RESTRICTION ESTIMATES TO COMPLEX HYPERSURFACES

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ABSTRACT. The restriction problem is better understood for hypersurfaces and recent progresses have been made by bilinear and multilinear approaches and most recently polynomial partitioning method which is combined with those estimates. However, for surfaces with codimension bigger than 1, bilinear and multilinear generalization of restriction estimates are more involved and effectiveness of these multilinear estimates is not so well understood yet. Regarding the restriction problem for the surfaces with codimensions bigger than 1, the current state of the art is still at the level of $TT^*$ method which is known to be useful for obtaining $L^p$–$L^2$ restriction estimates. In this paper, we consider a special type of codimension 2 surfaces which are given by graphs of complex analytic functions and attempt to make progress beyond the $L^2$ restriction estimates.

1. Introduction

Let $S$ be a smooth compact submanifold $\mathbb{R}^n (n \leq 3)$ with the usual surface measure $d\sigma$ (the induced Lebesgue measure) on $S$. The $L^p(\mathbb{R}^n) - L^q(d\sigma)$ boundedness of the restriction operator $f \rightarrow \hat{f}|_S$ has been extensively studied since the restriction phenomena was first observed by Stein in the late 1960s. As is standard in literature nowadays, it is more convenient to work with the dual operator $\int \hat{f} d\sigma$ which is called extension operator. The ultimate goal of the restriction problem is to characterize $L^p(S) - L^q(\mathbb{R}^n)$ boundedness of $f \rightarrow \int \hat{f} d\sigma$ in terms of geometric features of underlying (sub)-manifold $S$. Particularly, when $S$ is the sphere, it was conjectured that $\int \hat{f} d\sigma$ should map $L^p(S)$ boundedly to $L^q(\mathbb{R}^n)$ if and only if $q \geq \frac{2(n+1)}{n-1}$ and $q > \frac{2n}{n-1}$. There is a large body of literature which is devoted to this problem. Over the last couple of decades, the bilinear and multilinear approaches have been proved to be most effective, and via these new methods substantial progress has been made. Recently, polynomial partitioning method gave currently the best known result. We refer the reader to [4], [5], [12] for the latest developments.

Restriction phenomena to submanifolds other than hypersurfaces were also studied by some authors. Roughly, we may say that restriction problem is better understood when the dimension of manifold is 1 or its codimension is one. When the dimension of $S$ is 1(namely, $S$ is a curve), the restriction estimate is by now fairly well understood [7], [8], [9]. However, not much is known regarding the restriction estimate to surfaces of the intermediate dimensions, that is to say, when the codimension $k$ of the manifold is between 1 and $n-1$. The restriction problem for surfaces with codimension $k = 2$ was first studied by Christ[10] and later

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Mockenhaupt \cite{16} with general $k$. For certain types of surfaces they established the optimal $L^q \to L^2$ restriction estimates which can be regarded as an extension of the Stein-Tomas theorem which concerns the hypersurfaces with nonvanishing Gaussian curvature. There are some results \cite{1, 17} beyond the $L^q \to L^2$ restriction estimate for the surfaces with intermediate codimensions. However, it can be said that, for most surfaces with codimension between 1 and $n - 1$, the current state of the restriction problem is hardly beyond that of the Stein-Tomas theorem in the case of hypersurfaces.

To discuss the previous results for the restriction estimates for a surface of codimension $k \geq 2$, we consider the extension operator which is given by the surface $(\xi, \Phi(\xi))$. Let $D$ be a bounded region in $\mathbb{R}^d$ and $\Phi : D \to \mathbb{R}$ be a smooth function. Discarding harmless factor associated to parametrization of the surface measure, it is enough to consider the operator $E$ which is defined by

$$Ef(x, t) = \int_D e^{i(x \xi + t \Phi(\xi))} f(\xi) d\xi, \quad (x, t) \in \mathbb{R}^m \times \mathbb{R}^k.$$ 

Especially, if $\Phi$ is given by nontrivial quadratic forms, the optimal $L^q$ bound is that with $q = \frac{2(d+2k)}{d}$. This was shown to be true when $\Phi$ satisfies a suitable curvature condition \cite{10, 16}.

In this paper, we are concerned with restriction estimates for special type of surfaces of codimension 2 which are given by graphs of holomorphic functions. Identifying the complex number with a point in $\mathbb{R}^2$, complex hypersurface in $\mathbb{C}^n$ can be considered as a manifold of codimension 2 in $\mathbb{R}^{2n}$. We begin with introducing some notations. Let us set

$$\tilde{w} \odot w = \Re\left(\sum_{j=1}^n \bar{w}_j w_j\right),$$

where $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. For a function $\phi : \mathbb{C}^{n-1} \to \mathbb{C}$ and a bounded measurable set $D$ we define

$$\mathcal{E}^\phi_D f(w) = \int_D e^{i w \odot (z, \phi(z))} f(z) dz, \quad w \in \mathbb{C}^n,$$

where $dz$ denotes the usual Lebesgue measure on $(2d - 2)$-dimensional Euclidean space.

If we write $z = (z_1, \ldots, z_{n-1})$ and $z_j = x_j + iy_j$, $1 \leq j \leq n - 1$, the above operator is the extension operator given by the surface $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, \Re \phi(x + iy), \Im \phi(x + iy))$. We define the (complex) Hessian of $\phi$ by

$$H\phi := \left[\partial_{x_j} \partial_{y_j} \phi\right]_{1 \leq i,j \leq n-1}.$$ 

The natural non-degeneracy condition on $\phi$ is that $H\phi$ has nonzero determinant on $\mathbb{T}$. It is not difficult to see that $|\mathcal{E}^\phi_D \mathcal{P}_\chi(w)| \lesssim |w_n|^{-(n-1)}$ whenever $\phi$ is holomorphic and $\det(H\phi) \neq 0$ on the support of $\chi \in \mathcal{C}^\infty_c$ (see Lemma 2.4). Thus by the $TT^*$ argument it is not difficult to see that, for $q \geq \frac{2n+2}{n-1}$, $\|\mathcal{E}^\phi_D f\|_q \lesssim \|f\|_2$.

The main result of this paper is to show that this can be improved when $n$ is even.

\footnote{This is also true if $Q$ has not trivial second order term at a point in $D$.}
Theorem 1.1. Let $n$ be an even number $\geq 4$ and $D$ be a bounded region in $\mathbb{C}^{n-1}$. Suppose $\phi(z)$ is holomorphic on $\overline{D}$ and $\det H\phi \neq 0$ on $\overline{D}$. Then, for $p > \frac{2(n+2)}{n}$, $\|E^\phi_Df\|_p \leq C_p\|f\|_p$ for some constant $C_p$.

When $n = 2$, the result on the optimal range of $p,q$ was obtained by Christ, see [11, Theorem 3.2]. Our result relies on the multilinear restriction estimates for the complex surfaces and the induction argument due to Bourgain and Guth [5]. When $n$ is odd, as it was shown for the quadratic surfaces with principal curvatures of different signs [5], the induction argument based on multilinear restriction estimate is not enough to give estimate with the exponent $q < \frac{2n+2}{n-1}$. Since restriction of the complex analytic surface to subspace admits subsurface with no curved property, unlike the case of elliptic surfaces repeated use of multilinear restriction estimate is not allowed. See Remark 4.7. However, for $n = 3$, the $L^p - L^q$ estimates for $p,q$ satisfying $1/p + 2/q < 1$ and $q > 10/3$ were obtained in [6]. Our result doesn’t recover this and it is a manifestation that multilinear strategy has certain inefficiency in capturing the curvature property of the underlying surface.

In our proof of Theorem 1.1, the holomorphic assumption plays an important role. The assumption not only makes it possible to describe transversality condition in a simpler way but also provides good decay property of the Fourier transform of the surface measure. Though general forms of multilinear restriction estimates [2, 19] are known, not all of the restriction estimates we need for our purpose appear in literature. In Section 3 we prove these restriction estimates by following the argument in [2] and making use of general multilinear Kakeya and induction argument. In section 4, we prove our main result and discuss about surfaces given by almost complex structure.

