Gravitational Analog of Faraday’s Law via Torsion and a Metric with an Antisymmetric Part

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In this paper we show that in the presence of torsion and a metric with an antisymmetric part one can construct a gravitational analog of Faraday’s law of electromagnetism.

I. INTRODUCTION

In a curved Riemannian background the covariant equations of motion for the electromagnetic field \(F_{\mu\nu}\) take the form

\[
\nabla_\mu F^{\mu\nu} = J^{\nu},
\]

where \(J^\nu\) is the determinant of the metric \(g_{\mu\nu}\). Equation (2) can also be written in the convenient form

\[
(-g)^{-1/2}\epsilon^{\mu\nu\sigma\tau}\nabla_\nu F_{\sigma\tau} = 0,
\]

and for brevity, we shall refer to Eq. (2) as Faraday’s Law.

For pure Riemannian geometry the connection is given by the Levi-Civita connection (here and throughout we follow the notation given in [1])

\[
\Lambda^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}).
\]

II. THE LEVI-CIVITA, TORSION, AND SPIN CONNECTIONS

To establish our result we will need to consider the coupling of a fermion to a geometry with torsion, and to this end we first recall how torsion is introduced into gravity theory. In order to construct covariant derivatives in any metric theory of gravity one must introduce a connection \(\Gamma^\lambda_{\mu\nu}\), which has to transform under a coordinate transformation \(x^\mu \to x'^\mu\) as

\[
\Gamma^\lambda_{\mu\nu}(x') = \frac{dx^\lambda}{dx^\alpha} \frac{dx^\beta}{dx^\nu} \Gamma^\gamma_{\beta\gamma}(x) + \frac{dx^\lambda}{dx^\mu} \frac{dx^\beta}{dx^\nu} \Gamma^\gamma_{\beta\gamma}(x),
\]

For pure Riemannian geometry the connection is given by the Levi-Civita connection (here and throughout we follow the notation given in [1])

\[
\Lambda^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}).
\]

To introduce torsion one takes the connection to no longer be symmetric on its two lower indices, and defines the Cartan torsion tensor \(Q^\lambda_{\mu\nu}\) according to

\[
Q^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}.
\]

With this antisymmetry \(Q^\lambda_{\mu\nu}\) has 24 independent components. Unlike the Levi-Civita connection the torsion \(Q^\lambda_{\mu\nu}\) transforms as a true rank three tensor under general coordinate transformations. In terms of the torsion tensor one defines a contorsion tensor according to

\[
K^\lambda_{\mu\nu} = \frac{1}{2}Q^{\lambda\alpha}(Q_{\mu\alpha\nu} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}),
\]

and with \(K^\lambda_{\mu\nu}\) one constructs the generalized connection

\[
\tilde{\Gamma}^\lambda_{\mu\nu} = \Lambda^\lambda_{\mu\nu} + K^\lambda_{\mu\nu},
\]

To give a connection that now has 64 independent components, the maximum number possible in a 4-space. With this generalized connection also obeying Eq. (4) (since \(K^\lambda_{\mu\nu}\) transforms as a tensor), one can construct a covariant derivative operator \(\nabla_\mu\), with the metric now obeying a generalized metricity condition

\[
\nabla_\mu g^{\lambda\nu} = \partial_\mu g^{\lambda\nu} + \tilde{\Gamma}^{\lambda\mu\nu}g^{\alpha\nu} - \tilde{\Gamma}^{\nu\alpha\mu}g^{\lambda\alpha} = 0
\]

This renders the connection \(\tilde{\Gamma}^\lambda_{\mu\nu}\) to be a torsion theory.

A torsion theory is then defined to be one in which one replaces \(\Lambda^\lambda_{\mu\nu}\) by \(\tilde{\Gamma}^\lambda_{\mu\nu}\), with the Riemann tensor

