Frobenius Problem for Semigroups $S(d_1, d_2, d_3)$

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Abstract

The matrix representation of the set $\Delta(d^3)$, $d^3 = (d_1, d_2, d_3)$, of the integers which are un-representable by $d_1, d_2, d_3$ is found. The diagrammatic procedure of calculation of the generating function $\Phi(d^3; z)$ for the set $\Delta(d^3)$ is developed. The Frobenius number $F(d^3)$, genus $G(d^3)$ and Hilbert series $H(d^3; z)$ of a graded subring for non–symmetric and symmetric semigroups $S(d^3)$ are found. The upper bound for the number of non–zero coefficients in the polynomial numerators of Hilbert series $H(d^m; z)$ of graded subrings for non–symmetric semigroups $S(d^m)$ of dimension, $m \geq 4$, is established.

Key words: Restricted partitions, Frobenius problem, Non–symmetric and symmetric semigroups, Hilbert series of a graded subring.

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1 Introduction

Let $S(d_1, \ldots, d_m) \subset \mathbb{N}$ be the subsemigroup generated by a set of integers $\{d_1, \ldots, d_m\}$ such that

$$1 < d_1 < \ldots < d_m, \quad \gcd(d_1, \ldots, d_m) = 1.$$  \hspace{1cm} (1)

The set $\{d_1, \ldots, d_m\}$ is called minimal if there are no nonnegative integers $b_{i,j}$ for which the following linear dependence holds

$$d_i = \sum_{j \neq i} b_{i,j} d_j, \quad b_{i,j} \in \{0,1,\ldots\} \text{ for any } i \leq m.$$  \hspace{1cm} (2)

For short we denote the tuple $(d_1, \ldots, d_m)$ by $d^m$ where $m$ is the dimension of $d^m$. Henceforth $d^m$ will be a minimal generating set of $S(d^m)$. The conductor $c(d^m)$ of $S(d^m)$ is defined by

$$c(d^m) := \min \{ s \in S(d^m) \mid s + \mathbb{N} \cup \{0\} \subset S(d^m) \}.$$  \hspace{1cm} (3)

The semigroup ring $k[X_1, \ldots, X_m]$ over a field $k$ of characteristic 0 associated with $S(d^m)$ is a polynomial subring graded by deg $X_i = d_i, 1 \leq i \leq m$ and generated by all monomials $z^{d_i}$. The Hilbert series $H(d^m; z)$ of a graded subring $k[z^{d_1}, \ldots, z^{d_m}]$ is defined by [1]

$$\Phi(d^m; z) = \sum_{s \in S(d^m)} z^s,$$  \hspace{1cm} (4)

where $Q(d^m; z)$ is a polynomial in $z$. The number

$$F(d^m) := -1 + c(d^m)$$  \hspace{1cm} (7)

is referred to as Frobenius number in honor of G. Frobenius who, according to [2], repeatedly raised the following question in his lectures: determine (or bound) $F(d^m)$. Actually, all three entities, $F(d^m)$, $G(d^m)$ and $H(d^m; z)$, are originated by the same semigroup $S(d^m)$ and have a strong algebraic relationship (see Section 5). Due to this reason the determination of $F(d^m)$, $G(d^m)$ and $H(d^m; z)$ will be called the $m$-dimensional (mD) Frobenius problem.

Let $R = k[X_1, \ldots, X_m]$ be the ring of polynomials over a field $k$ and $\pi : R \longrightarrow k[S(d^m)]$ be the projection induced by $\pi(X_i) = z^{d_i}$. Then $k[S(d^m)]$ has a presentation $k[X_1, \ldots, X_m] / \mathcal{I}_m$ where $\mathcal{I}_m$ is the kernel of the map $\pi$. The semigroup $S(d^m)$ is called symmetric iff for all $s \in S(d^m)$ the following holds $F(d^m) - s \not\in S(d^m)$. This kind of semigroups is of high importance due to Kunz’s theorem [3] which asserts that $k[S(d^m)]$ is a Gorenstein ring iff $S(d^m)$ is symmetric. It is classically known that in small dimensions $m = 2,3$ the situation is even simpler. For every $d^2$, $S(d^2)$ is a
functions

The procedure defined in (9) completely determines the elements \(d, \ldots, d_m\) generated by three pairwise relatively prime elements \(d_i\) such is a symmetric semigroup \([2]\) and \(k [S(d^3)]\) is a complete intersection \([4]\). The kernel \(\mathcal{I}_2\) is principal and has the generator \(p = X_1^{c_1} - X_2^{c_2}\) where \(c_i = \text{lcm}(d_1, d_2)/d_i\). For \(m = 3\), Herzog \([5]\) has proved that \(k [S(d^3)]\) is a complete intersection if \(S(d^3)\) is symmetric.

For larger \(m\), the generic semigroup \(S(d^m)\) is mostly non–symmetric, e.g. \(S(d^3)\) minimally generated by three pairwise relatively prime elements \(d_i\) is such a semigroup \([6]\). Concerning the Frobenius numbers, a theorem of Curtis \([7]\) asserts that, for \(m = 3\), there is no non–zero polynomial \(P \in \mathbb{C}(Y_1, Y_2, Y_3, Z)\) such that \(P(d_1, d_2, d_3, F(d^3)) = 0\) for all minimal sets \((d_1, d_2, d_3)\) where \(d_1, d_2\) are primes not dividing \(d_3\). In other words, \(F(d^3)\) cannot be determined for all minimal sets \((d_1, d_2, d_3)\) by any set of closed formulas which could be reduced to a finite set of polynomials \(^{1}\).

As for Hilbert series, for any \(m \geq 4\), there is no way to write the rational function \(H(d^m; z)\) so that its polynomial numerator \(Q(d^m; z)\) has a bounded number of non–zero terms for all choices of \(d_1, \ldots, d_m\) \([10]\). The semigroup \(S(d^3)\) presents the first nontrivial and most elaborated case.

Our main results are the expressions for the Frobenius number \(F(d^3)\), genus \(G(d^3)\) and the numerator \(Q(d^3; z)\) of Hilbert series for both symmetric and non–symmetric semigroups \(S(d^3)\). In order to present them introduce auxiliary notions. Following Johnson \([11]\) define the first minimal relation \(\mathcal{R}_1(d^3)\) for given \(d^3 = (d_1, d_2, d_3)\) as follows

\[
\mathcal{R}_1(d^3) : \quad a_{11}d_1 = a_{12}d_2 + a_{13}d_3, \quad a_{22}d_2 = a_{21}d_1 + a_{23}d_3, \quad a_{33}d_3 = a_{31}d_1 + a_{32}d_2, \tag{8}
\]

where

\[
a_{11} = \min \{v_{11} | v_{11} \geq 2, \ v_{11}d_1 = v_{12}d_2 + v_{13}d_3, \ v_{12}, v_{13} \in \mathbb{N} \cup \{0\}\}, \quad a_{22} = \min \{v_{22} | v_{22} \geq 2, \ v_{22}d_2 = v_{21}d_1 + v_{23}d_3, \ v_{21}, v_{23} \in \mathbb{N} \cup \{0\}\}, \quad a_{33} = \min \{v_{33} | v_{33} \geq 2, \ v_{33}d_3 = v_{31}d_1 + v_{32}d_2, \ v_{31}, v_{32} \in \mathbb{N} \cup \{0\}\}. \tag{9}
\]

The uniquely defined values of \(v_{ij}, i \neq j\) which give \(a_{ij}\) will be denoted by \(a_{ij}, i \neq j\). Note that due to minimality of the set \((d_1, d_2, d_3)\) the elements \(a_{ij}, i, j \geq 3\) satisfy

\[
\gcd(a_{11}, a_{12}, a_{13}) = 1, \quad \gcd(a_{21}, a_{22}, a_{23}) = 1, \quad \gcd(a_{31}, a_{32}, a_{33}) = 1. \tag{10}
\]

The procedure defined in (9) completely determines the elements \(a_{ij}, i, j \geq 3\) of \(\mathcal{R}_1(d^3)\) as the functions \(a_{ij} = a_{ij}(d_1, d_2, d_3)\).

For \(F(d^3)\), \(G(d^3)\) and \(Q(d^3; z)\) we get the following formulas

\[
F(d^3) = \frac{1}{2} \left[ \langle a, d \rangle + J(d^3) \right] - \sum_{i=1}^{3} d_i, \quad \langle a, d \rangle = \sum_{i=1}^{3} a_{ii}d_i, \tag{11}
\]

\[
G(d^3) = \frac{1}{2} \left( 1 + \langle a, d \rangle - \prod_{i=1}^{3} a_{ii} - \sum_{i=1}^{3} d_i \right), \tag{12}
\]

\[
Q(d^3; z) = 1 - \sum_{i=1}^{3} z^{a_{ii}d_i} + z^{1/2}[\langle a, d \rangle - J(d^3)] + z^{1/2}[\langle a, d \rangle + J(d^3)], \tag{13}
\]

where \(J(d^3)\) is the positive integer

\[
J(d^3) = \sqrt{\langle a, d \rangle^2 - 4 \sum_{i>j}^{3} a_{ij}a_{ijd_i}d_j + 4d_1d_2d_3}. \tag{14}
\]

\(^{1}\)The words "all" and "polynomial" are essential here, since there exist infinitely many triples \(d_1, d_2, d_3\) of primes in arithmetic progression \([8]\) constituting a minimal set and, according to Roberts \([9]\), the Frobenius number associated to them can be presented in closed, but not polynomial formula \(F(d, d + p, d + 2p) = d \left[ \frac{d-2}{2} \right] + (d - 1)p, \ \gcd(d, p) = 1\). A standard notation \([a]\) is used for the integer part of a real number \(a\).
Formula (13) for the numerator $Q(d^3; z)$ of Hilbert series of non–symmetric semigroup $S(d^3)$ is new and was not obtained earlier. Formula (12) is in full agreement with the genus of generic monomial space curves found by Kraft [12] while formula (11) can be reduced to the known expressions for symmetric and non–symmetric semigroups $S(d^3)$ obtained by Herzog [5] and Fröberg [13].

We also prove a theorem (Theorem 10) on the upper bound of the number of non–zero coefficients in numerator $Q(d^m; z)$ of Hilbert series for non–symmetric semigroup $S(d^m)$, $m \geq 4$, that essentially enhances the result obtained in [10].

2 3D Frobenius problem: brief review

Start with the 2D Frobenius problem for which $F(d^2)$, $G(d^2)$ and $Q(d^2; z)$ were known already to J. Sylvester [14]

$$F(d^2) = d_1 d_2 - d_1 - d_2, \quad G(d^2) = \frac{1}{2}(d_1 - 1)(d_2 - 1), \quad Q(d^2; z) = 1 - zd_1 d_2 . \quad (15)$$

Recall some basic results on the Frobenius problem for semigroup $S(d^3)$ following [5], [12], [13] and [15]. Let $S(d^3)$ be a non–symmetric semigroup with the 1st minimal relation $R_1(d^3)$ defined by (8), (9). Such relations always exist due to the finiteness of the Frobenius number $F(d^3)$. Then by [5] the kernel $I_3$ is generated by $p_1, p_2, p_3$, where

$$p_1 = X_1^{a_{11}} - X_2^{a_{12}} X_3^{a_{13}}, \quad p_2 = X_2^{a_{22}} - X_1^{a_{21}} X_3^{a_{23}}, \quad p_3 = X_3^{a_{33}} - X_1^{a_{31}} X_2^{a_{32}}, \quad \tau(p_i) = 0 . \quad (16)$$

Represent (8) as a matrix equation

$$\hat{A}_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{A}_3 = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}, \quad \begin{cases} \gcd(a_{11}, a_{12}, a_{13}) = 1 \\ \gcd(a_{21}, a_{22}, a_{23}) = 1 \\ \gcd(a_{31}, a_{32}, a_{33}) = 1 \end{cases} \quad (17)$$

and establish the standard forms of the matrix $\hat{A}_3$ satisfying (9), (17).

2.1 3D non–symmetric semigroups

First, let all off–diagonal entries of $\hat{A}_3$ be negative integers, $a_{ij} \in \{1, 2, \ldots\}, i \neq j$ i.e. omitting 0. Then, as was shown by Johnson [11], it leads necessarily to the following

$$a_{11} = a_{21} + a_{31} , \quad a_{22} = a_{12} + a_{32} , \quad a_{33} = a_{13} + a_{23} , \quad (18)$$

$$\det \begin{pmatrix} a_{22} & -a_{23} \\ -a_{32} & a_{33} \end{pmatrix} = d_1 , \quad \det \begin{pmatrix} a_{11} & -a_{13} \\ -a_{31} & a_{33} \end{pmatrix} = d_2 , \quad \det \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix} = d_3 . \quad (19)$$

The ordering (1) of integers, $d_1 < d_2 < d_3$, imposes additional constraints on the elements $a_{ij}$

$$a_{11} > a_{12} + a_{13} , \quad a_{22} > a_{23} , \quad a_{33} < a_{31} + a_{32} . \quad (20)$$

Denote $\hat{A}_3$ satisfying (18) and (19) by $\hat{A}_3^{(n)}$ and call it the standard form for non–symmetric semigroup $S(d^3)$. Formula (8) together with (18) and (19) make it possible to show that at least one of the $a_{ii}$ exceeds 2. The proof is obtained by way of contradiction. Let all $a_{ii} = 2$. Then due to (18) we have $a_{ij} = 1, i \neq j$, or in accordance with (8)

$$2d_1 = d_2 + d_3 , \quad 2d_2 = d_3 + d_1 , \quad 2d_3 = d_1 + d_2 \quad \rightarrow \quad d_1 = d_2 = d_3 ,$$

that violates the minimality of $(d_1, d_2, d_3)$. This implies an inequality $a_{11}a_{22}a_{33} \geq 12$. Note that the above consideration does not exclude the possibility that the diagonal elements $a_{ii}$ coincide in
pairs, e.g. $a_{11} = a_{22}$, and, moreover, to completely coincide, $a_{ii} = a$. The latter kind of degeneration reduces significantly the number of different admissible triples $d_1, d_2, d_3$ being a minimal set and satisfying (18), (19) (see Appendix A).

The Frobenius number $F(\mathbf{d}^3)$ for non-symmetric semigroup was found for the first time in [5] (see also [13]) calculating only the largest degree of $H(\mathbf{d}^m; z)$ (without calculating Hilbert series itself)

$$F(\mathbf{d}^3) + \sum_{i=1}^{3} d_i = \max \{a_{11}d_1 + a_{32}d_2; a_{22}d_2 + a_{31}d_1\} = \max \{a_{22}d_2 + a_{13}d_3; a_{33}d_3 + a_{12}d_2\}$$

$$= \max \{a_{33}d_3 + a_{21}d_1; a_{11}d_1 + a_{23}d_3\}.$$ (21)

The genus $G(\mathbf{d}^3)$ of non-symmetric semigroup was calculated in algebraic geometry [12]. Dealing with the singularity degrees of the monomial space curve whose corresponding semigroup is $S(\mathbf{d}^3)$, Kraft [12] was able to calculate its Milnor number $\mu(\mathbf{d}^3)$ which in the unibranch case is twice larger than $G(\mathbf{d}^3)$ and given by (12). Thus, (12) gives a generalization of the Milnor number for the monomial plane curves presented in [16]

$$\mu(\mathbf{d}^2) = 1 + \sum_{i=1}^{2} a_{ii}d_i - \prod_{i=1}^{2} a_{ii} - \sum_{i=1}^{2} d_i = (d_1 - 1)(d_2 - 1), \quad \gcd(d_1, d_2) = 1.$$ (22)

As for Hilbert series, partial progress was achieved by Székely and Wormald [10] who proved that $Q(\mathbf{d}^3; z)$ consists of only a limited number of terms (at most twelve) independent of the values of $d_1, d_2$ and $d_3$. Recently this result was essentially refined by Denham [15] who gave an algorithm to compute the Hilbert series $H(\mathbf{d}^3; z)$ of a graded subring $k[S(\mathbf{d}^3)]$ for non-symmetric semigroups and established a universal property of these series: $Q(\mathbf{d}^3; z)$ has exactly six terms where the first four of them read $1 - \sum_{i=1}^{3} z^{a_{ii}d_i}$. The attempts [17] to extend further the algorithmic procedure to higher $m$ results only in the estimation of the polynomial time of computation of $H(\mathbf{d}^m; z)$.

### 2.2 3D symmetric semigroups

The number of independent entries $a_{ij}$ in (17) can be reduced if at least one off-diagonal element of $\tilde{A}_3$ vanishes, e.g. $a_{13} = 0$ and therefore $a_{11}d_1 = a_{12}d_2$. Due to minimality of the last relation we have from (8) the following equalities and consequently the matrix representation [5]

$$a_{11} = a_{21} = \frac{\lcm(d_1, d_2)}{d_1}, \quad a_{12} = a_{22} = \frac{\lcm(d_1, d_2)}{d_2}, \quad a_{23} = 0, \quad \tilde{A}_3^{(s)} = \begin{pmatrix} a_{11} & -a_{22} & 0 \\ -a_{11} & a_{22} & 0 \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}.$$ (23)

Call $\tilde{A}_3^{(s)}$ the standard form for the symmetric semigroup $S(\mathbf{d}^3)$. The kernel $I_3$ has 2 generators [5]

$$p_1 = -p_2 = X_1^{a_{11}} - X_2^{a_{32}}, \quad p_3 = X_3^{a_{33}} - X_1^{a_{31}}X_2^{a_{32}},$$ (24)

and the Frobenius number $F(\mathbf{d}^3)$ looks like [5]

$$F(\mathbf{d}^3) = a_{11}d_1 + a_{33}d_3 - \sum_{i=1}^{3} d_i = a_{22}d_2 + a_{33}d_3 - \sum_{i=1}^{3} d_i = \lcm(d_1, d_2) + a_{33}d_3 - \sum_{i=1}^{3} d_i.$$ (25)

The corresponding genus $G(\mathbf{d}^3)$ can be also simplified (see formula (142) in Section 5). If $S(\mathbf{d}^3)$ is symmetric semigroup then $k[S(\mathbf{d}^3)]$ is a complete intersection [5] and the Hilbert series $H_s(\mathbf{d}^3; z)$ reads [18]

$$H_s(\mathbf{d}^3; z) = \frac{(1 - z^{a_{22}d_2})(1 - z^{a_{33}d_3})}{(1 - z^{d_1})(1 - z^{d_2})(1 - z^{d_3})}.$$ (26)
It is interesting to interpret (25) in the sense of Johnson’s formula [11] when \( \gcd(d_1, d_2) = k \geq 1 \)

\[
F(d_1, d_2, d_3) = kF\left(\frac{d_1}{k}, \frac{d_2}{k}, d_3\right) + (k - 1)d_3. \tag{27}
\]

Comparison of (25) and (27) gives

\[
kF\left(\frac{d_1}{k}, \frac{d_2}{k}, d_3\right) = \text{lcm}(d_1, d_2) - d_1 - d_2 + (a_{33} - k)d_3. \tag{28}
\]

In case \( \gcd(d_1, d_2) = 1 \), this leads to \( F(d_1, d_2, d_3) = F(d_1, d_2) + (a_{33} - 1)d_3 \). Recalling the inequality (5) for conductors \( c(d^3) \leq c(d^2) \) and their connection with the Frobenius numbers we have \( F(d_1, d_2, d_3) \leq F(d_1, d_2) \) that results together with (15) and (28), in \( a_{33} = 1 \), i.e., \( d_3 \) is representable by \( d_1 \) and \( d_2 \). Thus, every semigroup generated by three pairwise relatively prime elements cannot be symmetric [6].

