Phi_{2n}. \quad (4.5)$$

They satisfy one constraint equation,

$$xy = z^{2n}, \quad v \text{ free}, \quad (4.6)$$

which describes \(\mathbb{C}^2/\mathbb{Z}_{2n} \times \mathbb{C}\) algebraically. We can see the same chiral ring structure from the toric diagram. We ignore the trivial \(\mathbb{C}\) factor and let \(\phi_k\) denote the GLSM fields associated to vertices on the ‘x-axis’ in the diagram on the left in Figure 5. Then, the gauge invariant monomials,

$$x = \prod_{k=0}^{2n} (\phi_k)^{2n-k}, \quad y = \prod_{k=0}^{2n} (\phi_k)^k, \quad z = \prod_{k=0}^{2n} \phi_k, \quad (4.7)$$

satisfy the same algebraic relation (4.6). Comparing (4.5) and (4.7), we find the correspondence between gauge theory variables and GLSM variables,

$$C_j = \prod_{k=0}^{j-1} \phi_k, \quad \tilde{C}_j = \prod_{k=j}^{2n} \phi_k. \quad (4.8)$$

We can apply the same methods to the IR theory. In the gauge theory, we have

$$x = \prod_{I=1}^{2n} C_I, \quad z = C_1 \tilde{C}_1 = C_3 \tilde{C}_3 = \ldots = C_{2n-1} \tilde{C}_{2n-1},$$

$$y = \prod_{I=1}^{2n} \tilde{C}_I, \quad \tilde{z} = C_2 \tilde{C}_2 = C_4 \tilde{C}_4 = \ldots = C_{2n} \tilde{C}_{2n}. \quad (4.9)$$
which satisfy the algebraic equation for $T^{1,1}/\mathbb{Z}_n$, 

$$xy = (z\bar{z})^n.$$  

(4.10)

In terms of the GLSM fields in the toric diagram, the generators of the chiral ring are

$$x = \prod_{k=0}^{n} (\phi_k \phi_k)^{n-k}, \quad y = \prod_{k=0}^{n} (\phi_k \bar{\phi}_k)^k, \quad z = \prod_{k=0}^{n} \phi_k, \quad \bar{z} = \prod_{k=0}^{n} \bar{\phi}_k.$$  

(4.11)

Comparing (4.9) and (4.11), we find the correspondence,

$$C_{2j-1} = \prod_{k=0}^{j-1} \phi_k, \quad C_{2j} = \prod_{k=0}^{j-1} \bar{\phi}_k, \quad \bar{C}_{2j-1} = \prod_{k=j}^{n} \phi_k, \quad \bar{C}_{2j} = \prod_{k=j}^{n} \bar{\phi}_k.$$  

(4.12)

### 4.2 Orbifolding the 3d flows

Having understood how the orbifolding works in four dimensions, it is straightforward to carry it over to the two flows in three dimensions via flavoring.

Figure 7. Toric diagrams for the flow from $\mathbb{C}^3/(\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}) \times \mathbb{C}$ to $D_3/(\mathbb{Z}_n \times \mathbb{Z}_n)$

$\mathbb{C}^3/(\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}) \times \mathbb{C} \rightarrow D_3/(\mathbb{Z}_n \times \mathbb{Z}_n)$

This flow is obtained by flavoring on all $C_i$ in the quiver diagram in Figure 6. In the UV theory, listing all elementary gauge invariant operators from the field theory and modding out by the F-term conditions, we find the generators of the chiral ring,

$$z_1 = T, \quad z_2 = \bar{T}, \quad z_3 = \prod_{I=1}^{2n} \bar{C}_I, \quad w = C_1 \bar{C}_1 = \ldots = C_{2n} \bar{C}_{2n}.$$  

(4.13)

With the help of the quantum F-term relation,

$$T\bar{T} = \prod_{I=1}^{2n} C_I.$$  

(4.14)
we obtain the algebraic description of $\mathbb{C}^3/(\mathbb{Z}_{2n} \times \mathbb{Z}_{2n})$,

\[ z_1 z_2 z_3 = w^{2n}, \quad v \text{ free}. \quad (4.15) \]

We can also see the chiral ring in the toric diagram. Again ignoring the trivial $\mathbb{C}$ factor and labeling the vertices at $(0, a, b)$ by $\phi_{a,b}$, we find the basic gauge invariant monomials,

\[ z_1 = \prod_{a,b} \phi_{a,b}^{2n-a-b}, \quad z_2 = \prod_{a,b} \phi_{a,b}^b, \quad z_3 = \prod_{a,b} \phi_{a,b}^a, \quad w = \prod_{a,b} \phi_{a,b}. \quad (4.16) \]

They clearly satisfy the algebraic relation (4.15). Comparing (4.13) and (4.16) and we find the correspondence between the gauge theory variables and the GLSM variables,

\[ C_j = \prod_{k=0}^{j-1} \hat{\phi}_k, \quad \tilde{C}_j = \prod_{k=j}^{2n} \phi_k \quad \left( \hat{\phi}_j \equiv \prod_{b} \phi_{j,b} \right). \quad (4.17) \]

Note the similarity between (4.8) and (4.17), which reflects the fact that the 3d theory has been obtained by the flavoring method.

In the IR theory, the generators of the chiral ring are

\[ z_1 = T, \quad z_2 = \tilde{T}, \quad z_3 = \prod_{I=1}^{2n} \tilde{C}_I, \]

\[ w = C_1 \tilde{C}_1 = C_3 \tilde{C}_3 = \cdots = C_{2n-1} \tilde{C}_{2n-1}, \]

\[ \tilde{w} = C_2 \tilde{C}_2 = C_4 \tilde{C}_4 = \cdots = C_{2n} \tilde{C}_{2n}. \quad (4.18) \]

With the help of the quantum F-term relation (4.14), we obtain the algebraic description of $D_3/(\mathbb{Z}_n \times \mathbb{Z}_n)$,

\[ z_1 z_2 z_3 = (w\tilde{w})^n \quad (4.19) \]

In the toric diagram, we label the GLSM variables at vertices $(0, a, b)$ by $\phi_{a,b}$ and those at $(1, a, b)$ by $\tilde{\phi}_{a,b}$. The gauge invariant monomials are

\[ z_1 = \prod_{a,b} (\phi_{a,b} \tilde{\phi}_{a,b})^{n-a-b}, \quad z_2 = \prod_{a,b} (\phi_{a,b} \tilde{\phi}_{a,b})^b, \quad z_3 = \prod_{a,b} (\phi_{a,b} \tilde{\phi}_{a,b})^a, \]

\[ w = \prod_{a,b} \phi_{a,b}, \quad \tilde{w} = \prod_{a,b} \tilde{\phi}_{a,b}. \quad (4.20) \]

