The Wave Function of a Collapsing Star
and Quantization Conditions

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ABSTRACT

A very simple minisuperspace describing the Oppenheimer-Snyder collapsing star is found. The semiclassical wave function of that model turns out to describe a bound state. For fixed initial radius of the collapsing star, the corresponding Bohr-Sommerfeld quantization condition implies mass quantization. An extension of this model, and some consequences, are considered.

\textsuperscript{1}This work is supported by the Fishbach Fellowship 92/3 and by the NSF grant PHY 88-04561.
The simplest model of a spherically symmetric homogenous dust collapsing star is that of Oppenheimer and Snyder [1]. In that model one connects continuously the Schwarzschild interior to the interior region, which is a slice of a Friedmann universe.

To construct the (mini) superspace of that model one needs the relevant Lagrangian and Hamiltonian. Generally one does not use the Lagrangian description of a dust distribution, because its density \( \rho(x) \), is not a dynamical variable. Instead one uses the energy momentum tensor, \( T_{\mu\nu} = \rho U_{\mu} U_{\nu} \), where \( U_{\mu} \) is the four-velocity of the matter particles. The Einstein equations are \( G_{\mu\nu} = 8\pi T_{\mu\nu} \). In [2] it was shown how one can define a matter dust Lagrangian that will lead to those equations of motion.

We will use geometrical units: \( G = c = 1 \) and signature \((- + + +)\). The interior region of the Oppenheimer-Snyder model is described by the Friedmann line element,

\[
ds^2 = -N^2(t)dt^2 + a^2(t)[d\chi^2 + \sin^2 \chi d\Omega^2] \tag{1}
\]

The range of \( \chi \) is \( 0 \leq \chi \leq \chi_0 \), where \( \chi_0 < \pi/2 \). At \( \chi = \chi_0 \) the interior is matched to the exterior Schwarzschild solution. At least classically, the energy momentum tensor must be conserved, \( \nabla_{\mu} T^{\mu\nu} = 0 \). In the case of a Friedmann universe, this implies that \( \rho = \rho_0/a^3 \), where \( \rho_0 \) is a constant determined by the initial conditions. In the collapse case one gets \( \rho_0 = 3a_0/8\pi \), where \( a_0 \) is the radius of the Friedmann ball at the beginning of the collapse.

So the density of the star is

\[
\rho = \begin{cases} 
3a_0/8\pi a^3(t) , & \chi \leq \chi_0 \ (r \leq r_s) \\
0 , & \chi > \chi_0 \ (r > r_s) 
\end{cases} \tag{2}
\]

where \( r_s \) is the surface radius of the collapsing star.