2. Preliminaries

In this section we review the known multilinear Kakeya and multilinear restriction estimates on which our results are to be based. These estimates generalize multilinear restriction estimates for hypersurfaces [4]. In fact, fairly general forms of these estimates can be found in [2]. Before stating their results, we introduce some notations and give the statement of the Brascamp-Lieb inequality, which we also use later.

Theorem 2.1 (Brascamp-Lieb inequality, [3]). Let $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be linear, onto, and $p_j \geq 0$ for $1 \leq j \leq m$. Then,

\[
\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^{d_j}} f_j^{p_j} \right)
\]

holds for some $C < \infty$ if and only if the following hold:

\[
\sum_{j=1}^m p_j d_j = d,
\]

\[
dim V \leq \sum_{j=1}^m p_j \dim(L_j V) \text{ for any subspace } V \subseteq \mathbb{R}^d.
\]
We denote \((L, p)\) by the collection of \(\{L_j\}_{1 \leq j \leq m}\) and \(\{p_j\}_{1 \leq j \leq m}\). Also, we denote \(\text{BL}(L, p)\) by the smallest constant \(C\) for which (2.1) holds for all input data \(f_1, \ldots, f_m\).

To prove theorem 2.3, we need the following generalization of multilinear Kakeya estimate, which can be viewed as perturbation of Brascamp-Lieb inequality.

**Theorem 2.2 ([2, 19]).** Suppose \((L, p)\) is a Brascamp-Lieb datum for which \(\text{BL}(L, p) < \infty, L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}\) for each \(j\). Then there exists \(\nu > 0\) such that, for every \(\epsilon > 0\),

\[
(2.4) \quad \int_{[-1,1]^d} \prod_{j=1}^m \left( \sum_{T_j \in \mathbb{T}_j} \chi_{T_j} \right)^{p_j} dx \leq C \delta^d \prod_{j=1}^m (\# \mathbb{T}_j)^{p_j}
\]

holds for all finite collections \(\mathbb{T}_j\) of \(\delta\)-neighborhoods of \((d - d_j)\)-dimensional affine subspaces of \(\mathbb{R}^d\) which, modulo translation, are within a distance \(\nu\) of the fixed subspace \(V_j := \ker L_j\).

The estimate (2.4) was proved with the bound \(C \delta^{d-\epsilon}\) in [2] and the \(\delta^{-\epsilon}\) loss in the bound was removed later in [19].

Let \(\Sigma_j : \overline{U}_j \to \mathbb{R}^d\) be smooth parametrizations of \(d_j\)-dimensional submanifolds \(S_j\), where \(U_j\) is a bounded open set in \(\mathbb{R}^{d_j}\). Then, the associated extension operator is defined by

\[
E_j g(z) = \int_{U_j} e^{i \xi \cdot \Sigma_j(u)} g(u) du, \quad \xi \in \mathbb{R}^d.
\]

The following is an easy consequence of [2, Theorem 1.3].

**Theorem 2.3.** Let \(a = \{a_j\}_{1 \leq j \leq m}, a_j \in \mathbb{R}^{d_j}\) and \(L(a) = \{d \Sigma_j(a_j)^\ast\}_{1 \leq j \leq m}\). Suppose that \(\text{BL}(L(a), p) \leq C_0\) for all \(a \in \overline{U}_1 \times \cdots \times \overline{U}_m\) and some constant \(C_0\). Then, for every \(\epsilon > 0\), there exists a constant \(C = C(\epsilon)\) such that

\[
(2.5) \quad \int_{B(0, R)} \prod_{j=1}^m \|E_j g_j\|_{L^2(U_j)}^{2p_j} \leq C R^\epsilon \prod_{j=1}^m \|g_j\|_{L^2(U_j)}^{2p_j}
\]

holds for all \(g_j \in L^2(U_j), 1 \leq j \leq m, \) and all \(R \geq 1\).

We only consider complex hypersurfaces which are given by \(\Sigma_j(z) = (z, \phi(z)), z \in U_j\), where \(\phi\) is a holomorphic function. In other words, we consider the case \(d_j = 2n - 2, d = 2n\), and regard \(U_j\) as a subset of \(\mathbb{C}^{n-1}\) rather than \(\mathbb{R}^{2n-2}\), and \(\mathbb{C}^n\) replaces \(\mathbb{R}^{2n}\) via the obvious identification. It is plausible to expect that \(R^\epsilon\) at the right hand side of (2.5) is removable. But this is known only for some special cases and the problem is left open in most of cases. \(R^\epsilon\) can be replaced by \((\log R)^\kappa\) for a suitable constant \(\kappa\), see [19].

**Lemma 2.4.** Let \(\phi : \mathbb{C}^{n-1} \to \mathbb{C}\) be a holomorphic function on the support of \(\chi \in C_c^\infty(\mathbb{C}^{n-1})\). Suppose \(\det H\phi \neq 0\) on the support of \(\chi\), then

\[
\left| \int e^{i w \cdot \phi(z)} \chi(z) dz \right| \lesssim |w_n|^{n-1}.
\]

**Proof.** Let us write \(w_n = s + it\), and \(z = x + iy, x, y \in \mathbb{R}^{n-1}\). For simplicity let us set

\[
(\phi_1(x, y), \phi_2(x, y)) := (\Re \phi(x + iy), \Im \phi(x + iy)).
\]
By the stationary phase method, it is sufficient to show that the determinant of the hessian matrix of $s\phi_1(x,y) + t\phi_2(x,y)$ is comparable to $|(s,t)|^{2n-2}$. That is to say,

$$\det \begin{pmatrix} s\phi_1'' + t\phi_2'' & s\phi_1'' + t\phi_2'' \\ s\phi_1'' + t\phi_2'' & s\phi_1'' + t\phi_2'' \end{pmatrix} = (s^2 + t^2)^{(n-1)} \det H(\phi(z))^2.$$

Here $\phi_1''$ denotes the matrix $((\partial_x, \partial_y) \phi_1)$, and similarly $\phi_1''$, $\phi_1''$ also denote the matrices $((\partial_x, \partial_y) \phi_1)$, respectively. In fact, we may write the phase function $w \circ (z, \phi(z)) = |w_n| \circ (z, \phi(z))$. Then (2.6) shows the determinant of the hessian matrix of $\frac{w}{|w_n|} \circ (z, \phi(z))$ as a function of $x, y$ is bounded away from zero. Thus the standard stationary phase method gives the desired estimate.

To see (2.6) note that $\nabla_z \phi = \phi_1' + i\phi_2'$, and $\det H \phi = \det(\phi_1'' + i\phi_2'')$ since $\phi$ is holomorphic. Meanwhile, $(sI - itI)H \phi = (s\phi_1'' + t\phi_2'') + i(s\phi_2'' - t\phi_1'')$. Hence, using (3.4),

$$\det((sI - itI)H \phi)^2 = \det \begin{pmatrix} s\phi_1'' + t\phi_2'' & s\phi_1'' - t\phi_2'' \\ s\phi_1'' - t\phi_2'' & s\phi_1'' + t\phi_2'' \end{pmatrix} = \det \begin{pmatrix} s\phi_1'' & s\phi_2'' \\ s\phi_1'' & s\phi_2'' \end{pmatrix} = 2^k \det \begin{pmatrix} s\phi_1'' & -s\phi_1'' \\ s\phi_1'' & -s\phi_1'' \end{pmatrix} = 2^k \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2^k.$$

By the Cauchy-Riemann equation it follows that $\phi_2'' = -\phi_1''$, $\phi_1'' = -\phi_1''$, $\phi_2'' = -\phi_1''$, and $\phi_1'' = 0$. Thus, the right hand side is equal to

$$\det \begin{pmatrix} s\phi_1'' & t\phi_2'' \\ s\phi_1'' & t\phi_2'' \end{pmatrix} = 2^k \det \begin{pmatrix} s\phi_1'' & -s\phi_1'' \\ s\phi_1'' & -s\phi_1'' \end{pmatrix} = 2^k \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2^k.$$

Therefore (2.6) follows.

3. Multilinear restriction estimates for complex hypersurfaces

Even though we have quite general multilinear restriction estimates in Theorem 2.3, applying those estimates to particular cases is another matter. We need to reformulate those estimates in favorable forms which suit for deducing linear restriction estimates for the complex surfaces. This is the place where the assumption that the function $\phi$ is holomorphic plays a role. The assumption significantly simplifies the description of the conditions which guarantee multilinear estimates. However, this is not enough for our purpose since we also need general $k$-linear estimates with $k < n$. These estimate can not be directly deduced from the $n$-linear estimates. In particular the condition (2.2) is not generally satisfied by these $k$-linear estimates. Nevertheless, the difficulty can be easily overcome by simple projection argument and the induction argument due to Guth [13].