They clearly satisfy the algebraic relation (4.19). Comparing (4.18) and (4.20) and we find the correspondence between gauge theory variables and GLSM variables,

\[ C_{2j-1} = \prod_{k=0}^{j-1} \hat{\phi}_k, \quad C_{2j} = \prod_{k=0}^{j-1} \hat{\phi}_k, \quad \tilde{C}_{2j-1} = \prod_{k=j}^{n} \hat{\phi}_k, \quad \tilde{C}_{2j} = \prod_{k=j}^{n} \hat{\phi}_k. \quad (4.21) \]

where $\hat{\phi}_k$ is defined as in (4.16) but with a different range of the $b$ index and similarly for $\tilde{\phi}_k$. The similarity between (4.12) and (4.21) is again a sign of the flavoring method.
This flow is obtained by flavoring on all $C_{2j-1}$ and $\tilde{C}_{2j}$ in the quiver diagram in Figure 6. To discuss the chiral ring structure, it is convenient to introduce the (gauge-non-invariant) composite fields,

$$
A_1 = \prod_{j=1}^{n} C_{2j-1}, \quad A_2 = \prod_{j=1}^{n} \tilde{C}_{2j}, \quad B_1 = \prod_{j=1}^{n} C_{2j}, \quad B_2 = \prod_{j=1}^{n} \tilde{C}_{2j-1}.
$$

(4.22)

In terms of these fields, the chiral ring generators are given by

$$
s_1 = TB_1, \quad s_2 = \tilde{T}B_2, \quad t_1 = \tilde{T}B_1, \quad t_2 = TB_2, \quad u_1 = A_2B_2, \quad u_2 = A_1B_1,
$$

$$
w = C_1\tilde{C}_1 = \cdots = C_{2n}\tilde{C}_{2n}, \quad v = \Phi_1 = \cdots = \Phi_{2n}.
$$

(4.23)

Note that, for $s_1, s_2, t_1, t_2$, the gauge non-invariance of the composite fields cancels against that of monopole operators $T, \tilde{T}$. Using the quantum F-term relation,

$$
TT = A_1A_2,
$$

(4.24)

we obtain the algebraic description of $dP_3/(\mathbb{Z}_n \times \mathbb{Z}_n) \times \mathbb{C}$,

$$
w^n s_1 = t_1 t_2 = u_1 u_2, \quad w^n s_2 = t_2 u_2, \quad w^n t_1 = u_2 s_2, \quad w^n u_1 = s_2 t_2, \quad w^n s_2 = t_1 u_1, \quad w^n t_2 = u_1 s_1, \quad w^n u_2 = s_1 t_1.
$$

(4.25)

In the toric diagram, we again ignore the trivial $\mathbb{C}$ factor and label the vertices at $(0, a, b)$ by $\phi_{a,b}$. The gauge invariant monomials are

$$
s_1 = \prod_{a,b} \phi_{a,b}^{n+b}, \quad t_1 = \prod_{a,b} \phi_{a,b}^{2n-a-b}, \quad u_1 = \prod_{a,b} \phi_{a,b}^a,
$$

$$
s_2 = \prod_{a,b} \phi_{a,b}^{-b}, \quad t_2 = \prod_{a,b} \phi_{a,b}^{a+b}, \quad u_2 = \prod_{a,b} \phi_{a,b}^{2n-a}, \quad w = \prod_{a,b} \phi_{a,b}.
$$

(4.26)
Comparing (4.23) and (4.26), we find the correspondence between the gauge theory variables and the GLSM variables,

\[
C_j = \prod_{k=0}^{j-1} \hat{\phi}_k, \quad \tilde{C}_j = \prod_{k=j}^{2n} \hat{\phi}_k \left( \hat{\phi}_j = \prod_b \hat{\phi}_{j,b} \right).
\] (4.27)

in agreement with (4.8).

In the IR theory, the generators of the chiral ring are

\[
s_1 = TB_1, \quad s_2 = \tilde{T}B_2, \quad t_1 = \tilde{T}B_1, \quad t_2 = TB_2, \quad u_1 = A_2B_2, \quad u_2 = A_1B_1,
\]
\[
w_1 = C_1\tilde{C}_1 = C_3\tilde{C}_3 = \cdots = C_{2n-1}\tilde{C}_{2n-1}, \quad w_2 = C_2\tilde{C}_2 = C_4\tilde{C}_4 = \cdots C_{2n}\tilde{C}_{2n}.
\] (4.28)

The quantum F-term relation (4.24) leads to the algebraic description of \(Q^{1,1,1}/(\mathbb{Z}_n \times \mathbb{Z}_n)\),

\[
(w_1w_2)^n = s_1s_2 = t_1t_2 = u_1u_2,
\]
\[
w_1^n s_1 = t_2u_2, \quad w_1^n t_1 = u_2s_2, \quad w_1^n u_1 = s_2t_2,
\]
\[
w_2^n s_2 = t_1u_1, \quad w_2^n t_2 = u_1s_1, \quad w_2^n u_2 = s_1t_1.
\] (4.29)

In the toric diagram, we again label the GLSM variables at vertices (0, a, b) by \(\phi_{a,b}\) and those at (1, a, b) by \(\hat{\phi}_{a,b}\). The gauge invariant monomials are

\[
s_1 = \prod_{a,b} \phi_{a,b}^{n+b}, \quad t_1 = \prod_{a,b} (\phi_{a,b}\tilde{\phi}_{a,b})^{n-a-b}, \quad u_1 = \prod_{a,b} (\phi_{a,b}\tilde{\phi}_{a,b})^a, \quad w_1 = \prod_{a,b} \phi_{a,b},
\]
\[
s_2 = \prod_{a,b} \phi_{a,b}^{-b}, \quad t_2 = \prod_{a,b} (\phi_{a,b}\tilde{\phi}_{a,b})^{a+b}, \quad u_2 = \prod_{a,b} (\phi_{a,b}\tilde{\phi}_{a,b})^{-a}, \quad w_2 = \prod_{a,b} \tilde{\phi}_{a,b}.
\] (4.30)

Comparing (4.28) and (4.30) and we find the correspondence between the gauge theory variables and the GLSM variables,

\[
C_{2j-1} = \prod_{k=0}^{j-1} \hat{\phi}_k, \quad C_{2j} = \prod_{k=0}^{j-1} \hat{\tilde{\phi}}_k, \quad \tilde{C}_{2j-1} = \prod_{k=0}^{n} \hat{\phi}_k, \quad \tilde{C}_{2j} = \prod_{k=0}^{n} \hat{\tilde{\phi}}_k.
\] (4.31)

with \(\hat{\phi}_k\) and \(\hat{\tilde{\phi}}_k\) defined in the same way as before.

5 Partition function and universal ratio \(16/27\)

Recent works have shown that the three-sphere partition function \(Z\) of \(N = 2\) supersymmetric theories can be exactly computed via localization techniques. The exact partition function on \(S^3\) provides a systematic and quantitative way to study the infrared dynamics of three-dimensional theories. In particular, it has been proposed in [34] that the free energy \(F = -\log|Z|\) on \(S^3\) can define a measure of the number of degrees of freedom that decreases monotonically along RG-flows. Ref. [36] recently proposed a possible proof of this conjectured ‘F-theorem’ using a relation between the free energy \(F\) on \(S^3\) and a certain entanglement entropy.
In this section, we provide a more refined test of the RG flows proposed in the previous sections by computing the exact three-sphere partition function. We will show that the partition functions of the UV CFT's at the UV with a certain relevant deformation exactly match those of the CFT's at the IR, which strongly supports our proposal.