To get the matter Lagrangian one defines the “weight one density” \( \tilde{\rho} \equiv \sqrt{-g}\rho \) [2] (which is the generalization of a delta function). Then the matter
Lagrangian is \( L_M = -8\pi \int \tilde{\rho} g^{\mu\nu} U_{\mu} U_{\nu} d^3 x \). The four-velocity satisfies (classically) \( U^\mu U_\mu = -1 \), so from (1) and (2) we get
\[
L_M = 32\pi^2 \int_0^{\chi_0} \sin^2 \chi d\chi N a^3 \rho .
\]
The gravitational Lagrangian may be split into its interior and exterior parts,
\[
L_G = 4\pi \int_0^{\chi_0} \sin^2 \chi d\chi \left[ \frac{3a}{N} \dot{a}^2 - 3Na \right] + \int_{r \geq r_s} \sqrt{-g} R d^3 x .
\]
The total Lagrangian is then
\[
L = L_G + L_M = 4\pi \int_0^{\chi_0} \sin^2 \chi d\chi \left[ \frac{a\dot{a}^2}{N} - N(a - a_0) \right] + \int_{r \geq r_s} \sqrt{-g} R d^3 x
\]
The matching conditions are [1]
\[
M = \frac{1}{2} a_0 \sin^3 \chi_0 ,
R_0 = a_0 \sin \chi_0
\]
where \( R_0 \) is the radius of the surface of the star at the beginning of the collapse, and \( M \) is the mass of the Schwarzschild solution. As one can see from (6), \( \chi_0 \) is determined by the initial condition: \( 2M/R_0 = \sin^2 \chi_0 \). For a reasonable cosmological initial condition we have \( R_0 >> M \) so: \( \chi_0 << 1 \). In that case the integral over the slice of \( S^3 \) is \( 4\pi \chi_0^3 / 3 \), and
\[
M \approx \frac{1}{2} R_0 \chi_0^2 .
\]
Hence
\[
L = 4\pi \chi_0^3 \left[ \frac{a\dot{a}^2}{N} - N(a - a_0) \right] + \int_{r \geq r_s} \sqrt{-g} R d^3 x
\]
The classical solution of (5) (or (8)) is easily obtained\(^2\). For \( r \geq r_s \), we have the Schwarzschild solution. For \( r \leq r_s \) (\( \chi \leq \chi_0 \)), it is convenient
\[^2\] Note that for our dust, the only variables in (5) (or (8)) are the metric components \((a, N)\) since the the matter field \( \rho \) is determined by \( a \); the two corresponding Euler-Lagrange equations are of course equivalent to the Einstein ones with the dust source.
to choose the gauge for which \( N(\eta) = a(\eta) \). Then the solution for which \( a(\eta = 0) = a_0 \) and \( \dot{a}(\eta = 0) = 0 \), is

\[
a(\eta) = a_0(1 + \cos \eta)/2 .
\] (9)

Our strategy is this: (5) (or (8)) is our model. This is our starting point. Classically it describes the Oppenheimer-Snyder collapse process; we will find the semiclassical wave function of this universe. This approach is consistent with the analysis of [3], in which a general (inhomogeneous) distribution of spherically symmetric dust matter is considered. The field describing the matter (the dust) is eliminated by gauge conditions and one ends up with only gravitational degrees of freedom.

Usually one starts from a known matter distribution and tries to find the metric. Here we go in the opposite direction: we first determine the geometry (the metric), and then find the matter distribution. Classically both directions are equivalent. Semiclassically we can use our model, because we need only the classical action. Higher quantum orders will require a modification of this model\(^\text{3}\). We will see that the WKB wave function of that universe describes a \textit{bound state}, so we will get quantization conditions.

One can use the same formulation to study closed Friedmann mini-superspace models filled with dust (and/or radiation) \([4]\). In that case \( 0 \leq \chi \leq 2\pi \), and there is no “outside” Schwarzschild region.

The semiclassical wave function\(^\text{4}\) of this collapsing star is

\[
\psi_{WKB}(t) = A(t) \exp[i S_{\text{Class.}}(0, t)/\hbar]
\] (10)

where

\[
S_{\text{Class.}}(0, t) = 4\pi \chi_0^3 \int_0^t L(t') dt' + \int_0^t dt' \int_{r \geq r_s} \sqrt{-g R_{\text{Schw.}}} d^3x .
\] (11)

\(^3\)Because for example \( U^\mu U^\nu_\mu \neq -1 \), and we cannot get (3).

\(^4\)We work in the Lorentzian section but one can get exactly the same results working in the Euclidian section. This is true also for the standard mini-superspace models, and one may argue that it should be the same in the full superspace.
Because $R_{\text{Schw.}} = 0$, only the first term in the r.h.s of (11) will contribute to the semiclassical wave function. We can calculate (11) very easily in the gauge $N(\eta) = a(\eta)$, for which

$$\psi_{WKB} = C\left(\pi \chi_0^3 a_0^2 (\eta - \frac{1}{2} \sin 2\eta)\right)^{-1/2} \exp \left(\frac{i}{\hbar} \pi \chi_0^3 a_0^2 (\eta - \frac{1}{2} \sin 2\eta)\right). \quad (12)$$