3.1. $n$-linear restriction estimate. For a point $a \in \mathbb{C}^{n-1}$ we set

$$n(\phi, a) = (\overline{\partial}_1 \phi(a), \overline{\partial}_2 \phi(a), \ldots, \overline{\partial}_{n-1} \phi(a), -1) \in \mathbb{C}^n,$$

where $\partial_j = \overline{\partial}_j$ is the complex derivative. This is the normal vector to the surface at $(a_1, a_2, \ldots, a_{n-1}, \phi(a))$ with respect to the usual Hermitian inner product on $\mathbb{C}^n$. We also set

$$n^\phi(a) = \frac{n(\phi, a)}{|n(\phi, a)|},$$

$^2\phi_1' = \phi_2', \phi_1' = -\phi_2'$. 

**References:**

1. Guth [13]
We will see that the complex line (real plane) generated by \( n^\phi \) is normal to the graph of \( \phi \) which has codimension 2. The following is a consequence of Theorem 2.3.

**Theorem 3.1.** Let \( U \subset \mathbb{C}^{n-1} \) be a bounded open set and \( U_j \subset U \), \( 1 \leq j \leq n \). Suppose \( \phi : \overline{U} \to \mathbb{C} \) is a holomorphic function and

\[
|\det (n^\phi(a_1), n^\phi(a_2), \cdots, n^\phi(a_n))| > c, \quad \forall a_i \in U_i, \; i = 1, \ldots, n
\]

for some \( c > 0 \). Then, \( \forall \varepsilon > 0, \forall R \geq 1, \) there is a constant \( C = C(\varepsilon) \) such that

\[
\left\| \prod_{j=1}^{n} \xi_{U_j}^\phi g_j \right\|_{L^\infty(B(0,R))} \leq C \varepsilon \prod_{j=1}^{n} \| g_j \|_{L^2(U_j)}.
\]

**Proof.** We rephrase the condition in the real valued form. We write \( z_j = x_j + iy_j \),

\[
\phi = \phi_1 + i\phi_2,
\]

and set

\[
\Sigma(x_1, y_1, x_2, \cdots, y_{n-1}) = (x_1, \cdots, y_{n-1}, \phi_1(x_1, \cdots, y_{n-1}), \phi_2(x_1, \cdots, y_{n-1})).
\]

Under this identification \( L_j = (d(\Sigma(a_j)))^* \) is given by

\[
L_j = \begin{pmatrix}
I_{2n-2} & \frac{\partial \phi_1}{\partial x_1}(a_j) & \frac{\partial \phi_2}{\partial y_1}(a_j) \\
\frac{\partial \phi_1}{\partial x_2}(a_j) & \cdots & \frac{\partial \phi_2}{\partial y_{n-1}}(a_j)
\end{pmatrix}.
\]

Then, by Theorem 2.3 it suffices to show that \( \text{BL}(L, p) < \infty \) for \( L = \{L_j\}_{1 \leq j \leq n}, \quad p = \{ \frac{1}{n-1} \}_{1 \leq j \leq n} \). That is to say, \((L, p)\) verifies (2.2) and (2.3). The condition (2.2) is clearly satisfied. For the condition 2.3, we need to show

\[
(3.2) \quad \dim V \leq \sum_{j=1}^{n} \frac{1}{n-1} \dim(L_j V) \text{ for any subspace } V \subseteq \mathbb{R}^{2n}.
\]

This follows from

**Lemma 3.2.** \( \sum_{j=1}^{n} \dim(L_j V) = \sum_{j=1}^{n} (2n - \dim(\ker(L_j | V))) \geq 2n^2 - \dim V \).

Indeed, by this lemma (3.2) is equivalent to \( \dim V \leq 2n \), which is trivially true. \( \square \)

**Proof of Lemma 3.2.** The first equality is obvious by the dimension theorem. For the second, it is enough to verify

\[
\dim V \geq \sum_{j=1}^{n} \dim(\ker(L_j | V)) = \sum_{j=1}^{n} \dim(\ker(L_j) \cap V).
\]

The kernel space of \( L_j \) is generated by two vectors

\[
v_{1,j} = \left( \frac{\partial \phi_1}{\partial x_1}(a_j), \frac{\partial \phi_1}{\partial y_1}(a_j), \cdots, \frac{\partial \phi_1}{\partial x_{n-1}}(a_j), \frac{\partial \phi_1}{\partial y_{n-1}}(a_j), -1, 0 \right),
\]

\[
v_{2,j} = \left( \frac{\partial \phi_2}{\partial x_1}(a_j), \frac{\partial \phi_2}{\partial y_1}(a_j), \cdots, \frac{\partial \phi_2}{\partial x_{n-1}}(a_j), \frac{\partial \phi_2}{\partial y_{n-1}}(a_j), 0, -1 \right).
\]
since these two vectors are orthogonal to all row vectors of \( L_j \). Since \( \phi \) is holomorphic, by the Cauchy-Riemann equation it follows that

\[
v_{2,j} = \left( -\frac{\partial \phi_1}{\partial y_1}(a_j), \ldots, -\frac{\partial \phi_1}{\partial y_{n-1}}(a_j), \frac{\partial \phi_1}{\partial x}(a_j), 0, -1 \right).
\]

Now we observe that

\[
| \det (v_{1,1} \cdots v_{1,n} \ v_{2,1} \cdots v_{2,n}) | = | \det (n(\phi, a_1) \ n(\phi, a_2) \cdots n(\phi, a_n)) |^2.
\]

Here we regard the vectors as column vectors. Indeed, if we denote by \( B \) the \( n \times n \) matrix with the \( j \)-th column \( (\frac{\partial \phi}{\partial x_j}(a_j), \ldots, \frac{\partial \phi}{\partial x_{n-1}}(a_j), -1), \ j = 1, \ldots, n \) and by \( D \) the \( n \times n \) matrix with the \( j \)-th column \( (\frac{\partial \phi}{\partial y_j}(a_j), \ldots, \frac{\partial \phi}{\partial y_{n-1}}(a_j), 0) \). Then after rearrangement we note that

\[
| \det (v_{1,1} \cdots v_{1,n} \ v_{2,1} \cdots v_{2,n}) | = \det \begin{pmatrix} B & D \\ -D & B \end{pmatrix}.
\]

Now recall the elementary identity

\[
\det \begin{pmatrix} B & D \\ -D & B \end{pmatrix} = | \det (B + iD) |^2
\]

which is valid for any square matrix \( B \) and \( D \). Note that \( B + iD \) is equal to the matrix \( (n(\phi, a_1), n(\phi, a_2) \cdots n(\phi, a_n)) \). Thus (3.3) follows.

From (3.3) and the condition (3.5) it follows that \( \{v_{1,1}, v_{2,1}, \ldots, v_{1,n}, v_{2,n}\} \) is a basis of \( \mathbb{R}^{2n} \). Consequently, we have the desired \( \dim V \geq \sum_{j=1}^{n} \dim (\ker (L_j) \cap V) \). \( \square \)

### 3.2. \( k \)-linear restriction estimate with \( k < n \). The above theorem is an \( n \)-linear restriction estimate while \( n \) is the complex dimension of the ambient space. Unfortunately, except the case \( n = 2 \) this type of multilinear restriction estimate alone is not sufficient to deduce linear estimate, and we also need multilinear estimates with intermediate multilinearity. However, these estimates are not straightforward from Theorem 3.1. In fact, since the multilinear estimates in [2] were obtained under assumption that the Bracamp-Lieb inequality is finite, the expected estimates are subject to the scaling condition (2.2), which is not satisfied with \( k < n \). To get around this, instead of deducing the desired estimate from the existing estimate we directly prove them by adopting the strategy [2] which was used for the proof of Theorem 3.1. For this purpose we first need to show suitable multilinear Kakeya estimates associated with the complex surfaces.