We will also observe that

$$\frac{F_{\text{IR}}}{F_{\text{UV}}} = \sqrt{\frac{16}{27}},$$

for all RG flows considered in this paper, a result consistent with the F-theorem. We will present in section 5.3 a general argument to explain the universal ratio 16/27.

5.1 Review of three-sphere partition function and large N limit

We begin by a brief review on the three-sphere partition function of 3d $\mathcal{N} = 2$ supersymmetric theories and its large $N$ limit. More details can be found in [30, 34, 35].

**Partition function** It has been shown in [34, 35] that the path integral for the partition function of an $\mathcal{N} = 2$ gauge theory coupled to charged matters on $S^3$ localizes to a matrix integral over the constant vev of the scalar fields in vector multiplets taking their values in the Cartan subalgebra $t$ of the given gauge group $G$. Explicitly, the partition function takes the following form,

$$Z = \frac{1}{|W(G)|} \int d\lambda 2\pi \exp \left( \frac{i}{4\pi} \text{tr}_k \lambda^2 - \text{tr}_m \lambda \right) \cdot \prod_{\alpha \in \Delta} \left( 2\sinh \frac{\alpha(\lambda)}{2} \right) \cdot Z_{\text{1-loop}}^{\text{matter}}(\Delta),$$

with

$$Z_{\text{1-loop}}^{\text{matter}}(\Delta) = \prod_a \prod_{\rho \in R_a} \exp \left[ l(1 - \Delta_a + i\rho(\lambda)) \right],$$

where $\Delta_a$ denote trial $R$-charges of matter chiral multiplets in representation $R_a$ of the gauge group $G$. Here $\alpha$ and $\rho$ denote roots of the gauge group $G$ and weights of the representation $R_a$, respectively. The trace $\text{tr}_k$ is normalized such that, if the gauge group $G$ is given by $\prod_I G_I$ with Chern-Simons level $k_I$, it equals to $\sum_I k_I \text{tr}_I$. The trace $\text{tr}_m$ is normalized with the bare monopole charge $\sum_I \Delta_{\text{mon}}^I \text{tr}_I$. $|W(G)|$ denotes the order of the Weyl group of $G$. The function $l(z)$ is defined by

$$l(z) = -z \log(1 - e^{2\pi i z}) + \frac{i}{2} \left[ \pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi i z}) \right] - \frac{i\pi}{12},$$

which satisfies $\frac{dl}{dz}(z) = -\pi z \cot(\pi z)$ and $l(0) = 0$. Note that, since $l(+ix) + l(-ix) = 0$ for any $x \in \mathbb{R}$, the contribution from an adjoint chiral multiplet of $R$-charge $\Delta_{\text{ad}} = 1$ becomes trivial.
**Large \( N \) limit**  Let us now discuss the general rules for evaluating the matrix integral in the large \( N \) limit [38–40]. Assuming that the eigenvalues \( \lambda_i \) scale as

\[
\lambda_i = N^{1/2} x_i + iy_i + o(N^0) ,
\]

in the large \( N \) limit and replacing \( x_i \) and \( y_i \) by continuous functions \( x(s) \) and \( y_I(s) \) with \( s \in [0,1] \), it turns out that the free energy functional becomes local in \( x(s) \) and \( y_I(s) \) for a large class of quiver gauge theories, including all examples in the present work. It leads to a significant simplification of the large \( N \) expression of the free energy functional. The general rules to construct the free energy functionals in the large \( N \) limit are as follows; from now on, we restrict our attention to unitary gauge groups only.

- **The classical part of the action can be expressed as**

  \[
  F_{cl} = \frac{k I}{2\pi} N^{3/2} \int dx \rho(x) x y_I(x) + \Delta_{\text{mon}} N^{3/2} \int dx \rho(x) x ,
  \]

  where the density function \( \rho(x) \) is defined as follows

  \[
  ds = \rho(x(s)) dx(s) , \quad \int dx \rho(x) = 1 .
  \]

- **Consider a pair of bifundamental matter fields, one of which in the \((\mathbb{N},\bar{\mathbb{N}})\) representation of \( U(N) \times U(N) \) with \( R \)-charge \( \Delta_{(I,J)} \) and another in \((\bar{\mathbb{N}},\mathbb{N})\) with \( R \)-charge \( \Delta_{(J,I)} \). The contribution to the free energy functional from such a pair of bifundamental matter fields is given by**

  \[
  F_{bf} = \frac{N^{3/2}}{2} \int dx \left[ \frac{\pi^2}{3} \left( \Delta^+_{(I,J)}(4 - \Delta^+_{(I,J)}) - \left( y_I - y_J + \pi \Delta^-_{(I,J)} \right) \right)^2 \right] ,
  \]

  where \( \Delta^\pm_{(I,J)} = \Delta_{(I,J)} \pm \Delta_{(J,I)} \). The above expression is valid only in the range,

  \[
  \left| y_I - y_J + \pi \Delta^-_{(I,J)} \right| \leq \pi \Delta^+_{(I,J)} .
  \]

- **If the gauge theory of our interest is coupled to an adjoint chiral multiplet of \( R \)-charge \( \Delta_{\text{ad}} \), the leading order contribution to the free energy functional from this multiplet becomes**

  \[
  F_{\text{ad}} = \frac{2\pi^2}{3} N^{3/2} \Delta_{\text{ad}} (1 - \Delta_{\text{ad}})(2 - \Delta_{\text{ad}}) \int dx \rho(x)^2 .
  \]

- **The leading order contribution to the free energy functional from a fundamental chiral multiplet of \( R \)-charge \( \Delta_f \) is**

  \[
  F_f = N^{3/2} \int dx \rho(x) |x| \left( \frac{1 - \Delta_f}{2} - \frac{1}{4\pi} y_I(x) \right) .
  \]

On the other hand, the contribution from an anti-fundamental chiral multiplet of \( R \)-charge \( \Delta_{\text{af}} \) is given by

\[
F_{\text{af}} = N^{3/2} \int dx \rho(x) |x| \left( \frac{1 - \Delta_{\text{af}}}{2} + \frac{1}{4\pi} y_I(x) \right) .
\]
Volume of Sasaki-Einstein sevenfold  It has been shown in [30, 38–40] that the free energy $F(\Delta)$ in the large $N$ limit as a function of trial $R$-charges $\Delta$ is related to the volume of a Sakakian seven-manifold $\text{Vol}(Y_7)[b]$ as a function of the Reeb vector $b$, provided that the gauge theory admits a gravity dual. The relation can be described as follows

$$ F(\Delta) = \frac{N^3}{2} \sqrt{\frac{2\pi^6}{27\text{Vol}(Y_7)[b]}} ,$$

where the Reeb vector $b$ is some linear combination of the trial $R$-charges $\Delta$. One can determine the $R$-charges $\Delta_*$ at the fixed point via maximizing the free energy $F$, which translates to minimizing the Sasakian volume of $Y_7$ on the geometry side. At the critical value of $\Delta_*$, the free energy $F(\Delta_*)$ is finally related to the Sasaki-Einstein volume $\text{Vol}(Y_7)[b_*]$.