One can easily see that (12) is the WKB solution of the one-dimensional Schrödinger equation

$$\left[-\hbar^2 \frac{d^2}{da^2} + V(a)\right] \psi(a) = 0 \quad (13)$$

where

$$V(a) = (8 \pi \chi_0^3)^2 a(a - a_0), \; 0 \leq a \leq a_0. \quad (14)$$

The semiclassical solution of (13), (14) is

$$\psi_{WKB}(a) = C(p(a))^{-1/2} \exp[ip(a)/\hbar] \quad (15)$$

where

$$p(a) = \int_{a_0}^{a} \sqrt{E - V(a')} da' = \int_{a_0}^{a} \sqrt{|V(a')|} da' \quad (16)$$

$$= \pi \chi_0^3 \left[2(a - a_0)(a_0 a - a^2)^{1/2} - a_0^2 \left(\arcsin(1 - 2a/a_0) + \frac{\pi}{2}\right)\right]$$

Using (9), one can see that (12) and (15) are identical.

The Schrödinger equation (13),(14) is just the Wheeler-DeWitt equation of this simple model [5,6,7]. To see this, we go back to (8) and construct the Hamiltonian in the standard way. One should remember that only the first term in the r.h.s of (8) contributes to the semiclassical dynamics. The Hamiltonian is

$$H = \dot{a}P_a - L \quad (17)$$

where $P_a = \partial L/\partial \dot{a} = 8 \pi \chi_0^3 a \dot{a}/N$, namely

$$H = N \left[\frac{1}{16 \pi \chi_0^3 a} P_a^2 + 4 \pi \chi_0^3 (a - a_0)\right] \quad (18)$$
The Wheeler-DeWitt equation is the quantum version of the classical Hamiltonian constraint, \( \partial H / \partial N = 0 \), in the coordinate representation \( |\psi> = \psi(a) \) and \( P_a \rightarrow -i\hbar \partial / \partial a \). Using (18) we get

\[
\frac{1}{16\pi\chi_0^3a} \left[ -\hbar^2 \frac{d^2}{da^2} + V(a) \right] \psi(a) = 0
\]

(19)

where the potential is (14). So (13) is (as we expected) the Wheeler-DeWitt equation of this model.

We see from (14) that there are two turning points, at \( a = 0 \) and \( a = a_0 \), between which there is a bound state\(^5\).

Consequently there will be quantization conditions. In our semiclassical limit (in which General Relativity should be a good approximation) the Bohr-Sommerfeld quantization condition says that

\[
\int_0^{a_0} \sqrt{|E - V(a)|} da = \int_0^{a_0} \sqrt{|V(a)|} da = (n + 1/2)\pi \hbar .
\]

(20)

Let \( R_0 \) be the initial radius of the collapsing star. Then (20) is a quantization condition on the mass of the star. From (7), (14) and (20) we get the mass quantization condition

\[
M(n) = \frac{1}{2\pi^2} \left( \frac{l_P}{R_0} \right)^3 (n + 1/2)^2 M_P
\]

(21)

where \( l_P \) and \( M_P \) are the Planck length and mass (in the geometrical units \( l_P = M_P = \hbar^{1/2} \)).

Before we consider some of the consequences of (21), it is important to verify its extension. Is it model dependent, and in what way?

\(^5\) The wave function of this bound state is of course real

\[
\Psi_{WK}(a) = c(p(a))^{-1/2}(e^{ip(a)/\hbar} + e^{-ip(a)/\hbar})
\]

which corresponds to the Hartle-Hawking wave function [7].
In the Appendix we consider in details the addition of a conformally invariant scalar field as a small perturbation. The resulting Wheeler-DeWitt equation is (see (41),(42))

\[
-\hbar^2 \frac{d^2}{da^2} + \tilde{V}_m(a) \psi(a) = 0
\]  (22)

where

\[
\tilde{V}_m(a) = (8\pi\chi_0^3)^2 \left[ a \left( a - a_0 - \frac{\alpha}{4\pi\chi_0^3} \right) - \frac{(m + 1/2)\hbar}{4\pi\chi_0^3} \right]
\]  (23)