**Definition 3.3.** For \( v = (a_1 + ib_1, \ldots, a_n + ib_n) \in \mathbb{C}^{2n} \), set \( I(v) = (a_1, b_1, \ldots, a_n, b_n) \in \mathbb{R}^{2n} \). For \( v_1, \ldots, v_k \in \mathbb{C}^{2n} \), we define

\[
|v_1 \wedge v_2 \wedge \cdots \wedge v_k| := | \det (I(v_1), I(iv_1), \ldots, I(v_k), I(iv_k), w_1, \ldots, w_{2n-2k}) |.
\]

where \( \{w_1, w_2, \ldots, w_1\} \) is an orthonormal basis of the orthonormal complement of the subspace \( \operatorname{span} \{I(v_1), I(iv_1), \ldots, I(v_k), I(iv_k)\} \).

Note that the definition does not depend on particular choices of bases \( w_1, w_2, \ldots, w_1 \). Clearly, the value \( |v_1 \wedge v_2 \wedge \cdots \wedge v_k| \) quantifies degree of transversality between subspaces \( \operatorname{span} \{v_i\} \) provided \( |v_i| \sim 1 \). Using this notion of transversality, we obtain multilinear restriction estimate with multiplicity smaller than \( n \).
Theorem 3.4. Let $2 \leq k \leq n-1$ be an integer. Let $U \subset \mathbb{C}^{n-1}$ be a bounded open set and $U_j \subset U$, $1 \leq j \leq k$. Suppose $\phi : U \to \mathbb{C}$ is a holomorphic function and

\begin{equation}
|n_1(a_1) \wedge n_2(a_2) \cdot \cdot \cdot \wedge n_k(a_k)| > c, \quad \forall a_i \in U_i, \; i = 1, \ldots, k
\end{equation}

for some $c > 0$. Then, $\forall \epsilon > 0, \forall R \geq 1$, there is a constant $C = C(\epsilon)$ such that

\begin{equation}
\left\| \prod_{j=1}^{k} \mathcal{E}_{U_j, \epsilon} g_j \right\|_{L^2(B(a, R))} \leq C \epsilon R^c \prod_{j=1}^{k} \|g_j\|_{L^2(U_j)}.
\end{equation}

To prove this, we need the following form of multilinear Kakeya estimate which we prove by adapting the argument in [2]. Once Theorem 3.5 below is obtained, one can prove Theorem 3.4 routinely following the argument in [2] which deduces mutilinear restriction estimate from general multilinear Kakeya estimate. So, we omit proof of Theorem 3.4.

Theorem 3.5. For $1 \leq j \leq k$, let $U_{j, \delta}$ be a collection of $\delta$-neighborhoods $U_{i,j}$ of 1-dimensional affine $\mathbb{C}$-subspaces span$\{v_{i,j}\}$ of $\mathbb{C}^n$. Suppose $|v_{i,j}| \sim 1$ and there is a fixed constant $c > 0$ such that

\begin{equation}
|v_{i,1} \wedge v_{i,2} \wedge \cdot \cdot \cdot \wedge v_{i,k}| \geq c.
\end{equation}

Then, for every $\epsilon > 0$,

\begin{equation}
\int_{[-1,1]^{2n}} \prod_{j=1}^{k} \left( \sum_{U_{i,j} \in U_{j, \delta}} \chi_{U_{i,j}} \right)^{\frac{1}{n}} \leq C \epsilon \delta^{2n-\epsilon} \prod_{j=1}^{k} (\#U_{j, \delta})^{-\frac{1}{n}}.
\end{equation}

It is likely that $\delta^{-\epsilon}$ can be removed but the current estimate is good enough for our purpose. A similar estimate of lower level of multilinearity was obtained in [4] (see Theorem 5.1) for the typical multilinear Kakeya case. It was shown by monotonicity of heat flow. However the following argument is quite flexible, so it can be used to deduce estimate of lower levelmultilinearity from various scaling invariant multilinear estimates.

By decomposing the collection $U_{j, \delta}$ along the directions and the stability of the Brascamp-Lieb constant (see Lemma 3.7, and [2]), in order to show Theorem 3.5 it suffices to prove the following reduced version.

Proposition 3.6. Let $L_j$ be linear maps from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$ whose kernels are 1-dimensional $\mathbb{C}$-subspaces of $\mathbb{C}^n$ spanned by $v_j$ satisfying $|v_j| = 1$, for $1 \leq j \leq k$. Suppose

\begin{equation}
|v_1 \wedge v_2 \wedge \cdot \cdot \cdot \wedge v_k| > c
\end{equation}

for some $c > 0$. Then there exists $\nu > 0$ such that, for every $\epsilon > 0$,

\begin{equation}
\int_{[-1,1]^{2n}} \prod_{j=1}^{k} \left( \sum_{U_{i,j} \in U_{j, \delta}} \chi_{U_{i,j}} \right)^{\frac{1}{n}} \leq C \epsilon \delta^{2n-\epsilon} \prod_{j=1}^{k} (\#U_{j, \delta})^{-\frac{1}{n}}
\end{equation}

holds for all finite collections $U_{j, \delta}$ of $\delta$-neighborhoods of 1-dimensional affine $\mathbb{C}$-subspaces span$\{v_{i,j}\}$ of $\mathbb{C}^n$ provided that the direction of $U_{i,j}$, $v_{i,j}$ is contained in the $\nu$-neighborhood of $V_j := \ker L_j$.

---

3Whenever we mention the direction vector, it is assumed to have unit length.
The significance of this form is that it no longer needs to satisfy the dimension condition \((2.2)\) which is necessary for the Brascamp-Lieb inequality. We first consider the case where \(v_{i,j}\) is contained in the \(\text{ker} L_j\).

**Lemma 3.7.** Suppose \(\nu = 0\) in Theorem 3.6, that is to say, all \(v_{i,j}\) are contained in the \(\text{ker} L_j\). Then, the following inequality holds.

\[
\int \prod_{j=1}^{k} \left( \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{U_{i,j}} \right)^{\frac{1}{k}} \leq C\delta^{2n} \prod_{j=1}^{k} (\# \mathcal{U}_{j,\delta})^{\frac{1}{k}}.
\]

The constant \(C\) remains uniformly bounded under small perturbation \(v_1, \ldots, v_k\).

**Proof.** We consider the integral over \(\mathbb{C}^n\) as a double integral over the product spaces of \(V = \text{span}\{v_1, v_2, \ldots, v_k\}\) and its orthonormal complement. After suitable change of coordinates, we may assume that \(V \times V^\perp = \mathbb{C}^k \times \mathbb{C}^{n-k}\). We write the left hand side of \((3.7)\) as follows:

\[
\mathcal{I} := \int_{\mathbb{C}^{n-k}} \int_{\mathbb{C}^k} \left( \prod_{j=1}^{k} \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{U_{i,j}}(x, y) \right)^{\frac{1}{k}} dx dy.
\]

We may write

\[
\sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{U_{i,j}} = \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{B_{i,j,\delta} \circ L_j}
\]

for some \(B_{i,j,\delta} \subset \mathbb{C}^{n-1}\) which are balls of radius \(\sim \delta\).