5.2 Applications to RG flows

The RG flows of our interest are triggered by adding a certain relevant deformation with chiral operators, i.e., superpotential deformation. Since the supersymmetry algebra on the three-sphere is given by $SU(2|1) \times SU(2)$ containing the $U(1)_R$ symmetry, the superpotential deformation should be of $R$-charge two. This requirement introduces a strong constraint on the $R$-charge assignment of matter fields. However, other than this constraint, there will be no changes in the expression for the free energy even after such a deformation.

We therefore expect that, given two CFT’s related by the RG flows of our interest, the free energy $F_{\text{IR}}$ of the CFT at IR should be the same as the free energy $F_{\text{UV}}$ of the CFT at UV with a restricted parameter space for the $R$-charges $\Delta$ due to the superpotential deformation. In general, it is not obvious that the free energy function $F_{\text{UV}}(\Delta)$ of an arbitrary gauge theory in three dimensions can have a critical point on such a restricted parameter space by relevant deformations. However, since the RG flow discussed in previous sections are proposed to have a fixed point, holographically dual to M-theory on Sasaki-Einstein sevenfolds, there must exist another critical value of $\Delta$ that maximizes the $F(\Delta)$ on this restricted parameter space of $R$-charges, different from the original critical value of $F_{\text{UV}}(\Delta)$ on the unrestricted parameter space.

Let us now carry out the computation of the free energy for the two basic examples of RG flows discussed in section 3. For both examples, the partition function of the UV CFT is given by the following matrix integral

$$ Z_{\text{UV}} = \frac{1}{(N!)^2} \int \prod_{I=1}^{2} \prod_{i=1}^{N} \frac{d\lambda_{I}^{i}}{2\pi} \text{Exp} \left( \sum_{I,j} \Delta_{\text{mon}}^{I} \lambda_{I}^{j} \right) \cdot \text{Exp} \left( -F_{\text{one-loop}} \right) ,$$

where $\Delta_{\text{mon}}^{I}$ denotes the bare $R$-charge of the monopole operators. The $F_{\text{one-loop}}$ represents the one-loop contributions from vector multiplets and chiral multiplets,

$$ \text{Exp} \left( -F_{\text{one-loop}} \right) = \prod_{l=1}^{2} \prod_{i \neq j} 2 \sinh \left( \frac{\lambda_{I}^{l} - \lambda_{J}^{l}}{2} \right) \cdot \text{Exp} \left( -F_{\text{ad}} - F_{\text{bf}} - F_{\text{i}} - F_{\text{af}} \right) .$$
The detailed form of each contribution \((F_{\text{ad}}, F_{\text{bf}}, F_{\text{f}}, F_{\text{af}})\) depends on the specifics of the theory. The RG flow is triggered by the relevant operator

\[
\mathcal{W}_{\text{def}} = \frac{m}{2} \text{tr} \left[ \Phi^2 - \Phi^2 \right].
\]  

which forces the \(R\)-charge of the adjoint fields to be 1. This is incorporated in the computation of the partition function in a simple way,

\[
Z_{\text{UV}}(\Delta_{\text{ad}} = 1, \Delta_{\text{other}}) = Z_{\text{IR}}(\Delta_{\text{other}}). 
\]  

\[
\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C} \rightarrow D_3
\]

CFT at UV The partition function of the orbifold CFT on the three-sphere is given by (5.14) and (5.15) with

\[
\begin{align*}
F_{\text{ad}} &= \sum_{i \neq j} l \left( 1 - \Delta_\Phi - i \frac{\lambda_1^i - \lambda_1^j}{2\pi} \right) + l \left( 1 - \Delta_\Phi - i \frac{\lambda_1^i - \lambda_1^j}{2\pi} \right), \\
F_{\text{bf}} &= \sum_{a=1}^{2} \sum_{i,j} l \left( 1 - \Delta_{A_a} + i \frac{\lambda_2^1 - \lambda_2^j}{2\pi} \right) + l \left( 1 - \Delta_{B_a} - i \frac{\lambda_1^i - \lambda_2^j}{2\pi} \right), \\
F_{\text{f}} &= \sum_i l \left( 1 - \Delta_\Phi + i \frac{\lambda_1^i}{2\pi} \right) + l \left( 1 - \Delta_{\tilde{\Phi}} + i \frac{\lambda_2^i}{2\pi} \right), \\
F_{\text{af}} &= \sum_i l \left( 1 - \Delta_\Phi - i \frac{\lambda_1^i}{2\pi} \right) + l \left( 1 - \Delta_{\tilde{\Phi}} - i \frac{\lambda_2^i}{2\pi} \right).
\end{align*}
\]  

(5.18)

Due to the superpotential, one should satisfy the following relations

\[
\begin{align*}
\Delta_\Phi + \Delta_{A_1} + \Delta_{B_2} &= 2, & \Delta_\Phi + \Delta_{A_2} + \Delta_{B_1} &= 2, \\
\Delta_\Phi + \Delta_{A_1} + \Delta_{B_2} &= 2, & \Delta_\Phi + \Delta_{A_2} + \Delta_{B_1} &= 2, \\
\Delta_{A_1} + \Delta_\rho + \Delta_q &= 2, & \Delta_{B_1} + \Delta_{\tilde{\rho}} + \Delta_{\tilde{q}} &= 2.
\end{align*}
\]  

(5.19)

Note that the orbifold theory has a \(\mathbb{Z}_2\) flip-symmetry exchanging

\[
A_a \leftrightarrow B_a, \quad \Phi \leftrightarrow \tilde{\Phi}, \quad p \leftrightarrow \tilde{p}, \quad q \leftrightarrow \tilde{q}.
\]

One can therefore naturally assume that

\[
\begin{align*}
\Delta_i &\equiv \Delta_{A_i} = \Delta_{B_i}, & \Delta_{\ell} &\equiv \Delta_\rho = \Delta_{\tilde{\rho}}, & \Delta_{\text{af}} &\equiv \Delta_q = \Delta_{\tilde{q}}, & \Delta_{\text{ad}} &\equiv \Delta_\Phi = \Delta_{\tilde{\Phi}},
\end{align*}
\]

which implies that

\[
\begin{align*}
\Delta_{\text{ad}} + \Delta_1 + \Delta_2 &= 2, & \Delta_1 + \Delta_{\ell} + \Delta_{\text{af}} &= 2.
\end{align*}
\]  

(5.20)

The free energy in the large \(N\) limit takes the following form

\[
\frac{F[\rho(x), y(x)]}{N^{3/2}} = \int dx \, \rho(x) \Delta_1 |x| + \int dx \, \rho^2(x)(\Delta_1 + \Delta_2 - 2)(y^2(x) - 4\pi^2 \Delta_1 \Delta_2).
\]  