where \(\alpha\) is a real number, and \(m\) is an integer (it is the quantum number of the scalar field inside the star). Comparing (14) with (23), we see that small perturbation requires that \(a_0 >> \alpha/4\pi\chi_0^3\) and \(n >> m\). The classical equation of motion that we get from (23) will be exactly like the unperturbed one, if we replace \(a_0 \to b_0 = a_0 + \alpha/(4\pi\chi_0^3)\). Now, one can easily see that for \(n >> m\), the quantization condition

\[
\int_0^{b_0} \sqrt{|\tilde{V}_m(a)|} da = (n + 1/2)\pi\hbar
\]  (24)

leads to the mass quantization condition

\[
M(n, m) \approx \frac{1}{2\pi^2} \left( \frac{l_P}{R_0} \right)^3 \left( \sqrt{n + m + 1} - \sqrt{m + 1/2} \right)^4 M_P
\]  (25)

which is slightly different from (21), (in the “limit”: \(m + 1/2 \to 0\) we of course recover (21)).

This example shows that one can get quantization condition in a more general case than the simple Oppenheimer-Snyder model. But if, for example, we make the same calculation for a minimally coupled massless scalar field, we get the potential

\[
W(a) = (8\pi\chi_0^3)^2 \left[ a \left( a - a_0 - \frac{\alpha}{4\pi\chi_0^3} \right) - \frac{k^2}{4\pi\chi_0^3a^2} \right]
\]  (26)

where \(k\) is a real number. But now the integral

\[
\int_0^{b_0} \sqrt{|W(a)|} da
\]  (27)
diverge logarithmically, and there are no quantization conditions! The problem is that when $a \to 0$, the potential (26) diverges. This means that the perturbations are not small at $a \to 0$, and we can no longer say that there is a collapse process. The classical solutions of (26) do not describe a collapse! This seems to contradict Price’s results [8] that the Oppenheimer-Snyder model is stable. However, one should remember that to get (23) or (26), we neglect a very important condition, continuity at the surface of the star. So as a matter of fact (23) and (26) are incorrect. We can see this from the following consideration: from the no-hair theorems we know that for late times the scalar field outside the horizon vanishes. From continuity (and the fact that we consider only constant scalar field inside the star), it should vanish also inside the star. In (26) $k$ (and in (23) $m$) is the quantum number of the scalar field inside the star. So if somehow (using the correct continuity conditions) we were able to get an effective one-dimensional Wheeler-DeWitt equation (like (22) with (23) (or (27))) then for late times $k \to 0$ ($m \to 0$).

Late times corresponds to small $a$, so $k$ (and $m$) should be $a$-dependent. In that case (27) will not diverge. It is not easy to find the whole superspace with the boundary conditions [9], and to solve the Wheeler-DeWitt equation. But if the Oppenheimer-Snyder model is stable, then for small perturbations, one should end up with an effective potential (such as we got in the oversimplified model of the conformally invariant scalar field\footnote{Probably the continuity conditions will not change (24) too much, because anyway $\alpha$ and $m$ are small and contribute a finite correction in (24).} which is close to (14). One is tempted to make the conjecture that the WKB wave function of a (general) collapsing star describes a bound state, leading to quantization conditions.

But one should find first the WKB solution of the full Wheeler-DeWitt equation, or at least the full linear perturbations, which is not an easy task!

The reason that there are no quantization condition in the standard cosmological model of Friedmann universe [7], is that in that model the potential
is (when there is a cosmological constant $\Lambda$):

$$V(a) = a^2 - H^2a^4$$

where $H^2 = \Lambda/3 \geq 0$. This potential does not describe a bound state.

We end with some comments:

1) It can be easily seen that our quantization conditions do not effect the classical collapse process, because of the correspondence limit.

2) We obtained a “strange” quantization condition. It depends strongly on the initial value $R_0$. As a matter of fact if $R_0$ is not much greater than $M$, one should replace $\chi_0^3$ in (14) (or (24)) with $3(\chi_0/2 - sin(2\chi_0)/4)$. Then the quantization condition (21) will be different, so the exact $M$ is a more complicated function of $R_0$. The strange feature of our quantization condition is that the mass quanta are generally much smaller then the Planck mass, a more “reasonable” quantization condition should not involve $R_0$ in an essential way.