We finish this Section noting that the minimal set \( \{d_1, d_2, d_3\} \), which generates the semigroup \( S(d^3) \), cannot include 2 as an element. Indeed, assume the opposite, that \( a_1 = 2 \) and the other two \( d_2 < d_3 \) are both odd integers. Then \( d_3 - d_2 \) is divisible by 2, and therefore such set is not minimal in accordance with (2). In the case, when one of \( d_2, d_3 \) does represent an even integer, the claim is clear. Henceforth, we assume that the elements of the minimal set \( d^3 \) satisfy

\[
3 \leq d_1 < d_2 < d_3. \tag{29}
\]

Further generalization of (29) to non–symmetric semigroups \( S(d^m) \) of higher dimension, \( m \geq 4 \), will be given in Section 7.

### 3 Matrix representation of the set \( \Delta(d^2) \)

In this Section we construct the matrix representation of the set \( \Delta(d^2) \) of integers \( t \) which are unrepresentable by \( d_1, d_2 \). We start with the important statement about matrix representation which dates back to A. Brauer [2] and results partly from his discussion with I. Schur.\(^2\)

**Lemma 1** ([2]) Let \( d_1 \) and \( d_2 \) be relatively prime positive integers. Then every positive integer \( s \) not divisible by \( d_1 \) or by \( d_2 \) is representable either in the form \( s = xd_1 + yd_2 \), \( x > 0, y > 0 \) or in the form \( s = d_1d_2 - pd_1 - qd_2 \), \( p > 0, q > 0; p, q \in \mathbb{N} \).

**Definition 1** Let integers \( 2 < d_1 < d_2 \) be given. Define function \( \sigma(p, q) \) as follows

\[
\sigma(p, q) := d_1d_2 - pd_1 - qd_2. \tag{30}
\]

The next Lemma specifies the bounds on the values of \( p \) and \( q \) introduced in Lemma 1 above.

**Lemma 2** Let \( t \) be an integer and \( d_2 > d_1 \), \( \gcd(d_1, d_2) = 1 \). Then \( t \in \Delta(d^2) \) iff \( t \) is uniquely representable as

\[
t = \sigma(p, q), \quad \text{where}
\]

\[
1 \leq p \leq \left\lfloor \frac{d_2 - d_1}{d_1} \right\rfloor, \quad 1 \leq q \leq d_1 - 1, \quad \text{and} \quad d_1 - 1 \leq \left\lfloor \frac{d_2 - d_1}{d_1} \right\rfloor. \tag{31}
\]

\(^2\)We quote from [2]: The Theorems in §3–5 result partly from discussions of Schur and the author. It was formerly intended to publish these results in a joint paper. I conform with Schur’s wishes that the publishing be not longer postponed and that I publish the paper alone. The paper [2] was submitted for publication in November 25, 1940, less than two months before Schur’s death, and was published two years after.
Proof  In accordance with Lemma 1 every integer \( t \) which is unrepresentable by \( d_1, d_2 \), \( \gcd(d_1, d_2) = 1 \) is representable by (30) and (31). Thus, the Frobenius number is \( F(d^2) = \sigma(1, 1) \). The restrictions (32) come from simple considerations

\[
\sigma(p, 1) > 0 \Rightarrow p \leq \left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor, \quad \sigma(1, q) > 0 \Rightarrow q \leq \left\lfloor \frac{d_1 - d_1}{d_2} \right\rfloor = d_1 - 1.
\]

The presentation of \( \sigma(p, q) \) by (30) is unique. A standard proof of uniqueness of (30) is to assume, by way of contradiction, that there are two such representations \( \sigma(p_1, q_1), \sigma(p_2, q_2) \) and consequently,

\[
(p_1 - p_2)d_1 = (q_2 - q_1)d_2, \quad 1 \leq p_1, p_2 \leq \left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor, \quad 1 \leq q_1, q_2 \leq d_1 - 1 \rightarrow |p_1 - p_2| < d_2, \quad |q_2 - q_1| < d_1.
\]

But this is impossible since \( d_1 \) and \( d_2 \) have no common factors.

Finally prove the last inequality in (32). Assuming \( d_2 \geq d_1 + 2 \) we get

\[
\left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor - (d_1 - 1) \geq d_2 - \frac{d_2}{d_1} - 1 -(d_1 - 1) = \left( \frac{d_2 - d_1}{d_1} \right) - 1 \geq 2 \frac{d_1 - 1}{d_1} - 1 = 1 - \frac{2}{d_1} > 0.
\]

In the case \( d_2 = d_1 + 1 \) we obtain

\[
\left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor - (d_1 - 1) = \left\lfloor d_1 + 1 - \frac{d_1 + 1}{d_1} \right\rfloor - (d_1 - 1) = \left\lfloor d_1 - \frac{1}{d_1} \right\rfloor - (d_1 - 1) = 0.
\]

Thus, combining both cases we arrive at (32). This completes the proof of the Lemma.

Show that the integers \( \sigma(p, q) \) given by (31), (32) exhaust all integers unrepresentable by \( d_1 \) and \( d_2 \), or, in other words, they give the genus \( G(d^2) \) obtained by Sylvester [14] and given in (15). Indeed, counting the number of integers \( \sigma(p, q) \) with the above properties (31) and (32) successively over the index set \( q = 1, \ldots, d_1 - 1 \) one gets

\[
G(d^2) = \sum_{q=1}^{d_1-1} \left( q \frac{d_2}{d_1} \right) = \sum_{q=1}^{d_1-1} \left( q \frac{d_2}{d_1} \right) = \frac{1}{2} \left( \sum_{q=1}^{d_1-1} \left( q \frac{d_2}{d_1} \right) + \sum_{q=1}^{d_1-1} \left( (d_1 - q) \frac{d_2}{d_1} \right) \right) = \frac{1}{2} \sum_{q=1}^{d_1-1} \left( \left\lfloor \frac{d_2}{d_1} \right\rfloor + \left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor \right) = \frac{(d_2 - 1)(d_1 - 1)}{2},
\]

that follows from the equalities

\[
\frac{d_2}{d_1} = \left\lfloor \frac{d_2}{d_1} \right\rfloor + \left\{ \frac{d_2}{d_1} \right\}, \quad \left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor = \left\lfloor \frac{d_2 - \left\{ \frac{d_2}{d_1} \right\} }{d_1} \right\rfloor = d_2 - \left\lfloor \frac{d_2}{d_1} \right\rfloor , \quad q < d_1. \quad (33)
\]

In (33) we denote by \( \{ b \} \) the fractional part of a real number \( b \).

The representation (31) of all integers \( \sigma(p, q) \in \Delta(d_1, d_2) \) is called the matrix representation of the set \( \Delta(d_1, d_2) \) and is denoted by \( M \{ \Delta(d^2) \} \) (see Figure 1)

\[
\sigma \{ M \{ \Delta(d^2) \} \} = \Delta(d^2). \quad (34)
\]

\( \sigma(p, q) \) is the integer which occurs in the row \( p \) and the column \( q \) of \( M \{ \Delta(d^2) \} \), e.g. \( \sigma(1, 1) = d_1d_2 - d_1 - d_2 \).

Based on \( M \{ \Delta(d^2) \} \) introduce two sets which will be important in the coming Sections. We call the totality of the lowest cells in every column of \( M \{ \Delta(d^2) \} \) the bottom layer of \( M \{ \Delta(d^2) \} \) and denote it by \( \mathcal{BL}_M \{ \Delta(d^2) \} \). We also call the totality of the highest cells in every column of \( M \{ \Delta(d^2) \} \) the top layer of \( M \{ \Delta(d^2) \} \) and denote it by \( \mathcal{TL}_M \{ \Delta(d^2) \} \).

We establish the structure of both sets \( \mathcal{BL}_M \{ \Delta(d^2) \} \) and \( \mathcal{TL}_M \{ \Delta(d^2) \} \). First, prove the following Lemma.
Figure 1: Typical matrix representation $M \{ \Delta(d^2) \}$ of the set $\Delta(d_1, d_2)$ for the case $d_1 < d_2 < 3/2d_1$. The bottom layer $BL_M \{ \Delta(d^2) \}$ is marked in gray color. The top layer $TL_M \{ \Delta(d^2) \}$ coincides with the highest row of diagram.

**Lemma 3** For every number $k$, $1 \leq k \leq d_1 - 1$ there exists $(p, q) \in BL_M \{ \Delta(d^2) \}$ such that $\sigma(p, q) = k$.

**Proof** Let $1 \leq q \leq d_1 - 1$ and define

$$p_b(q) = \max \{1 \leq p \mid \sigma(p, q) > 0\},$$

where subscript "b" stands for "bottom". We may consider $p_b$ as a function of $q$. Then $(p_b(q), q) \in BL_M \{ \Delta(d^2) \}$. Let us derive the function $p_b(q)$. It follows from (30) that

$$\sigma(p, q) > 0 \rightarrow p < d_2 - \frac{d_2}{d_1} q \rightarrow p_b(q) = \left\lfloor d_2 - \frac{d_2}{d_1} q \right\rfloor.$$

Hence, according to (33) we get $p_b(q) = d_2 - 1 - \left\lfloor \frac{q d_2}{d_1} \right\rfloor$, and further

$$\sigma(p_b(q), q) = d_1 - qd_2 + d_1 \left\lfloor \frac{d_2}{d_1} \right\rfloor = d_1 - d_1 \left\{ \frac{d_2}{d_1} \right\}.$$  

For $1 \leq q \leq d_1 - 1$ we have $1/d_1 \leq \left\{ \frac{q d_2}{d_1} \right\} \leq (d_1 - 1)/d_1$. Combining this with (36) gives the bounds for $\sigma(p_b(q), q)$

$$1 \leq \sigma(p_b, q) \leq d_1 - 1.$$  

Due to the uniqueness of the presentation of $\sigma(p, q)$ by (30), the bounds (37) lead to the conclusion that $BL_M \{ \Delta(d^2) \}$ is occupied exclusively by the integers $1, \ldots, d_1 - 1$ not in a necessarily consecutive order. This proves the Lemma.  

As for the top layer, $TL_M \{ \Delta(d^2) \}$ coincides with the highest row in $M \{ \Delta(d^2) \}$. Thus, finally we can write, in accordance with (34),

$$\sigma \{ BL_M \{ \Delta(d^2) \} \} = \{1, \ldots, d_1 - 1\}, \sigma \{ TL_M \{ \Delta(d^2) \} \} = \{\sigma(1, 1), \ldots, \sigma(1, d_1 - 1)\}.$$  

where $\sigma(1, 1) = F(d^2)$ and $\sigma(1, d_1 - 1) = d_2 - d_1$. 


The proof follows by way of contradiction. Let \( t = d_3 \) there exist \( p_{d_3} \) and \( q_{d_3} \) satisfying (32) such that
\[
d_3 = d_1 d_2 - p_{d_3} d_1 - q_{d_3} d_2 ,
\]
and \( d_3 \) is unrepresentable by \( d_1, d_2 \) (see (30)). Thus, the triple \( \{d_1, d_2, d_3\} \) represents the minimal set generating \( S(d_3) \) in accordance with (2). Define the set \( \Omega_{d_3}^1(d_2) \) of integers \( A_1 \) in \( \Delta(d_2) \) representable by \( d_1, d_2, d_3 \) as follows
\[
\Omega_{d_3}^1(d_2) = \{ A_1 : A_1 = u_1 d_1 + u_2 d_2 + d_3, \quad 0 \leq u_1 \leq p_{d_3} - 1, 0 \leq u_2 \leq q_{d_3} - 1 \} .
\]
(41)
A_1 depends on \( u_1 \) and \( u_2 \), hence we shall write \( A_1 = A_1(u_1, u_2) \). Since \( \Delta(d_3) \) consists of the integers unrepresentable by \( d_1, d_2, d_3 \) it is clear that
\[
\Omega_{d_3}^1(d_2) \cap \Delta(d_3) = \emptyset .
\]
(42)
It follows from (30) that \( A_1(u_1, u_2) = \sigma(p_{d_3} - u_1, q_{d_3} - u_2) \). By expressions (15) and (40) we have \( A_1(p_{d_3} - 1, q_{d_3} - 1) = F(d^2) \). In particular,
\[
F(d^2) \in \Omega_{d_3}^1(d_2) .
\]
(43)
Since \( F(d^2) \) \( \equiv \) \( \max \{ t \in \Delta(d^2) \} \) and \( F(d^2) \) \( \in \) \( \Omega_{d_3}^1(d_2) \) by (43), hence due to \( \Omega_{d_3}^1(d_2) \subset \Delta(d_2) \) we get
\[
\max \{ t \in \Omega_{d_3}^1(d_2) \} = \max \{ t \in \Delta(d^2) \} .
\]
By (5) we have \( \Delta(d^3) \subset \Delta(d^2) \), hence \( \max \{ t \in \Delta(d^3) \} \leq \max \{ t \in \Delta(d^2) \} \). However, \( F(d^2) \) \( \equiv \) \( \max \{ t \in \Delta(d^2) \} \) and \( F(d^2) \) \( \in \) \( \Omega_{d_3}^1(d_2) \) by (43), so it follows from (42) that
\[
\max \{ t \in \Delta(d^3) \} < \max \{ t \in \Delta(d^2) \}
\]
that proves Lemma. \( \square \)

It may happen that the set \( \Omega_{d_3}^1(d_2) \) described in Lemma 4 does not exhaust all elements of \( \Delta(d^2) \) which are representable by \( d_1, d_2, d_3 \).

**Lemma 5** If the integers \( d_3 \) and \( 2d_3 \) are unrepresentable by \( d_1, d_2 \) then \( 2d_3 \notin \Omega_{d_3}^1(d_2) \).

**Proof** The proof follows by way of contradiction. Let \( 2d_3 \in \Omega_{d_3}^1(d_2) \), then due to (41) there exist nonnegative integers \( u_1 \) and \( u_2 \) such that
\[
2d_3 = u_1 d_1 + u_2 d_2 + d_3 , \quad \text{or} \quad d_3 = u_1 d_1 + u_2 d_2 ,
\]
that violates the minimality of \( \{d_1, d_2, d_3\} \). \( \square \)
4.1 Associated sets \( \Omega_{d_3}^k (d^2) \)

In order to account for all integers which contribute to the construction of \( \Delta(d^3) \) we have to extend the set \( \Omega_{d_3}^1 (d^2) \). First, recall from (8) and (9) one of the 1st minimal relations \( \mathcal{R}_1 (d^3) \) for a given \( d^3 \): \( a_{33}d_3 = a_{31}d_1 + a_{32}d_2 \), where \( a_{33} \geq 2 \).

**Definition 2** Let \( d_3 \in \Delta(d^3) \) with representation \( d_3 = d_1d_2 - p_{d_3}d_1 - q_{d_3}d_2 \) where \( p_{d_3} \) and \( q_{d_3} \) satisfy (32). Let \( k \) be a positive integer, \( 1 \leq k < a_{33} \). Define the set \( \Omega_{d_3}^k (d^2) \) of integers \( A_k \) in \( \Delta(d^3) \)

\[
\Omega_{d_3}^k (d^2) = \{ A_k \mid A_k = u_1d_1 + u_2d_2 + kd_3 , 0 \leq u_1 \leq p_{kd_3} - 1, 0 \leq u_2 \leq q_{kd_3} - 1 \} .
\]

(44)

Call \( \Omega_{d_3}^k (d^2) \) a \( kd_3 \)-associated set.

\( A_k \) depends on \( u_1 \) and \( u_2 \), hence we shall write \( A_k = A_k(u_1, u_2) \). Taking \( u_1 = p_{kd_3} - 1, u_2 = q_{kd_3} - 1 \) gives

\[
F(d^2) = A_k(p_{kd_3} - 1, q_{kd_3} - 1) \in \Omega_{d_3}^k (d^2) , \quad 1 \leq k < a_{33} .
\]

(45)

Figure 2: Typical matrix representation \( M \{ \Delta(d^2) \} \) of the set \( \Delta(d^2) \). Matrix representations \( M \{ \Omega_{d_3}^1 (d^2) \} \) of the \( kd_3 \)-associated sets \( \Omega_{d_3}^1 (d^2) \) and \( \Omega_{d_3}^2 (d^2) \) are drawn by dashed lines. Their intersection (gray color) contains the Frobenius number \( F(d^2) \) which is marked by black oval. The integers \( kd_3 \notin \Delta(d^3), 1 \leq k < a_{33} \) are shown by black boxes.

It follows from (45) that the intersection of any two associated sets \( \Omega_{d_3}^j (d^2) \) and \( \Omega_{d_3}^k (d^2) \), \( 1 \leq j, k < a_{33} \) is non-empty set. As follows from (44) the matrix representation \( M \{ \Omega_{d_3}^j (d^2) \} \) of \( \Omega_{d_3}^k (d^2) \) is assigned by the rectangle \( [1, p_{kd_3}] \times [1, q_{kd_3}] \) inside \( M \{ \Delta(d^2) \} \) with cardinality \( \#\Omega_{d_3}^k (d^2) = p_{kd_3}q_{kd_3} \) (see Figure 2).

**Theorem 1** Let \( \{d_1, d_2, d_3\} \) be a minimal generating set of the semigroup \( S(d^3) \) and let \( \sigma \) be an integer \( \sigma \in \Delta(d^2) \) representable by \( d_1, d_2, d_3 \). Then there exists at least one \( k, 1 \leq k < a_{33} \) such that \( \sigma \in \Omega_{d_3}^k (d^2) \).

Before giving the proof of Theorem 1 let us present an auxiliary Lemma based on the theory of restricted partition function \( W(\sigma, d^m) \). Recall the main recursion relation [19] for \( W(\sigma, d^m) \) which gives the number of partitions of \( \sigma \) into positive integers \( d_1, \ldots, d_m \) each not greater than \( \sigma \). Then

\[
W(\sigma, d^m) - W(\sigma - d_m, d^m) = W(\sigma, d^{m-1}) , \quad d^{m-1} = \{d_1, \ldots, d_{m-1}\} .
\]

(46)

**Lemma 6** Let \( \{d_1, \ldots, d_m\} \) be a minimal generating set of \( S(d^m) \) and let \( \Delta(d^m) \) be the corresponding set of unrepresentable integers. If \( \sigma \in \Delta(d^m) \) and \( \sigma - d_k > 0, 1 \leq k \leq m \), then necessarily \( \sigma - d_k \in \Delta(d^m) \).
Proof Assume first that \( k = m \). If \( \overline{\sigma} \in \Delta(d^m) \) then \( W(\overline{\sigma}, d^m) = W(\overline{\sigma}, d^{m-1}) = 0 \) and consequently \( W(\overline{\sigma} - d_m, d^m) = 0 \) due to (46). The latter implies \( \overline{\sigma} - d_m \in \Delta(d^m) \).

Now let \( k \) be arbitrary, \( 1 \leq k < m \). The validity of the relation (46) does not depend on the position of \( d_k \) in the tuple \( d^m \). Thus, resorting the tuple \( d^m \) in such a way that \( d_k \) becomes the last in the list \( d_1, \ldots, d_m \) and repeating the above consideration, we come to the proof of the Lemma.

Note that Lemma 6 states the necessary but not sufficient requirement for \( \overline{\sigma} \), i.e. an opposite implication \( \overline{\sigma} - d_k \in \Delta(d^m) \rightarrow \overline{\sigma} \in \Delta(d^m) \) is not true.

Now we return to the proof of Theorem 1.

