The restriction theorem for the Grushin operators

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Abstract

Abstract. We study the Grushin operators acting on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and defined by the formula

$$L = -\sum_{j=1}^{d_1} \partial^2_{x_j} - \left( \sum_{j=1}^{d_1} |x_j|^2 \right) \sum_{k=1}^{d_2} \partial^2_{t_k}.$$

We establish a restriction theorem associated with the considered operators. Our result is an analogue of the restriction theorem on the Heisenberg group obtained by D. Muller.

1 Introduction

The restriction theorem for the Fourier transform plays an important role in harmonic analysis as well as in the theory of partial differential equations. The initial work on restriction theorem was given by E. M. Stein [4] on $\mathbb{R}^n$. The result is stated as follows:

Theorem 1.1 (Stein-Tomas) Let $1 \leq p \leq \frac{2n+2}{n+3}$. Then the estimate

$$||\hat{f}||_{L^2(S^{n-1})} \leq C||f||_{L^p(\mathbb{R}^n)}$$

holds for all functions $f \in L^p(\mathbb{R}^n)$.

A simple duality argument shows that Stein-Thomas theorem is equivalent to the following:

$$||f * \overline{d\sigma}_r||_{L^p} \leq C_r||f||_p$$

holds for all $f \in S(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $d\sigma_r$ is the surface measure on the sphere with radius $r$.

Moreover, according to the Knapp example [4], the restriction theorem fails if $\frac{2n+2}{n+3} < p \leq 2$.

From then on, the importance of the restriction theorem has become evident and various new restriction theorems has been proved. On the other hand, the restriction theorem can be generalized to many other spaces, such as Lie groups and compact manifolds (see [2][3][5][7][9]).

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The aim of this paper is to study the restriction theorem associated with the Grushin operators, that is,

$$L = -\Delta_x - |x|^2\Delta_t,$$

where $(x, t) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, while $\Delta_x$, $\Delta_t$ are the corresponding partial Laplacians, and $|x|$ is the Euclidean norm of $x$. It is obvious that $L$ is self-adjoint. The operator is closely related to the scaled Hermite operators $H(a) = -\Delta_x + a^2|x|^2$. Indeed, for a Schwartz function $f$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, let $f^\lambda(x) = \int_{\mathbb{R}^{d_2}} f(x, t)e^{i\lambda t} dt$ be the inverse Fourier transform of $f$ in the $t$ variable. Applying the operator $L$ to the Fourier expansion $f(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^{d_2}} f^\lambda(x)e^{-i\lambda t} d\lambda$, we see that

$$Lf(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^{d_2}} H(|\lambda|)f^\lambda(x)e^{-i\lambda t} d\lambda$$

$$= \frac{1}{2\pi} \int_0^\infty a^{d_2-1} \left( \int_{\mathbb{R}^{d_1}} H(a)f^{\alpha \epsilon}(x)e^{-ia\epsilon t} d\sigma(\epsilon) \right) da$$

Let us recall some results about the special scaled Hermite expansion. For $k \in \mathbb{N}$, the Hermite function $h_k$ of order $k$ is the function on $\mathbb{R}^d$ defined by

$$h_k(\tau) = (2^k k! \sqrt{\pi})^{-1/2} H_k(\tau)e^{-\tau^2/2}.$$  

Let $\nu$ be a multiindex and $x \in \mathbb{R}^{d_1}$, we define the $d_1$-dimensional Hermite functions $\Phi_\nu$ by

$$\Phi_\nu(x) = \prod_{j=1}^{d_1} h_k(x_j).$$

The eigenfunctions of the scaled Hermite operator $H(a)$ are given by $\Phi_\nu^a = |a|^{1/4} \Phi_\nu(\sqrt{|a|x})$ and $H(a)\Phi_\nu^a = (2k + d_1)|a|\Phi_\nu^a$. Let $P_k(a)$ stand for the projection of $L^2(\mathbb{R}^{d_1})$ onto the $k$-th eigenspace of $H(a)$. More precisely