\[
\nabla^\lambda_{\mu\nu\kappa} = \partial_\kappa \Lambda^\lambda_{\mu\nu} - \partial_\nu \Lambda^\lambda_{\mu\kappa} + \nabla^\rho_{\mu\nu} \Lambda^\lambda_{\kappa\rho} - \nabla^\rho_{\mu\kappa} \Lambda^\lambda_{\rho\nu}.
\]
being given by the antisymmetric, 24-component
then completely determined in terms of the vierbeins,
metricity in the form
to be spacetime dependent, to continue to maintain in-
under
V
ory without torsion one introduces a set of vierbeins
Dirac action in flat space, viz. the Poincare invariant
To make this action invariant under local translations
ω
refers to a fixed, special-relativistic reference coordinate
I
D
= 1
2
∫ d4x(−g)1/2iψγaV
aμ(∂μ + Σbcωμbc)ψ
(13)
that will transform as a tensor under both local trans-
systems with metric
I
D
= 1
2
∫ d4x(−g)1/2iψγaV
aμ(∂μ + Σbcωμbc)ψ
(19)
With this ω
ab
, we obtain a torsion-dependent Dirac ac-
to now generalize the spin connection and the Dirac
equation to a space with torsion one replaces Λ
ab
by
Γ
ab
, with the spin connection and metricity conditions generalizing to
\[ \bar{D}_\mu V^{a\lambda} = \partial_\mu V^{a\lambda} + (\Lambda^{a\lambda}_{\mu\nu} + K^{a\lambda}_{\nu\mu}) V^{\nu\mu} + \omega^{ab}_{\mu} V^b_{\lambda} = 0, \]
\[ -\bar{\omega}^{ab}_{\mu} = -\omega^{ab}_{\mu} + \bar{V}^b_{\lambda} K^{a\lambda}_{\nu\mu} V^{\nu\mu} = \bar{\omega}^{ba}_{\mu}. \]

to couple spinors to gravity in a Riemannian space
without torsion, one starts with the free massless
Hasen brook action in flat space, viz. the Poincare invariant
(1/2) ∫ d4x(−g)1/2iψγaV
aμ(∂μ + Σbcωμbc)ψ + H.c., where γaβγ = 2θabc
To make this action invariant under local translations one introduces a (−g)
/2 factor in the measure and replaces γ
\partial_\mu
by γ
\partial_\mu,V
aμ
. While the resulting action is then invariant against spacetime independent Lorentz
transformations of the form ψ → exp(ωabcΣab)ψ where
\[ \Sigma_{ab} = (1/8)(\gamma_\alpha 0_\gamma - 2\gamma_\alpha 0_\gamma), \]
and when the function ω
ab
is taken
to be spacetime dependent, to continue to maintain in-
variance one has to augment the action with the spin
connection of Eq. (10), to then obtain the curved space
Dirac action
\[ I_D = \frac{1}{2} \int d^4x(−g)^{1/2}i\bar{\psi}\gamma^aV^a_\mu(\partial_\mu + \Sigma_{bc}\omega^{bc}_\mu)\psi + H.c. \]
An integration by parts in I
D
and use of
\[ \partial_\mu V^{a\mu} + (−g)^{-1/2}\partial_\mu(−g)^{1/2}V^{a\mu} + \omega^{ab}_\mu V^b_{\lambda} = 0, \]
\[ \gamma^a[\gamma^b,\gamma^c] = [\gamma^b,\gamma^c]\gamma^a = 4\eta^{ab}\gamma^c - 4\eta^{bc}\gamma^a \]
(17)
yields
\[ I_D = \int d^4x(−g)^{1/2}i\bar{\psi}\gamma^aV^a_\mu(\partial_\mu + \Sigma_{bc}\omega^{bc}_\mu)\psi \]
\[ -\frac{1}{2} \int d^4x(−g)^{1/2}i\bar{\psi}\gamma^aV^a_\mu\omega^{ab}_\mu \psi \]
\[ + \frac{1}{2} \int d^4x(−g)^{1/2}i\bar{\psi}V^{a\mu}(\Sigma_{bc}\gamma_a - \gamma_a\Sigma_{bc})\omega^{bc}_\mu \psi \]
\[ = \int d^4x(−g)^{1/2}i\bar{\psi}\gamma^a V^a_\mu(\partial_\mu + \Sigma_{bc}\omega^{bc}_\mu)\psi. \]