Proof of Theorem 1 Let \( R_1(d^3) \) be the 1st minimal relation defined in (9). Then \( kd_3 \in \Delta(d^2) \), \( 1 \leq k < a_{33} \). Consider an integer \( \overline{\sigma} \in \Delta(d^2) \) representable by \( d_1, d_2, d_3 \)

\[
\overline{\sigma} = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \cup \{0\}.
\]  
(47)

It follows from (9) that \( \alpha_3 \) is not divisible by \( a_{33} \), otherwise

\[
\overline{\sigma} = \alpha_1 d_1 + \alpha_2 d_2 + \frac{\alpha_3}{a_{33}} a_{33} d_3 = \left( \alpha_1 + \frac{\alpha_3}{a_{33}} a_{33} \right) d_1 + \left( \alpha_2 + \frac{\alpha_3}{a_{33}} a_3 \right) d_2 \notin \Delta(d^2),
\]

that contradicts our assumption \( \overline{\sigma} \in \Delta(d^2) \).

We are going to show that \( \overline{\sigma} \in \Omega_{d_3}^{a_3}(d^2) \). To this end we have to show

\[
0 \leq \alpha_1 \leq p_{a_{33}d_3} - 1, \quad 0 \leq \alpha_2 \leq q_{a_{33}d_3} - 1.
\]  
(48)

Applying Lemma 6 with \( m = 3 \), \( \alpha_1 \) times with \( k = 1 \) and \( \alpha_2 \) times with \( k = 2 \) we get \( a_{33} d_3 \in \Delta(d^2) \). Consider 2 cases. First, let \( 1 \leq \alpha_3 < a_{33} \), then substituting (40) into (47) we obtain

\[
\overline{\sigma} = \alpha_1 d_1 + \alpha_2 d_2 + d_1 d_2 - p_{a_{33}d_3} d_1 - q_{a_{33}d_3} d_2 = d_1 d_2 - (p_{a_{33}d_3} - \alpha_1) d_1 - (q_{a_{33}d_3} - \alpha_2) d_2.
\]

Applying Lemma 2 to the last representation of \( \overline{\sigma} \) we get

\[
p_{a_{33}d_3} - \alpha_1 \geq 1, \quad q_{a_{33}d_3} - \alpha_2 \geq 1,
\]

and combining this with \( \alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\} \) in (47) one concludes that (48) does hold. This leads to \( \overline{\sigma} \in \Omega_{d_3}^{a_3}(d^2) \) in accordance with Definition 2.

In the second case, consider \( \alpha_3 > a_{33} \) and represent \( a_{33} d_3 \) as follows

\[
\alpha_3 d_3 = a_{33} d_3 \left\lfloor \frac{\alpha_3}{a_{33}} \right\rfloor + a_{33} d_3 \left\{ \frac{\alpha_3}{a_{33}} \right\}.
\]  
(49)

Substituting \( a_{33} d_3 \) from (8) into (49) we get

\[
\alpha_3 d_3 = \left\lfloor \frac{\alpha_3}{a_{33}} \right\rfloor (a_{31} d_1 + a_{32} d_2) + a_{33} \left\{ \frac{\alpha_3}{a_{33}} \right\} d_3.
\]

Further, substituting the above result into (47), we obtain

\[
\overline{\sigma} = \xi_1 d_1 + \xi_2 d_2 + \xi_3 d_3,
\]  
(50)

where

\[
\xi_1 = \alpha_1 + a_{31} \left\lfloor \frac{\alpha_3}{a_{33}} \right\rfloor, \quad \xi_2 = \alpha_2 + a_{32} \left\lfloor \frac{\alpha_3}{a_{33}} \right\rfloor, \quad \xi_3 = a_{33} \left\{ \frac{\alpha_3}{a_{33}} \right\} < a_{33}, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{N} \cup \{0\}.
\]  
(51)

Comparing (50), (51) with (47) one concludes that the second case \( \alpha_3 > a_{33}, \xi_3 < a_{33} \) is reduced to the first one \( \alpha_3 < a_{33} \) and therefore \( \overline{\sigma} \in \Omega_{d_3}^{\xi_3}(d^2) \) with \( \xi_3 \) instead of \( \alpha_3 \). This completes the proof of the Theorem. □

Finally we are ready to prove the main theorem of this Section.
Theorem 2 Let $d^3$ be given, $d^3 = (d_1, d_2, d_3)$, and the 1st minimal relation $R_1(d^3)$ is defined by (8). The set $\Delta(d^3)$ coincides with the complement of the union of all associated sets $\Omega^k_{d_3}(d^2)$ in the set of unrepresentable integers $\Delta(d^2)$ where $k = 1, \ldots, a_{33} - 1$.

$$\Delta(d^3) = \Delta(d^2) \setminus \left\{ \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2) \right\}.$$  \hspace{1cm} (52)

Proof First, we show that

$$\Delta(d^2) \setminus \left\{ \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2) \right\} \subseteq \Delta(d^3).$$  \hspace{1cm} (53)

Let $\sigma \in \Delta(d^2) \setminus \left\{ \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2) \right\}$ and suppose $\sigma \notin \Delta(d^3)$. Then by definition of $\Delta(d^3)$ $\sigma$ is representable by $d_1, d_2, d_3$. Hence, by Theorem 1, we have $\sigma \in \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2)$ that contradicts our assumption on $\sigma$. Consequently, (53) holds true.

Finally we show that

$$\Delta(d^2) \setminus \left\{ \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2) \right\} \supseteq \Delta(d^3).$$  \hspace{1cm} (54)

Let $\sigma \in \Delta(d^3)$. Then $\sigma \in \Delta(d^2)$ by (5). Suppose $\sigma \notin \Delta(d^2) \setminus \left\{ \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2) \right\}$. Then $\sigma \in \bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2)$. But then $\sigma$ is representable by $d_1, d_2, d_3$ by Definition 2 that again contradicts our assumption on $\sigma$. Hence (54) holds true and the Theorem is proved. \hfill \Box

Figure 3: Typical matrix representation $M \{ \Delta(d^3) \}$ of the set $\Delta(d^3)$ (striped area) inside $M \{ \Delta(d^2) \}$. TL$_M \{ \Delta(d^3) \}$ has $a_{33} + 1$ convex corners (gray boxes). The integers $kd_3 \notin \Delta(d^3), 1 \leq k < a_{33}$ occupy the concave corners (black boxes) of the union $\bigcup_{k=1}^{a_{33}-1} \Omega^k_{d_3}(d^2)$, which are adjacent to the concave corners of TL$_M \{ \Delta(d^3) \}$.

Generalizing BL$_M \{ \Delta(d^3) \}$ and TL$_M \{ \Delta(d^3) \}$ on $m = 3$ call the totalities of the lowest and top cells in every column of $M \{ \Delta(d^3) \}$ the bottom and top layers of $M \{ \Delta(d^3) \}$, respectively, with corresponding notations, BL$_M \{ \Delta(d^3) \}$ and TL$_M \{ \Delta(d^3) \}$

$$BL_M \{ \Delta(d^3) \} \subset M \{ \Delta(d^3) \}, \quad TL_M \{ \Delta(d^3) \} \subset M \{ \Delta(d^3) \}.$$  \hspace{1cm} (55)

In Figure 3 we present the typical matrix representation $M \{ \Delta(d^3) \}$ of the set $\Delta(d^3)$ inside $M \{ \Delta(d^2) \}$. The bottom layer BL$_M \{ \Delta(d^3) \}$ of this diagram coincides with BL$_M \{ \Delta(d^2) \}$ presented in Figure 1 (see (68) in Section 5). The top layer TL$_M \{ \Delta(d^3) \}$ of this diagram is much more intricate than TL$_M \{ \Delta(d^2) \}$ given in (38), e.g. TL$_M \{ \Delta(d^3) \}$ has $a_{33} + 1$ convex corners.
5 Diagrammatic calculation on the set $\Delta (d^3)$

A straightforward reconstruction of the Hilbert series $H(d^3; z)$ of a graded ring $k[z^{d_1}, z^{d_2}, z^{d_3}]$ out of the set $\Delta (d^3)$ is a difficult problem. In order to overcome this difficulty we develop the procedure of diagrammatic calculation in $\Delta (d^3)$ in the present Section. This procedure will be applied in Section 6 to calculate $Q(d^3; z)$ and to give a complete solution of the 3D Frobenius problem. The diagrammatic calculation is also useful in higher dimensions $m \geq 4$ and enables us to estimate the upper bound for the number of non–zero coefficients in the polynomial $Q(d^m; z)$ (see Section 7).

The algebraic approach to the Frobenius problem is based on a strong relationship between Hilbert series $H(d^m; z)$ of a graded ring $k[z^{d_1}, \ldots, z^{d_m}]$ over a field of characteristic 0, and the generating function $\Phi(d^m; z)$ for the set $\Delta(d^m)$ [1]

$$H(d^m; z) + \Phi(d^m; z) = \frac{1}{1 - z}, \quad (56)$$

where $\Phi(d^m; z)$ and $H(d^m; z)$ are defined in (4) and (6), respectively. Being evaluated at a special value of $z$ the function $\Phi(d^m; z)$ gives the Frobenius number and the genus of semigroups in any dimension $m$. Indeed, according to the definitions (3) and (7) we have

$$F(d^m) = \deg \Phi(d^m; z), \quad G(d^m) = \sum_{s \in \Delta(d^m)} 1^s = \Phi(d^m; 1). \quad (57)$$

Making use of (6) and (56) formulas (57) can be represented in more analytical way

$$F(d^m) = \deg Q(d^m; z) - \sum_{j=1}^{m} d_j, \quad (58)$$

$$G(d^m) = \lim_{z \to 1} \frac{\prod_{j=1}^{m} (1 - z^{d_j}) - (1 - z)Q(d^m; z)}{(1 - z)\prod_{j=1}^{m} (1 - z^{d_j})} = \frac{\partial^{m+1} \left[ \prod_{j=1}^{m} (1 - z^{d_j}) - (1 - z)Q(d^m; z) \right]}{\partial z^{m+1} \left[ (1 - z)\prod_{j=1}^{m} (1 - z^{d_j}) \right]} \bigg|_{z=1}, \quad (59)$$

where $\partial^{n} = d^n/dz^n$ stands for the usual derivative of $n$th order. As one can see from (58) and (59), the Frobenius problem is reduced to finding the numerator $Q(d^m; z)$ of Hilbert series which follows if one substitutes (6) into (56)

$$Q(d^m; z) = \prod_{j=2}^{m} (1 - z^{d_j}) \left[ \sum_{k=0}^{d_j-1} z^k - \left( 1 - z^{d_1} \right) \Phi(d^m; z) \right]. \quad (60)$$

A straightforward reconstruction of the numerator $Q(d^m; z)$ out of the set $\Delta(d^m)$ is a very difficult problem. The main difficulty arises when we are going to handle the term

$$\sum_{k=0}^{d_1-1} z^k - \left( 1 - z^{d_1} \right) \Phi(d^m; z). \quad (61)$$

However, it appears that in dimension $m = 3$ one can elaborate an effective procedure to calculate $Q(d^3; z)$ via geometrical transformations (shifts) of a diagram of the matrix representation $M \{ \Delta (d^3) \}$. We call such procedure diagrammatic calculation. It turns out, that diagrammatic calculation in dimensions $m = 3$ reduces the determination of $Q(d^3; z)$ to the calculation of $TL_M \{ \Delta (d^3) \}$ and $BL_M \{ \Delta (d^3) \}$ but not of the entire matrix $M \{ \Delta (d^3) \}$. 
First, introduce two functions, \( \tau \) and its inverse \( \tau^{-1} \), where \( \tau \) maps each polynomial \( \sum c_k z^k \in \mathbb{N}[z] \) with \( c_k \in \{0,1\} \) onto the set of degrees \( \{ k \in \mathbb{N} \mid c_k \neq 0 \} \). In particular, it follows from (4)

\[
\tau \left[ \Phi \left( d^3; z \right) \right] = \Delta \left( d^3 \right) \quad \text{and} \quad \tau^{-1} \left[ \Delta \left( d^3 \right) \right] = \Phi \left( d^3; z \right).
\]

(62)

Observe that since all coefficients of the polynomial \( \Phi \left( d^3; z \right) \) are 1 or 0, we can uniquely reconstruct \( \Phi \left( d^3; z \right) \) from \( \Delta \left( d^3 \right) \) and vice versa. In this sense \( \tau \) is an isomorphic map. The map \( \tau \) is also linear in the following sense:

Let \( d^3 \) be given and a set \( \Delta \left( d^3 \right) \) of all unrepresentable integers be related to its generating function \( \Phi \left( d^3; z \right) \) by the isomorphic map \( \tau \) defined in (62). Let two sets \( \Delta_1 \left( d^3 \right) \) and \( \Delta_2 \left( d^3 \right) \) be given such that

\[
\Delta_1 \left( d^3 \right), \Delta_2 \left( d^3 \right) \subset \Delta \left( d^3 \right), \quad \Delta_1 \left( d^3 \right) \cap \Delta_2 \left( d^3 \right) = \emptyset.
\]

(63)

Then the following holds

\[
\tau^{-1} \left[ \Delta_1 \left( d^3 \right) \cup \Delta_2 \left( d^3 \right) \right] = \tau^{-1} \left[ \Delta_1 \left( d^3 \right) \right] + \tau^{-1} \left[ \Delta_2 \left( d^3 \right) \right].
\]

(64)

Recalling (34) and (62) we present below the relations between three main entities \( M \{ \Delta \left( d^3 \right) \} \), \( \Delta \left( d^3 \right) \) and \( \Phi \left( d^3; z \right) \) which are concerned with the integers that are unrepresentable by \( d_1, d_2, d_3 \). These relations are carried out by two maps, \( \sigma \) and \( \tau \),

\[
M \{ \Delta \left( d^3 \right) \} \xrightarrow{\sigma} \Delta \left( d^3 \right) \xleftarrow{\tau} \Phi \left( d^3; z \right).
\]

(65)

5.1 Construction of the set \( \tau \left[ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi \left( d^3; z \right) \right] \)

Introduce an upward shift operator \( \hat{U}_1 \) which shifts the diagram of the matrix representation \( M \{ \Delta \left( d^3 \right) \} \) one step upwards. We define

\[
\hat{U}_1 \sigma(p,q) := \sigma(p-1,q).
\]

(66)

Thus, by (30) \( \sigma(p-1,q) = \sigma(p,q) + d_1 \) and if we denote by \( \hat{U}_1 \Delta \left( d^3 \right) \) the set of all integers \( \hat{U}_1 \sigma(p,q) \) such that \( \sigma(p,q) \in \Delta \left( d^3 \right) \) and define \( \Delta' \left( d^3 \right) = \hat{U}_1 \Delta \left( d^3 \right) \) then \( \Delta' \left( p,q \right) = \Delta \left( p-1,q \right) \) and

\[
\Delta' \left( d^3 \right) = \hat{U}_1 \Delta \left( d^3 \right) = \bigcup_{(p,q) \in M \{ \Delta \left( d^3 \right) \}} \hat{U}_1 \sigma(p,q).
\]

(67)

For the determination of the term (61) via diagrammatic calculation we need the following results.

Let \( \{d_1, d_2, d_3\} \) be the minimal generating set of \( S \left( d^3 \right) \) and let \( \Phi \left( d^3; z \right) \) be the generating function for the set \( \tau \left[ \Phi \left( d^3; z \right) \right] \) of unrepresentable integers. For our purpose here it is important that the construction of \( M \{ \Delta \left( d^3 \right) \} \) via Theorem 2 does not affect \( \text{BL}_M \{ \Delta \left( d^2 \right) \} \) (the gray cells in Figure 1). Indeed, it is clear that the \( d_1-1 \) integers \( 1, \ldots, d_1-1 \) are unrepresentable by \( d_1, d_2, d_3 \). Due to the first equality in (38) this leads to the important result about the bottom layer \( \text{BL}_M \{ \Delta \left( d^3 \right) \} \)

\[
\sigma \left\{ \text{BL}_M \left\{ \tau \left[ \Phi \left( d^3 \right) \right] \right\} \right\} = \sigma \left\{ \text{BL}_M \left\{ \tau \left[ \Phi \left( d^2 \right) \right] \right\} \right\} = \{1,2,\ldots,d_1-1\}.
\]

(68)

Consider the top layer \( \text{TL}_M \{ \tau \left[ \Phi \left( d^3 \right) \right] \} \). It is given by

\[
\sigma \left\{ \text{TL}_M \left\{ \tau \left[ \Phi \left( d^3 \right) \right] \right\} \right\} = \{\sigma \left( p_{33}(q), q \right) \mid q = 1, \ldots, d_1-1 \},
\]

(69)

where subscript "t3" stands for top of \( M \{ \Delta \left( d^3 \right) \} \) and \( p_{33}(q) \) is defined as

\[
p_{33}(q) = \min \left\{1 \leq p \mid \sigma(p,q) \in \Delta \left( d^3 \right) \right\}.
\]
Lemma 7

\[
\sigma \left\{ \text{TLM} \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} = \{ \sigma (p_{i3}(q) - 1, q) \} , \ q = 1, \ldots, d_1 - 1 . \tag{70}
\]

Proof  Consider the polynomial \( z^{d_1} \Phi (d^3) \). By (4), (66) and (67) we obtain

\[
z^{d_1} \Phi (d^3) = \sum_{s \in \Delta (d^3)} z^{s+d_1} = \sum_{s \in \hat{\Delta} (d^3)} z^s . \tag{71}
\]

Acting on it by the map \( \tau \) we get

\[
\tau \left[ z^{d_1} \Phi (d^3) \right] = \hat{\Delta} (d^3) = \hat{\Delta} \tau \left[ \Phi (d^3) \right] . \tag{72}
\]

By (66), (67) and (69) this leads to (70). \( \square \)

Corollary 1  Let \( d^3 \) be given, \( d^3 = (d_1, d_2, d_3) \), and the 1st minimal relation \( R_1 (d^3) \) is defined by (8). Then

\[
kd_3 \in \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} , \ k = 1, \ldots, a_{33} - 1 . \tag{73}
\]

Proof  Let \( R_1 (d^3) \) be the 1st minimal relation for the given \( d^3 \). Then \( kd_3 \notin \Delta (d^3) \), \( 1 \leq k < a_{33} \). First, consider one of such integers, \( kd_3 \), and show that \( kd_3 - d_1 \notin \Delta (d^3) \). Let, by way of contradiction, \( kd_3 - d_1 \notin \Delta (d^3) \), then there exist \( \rho_1, \rho_2, \rho_3 \in \mathbb{N} \cup \{0\} \) such that

\[
k d_3 - d_1 = \rho_1 d_1 + \rho_2 d_2 + \rho_3 d_3 \rightarrow (k - \rho_3)d_3 = (\rho_1 + 1)d_1 + \rho_2 d_2 , \ 1 \leq k \leq a_{33} - 1 ,
\]

violating the minimality of the relation \( R_1 (d^3) \) given by (8). Now for every \( k = 1, \ldots, a_{33} - 1 \) we have

\[
k d_3 \notin \Delta (d^3) \quad \text{and} \quad kd_3 - d_1 \notin \Delta (d^3) . \tag{74}
\]

Comparing (74) with (69) and (70) we conclude that the integers \( kd_3 - d_1, 1 \leq k < a_{33} \) occupy \( \text{TL}_M \left\{ \tau \left[ \Phi (d^3) \right] \right\} \) while the integers \( kd_3, 1 \leq k < a_{33} \) occupy \( \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \). This proves the Corollary. \( \square \)

