Stability of solitons under rapidly oscillating random perturbations of the initial conditions

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Abstract:
We use the Inverse Scattering Transform and a diffusion approximation limit Theorem to study the stability of soliton components of the solution of the nonlinear Schrödinger and Korteweg-de Vries equations under random perturbations of the initial conditions: for a wide class of rapidly oscillating random perturbations this problem reduces to the study of a canonical system of stochastic differential equations which depends only on the integrated covariance of the perturbation. We finally study the problem when the perturbation is weak, which allows us to analyze the stability of solitons quantitatively.

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1. Introduction

The aim of the present work is to study the stability of the soliton components of solutions of completely integrable systems under rapidly oscillating random perturbations of the initial condition. We will consider and compare two important examples of equations widely employed to model nonlinear and dispersive effects in wave propagation: the (1-dimensional) nonlinear Schrödinger (NLS) equation

$$\frac{\partial U}{\partial t} + i \frac{\partial^2 U}{2 \partial x^2} + i|U|^2 U = 0$$

(1)

and the Korteweg-de Vries (KdV) equation

$$\frac{\partial U}{\partial t} + 6U \frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0.$$ 

(2)

The NLS equation models in particular short pulse propagation in single-mode optical fibers (then $t$ is a propagation distance and $x$ is a time) [MN]. The KdV equation models shallow water wave propagation [Wh].

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Explicit computations are developed in the case of a square (box–like) initial condition perturbed with a zero mean, stationary, rapidly oscillating process $\nu_\varepsilon(x) := \nu(x/\varepsilon^2)$:

$$U_0(x) = \left(q + \frac{2}{\varepsilon} \nu_\varepsilon \right) 1_{[0,R]}(x),$$

but they can be extended to the case of perturbation of a more general initial condition defined by a bounded, compactly supported function $q(x)$. Our approach to both examples relies on the results of the Inverse Scattering Transform (IST), a powerful tool to study solutions of completely integrable nonlinear equations, see [APT]. In this framework, the problem is transformed into a linear system of differential equations where the initial condition enters as a potential and soliton components correspond to eigenvalues. Indeed, any initial condition may generate soliton components, that are solitary waves that propagate over arbitrarily large distances with constant velocity and constant profile, and radiation components, that decay in amplitude as a power law. The identification of the soliton components therefore characterizes the long–time behavior of the solution of the PDE.

We will show in Section 2 that for rapidly oscillating processes (small values of $\varepsilon$) the limit system governing the stability of the soliton components reads as a set of stochastic differential equations (SDEs) and it is formally equivalent to the system where the initial condition contains a white–noise perturbation:

$$U_0(x) = \left(q + \sqrt{2\alpha} \sigma \dot{W}_x \right) 1_{[0,R]}(x),$$

where $\alpha$ is the integrated covariance of the process $\nu$. This shows that to study the soliton components in the limit of rapid oscillations, the only required parameter of the statistics of $\nu$ is its integrated covariance. Notice that we cannot directly use a white noise to perturb the initial condition, as the IST requires some integrability conditions on the initial condition (for example, $U_0 \in L^1$), which are not satisfied by a white noise.

Our results state that solitons are stable under perturbations of the initial condition for both examples studied. However, a few interesting differences will be pointed out; in particular, thresholding effects for the creation of solitons are present in the NLS case and absent for KdV.

We also provide an easy way to compute the first order corrections to the parameters characterizing the soliton components of the solution.

In the IST framework, a direct scattering problem (known as the Zakharov–Shabat spectral problem - ZSSP) associated to the NLS equation is introduced

$$\begin{cases} \frac{\partial \psi_1}{\partial x} = iU_0(x) \psi_2 - i\zeta \psi_1 \\ \frac{\partial \psi_2}{\partial x} = iU_0^*(x) \psi_1 + i\zeta \psi_2 \end{cases},$$

where $\psi_n$ are the components of a vector eigenfunction $\Psi$ and $\zeta$ is the spectral parameter. For any localized initial profile $U_0$ the continuous spectrum consists
of the real axis. For \( \zeta \in \mathbb{R} \) the solutions of (5) with the boundary conditions

\[
\Psi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}, \quad \bar{\Psi} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}, \quad x \to -\infty
\]

\[
\Phi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}, \quad \bar{\Phi} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}, \quad x \to +\infty
\]

define two sets \((\Psi, \bar{\Psi})\) and \((\Phi, \bar{\Phi})\) of linearly independent solutions. Therefore, these functions are related through the system

\[
\begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix} = \begin{pmatrix} b(\zeta) & a(\zeta) \\ \bar{a}(\zeta) & \bar{b}(\zeta) \end{pmatrix} \begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix},
\]

where \(a, b\) are called Jost coefficients. The functions \(\Psi\) and \(\Phi\) are called Jost functions. If \(U_0 \in L^1(\mathbb{R})\), the function \(a(\zeta)\) is analytic in the upper half of the complex plane, where it can only have a countable number of simple zeros, see [APT, Lemma 2.1]. These zeros are the eigenvalues of the discrete spectrum. Each discrete eigenvalue \(\zeta = \xi + i\eta, \eta > 0\), corresponds to a soliton component of the solution. A pure soliton solution of the NLS equation has the form

\[
U(t, x) = 2i\eta \frac{\exp\left(-2i\xi x - 4i(\xi^2 - \eta^2)t\right)}{\cosh\left(2\eta(x + 4\xi t)\right)},
\]

up to a shift and a phase.

Similarly, one can link the existence of soliton components of solutions of the KdV equation to the spectral properties of an associated equation, the first equation of the Lax pair:

\[
\frac{\partial^2 \varphi}{\partial x^2} + (U_0 + \zeta^2) \varphi = 0,
\]

where \(\varphi : \mathbb{R} \to \mathbb{R}\). We need to assume that

\[
U_0 \in P_\mu := \left\{ f : \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{\infty} (1 + |x|^\mu) |U(x)| \, dx < \infty \right\}
\]

for \(\mu = 1\) or \(\mu = 2\), see [AC, chapter 2]. Consider the continuous part of the spectrum of equation (7), which is again the real axis. For \(\zeta \in \mathbb{R}\), there are two convenient complete sets of bounded functions associated to equation (7), defined by their asymptotic behavior:

\[
\phi(x, \zeta) \sim e^{-i\zeta x}, \quad \bar{\phi}(x, \zeta) \sim e^{i\zeta x} \quad \text{for } x \to -\infty
\]

\[
\psi(x, \zeta) \sim e^{i\zeta x}, \quad \bar{\psi}(x, \zeta) \sim e^{-i\zeta x} \quad \text{for } x \to +\infty.
\]

It follows from the above definitions that

\[
\phi(x, \zeta) = \bar{\psi}(x, -\zeta), \quad \psi(x, \zeta) = \bar{\phi}(x, -\zeta)
\]
and

\[ \phi(x, \zeta) = a(\zeta) \tilde{\psi}(x, \zeta) + b(\zeta) \psi(x, \zeta) \]

\[ \bar{\phi}(x, \zeta) = -\bar{a}(\zeta) \psi(x, \zeta) + \bar{b}(\zeta) \tilde{\psi}(x, \zeta) . \]

The function \( a \) is analytical in the upper half of the complex \( \zeta \)-plane, where it has only a finite number of simple zeros located on the imaginary axis \( \zeta = i\eta \), see [AC, Lemma 2.2.2]. These zeros are the eigenvalues of the discrete spectrum, and they correspond to the soliton component of the solution. A pure soliton solution is given by

\[ U(t, x) = 2\eta^2 \text{sech}^2 \left( \eta(x - x(t)) \right) , \]

where \( x(t) = x_0 + 4\eta^2t \) is the center of the soliton.

2. Limit of rapidly oscillating processes

This section contains a rigorous justification of the use of the IST. As remarked in the introduction, to be able to apply the IST, the initial condition \( U_0 \) needs to satisfy some integrability condition, \( L^1 \) for NLS and (8) for KdV. These hypotheses are satisfied by initial conditions of the form (3) if \( \nu \) is bounded. Our objective is to show that the IST applied to these random initial conditions gives a problem that reads as a canonical system of SDEs in the limit \( \varepsilon \to 0 \).

We make the following assumptions on the process \( \nu \):

**Hypothesis 2.1.** Let \( \nu(x) \) be a real, homogeneous, ergodic, centered, bounded, Markov stochastic process, with finite integrated covariance \( \int_0^\infty \mathbb{E}[\nu(0)\nu(x)]dx = \alpha < \infty \) and with generator \( L_\nu \) satisfying the Freedholm alternative.

Set

\[ U_0^\varepsilon := \left( q + \frac{\sigma}{\varepsilon} \nu(x/\varepsilon^2) \right) 1_{[0,R]}(x) \]

and remark that for \( x \in [0, R] \)

\[ \int_0^x U_0^\varepsilon(x')dx' \xrightarrow{\varepsilon \to 0} \int_0^x U_0(x')dx' = qx + \sqrt{2\alpha}\sigma W_x \]

in distribution, see [FGPS]. For every \( \varepsilon > 0 \), we apply the IST to the NLS and KdV equations with the initial condition \( U_0^\varepsilon \).

We consider the ZSSP associated to the NLS equation: our goal is to identify the points of the upper half of the complex plane, \( \zeta \in \mathbb{C}^+ \), for which there exists a solution \( \Psi \) of the first order system (5) for \( x \in [0, R] \), satisfying the boundary conditions

\[ \Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_1(R) = 0 \]
derived from the exponentially decaying conditions (6). These particular values of \( \zeta \) are the discrete eigenvalues of the ZSSP and correspond to the soliton components. The strategy employed is to consider the flow \( \Psi(x, \zeta) \), \( x \in [0, R] \), \( \zeta \in \mathbb{C}^+ \), solution of (5) with initial condition

\[
\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

(9)

and look for the values of \( \zeta \) for which the final condition is satisfied.

For a fixed value of \( \zeta \), we consider the solution \( \Psi^\varepsilon \) of the ZSSP obtained from the IST

\[
\begin{align*}
\frac{\partial \psi^\varepsilon_1}{\partial x} &= -i \zeta \psi^\varepsilon_1 + i U^\varepsilon_0(x) \psi^\varepsilon_2 \\
\frac{\partial \psi^\varepsilon_2}{\partial x} &= i \left(U^\varepsilon_0(x)\right)^* \psi^\varepsilon_1 + i \zeta \psi^\varepsilon_2 
\end{align*}
\]

(10)

with initial condition

\[
\Psi^\varepsilon(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Now, [FGPS, Theorem 6.1] states that the process \( \Psi^\varepsilon \) converges in distribution in \( \mathcal{C}([0, R]; \mathbb{C}^2) \) to the process \( \Psi \) solution of

\[
\begin{align*}
\frac{d \psi_1}{dx} &= \left( -i \zeta - \alpha \sigma^2 \right) \psi_1 + iq \psi_2 \\
\frac{d \psi_2}{dx} &= i \left( U^\varepsilon_0(x) \right)^* \psi_1 + i \zeta \psi_2 
\end{align*}
\]

(11)

with initial condition (9), which can be rewritten in Stratonovich form as

\[
d \Psi = i \begin{pmatrix} -\zeta & q \\ q & \zeta \end{pmatrix} \Psi dx + i \sqrt{2 \alpha \sigma} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi \circ dW_x.
\]

(12)

For NLS we can also consider perturbations produced by a complex process: let \( \nu_1, \nu_2 \) be two independent copies of the process \( \nu \) and set \( \widetilde{\nu} := \nu_1 + i \nu_2 \). One can define \( U^\varepsilon_0 \) using \( \widetilde{\nu} \) instead of \( \nu \); proceeding as above, from the IST one obtains again the system (10), and from [FGPS, Theorem 6.1] one gets that in this case the limit process is the solution of

\[
d \Psi = i \begin{pmatrix} -\zeta & q \\ q & \zeta \end{pmatrix} \Psi dx + i \sqrt{2 \alpha \sigma} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi \circ dW_x^{(1)} - i \sqrt{2 \alpha \sigma} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Psi \circ dW_x^{(2)},
\]

(13)

where the \( W^{(i)} \) are two independent Wiener processes, with the same initial condition (9).