$$P_k(a)\varphi = \sum_{|\nu|=k} (\varphi, \Phi_\nu^a)\Phi_\nu^a$$

Then the spectral decomposition of the operator $H(a)$ is explicitly known:

$$H(a) = \sum_{k=0}^\infty (2k + d_1)|a|P_k(a),$$

Hence the spectral decomposition of the Grushin operator $L$ is given by

$$Lf(x, t) = \frac{1}{2\pi} \int_0^\infty \left( \sum_{k=0}^\infty (2k + d_1)a^{d_2} \int_{\mathbb{R}^{d_1}} P_k(a)f^{\alpha \epsilon}(x)e^{-ia\epsilon t} d\sigma(\epsilon) \right) da$$

Thus the spectrum of $L$ consists of the half line $[0, \infty)$. The Grushin operator is a self-adjoint and positive operator. Furthermore, $e^{-ia\epsilon \cdot \phi}(x)$ is an eigenfunction of $L$ with the eigenvalue $(2|\nu| + d_1)|a|$. Therefore, we have

$$Lf(x, t) = \frac{1}{2\pi} \int_0^\infty \left[ \sum_{k=0}^\infty \frac{\mu^{d_2}}{(2k + d_1)^{d_2}} \int_{\mathbb{R}^{d_1}} P_k \left( \frac{\mu}{2k + d_1} \right) f^{\alpha \epsilon \mu}(x)e^{-i\mu t} d\sigma(\epsilon) \right] d\mu$$
where $P_\mu f(x,t) = \frac{1}{i} \sum_{k=0}^{\infty} \frac{e^{2it}}{2k+1} P_k \left( \frac{\mu}{2k+1} \right) f^{(2k+1)}(x) e^{-\frac{\mu t^2}{2k+1}} d\sigma(e)$ is an eigenfunction of $L$ with the eigenvalue $\mu$. $P_\mu$ is called the restriction operator.

Let $L_k^f$ be the Laguerre polynomial of type $\delta$ and degree $k$ defined by

$$L_k^f(t) = \frac{1}{k!} \frac{d}{dt} e^{-t} t^k e^{-\delta} \quad \forall k \in \mathbb{N}, \delta > -1$$

We define the normalized Laguerre functions by

$$L_k^f(\tau) = \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} \left( \frac{\tau}{\tau + \delta} \right)^\frac{1}{2} e^{-\frac{\tau}{2} \tau + \frac{\delta}{2} L_k^f(\tau)}$$

and set the Laguerre functions $\varphi_k(z) = L_k^f(1-\frac{1}{2}|z|^2) e^{-\frac{1}{2} |z|^2}, \forall z \in \mathbb{C}^d$.

Next we introduce the Weyl transform. Let $H^{d_1}$ be the $(2d_1 + 1)$-dimensional Heisenberg group. For each $a \in \mathbb{R} \setminus \{0\}$, $(z,s) \in H^{d_1}$, there is an infinite dimensional representation $\pi_a(z,s)$ which in the Schrödinger realization acts on $L^2(\mathbb{R}^{d_1})$ in the following way. For each $\varphi \in L^2(\mathbb{R}^{d_1})$, $z = x + iy$,

$$\pi_a(z,s) \varphi(x) = e^{iax} e^{ia(x \cdot \xi + \frac{i}{2} x \cdot y)} \varphi(x + y).$$

For any integrable function $g$ on $\mathbb{C}^{d_1}$, we define the Weyl transform of $g$ by

$$W_a(g) = \int_{\mathbb{C}^{d_1}} g(z) \pi_a(z,0) dz$$

For each $\varphi, \psi \in L^2(\mathbb{R}^{d_1})$,

$$|(W_a(g) \varphi, \psi)| = |\int_{\mathbb{C}^{d_1}} g(z) (\pi_a(z,0) \varphi, \psi) dz| \leq ||g||_1 ||\varphi||_2 ||\psi||_2$$