Figure 4: Matrix representation of two sets: \( \tau \left[ z^{d_1} \Phi (d^3; z) \right] \) (inside plain frame) and \( \tau \left[ \Phi (d^3; z) \right] \) (inside dashed frame). Their intersection \( \Pi_1 (d^3) \) is marked by bright gray color. The top \( \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \) and bottom layer \( \text{BL}_M \left\{ \tau \left[ \Phi (d^3) \right] \right\} \) are marked by dark gray and white colors, respectively. The integers \( kd_3 \in \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} , 1 \leq k < a_{33} \) are shown by black boxes.
In Figure 4 we show the matrix representations of two sets $\tau \left[ z^{d_1} \Phi (d^3; z) \right]$ and $\tau \left[ \Phi (d^3; z) \right]$ with their intersection

$$\Pi_1 (d^3) := \tau \left[ z^{d_1} \Phi (d^3; z) \right] \bigcap \tau \left[ \Phi (d^3; z) \right].$$  

(75)

$M \left\{ \tau \left[ z^{d_1} \Phi (d^3; z) \right] \right\}$ is shifted one step upwards with respect to $M \left\{ \tau \left[ \Phi (d^3; z) \right] \right\}$. From this presentation follows

$$\tau \left[ \Phi (d^3; z) \right] = \Pi_1 (d^3) \bigcup \sigma \left\{ \text{BL}_M \left\{ \tau \left[ \Phi (d^3) \right] \right\} \right\}, \quad \tau \left[ z^{d_1} \Phi (d^3; z) \right] = \Pi_1 (d^3) \bigcup \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\}, \quad \text{and}$$

(76)

(77)

$$\Pi_1 (d^3) \bigcap \sigma \left\{ \text{BL}_M \left\{ \tau \left[ \Phi (d^3) \right] \right\} \right\} = \emptyset, \quad \Pi_1 (d^3) \bigcap \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} = \emptyset.$$  

(78)

Denote the integers occupying the top layer $\text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\}$ by $\lambda_q$. Thus, we have

$$\sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} = \{ \lambda_q \mid \lambda_q = \sigma (p_{d_3}(q) - 1, q), \ 1 \leq q \leq d_1 - 1 \}.$$  

(79)

Now we are ready to prove the main Theorem of this Section

**Theorem 3**

$$\left( 1 - z^{d_1} \right) \Phi (d^3; z) = \sum_{q=1}^{d_1-1} z^q - \sum_{q=1}^{d_1-1} z^{\lambda_q}.$$  

(80)

**Proof** Consider the two polynomials

$$\Phi (d^3; z) = \tau^{-1} \left[ \Delta (d^3) \right], \quad z^{d_1} \Phi (d^3) = \tau^{-1} \left[ \hat{\Delta} (d^3) \right],$$

(81)

and construct their difference $K_1 = \left( 1 - z^{d_1} \right) \Phi (d^3; z)$ acting on (76) and (77) by $\tau^{-1}$

$$K_1 = \tau^{-1} \left[ \Pi_1 (d^3) \bigcup \sigma \left\{ \text{BL}_M \left\{ \tau \left[ \Phi (d^3) \right] \right\} \right\} \right] - \tau^{-1} \left[ \Pi_1 (d^3) \bigcup \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} \right].$$

Making use of (63), (64) and (78) we obtain

$$K_1 = \tau^{-1} \left[ \sigma \left\{ \text{BL}_M \left\{ \tau \left[ \Phi (d^3) \right] \right\} \right\} \right] - \tau^{-1} \left[ \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} \right].$$  

(82)

Substituting (68) and (79) into (82) we come to (80) that finishes the proof of the Theorem. \hfill \Box

In Figure 5 we show the matrix representation of the set $\tau \left[ \left( 1 - z^{d_1} \right) \Phi (d^3; z) \right]$. Finally we arrive at the term (61) which will be calculated in the next Theorem.

**Theorem 4**

$$\sum_{q=0}^{d_1-1} z^q - \left( 1 - z^{d_1} \right) \Phi (d^3; z) = \sum_{q=0}^{d_1-1} z^{\lambda_q}, \quad \lambda_0 = 0.$$  

(83)

**Proof** The proof follows immediately from Theorem 3. \hfill \Box

For application in the next Section we introduce the following notation

$$\Lambda (d^3) := \sigma \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_1} \Phi (d^3) \right] \right\} \right\} \cup \{0\}.$$  

(84)
Figure 5: Matrix representation of the set $\tau \left[ (1 - z^{d_1}) \Phi (d^3; z) \right]$. The signs "-" and "+" in the cells mark the corresponding terms $-z^{\lambda_q}, \lambda_q \in \sigma \{ T_M \{ \tau [z^{d_1} \Phi (d^3)] \} \}$ and $z^q, q = 1, \ldots, d_1 - 1$, respectively, which enter into polynomial (80). The integers $kd_3 \not\in \Delta(d^3), 1 \leq k < a_{33}$ are shown by black boxes.

The basic properties of the set $\Lambda (d^3)$ follow form (79) and (84)

$$\Lambda (d^3) \not\subset \Delta (d^3), \# \Lambda (d^3) = d_1, \Lambda (d^3) = \tau \left[ \sum_{q=0}^{d_1-1} z^{\lambda_q} \right]. \quad (85)$$

The structure of $\Lambda (d^3)$ is very intricate. The set $\Lambda (d^3)$ includes the integers $\lambda_q$ which do not even belong to the set $\Delta (d^2)$. Those are

$$\lambda_q = qd_2, \quad 1 \leq q \leq a_{22} - 1. \quad (86)$$

Indeed, from (38) and Lemma 7 follows $\lambda_q \in \Lambda (d^3)$. On the other hand, (86) means that $\lambda_q$ are representable by $d_2$ and therefore $\lambda_q \not\in \Delta (d^3)$.

5.2 The polynomial $(1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right\}$

In the previous Section we have found the set $\Lambda (d^3)$ of integers $\lambda_q$ which contribute to the polynomial $\sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z)$. Here we continue to construct the numerator $Q(d^3; z)$ according to (60). This will be done by further successive application of diagrammatic calculation on $\Lambda (d^3)$.

The diagrammatic representation of the set $\Lambda (d^3)$ (see Figure 5) is not convenient to deal with. We shall start with a matrix representation of the integers $\lambda_q \in \Lambda (d^3)$ which essentially simplifies the procedure of calculation.

Definition 3 Let integers $2 < d_1 < d_2 < d_3$ be given. Define the function $\lambda(v_2, v_3)$ as follows

$$\lambda(v_2, v_3) := v_2 d_2 + v_3 d_3, \quad v_2, v_3 \in \mathbb{N} \cup \{0\}. \quad (87)$$

The next Lemma specifies the restrictions on the domain of $(v_2, v_3)$ introduced in Definition 3. This is the hardest part of the paper.

Lemma 8 Let $d^3$ be given, $d^3 = (d_1, d_2, d_3)$, with the 1st minimal relation $R_1 (d^3)$ defined by (8). Let $r$ be an integer. Then $r \in \Lambda (d^3)$ iff $r$ is uniquely representable as

$$r = \lambda(v_2, v_3), \quad \text{where}$$

$$(v_2, v_3) \in ([0, a_{22} - 1] \times [0, a_{13} - 1]) \cup ([0, a_{12} - 1] \times [a_{13}, a_{33} - 1]) \quad (89)$$
Proof. Observe that according to (86) and Corollary 1 in Section 5.1 the following holds, respectively
\[ v_2 d_2 \in \Lambda (d^3), \quad 0 \leq v_2 < a_{22} \quad \text{and} \quad v_3 d_3 \in \Lambda (d^3), \quad 0 \leq v_3 < a_{33}. \] (90)

Fix \( v_3 \) such that \( 0 \leq v_3 < a_{33} \) and consider the sequence of integers
\[ v_3 d_3, d_2 + v_3 d_3, 2d_2 + v_3 d_3, \ldots, \overline{v_2} d_2 + v_3 d_3 \in \Omega_{d_3}^{v_3}(d^2), \] (91)
where the maximal element \( \overline{v_2} d_2 + v_3 d_3 \) of the sequence (91) is defined by
\[ \overline{v_2} d_2 + v_3 d_3 = \max \{ v_2 d_2 + v_3 d_3 \mid v_2 d_2 + v_3 d_3 \in \Lambda (d^3), 0 \leq v_2 < a_{22} \}. \] (92)
Note that the integers of the sequence (91) occupy continuously all the cells of the corresponding \( v_3 \)-th horizontal row in the diagram in Figure 5 (from the right to the left, without jumps).

In order to calculate \( \overline{v_2} d_2 + v_3 d_3 \), we should formulate the requirements it has to satisfy. They are based on two facts which follow from (92).

First, according to definition (92) the element \( \overline{v_2} d_2 + v_3 d_3 \) is contained in \( \Lambda (d^3) \).

Second, \( \overline{v_2} d_2 + v_3 d_3 + d_2 \) belongs neither to \( \Lambda (d^3) \) (since the element \( \overline{v_2} d_2 + v_3 d_3 \) is the maximal in the sequence (91)) nor to \( \Delta (d^3) \) (since \( \overline{v_2} d_2 + v_3 d_3 + d_2 \) is representable by \( d_2, d_3 \)).

Recalling definition (84) of the set \( \Lambda (d^3) \) and Lemma 7 we summarize the requirements as follow
\[ \begin{align*}
\overline{v_2} d_2 + v_3 d_3 & \in \Lambda (d^3) \quad \Rightarrow \quad \overline{v_2} d_2 + v_3 d_3 - d_1 \in \Delta (d^3), \quad (93) \\
\{ (\overline{v_2} + 1) d_2 + v_3 d_3 \notin \Lambda (d^3), \quad (\overline{v_2} + 1) d_2 + v_3 d_3 \notin \Delta (d^3) \} \quad \Rightarrow \quad (\overline{v_2} + 1) d_2 + v_3 d_3 - d_1 \notin \Delta (d^3). \quad (94)
\end{align*} \]

The requirement (94) provides the following representation
\[ (\overline{v_2} + 1) d_2 + v_3 d_3 - d_1 = \gamma_1 d_1 + \gamma_2 d_2 + \gamma_3 d_3 \quad \text{for some} \quad \gamma_1, \gamma_2, \gamma_3 \in \mathbb{N} \cup \{0\}. \] (95)

On the other hand, the integer \( \overline{v_2} d_2 + v_3 d_3 - d_1 \) is not representable by \( d_1, d_2, d_3 \) due to (93). This can happen only if \( \gamma_2 = 0 \), otherwise \( \overline{v_2} d_2 + v_3 d_3 - d_1 \) is always representable due to (95)
\[ \overline{v_2} d_2 + v_3 d_3 - d_1 = \gamma_1 d_1 + (\gamma_2 - 1) d_2 + \gamma_3 d_3, \quad \gamma_2 \geq 1. \]

Return to (95) and consider its solution, inserting \( \gamma_2 = 0 \).

First, consider \( v_3 \) in the interval \( 0 \leq v_3 < a_{13} \) and rewrite (95) in the form
\[ (\overline{v_2} + 1) d_2 = (\gamma_1 + 1) d_1 + (\gamma_3 - v_3) d_3. \] (96)
Comparing it with the 1st minimal relation \( a_{22} d_2 = a_{21} d_1 + a_{23} d_3 \) we get, by uniqueness (see (9)), the maximal value \( \overline{v_2} \)
\[ \overline{v_2} = a_{22} - 1, \quad \gamma_1 = a_{21} - 1, \quad \gamma_3 = v_3 + a_{23}. \] (97)

Thus, the first kind of the integers \( \lambda(v_2, v_3) \in \Lambda (d^3) \) has a representation
\[ \lambda(v_2, v_3) = v_2 d_2 + v_3 d_3 \quad \text{where} \quad ([0, a_{22} - 1] \times [0, a_{13} - 1]). \] (98)

Next, consider the solution of (95) for \( v_3 \) in the interval \( a_{13} \leq v_3 < a_{33} \). Summation of (95) and the 1st minimal relation \( a_{11} d_1 - a_{12} d_2 - a_{13} d_3 = 0 \) leads to the identity
\[ a_{11} d_1 + (\overline{v_2} - 1) d_2 + (v_3 - a_{13}) d_3 = (\gamma_1 + 1) d_1 + \gamma_3 d_3, \]
which, by uniqueness, gives the maximal value \( \overline{v_2} \)
\[ \overline{v_2} = a_{12} - 1, \quad \gamma_1 = a_{11} - 1, \quad \gamma_3 = v_3 - a_{13}. \] (99)
Thus, the second kind of the integers \( \lambda(v_2, v_3) \in \Lambda(d^3) \) has a representation
\[
\lambda(v_2, v_3) = v_2d_2 + v_3d_3 , \quad \text{where} \quad ([0, a_{12} - 1] \times [a_{13}, a_{33} - 1]) .
\]
(100)

Combining (98) and (100) we come to (89).

Finally, it remains to prove the uniqueness of (88). Indeed, the presentation of \( \lambda(v_2, v_3) \) by (88) is unique. A standard proof of uniqueness is to assume, by way of contradiction, that there are two such representations \( \lambda(v_2^b, v_3^b) \), and \( \lambda(v_2^a, v_3^a) \), \( v_2^b \neq v_2^a, v_3^a \neq v_3^b \), and consequently,
\[
(v_2^b - v_2^a)d_2 = (v_3^a - v_3^b)d_3 .
\]
(101)

Making use of (19) \( a_{22}a_{11} = d_3 - a_{12}a_{21} \rightarrow a_{22} < d_3 \) and \( a_{33}a_{11} = d_2 - a_{13}a_{31} \rightarrow a_{33} < d_2 \), and recalling the necessary constraints (90) imposed on \( v_2, v_3 \) we get
\[
|v_2^b - v_2^a| < d_3 , \quad |v_3^a - v_3^b| < d_2 .
\]

Thus, we come to the conclusion that (101) has no nontrivial solutions, since \( d_2 \) and \( d_3 \) have no common factors. This completes the proof of the Lemma. \( \square \)

Note that due to uniqueness of the matrix representation (87) of all integers \( \lambda(v_2, v_3) \in \Lambda(d^3) \) the following inequalities hold for non–symmetric semigroups
\[
a_{ii}d_i \neq a_{jj}d_j \quad \text{for} \quad i \neq j , \quad 1 \leq i, j \leq 3 .
\]
(102)

The case of symmetric semigroups admits only one equality in (102) (see Section 6.2 for details).

The representation (88) of all integers \( \lambda(v_2, v_3) \in \Lambda(d^3) \) is called the matrix representation of the set \( \Lambda(d^3) \) and is denoted by \( M\{\Lambda(d^3)\} \) (see Figure 6)
\[
\lambda \{M\{\Lambda(d^3)\}\} = \Lambda(d^3) .
\]
(103)

\( \lambda(v_2, v_3) \) is the integer which occurs in row \( v_2 \) and column \( v_3 \) of \( M\{\Lambda(d^3)\} \).

Figure 6: Typical matrix representation \( M\{\Lambda(d^3)\} \) of the set \( \Lambda(d^3) \). All cells in the diagram, which are marked in white color, are occupied by integers \( \lambda(v_2, v_3) \in \Lambda(d^3) \). The integers \( a_{ii}d_i \notin \Lambda(d^3) , i = 1, 2, 3 \), occupy three cells marked in gray color.

In Figure 6 we show the matrix representation \( M\{\Lambda(d^3)\} \) of the set \( \Lambda(d^3) \) for the non–symmetric semigroup \( S(d^3) \). This diagram appeared for the first time in [20] for algorithmic calculation of \( F(d^3) \). Later it was also used in [6] and [21] for the same purpose.

Lemma 8 has also two interesting corollaries related to the diagram in Figures 5 and 6.

**Corollary 2** The length of horizontal rows in the matrix representation \( T_L\{\tau[z^{d_1}\Phi(d^3)]\} \) in Figure 5, which are covered continuously by integers \( \lambda(v_2, v_3) \), is either \( a_{22} - 1 \), \( a_{22} \) or \( a_{12} \) (defined in unit cells).
Proof Consider the upper horizontal row of the matrix representation $\mathbf{T} \mathbf{L}_M \{ \tau \left[ z^d \Phi(d^3) \right] \}$ in Figure 5. According to (86) this is the unique row of the length $a_{22} - 1$ (in unit cells) covered continuously by integers $\lambda_q = qd_2, 1 \leq q < a_{22}$ which does not belong to $\Delta(d^2)$. All the other rows are contained in $\Delta(d^2)$ and in accordance with Corollary 1 their furthest right cells are occupied by one of the integers $kd_3, 1 \leq k < a_{33}$ (see Figure 5). Observe that all these horizontal rows are mapped in one to one manner into vertical columns in the diagram of the matrix representation $M \{ \Lambda(d^3) \}$ (see Figure 6). Thus, we conclude in accordance with Lemma 8 that their length is either $a_{22}$ or $a_{12}$ (defined in unit cells).

Corollary 3 Let $\{d_1, d_2, d_3\}$ be a minimal generating set of a non-symmetric semigroup $\mathfrak{S}(d^3)$ and let the 1st minimal relation be defined by (8). Then

$$a_{22} + a_{33} \leq d_1 + 1 \leq a_{22}a_{33} , \quad a_{33} + a_{11} \leq d_2 + 1 \leq a_{33}a_{11} , \quad a_{11} + a_{22} \leq d_3 + 1 \leq a_{11}a_{22} . \quad (104)$$

Proof The right hand sides of (104) follow from (19):

$$a_{22}a_{33} = d_1 + a_{23}a_{32} \geq d_1 + 1 , \quad a_{33}a_{11} = d_2 + a_{31}a_{13} \geq d_2 + 1 , \quad a_{11}a_{22} = d_3 + a_{12}a_{21} \geq d_3 + 1 .$$

The proof of the left hand sides of (104) follows from (18) and (19), e.g.

$$d_1 + 1 - (a_{22} + a_{33}) = a_{22}a_{33} - a_{23}a_{32} + 1 - (a_{22} + a_{33}) = a_{22}a_{33} - (a_{22} - a_{12})(a_{33} - a_{13}) + 1 - (a_{22} + a_{33}) = a_{22}(a_{13} - 1) + a_{33}(a_{12} - 1) - a_{12}a_{13} \geq 1 + (a_{12} + 1)(a_{13} - 1) + (a_{13} + 1)(a_{12} - 1) - a_{12}a_{13} = a_{12}a_{13} - 1 \geq 0 .$$

Thus, the Corollary is proved.

Notice that the relations (104) are survived as invariants under permutations of the elements $d_i$ in the generating set $\{d_1, d_2, d_3\}$. This is not completely obvious from the first glance since the ordering, $d_1 < d_2 < d_3$, should break such invariance.