We apply the same strategy to the KdV equation: the goal is to obtain the values of \( \zeta \in \mathbb{C}^+ \) for which there exists a solution \( \varphi \) of

\[
\varphi_{xx} + (U^\varepsilon_0 + \zeta^2) \varphi^\varepsilon = 0
\]

(14)

with the boundary conditions

\[
\varphi(0) = 1, \quad \varphi_x(0) = -i \zeta, \quad \varphi_x(R) - i \zeta \varphi(R) = 0.
\]
These conditions correspond to imposing exponential decay of the solution at infinity.

Setting $\Phi := (\varphi, \varphi_x)^T$ this equation can be transformed into
\[
\frac{d\Phi}{\varepsilon} = \begin{pmatrix} 0 & 1 \\ -U_0 - \zeta^2 & 0 \end{pmatrix} \Phi^x \, dx, \quad \Phi(0) = \begin{pmatrix} 1 \\ -i\zeta \end{pmatrix}, \quad \phi_2(R) - i\zeta \phi_1(R) = 0.
\]

We consider the flow $\Phi^\varepsilon(x, \zeta) \in [0, R]$, $\zeta \in \mathbb{C}^+$, defined by the above equation with only the initial condition, and look for the values of $\zeta$ s.t. the final condition is satisfied. Again by [FGPS, Theorem 6.1], $\Phi^\varepsilon$ converges in distribution to the solution of
\[
\frac{d\Phi}{x} = \begin{pmatrix} 0 & 1 \\ -q - \zeta^2 & 0 \end{pmatrix} \Phi \, dx + \sqrt{2\alpha} \sigma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi \, W_x
\]
which, in terms of the function $\varphi$, can be rewritten as
\[
\frac{d\varphi_x}{x} = -(q + \zeta^2) \varphi \, dx + \sqrt{2\alpha} \sigma \varphi \, W_x. \tag{15}
\]

The initial condition is
\[
\Phi(0) = \begin{pmatrix} 1 \\ -i\zeta \end{pmatrix}, \quad \text{or equivalently} \quad \varphi(0) = 1, \quad \varphi_x(0) = -i\zeta. \tag{16}
\]

Remark that in the last two differential equations above the Stratonovich and Itô stochastic integrals coincide.

The convergence obtained above is only for a (finite number of) fixed $\zeta$ and $\sigma$, but we will need a convergence in $C^0([0, R]; C^1(\mathbb{R}^3))$ to be able to differentiate the limit process with respect to the parameters. This is the main result of this section and it is provided by the following Theorem. We will focus on the problem of finding the values of $\zeta$ for which the limit flows $\Psi$ and $\Phi$ match the final conditions in Sections 3 and 4.

**Theorem 2.2.** Assume Hypothesis 2.1. Let $\Psi^\varepsilon := (\psi_1^\varepsilon, \psi_2^\varepsilon)^T$ be the solution of (10) with initial condition (9) and $\Psi$ the solution of (12) with the same initial condition. Let also $\varphi^\varepsilon$ be the solution of (14) with initial condition (16) and $\varphi$ be the solution of (15) with the same initial condition. Considering these as functions of the space variable $x$ and the parameters $\xi, \eta, \sigma$, we have in the limit of $\varepsilon \to 0$ that $\Psi^\varepsilon(x, \xi, \eta, \sigma) \to \Psi(x, \xi, \eta, \sigma)$ in $C^0([0, R]; C^1(\mathbb{R}^3; \mathbb{C}))$ and $\varphi^\varepsilon(x, \xi, \eta, \sigma) \to \varphi(x, \xi, \eta, \sigma)$ in $C^0([0, R]; C^1(\mathbb{R}^3; \mathbb{C}))$.

To prove this theorem we need the following standard tightness criteria, see [Me84].

**Lemma 2.3.** Let $(E, d)$ be a metric space, and $X^\varepsilon$ a process with paths in $D([0, R]; E)$. If for every $x$ in a dense subset of $[0, R]$ the family $\left(X^\varepsilon(x)\right)_{\varepsilon \in (0,1]}$ is tight in $E$ and $X^\varepsilon$ satisfy the Aldous property:
A. For any $\kappa > 0$, $\lambda > 0$, there exists $\delta > 0$ s.t.

$$\limsup_{\varepsilon \to 0} \sup_{\tau < R \theta < \delta (R - \tau)} \mathbb{P} \left( \| X^\varepsilon(\tau + \theta) - X^\varepsilon(\tau) \| > \lambda \right) < \kappa,$$

where $\tau$ is a stopping time;

then the family $\{X^\varepsilon\}_{\varepsilon \in (0,1]}$ is tight in $D([0,R]; E)$.

**Lemma 2.4.** Let $\mathcal{H}$ be an Hilbert space and $\mathcal{H}_n$ be an increasing sequence of finite-dimensional subspaces of $\mathcal{H}$ s.t., for any $h \in \mathcal{H}$, $\lim_{n \to \infty} \pi_{\mathcal{H}_n} h = h$. Let $X^\varepsilon$ be an $\mathcal{H}$-valued process. Then, $X^\varepsilon$ is tight iff for any $\kappa > 0$ and $\lambda > 0$, there exist $\rho_n$ and a subspace $\mathcal{H}_{n,\lambda}$ s.t.

$$\sup_{\varepsilon \in (0,1]} \mathbb{P}(\|X^\varepsilon\| \geq \rho_n) \leq \kappa$$

and

$$\sup_{\varepsilon \in (0,1]} \mathbb{P}(d_{\mathcal{H}}(X^\varepsilon, \mathcal{H}_{n,\lambda}) > \lambda) \leq \kappa.$$  \hfill (17)

**Proof of Theorem 2.2.** Since Propositions 3.2 and 4.2 ensure that the limit equations for $\Psi$ and $\Phi$ have a unique solution which is $C^0([0,R]; C^1(\mathbb{R}^3))$, it suffice to prove convergence in the space of CadLag processes $D([0,R]; C^1(\mathbb{R}^3))$, which is in turn provided by Lemma 2.3 if we take $E$ to be the Hilbert space $\mathcal{H} := W^{3,2}(G)$, which is imbedded in $C^1(G)$. Here, $G$ is an open, bounded subset of $\mathbb{R}^3$, the space of parameters. For simplicity we take $G = (-N,N)^3$ for some real positive constant $N$; a justification of the fact that it is not restrictive to assume that the set of parameters $G$ is bounded is given below in the proof of Proposition 3.2, where the convergences we are proving here will be used.

We start by using Lemma 2.4 to obtain the necessary tightness of the sequences $\{\Psi^\varepsilon\}$ and $\{\Phi^\varepsilon\}$. Let $d_{\mathcal{H}}$ be the distance induced on $\mathcal{H}$ by the scalar product and let $X^\varepsilon$ be the solution of the approximated system $X^\varepsilon$ is equal to either $\Psi^\varepsilon$ or $\Phi^\varepsilon$. The space $\mathcal{H}_n$ is constructed as follows. Divide $G$ into cubes with sides of length $1/n$ and add one extra layer of cubes around it: $A_{i,j,k} := [i/n, (i+1)/n] \times [j/n, (j+1)/n] \times [k/n, (k+1)/n]$, for $i,j,k = -(nN+1), \ldots, nN$. Define the piecewise (on every cube) polynomials of fourth degree as

$$\tilde{h}(x) := \sum_{i,j,k = -(Nn+1)}^{Nn} \sum_{m=0}^{4} \frac{1}{m!} \langle a_{i,j,k}^{(m)} | x - y_{i,j,k} \rangle^{(m)} \mathbb{1}_{A_{i,j,k}}(x),$$  \hfill (18)

where $y_{i,j,k}$ is the center of the cube $A_{i,j,k}$, $a_{i,j,k}^{(m)}$ are families of $m$-dimensional tensors and the brackets denote the relative tensor products (so that, for example, $\langle a_{i,j,k}^{(4)} | x - y_{i,j,k} \rangle^{(4)}$ denotes the product between the four-dimensional tensor $a_{i,j,k}^{(4)}$ and four copies of the vector $x-y_{i,j,k}$). With this definitions $\tilde{h}$ is a function defined on $[-N-\frac{1}{n}, N+\frac{1}{n}]^3$, but its restriction to $G$ does not belong to $\mathcal{H}$ in general, since it may not even be continuous. Let $\Gamma$ be a real, nonnegative, smooth function, with compact support contained in $[-1/2, 1/2]^3$ and such that $\int_{[-1/2, 1/2]^3} \Gamma(x) \, dx = 1$. Setting $\Gamma_n(x) := n^3 \Gamma(nx)$ we can finally define $\mathcal{H}_n$ as the finite dimensional space of functions of the form $h(x) := (\tilde{h} \ast \Gamma_n)(x)$.
Remark that \( \tilde{h} \) has been defined on a set larger than \( G \), so that the convolution product is well defined for \( x \in G \).

Since \( X^\varepsilon \) is the solution of a linear differential equation (if \( q \) is constant it even has constant coefficients in \( x \)) with coefficients smooth in the parameters \( \mu = (\xi, \eta, \sigma) \), from the explicit formula for the solution we get that \( X^\varepsilon(x, \mu) \) is smooth in the parameters. We will soon use its derivatives in the parameters: the vector of \( X^\varepsilon \) and its first derivatives in \( x \) still satisfy a linear system of ODEs whose coefficients depend linearly on the parameters and on the process \( \nu(x/\varepsilon^2) \), and the same result holds adding higher order derivatives.

The key step to show that \( \{ X^\varepsilon(x, \mu) \}_{\mu \in K} \) is tight in \( \mathcal{H} = W^{3,2}(G) \) for every \( x \in [0, R] \) is the proof of the bound

\[
\limsup_{\varepsilon \to 0} \mathbb{E}\left[ \|X^\varepsilon(x, \cdot)\|_{W^{6,2}(G)} \right] < \infty
\] (19)

uniformly in \( x \in [0, R] \). Indeed, the first part of condition (17) immediately follows, and for the second part we only need to use Sobolev’s imbedding \( W^{6,2} \hookrightarrow C^4(\mathbb{R}^3) \) and the following lemma. Corollary 2.6 will finally provide the last hypothesis of Lemma 2.4.

Lemma 2.5. For every \( g \in C^4(G) \), there exists a \( g_n \in \mathcal{H}_n \) s.t.

\[
\|g - g_n\|_{\mathcal{H}} \leq \frac{C}{n} \|g\|_{C^4(G)} .
\]

Proof. Step 1: (Construction of \( g_n \)). For \( i, j, k = -nN, \ldots, nN - 1 \) (which means that \( y_{i,j,k} \in G \)) set \( a_{i,j,k}^{(m)} := D^m g(y_{i,j,k}) \). For \( i, j, k \) such that \( y_{i,j,k} \notin G \) set \( a_{i,j,k}^{(m)} := D^m g(y') \), where \( y' \) is the nearest cube center; notice that the distance of these two points is at most the diameter of the cubes, which we call \( 2\delta := 2\sqrt{3}n^{-1} \). With the \( a_{i,j,k}^{(m)} \) thus defined we construct the piecewise polynomial function \( \tilde{g}_n \) as in (18): on the cubes the centers of which are not in \( G \), this function is just a copy of the function defined on the nearest cube with center in \( G \). Finally, we have \( g_n := \Gamma_n \ast \tilde{g}_n \).