(5.21)
Note that, for non-chiral theories, $\Delta_{\text{mon}} = 0$. One can show that two functions $\rho(x)$ and $y(x)$ maximizing the free energy $F$ is given by

$$\rho(x) = \sqrt{\frac{\Delta_1}{8\pi^2(2 - \Delta_1 - \Delta_2)\Delta_1\Delta_2}} - \frac{\Delta_1}{8\pi^2(2 - \Delta_1 - \Delta_2)\Delta_1\Delta_2}|x|, \quad y(x) = 0, \quad (5.22)$$

supported on $[-x_*, x_*]$ with $x_* = \sqrt{8 \pi^2 (2 - \Delta_1 - \Delta_2) \Delta_1 \Delta_2}$. Plugging the above result back to $F$, one obtains

$$F_{UV}(\Delta_1, \Delta_2) = \frac{4\pi}{3} N^{3/2} \sqrt{2 \Delta_1^2 \Delta_2 (2 - \Delta_1 - \Delta_2)}, \quad (5.23)$$

which is maximized at

$$\Delta_1 = 1, \quad \Delta_2 = \frac{1}{2}. \quad (5.24)$$

One can read off the Sasakian volume of the orbifold space from (5.23),

$$V_{UV}(\Delta) = \frac{\text{Vol}_{UV}(\Delta)}{\text{Vol}(S^7)} = \frac{1}{16\Delta_1^2 \Delta_2 (2 - \Delta_1 - \Delta_2)}, \quad (5.25)$$

as a function of the trial $R$-charges. It agrees exactly with the Sasakian volume (2.7) of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{C}$ geometry upon identification

$$2\Delta_1 = b_2 = b_3, \quad 2\Delta_2 = b_1. \quad (5.26)$$

**CFT at IR** As discussed earlier, the partition function of the IR theory follows straightforwardly from that of the UV theory by $Z_{IR}(\Delta_1, \Delta_2) = Z_{UV}(\Delta_{\text{ad}} = 1, \Delta_1, \Delta_2)$. The constraints due to the superpotential reads

$$\Delta_1 + \Delta_2 = 1, \quad \Delta_1 + \Delta_1 + \Delta_{\text{af}} = 2. \quad (5.27)$$

In the large $N$ limit, one can easily show that the free-energy becomes

$$F_{IR}(\Delta_1) = \frac{4\pi}{3} N^{3/2} \sqrt{2 \Delta_1^2 (1 - \Delta_1)}, \quad (5.28)$$

which leads to

$$V_{IR}(\Delta) = \frac{\text{Vol}_{IR}(\Delta)}{\text{Vol}(S^7)} = \frac{1}{16\Delta_1^2 (1 - \Delta_1)}. \quad (5.29)$$

The above volume function exactly matches with the volume of $D_3$ (2.12) as a function of the Reeb vector upon identification,

$$2\Delta_1 = b_2 = b_3, \quad b_1 = 2. \quad (5.30)$$

$dP_3 \times \mathbb{C} \rightarrow Q^{1,1,1}$
CFT at UV  The three-sphere partition function of the CFT for $dP_3 \times \mathbb{C}$ can be described by (5.14) and (5.15) with

$$ F_{ad} = \sum_{i \neq j} l \left( 1 - \Delta_{\Phi} + \frac{i \lambda_1^1 - \lambda_1^1}{2\pi} \right) + l \left( 1 - \Delta_{\Phi} + \frac{i \lambda_1^2 - \lambda_1^2}{2\pi} \right), $$

$$ F_{bf} = \sum_{a=1}^{2} \sum_{i,j} l \left( 1 - \Delta_{A_n} + \frac{i \lambda_1^1 - \lambda_1^2}{2\pi} \right) + l \left( 1 - \Delta_{B_n} - \frac{i \lambda_1^1 - \lambda_1^2}{2\pi} \right), $$

$$ F_{i} = \sum_{i} l \left( 1 - \Delta_{q_1} + \frac{i \lambda_2^2}{2\pi} \right) + l \left( 1 - \Delta_{q_2} + \frac{i \lambda_2^2}{2\pi} \right), $$

$$ F_{ad} = \sum_{i} l \left( 1 - \Delta_{p_1} - \frac{i \lambda_1^1}{2\pi} \right) + l \left( 1 - \Delta_{p_2} - \frac{i \lambda_1^1}{2\pi} \right). $$

(5.31)

Due to the superpotential, one should satisfy the following relations

$$ \Delta_{\Phi} + \Delta_{A_1} + \Delta_{B_2} = 2, \quad \Delta_{\Phi} + \Delta_{A_2} + \Delta_{B_1} = 2, $$

$$ \Delta_{\Phi} + \Delta_{A_1} + \Delta_{B_2} = 2, \quad \Delta_{\Phi} + \Delta_{A_2} + \Delta_{B_1} = 2, $$

$$ \Delta_{A_1} + \Delta_{p_1} + \Delta_{q_2} = 2, \quad \Delta_{A_2} + \Delta_{p_2} + \Delta_{q_1} = 2. $$

(5.32)

Note that the present model has a $\mathbb{Z}_2$ flip-symmetry exchanging

$$ A_1 \leftrightarrow A_2, \quad B_1 \leftrightarrow B_2, \quad \Phi \leftrightarrow -\Phi, \quad \tilde{\Phi} \leftrightarrow -\tilde{\Phi}, \quad p_1 \leftrightarrow p_2, \quad q_1 \leftrightarrow q_2. $$

One can therefore naturally assume that

$$ \Delta_A \equiv \Delta_{A_1}, \quad \Delta_B \equiv \Delta_{B_1}, \quad \Delta_p \equiv \Delta_{p_1}, \quad \Delta_q \equiv \Delta_{q_1}, $$

(5.33)

and then

$$ \Delta_{ad} = \Delta_{\Phi} = \Delta_{\Phi}, \quad \Delta_{ad} + \Delta_A + \Delta_B = 2, \quad \Delta_A + \Delta_p + \Delta_q = 2. $$

(5.34)

In the large $N$ limit, the free energy is given by

$$ F[\rho(x), y(x)] = \frac{F_{\text{mon}}}{N^{3/2}} \int dx \rho(x) x + \int dx |\rho(x)|^2 \left( \Delta_A + \frac{1}{2\pi} y(x) \right)^2 $$

$$ + (\Delta_A + \Delta_B - 2) \int dx \rho^2(x) (y(x) + 2\pi \Delta_A) (y(x) - 2\pi \Delta_B), $$

(5.35)

where $y(x) = y_1(x) - y_2(x)$. The above expression is valid only in the region

$$ \left| \frac{y(x)}{\pi} + \Delta_A - \Delta_B \right| \leq \Delta_A + \Delta_B. $$

(5.36)

One can show that two functions $\rho(x)$ and $y(x)$ maximizing the free energy $F$ are given by

$$ \rho(x) = \frac{(\Delta_A + \Delta_B) |x| + 2(x \Delta_{\text{mon}} + \mu)}{4\pi^2 (\Delta_A + \Delta_B) (\Delta_A + \Delta_B - 2)}, $$

$$ y(x) = -2\pi \frac{(\Delta_A - \Delta_B) (x \Delta_{\text{mon}} + \mu) + \Delta_A (\Delta_A + \Delta_B) |x|}{2(x \Delta_{\text{mon}} + \mu) + (\Delta_A + \Delta_B) |x|}. $$