We move on to the calculation of the polynomial $(1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1 - 1} z^k - (1 - z^{d_1}) \Phi(d^3; z) \right\}$ and apply the technique of diagrammatic calculation in the same way as it was done in Section 5.1. For this purpose call the totality of the lowest and top cells in every column of $M \{ \Lambda(d^3) \}$ the bottom and top layers of $M \{ \Lambda(d^3) \}$, respectively, and denote them $\mathbf{B} \mathbf{L}_M \{ \Lambda(d^3) \}$ and $\mathbf{T} \mathbf{L}_M \{ \Lambda(d^3) \}$, correspondingly. As one can see from Figure 6

$$\lambda \{ \mathbf{B} \mathbf{L}_M \{ \Lambda(d^3) \} \} = \{0, d_3, \ldots, (a_{33} - 1)d_3\} , \quad (105)$$

$$\lambda \{ \mathbf{T} \mathbf{L}_M \{ \Lambda(d^3) \} \} = \{ \lambda(a_{22} - 1, v_3); 0 \leq v_3 < a_{13} \} \cup \{ \lambda(a_{12} - 1, v_3); a_{13} \leq v_3 < a_{33} \} \quad (106)$$

Introduce an upward shift operator $\hat{U}_2$ which shifts the diagram of the matrix representation $M \{ \Lambda(d^3) \}$ one step upwards. We define

$$\hat{U}_2 \lambda(v_2, v_3) = \lambda(v_2 + 1, v_3) . \quad (107)$$

Thus, by (87) $\lambda(v_2 + 1, v_3) = \lambda(v_2, v_3) + d_2$ and if we denote by $\hat{U}_2 \Lambda(d^3)$ the set of all integers $\lambda(v_2, v_3)$ such that $\lambda(v_2, v_3) \in \Lambda(d^3)$ and define $\Lambda'(d^3) = \hat{U}_2 \Lambda(d^3)$ then $\Lambda'(v_2, v_3) = \Lambda(v_2 + 1, v_3)$ and

$$\Lambda'(d^3) = \hat{U}_2 \Lambda(d^3) = \bigcup_{(v_2, v_3) \in M(\Lambda(d^3))} \hat{U}_2 \lambda(v_2, v_3) . \quad (108)$$

Let $\{d_1, d_2, d_3\}$ be a minimal generating set of $\mathfrak{S}(d^3)$ and let $\Phi(d^3; z)$ be a generating function for the set $\tau \left[ \Phi(d^3; z) \right] \} of unrepresentable integers. This implies, by Theorem 4, that $\sum_{q=0}^{d_1 - 1} z^q - (1 - z^{d_1}) \Phi(d^3; z)$ is a generating function for the set $\Lambda(d^3)$.
Show that (115) necessarily follows from (116), (117) and (114). Indeed, a straightforward calculation gives

\[ \lambda \left\{ \text{TL}_M \left\{ \tau \left[ z^{d_2} \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right) \right] \right\} \right\} = \Gamma_1(d^3) \cup \Gamma_2(d^3), \]  

where

\[ \Gamma_1(d^3) = \{ \lambda(a_{22}, v_3); 0 \leq v_3 < a_{13} \}, \quad \Gamma_2(d^3) = \{ \lambda(a_{12}, v_3); a_{13} \leq v_3 < a_{33} \}. \]

Proof Consider the polynomial \( z^{d_2} \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right) \). By (83), (107) and (108) we obtain

\[ z^{d_2} \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right) = \sum_{q=0}^{d_1-1} z^{\lambda_q + d_2} = \sum_{\lambda \in \Lambda(d^3)} z^\lambda = \sum_{\lambda \in \tilde{\Lambda}(d^3)} z^\lambda. \]

Acting on it by the map \( \tau \) we get

\[ \tau \left[ z^{d_2} \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right) \right] = \tilde{\Lambda}(d^3). \]

Thus, the proof of (109) is reduced to finding a set of the integers \( \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\} \) occupying the top layer of the matrix representation \( \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \). Making successive use of (107), (108) and (106) we obtain

\[ \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\} = \{ \lambda(a_{22}, v_3); 0 \leq v_3 < a_{13} \} \cup \{ \lambda(a_{12}, v_3); a_{13} \leq v_3 < a_{33} \}. \]

Introducing in accordance with (110) the notations of two non–intersecting sets \( \Gamma_1(d^3) \) and \( \Gamma_2(d^3) \), \( \Gamma_1(d^3) \cap \Gamma_2(d^3) = \emptyset \), we arrive at the proof of the Lemma.

Note that according to (105) and (113)

\[ \lambda \left\{ \text{BL}_M \left\{ \Lambda(d^3) \right\} \right\} \cap \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\} = \emptyset. \]

Denote the intersection of the sets \( \Lambda(d^3) \) and \( \tilde{\Lambda}(d^3) \) by \( \Pi_2(d^3) \)

\[ \Pi_2(d^3) := \Lambda(d^3) \cap \tilde{\Lambda}(d^3). \]

Observe that the following presentation holds:

\[ \Lambda (d^3) = \Pi_2 (d^3) \cup \lambda \left\{ \text{BL}_M \left\{ \Lambda(d^3) \right\} \right\}, \]

\[ \tilde{\Lambda}(d^3) = \Pi_2 (d^3) \cup \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\}, \]

where

\[ \Pi_2 (d^3) \cap \lambda \left\{ \text{BL}_M \left\{ \Lambda(d^3) \right\} \right\} = \emptyset, \quad \Pi_2 (d^3) \cap \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\} = \emptyset. \]

Show that (115) necessarily follows from (116), (117) and (114). Indeed, a straightforward calculation gives

\[ \Lambda (d^3) \cap \tilde{\Lambda}(d^3) = \left( \Pi_2 (d^3) \cup \lambda \left\{ \text{BL}_M \left\{ \Lambda(d^3) \right\} \right\} \right) \cap \left( \Pi_2 (d^3) \cup \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\} \right) = \Pi_2 (d^3) \cap \lambda \left\{ \text{BL}_M \left\{ \Lambda(d^3) \right\} \right\} \cap \lambda \left\{ \text{TL}_M \left\{ \tilde{\Lambda}(d^3) \right\} \right\} = \Pi_2 (d^3). \]

Prove the important theorem.
\textbf{Theorem 5}

\[ (1 - z^{d_2}) \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right) = \sum_{\lambda \in \lambda(BL_M \{\Lambda(d^3)\})} z^\lambda - \sum_{\lambda \in \lambda(TL_M \{U_2\Lambda(d^3)\})} z^\lambda. \]  \hspace{1cm} (119)

\textbf{Proof} \hspace{0.5cm} The proof is similar to that given in Theorem 3. Consider the polynomials

\[ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) = \tau^{-1} \left[ \Lambda(d^3) \right], \quad z^{d_2} \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right) = \tau^{-1} \left[ \bar{U}_2\Lambda(d^3) \right] \]

and construct their difference \( K_2 = (1 - z^{d_2}) \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right) \) acting on (116) and (117) by \( \tau^{-1} \)

\[ K_2 = \tau^{-1} \left[ \Pi_2(d^3) \cup \lambda \{BL_M \{\Lambda(d^3)\}\} \right] - \tau^{-1} \left[ \Pi_2(d^3) \cup \lambda \{TL_M \{U_2\Lambda(d^3)\}\} \right]. \]

Making use of (118) and (63), (64) we obtain

\[ K_2 = \tau^{-1} \left[ \lambda \{BL_M \{\Lambda(d^3)\}\} \right] - \tau^{-1} \left[ \lambda \{TL_M \{U_2\Lambda(d^3)\}\} \right], \]  \hspace{1cm} (120)

that leads to (119) in accordance with definition (62) of the inverse map \( \tau^{-1} \). \hfill \Box

The result of diagrammatic calculation is shown in Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Matrix representation of the set \( \tau \left[ (1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right\} \right] \) for semigroup \( S(d^3) \). Positive and negative contributions to polynomial (119) of the terms \( z^{\lambda_q} \) with \( \lambda_q \) occupying the cells are marked in gray and white colors, respectively.}
\end{figure}

\subsection{5.3 The polynomial \( (1 - z^{d_3})(1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right\} \)}

In this Section we finish to calculate \( (1 - z^{d_3})(1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right\} \) relying on the results obtained in Section 5.2.

Let \( \{d_1, d_2, d_3\} \) be a minimal generating set of \( S(d^3) \) and let \( \vphi(d^3; z) \) be the generating function for the set \( \tau[\vphi(d^3; z)] \) of unrepresentable integers.

\textbf{Theorem 6}

\[ Q(d^3; z) = \left( 1 - z^{d_3} \right) \left( 1 - z^{d_2} \right) \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \vphi(d^3; z) \right) = 1 - \sum_{i=1}^{3} z^{a_i d_i} + z L_1 + z L_2 \]  \hspace{1cm} (121)

where

\[ L_1 = a_{12} d_2 + a_{33} d_3, \quad L_2 = a_{22} d_2 + a_{13} d_3. \]  \hspace{1cm} (122)
Proof. The first equality is due to (60). By Theorem 5 and Lemma 9 we obtain
\[
(1 - z^{d_3}) (1 - z^{d_2}) \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right) = T_1 - T_2 - T_3 ,
\]
where
\[
T_1 = (1 - z^{d_3}) \sum_{\lambda \in \lambda \{BL_M \{\Lambda (d^3)\}\}} z^\lambda , \quad T_2 = (1 - z^{d_3}) \sum_{\lambda \in \lambda (d^3)} z^\lambda , \quad T_3 = (1 - z^{d_3}) \sum_{\lambda \in \lambda (d^3)} z^\lambda ,
\]
and the sets \( \{BL_M \{\Lambda (d^3)\}\} \) and \( \Gamma_1 (d^3) \), \( \Gamma_2 (d^3) \) are given in (105) and (110), respectively. Calculating the terms \( T_1, T_2 \) and \( T_3 \) separately we get
\[
T_1 = (1 - z^{d_3}) \sum_{k=0}^{a_{33}-1} z^{kd_3} = 1 - z^{a_{33}d_3} , \quad T_2 = (1 - z^{d_3}) \sum_{k=0}^{a_{13}-1} z^{a_{22}d_2+kd_3} = z^{a_{22}d_2} - z^{a_{22}d_2+a_{13}d_3} ,
\]
\[
T_3 = (1 - z^{d_3}) \sum_{k=a_{13}}^{a_{22}d_2+a_{33}d_3} z^{a_{12}d_2+kd_3} = z^{a_{11}d_1} - z^{a_{12}d_2+a_{33}d_3} .
\]
Substituting (124) into (123) we arrive at (121). \( \square \)

The matrix representation of the set \( \tau \left[ (1 - z^{d_3}) (1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right\} \right] \) for non–symmetric semigroup \( S (d^3) \) is shown at Figure 8.

![Figure 8: Matrix representation of the set \( \tau \left[ (1 - z^{d_3}) (1 - z^{d_2}) \left\{ \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi (d^3; z) \right\} \right] \) for non–symmetric semigroup \( S (d^3) \). Positive and negative contributions of six terms \( z^\lambda \), \( \lambda = 0, a_{11}d_1, a_{22}d_2, a_{33}d_3, L_1, L_2 \) to the polynomial (121) are marked in gray and white colors, respectively.](image)

Observe that the number of the terms contributing to (121) coincides with the number of corners of the polygon, which assigned the matrix representation of the set \( \Lambda (d^3) \) (see Figure 6).

Below we consider two important results on the integers \( L_1, L_2 \) defined in (122). Let \( d^3 \) be given, \( d^3 = (d_1, d_2, d_3) \), and the 1st minimal relation \( \mathcal{R}_1 (d^3) \) for semigroup \( S (d^3) \) is defined by (8). We show that
\[
L_1 \neq L_2 .
\]
Assume, by way of contradiction, that the opposite is true, \( L_1 = L_2 \). Then by (122) \( a_{12}d_2 + a_{33}d_3 = a_{22}d_2 + a_{13}d_3 \), hence \( (a_{22} - a_{12})d_2 = (a_{33} - a_{13})d_3 \). By (18) \( a_{22} - a_{12} = a_{32} \) and \( a_{33} - a_{13} = a_{23} \), hence \( a_{32}d_2 = a_{23}d_3 \). Also by (18) \( a_{32} < a_{33} \) and \( a_{32} < a_{22} \). But then \( a_{32}d_2 \) and \( a_{23}d_3 \) be in \( \Lambda (d^3) \). Hence by the uniqueness of representation of elements of \( \Lambda (d^3) \) (see Lemma 8) we have \( a_{32} = a_{23} = 0 \). Consider the non–symmetric semigroup \( S (d^3) \) with \( a_{23} = a_{32} = 0 \). The matrix \( \mathcal{A}_3 \) of the 1st minimal relation \( \mathcal{R}_1 (d^3) \) has necessarily \( a_{13} = a_{12} = 0 \) that leads to \( a_{11} = 0 \) and contradicts (9). Note that \( L_1 \neq L_2 \) holds for non–symmetric and symmetric semigroups as well.

The next Lemma is related to non–symmetric semigroups \( S (d^3) \) only.
Lemma 10

\[ L_1 \geq a_{11}d_1 + d_3, \quad L_1 \geq a_{33}d_3 + d_2, \quad L_1 \geq a_{22}d_2 + d_1, \]
\[ L_2 \geq a_{11}d_1 + d_2, \quad L_2 \geq a_{33}d_3 + d_1, \quad L_2 \geq a_{22}d_2 + d_3. \]  

(126)

Proof  It clearly follows from diagram in Figure 8

\[ L_1 \geq a_{11}d_1 + d_3, \quad L_1 \geq a_{33}d_3 + d_2 \quad \text{and} \quad L_2 \geq a_{11}d_1 + d_2, \quad L_2 \geq a_{22}d_2 + d_3. \]

One can show that the rest two inequalities, \( L_1 \geq a_{22}d_2 + d_1 \) and \( L_2 \geq a_{33}d_3 + d_1 \), are also true. Indeed, in accordance with (122) we have

\[ L_1 - a_{22}d_2 = a_{12}d_2 + a_{33}d_3 - a_{22}d_2 = a_{33}d_3 - a_{32}d_2 = a_{31}d_1 \geq d_1. \]

In the last two equalities we used (18). The last inequality, \( L_2 \geq a_{33}d_3 + d_1 \), can be proved in the similar manner. \( \Box \)

Both results, (125) and Lemma 10, will be used later, in Section 6.

6 Hilbert series, Frobenius number and genus of monomial curve

In this Section we give a complete solution of the 3D Frobenius problem, i.e. calculate the numerator \( Q(d^3; z) \) of Hilbert series for both non–symmetric and symmetric semigroups \( S(d^3) \) and on its basis determine the Frobenius number and genus.

6.1 Frobenius problem for non–symmetric semigroup \( S(d^3) \)

Hilbert series \( H(d^3; z) \), the Frobenius number \( F(d^3) \) and genus \( G(d^3) \) of semigroup \( S(d^3) \) are invariants under permutations of elements \( d_i \) in the generating set \( \{d_1, d_2, d_3\} \). Therefore we have to find a more symmetrical representation for the integers \( L_1 \) and \( L_2 \) which were defined in (122). This leads to the main Theorem of the Section.

Theorem 7 Let \( d^3 \) be given, \( d^3 = (d_1, d_2, d_3) \), and the 1st minimal relation \( R_1(d^3) \) for non–symmetric semigroup \( S(d^3) \) be defined by (8). Then the numerator \( Q(d^3; z) \) of Hilbert series reads

\[ Q(d^3; z) = 1 - \sum_{i=1}^{3} z^{a_{ii}d_i} + z^{1/2[\langle a, d \rangle - J(d^3)]} + z^{1/2[\langle a, d \rangle + J(d^3)]}, \]  

(127)

\[ J^2(d^3) = \langle a, d \rangle^2 - 4 \sum_{i>j} a_{ii}a_{jj}d_id_j + 4d_1d_2d_3, \quad \langle a, d \rangle = \sum_{i=1}^{3} a_{ii}d_i. \]  

(128)

Proof  Making use of (18) and (19) for the matrix \( \widetilde{\mathcal{A}}_3^{(n)} \) of the 1st minimal relation for non–symmetric semigroup \( S(d^3) \) observe that the integers \( L_1 \) and \( L_2 \), defined in (122), satisfy

\[ L_1L_2 = \sum_{i>j} a_{ii}a_{jj}d_id_j - d_1d_2d_3, \quad L_1 + L_2 = \langle a, d \rangle, \quad \langle a, d \rangle = \sum_{i=1}^{3} a_{ii}d_i. \]  

(129)

In other words, \( L_1 \) and \( L_2 \) are the solutions of quadratic equation

\[ L_{1,2}^2 - \langle a, d \rangle L_{1,2} + \sum_{i>j} a_{ii}a_{jj}d_id_j - d_1d_2d_3 = 0. \]  

(130)
Therefore we have

$$L_{1,2} = \frac{1}{2} \left[ \langle \mathbf{a}, \mathbf{d} \rangle \pm J(d^3) \right] , \quad J(d^3) = \sqrt{\langle \mathbf{a}, \mathbf{d} \rangle^2 - 4 \sum_{i>j} a_i a_j d_i d_j + 4d_1 d_2 d_3} ,$$

(131)

that implies

$$\langle \mathbf{a}, \mathbf{d} \rangle - J(d^3) = \min \{2L_1, 2L_2\} , \quad \langle \mathbf{a}, \mathbf{d} \rangle + J(d^3) = \max \{2L_1, 2L_2\} .$$

(132)

Recalling the expression (60) for the numerator $Q(d^{3n}; z)$ of Hilbert series and inserting (129) into (121) we come to (127) that proves the Theorem. □

**Theorem 8** The Frobenius number $F(d^3)$ and genus $G(d^3)$ of non-symmetric semigroup $S(d^3)$ read, respectively

$$F(d^3) = \frac{1}{2} \left[ \langle \mathbf{a}, \mathbf{d} \rangle + J(d^3) \right] - \sum_{i=1}^{3} d_i , \quad G(d^3) = \frac{1}{2} \left( 1 + \langle \mathbf{a}, \mathbf{d} \rangle - \sum_{i=1}^{3} d_i - \prod_{i=1}^{3} a_i \right) .$$

(133)

Proof Implementation of formulas (58) and (59) for $Q(d^{3n}; z)$ given by (127) leads to (133). □

In Theorem 7 the number $J(d^3)$ was considered as a positive integer such that the degrees $L_1$ and $L_2$ of two last terms in (127) are positive integers. In other words, it was also presumed that the numbers $\langle \mathbf{a}, \mathbf{d} \rangle \pm J(d^3)$ are even positive integers. Here we are going to prove these statements in the case of non-symmetric semigroup $S(d^3)$.

**Lemma 11** Let $d^3$ be given, $d^3 = (d_1, d_2, d_3)$, and the 1st minimal relation $R_1(d^3)$ for non-symmetric semigroup $S(d^3)$ be defined by (8). Then the numbers $J(d^3)$ and $1/2 \left[ \langle \mathbf{a}, \mathbf{d} \rangle \pm J(d^3) \right]$ are non-negative and positive integers, respectively.

$$J(d^3) = |a_{12} a_{23} a_{31} - a_{13} a_{32} a_{21}| ,$$

(134)

$$\frac{1}{2} \left[ \langle \mathbf{a}, \mathbf{d} \rangle \pm J(d^3) \right] = a_{11} a_{22} a_{33} + \frac{1}{2} \left( a_{12} a_{23} a_{31} + a_{13} a_{32} a_{21} \pm |a_{12} a_{23} a_{31} - a_{13} a_{32} a_{21}| \right) .$$

(135)

Proof Inserting relations (18) and (19) into the expressions $J^2(d^3)$ and $\langle \mathbf{a}, \mathbf{d} \rangle$ given in (128) and making use of equality $\det \hat{\mathbf{A}}_3^{(3)} = 0$ which leads to

$$a_{11} a_{22} a_{33} - (a_{11} a_{23} a_{32} + a_{22} a_{13} a_{31} + a_{33} a_{12} a_{21}) = a_{12} a_{23} a_{31} + a_{13} a_{32} a_{21} ,$$

we obtain

$$J^2(d^3) = (a_{12} a_{23} a_{31} - a_{13} a_{32} a_{21})^2 , \quad \langle \mathbf{a}, \mathbf{d} \rangle = 2a_{11} a_{22} a_{33} + a_{13} a_{32} a_{21} + a_{12} a_{23} a_{31} .$$

(136)

The last relations lead to (134) and (135). □

**Corollary 4** Let $\{d_1, d_2, d_3\}$ be a minimal generating set of a non-symmetric semigroup. Then

$$J(d^3) \geq 1 .$$

(137)

Proof According to Lemma 11 the number $J(d^3)$ is a non-negative integer. On the other hand, due to (125) and (130) $J(d^3)$ does not vanish, that leads to (137). □

The unity in (137) is best possible, as the following Example shows.