Let us set \(W_j = \text{span}\{L_j(v_1), \ldots, L_j(v_k)\}\) and consider the map \(\tilde{L}_j : \mathbb{C}^k \to W\) which is given by \(\tilde{L}_j(x) = L_j(x, 0)\). Then, from (3.6) it is easy to see that \(L_1, \ldots, L_k\) satisfy (2.2) and (2.3) with \(m = k\) and \(p_j = 1/(k - 1)\). In fact we have already checked this in the proof of Theorem 3.1. Now taking \(f^\nu_j(u) = \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{B_{i,j,\delta} \circ L_j}(u + L_j(0, y)), j = 1, \ldots, k,\)

by Theorem 2.1 we have

\[
\int_{\mathbb{C}^k} \prod_{j=1}^{k} (f^\nu_j)^{\frac{1}{k}}(\tilde{L}_j(x)) dx \lesssim \prod_{j=1}^{k} \left( \int_{W_j} f^\nu_j(u) du \right)^{\frac{1}{k}}.
\]

Combining this with (3.8) yields

\[
\mathcal{I} \lesssim \prod_{j=1}^{k} \left( \int_{W_j} \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{B_{i,j,\delta} \circ L_j}(u + L_j(0, y)) du \right)^{\frac{1}{k}} dy.
\]

Using Hölder’s inequality,

\[
\mathcal{I} \lesssim \prod_{j=1}^{k} \left( \int_{W_j} \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{B_{i,j,\delta} \circ L_j}(u + L_j(0, y)) du \right)^{\frac{1}{k}} dy \lesssim \delta^{2n} (\# \mathcal{U}_{j,\delta})^{\frac{1}{k}}.
\]

Thus, to obtain (3.7) it is sufficient to show that

\[
\mathcal{II}_j := \int \left( \sum_{U_{i,j} \in \mathcal{U}_{j,\delta}} \chi_{B_{i,j,\delta} \circ L_j}(u + L_j(0, y)) du \right)^{\frac{1}{k}} dy \lesssim \delta^{2n} (\# \mathcal{U}_{j,\delta})^{\frac{1}{k}}.
\]
By decomposing $\mathbb{C}^{n-k}$ into boundedly overlapping balls $B$ of side length $\delta$, we have

$$II_j \lesssim \sum_B \delta^{2k} \left( \# \{ U_{i,j} : U_{i,j} \cap (\mathbb{C}^k \times B) \neq \emptyset \} \right)^{\frac{k}{k-1}} dy$$

$$\sim \delta^{2n} \sum_B \left( \# \{ U_{i,j} : U_{i,j} \cap (\mathbb{C}^k \times B) \neq \emptyset \} \right)^{\frac{k}{k-1}}$$

$$\sim \delta^{2n} \left( \sum_B \left( \# \{ U_{i,j} : U_{i,j} \cap (\mathbb{C}^k \times B) \neq \emptyset \} \right) \right)^{\frac{k}{k-1}}.$$

Clearly, $\sum_B \# \{ U_{i,j} : U_{i,j} \cap (\mathbb{C}^k \times B) \neq \emptyset \} \lesssim \# U_{j,\delta}$. Thus, we get the desired inequality.

Finally the last statement is consequence of the stability of Brascamp-Lieb constant $[2]$, which gives (3.9) with a uniform constant $C$ under small perturbation of the kernel of $v_1, \ldots, v_k$. Thus, from the argument above we see that (3.7) holds with a uniform $C$.  

We prove Theorem 3.6 by perturbing the directions in the above lemma. The idea is basically due to Guth [13].

Proof of Theorem 3.6. We continue to denote the direction of $U_{i,j}$ by $v_{i,j} \in \mathbb{C}^n$ as in the above, and we may also assume $|v_{i,j}| = 1$. Let $B(\delta, \nu)$ be the smallest bound $C$ such that

$$\int_{[-1,1]^2n} \prod_{j=1}^k \left( \sum_{U_{i,j} \in U_{j,\delta}} \chi_{U_{i,j}} \right)^{\frac{1}{k-1}} \leq C \delta^{2n} \prod_{j=1}^k \left( \# U_{j,\delta} \right)^{\frac{1}{k-1}}$$

holds. To complete proof, it is sufficient to show that $B(\delta, \nu) \leq C \delta^{-\epsilon}$. This can be shown by the following iterative inequality for $B(\delta, \nu)$.

Lemma 3.8. There exists a number $k$ independent of $\delta$ and $\nu$, such that

$$B(\delta, \nu) \leq kB\left(\frac{\delta}{\nu}, \nu\right). \tag{3.10}$$

Applying the inequality $l$ times, we have $B(\delta, \nu) \leq k^l B\left(\frac{\delta}{\nu^l}, \nu\right)$. We only need to choose $\nu$ such that $\log \frac{1}{\nu} = \log k$, then choose $l$ such that $\frac{k^l}{\nu^l} \sim 1$. So, $k^l \sim \delta^{-\epsilon}$ and the desired bound follows.  

Proof of Lemma 3.8. We partition $[-1,1]^2n = \bigcup Q$ by cubes of side length $\sim \frac{\delta}{\nu}$. Then, we write

$$\int_{[-1,1]^2n} \prod_{j=1}^k \left( \sum_{U_{i,j} \in U_{j,\delta}} \chi_{U_{i,j}} \right)^{\frac{1}{k-1}} = \sum_Q \int_{Q} \prod_{j=1}^k \left( \sum_{U_{i,j} \in U_{j,\delta,Q}} \chi_{U_{i,j}} \cap Q \right)^{\frac{1}{k-1}}, \tag{3.11}$$

where $U_{j,\delta,Q} = \{ U_{i,j} \in U_{j,\delta} : U_{i,j} \cap Q \} \neq \emptyset$.

We focus on a single $Q$. If $U_{i,j} \in U_{j,\delta,Q}$, we note that distance between $v_{i,j}$ and $kerL_j$ is less than $\nu$. So, since $Q$ has side length $\sim \frac{\delta}{\nu}$, there exists a $O(\delta)$—neighborhood
with a quadratic function.

Let \( U_{i,j} \) such that \( U_{i,j} \cap Q \subset U'_{i,j} \cap Q \) and \( U'_{i,j} \) is parallel to \( \ker L_j \). Now we can use Lemma 3.7 to get

\[
\int_Q \prod_j \left( \sum_{U_{i,j} \in U_j, \delta} \chi_{U'_{i,j}} \right)^{\frac{1}{2-n}} \lesssim \delta^{2n} \prod_j \left( \#\mathcal{U}_j, \delta, Q \right)^{\frac{1}{2-n}}.
\]

If \( U_{i,j} \in \mathcal{U}_j, \delta, Q \), let \( \tilde{U}_{i,j} = U_{i,j} + O(\frac{\delta}{\nu}) \) such that \( \tilde{U}_{i,j} \) contains \( Q \). So,

\[
\prod_j \left( \#\mathcal{U}_j, \delta, Q \right)^{\frac{1}{2-n}} \lesssim \frac{1}{|Q|} \int_Q \prod_j \left( \sum_{U_{i,j} \in U_j, \delta} \chi_{U_{i,j}}(x) \right)^{\frac{1}{2-n}} dx.
\]

Combining these two estimates gives

\[
\int_Q \prod_j \left( \sum_{U_{i,j} \in U_j, \delta} \chi_{U'_{i,j}} \right)^{\frac{1}{2-n}} \lesssim \delta^{2n} \frac{1}{|Q|} \int_Q \prod_j \left( \sum_{U_{i,j} \in U_j, \delta} \chi_{U_{i,j}}(x) \right)^{\frac{1}{2-n}} dx.
\]

Recalling (3.11), we put all the estimates over \( Q \) together. Since \( |Q| \sim (\delta/\nu)^{2n} \), recalling the definition of \( B(\delta, \nu) \), we get

\[
\int_{[-1,1]^{2n}} \prod_{j=1}^k \left( \sum_{U_{i,j} \in U_j, \delta} \chi_{U_{i,j}} \right)^{\frac{1}{2-n}} \lesssim \nu^{2n} \int_{[-1,1]^{2n}} \prod_j \left( \sum_{U_{i,j} \in U_j} \chi_{U_{i,j}} \right)^{\frac{1}{2-n}} \lesssim \nu^{2n} B(\frac{\delta}{\nu}, \nu) \left( \frac{\delta}{\nu} \right)^n \prod \left( \#\mathcal{U}_j, \delta, Q \right)^{\frac{1}{2-n}}.
\]

This gives the desired (3.10). \( \square \)

**Remark 3.9.** As mentioned before, it is an open problem whether it is possible to remove \( R^r \) in the above estimate in Theorem 3.4. But, if we consider the estimate for \( p > \frac{2k}{k-1} \), we may remove \( R^r \) if \( H \phi \neq 0 \) on \( U \). This can be shown by the epsilon removal argument, see [5, Appendix].

### 4. Restriction estimates for complex hypersurfaces

#### 4.1. Complex hypersurfaces given by quadratic polynomials.

Now we prove the linear restriction estimate. We first show Theorem 1.1 with a quadratic function \( \phi \). Once it is done, extension to general holomorphic function requires only small modification of the argument.

**Proposition 4.1.** Let \( n \geq 3 \) be an odd number and \( \phi(z) = z^t M z \) for a nonsingular symmetric matrix \( M \). Suppose \( D \) is a bounded domain. Then, for \( p > \frac{2(n+2)}{n} \), there is a constant \( C_p \) such that \( \|E_D^\phi f\|_p \leq C_p \|f\|_p \).