This shows that $W_a(g)$ is a bounded operator on $L^2(\mathbb{R}^{d_1})$ with $||W_a(g)|| \leq ||g||_1$. Furthermore, from the explicit description of $\pi(z,s)$ we see that

$$W_a(g) \varphi(\xi) = \int_{\mathbb{C}^{d_1}} e^{ia(x \cdot \xi + \frac{i}{2} x \cdot y)} g(x,y) \varphi(\xi + y) dx dy$$

From the above it follows that $W_a(g)$ is an integral operator whose kernel $K_a^g(x,y)$ is given by

$$K_a^g(x,y) = \int_{\mathbb{R}^{d_1}} g(\xi, y-x) e^{i\xi \cdot (x+y)} d\xi$$

Our main result is the following theorem:

**Theorem 1.2** For $1 \leq p \leq \frac{2(d_2+1)}{d_2+3}$ and $1 \leq q \leq 2 \leq r \leq \infty$, the following inequality holds

$$||P_\mu f||_{L^p_{\mu} L^q} \leq C \mu^{2d_2(\frac{1}{p} - \frac{1}{2}) + \frac{d_2}{2} (\frac{1}{q} - \frac{1}{2}) - 1} ||f||_{L^p_{\mu} L^q}$$

for any Schwartz function $f$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $\mu > 0$.

The theorem is stated in terms of the mixed Lebesgue norm

$$||f||_{L^p_{\mu} L^q} = \left( \int_{\mathbb{R}^{d_1}} \left( \int_{\mathbb{R}^{d_2}} |f(x,t)|^p dx \right)^{\frac{1}{p}} dt \right)^{\frac{1}{q}}, 1 \leq p, q \leq \infty$$

(with the obvious modifications when $p$ or $q$ are equal to $\infty$).
2 Restriction theorem

To prove the above theorem, first we state some asymptotic properties of the normalized Laguerre functions \( L_k^\delta(r) \) (see [10]). We let \( \nu = 4k + 2\delta + 2 \) and assume \( \delta > -1 \).

**Lemma 2.1** The Laguerre functions satisfy

\[
|L_k^\delta(\tau)| \leq C \begin{cases} 
(\tau \nu)^{\delta/2}, & 0 \leq \tau \leq 1/\nu \\
(\tau \nu)^{-1/4}, & 1/\nu \leq \tau \leq \nu/2 \\
\nu^{-1/4}(\nu^{1/3} + |\nu - \tau|)^{-1/4}, & \nu/2 \leq \tau \leq 3\nu/2 \\
e^{-\gamma \tau}, & \tau \geq 3\nu/2
\end{cases}
\]

where \( \gamma > 0 \) is a fixed constant.

Using the above estimates for the Laguerre functions, we can get a lower bound for the \( L^1 \) norm of them.

**Lemma 2.2**

\[
\int_0^\infty |L_k^{d_1-1}(\tau)|^q \tau^{-\frac{q}{2}} d\tau \leq C
\]

Moreover, there is an interesting result which connects the Laguerre function \( \varphi_{k,a}^{d_1-1}(z) = \varphi_k^{d_1-1}(\sqrt{a}z) \) with the spectral projection \( P_k(a) \) (see [11]).

**Lemma 2.3**

\[
W_a(\varphi_{k,a}^{d_1-1}) = (2\pi)^{d_1} |a|^{-d_1} P_k(a)
\]

In order to prove the restriction theorem, we need the estimates of the projections \( \varphi \to P_k(a)\varphi \) which are given in the following proposition.