**Example 1** The triple $\{3, 4, 5\}$ generates a non-symmetric semigroup (with minimal possible $d_i$).

$$\hat{\mathbf{A}}_3 = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix} , \quad \Delta(3,4,5) = \{1, 2\} , \quad \langle \mathbf{a}, \mathbf{d} \rangle = 27 , \quad J(d^3) = 1 ,$$

$$Q(d^3; z) = 1 - z^8 - z^9 - z^{10} + z^{13} + z^{14} , \quad G(d^3) = 2 , \quad F(d^3) = 2 .$$

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Note that $d_1 \geq 3$ by (29) and since $\hat{A}_3$ is containing no zeroes, hence the semigroup is non-symmetric (see Section 2.1).

Theorem 8 and Lemma 11 make it possible to express $F(d^3) + \sum_{i=1}^{3} d_i$ and $2G(d^3) + \sum_{i=1}^{3} d_i$ through the elements of the matrix $\tilde{A}^{(n)}_3$ of the 1st minimal relation for a non-symmetric semigroup $S(d^3)$ only.

$$F(d^3) + \sum_{i=1}^{3} d_i = a_{11}a_{22}a_{33} + \max \{a_{12}a_{23}a_{31}, a_{13}a_{32}a_{21}\}, \quad (138)$$

$$2G(d^3) + \sum_{i=1}^{3} d_i = 1 + a_{11}a_{22}a_{33} + a_{13}a_{32}a_{21} + a_{12}a_{23}a_{31}. \quad (139)$$

Formula (138) is in full agreement with formula (21) for the Frobenius number obtained in [5], [13]. This can be seen if one substitutes the relations (18) and (19) into (21). Let us point out the following inequality for non-symmetric semigroups.

**Lemma 12** Let $\{d_1, d_2, d_3\}$ be the minimal generating set for non-symmetric semigroup. Then

$$G(d^3) \geq 1 + \frac{1}{2} F(d^3). \quad (140)$$

**Proof** Consider (139) and (138) and take their difference

$$2G(d^3) - F(d^3) = 1 + a_{13}a_{32}a_{21} + a_{12}a_{23}a_{31} - \max \{a_{12}a_{23}a_{31}, a_{13}a_{32}a_{21}\}$$

$$= 1 + \min \{a_{12}a_{23}a_{31}, a_{13}a_{32}a_{21}\} \geq 2.$$

The last inequality proves the Lemma. \[\square\]

Note that (140) is slightly stronger than a similar inequality obtained by Nijenius and Wilf [22] for the mD Frobenius problem.

Below we illustrate formulas (127) and (133) obtained for the 3D Frobenius problem in example for three triples, $(23, 29, 44), (137, 251, 256)$ and $(1563, 2275, 2503)$, which were considered numerically in [17], [11], and [23], respectively.

**Example 2**

\[
\begin{align*}
\left( \begin{array}{c}
d_1 \\
d_2 \\
d_3 \\
\end{array} \right) &= \left( \begin{array}{c}
23 \\
29 \\
44 \\
\end{array} \right), \quad \hat{A}_3 = \left( \begin{array}{ccc}
7 & -1 & -3 \\
-5 & 7 & -2 \\
-2 & -6 & 5 \\
\end{array} \right), \quad \langle a, d \rangle = 584, \quad J(d^3) = 86, \\
Q(d^3; z) &= 1 - z^{161} - z^{203} + z^{220} + z^{249} + z^{335}, \quad F(d^3) = 239, \quad G(d^3) = 122.
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
d_1 \\
d_2 \\
d_3 \\
\end{array} \right) &= \left( \begin{array}{c}
137 \\
251 \\
256 \\
\end{array} \right), \quad \hat{A}_3 = \left( \begin{array}{ccc}
24 & -8 & -5 \\
-7 & 13 & -9 \\
-17 & -5 & 14 \\
\end{array} \right), \quad \langle a, d \rangle = 10135, \quad J(d^3) = 1049, \\
Q(d^3; z) &= 1 - z^{3263} - z^{3288} - z^{3584} + z^{4543} + z^{5592}, \quad F(d^3) = 4948, \quad G(d^3) = 2562.
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
d_1 \\
d_2 \\
d_3 \\
\end{array} \right) &= \left( \begin{array}{c}
1563 \\
2275 \\
2503 \\
\end{array} \right), \quad \hat{A}_3 = \left( \begin{array}{ccc}
23 & -7 & -8 \\
-17 & 114 & -93 \\
-6 & -107 & 101 \\
\end{array} \right), \quad \langle a, d \rangle = 548102, \quad J(d^3) = 10646, \\
Q(d^3; z) &= 1 - z^{35949} - z^{252803} - z^{250350} + z^{268728} + z^{279374}, \quad F(d^3) = 273033, \quad G(d^3) = 138470.
\end{align*}
\]

In the next Example we present a special kind of non-symmetric semigroups, the Pythagorean semigroups [12]. Their generators $d_1, d_2, d_3$ satisfy $d_1^2 + d_2^2 = d_3^2$.

**Example 3** Suppose $1 \leq k_2 < k_1$ such that gcd$(k_1, k_2) = 1$. Then

\[
\begin{align*}
\left( \begin{array}{c}
d_1 \\
d_2 \\
d_3 \\
\end{array} \right) &= \left( \begin{array}{c}
k_1^2 - k_2^2 \\
2k_1k_2 \\
k_1^2 + k_2^2 \\
\end{array} \right), \quad \hat{A}^{P_{123}}_3 = \left( \begin{array}{ccc}
k_1 + k_2 & -k_1 + k_2 & -k_1 + k_2 \\
-k_2 & k_1 & -k_2 \\
-k_1 & -k_2 & k_1 \\
\end{array} \right), \quad \langle a, d \rangle = (2k_1 - k_2)(k_1 + k_2)^3, \\
J^{P_{123}}(d^3) &= k_2(k_1 - k_2)^3, \\
Q^{P_{123}}(d^3; z) &= 1 - z^{(k_1+k_2)(k_1+k_2)} - z^{2k_1k_2} - z^{k_1(k_1+k_2)} - z^{k_1(k_1+k_2)^2} - z^{k_2(k_1+k_2)} - z^{k_1(k_1+k_2)k_2} - z^{k_2(k_1+k_2)k_1}, \\
F^{P_{123}}(d^3) &= k_1[k_1^2 - k_2^2 + 2(k_1k_2 - k_1 - k_2)], \quad G^{P_{123}}(d^3) = \frac{1 + k_1^2 - k_2^2}{2} + k_1(k_1k_2 - k_1 - k_2).
\end{align*}
\]
Note that the triple \( \{3,4,5\} \) from Example 1 generates the Pythagorean semigroup with minimal generators.

6.2 Frobenius problem for symmetric semigroup \( S(d^3) \)

Being a special type of non–symmetric semigroup \( S(d^3) \) the case of symmetric semigroup essentially simplifies formulas (127) and (133) for \( Q(d^3; z), F(d^3) \) and \( G(d^3) \).

First, a matrix representation of the set \( \tau \left[ (1 - z^{d_3}) (1 - z^{d_2}) \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi(d^3; z) \right) \right] \) looks much simpler (see Figure 9) and leads to the known Hilbert series (26) with four non–zero terms in the numerator \( Q(d^3; z) \). Denote the Frobenius number and the genus for symmetric semigroup \( S(d^3) \) by \( F_s(d^3) \) and \( G_s(d^3) \), respectively, and derive their expressions. The 1st minimal relation (23) for the given symmetric semigroup together with (8) and (19) yield

\[
a_{11}d_1 = a_{22}d_2, \quad a_{22}a_{33} = d_1, \quad a_{11}a_{33} = d_2, \quad a_{ii} \geq 2, \quad i = 1, 2, 3.
\]

![Figure 9: Matrix representation of the set \( \tau \left[ (1 - z^{d_3}) (1 - z^{d_2}) \left( \sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi(d^3; z) \right) \right] \)]

Substituting the relations (141) into (131) and (133) we obtain

\[
J_s(d^3) = a_{33}d_3, \quad L_{1,2} = \frac{1}{2} [2a_{11}d_1 + a_{33}d_3 \pm a_{33}d_3],
\]

\[
F_s(d^3) = a_{11}d_1 + a_{33}d_3 - \sum_{i=1}^{3} d_i, \quad G_s(d^3) = \frac{1}{2} \left[ 1 + F_s(d^3) \right].
\]

The latter formula in (142) has the following Corollary.

**Corollary 5** Let \( \{d_1, d_2, d_3\} \) be the minimal generating set for symmetric semigroup \( S(d^3) \). Then the Frobenius number \( F_s(d^3) \) is always an odd integer.

We finish this Section with an interesting observation. Recall that due to (29) all elements \( d_i \) of the minimal generating set \( \{d_1, d_2, d_3\} \) for a non–symmetric semigroup \( S(d^3) \) exceed 2. It appears that this restriction becomes even stronger for symmetric semigroup.

**Lemma 13** Let \( \{d_1, d_2, d_3\} \) be the minimal generating set for a symmetric semigroup \( S(d^3) \) and the 1st minimal relation be defined by (23). Then all elements \( d_i \) of the minimal set exceed 3.

**Proof** Let \( S(d^3) \) be a symmetric semigroup and the 1st minimal relation be defined by (23). Then according to (141) we have \( d_1, d_2 \geq 4 \), otherwise the generating set \( d_1, d_2, d_3 \) would be not minimal. Inserting the expressions (141) for \( d_1, d_2 \) into one of the 1st minimal relation \( a_{33}d_3 = a_{31}d_1 + a_{32}d_2 \) and keeping in mind \( a_{31}, a_{32} \geq 1 \) we get

\[
a_{33}d_3 = a_{31}a_{22}a_{33} + a_{32}a_{11}a_{33} \rightarrow d_3 = a_{31}a_{22} + a_{32}a_{11} \geq 4.
\]

Combining all restrictions \( d_i \geq 4 \) we come to the proof of the Lemma. \( \square \)
Example 4 The triple \( \{4, 5, 6\} \) generates a symmetric semigroup (with minimal possible elements \( d_i \)).

\[
\mathcal{A}_4^{(3)} = \begin{pmatrix} 3 & 0 & -2 \\ -1 & 2 & -1 \\ -3 & 0 & 2 \end{pmatrix}, \quad \Delta(4, 5, 6) = \{1, 2, 3, 7\} , \quad H(d^3; z) = \frac{(1 - z^{10})(1 - z^{12})}{(1 - z^5)(1 - z^3)(1 - z^6)}, \quad G(d^3) = 4 , \quad F(d^3) = 7 .
\]

6.3 Lower bounds of the Frobenius number \( F(d^3) \) and genus \( G(d^3) \)

The history of bounds for the Frobenius number \( F(d^3) \) dates back to Schur (see Theorem A in [20]) and has been the subject of intensive study for the last 30 years (see [21], [24], [25] and references therein). The subject is a very active research area till now. In particular, the main interest was devoted to the upper bound \( F(d^3) \) therein. The subject is a very active research area till now. In particular, the main interest was devoted to the upper bound \( F(d^3) \) of the Frobenius number \( d^3 \). Concerning the lower bound \( F^-(d^3) \), in 1994, Davison [23] obtained

\[
F(d^3) \geq F_{Dav}^-(d^3) , \quad F_{Dav}^-(d^3) = \sqrt{3} \sqrt{d_1 d_2 d_3} - \sum_{i=1}^{3} d_i , \quad (143)
\]

where \( \text{the constant } \sqrt{3} \text{ cannot be replaced by a larger value with the inequality remaining true for all } d_1, d_2, d_3 \) ([23], Theorem 2.3). Being obtained by combinatorial means it does not distinguish between the triples generating the non–symmetric and symmetric semigroups. In fact, the lower bound of \( F(d^3) \) for the set \( \{d_1, d_2, d_3\} \) generating symmetric semigroups is stronger than (143). Moreover, it appears that the case of non–symmetric semigroups permits also to enhance slightly the Davison’s bound (143). In order to show this we apply here the results of Sections 6.1 and 6.2, and start with the lower bound for non–symmetric semigroups \( S(d^3) \).

Lemma 14 Let \( \{d_1, d_2, d_3\} \) be the minimal generating set for non–symmetric semigroup. Then

\[
F(d^3) \geq \sqrt{3} \sqrt{d_1 d_2 d_3} + 1 - \sum_{i=1}^{3} d_i , \quad (144)
\]

Proof First, we find the lower bound for \( \langle a, d \rangle \). We start with inequalities which follow from (19)

\[
a_{11} a_{22} > d_3 , \quad a_{22} a_{33} > d_1 , \quad a_{33} a_{11} > d_2 \quad \Rightarrow \quad a_{11}^2 a_{22}^2 a_{33}^2 > d_1 d_2 d_3 . \quad (145)
\]

According to (145) and inequality for symmetric polynomials [27] we obtain

\[
\langle a, d \rangle \geq 3 \prod_{i=1}^{3} (a_{ii} d_i)^{1/3} > 3 \sqrt{d_1 d_2 d_3} . \quad (146)
\]

Making use of (102) we can write

\[
(a_{11} d_1 - a_{22} d_2)^2 + (a_{11} d_1 - a_{33} d_3)^2 + (a_{22} d_2 - a_{33} d_3)^2 \geq 1^2 + 1^2 + 2^2 = 6 ,
\]

or, in other words,

\[
\sum_{i > j}^{3} a_{ii} a_{jj} d_i d_j \leq \frac{1}{3} \langle a, d \rangle^2 - 1 . \quad (147)
\]

Consider the lower bound of \( F(d^3) \) in 2 regions for \( \langle a, d \rangle \):

1) \( \langle a, d \rangle > 2 \sqrt{3} \sqrt{d_1 d_2 d_3} + 1 \) and 2) \( \langle a, d \rangle \leq 2 \sqrt{3} \sqrt{d_1 d_2 d_3} + 1 . \)

\( ^3 \)Two conjectures on the upper bound \( F^+(d^3) \) were put forward recently [26]. Detailed description of the conjectures and their disproof will be given in Appendix B.
In the 1st region we immediately arrive at (144) according to the expression (133) for $F(d^3)$. Consider the 2nd region and observe that due to (147),

$$J^2(d^3) = 4d_1d_2d_3 + \langle a, d \rangle^2 - 4 \sum_{i>j}^3 a_ia_jd_id_j \geq 4d_1d_2d_3 + 4 - \frac{1}{3}\langle a, d \rangle^2 \geq 0.$$  

Thus, we arrive at

$$F(d^3) + \sum_{i=1}^3 d_i \geq \frac{1}{2} \left( \langle a, d \rangle + \sqrt{4d_1d_2d_3 + 4 - \frac{1}{3}\langle a, d \rangle^2} \right). \quad (148)$$

Denote $x = \langle a, d \rangle, c = \sqrt{d_1d_2d_3 + 1}$ and consider a function $f(x) = 1/2(x + \sqrt{4c^2 - x^2/3})$ in the interval $3\sqrt{c^2 - 1} < x \leq 2\sqrt{3c}$. It is easy to find its minimum: $\min f(x) = \sqrt{3c}$ when $x = 2\sqrt{3c}$. Comparing this with (148) we come to (144) in the 2nd region. Combining the bounds in both regions finishes the proof of the Lemma. \hfill \Box

In the next Lemma we find the lower bound of $F_s(d^3)$ for a symmetric semigroup $S(d^3)$.

**Lemma 15** Let $\{d_1, d_2, d_3\}$ be the minimal generating set for symmetric semigroup $S(d^3)$. Then

$$F_s(d^3) \geq 2\sqrt{d_1d_2d_3 - \sum_{i=1}^3 d_i}. \quad (149)$$

**Proof** Substituting $a_{11}a_{33} = d_2$ from (142) into the expression (141) for $F_s(d^3)$ obtain

$$F_s(d^3) + \sum_{i=1}^3 d_i = a_{11}d_1 + a_{33}d_3 = \frac{d_1d_2}{a_{33}} + a_{33}d_3 \geq 2\sqrt{d_1d_2d_3},$$

that proves the Lemma. \hfill \Box

The lower bound (149) is stronger than the Davison’s lower bound (144) for $F(d^3)$ in the generic case of a non–symmetric semigroup $S(d^3)$.

As for the lower bound $G^-(d^3)$ of the genus, to our knowledge, this question was not discussed earlier (see [22] though). Combining (140) with Lemma 14 for non–symmetric semigroups and (142) with Lemma 15 for symmetric semigroups one can obtain the lower bounds $G^-(d^3)$ and $G_s^-(d^3)$, respectively

**Corollary 6**

$$G^-(d^3) = 1 + \frac{\sqrt{3}}{2}\sqrt{d_1d_2d_3 + 1} - \frac{1}{2}\sum_{i=1}^3 d_i, \quad G_s^-(d^3) = \frac{1}{2} + \sqrt{d_1d_2d_3} - \frac{1}{2}\sum_{i=1}^3 d_i. \quad (150)$$

7 On semigroups $S(d^m)$ of higher dimensions, $m \geq 4$.

In 1975, Bresinsky [28] has shown that the complexity of the Frobenius problem changes qualitatively once $m$ exceeds 3: there exists monomial curve in mD space, $m \geq 4$, requiring arbitrary large number of generators for its defining ideal $\mathcal{I}_m$ (see Introduction). This led Székely and Wormald [10] to the following statement,

**Theorem 9** ([10]) The number of non–zero coefficients in the polynomials $Q(d^m; z)$ is not bounded by any function of $m$ for $m \geq 4$, although it is finite for every choice of the generators $d_i$. 


Later this Theorem was interpreted in [15]: 'for any \( m \geq 4 \), there is no way to write \( H(\mathbf{d}^m; z) \) so that the polynomial \( Q(\mathbf{d}^m; z) \) has a bounded number of non-zero terms for all choices of \( d_1, \ldots, d_m \).'

Making use of diagrammatic calculation developed for \( S(\mathbf{d}^3) \) in Section 6 we are going to refine the above statements here.

Denote the number of non-zero coefficients in the polynomial \( P(\mathbf{d}^m; z) \) by \( \# \{ P(\mathbf{d}^m; z) \} \). Thus, following (15), (26) and (127) respectively

\[
\# \{ Q(\mathbf{d}^2; z) \} = 2, \quad \# \{ Q(\mathbf{d}^3; z) \} = \begin{cases} 4, & \text{if } S(\mathbf{d}^3) \text{ is symmetric}, \\ 6, & \text{if } S(\mathbf{d}^3) \text{ is non-symmetric}. \end{cases}
\]

Estimate \( \# \{ Q(\mathbf{d}^m; z) \} \) for non-symmetric semigroup \( S(\mathbf{d}^m) \), \( m \geq 4 \). Before going to determination of an upper bound of \( \# \{ Q(\mathbf{d}^m; z) \} \) (see Section 7.2) we give a brief description of basic properties of the set \( \Delta(\mathbf{d}^m) \) and its matrix representation.