Let \( Q(a, r) \) be the cube centered at \( a \) with side length \( r \). For simplicity we also set

\[
\mathcal{E} = \mathcal{E}^{z^t M z}_{Q(0,1)}.
\]

Let \( R \geq 1 \) and, for given \( p \), we define

\[
\mathcal{A}_p(R) = \sup \{ \|\mathcal{E} f\|_{L^p(Q(0,R))} : \|f\|_p \leq 1 \}.
\]

To prove Proposition (4.1), by finite decomposition, translation and scaling, it is enough to show \( \mathcal{A}_p(R) \leq C \) for \( p > \frac{2(n+2)}{n} \). For this we make use of the following lemma.
Lemma 4.2 (Parabolic rescaling). Suppose $f$ is supported in $Q(a, r) \in Q(0, 1)$ and $r < (\sqrt{d(1 + 2\|M\|)})^{-1}$. Then,

$$\|\mathcal{E}f\|_{L^p_{Q(a, R)}} \leq Cr^{2(n-1)-d}\mathcal{A}_p(R)\|f_a\|_p.$$ 

Proof. We first note that

$$\mathcal{E}f(w) = \int_{Q(a, r)} e^{iw\cdot(z, z'Mz)} f(z)dz.$$ 

Let $\psi(z) = z'tMz$. Then $\psi(z-a) = \psi(z) - \psi(a) - \nabla \psi(a) \cdot (z-a)$. We also set $f_{a,r}(z) = r^{2n-2}f(a + rz)$. Using these notations, we may write

$$|\mathcal{E}f(w)| = \left| \int_{Q(a, r)} e^{i(w'+w_n\nabla \psi(a))\cdot z + w_n\psi(z-a)} f(z)dz \right|$$

$$= \left| \int_{Q(0,1)} e^{i(r(w'+w_n\nabla \psi(a))\cdot z + r^2w_n\psi(z))} f_{a,r}(z)dz \right|$$

$$= |\mathcal{E}f_{a,r}(r(w'+w_n\nabla \psi(a)), r^2w_n)|.$$ 

We integrate with respect to $w$. By making change of variables $r(w'+w_n\nabla \psi(a)) \rightarrow w'$ and $r^2w_n \rightarrow w_n$

$$\|\mathcal{E}f\|_{L^p_{Q(0, R)}} \leq r^{-2(n+2)} \int_{|w'| < \sqrt{d(1+2\|M\|)}rR, |w_n| < r^2R} |\mathcal{E}f_{a,r}(w)|^p dw$$

$$\leq r^{-2(n+2)} \int_{Q(0, R)} |\mathcal{E}f_{a,r}(w)|^p dw.$$ 

Now, using the definition of $\mathcal{A}_p$, we see

$$\|\mathcal{E}f\|_{L^p_{Q(0, R)}} \leq \mathcal{A}_p(R)r^{-\frac{2n+2}{p}}\|f_{a,r}\|_p.$$ 

This gives the desired bound by rescaling. \hfill \Box

Proof of Proposition 4.1. Let $R > 0$ be a large number and fix a large number $1 \ll K \ll R$. Let $\{q\}$ be a collection of essentially disjoint cubes with side length $\sim K^{-1}$ which partitions $Q(0, 1)$ and let $\{Q\}$ be a collection of essentially disjoint cubes with side length $\sim K$ which partitions $Q(0, R)$. Thus

$$Q(0, 1) = \bigcup_q q, \quad Q(0, R) = \bigcup_Q Q.$$ 

We set

$$f_q = f_{\chi_q}.$$ 

So, we have $f = \sum_q f_q$. We will denote by $C(K)$ some powers of $K$ and this may vary from line to line.

We first consider $\{\mathcal{E}f_q\}$ on each cube $Q$. An important observation is that, on each cube $Q$, $\mathcal{E}f_q$ behaves as if it were a constant. To make it precise we need a bit of manipulation. Let $\eta \in S(\mathbb{C}^n)$ such that $\hat{\eta}(w) = 1$ if $|w| \leq 1$ and $\hat{\eta}(w) = 0$ if $|w| \geq 2$. For $q$ let $z_q$ be the center of $q$ and set

$$\eta_q(w) = e^{2\pi i w\cdot (z_q, \psi(z_q))} K^{2n} \eta(w/K).$$
Thus, we have
\begin{equation}
|E_f_q(w) - E_f_q * \eta_q(w)|. \tag{4.1}
\end{equation}

Let us denote the center of \(Q\) by \(w_Q\). Put \(\zeta(x) = \max_{|x'| \leq \sqrt{d}} |\eta(x + x')|^\frac{1}{n}\) and set
\[
\zeta^Q := K^{-2n^2} \left( \int |E_f_q(w_Q - s)|^\frac{1}{n} \zeta(s) ds \right)^n.
\]

The following is a slight modification of the argument in [18] (see, p.1024).

**Lemma 4.3.** Suppose \(w \in Q\), then we have
\begin{equation}
|E_f_q(w)| \lesssim \zeta^Q \lesssim K^{-2n^2} \int |E_f_q(w - s)|^\frac{1}{n} \zeta(s) ds,
\end{equation}

where \(\zeta(x) = \max_{|x'| \leq \sqrt{d}} |\zeta(x + x')|\). Furthermore, let \(q_1, q_2, \ldots, q_k \in \{q\}. \) If \(p \geq 1\) and \(w \in Q\), we have
\begin{equation}
\left( \prod_{i=1}^{k} C^Q_{q_i} \right)^\frac{1}{n} \lesssim C(K) \prod_{j=1}^{k} \left| E_{f_{q_j}}(w - s_j) \right|^\frac{1}{n} \zeta^Q(s_j) ds_j.
\end{equation}

**Proof.** We first observe that
\[
\|E_f_q(w - \cdot)\eta_q(\cdot)\|_1 \leq \|E_f_q(w - \cdot)\eta_q(\cdot)\|_1^{\frac{1}{n}} \int \|E_f_q(w - s)\eta_q(s)\|_1^\frac{1}{n} ds \lesssim K^{-2n^2}\|E_f_q(w - \cdot)\eta_q(\cdot)\|_1^{\frac{1}{n}} \int \|E_f_q(w - s)\eta_q(s)\|_1^\frac{1}{n} ds.
\]

Since the Fourier transform of \(E_f_q(w - \cdot)\eta_q(\cdot)\) is supported on a ball of radius \(\lesssim \frac{1}{K}\), the last inequality follows from Bernstein’s inequality. This immediately yields
\[
\int \|E_f_q(w - s)\eta_q(s)\| ds \lesssim K^{-2n^2} \left( \int \|E_f_q(w - s)\eta_q(s)\|_1^\frac{1}{n} ds \right)^n.
\]

Theorefore, by (4.1) and the above, we get
\[
|E_f_q(w)| \lesssim K^{-2n^2} \left( \int |E_f_q(w - s)|^\frac{1}{n} |\eta(s)|^\frac{1}{n} ds \right)^n
= K^{-2n^2} \left( \int |E_f_q(w_Q - s)|^\frac{1}{n} |\eta(s + w - w_Q)|^\frac{1}{n} ds \right)^n.
\]

Since \(|w - w_Q| \leq K\sqrt{d}\), it follows that \(|\eta(s + w - w_Q)|^\frac{1}{n} \leq \zeta^Q(\frac{1}{K})\). Thus we have the desired \(|E_f_q(w)| \lesssim C^Q\). For (4.3), note that
\[
\left( \prod_{i=1}^{k} C^Q_{q_i} \right)^\frac{1}{n} \lesssim C(K) \prod_{j=1}^{k} \int |E_{f_{q_j}}(w - s_j)|^\frac{1}{n} \zeta^Q(s_j - w + w_Q) ds_j.
\]

For the last inequality we use Hölder’s inequality and, as before, (4.3) follows since \(|w - w_Q| \leq K\sqrt{d}\). The second inequality in (4.2) can be shown by the same argument. This completes proof. \(\Box\)
Fix $2 \leq k \leq n$ and $Q$. We set

$$Q_s^Q = \{ q : |C_q^{Q^s}| < K^{2-2n} \max_q |C_q^{Q^s}| \}, \quad Q_l^Q = \{ q : |C_q^{Q_l}| \geq K^{2-2n} \max_q |C_q^{Q_l}| \}.$$ 