**Proposition 2.1** For \( \varphi \in L^q(\mathbb{R}^{d_1}), 1 \leq q \leq 2 \),

\[
||P_k(a)\varphi||_2 \leq ||\varphi||_2 \cdot C |a|^{d_1} \left( \frac{1}{2} - \frac{1}{q} \right)(2k + d_1)^{\frac{d_1-1}{2} - \frac{1}{2}} ||\varphi||_q
\]

**Proof.** As \( ||P_k(a)\varphi||_2 \leq ||\varphi||_2 \), it is enough to prove the above estimate when \( q = 1 \). Since

\[
||P_k(a)\varphi||_2^2 = (P_k(a)\varphi, P_k(a)\varphi) = (P_k(a)\varphi, \varphi) \leq ||P_k(a)\varphi||_q ||\varphi||_q,
\]

it is enough to show that

\[
||P_k(a)\varphi||_\infty \leq |a|^{\frac{d_1}{2}} (2k + d_1)^{\frac{d_1-1}{2}} ||\varphi||_1. \tag{2.1}
\]

To prove (2.1) we use the fact that \( P_k(a) = (2\pi)^{-d_1} |a|^{d_1} W_a(\varphi_{k,a}^{d_1-1}) \). This shows that \( P_k(a) \) is an integral operator with the kernel \( F_{k,a}(x, y) \) given by

\[
F_{k,a}(x, y) = (2\pi)^{-d_1} |a|^{d_1} \int_{\mathbb{R}^{d_1}} e^{i\frac{\xi}{2} \cdot (x+y)} \varphi_{k,a}^{d_1-1}(\xi, x-y) d\xi
\]

\[
= (2\pi)^{-d_1} |a|^{d_1} \int_{\mathbb{R}^{d_1}} e^{i\frac{\xi}{2} \cdot (x+y)} L_{k,a}^{d_1-1} \left( \frac{|a|}{2} ||\xi||^2 + ||x-y||^2 \right) e^{-\frac{|a|}{4} (||\xi||^2 + ||x-y||^2)} d\xi
\]
Therefore, we have the estimate
\[
|F_{k,a}(x,y)| \leq (2\pi)^{-d_1} |a|^{d_1} \int_{\mathbb{R}^{d_1}} |L_k^{d_1-1} \left( \frac{|\alpha|}{2} (|\xi|^2 + |x-y|^2) \right) | e^{-\frac{|\alpha|}{4}(|\xi|^2+|x-y|^2)} \, d\xi \\
\leq C |a|^{d_1} \int_0^\infty |L_k^{d_1-1} \left( \frac{|\alpha|}{2} (r^2 + |x-y|^2) \right) | e^{-\frac{|\alpha|}{4}(r^2+|x-y|^2)} r^{d_1-1} \, dr \\
\leq C |a|^{d_1} \int_0^\infty |L_k^{d_1-1}(r) | r^{-\frac{1}{2}} \, dr 
\]

Using the estimate of Lemma 2.2 we get
\[
|F_{k,a}(x,y)| \leq C |a|^{d_1} (2k + d_1)^{-\frac{d_1-1}{2}}. 
\]
This proves (2.1) and hence the proposition.

Now we give a proof of Theorem 1.2.

**Proof of Theorem 1.2**  In order to simplify the notations, we write \(f\) as it were the product of two functions, that is \(f(x) = h(t)g(x)\), with \(f\) and \(g\) Schwartz functions. However, in the proof we will never use this fact. We take \(\alpha : \mathbb{R}^{d_1} \to \mathbb{C}\) and \(\beta : \mathbb{R}^{d_2} \to \mathbb{C}\), \(\alpha \in \mathcal{S}(\mathbb{R}^{d_1}), \beta \in \mathcal{S}(\mathbb{R}^{d_2})\). Because the spectral projections associated to the scaled Hermite operator are orthogonal, we have