### 7.1 Basic properties of the matrix representation of the set \( \Delta(\mathbf{d}^m) \)

Let \( \mathbf{d}^m \) be given, \( \mathbf{d}^m = (d_1, \ldots, d_m) \), and let \( \Delta(\mathbf{d}^m) \) be the set of integers which are unrepresentable by \( d_1, \ldots, d_m \), and let \( \Phi(\mathbf{d}^m; z) \) be a generating function for this set, \( \tau[\Phi(\mathbf{d}^m; z)] = \Delta(\mathbf{d}^m) \). In order to construct its matrix representation \( M\{\Delta(\mathbf{d}^m)\} \) we have to delete from \( \Delta(\mathbf{d}^2) \) a set \( \Xi(\mathbf{d}^m) \) of all integers \( s \) representable by \( d_1, \ldots, d_m \)

\[
\Delta(\mathbf{d}^m) = \Delta(\mathbf{d}^2) \setminus \Xi(\mathbf{d}^m), \quad \Xi(\mathbf{d}^m) = \left\{ s \mid s = \sum_{j=1}^{m} r_j d_j, \ r_j \in \mathbb{N} \cup \{0\} \right\}.
\]

In 3D case (see Section 3, Theorem 2) this procedure was reduced to the construction of the complement of the union of \( kd_j \)-associated sets \( \bigcup_{k=1}^{a_{j-1}} \Omega_{d_j}^k \) in \( \Delta(\mathbf{d}^2) \). However, in higher dimensions, \( m \geq 4 \), a construction of \( \Delta(\mathbf{d}^m) \) is not exhausted by the complement of the unions of all \( kd_j \)-associated sets \( \bigcup_{j=3}^{m} \left\{ \bigcup_{k=1}^{a_{j-1}} \Omega_{d_j}^k \right\} \) in \( \Delta(\mathbf{d}^2) \), where \( a_{jj} \) is an uniquely defined diagonal element of the matrix \( \hat{A}_m \) of the 1st minimal relation \( \mathcal{R}_1(\mathbf{d}^m) \) for given \( \mathbf{d}^m \)

\[
\hat{A}_m = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \cdots & d_j & 0 \\
\cdots & \cdots & \cdots & \cdots \\
d_m & 0 & \cdots & 0 \\
\end{pmatrix}, \quad \hat{A}_m = \begin{pmatrix} a_{11} & \cdots & -a_{1j} & \cdots & -a_{1m} \\
\cdots & \ddots & \cdots & \cdots & \cdots \\
-aj_1 & \cdots & a_{jj} & \cdots & -a_{jm} \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
-aj_{m1} & \cdots & -a_{mj} & \cdots & a_{mm} \\
\end{pmatrix},
\]

and

\[
a_{jj} = \min \left\{ v_{jj} \mid v_{jj} \geq 2, \ v_{jj} d_j = \sum_{i=1}^{j-1} v_{ij} d_i + \sum_{i=j+1}^{m} v_{ij} d_i, \ v_{ij} \in \mathbb{N} \cup \{0\} \right\}, \ q = 1, \ldots, m.
\]

The defined values of \( v_{ij}, i \neq j \) which give \( a_{ii} \) will be denoted by \( a_{ij}, i \neq j \). Due to minimality of the set \( (d_1, \ldots, d_m) \) the elements \( a_{ij} \) satisfy \( \gcd(a_{j1}, \ldots, a_{jm}) = 1, 1 \leq j \leq m \).

The reason of the complexity of the mD Frobenius problem in higher dimensions, \( m \geq 4 \), relies on the fact that there can appear [12] additional minimal relations \( \mathcal{R}_n(\mathbf{d}^m), n \geq 2 \), which are linearly independent. The problem is also complicated due to the reason that the off–diagonal matrix elements \( a_{ij}, i \neq j \) are not necessarily unique (see Example 6 in Section 7.2). We omit here the discussion of these properties which are unimportant for further consideration.

All this makes the construction of the matrix representation \( M \{ \Delta(\mathbf{d}^m) \} \), \( m \geq 4 \), extremely difficult and therefore such construction will not be a subject of the present paper. Nevertheless
we are in a position to get some positive answer to the question about the number of non–zero
coefficients in the polynomials $Q(d^m; \tau)$. 

Following (5), recall the first containment of the sets $\Delta(d^m) \subset \Delta(d^2)$ and construct the diagram of matrix representation $M\{\Delta(d^m)\}$ on the basis of $M\{\Delta(d_1, d_2)\}$ by deletion procedure described in (151). The obtained representation $M\{\Delta(d^m)\}$ (see Figure 10) is similar to $M\{\Delta(d^3)\}$ in Figure 3 and has two common features which are important to us.

Before discussing this let us generalize $BL_M\{\Delta(d^2)\}$ and $TL_M\{\Delta(d^2)\}$ for $m \geq 4$. Call the totalities of the lowest and top cells in every column of $M\{\Delta(d^m)\}$ the bottom and top layers of $M\{\Delta(d^m)\}$, respectively, with the corresponding notations, $BL_M\{\Delta(d^m)\}$ and $TL_M\{\Delta(d^m)\}$.

Figure 10: Typical matrix representation $M\{\Delta(d^m)\}$ of the set $\Delta(d^m)$ (gray color) inside $M\{\Delta(d^2)\}$. The integers $d_k \notin \Delta(d^m), \ 3 \leq k \leq m$ (black boxes) and the integer $d_2 \notin \Delta(d^2)$ (white box) are adjacent to the top layer $TL_M\{\Delta(d^m)\}$.

We also preserve the definition (66) of an upward shift operator $\hat{U}_1$ by its action on the matrix representation of the set $\Delta(d^m)$: $\hat{U}_1 \Delta(d^m) = \bigcup_{(p,q) \in M(\Delta(d^m))} \hat{U}_1 \sigma(p,q)$.

First, it is clear that the $d_1 - 1$ integers $1, \ldots, d_1 - 1$ are unrepresentable by $d_1, \ldots, d_m$ and therefore, in accordance with (68), we have

$$\sigma\{BL_M\{\tau[\Phi(d^m)]\}\} = \{1, \ldots, d_1 - 1\} = \sigma\{BL_M\{\Delta(d^2)\}\}.$$  \hspace{1cm} (154)

Second, consider the top layer $TL_M\{\tau[\Phi(d^m)]\}$. It is given by

$$\sigma\{TL_M\{\tau[\Phi(d^m)]\}\} = \{\sigma(p_{tm}(q), q)\}, \quad q = 1, \ldots, d_1 - 1,$$

where the subscript ”tm“ stands for the top of $M\{\Delta(d^m)\}$ and $p_{tm}(q)$ is defined as

$$p_{tm}(q) = \min\{1 \leq p \mid \sigma(p,q) \in \Delta(d^m)\}.$$  \hspace{1cm} (155)

Making use of an upward shift operator $\hat{U}_1$ which shifts the diagram of the matrix representation $M\{\Delta(d^m)\}$ one step upwards (66) it is easy to generalize Lemma 7 for $m \geq 4$

$$\sigma\{TL_M\{\tau[z^{d_1}\Phi(d^m)]\}\} = \{\sigma(p_{tm}(q) - 1, q)\}, \quad q = 1, \ldots, d_1 - 1.$$  \hspace{1cm} (156)

Apply diagrammatic calculation described in Section 5 in order to obtain the matrix representation of the set $\tau[(1 - z^{d_1})\Phi(d^m; \tau)]$. In full analogue with Theorem 3 its corresponding generating function looks like

$$(1 - z^{d_1})\Phi(d^m; \tau) = \tau^{-1}\left[\sigma\{BL_M\{\tau[\Phi(d^m)]\}\}\right] - \tau^{-1}\left[\sigma\{TL_M\{\tau[z^{d_1}\Phi(d^m)]\}\}\right] = \sum_{q=1}^{d_1-1} z^q - \sum_{q=1}^{d_1-1} z^q \lambda_q, \quad \lambda_q \in \sigma\{TL_M\{\tau[z^{d_1}\Phi(d^m)]\}\}.$$  \hspace{1cm} (157)
Comparing (160) with (155) and (156) we conclude that the integers that violate the minimality of the 1st relation $R_k$.

We will prove the Theorem in several steps.

**Theorem 10**

Now we are ready to prove the main Theorem of this Section.

**Lemma 16**  
Let $d^m$ be given, $d^m = (d_1, \ldots, d_m)$, and the 1st minimal relation $R_1(d^m)$ be defined by (152) and (153). Then

\[ k d_j \in \sigma \{ TL_M \left\{ \tau \left[ z^{d_i} \Phi(d^m) \right] \right\} \} , \quad k = 1, \ldots, a_{jj} - 1, \quad j = 2, \ldots, m . \]  

**Proof**  
Let $R_1(d^m)$ be the 1st minimal relation defined by (152) and (153). Then $k d_j \not\in \Delta(d^m)$ where $j = 2, \ldots, m$ and $1 \leq k < a_{jj}$. Consider one of such integers $k d_j$. Let, by way of contradiction, $k d_j - d_1 \not\in \Delta(d^m)$, then there exist $\rho_1, \ldots, \rho_m \in \mathbb{N} \cup \{0\}$ such that

\[ k d_j - d_1 = \sum_{i=1}^{m} \rho_i d_i \rightarrow (k - \rho_j)d_j = (\rho_1 + 1)d_1 + \sum_{i=2}^{j-1} \rho_i d_i + \sum_{i=j+1}^{m} \rho_i d_i , \quad 1 \leq k \leq a_{jj} - 1 , \]  

that violates the minimality of the 1st relation $R_1(d^m)$ given by (152) and (153). Hence, for every $k = 1, \ldots, a_{jj} - 1$ and $j = 2, \ldots, m$ we have the following pair of relations

\[ k d_j \not\in \Delta(d^m) \quad \text{and} \quad k d_j - d_1 \in \Delta(d^m) . \]  

Comparing (160) with (155) and (156) we conclude that the integers $k d_j - d_1, 1 \leq k < a_{jj}, j = 2, \ldots, m$ occupy $TL_M \{ \tau [ \Phi(d^m) ] \}$ while the integers $k d_j, 1 \leq k < a_{jj}, j = 2, \ldots, m$ occupy $TL_M \{ \tau [ z^{d_i} \Phi(d^m) ] \}$. This proves the Lemma. \qed

### 7.2 Upper bound for the number of non–zero coefficients in $Q(d^m; z)$

Now we are ready to prove the main Theorem of this Section.

**Theorem 10**  
The number of non–zero coefficients in the polynomial $Q(d^m; z), m \geq 4$ is bounded

\[ \# \{ Q(d^m; z) \} \leq 2^{m-1} \left( d_1 - \sum_{j=2}^{m}(a_{jj} - 2) \right) - 2(m - 1) . \]  

**Proof**  
We will prove the Theorem in several steps.

First, consider the expression (60) for $Q(d^m; z)$ and take into account (158) which implies $\# \Lambda(d^m) = d_1$. By assumption, that a successive multiplication in (60) does not lead to the partial cancellation of the terms, we can get $2^{m-1}d_1$ non–zero terms contributing to $Q(d^m)$ that gives the first preliminary bound

\[ \# \{ Q(d^m; z) \} \leq 2^{m-1}d_1 . \]
Next, if some of the diagonal elements $a_{ii}$ of the matrix $\hat{A}_m$ of the 1st minimal relation $\mathcal{R}_1(d^m)$ exceeds 2, this bound (162) can be actually enhanced. According to Lemma 16, the polynomial (158) can be presented as follows

$$\sum_{k=0}^{d_1-1} z^k - (1 - z^{d_1}) \Phi(d^m; z) = 1 + R_1(d^m; z) + R_2(d^m; z), \quad (163)$$

where

$$R_1(d^m; z) = \sum_{k=1}^{a_{22}-1} z^{kd_2} + \ldots + \sum_{k=1}^{a_{mm}-1} z^{kd_m}, \quad R_2(d^m; z) = \sum_{q=1}^{N} z^{\lambda_q}, \quad N = d_1 - 1 - \sum_{j=2}^{m} (a_{jj} - 1).$$

$N$ exponents $\lambda_q$, contributing to the term $R_2(d^m; z)$, do not have a simple representation $\lambda_q = kd_j, 1 \leq k < a_{jj}, 2 \leq j \leq m$. Denote by $Q_1(d^m; z)$ the following part of the numerator $Q(d^m; z)$

$$Q_1(d^m; z) = R_1(d^m; z) \prod_{j=2}^{m} (1 - z^{d_j}). \quad (164)$$

A straightforward calculation in (164) gives

$$Q_1(d^m; z) = \left(z^{d_2} - z^{a_{22}d_2} \right) \prod_{j=3}^{m} (1 - z^{d_j}) + \ldots + \left(z^{d_m} - z^{a_{mm}d_m} \right) \prod_{j=2}^{m-1} (1 - z^{d_j}). \quad (165)$$

Comparing the number of non–zero terms on the right hand sides of (164) and (165) we come to the conclusion that the entire number (162) of non–zero terms of the numerator $Q(d^m; z)$ can be diminished by $2^{m-1} \left(\sum_{j=2}^{m} a_{jj} - (m - 1)\right) - 2^{m-1}(m - 1) = 2^{m-1} \left(\sum_{j=2}^{m} (a_{jj} - 2)\right)$ that gives the second preliminary bound

$$\# \{Q(d^m; z)\} \leq 2^{m-1} \left(d_1 - \sum_{j=2}^{m} (a_{jj} - 2)\right). \quad (166)$$

Finally, return to (163) and consider the partial cancellation of the terms in the polynomial $(1 + R_1(d^m; z)) \prod_{j=2}^{m} (1 - z^{d_j})$. Due to (165) one can establish at least $m - 1$ such terms which appear twice with different signs. Namely, these are $\sum_{j=2}^{m} z^{d_j}$. Thus, one can diminish the second preliminary bound (166) by $2(m - 1)$. This proves the Theorem. $\square$

Note that, independently of the structure of the matrix $\hat{A}_m$ of the 1st minimal relation $\mathcal{R}_1(d^m)$, the following always holds

$$\# \{Q(d^m; z)\} \leq 2^{m-1} d_1 - 2(m - 1). \quad (167)$$

Theorem 10 leads to the following restriction on the diagonal elements $a_{ii}$ of a matrix $\hat{A}_m$ of the 1st minimal relation $\mathcal{R}_1(d^m)$.

**Corollary 7**

$$\sum_{j=2}^{m} a_{jj} \leq d_1 + 2(m - 1) \left(1 - \frac{1}{2^{m-1}} \right). \quad (168)$$
Proof The proof follows immediately from the fact that the right hand side in the formula (161) is positive. □

It is interesting to compare (168) for \( m = 3 \) with the left hand side of the inequality (104) which is definitely stronger.

We finish this Section with one more observation about the restrictions imposed by the dimension of a non–symmetric semigroup \( S(\mathbf{d}^m) \) on the minimal generating set \( \{d_1, \ldots, d_m\} \).

**Theorem 11** Let \( \{d_1, \ldots, d_m\} \) be a minimal set generating semigroup \( S(\mathbf{d}^m) \) such that \( d_1 < \ldots < d_m \), and let the 1st minimal relation \( R_1(\mathbf{d}^m) \) be defined by (152) and (153). Then the minimal element \( d_1 \) of \( \mathbf{d}^m \) exceeds \( m - 1 \),

\[
d_1 \geq m .
\]

**Proof** We will prove the Theorem in several steps. First, consider the distribution of the generators \( d_j, j = 3, \ldots, m \) inside the matrix representation \( M\{\Delta(d_1, d_2)\} \) in Figure 1. According to the definition (2) of the minimal generating set \( \{d_1, \ldots, d_m\} \) we have \( d_j \in \Delta(\mathbf{d}^2), j = 3, \ldots, m \). Let \( d_u, d_w, 3 \leq u < w \leq m \) be two of such generators with the corresponding representations (31)

\[
\begin{align*}
d_u &= d_1d_2 - p_u d_1 - q_u d_2 , \quad 1 \leq p_u, p_w \leq \frac{d_2 - d_1}{d_1} , \quad 1 \leq q_u, q_w \leq d_1 - 1 .
\end{align*}
\]

One can show that the minimality of the set \( \{d_1, \ldots, d_m\} \) does not allow to have at least one of the equalities, \( p_u = p_w \) or \( q_u = q_w \). Indeed, assume, by way of contradiction, that the first equality holds, \( p_u = p_w \). Then due to (170) we have

\[
d_u = d_w + (q_w - q_u)d_2 ,
\]

which leads to the linear dependence of the three elements \( d_2, d_u, d_w \) that contradicts (2). The other equality, \( q_u = q_w \), is also forbidden for the same reason. Thus, we come to the conclusion that the number \( d_1 - 1 \) of columns in the diagram in Figure 1 is at least not less than the number \( m - 2 \) of such elements, i.e.

\[
d_1 - 1 \geq m - 2 , \quad \text{or} \quad d_1 \geq m - 1 .
\]

Observe that this non–strict inequality was obtained by the assumption that every element \( d_j, j = 3, \ldots, m \) gives rise solely to one associated set \( \Omega^1_{d_j}(\mathbf{d}^2) \), i.e. \( a_{jj} = 2, j = 3, \ldots, m \).

Next, in order to prove (169) we have to show the existence of at least one generator \( d_j, 3 \leq j \leq m \) such that \( a_{jj} \geq 3 \). Indeed, if \( d_h \) is such a generator, then \( d_h \) gives rise to at least 2 associated sets, \( \Omega^1_{d_h}(\mathbf{d}^2) \) and \( \Omega^2_{d_h}(\mathbf{d}^2) \). Distributing all generators \( d_j \) inside \( M\{\Delta(d_1, d_2)\} \) we must account for \( d_h \) twice (\( d_h \) and \( 2d_h \), respectively). The final comparison between the number of columns in the diagram \( M\{\Delta(d_1, d_2)\} \) and the number of distributing generators gives \( d_1 - 1 \geq m - 1 \), or \( d_1 \geq m \).

Finally, it remains to prove the existence of an integer \( d_j, 3 \leq j \leq m \), such that \( a_{jj} \geq 3 \). Consider two last columns, \( q = d_1 - 1 \) and \( q = d_1 - 2 \), of the diagram \( M\{\Delta(d_1, d_2)\} \) and determine the numbers \( H_{d_1-1} \) and \( H_{d_1-2} \) of integers within, respectively. According to (31) the integers occupying these columns are of the form

\[
\sigma(p, d_1 - 1) = d_2 - pd_1 \quad \text{and} \quad \sigma(p, d_1 - 2) = 2d_2 - pd_1 .
\]

Therefore, the restriction \( \sigma(p, q) > 0 \) gives

\[
\sigma(H_{d_1-1}, d_1 - 1) > 0 \quad \Rightarrow \quad H_{d_1-1} = \left\lfloor \frac{d_2}{d_1} - 1 \right\rfloor \quad \text{and} \quad \sigma(H_{d_1-2}, d_1 - 2) > 0 \quad \Rightarrow \quad H_{d_1-2} = \left\lfloor \frac{2d_2}{d_1} - 1 \right\rfloor .
\]

Denote an integer from the last column \( \sigma(p, d_1 - 1) = d_h \) and prove that \( 2d_h \in \Delta(d_1, d_2) \). In accordance with (31) we have

\[
2d_h = 2(d_1d_2 - pd_1 - (d_1 - 1)d_2) = d_1d_2 - pd_1 - (d_1 - 2)d_2 = \sigma(2p, d_1 - 2) .
\]
Show that the integer \(2d_h\) is contained in the last but one column of the diagram \(M\{\Delta(d_1, d_2)\}\). In order to verify this we must prove that \(H_{d_1-2} \geq 2H_{d_1-1}\).

\[
H_{d_1-2} - 2H_{d_1-1} = \left(\frac{d_2}{d_1} - 1\right)^2 - 2 \left(\frac{d_2}{d_1} - 1\right) \geq 2 \left(\frac{d_2}{d_1} - 2\right) - 2 \left(\frac{d_2}{d_1} - 1\right) = 2 \left(\frac{d_2}{d_1} - 1\right) \geq 0.
\]

Thus, \(2d_h \in \Delta(d_1, d_2)\). Applying Lemma 5 we obtain \(2d_3 \notin \Omega(d^2)\), and therefore \(a_{ij} \geq 3\). In fact, we have proved a stronger statement, namely, that all integers \(d_h\) which belong to the last column of the diagram \(M\{\Delta(d_1, d_2)\}\) give rise to at least 2 associated sets, and therefore they all have \(a_{ij} \geq 3\). This completes the proof of the Theorem. □

It is easy to see that Theorem 11 generalizes the restriction (29) obtained for 3D non-symmetric semigroup. Combining now Theorem 10 and Theorem 11 we can find the minimal bound for non-symmetric semigroup \(S(4, d_2, d_3, d_4)\), namely, \# \(\{Q(4, d_2, d_3, d_4; z)\} \leq 26\).