Step 2: (Estimates). For every multiindex \( a \in \mathbb{N}^3 \) such that \( |a|_1 := a_1 + a_2 + a_3 \leq 3 \), we need to estimate

\[
\int_G \left| \partial^a (\Gamma_n \ast \tilde{g}_n)(x) - \partial^a g(x) \right|^2 \, dx .
\]

To clarify the procedure to obtain an estimate for the above term we first give explicit computations for the case \( a = e_1 = (1, 0, 0) \). Recall that, by definition,

\[
\sum_{i,j,k} \int_{A_{i,j,k}} \Gamma_n(x - y) \, dy = \int_{[-(N+1)/n, N+1/n]^3} \Gamma_n(x - y) \, dy = 1
\]
For \( a = (1, 0, 0) \), using twice the inequality \((a + b)^2 \leq 2(a^2 + b^2)\), we have

\[
|\partial_1 (\mathcal{I} + \tilde{g}_n)(x) - \partial_1 g(x)|^2 = \left| \sum_{i,j,k} \int_{A_{i,j,k}} -\partial_{y_1} \mathcal{I}^n(x-y)\tilde{g}_n(y)dy - \partial_1 g(x) \right|^2 \leq 2 \left\{ \sum_{i,j,k} \int_{A_{i,j,k}} \mathcal{I}^n(x-y)\partial_{y_1}\tilde{g}_n(y)dy - \partial_1 g(x) \right\}^2 \\
+ \left\{ \sum_{i,j,k} \int_{A_{i,j,k}} \mathcal{I}^n(x-y)\partial_{y_1}\tilde{g}_n(y) - \partial_1 \tilde{g}_n(x) \right\}^2 \\
+ \left\{ \sum_{i,j,k} \int_{A_{i,j,k}} \mathcal{I}^n(x-y)\partial_{y_1}\tilde{g}_n(y) - \partial_1 \tilde{g}_n(x) \right\} \left\{ \sum_{i,j,k} \int_{A_{i,j,k}} \mathcal{I}^n(x-y)\partial_{y_1}\tilde{g}_n(y) - \partial_1 \tilde{g}_n(x) \right\} \\
= 4 \left\{ S_1 + S_2 + S_3 \right\}, \tag{20}
\]

where \( \partial_1 A_{i,j,k} \) denotes the faces of the cubes orthogonal to the direction \( e_1 := (1, 0, 0) \). Notice that the number of non zero terms in the sums over \( i, j, k \) of this proof is limited to 8 because the support of \( \mathcal{I}^n \) can intersect at most 8 cubes. The term \( S_1 \) can be bounded by the square of

\[
\left( \sum_{i,j,k} \|\mathcal{I}^n(x-\cdot)\|_{L^1(A_{i,j,k})} \right) \|\partial_1 \tilde{g}_n(\cdot) - \partial_1 \tilde{g}_n(x)\|_{L^\infty(B_{\frac{1}{2^n}}(x))},
\]

where the second term is regarded as a function of \( y \) (\( x \) is fixed) and the \( L^\infty \)-norm is taken on the ball \( B_{\frac{1}{2^n}}(x) \), which is the support of \( \mathcal{I}^n(x-y) \). The first term above is 1, and to estimate the second term the worst case is when \( y \) does not belong to the same cube as \( x \); let us say that \( y \in A^{(1)} \) and \( x \in A^{(2)} \), where \( y^{(1)} \) and \( y^{(2)} \) are the centers of the cubes \( A^{(1)} \) and \( A^{(2)} \) respectively. We have therefore the bound

\[
\|\partial_1 \tilde{g}_n(\cdot) - \partial_1 \tilde{g}_n(x)\|_{L^\infty(B_{\frac{1}{2^n}}(x))} \leq \|\partial_1 \tilde{g}_n(\cdot) - \partial_{y_1} \tilde{g}_n(y^{(1)})\|_{L^\infty} + \|\partial_1 g(y^{(1)}) - \partial_1 g(y^{(2)})\| \\
+ \|\partial_{y_1} \tilde{g}_n(y^{(2)}) - \partial_{y_1} \tilde{g}_n(x)\| \\
\leq 4\delta\|D^2 y\|_{L^\infty(G)}.
\]

This provides the bound for the term \( S_1 \). Similarly, for \( S_2 \) we have the bound

\[
|\partial_1 \tilde{g}_n(x) - \partial_1 g(x)| \leq |\partial_1 \tilde{g}_n(x) - \partial_1 g(y^{(2)})| + |\partial_1 g(y^{(2)}) - \partial_1 g(x)| \\
\leq 2\delta\|D^2 y\|_{L^\infty(G)}.
\]

We still need to estimate \( S_3 \), which contains the boundary terms deriving from the discontinuities of \( \tilde{g}_n \) (and, in the general case, of its derivatives). This term
requires more careful estimates. With \( C_\Gamma := \sup_x \Gamma(x) \) and using the fact that 
\[
\| \Gamma^n(x - \cdot) \|_{L^1(\partial_1 A_{i,j,k})} \leq nC_\Gamma,
\]
we have
\[
\sum_{i,j,k} \| \Gamma^n(x - \cdot) \|_{L^1(\partial_1 A_{i,j,k})} \| \tilde{g}_n(y^1_{i,j,k}) - \tilde{g}_n(y^2_{i,j,k}) \|_{L^\infty(\partial_1 A_{i,j,k})}
\leq nC_\Gamma \sum_{i,j,k} \left\{ \| \tilde{g}_n(y^1_{i,j,k}) - g(y^1_{i,j,k}) \|_{L^\infty(\partial_1 A_{i,j,k})} + \| g(y^1_{i,j,k}) - \tilde{g}_n(y^2_{i,j,k}) \|_{L^\infty(\partial_1 A_{i,j,k})} \right\}
\leq 8nC_\Gamma \delta^4 \| D^4 g \|_{L^\infty(G)} = C\delta^3 \| D^4 g \|_{L^\infty(G)}.
\]
The sums over \( i, j, k \) above are meant as sums over the faces \( \partial_1 A_{i,j,k} \) intersecting the support of \( \Gamma^n(x - \cdot) \), which are at most 4. Collecting all these results, we have the uniform bound
\[
\left| \partial_1 (\Gamma^n \star \tilde{g}_n)(x) - \partial_1 g(x) \right|^2 \leq C\delta^2 \left( \| D^2 g \|_{L^\infty(G)}^2 + \delta^4 \| D^4 g \|_{L^\infty(G)}^2 \right).
\]
We proceed in the same way with higher order derivatives to get, for a generic derivative \( a \) of order \( 0 \leq |a| \leq 3 \), the estimate
\[
\left| \partial^a (\Gamma^n \star \tilde{g}_n)(x) - \partial^a g(x) \right|^2 \leq C \left\{ \sum_{i,j,k} \int_{A_{i,j,k}} \Gamma^n(x - y) \left| \partial^a \tilde{g}_n(y) - \partial^a \tilde{g}_n(x) \right| dy \right\}^2
+ \left| \partial^a \tilde{g}_n(x) - \partial^a g(x) \right|^2 + S_3^a
\leq C \left( \| D^{a+1} g \|_{L^\infty(G)}^2 + S_3^a \right).
\]
For \( a = 0, S_3^a = 0 \), but in the general case the estimate of the term \( S_3^a \) is a little bit more delicate, since one gets more boundary terms. In particular, when integrating by parts, derivatives along different directions result in terms containing discontinuities of \( \tilde{g}_n \) and its derivatives along the faces (that we denote for brevity \( \partial A \)), edges (denoted \( \partial^2 A \)) or vertices (denoted \( \partial^3 A \)) of the cubes, while multiple derivatives along the same direction result in derivatives of \( \Gamma^n \) appearing. For example, for \( a = (1,1,1) \) we get three kinds of terms:
\[
\int_{\partial A_{i,j,k}} \Gamma^n(x - y) \Delta \tilde{g}_n(y) dy, \quad \int_{\partial^2 A_{i,j,k}} \Gamma^n(x - y) \Delta \partial \tilde{g}_n(y) dy,
\int_{\partial A_{i,j,k}} \Gamma^n(x - y) \Delta \partial^2 \tilde{g}_n(y) dy,
\]
where \( \Delta \) denotes the jump of the function. For \( a = (3,0,0) \) we also have terms like
\[
\int_{\partial A_{i,j,k}} \partial^3 \Gamma^n(x - y) \Delta \tilde{g}_n(y) dy, \quad \int_{\partial_1 A_{i,j,k}} \partial_1 \Gamma^n(x - y) \Delta \tilde{g}_n(y) dy,
\int_{\partial_2 A_{i,j,k}} \partial_2 \Gamma^n(x - y) \Delta \tilde{g}_n(y) dy.
\]
and in the general case we find also terms like
\[\int_{\partial^2 A_{i,j,k}} \partial^m \Gamma^n (x-y) \Delta \tilde{g}_n(y) dy.\]

However, we can bound all these terms in the same way. We have that, for \(m \in \mathbb{N}\) and \(b, c \in \mathbb{N}^3\),
\[\int_{\partial^m A_{i,j,k}} \partial^m \Gamma^n (x) dx \leq C(m, n, |b|, |c|),\]
\[|\Delta \partial^c \tilde{g}_n(y)| = |\partial^c \tilde{g}_n(y^+) - \partial^c \tilde{g}_n(y^-)| \leq 2\|D^4 g\|_{L^\infty(G)} \delta^{2(|a| - |c|)}.
\]

Therefore
\[S_3^a \leq C\|D^4 g\|^2_{L^\infty(G)} \delta^{2(|a| - |a|)}.
\]

Remark that for all the terms composing \(S_3^a\) we always have \(m + |b| + |c| = |a| \leq 3.\)

Summing up, we have obtained the bound
\[\|g - g_0\|^2_{\mathcal{H}} = \sum_a \int_G \sum_{a} \left(\partial^a (\Gamma^n \ast \tilde{g}_n)(x) - \partial^a g(x)\right)^2 dx \leq C \sum_a \left(\|D^{|a| + 1} g\|^2_{L^\infty(G)} \delta^2 + \|D^4 g\|^2_{L^\infty(G)} \delta^{2(|a| - |a|)}\right) \leq C \frac{1}{n} \|g\|^2_{C^4(K)}.
\]

The Lemma is proved.

**Corollary 2.6.** For any \(h \in \mathcal{H}\), \(\lim_{n \to \infty} \pi_{\mathcal{H}, n} h = h.\)

**Proof.** Fix any \(\varepsilon > 0\). By density, there exist a \(h_\varepsilon \in C^4(G)\) s.t. \(\|h - h_\varepsilon\|_{\mathcal{H}} \leq \varepsilon/2.\)

Also, by the continuity of the projection, \(\|\pi_{\mathcal{H}, n} h - \pi_{\mathcal{H}, n} h_\varepsilon\|_{\mathcal{H}} \leq \|h - h_\varepsilon\|_{\mathcal{H}} \leq \varepsilon/2.\)

Since \(\|\pi_{\mathcal{H}, n} h - \pi_{\mathcal{H}, n} h_\varepsilon\|_{\mathcal{H}} \leq \|h_\varepsilon, n - h_\varepsilon\|_{\mathcal{H}}\), by the above lemma we get
\[\|\pi_{\mathcal{H}, n} h - h\|_{\mathcal{H}} \leq \|\pi_{\mathcal{H}, n} (h - h_\varepsilon)\|_{\mathcal{H}} + \|\pi_{\mathcal{H}, n} h_\varepsilon - h_\varepsilon\|_{\mathcal{H}} + \|h_\varepsilon - h\|_{\mathcal{H}} \leq \varepsilon + C \frac{1}{n} \|h_\varepsilon\|_{C^4(K)}.
\]

Therefore
\[\lim_{n \to \infty} \|\pi_{\mathcal{H}, n} h - h\|_{\mathcal{H}} \leq \varepsilon,
\]
and since \(\varepsilon\) is arbitrary, the Corollary is proved.