3) To see if the Hawking process can be affected by the quantization condition, one should compare the energy of a radiated particle to the mass difference $\Delta M$. The energy of a radiated particle is:

$$E_{r.p} \sim \frac{1}{2}k_BT = \frac{M^2_P}{16\pi M}.$$ (29)

The Hawking process should not be affected, if $E_{r.p} \gg \Delta M$. In that case one can consider the mass as continuous. In our case this leads to $n \ll R_0/l_P$, which (in our geometrical units) means that

$$M \ll R_0,$$ (30)

7 Otherwise the solution is not very interesting.
8 For our sun, for example, $n \sim 10^{100}$ !
9 For example $M \sim f(n)M_P$, such that $f$ is a function of $n$ but not of $R_0$. 

9
consistent with reasonable cosmological initial conditions. So our quantization condition should not affect the Hawking process. It is interesting to notice that the more “reasonable” quantization condition, $M(n) = f(n)M_P$, would (very much) affect the Hawking process, because (in that case) $E_{r,p} >> \Delta M$ leads to $M << M_P$. So perhaps our “strange” quantization condition is not so strange?

4) Can strong quantum effects drastically change our result? Quantum effects should be dominant when $a \to 0$. But if those effects should save us from the singularity, or make it less singular, then they must (drastically) increase the potential (near $a = 0$). This means that we will have an even stronger binding, and quantization conditions should remain.

Finally, it is very important to study consistently the effect of small perturbations around this minisuperspace.

Acknowledgment
I would like to thank Stanley Deser for very helpful discussions.

A  Addition of a conformally invariant scalar field:

In this Appendix we consider the addition of a conformally invariant scalar field, $\varphi(t, \vec{x})$. For $\chi \leq \chi_0$ ($r \leq r_s$) we take the scalar field to be only time dependent, $\varphi(t)$ (like in the standard minisuperspace models [7]). But for $r > r_s$ the scalar field must be a function of the radial coordinate too, $\varphi(t, r)$. The Lagrangian (8) is now:

$$L = L_G + L_M + L_\phi =$$
\[ a \dot{a}^2 - a \dot{\phi}^2 - N(a - a_0 + \phi^2/a) \]  
\[ + \int_{r > r_s} \sqrt{-g} \left[ R(1 - \phi^2/6) - (\nabla \phi)^2 \right] d^3x \]  
where \( \phi(t) = a(t) \varphi(t) \). We consider the scalar field as a small perturbation to the Oppenheimer-Snyder model. One can use “generalized Novikov coordinates” [10], for \( r > r_s \), whose line element is:

\[ ds^2 = -N^2(R, \xi)d\xi^2 + F^2(R, \xi)dR^2 + r^2(R, \xi)d\Omega_2^2. \]

The Hamiltonian related to (31) is

\[ H = H_1 + H_2 \]
\[ H_1 = N \left[ \frac{1}{16\pi \chi^3 a} (P_a^2 - P_{\phi}^2) + 4\pi \chi^3_{0}(a - a_0 - \phi^2/a) \right] \]
\[ H_2 = \int_{R_0}^{\infty} N(R, \xi) \left[ \frac{1}{32\pi} \left( -\frac{2}{Fr^2} P_{\phi}^2 + \frac{F}{r^2} P_F^2 + \frac{1}{r} P_F P_r \right) \right. \]
\[ + \left( \frac{3}{R} R^{\frac{1}{2}} \left( \frac{\partial \phi}{\partial R} \right)^2 - \frac{4}{R} R^{\frac{1}{2}} \right) F_{Fr}^{\frac{1}{2}} \right] dR \]

where \( P_\phi = \partial L/\partial \dot{\phi}, \ P_F = \partial L/\partial \dot{F}, \ P_r = \partial L/\partial \dot{r} \) and \( P_{\varphi} = \partial/\partial \varphi \). The corresponding Wheeler-DeWitt equation is