Below we present the results of numerical calculations for two tetrads, \((4,21,26,43)\) and \((43,37,50)\), which give rise to the corresponding \(\Delta(d^4)\)-sets and Hilbert series \(H(d^4; z)\).

**Example 5** \(\{d_1, d_2, d_3, d_4\} = \{4, 21, 26, 43\}\)

\[
\begin{align*}
\begin{cases}
a_{11}d_1 &= 52, \\
a_{22}d_2 &= 42, \\
a_{33}d_3 &= 52, \\
a_{44}d_4 &= 86,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Delta(4, 21, 26, 43) &= \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 23, 26, 27, 31, 35, 39\}, \\
Q(4, 21, 26, 43; z) &= \frac{Q(4, 21, 26, 43; z)}{(1 - z^4)(1 - z^{21})(1 - z^{26})(1 - z^{43})}, \\
H(4, 21, 26, 43; z) &= 21, \\
\# \{Q(4, 21, 26, 43; z)\} &= 18.
\end{align*}
\]

**Example 6** \(\{d_1, d_2, d_3, d_4\} = \{43, 37, 50\}\)

\[
\begin{align*}
\begin{cases}
a_{11}d_1 &= 68, \\
a_{22}d_2 &= 62, \\
a_{33}d_3 &= 74, \\
a_{44}d_4 &= 100,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Delta(43, 37, 50) &= \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27, 29, 30, 33, 34, 38, 42, 46\}, \\
Q(43, 37, 50; z) &= \frac{Q(43, 37, 50; z)}{(1 - z^4)(1 - z^{37})(1 - z^{38})(1 - z^{50})}, \\
H(43, 37, 50; z) &= 28, \\
\# \{Q(43, 37, 50; z)\} &= 18.
\end{align*}
\]

Note that the matrix \(\hat{A}_4\) of the 1st minimal relation \(R_1(d^4)\) in Example 6 is not unique. Indeed, \(2d_4 = 8d_1 + 2d_2 + d_3 \) and \(2d_4 = 25d_1\). Nevertheless, this does not affect the final result for the Hilbert series \(H(43, 37, 50; z)\).

Both numerators, \(Q(4, 21, 26, 35; z)\) and \(Q(43, 37, 50; z)\), have exactly 18 terms satisfying the above restriction (166), \# \(\{Q(4, d_2, d_3, d_4; z)\} \leq 26\). On the other hand, this may indicate that in the 4D Frobenius problem there exist more strong universal properties than the upper bound (161) for the number of non-zero coefficients in the polynomial \(Q(d^4; z)\).

\(^4\)A number 18 appears for \# \(\{Q(4, d_2, d_3, d_4; z)\}\) in numerical calculations for a dozen of tetrads \(4, d_2, d_3, d_4\) such that a tuple \((d_2, d_3, d_4)\) is built out of three pairwise relatively prime elements and the only one of them is an even integer not divisible by 4, e.g., \((4, 13, 15, 18)\), \((4, 17, 23, 26)\), \((4, 29, 31, 34)\), \((4, 41, 42, 51)\) etc. The author thanks G. Tchernikov for help with numerical calculations.
8 Genera of higher orders

The generating function of unrepresentable integers \( \Phi(d^m; z) \) is a source of another information about the set \( \Delta(d^m) \). We show how \( \Phi(d^m; z) \) can be used by computing the power series

\[
\begin{align*}
g_n(d^m) &= \sum_{s \in \Delta(d^m)} s^n, \\
g_0(d^m) &= G(d^m).
\end{align*}
\]

(172)

For the first time, the simplest series \( g_1(d^2) \) was calculated in [29]. In this Section we give a regular approach to that problem and compute some of \( g_n \) for 2D and 3D semigroups based on the results obtained in Section 6. Denoting the derivative \( d^n / dz^n = \partial^n_z \) we find

\[
\partial^n_z \Phi(d^m; 1) = g_n(d^m) - I_1 g_{n-1}(d^m) + I_2 g_{n-2}(d^m) - \ldots \pm I_{n-1} g_1(d^m),
\]

(173)

where the coefficients \( I_k \) appear as symmetric invariants of the set of the integers \( \{1, 2, \ldots, n - 1\} \)

\[
I_1 = \sum_{j=1}^{n-1} j = \frac{n(n-1)}{2}, \quad I_2 = \sum_{j_1 > j_2 = 1}^{n-1} j_1 j_2, \ldots \quad I_{n-2} = I_{n-1} - \sum_{j=1}^{n-1} \frac{1}{j}, \quad I_{n-1} = \prod_{j=1}^{n-1} j = (n-1)!
\]

and \( \partial^n_z \Phi(d^m; 1) = \partial^n_z \Phi(d^m; z)|_{z=1} \). Successive calculation of the first three terms gives

\[
g_1(d^m) = \partial_2 \Phi(d^m; 1), \quad g_2(d^m) = (\partial^2_z + \partial_z) \Phi(d^m; 1), \quad g_3(d^m) = (\partial^3_z + 3\partial^2_z + \partial_z) \Phi(d^m; 1).
\]

Below we present the first three genera of higher orders for the semigroup \( S(d^2) \)

\[
\begin{align*}
g_1(d^2) &= \frac{G(d^2)}{6} (2d_1d_2 - d_1 - d_2 - 1), \\
g_2(d^2) &= \frac{d_1d_2}{6} G(d^2) F(d^2), \\
g_3(d^2) &= \frac{G(d^2)}{60} [(1 + d_1d_2)(1 + d_1^2 + d_2^2 + 6d_1^2d_2^2) + (d_1 + d_2)(1 + d_1^2 + d_2^2 - 9d_1^2d_2^2)]
\end{align*}
\]

(174)

and the first genus \( g_1(d^3) \) for the non–symmetric semigroup \( S(d^3) \)

\[
\begin{align*}
g_1(d^3) &= \frac{1}{12} \left( -1 + \sum_{i=1}^{3} d_i + \sum_{i=1}^{3} A_i d_i^3 + 3 B_{ij} d_i d_j - \sum_{i=1}^{3} a_{ii} \sum_{i=1}^{3} C_{ij} d_j \right), \\
A_i &= (a_{ii} - 1)(2a_{ii} - 1), \quad B_{ij} = 3(a_{ii} - 1)(a_{jj} - 1) - a_{ii}a_{jj}, \quad C_{ij} = 2a_{jj} - 3.
\end{align*}
\]

(175)

In Example 7 we calculate \( g_1(d^3) \) for the triples presented in Example 2.

Example 7

\[
g_1(23, 29, 44) = 9526, \quad g_1(137, 251, 256) = 2380976, \quad g_1(1563, 2275, 2503) = 12178811815.
\]

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A Matrix $\tilde{A}_3^{(n)}$ of the 1st minimal relation with high degeneration

Consider a non–symmetric semigroup $S(d^3)$ which is minimally generated by a triple $d_1, d_2, d_3$ with the matrix $\tilde{A}_3^{(n)}$ of the 1st minimal relation where its diagonal elements $a_{ii}$ completely coincide, $a_{ii} = a$. For such kind of non–symmetric semigroups the expressions for $F(d^3)$ and $G(d^3)$ can be represented in a simple form

$$F(d^3) = \frac{a-2}{2}D_1 + \frac{1}{2}(D_1^2 - 4D_2)a^2 + 4D_3, \quad G(d^3) = \frac{1}{2}[1 + (a-1)D_1 - a^3], \quad (A1)$$

where $D_i$ denote the basic invariants of symmetric group $S_3$ acting on $d_1, d_2, d_3$

$$D_1 = d_1 + d_2 + d_3, \quad D_2 = d_1 d_2 + d_2 d_3 + d_3 d_1, \quad D_3 = d_1 d_2 d_3.$$ 

In this Appendix we present all possible different admissible triples $d_1, d_2, d_3$ generating the non–symmetric semigroups and corresponding to the matrix $\tilde{A}_3^{(n)}$ with a complete coincidence of their diagonal elements $a_{ii} = a$ for the first three values $a = 3, 4, 5$.

- $a=3$
  
  $$5 \rightarrow \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -2 \\ -2 & -2 & 3 \end{pmatrix}. \quad (A2)$$
  
  $$F(5, 7, 8) = 11, \quad G(5, 7, 8) = 7.$$ 

- $a=4$
  
  $$7 \rightarrow \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -3 \\ -3 & -3 & 4 \end{pmatrix}, \quad 10 \rightarrow \begin{pmatrix} 4 & -2 & -1 \\ -1 & 4 & -3 \\ -3 & -2 & 4 \end{pmatrix}. \quad (A3)$$
  
  $$F(7, 13, 15) = 38, \quad G(7, 13, 15) = 21, \quad F(10, 13, 14) = 45, \quad G(10, 13, 14) = 24.$$ 

- $a=5$
  
  $$9 \rightarrow \begin{pmatrix} 5 & -1 & -1 \\ -2 & 5 & -4 \\ -3 & -4 & 5 \end{pmatrix}, \quad 16 \rightarrow \begin{pmatrix} 5 & -2 & -2 \\ -1 & 5 & -3 \\ -4 & -3 & 5 \end{pmatrix}, \quad 17 \rightarrow \begin{pmatrix} 5 & -1 & -3 \\ -3 & 5 & -2 \\ -2 & -4 & 5 \end{pmatrix},$$
  
  $$22 \rightarrow \begin{pmatrix} 5 & -1 & -2 \\ -1 & 5 & -3 \\ -3 & -4 & 5 \end{pmatrix}, \quad 13 \rightarrow \begin{pmatrix} 5 & -2 & -1 \\ -2 & -4 & 5 \end{pmatrix}, \quad 19 \rightarrow \begin{pmatrix} 5 & -1 & -2 \\ -3 & 5 & -3 \end{pmatrix},$$
  
  $$23 \rightarrow \begin{pmatrix} 5 & -1 & -2 \\ -3 & -4 & 5 \end{pmatrix}, \quad 13 \rightarrow \begin{pmatrix} 5 & -2 & -2 \\ -4 & 5 \end{pmatrix}, \quad 17 \rightarrow \begin{pmatrix} 5 & -3 & -1 \\ -1 & 5 & -4 \end{pmatrix},$$
  
  $$24 \rightarrow \begin{pmatrix} 5 & -1 & -2 \\ -1 & 5 & -3 \end{pmatrix}, \quad 19 \rightarrow \begin{pmatrix} 5 & -2 & -2 \\ -3 & 5 \end{pmatrix}, \quad 21 \rightarrow \begin{pmatrix} 5 & -3 & -1 \\ -4 & 5 \end{pmatrix}. \quad (A4)$$

$$F(9, 22, 23) = 83, \quad F(16, 17, 23) = 93, \quad F(17, 19, 22) = 103, \quad F(13, 19, 23) = 86, \quad F(13, 21, 23) = 100$$

$$G(9, 22, 23) = 46, \quad G(16, 17, 23) = 50, \quad G(17, 19, 22) = 54, \quad G(13, 19, 23) = 48, \quad G(13, 21, 23) = 52$$

$$F(13, 21, 22) = 93, \quad F(13, 17, 24) = 83, \quad F(16, 19, 21) = 87, \quad F(17, 21, 22) = 113$$

$$G(13, 21, 22) = 50, \quad G(13, 17, 24) = 46, \quad G(16, 19, 21) = 50, \quad G(17, 21, 22) = 58$$

In fact, there is one more, the 10th tuple, $(9, 21, 24)$, generating a semigroup $S(d^3)$ with the matrix $\tilde{A}_3^{(n)}$ of the 1st minimal relation which is distinguished from those presented in (A4). However, the set $\{9, 21, 24\}$ is not minimal since gcd$(9, 21, 24) = 3$ and therefore is not included into (A4).

Note that there exist only two pairs of triples – $(9, 22, 23), (13, 17, 24)$ and $(16, 17, 23), (13, 21, 22)$ – which have at the same time the equal Frobenius numbers and genera in every pair. This fact may be interesting in the sense of the question posed in [6] on the number of semigroups $S(d^3)$ with the prescribed Frobenius number $F(d^3) = const$. Here we have two constraints, $F(d^3) = const_1$ and $Q(d^3) = const_2$, that must essentially diminish the number of addimisible semigroups $S(d^3)$. 


B  On two conjectures about the upper bound for $F(d^3)$

A recent paper [26] asserts two conjectures based on numerical calculations for more than ten thousands randomly chosen admissible triples $(d_1, d_2, d_3)$ such\(^5\) that $\sqrt{d_1d_2d_3} < 2 \cdot 10^4$. We quote from [26]:

\begin{quote}
For all admissible triples $(d_1, d_2, d_3)$ the Frobenius number $F(d^3)$ can be bounded from above,

$$F(d^3) \leq F^+_{C,\nu}(d^3), \quad F^+_{C,\nu}(d^3) = C(d_1d_2d_3)^\nu - (d_1 + d_2 + d_3).$$

(B1)
\end{quote}

where $C = \text{const}$ and $\nu < 2/3$,' and further,

\begin{quote}
In fact, our data suggests, more precisely, that for all admissible triples $(d_1, d_2, d_3)$, $F(d^3) \leq F^+_{1/5/8}(d^3)$, i.e. $F^+_{1/5/8}(d^3) = (d_1d_2d_3)^{5/8} - (d_1 + d_2 + d_3).$

(B2)
\end{quote}

In this Appendix we are going to falsify these both conjectures.

We start with (B2) by showing two counterexamples. Following [26] recall the terms which are necessary to discuss this conjecture. First, call the triple $(d_1, d_2, d_3)$ constituting an almost arithmetic sequence if there exist the integers $a, b$ such that

$$d_2 = ad_1 + b, \quad d_3 = ad_1 + 2b, \quad a \geq 1, \quad b \geq 1, \quad \gcd(d_1, b) = 1.$$  

(B3)

Next, call the triple $(d_1, d_2, d_3)$ excluded if at least one of the following holds:

1) one of the elements $d_i$ being representable by the other two; \hspace{1cm} (B4)
2) one element $d_i$ dividing the sum of the other two; \hspace{1cm} (B5)
3) the elements $d_i$ represent an almost arithmetic sequence. \hspace{1cm} (B6)

Following [26] define an admissible triple $(d_1, d_2, d_3)$ as a triple of pairwise coprime integers that is not excluded.

Now consider the triple of pairwise coprime integers $d_1, d_2, d_3$ generating a non–symmetric semigroup $S(d^3)$ with the matrix $\hat{A}_3$ (see (17))

\begin{equation*}
\begin{cases}
    d_1 &= 10001 = 73 \cdot 137 \\
    d_2 &= 10003 = 7 \cdot 1429 \\
    d_3 &= 20003 = 83 \cdot 241
\end{cases}
\end{equation*}

\begin{equation}
\hat{A}_3 = \begin{pmatrix}
    5003 & -5000 & -1 \\
    -5000 & 5001 & -1 \\
    -3 & -1 & 2
\end{pmatrix},
\end{equation}

(B7)

where $d_i$ are uniquely factorized into a product of primes. Notice that

$$2d_3 = 3d_1 + d_2, \quad d_2 - d_1 \ll d_1,$$

and

$$d_1 > 2^{13}.$$ \hspace{1cm} (B9)

Show that the triple (B7) is admissible. First, (B4) is not satisfied due to minimality of the set \{10001, 10003, 20003\} according to the matrix $\hat{A}_3$ of minimal relation in (B7) (see (2) and (9)). Next, (B5) is not satisfied, since

$$\frac{10001 + 10003}{20003} \notin \mathbb{N}, \quad \frac{10003 + 20003}{10001} \notin \mathbb{N}, \quad \frac{20003 + 10001}{10003} \notin \mathbb{N}.$$  

(B10)

\(^5\)In fact, the typical values of $d_i$ were not exceeding 750 [30].
In order to prove that (B6) is also not satisfied observe that it is sufficient to show, in accordance with (B3), that \( a = (2d_2 - d_3)/d_1 \) is not an integer. Indeed, a straightforward calculation gives \( a = \frac{3}{1001} \). Thus, the triple (B7) is not excluded and according to definition [26] is admissible.

Calculate the Frobenius number for the triple (B7). By (138) we obtain
\[
F(10001, 10003, 20003) = 50014999,
\]
while the conjectured bound (B2) reads
\[
F^+_{1.5/8}(d^3) = (10001 \cdot 10003 \cdot 20003)^{5/8} - (10001 + 10003 + 20003) = 48745742.422.
\]
Thus, the conjecture (B2) is disproved. The contradiction becomes even stronger if we increase the values of \( d_1, d_2, d_3 \) preserving (B8) and (B9), e.g.
\[
F(100001, 100003, 200003) = 5000149999,
\]
and the Frobenius number (see (138))
\[
F(2l + 1, 2l + 3, 4l + 3) = 2l^2 + 3l - 1.
\]
We prove an auxiliary Lemma.

**Lemma 17** The triple (B11) is admissible.

**Proof** First, (B4) is not satisfied due to minimality of the set \( \{2l + 1, 2l + 3, 4l + 3\} \) according to (B12). Next, (B5) is not satisfied, since
\[
\frac{4l + 4}{4l + 3} \notin \mathbb{N}, \quad \frac{6l + 4}{2l + 3} \notin \mathbb{N}, \quad \frac{6l + 6}{2l + 1} \notin \mathbb{N}.
\]
Finally, calculating \( a = (2d_2 - d_3)/d_1 \) we get \( a = 3/(2l + 1) \) and conclude that (B6) is also not satisfied. Thus, the triple (B11) is admissible and the Lemma is proved. \( \square \)

Finally we are ready to prove the main Lemma of this Appendix.

**Lemma 18** Let \( d^3 \) be given admissible triple, \( d^3 = (d_1, d_2, d_3) \). The Frobenius number \( F(d^3) \) can not be bounded from above by \( F^+_{C, \nu}(d^3) \) given by
\[
F^+_{C, \nu}(d^3) = C(d_1d_2d_3)\nu - (d_1 + d_2 + d_3), \quad \nu < 2/3,
\]
for any \( C = \text{const.} \)
Proof. Consider the triple (B11) which is admissible according to Lemma 17 and denote by $\delta_{C,\nu}(l)$ the ratio

$$\delta_{C,\nu}(l) = \frac{F_{C,\nu}^+(2l + 1, 2l + 3, 4l + 3)}{F(2l + 1, 2l + 3, 4l + 3)}.$$  \hspace{1cm} (B16)

In order to verify the conjecture B1 we have to find $C = \text{const}$ and $\nu < 2/3$ such that

$$\delta_{C,\nu}(l) \geq 1$$ \hspace{1cm} (B17)

holding for all $l > 1$. However, this is not true. Indeed, find a leading term of the asymptotics of $\delta_{C,\nu}(l)$ when $l \to \infty$

$$\delta_{C,\nu}(l) \simeq C 2^{4\nu - 1} l^{3\nu - 2}.$$ \hspace{1cm} (B18)

Observe that its growth with $l \to \infty$ is enough to break (B17) when $l$ exceeds a critical value $l_{cr}$

$$l > l_{cr}, \quad \log_2 l_{cr} = \frac{4\nu - 1}{2 - 3\nu} + \frac{\log_2 C}{2 - 3\nu}$$ \hspace{1cm} (B19)

for all $\nu < 2/3$. This is true for arbitrary large finite $C$.  \hspace{1cm} $\square$

Hence there follows the critical value $l_{cr} = 2^{12}$ for $\nu = 5/8$ and $C = 1$ that leads to (B9).