We return to the proof of Theorem 2.2. To show that in our case the hypothesis of Lemma 2.4 are met, we have to show (19). Define \(Y^\varepsilon\) as the vector process of \(X^\varepsilon\) and all its derivatives in the parameters \(\mu = (\xi, \eta, \sigma)\) up to order 6. As remarked above, this process is the solution of a linear system of ODEs with coefficients (the matrices \(M_1\) and \(M_2\)) linear in the parameters:
\[\frac{d}{dx} Y^\varepsilon = M_1 Y^\varepsilon + \frac{1}{\varepsilon} \nu(x/\varepsilon^2) M_2 Y^\varepsilon.\]
Since $G$ is bounded, we only need to check that the second moment of $Y^\varepsilon(x,\mu)$ is uniformly bounded with respect to $\varepsilon \in (0, 1]$, $\mu \in G$ and $x \in [0,R]$. Actually, we aim at a stronger result, which we will need later. We are going to show that (recall that $Y_0 = Y^\varepsilon_0$ is deterministic since it is defined by the equation for $x \leq 0$, which is deterministic, and the boundary condition at $x \to -\infty$)

$$E \left[ \sup_{x \in [0,R]} |Y^\varepsilon(x)|^2 \right] \leq C_R \left( 1 + |Y(0)|^2 \right) < \infty.$$ (21)

Following [FGPS, section 6.3.5], we show this bound with the perturbed function method. Let $\mathcal{L}^\varepsilon$ be the infinitesimal generator of the process $Y^\varepsilon$ and $\mathcal{L}$ the infinitesimal generator of the limit process $Y := \lim_{\varepsilon \to 0} Y^\varepsilon$. Let $m \in \mathbb{N}$ be such that $Y^\varepsilon(x) \in \mathbb{C}^m$ and let $K$ be a compact subset of $\mathbb{R}$ containing the image of the bounded process $\nu$. For every $y \in \mathbb{C}^m$ and $z \in K$, $\mathcal{L}^\varepsilon$ has the form

$$\mathcal{L}^\varepsilon g(y, z) = \frac{1}{\varepsilon^2} \mathcal{L}_\nu g(y, z) + \frac{\varepsilon}{\varepsilon} (M_2 y)^T \nabla_y g(y, z) + (M_1 y)^T \nabla_y g(y, z).$$

Let $f$ be the identity function on $\mathbb{C}^m$ and $f^\varepsilon(y, z) = y + \varepsilon f_1(y, z)$ be the associated perturbed function, which is solution of the Poisson equation $\mathcal{L}_\nu f_1(y, z) = -z (M_2 y)^T \nabla_y f(y)$. In this equation, $y$ plays the role of a frozen parameter, so that $f_1$ has linear growth in $y$, uniformly in $z$, and the same holds for

$$\mathcal{L}^\varepsilon f^\varepsilon(y, z) = z (M_2 y)^T \nabla_y f_1(y, z) + \varepsilon (M_1 y)^T \nabla_y f_1(y, z).$$

Since

$$Y^\varepsilon(x) = Y^\varepsilon(0) - \varepsilon \left[ f_1(Y^\varepsilon(x), \nu^\varepsilon(x)) - f_1(Y^\varepsilon(0), \nu^\varepsilon(0)) \right] + \int_0^x \mathcal{L}^\varepsilon f^\varepsilon(Y^\varepsilon(x'), \nu^\varepsilon(x')) dx' + M^\varepsilon_x,$$

where $M^\varepsilon_x$ is a vector valued martingale, we get the bound

$$\sup_{x \in [0,R]} |Y^\varepsilon(x)| \leq |Y^\varepsilon(0)| + \varepsilon C \left[ 1 + \sup_{x \in [0,R]} |Y^\varepsilon(x)| \right] + C \int_0^R 1 + \sup_{x' \in [0,x]} |Y^\varepsilon(x')| dx + C \sup_{x \in [0,R]} |M^\varepsilon_x|.$$ For $\varepsilon \leq \frac{1}{2C}$, applying Gronwall’s inequality and renaming constants we get

$$\sup_{x \in [0,R]} |Y^\varepsilon(x)| \leq C_R \left( 1 + |Y^\varepsilon(0)| + \sup_{x \in [0,R]} |M^\varepsilon_x| \right).$$ (22)

The quadratic variation of the martingale is given by

$$\langle M^\varepsilon \rangle_x = \int_0^x g^\varepsilon(Y^\varepsilon(x'), \nu^\varepsilon(x')) dx',$$
where
\[ g^\varepsilon(y, z) = (\mathcal{L} f^\varepsilon - 2 f^\varepsilon \mathcal{L} f^\varepsilon)(y, z) \]
\[ = (\mathcal{L}_v f_1^\varepsilon - 2 f_1 \mathcal{L}_v f_1)(y, z) \]
\[ + 2\varepsilon z \left[ (M_2 y)^T f_1(y, z) - (M_2 y)^T ((\nabla_y f_1)^T f_1)(y, z) \right] \]
\[ + 2\varepsilon^2 \left[ (M_1 y)^T f_1(y, z) - (M_1 y)^T ((\nabla_y f_1)^T f_1)(y, z) \right] \]
has quadratic growth in \( y \) uniformly in \( z \in K \). Therefore, by Doob’s inequality,
\[ \mathbb{E}\left[ \sup_{x \in [0, R]} |M^\varepsilon_x|^2 \right] \leq C \mathbb{E}\left[ \langle M^\varepsilon \rangle_R \right] \leq C \int_0^R 1 + \mathbb{E}\left[ |Y^\varepsilon(x)|^2 \right] dx . \]
Substituting into the expected value of the square of (22) and using again Gronwall’s inequality, we get (21), which gives (19). We can therefore apply Lemma 2.4 and obtain the tightness of \((X^\varepsilon(x, \mu))_{\mu \in K}\) in \( \mathcal{H} = W^{3,2}(G) \) for every \( x \in [0, R] \); thanks to Lemma 2.3, the tightness in \( \mathcal{D}([0, R]; E) \) follows if we show that the Aldous property \([\mathcal{A}] \) holds. Since \( G \) is bounded, we can prove the Aldous property showing that
\[ \lim_{\tau \to 0} \limsup_{\varepsilon \to 0} \sup_{\mu \in K} \sup_{\tau, \tau + \theta} \mathbb{E}\left[ |Y^\varepsilon(\tau + \theta, \mu) - Y^\varepsilon(\tau, \mu)|^2 \right] = 0 , \quad (23) \]
where \( Y^\varepsilon \) is the vector process having as components \( X^\varepsilon \) and its derivatives in \( \mu \) up to the third order only. We prove the above limit using again the perturbed test function method. With the notations introduced above, we have
\[ |Y^\varepsilon(\tau + \theta) - Y^\varepsilon(\tau)|^2 \leq C \left| M^\varepsilon_{\tau+\theta} - M^\varepsilon_\tau \right|^2 + C \int_\tau^{\tau+\theta} \left| \mathcal{L} f^\varepsilon (Y^\varepsilon(x), \nu^\varepsilon(x)) \right|^2 dx \]
\[ + C\varepsilon \left( 1 + \sup_{x \in [\tau, \tau+\theta]} |Y^\varepsilon(x)|^2 \right) \]
\[ \leq C \left| M^\varepsilon_{\tau+\theta} - M^\varepsilon_\tau \right|^2 + C \int_\tau^{\tau+\theta} \left| \mathcal{L} f^\varepsilon (Y^\varepsilon, \nu^\varepsilon) - \mathcal{L} f(Y^\varepsilon) \right|^2 dx \]
\[ + C \int_\tau^{\tau+\theta} \left| \mathcal{L} f(Y^\varepsilon) \right|^2 dx + C\varepsilon \left( 1 + \sup_{x \in [\tau, \tau+\theta]} |Y^\varepsilon(x)|^2 \right) . \]
We have that
\[ \mathbb{E}\left[ \left| M^\varepsilon_{\tau+\theta} - M^\varepsilon_\tau \right|^2 \right] = \mathbb{E}\left[ (M^\varepsilon_{\tau+\theta})^2 - (M^\varepsilon_\tau)^2 \right] = \mathbb{E}\left[ \int_\tau^{\tau+\theta} d\langle M^\varepsilon \rangle_x \right] . \]
Since \( |\mathcal{L} f^\varepsilon(y, z) - \mathcal{L} f(y)| \leq \varepsilon C(1 + |y|) \) and \( |\mathcal{L} f(y)| \leq C|y| \), for \( \theta \leq \delta \)
\[ \mathbb{E}\left[ |Y^\varepsilon(\tau + \theta) - Y^\varepsilon(\tau)|^2 \right] \leq C_R (\delta + \varepsilon) \left( 1 + \mathbb{E}\left[ \sup_{x \in [0, R]} |Y^\varepsilon(x)|^2 \right] \right) . \]
The right–hand side is independent of \( \tau \) and we can use estimate (21) to bound it uniformly in \( \varepsilon \) and \( \mu \). Therefore, (23) follows and the proof of Theorem 2.2 is completed. \( \square \)
Remark 2.7. As seen in [EK, Theorem 7.4.1], the conditions imposed on the driving process \( \nu(x) \) can be relaxed, assuming that it is just a mixing process.

3. Stability of NLS solitons

This section is devoted to the study of our first example, the NLS equation. We focus on the soliton components of the solution. In the previous section we have obtained the limit equation (12) and we have seen that every soliton component (soliton, in short) is identified by a complex number \( \zeta = \xi + i\eta \) s.t. the flow \( \Psi(x, \zeta) \) solution of (12) with initial condition (9) satisfy also a given final condition. The real and imaginary parts of \( \zeta \) define the velocity and amplitude of the soliton, respectively. We start by reporting, in Paragraph 3.1, some classical results on the background deterministic solution. Then we analyze how this solution is modified by the introduction of a real, in Paragraph 3.2, or complex, in Paragraph 3.3, small–amplitude white noise perturbation of the initial condition. The main results are contained in Propositions 3.1 and 3.6. We deal with the limit cases of „quiescent” solitons in Corollary 3.4 and Remark 3.8.

For simplicity of exposition, in the present and following sections we will choose the value of the integrated covariance of the process \( \nu \) to be \( \alpha = 1/2 \).

3.1. Deterministic background solution

Let \( U_0(x) = q \mathbf{1}_{0,R}(x) \). Burzlaff proved in [Bu88] that in this case the number of solitons generated is the integer part of \( 1/2 + qR/\pi \) (see also the relevant discussion and generalization of [Ki89]). They remark that physical intuition suggests that the first soliton created when increasing \( R \) corresponds to \( \zeta = 0 \) (this is a single soliton with zero amplitude and velocity, the quiescent soliton); this “soliton” is created for \( qR = \pi/2 \). For values of \( qR \) just over this critical threshold the created soliton has zero velocity and nonzero amplitude \( 2\eta \) which can be computed explicitly solving (5) for pure imaginary values of \( \zeta \).

In the first part of this paragraph we report some computations relative to this case, as the results and explicit formulas will be used below. We then conclude the paragraph providing the sketch of an analytical proof of the claimed fact that generated solitons correspond to purely imaginary values of \( \zeta \).

From the decaying condition at \( -\infty \) one obtains the initial condition

\[
\Psi(0) = \left. \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ux} \right|_{x=0} = \left. \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).
\]

The system (5) for \( x \in [0, R] \) reads

\[
\begin{align*}
\frac{\partial \psi_1}{\partial x} &= iq \psi_2 - i\zeta \psi_1 \\
\frac{\partial \psi_2}{\partial x} &= iq \psi_1 + i\zeta \psi_2
\end{align*}
\]
and $\Psi = (\psi_1, \psi_2)$ is a solution of the initial value problem for $\zeta \neq iq$ if

\[
\psi_1(x) = -\frac{i\zeta}{\sqrt{q^2 + \zeta^2}} \sin \left(\sqrt{q^2 + \zeta^2} x\right) + \cos \left(\sqrt{q^2 + \zeta^2} x\right);
\]
\[
\psi_2(x) = i\frac{q}{\sqrt{q^2 + \zeta^2}} \sin \left(\sqrt{q^2 + \zeta^2} x\right).
\]
(25)

To be a soliton solution, $\Psi$ needs to satisfy also the decaying condition at $+\infty$, which is to say $\psi_1(R) = 0$. The condition can be rewritten for $\zeta \neq 0$ as

\[
f = \tan(\sqrt{q^2 + \zeta^2} R) + i\frac{\sqrt{q^2 + \zeta^2}}{\zeta} = 0.
\]
(26)

Since $a(\zeta) = \psi_1(R, \zeta)e^{i\zeta R}$, the function $f$ is linked to the first Jost coefficient $a$ by the relation

\[
f = ia(\zeta)e^{-i\zeta R}\frac{e^{-i\zeta R}}{\zeta} \sqrt{\frac{q^2 + \zeta^2}{\cos(\sqrt{q^2 + \zeta^2} R)}}.
\]

from which we see that the zeros of $f$ coincide with those of $a$.