\[
\langle \mathcal{P}_\mu f, \alpha \otimes \beta \rangle = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \mathcal{P}_\mu f(x,t) \overline{\alpha(x)\beta(t)} \, dxdt \\
= \frac{1}{2\pi} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left[ \sum_{k=0}^\infty \frac{\mu^{d_2-1}}{(2k + d_1)^{d_2}} \int_{S^{d_2-1}} \hat{h} \left( \frac{\mu e}{2k + d_1} \right) \right. \\
\left. \times P_k \left( \frac{\mu}{2k + d_1} \right) g(x) e^{-\frac{\mu x^2}{2k + d_1}} \, d\sigma(\epsilon) \right] \overline{\alpha(x)\beta(t)} \, dxdt \\
= \frac{1}{2\pi} \sum_{k=0}^\infty \frac{\mu^{d_2-1}}{(2k + d_1)^{d_2}} \int_{\mathbb{R}^{d_1}} \int_{S^{d_2-1}} \hat{h} \left( \frac{\mu e}{2k + d_1} \right) \\
\times \left[ P_k \left( \frac{\mu}{2k + d_1} \right) g(x) \left( \int_{\mathbb{R}^{d_2}} \overline{\alpha(x)\beta(t)} e^{-i\frac{\mu x^2}{2k + d_1}} \, dt \right) d\sigma(\epsilon) \right] \\
= \frac{1}{2\pi} \sum_{k=0}^\infty \frac{\mu^{d_2-1}}{(2k + d_1)^{d_2}} \int_{S^{d_2-1}} \hat{h} \left( \frac{\mu e}{2k + d_1} \right) \beta \left( \frac{\mu e}{2k + d_1} \right) \\
\times \langle P_k \left( \frac{\mu}{2k + d_1} \right) g, P_k \left( \frac{\mu}{2k + d_1} \right) \alpha \rangle \\
= \frac{1}{2\pi} \sum_{k=0}^\infty \frac{\mu^{d_2-1}}{(2k + d_1)^{d_2}} \int_{S^{d_2-1}} \hat{h} \left( \frac{\mu e}{2k + d_1} \right) \beta \left( \frac{\mu e}{2k + d_1} \right) \\
\times \langle P_k \left( \frac{\mu}{2k + d_1} \right) g, P_k \left( \frac{\mu}{2k + d_1} \right) \alpha \rangle 
\]
Applying the Hölder’s inequality to the inner integral we deduce that

\[
\langle \mathcal{P}_\mu f, \alpha \otimes \beta \rangle \leq \frac{1}{2\pi} \sum_{k=0}^\infty \frac{\mu^{d_2-1}}{(2k+d_1)^{d_2}} \left( \int_{S^{d_2-1}} \left| \hat{h} \left( \frac{\mu \varepsilon}{2k+d_1} \right) \right|^2 d\sigma(e) \right)^{\frac{1}{2}}
\times \left( \int_{S^{d_2-1}} \left| \hat{\beta} \left( \frac{\mu \varepsilon}{2k+d_1} \right) \right|^2 d\sigma(e) \right)^{\frac{1}{2}} \left\| P_k \left( \frac{\mu}{2k+d_1} \right) g \right\|_{L^2} \left\| P_k \left( \frac{\mu}{2k+d_1} \right) \alpha \right\|_{L^2}
\]

By Proposition 2.1 we have for any \(1 \leq q \leq 2 \leq r \leq \infty\)

\[
\left\| P_k \left( \frac{\mu}{2k+d_1} \right) g \right\|_{L^q} \leq C \mu^{\frac{d_2}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} (2k+d_1)^{-\frac{d_2}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \|g\|_{L^q}
\quad (2.2)
\]

\[
\left\| P_k \left( \frac{\mu}{2k+d_1} \right) \alpha \right\|_{L^q} \leq C \mu^{\frac{d_2}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} (2k+d_1)^{-\frac{d_2}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} \|\alpha\|_{L^q}
\quad (2.3)
\]

For \(1 \leq p \leq \frac{2(d_2+1)}{d_2+3}\), it follows from the restriction theorem of the Fourier transform on \(S^{d_2-1}\) that

\[
\left( \int_{S^{d_2-1}} \left| \hat{h} \left( \frac{\mu \varepsilon}{2k+d_1} \right) \right|^2 d\sigma(e) \right)^{\frac{1}{2}} \leq C \left( \frac{2k+d_1}{\mu} \right)^{d_2(1-\frac{1}{p})} \|h\|_{L^p}
\quad (2.4)
\]

\[
\left( \int_{S^{d_2-1}} \left| \hat{\beta} \left( \frac{\mu \varepsilon}{2k+d_1} \right) \right|^2 d\sigma(e) \right)^{\frac{1}{2}} \leq C \left( \frac{2k+d_1}{\mu} \right)^{d_2(1-\frac{1}{q})} \|\beta\|_{L^q}
\quad (2.5)
\]