By Lemma 4.3, for $w \in Q$,

$$\sum_{q \in Q_l^Q} |E_{Fq}(w)| \leq \max_q C_q^{Q_l} \lesssim 1 := K^{-2n^2} \int \max_q |E_{Fq}(w-s)||\zeta(s/K)| ds.$$ 

We sort $Q_l^Q$ into two cases. It is clear that there are only the following two cases:

**Case 1:** There are $q_1, q_2, \ldots, q_k$ such that

$$|n^\phi(z_1) \wedge n^\phi(z_2) \wedge \cdots n^\phi(z_k)| > \frac{c}{K^{2k}}, \quad \forall z_i \in q_i, i = 1, \ldots, k. \quad (4.5)$$

**Case 2:** There is a $(k-1)$-dim $\mathbb{C}$-subspace $V_{k-1} = V_{k-1}(Q)$ such that

$$\text{dist}(n^\phi(q), V_{k-1}) \lesssim K^{-1}, \quad \forall q \in Q_l^Q. \quad (4.6)$$

Now let $q \in Q_l^Q$. In Case 1, it follows that $C_q^{Q_l} \leq K^{2n-2} \left( \prod_{j=1}^k C_{q_j}^{Q_l} \right)^{\frac{1}{k}}$. Thus from (4.2) and (4.3) we have, for $w \in Q$,

$$|E_{Fq}(w)|^p \lesssim C(K) \prod_{j=1}^k \int |E_{Fq_j}(w-s_j)| |\zeta(s_j/K)| ds_j$$

for some $q_1, q_2, \ldots, q_k$ satisfying (4.5). There are as many as $O(K^{2n-2})$ $q$. We make the right hand side independent of $Q$ by considering all the possible choices of $q_1, q_2, \ldots, q_k$ satisfying (4.5). Indeed, after a simple manipulation we have

$$\left( \sum_{q \in Q_l^Q} |E_{Fq}(w)|^p \right)^p \lesssim \Pi := C(K) \prod_{(q_1, q_2, \ldots, q_k) \text{ satisfying } (4.5)} \prod_{j=1}^k \int |E_{Fq_j}(w-s_j)| |\zeta(s_j/K)| ds_j. \quad (4.7)$$

Now we consider Case 2. Let us set $V_{k-1}' = \{ z : n^\phi(z) \in V_{k-1} \}$. Since the hessian matrix of $\phi$ is non-singular, from the inverse function theorem we see that $V_{k-1}'$ is a manifold of dimension $(2k-4)$. And observe that, for $q \in Q_l^Q$,

$$q \subset \{ z : \text{dist}(z, V_{k-1}') \leq C_{d,M} K^{-1} \}. \quad (4.8)$$

Clearly we have

$$|\tilde{w} - w| \gtrsim K^{-1} \iff |n^\phi(\tilde{w}) - n^\phi(w)| \gtrsim K^{-1}.$$

Since each $q$ is a cube of size $K^{-1}$ and contained in a $C K^{-1}$-neighborhood of the $(2k-4)$-manifold $V_{k-1}'$. It follows that

$$\# Q_l^Q \lesssim K^{2k-4}. \quad (4.9)$$

Since $E_{Fq}$ is bounded by $C_{q}^{Q_l}$ on $Q$, it is easy to see

$$\int_Q \left( \sum_{q \in Q_l^Q} |E_{Fq}|^p dw \right) \lesssim \left( \int_Q \left( \sum_{q \in Q_l^Q} |E_{Fq}|^2 dw \right)^{(p-2)} \right) \left( \sum_{q \in Q_l^Q} C_{q}^{Q_l} \right)^{p-2}.$$
Let \( \mu \) be a smooth function satisfying that \( \mu \sim 1 \) on \( Q \), and \( \hat{\mu} \) is supported in a ball of radius \( \sim K^{-1} \). Since \( \int_Q \sum_{q \in Q} |E_{f_q}|^2 \, dw \lesssim \int \left| \sum_{q \in Q} E_{f_q} \right|^2 |\mu|^2 \, dw \), Plancherel's theorem and orthogonality between \( E_{f_q} \cdot \mu \) give

\[
\int_Q \left| \sum_{q \in Q} E_{f_q} \right|^p \, dw \lesssim \left| Q \right| \left( \sum_{q \in Q} (C_Q^Q)^2 \right)^{\frac{p-2}{2}} \lesssim (\#Q)\left( 1 - \frac{1}{p} + (p-2)(1 - \frac{1}{p}) \right) \sum_q (C_Q^Q)^p |Q|.
\]

By (4.2) and Hölder's inequality, \( (C_Q^Q)^p \lesssim K^{-2n} \chi_Q(w) \int |E_{f_q}(w - z)|^p \zeta^{\frac{s}{K}}(\frac{s}{K}) \, dz \).

Thus, combining the above with (4.9), we have

\[
(4.10) \quad \int_Q \left| \sum_{q \in Q} E_{f_q} \right|^p \, dw \lesssim \int_Q \mathbb{I} \, dw,
\]

where

\[
\mathbb{I} = K^{(2k-4)p-2} \sum_j |E_{f_q}(w - s)|^p \zeta^{\frac{s}{K}}(\frac{s}{K}) \, ds.
\]

Therefore, putting (4.4), (4.7), and (4.10) together, we get

\[
\int_Q \left| \sum_{q \in Q} E_{f_q}(w) \right|^p \, dw \lesssim \int_Q \mathbb{I} + \mathbb{II} + \mathbb{III} \, dw.
\]

Note that I, II, and III are independent of \( Q \). Thus, by summation along \( Q \) we have

\[
\int_{Q(0, R)} |E f(w)|^p \, dw \lesssim \int_{Q(0, R)} \mathbb{I} + \mathbb{II} + \mathbb{III} \, dw.
\]

By (4.4), the imbedding \( \ell^p \hookrightarrow \ell^\infty \), and Lemma 4.2 we have

\[
\int_{Q(0, R)} \mathbb{I} \, dw \lesssim K^{-2n} \sum_j \int_{Q(0, R)} |E_{f_q}(w - s)|^p \, dw \zeta^{\frac{s}{K}}(\frac{s}{K}) \, ds 
\]

\[
\lesssim K^{4n - (2n-2)p} A_p^p(R) \sum_q \| f_q \|^p_p \leq K^{2n - (2n-2)p} A_p^p(R) \| f \|^p_p.
\]

By Theorem 3.4 we have, for \( p > \frac{2k}{k-1} \),

\[
\int_{Q(0, R)} \mathbb{II} \, dw \lesssim C(K) \sum_{(q_1, q_2, \ldots, q_k)} \int_{Q(0, R)} \prod_{j=1}^k |E_{f_{q_j}}(w - s_j)|^p \, dw \prod_{j=1}^k \zeta^{\frac{s_j}{K}}(\frac{s_j}{K}) \, ds \]

\[
\lesssim C(K) \| f \|^p_2.
\]

Repeating the same argument for \( \mathbb{I} \), it is easy to see that

\[
\int \mathbb{III} \, dw \lesssim K^{2(2(n-k+2) - p(n-k+1))} A_p^p(R) \| f \|^p_p.
\]

Combining all together we have

\[
\| E f \|^p_{L^p(\Omega(0, R))} \lesssim C(K) \| f \|^p + CK^{-\epsilon} A_p(R) \| f \|^p.
\]
for some $\epsilon_o > 0$ provided that
\[
p > \max_{2 \leq k \leq n-1} \left( \frac{2(n-k+2)}{n-k+1}, \frac{2k}{k-1} \right).
\]
The above inequality is valid for any $f$. With a sufficiently large $K$ such that $CK^{-\epsilon_o} \leq 1/2$, we have $A_p(R) \leq C(K)$. We may choose $k = \frac{n}{2} + 1$ since $n$ is even. This gives $A_p(R) \leq C(K)$ for $p > \frac{2(n+2)}{n}$, and completes the proof. \hfill \Box

4.2. General holomorphic function. Now we consider the general holomorphic function $\phi$ with nonzero $\det\phi$ and prove Theorem 1.1. The proof of Proposition 4.1 works with a slight modification. Since $\phi$ is no longer a quadratic polynomial, we need to modify the induction quantity $A_p(R)$.