Note that for $\zeta = iq$ the solution of (3.1) satisfying the initial conditions is

\[
\Psi = \begin{pmatrix} 1 + x \\ ix \end{pmatrix},
\]

which cannot satisfy the final conditions, so that no soliton can be created for this particular value of $\zeta$.

Here, plotting on the $(\xi, \eta)$–plane the level lines of the real and imaginary parts of $f$ at height zero, one can already see that the first soliton component of the solution corresponds to an imaginary value of $\zeta$. This fact can be proved in an analytic way as follows.

Recall that the zeros of $f$ coincide with those of $a$ and observe that the function $a(\xi, \eta, R)$ is analytic in the domain $\mathbb{R} \times (0, \infty) \times (0, \infty)$ and continuous in $\mathbb{R} \times [0, \infty) \times (0, \infty)$. We use the argument principle to study how the number of zeros in the upper half of the complex plane evolves with increasing $R$. For any fixed $R$ we proceed as in [DP08], taking a loop $C$ in the complex $\zeta$–plane composed of the (lower) real axis and the infinite semi–arc in the upper half plane. Then, the number of of zeros is given by

\[
N = \frac{1}{2\pi} \int_C \frac{1}{a} \frac{\partial a}{\partial \zeta} d\zeta.
\]

Since $a = 1 + O(1/\zeta)$ for $|\zeta| \gg 1$, the integral over the upper part of the loop is zero. Changing variables $a(\zeta) = \rho(\zeta) \exp(i\alpha(\zeta))$, after some computations one obtains that unless there is a zero on the real axis, also the integral on the lower part of the loop is zero. Therefore, the number of zeros changes for a given $R$ only if $a(\xi, 0, R) = 0$ admits a solution. But zeros of $a$ and $f$ coincide, and since
in equation (26) for real values of $\zeta = \xi \neq 0$ the tangent is real and the second term is purely imaginary and non zero, solutions of $f(\xi, 0, R) = 0$ can only be found at $\xi = 0$.

Explicit computations easily show that $\zeta = 0$ corresponds to a soliton solution only for $R = \frac{2n+1}{2q} \pi$, $n \in \mathbb{N}$. Computing explicitly the derivative of $a(\zeta)$ at $\zeta = 0$ we get

$$\partial_\zeta a(\zeta) = e^{i\zeta R} \left[ \left( 2 \frac{\zeta R}{\sqrt{q^2 + \zeta^2}} - i \frac{q^2}{\sqrt{q^2 + \zeta^2}} \right) \sin \left( \sqrt{q^2 + \zeta^2} R \right) + i R \frac{q^2}{q^2 + \zeta^2} \cos \left( \sqrt{q^2 + \zeta^2} R \right) \right]$$

$$\left. \partial_\zeta a(\zeta) \right|_{\zeta = 0} = - \frac{1}{q} \sin(qR) + iR \cos(qR),$$

so that for $Rq = \frac{2n+1}{2} \pi$ the derivative is equal to $\mp i/q$ and is never zero. Therefore, new solitons are generated one at a time and they are immediately pushed (as $R$ increases) towards the interior of the domain. [Ka79] showed that if $a(\zeta) = 0$ then $a'(\zeta) \neq 0$; from this fact it follows that zeros in the interior of the domain are always simple. Considering the complex conjugate $\Psi^*$, which is a solution whenever $\Psi$ is, one obtains that zeros not laying on the imaginary axis always come in pairs $\pm \xi + i\eta$. But since zeros move continuously (as $R$ grows) in the upper complex plane, cannot coalesce and cannot leave the imaginary axis ($\xi = 0$) unless they form a pair, we get that they must remain on the imaginary axis.

### 3.2. Small–intensity real white noise

In this paragraph we consider the example of an initial condition composed of a square function perturbed with a small real white noise. First, we use a perturbative approach to study the effects of the perturbation on the first soliton (Proposition 3.1); the effects on subsequent ”true” solitons are the same. Then, in the last part of this paragraph, we study the effect of this perturbation on ”quiescent” solitons (Corollary 3.4).

We have here $U_0(x) = (q + \sigma W_x)1_{[0,R]}(x)$. The initial condition is (24) and the system (5) for $x \in [0, R]$ reads

$$\begin{cases} d\psi_1 = i(q \psi_2 - \zeta \psi_1) \, dx + i\sigma \psi_2 \circ dW_x \nonumber \\
\psi_2 = i(q \psi_1 + \zeta \psi_2) \, dx + i\sigma \psi_1 \circ dW_x \nonumber \end{cases} \tag{27}$$

**Proposition 3.1.** For $qR > \frac{\pi}{2}$ and in the limit of a small real white noise type stochastic perturbation of the initial condition, the amplitude of the soliton component of the solution is perturbed at first order by a small, zero–mean, Gaussian random variable, which is given by

$$q \sin(c_0 R) W_R + \int_0^R 2 \frac{2q^2}{c_0} \sin \left( c_0(R - y) \right) \sin(c_0 y) dW_y 
onumber$$

$$2 \left[ \frac{q^2}{c_0^2} + Rq_0 \right] \sin \left( c_0 R \right) - R \frac{2q^2}{c_0} \cos \left( c_0 R \right), \tag{28}$$
where \( c_0 := \sqrt{q^2 - \eta_0^2} \), and \( \eta_0 \) is the amplitude of the soliton of the unperturbed system.

The velocity of the soliton remains unchanged.

The following proposition provides results that we will need in the following on the uniqueness and regularity of solutions.

**Proposition 3.2.** The stochastic differential equation (27) defines a stochastic flow \( \Psi_{\zeta,\sigma} = (\psi_1, \psi_2)^T \) of \( C^1 \)-diffeomorphisms, which is \( C^1 \) also in the parameters \( \zeta, \eta, \sigma \).

**Proof of Proposition 3.2.** Write the SDE in Itô and vector form:

\[
d\Psi = i \begin{pmatrix} -\zeta + i\sigma^2 \\ q \end{pmatrix} \Psi \, dx + i\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi \, dW_x .
\]

The coefficients of the SDE are independent of \( x \) and Lipschitz continuous in \( \Psi \) for every \( \zeta \) and \( \sigma \). Therefore, the existence of a unique solution to the SDE, which defines a stochastic flow of homeomorphisms \( \Psi_{\zeta,\sigma}(x) \), is a classical fact (see for example [Ku84]). Following the notation of [Ku], we define the local characteristic of the SDE as \( (a,b,x) \), where

\[
a(\zeta,\zeta',\sigma,\sigma',x) := -\sigma \sigma' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
b(\zeta,\sigma,x) := \begin{pmatrix} -\zeta \\ q \end{pmatrix} .
\]

Fix any \( n \in \mathbb{N} \), define the set \( G_n := \{ (\xi, \eta, \sigma) \in \mathbb{R}^3 \mid |\xi| < n, 0 < \eta < n, 0 < \sigma < n \} \) and consider the SDE only with parameters in \( G_n \). Then, both \( a \) and \( b \) are uniformly bounded and, together with their first derivatives, are Lipschitz continuous in the parameters. This means that the coefficients satisfy condition (A.5) of [Ku, Chapter 4.6]. It follows from [Ku, Theorem 4.6.4] that \( \Psi_{\zeta,\sigma}(x) \) is \( C^1 \) in the parameters almost surely on \( G_n \). Let \( \Omega_n \) be the set of \( \omega \in \Omega \) such that \( \Psi_{\zeta,\sigma}(x) \) is \( C^1(G_n) \); it is a set of full measure. Since \( n \) is arbitrary, \( \Psi_{\zeta,\sigma}(x) \) is actually \( C^1(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+) \) for every \( \omega \in \cap_n \Omega_n \), which is still a set of full measure. This proves the last statement of the proposition.

Since [Ku, Theorem 4.6.5] states that \( \Psi_{\zeta,\sigma}(x) \) is actually a stochastic flow of \( C^1 \)-diffeomorphisms, the proof is completed.

**Proof of Proposition 3.1.** Looking at the flow at point \( R \) we can define a complex-valued function of \( \Psi(R) \) as \( F(\xi, \eta, \sigma) := \psi_1(\zeta, \sigma) \). We look for the set of values of \( (\xi, \eta, \sigma) \) corresponding to zeros of the function \( F \): they are the parameters \( (\zeta = \xi + i\eta) \) defining the soliton components of the solution of the problem perturbed with a noise of amplitude \( \sigma \). We claim that a small stochastic perturbation has only the effect of a small variation in the value of \( \zeta = \xi + i\eta \) with respect to the value \( \zeta_0 = i\eta_0 \) of the corresponding soliton in the deterministic case. We will prove this using the implicit function Theorem: \( F(\xi, \eta, \sigma) \) has a unique zero in some open set containing the point \( (0, \eta_0, 0) \). The following lemma
ensures that the most important hypothesis of the implicit function Theorem is satisfied.

**Lemma 3.3.** Let $F(\xi, \eta, \sigma)$ be the function defined above. Then, whenever $\zeta_0 = i\eta_0$ is the value corresponding to a soliton component of the solution of the deterministic problem, the determinant of the Jacobian matrix

$$J := \begin{pmatrix} \frac{\partial \Re (F)}{\partial \xi} & \frac{\partial \Re (F)}{\partial \eta} \\ \frac{\partial \Im (F)}{\partial \xi} & \frac{\partial \Im (F)}{\partial \eta} \end{pmatrix}$$

at point $(0, \eta_0, 0)$ is not zero.

**Proof of Lemma 3.3.** For $\sigma = 0$ the system (27) becomes deterministic and the solution is given by (25). Then, setting $c := \sqrt{q^2 - \zeta^2}$,

$$i \partial_\xi \psi_1(\xi, \eta, 0) = \partial_\eta \psi_1(\xi, \eta, 0) = \left[ \frac{q^2}{c^3} - iR\frac{\zeta}{c} \right] \sin (\zeta R) + R\frac{\zeta^2}{c^2} \cos (\zeta R)$$

so that

$$i \partial_\xi F(0, \eta_0, 0) = \partial_\eta F(0, \eta_0, 0) .$$

We are left to verify that

$$0 \neq \det J(0, \eta_0, 0) = \left[ \partial_\xi \Re (F)\partial_\eta \Im (F) - \partial_\eta \Re (F)\partial_\xi \Im (F) \right] (0, \eta_0, 0)$$

$$= \left[ \Re (\partial_\xi F) \Im (\partial_\eta F) - \Re (\partial_\eta F) \Im (\partial_\xi F) \right] (0, \eta_0, 0)$$

$$= \left[ \left( \Re (\partial_\xi F) \right)^2 + \left( \Im (\partial_\xi F) \right)^2 \right] (0, \eta_0, 0)$$

or equivalently

$$\partial_\xi F(0, \eta_0, 0) = -i \left[ \frac{q^2}{c_0^3} + R\frac{\eta_0}{c_0} \right] \sin (c_0 R) + iR\frac{\eta_0^2}{c_0^3} \cos (c_0 R) \neq 0 , \quad (30)$$

where $c_0 = \sqrt{q^2 - \eta_0^2}$. Observe that condition (26) implies that $\eta_0 \leq q$; $c_0$ is therefore real. Indeed, for $\eta_0 > q$, $c_0$ would be imaginary pure and the function $f$ of equation (26) would become the sum of two imaginary pure terms of the same sign, so that it cannot be zero. In equation (30) the coefficient of the sinus is the sum of two non zero terms of the same sign, so that to ensure the condition we need to check that

$$\tan (c_0 R) = \frac{R\eta_0^2 c_0}{q + R\eta_0 c_0^2} \quad (31)$$

does not hold for $\eta_0$ solution of (26). The compatibility condition between (26) and (31) is

$$-\frac{c_0}{\eta_0} = \frac{R\eta_0^2 c_0}{q + R\eta_0 c_0^2}$$

or equivalently

$$\left( q + R\eta_0 c_0^2 + R\eta_0^3 \right) c_0 = 0 ,$$
which cannot be verified (recall that $\eta_0 \neq q$, so that $c_0 \neq 0$). The lemma is proved.