To now generalize the spin connection and the Dirac
equation to a space with torsion one replaces Λ
ab
by
Γ
ab
, with the spin connection and metricity conditions generalizing to
\[ \bar{D}_\mu V^{a\lambda} = \partial_\mu V^{a\lambda} + (\Lambda^{a\lambda}_{\mu\nu} + K^{a\lambda}_{\nu\mu}) V^{\nu\mu} + \omega^{ab}_\mu V^b_{\lambda} = 0, \]
\[ -\bar{\omega}^{ab}_\mu = -\omega^{ab}_\mu + \bar{V}^b_{\lambda} K^{a\lambda}_{\nu\mu} V^{\nu\mu} = \bar{\omega}^{ba}_\mu. \]
With this ω
ab
, we obtain a torsion-dependent Dirac ac-
to now generalize the spin connection and the Dirac
equation to a space with torsion one replaces Λ
ab
by
Γ
ab
, with the spin connection and metricity conditions generalizing to
\[ I_D = \int_0^{1/2} \gamma_a^b \gamma_a^b γ^c γ^d + 4i \epsilon^{abcd} γ^c \gamma^d, \]
\[ \epsilon^{abcd} V^a_\mu V^b_\nu V^c_\gamma V^d_\tau = (-g)^{-1/2} \epsilon^{abcd} g_\mu g_\nu g_\gamma g_\tau \]
(23)
yields
\[ \bar{I}_D = \int d^4x(−g)^{1/2}i\bar{\psi}\gamma^a V^a_\mu(\partial_\mu + \Sigma_{bc}\omega^{bc}_\mu - i\gamma^b S_\mu)\psi, \]
\[ S^\mu = \frac{1}{8}(-g)^{-1/2}e^{\mu \alpha \beta \gamma} Q_{\alpha \beta \gamma}. \]
(24)
Recalling that η^{0123} = 1, η_{0123} = −1, use of
\[ e^{0123} = 1 + (−g)^{-1/2} e^{\mu \alpha \beta \gamma} Q_{\alpha \beta \gamma}, \]
\[ -\bar{I}_D = \int d^4x(−g)^{1/2}i\bar{\psi}\gamma^a V^a_\mu(\partial_\mu + \Sigma_{bc}\omega^{bc}_\mu - i\gamma^b S_\mu)\psi, \]
\[ -\bar{I}_D = \frac{1}{4}[Q_{\alpha \beta \gamma} + Q_{\gamma \alpha \beta} + Q_{\beta \gamma \alpha}]. \]
(27)
In the action $I_D$, we note that even though the torsion is only antisymmetric on two of its indices, the only components of the torsion that appear in its torsion-dependent $S^\mu$ term are the four that constitute that part of the torsion that is antisymmetric on all three of its indices.

With $I_D$ admitting of an immediate generalization that incorporates electromagnetism, viz.

$$I_D = \int d^4 x (-g)^{1/2} i \bar{\psi} \gamma^\mu V_\mu (\partial_\mu + \Sigma_{\lambda c} \omega^{bc}_\mu)$$

$$- i A_\mu - i \bar{\psi} S_\mu \psi,$$  

we see that in Eq. (28) $S_\mu$ is an axial field that minimally couples to the axial fermion current in exactly the same minimally coupled way as the standard electromagnetic vector potential $A_\mu$ couples to the vector fermion current. Because of this minimal coupling $I_D$ is both locally gauge invariant under $\psi \rightarrow e^{i \alpha(x)} \psi, A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x)$, and locally chiral invariant \[1\] under $\psi \rightarrow e^{i \gamma^\beta(x)} \psi, S_\mu \rightarrow S_\mu + \partial_\mu \beta(x)$. Additionally, as noted in \[1\], $I_D$ is locally conformal invariant under $V_\mu(x) \rightarrow \Omega(x)V_\mu(x), \psi(x) \rightarrow \Omega^{-3/2}(x)\psi(x)$ since, just like the vector potential $A_\mu$, the equally minimally coupled $S_\mu$ also has zero conformal weight \[3\]. The $I_D$ action thus has a remarkably rich local invariance structure, as it is invariant under local translations, local Lorentz transformations, local gauge transformations, local axial transformations, and local conformal transformations.