(5.37)
Due to the condition (5.36), the density functions are supported on \([x_l, x_r]\) where

\[
x_l = \frac{\mu}{\Delta_A + \Delta_B - \Delta_{\text{mon}}}, \quad x_r = -\frac{\mu}{\Delta_A + \Delta_B + \Delta_{\text{mon}}},
\]

and

\[
\mu = -2\pi \left( (\Delta_A + \Delta_B)^2 - \Delta^2 \right) \sqrt{\frac{(\Delta_A + \Delta_B)(2 - \Delta_A - \Delta_B)}{3(\Delta_A + \Delta_B)^2 - \Delta_{\text{mon}}^2}}.
\]

Plugging the above result back to the free energy functional \(F\), one obtains

\[
\frac{F}{N^{3/2}} = \frac{4\pi}{3} (\Delta_A + \Delta_B) \left( (\Delta_A + \Delta_B)^2 - \Delta_{\text{mon}}^2 \right) \sqrt{\frac{2 - \Delta_A - \Delta_B}{3(\Delta_A + \Delta_B)^2 - \Delta_{\text{mon}}^2}} (\Delta_A + \Delta_B),
\]

which is maximized at

\[
\Delta_{\text{mon}} = 0, \quad \Delta_A + \Delta_B = \frac{3}{2}.
\]

As discussed in [30], one cannot determine the \(R\)-charge \(\Delta_A\) or \(\Delta_B\) for \(U(N) \times U(N)\) theory due to the flat direction of the partition function. For \(SU(N) \times SU(N) \times U(1)\) theory where the dibaryon operators are allowed, an extra condition can be used to determine \(\Delta_A\):

\[
\int dx \rho(x)y(x) = 0 \rightarrow \Delta_A = 1, \quad \Delta_B = \frac{1}{2}.
\]

At the critical value \(\Delta_A = 1\) and \(2\Delta_B = 1\), one finally confirms that the free energy \(F\) reproduces the volume of the base of the cone \(dP_3 \times \mathbb{C}\),

\[
F_{\text{UV}} \rightarrow V_{dP_3 \times \mathbb{C}} = \frac{\text{Vol}(dP_3 \times \mathbb{C})}{\text{Vol}(S^7)} = \frac{2}{9}.
\]

**CFT at IR** The partition function of the \(Q^{1,1,1}\) theory perfectly agrees with that of the \(dP_3 \times \mathbb{C}\) theory with \(\Delta_{\text{ad}} = 1\), i.e.,

\[
Z_{dP_3 \times \mathbb{C}}(\Delta_{\text{ad}} = 1) = Z_{Q^{1,1,1}}
\]

with

\[
\Delta_A + \Delta_B = 1, \quad \Delta_A + \Delta_p + \Delta_q = 2.
\]

The large \(N\) limit of the free energy now becomes

\[
\frac{F[\rho(x), y(x)]}{N^{3/2}} = \Delta_{\text{mon}} \int dx \, \rho(x)x + \int dx \, \rho(x)|x| \left( \Delta_A + \frac{1}{2\pi} y(x) \right)^2
- \int dx \, \rho^2(x) \left( y(x) + \Delta_A \right) (y(x) + 2\pi (\Delta_A - 1)) \nonumber \]

\[
- \int dx \, \rho^2(x) \left( y(x) + \Delta_A \right) (y(x) + 2\pi (\Delta_A - 1)) \nonumber \]

\[
\int dx \, \rho(x)x.
\]

\[
\int dx \, \rho(x)|x| \left( \Delta_A + \frac{1}{2\pi} y(x) \right)^2
- \int dx \, \rho^2(x) \left( y(x) + \Delta_A \right) (y(x) + 2\pi (\Delta_A - 1)) \nonumber \]

\[
\int dx \, \rho^2(x) \left( y(x) + \Delta_A \right) (y(x) + 2\pi (\Delta_A - 1)) \nonumber \]

\[
\int dx \, \rho(x)x.
\]
where \( y(x) = y_1(x) - y_2(x) \). The above expression is valid only in the region

\[
\left| \frac{y(x)}{\pi} + 2\Delta_A - 1 \right| \leq 1. \tag{5.47}
\]

One can show that two functions \( \rho(x) \) and \( y(x) \) maximizing the free energy \( F \) are given by

\[
\rho(x) = -\frac{|x| + 2(x\Delta_{\text{mon}} + \mu)}{4\pi^2},
\]

\[
y(x) = -2\pi \frac{(2\Delta_A - 1)(x\Delta_{\text{mon}} + \mu) + \Delta_A|x|}{2(x\Delta_{\text{mon}} + \mu) + |x|}. \tag{5.48}
\]

Due to the condition (5.47), the density functions are supported on \([x_l, x_r]\) where

\[
x_l = \frac{\mu}{1 - \Delta_{\text{mon}}}, \quad x_r = -\frac{\mu}{1 + \Delta_{\text{mon}}}, \quad \mu = \frac{2\pi (1 - \Delta^2)}{\sqrt{3 - \Delta_{\text{mon}}^2}}. \tag{5.49}
\]

Plugging the above result back to \( F \), one obtains

\[
\frac{F}{N^{3/2}} = \frac{4\pi}{3} (1 - \Delta_{\text{mon}}^2) \sqrt{\frac{1}{3 - \Delta_{\text{mon}}^2}}, \tag{5.50}
\]

which is maximized at

\[
\Delta_{\text{mon}} = 0. \tag{5.51}
\]

Again, one cannot determine the \( R \)-charge \( \Delta_A \) or \( \Delta_B \) for \( U(N) \times U(N) \) theory due to the flat direction of the partition function [30]. For \( SU(N) \times SU(N) \times U(1) \) theory where the dibaryon operators are allowed, the condition below determines \( \Delta_A \):

\[
\int dx \, \rho(x)y(x) = 0 \rightarrow \Delta_A = \frac{2}{3}, \quad \Delta_B = \frac{1}{3}. \tag{5.52}
\]

At this critical value \( \Delta_A = 2/3 \) and \( \Delta_B = 1/3 \), one finally confirms that the free energy \( F \) reproduces the volume of \( Q^{1,1,1} \),

\[
F_{\text{IR}} \rightarrow V_{Q^{1,1,1}} = \frac{\text{Vol}(Q^{1,1,1})}{\text{Vol}(S^7)} = \frac{3}{8}. \tag{5.53}
\]

### 5.3 Field theoretical proof of universal ratio

We consider a slightly broader generalization of the ‘necklace’ quiver gauge theory considered in section 4. The gauge theory of our interest has gauge group \( G = \prod_{I=1}^L U(N)_I \) with Chern-Simons levels \( k_I \) subject to \( \sum_I k_I = 0 \). Turning on Chern-Simons levels leads to a straightforward generalization of the RG flows discussed previous sections. While we do not intend to repeat the the analyses of previous sections for this larger class of theories, we will include the possibility of adding Chern-Simons levels in the computation of the partition function in this subsection.
The gauge fields couple to $U(N)_I \times U(N)_{I+1}$ bifundamental and anti-bifundamental chiral multiplets $C_I, \tilde{C}_I (I = 1, 2, \ldots, L)$ and one adjoint chiral multiplet $\Phi_I$ for each $U(N)_I$. The superpotential of the UV theory prior to flavoring is the same as before,

$$W_{UV}^{(0)} = \sum_{I=1}^{L} \text{tr} \left( \Phi_I C_I \tilde{C}_I - \Phi_I \tilde{C}_{I-1} C_{I-1} \right). \tag{5.54}$$