Therefore, by (2.2), (2.3), (2.4) and (2.5) we have

\[
\langle \mathcal{P}_\mu f, \alpha \otimes \beta \rangle \leq C \sum_{k=0}^\infty (2k+d_1)^{-2d_2 \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{d_2}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \mu^{2d_2 \left( \frac{1}{p} - \frac{1}{2} \right) + \frac{d_2}{2} \left( \frac{1}{q} - \frac{1}{2} \right) - 1} \|f\|_{L^p \mathbb{R}^d} \|\alpha \otimes \beta\|_{L^q \mathbb{R}^d}
\]

If \(d_2 \geq 2\) or \(1 \leq q < 2 \leq r \leq \infty\) or \(1 \leq q \leq 2 < r \leq \infty\), because of \(1 \leq p \leq \frac{2(d_2+1)}{d_2+3}\), we have

\[
2d_2 \left( \frac{1}{p} - \frac{1}{2} \right) + \frac{d_2}{2} \left( \frac{1}{q} - \frac{1}{2} \right) - 1 > 1. \]

Hence, the above sum converges and consequently we have

\[
\|\mathcal{P}_\mu f\|_{L^p \mathbb{R}^d} \leq C \mu^{2d_2 \left( \frac{1}{p} - \frac{1}{2} \right) + \frac{d_2}{2} \left( \frac{1}{q} - \frac{1}{2} \right) - 1} \|f\|_{L^p \mathbb{R}^d}
\]

If \(d_2 = 1\) and \(r = q = 2\), we have

\[
\mathcal{P}_\mu f(x,t) = \frac{1}{2\pi} \sum_{k=0}^\infty \frac{1}{2k+d_1} \left[ P_k \left( \frac{\mu}{2k+d_1} \right) f^{\frac{\mu}{2k+d_1}}(x) e^{-\frac{\mu}{2k+d_1} t} + P_k \left( \frac{\mu}{2k+d_1} \right) f^{-\frac{\mu}{2k+d_1}}(x) e^{\frac{\mu}{2k+d_1} t} \right].
\]

Since the operators \(P_k(a)\) are orthogonal projections, we have

\[
\|\mathcal{P}_\mu f\|_{L^p \mathbb{R}^d} \leq \frac{1}{2\pi} \sum_{k=0}^\infty \frac{1}{2k+d_1} \left[ \|P_k \left( \frac{\mu}{2k+d_1} \right) f^{\frac{\mu}{2k+d_1}}\|_{L^2} + \|P_k \left( \frac{\mu}{2k+d_1} \right) f^{-\frac{\mu}{2k+d_1}}\|_{L^2} \right]
\]
\[
\leq \frac{1}{2\pi} \sum_{k=0}^{\infty} \left( \|P_k \left( \frac{\mu}{2k + d_1} \right) f \|_{L^2} + \|P_k \left( \frac{\mu}{2k + d_1} \right) f \|_{L^2} \right)
\]

\[
\leq \frac{1}{2\pi} \left( \|f \|_{L^2} + \|f \|_{L^2} \right)
\]

\[
\leq \frac{1}{\pi} \|f\|_{L^2}
\]

### 3 Sharpness of the range \( p \)

In this section we only give an example to show that the range of \( p \) in the restriction theorem is sharp. The example is constructed similarly to the counterexample of Müller [3], which shows that the estimates between Lebesgue spaces for the operators \( P_\mu \) are necessarily trivial.