### III. GRAVITATIONAL ANALOG OF FARADAY’S LAW

We note immediately the similarity between the structure of the gravitational Eqs. \[25\] and \[27\] on the one hand and the electromagnetic Eqs. \[2\] and \[3\] on the other, with both involving a rank three tensor (viz. $\nabla_\mu F_{\nu \tau}$ and $Q_{\mu \nu \tau}$) that is antisymmetric on its last two indices. To make the analog complete, we need to posit that the torsion can be written as the covariant derivative of some, for the moment, general rank two tensor $A_\mu$, that is itself antisymmetric \[4\]. Thus, we posit that one can write the torsion tensor as

$$Q_{\mu \nu \tau} = \nabla_\mu A_{\nu \tau},$$  

where $\nabla_\mu$ denotes the Levi-Civita-based covariant derivative and not the generalized torsion-based covariant derivative $\nabla_\mu$. With Eq. \[29\] we then obtain a gravitational analog of the electromagnetic Faraday Law, viz.

$$\nabla_\nu A_{\sigma \tau} + \nabla_\sigma A_{\nu \tau} + \nabla_\tau A_{\nu \sigma} = -4(-g)^{1/2} \epsilon_{\nu \sigma \tau \mu} S^\mu,$$  

with $A_{\mu \nu}$ serving as a gravitational analog of the electromagnetic $F_{\mu \nu}$, and with $S^\mu$ serving as a gravitational analog of a magnetic monopole current should Faraday’s Law possess such a source. Finally, if we set $S^\mu$ equal to zero, we then obtain a Faraday-type law for gravity of the form

$$\nabla_\nu A_{\sigma \tau} + \nabla_\sigma A_{\nu \tau} + \nabla_\tau A_{\nu \sigma} = 0,$$  

to thus establish that via the coupling of torsion to fermions it is in principle possible to construct a gravitational analog of Faraday’s Law of electromagnetism.

### IV. PHYSICAL SIGNIFICANCE OF $A_{\mu \nu}$

While we have introduced $A_{\mu \nu}$, solely as a general antisymmetric rank two curved space tensor that is related to the torsion tensor via $Q_{\mu \nu \tau} = \nabla_\mu A_{\nu \tau}$, it is of interest to ask whether, given its antisymmetry, $A_{\mu \nu}$ could be identified with some specific antisymmetric rank two tensor that has previously been encountered in the literature. To this end we note that there are three physical possibilities for $A_{\mu \nu}$ that immediately come to mind, namely that $A_{\mu \nu}$ be identified with the electromagnetic Maxwell tensor $F_{\mu \nu}$, that it be identified with the Kalb-Ramond field $B_{\mu \nu}$ of string theory, or that it be identified with the antisymmetric part of what would then be a 16-component metric tensor.

In the past there has been much study in the literature of the possibility that the electromagnetic $F_{\mu \nu}$ could have a geometric origin, as it could potentially lead to a unification of gravitation and electromagnetism \[2\]. However, an identification of the $A_{\mu \nu}$ we have introduced with $F_{\mu \nu}$ cannot be immediate since not only do these two tensors have different engineering dimension ($A_{\mu \nu}$ is dimensionless while $F_{\mu \nu}$ has dimension of inverse length squared), they also have different conformal weights ($A_{\mu \nu}$ has conformal weight two while $F_{\mu \nu}$ has conformal weight zero). Thus to make an identification of $A_{\mu \nu}$ with $F_{\mu \nu}$ one would have to introduce a parameter such as Newton’s constant with the dimension of length squared. However that would not be compatible with the conformal invariance of the $I_D$ action from whence we obtained $S^\mu$ in the first place, and so we do not consider this possibility further.

A second possibility for $A_{\mu \nu}$ is that it would be identified with the Kalb-Ramond field, $B_{\mu \nu}$, a massless excitation in string theory \[6\]. This $B_{\mu \nu}$ field serves as a generalization of the electromagnetic vector potential $A_\mu$ to the string theory case, with $B_{\mu \nu}$ coupling to strings in a manner analogous to the way $A_\mu$ couples to point particles. In certain formulations of string theory $B_{\mu \nu}$ accompanies the graviton field that is associated with the symmetric part of the metric. Moreover, associated with $B_{\mu \nu}$ is a rank three field strength generalization of the rank two Maxwell tensor $F_{\mu \nu}$, viz. $H_{\mu \nu \lambda} = \partial_\mu A_{\nu \lambda} + \partial_\nu A_{\lambda \mu} + \partial_\lambda A_{\mu \nu}$. Just like the quantity $Q_{\alpha \beta \gamma} + Q_{\beta \gamma \alpha} + Q_{\gamma \alpha \beta}$ that appears on the right-hand side of Eq. \[27\], $H_{\mu \nu \lambda}$ is also antisymmetric on all three of its indices. It would thus be of interest to see if one could relate $H_{\mu \nu \lambda}$ to $Q_{\mu \nu \lambda}$, though that would be beyond the scope of this paper.
As to our third possibility, an identification of $A_{\mu\nu}$ with the antisymmetric part of the spacetime metric is actually quite feasible since, like $A_{\mu\nu}$, the metric also has zero engineering dimension and conformal weight two. Moreover, with such an identification we can then construct a geometrical analog of Faraday’s Law that fully respects the very same local conformal invariance that the electromagnetic Maxwell equations themselves possess. And as we now show, in a non-symmetric gravity theory it is actually possible to make an identification of $A_{\mu\nu}$ with the antisymmetric part of the metric just as we would want.