Next, we consider arbitrary flavoring by coupling the quiver theory with two types of paired chiral multiplets, denoted by $(p_I^a, q_I^{a+1})$ ($a = 1, 2, \ldots, n_I$) and $(\tilde{p}_I^a, \tilde{q}_I^{a+1})$ ($\tilde{a} = 1, 2, \ldots, \tilde{n}_I$), which introduces the superpotential term

$$W_{\text{flavor}} = \sum_{I=1}^{L} \left[ \sum_{a=1}^{n_I} \text{tr} \left( p_I^a C_I q_I^{a+1} \right) + \sum_{\tilde{a}=1}^{\tilde{n}_I} \text{tr} \left( \tilde{q}_I^{\tilde{a}} \tilde{C}_{I+1} \tilde{p}_I^{\tilde{a}} \right) \right]. \tag{5.55}$$

After the RG flow, the IR theory takes the bare superpotential

$$W_{IR}^{(0)} = \frac{1}{m} \sum_{I=1}^{L} (-1)^I \text{tr} \left( C_I \tilde{C}_I \tilde{C}_{I-1} C_{I-1} \right), \tag{5.56}$$

and shares the same flavoring superpotential (5.55).

The free energy of the UV theory in the large $N$ limit now takes the following form

$$\frac{F}{N^{3/2}} = \sum_{I} \frac{k_I}{2\pi} \int dx \rho(x) x y_I(x) + \Delta_{\text{mon}} \int dx \rho(x) x$$

$$+ \frac{2\pi^2}{3} \sum_{I} \Delta_{\Phi_I} (1 - \Delta_{\Phi_I}) (2 - \Delta_{\Phi_I}) \int dx \rho^2(x)$$

$$+ \sum_{I} \frac{2 - \Delta_{C_I} - \Delta_{\tilde{C}_I}}{2} \int dx \rho^2(x) \left[ \frac{\pi^2}{3} (\Delta_{C_I} + \Delta_{\tilde{C}_I}) (4 - \Delta_{C_I} - \Delta_{\tilde{C}_I}) \right.$$

$$- (y_I(x) - y_{I+1}(x) + \pi(\Delta_{C_I} - \Delta_{\tilde{C}_I}))^2]$$

$$+ \sum_{I} \sum_{a} \int dx \rho(x) |x| \left( \frac{2 - \Delta_{p_I^a} - \Delta_{q_I^{a+1}}}{2} + \frac{1}{4\pi} (y_I(x) - y_{I+1}(x)) \right)$$

$$+ \sum_{I} \sum_{\tilde{a}} \int dx \rho(x) |x| \left( \frac{2 - \Delta_{\tilde{p}_I^{\tilde{a}}} - \Delta_{\tilde{q}_I^{\tilde{a}+1}}}{2} - \frac{1}{4\pi} (y_I(x) - y_{I+1}(x)) \right), \tag{5.57}$$

which is valid when the following conditions are satisfied for all $I$

$$|y_I - y_{I+1} + \pi(\Delta_{C_I} - \Delta_{\tilde{C}_I})| \leq \pi(\Delta_{C_I} + \Delta_{\tilde{C}_I}), \tag{5.58}$$

and the density function $\rho(x)$ should satisfy the relation below

$$\int_{\mathbb{R}} dx \rho(x) = 1. \tag{5.59}$$
In order for the UV theory to be a CFT, all monomials in the superpotential must have $R$-charge 2. It follows that

\[
\Delta_\Phi = \Delta_{\Phi_I} \text{ for all } I, \quad \Delta_\Phi + \Delta_{C_I} + \Delta_{\tilde{C}_I} = 2 \text{ for all } I, \tag{5.60}
\]

\[
\Delta_{C_I} + \Delta_{p_I} + \Delta_{q_{I+1}} = 2 \text{ for all } a, \quad \Delta_{\tilde{C}_I} + \Delta_{\tilde{p}_I} + \Delta_{\tilde{q}_{I+1}} = 2 \text{ for all } \tilde{a}. \tag{5.61}
\]

Using (5.61), we can eliminate all dependences on $\left(\Delta_{p_I}, \Delta_{q_{I+1}}, \Delta_{\tilde{p}_I}, \Delta_{\tilde{q}_{I+1}}\right)$ from (5.57), so that $F$ becomes a function of $(\Delta_\Phi, \Delta_{C_I}, \Delta_{\tilde{C}_I})$ only. For notational simplicity, we will write $(\Delta_I, \tilde{\Delta}_I)$ to denote $(\Delta_{C_I}, \Delta_{\tilde{C}_I})$ in what follows.

One can rearrange the above free energy into the following form

\[
\frac{F}{N^{3/2}} = F_{\text{linear}} + F_{\text{quadratic}} \tag{5.62}
\]

with

\[
F_{\text{linear}} = \sum_I \frac{k_I}{2\pi} \int dx \, \rho(x) x y_I(x) + \Delta_{\text{mon}} \int dx \, \rho(x) x
\]

\[+ \sum_I n_I \int dx \, \rho(x) |x| \left[ \frac{\Delta_I}{2} + \frac{1}{4\pi} (y_I(x) - y_{I+1}(x)) \right]\]

\[+ \sum_I \tilde{n}_I \int dx \, \rho(x) |x| \left[ \frac{\tilde{\Delta}_I}{2} - \frac{1}{4\pi} (y_I(x) - y_{I+1}(x)) \right]. \tag{5.63}
\]

and

\[
F_{\text{quadratic}} = \sum_I \frac{2 - \Delta_I - \tilde{\Delta}_I}{2} \int dx \, \rho^2(x) \left[ 2\pi \Delta_I + (y_I(x) - y_{I+1}(x)) \right]
\]

\[\times \left[ 2\pi \tilde{\Delta}_I - (y_I(x) - y_{I+1}(x)) \right]. \tag{5.64}\]

It is easy to show that the above CFT can flow down to another CFT at the infrared once we turn on complex mass terms for adjoint chiral multiplets. In order to relate the partition function of the UV CFT to that of the IR CFT, it is useful to rescale the parameters entering the partition function:

- Rescale the conformal dimensions of bifundamental matters as follows

\[
\Delta'_I \equiv \frac{\Delta_I}{\Delta_I + \Delta_I} = \frac{\Delta_I}{2 - \Delta_\Phi}, \quad \tilde{\Delta}'_I \equiv \frac{\tilde{\Delta}_I}{\Delta_I + \tilde{\Delta}_I} = \frac{\tilde{\Delta}_I}{2 - \Delta_\Phi}, \tag{5.65}
\]

which guarantees, for all $I$,

\[
\Delta'_I + \tilde{\Delta}'_I = 1. \tag{5.66}
\]

The rescaled variables $(\Delta'_I, \tilde{\Delta}'_I)$ will later be identified with the $R$-charges of the CFT at IR, where the marginality of the superpotential will require (5.66).
Rescale functions $y_I(x)$ as follows
\begin{equation}
\hat{y}_I(\hat{x}) = \frac{y_I(x)}{\Delta_I + \Delta_I} = \frac{y_I(x)}{2 - \Delta_\phi}, \quad \hat{x} = \alpha x.
\end{equation}

The value of the constant $\alpha$ will be determined later. After this rescaling, (5.58) implies another relation one need to satisfy for the CFT at IR
\begin{equation}
\left| \hat{y}_I(\hat{x}) + \pi \left( \Delta'_I - \tilde{\Delta}'_I \right) \right| \leq \pi.
\end{equation}

It also guarantees that the rescaled density function $\hat{\rho}(\hat{x})$, defined below, is supported on the rescaled region $\hat{\mathcal{R}}$.