We return to the proof of Proposition 3.1. Since the previous Lemma guarantees that the Jacobian matrix $J$ of the derivatives of $F$ with respect to $\xi$ and $\eta$ is invertible, we can apply the implicit function Theorem at point $(\xi, \eta, \sigma) = (0, \eta_0, 0)$. Fix $\zeta = i\eta_0$. By Proposition 3.2 the flow defined by the system (27) is $C^1$ in the parameters, so that its derivative in $\sigma$ coincide with the first term of the Taylor expansion, denoted $\Psi^{(1)}$:

$$d\Psi^{(1)} = \begin{pmatrix} \eta_0 & iq \\ iq & -\eta_0 \end{pmatrix} \Psi^{(1)} dx + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi^{(0)} dW_x .$$

(32)

Here, $\Psi^{(0)}$ denoted the solution of the deterministic problem ($\sigma = 0$). Let $M$ be the matrix appearing in the drift term of the above equation; the solution can be computed explicitly:

$$\Psi^{(1)}(x) = i \int_0^x \exp(M(x-y)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi^{(0)}(y) dW_y ,$$

where

$$\left[ \exp(M(x-y)) \Psi^{(0)}(y) \right]_1 = i \frac{q}{c_0} \cos \left( c_0 (x-y) \right) \sin(c_0 y)$$

$$+ 2i \frac{iq}{c_0} \sin \left( c_0 (x-y) \right) \sin(c_0 y)$$

$$+ i \frac{q}{c_0} \sin \left( c_0 (x-y) \right) \cos(c_0 y)$$

$$= i \frac{q}{c_0} \sin(c_0 x) + 2i \frac{iq}{c_0} \sin \left( c_0 (x-y) \right) \sin(c_0 y) .$$

It follows that

$$\partial_\sigma F(0, \eta_0, 0) = i \int_0^R \left[ \exp(M(R-y)) \Psi^{(0)}_y \right]_1 dW_y .$$

Observe that $\partial_\xi F(0, \eta_0, 0)$ is imaginary and, symmetrically, $\partial_\eta F(0, \eta_0, 0)$ is real.

Set $\alpha := \partial_\eta F(0, \eta_0, 0) = \frac{q^2}{c_0^2} + R \frac{2q}{c_0^2} \sin \left( c_0 R \right) - R \frac{q^2}{c_0^2} \cos \left( c_0 R \right)$. We can write

$$J^{-1} = \begin{pmatrix} 0 & -\frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 \end{pmatrix}$$

and from the formula

$$v := \begin{pmatrix} \partial_\sigma \xi \\ \partial_\sigma \eta \end{pmatrix} (\sigma = 0) = -J^{-1} \begin{pmatrix} \Re(\partial_\sigma F) \\ \Im(\partial_\sigma F) \end{pmatrix} (0, \eta_0, 0)$$

we get that $\partial_\sigma \eta$ is given by (28) and $\partial_\sigma \xi = 0$. \qed
Corollary 3.4. When \( q = (2n + 1)\pi/2R \) for some \( n \in \mathbb{N} \), in the deterministic case we have the creation of what is sometimes called a "quiescent" soliton. Also in this case, the stochastic perturbation can modify, at first order, only the amplitude of the soliton. As a result, the "quiescent" soliton is destroyed and with probability \( 1/2 \) a true soliton is created.

Proof. In the case of a "quiescent" soliton Lemma 3.3 still holds. Indeed we have now \( c_0 = q \) and \( qR = (2n + 1)\pi/2 \), so that equation (30) becomes

\[
\partial \xi F (0, 0, 0) = -i \left[ \frac{q^2}{c_0^2} + R\frac{\sin (c_0 R)}{c_0} \right] \sin (c_0 R) + i R\sin (c_0 (R - y)) \sin (c_0 y) \frac{dW_y}{2} = \mp i \frac{1}{q} \neq 0
\]

and the determinant of the Jacobian is not zero. One has then \( \alpha = \pm 1/q \) and

\[
\partial \sigma F (0, 0, 0) = - \int_0^R \sin (qR) dW_y = \mp W_R,
\]

which is real. Therefore,

\[
v := \left( \frac{\partial \sigma \xi}{\partial \sigma \eta} \right) (\sigma = 0) = - \left( \begin{array}{cc} 0 & \mp q \\ \pm q & 0 \end{array} \right) \left( \begin{array}{c} \mp W_R \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ q W_R \end{array} \right).
\]

A true soliton is created whenever \( W_R > 0 \).

Remark 3.5. The result of the above Remark holds in greater generality. Indeed, substituting the white noise with a general process \( Q_x \) one obtains the same result: \( \partial \sigma \eta = q \int_0^R Q_x dx \), so that a true soliton is created whenever the integral is positive. We see that for this result to hold we only need the smallness assumption on the perturbing process, but this process needs not be rapidly oscillating.

3.3. Small–intensity complex white noise

The limit case of a small–amplitude complex white noise is very similar to the case of the real white noise treated above. Take \( U_0(x) = (q + \sigma W_x) 1_{[0, R]}(x) \) where \( W_x = W^{(1)}_x + iW^{(2)}_x \) is a complex Wiener process. We have

Proposition 3.6. For \( qR > \pi \) and in the limit of a small complex white noise type stochastic perturbation of the initial condition, the velocity and amplitude of solitons are perturbed at first order by small, zero–mean, Gaussian random variables, which are given by

\[
\partial \sigma \xi = - \frac{q \sin (c_0 R) W^{(2)}_R + \int_0^R 2 \frac{\sin (c_0 (R - y)) \sin (c_0 y) dW_y^{(2)}}{2 \left( \frac{c_0^2}{\nu} + R\frac{\sin (c_0 R)}{c_0} \sin (c_0 (R - y)) \sin (c_0 y) \right)}}{2 \left( \frac{c_0^2}{\nu} + R\frac{\sin (c_0 R)}{c_0} \sin (c_0 (R - y)) \sin (c_0 y) \right)},
\]

\[
\partial \sigma \eta = - \frac{q \sin (c_0 R) W^{(1)}_R + \int_0^R 2 \frac{\sin (c_0 (R - y)) \sin (c_0 y) dW_y^{(1)}}{2 \left( \frac{c_0^2}{\nu} + R\frac{\sin (c_0 R)}{c_0} \sin (c_0 (R - y)) \sin (c_0 y) \right)}}{2 \left( \frac{c_0^2}{\nu} + R\frac{\sin (c_0 R)}{c_0} \sin (c_0 (R - y)) \sin (c_0 y) \right)},
\]
where $c_0 := \sqrt{q^2 - \eta_0^2}$, and $\eta_0$ is the amplitude of the soliton of the unperturbed system.

**Proof.** This proof is similar to the one of Proposition 3.1. An analogous of Proposition 3.2 holds in this setting, and Lemma 3.3 remains unchanged (note that in Lemma 3.3 we work on the deterministic equation). From equation (32) onward one has just to remember that $W$ is now complex. 

**Remark 3.7.** In this case the first order perturbations of the velocity and amplitude of the soliton have the same law and are independent.

**Remark 3.8.** "Quiescent" solitons are perturbed in both amplitude and speed, leading to the possible creation of true solitons with nonzero velocity. The same result holds when perturbing the initial data with more general complex processes.

4. Stability of KdV solitons

In this section we study our second example, the KdV equation. We focus on the soliton components of the solution. Solitons of the KdV equation are identified by an imaginary number $\zeta = i\eta$, defining both the velocity and amplitude of the soliton, which are related. We start again by reporting in Paragraph 4.1 some classical results on the background deterministic solution. In Paragraph 4.2 we analyze how this solution is modified by the introduction of small–amplitude white noise perturbation of the initial condition: the main result is contained in Proposition 4.1, while Proposition 4.4 deals with the case of "quiescent" solitons. The last paragraph deals with the case of a perturbation of the zero initial condition.

4.1. Deterministic background solution

In [Mu78], Murray obtained a $C^\infty$ solution for the KdV equation with a deterministic "box–shaped" initial condition $U_0 = q \mathbf{1}_{[-R,R]}(x)$. He showed that in this case the Jost coefficient $a$ extends to an analytical function in the upper part of the $\zeta$–plane. Only in the case of positive values of $q$, $a$ has a finite number of zeros on the imaginary axis $\zeta = i\eta$ for $0 < \eta \leq \sqrt{q}$.

We report some explicit computations on our similar deterministic case, as the results will be used below, and study some properties of the soliton components of the solution. First, let us construct explicitly the solutions of the deterministic equation, which we call $\varphi_0$. Take here $U_0 = q \mathbf{1}_{[0,R]}(x)$ for some $q > 0$. Due to [AC, Lemma 2.2.2], we can assume $\zeta = i\eta$; we need to solve

$$
\varphi_{xx} = \begin{cases} 
\eta^2 \varphi & x < 0, x > R \\
(-q + \eta^2) \varphi & x \in [0, R]
\end{cases}.
$$

(33)
For \( \eta = 0 \) the only integrable solution is \( \varphi \equiv 0 \). Soliton components corresponding to \( \eta > 0 \) must satisfy

\[
\varphi = c_1 e^{\eta x} \quad x < 0, \tag{34}
\]
\[
\varphi = c_2 e^{-\eta x} \quad x > R. \tag{35}
\]

For \( x \in [0, R] \) one can rewrite the problem as

\[
\begin{cases}
\frac{d\varphi}{dx} = \tilde{\varphi} \\
\frac{d\tilde{\varphi}}{dx} = (-q + \eta^2)\varphi
\end{cases}
\]

Note that the sign of \( \eta \) is not relevant (we take it to be positive) and that for \( \eta \geq \sqrt{q} \) the solution of (33) is monotone, so that it cannot be a soliton (which must be integrable). We therefore look for solutions corresponding to \( 0 < \eta < \sqrt{q} \).

Set \( c = \sqrt{q} - \eta^2 \). Due to (34) and (35), we only need to solve (33) for \( x \in [0, R] \). From (33) and the initial conditions

\[ \varphi_0(0) = c_1, \quad \partial_x \varphi_0(0) = \eta c_1 \]

derived from (34), we get

\[
\varphi_0(x) = \alpha e^{ix} + \beta e^{-ix}, \quad \alpha = \frac{c_1}{2} (1 - i\frac{\eta}{c}), \quad \beta = \frac{c_1}{2} (1 + i\frac{\eta}{c}),
\]

which is to say

\[
\varphi_0(x) = c_1 \cosh(i\xi) - i\frac{\eta}{c} \sinh(i\xi) = c_1 \left[ \cos(\xi x) + \frac{\eta}{c} \sin(\xi x) \right]. \tag{36}
\]

We can set the global constant \( c_1 \) equal to 1. Considering also the final condition (35), we obtain an equation for \( \eta \):

\[
\begin{cases}
q \sin(\xi R) = 2 \xi \eta c_2 e^{-R\eta} \\
q \cos(\xi R) = (\xi^2 - \eta^2) c_2 e^{-R\eta} = q \cos(R\sqrt{q - \eta^2}) = c_2(q - \eta^2) e^{-R\eta}
\end{cases}
\]

For \( \eta = \sqrt{q}/2 \), the only possible solution is such that \( R\sqrt{q - \eta^2} = \pi/2 + k\pi \), which means that

\[
\sqrt{q} R = (2k + 1)\pi/\sqrt{2}. \tag{37}
\]

All other solutions can be found solving

\[
f(\eta) := \tan(R\sqrt{q - \eta^2}) - \frac{2\eta\sqrt{q - \eta^2}}{q - 2\eta^2} = 0 \tag{38}
\]

for \( \eta \in [0, \sqrt{q}] \setminus \{\sqrt{q}/2\} \). The existence and the number of solutions for the above equation depend on the quantity \( R\sqrt{q} \). Consider some fixed value of \( R \).