Now taking the spacetime metric to have a non-vanishing antisymmetric part would constitute a modification of standard, symmetric metric tensor gravity, and such a modification has been studied in and of itself quite extensively in the literature. Moreover, if we do wish to identify $A_{\mu\nu}$ with the antisymmetric part of the spacetime metric we would need to enlarge the theory we have developed above and embed it in a theory that is built on a 16-component metric tensor from the very beginning. Various approaches have been developed in the literature to treat such a 16-component metric the theory would need to have developed above and embed it in a theory that is built on a 16-component metric tensor from the very beginning. Various approaches have been developed in the literature to treat such a 16-component metric theory, since once the metric is not symmetric one can define differing covariant derivatives depending on the sequencing of indices. Moreover, one can either specify the connection to be the one that makes some specific covariant derivative of the metric zero, or one can leave the connection unspecified a priori and allow the equations of motion themselves to determine the connection by stationary functional variation with respect to it (Palatini approach). In this latter approach the resulting variation is not guaranteed to lead to a vanishing covariant derivative of the metric, and in general even in the symmetric metric case does not do so for actions other than the Einstein-Hilbert action itself.

Of particular interest for our purposes here is the specific non-symmetric gravity theory developed by Moffat. Moffat’s approach is based on the Einstein-Hilbert action, and using the Palatini formalism he found that in a weak gravity expansion around a flat Minkowski metric $\eta_{\mu\nu}$, the first-order term in the connection takes the form

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\alpha} \left( \partial_\mu g^S_{\alpha\nu} + \partial_\nu g^S_{\alpha\mu} - \partial_\alpha g^S_{\mu\nu} \right) + \frac{1}{2} \eta^{\lambda\alpha} \left( \partial_\mu g^A_{\alpha\nu} + \partial_\nu g^A_{\alpha\mu} - \partial_\alpha g^A_{\mu\nu} \right), \quad (32)$$

where $g^S_{\mu\nu}$ is the symmetric part of the 16-component metric tensor and $g^A_{\mu\nu}$ is its antisymmetric part. On comparing Eq. (32) with Eq. (0) we see that the antisymmetric part of the connection is written as none other the derivative of the antisymmetric part of the metric just as we want. Thus as we see, in non-symmetric gravity theory one is able to derive a relation between the torsion and the derivative of the antisymmetric part of the spacetime metric, and in so doing one is then able to derive a gravitational analog of Faraday’s Law of electromagnetism.

[1] L. Fabbri and P. D. Mannheim, Phys. Rev. D 90, 024042 (2014).
[2] I. L. Shapiro, Phys. Rept. 357, 113 (2002).
[3] Under a local conformal transformation the torsion transforms as $Q^\lambda_{\mu\nu} \to Q^\lambda_{\mu\nu} + q \Omega^{-1}(x)(\delta^\alpha_\mu \partial_\nu - \delta^\alpha_\nu \partial_\mu)\Omega(x)$ where $q$ is its conformal weight. No matter what the value of $q$, under a conformal transformation $S_c$ transforms into itself with all derivatives of $\Omega(x)$ dropping out.
[4] We are not of course asserting that $Q_{\mu\nu\tau}$ necessarily is given by the covariant derivative of an antisymmetric rank two tensor, but only noting that if it is, one is then able to construct a geometrical analog of Faraday’s Law.
[5] A. Einstein and E. G. Straus, Ann. Math. 47, 731 (1946).
[6] M. Kalb and P. Ramond Phys. Rev. D 9, 2273 (1974).
[7] For a possible role for torsion in Faraday’s Law of electromagnetism itself see P. D. Mannheim, Tor-