Let \( \varphi \in C_0^\infty(\mathbb{R}^{d_2}) \) be a radial function, such that \( \varphi(a) = \psi(|a|) \), where \( \psi \in C_0^\infty(\mathbb{R}) \), \( \psi = 1 \) on a neighborhood of the point \( \frac{1}{d_1} \) and \( \psi = 0 \) near 0. Let \( h \) be a Schwartz function on \( \mathbb{R}^{d_2} \) and define

\[
f(x, t) = \int_{\mathbb{R}^{d_2}} \varphi(\lambda) \hat{h}(\lambda) e^{-\frac{|x|^2}{2\lambda}} e^{-i\langle \lambda, t \rangle} |\lambda|^n d\lambda
\]

Denote \( g(x, t) = \int_{\mathbb{R}^{d_2}} \varphi(\lambda) e^{-\frac{|x|^2}{2\lambda}} e^{-i\langle \lambda, t \rangle} |\lambda|^n d\lambda = \int_{\mathbb{R}^{d_1+2d_2}} \varphi(\lambda) e^{-\frac{|x|^2}{2\lambda}} e^{-i\langle \lambda, t \rangle + \langle \xi, x \rangle} d\xi d\lambda. \)

Hence \( \hat{g}(\xi, a) = \varphi(a) e^{-\frac{|a|^2}{2\pi}} \), which shows that \( \hat{g} \) and consequently \( g \) are Schwartz functions. On the other hand, we have \( f = h *_t g \), where " *_t " denotes the involution about the second variable. Then,

\[
f(x, t) = \int_0^{+\infty} \left( \lambda^{d_1+2d_2-1} \psi(\lambda) e^{-\frac{|x|^2}{2\lambda}} \int_{S^{d_2-1}} \hat{h}(\lambda w) e^{-i\langle \lambda w, t \rangle} d\sigma(w) \right) d\lambda
\]

\[
= \int_0^{+\infty} \left( d_1^{-d_1-2d_2} \mu^{d_1+d_2-1} \psi(\mu d_1) e^{-\frac{|x|^2}{2\mu d_1}} \int_{S^{d_2-1}} \hat{h}(\mu w) e^{-i\langle \mu w, t \rangle} d\sigma(w) \right) d\mu
\]

\[
= \int_0^{+\infty} P_\mu f(x, t) d\mu
\]

where

\[
P_\mu f(x, t) = d_1^{-d_1-2d_2} \mu^{d_1+d_2-1} \psi(\mu d_1) e^{-\frac{|x|^2}{2\mu d_1}} \int_{S^{d_2-1}} \hat{h}(\mu w) e^{-i\langle \mu w, t \rangle} d\sigma(w)
\]

and it satisfies \( L(P_\mu f) = \mu P_\mu f \).

Therefore, specially let \( \mu = 1 \), we have

\[
P_1 f(x, t) = d_1^{-d_1-2d_2} e^{-\frac{|x|^2}{2\pi t}} \int_{S^{d_2-1}} \hat{h} \left( \frac{w}{d_1} \right) e^{-i\langle \frac{w}{d_1}, t \rangle} d\sigma(w)
\]

\[
= d_1^{-d_1-2d_2} e^{-\frac{|x|^2}{2\pi t}} h * \frac{d\sigma}{d_1}(t)
\]
From the restriction theorem associated the Grushin operators, we have the estimate \( \|P_1 f\|_{L^p_{x} L^q_{x}} \leq C \|f\|_{L^p L^q_{x}} \).

Because of
\[
\|P_1 f\|_{L^p_{x} L^q_{x}} = C \|h * \frac{d\sigma}{d_1} \|_{L^p_{x}} \tag{3.1}
\]
and
\[
\|f\|_{L^p_{x} L^q_{x}} \leq \|h\|_{L^p_{x}} |g|_{L^1 L^q_{x}} \lesssim \|h\|_{L^p_{x}} \tag{3.2}
\]
we have \( \|h * \frac{d\sigma}{d_1} \|_{L^p_{x}} \leq C \|h\|_{L^p_{x}} \).

From the sharpness of Stein-Tomas theorem which is guaranteed by the Knapp counterexample, it would imply \( p \leq 2 \left( \frac{d_2 + 1}{d_2 + 3} \right) \). Hence the range of \( p \) can not be extended.

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