As it is shown below, for small values of \( q \) a first soliton is created with \( \eta^{(1)} \sim 0 \). As \( q \) increases, the value of \( \eta^{(1)} \) increases too and tends to \( \sqrt{q}/2 \) as \( q \) tends to \( \pi^2/(2R^2) \). We have already found the solution for this specific value of \( q \).
(\(k = 0\) in equation (37)). For \(q \) larger than \(\pi^2/(2R^2)\), \(\eta^{(1)}\) continues to grow. A second solution appears \((\eta^{(2)} = 0)\) when \(q = \pi^2/R^2\). A third solution appears at \(q = (2\pi/R)^2\): the values of \(\eta^{(i)}\) (corresponding to the \(i^{th}\) soliton created) continuously increase as \(q\) grows, but remain ordered: \(\eta^{(i)} < \eta^{(j)}\) for \(i > j\). Therefore, the number of solitons created is \(\lfloor R\sqrt{q}/\pi \rfloor + 1\).

A few examples of \(f(\eta)\) are plotted below. We have taken \(R = 1\) and different values of \(q\). The first critical points (when new solitons are created) correspond here to \(q^{(2)} = \pi^2 \sim 9, 87, q^{(3)} = 4\pi^2 \sim 39, 48, q^{(4)} = 9\pi^2 \sim 88, 83, q^{(5)} = 16\pi^2\). The almost–vertical line appearing near \(\eta = \sqrt{q}/2\) for \(q = 44\) reflects the fact that we are near the critical points of (37): from (37) for \(k = 1\) we have \(q = 9\pi^2/2 \sim 44.41\).

Let us take a closer look at the case \(q \to 0\). We assume \(q = q_0 \varepsilon\) and look for the first terms of the expansion of \(\eta\) in \(\varepsilon\): \(\eta = \eta_0 + \eta_1 \varepsilon + \eta_2 \varepsilon^2 + O(\varepsilon^3)\). The first term of the expansion in \(\varepsilon\) of the function \(f\) defined by (38) gives

\[
(\text{ord. }0) \quad f(\eta) = \tan(R\sqrt{-\eta_0^2}) + \sqrt{-\eta_0^2}/\eta_0 + O(\varepsilon),
\]

Since the order–zero term cannot be made equal to zero, we must skip the first
term in the expansion of $\eta$: $\eta = \eta_1 \varepsilon + O(\varepsilon^2)$. Looking at equation (38) at first order
\[(\text{ord. } 1) \quad f(\eta) = \left( R q_0 - 2\eta_1 \right) \sqrt{\varepsilon} \varepsilon^{3/2} + O(\varepsilon^{5/2}) = 0 \]
we obtain $\eta_1 = \frac{R q_0}{2}$. Pushing the expansion further, one can obtain the following order coefficients. At order two we have:
\[(\text{ord. } 2) \quad f(\eta) = -\left( \frac{4}{24} R^3 q_0^2 + 2\eta_2 \right) \frac{\varepsilon^{3/2}}{\sqrt{q_0}} + O(\varepsilon^{5/2}). \]
One has then
\[
\eta = \frac{R q_0}{2} \varepsilon - \frac{R^3 q_0^2}{12} \varepsilon^2 + O(\varepsilon^3), \tag{39}
\]
showing that in the limit $\eta$ is of the same order of $q$.

4.2. Small–amplitude random perturbation with $q > 0$

We want now to add some random perturbation to this “box-shaped” initial profile.

Let $U_0 = (q + \sigma W_x) 1_{[0,R]}(x)$ be the initial condition of the KdV equation, where $W$ is a standard Wiener process. Since solitons correspond to zeros of the complex extension of $\alpha$, which in turn must be located on the imaginary axis, we look for bounded solutions of equation (7) for $\zeta = i\eta, \eta \in \mathbb{R}$. The first main result is contained in the following Proposition.

**Proposition 4.1.** In the limit of a small white noise type stochastic perturbation of the initial data, the parameter $\eta$ defining the velocity and amplitude of the generated soliton is perturbed at first order by a small, zero mean, Gaussian random variable, which is given by
\[
\frac{\int_0^R \phi_0(R-x)\phi_0(x)dW_x}{\cos(c_0 R) \left[ 2 + \eta_0 R - \frac{\eta_0^2 R}{c_0^2} \right] + \sin(c_0 R) \left[ 3\eta_0^2 + 2\eta_0^2 R \frac{1}{c_0^2} + \frac{\eta_0^3}{c_0^3} \right]}. \tag{40}
\]
Here, $\phi_0$ is the deterministic solution of (33), given by (36), $c_0 := \sqrt{q^2 - \eta_0^2}$, and $\eta_0$ is the amplitude of the soliton of the unperturbed system.

**Proof.** We need to solve
\[
\begin{aligned}
\frac{d\varphi}{dx} &= \tilde{\varphi} dx \\
\frac{d\tilde{\varphi}}{dx} &= (-q + \eta^2)\varphi dx + \sigma \varphi \circ dW_x
\end{aligned} \tag{41}
\]
for $x \in [0,R]$ with initial conditions $\varphi(0) = 1$, $\tilde{\varphi}(0) = \eta$. An analog of Proposition 3.2 holds, and it provides the uniqueness and regularity results for the solution.
Proposition 4.2. The stochastic differential equation (41) defines a stochastic flow $\Phi_{x}^{\eta,\sigma} = (\varphi_{x}, \tilde{\varphi}_{x})^{T}$ of $C^{1}$-diffeomorphisms, which is $C^{1}$ also in the parameters $\eta, \sigma$.

As we did for the NLS equation, we look at the flow provided by the above Proposition 4.2 and define a function of the flow at the point $x = R$ as $F(\eta, \sigma) := \tilde{\varphi}(R) + \eta \varphi(R)$: this remains a function of the two parameters $(\eta, \sigma)$. To prove Proposition 4.1 we use the implicit function Theorem to show that $F(\eta, \sigma)$ has a unique zero in some open set containing $(\eta_{0}, 0)$, the point corresponding to the deterministic solution. The following Lemma ensures that the hypothesis of the implicit function Theorem are satisfied.

Lemma 4.3. Let $F(\eta, \sigma)$ be the function defined above. Then, for all $\eta_{0}$ corresponding to soliton components of solutions of the deterministic problem, $\partial_{\eta} F(\eta_{0}, 0) \neq 0$.

Proof of Lemma 4.3. For $\eta = \eta_{0}$ and $\sigma = 0$ we have that $\varphi = \varphi_{0}$ and (recall that $c_{0} = \sqrt{q - \eta_{0}^{2}}$)

$$
\partial_{\eta} \varphi(R) = \sin(c_{0}R) \left[ \frac{1 + \eta_{0}R}{c_{0}} + \frac{\eta_{0}^{2}}{c_{0}^{2}} \right] - \frac{R\eta_{0}^{2}}{c_{0}^{2}} \cos(c_{0}R),
$$

$$
\partial_{\eta} \tilde{\varphi}(R) = \cos(c_{0}R) \left[ 1 + R\eta_{0} \right] + \frac{\eta_{0}}{c_{0}} \sin(c_{0}R) \left[ 1 + \eta_{0}R \right] = \left[ 1 + \eta_{0}R \right] \varphi_{0}(R).
$$

Since

$$
\partial_{\eta} F = \partial_{\eta} \tilde{\varphi} + \varphi + \eta \partial_{\eta} \varphi,
$$

$$
\partial_{\eta} F(\eta_{0}, 0) = \cos(c_{0}R) \left[ 2 + \eta_{0}R - \frac{\eta_{0}^{2}R}{c_{0}^{2}} \right] + \sin(c_{0}R) \left[ \frac{3\eta_{0} + 2\eta_{0}^{2}R}{c_{0}} + \frac{\eta_{0}^{3}}{c_{0}^{3}} \right].
$$

The coefficient of the sinus is strictly positive (the coefficient of the cosine has instead at least one zero for $\eta_{0} \in [0, \sqrt{q}]$, since it is positive for $\eta_{0} = 0$ and negative for $\eta_{0} \rightarrow \sqrt{q}$). Therefore, we only need to verify that the following equation

$$
g(R, q, \eta_{0}) := \tan(c_{0}R) + \frac{2 + \eta_{0}R - \frac{\eta_{0}^{2}R}{c_{0}^{2}}}{\frac{3\eta_{0} + 2\eta_{0}^{2}R}{c_{0}} + \frac{\eta_{0}^{3}}{c_{0}^{3}}} = 0 \quad (42)
$$

is not satisfied, knowing that $\eta_{0}$ is a value corresponding to a soliton solution of the deterministic equation. As we have seen, we can either have $\eta_{0} = \sqrt{q/2}$ if condition (37) is satisfied, or else $\eta_{0}$ is given as the solution of equation (38) in $(0, \sqrt{q}) \setminus \{\sqrt{q/2}\}$.

In the first case we have that $c_{0} = \sqrt{q/2}$, so that condition (37) implies that $\cos(c_{0}R) = 0$ and $\sin(c_{0}R) = \pm 1$. Therefore, $\partial_{\eta} F \neq 0$. 

Consider now the second case. We look for points \( \eta \in (0, \sqrt{q}) \setminus \{ \sqrt{q}/2 \} \) such that \( f(\eta) = g(\eta) = 0 \). If such a point exists, then

\[
\frac{2\eta c}{q - 2\eta^2} = \frac{2 + \eta R - \frac{\eta^3 R}{c}}{3\eta^2 + 2\eta^2 R^2 + \eta^4},
\]

which also reads

\[
\frac{2\eta}{q - 2\eta^2} = \frac{(2 + \eta R)c^2 - \eta^3 R}{(3\eta + 2\eta^2 R)c^2 + \eta^4}
\]

or equivalently

\[
\frac{(2\eta^2 + 2\eta^3 R)(q - \eta^2) + 2\eta^4(1 + \eta R) + (2 + \eta R)(q - \eta^2)q - \eta^3 Rq}{(q - 2\eta^2)((3\eta + 2\eta^2 R)(q - \eta^2) + \eta^4)} = 0. \tag{43}
\]

For \( \eta < \sqrt{q}/2 \) the denominator is positive and the numerator

\[
(2\eta^2 + 2\eta^3 R)(q - \eta^2) + 2\eta^4(1 + \eta R) + (2 + \eta R)(q - \eta^2)q - \eta^3 Rq > \eta R(q - \eta^2) - \eta^3 Rq > 0,
\]

so that the fraction (43) is positive and cannot be zero. For \( \sqrt{q}/2 < \eta < \sqrt{q} \) the denominator is negative and the numerator is larger than \( 2\eta^3 R - \eta^3 Rq = \eta^3 R(\eta^2 - q) > 0 \), so that the fraction (43) is negative and cannot be zero. The Lemma is proved.