Rescale the density function $\rho(x)$ as follows
\begin{equation}
\beta \hat{\rho}(\hat{x}) = \rho(x),
\end{equation}

where $\beta$ is a constant. One can fix this constant $\beta$ in terms of $\alpha$ by requiring that
\begin{equation}
\int_{\hat{\mathcal{R}}} \hat{d}\hat{x} \hat{\rho}(\hat{x}) = 1,
\end{equation}

which is again needed for the CFT at IR. From the relation (5.59), one can show that
\begin{equation}
1 = \int_{\mathcal{R}} d\hat{x} \rho(\hat{x}) = \alpha \beta \int_{\mathcal{R}} \hat{d}\hat{x} \hat{\rho}(\hat{x}) \implies \alpha \beta = 1.
\end{equation}

Under the above rescaling, the free energy of the CFT at UV scales as
\begin{align}
F_{\text{linear}} &= \alpha (2 - \Delta_\phi) \hat{F}_{\text{linear}}(\Delta'_I, \tilde{\Delta}'_I), \\
F_{\text{quadratic}} &= \alpha^{-1} (2 - \Delta_\phi)^2 \Delta_\phi \hat{F}_{\text{quadratic}}(\Delta'_I, \tilde{\Delta}'_I).
\end{align}

Choosing the constant $\alpha$ by
\begin{equation}
\alpha = \sqrt{\Delta_\phi (2 - \Delta_\phi)},
\end{equation}

one finally obtains
\begin{align}
F_{\text{UV}}(\Delta_\phi, \Delta_I, \tilde{\Delta}_I) &= \sqrt{\Delta_\phi (2 - \Delta_\phi)^3} \times \left( \hat{F}_{\text{linear}}(\Delta'_I, \tilde{\Delta}'_I) + \hat{F}_{\text{quadratic}}(\Delta'_I, \tilde{\Delta}'_I) \right) \\
&= \sqrt{\Delta_\phi (2 - \Delta_\phi)^3} \times F_{\text{IR}}(\Delta'_I, \tilde{\Delta}'_I),
\end{align}

where $F_{\text{IR}}$ can be identified as the free energy of the CFT at IR.

From (5.74), one can show that the free-energy of the CFT at UV should be extremized at
\begin{equation}
\Delta_\phi = \frac{1}{2}.
\end{equation}
which leads to the very universal ratio between the free energies at IR and UV:

\[ F_{UV}(\Delta_{\Phi}, \Delta_{I}, \tilde{\Delta}_{I}) = \sqrt{\frac{27}{16}} F_{IR}(\Delta'_{I}, \tilde{\Delta}'_{I}) . \] (5.76)

A few final remarks are in order. First, recall that the relevant term in the superpotential (4.2) which triggers the KW flow have a finely tuned relative ratio of mass-deformation parameters. However, the proof given above on the universal ratio 16/27 are rather insensitive to these ratios. It strongly suggests that there should exist a manifold of RG fixed points continuously connected to the KW flow. Second, we can slightly modify the above proof to incorporate other flows such as the PW flows. For instance, consider the RG flows induced by a cubic superpotential for adjoint chiral multiplets. For such a deformation, \( \Delta_{\Phi} = 2/3 \). We therefore need to choose a different rescaling for conformal dimensions,

\[ \Delta'_{I} = \frac{4\Delta_{I}}{3(2 - \Delta_{\Phi})}, \quad \tilde{\Delta}'_{I} = \frac{4\Delta_{I}}{3(2 - \Delta_{\Phi})}, \quad \Delta'_{I} + \tilde{\Delta}'_{I} = \frac{4}{3}. \] (5.77)

The rest of the proof can proceed in parallel with the one shown above, leading to another universal ratio

\[ F_{UV} = \frac{32\sqrt{2}}{27\sqrt{3}} F_{IR} . \] (5.78)

The cubic deformation and its universal ratio \( 2^{11}/3^{7} \) was first observed again in [30], and its supergravity solutions were discussed in a recent work [32].

6 Discussion

In this paper, we argued for the existence of two KW flows in M-theory and their generalization to orbifolds. All examples we considered in this paper, and further generalization with non-vanishing Chern-Simons levels, share some common features. The gauge theories are constructed by applying the flavoring method to dimensionally reduced KW flow. Geometrically, the flavoring lifts the toric diagram of a CY_3 to that of a CY_4.

A natural open question is whether there are more flows of KW type that are not in the same family as those considered in this paper. One possible approach would be to find a new KW-like flow in four dimensions (other than the orbifold generalization) and apply the flavoring method. In fact, that seems to be essentially the only viable approach one can take because all known methods to construct \( 2 \leq \mathcal{N} \leq 6 \) Chern-Simons-matter theories with large \( \mathcal{N} \) AdS dual involves KK reduction in one way or another.

For M2-branes probing a CY_4 cone, a genuinely three dimensional approach to construct the CFT was proposed in [42–44]. Being a three dimensional analog of the brane tiling model for four dimensional CFT’s [4, 5], it was later called a brane crystal model. The model was not sufficiently developed to fully specify the associated CFT for multiple M2 branes. But, in the abelian case, it did give some non-trivial information on the matter contents, superpotential terms and gauge groups, and successfully reproduced the chiral ring and the spectrum of baryons.
The brane crystal model can be used to study RG flows in M-theory. Ref. [44] gave a diagrammatic understanding of the original KW flow in terms of the brane tiling model and, by extending the idea to the brane crystal model, it predicted the existence of the new KW flows in M theory discussed in this paper. It would be interesting to improve the brane crystal model by using some of the recent developments in M2-brane CFT’s and use it to search for more new flows.

In the brane crystal model, the connection between the three dimensional gauge theory and its four dimensional counterpart can be understood as the projection of the crystal diagram onto a tiling diagram. There are two qualitatively different types of projections. There are relatively few cases where the projection can be done without double lines or crossings. There are many more cases where double lines or crossings are unavoidable. In the former case, it was shown in [19] that the Chern-Simons levels can encode the information on the ‘vertical’ moves of the crystal diagram. In the more subtle latter case, the flavoring method of [21, 22] was needed to account for the vertical moves. Some crystal diagrams can be projected down to two or more inequivalent tiling diagrams, leading to different field theory Lagrangians. It would be interesting to test the quantum equivalence among different projections by, for instance, computing the three-sphere partition function.

Finally, it would be interesting to find an explicit supergravity solution describing the new KW flow in M-theory, perhaps with the help of the recently developed methods for analyzing AdS4 solutions in M-theory [31, 32] and related works for the original KW flow in four dimensions [49].

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