\[ \left( \frac{\partial H}{\partial N} \right) |\Psi> = (\hat{A}_1 + \hat{A}_2)|\Psi> = 0 \]

where

\[ \hat{A}_1 = \frac{1}{16\pi \chi^3 a} \left[ \hat{P}_a^2 - \hat{P}_\phi^2 + (8\pi \chi^3_{0})^2 [a(a - a_0) - \phi^2] \right] \]
\[ \hat{A}_2 = \int_{R_0}^{\infty} \left[ \frac{1}{32\pi} \left( -\frac{2}{Fr^2} \hat{P}_{\phi}^2 + \frac{F}{r^2} \hat{P}_F^2 + \frac{1}{r} \hat{P}_F \hat{P}_r \right) \right. \]
\[ + \left( \frac{3}{R} R^{\frac{1}{2}} \left( \frac{\partial \phi}{\partial R} \right)^2 - \frac{4}{R} R^{\frac{1}{2}} \right) F_{Fr}^{\frac{1}{2}} \right] dR \]
and \( \hat{P}_\eta = -i\hbar \partial / \partial \eta \), \( \eta \equiv (a, \phi, F(R), r(R), \varphi(R)) \). We see that \( \hat{A}_1 \) is a second order differential operator acting on a two-dimensional space. But \( \hat{A}_2 \) acts on an infinite dimensional space (functional space). It is a very complicated space, with boundary conditions result from gauge fixing and continuity conditions at \( R = R_0 \). Because we are not going to solve the Wheeler-DeWitt equation in that region anyway, we will not consider all the details. Let \( \vec{z} \) be a coordinates in that space. And let \( x \equiv a \), \( y \equiv \phi \). Then we make a separation of variables\(^{10}\):

\[
\Psi(x, y, \vec{z}) = \Psi_1(x, y)\Psi_2(\vec{z})
\]

then we get from (34) and (37)

\[
\hat{A}_1 \Psi_1(x, y) = \alpha \Psi_1(x, y) \quad (38)
\]

\[
\hat{A}_2 \Psi_2(\vec{z}) = -\alpha \Psi_2(\vec{z}) \quad (39)
\]

where \( \alpha \) is a (positive) number. From (39) and (42) we get:

\[
\left\{ -\hbar^2 \frac{\partial^2}{\partial x^2} + \hbar^2 \frac{\partial^2}{\partial y^2} + (8\pi \chi_0^3)^2 \left[ x \left( x - a_0 - \frac{\alpha}{4\pi \chi_0^3} \right) - y^2 \right] \right\} \Psi_1(x, y) = 0 . \quad (40)
\]

The solution of (40) is very simple, \( \Psi(x, y) = \psi(x)\varrho(y) \). The \( y \)-part is just a harmonic oscillator\(^{11}\), for which the WKB is the exact solution. The frequency of this harmonic oscillator is \( \omega = 16\pi \chi_0^3 \). So the energy is \( \hbar \omega (m + 1/2) \), and from (40) we get

\[
\left[ -\hbar^2 \frac{d^2}{dx^2} + \tilde{V}_m(x) \right] \psi(x) = 0 \quad (41)
\]

\(^{10}\)The coordinates \( x, y \) and \( \vec{z} \), are not independent. They are related by the continuity conditions. We will work in this simplified model and then check if our results are “reasonable”.

\(^{11}\)with a different overall sign (relative to \( H_a \)). Remember that in Wheeler’s superspace, only the “volume coordinate” (in our case, \( a \)) is timelike, and all the other (infinite number of) dimensions are spacelike.
where
\[ \tilde{V}_m(x) = (8\pi\chi_0^3)^2 \left[ x \left( x - a_0 - \frac{\alpha}{4\pi\chi_0^3} \right) - \frac{(m + 1/2)\hbar}{4\pi\chi_0^3} \right] \] (42)

This is (22) and (23).

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