We return to the proof of Proposition 4.1. Since the above lemma guarantees that \( \partial_{\eta} F(\eta_0, 0) \neq 0 \), we can apply the implicit function Theorem. We have (at \( \eta_0 \))

\[
\left\{ \begin{array}{l}
d\partial_\sigma \varphi = \partial_\sigma \tilde{\varphi} \, dx \\
d\partial_\sigma \tilde{\varphi} = (-q + \eta_0^2)\partial_\sigma \varphi \, dx - (\varphi + \sigma \partial_\sigma \varphi) \circ dW_x.
\end{array} \right.
\]

By Proposition 4.2 the flow \( \Phi_{x > 0}^{(\eta, \sigma)} \) is \( C^1 \) in the parameters, so that its derivative in \( \sigma \) at \( \sigma = 0 \) coincide with the first term of the Taylor expansion, which is the solution of

\[
\left\{ \begin{array}{l}
d\partial_\sigma \varphi = \partial_\sigma \tilde{\varphi} \, dx \\
d\partial_\sigma \tilde{\varphi} = (-q + \eta_0^2)\partial_\sigma \varphi \, dx - \varphi_0 dW_x.
\end{array} \right.
\]

In matricial notation,

\[
d \begin{pmatrix} \partial_\sigma \varphi \\ \partial_\sigma \tilde{\varphi} \end{pmatrix} = d\Phi = \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix} \Phi \, dx \left( \begin{array}{c} 0 \\ \varphi_0 dW_x \end{array} \right),
\]

with

\[
M = \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}, \quad e^{Mx} = \begin{pmatrix} \cos(cx) & \frac{1}{c} \sin(cx) \\ -c \sin(cx) & \cos(cx) \end{pmatrix}.
\]
The solution is
\[ \Phi(x) = \Phi(0) - \int_0^x e^{M(x-y)} \left( \frac{\eta}{c} \phi_0(y) + \varphi_0(y) \right) dW_y. \]

We have
\begin{align*}
\partial_\sigma \varphi(x) &= -\frac{1}{c} \int_0^x \sin \left( c(x-y) \right) \left[ \cos(cy) + \frac{\eta}{c} \sin(cy) \right] dW_y, \\
\partial_\sigma \tilde{\varphi}(x) &= -\int_0^x \cos \left( c(x-y) \right) \left[ \cos(cy) + \frac{\eta}{c} \sin(cy) \right] dW_y.
\end{align*}

Therefore,
\begin{align*}
\partial_\sigma F(\eta_0,0) &= \partial_\sigma \tilde{\varphi}(R) + \eta_0 \partial_\sigma \varphi(R) = -\int_0^R \varphi_0(R-x) \varphi_0(x) dW_x
\end{align*}

and we obtain, at first order, the perturbation of the value of the parameter \( \eta \) defining the velocity and amplitude of the soliton: for every \((q,R,\eta_0)\),
\begin{align*}
\partial_\sigma \eta(\sigma) &= -\frac{\partial_\sigma F}{\partial \eta} F(\eta_0,0) \\
&= \frac{\int_0^R \left[ \cos(c_0(R-x)) + \frac{\eta_0}{c_0} \sin(c_0(R-x)) \right] \left[ \cos(c_0x) + \frac{\eta_0}{c_0} \sin(c_0x) \right] dW_x}{\cos(c_0R) \left[ 2 + \frac{\eta_0}{c_0} \right] + \sin(c_0R) \left[ \frac{3\eta_0 + 2\eta_0^2}{c_0} + \frac{\eta_0^2}{c_0^2} \right]}. 
\end{align*}

The Proposition is proved.

We turn now to consider the case when the stochastic perturbation can result in the creation of a new soliton. As for the NLS equation, this happens only for specific "critical" values of \( q \) (and \( R \)). The result is presented in the next Proposition.

**Proposition 4.4.** If we are at a critical point, which is to say \( \sqrt{q} R = n\pi \) for \( n \in \mathbb{N}_0 \), a small amplitude white noise type stochastic perturbation of the potential may create a new small-amplitude soliton. The condition for the creation of a new soliton is that the zero-mean, Gaussian random variable \((45)\) is positive.

**Proof.** The deterministic background solution is \((c = \sqrt{q} - \eta^2)\)
\[ \varphi_0 = \cos(cx) + \frac{\eta}{c} \sin(cx). \] (44)

We have \( U_0 = q + \sigma \hat{W}_x \); we want to apply the implicit function Theorem to the function \( F(\eta,\sigma) \) defined above, at the point \((0,0)\). At this point
\begin{align*}
\partial_\eta \varphi(R) &= \frac{1}{\sqrt{q}} \sin(\sqrt{q} R) = 0, \\
\partial_\eta \tilde{\varphi}(R) &= \cos(\sqrt{q} R) = \pm 1.
\end{align*}
Therefore
\[ \partial_\eta F(0,0) = \partial_\eta \tilde{\varphi} + \varphi + \eta \partial_\eta \varphi = 2 \cos(\sqrt{q} R) = \pm 2, \]
and the implicit function Theorem is applicable. We also have (again at \( \eta = 0 \))
\[
\begin{cases}
    d\partial_\sigma \varphi = \partial_\sigma \tilde{\varphi} \, dx \\
    d\partial_\sigma \tilde{\varphi} = -q \partial_\sigma \varphi \, dx - (\varphi + \sigma \partial_\sigma \varphi) \circ dW_x
\end{cases}
\]
As seen above, the solution of the above system coincide at \( \sigma = 0 \) with the solution of
\[
\begin{cases}
    d\partial_\sigma \varphi = \partial_\sigma \tilde{\varphi} \, dx \\
    d\partial_\sigma \tilde{\varphi} = -q \partial_\sigma \varphi \, dx - \varphi_0 \circ dW_x
\end{cases}
\]
which is given by (here, \( c = \sqrt{q} \))
\[
\begin{align*}
\partial_\sigma \varphi(x) &= -\frac{1}{\sqrt{q}} \int_0^x \sin \left( \sqrt{q}(x-y) \right) \cos \left( \sqrt{q} y \right) dW_y, \\
\partial_\sigma \tilde{\varphi}(x) &= -\int_0^x \cos \left( \sqrt{q}(x-y) \right) \cos \left( \sqrt{q} y \right) dW_y.
\end{align*}
\]
We have therefore
\[
\partial_\sigma F(0,0) = -\int_0^R \cos \left( \sqrt{q} (R-x) \right) \cos \left( \sqrt{q} x \right) dW_x,
\]
and since \( \cos(\sqrt{q} R) \pm 1 \)
\[
v = \partial_\eta \eta(\sigma) = -\frac{\partial_\eta F(0,0)}{\partial_\eta F} = \frac{\int_0^R \cos \left( \sqrt{q} (R-y) \right) \cos \left( \sqrt{q} y \right) dW_y}{2 \cos(\sqrt{q} R)}
\]
\[
= \frac{1}{2} \int_0^R \cos^2(\sqrt{q} x) dW_x \tag{45}
\]
In this case, a new small–amplitude soliton, corresponding to \( \eta = \sigma v \), is created whenever \( v > 0 \). \( \square \)

4.3. Small–amplitude random perturbations with \( q=0 \)

We analyze now the case in which the initial condition is the pure stochastic perturbation: contrarily to the NLS equation, even this small initial condition can generate a soliton. In this setting there is no (non trivial) solution in the deterministic case. This case is obtained at the critical point \( \sqrt{q} R = 0 \), where the first quiescent "soliton" is created.

**Proposition 4.5.** For \( q = 0 \), a small amplitude stochastic perturbation of the potential may create a new small–amplitude soliton. For a white noise type perturbation \( W_x \), the probability that a new soliton is created is 1/2 and the created soliton corresponds to the value \( \eta = \sigma W_R \).
Proof. Again, we want to use the implicit function Theorem; take the limit of 
the deterministic solution (44) as \( q \to 0 \):

\[
\varphi = 1, \quad \tilde{\varphi} = 0.
\]

We have

\[
\partial_\eta F(0, 0) = \partial_\eta \tilde{\varphi} + \varphi + \eta \partial_\eta \varphi = 1
\]

and \((\eta = 0)\)

\[
\begin{cases}
    d\varphi = \tilde{\varphi} \, dx \\
    d\tilde{\varphi} = -\sigma \varphi \circ dW_x
\end{cases},
\]

\[
\begin{cases}
    d\partial_\sigma \varphi = \partial_\sigma \tilde{\varphi} \, dx \\
    d\partial_\sigma \tilde{\varphi} = -(\varphi + \sigma \partial_\sigma \varphi) \circ dW_x
\end{cases}.
\]

Therefore,

\[
\partial_\sigma F(0, 0) = \partial_\sigma \tilde{\varphi} + \eta \partial_\sigma \varphi = -\int_0^R \varphi \, dW_x = -W_R.
\]

It follows that

\[
\partial_\sigma \eta(\sigma) = -\frac{\partial_\sigma F}{\partial_\eta F}(0, 0) = W_R,
\]

and a single soliton corresponding to \( \eta = \sigma W_R \) is generated whenever \( W_R > 0 \), 
which means with probability \( \frac{1}{2} \). \( \square \)

Remark 4.6. The same result of Proposition 4.5 holds with more general processes: 
if, instead of a white noise, we take \( U_0(x) = \sigma Q_x \) in (33), where \( Q_x \) is a 
generic stochastic process, we have \((\eta = 0)\)

\[
\begin{cases}
    d\partial_\sigma \varphi = \partial_\sigma \tilde{\varphi} \, dx \\
    d\partial_\sigma \tilde{\varphi} = -Q_x(\varphi + \sigma \partial_\sigma \varphi) \, dx
\end{cases},
\]

\[
\partial_\sigma F(0, 0) = \partial_\sigma \tilde{\varphi} + \eta \partial_\sigma \varphi = -\int_0^R Q_x \, dx
\]

and

\[
\partial_\sigma \eta(\sigma) = -\frac{\partial_\sigma F}{\partial_\eta F}(0, 0) = \int_0^R Q_x \, dx.
\]

In this case, a single soliton is created whenever the noise introduced has positive 
mean, and it corresponds to \( \eta = \sigma \int_0^R Q_x \, dx \).
Remark 4.7. The mass of a soliton of the KdV equation is \( M_\eta = \int_\mathbb{R} U(x) \, dx = 4\eta \). Here, we have introduced a perturbation of mass \( M_{U_0} = \sigma \int_\mathbb{R} Q_x \, dx = \eta \). We see therefore that the soliton created has a larger mass that the initial perturbation, implying that the radiative part (going in the direction opposite to the one of the soliton) has absorbed a total mass of \( 3\eta \).

The energy conversion efficiency from a small noise source to a soliton is very poor, since the input energy is of order \( \sigma^2 \) (\( E_{U_0} = \sigma^2 \int_\mathbb{R} Q_x^2 \, dx \)), while the energy of the created soliton is only \( E_\eta = \int_\mathbb{R} U^2(x) \, dx = \frac{16}{3} \eta^3 \sim \sigma^3 \), for a smooth source. For a white noise source, the input energy is infinite, while the energy of the created soliton is finite.

5. Conclusion

For the NLS and KdV equations, the study of soliton emergence from a localized, bounded initial condition perturbed by a wide class of rapidly oscillating random processes can be reduced to the study of a canonical system of SDEs, formally corresponding to the white noise perturbation of the initial condition. The integrated covariance is the only parameter of the perturbation process that influences the limit system of SDEs. From the study of this limit system, one obtains quantitative information on the modification of solitons due to the random perturbation.

For the NLS equation we have a threshold effect: if for the deterministic initial condition the integral \( \int_\mathbb{R} U_0(x) \, dx \) exceeds \( \pi/2 \) at least one soliton is created. In this case, a small amplitude random perturbation of the initial condition results in a small variation in amplitude for the created soliton. However, the speed of the created soliton is not substantially modified, unless the perturbation is complex.

On the other hand, since the KdV equation does not present a threshold phenomenon, a small amplitude random perturbation always results in a small variation in both speed and amplitude of the created soliton, for any non-negative initial condition.

As for "quiescent" soliton, for both NLS and KdV a stochastic perturbation has a positive probability (depending on the type of perturbation used) of creating a new real soliton.
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