Let denote $\mathcal{F}(\delta_o, r)$ by the collection of function $g$ which satisfies

1. $g$ is analytic on $Q(0, r)$.
2. $g(0) = 0$, and $\nabla g(0) = 0$.
3. $|\partial^n g(z) - \frac{1}{2} z \cdot z| < \delta_o$ for all $z \in \overline{Q(0, r)}$ and $2 \leq |\alpha| \leq 2n + 2$.

Considering the expansion $g(z + z_0) - g(z_0) = \nabla g(z_0) \cdot z = \frac{1}{2} z^t H g(z_0) z + O(|z|^3)$, we set
\[
g_{\delta_o}(z) := \frac{\epsilon^{-2}}{2} (g(\epsilon z + z_0) - g(z_0) - \nabla g(z_0) \cdot \epsilon z) = \frac{1}{2} z^t H g(z_0) z + O(\epsilon|z|^3).
\]
Since the matrix $H g(z_0)$ varies, it is desirable to normalize the second order term. The matrix $H g(z_0)$ is symmetric but not hermitian. However, by Takagi's decomposition we may write
\[
H g(z_0) = U^t(z_0) D(z_0) U(z_0)
\]
with a diagonal matrix $D(z_0)$ and a unitary matrix $U(z_0)$ (for example, see [14]). Let $\lambda_1(z_0), \ldots, \lambda_{n-1}(z_0)$ be the diagonal entries of $D(z_0)$. Since $H g$ is non singular, we clearly see there are $c, C > 0$ such that $c \leq |\lambda_1(z_0)|, \ldots, |\lambda_{n-1}(z_0)| \leq C$. Let us define
\[
[g]_{\delta_o} := g_{\delta_o}(\sqrt{D(z_0) U(z_0)})^{-1} z = \frac{1}{2} z \cdot z + O(c^{-1} \epsilon|z|^3).
\]
This shows that any analytic function with non-singular hessian matrix can be harmlessly transformed to a function contained in $\mathcal{F}(\delta_o)$ by an affine transform. Thus, by decomposing the operator we may regard the expansion operator $\mathcal{E}_{Q(0,1)}^n$ as a finite sum of $\mathcal{E}_{Q(0,1)}^n$ with $\phi \in \mathcal{F}(\delta_o, 1 + \epsilon_o)$ with sufficiently small $\delta_o, \epsilon_o > 0$.

Therefore, for the proof of Theorem 1.1 we only need to consider the extension operator given by this type of normalized $\phi \in \mathcal{F}(\delta_o, 1 + \epsilon_o)$. The above argument also shows

**Lemma 4.4.** Let $\delta_o, \epsilon_o > 0$. Suppose that $g \in \mathcal{F}(\delta_o, 1 + \epsilon_o)$ and $z_0 \in Q(0, \frac{1}{2})$. Then, there is a constant $c > 0$, independent of $g$ and $z_0$, such that $[g]_{z_0} \in \mathcal{F}(\delta_o, 1)$ provided that $0 < \epsilon < c$.

Let us set $Q = Q(0, 1)$. For $R \geq 1$ and a given $p \geq 1$, we define
\[
\tilde{A}_p(R) = \sup \{ \|\mathcal{E}^n_{Q} f\|_{L^p(Q(0, R))} : \|f\|_p \leq 1, \phi \in \mathcal{F}(\delta_o, 1 + \epsilon_o) \}.
\]
Then, we have the following which is a variant of Proposition 4.2.
**Lemma 4.5** (Parabolic rescaling). Let $\delta_0, \epsilon_0 > 0$. Suppose $\phi \in \mathfrak{F}(\delta_0, 1 + \epsilon_0)$, then there is an $r_\circ > 0$ and $C > 0$, independent of $\phi$, such that

$$\|\mathcal{E} f\|_{L^p_{\mathcal{Q}}(R)} \leq C r^{2(n-1)} \mathcal{P}(R) \|f\|_p$$

whenever $f$ is supported in $Q(z_0, r) \in Q(0, 1)$ with $r \leq r_\circ$.

This can be shown similarly as in the proof of Lemma 4.2 by making use of Lemma 4.4. So, we omit the proof. One may find a detailed argument of similar nature in [15].

**Remark 4.6.** Complex analyticity assumption plays important roles in our overall argument. It has been used various steps, so it is not clear at the moment how to generalize the result to general 2-dimension surfaces without complex analyticity.

**Remark 4.7.** The advantage of analyticity is compensated by new difficulty which results from lack of curved property of the complex surfaces. Unlike the case of elliptic surfaces, restriction of the surface to lower dimensional vector spaces does not necessarily guarantee persistence of the transversality since the complex analytic quadratic function can be factorized. For example, let $n = 2k + 1$ and consider a complex surface given by a holomorphic function

$$\phi(z_1, z_2) = z_1^2 + z_2^2 + \cdots + z_{n-1}^2 = (z_1 + iz_2)(z_1 - iz_2) + \cdots + (z_{n-2} + iz_{n-1})(z_{n-2} - iz_{n-1}).$$

If we take $V_k = \{z : (z_1 - iz_2) = c_1, \ldots, (z_{n-2} - iz_{n-1}) = c_k\}$. The restriction of the surface to $V_k$ does not have any curved property.

### 4.3. Almost complex structure.

Now we consider slightly more general manifolds of codimension 2. We recall the definition of almost complex structure.

**Definition 4.8.** Let $V$ be a real vector space of dimension $2n$. The automorphism $J : V \rightarrow V$ is called an almost complex structure if $J^2 = -id$.

This gives a complex structure on $V$. We can easily check that $V$ becomes a complex vector space with $i : v := J(v)$ for all $v \in V$. Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. We say the almost complex structure $J$ is compatible with the inner product if $\langle J(u), J(v) \rangle = \langle u, v \rangle$ holds for all $u, v \in V$. In what follows, by almost complex structure we mean an almost complex structure which is compatible with the inner product.

Consider $V = \mathbb{R}^{2n-2}$ with usual inner product. We may regard an almost complex structure $J$ as a matrix. Then, by using simple linear algebra, $J$ is an almost complex structure (compatible with the inner product) if and only if $J$ is a skew-symmetric orthogonal matrix. Using this $J$, we define our codimension 2 surface by a graph of $(\phi_1(z), \phi_2(z))$ for $z \in \mathbb{R}^{2n-2}$ which satisfies

$$\nabla \phi_2(z) = J \nabla \phi_1(z).$$

We call these kind of manifolds "almost complex hypersurfaces". For example, if

$$J = J_0 := \begin{pmatrix} P & 0 & \cdots & 0 \\ 0 & P & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P \end{pmatrix}$$
where \( P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), then (4.11) is just the Cauchy-Riemann equation and the case corresponds to the complex analytic case.

Since the minimal polynomial of \( P \) is \( x^2 + 1 \), it is diagonalizable. Moreover, the eigenspaces of \( i \) and \( -i \) have the same dimension. Using this fact, and by suitable linear changes, we indeed know that almost complex hypersurfaces are essentially complex hypersurfaces. In other words, imposing different almost complex structure just determines how we identify \( \mathbb{R}^{2n-2} \) with \( \mathbb{C}^{n-1} \).

Let \( v_1, v_2, \ldots, v_{n-1} \) be eigenvectors with respect to the eigenvalue \( i \). Since \( J \) is a real matrix, \( v_1, v_2, \ldots, v_{n-1} \in \mathbb{R}^{2n-2} \) are eigenvectors with respect to the eigenvalue \( -i \). Thus, the following \((2n-2) \times (2n-2)\) matrix is real and invertible:

\[
L = \begin{pmatrix}
\frac{v_1 + v_1^*}{2}, & \frac{v_1 - v_1^*}{2i}, & \ldots, & \frac{v_{n-1} + v_{n-1}^*}{2}, & \frac{v_{n-1} - v_{n-1}^*}{2i} \\
\end{pmatrix}.
\]

Clearly, \( L^{-1}JL = J_0 \). This means that we can always reduce the restriction problem for the almost complex hypersurfaces to that of the complex hypersurfaces.

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